ON IMAGES OF SOFIC SYSTEMS

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Abstract. Let \( \Sigma \) and \( \bar{\Sigma} \) be finite alphabets. For a topologically transitive sofic system \( X \subset \Sigma^\mathbb{Z} \) and a sofic system \( \bar{X} \subset \Sigma^\mathbb{Z} \) we give a necessary and sufficient condition for the existence of a homomorphism from \( X \) to \( \bar{X} \). For a topologically transitive sofic system \( X \subset \Sigma^\mathbb{Z} \) and a topologically transitive aperiodic sofic system \( \bar{X} \subset \bar{\Sigma}^\mathbb{Z} \) we give a necessary and sufficient condition for the existence of a homomorphism of \( X \) onto \( \bar{X} \).

1. Introduction

Let \( \Sigma \) be a finite alphabet. On the shift space \( \Sigma^\mathbb{Z} \) there acts the shift \( S \),

\[
S((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}.
\]

A closed shift invariant subset \( X \) of \( \Sigma^\mathbb{Z} \) together with the restriction \( S_X \) of the shift to \( X \) is a dynamical system that is called a subshift. For introductions to the theory of subshifts see the books by Lind and Marcus [LM] and by Kitchens [Ki].

A word is called admissible for the subshift \( X \subset \Sigma^\mathbb{Z} \) if it appears in a point of \( X \). An admissible word \( u \) of a subshift \( X \subset \Sigma^\mathbb{Z} \) is called synchronizing if for all words \( v \) and \( w \) such that the words \( vu \) and \( uw \) are admissible for \( X \), also the word \( vw \) is admissible. By a homomorphism \( \varphi : X \to \bar{X} \) of subshifts \( X \subset \Sigma^\mathbb{Z} \) and \( \bar{X} \subset \bar{\Sigma}^\mathbb{Z} \) is meant a continuous shift-commuting map from \( X \) into \( \bar{X} \). One of the basic classes of subshifts are the subshifts of finite type, that are defined as the subshifts that are obtained from a finite set \( \mathcal{F} \) of words by excluding from a shift space \( \Sigma^\mathbb{Z} \) the words in \( \mathcal{F} \). The class of subshifts of finite type is closed under topological conjugacy. Sofic systems [W] are the homomorphic images of subshifts of finite type.

For a sofic system \( X \) and a topologically transitive aperiodic subshift of finite type \( \bar{X} \), such that the topological entropy of \( X \) exceeds the topological entropy of \( \bar{X} \), Mike Boyle has shown that the periodic point condition is sufficient for the existence of a surjective homomorphism of \( X \) onto \( \bar{X} \) [B, Corollary (2.6)]. The periodic point condition says, that every \( \pi \in \mathbb{N} \) that appears as the period of a periodic point of \( X \) has a divisor that appears as a period of a periodic point of \( \bar{X} \). Boyle also gave a sufficient condition for the existence of a surjective homomorphism of a topologically transitive sofic system onto a topologically transitive aperiodic sofic system of lower entropy [B, Theorem 3.3]. This was extended by Klaus Thomsen [T, Theorem 9.13] and further extended by Jan Nielsen [N, Theorem 7.2.3].
After some preliminary consideration on subshifts and their periodic points in Section 2, we give for a topologically transitive sofic system \( X \subset \Sigma^\mathbb{Z} \) and a sofic system \( \tilde{X} \subset \tilde{\Sigma}^\mathbb{Z} \) a necessary and sufficient condition for the existence of a homomorphism from \( X \) to \( \tilde{X} \). In Section 4 we show that this condition is decidable. This is not unexpected, since the basic properties of regular languages are decidable [HU, Chapter 3], and also in view of the results on algorithms for sofic systems [CP].

The derived shift of a synchronizing subshift \( X \subset \Sigma^\mathbb{Z} \) was introduced by Thomsen in [T] as the subshift \( \partial X \) that is obtained by excluding from \( X \) the synchronizing words. For a topologically transitive sofic system \( X \) and a topologically transitive aperiodic sofic system \( \bar{X} \) such that the topological entropy of \( X \) exceeds the topological entropy of \( \bar{X} \) it was shown by Nielsen that there exists a surjective homomorphism of \( X \) onto \( \bar{X} \) if and only if there exists a homomorphism \( \varphi : X \to \bar{X} \) such that \( \varphi(X) \cap (\bar{X} - \partial \bar{X}) \neq \emptyset \) [N, Proposition 4.3.7]. We will reprove this result in section 5, and we will obtain for a topologically transitive sofic system \( X \) and a topologically transitive aperiodic sofic system \( \bar{X} \), such that the topological entropy of \( X \) exceeds the topological entropy of \( \bar{X} \), a necessary and sufficient condition for the existence of a homomorphism of \( X \) onto \( \bar{X} \).

As in [Kr] our arguments are based on the construction of a compact-open set such that the entries of any point into the set under the action of the shift are sufficiently spaced. This set is used to separate periodic events from non-periodic events. Also, we create sufficiently long periodic events by replacing words by words with the same context. At certain stages of our constructions we find it necessary to choose a time direction. This choice is always arbitrary.

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2. Preliminaries on subshifts and periodic points

We introduce notation and terminology for subshifts. With a finite alphabet \( \Sigma \) we set

\[ x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad i, k \in \mathbb{Z}, j \leq k, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}, \]

using similar notation also if indices range over semi-infinite intervals. For subshifts \( X \subset \Sigma^\mathbb{Z} \) we set

\[ X_{[i,k]} = \{x_{[i,k]} : x \in X\}. \]

The set of admissible words of a subshift \( X \subset \Sigma^\mathbb{Z} \) we denote by \( L(X) \). \( L(0) \) will denote the language that contains the empty word. The concatenation of words we write as a product. \( \ell \) denotes the length of a word. For cylinder sets we use the notation

\[ Z(b) = \{x \in X : x_{[0,\ell(b)]} = b\}, \quad b \in L(X). \]

For a compact-open set \( A \subset X \) and for \( x \in X \) we set

\[ \mathcal{I}_A(x) = \{i \in \mathbb{Z} : S^i_x(x) \in A\}, \]

and for \( x \in A \), in case that \( \mathcal{I}_A(x) \cap \mathbb{N} \) is not empty, we denote the smallest index in \( \mathcal{I}_A(x) \cap \mathbb{N} \) by \( I_A^+(x) \), setting \( I_A^+(x) \) equal to \( \infty \) otherwise. \( I_A^- \) has the symmetric meaning.

We recall that, given subshifts \( X \subset \Sigma^\mathbb{Z} \), \( \tilde{X} \subset \tilde{\Sigma}^\mathbb{Z} \), and a topological conjugacy \( \varphi : X \to \tilde{X} \), there is for some \( L \in \mathbb{Z}_+ \) a block map

\[ \phi : X_{[-L,L]} \to \tilde{\Sigma}, \]
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that implements the homomorphism \( \varphi \) with the action of \( \Phi \) on blocks (and words) given by

\[
\varphi(x)_{|_{I_-+L, I_+ - L}} = (\Phi(x|_{i-L, i+L}))_{I_-+L \leq i \leq I_+ - L}, \quad x \in X, I_-, I_+ \in \mathbb{Z}, I_+ - I_- > 2L.
\]

Let \( X \subset \Sigma^\mathbb{Z} \) be a subshift. We denote the set of periodic points of \( X \) by \( P(X) \), and we denote the smallest period of \( p \in P(X) \) by \( \pi(p) \). We set

\[
P_X\langle k \rangle = \{ p \in P(X) : \pi(p) \leq k \}, \quad k \in \mathbb{N}.
\]

We set

\[
\eta(p) = p_{[0, \pi(p)]}, \quad p \in P(X),
\]

and

\[
\mathcal{P} = \{ p_{[0, \pi(p)]} : p \in P(X) \},
\]

\[
P_X\langle k \rangle = \{ a \in \mathcal{P}(X) : \ell(a) \leq k \}, \quad k \in \mathbb{N}.
\]

We also write

\[
p^{(a)} = \eta^{-1}(a), \quad a \in \mathcal{P}(X).
\]

We say that \( a, a' \in \mathcal{P}(X) \) are conjugate and write \( a \sim (\mathcal{O}) a' \) if \( p^{(a)} \) and \( p^{(a')} \) are in the same orbit. Equivalently, \( a' \) is obtained from \( a \) by cyclically permuting the symbols of \( a \). For conjugate \( a, a' \in \mathcal{P}(X) \) we denote by \( u(a, a') \) the (non-empty) suffix of \( a' \) that is a prefix of \( a \). Here \( u(a, a) = a \). For subshifts \( X \subset \Sigma^\mathbb{Z} \) and \( \bar{X} \subset \bar{\Sigma}^\mathbb{Z} \) a homomorphism \( \varphi : X \rightarrow \bar{X} \) induces a map

\[
\tilde{\varphi} : a \mapsto \eta(\varphi(p^{(a)})) \ (a \in \mathcal{P}(X))
\]

from which \( \varphi \) can be reconstructed by

\[
\varphi(p) = \eta^{-1}(\tilde{\varphi}(\eta(p))), \quad p \in P(X).
\]

We note that, if \( a \sim (\mathcal{O}) a' \), then also \( \tilde{\varphi}(a) \sim (\mathcal{O}) \tilde{\varphi}(a') \).

We denote by \( \mathcal{A}_X \) the set of triples

\[
(a_-, c, a_+) \in \mathcal{P}(X) \times (\mathcal{L}_{(0)} \cup \mathcal{L}(X)) \times \mathcal{P}(X)
\]

such that

\[
a_-^{k_-} ca_+^{k_+} \in \mathcal{L}(X), \quad k_-, k_+ \in \mathbb{N},
\]

and such that, in the case that \( c \) is empty, one has that the last symbol of \( a_- \) is different from the last symbol of \( a_+ \), and in the case that \( c \in \mathcal{L}(X) \) one has that the last symbol of \( c \) is different from the last symbol of \( a_+ \), and the first symbol of \( c \) is different from the first symbol of \( a_- \). We associate with a triple \( (a_-, c, a_+) \in \mathcal{A}_X \) the point \( z^{(a_-, c, a_+)} \in X \) that is given by

\[
z_{(-\infty, -\ell(c))} = p^{(a_-)}_{(-\infty, 0)}, \quad z_{[-\ell(c), 0)} = c, \quad z_{[0, \infty)} = p^{(a_+)}_{[0, \infty)}.
\]
Lemma 2.1. Let $x \in X \setminus P(X)$ be left asymptotic to $p^- \in P(X)$ and right asymptotic to $p^+ \in P(X)$. Then there exist unique $t \in \mathbb{Z}$ and $(a_-, c, a_+) \in A_X$ such that

$$x = S_X^{-t}(x(a_-, c, a_+)).$$

Proof. Let $k^+(x) \in \mathbb{Z}$ be given by the condition

$$x_{[k^+(x), \infty)} = p^+_{[k^+(x), \infty)}, \quad x_{[k^+(x)-1, \infty)} \neq p^+_{[k^+(x)-1, \infty)}.$$

In case that

$$x_{(-\infty, k^+(x))} = p^-_{(-\infty, k^+(x))},$$

$c$ is empty, and

$$t = k^+(x), \quad a_- = x_{[k^+(x)-\pi(p^-), k^+(x))} \quad a_+ = x_{[k^+(x), k^+(x)+\pi(p^+))}.$$ 

In case that (2.1) does not hold, let $k^-(x) \in \mathbb{Z}$ be given by the condition

$$x_{(-\infty, k^-(x))} = p^-_{(-\infty, k^-(x))}, \quad x_{(-\infty, k^-(x)+1)} \neq p^-_{(-\infty, k^-(x)+1)},$$

and have

$$t = k^+(x), \quad a_- = x_{[k^-(x)-\pi(p^-), k^-(x))}, \quad c = x_{[k^-(x), k^+(x)]}, \quad a_+ = x_{(k^+(x), k^+(x)+\pi(p^+))}. \square$$

Words $w, w' \in \mathcal{L}(X)$ are said to have equal context, if for $b^-, b^+ \in \mathcal{L}(X)$ one has $b^- wb^+ \in \mathcal{L}(X)$ if and only if $b^- w'b^+ \in \mathcal{L}(X)$. The equality of context of $w, w' \in \mathcal{L}(X)$ we write as $w \sim w'$. With the product given by

$$[u]_\sim [v]_\sim = [uv]_\sim, \quad u, v \in \mathcal{L}(X),$$

the set

$$\mathcal{V}(X) = \{[w]_\sim : w \in \mathcal{L}(X)\}$$

of context classes of $X$ becomes a semigroup (the syntactic semigroup of $X$). Recall that the syntactic semigroup of a subshift is finite if and only if the subshift is sofic.

The set $\mathcal{V}(X)$ is also the vertex set of a Shannon graph $\mathcal{G}(X)$ with labeling alphabet $\Sigma$ that has an edge with label $\sigma \in \Sigma$ from $\delta \in \mathcal{V}(X)$ to $\delta' \in \mathcal{V}(X)$ precisely if

$$\delta[\sigma]_\sim = \delta'.$$

We denote by $\mathcal{V}_0(X)$ the set of $\delta \in \mathcal{V}(X)$ such that there exists a path in $\mathcal{G}(X)$ that starts and ends at $\delta$. For $\delta \in \mathcal{V}_0(X)$ we denote by $\Lambda(\delta)$ the set of lengths of these paths, and by $r_\delta$ the shortest length of such a path. Also we denote for $\delta \in \mathcal{V}_0(X)$ by $q_\delta$ the smallest $q \in \mathbb{Z}_+$ such that

$$\{l \in \Lambda(\delta) : l > q\} = \{l + mr_\delta : l \in \Lambda_\delta, q + r_\delta \geq l > q, m \in \mathbb{N}\}.$$
3. Homomorphisms

Let $X \subset \Sigma^\mathbb{Z}$ be a sofic system. For $a \in \mathcal{P}(X)$ let $R(a)$ be the smallest $R \in \mathbb{N}$ such that there exists a $Q \in \mathbb{N}$ such that

$$[a^Q]_\sim = [a^{Q+R}]_\sim,$$

and let $Q(a)$ be the smallest $Q \in \mathbb{N}$ such that

$$[a^{QR(a)}]_\sim = [a^{(Q+1)R(a)}]_\sim.$$

Denote the cardinality of $\mathcal{V}(X)$ by $\mathcal{V}(X)$ and the cardinality of $\mathcal{V}_\circ(X)$ by $\mathcal{V}_\circ(X)$. Also set

$$B_k(X) = \{b \in \mathcal{L}(X) : \ell(b) > k(V(X) + 2)\}, \quad k \in \mathbb{N},$$

and for $b \in B_k(X), k \in \mathbb{N}$, define the index $I(b, k)$ as the smallest index $I > 1$ such that there is an index $I' > I$ such that

$$[b_{[1,kI]}]_\sim = [b_{[1,kI']} ]_\sim,$$

the smallest such index $I'$ to be denoted by $I'(b, k)$. One has

$$I'(b, k) \leq k(V(X) + 2).$$

Further let $m(b, k)$ be the largest $m \in \mathbb{N}$ such that $b_{[1, kI(b,k)]}b_{m(b,k)}^{I'(b,k)}$ is a prefix of $b$. Note that

$$[b_{[1,kI(b,k)]}b_{m(b,k)}^{I'(b,k)}]_\sim = [b_{[1,kI(b,k)]}]_\sim [b_{m(b,k)}^{I'(b,k)}]_\sim.$$

Denote for $\delta \in \mathcal{V}_\circ(X)$ by $B_{k,\delta}(X)$ the set of $b \in B_k(X)$ with an index $J$ such that

$$k[I(b,k) + m(b,k)(I'(b,k) - I(b,k))] < J < \ell(b),$$

and

$$[b_{[1,J]}]_\sim = \delta,$$

and such that one has, denoting the smallest such index by $J(b, k)$ and the largest such index by $J'(b, k)$, that

$$J'(b, k) - J(b, k) > q_\delta + kr_\delta (I'(b,k) - I(b,k)).$$

Setting

$$H_\circ(X, k) = V(X) - V_\circ(X) + \sum_{\delta \in \mathcal{V}_\circ(X)} (q_\delta + kV_\circ(X)r_\delta), \quad k \in \mathbb{N},$$

one has

$$\ell(b) - k[I(b,k) - m(b,k)(I'(b,k) - I(b,k))] \leq H_\circ(X, k),$$

$$b \in B_k(X) \setminus \bigcup_{\delta \in \mathcal{V}_\circ(X)} B_{k,\delta}(X), \quad k \in \mathbb{N}.$$
We construct an auxiliary map 

$$\psi_k : B_k(X) \to B_k(X)$$

that respects length and context of words, that takes its values in the set 

$$B_k(X) \setminus \bigcup_{\delta \in \mathcal{V}_o(X)} B_{k,\delta}(X)$$

and that restricts to the identity on this set. For the construction we choose for \(\delta \in \mathcal{V}_o(X)\) and \(l \in \Lambda(\delta)\) a word \(w_\delta(l)\) of length \(l\) that is the label sequence of a path in \(G(X)\) that starts and ends at \(\delta\). Also for \(\delta \in \mathcal{V}_o(X), k \in \mathbb{N}\), we denote by \(\psi_{k,\delta}\) the map that carries the word \(b \in B_{k,\delta}(X)\) into the word that is obtained from \(b\) by simultaneously replacing in \(b\) the prefix 

$$b[1, kI(b,k)] \psi_{m(b,k)}^{kI(b,k)} b_{kI(b,k)}^{m(b,k)}$$

by the word 

$$b[1, kI(b,k)] \psi_{m(b,k)+r_\delta}^{kI(b,k)} b_{kI(b,k)}^{m(b,k)+r_\delta}$$

and the word 

$$b_{(J(b,k),J'(b,k)]}$$

by the word 

$$w_\delta(J'(b,k) - J(b,k) - kr_\delta[1 - I(b,k)])$$.

By (3.1) and (3.2) \(\psi_{k,\delta}\) does not change the context of a word, nor does it change the length of a word. Order the set \(\mathcal{V}_o(X)\) linearly,

$$\mathcal{V}_o(X) = \{\delta_m : 1 \leq m \leq V_o\},$$

and for a 

$$b \in B_k(X) \setminus \bigcup_{\delta \in \mathcal{V}_o(X)} B_{k,\delta}(X)$$

determine inductively an \(N \in \mathbb{N}\) as well as \(m_n, 1 \leq n \leq N,\)

$$1 \leq m_n \leq V_o,$$

together with words \(b^{(n)} \in \mathcal{L}(X)\) by

\[
\begin{align*}
    b^{(1)} &= b, \\
    m_n &= \min\{m : b^{(n-1)} \in B_{k,\delta_m}(X)\}, \quad 1 \leq n < N, \\
    b^{(n)} &= \psi_{k,\delta_{m_n}}(b^{(n-1)}), \\
    b^{(n)} &\in \bigcup_{\delta \in \mathcal{V}_o(X)} B_{k,\delta}(X), \quad 1 \leq n < N, \\
    b^{(N)} &\notin \bigcup_{\delta \in \mathcal{V}_o(X)} B_{k,\delta}(X).
\end{align*}
\]
and set
\[
\psi_k(b) = \begin{cases} 
  b^{(N)}, & \text{if } b \notin \bigcup_{\delta \in \mathcal{V}_o(X)} \mathcal{B}_{k,\delta}(X), \\
  b, & \text{if } b \in \bigcup_{\delta \in \mathcal{V}_o(X)} \mathcal{B}_{k,\delta}(X).
\end{cases}
\]

For sofic systems $X \subset \Sigma^\mathbb{Z}$, $\bar{X} \subset \bar{\Sigma}^\mathbb{Z}$ set
\[
T_o(X, \bar{X}) = \left( \max_{\bar{a} \in \mathcal{P}_{\bar{X}}(V_o(X))} R(\bar{a}) \right)^2,
\]
and
\[
H(X, \bar{X}) = V(X) + H_o(X) + 2T_o(X, \bar{X}) + \max(\{Q(a)\ell(\bar{a}) : a \in \mathcal{P}_X(V_o(X))\} \cup \{Q(\bar{a})\ell(\bar{a}) : \bar{a} \in \mathcal{P}_X(V_o(X))\}).
\]

Also set
\[
T(X, \bar{X}) = \left( \max_{\bar{a} \in \mathcal{P}_{\bar{X}}(H(X, \bar{X}))} R(\bar{a}) \right)^2,
\]
and set
\[
\mathcal{A}_X[H(X, \bar{X})] = \{(a_-, a_+): (a_-, a_+) \in \mathcal{A}_X : \ell(a_-), \ell(a_+) \leq H(X, \bar{X}), \psi_\ell(a_-)(a_-^{V(X)+2}c) \notin \bigcup_{\delta \in \mathcal{V}_o(X)} \mathcal{B}_{\ell(a_-), \delta}(X)\}.
\]

Given a homomorphism
\[
\varphi_o : \mathcal{P}_X(H(X, \bar{X})) \rightarrow \mathcal{P}_{\bar{X}}(H(X, \bar{X}))
\]
and
\[
a_{+, k} \in \mathcal{P}_X(H(X, \bar{X})), \quad 0 \leq k \leq K,
\]
\[
a_{-, k} \in \mathcal{P}_X(H(X, \bar{X})), \quad 0 < k \leq K + 1, \quad K \in \mathbb{Z}_+,
\]
such that
\[
a_{+, k} \sim (O) a_{-, k+1}, \quad 0 \leq k \leq K,
\]
set
\[
\bar{a}_+ = \eta(\varphi_o(p(a_{+, 0}))), \quad \bar{a}_- = \eta(\varphi_o(p(a_{-, K+1}))),
\]
and for $l, s \in \mathbb{Z}$,
\[
|l| \leq 2H(X, \bar{X}), \quad |s| < 2T(X, \bar{X}),
\]
and
\[
0 \leq R_k < R(a_{+, k}), \quad 0 < k \leq K,
\]
denote by
\[
\mathcal{R}(l, s, (a_{+, k}, R_k)_{0 \leq k \leq K}, (a_{-, k})_{0 < k \leq K + 1})
\]
the set of remainders that the integral parts of the ratios
\[
\ell(\bar{a}_+)^{-1}[s + l + 2(H(X, \bar{X}) + T(X, \bar{X}))\ell(\bar{a}_+)R(\bar{a}_+) + \\
\sum_{0 \leq k \leq K} \{\ell(a_{+, k})(Q(a_{+, k} + R_k + \bar{R}_k R(a_{+, k})) + \ell(u(a_{+, k}, a_{-, k+1}))\}]
\]
leave over after division by \( R(\bar{a}_+) \).

We will consider homomorphisms
\[
\varphi_0 : P_X(H(X, X)) \rightarrow P_X(H(X, \bar{X}))
\]

together with maps \( \Psi \) with a domain of definition \( A_{\varphi_0}^X[H(X, X)] \) that is a subset of \( A_{\varphi_0}^X[H(X, \bar{X})] \) such that
\[
A_{\varphi_0}^X[H(X, \bar{X})] \setminus A_{\varphi_0}^X[H(X, X)] \subset \{(a_-, c, a_+) \in A_{\varphi_0}^X[H(X, \bar{X})] : \varphi_0(S_{X}^{\ell(c)}(p^{(a_-)})) = \varphi_0(p^{(a_+)})\},
\]
and that take values in
\[
A_{\varphi_0}^X[H(X, \bar{X})] \times [-T(X, \bar{X}), T(X, \bar{X})],
\]
such that one has, setting
\[
((\bar{a}_-, \bar{c}, \bar{a}_+), t) = \Psi((a_-, c, a_+))
\]
that
\[
t \begin{cases} 
\geq -T_0(X, \bar{X}), & \text{if } \ell(a_-) \leq V_0(X), \\
\leq T_0(X, \bar{X}), & \text{if } \ell(a_+) \leq V_0(X),
\end{cases}
\]
and
\[
\bar{a}_- = \eta(\varphi_0(S_{X}^{\ell(c)-\ell(\bar{c})}(p^{(a_-)})), \quad \bar{a}_+ = \eta(\varphi_0(S_{X}^{\ell(c)}(p^{(a_+)})�).
\]

We say that the mapping \( \Psi \) accompanies the homomorphism \( \varphi_0 \).

Given a homomorphism
\[
\varphi_0 : P_X(H(X, X)) \rightarrow P_X(H(X, \bar{X}))
\]
and an accompanying map
\[
\Psi : A_{\varphi_0}^X[H(X, X)] \rightarrow A_{\varphi_0}^X[H(X, \bar{X})] \times [-T(X, \bar{X}), T(X, \bar{X})]
\]
we denote for \( N \in \mathbb{N} \) by \( \Delta_{\varphi_0, \Psi}^N \) the set of tuples
\[
((a_-, k(n))(n), c^{(k(n))}(n), a_{+, k(n)}(n), R_{k(n)}(n))_{0 \leq k \leq K(n), 0 \leq n \leq N},
\]
where
\[
K(n) \in \mathbb{Z}_+, \quad 0 \leq n < N, \quad K(N) = 0,
\]
such that
\[
(a_-, 0(n), c^{(0)}(n), a_{+, 0(n)}) \in A_{\varphi_0}^X[H(X, \bar{X})],
\]
\[
(a_-, k(n))(n), c^{(k(n))}(n), a_{+, k(n)}(n)) \in A_{\varphi_0}^X[H(X, \bar{X})] \setminus A_{\varphi_0}^X[H(X, X)], \quad 0 < k(n) \leq K(n),
\]
and
\[
0 \leq R_{k(n)}(n) < R(a_{+, k(n)}(n)), \quad 0 \leq k(n) \leq K(n)
\]
and such that one has, setting
\[
a_{-, K(n)+1(n)} = a_{-, 0}(n + 1), \quad 0 \leq n < N,
\]
that
\[
a_{+, k(n)}(n) \sim (\mathcal{O}) a_{-, k(n)+1}, \quad 0 \leq k(n) \leq K(n), \quad 0 \leq n < N.
\]

We will also set
\[
c^{(K(n)+1)}(n) = c^{(0)}(n + 1), \quad 0 \leq n < N.
Theorem 3.1. Let $X \subset \Sigma^\mathbb{Z}, \bar{X} \subset \Sigma^\mathbb{Z}$ be infinite sofic systems and let $X$ be topologically transitive. There exists a homomorphism $\varphi : X \to \bar{X}$ with $\varphi(X)$ infinite, if and only if there exists a homomorphism

$$\varphi : P_X(H(X, \bar{X})) \to P_{\bar{X}}(H(X, \bar{X}))$$

that is accompanied by a map

$$\Psi : A_{\bar{X}}^{\varphi}[H(X, \bar{X})] \to A_X^{\varphi}[H(X, \bar{X})] \times [-T(X, \bar{X}), T(X, \bar{X})]$$

such that the following holds:

For

$$((a_{-,k(n)}(n), c^{(k(n))}(n), a_{+,k(n)}(n), R_{k(n)}(n))_{0 \leq k \leq K(n)}_{0 \leq n \leq N} \in \Delta_N^{(\varphi, \Psi)}, n \in \mathbb{N},$$

the admissibility of

$$a_{-,0}(0)^{-c(0)(0)} \prod_{0 \leq n < N} \prod_{0 \leq k \leq K(n)} a_{+,k(n)}(n)^{Q(a_{+,k(n)})+R_{k(n)}(n)}$$

$$u(a_{+,k(n)}(n), a_{-,k(n)}(n)+1(n)) c^{(k(n)+1)}(n) a_{+,0}(n)^{l_+}, l_-, l_+ \in \mathbb{N},$$

for $X$ implies for

$$\bar{R}(n) \in \mathcal{R}(l(n), s(n), (a_{+,k(n)}(n), R_{k(n)}(n))_{0 \leq k < K(n)}, (a_{+,k(n)}(n))_{0 < k < K(n)}, 0 \leq n < N,$$

the admissibility of

$$\prod_{0 \leq n < N} \bar{a}_{+}(n)^{Q(\bar{a}_{+}(n))+\bar{R}(n)} u(\bar{a}_{+}(n), \bar{a}_{-}(n+1)) \bar{c}(n+1)$$

for $\bar{X}, N \in \mathbb{N}.$

Proof. We prove sufficiency. Introducing a linear order and writing

$$X_{[0,H(X,\bar{X})]} \setminus \{p_{[0,H(X,\bar{X})]}^{(a)} : a \in P_X(H(X,\bar{X}))\} = \{b_m : 1 \leq m \leq M\},$$

we set inductively

$$F_1 = Z(b_1),$$
\[ F_m = F_{m-1} \cup (Z(b_m) \setminus \bigcup_{-H(X,\bar{X}) < h < H(X,\bar{X})} S^h Z(F_{m-1})) , \]

and set 

\[ F = F_M . \]

In this way we have obtained a compact-open set \( F \subset X \) such that 

\[ F \cap S^h_X F = \emptyset , \quad 0 < h < H(X,\bar{X}) , \]

and

\[ Z(b_m) \subset \bigcup_{-H(X,\bar{X}) < h < H(X,\bar{X})} S^h Z(F) , \quad 1 \leq m \leq M . \tag{3.4} \]

Let

\[ \hat{H} > 2(T(X,\bar{X}) + H(X,\bar{X})) + \max_{a \in P_X(H(X,\bar{X}))} Q(a)\ell(a) , \]

be such that for \( x \in F \), such that 

\[ I_F^-(x) < -\hat{H} , \]

there exists \( p \in P_X(H(X,\bar{X})) \) such that

\[ p_{[I_F^-(x)+H(X,\bar{X}),-H(X,\bar{X})]} = x_{[I_F^-(x)+H(X,\bar{X}),-H(X,\bar{X})]} , \]

and extend the identity on \( P_X(H(X,\bar{X})) \) to an endomorphism \( x \to \tilde{x} (x \in X) \) of \( X \) by setting

\[ \tilde{x}_{[I_F^-(x),0]} = \begin{cases} \psi_1(x_{[I_F^-(x),0]}), & \text{if } I_F^-(x) > -\hat{H} , \\ x_{[I_F^-(x),0]}, & \text{if } I_F^-(x) \leq -\hat{H} , \end{cases} \quad x \in F , I_F^-(x) > -\infty , \]

and

\[ \tilde{x}_{(-\infty,0)} = x_{(-\infty,0)} , \quad x \in F , \quad I_F^+(x) = -\infty , \]

\[ \tilde{x}_{[0,\infty)} = x_{[0,\infty)} , \quad x \in F , \quad I_F^+(x) = \infty . \]

By the property (3.3) of the auxiliary mapping \( \psi \) and by the property (3.4) of the the set \( F \), and with the use of Lemma 2.1, we can determine for \( x \in F \) points \( p^{(x,-)}, p^{(x,+)} \in P(X) \), an \( (\tilde{a}_-(x), \tilde{c}(x), a_+(x)) \in \mathcal{A}_X \), and a \( J(x) \in \mathbb{Z} \) as follows:

For \( I_F^-(x) < \hat{H} , I_F^+(x) < \hat{H} \), set

\[ p^{(x,-)}(\bar{X},-2V_\delta(X),-H_\delta(X,1)) = \tilde{x}(\bar{X},-2V_\delta(X),-H_\delta(X,1)) , \]

\[ p^{(x,+)}(V(X),V(X)+2V_\delta(X)) = \tilde{x}(V(X),V(X)+2V_\delta(X)) , \]

and

\[ z^{(\tilde{a}_-(x), \tilde{c}(x), a_+(x))} = S^{(x)} X ([p^{(x,-)}_{(-\infty,-H_\delta(X,1))}, \tilde{x}_{[H_\delta(X,1),V(X)]}, p^{(x,+)}_{V(X),\infty}]). \]
For $I_F^-(x) \geq \hat{H}, I_F^+(x) < \hat{H}$, set
\[
P_{(-H(X,X),-2H(X,X),x)}^{(x,-)} = \tilde{x}(-4H(X,X),-2H(X,X)),
\]
\[
P_{[V(X),V(X)+2V_0(X)]}^{(x,+)} = \tilde{x}[V(X)+2V_0(X)],
\]
and
\[
z^{(\tilde{a},\tilde{c},x)} = S^J_X\left((p^{(x,-)}_{(\infty,-H(X,X))},\tilde{x}[2H(X,X)],p^{(x,+)}_{[H(X,X),2H(X,X)])})\right).
\]

For $I_F^-(x) < \hat{H}, I_F^+(x) \geq \hat{H}$, set
\[
P_{(-H(X,X),-2H(X,X),x)}^{(x,-)} = \tilde{x}(-H(X,X)-2V_0(X),-H(X,X)),
\]
\[
P_{[2H(X,X),4H(X,X)]}^{(x,+)} = \tilde{x}[2H(X,X),4H(X,X)],
\]
and
\[
z^{(\tilde{a},\tilde{c},x)} = S^J_X\left((p^{(x,-)}_{(\infty,\infty)},\tilde{x}[H(X,X),2H(X,X)],p^{(x,+)}_{[H(X,X),\infty])})\right).
\]

For $I_F^-(x) \geq \hat{H}, I_F^+(x) \geq \hat{H}$, set
\[
P_{(-4H(X,X),-2H(X,X),x)}^{(x,-)} = \tilde{x}(-4H(X,X),-2H(X,X)),
\]
\[
P_{[2H(X,X),4H(X,X)]}^{(x,+)} = \tilde{x}[2H(X,X),4H(X,X)],
\]
and
\[
z^{(\tilde{a},\tilde{c},x)} = S^J_X\left((p^{(x,-)}_{(\infty,-2H(X,X))},\tilde{x}[2H(X,X),2H(X,X)],p^{(x,+)}_{[H(X,X),\infty])})\right).
\]

Then we have for $x \in F$ an $(a_-(x),c(x),a_+(x)) \in A_X^0[H(X,X)]$ given by
\[
z^{(\tilde{a},\tilde{c},x)} = \Psi((a_-(x),c(x),a_+(x)),p^{(x,+)}_{[0,\infty]}).
\]

For $x \in F$, if
\[
(a_-(x),c(x),a_+(x)) \in A_X^0[H(X,X)],
\]
set
\[
((\tilde{a},\tilde{c},x),\tilde{a}_+(x)),t(x)) = \Psi((a_-(x),c(x),a_+(x)),t(x)),
\]
and if
\[
(a_-(x),c(x),a_+(x)) \in A_X^0[H(X,X)] \setminus A_X^0[H(X,X)],
\]
set $t(x)$ equal to zero, set $\tilde{c}(x)$ equal to the empty word, and set
\[
\tilde{a}_-(x) = \tilde{a}_+(x) = \eta(\varphi(p^{(a_+(x))})).
\]

We can then specify a homomorphism $x \to \tilde{x}$ ($x \in X$) that extends $\varphi$, by setting
\[
\tilde{x}[\ell(\tilde{c}(x)),0] = \tilde{c}(x), \quad x \in F.
\]
and

\[
\bar{x} = p^{\bar{a}_-}(x) = \begin{cases}
    I_F^-(x) + J(S_F^-(x)) + t(S_F^+(x)), & x < 0, \\
    [-I_F^-(x) - J(S_F^-(x)) + J(x) + t(x) + t(\bar{c}(x))] & x = 0,
\end{cases}
\]

\[
\bar{x}(\bar{a}_-, \bar{c}(\bar{c})(\bar{a}_+, \bar{a}_+)) = p^{(\bar{a}_-, \bar{c})} \in \mathcal{P}(\bar{X}),
\]

If \( x \in F \) has no index \( I \) in \( \mathcal{I}_F(x) \) such that

\[
(a_-(S_F^I(x)), c(S_F^I(x)), a_+(S_F^I(x))) \in \mathcal{A}^{\varphi}[H(X, \bar{X})],
\]

then \( \bar{x} \in P_X(H(X, \bar{X})) \), if \( x \in F \) is such that \( I = 0 \) is the only index in \( \mathcal{I}_F(x) \) such that (3.5) holds, then

\[
\bar{x} = S_X^{J(x) - t(x)}(\bar{a}_-, \bar{c}(x), \bar{a}_+),
\]

and if \( x \in F \) has more than one index \( I \) in \( \mathcal{I}_F(x) \) such that (3.5) holds, then \( \bar{x} \in X \) follows by an argument that uses condition (a) of the theorem and into which enter the remainders that the integral parts of the ratios

\[
\ell(a_-(x))^{-1}[I_F^-(x) - J(S_X^-(x)) - t(S_X^+(x) - J(x) + t(x))], \quad x \in F, I_F^-(x) > -\infty,
\]

leave over after division by \( R(a_-(x)) \). It follows that we can set \( \varphi(x) = \bar{x}, x \in X \).

We prove necessity. A homomorphism \( \varphi : X \to \bar{X} \) restricts to a homomorphism \( \varphi : P_X(H(X, \bar{X})) \to P_X(H(X, X)) \). We construct the map \( \Psi \) that accompanies \( \varphi \). We set

\[
\mathcal{A}_X^{\varphi}[H(X, \bar{X})] = \{ (a_-, c, a_+) \in \mathcal{A}_X[H(X, \bar{X})] : \varphi(p^{(a_-, c, a_+)}(X, X)) \neq \mathcal{P}(\bar{X}) \}
\]

The topological transitivity of \( X \) and the assumption that \( \varphi(X) \) is infinite imply that \( \mathcal{A}_X^{\varphi}[H(X, X)] \) is not empty. Given a \( (a_-, c, a_+) \in \mathcal{A}_X^{\varphi}[H(X, X)] \) one has by Lemma 2.1 a \((\bar{a}_-, \bar{c}, \bar{a}_+)\) and a \( \bar{t} \in \mathbb{Z} \) given by

\[
S_X^\bar{t}(\bar{a}_-, \bar{c}, \bar{a}_+) = \varphi((a_-, c, a_+)\).
\]

Let \( \bar{a}_- \) and \( \bar{c} \) be given by

\[
\bar{z}(\bar{a}_-, \bar{c}, \bar{a}_+) = p^{(\bar{a}_-, \bar{c})} = (p^{(\bar{a}_-)}, \psi_{\bar{t}}(\bar{a}_-) \bar{c}(X) + 2 \bar{c}, p^{(\bar{a}_+)})
\]

If \( \bar{t} = 0 \), set \( t = 0 \), and if \( \bar{t} \not= 0 \) and \( \ell(\bar{a}_-) \bar{c}(X) + 2 \bar{c} \geq V_0(X) \), or if \( \bar{t} < 0 \) and \( \ell(\bar{a}_-) \bar{c}(X) + 2 \bar{c} < V_0(X) \), or if \( \bar{t} > 0 \) and \( \ell(\bar{a}_-) \bar{c}(X) + 2 \bar{c} \geq V_0(X) \), give \( t \) the sign of \( \bar{t} \) and let the absolute value of \( t \) be equal to the remainder that is left over after division of \( |\bar{t}| \) by \( R(\bar{a}_-) R(\bar{a}_+) \), and if \( \bar{t} < 0 \) and \( \ell(\bar{a}_-) \bar{c}(X) + 2 \bar{c} < V_0(X) \), or if \( \bar{t} > 0 \) and \( \ell(\bar{a}_-) \bar{c}(X) + 2 \bar{c} \geq V_0(X) \), then give \( t \) the sign that is the opposite of the sign of \( \bar{t} \) and let the absolute value
of \( t \) be equal to \( R(\tilde{a}_-)R(\tilde{a}_+) \) minus the remainder that is left over after division of \( |\tilde{t}| \) by \( R(\tilde{a}_-)R(\tilde{a}_+) \), and set
\[
\Psi((a_-, c, a_+)) = ((\tilde{a}_-, \tilde{c}, \tilde{a}_+), t).
\]
To confirm that \( \varphi \circ \Psi \) and \( \Psi \) satisfy condition (a) of the theorem, let \( L \in \mathbb{Z}_+ \) be such that \([-L, L] \) is a coding window for \( \varphi \), and let
\[
\Phi : X_{[-L, L]} \to \Sigma
\]
be the block map that implements \( \varphi \). Observe, that for a tuple
\[
((a_-, k(n))(n), c^{(k(n))}(n), a_+, k(n))(n), R_{k(n)}(n))_{0 \leq k \leq K(n)}_{0 \leq n \leq N} \in \Delta_N^{(\varphi \circ \Psi), \Psi},
\]
if
\[
a_-, 0(0)^{l_1} - c^{(0)}(0) \prod_{0 \leq n < N} \left( \prod_{0 \leq k(n) \leq K(n)} a_+, k(n)Q(a_+, k(n)) + R_{k(n)}(n) \right)
\]
\[
u(a_+, k(n)(n), a_-, k(n)+1(n))c^{(k(n)+1)}(n)^{l_1}a_+, 0(N) \in \mathcal{L}(X), \quad l_-, l_+ \in \mathbb{N},
\]
then one has for
\[
R(n) \in \mathcal{R}(s(n), l(n), (a_+, k(n)(n), R_{k(n)}(n))_{0 \leq k(n) < K(n)}, (a_+, k(n))(n)_{0 < k \leq K(n)}),
\]
that
\[
a_-, 0(0)^{l_1} - c^{(0)}(0) \prod_{0 \leq n < N} \left( \prod_{0 \leq k(n) \leq K(n)} a_+, k(n)Q(a_+, k(n)) + R_{k(n)}(n) \right)
\]
\[
u(a_+, k(n)(n), a_-, k(n)+1(n))c^{(k(n)+1)}(n)^{l_1}a_+, 0(N) \in \mathcal{L}(X), \quad l_-, l_+ \in \mathbb{N}.
\]
Then also
\[
\prod_{0 \leq n < N} \left( \prod_{0 \leq k(n) \leq K(n)} a_+, k(n)2(L + H(X, \bar{X}) + T(X, \bar{X}))R(\tilde{a}_+)R(a_+, k(n)) + Q(a_+, k(n)) + R_{k(n)}(n) \right)
\]
\[
u(a_+, k(n)(n), a_-, k(n)+1(n))c^{(k(n)+1)}(n)^{l_1}a_+, 0(N)^{l_1} \in \mathcal{L}(X), \quad l_-, l_+ \in \mathbb{N},
\]
and from the action of \( \Phi \) on these words one can read off that
\[
\prod_{0 \leq n < N} \tilde{a}_+(n)Q(\tilde{a}_+(n)) + R(n)u(\tilde{a}_+(n), \tilde{a}_+(n) + 1)c(n + 1) \in \mathcal{L}(\bar{X}), N \in \mathbb{N}. \quad \square
\]

In Theorem 3.1 the set \( \mathcal{A}_X^\infty[H(X, \bar{X})] \) can be interpreted as the alphabet of a sofic system that is associated to \( X \subset \Sigma^\mathbb{Z} \). Instead of the set \( \mathcal{A}_X^\infty[H(X, \bar{X})] \) one can also use an alphabet that contains triples \((a_-, \gamma, a_+)\), where \( a_-, a_+ \in \mathcal{P}_X(H(X, \bar{X})) \), and where \( \gamma \) is a context class that is appropriately chosen with respect to \( a_- \) and \( a_+ \). Also, denoting for a subshift \( X \subset \Sigma^\mathbb{Z}_+ \) by \( \Pi(X) \) the subset of \( \mathbb{N} \), such that one has compact-open sets \( A_p, p \in \Pi(X) \), such that
\[
X = \bigcup_{p \in \Pi(X)} \bigcup_{0 < l_p < p} S_{X}^{l_p}A_p,
\]
\[
S_{X}^{p}A_p = A_p, \quad p \in \Pi(X),
\]
and
\[
A_p \cap S_{X}^{l_p}A_p = \emptyset, \quad 0 < l_p < p, \quad p \in \Pi(X),
\]
one observes that there exists a homomorphism of \( X \) into a finite dynamical system \( \bar{X} \) if and only if \( X \) has for every \( p \in \Pi(X) \) a periodic point whose period divides \( p \).
4. Decidability

**Theorem 4.1.** Let $X \subset \Sigma^\mathbb{Z}, \bar{X} \subset \bar{\Sigma}^\mathbb{Z}$ be infinite sofic systems and let $X$ be topologically transitive. Set

\begin{align}
\rho(X, \bar{X}) &= \max_{a \in \mathcal{P}_X(H(X, \bar{X}))} R(a), \quad \bar{\rho}(X, \bar{X}) = \max_{a \in \mathcal{P}_X(H(X, \bar{X}))} R(\eta(\varphi(p(\bar{a})))). \\
\end{align}

Writing $\nu^{\varphi_{N}}(X, \bar{X})$ for the cardinality of $\mathcal{A}^{\varphi_{N}}_{X}[H(X, \bar{X})]$ and $\nu(X, \bar{X})$ for the cardinality of $\mathcal{A}^{\varphi_{N}}_{X}[H(X, \bar{X})] \setminus \mathcal{A}^{\varphi_{N}}_{X}[H(X, \bar{X})]$, set

\begin{align}
K(X, \bar{X}) &= \bar{\rho}(X, \bar{X})V_{\circ}(X)\nu(X, \bar{X})(V(X)\nu(X, \bar{X})+1)(\rho(X, \bar{X})\nu(X, \bar{X}))V_{\circ}(X)\nu(X, \bar{X}), \\
\end{align}

and

\begin{align}
N(X, \bar{X}) &= V(X)V(\bar{X})\nu^{\varphi_{N}}(X, \bar{X}).
\end{align}

A homomorphism

\[ \varphi_{N} : P_{X}(H(X, \bar{X})) \to P_{\bar{X}}(H(X, \bar{X})) \]

and an accompanying map

\[ \Psi : \mathcal{A}^{\varphi_{N}}_{X}[H(X, \bar{X})] \to \mathcal{A}^{\varphi_{N}}_{X}[H(X, \bar{X})] \times [-T(X, \bar{X}), T(X, \bar{X})] \]

satisfy the condition of Theorem 3.1, provided they satisfy this condition for tuples

\[ ((a_{-,k(n)}(n), c^{(k(n))}(n), a_{+,k(n)}(n), R_{k(n)}(n))_{0 \leq k \leq K(n)}_{0 \leq n \leq N} \in \Delta^{(\varphi_{N}, \Psi)}_{N}, N \in \mathbb{N}, \]

with $K(n) \leq K(X, \bar{X}), 0 \leq n < N$, and $N \leq N(X, \bar{X})$.

**Proof.** Let there be given a tuple

\[ ((a_{-,k(n)}(n), c^{(k(n))}(n), a_{+,k(n)}(n), R_{k(n)}(n))_{0 \leq k \leq K(n)}_{0 \leq n \leq N} \in \Delta^{(\varphi_{N}, \Psi)}_{N}, N \in \mathbb{N}, \]

such that

\begin{align}
a_{-,0}(1)^{-}c^{(0)}(1) \prod_{1 \leq n < N} \prod_{0 \leq k(n) < K(n)} a_{+,k(n)}(n)^{Q(a_{+,k(n)})+R_{k(n)}(n)} \\
u(a_{+,k(n)}(n), a_{-,k(n)+1}(n))c^{(k(n)+1)}(n))a_{+,0}^{l_{+}}(N) \in \mathcal{L}(X), \quad l_{-}, l_{+} \in \mathbb{N}.
\end{align}

Let $n_{o}$ be the smallest index such that $K(n_{o})$ is maximal among the $K(n), 0 \leq n \leq N$, and assume that

\[ K(n_{o}) > K(X, \bar{X}). \]

By (4.1), (4.2) and (4.5) one can determine an $I$,

\[ 1 \leq I \leq V_{\circ}(X)\nu(X, \bar{X}), \]

and indices $k_{r}, 1 \leq r \leq R(\eta(\varphi(p^{(a_{+,o}(n_{o}))}))),$ such that

\[ 0 < k_{r}, k_{r} + I < k_{r+1} < K(n_{o}), \quad 1 \leq r \leq R(\eta(\varphi(p^{(a_{+,o}(n_{o}))}))), \]
together with
\[(a_-, i c^{(i)}, a_+, i) \in A_\beta^X [H(X, \bar{X})] \setminus \bar{A}_\beta^X [H(X, \bar{X})], \quad 0 \leq R_i < R(a_+, i), \quad 1 \leq i \leq I,\]
such that
\[(4.6) \quad \prod_{0 \leq k(n_0) < k_r} a_+, k(n_0)(n_0) Q(a_+, k(n_0)) + R_k(n_0) \]
\[u(a_+, k(n_0)(n_0), a_-, k(n_0) + 1(n_0)) c^{(k(n_0)+1)}(n_0)]_\sim = \]
\[\prod_{0 \leq k(n_0) < k_r + I} a_+, k(n_0)(n_0) Q(a_+, k(n_0)) + R_k(n_0) \]
\[u(a_+, k(n_0)(n_0), a_-, k(n_0) + 1(n_0)) c^{(k(n_0)+1)}(n_0)]_\sim \]
and such that
\[(4.7) \quad (a_+, k(n_0)(n_0), a_-, k(n_0)(n_0), R_k(n_0))_{k_r \leq k(n_0) \leq k_r + I} = \]
\[(a_-, i c^{(i)}, a_+, i, R_i)_{1 \leq i \leq I}, \quad 1 \leq r \leq R(\eta(p^{(a_+, 0(n_0))})).\]

By (4.6) and (4.7) the condition of Theorem 3.1 is satisfied for the tuple (4.4) if and only if it is satisfied for the tuple that is obtained from the tuple (4.4) by removing the segments
\[(a_-, k(n_0)(n_0), c^{(k(n_0))}(n_0), a_+, k(n_0)(n_0), R_k(n_0))_{k_r < k(n_0) \leq k_r + I}, \quad 1 \leq r \leq R(\eta(p^{(a_+, 0(n_0))})),\]
and this can be assumed to be the case by an induction hypothesis.

Let \(N_0\) be the smallest \(N > 1\) such that there is a tuple
\[((a_-, k(n))(n), c^{(k(n))}(n), a_+, k(n))(n), R_k(n)(n))_{0 \leq k \leq K(n) 0 \leq n \leq N} \in \Delta_N^{(\varphi, \Psi)}, N \in \mathbb{N},\]
such that
\[a_-, 0(1)^l - c^{(0)}(1) \prod_{1 \leq n < N} \prod_{0 \leq k(n) < K(n)} a_+, k(n)(n) Q(a_+, k(n)) + R_k(n)(n) \]
\[u(a_+, k(n)(n), a_-, k(n) + 1(n)) c^{(k(n)+1)}(n) a_{+, l_0}(N) \in \mathcal{L}(X), \quad l_-, l_+ \in \mathbb{N},\]
for which the condition of Theorem 3.1 is not satisfied. For a proof by contradiction assume that
\[(4.8) \quad N_0 > N(X, \bar{X}),\]
and let there be given a tuple
\[(4.9) \quad ((a_-, k(n))(n), c^{(k(n))}(n), a_+, k(n))(n), R_k(n)(n))_{0 \leq k \leq K(n) 0 \leq n \leq N_0} \in \Delta_N^{(\varphi, \Psi)}\]
such that
\[
\begin{align*}
& a_{-,0}(0)^{(l)} - c^{(0)}(0) \prod_{0 \leq k \leq K(n)} \left( \prod_{0 \leq n < N} a_{+,k(n)}(n)^{Q(a_{+,k(n)})+R_{k(n)}(n)} \right) \\
& u(a_{+,k(n)}(n), a_{-,k(n)+1}(n))c^{(k(n)+1)}(n)a_{+,0}(N) \in \mathcal{L}(X), \quad l_-, l_+ \in \mathbb{N},
\end{align*}
\]
and, with \( s(n), l(n) \) given by
\[
\begin{align*}
l(n) &= -\ell(\bar{c}(n+1)) + \sum_{0 \leq k \leq K(n)} \ell(c_{k(n)}(n)), \\
s(n) &= -t_n + t_{n+1}, \quad 0 \leq n < N,
\end{align*}
\]
let there also be given
\[
\bar{R}(n) \in \mathcal{R}(l(n), s(n), (a_{+,k_n}(n), R_{k_n}(n)))_{0 \leq k_n < K(n)}, (a_{+,k_n}(n))_{0 \leq k \leq K(n)}, \quad 0 \leq n < N_0,
\]
such that
\[
(4.10) \quad \prod_{0 \leq n < N_0} \bar{a}_{+}(n)^{Q(\bar{a}_{+}(n))^{+}\bar{R}(n)}u(\bar{a}_{+}(n), \bar{a}_{+}(n + 1))\bar{c}(n + 1) \notin \mathcal{L}(\bar{X}).
\]
By the choice of \( N_0 \) the condition of Theorem 3.1 is satisfied for the tuple
\[
((a_{-,k}(n), c^{(k(n))}(n), R_{k}(n), a_{+,k}(n)(n)))_{1 \leq k \leq K(n)}_{0 \leq n < N_0},
\]
Therefore
\[
(4.11) \quad \prod_{0 \leq n < N_0-1} \bar{a}_{+}(n)^{Q(\bar{a}_{+}(n))^{+}\bar{R}(n)}u(\bar{a}_{+}(n), \bar{a}_{+}(n + 1))\bar{c}(n + 1) \in \mathcal{L}(\bar{X}).
\]
By (4.3),(4.8) and (4.11) one can choose indices \( n_0, n_1, \)
\[
1 < n_0 < n_1 < N(X, \bar{X}),
\]
such that
\[
(4.12) \quad \prod_{0 \leq n < n_0} \prod_{0 \leq k_n \leq K(n)} a_{+,k_n}(n)^{Q(a_{+,k_n})(n)+R_{k_n}(n)}u(a_{+,k_n}(n), a_{-,k_n+1}(n))c^{(k_n+1)}(n) =
\]
\[
\prod_{0 \leq n < n_1} \prod_{0 \leq k_n \leq K(n)} a_{+,k_n}(n)^{Q(a_{+,k_n})(n)+R_{k_n}(n)}u(a_{+,k_n}(n), a_{-,k_n+1}(n))c^{(k_n+1)}(n),
\]
(13)
\[
\prod_{1 \leq n < n_0} \bar{a}_{+}(n)^{Q(\bar{a}_{+}(n))^{+}\bar{R}(n)}u(\bar{a}_{+}(n), \bar{a}_{+}(n + 1))\bar{c}(n + 1) =
\]
\[
\prod_{1 \leq n < n_1} \bar{a}_{+}(n)^{Q(\bar{a}_{+}(n))^{+}\bar{R}(n)}u(\bar{a}_{+}(n), \bar{a}_{+}(n + 1))\bar{c}(n + 1),
\]
and
\[
(14) \quad (a_{-,k}(n_0), c^{(k(n_0))}(n_0), a_{+,k}(n_0)(n_0)) = (a_{-,k}(n_1), c^{(k(n_1))}(n_1), a_{+,k}(n_1)(n_1)).
\]
By the choice of \( N_0 \) the condition of Theorem 3.1 is satisfied for the tuple that is obtained by removing from the tuple (4.9) the segment
\[
((a_{-,k}(n), c^{(k(n))}(n), R_{k}(n), a_{+,k}(n)(n)))_{1 \leq k \leq K(n)}_{n_0 < n \leq n_1},
\]
and, when restoring this segment, one derives from (4.12-14) a contradiction to (4.10).
5. SURJECTIVE HOMOMORPHISMS

Denote the set of synchronizing context classes of a sofic system $X \subset \Sigma^\mathbb{Z}$ by $\mathfrak{s}(X)$. We say that an expression $\gamma(-)w\gamma(+)$, where $\gamma(-), \gamma(+) \in \mathfrak{s}(X)$ and where $w \in \mathcal{L}(X)$, is admissible for $X$, if the words $c(-)wc(+)$, where $c(-) \in \gamma(-)$ and where $c(+) \in \gamma(+)$, are admissible for $X$. We denote by $\Omega^-(X)$ the set of pairs $(\gamma, a)$ where $\gamma \in \mathfrak{s}(X)$ and $a \in \mathcal{P}(X)$ such that for $c \in \gamma$ and $m \in \mathbb{N}$, the words $ca^m$ are admissible for $X$. $\Omega^-(X)$ is defined symmetrically.

**Theorem 5.1.** For a topologically transitive sofic system $X \subset \Sigma^\mathbb{Z}$ and a topologically transitive aperiodic sofic system $\bar{X} \subset \bar{\Sigma}^\mathbb{Z}$ such that the topological entropy of $X$ exceeds the topological entropy of $\bar{X}$, and such that every $\pi \in \mathbb{N}$ that appears as the period of a periodic point of $X$ has a divisor that appears as a period of a periodic point of $\bar{X}$, the following are equivalent:

(a) There exists a homomorphism $\varphi_\circ : P_X(H(X, \bar{X})) \to P_{\bar{X}}(H(X, \bar{X}))$ that is accompanied by a mapping

$$\Psi : \mathcal{A}_X^\circ[H(X, \bar{X})] \to \mathcal{A}_{\bar{X}}^\circ[H(X, \bar{X})] \times [-T(X, \bar{X}), T(X, \bar{X})]$$

together with maps

$$\Psi^- : \Omega^-(X) \to \mathfrak{s}(\bar{X}), \quad \Psi^+ : \Omega^+(X) \to \mathfrak{s}(\bar{X}),$$

such that the following statement holds:

For

$$(\gamma_-, a(+)) \in \Omega^-(X), \quad (a(-), \gamma_+) \in \Omega^+(X),$$

and for

$$(a_{-, k(0)}(0), c^{(k(0))}(0), a_{+, k(0)}(0)) \in \mathcal{A}_X^\circ[H(X, \bar{X})] \setminus \mathcal{A}_{\bar{X}}^\circ[H(X, \bar{X})],
0 \leq R_{k(0)}(0) < R(a_{+, k(0)}(0)), \quad 0 < k(0) \leq K(0), \quad K(0) \in \mathbb{Z}_+,$$

and

$$(a_{-, 0}(n), c^{(0)}(n), a_{+, 0}(n)) \in \mathcal{A}_X^\circ[H(X, \bar{X})],
0 \leq R_0(n) < R(a_{+, 0}(n)),
(a_{-, k(n)}(n), c^{(k(n))}(n), a_{+, k(n)}(n)) \in \mathcal{A}_X^\circ[H(X, \bar{X})] \setminus \mathcal{A}_{\bar{X}}^\circ[H(X, \bar{X})],
0 \leq R_{k(n)}(n) < R(a_{+, k(n)}(n)), \quad 0 < k(n) \leq K(n), \quad K(n) \in \mathbb{Z}_+,$$

such that one has, setting

$$a_{+, 0} = a(+),$$

$$a_{-, K(n)+1}(n) = a_{-, 0}(n+1), \quad 0 \leq n \leq N,$$

$$a_{-, K(N)}(N) = a(-),$$

that

$$a_{+, k(n)} \sim (\mathcal{O}) a_{-, k(n)+1}, \quad 0 \leq k \leq K(n), \quad 0 \leq n \leq N,$$
setting 
\[ \bar{\gamma}_- = \Psi^-(\gamma_-, a(+) ), \quad \bar{\gamma}_- = \Psi^-(a(+) , \gamma_+ ), \]

\[ c^{(K(n))(n)} = c^{(0)(n + 1)} , \quad 0 \leq n \leq N , \]

setting \( c^{(0)}(N + 1) \) equal to the empty word, and setting 
\[ t_0 = t_{N + 1} = 0 , \]

\[ (\bar{a}_-(n), \bar{c}(n), \bar{a}_+(n)), t_n) = \Psi(a_-, 0(n), c^{(0)}(n), a_-, 0(n)) , \quad 0 < n \leq N , \]

setting \( \bar{c}(N + 1) \) equal to the empty word, and setting 
\[ l(n) = -\ell(\bar{c}(n + 1)) + \sum_{0 \leq k(n) \leq K(n)} \ell(c^{(k(n))}(n)) , \]

\[ s(n) = -t_n + t_{n + 1} , \quad 0 \leq n \leq N , \]

the admissibility of 
\[ \gamma_-(\prod_{0 \leq n < N} a_+, k(n)(n)Q(a_+, k(n)) + R(k(n))\quad u(a_+, k(n)(n), a_-, 0(n + 1))c^{(k(n) + 1)}(n))\gamma_+ \]

for \( X \) implies for 
\[ \bar{R}(n) \in R(l(n), s(n), (a_+, k(n)(n), R(k(n))(n))_{0 \leq k(n) \leq K(n)} , (a_-, k(n)(n))_{0 \leq k \leq K(n) + 1}) , \quad 0 \leq n < N , \]

the admissibility of 
\[ \gamma_-(\prod_{0 \leq n < N} \bar{a}_+(n)Q(\bar{a}_+(n)) + \bar{R}(n)u(\bar{a}_+(n), \bar{a}_+(n + 1))\bar{c}(n + 1))\gamma_+ \]

for \( \bar{X} \), \( N \in \mathbb{Z}_+ \).

(b) There exists a homomorphism of \( X \) onto \( \bar{X} \).

(c) There exists a homomorphism \( \varphi : X \to \bar{X} \) such that \( \varphi(X) \cap (\bar{X} - \partial \bar{X}) \neq \emptyset \).

Proof. \( \psi_1, H_o(X, 1), H(X, \bar{X}), \bar{H} \), and \( F \) are as in Section 3. Also for \( x \in F, \bar{a}_-(x) \) and \( \bar{a}_+(x) \) are as in Section 3.

We show that (a) implies (b). Choose a topologically transitive subshift of finite type \( Y \subset X \) that projects onto the topological Markov chain of the right Fischer automaton \([F]\) of \( \bar{X} \), together with an \( M \in \mathbb{N} \) such that all words in \( \mathcal{L}(Y) \) of length \( M \) are synchronizing for \( X \). The task is to extend a projection \( \xi_o \) of \( Y \) onto \( \bar{X} \) to a homomorphism \( \xi : X \to \bar{X} \). Let \( K \in \mathbb{N} \) be such that for all synchronizing words \( b, b' \) of \( X \) one can choose words \( v(b, b') \) of length less than or equal to \( K \) such that \( bv(b, b')b' \in \mathcal{L}(X) \), and let \( \bar{K} \in \mathbb{N} \) be such that one can choose for synchronizing words \( \bar{b}, \bar{b}' \) of \( \bar{X} \) and \( l \geq \bar{K} \) a word \( v_l(\bar{b}, \bar{b}') \) of length \( l \) such that \( \bar{v}_l(\bar{b}, \bar{b}')b' \in \mathcal{L}(\bar{X}) \). Also let \( K_o \geq M \) be such that no point in \( P_X\langle H(X, \bar{X}) \rangle \setminus P(Y) \)
contains a word of length $K_\circ$ that is admissible for $Y$, and let $K_\circ \in \mathbb{N}$ be such out of every synchronizing context class of $\bar{X}$ one can choose a word $b(\bar{b})$ of length less than or equal to $K_\circ \in \mathbb{N}$.

Futher let $L \geq \bar{K} + K_\circ.2L + 1 \geq M$, be such that $[-L, L]$ is a coding window for $\xi_\circ$. We let $\xi_\circ$ be given by the block map

$$\Xi_\circ : Y([-L, L[ \to \bar{\Sigma}.$$  

We choose a synchronizing word $\bar{b}_\circ$ of $\bar{X}$ and a word $b_\circ \in \mathcal{L}(X)$ such that

$$\Xi_\circ(b_\circ) = \bar{b}_\circ,$$

and we set

$$N = K_\circ + \ell(b_\circ) + K + L + 1.$$  

Set

$$E_+ = \{ x \in X : x(0, N] \in \mathcal{L}(Y), x[0, M] \notin \mathcal{L}(Y) \},$$

and, symmetrically, set

$$E_- = \{ x \in X : x[-N, 0) \in \mathcal{L}(Y), x(-M, 0] \notin \mathcal{L}(Y) \}.$$  

Set

$$b^+(x) = x_{[M + \ell(b_\circ), M + \ell(b_\circ) + K + 2L]}, \quad x \in E_+,$$

and define for $x \in E_-$, $b^-(x)$ symmetrically. We formulate coding instructions that will determine for a point $x \in X$ a point $\bar{x} \in \bar{X}$ that will serve as the image under the extension $\xi$ of $\xi_\circ$. Set

$$(5.1) \quad \bar{x}_0 = \Xi_\circ(x[-L, L]), \quad x \in Z(Y[-N, N]).$$

We set

$$E'_+ = \{ x \in E_+ : I_{E_-} \cap [-2\bar{H} - 4H(X, \bar{X}), M] \neq \emptyset \},$$

and we set

$$j(x) = \max \ (I_{E_-} \cap [-2\bar{H} - 4H(X, \bar{X}), M]), \quad x \in E'_+,$$

and

$$b_-(x) = x_{[j(x) - M - \ell(b_\circ) - K - 2L, j(x) - M - \ell(b_\circ) - K]}, \quad x \in E'_+,$$

(5.2) $$\bar{x} =\bar{x}_{[j(x) - M - L - K + \ell(b_\circ), j(x) - M - L - K + \ell(v(b^-(x), b_\circ))]} \circ (v(b^-(x), b_\circ)) = \Xi_\circ(v(b^+(x), b_\circ)),$$

$$\bar{x} =\bar{x}_{[j(x) - M - L - K + \ell(v(b^-(x), b_\circ)), M + L + K - \ell(v(b_\circ, b^+(x)))] = \Xi_\circ(v(b_\circ, b^+(x))), \quad x \in E'_+.$$  

It is

$$I_F(x) \cap [-\bar{H} - H(X, \bar{X}), K_\circ + H(X, \bar{X})] \neq \emptyset, \quad x \in E_+ \setminus E'_+.$$
For otherwise one would have by the choice of \( \hat{H} \), and by property (3.4) of the set \( F \), a \( p \in P_X(H(X, \bar{X})) \) such that

\[
P_{[-\hat{H}, \mathcal{K}_o]} = \hat{p}_{[-\hat{H}, \mathcal{K}_o]},
\]

contradicting the choice of \( \mathcal{K}_o \). Set

\[
i^+(x) = \max(I_F(x) \cap [-\hat{H} - H(X, \bar{X}), \mathcal{K}_o + H(X, \bar{X})]), \quad x \in E_+ \setminus E'_+.
\]

We set

\[
E^0_+ = \{ x \in E_+ \setminus E'_+ : I^-(S_{X}^{-i^+(x)}(x)) > -\hat{H} \}.
\]

For \( x \in E_+ \) let \( a^+(x) \) be the word in \( P_X(H(X, \bar{X})) \) that precedes in

\[
\psi_1(x_{[i^+(x)-I^-(S_{X}^{-i^+(x)}(x)), i^+(x)]})
\]

the suffix of length \( H_0(X, 1) \) and that is conjugate to \( \bar{a}_-(S_{X}^{-i^+(x)}(x)) \). Also set

\[
\beta^+(x) = [x_{[i^+(x)-H(X, 1), \mathcal{K}_o + M]}]_{-}, \quad \bar{\beta}^+(x) = \Psi^+(a^+(x), \beta^+(x)), \quad x \in E^0_+.
\]

For \( x \in (E_+ \setminus E'_+) \setminus E^0_+ \) let \( a^+(x) \) be the word in \( P_X(H(X, \bar{X})) \) that precedes in

\[
(x_{[i^+(x)-I^-(S_{X}^{-i^+(x)}(x)), i^+(x)]})
\]

the suffix of length \( H(X, \bar{X}) \) and that is conjugate to \( \bar{a}_-(S_{X}^{-i^+(x)}(x)) \). Also set

\[
\beta^+(x) = [x_{[i^+(x)-H(X, \bar{X}), \mathcal{K}_o + M]}]_{-}, \quad \bar{\beta}^+(x) = \Psi^+(a^+(x), \beta^+(x)),
\]

\[
x \in (E_+ \setminus E'_+) \setminus E^0_+,
\]

and set

\[
(5.3)
\]

\[
\bar{x}_{[i^+(x)-H(X, \bar{X}), \mathcal{K}_o + L + K - \ell(v(b_o, b^+(x)))]} = \bar{b}(\bar{\beta}(x))v(\bar{b}(\bar{\beta}(x))), \bar{b}_o,
\]

\[
\bar{x}_{[\mathcal{K}_o + L + K - \ell(v(b_o, b^+(x))), \mathcal{K}_o + \ell(v(b_o, b^+(x)) + K + L]} =
\]

\[
\Xi_o(b_o v_{K_o + L + K - \ell(v(b_o, b^+(x)) - i^+(x)) + H(X, \bar{X}) - \ell(\bar{b}(\bar{\beta}(x))))(b_o, \bar{b}(\bar{\beta}(x))), \quad x \in E^0_+.
\]

Define \( E^0_- \) symmetrically to \( E^0_+ \) and define \( i^-(x), x \in E_- \setminus E'_- \) symmetrically. Define \( E^0_+ \) symmetrically to \( E^0_+ \). For \( x \in E^0_+ \) let \( a^-(x) \) be the word in \( P_X(H(X, \bar{X})) \) that follows in

\[
\psi_1(x_{[i^-(x), i^+(x) + I^+(S_{X}^{-i^-(x)}(x))])}
\]

the prefix of length \( V(X) \) and that is conjugate to \( \bar{a}_-(S_{X}^{-i^-(x)}(x)) \). Set

\[
\beta^-(x) = [x_{[i^-(x) - H(X, 1), \mathcal{K}_o + M]}]_{-}, \quad \bar{\beta}^-(x) = \Psi^-(\beta^-(x), a^-(x)), \quad x \in E^0_-.
\]

For \( x \in (E_- \setminus E'_-) \setminus E^0_- \) define \( a^-(x) \) and \( \beta^-(x), \bar{\beta}^-(x) \) symmetrically. Then one has for \( x \in E_- \setminus E'_- \) a coding instruction that (in its formulation) is symmetric to the coding instruction 4.3. for \( x \in E_+ \setminus E'_+ \). The coordinates of \( \bar{x} \) that are not yet
determined by the coding instructions for \( x \in E_- \cup E_+ \) and by shift commutativity are determined by the coding procedure that is given by \( \varphi_0 \), the accompanying map \( \Psi_0 \) and by \( F \). The condition (a) of the theorem ensures that \( \bar{x} \in \bar{X} \), and one can set \( \xi(x) = \bar{x}, x \in X \).

For the proof that (c) implies (a) we describe the construction of the accompanying map \( \Psi^+ \). Let \( L \in \mathbb{Z}_+ \) be such that \([-L, L] \) is a coding window for \( \varphi \) and let \( \varphi \) be given by the blockmap \( \Phi : X[-L,L] \to \bar{\Sigma} \).

Out of every synchronizing context class \( \gamma \) of \( \bar{X} \) choose a word \( c(\gamma) \), and let \( \zeta_\gamma \) be a map that assigns to \( b \in \mathcal{L}(X) \) a word with prefix \( c(\gamma) \) and suffix \( b \). By hypothesis there is a synchronizing word \( \bar{b}_o \) of \( \bar{X} \) and a synchronizing word \( b_o \) of \( X \) such that \( \bar{b}_o = \Phi(b_o) \).

Let \( (a, \gamma) \in \Omega^+(X), \bar{a} = \eta(\varphi(p(a))) \), and let \( y^- \in X(-\infty, \ell(\zeta_\gamma(b))) \) be given by

\[
y_{(\infty, 0)} = \varphi(p(a)),
\]

and

\[
y_{[0, \ell(\zeta_\gamma(b))] - \infty} = \zeta_\gamma(b),
\]

and set

\[
\bar{y}^- = \Phi(y^-).
\]

If

\[
\bar{y}_{(\infty, 0)} = \varphi(p(a)),
\]

then set

\[
\Psi((a, \gamma)) = [\bar{y}_{[0, \ell(\zeta_\gamma(b))] - L}]^-.
\]

Otherwise let \( I \in \mathbb{N} \) be given by

\[
\bar{y}_{(\infty, -1)} = \varphi(p(a)), \quad \bar{y}_{(\infty, -I + 1)} \neq \varphi(p(a)),
\]

and with \( J_o \) the smallest \( J \in \mathbb{N} \) such that \( J \ell(\bar{a}) > I \), set

\[
\Psi((a, \gamma)) = [\bar{y}_{[J_o \ell(\bar{a}), \ell(\zeta_\gamma(b))] - L}]^-.
\]

The construction of \( \Psi^- \) is symmetric. The proof is completed by arguments that are patterned after the argument that yielded the necessity of condition (a) of Theorem 3.1.

In Theorem 5.1 the set

\[
\Omega^-(X) \cup \mathcal{A}_X^+ \cup \Omega^-(X)
\]

can be interpreted as the alphabet of a regular language that is associated to the sofic system \( X \subset \Sigma^\mathbb{Z} \). Instead of this set one can also use an alphabet that contains besides \( \Omega^-(X) \cup \Omega^-(X) \) triples \( (a_-, \gamma, a_+) \), where \( a_-, a_+ \in \mathcal{P}_X(H(X, \bar{X})) \), and where \( \gamma \) is a context class that is appropriately chosen with respect to \( a_- \) and \( a_+ \).

We remark that, once the entropy condition and the periodic point condition have been verified, it decidable if condition (a) of Theorem 5.1 holds, as can be seen by an argument that is similar to the argument given in Section 4.
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