Moving Mesh with Streamline Upwind Petrov-Galerkin (MM-SUPG) Method for Convection-Diffusion Problems

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We investigate the effect of the streamline upwind Petrov-Galerkin method (SUPG) as it relates to the moving mesh partial differential equation (MMPDE) method for convection-diffusion problems in the presence of vanishing diffusivity. We first discretize in space using linear finite elements and then use a θ-scheme to discretize in time. On a fixed mesh, SUPG (FM-SUPG) is shown to enhance the stability and resolves spurious oscillations when compared to the classic Galerkin method (FM-FEM) when diffusivity is small. However, it falls short when the layer-gradient is large. In this paper, we develop a moving mesh upwind Petrov-Galerkin (MM-SUPG) method by integrating the SUPG method with the MMPDE method. Numerical results show that our MM-SUPG works well for these types of problems and performs better than FM-SUPG as well as MMPDE without SUPG.

1 Introduction

Denote the time-independent domain by Ω ⊂ ℝ² with Lipschitz boundary and Ωₜ = Ω × [0, T]. Consider the scalar convection-diffusion equation:

\begin{align}
    u_t + b \cdot \nabla u - \varepsilon \Delta u &= f, \quad \text{on} \quad \Omega_T \\
    u(x, y, t) &= g, \quad \text{on} \quad \partial \Omega \times [0, T] \\
    u(x, y, 0) &= u_0(x, y), \quad \Omega \times \{t = 0\},
\end{align}

where the field \( b \in \left[ L^\infty(0, T; W^{1,\infty}(\Omega)) \right]^2 \) is incompressible, \( g \in L^2(0, T; L^2(\partial \Omega)) \), \( f \in L^2(0, T; L^2(\Omega)) \), \( u \in L^2(0, T; H^1_g(\Omega)) \), and \( \varepsilon \in [0, 1) \) is the diffusivity coefficient. Such problems appear in pollutant dispersal in rivers, atmospheric pollution, the Stefan problem, turbulent transport, and are of particular interest when investigating the Navier–Stokes equations with large Reynolds number etc. For applications, see [1].

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There have been attempts at numerically solving (1a) for time-independent solutions using adaptive mesh methods in conjunction with streamline upwind Petrov-Galerkin methods (SUPG) for steady flows (see, for example, [2, 3]) when $\varepsilon \ll 1$. Mesh adaptation has been a strong force in tackling convection-dominated problems. The general class of problems undertaken in this area of research should have the following two properties: artificial (non-physical) oscillations and sharp layers (i.e., interior and boundary layers). Some other interesting results for solving (1) include Lagrange-Galerkin error analysis with first-order uniform convergence in $\varepsilon$ [4] and $\varepsilon$-uniform convergence theory for FE with homogeneous boundary conditions [5]. In [6], Carey et al. use so-called adaptive mesh refinement (AMR). In [7], Chinchapatnam et al. investigate unsymmetric and symmetric meshless collocation techniques with radial basis functions. The so-called SOLD (Spurious Oscillations at Layers Diminishing) schemes are of particular interest where one adds either isotropic or anisotropic artificial diffusion to the Galerkin formulation [8, 9, 10]. Recently, so-called local discontinuous Galerkin (LDG) on a class of layer-adapted Shishkin and Bakhvalov-type meshes were used, along with error analysis, to numerically approximate (1) in [11]. Singular boundary method with $f = 0$ was employed by Wang et al. in [12] using an exponential variable transform along with the Fourier transform.

We build on this area of research by exhibiting the robustness of our moving mesh finite element method with SUPG (MM-SUPG) (which is a consistently stabilized method) implementation for time-independent and time-dependent flows with initial sharp layers. It has been noted that even an adapted mesh can exhibit oscillations [13]. The method we use is based on equidistribution and alignment conditions with mesh of anisotropic-type. The complexity to the numerical solution of (1) becomes apparent when $\varepsilon \ll 1$ and the initial profile undergoes convection as time increases or when the profile has sharp regular, parabolic, corner, or interior layers that develop or are present initially. Non-physical oscillations appear in the solution unless an extremely fine spatial-temporal discretization is used in the numerical method, which is computationally expensive. To improve stability, it was encouraged in [14] that artificial diffusion in the direction of $b$ should be added to the standard Galerkin method. The SUPG method has been relatively useful in numerically solving these types of boundary/interior layer problems for time-dependent and time-independent problems [3, 15, 16, 17].

The three most commonly used weighting functions used in this area of research are given by

$$P_{SUPG}(\varphi_h) = b \cdot \nabla \varphi_h,$$

(2)

$$P_{MS}(\varphi_h) = P_{SUPG}(\varphi_h) + \varepsilon \Delta \varphi_h,$$

(3)

$$P_{GLS}(\varphi_h) = P_{SUPG}(\varphi_h) - \varepsilon \Delta \varphi_h,$$

(4)

or in a more general form, the weighted function is

$$P(\varphi_h) = b \cdot \nabla \varphi_h + \sigma \varepsilon \Delta \varphi_h,$$

(5)

where $\sigma = -1, 0, 1$ is the Galerkin least-squares operator, streamline upwind operator, and the multiscale operator, respectively. In our work, we use linear basis functions, hence all operators are equivalent to (2). The operators in (3) and (4) rely on using more complex basis functions than linear ones (see, for example, [18, 19]). The weighting function should supplement both sides of the Galerkin formulation so that the standard Galerkin projection property is preserved. This is paired in the $L^2$-sense with the differential equation and added to the classic Galerkin method as shown in Section 2.

The motivation is to upwind the test functions along the streamlines to enhance the diffusion of
solution. This can be most readily seen in the 1D time-independent case using finite difference stencils:

\[
\frac{\partial u}{\partial x} = -\varepsilon \frac{\partial^2 u}{\partial x^2}.
\]  

(6)

Applying the finite difference method with using only the center stencil one obtains (depending on \(\varepsilon\)) a numerical approximation that is under-diffusive. Applying only an upwind stencil has too much diffusion, called over-diffusive. Since the exact solution is somewhere in between, one is led to upwinding the center scheme [20]. Brooks and Hughes first proposed this method for incompressible Navier-Stokes problems [14] and it has since proven fruitful in studying the incompressible Navier-Stokes equations [21] [22].

Essentially, the test space is being altered so that one can think of working in the space of functions upwinding the trial functions along the streamlines, i.e. the space

\[
U_h^{\text{SUPG}} = \left\{ w_h + \sum_{K \in T_h} \tau_K b \cdot \nabla w_h \mid w_h \in U_h \right\}
\]  

(7)

(see Section 2.2). Hence, one may regard the space \(U_h^{\text{SUPG}}\) as an appropriate stabilized test space for singularly perturbed problems. A standard choice of \(\tau_K\) is

\[
\tau^K_{T_h} = \begin{cases} 
    c_0 \text{diam}(K)_{T_h} & \text{if } \text{Pe}_K > 1 \\
    c_1 \text{diam}(K)_{T_h} & \text{if } \text{Pe}_K \leq 1
\end{cases},
\]  

(8)

where \(\text{Pe}_K\) denotes the elementwise Péclet number (which determines local convection-dominated or diffusion-dominated regimes on the current mesh \(T_K\)) and is given by

\[
\text{Pe}_K = \frac{\|b\|_\infty \text{diam}(K)_{T_h}}{2\varepsilon},
\]  

(9)

and there is freedom with how one defines \(\text{diam}(K)_{T_h}\) and in choosing \(c_0\) and \(c_1\). This paper considers large \(\text{Pe}_K\) with respect to \(\varepsilon^{-1}\|b\|_\infty\). Typically, one estimates the individual bilinear forms in the SUPG formulation and derives \(\tau^K_{T_h}\) based off those errors. In [3] such a method was employed using error estimates in the norm

\[
\|w\|_{\varepsilon,k}^2 = \varepsilon \|\nabla w\|^2 + \sum_{K \in T_K} \tau^k_K \|b \cdot \nabla w\|_{L^2(K)}^2, \forall w \in H^1_0(\Omega).
\]  

(10)

However, our method relies on different error estimates for defining the metric tensor along with a mesh functional approach (see Section 2.1). If a source term is added, say \(c \in L^\infty(0,T;L^\infty(\Omega))\), then it has been proposed that the FE perturbation parameters should take into account this source term [23] [24] [25] [26].

In the case of anisotropic mesh adaptation, \(\text{diam}(K)_{T_h}\) will be need to be updated with each new mesh as the elements stretch and shrink. In this work, we define \(\tau^K_{T_h}\) as

\[
\tau^K_{T_h} = \frac{\text{diam}(K)_{T_h}}{\|b\|_\infty} \xi(\text{Pe}_K),
\]  

(11)

which is based off the stabilization parameter chosen in [2]. For each \(K \in T_h\), we define

\[
\text{diam}(K) = \sup_{x_1, x_2 \in K} \|x_2 - x_1\|
\]  

(12)
and
\[ \xi(Pe_K) = \min \left\{ 1, \frac{Pe_K}{3} \right\}. \tag{13} \]

One could also choose \( \text{diam}(K)_{\mathcal{T}_h} \) to be the length of the longest edge of the element \( K \) projected onto \( b \). Another choice is to choose \( \text{diam}(K)_{\mathcal{T}_h} \) to be the diameter of \( K \) in the direction of \( b \). Again, the exact choice of \( \text{diam}(K)_{\mathcal{T}_h} \) is an open question and appears to be problem-specific.

This paper is organized as follows. We first introduce the moving mesh partial differential equations method (MMPDE) and our MM-SUPG method in Section 2. Numerical examples are presented in Section 3 to demonstrate that our method works properly while highlighting the limits of the moving mesh method. Some conclusions are drawn in Section 4 along with potential future investigations.

## 2 Numerical Method

There are two types of moving mesh strategies that are typically employed in adaptive mesh methods – the location-based approach and the velocity-based approach. This paper uses the variational approach, which is a location-based approach. It defines the mesh equations as Euler-Lagrange equations of a mesh adaptation functional (see Section 2.1). Moving mesh methods are used to solve those PDEs whose gradient(s) at certain points within the domain, e.g. layers, become too large to economically calculate or there exist some other complexity that causes the physical domain to change per small change in time. Other standard means with uniform or Shishkin-type meshes such as fitted operator methods, finite difference methods, finite element methods, etc. fall short in fully resolving these issues without the need for a hyper-refined mesh or apriori knowledge about the solution. The purpose of the MMPDE is to resolve the PDE’s complex regions while still being economical about computational costs. Of the three main types of methods used in this area of research, the relocation or r-refinement method is utilized in this work. We work on an affine family of simplicial triangulations \( \{ \mathcal{T}_h \} \) for \( \Omega \). Denote by \((K, \mathcal{P}^K_1, \Sigma_K)\) the finite element space. In what follows, let \( N \), \( N_v \), and \( N_t \) denote the number of mesh elements, the number of vertices, and the number of time steps, respectively.

### 2.1 Review of Moving Mesh Partial Differential Equations (MMPDE)

In this subsection, we give a brief overview of MMPDE. More details can be found in [27]. Let \( \mathbb{M} = \mathbb{M}(x) \) be a symmetric and uniformly positive definite metric tensor defined on \( \Omega \) with \( c_1 \mathbb{I} \leq \mathbb{M}(x) \leq c_2 \mathbb{I} \) for all \( x \in \Omega \). The \( \mathbb{M} \)-uniform mesh method takes a nonuniform mesh and views it as a uniform one in the metric specified by \( \mathbb{M} \). In a more general sense, \( \mathbb{M} \) is used to control the concentration and alignment of the mesh elements.

We consider two affine-equivalent finite elements, one for the reference elements and another for the mesh element. Then there exists an invertible affine mapping between the finite elements, denoted by \( F_K : \end{align*}.

where \( \mathcal{P}^K_1 \) is the set of polynomials of degree 1 on element \( K \). That is, for each element \( K \in \mathcal{T}_h \), denote by \( F_K : \hat{K} \rightarrow K \) the affine mapping between \( K \) and the reference element \( \hat{K} \) so that \( |\hat{K}| = 1 \). Denote by \( x^K_j \) for \( j = 0, 1, \) and \( 2 \) the vertices of \( K \) and \( \xi^K_j \) the corresponding vertices of \( \hat{K} \). Then the affine map \( F_K \) maps \( \xi^K_j \) to \( x^K_j \), i.e. \( F_K(\xi^K_j) = x^K_j \) for \( j = 0, 1, \) and \( 2 \). \( \hat{K} \) is called the reference element and belongs to \( \mathcal{T}_h \), the reference mesh.
This work will use the metric tensor defined by
\[ M = \det (I + |H_K|)^{-\frac{1}{2}} (I + |H_K|), \]
where \( I \) is the identity matrix, \( H_K \) is the recovered Hessian using least squares fitting to the values of the function at the mesh vertices, and \( |H_K| = \text{Adiag}((\lambda_1, \lambda_2))\Lambda^T \) with \( \text{Adiag}(\lambda_1, \lambda_2)\Lambda^T \) being the eigen-decomposition of \( H_K \). This metric tensor is optimal in the \( L^2 \) norm of linear interpolation error on triangular meshes [28].

Denote by \( M_K \) the volume-average of \( M \) over element \( K \), i.e.
\[ M_K = \frac{1}{|K|} \int_K M(x) \, dx, \]
and
\[ \sigma_h = \sum_{K \in \mathcal{T}_h} |K| \sqrt{\det(M_K)} = \sum_{K \in \mathcal{T}_h} |K| \sqrt{\det(M_K)}. \]

\( F_K \) and \( M \) give rise to the so-called equidistribution and alignment conditions in \( \mathbb{R}^2 \):

1. equidistribution: \( |K| \sqrt{\det(M_K)} = \frac{\sigma_h}{N} \) for all \( K \in \mathcal{T}_h \) and

2. alignment: \( \frac{1}{2} \text{tr} ((F_K')^T M_K F_K') = \sqrt{(F_K')^T M_K F_K'} \) for all \( K \in \mathcal{T}_h \),

where the Jacobian matrix of \( F_K \) is \( F_K' \). These two completely characterize a non-uniform mesh: the equidistribution condition controls the size and shape of mesh elements, while the alignment condition controls the orientation of mesh elements. The new physical mesh \( \mathcal{T}_h \) is found by minimizing the energy functional (also called the mesh functional) \( I \) given in discrete form by
\[ I(\mathcal{T}_h) = \sum_{K \in \mathcal{T}_h} |K| G(\mathbb{J}_K, \det(\mathbb{J}_K), M_K), \tag{15} \]
where \( \mathbb{J}_K = (F_K')^{-1} \) and
\[ G(\mathbb{J}_K, \det(\mathbb{J}_K), M_K) = \alpha \sqrt{\det(M_K)} \left( \text{tr} (\mathbb{J}_K M_K \mathbb{J}_K^T) \right)^p + (1 - 2\alpha)^2 \sqrt{\det(M_K)} \left( \frac{\det(\mathbb{J}_K)}{\sqrt{\det(M_K)}} \right)^p, \tag{16} \]

where \( \alpha \in (0, \frac{1}{2}] \) and \( p > 1 \) are dimensionless parameters. For the minimization of \( I(\mathcal{T}_h) \), the integration of the so-called moving mesh PDE or MMPDE is equivalent to the steepest descent method [28]. Numerical experiments suggest \( \alpha = \frac{1}{4} \) and \( p = \frac{3}{2} \) work well for many problems [29].

The MMPDE is a partial differential equation (also called the mesh PDE) that represents the physical PDE as a coordinate transformation problem. The MMPDE used in our work is given by
\[ \frac{dx_i}{dt} = -\frac{P_j}{\gamma} \left( \frac{\partial I}{\partial x_i} \right)^T, \quad i = 1, \ldots, N_v, \tag{17} \]
where \( \gamma \) is a user parameter meant to adjust the time scaled of the mesh movement and \( P_j = \text{det}(M(x_j))^{\frac{p-1}{2}} \) is chosen so that the MMPDE is invariant under the scaling transformation of \( M \). With this choice of \( P \), one can view [17] as a gradient flow equation with diffusivity matrix
\[ D(x) = \frac{\sqrt{\text{det}(M(x))^p - 1}}{\gamma}. \tag{18} \]
As noted in [27], solutions to the physical PDE can be solved either simultaneously or alternatively. Simultaneous solutions couples the mesh solution and the physical solution, with the mesh responding quickly to any change in the physical solution. However, this procedure is highly nonlinear. The alternative approach, which is what we employ, generates a mesh based on the mesh solution at the previous time step and physical solution at the current time. The new solution is then obtained for the next time step. Thus, one disadvantage of this is a lag in time. One way to resolve this is to complete many iterations on the same time level before increasing to the next. The alternate procedure allows for flexibility and potential efficiency at each time level. We note that initial mesh adaptation is necessary, especially if the initial profile has layers present at the initial time. Additional mesh movement may also be necessary at each time step for more difficult problems.

2.2 Moving Mesh with Streamline Upwind Petrov-Galerkin (MM-SUPG) Method

The purpose of the so-called streamline upwind Petrov-Galerkin (SUPG) method is to add artificial diffusion in order to resolve the non-physical layers that appear due to resulting error from the underlying numerical method. An example of this is the pure convection ($\varepsilon = 0$) of a cylinder. The streamline upwind problem can be stated as follows. Decompose $\Omega$ into non-overlapping triangular elements $\Omega = \bigcup_{K \in \mathcal{T}_h} K$. Denote the Sobolev space $H^1(\Omega)$ by

$$H^1_g(\Omega) = \{ w : \Omega \to \mathbb{R} | w|_{\partial \Omega} = g, w, \nabla w \in L^2(\Omega) \}$$

and the usual linear finite dimensional finite element conforming space by

$$U_h = \{ w_h | w|_K \in P^K_1, \forall K \in \mathcal{T}_h \} \cap H^1_g(\Omega)$$

and $U^0_h$ the corresponding restriction of $U_h$ to functions with compact support. Denote by the Bochner-Sobolev space

$$H^1(0,T;H^1(\Omega)) = \{ w | w \in L^2(0,T;H^1(\Omega)) \text{ and } w_t \in L^2(0,T;H^{-1}(\Omega)) \}$$

where $L^p(0,T;X)$ is the Banach space of strongly measurable functions $w$ in the Banach space $X$ with norm $\|w\|_{L^p(0,T;X)} = \left( \int_0^T \|w(t)\|^p_X dt \right)^{1/p}$. Denote by $(u,v)_{0,\Omega}$ the usual $L^2(\Omega)$ inner product given by

$$(u,v)_{0,\Omega} = \int_{\Omega} u v dx.$$  

The streamline upwind Petrov-Galerkin formulation is to find $u_h \in H^1(0,T;U_h)$ such that for all $\varphi_h \in U^0_h$

$$G(u_h, \varphi_h) + G_\tau(u_h, \varphi_h) = F(u_h, \varphi_h) + F_\tau(u_h, \varphi_h),$$

where

$$G(u_h, \varphi_h) = ((u_h)_t + b \cdot \nabla u_h, \varphi_h)_{0,\Omega} + (\varepsilon \nabla u_h, \nabla \varphi_h)_{0,\Omega},$$

$$F(u_h, \varphi_h) = (f, \varphi_h)_{0,\Omega},$$

$$G_\tau(u_h, \varphi_h) = \sum_{K \in \mathcal{T}_h} \tau_K((u_h)_t + b \cdot \nabla u_h - \varepsilon \Delta u_h, b \cdot \nabla \varphi_h)_{0,K},$$

$$F_\tau(u_h, \varphi_h) = \sum_{K \in \mathcal{T}_K} \tau_K(f, b \cdot \nabla \varphi_h)_{0,K},$$

and

$$\tau_K = \left\{ \begin{array}{ll} C_K & \text{for } K \in \mathcal{T}_h, \\ \tau_K & \text{for } K \in \mathcal{T}_K \end{array} \right.$$
where $\tau_K$ is the perturbation parameter that is determined by the user. A general choice of $\tau_K$ is still an open question.

Write $u_h(x, y, t) = \sum_{j=1}^{N} u_j(t) \varphi_j(x, y)$, where $\{\varphi_j\}_{j=1}^{N}$ is the set of linear basis functions in $U_h^0$. The matrix formulation for (23) is given by

$$Mu'_h(t) + A(t)u_h(t) = f(t), \quad (28)$$

where $u = [u_1, \ldots, u_N]^T$, $M$, $A$, and $b$ are given by

$$M_{ij} = (\varphi_j, \varphi_i)_{0, \Omega} + \sum_{K \in T_h} \tau_K (\varphi_j, b \cdot \nabla \varphi_i)_{0, K} \quad (29)$$

$$A_{ij}(t) = (b \cdot \nabla \varphi_j, \varphi_i)_{0, \Omega} + \varepsilon (\nabla \varphi_j, \nabla \varphi_i)_{0, \Omega}$$

$$\sum_{K \in T_h} \tau_K ((b \cdot \nabla \varphi_j, b \cdot \nabla \varphi_i)_{0, K} + \varepsilon (\nabla \varphi_j, \nabla (b \cdot \nabla \varphi_i))_{0, K}) \quad (30)$$

$$f_i(t) = (f, \varphi_i)_{0, \Omega} + \sum_{K \in T_h} \tau_K (f, b \cdot \nabla \varphi_i)_{0, K} \quad (31)$$

### 2.3 Discretization in Time

Denote $N_t$ as the number of time steps. We use a $\theta$ scheme to discretize in time. For (28), we approximate

$$u'_h(t) \approx \frac{u_h(t_{m+1}) - u_h(t_m)}{\Delta t}, \quad m = 1, \ldots, N_t. \quad (32)$$

Then the corresponding $\theta$-scheme is given by

$$M \frac{u_h(t_{m+1}) - u_h(t_m)}{\Delta t} + \theta A(t_{m+1})u_h(t_{m+1}) + (1 - \theta)A(t_m)u_h(t_m)$$

$$= \theta f(t_{m+1}) + (1 - \theta)f(t_m), \quad m = 1, \ldots, N_t. \quad (33)$$

Grouping the $u_h(t_{m+1})$ terms, we may rewrite (33) as

$$\left( \frac{M}{\Delta t} + \theta A(t_{m+1}) \right) u_h(t_{m+1}) = \theta f(t_{m+1}) + (1 - \theta)f(t_m) + \left( \frac{M}{\Delta t} - (1 - \theta)A(t_m) \right) u_h(t_m), \quad (34)$$

for $m = 1, \ldots, N_t$. Hence, we may write a system of ODEs

$$\tilde{A}^{m+1} u_h(t_{m+1}) = \tilde{f}^{m+1}, \quad m = 1, \ldots, N_t, \quad (35)$$

where

$$\tilde{A}^{m+1} = \frac{M}{\Delta t} + \theta A(t_{m+1}) \quad (36)$$

$$\tilde{f}^{m+1} = \theta f(t_{m+1}) + (1 - \theta)f(t_m) + \left( \frac{M}{\Delta t} - (1 - \theta)A(t_m) \right) u_h(t_m). \quad (37)$$

In this work, we use $\theta = 0.5$, which corresponds to the Crank-Nicholson. Fixed time step is used in the computations.
3 Numerical Examples

We consider three different examples and analyze the $H^1$-seminorm error over the entire domain $\Omega = [0, 1]^2$ at $T = 0.5$, since this norm will detect the spurious, accumulated oscillatory behavior. We compare results from four different methods: fixed mesh finite element method (FM-FEM) without SUPG, fixed mesh finite element method with SUPG (FM-SUPG), moving mesh finite element method (MM-FEM) without SUPG (which is the MMPDE method), and our moving mesh with SUPG (MM-SUPG) method. We denote the MMPDE method as MM-FEM for consistency in what follows.

Example 1 In this example, we investigate a single traveling layer given by equation (1a) with $b = (1, 1)$ and $\varepsilon = 10^{-4}$, $f$ is chosen so that the manufactured solution is

$$u(x, y, t) = (1 + \exp (C(x + y - t)))^{-1}, \quad C > 0.$$  \hfill (38)

We consider Dirichlet boundary conditions provided by the exact solution. The layer is given by $y = t - x$ and travels from the lower-left corner to the upper-right corner and extends to the boundaries. In the exact solution, $C$ plays the central role in determining the sharpness in the wall that separates the upper shelf from the lower shelf. We observe $O(h^{1/2})$ convergence for $C = 100$ and $C = 150$ with $\varepsilon = 10^{-4}$ (cf. Figure 1 for all methods). When $C = 100$, the transition from the upper shelf to the lower shelf can be handled with even a fixed mesh, with small oscillations appearing. However, at $C = 150$, we see artificial oscillations more pronounced (cf. Figure 2). This example highlights the need for a discrete maximum principle (DMP) for time-dependent convection-diffusion-reaction problems with vanishing diffusivity. As indicated by Figure 2, this can be a problem for the moving mesh method; nodes are relocated to areas of the domain where non-physical oscillations occur. If spurious oscillations are not resolved, then we see false resolution of those non-physical oscillations. Interestingly, the SUPG method for a fixed mesh does not contain this overshoot, even though the $H^1$-seminorm is worse.

![Figure 1](image1.png)

Figure 1: Example 1 – $H^1$-seminorm errors for different methods with different $C$ values for $\Delta t = 0.001$.

Example 2 The second example is the convection of a cylinder in the presence of vanishing diffusivity. We consider (1) but replace $\partial \Omega = \partial^- \Omega \cup \partial^+ \Omega$, where the inflow boundary $\partial^- \Omega = \{x \in \partial \Omega \mid b \cdot n < 0\}$ has Dirichlet conditions and the outflow boundary $\partial^+ \Omega = \{x \in \partial \Omega \mid b \cdot n > 0\}$ has homogeneous
Figure 2: Example 1 – solution profiles for different methods with $C = 150$ and $\varepsilon = 10^{-4}$ and $\Delta t = 0.001$, $N = 32768$.

Neumann boundary conditions. That is,

$$u(x, y, t) = 0 \text{ on } \partial^- \Omega \quad (39)$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial^+ \Omega, \quad (40)$$

where $\frac{\partial u}{\partial n}$ is the normal derivative. The initial profile is given by the cylinder

$$u(x, y, 0) = \begin{cases} 1 \text{ if } \|x - (0.25, 0.25)\| \leq 0.2 \\ 0 \text{ otherwise} \end{cases} \quad (41)$$

This initial profile, along with the convection field, $b = (1, 0.7002075)$, was studied in [30] (see Figure 3 for a representation of the flow geometry along with the boundary conditions). In Figure 4 we observe that the $H^1$-seminorm for MM-FEM and MM-SUPG converge to the same values as $N$ increases for $\varepsilon = 10^{-4}$ and $10^{-8}$. However, no such convergence was observed for the fixed mesh. In Figure 5 we see artificial ripples in the FM-FEM. These ripples are somewhat resolved for FM-SUPG, but are completely resolved for MM-FEM and MM-SUPG for $N = 32768$ (see Figure 6). This example exhibits the need for SUPG for a fixed mesh as $\varepsilon \to 0$ while showing no significant difference between the MM-FEM and MM-SUPG for larger $N$, which agrees with the established theory since the added diffusion vanishes with smaller element sizes. For smaller $N$, we see that SUPG reduces the $H^1$-seminorm for both the fixed mesh and moving mesh cases (see Table 1 and Figure 4). Figure 7 shows
Table 1: Example 2 – $H^1$-seminorm for vanishing diffusivity with different $\Delta t$ values.

| $\varepsilon = 10^{-4}$ | $\varepsilon = 0$ |
|---------------------------|---------------------|
| $\Delta t$                |                     |
| Method                    | 0.1     | 0.01     | 0.001    | 0.0005   | Method | 0.1     | 0.01     | 0.001    | 0.0005   |
| $N = 512$                 |         |          |          |          |        |         |          |          |          |
| FM-FEM                    | 2.6858  | 2.6814   | 2.6814   | 2.6814   | FM-FEM | 2.7395  | 2.7534   | 2.7538   | 2.7539   |
| FM-SUPG                   | 2.2263  | 2.0988   | 2.0965   | 2.0965   | FM-SUPG| 2.2377  | 2.1122   | 2.1099   | 2.1099   |
| MM-FEM                    | 1.9672  | 1.492    | 1.436    | 1.4296   | MM-FEM | 1.9948  | 1.5088   | 1.4516   | 1.4453   |
| MM-SUPG                   | 1.692   | 1.3672   | 1.3319   | 1.3276   | MM-SUPG| 1.7044  | 1.3785   | 1.3547   | 1.3379   |
| $N = 8192$                |         |          |          |          |        |         |          |          |          |
| FM-FEM                    | 4.7327  | 4.108    | 4.1007   | 4.1003   | FM-FEM | 5.126   | 5.1881   | 5.1905   | 5.1905   |
| FM-SUPG                   | 4.3498  | 3.372    | 3.354    | 3.3539   | FM-SUPG| 4.431   | 3.522    | 3.5043   | 3.5042   |
| MM-FEM                    | 5.1765  | 2.8009   | 2.5094   | 2.4856   | MM-FEM | 5.827   | 3.1149   | 2.6714   | 2.6503   |
| MM-SUPG                   | 5.0066  | 2.8087   | 2.4847   | 2.4701   | MM-SUPG| 5.2908  | 3.0719   | 2.7376   | 2.6477   |

the meshes at $T = 0$ and $T = 0.5$. In Figure 7 (a) we see strong mesh adaptation for $T = 0$ and observe the diffusive effect of the SUPG method at $T = 0.5$ in Figure 7 (b).

Figure 3: Example 2 – Domain with inflow and outflow labeled along with time-independent flow $b$.

Example 3 Here we consider another example (taken from [8]) with exact solution

$$u(x, y, t) = 16 \sin(\pi t) x(1-x)y(1-y) \times \left( \frac{1}{2} + \frac{1}{\pi} \arctan \left( 2\varepsilon^{-1/2}(0.25^2 - (x - 0.5)^2) \right) \right), \quad (42)$$

with Dirichlet boundary conditions. The severity of the interior layer is proportional to $\varepsilon$ and its thickness is $\sqrt{\varepsilon}$. We also use $b = (2, 3)$ and $\varepsilon = 10^{-6}$. To obtain decent results, $\Delta t = 10^{-3}$ is used and $T = 0.5$. For smaller $N$ we observe artificial oscillations in the direction of $b$, even for the MM-FEM with $N = 8192$. Figure 8 shows that the MM-SUPG method gives a smoother surface than the MM-FEM method without SUPG. In Figures 8 and 9 we see significant smearing in the FM-FEM and FM-SUPG, though the latter shows an improvement. We see almost no real difference between
\( \varepsilon = 10^{-4} \)

\( \varepsilon = 10^{-8} \)

Fig. 4: Example 2 – \( H^1 \)-seminorm of the approximate solution with \( \Delta t = 0.001 \).

the MM-FEM and MM-SUPG at level \( N = 32768 \). This is supported by Table 2 and Fig. 8 (a). In Fig. 9, we see the DMP is violated more severely in the case of MM-FEM when compared to MM-SUPG. We observe pronounced mesh movement to the dominating streamline component \( b_1 = 2 \) for the MM-FEM and while observing significant smearing in the direction of \( b \) on the backside of the solution with SUPG turned on (cf. Fig. 10). The rate of convergence are shown in Fig. 11 (a).

The second flow we consider is time-dependent and incompressible, given by \( b = (y - t, x - t) \). In Table 3 we see similar results for larger \( N \) as for the flow \( b = (2, 3) \). However, for smaller \( N \), we note that all 4 methods fall short in reducing the \( H^1 \)-seminorm error. The rate of convergence can also be seen from Fig. 11 (b). The side profile for this flow is similar to that for the flow \( b = (2, 3) \).

We see small oscillations along the top of the surface in the direction of the flow in Fig. 12 (a) while this is not present in Fig. 12 (b). However, for larger \( N \), we see that the DMP is not violated (compare Figs. 9 and 12). Both methods converge to the same approximate solution as \( N \) increases. In Fig. 9, we see that the MM-SUPG method is a noticeable improvement compared to all methods. In Fig. 12 slight oscillations are still present on the backside of the surface (in the direction of the streamlines) for MM-FEM while this absent for MM-SUPG. This is more prominent for smaller \( N \) in the MM-FEM case.

Table 2: Example 3 – \( H^1 \)-seminorm errors for \( \varepsilon = 10^{-6} \) at \( \Delta t = 0.001 \) for \( b = (2, 3) \).

| \( N \)  | 128    | 512    | 2048   | 8192   | 32768   |
|---------|--------|--------|--------|--------|---------|
| FMFEM   | 27.2093253 | 27.6628344 | 26.7879375 | 34.9200595 | 39.3491284 |
| FMFEM-SUPG | 27.0641466 | 17.8225581 | 14.7917766 | 13.4083571 | 10.4239975 |
| MM-FEM  | 12.4336744 | 13.2555115 | 10.4479145 | 4.45803658 | 0.73479517 |
| MM-FEM-SUPG | 13.1810799 | 10.6414207 | 10.5061743 | 4.32948047 | 0.95505957 |

4 Conclusions

In this paper, we developed a moving mesh with streamline upwind Petrov-Galerkin (MM-SUPG) method for convection-diffusion problems with different homogeneous boundaries and both time-dependent and time-independent flows. After first discretizing in space by using the modified weak
Figure 5: Example 2 – top view with $\varepsilon = 0$ and $\Delta t = 0.001$, $N = 32768$.  

Table 3: Example 3 – $H^1$-seminorm errors for $\varepsilon = 10^{-6}$ at $\Delta t = 0.001$ for $b = (y - t, x - t)$. 

| $N$    | 128   | 512   | 2048  | 8192  | 32768 |
|--------|-------|-------|-------|-------|-------|
| FM-FEM | 26.8174 | 16.7382 | 13.0265 | 12.9658 | 11.0075 |
| FM-SUPG| 26.8244 | 16.5932 | 12.8416 | 12.2876 | 9.7351 |
| MM-FEM | 10.7011 | 11.7156 | 10.2839 | 5.0822  | 0.8617 |
| MM-SUPG| 9.2395  | 12.4767 | 9.9982  | 4.7775  | 0.9908 |

form that incorporates the perturbed element-wise integrals, we discretize in time using the Crank-Nicholson scheme. By using linear elements, our MM-SUPG method recovers $O(h^{1/2})$ convergence rate for larger $N$ as seen in Figures 4 and 11.

The presented examples all show the robustness of the MM-FEM (or MMPDE) for larger $N$. All the figures indicate that for larger $N$, the perturbation term goes to zero, which agrees with (12). However, for smaller $N$, Example 2 shows how MM-SUPG can reduce the $H^1$-seminorm considerably. Additionally, Figure 9 shows how the MM-SUPG can eliminate non-physical spikes that occur near interior layers. Our MM-SUPG improves the stability of the MM-FEM so that computational efficiency can be improved by using fewer mesh elements.

Future avenues of work include more interesting boundary conditions. For example, periodic boundary conditions are of great interest in passive scalar dynamics when the source term depends on the velocity field – specifically when the solution and the turbulent flow field have the same spatiotemporal periodicities. Investigation of the DMP condition also demands attention, as shown in Example 1.
Additionally, different choices and possible optimal choices of $\tau^K_h$ is still an open question and perhaps can be adapted to fit our MM-SUPG method by taking into account the mesh speed when the mesh velocity is nonzero. There is still work to be done in the area of error analysis for our functional approach to this class of problems as well.

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(a): MM-SUPG and $T = 0$

(b): MM-SUPG and $T = 0.5$

Figure 7: Example 2 – Mesh movement with $\varepsilon = 0$, $\Delta t = 0.001$, and $N = 32768$.

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Figure 11: Example 3 – $H^1$-seminorm errors with $\varepsilon = 10^{-6}$ and $\Delta t = 0.001$ with different flow fields $b$.

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