On the effect of reversing the direction of convection in a two-component fluid

L Kh Ingel¹,²

¹ Research and Production Association «Typhoon», Pobedy St. 4, 249038 Obninsk, Kaluga Reg.
² Obukhov Institute of Atmospheric Physics, Russian Academy of Sciences, Pyzhevskii per. 3, 119017, Moscow

E-mail: lev.ingel@gmail.com

Abstract. The theoretical problem of convection in a two-component medium (for example, in salt water) after "switching on" the sources/sinks of heat and admixture on an infinite vertical boundary is considered. Generally speaking, the degree of influence of the two components on convection is different due to the difference in their diffusion velocities. We suppose that heat diffuses much faster than the admixture and then the heated/cooled region at the lateral boundary is much thicker than the region of propagation of the impurity concentration disturbance. Therefore, convection due to temperature deviations is less affected by viscosity than due to deviations in admixture concentration. As a result, even a relatively weak "thermal" source of buoyancy can determine the direction of convection for some time in spite of the action of a more intense "concentration" source of buoyancy of a different sign. But if a vertical layer of a medium of finite thickness is considered, then a stationary regime is established over time, in which the direction of convection is determined by the total source of buoyancy, so that the direction of convection can change sign.

1. Introduction

A number of recent works have theoretically and experimentally discovered a new class of hydrodynamic phenomena in binary mixtures (for example, in salty seawater), associated with a significant difference in the values of the diffusion coefficients of two substances (heat and salt; see, for example, [1, 2] and bibliography to these articles). In this work, attention is drawn to another nontrivial effect: during the development of convection in such media, its direction can change sign.

2. Statement of the problem

Let us illustrate this using the simplest model of convection in an infinite flat vertical channel. Let \( x \) be one from horizontal coordinates, the axis \( z \) is directed upward vertically. A homogeneous two-component medium fills the area \( 0 < x < L \) and initially is at rest. We study convection caused in such a medium by constant (after "switching on" at the moment \( t = 0 \)) deviations of temperature and admixture on the surface \( x = 0 \) (Fig.1).
Figure 1. A schematic picture of the problem geometry. Due to the slowness of admixture diffusion, the disturbance of its concentration at the initial stage of convection development is contained in a relatively thin region 1 near the left boundary. A heat spreads much faster, so that the temperature disturbance covers regions 1 and 2. The vertical arrows show the direction of convection.

The density deviation of the medium $\rho$ in the commonly used approximation [3, 4] depends linearly on temperature perturbations $T$ and the concentration of admixture (salinity) $S$:

$$\rho = \rho_0 \left[ 1 - \alpha T + \beta S \right],$$

where $\rho_0$ is an average (reference) density of the medium; $\alpha$ is a thermal coefficient of expansion of the medium; the coefficient $\beta$ determines the dependence of the density of the medium on the concentration of admixture (in oceanology it is called the coefficient of salinity compression).

For simplicity, we consider the case of the absence of background stratifications of temperature and admixture concentration. In other words, at the initial moment, the medium is not only at rest, but also is homogeneous by temperature, admixture concentration and density.

Since the disturbances set at the boundary do not depend on a vertical coordinate $z$, we will look for solutions independent of $z$. The rationale and limits of applicability of such a one-dimensional convection regime are discussed, for example, in [4].

The system of equations of hydrodynamics, heat and admixture transfer in the Boussinesq approximation without taking into account the cross-kinetic effects (thermal diffusion and diffusion thermal conductivity [5]) for the given geometry of the problem has the form

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2} + gb,$$

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

$$\frac{\partial S}{\partial t} = \chi \frac{\partial^2 S}{\partial x^2}. \quad (1) \quad (2)$$

Here $w$ is the velocity component along the vertical axis $z$ (there are no other velocity components in this problem), $t$ is time, $\nu$, $\kappa$, $\chi$ and are the transfer coefficients;
\[ b = \alpha T - \beta S \]  

- Dimensionless buoyancy. We emphasize that the complete system of equations for hydrothermodynamics and admixture transfer has been reduced to the linear system (1), (2) only due to the symmetry of the problem, without any assumptions about the smallness of the amplitudes of the disturbances.

At the initial stage of development of the disturbance, it is distanced from the right boundary of the considered channel. Indeed, at times much shorter than the minimum of the "diffusion" time scales \( \frac{L^2}{\chi}, \frac{L^2}{\alpha}, \frac{L^2}{v} \), the perturbations practically do not reach the right boundary, the presence of which, therefore, has little effect on the solution. Therefore, we can approximately consider the problem for a semi-bounded medium \( 0 < x < \infty \), which is close to our work [2], and on sufficiently small time intervals we assume the fulfillment of conditions
\[
w, T, S \to 0, \quad x \to \infty. \tag{4}
\]

At the boundary \( x = 0 \), the adhesion condition \( w = 0 \) is satisfied. The boundary conditions are considered for the temperature and admixture concentration:
\[
T = T_h \eta(t), \quad S = S_b \eta(t) \quad \text{at} \quad x = 0, \tag{5}
\]
where \( \eta(t) \) is Heaviside unit function; \( T_h, S_b \) are given constant values of disturbances at the boundary.

The most attention below is paid to situations when \( 0, b_b < 0 \), i.e. when the medium is cooled and "freshened" from the lateral boundary \( x = 0 \), and the positive buoyancy inflow associated with freshening is significantly greater than the negative inflow associated with cooling (the ratio of these inflows, indicated \( \sigma \), is estimated below by relation (8)).

3. Solution

The solution of equations (1), (2) with boundary conditions (4), (5) is easy to derive by using an analysis of one-dimensional problems of convection represented in [4]:
\[
w = 4g \left[ \frac{\alpha x T_h}{v - \chi} \left[ i^2 \text{erfc} \left( \frac{x}{2(v\tau)^{1/2}} \right) - i^2 \text{erfc} \left( \frac{x}{2(\chi\tau)^{1/2}} \right) \right] \right. \left. \frac{\beta \delta b}{v - \chi} \left[ i^2 \text{erfc} \left( \frac{x}{2(\chi\tau)^{1/2}} \right) - i^2 \text{erfc} \left( \frac{x}{2(v\tau)^{1/2}} \right) \right] \right] =
\]
\[
= -\frac{4g \beta \delta b \chi}{v - \chi} \left[ i^2 \text{erfc} \left( \frac{x}{2(\chi\tau)^{1/2}} \right) - i^2 \text{erfc} \left( \frac{x}{2(v\tau)^{1/2}} \right) \right] - \Xi \left[ i^2 \text{erfc} \left( \frac{x}{2(\chi\tau)^{1/2}} \right) - i^2 \text{erfc} \left( \frac{x}{2(v\tau)^{1/2}} \right) \right], \tag{6}
\]

where the dimensionless parameter is introduced
\[
\Xi = \frac{\alpha T_h \kappa (v - \chi)}{\beta \delta b \chi} = \frac{\alpha T_h \kappa}{\beta \delta b \chi}; \tag{7}
\]
repeated integrals of the error function [6] are defined by the equality
\[
i^n \text{erfc} \ X = \int_X^\infty i^{-n} \text{erfc} \ X' \, dX', \quad n = 0, 1, 2, \ldots,
\]
\[
i \text{erfc} \ X \equiv i \text{erfc} \ X = \int_X^\infty 0 \text{erfc} \ X' \, dX',
\]
\[
i^0 \text{erfc} \ X = \text{erfc} \ X, \quad i^{-1} \text{erfc} \ X = \frac{2}{\sqrt{\pi}} \exp(-X^2).
\]

For calculations, it is convenient to use the recurrence relation [6]
\[ i^n \text{erfc} \ X = -\frac{X}{n} i^{n-1} \text{erfc} \ X + \frac{1}{2n} i^{n-2} \text{erfc} \ X. \]

Despite the rather cumbersome form of the solution, it is easy to analyze. After "switching on" the disturbances of temperature and salinity at the boundary \( x = 0 \), they extend to the area \( x > 0 \), accordingly to diffusion laws \( \Delta x \sim t^{1/2} \) (but with proportionality coefficients that are significantly different for different substances). The emergence of an expanding area of nonzero buoyancy leads to the emergence of a region of vertical movements, which also expands horizontally according to the diffusion law. The Fig. 2 shows examples of horizontal profiles of developing vertical movements (vertical velocity is normalized to \( -4 g \beta S_b \chi T/(\nu - \chi) \)).

\[ \text{Figure 2. Examples of dimensionless vertical velocity profiles at } \Xi = 2 \text{ and 3.33 (dashed and solid lines, respectively)} \]

The found solution has the following interesting feature [2]: even with a clear prevalence of a positive inflow of buoyancy into the medium, the arising movements nevertheless, can be mainly descending – as can be seen from the figure, relatively weak cooling at the lateral boundary at \( \chi << \kappa \) (sea water) can have a much stronger effect on convection than intensive freshening.

Formally, this follows from the fact that the "salinity" disturbance of buoyancy \( \beta S_b \) has a small weight \( \chi/\kappa \) as a member of the determining parameter \( \Xi \) in (7) as compared with "temperature" disturbance \( \alpha \chi T \).

Such a result can be explained as follows. The positive inflow of buoyancy is associated with freshening at the vertical boundary. Due to the slow diffusion of salt, the freshening area extends relatively slowly (its horizontal dimensions \( L_S \sim \sqrt{\kappa} \)). Therefore, the ascending jet, which arises in this thin vertical layer, is strongly hindered by viscosity (there is a vertical boundary nearby, on which the adhesion conditions are specified). But the cooling region, due to the relatively rapid diffusion of heat, is much wider than the freshening region (the thickness of the cooled vertical layer \( L_T \sim \sqrt{\kappa} \)). Therefore, the downward movements associated with cooling, are hindered by viscosity relatively weakly and can be dominant, despite the significant predominance of positive buoyancy inflow into the medium under a large value of the ratio assumed here

\[ \sigma \equiv \left| \frac{\beta \chi \partial S}{\partial x} / \frac{\alpha \chi \partial T}{\partial x} \right| \sim \frac{\beta \chi S_b L_T}{\alpha \chi T_b L_S} \left( \frac{\chi}{\kappa} \right)^{1/2}. \]  

(8)
In this meaning, cooling affects convection much more effectively than freshening (compared to their contributions to the change in buoyancy).

However, over time, the disturbances approach to the right boundary \( x = L \), the influence of which becomes significant. Then the solution becomes more complicated since it starts to depend on the conditions on the right boundary. At this boundary we assume an absence of disturbances: \( T = S = w = 0 \).

The temperature “cooling wave”, having reached the right boundary, obviously ceases to extend to the right and is gradually “overtaken” by the “freshening wave”. This means a decrease in the density of the medium and a tendency for upward movements in the case \( \beta S_b > |\alpha T_b| \). Finally, at sufficiently large times, obviously, a stationary solution should be established, which has the form

\[
T = T_b \left(1 - \frac{x}{L}\right), \quad S = S_b \left(1 - \frac{x}{L}\right), \quad w = \frac{gL}{v} \left(-\beta S_b + \alpha T_b\right) x \left(1 - \frac{x}{2L} + \frac{x^2}{6L^2}\right). \tag{9}
\]

Figure 3 shows the vertical velocity profile (9) normalized to \( gL^2 \left(-\beta S_b + \alpha T_b\right) / v \). As it could be expected, the positive buoyancy influx ultimately forms an upward movement.

Above, we assumed a fixed deviation of the admixture concentration at the left boundary at \( t > 0 \). In practice, the implementation of such a boundary condition is very difficult. But the results are qualitatively the same for a much broader class of boundary conditions. As an example, let us give a solution for the case when the conditions on the left boundary are specified not of the 1st, but of the 2nd kind:

\[
\begin{align*}
\rho_0 \kappa \frac{\partial T}{\partial x} &= Q \eta(t), \\
\rho_0 \chi \frac{\partial S}{\partial x} &= \Theta \eta(t), \quad \text{at} \quad x = 0, \tag{10}
\end{align*}
\]

where \( Q, \Theta \), up to signs and normalization, have the meaning of the corresponding horizontal flows.

The solution for the initial stage of disturbance development (while the influence of the right boundary is insignificant) has the form [2]:

\[
w = -\frac{8g}{\rho_0} \left(\kappa^3\right)^{1/2} \left\{ \frac{\alpha Q}{c(v - \kappa)} \left[ i^3 \text{erfc}\left(\frac{x}{2(\kappa^3)^{1/2}}\right) - i^3 \text{erfc}\left(\frac{x}{2(\kappa)\left(\kappa^3\right)^{1/2}}\right) \right] - \left(\frac{\kappa}{v - \kappa}\right)^{1/2} \left[ i^3 \text{erfc}\left(\frac{x}{2(v\left(\kappa^3\right)^{1/2}}\right) - i^3 \text{erfc}\left(\frac{x}{2(\kappa)\left(\kappa^3\right)^{1/2}}\right) \right] \right\} = \tag{11}
\]

\[
\begin{align*}
0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
0.01 & 0.02 & 0.03 & 0.04 & 0.05 & 0.06
\end{align*}
\]

\textbf{Figure 3. Vertical velocity profile (9)}
\[
\frac{8g\beta\Theta}{\rho_0(v-\chi)}  \left\{ \left[ i^3 \text{erfc} \left( \frac{x}{2(\chi \nu)^{1/2}} \right) - i^3 \text{erfc} \left( \frac{x}{2(\chi')^{1/2}} \right) \right] \right. \\
\left. - \Lambda \left[ i^3 \text{erfc} \left( \frac{x}{2(\chi \nu)^{1/2}} \right) - i^3 \text{erfc} \left( \frac{x}{2(\chi')^{1/2}} \right) \right] \right\},
\]

where the dimensionless parameter is introduced
\[
\Lambda = \frac{\alpha Q(v-\chi)}{c\beta \Theta(v-\chi)} \left( \frac{\kappa}{\chi} \right)^{1/2},
\]
determining the ratio of the contributions of temperature and admixture concentration to the vertical motion arising near the boundary. Figure 4 shows the examples of horizontal profiles of developing vertical motions for solution (11) at \( Q, \Theta > 0 \), when the medium, as in the previous solution, is weakly cooled and intensely freshened from the left boundary. The vertical velocity is plotted along the ordinate, normalized to \( 8g\beta\Theta(\chi')^{3/2}/\rho_0(v-\chi) \). It can be seen that in this case the direction of convection is mainly determined by the rapidly diffusing substance (cooling), although the latter makes a relatively small contribution to the buoyancy inflow. After a sufficiently long time, when both substances reach the right boundary, the direction of convection changes for a very wide class of boundary conditions at this boundary. This is easy to verify from the analysis of the stationary solution.

4. Conclusion
An effect, to which have not been paid attentions previously, is presented above. The direction of convection in a two-component medium can radically change over time due to the difference in the diffusion rates of the two components.

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