Diagram automorphisms and canonical bases for quantum affine algebras, II

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Abstract. Let $U_q^-$ be the negative part of the quantum enveloping algebra, and $\sigma$ the algebra automorphism on $U_q^-$ induced from a diagram automorphism. Let $U_q^{-}$ be the quantum algebra obtained from $\sigma$, and $\tilde{B}$ (resp. $\tilde{B}$) the canonical signed basis of $U_q^-$ (resp. $U_q^{-}$). Assume that $U_q^-$ is simply-laced of finite or affine type. In our previous papers [SZ1, 2], we have proved by an elementary method, that there exists a natural bijection $\tilde{B}_\sigma = \tilde{B}$ in the case where $\sigma$ is admissible. In this paper, we show that such a bijection exists even if $\sigma$ is not admissible, possibly except some small rank cases.

Introduction

This paper is a continuation of [SZ1, 2]. Let $X$ be a simply-laced Cartan datum with the vertex set $I$, and $U_q^-$ the negative part of the quantum enveloping algebra associated to $X$. We assume that $X$ is irreducible, of finite or affine type, and let $\sigma : I \to I$ be the diagram automorphism of $X$. We further assume that $\sigma$ is admissible (see 1.3 for the definition). Then the order $\varepsilon$ of $\sigma$ is 2, 3 or 4. $\sigma$ induces an algebra automorphism of $U_q^-$. Let $I$ be the set of $\sigma$-orbits in $I$. One can construct the Cartan datum $\tilde{X}$ with the vertex set $I$. We denote by $U_q^-$ the negative part of the quantum enveloping algebra associated to $\tilde{X}$. Let $B$ (resp. $B$) be the canonical basis of $U_q^-$ (resp. $U_q^-$). Then $\sigma$ acts on $B$ as a permutation. We denote by $B_\sigma$ the set of $\sigma$-fixed elements in $B$. It is known by Lusztig [L1] that there exists a natural bijection $B_\sigma = B$.

Let $A = \mathbb{Z}[q, q^{-1}]$, and $A U_q^-$ Lusztig’s integral form of $U_q^-$, which is an $A$-subalgebra of $U_q^-$. Assume that $\varepsilon = 2, 3$, and let $F = \mathbb{Z}/\varepsilon\mathbb{Z}$ be the finite field of $\varepsilon$ elements. Set $A' = F[q, q^{-1}]$, and define $A' U_q^{-,\sigma}$ as the $A'$-subalgebra of $A' \otimes_{A} A' U_q^-$ consisting of $\sigma$-fixed elements. The $A'$-algebra $A' U_q^-$ is defined as $A' \otimes_{A} A' U_q^-$.

In [SZ1, 2], it is shown that, in the case where $\sigma$ is admissible with $\varepsilon \neq 4$, there exists an $A'$-algebra isomorphism $\Phi : A' U_q^- \cong V_q$, where $V_q$ is a certain quotient algebra of $A' U_q^-$.

In this paper, we consider the case where $\sigma$ is not admissible. Already in [L2], Lusztig proved that in the case where $X$ is of finite type, the bijection $B_\sigma = B$ still holds. So, one can expect that our isomorphism $A' U_q^- \cong V_q$ will have a generalization for non-admissible cases. However, in the case where $X = A_1^{(1)}$, and $\sigma$ is the
one corresponding to the cyclic quiver, there does not exist \(\sigma\)-stable canonical basis (see Remark 1.9). So we must exclude this case. Then we show in Theorem 2.6, that similar results as in [SZ1, 2] hold for \(X\) of finite or affine type, possibly except \(X\) is of type \(A_2^{(2)}\) or \(A_1^{(1)}\). The latter cases are excluded simply because the computation becomes more complicated. It is likely that our results will hold in those cases.

Note that our discussion in [SZ1, 2] heavily depends on the property of PBW-bases. But in the non-admissible case, the action of \(\sigma\) on PBW-bases becomes quite different from the admissible case. The main ingredient for our discussion is the modified PBW-bases, which is defined by mixing the usual PBW-bases and the canonical bases for \(A_2\). However, in the non-admissible case, the action of \(\sigma\) becomes more complicated. It is likely that our results will hold in those cases.

1. Diagram Automorphisms

1.1. We basically follow the notation in [MSZ, 1]. Let \(X = (I, (, ))\) be a symmetric Cartan datum, where \(I\) is a finite set, and \((, )\) is a symmetric bilinear form on the vector space \(\bigoplus_{i\in I} Q\alpha_i\) with basis \(\{\alpha_i \mid i \in I\}\). Hence the Cartan matrix \(A = (a_{ij})\) is symmetric, where \(a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\). Let \(Q = \sum_{i \in I} \mathbb{Z}\alpha_i\) be the root lattice of \(X\). Set \(Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i\) and \(Q_- = -Q_+\). Let \(U_q^-\) be the negative part of the quantized enveloping algebra \(U_q\) over \(Q(q)\) associated to \(X\) with generators \(f_i (i \in I)\). It is known by Lusztig [L1] that \(U_q^-\) has the canonical basis \(B\).

Let \(\sigma : I \to I\) be a diagram automorphism on \(X\), namely \(\sigma\) satisfies the property \((\alpha_{\sigma(i)}, \alpha_{\sigma(j)}) = (\alpha_i, \alpha_j)\) for any \(i, j \in I\). Then \(\sigma\) induces an automorphism \(\sigma : U_q^- \to U_q^-\) as \(Q(q)\)-algebras, by \(f_i \mapsto f_{\sigma(i)}\). We denote by \(U_q^-\sigma\) the fixed point subalgebra of \(U_q^-\) by \(\sigma\). \(\sigma\) induces a permutation on the canonical basis \(B\). We denote by \(B^\sigma\) the set of \(\sigma\)-fixed elements in \(B\). The notion of canonical signed bases is given as in [MSZ, 1.18]. Then \(B = B \sqcup -B\) coincides with the canonical signed basis. We denote by \(\tilde{B}^\sigma\) the set of \(\sigma\)-fixed elements in \(\tilde{B}\).

1.2. Let \(A = \mathbb{Z}[q, q^{-1}]\). We denote by \(A U_q^-\) Lusztig’s integral form of \(U_q^-\), i.e., \(A U_q^-\) is the \(A\)-subalgebra of \(U_q^-\) generated by \(f_i^{(n)} = f_i^n/\lfloor n \rfloor_d\) for \(i \in I, n \in \mathbb{N}\), where \(d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{\geq 0}\). \(\sigma\) stabilizes \(A U_q^-\), and set \(A U_q^-\sigma = U_q^-\sigma \cap A U_q^-\), the \(A\)-subalgebra of \(A U_q^-\) consisting of \(\sigma\)-fixed elements.
We assume that the order of $\sigma$ is a power of a prime number $p$, and let $F = \mathbb{Z}/p\mathbb{Z}$ be the finite field of $p$-elements. Set $A' = F[q,q^{-1}]$, and consider an $A'$-algebra $A \cdot U_q^{-\sigma}$ by

$$A \cdot U_q^{-\sigma} = A' \otimes A \cdot U_q^{-\sigma} \simeq A \cdot U_q^{-\sigma}/p(A \cdot U_q^{-\sigma}).$$

For each $x \in U_q^{-\sigma}$, we denote by $O(x)$ the orbit sum of $x$ defined by $O(x) = \sum_{0 \leq i < k} \sigma^i(x)$, where $k$ is the smallest integer such that $\sigma^k(x) = x$. Hence $O(x) \in U_q^{-\sigma}$. $O(x)$ is similarly defined for $A \cdot U_q^{-\sigma}$. Let $J$ be the $A'$-submodule of $A \cdot U_q^{-\sigma}$ generated by $O(x)$ for $x \in A \cdot U_q^{-\sigma}/p(A \cdot U_q^{-\sigma})$ such that $\sigma(x) \neq x$. Then $J$ is the two-sided ideal of $A \cdot U_q^{-\sigma}$. We define an $A'$-algebra $V_q$ by $V_q = A \cdot U_q^{-\sigma}/J$. Let $\pi : A \cdot U_q^{-\sigma} \rightarrow V_q$ be the natural projection.

1.3. Let $\sigma : I \rightarrow I$ be the diagram automorphism. We denote by $L$ the set of $\sigma$-orbits in $I$. $\sigma : I \rightarrow I$ is called admissible, if for any $\eta \in L$, $(\alpha_i, \alpha_j) = 0$ for all $i \neq j \in \eta$. The algebra $V_q$ was defined in [MSZ, 3.2] under the assumption that $\sigma$ is admissible. But, as is seen from 1.2, the definition of $V_q$ makes sense even if $\sigma$ is not admissible.

Assume that $\sigma$ is admissible. Then by [MSZ, 2.1], one can construct a Cartan datum $\mathcal{X} = (L, (\cdot , \cdot ))$ induced from $(X, \sigma)$. We denote by $U_q^{-\sigma}$ the negative part of the quantized enveloping algebra $U_q$ associated to $\mathcal{X}$. The algebras $A \cdot U_q^{-\sigma}$ and $A \cdot U_q^{-\sigma}$ are defined similarly to the case of $U_q$.

For each $\eta \in L$, $a \in N$, set $\tilde{f}_\eta(a) = \prod_i f_i^\alpha(a)$. Since $\sigma$ is admissible, $\tilde{f}_\eta(a)$ does not depend on the order of the product, and $\tilde{f}_\eta(a) \in A \cdot U_q^{-\sigma}$. We denote its image in $A \cdot U_q^{-\sigma}$ also by $\tilde{f}_\eta(a)$. We define $g_\eta(a) \in V_q$ by

$$g_\eta(a) = \pi(\tilde{f}_\eta(a)).$$

Note that $A \cdot U_q^{-\sigma}$ is generated by $f_\eta(a) = f_\eta(a)/[a]_{d_\eta}$ for $\eta \in L$ and $a \in N$. We denote by the same symbol $f^\eta(a)$ its image in $A \cdot U_q^{-\sigma}$.

The following result was proved in Theorem 3.4 and Theorem 4.18 (see also Remark 4.19) in [MSZ]

**Theorem 1.4.** Assume that $\sigma$ is admissible, and the order of $\sigma$ is a power of a prime number $p$. Then

1. The assignment $f_\eta(a) \mapsto g_\eta(a)$ gives an $A'$-algebra isomorphism $\Phi : A \cdot U_q^{-\sigma} \simeq V_q$.

2. There exists the canonical signed basis $\tilde{B}$ of $U_q^{-\sigma}$, and a natural bijection $\xi : \tilde{B}^{\sigma} \simeq \tilde{B}$.

1.5. We consider the case where $X$ is finite or affine type. Let $\varepsilon$ be the order of $\sigma$. Assume that $X$ is irreducible, and $\sigma$ is admissible. In the case where $\varepsilon = 2, 3$, Theorem 1.4 was proved by [SZ1, 2], by an elementary method.

In the remaining cases, $X$ are affine, and they are given as follows;
\[(A_a 1) \quad X = D_{2n}^{(1)}, \quad X = A_{2n-2}^{(2)}, \quad (n \geq 3), \quad \varepsilon = 4,\]
\[(A_a 1') \quad X = D_{4}^{(1)}, \quad X = A_{2}^{(2)}, \quad \varepsilon = 4,\]
\[(A_a 2) \quad X = A_{n-1}^{(1)}, \quad X = A_{m-1}^{(1)}, \quad (n = mc, 1 < c < n), \quad \varepsilon = c.\]

The diagrams are given as follows:

\[(A_a 1) \quad X = D_{2n}^{(1)} : \quad \begin{array}{c}
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\[(A_a 1') \quad X = D_{4}^{(1)} : \quad \begin{array}{c}
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\end{array}

\[(A_a 2) \quad X = A_{n-1}^{(1)} : \quad \begin{array}{c}
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\[(A_a 1) \quad X = D_{2n}^{(1)} : \quad \begin{array}{c}
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\[(A_a 1') \quad X = D_{4}^{(1)} : \quad \begin{array}{c}
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\[(A_a 2) \quad X = A_{n-1}^{(1)} : \quad \begin{array}{c}
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In the case \((A_a 1)\), \(I = \{0, 1, \ldots, 2n\}\) and \(\sigma\) is given by \(0 \rightarrow 2n - 1 \rightarrow 1 \rightarrow 2n \rightarrow 0\), \(i \leftarrow 2n - i\) for \(i = 2, \ldots, n\). Thus \(\varepsilon = 4\), and \(I = \{0, \overline{i} \ (2 \leq i \leq n - 1), 2n\}\). In the case \((A_a 1')\), \(I = \{1, 2, 2', 2'', 2'''\}\) and \(\sigma\) is given by \(\sigma : 2 \rightarrow 2' \rightarrow 2'' \rightarrow 2''' \rightarrow 2\), and \(\sigma(1) = 1\). Thus \(\varepsilon = 4\) with \(I = \{1\}\) and \(I = \{2, 2', 2'', 2'''\}\). In the case \((A_a 2)\), we identify \(I = \{0, 1, \ldots, n - 1\}\) with \(\mathbb{Z}/n\mathbb{Z}\). Then \(\sigma\) is given by \(i \mapsto i + m\) for \(i \in \mathbb{Z}/n\mathbb{Z}\). Thus \(\varepsilon = n/m = c\), and \(I = \{0, \ldots, m - 1\} \simeq \mathbb{Z}/m\mathbb{Z}\).
1.6. Assume that $X$ is of finite type. In [L2], Lusztig proved the existence of a natural bijection $B^\sigma \cong B$ even in the case where $\sigma$ is not admissible. Thus one can expect that Theorem 1.4 can be generalized to the case where $\sigma$ is not admissible. In the remainder of this section, we consider the case where $X$ is finite or affine, and $\sigma$ is not admissible. Note that $X$ is simply-laced, i.e., $(\alpha_i, \alpha_j) \in \{0, -1\}$ for $i \neq j$, and $(\alpha_i, \alpha_j) = 2$ for any $i \in I$. For each $\eta \in \mathcal{L}$, set $\delta_\eta = 1$ if $(\alpha_i, \alpha_j) = 0$ for any $i \neq j$ in $\eta$, and $\delta_\eta = 2$ otherwise. Note that $\sigma$ is admissible if and only if $\delta_\eta = 1$ for any $\eta \in \mathcal{L}$. We define a symmetric bilinear form $(\cdot, \cdot)_1$ on the vector space $\bigoplus_{\eta \in \mathcal{L}} \mathbb{Q} \alpha_\eta$ over $\mathbb{Q}$ by

\[(\alpha_\eta, \alpha_{\eta'})_1 = \begin{cases} 2\delta_{\eta} |\eta| & \text{if } \eta = \eta', \\ -\delta_{\eta} \delta_{\eta'} \{(i, j) \in \eta \times \eta' \mid (\alpha_i, \alpha_j) \neq 0\} & \text{if } \eta \neq \eta'. \end{cases}\]

Then $X = (I, (\cdot, \cdot))$ gives a Cartan datum since $(\alpha_\eta, \alpha_\eta)_1 \in 2\mathbb{Z}_{>0}$, and

\[\frac{2(\alpha_\eta, \alpha_{\eta'})_1}{(\alpha_{\eta'}, \alpha_{\eta'})_1} = \frac{1}{\delta_{\eta'} |\eta'|} \sum_{j \in \eta'} \delta_{\eta} \delta_{\eta'} \sum_{i \in \eta} (\alpha_i, \alpha_j) \in \mathbb{Z}_{\leq 0}.\]

1.7. Assume that $\sigma$ is non-admissible. If $X$ is irreducible of finite type, only the following case occurs: $X$ is of type $A_{2n}$ with $I = \{1, 2, \ldots, 2n\}$, and $\sigma: i \leftrightarrow 2n - i + 1$ with $\varepsilon = 2$. Then $X$ is of type $C_n$ with $I = \{1, \ldots, n\}$.

\[(F_n1) \quad X = A_{2n}: \quad \bullet \ldots \bullet \bullet \bullet \bullet \ldots \bullet \]

\[\quad 1 \quad n \quad n + 1 \quad 2n \]

\[X = C_n: \quad \bullet \ldots \bullet \bullet \bullet \bullet \ldots \bullet \bullet \]

\[\quad 1 \quad 2 \quad \ldots \quad n - 1 \quad n \]

with $\sigma: i \leftrightarrow 2n + 1 - i$ for $1 \leq i \leq 2n$.

If $X$ is irreducible of affine type, the following cases occur.

\[(A_n1) \quad X = D^{(1)}_{2n+1}, \quad X = A_{2n-1}^{(2)}, \quad (n \geq 3), \quad \varepsilon = 2,\]

\[(A_n2) \quad X = D^{(1)}_{2n+1}, \quad X = C_{n-1}^{(1)}, \quad (n \geq 3), \quad \varepsilon = 4,\]

\[(A_n3) \quad X = A_{2n}^{(1)}, \quad X = A_{2n}^{(2)}, \quad (n \geq 2), \quad \varepsilon = 2,\]

\[(A_n4) \quad X = A_{2n+1}^{(1)}, \quad X = C_{n}^{(1)}, \quad (n \geq 2), \quad \varepsilon = 2,\]

\[(A_n2') \quad X = D_{5}^{(1)}, \quad X = A_{1}^{(1)}, \quad \varepsilon = 4,\]

\[(A_n3') \quad X = A_{2}^{(1)}, \quad X = A_{2}^{(2)}, \quad \varepsilon = 2,\]

\[(A_n4') \quad X = A_{3}^{(1)}, \quad X = A_{1}^{(1)}, \quad \varepsilon = 2,\]

\[(A_n5) \quad X = A_{n-1}^{(1)}, \quad X = A_{1}, \quad \varepsilon = n.\]

The diagrams are given as follows.
\((A_n)\)  \(X = D_{2n+1}^{(1)}:\)

\[ X = A_{2n-1}^{(2)} : \]

\[ X = C_{n-1}^{(1)} : \]

\[(A_n)  X = A_{2n}^{(1)} : \]

\[ X = A_{2n}^{(2)} : \]

\[(A_n)  X = A_{2n+1}^{(1)} : \]

\[ X = C_n^{(1)} : \]
The actions of $\sigma$ on $I$ are as follows. In $(A_n1)$, $\sigma: i \leftrightarrow 2n + 1 - i$ for $0 \leq i \leq 2n + 1$. In $(A_n2)$, $\sigma: 0 \mapsto 2n \mapsto 1 \mapsto 2n + 1 \mapsto 0$ and $i \leftrightarrow 2n + 1 - i$ for $2 \leq i \leq 2n - 1$. In $(A_n3)$, $\sigma: i \leftrightarrow 2n + 1 - i$ for $1 \leq i \leq 2n$, and $\sigma(0) = 0$. In $(A_n4)$, $\sigma: i \leftrightarrow 2n + 1 - i$ for $0 \leq i \leq 2n + 1$. In $(A_n2')$, $\sigma: 0 \mapsto 4 \mapsto 1 \mapsto 5 \mapsto 0$ and $2 \leftrightarrow 3$. In $(A_n3')$, $\sigma: 1 \leftrightarrow 2$ and $\sigma(0) = 0$. In $(A_n4')$, $\sigma: 0 \leftrightarrow 3, 1 \leftrightarrow 2$. In $(A_n5)$, $\sigma: 0 \mapsto 1 \mapsto 2 \mapsto \cdots \mapsto n - 1 \mapsto 0$.

**Remarks 1.8.**

(i) In [L2], Lusztig defines a Cartan datum $X$ in the non-admissible case, by using the inner product which corresponds to $(\alpha, \alpha')$, where $\delta = \max\{\delta, \delta'\}$. Although the Cartan matrix is the same, (1.6.1) is more convenient for our later discussion.

(ii) The numbering of $A_{2n}^{(2)}$ obtained from $A_{2n+1}^{(2)}$ by non-admissible $\sigma$ (the cases $(A_n2), (A_n3)$) is reverse to the order obtained from $D_{2n+1}^{(1)}$ by admissible $\sigma$ (see 5.1 (5), (6) in [SZ2]). The numbering of $(A_n3), (A_n3')$ coincides with those in Kac's textbook [K], but is reverse to the one in Beck-Nakajima [BN].

**Remark 1.9.** In the case $(A_n5)$, the $\sigma$-invariant canonical basis does not exist. We follow the description in the lecture note by Schiffmann [S, 2.4]. Consider the cyclic quiver $\overrightarrow{Q}$ associated to $X$, where the orientation is given by $\sigma: 0 \mapsto 1 \mapsto 2 \mapsto \cdots \mapsto n - 1 \mapsto 0$. Then $\sigma$ gives an automorphism of the quiver $\overrightarrow{Q}$. We identify $I$ with $\mathbb{Z}/n\mathbb{Z}$. We denote by $\overrightarrow{I}_{i,m}$ ($i \in I, m \in \mathbb{N}$) the unique nilpotent indecomposable representation of $\overrightarrow{Q}$ with socle $\varepsilon_i$ and length $m$. Let $\Pi^n$ be the set of $n$-tuple of
partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)})$, where $\lambda^{(i)} = (\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdots)$ is a partition. The set of isomorphism classes of nilpotent representations of $\overline{Q}$ is identified with $\Pi^n$ by the correspondence

$$\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \mapsto M_\lambda = \bigoplus_{i \in I} \bigoplus_{j} I_{[i] \lambda_j^{(i)}}.$$ 

Let $\nu = \text{Dim} M_\lambda$, and consider the representation space $E(\overline{Q}, \nu)$ of $\overline{Q}$. We denote by $\sigma_\lambda$ the nilpotent orbit in $E(\overline{Q}, \nu)$ corresponding to $M_\lambda$. Let $P_\lambda = \text{IC}(\overline{O}_\lambda, \overline{Q}_I)$ be the simple perverse sheaf (up to shift) on $E(\overline{Q}, \nu)$ corresponding to the orbit $\sigma_\lambda$. Let $\mathcal{P}_{\overline{Q}}$ be the set of simple perverse sheaves as defined in [S, 1.4]. The following result was proved by Lusztig [L3], [L4].

(1.9.1) Assume that $n > 1$. Then

$$\mathcal{P}_{\overline{Q}} = \{ P_\lambda \mid \lambda \in \Pi^n, \lambda : \text{aperiodic} \},$$

where $\lambda = (\lambda_j^{(i)}) \in \Pi^n$ is called aperiodic if $\lambda^{(1)}, \ldots, \lambda^{(n)}$ have no common parts, namely for any integer $c > 0$, there exists $i$ such that $c$ does not appear as $\lambda_j^{(i)}$.

$\sigma$ maps $M_\lambda$ onto $M_{\sigma(\lambda)}$, where $\sigma(\lambda) = (\lambda^{(2)}, \lambda^{(3)}, \ldots, \lambda^{(n)}, \lambda^{(1)})$. Hence $\sigma$ acts on the set of $\{ P_\lambda \mid \lambda \in \Pi^n \}$ as a permutation, $P_\lambda \mapsto P_{\sigma(\lambda)}$. In particular, $P_\lambda$ is $\sigma$-invariant if and only if $\lambda^{(1)} = \lambda^{(2)} = \cdots = \lambda^{(n)}$, in which case $\lambda$ is not aperiodic. Thus (1.9.1) implies that $\sigma$ acts on $\mathcal{P}_{\overline{Q}}$ as a permutation, and there does not exist $\sigma$-invariant element. From the definition of canonical basis, there exits a bijection $\mathcal{P}_{\overline{Q}} \cong B$, which is compatible with the action of $\sigma$. It follows that $B^\sigma = \emptyset$. Since $\pi(B^\sigma)$ gives a basis of $V_q$, we conclude that $V_q = 0$. Summing up the above discussion, we have

(1.9.2) In the case of $(A_n, 5)$, an analogue of Theorem 1.4 does not hold.

1.10. In the rest of this paper, we assume that $X$ is irreducible of finite or affine type, and by Remark 1.9, we exclude the case $(A_n, 5)$. Let $U_q^-$ (resp. $\overline{U}_q^-$) the quantum enveloping algebra associated to $X$ (resp. $\overline{X}$). We follow the notation in [SZ1, 2]. In particular, for each $i \in I$, let $T_i : U_q \to U_q$ be the braid group action, and similarly consider $T_i : \overline{U}_q \to \overline{U}_q$ for $i \in \overline{I}$.

Let $W$ be the Weyl group associated to $X$. Assume that $X$ is of finite type, and $w_0$ is the longest element in $W$. Let $h = (i_1, \ldots, i_N)$ be a sequence of $i \in I$ such that $s_{i_1} \cdots s_{i_N}$ is a reduced expression of $w_0$. For a given sequence $h$, the PBW-basis $\mathcal{H}_h = \{ L(c, h) \mid c \in \mathbb{N}^N \}$ for $U_q^-$ is defined as in [SZ1, 1.7]. In the case where $X$ is of affine type, let $h = (\ldots, i_{-1}, i_0, i_1, \ldots)$ be a doubly infinite sequence of $i \in I$ as defined in [BN, 3.1]. For a given $h$, and an integer $p$, the PBW-basis $\mathcal{H}_{h, p} = \{ L(c, p) \mid c \in \mathcal{C} \}$ of $U_q^-$ is defined as in [SZ2, 1.5], where $\mathcal{C}$ is a certain parameter set.
2. THE ALGEBRA $\mathbf{V}_q$

2.1. In this section, we assume that $\sigma$ is not admissible. First we consider the simplest case, namely, $X = A_2$ and $X = C_1$, hence $I = \{1, 2\}$ with $\sigma : 1 \leftrightarrow 2$ and $\mathcal{L} = \{1\}$. $\mathbf{U}_q^-$ has two generators $f_1, f_2$. Set

$$
\begin{align*}
\psi_{12} &= T_1(f_2) = f_2 f_1 - q f_1 f_2, \\
\psi'_{12} &= T_2(f_1) = f_1 f_2 - q f_2 f_1.
\end{align*}
$$

The action of $\sigma$ on $\mathbf{U}_q^-$ is given by $\psi_1 \leftrightarrow \psi_2$. Thus $\sigma(\psi_{12}) = \psi'_{12}$. For $h = (1, 2, 1)$, $\mathbf{h}' = (2, 1, 2)$, (see 1.10)

$$
\mathcal{R}_h = \{ L(c, h) = f_1^{(c_1)} f_2^{(c_2)} f_2^{(c_3)} | c = (c_1, c_2, c_3) \in \mathbb{N}^3 \}
$$

$$
\mathcal{R}_{h'} = \{ L(c, h') = f_2^{(c_1)} f_1^{(c_2)} f_1^{(c_3)} | c = (c_1, c_2, c_3) \in \mathbb{N}^3 \}
$$

give two kinds of PBW-bases of $\mathbf{U}_q^-$. The following formulas are known.

$$
\begin{align*}
\psi_{12} &= \sum_{k=0}^{\ell} q^{(\ell-k)(m-k)} \left[ \ell - k + n \atop \ell - k \right] f_1^{(m-k)} f_2^{(k)} f_2^{(\ell-k+n)}, \\
\psi'_{12} &= \sum_{k=0}^{n} q^{(m-k)(n-k)} \left[ n - k + \ell \atop n - k \right] f_1^{(n-k+\ell)} f_2^{(k)} f_2^{(m-k)}.
\end{align*}
$$

By (2.1.2) and (2.1.3), we see that, if $m \geq \ell + n$, $f_2^{(\ell)} f_1^{(m)} f_2^{(n)}$ is the canonical basis corresponding to the PBW-basis $f_1^{(m-\ell)} f_2^{(\ell)} f_1^{(n)}$, and $f_1^{(\ell)} f_2^{(m)} f_1^{(n)}$ is the canonical basis corresponding to the PBW-basis $f_1^{(\ell)} f_2^{(n)} f_2^{(m-n)}$. Thus

$$
\mathcal{B} = \{ f_2^{(\ell)} f_1^{(m)} f_2^{(n)} | m \geq \ell + n \} \cup \{ f_1^{(\ell)} f_2^{(m)} f_1^{(n)} | m \geq \ell + n \}
$$

gives the canonical basis of $\mathbf{U}_q^-$, and the overlapping occurs only when $f_2^{(\ell)} f_1^{(m)} f_2^{(n)} = f_1^{(n)} f_2^{(m)} f_1^{(\ell)}$ with $m = \ell + n$. Note that $\sigma(f_2^{(\ell)} f_1^{(m)} f_2^{(n)}) = f_1^{(\ell)} f_2^{(m)} f_1^{(n)}$. Let $\mathcal{B}^\sigma$ be the set of $\sigma$-invariant elements in $\mathcal{B}$. It follows from the above discussion that

$$
\mathcal{B}^\sigma = \{ f_2^{(\ell)} f_1^{(2a)} f_2^{(a)} | a \in \mathbb{N} \}.
$$

More generally, the following result is also obtained from the previous discussion.

(2.1.6) Assume that $\nu = -m(\alpha_1 + \alpha_2) \in \mathcal{Q}_-$. Then $\{ f_1^{(\ell)} f_2^{(m-\ell)} f_2^{(\ell)} | 0 \leq \ell \leq m \}$ gives the PBW -basis, and $\{ f_2^{(\ell)} f_1^{(m)} f_2^{(m-\ell)} | 0 \leq \ell \leq m \}$ gives the canonical basis of $(\mathbf{U}_q^-)_\nu$, under the correspondence $f_1^{(\ell)} f_2^{(m-\ell)} f_2^{(\ell)} \leftrightarrow f_2^{(\ell)} f_1^{(m)} f_2^{(m-\ell)}$.

Concerning the action of $\sigma$ on PBW-basis, we have the following..
Lemma 2.2. Assume that $X$ is of type $A_2$. Then $\sigma$-invariant PBW-bases do not exist (except the trivial one). The PBW-basis corresponding to the $\sigma$-invariant canonical basis $f_2^{(a)} f_1^{(2a)} f_1^{(a)}$ is given by $f_1^{(a)} f_2^{(a)} f_1^{(a)}$ for $h = (1, 2)$, and by $f_2^{(a)} f_1^{(2a)} f_1^{(a)}$ for $h' = (2, 1)$. We have, for $a \geq 1$,

$$\sigma(f_1^{(a)} f_2^{(a)}) = f_2^{(a)} f_1^{(a)} f_2^{(a)} \neq f_1^{(a)} f_2^{(a)} f_1^{(a)}.$$ 

Proof. The latter assertion is clear from the previous discussion. We show that $\sigma$-invariant PBW-basis does not exist. If a PBW-basis $x$ is $\sigma$-invariant, then the corresponding canonical basis is $\sigma$-invariant, hence has the form $f_2^{(a)} f_1^{(2a)} f_1^{(a)}$ for some $a \in \mathbf{N}$. It follows that $x = f_1^{(a)} f_2^{(a)} f_1^{(a)}$, and $y = \sigma(x) = f_2^{(a)} f_1^{(a)} f_1^{(a)}$. It is enough to show that $x \neq y$ if $a \geq 1$. By (2.1.2), one can write as

$$f_2^{(a)} f_1^{(2a)} f_1^{(a)} = f_1^{(a)} f_2^{(a)} f_1^{(a)} + \sum_{0 \leq k < a} c_k f_2^{(k)} f_1^{(2a-k)}$$

with $c_k \in \mathbf{Q}(q)$, where $c_k \neq 0$ for some $k$. If $f_1^{(a)} f_2^{(a)} f_2^{(a)}$ is $\sigma$-invariant, then the sum in the right hand side of (2.2.1) is $\sigma$-invariant. It follows by (2.1.5) that $c_k = c_{a-k}$ for any $k$. But since $0 \leq k < a$, we have $2a - k > a$, and $c_{a-k}$ does not appear in this sum. This is absurd, and $f_1^{(a)} f_2^{(a)} f_1^{(a)}$ is not $\sigma$-invariant. The lemma is proved. \qed

2.3. We define $V_q = A^\ast U_q^{-\sigma} / J$ as in 1.2, and let $\pi : A^\ast U_q^{-\sigma} \to V_q$ be the natural projection. Let $B$ be the canonical basis of $U_q^{-}$ (of type $A_2$). Then $\sigma$ permutes $B$, and $B^\sigma$ is given as in (2.1.5). For each $a \geq 0$, set

$$g_1^{(a)} = \pi(f_2^{(a)} f_1^{(2a)} f_1^{(a)}) = \pi(f_1^{(a)} f_2^{(2a)} f_1^{(a)}) \in V_q$$

Then $\pi(B^\sigma) = \{g_1^{(a)} \mid a \in \mathbf{N}\}$ gives an $A^\ast$-basis of $V_q$.

We consider $U_q^{-}$ associated to the Cartan datum $X$ of type $C_1$. The canonical basis of $U_q^{-}$ is given by $\{f_{-1}^{(a)} \mid a \in \mathbf{N}\}$, where $f_{-1}^{(a)} = ([a]_q^{-1})^{1/2} f_{-1}^{(a)}$. (Note that $d_{-1} = (\alpha_1, \alpha_1)/2 = 4$ by (1.6.1).) $A^\ast U_q^{-}$ is an $A$-subalgebra of $U_q^{-}$ generated by $f_{-1}^{(a)}$ $(a \in \mathbf{N})$. $A^\ast$-algebra $A^\ast U_q^{-}$ is defined as before. We show the following.

Proposition 2.4. Assume that $X$ is of type $A_2$. Then we have

$$[a]_q^{1/2} g_{-1}^{(a)} = g_{-1}^{(a)}.$$ 

The correspondence $f_{-1}^{(a)} \mapsto g_{-1}^{(a)}$ gives an isomorphism $\Phi : A^\ast U_q^{-} \cong V_q$ of $A^\ast$-algebras.

Proof. Since $ch F = 2$, in order to prove the proposition, it is enough to show that

$$(f_1 f_2^{(2a)} f_1^{(a)}) \equiv ([a]_q^4 f_1^{(a)} f_2^{(2a)} f_1^{(a)}) \mod J.$$ 

For $a \geq 1$, we consider the following statement.
(2.4.3) \[ a (a-1) f_1^{(2a)} f_1^{(2a)} \equiv (f_1^{(a-1)} f_2^{(2a-2)} f_1^{(a-1)})(f_1 f_2^{(2a)}) \mod J. \]

Note that if (2.4.3) holds, then (2.4.2) follows by induction. We prove (2.4.3). Let \( Z \) be the right hand side of (2.4.3). By using the formula (2.1.3), we can express \( Z \) as a linear combination of PBW-bases \( f_1^{(a)} f_2^{(b)} f_1^{(c)} \),

\[
\sum_{k=0}^{a-1} (A_k f_1^{(2a-2-k)(a-1-k)-k} f_2^{(k+1)} f_2^{(2a-1-k)} + B_k f_1^{(2a-2-k)} f_2^{(k+2)} f_2^{(2a-2-k)} + C_k f_1^{(2a-k)} f_2^{(k)} f_2^{(2a-k)}),
\]

where

\[
A_k = q^{(2a-2-k)(a-1-k)-k} \left[ \frac{2a - 2 - k}{a - 1 - k} \right] [2a - 1 - k]^2[k + 1]
+ q^{(2a-2-k)(a-1-k)+(2a-1-2k)} \left[ \frac{2a - 2 - k}{a - 1 - k} \right] [2a - 1 - k]^2[k + 1][2a - 2 - k],
\]

\[
B_k = q^{(2a-2-k)(a-1-k)-(2a-3-k)} \left[ \frac{2a - 2 - k}{a - 1 - k} \right] [2a - 2 - k][k + 1][k + 2]
+ q^{(2a-2-k)(a-1-k)+2} \left[ \frac{2a - 2 - k}{a - 1 - k} \right] [2a - 3 - k][2a - 2 - k][k + 1][k + 2],
\]

\[
C_k = q^{(2a-2-k)(a-1-k)+(4a-2-4k)} \left[ \frac{2a - 2 - k}{a - 1 - k} \right] [2a - 1 - k]^2[2a - 2 - k]^2.
\]

From this formula, we see that the coefficient of \( f_1^{(2a-k)} f_1^{(k)} f_2^{(2a-k)} \) in \( Z \) is equal to

\[
\begin{cases} 
  A_{a-1} + B_{a-2} & \text{if } k = a, \\
  B_{a-1} & \text{if } k = a + 1, \\
  0 & \text{if } k \geq a + 2.
\end{cases}
\]

On the other hand, by using (2.1.6), \( Z \) can be expressed as

\[
Z = \sum_{0 \leq i \leq 2a} m_i f_1^{(i)} f_2^{(2a)} f_1^{(2a-i)}.
\]

Note that since \( Z \) is \( \sigma \)-invariant, we see, by (2.1.5), that \( m_i = m_{2a-i} \) for each \( i \). By using (2.1.3), each \( f_1^{(i)} f_2^{(2a)} f_1^{(2a-i)} \) is written by the PBW-basis. Then the coefficient of \( f_1^{(2a-k)} f_1^{(k)} f_2^{(2a-k)} \) in \( Z \) is given as

\[
\sum_{i=0}^{2a-k} m_i q^{(2a-k)(2a-k-i)} \left[ \frac{2a - k}{2a - k - i} \right].
\]
By comparing (2.4.4) with (2.4.6), we see that $m_0 = \cdots = m_{a-2} = 0$, and

$$m_{a-1} = B_{a-1}, \quad m_{a-1}q^a[a] + m_a = A_{a-1} + B_{a-2},$$

where

$$B_{a-1} = q^{-a+2}[a-1][a+1] + q^2[2]^{-1}[a-2][a-1][a][a+1],$$

$$A_{a-1} + B_{a-2} = (q^{-a+1}[a]^3 + q[a]^{3}[a-1]) + (q[a]^{3}[a-1] + q^{a+2}[2]^{-1}[a-1]^2[a]^3).$$

This implies that $m_a = [a]^4$. $Z$ is $\sigma$-invariant, and in the expansion of $Z$ in (2.4.5) the only $\sigma$-invariant term is $f_1^{(a)}f_2^{(2a)}f_1^{(a)}$, whose coefficient is equal to $m_a = [a]^4$. It follows that $Z \equiv [a]^4f_1^{(a)}f_2^{(2a)}f_1^{(a)} \mod J$, and (2.4.3) holds. The lemma is proved.

2.5. We now consider the general case, namely, $X$ is irreducible of finite or affine type, and $\sigma$ is non-admissible. We exclude the case $(A_n5)$. $V_q$ is defined as in 1.2, and we can consider $\mathcal{A}/U_q$ corresponding to $X$. But as observed in the case where $X = A_2$ (Proposition 2.4), the definition of $\Phi : \mathcal{A}/U_q \to V_q$ for the admissible case in 1.3 is not applicable for the non-admissible case. We modify the definition of $\Phi$ as follows. Note that if $X$ is irreducible and $\delta_\eta = 2$, then $\varepsilon = 2$ or 4, and $\eta = \{i, i'\}$ with $(\alpha_i, \alpha_{i'}) = -1$ (see 1.7).

For $\eta \in L$, and $a \in N$, we define $\tilde{f}_\eta \in \mathcal{A}/U_q$ by

$$\tilde{f}_\eta^{(a)} = \begin{cases} f_i^{(a)}f_{i'}^{(2a)}f_i^{(a)} & \text{if } \delta_\eta = 2 \text{ and } \eta = \{i, i'\}, \\ \prod_{i \in \eta} f_i^{(a)} & \text{if } \delta_\eta = 1, \end{cases}$$

(2.5.1)

and set

$$g_\eta^{(a)} = \pi(\tilde{f}_\eta^{(a)}).$$

(2.5.2)

Note that $f_i^{(a)}f_{i'}^{(2a)}f_i^{(a)} = f_i^{(a)}f_{i'}^{(2a)}f_i^{(a)}$ if $\delta_\eta = 2$, and $\tilde{f}_\eta^{(a)}$ does not depend on the order of the product if $\delta_\eta = 1$.

We prove the following.

**Theorem 2.6.** Assume that $X$ is irreducible of finite or affine type, and $\sigma$ is not admissible. We exclude the cases $(A_n2')$, $(A_n3')$, $(A_n4')$, $(A_n5)$ in 1.8. Then the correspondence $f_\eta^{(a)} \mapsto g_\eta^{(a)}$ gives rise to an isomorphism $\Phi : \mathcal{A}/U_q \cong V_q$.

**Remark 2.7.** The theorem does not hold for the case $(A_n5)$ since $V_q = \{0\}$ by Remark 1.9. It is likely that the theorem holds for the cases $(A_n2') \sim (A_n4')$. Actually, in those three cases, the verification of the fact that $\Phi : \mathcal{A}/U_q \to V_q$ is a homomorphism (Proposition 2.8) becomes more complicated, and it is not yet done. If we can find a combinatorial discussion as in [MSZ, 5], this might be possible.

The first step towards the proof of the theorem is the following.
Proposition 2.8. Under the assumption of Theorem 2.6, the assignment $f^{(a)}_i \mapsto g^{(a)}_i$ gives rise to a homomorphism $\Phi : A[U^-] \to V_q$.

2.9. Proposition 2.8 will be proved in Section 4. Here, assuming Proposition 2.8, we continue the discussion. The algebra homomorphism $r : U^- \to U^- \otimes U^-$ is defined by $r(f_i) = f_i \otimes 1 + 1 \otimes f_i$, with respect to the twisted algebra structure of $U^\otimes \otimes U^-$. The symmetric bilinear form $(\ ,\ )$ on $U^-$ is defined by using the property of $r$. The bilinear form $(\ ,\ )$ is non-degenerate. A similar result also hold for $U^-_q$ (see [MSZ, 1.3]). Let $F(q)$ be the field of rational functions of $q$ with coefficients in $F$, which contains $A' = F[q, q^{-1}]$. Set $F(q)V_q = F(q) \otimes A' V_q$. Then by [MSZ, 3.6], the bilinear form $(\ ,\ )$ on $U^-_q$ induces a non-degenerate symmetric bilinear form $(\ ,\ )$ on $F(q)V_q$. (Note that the discussion there holds even if $\sigma$ is not admissible). On the other hand, the bilinear form $(\ ,\ )$ on $U^-_q$ induces a symmetric bilinear form on $F(q)U^-_q$. This bilinear form is non-degenerate by [MSZ, Prop. 3.7]. We have the following proposition.

Proposition 2.10.  
(i) For any $x, y \in A[U^-_q]$, we have $(\Phi(x), \Phi(y)) = (x, y)$.
(ii) The map $\Phi : A[U^-_q] \to V_q$ is injective.

Proof. The proposition is proved in a similar way as in [SZ2, Prop. 2.8], [MSZ, Prop. 3.8]. The map $\Phi$ can be extended to a homomorphism $\tilde{\Phi} : F(q)U^-_q \to F(q)V_q$. In order to prove (ii), it is enough to show that $\tilde{\Phi}$ is injective. This follows from (i) since the bilinear form on $F(q)U^-_q$ is non-degenerate. We prove (i). The $A$-submodule $J_1$ of $(A[U^-] \otimes A[U^-])^\sigma$ is defined as in the proof of Proposition 3.8 in [MSZ]. Note that in our case, $\varepsilon = 2$ or 4, $J_1$ coincides with the $A$-submodule generated by $x + \sigma(x)$ such that $\sigma^2(x) = x, \sigma(x) \neq x$. We note the formula, for $\eta \in I$.

\[(2.10.1)\quad r(\tilde{f}_\eta) \equiv \tilde{f}_\eta \otimes 1 + 1 \otimes \tilde{f}_\eta \mod J_1.\]

In fact, (2.10.1) was proved in [MSZ] if $\delta_\eta = 1$. We verify this for $\eta$ such that $\delta_\eta = 2$. Assume that $\eta = \{i, i'\}$ with $(\alpha_i, \alpha_{i'}) = -1$. We have

\[(2.10.2)\quad r(f_i f^{(2)}_{i'} f_i) = [2]^{-1}(f_i \otimes 1 + 1 \otimes f_i)(f_{i'} \otimes 1 + 1 \otimes f_{i'})^2(f_i \otimes 1 + 1 \otimes f_i)\]

\[= f_i f^{(2)}_{i'} f_i \otimes 1 + 1 \otimes f_i f^{(2)}_{i'} f_i + Z,\]

where

\[Z = q^{-1}(f_i f_{i'} \otimes f_{i'} f_i + f_{i'} f_i \otimes f_i f_{i'})\]

\[+ q^2(f_i f^{(2)}_{i'} \otimes f^{(2)}_{i'} + f^{(2)}_{i'} \otimes f^{(2)}_i)\]

\[+ f_i \otimes f_i f^{(2)}_{i'} + f_{i'} \otimes f^{(2)}_{i'} f_i + f_{i'} \otimes f_i f_{i'} f_i\]

\[+ f^{(2)}_{i'} f_i \otimes f_i + f_i f^{(2)}_{i'} \otimes f_i + f_i f_{i'} f_i \otimes f_{i'}.\]
The first two terms in $Z$ are of the form $x + \sigma(x)$, thus they are contained in $J_1$. In the third term, by using the Serre relation $f_i f_i^{(2)} + f_i^{(2)} f_i = f_i f_i f_i$, we have

$$f_i \otimes f_i f_i^{(2)} + f_i \otimes f_i^{(2)} f_i = f_i \otimes f_i f_i f_i.$$ 

Thus the third term is contained in $J_1$. Similarly, the 4th term is contained in $J_1$. Hence $Z \in J_1$. and (2.10.1) holds. Then the proof of (2.8.2) and (2.8.7) in [SZ2] is done in a similar way as in the proof of Proposition 3.8 in [MSZ]. As in [SZ2], [MSZ], the proof of the proposition is reduced to the computation of $(\tilde{f}_{\eta}, \tilde{f}_{\eta'})$. If $\delta_\mu = \delta_\nu' = 1$, $(\tilde{f}_{\eta}, \tilde{f}_{\eta'})$ was computed in (3.8.1) in [MSZ]. If $\delta_\eta \neq \delta_\eta'$, $(\tilde{f}_{\eta}, \tilde{f}_{\eta'}) = 0$ since the weights are different. By the same reason, $(\tilde{f}_{\eta}, \tilde{f}_{\eta'}) = 0$ if $\delta_\eta = \delta_\eta' = 2$ and $\eta \neq \eta'$. Hence we may assume that $\eta = \eta'$ with $\delta_\eta = 2$. We have

$$(\tilde{f}_{\eta}, \tilde{f}_{\eta'}) = (f_i f_i^{(2)} f_i, f_i f_i^{(2)} f_i) = [2]^{-1} (r(f_i f_i^{(2)} f_i), f_i f_i^{(2)} \otimes f_i f_i).$$

$r(f_i f_i^{(2)} f_i)$ is computed in (2.10.2). By comparing the weights, the last term is equal to

$$q^{-1}[2]^{-1}(f_i f_i \otimes f_i f_i + f_i f_i \otimes f_i f_i + f_i f_i \otimes f_i f_i) = q^{-1}[2]^{-1}((f_i f_i, f_i f_i)(f_i f_i, f_i f_i) + (f_i f_i, f_i f_i)(f_i f_i, f_i f_i)) = q^{-1}[2]^{-1}(q^2 + 1)(f_i, f_i)(f_i, f_i)(f_i, f_i)(f_i, f_i) = (1 - q^2)^{-4}.$$ 

Thus we have

$$(\tilde{f}_{\eta}, \tilde{f}_{\eta'}) = \begin{cases} (1 - q^2)^{-4} & \text{if } \eta = \eta', \delta_\eta = 2, \\ (1 - q^2)^{-|\eta|} & \text{if } \eta = \eta', \delta_\eta = 1, \\ 0 & \text{if } \eta \neq \eta'. \end{cases}$$ 

Since $q_\eta = q^{d_\eta} = q^4$ if $\delta_\eta = 2$, and $q_\eta = q^{|\eta|}$ if $\delta_\eta = 1$ by (1.6.1), by comparing with

$$(\tilde{f}_{\eta}, \tilde{f}_{\eta'})_1 = \begin{cases} (1 - q^2)^{-1} & \text{if } \eta = \eta', \\ 0 & \text{if } \eta \neq \eta'. \end{cases}$$

we obtain the result corresponding to (2.8.3) in [SZ2]. Now the assertion (i) follows from (2.8.3) as in [SZ2]. The proposition is proved.

\[\Box\]

2.11. For the surjectivity of $\Phi$, the discussion in [SZ2] or [MSZ] cannot be applied directly. The main ingredient for the proof is the modified PBW-bases. In the case where $X$ is of finite type, $\sigma$ acts on the set of modified PBW-bases as a permutation. In this case, the surjectivity will be proved in Theorem 3.7, in a similar way as in [SZ1]. While in the case where $X$ is of affine type, it is not certain whether $\sigma$ acts on the set of modified PBW-bases. In this case, we prove the surjectivity in
Proposition 6.21, by applying the properties of Kashiwara operators. The discussion is similar to [SZ2]. However, in contrast to [SZ2], we construct Kashiwara operators by making use of modified PBW-bases.

3. Modified PBW-basis - finite case

3.1. In this section, we assume that X is of finite type. Let W (resp. \(\overline{W}\)) be the Weyl group of X (resp. of \(\overline{X}\)). Let \(\mathbf{h}\) be a sequence of \(I\) with respect to \(W\). Then there exists a sequence \(\mathbf{h}\) of \(I\) obtained from \(\mathbf{h}\), unique up to the permutation of \(i\) and \(j\) such that \((\alpha_i, \alpha_j) = 0\) (see [SZ1, 1.6]). Let \(\mathcal{P}_h\) be the PBW-basis of \(U_q^\omega\) and \(\mathcal{P}_h^\sigma\) the PBW-basis of \(U_q^\omega\) as in 1.10. It is shown in [SZ1, Lemma 1.8], in the case where \(\sigma\) is admissible, that \(\sigma\) induces a permutation on the set \(\mathcal{P}_h\), and the set of \(\sigma\)-invariant PBW-basis \(\mathcal{P}_h^\sigma\) corresponds bijectively to the set \(\mathcal{P}_h\). If \(\sigma\) is not admissible, this does not hold, since for \(X = A_2\), there does not exist \(\sigma\)-invariant PBW-basis (Lemma 2.2). In order to extend Lemma 1.8 in [SZ1] to the non-admissible case, we will modify the definition of PBW-basis.

3.2. We recall the definition of PBW-bases in [SZ1]. Let \(h = (i_1, \ldots, i_p)\) be a sequence of \(I\) (called the reduced sequence) corresponding to a reduced expression \(w_0 = s_{i_1} \cdots s_{i_N}\) of \(w_0 \in W\), where \(N = l(w_0)\). Let \(\Delta \subset Q\) be the set of roots, and \(\Delta^+ \subset Q_+\) the set of positive roots. Then \(h\) gives a total order \(\{\beta_1, \ldots, \beta_N\}\) of \(\Delta^+\), where \(\beta_i\) is defined by \(\beta_{i_k} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\). We define a root vector \(f_{\beta_k}^{(c)}\) corresponding to \(\beta_k \in \Delta^+\) and \(c \in \mathbb{N}\) by

\[
(3.2.1) \quad f_{\beta_k}^{(c)} = T_{i_1} \cdots T_{i_{k-1}}(f_{\beta_k}^{(c)}).
\]

Note that \(\beta_{i_1} = \alpha_{i_1}\) and so \(f_{\beta_{i_1}}^{(c)} = f_{i_1}^{(c)}\). For \(c = (c_1, \ldots, c_N) \in \mathbb{N}^N\), set \(L(c, h) = f_{\beta_1}^{(c_1)} \cdots f_{\beta_N}^{(c_N)}\). Then \(\mathcal{P}_h = \{L(c, h) \mid c \in \mathbb{N}^N\}\) gives a PBW-basis of \(U_q^\omega\) associated to \(h\). For a sequence \(\mathbf{h}\) of \(I\), the PBW-basis \(\mathcal{P}_h\) of \(U_q^\omega\) is defined similarly.

Let \(\sigma : I \to I\) be the diagram automorphism. Here we assume that \(\sigma\) is not necessarily admissible. \(\sigma\) induces an automorphism \(\sigma : W \to \overline{W}\). Let \(\overline{W}\) be the Weyl group associated to \(\overline{I}\). For each \(\eta \in \overline{I}\), we denote by \(t_\eta \in W\) the longest element in the Weyl subgroup \(W_\eta\) of \(W\) generated by \(\{s_i \mid i \in \eta\}\). Then \(t_\eta \in W^\sigma\), and the correspondence \(s_\eta \mapsto t_\eta\) gives an isomorphism \(W \cong W^\sigma\).

Let \(h = (\eta_1, \ldots, \eta_N)\) be a reduced sequence of \(\overline{I}\), where \(N = l(w_0)\). For each \(\eta \in \overline{I}\), we denote by \(h_\eta\) a reduced sequence of \(W_\eta\). Hence \(h_\eta = (i_1, \ldots, i_{|\eta|})\) for \(\eta = \{i_1, \ldots, i_{|\eta|}\}\) if \(|\eta| = 1\), and \(h_\eta = (i, j, i)\) or \((j, i, j)\) if \(\eta = \{i, j\}\) with \(|\eta| = 2\). Then by a juxtaposition of \(h_\eta\) for each \(\eta \in \overline{I}\), we obtain a reduced sequence \(h\) of \(I\), namely,

\[
(3.2.2) \quad h = (i_{\underbrace{1, \ldots, i_a, i_{a+1}, \ldots, i_b, i_{b+1}, \ldots}})\]

For each \( \eta \in \mathcal{L} \), we define \( \tilde{T}_\eta \) by
\[
(3.2.3) \quad \tilde{T}_\eta = \begin{cases} 
\prod_{i \in \eta} T_i & \text{if } \delta_\eta = 1, \\
T_i T_j T_i = T_j T_i T_j & \text{if } \delta_\eta = 2 \text{ with } \eta = \{i, j\}.
\end{cases}
\]

Then \( \tilde{T}_\eta \) does not depend on the choice of the order in \( \eta \). Since \( \sigma \circ T_i \circ \sigma^{-1} = T_{\sigma(i)} \), \( \tilde{T}_\eta \) commutes with \( \sigma \).

Now assume that \( \sigma \) is admissible. For each \( h \) and \( c \in \mathbb{N}^N \), set
\[
(3.2.4) \quad \tilde{f}(c_k)_{\eta_k} = \prod_{i \in \eta_k} f_{\beta_i}^{(c_i)},
\]
where \( c_k \) is a subsequence of \( c \) consisting of \( c_\ell \) such that \( i_\ell \in \eta_k \). Thus we have \( L(c, h) = \tilde{f}(c_k)_{\eta_k} \cdots \tilde{f}(c_1)_{\eta_1} \). Note that \( \tilde{f}(c_k)_{\eta_k} \) can be expressed as
\[
(3.2.5) \quad \tilde{f}(c_k)_{\eta_k} = \tilde{T}_{\eta_1} \cdots \tilde{T}_{\eta_{k-1}} (\prod_{i \in \eta_k} f_{i_\ell}^{(c_i)}).
\]
It follows from (3.2.5) that
\[
\sigma(\tilde{f}(c_k)_{\eta_k}) = \tilde{T}_{\eta_1} \cdots \tilde{T}_{\eta_{k-1}} (\prod_{i \in \eta_k} f_{\sigma(i)}^{(c_i)}).
\]
This implies that \( \sigma(L(c, h)) = L(\sigma(c), h) \), where the action of \( \sigma \) on \( c \in \mathbb{N}^N \) is given, for each \( c_k = (a_1, \ldots, a_{m-1}, a_m) \) by \( \sigma(c_k) = (a_m, a_1, \ldots, a_{m-1}) \).

For each \( c = (c_1, \ldots, c_N) \in \mathbb{N}^N \), define \( c \in \mathbb{N}^N \) by
\[
(3.2.6) \quad c = (c_1, \ldots, c_1, c_2, \ldots, c_2, c_3, \ldots),
\]
where \( \underbrace{\eta_1}_{\eta_2} \). Then \( L(c, h) \) is \( \sigma \)-invariant. Summing up the above discussion we have

**Lemma 3.3** ([SZ1, Lemma 1.8]). Assume that \( \sigma \) is admissible. Then \( \sigma \) acts on \( \mathcal{X}_h \), as a permutation, \( \sigma : L(c, h) \mapsto L(\sigma(c), h) \). Let \( \mathcal{X}_h^\sigma \) be the set of \( \sigma \)-invariant PBW-basis of \( U_q^{-} \). For each \( c \in \mathbb{N}^N \), define \( c \in \mathbb{N}^N \) as in (3.2.6). Then \( L(c, h) \mapsto L(c, h) \) gives a bijection \( \mathcal{X}_h \sim \mathcal{X}_h^\sigma \).

### 3.4
We now assume that \( \sigma \) is not admissible. We modify the definition of \( \tilde{f}(c_k)_{\eta_k} \) as follows. Assume that \( \delta_\eta = 2 \) with \( \eta = \{i, j\} \). Let \( h_\eta = (i, j, i) \) or \( (j, i, j) \) be a reduced sequence associated to \( W_\eta \). Let \( \mathcal{X}_{h_\eta} \) be the set of PBW-basis of the subalgebra \( L_\eta^{-} \) of \( U_q^{-} \) isomorphic to \( U_q(s_k)^{-} \). For each PBW-basis \( L(c, h_\eta) \in \mathcal{X}_{h_\eta} \) in \( L_\eta^{-} \), the canonical basis \( b(c, h_\eta) \in L_\eta^{-} \) can be assigned, and \( B_\eta = \{b(c, h_\eta) \mid c \in \mathbb{N}^3\} \) gives the canonical basis of \( L_\eta^{-} \). We define \( \tilde{f}(c_k)_{\eta_k} \) by
\[
(3.4.1) \quad \tilde{f}(c_k)_{\eta_k} = \tilde{T}_{\eta_1} \cdots \tilde{T}_{\eta_{k-1}} (b(c_k, h_{\eta_k})).
\]
In the case where \( \delta_\eta = 1 \), define \( \tilde{f}^{(c_k)}_{\eta_{\ell}} \) by using (3.2.5). For any \( c \in N^n \), define

\[
L^\natural(c, h) = \tilde{f}^{(c_1)}_{\eta_1} \ldots \tilde{f}^{(c_k)}_{\eta_k},
\]

and set \( \mathcal{L}^\natural_h = \{ L^\natural(c, h) \mid c \in N^n \} \). In the case where \( \delta_\eta = 2 \), \( \sigma \) leaves \( L^\natural_\eta \) invariant, and \( \sigma \) permutes the canonical basis \( B_\eta \). It follows that \( \sigma(\tilde{f}^{(c_k)}_{\eta_{\ell}}) = \tilde{f}^{(c'_k)}_{\eta_{\ell}} \), where \( c'_k \) is determined by the condition that \( \sigma(b(c_k, h_{\eta_k})) = b(c'_k, h_{\eta_k}) \). From this, we see that \( \sigma \) acts on \( \mathcal{L}^\natural_h \) as a permutation. Note that, for a given \( c \), the definition of \( c \) in (3.2.6) makes sense even for the non-admissible case (under the assignment \( c \mapsto (c, c, c) \) for the case \( \delta_\eta = 2 \)), and one can define \( L^\natural(c, h) \).

The following result is a generalization of Lemma 3.3 to the case where \( \sigma \) is non-admissible.

**Lemma 3.5.** Assume that \( \sigma \) is not necessarily admissible. Then \( \mathcal{L}^\natural_h \) is a \( Q(q) \)-basis of \( U_q^\natural \), and an \( A \)-basis of \( AU_q^\natural \). \( \sigma \) acts as a permutation on \( \mathcal{L}^\natural_h \). The correspondence \( L(c, h) \mapsto L^\natural(c, h) \) gives a bijection \( \mathcal{L}^\natural_h \cong (\mathcal{L}^\natural_h)^\sigma \).

**Proof.** Assume that \( \eta_k = \{ i, j \} \) with \( \delta_{\eta_k} = 2 \), and take \( h_{\eta_k} = (i, j, i) \). Then \( b(c_k, h_{\eta_k}) \) is written as \( b(c_k, h_{\eta_k}) = L(c_k, h_{\eta_k}) + z \), where \( z \) is a \( qZ[q] \)-linear combination of PBW-bases \( L(c_k', h_{\eta_k}), \) with \( c'_k > c_k \) (the lexicographic order on \( N^3 \) from the left).

Furthermore, \( L(c_k', h_{\eta_k}) = f^{(c_k')}_{1,i} T_i f^{(c_k')}_{2,j} T_j f^{(c_k')}_{3,i} \) for \( c' = (c_1', c_2', c_3') \). It follows that

\[
\tilde{T}_{\eta_{\ell}} \ldots \tilde{T}_{\eta_{\ell-1}}(L(c_k', h_{\eta_k})) = T_{i_1} \ldots T_{i_{\ell-3}} (f^{(c_k')}_{i_{\ell-2}} T_{i_{\ell-2}} (f^{(c_k')}_{i_{\ell-1}} T_{i_{\ell-1}} (f^{(c_k')}_{i_{\ell}})))
\]

\[
= f^{(c_k')}_{\beta_{i_{\ell-2}}} f^{(c_k')}_{\beta_{i_{\ell-1}}} f^{(c_k')}_{\beta_{i_{\ell}}},
\]

where we write \( h_{\eta_k} = (i_{\ell-2}, i_{\ell-1}, i_{\ell}) \) as a subsequence of \( h \). Thus \( \tilde{f}^{(c_k)}_{\eta_{\ell}} \) is a linear combination of \( f^{(c_k')}_{\beta_{i_{\ell-2}}} f^{(c_k')}_{\beta_{i_{\ell-1}}} f^{(c_k')}_{\beta_{i_{\ell}}} \) for various \( c'_1, c'_2, c'_3 \). If \( \eta_k \) is such that \( \delta_{\eta_k} = 1 \), \( f^{(c_k)}_{\eta_{\ell}} \) is the same as in 3.2. Hence \( L^\natural(c, h) \) can be written as \( L^\natural(c, h) = L(c, h) + z' \), where \( z' \) is a \( qZ[q] \)-linear combination of PBW-bases \( L(c', h) \) with \( c' > c \), the lexicographic order on \( N^3 \) from the left. Here the transition matrix between \( \mathcal{L}_h \) and \( \mathcal{L}^\natural_h \) is expressed as a triangular matrix where the diagonal entries are 1, and off diagonals are contained in \( qZ[q] \). It follows that \( \mathcal{L}^\natural_h \) gives an \( A \)-basis of \( AU_q^\natural \), and an \( Q(q) \)-basis of \( U_q^- \). We know already that \( \sigma \) acts on \( \mathcal{L}^\natural_h \) as a permutation. From the discussion in 3.4, for \( \eta_k \) such that \( \delta_{\eta_k} = 2 \), \( \tilde{f}^{(c_k)}_{\eta_{\ell}} \) is \( \sigma \)-invariant if and only if \( b(c_k, h_{\eta_k}) \) is \( \sigma \)-invariant. By Lemma 2.2, this condition is given by \( c_k = (c, c, c) \) for some \( c \in N \). Hence the condition that \( L^\natural(c, h) \) is \( \sigma \)-invariant is given by \( c \in N^3 \) obtained from \( c \) as in (3.2.6). The lemma is proved.

### 3.6. \( \mathcal{L}^\natural_h = \{ L^\natural(c, h) \} \) is called the modified PBW-basis associated to \( h \). Concerning the relation with the canonical basis, the modified PBW-basis satisfies a
similar property as the PBW-basis. The proof is straightforward from the construction in 3.4. Let \(b(c, h)\) be the canonical basis corresponding to \(L(c, h)\). Then

\[
(3.6.1) \quad b(c, h) = L^2(c, h) + \sum_{d < c} a_d L^2(d, h),
\]

where \(a_d \in q \mathbb{Z}[q]\). Note that the canonical bases are characterized by (3.6.1) and by the property \(\overline{b(c, h)} = b(c, h)\).

We consider the map \(\pi : A' U^{-\sigma} \to V_q\). The image of the \(A'\)-basis of \(V_q\) gives an \(A'\)-basis of \(V_q\), which we denote by the same symbol. For \(L(c, h) \in \mathcal{P}_h\), we denote by \(b(c, h)\) the corresponding canonical basis in \(U_q^-\).

In the case where \(X\) is of finite type, we can prove Theorem 2.6, assuming Proposition 2.8, in a similar way as in [SZ1]. The following result is a generalization of [SZ1, Thm. 1.14] to the non-admissible case.

**Theorem 3.7.** Assume that \((X, \chi) = (A_{2n}, C_n)\), i.e. \((F_n1)\) in 1.8. Let \(L(c, h) \mapsto L^2(c, h)\) be the bijection \(\mathcal{P}_h \simeq (\mathcal{P}_h^\sigma)\) given in Lemma 3.5. Let \(\Phi : A' U^- \to V_q\) be the homomorphism given in Proposition 2.8. Then

(i) The map \(\Phi\) gives an isomorphism \(A' U^- \simeq V_q\).

(ii) \(\Phi(L(c, h)) = L^2(c, h)\).

(iii) \(\Phi(b(c, h)) = b(c, h)\).

**Proof.** We know already that \(\Phi\) is injective. The surjectivity of \(\Phi\) will follow from (ii). Thus (i) follows from (ii). We prove (ii). By (3.4.1) and (3.4.2), in order to prove (ii), it is enough to show that

\[
(3.7.1) \quad \Phi(T_{\eta_1} \cdots T_{\eta_{k-1}}(f_{\eta_k})) = \pi(\tilde{T}_{\eta_1} \cdots \tilde{T}_{\eta_{k-1}}(\tilde{f}_{\eta_k})),
\]

where \(T_{\eta}\) denotes the braid group action on \(U_q^-\). By a similar argument as in the proof of Theorem 1.14 in [SZ1], the proof of (3.7.1) is reduced to the case where the rank of \(X\) is 2. Thus we may assume that \(X\) is of type \(A_2\) and \(X\) is of type \(C_2\). We use the notation \(I = \{1, 1', 2, 2'\}\) with \(\sigma : 1 \leftrightarrow 1', 2 \leftrightarrow 2'\), \(\bar{I} = \{1, 2\}\). By considering the \(*\)-action on \(U_q^-\) and \(U_q^-\), we may assume that \(h = (1, 2, 1, 2)\) (see [SZ1, 4.1]). It is enough to check the following equalities,

(a) \(\Phi(f_1) = \pi(\tilde{f}_1)\),

(b) \(\Phi(T_1(f_2)) = \pi(\tilde{T}_1(\tilde{f}_2))\),

(c) \(\Phi(T_1 T_2(f_3)) = \pi(\tilde{T}_1 \tilde{T}_2(\tilde{f}_3))\),

(d) \(\phi(T_1 T_2 T_1(f_3)) = \pi(\tilde{T}_1 \tilde{T}_2 \tilde{T}_1(\tilde{f}_3))\).

In the discussion below, we use the notation in Section 4 for the description of PBW-basis. Since \(T_1 T_2 T_1(f_3) = f_3, \tilde{T}_1 \tilde{T}_2 \tilde{T}_1(\tilde{f}_3) = \tilde{f}_3\), (a) and (d) follows from the definition of \(\Phi\). We verify (b) and (c). First we show (c). It is easy to see that
\[ T_1 T_2 (f_\perp) = f_\perp f_\perp - q^4 f_\perp f_2. \] Hence we have
\[ \Phi(T_1 T_2 (f_\perp)) = \pi(f_2 f_2^{(2)} f_2 f_1 f_1' - q^4 f_1 f_1' f_2 f_2^{(2)} f_2). \]

On the other hand, \( \tilde{T}_1 \tilde{T}_2 (f_\perp) = T_1 T_1' T_2 T_2' (f_1 f_1') = f_{1'22'} f_{122'} \). By a similar computation as in Section 4, one can check that
\[ f_2 f_2^{(2)} f_2 f_1 f_1' \equiv q^4 f_1 f_1' f_2 f_2^{(2)} f_2 + f_{1'22'} f_{122'} \mod J. \]
Hence (c) holds. For (b), we have \( T_1 (f_\perp) = f_\perp f_\perp^{(2)} - q^2 f_\perp f_\perp f_\perp + q^4 f_\perp^{(2)} f_\perp. \) Hence
\begin{equation}
\Phi(T_1 (f_\perp)) = \pi(f_2 f_2^{(2)} f_2 f_1 f_1' f_1^{(2)} - q^2 f_1 f_1' f_2 f_2^{(2)} f_2 f_1 f_1' + q^4 f_1 f_1' f_2 f_2^{(2)} f_2). \tag{3.7.2}
\end{equation}

On the other hand, we have \( T_1 T_1' (f_2 f_2^{(2)} f_2) = f_{12} f_{1'2} f_{12}. \) By a direct computation, we have
\begin{align*}
f_2 f_2^{(2)} f_2 f_1 f_1' f_1^{(2)} &\equiv q^8 f_1 f_1' f_2 f_2^{(2)} f_2 + f_{12} f_{1'2} f_{12} + q^2 f_1 f_1' f_{1'22'} f_{122'} \mod J, \\
f_1 f_1 f_2 f_2^{(2)} f_2 f_1 f_1' &\equiv f_1 f_1 f_{1'22'} f_{122'} + q^4 (2) f_1 f_1' f_2 f_2^{(2)} f_2 \mod J.
\end{align*}
Substituting this into the right hand side of (3.7.2), we obtain the equality (b). Thus (3.7.1) is verified, and (ii) follows.

Since the bar-involution is compatible with the map \( \Phi_1 \), (iii) follows from (ii) by using (3.6.1) and the corresponding formula for \( U_{\tau}^- \). The theorem is proved. \( \square \)

4. The proof of Proposition 2.8

4.1. We assume that \( X \) is irreducible, \( \sigma \) is not admissible, and that \( (X, \bar{X}) \) satisfies the condition in Theorem 2.6. For the proof of Proposition 2.8, we need to show, for \( \eta, \eta' \in \mathcal{L} \), that
\begin{align}
\sum_{k=0}^{1-a_{\eta \eta'}} (-1)^k g_\eta^{(k)} g_{\eta'}^{(1-a_{\eta \eta'}-k)} &= 0, \quad (\eta \neq \eta'), \\
[a]_{\eta \eta'} g_\eta^{(a)} &= g_{\eta}^{a}, \quad (a \in \mathbb{N}).
\end{align}

(4.1.2) is known if \( \delta_\eta = 1 \) (see [SZ1, 3.1]), and it was proved in Proposition 2.4 if \( \delta_\eta = 2 \). If \( \delta_\eta = \delta_{\eta'} = 1 \), (4.1.1) was already verified in [SZ1, 3.2]. Thus from the tables in 1.8, for the proof of (4.1.1), it is enough to consider the case where \( X = A_4, \bar{X} = C_2 \) (note that we have excluded the cases \( (A_n, 2') \sim (A_n, 4') \) and \( (A_n, 5) \) in 1.8). We write \( I = \{1, 2, 2', 1'\} \) with \( \sigma : 1 \leftrightarrow 1', 2 \leftrightarrow 2' \). Set \( \mathcal{L} = \{1, 2\} \) with \( \delta_1 = 1, \delta_2 = 2 \). We have \( (\alpha_1, \alpha_1) = 4, (\alpha_2, \alpha_2) = 8 \). The Cartan matrix \( \mathbf{A} \) is given
Correspondingly, we have a total order of $\Delta$. Then

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$ 

Thus we have only to prove the following two formulas.

\begin{align*}
(4.1.3) & \quad g_1g_2^{(2)} - g_2g_1g_2 + g_2^{(2)}g_1 = 0, \\
(4.1.4) & \quad g_2g_1^{(3)} - g_1g_2g_1^{(2)} + g_1^{(2)}g_2g_1 - g_1^{(3)}g_2 = 0.
\end{align*}

4.2. Let $W = S_5$ be the Weyl group of type $A_4$, and $w_0$ the longest element in $W$. Then $l(w_0) = 10$. We consider the following reduced sequence $h$ of $I$.

$$h = (i_1, i_2, \ldots, i_{10}) = (1, 1', 2, 2', 2, 1, 1', 2, 2', 2).$$

Correspondingly, we have a total order of $\Delta^+$ as follows;

$$\Delta^+ = \{ \beta_1, \ldots, \beta_{10} \} = \{ 1, 1', 12, 11'22', 1'2', 1'2', 122', 2, 22', 2' \},$$

where $\beta_k = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$. We define $f^{(c)}_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f^{(c)}_{i_k})$, and set $L(c, h) = f^{(c_1)}_{\beta_1} \cdots f^{(c_{10})}_{\beta_{10}}$ for $c = (c_1, \ldots, c_{10}) \in \mathbb{N}^{10}$. Then $\{ L(c, h) | c \in \mathbb{N}^{10} \}$ gives a PBW-basis of $U_q^{-} A$, $f_{\beta_1}, \ldots, f_{\beta_{10}}$ are written explicitly as follows;

\begin{align*}
 f_{\beta_1} &= f_1, \quad f_{\beta_2} = f_1', \quad f_{\beta_{10}} = f_2, \quad f_{\beta_8} = f_2, \quad f_{\beta_3} = f_2f_1 - qf_1f_2, \\
 f_{\beta_5} &= f_2f_1 - qf_1f_2', \quad f_{\beta_6} = f_2f_2 - qf_2f_2', \quad f_{\beta_4} = f_2f_1 - qf_1f_2, \\
 f_{\beta_8} &= f_2f_1' - qf_1f_2', \quad f_{\beta_9} = f_2f_1' - qf_1f_2, \quad f_{\beta_7} = f_2f_2 - qf_2f_2'.
\end{align*}

(4.1.3) and (4.1.4) are equivalent to the formulas

\begin{align*}
(4.2.1) & \quad (f_1f_1')f_2^{(2)}f_2'^{(2)}f_2' - f_2f_2'^{(2)}f_2f_1f_1'f_2f_2'^{(2)}f_2 + f_2^{(2)}f_2'^{(2)}f_2f_1f_1'f_2^{(2)}(f_1f_1') \equiv 0 \mod J, \\
(4.2.2) & \quad (f_2f_2^{(2)}f_2)f_1^{(3)}f_1'^{(2)}f_1' - f_1f_1'(f_2f_2^{(2)}f_2)f_1^{(2)}f_1'f_1'^{(2)} \\
 & \quad + f_1^{(2)}f_1'^{(2)}(f_2f_2^{(2)}f_2)f_1f_1'f_1 - f_1^{(3)}f_1'^{(3)}(f_2f_2^{(2)}f_2) \equiv 0 \mod J.
\end{align*}

We shall prove (4.2.1) and (4.2.2) by expressing them in terms of modified PBW-bases. We compute them by making use of various commutation relations for $f_{\beta_k}$.

4.3. We prove (4.2.2). First we compute

\begin{align*}
f_1^{(2)}f_1'^{(2)}(f_2f_2^{(2)}f_2)f_1f_1' \\
&= q^2[3](f_1^{(2)}f_1'^{(3)}f_1f_2f_2'f_2 + f_1^{(3)}f_1'^{(2)}f_1f_1'f_1f_2f_2) \\
&\quad + q(f_1^{(2)}f_1'^{(2)}f_1f_2f_2'f_2' + f_1^{(2)}f_1'^{(2)}f_1f_1'f_1f_2f_2)
\end{align*}
\[ + q(f^{(2)}_1 f^{(2)}_v f_{12} f_2 f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'2} f_2 f_2) \\
+ q^3[3](f^{(2)}_1 f^{(3)}_v f_{12} f_{2} f_2 + f^{(2)}_1 f^{(3)}_v f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(3)}_v f_{12} f_{12'} f_2 f_2) \\
+ q^4[3] f^{(2)}_1 f^{(3)}_v f_{2} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2. \]

The first three terms in the right hand side are of the form \(x + \sigma(x)\), hence contained in \(J\). Using the relation \(f^{(2)}_2 f_{2'} - f_{2} f_{2'} f_2 + f_{2'} f^{(2)}_2 = 0\), the fourth term can be rewritten as \(q^3[3](f^{(2)}_1 f^{(3)}_v f_{12} f_{2} f_2 + f^{(3)}_1 f^{(2)}_v f_{12} f_{2} f_2 f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2)\), hence again contained in \(J\). It follows that

\[ f^{(2)}_1 f^{(3)}_v f_{2} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 = \mod J. \tag{4.3.1} \]

Next we compute

\[ f_2 f^{(2)}_2 f_2 f^{(3)}_1 f^{(3)}_v \]
\[ = q^6(f^{(3)}_1 f_{12} f_{12'} f_2 + f^{(3)}_1 f_{12} f_{12} f_{12'} f_2) + q^8[2](f^{(3)}_1 f_{12} f^{(2)}_2 + f^{(3)}_1 f_{12} f^{(2)}_2 f_2) \\
+ q^8(f^{(2)}_1 f^{(3)}_v f_{12} f_{2} f_2 + f^{(2)}_1 f^{(3)}_v f_{12} f_{12'} f_2) \\
+ q^5(f^{(2)}_1 f^{(2)}_v f_{12} f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2) \\
+ q^2(f^{(2)}_1 f^{(2)}_v f_{12} f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2) \\
+ q^5(f^{(2)}_1 f^{(2)}_v f_{12} f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2) \\
+ q^9(f^{(2)}_1 f^{(2)}_v f_{12} f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2) \\
+ q^3(f^{(2)}_1 f^{(2)}_v f_{12} f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2 + f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 f_2) \\
+ q^{12} f^{(3)}_1 f^{(3)}_v f_{2} f_2 + q^4 f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 \mod J. \tag{4.3.2} \]

The first 6 terms in the right hand side are of the form \(x + \sigma(x)\), hence are contained in \(J\). The 7th term is contained in \(J\) by the same reason as the previous computation. By applying \(T_1 T_v\) to the relation \(f_2 f_{2'} f_2 = f_{2} f^{(2)}_2 + f^{(2)}_2 f_{2'}\), we obtain \(f^{(2)}_v f_{12} f^{(2)}_v = f^{(2)}_v f_{12} f^{(2)}_v\). Thus, by a similar reason, the 8th term is also contained in \(J\). It follows that

\[ f_2 f^{(2)}_2 f_2 f^{(3)}_1 f^{(3)}_v \equiv q^{12} f^{(3)}_1 f^{(3)}_v f_{2} f_2 + q^4 f^{(2)}_1 f^{(2)}_v f_{12} f_{12'} f_2 + f^{(2)}_1 f^{(3)}_v f_{12} f_{12'} f_2 \mod J. \]

Finally we compute

\[ f_1 f^{(2)}_v (f^{(2)}_2 f^{(2)}_2 f_2) f^{(2)}_1 f^{(2)}_v \]
= q^3[2]^2(f_1^{(2)}f_{1^V}^{(2)}f_{12}f_{1V'22'}f_2 + f_1^{(2)}f_{1^V}^{(2)}f_{1122'}f_2) \\
+ q^5[2][3](f_1^{(2)}f_{1^V}^{(3)}f_{122}f_{12}f_{2^2} + f_1^{(3)}f_{1^V}^{(2)}f_{1V'22'}f_2) \\
+ q^3[2]^2(f_1^{(2)}f_{1^V}^{(2)}f_{12}f_{1V'22'}f_2 + f_1^{(2)}f_{1^V}^{(2)}f_{1V'22'}f_1f_2f_2) \\
+ q^4[3](f_1f_{1^V}^{(3)}f_{12}f_{122}f_2 + f_1^{(3)}f_{1^V}^{(2)}f_{1V'22'}f_2) \\
+ q^6[2][3](f_1f_{1^V}^{(3)}f_{12}f_{2^2} + f_1^{(3)}f_{1^V}^{(2)}f_{1V'22'}f_2) \\
+ q^2[2](f_1f_{1^V}^{(2)}f_{12}f_{1V'22'}f_2 + f_1^{(2)}f_{1^V}^{(2)}f_{1122'}f_2 + f_1f_{1^V}^{(2)}f_{12}f_{1V'22'}f_2) \\
+ q^8[3]f_1^{(3)}f_{1^V}^{(3)}f_{12}f_{2^2} + q^2[2]^2f_1^{(2)}f_{1^V}^{(2)}f_{1V'22'}f_{122} + f_1f_{1^V}^{(2)}f_{12}f_{12}f_{2^2}^{(2)}f_{12}.

The first 6 terms in the right hand side are of the form \( x + \sigma(x) \), hence are contained in \( J \). The 7th and 8th terms are contained in \( J \) by the same reason as before. It follows that

\[
(4.3.3) \quad f_1f_{1^V}(f_2f_{2^2}^{(2)}f_2)f_1^{(2)}f_{1^V}^{(2)} \equiv q^8[3]^2f_1^{(3)}f_{1^V}^{(3)}f_{12}f_{2^2}^{(2)}f_2 \\
+ q^2[2]^2f_1^{(2)}f_{1^V}^{(2)}f_{1V'22'}f_{122} + f_1f_{1^V}^{(2)}f_{12}f_{1V'22'}f_{12} \mod J.
\]

Now putting together (4.3.1), (4.3.2) and (4.3.3), we see that the left hand side of (4.2.2) is equal to

\[
-F_3(q^2)f_1^{(3)}f_{1^V}^{(3)}f_{2^2}^{(2)}f_2 + F_2(q^2)f_1^{(2)}f_{1^V}^{(2)}f_{1V'22'}f_{122} - F_1(q^2)f_1f_{1^V}f_{12}f_{1V'22'}f_{12} \mod J,
\]

where, for \( a \geq 1 \), \( F_a(q) \) is a polynomial in \( q \) defined by

\[
F_a(q) = \sum_{t=0}^{a} (-1)^t q^{t(a-t)} {a \choose t}.
\]

Since \( F_a(q) \) is identically zero by [L1, 1.3.4 (a)], (4.2.2) holds.

**4.4** We prove (4.2.1). We compute

\[
f_2f_{2^2}^{(2)}f_2f_1f_{1^V} = q^2(f_1f_{1V'22'}f_2f_2 + f_{1^V}f_{122'}f_2f_2) \\
+ q(f_{12}f_{1V'}f_2f_2 + f_{1V'}f_{12}f_2f_2) \\
+ q(f_{12}f_{1V'22'}f_2 + f_{1V'}f_{122}f_2) \\
+ q^3(f_1f_{12}f_2f_{2^2}^{(2)}f_2 + f_{1V'}f_{12}f_2f_2) \\
+ q^4f_1f_{1V'}f_2f_{2^2}^{(2)}f_2 + f_{1V'22'}f_{122}.
\]
(Actually, this is a part of the computation of \( f_1^{(2)} f_{1'}^{(2)} (f_2 f_2^{(2)} f_2) f_1 f_1' \) in 4.3.) Since the first three terms are written as \( x + \sigma(x) \), they are contained in \( J \). The 4th term is contained in \( J \) by a similar discussion as in 4.3. It follows that

\[
(4.4.1) \quad f_2 f_2^{(2)} f_1 f_1' \equiv q^4 f_1 f_1' f_2 f_2^{(2)} f_2 + f_1' f_2 f_1 f_1'.
\]

On the other hand, we have

\[
|\begin{array}{l}
f_2 f_2^{(2)} f_2^{(4)} f_1 f_1' = q^3 (f_1' f_2 f_2^{(2)} f_2^{(3)} f_2 + f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2) \\
+ q^4 (f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2 + f_1 f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2)
\end{array} |
\]

\[
\equiv q^2 (f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2 + f_1 f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2)
\]

\[
+ q^4 (f_1 f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2 + f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2)
\]

\[
+ q^6 (f_1 f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2 + f_1 f_1 f_1' f_2 f_2^{(2)} f_2^{(3)} f_2)
\]

The first two terms are written as \( x + \sigma(x) \), hence they are contained in \( J \). For the third term and the 4th term, we note the relations

\[
(4.4.2) \quad f_2 f_2^{(2)} f_2^{(3)} f_2 + f_2^{(3)} f_2 = f_2^{(2)} f_2^{(2)} f_2^{(2)}
\]

\[
(4.4.3) \quad f_2 f_2^{(2)} f_2^{(4)} f_2 + f_2^{(4)} f_2 = f_2^{(2)} f_2^{(2)} f_2^{(2)}
\]

(4.4.2) and (4.4.3) are verified by using (2.1.2) and (2.1.3). Hence the third and the 4th terms are also contained in \( J \). It follows that

\[
(4.4.4) \quad f_2^{(2)} f_2^{(4)} f_2 f_1 f_1' \equiv q^4 f_1 f_1 f_2^{(2)} f_2^{(4)} f_2^{(2)} + f_1 f_1 f_2 f_2^{(2)} f_2^{(2)} f_2 \mod J.
\]

By multiplying \( f_2 f_2^{(2)} f_2 \) on (4.4.1) from the right, we have

\[
f_2 f_2^{(2)} f_2^{(2)} f_2 = q^4 f_1 f_1 f_2^{(2)} f_2^{(2)} f_2^{(2)} + f_1' f_2 f_2^{(2)} f_2
\]

\[
\equiv q^4 f_1 f_1 f_2^{(2)} f_2^{(2)} f_2^{(2)} + f_1 f_1 f_2 f_2^{(2)} f_2^{(2)} f_2 \mod J.
\]

The second identity follows from (2.4.2) for \( a = 2 \). Combining this formula with (4.4.4), we see that the left hand side of (4.2.1) is equal to

\[
F_2(q^4) f_1 f_1 f_2^{(2)} f_2^{(2)} f_2^{(2)} + F_1(q^4) f_1 f_1 f_2 f_2^{(2)} f_2^{(2)} f_2 \equiv 0 \mod J.
\]

Hence (4.2.1) holds. Proposition 2.8 is proved.
5. $\sigma$-ACTION ON ROOT SYSTEMS

5.1. In this section, we assume that $X$ is irreducible of affine type, and consider the pair $(X, X)$ for $\sigma$: admissible or non-admissible (but exclude the case $(A_n, 5)$). Let $g$ be the affine Kac-Moody algebra associated to $X$. We write $I = \{0, 1, \ldots, n\}$, and set $I_0 = I - \{0\}$, where $I_0$ corresponds to the subalgebra $g_0$ of $g$ of finite type. Recall that $Q = \bigoplus_{i \in I} Z \alpha_i$ and let $\Delta \subset Q$ be the affine root system for $g$. Since $\sigma : I \to I$, $\sigma$ acts on $Q$ by $\alpha_i \mapsto \alpha_{\sigma(i)}$. $\Delta$ is decomposed as $\Delta = \Delta^{re} \sqcup \Delta^{im}$, where $\Delta^{re}$ is the set of real roots, and $\Delta^{im} = (Z - \{0\}) \delta$ is the set of imaginary roots. $\delta$ is written as $\delta = \sum_{i \in I} a_i \alpha_i$. From the explicit table of coefficients $a_i$ (see [K]), we see $a_i$ is symmetric with respect to $\sigma$. Thus $\delta$ is stable by $\sigma$. On the other hand, $\Delta^{re} = W \Pi$, where $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots. Since $\Pi$ is $\sigma$-stable, and $W$ is $\sigma$-stable, we see that $\Delta^{re}$ is $\sigma$-stable. Hence the action of $\sigma$ on $Q$ leaves $\Delta$ invariant. Let $\Delta_0$ be the subsystem of $\Delta$ corresponding to $I_0$.

5.2. Let $\eta_0 \subset I$ be the $\sigma$-orbit containing 0. Set $I_0' = I - \eta_0$ and let $\Delta_0'$ be the subsystem of $\Delta_0$ obtained from $I_0'$. Then $I_0'$ and $\Delta_0'$ are $\sigma$-stable. We have $I = I_0' \cup \{\eta_0\}$, and $I_0'$ corresponds to the Cartan datum $X_0$ of finite type.

For each $\beta \in \Delta$, let $\theta_\beta$ be the $\sigma$-orbit of $\beta$ in $\Delta$. If $\theta_\beta = \{\beta, \sigma(\beta)\}$ with $\beta + \sigma(\beta) \in \Delta^{re}$, set $O(\beta) = O(\sigma(\beta)) = O(\beta + \sigma(\beta)) = 2(\beta + \sigma(\beta))$ (this only occurs in the case where $\sigma$ is non-admissible). Otherwise, set $O(\beta) = \sum_{\gamma \in \theta_\beta} \gamma$. Let $Q^\sigma$ be the set of $\sigma$-fixed elements in $Q$, and $(Q^\sigma)'$ the $Z$-submodule of $Q^\sigma$ spanned by $\tilde{\alpha}_\eta$ with $\eta \in I$ where $\tilde{\alpha}_\eta = O(\alpha_i)$ for $i \in \eta$. Recall that $\overline{Q} = \bigoplus_{\eta \in I} Z \tilde{\alpha}_\eta$. The map $\tilde{\alpha}_\eta \mapsto O(\tilde{\alpha}_\eta)$ gives an isomorphism $(Q^\sigma)' \cong \overline{Q}$. This map is compatible with the inner product $(, )$ on $Q$ and the inner product $O(, )_1$ on $Q$ defined in (1.6.1).

Set $\Delta_0 = \{O(\beta) \mid \beta \in \Delta_0'\}$, which we consider as a subset of $\overline{Q}$ under the identification $(Q^\sigma)' \cong \overline{Q}$. Then it is easy to see that $\Delta_0$ gives a root system of type $X_0$ with simple system $\{O(\tilde{\alpha}_\eta) \mid \eta \in I_0'\}$, where $O(\tilde{\alpha}_\eta) = O(\alpha_i)$ for $i \in \eta$.

5.3. Hereafter we assume that $X$ is irreducible and simply-laced, hence it is of untwisted type. Set $\Sigma_0^+ = \{\alpha \in \Delta_0^+ \mid O(\alpha) \in Z \delta\}$. We shall determine the set $\Sigma_0^+$ for each $(X, X)$. If $\sigma$ leaves $\Delta_0$ invariant, namely, $\eta_0 = \{0\}$, then $\Delta_0 = \Delta_0'$, and in this case, $\Sigma_0^+ = \emptyset$. Thus we assume that $\Delta_0' \neq \Delta_0$. In the case where $\sigma$ is admissible, and $\varepsilon \neq 4$, $\Sigma_0^+$ is already determined in [SZ2]. In the remaining cases $(X, X)$ are given as follows;

(A) $(D_{2n}^{(1)}, A_{2n-2}^{(2)})$, $(n \geq 2, \sigma: \text{admissible}, \varepsilon = 4)$,
(B) $(D_{2n+1}^{(1)}, C_{n-1}^{(1)})$, $(n \geq 2, \sigma: \text{non-admissible}, \varepsilon = 4, X = A_1^{(1)} \text{ if } n = 2)$,
(C) $(D_{2n+1}^{(1)}, A_{2n-1}^{(2)})$, $(n \geq 2, \sigma: \text{non-admissible}, \varepsilon = 2)$,
(D) $(A_{2n+1}^{(1)}, C_n^{(1)})$, $(n \geq 1, \sigma: \text{non-admissible}, \varepsilon = 2, X = A_1^{(1)} \text{ if } n = 1)$.

5.4. First consider the case (A), (B). Let $X = D_m^{(1)}$, where $m = 2n$ in the case (A), and $m = 2n + 1$ in the case (B). Set $I = \{0, 1, \ldots, m\}$, then $\sigma : I \to I$ is given by $\sigma : 0 \mapsto m - 1 \mapsto 1 \mapsto m \mapsto 0, i \mapsto m - i$ for $2 \leq i \leq m - 2$. $\eta_0 = \{0, 1, m - 1, m\}$ and $I_0' = \{2, \ldots, m - 2\}$. $\Delta_0$ is of type $D_m$, and $\Delta_0'$ is of type $A_{m-3}$. The highest root $\theta \in \Delta_0^+$ is given by $\theta = 2\alpha_1 + \cdots + 2\alpha_{m-2} + \alpha_{m-1} + \alpha_m$. Write $\alpha = \sum_{1 \leq i \leq m} c_i \alpha_i$. 

and \( z = \sum_{2 \leq i \leq m-2} c_i \alpha_i \). We have

\[
\begin{align*}
(5.4.1) & \quad \sigma(\alpha) = c_1 \alpha_m + c_{m-1} \alpha_1 + c_m (\delta - \theta) + \sigma(z) \\
(5.4.2) & \quad \sigma^2(\alpha) = c_1 (\delta - \theta) + c_{m-1} \alpha_m + c_m \alpha_{m-1} + z,
\end{align*}
\]

since \( \sigma^2 \) acts trivially on \( \Delta_0 \).

For \( i = 1, 2, 4 \), set \( Z_i = \{ \alpha \in \Delta_0 \mid \{ \delta_i \} = i \} \). First assume that \( \alpha \in Z_1 \). Then \( c_1 = c_m = 0 \), and so \( \alpha = z \in \Delta_0 \). If we write \( I'_0 = \{ \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-2} - \varepsilon_{m-1} \} \), then

\[
Z_1 = \begin{cases} 
\{ \varepsilon_2 - \varepsilon_{m-1}, \varepsilon_3 - \varepsilon_{m-2}, \ldots, \varepsilon_n - \varepsilon_{n+1} \} & \text{if } m = 2n, \\
\{ \varepsilon_2 - \varepsilon_{m-1}, \varepsilon_3 - \varepsilon_{m-2}, \ldots, \varepsilon_n - \varepsilon_{n+2} \} & \text{if } m = 2n + 1,
\end{cases}
\]

Note that \( Z_1 \cap \Sigma_0^+ = \emptyset \).

Next assume that \( \alpha \in Z_2 \). From (5.4.2), the condition \( \alpha \in Z_2 \) is given by \( c_1 = 0 \) and \( c_m = c_{m-1} \). If we write \( \Delta_0 = \{ \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq m \} \), this condition implies that

\[
Z_2 = \{ \varepsilon_i \pm \varepsilon_j \mid 2 \leq i < j \leq m - 1 \} - Z_1.
\]

The action of \( \sigma \) on \( Z_2 \) is given as follows;

\[
(5.4.3) \quad \varepsilon_i - \varepsilon_j \leftrightarrow \varepsilon_{m+1-j} - \varepsilon_{m+1-i}, \\
\varepsilon_i + \varepsilon_j \leftrightarrow \delta - (\varepsilon_{m+1-j} + \varepsilon_{m+1-i}).
\]

From this, we see that

\[
(5.4.4) \quad Z_2 \cap \Sigma_0^+ = \begin{cases} 
\{ \varepsilon_2 + \varepsilon_{m-1}, \varepsilon_3 + \varepsilon_{m-2}, \ldots, \varepsilon_n + \varepsilon_{n+1} \} & \text{if } m = 2n, \\
\{ \varepsilon_2 + \varepsilon_{m-1}, \varepsilon_3 + \varepsilon_{m-2}, \ldots, \varepsilon_n + \varepsilon_{n+2} \} & \text{if } m = 2n + 1.
\end{cases}
\]

Next consider \( Z_4 \). From the above discussion, we have

\[
(5.4.5) \quad Z_4 = \{ \varepsilon_i \pm \varepsilon_j \mid i = 1 \text{ or } j = m \}.
\]

We shall determine \( Z_4 \cap \Delta_0 \). We consider the formula (5.4.1). Since \( \alpha \in Z_4 \), we have \( c_1 = 1 \) or \( (c_m, c_{m-1}) = (1, 0), (0, 1) \). In the expression of (5.4.1), we only consider the first part, \( x = c_1 \alpha_1 + c_{m-1} \alpha_{m-1} + c_m \alpha_m \). Then

\[
x + \sigma(x) + \sigma^2(x) + \sigma^3(x) = (c_1 + c_{m-1} + c_m)(\delta - \theta) + z',
\]

where \( z' \) is a sum of elements in \( \Delta_0 \).

Hence if \( 3\delta \) appears in this sum, then \( (c_1, c_{m-1}, c_m) = (1, 1, 1) \). If \( 2\delta \) appears, then \( (c_1, c_{m-1}, c_m) = (1, 1, 0) \) or \( (1, 0, 1) \). If \( \delta \) appears, then \( (c_1, c_{m-1}, c_m) = (1, 0, 0), (0, 1, 0), (0, 0, 1) \). Write \( \alpha = x + z \), where \( x \) and \( z \) are as above. Then we have

\[
O(\alpha) = (c_1 + c_{m-1} + c_m)O(\alpha_0) + 2(z + \sigma(z)).
\]
Note that \(O(\alpha_0) = \delta - 2(\varepsilon_2 - \varepsilon_{m-1})\). Set \(\Sigma_0^+(k) = \{\alpha \in \Delta_0^+ \mid O(\alpha) = k\delta\}\). The condition that \(\alpha \in \Sigma_0^+(k)\) is given by

\[
(5.4.6) \quad z + \sigma(z) = k(\varepsilon_2 - \varepsilon_{m-1}) = k \left( \sum_{2 \leq i \leq m-2} \alpha_i \right).
\]

Since \(z + \sigma(z) = \sum_{2 \leq i \leq m-2} (c_i + c_{m-i})\alpha_i\), \((5.4.6)\) implies a relation

\[
(5.4.7) \quad c_i + c_{m-i} = k, \quad (2 \leq i \leq m-2).
\]

Note that in the case where \(m = 2n, k = 2c_n\). Since \(1 \leq k \leq 3\), we must have \(k = 2\). Thus \((c_1, c_{2n-1}, c_{2n}) = (1, 1, 0)\) or \((1, 0, 1)\). Summing up the above discussion, we have the following.

Case \((A)\) : \((X, X) = (D_{2n}^{(1)}, A_{2n-2}^{(2)})\).

\[
(5.4.8) \quad Z_4 \cap \Delta_0^+ = \{\varepsilon_1 - \varepsilon_{2n}, \varepsilon_1 + \varepsilon_{2n}\}.
\]

Combining \((5.4.8)\) with \((5.4.4)\), we have

\[
(5.4.9) \quad \Sigma_0^+ = \{\varepsilon_1 - \varepsilon_{2n}, \varepsilon_1 + \varepsilon_{2n}, \varepsilon_2 + \varepsilon_{2n-1}, \varepsilon_3 + \varepsilon_{2n-2}, \ldots, \varepsilon_n + \varepsilon_{n+1}\}.
\]

Hence \(\Sigma_0^+\) is a positive subsystem of \(\Delta_0^+\) of type \((n + 1)A_1\).

Case \((B)\) : \((X, X) = (D_{2n+1}^{(1)}, C_n^{(1)})\).

By using \((5.4.7)\), one can determine \(Z_4 \cap \Sigma_0^+(k)\) as follows.

\[
(5.4.10) \quad Z_4 \cap \Delta_0^+(3) = \{\varepsilon_1 + \varepsilon_{n+1}\},
\]

\[
Z_4 \cap \Delta_0^+(2) = \{\varepsilon_1 - \varepsilon_{2n+1}, \varepsilon_1 + \varepsilon_{2n+1}\},
\]

\[
Z_4 \cap \Delta_0^+(1) = \{\varepsilon_1 - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{2n+1}, \varepsilon_{n+1} + \varepsilon_{2n+1}\}.
\]

It follows that

\[
(5.4.11) \quad \Sigma_0^+ = \{\varepsilon_2 + \varepsilon_{2n}, \varepsilon_3 + \varepsilon_{2n-1}, \ldots, \varepsilon_n + \varepsilon_{n+2}\}
\]

\[
\cup \{\varepsilon_1 - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{2n+1}, \varepsilon_{n+1} + \varepsilon_{2n+1}, \varepsilon_1 - \varepsilon_{2n+1}, \varepsilon_1 + \varepsilon_{2n+1}, \varepsilon_1 + \varepsilon_{n+1}\}.
\]

Hence \(\Delta_0^+\) is a positive subsystem of \(\Delta_0^+\) of type \((n - 1)A_1 + A_3\).

5.5. Next consider the case \((C)\) : \((X, X) = (D_{2n+1}^{(1)}, A_{2n-1}^{(2)})\). \(\Delta_0\) : type \(D_{2n+1}\),

where \(\sigma : \alpha_0 \leftrightarrow \alpha_{2n+1}, \alpha_i \leftrightarrow \alpha_{2n+1-i}\) for \(i = 2, \ldots, 2n - 1\). In this case, \(\eta_0 = \{0, 2n + 1\}, I_0' = \{1, 2, \ldots, 2n\}\), and \(\Delta_0'\) is of type \(A_{2n}\). Here \(\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}\). We write \(\alpha \in \Delta_0^+\) as \(\alpha = \sum_{i \in I_0'} c_i\alpha_i\), with \(\eta_0 = \{0, i_1\}\). By the discussion in [SZ2, (5.3.1)], we see that \(\Sigma_0^+ = \{\alpha \in \Delta_0^+ \mid O(\alpha) = \delta\}\), and the condition \(\alpha \in \Sigma_0^+\) is given by

\[
(5.5.1) \quad c_i + c_{\sigma(i)} = a_i, \quad (i \in I_0')
\]
where we write $\delta = \sum_{i \in I} a_i \alpha_i$. Here $\Delta_0$ is of type $C_n$ with $\alpha_2$ long root. We have $c_{2n+1} = 1$ and
\[ c_i + c_{\sigma(i)} = 2, \quad (2 \leq i \leq 2n - 1), \quad c_1 + c_2 = 1. \]
This implies that
\begin{equation}
\Sigma_0^+ = \{ \varepsilon_1 + \varepsilon_{2n+1}, \varepsilon_2 + \varepsilon_{2n}, \ldots, \varepsilon_n + \varepsilon_{n+2} \}.
\end{equation}

Hence $\Sigma_0^+$ is a positive subsystem of $\Delta_0^+$ of type $nA_1$.

5.6. Finally, consider the case (D) : $(X, \vec{X}) = (A_{2n+1}^{(1)}, C_n^{(1)})$. Here $I = \{0, 1, \ldots, 2n+1\}$ with $\sigma : i \mapsto 2n+1-i$. Hence $I = \{0, 1, \ldots, n\}$, $I_0' = \{1, 2, \ldots, 2n\}$ and $\Delta_0'$ is of type $A_{2n}$ on which $\sigma$ acts. $\Delta_0'$ is of type $C_n$ with $\alpha_n$ long root. Take $\alpha \in \Delta_0$ and write it as $\alpha = \sum_{i \in I_0} c_i \alpha_i$. Here $\theta = \alpha_1 + \cdots + \alpha_{2n+1}$, and $\sigma(\alpha_{2n+1}) = \delta - \theta$. Assume that $\alpha \in \Sigma_0^+$. We have
\[ \alpha + \sigma(\alpha) = c_{2n+1}(\delta - \theta + \alpha_{2n+1}) + \sum_{1 \leq i \leq n} (c_i + c_{2n+1-i})(\alpha_i + \alpha_{2n+1-i}) \]
Hence $c_{2n+1} = 1$ and the condition $\alpha \in \Sigma_0^+$ is given by
\[ c_i + c_{2n+1-i} = 1, \quad (1 \leq i \leq n). \]
This implies that $\alpha = \alpha_{n+1} + \cdots + \alpha_{2n+1} = \varepsilon_{n+1} - \varepsilon_{2n+2}$. If we write $\Delta_0^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq 2n+2 \}$, we have
\begin{equation}
\Sigma_0^+ = \{ \varepsilon_{n+1} - \varepsilon_{2n+2} \}.
\end{equation}

Hence $\Sigma_0^+$ is a positive subsystem of $\Delta_0^+$ of type $A_1$.

5.7. We shall describe the set of positive real roots for $\Delta$ appearing as $\Delta$ under the isomorphism $(Q^r)' \cong \overline{Q}$ in 5.1. Let $\Delta$ be the irreducible root system of type $X_n^{(r)}$ with vertex set $I = \{0, 1, \ldots, n\}$, where $I_0 = \{1, \ldots, n\}$ corresponds to the finite system $I_0$. In the case where $X_n^{(r)} \neq A_{2n}^{(2)}$, the choice of 0 is determined uniquely, up to the diagram automorphisms, in which case, we follow the ordering of [K]. While in the case where $X_n^{(r)} = A_{2n}^{(2)}$, we use the ordering as in (A_0) in 1.5, which is the ordering used in [BN], and is opposite to the one in [K]. We denote by $\Delta_0^+$ the set of positive long roots in $\Delta_0$, and $\Delta_{0,s}^+$ the set of positive short roots in $\Delta_0$. Let $\Delta_{re,+}$ be the set of positive real roots in $\Delta$.

Assume that $X_n^{(r)} \neq A_{2n}^{(2)}$. In this case, $\Delta_{re,+}$ is written as $\Delta_{re,+} = \Delta_{0,+} \cup \Delta_{re,+}$, where
\begin{equation}
\Delta_{re,+} = \{ \alpha + m\delta \mid \alpha \in \Delta_{0,s}^+, m \in \mathbb{Z}_{\geq 0} \} \cup \{ \alpha + m\delta \mid \alpha \in \Delta_{0,l}^+, m \in \mathbb{Z}_{\geq 0} \},
\end{equation}
\[ \Delta^{\text{re,+}}_\leq = \{ \alpha + m\delta \mid \alpha \in -\Delta^+_{0,s}, m \in \mathbb{Z}_{>0} \} \cup \{ \alpha + m\delta \mid \alpha \in -\Delta^+_{0,l}, m \in \mathbb{Z}_{>0} \}. \]

Next assume that \( X_n^{(r)} = A_{2n}^{(2)} \). In this case, \( \Delta^{\text{re,+}} \) is given as follows. Note that since our ordering is opposite to \([K]\), this description of \( \Delta^{\text{re,+}} \) is different from the one in \([K]\). We fix the inner product \((\ , \) ) so that \((\alpha, \alpha) = 1, 2, 4\). We give the ordering of \( A_{2n}^{(2)} \) as in 1.5, namely \( \alpha_0 \) is such that \((\alpha_0, \alpha_0) = 4\). The finite root system \( \Delta_0 \) corresponding to \( I_0 \) is of type \( B_n \). Then \( \Delta^{\text{re,+}} \) is given as \( \Delta^{\text{re,+}} = \Delta^+_s \cup \Delta^+_l \cup \Delta^+_2s \), where

\begin{align*}
\Delta^+_s &= \{ m\delta + \alpha \mid \alpha \in \Delta^+_{0,s}, m \in \mathbb{Z}_{\geq 0} \} \cup \{ m\delta - \alpha \mid \alpha \in \Delta^+_{0,s}, m \in \mathbb{Z}_{>0} \}, \\
\Delta^+_l &= \{ m\delta + \alpha \mid \alpha \in \Delta^+_{0,l}, m \in \mathbb{Z}_{\geq 0} \} \cup \{ m\delta - \alpha \mid \alpha \in \Delta^+_{0,l}, m \in \mathbb{Z}_{>0} \}, \\
\Delta^+_2s &= \{ m\delta + 2\alpha \mid \alpha \in \Delta_{0,s}, m : \text{odd integer} \geq 0 \},
\end{align*}

where \( \Delta_{0,s} \) is the set of short roots in \( \Delta_0 \). Note that

\begin{align*}
\Delta^+_s &= \{ \alpha \in \Delta^+ \mid (\alpha, \alpha) = 1 \}, \\
\Delta^+_l &= \{ \alpha \in \Delta^+ \mid (\alpha, \alpha) = 2 \}, \\
\Delta^+_2s &= \{ \alpha \in \Delta^+ \mid (\alpha, \alpha) = 4 \}.
\end{align*}

5.8. Let \( \Delta^+_{0,l} \) (resp. \( \Delta^+_{0,s} \)) be the set of long roots (resp. short roots) in \( \Delta_0 \). We denote by \( \Delta^+_{0,l}, \Delta^+_{0,s} \) the corresponding positive roots. Set

\begin{align*}
\Sigma^+_l &= \{ \alpha \in \Delta^+_l \mid O(\alpha) \in \Delta^+_{0,l} \}, \\
\Sigma^+_s &= \{ \alpha \in \Delta^+_s \mid O(\alpha) \in \Delta^+_{0,s} \}, \\
\Sigma'_l &= \{ \alpha \in \Delta^+_l \mid O(\alpha) \in \delta + \Delta^+_{0,l} \}, \\
\Sigma'_s &= \{ \alpha \in \Delta^+_s \mid O(\alpha) \in \delta + \Delta^+_{0,s} \}, \\
\Sigma''_l &= \{ \alpha \in \Delta^+_l \mid O(\alpha) \in 2\delta + \Delta^+_{0,l} \}, \\
\Sigma''_s &= \{ \alpha \in \Delta^+_s \mid O(\alpha) \in 2\delta + \Delta^+_{0,s} \}, \\
(5.8.1) & \Sigma^+_{2s'} = \{ \alpha \in \Delta^+_2s' \mid O(\alpha) \in 3\delta + \Delta^+_{0,s} \}, \\
\Sigma^+_{s''} &= \{ \alpha \in \Delta^+_0 \mid O(\alpha) \in \mathbb{Z}_{>0}\delta \}. \\
\Sigma^+_0 &= \{ \alpha \in \Delta^+_0 \mid O(\alpha) \in \Delta^+_{0,l} \}.
\end{align*}

\( \Sigma^+_0 \) is already defined, and was determined in 5.3 \~ 5.6. Note that \( \Sigma_{2s'}, \Sigma_{2s''} \) occurs only in the case where \((X, X) = (D_2^{(1)}, A_{2n-2}^{(2)})\). We define \( \Sigma^+_l, \Sigma^+_s, \Sigma^+_{2s'} \) (resp. \( \Sigma^+_{s''} \)) by replacing \( \Delta^+_{0,l} \) by \( \Delta^+_{0,l} \) (resp. by replacing \( \Delta^+_{0,s} \) by \( \Delta^+_{0,s} \)) in the corresponding formulas.

5.9. Let \( \theta \) be the highest root in \( \Delta^+_0 \). Then we have \( \alpha_0 = \delta - \theta \). Let \( \tilde{\theta} \) be the highest root in \( \Delta^+_0 \), and \( \tilde{\theta} \), the highest short root in \( \Delta^+_0 \). The following relations are verified by a direct computation.
\begin{align*}
O(\alpha_0) &= \begin{cases} 
\delta - 2\theta, & \text{Case (A)}, \\
\delta - \theta, & \text{Case (B)}, \\
\delta - \theta, & \text{Case (C)}, \\
2\delta - \theta, & \text{Case (D)}. 
\end{cases} 
\tag{5.9.1} 
\end{align*}

This implies that
\begin{align*}
O(\theta) &= \begin{cases} 
3\delta + 2\theta, & \text{Case (A)}, \\
3\delta + \theta, & \text{Case (B)}, \\
\delta + \theta, & \text{Case (C)}, \\
2\delta + \theta, & \text{Case (D)}. 
\end{cases} 
\tag{5.9.2} 
\end{align*}

Note that \(O(\theta) = 2(\theta + \sigma(\theta))\) in Case (D).

Let \(W\) be the Weyl group of \(\Delta\), and \(W_\sigma\) the Weyl group of \(\Delta\). As in 3.2, the correspondence \(s_\eta \mapsto t_\eta\) gives an isomorphism \(W \cong W_\sigma\). Let \(W_0\) be the Weyl group associated to \(\Delta_0\). Then \(W_0\) is regarded as a subgroup of \(W\), and under the identification \(W_0 \cong W_\sigma\), we regard \(W_0\) as a subgroup of \(W\).

We prove the following lemma.

**Lemma 5.10.** Under the notation in (5.8.1), we have
\begin{align*}
\Delta_0^+ &= \begin{cases} 
\Sigma_+^s \cup \Sigma_i^+ \cup \Sigma_{\tau}^s \cup \Sigma_{2\tau}^s \cup \Sigma_{2\tau'}^s \cup \Sigma_0^+ & \text{Case (A)}, \\
\Sigma_+^s \cup \Sigma_i^+ \cup \Sigma_{\tau}^s \cup \Sigma_{\tau'}^s \cup \Sigma_{\tau''}^s \cup \Sigma_{\tau'''}^s \cup \Sigma_0^+ & \text{Case (B)}, \\
\Sigma_+^s \cup \Sigma_i^+ \cup \Sigma_{\tau}^s \cup \Sigma_{\tau''}^s \cup \Sigma_{\tau'''}^s \cup \Sigma_0^+ & \text{Case (C)}, \\
\Sigma_+^s \cup \Sigma_i^+ \cup \Sigma_{\tau}^s \cup \Sigma_{\tau''}^s & \text{Case (D)}. 
\end{cases} 
\end{align*}

**Proof.** Set
\[Y = \{ \alpha \in \Delta_0^+ \mid O(\alpha) \in \Delta_0^+ \} = \Sigma_+^s \cup \Sigma_i^+.
\]
We prove the lemma in a similar way as in [SZ2, Lemma 5.5], separately for each case.

Case (A). Let \(E\) be the union of \(W_0\)-orbits of \(\sigma(\alpha_0)\) and of \(\theta\), and set \(E^+ = E \cap \Delta_0^+\). We know by (5.9.1) and (5.9.2) that \(O(\alpha_0) = \delta - 2\theta\) and \(O(\theta) = 3\delta + 2\theta\). Since any short root in \(\Delta_0\) is \(W_0\)-conjugate to \(\theta\), we see that for any \(\alpha \in E^+\), \(O(\alpha)\) can be written as \(\delta + 2\beta\) or \(3\delta + 2\beta\), where \(\beta \in \Delta_{0,\delta}\). Hence \(\alpha \in \Sigma_{2\tau}^s \cup \Sigma_{2\tau'}^s\). It follows that
\[E^+ \subset \Sigma_2^s \cup \Sigma_2^s' \subset \Delta_0^+ - Y - \Sigma_{\tau}^s - \Sigma_0^+.
\]
We show that these inclusion relations are all equalities. For any \(\beta \in \Delta_{0,\delta}\), there exists at least three \(\alpha \in E^+\) such that \(O(\alpha) = \delta + 2\beta\), and at least one \(\alpha \in E^+\) such that \(O(\alpha) = 3\delta + 2\beta\). It follows that \(4|\Delta_{0,\delta}| \leq |E^+|\). On the other hand, under the
notation of 5.4, we see that \( \Delta_0^+ = \{ \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 2n \} \), and

\[
\Sigma_0^+ - Y - \Sigma_{\ell'} = \{ \varepsilon_i \pm \varepsilon_j \mid 2 \leq j \leq 2n \} \cup \{ \varepsilon_i \pm \varepsilon_{2n} \mid 2 \leq i \leq 2n - 1 \}.
\]

By (5.4.7), \( (\Sigma_0^+ - Y - \Sigma_{\ell'}) \cap \Sigma_0^+ = \{ \varepsilon_1 \pm \varepsilon_{2n} \} \). Hence \( |\Sigma_0^+ - Y - \Sigma_{\ell'} - \Sigma_0^+| = 8n - 8 \). As \( \Delta_0 \) is of type \( B_{n-1} \), we have \( |\Delta_0| = 2(n-1) \), hence \( |E^+| = |\Delta_0^+ - Y - \Sigma_{\ell'} - \Sigma_0^+| \).

It follows that

\[
E^+ = \Sigma_{2\ell'} \cup \Sigma_{2\ell''} = \Delta_0^+ - Y - \Sigma_{\ell'} - \Sigma_0^+.
\]

The required formula follows from this.

Case (B) First note that, under the notation in 5.4, for \( \alpha = \varepsilon_i + \varepsilon_j \in Z_2 \), we have \( \alpha + \sigma(\alpha) = \delta + (\varepsilon_i + \varepsilon_j - \varepsilon_{2n+2-i} - \varepsilon_{2n+2-j}) \). This comes from (5.4.3). The condition that \( \alpha + \sigma(\alpha) \in \Delta^e \) is given by \( 2n+2-i = i \), namely, \( i = n+1 \). Hence we have

\[
\{ \alpha \in Z_2 \mid O(\alpha) \in 2\delta + \Delta_{0,l} \} = \{ \varepsilon_i + \varepsilon_{n+1} \mid 2 \leq i \leq 2n, i \neq n+1 \}.
\]

Let \( E \) be the union of \( W_{0} \)-orbits of \( \beta_1, \beta_2, \beta_3 \in \Delta_0 \), where \( O(\beta_k) \in k\delta + \Delta_{0,l} \) for \( k = 1, 2, 3 \), and set \( E^+ = E \cap \Delta_0^+ \). By (5.9.1) and (5.9.2), we have \( O(\alpha_0) = \delta - \theta, O(\theta) = 3\delta + \theta \), we may choose \( \beta_1 = \sigma(\alpha_0), \beta_3 = \theta \). It follows from (5.10.2), we may choose, for example, \( \beta_2 = \varepsilon_2 + \varepsilon_{n+1} \). Then we have

\[
E^+ \subset \Sigma_{\ell'} \cup \Sigma_{\ell''} \cup \Sigma_{\ell'''} \subset \Delta_0^+ - Y - \Sigma_{\ell'} - \Sigma_0^+.
\]

We show that these inclusion relations are actually equalities. For each \( \beta \in \Delta_{0,l} \), there exist, at least three \( \alpha \in E^+ \) such that \( O(\alpha) = \delta + \beta \), at least one \( \alpha \in E^+ \) such that \( O(\alpha) = 2\delta + \beta \), and at least one \( \alpha \in E^+ \) such that \( O(\alpha) = 3\delta + \beta \) (note that \( \beta_2 \in Z_2 \)). It follows that \( 5|\Delta_{0,l}| \leq |E^+| \). On the other hand, by the computation for \( Z_2, Z_4 \) in 5.4, and by (5.10.2),

\[
\Delta_0^+ - Y - \Sigma_{\ell'} = \{ \varepsilon_i \pm \varepsilon_j \mid 2 \leq j \leq 2n + 1 \}
\]

\[
\cup \{ \varepsilon_i \pm \varepsilon_{2n+1} \mid 2 \leq i \leq 2n \}
\]

\[
\cup \{ \varepsilon_i + \varepsilon_{n+1} \mid 2 \leq i \leq 2n, i \neq n+1 \}.
\]

The first two sets are contained in \( Z_4 \) and the third one is contained in \( Z_2 \) (see (5.4.5) and (5.10.2)). By (5.4.11), we have

\[
(\Delta_0^+ - Y - \Sigma_{\ell'}) \cap \Sigma_0^+ = \{ \varepsilon_i \pm \varepsilon_{2n+1}, \varepsilon_1 \pm \varepsilon_{n+1}, \varepsilon_{n+1} \pm \varepsilon_{2n+1} \}.
\]

It follows that

\[
|\Delta_0^+ - Y - \Sigma_{\ell'} - \Sigma_0^+| = 10(n-1).
\]

Since \( \Delta_0 \) is of type \( C_{n-1} \), we have \( |\Delta_{0,l}| = 2(n-1) \), and so \( |\Delta_0^+ - Y - \Sigma_{\ell'} - \Sigma_0^+| = |E^+| \). Hence
(5.10.3) \[ E^+ = \Sigma_{\nu} \cup \Sigma_{\nu'} \cup \Sigma_{\nu''} = \Delta^+_0 - Y - \Sigma^+_s - \Sigma^+_0. \]

The required formula follows from this.

Case (C) In this case, \(\Delta^+_0 = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq 2n + 1 \}\). The action of \(\sigma\) on \(\alpha \in \Delta^+_0\) is given as in (5.4.3). It follows that

\[ (5.10.4) \quad \{ \alpha \in \Delta^+_0 | O(\alpha) \in 2\delta + \Delta_{0,l} \} = \{ \varepsilon_i + \varepsilon_{n+1} | 1 \leq i \leq 2n + 1, i \neq n + 1 \}. \]

Let \(E\) be the union of \(W_0\)-orbit of \(\beta_1, \beta_2 \in \Delta^+_0\) such that \(O(\beta_1) \in \delta + \Sigma_{0,s}\), \(\beta_2 \in 2\delta + \Delta_{0,l}\), and set \(E^+ = E \cap \Delta^+\). By (5.9.2), we may choose \(\beta_1 = \theta\). By (5.10.4), we may choose, for example, \(\beta_2 = \varepsilon_1 + \varepsilon_{n+1}\). Then we have

\[ E^+ \subset \Sigma_{\nu} \cup \Sigma_{\nu'} \subset \Delta^+_0 - Y - \Sigma^+_0. \]

We show that these inclusion relations are actually equalities. For each \(\beta \in \Delta_{0,s}\), there exists at least one \(\alpha \in E^+\) such that \(O(\alpha) \in \delta + \beta\), and for each \(\beta' \in \Delta_{0,l}\), there exists at least one \(\alpha \in E^+\) such that \(O(\alpha) \in 2\delta + \beta'\). It follows that \(|E^+| \geq |\Delta_0|\). On the other hand, we have

\[ \Delta^+_0 - Y = \{ \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq 2n + 1 \}. \]

By (5.5.2), \(\Sigma^+_0 \subset \Delta^+_0 - Y\), and \(|\Sigma^+_0| = n\). It follows that \(|\Delta^+_0 - Y - \Sigma^+_0| = 2n^2\). Since \(\Delta_0\) is of type \(C_n\), \(|\Delta_0| = 2n^2\). Thus we have

\[ (5.10.5) \quad E^+ = \Sigma_{\nu} \cup \Sigma_{\nu'} = \Delta^+_0 - Y - \Sigma^+_0. \]

The required formula follows from this.

Case (D) Let \(E\) be the \(W_0\)-orbit of \(\theta\), and set \(E^+ = E \cap \Delta^+_0\). By (5.9.2), \(O(\theta) = 2\delta + \theta\). Then we have

\[ E^+ \subset \Sigma_{\nu'} \subset \Delta^+_0 - Y - \Sigma^+_0. \]

We show that these inclusions are actually equalities. We have \(|E^+| \geq |\Delta_{0,l}|\). On the other hand,

\[ \Delta^+_0 - Y = \{ \varepsilon_i - \varepsilon_{2n+2} \}. \]

Since \(\Sigma^+_0 = \{ \varepsilon_{n+1} - \varepsilon_{2n+2} \}\), we see that \(|\Delta^+_0 - Y - \Sigma^+_0| = 2n\). Since \(\Delta_0\) is of type \(C_n\), we have \(|\Delta_{0,l}| = 2n\). Thus we have

\[ E^+ = \Sigma_{\nu'} = \Delta^+_0 - Y - \Sigma^+_0. \]

The required formula follows from this.

The lemma is proved. \(\Box\)
For each $m \geq 0$, $x \in \{s, 2s', 2s'', l, l', l'', l'''\}$, we define subsets $\Sigma_x(m)$, $\Sigma'_x(m)$ of $\Delta^{\text{re,}+}$ as

\[
\Sigma_x(m) = \bigcup_i \sigma^i \{ m\delta + \alpha \mid \alpha \in \Sigma^+_x \},
\]

\[
\Sigma'_x(m) = \bigcup_i \sigma^i \{ m\delta - \alpha \mid \alpha \in \Sigma^+_x \},
\]

where we assume that $m \geq 1$ for $\Sigma'_x(m)$. We define an equivalence relation on $\Delta^{\text{re}}$ by the condition that $\beta, \beta' \in \Delta^{\text{re}}$ are equivalent if $O(\beta) = O(\beta')$. We denote by $\hat{\Sigma}_x(m)$ the set of equivalence classes in $\Sigma_x(m)$, and define $\hat{\Sigma}'_x(m)$ similarly. Let $\hat{\Delta}^{\text{re,}+}$ be the set of equivalence classes in $\Delta^{\text{re,}+} - \bigcup_{m \geq 0} \{ m\delta \pm \alpha \mid \alpha \in \Sigma^+_0 \}$. For $x = 0$ and $m \geq 0$, we set

\[
\Sigma_0(m) = \{ m\delta + \alpha \mid \alpha \in \Sigma^+_0 \}, \quad \Sigma'_0(m) = \{ m\delta - \alpha \mid \alpha \in \Sigma^+_0 \}.
\]

Also for each $m \geq 0$, we define subsets of $\hat{\Delta}^{\text{re,}+}$ by

\[
\Sigma_{2s}(m) = \{ m\delta + 2\alpha \mid \alpha \in \Sigma^+_{2s} \}, \quad \Sigma'_{2s}(m) = \{ m\delta - 2\alpha \mid \alpha \in \Sigma^+_{2s} \},
\]

\[
\Sigma_{l''}(m) = \{ m\delta + \alpha \mid \alpha \in \Sigma^+_{l''} \}, \quad \Sigma'_{l''}(m) = \{ m\delta - \alpha \mid \alpha \in \Sigma^+_{l''} \},
\]

Note that we assume $m \geq 1$ in the right hand sides of (5.11.2) and (5.11.3).

The following result is a generalization of [SZ2, Prop. 5.7]. The proof is straightforward from Lemma 5.10, by using the description of the root system $\hat{\Delta}^{\text{re,}+}$ given in 5.7.

Proposition 5.12. Under the identification $(Q^\sigma)' \simeq Q$, the followings hold.

(i) For each $\beta \in \Delta^+$, $O(\beta) \in \Delta^+$.

(ii) In the cases (A), (B), (C), we have a partition

\[
\hat{\Delta}^{\text{re,}+} = \bigsqcup_{x, m \geq 0} \hat{\Sigma}_x(m) \sqcup \bigsqcup_{x, m \geq 1} \hat{\Sigma}'_x(m).
\]

The map $f : \beta \mapsto O(\beta)$ induces a bijection $\hat{\Delta}^{\text{re,}+} \simeq \Delta^{\text{re,}+}$. The bijection is given more precisely as follows. (In the formulas below, we only give the part corresponding to $\hat{\Sigma}_x(m)$. The part corresponding to $\hat{\Sigma}'_x(m)$ is described similarly.)
Case (A)
\[
\mathring{\Sigma}_s(m) \simeq \Sigma_s(m), \quad (m \equiv 0 \mod 2),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 1 \mod 2),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 3 \mod 2),
\]
\[
\mathring{\Sigma}_2(m) \simeq \Sigma_2(m), \quad (m \equiv 1 \mod 4),
\]
\[
\mathring{\Sigma}_2(m) \simeq \Sigma_2(m), \quad (m \equiv 3 \mod 4).
\]

Case (B)
\[
\mathring{\Sigma}_s(m) \simeq \Sigma_s(m), \quad (m \equiv 0 \mod 2),
\]
\[
\mathring{\Sigma}_s(m) \simeq \Sigma_s(m), \quad (m \equiv 1 \mod 2),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 0 \mod 4),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 2 \mod 4),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 3 \mod 4).
\]

Case (C)
\[
\mathring{\Sigma}_s(m) \simeq \Sigma_s(m), \quad (m \equiv 0 \mod 2),
\]
\[
\mathring{\Sigma}_s(m) \simeq \Sigma_s(m), \quad (m \equiv 1 \mod 2),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 0 \mod 4),
\]
\[
\mathring{\Sigma}_l(m) \simeq \Sigma_l(m), \quad (m \equiv 2 \mod 4).
\]

(iii) In the case (D), we have a partition
\[
\widehat{\Delta}_{\text{re}, +} = \bigcup_{x, m \geq 0} \mathring{\Sigma}_x(2m) \sqcup \bigcup_{x, m \geq 0} \mathring{\Sigma}_x(2m + 1).
\]

For \( \beta \in \Sigma_x(2m) \) (resp. \( \beta \in \Sigma_x(2m + 1) \)), \( O(\beta) \) is written as \( O(\beta) = 2m\delta + \beta \) (resp. \( O(\beta) = (2m + 1)\delta + \beta \)) with \( \beta \in \underline{\Delta}^\pm \). Define \( f : \mathring{\Sigma}_x(2m) \to \underline{\Delta} \) (resp. \( f : \mathring{\Sigma}_x(2m + 1) \to \underline{\Delta} \)) by \( f(\beta) = m\delta + \beta \). Then \( f \) induces a bijection \( \widehat{\Delta}_{\text{re}, +} \simeq \underline{\Delta}_{\text{re}, +} \). More precisely, the map is given as follows;
\[
\mathring{\Sigma}_s(2m) \simeq \Sigma_s(m),
\]
\[
\mathring{\Sigma}_l(2m) \sqcup \mathring{\Sigma}_l(2m) \simeq \Sigma_l(m).
\]

The case for \( \mathring{\Sigma}_x(2m + 1) \) is given similarly.
We keep the assumption in 5.1. We consider the doubly infinite sequence $h = (\ldots, \eta_{-1}, \eta_0, \eta_1, \ldots)$ for $W$ associated to $\xi \in P^\vee_\xi$ as defined in [SZ2, 1.5], applied for $U_{-q}^-$. We define a sequence $h' = (\ldots, i_{-1}, i_0, i_1, \ldots)$ by replacing $\eta = \eta_k$ by a sequence $J_\eta$ in $I$, where

$$J_\eta = \begin{cases} (j_1, \ldots, j_{|\eta|}) & \text{if } \delta_\eta = 1 \text{ and } \eta = \{j_1, \ldots, j_{|\eta|}\}, \\ (i, j, i) & \text{if } \delta_\eta = 2 \text{ and } \eta = \{i, j\}. \end{cases}$$

Thus

$$h' = (\ldots, i_{-k-|\eta|-1}, \ldots, i_{-k}, i_{-k+1}, i_{-k+2}, \ldots).$$

For any $k < \ell$, $w = s_{\eta_k} \cdots s_{\eta_\ell}$ is a reduced expression of $w \in W$. We define $w \in W$ by

$$w = t_{\eta_k} \cdots t_{\eta_\ell},$$

where $t_{\eta_k}$ is defined as in 3.2. In view of Proposition 5.12,

$$\Delta^+ \cap w^{-1}(-\Delta^+) = f^{-1}(\Delta^+ \cap w^{-1}(-\Delta^+)).$$

It follows that (6.1.2) is a reduced expression of $w$, and $h'$ gives an infinite reduced word. We define $\beta_k \in \Delta^{re,+}$ for any $k \in \mathbb{Z}$ by

$$\beta_k = \begin{cases} s_{\eta_0}s_{\eta_1} \cdots s_{\eta_k+1}(\alpha_{\eta_k}) & \text{if } k \leq 0, \\ s_{\eta_1}s_{\eta_2} \cdots s_{\eta_k-1}(\alpha_{\eta_k}) & \text{if } k > 0. \end{cases}$$

Then by [BN, 3.1], we have

$$\Delta^{re,+} = \{\beta_k \mid k \leq 0\}, \quad \Delta^{re,+} = \{\beta_k \mid k > 0\}.$$

Also define $\beta_k \in \Delta^{re,+}$ for $k \in \mathbb{Z}$ by

$$\beta_k = \begin{cases} s_{i_0}s_{i_1} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0, \\ s_{i_1}s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0. \end{cases}$$

Recall $\Sigma_x(m), \Sigma'_x(m)$ in (5.11.1) for $x \in \mathcal{X} = \{s, 2s', 2s'', l, l', l'', l'''\}$ and $\Sigma_0(m)$, $\Sigma'_0(m)$ in (5.11.2). We define $\Delta^{(0)}_>, \Delta^{(0)}_<$ by

$$\Delta^{(0)}_> = \bigsqcup_{m \geq 0} \bigsqcup_{x \in \mathcal{X}} \Sigma_x(m), \quad \Delta^{(0)}_< = \bigsqcup_{m \geq 1} \bigsqcup_{x \in \mathcal{X}} \Sigma'_x(m).$$
Also set \( \Delta_\prec^{(1)} = \bigcup_{m \geq 0} \Sigma_0(m) \), \( \Delta_\succ^{(1)} = \bigcup_{m \geq 1} \Sigma'_0(m) \).

Then \( \Delta_\prec^{(0)} \) and \( \Delta_\succ^{(0)} \) are \( \sigma \)-stable, and \( \Delta_\prec^{(1)} \sqcup \Delta_\succ^{(1)} \) is \( \sigma \)-stable. By Proposition 5.12 and (6.1.4), we see that

\[
\{ \beta_k \mid k \leq 0 \} = \Delta_\prec^{(0)}, \quad \{ \beta_k \mid k > 0 \} = \Delta_\succ^{(0)}.
\]

Moreover, we have \( \Delta^{\text{re} +} = \Delta_\prec^{(0)} \sqcup \Delta_\succ^{(1)} \sqcup \Delta_\prec^{(1)} \sqcup \Delta_\succ^{(0)} \).

6.2. Hereafter we apply the theory of PBW-basis associated to the convex order developed by Muthiah-Tingley [MT] to our setup. We follow the notation in [SZ2, Section 4]. For any real root \( \beta = \alpha + m\delta \in \Delta^{\text{re}} \), we denote by \( \bar{\beta} = \alpha \in \Delta_0 \). Set \( \Delta_+^{\text{min}} = \Delta^{\text{re} +} \sqcup \{ \delta \} \). For any subset \( Z \) of \( \Delta_+^{\text{min}} \), we define a total order \( \alpha \preceq \beta \), called a convex order, by the following two conditions,

(6.2.1) If \( \alpha, \beta \in Z \) with \( \alpha + \beta \in Z \), then \( \alpha + \beta \) is in between \( \alpha \) and \( \beta \).

(6.2.2) If \( \alpha \in Z, \beta \in \Delta_+^{\text{min}} - Z \) with \( \alpha + \beta \in Z \), then \( \alpha \prec \alpha + \beta \).

If in (6.2.2), the condition “\( \alpha \prec \alpha + \beta \)” is replaced by “\( \alpha + \beta \prec \alpha \)” such a total order is called a reverse convex order.

We define a total order \( \prec \) on \( \Delta_\prec^{(0)} \) by \( \beta_0 \prec \beta_{-1} \prec \beta_{-2} \cdots \), and a total order \( \prec \) on \( \Delta_\succ^{(0)} \) by \( \cdots \prec \beta_2 \prec \beta_1 \). Then by [SZ2, Lemma 4.3], \( \prec \) gives a convex order on \( \Delta_\prec^{(0)} \) and a reverse convex order on \( \Delta_\succ^{(0)} \). This implies that

(6.2.3) \( \Delta_\prec^{(0)} \prec \delta \prec \Delta_\succ^{(0)} \) satisfies the condition (6.2.1). In particular, \( \beta \prec \delta + \beta \) for \( \beta \in \Delta_\prec^{(0)} \) and \( \beta + \delta \prec \beta \) for \( \beta \in \Delta_\succ^{(0)} \).

We shall define a total order on \( \Delta_\prec^{(1)} \) and on \( \Delta_\succ^{(1)} \). In the case where \( (X, X) \) is in the case (A), (C), or (D), \( \Sigma_0^+ \) is a positive subsystem of \( \Delta^+ \) of type \( nA_1 \) for some \( n \geq 1 \). In this case, write \( \Sigma_\gamma^+ = \{ \gamma_1, \ldots, \gamma_i \} \) in any order. We define a total order on \( \Sigma_\gamma^{(1)} \) by the condition that \( m\delta + \gamma_i \prec m'\delta + \gamma_j \) if \( m < m' \) or if \( m = m' \) and \( i < j \).

Similarly, we define a total order on \( \Sigma_\gamma^{(1)} \) by the condition that \( m\delta - \gamma_i \prec m'\delta - \gamma_j \) if \( m > m' \) or if \( m = m' \) and \( i > j \).

We now consider the case (B). In this case \( \Delta_0^+ \) is a positive subsystem in \( \Delta^+ \) of type \( (n - 1)A_1 + A_3 \) by (5.4.11). Thus \( \Delta_0^+ = X_1 \cup X_2 \), where \( X_1 \) is a positive system of type \( A_3 \) and \( X_2 \) is a positive system of type \( (n - 1)A_1 \). Let \( \Delta^{(1)} \) be the subsystem of \( \Delta \) spanned by \( \gamma \in X_1, \gamma_0 = \delta - \theta_1 \), and \( \gamma', \delta - \gamma' \) for \( \gamma' \in X_2 \), where \( \theta_1 \) is the highest root in \( X_1 \). Then \( \Delta^{(1)} \) is an affine root system of type \( A_3^{(1)} + (n - 1)A_1^{(1)} \). We denote by \( W_1 \) the Wyl group of \( \Delta^{(1)} \). We consider the infinite reduced word \( h^{(1)} = (\ldots, i_{-1}, i_0, i_1, \ldots) \) of \( W_1 \) defined in [BN] as in [SZ, 1.5], and define \( \beta_k \in (\Delta^{(1)})^{\text{re}, +} \) for \( k \in \mathbb{Z} \) as in (6.1.5). Then we have

(6.2.4) \( \{ \beta_k \mid k \leq 0 \} = \Delta_\prec^{(1)}, \quad \{ \beta'_k \mid k > 0 \} = \Delta_\succ^{(1)}. \)
Note that the discussion in the case (B) works also for the cases (A), (C), (D). Thus again by [SZ2, Lemma 4.3], we see that

(6.2.5) \( \Delta_+^{(1)} \prec \delta \prec \Delta_+^{(1)} \) satisfies the condition (6.2.1).

We define a total order \( \prec \) on \( \Delta_+^{\text{min}} = \Delta_+ \cup \{ \delta \} \) by extending the total order on \( \Delta_+^{(i)} \) and \( \Delta_+^{(i)} \) for \( i = 1, 2 \), under the condition

(6.2.6) \( \Delta_+^{(0)} \prec \Delta_+^{(1)} \prec \delta \prec \Delta_+^{(1)} \prec \Delta_+^{(0)} \).

Recall that the coarse type of the convex order \( \preceq \) on \( \Delta_+^{\text{min}} \) is defined as the unique \( \overline{w} \in W_0 \) such that \( \overline{w}(\Delta^+_+) = \{ \beta \in \Delta_0 \mid \beta \prec \delta \} \) (see [SZ2, 4.8]). The convex order is called the standard type if the coarse type \( \overline{w} \) coincides with the identity element in \( W_0 \).

**Lemma 6.3.** The total order \( \prec \) gives a convex order on \( \Delta_+^{\text{min}} \).

**Proof.** We show that \( \prec \) satisfies the condition (6.2.1) for \( X = \Delta_+^{\text{min}} \). Note that in this case, the condition (6.2.2) is redundant. It is enough to consider the case where \( \beta \in \Delta_+^{(0)} \cup \Delta_+^{(0)} \) and \( \beta' \in \Delta_+^{(1)} \cup \Delta_+^{(1)} \). Take \( \beta \in \Delta_+^{(0)} \), and assume that \( \gamma = \beta + \beta' \in \Delta_+^{(1)} \). We show that \( \gamma \in \Delta_+^{(0)} \) and \( \beta \prec \gamma \). We have \( O(\gamma) = O(\beta) + O(\beta') \notin \mathbb{Z}_{>0} \delta \) since \( O(\beta) \notin \mathbb{Z}_{>0} \delta \) and \( O(\beta') \notin \mathbb{Z}_{>0} \delta \). Thus \( \gamma \in \Delta_+^{(0)} \cup \Delta_+^{(0)} \). If \( \beta' \in \Delta_+^{(1)} \), we must have \( \gamma \in \Delta_+^{(0)} \), and \( \beta \prec \gamma \) (see (6.2.3)). So consider the case where \( \beta' \in \Delta_+^{(1)} \). Suppose that \( \gamma \in \Delta_+^{(0)} \). We can write as \( \beta = m\delta + \alpha, \beta' = m'\delta - \alpha' \) and \( \beta + \beta' = m''\delta - \alpha'' \), where \( \alpha, \alpha'' \in \Delta_+^{(0)} \) and \( \alpha' \in \Delta_+^{(1)} \). Then we have \( \alpha + \alpha'' = \alpha' \). This contradicts \( \alpha' \in \sum_0^{(1)} \). Hence \( \gamma \in \sum_0^{(0)} \), and we have \( \beta \prec \gamma \) by (6.2.3). Thus \( \prec \) gives a convex order on \( \Delta_+^{\text{min}} \). \( \square \)

6.4. Let \( \prec \) be the convex order on \( \Delta_+^{\text{min}} \) defined in Lemma 6.3. Let \( \mathcal{C} \) be the set of triples \( \mathbf{c} = (\mathbf{c}_+, \mathbf{c}_0, \mathbf{c}_-) \), where \( \mathbf{c}_+ = (c_\beta)_{\beta \prec \delta}, \mathbf{c}_- = (c_\gamma)_{\gamma \prec \delta} \), and \( \mathbf{c}_0 = (\rho^i)_{i \in I_0} \) is an \( I_0 \)-tuple of partitions. For each finite sequence

\[ \beta_1 \prec \beta_2 \prec \cdots \prec \beta_N \prec \delta \prec \gamma_M \prec \cdots \prec \gamma_2 \prec \gamma_1 \]

in \( \Delta_+^{\text{min}} \), we define

(6.4.1) \[ L(\mathbf{c}, \prec) = \prod_{i=1}^N f_{\beta_i}^{\prec,c_{\beta_i}} \cdots f_{\beta_1}^{\prec,c_{\beta_1}} S_{\mathbf{c}_0} f_{\gamma_M}^{\prec,c_\gamma M} \cdots f_{\gamma_1}^{\prec,c_{\gamma_1}}. \]

as in [SZ2, 4.19]. Then by the discussion in [SZ2, Section 4], \( \mathcal{X}_\prec = \{ L(\mathbf{c}, \prec) \mid \mathbf{c} \in \mathcal{C} \} \) gives a PBW-basis of \( U^-_q \). The coarse type \( \overline{w} \in W_0 \) of the convex order \( \prec \) is given by \( \overline{w} = \prod_{i \in I_0} s_i \). Hence by [MT, Theorem 4.13] that \( S_{\mathbf{c}_0}^\prec = T_{\overline{w}} S_{\mathbf{c}_0} \), where \( S_{\mathbf{c}_0} \) is the one defined by [BN] (see [SZ2, 1.5]), and \( w \in W \) is the minimal length lift of \( \overline{w} \in W_0 \). Also by [SZ2, Lemma 6.7], for \( \beta \in \Delta_+^{(0)} \cup \Delta_+^{(0)} \), \( f_{\beta}^{\prec,c} \) coincides with \( f_{\beta_k}^{h_\beta^k} \), where for \( \beta = \beta_k \) in the notation of (6.1.6), we define the root vector \( f_{\beta_k}^{h_\beta^k} \) (with
respect to the infinite reduced word \( h' \) in (6.1.1)) by

\[
f_{h'} = \begin{cases} \prod_{i=0}^{k} T_i T_{i-1} \cdots T_{k+1} f_{i_k}^{(c)} & \text{if } k \leq 0, \\ \prod_{i=0}^{k} T_i^{-1} T_{i-1}^{-1} \cdots T_{k+1}^{-1} f_{i_k}^{(c)} & \text{if } k > 0. \end{cases}
\]

**Remark 6.5.** In [SZ2, Lemma 6.4], it is stated that the convex order \( \prec \) defined there is of standard type. But this is incorrect, and the coarse type \( \bar{w} \in W_0 \) is given in a similar way as above.

6.6. Let \( \prec \) be a convex order of \( \Delta^\text{min}_p \) such that \( \beta_0 \) is minimal, i.e., \( \beta_0 \prec \beta_{-1} \prec \beta_{-2} \cdots \). We assume further that \( \beta_0 = \alpha_i \) is a simple root. Let \( s_i = s_{\alpha_i} \in W \) be the simple reflection. We can define a new convex order \( \prec s_i \) by moving \( \beta_0 \) to the right end, and by acting \( s_i \) on the remaining part, namely,

\[s_i(\beta_{-1}) \prec s_i(\beta_{-2}) \prec \cdots \prec s_i(\beta_1) \prec s_i(\beta_0),\]

where \( \beta_0 \) is maximal in \( \prec s_i \). Similarly, if \( \beta_1 = \alpha_i \) is maximal in \( \Delta^\text{min}_p \), i.e., \( \beta_0 \prec \beta_2 \prec \beta_1 \), we can define a new convex order \( \prec s_i \) in a similar way, by moving \( \beta_1 \) to the left end.

We apply this construction to our convex order on \( \Delta^\text{min}_p \). In this case, the part in \( \Delta^0_s \cup \Delta^0_p \) is determined by \( h' \), namely for \( h' = (\ldots, i_{-2}, i_{-1}, i_0, i_1, i_2, \ldots) \), the order is given by

\[\beta_0 \prec \beta_{-1} \prec \beta_{-2} \prec \cdots \prec \beta_2 \prec \beta_1,\]

where \( \beta_0 = \alpha_{i_0}, \beta_1 = \alpha_{i_1}, \) and \( \beta_k \) are given as in (6.1.5). Thus the above construction of \( \prec s_i \) can be applied successively for \( s_{i_0}, s_{i_{-1}}, \ldots, s_{i_{k+1}} \), and one can define a convex order \( \prec w \) for \( w = s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}} \) by \( \prec w = (\cdots((\prec s_{i_0}) s_{i_{-1}}) \cdots) s_{i_{k+1}} \). We obtain

\[
\alpha_k \prec w s_{i_k}(\alpha_{i_{k-1}}) \prec w s_{i_k} s_{i_{k-1}}(\alpha_{i_{k-2}}) \prec w \cdots
\]

Similarly, one can define a convex order \( \prec w \) for \( w = s_{i_1} \cdots s_{i_{k+1}}(k \geq 1) \), and we obtain

\[
\cdots \prec w s_{i_k} s_{i_{k+1}}(\alpha_{i_{k+2}}) \prec w s_{i_k}(\alpha_{i_{k+1}}) \prec w \alpha_k
\]

Here for each \( p \leq 0 \), we consider the subsequence \( (\eta_p, \ldots, \eta_{-1}, \eta_0) \) of \( h \), and let \( (i_{k+1}, \ldots, i_{-1}, i_0) (k \leq 0) \) be the subsequence of \( h' \) corresponding to \( (J_{\eta_p}, \ldots, J_{\eta_0}). \) Similarly, we define a subsequence \( (i_1, \ldots, i_{-1}) \) of \( h' \) associated to the subsequence \( (\eta_1, \eta_2, \ldots, \eta_p) \) for \( p \geq 1 \). Let \( w = s_{i_0} \cdots s_{i_{k+1}} \) or \( w = s_{i_1} \cdots s_{i_{k+1}} \), and consider the convex order \( \prec w \) defined above, which we denote by \( \prec_p \). Thus \( \prec_p \) gives a two-row order

\[
\Delta^0_p \prec_p \Delta^{(1)}_p \prec_p \Delta^{(1)}_p \prec_p \Delta^{(0)}_p.
\]
Now we define \( L(c, \prec_p) \) by replacing \( \prec \) by \( \prec_p \) in (6.4.1). We note, by [MT, Cor. 4.14], that
\[
S_{c_0}^{\prec_p} = \begin{cases} 
T_{i_p}^{-1} \cdots T_{i_0}^{-1} (S_{c_0}^{\prec}), & \text{if } p \leq 0, \\
T_{i_p-1} \cdots T_{i_1} T_{i_1} (S_{c_0}^{\prec}), & \text{if } p \geq 1.
\end{cases}
\] (6.6.3)

Set
\[
\mathcal{X}_{\prec_p} = \{ L(c, \prec_p) \mid c \in \mathcal{C}_p \},
\]

where \( \mathcal{C}_p \) is the parameter set for \( \prec_p \). Then \( \mathcal{X}_{\prec_p} \) gives a PBW-basis of \( U_q^- \).

By [MT, Prop.4.18], the following holds.
\[
T_{i_p}^{-1} L(c, \prec_p) = L(c', \prec_{p-1}), \quad \text{if } p \leq 0 \text{ and } c_\alpha = 0,
\]
\[
T_{i_p+1} L(c, \prec_p) = L(c', \prec_{p+1}), \quad \text{if } p \geq 1 \text{ and } c_\alpha = 0.
\] (6.6.4)

Here \( \alpha = \alpha_{i_p} \) or \( \alpha = \alpha_{i_p+1} \) according to \( p \leq 0 \) or \( p \geq 1 \). \( c' \) is defined by \( c'_\beta = c_{s_{i_p}(\beta)} \) for \( \beta \neq \alpha \), \( c'_\alpha = 0 \), and the imaginary part is defined by the action of \( T_{i_p}^{-1} \) or \( T_{i_p+1} \) accordingly (see (6.6.3)).

### 6.7
Following the discussion in 3.4, we modify the root vectors corresponding to \( \Delta^{(0)}_{>,p} \) and \( \Delta^{(0)}_{<,p} \). Note that for \( \beta \in \Delta^{(0)}_{>,p} \) or \( \beta \in \Delta^{(0)}_{<,p} \), \( f^\prec_\beta \) coincides with \( f^{h'(w)}_\beta \), where \( h'(w) \) is the doubly infinite sequence obtained from \( h' \) by the action of \( w \in W \) as in 6.6. In the case where \( \delta_{i_k} = 1 \), set \( d_k = (d_j)_{j \in \eta_k} \in N^{\eta_k} = N^{J_k} \), where \( J_k \) is as in 6.1. We define
\[
\tilde{f}_{i_k}^{(d_k)} = \begin{cases} 
\tilde{T}_{i_p} \cdots \tilde{T}_{i_0} (\prod_{j \in \eta_k} f^{(d_j)}_j) & \text{if } p \leq 0, \\
\tilde{T}_{i_p-1} \cdots \tilde{T}_{i_2} \tilde{T}_{i_1} (\prod_{j \in \eta_k} f^{(d_j)}_j) & \text{if } p > 0.
\end{cases}
\] (6.7.1)

While in the case where \( \delta_{i_k} = 2 \) with \( \eta_k = \{i, j\} \), we consider the subalgebra \( \mathbf{L}_{\eta_k}^- \) of \( U_q^- \) isomorphic to \( U_q(\mathfrak{sl}_3)^- \), and consider the canonical basis \( B_{\eta_k} = \{ b(d_k, h_{\eta_k}) \} \) as defined in 3.4, where \( d_k \in N^3 = N^{J_k} \). We define
\[
\tilde{f}_{i_k}^{(d_k)} = \begin{cases} 
\tilde{T}_{i_p} \cdots \tilde{T}_{i_0} (b(d_k, h_{\eta_k})) & \text{if } p \leq 0, \\
\tilde{T}_{i_p-1} \cdots \tilde{T}_{i_2} \tilde{T}_{i_1} (b(d_k, h_{\eta_k})) & \text{if } p > 0.
\end{cases}
\] (6.7.2)

In the definition of \( L(c, \prec_p) \), we replace the root vectors belonging to \( \Delta^{(0)}_{>,p} \) and \( \Delta^{(0)}_{<,p} \) by (6.7.1) and (6.7.2), and leave the root vectors belonging to \( \Delta^{(1)}_< \sqcup \Delta^{(1)}_> \) unchanged. Thus one can define \( L^\prec(d, \prec_p) \) as in 3.4. Set
\[
\mathcal{X}_{\prec_p} \supseteq \{ L^\prec(d, \prec_p) \mid d \in \mathcal{C}_p \},
\] (6.7.3)
where \( \mathcal{C}^2_p = \{ d = (d^1, d_0, d^2) \} \) is a parameter set for \( \mathcal{X}^2_{\prec p} \). Here if we define \( d^{(0)}_+ \) (resp. \( d^{(0)}_- \)) as the part in \( d^1 \) (resp. \( d^2 \)) corresponding to \( \Delta^{(0)}_{\prec p} \) (resp. \( \Delta^{(0)}_{\succ p} \)), we have

\[
\begin{align*}
d^{(0)}_+ &= (d_k)_{k \leq p} \text{ with } d_k \in \mathbf{N}^{J_k}, \quad d^{(0)}_- = (d_k)_{k > p} \text{ with } d_k \in \mathbf{N}^{J_k},
\end{align*}
\]

where \( J_k = J_{\eta_k} \). Note that the parameter set \( \mathcal{C}^2_p \) is naturally identified with \( \mathcal{C}_p \) under the correspondence \( d_k \leftrightarrow (\cdots, c_{\beta_j}, \cdots)_{j \in J_k} \) (see 2.1 for the case where \( \delta_{\eta_k} = 2 \)). \( \mathcal{X}^2_{\prec p} \) gives an \( A \)-basis of \( A U_q^- \), which we call the modified PBW-basis of \( U_q^- \). As in (6.6.4), we have the following.

\[
\begin{align*}
\tilde{T}^{-1}_{mp} L^2(c, \prec_p) &= L^2(c', \prec_{p-1}), & \text{if } p \leq 0 \text{ and } c_p = 0, \\nn\tilde{T}^{-1}_{mp+1} L^2(c, \prec_p) &= L^2(c', \prec_{p+1}), & \text{if } p > 0 \text{ and } c_p = 0, \n\end{align*}
\]

6.8. Let \( \beta_1 < \beta_2 < \cdots < \delta < \cdots < \beta_1 \) be the convex order associated to \( h \).

We know that \( \mathcal{A}^\text{re.} \times \mathcal{T}_\text{re.} = \{ \beta_k \mid k \leq 0 \} \) and \( \mathcal{A}^\text{re,+} = \{ \beta_k \mid k > 0 \} \). Thus for each \( \eta \in L \), there exists \( p \in \mathbf{Z} \) such that \( \eta = \eta_p \). We consider the modified PBW-basis \( \mathcal{X}^2_{\prec p} \) of \( U_q^- \). In the case where \( p \leq 0 \), we denote by \( U_q^- [\eta] \) the \( Q(q) \)-subspace of \( U_q^- \) spanned by \( L^2(c, \prec_p) \) such that \( c_p = 0 \), and set \( ^* U_q^- = *(U_q^- [\eta]) \), where \( * : U_q^- \rightarrow U_q^- \) is the anti-algebra automorphism. While in the case where \( p > 0 \), we define \( ^* U_q^- [\eta] \) as the \( Q(q) \)-subspace of \( U_q^- \) spanned by \( L^2(c, \prec_p) \) such that \( c_p = 0 \), and set \( U_q^- [\eta] = *(^* U_q^- [\eta]) \). Note that in the case where \( \delta_{\eta} = 1 \) with \( \eta = \{ \alpha_{i_1}, \ldots, \alpha_{i_k} \} \), where \( k = |\eta| \), we consider \( L_{\eta}^- \) the subalgebra of \( U_q^- \) generated by \( f_{i_1}, \ldots, f_{i_k} \). Then \( L_{\eta}^- \) is of type \( A_1 \times \cdots \times A_1 \) (\( k \)-times), and the set \( \{ f_{i_1}^{(c_1)} \cdots f_{i_k}^{(c_k)} \mid c = (c_1, \ldots, c_k) \in \mathbf{N}^k \} \) coincides with the canonical basis \( B_\eta = \{ b(c, h_\eta) \mid c \in \mathbf{N}^{[n]} \} \). In the following discussion, we write \( b(c, h_\eta) \) simply as \( b(c_\eta) \), where

\[
c_\eta = \begin{cases} (c_1, \ldots, c_{|\eta|}) \in \mathbf{N}^{[n]} & \text{if } \delta_\eta = 1, \\ (c_1, c_2, c_3) \in \mathbf{N}^3 & \text{if } \delta_\eta = 2. \end{cases}
\]

Thus even for the case (6.7.1), we use the same notation as in (6.7.2), by using the canonical basis \( B_\eta \).

In the case where \( \eta = \eta_p \) with \( p \leq 0 \), let \( A U_q^- [\eta] \) be the \( A \)-submodule of \( A U_q^- \) spanned by \( L^2(c, \prec_p) \) such that \( c_p = 0 \), and set \( ^* A U_q^- [\eta] = *(A U_q^- [\eta]) \). In the case where \( \eta = \eta_p \) with \( p > 0 \), let \( ^* A U_q^- [\eta] \) be the \( A \)-submodule of \( A U_q^- \) spanned by \( L^2(c, \prec_p) \) such that \( c_p = 0 \), and set \( A U_q^- [\eta] = *(A U_q^- [\eta]) \).

Lemma 6.9. (i) We have a direct sum decomposition

\[
\begin{align*}
U_q^- &= \bigoplus_{b \in B_\eta} b U_q^- [\eta], \\
U_q^- &= \bigoplus_{b \in B_\eta} ^* U_q^- [\eta] b.
\end{align*}
\]
While in the case where $\eta = \eta_p$ for $p \leq 0$. Then by using the PBW-basis $\mathcal{D}_q^2$, we obtain the first identity in (6.9.1). Note that $\sigma$ acts on $L_q^-$, and $\sigma$ gives a permutation on the canonical basis $B_q$ of $L_q^−$, which is the same as the $*$-operation on $B_q$. Thus the second identity follows from the first by applying $*$ on both sides. In the case where $\eta = \eta_p$ with $p > 0$, by using $\mathcal{D}_q^2$, we obtain the second identity in (6.9.1). By applying $*$, the first identity follows. (i) is proved. (ii) follows from (i) since $\mathcal{D}_q^2$ is an $A$-basis of $A U_q^−$. 

**Proposition 6.10.** Under the notation above, we have

\[
\begin{align*}
U_q^−[\eta] &= \{ x \in U_q^− \mid \tilde{T}_q^{-1}(x) \in U_q^− \}, \\
\ast U_q^−[\eta] &= \{ x \in U_q^− \mid \tilde{T}_q(x) \in U_q^− \}.
\end{align*}
\]

**Proof.** We consider the case where $\eta = \eta_p$ with $p \leq 0$. By (6.7.4), we have

\[
U_q^−[\eta] \subset \{ x \in U_q^− \mid \tilde{T}_q^{-1}(x) \in U_q^− \}.
\]

We show the converse inclusion. Lemma 6.9 implies that the map $u \otimes v \mapsto uv$ gives an isomorphism as vector spaces $U_q^− \simeq L_q^- \otimes U_q^−[\eta]$. Hence the proof of (6.10.1) is reduced to the case where $U_q^− = L_q^-$. It is enough to see that $\{ x \in L_q^- \mid \tilde{T}_q^{-1}(x) \in L_q^- \} = \{ 0 \}$. In general, we consider the braid group action $T_w : U_q \rightarrow U_q$. We have a weight space decomposition $U_q = \bigoplus_{\nu \in q} (U_q)_{\nu}$, and we know that $T_w : (U_q)_{\nu} \rightarrow (U_q)_{w(\nu)}$. Hence $L_q^- \cap \tilde{T}_q(L_q^-) = \{ 0 \}$ since $\tilde{T}_q(L_q^-) = L_q^+$, where $L_q^+ = U_q^+$ is the positive part of $U_q$. This proves (6.10.1). Then (6.10.2) is obtained from (6.10.1) by applying the $*$-operation, since $\ast \circ \tilde{T}_q \circ * = \tilde{T}_q^{-1}$. In the case where $\eta = \eta_p$ with $p > 0$, (6.10.2) is proved in a similar way. Then (6.10.1) follows from (6.10.2). The proposition is proved. 

**6.11.** Proposition 6.10 shows that $U_q^−[\eta]$ does not depend on the choice of $h$. Note that $\tilde{T}_q = \prod_{i \in I} T_i$ for $\eta \in I$ with $\delta_q = 1$, and $\tilde{T}_q = T_i T_j T_i = T_j T_i T_j$ if $\eta = \{ i, j \}$ with $\delta_q = 2$. Hence $\sigma \circ \tilde{T}_q = \tilde{T}_q \circ \sigma$. Then Proposition 6.10 implies that $U_q^−[\eta]$ is $\sigma$-stable. Note that this is not directly obtained from the definition since we don’t know that $\mathcal{D}_q^2$ is stable under the action of $\sigma$.

In the case where $\delta_q = 1$, the set $B_q^\sigma$ is given as $B_q^\sigma = \{ \tilde{f}^{(a)}_{\eta} = \prod_{i \in I} f_i^{(a)} \mid a \in N \}$. While in the case where $\delta_q = 2$ with $\eta = \{ i, j \}$,

\[
B_q^\sigma = \{ \tilde{f}^{(a)}_{\eta} = f_i^{(a)} f_j^{(2a)} f_i^{(a)} = f_j^{(a)} f_i^{(2a)} f_j^{(a)} \mid a \in N \}.
\]
The following result is obtained immediately from Lemma 6.9. Note that the uniqueness property follows from the property of PBW-basis $\mathcal{D}_{\prec_p}^\times$.

**Proposition 6.12.** Concerning the first formula in (6.9.1), the following holds.

(i) We have a direct sum decomposition

$$U_q^{-\sigma} = \bigoplus_{a \in \mathbb{N}} \mathcal{J}_a^{(a)}(U_q^-[\eta])^\sigma \oplus K_\eta,$$

where $K_\eta$ is a sum of $bx + \sigma(bx)$ for $b \in B_\eta$ such that $\sigma(b) \neq b$, and $x \in U_q^-[\eta]$. $x \in U_q^{-\sigma}$ can be written uniquely as $x = \sum_{a \in \mathbb{N}} \mathcal{J}_a^{(a)} x_a + y$ with $x_a \in (U_q^-[\eta])^\sigma$ and $y \in K_\eta$.

(ii) We also have a decomposition as $A$-modules

$$A U_q^{-\sigma} = \bigoplus_{a \in \mathbb{N}} \mathcal{J}_a^{(a)}(A U_q^-[\eta])^\sigma \oplus A K_\eta,$$

where $A K_\eta$ is the $A$-submodule of $K_\eta$ generated by $bx + \sigma(bx)$ as above with $x \in A U_q^-$.

Similar results hold also for the second formula in (6.9.1).

**Remark 6.13.** The above results correspond to the results Lemma 3.6 ∼ Lemma 3.11 in [SZ2]. But the discussion here becomes much simpler than those used in [SZ2] once we use the PBW-basis $\mathcal{D}_{\prec_p}^\times$.

**Lemma 6.14.** For each $\eta \in L^+$, we have

(6.14.1) $U_q^-[\eta] \subseteq \bigcap_{i \in \eta} U_q^-[i]$ if $\delta_\eta = 1$, $U_q^-[i] \cap U_q^-[j]$ if $\delta_\eta = 2$ with $\eta \neq \{i, j\}$.

(6.14.2) $^*U_q^-[\eta] \subseteq \bigcap_{i \in \eta} ^*U_q^-[i]$ if $\delta_\eta = 1$, $^*U_q^-[i] \cap ^*U_q^-[j]$ if $\delta_\eta = 2$ with $\eta \neq \{i, j\}$.

**Proof.** Assume that $\eta = \eta_p$ with $p \leq 0$. We consider the case where $\delta_\eta = 2$ with $\eta = \{i, j\}$. Then $U_q^-[\eta]$ is spanned by $L^i(c, \prec_p)$ with $c_p = 0$. This basis is also expressed as a sum of the ordinary PBW-bases $L(c', \prec_p)$. Since $\eta_p$ corresponds to $(i, j, i) \in h'$, by the property of $\mathcal{D}_{\prec_p}$ similar to (6.7.4), we see that $T_i^{-1}(x) \in U_q^-$ for $x \in U_q^-[\eta]$. By the discussion using $(j, i, j)$, we also have $T_j^{-1}(x) \in U_q^-$. Hence $x \in U_q^-[i] \cap U_q^-[j]$. The proof for the case $\delta_\eta = 1$ is similar. Thus (i) holds. Then (ii) follows from (i) by applying *-operation.

Next assume that $\eta = \eta_p$ with $p > 0$. In this case, by a similar argument as above, one can prove (ii). Then (i) follows from (ii) by applying *. The lemma is proved. □
Proposition 6.15. For each \( \eta \in L \), we have a direct sum decomposition

\[
U_q^{-\sigma} = (U_q^{-[\eta]})^\sigma \oplus \tilde{f}_\eta U_q^{-\sigma} \oplus K_\eta.
\]

The first and the second terms are orthogonal each other with respect to the inner product \( (\ , \ ) \).

Proof. We show the first statement. This is clear if \( \delta_\eta = 1 \). We assume that \( \delta_\eta = 2 \). By the proof of Proposition 2.4, \( \tilde{f}_\eta f^{(n)}_\eta \equiv f^{(n+1)}_\eta \mod K_\eta \), up to scalar. Hence, \( \bigoplus_{n > 0} f^{(n)}_\eta U_q^{-[\eta]} \subset \tilde{f}_\eta U_q^{-} \mod \eta \). Then (6.15.1) follows from Proposition 6.12. We show the second statement. Assume that \( \eta = \{i, j\} \) with \( \delta_\eta = 2 \). Then \( \tilde{f}_\eta = f_i f_j f_i \), and \( \tilde{f}_\eta U_q^{-\sigma} \subset f_i U_q^{-} \). By Lemma 6.14, \( U_q^{-[i]} \subset U_q^{-} \). We know that \( (U_q^{-[i]}, f_i U_q^{-}) = 0 \), and this implies that \( (U_q^{-[i]}, \tilde{f}_\eta U_q^{-}) = 0 \). The lemma holds. \( \square \)

6.16. We define a partial order \( c < c' \) in \( C_q \) by the condition as follows; for \( c = (c_+, c_0, c_-), c' = (c'_+, c'_0, c'_-), c < c' \) if and only if \( c_+ \leq c'_+ \) and \( c_- \leq c'_- \), and at least one of the inequalities is strict, with respect to the lexicographic order \( \leq \) on \( c_+ \) (resp. on \( c_- \)) from the left (resp. the right). Then for each \( L(c, \prec_p) \in \mathcal{X}_\prec_q \), there exists a unique \( b(c, \prec_p) \in \tilde{B} \) satisfying the property that \( b(c, \prec_p) \) is \( \prec \)-invariant, and that

\[
b(c, \prec_p) = L(c, \prec_p) + \sum_{d \succ c} a_d L(d, \prec_p),
\]

where \( a_d \in qZ[q] \). Then \( \pm b(c, \prec_p) \in B \). A similar construction also works for \( L^\natural(c, \prec_p) \), and one can define the canonical (signed) basis \( b^\natural(c, \prec_p) \) by the property that \( b^\natural(c, \prec_p) \) is \( \prec \)-invariant and that

\[
b^\natural(c, \prec_p) = L^\natural(c, \prec_p) + \sum_{d \succ c} a_d L^\natural(d, \prec_p),
\]

where \( a_d \in qZ[q] \).

The \( Z[q] \)-submodule \( \mathcal{L}_Z(\infty) \) of \( U_q^{-} \) is defined as in [MSZ, 1.10]. The set \( \{ b^\natural(c, \prec_p) \mid c \in C_q \} \) gives a \( Z[q] \)-basis of \( \mathcal{L}_Z(\infty) \), and the modified PBW-bases also \( \mathcal{X}^\natural_{\prec_q} \) gives a \( Z[q] \)-basis of \( \mathcal{L}_Z(\infty) \).

Note that the set of \( \sigma \)-orbit sums of \( B \) gives a basis of \( U_q^{-\sigma} \), and an \( A \)-basis of \( A U_q^{-\sigma} \). The set \( \tilde{B}^\sigma \) is contained in \( \bigoplus_{n \geq 0} f^{(n)}_\eta (U_q^{-[\eta]})^\sigma \) or in \( \bigoplus_{n \geq 0} (U_q^{-[\eta]})^\sigma f^{(n)}_\eta \) by Proposition 6.12. For \( b \in \tilde{B}^\sigma \), we define \( \varepsilon_\eta(b) \) as the largest integer \( a \) such that \( b \in f^{(n)}_\eta U_q^{-\sigma} \), and \( \varepsilon^*_\eta(b) \) as the largest integer \( a \) such that \( b \in U_q^{-\sigma} f^{(a)}_\eta \).

Proposition 6.17. (i) Let \( b \in \tilde{B}^\sigma \) be such that \( \varepsilon_\eta(b) = 0 \). There exists a unique \( b' \in \tilde{B}^\sigma \) such that

\[
f^{(n)}_\eta b \equiv b' \mod f^{(n+1)}_\eta U_q^{-\sigma} \oplus K_\eta.
\]
Then \( \varepsilon_\eta(b') = n \). The correspondence \( b \mapsto b' \) gives a bijection

\[
\pi_{n, \eta} : \{ b \in \tilde{B}^\sigma \mid \varepsilon_\eta(b) = 0 \} \simeq \{ b' \in \tilde{B}^\sigma \mid \varepsilon_\eta(b') = n \}.
\]

(ii) Let \( b \in \tilde{B}^\sigma \) be such that \( \varepsilon_\eta^*_n(b) = 0 \). There exists a unique \( b' \in \tilde{B}^\sigma \) such that

\[
b \tilde{f}_n^{(n)} \equiv b' \mod U_{q}^{-\sigma} \tilde{f}_{\eta}^{(n+1)} \oplus K_{\eta}.
\]

Then \( \varepsilon_\eta^*(b') = n \). The correspondence \( b \mapsto b' \) gives a bijection

\[
\pi_{n, \eta}^* : \{ b \in \tilde{B}^\sigma \mid \varepsilon_\eta^*(b) = 0 \} \simeq \{ b' \in \tilde{B}^\sigma \mid \varepsilon_\eta^*(b') = n \}.
\]

Proof. We assume that \( \eta = \eta_p \) with \( p \leq 0 \), and consider \( \mathcal{A}_{<n}^\mathcal{A} \). We also assume that \( \eta = \{ i, j \} \) with \( \delta_\eta = 2 \). We prove (i). Take \( b \in \tilde{B}^\sigma \) such that \( \varepsilon_\eta(b) = 0 \). Then there exists \( \tilde{b}(c, <_{\eta}) \) such that \( b = \pm \tilde{b}(c, <_{\eta}) \). We write

\[
\tilde{b}(c, <_{\eta}) = L^\sigma(c, <_{\eta}) + \sum_{d \in c} a_d L^\sigma(d, <_{\eta})
\]

with \( a_d \in q\mathbb{Z}[q] \). Since \( b \in \tilde{B}^\sigma \), \( c = (c_+, c_0, c_-) \) with \( c_p = 0 = (0, 0, 0) \), and \( d_p = (d_1, d_1, d_1) \) for some \( d_1 \geq 0 \), namely, \( L^\sigma(d, <_{\eta}) = \tilde{f}_{\eta}^{(d_1)} L^\sigma(d', <_{\eta}) \) with \( d' = 0 \).

Since \( \tilde{f}_{\eta}^{(n)} \tilde{b}(c, <_{\eta}) \in U_{q}^{-\sigma} \), by Proposition 6.12 it is written as a linear combination of \( \tilde{f}_{\eta}^{(a)} L^\sigma(c', <_{\eta}) \) modulo \( K_{\eta} \) for various \( a \) and \( c' \), where \( c_p' = 0 \). It follows from the computation for \( U_{q}(\mathfrak{sl}_n) \) (see the proof of Proposition 2.4) that \( \tilde{f}_{\eta}^{(n)} \tilde{f}_{n}^{(d_1)} \) is a linear combination of canonical basis in \( B_{\eta} \), among them, the \( \sigma \)-invariant one is only \( \tilde{f}_{\eta}^{(d_1+n)} \). Therefore, we have

\[
\tilde{f}_{\eta}^{(n)} b^\sigma(c, <_{\eta}) = L^\sigma(c', <_{\eta}) + \sum_{d \in c} a_d \left[ \begin{array}{c} d_1 + n \\ n \end{array} \right] \tilde{f}_{\eta}^{(d_1+n)} L^\sigma(d', <_{\eta}) + z
\]

\[
= L^\sigma(c', <_{\eta}) + \sum_{d \in c} a_d L^\sigma(d'', <_{\eta}) + y + z,
\]

with

\[
y = \sum_{d_1 > 0} a_d \left( \left[ \begin{array}{c} d_1 + n \\ n \end{array} \right] - 1 \right) L^\sigma(d'', <_{\eta}), \quad z \in A K_{\eta},
\]

where \( L^\sigma(c', <_{\eta}) = \tilde{f}_{\eta}^{(n)} L^\sigma(c, <_{\eta}) \), and \( L^\sigma(d'', <_{\eta}) = \tilde{f}_{\eta}^{(d_1+n)} L^\sigma(d', <_{\eta}) \) are elements in \( \mathcal{A}_{<n}^\mathcal{A} \). Write \( y + z = \sum_e c_e b^\sigma(e, <_{\eta}) \) with \( c_e \in A \). Since \( d_1 > 0 \), we have \( e_p > n = (n, n, n) \) if \( e_p \) is \( \sigma \)-stable. We write \( c_e \) as \( c_e = c_e^+ + c_e^0 + c_e^- \), where \( c_e^+ \in q \mathbb{Z}[q] \), \( c_e^- \in q^{-1} \mathbb{Z}[q^{-1}] \) and \( c_e^0 \in \mathbb{Z} \). Then we have

\[
\tilde{f}_{\eta}^{(n)} b^\sigma(c, <_{\eta}) = \sum_e \left( c_e^0 + c_e^- + c_e^+ \right) b^\sigma(e, <_{\eta})
\]
\[ L^\natural(c', \prec_p) \equiv \sum_{d < e} a_d L^\natural(d'', \prec_p) + \sum_{e} (c^+_e - c^-_e) b^\natural(d'', \prec_p). \]

The left hand side of (6.17.5) is \(-\)-invariant, and the right hand side belongs to \( L^\natural, \) and is equal to \( L^\natural(c', \prec_p) \mod qL^\natural, \) Hence by (6.16.2), this coincides with \( b^\natural(c', \prec_p), \) It follows that

\[ f_\eta^{(n)} b^\natural(c, \prec_p) = b^\natural(c', \prec_p) \equiv \sum_{e} (c^+_e + c^-_e) b^\natural(e, \prec_p) \mod K_\eta, \]

where \( b^\natural(e, \prec_p) \) in the last sum corresponds to \( L^\natural(e, \prec_p) \in f_\eta^{(n')} U_q^{-}[\eta] \) for some \( n' > n, \) Hence \( b^\natural(e, \prec_p) \in f_\eta^{(n+1)} U_q^{-}, \) and the sum in (6.17.6) is contained in \( f_\eta^{(n+1)} U_q^{-}. \)

Note that since \( f_\eta^{(n)}(b^\natural(c, \prec_p)) \) is \( \sigma \)-stable, \( b^\natural(c', \prec_p) \) and \( b^\natural(e, \prec_p) \) in the last sum are \( \sigma \)-stable (or its \( \sigma \)-orbit is contained in \( K_\eta \)). Thus \( b' = b^\natural(c', \prec_p) \) satisfies the condition in (6.17.1). It is clear that \( \varepsilon_\eta(b') = n. \)

Note that \( b' = b^\natural(c', \prec_p) \) is determined uniquely from \( L^\natural(c, \prec_p), \) which is the canonical (signed) basis corresponding to the PBW-basis \( f_\eta^{(n)} L^\natural(c, \prec_p). \)

Conversely, assume that \( b' \in \tilde{B}^\sigma \) is such that \( \varepsilon_\eta(b') = n. \) Take \( b^\natural(c', \prec_p) \) such that \( b^\natural(c', \prec_p) = \pm b. \) Then \( L^\natural(c', \prec_p) = f_\eta^{(n')} L^\natural(c'', \prec_p) \) with \( c'_p = 0. \) Let \( x \) be the projection of \( b^\natural(c', \prec_p) \) onto \( f_\eta^{(n')} U_q^{-}[\eta]. \) Then \( x \equiv L^\natural(c', \prec_p) \mod qL^\natural, \) and \( x \) is \( \sigma \)-stable. Write \( x = f_\eta^{(n')} x_0 \) with \( x_0 \in (U_q^{-}[\eta])^\sigma. \) Then \( x_0 \equiv L^\natural(c'', \prec_p) \mod qL^\natural, \) Consider \( b = b^\natural(c'', \prec_p). \) Then \( b \) is determined uniquely from \( x_0 \) by a similar condition as in (6.16.2), hence \( b \) is \( \sigma \)-stable. Clearly \( \varepsilon_\eta(b) = 0. \) It follows from the previous construction that \( \pi_{\eta,n}(b) = b'. \) The map \( b' \mapsto b \) gives the inverse map of \( \pi_{\eta,n}. \) Hence \( \pi_{\eta,n} \) is a bijection. This proves (i). The case where \( \delta_\eta = 1 \) is proved similarly, and is simpler. Now (ii) follows from (i) by applying the \( * \)-operation. Next assume that \( \eta = \eta_p \) with \( p > 0. \) In this case, by a similar argument as above, we can prove (ii). Then (i) is obtained from this by applying the \( \ast \)-operation. The proposition is proved.

\[ \text{The \( A \)-version of Proposition 6.17 is given as follows;} \]

\[ \text{Corollary 6.18. \ Let \( b \in \tilde{B}^\sigma \) be such that \( \varepsilon_\eta(b) = 0. \) Then there exists a unique \( b' \in \tilde{B}^\sigma \) such that} \]

\[ f_\eta^{(n)} b \equiv b' \mod \sum_{n' > n} f_\eta^{(n')} A U_q^{-,\sigma} \oplus A K_\eta. \]

\[ \text{6.19. \ For each} \ \eta \in \bar{L}, \ \text{by making use of the bijection (6.17.2), we define a map} \]

\[ \tilde{F}_\eta \text{ as a composite of bijections} \ \pi_{\eta,n}^{-1} \text{ and} \ \pi_{\eta,n+1} \]
Proposition 6.20. For each \( b \in \tilde{\mathcal{B}}^\sigma \), there exists a sequence \( \eta_1, \ldots, \eta_N \in \mathcal{I} \) such that \( b = \pm \tilde{F}_{\eta_1} \cdots \tilde{F}_{\eta_N} 1 \).

Proof. Take \( b \in \tilde{\mathcal{B}}^\sigma \), and assume that \( b \neq \pm 1 \). Then there exists \( \eta \in \mathcal{I} \) such that \( \varepsilon_\eta(b) \neq 0 \). In fact assume that \( \varepsilon_\eta(b) = 0 \) for any \( \eta \). By Lemma 6.14, if \( \varepsilon_\eta(b) = 0 \), then \( \varepsilon_i(b) = 0 \) for any \( i \in \eta \). Thus \( \varepsilon_i(b) = 0 \) for any \( i \in I \). By a well-known result (see [L1, Lemma 1.2.15]), this implies that \( b = \pm 1 \).

Now take \( \eta \) such that \( \varepsilon_\eta(b) \neq 0 \). Then \( b' = \tilde{E}_\eta(b) \in \tilde{\mathcal{B}}^\sigma \) satisfies the condition that \( \varepsilon_\eta(b') < \varepsilon_\eta(b) \). Then the proposition follows by induction on the weight of \( b \).

We are now in a position to prove the surjectivity of \( \Phi \) in the affine case. The proof is almost parallel to the one in [SZ2, Proposition 2.10].

Proposition 6.21. Assume that \( (X, X) \) satisfies the condition in Theorem 2.6. Then the map \( \Phi : \bigoplus_{\mathcal{I}} U^-_{\eta} \rightarrow V_q \) is surjective, hence Theorem 2.6 holds.

Proof. The image of \( \tilde{\mathcal{B}}^\sigma \) gives a signed basis of \( V_q \). Thus it is enough to show, for each \( b \in \tilde{\mathcal{B}}^\sigma \), that

\[
\pi(b) \in \text{Im } \Phi. \tag{6.21.1}
\]

Take \( b \in \tilde{\mathcal{B}}^\sigma \). Let \( \nu = \nu(b) \in Q_+ \) be the weight of \( b \). We prove (6.21.1) by induction on \( |\nu| \). If \( |\nu| = 0 \), it certainly holds. So assume that \( |\nu| \geq 1 \), and assume that (6.21.1) holds for \( b' \) such that \( |\nu(b')| < |\nu| \). By Proposition 6.20, there exists \( \eta \in \mathcal{I} \) such that \( b = \tilde{F}_\eta(b') \) for some \( b' \in \tilde{\mathcal{B}}^\sigma \) with \( \varepsilon_\eta(b') = 0 \). Thus \( b' \) satisfies (6.21.1) by induction hypothesis. We may further assume that (6.21.1) holds for \( b'' \in \tilde{\mathcal{B}}^\sigma \) such that \( \nu(b'') = \nu \) and that \( \varepsilon_\eta(b'') > n \) (note that since \( \dim(U^-_{\eta})_\nu < \infty \), if \( n' > 0 \), there does not exist \( b'' \) such that \( \nu(b'') = \nu \) and \( \varepsilon_\eta(b'') = n' \)). By Corollary 6.18, \( b \) can be written as \( b = \tilde{f}_\eta^{(n)}y + z \), where \( z \in \mathcal{A}K_n \), and \( y \) is an \( \mathcal{A} \)-linear combination of \( b' \in \tilde{\mathcal{B}}^\sigma \) such that \( \nu(b'') = \nu \) and \( \varepsilon_\eta(b'') > n \). By induction, \( \pi(b') \in \text{Im } \Phi \). Since \( \Phi \) is a homomorphism by Proposition 2.8, we have \( \pi(\tilde{f}_\eta^{(n)}y) = \Phi(f_\eta^{(n)})\pi(b') \in \text{Im } \Phi \). Since \( \mathcal{A}K_n \subset J \), we have \( \pi(z) = 0 \). By induction hypothesis, \( \pi(y) \in \text{Im } \Phi \). This implies that \( \pi(b) \in \text{Im } \Phi \). Hence (6.21.1) holds for \( \nu \). The proposition is proved. \( \square \)

### 6.22

We consider the isomorphism \( \Phi : \bigoplus_{\mathcal{I}} U^-_{\eta} \cong V_q \) as in Theorem 2.6. Recall the \( \mathbb{Z} \)-module \( (Q^\sigma)' = \bigoplus_{\eta \in \mathcal{I}} \mathbb{Z}\alpha_\eta \), and we identify this with \( Q = \bigoplus_{\eta \in \mathcal{I}} \mathbb{Z}\alpha_\eta \) as in 5.2. We have weight space decompositions

\[
U^-_{\eta} = \bigoplus_{\nu \in Q_-} (U^-_{\eta})_\nu \quad \text{and} \quad U^-_{\eta} = \bigoplus_{\nu \in Q_-} (U^-_{\eta})_\nu.
\]
Under the identification $(Q)_- \simeq (Q_\sigma)' \subset Q_-$, Theorem 2.6 implies that we have a weight space decomposition $V_q = \bigoplus_{\nu \in (Q_\sigma)'} (V_q)_\nu$, and $\Phi$ induces an isomorphism $A'(U_q)^\sigma \cong (V_q)_\nu$ for each $\nu \in Q_-$. Note that $\{\pi(b) \mid b \in \tilde{B}_\sigma\}$ gives a signed basis of $V_q$. Hence $\{\pi(b) \mid b \in \tilde{B}_\sigma(\nu)\}$ gives a signed basis of $(V_q)_\nu$, where $\tilde{B}_\sigma(\nu) = \{b \in \tilde{B}_\sigma \mid \nu(b) = \nu\}$. In particular, we have
\[
(6.22.1) \quad |\tilde{B}_\sigma(\nu)|/2 = \dim(U_q^-)_\nu \quad \text{for} \quad \nu \in (Q_\sigma)' \simeq Q_-.
\]

### 6.23. We consider the modified PBW-basis $X_\sigma = \{L^\pm(c, \prec) \mid c \in \mathcal{C}\}$ for the case where $\rho = 0$. Note that it is not verified that $\sigma$ leaves the set $X_\sigma$ invariant. By 6.4, $S_{\sigma_0}^\leq = T_w S_{c_0}$, and $w$ is the minimal lift of $w = \prod_{i \in \mathbb{N} - \{0\}} s_i$. Recall that $c_0 = (\rho(i))_{i \in I_0}$ is an $I_0$-tuple of partitions. Let $\mathcal{C}_0$ be the set of $I_0$-tuple of partitions, and we regard $c_0 \in \mathcal{C}_0$ as an $I_0$-tuple of partition such that $\rho(i) = \emptyset$ for $i \notin I_0$. It follows that $S_{\sigma_0}^\leq = S_{c_0}$ for $c_0 \in \mathcal{C}_0$. $\sigma$ acts naturally on $\mathcal{C}_0$ by $\sigma(\rho(i)) = \rho(\sigma(i))$, and we have $\sigma(S_{\sigma_0}) = S_{\sigma(c_0)}$ for $c_0 \in \mathcal{C}_0$. We denote by $\mathcal{C}_0^{\sigma}$ the set of $\sigma$-fixed elements in $\mathcal{C}_0$. Let $c_+ = (c_\beta)_{\beta < \delta}$, $c_- = (c_\gamma)_{\gamma > \delta}$ be as in 6.4. Let $\mathcal{C}_+$ be the set of $c_+$ such that $c_\beta = 0$ for $\beta \in \Delta(1)$, and $\mathcal{C}_-$ the set of $c_-$ such that $c_\gamma = 0$ for $\gamma \in \Delta(2)$. We denote by $\mathcal{C}_\sigma$ the set of $\sigma$-fixed elements in $\mathcal{C}_\pm$. Let $\mathcal{C}_\sigma$ be the set of triples $c = (c_+, c_0, c_-)$ such that $c_+ \in \mathcal{C}_\sigma^+$, $c_0 \in \mathcal{C}_\sigma$, $c_- \in \mathcal{C}_\sigma^-$. The set of triples $\mathcal{C} = \{c = (c_+, c_0, c_-)\}$, and the set of PBW-basis $\mathcal{X}_h = \{L(c, 0) \mid c \in \mathcal{C}\}$ are defined as in [BN]. As in [SZ2], one can check that we have a natural bijection $\mathcal{C}_\sigma \simeq \mathcal{C}$. Let $(\mathcal{X}_\sigma^\pm)^\sigma$ be the set of $\sigma$-fixed PBW-bases contained in $\mathcal{X}_\sigma$. The following result was proved in [SZ2, Theorem 6.13] in the case where $\sigma$ is admissible with $\varepsilon \neq 4$. We can prove it in the general case by using (6.22.1) in a similar way.

**Theorem 6.24.** Assume that $\sigma$ is admissible with $\varepsilon = 4$, or not admissible satisfying the condition in Theorem 2.6. Then
\[
\text{(i)} \quad (\mathcal{X}_\sigma^\pm)^\sigma = \{L^\pm(c, \prec) \mid c \in \mathcal{C}_\sigma\},
\]
\[
\text{(ii)} \quad \tilde{B}_\sigma = \pm \{b^\sigma(c, \prec) \mid c \in \mathcal{C}_\sigma\}.
\]

### 6.25. We denote by $\mathcal{C} \mapsto c$ the bijection $\mathcal{C} \cong \mathcal{C}_\sigma$. For each $c \in \mathcal{C}_\sigma$, $L^\pm(c, \prec)$ gives an element in $A'(U_q)^\sigma$, hence $\pi(L^\pm(c, \prec)) \in V_q$. On the other hand, for each $c \in \mathcal{C}$, one can consider $L(c, 0) \in \mathcal{X}_h$ and its image $\Phi(L(c, 0)) \in V_q$. We have the following result.

**Theorem 6.26.** Let $\Phi: A'(U_q)^- \cong V_q$ be the isomorphism given in Theorem 1.4 or Theorem 2.6. In particular $\mathcal{X}$ is not of type $A_2^{(2)}$ nor $A_1^{(1)}$. Then
\[
\text{(i)} \quad \Phi(L(c, 0)) = \pi(L^\pm(c, \prec)) \text{ for each } c \in \mathcal{C}.
\]
\[
\text{(ii)} \quad \Phi(b^\sigma(c, 0)) = \pi(b^\sigma(c, \prec)) \text{ for each } c \in \mathcal{C}_\sigma.
\]
\[
\text{(iii)} \quad \text{The natural map}
\]
\[
A'(U_q)^- \pi \Phi^{-1} \rightarrow A'(U_q)^- \mathcal{X}
\]
induces a bijection $\bar{\mathcal{B}}^\sigma \cong \bar{\mathcal{B}}$, which is explicitly given by $b^\sigma(c, \prec) \mapsto b(c, 0)$.

**Proof.** Recall that $h = (\cdots, \eta_{-1}, \eta_0, \eta_1, \ldots)$ is the doubly infinite sequence given in 6.1. In order to prove (i), it is enough to see that

\begin{equation}
\Phi(T_{\eta_0} T_{\eta_1} \cdots T_{\eta_k+1}(f_{\eta_k})) = \pi(\bar{\mathcal{T}}_{\eta_0} \bar{\mathcal{T}}_{\eta_1} \cdots \bar{\mathcal{T}}_{\eta_k+1}(f_{\eta_k})), \quad \text{if } k \leq 0,
\end{equation}

\begin{equation}
\Phi(T_{\eta_n-1}^{-1} T_{\eta_{n-2}}^{-1} \cdots T_{\eta_{k+1}}^{-1}(f_{\eta_k})) = \pi(\bar{\mathcal{T}}_{\eta_n-1}^{-1} \bar{\mathcal{T}}_{\eta_{n-2}}^{-1} \cdots \bar{\mathcal{T}}_{\eta_{k+1}}^{-1}(f_{\eta_k})), \quad \text{if } k \geq 1.
\end{equation}

This is reduced to the case where the rank of $X$ is 2. As we have excluded the case where $X$ is of type $A_2^{(2)}$ or $A_1^{(1)}$, $X$ is of finite type of rank 2. We have verified (6.26.1) in [SZ1] in the case where $\sigma$ is admissible with $\varepsilon \neq 4$. Thus the remaining one is the case where $(X, \bar{X})$ is of type $(A_4, C_2)$, which was already verified in the proof of Theorem 3.7. Hence (6.26.1) holds, and (i) follows. (ii) follows from (i), and (iii) follows from (ii). The theorem is proved. \(\square\)

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