Enumerating perfect forms

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Perfect Forms

Consider the space $S^n_{>0}$

of positive definite quadratic forms $Q : \mathbb{R}^n \to \mathbb{R}$

( of sym. pos. def. matrices in $\mathbb{R}^{n \times n}$ )
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DEF: $\lambda(Q) = \min_{x \in \mathbb{Z}^n \setminus \{0\}} Q[x]$ is the arithmetical minimum

DEF: $Q \in S_{>0}^n$ perfect $\iff$ $Q$ is uniquely determined by $\lambda(Q)$ and $\text{Min } Q = \{ x \in \mathbb{Z}^n : Q[x] = \lambda(Q) \}$
Extreme Forms

**THM:** (Hermite, 1850)

\[ \lambda(Q) \leq \left( \frac{4}{3} \right)^{(n-1)/2} (\text{det } Q)^{1/n} \]

(Hermite, 1822–1901)
Extreme Forms

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\[ \lambda(Q) \leq \left( \frac{4}{3} \right)^{(n-1)/2} \left( \det Q \right)^{1/n} \]

Hermite’s constant

\[ \mathcal{H}_n = \sup_{Q \in \mathcal{S}^n_{>0}} \frac{\lambda(Q)}{(\det Q)^{1/n}} \]

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Hermite’s constant

\[ H_n = \sup_{Q \in S^n_{\geq 0}} \frac{\lambda(Q)}{(\det Q)^{1/n}} \]

**DEF:**

\( Q \) is (geometric) extreme

if it attains a local maximum of \( \lambda(Q)/(\det Q)^{1/n} \) on \( S^n_{\geq 0} \)
Sphere packings

$$\delta_n = \mathcal{H}_n^{n/2} \frac{\text{vol } B^n}{2^n}$$

density of densest lattice sphere packing
Sphere packings

\[ \delta_n = \mathcal{H}_{n/2} \frac{\text{vol } B^n}{2^n} \]

density of densest lattice sphere packing

- \( \lambda(Q) \) — squared length of shortest non-zero lattice vector
- \( \det(Q) \) — squared volume of a fundamental cell
**Known results**

| $n$ | PQF/lattice | $\delta_n$ | $\mathcal{H}_n$ | author(s) |
|-----|-------------|------------|------------------|-----------|
| 2   | $A_2$       | 0.9069...  | $(\frac{4}{3})^{1/2}$ | Lagrange, 1773 |
| 3   | $A_3 = D_3$ | 0.7404...  | $2^{1/3}$        | Gauß, 1840 |
| 4   | $D_4$       | 0.6168...  | $4^{1/4}$        | Korkine & Zolotarev 1877 |
| 5   | $D_5$       | 0.4652...  | $8^{1/5}$        | Korkine & Zolotarev 1877 |
| 6   | $E_6$       | 0.3729...  | $(\frac{64}{3})^{1/6}$ | Blichfeldt, 1935 |
| 7   | $E_7$       | 0.2953...  | $64^{1/7}$       | Blichfeldt, 1935 |
| 8   | $E_8$       | 0.2536...  | 2                | Blichfeldt, 1935 |
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| 24  | $\Lambda_{24}$ | 0.0019...   | 4                | Cohn & Kumar, 2004 |

Densest lattice sphere packings known
## Known results

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### Densest lattice sphere packings known

**OPEN:** What are the densest sphere packings for \( n \geq 4 \)?
Voronoi’s characterization

**THM:** (Voronoï, 1907)

\[ Q \text{ extreme } \iff Q \text{ perfect and eutactic} \]
Voronoï’s characterization

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**DEF:** \( Q \in S^n_{>0} \) is eutactic, if

\[ Q^{-1} = \sum_{v \in \text{Min } Q}^{\alpha_v} v v^t \quad \text{if } \alpha_v > 0 \]
Determinant minimization

Extreme forms are local minima of \( (\det Q)^{\frac{1}{n}} \)

on \( \mathcal{R} = \{ Q \in S^n_\succ 0 : \lambda(Q) \geq 1 \} \)
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on \( \mathcal{R} = \{ Q \in S_{>0}^n : \lambda(Q) \geq 1 \} \)

\[= \{ Q \in S_{>0}^n : Q[x] \geq 1 \text{ for all } x \in \mathbb{Z}^n \setminus \{0\} \} \]
Determinant minimization

Extreme forms are local minima of \( (\det Q)^{\frac{1}{n}} \) on

\[ R = \{ Q \in S^n_{>0} : \lambda(Q) \geq 1 \} \]

\[ = \{ Q \in S^n_{>0} : Q[x] \geq 1 \text{ for all } x \in \mathbb{Z}^n \setminus \{0\} \} \]

\[ Q[x] = \langle Q, xx^t \rangle = \text{trace}(Q xx^t) \]

is for fixed \( x \in \mathbb{R}^n \)

linear in the \( \binom{n+1}{2} \) parameters \( q_{ij} \) of \( Q \)
Ryshkov Polyhedra

- $\mathcal{R}$ is a locally finite polyhedron
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- Vertices of $\mathcal{R}$ are perfect forms
Ryshkov Polyhedra

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• Vertices of $\mathcal{R}$ are perfect forms

• $\alpha \mapsto \left( \det(Q + \alpha Q') \right)^{\frac{1}{n}}$ is strictly concave on $S^n_{>0}$
Voronoi Cones
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- \( \text{grad} \, \det Q = (\det Q)Q^{-1} \) for \( Q \in S_{>0}^n \)
Voronoi Cones

\[ \text{grad} \det Q = (\det Q)Q^{-1} \quad \text{for} \quad Q \in S_{>0}^n \]

\[ V(Q) = \text{cone}\{uv^t : v \in \text{Min } Q\} \]
**Voronoi Cones**

- \( \text{grad det } Q = (\det Q)Q^{-1} \) for \( Q \in S_{>0}^n \)

\[
\mathcal{V}(Q) = \text{cone}\{uv^t : v \in \text{Min } Q\}
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- \( Q \text{ eutactic} \iff Q^{-1} \in \text{relint } \mathcal{V}(Q) \)
Voronoi Cones

- $\text{grad} \det Q = (\det Q)Q^{-1}$ for $Q \in S_{>0}^n$

$V(Q) = \text{cone}\{vv^t : v \in \text{Min } Q\}$

- $Q$ eutactic $\iff Q^{-1} \in \text{relint } V(Q)$

- $Q$ perfect $\iff V(Q)$ is $(\frac{n+1}{2})$-dimensional
Arithmetic equivalence
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$Q$ and $U^t QU$ with $U \in \text{GL}_n(\mathbb{Z})$ are arithmetical equivalent

$\text{GL}_n(\mathbb{Z})$ operates on $\mathcal{R}$ and its vertices and edges by

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\[\Rightarrow\] Enumeration of perfect and extreme forms is possible
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$\Rightarrow$ Enumeration of perfect and extreme forms is possible

Voronoi’s algorithm : Vertex enumeration up to arithmetical equivalence
Voronoi’s algorithm

Start with a perfect form $Q$
Voronoi’s algorithm

Start with a perfect form $Q$

1. **SVP**: Compute $\text{Min } Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) = \{ Q' \in S^n : Q'[x] \geq 1 \text{ for all } x \in \text{Min } Q \}$$
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5. Repeat steps 1.–4. for new perfect forms
Enumeration of perfect forms

- **BOTTLENECK**: Computing rays of polyhedra!

  **EX:** Rays of a 36-dim. polyhedral cone given by 120 linear inequalities yield „neighbors“ of $E_8$.
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**EX:** Rays of a 36-dim. polyhedral cone given by 120 linear inequalities yield „neighbors“ of $E_8$

| $n$ | # perfect forms | # extreme forms | author(s)                      |
|-----|-----------------|-----------------|--------------------------------|
| 2   | 1               | 1               | Lagrange, 1773                 |
| 3   | 1               | 1               | Gauß, 1840                     |
| 4   | 2               | 2               | Korkine & Zolotareff, 1877     |
| 5   | 3               | 3               | Korkine & Zolotareff, 1877     |
| 6   | 7               | 6               | Barnes, 1957                   |
| 7   | 33              | 30              | Jaquet-Chiffelle, 1991         |
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**EX:** Rays of a 36-dim. polyhedral cone given by 120 linear inequalities yield “neighbors“ of $E_8$

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| 7   | 33             | 30             | Jaquet-Chiffelle, 1991 |
| 8   | 10916          | 2408           | Dutour Sikirić, Sch. & Vallentin, 2005; Riener, 2005 |
| 9   | > 500000       |                |            |

**Computer assisted proof** with *Recursive Adj. Decomp. Method* for ray enumeration under symmetries

( showing that the “$E_8$-cone” has $25075566937584$ rays in $83092$ orbits )
Equivariant theory

For a finite group $G \subset \text{GL}_n(\mathbb{Z})$ the space of invariant forms

$$T_G := \{ Q \in S^n : G \subset \text{Aut} \ Q \}$$

is a linear subspace of $S^n$; $T_G \cap S^n_{>0}$ is called Bravais space.
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IDEA (Bergé, Martinet, Sigrist, 1992):
Intersect Ryshkov polyhedron $\mathcal{R}$ with a linear subspace $T \subset S^n$
T-perfect and T-extreme forms

DEF: \( Q \in T \cap S^n_{>0} \)

- is \( T \)-extreme if it attains a loc. max. of \( \delta \) within \( T \)
T-perfect and T-extreme forms

DEF: $Q \in T \cap S^n_{>0}$

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- is **T-eutactic** if \( Q^{-1} | T \in \text{relint}(\mathcal{V}(Q) | T) \)
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**THM (BMS, 1992):** \( Q \) **T-extreme** \( \iff \) \( Q \) **T-perfect and T-eutactic**
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- \( Q, Q' \in T \cap S_{>0} \) are called **T-equivalent**, if \( \exists U \in \text{GL}_n(\mathbb{Z}) \) with
\[
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**THM (Jaquet-Chiffelle, 1995):** \( \{ T_G\text{-perfect } Q : \lambda(Q) = 1 \} / \sim_{T_G} \) finite
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\[\Rightarrow\] Voronoi's algorithm can be applied to \( \mathcal{R} \cap T_G \)
T-Algorithm

SVPs: Obtain a $T$-perfect form $Q$
**T-Algorithm**

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1. **SVP**: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

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5. Repeat steps 1.–4. for new perfect forms
Examples/Applications

| $n$ | 2    | 4    | 6    | 8    | 10    | 12   |
|-----|------|------|------|------|-------|------|
| # $E$-perfect | 1    | 1    | 2    | 5    | 1628  | ?    |
| maximum $\delta$ | 0.9069... | 0.6168... | 0.3729... | 0.2536... | 0.0360... |     |

Perfect Eisenstein forms
## Examples/Applications

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**Perfect Eisenstein forms**

| $n$ | 2   | 4   | 6   | 8   | 10  | 12  |
|-----|-----|-----|-----|-----|-----|-----|
| $\# G$-perfect | 1   | 1   | 1   | 2   | $\geq$ 8192 | ?   |
| maximum $\delta$ | 0.7853$\ldots$ | 0.6168$\ldots$ | 0.3229$\ldots$ | 0.2536$\ldots$ |          |          |

**Perfect Gaussian forms**
Examples/Applications

| $n$ | 2  | 4  | 6  | 8  | 10 | 12 |
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| # $E$-perfect | 1  | 1  | 2  | 5  | 1628 | ?  |
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Perfect Eisenstein forms

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Perfect Gaussian forms

| $n$ | 4  | 8  | 12 | 16 |
|-----|----|----|----|----|
| # $Q$-perfect | 1  | 1  | 8  | ?  |
| maximum $\delta$ | 0.6168… | 0.2536… | 0.03125… |

Perfect Quaternion forms
Extension from Lattices to Periodic Sets
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$$\Lambda = A \left( \bigcup_{i=1}^{m} t_i + \mathbb{Z}^n \right)$$ with $A \in \text{GL}_n(\mathbb{R})$, $t_i \in \mathbb{R}^n$ and $t_m = 0$

is identified (up to orthogonal transformations) with

$$(A^t A, t_1, \ldots, t_{m-1}) \in S_{>0}^{n,m} := S_{>0}^n \times \mathbb{R}^{n \times (m-1)}$$
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is identified (up to orthogonal transformations) with

\[ \left( A^t A, t_1, \ldots, t_{m-1} \right) \in \mathcal{S}^{n, m}_{>0} := \mathcal{S}^n_{>0} \times \mathbb{R}^n \times (m-1) \]

- For fixed \( m \) and \( t = (t_1, \ldots, t_{m-1}) \), the set of periodic sets with points at min. dist. \( \geq \lambda > 0 \) is identified with a locally finite polyhedron \( \mathcal{R} \) in \( \mathcal{S}^n_{>0} \).
Extension from Lattices to Periodic Sets

\[ \Lambda = A \left( \bigcup_{i=1}^{m} t_i + \mathbb{Z}^n \right) \] with \( A \in \text{GL}_n(\mathbb{R}) \), \( t_i \in \mathbb{R}^n \) and \( t_m = 0 \)

is identified (up to orthogonal transformations) with

\[ \left( A^t A, t_1, \ldots, t_{m-1} \right) \in S_{>0}^{n,m} := S_{>0}^n \times \mathbb{R}^n \times (m-1) \]

- For fixed \( m \) and \( t = (t_1, \ldots, t_{m-1}) \), the set of periodic sets with points at min. dist. \( \geq \lambda > 0 \) is identified with a locally finite polyhedron \( \mathcal{R} \) in \( S_{>0}^n \)

**THM:** For rational and fixed \( t \),
there exist only finitely many *inequivalent* vertices of \( \mathcal{R} \)
Periodic extreme sets
Periodic extreme sets

DEF: \( X = (Q, t) \in S_{>0}^{n,m} \) (and a corresponding periodic pointset) is called periodic extreme, if it is \( m' \)-extreme for all possible representations \( X' \in S_{>0}^{n,m'} \) (attains a local maximum of \( \delta \) on \( S_{>0}^{n,m'} \) )
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**DEF:** $Q \in S_{>0}^{n}$ (and a corresponding lattice) is called strongly eutactic, if

$$Q^{-1} = \sum_{v \in \text{Min } Q} v v^t$$
Periodic extreme sets

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DEF: \( Q \in S_{>0}^{n} \) (and a corresponding lattice) is called strongly eutactic, if

\[
Q^{-1} = \alpha \sum_{v \in \text{Min } Q} v v^t
\]

THM: (Sch. 2007) Perfect and strongly eutactic forms are periodic extreme.
Periodic extreme sets

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\]

**THM:** (Sch. 2007) Perfect and strongly eutactic forms are periodic extreme

**COR:** \( A_n, D_n, E_n \) and \( \Lambda_{24} \) are periodic extreme
ToDo

- Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)
Todo

• Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)

• Systematic searches for dense periodic (non-lattice) sets
**ToDo**

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**Challenges**

- Prove for some non-lattice sphere packing that it is denser than any lattice packing in its dimension
**ToDo**

- Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)

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**Challenges**

- Prove for some non-lattice sphere packing that it is denser than any lattice packing in its dimension

- Determine Hermite’s constant for some $n \geq 9 \ (n \neq 24)$
Muchas Gracias!

http://www.math.uni-magdeburg.de/lattice_geometry/