Safety Third: Roy’s Criterion and Higher Order Moments

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June 16, 2015

Abstract

Roy’s ‘Safety First’ criterion for selecting one risky asset from many is adapted to the case of non-normal returns, via Cornish Fisher expansion. The resulting investment objective is consistent with first order stochastic dominance, and is equal to the Sharpe ratio for the case of normal returns. An investor selecting assets via this objective is not universally attracted to positive skew, rather the preference for skew depends on term, the expected return and the disastrous rate of return.

1 Introduction

Mathematical economic theory posits that agents seek to maximize some utility function. [5] In practice, however, real investors can rarely evoke their own utility functions. Rather, when selecting from a number of risky assets, investors (and quantitative-minded asset managers) often rank their choices based on the moments of the returns stream, preferring, e.g., higher expected returns for a fixed level of volatility, ceterus paribus. Arguably the most commonly used measure of investment opportunities is the Sharpe ratio, here defined as

\[ \zeta = \frac{\mu - r_0}{\sigma}, \]

where \( r_0 \) is the ‘disastrous’ or ‘risk-free’ rate of return, and \( \mu \) and \( \sigma^2 \) are the expected value and variance of the returns stream, assumed to be known.

One objection to the use of the Sharpe ratio as an investment objective is that it is generally not consistent with first order stochastic dominance. [7, 16, 20] That is, one can construct two random variables, say \( x \) and \( y \), such that \( x \) stochastically dominates \( y \), but the Sharpe ratio of \( x \) is lower than that of \( y \). Moreover this deficiency cannot be solved by assuming away the \( \mu < 0 \) case.

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1It might be more accurate to call \( \zeta \) the signal-noise ratio, and reserve the term Sharpe ratio for the analogous quantity constructed from sample estimates. Sharpe himself notes, “Since the predictions cannot be obtained in any satisfactory manner, . . . ex post values must be used—the average rate of return of a portfolio must be substituted for its expected rate of return, and the actual standard deviation of its rate of return for its predicted risk.” [15, p. 122] However, we will follow common usage in calling \( \zeta \) the Sharpe ratio, without much risk of confusion.

2The Sharpe ratio as an objective ‘prefers’ higher volatility in the case \( \mu < 0 \), and is thus clearly inconsistent with second-order stochastic dominance. It is not clear, however, that the sample analogue shares this deficiency.
Hodges’ provides the classical counterexample, but such pathological cases are easy to construct, as shown in the appendix.

There have been numerous attempts to generalize the Sharpe ratio to remedy these deficiencies, making it suitable for the case of non-normal returns by including higher order moments. \cite{7, 16, 20} Hodges assumes an investor with the CARA utility function, $U(w) = -e^{-\lambda w}$. For an asset with normally distributed returns, the optimal amount to invest, long or short, in the asset is $\mu/\lambda\sigma^2$, in the sense of maximizing the expected utility. The maximum expected utility at this allocation is $-e^{-\lambda(\mu/\lambda\sigma^2)^2}$, ignoring the time term for simplicity. This leads Hodges to define the “Generalized Sharpe ratio” as

$$\zeta_g = \sqrt{-2 \log \left(-U^*\right)}.$$  

where $U^*$ is the maximum expected utility under the CARA utility function. \cite{7} That is

$$U^* = \max_{x} \mathbb{E} \left[-e^{-\lambda xw}\right],$$

and so

$$\zeta_g = \sqrt{\max_{x} -2 \log \left(\mathbb{E}[e^{-\lambda xw}]\right)}.$$  

As Hodges’ objective is difficult to compute, Zakamouline and Koekebakker carry his analysis to its logical conclusion, using Taylor’s theorem to describe the Generalized Sharpe ratio in terms of investor’s relative preferences for higher order moments of wealth. \cite{20} They derive an “adjusted for skew Sharpe ratio”, defined as

$$\zeta_3 = \zeta \sqrt{1 + b_3 \gamma_3 \frac{\zeta}{3}},$$

where $\gamma_3$ is the skewness of the returns distribution, and $b_3$ is the investor’s relative preference for third order moments:

$$b_3 = \frac{a_3}{a_2^2}, \quad \text{where} \quad a_k = \frac{U^{(k)}(w_r)}{U^{(1)}(w_r)}.$$  

and $U^{(k)}(w_r)$ denotes the $k^{th}$ derivative of the investor’s utility function at the zero dollar allocation in the risk asset, denoted as $w_r$. For an investor with HARA utility, the quantity $b_3$ is generally positive, and thus the skew adjusted Sharpe ratio has positive derivative with respect to skewness (assuming $\zeta > 0$). In fact, a necessary condition for the investor to demonstrate decreasing risk aversion is that $b_3 \geq 1$, a result due to Pratt. \cite{20, 13}

Smetters and Zhang carry this line of analysis further, showing that a valid ranking of investments must take into account investor’s preferences and cannot be a function only of the distributions of returns. \cite{16} Moreover, they develop a ranking measure like the Sharpe ratio expressed in terms of the cumulants of the returns distribution and the derivatives of the utility. Their Theorem 9 establishes positive derivative of their objective with respect to odd cumulants and negative derivative with respect to even cumulants of the returns distribution, in accordance with the usual interpretations of ‘temperance’, ‘prudence’, ‘edginess’, etc. \cite{16, 4} Smetters and Zhang describe how to approximately compute their objective, showing that their third order approximation matches that of Zakamouline and Koekebakker.

\textit{n.b.}, this is essentially the Markowitz portfolio on one asset.
It is only by Stigler’s Law of Eponymy that we know the quantity $\zeta$ as “the Sharpe ratio,” instead of “Roy’s criterion.” [17] Sharpe first described his “reward-to-variability ratio” in 1966 as a yardstick for comparing mutual funds, but Roy described the same quantity in 1952 as a means of choosing among risky assets, under the moniker of “Safety First.” [15, 14, 18] Roy’s justification for this objective followed from Chebyshev’s inequality, which states that
\[
\Pr\{|x - \mu| \geq \sqrt{k}\sigma\} \leq \frac{1}{k}.
\]
(5)

For a given $r_0 < \mu$, let $\sqrt{k} = (\mu - r_0) / \sigma$. Then since $\Pr\{x - \mu \leq -\sqrt{k}\sigma\} \leq \Pr\{|x - \mu| \geq \sqrt{k}\sigma\}$, we have
\[
\Pr\{x \leq r_0\} = \Pr\{x - \mu \leq -\frac{\mu - r_0}{\sigma}\sigma\} \leq \left(\frac{\sigma}{\mu - r_0}\right)^2 = \frac{1}{\zeta^2}.
\]
(6)

Thus to minimize the probability of a loss (relative to $r_0$), one should maximize $\zeta$.

## 2 Safety First

The crux of Roy’s justification for the ‘Safety-First’ objective, which is just the signal-noise ratio, is that it bounds the probability of a loss, defined as a return less than $r_0$. The argument, based on Chebyshev’s inequality, is only a rough upper bound. There are some situations, however, where the signal-noise ratio is exactly monotonic in the probability of a loss. For example, if the returns are drawn from a scale-location family, like the Gaussian family. Note that the central limit theorem tells us that, conditional on finite variance, the sample mean of some random variable converges to a normal distribution, and thus for the case of log returns, since the mean return is just the total log return rescaled, the long term log return is approximately drawn from a scale-location family.

We can maintain the spirit of Roy’s criterion by directly optimizing the quantity he sought to maximize, viz. the probability of exceeding $r_0$. To match the Sharpe ratio in the case of Gaussian returns, we need only invert the normal CDF, resulting in the quantity:
\[
\zeta_h = \Phi^{-1}\left(\Pr\{x \leq r_0\}\right),
\]
(7)

where $\Phi(\cdot)$ is the CDF of the normal distribution. When $x \sim \mathcal{N}(\mu, \sigma^2)$, the probability that $x \leq r_0$ is $\Phi((r_0 - \mu) / \sigma)$, and so $\zeta_h$ equals the Sharpe ratio, $(\mu - r_0) / \sigma$. This objective is legitimately a ‘generalized Sharpe ratio’, since it agrees with the Sharpe ratio exactly for normal returns. [20]

It is trivial to verify that $\zeta_h$ is consistent with first order stochastic dominance, or at least not inconsistent with it. Since if $x$ stochastically dominates $y$, $\Pr\{x \leq r_0\} \leq \Pr\{y \leq r_0\}$ for all $r_0$. By monotonicity of $\Phi^{-1}(\cdot)$, $\zeta_h$ is no

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4This statement is weak, but cannot be strengthened; it must be admitted, for example, that for most $r_0$, $\zeta_h$ makes no distinction between the two assets of Hodges’ classic counterexample.
smaller for \( x \) than \( y \). It should be clear, however, that the converse does not, indeed can not, hold: if \( \zeta_h \) is higher for \( x \) than \( y \), for a single \( r_0 \), it need not be the case that \( x \) stochastically dominates \( y \). The simple proof is that since stochastic dominance does not form a total ordering on probability distributions, but generalized Roy’s criterion (for one choice of \( r_0 \)) does form a total ordering, the latter ordering cannot imply the former.

Roy’s approximation is based on Chebyshev’s inequality. We can construct tighter approximations to the probability of a loss via some classical approximations to the central limit theorem. Suppose that one will observe \( n \) independent draws from the returns stream, \( x \). Without loss of generality, let the disastrous event be that the observed sample mean return, \( \hat{\mu} \), is less than \( r_0 \). This is equivalent to

\[
\sqrt{n} \frac{\hat{\mu} - \mu}{\sigma} \leq \sqrt{n} r_0 - \frac{\mu}{\sigma}.
\]

The cumulative distribution function of the quantity on the left hand side can be approximated via some truncation of the Edgeworth expansion. \(^2\)

Define \( \delta = \frac{\sqrt{n}(\mu - r_0)}{\sigma} \). The Edgeworth expansion is \([1, 26.2.48]\)

\[
\Pr \left\{ \sqrt{n} \frac{\hat{\mu} - \mu}{\sigma} \leq -\delta \right\} = \Phi (-\delta) - \Phi (-\delta) \left[ \frac{\gamma_3}{6 \sqrt{n}} H_{6}(\delta) \right] \\
+ \Phi (-\delta) \left[ \frac{\gamma_4}{24n} H_{6}(\delta) + \frac{\gamma_3^2}{72n} H_{6}(\delta) \right] \\
- \Phi (-\delta) \left[ \frac{\gamma_5}{120n^{3/2}} H_{5}(\delta) + \frac{\gamma_3 \gamma_4}{144n^{3/2}} H_{5}(\delta) + \frac{\gamma_3^3}{1296n^{3/2}} H_{8}(\delta) \right] \ldots \quad (8)
\]

where \( \Phi (x) \) and \( \phi (x) \) are the cumulative distribution and density functions of the standard unit normal, \( H_i(x) \) is the probabilist’s Hermite polynomial \([1, 26.2.31]\), and \( \gamma_i \) is the standardized \( i \)th cumulant, defined as the \( i \)th cumulant of the distribution divided by \( \sigma^i \). It happens to be the case that \( \gamma_3 \) is the skewness, and \( \gamma_4 \) is the excess kurtosis of the distribution.

Truncating beyond the \( n^{-1/2} \) term and applying basic facts of probability yields

\[
\Pr \{ \hat{\mu} \geq r_0 \} \approx \Phi (\delta) + \frac{\Phi (\delta)}{\sqrt{n}} \left[ \frac{\gamma_3}{6} (\delta^2 - 1) \right]. \quad \quad \quad (9)
\]

The implication is that the probability that \( \hat{\mu} \) exceeds \( r_0 \) will be increased if \( \delta \) is large. Moreover, for a fixed \( \delta \), the probability that \( \hat{\mu} \) exceeds \( r_0 \) is increased for large positive skew if \( \delta^2 > 1 \), but for large negative skew when when \( \delta^2 < 1 \). The implication is that when \( \delta^2 \) is ‘large’ (larger than one unit), one has positive preference for skewed returns, otherwise one has negative preference. As long as \( \mu > r_0 \), this is asymptotically compatible with \( n \to \infty \) with the commonly held belief that investors universally value positive skew.

2.1 Approximating Roy’s criterion

The generalized Roy’s criterion of Equation 7 is now expressed as

\[
\zeta_h = \frac{1}{\sqrt{n}} \Phi^{-1} \left( \Pr \left\{ \sqrt{n} \frac{\hat{\mu} - \mu}{\sigma} \leq -\delta \right\} \right). \quad \quad \quad (10)
\]

\(^5\)Here we assume the returns are log returns. Then the sample mean is just the rescaled total return. By similarly rescaling the disastrous return, we arrive at the formulation here.
This implicit definition is a bit unwieldy for use as an objective. One would prefer a definition in terms of the cumulants of the returns stream. Rather than use the Taylor series expansion of $\Phi^{-1}(x)$, one can instead use the Cornish Fisher expansion of the sample quantile. \[9, 8, 19\]

Let $Y = \sqrt{n}(\hat{\mu} - \mu)/\sigma$. This is a random variable with zero mean and unit standard deviation. Let $\gamma_i$ be the $i^{th}$ standardized cumulant of $x$. The $i^{th}$ standardized cumulant of $Y$ is $n^{1-i/2}\gamma_i$. The Cornish Fisher expansion \[1, 26.2.49\] finds $w$ in

$$\Pr\{Y \leq w\} = \Phi(z),$$

in terms of $z$ and the higher order cumulants of the distribution. Setting $w = -\delta$, we have $z = -\sqrt{n}\zeta_h$, and the Cornish Fisher expansion reduces to

$$\zeta_h = \frac{(\mu - r_0)}{\sigma} + \frac{1}{n} \left( \frac{\gamma_3}{6} He_2 (\sqrt{n}\zeta_h) \right) - \frac{1}{n^{1/2}} \left( \frac{\gamma_4}{24} He_3 (\sqrt{n}\zeta_h) - \frac{\gamma_3^2}{36} [2He_3 (\sqrt{n}\zeta_h) + He_1 (\sqrt{n}\zeta_h)] \right) + \frac{1}{n^2} \left( \frac{\gamma_5}{120} He_4 (\sqrt{n}\zeta_h) - \frac{\gamma_3 \gamma_4}{24} [He_4 (-\sqrt{n}\zeta_h) + He_2 (-\sqrt{n}\zeta_h)] \right) + \frac{\gamma_3^3}{324} \left[ 12He_4 (-\sqrt{n}\zeta_h) + 19He_2 (-\sqrt{n}\zeta_h) \right] + \ldots$$

While this defines $\zeta_h$ implicitly, truncation gives polynomial equations, whose roots can be found analytically or numerically. Noting that derivatives of Hermite polynomials can be easily computed, solving iteratively for $\zeta_h$ via Newton's method should be simple.

Truncating at two terms gives an equation which is quadratic in $\zeta_h$, yielding the (aesthetically unpleasing) solution:

$$\zeta_h \approx \frac{3}{\gamma_3} \pm \sqrt{\frac{9}{\gamma_3^2} + \frac{1}{n} - \frac{6\zeta}{\gamma_3}}.$$ \[(12)\]

As an example, for garden variety applications in asset management, setting $\zeta = 0.07\text{day}^{-1/2}$, $\gamma_3 = -1$, $n = 60\text{day}$, we have $\zeta_h \approx 0.0719\text{day}^{-1/2}$. If we consider a longer horizon, say $n = 252\text{day}$, one observes $\zeta_h \approx 0.0698\text{day}^{-1/2}$. Thus the difference between $\zeta_h$ and $\zeta$ is modest at the quarter year time scale, but negligible at the annual time scale. Note that at the shorter time scale, $\sqrt{n}\zeta < 1$, resulting in a boost to $\zeta_h$ due to negative skew, while at the longer time horizon, $\zeta_h < \zeta$ since $\sqrt{n}\zeta > 1$.

3 Discussion

It is not the purpose of this note to suggest that investors should optimize $\zeta_h$. Prima facie, the generalized Roy’s criterion appears inconsistent with the received wisdom that investors should maximize expected utility, or corresponds somehow to decreasing risk aversion\[6\]. Moreover, since Roy’s criterion

\[6\] Perhaps Roy’s criterion can be expressed in the classical framework as a Heaviside utility function.
dichotomizes future returns, it shares some of the hallmark failings of the Value at Risk measure, viz. that it does not control for severe tail losses, may not promote diversification, etc. [3] Note, however, that Roy was decidedly unenthusiastic about the prospect of maximizing expected utility, for pragmatic and philosophical reasons, writing, “a man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility.” [14, p. 433]

While we do not have positive proof of investors who do maximize Roy’s criterion, we can easily imagine there are some who might. For example, at times a professional portfolio manager might try to maximize the probability of beating their benchmark over the next month, fearing withdrawals from their fund. While investors cannot easily estimate, ex post, what the ex ante expected return of an investment should have been, they do exhibit a tendency to dichotomize their holdings as ‘winners’ or ‘losers’.

Optimization of Roy’s criterion provides an interesting mechanism by which fully informed agents can agree on all moments of returns of an instrument, yet rank the instrument differently based entirely on term. The short term investor essentially sells (or leases, really) positive skew to the long term investor. It is not at all clear, however, that this differential preference for skew drives the classical narrative of ‘investors’ versus ‘speculators’; perhaps these two mythical groups can be separated by their appetite for kurtosis.

Finally, as a practical matter, it must be noted that maximization of Roy’s criterion is largely a quixotic pursuit. As illustrated in the sample calculation above, the difference between ζ and ζh tends to be small, much smaller in the estimation error around ζh. Invoking estimates of the higher order moments of the returns distribution will only increase that estimation error. [10, 11, 12]

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The title of this paper alludes to this possible mismatch between goals of a fund investor and the fund manager: in occupational safety, the “Safety Third” principle states that no party is as concerned with your personal wellbeing as you yourself are, with the implication that overreliance on implicit workplace safeguards can be hazardous.
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A A counterexample

Let $x$ have mean and variance $\mu > 0$ and $\sigma^2$, respectively. Let $y$ have the same distribution as $x$, except with probability $p > 0$ has an additional ‘bonus’ return of a constant $B > 0$. Clearly $y$ (first-order) stochastically dominates $x$. The mean of $y$ is equal to $\mu + pB$. The uncentered second moment of $y$ is equal to $\sigma^2 + \mu^2 + pB^2$. The Sharpe ratio of $y$ is thus equal to

$$\frac{\mu + pB}{\sqrt{\sigma^2 - 2\mu pB - p^2 B^2 + pB^2}}.$$

Then if, for example, $\mu = 0.001, \sigma = 0.01, p = 10^{-4}$, and $B = 0.25$, the Sharpe ratio of $x$ is 0.1, while the Sharpe ratio of $y$ is 0.0995.

In fact, we can construct a sufficient condition for the Sharpe ratio to be reversed in this case. Since $\mu, p$ and $B$ are assumed positive,

$$\frac{\mu + pB}{\sqrt{\sigma^2 - 2\mu pB - p^2 B^2 + pB^2}} \leq \frac{\mu}{\sigma},$$

$$\Leftrightarrow \frac{(\mu + pB)^2}{\sigma^2 - 2\mu pB - p^2 B^2 + pB^2} \leq \frac{\mu^2}{\sigma^2},$$

$$\Leftrightarrow \sigma^2 (\mu + pB)^2 \leq \mu^2 (\sigma^2 - 2\mu pB - p^2 B^2 + pB^2),$$

$$\Leftrightarrow \sigma^2 (2pB(\mu + pB)) \leq \mu^2 (-2\mu pB - p^2 B^2 + pB^2),$$

$$\Leftrightarrow \sigma^2 (2\mu + pB) \leq \mu^2 (-2\mu - pB + B),$$

$$\Leftrightarrow (\sigma^2 + \mu^2) (2\mu + pB) \leq B \mu^2,$$

$$\Leftrightarrow 2\mu + pB \leq \frac{\mu^2}{\sigma^2 + \mu^2},$$

$$\Leftrightarrow p \leq \frac{\mu^2}{\sigma^2 + \mu^2} - \frac{2\mu}{B}.$$
In order for this last inequality to admit a solution with positive $p$, one must have
\[ \frac{B}{2} \geq \mu + \frac{\sigma^2}{\mu}. \]
For the example above, this ‘minimum’ value of $B$ is 0.202, while the maximum acceptable value for $p$ is 0.0019.