Black holes with Skyrmion-anti-Skyrmion hairs

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We construct static axially symmetric black holes in multi-Skyrmion configurations coupled to Einstein gravity in four dimensional asymptotically flat space-time. In a simplest case the event horizon is located in-between a Skyrmion-anti-Skyrmion pair, other solutions represent black holes with gravitationally bounded chains of Skyrmions and anti-Skyrmions placed along the axis of symmetry in alternating order. We discuss the properties of these hairy black holes and exhibit their domain of existence.

I. INTRODUCTION

Various black holes with scalar hair, which circumvent the well-known no-hair theorem (see, e.g., [1, 2] and references therein), are rather a common presence in the landscape of gravity solutions. Historically, one of the first counter-examples to the "no-hair" theorem was found in the Skyrme model coupled to the Einstein gravity [3–5]. It was shown that a small Schwarzschild black hole can be continuously connected to the self-gravitating Skyrmion with some amount of the scalar field absorbed into horizon. Another example of the primary hairy black holes was obtained in the SU(2) Einstein-Yang-Mills (EYM) theory [6, 7] and in the Einstein-Yang-Mills-Higgs (EYMH) model [8, 9]. Interestingly, these static spherically symmetric solutions share many common features with the corresponding black holes with Skyrme hairs, see e.g. [2, 10].

There are various extensions of this type of the solutions. First, there are static axially symmetric black holes both in the Einstein-Skyrme theory [11] and in the EYM model [12]. Secondly, there are stationary spinning asymptotically flat hairy black holes in the Einstein-Skyrme model [13] and in the non-linear O(3) sigma model [14] which belong to the same class of solutions, as the spinning black holes with primary scalar hair in the Einstein-Klein-Gordon theory [15, 16]. Further, both the Skyrme theory [17, 19] and the EYMH model [20, 30] admit axially symmetric equilibrium configurations with a number of constituents located symmetrically with respect to the origin along the symmetry axis. An example of such a configuration in the flat space is a sphaleron solutions that represent a monopole-antimonopole pair in a static equilibrium [20, 21]. Similar Skyrmion-anti-Skyrmion solution exists in the Skyrme model [17]. Further, there are generalizations of these solutions that represent chains of interpolating Skyrmion-anti-Skyrmion [18] and monopole-antimonopole
chains \[22, 24\]. Notably, inclusion of gravity allows for an attractive phase of Skyrmions not present in flat space \[19\], flat space Skyrmion-anti-Skyrmion chains do not exist if the topological charge of a constituent is less than two \[17, 18\].

The similarity between the regular gravitating soliton configurations of the non-abelian YMH theory and the Skyrme models can also be noted in the pattern of their evolution. In both cases, there are two branches of solutions, one of which emerges smoothly from the corresponding flat space configuration. This branch is extended up to some critical value of the effective gravitational coupling at which it merges the second, backward branch leading to some limiting rescaled Bartnik-McKinnon type solutions \[31\]. Heuristically, this property can be attributed to the structure of the emerging effective gravitational constant, which may approach zero in two different limits. First, it is vanishing as the Newton constant tends to zero, secondly, it approaches zero as the vacuum expectation value of the scalar field in the YMH theory, or the pion decay constant in the Skyrme model, becomes zero.

There are also hairy black holes supporting the Yang-Mills-Higgs hair of the monopole-antimonopole pairs, chains, and vortex ring solutions with a small black hole placed at the center \[32, 33\]. These static black holes possess non-trivial non-Abelian magnetic field outside their regular event horizon, furthermore, they provide a counter-example to the black holes uniqueness theorem \[8, 9\]. Similar to the case of the hairy black holes in the Einstein-Skyrme model, there are two branches of solutions, which emerge from the two globally regular solutions and bifurcate at a maximal value of the horizon size \[8, 9, 32, 33\]. In a contrary, the Einstein-Yang-Mills hairy black hole solutions exist for arbitrarily large horizon size \[6, 7, 12, 34, 35\].

Following the strategy of \[32, 33\] we shall, in this paper, study the existence of a new type of hairy black hole solutions, which correspond to the static axially symmetric Skyrmion-anti-Skyrmion chains with an event horizon at the center. For the chains with an odd number of constituents a small black hole is placed into the central Skyrmion, such configuration can be considered as a deformation of the usual spherically symmetric black hole with Skyrmion hairs. Similarly, for the chains with even number of components, a black hole is immersed in-between the central Skyrmion-anti-Skyrmion pair.

This paper is organised as follows. In Section II we describe the Einstein-Skyrme model and construction of the axially-symmetric solutions representing chains of interpolating Skyrmion-anti-Skyrmion. We restrict our consideration to the configuration with constituents of topological degree one and two and consider the configurations with number of components \(k \leq 3\). We found that the solutions possess a branch structure for their global quantities both in terms of the gravitational coupling and in event horizon radius. Numerical results are presented in Section III, while the conclusions and remarks are formulated in the last Section.
II. THE MODEL AND FIELD EQUATIONS

We consider the Einstein-Skyrme theory in asymptotically flat 3+1 dimensional space. The action of the model reads

\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + L_{Sk} \right), \tag{1} \]

where \( R \) is the curvature scalar, \( g \) denotes the metric determinant and \( G \) represents Newton's constant. The matter part of the action \( L_{Sk} \) chosen as the Skyrme model

\[ L_{Sk} = \frac{F_\pi^2}{16} g^{\mu\nu} \text{Tr} \left( L_\mu L_\nu \right) + \frac{1}{32e^2} g^{\mu\nu} g^{\rho\sigma} \left[ L_\mu, L_\rho \right] \left[ L_\nu, L_\sigma \right] + \frac{\mu_\pi^2 F_\pi^2}{8} \text{Tr} \left( U - 1 \right) \tag{2} \]

with a potential term with a mass parameter \( \mu_\pi^2 \). Here \( F_\pi \) and \( e \) are positive coupling constants and

\[ L_\mu = U^\dagger \partial_\mu U \tag{3} \]

is the \( \mathfrak{su}(2) \)-valued left-invariant current, associated with the \( \mathfrak{su}(2) \)-valued scalar field \( U = \sigma \mathbb{I} + i \pi \tau \). It can be represented in terms of the quartet of scalar fields \( \phi^a = (\sigma, \pi) \) restricted to the surface of the unit sphere \( S^3 \), \((\phi^a)^2 = \sigma^2 + \pi \cdot \pi = 1 \).

The field of the model is required to satisfy the boundary condition \( U(x) \to \mathbb{I} \) as \( x \to \infty \), the field is a map \( U: S^3 \to S^3 \) labeled by the topological invariant \( B = \pi_3(S^3) \). Explicitly,

\[ B = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr} \left[ \left( U^\dagger \partial_i U \right) \left( U^\dagger \partial_j U \right) \left( U^\dagger \partial_k U \right) \right] = \int d^3x |g|^{1/2} B_0 \tag{4} \]

where \( B_0 \) is the temporal component of the topological current

\[ B^\mu = \frac{1}{24\pi^2 |g|^{1/2}} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left( L_\nu L_\rho L_\sigma \right). \tag{5} \]

We note that a rescaling of the radial coordinate \( r \to er F_\pi/2 \) transforms the action of the Einstein-Skyrme model \( (1) \) to the form

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{R}{\alpha^2} + \frac{1}{2} g^{\mu\nu} \text{Tr} \left( L_\mu L_\nu \right) + \frac{1}{16} g^{\mu\nu} g^{\rho\sigma} \text{Tr} \left[ \left[ L_\mu, L_\rho \right] \left[ L_\nu, L_\sigma \right] \right] + \mu^2 \text{Tr} \left( U - 1 \right) \right\}, \tag{6} \]

where \( \mu = 2\mu_\pi/(e F_\pi) \) is the rescaled mass parameter and \( \alpha^2 = 4\pi GF_\pi \) is the effective gravitational coupling. In our numerical simulations we set \( \mu = 1 \).

Variation of the rescaled action \( (6) \) with respect to the metric leads to the Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \alpha^2 T_{\mu\nu} \tag{7} \]
where the Skyrme stress-energy tensor is
\[ T_{\mu\nu} = \text{Tr} \left( \frac{1}{2} g_{\mu\nu} L^\alpha L_\alpha - L_\mu L_\nu \right) + \text{Tr} \left( g_{\mu\nu} [L_\alpha, L_\beta] [L^\alpha, L^\beta] - 4 g_{\alpha\beta} [L_\mu, L_\nu] [L^\alpha, L^\beta] \right) + \mu^2 g_{\mu\nu} \text{Tr}(U - 1). \] (8)

Both the regular self-gravitating Skyrmions and the static axially symmetric black hole solutions can be constructed in isotropic coordinates with the Lewis-Papapetrou metric
\[ ds^2 = -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2 \theta d\varphi^2, \] (9)
where the metric functions $f$, $m$ and $l$, are functions of the radial variable $r$ and polar angle $\theta$, only.

Note that, making use of the axially symmetric parametrization of the Skyrme fields
\[ \pi_1 = \phi_1 \cos(n\varphi); \quad \pi_2 = \phi_1 \sin(n\varphi); \quad \pi_3 = \phi_2; \quad \sigma = \phi_3 \] (10)
where $n \in \mathbb{Z}$ is the azimuthal winding number and $\phi_a$ is a triplet of field variables on the unit sphere $S^2$, we can implement so-called ”trigonometric” parametrization, see e.q. \[11\]
\[ \pi_1 = \sin P \sin Q; \quad \pi_2 = \sin P \cos Q; \quad \pi_3 = \cos P, \] (11)
where the Skyrmion’s profile functions $P(r, \theta)$ and $Q(r, \theta)$ depend on the radial coordinate $r$ and polar angle $\theta$. The value of the topological charge \[14\] depends on the boundary conditions on these two functions \[17-19\]. Imposing
\[ \lim_{r \to \infty} Q(r, \theta) = k \theta, \] (12)
where the second integer $k$ specifies the asymptotic value of the field $Q(r, \theta)$, we obtain $B = \frac{n}{2} \left( 1 - (-1)^k \right)$. Thus, the case $k = 1$ corresponds to the multi-Skyrmions of topological degree $B = n$, while $k = 2$ yields the Skyrmion-anti-Skyrmion (S-A) static sphaleron solution of the Einstein-Skyrme model, consisting of a charge $n$ Skyrmion and a charge $-n$ anti-Skyrmion \[17-19\]. Configurations with $k \geq 3$ correspond to the Skyrmion-anti-Skyrmion chains with $k$ constituents placed along the axis of symmetry in alternating order.

### III. RESULTS

#### A. Boundary conditions

To obtain asymptotically flat solutions of the Einstein-Skyrme equations, which are either globally regular or possess a regular event horizon, we must impose appropriate boundary conditions \[3, 5, 11-19, 36\].
Here we consider static axially symmetric black hole solutions with Skyrmion-anti-Skyrmion hair, which are asymptotically flat, and possess a finite mass. Then the corresponding boundary conditions at spatial infinity and along the symmetry axis are the same as those of the regular self-gravitating Skyrmions [5, 11, 19]. In particular, as \( r \to \infty \), the asymptotic value of the Skyrme field is restricted to the vacuum and the metric functions must approach unity. Explicitly, we impose

\[
\begin{align*}
\phi_1 \bigg|_{r \to \infty} &\to 0, \quad \phi_2 \bigg|_{r \to \infty} \to 0, \quad \phi_3 \bigg|_{r \to \infty} \to 1, \\
f \bigg|_{r \to \infty} &\to 1, \quad l \bigg|_{r \to \infty} \to 1, \quad m \bigg|_{r \to \infty} \to 1.
\end{align*}
\]

The condition of regularity of the functions on the symmetry axis yields

\[
\begin{align*}
\phi_1 \bigg|_{\theta = 0} &= 0, \quad \partial_{\theta} \phi_2 \bigg|_{\theta = 0} = 0, \quad \partial_{\theta} \phi_3 \bigg|_{\theta = 0} = 0, \\
\partial_{\theta} f \bigg|_{\theta = 0} &= 0, \quad \partial_{\theta} l \bigg|_{\theta = 0} = 0, \quad \partial_{\theta} m \bigg|_{\theta = 0} = 0.
\end{align*}
\]

We further impose, that the two metric functions \( m(r, \theta), l(r, \theta) \) on the symmetry axis satisfy

\[
m(r, \theta = 0, \pi) = l(r, \theta = 0, \pi)
\]

This condition secures the absence of a conical singularity, it requires that the deficit angle should vanish.

Note that the non-linear system of the Einstein-Skyrme equations includes three equations on the matter fields \( \phi_a \). One can try to reduce the number of equations, considering the trigonometric parametrization of the triplet \( \phi_a \) given by ansatz [11]. However, in such a case, there are obstacles related with regularity conditions we have to impose on the angular function \( Q(r, \theta) \) for \( k \geq 2 \) [13], thus we just make use of the parametrization [11] to generate an appropriate input configuration with \( k \) components.

To obtain globally regular solutions, we must impose appropriate boundary conditions at the origin. For odd values of the integer \( k \) the boundary conditions are identical to those for the case of the spherically symmetric fundamental gravitating Skyrmion

\[
\begin{align*}
\phi_1 \bigg|_{r \to 0} &\to 0, \quad \phi_2 \bigg|_{r \to 0} \to 0, \quad \phi_3 \bigg|_{r \to 0} \to -1, \\
\partial_r f \bigg|_{r \to 0} &\to 0, \quad \partial_r l \bigg|_{r \to 0} \to 0, \quad \partial_r m \bigg|_{r \to 0} \to 0.
\end{align*}
\]

while for the S-A pair and other Skyrmion-anti-Skyrmion chains with even number of components, the boundary conditions on the matter field are different [17] [19]:

\[
\begin{align*}
\phi_1 \bigg|_{r \to 0} &\to 0, \quad \partial_r \phi_2 \bigg|_{r \to 0} \to 0, \quad \partial_r \phi_3 \bigg|_{r \to 0} \to 0
\end{align*}
\]

The event horizon of the static black hole resides at a surface of constant radial coordinate, \( r = r_h \). The boundary conditions are obtained from the asymptotic expansion of the
corresponding field equations near the horizon. Regularity at $r = r_h$ requires that

$$
\begin{align*}
  f \big|_{r \to r_h} &\to 0, \quad m \big|_{r \to r_h} \to 0, \quad l \big|_{r \to r_h} \to 0, \\
  \partial_r \phi_1 \big|_{r \to r_h} &\to 0, \quad \partial_r \phi_2 \big|_{r \to r_h} \to 0, \quad \partial_r \phi_3 \big|_{r \to r_h} \to 0.
\end{align*}
$$

(17)

In our numerical calculations we make use of the parametrization of the metric functions

$$
\begin{align*}
  f(r, \theta) &= f_2(r, \theta) \left(1 - \frac{r_h}{r}\right)^2, \\
  l(r, \theta) &= l_2(r, \theta) \left(1 - \frac{r_h}{r}\right)^2, \\
  m(r, \theta) &= m_2(r, \theta) \left(1 - \frac{r_h}{r}\right)^2
\end{align*}
$$

and impose the following boundary conditions at $r = r_h$

$$
\begin{align*}
  \partial_r f_2 \big|_{r \to r_h} &\to 0, \quad \partial_r m_2 \big|_{r \to r_h} \to 0, \quad \partial_r l_2 \big|_{r \to r_h} \to 0
\end{align*}
$$

(18)

We have solved the boundary value problem for the coupled system of six nonlinear partial differential equations with boundary conditions (13)-(17) using a six-order finite difference scheme.

Within our formulation, the numerical problem possesses five input parameters: $r_h, \alpha, n, k$ and the mass parameter $\mu$. The emerging overall system becomes rather complicated and we did not attempt to explore in a systematic way the entire parameter space of all solutions. In particular, we restricted our analysis to the configurations with winding $n = 1, 2$ and consider only chains with two and three components. We also did not study the dependency of the solutions on the value of the mass parameter setting $\mu = 1$.

To facilitate the calculations in the near horizon area, we have introduced the new compact radial coordinate $x = \frac{r - r_h}{r + c}$, which maps the semi-infinite region $r \in [r_h, \infty)$ onto the unit interval $x \in [0, 1]$. Here $c$ is an arbitrary constant used to adjust the contraction of the grid. The system of equations is discretized on a grid with typical number of points $89 \times 69$. The underlying linear system is solved with the packages FIDISOL/CADSOL [39]. The typical errors are of order of $10^{-4}$.

B. Quantities of interest and horizon properties

Asymptotic expansions of the metric functions at the horizon and at spatial infinity yield important physical properties of the BHs. The total ADM mass of the configuration can be read off from the asymptotic subleading behavior of the metric functions as $r \to \infty$:

$$
M = \frac{1}{2G} \lim_{r \to \infty} r^2 \partial_r f.
$$

(19)

It is convenient to introduce the rescaled coordinate $\hat{x} = x/\alpha$, horizon radius $\hat{r}_h = r_h/\alpha$ and rescaled mass $\hat{M} = \alpha M$. 
The physically interesting horizon properties include the surface gravity

$$\kappa^2 = -\frac{1}{4}g^{00}g^{ij}(\partial_i g^{00})(\partial_j g^{00}).$$

Taking into account the parametrization (18) and the expansion of the metric functions in the near-horizon region we obtain the dimensionless surface gravity

$$\hat{\kappa} = \frac{\kappa}{\alpha} = \frac{f_2(\theta)}{8r_h \sqrt{m_2(\theta)}}.$$  \hspace{1cm} (20)

The surface gravity, as well as the Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, is finite at the event horizon. The Hawking temperature is proportional to the surface gravity, $T = \frac{\kappa}{2\pi}$, see e.g. [40]. Further, taking into account the parametrization of the metric functions (18), we can see that the dimensionless event horizon area is defined as

$$A = 32\pi r_h^2 \int_0^\pi d\theta \sin \theta \frac{l_2 m_2}{f_2},$$  \hspace{1cm} (21)

it is proportional to the entropy $S = A/4$.

Presence of the axially-symmetric field of the gravitationally bound Skyrmions deforms the event horizon. As usually, small deformation is revealed, when measuring the ratio of circumferences of the horizon along the equator

$$L_e = 16r_h \int_0^{2\pi} d\varphi \sqrt{\frac{l_2}{f_2}},$$  \hspace{1cm} (22)

and along the poles

$$L_p = 32\pi r_h \int_0^\pi d\theta \sqrt{\frac{m_2}{f_2}}.$$  \hspace{1cm} (23)

Note that, in the case of the black holes with Skyrmie hair, strictly speaking, the solutions cannot be classified according to the topological mapping (4) between physical and internal spaces. Still, one defines a baryon charge density performing the integration (4) in the exterior region. The event horizon absorbs a part of the baryon charge but the integral (4) never vanishes [4, 11]. As we shall see below, the same holds for the black holes with Skyrmion-anti-Skyrmion hairs.

We construct numerically the black hole solutions for the Skyrmion-anti-Skyrmion pairs and Skyrmion-anti-Skyrmion-Skyrmion chains with constituents of degrees one and two. These asymptotically flat solutions are associated with the regular self-gravitating solutions discussed in [19]. Further, they have many features in common with the EYMH black holes.
FIG. 1: Component $\phi_3$ of the Skyrmion field of the hairy BH solutions (left column), the baryon charge density distribution (middle column) and the metric function $f$ (right column) of the BHs with Skyrmion-anti-Skyrmion hairs are plotted for the single black hole with Skyrmion hair (upper row), the S-A pair (middle row) and for the S-A-S chain (bottom row) on the lower branches of solutions as functions of the coordinates $z = r \cos \theta$ and $\rho = r \sin \theta$ at $\alpha = 0.15$, $|n| = 1$ and $r_h = 0.01$.

with magnetic dipole hair [32] and the static axially symmetric black hole solutions of the EYM theory [41]. In particular, for a fixed value of the effective gravitational coupling $0 \leq \alpha \leq \alpha_{cr}$, in both cases there are two branches of solutions, which are linked to the corresponding flat space sphaleron configurations and the generalized Bartnik-McKinnon solutions, respectively. The difference is that, as the gravitational interaction remains relatively weak, the non-Abelian interaction between the constituents of the EYM is stronger than the dipole-dipole interaction between the Skyrmions [17, 18]. As a result, the Skyrmion-anti-Skyrmion chains may exist in the flat space only when each of the constituents carries charge larger than two. As the gravitational attraction becomes stronger, the S-A chains with constituents of unit charge arise forming the lower branch of solutions, which may be not linked to the flat space [19].

The lower in energy branch of the solutions emerges from the corresponding regular self-gravitating solution with a nonzero mass as the horizon radius $r_h$ increases from zero. This configuration can be viewed as a small Schwarzschild black hole, immersed into the
FIG. 2: The scaled mass $M_\alpha$ (divided by the number of components $k$) of the static axially symmetric $|n|=1$ Skyrmion bound states (left plot) and the Hawking temperature (right plot) are shown as functions of the scaled event horizon radius $r_h/\alpha$ for the single black hole with Skyrmion hair ($k=1$), the S-A pair ($k=2$) and for the S-A-S chain ($k=3$) at $\alpha=0.15$.

FIG. 3: The value of the metric function $f_2$ on the event horizon for the black holes bounded with static $|n|=1$ axially symmetric Skyrmions (left plot) and the ratio $L_e/L_p$ (which gives a measure of the horizon deformation, right plot) are shown as functions of the scaled event horizon radius $r_h/\alpha$ for the single black hole with Skyrmion hair ($k=1$), the S-A pair ($k=2$) and for the S-A-S chain ($k=3$) at $\alpha=0.15$.

Many features of the hairy black holes usually become more transparent if we introduce Skyrmion-anti-Skyrmion chains. In Fig. 1 we display the distribution of the charge density $B_0$, the third component of the Skyrme field $\phi_3$ and the metric function $f$ for the $n=1$ single black hole with Skyrmion hair ($k=1$), the S-A pair ($k=2$) and for the S-A-S chain ($k=3$) at $\alpha=0.15$. Many features of the hairy black holes usually become more transparent if we introduce
FIG. 4: The scaled mass $M_\alpha$ (left plot) and the Hawking temperature (right plot) of the $|n| = 2$ S-A pair are shown as functions of the scaled event horizon radius $r_h/\alpha$ for some set of values of $\alpha$.

FIG. 5: The value of the metric function $f_2$ on the event horizon and the ratio $L_e/L_p$ (right plot) of the $|n| = 2$ S-A pair are shown as functions of the scaled event horizon radius $r_h/\alpha$ for some set of values of $\alpha$.

the scaled ADM mass $M_\alpha$ and scaled radial coordinate $r/\alpha$. The scaled mass of the configuration remains almost constant along the lower-in-mass branch while the Hawking temperature and the value of the metric function $f_2(r_h)$ at the horizon decreases, as the horizon radius $r_h$ increases, see Figs. 2-5.

An interesting feature shared by all BHs with Skyrmion-anti-Skyrmion hairs, is that they always possess a prolate horizon, $L_e/L_p < 1$, although the horizons deformation from sphericity remains very small, as seen in Figs. 3-5. A tiny deformation is observed for the S-A-S chain with the winding $n = 1$, the strongest prolate deformation is seen for the $n = 1$ Skyrmion-anti-Skyrmion pair. We observe that increasing of both the gravitational coupling
α and the winding n reduces the deformation of the horizon. Also, both the Skyrme field and the energy density do not vanish on the horizon, possessing an angular dependence which is relatively large for most solutions we found.

Similar to the case of the spherically symmetric black holes with Skyrme hair [3, 4], the lower branch always terminates at some critical maximal value of the horizon radius \( r_h^{(cr)} \). There it merges a secondary, upper (higher-mass) branch, which extends backward in \( r_h \), see Figs. 2. The upper-branch solutions always have higher entropy than the corresponding solutions on the lower-in-mass branch. For the same value of the horizon radius, the deformation of the horizon is stronger on the of the second branch, see Figs. 3,5.

As expected, the value of the critical horizon radius \( r_h^{(cr)} \) decreases as the gravitational coupling \( \alpha \) grows, see Figs. 4,5. In the limit \( r_h \to 0 \) the solution of the upper branch approaches the corresponding solution on the upper branch of the regular self-gravitating Skyrmion-anti-Skyrmion chains. The black holes with Skyrmion hairs cease to exist at some critical maximal value of the gravitational coupling \( \alpha \). The maximal value of the horizon radius \( r_h^{(cr)} \) weakly depends on the winding number n, it sightly decreases as n increases.

**IV. CONCLUSIONS**

We have considered new families of asymptotically flat static hairy BHs in Einstein-Skyrme theory, which represent a black hole immersed into the center of a chain of Skyrmions and anti-Skyrmions in interpolating order. Analogous to their counterparts in the EYM theory [41] and in the EYMH theory [32], these solutions emerge from the corresponding regular self-gravitating axially symmetric Skyrmion-anti-Skyrmion chains, as a small event horizon radius \( r_h \) is imposed via the boundary conditions. Their domain of existence is restricted both by some maximal value of the \( r_h \) and by a maximal value of the effective gravitational coupling \( \alpha \).

In summary, concerning the dependence of the black holes with Skyrmion-anti-Skyrmion hairs on the gravity coupling constant and on the horizon radius \( r_h \), we generally observe the picture that is very similar to the pattern observed for the Einstein-Yang-Mills hairy black hole sphaleron solutions [32, 33]. The most important difference is that for the black hole solution in the Einstein-Skyrme theory we never observe oblate deformations of the horizon.

It would be interesting to investigate the stability of these new black hole solutions with Skyrmion-anti-Skyrmion hairs. By analogy with the usual stability analysis of the spherically symmetric black holes in the Einstein-Skyrme theory [43, 44], one can expect the existence of unstable fluctuations on the upper branch. However, the problem of systematic study of the spectrum of fluctuations of the fields in the presence of the event horizon is a nontrivial technical task, which we leave for future study.
In the present work we have focused on the Skyrmion-anti-Skyrmion chains with two and three constituents and with the winding number of each individual component restricted to the lowest values \( n = 1, 2 \). As a direction for future work, it would be interesting to study the higher charge solutions, in particular the chains with charge 3 and 4 Skyrmions, which do not possess axial symmetry \([12]\).

The work here should be taken further by considering the isorotating generalization of the Skyrmion-anti-Skyrmion chains, which will extend the study of the spinning black holes with Skyrme hairs \([13]\). It would be also interesting to address the question of how the properties of the static axially symmetric black hole solutions with chainlike Skyrmion-anti-Skyrmion hairs will be modified in the asymptotically AdS spacetime. We hope to return elsewhere with a discussion of some of these interesting problems.

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[1] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 24 (2015) no.09, 1542014
[2] M. S. Volkov, arXiv:1601.08230 [gr-qc].
[3] H. Luckock and I. Moss, Phys. Lett. B 176 (1986) 341.
[4] S. Droz, M. Heusler and N. Straumann, Phys. Lett. B 268 (1991) 371.
[5] P. Bizon and T. Chmaj, Phys. Lett. B 297 (1992) 55.
[6] M. S. Volkov and D. V. Galtsov, JETP Lett. 50 (1989), 346-350
[7] M. S. Volkov and D. V. Galtsov, Sov. J. Nucl. Phys. 51 (1990), 747-753
[8] K. M. Lee, V. P. Nair and E. J. Weinberg, Phys. Rev. D 45 (1992), 2751-2761
[9] P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. B 442 (1995), 126-156
[10] M. S. Volkov and D. V. Galtsov, Phys. Rept. 319 (1999), 1-83
[11] N. Sawado, N. Shiiki, K. i. Maeda and T. Torii, Gen. Rel. Grav. 36 (2004), 1361-1371
[12] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 79 (1997), 1595-1598
[13] C. Herdeiro, I. Perapechka, E. Radu and Y. Shnir, JHEP 1810 (2018) 119
[14] C. Herdeiro, I. Perapechka, E. Radu and Y. Shnir, JHEP 1902 (2019) 111
[15] S. Hod, Phys. Rev. D 86 (2012) 104026 Erratum: [Phys. Rev. D 86 (2012) 129902]
[16] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. 112 (2014) 221101
[17] S. Krusch and P. Sutcliffe, J. Phys. A 37 (2004) 9037
[18] Y. Shnir and D. H. Tchrakian, J. Phys. A 43 (2010) 025401
[19] Y. Shnir, Phys. Rev. D 92 (2015) no.8, 085039
[20] B. Kleihaus and J. Kunz, Phys. Rev. D 61 (2000) 025003
[21] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 85 (2000), 2430-2433
[22] B. Kleihaus, J. Kunz and Y. Shnir, Phys. Lett. B 570 (2003) 237
[23] B. Kleihaus, J. Kunz and Y. Shnir, Phys. Rev. D 68 (2003) 101701
[24] B. Kleihaus, J. Kunz and Y. Shnir, Phys. Rev. D 70 (2004) 065010
[25] B. Kleihaus, J. Kunz and Y. Shnir, Phys. Rev. D 71 (2005) 024013
[26] R. Teh and K. Wong, J. Math. Phys. 46 (2005), 082301
[27] V. Paturyan, E. Radu and D. Tchrakian, Phys. Lett. B 609 (2005), 360-366
[28] B. Kleihaus, J. Kunz and U. Neemann, Phys. Lett. B 623 (2005) 171
[29] J. Kunz, U. Neemann and Y. Shnir, Phys. Lett. B 640 (2006) 57
[30] J. Kunz, U. Neemann and Y. Shnir, Phys. Rev. D 75 (2007) 125008
[31] R. Bartnik and J. Mckinnon, Phys. Rev. Lett. 61 (1988) 141.
[32] B. Kleihaus and J. Kunz, Phys. Lett. B 494 (2000), 130-134
[33] R. Ibadov, B. Kleihaus, J. Kunz and M. Wirschins, Phys. Lett. B 627 (2005), 180-187
[34] P. Bizon, Phys. Rev. Lett. 64 (1990), 2844-2847
[35] H. P. Kuenzle and A. K. M. Masood- ul- Alam, J. Math. Phys. 31 (1990), 928-935
[36] T. Ioannidou, B. Kleihaus and J. Kunz, Phys. Lett. B 643 (2006) 213
[37] C. Adam, O. Kichakova, Y. Shnir and A. Wereszczynski, Phys. Rev. D 94 (2016) no.2, 024060
[38] S. B. Gudnason, M. Nitta and N. Sawado, JHEP 1609 (2016) 055
[39] W. Schönauer and R. Weiß, J. Comput. Appl. Math. 27, 279 (1989) 279;
  M. Schauder, R. Weiß and W. Schönauer, Universität Karlsruhe, Interner Bericht Nr. 46/92 (1992).
[40] R.M. Wald, General Relativity , University of Chicago Press, Chicago, 1984.
[41] B. Kleihaus, J. Kunz, F. Navarro-Lerida and U. Neemann, Gen. Rel. Grav. 40 (2008), 1279-1310
[42] C. J. Houghton, N. S. Manton and P. M. Sutcliffe, Nucl. Phys. B 510 (1998), 507-537
[43] M. Heusler, S. Droz and N. Straumann, Phys. Lett. B 271 (1991), 61-67
[44] K. I. Maeda, T. Tachizawa, T. Torii and T. Maki, Phys. Rev. Lett. 72 (1994), 450-453