LEVEL OPTIMIZATION IN THE TOTALLY REAL CASE

KAZUHIRO FUJIIWARA

ABSTRACT. In this paper, congruences between holomorphic Hilbert modular forms are studied. We show the best possible level optimization result outside \( \ell \) for \( \ell \geq 3 \) by solving the remaining case of Mazur principle when the degree of the totally real field is even.

1. Introduction

The aim of this paper is to discuss mod \( \ell \) congruences between Hilbert modular forms. Here \( \ell \) is a fixed prime. More precisely, given a holomorphic Hilbert modular form \( f \) of some level and weight, we look for another modular form \( g \) with a smaller level and the same weight whose Fourier coefficient are congruent to that of \( f \). For elliptic modular forms, i.e., when \( F = \mathbb{Q} \), such studies have been done by J. P. Serre, B. Mazur, K. A. Ribet, F. Diamond, and by other authors (see the references of [18]). Amazingly, to obtain some optimal \( g \), the correct condition imposed on \( f \) involves an information from the Galois representation attached to \( f \), so we need to use Galois representations to study congruences.

We use a representation theoretical terminology in the adelic setting, since it is essential in the local analysis. For a totally real field \( F \), let \( I_F = \{ \iota : F \hookrightarrow \mathbb{R} \} \) be the set of the infinite places of \( F \). We take an element \( k = (k_\iota)_{\iota \in I_F} \in \mathbb{Z}^{I_F} \) and \( w \in \mathbb{Z} \), where \( k_\iota \geq 2 \), and \( k_\iota \equiv w \mod 2 \) for all \( \iota \). Let \( \pi \) be a cuspidal representation of \( \text{GL}_2(\mathbb{A}_F) \) having infinity type \((k, w)\) (see notations for our normalization). Those types of cuspidal representations are generated by holomorphic Hilbert cusp forms.

Fix a prime \( \ell \), and an isomorphism \( \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell \) by the axiom of choice. It is known that the finite part \( \pi^\infty \) of \( \pi \) is defined over some algebraic number field, and hence over some finite extension \( E_\lambda \) of \( \mathbb{Q}_\ell \). In this case we say that \( \pi \) is defined over \( E_\lambda \). For the integer ring \( \mathcal{O}_\lambda \) of \( E_\lambda \), there is a two dimensional continuous \( \lambda \)-adic Galois representation \( \rho_{\pi, \lambda} : G_F \rightarrow \text{GL}_2(\mathcal{O}_\lambda) \) associated to \( \pi \) (see [17], [5], [22], [21], [1] for \( F \neq \mathbb{Q} \)). Here \( G_F = \text{Gal}(\bar{F}/F) \) is the absolute Galois group of \( F \). Two cuspidal representations \( \pi \) and \( \pi' \) which are defined over some \( E_\lambda \) are congruent mod \( \lambda \) if the semi-simplifications of the mod \( \lambda \)-reductions of their \( \lambda \)-adic representations are isomorphic over \( k_\lambda \):

\[
(\rho_{\pi, \lambda} \mod \lambda)\beta \simeq (\rho_{\pi', \lambda} \mod \lambda)\beta.
\]

Here \((-)\beta\) denotes the semi-simplification. We note that this notion of congruences is equivalent to the congruences between the Fourier coefficients of the corresponding normalized Hilbert modular newforms by the Chebotarev density theorem.

We call an absolutely irreducible mod \( \ell \) continuous representation \( \bar{\rho} : G_F \rightarrow \text{GL}_2(\bar{k}_\lambda) \) modular if it is isomorphic to some \( \rho_{\pi, \lambda} \mod \lambda \) over \( \bar{k}_\lambda \). Even if \( \bar{\rho} \) is modular, there are many cuspidal representations giving the same \( \bar{\rho} \), which are all seen as deformations of \( \bar{\rho} \) from the viewpoint of Mazur, and the purpose of this paper is restated as to find a good cuspidal representation \( \pi' \) which is optimal for modular representation \( \bar{\rho} \) in a suitable sense.
As is already mentioned, $F = \mathbb{Q}$ case is studied well, so we restrict our attention to general totally real $F$ other than $\mathbb{Q}$.

Here is our main theorem, which treats the level optimization at $v \nmid \ell$.

**Theorem 1.1.** (Theorem A) Let $\bar{\rho} : G_F \to \text{GL}_2(k_\lambda)$ be a continuous absolutely irreducible mod $\ell$-representation satisfying A1)-A3):

A-1) $\ell \geq 3$, and $\bar{\rho}\big|_{F(\zeta_\ell)}$ is absolutely irreducible if $|F(\zeta_\ell) : F| = 2$.

A-2) $\bar{\rho} \simeq \rho_{\pi, \lambda} \mod \lambda$. Here $\pi$ is a cuspidal representation of $\text{GL}_2(\mathbb{A}_F)$ of infinity type $(k, w)$ defined over $E_\lambda$, satisfying $(\pi^\infty)^K \neq \{0\}$ for some compact open subgroup $K = \prod_u K_u$ of $\text{GL}_2(\mathbb{A}_F^\infty)$.

A-3) for a place $v \nmid \ell$, $\bar{\rho}$ is either ramified at $v$, or $q_v \equiv 1 \mod \ell$.

Then there is a cuspidal representation $\pi'$, having the same infinity type $(k, w)$ as $\pi$, defined over a finite extension $E'_\lambda \supset E_\lambda$ such that the following conditions hold:

1. The associated $\lambda'$-adic representation $\rho_{\pi', \lambda'}$ gives $\bar{\rho}$. $(\pi^\prime_{\nu, \infty}\backslash K^\nu) \neq \{0\}$,
2. The conductor $\text{cond}(\pi'_v)$ of $\pi'_v$ is equal to $\text{Art}\bar{\rho}|_{G_v}$, where $G_v$ is the decomposition group at $v$, and $\text{Art} \in \mathbb{Z}$ means the Artin conductor,
3. $\det \rho_{\pi', \lambda'} \cdot \chi^{w+1}_{\text{cycle}}$ is the Teichmüller lift of $\det \bar{\rho} \cdot \chi^{w+1}_{\text{cycle}}$.

If $\pi'$ gives $\bar{\rho}$, we always have the basic inequality

$$\text{cond}\pi_v \geq \text{Art}\bar{\rho}|_{G_v},$$

so the equality in (1) (2) is optimal. When we remove condition A-3), i.e., even when $\bar{\rho}$ is unramified at $v$ and $q_v \equiv 1 \mod \ell$, we still have some $\pi'$ giving $\bar{\rho}$ with the property that $\pi'_v$ is spherical, or a special representation twisted by an unramified character. This missing case in theorem A, i.e., the case when $\bar{\rho}$ is unramified at $v$ and $q_v \equiv 1 \mod \ell$, was treated by K. A. Ribet in case of $\mathbb{Q}$ \[18\], and by A. Rajaei in the totally real case.

Especially, we note that theorem A includes

**Corollary 1.2** (Corollary A’ (Mazur principle)). There exists $\pi'$ as in theorem A when $\bar{\rho}$ is unramified at $v$, and $q_v \neq 1 \mod \ell$.

We should note that theorem A is a stronger form of theorem B below, which was obtained earlier by F. Jarvis \[12\], \[15\] (some additional condition on $\bar{\rho}$ is put in the references compared to theorem B, but it is easily removed, see lemma \[14\] in \[15\].

**Theorem 1.3** (Theorem B (Jarvis)). In addition to assumptions A-1)-A-3) of theorem A, we assume

A-4) If the degree $[F : \mathbb{Q}] = g$ is even, assume that $\pi_u$ is essentially square integrable for some $u \neq v, v \nmid \ell$, with $K_u = K_1(m_u^{\text{cond}\pi_u})$.

Then we have the same conclusion as in theorem A.

Condition A-4 in theorem B especially excludes the unramified case when the degree $[F : \mathbb{Q}]$ is even. Logically, theorem A is a consequence of theorem B and corollary A’ in the full form. Corollary A’ in the even degree case, i.e., when $[F : \mathbb{Q}]$ is even, is one of the main contributions of this paper. We also give a new proof to theorem B in this article.

Let us explain the method of proofs. In \[14\], Jarvis proved a part of corollary A’, i.e., the Mazur principle under A-1)-A-4), by a detailed study on the arithmetic models of Shimura curves. Note that assumption A-4) in theorem B is used to relate $\bar{\rho}$ to the cohomology of a Shimura curve associated with a division algebra which is unramified at $v$. When $\bar{\rho}$ is ramified at $v$ (\[15\]), Jarvis used the argument of Carayol in \[6\], which does not use any arithmetic models. The methods are cohomological.
Our corollary A', especially the Mazur principle in the even degree case, has been thought difficult since the \( \lambda \)-adic representation \( \rho_{\pi, \lambda} \) may not be obtained from a Shimura curve in general \[21\]. Surprisingly, we can use a Shimura curve attached to a division quaternion algebra which is ramified at \( v \) in our proof.

Our method to prove corollary A', and to give a new proof to the orem B, is also cohomological and summarized in the following way.

For any finite \( \mathcal{O}_\lambda \)-algebra \( R \), we define a cohomological functor \( H^i_R \) from Shimura curves, on which \( G_F \) acts, and behaves nicely under any scalar extension \( R \rightarrow R' \). \( H^i_{\mathcal{O}_\lambda} \otimes_{\mathcal{O}_\lambda} E_\lambda \) consists of representations \( \rho_{\pi', \lambda}' \)'s giving \( \bar{\rho} \). The main observation is that the inertia fixed part \( H^i_{I_v} \) also commutes with scalar extensions, which shows theorem B when \( \bar{\rho} \) is ramified at \( v \), since we have some \( \pi' \) with \( \rho_{\pi', \lambda}' \neq \{0\} \) under the assumption \( \bar{\rho} \neq \{0\} \). The equality of Artin conductors follows easily from this.

To show the good property of \( H^i_{I_v} \) with respect to scalar extensions in case of theorem B, we use a deep arithmetic geometrical result, namely the regularity of the arithmetic models of Shimura curves using Drinfeld level structures. By this result and the purity theorem of Zariski-Nagata for étale coverings of a regular scheme, we analyze the cohomology groups directly, without any calculation of vanishing cycles. Our result is seen as a mod \( \ell \)-version of the local invariant cycle theorem.

In the case of corollary A', the analysis of the cohomology is done with the help of a Cerednik-Drinfeld type theorem obtained by Boutot-Zink \[2\], based on the principle used for proving theorem B. Also in \[13\] we give an interpretation of Carayol’s lemma \[6\] by a standard homological algebra. It is an application of a property of perfect complexes, and also cohomological. The methods developed in this article are new even for \( F = \mathbb{Q} \). It is our belief that a level optimization in easier situation is a consequence of a homological algebra on Shimura varieties. We hope that the method in this paper is effective in some higher dimensional cases as well. One may use O. Gabber’s absolute purity theorem \( \[9\] \) instead of the purity theorem of Zariski-Nagata.

Acknowledgment: A part of the work was done during the author’s stay at Caltech in 1996, at Université Paris-Nord in 1998, and at the Institute for Advanced Study for the academic year 1998-1999. The author thanks D. Ramakrishnan and J. Tilouine for the hospitality at Caltech and Paris-Nord. The author also thanks P. Deligne for the discussion on some cohomological operations used in the paper.

Finally, the author would like to dedicate this paper to professor Kazuya Kato. The author has leaned so many mathematical insights from him, and also the beauty of Mathematics.

2. Notations

For a number field \( F \), \( \mathcal{O}_F \) is the integer ring, and \( G_F = \text{Gal}(\bar{F}/F) \) is the absolute Galois group. For a place \( v \) of \( F \), \( F_v \) means the local field at \( v \). For a finite place \( v \), \( \alpha_v \) is the integer ring of \( F_v \), with the maximal ideal \( m_v \), \( k(v) = \alpha_v/m_v \) is the residue field with the cardinal \( q_v \). \( p_v \) is a uniformizer of \( m_v \). \( G_v \) and \( I_v \) mean the decomposition and the inertia groups at place \( v \), respectively.

\( \mathbb{A}_F \) means the adèle ring of \( F \), \( \mathbb{A}^\infty_F \), and \( (\mathbb{A}_F)_{\infty} \) the finite part and the infinite part, respectively. For a non-zero ideal \( \mathfrak{f} \) of \( \mathcal{O}_F \), we define compact open subgroups of \( \text{GL}_2(\mathbb{A}^\infty_F) \) by

\[ K(\mathfrak{f}) = \{ g \in \text{GL}_2(\prod_{\text{finite}} \alpha_u), \ g \equiv 1 \ mod \ \mathfrak{f} \}, \]
\[ K_{11}(f) = \{ g \in \text{GL}_2(\prod_{u: \text{finite}} o_u), \ g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{f} \}, \]
\[ K_1(f) = \{ g \in \text{GL}_2(\prod_{u: \text{finite}} o_u), \ g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{f} \}, \]
\[ K_0(f) = \{ g \in \text{GL}_2(\prod_{u: \text{finite}} o_u), \ g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{f} \}. \]

We use the similar notation for the corresponding local group.

For an infinity type \((k, w), k \in \mathbb{Z}, w \in \mathbb{Z}\), which satisfies \(k_i \equiv w \pmod{2}\), \(k' \in \mathbb{Z}\) is defined by the formula
\[ k + 2k' = (w + 2) \cdot (1, \ldots, 1). \]

As in the introduction, we fix an isomorphism \(\mathbb{C} \simeq \mathbb{Q}_\ell\) by the Axiom of Choice. The local Langlands correspondence for \(\text{GL}_2(F_v)\) defines a bijection between isomorphism classes of \(F\)-semisimple representation \(\rho_v\) and admissible representation \(\pi_v\) of \(\text{GL}_2(F_v)\).

Our normalization of the local Langlands correspondence is as follows. For the local class \(\pi, \lambda\) seen as a semi-simple conjugacy class in the dual group \(\text{GL}_2^\vee(\mathbb{Q}_\ell)\), and \(\pi_v\) is a constituent of the non-unitary induction
\[ \text{Ind}^{G(F_v)}_{B(F_v)} \chi_{\alpha_v, \beta_v} = \{ f : \text{GL}_2(F_v) \to \mathbb{Q}_\ell, \ f(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \chi_{\alpha_v, \beta_v}(a, d)|a_v f(g) \}. \]

Here \(B\) is the standard Borel subgroup consisting of the upper triangular matrices,
\[ \chi_{\alpha_v, \beta_v} : B(F_v) \to F_v \times F_v \to \mathbb{Q}_\ell \]
is the unramified character given by \(\chi_{\alpha_v, \beta_v}(a, b) = \alpha_v^{\text{ord}_a} \beta_v^{\text{ord}_a b}\).

At infinite places, \(\text{GL}_2(\mathbb{R})\)-representation \(D_{k, w}\) corresponds to the unitarily induced representation
\[ \text{Ind}^{G(\mathbb{R})}_{B(\mathbb{R})} (\mu_{k, w}, \nu_{k, w})_u \]
for two characters of split maximal torus
\[ \mu_{k, w}(a) = |a|^{\frac{1}{2} - k'} (\text{sgn} a)^{-w} \]
\[ \nu_{k, w}(d) = |d|^{\frac{1}{2} - w + k'} \]
for \(k'\) satisfying \(k - 2 + 2k' = w\). This normalization, which is the \(|\cdot|_\psi\)-twist of unitary normalization, preserves the field of definition. The central character of \(\pi\) corresponds to \(\det \rho_{\pi, \lambda}(1)\). Our normalization is basically the same as in [3], except one point. In [2], an arithmetic Frobenius element corresponds to a uniformizer.

The global correspondence \(\pi \mapsto \rho_{\pi, \lambda}\) is compatible with the local Langlands correspondence for \(v \nmid \ell\) ([3] théorème (A), see [22], [21], theorem 2, for the missing even degree cases): the \(F\)-semisimplification of \(\rho_{\pi, \lambda}|_{G_v}\) corresponds to \(\pi_v\) by the local correspondence normalized as above.
3. Preliminaries

3.1. Shimura curves and Hida varieties. We assume \([\mathbb{F} : \mathbb{Q}] > 1\) in the following (cf. introduction). One can include \(\mathbb{Q}\)-case with a slight modification. Fix an element \(\iota_1 \in I_F\).

Take a division quaternion algebra \(D\) over \(\mathbb{F}\) which ramifies at all infinite places other than \(\iota_1\), and ramifies possibly at \(\iota_1\). We fix an identification

\[ D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^{g'} \times H^{g-g'}, \]

where \(g' = 1\) or 0 according to \(D\) is split at \(\iota_1\) or not. Here \(H\) is the Hamilton quaternion algebra over \(\mathbb{R}\). We put

\[ G_D = \text{Res}_{F/Q} D^\times. \]

Here \(\text{Res}\) means the Weil restriction of scalars. For a compact open subgroup \(K \subset D^\times(\mathbb{A}^\infty)\), we define the associated modular variety with a complex structure, by

\[ S_K = D^\times \backslash D^\times(\mathbb{A})/K \times K_{\mathbb{F}}. \]

Here \(K_{\mathbb{F}}\) is the maximal compact subgroup of \(D^\times(\mathbb{R})\) modulo center. When \(g'\) is one, \(S_K\) is a Shimura curve, and it has a canonical model \(S_K,F\) defined over \(\mathbb{F}\). When \(g'\) is zero, the zero-dimensional variety was used extensively by H. Hida \([1,2]\) in his study of \(\ell\)-adic Hecke algebras. Note that this is not a Shimura variety in the sense of Deligne.

3.2. \(\ell\)-adic local systems. Fix a prime \(\ell\), and an \(\ell\)-adic field \(E_\ell\) with the integer ring \(\mathcal{O}_\ell\). We denote the maximal ideal of \(\mathcal{O}_\ell\) by \(\lambda\) and the residue field by \(k_\ell\).

For a pair \((k_\ell, w), k_\ell \in \mathbb{Z}^{I_F}, w \in \mathbb{Z}\) as in the introduction, we define an \(\ell\)-adic sheaf \(\mathcal{H}_{k_\ell, w}\) on \(S_K\) \([5]\, p.418-419\) for \(g' = 1\).

Since we need a \(\mathbb{Z}_\ell\)-structure for this sheaf, we briefly review the construction, assuming \(D\) splits at all \(v|\ell\), which is sufficient for our later work.

Let \(\pi_\ell : \hat{S}_\ell \to S_{D,K}\) be the Galois covering corresponding to \(\prod_{v|\ell} K_v/K \cap F^\times\). Here \(F^\times\) is the closure inside \(D^\times(k^{\text{F}}_\ell)\).

We choose an isomorphism \(D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \prod_{v|\ell} M_2(F_v)\). This determines a maximal hyperspecial subgroup of \(D^\times(\mathbb{Q}_\ell)\), which we identify with \(\prod_{v|\ell} \text{GL}_2(\mathcal{O}_v)\). We take \(E_\ell\) large enough so that all \(F_v, v|\ell\) are embedd into \(E_\ell\) over \(\mathbb{Q}_\ell\). So the representation

\[ V_{k_\ell, w} = \otimes_{\iota \in I_F} (\iota \det)^{-k_\ell} \text{Sym}^{\otimes (k_\ell - 2)} \]

of \(D^\times(\mathbb{Q}_\ell)\) is defined over \(E_\ell\), and has an \(\mathcal{O}_\ell\)-lattice

\[ V_{k_\ell, w, \mathcal{O}_\ell} = \otimes_{\iota \in I_F} (\iota \det)^{-k_\ell} \text{Sym}^{\otimes (k_\ell - 2)} \mathcal{O}_\ell^{\otimes 2} \}

\[ \otimes \mathcal{O}_\ell. \]

The \(\mathcal{O}_\ell\)-smooth sheaf \(\mathcal{H}_{D, k_\ell, w}\) is obtained from the covering \(\pi_\ell\) and the representation \(V_{k_\ell, w, \mathcal{O}_\ell}\) by contraction. Note that the action of \(K \cap F^\times\) on the representation \(V_{k_\ell, w}\) is trivial if \(K\) is sufficiently small, so that the sheaf is well-defined.

See \([5]\, p.418-419\) for the Betti-version of \(\mathcal{H}_{k_\ell, w}\) (note that we have taken the dual of the sheaf in the reference). By the comparison theorem in étale cohomology, those two cohomologies are canonically isomorphic, so we do not make any distinction unless otherwise specified. The projective limit \(S = \varprojlim_K S_K\) admits a right \(D^\times(\mathbb{A}^\infty)\)-action by the multiplication from the right, and the \(E_\ell\)-sheaf

\[ \mathcal{H}_{D, k_\ell, w, E_\ell} = \mathcal{H}_{D, k_\ell, w} \otimes_{\mathcal{O}_\ell} E_\ell \]

is \(D^\times(\mathbb{A}^\infty)\)-equivariant by the construction.

The lattice structure \(\mathcal{H}_{D, k_\ell, w}\) is preserved by the \(D^\times(\mathbb{A}^\infty)\)-action.
When \( g' \) is one, the sheaf \( \mathcal{F}_{k,w}^D \) is canonically defined over \( F \) by the theory of canonical models, which we denote by \( \mathcal{F}_{k,w}^D \). By [4] 2.6, \( \mathcal{F}_{k,w}^D \) is pure of weight \( w \). This canonical \( F \)-structure gives a continuous \( G_F \)-action on the cohomology, giving the decomposition

\[
H^1 \ker(S_{K,F}, \mathcal{F}_{k,w} \otimes E_\lambda) = \oplus_{\pi \in \rho_{\sigma,\lambda} \otimes \sigma_{\lambda}} (\pi_\infty)\hat{K}
\]

for cuspidal representations \( \pi \) of \( D^\times (\mathbb{A}_F) \) with infinity type \((k,w)\) which does not factor through the reduced norm ([5], 2.2) assuming that \( E_\lambda \) is sufficiently large. By infinity type \((k,w)\), we mean that \( \pi_\infty \) has the form

\[
\pi_\infty = D_{k_1,w} \otimes_{v \in \mathcal{I}_F, v \neq \iota} \bar{D}_{k_v,w}
\]

(see [5] §0 for the notation \( \bar{D}_{k_v,w} = (\text{idet})^{-k_v} \text{Sym}^{\nu \otimes k_v} - 2 \), which corresponds to \( D_{k_v,w} \) by the Jacquet-Langlands correspondence [16]).

3.3. **Hecke algebras and correspondences.** We keep the same notations as in 3.2. We assume that \( K \) is locally factorizable, i.e., \( K = \prod_l K_\ell \), and \( D \) is split at all \( v \mid \ell \). For a finite set of finite places \( \Sigma \) containing \( \{v; v \mid \ell\} \), let \( H(D^\times (\mathbb{A}_F^\Sigma), K_\Sigma) \) be the convolution algebra over \( \sigma_\lambda \). Namely, \( H(D^\times (\mathbb{A}_F^\Sigma), K_\Sigma) \) is the set of the compactly supported \( \sigma_\lambda \)-valued smooth \( K_\Sigma \)-bi-invariant functions on \( D^\times (\mathbb{A}_F^\Sigma) \). The algebra structure is given by the convolution.

We consider the action of \( H(D^\times (\mathbb{A}_F^\Sigma), K_\lambda) \) on the cohomology of \( S_{K} \) induced from the adelic right action on \( S = \lim_{\leftarrow} K \). We briefly review the basic facts since the relationship with the Verdi dual duality is subtle and important for our purpose.

For two compact open subgroups \( K, K' \subset D^\times (\mathbb{A}_F) \), \( g \in D^\times (\mathbb{A}_F^\Sigma) \) define an algebraic correspondence \([KgK']\): the first projection \( S_{K \cap g^{-1}K'g} \rightarrow S_K \) and the second \( S_{K \cap g^{-1}K'g} \rightarrow S_{g^{-1}K'g} \cong S_{K'} \).

The correspondence induced by \( KgK' \) from \( S_K \) to \( S_{K'} \) is dual to \( K'g^{-1}K \) from \( S_{K'} \) to \( S_K \) by the definition.

Since \( \mathcal{F}_{k,w,E_\lambda} \) is \( D^\times (\mathbb{A}_F^\Sigma) \)-equivariant by the construction, \([KgK']\) gives

\[
[KgK'] : R\Gamma(S_K, \mathcal{F}_{k,w}) \otimes_{\sigma_\lambda} E_\lambda \rightarrow R\Gamma(S_{K'}, \mathcal{F}_{k,w}) \otimes_{\sigma_\lambda} E_\lambda
\]

via cohomological correspondences they define.

We fix a uniformizer \( p_v \) of \( F_v \) at each finite place \( v \). If the \( v \)-component \( g_v \) of \( g \in D^\times (\mathbb{A}_F^\Sigma) \) satisfies

\[
\text{GL}_2(\mathcal{O}_v)g_v \text{GL}_2(\mathcal{O}_v) \text{ is represented by } \begin{pmatrix} p_v^a & 0 \\ 0 & p_v^b \end{pmatrix} \text{ with } a, b \geq 0 \text{ for all } v \mid \ell
\]

the \( \sigma_\lambda \)-lattice structure \( \mathcal{F}_{k,w} \) is preserved, and

\[
[KgK'] : R\Gamma(S_K, \mathcal{F}_{k,w}) \rightarrow R\Gamma(S_{K'}, \mathcal{F}_{k,w})
\]

is induced. We call this the **standard** action of \([KgK']\). Especially, we have the action of \( H(D^\times (\mathbb{A}_F^\Sigma), K_\Sigma) \) for \( K = K_\Sigma \cdot K_\Sigma \) by extending \( K_\Sigma g K_\Sigma \) to \( K g K \), \( g = ((1_{D(F_v)})_{v \in \Sigma}, g), g \in D^\times (\mathbb{A}_F^\Sigma) \).

It is a natural question to ask whether there is a complex of \( H(D^\times (\mathbb{A}_F^\Sigma), K_\Sigma) \)-modules which represents \( R\Gamma(S_K, \mathcal{F}_{k,w}) \) or not. The answer is affirmative, which we state it in the form of proposition. In section 3 it becomes quite important.

**Proposition 3.1.** There is a complex \( L' \) of \( H(D^\times (\mathbb{A}_F^\Sigma), K_\Sigma) \)-modules bounded from below which represents \( R\Gamma(S_K, \mathcal{F}_{k,w}) \) in \( D^+(S_K, \sigma_\lambda) \). The induced action of \( H(D^\times (\mathbb{A}_F^\Sigma), K_\Sigma) \) on \( H^q(S_K, \mathcal{F}_{k,w}) \) coincides with the induced action by the Hecke correspondences.
Proof. We work with the Betti version for simplicity (this case is sufficient for our later purpose). Let \( L \) be the Godement’s canonical resolution of \( \mathcal{F}_{k,w} \). Since all the maps in defining the algebraic correspondences are finite and étale, cohomological operations defining the action of a cohomological correspondence are actually defined on \( L \): for example, for a finite étale morphism \( f : X \to Y \), the trace map \( f_* f^* \mathcal{F} = f_* f^* \mathcal{F} \to \mathcal{F} \) for a sheaf \( \mathcal{F} \) is defined by
\[
\sum_{y \in f^{-1}(x)} \mathcal{F}_y \to \mathcal{F}_x, \quad (f_y)_{y \in f^{-1}(x)} \to \sum_{y \in f^{-1}(x)} f_y
\]
on the level of stalk, which gives a lift of the trace map to Godement resolution \( L \). In this way one has an action of the convolution algebra on \( \Gamma(L) \), and \( R\Gamma(S_K, \mathcal{F}_{k,w}) \) belongs to the derived category of \( H(D^\times(\mathbb{A}^\Sigma, \infty), K) \)-modules bounded below. Especially, for \( K = \prod K_v \), the actions at two different places commute.

\[\square\]

Remark 3.2. For the étale cohomology with finite coefficients, the claim is proved by the same argument.

3.4. Duality formalism. Take a finite set of finite places \( \Sigma \) containing \( \{ v; v|\ell \} \). Assume \( \overline{\mathbb{K}} = \prod \mathbb{K}_v \), and \( D \) is split at all finite \( \not\in \Sigma \).

Note that
\[(\dagger)\quad \mathcal{F}_{k,w}' \cong \mathcal{F}_{k,-w}.
\]
giving a perfect pairing in the derived category of \( \mathcal{O}_\lambda \)-modules
\[R\Gamma(S_K, \mathcal{F}_{k,w}) \otimes_{\mathcal{O}_\lambda} R\Gamma(S_K, \mathcal{F}_{k,-w}) \to \mathcal{O}_\lambda(-g')[2g'] \]
by the Poincaré duality. For the \( D^\times(\mathbb{A}^\Sigma, \infty) \)-action, if we consider the standard action of \( g \) on \( R\Gamma(S_K, \mathcal{F}_{k,w}) \), by the Poincaré duality, this corresponds to the standard action of \( g^{-1} \) on \( R\Gamma(S_K, \mathcal{F}_{k,-w}) \) since the isomorphism (\( \dagger \)) sends \( g \)-action to \( g^{-1} \).

For the relation between \( H(D^\times(\mathbb{A}^\Sigma, \infty), K) \)-action and the Verdier duality, we have the following proposition by discussion of section 3.3.

Proposition 3.3. The standard action \( R\Gamma(S_K, \mathcal{F}_{k,w}) \to R\Gamma(S_{K'}, \mathcal{F}_{k,w}) \) induced by \([KgK']\) is dual to the standard action \( R\Gamma(S_{K'}, \mathcal{F}_{k,-w})(g)[2g'] \to R\Gamma(S_K, \mathcal{F}_{k,-w})(g')[2g'] \) by \([K'g^{-1}K]\).

We have two geometric actions of the convolution algebra, which we call the standard action and the dual action. By the dual action of \([KgK]\) on \( R\Gamma(S_K, \mathcal{F}_{k,w}) \), we mean the standard action of \([K g^{-1} K]\).

By proposition 3.3 the standard action of \( T_\Sigma \) becomes the dual action by the Poincaré duality.

We define standard Hecke operators. We choose a uniformizer \( p_v \) of \( F_v \) for any finite place \( v \). \( a(p_v) \) (resp. \( b(p_v) \)) is the element in \( D^\times(\mathbb{A}^\Sigma) \) having \( \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix} \) (resp. \( \begin{pmatrix} p_v & 0 \\ 0 & p_v \end{pmatrix} \)) as its \( v \)-component, and 1 as other components.

Then the Hecke algebra \( T_\Sigma = H(D^\times(\mathbb{A}^\Sigma, \infty), K^\Sigma) \) over \( \mathcal{O}_\lambda \) is isomorphic to the \( \mathcal{O}_\lambda \)-algebra generated by indeterminates \([T_v], [T_{v,v}], v \not\in \Sigma\), adding \([T_{v,v}]^{-1}\) for \( v \not\in \Sigma \). Here \([T_v]\) is given by the characteristic function of \( \mathcal{K}^\Sigma a(p_v) \mathcal{K}^\Sigma \), \([T_{v,v}] \) is given by the characteristic function of \( \mathcal{K}^\Sigma b(p_v) \mathcal{K}^\Sigma \). For a \( T_\Sigma \)-module \( M \), we define the dual \( T_\Sigma \)-action on \( M \) by
\[
\begin{cases}
[T_v] \mapsto [T_v]^{-1} [T_v], \\
[T_{v,v}] \mapsto [T_{v,v}]^{-1}.
\end{cases}
\]
Example 3.4. For any continuous mod ℓ Galois representation \( \tilde{\rho} : G_\Sigma \to \text{GL}_2(\overline{k}_\lambda) \), we define a maximal ideal \( m_\rho \) of \( T_\Sigma \) by

\[
[T_v] \mapsto \text{trace}\tilde{\rho}(\text{Fr}_v), \quad [T_{v,\nu}] \mapsto q_v^{-1}\det(\text{Fr}_v).
\]

The maximal ideal corresponding to the dual action is \( m_{(\det(\tilde{\rho}))^{-1} \otimes \tilde{\rho}(1)} \).

In case of Shimura curves, the definition is compatible with the \( T_\Sigma \)-action on cohomology groups:

By choosing the canonical resolution, \( RT(S_\tilde{K}, \tilde{\mathcal{F}}_{k,w}) \) belongs to the derived category of \( T_\Sigma \)-complexes bounded below, sending \([T_v]\) and \([T_{v,\nu}]\) to the standard actions of \([\tilde{K}\left(\begin{smallmatrix} 1 & 0 \\ 0 & p_v \end{smallmatrix}\right)\tilde{K}]\)

and \([\tilde{K}\left(\begin{smallmatrix} p_v & 0 \\ 0 & p_v \end{smallmatrix}\right)\tilde{K}]\). Then the dual action defined by the Poincaré duality is given by the above dual \( T_\Sigma \)-action since they are defined by \([\tilde{K}\left(\begin{smallmatrix} 1 & 0 \\ 0 & p_v^{-1} \end{smallmatrix}\right)\tilde{K}]\) and \([\tilde{K}\left(\begin{smallmatrix} p_v^{-1} & 0 \\ 0 & p_v^{-1} \end{smallmatrix}\right)\tilde{K}]\).

3.5. Modules of type \( \omega \). We keep the notations, especially \( T_\Sigma = H(D^\infty(A_{\Sigma,\infty}), K_\Sigma^\infty) \). We define a class of \( T_\Sigma \)-modules.

Definition 3.5. Consider the category \( \mathcal{C}_{T_\Sigma} \) of \( T_\Sigma \)-modules which are finitely generated as \( \mathcal{O}_\chi \)-modules. We call an object \( N \) in \( \mathcal{C}_{T_\Sigma} \) of type \( \omega \)

(1) if it has a finite length, then for any constituent \( N' \) appearing in the Jordan-Hölder sequence

\[
[T_v]^2 = [T_{v,\nu}](1 + q_v)^2
\]

holds on \( N' \) and for \( v \notin \Sigma \),

(2) in general \( N \) is of type \( \omega \) if and only if the graded modules \( \lambda^n N/\lambda^{n+1} N \) (\( n \in \mathbb{N} \)) for the \( \lambda \)-adic filtration are all of type \( \omega \).

(3) A maximal ideal \( m \) of \( T_\Sigma \) is of type \( \omega \) if \( T_\Sigma/m \) is of type \( \omega \).

By \( \mathcal{C}_\Omega \) we denote the subcategory of \( \mathcal{C}_{T_\Sigma} \) consisting of the \( T_\Sigma \)-modules of type \( \omega \).

It is easy to see that \( \mathcal{C}_\Omega \) forms a Serre subcategory of \( \mathcal{C}_{T_\Sigma} \) and is stable under the dual action of \( T_\Sigma \). By \( \mathcal{C}_{N_\Omega} \), we mean the quotient category of \( \mathcal{C}_{T_\Sigma} \) by \( \mathcal{C}_\Omega \).

A typical example of modules of type \( \omega \) is obtained by a one dimensional representation \( \chi : D^\infty(A_{\Sigma,\infty}) \to E_\lambda^\infty \). The induced \( T_\Sigma \)-action gives a module of type \( \omega \).

The following proposition shows that the maximal ideals of \( T_\Sigma \) of type \( \omega \) correspond to very reducible representations.

Proposition 3.6. Assume that a continuous representation \( \tilde{\rho} : G_\Sigma \to \text{GL}_2(\overline{k}_\lambda) \) satisfies

\[
\text{trace}\tilde{\rho}(\text{Fr}_v)^2 = (1 + q_v)^2 q_v^{-1}\det(\text{Fr}_v)
\]

for all \( v \notin \Sigma \). Then \( \tilde{\rho} \) is reducible, and the semi-simplification \( \rho^\oplus \) satisfies

\[
\rho^\oplus \simeq \tilde{\chi} \oplus \tilde{\chi}(1)
\]

for some one dimensional character \( \tilde{\chi} : G_\Sigma \to k_\lambda^\times \). In other words, the maximal ideal \( m_\rho \) corresponding to \( \rho^\oplus \) is of type \( \omega \) if and only if \( \rho^\oplus \simeq \tilde{\chi} \oplus \tilde{\chi}(1) \) for some \( \tilde{\chi} \).

Proof. Let \( \omega = \chi_{\text{cycle}} \mod \ell \) be the Teichmüller character. By the Chebotarev density theorem, 3.5 (\( \dagger \)) is equivalent to

\[
\text{trace}(g; \text{ad}^0 \tilde{\rho}) = \text{trace}(g; 1 \oplus \omega \oplus \omega^{-1})
\]
for any $g \in G_S$. Since $\text{ad}^0 \bar{\rho}$ satisfies $\Lambda^3 \text{ad}^0 \bar{\rho} \simeq k$ and self-dual, the above equality of the traces implies the equality of characteristic polynomials

$$\det(1 - gT; \text{ad}^0 \bar{\rho}) = \det(1 - gT; 1 \oplus \omega \oplus \omega^{-1})$$

for any $g \in G_S$. By the Brauer-Nesbit theorem, the equality of semi-simplifications

$$(\text{ad}^0 \bar{\rho})^g \simeq 1 \oplus \omega \oplus \omega^{-1}$$

as $G_S$-modules follows. If $\bar{\rho}$ is irreducible, the only possibility is to have $\omega$ or $\omega^{-1}$ as a submodule of $\text{ad}^0 \rho$, and hence $[F(\zeta) : F] = 2$ and $\rho$ is induced from $F(\zeta)$. But this type of representations do not have the adjoint representation of the above form. We conclude that $\bar{\rho}$ is reducible, and the claim follows.

\[\square\]

**Remark 3.7.** The notion of modules of type $\omega$ is stronger than the notion of Eisenstein modules in $\mathbb{F}$.

### 3.6. Cohomological lemmas

Let $S$ be a strict trait with the generic point $\eta$ and the closed point $s$. $\tilde{I} = \text{Gal}(\bar{\eta}/\eta)$. Here $\bar{\eta}$ is a geometric point over $\eta$.

**Lemma 3.8.** Let $X$ be a flat regular scheme of finite type over $S$. Assume $\ell$ is prime to the residual characteristic of $S$, and take a coefficient ring $\Lambda$ which is finite over $\mathbb{Z}_{\ell}$. For a $\Lambda$-smooth sheaf $\mathcal{F}$ on $X$, there are two exact sequences

(a) 
$$0 \to H^0_{\text{et}}(X, \mathcal{F})(-1)_{\tilde{I}} \to H^1_{\text{et}}(X, \mathcal{F}_{\eta}) \to H^1_{\text{et}}(X, \mathcal{F})^I \to 0,$$

(b) 
$$0 \to H^1_{\text{et}}(X, \mathcal{F}) \to H^1_{\text{et}}(X, \mathcal{F}_{\eta}) \to \prod_{Y \in J} H^0_{\text{et}}(Y_{\text{reg}}, \mathcal{F}|_{Y_{\text{reg}}})(-1).$$

Here $J$ is the set of irreducible components of $X_s$, and $(-)_{\text{reg}}$ means the regular locus of $(-)$.

**Proof.** The exactness of (a) follows from the Hochshield-Serre spectral sequence applied to the morphism from $X_{\bar{\eta}}$ to $X_{\eta}$. We show (b). $H^0_{\text{et}}(Y_{\text{reg}}, \mathcal{F}|_{Y_{\text{reg}}})(-1)$ is canonically isomorphic to $H^1_{\text{et}}(\text{Frac} o_{X, \bar{\eta}}, \mathcal{F}_{\bar{\eta}})$, where $\bar{\eta}$ is a geometric generic point of $Y$. Then the exactness of (b) is equivalent to the following claim.

**Claim 3.9.** An $\mathcal{F}_{\eta}$-torsor over $X_{\eta}$ which becomes unramified at all maximal points of $X_s$ extends to an $\mathcal{F}$-torsor over $X$ uniquely up to isomorphism.

Claim 3.9 is a consequence of the purity theorem of Zariski-Nagata [11], which says that, if $X$ is regular, the category of the étale coverings of $X$ is equivalent to the subcategory of the étale coverings of $X_{\eta}$ which becomes unramified at all maximal points of $X_s$. \[\square\]

Fix a commutative $\mathcal{O}_X$-algebra $A$. We assume the following conditions on $(A, X, \mathcal{F})$.

**Assumption 3.10.** For any $i \in \mathbb{Z}$,

1. $H^i_{\mathcal{R}} = H^i_{\text{et}}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}), \mathcal{H}^i_{\mathcal{R}} = H^i_{\text{et}}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R})$ are $(A, \mathcal{R})$-bimodules for any commutative finite $\mathcal{O}_X$-algebra $\mathcal{R}$ which commutes with $I$-actions.

2. $H^1_{\mathcal{R}} \to H^1_{\mathcal{R}'}$ is an $A$-module homomorphism for any $\mathcal{O}_X$-algebra homomorphism $\mathcal{R} \to \mathcal{R}'$.

3. All homomorphisms in the long cohomology exact sequence $\{H^i_{\mathcal{R}}\}_{i \in \mathbb{Z}}$ are $A$-module homomorphisms, functorial for any ring extension $\mathcal{R} \to \mathcal{R}'$. 

Lemma 3.11. Fix a maximal ideal $m$ of $A$. Let $\mathcal{E}$ be a Serre subcategory of the category of $A$-modules ($A$-mod), consisting of $O_\lambda$-modules of finite length satisfying the following conditions.

1. $\mathcal{E}$ is stable under $(-) \otimes_{O_\lambda} R$ for any finite $O_\lambda$-algebra $R$.
2. The localization $M_m$ at $m$ of any element $M$ in $\mathcal{E}$ vanishes.
3. $X$ is a proper flat regular curve over $S$ with the smooth generic fiber, and $\mathcal{F}$ is an $O_\lambda$-smooth sheaf on $X$. Assumption 3.10 is satisfied for $(A, X, \mathcal{F})$.
4. $H^0_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R), H^1_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R) \in \mathcal{E}$ for any $O_\lambda$-module $R$ of finite length.
5. $H^2_{\text{ét}}(X_s, \mathcal{F}|_{X_s} \otimes_{O_\lambda} R), H^2_{\text{ét}}(Y_{\text{reg}}, \mathcal{F}|_{Y_{\text{reg}}} \otimes_{O_\lambda} R) \in \mathcal{E}$ for any irreducible component $Y$ of $X_s$ and for any $R$ of finite length.

Put $H_R = H^1_{\text{ét}}$, $\mathcal{H}_R = \mathcal{H}^1_{\text{ét}}$. Then the following hold:

(a) $H_{R,m}$ is $O_\lambda$-free, and $H_{R,m} \otimes_{O_\lambda} k_\lambda \simeq H_{R/\lambda R,m}$ if $R$ is $O_\lambda$-flat.
(b) $\mathcal{H}_{R,m}$ is $O_\lambda$-free, and $(\mathcal{H}_{R,m}) \otimes_R k_\lambda \simeq \mathcal{H}_{R/\lambda R,m}$ if $R$ is $O_\lambda$-flat.
(c) $\mathcal{H}_{R,m} \simeq (H_{R,m})^I$ if $R$ is $O_\lambda$-flat.

Proof. First note that $M_m$ is finitely generated as an $O_\lambda$-module for any $A$-module $M$, finite over $O_\lambda$, since the $A$-action factors through a subalgebra in $\text{End}_{O_\lambda} M$, and hence a finite $O_\lambda$-algebra, which is a product of local rings.

We prove (a). By the exact sequence

$$H^0_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R/\lambda R) \rightarrow H^1_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R) \rightarrow H^1_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R)$$

$H_{R,m}$ is an $O_\lambda$-flat module since $H^0_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R/\lambda R)$ vanishes after localization at $m$.

By (2), $H^1_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R) \rightarrow H^1_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R/\lambda R) \rightarrow H^2_{\text{ét}}(X_\eta, \mathcal{F} \otimes_{O_\lambda} R/\lambda R)$ is exact, and the claim for $H_R$ follows.

We prove (b). By the proper base change theorem. For the rest of (b), it suffices to see $H^2_{\text{ét}}(X_s, \mathcal{F}|_{X_s} \otimes_{O_\lambda} R/\lambda R) \in \mathcal{E}$. This follows from

$$H^0_{\text{ét}}(X_s, \mathcal{F}|_{X_s} \otimes_{O_\lambda} R) \rightarrow H^0_{\text{ét}}(X_s, \mathcal{F} \otimes_{O_\lambda} R) \rightarrow H^0_{\text{ét}}(X_s, \mathcal{F} \otimes_{O_\lambda} R)$$

by (2). Hence (1) is an isomorphism by the proper base change theorem. For the rest of (b), it suffices to see $H^2_{\text{ét}}(X_s, \mathcal{F}|_{X_s} \otimes_{O_\lambda} R/\lambda R) \in \mathcal{E}$. This follows from

$$H^2_{\text{ét}}(X_s, \mathcal{F}|_{X_s} \otimes_{O_\lambda} R/\lambda R) \rightarrow H^2_{\text{ét}}(Y_{\text{reg}}, \mathcal{F}|_{Y_{\text{reg}}} \otimes_{O_\lambda} R/\lambda R).$$

For (c), consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_{R,m} \otimes_{O_\lambda} k_\lambda & \rightarrow & H^1_{R,m} \otimes_{O_\lambda} k_\lambda \\
\downarrow & & \downarrow \\
\mathcal{H}_{R/\lambda R,m} & \rightarrow & H^1_{R/\lambda R,m}.
\end{array}$$

(2) is an isomorphism by (b). (3) is an isomorphism by lemma 3.12. The composition

$$H^1_{R,m} \otimes_{O_\lambda} k_\lambda \rightarrow H^1_{R/\lambda R,m} \subset H^1_{R/\lambda R,m} = H_{R,m} \otimes_{O_\lambda} k_\lambda$$

is injective since $H^1_{R,m} \subset H_{R,m}$ is an $O_\lambda$-direct summand by the definition and (a). So the map (1) is an isomorphism. $\mathcal{H}_{R,m} \simeq H^1_{R,m}$ follows since both modules are $O_\lambda$-finite free.

Remark 3.12. Since $\mathcal{E}$ is a Serre subcategory, it suffices to check the conditions for $R = A/m$. 

\[\square\]
4. Proof of theorem B in the ramified case

4.1. Auxiliary places. Technically, we need to choose some auxiliary place $y$ such that the discrete subgroups are torsion-free, and the Hecke algebra does not introduce essentially new component at $y$, the idea introduced by F. Diamond and R. Taylor [3, lemma 11]. The extra assumption on $\bar{\rho}$ when $[F(\zeta) : F] = 2$ is necessary to make this change possible (this condition seems to be natural since $\text{PGL}_2(F)$ has a non-trivial $\ell$-torsion elements in this case).

**Lemma 4.1.** [existence of an auxiliary place] Assume that $\ell \neq 2$ and $\bar{\rho}$ is absolutely irreducible. Assume moreover that $\bar{\rho}|_{F(\zeta)}$ is absolutely irreducible if $[F(\zeta) : F] = 2$. Then there are infinitely many finite places $y$ such that $q_y \equiv 1 \pmod{\ell}$ and for the eigenvalues $\bar{\alpha}_y, \bar{\beta}_y$ of $\bar{\rho}(\text{Fr}_y)$, $\bar{\alpha}_y \neq q_y^{\pm 1} \bar{\beta}_y$ holds.

As in [8], this follows from the Chebotarev density theorem and the following lemma which we give a proof since the linear disjointness of $F$ and $\mathbb{Q}(\zeta_\ell)$ is assumed in the reference. The proof given here does not use the classification of subgroups of $\text{GL}_2(\mathbb{F}_p)$.

In the following, we only consider representations defined over a field of characteristic different from 2. We say an absolutely irreducible two dimensional representation $\rho$ is monomial if it induced from an index 2 subgroup of $G$. An absolutely irreducible representation $\rho$ is monomial if the restriction $\rho|_H$ to a normal subgroup $H$ of $G$ is a sum of two distinct characters. Equivalently, an absolutely irreducible representation $\rho$ is monomial if $\text{ad}^{\rho} \rho$ is absolutely reducible.

**Lemma 4.2.** Let $k$ be a field of characteristic $\ell \neq 2$, $G$ be a finite group, $\rho : G \to \text{GL}_2(k)$ be an absolutely irreducible representation, and $\chi : G \to k^\times$ be an even order character. Assume that

(*) For any $g \in G$ with $\chi(g) \neq 1$, $\rho(g)$ has eigenvalues of the form $\{\alpha, \chi(g)\alpha\}$. Then $\chi$ has order 2, and $\rho$ is induced from a character of $H$. Here $H = \ker \chi$.

**Proof of lemma 4.2.** We enlarge $k$ so that all eigenvalues of $\rho(g)$, $g \in G$, belong to $k$. Put $Z = \{g \in G, \rho(g) \text{ acts as a scalar}\}$. By Schur’s lemma, $Z$ is the center of $G$. $Z$ is a normal subgroup of $H = \ker \chi$, since $\chi$ factors through $G \to G/Z \to k^\times$ by (*). $H \neq Z$ since $\chi$ is a character of an even order.

We need the following sublemma.

**Sublemma 4.3.** Assumptions are as in 4.2. Then the following hold:

(a) If $\chi(g)$ has order $d > 1$, $g^d \in Z$.
(b) $H/Z$ is an abelian group.
(c) Let $G' = \chi^{-1}(\{\pm 1\})$. Then $G'/Z$ is also abelian if $H/Z$ is of type $(2, \ldots, 2)$.

**Proof of 4.3.** For (a), note that $\rho(g)$ is semi-simple, and $\rho(g)^d$ is a scalar matrix. For (b), take an element $c \in G'$ such that $\chi(c) = -1$. For any $h \in H$, $(c \cdot h)^2 \in Z$, especially $c^2 \in Z$, by (a). This means that in $H/Z$, $cch^{-1} = h^{-1}$ holds, and hence $H/Z$ is abelian because the map $h \mod Z \mapsto h^{-1} \mod Z$ is a group homomorphism. $G'$ is a semi-direct product of $\mathbb{Z}/2$ by $H/Z$. (c) is clear since the adjoint action of $c$ is trivial on $H/Z$ by the assumption. 

We return to the proof of lemma 4.2. We show $\rho|_H$ is reducible. Assume contrary. Then the adjoint representation $\text{ad}^{\rho} \rho|_H$ is reducible, and the irreducible constituents are one dimensional characters since it factors through an abelian group $H/Z$. Thus $\rho|_H$ is monomial. $H/Z$ is isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ since it is dihedral and abelian. Since $H/Z$ is of type $(2, \ldots, 2)$, $G'/Z$ is also abelian by 4.3 (c). By the same argument, $G'/Z$ is
for \( g \in G \setminus H \), and hence \( \rho \) is reducible by the argument of 3.6 which leads to a contradiction.

We finish the proof of 4.2. Since \( \rho \) is a sum of two distinct characters, \( \rho \) is monomial. \( G/Z \) is dihedral, and the image of \( \chi \) is a quotient of a dihedral group. This implies that \( \chi \) has order 2, and \( H \) is an index 2 subgroup of \( G \). \( \rho \) is induced from a character of \( H \). \( \square \)

4.2. Proof of theorem B. We prove theorem B in the introduction.

**Theorem 4.4.** [Theorem B] Let \( \bar{\rho} : G_F \to \text{GL}_2(\bar{k}_\lambda) \) be a continuous absolutely irreducible mod \( \ell \)-representation satisfying A1)-A4):

A-1) \( \ell \geq 3 \), and \( \bar{\rho}_{F(\xi)} \) is absolutely irreducible if \( |F(\xi) : F| = 2 \).
A-2) \( \bar{\rho} \cong \rho_{\pi,\lambda} \mod \lambda \), \( \pi \) a cuspidal representation of \( \text{GL}_2(\mathbb{A}_F) \) of infinity type \((k,w)\) defined over \( E_\lambda \), satisfying \( (\pi^{\infty})^K \neq \{0\} \) for some compact open subgroup \( K = \prod_u K_u \).
A-3) for a place \( v /\ell \), \( \bar{\rho} \) is either ramified at \( v \), or \( q_v \equiv \ell \) mod \( \ell \).
A-4) If \( g \) is even, assume that \( \pi_u \) is essentially square integrable for some \( u \neq v, v /\ell \), with \( K_u = K_1(m_u^{\text{cond}}) \).

Then there is a cuspidal representation \( \pi' \), having the same infinity type \((k,w)\) as \( \pi \), defined over the finite extension \( E'_{\lambda,\pi'} \supset E_\lambda \) such that the following conditions hold:

1. The associated \( \lambda' \)-adic representation \( \rho_{\pi',\lambda'} \) gives \( \bar{\rho} \). \((\pi^{\infty})^{K'} \neq \{0\}\).
2. The conductor \( \text{cond}(\pi'_v) \) of \( \pi'_v \) is equal to \( \text{Art}\bar{\rho}|_{G_v} \), where \( G_v \) is the decomposition group at \( v \), and \( \text{Art} \in \mathbb{Z} \) means the Artin conductor.
3. \( \det\rho_{\pi',\lambda'} \cdot \chi^{w+1}_{\text{cycle}} \) is the Teichmüller lift of \( \det\bar{\rho} \cdot \chi^{w+1}_{\text{cycle}} \).

**Proof.** We prove (3) in the next subsection (proposition 4.11). Under the assumption of theorem B, \( \bar{\rho} = \rho_{\pi,\lambda} \mod \lambda \) for a cuspidal representation \( \pi = \otimes_u \pi_u \) with \( (\pi^{\infty})^{K'} \neq \{0\} \) for a compact open subgroup \( K = K_v \cdot K' \subset \text{GL}_2(\mathbb{A}_F) \). Take an integer \( n \geq 0 \) so that \( K' = K(m_u^n) \cdot K' \subset K \). \( \bar{\rho}^{\ell_n} \neq \{0\} \). We seek for a cuspidal representation \( \pi' \) with \( \ell \)-coefficient satisfying \( (\pi^{\infty})^{K'} \neq \{0\} \), and \( \rho_{\pi',\lambda'}^{K'} \neq \{0\} \).

We make use of a Shimura curve. Fix \( \xi_1 \in I_F \). Choose a division quaternion algebra \( D \) which is unramified outside \( \xi_1 \) when \( g \) is odd, unramified outside \( u \) and \( \xi_1 \) when \( g \) is even. \( S_K \) denotes the canonical model over \( F \) of the Shimura curve associated to \( D \) and a compact open subgroup \( \bar{K} \) of \( D^\times(\mathbb{A}_F) \). Our choice of \( \bar{K} \) is as follows.

By a Chebotarev density argument using lemma 4.1 and lemma 7.1, we can take an auxiliary place \( y \) so that \( g^{-1}K_1(y)g \cap \text{SL}(D)_+ \), \( g \in D^\times(\mathbb{A}_F) \), are torsion free, \( q_y \neq \ell \) mod \( \ell \), and \( \bar{\alpha}_y \neq q_y^{w+1} \beta_y \). Then we put

\[
\bar{K} = \begin{cases} 
K' \cap K_{11}(y) & (g : \text{odd}) \\
K = (K_u \cdot K'') \cap K_{11}(y) & (g : \text{even}).
\end{cases}
\]

Here we fix a maximal order \( o_{D_u} \) of \( D_u \), and \( \bar{K}_u = 1 + \Pi_u^{\text{cond}} \cdot o_{D_u} \) where \( \Pi_u \) is a uniformizer of \( o_{D_u} \).
We take a finite set of finite places $\Sigma$ containing $\{v,v'|\ell\}$, $\{v,v|\cond\pi\}$, $y$ and $u$ when $g$ is even. We put $T_\Sigma = H(D^\times(\mathbb{A}_F^\Sigma,\mathbb{A}_E^\Sigma,\mathbb{A}_C)), \tilde{K}^\Sigma$.

By replacing $E_\lambda$ by a bigger $\ell$-adic field $E'_\lambda$, $\mathcal{F}_{k,w}$ defined in 3.2 we have the decomposition

$$H_{et}^1(S_{\tilde{K},\bar{F}}, \mathcal{F}_{k,w} \otimes \mathcal{O}_\lambda E'_{\lambda}) = \oplus \pi_{\mathcal{O}_\lambda} \otimes \mathcal{O}_\lambda (\pi_{\infty})^\mathbb{R}$$

Here we identify the index set as a set of cuspidal representations of $GL_2(\mathbb{A}_E)$ (not of $D^\times(\mathbb{A}_F)$) by the Jacquet-Langlands correspondence. By our assumption 4.3 A-2) and 4.3 A-4), we may assume that $\pi_{\mathcal{O}_\lambda}$ appears in the decomposition of $H_{et}^1(S_{\tilde{K},\bar{F}}, \mathcal{F}_{k,w})$.

By $X$ we mean the arithmetic model of $S_{\tilde{K}}$ over $o_v$ defined by Carayol [4] using Drinfeld basis, $\mathcal{F} = \mathcal{F}_{k,w}$. $I = I_v$ is the inertia group at $v$.

Note that the arithmetic model $X$ is available for our $\tilde{K}$: In [4], Carayol assumes that the compact open subgroup defining a Shimura curve is sufficiently small. Note that this condition of smallness is satisfied if we replace $\tilde{K}$ by a smaller subgroup $U$ with $U_v = \tilde{K}_v$, so it is also true for our $\tilde{K}$ since $S_{\tilde{K}}$ is obtained by a quotient of $S_U$ by a finite group which acts freely by our choice of auxiliary place $y$. See also the remark on page 62 of [14].

For a finite $\mathcal{O}_\lambda$-algebra $R$, we put

$$H_R = H_{et}^1(X_{F_v}, \mathcal{F} \otimes \mathcal{O}_\lambda R).$$

Note that $H_R$ carries a natural $\mathcal{O}_\lambda[I] \times T_\Sigma$-module structure given by the standard action of Hecke operators.

**Lemma 4.5.** For a maximal ideal $m$ of $T$ which is not of type $\omega$, $H_{\mathcal{O}_\lambda,m}$ is $\mathcal{O}_\lambda$-free, and

$$H_{\mathcal{O}_\lambda,m}^1 \otimes \mathcal{O}_\lambda k_\lambda = H_{k_\lambda,m}^1$$

holds.

**Proof of lemma 4.5.** We check the conditions of lemma 3.1 by taking $\mathcal{E}$ to be the category $\mathcal{C}_\Omega$ of modules of type $\omega$.

**Sublemma 4.6.** The $T_\Sigma$-actions on $H_{et}^0(X_{F_v}, \mathcal{F} \otimes \mathcal{O}_\lambda k_\lambda)$ and $H_{et}^0(X_{F_v}, \mathcal{F}^\vee \otimes \mathcal{O}_\lambda k_\lambda)$ are both of type $\omega$.

**Proof of sublemma 4.6.** The sheaf $\mathcal{F} \otimes \mathcal{O}_\lambda k_\lambda$ is trivialized in a $D^\times(\mathbb{A}_E^\Sigma,\mathbb{A}_E^\Sigma)$-equivariant way by a finite covering $X'_{F_v} \to X_{F_v}$ corresponding to $\tilde{K}' = \prod_{v|\ell} K(m_v) \cdot \tilde{K}_v$ by the definition 3.2. Over $X'_{F_v}$, $D^\times(\mathbb{A}_E^\Sigma,\mathbb{A}_E^\Sigma)$-action induces the action of the convolution algebra of type $\omega$, since it is obtained, on any constituents, from one dimensional actions of $D^\times(\mathbb{A}_E^\Sigma,\mathbb{A}_E^\Sigma)$ by

$$\pi_0(X_{F_v}) \simeq \pi_0(F^\times \backslash \mathbb{A}_F^\Sigma \mathbb{A}_E^\Sigma / \det \tilde{K}'(\mathbb{A}_E^\Sigma,\mathbb{A}_E^\Sigma)).$$

So the claim follows for $H^0$. Since the standard action on $H^2(X_{F_v}, \mathcal{F} \otimes \mathcal{O}_\lambda k_\lambda)$ is obtained from the dual action on $H^0(X_{F_v}, \mathcal{F}^\vee \otimes \mathcal{O}_\lambda k_\lambda)$ by the Poincaré duality, by the same argument the claim also follows for $H^2$. $\blacksquare$

By the proof of lemma 3.1 we have that $H_{\mathcal{O}_\lambda}$ is $\mathcal{O}_\lambda$-free ignoring modules of type $\omega$. $H_{\mathcal{O}_\lambda} \otimes \mathcal{O}_\lambda k_\lambda = H_{k_\lambda}$ in $\mathcal{C}_\Omega \Sigma$ since the $T_\Sigma$-actions on $H_{et}^0(X_{F_v}, \mathcal{F} \otimes \mathcal{O}_\lambda k_\lambda)$ and $H_{et}^0(X_{F_v}, \mathcal{F}^\vee \otimes \mathcal{O}_\lambda k_\lambda)$ are of type $\omega$.

By 4 9.4.3, the arithmetic model $X = X_{K(v^u),K^v}$ over $S = \Spec o_v$ of $S_{K(v^u),K^v}$ is regular, proper and flat over $S$, if $K^v$ is sufficiently small. By the same holds for our $\tilde{K}$.

Here the set of irreducible components $J_{K^v}$ of the geometric special fiber $X_{\tilde{K}}$ is isomorphic to $\oplus_{E \mathbb{P}^1(o_v/v^u)} Y_{L,K^v}$. $Y_{L,K^v}$ is a smooth curve, and there is a $D^\times(\mathbb{A}_E^\Sigma,\mathbb{A}_E^\Sigma)$-equivariant isomorphism

$$\pi_0(Y_{L,K^v}) \simeq \pi_0(F^\times \backslash \mathbb{A}_F^\Sigma / o_v^\times \cdot \det K^v).$$
From this description, it follows that conditions of b), c) of lemma 3.11 are satisfied for \( \mathcal{F} = \mathcal{F}_{k,w} \), since \( \mathcal{F} / \lambda^n \mathcal{F} \) is trivialized in an equivariant way by a finite covering \( Y_{L,K'} \to Y_{L,K} \), by the same argument as above.

So lemma 3.11 is applied, and lemma 3.5 is shown.

We return to the proof of theorem 4.4. By lemma 4.5, \( \bar{\rho} \) appears in \( H^1_{\omega(m)} \otimes_{\Theta} k_\lambda \), and for the corresponding maximal ideal \( m \), which is not of type \( \omega \), \( T_m \)-module \( H^1_{\omega(m)} = H^1_{\omega} \otimes_T T_m \), \( H^1_{\omega} \) localized at \( m \), is \( \Theta_\lambda \)-free and non-zero, implying that there is a cuspidal representation \( \pi' \) with coefficient in \( \Theta'_\lambda \) such that \( (\pi')^\infty K' \neq \{0\} \), \( \rho_{\pi',\lambda} \) gives \( \bar{\rho} \) and \( \rho'_{\omega',\lambda} \neq \{0\} \).

The \( y \)-component of \( \pi' \) is spherical: since \( \pi'_y \) has a non-zero fixed vector by \( K_{11}(m_y) \), \( \pi'_y \) belongs to principal series or (twisted) special representation. By our condition on \( y \), the latter case does not occur since \( Fr_y \)-eigenvalues satisfies \( \alpha_y \neq q_y^1 \beta_y \). By condition \( q_y \neq 1 \) mod \( \ell \), \( \pi'_y \) is spherical.

By [5], th\'eor\'eme A, the Artin conductor of \( \rho'_v = \rho_{\pi'_v,\lambda}|_{G_v} \) and the conductor of \( \pi'_v \) is the same for \( v \nmid \ell \). By the formula for Artin conductors,

\[
\text{Art}\rho'_v = 2 - \dim E'_\lambda, \rho'^I_v + \text{sw}\rho'_v
\]

\[
\text{Art}\bar{\rho}|_{G_v} = 2 - \dim E'_\lambda, \bar{\rho}'_v + \text{sw}\bar{\rho}|_{G_v}
\]

hold. Here sw means the swan conductor. Since the swan conductor does not change under mod \( \ell \)-reduction, \( \text{Art}\rho'_v = \text{Art}\bar{\rho}|_{G_v} \) if and only if \( \dim E'_\lambda, \rho'^I_v = \dim E'_\lambda, \bar{\rho}'_v \). We conclude that \( \pi' \) satisfies the desired equality

\[
\text{cond}(\pi'_v) = \text{Art}\bar{\rho}|_{G_v}
\]

since \( \bar{\rho} \) is ramified at \( v \). So \( \pi'^\infty \) has a non-zero \( K_1(m_{\text{Art}\bar{\rho}|_{G_v}}) \cdot \tilde{K}^v \)-fixed vector in the ramified cases.

We have proved theorem B, except for the claim on the determinant.

**Remark 4.7.**

a) The equality \( \mathcal{H}_R = H^1_R \) is seen as a local invariant cycle theorem for characteristic \( \ell \) coefficient. We are mimicking the proof in the \( \mathbb{Q}_\ell \)-case, using Carayol’s result for \( K(v^n) \) that the corresponding arithmetic model is regular, plus the determination of the ad\'ele action on the set of irreducible components of the special fiber. The regularity assures a purity property, in our case the Zarski-Nagata’s purity theorem for étale coverings suffices.

b) Even in the unramified case, we get a cuspidal representation \( \pi' \) whose \( v \)-component \( \pi'_v \) has conductor at most one. By the determinant normalization in 4.3, we may replace \( \pi' \) again, and obtain \( \pi' \) with \( \pi'_v K_0(m_v) \neq \{0\} \). In this case we analyze a filtration on \( \mathcal{H}_R \), and show the Mazur’s principle in [5,7].

c) We may add \( U_w \)-operators for \( w | \ell \) (or their modification) to \( T_\Sigma \). By the same method, one can get a nearly ordinary \( \pi' \) starting from nearly ordinary \( \pi \).

4.3. **Perfect complex argument.**

**Lemma 4.8.** Let \( A \) be a noetherian local ring with maximal ideal \( m_A \) and the residue field \( k_A, B \) be an \( A \)-algebra. Let \( L \) be a complex of \( B \)-modules bounded below, having finitely generated cohomologies \( H^i(L) \) as \( A \)-modules. For a maximal ideal \( m \) of \( B \) above \( m_A \), assume \( H^i(L \otimes_A^L k_A) \otimes_B B_m \), the localization at \( m \), is zero for \( i \neq 0 \). Then \( H^0(L) \otimes_B B_m \) is \( A \)-free.
Proof. Since $B_m$ is $B$-flat, we may assume $B = B_m$ (finite generation assumption is satisfied after localization since $m$ is above $m_A$). Then $H^i(L \otimes_A k_A)$ is zero except $i = 0$. By taking the minimal free resolution as $A$-complexes, the claim follows. □

**Lemma 4.9.** Let $\pi : X \to Y$ be an étale Galois covering with Galois group $G$. Let $\mathcal{F}$ be a smooth $\Lambda$-sheaf on $Y$. Then $R\Gamma(X, \pi^*\mathcal{F})$ is a perfect complex of $\Lambda[G]$-modules, and

$$R\Gamma(X, \pi^*\mathcal{F}) \otimes^L_{\Lambda[G]} \Lambda[G]/I_G \simeq R\Gamma(Y, \mathcal{F})$$

holds. Here $I_G$ is the augmentation ideal, and the map is induced by the trace map.

This is known (especially in the dual form) in any standard cohomology theory.

**Remark 4.10.** The above canonical morphism $R\Gamma(X, \pi^*\mathcal{F}) \to R\Gamma(Y, \mathcal{F})$ in $D_c^b(\Lambda)$ obtained by forgetting the $G$-action is given by the trace map.

In case of Shimura curves, which is our main application of lemma 4.9, the above morphism is compatible with the standard action of Hecke operators: it suffices to see the dual morphism $R\Gamma(Y, \mathcal{F}^\vee) \to R\Gamma(X, \mathcal{F}^\vee)$ is compatible with dual action of $[Kg^{-1}K]$. But this is the standard action of $[Kg^{-1}K]$ by proposition 3.5, and for the standard action, the compatibility is clear.

We apply the lemma to adjust the determinant, which was proved by Carayol [6] in case of $\mathbb{Q}$ (see [15] for the generalization).

**Proposition 4.11.** [Determinant optimization] Let $\pi$ be a cuspidal representation of infinite type $(k, w)$, having a non-zero fixed vector under $K_1(m^n_v) : K^v$. Then there is $\pi'$, $\pi' \infty$ has a non-zero fixed vector under $(K_1(m^n_v) \cdot H) \cdot K^v$. Here $H = \{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha \mod m^n_v \text{ has an } \ell \text{-power order} \}$. 

Proof. We take an auxiliary place $y$ by lemma 4.7 and replace $K$ by $K \cap K_1(y)$. Put $X = S_{K_1(w^n)_v} \cdot K^v$, $Y = S_{K_1(w^n)_u} \cdot K^v$, $\mathcal{F} = \mathcal{F}_{k, w}$. We view $X$ and $Y$ as complex varieties. $\pi : X \to Y$ is an étale Galois covering with group $G = H/H'$, $H'$ is the inverse image of a subgroup of $(\alpha_v/m^n_v)\times$ represented by units determined by $K^v$. We consider the complex $L$ associated to Godement’s canonical resolution of $\pi^*\mathcal{F}$ on $X$, which calculates (usual, not étale) sheaf cohomology $H^i(X, \pi^*\mathcal{F})$, and represents $R\Gamma(X, \pi^*\mathcal{F})$ in the derived category. Put $A = \mathcal{O}_\Lambda[G]$, $B = A[[T_u], [T_{u, u}], [T_{u, u}]^{-1}, u \notin \Sigma]$. The complex $L$ admits a commuting action of Hecke operators, and of $G$ by the discussion in 3.5 so we view $L$ as a complex of $B$-modules, sending $[T_u], [T_{u, u}]$ to the corresponding standard Hecke action at $u$. Then it follows that $H^1(S_{K_1(w^n)_v} \cdot K^v, \mathcal{F}_{w, k})_m$ is $\mathcal{O}_\Lambda[G]$-free by the previous lemmas ($i = 0$ or 1 according to $g$ is even or not), since $H^0(Y, \mathcal{F} \otimes_{\mathcal{O}_\Lambda} k_\lambda)$ and $H^2(Y, \mathcal{F} \otimes_{\mathcal{O}_\Lambda} k_\lambda)$ are of type $\omega$ when $g$ is odd as in 4.6. The $G$-invariant part is non-zero by the freeness. □

5. Mazur principle

In this section, we prove the following corollary $A'$ in the introduction:

**Claim 5.1.** [Corollary $A'$ (Mazur principle)] Assumptions are as in theorem $A$. Then there exists $\pi' \infty$ as in theorem $A$ when $\tilde{\rho}$ is unramified at $v$, and $q_v \not\equiv 1 \mod \ell$.

5.1. The odd degree case. In this subsection, we discuss 5.1 under condition $A$-1)-$A$-4) proved by Jarvis in [14]. We include this as a toy model for the Mazur principle in the even degree case in 5.3. We assume that $q_v \not\equiv 1 \mod \ell$. We may also assume that $\pi_v$ is an unramified special representation by remark 4.7(b).

We return to the general setting as in 3.6. In addition to the assumption of lemma 4.7(b) we make the following additional assumption further:
Assumption 5.2. (1) Any irreducible component $Y$ of $X_s$ is smooth.
(2) $\mathcal{F}_Y$ is pure of weight $w$ for some integer $w$ independent of $Y$.

We define and analyze a standard filtration $W_R \subset \mathcal{H}_R = H^1_{et}(X_s, \mathcal{F}_{X_s} \otimes_{\mathcal{O}_\lambda} R)$. Let $J$ be the set of irreducible component of $X_s$, and $Z$ be the set of singular points on $(X_s)_{red}$.

We define a skyscraper sheaf $\mathcal{G}_R$ supported on $Z$ by

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_\lambda} R \to \bigoplus_{Y \in J_0} (\mathcal{F}_Y) \otimes_{\mathcal{O}_\lambda} R \to \mathcal{G}_R \to 0.$$ 

Then $W_R \subset \mathcal{H}_R$ is defined by

$$0 \to H^0_{et}(X_s, \mathcal{F}_{X_s} \otimes_{\mathcal{O}_\lambda} R) \to \bigoplus_{Y \in J} H^0_{et}(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_\lambda} R) \to H^0_{et}(Z, \mathcal{G}_R) \to W_R \to 0.$$ 

$\mathcal{H}_R/W_R = \bigoplus_{Y \in J} H^1(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_\lambda} R)$ follows from the definition. In $\mathcal{E}_{NO}$, we have $W_R = H^0(Z, \mathcal{G}_R)$. Since $\mathcal{G}_\lambda$ is $\mathcal{O}_\lambda$-smooth, it follows that the formation of $W_R$ commutes with base change after localization at maximal ideal $m$, and we conclude that $\mathcal{H}_R/W_R$ has the same property.

We apply the formalism in 3.6 to Shimura curves $S_{D, \tilde{K}}$ and $\mathcal{F}_{k,w}$ with additional assumption 5.2. As in 4.5, we take $\mathcal{E}$ in lemma 3.11 to be the category of $\omega$-type modules.

We choose $D$ as in the proof of theorem 4.4. Our choice of compact open subgroup $\tilde{K} = \tilde{K}^v \cdot \tilde{K}_v$ is $\tilde{K}^v = K^v$, and for $\tilde{K}_v$

$$\tilde{K}_v = K(m_v^N) \cdot A, \quad A = \{ \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha \end{array} \right), \alpha \in o_v^\times, \alpha \equiv 1 \mod m_v^N \}, \quad N \geq 1.$$ 

The assumption 5.2 (1) is satisfied for $K_v = K(m_v^N)$ by 4.4.9.4.3, and the action of $A/A \cap K(m_v^N)$ is étale on the arithmetic model, so it is also satisfied with our $\tilde{K}$. The assumption 5.2 (2) on the weight of $\mathcal{F}$ is satisfied with weight $w$, since over a quadratic extension of $F$ unramified at $v$, $\mathcal{F}$ is a subquotient of a (possibly higher) cohomology sheaf of an abelian scheme. This follows from 4.4, 2.6. See also 20 for the discussion.

Let $T$ be the Hecke algebra generated by $[T_u], [T_{u,u}], [T_{u,u}]^{-1}$ for $u \notin \Sigma$ over $\mathcal{O}_\lambda$, $m$ be the maximal ideal of $T$ corresponding to $\tilde{\rho}$.

Let $P$ be the $p$-Sylow subgroup of $K_v/K(m_v^N)$, where $p$ is the residue characteristic of $v$. It contains $K_{11}(m_v)/K(m_v^N)$. Since $p$ is different from $\ell$, the operation of taking the $P$-invariants of $\mathcal{H}_R$ commutes with any scalar extension. We define $U(p_v)$-operator for a fixed uniformizer $p_v$ at $v$ acting on $\mathcal{H}_R^P$ in the usual way: it is defined by double coset $K_0(m_v)\alpha(p_v)K_0(m_v) \cdot K^v$ (see section 2 for the notation). $U(p_v)$-operator thus defined commutes with $G_F$-action since all group actions are defined over $F$.

(By the vanishing of $H^s(P, -)$,

$$H^P_R = H^1_{et}(S_{K^v}, \mathcal{F}_{(k,w)} \otimes_{\mathcal{O}_\lambda} R)_m, \quad K' = K^v \cdot (K_{11}(m_v)A),$$ 

and the $U(p_v)$-operator is equal to the one defined geometrically.)

By our assumption, $\mathcal{H}_R^P \neq \{ 0 \}$. Note that $(H^P_R)_m/m(H^P_R)_m$ as a Galois module is isomorphic to $\bar{\rho}^{\alpha\omega}$ for some $\alpha \geq 1$ by the Eichler-Shimura relation 4.10.3 and by the Boston-Lenstra-Ribet theorem 3 using the irreducibility of $\bar{\rho}$. As a $T$-module, it is isomorphic to $(T/m)^{2\alpha}$.

Assume the Hecke module $T/m$ corresponding to $\bar{\rho}$ occurs in $\mathcal{H}_{k\lambda}^P/W_{k\lambda}^P$. Then it occurs in $(\mathcal{H}_{k\lambda}^P/W_{k\lambda}^P)_m \neq \{ 0 \}$, which is $\mathcal{O}_\lambda$-free. Let $\pi$ be the corresponding cuspidal representation. Since $\mathcal{F}_Y$ is pure of weight $w$ for each irreducible component $Y$ of $X_s$, the weight here is $w + 1$ by the Weil conjecture. If we assume that $\pi_v$ is special, $\det \rho_{k\lambda}(Fr_v) = q_v^{\beta_1}$, with $|\beta| = q_v^{\frac{N-1}{2}}$, which is impossible by the weight reason. It follows that $\pi_v$ must be a principal series. Moreover, $\rho_v = \rho_{k\lambda}(Fr_v)$, associated to $\pi_v$ by the local Langlands correspondence,
has a non-zero $I_\nu$-fixed part and $\det p_\nu|_{I_\nu}$ is the Teichmüller lift of $\det \bar{\rho}|_{I_\nu}$ by our choice of $\bar{K}_\nu$. We conclude that $\pi_\nu$ is an unramified principal series.

So we may assume all $T/m$ appear in $W_{k_\lambda}^P$ and hence $\bar{\rho}^{\otimes \alpha} \simeq W_{k_\lambda}^P/mW_{k_\lambda}^P$ as a Galois-Hecke bimodule. For any lift $\pi$ found in $W_{E_\lambda}^P$, $\pi$ has an unramified special component at $v$. $q_vU(p_v) \cdot Fr_v^{-1}$ is identity on $W_{E_\lambda}^P$ by the compatibility of local and global Langlands correspondence recalled in section 2 (by our normalization, the eigenvalue of $U(p_v)$-operator is equal to the Frobenius eigenvalue on $\rho_{p,\lambda}/\rho_{E_\lambda}^P$), and hence on $W_{E_\lambda}^P$ also by the $G_{\lambda}$-freeness.

Since $U(p_v)$ commutes with global Galois action on $\bar{\rho}$, by Schur’s lemma we may assume that there is a scalar $\gamma \in \bar{k}_\lambda \setminus \{0\}$ such that $\gamma^{-1} Fr_v$ acts trivially on $W_{k_\lambda}^P/mW_{k_\lambda}^P = (H_{k_\lambda})^P/m(H_{k_\lambda})_m \otimes_{k_\lambda} \bar{k}_\lambda \simeq \bar{\rho}^{\otimes \alpha}$.

It follows that any $Fr_v$-eigenvalues of $\bar{\rho}$ are the same, which leads to a contradiction since two eigenvalues of $Fr_v$ on $\bar{\rho}$ are of the form $\alpha_v$, $q_v\alpha_v$, and $q_v \neq 1 \mod \ell$.

5.2. Cerednik-Drinfeld type theorem for totally real fields. Let $D$ be a quaternion algebra over $F$. Assume $D$ defines a Shimura curve, i.e., $D \otimes _Q \mathbb{R} \simeq M_2(\mathbb{R}) \times \mathbb{H}^2$.

Choose a finite place $v$ where $\text{inv}_v D = 1/2$. Let $\bar{D}$ be the definite quaternion with $\text{inv}_v D = 0$, and other invariants at finite places are the same as $D$. We need a generalization of the Cerednik-Drinfeld theorem for Shimura curves over $\mathbb{Q}$. For general totally real fields, such a result follows from the work of Boutot-Zink [2], which we recall in the following.

We choose a compact open subgroup $K \subset D^\times (\mathbb{A}^{\text{unr}})$ such that $\bar{K} = \sigma_D^v \cdot K^v$. Put $\bar{K} = GL_2(\sigma_v) \cdot K^v$. Then the main result of [2] claims:

**Theorem 5.3.** [2, theorem 0.1] There is a canonical isomorphism

$$S_{D,K, o_v} \simeq \bar{D}^\times \backslash D^\times (\mathbb{A}^{\text{unr}}) \times \hat{\Omega}_{o_v} \hat{\hat{o}}_v^{\text{unr}} / K^v.$$ 

Here $\hat{\Omega}_{o_v}$ is the Deligne model of the Drinfeld upper half plane, and $\hat{\hat{o}}_v^{\text{unr}}$ the maximal unramified extension of $\sigma_v$. The action of $GL_2(F_v)$ on $\hat{\hat{\Omega}}_{o_v} \hat{\hat{o}}_v^{\text{unr}}$ is

$$g \mapsto (g, Fr_v(g)), \quad n(g) = \text{ord}_v(\det g), \quad Fr_v : \text{Frobenius at } v$$

and $\bar{D}^\times$ acts diagonally. Especially, $S_{D,K, o_v}$ is regular.

**Corollary 5.4.** The set of irreducible components $J_K$ of $S_K, o_v^{\text{unr}}$ is identified with two copies of the double cosets $S_{\bar{D},\bar{K}} = \bar{D}^\times \backslash \bar{D}^\times (\mathbb{A}^{\text{unr}}) / \bar{K}$, and the identification is $\bar{D}^\times (\mathbb{A}^{\text{unr}})$-equivariant.

**Proof.** The set of irreducible components $I$ of $\hat{\hat{\Omega}}_{o_v}$ is isomorphic to the set of all lattices up to homothety in $F_v^2$ by the structure of the Bruhat-Tits building, and hence identified with $GL_2(F_v)/F_v^\times \cdot GL_2(\sigma_v)$.

By theorem 5.3, $J_K$ is canonically isomorphic to

$$\bar{D}^\times \backslash D^\times (\mathbb{A}^{\text{unr}}) \times I \times \mathbb{Z} / K^v.$$ 

Put $\bar{D}_+^\times = \{ g \in \bar{D}^\times \mid \text{ord}(\det g) \text{ is even} \}$.

The map $\alpha : \bar{D}_+^\times \to 2\mathbb{Z}$ given by $\text{ord}(\det g)$ is surjective. So the pieces $J_{K,+} = \bar{D}_+^\times / D^\times (\mathbb{A}^{\text{unr}}) \times I \times 2\mathbb{Z} / K^v$ and $J_{K,-} = \bar{D}^\times _+ / \bar{D}^\times (\mathbb{A}^{\text{unr}}) \times I \times (\mathbb{Z} \setminus 2\mathbb{Z}) / K^v$ inside $J_K$ are isomorphic to $\alpha^{-1}(0) \backslash GL_2(F_v)/(F_v^\times \cdot GL_2(\sigma_v))/K^v \simeq S_{\bar{D},\bar{K}}$. $\bar{D}^\times (\mathbb{A}^{\text{unr}})$-equivariance follows from the description. Moreover, the construction of $J_{K,\pm}$ is canonical.

We need to calculate the fibers of our sheaves $\mathcal{F}_{k,w}^D$ as well. Note that we also have a sheaf $\mathcal{F}_{k,w}^\bar{D}$ which is an analogue of $\mathcal{F}_{k,w}^D$ on $S_{D,K}$. 

Corollary 5.5. The sum of geometric generic fibers \( \oplus_{\eta \in I_K} (\mathcal{F}_{k,w})^\eta \) is identified with the sheaf \( \mathcal{F}_{D,K}^D \) on \( S_{D,K} \) on each \( J_K, \pm \), and the identification is \( D^\times (A_{v,\ell,\infty}) \)-equivariant.

Proof. Let \( \pi_\ell : \hat{S}_\ell \rightarrow S_{D,K} \) be the Galois covering corresponding to \( \prod_{u \mid \ell} K_u / K \cap (F^\times) \). By corollary 5.4, the set of irreducible components of \( (\hat{S}_\ell)_{k,0} \) is identified with two copies of \( S_{D,\hat{R}_\ell} \), preserving \( D^\times (A_{v,\ell,\infty}) \)-action. Since the sheaf \( \mathcal{F}_{D,K}^D \) is obtained from the covering \( \pi_\ell \) and the representation \( \otimes_{\eta \in I_K} (\det)^{\ell/2} \Sym^{k-2}_v \) of \( D^\times (Q_{\ell}) \) by contracted product, their geometric generic fibers are identified with \( \mathcal{F}_{D,K}^D \) on each \( J_K, + \) and \( J_K, - \) by the \( D^\times (A_{v,\ell,\infty}) \)-equivariance. \( \square \)

We need \( U_v \)-operator in the following, which is defined as follows. Take a uniformizer \( \Pi_v \) of \( \mathcal{O}_{D,v} \), and consider the double coset \( K^v \cdot \mathcal{O}_v \cdot K_u \). This defines a correspondence, and the action on cohomology is \( U_v \). \( U_v \)-operator thus defined exists over \( F \), not only over \( F_v \), and by the local Jacquet-Langlands correspondence [16], it corresponds to \( U(p_v) \)-operator defined in 5.1 for \( GL_2 (F_v) \) (this is proved easily).

5.3. The even degree case. By 5.3 and the method of section 5.1, we prove 5.1 without assuming A-4). By subsection 5.1 and remark 4.7(b), we may assume that the degree \([F : \mathbb{Q}]\) is even, and \( \pi_v \) is a special representation twisted by an unramified character.

Take \( D \), the definite quaternion algebra which is unramified at all finite places, and \( D \) be an indefinite quaternion algebra corresponding to \( \hat{D} \) as above. By the argument of 1.3, we take an auxiliary place \( y \), and take a compact open subgroup \( K = \prod_u K_u \) with \( K_v = \mathcal{O}_{D,v} \), \( K_y = K_{11} (m_y) \). Then \( \rho_{\pi_v} \) occurs in \( H^0_{\text{ét}} (S_{D,K}, \mathcal{F}_{D,K}^D) \) by the Jacquet-Langlands correspondence, since \( \pi_v \) is an unramified special representation.

We choose \( \Sigma \) so that \( u \not\in \Sigma \Rightarrow u \not\mid \ell \) and \( K_u = GL_2 (\mathcal{O}_u) \). Let \( T \) be the Hecke algebra generated by \( [T_u] \), \( [T_{u,u}] \), \( [T_{u,u}]^{-1} \) for \( u \not\in \Sigma \) over \( \mathcal{O}_\lambda \), \( m \) be the maximal ideal of \( T \) corresponding to \( \bar{\rho} \). Assume contrary, so \( \bar{\rho} \) does not appear in \( H^0 (S_{D,K}, \mathcal{F}_{D,K}^D \otimes \mathcal{O}_\lambda, k_\lambda) \).

Proposition 5.6.

\[ H^1_{\text{ét}} (S_{D,K,\mathcal{O}_w^\text{un}}, \mathcal{F}_{k,w}^D \otimes \mathcal{O}_\lambda, R) \simeq H^1_{\text{ét}} (S_{D,K,F_v}, \mathcal{F}_{D,K}^D \otimes \mathcal{O}_\lambda, R) \]

holds for a finite local \( \mathcal{O}_\lambda \)-algebra \( R \), and hence \( R \mapsto H^1_{\text{ét}} (S_{D,K,\mathcal{F}_v}, \mathcal{F}_{k,w}^D \otimes \mathcal{O}_\lambda, R) \) commutes with scalar extensions.

Proof. We adopt the method used in previous sections. Since our \( T \)-action annihilates \( (\oplus_{Y \in J} H^1_{\text{ét}} (Y, \mathcal{F}_Y))^m \simeq H^0 (S_{D,K}, \mathcal{F}_{D,K}^D \otimes \mathcal{O}_\lambda, k_\lambda)^{\otimes 2}_m \) by our assumption that \( \bar{\rho} \) does not come from \( H^0 (S_{D,\hat{R}}, \mathcal{F}_{D,K}^D \otimes \mathcal{O}_\lambda, k_\lambda) \), the claim follows from 5.5 and 5.11. \( \square \)

The rest of the argument is the same as in section 5.1. By proposition 5.6, \( \bar{\rho} \) comes from \( H^1_{\text{ét}} (S_{D,K,F_v}, \mathcal{F}_{D,K}^D) \) as a Hecke module. As in section 5.1, \( q_v U_v : F_v^{-1} \) is identity on \( H^1_{\text{ét}} (S_{D,K,F_v}, \mathcal{F}_{D,K}^D)^{U_v} \), and \( U_v \)-operator on \( H^1_{\text{ét}} (S_{D,K,F_v}, \mathcal{F}_{D,K}^D \otimes \mathcal{O}_\lambda, k_\lambda) \) acts as a scalar since it commutes with the global Galois group \( G_F \) by using Boston-Lenstra-Ribet theorem [3]. Two eigenvalues of \( Fr_v \) on \( \bar{\rho} \) are of the form \( \alpha_v, q_v \alpha_v \), this implies \( q_v \equiv 1 \mod \ell \).

6. Proof of theorem A

Now we finish the proof of theorem A in the introduction. We may assume that the degree \([F : \mathbb{Q}]\) is even. By 21, theorem 1, there is a finite place \( z \not\equiv v \) of \( F \) where \( \bar{\rho} \) is unramified, \( q_z \equiv -1 \mod \ell \), and there is a cuspidal representation \( \dot{\pi} \) which gives \( \bar{\rho} \) such that \( \dot{\pi}_z \) is an unramified special representation at \( z \), \( (\dot{\pi}_z)^{K \cap K_0 (z)} \not\equiv 0 \). We can now apply
theorem B to $\tilde{\pi}$, and optimize the level at $v$. When $\tilde{\rho}$ is unramified at $v$, we first apply remark [17] b) to get $\pi'$ which has an unramified special component $\pi'_v$ at $v$, then apply the Mazur principle in the form of section [5.1]. Finally, the auxiliary place $z$ can be removed by the Mazur principle in [5.3] since $q_z \not\equiv 1 \mod \ell$, and the component at $z$ is an unramified special representation.

References

[1] Blasius, D., Rogawski, J.: Galois representations for Hilbert modular forms, Bull. Amer. Math. Soc. 21 (1989), 65–69.
[2] Boutot, J. F., Zink, T.: The $p$-adic uniformization of Shimura curves, preprint series of Universität Bielefeld, (1995), 95–107.
[3] Boston, N., Lenstra, H.W., Ribet, K.A.: Quotients of group rings arising from two dimensional representations, Comptes Rendus de l’académie des Sciences, serie I 312 (1991), 323–328.
[4] Carayol, H.: Sur la mauvaise réduction des courbes de Shimura, Compositio Math., 59 (1986), 151–230.
[5] Carayol, H.: Sur les représentations $p$-adiques associées aux formes modulaires de Hilbert, Ann. Sci. Ec. Norm. Sup. IV, Ser 19 (1986), 409–468.
[6] Fujiwara, K.: Deformation rings and Hecke algebras in the totally real case, preprint.
[7] Grothendieck, A.: Revêtement étales et groupe fondamental (SGA1), Lecture Notes in Math., 224, Springer Verlag (1971).
[8] Hida, H.: On $p$-adic Hecke algebras for $GL_2$ over totally real fields, Ann. of Math., 128, (1988), 295–384.
[9] Hida, H.: Nearly ordinary Hecke algebras and Galois representations of several variables, Amer. J. of Math., special issue, (1989).
[10] Jarvis, F.: Mazur’s principle for totally real fields of odd degree, Compositio Math., 116, (1999), 39–79.
[11] Jacquet, H., Langlands, R.P.: Automorphic forms on $GL_2$ Lecture Notes in Math., 114, Springer Berlin, (1970).
[12] Ohta, M.: On the zeta function of an abelian scheme over the Shimura curve, Jpn. J. Math. 9, (1983), 1–26.
[13] Ribet, K.A.: On modular representations of $Gal(\overline{Q}/Q)$ arising from modular forms, Invent. Math., 100, (1990), 431–476.
[14] Ribet, K.A.: Report on mod $\ell$ representations of $Gal(\overline{Q}/Q)$, Proc. Symp. in Pure Math. AMS, 639–676.
[15] Saito, T.: Hilbert modular forms and $p$-adic Hodge theory, (1999).
[16] Taylor, R.: On Galois representations associated to Hilbert modular forms, Invent. Math., 98, (1989), 265–280.
[17] Wiles, A.: On ordinary $\lambda$-adic representations associated to Hilbert modular forms, Invent. Math., 94, (1988), 529–573.

E-mail address: fujiwara@math.nagoya-u.ac.jp