Robustness of spectral methods for community detection

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Abstract

The present work is concerned with community detection. Specifically, we consider a random graph drawn according to the stochastic block model: its vertex set is partitioned into blocks, or communities, and edges are placed randomly and independently of each other with probability depending only on the communities of their two endpoints. In this context, our aim is to recover the community labels better than by random guess, based only on the observation of the graph.

In the sparse case, where edge probabilities are in $O(1/n)$, we introduce a new spectral method based on the distance matrix $D^{(\ell)}$, where $D^{(\ell)}_{ij} = 1$ if the graph distance between $i$ and $j$, noted $d(i,j)$ is equal to $\ell$. We show that when $\ell \sim c \log(n)$ for carefully chosen $c$, the eigenvectors associated to the largest eigenvalues of $D^{(\ell)}$ provide enough information to perform non-trivial community recovery with high probability, provided we are above the so-called Kesten-Stigum threshold. This yields an efficient algorithm for community detection, since computation of the matrix $D^{(\ell)}$ can be done in $O(n^{1+\kappa})$ operations for a small constant $\kappa$.

We then study the sensitivity of the eigendecomposition of $D^{(\ell)}$ when we allow an adversarial perturbation of the edges of $G$. We show that when the considered perturbation does not affect more than $O(n^\varepsilon)$ vertices for some small $\varepsilon > 0$, the highest eigenvalues and their corresponding eigenvectors incur negligible perturbations, which allows us to still perform efficient recovery.

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1 Introduction

1.1 Background

Community detection is the task of finding large groups of similar items inside a large relationship graph, where it is expected that related items are (in the assortative case) more likely to be linked together. The Stochastic Block Model (abbreviated in SBM) has been designed by Holland et al. [6] to analyze the performance of algorithms for this task; it consists in a random graph \( G \) whose edge probabilities depend only on the community membership of their endpoints. Since then, a large number of articles has been devoted to the study of this model; a survey of these results can be found in Abbe [2], or in Fortunato [5] for a more general view on community detection.

The sparse case, when edge probabilities are in \( O(1/n) \), is known to be much harder to study than denser models; the existence of a positive portion of isolated vertices makes complete reconstruction impossible, and studies usually focus on partial recovery of the community structure. Insights on this topic often come from statistical physics; in the two-community case, Decelle et al. conjectured in [4] the existence of a threshold for reconstruction, which was then proved in Mossel et al. [13] for the first part, Massoulié [9] and Mossel et al. [12] for the converse part. Similarly, in the general case, a method was first presented in Krzakala et al. [7] and then proven to work in Bordenave et al. [3] – bar a technical condition – and Abbe and Sandon [1].

Notably, in the sparse setting, the usual method relying on the eigenvectors of the adjacency matrix of \( G \) fails due to the lack of separation of the eigenvalues. Consequently, a wide array of alternative spectral methods have been designed, relying on the spectrum of a matrix associated to \( G \). More precisely, the eigenvectors associated to the highest eigenvalues will often carry some information about the community structure of \( G \), enough for partial reconstruction. Examples include the path expansion matrix used in [9], or the non-backtracking matrix in [7].

Additionally, other types of methods can be used in this setting: for example, the semi-definite programming (or SDP) algorithm relaxes the problem into a convex optimization one, which can then be approximately solved (see for example Montanari and Sen [11]).

An important feature of real-life networks that is missing from the SBM is the existence of small-scale regions of higher density, that arise from phenomena unrelated to the community structure. For this reason, a common variant of the SBM is the addition of small cliques to the generated random graph. Commonly-used spectral methods, for example those relying on the non-backtracking matrix in [3], are known to fail in this setting, due to the apparition of localized eigenvectors, with no ties to the community structure, and corresponding to large eigenvalues – see Zhang [16] for a comparison of those methods, as well as a proposed heuristic to deal with those localized vectors by lowering their associated eigenvalues. SDP methods are the most studied for this problem, due to their natural stability; in particular, Makarychev et al. [8] show a reconstruction algorithm that is robust to the adversarial addition of \( o(n) \) edges, in the case of an arbitrary number of communities; this was also shown independently by Moitra et al. [10]. However, all the SDP methods mentioned here fail to reach the KS threshold by at least a large constant, with only [11] approaching it as the average degree increases.
1.2 Setting and main results

Stochastic block model

Let \( r > 0 \) be a given integer, and \( \pi = (\pi_1, \ldots, \pi_r) \) a probability vector. We consider a random graph \( G = (V, E) \) as follows. The vertex set \( V \) is taken to be \( [n] = \{1, \ldots, n\} \), and each vertex \( v \in V \) is assigned a type \( \sigma(v) \) sampled independently from distribution \( \pi \).

Given a symmetric \( r \times r \) matrix \( W \) with positive coefficients, two vertices \( u, v \in V \) are joined with an edge randomly and independently with probability

\[
\min\left(\frac{W_{\sigma(u), \sigma(v)}}{n}, 1\right).
\]

Following [3], we introduce \( \Pi = \text{diag}(\pi_1, \ldots, \pi_r) \) and define the mean progeny matrix \( M = \Pi W \); the eigenvalues of \( M \) are the same as those of \( S = \Pi^{1/2}W\Pi^{1/2} \) and in particular are real. We denote them by

\[
\mu_1 \geq |\mu_2| \geq \ldots \geq |\mu_r|.
\]

We shall make the following regularity assumptions: first,

\[
\mu_1 > 1 \quad \text{and} \quad M \text{ is positive regular}, \quad (1)
\]

i.e. the coefficients of \( M^t \) are all positive for some \( t \). Secondly, each type of vertex has the same asymptotic average degree, that is

\[
\sum_{i=1}^r M_{ij} = \sum_{i=1}^r \pi_i W_{ij} = \alpha \quad \text{for all} \quad j \in [r]. \quad (2)
\]

In this case, the matrix \( M^* = M/\alpha \) is a stochastic matrix and therefore

\[
\mu_1 = \alpha > 1. \quad (3)
\]

Since \( M = \Pi^{-1/2}S\Pi^{1/2} \), \( M \) is diagonalizable; let \( (\phi_1, \ldots, \phi_r) \) be a basis of normed left eigenvectors for \( M \), that is

\[
^t \phi_i M = \mu_i ^t \phi_i \quad \text{for all} \quad i \in [r]. \quad (4)
\]

Condition (2) implies that \( \phi_i = 1/\sqrt{k} \), where \( 1 \) is the all-ones vector. Finally, let \( \tilde{\pi} \) be the actual distribution of types in the graph \( G \); we know that with high probability

\[
\|\tilde{\pi} - \pi\|_\infty = \max_{i \in [r]} |\tilde{\pi}_i - \pi_i| = O(n^{-\gamma}) \quad \text{for all} \quad \gamma < 1/2. \quad (5)
\]

Our measure of success for a type estimate \( \hat{\sigma} \) is the empirical overlap between \( \hat{\sigma} \) and the true types \( \sigma \), defined as:

\[
\text{ov(}\sigma, \hat{\sigma}) = \left( \max_{p \in S_r} \frac{1}{n} \sum_{v=1}^n \left[ \hat{\sigma}(v) = p \sigma(v) \right] - \max_{k \in [r]} \pi_k \right) \left( 1 - \max_{k \in [r]} \pi_k \right)^{-1}, \quad (6)
\]
where $S_r$ is the set of permutations of $[r]$.

Since there is a positive proportion of isolated vertices, we can’t hope to achieve an asymptotic overlap of 1; we will say that partial reconstruction is possible if there exists an algorithm leading to estimates $\hat{\sigma}$ such that:

$$\lim_{n \to \infty} \text{ov}(\sigma, \hat{\sigma}) > 0 \text{ w.h.p.}\tag{7}$$

It has been shown in [3] and [1] that polynomial-time algorithms achieve partial reconstruction when the following condition, called the Kesten-Stigum threshold, is verified:

$$\mu_2 > \mu_1.\tag{8}$$

Alternatively, we define $r_0$ such that

$$\mu_2 r_0 + 1 < \mu_1 < \mu_2 r_0.\tag{9}$$

Therefore, the condition (8) is equivalent to $r_0 > 1$.

In the two-community case, the above condition is equivalent to the possibility of reconstruction (see [3], [13]). However, in the general setting ($r > 4$), non-polynomial algorithms can achieve partial reconstruction even below this threshold. This was originally conjectured in [4], and more recently proven in [1].

**Path expansion matrix**

In [9], an algorithm for partial reconstruction in the two-community case makes use of the path expansion matrix $B^{(\ell)}$, where $B^{(\ell)}_{ij}$ counts the number of self-avoiding paths (that is, paths that do not go through the same vertex twice) of length $\ell$ between $i$ and $j$.

Our first aim is to extend the result from this paper to the general case; we first define for all $k \in [r]$ the vectors $\chi_k$ and $\varphi_k$ by

$$\chi_k(v) = \phi_k(\sigma(v)) \quad \text{and} \quad \varphi_k = \frac{B^{(\ell)} \chi_k}{\|B^{(\ell)} \chi_k\|}.\tag{10}$$

Let $\lambda_1(B^{(\ell)}) \geq |\lambda_2(B^{(\ell)})| \geq |\lambda_n(B^{(\ell)})|$ be the eigenvalues of $B^{(\ell)}$ ordered by absolute value; our first theorem is an extension of Theorem 2.1 in [9]:

**Theorem 1.1.** Consider a graph $G$ generated as above, and let $\ell \sim \kappa \log_\alpha(n)$, with $\kappa < 1/12$. Then, with probability going to 1 as $n$ goes to $+\infty$:

\begin{itemize}
  \item[(i)] $\lambda_k(B^{(\ell)}) = \Theta(\mu_k^\ell)$ for $k \in [r_0]$;
  \item[(ii)] For $k > r_0$, $\lambda_k(B^{(\ell)}) = O((\log(n))^c \alpha^{\ell/2})$ for some constant $c > 0$.
\end{itemize}

Furthermore, suppose that $k \leq r_0$ is such that $\mu_k$ is a simple eigenvalue of $M$. Then the eigenvector $\xi_k$ associated to $\lambda_k(B^{(\ell)})$ is asymptotically parallel to $\varphi_k$.

The above theorem does not yield immediately an algorithm for community reconstruction; however, an analysis identical to the one in [3] leads us to:
**Theorem 1.2.** Assume now that \( \pi_i \equiv 1/r \), that \( r_0 > 1 \) and that there exists \( k \in [r_0] \) such that \( k \neq 1 \) and \( \mu_k \) is a simple eigenvalue of \( M \).

Let \( \xi_k \in \mathbb{R}^V \) be a normed eigenvector of \( B^{(\ell)} \) associated to the eigenvalue \( \lambda_k(B^{(\ell)}) \).

Then there exists a deterministic threshold \( \tau \in \mathbb{R} \), a partition \( (I^+, I^-) \) of \( [r] \) and a random signing \( \omega \in \{-1, 1\}^V \) such that the following holds: assign to each vertex \( v \) a label \( \hat{\sigma}(v) \) picked uniformly from \( I^+ \) if \( \omega(v)\xi_k(v) > \tau/\sqrt{n} \) and from \( I^- \) otherwise. Then \( \text{ov}(\sigma, \hat{\sigma}) \) goes to a positive constant as \( n \to \infty \).

**The distance matrix**

We introduce now the distance matrix \( D^{(\ell)} \), defined by \( D^{(\ell)}_{ij} = 1 \) if and only if \( d(i, j) = \ell \), where \( d \) is the distance in \( G \). This matrix, while sparser than \( B^{(\ell)} \), retains much of the desired spectral properties. In particular, we have the following theorem:

**Theorem 1.3.** Assume that condition (8) holds, and set \( \ell \) such that \( \ell \sim \kappa \log \alpha(n) \), where \( \kappa \) is a constant such that \( \kappa < 1/12 \). Then, with high probability:

(i) The \( r_0 \) leading eigenvalues of \( D^{(\ell)} \) are asymptotically equivalent to those of \( B^{(\ell)} \).

(ii) All other eigenvalues of \( D^{(\ell)} \) are of order \( (\log(n))^{c_\alpha} \ell \) for some constant \( c_\alpha \).

Furthermore, suppose that \( k \leq r_0 \) is such that \( \mu_k \) is a simple eigenvalue of \( M \). Then the eigenvector associated to \( \lambda_k(D^{(\ell)}) \) is asymptotically parallel to \( \xi_k \).

As a result, the algorithm defined in Theorem 1.2 will still work when applied to the matrix \( D^{(\ell)} \).

**Graph perturbation**

As mentioned in the introduction, community detection algorithms have to be resilient to the presence of small cliques (or denser subgraphs) to be useful in practice, since this kind of pattern is often present in real-life networks. We will here consider a more general type of perturbation, classifying it by the number of affected vertices:

**Definition 1.1.** (SBM with perturbation) Consider a graph \( G \) drawn according to the SBM defined above, with parameters \( \pi, W \) and \( n \).

Let \( \gamma = \gamma(n) \) be a positive number; an adversary can then replace any subgraph of \( G \) of size at most \( \gamma \) with any other graph of the same size.

The goal is then to recover the original communities with asymptotically positive overlap.

As shown in [16], the usual spectral methods do not fare well against this type of perturbation, especially when the added subgraph contains several cliques. This is especially the case for the non-backtracking matrix in [3], but also the path expansion matrix in [9].

However, the distance matrix is more stable to clique addition, since it does not count the number of paths between two vertices – which is affected significantly by small perturbations. We can therefore allow a perturbation of size up to a small power of \( n \), as stated in the following theorem:
Theorem 1.4. Let $G$ be an SBM as above, with $\pi_i \equiv 1/r$. Assume that $r_0 > 1$, and that there exists $1 < k \leq r_0$ such that $\mu_k$ is a simple eigenvalue of $M$. Let $\tau_i = \mu_k^2 / \mu_1 > 1$ for $i \in [r_0]$.

Then, if
\[
\gamma = o(\tau_k \log(n)) = o(n^{\kappa \log_\alpha(\tau_k) / \log(n)})
\]
the algorithm defined in Theorem 1.2 based on $D^{(\ell)}$ recovers the original communities with asymptotically positive overlap, even after a perturbation affecting at most $\gamma$ vertices.

To our knowledge, the current best result about stability of community detection algorithms in the sparse setting is Theorem 6. in [8], which states that the SDP approach in community reconstruction or a SBM is robust to perturbation of $o(n)$ edges. Unfortunately, spectral methods on the distance matrix fail to reach this threshold, as evidenced by the following theorem:

Theorem 1.5. With the same assumptions as above, let $D^{(\ell)}$ be the distance matrix of $G$ and $\tilde{D}^{(\ell)}$ the one of the perturbed graph.

If $\gamma = \Omega(\tau_2^k)$, then there exists a perturbation of size at most $\gamma$ such that $D^{(\ell)}$ has an eigenvalue of size $\Omega(\mu_2^k)$, with associated eigenvector asymptotically perpendicular to the first $r_0$ ones of $D^{(\ell)}$.

Therefore, we cannot guarantee the stability of the eigenvectors of $D^{(\ell)}$ when the perturbation affects too many vertices. Note that $\tau_2$ goes to 1 as we approach the KS threshold, so we can only guarantee a perturbation of size $O(n^\varepsilon)$, with $\varepsilon$ going to 0 the closer we are from the KS bound.

Remark. We notice that $\tau_2^k \leq n^\kappa < n^{1/12}$, which is far from the aimed bound of $o(n)$; optimizing the constant $\kappa$ could maybe raise the threshold a little, but will not reach the result from [11]. As mentioned in that article, it is likely because spectral methods rely on more precise properties of a graph, that are more disrupted by a perturbation.

However, our result still has several advantages compared to the one in [11], namely:

(i) the threshold for partial reconstruction in our method is exactly the KS threshold, whereas the SDP-based method requires a slightly stronger condition, especially when the mean degree $\alpha$ is low.

(ii) As mentioned afterwards, the running time of our algorithm is $O(n^{1+\kappa}) = O(n^{13/12})$, which is much faster than the usual methods for SDP algorithms.

(iii) Finally, all the SDP methods mentioned throughout this paper only consider the symmetric case even in the case of multiple communities.

1.3 Notations and outline of the paper

Throughout this paper, we will make use of the following notation: for two functions $f, g$, we say that $f = O'(g)$ if there exists a constant $c$ such that $f = O(\log(n)^c \cdot g)$. We similarly define the notations $\tilde{\Theta}$ and $\tilde{\Omega}$.

The next Section is devoted to the study of the spectral structure of $B^{(\ell)}$; we also state there an important theorem on spectral perturbation that will be useful for the study of matrix $D^{(\ell)}$ as well. In Section 3, we study the distance matrix $D^{(\ell)}$ and introduce a method to deal with perturbations of this matrix. We then leverage this method to obtain bounds on the size of allowed perturbations.
2 Spectral structure of $B^{(\ell)}$

2.1 A theorem on eigenspace perturbation

In the following, we’ll need a way to link the operator norm of a matrix perturbation to the consequent perturbation of its eigenvectors. This is provided by the following variant of the Davis-Kahan sin $\theta$ theorem (Yu et al. [15], Theorem 2):

**Theorem 2.1.** Let $\Sigma, \hat{\Sigma}$ be symmetric $n \times n$ matrices, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_n$ respectively. Fix $1 \leq r \leq s \leq n$ and assume that $\min(\lambda_{r-1} - \lambda_r, \lambda_s - \lambda_{s+1}) > 0$, where we define $\lambda_0 = +\infty$ and $\lambda_{n+1} = -\infty$.

Let $d = s - r + 1$, and let $V = (v_r, \ldots, v_s)$ and $\hat{V} = (\hat{v}_r, \ldots, \hat{v}_s)$ have orthonormal columns satisfying $\Sigma v_j = \lambda_j v_j$ and $\hat{\Sigma} \hat{v}_j = \hat{\lambda}_j \hat{v}_j$ for $j \in \{r, \ldots, s\}$.

Then there exists an orthogonal matrix $\hat{O} \in \mathbb{R}^{d \times d}$ such that

$$\|\hat{O} \hat{V} - V\|_F \leq \frac{2\sqrt{2d}\|\hat{\Sigma} - \Sigma\|_{op}}{\min(\lambda_{r-1} - \lambda_r, \lambda_s - \lambda_{s+1})}$$

(12)

In particular, when $r = s = j$ (i.e. $d = 1$), choosing $\hat{v}_j$ such that $^t \hat{v}_j v_j > 0$, we have

$$\|\hat{v}_j - v_j\| \leq \frac{2\sqrt{2d}\|\hat{\Sigma} - \Sigma\|_{op}}{\min(\lambda_{j-1} - \lambda_j, \lambda_j - \lambda_{j+1})}$$

(13)

2.2 Strategy of proof

We present here the main ideas of the proof, and defer its full version to the appendix. The first step is an adaptation of Proposition 19 from [3]:

**Proposition 2.1.** Let $\ell \sim \kappa \log_\alpha(n)$ with $\kappa = 1/12$. Define, for $k \in [r]$,

$$\theta_k = \|B^{(\ell)} \varphi_k\| \text{ and } \zeta_k = \frac{B^{(\ell)} \varphi_k}{\theta_k},$$

(14)

with $\varphi_k$ as in (10).

Then, with high probability, we have the following estimations for every $\gamma < 1/2$:

(i) $\theta_k = \Theta(\mu_k^{\ell})$ for $k \in [r_0]$,

(ii) $|\langle \varphi_j, \varphi_k \rangle| = \tilde{O}(\alpha^{3/2} n^{-\gamma/2})$ for $j \neq k \in [r_0]$,

(iii) $|\langle \zeta_j, \varphi_k \rangle| = \tilde{O}(\alpha^{3/2} n^{-\gamma/2})$ for $j \neq k \in [r_0]$.

Now, let $(z_1, \ldots, z_{r_0})$ be the Gram-Schmidt orthonormalization of $(\varphi_1, \ldots, \varphi_{r_0})$, and define

$$D = \sum_{k=1}^{r_0} \theta_k z_k^t z_k.$$  

The non-zero eigenvalues of $D$ are thus the $\theta_k$, with corresponding eigenvectors $z_k$. Then, using the asymptotic orthogonality properties of Proposition [2.1] we prove the following:
Proposition 2.2. For all $k \in [r_0]$, $z_k$ is asymptotically parallel to $\varphi_k$.

Furthermore,
\[
\|B^{(\ell)} - D\|_{op} = \tilde{O}(\alpha^{\ell/2}).
\] (15)

Theorem 1.1 then results from a simple application of the Weyl inequality (14) and Theorem 12:

Proof. (of Theorem 1.1) : let $\theta_k = 0$ for $k > r_0$; the eigenvalues of $D$ are then exactly the $\theta_i$ for $i \leq n$.

By Weyl's inequality, we have for all $i \in [n]$
\[
|\lambda_i(B^{(\ell)}) - \theta_i| = \tilde{O}(\alpha^{\ell/2}).
\]

Since $\theta_k = \Theta(\mu_k^2)$ for $k \in [r_0]$, this implies the statements (i) and (ii) of the Theorem.

Assume now that $k$ is such that $\mu_k$ is a simple eigenvalue of $M$. Applying Theorem 12 to $r = s = k$ gives the existence of a sign $\varepsilon = \pm 1$ such that
\[
\|\varepsilon\xi_k - z_k\| \leq 2\sqrt{2}\|B^{(\ell)} - D\|_{op} = \tilde{O}\left(\frac{\alpha^{\ell/2}}{\mu_k^2}\right) = o(1),
\] (16)

where the last equality stems from $\mu_k^2 > \alpha$ since $k \leq r_0$. This implies the last statement of Theorem 1.1.

\[\square\]

3 Study of the matrix $D^{(\ell)}$

3.1 From $B^{(\ell)}$ to $D^{(\ell)}$

The first aim of this section is to prove Theorem 1.3, i.e. that we can replace matrix $B^{(\ell)}$ by $D^{(\ell)}$ in the algorithm from Theorem 1.2. In view of the proof of Theorem 1.1 above, it is sufficient to prove the following proposition:

Proposition 3.1. Let $G$ be a SBM as above, and $\ell \sim \kappa \log_\alpha(n)$ with $\kappa < 1/12$. Let $B^{(\ell)}$ be the path expansion matrix of $G$, and $D^{(\ell)}$ its distance matrix. Then, with high probability:
\[
\rho(B^{(\ell)} - D^{(\ell)}) = \tilde{O}(\alpha^{\ell/2}),
\] (17)

where $\rho$ is the spectral radius of a matrix.

We first recall some results about the neighbourhoods of vertices, whose proofs can be found in [3] :

Lemma 3.1. For a vertex $i$, define $S_t(i)$ as the number of vertices at distance $t$ of $i$.

Then there exist constants $C$ and $\varepsilon > 0$ such that with probability $1 - O(n^{-\varepsilon})$, for all $i \in \{1, \ldots, n\}$ and $\ell = O(\log(n))$:
\[
S_t(i) \leq C \cdot \log(n) \cdot \alpha^t, \quad t \in \{1, \ldots, \ell\}.
\] (18)
On the other hand, with high probability, when \( \ell = \kappa \log_\alpha(n) \) with \( \kappa < 1/2 \):

\[
\sum_{i=1}^{n} S_\ell(i)^2 = \Theta(na^2\ell) .
\]

(19)

In the same vein, for a vertex set \( \mathcal{X} \), define \( S_t(\mathcal{X}) \) as the number of vertices at distance \( t \) of \( \mathcal{X} \). By taking the union on all vertices of \( \mathcal{X} \), we easily get the following corollary:

**Corollary 3.1.** For the same constants \( C \) and \( \varepsilon \) as above, with probability \( 1 - O(n^{-\varepsilon}) \), we have for all vertex subsets \( \mathcal{X} \in \mathcal{P}([1, \ldots, n]) \) and \( \ell = O(\log(n)) \):

\[
S_t(\mathcal{X}) \leq C \cdot |\mathcal{X}| \log(n) \cdot \alpha^t , \quad t \in \{1, \ldots, \ell\}.
\]

Finally, a result about the almost tree-like structure of vertex neighbourhoods:

**Lemma 3.2.** Assume \( \ell = \kappa \log(n) \), with \( \kappa \log(\alpha) < 1/4 \). Then with high probability no node \( i \) has more than one edge cycle in its \( \ell \)-neighbourhood ; we say that \( G \) is \( \ell \)-tangle-free.

We can now set out to prove Proposition 3.1. For ease of notation, let \( \Delta^{(\ell)} = B^{(\ell)} - D^{(\ell)} \); we first notice that \( \Delta^{(\ell)} \) is a \( 0-1 \) matrix:

**Lemma 3.3.** Let \( \ell \sim \kappa \log(n) \) with \( \kappa < 1/12 \). For all vertices \( i, j \in \{1, \ldots, n\} \),

\[
0 \leq \Delta^{(\ell)}_{ij} \leq 1.
\]

Furthermore, if \( \Delta^{(\ell)}_{ij} = 1 \), then there exists a cycle \( C \) such that:

\[
d(i, C) + d(j, C) \leq \ell.
\]

(21)

Define now a matrix \( P^{(\ell)} \) by \( P^{(\ell)}_{ij} = 1 \) if there is a cycle \( C \) such that \( d(i, C) + d(j, C) \leq \ell \). By the previous lemma, we have \( \Delta^{(\ell)}_{ij} \leq P^{(\ell)}_{ij} \) for all \( (i, j) \), and the Perron-Frobenius theorem implies:

\[
\rho(\Delta^{(\ell)}) \leq \rho(P^{(\ell)}).
\]

(22)

It remains then to bound the spectral radius of \( P^{(\ell)} \); the key lemma is the following:

**Lemma 3.4.** For a given cycle \( C \), let \( P^{(\ell)}_C \) be the matrix defined by \( P^{(\ell)}_{C,ij} = 1 \) if \( d(i, C) + d(j, C) \leq \ell \), and \( V_C \) the set of vertices such that \( d(i, C) \leq \ell \). Then:

(i) \( P^{(\ell)}_C \) is zero outside of \( V_C \times V_C \),

(ii) \( \rho(P^{(\ell)}) = \max_C \rho(P^{(\ell)}_C) \).

By part (ii) of the above lemma, it is sufficient to bound \( \rho(P^{(\ell)}_C) \) for a given cycle \( C \) in \( \mathcal{G} \); using part (i) of the above lemma, we can restrict our study to the subspace spanned by the vertices in \( V_C \).
Let \( v \) be a normed vector of size \(|V_C|\) corresponding to the highest eigenvalue of \( P_C^{(\ell)} \); as the coefficient \((i, j)\) of \( P_C^{(\ell)} \) only depends on the distance of \( i \) and \( j \) to \( C \), we likewise group the coefficients of \( v \) by their distance \( t \) to \( C \), and write

\[
v = (v_{ij})_{0 \leq t \leq \ell, 1 \leq j \leq S_t(C)}.
\]

We then have:

\[
t^* P_C v = \sum_{t+u \leq \ell} \sum_{i,j} v_{ti} v_{uj} = \sum_{t+u \leq \ell} \left( \sum_{i} v_{ti} \right) \left( \sum_{j} v_{uj} \right).
\]

By the Perron-Frobenius theorem, the coefficients of \( v \) are non-negative. For a given \( t \), the coefficients \( v_{ti} \) are necessarily equal; otherwise, we could increase \( \sum v_{ti} \) while leaving \( \sum v_{ti}^2 \) fixed, which leads to increasing \( t^* P_C v \) while keeping \( \|v\|^2 \) constant: this contradicts the definition of \( v \).

Writing \( v_{ti} = v_t \) for all \( 1 \leq i \leq S_t(C) \); we get:

\[
t^* P_C v = \sum_{t+u \leq \ell} S_t(C) S_u(C) v_t v_u \quad \text{and} \quad \|v\|^2 = \sum_t S_t(C) v_t^2.
\]

(23)

Let \( w \) be the size \( \ell \) vector defined by \( w_t = \sqrt{S_t(C)} v_t \). Rewriting the above expression in terms of \( w \) yields

\[
t^* P_C v = \sum_{t+u \leq \ell} \sqrt{S_t S_u} w_t w_u \quad \text{and} \quad \|v\|^2 = \|w\|^2,
\]

(24)

where we omit the dependency of \( S_t \) in \( C \).

As a result, the spectral radius of \( P_C \) is equal to that of the \( \ell \times \ell \) matrix \( Q_C \) defined by:

\[
Q_C = \begin{pmatrix}
S_0 & \sqrt{S_0 S_1} & \cdots & \sqrt{S_0 S_{\ell-1}} & \sqrt{S_0 S_\ell} \\
\sqrt{S_0 S_1} & S_1 & \cdots & \sqrt{S_1 S_{\ell-1}} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sqrt{S_0 S_{\ell-1}} & \sqrt{S_1 S_{\ell-1}} & \cdots & 0 & 0 \\
\sqrt{S_0 S_\ell} & \sqrt{S_1 S_\ell} & \cdots & 0 & 0 \\
\end{pmatrix}.
\]

We now finally use the row sum bound to get:

\[
\rho(P_C^{(\ell)}) = \rho(Q_C) \leq \max_t \sum_{u \leq \ell-t} \sqrt{S_t S_u} \leq \max_t \sum_{u \leq \ell-t} \log(n) \alpha^{t+u/2} \quad \text{via lemma 3.1}
\]

(26)

\[
= O(\log(n) \alpha^{\ell/2}).
\]

(27)
Combining the above inequality with Lemma 3.4 and inequality (22) eventually leads to \[
\rho(\Delta^{(\ell)}) = \tilde{O}(\alpha^{\ell/2}),
\] (28)
which completes the proof of Proposition 3.1.

### 3.2 Stability to graph perturbation

In this subsection, we sketch the proofs for Theorems 1.4 and 1.5.

**A note about computational complexity**

In the original algorithm, the computation of \(B^{(\ell)}\) in polynomial time relies on the almost tree-like, tangle-free structure of the random graph \(G\); this structure may be lost when we add cliques, and increase the algorithm complexity. As we want to devise polynomial algorithms in every case, this may be a hindrance.

Conversely, the computation of the distance matrix \(D^{(\ell)}\) can be done in polynomial time (for example breadth-first search of the \(\ell\)-neighbourhood of each vertex in \(G\) yields an algorithm in \(O(n^{1+\kappa}) = O(n^{13/12})\) in the case of SBM, \(O(n^2)\) in general) for any graph, which makes it all the more adapted to the problem at hand.

In order to prove Theorem 1.4, we need a less restrictive version of Proposition 3.1; indeed, bounding the spectral radius of the perturbation by \(O(\alpha^{\ell/2})\) not only preserves the highest eigenvalues, but also bounds the remaining eigenvalues of \(D^{(\ell)}\) by \(\lambda_1(D^{(\ell)})\). This bound is commonly referred to as a Ramanujan-like property of \(G\).

This property, although interesting on its own, is not specifically needed for the reconstruction algorithm to work; rather, we only need the eigenvector associated to the simple eigenvalue \(\mu_k\) to remain asymptotically parallel to the unperturbed one.

We’ll therefore only need the following proposition:

**Proposition 3.2.** We consider the same setting as Theorem 1.4. Let \(D^{(\ell)}\) be the distance matrix of \(G\), and \(\tilde{G}\) and \(\tilde{D}^{(\ell)}\) be the perturbed versions of \(G\) and \(D^{(\ell)}\), respectively. Then
\[
\rho(\tilde{D}^{(\ell)} - D^{(\ell)}) = o(\mu_k^\ell).
\] (29)

The proof relies on a bound similar to the one in Theorem 1.3 replacing matrices \(P_C\) and \(Q_C\) by matrices \(P_K\) and \(Q_K\) also depending only on the distance to the perturbed vertex set \(K\). The details can be found in the appendix.

### 3.3 Optimality of the bound

In order to prove Theorem 1.5 we need to show that the controls in the proof of Theorem 1.4 are actually sharp. We begin with the following lemma, which comes from the fact that \(\ell\)-neighbourhoods of the vertices of \(G\) are roughly of the same size:

**Lemma 3.5.** Assume that \(\gamma = \Theta(\tau_2^{\ell/2})\). Then there exists a set of vertices \(K\) of size \(\gamma\) such that:
\[
S_\ell(K) = \Omega(\alpha^{\ell} \cdot \gamma).
\] (30)
Proof. (of Theorem 1.5) Consider the vector $v$ such that:

$$v_i = \begin{cases} 
\gamma^{-1/2} & \text{if } i \in K \\
S_\ell(K)^{-1/2} & \text{if } d(i, K) = \ell \\
0 & \text{otherwise}.
\end{cases}$$

(31)

We then have $\|v\|^2 = 2$, and:

$$\ell v D(\ell)v = \sum_{i,j} v_i D_{ij}^{(\ell)} v_j$$

(32)

$$\geq 2 \sum_{i \in S_\ell(K)} \sum_{j \in K} v_i v_j$$

(33)

$$= 2 \gamma S_\ell(K) \gamma^{-1/2} S_\ell(K)^{-1/2}$$

(34)

$$= 2 \sqrt{\gamma S_\ell(K)}$$

(35)

$$= \Omega(\mu_2^\ell)$$

(36)

It remains then to prove that $v$ is asymptotically orthogonal to $B^{(\ell)}\chi_k$ for $k \in [r_0]$ : noticing that $v_i \leq 1$ for all $i$ and $\|v\|_0 = \gamma + S_\ell(K)$, we find, using Corollary 3.1:

$$\langle v, B^{(\ell)}\chi_k \rangle \leq (\gamma + S_\ell(K)) \cdot \|B^{(\ell)}\chi_k\|_\infty$$

$$\leq (\gamma + S_\ell(K)) \cdot \tilde{O}(\alpha^{\ell})$$

$$= \tilde{O}(\gamma \alpha^{2\ell})$$

$$= o(\sqrt{n \mu_2^{\ell}}) \quad \text{since } \kappa < 1/4$$

$$= o(\|v\| \|B^{(\ell)}\chi_k\|),$$

where we used part (ii) of proposition A.5 in the appendix to bound $\|B^{(\ell)}\chi_k\|_\infty$.

The Courant-Fisher theorem then implies Theorem 1.5. \qed
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A Proof or Propositions 2.1 and 2.2

A.1 Outline of the proof and similarities with [3]

The main arguments of the proof rely on the study of three quantities:

(i) a multi-type branching process $Z_t$,

(ii) a similar process based on exploring the neighbourhood of a vertex $v$ in $G$, named $Y_t(v)$,

(iii) the actual vectors we’re aiming to study, $B^{(\ell)}\chi_k$.

When the $\ell$-neighbourhood of $v$ is cycle-free, we have that $B^{(\ell)}\chi_k = \langle \phi_k, Y_t(v) \rangle$ for $k \in [r_0]$; and there is a coupling between the laws of $Z_t$ and $Y_t(v)$ for almost every $v$, which allows us to translate results on $Z_t$ to results on $B^{(\ell)}\chi_k$.

The proof in [3] studies the matrix $B^{\ell}$, where $B$ is the non-backtracking matrix; $B^{\ell}_{ij}$ therefore counts the number of non-backtracking walks between $i$ and $j$. When the $\ell$-neighbourhood of $i$ is tree-like, $(B^{(\ell)}\chi_k)_i = (B^{\ell}\check{\chi}_k)_i$, where $\check{\chi}_k$ is a similarly defined vector; most of the results from [3] can therefore be applied to this setting without further work. We will simply lay out the main steps of the proof, highlighting the main differences with [3] when necessary.

A.2 Local structure of $G$

For an integer $t \geq 0$, we introduce the vector $Y_t(v) = (Y_t(v)(i))_{i\in[r]}$, where

$$Y_t(v)(i) = |\{w \in V \mid d(v, w) = t, \sigma(w) = i\}|.$$ 

The proof of our first proposition, although quite lengthy, is completely identical to its equivalent in [3]; we therefore omit it.

**Proposition A.1.** Let $\ell \sim \kappa \log_{\alpha}(n)$ with $\kappa < 1/8$; then, for all $\gamma < 1/2$:

(i) for any $k \in [r_0]$, there exists $\rho_k > 0$ such that in probability,

$$
\frac{1}{n} \sum_{v \in V} \frac{\langle \phi_k, Y_t(v) \rangle^2}{\mu_k^{2\ell}} \to \rho_k.
$$

(ii) for any $j \neq k \in [r]$,

$$
E \left| \frac{1}{n} \sum_{v \in V} \langle \phi_j, Y_t(v) \rangle \langle \phi_k, Y_t(v) \rangle \right| = O \left( \alpha^{5\ell/2} n^{-\gamma/2} (\log(n))^{5/2} \right).
$$

(iii) for any $j \neq k \in [r]$,

$$
E \left| \frac{1}{n} \sum_{v \in V} \langle \phi_j, Y_{2\ell}(v) \rangle \langle \phi_k, Y_{\ell}(v) \rangle \right| = O \left( \alpha^{7\ell/2} n^{-\gamma/2} (\log(n))^{5/2} \right).
$$
For \( t \geq 0 \), define \( \mathcal{Y}_t(v) = \{w \in V \mid d(v, w) = t\} \); for \( k \in [r] \), we set
\[
P_{k, \ell}(v) = \sum_{t=0}^{\ell-1} \sum_{w \in \mathcal{Y}_t(v)} L_k(w),
\]
where
\[
L_k(w) = \sum_{(x, y) \in \mathcal{Y}_1(w) \setminus \mathcal{Y}_1(v)} \langle \phi_k, \tilde{Y}_t(x) \rangle \tilde{S}_{t-l-1}(y),
\]

\( \tilde{Y}_t(x) \) is the equivalent of \( Y_t(x) \) when all vertices in \( (G, v)_t \) (i.e. vertices at distance at most \( t \) from \( v \)) are removed and \( \tilde{S}_{t-l-1}(y) = \|\tilde{Y}_{t-l-1}(y)\|_1 \).

It can be seen from [3] that when \( (G, v)_{2\ell} \) is a tree, then
\[
(B^{(\ell)}B^{(\ell)}\chi_k)v = P_{k, \ell}(v) + \chi_k(v)S_{\ell}(v) + \langle \phi_k, Y_{2\ell}(v) \rangle.
\]

One main difference with the proof in [3] is the presence of the last term in the above sum, as well as the fact that dealing with \( B^{(\ell)}B^{(\ell)}\chi_k \) is a little more difficult. The next proposition is an adaptation of Proposition 38 from [3], with an identical – and thus omitted – proof :

**Proposition A.2.** Let \( \ell \sim \kappa \log_\alpha(n) \) with \( \kappa < 1/10 \). Then, for all \( \gamma < 1/2 \) :

(i) for all \( k \in [r_0] \), there exists \( \rho^*_k \) such that w.h.p
\[
\frac{1}{n} \sum_{v \in V} \left( P_{k, \ell}(v) + \langle \phi_k, Y_{2\ell}(v) \rangle \right)^2 / \mu^{4\ell}_k \rightarrow \rho^*_k.
\]

(ii) for any \( j \neq k \in [r] \), for some \( c > 0 \):
\[
\frac{1}{n} \sum_{v \in V} P_{k, \ell}(v) \langle \phi_j, Y_{\ell}(v) \rangle = O \left( \alpha^{7/2} n^{-\gamma/2}(\log(n))^c \right).
\]

**A.3 From local neighbourhoods to the matrix \( B^{(\ell)} \)**

For ease of notation, we define \( N_{k, \ell}(v) = \langle \phi_k, Y_{\ell}(v) \rangle \); using the same methods as in [3], we have the following estimates :

**Proposition A.3.** Let \( \ell \sim \kappa \log_\alpha(n) \) with \( \kappa < 1/4 \). Then w.h.p :
\[
\|B^{(\ell)}\chi_k - N_{k, \ell}\| = o(\alpha^{\ell/2} \sqrt{n}) \text{ and } \|B^{(\ell)}B^{(\ell)}\chi_k - P_{k, \ell} - N_{k, 2\ell}\| = O(\alpha^\ell \sqrt{n}).
\]

It then remains to follow the proof of Proposition 19 from [3] ; we simply highlight the proof for estimation (iii) of Proposition 2.1 since it is the only difference :

**Proof.** (Proposition 2.1(iii)) : We have by definition
\[
\langle \varphi_j, \zeta_k \rangle = \frac{\langle B^{(\ell)}\chi_j, B^{(\ell)}B^{(\ell)}\chi_k \rangle}{\|B^{(\ell)}\chi_j\| \|B^{(\ell)}B^{(\ell)}\chi_k\|}.
\]
But \( \|B^{(\ell)}\chi_j\| = \Theta(\sqrt{n\mu_k^\ell}) \), \( \|B^{(\ell)}B^{(\ell)}\chi_k\| = \|B^{(\ell)}\chi_k\| \theta_k = \Theta(\sqrt{n\mu_k^{2\ell}}) \) and :
\[
\left| \langle B^{(\ell)}\chi_j, B^{(\ell)}B^{(\ell)}\chi_k \rangle - \langle N_{j,\ell}, P_{k,\ell} + N_{k,2\ell} \rangle \right| \leq \|N_{j,\ell}\| \|B^{(\ell)}B^{(\ell)}\chi_k\| \|B^{(\ell)}\chi_j - N_{j,\ell}\| + \|B^{(\ell)}B^{(\ell)}\chi_k\| \|B^{(\ell)}\chi_j - N_{j,\ell}\| = \tilde{O}(\alpha^4 \sqrt{n}).
\]

Furthermore, from Propositions A.1 and A.2, we get
\[
\langle N_{j,\ell}, P_{k,\ell} + N_{k,2\ell} \rangle = \tilde{O}(\alpha^7 \ell^{\gamma/2} n^{1-\gamma/2}).
\]
This gives the desired result. \(\square\)

A.4 Ramanujan property of \(B^{(\ell)}\)

In order to complete the proof of Theorem 1.1, we need a control on the other eigenvalues of \(B^{(\ell)}\). This is covered by the following proposition :

**Proposition A.4.** Let \(H = \langle \varphi_1, \ldots, \varphi_{\ell_0} \rangle\), and \(\ell \sim \kappa \log_\alpha(n)\) with \(\kappa < 1/12\). Then with high probability
\[
\sup_{x \in H^\perp, \|x\|=1} \|B^{(\ell)}x\| = \tilde{O}(\alpha^{\ell/2}).
\]

The proof of this result relies on the following decomposition of \(B^{(\ell)}\), whose proof can be found in [9] :

**Lemma A.1.** Matrix \(B^{(\ell)}\) verifies the identity
\[
B^{(\ell)} = \Delta^{(\ell)} + \sum_{m=1}^{\ell} \Delta^{(\ell-m)} \bar{A} B^{(m-1)} - \sum_{m=0}^{\ell} \Gamma^{\ell,m},
\]
for matrices \(\Delta^{(j)}, \Gamma^{\ell,m}\) such that for \(\ell = O(\log(n))\) and with high probability, for all \(\varepsilon > 0\),
\[
\rho(\Delta^{(j)}) = \tilde{O}(\alpha^2), \ j = 1, \ldots, \ell, \quad (39)
\]
\[
\rho(\Gamma^{\ell,m}) = n^{\varepsilon-1} \alpha^{(\ell-m)/2}, \ m = 1, \ldots, \ell. \quad (40)
\]

Here, \(\bar{A}\) refers to the expected value of the adjacency matrix \(A\) of \(G\).

The next step is therefore to control \(B^{(m-1)}x\) for \(x \in H^\perp\); in what follows \(\gamma\) will be any constant below 1/2. We begin with the following proposition from [3] :
**Proposition A.5.** Let $\ell \sim \kappa \log_\alpha(n)$ with $\kappa < \gamma/2$. There exists a subset $B \subset V$, constants $C$ and $c$ such that w.h.p the following holds:

(i) for all $i \in V \setminus B$, $0 \leq m \leq \ell$,

\[
|(B^{(m)} \chi_k)_i - \mu_k^{t-\ell}(B^{(\ell)} \chi_k)_i| \leq C \log(n)\alpha^{m/2}
\]

if $k \in \lfloor r \rfloor$,

\[
|(B^{(m)} \chi_k)_i| \leq C \log(n)\alpha^{m/2}
\]

if $k \in \lceil r \rceil \setminus \lfloor r \rfloor$.

(ii) for all $i \in B$, $0 \leq m \leq \ell$ and $k \in \lfloor r \rfloor$,

\[
|(B^{(\ell)} \chi_k)_i| \leq C \log(n)\alpha^m.
\]

(iii) $|B| = \tilde{O}(\alpha^\ell n^{1-\gamma})$.

From this, we get the following corollary:

**Corollary A.1.** Let $\ell \sim \kappa \log_\alpha(n)$ with $\kappa < \gamma/2$; then, with high probability, for $0 \leq m \leq \ell - 1$ and $k \in \lfloor r \rfloor$:

\[
\sup_{x \perp B^{(\ell)} \chi_k, \|x\|=1} \langle B^{(m)} \chi_k, x \rangle = \tilde{O}(\sqrt{n} \alpha^{m/2}).
\]

Additionally, for $k \in \lceil r \rceil \setminus \lfloor r \rfloor$,

\[
\|B^{(m)} \chi_k\| = \tilde{O}(\sqrt{n} \alpha^{m/2}).
\]

**Proof.** Write

\[
\langle B^{(m)} \chi_k, x \rangle = \sum_{i \in B} x_i (B^{(m)} \chi_k)_i + \sum_{i \not\in B} x_i (B^{(m)} \chi_k)_i = s_1 + s_2.
\]

Using the Cauchy-Schwarz inequality, the first sum is bounded by

\[
|s_1| \leq \log(n)\alpha^m \sqrt{|B|} \leq \log(n)\alpha^m \alpha^{\ell/2} n^{(1-\gamma)/2} = o(\sqrt{n} \alpha^{m/2}),
\]

while the second can be treated using Proposition A.5 and the fact that $\langle B^{(\ell)} \chi_k, x \rangle = 0$:

\[
|s_2| \leq \mu_k^{t-\ell} \sum_{i \in B} |x_i| (B^{(\ell)} \chi_k)_i + \sum_{i \not\in B} |x_i| (B^{(m)} \chi_k)_i - \mu_k^{t-\ell} (B^{(\ell)} \chi_k)_i \\
\leq \log(n)\alpha^{t-\ell} \alpha^{\ell/2} n^{(1-\gamma)/2} + \log(n)\alpha^{t/2} \sqrt{n} \alpha^{m/2} \\
= \tilde{O}(\sqrt{n} \alpha^{m/2}),
\]

where we used again the Cauchy-Schwarz inequality as before.
Let now \( k \in [r] \setminus [r_0] \); as before, we write
\[
\|B^{(m)}\chi_k\|^2 = \sum_{i \in B} (B^{(m)}\chi_k)^2 + \sum_{i \notin B} (B^{(m)}\chi_k)^2
\]
\[
\leq |B| \log(n)\alpha^{2m} + n \log(n)\alpha^m
\]
\[
= n \log(n)\alpha^{(\ell+2m)n^{-\gamma} + \alpha^m})
\]
\[
= \tilde{O}(n\alpha^m),
\]
and the result follows.

We are now ready to prove Proposition A.4:

Proof. Let \( x \in H^\perp \) such that \( \|x\| = 1 \) and the supremum in (37) is reached; using the decomposition from Lemma A.1, we have
\[
\|B^{(\ell)}x\| \leq \rho(\Delta^{(\ell)}) + \sum_{m=1}^{\ell} \rho(\Delta^{(\ell-m)})\|\tilde{A}B^{(m-1)}x\| + \sum_{m=1}^{\ell} \rho(\Gamma^{\ell,m}).
\]

The first and third terms are bounded by \( \tilde{O}(\alpha^{\ell/2}) \). For the second term, we notice that defining the matrix \( P \) by
\[
P = \frac{1}{n} \sum_{k=1}^{r} \mu_k \chi_k \chi_k,
\]
we have \( \tilde{A} = P - \text{diag}(P) \) since \( W = \sum \mu_k \phi_k \phi_k \).

Therefore, for fixed \( 1 \leq m \leq \ell \), we have:
\[
\|\tilde{A}B^{(m-1)}x\| = \left| \sum_{k=1}^{r} \mu_k \chi_k B^{(m-1)}x - \text{diag}(P)B^{(m-1)}x \right|
\]
\[
\leq \frac{\sup_i W_{ii}}{n} \|B^{(m-1)}x\| + \sum_{k \in [r_0]} \frac{\mu_k}{n} \|\chi_k B^{(m-1)}x\| + \sum_{k \notin [r_0]} \frac{\mu_k}{n} \|\chi_k B^{(m-1)}x\|
\]
\[
= I + J + K.
\]

Notice first that \( B_{ij}^{(\ell)} \leq 2 \) for all \( i, j \) by the tangle-free property, so \( I = O(1) \). Now, for \( k \in [r_0] \), we have
\[
\|\chi_k B^{(m-1)}x\| = \|\chi_k \langle B^{(m-1)}\chi_k, x \rangle\|
\]
\[
\leq \tilde{O}(\sqrt{n} \times \sqrt{n} \alpha^{m/2}).
\]

Therefore, \( J = \tilde{O}(\alpha^{m/2}) \); finally, using the Cauchy-Schwarz inequality, we have for \( k \in [r] \setminus [r_0] \)
\[
\|\chi_k B^{(m-1)}x\| \leq \|\chi_k\| \|B^{(m-1)}\chi_k\| \|x\|
\]
\[
= \tilde{O}(\sqrt{n} \times \sqrt{n} \alpha^{m/2} \times 1).
\]
Putting this all together, we find that for $1 \leq m \leq \ell$

$$\|\bar{A}B^{(m-1)}x\| = \tilde{O}(\alpha^{m/2}).$$

Since $\rho(\Delta^{(\ell-m)}) = \tilde{O}(\alpha^{(\ell-m)/2})$, we get $\|B^{(\ell)}x\| = \tilde{O}(\alpha^{\ell/2})$, which proves the desired result. \qed

### A.5 Proof of Proposition 2.2

Using Proposition 37, we are now able to prove our last result. Note that if $\kappa < 1/12$, there exists a $\gamma < 1/2$ such that $\kappa < \gamma/6$.

Let $z_k$ be the Gram-Schmidt orthonormalization of $\varphi_k$; using Lemma 9 from [3], we know that

$$\|\varphi_k - z_k\| = \tilde{O}(\alpha^{3\ell/2}n^{-\gamma/2}),$$

and thus $z_k$ is asymptotically parallel to $\varphi_k$.

We only need a final lemma to complete our proof:

**Lemma A.2.** Assume that $\ell \sim \kappa \log_\alpha(n)$ with $\kappa < \gamma/6$. Then

$$\|z_k - \zeta_k\| = \tilde{O}(\theta_k^{-1}\alpha^{\ell/2}).$$

**Proof.** Write

$$\zeta_k = \sum_{j \in [r_0]} \langle \zeta_k, z_j \rangle z_j + x,$$

where $x \in H^\perp$.

We have, for $j \neq k$, $\langle \zeta_k, z_j \rangle = \tilde{O}(\alpha^{2\ell}n^{-\gamma/2})$ by the above bound of $\|\varphi_j - z_j\|$; furthermore,

$$\|x\|^2 = \langle \zeta_k, x \rangle = \theta_k^{-1} \langle B^{(\ell)} \varphi_k, x \rangle \leq \theta_k^{-1} \|B^{(\ell)}x\| = \tilde{O}(\theta_k^{-1}\alpha^{\ell/2}) \times \|x\|.$$ 

Therefore, we can write

$$1 = \|\zeta_k\|^2 = \langle \zeta_k, z_k \rangle^2 + \sum_{j \neq k} \langle \zeta_k, z_j \rangle^2 + \|x\|^2$$

$$= \langle \zeta_k, z_k \rangle^2 + \tilde{O}(\alpha^{2\ell}n^{-\gamma/2}) + \tilde{O}(\theta_k^{-2}\alpha^\ell)$$

$$= \langle \zeta_k, z_k \rangle^2 + \tilde{O}(\theta_k^{-2}\alpha^\ell),$$

since $\kappa < \gamma/6$.

Then,

$$\|z_k - \zeta_k\|^2 = 2(1 - \langle \zeta_k, z_k \rangle) = \tilde{O}(\theta_k^{-2}\alpha^\ell),$$

which yields the desired result. \qed

20
Proof. (of Proposition 2.2) : We first bound $\|B^{(\ell)} z_k - D z_k\|$ for $k \in [r_0]$. Notice that $D z_k = \theta_k z_k$ ;

this gives

$$\|B^{(\ell)} z_k - D z_k\| \leq \|B^{(\ell)} z_k - B^{(\ell)} \varphi_k\| + \|B^{(\ell)} \varphi_k - \theta_k z_k\|$$
$$\leq \rho(B^{(\ell)}) \|z_k - \varphi_k\| + \theta_k \|\zeta_k - z_k\|$$
$$= O(\alpha^{\ell}) \times \tilde{O}(\alpha^{3d/2n - \gamma/2}) + \tilde{O}(\alpha^{\ell/2})$$
$$= \tilde{O}(\alpha^{\ell/2}).$$

Consider now $x \in \mathbb{R}^V$ such that $\|x\| = 1$. Decomposing $x$ as $\sum x_k z_k + x'$ where $x' \in H^\perp$, we have :

$$\|B^{(\ell)} x - Dx\| \leq \sum_{k \in [r_0]} x_k \|B^{(\ell)} z_k - D z_k\| + \|B^{(\ell)} x' - Dx'\|$$
$$\leq \tilde{O}(\alpha^{\ell/2}) + \|B^{(\ell)} x'\|$$
$$= \tilde{O}(\alpha^{\ell/2}),$$

which completes the proof.

\[\square\]

B Proof of Lemma 3.3

Proof. From Lemma 3.2 we can deduce that if $d(i, j) \leq \ell$, there are at most two distinct paths between $i$ and $j$. Therefore, $B_{ij}^{(\ell)} \leq 2$ for all $i, j$.

Additionally, if $D_{ij}^{(\ell)} = 1$, then there is a self-avoiding path of length $\ell$ between $i$ and $j$, and thus $B_{ij}^{(\ell)} = 1$, so $\Delta_{ij}^{(\ell)} \geq 0$ for all $i, j$.

Finally, assume that there exists a pair $i, j$ such that $D_{ij}^{(\ell)} = 0$ and $B_{ij}^{(\ell)} = 2$ ; then there are two paths of length $\ell$ between $i$ and $j$ and $d(i, j) < \ell$ so there is also a path of length less than $\ell$. This contradicts Lemma 3.2.

Consider now two vertices $i$ and $j$ such that $\Delta_{ij}^{(\ell)} = 1$, there are two possibilities :

(i) $D_{ij}^{(\ell)} = 0$ and $B_{ij}^{(\ell)} > 0$ : then $d(i, j) < \ell$ and there is a path of length $< \ell$ and at least a path of length $\ell$ between $i$ and $j$.

(ii) $D_{ij}^{(\ell)} = 1$ and $B_{ij}^{(\ell)} > 1$ : then there are at least two paths of length $\ell$ between $i$ and $j$.

In both cases, there are at least two paths of length at most $\ell$ connecting $i$ and $j$, which implies the statement of the lemma.

\[\square\]

C Proof of Lemma 3.4

Proof. (i) is obvious since $d(i, C) + d(j, C) \leq \ell$ implies $d(i, C) \leq \ell$. 

21
For (ii), note first that $V_C$ and $V_{C'}$ are disjoint for $C \neq C'$ : if $i \in V_C \cap V_{C'}$, then $C$ and $C'$ are in the $\ell$-neighbourhood of $i$, which contradicts Lemma 3.2.

Let $\pi_C$ be the projection on $V_C$ for all $C$; the $\pi_C$ are mutually orthogonal and for a vector $v$, we have:

$$t_vP^{(\ell)}v = \sum_C t_v\pi_C P^{(\ell)}\pi_C v = \sum_C t(\pi_C v)P^{(\ell)}_C(\pi_C v) \leq \sum_C \rho(P^{(\ell)}_C) \cdot \|\pi_C v\|^2 \leq \max_C \rho(P^{(\ell)}_C) \cdot \sum_C \|\pi_C v\|^2. \quad (41)$$

On the other hand,

$$\|v\|^2 \geq \sum_C \|\pi_C v\|^2. \quad (44)$$

Combining inequalities (43) and (44) yields $\rho(P^{(\ell)}) \leq \max_C \rho(P^{(\ell)}_C)$; the reverse inequality comes from the decomposition $P^{(\ell)} = \sum P^{(\ell)}_C$.

D Proof of Proposition 3.2

Proof. Let $\mathcal{K}$ be the modified vertex set, and consider vertices $i$ and $j$ such that $D^{(\ell)}_{ij} \neq \tilde{D}^{(\ell)}_{ij}$. Then we have one of four possibilities:

(i) $\tilde{d}(i, j) = \ell$ and $d(i, j) < \ell$
(ii) $\tilde{d}(i, j) > \ell$ and $d(i, j) = \ell$
(iii) $\tilde{d}(i, j) = \ell$ and $d(i, j) > \ell$
(iv) $\tilde{d}(i, j) < \ell$ and $d(i, j) = \ell$

In cases (i) and (ii), there is a path between $i$ and $j$ in $G$ through $\mathcal{K}$ of length at most $\ell$, and in cases (iii) and (iv) there is a path between $i$ and $j$ in $\tilde{G}$ through $\mathcal{K}$. Therefore, in all cases, we have that

$$d(i, \mathcal{K}) + d(j, \mathcal{K}) \leq \ell.$$ 

Write $|\tilde{D}^{(\ell)} - D^{(\ell)}|$ for the matrix whose $(i, j)$ coefficient is $|\tilde{D}^{(\ell)}_{ij} - D^{(\ell)}_{ij}|$, and $P_{\mathcal{K}}$ for the matrix such that $P_{\mathcal{K}, i,j} = 1 \{d(i, \mathcal{K}) + d(j, \mathcal{K}) \leq \ell\}$; the previous analysis and the Perron-Frobenius theorem imply that

$$\rho(\tilde{D}^{(\ell)} - D^{(\ell)}) \leq \rho(|\tilde{D}^{(\ell)} - D^{(\ell)}|) \leq \rho(P_{\mathcal{K}}). \quad (45)$$

We can then perform the same analysis as in the proof of Proposition 3.1 to find that the spectral
radius of \( P_K \) is the same as that of

\[
Q_K = \begin{pmatrix}
S_0 & \sqrt{S_0 S_1} & \cdots & \sqrt{S_0 S_{\ell-1}} & \sqrt{S_0 S_\ell} \\
\sqrt{S_0 S_1} & S_1 & \cdots & \sqrt{S_1 S_{\ell-1}} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sqrt{S_0 S_{\ell-1}} & \sqrt{S_1 S_{\ell-1}} & \cdots & 0 & 0 \\
\sqrt{S_0 S_\ell} & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

where we write \( S_t \) instead of \( S_t(K) \) for ease of notation.

Corollary 3.1 then gives \( S_t(K) = O(\alpha^t \log(n)|K|) = o(\alpha^t \tau_{2k}^\ell) \), and the same calculation as in Proposition 3.1 yields:

\[
\rho(Q) = o(\alpha^t \tau_{2k}^\ell) = o(\mu_{2k}^\ell),
\]

and the theorem follows.

\[ E \text{ Proof of Lemma 3.5} \]

\textbf{Proof.} Let \( \varepsilon > 0 \) to be determined later, \( S \) be the set consisting of the \( n^{1-\varepsilon} \) vertices \( i \) with the largest values \( S_i(i) \); we first show that, for all \( i \in S \)

\[
S_i(i) = \Theta(\alpha^\ell).
\]  

Indeed, from Lemma 3.1 we have the following inequalities:

\[
K \alpha^{2\ell} \leq \sum_{i=1}^{n} S_i(i)^{2} \leq n \min_{i \in S} S_i(i)^{2} + \left| \mathcal{S} \right| (C \log(n) \alpha^{\ell})^{2},
\]

and the second term is negligible before the two others, which implies (47).

We then build a set \( K \) of size \( \gamma \) as follows: begin with any member of \( S \), and at each step add a vertex \( x \) such that \( d(x, K) > 2\ell \). This is possible as long as the \( 2\ell \)-neighbourhood of \( K \) does not cover \( S \), i.e. as long as:

\[
\gamma \cdot C \log(n) \alpha^{2\ell} < n^{1-\varepsilon}.
\]

But the LHS of this inequality is bounded by \( C \log(n)n^{3/4} \), so this condition is satisfied as long as \( \varepsilon < 1/4 \).

By this construction, the vertices of \( K \) have \( \ell \)-neighbourhoods that are pairwise disjoint, so by equation (3.5) we have:

\[
S_{\ell}(K) = \Omega(\alpha^\ell \times \gamma).
\]  

\[ \square \]