COMPLEX MULTIPLICATION IN TWISTOR SPACES

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Abstract. Despite the transcendental nature of the twistor construction, the algebraic fibres of the twistor space of a K3 surface share certain arithmetic properties. We prove that for a polarized K3 surface with complex multiplication, all algebraic fibres of its twistor space away from the equator have complex multiplication as well.

Let $S$ be a complex projective K3 surfaces with an ample class $\ell = c_1(L) \in H^2(S, \mathbb{Z})$. Viewing $\ell$ as a Kähler class and using the existence of a Ricci-flat Kähler form representing $\ell$, one constructs the twistor space which consists of a non-algebraic complex threefold $S$ and a natural holomorphic projection $S \rightarrow \mathbb{P}^1$. All fibres $S_t$ are K3 surfaces, but only countably many of them are again algebraic. However, the set of algebraic fibres is dense and away from the equator of $\mathbb{P}^1 \cong S^2$ all algebraic fibres $S_t$ have the same Picard number $\rho(S)$.

The main result of this paper proves that despite the transcendental nature of the twistor construction, which relies on Yau's solution of the Calabi conjecture [Yau78], the K3 surface $S$ passes on certain arithmetic features to other algebraic fibres of the twistor family. We will demonstrate this for K3 surfaces with complex multiplication (CM).

A K3 surfaces $S$ is said to have CM if the endomorphism ring $K_S := \text{End}_{\text{Hdg}}(T(S))$ of the Hodge structure provided by the rational transcendental lattice $T(S) := \text{NS}(S)^\perp \otimes \mathbb{Q}$ is a CM field with $\dim_{K_S} T(S) = 1$, see Sections 2.2 and 5.3.

K3 surfaces with CM are defined over number fields and they are exactly those K3 surfaces that are defined over number fields and have algebraic periods [Tre15]. Examples include all K3 surfaces of maximal Picard number 20, for which the theorem is immediate, but there exist many K3 surfaces with CM of arbitrary even Picard number $\leq 20$.

Theorem 0.1. Let $(S, L)$ be a polarized, complex K3 surface with complex multiplication and assume $S'$ is an algebraic fibre of the associated twistor space $S \rightarrow \mathbb{P}^1$ over a point not contained in the equator $S^1 \subset S^2$ (cf. Sections 1.1, 3.1, and 5.1).

(i) Then $S'$ has complex multiplication.
(ii) The maximal totally real subfields of the CM endomorphism fields $K_S$ and $K_{S'}$ coincide.

More geometrically, the endomorphism field $K_S = \text{End}_{\text{Hdg}}(T(S))$ can be viewed, using Poincaré duality, as the non-trivial part of the space of Hodge classes on $S \times S$, cf. Remark 5.6

$$K_S \simeq (T(S) \otimes T(S))^{2,2} \subset H^{2,2}(S \times S, \mathbb{Q}) \simeq (\text{NS}(S) \otimes \text{NS}(S))_\mathbb{Q} \oplus (T(S) \otimes T(S))^{2,2}. $$
The maximal totally real subfield of $K_S$ is the subspace $(S^2T(S))^2.2$, which is at least half-dimensional. Although classes in this subspace do not deform along the twistor family $S \mapsto \mathbb{P}_1$, the same classes recur at all other algebraic fibre $S'$ away from the equator. More precisely, there is a multiplicative (with respect to convolution) isomorphism

$$(S^2T(S))^2.2 \simeq (S^2T(S'))^2.2.$$ 

**Outline:** In Section 1 one finds a quick reminder of the twistor construction. In the subsequent Sections 2 and 3 we discuss abstract Hodge structures of K3 type and explain the twistor construction in this setting. In particular, we show that the CM property is equivalent to the equality $k_T = K_T$ of the period and the endomorphism field (Proposition 2.12) and that excessive Picard jumps only occur along the equator (Proposition 3.2). The main result in this part is Proposition 3.8 which is the Hodge theoretic version of Theorem 0.1. The final Corollary 3.10 explains how to reconstruct the CM field $K_{S'}$ from its maximal totally real subfield $K^0_{S'} = K^0_S$. In Section 4 we discuss the notion of the period value of a K3 surfaces defined over a number field in the abstract Hodge theoretic setting and compute the period values of all CM twistor fibres. Section 5 translates the abstract Hodge theory into geometric results and proves Theorem 0.1. The section also contains a proof of the known fact that K3 surfaces with CM are defined over number fields that does not use the Kuga–Satake construction (Proposition 5.3), and a discussion of transcendental periods.

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1. Review of the twistor construction for K3 surfaces

For basic facts about hyperkähler geometry and the twistor construction, the reader may consult the survey [GHJ02, Hit92] or the extensive [Bes87]. Here, we merely sketch the main features and stress the analytic nature of the construction.

1.1. Ricci flat metrics. Let $S$ be a K3 surface and think of it as a differentiable manifold $M$ endowed with a complex structure $I$. Then, according to [Yau78], any Kähler class $\alpha \in H^{1,1}(S, \mathbb{R})$ is represented by a unique Kähler form $\omega$ such that the two real volume forms $\omega^2$ and $\sigma \bar{\sigma}$, where $0 \neq \sigma \in H^{2,0}(S)$, differ only by a constant scalar. After normalizing $\sigma$ appropriately, we may assume $\omega^2 = \sigma \bar{\sigma}$. In terms of the complex structure $I$ and the underlying Ricci-flat Kähler metric $g$, the Kähler form $\omega$ can be written as $\omega = g(I(\ ), \ )$.

As the holomorphic volume form $\sigma$ is actually a holomorphic symplectic structure, the holonomy group of $g$ is Sp(1). In particular, there are two other complex structures $J$ and $K$ compatible with the metric $g$ and such that $I = J \cdot K = -K \cdot J$. In fact, any linear combination $I_t := aI + bJ + cK$ with $t = (a, b, c) \in \mathbb{R}^2$ defines a complex structure on $M$ compatible with the
given Kähler metric $g$. We denote the corresponding Kähler forms by $\omega_t = \omega_{I_t} = g(I_t(\cdot,\cdot))$, for example, $\omega = \omega_I = \omega_{(1,0,0)}$. In fact, all the complex surfaces $(M,I_t)$ constructed in this way are K3 surfaces and, in particular, come with a holomorphic volume form $\sigma_t := \sigma_{I_t}$, for which we may assume $\omega_t^2 = \sigma_t \bar{\sigma}_t$. Note also that $\sigma = \sigma_I = \omega_J + i \omega_K$ and that in fact all $\omega_t$ are contained in the $\mathbb{R}$-linear span of $\omega_J, \omega_K$, and $\omega_K$. Furthermore, the real and imaginary parts of $\sigma_t$ span the orthogonal complement of $\omega_t$ in $\langle \omega_I, \omega_J, \omega_K \rangle_{\mathbb{R}}$:

$$\mathbb{R} \cdot \text{Re}(\sigma_t) \oplus \mathbb{R} \cdot \text{Im}(\sigma_t) \oplus \mathbb{R} \cdot \omega_t = \langle \omega_I, \omega_J, \omega_K \rangle_{\mathbb{R}}.$$  

On the differentiable manifold $M \times S^2$ one defines the almost complex structure $I$ at a point $(x,t) \in M \times S^2$ as $I_t \times I_{\bar{t}}$, where $S^2$ is interpreted as the complex projective line $\mathbb{P}^1$. A natural identification in this context will be explained below, cf. (3.1). It turns out that $I$ is in fact integrable. The resulting complex manifold defined in this way together with the holomorphic projection to the second factor shall be denoted by

$$S := (M \times S^2, I) \longrightarrow S^2 \simeq \mathbb{P}^1.$$  

The fibres $S_t$ of the projection are the K3 surfaces $(M,I_t)$. We think of $(1,0,0)$, which corresponds to the original complex structure $I$, as the north pole of $S^2$. Then the fibre over the south pole $(-1,0,0)$ is the K3 surface $(M,-I)$, the complex conjugate of $S$. The equator $\{(0,b,c) \mid b^2 + c^2 = 1\}$ parametrizes all complex structures $I_t$ for which $\omega$ is contained in the plane $\langle \text{Re}(\sigma_t), \text{Im}(\sigma_t) \rangle_{\mathbb{R}}$.

In Section 3 we provide a description of the twistor construction in terms of the involved Hodge structures, which in Section 5 will then be translated back into results for $S \longrightarrow \mathbb{P}^1$.

1.2. Twistor spaces associated with ample classes. The twistor construction relies on the existence of the Ricci-flat metric $g$. The existence of $g$ is guaranteed by [Yau78], but it cannot be constructed in an explicit way and is thought of as a transcendental structure. For example, if $S$ is embedded into $\mathbb{P}^N$, so in particular $S$ is projective, then the restriction of the Fubini–Study metric on $\mathbb{P}^N$ to $S$ is never Ricci-flat [Him90]. One may want to compare this result to [Don91], where it is shown that for appropriately chosen projective embeddings $\varphi_n : S^\prime \longrightarrow \mathbb{P}^{N_n}$ associated with the linear systems $|L^n|, n \longrightarrow \infty$, the pull-backs $(1/n)\varphi_{n*}\omega_{FS}$ of the Fubini–Study Kähler forms on $\mathbb{P}^{N_n}$ approach the Ricci-flat Kähler form representing the ample class $c_1(L)$.

For an arbitrary Kähler class $\alpha \in H^{1,1}(S,\mathbb{R})$ there is very little one can say about the various fibres $S_t$ of the associated twistor space. However, more structure emerges when $\alpha$ is an ample class $\ell = c_1(L)$.

Remark 1.1. Note that the twistor space $S \longrightarrow \mathbb{P}^1$ associated with $S$ and a Kähler class $\alpha = [\omega]$ can be viewed as the twistor space associated with an arbitrary fibre $S_t$ endowed with the Kähler class $[\omega_t]$. However, the property that $\alpha$ is an ample class $\ell$ is not preserved, i.e. $[\omega_t]$ will rarely be integral or rational again.
1.3. Hyperkähler manifolds. In the case of higher-dimensional hyperkähler manifolds, i.e. simply-connected, compact Kähler manifolds $X$ for which $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate form, the condition $\omega^2 = \sigma \bar{\sigma}$, up to constant scaling, has to be replaced by $\omega^{2n} = (\sigma \bar{\sigma})^n$. Here, $2n$ is the complex dimension of $X$. Again, the situation is controlled by the Hodge structure of weight two $H^2(X, \mathbb{Z})$ which is endowed with the Beauville–Bogomolov form. Theorem 0.1 remains valid in higher dimensions, for the result is ultimately deduced from purely Hodge theoretic arguments in Section 3 and those apply to the transcendental part $T(S) \subset H^2(S, \mathbb{Q})$ of a projective K3 surface $S$ as well as to the transcendental part $T(X) \subset H^2(X, \mathbb{Q})$ of a projective hyperkähler manifold $X$.

2. Hodge structures of K3 type: Periods and endomorphisms

In the following $T$ will denote a rational Hodge structure of K3 type. Concretely, this means that $T$ is a $\mathbb{Q}$-vector space of finite dimension $r$ endowed with a symmetric bilinear form $( , )$ of signature $(2, r - 2)$ and a decomposition

\[(2.1) \quad T_C := T \otimes_{\mathbb{Q}} \mathbb{C} = T^{2,0} \oplus T^{1,1} \oplus T^{0,2}\]

such that with respect to the $\mathbb{C}$-linear extension of $( , )$ the following conditions are satisfied:

(i) The subspaces $T^{1,1}$ and $T^{2,0} \oplus T^{0,2}$ are orthogonal.

(ii) $( , )$ is positive definite on $P_T := (T^{2,0} \oplus T^{0,2}) \cap T_\mathbb{R}$ and $T^{2,0}, T^{0,2} \subset T_C$ are isotropic.

(iii) Complex conjugation on $T_C$ preserves $T^{1,1}$ and exchanges $T^{2,0}$ and $T^{0,2}$, i.e. $\overline{T^{2,0}} = T^{0,2}$.

(iv) $\dim_{\mathbb{C}} T^{2,0} = 1$.

A generator of $T^{2,0}$ will usually be called $\sigma$, i.e. $T^{2,0} = \mathbb{C} \cdot \sigma$. Note that giving a decomposition \[(2.1)\] satisfying (i)-(iv) is equivalent to giving $\sigma \in T_C$ with $(\sigma, \sigma) = 0$ and $(\sigma, \bar{\sigma}) > 0$. The two classes $\text{Re}(\sigma)$ and $\text{Im}(\sigma)$, which span the positive real plane

\[P_T = \mathbb{R} \cdot \text{Re}(\sigma) \oplus \mathbb{R} \cdot \text{Im}(\sigma),\]

are orthogonal to each other $(\text{Re}(\sigma), \text{Im}(\sigma)) = 0$ and of the same norm $(\text{Re}(\sigma))^2 = (\text{Im}(\sigma))^2$.

We recall that the Hodge structure $T$ is called irreducible if there is no proper subvector space $T' \subset T$ with $T^{2,0} \subset T'_C$. Alternatively, irreducible Hodge structures can be described by the condition $T \cap T^{1,1} = 0$.

**Remark 2.1.** Note that any $\mathbb{Q}$-linear subspace $T' \subset T$ with $T^{2,0} \subset T'_C$ is a sub-Hodge structure, i.e. the inclusion $(T'_C \cap T^{2,0}) \oplus (T'_C \cap T^{0,2}) \oplus (T'_C \cap T^{1,1}) \subset T'_C$ is an equality. Indeed, in this case $T'_C \cap T^{2,0} = T^{2,0}$ and, applying complex conjugation, one then also has $T'_C \cap T^{0,2} = T^{0,2}$. This shows that for any $\gamma \in T'_C$ the classes $\gamma^{2,0}, \gamma^{0,2}$, and $\gamma^{1,1} = \gamma - \gamma^{2,0} - \gamma^{0,2}$ are contained in $T'_C$. Also observe that for a $\mathbb{Q}$-linear subspace $T' \subset T$ the condition $T^{2,0} \subset T'_C$ is equivalent to $T^{0,2} \subset T'_C$. 

Remark 2.2. Generically the positive plane $P_T \subset T_\mathbb{R}$ does not contain non-trivial rational classes, i.e. $P_T \cap T = 0$. In the non generic case, one distinguishes two cases:

(i) The real plane $P_T$ is defined over $\mathbb{Q}$, i.e. $\dim_\mathbb{Q}(P_T \cap T) = 2$ or, equivalently, $(P_T \cap T) \otimes \mathbb{Q} \cong P_T$. For irreducible $T$, the condition is equivalent to $\dim_\mathbb{Q} T = 2$, i.e. $P_T = T_\mathbb{R}$.

(ii) The real plane $P_T$ contains exactly one rational line, i.e. $\dim_\mathbb{Q}(P_T \cap T) = 1$.

Geometrically, (i) corresponds to K3 surfaces (or hyperkähler manifolds) of maximal Picard number. The second case (ii) will come up, but it cannot be described algebro-geometrically.

Lemma 2.3. For an irreducible Hodge structure $T$ of K3 type, orthogonal projection defines an injection

$$T' \longrightarrow P_T.$$ 

In other words, if $T$ is irreducible, $0 \neq \sigma \in T^{2,0}$, and $0 \neq \gamma \in T$, then $(\sigma, \gamma) \neq 0$.

Proof. For any class $\gamma \in T$ in the kernel of the orthogonal projection $T \longrightarrow P_T$, one has $P_T \subset \gamma_\perp R$. Hence, $0 \neq T' := \gamma_\perp \subset T$ is a sub-Hodge structure. Irreducibility of $T$ then implies $T' = T$ and, therefore, $\gamma = 0$. \hfill $\square$

2.1. The period field $k_T$. The period of the Hodge structure $T$ is the point

$$x_T := [T^{2,0}] \in \mathbb{P}(T_\mathbb{C}) = \mathbb{P}(T) \times_\mathbb{Q} \mathbb{C}.$$ 

We will denote its residue field by

$$k_T := k(x_T)$$

and call it the period field of the Hodge structure $T$.

To make this more concrete, fix a basis $\gamma_1, \ldots, \gamma_r$ of the $\mathbb{Q}$-vector space $T$ and consider the induced isomorphism $T \sim \longrightarrow \mathbb{Q}^r$, $\gamma \longrightarrow ((\gamma, \gamma_i))$. This is the composition of $T \sim \longrightarrow T^*$, $\gamma \longrightarrow (\gamma_\cdot)$, or, equivalently, $\gamma \longrightarrow \sum (\gamma, \gamma_i) \gamma_i^*$, and the natural isomorphism $T^* \sim \longrightarrow \mathbb{Q}^r$ given by the dual basis $(\gamma_i^*)$. The $\mathbb{C}$-linear extension $T_\mathbb{C} \sim \longrightarrow \mathbb{C}^r$ maps a generator $\sigma$ of $T^{2,0}$ to $(x_1 := (\sigma, \gamma_1), \ldots, x_r := (\sigma, \gamma_r))$ and if $x_1 \neq 0$

$$k_T = \mathbb{Q}(x_i/x_1) \subset \mathbb{C}.$$ 

Lemma 2.3 then yields the following consequence for irreducible Hodge structures.

Corollary 2.4. Let $T$ be an irreducible Hodge structure of K3 type with a fixed basis $(\gamma_i)$ and $0 \neq \sigma \in T^{2,0}$.

(i) The coordinates $x_i := (\sigma, \gamma_i) \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$. In particular, $x_i \neq 0$ for all $i$.

(ii) The affine coordinates $x_i/x_1$ are linearly independent over $\mathbb{Q}$, i.e. $\bigoplus_{i=1}^r \mathbb{Q}(x_i/x_1) \longrightarrow k_T$. In particular, $[k_T : \mathbb{Q}] \geq \dim_\mathbb{Q} T$. \hfill $\square$
2.2. The endomorphism field $K_T$. Consider endomorphisms $\varphi \colon T \rightarrow T$ of the Hodge structure $T$, i.e. $\mathbb{Q}$-linear maps $\varphi$ whose $\mathbb{C}$-linear extensions $\varphi_\mathbb{C}$ (or, simply, $\varphi$) satisfy $\varphi_\mathbb{C}(T^{p,q}) \subset T^{p,q}$. They form an algebra

$$K_T := \text{End}_{\text{Hdg}}(T).$$

If $T$ is irreducible, any non-zero $\varphi$ is in fact an isomorphism, for either its kernel or its image defines a subspace $T' \subset T$ with $T^{2,0} \subset T'$. Hence, $K_T$ is a division algebra. For the same reason, $\varphi = \text{id}$ if and only if $\varphi_\mathbb{C}|_{T^{2,0}} = \text{id}$. Moreover, the image $\varphi_\mathbb{C}(\sigma)$ of a generator $\sigma \in T^{2,0}$ is always a scalar multiple of $\sigma$ and mapping $\varphi$ to the scalar factor, therefore, defines an injection

$$K_T \hookrightarrow \mathbb{C}.$$

We will denote the image of $\varphi \in K_T$ under this morphism again by $\varphi$, i.e. $\varphi_\mathbb{C}(\sigma) = \varphi \cdot \sigma$. This immediately shows that for an irreducible Hodge structure $T$ of K3 type its endomorphism algebra $K_T$ is a field $\text{[Zar80]}$.

**Remark 2.5.** As subfields of $\mathbb{C}$, the endomorphism field $K_T$ and the period field $k_T$ of an irreducible Hodge structure $T$ are contained in each other:

(2.2) \quad K_T \subset k_T.

Indeed, for $\varphi \in K_T$ with $\varphi(\sigma) = \varphi \cdot \sigma$, one has $\varphi \cdot (\sigma, \gamma_1) = (\varphi(\sigma), \gamma_1) = (\sigma, \varphi'(\gamma_1))$, where the transpose $\varphi'$ maps $\gamma_1$ to a $\mathbb{Q}$-linear combination of the $\gamma_i$. Dividing by $(\sigma, \gamma_1)$ yields (2.2).

Note that $K_T$ is always a number field (of degree at most $\dim_{\mathbb{Q}} T$), whereas for the very general Hodge structure on $T$ the period field $k_T$ is of transcendence degree $r - 2$. Also note that frequently $K_T \subset \mathbb{R}$, whereas this is never the case for $k_T$, because it would contradict the two conditions $(\sigma, \sigma) = 0$ and $(\sigma, \bar{\sigma}) > 0$.

As shown by Zarhin $\text{[Zar80]}$, $K_T$ is either a totally real field or a CM field. Recall that a number field $K$ is totally real if the image of any embedding $K \hookrightarrow \mathbb{C}$ takes image in $\mathbb{R} \subset \mathbb{C}$. It is a CM field if it is a quadratic extension $K = K_0(\sqrt{\lambda})$ of a totally real field $K_0$ such that $\lambda \in \mathbb{R}_{<0}$ for any embedding $K_0 \hookrightarrow \mathbb{C}$.

**Remark 2.6.** (i) Complex conjugation in $\mathbb{C}$ corresponds to taking adjoint with respect to $(\cdot, \cdot)$, i.e. $(\psi(\gamma_1), \gamma_2) = (\gamma_1, \varphi(\gamma_2))$ for all $\gamma_1, \gamma_2 \in T$ if and only if $\bar{\psi} = \psi$ for the images of $\varphi, \psi$ under one, or equivalently any, embedding $K_T \hookrightarrow \mathbb{C}$, cf. $\text{[Zar80]}$ or $\text{[Huy16]}$ Ch. 3. In particular, $\varphi$ is an isometry if and only if its image in $\mathbb{C}$ has norm one. In other words, $(\varphi(\gamma_1), \varphi(\gamma_2)) = (\gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in T$ if and only if $\varphi \cdot \varphi = 1$.

(ii) We will need a slightly stronger variant of the last fact. Assume $\varphi, \psi : T \rightarrow T$ are just $\mathbb{Q}$-linear maps such that $\varphi(\sigma) = \lambda \cdot \sigma$ and $\psi(\sigma) = \bar{\lambda} \cdot \sigma$ for some $\lambda \in \mathbb{C}$. Then $\psi$ is the transpose $\varphi'$ of $\varphi$, i.e. $(\psi(\gamma_1), \gamma_2) = (\gamma_1, \varphi(\gamma_2))$ for all $\gamma_1, \gamma_2 \in T$. To prove this, let $T' \subset T$ be the kernel of $\psi - \varphi'$. Then $T^{2,0} \subset T'_C$ if $((\psi - \varphi')(\sigma), \gamma) = 0$ for all $\gamma \in T$ or, equivalently, $\bar{\lambda} \cdot (\sigma, \gamma) = (\psi(\sigma), \gamma) = (\sigma, \varphi(\gamma))$ for all $\gamma \in T$. However, as $\varphi(\bar{\sigma}) = \bar{\lambda} \cdot \bar{\sigma}$, the subspace $T'' \subset T$ of
all such \( \gamma \) satisfies \( T^{0,2} \subset T''_C \). By irreducibility of \( T \), this yields \( T'' = T \) and, therefore, \( T' = T \), i.e. \( \psi = \varphi^1 \).

**Definition 2.7.** An irreducible Hodge structure \( T \) of K3 type has complex multiplication (CM) if its endomorphism ring \( K_T = \text{End}_{\text{Hdg}}(T) \) is a CM field and \( \text{dim}_{K_T} T = 1 \).

Sometimes the less restrictive notion is used, where \( K_T \) is required to be a CM field but one allows \( \text{dim}_{K_T} T > 1 \). Note that the condition \( \text{dim}_{K_T} T = 1 \) can equivalently be phrased as \( [K_T : \mathbb{Q}] = \text{dim}_\mathbb{Q} T \).

**Remark 2.8.** Any CM field \( K \) admits a primitive element of norm one, i.e. \( K = \mathbb{Q}(\alpha) \) with \( \bar{\alpha} \cdot \alpha = 1 \). For \( K = K_T \) this can be rephrased by saying that whenever \( T \) has CM then there exists a Hodge isometry \( \alpha : T \cong T \), i.e. one in addition has \( (\alpha(\gamma_1), \alpha(\gamma_2)) = (\gamma_1, \gamma_2) \) for all \( \gamma_1, \gamma_2 \in T \), such that every Hodge endomorphism \( \varphi : T \rightarrow T \) can be written as \( \varphi = \sum_{i=1}^r a_i \alpha^{i-1}, a_i \in \mathbb{Q} \), cf. [Huy16, Thm. 3.3.7] for an elementary proof and references. The primitive element \( \alpha \) has degree \( \text{dim}_\mathbb{Q} T \) over \( \mathbb{Q} \).

**Remark 2.9.** In the totally real case, the only elements \( \varphi \in K_T \) that correspond to Hodge isometries of \( T \) are \( \varphi = \pm 1 \). Note also that in the totally real case, the analogue of the condition \( \text{dim}_{K_T} T = 1 \) is never realized, see [vGe08, Lem. 3.2]: If \( K_T \) is a totally real field, then \( \text{dim}_{K_T} T \geq 3 \). Roughly, if \( \text{dim}_{K_T} T = 1 \), then \( K_T \otimes_\mathbb{Q} \mathbb{R} \) splits as \( \bigoplus V_\varepsilon \), where \( \varepsilon \) runs through all embeddings \( K_T \hookrightarrow \mathbb{R} \) and hence \( \text{dim}_\mathbb{R} V_\varepsilon = 1 \). On the other hand, the real field \( K_T \) acts on the plane \( P_T \) by multiplication, which yields the contradiction \( \text{dim}_\mathbb{R} V_{id} = 2 \). In the case of a totally real field acting on a two-dimensional \( T \), Zarhin’s classification of the possible Mumford–Tate groups implies that there must exist an action of a further quadratic extension. For example, any Hodge structure of K3 type \( T \) with \( \text{dim}_\mathbb{Q} T = 2 \) has complex multiplication, a Hodge isometry can be written down explicitly, cf. [Huy16, Rem. 3.3.10].

### 2.3. Complex multiplication

It turns out that complex multiplication cannot only be read off from the endomorphism field \( K_T \) but also from its period field.

**Lemma 2.10.** Assume \( T \) is an irreducible Hodge structure of K3 type with complex multiplication.

(i) Then the endomorphism field \( K_T \) and the period field \( k_T \) coincide (as subfields of \( \mathbb{C} \)):

\[
K_T = k_T.
\]

(ii) For any basis \( (\gamma_i) \) of \( T \) and any \( 0 \neq \sigma \in T^{2,0} \) the coordinates \( x_i := (\sigma, \gamma_i) \) satisfy

\[
K_T = k_T = \bigoplus_{i=1}^r \mathbb{Q} \cdot (x_i/x_1).
\]

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1 It may be interesting to study the possibly larger number field \( L_T \) of all \( \mathbb{Q} \)-linear endomorphisms \( \varphi \) of \( T \) with just \( \varphi(T^{2,0}) \subset T^{2,0} \). It is indeed a field, but it may not be closed under complex conjugation as such \( \varphi \)'s do not necessarily preserve \( T^{1,1} \).
Proof. Pick a primitive element of norm one, i.e. we write \( K_T = \mathbb{Q}(\alpha) \) with a Hodge isometry \( \alpha \). Applying powers of \( \alpha \) to a fixed \( 0 \neq \gamma_1 \in T \) yields a basis of \( T \). More precisely, as \( \deg(\alpha) = \dim_\mathbb{Q} T \), the elements \( \gamma_i := \alpha^{1-i}(\gamma_1) \), \( i = 1, \ldots, r \), form a basis of \( T \). As \( \alpha \) is an isometry, one has \( (\sigma.\gamma_i) = (\sigma.\alpha^{1-i}(\gamma_1)) = (\alpha^{j-1}(\sigma).\gamma_1) = \alpha^{i-1} \cdot (\sigma.\gamma_1) \). Hence, \( (\sigma.\gamma_i)/(\sigma.\gamma_1) = \alpha^{i-1} \) and, therefore, \( k_T = K_T \).

To see (ii), we use Corollary 2.4. We know that \( x_1 \neq 0 \), hence the \( x_i/x_1 \) are well defined, and that the \( x_i/x_1, i = 1, \ldots, r \) are linearly independent over \( \mathbb{Q} \). As \( r = [K_T : \mathbb{Q}] = [k_T : \mathbb{Q}] \), this proves the second assertion.

The following result is a partial converse. We consider an irreducible Hodge structure \( T \) of K3 type and pick a generator \( \sigma \in T^{2,0} \) and a basis \((\gamma_i)\) of \( T \). As before, we write \( x_i := (\sigma.\gamma_i) \). Then \( k_T = \mathbb{Q}(x_i/x_1) \) and the natural map \( \bigoplus \mathbb{Q}(x_i/x_1) \to k_T \) is injective by Corollary 2.4.

**Lemma 2.11.** Let \( T \) be an irreducible Hodge structure of K3 type with period field \( k_T \). Assume \( L \subset k_T \cap \mathbb{R} \) is a subfield with \( [L : \mathbb{Q}] \geq (1/2) \dim_\mathbb{Q} T \) and such that multiplication with elements in \( L \) preserves the subspace \( \bigoplus \mathbb{Q} \cdot (x_i/x_1) \subset k_T \). Then \( T \) has complex multiplication.

**Proof.** In most of the argument we only use \( L = \bar{L} \) and \( L \subset k_T \). Only at the very end, \( L \subset \mathbb{R} \) becomes important.

By assumption, for any \( 0 \neq \lambda \in L \) there exists a unique invertible matrix \( (b(\lambda)_{ij}), b(\lambda)_{ij} \in \mathbb{Q} \), with \( \lambda \cdot x_i = \sum b(\lambda)_{ij} x_j \). Using the isomorphism \( T \cong \mathbb{Q}^r, \gamma \mapsto ((\gamma.\gamma_i)) \), we interpret \( (b(\lambda)_{ij}) \) as a linear map \( \varphi: T \to T \). Its \( \mathbb{C} \)-linear extension satisfies \( (\varphi(\sigma).) = \sum_j x_j \left( \sum_i b(\lambda)_{ij} \gamma_i^* \right) = \sum_i (\sum_j b(\lambda)_{ij} x_j) \gamma_i^* = \sum_i (\lambda \cdot x_i) \gamma_i^* = \lambda \cdot \sum_i x_i \gamma_i^* \), i.e. \( \varphi(\sigma) = \lambda \cdot \sigma \). This is enough to conclude that \( \varphi \) is an endomorphism of Hodge structures. Indeed, if \( \gamma \in T^{1,1} \cap T_\mathbb{R} \), then \( (\sigma.\varphi(\gamma)) = (\varphi'(\sigma).\gamma) = \lambda \cdot (\sigma.\gamma) = 0 \), as by (ii) in Remark 2.10 \( \varphi' \) is given by \( (b(\lambda)_{ij}) \), where we use \( \lambda \in L \), and hence \( \varphi(\gamma) \in T^{1,1} \). Therefore, \( \varphi \in K_T \) and \( \varphi \) is mapped to \( \lambda \) under the natural inclusion \( K_T \to \mathbb{C} \). Hence, \( L \subset K_T \) as subfields of \( \mathbb{C} \) and, therefore, \( [K_T : \mathbb{Q}] \geq (1/2) \dim_\mathbb{Q} T \) or, equivalently, \( \dim_{K_T} T \leq 2 \). However, according to [vG08, Lem. 3.2], if \( K_T \) is a totally real field, then \( \dim_{K_T} T \geq 3 \). Hence, \( K_T \) is a CM field.

Eventually, now using \( L \subset \mathbb{R} \), the inclusion \( L \subset K_T \) is in fact proper, and, therefore, \( \dim_{K_T} T = 1 \).

We summarize the discussion as follows.

**Proposition 2.12.** For an irreducible Hodge structure \( T \) of K3 type the following conditions are equivalent:

(i) \( T \) is of CM type, i.e. \( K_T \) is a CM field and \( [K_T : \mathbb{Q}] = \dim_\mathbb{Q} T \).

(ii) \( k_T \) is a CM field and \( [k_T : \mathbb{Q}] = \dim_\mathbb{Q} T \).

(iii) \( K_T = k_T \).
Proof. Let us assume (ii). Then, for dimension reasons, $\bigoplus \mathbb{Q} \cdot (x_i/x_1) \hookrightarrow k_T$ is a bijection. In particular, the space is preserved by multiplication with elements in the maximal totally real subfield $L := k_T \cap \mathbb{R}$. Now apply Lemma 2.11 using $[k_T \cap \mathbb{R} : \mathbb{Q}] = (1/2) \dim_{\mathbb{Q}} T$ for the CM field $k_T$. Hence, (ii) implies (i) and (iii). As by virtue of Lemma 2.10 (i) implies (ii) and (iii), it remains to prove that (iii) implies (i) or (ii). However, (iii) together with $[K_T : \mathbb{Q}] \leq r \leq [k_T : \mathbb{Q}]$ yields $\dim_{K_T} T = 1$ and, as above, this proves that $K_T$ is a CM field. \qed

3. Hodge theory of the twistor space

As before $T$ denotes an irreducible Hodge structure of K3 type with its positive real plane $P_T = \langle \text{Re}(\sigma), \text{Im}(\sigma) \rangle_\mathbb{R} \subset T_\mathbb{R}$, where $0 \neq \sigma \in T^{2,0}$. Associated with $T$ and an abstract class $\ell$ of positive square, there exists a sphere of related Hodge structures.

3.1. The twistor construction. For any $d \in \mathbb{Z}_{>0}$ we extend $T$ to the Hodge structure of K3 type $T \oplus \mathbb{Q} \cdot \ell$ by declaring $\ell$ to be of type $(1,1)$, orthogonal to $T$, and to satisfy $(\ell)^2 = d$. Note that then $P_T \oplus \mathbb{R} \cdot \ell \subset T_\mathbb{R} \oplus \mathbb{R} \cdot \ell$ is a positive three-space.

The associated twistor base $\mathbb{P}_\ell^1 \subset \mathbb{P}(T^{2,0} \oplus T^{0,2} \oplus \mathbb{C} \cdot \ell) = \mathbb{P}(P_T \mathbb{R} \oplus \mathbb{C} \cdot \ell) \subset \mathbb{P}(T_\mathbb{C} \oplus \mathbb{C} \cdot \ell)$ is the conic

$$\mathbb{P}_\ell^1 := \{ z \mid (z)^2 = 0 \}.$$ 

Any $z \in \mathbb{P}_\ell^1$ defines a Hodge structure of K3 type on $T \oplus \mathbb{Q} \cdot \ell$. Its $(2,0)$-part is the line corresponding to $z$, the complex conjugate of the line is the $(0,2)$-part, and the $(1,1)$-part is given as the orthogonal complement of the former two.

Mapping $z \in \mathbb{P}_\ell^1$ to the oriented, positive real plane $P_z := \langle \text{Re}(z), \text{Im}(z) \rangle_\mathbb{R} \subset P_T \oplus \mathbb{R} \cdot \ell$ yields the usual identification $\mathbb{P}_\ell^1 \simeq \text{Gr}^{\text{po}}(P_T \oplus \mathbb{R} \cdot \ell)$ with the Grassmannian of oriented, positive planes. The complex conjugate $\bar{z}$ corresponds to the same plane with the reversed orientation $P_{\bar{z}} = P_z := \langle \text{Im}(z), \text{Re}(z) \rangle_\mathbb{R}$. We will consider the period point of $T$ and its complex conjugate $x_T, \bar{x}_T \in \mathbb{P}(T_\mathbb{C})$ as points in $\mathbb{P}_\ell^1$ via the natural inclusion $\mathbb{P}(T_\mathbb{C}) \subset \mathbb{P}(T_\mathbb{C} \oplus \mathbb{C} \cdot \ell)$.

Thinking of $P_z$ with its orientation being given as the orthogonal complement of a generator $\alpha_z$ of the line $P_z \subset P_T \oplus \mathbb{R} \cdot \ell$ provides a natural identification

$$\mathbb{P}_\ell^1 \simeq \text{Gr}^{\text{po}}(P_T \oplus \mathbb{R} \cdot \ell) \simeq S_\ell^2 := \{ \alpha \mid (\alpha)^2 = 1 \} \subset P_T \oplus \mathbb{R} \cdot \ell.$$ 

With this identification, $x_T$ and $\bar{x}_T$ correspond to (the normalization of) $\ell$ and $-\ell$. We think of them as the north and south pole of $S_\ell^2$. The equator is the circle

$$S_\ell^1 := \{ z \in \mathbb{P}_\ell^1 \mid \ell \in P_z \} \simeq \{ \alpha \mid (\alpha, \ell) = 0 \} \subset S_\ell^2.$$ 

If for $z \in \mathbb{P}(P_T \mathbb{C} \oplus \mathbb{C} \cdot \ell)$ we write $z = [\alpha \sigma + b \bar{\sigma} + c \ell], a, b, c \in \mathbb{C}$, then $z \in \mathbb{P}_\ell^1$ if and only if

$$2ab(\sigma, \bar{\sigma}) + c^2 d = 0.$$ 

The only points with $c = 0$ are the north and the south pole. For all other points, $c \neq 0$ and, after scaling, we may assume $c = 1$. 

3.2. Picard number jumping. The Picard number $\rho_z$ of the Hodge structure on $T \oplus \mathbb{Q} \cdot \ell$ corresponding to $z \in \mathbb{P}^1_\ell$ is defined to be the dimension of the $\mathbb{Q}$-vector space of classes of type $(1,1)$, i.e.

$$\rho_z := \dim_{\mathbb{Q}}(P_z^\perp \cap (T \oplus \mathbb{Q} \cdot \ell)).$$

If the original Hodge structure $T$ was irreducible, then $P_x^\perp \cap (T \oplus \mathbb{Q} \cdot \ell) = \mathbb{Q} \cdot \ell$ and, therefore, $\rho_x = 1$. For very general $z \in \mathbb{P}^1_\ell$, the corresponding Hodge structure on $T \oplus \mathbb{Q} \cdot \ell$ is irreducible, i.e. $\rho_z = 0$. The Hodge structure on $T \oplus \mathbb{Q} \cdot \ell$ corresponding to $z \in \mathbb{P}^1_\ell$ is not irreducible, so $\rho_z \geq 1$, if and only if there exists a non-zero class $\ell' \in T \oplus \mathbb{Q} \cdot \ell$ orthogonal to $z$. If $z$ is different from $x$ and $\bar{x}$ and is written in the form $z = [a\sigma + b\bar{\sigma} + \ell]$, then orthogonality of $z$ and $\ell'$ is expressed by the equation

$$(3.3) \quad a(\sigma,\ell') + b(\bar{\sigma},\ell') + (\ell,\ell') = 0.$$

The following observation will be made more precise in the CM case later.

**Lemma 3.1.** Let $k_z$ be the period field of a point $z \in \mathbb{P}^1_\ell$ with $\rho_z > 0$. Then $k_T \subset \bar{\mathbb{Q}}$ if and only if $k_z \subset \bar{\mathbb{Q}}$.

**Proof.** If $k_T$ is algebraic, then after rescaling $\sigma$, we may assume $\sigma \in T_{\mathbb{Q}} \subset T_{\mathbb{Q}} \oplus \mathbb{Q} \cdot \ell$. Then the twistor conic is defined by the quadratic equation $2ab(\sigma,\bar{\sigma}) + c^2d = 0$ with coefficients $(\sigma,\bar{\sigma})$ and $d$ contained in $\bar{\mathbb{Q}}$. Clearly, the intersection with the rational hyperplane $\ell'^\perp$ consists of two points with affine coordinates in $\bar{\mathbb{Q}}$, one of which corresponds to $z$. \hfill $\square$

**Proposition 3.2.** Assume $T$ is an irreducible Hodge structure of K3 type. Then for the twistor base $\mathbb{P}^1_\ell \simeq S^2_\ell$ one has

(i) The set $\{z \ | \ \rho_z \geq 1\} \subset \mathbb{P}^1_\ell \simeq S^2_\ell$ is countable and dense (in the classical topology).

(ii) The set $\{z \ | \ \rho_z > 1\}$ is at most countable and contained in the equator $S^1_\ell \subset S^2_\ell$.

**Proof.** The first assertion is well known. We restrict to the second. Again, the assertion that the set is at most countable is a standard Hodge theoretic fact. We have to show it is contained in the equator $S^1_\ell$ of all $z \in \mathbb{P}^1_\ell$ with $\ell \in F_z$. Assume $\ell_i = \ell'_i + \mu_i \cdot \ell \in T \oplus \mathbb{Q} \cdot \ell$, $i = 1, 2$, are two classes orthogonal to $z$, i.e. their orthogonal projections $\ell_i = \bar{\ell}_i \perp \mu_i \cdot \ell$ in $P_T \oplus \mathbb{R} \cdot \ell$ both

\[ S^2_\ell \]

\[ S^1_\ell \]

\[ x \]

\[ \rho > \rho_x \]
span $P_x^\perp$, i.e. $\langle \tilde{\ell}_1 \rangle_R = P_x^\perp = \langle \tilde{\ell}_2 \rangle_R$. If $\mu_1 \neq 0 \neq \mu_2$, then $\langle \tilde{\ell}_1 \rangle_R = \langle \tilde{\ell}_2 \rangle_R$ implies $\mu_2 \cdot \tilde{\ell}_1 = \mu_1 \cdot \tilde{\ell}_2$ in $P_T$. As the orthogonal projection $T^\perp \rightarrow P_T$ is injective by Lemma 2.3, we find $\mu_2 \cdot \tilde{\ell}_1 = \mu_1 \cdot \tilde{\ell}_2$, i.e. $\tilde{\ell}_1, \tilde{\ell}_2$ are linearly dependent. If only $\mu_1 \neq 0$ but $\mu_2 = 0$, then $\tilde{\ell}_1, \tilde{\ell}_2 = \tilde{\ell}_2^\perp \in P \oplus \mathbb{R} \cdot \ell$ cannot be linearly dependent over $\mathbb{R}$. In the remaining case $\mu_1 = 0 = \mu_2$, i.e. $\ell_1, \ell_2 \in T$, clearly $\ell$ is orthogonal to $\tilde{\ell}_1$, i.e. $\ell \in \mathbb{P}_x = \tilde{\ell}_1^\perp = \tilde{\ell}_2^\perp$.

**Remark 3.3.** Clearly, the inclusion in (ii) above is strict, i.e. the very general couple $(\ell, \gamma)$ does not show up in $S_\ell^1$. Also, there may be points $z \in S_\ell^1$ where $\rho_z \neq 0$. Also, there may be no $\ell \in \mathbb{P}_x^1$ at all with $\rho_z > 1$. For example, when $\dim Q T = 2$, then $P_x^\perp \subset T \oplus \mathbb{R} \cdot \ell$ can contain at most one rational class up to scaling. From Remark 3.4 below, one can also deduce that for $\dim Q T \geq 5$, there always exists a $z \in \mathbb{P}_x^1$ with $\rho_z > 1$.

Note however, that in the main result of this paper we have to exclude all points in the equator $S_\ell^1$ and not only those with an excessive Picard number, cf. Lemma 3.9 and Corollary 3.4.

Let us fix a class $0 \neq \ell' \in T \oplus Q \cdot \ell$ and consider $x' \in \mathbb{P}_x^1$ orthogonal to $\ell'$. In fact, due to (3.2) and (3.3), $x'$ is uniquely determined by $\ell'$ up to conjugation.

Assume now that $x'$ is not contained in the equator $S_\ell^1$, i.e. $\ell' \notin T$, and that $x' \neq x, \bar{x}$, i.e. $\ell' \notin Q \cdot \ell$. To simplify notations we introduce $0 \neq m := (\ell, \ell') \in \mathbb{Z}$ and pick $\sigma \in T^{2,0}$ such that $(\sigma, \ell') = 1$. Writing $x' = [\sigma' := a\sigma + b\bar{\sigma} + \ell] \in \mathbb{P}_x^1$, then (3.3) becomes

$$a + b + m = 0.$$  

### 3.3. The twistor construction in the CM case.

With the same assumptions as before, we now additionally assume that $T$ itself has CM with CM field $K_T = K_T$. Pick a primitive element $\alpha$ of norm one, i.e. $K_T = \mathbb{Q}(\alpha)$ with $\alpha \cdot \bar{\alpha} = 1$. Furthermore, we consider a basis $(\gamma_i)$ of $T$ that is given by letting $\gamma_1$ be the $\ell$-component of $\ell'$ and setting $\gamma_i := \alpha^{-1}(\gamma_{i-1})$ for $i > 1$. As we have seen earlier (cf. proof of Lemma 2.10), then $\sigma$ corresponds to the vector $(1, \alpha, \ldots, \alpha^{r-1})$ under $T \overset{\sim}{\rightarrow} \mathbb{Q}^r$, $\gamma \mapsto ((\gamma, \gamma))$.

A basis of $T' := \ell^\perp$ is then given by $\gamma_i' := \gamma_i - m^{-1}(\ell', \gamma_i) \cdot \ell$. With respect to this basis, the period $x'$ is represented by $((\sigma', \gamma_i'))$. The coefficients can be computed as follows:

$$(\sigma', \gamma_i') = (a\sigma + b\bar{\sigma} + \ell \cdot \gamma_i - m^{-1}(\ell' \cdot \gamma_i) \cdot \ell)$$

$$= a(\sigma \cdot \gamma_i) + b(\bar{\sigma} \cdot \gamma_i) - dm^{-1}(\ell' \cdot \gamma_i)$$

$$= a\alpha_i + b\alpha_i^{-1} + m_i$$

$$= a(\alpha_i^{-1} - \alpha^{-1}) - m\alpha_i - m_i$$

with $m_i := -dm^{-1}(\ell' \cdot \gamma_i) \in \mathbb{Q}$. Use (3.4) for the last equality. Note that $(\sigma', \gamma_1') \in \mathbb{Q}$.

**Remark 3.4.** The Hodge structure $T'$ is not necessarily irreducible. For example, for $(\ell')^2 = 0$ one finds $(\sigma', \gamma_i') = 0$, which implies that $T'$ cannot be irreducible by Corollary 2.3.
Lemma 3.5. Using the above notation, we assume that $T'$ is an irreducible Hodge structure. Then the period field of $T'$ is described by

$$k_{T'} = \mathbb{Q}(x'_1).$$

Here, $x'_1 := a(\alpha^{i-1} - \alpha^{1-i}) - m\alpha^{1-i}$, $i = 1, \ldots, r$, which are all non-zero. Moreover, the natural map yields an injection $\bigoplus \mathbb{Q} \cdot x'_i \longrightarrow k_{T'}$.

Proof. This is a direct consequence of Corollary 2.4 and the fact that $x'_1 \in \mathbb{Q}$. \hfill $\Box$

The next technical result together with Lemma 2.11 will lead to an action of the maximal totally real subfield $K_T^0 := K_T \cap \mathbb{R}$ of the CM field $K_T$ on the Hodge structures $T' = \ell'^\perp$ away from the equator. As $K_T^0$ is rather big, recall it satisfies $[K_T^0 : \mathbb{Q}] = r/2$, this will suffice to conclude.

Lemma 3.6. Assume the Hodge structure $T'$ is irreducible. Then $K_T^0 := K_T \cap \mathbb{R}$ satisfies the following properties.

(i) $K_T^0 \subset \bigoplus \mathbb{Q} \cdot x'_i \subset k_{T'}$.

(ii) Multiplication in $k_{T'}$ with elements of $K_T^0$ preserves the subspace $\bigoplus \mathbb{Q} \cdot x'_i$.

Proof. As $0 \neq x_1 \in \mathbb{Q}$, the second assertion implies the first, but for clarity reasons we state and prove them separately.

Any element $f \in K_T$ can be uniquely written as $f = \sum_{i=1}^r a_i \alpha^{i-1}$ with $a_i \in \mathbb{Q}$. As $|\alpha| = 1$, the complex conjugate of $f$ is given by $\bar{f} = \sum_{i=1}^r a_i \alpha^{1-i}$. Hence, $f \in K_T^0$ if and only if $\sum_{i=1}^r a_i(\alpha^{i-1} - \alpha^{1-i}) = 0$. In this case, one finds $\sum_{i=1}^r a_i x'_i = a \sum_{i=1}^r a_i(\alpha^{i-1} - \alpha^{1-i}) - m \sum_{i=1}^r a_i \alpha^{1-i} = -m \bar{f} = -m f$, and, therefore, $f \in \bigoplus \mathbb{Q} \cdot x'_i$. Hence, $K_T^0 \subset \bigoplus \mathbb{Q} \cdot x'_i$.

For any $f = 2 \sum_{i=1}^r a_i \alpha^{i-1}$, the decomposition in its real and imaginary part is of the form $f = (1/2)(f + \bar{f}) + (1/2)(f - \bar{f}) = \sum_{i=1}^r a_i(\alpha^{i-1} + \alpha^{1-i}) + \sum_{i=1}^r a_i(\alpha^{i-1} - \alpha^{1-i})$. In particular, the elements $\alpha^{i-1} + \alpha^{1-i}$, $i = 1, \ldots, r$, generate $K_T^0$ (but, for dimension reasons, they are not linearly independent). Observe that $(\alpha + \alpha^{-1})^{i-1} = \alpha^{i-1} + \alpha^{1-i} + M$, where $M$ is a linear combination of $\alpha^i + \alpha^{-j}$, $j = 0, \ldots, i - 2$. In particular, $\alpha + \alpha^{-1}$ is a primitive element of $K_T^0$. Thus, by induction, in order to show that $K_T^0$ preserves $\bigoplus \mathbb{Q} \cdot x'_i$, it suffices to prove that multiplication with $\alpha + \alpha^{-1}$ does.

A computation yields that $(\alpha + \alpha^{-1})x'_i = x'_{i-1} + x'_{i+1}$ for $2 \leq i \leq r - 1$. For $i = 1$ observe $(\alpha + \alpha^{-1})x'_1 = -m(\alpha + \alpha^{-1}) \in K_T^0 \subset \bigoplus \mathbb{Q} x'_i$ using (i). To deal with the case $i = r$, we write $\alpha^r = \sum_{i=1}^r c_i \alpha^{-1}$ for some $c_i \in \mathbb{Q}$ and, thus, $\alpha^{-r} = \sum_{i=1}^r c_i \alpha^{1-i}$. Hence, $(\alpha + \alpha^{-1})x'_r = x'_{r-1} + a(\alpha^r - \alpha^{-r}) - m\alpha^{-r} = x'_{r-1} + \sum_{i=1}^r c_i x'_i \in \bigoplus \mathbb{Q} \cdot x'_i$. \hfill $\Box$

Remark 3.7. (i) We stress that both assumptions, $m = (\ell, \ell') \neq 0$ and $m \in \mathbb{Q}$, are used in the proof of the proposition. Indeed, the equation $a_i x'_i = -mf$ would otherwise not yield the desired inclusion $K_T^0 \longrightarrow \bigoplus \mathbb{Q} \cdot x'_i$. Similarly, one would not expect $K_T^0$ to be contained in $k_{T'}$, when $T'$ is not irreducible.
(ii) Although the proposition only establishes properties of the totally real field $K_T^0$, its proof uses the action of the CM field $K_T$ on $T$. It is unclear whether also in the totally real case $K_T = K_T^0$ is a subfield of $k_T$.

**Proposition 3.8.** Assume $T$ is an irreducible Hodge structure with CM by $K_T$ and let $x' \in \mathbb{P}^1_\ell \setminus S^1_\ell$ be a point in the twistor base orthogonal to a non-trivial class $\ell' \in T \oplus \mathbb{Q} \cdot \ell$.

Then the corresponding Hodge structure on $T' = \ell'^\perp \subset T \oplus \mathbb{Q} \cdot \ell$ is irreducible with CM. The maximal totally real subfields of the CM fields $K_T$ and $K_{T'}$ coincide: $K_T^0 = K_{T'}^0$.

**Proof.** The irreducibility of $T'$ follows from Proposition 3.2. Lemma 3.6 shows that $K_T^0$ is contained in $k_T \cap \mathbb{R}$ and that it preserves the subspace $\bigoplus \mathbb{Q} \cdot x'_i \subset k_T$. As $2[K_T^0 : \mathbb{Q}] = [K_T : \mathbb{Q}] = \dim_{\mathbb{Q}} T = \dim_{\mathbb{Q}} T'$, Lemma 2.11 applies and yields the result.

The assumption that only fibres away from the equator are allowed is not an artefact of our technique unless we are in the case $r = \dim_{\mathbb{Q}} T = 2$.

**Lemma 3.9.** Assume $\dim_{\mathbb{Q}} T > 2$. Then a Hodge structure $T' = \ell'^\perp$ with $(\ell')^2 \neq 0$ for which the corresponding point $x'$ is contained in the equator $S^1_\ell$ is not of CM type. In fact, in the case that $T'$ is not irreducible, also the minimal Hodge structure of $K3$ type $T'' \subset T'$ with $T''^0 \subset T''_\mathbb{C}$ is not of CM type.

**Proof.** Observe that for such a Hodge structure, $\ell$ is contained in $P_{T'}$. If $T'$ were of CM type, then its endomorphism field $K_{T'}$ would act transitively on $T'$. In particular, $T'$ is irreducible and $K_{T'} \cdot \ell = T'$. However, all elements of $K_{T'}$ act as endomorphisms of the Hodge structure and, in particular, preserve $P_{T'}$. This only leaves the possibility that $P_{T'} = T'_R$, which yields the contradiction $\dim_{\mathbb{Q}} T = \dim_{\mathbb{Q}} T' = 2$.

If $T'$ is not irreducible, then the same arguments apply to $T'' \subset T'$ and prove $\dim_{\mathbb{Q}} T'' = 2$.

As $T' = T'' \oplus T''^\perp$ and $T''^\perp$ is pure of type $(1,1)$, the residue fields $k_{T'}$ and $k_{T''}$ of the two Hodge structures $T'$ and $T''$ coincide. To get a contradiction, it is enough to show that $[k_{T''} : \mathbb{Q}] = [k_{T'} : \mathbb{Q}] > 2$. For this we adapt the approach explained before to the case $m = 0$.

The classes $\ell$ and $\gamma'_i := \gamma_i - ((\ell' \cdot \gamma_i)/(\ell')^2) \cdot \ell'$, $i = 1, \ldots, r$, generate $T'$. Hence, the period field $k_{T'}$ is generated by $(\sigma', \ell') = d \in \mathbb{Z}$ and $(\sigma' \cdot \gamma'_i) = a(\alpha^{i-1} - \alpha^{1-i})$, where we use that $a + b = 0$ and $a = \sqrt{d/(2(\sigma' \cdot \sigma'))} \in \mathbb{R}$ under the assumption $m = (\ell, \ell') = 0$. However, the elements $\alpha^{i} - \alpha^{-i}$ generate the subspace of all purely imaginary elements in $K_T$, which is of dimension $r/2$. Thus, the elements $a(\alpha^{i-1} - \alpha^{1-i}) = (\sigma' \cdot \gamma'_i) \in k_{T''}$, which are all purely imaginary, already span a sub-vector space of dimension $r/2 \geq 2$. Therefore, one finds the contradiction $[k_{T'} : \mathbb{Q}] > 2$.

**3.4. CM fields of twistor fibres.** In the situation of Proposition 3.8 we have $K_T^0 = K_{T'}^0$, but the totally imaginary quadratic extensions of this field

$K_T^0 \subset K_T$ and $K_{T'}^0 \subset K_{T'}$.
will usually be distinct. This can be made precise as follows. As before, pick a primitive
element of norm one of \( K_T \) and write \( K_T = \mathbb{Q}(\alpha) \). Then the maximal totally real field is
\( K_T^0 = \mathbb{Q}(\alpha + \alpha^{-1}) \), see the arguments in the proof of Lemma 3.6.

**Corollary 3.10.** Under the assumption of Proposition 3.8 and additionally assuming \( x' \neq x, \bar{x} \),
the endomorphism field \( K_{T'} \) of the CM Hodge structure \( T' \) is the quadratic extension of \( K_T^0 = K_T^{0} \),
described by

\[
X'^2 - (\alpha + \alpha^{-1})X + d(\sigma, \bar{\sigma})^{-1} - m.
\]

Here, \( d = (\ell)^2 \), \( m = (\ell, \ell') \) and \( \sigma \in T^{2,0} \) is chosen such that \( (\sigma, \ell') = 1 \).

**Proof.** The first step consists of observing that the arguments in the proof of Lemma 3.6 prove
\( K_{T'} = K_T^0(a\alpha + b\alpha^{-1}) \).

Therefore, it suffices to show that \( a\alpha + b\alpha^{-1} \) satisfies (3.5).

As before, we represent \( x' \) by \( a\sigma + \bar{b}\sigma + \ell \). According to (3.2) and (3.4), one has \( 2ab(\sigma, \bar{\sigma}) + d = 0 \)
and \( a + b + m \), which readily imply \( (a\alpha + b\alpha^{-1})^2 = a^2\alpha^2 + b^2\alpha^{-2} - d(\sigma, \bar{\sigma})^{-1} \) and \( (\alpha + \alpha^{-1})(a\alpha + b\alpha^{-1}) = a\alpha^2 + b\alpha^{-2} - m \). A straightforward computation concludes the proof.

It is an exercise to directly check that (3.5) does indeed define a totally imaginary quadratic
extension of \( K_T^0 \), as we know it should.

**Remark 3.11.** In (3.5) only \( m \) seems to depend on the actual class \( \ell' \) or the point \( t \). But
implicitly \( \ell' \) is involved once more via the condition \( (\sigma, \ell') = 1 \). There are of course many
classes \( \ell'_1, \ell'_2 \) with the same \( m \), i.e. \( (\ell, \ell'_1) = (\ell, \ell'_2) \). However, if \( \sigma \) can be chosen such that for
both \( (\sigma, \ell'_i) = 1, i = 1, 2 \), then by virtue of the irreducibility of the Hodge structure \( T \) one has
\( \ell'_1 - \ell'_2 \in \mathbb{Q} \cdot \ell \) and \( (\ell, \ell'_1) = (\ell, \ell'_2) \) would in fact show \( \ell'_1 = \ell'_2 \). Hence, (3.5) will be the same only
for conjugate pairs of points \( x', \bar{x}' \). Of course, the CM fields of two even non-conjugate points
can be isomorphic without (3.5) being identical.

**4. Period values**

So far we have encoded a K3 Hodge structure \( T \) by the line \( T^{2,0} \subset T_\mathbb{C} \) or, equivalently, by its
period point \( x \in \mathbb{P}(T_\mathbb{C}) \). In the applications to K3 surfaces with CM, which are always defined
over \( \mathbb{Q} \), a finer structure is present, namely a generator \( \sigma \in T^{2,0} \) that is unique up to scalars in
\( \mathbb{Q}^* \). In this section we formalize the situation. We introduce the period value and explain how
it behaves in a twistor family.

**4.1. Period value.** Let \( T \) be an irreducible Hodge structure of K3 type and let \( \Sigma \subset T^{2,0} \) be a
\( \mathbb{Q} \)-line, i.e. a \( \mathbb{Q} \)-linear subspace of dimension one. Then we define the period value
\[
r_{T, \Sigma} := (\sigma, \gamma) \in \mathbb{C}^*/\mathbb{Q}^*.
\]

Here, \( 0 \neq \sigma \in \Sigma \) and \( 0 \neq \gamma \in T \) are chosen arbitrarily.
Lemma 4.1. If $T$ has complex multiplication, then $r_{T,\Sigma} \in \mathbb{C}^*/\bar{\mathbb{Q}}^*$ is well-defined, i.e. it is independent of the choices of $\sigma \in \Sigma$ and $\gamma \in T$.

Proof. Indeed, for $\lambda \in \bar{\mathbb{Q}}^*$ one clearly has $(\lambda \cdot \sigma) \equiv (\sigma \cdot \gamma) \in \mathbb{C}^*/\bar{\mathbb{Q}}^*$. To prove independence of $\gamma$, choose a primitive element $\alpha \in K$ of norm one of the CM field $K$ of $T$. Then any other element $\gamma' \in T$ can be written as $\gamma' = (\sum a_i \alpha^i)(\gamma)$ with $a_i \in \mathbb{Q}$. Then $(\sigma \cdot \gamma') = \sum a_i \cdot (\alpha^{-i}(\sigma) \cdot \gamma) = (\sum a_i \alpha^{-i}) \cdot (\sigma \cdot \gamma) \equiv (\sigma \cdot \gamma) \in \mathbb{C}^*/\bar{\mathbb{Q}}^*$.

Clearly, any two $\bar{\mathbb{Q}}$-lines $\Sigma_1, \Sigma_2 \subset T^{2,0}$ differ by some complex scalar $\lambda \in \mathbb{C}^*$, i.e. $\Sigma_2 = \lambda \cdot \Sigma_1$.

The effect on the period value is expressed by

$$r_{T,\lambda \Sigma} = \lambda \cdot r_{T,\Sigma}.$$ 

In the following we will often write $r_\sigma := r_{T,\sigma} := r_{T,\Sigma}$.

Lemma 4.2. If $T$ has complex multiplication and $0 \neq \sigma \in T^{2,0}$, then

$$r_\sigma^{-1} \cdot \sigma \in T \otimes \bar{\mathbb{Q}}$$ and $(\sigma, \bar{\sigma}) \equiv r_\sigma \cdot \bar{r}_\sigma$ in $\mathbb{C}^*/\bar{\mathbb{Q}}^*$.

Proof. For the first assertion observe that for all $\gamma \in T$ one has $(r_\sigma^{-1} \cdot \sigma, \gamma) \in \bar{\mathbb{Q}}$. The second assertion follows from the first.

4.2. Period value of twistor fibres. We are using the notation of Section 3.3. In particular, $T$ is assumed to have complex multiplication.

Let $x' \in \mathbb{P}^1_\ell \setminus S^1_\ell$ be orthogonal to some fixed $\ell' \in T \oplus \mathbb{Q} \cdot \ell$ and consider the induced natural Hodge structure on $T' := \ell'^\perp \subset T \oplus \mathbb{Q} \cdot \ell$.

Proposition 4.3. Assume that $x'$ is represented by $\sigma' = a \sigma + b \bar{\sigma} + c \ell$. Then in $\mathbb{C}^*/\bar{\mathbb{Q}}^*$

$$b \equiv a \cdot r_\sigma/\bar{r}_\sigma$$ and $r_{\sigma'} \equiv c \equiv a \cdot r_\sigma$.

Furthermore, $r_{\sigma'} = a \cdot r_\sigma = \bar{r}_\sigma$ if $b = 1$ and $r_{\sigma'} \equiv 1$ if $c = 1$.

Proof. Note that by virtue of Proposition 3.8 the Hodge structure $T'$ has complex multiplication and, therefore, its period values $r_{\sigma'} := r_{T',\Sigma'}$ are well defined for any $0 \neq \sigma' \in T^{2,0}$.

Let us first look at the case that $r_\sigma \equiv 1$, i.e. $\sigma \in T \otimes \bar{\mathbb{Q}}$, and $a = 1$. Then, (3.2) and (3.3) imply $b, c \in \bar{\mathbb{Q}}$, cf. Lemma 3.1. In the general case, rewrite $r_{\sigma}^{-1} \sigma'$ as follows

$$r_{\sigma}^{-1} \sigma' = \frac{r_{\sigma}^{-1}(a \sigma)}{b/\bar{a}} \cdot (\bar{r}_{\sigma}^{-1}(a \sigma)) + (r_{\sigma}^{-1} c) \ell.$$ 

Then by the first step $b \equiv a \cdot r_\sigma/\bar{r}_\sigma \equiv a \cdot r_\sigma/\bar{r}_\sigma$ and $c \equiv r_{\sigma} \equiv a \cdot r_\sigma$ in $\mathbb{C}^*/\bar{\mathbb{Q}}^*$.

To determine $r_{\sigma'}$ we pick an arbitrary $0 \neq \gamma \in T'$ and compute $(\sigma', \gamma) = a(\sigma, \gamma) + b(\bar{\sigma}, \gamma) + c(\ell, \gamma) = a(u \cdot r_\sigma + v \cdot (r_{\sigma}/\bar{r}_\sigma) \cdot r_{\sigma} + w \cdot r_{\sigma}), u, v, w \in \bar{\mathbb{Q}}$, which suffices to conclude.

The assertion for $c = 1$ is immediate and for $b = 1$ let $\gamma \in T$ and compute $r_{\sigma'} \equiv (\sigma', \gamma) = a(\sigma, \gamma) + (\bar{r}_{\sigma}) \equiv a \cdot r_\sigma + \bar{r}_\sigma \equiv a \cdot r_\sigma$. □
In the geometric situation, where all CM fibres \( S_t \) come equipped with a natural \( \mathbb{Q} \)-line in \( H^{2,0}(S_t) \), we expect the natural period values to change. As the three natural choices for \( \sigma' \) consisting of setting \( a = 1, b = 1, \) resp. \( c = 1 \), all lead to constant period values \( \gamma_{\sigma'} \) no preferred choice for \( \sigma' \) seems to suggest itself from a purely Hodge theoretic point of view.

5. K3 surfaces with CM

The results of the previous sections can be applied to the transcendental part

\[
T := T(S) := \text{NS}(S)_{±} \otimes \mathbb{Q} \subset H^2(S, \mathbb{Q})
\]

of a complex projective K3 surface \( S \). In fact, everything remains valid if instead of a K3 surface \( S \), one considers a hyperkähler manifold, cf. Section 1.3. For simplicity we restrict to the case of K3 surfaces and leave the necessary modifications in the hyperkähler case to the reader.

5.1. Dictionary. The intersection form on \( H^2(S, \mathbb{Z}) \) provides the bilinear form \(( , )\) on \( T = T(S) \). It is positive definite on the plane

\[
P_T = (H^{2,0}(S) \oplus H^{0,2}(S)) \cap H^2(S, \mathbb{R})
\]

and, by virtue of the projectivity of \( S \), negative definite on its orthogonal complement \( P_T^\perp \subset T_{\mathbb{R}} \).

The period field and the endomorphism field of \( S \) are introduced as the corresponding fields of the transcendental part \( T(S) \):

\[
k_S := k_{T(S)} \text{ and } K_S := K_{T(S)}.
\]

So, \( k_S \) is the residue field of the period point of \( S \) taken either in \( \mathbb{P}(T(S)_{\mathbb{Q}}) \times \mathbb{C} \) or \( \mathbb{P}(H^2(S, \mathbb{Q})) \times \mathbb{Q} \mathbb{C} \) and for \( K_S \) one knows \( \dim k_S \cdot T = (22 - \rho(S)) : |K_S : \mathbb{Q}|^{-1} \).

Fix an ample class \( \ell = c_1(L) \in H^2(S, \mathbb{Z}) \) and consider \( T(S) \oplus \mathbb{Q} \cdot \ell \subset H^2(S, \mathbb{Q}) \) is positive definite of dimension three. As in the abstract setting, one can think of the twistor base \( \mathbb{P}^1_{\ell} \) as the set of all \( \sigma_t \in H^{2,0}((M, I_t)) \) up to scaling or as the set of all oriented positive planes \( P_t = (\text{Re}(\sigma_t), \text{Im}(\sigma_t))_{\mathbb{R}} \subset P_T \oplus \mathbb{R} \cdot \ell \) or as the sphere \( S^2_\ell \subset P_T \oplus \mathbb{R} \cdot \ell \subset H^2(S, \mathbb{R}) \) of Kähler classes \( \omega_t \) that span the line orthogonal to \( P_t \). The Picard number \( \rho_\ell \) (cf. Section 3.2) of a point \( z \in \mathbb{P}^1_{\ell} \) and the Picard number \( \rho(S_t) \) of the twistor fibre \( S_t = (M, I_t) \), with \( t \) corresponding to \( z \) under \( S^2_\ell \simeq \mathbb{P}^1_{\ell} \), compare as follows

\[
\rho_\ell + \rho(S) - 1 = \rho(S_t).
\]

The equator

\[
S^1_\ell \subset S^2_\ell \subset P_T \oplus \mathbb{R} \cdot \ell \subset H^2(S, \mathbb{R})
\]

can be thought of as the set of Kähler classes \( \omega_t \) orthogonal to \( \ell \) or as the set of complex structures \( I_t \) such that \( \ell \) is contained in the positive plane \( P_t \). As an immediate consequence of the abstract Proposition 3.2 we state the following.
Corollary 5.1. Assume $S \rightarrow \mathbb{P}^1$ is the twistor space associated with a polarized K3 surface $(S, L)$. If $\rho(S_t) > \rho(S)$, then $t$ is contained in the equator $S^1 \subset S^2$. □

Remark 5.2. Note however that there are cases where $\rho(S)$ is maximal, i.e. for all fibres one has $\rho(S_t) \leq \rho(S)$. Clearly, this is the case when $\rho(S) = 20$. It would be interesting to work out geometric conditions on $(S, L)$ such that $\rho(S_t) \leq \rho(S)$ holds for all twistor fibres.

The main Theorem 0.1 is then an immediate consequence of Proposition 3.8. We rephrase it in the following alternative but equivalent form.

Theorem 5.3. Consider the twistor space $S \rightarrow \mathbb{P}^1$ associated with a polarized, complex K3 surface $(S, L)$ with complex multiplication. Then every algebraic fibre $S_t$ such that $\ell = c_1(L)$ is not contained in $H^{2,0}(S_t) \oplus H^{0,2}(S_t)$ has complex multiplication as well. Moreover, the maximal totally real subfields of the two endomorphism fields of $S$ and $S_t$ coincide. □

Using Lemma 3.9 one sees that in the case of $\rho(S) < 20$ the fibres over the equator have indeed to be avoided. This is the following corollary.

Corollary 5.4. Assume $(S, L)$ is a polarized K3 surfaces with complex multiplication and Picard number $\rho(S) < 20$. Then no algebraic fibre $S_t$ of the associated twistor space with $t$ contained in the equator has complex multiplication.

Proof. As we assume $S_t$ is algebraic, there must exist an $\ell' \in T \oplus \mathbb{Q} \cdot \ell$ orthogonal to the period $a\sigma + b\bar{\sigma} + \ell$ with $(\ell')^2 > 0$. Hence, Lemma 3.9 applies. □

Remark 5.5. The last result in particular shows that no twistor fibre $S_t$ of a twistor space associated with a polarized K3 surface $(S, L)$ with complex multiplication and Picard number $\rho(S) < 20$ will ever have maximal Picard number $\rho(S_t) = 20$.

5.2. **Endomorphisms as Hodge classes on the product.** Recall that a projective K3 surface $S$ has CM if its transcendental part $T = T(S)$ has CM, i.e. the endomorphism field $K_S$ of all endomorphisms of the Hodge structure $T$ is a CM field with $\dim_{K_S} T = 1$. 
Remark 5.6. The endomorphism field $K_S = \text{End}_{Hdg}(T(S))$ can be interpreted geometrically in terms of Hodge classes on the product $S \times S$. More precisely, via Poincaré duality the space of rational $(2,2)$ classes $H^{2,2}(S \times S, \mathbb{Q})$ splits as a direct sum of $\text{End}(\text{NS}(S)_{\mathbb{Q}}) \cong \text{NS}(S)_{\mathbb{Q}} \otimes \text{NS}(S)_{\mathbb{Q}}$ and

$$K_S = \text{End}_{K_S}(T(S)) \cong (T(S) \otimes T(S))^{2,2}.$$ 

Multiplication in $K_S$ can be understood in terms of convolution of classes on $S \times S$. As $T(S) \otimes T(S) \simeq S^2 T(S) \oplus \wedge^2 T(S)$, one also has

$$K_S \simeq (S^2 T(S))^{2,2} \oplus (\wedge^2 T(S))^{2,2}.$$ 

This is the eigenspace decomposition with respect to complex conjugation acting on $K_S$. In particular, the maximal totally real subfield is given by

$$K_S^0 \simeq (S^2 T(S))^{2,2} \subset H^{2,2}(S^{[2]}, \mathbb{Q}).$$

As alluded to in the introduction, although $K_S^0 = K_S^{0'}$ for all algebraic twistor fibres away from the equator, there is no class $\varphi \in H^1(S \times_{\mathbb{P}^1} S, \mathbb{Q})$ that would restrict to a given class $\varphi \neq \pm 1$ in $K_S^0$ on all the CM fibres.

So, it seems something geometrically happens behind the curtain of the transcendental twistor space, that is not completely explained by Hodge theory.

5.3. Defined over number fields. The following important fact was proved in [PSS75, Thm. 4] and a more precise version can be found in [Riz05, Cor. 3.3.19]. The original arguments involve the Kuga–Satake construction reducing the problem to the corresponding problem for abelian varieties. The proof below is more geometric relying on the Hodge conjecture for the square $S \times S$ of a K3 surface $S$ with CM.

Proposition 5.7. Any K3 surface with CM is defined over $\overline{\mathbb{Q}}$. Equivalently, if $K_S = k_S$ for a projective K3 surface $S$, then $S$ is defined over $\overline{\mathbb{Q}}$.

Proof. First note that by virtue of Proposition 2.12, CM is indeed equivalent to the equality $K_S = k_S$.

Pick a primitive element $\alpha$ of $K_S$ of norm one, i.e. $\alpha$ generates $K_S$ and, viewed as an endomorphism of the Hodge structure $T(S)$, is an isometry. As it has been proved recently in [Bus19], $\alpha$ as a class in $H^{2,2}(S \times S, \mathbb{Q})$ is algebraic, see also [Huy19] for a motivic interpretation.

Now one applies the usual ‘spread out’ technique to $S$ and a cycle $Z$ on $S \times S$ representing $\alpha$. More precisely, $S$ and $Z$ are both defined over a finitely generated field extension $L/\overline{\mathbb{Q}}$, i.e. $S \simeq S_0 \times_L \mathbb{C}$ and $Z = Z_0 \times_L \mathbb{C}$. Viewing $L$ as a function field of a variety $Y$ over $\overline{\mathbb{Q}}$ and inverting denominators, we may consider $S_0$ as the scheme theoretic fibre $S_0$ of a family $S \rightarrow Y$. Taking the closure $Z$ of $Z_0$ in $S \times_Y S$ and shrinking $Y$ if necessary produces a relative flat cycle. Next, let $S_C \rightarrow Y_C$ be the base change to a family of complex K3 surfaces and consider the action of the fibres of the cycle $Z_C$ on the transcendental part $T(S_t)$, $t \in Y(\mathbb{C})$. For very general
t ∈ Y(ℂ), the fibre 𝑆𝑡 as a scheme is isomorphic to the original S via a base change with respect to a chosen embedding L ⊂ ℂ. The isomorphism is compatible with the ℓ-adic action of ℤ𝑡 and, therefore, the action of [𝑍𝑡] and α on the ℓ-adic transcendental part 𝑇̃(𝑆𝑡)𝑄𝑡 coincide. In particular, on that fibre the action of ℤ𝑡 coincides with the action of α and, therefore, the classes [𝑍𝑡]∗ 𝜅, 𝑘 = 1, . . . , span a subspace of dimension [𝐾_S : ℚ] (namely 𝑇(𝑆𝑡)) for any 0 ≠ 𝜅 ∈ 𝑇(𝑆𝑡).

This remains valid for any nearby fibre 𝑆′ and, hence, dim_ℚ 𝑇(𝑆′) ≥ dim_ℚ 𝑇(𝑆).

Remark 5.8. A converse of Proposition 5.7 was proved in [Tre15]: If a projective K3 surface S is defined over ℤ̄ and its period field k_S is algebraic, i.e. k_S ⊂ ℤ̄, then S has CM.

This is a K3 analogue of the classical result for elliptic curves that the only elliptic curves ℂ/(ℤ ⊕ ℤτ) with τ and j(τ) algebraic are CM elliptic curves, i.e. when τ is imaginary quadratic.

Via the Kuga–Satake construction, the problem is reduced to the analogous statement for abelian varieties which had been settled by Cohen and Shiga–Wolfart [Coh96, SW95], see also [UY11, Thm. 1.3]. It would certainly be interesting to find a geometric argument not relying on the Kuga–Satake construction.

According to Corollary 5.4 we know that the algebraic fibres 𝑆𝑡 over the equator do not have complex multiplication. This is strengthened by the following result.

Corollary 5.9. Assume (S, ℒ) is a polarized K3 surfaces with complex multiplication and Picard number ρ(S) < 20. Then no algebraic fibre 𝑆𝑡 of the associated twistor space with t contained in the equator is defined over ℤ̄. 

Proof. Suppose 𝑆𝑡 were defined over ℤ̄. According to Lemma 3.1 the period field k_S is also algebraic. Then use [Tre15] to conclude that 𝑆𝑡 has complex multiplication which is excluded by Corollary 5.4.

5.4. Period values of K3 surfaces with CM. Assume that a K3 surface S can be defined over ℤ̄, i.e. there exists a K3 surface S⁰ over ℤ̄ such that S ≃ S⁰ × ℤ̄ ℂ. Pick a regular form 0 ≠ σ⁰ ∈ 𝑉⁰(S⁰, 𝜋_S⁰/ℤ̄) and consider (σ⁰, 𝜅) for a class 0 ≠ 𝜅 ∈ 𝑇(S). These values are all expected to be transcendental which would be the weight-two analogue of the classical fact that for an elliptic curve E⁰ over ℤ̄ the integrals ∫_Ω ω⁰ with ω⁰ ∈ 𝑉⁰(E⁰, 𝜋_E⁰/ℤ̄) and δ ∈ 𝑉₁(E, ℤ) are transcendental numbers. The problem for K3 surfaces seems open, but see [Wue86, Exa. 3]. Questions of irrationality have been dealt with in [BC16].

Remark 5.10. Often, in this context, the field ℤ̄((σ⁰, 𝜅)) is called the period field, but it should not be confused with the period field k_S in the sense of Sections 2.1 and 5.1, which for example for K3 surfaces with CM is algebraic.

Note that for a different choice of σ⁰ in 𝑉⁰(S⁰, 𝜋_S⁰/ℤ̄) the value of (σ⁰, 𝜅) changes by an algebraic factor. As a consequence of Lemma 4.1 one has the following stronger fact.
Corollary 5.11. Assume $S$ is a complex projective K3 surface with complex multiplication. Then the period value of $S$
\[ r_S := r_{T(S), \sigma^o} \in \mathbb{C}^*/\bar{\mathbb{Q}}^* \]
is well-defined. Here, $S^o$ is a model of $S$ over $\bar{\mathbb{Q}}$ and $0 \neq \sigma^o \in H^0(S^o, \Omega^2_{S^o/\bar{\mathbb{Q}}})$.

Proof. The model $S^o$ is unique up to isomorphism and so is the induced $\bar{\mathbb{Q}}$-line $H^0(S^o, \Omega^2_{S^o/\bar{\mathbb{Q}}}) \subset H^0(S, \Omega^2_{S/\mathbb{C}})$.

Similarly, Lemma 4.2 implies that $r_S^{-1} \cdot \sigma^o \in T(S) \otimes \bar{\mathbb{Q}}$ and $(\sigma^o \cdot \bar{\sigma}^o) \equiv r_S \cdot \bar{r}_S \in \mathbb{C}^*/\bar{\mathbb{Q}}^*$.

Currently, there are no techniques to compute or even guess the natural $\bar{\mathbb{Q}}$-line $H^0(S^o, \Omega^2_{S^o/\bar{\mathbb{Q}}}) \subset H^0(S, \Omega^2_{S/\mathbb{C}})$ for a K3 surface $S$ with CM. In the case $\rho(S) = 20$, the problem reduces to the computation of $\left(\int \omega^o\right)^2$ for $\omega^o \in H^0(E^o, \Omega_{E^o/\mathbb{Q}})$. Here, $S$ is covered by the Kummer surface associated with a product $E_1 \times E_2$ of two elliptic curves both isogenous to $E^o \times \bar{\mathbb{Q}} \mathbb{C}$.

Remark 5.12. In principle one could hope that once the natural $\bar{\mathbb{Q}}$-line for $S$ is found, the ones for all other twistor fibres $S_T$ with CM can be predicted. However, at this point the results of Section 4.2 only tell us that $r_{S_t} = a_t \cdot r_S$ assuming $\sigma_t^o = a_t \sigma^o + b_t \bar{\sigma}^o + c_t \ell$, but how to predict $a_t$ seems unclear.

As according to Grothendieck’s period conjecture any $\bar{\mathbb{Q}}$-algebraic relation between $r_S$ and $r_{S_t}$ should be motivic, we actually expect that infinitely many of the $r_{S_t}$ are independent of $r_S$, cf. [Huy18, Sec. 2.3]

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