A new Improvement of Ambarzumyan’s Theorem

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Abstract We extend the classical Ambarzumyan’s theorem to the quasi-periodic boundary value problems by using only a part knowledge of one spectrum. We also weaken slightly the Yurko’s conditions on the first eigenvalue.

Keywords Ambarzumyan theorem · Inverse spectral theory · Hill operator · eigenvalue asymptotics

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1 Preliminary Improvement

In 1929, Ambarzumyan [1] proved that if \{((n\pi)^2 : n = 0, 1, 2\ldots}\} is the spectrum of the boundary value problem

\[-y''(x) + q(x)y(x) = \lambda y(x), \quad y'(0) = y'(1) = 0\]  \hspace{1cm} (1)

with real potential \(q \in L^1[0, 1]\), then \(q = 0\) a.e. Clearly, if \(q = 0\) a.e., then the eigenvalues \(\lambda_n = (n\pi)^2\), \(n \geq 0\).

Freiling and Yurko [2] proved that it is enough to specify only the first eigenvalue rather than the whole spectrum. More precisely, the first eigenvalue denoted by \(\lambda_0\) is a mean value of the potential, that is, they proved the following Ambarzumyan-type theorem:

**Theorem 1** If \(\lambda_0 = \int_0^1 q(x) \, dx\), then \(q = \lambda_0\) a.e.
In [13], Yurko also provided generalizations of Theorem 1 on wide classes of self-adjoint differential operators. Some of the inverse results in [13] are in the following theorem:

**Theorem 2**

(a) Let

\[ \lambda_0 = \int_0^1 q(x) \, dx \]

be the first eigenvalue of the periodic boundary value problem

\[ -y''(x) + q(x)y(x) = \lambda y(x), \quad y(1) = y(0), \quad y'(1) = y'(0), \]

(2)

then \( q = \lambda_0 \) a.e.

(b) Let

\[ \lambda_0 = \pi^2 + \frac{2}{\alpha^2 + \beta^2} \int_0^1 q(x)(\alpha \sin \pi x + \beta \cos \pi x)^2 \, dx, \]

(3)

for some fixed \( \alpha \) and \( \beta \), be the first eigenvalue of the anti-periodic boundary value problem

\[ -y''(x) + q(x)y(x) = \lambda y(x), \quad y(1) = -y(0), \quad y'(1) = -y'(0). \]

(4)

Then

\[ q = \frac{2}{\alpha^2 + \beta^2} \int_0^1 q(x)(\alpha \sin \pi x + \beta \cos \pi x)^2 \, dx \text{ a.e.} \]

Consider the boundary value problems \( L_t(q) \) generated in the space \( L^2[0,1] \) by the following differential equation and quasi-periodic boundary conditions

\[ -y''(x) + q(x)y(x) = \lambda y(x), \quad y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0), \]

(5)

where \( q \in L^1[0,1] \) is a real-valued function and \( t \in [0, \pi) \cup [\pi, 2\pi) \). The operator \( L_t(q) \) is self-adjoint and the cases \( t = 0 \) and \( t = \pi \) correspond to the periodic and anti-periodic problems, respectively. Let \( \{\lambda_n(t)\}_{n \in \mathbb{Z}} \) be the eigenvalues of the operator \( L_t(q) \). In the case when \( q = 0 \), \( (2\pi n + t)^2 \) for \( n \in \mathbb{Z} \) is the eigenvalue of the operator \( L_t(0) \) for any fixed \( t \in [0, 2\pi) \) corresponding to the eigenfunction \( e^{i(2\pi n + t)x} \).

The result of Ambarzumyan [1] is an exception to the general rule. In general, Borg [2] proved that one spectrum does not determine the potential. Also, Borg showed that two spectra determine it uniquely. Many generalizations of the Ambarzumyan’s theorem can be found in [4,7,5,12,8,3]. But as far as we know, for the first time, we extend the classical Ambarzumyan’s theorem to the quasi-periodic boundary value problems \( L_t(q) \) by using only a part knowledge of one spectrum (see Theorem 4). More precisely, the aim of this paper is to weaken slightly the conditions in Theorem 1 and Theorem 2 for the first eigenvalue (see Theorem 3) and to prove a new generalization of Ambarzumyan’s theorem, without using any additional conditions on the potential. We also extend Theorem 1 of the previous paper [9].

Our new uniqueness-type results read as follows:
Theorem 3 Let $\lambda_0(t)$ be the first eigenvalue of $L_t(q)$. Then:
(a) If $\lambda_0(t) \geq t^2 + \int_0^1 q(x) \, dx$ for $t \in [0, \pi)$, then $q = \int_0^1 q(x) \, dx \ a.e.$
(b) If $\lambda_0(t) \geq (2\pi - t)^2 + \int_0^1 q(x) \, dx$ for $t \in [\pi, 2\pi)$, then $q = \int_0^1 q(x) \, dx \ a.e.$

Theorem 4 Let $\lambda_0(t)$ be the first eigenvalue of $L_t(q)$, and suppose that $n_0$ is a sufficiently large positive integer. Then the following two assertions hold:
(a) If $\lambda_0(t) \geq t^2$ and $\lambda_n(t) = (2n\pi + t)^2$ for $t \in [0, \pi)$ and $n > n_0$, then $q = 0 \ a.e.$
(b) If $\lambda_0(t) \geq (2\pi - t)^2$ and $\lambda_n(t) = (2n\pi + t)^2$ for $t \in [\pi, 2\pi)$ and $n > n_0$, then $q = 0 \ a.e.$

In Theorem 3 (a), the case $t = 0$ corresponds to the periodic problem and the assertion of Theorem 2 (a) holds by using $\lambda_0(0) \geq \int_0^1 q(x) \, dx$ instead of $\lambda_0(0) = \int_0^1 q(x) \, dx$. In Theorem 3 (b), the case $t = \pi$ corresponds to the anti-periodic problem and the assertion of Theorem 3 (b) holds by using $\lambda_0(\pi) \geq \pi^2 + \int_0^1 q(x) \, dx$ instead of (3), namely the first eigenvalue depends only on the mean value of the potential as in Theorem 1 and Theorem 2 (a).

And, for the boundary value problem, the form with reduced spectrum of Ambarzumyan’s theorem read as follows:

Theorem 5 (a) If $\lambda_0 \geq \int_0^1 q(x) \, dx$, then $q = \int_0^1 q(x) \, dx \ a.e.$
(b) If $\lambda_0 \geq 0$ and $\lambda_n = (n\pi)^2$ for $n > n_0$, then $q = 0 \ a.e.$, where $n_0$ is a sufficiently large positive integer.

Note that, for example, in Theorem 1 if the first eigenvalue $\lambda_0 = 0$ and $\int_0^1 q(x) \, dx = 0$, then $q = 0 \ a.e.$ The first eigenvalue-type of Ambarzumyan’s theorem, such as Theorem 1[3] and 5 (a), depends on a mean value of the potential. Hence, to prove Theorem 4 and 5 (b), without imposing an additional condition on the potential such as $\int_0^1 q(x) \, dx = 0$, we have information about the first eigenvalue with a less restrictive one and a subset of the sufficiently large eigenvalues of the spectrum.

2 Proofs

Proof of Theorem 3 (a) We show that $y = e^{itx}$ is the first eigenfunction corresponding to the first eigenvalue $\lambda_0(t)$ of the operator $L_t(q)$ for $t \in [0, \pi)$. Since the test function $y = e^{itx}$ satisfies the quasi-periodic boundary conditions in (3), by the variational principle, we get

$$t^2 + \int_0^1 q(x) \, dx \leq \lambda_0(t) \leq \frac{\int_0^1 y'' \, dx + \int_0^1 q(x) |y|^2 \, dx}{\int_0^1 |y|^2 \, dx} = t^2 + \int_0^1 q(x) \, dx. \quad (6)$$

This implies that $\lambda_0(t) = (t^2 + \int_0^1 q(x) \, dx)$ is the first eigenvalue corresponding to the first eigenfunction $y = e^{itx}$. Substituting this into the equation

$$-y'' + q(x)y = \lambda y,$$
we get \( q = \int_0^1 q(x) \, dx \) a.e.

(b) Similarly, for \( t \in [\pi, 2\pi) \), the test function \( y = e^{i(-2\pi + t)x} \) is the first eigenfunction corresponding to the first eigenvalue \( \lambda_0(t) = (2\pi - t)^2 + \int_0^1 q(x) \, dx \). Thus, \( q = \int_0^1 q(x) \, dx \) a.e. □

**Proof of Theorem 4.** Note that by [10] (see also [11]), without using the assumption \( \int_0^1 q(x) \, dx = 0 \), the eigenvalues \( \lambda_n(t) \) of the operator \( L_t(q) \) for \( t \neq 0, \pi \) are asymptotically located such that

\[
\lambda_n(t) = (2\pi n + t)^2 + \int_0^1 q(x) \, dx + O\left(n^{-1}|n|\right),
\]

for \( |n| \geq n_0 \), where \( n_0 \) is a sufficiently large positive integer. And, by Theorem 1.1 of [3], there is a similar asymptotic formulas for the sufficiently large eigenvalues \( \lambda_{-n}(t) \) of the operator \( L_t(q) \) for \( t = 0, \pi \). Thus, for all \( t \in [0, \pi] \cup [\pi, 2\pi) \), if \( \lambda_n(t) = (2n\pi + t)^2 \) for \( n > n_0 \), then \( \int_0^1 q(x) \, dx = 0 \). From Theorem 3, \( q = 0 \) a.e. Thus Theorem 4 (a) and (b) are proved. □

**Remark.** \( n < 0 \) implies that \( -n > 0 \). In Theorem 4 (a) and (b), if we use the large eigenvalues \( \lambda_{-n}(t) = (2n\pi - t)^2 \) for \( n > n_0 \) instead of \( \lambda_n(t) = (2n\pi + t)^2 \), then the assertions of Theorem 4 remain valid. Moreover, for all \( n > n_0 \), it is enough to set either of the eigenvalues \( \lambda_n(t), \lambda_{-n}(t) \).

**Proof of Theorem 5.** (a) Arguing as in the proof of Theorem 3 (a), for \( \lambda_0 \geq \int_0^1 q(x) \, dx \), the test function \( y = 1 \) is the first eigenfunction corresponding to the first eigenvalue \( \lambda_0 = \int_0^1 q(x) \, dx \). Thus, \( q = \int_0^1 q(x) \, dx \) a.e.

(b) It follows from the asymptotic formula (1.6.6) in [8] that if \( \lambda_n = (n\pi)^2 \) for \( n > n_0 \), then certainly \( \int_0^1 q(x) \, dx = 0 \). From (a), \( q = 0 \) a.e. □

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