On Derivation of Equations of Electrodynamics and Gravitation from the Principle of Least Action, the Hamilton–Jacobi Method, and Cosmological Solutions

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Abstract—In classical texts, equations for fields are proposed without derivation of right-hand sides. Below, the right-hand sides of the Maxwell and Einstein equations are derived within the framework of the Vlasov–Maxwell–Einstein equations from a classical, but slightly more general principle of least action and Hamilton–Jacobi equations are used to obtain cosmological solutions.

Keywords: Vlasov equation, Vlasov–Einstein equation, Vlasov–Maxwell equation, Vlasov–Poisson equation

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1. INTRODUCTION

In classical texts (see [1–4]), equations for fields are proposed without deriving right-hand sides. Below, the right-hand sides of the Maxwell and Einstein equations are derived within the framework of the Vlasov–Maxwell–Einstein equations from a classical, but slightly more general principle of least action and Hamilton–Jacobi equations are used to obtain cosmological solutions. Thus, the energy–momentum tensor and a closed system of gravitation and electromagnetic equations have been derived from the principle of least action for the first time.

2. ACTION IN GENERAL RELATIVITY AND EQUATIONS FOR FIELDS

Let \( f(t, x, v, m, e) \) be the distribution function of particles over space \( x \in \mathbb{R}^3 \), velocities \( v \in \mathbb{R}^3 \), masses \( m \in \mathbb{R} \), and charges \( e \in \mathbb{R} \) at time \( t \in \mathbb{R} \). This means that the number of particles in the volume \( dx dv dm de \) is equal to \( f(t, x, v, m, e) dx dv dm de \). Consider the action

\[
S = -c \int m f(t, x, v, m, e) \sqrt{g_{\mu\nu}} u^\mu u^\nu dx dv dm de dt
- \frac{1}{c} \int e f(t, x, v, m, e) A_\mu u^\mu dx dv dm de dt
+ k_1 \int (R + \Lambda) \sqrt{-g} dx + k_2 \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} dx,
\]

where \( c \) is the speed of light, \( u^0 = c, u^i = v^i \, (i = 1–3) \) is the three-dimensional velocity, \( x^0 = ct, x^i \, (i = 1–3) \) is the coordinate, \( g_{\mu\nu}(x, t) \) is the metric \((\mu, \nu = 0–3)\), \( A_\mu(x, t) \) is the electromagnetic field 4-potential, \( F_{\mu\nu}(x, t) = \partial A_\mu(x, t)/\partial x^\nu - \partial A_\nu(x, t)/\partial x^\mu \) are electromagnetic fields, \( R \) is the total curvature, \( \Lambda \) is Einstein’s lambda term, \( k_1 = -c^3/16\pi \) and \( k_2 = -1/16\pi c \) are constants [1–4], \( g \) is the determinant of the metric \( g_{\mu\nu} \), and \( \gamma \) is the gravitational constant; as usual, summation is implied over repeated indices.

The form of action (1) is convenient for obtaining the Einstein and Maxwell equations by variation over the fields \( g_{\mu\nu} \) and \( A_\mu \). This method for deriving the Vlasov–Maxwell and Vlasov–Einstein equations was used in [5–11]. The variation of (1) over \( g_{\mu\nu} \) yields the Einstein equation

\[
k_1 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (R + \Lambda) \right) = k_2 F_{\mu\nu} F^{\mu\nu} \sqrt{-g}.
\]

By Hilbert’s definition, the first term on the right-hand side of this equation is the energy–momentum tensor. It was first given in [9–11] in a less general form with no mass or charge distribution. To the best of our knowledge, attempts to write the energy–momentum tensor in terms of the distribution function were made only in relativistic kinetic theory for the Vlasov–Einstein equation [5–15]. The electromagnetic field
ON DERIVATION OF EQUATIONS OF ELECTRODYNAMICS

179

For momenta, we obtain the expression

$$q_\mu = \frac{\partial L}{\partial u^\mu} = -mc \frac{g_{\mu\alpha} u^\alpha}{\sqrt{g_{\nu\nu} u^\nu}} + \frac{e}{c} A_\mu.$$  \hspace{1cm} (6)

Here, an expression for $q_0$ is derived by formal differentiation with respect to $u^0 = c$.

These are expressions for long, or canonical momenta, but we will also need the small momenta

$$p_\mu = q_\mu - \frac{e}{c} A_\mu = -mc \frac{g_{\mu\alpha} u^\alpha}{\sqrt{g_{\nu\nu} u^\nu}}.$$ Formulas for the relation to velocities are simpler for small momenta, but, in the transition to the Hamilton–Jacobi equation, we have to use the canonical momenta.

Passing to upper indices via multiplication by the inverse matrix $g^{\mu\beta}$ yields

$$p_\beta = -mc \frac{u_\beta}{\sqrt{g_{\nu\nu} u^\nu}}.$$  \hspace{1cm} (7)

Now, we need to invert this formula, expressing the velocities in terms of momenta, in order to write the action in terms of momenta. For this purpose, in the last formula, the $\beta$th component is divided by the zeroth one:

$$\frac{p_\beta}{p^0} = \frac{u_\beta}{c}.$$  \hspace{1cm} (8)

The momentum with zeroth component is eliminated from this formula by using the mass equation $\dot{p}_0 = p_0 c^2$ and its solution for $p_0$:

$$p_0 = -b \pm \sqrt{b^2 - 4ac},$$

where $a = g^{00}$, $b = 2p_0 g^{0i}$, $c = p_0 p_j g^{0j} - (mc)^2$. Here, a minus sign is taken for consistency with non-relativistic dynamics.

The mass equation is obtained by substituting the same relations (for eliminating velocities)

$$\frac{p_\beta}{p^0} = \frac{u_\beta}{c}$$

into formula (6) with $\mu = 0$ and taking into account that $u^0 = c$ (cf. [1–4]).

The field equation (2) remains the same after changing to integration with respect to momenta by using the formulas $f(t, x, v, m, e) dw dm de = f(t, x, q, m, e) dq dm de$. Each of these three quantities is the number of particles in the volume element, which is invariant under the change of variables:

DOKLADY MATHEMATICS  Vol. 105  No. 3  2022
\[ k_1 \left( R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} (R + \Lambda) \right) \sqrt{-g} \]

\[ = c \int m f(t, x, q, m, e) u^\mu d^3 q d m d e - \frac{1}{2} k_2 F_{\mu \nu} F^{\mu \nu} g^{\mu \nu} \sqrt{-g} \]

\[ k_2 \frac{\partial F^{\mu \nu}}{\partial x_\nu} \sqrt{-g} = \frac{1}{c^2} \int e \frac{c^\mu}{p^0} f(t, x, q, m, e) d^3 q d m d e. \]

Here, we mean that the velocities in the first equation and the momenta \( p^\mu \) in the second equation have to be expressed in terms of the canonical momenta \( q_0 \).

The equation of motion for particles is obtained in Hamiltonian form, where the Hamiltonian function is

\[ H = -\frac{c \partial L}{\partial u^0} = -c q_0. \]

This formula is derived taking into account that the Lagrangian for the action

\[ S = \int \sqrt{g_{\mu \nu} u^\nu} dt - \frac{c}{e} \int A_\mu u^\mu dt \]

is a function of the first degree with respect to velocities and using the Euler formula \( u^\mu \frac{\partial L}{\partial u^\mu} - L = 0 \). Since, by definition,

\[ H = u^i \frac{\partial L}{\partial u^i} - L, \]

we obtain \( c \frac{\partial L}{\partial u^0} + H = 0 \). Here, summation over \( i = 1 - 3 \) and \( \mu = 0 - 3 \) is implied. From this equation, we find the velocity expressions

\[ u^i = \frac{\partial H}{\partial q_i} = u^i(q) = -\frac{c q_0}{\partial q_i}. \]

In terms of this Hamiltonian, we write the Liouville equation

\[ \frac{\partial f}{\partial t} - c \frac{\partial f}{\partial q_i} \frac{\partial f}{\partial x^i} - c \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial q_i} = 0. \]  

As a result, we obtain the closed system (7), (8) of Vlasov–Maxwell–Einstein equations of gravitation and electrodynamics in terms of momenta. According to the general scheme of [16–20], a hydrodynamic consequence of system (7), (8) is derived by making the hydrodynamic substitution \( f(t, x, q, m, e) = \rho(x, t, m, e) \delta(q - Q(x, t, m, e)) \):

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i} \left( \rho u^i \right) = - \frac{1}{8 \pi \gamma} \int (\nabla U)^2 dx dt - \frac{c^2 A}{8 \pi} \int U dx dt. \]

Varying \( S \) with respect to the coordinates \( x(t) \) yields the usual relativistic equations in the Lorentz metric with the Hamiltonian [1–4].
\( H(x,q) = c\sqrt{(mc)^2 + q^2} + U. \)

Now we pass to an action that can be varied with respect to fields according to our usual scheme:

\[
S = -c \int m \left( \sqrt{c^2 - \left(\frac{dx}{dt}\right)^2} + U \right) f(x,p,t,m)dpdmdxdt - \frac{1}{8\pi\gamma} \int (\nabla U)^2 dxdt - \frac{c^2\Lambda}{8\pi\gamma} \int Udxdtdt.
\]

Varying \( S \) with respect to the potential \( U \) yields equations for fields:

\[
\Delta U = 4\pi\gamma \int mf(t,x,q,m,e)dqdmde - \frac{1}{2}c^2\Lambda.
\]

Immediately passing to the Hamilton–Jacobi equation, we obtain the system of equations

\[
\left\{ \begin{array}{l}
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x^\alpha} (v^\alpha (\nabla W) p) = 0 \\
\frac{\partial W}{\partial t} + c\sqrt{(mc)^2 + (\nabla W)^2} + U = 0 \\
\Delta U = 4\pi\gamma \int mpdmde - \frac{c^2\Lambda}{2},
\end{array} \right.
\]

where \( v^\alpha (q) = \frac{\partial H}{\partial q^\alpha} = \frac{cq^\alpha}{\sqrt{(mc)^2 + q^2}}. \)

We have obtained a velocity expression showing the Hubble expansion, a closed system of equations, and the possibility of passing to cosmological solutions in the isotropic case and when the density is independent of space.

**Example 2.** Consider another relativistic action with the weakly relativistic (rather than Lorentz) metric

\[
g_{\alpha\beta} = \text{diag} \left[ 1 + \frac{2U}{c^2}, -1, -1, -1 \right].
\]

The potential in the action is taken inside the square root:

\[
S = -cm \left( \sqrt{c^2 - \left(\frac{dx}{dt}\right)^2} + U \right) dt - \frac{1}{8\pi\gamma} \int (\nabla U)^2 dxdt - \frac{c^2\Lambda}{8\pi\gamma} \int Udxdtdt.
\]

Proceeding as above, we obtain the Hamiltonian

\[
H = -c\rho_0(x,q,t) = c\sqrt{(mc)^2 + q^2} \left( 1 + \frac{2U}{c^2} \right)
\]

and the system of equations

\[
\left\{ \begin{array}{l}
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x^\alpha} (v^\alpha (\nabla W) p) = 0 \\
\frac{\partial W}{\partial t} + c\sqrt{(mc)^2 + (\nabla W)^2} \left( 1 + \frac{2U}{c^2} \right) = 0 \\
\Delta U = 4\pi\gamma \int \frac{mp}{\sqrt{c^2 - (\nabla W)^2}} dm - \frac{c^2\Lambda}{2},
\end{array} \right.
\]

where \( v^\alpha (q) = \frac{\partial H}{\partial q^\alpha} = \frac{cq^\alpha}{\sqrt{(mc)^2 + q^2}}. \)

Once again, we have obtained a closed system of equations, which shows the origin of the root on the right-hand side of the Einstein equations. Additionally, we have derived a velocity expression showing the Hubble expansion. Moreover, the possibility of passing to cosmological solutions in the isotropic case and when the density is independent of space has been demonstrated.

**CONCLUSIONS**

Closed equations of electrodynamics and gravitation have been derived from the principle of least action in the form of the Vlasov equation (cf. [5–15]). The meaning of Vlasov-type equations was clarified. Specifically, this is still the only way of deriving both equations of gravitation and electrodynamics from the principle of least action. Moreover, this is still the only method for closing the system of electrodynamic and gravitation equations with the help of the principle of least action by using the velocity and space distribution functions of objects (such as electrons, ions, stars in galaxies, and galaxies in supergalaxies or the Universe). Corresponding hydrodynamic equations (e.g., magnetohydrodynamics equations or gravitating gas dynamics equations) are also naturally obtained from Vlasov-type equations by making the hydrodynamic substitution (still the only way of relation to the classical action for these equations as well). In [20, 21], the system of Vlasov–Maxwell–Einstein equations was obtained for velocities, while, in this paper, we derived it for momenta, which makes it possible to investigate cosmological solutions via the transition to the Hamilton–Jacobi equation. An issue of interest is to study stationary solutions of the resulting equations, as was done for the Vlasov–Poisson equations in [22]. In [20, 21] cosmological solutions in the nonrelativistic case were obtained and the Milne–McCrea model [23, 24] was derived and generalized. On this basis, Gurzadyan’s potential \( U(r) = -\frac{1}{r} + ar^2 \) [25] was justified, where the second term is related to Einstein’s lambda term. A task of great interest to do the same work for the above-proposed models in order to estimate Einstein’s lambda term and various relativistic and weakly relativistic approximations.
CONFLICT OF INTEREST
The author declares that he has no conflicts of interest.

REFERENCES
1. V. A. Fock, *The Theory of Space, Time, and Gravitation* (Pergamon, Oxford, 1964).
2. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth–Heinemann, Oxford, 1980).
3. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
4. B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry: Methods and Applications* (Springer–Verlag, New York, 1984, 1985, 1990), Parts I–III.
5. V. V. Vedenyapin and M. A. Negmatov, “Derivation and classification of Vlasov-type and magnetohydrodynamics equations: Lagrange identity and Godunov’s form,” Theor. Math. Phys. 170 (3), 394–405 (2012).
6. V. V. Vedenyapin, M. A. Negmatov, and N. N. Fimin, “Vlasov–type and Liouville-type equations, their microscopic, energetic, and hydrodynamical consequences,” Izv. Math. 81 (3), 505–541 (2017).
7. V. V. Vedenyapin and M. A. Negmatov, “On derivation and classification of Vlasov type equations and equations of magnetohydrodynamics: The Lagrange identity, the Godunov form, and critical mass,” J. Math. Sci. 202, 769–782 (2014).
8. Y. Choquet–Bruhat, *General Relativity and Einstein’s Equations* (Oxford Univ. Press, New York, 2009).
9. Yu. G. Ignat’ev, *Relativistic Kinetic Theory of Nonequilibrium Processes* (Foliants, Kazan, 2010) [in Russian].
10. T. Okabe, P. J. Morrison, J. E. Friedrichsen III, and L. C. Shepley, “Hamiltonian dynamics of spatially-homogeneous Vlasov–Einstein systems,” Phys. Rev. D 84, 024011 (2011).
11. F. Pegoraro, F. Califano, G. Manfredi, and P. J. Morrison, “Theory and applications of the Vlasov equation,” Eur. Phys. J. D 69, Article No. 68 (2015).
12. C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhäuser, Boston, 2002).
13. Y. Choquet–Bruhat and T. Damour, *Introduction to General Relativity, Black Holes, and Cosmology* (Oxford Univ. Press, New York, 2015).
14. G. Rein and A. D. Rendall, “Global existence of solutions of the spherically symmetric Vlasov–Einstein system with small initial data,” Commun. Math. Phys. 150, 561–583 (1992).
15. H. E. Kandrup and P. J. Morrison, “Hamiltonian structure of the Vlasov–Einstein system and the problem of stability for spherical relativistic star clusters,” Ann. Phys. 225, 114–166 (1993).
16. V. V. Kozlov, *General Theory of Vortices* (Udmurt. Univ., Izhevsk, 1998) [in Russian].
17. V. V. Kozlov, “The hydrodynamics of Hamiltonian systems,” Moscow Univ. Mech. Bull. 38 (6), 9–23 (1983).
18. I. S. Arzhanykh, *Momentum Fields* (Nauka, Tashkent, 1965; Nat. Lending Lib., Boston Spa, Yorkshire, 1971).
19. V. V. Vedenyapin and M. A. Negmatov, “On the topology of steady-state solutions of hydrodynamic and vortex consequences of the Vlasov equation and the Hamilton–Jacobi method,” Dokl. Math. 87 (2), 240–244 (2013).
20. V. V. Vedenyapin, M. Yu. Voronina, and A. A. Russkov, “Derivation of the equations of electrodynamics and gravitation from the principle of least action,” Dokl. Phys. 65 (12), 413–417 (2020).
21. V. V. Vedenyapin, N. N. Fimin, and V. M. Chechetkin, “The generalized Friedmann model as a self-similar solution of Vlasov–Poisson equation system,” Eur. Phys. J. Plus 136, Article No. 670 (2021).
22. V. V. Vedenyapin, “Boundary value problems for the steady-state Vlasov equation,” Sov. Math. Dokl. 34 (2), 335–338 (1987).
23. E. A. Milne, *Relativity, Gravitation, and World-Structure* (Oxford Univ. Press, Oxford, 1935).
24. W. H. McCrea and E. A. Milne, “Newtonian universes and the curvature of space,” Q. J. Math. 5, 73–80 (1934).
25. V. G. Gurzadyan, “The cosmological constant in the McCrea–Milne cosmological scheme,” Observatory 105, 42 (1985).

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