Coin tossing is a cryptographic task in which two parties who do not trust each other aim to generate a common random bit. Using classical communication this is impossible, but non trivial coin tossing is possible using quantum communication. Here we consider the case when the parties do not want to toss a single coin, but many. This is called bit string generation. We report the experimental generation of strings of coins which are provably more random than achievable using classical communication. The experiment is based on the “plug and play” scheme developed for quantum cryptography, and therefore well suited for long distance quantum communication.

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cheating. However the corresponding classical bound is not known, although it is conjectured in [12] to be of the form \( H_A + H_B \leq n + o(n) \). We refer to [12] for a more detailed discussion of security conditions and for formal definitions of \( H_{A(B)} \).

The protocol we shall use, inspired by that of [10, 12] is as follows. Choose a security parameter \( 0 < \kappa < 1 \). The detailed discussion of security conditions and for formal definitions of \( H_{A(B)} \).

1. For \( i = 1 \) to \( n \).

2. Alice chooses a random bit \( a_i \). If \( a_i = 0 \), she prepares a coherent state of the electromagnetic field with amplitude \( \alpha: \psi_0 = |\alpha\rangle \). If \( a_i = 1 \), she prepares a coherent state with amplitude \( -\alpha: \psi_1 = |-\alpha\rangle \). She sends the coherent state \( \psi_{a_i} \) to Bob. After receiving the quantum state from Alice, Bob chooses a random bit \( b_i \). Bob tells Alice the value of \( b_i \).

3. After learning the value of \( b_i \), Alice reveals the value of \( a_i \) to Bob.

4. Bob now verifies whether the state Alice sent him is indeed the coherent state \( |(-1)^i\alpha\rangle \). He does this by using a Local Oscillator (LO) to carry out the displacement \( D((-1)^i\alpha) \). If Alice was honest, the displaced state should be the vacuum state. Bob checks that this is the case by sending the state onto a single photon detector. If the detector clicks, Bob sets \( k_i = 1 \). If the detector does not click, Bob sets \( k_i = 0 \).

5. Next \( i \).

6. If \( \frac{1}{n} \sum k_i > \kappa \), Bob aborts. Otherwise the output of the protocol is the bit string \( x_i = (a_i + b_i) \mod 2 \).

When Bob is dishonest his best strategy is to measure the state sent to him by Alice as soon as he receives it (i.e. before carrying out step 3 above). One easily shows, see [12], that

\[
e_{\theta} \leq \frac{\sin \theta}{2}, \quad \text{where} \quad \cos \theta = |\langle \psi_0 | \psi_1 \rangle| = e^{-2|\alpha|^2}.
\] (3)

If Alice is dishonest she may not send Bob the state \( \psi_{a_i} \), but an arbitrary state \( \rho \). In general she may prepare an entangled state, keeping half of it and sending the other half to Bob. Furthermore, she may correlate and even entangle her strategy over different runs. In [12] however, it is shown that strategies correlated over different runs cannot help Alice for large \( n \). A bound on \( e_{\theta} \) is proven that depends on the average value of the fidelity \( f_i = |\langle \psi_{a_i} | \rho | \psi_{a_i} \rangle| \), as estimated by Bob. Since the probability that Bob’s detector clicks (assuming his detector is perfect) is related to the fidelity by \( P(k_i = 1) \geq 1 - f_i \), the result of [12] then implies that, assuming large \( n \), the bias if Alice is dishonest is bounded by \( e_{\theta} \leq F(\kappa) \), where

\[
F(x) = \frac{\sqrt{x}}{\sqrt{2 \sin^2 \theta}} + \frac{x}{\sin^2 \theta}.
\]

Below we show how this relation must be modified to take into account imperfections in Bob’s measuring apparatus.

Note that due to such imperfections, Bob’s detector may click even if Alice is honest. Alice and Bob should choose \( \kappa \) such that it is larger than the expected number of clicks if both parties are honest. When this is the case, the probability \( \delta_n \) that the protocol aborts if both parties are honest decreases exponentially fast to zero and the protocol is correct.

Our experimental setup, depicted in Fig. 1 is based on the plug and play system developed for long distance QKD [13]. The advantage of the plug and play system is that it constitutes an all-fiber (standard SMF-28), automatically balanced interferometer, and hence is well suited to long distance quantum communication. However plug and play system has a number of specific features which must be carefully taken into account.

**Bob to Alice and Bob’s cheating.** Each round of the protocol begins with Bob producing a short (20ns) intense (25mW) laser pulse at \( \lambda = 1.55 \mu m \). The pulse is split in two by the 50/50 coupler \( C_1 \). The two pulses acquire a relative time delay of 100ns and then impinge with orthogonal polarization on a Polarizing Beam Splitter (PBS) whereupon they are sent to Alice. Between \( C_1 \) and the PBS, along the long path, are an attenuator, a 99/1 coupler \( C_2 \) and a phase modulator. The role of these elements will be explained later. The relative attenuation of the two pulses is \( A \approx 45dB \). The first pulse to reach Alice is intense and contains \( N_0 \approx 10^9 \) photons. This pulse will play the role of LO. The second pulse to reach Alice is attenuated and contains \( A N_0 \) photons. The second pulse will play the role of signal.

Upon receiving the pulses, Alice measures the intensity of the signal pulse (using the 80/20 coupler \( C_3 \) and a classical detector \( D_3 \) and attenuates both pulses. The two pulses are reflected by the Faraday mirror and travel back to Bob. The total attenuation at Alice’s site is \( A' \approx 50dB \). Thus the two pulses now contain \( A' N_0 \) and \( A A' N_0 \) photons respectively. In particular the signal pulse now contains only a few photons \( \langle |a|^2 = O(1) \rangle \). Alice also adds a phase \( \phi_A = a, \pi \) to the signal pulse, thereby encoding the value of her bit \( a_i \).

The fact that Bob provides Alice with the signal state seems to provide him with some simple cheating strategies. For instance he could provide Alice with a signal state that is squeezed in phase in order to decrease the overlap between \( |\psi_0\rangle \) and \( |\psi_1\rangle \). This apparently allows him to discriminate much better \( |\psi_0\rangle \) from \( |\psi_1\rangle \) and hence the value of \( a_i \). The role of the attenuation is to prevent this kind of cheating. Indeed under strong attenuation any quantum state tends towards a mixture of coherent states.

To show this we describe the state by its generalized Wigner function \( W(q,p,s) \). We recall that \( W(s = -1) \) is the \( Q \) function which is always positive, \( W(s = 0) \) is the Wigner function, and \( W(s = +1) \) is the \( P \) function.
If the $P$ function is positive, then the state is a mixture of coherent states. Under attenuation by $A$ we have (see [15]): 

$$W^{\text{out}}(q, p, s) = \frac{1}{\sqrt{A}} W^{\text{in}}\left( \sqrt{\frac{q}{A}}, \sqrt{\frac{p}{A}}, s - \frac{\beta^2 - 2\beta + 1}{A} \right)$$

and in particular if $n = A$ we have $W^{\text{out}}(q, p, s = +1) = \frac{1}{\sqrt{A}} W^{\text{in}}\left( \sqrt{\frac{q}{A}}, \sqrt{\frac{p}{A}}, s = -1 \right)$, i.e. the output $P$ function is positive since it is given in terms of the input $Q$ function. Thus after strong attenuation, say $A = 10^{-3}$, a quantum state is very well approximated by a mixture of coherent states since a very small amount of Gaussian noise with mean number of chaotic photons $n = 10^{-3}$ transforms the state into a mixture of coherent states.

Another simple cheating strategy is for Bob to increase the intensity of the signal state since it is then much easier for him to estimate the phase $\phi_A$. The role of the classical intensity measurement is to ensure that the signal state Alice sends back is not too intense. In fact it is impossible for Bob to exploit the fact that he provides Alice with the light pulse which will become the signal state, since by measuring the intensity of the pulse Bob sends her and then attenuating it, Alice ensures that she sends back to Bob a coherent state of known intensity.

Note that the classical intensity measurement of Alice will be affected by noise because $\Delta N_0$ is close to the sensitivity limit of Alice’s detector. We circumvent this technical problem by letting Alice carry out statistical tests on the $n$ intensity measurements (one for each round of the protocol). More precisely she checks whether the distribution of measured intensities is consistent with the Gaussian distribution she expects from instrumental noise. If it is she has a precise estimate of $|\alpha|^2$, and hence of $\tau_B$ through eq. (3). If it is not she aborts.

From Alice to Bob and Alice’s cheating. Upon receiving the two pulses from Alice, Bob uses coupler $C_2$ to measure the intensity of the LO, attenuates it by $A$, and adds a phase $e^{i\phi_B}$, with $\phi_B = a_i \pi$. Note that by measuring the intensity of the LO state provided by Alice and then attenuating it, Bob ensures that the LO he uses is a coherent state (or a mixture of coherent states) of known intensity $|\beta|^2$. (The argument is exactly the same as that given above in the case of Alice).

Let us consider the two states that interfere at coupler $C_1$. On the one hand there is the LO which as we have just argued is a coherent state of known intensity $|\beta|^2$. On the other hand there is the signal state. The signal state travels through the PBS where it gets attenuated by $A_0$. It then interferes with the LO at coupler $C_1$. This coupler has transmission and reflection coefficients $T$ and $R$ (both are approximately 50%). Finally one of the outputs of the coupler is sent to a single photon detector (Id Quantique) with efficiency $\eta$. In our experiment $A_0 T = 4.3\text{dB}$ and $\eta = 10.5\%$. We can therefore model the whole of Bob’s detection system by the scheme depicted in Fig. 2. It is composed of the LO (a coherent state of amplitude $\beta$), the signal state $\Psi$, the attenuator $A_0$, a beam splitter with transmission and reflection coefficients $T$ and $R$. The imperfect detector is modeled by an attenuation of $\eta$ followed by a perfect detector.

Let us denote by $\alpha$ the amplitude of the coherent state that would give rise to destructive interference at the single photon detector. It satisfies $\alpha \sqrt{A_0 T} + i \beta \sqrt{R} = 0$. When $a_i = 0$, the state Alice should send if she is honest is the coherent state $|\alpha\rangle$. (If $a_i = 1$ she should send the state $|-\alpha\rangle$.) By using the phase modulator Bob can cancel this phase. But if Alice is dishonest she will send another state $|\Psi\rangle$. We expand $|\Psi\rangle$ in the basis of displaced Fock states $|\Psi\rangle = D_\alpha(\alpha) \sum_n c_n |n\rangle$ where $D_\alpha(\alpha)$ is the displacement operator acting on mode $a$, ie.
\[ D_0(\alpha)D_0(\alpha) = a - \alpha, \text{ and } |n\rangle = (a\dagger)^\dagger/\sqrt{n!}|0\rangle \] are the Fock states. The fidelity of the state sent by Alice is thus
\[ f = |\langle \alpha | \Psi \rangle|^2 = |\alpha|^2. \]

We model the effect of the attenuation by the transformation \( a \rightarrow \sqrt{A_0}d + \sqrt{1 - A_0}e_1 \) where \( e_1 \) is a mode of the environment; the effect of the BS by the transformations \( d' \rightarrow \sqrt{T}d - i\sqrt{R}c, b \rightarrow \sqrt{T}c - i\sqrt{R}d' \); and the effect of the detector inefficiency by \( d' \rightarrow \sqrt{\eta}d + \sqrt{1 - \eta}e_2 \) where \( e_2 \) is another mode of the environment (the modes \( a', b, c, d' \) are all described in the figure). One then finds that the state just before entering the single photon detector is \( \sum_n \frac{\epsilon_n}{\sqrt{n!}}[\sqrt{A_0}T\eta^|d| + \sqrt{1 - A_0}e_1^|d| + \sqrt{1 - \eta}TA_0e_2^|d|\rangle|0\rangle \) where \( \gamma = \beta\sqrt{T} + i\alpha\sqrt{A_0}R \). From this one easily computes that the probability that the detector does not register a single click is
\[ P(\text{no click}) = \sum_{n=0}^\infty |\epsilon_n|^2(1 - A_0T\eta)^n. \] (4)

The probability of registering a click is thus bounded by
\[ P(\text{click}) \geq (1 - |\epsilon_0|^2)A_0T\eta. \] Thus the number of clicks on Bob’s detector divided by \( A_0T\eta \) gives a bound on the fidelity \(|\epsilon_0|^2\).

A final inefficiency that must be taken into account is that Bob’s detector will have a non-zero dark rate \( \kappa_{\text{dark}} = 9 \times 10^{-4} \). Putting all this together we deduce the bound on the average bias if Alice is dishonest:
\[ e_A \leq \frac{F}{A_0T\eta} \left( \kappa - \kappa_{\text{dark}} \right) \] (5)

Note that this bound on \( e_A \) is given entirely by parameters which can be measured by Bob.

Using this protocol, and taking into account experimental imperfections as described below, a typical run of our experiment generates 10^7 coins. Some results for different values of \(|\alpha|^2\) are presented in Fig. 3. For instance when \(|\alpha|^2 = 0.03\), we obtained \( e_A + e_B = 0.32 \pm 0.04 \), which is significantly better than the classical bound \( e_A + e_B \geq 1/2 \).

An important property of this protocol and of its experimental implementation is that we do not have to make any hypothesis about the Hilbert space Alice or Bob use if they are dishonest -for instance it is not necessary to restrict them to the single photon subspace-, nor do we have to make any hypothesis about the kind of technology they can use if they are dishonest. Thus the randomness of the bit string when one of the parties is dishonest is guaranteed by the laws of physics.

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FIG. 3: Measured bounds on average bias and on entropy of bit strings for different values of the average photon number \( \alpha^2 \). Open squares: bounds on \( e_A \) obtained using eq. 5; open circles: bounds on \( e_A \) obtained using eq. 6; filled squares: bounds on \( e_A + e_B \). Classically the sum is always greater than 1/2. The bit strings are clearly more random than is allowed by the best classical protocol. The same expressions which give bounds on \( e_A, e_B \) also give lower bounds on the entropies \( H_{A(B)} \) of the bit string if Alice (Bob) is dishonest (see 12). Filled triangles: bounds on the entropy per bit \( (H_A + H_B)/n \). It is conjectured in 12 that for any classical protocol \( (H_A + H_B)/n \) is bounded by 1 for large \( n \). The experimental points are clearly above this bound. The error bars for \( e_A + e_B \) and \( (H_A + H_B)/n \) describe systematic errors arising from incorrect calibration of detector efficiency \( \eta \) and incorrect estimation of \( \alpha^2 \). The plotted curves are theoretical predictions based on the observed optical visibility of 96.5%. For \( e_A \) the curve is given by eq. 4 and for \( e_A \) it is given by eq. 5 using the fact that for small \( \alpha^2 \), \( (\kappa - \kappa_{\text{dark}})/A_0T\eta \approx (1 - V)\alpha^2/2 \).

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