THE PARTITION RANK OF A TENSOR AND k-RIGHT CORNERS
IN $\mathbb{F}_q^n$

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ABSTRACT. Following the breakthrough of Croot, Lev, and Pach [3], Tao [10] introduced a symmetrized version of their argument, which is now known as the slice rank method. In this paper, we introduce a more general version of the slice rank of a tensor, which we call the Partition Rank. This allows us to extend the slice rank method to problems that require the variables to be distinct. Using the partition rank, we generalize a recent result of Ge and Shangguan [5], and prove that any set $A \subset \mathbb{F}_q^n$ of size

$$|A| > (k + 1) \cdot \left( \frac{n + (k - 1)q}{(k - 1)(q - 1)} \right),$$

contains a $k$-right-corner, that is distinct vectors $x_1, \ldots, x_k, x_{k+1}$ where $x_1 - x_{k+1}, \ldots, x_k - x_{k+1}$ are mutually orthogonal, for $q = p^r$, a prime power with $p > k$.

1. Introduction

In the spring of 2016, Croot, Lev and Pach [3] proved a breakthrough result on progression-free subsets of $(\mathbb{Z}/4\mathbb{Z})^n$. They proved that if $A \subset (\mathbb{Z}/4\mathbb{Z})^n$ contains no non-trivial three term arithmetic progression, then

$$|A| \leq 3.60172^n.$$

Ellenberg and Gijswijt [4] used Croot Lev and Pach’s method to prove that any progression-free subset $A \subset (\mathbb{Z}/p\mathbb{Z})^n$, where $p$ is a prime, satisfies

$$|A| \leq (J(p)p)^n,$$

where

$$J(p) = \frac{1}{p} \min_{0 < t < 1} \frac{1 - tp}{(1-t)^{p-1}},$$

is an explicit constant less than 1. This was extended to $(\mathbb{Z}/k\mathbb{Z})^n$ for any integer $k$ in [2]. Following the work of Ellenberg and Gijswijt, Tao, in his blog [10], symmetrized Croot, Lev and Pach’s argument, introducing the notion of the slice rank of a tensor. The slice rank method has seen numerous applications to a variety of problems, such as the sunflower problem [7], right angles in $\mathbb{F}_p^n$ [5], and approaches to fast matrix multiplication [2], and we refer the reader to [2, Section 4] for an in depth discussion of the slice rank as well as its connection to geometric invariant theory.

In this paper, we introduce the partition rank of a tensor, which generalizes the slice rank, and allows us to handle problems that require the variables to be distinct. Using this new notion of rank, we generalize the work of Ge and Shangguan [5] to $k$-right corners in $\mathbb{F}_q^n$. In a different direction, in [6], we use the partition rank to provide an exponential improvement to bounds for the Erdos-Ginzburg Ziv constant of $(\mathbb{Z}/p\mathbb{Z})^n$, which are the first non-trivial upper bounds for high rank abelian groups.

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Let \( q \) be an odd prime power. A right angle in \( \mathbb{F}_q^n \) is a triple \( x, y, z \in \mathbb{F}_q^n \) of distinct elements satisfying

\[
\langle x - y, x - z \rangle = 0.
\]

Bennett [1] proved that any subset \( A \subset \mathbb{F}_q^n \) of size

\[
|A| \gg q^{n+2}
\]

contains a right angle. Ge and Shangguan [5] used the slice rank method to improve this result for very large \( n \), showing that any set \( A \subset \mathbb{F}_q^n \) of size

\[
|A| > \left( \frac{n + q}{q - 1} \right) + 3
\]

contains a right angle. In section 4, we improve Ge and Shangguan’s bound, and prove the following theorem:

**Theorem 1.** Let \( q \) be an odd prime power. If \( A \subset \mathbb{F}_q^n \) satisfies

\[
|A| > 2 \left( \frac{n + q - 1}{q - 1} \right) + 2 \left( \frac{n + \frac{q-1}{2}}{\frac{q-1}{2}} \right) + 2
\]

then \( |A| \) contains a right angle.

Generalizing the notion of a right angle in \( \mathbb{F}_q^n \), we say that the vectors \( x_1, \ldots, x_k, x_{k+1} \) form a \( k \)-right corner if they are distinct, and if the \( k \) vectors \( x_1 - x_{k+1}, \ldots, x_k - x_{k+1} \) form a mutually orthogonal \( k \)-tuple. In other words, \( x_1, \ldots, x_k \) must meet \( x_{k+1} \) at mutually right angles. Our main result is a bound, polynomial in \( n \), for the size of the largest subset of \( \mathbb{F}_q^n \) that does not contain a \( k \)-right corner.

**Theorem 2.** Let \( k \) be given, and let \( q = p^r \) with \( p > k \). If \( A \subset \mathbb{F}_q^n \) has size

\[
|A| > (k + 1) \cdot \left( \frac{n + (k - 1)q}{(k - 1)(q - 1)} \right),
\]

then \( A \) contains a \( k \)-right corner.

The proof of theorem 2 relies on the flexibility of the partition rank for \( k \geq 3 \), as the tensors involved will have polynomially small partition rank yet exponentially large slice rank.

In subsection 2.1, we review the slice rank, as defined in [10] and [2, Section 4.1], and in subsection 2.2 we introduce the partition rank and prove the critical lemma, which states that the partition rank of a diagonal tensor is equal to the number of non-zero diagonal entries. In section 3, we introduce the indicator function \( H_k(x_1, \ldots, x_k) \), which satisfies

\[
H_k(x_1, \ldots, x_k) = \begin{cases} 
1 & \text{if } x_1, \ldots, x_k \text{ are distinct} \\
(-1)^{k-1}(k - 1)! & \text{if } x_1 = \cdots = x_k \\
0 & \text{otherwise}
\end{cases}
\]

For \( k \geq 4 \), \( H_k \) will have low partition rank, but large slice rank. This function allows us to modify our tensor so that it picks up only \( k \)-tuples of distinct elements. In section 5, we use the partition rank and the indicator function \( H_k \) to prove theorem 2.
2. The Slice Rank and the Partition Rank

2.1. The Slice Rank. We begin by recalling the definition of the rank of a two variable function. Let \( X, Y \) be finite sets, and suppose that \( F \) is a field. The rank of

\[ F : X \times Y \to F \]

is defined to be the smallest \( r \) such that

\[ F(x, y) = \sum_{i=1}^{r} f_i(x)g_i(y) \]

for some functions \( f_i, g_i \). The function \( F \) is given by an \( |X| \times |Y| \) matrix with entries in \( F \), and the products \( f_i(x)g_i(y) \) correspond to the outer products of vectors. For finite sets \( X_1, \ldots, X_n \), a function

\[ h : X_1 \times \cdots \times X_n \to F \]

of the form

\[ h(x_1, \ldots, x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n) \]

is called a rank 1 function, and the tensor rank of

\[ F : X_1 \times \cdots \times X_n \to F \]

is defined to be the minimal \( r \) such that

\[ F = \sum_{i=1}^{r} g_i \]

where the \( g_i \) are rank 1 functions. Following the breakthrough of Croot, Lev and Pach [3], Tao in his blog [10] introduced the notion of the slice rank of a tensor:

**Definition 3.** Let \( X_1, \ldots, X_n \) be finite sets. We say that the function

\[ h : X_1 \times \cdots \times X_n \to F \]

has slice rank \( 1 \) if

\[ h(x_1, \ldots, x_n) = f(x_i)g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \]

for some \( 1 \leq i \leq n \). The slice rank of

\[ F : X_1 \times \cdots \times X_n \to F \]

is the smallest \( r \) such that

\[ F = \sum_{i=1}^{r} g_i \]

where the \( g_i \) have slice rank 1.

The following lemma was proven by Tao, and used to great effect:

**Lemma 4.** Let \( X \) be a finite set, and let \( X^n \) denote the \( n \)-fold Cartesian product of \( X \) with itself. Suppose that

\[ F : X^n \to F \]

is a diagonal tensor, that is

\[ F(x_1, \ldots, x_n) = \sum_{a \in A} c_a \delta_a(x_1) \cdots \delta_a(x_n) \]
for some $A \subset X$, $c_a \neq 0$, where

$$
\delta_a(x) = \begin{cases} 
1 & x = a \\
0 & \text{otherwise}
\end{cases}.
$$

Then

$$\text{slice-rank}(F) = |A|.$$

Proof. We refer the reader to [10, Lemma 1] or [2, Lemma 4.7].

2.2. The Partition Rank. We introduce a new more general definition of the slice rank, that we call the partition rank. Given variables $x_1, \ldots, x_n$ and a set $S \subset \{1, \ldots, n\}$, $S = \{s_1, \ldots, s_k\}$, we use the notation $\vec{x}_S$ to denote the subset of variables $x_{s_1}, \ldots, x_{s_k}$, and so for a function $g$ of $k$ variables, we have that

$$g(\vec{x}_S) = g(x_{s_1}, \ldots, x_{s_k}).$$

For example, if $S = \{1, 3, 4\}$ and $T = \{2, 5\}$ then

$$g(\vec{x}_S) f(\vec{x}_T) = g(x_1, x_3, x_4)f(x_2, x_5).$$

A partition of $\{1, 2, \ldots, n\}$ is a collection $P$ of non-empty, pairwise disjoint, subsets of $\{1, \ldots, n\}$, satisfying

$$\bigcup_{A \in P} A = \{1, \ldots, n\}.$$ We say that $P$ is the trivial partition if it consists only of a single set, $\{1, \ldots, n\}$.

Definition 5. Let $X_1, \ldots, X_n$ be finite sets, and suppose that

$$h : X_1 \times \cdots \times X_n \to \mathbb{F}.$$ If there exists some non-trivial partition $P$ such that

$$h(x_1, \ldots, x_n) = \prod_{A \in P} f_A(\vec{x}_A)$$

for some functions $f_A$, then $h$ is said to have partition rank 1.

Equivalently, the tensor $h : X_1 \times \cdots \times X_n \to \mathbb{F}$ has partition rank 1 if the variables can be split into disjoint non-empty sets $S_1, \ldots, S_t$, with $t \geq 2$, such that

$$S_1 \cup \cdots \cup S_t = \{1, 2, \ldots, n\}$$

and

$$h(x_1, \ldots, x_n) = f_1(\vec{x}_{S_1})f_2(\vec{x}_{S_2}) \cdots f_n(\vec{x}_{S_t})$$

for some functions $f_1, \ldots, f_t$. In particular, $h : X_1 \times \cdots \times X_n \to \mathbb{F}$ will have partition rank 1 if and only if it can be written as

$$h(x_1, \ldots, x_n) = f(\vec{x}_S)g(\vec{x}_T)$$

for some $f, g$ and some $S, T \neq \emptyset$ with $S \cup T = \{1, \ldots, n\}$. $h$ will have slice rank 1 if it can be written in the above form with either $|S| = 1$ or $|T| = 1$. In other words, $h$ has partition rank 1 if the tensor can be written as a non-trivial outer product, and it has slice rank 1 if it can be written as the outer product between a vector and a $k - 1$ dimensional tensor.
Example 6. Let $X$ be a finite set. The function $h : X^7 \to \mathbb{F}$ given by
\[ h(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = f_1(x_2, x_5, x_7)f_2(x_1, x_3)f_3(x_4, x_6) \]
will have partition rank 1, with partition $P$ given by the sets with $S_1 = \{2, 5, 7\}$, $S_2 = \{1, 3\}$ and $S_3 = \{4, 6\}$.

This leads us to the definition of the partition rank:

**Definition 7.** Let $X_1, \ldots, X_n$ be finite sets. The **partition rank** of $F$ is the minimal $r$ such that
\[ F = \sum_{i=1}^{r} g_i \]
where the $g_i$ have partition rank 1.

The partition rank is the minimal rank among all possible ranks obtained from partitioning or separating the variables. It will be convenient to have a notation for the rank corresponding to a specific subset of partitions $\mathcal{P}$.

**Definition 8.** Let $X_1, \ldots, X_n$ be finite sets, and let $\mathcal{P}$ be a collection of non-trivial partitions of $\{1, \ldots, n\}$, and let
\[ h : X_1 \times \cdots \times X_n \to \mathbb{F}. \]
We say that $h$ has $\mathcal{P}$-rank 1 if there exists a partition $P \in \mathcal{P}$ such that
\[ h(x_1, \ldots, x_n) = \prod_{A \in \mathcal{P}} f_A (\bar{x}_A) \]
for some functions $f_A$. The **$\mathcal{P}$-rank** of a function
\[ F : X_1 \times \cdots \times X_n \to \mathbb{F}, \]
is defined to be the minimal $r$ such that
\[ F = \sum_{i=1}^{r} g_i \]
where the $g_i$ have $\mathcal{P}$-rank 1.

The partition rank is given by the $\mathcal{P}$-rank when $\mathcal{P}$ is the set of all non-trivial partitions. Let $\mathcal{P}_{\text{slice}}$ denote the set of partitions of $\{1, \ldots, n\}$ into a set of size 1 and a set of size $n - 1$. Then the $\mathcal{P}_{\text{slice}}$-rank will be equal to the slice-rank, and it follows that
\[ \text{partition-rank} \leq \text{slice-rank}. \]
Letting $\mathcal{P}_{\text{tensor}}$ denote the set containing only the partition of $\{1, \ldots, n\}$ into $n$ sets each of size 1. Then the $\mathcal{P}_{\text{tensor}}$-rank will be equal to the tensor rank. This partition of $\{1, \ldots, n\}$ is a refinement of every partition in $\mathcal{P}_{\text{slice}}$, and so
\[ \text{partition-rank} \leq \text{slice-rank} \leq \text{tensor-rank}. \]
Generalizing this relation between a refinement of a partition and the $\mathcal{P}$-rank, we have the following proposition:

**Proposition 9.** Let $\mathcal{P}, \mathcal{P}'$ be two collections of non-trivial partitions of $\{1, \ldots, n\}$. Suppose that every partition $P \in \mathcal{P}$ is refined by some partition $P' \in \mathcal{P}'$. Then we have
\[ \mathcal{P}\text{-rank} \leq \mathcal{P}'\text{-rank}. \]

In particular, this proposition implies that the partition rank is equal to the bipartition rank, the $\mathcal{P}$-rank when $\mathcal{P}$ is the set of all bipartitions.
Theorem. Suppose that \( g \) has \( P' \)-rank 1. Then there exists \( P' \in P' \), and \( f_A \) such that
\[
h = \prod_{A \in P'} f_A.
\]
Since \( P' \) refines some \( P \in P \), we may write
\[
h = \prod_{B \in P} g_B,
\]
where
\[
g_B = \prod_{A \in P} f_A, \quad A \subset B
\]
\( \square \)

When \( k = 2 \), the slice rank, partition rank and tensor rank are identical, since there is only one non-trivial partition of 2. When \( k = 3 \), the slice rank and partition rank are identical, but differ from the tensor-rank, and when \( k \geq 4 \), all three are different. However, the partition rank can be substantially lower than the slice rank.

Example 10. Consider \( k = 4 \). The only partitions of \( \{1, 2, 3, 4\} \) that do not refine partitions appearing in \( P_\text{slice} \) are given by the additive partition \( 2 + 2 = 4 \), that is \( \{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\} \). For a finite set \( X \) and a field \( \mathbb{F} \), consider the function
\[
F : X \times X \times X \times X \to \mathbb{F}
\]
given by
\[
F(x, y, z, w) = \begin{cases} 1 & x = y \text{ and } z = w \\ 0 & \text{otherwise} \end{cases},
\]
that is
\[
F(x, y, z, w) = \delta(x, y)\delta(z, w)
\]
where \( \delta(x, y) \) is the function that is 1 when \( x = y \), and 0 otherwise. Then \( F \) satisfies partition-rank\( (F) = 1 \) and slice-rank\( (F) = |X| \) by the lower bounds of [8].

Our key observation is that the partition rank of a diagonal tensor is maximal.

Lemma 11. Let \( X \) be a finite set, and let \( X^n \) denote the \( n \)-fold Cartesian product of \( X \) with itself. Suppose that
\[
F : X^n \to \mathbb{F}
\]
is a diagonal tensor, that is
\[
F(x_1, \ldots, x_n) = \sum_{a \in A} c_a \delta_a(x_1)\cdots \delta_a(x_n)
\]
for some \( A \subset X \) where \( c_a \neq 0 \), and
\[
\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}.
\]
Then
\[
\text{partition-rank}(F) = |A|.
\]
Proof. It's evident that the partition rank is at most \(|A|\), and so our goal is to prove the lower bound. The proof proceeds by induction on the number of variables. When \(n = 2\), this is the usual notion of rank, and so the result follows. Suppose that \(F\) has partition rank \(r < |A|\), that is suppose that we can write

\[
F(x_1, \ldots, x_n) = \sum_{i=1}^{r} f_i(\vec{x}_{S_i})g_i(\vec{x}_{T_i})
\]

for some sets \(S_i, T_i\) with \(S_i \cap T_i = \emptyset\) and \(S_i \cup T_i = \{1, \ldots, n\}\). Without loss of generality that \(|S_i| \leq \frac{n}{2}\) for each \(i\). If there is no \(i\) such that \(|S_i| = 1\), then choose an arbitrary variable, say \(x_1\), and average over than coordinate. Then

\[
\sum_{x_1 \in X} F(x_1, \ldots, x_n) = \sum_{a \in A} c_a \delta_a(x_2) \cdot \delta_a(x_n) = \sum_{i=1}^{r} \tilde{f}_i(\vec{x}_{S_i \setminus \{1\}}) \tilde{g}_i(\vec{x}_{\{2, \ldots, n\} \setminus S_i}),
\]

for functions \(\tilde{f}, \tilde{g}\) given by averaging \(f, g\) over \(x_1\). This contradicts the inductive hypothesis since \(\sum_{a \in A} c_a \delta_a(x_2) \cdot \delta_a(x_n)\) will have partition rank equal to \(|A| > r\). Suppose that there exists some \(S_i\) such that \(|S_i| = 1\). Then \(S_i = \{j\}\) for some \(j \in \{1, \ldots, n\}\). Let \(U\) be the set of indices \(u\) for which \(S_u = \{j\}\). Consider the annihilator of \(U\), defined to be

\[
V = \left\{ h : X \to \mathbb{F} : \sum_{x_j \in X} f_u(x_j)h(x_j) = 0 \text{ for all } u \in U \right\}.
\]

This vector space has dimension at least \(|X| - |U|\). Let \(v \in V\) have maximal support, and set \(\Sigma = \{x \in X : v(x) \neq 0\}\). Then \(|\Sigma| \geq \dim V \geq |X| - |U|\), since otherwise there exists nonzero \(w \in V\) vanishing on \(\Sigma\), and the function \(v + w\) would have a larger support than \(v\). Multiplying both sides of our expression by \(v(x_j)\) and summing over \(x_j\) reduces the dimension by 1. Indeed

\[
\sum_{x_j \in X} v(x_j)F(x_1, \ldots, x_n) = \sum_{a \in A} c_a \delta_a(x_1) \cdot \delta_a(x_1-1) \delta_a(x_{j+1}) \delta_a(x_n) \left( \sum_{x_j \in X} v(x_j) \delta_a(x_j) \right),
\]

and since the sum \(\sum_{x_j \in X} u(x_j) \delta_a(x_j)\) will be non-zero for at least \(|X| - |U|\) values of \(a \in X\), the partition rank of the above must be at least \(|A| - |U|\) by the inductive hypothesis. Since

\[
\sum_{x_j \in X} v(x_j)f_i(\vec{x}_{S_i}) = 0
\]

for each \(i \in U\), it follows that

\[
\sum_{x_j \in X} v(x_j) \sum_{i=1}^{r} c_i f_i(\vec{x}_{S_i})g_i(\vec{x}_{T_i})
\]

will be a sum of at most \(k - |U|\) partition rank 1 functions, and hence it has partition rank at most \(k - |U|\). This implies that \(|A| - |U| < k - |U|\), which is a contradiction, and the lemma is proven.

Since the partition rank is minimal over all \(\mathcal{P}\)-ranks, the \(\mathcal{P}\)-rank for any set of non-trivial partitions \(\mathcal{P}\) of a diagonal tensor will equal to the number of non-zero entries on that diagonal tensor.

We conclude this section by rephrasing an open problem in non-commutative circuits in terms of the \(\mathcal{P}\)-rank:
Problem 12. Let $X$ be a finite set of size $n$, and let $k = 4$. Let $\mathcal{P}$ be the set that contains the two partitions $\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}$. Does $\delta(x_1, x_4)\delta(x_2, x_3)$ have superlinear $\mathcal{P}$-rank? That is, does $\delta(x_1, x_4)\delta(x_2, x_3)$ have $\mathcal{P}$-rank $\gg n^{1+\epsilon}$ for some $\epsilon > 0$ as $n$ grows?

A counting argument shows that there will exist many tensors of $\mathcal{P}$-rank $\gg n^2$, however no explicit superlinear lower bounds are known for any tensor. A positive answer to problem 12 would lead to improved lower bounds for non-commutative circuits, see [9, Theorem 3.6]. In general, we can ask about the $\mathcal{P}$-rank of a product of $\delta$ functions that is given by a partition that is not a refinement of any $P \in \mathcal{P}$.

Problem 13. Let $X$ be a finite set. Let $\mathcal{P}$ be a collection of non-trivial partitions of $\{1, \ldots, n\}$ and suppose that $P$ is not a refinement of any $P' \in \mathcal{P}$. Let

$$\delta_P(x_1, \ldots, x_n) = \prod_{A \in P} \delta(\overrightarrow{x}_A),$$

where for a singleton set $A = \{j\}$, we use the convention $\delta(\overrightarrow{x}_A) = \delta(x_j) = 1$. What is the $\mathcal{P}$-rank of $\delta_P$?

3. The Distinctness Indicator Function

Let $X$ be a finite set, $\mathbb{F}$ a field, and let $X^k = X \times \cdots \times X$ denote the Cartesian product of $X$ with itself $k$ times. For every $\sigma \in S_k$, define

$$f_\sigma : X \times \cdots \times X \to \mathbb{F}$$

to be the function that is 1 if $(x_1, \ldots, x_k)$ is a fixed point of $\sigma$, and 0 otherwise. Using these functions $f_\sigma$ we will construct an indicator function for the distinctness of $k$ variables.

Lemma 14. We have the identity

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) f_\sigma(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } x_1, \ldots, x_k \text{ are distinct} \\ 0 & \text{otherwise} \end{cases},$$

where $\text{sgn}(\sigma)$ is the sign of the permutation.

Proof. By definition,

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) f_\sigma(x_1, \ldots, x_k) = \sum_{\sigma \in \text{Stab}(\overrightarrow{x})} \text{sgn}(\sigma)$$

where $\text{Stab}(\overrightarrow{x}) \subset S_k$ is the stabilizer of $\overrightarrow{x}$. Since the stabilizer is a product of symmetric groups, this will be non-zero precisely when $\text{Stab}(\overrightarrow{x})$ is trivial, and hence $x_1, \ldots, x_k$ must be distinct. This vector is then fixed only by the identity element, and so the sum equals 1. \hfill \Box

Lemma 15. Let $C \subset S_k$, $C = \text{Cl}(1 2 \cdots k)$, be the conjugacy class of the $k$-cycles in $S_k$, and define

$$H_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k, \sigma \notin C} \text{sgn}(\sigma) f_\sigma(x_1, \ldots, x_k).$$
Then

\[ H_k(x_1, \ldots, x_k) = \begin{cases} 
1 & \text{if } x_1, \ldots, x_k \text{ are distinct} \\
(-1)^{k-1}(k-1)! & \text{if } x_1 = \cdots = x_k \\
0 & \text{otherwise}
\end{cases} \]

Proof. This follows from lemma 14 and the fact that the conjugacy class of the \( k \)-cycle in \( S_k \) has \((k-1)! \) elements with sign equal to \((-1)^{k-1}\). \( \square \)

This function can be used to zero-out those tuples of vectors with repetitions. Suppose that we are interested in the size of the largest set \( A \subset X \) that does not contain \( k \) distinct vectors satisfying some condition \( \mathcal{K} \). Then if \( F_k : X^k \rightarrow \mathbb{F} \) is some function satisfying

\[ F_k(x_1, \ldots, x_k) = \begin{cases} 
c_1 & \text{if } x_1, \ldots, x_k \text{ satisfy } \mathcal{K} \\
c_2 & \text{if } x_1 = \cdots = x_k \\
0 & \text{otherwise}
\end{cases} \]

where \( c_2 > 0 \), then

\[ I_k(x_1, \ldots, x_k) := F_k(x_1, \ldots, x_k)H_k(x_1, \ldots, x_k) \]

when restricted to \( A^k \) will be a diagonal tensor, and hence by lemma 11

\[ |A| \leq \text{partition-rank}(I_k). \]

Let

\[ \delta(x_1, \ldots, x_k) = \begin{cases} 
1 & \text{if } x_1 = \cdots = x_k \\
0 & \text{otherwise}
\end{cases} \]

and for a partition \( P \) of \( \{1, \ldots, k\} \) define \( \delta_P(x_1, \ldots, x_k) \) as in (2.1). Suppose that \( \sigma \) is a permutation of \( \{1, \ldots, k\} \) with \( t \) cycles in its disjoint cycle decomposition, and these cycles permute the sets \( S_1, S_2, \ldots, S_t \). Then \( P = \{S_1, \ldots, S_t\} \) will be a partition of \( \{1, \ldots, k\} \) and we have that

\[ f_\sigma = \delta_P. \]

It follows that \( H_k(x_1, \ldots, x_k) \) can be written as a sum product of delta functions. When \( k = 2 \), we have that

\[ H_2(x_1, x_2) = 1, \]

when \( k = 3 \)

\[ H_2(x_1, x_2, x_3) = 1 - \delta(x_1, x_2) - \delta_2(x_2, x_3) - \delta(x_3, x_2) \]

and when \( k = 4 \)

\[ H_4(x_1, x_2, x_3, x_4) = 1 - \delta(x_1, x_2) - \delta_2(x_2, x_3) - \delta(x_3, x_4) - \delta(x_4, x_1) - \delta(x_1, x_3) - \delta(x_2, x_4) + 2\delta(x_1, x_2, x_3) + 2\delta(x_2, x_3, x_4) + 2\delta(x_3, x_4, x_1) + 2\delta(x_4, x_1, x_2) + \delta(x_1, x_2)\delta(x_3, x_4) + \delta(x_1, x_3)\delta(x_2, x_4) + \delta(x_1, x_4)\delta(x_2, x_3). \]

Note that as a function on \( X^3 \), \( \text{slice-rank}(\delta(x_1, x_2)) = 1 \), but on \( X^4 \) the function

\[ \delta(x_1, x_2)\delta(x_3, x_4) \]

has slice rank equal to \( |X| \). This function does however have partition rank equal to 1. When writing \( H_k(x_1, \ldots, x_k) \) as a linear combination of \( \delta \) functions, it will always have partition rank at most \( 2^k - 1 \) since every term will have a \( \delta \)-function corresponding to some subset strict \( S \subset \{1, \ldots, k\} \). Starting from \( k = 4 \), there will be functions in the
sum whose slice rank is maximal, and this is why the partition rank is needed to handle a function of the form
\[ I_k(x_1, \ldots, x_k) := F_k(x_1, \ldots, x_k)H_k(x_1, \ldots, x_k). \]

4. Right Angles in \( \mathbb{F}_q^n \)

In this section we provide a proof of theorem \( \square \). We begin with a lemma concerning the dimension of the space of polynomials of degree \( \leq d \) in \( n \) variables over \( \mathbb{F}_q \):

**Lemma 16.** Suppose that \( d \leq (q - 1)n \). The number of monomials of degree at most \( d \) in \( n \) variables over \( \mathbb{F}_q \) is at most
\[
(4.1) \quad C_d := \# \left\{ v \in \{0, 1, \ldots, q - 1\}^n : \sum_{i=1}^n v_i \leq d \right\}.
\]
Furthermore,
\[
(4.2) \quad C_d \leq \binom{n + d}{d}
\]
and
\[
(4.3) \quad C_d \leq \left( \min_{0 < t < 1} \frac{1 - t^q}{(1-t) \cdot t^q} \right)^n.
\]

**Proof.** Every monomial \( z_1^{e_1} \cdots z_n^{e_n} \) of degree at most \( d \) over \( n \) variables, \( z_1, \ldots, z_n \), corresponds to an exponent vector \( (e_1, \ldots, e_n) \), where \( 0 \leq e_i \leq q - 1 \) and \( \sum_{i=1}^n e_i \leq d \), which proves (4.1). Equation (4.2) follows from the buckets and balls theorem, and equality holds only when \( d \leq q - 1 \). The final equation follows from a Chernoff type bound. Let \( X_1, \ldots, X_n \) be i.i.d. uniform random variables on \( \{0, 1, \ldots, q - 1\} \). Then for \( 0 < t < 1 \) we have
\[
C_d = q^n \mathbb{P} (X_1 + \cdots + X_n \leq d)
\]
\[
= q^n \mathbb{P} (t^{X_1 + \cdots + X_n} \geq t^d)
\]
\[
\leq q^n \mathbb{E} \left( t^{X_1 + \cdots + X_n} \right) / t^d
\]
where the last inequality follows from Markov’s inequality. By independence, along with the fact that
\[
\mathbb{E} \left( t^{X_i} \right) = \frac{1}{q} \left( 1 + t + \cdots + t^{q-1} \right) = \frac{1}{q} \cdot \frac{1 - t^q}{(1-t)},
\]
equation (4.3) follows. \( \square \)

Using this lemma, and the slice rank, we prove theorem \( \square \).

**Proof.** (proof of theorem \( \square \)) Consider the function
\[
F : \mathbb{F}_q^n \times \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q
\]
defined by
\[
F(x, y, z) = (1 - \delta(x, y) - \delta(y, z) - \delta(z, x)) \left( 1 - \langle x - z, y - z \rangle^{q-1} \right).
\]
We have that
\[ F(x, y, z) = \begin{cases} -2 & \text{if } x = y = z \\ 1 & \text{if } x, y, z \text{ are distinct and form a right corner} \\ 0 & \text{otherwise} \end{cases}. \]

If \( A \subseteq \mathbb{F}_q^n \) has no right-corners, then \( F|_{A \times A \times A} \) is a diagonal tensor, and so
\[ |A| \leq \text{slice-rank}(F). \]

Note that
\[ \delta(x, z) \left( 1 - (x - z, y - z)^{q-1} \right) = \delta(x, z) \]
and
\[ \delta(y, z) \left( 1 - (x - z, y - z)^{q-1} \right) = \delta(y, z), \]
and both of these functions have slice rank 1. We have that
\[ \delta(x, y) \left( 1 - \langle x - z, y - z \rangle^{q-1} \right) = \delta(x, y) \left( 1 - \langle x - z, x - z \rangle^{q-1} \right), \]
and by expanding \( \langle x - z, x - z \rangle \) as a degree \( 2(q-1) \) polynomial, the above can be written as a linear combination of terms of the form
\[ \delta(x, y)x_1^{d_1} \cdots x_n^{d_n} z_1^{e_1} \cdots z_n^{e_n} \]
where
\[ \sum_{i=1}^n d_i + \sum_{i=1}^n e_i \leq 2(q-1), \]
and so for each term, one of \( \sum_{i=1}^n d_i, \sum_{i=1}^n e_i \) will be at most \( (q-1) \). The term
\[ \left( 1 - \langle x - z, y - z \rangle^{q-1} \right) \]
is a degree \( 2(q-1) \) polynomial and can be expanded and written as a linear combination of terms of the form
\[ x_1^{d_1} \cdots x_n^{d_n} z_1^{e_1} \cdots z_n^{e_n} y_1^{f_1} \cdots y_n^{f_n} \]
where
\[ \sum_{i=1}^n d_i + \sum_{i=1}^n e_i + \sum_{i=1}^n f_i \leq 2(q-1). \]

It follows that for each term, either
\[ \sum_{i=1}^n e_i \leq (q-1) \quad \text{or} \quad \sum_{i=1}^n d_i \leq \frac{1}{2}(q-1) \quad \text{or} \quad \sum_{i=1}^n f_i \leq \frac{1}{2}(q-1). \]

Hence, by decomposing \( \delta(x, y) \langle x - z, x - z \rangle^{q-1} \) and \( \langle x - z, y - z \rangle^{q-1} \) simultaneously, we obtain the bound
\[ \text{slice-rank}(F) \leq 2 \cdot \# \left\{ \vec{v} \in \{0, 1, \ldots, q-1\}^n : \sum_{i=1}^n v_i \leq (q-1) \right\}, \]
\[ + 2 \cdot \# \left\{ \vec{v} \in \{0, 1, \ldots, q-1\}^n : \sum_{i=1}^n v_i \leq \frac{(q-1)}{2} \right\}\]
and hence
\[ \text{slice-rank}(F) \leq 2 \left( n + q - 1 \right) + 2 \left( \frac{n + q - 1}{q - 1} \right) + 2 \]
as desired. \( \square \)
5. $k$-Right Corners in $\mathbb{F}_q^n$

In this section, we use the partition rank to prove theorem 2. Our goal is to construct a function

$$J_k : \mathbb{F}_q^{k+1} \to \mathbb{F}_q,$$

of low partition rank satisfying

$$J_k(x_1, \ldots, x_{k+1}) = \begin{cases} c_1 & x_1, \ldots, x_{k+1} \text{ form a } k \text{-right corner} \\ c_2 & x_1 = \cdots = x_k \\ 0 & \text{otherwise} \end{cases}$$

(5.1)

where $c_2 \neq 0$. Suppose that $E \subset \mathbb{F}_p$ is a set without any $k$-right corners. Then $J_k$ restricted to $E^{k+1}$ will be a diagonal tensor taking the value $c_2$ on the diagonal, and so by lemma 11, $J_k$ must have partition rank at least $|E|$. It then follows that any set $E \subset \mathbb{F}_p$ of size

$$E > \text{partition-rank}(J_k),$$

must contain a $k$-right corner. Define

$$F_k : \mathbb{F}_q^k \to \mathbb{F}_q$$

by

$$F_k(x_1, \ldots, x_k) = \prod_{j<l \leq k} (1 - \langle x_j, x_l \rangle^{q-1}).$$

(5.2)

This polynomial has degree

$$\text{deg } F = k(k-1)(q-1)$$

(5.3)

and satisfies

$$F(x_1, \ldots, x_k) = \begin{cases} 1 & x_1, \ldots, x_k \text{ are mutually orthogonal} \\ 0 & \text{otherwise} \end{cases}.$$  

Note that $x_1, \ldots, x_k$ are not required to be unique for $F(x_1, \ldots, x_k)$ to equal 1. It follows that

$$F_k(x_1-x_{k+1}, x_2-x_{k+1}, \ldots, x_k-x_{k+1}) = \begin{cases} 1 & x_i-x_{k+1} \text{ are mutually orthogonal} \\ 0 & \text{otherwise} \end{cases}.$$  

This function is not an indicator function for $k$-right corners since there are trivial off-diagonal solutions that arise from repeated variables. For example, suppose that $x_1-x_{k+1}$ and $x_2-x_{k+1}$ are orthogonal, and that $x_2-x_{k+1}$ is self orthogonal. Then if we take $x_i = x_2$ for $2 \leq i \leq k$, the $(k+1)$-tuple $(x_1, \ldots, x_{k+1})$ will satisfy $F(x_1, \ldots, x_{k+1}) = 1$, despite not being a $k$-right corner. To handle this issue, we use the distinctness indicator function $H_{k+1}$ from lemma 15 and define

$$J_k(x_1, \ldots, x_k, x_{k+1}) = H_{k+1}(x_1, \ldots, x_{k+1})F_k(x_1-x_{k+1}, \ldots, x_k-x_{k+1})$$

(5.4)

The function $J_k$ will satisfy

$$J_k(x_1, \ldots, x_k, x_{k+1}) = \begin{cases} 1 & \text{if } x_1, \ldots, x_{k+1} \text{ form a } k\text{-right corner} \\ (-1)^k k! & \text{if } x_1 = \cdots = x_{k+1} \\ 0 & \text{otherwise} \end{cases}.$$
If $A \subset \mathbb{F}_q^n$ contains no $k$-right corners, then the restriction of $J_k$ to $A^{k+1}$ will be a diagonal tensor with $(-1)^k k!$ on the diagonal. Lemma 11 implies that

$$|A| \leq \text{partition-rank}(J_k)$$

as long as $q = p^r$ satisfies $p > k$ so that $k!$ is non-zero. To prove theorem 2 all that remains is to bound the partition rank of $J_k$.

**Proposition 17.** The function $J_k$ defined in (5.4) satisfies

$$\text{partition-rank}(J_k) \leq (k + 1) \cdot \left( \frac{n + (k - 1)q}{(k - 1)(q - 1)} \right).$$

**Proof.** By lemma 15 and (5.2), $H_{k+1}$ can be written as a sum product of delta functions $\delta(\bar{x}_S)$ where $S \neq \{1, \ldots, k + 1\}$. For a set $S \subset \{1, \ldots, k + 1\}$, let $s$ denote the smallest element of $S$. We may expand $F_k$ as a polynomial and split up the product $F_k H_{k+1}$ into a linear combination of terms that are monomials multiplied by delta functions. Any function of the form $\delta(\bar{x}_S) P(\bar{x}_S)$ will be exactly equal to $\delta(\bar{x}_S) P(x_s, x_{s+1}, \ldots, x_n) = \delta(\bar{x}_S) P(x_s)$ for some polynomial $P$ of degree $dP$, and so we may write $J_k$ as

$$J_k = \sum_{S \subset \{1, \ldots, n\}} \delta(\bar{x}_S) \sum_{\substack{P \text{ monomial} \\ \deg P \leq d(S)}} P(x_s) G_P(\bar{x}_{\{1, \ldots, n\} \setminus S}), \tag{5.5}$$

where $G_P$ is some polynomial function of the variables $\bar{x}_{\{1, \ldots, n\} \setminus S}$, depending on $P$, and the degree $d(S)$ is a function depending on the set $S$. Let $C_d$, as defined in (4.1), denote the number of monomials of degree at most $d$ in $n$ variables over $\mathbb{F}_q$. It follows from (5.5) that

$$\text{partition-rank}(J_k) \leq \sum_{\substack{S \subset \{1, \ldots, n\} \\ S \neq \emptyset, S \neq \{1, \ldots, n\}}} C_{d(S)},$$

and so we need only determine $d(S)$ for an explicit decomposition of this form.

Let $f_\sigma$ for $\sigma \in S_{k+1} \setminus C$ be given, and express $f_\sigma$ as

$$f_\sigma(x_1, \ldots, x_{k+1}) = \delta(\bar{x}_S_1) \delta(\bar{x}_S_2) \cdots \delta(\bar{x}_S_t).$$

For each $i \in \{1, \ldots, t - 1\}$, let $s_i$ be a representative in the set $S_i$, and without loss of generality, suppose that $k + 1$ is the representative for the set $S_t$. Then, since

$$(1 - \langle x, y \rangle^{q-1})^2 = (1 - \langle x, y \rangle^{q-1}),$$

we may remove any duplicates that appear, as well as any terms that are identically equal to the constant function 1. Thus we obtain the equality of functions

$$f_\sigma(x_1, \ldots, x_{k+1}) F_k(x_1, \ldots, x_{k+1}) = \prod_{i=1}^t \delta(\bar{x}_S_i) \Pi_1 \Pi_2 \tag{5.6}$$

where

$$\Pi_1 = \prod_{1 \leq j < t \leq t-1} (1 - \langle x_{s_j} - x_k, x_{s_t} - x_k \rangle^{q-1})$$

and

$$\Pi_2 = \prod_{1 \leq j \leq t-1 \atop \text{s.t. } |S_j| \geq 2} (1 - \langle x_{s_j} - x_k, x_{s_j} - x_k \rangle^{q-1}).$$
Expanding the polynomial above, it will be a linear combination of monomials of the form
\[
\left[ \delta \left( \vec{x}_{S_1} \right) x_{s_{11}}^{e_{11}} \cdots x_{s_{1n}}^{e_{1n}} \right] \left[ \delta \left( \vec{x}_{S_2} \right) x_{s_{21}}^{e_{21}} \cdots x_{s_{2n}}^{e_{2n}} \right] \cdots \left[ \delta \left( \vec{x}_{S_t} \right) x_{s_{t1}}^{e_{t1}} \cdots x_{s_{tn}}^{e_{tn}} \right],
\]
where \( e_{ij} \leq q - 1 \) and
\[
\sum_{i=1}^{t} \sum_{j=1}^{n} e_{ij} \leq (t-1)(t-2)(q-1) + 2 \cdot \# \{ 1 \leq j \leq t-1 : |S_j| \geq 2 \}
\]
by (5.2) and (5.6). For every monomial appearing in the expansion of \( f_\sigma F_k \), there will be some index \( i \) such that the monomial
\[
\delta \left( \vec{x}_{S_i} \right) x_{s_{i1}}^{e_{i1}} \cdots x_{s_{in}}^{e_{in}}
\]
has degree less than or equal to the average, that is degree at most
\[
(t - 1)(t - 2) + \frac{2}{t} \# \{ 1 \leq j \leq t-1 : |S_j| \geq 2 \}(q - 1).
\]
Looking at all possible permutations, we need to determine the highest possible degree a \( \delta \left( \vec{x}_S \right) \) term can have while still having degree less than the average. That is, for a fixed \( S \), we are trying to find the maximum of (5.7) varying over \( \sigma \). This occurs when \( \sigma \) has one cycle of length \( |S| \) and \( k + 1 - |S| \) fixed points. In this case, we have \( t = k + 2 - |S| \), and degree at most
\[
r_1(t) := \left( \frac{(t - 1)(t - 2) + 2}{t} \right)(q - 1)
\]
if \( k + 1 \notin S \) and \( |S| \geq 2 \), and
\[
r_2(t) := \left( \frac{(t - 1)(t - 2)}{t} \right)(q - 1)
\]
if \( k + 1 \in S \) or if \( |S| = 1 \). Summing over all possible subsets \( S \subset \{1, \ldots, k+1\} \), with \( C_d \) defined in (4.1), we have that
\[
\text{partition-rank}(J_k) \leq \sum_{S \subset \{1, \ldots, k+1\}} C_{r_2(k+1)} + \sum_{\substack{S \subset \{1, \ldots, k+1\} \\kappa + 1 \in S, \ |S| \geq 2}} C_{r_2(k+2-|S|)} + \sum_{\substack{S \subset \{1, \ldots, k+1\} \\kappa + 1 \notin S, \ |S| \geq 2}} C_{r_1(k+2-|S|)}.
\]
Letting \( t = k + 1 - |S| \), this becomes
\[
(5.8) \quad \text{partition-rank}(J_k) \leq (k + 1)C_{r_2(k+1)} + \sum_{t=1}^{k-1} \binom{k}{t} C_{r_1(t+1)} + \sum_{t=1}^{k-1} \binom{k}{t-1} C_{r_2(t+1)}.
\]
The above is a precise version of the bound achieved by the partition rank method. In what follows we will make the bound slightly weaker and put it into a nicer form. For \( t \geq 3 \), both \( r_1(t+1) \) and \( r_2(t+1) \) will be bounded above by \( (t - 1)(q - 1) \). For \( t \in \{1, 2\} \), \( r_1(t+1) \) will be at most \( (q - 1) \) and \( \frac{4}{3}(q - 1) \) respectively. Hence,
\[
\text{partition-rank}(J_k) \leq \sum_{t=1}^{k} \binom{k + 1}{t} C_{(t-1)(q-1)} + \binom{k}{2} C_{\frac{4}{3}(q - 1)}.
\]
For $k \geq 3$, this extra term will contribute little. By equation (4.2), we have the upper bound

\begin{equation}
\text{partition-rank}(J_k) \leq \sum_{t=0}^{k-1} \frac{(k+1)}{t+1} \left(\frac{n + t(q-1)}{t(q-1)}\right) + \frac{k}{2} \left(\frac{n + \frac{4}{3}(q-1)}{4/3(q-1)}\right).
\end{equation}

Since $(k-1)q = (k-1)(q-1) + (k-1)$, it follows from a standard binomial identity that

\[
\frac{(n + (k-1)q)}{(k-1)(q-1)} = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(n + (k-1)(q-1))}{(k-1)(q-1) - i},
\]

and if we multiply this by $(k+1)$ it will strictly dominate the sum in (5.9), and hence the proposition follows.

\[\blacksquare\]

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