Weak Hopf Algebras and Singular Solutions of Quantum Yang-Baxter Equation

Fang Li and Steven Duplij

Abstract. We investigate a generalization of Hopf algebra \( \mathfrak{sl}_q(2) \) by weakening the invertibility of the generator \( K \), i.e. exchanging its invertibility \( KK^{-1} = 1 \) to the regularity \( KK = K \). This leads to a weak Hopf algebra \( w\mathfrak{sl}_q(2) \) and a \( J \)-weak Hopf algebra \( v\mathfrak{sl}_q(2) \) which are studied in detail. It is shown that the monoids of group-like elements of \( w\mathfrak{sl}_q(2) \) and \( v\mathfrak{sl}_q(2) \) are regular monoids, which supports the general conjecture on the connection between weak Hopf algebras and regular monoids. Moreover, from \( w\mathfrak{sl}_q(2) \) a quasi-braided weak Hopf algebra \( \mathcal{U}_w \) is constructed and it is shown that the corresponding quasi-\( R \)-matrix is regular \( R^w \hat{R}^w R^w = R^w \).

1. Introduction

The concept of a weak Hopf algebra as a generalization of a Hopf algebra \([21, 1]\) was introduced in \([14]\) and its characterizations and applications were studied in \([16]\). A \( k \)-bialgebra \( H = (H, \mu, \eta, \Delta, \varepsilon) \) is called a weak Hopf algebra if there exists \( T \in \text{Hom}_k(H, H) \) such that \( \text{id} \ast T \ast \text{id} = \text{id} \) and \( T \ast \text{id} \ast T = T \) where \( T \) is called a weak antipode of \( H \). This concept also generalizes the notion of the left and right Hopf algebras \([18, 9]\).

The first aim of this concept is to give a new sub-class of bialgebras which includes all of Hopf algebras such that it is possible to characterize this sub-class through their monoids of all group-like elements \([14, 16]\). It was known that for every regular monoid \( S \), its semigroup algebra \( kS \) over \( k \) is a weak Hopf algebra as the generalization of a group algebra \([13]\).

The second aim is to construct some singular solutions of the quantum Yang-Baxter equation (QYBE) and research QYBE in a larger scope. On this hand, in \([16]\) a quantum quasi-double \( D(H) \) for a finite dimensional cocommutative perfect weak Hopf algebra with invertible weak antipode was built and it was verified that its quasi-\( R \)-matrix is a regular solution of the QYBE. In particular, the quantum quasi-double of a finite Clifford monoid as a generalization of the quantum double of a finite group was derived \([16]\).

In this paper, we will construct two weak Hopf algebras in the other direction as a generalization of the quantum algebra \( \mathfrak{sl}_q(2) \) \([14, 16]\). We show that \( w\mathfrak{sl}_q(2) \) possesses a quasi-\( R \)-matrix which becomes a singular (in fact, regular) solution of the QYBE, with a parameter \( q \). In this reason, we want to treat the meaning of \( w\mathfrak{sl}_q(2) \) and its quasi-\( R \)-matrix just as \( \mathfrak{sl}_q(2) \) \([20, 12]\). It is interesting to note that \( w\mathfrak{sl}_q(2) \) is a natural and non-trivial example of weak Hopf algebras.

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In this paper, \( k \) always denotes a field.
2. Weak Quantum Algebras

For completeness and consistency we remind the definition of the enveloping algebra $U_q = U_q(\mathfrak{sl}_q(2))$ (see e.g. [12]). Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. The algebra $U_q$ is generated by four variables (Chevalley generators) $E, F, K, K^{-1}$ with the relations

\begin{align}
K^{-1}K &= KK^{-1} = 1, \\
KEK^{-1} &= q^2 E, \\
KFK^{-1} &= q^{-2} F, \\
EF - FE &= K - K^{-1} q - q^{-1}.
\end{align}

(1)\hspace{2cm} (2)\hspace{2cm} (3)\hspace{2cm} (4)

Now we try to generalize the invertibility condition (1). The first thought is weaken the invertibility to regularity, as it is usually made in semigroup theory [13] (see also [4, 5, 6] for higher regularity). So we will consider such weakening the algebra $U_q(\mathfrak{sl}_q(2))$, in which instead of the set $\{K, K^{-1}\}$ we introduce a pair $\{K_w, \overline{K}_w\}$ by means of the regularity relations

\begin{align}
K_w \overline{K}_w K_w &= K_w, \\
\overline{K}_w K_w \overline{K}_w &= \overline{K}_w.
\end{align}

(5)

If $\overline{K}_w$ satisfying (5) is unique for a given $K_w$, then it is called inverse of $K_w$ (see e.g. [19, 8]). The regularity relations (5) imply that one can introduce the variables

\begin{align}
J_w &= K_w \overline{K}_w, \\
J_\bar{w} &= \overline{K}_w K_w.
\end{align}

(6)

In terms of $J_w$ the regularity conditions (5) are

\begin{align}
J_w K_w &= K_w, \\
\overline{K}_w J_w &= \overline{K}_w, \\
K_w J_\bar{w} &= K_w.
\end{align}

(7)\hspace{2cm} (8)

Since the noncommutativity of generators $K_w$ and $\overline{K}_w$ very much complexifies the generalized construction\footnote{This case will be considered elsewhere.} we first consider the commutative case and imply in what follow that

\begin{align}
J_w &= J_\bar{w}.
\end{align}

(9)

Let us list some useful properties of $J_w$ which will be needed below. First we note that commutativity of $K_w$ and $\overline{K}_w$ leads to idempotency condition

\begin{align}
J_w^2 &= J_w.
\end{align}

(10)

which means that $J_w$ is a projector (see e.g. [11]).

Conjecture 1. In algebras satisfying the regularity conditions (5) there exists as minimum one zero divisor $J_w - 1$.

Remark 1. In addition with unity 1 we have an idempotent analog of unity $J_w$ which makes the structure of weak algebras more complicated, but simultaneously more interesting.

For any variable $X$ we will define “$J$-conjugation” as

\begin{align}
X_{J_w} \overset{def}{=} J_w X J_w
\end{align}

(11)
and the corresponding mapping will be written as $e_w(X): X \to X_{J_w}$. Note that the mapping $e_w(X)$ is idempotent

\[(12) \quad e_w^2(X) = e_w(X).\]

**Remark 2.** In the invertible case $K_w = K, \overline{K}_w = K^{-1}$ we have $J_w = 1$ and $e_w(X) = X = \text{id}(X)$ for any $X$, so $e_w = \text{id}$.

It is seen from (4) that the generators $K_w$ and $\overline{K}_w$ are stable under “$J_w$-conjugation”

\[(13) \quad K_{J_w} = J_w K_w J_w = K_w, \quad \overline{K}_{J_w} = J_w \overline{K}_w J_w = \overline{K}_w.\]

Obviously, for any $X$

\[(14) \quad K_w X \overline{K}_w = K_w X_{J_w} \overline{K}_w,\]

and for any $X$ and $Y$

\[(15) \quad K_w X \overline{K}_w = Y \Rightarrow K_w X_{J_w} \overline{K}_w = Y_{J_w},\]

Another definition connected with the idempotent analog of unity $J_w$ is “$J_w$-product” for any two elements $X$ and $Y$, viz.

\[(16) \quad X \circ_{J_w} Y \overset{def}{=} X_{J_w} Y.\]

**Remark 3.** From (5) it follows that “$J_w$-product” coincides with usual product, if $X$ ends with generators $K_w$ and $\overline{K}_w$ on right side or $Y$ starts with them on left side.

Let $J^{(ij)} = K_w^i \overline{K}_w^j$ then we will need a formula

\[(17) \quad J^{(ij)}_w = K_w^i \overline{K}_w^j = \begin{cases} K_w^{i-j} & i > j, \\ J_w & i = j, \\ \overline{K}_w^{i-j} & i < j, \end{cases}\]

which follows from the regularity conditions (5). The variables $J^{(ij)}$ satisfy the regularity conditions

\[(18) \quad J^{(ij)}_w J^{(ji)}_w J^{(ij)}_w = J^{(ij)}_w\]

and stable under “$J$-conjugation” \[(19) \quad J^{(ij)}_w J^{(ji)}_w J^{(ij)}_w = J^{(ij)}_w.\]

The regularity conditions (5) lead to the noncancellativity: for any two elements $X$ and $Y$ the following relations hold valid

\[(19) \quad X = Y \Rightarrow K_w X = K_w Y\]
\[(20) \quad K_w X = K_w Y \not\Rightarrow X = Y\]
\[(21) \quad X = Y \Rightarrow \overline{K}_w X = \overline{K}_w Y\]
\[(22) \quad \overline{K}_w X = \overline{K}_w Y \not\Rightarrow X = Y\]
\[(23) \quad X = Y \Rightarrow X_{J_w} = Y_{J_w},\]
\[(24) \quad X_{J_w} = Y_{J_w} \not\Rightarrow X = Y.\]

The generalization of $U_q(\mathfrak{sl}_2(2))$ by exploiting regularity (5) instead of invertibility (6) can be done in two different ways.
Definition 1. Define $U^w_q = w\mathfrak{sl}_q(2)$ as the algebra generated by the four variables $E_w, F_w, K_w, \overline{K}_w$ with the relations:

\begin{align}
(25) & \quad K_w\overline{K}_w = \overline{K}_wK_w, \\
(26) & \quad K_w\overline{K}_wK_w = K_w, \quad \overline{K}_wK_w\overline{K}_w = \overline{K}_w, \\
(27) & \quad K_wE_w = q^2E_wK_w, \quad \overline{K}_wE_w = q^{-2}E_w\overline{K}_w, \\
(28) & \quad K_wF_w = q^{-2}F_wK_w, \quad \overline{K}_wF_w = q^2F_w\overline{K}_w, \\
(29) & \quad E_wF_w - F_wE_w = k_w - \overline{k}_w, \\
\end{align}

We call $w\mathfrak{sl}_q(2)$ a weak quantum algebra.

Definition 2. Define $U^q_v = v\mathfrak{sl}_q(2)$ as the algebra generated by the four variables $E_v, F_v, K_v, \overline{K}_v$ with the relations ($J_v = K_v\overline{K}_v$):

\begin{align}
(30) & \quad K_v\overline{K}_v = \overline{K}_vK_v, \\
(31) & \quad K_v\overline{K}_vK_v = K_v, \quad \overline{K}_vK_v\overline{K}_v = \overline{K}_v, \\
(32) & \quad K_vE_v\overline{K}_v = q^2E_v, \\
(33) & \quad K_vF_v\overline{K}_v = q^{-2}F_v, \\
(34) & \quad E_vJ_vF_v - F_vE_vJ_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}.
\end{align}

We call $v\mathfrak{sl}_q(2)$ a J-weak quantum algebra.

In these definitions indeed the first two lines (25)–(26) and (30)–(31) are called to generalize the invertibility $KK^{-1} = K^{-1}K = 1$. Each next line (27)–(29) and (32)–(33) generalizes the corresponding line (2)–(4) in two different ways respectively. In the first almost quantum algebra $w\mathfrak{sl}_q(2)$ the last relation (29) between $E$ and $F$ generators remains unchanged from $\mathfrak{sl}_q(2)$, while two $EK$ and $FK$ relations are extended to four ones (27)–(28). In $v\mathfrak{sl}_q(2)$, oppositely, two $EK$ and $FK$ relations remain unchanged from $\mathfrak{sl}_q(2)$ (with $K^{-1} \rightarrow \overline{K}$ substitution only), while the last relation (34) between $E$ and $F$ generators has additional multiplier $J_v$ which role will be clear later. Note that the $EK$ and $FK$ relations (32)–(33) can be written in the following form close to (27)–(28)

\begin{align}
(35) & \quad K_vE_vJ_v = q^2J_vE_vK_v, \quad \overline{K}_vE_vJ_v = q^{-2}J_vE_v\overline{K}_v, \\
(36) & \quad K_vF_vJ_v = q^{-2}J_vF_vK_v, \quad \overline{K}_vF_vJ_v = q^2J_vF_v\overline{K}_v.
\end{align}

Using (10) and (11) in the case of $J_v$ we can also present the $v\mathfrak{sl}_q(2)$ algebra as an algebra with “$J_v$-product”

\begin{align}
(37) & \quad K_v \circ J_v \overline{K}_v = \overline{K}_v \circ J_v K_v, \\
(38) & \quad K_v \circ J_v \overline{K}_v \circ J_v K_v = K_v, \quad \overline{K}_v \circ J_v K_v \circ J_v \overline{K}_v = \overline{K}_v, \\
(39) & \quad K_v \circ J_v E_v \circ J_v \overline{K}_v = q^2E_v, \\
(40) & \quad K_v \circ J_v F_v \circ J_v \overline{K}_v = q^{-2}F_v, \\
(41) & \quad E_v \circ J_v F_v - F_v \circ J_v E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}.
\end{align}
Remark 4. Due to [7], the only relation where “$J_w$-product” is really plays its role is the last relation [41].

From the following proposition, one can find the connection between $U^w_q = w\mathfrak{sl}_q(2)$, $U^v_q = \mathfrak{sl}_q(2)$ and the quantum algebra $\mathfrak{sl}_q(2)$.

**Proposition 2.** $\mathfrak{sl}_q(2)/(J_w - 1) \cong \mathfrak{sl}_q(2)$; $\mathfrak{sl}_q(2)/(J_v - 1) \cong \mathfrak{sl}_q(2)$.

Proof. For cancellative $K_w$ and $K_v$ it is obvious.

**Proposition 3.** Quantum algebras $\mathfrak{wsl}_q(2)$ and $\mathfrak{vsl}_q(2)$ possess zero divisors, one of which is $\mathfrak{i} (J_{w,v} - 1)$ which annihilates all generators.

Proof. From regularity (26) and (31) it follows $K_{w,v}(J_{w,v} - 1) = 0$ (see also [1]). Multiplying (27) on $J_w$ gives $K_w E_w J_w = q^2 E_w K_w J_w \Rightarrow K_w (E_w K_w) K_w = q^2 E_w K_w$. Using second equation in (27) for term in bracket we obtain $K_w (q^2 E_w K_w) K_w = q^2 E_w K_w \Rightarrow (J_w - 1) E_w K_w = 0$. For $F_w$ similarly, but using equation (28). By analogy, multiplying (32) on $J_v$, we have $K_v E_v K_v F_v = q^2 E_v J_v \Rightarrow K_v E_v F_v = q^2 E_v J_v \Rightarrow q^2 E_v = q^2 E_v J_v$, and so $E_v (J_v - 1) = 0$. For $F_v$ similarly, but using equation (33).$$

Remark 5. Since $\mathfrak{sl}_q(2)$ is an algebra without zero divisors, some properties of $\mathfrak{sl}_q(2)$ cannot be upgraded to $\mathfrak{wsl}_q(2)$ and $\mathfrak{vsl}_q(2)$, e.g. the standard theorem of Ore extensions and its proof (see Theorem 1.7.1 in [12]).

Remark 6. We conjecture that in $U^w_q$ and $U^v_q$ there are no other than $(J_{w,v} - 1)$ zero divisors which annihilate all generators. In other case thorough analysis of them will be much more complicated and very different from the standard case of non-weak algebras.

We can get some properties of $U^w_q$ and $U^v_q$ as follows.

**Lemma 4.** The idempotent $J_w$ is in the center of $\mathfrak{wsl}_q(2)$.

Proof. For $K_w$ it follows from (33). Multiplying first equation in (27) on $K_w$ we derive $K_w (E_w K_w) = q^2 E_w J_w$, and the applying second equation in (27) obtain $E_w J_w = J_w E_w$. For $F_w$ similarly, but using equation (28).$$

**Lemma 5.** There are unique algebra automorphism $\omega_w$ and $\omega_v$ of $U^w_q$ and $U^v_q$ respectively such that

\begin{align}
\omega_w(K_w, v) &= K_w, v, \\
\omega_w(E_w, v) &= F_w, v, \\
\omega_v(K_w, v) &= K_w, v, \\
\omega_v(E_w, v) &= E_w, v.
\end{align}

Proof. The proof is obvious, if we note that $\omega_w^2 = \text{id}$ and $\omega_v^2 = \text{id}$.

As in case of automorphism $\omega$ for $\mathfrak{sl}_q(2)$ [12], the mappings $\omega_w$ and $\omega_v$ can be called the weak Cartan automorphisms.

Remark 7. Note that $\omega_w \neq \omega$ and $\omega_v \neq \omega$ in general case.

The connection between the algebras $\mathfrak{wsl}_q(2)$ and $\mathfrak{vsl}_q(2)$ can be seen from the following

\footnote{We denote by $X_{w,v}$ one of the variables $X_w$ or $X_v$.}
Proposition 6. There exist the following partial algebra morphism \( \chi : \mathfrak{wsl}_q(2) \to \mathfrak{wsl}_q(2) \) such that

\begin{equation}
\chi (X) = e_v (X)
\end{equation}

or more exactly: generators \( X_{w}^{(v)} = J_v X_v J_v = X_{vJ_v} \) for all \( X_v = K_v, K_v, E_v, F_v \) satisfy the same relations as \( X_w \) \( (24) \)–\( (29) \).

Proof. Multiplying the equation \( (32) \) on \( K_v \) we have \( K_v E_v J_v = q^2 E_v J_v K_v \), and using \( (2) \) we obtain \( K_v E_v J_v = q^2 E_v J_v K_v \Rightarrow K_v J_v E_v J_v = q^2 J_v E_v J_v K_v \), and so

\begin{equation}
K_v J_v E_v J_v = q^2 E_v J_v K_v
\end{equation}

which has shape of the first equation in \( (27) \). For \( F_v \) similarly using equation \( (33) \) we obtain

\begin{equation}
K_v J_v F_v J_v = q^2 F_v J_v K_v J_v.
\end{equation}

The equation \( (24) \) can be modified using \( (2) \) and then applying \( (11) \), then we obtain

\begin{equation}
E_v J_v F_v J_v - F_v J_v E_v J_v = \frac{K_v J_v - K_v J_v}{q - q^{-1}}
\end{equation}

which coincides with \( (29) \).

For conjugated equations (second ones in \( (27) \)–\( (28) \)) after multiplication of \( (32) \) on \( K_v \) we have \( K_v K_v E_v K_v = q^2 K_v E_v K_v \Rightarrow J_v E_v J_v K_v = q^2 K_v J_v E_v J_v \) or using definition \( (11) \) and \( (2) \)

\begin{equation}
K_v J_v E_v J_v = q^2 E_v J_v K_v J_v.
\end{equation}

By analogy from \( (33) \) it follows

\begin{equation}
K_v J_v F_v J_v = q^2 F_v J_v K_v J_v.
\end{equation}

Note that the generators \( X_{w}^{(v)} \) coincide with \( X_w \) if \( J_v = 1 \) only. Therefore, some (but not all) properties of \( \mathfrak{wsl}_q(2) \) can be extended on \( \mathfrak{wsl}_q(2) \) as well, and below we mostly will consider \( \mathfrak{wsl}_q(2) \) in detail.

Lemma 7. Let \( m \geq 0 \) and \( n \in \mathbb{Z} \). The following relations hold in \( U_q^{(w)} \):

\begin{equation}
E_w^m K_w^n = q^{-2mn} K_w^m E_w^n, \quad F_w^m K_w^n = q^{2mn} K_w^m F_w^n.
\end{equation}

\begin{equation}
E_w^m K_w^n = q^{2mn} K_w^m E_w^n, \quad F_w^m K_w^n = q^{-2mn} K_w^m F_w^n.
\end{equation}

\begin{equation}
[E_w, F_w^m] = [m] E_w^{m-1} \frac{q^{-m-1} K_w - q^{m-1} K_w}{q - q^{-1}}
\end{equation}

\begin{equation}
= [m] q^{-m-1} K_w - q^{m-1} K_w E_w^{m-1},
\end{equation}

\begin{equation}
[E_w^m, F_w] = [m] E_w^{m-1} \frac{q^{-m-1} K_w - q^{m-1} K_w}{q - q^{-1}}
\end{equation}

\begin{equation}
= [m] q^{-m-1} K_w - q^{m-1} K_w E_w^{m-1}.
\end{equation}
Proof. The first two relations can be resulted easily from Definition 1. The third one follows by induction using Definition 1 and

\[ [E_w, F_w^n] = [E_w, F_w^{m-1}] F_w + F_w^{m-1} E_w, \]
\[ = [E_w, F_w^{m-1}] F_w + F_w^{m-1} E_w. \]

Applying the automorphism \( \omega_w \) to (46), one gets (47).

Note that the commutation relations (44)–(47) coincide with \( \mathfrak{sl}_q(2) \) case. For \( \mathfrak{vs}(2) \) the situation is more complicated, because the equations (42)–(46) cannot be solved under \( K_v \) due to noncancellativity (see also (19)–(24)). Nevertheless, some analogous relations can be derived. Using the morphism (43) one can conclude that the similar as (44)–(47) relations hold for \( X_v^{(e)} = J_v X_v J_v \), from which we obtain for \( \mathfrak{vs}(2) \)

\[ J_v E_w^n K_v = q^{-2mn} K_v E_w^n J_v, \]
\[ J_v E_w^n K_v = q^{2mn} K_v E_w^n J_v, \]
\[ J_v E_v J_v F_w^n J_v - J_v F_v E_v J_v J_v = [m] J_v F_v^{m-1} q^{-(m-1)K_v} - q^{m-1K_v} J_v, \]
\[ = [m] J_v F_v^{m-1} q^{-(m-1)K_v} - q^{m-1K_v} J_v, \]
\[ J_v E_v J_v F_v J_v - J_v F_v E_v J_v J_v = [m] J_v F_v^{m-1} q^{-(m-1)K_v} - q^{m-1K_v} J_v, \]
\[ = [m] J_v F_v^{m-1} q^{-(m-1)K_v} - q^{m-1K_v} J_v. \]

It is important to stress that due to noncancellativity of weak algebras we cannot cancel these relations on \( J_v \) (see (19)–(24)).

In order to discuss the basis of \( U_q^w = \mathfrak{vs}(2) \), we need to generalize some properties of Ore extensions (see 13).

3. Weak Ore extensions

Let \( R \) be an algebra over \( k \) and \( R[t] \) be the free left \( R \)-module consisting of all polynomials of the form \( P = \sum a_i t^i \) with coefficients in \( R \). If \( a_0 \neq 0 \), define \( \deg(P) = n \); say \( \deg(0) = -\infty \).

Let \( \alpha \) be an algebra morphism of \( R \). An \( \alpha \)-derivation of \( R \) is a \( k \)-linear endomorphism \( \delta \) of \( R \) such that \( \delta(ab) = \alpha(a) \delta(b) + \delta(a)b \) for all \( a, b \in R \). It follows that \( \delta(1) = 0 \).

Theorem 8. (i) Assume that \( R[t] \) has an algebra structure such that the natural inclusion of \( R \) into \( R[t] \) is a morphism of algebras and \( \deg(PQ) \leq \deg(P) + \deg(Q) \) for any pair \( P, Q \) of elements of \( R[t] \). Then there exists a unique injective algebra endomorphism \( \alpha \) of \( R \) and a unique \( \alpha \)-derivation \( \delta \) of \( R \) such that \( \alpha(a) = \alpha(a) t + \delta(a) \) for all \( a \in R \).

(ii) Conversely, given an algebra endomorphism \( \alpha \) of \( R \) and an \( \alpha \)-derivation \( \delta \) of \( R \), there exists a unique algebra structure on \( R[t] \) such that the inclusion of \( R \) into \( R[t] \) is an algebra morphism and \( \alpha(a) t + \delta(a) \) for all \( a \in R \).

Proof. (i) Take any \( 0 \neq a \in R \) and consider the product \( ta \). We have \( \deg(ta) \leq \deg(t) + \deg(a) = 1 \). By the definition of \( R[t] \), there exists uniquely determined elements \( \alpha(a) \) and \( \delta(a) \) of \( R \) such that \( \alpha(a) t + \delta(a) \). This defines maps \( \alpha \) and
proof. Let $\delta$ in a unique fashion. The left multiplication by $t$ being linear, so are $\alpha$ and $\delta$. Expanding both sides of the equality $(ta)b = t(ab)$ in $R[t]$ using $ta = \alpha(a)t + \delta(a)$ for $a, b \in R$, we get

$$\alpha(a)\alpha(b)t + \alpha(a)\delta(b) + \delta(a)b = \alpha(ab)t + \delta(ab).$$

It follows that $\alpha(ab) = \alpha(a)\alpha(b)$ and $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$. And, $\alpha(1)t + \delta(1) = t_1 + t$. So, $\alpha(1) = 1$, $\delta(1) = 0$. Therefore, we know that $\alpha$ is an algebra endomorphism and $\delta$ is an $\alpha$-derivation. The uniqueness of $\alpha$ and $\delta$ follows from the freeness of $R[t]$ over $R$.

(ii) We need to construct the multiplication on $R[t]$ as an extension of that on $R$ such that $ta = \alpha(a)t + \delta(a)$. For this, it needs only to determine the multiplication $ta$ for any $a \in R$.

Let $M = \{(f_{ij})_{i,j \geq 1} : f_{ij} \in \text{End}_k(R)\}$ and each row and each column has only finitely many $f_{ij} \neq 0$ and $I = \begin{pmatrix} 1 \\ \vdots \end{pmatrix}$ is the identity of $M$.

For $a \in R$, let $\hat{\alpha} : R \rightarrow R$ satisfying $\hat{\alpha}(r) = ar$. Then $\hat{\alpha} \in \text{End}_k(R)$; and for $r \in R$, $(\alpha\hat{\alpha})(r) = \alpha(ar) = \alpha(a)\alpha(r) = \alpha(a)\alpha(r)$, $\hat{\delta}(r) = \delta(ar) = \delta(a)r + \delta(a)r = (\alpha\delta + \delta(a))(r)$, thus $\hat{\alpha} = \alpha(a)\alpha, \hat{\delta} = \alpha(a)\delta + \delta(a)$ in $\text{End}_k(R)$. And, obviously, for $a, b \in R$, $ab = \hat{ab}; a + b = \hat{a} + \hat{b}$.

Let $T = \begin{pmatrix} \delta & \alpha & \delta & \cdots \\ \alpha & \delta & \alpha & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in M$ and define $\Phi : R[t] \rightarrow M$ satisfying $\Phi(\sum_{i=0}^{n} a_{i}t^{i}) = \sum_{i=0}^{n}(\hat{a}_{i}I)T^{i}$. It is seen that $\Phi$ is a $k$-linear map.

**Lemma 9.** The map $\Phi$ is injective.

**Proof.** Let $p = \sum_{i=0}^{n} a_{i}t^{i}$. Assume $\Phi(p) = 0$.

$$e_{i} = \begin{pmatrix} 0_{1} \\ \vdots \\ 0_{i-1} \\ 1_{i} \\ 0_{i+1} \\ \vdots \\ 0_{n} \end{pmatrix},$$

For $e_{i}$, obviously, $\{e_{i}\}_{i \geq 1}$ are linear independent. Since $\delta(1) = 0$ and $\alpha(1) = 1$, we have $Te_{i} = \begin{pmatrix} 0_{1} \\ \vdots \\ 0_{i-1} \\ \delta(1)_{i} \\ \alpha(1)_{i+1} \\ 0_{i+2} \\ \vdots \\ 0_{n} \end{pmatrix} = e_{i+1}$ and $T^{i}e_{1} = e_{i+1}$ for any $i \geq 0$. 


Thus, \(0 = \Phi(P)e_1 = \sum_{i=0}^{n}(\hat{a}_i I)T_e e_1 = \sum_{i=0}^{n}\hat{a}_i e_{i+1}\). It means that \(\hat{a}_i = 0\) for all \(i\), then \(a_i = a_{i+1} = \hat{a}_i = 0\). Hence \(P = 0\). □

**Lemma 10.** The following relation holds \(T(\hat{a}I) = (\alpha(\hat{a})I)T + \hat{\delta(a)}I\).

**Proof.** We have \(T(\hat{a}I) = \left(\begin{array}{c}
\delta \\
\alpha \\
\delta \\
\alpha \\
\vdots \\
\hat{a} \\
\hat{\alpha} \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
\delta \\
\alpha \\
\delta \\
\alpha \\
\alpha \\
\vdots \\
\alpha
\end{array}\right)\left(\begin{array}{c}
\delta \\
\alpha \\
\alpha \\
\alpha \\
\vdots \\
\hat{a} \\
\hat{\alpha} \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\vdots \\
\alpha
\end{array}\right) = \left(\begin{array}{c}
\alpha(\hat{a})\delta + \hat{\delta(a)} \\
\alpha(a)\alpha \\
\alpha(\hat{a})\delta + \hat{\delta(a)} \\
\alpha(a)\alpha \\
\vdots \\
\hat{\alpha} \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\alpha \\
\vdots \\
\alpha
\end{array}\right) = \alpha(a)T + \hat{\delta(a)}I = (\alpha(\hat{a})I)T + \hat{\delta(a)}I.

Now, we complete the proof of Theorem 8. Let \(S\) denote the subalgebra generated by \(T\) and \(\hat{a}I\) (all \(a \in R\)) in \(M\). From Lemma 10, we see that every element of \(S\) can be generated linearly by some elements in the form as \((\hat{a}I)T^n\) \((a \in R, n \geq 0)\).

But \(\Phi(at^n) = (\hat{a}I)T^n\), so \(\Phi(R[t]) = S\), i.e. \(\Phi\) is surjective. Then by Lemma 8, \(\Phi\) is bijective. It follows that \(R[t]\) and \(S\) are linearly isomorphic.

Define \(t\alpha = \Phi^{-1}(T(\hat{a}I))\), then we can extend this formula to define the multiplication of \(R[t]\) with \(fg = \Phi^{-1}(xy)\) for any \(f, g \in R[t]\) and \(x = \Phi(f), y = \Phi(g)\). Under this definition, \(R[t]\) becomes an algebra and \(\Phi\) is an algebra isomorphism from \(R[t]\) to \(S\). And, \(ta = \Phi^{-1}(T(\hat{a}I)) = \Phi^{-1}((\alpha(\hat{a})I)T + \hat{\delta(a)}I) = \alpha(a)t + \hat{\delta(a)}\) for all \(a \in R\). Obviously, the inclusion of \(R\) into \(R[t]\) is an algebra morphism. □

**Remark 8.** Note that Theorem 8 can be recognized as a generalization of Theorem I.7.1 in [3], since \(R\) does not need to be without zero divisors, \(\alpha\) does not need to be injective and only \(\deg(PQ) \leq \deg(P) + \deg(Q)\).

**Definition 3.** We call the algebra constructed from \(\alpha\) and \(\delta\) a weak Ore extension of \(R\), denoted as \(R_w[t, \alpha, \delta]\).

Let \(S_{n,k}\) be the linear endomorphism of \(R\) defined as the sum of all \(\binom{n}{k}\) possible compositions of \(k\) copies of \(\alpha\) and of \(n-k\) copies of \(\delta\). By induction, from \(ta = \alpha(a)t + \hat{\delta(a)}\) under the condition of Theorem 8(ii), we get \(t^n a = \sum_{k=0}^{n} S_{n,k}(a)t^{n-k}\) and moreover, \(\sum_{i=0}^{m} a_i S_{i} = \sum_{i=0}^{m} c_i t^i\) where \(c_i = \sum_{p=0}^{\alpha} t_p \sum_{k=0}^{n} S_{p,k}(b_{i},p+k+1)\).

**Corollary 11.** Under the condition of Theorem 8(ii), the following statements hold:

(i) As a left \(R\)-module, \(R_w[t, \alpha, \delta]\) is free with basis \(\{t^i\}_{i \geq 0}\);

(ii) If \(\alpha\) is an automorphism, then \(R_w[t, \alpha, \delta]\) is also a right free \(R\)-module with the same basis \(\{t^i\}_{i \geq 0}\).

**Proof.** (i) It follows from the fact that \(R_w[t, \alpha, \delta]\) is just \(R[t]\) as a left \(R\)-module.

(ii) Firstly, we can show that \(R_w[t, \alpha, \delta] = \sum_{i \geq 0} t^i R\), i.e. for any \(p \in R_w[t, \alpha, \delta]\), there are \(a_0, a_1, \ldots, a_n \in \alpha\) such that \(p = \sum_{i=0}^{n} t^i a_i\). Equivalently, we show by induction on \(n\) that for any \(b \in R\), \(bt^n\) can be in the form \(\sum_{i=0}^{n} t^i a_i\) for some \(a_i\).
When $n = 0$, it is obvious. Suppose that for $n \leq k - 1$ the result holds. Consider the case $n = k$. Since $\alpha$ is surjective, there is $a \in R$ such that $b = \alpha^n(a) = S_{n,0}(a)$. But $t^n a = \sum_{k=0}^{n} S_{n,k}(a) t^{n-k}$, we get $bt^n = t^n a - \sum_{k=1}^{n} S_{n,k}(a) t^{n-k} = \sum_{i=0}^{n} t^i a_i$ by the hypothesis of induction for some $a_i$ with $a_n = a$. For any $i$ and $a,b \in R$, $(t^i a) b = t^i (ab)$ since $R[t,\alpha,\delta]$ is an algebra. Then $R_w[t,\alpha,\delta]$ is a right $R$-module.

Suppose $f(t) = t^n a_n + \cdots + t a_1 + A_0 = 0$ for $a_i \in R$ and $a_n \neq 0$. Then $f(t)$ can be written as an element of $R[t]$ by the formula $t^n a = \sum_{k=0}^{n} S_{n,k}(a) t^{n-k}$ whose highest degree term is just that of $t^n a_n = \sum_{k=0}^{n} S_{n,k}(a_n) t^{n-k}$, i.e. $\alpha^n(a_n t^n)$. From (1), we get $\alpha^n(a_n) = 0$. It implies $a_n = 0$. It is a contradiction. Hence $R_w[t,\alpha,\delta]$ is a free right $R$-module.

We will need the following:

**Lemma 12.** Let $R$ be an algebra, $\alpha$ be an algebra automorphism and $\delta$ be an $\alpha$-derivation of $R$. If $R$ is a left (resp. right) Noetherian, then so is the weak Ore extension $R_w[t,\alpha,\delta]$.

The proof can be made as similarly as for Theorem I.8.3 in [2].

**Theorem 13.** The algebra $wsl_2(2)$ is Noetherian with the basis

$$P_w = \{ E_w^i F_w^j K_w^l, E_w^i F_w^j \overline{K}_w^m, E_w^i F_w^j J_w \},$$

where $i,j,l$ are any non-negative integers, $m$ is any positive integer.

Proof. As is well known, the two-variable polynomial algebra $k[K_w,\overline{K}_w]$ is Noetherian (see e.g. [13]). Then $A_0 = k[K_w,\overline{K}_w]/(J_w K_w - K_w, \overline{K}_w J_w - \overline{K}_w)$ is also Noetherian. For any $i,j \geq 0$ and $a,b,c \in k$, if at least one element of $a,b,c$ does not equal 0, $aK_w + b\overline{K}_w + cJ_w$ is not in the ideal $(J_w K_w - K_w, \overline{K}_w J_w - \overline{K}_w)$ of $k[K_w,\overline{K}_w]$. So, in $A_0$, $aK_w + b\overline{K}_w + cJ_w \neq 0$. It follows that $\{ K_w^i, \overline{K}_w^m, J_w : i,j \geq 0 \}$ is a basis of $A_0$.

Let $\alpha_1$ satisfies $\alpha_1(K_w) = q^2 K_w$ and $\alpha_1(\overline{K}_w) = q^2 \overline{K}_w$. Then $\alpha_1$ can be extended to an algebra automorphism on $A_0$ and $A_1 = A_0[F_w,\alpha_1,0]$ is a weak Ore extension of $A_0$ from $\alpha = \alpha_1$ and $\delta = 0$. By Corollary [1], $A_1$ is a free left $A_0$-module with basis $\{ F_w^i J_w \}_{i \geq 0}$. Thus, $A_1$ is a $k$-algebra with basis $\{ K_w^i F_w^j,\overline{K}_w^m F_w^j, J_w F_w^j \}$ and $i,j$ run respectively over all non-negative integers, $m$ runs over all positive integers. But, from the definition of the weak Ore extension, we have $K_w^i F_w^j = q^{-2i} F_w^i K_w^l, \overline{K}_w^m F_w^j = q^{2m} F_w^j \overline{K}_w^m, J_w F_w^j = F_w^j J_w$. Thus, we can conclude that $\{ F_w^i K_w^l, F_w^j \overline{K}_w^m, F_w^j J_w : i,j \geq 0 \}$ is a basis of $A_1$.

Let $\alpha_2$ satisfies $\alpha_2(F_w^i K_w^l) = q^{-2l} F_w^i K_w^l, \alpha_2(F_w^j \overline{K}_w^m) = q^{2m} F_w^j \overline{K}_w^m, \alpha_2(F_w^i J_w) = F_w^i J_w$. Then $\alpha_2$ can be extended to an algebra automorphism on $A_1$. Let $\delta$ satisfies

$$\delta(1) = \delta(K_w) = \delta(\overline{K}_w) = 0,$$

$$\delta(F_w^i K_w^l) = \sum_{i=0}^{j-1} F_w^{j-i} q^{-2l} K_w^l - q^{2l} K_w^l,$$

$$\delta(F_w^i \overline{K}_w^m) = \sum_{i=0}^{j-1} F_w^{j-i} q^{2j} \overline{K}_w^m - q^{-2j} \overline{K}_w^m,$$

$$\delta(F_w^i J_w) = \sum_{i=0}^{j-1} F_w^{j-i} q^{-2l} K_w^l - q^{2l} K_w^l.$$
for $j > 0$ and $l \geq 0$. Then just as in the proof of Lemma VI.1.5 in [12], it can be shown that $\delta$ can be extended to an $\alpha_2$-derivation of $A_1$ such that $A_2 = A_1[E_w, \alpha_2, \delta]$ is a weak Ore extension of $A_1$. Then in $A_2$,

\[ E_wK_w = \alpha_2(K_w)E_w + \delta(K_w) = q^{-2}K_wE_w, \quad E_wK_w = q^2K_wE_w, \]

\[ E_wF_w = \alpha_2(F_w)E_w + \delta(F_w) = F_wE_w + \frac{K_w - \rho K_w}{q - q^{-1}}. \]

From these, we conclude that $A_2 \cong U_w^q$ as algebras. Thus, from Lemma [12] $U_w^q$ is Noetherian. By Corollary [11] $U_q^w$ is free with basis $\{E_w\}_{i \geq 0}$ as a left $A_1$-module. Thus, as a $k$-linear space, $U_w^q$ has the basis $Q_w = \{F_w^i K_w^j, F_w^j K_w^m, F_w^j J_w E_w^l : i, j, l \text{ run over all non-negative integers, } m \text{ runs over all positive integers}\}$. By Lemma 7 any $x \in P_w$ (resp. $Q_w$) can be $k$-linearly generated by some elements of $Q_w$ (resp. $P_w$), and therefore $P_w$ and $Q_w$ generate the same space $U_q^w$. \[ \square \]

The similar theorem can be proved for $\mathfrak{vs}^q(2)$ as well.

**Theorem 14.** The algebra $\mathfrak{vs}^q(2)$ is Noetherian with the basis

\[ P_v = \{J_v E_v^i J_v F_v^j K_v^l, J_v E_v^i J_v F_v^j K_v^m : J_v E_v^i J_v F_v^j J_v \}, \]

where $i, j, l$ are any non-negative integers, $m$ is any positive integer.

**Proof.** The two-variable polynomial algebra $k[K_v, \overline{K_v}]$ is Noetherian (see e.g. [11]). Then $A_0 = k[K_v, \overline{K_v}]/(J_v K_v - K_v J_v, \overline{K_v} J_v - \overline{K_v})$ is also Noetherian. For any $i, j \geq 0$ and $a, b, c \in k$, if at least one element of $a, b, c$ does not equal $0$, $aK_v^i + b\overline{K_v}^j + cJ_v$ is not in the ideal $(J_v K_v - K_v, \overline{K_v} J_v - \overline{K_v})$ of $k[K_v, \overline{K_v}]$. So, in $A_0$, $aK_v^i + b\overline{K_v}^j + cJ_v \neq 0$. It follows that $\{K_v^i, \overline{K_v}^j, J_v : i, j \geq 0\}$ is a basis of $A_0$.

Let $\alpha_1$ satisfies $\alpha_1(K_v) = q^2K_v$ and $\alpha_1(K_v) = q^{-2}\overline{K_v}$. Then $\alpha_1$ can be extended to an algebra automorphism on $A_0$ and $A_1 = A_0[J_v F_v^j J_v, \alpha_1, 0]$ is a weak Ore extension of $A_0$ from $\alpha = \alpha_1$ and $\delta = 0$. By Corollary 7, $A_1$ is a free left $A_0$-module with basis $\{J_v F_v^j J_v \}_{i \geq 0}$. Thus, $A_1$ is a $k$-algebra with basis $\{K_v^i, \overline{K_v}^j, J_v F_v^j J_v, J_v F_v^j \overline{K_v}^m : i, j \text{ and } l \text{ run respectively over all non-negative integers, } m \text{ runs over all positive integers}\}$. From the definition of the weak Ore extension, we have $K_v^i, \overline{K_v}^j, J_v F_v^j J_v = q^{-2l}J_v F_v^j K_v^l, J_v F_v^j \overline{K_v}^m = q^{2m}J_v F_v^j \overline{K_v}^m, J_v F_v^j = F_v^j J_v$.

So, we conclude that $\{F_v^i K_v^j, F_v^j K_v^m, F_v^j J_v, J_v F_v^j J_v : i, j \text{ and } l \text{ run respectively over all non-negative integers, } m \text{ runs over all positive integers}\}$ is a basis of $A_1$.

Let $\alpha_2$ satisfies $\alpha_2(J_v F_v^j K_v^l) = q^{-2l}J_v F_v^j K_v^l$, $\alpha_2(J_v F_v^j \overline{K_v}^m) = q^{2m}J_v F_v^j \overline{K_v}^m$, $\alpha_2(J_v F_v^j J_v) = J_v F_v^j J_v$. Then $\alpha_2$ can be extended to an algebra automorphism on $A_1$. Let $\delta$ satisfies

\[ \delta(1) = \delta(K_v) = \delta(\overline{K_v}) = 0, \]

\[ \delta(J_v F_v^j K_v^l) = \sum_{i=0}^{j-1} J_v F_v^{i-j} q^{2i}K_v - q^{2i}\overline{K_v}^m, \]

\[ \delta(J_v F_v^j \overline{K_v}^m) = \sum_{i=0}^{j-1} J_v F_v^{i-j} q^{2i}K_v - q^{2i}\overline{K_v}^m, \]

\[ \delta(J_v F_v^j J_v) = \sum_{i=0}^{j-1} J_v F_v^{i-j} q^{2i}K_v - q^{2i}\overline{K_v}^m. \]


for \( j > 0 \) and \( l \geq 0 \). Then just as in the proof of Lemma VI.1.5 in \([2]\), it can be shown that \( \delta \) can be extended to an \( \alpha_2 \)-derivation of \( A_1 \) such that \( A_2 = A_1[J_vE_vJ_v, \alpha_2, \delta] \) is a weak Ore extension of \( A_1 \). Then in \( A_2 \),

\[
J_vE_vK_v = \alpha_2(K_v)J_vE_vJ_v + \delta(K_v) = q^{-2}K_vE_vJ_v, \quad J_vE_vJ_vF_v = \alpha_2(F_v)J_vE_vJ_v + \delta(J_vF_vJ_v) = J_vF_vJ_vE_vJ_v + \frac{K_v - K_v}{q - q^{-1}}.
\]

From these, we conclude that \( A_2 \cong U_q^\varphi \) as algebras. Thus, from Lemma \([2]\), \( U_q^\varphi \) is Noetherian. By Corollary \([1]\), \( U_q^\varphi \) is free with basis \( \{ J_vE_vJ_v \}_{j \geq 0} \) as a left \( A_1 \)-module. Thus, as a \( k \)-linear space, \( U_q^\varphi \) has the basis

\[
Q_v = \{ J_vF_v^iF_v^jE_v^mJ_v, J_vF_v^i\overline{K}_vE_v^mJ_v, J_vF_v^iJ_vE_v^mJ_v \},
\]

where \( i, j, l \) run over all non-negative integers, \( m \) runs over all positive integers. By \([8]–[11]\) any \( x \in P_v \) (resp. \( Q_v \)) can be \( k \)-linearly generated by some elements of \( Q_v \) (resp. \( P_v \)), and therefore \( P_v \) and \( Q_v \) generate the same space \( U_q^\varphi \).

### 4. Extension to \( q = 1 \) Case

Let us discuss the relation between \( U_q^w = w\mathfrak{sl}_q(2) \) and \( U(\mathfrak{sl}_q(2)) \). Just like the quantum algebra \( \mathfrak{sl}_q(2) \), we first have to give another presentation for \( U_q^w \).

Let \( q \in \mathbb{C} \) and \( q \neq \pm 1,0 \). Define \( U_q^w \) as the algebra generated by the five variables \( E_w, F_w, K_w, \overline{K}_w, L_w \) with the relations (for \( U_q^w \) the equations \([56] \) and \([57] \) should be exchanged with \([12] \) and \([13] \) respectively):

\[
\begin{align*}
(54) & \quad K_w\overline{K}_w = \overline{K}_wK_w, \\
(55) & \quad K_w\overline{K}_wK_w = K_w, \quad \overline{K}_wK_w\overline{K}_w = \overline{K}_w, \\
(56) & \quad K_wE_w = q^2E_wK_w, \quad K_wE_w = \overline{K}_wE_w = q^{-2}E_wK_w, \\
(57) & \quad K_wF_w = q^{-2}F_wK_w, \quad K_wF_w = \overline{K}_wF_w = q^2F_wK_w, \\
(58) & \quad [L_w, E_w] = q(E_wK_w + \overline{K}_wE_w), \\
(59) & \quad [L_w, F_w] = -q^{-1}(F_wK_w + \overline{K}_wF_w), \\
(60) & \quad E_wF_w - F_wE_w = L_w, \quad (q - q^{-1})L_w = (K_w - \overline{K}_w).
\end{align*}
\]

For \( w\mathfrak{sl}_q(2) \) we can similarly define the algebra \( U_q^w \)

\[
\begin{align*}
(61) & \quad K_vK_w = \overline{K}_vK_w, \\
(62) & \quad K_v\overline{K}_vK_w = K_v, \quad \overline{K}_vK_v\overline{K}_v = \overline{K}_v, \\
(63) & \quad K_vE_v\overline{K}_v = q^2E_v, \\
(64) & \quad K_vF_v\overline{K}_v = q^{-2}F_v, \\
(65) & \quad L_vJ_vE_v - E_vJ_vL_v = q(E_vK_v + \overline{K}_vE_v), \\
(66) & \quad L_vJ_vF_v - F_vJ_vL_v = -q^{-1}(F_vK_v + \overline{K}_vF_v), \\
(67) & \quad E_vJ_vF_v - F_vJ_vE_v = L_v, \quad (q - q^{-1})L_v = (K_v - \overline{K}_v).
\end{align*}
\]

Note that contrary to \( U_q^w \) and \( U_q^\varphi \), the algebras \( U_q^w \) and \( U_q^w \) are defined for all invertible values of the parameter \( q \), in particular for \( q = 1 \).

**Proposition 15.** The algebra \( U_q^w \) is isomorphic to the algebra \( U_q^w \) with \( \varphi_w \) satisfying \( \varphi_w(E_w) = E_w, \varphi_w(F_w) = F_w, \varphi_w(K_w) = K_w, \varphi_w(\overline{K}_w) = \overline{K}_w \).
Proof. The proof is similar to that of Proposition VI.2.1 in [12] for \( \mathfrak{sl}_q(2) \). It suffices to check that \( \varphi_w \) and the map \( \psi_w : U_q(\mathfrak{sl}_q(2)) \to U_q(\mathfrak{sl}_q(2)) \) satisfying \( \psi_w(E_w) = E_w, \psi_w(F_w) = F_w, \psi_w(K_w) = K_w, \psi_w(L_w) = [E_w, F_w] \) are reciprocal algebra morphisms. \( \square \)

On the otherwise, we can give the following relationship between \( U_q(\mathfrak{sl}_q(2)) \) and \( U(\mathfrak{sl}_q(2)) \) whose proof is easy.

Proposition 16. For \( q = 1 \)

(i) the algebra isomorphism \( U(\mathfrak{sl}_q(2)) \cong U_q(\mathfrak{sl}_q(2))/\{K_w - 1\} \) holds;

(ii) there exists an injective algebra morphism \( \pi \) from \( U_q(\mathfrak{sl}_q(2)) \) to \( U(\mathfrak{sl}_q(2))[K_w]/(K^3 - K) \) satisfying \( \pi(E_w) = XK_w, \pi(F_w) = Y, \pi(K_w) = K_w, \pi(L) = HK_w \).

Remark 9. In Proposition 16(ii), \( \pi \) is only injective, but not surjective since \( K^2 \neq 1 \) in \( U(\mathfrak{sl}_q(2))[K]/(K^3 - K) \) and then \( X \) does not lie in the image of \( \pi \).

5. Weak Hopf Algebras Structure

Here we define weak analogs in \( \mathfrak{wsl}_q(2) \) and \( \mathfrak{vsl}_q(2) \) for the standard Hopf algebra structures \( \Delta, \varepsilon, S \) — comultiplication, counit and antipod, which should be algebra morphisms.

For the weak quantum algebra \( \mathfrak{wsl}_q(2) \) we define the maps \( \Delta_w : \mathfrak{wsl}_q(2) \to \mathfrak{wsl}_q(2) \otimes \mathfrak{wsl}_q(2), \varepsilon_w : \mathfrak{wsl}_q(2) \to k \) and \( T_w : \mathfrak{wsl}_q(2) \to \mathfrak{wsl}_q(2) \) satisfying respectively

\[
\begin{align*}
\Delta_w(E_w) &= 1 \otimes E_w + E_w \otimes K_w, \\
\Delta_w(F_w) &= F_w \otimes 1 + \overline{K_w} \otimes F_w, \\
\Delta_w(K_w) &= K_w \otimes K_w, \\
\Delta_w(\overline{K_w}) &= \overline{K_w} \otimes \overline{K_w}, \\
\varepsilon_w(E_w) &= \varepsilon_w(F_w) = 0, \\
\varepsilon_w(K_w) &= \varepsilon_w(\overline{K_w}) = 1, \\
T_w(E_w) &= -\varepsilon_w(\overline{K_w}), \\
T_w(F_w) &= -K_w F_w, \\
T_w(K_w) &= \overline{K_w}, \\
T_w(\overline{K_w}) &= K_w.
\end{align*}
\]

The difference with the standard case (we follow notations of [12]) is in substitution \( K^{-1} \) with \( \overline{K_w} \) and the last line, where instead of antipod \( S \) the weak antipod \( T_w \) is introduced [14].

Proposition 17. The relations (68)–(71) endow \( \mathfrak{wsl}_q(2) \) with a bialgebra structure.

Proof. It can be shown by direct calculation that the following relations hold valid.

\[
\begin{align*}
\Delta_w(K_w) \Delta_w(\overline{K_w}) &= \Delta_w(\overline{K_w}) \Delta_w(K_w), \\
\Delta_w(K_w) \Delta_w(\overline{K_w}) \Delta_w(K_w) &= \Delta_w(K_w), \\
\Delta_w(\overline{K_w}) \Delta_w(K_w) \Delta_w(\overline{K_w}) &= \Delta_w(\overline{K_w}), \\
\Delta_w(K_w) \Delta_w(\overline{K_w}) \Delta_w(K_w) &= q^2 \Delta_w(E_w) \Delta_w(K_w), \\
\Delta_w(\overline{K_w}) \Delta_w(\overline{K_w}) \Delta_w(E_w) &= q^{-2} \Delta_w(E_w) \Delta_w(\overline{K_w}), \\
\Delta_w(K_w) \Delta_w(F_w) &= q^{-2} \Delta_w(F_w) \Delta_w(K_w), \\
\Delta_w(\overline{K_w}) \Delta_w(F_w) &= q^2 \Delta_w(F_w) \Delta_w(\overline{K_w}), \\
\Delta_w(E_w) \Delta_w(F_w) - \Delta_w(F_w) \Delta_w(E_w) &= \frac{(q^{-1} - q) \Delta_w(K_w) - \Delta_w(\overline{K_w})}{q-q^{-1}}.
\end{align*}
\]
(80) \[ \varepsilon_w(K_w)\varepsilon_w(\overline{K}_w) = \varepsilon_w(\overline{K}_w)\varepsilon_w(K_w), \]
(81) \[ \varepsilon_w(K_w)\varepsilon_w(K_w) = \varepsilon_w(K_w), \]
(82) \[ \varepsilon_w(\overline{K}_w)\varepsilon_w(\overline{K}_w) = \varepsilon_w(\overline{K}_w), \]
(83) \[ \varepsilon_w(K_w)\varepsilon_w(E_w) = q^2\varepsilon_w(E_w)\varepsilon_w(K_w), \]
(84) \[ \varepsilon_w(\overline{K}_w)\varepsilon_w(E_w) = q^{-2}\varepsilon_w(E_w)\varepsilon_w(\overline{K}_w), \]
(85) \[ \varepsilon_w(K_w)\varepsilon_w(F_w) = q^{-2}\varepsilon_w(F_w)\varepsilon_w(K_w), \]
(86) \[ \varepsilon_w(\overline{K}_w)\varepsilon_w(F_w) = q^2\varepsilon_w(F_w)\varepsilon_w(\overline{K}_w), \]
(87) \[ \varepsilon_w(E_w)\varepsilon_w(F_w) - \varepsilon_w(F_w)\varepsilon_w(E_w) = \frac{(\varepsilon_w(K_w) - \varepsilon_w(\overline{K}_w))}{(q - q^{-1})}; \]
(88) \[ T_w(\overline{K}_w)T_w(K_w) = T_w(K_w)T_w(\overline{K}_w), \]
(89) \[ T_w(K_w)T_w(\overline{K}_w)T_w(K_w) = T_w(K_w), \]
(90) \[ T_w(\overline{K}_w)T_w(K_w)T_w(\overline{K}_w) = T_w(\overline{K}_w), \]
(91) \[ T_w(E_w)T_w(K_w) = q^2T_w(K_w)T_w(E_w), \]
(92) \[ T_w(E_w)T_w(\overline{K}_w) = q^{-2}T_w(\overline{K}_w)T_w(K_w), \]
(93) \[ T_w(F_w)T_w(K_w) = q^{-2}T_w(K_w)T_w(F_w), \]
(94) \[ T_w(F_w)T_w(\overline{K}_w) = q^2T_w(\overline{K}_w)T_w(F_w), \]
(95) \[ T_w(F_w)T_w(E_w) - T_w(E_w)T_w(F_w) = \frac{(T_w(K_w) - T_w(\overline{K}_w))}{(q - q^{-1})}. \]

Therefore, through the basis in Theorem 13, \(\Delta\) and \(\varepsilon_w\) can be extended to algebra morphisms from \(\mathfrak{wsl}_q(2)\) to \(\mathfrak{wsl}_q(2) \otimes \mathfrak{wsl}_q(2)\) and from \(\mathfrak{wsl}_q(2)\) to \(k\). \(T_w\) can be extended to an anti-algebra morphism from \(\mathfrak{wsl}_q(2)\) to \(\mathfrak{wsl}_q(2)\) respectively.

Using (72)–(77) it can be shown that

(96) \[ (\Delta_w \otimes \text{id})\Delta_w(X) = (\text{id} \otimes \Delta_w)\Delta_w(X), \]
(97) \[ (\varepsilon_w \otimes \text{id})\Delta_w(X) = (\text{id} \otimes \varepsilon_w)\Delta_w(X) = X \]

for any \(X = E_w, F_w, K_w\) or \(\overline{K}_w\). Let \(\mu_w\) and \(\eta_w\) be the product and the unit of \(\mathfrak{wsl}_q(2)\) respectively. Hence \((\mathfrak{wsl}_q(2), \mu_w, \eta_w, \Delta_w, \varepsilon_w)\) becomes into a bialgebra. \(\square\)

Next we introduce the star product in the bialgebra \((\mathfrak{wsl}_q(2), \mu_w, \eta_w, \Delta_w, \varepsilon_w)\) in the similar to the standard way (see e.g. [12]).

(98) \[ (A \ast_w B)(X) = \mu_w [A \otimes \overline{B}] \Delta_w(X). \]

**Proposition 18.** \(T_w\) satisfies the regularity conditions

(99) \[ (\text{id} \ast_w T_w \ast_w \text{id})(X) = X, \]
(100) \[ (T_w \ast_w \text{id} \ast_w T_w)(X) = T_w(X) \]

for any \(X = E_w, F_w, K_w\) or \(\overline{K}_w\). It means that \(T_w\) is a weak antipode
Proof. Follows from (102)−(103) by tedious calculations. For \( X = K_w, K_w \) it is easy, and so we consider \( X = E_w, \) as an example. We have

\[
(id \ast_w T_w \ast_w \text{id})(E_w) = \mu_w [(id \ast_w T_w) \otimes \text{id}] \Delta_w(E_w)
\]

\[
= \mu_w [(id \ast_w T_w) \otimes \text{id}] (1 \otimes E_w + E_w \otimes K_w)
\]

\[
= (id \ast_w T_w) \text{id} (E_w) + (id \ast_w T_w)(E_w) \text{id} (K_w)
\]

\[
= \mu_w [id \otimes T_w] \Delta_w(1) \text{id} (E_w) + \mu_w [id \otimes T_w] \Delta_w(K_w) \text{id} (E_w)
\]

\[
= T_w(1) \text{id} (E_w) + \mu_w [id \otimes T_w](1 \otimes E_w + E_w \otimes K_w) \text{id} (K_w)
\]

\[
= E_w - E_w K_w \cdot K_w + E_w \cdot K_w \cdot K_w = E_w = \text{id} (E_w).
\]

By analogy, for (100) and \( X = E_w \) we obtain

\[
(T_w \ast_w \text{id} \ast_w T_w)(E_w) = \mu_w [(T_w \ast_w \text{id}) \otimes T_w] \Delta_w(E_w)
\]

\[
= \mu_w [(T_w \ast_w \text{id}) \otimes T_w](1 \otimes E_w + E_w \otimes K_w)
\]

\[
= (T_w \ast_w \text{id})T_w(E_w) + (T_w \ast_w \text{id})(E_w)T_w(K_w)
\]

\[
= \mu_w [T_w \otimes \text{id}](1 \otimes 1)T_w(E_w) \text{id} + \mu_w [T_w \otimes \text{id}](1 \otimes E_w + E_w \otimes K_w)T_w(1)
\]

\[
= T_w(1)T_w(E_w) + T_w(1) \text{id}(E_w)T_w(K_w) + T_w(E_w) \text{id}(K_w)T_w(K_w)
\]

\[
= -E_w K_w - E_w K_w - E_w K_w K_w = -E_w K_w = T_w(E_w).
\]

Corollary 19. The bialgebra \( \mathfrak{wsl}_q(2) \) is a weak Hopf algebra with the weak antipode \( T_w. \)

We can get an inner endomorphism as follows.

Proposition 20. \( T_w^2 \) is an inner endomorphism of the algebra \( \mathfrak{wsl}_q(2) \) satisfying for any \( X \in \mathfrak{wsl}_q(2) \)

\[
T_w^2(X) = K_w X K_w.
\]

especially

\[
T_w^2(K_w) = \text{id}(K_w), \quad T_w^2(K_w) = \text{id}(K_w).
\]

Proof. Follows from (71).

Assume that with the operations \( \mu_w, \eta_w, \Delta_w, \varepsilon_w \) the algebra \( \mathfrak{wsl}_q(2) \) would possess an antipode \( S \) so as to become a Hopf algebra, which should satisfy \( (S \ast_w \text{id})(K_w) = \eta_w \varepsilon_w(K_w) \), and so it should follow that \( S(K_w)K_w = 1 \). But, it is not possible to hold since \( S(K_w) \) can be written as a linearly sum of the basis in Theorem 13. It implies that \( \mathfrak{wsl}_q(2) \) is impossible to become a Hopf algebra about the operations above.

Corollary 21. \( \mathfrak{wsl}_q(2) \) is an example for a non-commutative and non-cocommutative weak Hopf algebra which is not a Hopf algebra.

In order to become \( U_q^w \) into a weak Hopf algebra, it is enough to define \( \Delta_w(E_w), \Delta_w(F_w), \Delta_w(K_w), \Delta_w(K_w), \Delta_w(K_w), \varepsilon_w(E_w), \varepsilon_w(F_w), \varepsilon_w(K_w), \varepsilon_w(K_w), \varepsilon_w(K_w), T_w(E_w), T_w(F_w), T_w(K_w), T_w(K_w) \) just as in \( \mathfrak{wsl}_q(2) \) and define

\[
\Delta_w(L_w) = \frac{1}{q - q^{-1}}(K_w \otimes K_w - K_w \otimes K_w), \quad \varepsilon_w(L_w) = 0, \quad T_w(L_w) = \frac{K_w - K_w}{q - q^{-1}}.
\]
From Proposition 22, we conclude that \( \text{vs} \mathfrak{l}_q(2) \) is isomorphic to the algebra \( U_q^{\text{wt}}(2) \) with \( \varphi_w \). Moreover, one can see easily that \( \varphi_w \) is an isomorphism of weak Hopf algebras from \( \text{vs} \mathfrak{l}_q(2) \) to \( U_q^{\text{wt}}(2) \).

For \( J \)-weak quantum algebra \( \text{vs} \mathfrak{l}_q(2) \) we suppose that some additional \( J_v \) should appear even in the definition of comultiplication and antipod. A thorough analysis gives the following nontrivial definitions:

\[
\begin{align*}
(103) & \quad \Delta_v(E_v) = J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v, \\
(104) & \quad \Delta_v(F_v) = J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v, \\
(105) & \quad \Delta_v(K_v) = K_v \otimes K_v, \quad \Delta_v(\overline{K}_v) = \overline{K}_v \otimes \overline{K}_v, \\
(106) & \quad \varepsilon_v(E_v) = \varepsilon_v(F_v) = 0, \quad \varepsilon_v(K_v) = \varepsilon_v(\overline{K}_v) = 1, \\
(107) & \quad T_v(E_v) = -J_v E_v \overline{K}_v, \quad T_v(F_v) = -K_v F_v J_v, \\
(108) & \quad T_v(K_v) = \overline{K}_v, \quad T_v(\overline{K}_v) = K_v.
\end{align*}
\]

Note that from (105) it follows that

\[
\Delta_v(J_v) = J_v \otimes J_v,
\]

and so \( J_v \) is a group-like element.

**Proposition 22.** The relations (103)–(108) endow \( \text{vs} \mathfrak{l}_q(2) \) with a bialgebra structure.

**Proof.** First we should prove that \( \Delta_v \) defines a morphism of algebras from \( \text{vs} \mathfrak{l}_q(2) \otimes \text{vs} \mathfrak{l}_q(2) \) into \( \text{vs} \mathfrak{l}_q(2) \). We check that

\[
\begin{align*}
(110) & \quad \Delta_v(K_v) \Delta_v(\overline{K}_v) = \Delta_v(\overline{K}_v) \Delta_v(K_v), \\
(111) & \quad \Delta_v(K_v) \Delta_v(\overline{K}_v) \Delta_v(K_v) = \Delta_v(K_v), \\
(112) & \quad \Delta_v(\overline{K}_v) \Delta_v(K_v) \Delta_v(\overline{K}_v) = \Delta_v(\overline{K}_v), \\
(113) & \quad \Delta_v(K_v) \Delta_v(E_v) \Delta_v(\overline{K}_v) = q^2 \Delta_v(E_v), \\
(114) & \quad \Delta_v(K_v) \Delta_v(F_v) \Delta_v(\overline{K}_v) = q^{-2} \Delta_v(F_v), \\
(115) & \quad \Delta_v(E_v) \Delta_v(J_v) \Delta_v(F_v) - \Delta_v(F_v) \Delta_v(J_v) \Delta_v(E_v) = \frac{\Delta_v(K_v) - \Delta_v(\overline{K}_v)}{q - q^{-1}}.
\end{align*}
\]

The relations (110)–(112) are clear from (103). For (113) we have

\[
\begin{align*}
\Delta_v(K_v) \Delta_v(E_v) \Delta_v(\overline{K}_v) & = (K_v \otimes K_v)(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)(\overline{K}_v \otimes \overline{K}_v) \\
& = J_v \otimes K_v E_v \overline{K}_v + K_v E_v \overline{K}_v \otimes K_v \\
& = q^2 (J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v) = q^2 \Delta_v(E_v).
\end{align*}
\]
Moreover, it can be shown that
\[ \Delta_v (E_v) \Delta_v (F_v) - \Delta_v (F_v) \Delta_v (E_v) \]
\[ = (J_v \otimes J_v E_v J_v + J_v E_v J_v) \otimes (J_v \otimes J_v E_v J_v + \bar{K}_v \otimes \bar{K}_v) \]
\[ = J_v F_v E_v J_v - J_v F_v E_v J_v \]
\[ = J_v (E_v J_v F_v - F_v J_v E_v) J_v \otimes \bar{K}_v \]
\[ = \epsilon_v (K_v) \epsilon_v (\bar{K}_v) = \epsilon_v (K_v) \]
\[ \epsilon_v (K_v) \epsilon_v (\bar{K}_v) = \epsilon_v (K_v) \]
\[ \epsilon_v (\bar{K}_v) \epsilon_v (K_v) = \epsilon_v (\bar{K}_v) \]
\[ \epsilon_v (K_v) \epsilon_v (E_v) \epsilon_v (\bar{K}_v) = q^2 \epsilon_v (E_v) \]
\[ \epsilon_v (F_v) \epsilon_v (\bar{K}_v) = q^{-2} \epsilon_v (F_v) \]
\[ \epsilon_v (E_v) \epsilon_v (J_v) \epsilon_v (F_v) - \epsilon_v (F_v) \epsilon_v (J_v) \epsilon_v (E_v) = \frac{\epsilon_v (K_v) - \epsilon_v (\bar{K}_v)}{q - q^{-1}}. \]

Moreover, it can be shown that
\[ (\epsilon_v \otimes \text{id}) \Delta_v (X) = (\text{id} \otimes \epsilon_v) \Delta_v (X) = X \]
for \( X = E_v, F_v, K_v, \bar{K}_v \).
Further we check that $T_v$ defines an anti-morphism of algebras from $\mathfrak{sl}_q(2)$ to $\mathfrak{vsf}_q(2)$ as follows

\begin{align}
T_v (K_v) T_v (\overline{K}_v) &= T_v (\overline{K}_v) T_v (K_v), \\
T_v (K_v) T_v (\overline{K}_v) T_v (K_v) &= T_v (K_v), \\
T_v (\overline{K}_v) T_v (K_v) T_v (\overline{K}_v) &= T_v (\overline{K}_v), \\
T_v (\overline{K}_v) T_v (E_v) T_v (K_v) &= q^2 T_v (E_v), \\
T_v (\overline{K}_v) T_v (F_v) T_v (K_v) &= q^{-2} T_v (F_v), \\
T_v (F_v) T_v (J_v) T_v (E_v) - T_v (E_v) T_v (J_v) T_v (F_v) &= T_v (K_v) - T_v (\overline{K}_v),
\end{align}

The first three relations are obvious. For (126) using (107) and (35)–(36) we have

\begin{align*}
T_v (\overline{K}_v) T_v (E_v) T_v (K_v) &= K_v (-J_v E_v \overline{K}_v) \overline{K}_v = -q^2 K_v (-\overline{K}_v E_v J_v) \overline{K}_v \\
&= -q^2 J_v E_v J_v \overline{K}_v = q^2 J_v E_v \overline{K}_v = q^2 T_v (E_v).
\end{align*}

For last relation (128) using (45)–(46) we obtain

\begin{align*}
T_v (F_v) T_v (J_v) T_v (E_v) - T_v (E_v) T_v (J_v) T_v (F_v) &= (K_v F_v J_v) J_v (-J_v E_v \overline{K}_v) - (-J_v E_v \overline{K}_v) J_v (K_v F_v J_v) \\
&= J_v (F_v J_v E_v - E_v J_v F_v) J_v = J_v \overline{K}_v - K_v \overline{K}_v \frac{J_v}{q - q^{-1}} J_v = \frac{T_v (K_v)}{q - q^{-1}} - T_v (\overline{K}_v).
\end{align*}

Therefore, we conclude that $(\mathfrak{vsf}_q(2), \mu_v, \eta_v, \Delta_v, T_v)$ has a structure of a bialgebra.

The following property of $T_v$ is crucial for understanding the structure of the bialgebra $(\mathfrak{vsf}_q(2), \mu_v, \eta_v, \Delta_v, T_v)$.

**Proposition 23.** For any $X \in \mathfrak{sl}_q(2)$ we have (cf. [101]–[103])

\begin{align}
T_v^2 (K_v) &= e_v (K_v), \\
T_v^2 (\overline{K}_v) &= e_v (\overline{K}_v), \\
T_v^2 (E_v) &= K_v E_v \overline{K}_v, \\
T_v^2 (F_v) &= K_v F_v \overline{K}_v,
\end{align}

where $e_v (X)$ is defined in (47).

**Proof.** Follows from [5] and [107]–[108]. As an example for $E_v$ we have $T_v^2 (E_v) = T_v (-J_v E_v \overline{K}_v) = -T_v (\overline{K}_v) T_v (E_v) T_v (J_v) = K_v (J_v E_v \overline{K}_v) J_v = K_v E_v \overline{K}_v$.

The star product in $(\mathfrak{vsf}_q(2), \mu_v, \eta_v, \Delta_v, T_v)$ has the form

\begin{align}
(A \ast_v B) (X) &= \mu_v [A \otimes B] \Delta_v (X).
\end{align}

**Proposition 24.** $T_v$ satisfies the regularity conditions

\begin{align}
(e_v \ast_v T_v \ast_v e_v) (X) &= e_v (X), \\
(T_v \ast_v e_v \ast_v T_v) (X) &= T_v (X)
\end{align}

for any $X = E_v, F_v, K_v$ or $\overline{K}_v$. 
Proof. Follows from (103)–(108) and (131). For \( X = K_v \), it is easy, and so we consider \( X = E_v \), as an example. We have
\[
\begin{align*}
(e_v \ast_v T_v \ast_v e_v)(E_v) &= \mu_v \left[ (e_v \ast_v T_v) \otimes e_v \right] \Delta_v(E_v) \\
&= \mu_v \left[ (e_v \ast_v T_v) \otimes e_v \right] (J_v \otimes J_v E_v J_v + J_v \otimes E_v J_v \otimes K_v) \\
&= (e_v \ast_v T_v) (J_v) e_v (J_v E_v J_v) + (e_v \ast_v T_v) (J_v E_v J_v) e_v (K_v) \\
&= \mu_v [e_v \otimes T_v] \Delta_v(J_v) e_v (J_v E_v J_v) + \mu_v [e_v \otimes T_v] \Delta_v(E_v) e_v (K_v) \\
&= e_v (J_v) T_v (J_v) e_v (E_v) + \mu_v [e_v \otimes T_v] (J_v \otimes J_v E_v J_v + J_v \otimes E_v J_v \otimes K_v) e_v (K_v) \\
&= J_v \cdot J_v \cdot J_v E_v J_v - J_v \cdot J_v \cdot J_v E_v K_v \cdot J_v E_v J_v + J_v E_v J_v \cdot K_v = J_v E_v J_v = e_v (E_v).
\end{align*}
\]

By analogy, for (133) and \( X = E_v \), we obtain
\[
\begin{align*}
(T_v \ast_v e_v \ast_v T_v)(E_v) &= \mu_v \left[ (T_v \ast_v e_v) \otimes T_v \right] \Delta_v(E_v) \\
&= \mu_v (T_v \ast_v e_v) \otimes T_v (J_v \otimes J_v E_v J_v + J_v \otimes E_v J_v \otimes K_v) \\
&= (T_v \ast_v e_v) (J_v) T_v (J_v E_v J_v) + (T_v \ast_v e_v) (E_v) T_v (K_v) \\
&= \mu_v T_v \otimes e_v (J_v \otimes J_v E_v J_v + J_v \otimes E_v J_v \otimes K_v) T_v (K_v) \\
&= T_v (J_v E_v J_v) e_v (J_v) T_v (J_v E_v J_v) + T_v (J_v) e_v (J_v E_v J_v) T_v (K_v) \\
&= T_v (J_v E_v J_v) e_v (K_v) T_v (K_v) - J_v \cdot J_v \cdot J_v (J_v E_v K_v) J_v + J_v \cdot J_v E_v J_v \cdot K_v = -J_v E_v K_v = T_v (E_v).
\end{align*}
\]

\( \square \)

From (132)–(133) it follows that \( \mathfrak{wsl}_q(2) \) is not a weak Hopf algebra in the definition of [14]. So we will call it \( J \)-weak Hopf algebra and \( T_r \) a \( J \)-weak antipode. As it is seen from (19)–(20) and (132)–(133) the difference between them is in the exchange id with \( e_v \).

Remark 10. The variable \( e_v \) can be treated as \( n = 2 \) example of the “tower identity” \( e^{(n)}_v \) introduced for semisupermanifolds in [3, 4] or the “obstructor” \( e^{(n)}_X \) for general mappings, categories and Yang-Baxter equation in [3, 4, 5].

Comparing (158)–(171) with (103)–(108) we conclude that the connection of \( \Delta_w, T_w, \varepsilon_w \) and \( \Delta_v, T_v, \varepsilon_v \) can be written in the following way
\[
\begin{align*}
\Delta_v(X) &= \Delta_w(e_v(X)), \\
T_v(X) &= T_w(e_v(X)), \\
\varepsilon_v(X) &= \varepsilon_w(e_v(X)),
\end{align*}
\]
which means that additionally to the partially algebra morphism \( (43) \) there exists a partial coalgebra morphism which is described by (134)–(136).

6. Group-like Elements

Now, we discuss the set \( G(\mathfrak{wsl}_q(2)) \) of all group-like elements of \( \mathfrak{wsl}_q(2) \). As is well-known (see e.g. [10]) a semigroup \( S \) is called an inverse semigroup if for every \( x \in S \), there exists a unique \( y \in S \) such that \( x y x = x \) and \( y x y = y \), and a monoid is a semigroup with identity. We will show the following
Proposition 25. The set of all group-like elements \( G(\mathfrak{vsl}_q(2)) = \{ J^{(ij)} = K^i_w K^j_w : i, j \text{ run over all non-negative integers} \} \), which forms a regular monoid under the multiplication of \( \mathfrak{vsl}_q(2) \).

Proof. Suppose \( x \in \mathfrak{vsl}_q(2) \) is a group-like element, i.e. \( \Delta_w(x) = x \otimes x \). By Theorem 13, \( x \) can be written as \( x = \sum_{i,j,l,m} \alpha_{ijl} \beta^{i,j,m} (\beta^{i,j,m} E_w F_w K_w^l + \beta^{i,j,m} E_w F_w K_w^m) \). Here and in the sequel, every \( \alpha \) and \( \beta \) with subscripts is in the field \( K \) and does not equal zero. Then

\[
\Delta_w(x) = \sum_{i,j,l,m} \alpha_{ijl} \beta^{i,j,m} (\beta^{i,j,m} E_w F_w K_w^l + \beta^{i,j,m} E_w F_w K_w^m) \]

Moreover, it is easy to see that \( \alpha \) and \( \beta \) with subscripts is in the field \( K \) and does not equal zero. Then

\[
\Delta_w(x) = \sum_{i,j,l,m} \alpha_{ijl} \beta^{i,j,m} (\beta^{i,j,m} E_w F_w K_w^l + \beta^{i,j,m} E_w F_w K_w^m) \]

It is seen that if \( i \neq 0 \) or \( j \neq 0 \), \( \Delta_w(x) \) is impossible to equal \( x \otimes x \). So, \( i = 0 \) and \( j = 0 \). We get \( x = \sum_{l,m} \alpha_l K_w^l + \beta_m K_w^m + J_w \). Then

\[
\Delta_w(x) = \sum_{l,m} \alpha_l K_w^l \otimes K_w^l + \beta_m K_w^m \otimes K_w^m \]

If there exists \( l \neq l' \), then \( x \otimes x \) possesses the monomial \( K_w^l \otimes K_w^{l'} \), which does not appear in \( \Delta_w(x) \). It contradicts to \( \Delta_w(x) = x \otimes x \). Hence we have only a unique \( l \). Similarly, there exists a unique \( m \). Thus \( x = \alpha_l K_w^l + \beta_m K_w^m + J_w \). Moreover, it is easy to see that \( \alpha_l K_w^l \), \( \beta_m K_w^m \) and \( J_w \) cannot appear simultaneously in the expression of \( x \). Therefore, we conclude that \( x = \alpha_l K_w^l \), \( \beta_m K_w^m \) or \( J_w \) (no summation) and we have

(137)

\[
\Delta_w(J_w^{(ij)}) = J_w^{(ij)} \otimes J_w^{(ij)}.
\]

It follows that \( G(\mathfrak{vsl}_q(2)) = \{ J_w^{(ij)} = K_w^i K_w^j : i, j \text{ run over all non-negative integers} \} \).

For any \( J_w^{(ij)} = K_w^i K_w^j \in G(\mathfrak{vsl}_q(2)) \), one can find \( J_w^{(ij)} = K_w^i K_w^j \in G(\mathfrak{vsl}_q(2)) \) such that the regularity \( \{138\} \) takes place \( J_w^{(ij)} J_w^{(ij)} J_w^{(ij)} = J_w^{(ij)} \), which means that \( G(\mathfrak{vsl}_q(2)) \) forms a regular monoid under the multiplication of \( \mathfrak{vsl}_q(2) \).

For \( \mathfrak{vsl}_q(2) \) we have a similar statement.
Proposition 26. The set of all group-like elements $G(\mathfrak{sl}_2(2)) = \{J_{i,j}^{(ij)} = K_v^{ij}T_v^{ij} : i, j \text{ run over all non-negative integers}\}$, which forms a regular monoid under the multiplication of $\mathfrak{sl}_2(2)$.

Proof. Suppose $x \in \mathfrak{sl}_2(2)$ is a group-like element, i.e. $\Delta_v(x) = x \otimes x$. By Theorem 24, $x$ can be written as $x = \sum_{i,j,l,m} \alpha_{ijl}J_v E_v^i J_v F_v^j K_v^{ij} + \beta_{ijlm}J_v E_v^i J_v F_v^j \overline{K}_v^{im} + \gamma_{ijlm}J_v E_v^i J_v F_v^j J_v$. Here and in the sequel, every $\alpha$, $\beta$ and $\gamma$ with subscripts is in the $\mathfrak{sl}_2(2)$ is a group-like element, i.e. $\Delta_v(x) = x \otimes x$. So, $i = 0$ and $j = 0$. We get $x = \sum_{l,m} \alpha_{l}K_v^l + \beta_{l} \overline{K}_v^m + J_v$. Then

$$\Delta_v(x) = \sum_{l,m} [\alpha_{l}K_v^l \otimes K_v^l + \beta_{l} \overline{K}_v^m \otimes \overline{K}_v^m + J_v \otimes J_v];$$

and

$$x \otimes x = \sum_{l',m',m''} \alpha_{l'}K_v^{l'} \otimes K_v^{l'} + \alpha_{l'} \beta_{m'} K_v^{l'} \otimes \overline{K}_v^{m''} + \alpha_{l'} K_v^{l'} \otimes J_v + \alpha_{l'} \beta_{m'} K_v^{l'} \otimes \overline{K}_v^{m''} + \beta_{m'} K_v^{l'} \otimes J_v + \alpha_{l'} J_v \otimes K_v^{l'} + \alpha_{l'} \beta_{m'} J_v \otimes \overline{K}_v^{m''} + \beta_{m'} J_v \otimes J_v].$$

If there exists $l \neq l'$, then $x \otimes x$ possesses the monomial $K_v^{l} \otimes K_v^{l'}$, which does not appear in $\Delta_v(x)$ (it contradicts to $\Delta_v(x) = x \otimes x$). Hence we have only a unique $l$. Similarly, there exists a unique $m$. Thus $x = \alpha_{l}K_v^l + \beta_{l} \overline{K}_v^m + J_v$. Moreover, it is easy to see that $\alpha_{l}K_v^l$, $\beta_{l} \overline{K}_v^m$ and $J_v$ can not appear simultaneously in the expression of $x$. Therefore, we conclude that $x = \alpha_{l}K_v^l + \beta_{l} \overline{K}_v^m$ or $J_v$ (no summation) and we have

$$\Delta_v(J_v^{(ij)}) = J_v^{(ij)} \otimes J_v^{(ij)}.$$ 

It follows that $G(\mathfrak{sl}_2(2)) = \{J_v^{(ij)} = K_v^{ij}T_v^{ij} : i, j \text{ run over all non-negative integers}\}$. 


For any \( J_{i}^{(j)} = K^{-1}_{i,j} \in G(\mathfrak{sl}_q(2)) \), one can find \( J_{i}^{(j)} = K^{-1}_{i,j} \in G(\mathfrak{sl}_q(2)) \) such that the regularity \( \{ \mathfrak{sl}_q(2) \} \) takes place \( J_{i}^{(j)} J_{j}^{(i)} = J_{i}^{(j)} \), which means that \( G(\mathfrak{sl}_q(2)) \) forms a regular monoid under the multiplication of \( \mathfrak{sl}_q(2) \).

These results show that \( \mathfrak{wsl}_q(2) \) and \( \mathfrak{vsl}_q(2) \) are examples of a weak Hopf algebra whose monoid of all group-like elements is a regular monoid. It incarnates further the corresponding relationship between weak Hopf algebras and regular monoids.

7. Regular Quasi-\( R \)-Matrix

From Proposition \( 2 \) we have seen that \( \mathfrak{wsl}_q(2)/(J_w - 1) = \mathfrak{sl}_q(2) \). Now, we give another relationship between \( \mathfrak{wsl}_q(2) \) and \( \mathfrak{sl}_q(2) \) so as to construct a non-invertible universal \( R^w \)-matrix from \( \mathfrak{wsl}_q(2) \).

**Theorem 27.** \( \mathfrak{wsl}_q(2) \) possesses an ideal \( W \) and a sub-algebra \( Y \) satisfying \( \mathfrak{wsl}_q(2) = Y \oplus W \) and \( W \cong \mathfrak{sl}_q(2) \) as Hopf algebras.

**Proof.** Let \( W \) be the linear sub-space generated by \( \{ E^i_w, F^l_w, K^m_w, E^i_w F^l_w, F^l_w E^i_w, J_w : \text{for all } i \geq 0, j \geq 0, l > 0 \text{ and } m > 0 \} \), and \( Y \) is the linear sub-space generated by \( \{ E^i_w, F^l_w : \text{for all } i \geq 0, l > 0 \} \). It is easy to see that \( \mathfrak{wsl}_q(2) = Y \oplus W \); \( \mathfrak{wsl}_q(2) W \mathfrak{vsl}_q(2) \subseteq W \), thus, \( W \) is an ideal; and, \( Y \) is a sub-algebra of \( \mathfrak{wsl}_q(2) \).

Note that the identity of \( W \) is \( J_w \). Moreover, \( W \) is a Hopf algebra with the unit \( J_w \), the comultiplication \( \Delta_w \) satisfying

\[
\Delta_w(E_w) = E_w \otimes E_w + E_w \otimes K_w,
\]

\[
\Delta_w(F_w) = F_w \otimes J_w + K_w \otimes F_w,
\]

and the same counit, multiplication and antipode as in \( \mathfrak{wsl}_q(2) \). Let \( \rho \) be the algebra morphism from \( \mathfrak{sl}_q(2) \) to \( W \) satisfying \( \rho(E) = E_w, \rho(F) = F_w, \rho(K) = K_w \). Then \( \rho \) is, in fact, a Hopf algebra isomorphism since \( \{ E^i_w, F^l_w, K^m_w, E^i_w F^l_w, F^l_w E^i_w, J_w : \text{for all } i \geq 0, j \geq 0, l > 0 \text{ and } m > 0 \} \) is a basis of \( Y \) by Theorem \( 3 \).

Let us assume here that \( q \) is a root of unity of order \( d \) in the field \( k \) where \( d \) is an odd integer and \( d > 1 \).

Set \( I = (E^d_w, F^d_w, K^d_w - J_w) \) the two-sided ideal of \( U_q^w \) generated by \( E^d_w, F^d_w, K^d_w - J_w \). Define the algebra \( U_q^w/I \).

**Remark 11.** Note that \( K^d_w = J_w \) in \( U_q^w/I = U_q^w/I \) since \( K^d_w = J_w \).

It is easy to prove that \( I \) is also a coideal of \( U_q^w \) and \( T_w(I) \subseteq I \). Then \( I \) is a weak Hopf ideal. It follows that \( U_q^w \) has a unique weak Hopf algebra structure such that the natural morphism is a weak Hopf algebra morphism, so the comultiplication, the counit and the weak antipode of \( U_q^w \) are determined by the same formulas with \( U_q^w \). We will show that \( U_q^w \) is a quasi-braided weak Hopf algebra. As a generalization of a braided bialgebra and \( R \)-matrix we have the following definitions.

**Definition 4.** Let in a \( k \)-linear space \( H \) there are \( k \)-linear maps \( \mu : H \otimes H \to H, \eta : k \to H, \Delta : H \to H \otimes H, \varepsilon : H \to k \) such that \( (H, \mu, \eta) \) is a \( k \)-algebra and \( (H, \Delta, \varepsilon) \)
is a $k$-coalgebra. We call $H$ an almost bialgebra, if $\Delta$ is a $k$-algebra morphism, i.e. $\Delta(xy) = \Delta(x)\Delta(y)$ for every $x, y \in H$.

**Definition 5.** An almost bialgebra $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called quasi-braided, if there exists an element $R$ of the algebra $H \otimes H$ satisfying

\[
\Delta^{op}(x)R = R\Delta(x)
\]

for all $x \in H$ and

\[
(\Delta \otimes \text{id}_H)(R) = R_{13}R_{23},
\]

\[
(\text{id}_H \otimes \Delta)(R) = R_{13}R_{12}.
\]

Such $R$ is called a quasi-$R$-matrix.

By Theorem 27, we have

\[
U^w_q = U^w_q/I = Y/I \oplus W/I \cong Y/(E^d_w, F^d_w) \oplus \tilde{U}_q \text{ where } \tilde{U}_q = \mathfrak{sl}_q(2)/(E^d_w, F^d_w, K^d - 1) \text{ is a finite Hopf algebra. We know in [12] that the sub-algebra } \tilde{B}_q \text{ of } \tilde{U}_q \text{ generated by } \{E^m_wK^n_w : 0 \leq m, n \leq d - 1\} \text{ is a finite dimensional Hopf sub-algebra and } \tilde{U}_q \text{ is a braided Hopf algebra as a quotient of the quantum double of } \tilde{B}_q. \text{ The } R\text{-matrix of } \tilde{U}_q \text{ is}
\]

\[
\tilde{R} = \frac{1}{d} \sum_{0 \leq k \leq d - 1; 1 \leq i,j \leq d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k_wK^k_i \otimes F^k_wK^k_j.
\]

Since $\mathfrak{sl}_q(2) \cong W$ was Hopf algebras and $(E^d_w, F^d_w, K^d - 1) \cong I$, we get $\tilde{U}_q \cong W/I$ as Hopf algebras under the induced morphism of $\rho$. Then $W/I$ is a braided Hopf algebra with a $R$-matrix

\[
R^w = \frac{1}{d} \sum_{0 \leq k \leq d - 1; 1 \leq i,j \leq d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k_wK^k_i \otimes F^k_wK^k_j.
\]

Because the identity of $W/I$ is $J_w$, there exists the inverse $\hat{R}^w$ of $R^w$ such that

\[
\hat{R}^wR^w = R^w, \quad R^w\hat{R}^w = J_w.
\]

Then we have

\[
R^w\hat{R}^wR^w = R^w, \quad \hat{R}^wR^w = \hat{R}^w,
\]

which shows that this $R$-matrix is regular in $U^w_q$. It obeys the following relations

\[
\Delta^{op}_w(x)R^w = R^w\Delta_w(x)
\]

for any $x \in W/I$ and

\[
(\Delta_w \otimes \text{id})(R^w) = R^w_{13}R^w_{23},
\]

\[
(\text{id} \otimes \Delta_w)(R^w) = R^w_{13}R^w_{12},
\]

which are also satisfied in $U^w_q$. Therefore $R^w$ is a von Neumann’s regular quasi-$R$-matrix of $U^w_q$. So, we get the following
Theorem 28. $U_q$ is a quasi-braided weak Hopf algebra with

$$R_w = \frac{1}{d} \sum_{0 \leq k \leq d-1; 1 \leq i,j \leq d} \frac{(q - q^{-1})^k}{k!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^i K_w^j \otimes F_w^i K_w^j$$

as its quasi-$R$-matrix, which is regular.

The quasi-$R$-matrix from $J$-weak Hopf algebra $\mathfrak{sl}_q(2)$ has more complicated structure and will be considered elsewhere.

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Department of Mathematics, Zhejiang University (Xixi Campus), Hangzhou, Zhejiang 310028, China
E-mail address: fangli@mail.hz.zj.cn

Kharkov National University, Kharkov 61077, Ukraine
E-mail address: Steven.A.Duplij@univer.kharkov.ua
URL: http://gluon.physik.uni-kl.de/~duplij