GUP Effects on Chaotic Motion Near Schwarzschild Black Hole Horizon

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The paper is devoted to a detailed study of chaotic behavior in the dynamics of a (massless) probe particle near the horizon of a generalized Schwarzschild black hole. Two possible origins inducing the modification of black hole metric are considered separately; the noncommutative geometry inspired metric (suggested by Nicolini, Smailagic and Spallucci) and the metric with quantum field theoretic corrections (derived by Donoghue). Our results clearly show that in both cases, the metric extensions favour chaotic behavior, namely chaos is attained for relatively lower particle energy. This is demonstrated numerically by exhibiting the breaking of the KAM tori in Poincare sections of particle trajectories and also via explicit computation of the (positive) Lyapunov exponents of the trajectories.

I. INTRODUCTION

A universal upper bound on chaos in quantum field theory at temperature \( T \) has been discovered by Maldacena, Shenker and Stanford [1], in terms of the Lyapunov exponent \( \lambda_L \) of out-of-time-ordered correlators (OTOC)

\[
\lambda_L \leq \frac{2\pi T}{\hbar}.
\]  

The bound is saturated for the Sachdev-Ye-Kitaev models [2, 3]. This can be exploited to study the effect of temperature on gravity since black hole thermodynamics connects black hole surface gravity \( \kappa \) to Hawking temperature

\[
\kappa = \frac{2\pi T}{\hbar}.
\]  

Now, the Lyapunov exponent can be calculated independently for a black hole from the study of chaotic behavior in particle motion near a black hole horizon [4, 5]. Thus, the universality of the above bound (1) can be established in a different setting. This has triggered an immediate interest in particle motion near black hole horizon. Analysis of shock waves near black hole horizons [6] and AdS/CFT correspondence [7] originally provided \( T \)-dependence of \( \lambda \). Of topical interest is the possibility that signatures of chaotic behavior around black holes on gravitational wave emitted from it [8] might be observed in recently realized LIGO experiment [9]. Recently it has been shown (in works involving one of us) that in the presence of the Rindler horizon where the intrinsic curvature of the space-time is absent, unlike the case of black holes, the particle dynamics becomes chaotic in the near horizon region [10, 11].

Studies on the influence of black hole horizon in inducing chaos on particle motion near it has a long history [4, 5, 10, 12–21]. This feature has led many researchers to the study of near horizon chaotic dynamics both in classical as well as in quantum regime. In the former, it has been shown that in the presence of event horizon for different kinds of black holes, either static spherically symmetric [13, 15] or rotational [18], or magnetized [21], the particle dynamics becomes chaotic in the vicinity of the horizon. In all those cases, the considered test particle was either massive, charged, spinning [19, 20] or massless. This shows the fact that in the classical scenario, the horizon has an inherent property of inducing chaos in a system [12, 15–17, 20]. Exploration of chaotic dynamics in the latter, in the context of quantum chaos, the characteristics OTOC has been mainly studied [1, 22]. The exponential growth of OTOC is the main signature of quantum chaos [1, 22]. Therefore, all these analyses indicate towards one conclusion that a horizon is the nest of chaos.

Let us come to our work in this perspective. In the extreme environment near black hole horizon, it is expected that quantum gravity will play a decisive role. In the absence of a fundamental theory one considers physically motivated and viable models that are extensions of conventional theories that incorporate quantum gravity effects in a phenomenological way. One such extension is the Generalized Uncertainty Principle (GUP) [23]
that takes into account the existence of a minimal length scale, which is advocated, through diverse models, as a characteristic feature of quantum gravity scenario. The GUP framework has been very productive in generating a plethora of quantum gravity signatures in conventional physics (see for example [24–26]). In the context of the theory of gravitation, a significant application of GUP effect is a modified black hole metric, derived by Nicolini, Smailagic and Spallucci [27]. This metric appears as a solution of Einstein field equations where the matter density is given by a distribution with an inbuilt minimal length (of quantum gravity origin). This metric has the cherished feature that, on one hand, the essential black hole singularity is removed, whereas on the other hand, at large distance the GUP effects weaken and eventually the standard black hole metric is recovered (see [28] for a review). Because of its connection to quantum gravity inspired Non-Commutative (NC) geometry the above metric [27] is also referred to as NC inspired metric.

In an alternative scheme, quantum field theoretic effects in general relativity have also been considered by Donoghue [29] that are manifested in a generalized form of black hole metric, that is distinct from [27]. We will refer to this deformed metric as quantum corrected metric.

In the present work we will study the near horizon chaotic behavior of probe particles in the background of the two types of generalized metrics mentioned above [27, 29]. In a nutshell our results indicate that both NC and quantum effects tend to increase the chaotic nature of particle motion that is chaos appears at a lower particle energy if NC or quantum effects are present. In fact it is natural to ask whether these two deformations can be related. An interesting option is to exploit the approach in the work (involving one of us) [30] that estimated the NC parameter in generalized uncertainty principle by comparing the two results: NC parameter corrected Hawking temperature for a black hole on one hand and the same temperature computed from Newtonian dynamics for an effective potential arising from a deformed black hole metric that incorporated quantum field theoretic corrections [29]. We will come back to this question at the end.

The paper is organized as follows: In Section II A, the NC extended metric, derived in [27], together with the Hamiltonian equations of motion for the probe particle is given. In Section II B, the Poincare sections are plotted numerically to reveal the near horizon chaotic behavior. In Section II C, an approximate analytic form of the metric, derived in [27] is introduced which induces qualitatively similar chaotic behavior similar to the exact metric, as demonstrated numerically. Subsequently, the Lyapunov exponents for the chaotic paths are derived to reveal the chaotic features quantitatively. In Section III, the metric with quantum field theoretic corrections, derived in [29] is considered and the corresponding chaotic trajectories are studied via Poincare sections (subsection IIIA) and via Lyapunov exponents (subsection IIIB). The paper concludes in Section IV with discussion of results and future directions of research.

II. NEAR HORIZON CHAOS FOR NC INSPIRED SCHWARZSCHILD METRIC

A. Exact form of NC-deformed metric

In this section we shall start with the NC inspired Schwarzschild metric given in [28]

\[ ds^2 = -f_{nc}(r)dt^2 + f_{nc}(r)^{-1}dr^2 + r^2d\Omega^2 \]  

(3)

where \( f_{nc}(r) = (1 - (4M/r\sqrt{\pi})\gamma(3/2, r^2/4\theta_{nc})) \) and

\[ \gamma(3/2, r^2/4\theta_{nc}) = \int_0^{\frac{r^2}{4\theta_{nc}}} x^{1/2}e^{-x}dx. \]

The dimensional constant \( \sqrt{\theta_{nc}} \sim \text{length} \) corresponds to the NC parameter and fixes the scale for NC effects to be appreciable. For \( r > > \sqrt{\theta_{nc}} \), \( f_{nc} \) reduces to the undeformed Schwarzschild form \( f_{nc} \approx 1 - 2M/r \). The horizon \( r = r_H \) is determined by \( f_{nc}(r = r_H) = 0 \) and \( d\Omega^2 = (d\phi^2 + \sin^2\theta d\theta^2) \). This line element is a solution of the Einstein equation where the energy density distribution of a static, spherically symmetric, particle-like gravitational source is diffused due to the presence of the inherent length scale in \( \theta_{nc} \). The generalized density is given by [28]

\[ \rho_{\theta_{nc}}(r) = \frac{M}{(4\pi\theta_{nc})^{3/2}} \exp\left(-\frac{r^2}{4\theta_{nc}}\right). \]

(4)

We point out that as we decrease the value of \( \theta_{nc} \), the diffusive nature of energy density gets reduced so that the results tend towards Schwarzschild one (without noncommutative correction) [10]. As we decrease the value of \( \theta_{nc} \), the expression of \( \rho_{\theta_{nc}}(r) \) becomes the energy density distribution of a \( \delta \)-functional like point gravitational source [28]. So, it is evident that with the decrease of \( \theta_{nc} \), the result will tend towards the results of the Schwarzschild one. Profile of the NC modified metric is shown in Fig. 1. Here all the graphs are for different \( \theta_{nc} \) but fixed black hole mass \( M \). It is interesting to note that there is a critical black hole mass, (function of \( \theta_{nc} \)) below which there is no horizon and above which there are two horizons, as is the present case. The two values qualesce at the critical mass. For still larger \( M \) the horizons move apart until in the limit the inner horizons shrinks to zero and the outer one equals the Schwarzschild horizon \( 2M \). Within the inner horizon the original singularity is replaced by a de Sitter core of constant curvature (for details see [27, 28]) as a result of noncommutative effects. In our work, we will always consider the region near and outside of the outer horizon since it is a closer analogue of the Schwarzschild horizon.

Similar to normal Schwarzschild, the NC corrected metric (3) has a coordinate singularity at the horizon \( r = r_H \). To remove this, we shall adopt the Painleve
coordinate transformation [31, 32]
\[ dt \rightarrow dt - \frac{1 - f_{nc}(r)}{f_{nc}(r)} \, dr . \] (5)

Implementing this transformation, the metric (3) takes the following form
\[ ds^2 = -f_{nc}(r)dt^2 + 2\sqrt{1 - f_{nc}(r)}dtdr + dr^2 + r^2d\Omega^2 . \] (6)

The above metric has a timelike Killing vector \( \chi^a = (1, 0, 0, 0) \). With the help of this Killing vector \( \chi^a \), we can define the energy of the particle, moving under this background as \( E = -\chi^ap_a = -p_t \) where \( p_a \) is the four momentum vector, i.e., \( p_a = (p_t, p_r, p_\theta, p_\phi) \). In the present context, our aim is to study the particle motion near the horizon. To do that first we need to formulate the expression of energy of the particle in the background (6). With the help of the dispersion relation \( g^{\alpha\beta}p_\alpha p_\beta = -m^2 \), \( m \) being the mass of the particle, we obtain
\[ E^2 + 2\sqrt{1 - f_{nc}(r)}p_tE - \left( f_{nc}(r)p_r^2 + \frac{p_\theta^2}{r^2} \right) = m^2 . \] (7)

We have considered the motion of the particle in the radial, i.e., \( r \), and \( \theta \) directions. Our entire calculation will be done for the case of a massless particle, i.e., \( m = 0 \), and with this, we obtain the two solutions of energy,
\[ E = -\sqrt{1 - f_{nc}(r)}p_t \pm \frac{\sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}}{r} \] (8)

where the positive sign corresponds to the outgoing particle, and the negative sign corresponds to the ingoing one. In the present instance, we shall be considering only the case of outgoing particle having the positive sign solution.

Our next task is to find out the trajectory of the outgoing particle, and it will be computed using Hamilton’s equation of motion. However, before that, let us concentrate for a moment on our particular system. In many contexts, it has been shown that the particle trajectory experiences instability in the near horizon regions [33–35]. In different parameter values this instability leads to chaotic motion of the particle [10, 11]. Keeping in mind that the particle must not fall into the black holes, an external potential (like harmonic potential [10] or any other effective potential [4]) has been applied in order to keep the particle bounded in the near-horizon region. Here also, we shall introduce an external harmonic potential to make sure that the particle must not fall into the black hole. Therefore, introducing the harmonic potentials into the picture, we obtain the total energy of the system as
\[ E = -\sqrt{1 - f_{nc}(r)}p_t + \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}} + \frac{1}{2}K_r(r - r_c)^2 + \frac{1}{2}K_\theta(y - y_c)^2 \] (9)

where \( y = r_H\theta \), \( K_r \) and \( K_\theta \) are spring constants while \( r_c \) and \( y_c \) are the equilibrium position of the two harmonic potentials. Using Hamilton’s equations of motion, we obtain the particle dynamics
\[ \dot{r} = \frac{\partial E}{\partial p_r} = -\sqrt{1 - f_{nc}(r)} + \frac{p_r}{\sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}} \] (10)
\[ \dot{p}_r = -\frac{\partial E}{\partial r} = -\frac{f_{nc}(r)}{2\sqrt{1 - f_{nc}(r)}}p_r + \frac{p_\theta^2/r^3}{\sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}} - K_r(r - r_c) \] (11)
\[ \dot{\theta} = \frac{\partial E}{\partial p_\theta} = \frac{p_\theta/r^2}{\sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}} \] (12)
\[ \dot{p}_\theta = -\frac{\partial E}{\partial \theta} = -K_\theta r_H(y - y_c) \] (13)

where the derivative is taken with respect to some affine parameter. In the next subsection, we shall study these equations with the help of numerical analysis to reveal the characteristics of the particle motion.

B. Numerical analysis for NC-deformed metric

To demonstrate the characteristics of the motion of the particle, we first solve the equations of motion (Eqs. (10)-(13)) numerically. Next, we present the Poincaré sections for our composite system in order to see the change in the dynamics of the particle motion with the variation of different parameters like the noncommutative parameter \( \theta_{nc} \) and \( E \), energy of the system. Although exact form of the NC-deformed metric (3) is valid for all \( r \), the near horizon condition is imposed in the numerical computation where, for our exterior black hole horizon at \( r_H \approx 2 \) (see Fig. 1), the range of \( r \) is fixed at \( 3.5 > r > 3.0 \) (to make sure that the particle resides near the exterior horizon).

1. Poincaré sections for NC-deformed metric

In the following figure (Fig. 2), we show the Poincaré section of the particle trajectory projected over the \((r, p_r)\) plane for different energies but for a constant value of \( \theta_{nc} \). These sections are plotted with the condition \( p_\theta > 0 \) and \( \theta = 0 \). For the present, we have also considered \( M = 1.0 \), \( K_r = 100 \), \( K_\theta = 25 \), \( r_c = 3.2 \) and \( y_c = 0 \) solving the dynamical equations of motion of the particle (Eqs. (10)-(13)). We have considered the energies \( E = 50, 55, 60 \) and 65 as indicated in the plots. Now, looking at these plots, it can be seen that for the lower energy value \( E = 50 \), the Poincaré section exhibits the regular KAM tori, which suggests that our system is still periodic as only a single frequency is present in the system. As the total energy of the system is increased (\( E = 55 \) and 60), the trajectory approaches the horizon, and as a consequence of that, this tori starts getting distorted and finally breaks down, which indicates the appearance of chaos into the system.


Finally, the further increase in the energy ($E = 65$) results in the complete breaking of regular tori and the appearance of the scattered points in the plane. This emergence of the scattered points suggests that our system has reached a completely chaotic situation. Next we plot try to analyze them in the following figures.

FIG. 1: The figures show the variation of $f_{nc}(r)$ with $r$ for different values of $\theta_{nc}$ where the exterior horizon is at $r_H \approx 2$.

FIG. 2: The Poincaré sections for $\theta_{nc} = 0.14$ in the ($r, p_r$) plane with $\theta = 0$ and $p_\theta > 0$ at different energies for quantum corrected Schwarzschild black hole. The horizontal and vertical axis in each of the graphs corresponds to $r$ and $p_r$, respectively.

FIG. 3: The Poincaré sections for $\theta_{nc} = 0.16$ in the ($r, p_r$) plane with $\theta = 0$ and $p_\theta > 0$ at different energies for quantum corrected Schwarzschild black hole. The horizontal and vertical axis in each of the graphs corresponds to $r$ and $p_r$, respectively.

FIG. 4: The Poincaré sections for $\theta_{nc} = 0.18$ in the ($r, p_r$) plane with $\theta = 0$ and $p_\theta > 0$ at different energies for quantum corrected Schwarzschild black hole. The horizontal and vertical axis in each of the graphs corresponds to $r$ and $p_r$, respectively.
In order to see the effect of \( \theta_{nc} \) on the particle dynamics, we have gradually increased the value of \( \theta_{nc} \) in the above figures (Fig: 3, Fig: 4 and Fig: 5). Interestingly, we notice that with the increase of the value of \( \theta_{nc} \), the chaotic fluctuations start to appear in the lower energy ranges. This means that the increment in the value of \( \theta_{nc} \) induces more chaos into the system, which is evident from the plots. The tori get more deformed for higher values of \( \theta_{nc} \) in the lower energy regime.

An important point to be noted here is that there is an upper bound on \( \theta_{nc} \) for a given value of black hole mass \( M \). According to the metric structure (3), there is a limiting value in \( M = M_0 \) and its relation with \( \theta_{nc} \) is \( M_0 \approx 1.90 \sqrt{\theta_{nc0}} \) where \( M_0 \) corresponds to that mass limit value below which there will be no horizon [28]. In our case, we have considered \( M = 1.0 \). Therefore, there is a limiting value of \( \theta_{nc} = \theta_{nc0} \) also and in our case the limiting value is \( \theta_{nc0} \sim 0.27 \). That means the value of \( \theta_{nc} \) cannot be increased beyond 0.27.

**C. Approximate form of NC-deformed metric**

At large distance, for \( \frac{x^2}{4\theta_{nc}} \gg 1 \) the behavior of incomplete \( \gamma \) function is given by

\[
\gamma \left( \frac{3}{2}, \frac{r^2}{4\theta_{nc}} \right) \approx \frac{\sqrt{\pi}}{2} + \frac{1}{2} \frac{r}{\sqrt{\theta_{nc}}} e^{-\frac{r^2}{\theta_{nc}}}.
\]

Now, incorporating this into the exact metric (3) we obtain an analytic form [28]

\[
ds^2 = - \left( 1 - \frac{2M}{r} - \frac{2M}{\sqrt{\pi \theta_{nc}}} e^{-\frac{r^2}{\theta_{nc}}} \right) dt^2 + \left( 1 - \frac{2M}{r} - \frac{2M}{\sqrt{\pi \theta_{nc}}} e^{-\frac{r^2}{\theta_{nc}}} \right)^{-1} dr^2 + r^2 d\Omega^2.
\]

In order to see the validity of the approximate form of the metric (15), one needs to see whether the physical phenomena in both cases remain the same or not. First of all, in the following plots (Fig. 6), we show that if we consider the approximated form of the metric (Eq. 15), the position of the outer horizon does not change with the changing value of \( \theta_{nc} \) for some constant value of \( M \) (\( M = 1.0 \) in this case). However, as we mentioned before, there is a limiting value of \( M = M_0 = 1.90 \sqrt{\theta_{nc0}} \), therefore in this case also we cannot increase the value of \( \theta_{nc} \) beyond the allowed range. The limiting value of \( \theta_{nc0} \) is \( \sim 0.27 \) as we mentioned earlier due to our consideration of the value of the mass \( M = 1.0 \) in this case.

1. **Poincaré sections for approximate NC-deformed metric**

Let us concentrate on studying the particle dynamics in the near horizon region and let us find out whether this approximated metric leads us to the same physical phenomenology or not. In order to investigate that, we plot the following Poincaré sections following the same procedure as before but this time for the approximated metric (15). We have plotted these Poincaré sections for different values of energy of our system consisting of the massless test particle. Fig. (7) and Fig. (8) are plotted for some constant values of energy of our system consisting of the massless test particle. Fig. (7) and Fig. (8) are plotted for some constant values of energy of our system consisting of the massless test particle.
The appearance of chaos is evident whether we increase the value of the energy of the system \((E)\) or the value of \(\theta_{nc}\) and that we have learned from the study of the

2. **Lyapunov exponent for approximated NC-deformed metric**

Poincare sections in the previous section. Now, this gradual emergence of chaotic behavior in the system can be shown more unambiguously in a quantifying manner with the help of studying the largest Lyapunov exponent of the system. We adopted the standard algorithm to compute the largest Lyapunov exponent which is related to the rate of separation of the trajectories for two nearby points [36]. If we consider two trajectories with initial separation \(\delta x_0\), the rate of divergence within the linearized approximation is given by

\[
|\delta x(t)| \approx e^{\lambda_L t} |\delta x_0|
\]

\[(16)\]

where \(\lambda_L\) is the Lyapunov exponent. Here we have reproduced two graphs of the Lyapunov exponents (Fig. (9) and Fig. (10)). In order to plot these graphs, first we numerically solve the equations of motion of the particle (Eq. (10) - Eq. (13)) for \(r\) with two initial conditions which are initially separated infinitesimally \((\delta r_0)\). Then, the separation of these two trajectories has been studied for a long period of time which essentially gives us the saturated value of the maximum Lyapunov exponent. Here, we have studied the characteristics of the Lyapunov exponent for two cases. In Fig. (9), we have plotted for different values of the system energies but for a constant value of \(\theta_{nc}\) (for \(\theta_{nc} = 0.2\)). Fig. (10) is plotted for different values of \(\theta_{nc}\) but for a particular value of the system energy \(E\) (for \(E = 55\)).

From Fig. (9), we find that with the increase in the value of energy \(E\) for a particular value of \(\theta_{nc} = 0.2\) the value of the largest Lyapunov exponent also increases. At
FIG. 9: Largest Lyapunov exponents for different values of energy of the system $E$ but for a particular value of $\theta_{nc} = 0.2$.

$E = 50$, the value of the largest Lyapunov exponent saturates at a value around 0 which suggests that our system is still periodic. Whereas with the increase in the energy value, i.e., for $E = 55$, 60 and 65, the largest Lyapunov exponent value saturates at the value $\sim 0.015$, $\sim 0.02$ and $\sim 0.04$, respectively. The positive increment in the largest Lyapunov exponent values with the increase in the energy in the system indicates that our system becomes more chaotic with the increase in energy. One important point to be noted here is that the Lyapunov exponent has an upper bound [1] which in this case is $\lambda_{L_{\text{max}}} = \kappa = 1/2 f'(r_H)$, where $\kappa$ is the surface gravity of the black hole. Analysing the plots of the Lyapunov exponents in Fig. 9, we find that the obtained value of the Lyapunov exponents for different values of energy ($E$) of the system are much lower than the corresponding upper bound (in this case the upper bound is $\lambda_{L_{\text{max}}} \approx 0.21$).

Similarly, from Fig. (10), we can see that for a particular value of the energy of the system, i.e., $E = 55$, with the changing value in $\theta_{nc}$ some changes occur into the system. With the increment in the value of $\theta_{nc} = 0.19$, 0.22, 0.25 and 0.27, our system turns into a more chaotic one which suggests that the increased value of this parameter $\theta_{nc}$ inducing more chaos into the system. In addition, in this case we also found out that the acquired value of the Lyapunov exponents for different values of $\theta_{nc}$ are lower than the upper bounds discovered in [1] ($\lambda_{L_{\text{max}}} \approx 0.22$, 0.20, 0.18 and 0.16 for $\theta_{nc} = 0.19$, 0.22, 0.25 and 0.27, respectively). Notice that the upper bound depends on $\theta_{nc}$.

FIG. 10: Largest Lyapunov exponents for different values of $\theta_{nc}$ but for a particular value of the system energy $E = 55$. 

(a) $\theta_{nc} = 0.19$

(b) $\theta_{nc} = 0.22$

(c) $\theta_{nc} = 0.25$

(d) $\theta_{nc} = 0.27$
where \( \epsilon(r) \) is the quantum correction term given by
\[
\epsilon(r) = -6 \left( G^2 M^2 / c^4 r^2 \right) \left( 1 + (m/M) \right) - (41/5\pi)(GM/c^2)(l_p^2/r^3),
\]
where \( l_p \) is the Planck length and \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \). \( M \) represents the mass of the black hole and \( m \) stands for the particle mass. However, in our analysis our main focus is on the dynamics of the massless particle, so, \( m = 0 \) in our case.

Due to smallness, the last term has not been taken into account in our analysis and for massless particle quantum corrected metric takes the form

\[
ds^2 = - \left( 1 - \frac{2GM}{c^2 r} - \frac{6k^2}{r^2} \right) dt^2 + \left( 1 - \frac{2GM}{c^2 r} - \frac{6k^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2
\]  

where, \( k = GM/c^2 \) and \( k \) is taken to be unity for further analysis according to the unit convention (\( G = c = \hbar = 1 \)) that we have been following so far and for the mass of the black hole, i.e., \( M = 1.0 \). Next, we shall study the dynamics of massless particle near the horizon of the quantum corrected Schwarzschild metric (Eq. 18). Considering all these facts we obtain the position of the horizon at \( r_H \approx 3.6 \). In this case we shall also follow the same formalism as we performed in the previous case of NC-deformed Schwarzschild metric in Section II. Solving the equations of motion of the particle numerically, we shall plot the Poincaré sections and Lyapunov exponents for the particle motion in the near horizon region and then we shall try to analyze both of them.

### A. Poincare sections for quantum corrected metric

In this case, the Poincare sections (Fig. 11) are also plotted for \( p_\theta > 0 \) and \( \theta = 0 \) just like before. We have considered \( M = 1.0, K_r = 100, K_\theta = 25, r_c = 4.5 \) and \( y_c = 0 \). In order to make sure that the particle resides near the horizon we have restricted the range of \( r \) at \( 4.5 < r < 5.5 \). Similar nature can be found in these plots also. As we increase the value of the energy of the system \( E = 55.0, 60, 65, 70, 72 \) the system gradually goes from the periodic state to the chaotic one. The appearance of the scattered points in the higher energy ranges \( (E = 70, 72) \) are indication of the chaotic fluctuations into the system. With the increase in the energy of the system, our massless test particle reaches nearer to the horizon and the more our system comes under the influence of horizon the more it becomes chaotic. Some more examples of Poincare sections for different initial conditions for this case are provided in the Appendix.

FIG. 11: The Poincaré sections in the \((r, p_r)\) plane with \( \theta = 0 \) and \( p_\theta > 0 \) at different energies for quantum corrected Schwarzschild black hole. The horizontal and vertical axis in each of the graphs corresponds to \( r \) and \( p_r \), respectively.
B. Lyapunov exponents for quantum corrected metric

Here, we also quantify the chaos in our system. From Fig. (12) we can see that with the increase in the value of the energy of the system, the largest Lyapunov value of the system gets increased. As we increase the energy, the particle moves closer towards the horizon, subsequently influence of the horizon increases making the system more chaotic. Moreover, it is worth to mention that the obtained value of the Lyapunov exponents for different values of energy of the system are lower than the upper bound [1].

IV. CONCLUSION

Let us summarize the results obtained in the present work. The chaotic behavior of particle dynamics near Schwarzschild black hole has been under study for quite some time. In recent times, this research has received new impetus from deep results a very different area; existence of a universal upper bound of Lyapunov exponent for a finite temperature quantum field theory. These two very distinct phenomena are intimately connected by the fact that black holes can be treated as thermodynamic systems with a Hawking temperature. Another area of recent interest in black hole physics with quantum corrections incorporated in the metric, arising from possible quantum gravity effects via noncommutative geometry or generated through conventional quantum field theoretic effects. In the present work we have analysed in detail the effects of these metric extensions on the chaotic behavior of particles close to the black hole horizon.

Our results clearly show that in both cases, the metric extensions favour chaotic behavior that is chaos is attained for relatively smaller particle energy. This is demonstrated numerically by exhibiting the breaking of the KAM tori in Poincare sections of particle trajectories and also via explicit computation of the (positive) Lyapunov exponents of the trajectories. Also of interest is that in all the cases considered here, numerical value of the Lyapunov exponent is well within the universal saturation value.

Appendix A: Poincaré sections for quantum corrected Schwarzschild metric

The following figures represent the Poincaré sections for the particle motion near the QC Schwarzschild horizon. Here the dynamical equations of motion of the particle has been solved for different initial conditions of $r$ and $p_r$. Different colours in the figures represent different initial conditions. However, the initial conditions of $r$ are restricted in between $4.5 < r < 5.5$. The scattered points in the Poincaré sections for the higher energy values show the chaos emerges with the increase in the value of the energy of the system ($E$).
FIG. 13: The Poincaré sections in the \((r, p_r)\) plane with \(\theta = 0\) and \(p_\theta > 0\) at different energies for quantum corrected Schwarzschild black hole. The horizontal and vertical axis in each of the graphs corresponds to \(r\) and \(p_r\), respectively. Different coloured lines indicate the different initial conditions.

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