Dynamical construction of Horava-Lifshitz geometry

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Abstract

We derive the projectable version of Horava-Lifshitz gravity from the localisation of the Galilean symmetry. Specifically we provide a dynamical construction of the metric, from first principles, that reproduces the transformations of the physical variables - lapse, shift and spatial component of the metric. Also, the measure defining the action is reproduced. The geometrical basis of the Horava-Lifshitz gravity is thereby revealed which also elucidates its difference from the Newton-Cartan geometry - the spacetime of Newtonian gravity. The connection of Newton’s gravity with Horava-Lifshitz gravity is elucidated.

1 Introduction

The proposal of Horava to find a UV completion of quantum perturbative gravity [1] borrows Lifshitz technique of changing the relative scaling of space and time in the arena of critical phenomena and then invoking an additional term at the fixed point to cause flow to the actual scaling [2]. Horava’s gravity is formulated on an essentially non-relativistic manifold which is invariant under the foliation preserving diffeomorphism $\text{Diff}_F$. Note that Horava’s motivation was to attribute asymmetric scaling properties to space and time which was inspired by Lifshitz’s argument. Time was given a distinctive role with respect to space. Hence Lorentz symmetry is violated. While this explained the non-relativistic nature of the theory, nevertheless, the interpretation of Horava space time remained rather obscure. Incidentally, the structure of the transformations of the physical variables could be obtained from the ADM foliation of space time in general relativity and by going to the $c \to \infty$ limit [3].

The privileged role of time is the cornerstone of the Galileo - Newton concept of relative space and universal time. The motion of a particle under gravity is viewed as motion in a curved path in flat Euclidean space with time running universally. This is the natural point of view with which we are familiar from our ‘experience’. However there exists a geometrical theory of Newtonian gravity [4] where the same process may be seen as a geodesic motion in the Newton - Cartan spacetime [5]. But the Newton - Cartan geometry is characterised by the existence of two different degenerate metric and is not useful to explain the foliation preserving diffeomorphism on which Horava - Lifshitz gravity is formulated. This implies that though Horava - Lifshitz theory is non-relativistic it is not equivalent to Newton’s theory. Also the geometrical content of Horava gravity, unlike that of Newtonian gravity, is unclear. A first principle derivation of Horava’s theory should follow from some other kind of non-relativistic diffeomorphism that would illuminate the role of its geometry.
In recent times the idea of non-relativistic diffeomorphism has surfaced in a wide variety of applications, particularly in the context of fractional quantum Hall effect [6], [7]. Again, the theory of fractional quantum Hall effect has been connected with Horava-Lifshitz gravity [8]-[9]. Note however that in these theories the concept of non-relativistic diffeomorphism invariance has been used in a minimal way. The power and utility of this symmetry is neither highlighted nor exploited in a holistic manner.

Recently, in a series of papers [10] - [12] we have shown that the non-relativistic diffeomorphism can be systematically achieved by gauging the space time transformations of a Galilean invariant theory. This can be understood from the analogy with the Poincare gauge theory [13]-[15] where localisation of the Poincare symmetry in the Minkowski space is known to lead to the Riemann-Cartan spacetime. However details of the technique differ significantly due to the inherent difference between euclidean space and universal time with the Minkowski space - time. Inspite of this difference, we have successfully formulated spatial non-relativistic diffeomorphism [10, 12] and derived Newton-Cartan space [11] from the localisation of Galilean symmetry.

In this paper we will show that the space time manifold on which the Horava gravity is based has a deep connection with the gauging of the non-relativistic symmetries of Galilean space and time. We will provide a dynamical construction of the metric. From this construction we will reproduce the transformations of the physical observables of Horava gravity. Also we show the emergence of the invariant volume element appearing in the action. It may be mentioned that there are two classes of Horava gravity – projectable and non projectable. The projectable version considers the case where the lapse is a function of time only whereas in the non projectable version the lapse is a function of both time and space. Here we will restrict our attention to the projectable version only.

The Galilean transformations consist of constant time translation \( \delta^0 \), space translation \( \delta^i \), spatial rotation \( \omega^i_j \) and Galilean boost \( v^i \). The localised version may conveniently written as

\[
x^\mu \rightarrow x^\mu + \xi^\mu
\]

where

\[
\xi^\mu = (\eta^\mu - v^\mu t)(t, r)
\]

with \( \eta^\mu = \delta^\mu + \omega^i_j x^j \). Note the nature of \( \xi^0 \) which is assumed to be independent of \( r \). In the context of Galilean relativity this is only natural [16].

The transformations (2) vary from point to point. Thus it is necessary to introduce local coordinates to give the local Galilean transformations a meaning. In the following we will use two types of coordinates, the global coordinates and the local coordinates. The global coordinates will be denoted by \( x^\mu (\mu = 0, 1, 2, 3 \equiv 0, i) \) and the local coordinates by \( x^\alpha (\alpha = 0, 1, 2, 3 \equiv 0, a) \). However note that during the gauging of the Galilean symmetries, space is flat and time is universal. So the local basis is trivially connected with the global basis at this stage.

We provided an algorithm [10] [12] to convert a globally Galilean symmetric model

\[
S = \int dt d^3 x L (\phi, \partial_0 \phi, \partial_k \phi)
\]

to one symmetric under (2) as,

\[
S = \int dt d^3 x \frac{M}{\theta} L (\phi, \nabla_0 \phi, \nabla_a \phi)
\]
where ordinary derivatives are replaced by covariant derivatives and \( M, \theta \) are related to the new fields that were introduced by the gauging process. Note that local Galilean symmetry can only be referred to a local basis which we denote by \('b\). On the other hand a global basis is denoted by \('i\). In the initial stage they are trivially connected: \( x^i = \delta^i_0 x^b \). Also \( x^0 = x^0 \). Still the distinction is inbuilt in the definition of the local covariant derivatives \( \nabla_{\theta} \phi \) and \( \nabla_a \phi \). \[10\]

The covariant derivatives \( \nabla_{\theta} \phi, \nabla_a \phi \) are defined as,

\[
\begin{align*}
\nabla_{\theta} \phi &= \theta (D_0 \phi + \Psi^k D_k \phi) \\
\nabla_a \phi &= \Sigma_a^k D_k \phi.
\end{align*}
\]  

where,

\[
\begin{align*}
D_k \phi &= \partial_k \phi + i A_k \phi \\
D_0 \phi &= \partial_t \phi + i A_0 \phi
\end{align*}
\]  

To ensure the appropriate covariant transformation properties of \( \nabla_{\theta} \phi \) and \( \nabla_a \phi \), the new fields must transform as \[10\],

\[
\begin{align*}
\delta_0 \theta &= -\theta \dot{\epsilon} + \epsilon \dot{\theta} \\
\delta_0 \Psi^k &= \epsilon \dot{\Psi}^k + \dot{\epsilon} \Psi^k + \frac{\partial}{\partial x^0} (\eta^k - x^0 v^k) \partial_t \Psi^k - x^0 \dot{\Psi}^i \partial_i v^k + \Psi^i \partial_i \eta^k + \frac{1}{\theta} v^b \Sigma_b^k \\
\delta_0 \Sigma_a^k &= \epsilon \dot{\Sigma}_a^k + \Sigma_a^i \partial_i \left( \eta^k - x^0 v^k \right) - \left( \eta^i - x^0 v^i \right) \partial_i \Sigma_a^k + \omega_a^b \Sigma_b^k
\end{align*}
\]  

In gauging the global Galilean invariance two classes of new fields have been introduced, namely \( A_0(t, r), A_k(t, r) \) and \( \theta(t), \Psi^k(t, r), \Sigma_a^k(t, r) \). Their required transformations have been reported earlier \[10\]. Only the second kind of fields are involved in the measure correction of the volume (see \[1\]) as the correction factor \( M/\theta \) is constructed by taking fields from this set:

\[
M = det \Lambda_k^a.
\]  

where \( \Lambda_k^a \) is the inverse of \( \Sigma_a^k \).

The above development can be given a crucial turn. The theory \[1\] is a gauge theory which can entirely be written in terms of global flat coordinates. But the way the measure changes from \[3\] to \[4\] invites a geometric interpretation. This is further buttressed by the structure of the local Galilean transformations \[1\ \[2\]. They can be reinterpreted as the foliation preserving diffeomorphism of the background space time labelled by the coordinates \( x^\mu \). The local coordinates \( x^\alpha \) span the tangent space with respect to a non coordinate basis. The two bases are now comparable at the overlap.

That this geometric analogy is deep rooted will now be established. We introduce a \( 4 \times 4 \) matrix \( \Sigma_{\alpha}^\mu \) by defining

\[
\Sigma_0^0 = \theta, \quad \Sigma_0^k = \theta \Psi^k \quad \text{and} \quad \Sigma_a^0 = 0
\]  

in addition to the \( \Sigma_a^k \) already defined in \[5\]. This matrix is invertible. The inverse matrix is denoted by \( \Lambda_{\mu}^\alpha \). A simple calculation gives

\[
\Lambda_0^0 = \frac{1}{\theta}, \quad \Lambda_0^a = -\Psi^k \Lambda_k^a \quad \text{and} \quad \Lambda_0^0 = 0
\]  

where \( \Lambda_k^a \) is the inverse of \( \Sigma_a^k \),

\[
\Lambda_k^a \Sigma_b^k = \delta_b^a
\]  

3
From (9, 10, 11) and using the transformations (7) of \( \theta, \Psi^k \) and \( \Sigma^a_k \) we can deduce that

\[
\delta_0 \Sigma^0_k = -\xi^\nu \partial_\nu \Sigma^0_k + \Sigma^0_\nu \partial_\nu \xi^k - v^b \Sigma^0_k
\]

\[
\delta_0 \Sigma^a_k = -\xi^\nu \partial_\nu \Sigma^a_k + \Sigma^a_\nu \partial_\nu \xi^k - \omega^b_a \Sigma^b_k
\]

\[
\delta_0 \Lambda_0^a = -\xi^\nu \partial_\nu \Lambda_0^a - \Lambda_\nu^a \partial_\nu \xi^0 + v^a \Lambda_0^0 - \omega^a_c \Lambda_0^c
\]

\[
\delta_0 \Lambda_k^a = -\xi^\nu \partial_\nu \Lambda_k^a - \Lambda_\nu^a \partial_\nu \xi^k - \omega^a_c \Lambda_k^c
\]

(12)

Note that elements of these matrices converted the global covariant derivatives to their local counterparts and conversely, during the gauging process. As remarked earlier they carry two indices: the local index \( \alpha(0, a) \) and the global index \( \mu(0, i) \). The transformations (12) are like a co (contra)variant vector under the foliation preserving diffeomorphism (2) corresponding to a global index whereas those corresponding to the local indices are Galilean boost or rotation in flat euclidean space and universal time. The matrices \( \Sigma^\mu_\alpha \) (or its inverse) can thus be identified with the vielbeins that link the local basis in the tangent space with the global coordinate basis.

The geometric interpretation of the Galilean gauge theory brings us close to the space time of Horava gravity. Before we bring more geometric objects it is worth to note an important point. In the locally Minkowskian tangent space both boost and rotations are transformations that keep the origin of the coordinate system invariant. But in the Galilean space time no natural space time metric exists and the boosts affect time and space asymmetrically. If we keep this asymmetry a single non degenerate spacetime metric (which is necessarily symmetric) does not exist [4]. This line of analysis leads to the Newton - Cartan spacetime which is characterised by two degenerate metrics eventually leading to the geometric formulation of Newton’s gravity. On the other hand a single nondegenerate metric must be assumed for the Horava geometry. This is easily achieved by putting the boost parameter vanishing. The implication of this assumption will be discussed below. Such a choice immediately leads to Horava construction as will be demonstrated below.

We now propose the following ‘metric’

\[
g_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu
\]

(13)

and work out its variation under (2). Both \( g_{00} \) and \( g_{ij} \) transform as second rank tensor under it

\[
\delta_0 g_{00} = -\xi^\nu \partial_\nu g_{00} - \partial_0 \xi^\rho g_{0\rho} - \partial_0 \xi^\rho g_{0\rho}
\]

\[
\delta_0 g_{ij} = -\xi^\nu \partial_\nu g_{ij} - \partial_i \xi^\rho g_{j\rho} - \partial_j \xi^\rho g_{i\rho}
\]

(14)

In the above deduction the definition (13) and the transformations (12) are used. The transformation of \( g_{0k} \) comes out as

\[
\delta_0 g_{0j} = -\xi^\nu \partial_\nu g_{0j} - \partial_0 \xi^\rho g_{0\rho} - \partial_j \xi^\rho g_{0\rho} + v_b \Lambda_0^0 \Lambda_b^k
\]

(15)

The term containing the boost parameter will be dropped as per our argument in the above. Thus the coefficients \( g_{0j} \) also satisfy the required transformation properties.

The significance of the ‘metric’ is to be understood properly. The coordinates \( x^\mu \) define a four dimensional differentiable manifold whose physical structure is \( \mathbb{R}^3 \times R \). The ‘metric’ imposes a Riemannian structure on this manifold which is constructed from the vielbeins arising out of our localisation procedure. It is nonsingular and symmetric. The inverse of \( g_{\mu\nu} \) can be easily constructed as

\[
(4) g^{\mu\nu} = \eta^{\alpha\beta} \Sigma_\alpha^\mu \Sigma_\beta^\nu
\]

(16)
The invariant 'interval' corresponding to it is not the same as that of the Galileo - Newton space time. But it helps us to implement a foliation through the Arnowit - Deser - Misner (ADM) construction in general relativity. So we define the lapse \( N \) and the shift variables in the usual way

\[
N = (-g^{00})^{1/2} \\
N^j = g^{ij} g_{0i}
\]

(17)

\( g^{ij} \) is the inverse of the spatial part of the metric \( g_{\mu \nu} \). We thus get the physical variables of the Horava gravity. Their transformation laws are easily calculated from the above relations,

\[
\delta g_{ij} = -\partial_i \xi^k g_{jk} - \partial_j \xi^k g_{ik} - \xi^k \partial_k g_{ij} - \xi^0 \dot{g}_{ij}, \\
\delta N_i = -\partial_i \xi^j N_j - \xi^j \partial_j N_i - \dot{\xi}^j g_{ij} - \dot{\xi}^0 N_i - \xi^0 \dot{N}_i, \\
\delta N = -\xi^j \partial_j N - \dot{\xi}^0 N - \dot{\xi}^0 \dot{N}.
\]

(18)

They are exactly the same as found by taking the \( c \to \infty \) limit of the ADM decomposition of the metric in general relativity, which is the prescription followed by Horava.

Let us pause to think what we have achieved. We have reinterpreted the Galilean gauge theory in flat Euclidean space with absolute time constructed by us \([10, 12]\) as a geometric theory over a curved manifold. It is a differentiable manifold which is left invariant by the foliation preserving diffeomorphism \([2]\). The space is converted to a metric space by constructing a metric which has all its desired properties, namely, it is a second rank covariant tensor under Diff\(_F\) on \( M \), nonsingular and symmetric. An ADM splitting of this manifold exists and as usual the physical variables are identified as \( g_{ij} \), the spatial part of the metric, \( N \), the lapse and \( N^i \), the shift variables. The transformation rules of these variables are given by \([13]\). These are the same transformations obtained in \([1]\). Naturally we would like to identify the space time given by the metric \([13]\) with that of Horava - Lifshitz gravity. But one piece of dictionary is still left. It is the invariant measure of the volume. From Galilean gauge theory we have identified the measure as (see equation \([4]\))

\[
\int dtd^3x \frac{M}{\theta} = \int dtd^3x \sqrt{\det g_{ij} N}
\]

(19)

which reproduces the measure of Horava-Lifshitz gravity.

In this paper we have used techniques of non-relativistic diffeomorphism invariance developed by us \([10, 11, 12]\) to provide a geometric basis for the construction of Horava - Lifshitz theory of gravity which was unclear in its original formulation \([1]\). The genesis of the foliation preserving diffeomorphism invariant space time of Horava is shown to originate from the localisation of non-relativistic symmetry subject to a particular condition. This condition is the vanishing of the boost parameter. This is done on the ground that in the non-relativistic case, there is no single nondegenerate space time metric. Indeed, if we had retained the boost parameter it would have led to the Newton - Cartan space time as we have shown elsewhere \([11]\). The Newton - Cartan space time, as is well known, is the basis for the construction of Newton’s gravity as a space time phenomenon. This clearly shows the difference between the geometric aspects of Newton’s formulation vis-a-vis Horava's formulation. However, we must note the common origin of both these types of non-relativistic gravity. The non-relativistic diffeomorphism associated with these theories emanate from a gauging of the non-relativistic space time symmetries. Retaining the boost parameter would lead to Newton’s gravity.
as shown by us, while setting it to zero leads to Horava - Lifshitz formulation as shown here. The present analysis demonstrates the versatility of the gauging technique developed by us \[10-12\] as it is useful in discussing diverse topics like fractional quantum Hall effect \[10\], Newton-Cartan geometry \[11\] and, as shown here, the geometry of Horava-Lifshitz gravity.

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