Optimisation estimation of uncertainty integrated with production information based on Bayesian fusion method

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Abstract: The new generation Geometrical Product Specifications require consideration of the effects of measurement uncertainty in the product inspection. This study estimated the measurement results and the uncertainty by integrating the statistical production information into the product detection results to rationally and fairly narrow the uncertainty area of qualification determination. Based on the Bayesian information fusion and statistical inference principle, the model of uncertainty evaluation is established. The Bayesian information fusion model integrated measuring information with manufacturing information was built, with which the uncertainty of product inspection was reappraised based on posterior distribution function. The validity of the proposed method and theory was demonstrated by the example analysis.

1 Introduction

Uncertainty is an important parameter in measurement results. To give scientific and proper evaluation of measurement uncertainty is an important factor to guarantee the development of modern measuring science. Influenced by the uncertainty of product inspection, there is a certain risk for product detection results located near the tolerance limit, whether the inspection is determined as qualified or unqualified. The ISO standards in the new generation of Geometrical Product Specifications (GPS) provide the guidance for product inspection based on uncertainty.

The GPS standard ISO 14253 [1–3] gives determination rules for qualified or unqualified product detection results. It is clear that the measurement uncertainty should be taken into account in the determination rules. Hinrichs [4] proposed that the product inspection method should be carefully selected based on the inspection ability, and pointed out that the uncertainty should be made clear at the same time when the inspection result was given, along with a detailed traceability description of the measurement method. Forbes [5] used the Bayesian decision method to establish the loss function and quantify the cost of wrong decisions in product inspection, deducing the optimal decision criterion to minimise the economic loss during product inspection. Djapic et al. [6] suggested that the risk management in the product test should be standardised, that the uncertainty of the product test should be determined comprehensively using the statistical and non-statistical techniques combined with the Bayesian method. The authors in [7, 8] pointed out that the conformity of the product should be determined quantitatively based on measurement uncertainty. The selection of uncertainty should be guided by establishing a cost model which included the test cost and user risk cost.

In view of the limitations and shortcomings of the current studies, this paper integrates the statistical production information in the product detection results to narrow the inspection uncertainty assessment results and expand the qualified area of the product inspection through the Bayesian information fusion technology.

2 Relationship between product tolerance and uncertainty

Product tolerance limits can be divided into the unilateral tolerance limit and the bilateral tolerance limit. In this paper, the more representative dimensional tolerance is taken as example. The tolerance limit of the dimension measurement is given in the form of a bilateral tolerance limit, \([T_L, T_U]\), among which \(T_L\) is lower limit of the tolerance and \(T_U\) is the upper limit. For the measurement result \(x\), when \(x \in [T_L, T_U]\), the product is determined to be qualified, when \(x > T_U\) or \(x < T_L\), the product is unqualified. The GPS standards state that the influence of measurement uncertainty must be considered in the geometric dimensioning and tolerancing inspection. The product inspection which is carried out only based on the tolerance limits may result in misjudgement of the dimensions or geometrical tolerances of the workpiece. Fig. 1 shows the way product qualification is determined when the uncertainty is considered. In Fig. 1, \(U\) means the extended uncertainty of the measurement results, and \(2U\) is the distribution interval of the measurement results. When \(2U\) is located in the tolerance interval (such as \(x_1\)), it can be directly determined as qualified. When \(2U\) is located outside the tolerance interval (such as \(x_3\)), it can be directly determined as unqualified. When \(2U\) is located inside the tolerance interval (such as \(x_2\)), it cannot be determined as qualified nor qualified.

Fig. 1 Relationship between product qualification and uncertainty
3 Bayesian uncertainty estimation method

3.1 Bayesian principle

The Bayesian principle takes measurement parameter $\theta$ as random variable, and identifies its prior distribution according to the historical information of $\theta$. After obtaining the measurement sample $X = (x_1, x_2, x_3, ... , x_n)$, it integrates prior information and current sample information based on the Bayes formula $\pi(\theta|x) \propto l(x|\theta)\pi(\theta)$ to obtain $\theta$ posterior distribution, thus achieving statistical inference of $\theta$. Where, $\pi(\theta)$ is the prior density function of $\theta$, $\pi(\theta|x)$ is the posterior density function and $l(x|\theta)$ is the sample likelihood function [9, 10].

Probability density function of posterior distribution includes all information related to $\theta$ in the population, sample and prior information; it is more reliable to carry out a statistical inference of $\theta$ based on the posterior distribution $\pi(\theta|x)$.

The prior and posterior distributions are important constituents of the Bayesian statistical model. The key points for the Bayesian evaluation method are to identify prior distribution according to historical information, and select proper methods to determine the posterior distribution. Based on different prior distributions, this paper studies and compares the Bayesian evaluation methods and their applications. It is of great practical significance for researchers to optimise the modern uncertainty evaluation technology.

3.2 Bayesian prior distribution

The subjective prior distribution refers to the prior distribution determination method which only uses the subjective prior information, including subjective belief, experience, historical data and so on. The method uses no other information. The subjective prior distribution method can use classical methods to solve statistical distribution, such as the subjective probability method, the histogram method, the relative likelihood method, the function-based hyper-parameter estimation method and the cumulative distribution function method.

The non-subjective prior distribution method only uses non-subjective information, such as population or sample information, to identify the prior distribution; in the non-subjective prior determination method, the non-informative prior distribution methods are the most commonly used methods. A key feature of the Bayesian method is that prior information is required in its statistical inference. In using the Bayesian method for parameter estimation, if no or little prior information is available, it is possible to use the population information and the non-informative prior distribution. Non-informative prior distribution methods include the hypothesis-based Bayesian prior distribution method, position or the scale parameter prior distribution method and the Jeffreys prior distribution method.

4 Bayesian uncertainty estimation based on conjugate prior

When certain prior information and sample distribution are known, uncertainty evaluation can be performed by the conjugate Bayesian method. The conjugate prior distribution can make the form of the prior distribution and the posterior distribution identical and categorise them in the same probability distribution group. The prior information, after integrating sample information, just changes the distribution of parameter values, which is consistent with a subjective judgement; furthermore, the integration of prior information and the sample information can form a prior chain. Namely, posterior distribution obtained after every information integration can serve as the prior distribution for the next
calculation. Performed repeatedly, this will form a chain which enables continuous integration and updating of measurement information.

4.1 Conjugate prior distribution.

Suppose prior data \((X_{i1}, X_{i2}, X_{i3}, \ldots, X_{i_{m}})\) follows normal distribution \(N(\mu, \sigma^2)\). It can be learned from mathematical statistics that

\[
\bar{X}_i = \frac{1}{n_i} \sum_{i=1}^{n_i} X_{ij}, \quad X_i \sim N(\mu, \sigma^2) \\
S_i = \sum_{i=1}^{n_i} (X_{ij} - \bar{X}_i)^2, \quad \frac{S_i}{\sigma^2} \sim \chi^2_{n_i - 1}
\]

where \(\bar{X}_i\) and \(S_i\) are independent of each other.

Conjugate prior distributions of \(\mu\) and \(\sigma^2\) are

\[
\mu \sim N\left(\frac{\bar{X}_i}{n_i}, \frac{\sigma^2}{n_i}\right) \\
\sigma^2 \sim \Gamma^{-1}\left(n_i - 1, \frac{S_i}{2}\right)
\]

So, the joint prior distribution density of \((\mu, \sigma^2)\) is

\[
\pi(\mu, \sigma^2) = \pi(\mu)\pi(\sigma^2)
\]

\[
= \frac{\sqrt{2\pi n_i}}{\sqrt{\Sigma_{n_i}}} \Gamma\left((n_i - 1)/2\right) \left(1/n_i\right)^{n_i/2 + 1} \cdot \exp\left[-\frac{S_i + n_i(\mu - \bar{X}_i)^2}{2\sigma^2}\right]
\]

4.2 Posterior distribution and its uncertainty of conjugate prior

Suppose \((X_{1j}, X_{2j}, X_{3j}, \ldots, X_{nj})\) are samples drawn from the normal population \(N(\theta, \sigma^2)\), where \(\theta\) and \(\sigma^2\) are unknown. The joint conjugate prior density function of \(\theta\) and \(\sigma^2\) can be provided with a reference to formula (5). Use \(\bar{X} = 1/n_i \sum_{i=1}^{n_i} X_{ij}\) and \(S_i = \sum_{i=1}^{n_i} (X_{ij} - \bar{X})^2\) as the sufficient statistic of \(\theta\) and \(\sigma^2\), and its sample likelihood function is

\[
h(\theta, \sigma^2 | X) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{S_i + n_i(\theta - \bar{X}_i)^2}{2\sigma^2}\right]
\]

It can be obtained by the Bayesian formula that the joint posterior PDF of \((\theta, \sigma^2)\) is

\[
\pi(\theta, \sigma^2 | X) \propto \left(\frac{\sigma^2}{m}\right)^{n/2} \exp\left[-\frac{m(\theta - \bar{X}_i)^2}{2\sigma^2}\right]
\]

\[
\cdot \left(\sigma^2\right)^{-n_i/2 - 1} \exp\left[-\frac{S_i}{2\sigma^2}\right]
\]

where

\[
m = n_i + n
\]

\[
\bar{X} = \frac{n_i\bar{X}_i + n\bar{X}}{n_i + n}
\]

\[
S = \frac{n_iS_i}{n_i + n}(\bar{X}_i - \bar{X})^2 + S + S_i
\]

Namely, the posterior distributions of \((\theta, \sigma^2)\) are as follows:

\[
\pi(\sigma^2 | X) \sim \Gamma^{-1}\left(m - 1, \frac{S}{2}\right), \pi(\theta | \sigma^2, X) \sim N\left(\bar{X}_i, \frac{\sigma^2}{m}\right)
\]

5 Uncertainty estimation with the fusion of production information

To reasonably reduce the uncertainty area of the conformity assessment in the product inspection, the current conventional practice is to increase the number of repetitive measurements or to use a more accurate measuring instrument. However, the increase in the number of repetitive measurements only reduces the impact of measurement repeatability while the use of high-precision instruments will greatly increase the measurement costs. Based on the Bayesian method, this paper presents an information fusion model that re-estimates the measurement estimated values and uncertainties by fusing production information and measurement information. The method expands the qualified area of the product inspection and reduces the production and inspection costs without changing the measurement conditions.

Assuming that the true value of the measured parameter is \(x\), when the producing process is controlled, according to the central limit theorem, \(x\) is subject to the normal distribution and the probability density function is

\[
p(x) = \frac{1}{\sqrt{2\pi\sigma_m}} \exp\left[-\frac{(x - \mu)^2}{2\sigma_m^2}\right]
\]

where \(\mu\) is the central value of product quality control, \(\sigma_m\) is the dispersion of the workpiece processing which is determined by the processing accuracy.

Assuming that the actual measurement result is \(y_m\) while considering the processing distribution of the measured parameters as the priori information, the systematic error has been corrected. Based on \(y_m\), carry out the statistical inference to \(x\) again, and the likelihood function of \(x\) is

\[
f(y_m | x) = \frac{1}{\sqrt{2\pi\sigma_m}} \exp\left[-\frac{(y_m - x)^2}{2\sigma_m^2}\right]
\]

where \(y_m\) is the measurement uncertainty of \(y_m\).

Based on the conjugate distribution, combined with (13) and (14), the kernel of the posterior distribution can be obtained as

\[
h(x | y_m) \propto p(x) f(y_m | x)
\]

\[
\propto \exp\left[-\frac{(x - \mu)^2}{2\sigma_m^2} \cdot \frac{(y_m - x)^2}{2\sigma_m^2}\right]
\]

As the posterior distribution of \(x\) is still subject to the normal distribution, the posterior distribution can be set as

\[
h(x | y_m) = \frac{1}{\sqrt{2\pi \sigma_m}} \exp\left[-\frac{(x - y)^2}{2\sigma_m^2}\right]
\]

where \(y_i\) is the best estimated result after the fusion of the production apriori information and the sample measurement information; \(u_i\) is the measurement uncertainty of \(y_i\) after the fusion.
From (15) and (16), the following can be obtained:

\[ y_f = \mu + \sigma_m^2 + \mu + \sigma_m^2 \]  
\[ u_f^2 = \mu + \sigma_m^2 + \sigma_m^2 \]  

Assuming that \( \lambda = \sigma_m^2 / \sigma_m^2 \), then (17) and (18) can be simplified as

\[ y_f = \frac{1}{1 + \lambda} \mu + \frac{\lambda}{1 + \lambda} \sigma_m \]  
\[ u_f = \sqrt{\frac{1}{1 + \lambda} \sigma_m} \]  

After the Bayesian information fusion, the best estimated result \( y_f \) of the measurement results is the weighted average value of the production apriori information \( \mu \) and the sample measurement information \( \sigma_m \), reflecting the production information and the measurement information comprehensively. The fusion of the production apriori information reduces the uncertainty effectively.

Before the fusion of the production apriori information, the qualified area of bilateral tolerance test is

\[ T_L + \frac{\lambda}{1 + \lambda} 2u_m \leq y_m \leq T_U - \frac{\lambda}{1 + \lambda} 2u_m \]  

Equation (21) shows that the qualified area after the information fusion is larger than the qualified area before the fusion, effectively reducing the uncertainty area.

6 Experimental analysis

This experiment used the aperture of the workpiece measured by coordinate measuring machine to demonstrate the validity of the theory mentioned. The estimation of the measurement uncertainty is shown in Table 1. The combined standard uncertainty \( u_c \) is

\[ u_c = \sqrt{u_E^2 + u_{rp}^2 + u_{rd}^2 + u_{temp}^2} = 0.0022 \text{ mm} \]  

According to the manufacturing department, the workpiece processing is statistically controlled. The central value of workpiece quality control is \( \mu = 32.0113 \text{ mm} \), while the batch variation is \( \sigma = 0.0038 \text{ mm} \), the following can be obtained:

\[ \lambda = \frac{\sigma^2}{\sigma_m^2} = 1.98 \]  

Substitute the value into (19) and (20) and we have

\[ y_f = \frac{1}{2.98} (32.0113 + 1.98 \sigma_m) \]  
\[ u_f = 0.0018 \text{ mm} < 0.0022 \text{ mm} \]  

The experiment has been carried out on eight workpieces. The data before and after the information fusion is shown in Table 2. The comparison of the complete measurement results including the measurement uncertainty before and after the information fusion is as shown in Fig. 3. In Fig. 3, before the information fusion, part of the measurement intervals of the workpieces No.4 and No.6 are outside the product tolerance limit. There is a risk of misjudgement. After the information fusion, the measurement intervals of the workpieces No.4 and No.6 are completely within the tolerance limit, which can be directly determined as qualified.
also narrowed the uncertainty area and expanded the qualified area of the product inspection effectively.

Experimental results show that the best estimated value of the measurement results is close to the central value \( \mu \) of product quality control after the integration of the statistically controlled production information. The production and inspection costs of further measurement and determination can be saved. The relevant suggestions for product inspection based on experimental results can be provided. The reliability of the product inspection results will be ensured.

8 Acknowledgments

This project is supported by National Key Research and Development Plan of China (grant no. 2016YFF0203801), Certification and Accreditation Standard Funded Project of Certification and Accreditation Administration of China (grant no. 2017RB076) and National Natural Science Foundation of China (grant no. 51275148).

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