GENERALIZED METALLIC STRUCTURES

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Abstract. We study the properties of a generalized metallic, a generalized product and a generalized complex structure induced on the generalized tangent bundle of a smooth manifold \( M \) by a metallic Riemannian structure \((J, g)\) on \( M \), providing conditions for their integrability with respect to a suitable connection. Moreover, by using methods of generalized geometry, we lift \((J, g)\) to metallic Riemannian structures on the tangent and cotangent bundles of \( M \), underlying the relations between them.

1. Preliminaries

On a smooth manifold \( M \), besides the almost complex, almost tangent, almost product structures, etc., some other polynomial structures can be considered as \( \mathcal{C}^\infty \)-tensor fields \( J \) of \((1,1)\)-type which are roots of the algebraic equation

\[
Q(J) := J^n + a_n J^{n-1} + \cdots + a_2 J + a_1 I = 0,
\]

where \( I \) is the identity operator on the Lie algebra of vector fields on \( M \). In particular, if \( Q(J) := J^2 - pJ - qI \), with \( p \) and \( q \) positive integers, its solution \( J \) will be called metallic structure \([2]\). The name is motivated by the fact that the \((p,q)\)-metallic number introduced by Vera W. de Spinadel \([8]\) is precisely the positive root of the quadratic equation \( x^2 - px - q = 0 \), namely \( \sigma_{p,q} := \frac{p + \sqrt{p^2 + 4q}}{2} \). For example: if \( p = q = 1 \) we get the golden number \( \sigma = \frac{1 + \sqrt{5}}{2} \); if \( p = 2 \) and \( q = 1 \) we get the silver number \( \sigma_{2,1} = 1 + \sqrt{2} \); if \( p = 3 \) and \( q = 1 \) we get the bronze number \( \sigma_{3,1} = \frac{3 + \sqrt{13}}{2} \); if \( p = 1 \) and \( q = 2 \) we get the copper number \( \sigma_{1,2} = 2 \); if \( p = 1 \) and \( q = 3 \) we get the nickel number \( \sigma_{1,3} = \frac{1 + \sqrt{13}}{2} \), and so on.

We shall briefly recall the basic notions of metallic (Riemannian) geometry.

Definition 1.1 \([3]\). A metallic structure \( J \) on \( M \) is an endomorphism \( J : TM \to TM \) satisfying

\[
J^2 = pJ + qI,
\]

for some \( p, q \in \mathbb{N}^* \). The pair \((M, J)\) is called a metallic manifold. Moreover, if a Riemannian metric \( g \) on \( M \) is compatible with \( J \), that is \( g(JX, Y) = g(X, JY) \),

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for any \( X, Y \in C^\infty(TM) \), we call the pair \((J, g)\) a \textit{metallic Riemannian structure} and \((M, J, g)\) a \textit{metallic Riemannian manifold}.

The concept of integrability for a metallic structure is defined in the classical manner.

**Definition 1.2.** A metallic structure \( J \) is called \textit{integrable} if its Nijenhuis tensor field

\[
N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]
\]

vanishes for all \( X, Y \in C^\infty(TM) \).

It is known \[^3\] that an almost product structure \( F \) on \( M \) induces two metallic structures:

\[
J^\pm = \pm \frac{2\sigma_{p,q} - p}{2} F + \frac{p}{2} I
\]

and, conversely, every metallic structure \( J \) on \( M \) induces two almost product structures:

\[
F^\pm = \pm \left( \frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I \right),
\]

where \( \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2} \) is the metallic number, for \( p, q \in \mathbb{N}^* \).

In particular, if the almost product structure \( F \) is compatible with a Riemannian metric \( g \), then \((J^+, g)\) and \((J^-, g)\) are metallic Riemannian structures.

The analogue concept of locally product manifold is considered in the context of metallic geometry.

**Definition 1.3** \(^{[1]}\). A metallic Riemannian manifold \((M, J, g)\) is called \textit{locally metallic} if \( J \) is parallel with respect to the Levi-Civita connection \( \nabla \) of \( g \), that is \( \nabla J = 0 \).

In the following, we shall extend the definition of a metallic structure for any real numbers \( p \) and \( q \). In this way, we also include some other well-known structures; for instance, if \((p, q)\in\{(0, -1), (0, 0), (0, 1), (1, 0)\}\), the solution of \((1.1)\) would yield an almost complex, an almost tangent, an almost product and a \( J(2,1) \)-structure, respectively.

2. **Generalized structures induced by metallic structures**

Let \( TM \oplus T^*M \) be the generalized tangent bundle of a smooth manifold \( M \).

**Definition 2.1.** A \textit{generalized metallic structure} \( \hat{J} \) on \( M \) is an endomorphism \( \hat{J} : TM \oplus T^*M \to TM \oplus T^*M \) satisfying

\[
\hat{J}^2 = p\hat{J} + qI,
\]

for some real numbers \( p \) and \( q \).

For a linear connection \( \nabla \) on \( M \), we consider the bracket \([\cdot, \cdot]_\nabla \) on \( C^\infty(TM \oplus T^*M) \) \[^6\]:

\[
[X + \alpha, Y + \beta]_\nabla := [X, Y] + \nabla_X \beta - \nabla_Y \alpha,
\]

for all \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \).
Definition 2.2. A generalized metallic structure \( \hat{J} \) is called \( \nabla \)-integrable if its Nijenhuis tensor field \( N^\nabla \) with respect to \( \nabla \),
\[
N^\nabla_\sigma(\tau, \nu) := [\hat{J}\sigma, \hat{J}\tau]_{\nabla} - \hat{J}[\hat{J}\sigma, \hat{J}\tau]_{\nabla} - \hat{J}[\sigma, \tau]_{\nabla} + \hat{J}^2[\sigma, \tau]_{\nabla},
\]
vanishes for all \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \).

2.1. Generalized metallic structure induced by \( (J,g) \). Let \( (J,g) \) be a metallic Riemannian structure on \( M \) such that \( J^2 = pJ + qI \), \( p, q \in \mathbb{R} \). If we denote by \( \sharp_g : T^*M \to TM \) the inverse of the isomorphism \( \flat_g : TM \to T^*M \), \( \flat_g(X) := i_Xg \), from the \( g \)-symmetry of \( J \) we have \( \sharp_g \circ J^* = J \circ \sharp_g \) and \( \flat_g \circ J = J^* \circ \flat_g \), where \( (J^*\alpha)(X) := \alpha(JX) \). Also notice that \( J^* \) is a metallic structure too, namely, \( (J^*)^2 = pJ^* + qI \), and easily get that \( \sharp_g \circ (J^*)^k = J^k \circ \sharp_g \) and \( \flat_g \circ J^k = (J^*)^k \circ \flat_g \), for any \( k \in \mathbb{N} \).

On \( TM \oplus T^*M \) we consider the Riemannian metric \( \hat{g}(X + \alpha, Y + \beta) := g(X, Y) + g(\sharp_g\alpha, \sharp_g\beta) \), (2.1) for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \).

Definition 2.3. A pair \( (\hat{J}, \hat{g}) \) of a generalized metallic structure \( \hat{J} \) and a Riemannian metric \( \hat{g} \) such that \( \hat{J} \) is \( \hat{g} \)-symmetric is called generalized metallic Riemannian structure.

Remark that the generalized metallic structure \( \hat{J}_m := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \) induced by the metallic Riemannian structure \( (J,g) \) is \( \hat{g} \)-symmetric; hence, \( (\hat{J}_m, \hat{g}) \) is a generalized metallic Riemannian structure.

Proposition 2.4. The generalized metallic structure \( \hat{J}_m \) induced by the metallic Riemannian structure \( (J,g) \) on \( M \) is \( \nabla \)-integrable if and only if \( J \) is integrable and \( (\nabla \nabla_J J) = (\nabla_X J)J \), for any \( X \in C^\infty(TM) \).

Proof. We have:
\[
N^\nabla_{\hat{J}_m}(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] = N_J(X,Y)
\]
\[
N^\nabla_{\hat{J}_m}(X,\beta) = [JX, J^*\beta]_{\nabla} - J^*[JX, \beta]_{\nabla} - J^*[X, J^*\beta]_{\nabla} + (J^*)^2[X, \beta]_{\nabla}
\]
\[
= \nabla_{JX}J^*\beta - J^*\nabla_{JX}\beta - J^*\nabla_X J^*\beta + (J^*)^2\nabla_X \beta
\]
\[
= ((\nabla_{JX} J^*) - J^*(\nabla_X J^*)) (\beta)
\]
\[
= \beta((\nabla_{JX} J) - (\nabla_X J) J)
\]
\[
N^\nabla_{\hat{J}_m}(\alpha, \beta) = 0,
\]
for all \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \). The proof is thus complete. \( \square \)

Remark that if \( \nabla \) is a \( J \)-connection, that is \( \nabla J = 0 \), then \( \hat{J}_m \) is \( \nabla \)-integrable if and only if \( J \) is integrable. Moreover, if \( T^\nabla \) is the torsion of \( \nabla \), \( T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \), then a direct computation gives:
\[
N_J(X, Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X - J(\nabla_X J)Y + \Phi(T^\nabla)(X,Y),
\]
where

\[ \Phi(T^\nabla)(X,Y) := -T^\nabla(JX,JY) + JT^\nabla(JX,Y) + JT^\nabla(X,JY) - J^2T^\nabla(X,Y). \]

In particular, if \( \nabla \) is a torsion free \( J \)-connection, then \( \hat{J}_m \) is \( \nabla \)-integrable.

Let \( \nabla^g \) be the Levi-Civita connection of \( g \) and define a linear connection \( D \) on \( M \) by \( D := \nabla^g + F \), where \( F \) is a \((1,2)\)-type tensor field such that

\[
\begin{align*}
DJ &= 0 \\
g &= 0.
\end{align*}
\]

This is equivalent to

\[
\begin{align*}
(\nabla^g_X J)Y &= J(F(X,Y)) - F(X,JY) \\
g(F(X,Y),Z) + g(Y,F(X,Z)) &= 0
\end{align*}
\]

for any \( X, Y, Z \in C^\infty(TM) \).

Consider the bracket \([\cdot,\cdot]_D\) on \( C^\infty(TM \oplus T^*M) \):

\[
[X + \alpha, Y + \beta]_D := [X, Y] + DX\beta - DY\alpha,
\]

for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \).

Define the connection \( \hat{D} \) on \( TM \oplus T^*M \) by

\[
\hat{D}X(Y + \beta) := DXY + DX\beta,
\]

for any \( X, Y \in C^\infty(TM) \) and \( \beta \in C^\infty(T^*M) \). It follows that

\[
\hat{D}X(Y + \beta) = \nabla^g_X Y + F(X,Y) + \nabla X\beta - \beta \circ F(X, \cdot).
\]

Let \( n \) be the dimension of \( M \) and assume that \( q \neq 0 \). Denote by \( \{x^1, \ldots, x^n\} \) the local coordinates on \( M \) and let \( \{X_1, \ldots, X_n\} \) be the corresponding local frame for \( TM \). Following [4] we define:

\[
F(X_i, X_j) := \omega(X_j)X_i - \omega(X_i)g^{lk}g_{ij}X_k + \frac{1}{q}\omega(JX_j)JX_i - \frac{1}{q}\omega(JX_i)g^{lk}J_l^sJ_sX_k,
\]

where \( \omega \) is a 1-form on \( M \) and we use Einstein’s convention of summation.

We immediately have that \( g(F(X_i, X_j), X_r) + g(X_j, F(X_i, X_r)) = 0 \), for all \( i, j, r \); therefore, \( Dg = 0 \), for any 1-form \( \omega \). Moreover, the torsion of \( D \) is given by

\[
T^D(X,Y) = \omega(Y)X - \omega(X)Y + \frac{1}{q}\omega(JY)JX - \frac{1}{q}\omega(JX)JY,
\]

for any \( X, Y \in C^\infty(TM) \).

**Lemma 2.5.** \( T^D \) satisfies the following properties:

\[
T^D(JX,Y) = JT^D(X,Y) = T^D(X,JY)
\]

\[
\Phi(T^D)(X,Y) = 0,
\]

for any \( X, Y \in C^\infty(TM) \).
\textit{Proof.} From a direct computation we get
\[ JT_D(X, Y) = \omega(Y)JX - \omega(X)JY + \frac{p}{q}\omega(JY)JX - \frac{p}{q}\omega(JX)JY + \omega(JY)X - \omega(JX)Y, \]
which is equal to \( T^D(JX, Y) \) and \( T^D(X, JY) \).
Consequently, we have \( \Phi(T^\nabla)(X, Y) = 0 \). \( \square \)

Recently, C. Karaman \[4\] constructed metallic semi-symmetric metric \( J \)-connections \( D \) on locally decomposable metallic Riemannian manifolds \((M, J, g)\). These connections satisfy:
\[ DJ = 0, \ Dg = 0, \ T^D(X, Y) = \omega(Y)X - \omega(X)Y + \frac{1}{q}\omega(JY)JX - \frac{1}{q}\omega(JX)JY, \]
for any \( X, Y \in C^\infty(TM) \). In particular, we can state the following:

\textbf{Proposition 2.6.} Let \((M, J, g)\) be a locally decomposable metallic Riemannian manifold and let \( D \) be a metallic semi-symmetric metric \( J \)-connection. Then \( \hat{J}_m \) is \( D \)-integrable.

\textbf{Proposition 2.7.} Let \( \left( \hat{J}_m := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \hat{g} \right) \) be the generalized metallic Riemannian structure induced by the metallic Riemannian structure \((J, g)\) on \( M \) with \( \hat{g} \) the Riemannian metric defined by (2.1). Then:
\begin{enumerate}
  \item \( \hat{D}\hat{J}_m = 0 \) if and only if \( DJ = 0 \);
  \item \( \hat{D}\hat{g} = 0 \) if and only if \( Dg \).
\end{enumerate}

\textit{Proof.} Remark that \( \hat{D}\hat{J}_m = 0 \) is equivalent to \((DX J)Y + \beta \circ DX J = 0\), for any \( X, Y \in C^\infty(TM) \) and \( \beta \in C^\infty(T^*M) \), and \( \hat{D}\hat{g} = 0 \) is equivalent to \((DX g)(Y, Z) - (DX g)(\sharp g \beta, \sharp g \gamma) = 0\), for any \( X, Y, Z \in C^\infty(TM) \) and \( \beta, \gamma \in C^\infty(T^*M) \). \( \square \)

\textbf{Definition 2.8.} A smooth map \( f \) between two metallic manifolds \((M_1, J_1)\) and \((M_2, J_2)\) is called metallic if \( f_* \circ J_1 = J_2 \circ f_* \).

\textbf{Remark 2.9.} A metallic diffeomorphism \( f \) between two metallic manifolds \((M_1, J_1)\) and \((M_2, J_2)\) naturally induces an isomorphism \( \hat{f} \) between their generalized tangent bundles defined by
\[ \hat{f} : TM_1 \oplus T^*M_1 \rightarrow TM_2 \oplus T^*M_2, \quad \hat{f}(X + \alpha) := f_*X + ((f_*)^{-1})\alpha, \]
where \( f_* : TM_1 \rightarrow TM_2 \) is the tangent map of \( f \) and \((f_*)^* : T^*M_2 \rightarrow T^*M_1 \) is the dual map of \( f_* \), that is, \((f_*)^*\alpha)(X) := \alpha(f_*X)\), for all \( \alpha \in C^\infty(T^*M_2) \) and \( X \in C^\infty(TM_1) \), which preserves the generalized metallic structures \( \hat{J}_{i,m} := \begin{pmatrix} J_i & 0 \\ 0 & J^*_i \end{pmatrix} \), \( i = 1, 2 \). Indeed, from \( f_* \circ J_1 = J_2 \circ f_* \) follows that \((f_*)^* \circ J^*_2 = J_1^* \circ (f_*)^*\), hence \( \hat{f} \circ \hat{J}_{1,m} = \hat{J}_{2,m} \circ \hat{f} \).
In particular, if \( f : M \rightarrow M \) is a diffeomorphism which preserves the metallic structure \( J \), then \( \hat{f} \) can be defined by
\[ \hat{f}(X + \alpha) := f_*X + (f_*)^*\alpha, \]
which coincides with the generalized metallic structure $\hat{J}_m$ when $J = f_*$. In this case, $J$ is invertible and $J^{-1} = \frac{1}{q} J - \frac{p}{q} I$, for $q \neq 0$.

2.2. Generalized product structure induced by $(J,g)$. Let $(J,g)$ be a metallic Riemannian structure on $M$ such that $J^2 = pJ + qI$, $p, q \in \mathbb{R}$. Then $\hat{J}_p := \left( \begin{array}{cc} J & (I - J^2) b_g \\ b_g & -J^* \end{array} \right)$ is a generalized product structure on $M$, that is $\hat{J}_p^2 = I$.

A direct computation gives the following.

Proposition 2.10. The generalized product structure $\hat{J}_p$ induced by the metallic Riemannian structure $(J,g)$ on $M$ is $\nabla$-integrable if and only if the following conditions are satisfied:

\[
\begin{align*}
N_J - (I - J^2) b_g(d\nabla g) &= 0 \\
(\nabla_{JX} g)Y - (\nabla_{JY} g)X + J^*((\nabla_{X} g)Y - (\nabla_{Y} g)X) + g((\nabla_Y J)X - (\nabla_X J)Y) \\
&+ g(T^\nabla(X, JY) + T^\nabla(JX, Y)) = 0 \\
(d^\nabla)(I - J^2)Y, X) - (\nabla_X J^*)g(Y) + (\nabla_{X} g)(JY) &= 0 \\
(\nabla_{(I-J^2)X} J^*)g(Y) - (\nabla_{(I-J^2)Y} J^*)g(X) &= 0 \\
(\nabla_{(I-J^2)X} J^2)Y - (\nabla_{(I-J^2)Y} J^2)X + (\nabla_{X} g)(JY) + (\nabla_{X} J^2)Y - J^2(\nabla_{X} J)Y &- (I - J^2) b_g((\nabla_{JX} g)Y - (\nabla_{JX} g)Y) \\
&+ \nabla(JX, (I - J^2)Y) + J\nabla(JX, (I - J^2)Y) = 0,
\end{align*}
\]

for all $X, Y \in C^\infty(TM)$, where we denoted $b_g$ by $g$ and the exterior differential associated to $\nabla$ acting on $g$ by $(d^\nabla g)(X,Y) := (\nabla_X g)(Y) - (\nabla_Y g)(X) + g(T^\nabla(X, Y))$.

Proposition 2.11. Let $(M, J, g)$ be a locally metallic Riemannian manifold. Then $\hat{J}_p$ is $\nabla$-integrable, for $\nabla$ the Levi-Civita connection of $g$.

Proof. From the previous proposition, we have that the generalized product structure $\hat{J}_p$ is $\nabla$-integrable if and only if the following conditions are satisfied:

\[
\begin{align*}
N_J &= 0 \\
(\nabla_Y J)X - (\nabla_X J)Y &= 0 \\
(\nabla_X J^*)Y - (\nabla_{X} J^*)Y &= 0 \\
(\nabla_{(I-J^2)X} J^*)g(Y) - (\nabla_{(I-J^2)Y} J^*)g(X) &= 0 \\
(\nabla_{(I-J^2)X} J^2)Y - (\nabla_{(I-J^2)Y} J^2)X &= 0 \\
-(\nabla_{JX} J^2)Y - (\nabla_{(I-J^2)Y} J^2)X + (\nabla_{X} J)Y + J(\nabla_{X} J^2)Y - J^2(\nabla_{X} J)Y &= 0 \\
-(\nabla_{JX} J^2)Y + (\nabla_{(I-J^2)Y} J^2)X - (\nabla_{X} J)Y + J(\nabla_{X} J^2)Y - J^2(\nabla_{X} J)Y &= 0,
\end{align*}
\]

for all $X, Y \in C^\infty(TM)$. In particular, if $\nabla J = 0$, then $\hat{J}_p$ is $\nabla$-integrable. \qed
Definition 2.12. A generalized product structure \( \hat{J} \) on \( M \) is called _anti-pseudo-calibrated_ if it is \( (\cdot, \cdot) \)-anti-invariant and the bilinear symmetric form defined by \( (\cdot, \hat{J} \cdot) \) on \( TM \) is non-degenerate, where

\[
(X + \alpha, Y + \beta) := -\frac{1}{2}(\alpha(Y) - \beta(X))
\]

is the natural symplectic structure on \( TM \oplus T^*M \).

Remark 2.13. The generalized product structure \( \hat{J}_p \) is anti-pseudo-calibrated with respect to \( (\cdot, \cdot) \).

Proposition 2.14. Let \( \hat{J}_p \) be the generalized product structure defined by the metallic Riemannian structure \( (J, g) \) on \( M \). Then

\[
G(\sigma, \tau) := (\sigma, \hat{J}_p(\tau)),
\]

with \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \), is a neutral metric.

Proof. Locally we can write \( 2G \) in block matrix form as:

\[
\begin{pmatrix}
g & -J \\
-J & -(I - J^2)\sharp_g
\end{pmatrix}.
\]

As \( J \) is \( g \)-symmetric, pointwise, we can take \( g = I \) and \( J = \Lambda \) the diagonal matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) which are solutions of the metallic equation \( \lambda^2 - p\lambda - q = 0 \). Then we get:

\[
\begin{pmatrix}
I & -\Lambda \\
-\Lambda & p\Lambda + (q - 1)I
\end{pmatrix}.
\]

In order to compute the indices of \( 2G \), we can use the Gauss–Lagrange algorithm and by elementary operations on rows and columns of the matrix we get the form

\[
\begin{pmatrix}
I & 0 \\
0 & -I + (\Lambda^2 - p\Lambda - qI)
\end{pmatrix};
\]

therefore

\[
\begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}.
\]

hence \( 2G \) has \( n \) positive and \( n \) negative eigenvalues and the proof is complete. \( \square \)

Proposition 2.15. Let \( \hat{J}_p := \left( J, (I - J^2)^\sharp_g \right), \hat{g} \) be the generalized product structure induced by the metallic Riemannian structure \( (J, g) \) on \( M \), with \( \hat{g} \) the Riemannian metric defined by \( (2.1) \). Then

\[
\hat{D}\hat{J}_p = 0 \text{ if and only if } DJ = 0 \text{ and } Dg = 0.
\]

Proof. Remark that \( (\hat{D}_Y \hat{J}_p) X = (D_Y J) X + (D_Y g) X \), for any \( X, Y \in C^\infty(TM) \) and \( (\hat{D}_Y \hat{J}_p) \alpha = -p(D_Y (J^\sharp g)) \alpha - (q - 1)(D_Y \sharp g) \alpha - (D_Y J^* \alpha), \) for any \( Y \in C^\infty(TM) \) and \( \alpha \in C^\infty(T^*M) \), therefore the statement. \( \square \)
Remark 2.16. Starting with a metallic structure on a manifold, with minimal restrictions on $p$ and $q$, some other generalized metallic structures on its generalized tangent bundle can be constructed as follows.

The metallic structure $J$ on $M$ induces two almost product structures on $M$:

$$F^\pm := \pm \left( \frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I \right);$$

the almost product structures $F^\pm$ induce two generalized product structures on $TM \oplus T^*M$:

$$\hat{F}^\pm := \begin{pmatrix} F^\pm & 0 \\ 0 & (F^\pm)^* \end{pmatrix};$$

and the generalized product structures $\hat{F}^\pm$ induce two generalized metallic structures on $TM \oplus T^*M$:

$$\hat{\sigma}_{+,-m} := \pm \frac{2\sigma_{p,q} - p}{2} \hat{F}^+ + p \frac{I}{2}, \quad \hat{\sigma}_{-,m} := \pm \frac{2\sigma_{p,q} - p}{2} \hat{F}^- + p \frac{I}{2},$$

where

$$\hat{\sigma}_{+,-m} = \frac{J}{\sigma} \left( \begin{array}{c} 0 \\ \sigma \end{array} \right),$$

and

$$\hat{\sigma}_{-,m} = \frac{J}{\sigma} \left( \begin{array}{c} -J + pI \\ \sigma \end{array} \right).$$

The metallic structure $J$ on $M$ induces a generalized product structure on $TM \oplus T^*M$:

$$\hat{\sigma}_p := \begin{pmatrix} J & (I - J^2)\sigma_g \\ -J^* & -\sigma \end{pmatrix},$$

and the generalized product structure $\hat{\sigma}_p$ induces two generalized metallic structures on $TM \oplus T^*M$:

$$\hat{\sigma}_m := \pm \frac{2\sigma_{p,q} - p}{2} \hat{\sigma}_p + p \frac{I}{2},$$

namely,

$$\hat{\sigma}_m = \left( \begin{array}{c} \frac{2\sigma_{p,q} - p}{2} J + p \frac{I}{2} \\ \sigma \end{array} \right),$$

and

$$\hat{\sigma}_m = \left( \begin{array}{c} -\frac{2\sigma_{p,q} - p}{2} J + p \frac{I}{2} \\ \sigma \end{array} \right).$$

2.3. Generalized complex structure induced by $(J, g)$. Let $(J, g)$ be a metallic Riemannian structure on $M$ such that $J^2 = pJ + qI$, $p, q \in \mathbb{R}$. Then $\hat{\sigma}_c := \begin{pmatrix} J & -(I + J^2)\sigma_g \\ -J^* & -\sigma \end{pmatrix}$ is a generalized complex structure on $M$, that is, $\hat{\sigma}_c^2 = -I$. A direct computation gives the following.
The generalized complex structure \( \hat{J}_c \) induced by the metallic Riemannian structure \((J, g)\) on \(M\) is \(\nabla\)-integrable if and only if the following conditions are satisfied:

\[
N_J + (I + J^2)\alpha_g (d\nabla g) = 0
\]
\[
(\nabla_J X) Y - (\nabla_J Y) X + J^*((\nabla_X g) Y - (\nabla_Y g) X) + g((\nabla_Y J) X - (\nabla_X J) Y)
\]
\[
+ g(T\nabla (X, JY) + T\nabla (JX, Y)) = 0
\]
\[
(d\nabla g)((I + J^2) X, Y) + (\nabla_X J^*) g(Y) - (\nabla_J X J^*) g(Y) = 0
\]
\[
(\nabla_{(I + J^2) X} J^2) Y - (\nabla_{(I + J^2) Y} J^2) X - T\nabla ((I + J^2) X, (I + J^2) Y)
\]
\[
- (I + J^2)\alpha_g ((\nabla_{(I + J^2) X} g) Y - (\nabla_{(I + J^2) Y} g) X) = 0
\]
\[
- (\nabla_J X J^2) Y + (\nabla_{(I + J^2) Y} J) X - (\nabla_X J) Y + J((\nabla_X J^2) Y - J^2(\nabla_X J) Y)
\]
\[
+ (I + J^2)\alpha_g ((\nabla_J X g) Y - (\nabla_Y g) J Y)
\]
\[
+ T\nabla (J X, (I + J^2) Y) - JT\nabla (X, (I + J^2) Y) = 0,
\]

for all \(X, Y \in C^\infty(TM)\), where we denoted \(\alpha_g\) by \(g\) and the exterior differential associated to \(\nabla\) acting on \(g\) by \((d\nabla g)(X, Y) := (\nabla_X g)(Y) - (\nabla_Y g)(X) + g(T\nabla (X, Y)).\)

**Proposition 2.18.** Let \((M, J, g)\) be a locally metallic Riemannian manifold. Then \(\hat{J}_c\) is \(\nabla\)-integrable, for \(\nabla\) the Levi-Civita connection of \(g\).

**Proof.** From the previous proposition, we have that the generalized complex structure \(\hat{J}_c\) is \(\nabla\)-integrable if and only if the following conditions are satisfied:

\[
N_J = 0
\]
\[
(\nabla_J X) Y - (\nabla_X J) Y = 0
\]
\[
(\nabla_X J^*) J^* - (\nabla_J X J^*) = 0
\]
\[
(\nabla_{(I + J^2) X} J^2) Y - (\nabla_{(I + J^2) Y} J^2) X = 0
\]
\[
- (\nabla_J X J^2) Y + (\nabla_{(I + J^2) Y} J) X - (\nabla_X J) Y + J((\nabla_X J^2) Y - J^2(\nabla_X J) Y) = 0,
\]

for all \(X, Y \in C^\infty(TM)\). In particular, if \(\nabla J = 0\), then \(\hat{J}_c\) is \(\nabla\)-integrable. \(\square\)

**Definition 2.19.** A generalized complex structure \(\hat{J}\) on \(M\) is called *calibrated* if it is \((\cdot, \cdot)\)-invariant and the bilinear symmetric form defined by \((\cdot, \hat{J}\cdot)\) on \(TM\) is non-degenerate and positive definite, where

\[
(X + \alpha, Y + \beta) := -\frac{1}{2}(\alpha(Y) - \beta(X))
\]

is the natural symplectic structure on \(TM \oplus T^*M\).

**Remark 2.20.** The generalized complex structure \(\hat{J}_c\) is calibrated with respect to \((\cdot, \cdot)\).
Proposition 2.21. Let \((\hat{J}_c := \begin{pmatrix} J & -(I + J^2) \hat{g} \\ \hat{g} & -J^* \end{pmatrix}, \hat{g})\) be the generalized complex structure induced by the metallic Riemannian structure \((J, g)\) on \(M\) with \(\hat{g}\) the Riemannian metric defined by \((2.1)\). Then:

\[
\hat{D}\hat{J}_c = 0 \quad \text{if and only if} \quad DJ = 0 \quad \text{and} \quad Dg = 0.
\]

**Proof.** Remark that \((\hat{D}_Y \hat{J}_c)X = (D_Y J)X + (D_Y g)X\), for any \(X, Y \in C^\infty(TM)\) and \((\hat{D}_Y \hat{J}_c)\alpha = -p(D_Y (J^*_g)g)\alpha - (q + 1)(D_Y \hat{g})\alpha - (D_Y J^*)\alpha\), for any \(Y \in C^\infty(TM)\) and \(\alpha \in C^\infty(T^*M)\), therefore the statement. \(\square\)

**Definition 2.22.** A pair \((\hat{J}_c, \hat{J}_p)\) of a generalized complex structure and a generalized product structure is called generalized complex product structure if \(\hat{J}_c\hat{J}_p = -\hat{J}_p\hat{J}_c\).

**Remark 2.23.** If \((J, g)\) is a metallic Riemannian structure on \(M\), then \((\hat{J}_c, \hat{J}_p)\), for \(\hat{J}_c := \begin{pmatrix} J & -(I + J^2) \hat{g} \\ \hat{g} & -J^* \end{pmatrix}\) and \(\hat{J}_p := \begin{pmatrix} J & (I - J^2) \hat{g} \\ \hat{g} & -J^* \end{pmatrix}\), is a generalized complex product structure.

3. Metallic structures on tangent and cotangent bundles

3.1. Metallic structure on the tangent bundle. Let \((M, J, g)\) be a metallic Riemannian manifold and let \(\nabla\) be a linear connection on \(M\). \(\nabla\) defines the decomposition into the horizontal and vertical subbundles of \(T(TM)\):

\[
T(TM) = TH(TM) \oplus TV(TM).
\]

Let \(\pi : TM \rightarrow M\) be the canonical projection and \(\pi_* : T(TM) \rightarrow TM\) be the tangent map of \(\pi\). If \(a \in TM\) and \(A \in T_a(TM)\), then \(\pi_*(A) \in T_{\pi(a)}M\) and we denote by \(\chi_a\) the standard identification between \(T_{\pi(a)}M\) and its tangent space \(T_a(T_{\pi(a)}M)\).

Let \(\Psi^V : TM \oplus T^*M \rightarrow T(TM)\) be the bundle morphism defined by

\[
\Psi^V(X + \alpha) := X_a^H + \chi_a(\hat{g}^\alpha),
\]

where \(a \in TM\) and \(X_a^H\) is the horizontal lifting of \(X \in T_{\pi(a)}M\).

Let \(\{x^1, \ldots, x^n\}\) be local coordinates on \(M\), let \(\{\tilde{x}^1, \ldots, \tilde{x}^n, y^1, \ldots, y^n\}\) be respectively the corresponding local coordinates on \(TM\), and let \(\{X_1, \ldots, X_n, \frac{\partial}{\partial \tilde{x}^1}, \ldots, \frac{\partial}{\partial y^n}\}\) be a local frame on \(T(TM)\), where \(X_i = \frac{\partial}{\partial \tilde{x}^i}\). We have:

\[
X_i^H = X_i - y^k \Gamma_{ik}^l \frac{\partial}{\partial y^l},
\]

\[
X_i^V = y^k \Gamma_{ik}^l \frac{\partial}{\partial y^l},
\]

\[
\left(\frac{\partial}{\partial y^i}\right)^H = 0.
\]
where $i, k, l$ run from 1 to $n$ and $\Gamma^k_{il}$ are the Christoffel symbols of $\nabla$.

Let $\Psi \nabla : TM \oplus T^* M \to T(TM)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$
\Psi \nabla \left( \frac{\partial}{\partial x^i} \right) = X_i^H
$$

$$
\Psi \nabla (dx^j) = g^{jk} \frac{\partial}{\partial y^k}.
$$

Let $\tilde{\mathcal{J}}_m$, $\tilde{\mathcal{G}}$ be the generalized metallic structure defined in the previous section. The isomorphism $\Psi \nabla$ allows us to construct a natural metallic structure $\tilde{\mathcal{J}}_m$ and a natural Riemannian metric $\tilde{\mathcal{G}}$ on $TM$ in the following way.

We define $\tilde{\mathcal{J}}_m : T(TM) \to T(TM)$ by

$$
\tilde{\mathcal{J}}_m := (\Psi \nabla) \circ \dot{\mathcal{J}}_m \circ (\Psi \nabla)^{-1}
$$

and the Riemannian metric $\tilde{\mathcal{G}}$ on $TM$ by

$$
\tilde{\mathcal{G}} := ((\Psi \nabla)^{-1})^* (\dot{\mathcal{G}}).
$$

**Proposition 3.1.** $(TM, \tilde{\mathcal{J}}_m, \tilde{\mathcal{G}})$ is a metallic Riemannian manifold.

*Proof.* From the definition it follows that $\tilde{\mathcal{G}}^2_m = p\tilde{\mathcal{J}}_m + qI$ and $\tilde{\mathcal{G}}(\tilde{\mathcal{J}}_m X, Y) = \tilde{\mathcal{G}}(X, \tilde{\mathcal{J}}_m Y)$, for any $X, Y \in C^\infty (T(TM))$. □

In local coordinates, we have the following expressions for $\tilde{\mathcal{J}}_m$ and $\tilde{\mathcal{G}}$:

$$
\left\{ \begin{array}{l}
\tilde{\mathcal{J}}_m (X_i^H) = J^k_i X_k^H \\
\tilde{\mathcal{J}}_m \left( \frac{\partial}{\partial y^i} \right) = g_{ji} J^k_j g^{kl} \frac{\partial}{\partial y^l} = J^k_j \frac{\partial}{\partial y^k} \\
\tilde{\mathcal{G}}(X_i^H, X_j^H) = g_{ij} \\
\tilde{\mathcal{G}}\left( X_i^H, \frac{\partial}{\partial y^j} \right) = 0 \\
\tilde{\mathcal{G}}\left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}.
\end{array} \right.
$$

Moreover,

$$
\tilde{\mathcal{J}}_m (X_i) = J^k_i X_k - y^l (J^k_i \Gamma^s_{kl} - J^s_i \Gamma^k_{il}) \frac{\partial}{\partial y^s}
$$

$$
\left\{ \begin{array}{l}
\tilde{\mathcal{G}}(X_i, X_j) = g_{ij} + y^k y^h \Gamma^l_{ik} \Gamma^s_{jh} g_{hk} \\
\tilde{\mathcal{G}}\left( X_i, \frac{\partial}{\partial y^j} \right) = y^k \Gamma^l_{ik} g_{lj}.
\end{array} \right.
$$
Computing the Nijenhuis tensor of $\bar{J}_m$, we get the following:

$$N_{\bar{J}_m} \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0$$

$$N_{\bar{J}_m} \left( X^H_i, \frac{\partial}{\partial y^j} \right) = \left( \nabla_{JX^i} J \right) X_j \left( \nabla_{X^j} J \right) X^k \frac{\partial}{\partial y^k}$$

$$N_{\bar{J}_m} \left( X^H_i, X^H_j \right) = \left( N_J (X_i, X_j) \right)^k X^H_k$$

$$- y^s \left( J^k_j J^h_i R^r_{kh} \right) - J^r_i J^k_j R^l_{kjs} - J^h_j J^r_i R^l_{ih} + p J^r_i R^l_{ij} + q R^r_{ij} \frac{\partial}{\partial y^r}.$$

Therefore we can state the following.

**Proposition 3.2.** Let $(M, J, g)$ be a flat locally metallic Riemannian manifold. If $\nabla$ is the Levi-Civita connection of $g$, then $(\bar{J}_m, \bar{g})$ is an integrable metallic Riemannian structure on $TM$.

### 3.2. Metallic structure on the cotangent bundle

Let $(M, J, g)$ be a metallic Riemannian manifold and let $\nabla$ be a linear connection on $M$. $\nabla$ defines the decomposition into the horizontal and vertical subbundles of $T(T^* M)$:

$$T(T^* M) = T^H(T^* M) \oplus T^V(T^* M).$$

Let $\pi : T^* M \to M$ be the canonical projection and $\pi_* : T(T^* M) \to TM$ be the tangent map of $\pi$. If $a \in T^* M$ and $A \in T_a(T^* M)$, then $\pi_*(A) \in T_{\pi(a)} M$ and we denote by $\chi_a$ the standard identification between $T_{\pi(a)} M$ and its tangent space $T_{\pi(a)} T^* M$.

Let $\Phi^{\nabla} : TM \oplus T^* M \to T(T^* M)$ be the bundle morphism defined by [5]:

$$\Phi^{\nabla}(X + \alpha) := X^H_a + \chi_a(\alpha),$$

where $a \in T^* M$ and $X^H_a$ is the horizontal lifting of $X \in T_{\pi(a)} M$.

Let $\{x^1, \ldots, x^n\}$ be local coordinates on $M$, let $\{\tilde{x}^1, \ldots, \tilde{x}^n, y_1, \ldots, y_n\}$ be respectively the corresponding local coordinates on $T^* M$ and let $\{X_1, \ldots, X_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ be a local frame on $T(T^* M)$, where $X_i = \frac{\partial}{\partial \tilde{x}^i}$. We have:

$$X^H_i = X_i + y_k \Gamma^k_{il} \frac{\partial}{\partial y_l}$$

$$X^V_i = - y_k \Gamma^k_{il} \frac{\partial}{\partial y_l}$$

$$\begin{pmatrix} \frac{\partial}{\partial y_i} \end{pmatrix}^H = 0$$

$$\begin{pmatrix} \frac{\partial}{\partial y_i} \end{pmatrix}^V = \frac{\partial}{\partial y_i},$$

where $i, k, l$ run from 1 to $n$ and $\Gamma^k_{il}$ are the Christoffel symbols of $\nabla$. 

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Let $Φ^\nabla : TM \oplus T^*M \to T(T^*M)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$Φ^\nabla \left( \frac{\partial}{\partial x^i} \right) = X_i^H$$
$$Φ^\nabla (dx^j) = \frac{\partial}{\partial y_j}.$$

Let $(\hat{J}_m, \hat{g})$ be the generalized metallic structure defined in the previous section. The isomorphism $Φ^\nabla$ allows us to construct a natural metallic structure $\tilde{J}_m$ and a natural Riemannian metric $\tilde{g}$ on $T^*M$ in the following way.

We define $\tilde{J}_m : T(T^*M) \to T(T^*M)$ by

$$\tilde{J}_m := (Φ^\nabla) \circ \hat{J}_m \circ (Φ^\nabla)^{-1}$$

and the Riemannian metric $\tilde{g}$ on $T^*M$ by

$$\tilde{g} := ((Φ^\nabla)^{-1})^*(\hat{g}).$$

**Proposition 3.3.** $(T^*M, \tilde{J}_m, \tilde{g})$ is a metallic Riemannian manifold.

**Proof.** From the definition it follows that $\tilde{J}_m^2 = p\tilde{J}_m + qI$ and $\tilde{g}(\tilde{J}_m X, Y) = \hat{g}(X, \hat{J}_m Y)$, for any $X, Y \in C^\infty(T(T^*M)).$ \hfill \qed

In local coordinates, we have the following expressions for $\tilde{J}_m$ and $\tilde{g}$:

\[
\begin{cases}
\tilde{J}_m (X_i^H) = J_i^k X_k^H \\
\tilde{J}_m \left( \frac{\partial}{\partial y_j} \right) = J_k^j \frac{\partial}{\partial y_k} \\
\tilde{g} (X_i^H, X_j^H) = g_{ij} \\
\tilde{g} \left( X_i^H, \frac{\partial}{\partial y_j} \right) = 0 \\
\tilde{g} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = g^{ij}.
\end{cases}
\]

Moreover,

\[
\begin{cases}
\tilde{J}_m (X_i) = J_i^k X_k + y_l (J_i^k \Gamma_{kr}^l - J_r^s \Gamma_{is}^l) \frac{\partial}{\partial y_r} \\
\tilde{g} (X_i, X_j) = g_{ij} + y_k y_h \Gamma_{ii}^k \Gamma_{hr}^j \tilde{g}^{lr} \\
\tilde{g} \left( X_i, \frac{\partial}{\partial y_j} \right) = -y_k \Gamma_{ik}^d g^{dj}.
\end{cases}
\]

Computing the Nijenhuis tensor of $\tilde{J}_m$, we get the following:

$$N_{\tilde{J}_m} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = 0.$$
\[ N_{\tilde{J}_m} \left( X^H_i, \frac{\partial}{\partial y_j} \right) = \left( (\nabla_{JX_i} J) X_k - J (\nabla_{X_i} J) X_k \right) \frac{\partial}{\partial y_k} \]

\[ N_{\tilde{J}_m} \left( X^H_i, X^H_j \right) = (N_J (X_i, X_j))^k X^H_k \]

\[ + y^l \left( J^k J^j R^l_{kh}s - J^r_s J^k J^j R^l_{kr} - J^r_s J^k J^j R^l_{ijr} + p J^k J^j R^l_{ikr} + q R^l_{ijr} \right) \frac{\partial}{\partial y_s} . \]

Therefore we can state the following.

**Proposition 3.4.** Let \((M, J, g)\) be a flat locally metallic Riemannian manifold. If \(\nabla\) is the Levi-Civita connection of \(g\), then \((\tilde{J}_m, \tilde{g})\) is an integrable metallic Riemannian structure on \(T^* M\).

**Remark 3.5.** The metallic structures \(\tilde{J}_m\) and \(\tilde{J}_m\) on the tangent and cotangent bundles respectively, satisfy:

\[ \tilde{J}_m \circ (\Psi \nabla \circ (\Phi \nabla)^{-1}) = (\Psi \nabla \circ (\Phi \nabla)^{-1}) \circ \tilde{J}_m. \]

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