GLOBAL SOLUTIONS TO THE CHEMOTAXIS–NAVIER–STOKES EQUATIONS WITH SOME LARGE INITIAL DATA

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Abstract. In this paper, we mainly study the Cauchy problem of the Chemotaxis-Navier-Stokes equations with initial data in critical Besov spaces. We first get the local wellposedness of the system in \( \mathbb{R}^d \) \((d \geq 2)\) by the Picard theorem, and then extend the local solutions to be global under the only smallness assumptions on \( \|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} \), \( \|n_0\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}} \) and \( \|c_0\|_{\dot{B}^{-r+\frac{d}{r}}_{r,1}} \). This obtained result implies the global wellposedness of the equations with large initial vertical velocity component. Moreover, by fully using the global wellposedness of the classical 2D Navier–Stokes equations and the weighted Chemin-Lerner space, we can also extend the obtained local solutions to be global in \( \mathbb{R}^2 \) provided the initial cell density \( n_0 \) and the initial chemical concentration \( c_0 \) are doubly exponential small compared with the initial velocity field \( u_0 \).

1. Introduction. In the paper, we consider the following Chemotaxis–Navier–Stokes equations in \( \mathbb{R}^d \) \((d \geq 2)\):

\[
\begin{align*}
\text{div} u &= 0, \\
c_t - \nu \Delta c + u \cdot \nabla c &= -\kappa(c)n, \\
u_t - \mu \Delta u + u \cdot \nabla u + \nabla P &= -n \nabla \phi, \\
n_t - \gamma \Delta n + u \cdot \nabla n &= -\nabla \cdot (\chi(c)n \nabla c),
\end{align*}
\]

with the initial conditions

\[
u|_{t=0} = u_0(x), \quad n|_{t=0} = n_0(x), \quad c|_{t=0} = c_0(x), \quad \text{in} \quad \mathbb{R}^d,
\]

where \( u(x, t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d \) is the fluid velocity field, \( n = n(x, t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+ \) is the cell density, \( c = c(x, t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+ \) is the chemical concentration, and \( P = P(x, t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R} \) is the pressure of the fluid. Positive constants \( \gamma, \mu \) and \( \nu \) are the corresponding diffusion coefficients for the cells, chemical and fluid.

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respectively. \( \chi(c) \) is the chemotactic sensitivity and \( \kappa(c) \) is the consumption rate of the chemical by the cells. \( \phi = \phi(x) \) is a given potential function accounting the effects of external forces such as gravity. Different functional forms of \( \chi(c) \) and \( \kappa(c) \) are meaningful, according to various conceivable threshold effects and saturation mechanism. In general, \( \chi(c) \) and \( \phi(x) \) are supposed to be sufficiently smooth given functions. In the present paper, we consider a simplified model with \( \gamma = \mu = \nu = 1 \), \( \chi(c) = 1 \) and \( \kappa(c) \leq Ac \) for some constant \( A \). Throughout the paper, we write \( u(x,t) = (u^1(x,t),u^2(x,t),\cdots,u^{d-1}(x,t),u^d(x,t)) \) and \( u^h \) the horizontal component of \( u \), and \( u^d \) the vertical component of \( u \). Here “horizontal” and “vertical” should be understood as a convention.

The system (1.1)–(1.4) describes a biological process, in which cells (e.g. bacteria) move towards a chemically more favorable environment. Specifically, the mechanism is a chemotactic movement of bacteria often towards higher concentration of oxygen which they consume, a gravitational effect on the motion of the fluid by the heavier bacteria, and a convective transport of both cells and oxygen through the water. One can refer to [14], [22] for more details. This model has been studied by many researchers due to the significance of the biological background. For the case \( \chi(c) = c \), Lorz [14] got the local existence of the solutions for the Keller-Segel-Stokes system in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). Duan, Lorz, and Markowich [8] obtained the global-in-time existence of the \( H^3(\mathbb{R}^d) \)-solutions, near constant states, to (1.1)–(1.4), i.e., if the initial data \( \| (n_0 - n_\infty, c_0, u_0) \|_{H^3} \) is sufficiently small, then there exists a unique global solution. Moreover, they showed the global existence of the weak solutions to the Keller-Segel-Stokes equations in \( \mathbb{R}^2 \), where the nonlinear convective term \( u \cdot \nabla u \) is ignored in (1.3), with the small data assumptions on either \( \nabla \phi \) or the initial data \( c \). The key ingredient of their proof is to establish a priori estimates involving energy type functionals. Subsequently, under the same assumptions on \( \chi \) and \( \kappa \) [8], Liu and Lorz [13] removed the above smallness conditions and proved the global existence of weak solutions to the 2D Keller-Segel-Navier–Stokes equations for arbitrarily large initial data. In [23], Winkler proved that the system (1.1)–(1.4) admits a unique global classical solution in a bounded convex domain with smooth boundary in \( \mathbb{R}^2 \) under the assumptions on \( \chi, \kappa, \phi \) which are weaker than those in [8], [13]. Chae, Kang and Lee [6] established the local existence of regular solutions and presented some blow-up criteria in \( H^m(\mathbb{R}^d) \) (\( d = 2, 3 \)) with \( m \geq 3 \). They also got the existence of global weak solutions in \( \mathbb{R}^3 \) with stronger restrictions on the consumption rate and the chemotactic sensitivity. However, whether the solutions to (1.1)–(1.4) with large initial data exist globally or blow up in finite time remains an open problem.

The main goal of this paper is to show the local and global wellposedness of (1.1)–(1.4) with initial data in critical Besov spaces. Here “the critical space” means that we want to solve the system (1.1)–(1.4) in functional spaces with invariant norms by the changes of scales that leave (1.1)–(1.4) invariant. For (1.1)–(1.4), it is easy to see that the transformations

\[ (u_\lambda, n_\lambda, c_\lambda)(t, x) = (\lambda u(\lambda^2 \cdot, \lambda \cdot), \lambda^2 n(\lambda^2 \cdot, \lambda \cdot), c(\lambda^2 \cdot, \lambda \cdot)) \]

have this “critical” property provided that the pressure term \( P \) and the potential function \( \phi(x) \) have been changed accordingly. We can also verify that the product space \( \dot{B}^{-1+\frac{d}{r}}_{p,1}(\mathbb{R}^d) \times \dot{B}^{-2+\frac{d}{r}}_{q,1}(\mathbb{R}^d) \times \dot{B}^\frac{d}{r}_{r,\infty}(\mathbb{R}^d), 1 \leq p, q, r \leq \infty \), is the critical space for the system (1.1)–(1.4). By fully using Bony’s decomposition, Bernstein’s lemma and interpolation inequality in the Chemin-Lerner space, we get respectively different estimates of couple terms appeared in (1.1)–(1.4). Then we can establish the
Corollary 1.2. Let $\omega_0 = \nabla \times u$ and let $T^*$ be the maximal local existence time of $(u, n, c)$ in Theorem 1.1. If $T^* < \infty$, then

$$\int_0^{T^*} (\|\omega\|_{B^{\frac{d}{d}-1}_p} + \|n\|_{B^{\frac{d}{d}}_{q,1}}) dt = \infty.$$

We now present our second main result of the paper. More precisely, we can obtain the following global wellposedness result without any size restrictions on the initial velocity $u_0$ in $\mathbb{R}^2$.

Theorem 1.3. Let $1 < p, q \leq \min\{4, r\} < \infty$, $\frac{1}{q} - \frac{1}{r} \leq \frac{1}{2} < \frac{1}{q} + \frac{1}{r}$, and $-\frac{1}{2} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1-\varepsilon}{2}$. Suppose that $u_0 \in \dot{B}^{-1+\frac{d}{d}}_{p,1}(\mathbb{R}^2)$ with $\text{div} u_0 = 0$, $n_0 \in \dot{B}^{-2+\frac{d}{d}}_{q,1}(\mathbb{R}^2)$, $c_0 \in \dot{B}^{-\frac{d}{d}}_{r,1}(\mathbb{R}^2)$, $\phi \in \dot{B}^{-\frac{d}{d}+\varepsilon}_{r,1}(\mathbb{R}^2)$. Then (1.1)–(1.4) has a unique local solution $(u, n, c)$ on $[0, T]$ such that

$$u \in C([0, T]; \dot{B}^{-1+\frac{d}{d}}_{p,1}(\mathbb{R}^2)) \cap L^\infty_T (\dot{B}^{-1+\frac{d}{d}}_{p,1}(\mathbb{R}^2)) \cap L^1_T (\dot{B}^{-1+\frac{d}{d}}_{p,1}(\mathbb{R}^2)),$$

$$n \in C([0, T]; \dot{B}^{-2+\frac{d}{d}}_{q,1}(\mathbb{R}^2)) \cap L^\infty_T (\dot{B}^{-2+\frac{d}{d}}_{q,1}(\mathbb{R}^2)) \cap L^1_T (\dot{B}^{-2+\frac{d}{d}}_{q,1}(\mathbb{R}^2)),$$

$$c \in C([0, T]; \dot{B}^{-\frac{d}{d}}_{r,1}(\mathbb{R}^2)) \cap L^\infty_T (\dot{B}^{-\frac{d}{d}}_{r,1}(\mathbb{R}^2)) \cap L^1_T (\dot{B}^{-\frac{d}{d}}_{r,1}(\mathbb{R}^2)).$$

Moreover, if $1 < q \leq d$, $\|\phi\|_{\dot{B}^{\frac{d}{d}}_{r,1}}$ is small enough, and there exists a positive constant $C_0$ such that

$$C_0(\|u_0\|^{1-\alpha}_{\dot{B}^{-1+\frac{d}{d}}_{p,1}} + \|n_0\|^{1-\alpha}_{\dot{B}^{-2+\frac{d}{d}}_{q,1}} + \|c_0\|^{1-\alpha}_{\dot{B}^{-\frac{d}{d}}_{r,1}} + \|u_0\|^{\alpha}_{\dot{B}^{-1+\frac{d}{d}}_{p,1}} + \|u_0\|^{\alpha}_{\dot{B}^{-2+\frac{d}{d}}_{q,1}} + \|u_0\|^{\alpha}_{\dot{B}^{-\frac{d}{d}}_{r,1}}) \leq 1,$$

where

$$\alpha = \begin{cases} \frac{1}{p}, & 1 < p < 2d - 1, \\ \varepsilon, & 2d - 1 \leq p < 2d, \end{cases}$$

for $0 < \varepsilon < \frac{2d}{p} - 1$, then the local solution can be extended to be global.
$c_0 \in \dot{B}^\frac{\delta}{r_1}_p(R^2)$ and $\phi \in \dot{B}^\frac{\delta}{p_1}_1(R^2) \cap \dot{B}^{\frac{\delta}{p_1}+\epsilon}(R^2)$ is small enough. If there exist positive constants $C_0$, $\varepsilon_0$ such that the initial data $(u_0, n_0, c_0)$ satisfies

$$C_0(\|n_0\|_{\dot{B}^\frac{-2+\frac{\delta}{q}}{q_1}_1} + \|c_0\|_{\dot{B}^\frac{-2+\frac{\delta}{q}}{q_1}_1}) \exp \left( C_0(1 + \|u_0\|_{\dot{B}^{-1+\frac{\delta}{r_1}}_{p_1}})^2 \exp(\|u_0\|_{\dot{B}^{-1+\frac{\delta}{r_1}}_{p_1}}^2) \right) \leq \varepsilon_0,$$

(1.9)

then \([1.1]-[1.4]\) has a unique global solution $(u, n, c)$ satisfying

$$u \in C([0, \infty); \dot{B}^{-1+\frac{\delta}{r_1}}_{p_1}(R^2)) \cap \dot{L}^\infty(R^+; \dot{B}^{-1+\frac{\delta}{r_1}}_{p_1}(R^2)) \cap L^1(R^+; \dot{B}^{\frac{2}{p_1}+1}_{p_1}(R^2)),$$

$$n \in C([0, \infty); \dot{B}^{-2+\frac{\delta}{r_1}}_{q_1}(R^2)) \cap \dot{L}^\infty(R^+; \dot{B}^{-2+\frac{\delta}{r_1}}_{q_1}(R^2)) \cap L^1(R^+; \dot{B}^{\frac{2}{q_1}}_{q_1}(R^2)),$$

$$c \in C([0, \infty); \dot{B}^\frac{\delta}{r_1}_1(R^2)) \cap \dot{L}^\infty(R^+; \dot{B}^{\frac{\delta}{r_1}+\frac{\delta}{r_1}}_{r_1}(R^2)) \cap L^1(R^+; \dot{B}^{\frac{2}{r_1}+\frac{2}{r_1}}_{r_1}(R^2)).$$

(1.10)

The remainder of the paper is organized as follows. In Section 2, we recall some basic facts on the Littlewood-Paley theory and various product laws in Besov spaces. In Section 3, we present the proof of the local wellposedness result in Theorem 1. In Section 4, by anisotropic Bernstein’s lemma and Gagliardo-Nirenberg’s inequality, we can extend our local solutions to be global under the some smallness condition on initial data. In Section 5, we give the proof of Corollary 1.4. In the last section, by fully using the global wellposedness of the classical 2D Navier–Stokes system and the weighted Chemin-Lerner space, we also get the global wellposedness of \([1.1]-[1.4]\) in $R^2$ under another type of initial data.

**Notations.** Given $A$, $B$ are two operators, we denote the commutator $[A, B] = AB - BA$. For $a \leq b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq Cb$. For a Banach space $X$ and an interval $I$ of $R$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$. For $q \in [1, +\infty]$, the notation $L^q(I; X)$ stands for the set of measurable functions on $I$ with values in $X$, such that $t \rightarrow \|f(t)\|_X$ belongs to $L^q(I)$. Finally, we let $(d_j)_{j \in Z}$ denote a generic element of $\ell^1(Z)$ such that $\sum_{j \in Z} d_j = 1$.

2. Littlewood-Paley theory. In this section, we recall some basic facts on Littlewood-Paley theory (see \([1]\) for instance). Let $\chi, \varphi$ be two smooth radial functions valued in the interval $[0, 1]$, the support of $\chi$ be the ball $B = \{x \in R^d : |x| \leq \frac{3}{4}\}$ while the support of $\varphi$ be the annulus $C = \{x \in R^d : \frac{3}{4} \leq |x| \leq \frac{8}{3}\}$, and

$$\sum_{j \in Z} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \neq 0.$$

Let $h \overset{\text{def}}{=} \varphi^{-1} \varphi$ and $\tilde{h} \overset{\text{def}}{=} \varphi^{-1} \chi$. The homogeneous dyadic blocks $\Delta_j$ and the homogeneous low-frequency cutoff operators $\dot{S}_j$ are defined for all $j \in Z$ by

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{dj} \int_{R^d} h(2^jy)u(x-y)dy,$$

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{dj} \int_{R^d} \tilde{h}(2^jy)u(x-y)dy.$$

Denote by $\mathcal{S}'_h(R^d)$ the space of tempered distributions $u$ such that

$$\lim_{j \to -\infty} \dot{S}_j u = 0 \quad \text{in} \quad \mathcal{S}'.$$
Then we have the formal decomposition
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \forall u \in \mathcal{S}^\prime_0(\mathbb{R}^d). \]

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:
\[ \Delta_k \Delta_j u = 0 \text{ if } |k - j| \geq 2, \quad \text{and } \Delta_k(\dot{S}_{j-1} u \Delta_j v) = 0 \text{ if } |k - j| \geq 5. \]

**Definition 2.1.** Let \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \). The homogeneous Besov space \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) consists of all the distributions \( u \in \mathcal{S}^\prime_0(\mathbb{R}^d) \) such that
\[ \|u\|_{\dot{B}^s_{p,r}} \equiv \left\| \left(2^{js}\|\Delta_j u\|_{L^p}\right)_{j \in \mathbb{Z}}\right\|_{\ell^r} < \infty. \]

**Remark 3.** Let \( s \in \mathbb{R} \), \( 1 \leq p, r \leq \infty \) and \( u \in \mathcal{S}^\prime_0(\mathbb{R}^d) \). Then there exists a positive constant \( C \) such that \( u \) belongs to \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) if and only if there exists \( \{c_{j,r}\}_{j \in \mathbb{Z}} \) such that \( c_{j,r} \geq 0 \), \( \|c_{j,r}\|_{\ell^r} = 1 \) and
\[ \|\Delta_j u\|_{L^p} \leq Cc_{j,r}2^{-js}\|u\|_{\dot{B}^s_{p,r}}, \quad \forall j \in \mathbb{Z}. \]

If \( r = 1 \), we denote by \( d_j \equiv c_{j,1} \).

We also need to use Chemin-Lerner type Besov spaces (see \([1, 17]\)).

**Definition 2.2.** Let \( s \in \mathbb{R} \) and \( 0 < T \leq +\infty \). We define
\[ \|u\|_{\dot{L}^s_T(\dot{B}^s_{p,r})} \equiv \left\| \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T \|\Delta_j u(t)\|_{\dot{L}^p_T}^\theta \|\Delta_j u(t)\|_{\dot{L}^r_T}^{1-\theta} \right)^{\frac{s}{\theta}} \right\|^\frac{1}{\theta}. \]

for \( p \in [1, \infty] \), \( r, \sigma \in [1, \infty) \), and with the standard modification for \( r = \infty \) or \( \sigma = \infty \).

**Remark 4.** It is easy to observe that for \( 0 < s_1 < s_2 \), \( \theta \in [0, 1] \), \( p, r, \lambda, \lambda_1, \lambda_2 \in [1, +\infty) \), we have the following interpolation inequality in the Chemin-Lerner space (see \([1]\)):
\[ \|u\|_{\dot{L}^{s_1}_T(\dot{B}^{s_1}_{p,r})} \leq \|u\|_{\dot{L}^{s_2}_T(\dot{B}^{s_2}_{p,r})} \leq \|u\|_{\dot{L}^{s_1}_T(\dot{B}^{s_1}_{p,r})} \|u\|_{\dot{L}^{s_2}_T(\dot{B}^{s_2}_{p,r})} \]
with \( \frac{1}{s} = \frac{\theta}{s_1} + \frac{1-\theta}{s_2} \) and \( s = \theta s_1 + (1-\theta)s_2 \).

Let us emphasize that, according to the Minkowski inequality, we have
\[ \|f\|_{\dot{L}^s_T(\dot{B}^s_{p,r})} \leq \|f\|_{\dot{L}^s_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda \leq r, \quad \|f\|_{\dot{L}^s_T(\dot{B}^s_{p,r})} \leq \|f\|_{\dot{L}^s_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda \geq r. \]

In order to prove Theorem 1.3, we need to introduce the following weighted Chemin-Lerner type norm \([1, 17]\):**

**Definition 2.3.** Let \( f(t) \in L^1_{loc}(\mathbb{R}^+), f(t) \geq 0 \). We define
\[ \|u\|_{\dot{L}^s_T,f(\dot{B}^s_{p,r})} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jr_1} \left( \int_0^T (f(t)\|\Delta_j u(t)\|_{\dot{L}^p_T})^{r_1} \right)^{\frac{s}{r_1}} \right\}^{\frac{1}{r_1}} \]
for \( s \in \mathbb{R}, p \in [1, \infty], q, r \in [1, \infty) \), and with the standard modification for \( q = \infty \) or \( r = \infty \).

The following anisotropic Bernstein’s lemma will be repeatedly used throughout this paper.
Lemma 2.4. (see Lemma 2.1 in [16]) Let \(B_x\) (resp. \(B_v\)) be a ball of \(\mathbb{R}^{d-1}\) (resp. \(\mathbb{R}_v\)) and \(C_h\) (resp. \(C_v\)) a ring of \(\mathbb{R}^{d-1}\) (resp. \(\mathbb{R}_v\)). Let \(1 \leq p_2 \leq p_1 \leq \infty\) and \(1 \leq q_2 \leq q_1 \leq \infty\). Then there hold:

If the support of \(f\) is included in \(2^k B_h\), then
\[
\|\partial_x^a f\|_{L^p_h(L^q_v)} \lesssim 2^k \left(\left|a\right| + \left(\frac{d-1}{2} - \frac{k-1}{4} \right)\right) \|f\|_{L^p_h(L^q_v)}.
\]

If the support of \(\hat{f}\) is included in \(2^k B_v\), then
\[
\|\partial^\beta_x f\|_{L^p_h(L^q_v)} \lesssim 2^k \left(\left|\beta\right| + \left(\frac{d-1}{2} - \frac{k-1}{4} \right)\right) \|f\|_{L^p_h(L^q_v)}.
\]

If the support of \(\hat{f}\) is included in \(2^k C_h\), then
\[
\|f\|_{L^p_h(L^q_v)} \lesssim 2^{-kN} \sup_{|\alpha| = N} \|\partial_x^\alpha f\|_{L^p_h(L^q_v)}.
\]

If the support of \(\hat{f}\) is included in \(2^k C_v\), then
\[
\|f\|_{L^p_h(L^q_v)} \lesssim 2^{-kN} \|\partial^\alpha_x f\|_{L^p_h(L^q_v)}.
\]

On the other hand, it has been demonstrated that the Bony’s decomposition [11, 2] is very effective to deal with nonlinear problems. Here, we recall the Bony’s decomposition in the homogeneous context:

\[
uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v) = \hat{T}_u v + \hat{R}(u, v), \tag{2.11}
\]

where
\[
\hat{T}_u v \triangleq \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u, v) \triangleq \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \hat{\Delta}_j v,
\]

and
\[
\hat{\Delta}_j v \triangleq \sum_{|j-j'| \leq 1} \hat{\Delta}_j v, \quad \hat{\Delta}_j u \triangleq \sum_{j \in \mathbb{Z}} \hat{S}_{j+2} u \hat{\Delta}_j u.
\]

We shall also use the following lemmas to prove our theorems.

Lemma 2.5. Let \(1 \leq p, q \leq \infty\), \(s_1 \leq \frac{d}{q}\), \(s_2 \leq d \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\) and \(s_1 + s_2 > d \max\{0, \frac{1}{p} + \frac{1}{q} - 1\}\). For \(v(a, b) \in \dot{B}^s_{q,1}(\mathbb{R}^d) \times \dot{B}^s_{p,1}(\mathbb{R}^d)\), we have
\[
\|ab\|_{\dot{B}^{s_1 + s_2 - \frac{d}{q}}_{p,1}} \lesssim \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}. \tag{2.12}
\]

Proof. This lemma is proved in [17] for the case when \(q \leq p\). We shall only prove (2.12) for the case when \(q > p\). Applying Bony’s decomposition, we have
\[
ab = \hat{T}_a b + T_b a + \hat{R}(a, b).
\]

Then by Lemma 2.4, we get for \(s_1 \leq \frac{d}{q}\)
\[
\|\hat{\Delta}_j(\hat{T}_a b)\|_{L^p} \lesssim \sum_{|j-j'| \leq 4} \|\hat{S}_{j'-1} a\|_{L^q} \|\hat{\Delta}_j b\|_{L^p} \lesssim d_j 2^{-j(s_1 + s_2 - \frac{d}{q})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}},
\]

and for \(s_2 \leq \frac{d}{q}\)
\[
\|\hat{\Delta}_j(\hat{T}_b a)\|_{L^p} \lesssim \sum_{|j-j'| \leq 4} \|\hat{\Delta}_j a\|_{L^q} \|\hat{S}_{j'-1} b\|_{L^q} \lesssim d_j 2^{-j(s_1 + s_2 - \frac{d}{q})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}.
\]
If \( \frac{1}{p} + \frac{1}{q} \geq 1 = \frac{1}{p} + \frac{1}{\tilde{p}} \), we infer

\[
\| \tilde{\Delta}_j \left( \tilde{R}(a, b) \right) \|_{L^p} \lesssim 2^{q j \left( 1 - \frac{1}{p} \right)} \sum_{j' \geq j - 3} \| \tilde{\Delta}_{j'} a \|_{L^q} \| \tilde{\Delta}_{j'} b \|_{L^p}
\]

\[
\lesssim 2^{q j \left( 1 - \frac{1}{p} \right)} \| a \|_{\dot{B}_{q,1}^{s_1}} \| b \|_{\dot{B}_{p,1}^{s_2}} \sum_{j' \geq j - 3} d_{j'} 2^{-j' \left( s_1 + s_2 - d \left( \frac{1}{p} + \frac{1}{\tilde{p}} \right) \right)}
\]

\[
\lesssim d_j 2^{-j \left( s_1 + s_2 - \frac{d}{2} \right)} \| a \|_{\dot{B}_{q,1}^{s_1}} \| b \|_{\dot{B}_{p,1}^{s_2}}.
\]

for \( s_1 + s_2 > d \left( \frac{1}{p} + \frac{1}{q} \right) - 1 \). Finally, in the case when \( \frac{1}{p} + \frac{1}{q} \overset{\text{def}}{=} \frac{1}{\tilde{p}} < 1 \), noticing that \( s_1 + s_2 > 0 \), one has

\[
\| \tilde{\Delta}_j \left( \tilde{R}(a, b) \right) \|_{L^p} \lesssim 2^{q j \left( 1 - \frac{1}{p} \right)} \sum_{j' \geq j - 3} \| \tilde{\Delta}_{j'} a \|_{L^q} \| \tilde{\Delta}_{j'} b \|_{L^p}
\]

\[
\lesssim d_j 2^{-j \left( s_1 + s_2 - \frac{d}{2} \right)} \| a \|_{\dot{B}_{q,1}^{s_1}} \| b \|_{\dot{B}_{p,1}^{s_2}}.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 2.6.** (see Lemma 2.100 in [11] for instance) Let \( 1 \leq p, q \leq \infty \), \( s \leq 1 + d \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \), \( a \in \dot{B}_{q,1}^s (\mathbb{R}^d) \) and \( u \in \dot{B}_{p,1}^{1 + \frac{d}{q} + 1} (\mathbb{R}^d) \). Assume that

\[
s > -d \min \left\{ \frac{1}{p}, 1 - \frac{1}{q} \right\}, \quad \text{or} \quad s > -1 - d \min \left\{ \frac{1}{p}, 1 - \frac{1}{q} \right\} \quad \text{if} \quad \text{div} u = 0.
\]

Then there holds

\[
\| [u \cdot \nabla, \tilde{\Delta}_j] a \|_{L^q} \lesssim d_j 2^{-j s} \| a \|_{\dot{B}_{p,1}^{1 + \frac{d}{q} + 1}}.
\]

**Lemma 2.7.** Let \( 1 \leq p \leq m_1, m_2 \leq \infty \), \( u = (u^h, u^d) \in \tilde{L}^\infty ((0, +\infty); \dot{B}_{p,1}^{1 + \frac{d}{q}}) \cap L^1((0, +\infty); \dot{B}_{p,1}^{1 + \frac{d}{q}}) \) with \( \text{div} u = 0 \), there hold

\[
\| \tilde{\Delta}_j u^d \|_{L^1_2 (L_m^1(L_m^2))} \lesssim d_j 2^{-j \left( \frac{d}{m_1} + \frac{1}{m_2} \right)} \| u^d \|^{1 - \frac{1}{p} + \frac{d}{m_2}}_{L^1_2(\dot{B}_{p,1}^{\frac{d}{q}})} \| u^h \|^{\frac{1}{p} - \frac{d}{m_2}}_{L^1_2(\dot{B}_{p,1}^{\frac{d}{q}})} \tag{2.13}
\]

\[ \text{and} \]

\[
\| \tilde{\Delta}_j u^d \|_{L^1_2 (L_m^1(L_m^2))} \lesssim d_j 2^{-j \left( 1 + \frac{d}{m_1} + \frac{1}{m_2} \right)} \| u^d \|^{1 - \frac{1}{p} + \frac{d}{m_2}}_{L^1_2(\dot{B}_{p,1}^{1 + \frac{d}{q}})} \| u^h \|^{\frac{1}{p} - \frac{d}{m_2}}_{L^1_2(\dot{B}_{p,1}^{1 + \frac{d}{q}})} \tag{2.14}
\]

**Proof.** The proof of this lemma is similar to that of Lemma 2.4 in [27], we also outline its proof here for completeness. Firstly, using \( \text{div} u = 0 \) and the Gagliardo-Nirenberg inequality, we have

\[
\| \tilde{\Delta}_j u^d \|_{L_m^2} \lesssim \| \tilde{\Delta}_j u^d \|^{1 - \left( \frac{1}{p} - \frac{1}{m_2} \right)}_{L^p} \| \partial_d \tilde{\Delta}_j u^d \|^{\frac{1}{p} - \frac{1}{m_2}}_{L^p}
\]

\[
\lesssim \| \tilde{\Delta}_j u^d \|^{1 - \left( \frac{1}{p} - \frac{1}{m_2} \right)}_{L^p} \| \tilde{\Delta}_j (\text{div} u^h) \|^{\frac{1}{p} - \frac{1}{m_2}}_{L^p}
\]

\[
\lesssim 2^{j \left( \frac{1}{p} - \frac{1}{m_2} \right)} \| \tilde{\Delta}_j u^d \|^{1 - \left( \frac{1}{p} - \frac{1}{m_2} \right)}_{L^p} \| \tilde{\Delta}_j u^h \|^{\frac{1}{p} - \frac{1}{m_2}}_{L^p}. \tag{2.15}
\]
Thus, by Lemma 2.4, we have
\[ \| \tilde{\Delta}_j u^d \|_{L^2_t(L^p_w(L^{m_2}))} \lesssim 2^j \left( \frac{d+1}{p'} - \frac{d+1}{m_1} \right) \| \tilde{\Delta}_j u^d \|_{L^2_t(L^p_w(L^{m_2}))} \]
\[ \lesssim 2^{j \left( \frac{d}{p'} - \frac{d}{m_1} - \frac{1}{m_2} \right)} \| \tilde{\Delta}_j u^d \|_{L^2_t(L^p_w(L^\infty))} \frac{1}{L^2_t(B_{p,1}^1)} \|	ilde{\Delta}_j u^h\|_{L^2_t(L^p_w(L^\infty))} \frac{1}{L^2_t(B_{p,1}^1)}. \]

(2.16)

Similarly,
\[ \| \tilde{\Delta}_j u^d \|_{L^1_t(L^p_w(L^{m_2}))} \lesssim d_2 2^{-j \left( \frac{d+1}{p'} + \frac{1}{m_2} \right)} \| u^d \|_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| u^h \|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} \]. \]

(2.17)

This completes the proof of the lemma.

Lemma 2.8. Let 1 < p < 2d and (v, w) ∈ $\tilde{L}^\infty_t(B_{p,1}^{1+\frac{d}{p}}) \cap L^1_t(B_{p,1}^{1+\frac{d}{p}})$, then one has
\[ 2^j \| \tilde{\Delta}_j (v^d w^h) \|_{L^2_t(L^p_w(L^r))} + \| \tilde{\Delta}_j (v^d \text{div}_h w^h) \|_{L^2_t(L^p_w(L^r))} \]
\[ \lesssim d_2 2^{-(\frac{d}{p} - \frac{1}{2}) j} \left( \| v^d \|^{1 - \frac{1}{p}}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| v^h \|^{\frac{1}{p}}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| w^h \|^{1 - \frac{1}{p}}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \right. \]
\[ \left. + \| v^d \|^{1 - \alpha}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| v^h \|^{\alpha}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| w^h \|^{\frac{1}{p}}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \right), \]

(2.18)

where
\[ \alpha = \begin{cases} \frac{1}{p'}, & 1 < p < 2d - 1, \\ \varepsilon, & 2d - 1 \leq p < 2d, \end{cases} \]

(2.19)

for 0 < \varepsilon < \frac{2d}{p} - 1.

Proof. Firstly, thanks to Bony’s decomposition, we have
\[ v^d w^h + v^d \text{div}_h w^h = \tilde{T}_v w^h + \tilde{T}_v \text{div}_h w^h + \tilde{T}_v \text{div}_h w^h \]
\[ + \tilde{T}_v \text{div}_h w^h + \tilde{T}(v^d, w^h) + \tilde{T}(v^d, \text{div}_h w^h). \]

Applying Lemma 2.4 and (2.13) with $m_1 = m_2 = \infty$ gives
\[ 2^j \| \tilde{\Delta}_j (\tilde{T}_v w^h) \|_{L^2_t(L^p_w(L^r))} + \| \tilde{\Delta}_j (\tilde{T}_v \text{div}_h w^h) \|_{L^2_t(L^p_w(L^r))} \]
\[ \lesssim \sum_{|j' - j| \leq 5} \left( 2^j \| \tilde{\Delta}_{j'} v^d \|_{L^2_t(L^\infty)} \| \tilde{\Delta}_{j'} w^h \|_{L^2_t(L^r)} \right) \]
\[ + \| \tilde{\Delta}_{j'} v^d \|_{L^2_t(L^\infty)} \| \tilde{\Delta}_{j'} \text{div}_h w^h \|_{L^2_t(L^r)} \right) \]
\[ \lesssim \sum_{|j' - j| \leq 5} \sum_{j'' \leq j'} \left( 2^j \| \tilde{\Delta}_{j'} v^d \|_{L^2_t(L^\infty)} \| \tilde{\Delta}_{j'} w^h \|_{L^2_t(L^r)} \right) \]
\[ + \| \tilde{\Delta}_{j''} v^d \|_{L^2_t(L^\infty)} \| \tilde{\Delta}_{j''} \text{div}_h w^h \|_{L^2_t(L^r)} \right) \]
\[ \lesssim d_2 2^{(\frac{d}{p} - \frac{1}{2}) j} \| v^d \|^{1 - \frac{1}{p}}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| v^h \|^{\frac{1}{p}}_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \| w^h \|_{L^2_t(B_{p,1}^{1+\frac{d}{p}})} \]

(2.20)
Similarly, we have
\[
\|\Delta_j (\tilde{T}_{\text{div} h} v^d)\|_{L^1_t(L^p)} \lesssim \sum_{|j'| - j| \leq 5} \left( \|\hat{S}_{j'-1} (\text{div}_h w^h)\|_{L^\infty_t(L^\infty)} \|\Delta_j v^d\|_{L^2_t(L^p)} \right)
\]
\[
\lesssim \sum_{|j'| - j| \leq 5} \sum_{j'' \leq j'} \left( 2^{\frac{d-1}{p} j''} \|\hat{\Delta}_{j''} (\text{div}_h w^h)\|_{L^2_t(L^p)} \|\Delta_j v^d\|_{L^2_t(L^p)} \right)
\]
\[
\lesssim d_j 2^{(1 - \frac{d}{p})j} \|v^d\|^{1 - \frac{p}{2}}_{L^1_t(\tilde{B}_{p,1} \frac{d}{2})} \|w^h\|_{L^2_t(\tilde{B}_{p,1} \frac{d}{2})} \|w^h\|_{L^2_t(\tilde{B}_{p,1} \frac{d}{2})} .
\] (2.21)

And applying (2.14) with \(m_1 = p, m_2 = \infty\) yields
\[
2^d \|\Delta_j \hat{R}(v^d, w^h)\|_{L^1_t(L^p)} \lesssim \sum_{j' \geq j - N_0} \left( 2^j \|\Delta_j w^h\|_{L^\infty_t(L^\infty)} \|\Delta_j v^d\|_{L^1_t(L^p)} \right)
\]
\[
\lesssim \sum_{j' \geq j - N_0} \left( 2^{j' + \frac{d-1}{p} j} \|\Delta_j w^h\|_{L^\infty_t(L^\infty)} \|\Delta_j v^d\|_{L^1_t(L^p)} \right)
\]
\[
\lesssim d_j 2^{(1 - \frac{d}{p})j} \|v^d\|^{1 - \frac{p}{2}}_{L^1_t(\tilde{B}_{p,1} \frac{d}{2})} \|w^h\|_{L^2_t(\tilde{B}_{p,1} \frac{d}{2})} \|w^h\|_{L^2_t(\tilde{B}_{p,1} \frac{d}{2})} .
\] (2.22)

To estimate the remaining term \(\hat{R}(v^d, \text{div}_h w^h)\), we consider the following two cases.
In the case \(1 < p < 2d - 1\), applying (2.13) with \(m_1 = p, m_2 = \infty\) gives
\[
\|\Delta_j \hat{R}(v^d, \text{div}_h w^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{(1 - \frac{d}{p})j} \|v^d\|^{1 - \frac{p}{2}}_{L^1_t(\tilde{B}_{p,1} \frac{d}{2})} \|w^h\|_{L^2_t(\tilde{B}_{p,1} \frac{d}{2})} \|w^h\|_{L^2_t(\tilde{B}_{p,1} \frac{d}{2})} .
\] (2.24)
In the case \(2d - 1 \leq p < 2d\), we get by using (2.13) with \(m_1 = p, \frac{1}{m_2} = \frac{1}{p} - \varepsilon\) that
\[
\| \Delta_j \tilde{R}(v^d, \text{div}_h w^h) \|_{L^1(L^p)} \\
\lesssim 2^{\frac{j}{2} - \varepsilon} \sum_{j' \geq j - N_0} \| \Delta_{j'}(\text{div}_h w^h) \Delta_{j'} v^d \|_{L^1(L^p(L^{\frac{p}{p-\varepsilon}}))}
\]
\[
\lesssim 2^{\frac{j}{2} - \varepsilon} \sum_{j' \geq j - N_0} \| \Delta_{j'}(\text{div}_h w^h) \|_{L^1(L^p(L^{\frac{p}{p-\varepsilon}}))} \| \Delta_{j'} v^d \|_{L^1(L^p(L^{\frac{p}{p-\varepsilon}}))}
\]
\[
\lesssim d_j 2^{\frac{j}{2} - \varepsilon} \| v^d \|_{L^1} \left( \| v^h \|_{L^1(B_{p,1})} \| w^h \|_{L^1(B_{p,1})} \right) .
\]  
(2.25)

Combining with the estimates (2.20)–(2.25), we can finally get (2.18). Thus, the proof of Lemma 2.8 is completed.

\[\square\]

**Lemma 2.9.** (See (3.39) on page 157 in [1]). Let \(S(t) = e^{t\Delta}, T > 0\) and \(s \in \mathbb{R}\). If \(f_0(x) \in B_{p,r}(\mathbb{R}^d)\) and \(g(t,x) \in L^1_t(B_{p,r})\), then
\[
f(t,x) = S(t)f_0(x) + \int_0^t S(t-\tau)g(\tau,x)d\tau
\]
satisfies for \(1 \leq p \leq \infty\),
\[
\| f \|_{L^p_t(B_{p,r}^s)} \lesssim (\| f_0 \|_{B_{p,r}^s} + \| g \|_{L^p_t(B_{p,r}^{-2s+\frac{2}{p}})}).
\]

**Lemma 2.10.** (The Picard theorem on a Banach space) (see [15]). Let \(O \subset B\) be an open subset of a Banach space \(B\) and \(F : O \to B\) be a mapping that satisfies the following parameters:
1. \(F(X)\) maps \(O\) to \(B\).
2. \(F\) is locally Lipschitz continuous, i.e., for any \(X \in O\) there exists \(L > 0\) and an open neighborhood \(U_X \subset O\) of \(X\) such that for any \(\tilde{X}, \tilde{X} \in U_X\),
\[
\| F(\tilde{X}) - F(\tilde{X}) \|_B \leq L \| \tilde{X} - \tilde{X} \|_B
\]

Then for any \(X_0 \in O\), there exists a time \(T\) such that the ODE
\[
\frac{dX}{dt} = F(x), \quad X|_{t=0} = X_0 \in O,
\]
has a unique local solution \(X \in C^1([\frac{-T}{T}; O])\).

3. **Local wellposedness of Theorem 1.1.** Now we are in a position to prove Theorem 1.1. Let \(p, q, r\) satisfy the conditions in Theorem 1.1 with \(\text{div}u_0 = 0\), \(u_0 \in \dot{B}^{-1+\frac{d}{2}}_{p,1}(\mathbb{R}^d), n_0 \in \dot{B}^{-2+\frac{d}{2}}_{q,1}(\mathbb{R}^d), c_0 \in \dot{B}^0_{r,1}(\mathbb{R}^d)\). Then we introduce a vector space \(\mathfrak{X}_T \triangleq \mathfrak{X}_T \times \mathcal{Y}_T \times \mathcal{Z}_T\) with the following product norm:
\[
\mathfrak{X}_T \triangleq \{ u : u \in \tilde{L}^{\infty}(0,T; \dot{B}_{p,1}^{-1+\frac{d}{2}}(\mathbb{R}^d)) \cap L^1(0,T; \dot{B}_{p,1}^{1+\frac{d}{2}}(\mathbb{R}^d)) \},
\]
\[
\mathcal{Y}_T \triangleq \{ n : n \in \tilde{L}^{\infty}(0,T; \dot{B}_{q,1}^{-2+\frac{d}{2}}(\mathbb{R}^d)) \cap L^1(0,T; \dot{B}_{q,1}^{0+\frac{d}{2}}(\mathbb{R}^d)) \},
\]
\[
\mathcal{Z}_T \triangleq \{ c : c \in \tilde{L}^{\infty}(0,T; \dot{B}_{r,1}^{0+\frac{d}{2}}(\mathbb{R}^d)) \cap L^1(0,T; \dot{B}_{r,1}^{2+\frac{d}{2}}(\mathbb{R}^d)) \},
\]
(3.26)
We shall denote the heat semiflow by $S(t) = e^{t\Delta}$, and the projector by $\mathbb{P} = \text{Id} + (-\Delta)^{-1} \nabla \text{div}$. Let $\Psi : \mathcal{X}_T \to \mathcal{X}_T$ be a map given by

$$
\Psi((u, n, c)) = S(t)(u_0, n_0, c_0) - \int_0^t S(t - \tau)(f_1, f_2, f_3)d\tau,
$$

where

$$
\begin{align*}
&f_1 = \mathbb{P}[u \cdot \nabla u + n \nabla \phi], \\
&f_2 = u \cdot \nabla n + \nabla \cdot (n \nabla c), \\
&f_3 = u \cdot \nabla c + \kappa(c)n.
\end{align*}
$$

Let $\bar{B}_{\delta}$ be a closed ball centered at 0 in $\mathcal{X}_T$ and the radius $\delta$ will be specified later. Hence, the proof of Theorem 1.1 is reformulated to prove that the map $\Psi$ is a contraction mapping in $\bar{B}_{\delta}$ for some $\delta$. By Lemma 2.9 for any $0 < t \leq T$, $u_0 \in \dot{B}_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$, $n_0 \in \dot{B}_{q,1}^{-2+\frac{d}{q}}(\mathbb{R}^d)$, $c_0 \in \dot{B}_{r,1}^{\frac{d}{2}}(\mathbb{R}^d)$, we have

$$
S(t)(u_0, n_0, c_0) \in \mathcal{X}_T \times \mathcal{Y}_T \times \mathcal{Z}_T,
$$

and

$$
\|S(t)(u_0, n_0, c_0)\|_{\mathcal{X}_T} \lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{d}{p}}} + \|n_0\|_{\dot{B}_{q,1}^{-2+\frac{d}{q}}} + \|c_0\|_{\dot{B}_{r,1}^{\frac{d}{2}}}.
$$

(3.27)

In what follows, we will present three lemmas to deal with the couple terms in (1.1)–(1.4).

**Lemma 3.1.** Let $1 < p < \infty, 1 < q < \infty, 0 < \varepsilon < 1$, $\frac{1}{p} - \frac{1}{q} \leq \frac{1 - \varepsilon}{d}$, $\frac{1}{p} + \frac{1}{q} > \frac{1}{d}$, $u \in \mathcal{X}_T, n \in \mathcal{Y}_T, \nabla \phi \in \dot{B}_{p,1}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$. Then

$$
\|u \cdot \nabla u\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})} + \|n \nabla \phi\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})}
\lesssim \||u\|_{L^\infty_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})} \|u\|_{L^2_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})} + T^{\varepsilon/2} \|n\|_{L^2_t(\dot{B}_{q,1}^{-2+\frac{d}{q}})} \|n\|_{L^2_t(\dot{B}_{q,1}^{-2+\frac{d}{q}})} \|\nabla \phi\|_{\dot{B}_{r,1}^{\frac{d}{2}}}.
$$

**Proof.** By Lemma 2.8, one can get easily

$$
\|u \cdot \nabla u\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})} \lesssim \|u\|_{L^\infty_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})} \|u\|_{L^2_t(\dot{B}_{p,1}^{-1+\frac{d}{p}})}.
$$
Again by Lemma 2.5 and interpolation inequality in the Chemin-Lerner space, we have
\[ \| n \nabla \phi \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \lesssim \int_0^T \| n \|_{B_{q,1}^{\frac{d}{q}-\varepsilon}} \| \nabla \phi \|_{B_{p,1}^{\frac{d}{q}-1+\varepsilon}} d\tau \]
\[ \lesssim \int_0^T \| n \|_{B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{q}}} \| n \|^{(2-\varepsilon)/2} \| \nabla \phi \|_{B_{p,1}^{\frac{d}{q}-1+\varepsilon}} d\tau \]
\[ \lesssim T^{\varepsilon/2} \| n \|_{B_{q,1}^{\frac{d}{q}}}^{(2-\varepsilon)/2} \| \nabla \phi \|_{B_{p,1}^{\frac{d}{q}}}^{1-1+\varepsilon}, \]
This complete the proof of the lemma.

**Lemma 3.2.** Let \( 1 < p < \infty, 1 < q \leq r < \infty, \frac{1}{q} - \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}, \frac{1}{p} - \frac{1}{r} \leq \frac{1}{d} < \frac{1}{q} + \frac{1}{r} \), \( u \in \mathcal{X}_T, n \in \mathcal{Y}_T, c \in \mathcal{Z}_T \). Then
\[ \| u \cdot \nabla n \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} + \| \nabla \cdot (n \nabla c) \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \lesssim \| n \|_{L^T_t(B_{q,1}^{\frac{d}{q}})} \| c \|_{L^T_t(B_{r,1}^{\frac{d}{r}})} \]
\[ + \| u \|_{L^T_t(B_{p,1}^{\frac{d}{p}+\frac{2}{r}} \dot{B}_{q,1}^{-1+\frac{2}{q}} \dot{B}_{r,1}^{\frac{d}{r}})} + \| n \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \| n \|_{L^T_t(B_{p,1}^{\frac{d}{p}+\frac{2}{r}} \dot{B}_{q,1}^{-1+\frac{2}{q}} \dot{B}_{r,1}^{\frac{d}{r}})}. \]

**Proof.** According to div \( u = 0 \), one can deduce that \( u \cdot \nabla n = \text{div}(nu) \). Thus by Lemma 2.5 Remark 4 and Young’s inequality, we have
\[ \| u \cdot \nabla n \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \lesssim \| nu \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \lesssim \int_0^T \| u \|_{B_{p,1}^{\frac{d}{p}}} \| n \|_{B_{q,1}^{\frac{d}{q}}} d\tau \]
\[ \lesssim \int_0^T \| u \|^{\frac{1}{2}}_{B_{p,1}^{\frac{d}{p}}} \| u \|_{B_{p,1}^{\frac{d}{p}}} \| n \|^{\frac{1}{2}}_{B_{q,1}^{\frac{d}{q}}} \| n \|^{\frac{1}{2}}_{B_{q,1}^{\frac{d}{q}}} d\tau \]
\[ \lesssim \| u \|_{L^T_t(B_{p,1}^{\frac{d}{p}+\frac{2}{r}} \dot{B}_{q,1}^{-1+\frac{2}{q}} \dot{B}_{r,1}^{\frac{d}{r}})} + \| n \|_{L^T_t(B_{q,1}^{\frac{d}{q}})} \| n \|_{L^T_t(B_{p,1}^{\frac{d}{p}+\frac{2}{r}} \dot{B}_{q,1}^{-1+\frac{2}{q}} \dot{B}_{r,1}^{\frac{d}{r}})}, \]
and when \( \frac{1}{q} - \frac{1}{r} < \frac{1}{d} \leq \frac{1}{q} + \frac{1}{r} \), we obtain
\[ \| \nabla \cdot (n \nabla c) \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \lesssim \| n \nabla c \|_{L^T_t(B_{q,1}^{-1+\frac{2}{q}} \dot{B}_{p,1}^{\frac{d}{q}})} \]
\[ \lesssim \int_0^T \| \nabla c \|_{B_{r,1}^{\frac{d}{r}}} \| n \|_{B_{q,1}^{\frac{d}{q}}} d\tau \]
\[ \lesssim \int_0^T \| c \|^{\frac{1}{2}}_{B_{r,1}^{\frac{d}{r}}} \| c \|^{\frac{1}{2}}_{B_{r,1}^{\frac{d}{r}}} \| n \|^{\frac{1}{2}}_{B_{q,1}^{\frac{d}{q}}} \| n \|^{\frac{1}{2}}_{B_{q,1}^{\frac{d}{q}}} d\tau \]
\[ \lesssim \| c \|_{L^T_t(B_{r,1}^{\frac{d}{r}})} \| c \|_{L^T_t(B_{r,1}^{\frac{d}{r}})} + \| n \|_{L^T_t(B_{q,1}^{\frac{d}{q}})} \| n \|_{L^T_t(B_{q,1}^{\frac{d}{q}})}. \]

This complete the proof of the lemma.

**Lemma 3.3.** Let \( 1 < p < \infty, 1 < q \leq r < \infty, \frac{1}{r} - \frac{1}{p} \leq \frac{1}{d} \), \( u \in \mathcal{X}_T, n \in \mathcal{Y}_T, c \in \mathcal{Z}_T \), then we have
\[ \| u \cdot \nabla c \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{p}})} + \| \kappa(c)n \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{p}})} \]
\[ \lesssim \| u \|_{L^T_t(B_{p,1}^{-1+\frac{2}{q}} \dot{B}_{q,1}^{\frac{d}{q}})} \| c \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{p}})} + \| c \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{q,1}^{\frac{d}{p}})} \| n \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{p}})}, \]

**Proof.** Similar to the proof of the above lemma, one can infer from Lemma 2.5 that
\[ \| \kappa(c)n \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{p}})} \lesssim \| c \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{q,1}^{\frac{d}{p}})} \| n \|_{L^T_t(B_{q,1}^{\frac{d}{q}} \dot{B}_{p,1}^{\frac{d}{p}})}. \]
Thus, in the following we only pay attention to the case that $r \geq p$. By Lemma \ref{lem:2.4} and Hölder’s inequality, one has
\[
\| u \cdot \nabla c \|_{L^r_t(B^\#_{p,1})} \lesssim \int_0^T \| u \|_{B^\#_{p,1}} \| \nabla c \|_{B^\#_{r,1}} \, d\tau
\]
\[
\lesssim \int_0^T \| u \|_{B^\#_{p,1}} \| \nabla c \|_{B^\#_{r,1}} \| c \|_{B^\#_{r,1}} \, d\tau
\]
\[
\lesssim \| u \|_{L^\infty_t(B^\#_{p,1})} \| c \|_{L^r_t(B^\#_{r,1})} + \| c \|_{L^r_t(B^\#_{r,1})} \| u \|_{L^r_t(B^\#_{r,1})}.
\] (3.29)

Thus, in the following we only pay attention to the case $r \leq p$. According to Bony’s decomposition, we have
\[
u \cdot \nabla c = \hat{T}_u \nabla c + \hat{T}_\nabla u + \hat{R}(u, \nabla c).
\]

By Lemma \ref{lem:2.4} and Hölder’s inequality, one has
\[
\| \hat{\Delta}_j (\hat{T}_u \nabla c) \|_{L^r_t(L^r)} \lesssim \sum_{|j'|-|j| \leq 4} \int_0^T \| \hat{S}_{j'-1} u \|_{L^\infty} \| \hat{\Delta}_j \nabla c \|_{L^r_t} \, d\tau
\]
\[
\lesssim \sum_{|j'|-|j| \leq 4} \int_0^T \nu_j 2^{-\frac{j'}{2} j} \| u \|_{B^\#_{p,1}} \| \nabla c \|_{B^\#_{r,1}} \, d\tau
\]
\[
\lesssim \sum_{|j'|-|j| \leq 4} \int_0^T \nu_j 2^{-\frac{j'}{2} j} \| u \|_{B^\#_{p,1}} \| \nabla c \|_{B^\#_{r,1}} \| c \|_{B^\#_{r,1}} \| u \|_{B^\#_{p,1}} \| \hat{\Delta}_j \nabla c \|_{L^r_t(L^r)}.
\] (3.30)

Noticing $\frac{1}{r} - \frac{1}{p} \leq \frac{1}{4}$, we have
\[
\| \hat{\Delta}_j (\hat{T}_\nabla u) \|_{L^r_t(L^r)} \lesssim \sum_{|j'|-|j| \leq 4} \| \hat{\Delta}_j \nabla c \|_{L^r_t(L^r)} \| \hat{\Delta}_j u \|_{L^r_t(L^r)}
\]
\[
\lesssim \sum_{|j'|-|j| \leq 4} \sum_{k \leq j'-2} 2^{(1+\frac{d}{p}) k} \| \hat{\Delta}_j \nabla c \|_{L^r_t(L^r)} \| \hat{\Delta}_j u \|_{L^r_t(L^r)}
\]
\[
\lesssim \sum_{|j'|-|j| \leq 4} \sum_{k \leq j'-2} \nu_j \nu_j 2^{(1+\frac{d}{p}) k} \| \hat{\Delta}_j \nabla c \|_{L^r_t(L^r)} \| \hat{\Delta}_j u \|_{L^r_t(L^r)}
\]
\[
\lesssim \nu_j \nu_j 2^{-\frac{j'}{2} j} \| \hat{\Delta}_j \nabla c \|_{L^r_t(B^\#_{p,1})} \| \hat{\Delta}_j u \|_{L^r_t(B^\#_{p,1})}.
\] (3.31)

For the remainder term, when $\frac{1}{r} + \frac{1}{p} \geq 1 = \frac{1}{r} + \frac{1}{r}$, by Lemma \ref{lem:2.4} and div u = 0, we have
\[
\| \hat{\Delta}_j \hat{R}(u, \nabla c) \|_{L^r_t(L^r)} \lesssim 2^{j+1-\frac{1}{2} j} \sum_{j' \geq j-N_0} \| \hat{\Delta}_j \nabla c \|_{L^r_t(L^r)} \| \hat{\Delta}_j u \|_{L^r_t(L^r)}
\]
\[
\lesssim 2^{j+1-\frac{1}{2} j} \sum_{j' \geq j-N_0} 2^{\left(\frac{1}{r}-\frac{1}{2} j\right) j'} \| \hat{\Delta}_j \nabla c \|_{L^r_t(L^r)} \| \hat{\Delta}_j u \|_{L^r_t(L^r)}
\]
\[
\lesssim 2^{j+1-\frac{1}{2} j} \sum_{j' \geq j-N_0} 2^{-(d+1) j'} \| \hat{\Delta}_j \nabla c \|_{L^r_t(B^\#_{p,1})} \| \hat{\Delta}_j u \|_{L^r_t(B^\#_{p,1})}
\]
\[
\lesssim \nu_j \nu_j 2^{-\frac{j'}{2} j} \| \hat{\Delta}_j \nabla c \|_{L^r_t(B^\#_{p,1})} \| \hat{\Delta}_j u \|_{L^r_t(B^\#_{p,1})}.
\] (3.32)
we can deduce that
\[ 2^{1+\frac{\delta}{16}} \sum_{j' \geq j-N_0} 2^{-\frac{1}{16}} 2^{-\frac{j}{2} \frac{1}{16}} 2^{-\frac{j'}{2} \frac{1}{16}} \| c \|_{L^2(B_{r,1}^{\frac{1}{2}})} \| u \|_{L^2(B_{r,1}^{\frac{1}{2}})} \| \Delta_j \hat{R}(u, \nabla c) \|_{L^2(L^r)} \]
\[ \lesssim 2^{1+\frac{\delta}{16}} \sum_{j' \geq j-N_0} \| \Delta_j \hat{R}(u, \nabla c) \|_{L^2(L^r)} \lesssim d_j 2^{1+\frac{\delta}{16}} \| u \|_{L^2(B_{r,1}^{\frac{1}{2}})} \| c \|_{L^2(B_{r,1}^{\frac{1}{2}})}. \] (3.33)

Combining with estimates (3.29)–(3.33) gives
\[ \| u \cdot \nabla c \|_{L^2(B_{r,1}^{\frac{1}{2}})} \lesssim \| u \|_{L^2(B_{r,1}^{\frac{1}{2}})} \| c \|_{L^2(B_{r,1}^{\frac{1}{2}})} + \| c \|_{L^2(B_{r,1}^{\frac{1}{2}})} \| u \|_{L^2(B_{r,1}^{\frac{1}{2}})}. \]

This completes the proof Lemma 3.3

Now, we continue to prove the local wellposedness of Theorem 1.1.

By Lemma 3.1 and the fact that \( \mathcal{P} \) is a multiplier operator of order of zero, one has
\[ \| f_1 \|_{L^2(B_{r,1}^{\frac{1}{2}})} \lesssim \| u \cdot \nabla u \|_{L^2(B_{r,1}^{\frac{1}{2}})} + \| n \nabla \phi \|_{L^2(B_{r,1}^{\frac{1}{2}})} \]
\[ \lesssim \| u \|_{X_T} + T \| \nabla \phi \|_{B^{\frac{1}{2} + \epsilon}_{r,1}} \| n \|_{Y_T}. \]

Similarly, by Lemma 3.2, one has
\[ \| f_2 \|_{L^2(B_{r,1}^{\frac{1}{2}})} \lesssim \| u \cdot \nabla n \|_{L^2(B_{r,1}^{\frac{1}{2}})} + \| \nabla \cdot (n \nabla c) \|_{L^2(B_{r,1}^{\frac{1}{2}})} \]
\[ \lesssim \| u \|_{X_T} + \| n \|_{Y_T} + \| c \|_{Z_T}. \]

and by Lemma 3.3 one has
\[ \| f_3 \|_{L^2(B_{r,1}^{\frac{1}{2}})} \lesssim \| u \cdot \nabla c \|_{L^2(B_{r,1}^{\frac{1}{2}})} + \| \kappa(c) n \|_{L^2(B_{r,1}^{\frac{1}{2}})} \]
\[ \lesssim 2 \| u \|_{X_T} \| c \|_{Z_T} + \| n \|_{Y_T} \| c \|_{Z_T} \]
\[ \lesssim \| u \|_{X_T}^2 + \| n \|_{Y_T}^2 + \| c \|_{Z_T}^2. \]

Thus, we have
\[ \| \Psi((u, n, c)) \|_{X_T} \lesssim \| S(t)(u_0, n_0, c_0) \|_{X_T} + T \| \nabla \phi \|_{B^{\frac{1}{2} + \epsilon}_{r,1}} \| n \|_{Y_T} \]
\[ + \| u \|_{X_T}^2 + \| n \|_{Y_T}^2 + \| c \|_{Z_T}^2. \]

Hence, if \((u, n, c) \in B_3^D\) and \( \nabla \phi \in B_{r,1}^{\frac{1}{2} + \epsilon}(\mathbb{R}^d) \), we can get
\[ \| \Psi((u, n, c)) \|_{X_T} \lesssim \| S(t)(u_0, n_0, c_0) \|_{X_T} + T \delta + \delta^2. \]

Moreover, if we choose \( T \) small enough and
\[ 0 < \delta \leq \frac{(1 - CT) - \sqrt{(1 - CT)^2 - 4C \| S(t)(u_0, n_0, c_0) \|_{X_T}}}{2C} \]
we can deduce that
\[ \| \Psi((u, n, c)) \|_{X_T} \lesssim \delta. \]

In what follows, we only need to prove that the map \( \Psi \) is a contraction mapping in \( B_3^D \) for some \( \delta \). For that purpose, let \((u_1, n_1, c_1), (u_2, n_2, c_2) \) be inside the ball \( B_3^D \).
By Lemmas 2.5 [3.1, 3.2, 3.3] we can easily get
\[
\|u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2\|_{L^1_T(B_{p,1}^{1+\frac{d}{p}})} + \|u_1 \cdot \nabla n_1 - u_2 \cdot \nabla n_2\|_{L^1_T(B_{q,1}^{2+\frac{d}{q}})} + \|u_1 \cdot \nabla c_1 - u_2 \cdot \nabla c_2\|_{L^1_T(B_{q,1}^{-1+\frac{d}{q}})} + \|n_1 - n_2\|_{L^1_T(B_{r,1}^{-1+\frac{d}{r}})} + \|(n_1 - n_2)\nabla c_1 - (n_2 - n_2)\nabla c_2\|_{L^1_T(B_{q,1}^{-2+\frac{d}{q}})} + \|\kappa(c_1) n_1 - \kappa(c_2) n_2\|_{L^1_T(B_{q,1}^{-\frac{d}{q}})}
\]
\[
= \|(u_1 - u_2) \cdot \nabla u_1 - u_2 \cdot \nabla (u_1 - u_2)\|_{L^1_T(B_{p,1}^{1+\frac{d}{p}})} + \|(u_1 - u_2) \cdot \nabla n_1 - u_2 \cdot \nabla (n_1 - n_2)\|_{L^1_T(B_{q,1}^{2+\frac{d}{q}})} + \|(u_1 - u_2) \cdot \nabla c_1 - u_2 \cdot \nabla (c_1 - c_2)\|_{L^1_T(B_{q,1}^{-1+\frac{d}{q}})} + \|(n_1 - n_2)\nabla c_1 - (n_2 - n_2)\nabla c_2\|_{L^1_T(B_{q,1}^{-2+\frac{d}{q}})} + \|\kappa(c_1) n_1 - \kappa(c_2) (n_1 - n_2)\|_{L^1_T(B_{q,1}^{-\frac{d}{q}})}
\]
\[
\lesssim \delta \|u_1 - u_2\|_{C_T^1} + T \|\nabla \phi\|_{L^1_T(B_{p,1}^{1+\frac{d}{p}})} + \|n_1 - n_2\|_{C_T^1} + \|c_1 - c_2\|_{C_T^1}.
\]
(3.34)

By the estimate (3.27) and the dominated convergence theorem, we can deduce, as \(T\) goes to 0, that \(\|S(t)(u_0, n_0, c_0)\|_{X_T}\) tends to 0. Thus we get from (3.34) that
\[
\|\Psi((u_1, n_1, c_1)) - \Psi((u_2, n_2, c_2))\|_{X_T} \lesssim (\delta + T)\|\Psi((u_1 - u_2, n_1 - n_2, c_1 - c_2))\|_{X_T},
\]
which implies that we can choose \(\delta\) and \(T\) small enough such that the map \(\Psi\) is a contraction mapping in \(B_\delta\).

By Lemma 2.10 we can deduce that the system (1.1)–(1.3) has a unique local solution. By a standard unique continuation argument, we can obtain that
\[
u \in C([0, T]; B_{p,1}^{1+\frac{d}{p}}(\mathbb{R}^d)) \cap L^\infty_T(B_{p,1}^{1+\frac{d}{p}}(\mathbb{R}^d)) \cap L^1_T(B_{p,1}^{1+\frac{d}{p}}(\mathbb{R}^d)),
\]
\[
\nu \in C([0, T]; B_{q,1}^{-2+\frac{d}{q}}(\mathbb{R}^d)) \cap L^\infty_T(B_{q,1}^{-2+\frac{d}{q}}(\mathbb{R}^d)) \cap L^1_T(B_{q,1}^{-2+\frac{d}{q}}(\mathbb{R}^d)),
\]
\[
c \in C([0, T]; B_{q,1}^{\frac{d}{q}}(\mathbb{R}^d)) \cap L^\infty_T(B_{q,1}^{\frac{d}{q}}(\mathbb{R}^d)) \cap L^1_T(B_{q,1}^{\frac{d}{q}}(\mathbb{R}^d)).
\]
(3.35)

4. Global wellposedness of Theorem 1.1 The goal of this section is to prove global wellposedness of Theorem 1.1. We will divide the proof into six subsections. We first use the equation (1.3) to estimate the pressure function which will be used in the estimates of \(u^h, u^d\) in second and third subsections. In Subsections 4 and 5, we apply our key Lemma 2.7 to estimate \(u, c\) respectively. Finally, we complete the proof of the global wellposedness of Theorem 1.1 in Subsection 6.

The main idea used in this section comes from papers [11], [17], [27]. As the authors pointed out in [17], they mainly used the algebraical structure of the momentum equation, i.e., the equation on the vertical component of the velocity is a linear equation with coefficients depending on the horizontal components. Therefore, the equation on the vertical component does not demand any smallness condition. While the equation on the horizontal component contains bilinear terms in the horizontal components and also terms associated with the interactions between the horizontal components and the vertical one. In order to solve this equation, we need a smallness condition on \(n, c\) and the horizontal component (amplified by the vertical component) of the initial data.
4.1. The estimate of the pressure. It is well known, the main difficulty in the study of the well-posedness of incompressible system is to derive the estimate for the pressure term. We first get by taking div to the equation \((1.3)\) and using div \(u = 0\) that

\[-\Delta \Pi = \text{div}_h \text{div}_h (u^h \otimes u^h) + 2 \partial_t \text{div}_h (u^d u^h) - 2 \partial_t (u^d \text{div}_h u^h) + \text{div}(n \nabla \phi), \tag{4.36}\]

where, for a vector field \(u = (u^h, u^d)\), div \(_h u^h = \partial_1 u^1 + \partial_2 u^2 + \cdots + \partial_{d-1} u^{d-1}\). Next, we will give the estimate of the pressure which will be used in the estimates of \(u^h\) and \(u^d\).

**Proposition 4.1.** Let \(1 < p < \infty\), \(1 < q < \infty\), \(-\frac{1}{2} \leq \frac{1}{p} - \frac{1}{2} \leq \frac{1}{2} < \frac{1}{q} + \frac{1}{2}\), and let \(u = (u^h, u^d) \in L^\infty_t (\dot{B}^{-1+\frac{d}{p}}_{p,q}(\mathbb{R}^d)) \cap L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q}(\mathbb{R}^d)), \ n \in L^\infty_t (\dot{B}^{-2+\frac{d}{p}}_{q,1}(\mathbb{R}^d)) \cap L^1_t (\dot{B}^{\frac{d}{q}}_{q,1}(\mathbb{R}^d)), \ \phi \in \dot{B}^{\frac{d}{q}}_{q,1}(\mathbb{R}^d). \) Then \((4.36)\) has a unique solution \(\nabla \Pi \in L^1_t (\dot{B}^{-1+\frac{d}{p}}_{p,q}(\mathbb{R}^d))\) which decays to zero when \(|x| \to \infty\) so that for all \(t \in [0, T]\), there holds

\[
\|\nabla \Pi\|_{L^1_t (\dot{B}^{-1+\frac{d}{p}}_{p,q})} \lesssim \|u^h\|_{L^\infty_t (\dot{B}^{-1+\frac{d}{p}}_{p,q})} + \|u^h\|_{L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q})} + \|\nabla \phi\|_{\dot{B}^{-1+\frac{d}{q}}_{p,q}} + \|u^d\|_{L^1_t (\dot{B}^{-1+\frac{d}{q}}_{p,q})} + \|u^d\|_{L^\infty_t (\dot{B}^{1+\frac{d}{q}}_{p,q})} + \|u^h\|_{L^\infty_t (\dot{B}^{1+\frac{d}{q}}_{p,q})} + \|u^h\|_{L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q})}. \tag{4.37}\]

**Proof.** As both the existence and uniqueness parts of Proposition 4.1 basically follow from the uniform estimate \((4.37)\) for appropriate approximate solutions of \((4.36)\). For simplicity, we just prove \((4.37)\) for smooth enough solutions of \((4.36)\). Firstly, according to \((4.36)\), we have

\[
\nabla \Pi = \nabla (-\Delta)^{-1} \left( \text{div}_h \text{div}_h (u^h \otimes u^h) + 2 \partial_t \text{div}_h (u^d u^h) \right) - 2 \partial_t (u^d \text{div}_h u^h) + \text{div}(n \nabla \phi). \tag{4.38}\]

Applying the operator \(\tilde{\Delta}_j\) to the above equation \((4.38)\), taking \(L^1_t (L^p)\)-norm and using Lemma 2.4, we obtain

\[
\|\tilde{\Delta}_j(\nabla \Pi)\|_{L^1_t (L^p)} \lesssim 2^j \|\tilde{\Delta}_j (u^h \otimes u^h)\|_{L^1_t (L^p)} + 2^j \|\tilde{\Delta}_j (u^d u^h)\|_{L^1_t (L^p)} + \|\tilde{\Delta}_j (n \nabla \phi)\|_{L^1_t (L^1)}. \tag{4.40}\]

By Lemma 2.5 and Lemma 2.8 one has

\[
2^j \|\tilde{\Delta}_j (u^h \otimes u^h)\|_{L^1_t (L^p)} \lesssim d_j 2^{(1-\frac{d}{2})j} \|n\|_{L^1_t (\dot{B}^{\frac{d}{q}}_{p,q})} \|\nabla \phi\|_{\dot{B}^{-1+\frac{d}{q}}_{p,q}}, \tag{4.41}\]

\[
2^j \|\tilde{\Delta}_j (u^d u^h)\|_{L^1_t (L^p)} \lesssim d_j 2^{(1-\frac{d}{2})j} \|u^h\|_{L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q})} \|u^h\|_{L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q})}, \tag{4.42}\]

\[
2^j \|\tilde{\Delta}_j (n \nabla \phi)\|_{L^1_t (L^1)} \lesssim d_j 2^{(1-\frac{d}{2})j} \|u^d\|_{L^1_t (\dot{B}^{-1+\frac{d}{q}}_{p,q})} \|u^h\|_{L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q})} + \|u^d\|_{L^\infty_t (\dot{B}^{1+\frac{d}{q}}_{p,q})} \|u^h\|_{L^1_t (\dot{B}^{1+\frac{d}{q}}_{p,q})}, \tag{4.43}\]

with

\[
\alpha = \begin{cases} 
1, & 1 < p < 2d - 1, \\
\varepsilon, & 2d - 1 < p < 2d, 
\end{cases} \tag{4.44}\]
for $0 < \varepsilon < \frac{2d}{p} - 1$.

Inserting (4.41)–(4.43) into (4.40) and performing an $\ell^1$ summatiting give the estimate (4.37).

4.2. The estimate of $u^h$. Thanks to (1.3), we have

$$\partial_t u^h - \Delta u^h + u \cdot \nabla u^h + \nabla \Pi = -n \nabla \phi.$$  \hspace{1cm} (4.45)

Applying $\hat{\Delta}_j$ to (4.45) and taking $L^2$ inner product with $|\hat{\Delta}_j u^h|^p - 2 \hat{\Delta}_j u^h$ (when $1 < p < 2$, we need to make some modification, see [7]), we obtain

$$\frac{d}{dt} \|\hat{\Delta}_j u^h\|_{L^p} + C_1 2^j \|\hat{\Delta}_j u^h\|_{L^p}$$

$$\lesssim \|\hat{\Delta}_j (u \cdot \nabla u^h)\|_{L^p} + \|\hat{\Delta}_j (n \nabla \Pi)\|_{L^p} + \|\hat{\Delta}_j (n \nabla \phi)\|_{L^p},$$  \hspace{1cm} (4.46)

where we have used the following fact (it can be found in the Appendix of [7]): there exists a positive constant $C_1$ so that

$$- \int_{\mathbb{R}^d} \hat{\Delta}_j u^h |\hat{\Delta}_j u^h|^p - 2 \hat{\Delta}_j u^h dx \geq C_1 2^j \|\hat{\Delta}_j u^h\|^p_{L^p}.$$

By using div $= 0$, we get

$$u \cdot \nabla u^h = \sum_{i=1}^d \partial_i (u^h u^i) = \sum_{i=1}^{d-1} \partial_i (u^h u^i) + \partial_d (u^d u^h) = \text{div}_h(u^h \otimes u^h) + \partial_d (u^d u^h).$$

Thus, by Lemmas 4.2, 4.5, 4.8 and the estimates (4.37), (4.41)–(4.43), we have

$$\|\hat{\Delta}_j (u \cdot \nabla u^h)\|_{L^p} + \|\hat{\Delta}_j (n \nabla \Pi)\|_{L^p} + \|\hat{\Delta}_j (n \nabla \phi)\|_{L^p}$$

$$\lesssim d 2^{j(1 - \frac{2}{p})} \left( \|u^h\|_{L^\infty(B_{p,1}^{-1,\frac{d}{p}})} \|u^h\|_{L_1^1(B_{p,1}^{1,\frac{d}{p}})} + \|u^d\|_{L_1^1(B_{p,1}^{1,\frac{d}{p}})} \right).$$  \hspace{1cm} (4.47)

Hence, integrating (4.46) over $[0, t]$ and taking the above estimate into the resulting inequality, we can infer that

$$\|u^h\|_{L^\infty(B_{p,1}^{-1,\frac{d}{p}})} + \|u^h\|_{L_1^1(B_{p,1}^{1,\frac{d}{p}})}$$

$$\lesssim \|u_0^h\|_{B_{p,1}^{-1,\frac{d}{p}}} + \|u_0^d\|_{B_{p,1}^{-1,\frac{d}{p}}} \|\nabla \phi\|_{B_{p,1}^{1,\frac{d}{p}}} + \|u^h\|_{L^\infty(B_{p,1}^{-1,\frac{d}{p}})} \|u^h\|_{L_1^1(B_{p,1}^{1,\frac{d}{p}})}$$

$$+ \|u^d\|_{L_1^1(B_{p,1}^{1,\frac{d}{p}})} \|u^h\|_{L^\infty(B_{p,1}^{-1,\frac{d}{p}})} + \|u^d\|_{L_1^1(B_{p,1}^{1,\frac{d}{p}})} \|u^h\|_{L^\infty(B_{p,1}^{-1,\frac{d}{p}})}.$$  \hspace{1cm} (4.48)

4.3. The estimate of $u^d$. In this subsection, we will use the fact that the velocity field equation on the vertical component is a linear equation with coefficients depending only on the horizontal components. We first consider the vertical components of equation (1.3) to get

$$\partial_t u^d - \Delta u^d + u \cdot \nabla u^d + \partial_d \Pi = -n \partial_d \phi.$$  \hspace{1cm} (4.49)

Applying the operator $\hat{\Delta}_j$ to the above equation and taking $L^2$ inner product of the resulting equation with $|\hat{\Delta}_j u^d|^p - 2 \hat{\Delta}_j u^d$, respectively, we get by a similar derivation
of (4.46) that
\[
\frac{d}{dt} \| \Delta_j u^d \|_{L^p} + C_2 2^j \| \Delta_j u^d \|_{L^p} \\
\lesssim \| \Delta_j (u \cdot \nabla u^d) \|_{L^p} + \| \Delta_j (\partial_d \Pi) \|_{L^p} + \| \Delta_j (n \partial_d \phi) \|_{L^p}. \tag{4.50}
\]

By using \( \text{div} u = 0 \), (4.43) and Lemma 2.4 we have
\[
\| \Delta_j (u \cdot \nabla u^d) \|_{L^p_t(L^p)} \\
\lesssim 2^j \| \Delta_j (u^h u^d) \|_{L^p_t(L^p)} + \| \Delta_j (u^d \text{div}_h u^h) \|_{L^p_t(L^p)} \\
\lesssim d_j 2^{(1-\frac{2j}{q})} \left( \| u^d \|_{L^2_t(B_{p,1}^{-1+j/q})} \right) \left( \| u^h \|_{L^2_t(B_{p,1}^{-1+j/q})} \right) + \| u^h \|_{L^2_t(B_{p,1}^{-1+j/q})} \| u^d \|_{L^2_t(B_{p,1}^{-1+j/q})}. \tag{4.51}
\]

Hence, integrating (4.50) over \([0,t]\) and taking estimates (4.37), (4.51) into the resulting inequality, we can deduce that
\[
\left\| \Delta_j n^{q-2} \Delta_j n \right\|_{L^q_t} \\
\lesssim d_j 2^{(1-\frac{2j}{q})} \left( \| u^d \|_{L^2_t(B_{p,1}^{-1+j/q})} \right) \left( \| u^h \|_{L^2_t(B_{p,1}^{-1+j/q})} \right) + \| u^h \|_{L^2_t(B_{p,1}^{-1+j/q})} \| u^d \|_{L^2_t(B_{p,1}^{-1+j/q})}. \tag{4.52}
\]

### 4.4. The estimate of \( n \)

In this subsection, we shall derive the desired estimates of \( n \). Applying \( \Delta_j \) to the equation (1.4) and taking \( L^2 \) inner product with \( \Delta_j n^{q-2} \Delta_j n \), we obtain
\[
\frac{1}{q} \frac{d}{dt} \left\| \Delta_j n^{q-2} \Delta_j n \right\|_{L^q} - \int_{\mathbb{R}^d} \Delta_j n \Delta_j n^{q-2} \Delta_j n dx = - (\Delta_j (\hat{T}_u \nabla n)) \| \Delta_j n^{q-2} \Delta_j n \) \\
- (\Delta_j (\hat{R}(u, \nabla n))) \| \Delta_j n^{q-2} \Delta_j n \) - \int_{\mathbb{R}^d} \Delta_j \nabla \cdot (n \nabla c) \Delta_j n^{q-2} \Delta_j n dx, \tag{4.53}
\]

in which we have used the Bony’s decomposition (2.11):
\[
u \cdot \nabla n = \hat{T}_u \nabla n + \hat{R}(u, \nabla n). \tag{4.54}
\]

Using a standard commutator’s argument and the energy estimate, we have
\[
\left\| \Delta_j n \right\|_{L^q_t(L^q)} + 2^{2j} \left\| \Delta_j n \right\|_{L^p_t(L^p)} \\
\lesssim \| \Delta_j n_0 \|_{L^q} + \int_0^t \| \Delta_j \nabla \cdot (n \nabla c) \|_{L^q} \, dt + \int_0^t \| \Delta_j n \|_{L^q_t(L^q)} \, dt \\
+ \int_0^t \left( \sum_{|j-j'| \leq 5} (\| \Delta_j \hat{S}_{j-1} n \|_{L^q} + \| (\hat{S}_{j-1} n - \hat{S}_{j-1} n) \Delta_j \nabla n \|_{L^q}) \right) \, dt. \tag{4.55}
\]

In view of Lemma 3.2 one has
\[
\left\| \Delta_j (\nabla \cdot (n \nabla c)) \right\|_{L^q_t(L^q)} \\
\lesssim d_j 2^{(1-\frac{2j}{q})} \left( \| c \|_{L^2_t(B_{q,1}^{-1+j/q})} \right) \left( \| c \|_{L^2_t(B_{q,1}^{-1+j/q})} \right) + \| n \|_{L^2_t(B_{q,1}^{-1+j/q})} \| n \|_{L^2_t(B_{q,1}^{-1+j/q})}. \tag{4.56}
\]
We get by applying the estimate on commutators and (2.14) with \( m_1 = m_2 = \infty \) that

\[
\sum_{|j-j'| \leq 5} (\|\hat{\Delta}_j \hat{\Delta}_j u\|_{L^1_t(L^q)} + \|\hat{\Delta}_j u\|_{L^1_t(L^q)}) + (\|\hat{\Delta}_j u\|_{L^1_t(L^q)} + \|\hat{\Delta}_j u\|_{L^1_t(L^q)})
\]

\[
\lesssim \sum_{|j-j'| \leq 5} \left[ (\|\hat{\Delta}_j u\|_{L^1_t(L^q)} + \|\hat{\Delta}_j u\|_{L^1_t(L^q)}) \right] \|\hat{\Delta}_j n\|_{L^\infty_t(L^q)}
\]

\[
\lesssim \sum_{|j-j'| \leq 5} \left( \|\hat{\Delta}_j u\|_{L^1_t(L^q)} + \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \right) \|\hat{\Delta}_j n\|_{L^\infty_t(L^q)}
\]

\[
\lesssim d_j 2^{(2 - \frac{4}{d} - \frac{1}{q})j} \|n\|_{L^\infty_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \left( \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} + \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \right)^{\frac{1}{d} + \frac{1}{q} - \frac{1}{p}} + \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \right) \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \right)
\]

\[
\lesssim d_j 2^{(2 - \frac{4}{d} - \frac{1}{q})j} \|n\|_{L^\infty_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \left( \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} + \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \right)^{\frac{1}{d} + \frac{1}{q} - \frac{1}{p}} + \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \right) \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \right)
\]

When \( p \geq q \) and \( \frac{1}{p} - \frac{1}{q} = \frac{1}{pd} - \frac{2}{q} \), by Lemma 2.4 and (2.14) with \( m_1 = p, m_2 = \infty \), we deduce that

\[
\|\hat{\Delta}_j \hat{\Delta}_j u \|_{L^1_t(L^q)} \leq \|\hat{\Delta}_j u \|_{L^1_t(L^q)}
\]

\[
\lesssim \|\hat{\Delta}_j u \|_{L^1_t(L^q)} \|\hat{\Delta}_j n \|_{L^1_t(L^q)}
\]

\[
\lesssim d_j 2^{(2 - \frac{4}{d} - \frac{1}{q})j} \|n\|_{L^\infty_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \left( \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} + \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \right)^{\frac{1}{d} + \frac{1}{q} - \frac{1}{p}} + \|u\|_{L^1_t(B_1^{-\frac{2}{d} - \frac{2}{q}})} \right) \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \|\hat{\Delta}_j u\|_{L^1_t(L^q)} \right)
\]
\[
\lesssim 2^j \sum_{j' \geq j-N_0} \left( \sum_{j'' \leq j' + 1} 2^{\frac{3j''}{2}} \|\tilde{\Delta}_{j''} n\|_{L^\infty_t(L^2)} \right) \|\tilde{\Delta}_{j'} u^h\|_{L^1_t(L^p)} \\
+ 2^j \sum_{j' \geq j-N_0} \left( \sum_{j'' \leq j' + 1} 2^{\frac{3j''}{2}} \|\tilde{\Delta}_{j''} n\|_{L^\infty_t(L^2)} \right) \|\tilde{\Delta}_{j'} u^d\|_{L^1_t(L^p)}(L^\infty_t(L^r))
\]
\[
\lesssim 2^j \sum_{j' \geq j-N_0} d_j 2(2^{\frac{3}{4} + \frac{1}{p}} + \frac{1}{4})' \|n\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \|\tilde{\Delta}_{j'} u^d\|_{L^1_t(L^p)}(L^\infty_t(L^r)) \\
+ 2^j \sum_{j' \geq j-N_0} d_j 2(2^{\frac{3}{4} + \frac{1}{p}} + \frac{1}{4})' \|n\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \|\tilde{\Delta}_{j'} u^d\|_{L^1_t(L^p)}(L^\infty_t(L^r))
\]
\[
\lesssim d_j 2(2^{\frac{3}{4} + \frac{1}{p}} + \frac{1}{4})' \|n\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \left( \|u^d\|_{L^1_t(B_{1/2}^{1+s}\frac{q}{2})}^{1-\frac{1}{p}} + \|u^h\|_{L^1_t(L^p)}^{\frac{1}{p}} \right) + \|u^h\|_{L^1_t(L^p)}(L^\infty_t(L^r)) . \quad (4.59)
\]

Inserting the estimates (4.57)–(4.59) into (4.55), we can infer that

\[
\|n\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \lesssim \|n_0\|_{L^2(B_{q,1}^{2-s}\frac{q}{2})} + \|n\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \left( \|u^d\|_{L^1_t(B_{1/2}^{1+s}\frac{q}{2})}^{1-\frac{1}{p}} \|u^h\|_{L^1_t(L^p)}^{\frac{1}{p}} \right) + \|u^h\|_{L^1_t(L^p)}(L^\infty_t(L^r)) \\
+ \|c\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \|c\|_{L^1_t(B_{1/2}^{1+s}\frac{q}{2})} + \|n\|_{L^\infty_t(B_{q,1}^{2-s}\frac{q}{2})} \|n\|_{L^1_t(L^p)}(L^\infty_t(L^r)) . \quad (4.60)
\]

4.5. The estimate of $c$. In this subsection, we will give the estimates of $c$. Applying $\tilde{\Delta}_j$ to (1.2) and taking $L^2$ inner product with $|\tilde{\Delta}_j c|^{-2} \tilde{\Delta}_j c$, we get by a similar derivation of \((4.53)\) that

\[
\frac{1}{r} \int d \int \Delta c^2 \|\Delta c\|_{L^r} - \int \Delta \Delta c^2 \|\Delta c\|_{L^r} - \int \Delta \Delta c^2 \|\Delta c\|_{L^r}
\]
\[
= - (\tilde{\Delta}_j (\hat{T}_n \nabla n)) |\tilde{\Delta}_j c|^{-2} \tilde{\Delta}_j c - (\tilde{\Delta}_j (\hat{R}(u, \nabla n))) |\tilde{\Delta}_j c|^{-2} \tilde{\Delta}_j c
\]
\[
- \int \Delta \Delta (\kappa(c)n)|\Delta c|^{-2} \tilde{\Delta}_j c . \quad (4.61)
\]

Integrating the above estimate from 0 to $t$ and using a standard commutator’s argument and the basic energy estimate, one has

\[
\|\Delta c\|_{L^r} + 2^j \|\Delta c\|_{L^1_t(L^r)} \\
\lesssim \|\Delta c_0\|_{L^r} + \int \Delta \hat{R}(u, \nabla n)\|L^2\| + \int \Delta (\kappa(c)n)\|L^r\| \\
+ \int (\sum_{j-j' \leq 5} (||\Delta_j, S_{j-1} u|\tilde{\Delta}_j \nabla n|_{L^r} + ||\hat{S}_{j-1} u - S_{j-1} u|\tilde{\Delta}_j \nabla n|_{L^r}) \|L^r\| d\tau . \quad (4.62)
\]
On the one hand, we get by applying the classical estimate on commutators and (2.1.4) with $m_1 = m_2 = \infty$ that

$$
\sum_{|j-j'| \leq 5} (\| \hat{\Delta}_j \hat{S}_{j'-1} u \|_{L^1_t(L^r)} \| \hat{\Delta}_j \nabla c \|_{L^1_t(L^r)} + \| (\hat{S}_{j'-1} u - \hat{S}_{j-1} u) \hat{\Delta}_j \nabla u \|_{L^1_t(L^r)})
$$

$$
\lesssim \sum_{|j-j'| \leq 5} \left[ (\| \hat{S}_{j'-1} \nabla u^h - \hat{S}_{j-1} \nabla u^h \|_{L^1_t(L^\infty)} + \| \hat{S}_{j'-1} \nabla u^d - \hat{S}_{j-1} \nabla u^d \|_{L^1_t(L^\infty)})
\right.

$$

$$
+ \| (\hat{S}_{j'-1} \nabla u^h \|_{L^1_t(L^\infty)} + \| \hat{S}_{j'-1} \nabla u^d \|_{L^1_t(L^\infty)}) \| \hat{\Delta}_j c \|_{L^\infty_t(L^r)}
$$

$$
\lesssim \sum_{|j-j'| \leq 5} \sum_{j'' \leq j'-2} \left( \| \hat{\Delta}_j \nabla u^h \|_{L^1_t(L^\infty)} + 2^{j''} \| \hat{\Delta}_j u^d \|_{L^1_t(L^\infty)} \right) \| \hat{\Delta}_j c \|_{L^\infty_t(L^r)}
$$

$$
\lesssim d_2 2^{-\#j} \| c \|_{L^\infty_t(\dot{B}_{q,1}^{2\#})} \left( \| u^d \|_{L^1_t(\dot{B}_{q,1}^{1+\#})} \right) \| u^h \|_{L^1_t(\dot{B}_{q,1}^{1+\#})} + \| u^h \|_{L^1_t(\dot{B}_{q,1}^{1+\#})}. \tag{4.63}
$$

Similar to the estimate (4.58), we will divide the proof into two cases to estimate the rest term $\mathcal{R}(u, \nabla c)$. When $r \geq p$, thanks to Lemma 2.4 and (2.1.4) with $m_1 = p, m_2 = \infty$, we obtain

$$
\| \hat{\Delta}_j \mathcal{R}(u, \nabla c) \|_{L^1_t(L^r)}
$$

$$
\lesssim \sum_{j' \geq j-N_0} \left( \| \hat{S}_{j'+2} \nabla c \|_{L^\infty_t(L^r)} \| \hat{\Delta}_j u^h \|_{L^1_t(L^r)}
\right.

$$

$$
+ \| \hat{S}_{j'+2} \partial_x c \|_{L^\infty_t(L^r)} \| \hat{\Delta}_j u^d \|_{L^1_t(L^\infty)} \right)
$$

$$
\lesssim \sum_{j' \geq j-N_0} \sum_{j'' \leq j'+1} \left( 2^{(1+\#)j''} \| \hat{\Delta}_j c \|_{L^\infty_t(L^r)} \| \hat{\Delta}_j u^h \|_{L^1_t(L^\infty)}
\right.

$$

$$
+ 2^{(1+\frac{1}{p}+\#)j''} \| \hat{\Delta}_j c \|_{L^\infty_t(L^r)} \| \hat{\Delta}_j u^d \|_{L^1_t(L^\infty)} \left)
$$

$$
\lesssim \sum_{j' \geq j-N_0} d_j 2^{j' j''} \| c \|_{L^\infty_t(\dot{B}_{q,1}^{2\#})} \| \hat{\Delta}_j u^h \|_{L^1_t(L^r)}
$$

$$
+ \sum_{j' \geq j-N_0} d_j 2^{(1+\frac{1}{p})j''} \| c \|_{L^\infty_t(\dot{B}_{q,1}^{2\#})} \| \hat{\Delta}_j u^d \|_{L^1_t(L^\infty)} \right)
$$

$$
\lesssim d_2 2^{-\#j} \| c \|_{L^\infty_t(\dot{B}_{q,1}^{2\#})} \left( \| u^d \|_{L^1_t(\dot{B}_{q,1}^{1+\#})} \right) \| u^h \|_{L^1_t(\dot{B}_{q,1}^{1+\#})} + \| u^h \|_{L^1_t(\dot{B}_{q,1}^{1+\#})}. \tag{4.64}
$$

When $r \leq p$ and $\frac{1}{p} - \frac{1}{r} \leq \frac{1}{2}$, by Lemma 2.4 and (2.1.4) with $m_1 = p, m_2 = \infty$, one has

$$
\| \hat{\Delta}_j \mathcal{R}(u, \nabla c) \|_{L^1_t(L^r)}
$$

$$
\lesssim \| \hat{\Delta}_j \sum_{j' \geq j-N_0} \hat{\Delta}_j \hat{\Delta}_j u \|_{L^1_t(L^r)}
$$

$$
\lesssim \sum_{j' \geq j-N_0} \| \hat{S}_{j'+2} \nabla c \|_{L^\infty_t(L^p)} \| \hat{\Delta}_j u^h \|_{L^1_t(L^r)}
$$

$$
+ \sum_{j' \geq j-N_0} \| \hat{S}_{j'+2} \nabla c \|_{L^\infty_t(L^p)} \| \hat{\Delta}_j u^d \|_{L^1_t(L^\infty)} \right)
In what follows, we will prove that the existence of equations (1.1)–(1.4). Then we define $T^*$ by

\[ T^* = \sup \left\{ t \in [0, T^*) : \|u\|_{L^\infty_t(B_{r,1}^{1+\frac{2}{p}})}, \|u\|_{L^2_t(B_{r,1}^{1+\frac{2}{p}})}, \|n\|_{L^2_t(B_{r,1}^{1+\frac{2}{p}})}, \|\xi\|_{L^2_t(B_{r,1}^{1+\frac{2}{p}})} \leq 8(\|\xi\|_{B_{r,1}^{-1+\frac{2}{p}}} + \|\xi\|_{B_{r,1}^{-2+\frac{2}{p}}}) \right\}. \]

In what follows, we will prove that $T^* = T^* = \infty$ under the assumption of (1.6).

If not, we assume that $T^* < T^*$. For all $t \leq T^*$, we get from (4.52) that

\[ \|u\|_{L^\infty_t(B_{r,1}^{1+\frac{2}{p}})}, \|u\|_{L^2_t(B_{r,1}^{1+\frac{2}{p}})}, \|u\|_{L^2_t(B_{r,1}^{1+\frac{2}{p}})} \leq 8(\|\xi\|_{B_{r,1}^{-1+\frac{2}{p}}} + \|\xi\|_{B_{r,1}^{-2+\frac{2}{p}}}) \]

Let $c_1$ and $\xi$ be small enough such that

\[ Cc_1 \xi + Cc_1^2 + 2Cc_1 \leq \alpha. \]
Then we have
\[
\|u^d\|_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} + \|u^d\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} \leq \|u_0^d\|_{B_{p,1}^{1+\frac{d}{p}}} + \alpha. \tag{4.71}
\]

For all \(t \leq T^*\), one can deduce from the estimates (4.48), (4.60), (4.67) respectively that
\[
\begin{align*}
\|u^h\|_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} + \|u^h\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} & \leq \|u_0^h\|_{B_{\alpha,1}^{1+\frac{d}{p}}} + C\|\n\|_{L^1_t(B_{\alpha,1}^{\frac{d}{p}})} \|\nabla \phi\|_{B_{\alpha,1}^{-1+\frac{d}{p}}} + C\|\n\|_{L^\infty_t(B_{\alpha,1}^{-1+\frac{d}{p}})} \|u^h\|_{L^\infty_t(B_{\alpha,1}^{1+\frac{d}{p}})} + C\|u^d\|^{1-\alpha}_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} \|\n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + C\|u^d\|^{1-\alpha}_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} \|u^h\|_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} \\
& \leq \|u_0^h\|_{B_{\alpha,1}^{1+\frac{d}{p}}} + C\|\n\|_{L^1_t(B_{\alpha,1}^{\frac{d}{p}})} + C\|\n\|_{L^\infty_t(B_{\alpha,1}^{-1+\frac{d}{p}})} + \alpha \|u^d\|^{1-\alpha}_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} + C\|\n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + C\|u^d\|^{1-\alpha}_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} \|\n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}, \tag{4.72}
\end{align*}
\]

\[
\begin{align*}
\|n\|_{L^\infty_t(B_{\alpha,1}^{-2+\frac{d}{p}})} + \|n\|_{L^1_t(B_{\alpha,1}^{\frac{d}{p}})} & \leq \|n_0\|_{B_{\alpha,1}^{-2+\frac{d}{p}}} + Cc_1 \|\n\|_{L^1_t(B_{\alpha,1}^{\frac{d}{p}})} + Cc_1 \|n\|_{L^\infty_t(B_{\alpha,1}^{\frac{d}{p}})} + Cc_1 \|n\|_{L^1_t(B_{\alpha,1}^{\frac{d}{p}})} + C(c_1 + c_1 \|u^d\|^{1-\frac{1}{p}_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}) \|n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}, \tag{4.73}
\end{align*}
\]

and
\[
\begin{align*}
\|c\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})} + \|c\|_{L^1_t(B_{p,1}^{\frac{d}{p}})} & \leq \|c_0\|_{B_{p,1}^{\frac{d}{p}}} + C(c_1 + c_1 \|u^d\|^{1-\frac{1}{p}_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}) \|c\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})} + Cc_1 \|n\|_{L^1_t(B_{p,1}^{\frac{d}{p}})}, \tag{4.74}
\end{align*}
\]

Summing up the estimates (4.72)–(4.74) and using (4.68), one can finally get from (4.71) that
\[
\begin{align*}
\|u^h\|_{L^\infty_t(B_{p,1}^{1+\frac{d}{p}})} + \|u^h\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + \|n\|_{L^\infty_t(B_{\alpha,1}^{-2+\frac{d}{p}})} + \|\n\|_{L^1_t(B_{\alpha,1}^{\frac{d}{p}})} + \|c\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})} + \|c\|_{L^1_t(B_{p,1}^{\frac{d}{p}})} & \leq \|u_0^h\|_{B_{p,1}^{1+\frac{d}{p}}} + \|u_0^d\|_{B_{p,1}^{1+\frac{d}{p}}} + \|\n\|_{B_{\alpha,1}^{-2+\frac{d}{p}}} + \|\n\|_{B_{\alpha,1}^{\frac{d}{p}}} + \|c\|_{B_{p,1}^{\frac{d}{p}}} + \|c\|_{B_{p,1}^{\frac{d}{p}}} + C(c_1 + c_1 \|u^d\|^{1-\frac{1}{p}_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}) \|n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + C\|\n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + C(c_1 + c_1 \|u^d\|^{1-\frac{1}{p}_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}) \|c\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + C\|\n\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})} + 2Cc_1 \|n\|_{L^1_t(B_{p,1}^{\frac{d}{p}})} + C(c_1 + c_1 \|u^d\|^{1-\frac{1}{p}_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}) \|c\|_{L^1_t(B_{p,1}^{1+\frac{d}{p}})}, \tag{4.75}
\end{align*}
\]
While taking
\[ C\xi \leq \frac{1}{6}, \quad C(c_1 + c_1^{\frac{1}{\alpha}} (\|u_0\|_{B_{\infty}^{-1+\frac{d}{p}}} + \alpha)^{1-\frac{1}{\alpha}}) \leq \frac{1}{2}, \]
we have
\[ C(\|u_0\|_{B_{\infty}^{-1+\frac{d}{p}}} + \alpha)^{1-\alpha} \leq \frac{1}{3}, \]  \hspace{1cm} (4.76)
\]
Combining (4.70) with (4.76), we can get that (4.71) and (4.77) hold if we take \( C_0 \) large enough in (1.6). This contradicts the definition (4.68). Thus we conclude that \( T^{**} = T^* \). Consequently, we complete the proof of Theorem 1.1 by the standard continuation argument.

5. Proof of Corollary 1.2 We prove Corollary 1.2 by contradiction. Let \( 0 < T^* < \infty \) be the maximum time for the existence of strong solutions \((u, n, c)\) to the system (1.1)–(1.4). Assume that (1.8) is not true. Then there exists a positive constant \( M_0 \) such that
\[ \int_0^{T^*} (\|\omega\|_{B_{\infty}^{\frac{d}{p}}} + \|n\|_{B_{\infty}^{\frac{d}{p}}}) dt \leq M_0. \]  \hspace{1cm} (5.78)

The goal is to show that if assumption (5.78) holds, there is a bound \( C \) depending only on \( u_0, n_0, c_0, T^* \) and \( M_0 \) such that
\[ \sup_{0 < t \leq T^*} (\|u\|_{L_t^p(B_{\infty}^{-1+\frac{d}{p}})} + \|n\|_{L_t^p(B_{\infty}^{-2+\frac{d}{p}})} + \|c\|_{\tilde{L}_t^\infty(B_{\infty}^{-\frac{d}{p}})}) \leq C. \]  \hspace{1cm} (5.79)
In order to do so, on one hand, by Lemmas 2.6, 3.1, we can get from the equation (1.3) and the basic energy method that
\[ \|u\|_{L_t^p(B_{\infty}^{-1+\frac{d}{p}})} + \|u\|_{L_t^1(B_{\infty}^{1+\frac{d}{p}})} \lesssim \|u_0\|_{B_{\infty}^{-1+\frac{d}{p}}} + \int_0^{T^*} \|\nabla u\|_{B_{\infty}^{\frac{d}{p}}} \|u\|_{B_{\infty}^{-1+\frac{d}{p}}} dt + \int_0^{T^*} \|n\|_{B_{\infty}^{\frac{d}{p}}} \|\nabla \phi\|_{B_{\infty}^{-1+\frac{d}{p}}} dt. \]  \hspace{1cm} (5.80)

On the other hand, we recall the well known fact that the elliptic system, \( \text{div} u = 0 \) and \( \omega = \nabla \times u \) imply the relation between the gradient of velocity and the vorticity as
\[ \nabla u = \mathcal{P}(\omega) + \tilde{c}\omega, \]
where \( \mathcal{P} \) is a singular integral operator of the Calderón-Zygmund type, and \( \tilde{c} \) is a constant matrix. By the boundedness of the singular integral operator, we have
\[ \|\nabla u\|_{B_{\infty}^{\frac{d}{p}}} \lesssim \|\omega\|_{B_{\infty}^{\frac{d}{p}}}. \]
Therefore, if
\[ \int_0^{T^*} \|\omega(t)\|_{B_{\infty}^{\frac{d}{p}}} dt < \infty, \]
then (5.80) and Gronwall’s lemma imply that
\[ \|u\|_{L^\infty(B_{p,1}^{-1+\frac d q})} + \|u\|_{L^1_t(B_{p,1}^{1+\frac d q})} \leq \left( \|u_0\|_{B_{p,1}^{-1+\frac d q}} + \int_0^{T^*} \|n\|_{B_{q,1}^1} \, dt \right) \exp\left( \int_0^{T^*} \|\omega(t)\|_{B_{q,1}^{\frac d q}} \, dt \right). \] (5.81)

Similarly, by Lemmas 2.6, 3.2, 3.3 and the equations (1.2), (1.4), one has
\[ \|c\|_{L^\infty(B_{q,1}^{1+\frac d q})} + \|c\|_{L^1_t(B_{q,1}^{2+\frac d q})} \leq \left( \|c_0\|_{B_{q,1}^{1+\frac d q}} + \int_0^T \|\nabla u\|_{B_{p,1}^1} \, dt + \int_0^T \|n\|_{B_{q,1}^1} \|c\|_{B_{q,1}^{\frac d q}} \, dt \right) \exp\left( \int_0^T \|\omega(t)\|_{B_{q,1}^{\frac d q}} + \|n\|_{B_{q,1}^{\frac d q}} \, dt \right), \] (5.82)
and
\[ \|n\|_{L^\infty(B_{q,1}^{-2+\frac d q})} + \|n\|_{L^1_t(B_{q,1}^{-1+\frac d q})} \leq \left( \|n_0\|_{B_{q,1}^{-2+\frac d q}} + \int_0^T \|\nabla u\|_{B_{p,1}^1} \|n\|_{B_{q,1}^{-2+\frac d q}} \, dt \right) \exp\left( \int_0^T \|\omega(t)\|_{B_{q,1}^{\frac d q}} + \|n\|_{B_{q,1}^{\frac d q}} \, dt \right). \] (5.83)

Therefore, combining with (5.81), (5.82), (5.83), and using a standard argument, we can extend the solutions \((u, n, c)\) beyond \(T^*\), which leads to the desired contradiction. This completes the proof of Corollary 1.2

6. **Proof of Theorem 1.3.** In this section, we will prove Theorem 1.3. The main tool we are going to use is the weight Chemin-Lerner space.

Let \( u = u_R + \bar{u} \), \( P = \pi + \Pi \), where \( u_R \) solves the following incompressible Navier–Stokes equations:
\[ \begin{cases} 
\partial_t u_R - \Delta u_R + u_R \cdot \nabla u_R + \nabla \pi = 0, \\
\text{div} u_R = 0, \\
u_R|_{t=0} = u_0.
\end{cases} \] (6.84)

We can get from the equations (1.1)–(1.4) and (6.84) that \((\bar{u}, n, c)\) satisfies
\[ \begin{align*}
\text{div} \bar{u} &= 0, \quad (6.85) \\
c_t - \Delta c + (u_R + \bar{u}) \cdot \nabla c &= -\kappa(c) n, \quad (6.86) \\
n_t - \Delta n + (u_R + \bar{u}) \cdot \nabla n &= -\nabla \cdot (n \nabla c), \quad (6.87) \\
\bar{u}_t - \Delta \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \Pi &= -\bar{u} \cdot \nabla u_R - u_R \cdot \nabla \bar{u} - n \nabla \phi, \quad (6.88)
\end{align*} \]

with the initial conditions
\[ \bar{u}|_{t=0} = 0, \quad n|_{t=0} = n_0(x), \quad c|_{t=0} = c_0(x), \quad \text{in} \quad \mathbb{R}^2. \]

The global infinite energy solutions to (6.84) have been proved in [11], more precisely, they proved the following proposition:
Proposition 6.1. For any $1 < p < 4$, $u_0 \in \dot{B}_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)$, (6.84) has a unique solution $u_R$ with

$$u_R \in C((0, +\infty); \dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap \dot{L}^\infty((0, +\infty); \dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap L^1((0, +\infty); \dot{B}_{p,1}^{1+\frac{2}{p}})$$

and there holds

$$\|u_R\|_{L^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|u_R\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla \pi\|_{L^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}$$

$$\leq C\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} (1 + \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}) \exp\left\{ C\|u_0\|^2_{\dot{B}_{p,1}^{1+\frac{2}{p}}} \right\}. \quad (6.89)$$

6.1. The estimate of $\bar{u}$. Denote $f(t) = \|u_R(t)\|_{\dot{B}_{p,1}^{\frac{2}{p}}}$ and

$$u_\lambda(t, x) \overset{\text{def}}{=} u(t, x) \exp \left\{ -\lambda \int_0^t f(t') dt' \right\},$$

$$n_\lambda(t, x) \overset{\text{def}}{=} n(t, x) \exp \left\{ -\lambda \int_0^t f(t') dt' \right\},$$

$$c_\lambda(t, x) \overset{\text{def}}{=} c(t, x) \exp \left\{ -\lambda \int_0^t f(t') dt' \right\}. \quad (6.90)$$

with $\lambda \geq 0$. Recalling that the Leray projection operator to the divergence vector field space is

$$\mathbb{P} = I + \nabla (-\Delta)^{-1} \text{div},$$

we can deduce from the equation (6.88) that

$$\partial_t \bar{u} - \Delta \bar{u} + \mathbb{P} [\bar{u} \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_R + u_R \cdot \nabla \bar{u} + n \nabla \phi] = 0. \quad (6.91)$$

Multiplying the above equation by $\exp \left\{ -\lambda \int_0^t f(t') dt' \right\}$ and applying $\hat{\Delta}_j$ to the resulting equation give

$$\partial_t \hat{\Delta}_j \bar{u}_\lambda + \lambda f(t) \bar{u}_\lambda - \Delta \hat{\Delta}_j \bar{u}_\lambda + \hat{\Delta}_j \mathbb{P} [\bar{u}_\lambda \cdot \nabla \bar{u} + \bar{u}_\lambda \cdot \nabla u_R + u_R \cdot \nabla \bar{u}_\lambda + n_\lambda \nabla \phi] = 0. \quad (6.92)$$

Taking the $L^2$ inner product with $|\hat{\Delta}_j \bar{u}_\lambda|^{q-2} \hat{\Delta}_j \bar{u}_\lambda$, one can finally get

$$\|\bar{u}_\lambda\|_{L^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\bar{u}_\lambda\|_{L^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\bar{u}_\lambda\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}$$

$$\lesssim \|\mathbb{P} [\bar{u}_\lambda \cdot \nabla \bar{u}]\|_{L^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\mathbb{P} [\bar{u}_\lambda \cdot \nabla u_R]\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}$$

$$+ \|\mathbb{P} [u_R \cdot \nabla \bar{u}_\lambda]\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\mathbb{P} [n_\lambda \nabla \phi]\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}. \quad (6.93)$$

By Lemma 2.5, one has

$$\|\mathbb{P} [\bar{u}_\lambda \cdot \nabla \bar{u}]\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \lesssim \|\bar{u}\|_{L^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \|\bar{u}\|_{L^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}. \quad (6.94)$$
By Lemma 2.5 and the definition of the weighed Chemin-Lerner space, we have
\[ \left\| \mathbb{P}[\hat{u}_\lambda \cdot \nabla u_R] \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} + \left\| \mathbb{P}[u_R \cdot \nabla \bar{u}_\lambda] \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \]
\[ \lesssim \left\| \text{div}(\bar{u}_\lambda \otimes u_R + u_R \otimes \bar{u}_\lambda) \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \]
\[ \lesssim \int_0^t \left\| \bar{u}_\lambda \right\|_{B_{p,1}^{\frac{2}{p}+1}} \left\| u_R \right\|_{B_{p,1}^{\frac{2}{p}+1}} \, dt \]
\[ \lesssim \int_0^t \left\| \bar{u}_\lambda \right\|_{B_{p,1}^{\frac{2}{p}+1}} \left\| u_R \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \, dt \]
\[ \lesssim \varepsilon \left\| \bar{u}_\lambda \right\|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \left\| u_R \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \left\| \bar{u}_\lambda \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})}. \quad (6.95) \]

When \( p \geq q \), in view of \( \mathbb{P}[\nabla \phi] = \mathbb{P}[\nabla \lambda \phi] \), we can get from Lemma 2.5 that
\[ \left\| \mathbb{P}[\nabla \phi] \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \lesssim \int_0^t \left\| \nabla \nabla \right\|_{B_{p,1}^{\frac{2}{p}+1}} \left\| \phi \right\|_{B_{p,1}^{\frac{2}{p}+1}} \, dt \lesssim \left\| \phi \right\|_{B_{p,1}^{\frac{2}{p}+1}} \left\| \nabla \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})}. \quad (6.96) \]

When \( p < q \) and \( \frac{1}{p} - \frac{1}{q} \leq \frac{1}{2} < \frac{1}{p} + \frac{1}{q} \), we have
\[ \left\| \mathbb{P}[\nabla \phi] \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \lesssim \int_0^t \left\| \nabla \nabla \right\|_{B_{p,1}^{\frac{2}{p}+1}} \left\| \phi \right\|_{B_{p,1}^{\frac{2}{p}+1}} \, dt \lesssim \left\| \phi \right\|_{B_{p,1}^{\frac{2}{p}+1}} \left\| \nabla \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})}. \quad (6.97) \]

Inserting the estimates (6.94)–(6.97) into (6.93), we can deduce that
\[ \left\| u_R \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} + \left\| \bar{u}_\lambda \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \lesssim \varepsilon \left\| \bar{u}_\lambda \right\|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \left\| u_R \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \left\| \bar{u}_\lambda \right\|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \left\| u_R \right\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \left\| \bar{u}_\lambda \right\|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} \quad (6.98) \]

6.2. The estimate of \( n \). We first get from the equation (6.87) that
\[ \partial_t n + \lambda f(t) n - \Delta n + (u_R + \bar{u}) \cdot \nabla n = -\nabla \cdot (n \nabla c). \quad (6.99) \]

Acting \( \Delta_j \) to the above equation and taking \( L^2 \) inner product with \( |\Delta_j n| |\Delta_j n| n \), we infer that
\[ \left\| n \right\|_{L^1_t(B_{q,1}^{2+\frac{2}{q}})} + \lambda \left\| n \right\|_{L^1_t(B_{q,1}^{2+\frac{2}{q}})} + \left\| n \right\|_{L^1_t(B_{q,1}^{2+\frac{2}{q}})} \]
\[ \lesssim n_0 \left| B_{q,1}^{2+\frac{2}{q}} \right| + \sum_{j \in \mathbb{Z}} 2^{-(2+\frac{2}{q})j} \left| \left| \Delta_j, u_R \cdot \nabla n \right|_{L^1_t(L^q)} \quad (6.100) \]

By Lemma 2.6, we have
\[ \sum_{j \in \mathbb{Z}} 2^{-(2+\frac{2}{q})j} \left| \left| \Delta_j, u_R \cdot \nabla n \right|_{L^1_t(L^q)} \lesssim \left\| n \right\|_{L^1_t(B_{q,1}^{2+\frac{2}{q}})}. \quad (6.101) \]

and
\[ \sum_{j \in \mathbb{Z}} 2^{-(2+\frac{2}{q})j} \left| \left| \Delta_j, \bar{u} \cdot \nabla \right|_{n \right\|_{L^1_t(L^q)} \lesssim \left\| \bar{u} \right\|_{L^1_t(B_{q,1}^{2+\frac{2}{p}})} \left\| n \right\|_{L^1_t(B_{q,1}^{2+\frac{2}{q}})}. \quad (6.102) \]
By Lemma 3.2, one has

\[
\| \nabla \cdot (n_\lambda \nabla c) \|_{\dot{L}_1^1(B_{\eta,1}^{-2+\frac{2}{\eta}})} \lesssim \| n_\lambda \nabla c \|_{\dot{L}_1^1(B_{\eta,1}^{-1+\frac{2}{\eta}})} \leq \int_0^T \| \nabla c \|_{B_{\tau,1}^\frac{2}{2}} \| n_\lambda \|_{B_{\eta,1}^{-1+\frac{2}{\eta}}} d\tau \\
\lesssim \int_0^T \| c \|_{B_{\tau,1}^\frac{2}{2}} \| c \|_{B_{\tau,1}^\frac{2}{2}} \| n_\lambda \|_{B_{\eta,1}^{-1+\frac{2}{\eta}}} \| n_\lambda \|_{\dot{L}_1^1(B_{\eta,1}^{-1+\frac{2}{\eta}})} d\tau \\
\lesssim \| c \|_{L_t^\infty(B_{r,1}^{-2+\frac{2}{\eta}})} \| \lambda c \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{\dot{L}_1^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{\dot{L}_1^1(B_{r,1}^{-2+\frac{2}{\eta}})}.
\]

(6.103)

Plugging the inequalities (6.101)–(6.103) into (6.100), we can finally get

\[
\| n_\lambda \|_{L_t^\infty(B_{r,1}^{-2+\frac{2}{\eta}})} + \lambda \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} \lesssim \| n_0 \|_{B_{\eta,1}^{-2+\frac{2}{\eta}}} + \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| \tilde{u} \|_{L_t^2(B_{r,1}^{-2+\frac{2}{\eta}})} \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})} + \| n_\lambda \|_{L_t^1(B_{r,1}^{-2+\frac{2}{\eta}})}.
\]

(6.104)

6.3. The estimate of $c$. Similar to the estimates (6.93), (6.100), applying the usual energy method to the equation (6.86) yields

\[
\| c_\lambda \|_{L_t^\infty(B_{r,1}^{\frac{2}{\eta}})} + \lambda \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} \lesssim \| c_0 \|_{B_{r,1}^{-2+\frac{2}{\eta}}} + \sum_{j \in \mathbb{Z}} \| \Delta_j \tilde{u} \cdot \nabla |c_\lambda|^2_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})}.
\]

(6.105)

By Lemmas 2.6, 3.3 we have

\[
\sum_{j \in \mathbb{Z}} \| \Delta_j \tilde{u} \cdot \nabla |c_\lambda|^2_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \sum_{j \in \mathbb{Z}} \| \Delta_j \tilde{u} \cdot \nabla |c_\lambda|^2_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} \lesssim \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \tilde{u} \|_{L_t^2(B_{r,1}^{\frac{2}{\eta}})} \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})}.
\]

(6.106)

Taking the above estimate into (6.105), one can end up with

\[
\| c_\lambda \|_{L_t^\infty(B_{r,1}^{\frac{2}{\eta}})} + \lambda \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} \lesssim \| c_0 \|_{B_{r,1}^{-2+\frac{2}{\eta}}} + \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \tilde{u} \|_{L_t^2(B_{r,1}^{\frac{2}{\eta}})} \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| c_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})} + \| \kappa(c) n_\lambda \|_{L_t^1(B_{r,1}^{\frac{2}{\eta}})}.
\]

(6.107)

6.4. Proof of Theorem 1.3 The goal of this subsection is to prove Theorem 1.3. In fact, when $u_0 \in B_{p,1}^{-1+\frac{2}{\eta}}(\mathbb{R}^2)$, $n_0 \in B_{q,1}^{-2+\frac{2}{\eta}}(\mathbb{R}^2)$, $c_0 \in B_{r,1}^{\frac{2}{\eta}}(\mathbb{R}^2)$, $p, q, r$ satisfy the conditions listed in Theorem 1.3, according to Theorem 1.1, there exists a positive time $T$ so that the equations (1.1)–(1.4) have a unique solution $(u, n, c)$.
with
\[ u \in C([0, \bar{T}); \dot{B}_{p,1}^{-1+\frac{s}{2}}(\mathbb{R}^2)) \cap \dot{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{s}{2}}(\mathbb{R}^2)) \cap L_t^1(\dot{B}_{p,1}^{2+\frac{s}{2}}(\mathbb{R}^2)), \quad (6.108) \]
\[ n \in C([0, \bar{T}); \dot{B}_{q,1}^{-2+\frac{s}{2}}(\mathbb{R}^2)) \cap \dot{L}_t^\infty(\dot{B}_{q,1}^{-2+\frac{s}{2}}(\mathbb{R}^2)) \cap L_t^1(\dot{B}_{q,1}^{2+\frac{s}{2}}(\mathbb{R}^2)), \quad (6.109) \]
\[ c \in C([0, \bar{T}); \dot{B}_{r,1}^{\frac{s}{2}}(\mathbb{R}^2)) \cap \dot{L}_t^\infty(\dot{B}_{r,1}^{\frac{s}{2}}(\mathbb{R}^2)) \cap L_t^1(\dot{B}_{r,1}^{2-\frac{s}{2}}(\mathbb{R}^2)). \quad (6.110) \]

We let \( T^* \) to be the maximal existence time so that (6.108)–(6.110) hold. Hence to prove Theorem 1.3, we only need to prove that \( T^* = \infty \). To complete the proof, we shall use the method of continuity. For this, we define
\[ T^{**} := \sup \{ t \in [0, T^*) : \| \vec{u} \|_{L_t^\infty(B_{p,1}^{-1+\frac{s}{2}})} + \| n \|_{L_t^\infty(B_{q,1}^{-2+\frac{s}{2}})} + \| c \|_{L_t^\infty(B_{r,1}^{\frac{s}{2}})} < \eta \}, \quad (6.111) \]

In what follows, we will prove that \( T^{**} = T^* \) under the assumptions of (1.9).

If not, we assume that \( T^{**} < T^* \), and for all \( t \leq T^{**} \), we get from (6.98), (6.104) and (6.107), taking \( \varepsilon \) small enough, that
\[
\| \vec{u} \|_{L_t^\infty(B_{p,1}^{-1+\frac{s}{2}})} + \| n \|_{L_t^\infty(B_{q,1}^{-2+\frac{s}{2}})} + \| c \|_{L_t^\infty(B_{r,1}^{\frac{s}{2}})} \leq \eta.
\]

Choosing \( \eta < \frac{1}{6} \) and \( \lambda \geq 2C + C\| u_0 \|_{B_{p,1}^{-1+\frac{s}{2}}} \left( 1 + \| u_0 \|_{B_{p,1}^{-1+\frac{s}{2}}} \right) \exp \left\{ C\| u_0 \|_{B_{p,1}^{-1+\frac{s}{2}}}^2 \right\} \), we have
\[
\| \vec{u} \|_{L_t^\infty(B_{p,1}^{-1+\frac{s}{2}})} + \| n \|_{L_t^\infty(B_{q,1}^{-2+\frac{s}{2}})} + \| c \|_{L_t^\infty(B_{r,1}^{\frac{s}{2}})} \leq 2\left( \| u_0 \|_{B_{p,1}^{-1+\frac{s}{2}}} + \| c_0 \|_{B_{r,1}^{\frac{s}{2}}} \right), \quad \text{for } t \leq T^{**}. \quad (6.113)\]
This implies that, for $\forall t \leq T^{**}$

\[
\|\bar{u}\|_{L^\infty_t(B^{1+\frac{\alpha}{p}}_{\infty,p})} + \|n\|_{L^\infty_t(B^{2+\frac{\alpha}{q}}_{\infty,q})} + \|c\|_{L^\infty_t(B^{\frac{\alpha}{q}}_{\infty,q})} + \|\bar{u}\|_{L^1_t(B^{\frac{\alpha}{q}}_{\infty,q})} + \|n\|_{L^1_t(B^{\frac{\alpha}{q}}_{\infty,q})} + \|c\|_{L^1_t(B^{\frac{\alpha}{q}}_{\infty,q})} 
\leq 2C(\|n\|_{B^{2+\frac{\alpha}{q}}_{\infty,q}} + \|c\|_{B^{\frac{\alpha}{q}}_{\infty,q}}) \exp(\lambda \int_0^t f(t')dt')
\]

\[
\leq 2C(\|n\|_{B^{2+\frac{\alpha}{q}}_{\infty,q}} + \|c\|_{B^{\frac{\alpha}{q}}_{\infty,q}}) \times \exp(\lambda \|n\|_{B^{2+\frac{\alpha}{q}}_{\infty,q}} (1 + \|u_0\|_{B^{1+\frac{\alpha}{p}}_{p,1}}) \exp \{C\|u_0\|_{B^{1+\frac{\alpha}{p}}_{p,1}}^2\}). \quad (6.114)
\]

In particular, (6.114) implies that if we take $C_0$ large enough and $\varepsilon_0$ sufficiently small in (1.9), there holds

\[
\|\bar{u}\|_{L^\infty_t(B^{1+\frac{\alpha}{p}}_{\infty,p})} + \|n\|_{L^\infty_t(B^{2+\frac{\alpha}{q}}_{\infty,q})} + \|c\|_{L^\infty_t(B^{\frac{\alpha}{q}}_{\infty,q})} + \|\bar{u}\|_{L^1_t(B^{\frac{\alpha}{q}}_{\infty,q})} + \|n\|_{L^1_t(B^{\frac{\alpha}{q}}_{\infty,q})} + \|c\|_{L^1_t(B^{\frac{\alpha}{q}}_{\infty,q})} \leq \frac{\eta}{2} \quad (6.115)
\]

for $t \leq T^{**}$, which contradicts (6.111). Thus we conclude that $T^{**} = \infty$. This completes the proof of Theorem 1.3. \(\square\)

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