Yang–Mills-scalar-matter fields in the quantum Hopf fibration

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Abstract
In this paper we present solutions to the non-commutative geometrical version of the Yang–Mills-scalar-matter theory in the Hopf fibration $S^1 \leftrightarrow S^3 \to S^2$ using the 3D-calculus.

Keywords Fields · Quantum Hopf fibration · Canonical quantum principal connection

Mathematics Subject Classification 32C05

1 Introduction
In Differential Geometry, the Hopf fibration is perhaps maybe one of the most basic and well-established examples of principal bundles. This bundle is particularly important in Physics since its connections can describe magnetic monopoles. Even more, gauge theory allows us to study matter fields in the presence of magnetic monopoles. In Non-Commutative Geometry there is an analog concept known as the quantum Hopf fibration or the $q$-deformed Dirac monopole bundle [1].

This paper aims to show in a concrete example that the theory presented on [12, 13] is non-trivial, i.e., we are going to present a non-commutative geometrical version of magnetic monopoles and its interaction with space-time scalar matter fields. Unlike examples shown in [13, 14] where we used trivial quantum principal bundles; here we will use the quantum Hopf fibration which relates the structure group and the base space in a non–trivial way. The importance of this work lies in the fact it provides a better support of the general theory showing explicit and interesting solutions of the field equations.

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To accomplish our purpose, we will consider the quantum Hopf fibration together with its 3D-calculus, which represent the non-commutative counterpart of the classical differential calculus, but of course, it is possible to use others, for example, the 4D-calculus like on [10]. It is worth remarking (again) that the theory presented here is just an application of the general theory [12, 13] not an ad hoc theory created for this space; and this work follows the research line of M. Durdevich shown on [2–4], among other papers. If the reader wants to check another concrete example of the general theory, we recommend [14].

The paper is organized into five sections. In Sect. 2 we are going to develop our study of the quantum Hopf fibration using the 3D -*first order differential calculus ( -*FODC) to create the differential calculus in the quantum Hopf fibration shown on [4]. This differential calculus arises from the classical differential geometry and like in that case, one can define a particular quantum principal connection (qpc) which will play the role of the canonical principal connection on the Hopf fibration. In Sect. 3 we will talk about the associated quantum vector bundles (associated qvb) as well as all the necessary conditions to work with Yang–Mills fields and space-time scalar matter fields [12, 13]. In Sect. 4 we are going to present solutions to the fields equations in this space as well as the spectrum of the left and right Laplace-de Rham operator for qvbs [13]. The Sect. 5 is about some concluding comments. It is worth mentioning that this paper is not self-contained, so we strongly recommend to read [4, 12, 13] before this paper.

2 The quantum Hopf fibration

Let us take the compact matrix quantum group (cmqg) $SU_q(2)$ (the quantum $SU(2)$ group [17]) with $q \in (-1, 1) - \{0\}$. The dense *-algebra $SU_q(2)$ is generated by two letters \{\(\alpha, \gamma\)\} and they satisfy
\[
\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha \alpha^* + q^2 \gamma \gamma^* = 1,
\]
\[
\gamma \gamma^* = \gamma^* \gamma, \quad q \gamma \alpha = \alpha \gamma, \quad q \gamma^* \alpha = \alpha \gamma^*,
\]
and its *-Hopf algebra structure is given by
\[
\phi(\alpha) = \alpha \otimes \alpha - q \gamma^* \otimes \gamma, \quad \phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,
\]
\[
\epsilon(\alpha) = 1, \quad \epsilon(\gamma) = 0,
\]
\[
\kappa(\alpha) = \alpha^*, \quad \kappa(\alpha^*) = \alpha, \quad \kappa(\gamma) = -q \gamma, \quad \kappa(\gamma^*) = -q^{-1} \gamma^*.
\]

In an abuse of notation, we will identify the group $U(1)$ with the Laurent polynomial algebra, i.e.,
\[
U(1) := \mathbb{C}[z, z^*] = \mathbb{C}[z, z^{-1}]
\]
and in this way, its *-Hopf algebra structure is defined by
\[
\phi'(z) = z \otimes z, \quad \epsilon'(z) = 1, \quad \kappa'(z) = z^*, \quad \kappa'(z^*) = z.
\]
This algebra is commutative and $\kappa'$ is a $\ast$-algebra morphism. By defining the $\ast$-Hopf algebra epimorphism

$$j : SU_q(2) \rightarrow U(1)$$

such that $j(\alpha) = z$, $j(\gamma) = 0$, we can consider

$$SU_q(2)\Phi := (\text{id}_{SU_q(2)} \otimes j) \circ \Phi : SU_q(2) \rightarrow SU_q(2) \otimes U(1).$$

Now we define the quantum 2-sphere as (the quantum space whose $\ast$-algebra of $\mathbb{C}$-valued functions is given by) the $\ast$-subalgebra of $SU_q(2)$

$$(\mathcal{S}_q^2, \cdot, 1, \ast),$$

where

$$\mathcal{S}_q^2 := \{x \in SU_q(2) \mid SU_q(2)\Phi(x) = x \otimes 1\}$$

which can be viewed as the $\ast$-algebra generated by $\{\alpha\alpha^*, \alpha\gamma^*\}$ and it is a quantum sphere in the sense of [11]. In this way, the quantum principal $U(1)$-bundle over $\mathcal{S}_q^2$ given by

$$\zeta_{HF} = (SU_q(2), \mathcal{S}_q^2, SU_q(2)\Phi)$$

is usually called the quantum Hopf fibration [1, 3].

Now to accomplish our purpose we will take the differential calculus on $\zeta_{HF}$ shown [4]; however, we are going to use the notation presented on [12, 13]. The bicovariant $\ast$-FODC of $U(1)$ will be denoted by $(\Gamma, d)$ and we shall take

$$\beta' := \{\zeta := \pi'(z - z^*)\}$$

as a Hamel basis of $\text{inv} \Gamma$. In this case, the quantum germs map and the adjoint corepresentation will be denoted by $\pi' : U(1) \rightarrow \text{inv} \Gamma$ and $\text{ad}' : \text{inv} \Gamma \rightarrow \text{inv} \Gamma \otimes U(1)$, respectively. For this space, the universal differential envelope $\ast$-calculus of $(\Gamma, d)$ [2], $(\Gamma^\wedge, d, \ast)$, satisfies

$$\Gamma^\wedge k = \{0\} \text{ for } k \geq 2;$$

however, it differs from the classical differential calculus on $U(1)$ [4].

**Definition 1** With the previous differential calculus, the linear map

$$\omega^c : \text{inv} \Gamma \rightarrow \mathcal{O}^1(SU_q(2))$$

$$\theta \mapsto 1 \otimes \theta$$

is a real, regular and multiplicative quantum principal connection (qpc) and it is called the canonical quantum principal connection in $\zeta_{HF}$. 

\[\text{Birkhäuser}\]
Proposition 2 The only regular qpc is $\omega^c$.

Proof According to the general theory of qpcs we know that every regular qpc has to have the form\cite{15}

$$\omega^c + \lambda$$

such that $\varphi \lambda(\theta) = (-1)^k \lambda(\theta \circ \kappa^{-1}((\varphi)^{(1)}))\varphi(0)$ for all $\varphi \in \text{Hor}^k SU_q(2)$, $\theta \in \text{inv } \Gamma$ with $\mathcal{H}(\varphi) = \varphi(0) \otimes \varphi(1)$ (in Sweedler’s notation). We are going to prove that $\lambda = 0$. A direct calculation shows that

$$\mathcal{H}(\lambda(\varsigma)) = (\mathcal{H} \otimes \text{id}_{U(1)})ad' \lambda(\varsigma) \iff \lambda(\varsigma) \in \Omega^1(\mathfrak{g}_q^2);$$

so $\lambda(\varsigma) = x\eta_+ + y\eta_-$ with $x = \sum_{m+k-l=2} \lambda_{mkl} \alpha^m \gamma^k \gamma^{*l}$, $y = \sum_{p+q-r=-2} \mu_{pqr} \alpha^p \gamma^q \gamma^{*r}$, $\lambda_{mkl}$, $\mu_{pqr} \in \mathbb{C}$ (these elements form a Hamel basis). Due to the fact that $\lambda$ has to satisfies $\lambda(\varsigma) \alpha = \alpha \lambda(\varsigma \circ z) = q^{-2} \alpha \lambda(\varsigma)$, we get

$$x\eta_- \alpha = q^{-2} \alpha x\eta_+ \implies x = \sum_{m+k-l=2} \lambda_{mkl} \alpha^m \gamma^k \gamma^{*l} \text{ with } k + l = 1, \ m \geq 0.$$

Applying the same process to $\gamma$ we find that $m = -1$ which is a contradiction, so $x = 0$. A similar calculation shows $y = 0$ and hence $\lambda = 0$. \hfill $\square$

A quick calculation shows that the operator $D$ presented on \cite{4} is the covariant derivative of $\omega^c$ and its curvature fulfills

$$R^{\omega^c}(\varsigma) = (1 + q^2)q \eta_- \eta_+.$$  

(3)

It is worth mentioning that for the form of the differential calculus, the only possible embedded differential \cite{3, 15} is $\delta = 0$ and for $q = 1$, $\omega^c$ is the principal connection associated to the Levi-Civita connection.

3 Associated quantum vector bundles and the quantum Hodge operator

In accordance with the general theory of associated qvbs, we need to check that Equations 34 and 35 of \cite{12} hold.

It is well-known that a complete set of mutually inequivalent irreducible unitary finite dimensional (smooth) $U(1)$-corepresentations $\mathcal{T}$ is in bijection with $\mathbb{Z}$. These corepresentations are unitary with the canonical inner product of $\mathbb{C}$ and it is worth mentioning that for all $n \in \mathbb{Z}$, the matrix of the canonical corepresentation morphism between the corepresentation and its double contragradient is the identity matrix (see Equation 35 of \cite{12}).

Proposition 3 For every $n \in \mathbb{Z}$, Equations 34 and 35 of \cite{12} are satisfied.
**Proof** By taking

\[ T^{\text{triv}} : \mathbb{C} \longrightarrow \mathcal{S}^2_q \]

\[ w \longmapsto w \mathbb{1} \]

it follows that the statement is true for \( n = 0 \). Now let us take \( n \in \mathbb{N} \) and consider the linear maps

\[ T^n_{k+1} : \mathbb{C} \longrightarrow \text{SU}_q(2) \]

defined by

\[ T^n_{k+1}(1) = \left[ \begin{array}{c} n \\ k \end{array} \right]_{q^{-2}} \alpha^{n-k} \gamma^k =: x^n_{k+1} \]

with \( k = 0, \ldots, n \), where \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{q^{-2}} \) is the Gaussian binomial coefficient also known as the \( q \)-binomial coefficient [6]. Due to the fact that \( \text{SU}_q(2) \Phi(\alpha) = \alpha \otimes z \), \( \text{SU}_q(2) \Phi(\gamma) = \gamma \otimes z \) we get

\[ T^n_k \in \text{Mor}(n, \text{SU}_q(2) \Phi) \]

According to [6], these elements form the first column of the \( \text{SU}_q \)-representation matrix for \( \text{spin} \ l = \frac{n}{2}, u^l \). Since \( u^l \dagger u^l = \text{Id}_{n+1} \in M_{n+1}(\text{SU}_q(2)) \) (here \( \dagger \) is denoting the transpose conjugate matrix) we get that Equation 34 holds. Taking

\[ Z^n = (q^{2(i-1)}\delta_{ij}) \in M_{n+1}(\mathbb{C}) \]

where \( \delta_{ij} \) is the Kronecker delta, Equation 35 holds since in this case

\[ W^n X^n = \text{Id}_{n^a} \quad \text{with} \quad W^n = (w^n_{ij}) = Z^n X^n, \quad X^n = (x^n_{k+1}) \]

is the (1, 1)-entry of \( u^l u^{l\dagger} = \text{Id}_{n+1} \) [6]. For negative integers \( n \) it is enough to take the last column of \( u^l \) with \( l = \frac{|n|}{2} \) to ensure that the Equation 34 is holds and taking

\[ Z^n = (q^{-2(|n|+1-i)}\delta_{ij}) \in M_{|n|+1}(\mathbb{C}) \]

we get that Equation 35 holds since in this case \( W^n X^n = \text{Id}_{n^a} \) will be the (\( |n| + 1, |n| + 1 \))-entry of \( u^l u^{l\dagger} = \text{Id}_{|n|+1} \) [6].

We have to remark that Equations 36 of [12] can be viewed in terms of Hopf–Galois extension’s theory [5]. However, an advantage of having proven Proposition 3 is that it provides us with the left and right generators of the associated qvbs [12, 13]. In this way, it is possible to take left and right associated quantum vector bundles.
(qvbs) and induced quantum linear connections (qlc) for all \( n \in \mathbb{Z} \). The left/right qvb associated to the corepresentation \( n \) will be denoted by \( \zeta^L_n := (\Gamma^L(\mathfrak{q}^2, \mathbb{C}^n\mathfrak{q}^2), +, \cdot) \), \( \zeta^R_n := (\Gamma^R(\mathfrak{q}^2, \mathbb{C}^n\mathfrak{q}^2), +, \cdot) \), respectively. Moreover, the induced qlc by any \( \omega \) will be denote by \( \nabla^\omega_n, \hat{\nabla}^\omega_n \), respectively.

Now we have to verify if \( \zeta_{HF} \) satisfies all the conditions written in Definition 2.1 and Remark 2.3 of [13]. In order to do this, we need the following lemma.

**Lemma 4** Let us consider the linear functional

\[
\int_{\mathfrak{q}^2} : \Omega^2(\mathfrak{q}^2) \longrightarrow \mathbb{C},
\]

\[
p \eta_- \eta_+ \longmapsto h_q(p),
\]

where \( h_q \) is the quantum Haar measure of \( SU_q \) [16]. Then

\[
d(\Omega^1(\mathfrak{q}^2)) \subseteq \text{Ker}\left(\int_{\mathfrak{q}^2}\right).
\]

**Proof** Let us start by remembering the definition of \( D \) for degree zero:

\[
D(a) = a^{(0)}(\pi_-(a^{(1)}) + \pi_+(a^{(1)})),
\]

where \( \pi_\pm := \rho_\pm \circ \pi \) with \( \pi : SU_q(2) \longrightarrow \text{inv}\mathfrak{G} \) the quantum germs map and \( \rho_\pm : \text{inv}\mathfrak{G} \longrightarrow \mathbb{C}\eta_\pm \) the canonical projection. In this way, we define the linear functional

\[
\lambda_- : SU_q(2) \longrightarrow \mathbb{C}
\]

such that \( \pi_-(a) = \lambda_-(a)\eta_- \). Notice that \( 1 \in \text{Ker}(\lambda_-) \).

Consider \( y\eta_+ \in \Omega^1(\mathfrak{q}^2) \). Hence

\[
\int_{\mathfrak{q}^2} d(y\eta_+) = h_q(y^{(1)})\lambda_-(y^{(2)}) = \lambda_-(h_q(y^{(1)})y^{(2)}) = \lambda_-(h_q \ast y) = \lambda_-(h_q(y)1) = \lambda_-(1)h_q(y) = 0.
\]

In an analogous way it can be proved that

\[
\int_{\mathfrak{q}^2} d(x\eta_-) = 0
\]

and therefore the lemma follows. ∎

**Proposition 5** The quantum 2-sphere satisfies all the conditions written in Definition 2.1 and Remark 2.3 of [13] with respect to the graded differential \( \ast \)-algebra of base forms.
Proof First of all let us observe that $\mathbb{S}^2_q$ is (obviously) $C^*$-closeable.

1. $\mathbb{S}^2_q$ is oriented since for $k > 2$, $\mathcal{O}^k(\mathbb{S}^2_q) = 0$ and

$$
dvol := \eta_- \eta_+
$$

is a quantum 2-volume form.

2. A direct calculation shows that a lqrm can be defined on $\mathbb{S}^2_q$ by means of

$$
\langle - , - \rangle : \mathbb{S}^2_q \times \mathbb{S}^2_q \longrightarrow \mathbb{S}^2_q
$$

$$( \hat{p}, \hat{p}^* ) \longmapsto \hat{p} \hat{p}^* ,$$

$$
\langle - , - \rangle : \mathcal{O}^1(\mathbb{S}^2_q) \times \mathcal{O}^1(\mathbb{S}^2_q) \longrightarrow \mathbb{S}^2_q
$$

$$
((\hat{x} \eta_- + \hat{y} \eta_+), (x \eta_- + y \eta_+)) \longmapsto \frac{1}{2} \left( q^2 \hat{x} x^* + \hat{y} y^* \right)
$$

and finally

$$
\langle - , - \rangle : \mathcal{O}^2(\mathbb{S}^2_q) \times \mathcal{O}^2(\mathbb{S}^2_q) \longrightarrow \mathbb{S}^2_q
$$

$$( \hat{p} \text{ dvol}, \text{ p dvol} ) \longmapsto \hat{p} \hat{p}^* .
$$

With this lqrm, dvol is actually a lqr 2-form. Taking into account Remark 2.2, we get a rqrm with a rqr 2-form.

3. According to [16], $h_q$ is a faithful state on $\text{SU}_q(2)$ and hence the linear functional of Eq. 4 is actually a quantum integral. In this way, by the previous lemma we conclude that $(\mathbb{S}^2_q, \cdot, \frac{1}{2}, \ast)$ is a quantum space without boundary.

4. A direct calculation shows

$$\ast_L p = p^* \text{ dvol}$$

for all $p \in \mathbb{S}^2_q$;

$$\ast_L (p \text{ dvol}) = p^*$$

for all $p \text{ dvol} \in \mathcal{O}^2(\mathbb{S}^2_q)$ and finally

$$\ast_L \mu = \frac{1}{2} \left( - y^* \eta_- + x^* \eta_+ \right),$$

for all $\mu = x \eta_- + y \eta_+ \in \mathcal{O}^1(\mathbb{S}^2_q)$. To define $\ast_R$ we can use Remark 2.2 of [13].

$\Box$

It is worth mentioning that for this differential calculus, the only possible embedded differential is $\delta = 0$ [3, 13].
4 Yang–Mills-scalar-matter fields

In this section we are going to show solutions of the field equations of the Yang–Mills-scalar-matter theory \([13]\) using \((\zeta_{HF}, \omega^c)\) and all the structures that we have just defined.

4.1 Non-commutative geometrical Yang–Mills fields

We know that every single qpc \(\omega\) has the form \([3]\)

\[
\omega = \omega^c + \lambda \quad \text{with} \quad \lambda(\zeta) = x\eta_+ + y\eta_+ \in \Omega^1(\mathbb{S}^2_q).
\]

**Proposition 6** Every YM qpc is of the form \(\omega^c + \lambda\), where \(\lambda(\zeta) = dp\) for some \(p \in \mathbb{S}^2_q\).

**Proof** First, notice that for all qpc \(\omega = \omega^c + \lambda\) (see Eq. 3)

\[
R^{\alpha}(\zeta) = (1 + q^2)q\eta_-\eta_+ + d\lambda(\zeta);
\]

so

\[
\left.\frac{\partial}{\partial z}\right|_{z=0} \mathcal{S}_{YM}(\omega + z\lambda') = -\frac{1}{4} \left( (\lambda'(\zeta) | d^* R^{\alpha}(\zeta))_L + (\lambda'(\zeta)^* | d^R R^{\alpha}(\zeta)^*)_R \right)
\]

\[
= -\frac{1}{4} \left( (d\lambda'(\zeta) | R^{\alpha}(\zeta))_L + (d\lambda'(\zeta)^* | R^{\alpha}(\zeta)^*)_R \right)
\]

\[
= -\frac{1}{4} \left( (d\lambda'(\zeta) | d\lambda(\zeta))_L + (d\lambda'(\zeta)^* | d\lambda(\zeta)^*)_R \right)
\]

\[
= -\frac{1}{2} (d\lambda'(\zeta) | d\lambda(\zeta))_L.
\]

Since \((− | −)_L\) is an inner product we conclude that every YM qpc has the form \(\omega^c + \lambda\) with \(d\lambda(\zeta) = 0\).

In accordance with \([17]\), the zero cohomology group of SU\(_q\)(2) is \(\mathbb{C}\); while the first cohomology group is \([0]\). Hence, since \(\lambda(\zeta) \in \Omega^1(\mathbb{S}^2_q)\) is exact, there exists \(p \in \mathbb{S}^2_q\) such that \(\lambda(\zeta) = dp\).

In this case, the quantum gauge group (qgg) of the Lagrangian \([12]\) satisfies

\[q\mathcal{G}_YM := \{ f \in q\mathcal{G} | f^\circ \omega^c = \omega^c + \lambda \text{ with } d\lambda = 0\}.\]

We can deduce that U(1) \(\subseteq q\mathcal{G}_YM\) and all YM qpcs are in the same orbit, just like in the classical case.
4.2 Non-commutative geometrical n-multiple of space-time scalar matter fields

According to [13], it is enough to look for eigenvectors of the left quantum Laplace–de Rham operator $\triangle L$ with $\mathbb{S}_q^2$-valued eigenvalues. A direct calculation shows that

$$d^*d p = \frac{1}{2}(1 + q^2)^2 p \quad \text{with} \quad p = \mathbb{1} - (1 + q^2)\gamma \gamma^*, \quad \alpha \gamma^*, \quad \alpha^* \gamma.$$

It is important to mention that for $q \in (-1, 1) - \{0\}$, these eigenvalues are not 0. In this way, taking a potential such that

$$V' = \frac{1}{2}(1 + q^2)^2$$

it is easy to find non-commutative geometrical space-time scalar matter fields. Of course, there are more solutions but they depend on the form of the potential $V$.

4.3 Non-commutative geometrical Yang–Mills-scalar-matter equations

Let us take $n \in \mathbb{Z}$. If $n = 0$, YMSM fields are triplets $(\omega, T_1, T_2)$ where $\omega$ is a YM qpc and $(T_1, T_2)$ is an stationary point of $\mathcal{S}_\text{SM}$.

Consider now $n \neq 0$. It is easy to see that

$$d^* L R^\omega(\zeta) = 0,$$

so we have to look for $T_1 \in \Gamma^L(\mathbb{S}_q^2, \mathbb{C}_n \mathbb{S}_q^2), \quad T_2 \in \Gamma^R(\mathbb{S}_q^2, \mathbb{C}_{-n} \mathbb{S}_q^2)$ such that

$$\langle \nabla_n^\omega(\gamma T_1) | \nabla_{-n}^\omega(\gamma T_2) \rangle_L - \langle \nabla_{-n}^\omega(\gamma T_2) | \nabla_n^\omega(\gamma T_1) \rangle_R = 0 \quad (7)$$

for all $\lambda \in \mathcal{qpc}(\zeta_{HF}),$ and

$$\nabla_n^\omega(\gamma T_1) - V'_L(T_1)^* T_1 = 0, \quad \nabla_{-n}^\omega(\gamma T_2) - T_2 V_R(T_2)^* = 0. \quad (8)$$

Now it is possible to explicitly find solutions. For example, for $n > 0$ the triplet $(\omega^\mathcal{N}, T_1, T_2)$ such that

$$T_1(1) = \alpha^n, \quad T_2(1) = \alpha^* n \quad \text{or} \quad T_1(1) = \gamma^n, \quad T_2(1) = \gamma^* n$$

is a YMSM field for a potential such that

$$V' = \frac{1}{2} \left( q^4(1 - q^{2n}) \frac{1}{1 - q^2} \right).$$

It is worth mentioning that $q \longrightarrow 1$ implies $V' \longrightarrow n/2$, so we recover the winding number $n$. Of course, there are more solutions; however, they depend on the form of the potential $V$. 
The spectrums of
\[ \nabla_n^{\omega^*} \nabla_n^{\omega^*} : \Gamma^L(\mathbb{S}_q^2, \mathbb{C}_n \mathbb{S}_q^2) \rightarrow \Gamma^L(\mathbb{S}_q^2, \mathbb{C}_n \mathbb{S}_q^2) \]
and
\[ \hat{\nabla}_n^{\omega^*} \hat{\nabla}_n^{\omega^*} : \Gamma^R(\mathbb{S}_q^2, \mathbb{C}_n \mathbb{S}_q^2) \rightarrow \Gamma^R(\mathbb{S}_q^2, \mathbb{C}_n \mathbb{S}_q^2) \]
for all \( n \in \mathbb{Z} \) are shown in the following tables. In the second row of the Table 1 and the fifth row of the Table 2, \( m, k \in \mathbb{N}_0 \) (in the other cases, \( m, k, l \in \mathbb{N} \)) and they cannot be both 0 at the same time. On the other hand, \( p(\gamma^k \gamma^* l), \hat{p}(\gamma^k \gamma^* l) \) are polynomials with coefficients in \( \mathbb{C} \) such that their terms are \( \gamma^k \gamma^* l, \gamma^{k-1} \gamma^* l^{-1} \), etc. Until \( \gamma \) or \( \gamma^* \) disappear. For example
\[ p(\gamma \gamma^*) = \hat{p}(\gamma \gamma^*) = 1 - (1 + q^2)\gamma \gamma^*. \]

Polynomials \( p(\alpha^m \gamma^k \gamma^* l), p(\alpha^m \gamma^k \gamma^* l), \hat{p}(\alpha^m \gamma^k \gamma^* l), \hat{p}(\alpha^m \gamma^k \gamma^* l) \) follow an analogous rule. For example
\[ p(\alpha \gamma \gamma^*) = - \frac{(q^6 + 3q^4 + 2q^2 + 1)(q^2 + q^4)}{q^6 + 2q^4 + 2q^2 + 1} \alpha + (q^6 + 3q^4 + 2q^2 + 1)\alpha \gamma \gamma^* \]
and
\[ \hat{p}(\alpha \gamma \gamma^*) = - \frac{(q^4 + 2q^2 + q^{-2} + 3)(1 + q^2)}{q^4 + 2q^2 + q^{-2} + 2} \alpha + (q^4 + 2q^2 + q^{-2} + 3)\alpha \gamma \gamma^*. \]

In addition, let us define the \textit{the} \( q^2 \)-\textit{number}
\[ [r] := [r]_{q^2} = \frac{1 - q^{2r}}{1 - q^2} \]
for all \( r \in \mathbb{N} \). Then let us take
\[ \lambda_{m,k,l} := \frac{1}{2} \left( [m] [l + 1] q^{2(2-l)} + [k] [l + 1] q^{4+2m-2l} + [l] [m + 1] q^{2(1-l)} + [l] [k] q^{4+2m-2l} \right), \]
\[ \lambda_{-m,k,l} := \frac{1}{2} \left( [m] [k + 1] q^{2(1-m)} + [l] [k + 1] q^{2-2m-2l} + [k] [m + 1] q^{2(2-m)} + [l] [k] q^{4-2m-2l} \right), \]
\[ \hat{\lambda}_{m,k,l} := \frac{1}{2} \left( [m] [l + 1] q^{2-2m-2k} + [k] [l + 1] q^{2(1-k)} + [l] [m + 1] q^{4-2m-2k} + [l] [k] q^{2(3-k)} \right). \]
and
\[
\hat{\lambda}_{-m,k,l} := \frac{1}{2} \left( [m][k + 1] q^{4-2k+2l} + [l][k + 1] q^{2(2-k)} + [k][m + 1] q^{2-2k+2l} + [k][l] q^{2(1-k)} \right).
\]

Values of the first columns form linear basis of SU\(_q(2)\), thus for each \( n \in \mathbb{Z} \), these sections form a basis of eigenvectors.

**Proposition 7** Considering \( \text{Mor}(n, SU_q(2))\Phi) = \Gamma^L(\frac{\mathbb{S}^2_q}{\mathbb{C}_n \mathbb{S}^2_q}) = \Gamma^R(\frac{\mathbb{S}^2_q}{\mathbb{C}_n \mathbb{S}^2_q}) \) just as a vector space, the operators \( \nabla_n^{\omega^c} \star L_n \nabla_n^{\omega^c} \) and \( \hat{\nabla}_n^{\omega^c} \star R_n \hat{\nabla}_n^{\omega^c} \) are not simultaneously diagonalizable for each \( n \in \mathbb{Z} \).

**Proof** We are going to prove that these operators do not commute each other. In fact
\[
(\hat{\nabla}_n^{\omega^c} \star R_n \hat{\nabla}_n^{\omega^c}) (\nabla_n^{\omega^c} \star L_n \nabla_n^{\omega^c}) T \neq (\nabla_n^{\omega^c} \star L_n \nabla_n^{\omega^c}) (\hat{\nabla}_n^{\omega^c} \star R_n \hat{\nabla}_n^{\omega^c}) T,
\]
where \( T(1) = \alpha^n \gamma \gamma^* \) for \( n > 0 \); \( T(1) = \alpha^* n \gamma \gamma^* \) for \( n < 0 \) and \( T(1) = \alpha \gamma \gamma^* \) for \( n = 0 \).

Furthermore
\[
\hat{\nabla}_n^{\omega^c} \star R_n \hat{\nabla}_n^{\omega^c} \neq \ast \circ \nabla_n^{\omega^c} \star L_n \nabla_n^{\omega^c} \circ \ast. \quad (9)
\]

In accordance with [13], the operators \( \nabla_n^{\omega^c} \star L_n \nabla_n^{\omega^c} \) and \( \hat{\nabla}_n^{\omega^c} \star R_n \hat{\nabla}_n^{\omega^c} \) are symmetric and non-negative. Moreover, according to the tables below, the eigenvalues are not symmetric under the change \( n \leftarrow -n \), which is a difference with the classical case [7] and in agreement with the classical case, both operators are not bounded. Finally, by taking the classical limit \( q \rightarrow 1 \), both operators reproduces the spectrum of the Laplacian on associated vector bundles of the Hopf fibration [7].

**5 Concluding comments**

There are a lot of interesting papers about the quantum Hopf fibration and its associated qvbs, as well as a treatment of gauge theory in this space, for example, [5, 8–10, 18]. All of them follow the line of research of Brzezinski and Majid shown on [1]. Unlike all these papers, the work shown here follows the line of research of M. Đurđevich in which we deal with two kind of covariant derivatives for any qpc (both agree in the classical case); this allows us to define induced qlc in left/right associated qvb as well as the Lagrangians and their respective field equations. For example, if we do not consider the right structure, Eq. 7 becomes into
\[
\langle T_n \circ K^L(T_1) | \nabla_n^{\omega^c} T_1 \rangle_L = 0,
\]
which does not have no-trivial solutions for an arbitrary \( n \). Furthermore, the operators \( \nabla_n^{\omega^c \cdot k} \), \( \nabla_n^{\omega^c} \), \( \nabla_n^{\omega^c \cdot k} \), \( \nabla_n^{\omega^c} \) are not the same (see Eq. 9), they do not even commute between them!

The general theory is, at first sight, too restrictive in the sense that too many conditions are necessary; however this example and the others developed show that these conditions are (relatively) easy to satisfy. This theory can be applied perfectly to other spaces; there are a lot of illustrative and rich examples to study.

In terms of a physical interpretation, this example models left space–time scalar matter fields and right space–time scalar antimatter fields coupled to a magnetic

Table 1 Values for \( \nabla_n^{\omega^c \cdot k} \nabla_n^{\omega^c} T = \lambda T \)

| \( T(1) \) | \( n \) | \( \lambda \) |
|---|---|---|
| 1 | 0 | 0 |
| \( \alpha^m \gamma^k \) | \( m + k = n \) | \( \frac{[n] q^4}{2} \) |
| \( \alpha^n \gamma^m \) | \( n > 0 \) | \( \frac{[n] q^{2(1-n)}}{2} \) |
| \( \alpha^m \gamma^l \) | \( m + l = n \) | \( \frac{[-n] q^2}{2} \) |
| \( \alpha^m \gamma^l \) | \( m - l = n \) | \( \frac{1}{2} \left( [l][m + 1] q^{2(1-n)} + [m][l + 1] q^{2(2-n)} \right) \) |
| \( p(y^k \gamma^l) \) | \( k - l = n \) | \( \frac{1}{2} \left( [l] q^{2(1-k)} + [k] q^{4} + 2[l][k] q^{2(2-k)} \right) \) |
| \( p(\alpha^m y^k \gamma^l) \) | \( m + k - l = n \) | \( \lambda_{m,k,l} \) |
| \( p(\alpha^m y^k \gamma^l) \) | \( -m + k - l = n \) | \( \lambda_{-m,k,l} \) |

Table 2 Values for \( \tilde{\nabla}_n^{\omega^c \cdot k} \tilde{\nabla}_n^{\omega^c} \tilde{T} = \tilde{\lambda} \tilde{T} \)

| \( \tilde{T}(1) \) | \( n \) | \( \tilde{\lambda} \) |
|---|---|---|
| 1 | 0 | 0 |
| \( \alpha^m \gamma^k \) | \( m + k = n \) | \( \frac{[-n] q^2}{2} \) |
| \( \alpha^n \gamma^m \) | \( n > 0 \) | \( \frac{[n] q^{2(1-n)}}{2} \) |
| \( \alpha^m \gamma^l \) | \( m + l = n \) | \( \frac{[n] q^4}{2} \) |
| \( \alpha^m \gamma^l \) | \( m - l = n \) | \( \frac{1}{2} \left( [m][l + 1] q^{2(1-m)} + [l][m + 1] q^{2(2-m)} \right) \) |
| \( \alpha^m \gamma^k \) | \( -m + k = n \) | \( \frac{1}{2} \left( [k][m + 1] q^{2(1-k)} + [m][k + 1] q^{2(2-k)} \right) \) |
| \( \tilde{p}(y^k \gamma^l) \) | \( k - l = n \) | \( \frac{1}{2} \left( [l] q^{2(2-k)} + [k] q^{2(1-n)} + [l][k] (1 + q^4) q^{2(1-k)} \right) \) |
| \( \tilde{p}(\alpha^m y^k \gamma^l) \) | \( m + k - l = n \) | \( \tilde{\lambda}_{m,k,l} \) |
| \( \tilde{p}(\alpha^m y^k \gamma^l) \) | \( -m + k - l = n \) | \( \tilde{\lambda}_{-m,k,l} \) |
monopole in the quantum sphere. Since the spectrums of $\nabla_n^\omega \star L_n \nabla_n^\omega$ and $\hat{\nabla}_n^\omega \star R_n \hat{\nabla}_n^\omega$ are discrete, the eigenvalues could be interpreted as quantum numbers.

This theory works for any qpc $\omega$ [13], for example, there is not necessary to assume that $\omega$ is strong [5], which is a big difference with almost all references in literature; although to develop this paper we worked with $\omega^c$ just to show that even in well-studied cases [8] our theory gets more information about the quantum spaces. It is worth mentioning that by considering all the fixed elements of [8] instead of ours (for example $\omega_{\pm}$ instead of $\eta_{\pm}$), the operator $\nabla_n^\omega \star L_n \nabla_n^\omega$ is exactly the gauge Laplacian operator shown on [8] and all its results can be reproduced; however (and as we mentioned before) in this paper we also deal with $\hat{\nabla}_n^\omega \star R_n \hat{\nabla}_n^\omega$ and with the Yang–Mills part as a functional on the space of (quantum) principal connections, just like in the classical case.

**Declarations**

**Conflict of interest** There is no financial interest.

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