THE BLACK-AND-WHITE COLORING PROBLEM ON DISTANCE-HEREDITARY GRAPHS AND STRONGLY CHORDAL GRAPHS

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Abstract. Given a graph $G$ and integers $b$ and $w$, the black-and-white coloring problem asks if there exist disjoint sets of vertices $B$ and $W$ with $|B| = b$ and $|W| = w$ such that no vertex in $B$ is adjacent to any vertex in $W$. In this paper we show that the problem is polynomial when restricted to cographs, distance-hereditary graphs, interval graphs and strongly chordal graphs. We show that the problem is NP-complete on splitgraphs.

1 Introduction

Definition 1. Let $G = (V, E)$ be a graph and let $b$ and $w$ be two integers. A black-and-white coloring of $G$ colors $b$ vertices black and $w$ vertices white such that no black vertex is adjacent to any white vertex.

In other words, the black-and-white coloring problem asks for a complete bipartite subgraph $M$ in the complement $ar{G}$ of $G$ with $b$ and $w$ vertices in the two color classes of $M$.

The black-and-white coloring problem is NP-complete for graphs in general [22]. That paper also shows that the problem can be solved for trees in $O(n^3)$ time. In a recent paper [6] the worst-case timebound for an algorithm on trees was improved to $O(n^2 \log^3 n)$ time [6]. The paper [6] mentions, among other things, a manuscript by Kobler, et al., which shows that the problem can be solved in polynomial time for graphs of bounded treewidth.

In this paper we investigate the complexity of the problem for some graph classes. We start our analysis for the class of cographs.

A $P_4$ is a path with four vertices.

* National Science Council of Taiwan Support Grant NSC 99–2218–E–007–016.
Definition 2 ([13]). A graph is a cograph if it has no induced $P_4$.

There are various characterizations of cographs. For algorithmic purposes the following characterization is suitable.

Theorem 1. A graph is a cograph if and only if every induced subgraph $H$ is disconnected or the complement $\overline{H}$ is disconnected.

It follows that a cograph has a tree decomposition which is called a cotree. A cotree is a pair $(T, f)$ comprising a rooted binary tree $T$ together with a bijection $f$ from the vertices of the graph to the leaves of the tree. Each internal node of $T$, including the root, has a label $\otimes$ or $\oplus$. The $\otimes$ operation is called a join operation, and it makes every vertex that is mapped to a leaf in the left subtree adjacent to every vertex that is mapped to a leaf in the right subtree. The operator $\oplus$ is called a union operation. In that case the graph is the union of the graphs defined by the left - and right subtree. A cotree decomposition can be obtained in linear time [14].

2 Black-and-white colorings of cographs

In this section we show that the black-and-white coloring problem can be solved in polynomial time for cographs.

Lemma 1. There exists an $O(n^5)$ algorithm which solves the black-and-white coloring problem on cographs.

Proof. Let $\gamma_G(b, w)$ be a boolean variable which indicates if the graph $G$ has a black-and-white coloring with $b$ black and $w$ white vertices. Obviously, we have

$$\gamma_G(b, 0) = \begin{cases} 
\text{true} & \text{if } 0 \leq b \leq n, \\
\text{false} & \text{otherwise},
\end{cases} \quad (1)$$

where $n$ is the number of vertices in $G$. A similar formula holds for $\gamma_G(0, w)$. The algorithm uses dynamic programming on the cotree and it derives $\gamma_G(b, w)$ for every node from a table of values stored at the children as follows.

When $G$ has only one vertex then we have

$$\gamma_G(b, w) = \begin{cases} 
\text{true} & \text{if } (b, w) \in \{(0, 0), (1, 0), (0, 1)\}, \\
\text{false} & \text{in all other cases.}
\end{cases}$$

Assume that $G$ is the join of two cographs $G_1$ and $G_2$. Let $n_1$ be the number of vertices of $G_1$. Then $n = n_1 + n_2$. If there is a black-and-white coloring for $G$ with at least one black vertex and at least one white vertex then all the black and white vertices must be contained in the same graph $G_i$. It follows that, when $b \geq 1$ and $w \geq 1$,

$$\gamma_G(b, w) = \text{true} \quad \text{if and only if} \quad \gamma_{G_1}(b, w) = \text{true} \quad \text{or} \quad \gamma_{G_2}(b, w) = \text{true}.$$
The cases where $b = 0$ or $w = 0$ follow from Equation 1.

Finally assume that $G$ is the union of two cographs $G_1$ and $G_2$. Let $n_i$ be the number of vertices in $G_i$ and let $n = n_1 + n_2$ be the number of vertices in $G$. Then we have

$$\gamma_G(b, w) = \text{true} \quad \text{if and only if} \quad \exists k \exists \ell \quad \gamma_{G_1}(k, \ell) = \text{true} \quad \text{and} \quad \gamma_{G_2}(b - k, w - \ell) = \text{true}.$$  

A table containing the boolean values $\gamma_G(b, w)$ has $n^2$ entries. By the formulas above, each entry can be computed in $O(n^2)$ time. Thus a complete table for each node in the cotree can be obtained in $O(n^4)$ time. Since a cotree has $O(n)$ nodes, this algorithm can be implemented to run in $O(n^5)$ time. \hfill \Box

The following theorem improves the timebound.

**Theorem 2.** There exists an $O(n^3)$ algorithm which solves the black-and-white coloring problem on cographs.

**Proof.** Let $f_G(b)$ be the maximum number of white vertices in a black-and-white coloring of $G$ with $b$ black vertices. We prove that the function $f_G$ can be computed in $O(n^3)$ time for cographs.

Let $G$ be a cograph with $n$ vertices. We write $f$ instead of $f_G$. By convention,

$$f(b) = 0 \quad \text{when} \quad b < 0 \quad \text{or} \quad b > n.$$  

Assume that $G$ has one vertex. Then

$$f(b) = \begin{cases} 1 & \text{if} \quad b = 0 \\ 0 & \text{in all other cases}. \end{cases}$$

Assume that $G$ is the join of two cographs $G_1$ and $G_2$. We write $f_i$ instead of $f_{G_i}$, for $i \in \{1, 2\}$. We have that $f(0) = n$, where $n$ is the number of vertices in $G$. When $b > 0$ we have

$$f(b) = \max \{ f_1(b), \ f_2(b) \}.$$  

Assume that $G$ is the union of two cographs $G_1$ and $G_2$. Then

$$f(b) = \max_{0 \leq k \leq b} f_1(k) + f_2(b - k).$$

A cotree $T$ has $O(n)$ nodes and it can be computed in linear time [13]. Consider a node $i$ in $T$. Let $G_i$ be the subgraph of $G$ induced by the vertices that are mapped to leaves in the subtree rooted at $i$. By the previous observations, the function $f_i$ for the graph $G_i$ can be computed in $O(n^2)$ time. Since $T$ has $O(n)$ nodes this proves the theorem. \hfill \Box
2.1 Threshold graphs

A subclass of the class of cographs are the threshold graphs.

**Definition 3 ([12]).** A graph $G = (V, E)$ is a threshold graph if there is a real number $T$ and a real number $w(x)$ for every vertex $x \in V$ such that a subset $S \subseteq V$ is an independent set if and only if

$$\sum_{x \in S} w(x) \geq T.$$

There are many ways to characterize threshold graphs [29]. For example, a graph is a threshold graph if it has no induced $P_4$, $C_4$ or $2K_2$.

![Fig. 1. A graph is a threshold graph if it has no induced $C_4$, $P_4$ or $2K_2$.](image)

Another characterization is that a graph is a threshold graph if every induced subgraph has a universal vertex or an isolated vertex [12, Theorem 1]. In [12, Corollary 1B] appears also the following characterization. A graph $G = (V, E)$ is a threshold graph if and only if there is a partition of $V$ into two sets $A$ and $B$, of which one is possibly empty, such that

1. $A$ induces a clique,
2. $B$ induces an independent set, and
3. there is an ordering $b_1, \ldots, b_k$ of the vertices in $B$ such that

$$N(b_1) \subseteq \ldots \subseteq N(b_k).$$

We use the notation $N[x]$ to denote the closed neighborhood of a vertex $x$. Thus $N[x] = N(x) \cup \{x\}$.

**Theorem 3.** There exists a linear-time algorithm which, given a threshold graph $G$ and integers $b$ and $w$, decides if there is a black-and-white coloring of $G$ with $b$ vertices colored black and $w$ vertices colored white.

**Proof.** Let $x_1, \ldots, x_n$ be an ordering of the vertices in $G$ such that for all $i < n$

(a) $N(x_i) \subseteq N(x_{i+1})$ if $x_i$ and $x_{i+1}$ are not adjacent, and
(b) $N[x_i] \subseteq N[x_{i+1}]$ if $x_i$ and $x_{i+1}$ are adjacent.
Assume that there exists a black-and-white coloring which colors b vertices black and w vertices white. Assume that there is an index \( k \leq b + w \) such that \( x_k \) is uncolored. Then there exists an index \( \ell > b + w \) such that \( x_\ell \) is colored black or white. Then we may color \( x_k \) with the color of \( x_\ell \) and uncolor \( x_\ell \) instead. Thus we may assume that there exists a coloring such that \( x_1, \ldots, x_{b+w} \) are colored and all other vertices are uncolored.

Assume that \( b \leq w \). We prove that there exists a coloring \( f \) such that

\[
f(x_i) = \begin{cases} 
  \text{black} & \text{if } 1 \leq i \leq b, \\
  \text{white} & \text{if } b + 1 \leq i \leq b + w.
\end{cases}
\]

We may assume that \( b \geq 1 \) and that \( w \geq 1 \). Assume that \( x_i \) is adjacent to \( x_j \) for some \( i \leq b < j \). Then

\[
\{x_j, \ldots, x_{b+w}\} \subseteq N(x_i) \quad \text{and} \quad \{x_i, \ldots, x_j\} \subseteq N[x_j].
\]

Thus all vertices in

\[
\{x_i, \ldots, x_{b+w}\}
\]

are the same color. If they are all black then there are at least \( w+1 \) black vertices in the coloring, which contradicts \( b \leq w \). If they are all white then we have at least \( w+1 \) white vertices, which is a contradiction as well. Thus no two vertices \( x_i \) and \( x_j \) with \( i \leq b < j \) are adjacent, which proves that the coloring above is valid.

This proves the theorem, since an algorithm only needs to check if \( x_b \) is adjacent to \( x_{b+w} \) or not. \( \square \)

### 2.2 Difference graphs

**Definition 4 ([21]).** A graph \( G = (V, E) \) is a difference graph if there exists a positive real number \( T \) and a real number \( w(x) \) for every vertex \( x \in V \) such that \( w(x) \leq T \) for every \( x \in V \) and such that for any pair of vertices \( x \) and \( y \)

\[
\{x, y\} \in E \quad \text{if and only if} \quad |w(x) - w(y)| \geq T.
\]

![Fig. 2. A graph is a difference graph if it has no induced triangle, 2K₂ or C₅.](image)

Difference graphs are sometimes called chain graphs [33].
Difference graphs can be characterized in many ways [21]. For example, a graph is a difference graph if and only if it has no induced $K_3$, $2K_2$ or $C_5$ [21, Proposition 2.6]. Difference graphs are bipartite. Let $X$ and $Y$ be a partition of $V$ into two color classes. Then the graph obtained by making a clique of $X$ is a threshold graph and this property characterizes difference graphs [21, Lemma 2.1].

**Theorem 4.** There exists a linear-time algorithm which, given a difference graph $G$ and integers $b$ and $w$, decides if there is a black-and-white coloring of $G$ with $b$ black vertices and $w$ white vertices.

**Proof.** An argument, similar to the one given in Theorem 3, provides the proof. \(\square\)

### 3 Distance-hereditary graphs

**Definition 5 ([24]).** A graph $G$ is distance hereditary if for every pair of nonadjacent vertices $x$ and $y$ and for every connected induced subgraph $H$ of $G$ which contains $x$ and $y$, the distance between $x$ and $y$ in $H$ is the same as the distance between $x$ and $y$ in $G$.

In other words, a graph $G$ is distance hereditary if for every nonadjacent pair $x$ and $y$ of vertices, all chordless paths between $x$ and $y$ in $G$ have the same length.

There are various characterizations of distance-hereditary graphs. One of them states that a graph is distance hereditary if and only if it has no induced house, hole, domino or gem [4, 24]. Distance-hereditary graphs are also characterized by the property that every induced subgraph has either an isolated vertex, or a pendant vertex, or a true or false twin [4].

![Fig. 3. A graph is distance hereditary if it has no induced house, hole, domino or gem.](image)

Distance-hereditary graphs are the graphs of rankwidth one. This implies that they have a special decomposition tree which we describe next.

A decomposition tree for a graph $G = (V, E)$ is a pair $(T, f)$ consisting of a rooted binary tree $T$ and a bijection $f$ from $V$ to the leaves of $T$.

When $G$ is distance hereditary it has a decomposition tree $(T, f)$ with the following three properties [11].
Consider an edge \( e = \{p, c\} \) in \( T \) where \( p \) is the parent of \( c \). Let \( W_e \subset V \) be the set of vertices of \( G \) that are mapped by \( f \) to the leaves in the subtree rooted at \( c \). Let \( Q_e \subset W_e \) be the set of vertices in \( W_e \) that have neighbors in \( G - W_e \). The set \( Q_e \) is called the twinset of \( e \). The first property is that the subgraph of \( G \) induced by \( Q_e \) is a cograph for every edge \( e \) in \( T \).

Consider an internal vertex \( p \) in \( T \). Let \( c_1 \) and \( c_2 \) be the two children of \( p \). Let \( e_1 = \{p, c_1\} \) and let \( e_2 = \{p, c_2\} \). Let \( Q_1 \) and \( Q_2 \) be the twinsets of \( e_1 \) and \( e_2 \). The second property is that there is a join- or a union-operation between \( Q_1 \) and \( Q_2 \). Thus every vertex of \( Q_1 \) has the same neighbors in \( Q_2 \).

Let \( p \) be an internal vertex of \( T \) which is not the root. Let \( e \) be the line that connects \( p \) with its parent. Let \( Q_e \) be the twinset of \( e \). Let \( c_1 \) and \( c_2 \) be the two children of \( p \) in \( T \). Let \( e_1 = \{p, c_1\} \) and let \( e_2 = \{p, c_2\} \). Let \( Q_i \) be the twinset of \( e_i \), for \( i \in \{1, 2\} \). The third, and final, property is that

\[
Q_e = Q_1 \quad \text{or} \quad Q_e = Q_2 \quad \text{or} \quad Q_e = Q_1 \cup Q_2.
\]

When \( G \) is distance hereditary then a tree-decomposition for \( G \) with the three properties described above can be obtained in linear time [11].

Notice that the first property is a consequence of the other two. As an example, notice that cographs are distance hereditary. A cotree is a decomposition tree for a cograph with the three properties mentioned above.

**Theorem 5.** There exists a polynomial-time algorithm that solves the black-and-white coloring problem on distance-hereditary graphs.

**Proof.** Let \( (T, f) \) be a tree-decomposition for \( G \) which satisfies the properties mentioned above.

Define a boolean variable

\[
\gamma_e(b, w, b', w')
\]

for a subgraph induced by a branch rooted at a line \( e \) of \( T \). This variable is true if there exists a black-and-white coloring of the subgraph, induced by the vertices that are mapped to the leaves in the branch, with \( b \) black vertices and \( w \) white vertices, such that \( b' \) black vertices and \( w' \) white vertices are contained in the twinset of the branch.

First assume that \( e = \{p, c\} \) is an edge of \( T \) which connects a leaf \( c \) with its parent. Let \( Q \) be the twinset of \( e \), that is, \( Q = \emptyset \) or \( Q = \{c\} \). Then we have

(a) \( \gamma_e(0, 0, 0, 0) = true \),
(b) \( \gamma_e(1, 0, 0, 0) = \gamma_e(0, 1, 0, 0) = true \) if \( Q = \emptyset \),
(c) \( \gamma_e(1, 0, 1, 0) = \gamma_e(0, 1, 0, 1) = true \) if \( Q = \{c\} \), and
(d) \( \gamma_e(b, w, b', w') = false \) in all other cases.
Consider a node $p$ in $T$ with two children $c_1$ and $c_2$. Let $e_i = \{p, c_i\}$ for $i \in \{1, 2\}$. Let $Q_1$ and $Q_2$ be the twinsets of $e_1$ and $e_2$. If $p$ is not the root then let $Q$ be the twinset for the line that connects $p$ with its parent. We consider the following cases. First assume that there is a join between $Q_1$ and $Q_2$ and that $Q = Q_1 \cup Q_2$. Consider the case where there are no white vertices in the twinset $Q$. Then we have, for all $p, b, w$

$$\gamma_{e_i}(b, w, p, 0) = \text{true}$$

if and only if there exists partitions $p = p_1 + p_2$, $w = w_1 + w_2$ and $b = b_1 + b_2$ such that

$$\gamma_{e_1}(b_1, w_1, p_1, 0) = \text{true} \quad \text{and} \quad \gamma_{e_2}(b_2, w_2, p_2, 0) = \text{true}.$$ 

A similar formula holds for the case where there are no black vertices in the twinset.

Next, consider black-and-white colorings where there is at least one black, and at least one white vertex in the twinset $Q$. Then we have that all the black and white vertices of $Q$ must be in one of $Q_1$ and $Q_2$. In that case we have, for all $b, w$, and for all $p > 0$ and $q > 0$

$$\gamma_{e_i}(b, w, p, q) = \text{true}$$

if and only if there exist partitions $b = b_1 + b_2$, $w = w_1 + w_2$ such that

$$(\gamma_{e_1}(b_1, w_1, p, q) = \text{true} \quad \text{and} \quad \gamma_{e_2}(b_2, w_2, 0, 0) = \text{true}) \quad \text{or} \quad (\gamma_{e_1}(b_1, w_1, 0, 0) = \text{true} \quad \text{and} \quad \gamma_{e_2}(b_2, w_2, p, q) = \text{true}).$$

In the second case we assume that there is a join between $Q_1$ and $Q_2$ and that $Q = Q_1$. Obviously, we obtain the same formulas as above, except that the numbers of black and white vertices in the twinset $Q$ are copied from those numbers in $Q_1$. Thus we obtain that

$$\gamma_{e_i}(b, w, p, q) = \text{true}$$

if and only if there exist partitions $b = b_1 + b_2$, and $w = w_1 + w_2$ such that the following hold.

If $p > 0$ and $q > 0$:

$$\gamma_{e_1}(b_1, w_1, p, q) = \text{true} \quad \text{and} \quad \gamma_{e_2}(b_2, w_2, 0, 0) = \text{true}$$

if $p > 0$ and $q = 0$:

$$\gamma_{e_1}(b_1, w_1, p, 0) = \text{true} \quad \text{and} \quad \exists_{p_2} \gamma_{e_2}(b_2, w_2, p_2, 0) = \text{true}$$

if $p = 0$ and $q > 0$:

$$\gamma_{e_1}(b_1, w_1, 0, q) = \text{true} \quad \text{and} \quad \exists_{q_2} \gamma_{e_2}(b_2, w_2, 0, q_2) = \text{true}$$

and, if $p = 0$ and $q = 0$:

$$\gamma_{e_1}(b_1, w_1, 0, 0) = \text{true} \quad \text{and} \quad \exists_{p_2} \exists_{q_2} \gamma_{e_2}(b_2, w_2, p_2, q_2) = \text{true}.$$
Now assume that there is a union between $Q_1$ and $Q_2$. First assume that $Q = Q_1 \cup Q_2$. Then we have for all $b, w, p$ and $q$,

$$\gamma_e(b, w, p, q) = \text{true}$$

if and only if there exist partitions $b = b_1 + b_2$, $w = w_1 + w_2$, $p = p_1 + p_2$ and $q = q_1 + q_2$ such that

$$\gamma_{e_1}(b_1, w_1, p_1, q_1) = \text{true} \quad \text{and} \quad \gamma_{e_2}(b_2, w_2, p_2, q_2) = \text{true}.$$ 

Finally, assume that there is a union between $Q_1$ and $Q_2$ and that $Q = Q_1$. Then

$$\gamma_e(b, w, p, q) = \text{true}$$

if and only if there exist partitions $b = b_1 + b_2$ and $w = w_1 + w_2$, such that

$$\gamma_{e_1}(b_1, w_1, p, q) = \text{true} \quad \text{and} \quad \exists_{p_2} \exists_{q_2} \gamma_{e_2}(b_2, w_2, p_2, q_2) = \text{true}.$$ 

By symmetry, the remaining cases are similar.

When $p$ is the root, then the twinset $Q$ is not defined. To get around this obstacle we may simply add an edge $\hat{e}$ in the tree adjacent to $p$ and define the twinset $Q$ for this edge, arbitrarily, as $Q = Q_1 \cup Q_2$, or $Q = Q_1$, or $Q = Q_2$. There exists a black-and-white coloring of $G$ with $b$ black and $w$ white vertices if there are $p$ and $q$ such that

$$\gamma_{\hat{e}}(b, w, p, q) = \text{true}.$$ 

A table consists of $O(n^4)$ entries for values of $b$, $w$, $p$ and $q$ ranging from 0 up to $n$. For each node in the tree-decomposition, the value of each entry in the table can be computed in $O(n^8)$ time from the tables that are stored at the two children of the node. Therefore, a table at each node can be computed in $O(n^{12})$ time. Since the tree-decomposition has $O(n)$ nodes, this gives an upperbound of $O(n^{13})$ for solving the black-and-white coloring problem on distance-hereditary graphs. \(\square\)

### 4 Interval graphs

In this section we show that there is an efficient algorithm to solve the black-and-white coloring problem on interval graphs.

**Definition 6 ([28]).** A graph $G$ is an interval graph if it is the intersection graph of a collection of intervals on the real line.

There are various characterizations of interval graphs. For example, a graph is an interval graph if and only if it is chordal and it has no asteroidal triple. Also, a graph is an interval graph if and only if it has no $C_4$ and the complement $\overline{G}$ has a transitive orientation [19].

For our purposes the following characterization of interval graphs is suitable.
Theorem 6 ([19]). A graph $G$ is an interval graph if and only if there is a linear ordering $L$ of its maximal cliques such that for every vertex, the maximal cliques that contain that vertex are consecutive in $L$.

Interval graphs can be recognized in linear time. When $G$ is an interval graph then $G$ is chordal and so it has at most $n$ maximal cliques. A linear ordering of the maximal cliques can be obtained in $O(n^2)$ time [7].

Theorem 7. There exists an $O(n^6)$ algorithm that solves the black-and-white coloring problem on interval graphs.

Proof. Let $[C_1, \ldots, C_t]$ be a linear ordering of the maximal cliques of an interval graph $G = (V, E)$ such that for every vertex $x$, the maximal cliques that contain $x$ appear consecutively in this ordering.

Consider a black-and-white coloring of $G$. First assume that the first clique $C_1$ contains no black or white vertices. Then we may remove the vertices that appear in $C_1$ from the graph and consider a black-and-white coloring of the vertices in cliques of the linear ordering $[C_2, \ldots, C_t]$, where, for $i > 1$, $C_i = C_i \setminus C_1$.

Now assume that $C_1$ contains some black vertices. Then, obviously, $C_1$ contains no white vertices. Let $i$ be the maximal index such that all the cliques $C_\ell$ with $1 \leq \ell \leq i$ contain no white vertices. Remove all the vertices that appear in $C_1, \ldots, C_i$ from the remaining cliques and consider the ordering $[C_{i+1}^*, \ldots, C_t^*]$ where, for $\ell > i$, $C_\ell^* = C_\ell \setminus \bigcup_{k=1}^i C_k$.

Then we may take an arbitrary black-and-white coloring of the graph induced by the vertices $\bigcup_{\ell=i+1}^t C_\ell^*$ and color an arbitrary number of vertices in $\bigcup_{\ell=1}^i C_\ell$ black.

For this purpose define, for $p \leq q$,

$$X_{p,q} = \{ x \in V \mid x \in C_k \text{ if and only if } p \leq k \leq q \}.$$ 

Thus $X_{p,q}$ consists of the vertices of which the indices of the first and the last clique that contain the vertex are both in the interval $[p, q]$.

For $i \geq 1$ let $G_i$ be the graph with vertices in $\bigcup_{k=i}^t C_k$, where, for $k \geq i$, $C_k = C_k \setminus \bigcup_{\ell=1}^{i-1} C_\ell$.

The algorithm keeps a table with entries $b, w \in \{1, \ldots, n\}$ and the boolean value $\gamma_i(b, w)$ which is true if and only if there exists a black-and-white coloring.
of $G_i$ with $b$ black vertices and $w$ white vertices. Then we have, for $i = 1, \ldots, t$,
\[
\gamma_i(b, w) = \text{true} \quad \text{if and only if} \quad \exists j \geq i \exists k \quad 0 \leq k \leq |X_{i,j}| \quad \text{and} \quad \gamma_j(k) = \text{true}.
\]
\[
\{ (b, w) \in \{(k, 0), (0, k) \} \quad \text{if} \quad j = t, \quad \text{and} \quad 
\gamma_{j+1}(b-k, w) \quad \text{or} \quad \gamma_{j+1}(b, w-k) \quad \text{if} \quad j < t.
\]

To implement this algorithm one needs to compute the cardinalities $|X_{p,q}|$. Initialize $|X_{p,q}| = 0$. We assume that we have, for each vertex $x$, the index $F(x)$ of the first clique that contains $x$ and the index $L(x)$ of the last clique that contains $x$. Consider the vertices one by one. For a vertex $x$, add one to $|X_{p,q}|$ for all $p \leq F(x)$ and all $q \geq L(x)$. For each vertex $x$ we need to update $O(n^2)$ cardinalities $|X_{p,q}|$. Thus computing all cardinalities $|X_{p,q}|$ can be done in $O(n^3)$ time.

For each $i = 1, \ldots, t$, the table for $G_i$ contains $O(n^2)$ boolean values $\gamma_i(b, w)$. For the computation of each $\gamma_i(b, w)$ the algorithm searches the tables of $G_i$ for all $j > i$. Thus the computation of $\gamma_i(b, w)$ takes $O(n^3)$ time. Thus the full table for $G_i$ can be obtained in $O(n^5)$ time and it follows that the algorithm can be implemented to run in $O(n^6)$ time.

There exists a black-and-white coloring of $G$ with $b$ black vertices and $w$ white vertices if and only if $\gamma_1(b, w) = \text{true}$. This proves the theorem. \qed

5 Strongly chordal graphs

The class of interval graphs is contained in the class of strongly chordal graphs. In this section we generalize the results of Section 4 to the class of strongly chordal graphs.

**Definition 7.** Let $C = [x_1, \ldots, x_{2k}]$ be a cycle of even length. A chord $(x_i, x_j)$ in $C$ is an odd chord if the distance in $C$ between $x_i$ and $x_j$ is odd.

Recall that a graph is chordal if it has no induced cycle of length more than three [15, 20].

**Definition 8 ([16]).** A graph $G$ is strongly chordal if $G$ is chordal and each cycle in $G$ of even length at least six has an odd chord.

Farber discovered the strongly chordal graphs as a subclass of chordal graph for which the weighted domination problem is polynomial. The class of graphs is closely related to the class of chordal bipartite graphs [9].

There are many ways to characterize strongly chordal graphs. For example, a graph is strongly chordal if and only if its closed neighborhood matrix, or also, its clique matrix, is totally balanced [2, 3, 9, 16, 23, 27]. Strongly chordal graphs are also characterized by the property that they have no induced cycles of length more than three and no induced suns [9, 16]. For $k \geq 3$, a $k$-sun consists of a clique $C = \{c_1, \ldots, c_k\}$ and an independent set $S = \{s_1, \ldots, s_k\}$. Each vertex $s_i$, with $1 \leq i \leq k$, is adjacent to $c_1$ and to $c_{i-1}$ and $s_k$ is adjacent to $c_k$ and $c_1$.

Another way to characterize strongly chordal graphs is by the property that every induced subgraph has a simple vertex.
**Fig. 4.** A chordal graph is strongly chordal if it has no sun. The figure shows a 3-sun and a 4-sun.

**Definition 9.** A vertex $x$ in a graph $G$ is simple if for all $y, z \in N[x]$

$$N[y] \subseteq N[z] \quad \text{or} \quad N[z] \subseteq N[y].$$

Notice that a simple vertex is simplicial, that is, its neighborhood is a clique.

**Theorem 8 ([10, 16]).** A graph is strongly chordal if and only if every induced subgraph has a simple vertex.

### 5.1 Strongly chordal $k$-trees

We use Lehel's decomposition which decomposes a strongly chordal graph $G$ into a sequence of strongly chordal $k$-trees, for $k = 1, \ldots, n - 1$ such that every maximal clique of $G$ with cardinality $\ell + 1$ is a maximal clique in the strongly chordal $\ell$-tree.

In this section we show that there is a polynomial-time algorithm which solves the black-and-white coloring problem on strongly chordal $k$-trees.

**Definition 10 ([5, 26, 30, 32]).** A $k$-tree is a connected chordal graph which is either a $k$-clique or in which in which every maximal clique has cardinality $k + 1$.

Let $H = (V, E)$ be a strongly chordal $k$-tree and let $x \in V$. Let $C$ be a component of $H - N[x]$ and let

$$S = N(C) = \{x_1, \ldots, x_k\}.$$ 

Since $H$ has no sun, the neighborhoods in $C$ of any two vertices $x_i$ and $x_j$ are comparable. We assume that the vertices of $S$ are ordered such that

$$N(x_1) \cap C \subseteq \ldots \subseteq N(x_k) \cap C.$$ 

The component $C$ has exactly one vertex $c$ which is adjacent to all vertices of $S$. Let $C_1, \ldots, C_t$ be the components of $H[C] - c$. For each component $C_i$ we have that

$$N(C_i) = \{c\} \cup \{x_2, \ldots, x_k\}.$$
Obviously, the neighborhoods in $C_i$ of $x_2, \ldots, x_k$ are ordered by inclusion as above and there exists some function $f : \{1, \ldots, t\} \to \{1, \ldots, k\}$ such that

$$\forall 2 \leq i \leq k \ N(x_i) \cap C_i \begin{cases} \subseteq N(c) \cap C_i & \text{if } \ell \leq f(i), \text{ and} \\
\supseteq N(c) \cap C_i & \text{if } \ell > f(i). \end{cases}$$

Consider a black-and-white coloring of the vertices in $C$ with $b$ black vertices and $w$ white vertices. Assume that the vertex $c$ is colored black. For $i = 1, \ldots, t$, let $\gamma_i(b_i, w_i) = \text{true}$ if there exists a black-and-white coloring of $H[C_i]$ with $b_i$ black vertices and $w_i$ white vertices such that $c$ is not adjacent to any white vertex in $C$. Let $\gamma(b, w) = \text{true}$ if there exists a black-and-white coloring of the vertices in $C$ such that $c$ is colored black. Then

$$\gamma(b, w) = \text{true} \quad \text{if and only if}$$

$$\exists b_1 \cdots \exists b_t \exists w_1 \cdots \exists w_t$$

$$b = 1 + \sum_{i=1}^t b_i \quad \text{and} \quad w = \sum_{i=1}^t w_i \quad \text{and}$$

$$\forall 1 \leq i \leq t \ \gamma_i(b_i, w_i) = \text{true}.$$ 

Similar formulas can be obtained for the cases where $c$ is colored white and where $c$ is uncolored.

Notice that, in order to maintain $\gamma_i(b_i, w_i)$, it is sufficient to keep a table for each possible position $f(i)$ which the vertex $c$ can occupy in the neighborhood ordering of the component $C_i$.

Obviously, when $c$ is colored black, no vertex of $S$ can be colored white, since $c$ is adjacent to all vertices of $S$. Assume that a vertex $x_t \in S$ is colored black. Then $x_t$ is not adjacent to any white vertex in $C$. In order to know whether we can color $x_t$ black, it is sufficient to have the index in the neighborhood ordering of each $N(C_i)$, of the vertex (if any) with the smallest neighborhood in $C_i$ which is adjacent to any white vertex.

Summarizing, it suffices to keep a table of boolean values $\gamma(b, w, p, q)$. The value of $\gamma(b, w, p, q)$ is true if there exists a black-and-white coloring of $H[C]$ with $b$ black vertices and $w$ white vertices such that

1. $x_1, \ldots, x_p$ is not adjacent to any black vertex and, if $p < k$, $x_{p+1}$ is adjacent to a black vertex, and
2. $x_1, \ldots, x_q$ is not adjacent to any white vertex and, if $q < k$, $x_{q+1}$ is adjacent to a white vertex.

Notice that, e.g., $p = 0$ implies that $x_1$ is adjacent to a black vertex in $C$, that is, the vertex $c$ is colored black.

We omit the lengthy description of the recursive formula for $\gamma(b, w, p, q)$. 

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Consider a vertex \( x \). Possibly, \( H - N[x] \) contains more than one component. In order to combine the black-and-white colorings of different components in a table, we proceed as follows.

Choose a simplicial vertex \( r \) in \( H \) a ‘root.’ Let \( x \) be a vertex which is not adjacent to \( r \). Let \( C_x(r) \) be the component of \( H - N[x] \) that contains \( r \). Let the vertices in \( N(C_x(r)) \) be ordered

\[
N(C_x(r)) = \{ y_1, \ldots, y_k \} \quad \text{such that} \quad N(y_1) \cap C_x(r) \subseteq \cdots \subseteq N(y_k) \cap C_x(r).
\]

From now on, we consider only pairs \( x \) and \( C \) such that \( x \) is not adjacent to \( r \) and such that \( C \) is a component of \( G - N[x] \) that does not contain \( r \). For such a pair consider the vertex \( c \in C \) which is adjacent to all vertices of \( S = N(C) \). Let \( S = \{ y_1, \ldots, y_k \} \) be the vertices of \( S \) ordered such that

\[
N(y_1) \cap C_c(r) \subseteq \cdots \subseteq N(y_k) \cap C_c(r).
\]

Suppose we want to color a vertex \( y \in C_x(r) \) black. Let \( \ell \) be the smallest index such that \( y_\ell \) is adjacent to \( y \). We need to make sure that no vertex of \( \{ y_\ell, \ldots, y_k \} \) is colored black.

For that purpose, define the boolean variable \( \gamma'(b, w, p, q) \) as true if there exists a black-and-white coloring of the vertices in \( C \cup S \) with \( b \) black vertices and \( w \) white vertices such that

1. \( y_p, \ldots, y_k \) are not colored black, and
2. \( y_q, \ldots, y_k \) are not colored white.

Notice that the values \( \gamma'(b, w, p, q) \) can easily be deduced from the \( \gamma \)-table(s).

**Theorem 9.** There exists a polynomial-time algorithm which solves the black-and-white coloring problem on \( k \)-trees.

**Proof.** The algorithm sorts the pairs \( (x, C) \), where \( x \in V \) is a vertex not adjacent to \( r \) and where \( C \) is a component of \( H - N[x] \) that does not contain \( r \), in increasing order of \( |C| \). There are \( O(n^2) \) such pairs since each pair \( (x, C) \) is fixed by the pair of vertices \( x \) and \( c \), where \( c \in C \) is the vertex in \( C \) with

\[
N(C) = N(x) \cap N(c).
\]

For each pair compute a table of boolean values \( \gamma(b, w, p, q) \) from the tables at the components \( C_1, \ldots, C_t \) as outlined above. The components are added one by one. When a component \( C_t \) is handled an update is made for the suitable table entries of \( C \) by going through the table entries \( \gamma_t(b_i, w_i, p_i, q_i) \) of the component \( C_t \). There are \( O(n^2k^2) \) entries in each table, and since \( t \leq n \), a table for \( (x, C) \) is computed in \( O(n^3k^2) \) time.

In a similar manner compute the tables with boolean values \( \gamma'(b, w, p, q) \). As above, it is easy to update a table for a vertex \( x \) when there are two or more components of \( G - N[x] \) that do not contain \( r \). \( \square \)
5.2 The transition from k-trees to \((k + 1)\)-trees

Lehel’s decomposition for strongly chordal graphs \(G = (V, E)\) is a sequence of strongly chordal k-trees with vertex set \(V\) for \(k = 1, \ldots, n - 1\) such that every maximal clique of \(G\) is a maximal clique in one of the strongly chordal k-trees. The \((k + 1)\)-tree in this sequence is obtained from the k-tree by a construction that we describe next.

Let \(x\) and \(y\) be nonadjacent vertices in a graph \(G\). An \(x, y\)-separator is a set of vertices \(S\) such that \(x\) and \(y\) are in different components of \(G - S\). The \(x, y\)-separator is minimal if no proper subset of \(S\) separates \(x\) and \(y\) in different components. A set \(S\) is a minimal separator in \(G\) if there exist nonadjacent vertices \(x\) and \(y\) such that \(S\) is a minimal \(x, y\)-separator.

Rose characterizes chordal graphs by the property that every minimal separator is a clique [31]. In a k-tree \(H\) every minimal separator is a k-clique [32]. Consider a pair \((x, C)\) where \(x\) is a vertex in \(H\) and where \(C\) is a component of \(H - N[x]\). Then \(N(C)\) is a minimal separator since it separates \(x\) from every vertex in \(C\). If \(c \in C\) is adjacent to all vertices of \(N(C)\) then \(N(C)\) is the common neighborhood of \(x\) and \(c\) and so, no proper subset of \(N(C)\) separates \(x\) and \(c\). Furthermore, it is easy to see that every minimal separator in a k-tree is of this form.

Let \(T_k\) be a clique tree for a k-tree \(H\). A clique tree \(T_k\) for \(H\) is a tree of which the vertices are the maximal cliques in \(H\). The tree \(T_k\) satisfies the following property.

For every vertex \(x\) in \(H\) the maximal cliques that contain \(x\) form a subtree of \(T_k\).

Notice that, if \(C_1\) and \(C_2\) are adjacent cliques in \(T_k\) then \(C_1 \cap C_2\) is a minimal separator in \(H\). Since every minimal separator in \(H\) is a k-clique,

\[|C_1 \cap C_2| = k.\]

Thus each edge in \(T_k\) corresponds with a minimal separator in \(H\) and it is easy to see that this collection of minimal separators is the set of all the minimal separators in \(H\).

Define a \((k + 1)\)-tree \(H'\) as follows [27]. The maximal cliques of \(H'\) are the unions of maximal cliques that are endpoints of edges in \(T_k\). By the observation above, these maximal cliques have cardinality \(k + 2\). A clique tree \(T_{k+1}\) for \(H'\) is a spanning tree of the linegraph \(L(T_k)\).

Lehel proves that for every strongly chordal graph \(G\) there is a sequence \(H_k\) of k-trees such that every maximal clique in \(G\) is a maximal clique in one of the \(H_k\). Furthermore, each \(H_{k+1}\) with clique tree \(T_{k+1}\) is obtained from \(H_k\) with clique tree \(T_k\) by an operation as described above [27]. The starting clique tree \(T_0\) is called the basic tree in [27]. This basic tree \(T_0\) is any tree with vertex set \(V\) such that every maximal clique in \(G\) induces a subtree of \(T_0\).
We shortly analyze the transition of the k-tree $H$ into the $(k + 1)$-tree $H'$.

Let $H$ be a k-tree with clique tree $T_k$. Let $x$ be a vertex in $H$ and let $C$ be a component of $H - N_H[x]$. Let $S = N(C) = \{ x_1, \ldots, x_k \}$. We assume that

$$N_H(x_1) \cap C \subseteq \ldots \subseteq N_H(x_k) \cap C.$$

Let $c$ be the unique vertex in $C$ which is adjacent to $S$ in $H$. Let $C_1, \ldots, C_t$ be the components of $H[C] - c$. Each component $C_i$ has a unique vertex $c_i$ such that

$$N_H(C_i) = N_H(c_i) \cap N_H(x_1) = \{ c \} \cup \{ x_2, \ldots, x_k \}.$$

Assume that $S \cup \{ x \}$ is the parent of $S \cup \{ c \}$ in $T_k$. Every edge in $T_k$ merges into one clique of $H'$. Thus the two $(k + 1)$-cliques

$$S \cup \{ x \} \quad \text{and} \quad S \cup \{ c \}$$

merge into $S \cup \{ x, c \}$ in $H'$.

The component $C$ that contains $c$ consists of vertices that appear in maximal cliques in the subtree of $S \cup \{ c \}$.

Consider all the maximal cliques in $T_k$ that contain $\{ c, x_2, \ldots, x_k \}$. Notice that this includes the $(k + 1)$-cliques

$$\{ c_i \} \cup \{ c, x_2, \ldots, x_k \}.$$

By the Helly property (see, e.g., [18, 26]), these maximal cliques form a subtree of $T_k$ rooted at $S \cup \{ c \}$. It follows that Lehel's construction of the $(k + 1)$-tree $H'$ creates a tree $R$, with vertex set

$$\{ x_1, c_1, \ldots, c_t \},$$

rooted at $x_1$. Each edge in $R$ forms a $k + 2$-clique with $S$ in $H'$.

Obviously, not every maximal clique in $H'$ is a clique in $G$. Assume that $x$ and $c$ are not adjacent in $G$. Consider a vertex $y$ in $H'$ which is adjacent to $S \cup \{ x \}$ and which is not in $C$. Consider the computation of the table for the pair $y$ and $C$ in $H'$. The vertex $x$ is not adjacent to $c$ and so it is not adjacent to any vertex in $C$ in $H'$. In that case the variables $p$ and $q$ in $\gamma(b, w, p, q)$ are at least one, since $x$ is the smallest vertex in the neighborhood ordering and $x$ is not adjacent to any black or white vertex in $C$. Since $x$ enters the separator as a minimal vertex in the neighborhood ordering, the table for the pair $y$ and $C$ can be determined in the same manner as described in the previous section.

**Theorem 10.** There exists a polynomial-time algorithm which solves the black-and-white coloring problem on strongly chordal graphs.

**Proof.** An analysis of Lehel's decomposition of the strongly chordal graph into a sequence of $k$-trees shows that this decomposition can be obtained $O(n^4)$ time [27]. By the result of the previous section and the observations above, the tables for each $k$-tree can be obtained in $O(n^5k^2) = O(n^7)$ time. Since the list contains at most $n$ $k$-trees this proves the theorem. \hfill $\square$

---

4 Possibly, when $G$ is disconnected, the subtree contains also the vertices of some other components of $G$. Notice that a clique tree for a chordal graph may connect the clique trees of its components in some arbitrary way.
6 Splitgraphs

In this section we show that the black-and-white coloring problem on splitgraphs is NP-complete.

**Definition 11.** A graph \( G = (V, E) \) is a splitgraph if there exists a partition of the vertices in two sets \( C \) and \( S \) such that \( C \) induces a clique in \( G \) and \( S \) induces an independent set in \( G \). Here, one of the two sets \( C \) and \( S \) may be empty.

A splitgraph can be characterized in various ways. Notice that, if \( G \) is a splitgraph then \( G \) is chordal and, furthermore, its complement \( \overline{G} \) is also a splitgraph. Actually, this property characterizes splitgraphs [17]; a graph \( G \) is a splitgraphs if and only if \( G \) and its complement \( \overline{G} \) are both chordal. Splitgraphs are exactly the graphs that have no induced \( C_4 \), \( C_5 \) or \( 2K_2 \) [17].

![Fig. 5. A graph is a splitgraph if it has no \( C_4 \), \( C_5 \) or \( 2K_2 \).]

**Theorem 11.** The black-and-white coloring problem is NP-complete for the class of splitgraphs.

**Proof.** Since splitgraphs are closed under complementation, we can formulate the problem as a black-and-white coloring problem with all black vertices adjacent to all white vertices. We call this the ‘inverse B&W-coloring problem.’

We adapt a proof of Johnson, which proves the NP-completeness of finding a balanced complete bipartite subgraph in a bipartite graph [25, Page 446].

Let \( G = (V, E) \) be a graph with \( |V| = n \). Construct a splitgraph \( H \) as follows.

The clique of the splitgraph consists of the set \( V \). The independent set of the splitgraph consists of the set \( E \). In the splitgraph, make a vertex \( x \in V \) adjacent to an edge \( \{y, z\} \in E \) if and only if \( x \) is not an endpoint of \( \{y, z\} \).

This completes the description of \( H \).

Assume that the clique number of \( G \) is \( \omega \). We may assume that \( n \) is even and \( n > 6 \), and that \( \omega = \frac{n}{2} \) [25].

Then we have an inverse B&W-coloring of \( H \) with

\[
\begin{align*}
    b &= \omega, \\
    w &= \omega + \left( \omega + 1 \right) = \left( \frac{\omega + 1}{2} \right).
\end{align*}
\]

(2)

For the converse, assume that \( H \) has an inverse B&W-coloring with the numbers of black and white vertices as in Equation 2. Since \( E \) is an independent set
in $H$ the colored vertices in $E$ must all have the same color. First assume that $E$ contains no white vertices. Then $V$ contains a set $W$ of white vertices, and $V \setminus W$ is black. Since
\[
w = \omega + \binom{\omega}{2} > n = 2\omega \quad \text{if } n > 6,
\]
this is not possible. Thus the inverse black-and-white coloring has white vertices in $E$.

Assume that the inverse B&W-coloring has a set $E'$ of white vertices in $E$ and a set of $V'$ of $\omega$ black vertices in $V$. By the construction, no edge of $E'$ has an endpoint in $V'$. Now $|V \setminus V'| = \omega$ and all the endpoints of $E'$ are in $V \setminus V'$. The only possibility is that $E'$ is the set of edges of a clique $V \setminus V'$ of cardinality $\omega$ in $G$.

This proves the theorem. $\square$

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