Research article

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**Blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation**

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**Abstract:** In this paper, we study blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation

\[i\partial_t \psi - (-\Delta)^s \psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0.\]

By using localized virial estimates, we firstly establish general blow-up criteria for non-radial solutions in both \(L^2\)-critical and \(L^2\)-supercritical cases. Then, we show existence of normalized standing waves by using the profile decomposition theory in \(H^s\). Combining these results, we study the strong instability of normalized standing waves. Our obtained results greatly improve earlier results.

**Keywords:** Fractional Schrödinger-Choquard equation; Blow-up criteria; Strong instability; Normalized standing waves

**MSC:** 35Q55; 35J10

**1 Introduction**

Over the past decade, there has been a great deal of interest in studying the fractional Schrödinger equation (NLS)

\[i\partial_t \psi = (-\Delta)^s \psi + f(\psi),\]

where \(0 < s < 1\) and \(f(\psi)\) is the nonlinearity. The fractional differential operator \((-\Delta)^s\) is defined by \((-\Delta)^s \psi = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(\psi)]\), where \(\mathcal{F}\) and \(\mathcal{F}^{-1}\) are the Fourier transform and inverse Fourier transform, respectively. The fractional NLS (1.1) was first deduced by Laskin in [29, 30] by extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The fractional NLS also arises in the description of Bonson stars as well as in water wave dynamics (see e.g. [22]) and in the continuum limit of discrete models with long-range interactions (see e.g. [28]).

In this paper, we consider blow-up criteria and instability of normalized standing waves for the fractional nonlinear Schrödinger-Choquard equation

\[
\begin{aligned}
\left\{
\begin{array}{l}
 i\partial_t \psi - (-\Delta)^s \psi + (I_\alpha * |u|^p)|\psi|^{p-2}\psi = 0, \\
 \psi(0, x) = \psi_0(x),
\end{array}
\right. & \quad (t, x) \in [0, T^*) \times \mathbb{R}^N,
\end{aligned}
\]

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where $\psi : [0, T^*) \times \mathbb{R}^N \to \mathbb{C}$ is the complex valued function, $N \geq 1$, $\psi_0 \in H^s$, $0 < s < 1$, $0 < T^* \leq \infty$, $1 + \frac{a}{N} < p < \frac{N+a}{N-2s}$, \( I_a : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential defined by
\[
I_a(x) = \frac{A(a)}{|x|^{N-a}}, \quad A(a) := \frac{\Gamma(\frac{N-a}{2})}{\Gamma(\frac{N}{2})m^{N/2}2^a}
\]
with $a \in (0, N)$ and $\Gamma$ is the Gamma function.

Equation (1.2) enjoys the scaling invariance. That is, if $\psi$ is a solution of (1.2) with initial data $\psi_0$, then $\psi_\mu(t, x) := \mu^{\frac{N-2s}{2}} \psi(\mu t, \mu x)$ for all $\mu > 0$ is also a solution of (1.2) with initial data $\mu^{\frac{N-2s}{2}} \psi_0(\mu x)$. In particular, $\|\psi_\mu(t)\|_{H^s} = \|\psi(\mu t)\|_{H^s}$, where
\[
s_c := \frac{N}{2} - \frac{\alpha + 2s}{2p - 2}.
\]
Thus, $s_c$ is referred as the critical Sobolev exponent of (1.2). If the initial data $\psi_0 \in H^s$, then equation (1.2) enjoys mass and energy conservation laws:
\[
\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad E(\psi(t)) = E(\psi_0),
\]
where the energy $E$ is defined by
\[
E(\psi(t)) = \frac{1}{2}\|\psi(t)\|^2_{H^s} - \frac{1}{2p} \int_{\mathbb{R}^N} (I_a * |\psi(t)|^p)(x)|\psi(t, x)|^p \, dx.
\]

Before entering our main results, we firstly recall some known blow-up results for NLS. For the classical NLS, i.e., $s = 1$, when initial data $\psi_0 \in \Sigma := \{\psi_0 \in H^1 \text{ and } x\psi_0 \in L^2\}$, the following Variance-Virial Law holds
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |\psi(t, x)|^2 \, dx = 2\text{Im} \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \psi(t, x) \, dx.
\]
By using (1.5) and the virial identity, ones can prove existence of blow-up solutions for the classical NLS with negative energy $E(\psi_0) < 0$, see [7]. However, since identity (1.5) fails for $s < 1$, which readily checks by dimensional analysis, this argument cannot work. Rather, a possible generalization of the variance for the fractional NLS is given by the nonnegative quantity
\[
\gamma^{(s)}[\psi(t)] := \int_{\mathbb{R}^N} \bar{\psi}(t, x) x \cdot (-\Delta)^{1-s} x \psi(t, x) \, dx = \|x(-\Delta)^{1-s} \psi(t)\|^2_{L^2}.
\]
Let $\psi(t)$ be a sufficiently regular and spatially localized solution of equation $i\partial_t \psi = (-\Delta)^s \psi$, it follows that
\[
\frac{1}{2} \frac{d}{dt} \gamma^{(s)}[\psi(t)] := 2\text{Im} \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \psi(t, x) \, dx.
\]
This method has been successfully applied to prove the existence of radial blow-up solutions of (1.1) with focusing Hartree-type nonlinearities, i.e., $f(\psi) = -(|x|^s \ast |\psi|^2)\psi$ with $y \geq 1$, see [8, 9, 48]. But this method can not work due to the nontrivial error terms which seem very hard to control for the local nonlinearities $f(\psi) = -|\psi|^p \psi$, see [6]. In [6], Boulenger, Himmelsbach and Lenzmann applied the Balakrishnan's formula
\[
(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} \, dm,
\]
and obtained the differential inequality
\[
\frac{d}{dt} \left( \text{Im} \int_{\mathbb{R}^N} \bar{\psi}(t) \nabla \varphi_R \cdot \nabla \psi(t) \, dx \right) \leq 4pN E(\psi_0) - 2\delta \|(-\Delta)^{1/2} \psi(t)\|^2_{L^2} + c_R(1 + \|(-\Delta)^{1/2} \psi(t)\|^p_{L^1}),
\]
where $\delta = pN - 2s$. Based on this key estimate, they proved the existence of radial blow-up solutions by applying a standard comparison ODE argument.

For the fractional Schrödinger-Choquard equation (1.2), Saanouni in [39] proved the existence of radial blow-up solutions by using the method in [6]. In this paper, we will further study the existence of blow-up solutions of (1.2) for non-radial initial data by using the idea of Du, Wu and Zhang in [13]. The main difficulty is the appearance of the fractional order Laplacian $(-\Delta)^{s}$. When $s = 1$, the time derivative of the virial action can be easily obtained, that is

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \varphi(x)||\psi(t, x)||^2 dx = 2\text{Im} \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \varphi(x) \cdot \nabla \psi(t, x) dx.
$$

(1.9)

Using this identity, Du, Wu and Zhang in [13] derived an $L^2$-estimate in the exterior ball. Combining this $L^2$-estimate and the virial estimates, they established blow-up criteria for the classical NLS. When $s \in (\frac{1}{2}, 1)$, the identity (1.9) does not hold. However, by exploiting the ideas in [6, 12] and using the Balakrishnan’s formula (1.8), we can obtain the time derivative of the virial action, see Lemma 2.9. Thus, we can establish the blow-up criteria for (1.2).

**Theorem 1.** Let $N \geq 1$, $s \in (\frac{1}{2}, 1)$, $1 + \frac{2s+n}{N} \leq p < \frac{N+2s}{N-2s}$, $\psi_0 \in H^s$ and $\psi \in C([0, T^*), H^s)$ be the corresponding solution of (1.2). Furthermore, we suppose either $E(\psi_0) < 0$, or, if $E(\psi_0) \geq 0$ and

$$
\begin{cases}
E(\psi_0)\|\psi_0\|_{L^2}^{2(2-s_c)} < E(\psi)\|\psi\|_{L^2}^{2(2-s_c)}, \\
\|(-\Delta)^{s_c/2}\psi_0\|_{L^2}^{2s} \|\psi_0\|_{L^2}^4 > \|(-\Delta)^{s/2}\psi\|_{L^2}^{2s} \|\psi\|_{L^2}^4,
\end{cases}
$$

(1.10)

where $s_c$ is defined by (1.3) and $u$ is a ground state of the following elliptic equation

$$
(-\Delta)^s u + u - (I_a * |u|^p)|u|^{p-2} u = 0.
$$

(1.11)

Then one of the following statements holds true:

- $\psi(t)$ blows up in finite time, i.e. $T^* < +\infty$;
- $\psi(t)$ blows up infinite time and there exists a time sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to +\infty$ and

$$
\lim_{n \to \infty} \|(-\Delta)^{\frac{s}{2}} \psi(t_n)\|_{L^2} = \infty.
$$

**Remark 1.** The uniqueness of ground state solutions to (1.11) is still unknown. However, it follows from the optimal constant in (2.2) that all ground states have the same $L^1$-norm. Moreover, we see from Pohozaev’s identities (2.3) that all ground states have the same $H^s$-norm and energy. Therefore, for different ground states, the quantities $E(u)\|u\|_{L^2}^{2(2-s_c)}$ and $\|(-\Delta)^{s_c/2} u\|_{L^2}^{2s} \|u\|_{L^2}^4$ are same. These imply that the assumption (1.10) is reasonable.

**Remark 2.** When $p = 2$, similar blow-up criteria for (1.2) with radial solutions have been established in [8, 9, 25, 33, 37, 40, 41, 47, 48]. Here, we remove the assumption of radial solutions and extend these results to more general Choquard-type nonlinearity.

Based on blow-up criteria (1.10), we study the strong instability of normalized standing waves of (1.2). Firstly, we introduce some notations. Equation (1.2) enjoys a class of special solutions, which are called standing waves, namely solutions of the form $e^{i\omega t} u_\omega$, where $\omega \in \mathbb{R}$ is a frequency and $u_\omega \in H^s$ is a nontrivial solution to the elliptic equation

$$
(-\Delta)^s u_\omega + \omega u_\omega - (I_a * |u_\omega|^p)|u_\omega|^{p-2} u_\omega = 0.
$$

(1.12)

At this moment, our intention is reduced to study (1.12). To do this, there exist two substantially different choices in terms of the frequency $\omega$. One is to fix the frequency $\omega \in \mathbb{R}$. In this situation, every solution to (1.12) corresponds to a critical point of the action functional $S_\omega(u)$ on $H^s$, where

$$
S_\omega(u) := \frac{1}{2} \|u\|_{H^s}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_a * |u|^p)(x) u(x)^p dx.
$$

(1.13)
Alternatively, it is interesting to study solutions of (1.12) having prescribed $L^2$-norm. That is, for any given $c > 0$, ones study solutions of (1.12) satisfying the $L^2$-norm constraint

$$S(c) = \{ u \in H^s : \|u\|_{L^2}^2 = c \}, \quad c > 0.$$ \hfill (1.14)

Physically, such solutions are called normalized solutions of (1.12), which formally corresponds to critical points of the energy functional $E(u)$ restricted on $S(c)$, where $E(u)$ is defined by (1.4). In particular, in this situation, the frequency $\omega \in \mathbb{R}$ is an unknown part, which corresponds to the associated Lagrange multiplier. Recently, these questions have received more attention, see \cite{1–5, 26, 32, 42, 45}.

In the $L^2$-subcritical case, i.e., $1 + \frac{a}{N} < p < 1 + \frac{2s+a}{N}$, the energy $E(u)$ is bounded from below on $S(c)$. Feng and Zhang in \cite{21} studied existence of normalized ground states to (1.12) by using the profile decomposition theory. To prove this theorem, we need to construct a submanifold of $S(c)$, on which $E(u)$ is bounded from below and coercive, and then we look for minimizers of $E(u)$ on such a submanifold. Precisely, we introduce the following minimizing problem

$$m(c) := \inf_{u \in V(c)} E(u),$$ \hfill (1.15)

where the constraint $V(c)$ is defined by

$$V(c) := \{ u \in S(c) : K(u) = 0 \},$$ \hfill (1.16)

and the functional $K(u)$ is defined by

$$K(u) := \partial_1 S_\omega(u^1)|_{1=1} = s\|u\|^2_{H^s} - \theta \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(x)|u(x)|^p \, dx,$$ \hfill (1.17)

where

$$\theta = Np - N - a, \quad u^1(x) := \lambda^{N/2} u(\lambda x).$$ \hfill (1.18)

Indeed, the identity $K(u) = 0$ is the Pohozaev identity related to (1.12). The constraint $V(c)$ is the so-called Pohozaev manifold related to (1.12)-(1.14). In the following theorem, we can prove the existence of minimizers of (1.15).

**Theorem 1.2.** Let $1 + \frac{a+2s}{N} < p < \frac{N+a}{N-2s}$ and $c > 0$. Then there exists $u_c \in V(c)$ such that $E(u_c) = m(c)$.

**Remark.** This theorem can be proved by using the method in \cite{18}. Here, we will use the profile decomposition of bounded sequences in $H^s$ to prove this theorem. The profile decomposition theory has been extensively applied to study existence of normalized standing waves in the $L^2$-subcritical case, see, e.g., \cite{20, 21, 49}. Here, we successfully apply it to study existence of normalized standing waves in the $L^2$-supercritical case. Therefore, our approach is of particular interest.

Next, we denote the set of minimizers of $E$ on $V(c)$ as

$$\mathcal{M}_c := \{ u \in V(c) : E(u) = \inf_{v \in V(c)} E(v) \}.$$ \hfill (1.19)

In the following theorem, we can show any minimizer to (1.15) is a ground state to (1.12)-(1.14).

**Theorem 1.3.** Let $1 + \frac{2s+a}{N} < p < \frac{N+a}{N-2s}$. Then for any $u_c \in \mathcal{M}_c$, there exists $\omega_c > 0$ such that $(u_c, \omega_c) \in H^s \times \mathbb{R}$ is a weak solution to problem (1.12). Furthermore, $u_c$ is a ground state solution to problem (1.12) with $\omega = \omega_c$.

Finally, we consider the strong instability of normalized standing waves. The usual strategy to study the strong instability of standing waves for the classical NLS ($s=1$) is to use the variational characterization of the
ground states as minimizers of the action functional and obtain the key estimate \( K(\psi(t)) \leq 2(S_\omega(\psi_0) - S_\omega(u_\omega)) \). Then, it follows from the virial identity that
\[
\frac{d^2}{dt^2} \| x\psi(t) \|_{L^2}^2 = 8K(\psi(t)) \leq 16(S_\omega(\psi_0) - S_\omega(u_\omega)) < 0,
\]
where \( K(\psi(t)) \) is defined by (1.17) with \( s = 1 \). This implies that the solution \( \psi(t) \) of (1.1) with \( s = 1 \) blows up in finite time. Thus, one can prove the strong instability of ground state standing waves, see [7, 11, 16, 17, 23, 24, 31, 34–36, 38, 43, 44].

Here, we only need to use the blow-up criterion (1.10) to study the strong instability of normalized standing waves.

**Theorem 1.4.** Let \( N \geq 1, s \in \left( \frac{1}{2}, 1 \right), 1 + \frac{2s\alpha}{N} < p < \frac{N\alpha}{N-2s}, c > 0 \). Then for any \( u_c \in \mathcal{M}_c \), the standing wave \( \psi(t, x) = e^{it\omega}u_c(x) \) is strongly unstable in the following sense: there exists \( \{\psi_{0,n}\} \subset H^s \) such that \( \psi_{0,n} \rightarrow u_c \) in \( H^s \) as \( n \rightarrow \infty \) and the corresponding solution \( \psi_n \) of (1.2) with initial data \( \psi_{0,n} \) blows up in finite or infinite time for any \( n \geq 1 \).

**Remark 1.** In previous results, in order to construct blow-up solutions around the ground state solution, one need to assume that the ground state solution \( u_\omega \) is radial or \( u_\omega \in \Sigma := \{ v \in H^s \text{ and } xv \in L^2 \} \). Here, we remove these assumptions, so our result greatly improve some previous results.

**Remark 2.** When \( p = 2 \) and \( N - \alpha = 2s \), i.e., in the \( L^2 \)-critical case, Zhang and Zhu in [46] proved the strong instability of radial ground state standing waves of (1.2). Here, we remove this radial assumption and extend this result to the \( L^2 \)-supercritical case and more general Choquard-type nonlinearity.

This paper is organized as follows: in Section 2, we will recall and prove some lemmas such as the local well-posedness theory of (1.2), a sharp Gagliardo-Nirenberg type inequality and the localized virial estimate.

Notations. In this paper, we use the following notations. For any \( s \in (0, 1) \), the fractional Sobolev space \( H^s(\mathbb{R}^N) \) is defined by
\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi < \infty \right\},
\]
edowed with the norm
\[
\| u \|_{H^s(\mathbb{R}^N)} = \| u \|_{L^2(\mathbb{R}^N)} + \| u \|_{H^s(\mathbb{R}^N)},
\]
where up to a multiplicative constant
\[
\| u \|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}}
\]
is the so-called Gagliardo semi-norm of \( u \). In this paper, we often use the abbreviations \( L^r = L^r(\mathbb{R}^N), H^s = H^s(\mathbb{R}^N) \). For \( J \subset \mathbb{R} \) and \( q, r \in [1, \infty] \), we define the mixed norm
\[
\| u \|_{L^q(J \times \mathbb{R}^N)} := \left( \int_J \left( \int_{\mathbb{R}^N} |u(t, x)|^r dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}
\]
with the usual modification when either \( q \) or \( r \) are infinity. In the case \( q = r \), we shall use \( L^q(J \times \mathbb{R}^N) \) instead of \( L^q(J, L^r) \).
2 Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, we recall the local well-posedness for the Cauchy problem (1.2). Hong and Sire in [27] first studied the local well-posedness of the fractional NLS in $H^s$ by using Strichartz’s estimates and the contraction mapping argument. Since Strichartz’s estimates for non-radial data have a loss of derivatives, a weak local well-posedness holds in the energy space compared to the classical nonlinear Schrödinger equation, see [10, 27] for more details. One can remove the loss of derivatives in Strichartz’s estimates by considering radially symmetric data. However, it needs a restriction on the validity of $s$, namely $\frac{N}{2N-4} \leq s < 1$.

**Proposition 2.1.** [15, Proposition 2.3][Non-radial $H^s$ LWP] Let $s \in (0, 1) \setminus \{1/2\}$, $2 \leq p < \frac{N+a}{N-2s}$ and $\max\{0, N-4s\} < \alpha < N$ be such that

\[
S > \begin{cases} 
\frac{1}{2} - \frac{2s}{\max\{2p-2, \alpha\}} & \text{if } N = 1, \\
\frac{N}{2} - \frac{s}{p} & \text{if } N \geq 2.
\end{cases}
\]

(2.1)

Then for all $\psi_0 \in H^s$, there exist $T^* \in (0, +\infty)$ and a unique solution $\psi \in C([0, T^*), H^s) \cap L^q_{loc}((0, T^*), L^\infty)$, for some $q > \max\{2p-2, 4\}$ when $N = 1$ and some $q > 2p-2$ when $N \geq 2$. Moreover, the following properties hold:

- If $T^* < +\infty$, then $\|\psi(t)\|_{H^s} \to \infty$ as $t \uparrow T^*$.

- The solution enjoys conservation of mass and energy, i.e. $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ and $E(u(t)) = E(\psi_0)$ for all $t \in [0, T^*)$, where $E(\psi(t))$ defined by (1.4).

**Remark.** When $1 + \frac{a}{N} < p < \frac{N+a}{N-2s}$, it follows from the Hardy-Littlewood-Sobolev inequality that $\int_{\mathbb{R}^N} (I_a * |\psi|^p)(x)|\psi(x)|^p \, dx$ is well-defined for $\psi \in H^s$. Therefore, we guess that these results also hold for $1 + \frac{a}{N} < p < 2$. However, we cannot prove these results since the nonlinearity $(I_a * |\psi|^p)|\psi|^{p-2} \psi$ is singular when $1 + \frac{a}{N} < p < 2$, see [19].

**Proposition 2.2.** [15, Proposition 2.3][Radial $H^s$ LWP] Let $N \geq 2$, $\frac{N}{2N-4} \leq s < 1$, $2 \leq p < \frac{N+a}{N-2s}$ and $\max\{0, N-4s\} < \alpha < N$. Then for any $\psi_0 \in H^s$ radial, there exist $T^* \in (0, +\infty)$ and a unique solution $\psi \in C([0, T^*), H^s)$ to (1.2). Moreover, the following properties hold:

- $\psi \in L^q_{loc}([0, T), W^{a,b})$ for any fractional admissible pair $(a, b)$.

- If $T^* < +\infty$, then $\|\psi(t)\|_{H^s} \to \infty$ as $t \uparrow T^*$.

- The solution enjoys conservation of mass and energy, i.e. $M(\psi(t)) = M(\psi_0)$ and $E(\psi(t)) = E(\psi_0)$ for all $t \in [0, T^*)$.

Next, we recall a sharp Gagliardo-Nirenberg type inequality established in [21].

**Lemma 2.3.** [21, Theorem 2.3] Let $0 < s < 1$ and $1 + \frac{a}{N} < p < \frac{N+a}{N-2s}$. Then, for all $u \in H^s$,

\[
\int_{\mathbb{R}^N} (I_a * |u|^p)|u|^p \, dx \leq C_{opt} \|(-\Delta)^{s/2} u\|_{L^{N/p-a}}^{Np-a} \|u\|_{L^2}^{Np-N+a},
\]

(2.2)

where the optimal constant $C_{opt}$ is given by

\[
C_{opt} = \frac{2sp}{2sp - Np + N + a} \left( \frac{2sp - Np + N + a}{Np - N - a} \right)^{\frac{Np-N+a}{2s}} \|Q\|_{L^2}^{2-2p},
\]

where $Q$ is the ground state of the elliptic equation (1.11). In particular, in the $L^2$-critical case, i.e., $p = 1 + \frac{2s+a}{N}$, $C_{opt} = p \|Q\|_{L^2}^{2-2p}$.

Moreover, the following Pohozaev’s identities hold true:

\[
\|Q\|_{H^s}^2 = \frac{Np - N - a}{2sp} \int_{\mathbb{R}^N} (I_a * |Q|^p)(x) Q(x) |\nabla Q|^2 \, dx = \frac{Np - N - a}{2sp - Np + N + a} \|Q\|_{L^2}^2.
\]

(2.3)
Next, we recall the profile decomposition of bounded sequences in $H^s$, which has been established in [48].

**Lemma 2.4.** Let $N \geq 3$, $0 < s < 1$ and $1 + \frac{a}{N} < p < \frac{N + a}{N - 2}$. If $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $H^s$, then there exist a subsequence of $\{u_n\}_{n=1}^\infty$ (still denoted by $\{u_n\}_{n=1}^\infty$), a family $\{x_n^j\}_{j=1}^\infty$ of sequences in $\mathbb{R}^N$ and a sequence $\{U_j\}_{j=1}^\infty$ in $H^s$ such that

(i) for every $k \neq j$, $|x_n^k - x_n^j| \to +\infty$ as $n \to \infty$;

(ii) for every $l \geq 1$ and every $x \in \mathbb{R}^N$, we have

$$u_n(x) = \sum_{j=1}^l U_j(x - x_n^j) + r_n^l,$$  

(2.4)

with $\limsup_{n \to \infty} |r_n^l|_{L^q} \to 0$ as $l \to \infty$ for every $q \in (2, \frac{2N}{N-2s})$. Moreover,

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^l \|U_j\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + o(1),$$  

(2.5)

$$\|(-\Delta)^{s/2}u_n\|_{L^2}^2 = \sum_{j=1}^l \|(-\Delta)^{s/2}U_j\|_{L^2}^2 + \|(-\Delta)^{s/2}r_n^l\|_{L^2}^2 + o(1),$$  

(2.6)

$$\int_{\mathbb{R}^N} I_\alpha \ast \sum_{j=1}^l U_j(\cdot - x_n^j)|^p \sum_{j=1}^l U_j(\cdot - x_n^j)|^p \, dx$$

$$= \sum_{j=1}^l \int_{\mathbb{R}^N} I_\alpha \ast |U_j(\cdot - x_n^j)|^p \, dx + o(1),$$  

(2.7)

where $o(1) = o_n(1) \to 0$ as $n \to \infty$.

Finally, we recall and prove some virial estimates related to (1.2) which is the main ingredient in the proof of Theorem 1.1.

**Lemma 2.5** ([6]). Let $N \geq 1$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ is such that $\nabla \varphi \in W^{1,\infty}(\mathbb{R}^N)$. Then, for all $u \in H^\frac{1}{2}$, it holds that

$$\left| \int_{\mathbb{R}^N} \nabla \varphi(x) \cdot \nabla u(x) \, dx \right| \leq C \|
abla \varphi\|_{W^{1,\infty}} \left( \|u\|_{H^\frac{1}{2}}^2 + \|u\|_{L^2} \|u\|_{H^\frac{1}{2}} \right),$$

for some constant $C > 0$ that depends only on $N$.

In order to study localized virial estimates for (1.2), we need to introduce the auxiliary function

$$u_m(x) := c_s \frac{1}{-\Delta + m} u(x) = c_s \beta^{-1} \left( \frac{\beta(\xi)}{|\xi|^2 + m} \right), \quad m > 0,$$

(2.8)

where

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$

**Lemma 2.6** ([6]). Let $N \geq 1$, $s \in (0, 1)$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ with $\Delta \varphi \in W^{2,\infty}(\mathbb{R}^N)$. Then, for all $u \in L^2$, it holds that

$$\int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi) |u_m|^2 \, dx \, dm \leq C \|\Delta^2 \varphi\|_{L^\infty}^s \|\Delta \varphi\|^\frac{s}{2} \|\Delta \varphi\|^\frac{1-s}{2} \|u\|_{L^2}^2,$$

for some constant $C > 0$ that depends only on $s$ and $N$. 
We refer the reader to [6, Appendix A] for the proof of Lemma 2.5 and Lemma 2.6. Using the fact

$$\frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s}{(|\xi|^2 + m)^2} \, dm = s|\xi|^{2s-2},$$

the Plancherel’s and Fubini’s theorems imply

$$\int_\mathbb{R}^N m^s |\nabla u_m|^2 \, dx \leq \int_\mathbb{R}^N \left( \frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s \, dm}{(|\xi|^2 + m)^2} \right) |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi$$

$$= \int_\mathbb{R}^N (s|\xi|^{2s-2}) |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi = s(-\Delta \tilde{u})^2_{L^2},$$

for any $u \in \dot{H}^s$.

**Lemma 2.7.** [12, Lemma 4.2] Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla \varphi \in W^{1,\infty}$. Then for any $u \in L^2$, it holds that

$$\int_0^{\infty} \int_\mathbb{R}^N m^s (\Delta \varphi) |u_m|^2 \, dx \, dm \leq C \|\Delta \varphi\|_{L^{2s-1}}^2 \|\nabla \varphi\|_{L^{2s}}^2 \|u\|_{L^2}^2,$$

for some constant $C > 0$ that depends only on $s$ and $N$.

By the same argument as in Lemma 2.7 and using in addition Lemma 2.5, we obtain the following estimate.

**Lemma 2.8.** Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla \varphi \in W^{1,\infty}$. Then for any $u \in H^{1/2}$, it holds that

$$\int_0^{\infty} \int_\mathbb{R}^N m^s |\bar{u}_m \nabla \varphi \cdot \nabla u_m| \, dx \, dm \leq C \|\nabla \varphi\|_{W^{1,\infty}}^2 \|u\|_{H^{1/2}}^2,$$

for some constant $C > 0$ depending only on $N$.

Let $N \geq 1$, $1/2 < s < 1$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C((0, T^*), H^s)$ is a solution to (1.2). We define the localized virial action of $\psi$ associated to $\varphi$ by

$$V_\varphi[\psi(t)] := \int_\mathbb{R}^N \varphi(x) |\psi(t, x)|^2 \, dx.$$

**Lemma 2.9.** [12, Lemma 4.5] [Virial identity] Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C((0, T^*), H^s)$ is a solution to (1.2). Then for any $t \in [0, T^*)$, it holds that

$$\frac{d}{dt} V_\varphi[\psi(t)] = -i \int_\mathbb{R}^N (\Delta \varphi) |\psi_m(t)|^2 \, dx \, dm - 2i \int_\mathbb{R}^N \bar{\psi}_m(t) \nabla \varphi \cdot \nabla \psi_m(t) \, dx \, dm,$$

where $\psi_m(t) = c_s (-\Delta + m)^{-1} \psi(t)$.

A direct consequence of Lemmas 2.7, Lemma 2.8 and 2.9 is the following estimate.

**Corollary 2.10.** Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C((0, T^*), H^s)$ is a solution to (1.2). Then for any $t \in [0, T^*)$,

$$\left| \frac{d}{dt} V_\varphi[\psi(t)] \right| \leq C \|\nabla \varphi\|_{W^{1,\infty}} \|\psi(t)\|_{H^s}^2,$$

for some constant $C > 0$ depending only on $s$ and $N$. 
We next define the localized Morawetz action of $\psi$ associated to $\varphi$ by

$$M_\varphi[\psi(t)] := 2 \text{Im} \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \varphi(x) \cdot \nabla \psi(t, x) \, dx.$$  \hspace{1cm} (2.10)

By Lemma 2.5, we obtain the bound

$$|M_\varphi[\psi(t)]| \leq C (\|\nabla \varphi\|_{L^\infty}, \|\Delta \varphi\|_{L^\infty}) \|\psi(t)\|^2_{H^{1/2}}.$$  

Hence the quantity $M_\varphi[\psi(t)]$ is well-defined, since $\psi(t) \in H^{s}$ with some $s > \frac{1}{2}$ by assumption.

By a similar argument as that in [6, Lemma 2.1], we have the following time evolution of $M_\varphi[u(t)]$.

**Lemma 2.11 (Morawetz identity).** Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla \varphi \in W^{3, \infty}$. Assume that $\psi \in C([0, T^*), H^{s})$ is a solution to (1.2). Then for any $t \in [0, T^*)$, it holds that

$$\frac{d}{dt} M_\varphi[\psi(t)] = \int_{\mathbb{R}^N} m^s \int \left\{ 4 \partial_x \nabla \psi m(t) \partial_t \psi m(t) - (\Delta^2 \varphi) \psi m(t) \right\} \, dx + \int_{\mathbb{R}^N} \Delta \varphi (I_1 * |\psi(t)|^p) \psi(t)^p \, dx$$

$$- \frac{2p - 4}{p} \int_{\mathbb{R}^N} \Delta \varphi (I_1 * |\psi(t)|^p) \psi(t)^p \, dx - \frac{2N - 2a}{p} A(a) \int_{\mathbb{R}^N} \frac{|\psi(t, x)|^p \{|\psi(t, y)|^p (x - y) \cdot (\nabla \varphi(x) - \nabla \varphi(y))\}}{|x - y|^{N-a+2}} \, dx dy,$$

where $\psi_m(t) = \psi_m(t, x)$ is defined in (2.8).

**Proof.** It follows from an integration by parts that

$$\langle \psi(t), -\frac{1}{p} \psi(t) \rangle = -\langle \psi(t), [I_1 * |\psi(t)|^p] \psi(t)^{p-2}, \Delta \varphi \rangle$$

$$= 2 \int_{\mathbb{R}^N} \nabla \varphi \cdot \nabla (I_1 * |\psi(t)|^p) \psi(t)^p \, dx + 2 \int_{\mathbb{R}^N} (I_1 * |\psi(t)|^p) \psi(t)^2 \nabla \varphi \cdot \nabla (|\psi(t)|^{p-2}) \, dx$$

$$= \int_{\mathbb{R}^N} \frac{|\psi(t, x)|^p \{|\psi(t, y)|^p (x - y) \cdot (\nabla \varphi(x) - \nabla \varphi(y))\}}{|x - y|^{N-a+2}} \, dx dy$$

Therefore, following the method used in [6], we prove Lemma 2.11. \hfill \Box

### 3 Blow-up criteria

In this section, we will prove Theorem 1.1. To this end, we will establish the following blow-up criterion for (1.2).

**Lemma 3.1.** Let $N \geq 1$, $s \in (1/2, 1)$, $1 + \frac{2as}{N} < p < \frac{N + s}{N - 2s}$. Assume that $\psi_0 \in H^{s}$ and $\psi \in C([0, T^*), H^{s})$ is the corresponding solution of (1.2). If there exists $\delta > 0$ such that

$$\sup_{t \in [0, T^*)} K(\psi(t)) < -\delta < 0,$$

then one of the following statements holds true:

- $\psi(t)$ blows up in finite time, i.e. $T^* < +\infty$;
\[ \psi(t) \text{ blows up infinite time and there exists a time sequence } (t_n)_{n \geq 1} \text{ such that } t_n \to +\infty \text{ and} \]
\[ \lim_{n \to \infty} \|(-\Delta)\hat{\psi}(t_n)\|_{L^2} = \infty. \tag{3.2} \]

**Proof.** If \( T^* < +\infty \), then the proof is completed. If \( T^* = +\infty \), then we show (3.2). Assume by contradiction that the solution \( \psi(t) \) exists globally and there exists \( C_0 > 0 \) such that
\[ C_0 := \sup_{t \in [0, +\infty)} \|(-\Delta)\hat{\psi}(t)\|_{L^2} < \infty. \tag{3.3} \]
Combining this and the conservation of mass, we have
\[ C_1 := \sup_{t \in [0, +\infty)} \|\psi(t)\|_{H^1} < \infty. \tag{3.4} \]
Next, we introduce a smooth function \( \theta : [0, \infty) \to [0, 1] \) and satisfy
\[ \theta(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq 1/2, \\ 1 & \text{if } r \geq 1. \end{cases} \]
For \( R > 1 \), we define the radial function
\[ \phi_R(x) = \phi_R(r) := \theta(r/R), \quad r = |x|. \]
After some simple calculations, we can obtain
\[ \nabla \phi_R(x) = \frac{x}{rR} \theta'(r/R), \quad \Delta \phi_R(x) = \frac{1}{r^2} \theta''(r/R) + \frac{(N-1)}{rR} \theta'(r/R). \]
These imply
\[ \|\nabla \phi_R\|_{W^{1,\infty}} \sim \|\nabla \phi_R\|_{L^\infty} + \|\Delta \phi_R\|_{L^\infty} \lesssim R^{-1}. \tag{3.5} \]
Thus, we can define the localized virial function
\[ \mathcal{V}_{\phi_R}[\psi(t)] := \int_{\mathbb{R}^N} \phi_R(x)|\psi(t, x)|^2 \, dx. \]
It easily follows that
\[ \mathcal{V}_{\phi_R}[\psi(t)] = \mathcal{V}_{\phi_R}[\psi_0] + \int_0^t \frac{d}{d\tau} \mathcal{V}_{\phi_R}[\psi(\tau)] \, d\tau \]
\[ \leq \mathcal{V}_{\phi_R}[\psi_0] + \left( \sup_{\tau \in [0, t]} \left| \frac{d}{d\tau} \mathcal{V}_{\phi_R}[\psi(\tau)] \right| \right) t. \]
Combining Corollary 2.10, (3.4) and (3.5), we can obtain
\[ \sup_{\tau \in [0, t]} \left| \frac{d}{d\tau} \mathcal{V}_{\phi_R}[\psi(\tau)] \right| \lesssim \|\nabla \phi_R\|_{W^{1,\infty}} \sup_{\tau \in [0, t]} \|\psi(\tau)\|_{H^1}^2 \leq CC_1^2 R^{-1}, \]
for some constant \( C > 0 \) independent of \( R \) and \( C_1 \). We consequently obtain
\[ \mathcal{V}_{\phi_R}[\psi(t)] \leq \mathcal{V}_{\phi_R}[\psi_0] + CC_1^2 R^{-1} t, \]
for all \( t \geq 0 \). We infer from the definition of \( \theta \) that
\[ \mathcal{V}_{\phi_R}[\psi_0] = \int_{\mathbb{R}^N} \phi_R(x)|\psi_0(x)|^2 \, dx \leq \int_{|x| > R/2} |\psi_0(x)|^2 \, dx \to 0, \]
Collecting the above estimates, we can obtain the following control about the $L^2$-norm of the solution $\psi(t)$ outside a large ball.

**Lemma 3.2.** Let $\eta > 0$, $R > 1$ and $C_1$ be as in (3.4). Then there exists a constant $C > 0$ independent of $R$ and $C_1$ such that for any $t \in [0, T_0]$ with $T_0 := \frac{R}{C_1}$,

$$\int_{|x|>R} |\psi(t, x)|^2 dx \leq \eta + o_R(1).$$

Next, we introduce a radial function $\varphi(x) = \varphi(r)$ which satisfies

$$\varphi(r) = \begin{cases} \frac{r^2}{2} & \text{for } r \leq 1, \\ \text{const.} & \text{for } r \geq 10, \end{cases}$$

and $\varphi''(r) \leq 1$ for $r \geq 0$. For any $R > 0$, we define the rescaled function $\varphi_R : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\varphi_R(x) := R^2 \varphi \left( \frac{x}{R} \right).$$

It easily follows that

$$1 - \varphi''(r) \geq 0, \quad 1 - \frac{\varphi'(r)}{r} \geq 0, \quad N - \Delta \varphi_R(x) \geq 0,$$

for all $r \geq 0$ and all $x \in \mathbb{R}^N$. It is easy to see that

$$\|\nabla^k \varphi_R\|_{L^\infty} \lesssim R^{-k}, \quad k = 0, \ldots, 4,$$

and

$$\text{supp}(\nabla^k \varphi_R) \subset \begin{cases} \{ |x| \leq 10R \} & \text{for } k = 1, 2, \\ \{ R \leq |x| \leq 10R \} & \text{for } k = 3, 4. \end{cases}$$

Applying Lemma 2.11, we can obtain

$$\frac{d}{dt} M_{\varphi_R}[\psi(t)] = \int_{\mathbb{R}^N} \left\{ 4\partial_\alpha \psi_m(t)(\partial_\alpha \varphi_R)\partial_\alpha \psi_m(t) - (\Delta^2 \varphi_R)\psi_m(t) \right\} dx dm$$

$$- \frac{2p-\Delta}{p} \int_{\mathbb{R}^N} \Delta \varphi_R(I_6 \ast |\psi(t)|^p)|\psi(t)|^p dx$$

$$- \frac{2N-2n}{p} A(n) \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\psi(t, x)|^p |\psi(t, y)|^p (x-y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))}{|x-y|^{N+n+2}} dxdy$$

where $\psi_m(t) = \psi_m(t, x)$ is defined in (2.8). Due to $\text{supp}(\Delta^2 \varphi_R) \subset \{ |x| \geq R \}$, we infer from Lemma 2.6 that

$$\left| \int_0^\infty \int_{\mathbb{R}^N} (\Delta^2 \varphi_R) |\psi_m(t)|^2 dx dm \right| \lesssim \|\Delta^2 \varphi_R\|_{L^\infty} \|\Delta \varphi_R\|_{L^\infty} \|\psi(t)\|_{L^2(|x| \geq R)}^2$$

$$\lesssim R^{-2s} \|\psi(t)\|_{L^2(|x| \geq R)}^2.$$  \hfill (3.8)

Since $\varphi_R$ is a radial function, applying

$$\partial^2_{|x|^2} \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r + \frac{x_j x_k}{r^2} \partial_r^2,$$
we can obtain
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} \partial_k \psi_m(t) (\partial_k \phi_m) \partial_t \psi_m(t) \, dx \, dm = \int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\phi''}{r} |\nabla \psi_m(t)|^2 \, dx \, dm
\]
\[
+ \int_0^\infty m^s \int_{\mathbb{R}^N} \left( \frac{\phi''}{r^2} - \frac{\phi''}{r^3} \right) |x \cdot \nabla \psi_m(t)|^2 \, dx \, dm.
\]
It follows from (2.9) that
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\phi''}{r} |\nabla \psi_m(t)|^2 \, dx \, dm = s\|(-\Delta)^{s/2} \psi(t)\|_{L^2}^2 + \int_0^\infty m^s \int_{\mathbb{R}^N} \left( \frac{\phi''}{r} - 1 \right) |\nabla \psi_m(t)|^2 \, dx \, dm.
\]
Since \(\phi'' \leq 1\), we deduce from Cauchy-Schwarz inequality that
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} \left( \frac{\phi''}{r} - 1 \right) |\nabla \psi_m(t)|^2 \, dx \, dm + \int_0^\infty m^s \int_{\mathbb{R}^N} \left( \phi'' - \frac{\phi''}{r} \right) \frac{|x \cdot \nabla \psi_m(t)|^2}{r^2} \, dx \, dm \leq 0.
\]
Thus, we can obtain
\[
4 \int_0^\infty m^s \int_{\mathbb{R}^N} \partial_k \psi_m(t) (\partial_k \phi_m) \partial_t \psi_m(t) \, dx \, dm \leq 4s\|(-\Delta)^{s/2} \psi(t)\|_{L^2}^2.
\] (3.9)
Next, we write
\[
- \frac{2p - 4}{p} \int_{\mathbb{R}^N} \Delta \phi(t) (I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx = - \frac{2p - 4}{p} \int_{\mathbb{R}^N} (I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx
\]
\[
+ \frac{2p - 4}{p} \int_{\mathbb{R}^N} (N - \Delta \phi)(I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx.
\]
By the Hardy-Littlewood-Sobolev inequality and the conservation of mass, we can estimate as follows
\[
\int_{\mathbb{R}^N} (N - \Delta \phi)(I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx \lesssim \int_{|x| \leq R} (I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx
\]
\[
\lesssim \|I_{\alpha} * |\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(|x| \leq R) \|\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(\mathbb{R}^N),
\]
\[
\lesssim \|\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(\mathbb{R}^N) \|\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(|x| \leq R)
\]
\[
\lesssim \|\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(|x| \leq R)
\]
\[
\lesssim \|\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(|x| \leq R).
\]
We consequently obtain
\[
- \frac{2p - 4}{p} \int_{\mathbb{R}^N} \Delta \phi(t) (I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx \leq - \frac{2p - 4}{p} \int_{\mathbb{R}^N} (I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx
\]
\[
+ \frac{2p - 4}{p} \|\psi(t)|^p\|_{L^\frac{2N}{N-2p}}(|x| \leq R).
\] (3.10)
Denote the last term in (3.7) by \(J\). We can obtain
\[
J = - \frac{2N - 2\alpha}{p} A(\alpha) \int_{\mathbb{R}^N} (I_{\alpha} * |\psi(t)|^p) |\psi(t)|^p \, dx
\]
+ \frac{2N - 2\alpha}{p} A(\alpha) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left\{ |x - y|^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) \right\} \frac{|\psi(t, x)|^p |\psi(t, y)|^p}{|x - y|^{N-\alpha+2}} dxdy.

Note that
\[ \text{supp}(|x - y|^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y))) \subset \{|x| \geq R\} \cup \{|y| \geq R\}. \]
In the region \{|x| \geq R\}, it follows that
\[ ||x - y||^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) \lesssim ||x - y||^2. \]
This implies that
\[ A(\alpha) \int_{|x| \geq R} \int_{\mathbb{R}^N} \left\{ |x - y|^2 - (x - y) \cdot (\nabla \varphi_R(x) - \nabla \varphi_R(y)) \right\} \frac{|\psi(t, x)|^p |\psi(t, y)|^p}{|x - y|^{N-\alpha+2}} dxdy \lesssim \int_{|x| \geq R} (I_\alpha * |\psi(t)|^p)|\psi(t)|^p dx. \]
We have a similar control in the region \{|y| \geq R\}. By a similar argument as above, we can obtain
\[ J \leq - \frac{2N - 2\alpha}{p} \int_{\mathbb{R}^N} (I_\alpha * |\psi(t)|^p)|\psi(t)|^p dx + C_1 \frac{N - N_\text{p}}{p} \|\psi(t)\|_{L^p((|x| \geq R))}^p. \]
Combining (3.8) – (3.11), we obtain
\[ \frac{d}{dt} \mathcal{M}_{\varphi_R}[\psi(t)] \leq 4s \|(-\Delta)^{s/2} \psi(t)\|_{L^2}^2 - \frac{2Np - 2\alpha}{p} \int_{\mathbb{R}^N} (I_\alpha * |\psi(t)|^p)|\psi(t)|^p dx + CR^{-2s} \|\psi(t)\|_{L^p((|x| \geq R))}^2 + CC_1 \frac{2p - N_\text{p}}{p} \|\psi(t)\|_{L^p((|x| \geq R))}^p. \]
Applying Lemma 3.2, for any \( \eta > 0 \) and any \( R > 1 \), there exists \( C > 0 \) independent of \( R \) and \( C_1 \) such that for any \( t \in [0, T_0) \) with \( T_0 = \frac{\eta R}{C_1} \),
\[ \frac{d}{dt} \mathcal{M}_{\varphi_R}[\psi(t)] \leq 4K(\psi(t)) + CR^{-2s}(\eta + o_R(1))^2 + CC_1 \frac{2p - N_\text{p}}{p} \eta \eta + o_R(1)) \frac{N - N_\text{p}}{2} \]
\[ \leq -4\delta + CR^{-2s}(\eta^2 + o_R(1)) + CC_1 \frac{2p - N_\text{p}}{p} \eta \eta \frac{N - N_\text{p}}{2} + o_R(1). \]
We first choose \( \eta > 0 \) small enough so that
\[ CC_1 \frac{2p - N_\text{p}}{p} \frac{\eta}{\eta} \frac{N - N_\text{p}}{2} \leq 2\delta. \]
We next choose \( R > 1 \) large enough so that
\[ \frac{d}{dt} \mathcal{M}_{\varphi_R}[\psi(t)] \leq -\delta < 0, \]
for any \( t \in [0, T_0] \) with \( T_0 = \frac{\eta R}{CC_1} \). Note that \( \eta > 0 \) is fixed, so we can choose \( R > 1 \) large enough so that \( T_0 \) is as large as we want. By (3.13), it follows that
\[ \mathcal{M}_{\varphi_R}[\psi(t)] \leq -ct, \]
for all \( t \in [t_0, T_0) \) with some sufficiently large \( t_0 \in [0, T_0] \). The constant \( c > 0 \) depends only on \( \delta \). On the other hand, we deduce from Lemma 2.5 and the conservation of mass that for any \( t \in [0, +\infty) \),
\[ \left| \mathcal{M}_{\varphi_R}[\psi(t)] \right| \lesssim C(\varphi_R) \left( \|\psi(t)\|_{H^{1/2}}^2 + \|\psi(t)\|_{L^2} \|\psi(t)\|_{H^{1/2}} \right) \]
\[ \lesssim C(\varphi_R) \left( \|\psi(t)\|_{H^{1/2}}^2 + \|\psi(t)\|_{L^2}^2 \right). \]
By interpolating between $L^2$ and $\dot H^s$, we get for any $t \in [t_0, T_0]$, 
\[
ct \leq -\mathcal{M}_R(\theta(t)) = |\mathcal{M}_R(\theta(t))| \lesssim c(\mathcal{R}) \left( \|(-\Delta)^{s/2} \psi(t)\|_{L^2} + 1 \right).
\]
This implies that
\[
\|(-\Delta)^{s/2} \psi(t)\|_{L^2} \geq Ct^s,
\]
for all $t \in [t_1, T_0]$ with some sufficiently large $t_1 \in [t_0, T_0]$. Taking $t$ close to $T_0 = \frac{NR}{cT}$, we see that $\|(-\Delta)^{s/2} \psi(t)\|_{L^2} \to \infty$ as $R \to \infty$. Taking $R > 1$ sufficiently large, we have a contradiction with (3.4). The proof is complete.

Applying Lemma 3.1, we can prove Theorem 1.1.

**Proof of Theorem 1.1.** We need only to check that (3.1) follows. In the $L^2$-critical case, i.e., $s_c = 0$, we infer from (1.10) that $\|\psi_0\|_{L^2} < \|u\|_{L^2}$ and $\|\psi_0\|_{L^2} > \|u\|_{L^2}$, which is an contradiction. Thus, when $s_c = 0$, we have $E(\psi_0) < 0$. Applying the conservation of energy and $1 + \frac{2s_d}{N} \leq p < \frac{N+s_d}{N-2}$, it follows that
\[
K(\psi(t)) = \int_{\mathbb{R}^N} (I_\alpha \ast |\psi(t)|^p) \psi(t, x)^p \, dx
\]
\[= 2sE(\psi(t)) + \frac{2sp - \theta}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |\psi(t)|^p) \psi(t, x)^p \, dx \lesssim 2sE(\psi_0),
\]
for all $t \in [0, T^*)$. Hence, (3.1) follows with $\delta = -2sE(\psi_0)$.

Next, we consider the case $E(\psi_0) \geq 0$. We deduce from the assumption (1.10) that
\[
\begin{align*}
E(\psi_0)\|\psi_0\|_{L^2}^{2}\leq & E(u)\|u\|_{L^2}^{2}, \\
\|(-\Delta)^{s/2} \psi_0\|_{L^2} \geq & \|(-\Delta)^{s/2} u\|_{L^2}.
\end{align*}
\]
where
\[
\sigma := \frac{s-s_c}{s_c} = \frac{2sp - \theta}{\theta - 2s}.
\]
Notice that the sharp constant in Gagliardo-Nirenberg inequality (2.2) is
\[
C_{\text{opt}} = \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) \, dx
\]
\[\int_{\mathbb{R}^N} |u|_{H^s}^\sigma \|u\|_{L^}\frac{2p}{s} \, dx.
\]
By (2.3), we can rewrite $C_{\text{opt}}$ as
\[
C_{\text{opt}} = \frac{2sp}{\theta} \left( \|u\|_{H^s} \|u\|_{L^2}^\sigma \right)^{2}.\]
By a direct calculation, we also have
\[
E(u)\|u\|_{L^2}^{2}\leq \frac{2s}{\theta} \|u\|_{H^s}^{\sigma} \|u\|_{L^2}^{2}.
\]
Multiplying both sides of $E(\psi(t))$ by $\|\psi(t)\|_{L^2}^{2\sigma}$ and use the sharp Gagliardo-Nirenberg inequality (2.2), we obtain
\[
E(\psi(t))\|\psi(t)\|_{L^2}^{2\sigma} = \frac{1}{2} \|\psi(t)\|_{H^s}^{2\sigma} \|\psi(t)\|_{L^2}^{2\sigma} - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |\psi(t)|^p) \psi(t, x)^p \, dx \|u(t)\|_{L^2}^{2\sigma}
\]
\[\geq \frac{1}{2} \|\psi(t)\|_{H^s} \|\psi(t)\|_{L^2}^{2\sigma} - \frac{C_{\text{opt}}}{2p} \|\psi(t)\|_{H^s} \|\psi(t)\|_{L^2}^{2\sigma} \frac{\theta}{\theta - 2s}.
\]
= f(||\psi(t)||_{H^1}, ||\psi(t)||_{L^2}^2),

where $f(x) = \frac{1}{2}x^2 - \frac{C_{opt}}{2p}x^{\frac{p}{2}}$. It easily follows that $f$ is increasing on $(0, x_0)$ and decreasing on $(x_0, \infty)$, where

$$x_0 = \left(\frac{2sp}{C_{opt}2p}\right)^{\frac{1}{p-2}} = ||u||_{H^1}||u||_{L^2}^2,$$

where the last equality follows from (3.17). It follows from (3.17) and (3.18) that $f(||u||_{H^1}, ||u||_{L^2}^2) = E(u)||u||_{L^2}^2$. Thus, by the conservation of energy, (3.18) and (3.19), we have

$$f(||u||_{H^1}, ||u||_{L^2}^2) = E(u)||u||_{L^2}^2 < E(u)||u||_{H^1},$$

for all $t \in [0, T^*)$. Using the second condition (1.10), the continuity argument shows that

$$||\psi(t)||_{H^1}||\psi(t)||_{L^2}^2 > ||u||_{H^1}, ||u||_{L^2}^2$$

for any $t \in [0, T^*)$. On the other hand, since $E(\psi_0)||\psi_0||_{L^2}^2 < E(u)||u||_{L^2}^2$, we pick $\eta > 0$ small enough so that

$$E(\psi_0)||\psi_0||_{L^2}^2 \leq (1 - \eta)E(u)||u||_{L^2}^2.$$

Thus, by the conservation of energy, (3.18) and (3.19), we have

$$K(\psi(t))||\psi(t)||_{L^2}^2 \leq E(\psi(t))||\psi(t)||_{L^2}^2 - \frac{\theta - 2s}{2}||\psi(t)||_{H^1}^2||\psi(t)||_{L^2}^2$$

$$= E(\psi_0)||\psi_0||_{L^2}^2 - \frac{\theta - 2s}{2}||\psi(t)||_{H^1}^2||\psi(t)||_{L^2}^2$$

$$\leq (1 - \eta)E(u)||u||_{H^1}^2||\psi(t)||_{L^2}^2 - \frac{\theta - 2s}{2}||u||_{H^1}^2||u||_{L^2}^2$$

$$= - \eta E(u)||u||_{L^2}^2,$$

for all $t \in [0, T^*)$. This implies (3.1) with $\delta = \eta \theta E(u)||u||_{L^2}^2$. Thus, the solution $\psi(t)$ of (1.2) blows up in finite or infinite time. This completes the proof.

### 4 Existence and instability of normalized standing waves

In this section, we will prove the existence and instability of normalized standing waves of (1.2). Firstly, we prove Theorem 1.2.

**Proof of Theorem 1.2.** We first show $m(c) > 0$. By $K(v) = 0$ and the inequality (2.2), we have

$$s||v||_{H^1}^{\frac{2p-4}{2}} \leq \frac{\theta}{2p} \left( \int_{\mathbb{R}^N} (I_A * |v|^p)|v|^p \, dx \right) \leq C ||v||_{L^2}^{\frac{2p-4}{2}} = C ||v||_{H^1}^{\frac{2p-4}{2}},$$

where $\theta = np - N - \alpha$, which implies that there exists $C_1 > 0$ such that $||v||_{H^1} \geq C_1 > 0$. Thus, it follows from $K(v) = 0$ that

$$E(v) = \frac{1}{2s}K(v) + \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_A * |v|^p)|v|^p \, dx = \frac{\theta - 2s}{2\theta} ||v||_{H^1}^2 \geq \frac{\theta - 2s}{2\theta} C_1.$$  (4.1)

Taking the infimum over $v \in V(c)$, we have $m(c) > 0$.

Next, let $\{v_n\} \subseteq V(c)$ be a minimizing sequence of (1.15), i.e., $K(v_n) = 0$, $||v_n||_{L^2}^2 = c$ and $E(v_n) \to m(c)$ as $n \to \infty$. Thus, it follows from (4.1) that

$$||v_n||_{H^1} = \frac{2\theta}{\theta - 2s} E(v_n) \to \frac{2\theta m(c)}{\theta - 2s},$$
which implies that \( \{v_n\} \) is bounded in \( H^s \).

Now, applying the profile decomposition in \( H^s \), there exists a subsequence, still denoted by \( \{v_n\} \), a family \( \{x_n^j\}_{j=1}^{\infty} \) of sequences in \( \mathbb{R}^N \) and a sequence \( \{U^j\}_{j=1}^{\infty} \) in \( H^s \) such that for every \( l \geq 1 \) and every \( x \in \mathbb{R}^N \), we have

\[
v_n(x) = \sum_{j=1}^{l} U^j(x - x_n^j) + r_n^l, \quad (4.2)
\]

and (2.5)-(2.7) hold. Moreover, we deduce from (2.5)-(2.6) that

\[
0 = K(v_n) = s\|v_n\|_{H^s}^2 - \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_a * |v_n|^p)|v_n|^p \, dx
\]

\[
= s \sum_{j=1}^{l} \|U^j\|_{H^s}^2 + s\|v_n\|_{H^s}^2 - \frac{\theta}{2p} \int_{\mathbb{R}^N} (I_a * |v_n|^p)|v_n|^p \, dx + o_n(1)
\]

\[
= \sum_{j=1}^{l} K(U^j) + \frac{\theta}{2p} \sum_{j=1}^{l} \int_{\mathbb{R}^N} (I_a * |U^j|^p)|U^j|^p \, dx + s\|v_n\|_{H^s}^2
\]

\[
- \frac{\theta}{2p} \sum_{j=1}^{l} \int_{\mathbb{R}^N} (I_a * |v_n|^p)|v_n|^p \, dx + o_n(1), \quad (4.3)
\]

where \( o_n(1) \to 0 \) as \( n \to \infty \). Since \( K(v_n) = 0 \),

\[
\int_{\mathbb{R}^N} (I_a * |v_n|^p)|v_n|^p \, dx = \frac{4sp}{\theta - 2s} E(v_n) \to \frac{4spm(c)}{\theta - 2s}, \quad \text{as} \quad n \to \infty
\]

and \( s\|v_n\|_{H^s}^2 \geq 0 \) for all \( n \geq 1 \), we infer that

\[
\sum_{j=1}^{l} K(U^j) + \frac{\theta}{2p} \sum_{j=1}^{l} \int_{\mathbb{R}^N} (I_a * |U^j|^p)|U^j|^p \, dx - \frac{\theta}{2p} \frac{4spm(c)}{\theta - 2s} \leq 0, \quad (4.4)
\]

or equivalently,

\[
s \sum_{j=1}^{l} \|U^j\|_{H^s}^2 - \frac{2sm(c)\theta}{\theta - 2s} \leq 0. \quad (4.5)
\]

On the other hand, we see from (2.7) that

\[
\frac{4spm(c)}{\theta - 2s} = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_a * |v_n|^p)|v_n|^p \, dx = \sum_{j=1}^{\infty} \int_{\mathbb{R}^N} (I_a * |U^j|^p)|U^j|^p \, dx. \quad (4.6)
\]

Combining (4.4)-(4.6), we obtain

\[
\sum_{j=1}^{\infty} K(U^j) \leq 0, \quad \sum_{j=1}^{\infty} \|U^j\|_{H^s}^2 \leq \frac{2m(c)\theta}{\theta - 2s}. \quad (4.7)
\]

We claim that \( K(U^j) = 0 \) for all \( j \geq 1 \). Indeed, suppose that there exists \( j_0 \geq 1 \) such that \( K(U^{j_0}) < 0 \). Notice that

\[
K(\lambda U^{j_0}) = \lambda^2 s\|U^{j_0}\|_{H^s}^2 - \frac{\theta}{2p} \lambda^2 p \int_{\mathbb{R}^N} (I_a * |U^{j_0}|^p)|U^{j_0}|^p \, dx > 0,
\]

for sufficiently small \( \lambda > 0 \). There exists \( \lambda_0 \in (0, 1) \) such that \( K(\lambda_0 U^{j_0}) = 0 \). Let \( V = \lambda_0 U^{j_0} \), then we have

\[
K(V) = 0, \quad \int_{\mathbb{R}^N} |V(x)|^2 \, dx < c. \quad (4.9)
\]
let $V_\mu = \mu^{\frac{2s}{N-2s}} V(\mu x)$, 

$$K(V_\mu) = \mu^{\frac{2s}{N-2s}+2s-N} K(V) = 0.$$  

(4.10)

Since $p > \frac{N+2s}{N-2s}$ and

$$\int_{\mathbb{R}^N} |V_\mu(x)|^2 dx = \mu^{\frac{2s}{N-2s}} \int_{\mathbb{R}^N} |V(x)|^2 dx = c.$$  

(4.11)

there exists $\mu_0 \in (0, 1)$ such that $\|V_{\mu_0}\|_{L^2} = c$. We consequently estimate as follows:

$$m(c) \leq E(V_{\mu_0}) = \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_\alpha * |V_{\mu_0}|^p) |V_{\mu_0}|^p dx$$

$$= \frac{\theta - 2s}{4sp} \mu_0^{\frac{2s}{N-2s}+2s-N} \int_{\mathbb{R}^N} (I_\alpha * |U_0|)|U_0|^p dx$$

$$< \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_\alpha * |U_0|)|U_0|^p dx$$

$$\leq \frac{\theta - 2s}{4sp} \mu_0^{\frac{2s}{N-2s}} = m(c),$$  

(4.12)

which is a contraction. Finally, we claim that there exists only one term $U^j \neq 0$. Indeed, if there exist two terms $U^{j_1} \neq 0$ and $U^{j_2} \neq 0$, it follows from (4.7) that $K(U^{j_1}) = 0, K(U^{j_2}) = 0$ and

$$\int_{\mathbb{R}^N} (I_\alpha * |U^{j_1}|)|U^{j_1}|^p dx < \frac{4spm(c)}{\theta - 2s}$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |U^{j_2}|)|U^{j_2}|^p dx < \frac{4spm(c)}{\theta - 2s}.$$  

(4.13)

Next, we set

$$U_{\mu_1}^j = \mu^{\frac{2s}{N-2s}} U^j(\mu x), \quad U_{\mu_2}^j = \mu^{\frac{2s}{N-2s}} U^j(\mu x).$$  

(4.14)

It follows from that $K(U_{\mu_1}^j) = K(U_{\mu_2}^j) = 0$, and $K(U_{\mu_1}^j) = K(U_{\mu_2}^j) = 0$ for all $\mu > 0$. By $\int_{\mathbb{R}^N} |U_{\mu}^j|^2 dx < c$ and $\int_{\mathbb{R}^N} |U_{\mu}^j|^2 dx < c$, we obtain that there exist $\mu_1, \mu_2 \in (0, 1)$ such that

$$\int_{\mathbb{R}^N} |U_{\mu_1}^j|^2 dx = c, \quad \int_{\mathbb{R}^N} |U_{\mu_2}^j|^2 dx = c.$$  

(4.15)

Thus, we can estimate as follows:

$$m(c) \leq E(U_{\mu_1}^j) = \frac{\theta - 2s}{4sp} \int_{\mathbb{R}^N} (I_\alpha * |U_{\mu_1}^j|^p) |U_{\mu_1}^j|^p dx$$

$$= \frac{\theta - 2s}{4sp} \mu_1^{\frac{2s}{N-2s}} \int_{\mathbb{R}^N} (I_\alpha * |U^j|)|U^j|^p dx$$

$$\leq \frac{\theta - 2s}{4sp} \mu_1^{\frac{2s}{N-2s}} = m(c),$$  

which is a contradiction. Therefore, there exists only one term $U_0 \neq 0$ in the decomposition (4.2) and $K(U_0) = 0$, which together with (2.5) implies that the infimum of the variational problem (1.15) is attained at $U_0$. This completes the proof.

**Proof of Theorem 1.3.** With Theorem 1.2 in hand, ones can prove Theorem 1.3 by a similar argument as Theorem 1.3 in [18]. So we omit it.

**Proof of Theorem 1.4.** Let $u_c \in M_c$, a direct computation shows

$$E(u_c^j) = \frac{1}{2} A^2 \|u_c\|^2_{H^s} - \frac{\lambda^2}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)(x)|u_c(x)|^p dx,$$
and
\[ \partial_t E(u_c^\lambda) = s\lambda^{2s-1} \|u_c\|_{H^s}^2 - \frac{\theta\lambda^{\theta-1}}{2p} \int (|I_a * |u_c|^p)(x)|u_c(x)|^p \, dx = \frac{K(u_c^\lambda)}{\lambda}. \]

It is easy to see that the equation \( \partial_t E(u_c^\lambda) = 0 \) has a unique non-zero solution
\[ \left( \frac{2sp\|u_c\|_{H^s}^2}{\theta \int_{\mathbb{R}^n} (|I_a * |u_c|^p)(x)|u_c(x)|^p \, dx} \right)^{\frac{1}{p-1}} = 1. \]

The last inequality comes from the fact that \( K(u_c) = 0 \), which follows from Pohozaev’s identities (2.3). We thus obtain
\[ \begin{cases} 
\partial_t E(u_c^\lambda) > 0 & \text{if } \lambda \in (0, 1), \\
\partial_t E(u_c^\lambda) < 0 & \text{if } \lambda \in (1, \infty). 
\end{cases} \]

This implies that for any \( \lambda > 1 \)
\[ E(u_c^\lambda) < E(u_c). \quad (4.16) \]

Let \( \lambda_n > 1 \) such that \( \lim_{n \to \infty} \lambda_n = 1 \). We take the initial data
\[ \psi_{0,n}(x) = u_{c_n}^\lambda(x) = \lambda_n^\frac{N}{2} u_c(\lambda_n x). \]

By Brezis-Lieb’s lemma, we have \( \psi_{0,n} \to u_c \) in \( H^s \) as \( n \to \infty \). We deduce from (4.16) that
\[ E(\psi_{0,n}) < E(u_c), \]
and
\[ \|(-\Delta)^{s/2} \psi_{0,n}\|_{L^2} = \lambda_n^s \|(-\Delta)^{s/2} u_c\|_{L^2} > \|(-\Delta)^{s/2} u_c\|_{L^2}. \]

On the other hand, let \( u_c(x) = \omega_{\frac{N+2s}{2}} u(\omega_{\frac{N+2s}{2}} x) \) in (1.12), then \( u \) satisfies equation (1.11). In particular, by some basic calculations, we have
\[ E(u_c)^{s_c} \|u_c\|_{L^2}^{2(s_c-s_c)} = E(u)^{s_c} \|u\|_{L^2}^{2(s_c-s_c)}, \quad (4.17) \]
and
\[ \|(-\Delta)^{\frac{s}{2}} u_c\|_{L^2}^{s_c} \|u_c\|_{L^2}^{s-s_c} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{s_c} \|u\|_{L^2}^{s-s_c}. \quad (4.18) \]

Thus, by (4.17), (4.18) and \( \|\psi_{0,n}\|_{L^2} = \|u_c\|_{L^2} \), we have
\[ E(\psi_{0,n})^{s_c} \|\psi_{0,n}\|_{L^2}^{2(s_c-s_c)} < E(u_c)^{s_c} \|u_c\|_{L^2}^{2(s_c-s_c)} = E(u)^{s_c} \|u\|_{L^2}^{2(s_c-s_c)}, \]
and
\[ \|(-\Delta)^{s/2} \psi_{0,n}\|_{L^2}^{s_c} \|\psi_{0,n}\|_{L^2}^{s-s_c} > \|(-\Delta)^{s/2} u_c\|_{L^2}^{s_c} \|u_c\|_{L^2}^{s-s_c} = \|(-\Delta)^{s/2} u\|_{L^2}^{s_c} \|u\|_{L^2}^{s-s_c}, \]
where \( s_c = \frac{N}{2} - \frac{a^{s+2s}}{2p-2} \). Applying Theorem 1.1, the solution \( \psi_n \) of (1.2) and initial data \( \psi_{0,n} \) blows up in finite or infinite time. This completes the proof.

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