MONOIDAL INTERVALS OF CLONES ON INFINITE SETS

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Abstract. We show that on an infinite set $X$ of cardinality $\kappa$, if $\mathcal{L}$ is the lattice of order ideals of some partial order $\mathcal{P}$ with smallest element such that $|\mathcal{P}| \leq 2^\kappa$, then there is a monoidal interval in the clone lattice on $X$ which is isomorphic to $\mathcal{L}$. In particular, we find that if $\mathcal{L}$ is any chain with smallest element which is an algebraic lattice, and if $|\mathcal{L}| \leq 2^\kappa$, then $1 + \mathcal{L}$ appears as a monoidal interval; also, if $\lambda \leq \kappa^+$, then the power set of $\lambda$ with an additional smallest element is a monoidal interval. Concerning cardinalities of monoidal intervals these results imply that there are monoidal intervals of all cardinalities smaller than $2^\kappa$, as well as monoidal intervals of cardinality $2^\lambda$, for all $\lambda \leq 2^\kappa$.

1. The problem

Let $X$ be a set of cardinality $\kappa$, and denote for all $n \geq 1$ the $n$-ary operations on $X$ by $\mathcal{O}^{(n)}$. Then $\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)}$ is the set of all finitary operations on $X$. A set of operations $\mathcal{C} \subseteq \mathcal{O}$ is called a clone iff it is closed under composition and contains all projections, that is, all functions of the form $\pi_k^n(x_1, \ldots, x_n) = x_k$ ($1 \leq k \leq n$). The set of all clones on $X$ equipped with the order of set-theoretical inclusion forms a complete algebraic lattice $Cl(X)$. After this introductory section, we are going to work exclusively with an infinite base set $X$, in which case the cardinality of $Cl(X)$ is $2^{2^\kappa}$. For finite $X$ with at least three elements we have $|Cl(X)| = 2^{\aleph_0}$, and $|Cl(X)| = \aleph_0$ if the base set has two elements. Only in the last case the structure of the clone lattice has been completely resolved [?]. If $X$ has at least three elements, then $Cl(X)$ seems to be too large and complicated to be fully understood. One approach to this problem is to partition the clone lattice into so-called monoidal intervals.

Let $\mathcal{G}$ be a submonoid of the monoid of unary operations $\mathcal{O}^{(1)}$. The set of all clones $\mathcal{C}$ with unary part $\mathcal{G}$ (that is, with $\mathcal{C}^{(1)} = \mathcal{G}$, where $\mathcal{C}^{(1)} = \mathcal{C} \cap \mathcal{O}^{(1)}$) forms an interval $\mathcal{I}_q$ of the clone lattice; such intervals are referred to as monoidal. The smallest element of $\mathcal{I}_q$ is obviously $\langle \mathcal{G} \rangle$, the clone generated by $\mathcal{G}$ which in this case consists of all essentially unary functions.

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(i.e. functions depending on only one variable) whose corresponding unary function is an element of \( G \). The largest element of \( \mathcal{I} \) is easily seen to be \( \text{Pol}(G) \), defined to contain precisely those functions \( f \in \mathcal{O} \) for which \( f(g_1, \ldots, g_n) \in G \) whenever \( g_1, \ldots, g_n \) are functions in \( G \). Functions with this property are called polymorphisms of \( G \).

We are interested in the structure of monoidal intervals, in particular in the cardinalities monoidal intervals can have; this question was first posed by Szendrei [1]. One motivation behind this is that if all monoidal intervals were similar in some sense, then the problem of describing the clone lattice would, up to that similarity, be reduced to the description of one monoidal interval, as well as the description of the lattice of all submonoids of \( \mathcal{O}(1) \).

If on the other hand monoidal intervals could take many forms, then this would be another indication that the clone lattice is very complicated.

There is a deeper concept behind the partition of the clone lattice into monoidal intervals. If \( C, D \subseteq \mathcal{O} \) are two distinct clones, then there exists \( n \geq 1 \) such that \( C^{(n)} \neq D^{(n)} \), where \( C^{(n)} = C \cap \mathcal{O}^{(n)} \). Moreover, if this is the case and \( m \geq n \), then also \( C^{(m)} \neq D^{(m)} \). Therefore, we can say that two clones are closer the later their \( n \)-ary parts start to differ. More precisely, the function

\[
d(C, D) = \begin{cases} \frac{1}{2n-1}, & C \neq D \wedge n = \min \{ k : C^{(k)} \neq D^{(k)} \}, \\ 0, & C = D \end{cases}
\]

defines a metric on the clone lattice, first introduced by Machida [2]. Formulated in this metric, a monoidal interval is just an open sphere of radius 1 in the metric space \( (\text{Cl}(X), d) \). It also makes sense to consider refinements of this partition, for example open spheres of radius \( \frac{1}{2} \), or equivalently sets of clones with identical binary parts; they are of the form \( [\mathcal{H}, \text{Pol}(\mathcal{H})] \), where \( \mathcal{H} \subseteq \mathcal{O}^{(2)} \) is a set of binary functions closed under composition and containing the two binary projections.

For a finite base set \( X \) it has been observed by Rosenberg and Sauer [3] that all intervals are either at most countably infinite or of size continuum. We shall give a short argument proving this: On a finite base set, the clone lattice equipped with Machida’s metric is homeomorphic to a closed subset of the Cantor space \( 2^\omega \). To see this, notice first that \( \mathcal{O} \) is countably infinite, and let \( (f_i)_{i < \omega} \) be an enumeration of \( \mathcal{O} \) with the property that for all \( i < j \) the arity of \( f_i \) is not greater than the arity of \( f_j \); this is possible, since \( \mathcal{O}^{(n)} \) is finite for all \( n \geq 1 \). Now we can assign to every set of operations \( \mathcal{C} \subseteq \mathcal{O} \) a sequence \( s(\mathcal{C}) \in 2^\omega \) by defining \( s(\mathcal{C})(i) = 1 \), if \( f_i \in \mathcal{C} \), and \( s(\mathcal{C})(i) = 0 \) otherwise. This gives a bijection from the power set \( \mathcal{P}(\mathcal{O}) \) of \( \mathcal{O} \) onto \( 2^\omega \), and if we extend Machida’s metric from the clone lattice to \( \mathcal{P}(\mathcal{O}) \) (with the same definition), this mapping is easily seen to be a homeomorphism. The set of sequences of \( 2^\omega \) that correspond to clones is a closed subset of \( 2^\omega \). Indeed, set for \( i \in \omega \) and \( j \in 2 \) a set \( A^j_i \) to consist of all \( s \in 2^\omega \) with \( s(i) = j \); the \( A^j_i \) form a clopen subbasis of the topology of \( 2^\omega \). Now the property that
\[ \mathcal{C} \subseteq \mathcal{O} \text{ contains all projections is equivalent to } s(\mathcal{C}) \text{ being an element of } \Lambda_1 = \bigcap \{ A^1_1 : f_1 \text{ projection} \}. \] Moreover, that \( \mathcal{C} \) is closed under composition can be stated in the language of sequences by saying that \( s(\mathcal{C}) \) is an element of
\[ \Lambda_1 = \bigcap \{ A^1_1 \cup A^0_{i_0} \cup A^1_j : f_j = f_{i_0}(f_{i_1}, \ldots, f_{i_n}) \}. \]
Thus \( \mathcal{C} \subseteq \mathcal{O} \) is a clone iff \( s(\mathcal{C}) \) is an element of \( \Lambda = \Lambda_1 \cap \Lambda_2 \), a closed set since both \( \Lambda_i \) are intersections of closed sets and hence closed themselves.

\( (\text{Cl}(\mathcal{X}), d) \) is indeed homeomorphic to a closed subset of \( 2^{\omega} \), which immediately yields the topological properties of the clone space proven in \([?]\).

Now if \( \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{O} \), then the interval \([\mathcal{C}_1, \mathcal{C}_2]\) in the power set of \( \mathcal{O} \) corresponds to the interval \([s(\mathcal{C}_1), s(\mathcal{C}_2)]\) in \( 2^{\omega} \) with the pointwise order, a closed set. Therefore it satisfies the continuum hypothesis (see \([?]\) for basics of descriptive set theory). Also, if \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are clones, then the interval \([\mathcal{C}_1, \mathcal{C}_2]\) in \( \text{Cl}(\mathcal{X}) \) corresponds to \([s(\mathcal{C}_1), s(\mathcal{C}_2)] \cap \Delta \) in \( 2^{\omega} \), again a closed set.

We conclude that all intervals of the clone lattice on a finite set satisfy \( \text{CH} \).

The same argument does not work for infinite sets, and we shall prove that on a countably infinite set there exist monoidal intervals of all cardinalities between \( \aleph_0 \) and \( 2^{\aleph_0} \).

Of the possible sizes finite, \( \aleph_0 \), and \( 2^{\aleph_0} \) for monoidal intervals over a finite set with at least three elements, all possibilities occur: There must be a monoidal interval of size continuum, since there exist only finitely many monoids and \( |\text{Cl}(\mathcal{X})| = 2^{\aleph_0} \). Also finite sizes appear, for example the interval corresponding to the monoid \( \mathcal{O}^{(1)} \) is of size \( |X| + 1 \) (\([?]\)), and we will see in this paper that the permutation group is an example of a monoid whose monoidal interval has only one element (for infinite \( X \), but the same proof works on finite sets). See \([?], [?], [?]\) for more examples. However, for a fixed set, only finitely many finite numbers appear as sizes of monoidal intervals, again because there exist only finitely many monoids. Krokhin \([?]\) proved that there exist countably infinite monoidal intervals over a finite set.

Goldstern and Shelah \([?]\) showed that on a countably infinite base set, many monoids define a monoidal interval which is as large as the clone lattice \( (2^{\aleph_0}) \). Starting from this result, we investigated the question whether all monoidal intervals on infinite sets are that large, and found that the situation is much more diverse.

2. Results

Let \( \mathcal{P} \) be a partial order. The set of all order ideals (also called lower subsets) on \( \mathcal{P} \) with the operations of set-theoretical intersection and union is a complete algebraic lattice, a sublattice of the power set of \( \mathcal{P} \). We are going to prove the following
Theorem 1. Let $X$ be an infinite set of size $\kappa$. If $\mathcal{P}$ is any partial order with smallest element which has cardinality at most $2^\kappa$, and if $\mathcal{L}$ is the lattice of order ideals on $\mathcal{P}$, then there exists a monoidal interval in the clone lattice over $X$ which is isomorphic to $\mathcal{L}$.

It is well-known that the class of lattices of order ideals is exactly the class of completely distributive algebraic lattices. Therefore we have

Corollary 2. Let $\mathcal{L}$ be a completely distributive algebraic lattice with at most $2^\kappa$ completely join irreducible elements. Then there is a monoidal interval in $\text{Cl}(X)$ isomorphic to $1 + \mathcal{L}$, which is to denote $\mathcal{L}$ plus a new smallest element added.

As an immediate consequence we obtain

Corollary 3. Let $\lambda \leq 2^\kappa$. Then there is a monoidal interval isomorphic to $1 + \mathcal{P}(\lambda)$, where $1 + \mathcal{P}(\lambda)$ is the power set of $\lambda$ with a new smallest element added.

Let $\mathcal{L}$ be a chain which is complete as a lattice. An element $p \in \mathcal{L}$ is called a successor iff there exists $q \in \mathcal{L}$ with $q <_\mathcal{L} p$ such that the interval $[q, p]_\mathcal{L}$ contains only $p$ and $q$. Obviously, the compact elements of $\mathcal{L}$ are exactly the successors and the smallest element of $\mathcal{L}$. Therefore, $\mathcal{L}$ is a complete algebraic lattice iff the successors are unbounded below every $p \in \mathcal{L}$.

Corollary 4. Let $\mathcal{L}$ be any chain of size at most $2^\kappa$ which is a complete algebraic lattice. Then there is a monoidal interval isomorphic to $1 + \mathcal{L}$, which is $\mathcal{L}$ plus a new smallest element added.

Remark 5. Since $\text{Cl}(X)$ is an algebraic lattice, all its intervals are algebraic. Also, $\text{Cl}(X)$ cannot contain any chains larger that $2^\kappa$, since there exist only $2^\kappa$ finitary functions on $X$. Hence, these chains are all chains which can occur as monoidal intervals (up to the additional smallest element).

Corollary 6. If $1 \leq \mu \leq 2^\kappa$ is an ordinal, then there is a monoidal interval with the order of $\mu$.

Corollary 7. On infinite $X$ of size $\kappa$, there exist at least monoidal intervals of the following cardinalities:

- $\lambda$ for all $\lambda \leq 2^\kappa$.
- $2^\lambda$ for all $\lambda \leq 2^\kappa$.

Being complete sublattices of the power set of the base set of the partial order, the monoidal intervals exposed in our theorem are all completely distributive, and therefore still quite special lattices. Therefore not surprisingly, they are not all monoidal intervals that can appear.

Proposition 8. There exists a nonmodular monoidal interval.

Proof. Let $X$ be linearly ordered, and write $\min(x_1, x_2)$ for the minimum function, $\text{med}(x_1, x_2, x_3)$ for the median function, and $\max(x_1, x_2)$ for the
maximum function with respect to that linear order. Denote by Proj the clone of projections. Then

\[
\langle\{\min, \max\}\rangle \\
\langle\{\min, \text{med}\}\rangle \\
\langle\{\min\}\rangle \\
\langle\{\max\}\rangle
\]

is a sublattice of the monoidal interval corresponding to the trivial monoid \(\{\pi_1\}\). That \(\langle\{\min, \text{med}\}\rangle \cap \langle\{\max\}\rangle = \text{Proj}\) follows from [?] but is also not difficult to verify. □

The fact that monoidal intervals must be algebraic lattices with no more than \(2^\kappa\) compact (in the clones lattice, this means finitely generated) elements is the only restriction for them we know of. Therefore we pose the following problem.

**Problem 9.** If \(\mathcal{L}\) is any algebraic lattice with at most \(2^\kappa\) compact elements, is there a monoidal interval isomorphic to \(\mathcal{L}\)?

Concerning cardinalities our theorem leaves the following cases open:

**Problem 10.** Are the cardinalities of Corollary all possible sizes of monoidal intervals? That is, if \(2^\kappa < \lambda < 2^{2^\kappa}\) and \(\lambda\) is not a cardinality of a power set, does there exist a monoidal interval of size \(\lambda\)?

2.1. **Notation.** The smallest clone containing a set \(\mathcal{F} \subseteq \mathcal{O}\) shall be denoted by \(\langle\mathcal{F}\rangle\); moreover, we write \(\mathcal{F}^*\) for the set of all functions which arise from functions of \(\mathcal{F}\) by identification of variables, addition of fictitious variables, or permutation of variables. For \(n \geq 1\) we denote the \(n\)-ary operations on \(X\) by \(\mathcal{O}^{(n)}\); if \(\mathcal{F} \subseteq \mathcal{O}\), then \(\mathcal{F}^{(n)}\) will stand for \(\mathcal{F} \cap \mathcal{O}^{(n)}\). We will see \(X\) equipped with a vector space structure; then we write \(\text{span}(S)\) for the subspace of \(X\) generated by a set of vectors \(S \subseteq X\). We shall denote the zero vector of \(X\) by \(0\), and use the same symbol for the constant function with value \(0\). We write \(\mathcal{L}\) for the set of linear functions on \(X\). The sum \(f + g\) of two linear functions \(f, g\) on \(X\) is defined pointwise, as is the binary function \(f(x) + g(y)\) obtained by the sum of two unary functions of different variables. The range of a function \(f \in \mathcal{O}\) is given the symbol \(\text{ran} f\). For a set \(Y\) we write \(\mathcal{P}(Y)\) for the power set of \(Y\) and \(\mathcal{P}_{\text{fin}}(Y)\) for the set of finite subsets of \(Y\).
3. Monoids of linear functions

Given any partial order $\mathcal{P}$ with $|\mathcal{P}| = \lambda \leq 2^\kappa$, we construct a monoid $\mathcal{M}$ such that $\mathcal{I}_\mathcal{M}$ is isomorphic to $1 + \mathcal{L}$, where $\mathcal{L}$ is the lattice of order ideals of $\mathcal{P}$.

Equip $X$ with a vector space structure of dimension $\kappa$ over any field $K$ of characteristic $\neq 2, 3$ and fix a basis $B$ of $X$. Fix moreover three distinguished elements $a, b, c \in B$ and write $A = B \setminus \{a, b, c\}$.

Next we want to introduce a preferably natural notion of “small” for subsets of $A$; in fact, we are looking for an order ideal $\mathcal{I}$ in $\mathcal{P}(A)$ extending the ideal $\mathcal{P}_{fin}(A)$ which is invariant under permutations of $A$ (i.e., if $S \in \mathcal{I}$ then also $\alpha[S] \in \mathcal{I}$ for all permutations $\alpha$ of $X$), such that if we factorize $\mathcal{P}(A)$ by this ideal, then the resulting partial order has an antichain of length $\lambda$.

Since we want to prove our theorem for all $\lambda \leq 2^\kappa$, we need the existence of an antichain of length $2^\kappa$, i.e., as large as $\mathcal{P}(A)$. It is quite obvious that the only order ideals in $\mathcal{P}(A)$ that are invariant under permutations are the $\mathcal{I}_\xi = \{S \subseteq A : |S| < \xi\}$, and the $\mathcal{I}_\xi = \{S \subseteq A : |X \setminus S| \geq \xi\}$, where $\xi \leq \kappa$ is a cardinal. For $X$ countably infinite, the ideal $\mathcal{P}_{fin}(A) = \mathcal{I}_{\aleph_0}$ satisfies our requirement for the antichain. For there exists an almost disjoint family $\mathcal{A}$ of subsets of $A$ of size $2^{\aleph_0}$, meaning that all sets of $\mathcal{A}$ are infinite and whenever $A_1, A_2 \in \mathcal{A}$ are distinct, then $A_1 \cap A_2$ is finite (see the textbook [?]). The reader interested in countably infinite base sets only can imagine this ideal in the following. However, it is consistent with ZFC that almost disjoint families of size $2^\kappa$ fail to exist on uncountable $\kappa$, even if we consider $\mathcal{I}_\kappa$ instead of $\mathcal{I}_{\aleph_0}$ and replace “$A_1 \cap A_2$ is finite” by the weaker “$|A_1 \cap A_2| < \kappa$”. Moreover, if $\mathcal{I}_\kappa$ does not give us an antichain of desired length, then the other ideals will not work either, so we have to do something less elegant: Fix any family $\mathcal{A} \subseteq \mathcal{P}(A)$ of subsets of $A$ of cardinality $\kappa$ such that $|\mathcal{A}| = \lambda$, and such that $A_1 \not\subseteq A_2$ for all distinct $A_1, A_2 \in \mathcal{A}$. Such families exist; see the textbook [?, Lemma 7.7] for a proof of this. Now we set the ideal $\mathcal{I}$ to consist of all proper subsets of sets in $\mathcal{A}$, plus all finite sets, and call the sets of $\mathcal{I}$ small. Obviously, $\mathcal{I}$ is only an order ideal (no lattice ideal) and quite arbitrary compared to the ideal of finite subsets of $A$ which we can use for countably infinite $X$. Note also that we had to give up invariance under permutations of $A$; however, it will be sufficient that if $\alpha$ maps $A_1$ bijectively onto $A_2$, where $A_1, A_2 \in \mathcal{A}$, and if $S \subseteq A_1$ is small, then $\alpha[S]$ is small. Clearly, the sets of $\mathcal{A}$ are not elements of $\mathcal{I}$, but their nontrivial intersections are. We index the family $\mathcal{A}$ by the elements of $\mathcal{P}$: $\mathcal{A} = (A_p)_{p \in \mathcal{P}}$.

The monoid $\mathcal{M}$ we are going to construct will be one of linear functions on the vector space $X$, the set of which we denote by $\mathcal{L}$. We shall sometimes speak of the support of a linear function $f$, by which we mean the subset of $A$ of those basis vectors which $f$ does not send to 0. The monoid $\mathcal{M}$ will be the union of seven classes of functions, plus the zero function. Three classes, namely $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$, do “almost nothing”, in the sense that
they have small support; $\mathcal{N}$ essentially guarantees that the polymorphisms $\text{Pol}(\mathcal{M})$ of the monoid $\mathcal{M}$ are sums of linear functions, and $\mathcal{N}'$ and $\mathcal{N}''$ are auxiliary functions necessary for the monoid to be closed under composition. The class $\Phi$ represents the elements of the partial order $\mathcal{P}$, the class $\Psi$ its order. Finally, the classes $\mathcal{L}_\Phi$ and $\mathcal{L}_\Psi$ ensure that there exist nontrivial polymorphisms of the monoid, and that they correspond to elements of the partial order.

We start with the set $\mathcal{N}$ of those linear functions $n \in \mathcal{L}$ which satisfy the following conditions:

- $n(a) = a$
- $n(b) = 0$
- $n(c) = c$
- $n$ has small support.

Next we add the set $\mathcal{N}' \subseteq \mathcal{L}$ consisting of all linear functions $n'$ for which:

- $n'(a) = 0$
- $n'(b) = 0$
- $n'(c) = b$
- $n'$ has small support
- $\text{ran } n' \subseteq \text{span}\{b\}$.

The class $\mathcal{N}''$ contains all $n'' \in \mathcal{L}$ with

- $n''(a) = a$
- $n''(b) = 0$
- $n''(c) = 0$
- $n''$ has small support
- $\text{ran } n'' \subseteq \text{span}\{a\}$.

Observe that all functions $f$ in these three classes have small support, and that the range of the functions of $\mathcal{N}'$ and $\mathcal{N}''$ is only a one-dimensional subspace of $X$.

Now we define for all $p \in \mathcal{P}$ a function $\phi_p \in \mathcal{L}$ by setting

- $\phi_p(a) = 0$
- $\phi_p(b) = 0$
- $\phi_p(c) = b$
- $\phi_p(d) = b$ for all $d \in A_p$
- $\phi_p(d) = 0$ for all other $d \in B$.

So $\phi_p$ is essentially the characteristic function of $A_p$. Observe that $\text{ran } \phi_p \subseteq \text{span}\{b\}$. We write $\Phi = \{\phi_p : p \in \mathcal{P}\}$.

We fix for all $p, q \in \mathcal{P}$ with $q \leq p$ a function $\psi_{p,q} \in \mathcal{L}$ such that

- $\psi_{p,q}$ maps $A_q$ bijectively onto $A_p$
- $\psi_{p,q}(a) = a$
- $\psi_{p,q}(b) = 0$
- $\psi_{p,q}(c) = c$
\( \psi_{p,q}(d) = 0 \) for all other \( d \in B \)

- If \( q \leq p \), then \( \psi_{p,r} \circ \psi_{r,q} = \psi_{p,q} \).

This is possible: Let \( Y \) be a set of cardinality \( \kappa \) and choose for all \( p \in \Psi \) a bijection \( \mu_p \) mapping \( A_p \) onto \( Y \). Then setting \( \psi_{p,q}(d) = \mu_p^{-1} \circ \mu_q(d) \) for all \( d \in A_q \), \( \psi_{p,q}(a) = a \), \( \psi_{p,q}(c) = c \), and \( \psi_{p,q}(d) = 0 \) for all remaining \( d \in B \) yields the required functions. We set \( \Psi = \{ \psi_{p,q} : p, q \in \Psi, q \leq p \} \). The idea behind \( \psi_{p,q} \) is that it "translates" the function \( \phi_p \) of \( \Phi \) into the function \( \phi_q \), and that such a translation function exists only if \( q \leq p \). More precisely we have

**Lemma 11.** Let \( \phi_r \in \Phi \) and \( \psi_{p,q} \in \Psi \). If \( r = p \), then \( \phi_r \circ \psi_{p,q} = \phi_q \); otherwise, \( \phi_r \circ \psi_{p,q} \in \mathcal{N}' \).

**Proof.** Assume first that \( r = p \). Then in the composite \( \phi_r \circ \psi_{p,q} \), first \( \psi_{p,q} \) maps \( A_q \) onto \( A_p \), and all other vectors of \( A \) to 0, and then \( \phi_r \) sends \( A_r = A_p \) to \( b \), so that the composite indeed sends \( A_q \) to \( b \) and all other vectors of \( A \) to 0, as does \( \phi_q \); one easily checks that also the extra conditions on \( a, b, c \in B \) are satisfied. If on the other hand \( r \neq p \), then the only basis vectors in \( A \) which \( \phi_r \circ \psi_{p,q} \) does not send to zero are those in \( \psi_{p,q}^{-1}[A_r \cap A_p] \), a small set since \( \psi_{p,q} \) is one-one on its support and by the properties of the family \( \mathcal{A} \). Moreover, \( \text{ran}(\phi_r \circ \psi_{p,q}) \subseteq \text{ran}(\phi_r) \subseteq \text{span}(\{b\}) \). Hence, since also the respective additional conditions on \( a, b, c \in B \) are satisfied we have \( \phi_r \circ \psi_{p,q} \in \mathcal{N}' \).

The remaining functions to be added to our monoid are those of the form \( \phi_p + n'' \), where \( \phi_p \in \Phi \) and \( n'' \in \mathcal{N}'' \), the set of which we denote by \( \mathcal{I}_{\Phi} \), and all functions of the form \( n' + n'' \), where \( n' \in \mathcal{N}' \) and \( n'' \in \mathcal{N}'' \); this set we call \( \mathcal{I}_{\mathcal{N}'} \). The elements \( f \) of \( \mathcal{I}_{\Phi} \) and \( \mathcal{I}_{\mathcal{N}'} \) both satisfy

- \( f(a) = a \)
- \( f(b) = 0 \)
- \( f(c) = b \).

We set \( M = \mathcal{N} \cup \mathcal{N}' \cup \mathcal{N}'' \cup \Phi \cup \Psi \cup \mathcal{I}_{\Phi} \cup \mathcal{I}_{\mathcal{N}'} \cup \{0\} \). Observe the following properties which hold for all \( f \in M \) and which will be useful:

- \( f(a) \in \{0, a\} \)
- \( f(b) = 0 \)
- \( f(c) \in \{0, b, c\} \).

**Lemma 12.** \( M \) is a monoid.

**Proof.** The following table describes the composition of the different classes of functions in \( M \). Here, the meaning of \( \mathcal{X} \circ \mathcal{Y} = \mathcal{Z} \) is: Whenever \( f \in \mathcal{X} \) and \( g \in \mathcal{Y} \), then \( f \circ g \in \mathcal{Z} \).
If we get the Φ-column.

We check the fields of the table. The fact that ran $n' \subseteq \text{span}(\{b\})$ for all $n' \in \mathcal{N}'$ and $f(b) = 0$ for all $f \in \mathcal{M}$ yields the $\mathcal{N}'$-column; in the same way we get the Φ-column.

If $g = \phi_p + n'' \in \mathcal{I}_\Phi$ and $f \in \mathcal{M}$, then $f \circ g = f \circ \phi_p + f \circ n'' = f \circ n''$, so the $\mathcal{I}_\Phi$-column is equal to the $N''$-column, and the same holds for the $\mathcal{I}_{N''}$-column.

We turn to the $\mathcal{N}$- and $\mathcal{N}''$-columns. The $\mathcal{I}_\Phi$- and the $\mathcal{I}_{N'}$-row are the sum of the $\Phi$- and the $\mathcal{N}'$-row with the $\mathcal{N}''$-row, respectively, since $(f + g) \circ h = (f \circ h) + (g \circ h)$ for all $f, g, h \in \mathcal{O}(1)$. For the other rows of those columns, note that if $f, g \in \mathcal{L}$ and $g$ has small support, then also $f \circ g$ has small support. It is left to the reader to check the conditions on $a, b, c \in B$ and on the range for the composites.

It remains to verify the $\Psi$-column. For the first row, observe that since all $n \in \mathcal{N}$ have small support and since $\psi_{p,q}^{-1}(S)$ is small for all small $S \subseteq A$ and all $\psi_{p,q} \in \Psi$ by the properties of $\mathcal{A}$, any composition $n \circ \psi_{p,q}$ will have small support. Thus, together with the readily checked fact that the extra conditions on $a, b, c \in B$ are satisfied we get that $n \circ \psi_{p,q} \in \mathcal{N}$. The same argument yields the $\mathcal{N}'$- and $\mathcal{N}''$-rows.

The Φ-row is a consequence of Lemma 11. Similarly to the proof of that lemma, we show that $\psi_{p,s} \circ \psi_{t,q}$ is an element of $\mathcal{N}$ unless $s = t$, in which case it is $\psi_{p,q}$ by construction. Indeed, assume $s \neq t$; then $\psi_{t,q}$ takes $A_t$ to $A_t$, but $\psi_{p,s}$ has support $A_s$; therefore, the composite $\psi_{p,s} \circ \psi_{t,q}$ has support $\psi_{t,q}^{-1}[A_t \cap A_s]$, a small set since $\psi_{t,q}$ is injective on its support and by the properties of the family $\mathcal{A}$. The conditions on $a, b, c$ for the composite to be in $\mathcal{N}$ are left to the reader, and we are done with the $\Psi$-row.

The $\mathcal{I}_\Phi$- and $\mathcal{I}_{N'}$-rows are the sums of the $\mathcal{N}''$-row with the $\Phi$-row and the $\mathcal{N}'$-row respectively, by the definitions of $\mathcal{I}_\Phi$ and $\mathcal{I}_{N'}$.

Recall that if $\mathcal{F} \subseteq \mathcal{O}$, then $\mathcal{F}^*$ consists of all functions which arise from functions of $\mathcal{F}$ by identification of variables, adding of fictitious variables, as well as by permutation of variables. Functions in $\mathcal{F}^*$ are called polymers of functions on $\mathcal{F}$. Set

$$\mathcal{V} = \{n'(x) + n''(y) : n' \in \mathcal{N}', n'' \in \mathcal{N}'\}.$$  

Moreover, define for all $I \subseteq \Psi$ sets of functions

$$\mathcal{D}_I = \{\phi_p(x) + n''(y) : p \in I, n'' \in \mathcal{N}''\}.$$
and
\[ \mathcal{C}_I = (\mathcal{M} \cup \mathcal{V} \cup \mathcal{D}_I)^*. \]

Observe that \( \mathcal{D}_\Phi \) is the set of all functions of the form \( \phi_p(x) + n''(y) \), where \( \phi_p \in \Phi \) and \( n'' \in \mathcal{N}'' \).

**Lemma 13.** Let \( I \subseteq \Psi \) be an order ideal. Then \( \mathcal{C}_I \) is a clone in \( \mathcal{I}_\Phi \).

**Proof.** We first show that \( \mathcal{C}_I^{(1)} = \mathcal{M} \). Indeed, by its definition the unary functions in \( \mathcal{C}_I \) are exactly \( \mathcal{M} \) and those functions which arise when one identifies the two variables of a function in \( \mathcal{V} \cup \mathcal{D}_I \). If \( f \in \mathcal{V} \cup \mathcal{D}_I \), then \( f = n'(x) + n''(y) \) or \( f = \phi_p(x) + n''(y) \). Identifying its variables, we obtain a function of \( \mathcal{I}_\Psi \) in the first and of \( \mathcal{D}_\Phi \) in the second case, and in either case an element of \( \mathcal{M} \). Therefore, the unary part of \( \mathcal{C}_I \) is exactly \( \mathcal{M} \) and \( \mathcal{C}_I \), if a clone, is indeed an element of \( \mathcal{I}_\Phi \).

\( \mathcal{C}_I \) contains \( \pi_1 \in \mathcal{M} \) and therefore all projections, as it is by definition closed under the addition of fictitious variables.

We prove that \( \mathcal{C}_I \) is closed under composition. To do this it suffices to prove that if \( f(x_1, \ldots, x_n), g(y_1, \ldots, y_m) \in \mathcal{C}_I \), then \( f(x_1, \ldots, x_{i-1}, g(y_1, \ldots, y_m), x_{i+1}, \ldots, x_n) \in \mathcal{C}_I \), for all \( 1 \leq i \leq n \). Moreover, since \( \mathcal{C}_I \) is closed under the addition of fictitious variables, we may assume that \( f, g \) depend on all of their variables, so by the definition of \( \mathcal{C}_I \) they are at most binary; since within \( \mathcal{C}_I \) we can freely permute variables, we can assume \( f, g \in \mathcal{M} \cup \mathcal{V} \cup \mathcal{D}_I \). Also, since \( \mathcal{C}_I \) is by definition closed under identification of variables, we may assume that \( y_i \) and \( x_j \) are different variables, for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

Let first \( f \in \mathcal{M} \). If we substitute any \( g \in \mathcal{M} \) for the only variable of \( f \), then we stay in \( \mathcal{M} \subseteq \mathcal{C}_I \) since \( \mathcal{M} \) is a monoid by Lemma 12. If \( g \) is binary and of the form \( m'(x) + m''(y) \in \mathcal{V} \), then by Lemma 12 we have \( f(m'(x) + m''(y)) = (f(m'(x)) + f(m''(y))) = (m'(x)) + (m''(y)) \in \mathcal{M} \subseteq \mathcal{C}_I \), since the unary function \( f \circ m'' \in \mathcal{M} \) as \( \mathcal{M} \) is a monoid. Similarly, if \( g = \phi_p(x) + m''(y) \in \mathcal{D}_I \) we get \( f(\phi_p(x) + m''(y)) = f(\phi_p(x)) + f(m''(y)) = (m''(y)) \in \mathcal{M}^* \).

We proceed with the case where \( f \) is binary, so \( f \in \mathcal{V} \cup \mathcal{D}_I \). Assume \( f = n'(x) + n''(y) \in \mathcal{V} \), and that we substitute a unary \( g(z) \in \mathcal{M} \) for \( x \). By Lemma 12, \( n' \circ g \in \mathcal{N} \cup \{0\} \); hence, \( f(g(z), y) \) is a function of the form \( m'(z) + n''(y) \in \mathcal{V} \) if \( n' \circ g \in \mathcal{N}' \), and the essentially unary function \( n''(y) \in \mathcal{N}'' \) if \( n' \circ g = 0 \). If we substitute a unary \( g(z) \in \mathcal{M} \) for \( y \), then \( n'' \circ g \in \mathcal{N}'' \cup \{0\} \), so that again we stay in \( \mathcal{V} \cup \mathcal{M}^* \). So say that \( f = \phi_p(x) + n''(y) \in \mathcal{D}_I \), and that we substitute a unary \( g(z) \in \mathcal{M} \) for \( x \).

From Lemma 12 we know that \( \phi_p \circ g \in \mathcal{N}' \cup \Phi \cup \{0\} \). If \( g \) vanishes, then we obtain an essentially unary function in \( \mathcal{N}''^* \subseteq \mathcal{M}^* \) for \( f(g(z), y) \). If \( \phi_p \circ g \in \mathcal{N}' \), then the sum with \( n''(y) \) is in \( \mathcal{V} \). The interesting case is the one where \( \phi_p \circ g \in \Phi \); from the proof of Lemma 12 we know that this can only happen if \( g \) equals some \( \psi_{s,t} \in \Psi \). Moreover, from Lemma 11 we infer that the composition is only in \( \Phi \) if \( s = p \), and then we have \( \phi_p \circ \psi_{p,t} = \phi_t \). Hence in this case, \( f(g(z), y) = \phi_t(z) + n''(y) \in \mathcal{D}_I \) since \( t \leq p \in I \). To finish the case where we substitute a unary function for a variable of a binary function, let \( f = \phi_p(x) + n''(y) \) and substitute \( g(z) \in \mathcal{M} \) for \( y \). Then, since
Proof. Given remaining basis vectors to 0. \( \square \)

Lemma is satisfied for all \( k \) which contains

\( 0 \leq k \) for 1 \( \square \)

Let

We now substitute binary functions \( g(v, w) \in \mathcal{V} \cup \mathcal{D}_I \) into one variable of a binary \( f(x, y) \in \mathcal{V} \cup \mathcal{D}_I \). Let \( g(v, w) = m'(v) + m''(w) \in \mathcal{V} \). Since \( h \circ m' = 0 \) for all \( h \in \mathcal{M} \), and \( f(x, y) \) is of the form \( f_1(x) + f_2(y) \) for some \( f_1, f_2 \in \mathcal{M} \), and since all involved functions are linear, \( m' \) will vanish in any substitution with \( g \). Therefore substituting \( g \) is the same as substituting only an essentially unary function, which we already discussed. So let \( g(v, w) = \phi_q(v) + m''(w) \). Then again, \( h \circ \phi_q = 0 \) for all \( h \in \mathcal{M} \), so substitution of \( g \) is equivalent to substituting only \( m''(y) \) and we are done.

\( \square \)

We now prove that \( \langle \mathcal{M} \rangle \) and the \( \mathcal{G}_I \) are the only clones in \( \mathcal{I}_\mathcal{M} \).

**Lemma 14.** Let \( \mathcal{G} \) be a monoid of linear functions on the vector space \( X \) which contains the constant function 0, and let \( k \geq 1 \) be a natural number. If for any finite sequence of vectors \( d_1, \ldots, d_k \in X \) there exist \( e_1, \ldots, e_k \in X \) and \( h_1, \ldots, h_k \in \mathcal{G} \) such that \( h_j(e_j) = d_j \) and \( h_j(e_i) = 0 \) for all \( 1 \leq i, j \leq k \) with \( i \neq j \), then all functions in \( \text{Pol}(\mathcal{G})^{(k)} \) are of the form \( g_1(x_1) + \ldots + g_k(x_k) \), with \( g_1, \ldots, g_k \in \mathcal{G} \).

Proof. Let \( F(x_1, \ldots, x_k) \in \text{Pol}(\mathcal{G})^{(k)} \). Since 0 \( \in \mathcal{G} \), the functions \( g_j(x_j) = F(0, \ldots, 0, x_j, 0, \ldots, 0) \) are elements of \( \mathcal{G} \) for all \( 1 \leq j \leq k \). We claim \( F(d_1, \ldots, d_k) = g_1(d_1) + \ldots + g_k(d_k) \) for all \( d_1, \ldots, d_k \in X \). Indeed, let \( e_1, \ldots, e_k \in X \) and \( h_1, \ldots, h_k \in \mathcal{G} \) be provided by the assumption of the lemma. Then \( h(x) = F(h_1(x), \ldots, h_k(x)) \) is an element of \( \mathcal{G} \); therefore it is linear. Hence,

\[
\begin{align*}
    h(e_1 + \ldots + e_k) &= h(e_1) + \ldots + h(e_k) \\
                      &= F(h_1(e_1), \ldots, h_k(e_1)) + \ldots + F(h_1(e_k), \ldots, h_k(e_k)) \\
                      &= F(d_1, 0, \ldots, 0) + \ldots + F(0, \ldots, 0, d_k) \\
                      &= g_1(d_1) + \ldots + g_k(d_k).
\end{align*}
\]

On the other hand,

\[
\begin{align*}
    h(e_1 + \ldots + e_k) &= F(h_1(e_1 + \ldots + e_k), \ldots, h_k(e_1 + \ldots + e_k)) \\
                        &= F(h_1(e_1) + \ldots + h_1(e_k), \ldots, h_k(e_1) + \ldots + h_k(e_k)) \\
                        &= F(d_1, \ldots, d_k).
\end{align*}
\]

This proves the lemma. \( \square \)

**Lemma 15.** Let \( \mathcal{G} \) be a monoid of linear functions on the vector space \( X \) which contains 0. If \( \mathcal{G} \) contains \( \mathcal{N} \), then the condition of the preceding lemma is satisfied for all \( k \geq 1 \).

Proof. Given \( d_1, \ldots, d_k \in X \) we choose any distinct \( e_1, \ldots, e_k \in A \). Now for \( 1 \leq j \leq k \) we define \( h_j \in \mathcal{N} \) to map \( e_j \) to \( d_j \), a to a, c to c, and all remaining basis vectors to 0. \( \square \)
Lemma 16. Let $f, g \in \mathcal{M}$ be nonconstant. If $f + g \in \mathcal{M}$, then $f \in \mathcal{N}' \cup \Phi$ and $g \in \mathcal{N}''$ (or the other way round).

Proof. Observe where the nontrivial functions of $\mathcal{M}$ map $a, c \in B$:

| $\mathcal{N}$ | $a$ | $c$ |
|---------------|-----|-----|
| $\mathcal{N}'$ | 0   | $b$ |
| $\mathcal{N}''$ | $a$ | 0   |
| $\Phi$         | 0   | $b$ |
| $\Psi$         | $a$ | $c$ |
| $\mathcal{L}_\Phi$ | $a$ | $b$ |

All functions $f \in \mathcal{M}$ satisfy $f(a) \in \{a, 0\}$ and $f(c) \in \{b, c, 0\}$. Hence, if $f + g \in \mathcal{M}$, then $f + g(a) = f(a) + g(a) \in \{a, 0\}$ and $f + g(c) = f(c) + g(c) \in \{b, c, 0\}$. Since the field $K$ has characteristic $\neq 2, 3$ we have that $a + a, b + b, c + c, b + c \notin \{0, a, b, c\}$. Thus it can be seen from the table that if $f(a) + g(a) \in \{a, 0\}$, then at least one of the functions must map $a$ to 0 and thereby be an element of $\mathcal{N}' \cup \Phi$. From the condition $f(c) + g(c) \in \{b, c, 0\}$ we infer that either $f$ or $g$ must map $c$ to 0 and hence belong to $\mathcal{N}''$. This proves the lemma. \hfill $\square$

Lemma 17. Let $f, g, h \in \mathcal{M}$ be nonconstant. Then $f + g + h \notin \mathcal{M}$.

Proof. Since $K$ has characteristic $\neq 2, 3$ we have that no sum of two or three elements of $\{a, b, c\}$ is an element of $\{0, a, b, c\}$. If $f + g + h \in \mathcal{M}$, then $f(a) + g(a) + h(a) \in \{a, 0\}$. This implies that at least two of the three functions have to map $a$ to 0 and therefore belong to $\mathcal{N}' \cup \Phi$. Also, $f(c) + g(c) + h(c) \in \{b, c, 0\}$, from which we conclude that at least two functions must map $c$ to 0 and thus be elements of $\mathcal{N}''$. So one function would have to be both in $\mathcal{N}' \cup \Phi$ and in $\mathcal{N}''$ which is impossible. Hence, $f + g + h \notin \mathcal{M}$. \hfill $\square$

Lemma 18. $\text{Pol}(\mathcal{M}) = \mathcal{G}_\Phi$. In particular, all functions in $\text{Pol}(\mathcal{M})$ depend on at most two variables.

Proof. Since $\mathcal{G}_\Phi$ is a clone with unary part $\mathcal{M}$ by Lemma 16, we have that $\mathcal{G}_\Phi \subseteq \text{Pol}(\mathcal{M})$. To see the other inclusion, let $F(x_1, \ldots, x_k) \in \text{Pol}(\mathcal{M})^{(k)}$. Then by Lemma 15 $F(x_1, \ldots, x_k) = f_1(x_1) + \ldots + f_k(x_k)$, with $f_i \in \mathcal{M}$, $1 \leq i \leq k$. We show $F \in \mathcal{G}_\Phi$; since clones are closed under the addition of fictitious variables, we may assume that $F$ depends on all of its variables, i.e. $f_i$ is nontrivial for all $1 \leq i \leq k$. If $k = 1$, then $F \in \mathcal{M}$, so $F \in \mathcal{G}_\Phi$. If $k = 2$, then since $F(x, x) = (f_1 + f_2)(x)$ has to be an element of $\mathcal{M}$, Lemma 16 implies that up to permutation of variables, $F \in \mathcal{V} \cup \mathcal{D}_I \subseteq \mathcal{G}_\Phi$. To conclude, observe that $k \geq 3$ cannot occur by Lemma 14, since $F(x, x, x, 0, \ldots, 0) = f_1(x) + f_2(x) + f_3(x)$ must be an element of $\mathcal{M}$ if $F \in \text{Pol}(\mathcal{M})$. \hfill $\square$
Lemma 19. Let $C$ be a clone containing $M$ and any function of $V$. Then $C$ contains $V$.

Proof. Let $n'(x) + n''(y) \in V \cap C$, where $n' \in N'$ and $n'' \in N''$, and let $m'(x) + m''(y)$ with $m' \in N'$ and $m'' \in N''$ be an arbitrary function in $V$. Since $\text{ran } m' = \text{ran } n' = \text{span}\{b\}$, there is $n_1 \in L$ with $m' = n' \circ n_1$. This $n_1$ can be chosen to satisfy $n_1(a) = a$, $n_1(b) = 0$, and $n_1(c) = c$; also, since $m'$ has small support, we can choose $n_1$ to have small support too. Then $n_1 \in N \subseteq M \subseteq C$. Similarly, there is $n_2 \in N$ such that $m'' = n'' \circ n_2$. Hence, $m'(x) + m''(y) = n'(n_1(x)) + n''(n_2(y)) \in C$. 

Lemma 20. Let $C$ be a clone containing $M$ and any function of $D_{\Phi}$. Then $C$ contains $V$.

Proof. Let $\phi_p(x) + n''(y) \in C \cap D_{\Phi}$, where $\phi_p \in \Phi$ and $n'' \in N''$. Taking any $n \in N$ we set $n' = \phi_p \circ n \in N'$. Then $C$ contains $n'(x) + n''(y) \in V$ and hence all functions of $V$ by the preceding lemma.

Lemma 21. Let $C$ be a clone containing $M$ and a function $\phi_p(x) + n''(y) \in D_{\Phi}$, where $\phi_p \in \Phi$ and $n'' \in N''$. If $q \leq p$ and $m'' \in N''$, then $C$ contains the function $\phi_q(x) + m''(y)$.

Proof. As discussed in the proof of Lemma 19, there is $n \in N$ such that $m'' = n'' \circ n$. Therefore $C$ contains $\phi_p(\psi_{p,q}(x)) + n''(n(y)) = \phi_q(x) + m''(y)$.

Proposition 22. If $C \in J_{\Phi}$ is a clone, then $C = M^* = \langle M \rangle$, or $C = C_1$, where $I \subseteq \Phi$ is an order ideal on $\Phi$.

Proof. Let $C \neq \langle M \rangle$, that is, $C$ contains an essentially binary function. Set $I = \{p \in \Phi : \exists n'' \in N'' (\phi_p(x) + n''(y) \in C)\}$. By Lemma 21, $I$ is an order ideal of $\Phi$. We claim $C = C_1$. Being elements of $J_{\Phi}$, both $C$ and $C_1$ have $M$ as their unary part. Let $f(x,y) \in C^{(2)}$ be essentially binary, i.e. depending on both of its variables; then up to permutation of variables, $f(x,y) \in V \cup D_{\Phi}$ by Lemma 18. If $f \in V$, then $f \in C_1$ by definition of $C_1$. If $f \in D_{\Phi}$, then $f(x,y) = \phi_p(x) + n''(y)$, where $p \in \Phi$ and $n'' \in N''$. But then $p \in I$ by definition of $I$ and so $f \in C_1$. Hence, $C^{(2)} \subseteq C_1^{(2)}$. Because $C$ contains a binary function from $V \cup D_{\Phi}$, Lemmas 19 and 20 imply $C^{(2)} \supseteq V$. Also, $\phi_q(x) + m''(y) \in C^{(2)}$ for all $q \in I$ and all $m'' \in N''$ by Lemma 21 so that we have $C^{(2)} \supseteq C_1^{(2)}$ and thus $C^{(2)} = C_1^{(2)}$. Lemma 18 implies that clones in $J_{\Phi}$ are uniquely determined by their binary parts, so that we conclude $C = C_1$.

Proposition 23. Let $L$ be the lattice of order ideals on the partial order $\Phi$. The monoidal interval $J_{\Phi}$ is isomorphic to $1 + L$, which is to denote $L$ with a new smallest element (which corresponds to $\langle M \rangle$) added to $L$.

Proof. The mapping $\sigma : 1 + L \to J_{\Phi}$ taking an order ideal $I \in L$ to $C_I$, as well as the smallest element of $1 + L$ to $\langle M \rangle$, is obviously a lattice homomorphism and injective. By the preceding proposition it is also surjective.
Proof of Theorem 4. Given a partial order \( \mathcal{P} \) with smallest element, we consider the partial order \( \mathcal{P}' \) obtained from \( \mathcal{P} \) by taking away the smallest element. By the preceding proposition, we can construct a monoid \( \mathcal{M} \) such that \( \mathcal{I}_\mathcal{M} \) is isomorphic to \( 1 + \mathcal{L}' \), where \( \mathcal{L}' \) is the lattice of order ideals on \( \mathcal{P}' \). Now it is enough to observe that \( 1 + \mathcal{L}' \) is isomorphic to the lattice \( \mathcal{L} \) of order ideals on \( \mathcal{P} \).

\( \square \)

Proof of Corollary 3. Let \( \mathcal{L} \) be a completely distributive algebraic lattice with at most \( 2^\kappa \) completely join irreducibles. Write \( \mathcal{P} \) for the partial order of completely join irreducibles of \( \mathcal{L} \) (with the induced order), and write \( \mathcal{L}' \) for the lattice of order ideals on \( \mathcal{P} \). The mapping

\[
\sigma : \mathcal{L} \rightarrow \mathcal{L}' \quad p \mapsto \{ q \in \mathcal{P} : q \leq p \}
\]

is easily seen to be a homomorphism; \( \sigma \) is bijective because in a completely distributive algebraic lattice, every element is a join of completely join irreducibles.

\( \square \)

Proof of Corollary 4. The completely join irreducibles of \( \mathcal{P}(\lambda) \) are exactly the singleton sets, so there are exactly \( \lambda \leq 2^\kappa \) of them and we can refer to Corollary 2.

\( \square \)

Proof of Corollary 5. \( \mathcal{L} \) is completely distributive algebraic, so this is a direct consequence of Corollary 2.

\( \square \)

Definition 24. A monoid \( \mathcal{G} \subseteq \mathcal{G}^{(1)} \) is called collapsing iff its monoidal interval has only one element, i.e. \( \langle \mathcal{G} \rangle = \operatorname{Pol}(\mathcal{G}) \).

Denote by \( \mathcal{I} \) the monoid of all permutations of \( X \).

Proposition 25. \( \mathcal{I} \) is collapsing.

Proof. Let \( f \in \operatorname{Pol}(\mathcal{I})^{(2)} \). Then \( \gamma(x) = f(x, x) \) is a permutation. Now let \( x, y \in X \) be distinct. There exists \( z \in X \) with \( \gamma(z) = f(x, y) \). If \( z \notin \{x, y\} \), then we can find \( \alpha, \beta \in \mathcal{I} \) with \( \alpha(x) = x, \alpha(y) = z, \beta(x) = y \), and \( \beta(y) = z \). But then \( f(\alpha, \beta)(x) = f(x, y) = f(z, z) = f(\alpha, \beta)(y) \), so \( f(\alpha, \beta) \) is not a permutation. Thus, \( z \in \{x, y\} \), and we have shown that \( f(x, y) \in \{f(x, x), f(y, y)\} \) for all \( x, y \in X \).

Next we claim that for all \( x, y \in X \), if \( f(x, y) = f(x, x) \), then \( f(y, x) = f(y, y) \). Indeed, consider any permutation \( \alpha \) which has the cycle \( (xy) \). Then \( f(x, \alpha(x)) = f(x, y) = f(x, x) \), so \( f(y, \alpha(y)) = f(y, x) \) has to be different from \( f(x, x) \), because otherwise the function \( \delta(x) = f(x, \alpha(x)) \in \mathcal{I} \) is not injective. Hence, \( f(y, x) = f(y, y) \).

Assume without loss of generality that \( f(a, b) = f(a, a) \) for some distinct \( a, b \in X \). We claim that \( f(a, c) = f(a, a) \) for all \( c \in X \). For assume not; then \( f(a, c) = f(c, c) \) for some \( c \in X \), and therefore \( f(c, a) = f(a, a) \). Let \( \beta \in \mathcal{I} \) map \( a \) to \( b \) and \( c \) to \( a \). Then \( f(a, \beta(a)) = f(a, b) = f(a, a) \), but also \( f(c, \beta(c)) = f(c, a) = f(a, a) \), a contradiction since \( f \) preserves \( \mathcal{I} \). Hence, \( f(a, c) = f(a, a) \) for all \( c \in X \).
Now if \( f(\tilde{a}, \tilde{b}) \neq f(\tilde{a}, \tilde{a}) \) for some \( \tilde{a}, \tilde{b} \in X \), then \( f(\tilde{a}, \tilde{b}) = f(\tilde{b}, \tilde{b}) \) and as before we conclude \( f(c, \tilde{b}) = f(\tilde{b}, \tilde{b}) \) for all \( c \in X \). But then \( f(a, a) = f(a, \tilde{b}) = f(\tilde{b}, \tilde{b}) \), so \( a = \tilde{b} \); furthermore, \( f(a, \tilde{a}) = f(\tilde{b}, \tilde{a}) = f(\tilde{a}, \tilde{a}) \neq f(a, a) \) since we must have \( a \neq \tilde{a} \), contradicting \( f(a, c) = f(a, a) \) for all \( c \in X \). Hence, \( f(x, y) = f(x, x) \) for all \( x, y \in X \) so that \( f \) is essentially unary.

Therefore, all binary functions of \( \text{Pol}(\mathcal{S}) \) are essentially unary. By a result of Grabowski [?], this implies that \( \mathcal{S} \) is collapsing. (The mentioned result was proved for finite base sets with at least three elements, but the same proof works on infinite sets.)

Proof of Corollary 2. The preceding proposition gives us the ordinal 1. For larger ordinals, we can refer to Corollary 4.

Proof of Corollary 7. This is the direct consequence of Corollaries 3 and 6.