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Anomalous localized resonance using a folded geometry in three dimensions

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If a body of dielectric material is coated by a plasmonic structure of negative dielectric material with non-zero loss parameter, then cloaking by anomalous localized resonance (CALR) may occur as the loss parameter tends to zero. If the coated structure is circular (two dimensions) and the dielectric constant of the shell is a negative constant (with loss parameter), then CALR occurs, and if the coated structure is spherical (three dimensions), then CALR does not occur. The aim of this paper is to show that CALR takes place if the spherical coated structure has a specially designed anisotropic dielectric tensor. The anisotropic dielectric tensor is designed by unfolding a folded geometry.

1. Introduction

If a body of dielectric material (core) is coated by a plasmonic structure of negative dielectric constant with non-zero loss parameter (shell), then anomalous localized resonance may occur as the loss parameter tends to zero. To be precise, let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, and $D$ be a domain whose closure is contained in $\Omega$. In other words, $D$ is the core and
\( \Omega \setminus \bar{D} \) is the shell. For a given loss parameter \( \delta > 0 \), the permittivity distribution in \( \mathbb{R}^d \) is given by

\[
\epsilon_{\delta} = \begin{cases} 
1 & \text{in } \mathbb{R}^d \setminus \Omega, \\
\epsilon_s + i\delta & \text{in } \Omega \setminus \bar{D}, \\
\epsilon_c & \text{in } D.
\end{cases}
\] (1.1)

Here \( \epsilon_c \) is a positive constant, but \( \epsilon_s \) is a negative constant representing the negative dielectric constant of the shell. For a given function \( f \) compactly supported in \( \mathbb{R}^d \setminus \bar{D} \) satisfying

\[
\int_{\mathbb{R}^d} f \, dx = 0
\] (1.2)

(which is required by conservation of charge), we consider the following dielectric problem:

\[
\nabla \cdot \epsilon_{\delta} \nabla V_{\delta} = f \quad \text{in } \mathbb{R}^d,
\] (1.3)

with the decay condition \( V_{\delta}(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \). Equation (1.3) is known as the quasi-static equation and the real part of \( -\nabla V_{\delta}(x) e^{-i\omega t} \), where \( \omega \) is the frequency and \( t \) is the time, represents an approximation for the physical electric field in the vicinity of \( \Omega \), when the wavelength of the electromagnetic radiation is large compared with \( \Omega \).

Let

\[
E_{\delta} := \Im \int_{\mathbb{R}^d} \epsilon_{\delta} |\nabla V_{\delta}|^2 \, dx = \int_{\Omega \setminus D} |\nabla V_{\delta}|^2 \, dx
\] (1.4)

(\( \Im \) for the imaginary part), which, within a factor proportional to the frequency, approximately represents the time-averaged electromagnetic power produced by the source dissipated into heat. (Note that, for the quasi-static approximation to be valid, it is not necessary for the frequency to be small, only that \( \Omega \) is sufficiently small compared with the wavelength.) Also for any region \( \Upsilon \), let

\[
E_{\delta}^{0}(\Upsilon) = \int_{\Upsilon} |\nabla V_{\delta}|^2 \, dx,
\] (1.5)

where, when \( \Upsilon \) is outside, \( \Omega \) approximately represents, within a proportionality constant, the time-averaged electrical energy stored in the region \( \Upsilon \). Anomalous localized resonance is the phenomenon of field blow-up in a localized region. It may (and may not) occur depending upon the structure and the location of the source. Quantitatively, it is characterized by \( E_{\delta}^{0}(\Upsilon) \rightarrow \infty \), as \( \delta \rightarrow 0 \) for all regions \( \Upsilon \) that overlap the region of anomalous resonance, and this defines that region. Cloaking by anomalous localized resonance (CALR) may occur when the support of the source, or part of it, lies in the anomalously resonant region. Physically the enormous fields in the anomalously resonant region interact with the source to create a sort of optical molasses, against which the source has to do a tremendous amount of work to maintain its amplitude, and this work tends to infinity as \( \delta \rightarrow 0 \). Quantitatively it is characterized by \( E_{\delta} \rightarrow \infty \) as \( \delta \rightarrow \infty \).

This phenomenon of anomalous resonance was first discovered by Nicorovici et al. [1] and is related to invisibility cloaking [2]; the localized resonant fields created by a source can act back on the source and mask it (assuming the source is normalized to produce fixed power). It is also related to superlenses [3,4] because, as shown by Nicorovici et al. [1], the anomalous resonance can create apparent point sources. For these connections and further developments tied to this form of invisibility cloaking, we refer to [5–9] and references therein. Anomalous resonance is also presumably responsible for cloaking owing to complementary media [10–12], although we do not study this here.

The problem of CALR can be formulated as the problem of identifying the sources \( f \) such that, first,

\[
E_{\delta} := \int_{\Omega \setminus D} \delta |\nabla V_{\delta}|^2 \, dx \rightarrow \infty \quad \text{as } \delta \rightarrow 0,
\] (1.6)

and, second, \( V_{\delta}/\sqrt{E_{\delta}} \) goes to zero outside some radius \( a \), as \( \delta \rightarrow 0 \),

\[
|V_{\delta}(x)/\sqrt{E_{\delta}}| \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \quad \text{when } |x| > a.
\] (1.7)
Because the quantity $E_\delta$ is proportional to the electromagnetic power dissipated into heat by the time-harmonic electric field averaged over time, (1.6) implies an infinite amount of energy dissipated per unit time in the limit $\delta \to 0$ that is unphysical. If we rescale the source $f$ by a factor of $1/\sqrt{E_\delta}$, then the source will produce the same power independently of $\delta$ and the new associated potential $V_\delta/\sqrt{E_\delta}$ will, by (1.7), approach zero outside the radius, $a$. Hence, CALR occurs. The normalized source is essentially invisible from the outside, yet the fields inside are very large. We also say that the weak CALR occurs if

$$\limsup_{\delta \to 0} E_\delta = \infty,$$

(1.8)

which is weaker than (1.6), and the limit in (1.7) is replaced by $\limsup$.

In recent papers [5,6], the authors developed a spectral approach to analyse the CALR phenomenon. In particular, they showed that if $D$ and $\Omega$ are concentric discs in $\mathbb{R}^2$ of radii $r_i$ and $r_c$, respectively, and $\epsilon_s = -1$, then there is a critical radius $r_\varepsilon$ such that for any source $f$ supported outside $r_\varepsilon$ CALR does not occur, and for sources $f$ satisfying a mild (gap) condition CALR takes place. The critical radius $r_\varepsilon$ is given by $r_\varepsilon = \sqrt{3}r_i/r_c$ if $\epsilon_c = 1$, and by $r_\varepsilon = r_i^2/r_c$, if $\epsilon_c \neq 1$. It is also proved that if $\epsilon_s \neq -1$, then CALR does not occur: $E_\delta$ is bounded regardless of $\delta$ and the location of the source. It is worth mentioning that these results (when $\epsilon_c = -\epsilon_s = 1$) were extended in Kohn et al. [13] to the case when the core $D$ is not radial by a different method based on a variational approach. There the source $f$ is assumed to be supported on circles.

The situation in three dimensions is completely different. If $D$ and $\Omega$ are concentric balls in $\mathbb{R}^3$, CALR does not occur whatever $\epsilon_s$ and $\epsilon_c$ are, as long as they are constants. We emphasize that this discrepancy comes from the convergence rate of the singular values of the Neumann–Poincaré-type operator associated with the structure. In two dimensions, they converge to 0 exponentially fast, but in three dimensions they converge only at the rate of $1/n$ [6]. The absence of CALR in such coated sphere geometries is also linked with the absence of perfect plasmon waves: see the appendix in Kohn et al. [13]. On the other hand, in a slab geometry CALR is known to occur in three dimensions with a single dipolar source [2]. (CALR is also known to occur for the full time-harmonic Maxwell equations with a single dipolar source outside the slab superlens [2,14,15].)

The purpose of this paper is to show that we are able to make CALR occur in three dimensions by using a shell with a specially designed anistropic dielectric constant. In fact, let $D$ and $\Omega$ be concentric balls in $\mathbb{R}^3$ of radii $r_i$ and $r_c$, and choose $r_\delta$ so that $r_0 > r_c$. For a given loss parameter $\delta > 0$, define the dielectric constant $\epsilon_\delta$ by

$$\epsilon_\delta(x) = \begin{cases} I, & |x| > r_\varepsilon, \\ (\epsilon_s + i\delta)a^{-1} \left( I + \frac{b(|x| - 2|\hat{x}|)}{|x|^2} \hat{x} \otimes \hat{x} \right), & r_i < |x| < r_\varepsilon, \\ \epsilon_c \frac{r_0}{r_i} I, & |x| < r_i, \end{cases}$$

(1.9)

where $I$ is the $3 \times 3$ identity matrix, $\epsilon_s$ and $\epsilon_c$ are constants, $\hat{x} = x/|x|$, and

$$a := \frac{r_\varepsilon - r_i}{r_0 - r_\varepsilon} > 0 \quad \text{and} \quad b := (1 + a)r_\varepsilon.$$

(1.10)

Note that $\epsilon_\delta$ is anisotropic and variable in the shell. This dielectric constant is obtained by push-forwarding (unfolding) that of a folded geometry, as in figure 1. (See the next section for details.) It is worth mentioning that this idea of a folded geometry has been used in Milton et al. [16] to prove CALR in the analogous two-dimensional cylinder structure for a finite set of dipolar sources. Folded geometries were first introduced in Leonhardt & Philbin [17] to explain the properties of superlenses, and their unfolding map was generalized in Milton et al. [16] to allow for three different fields, rather than a single one, in the overlapping regions. Folded cylinder structures were studied as superlenses in Yan et al. [18] and folded geometries using bipolar coordinates were introduced in Chen & Chan [19] to obtain new complementary media cloaking structures. More general folded geometries were rigorously investigated in Nguyen [12].
Figure 1. Unfolding map.

For a given source $f$ supported outside $B_r$, let $V_\delta$ be the solution to
\begin{align}
\nabla \cdot (\epsilon_\delta \nabla V_\delta) &= f \quad \text{in } \mathbb{R}^3, \\
V_\delta(x) &\to 0 \quad \text{as } |x| \to \infty,
\end{align}
and define
\begin{align}
E_\delta &= \Im \int_{\mathbb{R}^3} \epsilon_\delta \nabla V_\delta \cdot \nabla \overline{V_\delta} \, dx,
\end{align}
where $\overline{V_\delta}$ is the complex conjugate of $V_\delta$. Let $F$ be the Newtonian potential of the source $f$, i.e.
\begin{align}
F(x) &= \int_{\mathbb{R}^3} G(x-y)f(y) \, dy,
\end{align}
with $G(x-y) = -1/4\pi|x-y|$. Because $f$ is supported in $\mathbb{R}^3 \setminus \overline{B_r}$, $F$ is harmonic in $|x| < R$ for some $R > r$, and can be expressed there as
\begin{align}
F(x) &= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} f^k_n |x|^n Y^k_n(\hat{x}),
\end{align}
where $Y^k_n(\hat{x})$ is the (real) spherical harmonic of degree $n$ and order $k$. The coefficients $f^k_n$ can be calculated by
\begin{align}
f^k_n &= \frac{1}{4\pi r^{n+2}} \int_{|x|=r} F(x) Y^k_n(\hat{x}) \, d\sigma(x),
\end{align}
for any $r < R$. The following is the main result of this paper.

**Theorem 1.1.** Let $\epsilon_\delta$ be the permittivity profile in $\mathbb{R}^3$ given by (1.9).

(i) If $\epsilon_c = -\epsilon_s = 1$, then weak CALR occurs and the critical radius is $r_* = \sqrt{r_0 r}$, i.e. if the source function $f$ is supported inside the sphere of radius $r_*$ (and the series in (1.14) does not extend harmonically to $|x| < r_*$), then the weak CALR occurs, i.e.
\begin{align}
\limsup_{\delta \to 0} E_\delta &= \infty,
\end{align}
and there exists a constant $C$ such that
\begin{align}
|V_\delta(x)| < C,
\end{align}
for all x with |x| > r_0^2 r_e^{-1}. If, in addition, the Fourier coefficients f^k_n of F satisfy the following gap condition:

**[GC1]:** There exists a sequence \{n_j\} with n_1 < n_2 < \cdots such that

\[
\lim_{j \to \infty} \rho^{n_{j+1} - n_j} \sum_{k=-n_j}^{n_j} n_j^2 r_k^2 |f^k_n|^2 = \infty,
\]

where \( \rho := r_e/r_0 \), then CALR occurs, i.e.

\[
\lim_{\delta \to 0} E_\delta = \infty, \quad (1.18)
\]

and \( V_\delta/\sqrt{E_\delta} \) goes to zero outside the radius \( r_0^2/r_e \), as implied by (1.17).

(ii) If \( \epsilon_c \neq -\epsilon_0 = 1 \), then weak CALR occurs and the critical radius is \( r_{\ast \ast} = r_0 \). If, in addition, the Fourier coefficients \( f^k_n \) of F satisfy

**[GC2]:** There exists a sequence \{n_j\} with n_1 < n_2 < \cdots such that

\[
\lim_{j \to \infty} \rho^{2(n_{j+1} - n_j)} \sum_{k=-n_j}^{n_j} n_j^2 r_k^2 |f^k_n|^2 = \infty,
\]

then CALR occurs.

(iii) If \(-\epsilon_0 \neq 1\), then CALR does not occur.

We remark that, even if the source \( f \) is located inside in \( |x| < r_s \), the corresponding series (1.14) may be harmonic in \( |x| < r_s \). For example, the Newtonian potential of the form \( f = c_0 \chi_{r_1 < |x| < r_2} - c_2 \chi_{r_3 < |x| < r_4} \) with \( r_e < r_j < r_s, \ 1 \leq j \leq 4 \), is quadratic in \( |x| < r_e \). We also emphasize that [GC1] and [GC2] are mild conditions on the Fourier coefficients of the Newtonian potential of the source function. For example, if the source function is a dipole in \( B_{r_e} \setminus \bar{B}_e \), i.e. \( f(x) = a \cdot \nabla \delta_y(x) \) for a vector \( a \) and \( y \in B_{r_e} \setminus \bar{B}_e \), where \( \delta_y \) is the Dirac delta function at \( y \), [GC1] and [GC2] hold, and CALR takes place. A proof of this fact is provided in appendix A. Similarly one can show that if \( f \) is a quadrupole, \( f(x) = A : \nabla \nabla \delta_y(x) = \sum_{i,j=1}^2 a_{ij} (\partial^2/\partial x_i \partial x_j) \delta_y(x) \) for a \( 3 \times 3 \) matrix \( A = (a_{ij}) \) and \( y \in B_{r_e} \setminus \bar{B}_e \), then [GC1] and [GC2] hold.

### 2. Proof of theorem 1.1

Let \( r_i, r_e \) and \( r_0 \) be positive constants satisfying \( r_i < r_e < r_0 \), as before. In terms of spherical coordinates \((r, \theta, \phi)\), we define a mapping \( \Phi = \{ \Phi_e, \Phi_\delta, \Phi_m \} \), called the unfolding map, by

\[
\Phi_m(r, \theta, \phi) = (r, \theta, \phi), \quad r \geq r_e, \\
\Phi_\delta(r, \theta, \phi) = (b - ar, \theta, \phi), \quad r_e \leq r \leq r_0 \\
\Phi_e(r, \theta, \phi) = \left( \frac{r_i}{r_0} r, \theta, \phi \right), \quad r \leq r_0,
\]

where \( a \) and \( b \) are constants defined in (1.10). Then, the folding map can be written (with an abuse of notation) as

\[
\Phi^{-1}(x) = \begin{cases} 
  x, & |x| > r_e, \\
  -a^{-1}x + a^{-1}b \hat{x}, & r_i < |x| < r_e, \\
  \frac{r_0}{r_i} x, & |x| < r_i.
\end{cases} \quad (2.2)
\]
Let \( \kappa(x) \) be a permittivity profile (in the folded geometry) given by

\[
\kappa(x) = \begin{cases} 
\kappa_m, & |x| \geq r_e, \\
\kappa_s, & r_e \leq |x| \leq r_0, \\
\kappa_c, & |x| \leq r_0,
\end{cases}
\] (2.3)

and let \( \epsilon \) be the push-forward of \( \kappa \) by the unfolding map \( \Phi \), namely

\[
\epsilon(x) = \begin{cases} 
\kappa_m(\det \nabla \Phi_m(y))^{-1} \nabla \Phi_m(y) \nabla \Phi_m(y)^T, & |x| > r_e, \\
\kappa_s(\det \nabla \Phi_s(y))^{-1} \nabla \Phi_s(y) \nabla \Phi_s(y)^T, & r_1 < |x| < r_e, \\
\kappa_c(\det \nabla \Phi_c(y))^{-1} \nabla \Phi_c(y) \nabla \Phi_c(y)^T, & |x| < r_1,
\end{cases}
\] (2.4)

where \( x = \Phi(y) \). By straight-forward computations one can see

\[
\epsilon(x) = \begin{cases} 
\kappa_m \mathbf{I}, & |x| > r_e, \\
-\kappa_s \delta^{-1} \left( \mathbf{I} + \frac{b(b - 2|x|)}{|x|^2} \hat{x} \otimes \hat{x} \right), & r_1 < |x| < r_e, \\
\frac{r_0}{r_i} \mathbf{I}, & |x| < r_i,
\end{cases}
\] (2.5)

and \( \epsilon = \epsilon_\delta \) in (1.9) if we set

\[
\kappa_m = 1, \quad \kappa_s = -(\epsilon_\delta + i\delta) \quad \text{and} \quad \kappa_c = \epsilon_c. \] (2.6)

For a source \( f \) supported outside \( \overline{B_{r_e}} \) and the solution \( V_\delta \) to (1.11), we define

\[
\begin{align*}
\upsilon_m(x) &= V_\delta \circ \Phi_m(x), & \text{if } |x| > r_e, \\
\upsilon_s(x) &= V_\delta \circ \Phi_s(x), & \text{if } r_e < |x| < r_0, \\
\upsilon_c(x) &= V_\delta \circ \Phi_c(x), & \text{if } |x| < r_0.
\end{align*}
\] (2.7)

Then \( \upsilon_c, \upsilon_s \) and \( \upsilon_m \) satisfy

\[
\begin{align*}
\Delta \upsilon_c &= 0 \quad \text{in } B_{r_0}, \\
\Delta \upsilon_s &= 0 \quad \text{in } B_{r_0} \setminus \overline{B_{r_e}}, \\
\Delta \upsilon_m &= f \quad \text{in } \mathbb{R}^3 \setminus \overline{B_{r_e}}, \\
\upsilon_c &= \upsilon_s, & \text{on } \partial B_{r_0}, \\
\kappa_c \frac{\partial \upsilon_c}{\partial r} &= \kappa_s \frac{\partial \upsilon_s}{\partial r} \quad \text{on } \partial B_{r_e}, \\
\upsilon_s &= \upsilon_m, & \text{on } \partial B_{r_e}, \\
\upsilon_m(x) \to 0 \quad &\text{as } |x| \to \infty.
\end{align*}
\] (2.8)

and

We emphasize that the domains of \( \upsilon_c, \upsilon_s \) and \( \upsilon_m \) are overlapping on \( r_e \leq |x| \leq r_0 \), so that the solutions combined may be considered as the solution of the transmission problem with dielectric constants \( \kappa_c, \kappa_s \) and \( \kappa_m \) in the folded geometry, as shown in figure 1. We obtain \( V_\delta \) by unfolding the solution \( (\upsilon_m, \upsilon_s, \upsilon_c) \) into one whose domain is not overlapping, following the idea in Milton et al. [16].

By the change of variables \( x = \Phi_s(y) \) and (2.4), we have

\[
E_\delta = \Im \int_{\mathbb{R}^3} \epsilon(x) \nabla V_\delta(x) \cdot \nabla V_\delta(x) = \delta \int_{r_e < |y| < r_0} |\nabla \upsilon_s(y)|^2. \] (2.9)
Suppose that the source \( f \) is supported in \( |x| > R \) for some \( R > r_e \). Then, the solution \( u \) to (2.8) can be expressed in \( |x| < R \) as follows:

\[
\begin{align*}
  u_c(x) &= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} a_n^k |x|^n Y_n^k(\hat{x}), \quad \text{if } |x| < r_0, \\
  u_s(x) &= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} (b_n^k |x|^n + c_n^k |x|^{-n-1}) Y_n^k(\hat{x}), \quad \text{if } r_e < |x| < r_0, \\
  u_m(x) &= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} (e_n^k |x|^n + d_n^k |x|^{-n-1}) Y_n^k(\hat{x}), \quad \text{if } r_e < |x| < R,
\end{align*}
\]

(2.10)

and the coefficients satisfy the following relations resulting from the interface conditions:

\[
\begin{align*}
a_n^k r_0^n &= b_n^k r_0^n + c_n^k r_0^{-n-1}, \\
e_n^k r_e^n + d_n^k r_e^{-n-1} &= b_n^k r_e^n + c_n^k r_e^{-n-1}, \\
\kappa_n a_n^k r_0^n &= \kappa_s (b_n^k r_0^n - c_n^k (n+1) r_0^{-n-1}), \\
\kappa_s (b_n^k r_e^n - c_n^k (n+1) r_e^{-n-1}) &= \kappa_m (e_n^k r_e^n - d_n^k (n+1) r_e^{-n-1}).
\end{align*}
\]

By solving this system of linear equations one can see that

\[
\begin{align*}
a_n^k &= a_n e_n^k, \\
b_n^k &= b_n e_n^k, \\
c_n^k &= c_n e_n^k \quad \text{and} \quad d_n^k &= d_n e_n^k,
\end{align*}
\]

where

\[
\begin{align*}
a_n &= -\rho^{2n+1}(2n+1)^2 \kappa_n \kappa_s \left( \frac{n^2+n}{(n^2+n)(\kappa_s-\kappa_c)(\kappa_s-\kappa_m)} - \rho^{2n+1}((n+1)\kappa_s+n\kappa_c)((n+1)\kappa_m+n\kappa_s) \right), \\
b_n &= -\rho^{2n+1}(2n+1)\kappa_m (2n+1)((n+1)\kappa_s+n\kappa_c)((n+1)\kappa_m+n\kappa_s), \\
c_n &= -\rho^{2n+1}(2n+1)(\kappa_s-\kappa_c) \left( \frac{n^2+n}{(n^2+n)(\kappa_s-\kappa_c)(\kappa_s-\kappa_m)} - \rho^{2n+1}((n+1)\kappa_s+n\kappa_c)((n+1)\kappa_m+n\kappa_s) \right), \\
d_n &= -\rho^{2n+1}(2n+1)(\kappa_s-\kappa_c) \left[ \rho^{2n+1}(2n+1)((n+1)\kappa_s+n\kappa_c)((n+1)\kappa_m+n\kappa_s) \right],
\end{align*}
\]

(2.11-2.14)

Here \( \rho \) is defined to be \( r_e/r_0 \).

Let \( F \) be the Newtonian potential of \( f \), as before. Because \( u - F \) is harmonic in \( |x| > r_e \) and tends to 0 as \( |x| \to \infty \), we have

\[
ev_n^k = \hat{f}_n^k.
\]

(2.15)

So \( u_m \) (the solution in the matrix) is given by

\[
u_m(x) = F(x) + \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}_n^k d_n |x|^{-n-1} Y_n^k(\hat{x}).
\]

(2.16)

Because \( |d_n| \leq C r_0^{2n} \), we have

\[
u_m(x) - F(x) \leq C \sum_{n=0}^{\infty} \sum_{k=-n}^{n} |\hat{f}_n^k| r_0^{2n} |x|^{-n-1} < \infty,
\]

(2.17)

if \( |x| = r > r_0^2 r_e^{-1} \). This proves (1.17).
The solution in the shell $u_{n}$ is given by
\[ u_{n}(y) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} f_{n}^{k} (b_{n} \left| y \right|^{n} + c_{n} \left| y \right|^{-n-1}) Y_{n}^{k}(\hat{y}). \]  
(2.18)

Using Green’s identity and the orthogonality of $Y_{n}^{k}$, we obtain that
\[
\int_{r_{e}<|y|<r_{0}} \left| \nabla u_{n}(y) \right|^{2} = \int_{|y|=r_{0}} u_{n} \frac{\partial u_{n}}{\partial r} - \int_{|y|=r_{e}} u_{n} \frac{\partial u_{n}}{\partial r}
\]
\[= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left| f_{n}^{k} \right|^{2} \left( (b_{n} r_{n}^{n} + c_{n} r_{n}^{-n-1}) (n b_{n} r_{n}^{n} - (n + 1) \bar{c}_{n} r_{n}^{-n-1}) r_{0} \right)
\]
\[\quad - \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left| f_{n}^{k} \right|^{2} \left( (b_{n} r_{c}^{n} + c_{n} r_{c}^{-n-1}) (n b_{n} r_{c}^{n} - (n + 1) \bar{c}_{n} r_{c}^{-n-1}) r_{c} \right)
\]
\[= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left| f_{n}^{k} \right|^{2} \left( (b_{n} r_{n}^{2n+1} - r_{c}^{2n+1} - (n + 1) |c_{n}|^{2} (r_{n}^{-2n-1} - r_{c}^{-2n-1}) \right)
\]
\[\approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n \left| f_{n}^{k} \right|^{2} (b_{n}^{2} r_{n}^{2n+1} + |c_{n}|^{2} r_{c}^{-2n-1}).
\]

The estimate (2.9) yields
\[ E_{\delta} \approx \delta \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n \left| f_{n}^{k} \right|^{2} (b_{n}^{2} r_{n}^{2n+1} + |c_{n}|^{2} r_{c}^{-2n-1}). \]  
(2.19)

Here and afterwards, $a \approx b$ means that there exist constants $C_{1}$ and $C_{2}$ independent of $n$ and $\delta$ such that
\[ C_{1} a \leq b \leq C_{2} a. \]

(i) Suppose that $\epsilon_{c} = -\epsilon_{s} = 1$. With the notation in (2.6), we have
\[ \left| (n^{2} + n) (\kappa_{s} - \kappa_{c}) (\kappa_{s} - \kappa_{m}) - \rho^{2n+1} (n + 1) \kappa_{s} + n \kappa_{c}) (n + 1) \kappa_{m} + n \kappa_{s}) \right| \approx n^{2} (\delta^{2} + \rho^{2n+1}), \]
and, hence,
\[ |b_{n}| \approx \frac{\rho^{2n}}{\delta^{2} + \rho^{2n}} \quad \text{and} \quad |c_{n}| \approx \frac{\delta r_{e}^{2n}}{\delta^{2} + \rho^{2n}}. \]  
(2.20)

It then follows from (2.19) that
\[ E_{\delta} \approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta n r_{e}^{2n} \left| f_{n}^{k} \right|^{2}}{\delta^{2} + \rho^{2n}}. \]  
(2.21)

Let
\[ N_{\delta} = \frac{\log \delta}{\log \rho}. \]  
(2.22)

If $n \leq N_{\delta}$, then we know that $\delta \leq \rho^{n}$ and $r_{e}^{2n}/(\delta^{2} + \rho^{2n}) \geq \frac{1}{2} r_{0}^{-2n}$. Hence,
\[ \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta n r_{e}^{2n} \left| f_{n}^{k} \right|^{2}}{\delta^{2} + \rho^{2n}} \geq \sum_{n \leq N_{\delta}} \sum_{k=-n}^{n} \frac{\delta n r_{e}^{2n} \left| f_{n}^{k} \right|^{2}}{\delta^{2} + \rho^{2n}} \]
\[\geq \frac{\delta m r_{0}^{-2m}}{2} \sum_{k=-m}^{m} \left| f_{n}^{k} \right|^{2} \geq \frac{\delta m r_{0}^{-2m}}{2 (2m + 1) r_{0}^{-2m}} \left( \sum_{k=-m}^{m} \left| f_{n}^{k} \right|^{2} \right)^{2}, \]
for any $m \leq N_{\delta}$. By taking $\delta$ to be $\rho^{n}$, $n = 1, 2, \ldots$, we see that if the following holds
\[ \lim_{n \to \infty} (r_{e} r_{0})^{n/2} \sum_{k=-n}^{n} \left| f_{n}^{k} \right| = \infty, \]  
(2.23)
then there is a sequence \( \{ n_k \} \) such that
\[
\lim_{k \to \infty} E_{\rho|n_k|} = \infty,
\]
i.e. weak CALR occurs.

Suppose that the source function \( f \) is supported inside the critical radius \( r_0 = \sqrt{r_c r_0} \) (and outside \( r_0 \)) and its Newtonian potential \( F \) cannot be extended harmonically in \( |x| < r_0 \). Then we have
\[
\lim_{n \to \infty} \left( \sum_{k=-n}^{n} |f_k| \right)^{1/n} > \frac{1}{\sqrt{r_c r_0}},
\]
because, otherwise, \( F \) given by (1.14) converges in \( |x| < r_0 \) because \( |Y_k| \leq \sqrt{2n+1} \). Consequently, (2.23) holds. This proves that if the source function \( f \) is supported inside the sphere of critical radius \( r_0 \), then weak CALR occurs.

If the source function \( f \) is supported outside the sphere of critical radius \( r_0 = \sqrt{r_c r_0} \), then its Newtonian potential \( F \) can be extended harmonically in \( |x| < r_0 + 2\eta \) for \( \eta > 0 \) and
\[
\sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\rho r_{n_k}^2 |Y_k|^2}{\delta^2 + \rho 2n} \leq \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n r_{n_k}^2 |f_k|^2 \leq C\|F\|_{L^2(B_{r_0} + \eta)}^2 < \infty.
\]
So \( E_{\delta} \) is bounded regardless of \( \delta \) and CALR does not occur.

Suppose that \( f \) is supported inside \( r_0 \) and [GC1] holds. Let \( \{ n_j \} \) be the sequence satisfying
\[
\lim_{j \to \infty} \rho^{n_j+1} \sum_{k=-n_j}^{n_j} n_j^2 |f_k|^2 = \infty.
\]
If \( \delta = \rho^\alpha \) for some \( \alpha \), let \( j(\alpha) \) be the number in the sequence such that
\[
n_j(\alpha) \leq \alpha < n_j(\alpha)+1.
\]
Then, we have
\[
E_{\delta} \approx \sum_{n \leq n_j} \sum_{k=-n}^{n} \frac{\rho n_j^2 |f_k|^2}{\delta^2 + \rho 2n} \geq \rho^\alpha \sum_{n \leq n_j} \sum_{k=-n}^{n} \frac{n_j^2 |f_k|^2}{\rho 2n} = \rho^{n_j(\alpha)} \sum_{k=-n_j(\alpha)}^{n_j(\alpha)} n_j(\alpha) r_{n_j(\alpha)}^2 |f_k|^2 \to \infty
\]
as \( \alpha \to \infty \). So CALR takes place.

To prove (ii) assume that \( \epsilon_c \neq -\epsilon_0 = 1 \). In this case, we have
\[
|b_n| \approx \frac{\rho^{2n}}{\delta + \rho 2n} \quad \text{and} \quad |c_n| \approx \frac{\rho^{2n}}{\delta + \rho 2n},
\]
and
\[
E_{\delta} \approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\rho n_j^2 |f_k|^2}{\delta^2 + \rho 4n}.
\]
The rest of proof of (ii) is the same as that for (i).
Suppose now that $-\epsilon_s \neq 1$. If $\epsilon_c = 1$, then we have

$$|b_n| \approx \frac{\rho^{2n}}{\delta + \rho^{2n}} \quad \text{and} \quad |c_n| \approx \frac{\delta^{2n}}{\delta + \rho^{2n}},$$

and

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta (\delta^2 + \rho^{2n}) n^2 c_n |f_n|^2}{(\delta + \rho^{2n})^2} \leq \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n^2 c_n |f_n|^2 \leq \left\| \frac{\partial F}{\partial r} \right\|_{L^2(\partial B_e)}^2.$$

Because the source function $f$ is supported outside the radius $r_e$, we have

$$\left\| \frac{\partial F}{\partial r} \right\|_{L^2(\partial B_e)} \leq C \|f\|_{L^2(\mathbb{R}^3)},$$

and $E_\delta$ is bounded independently of $\delta$. The case when $\epsilon_c \neq 1$ can be treated similarly.

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### Appendix A. Gap property of dipoles

In this appendix, we show that the Newtonian potentials of dipole source functions satisfy the gap conditions [GC1] and [GC2]. We only prove [GC1], because the other can be proved similarly.

Let $f$ be a dipole in $B_r \setminus \overline{B}_e$, i.e., $f(x) = a \cdot \nabla \delta(x)$ for a vector $a$ and $y \in B_r \setminus \overline{B}_e$. Then its Newtonian potential is given by $F(x) = -a \cdot \nabla G(x - y)$. It is well known (see [20]) that the fundamental solution $G(x - y)$ admits the following expansion if $|y| > |x|:

$$G(x - y) = -\sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{1}{2n+1} Y_n^k(\hat{x}) Y_n^k(\hat{y}) \frac{|x|^{n}}{|y|^{n+1}}.$$

So we have

$$F(x) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{1}{2n+1} |x|^{n} Y_n^k(\hat{x}) \mathbf{a} \cdot \nabla \left( \frac{1}{|y|^{n+1}} Y_n^k(\hat{y}) \right),$$

and, hence,

$$j_n^k = \frac{1}{2n+1} \mathbf{a} \cdot \nabla \left( \frac{1}{|y|^{n+1}} Y_n^k(\hat{y}) \right). \quad (A1)$$

We show that

$$\sum_{k=-n}^{n} n^2 |j_n^k|^2 \to \infty \quad \text{as} \quad n \to \infty, \quad (A2)$$

and hence [GC1] holds. The following lemma is needed.

**Lemma A.1.** For any $\mathbf{a}$ and $\hat{y}$ on $S^2$ and for any positive integer $n$ there is a homogeneous harmonic polynomial $h$ of degree $n$ such that

$$\mathbf{a} \cdot \nabla h(\hat{y}) = 1 \quad (A3)$$

and

$$\max_{|\hat{x}|=1} |h(\hat{x})| \leq \frac{\sqrt{3}}{n}. \quad (A4)$$

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Proof. After rotation, if necessary, we may assume that \( \hat{y} = (1, 0, 0) \). We introduce three homogeneous harmonic polynomials of degree \( n \),

\[
h_1(x) := \frac{1}{2n} [(x_1 + ix_2)^n + (x_1 - ix_2)^n],
\]

\[
h_2(x) := \frac{1}{2ni} [(x_1 + ix_2)^n - (x_1 - ix_2)^n]
\]

and

\[
h_3(x) := \frac{1}{2ni} [(x_1 + ix_3)^n - (x_1 - ix_3)^n].
\]

Then one can easily see that

\[
\nabla h_1(\hat{y}) = (1, 0, 0), \quad \nabla h_2(\hat{y}) = (0, 1, 0) \quad \text{and} \quad \nabla h_3(\hat{y}) = (0, 0, 1).
\]

So if we define

\[
h = a_1h_1 + a_2h_2 + a_3h_3,
\]

then (A 3) holds.

Since

\[
\max_{|\hat{x}|=1} |h_j(\hat{x})| \leq \frac{1}{n} \quad \text{for} \quad j = 1, 2, 3,
\]

we obtain (A 4) using the Cauchy–Schwartz inequality. This completes the proof.

Let \( a \) and \( \hat{y} \) be two unit vectors, and let \( h \) be a homogeneous harmonic polynomial of degree \( n \) satisfying (A 3) and (A 4). Then \( h \) can be expressed as

\[
h(x) = \sum_{k=-n}^{n} \alpha_k |x|^n Y_n^k(\hat{x}),
\]

where

\[
\alpha_k = \frac{1}{4\pi} \int_{S^2} h(\hat{x}) Y_n^k(\hat{x}) \, dS. \tag{A 5}
\]

Because of (A 3), we have

\[
1 = a \cdot \nabla h(\hat{y}) \leq \sum_{k=-n}^{n} |\alpha_k| |a \cdot \nabla (|x|^n Y_n^k(\hat{x})).
\]

So there is \( k_n \), between \(-n \) and \( n \) such that

\[
|\alpha_{k_n}| |a \cdot \nabla (|x|^n Y_n^{k_n}(\hat{x}))| \geq \frac{1}{2n + 1}. \tag{A 6}
\]

On the other hand, from (A 4) and (A 5), it follows by using Jensen’s inequality that

\[
|\alpha_{k_n}|^2 \leq \frac{1}{4\pi} \int_{S^2} |h(\hat{x})|^2 |Y_n^{k_n}(\hat{x})|^2 \, dS \leq \frac{3}{n^2}.
\]

Thus, we have

\[
|a \cdot \nabla (|x|^n Y_n^{k_n}(\hat{x}))| \geq \frac{n}{\sqrt{3}(2n + 1)} \geq C \tag{A 7}
\]

for some \( C \) independent of \( n \).

Now one can see from (A 1) that

\[
|f_n^{k_n}| \geq \frac{C}{n|y|^{n+1}} \tag{A 8}
\]

for some \( C \) independent of \( n \). Because \( |y| < r_\ast \), we obtain that

\[
\sum_{k=-n}^{n} m_{2n}^n |f_n^{k_n}|^2 \geq m_{2n}^n |f_n^{k_n}|^2 \geq \frac{C}{n \left( \frac{r_\ast}{|y|} \right)^{2n}} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty,
\]

as desired. It is worth mentioning that the constants \( C \) in the above may be different at each occurrence, but are independent of \( n \).
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