A SCHUR TYPE LEMMA FOR THE MEAN BERWALD CURVATURE IN FINSLER GEOMETRY

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ABSTRACT. In this short paper, we study a symmetric covariant tensor in Finsler geometry, which is called the mean Berwald curvature. We first investigate the geometry of the fibres as the submanifolds of the tangent sphere bundle on a Finsler manifold. Then we prove that if the mean Berwald curvature is isotropic along fibres, then the Berwald scalar curvature is constant along fibres.

INTRODUCTION

Let \((M, F)\) be a Finsler manifold of dim \(M = n\). Let \(SM = \{ (x, y) \in TM | F(x, y) = 1 \}\) be the unit tangent sphere bundle of \(M\) with the natural projection \(\pi : SM \rightarrow M\). The tangent bundle \(T(SM)\) admits a natural horizontal subbundle \(H(SM)\) and a Sasaki-type metric \(g^T(SM)\). The Hilbert form \(\omega = F_y dx^i\) defines a contact structure on \(SM\). The Reeb field is just the spray \(G\).

Let \(D\) be the contact distribution \(\{ \omega = 0 \}\). There is an almost complex structure \(J\) on \(D\). \(J\) extends to be an endomorphism of \(SM\) by defining \(J(G) = 0\). It is clear that \((SM, J, G, \omega, g^T(SM))\) gives rise to a contact metric structure.

Let \(F := V(SM)\) be the integrable distribution given by the tangent spaces of the fibres of \(SM\). Let \(p : T(SM) \rightarrow F\) be the natural projection. Then the tangent bundle \(T(SM)\) admits a splitting

\[(0.1)\quad T(SM) = \pi + JF \oplus F =: H(SM) \oplus F.\]

where \(\pi\) is the trivial bundle generated by \(G\). The cotangent bundle \(T^*(SM)\) has the corresponding splitting.

The Berwald connection \(\nabla^{Be}\) on the horizontal subbundle \(H(SM)\) is a linear connection without torsion. The curvature \(R^{Be} = (\nabla^{Be})^2\) of the Berwald connection is an \(\text{End}(H(SM))\)-valued two form on \(SM\). With respect to the natural splitting \((0.1)\), the curvature \(R^{Be}\) is divided into two parts

\[(0.2)\quad R^{Be} = \tilde{R} + \tilde{P},\]

where \(\tilde{R}\) contains pure \(hh\)-forms, \(\tilde{P}\) contains \(hv\)-forms. The mean Berwald curvature or the \(E\)-curvature is defined as the trace of the Berwald curvature \(\tilde{P}\).

Let \(\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n-1}\}\) be a local adapted orthonormal frame with respect to the Sasaki-type Riemannian metric

\[(0.3)\quad g^T(SM) = \sum_i \omega^i \otimes \omega_i^i + \sum_{\bar{\alpha}} \omega^{\bar{\alpha}} \otimes \omega^{\bar{\alpha}} =: g + \hat{g}\]

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on \( SM \), where \( \{e_1, \ldots, e_n\} \) spans \( H(SM) \), \( e_n = G \) and \( e_{\bar{a}} = -J e_a \). Let \( \theta = \{\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{2n-1}\} \) be the dual frame, then \( \omega^n = \omega \) is the Hilbert form.

Under the adapted dual frame, the mean Berwald curvature has the following expression
\[
(0.4) \quad E := \text{tr} \hat{P} =: E_{\gamma \mu} \omega^\gamma \wedge \omega^\mu.
\]

Using the almost complex structure \( J \) and the \( E \)-curvature (0.4), we define the following symmetric vertical tensor
\[
(0.5) \quad \dot{E} := E_{\gamma \mu} \omega^\gamma \otimes \omega^\mu.
\]

The trace of \( \dot{E} \) with respect to the vertical metric \( \dot{g} \) is called the *Berwald scalar curvature*
\[
(0.6) \quad e := \text{tr}_\dot{g} \dot{E}.
\]

In this study, a Finsler manifold is called to have *isotropic mean Berwald curvature* if the following condition is satisfied
\[
(0.7) \quad \dot{E} = \frac{e}{n-1} \dot{g}.
\]

One notices that the Berwald scalar curvature \( e \) is a smooth function on \( SM \) for a Finsler manifold. In the previous papers [5, 6], a Finsler manifold \( (M, F) \) is called to have *isotropic Berwald scalar curvature* if \( e \in C^\infty(M) \).

The main result of this short note is a Schur type lemma for the mean Berwald curvature.

**Theorem 1.** Let \( (M, F) \) be a Finsler manifold and \( \dim M = n \geq 3 \). If the mean Berwald curvature is isotropic, then the Berwald scalar curvature is isotropic.

**Remark 1.** In literature [2, 8], a Finsler manifold is called to have isotropic mean Berwald curvature if both the conditions (0.7) and \( e \in C^\infty(M) \) are satisfied. By Theorem 1, one notices that the assumption about the isotropic Berwald scalar curvature is unnecessary.

Combining the results in [5] and Theorem 1, we obtain the following corollary.

**Corollary 1.** Let \( (M, F) \) be a Finsler manifold and \( \dim M \geq 3 \). The following conditions are equivalent:

1. \( F \) has isotropic mean Berwald curvature, i.e., the condition (0.7) holds;
2. the Berwald scalar curvature is isotropic, i.e., \( e \in C^\infty(M) \);
3. \( F \) has weakly isotropic \( S \)-curvature, and

\[
S = \frac{1}{n-1} e + \pi^* \xi(G),
\]

where \( e \in C^\infty(M) \) and \( \xi \in \Omega^1(M) \).

**Remark 2.** In the recent paper [3], the author proves an attractive property about the the \( S \)-curvature based on his formula involved in the derivative of the coefficient of the Busemann-Hausdorff volume form.
To emphasize this important result, we would like to state it alone as follows: a Finsler structure \( F \) has weakly isotropic \( S \)-curvature, i.e., (3) in Corollary 1 holds if and only if the following statement is true:

(4). The \( S \)-curvature satisfies

\[
S = \frac{1}{n-1} e + \pi^* df(G),
\]

where \( e, f \in C^\infty(M) \). If the volume form on \( M \) in the definition of the distortion is the Bussmann-Hausdorff one, then \( f = 0 \) and \( S \) is isotropic.

In this paper we adopt the index range and notations:

\[
1 \leq i, j, k, \ldots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq n-1, \quad 1 \leq A, B, C, \ldots \leq 2n-1,
\]

and \( \bar{\alpha} := n + \alpha, \bar{\beta} := n + \beta, \bar{\gamma} := n + \gamma, \ldots \).

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1. **The geometry of the fibres of the sphere bundle of a Finsler manifold**

Let \((M, F)\) be a \( n \)-dimensional Finsler manifold. Let

\[
\nabla^\text{Ch} : \Gamma(H(SM)) \to \Omega^1(SM; H(SM))
\]

be the Chern connection, which can be extended to a map

\[
\nabla^\text{Ch} : \Omega^*(SM; H(SM)) \to \Omega^{*+1}(SM; H(SM)),
\]

where \( \Omega^*(SM; H(SM)) := \Gamma(\Lambda^*(T^*SM) \otimes H(SM)) \) denotes the horizontal valued differential forms on \( SM \). It is well known that the symmetrization of Chern connection \( \hat{\nabla}^\text{Ch} \) is the Cartan connection. The difference between \( \hat{\nabla}^\text{Ch} \) and \( \nabla^\text{Ch} \) will be referred as the Cartan endomorphism,

\[
H = \hat{\nabla}^\text{Ch} - \nabla^\text{Ch} \in \Omega^1(SM, \text{End}(H(SM))).
\]

Set \( H = H_{ij} \omega^i \otimes e_i \). By Lemma 3 and Lemma 4 in [4], \( H_{ij} = H_{ji} = H_{ij}; \gamma \omega^\gamma \) has the following form under natural coordinate systems

\[
H_{ij}; \gamma = -A_{pqk} u^p_i u^q_j u^k_\gamma,
\]

where \( A_{ijk} = \frac{1}{4} F[F^2]_{y^i y^j y^k} \) and \( u^j_i \) are the transformation matrix from adapted orthonormal frames to natural frames.

Let \( \omega = (\omega^i_j) \) be the connection matrix of the Chern connection with respect to the local adapted orthonormal frame field, i.e.,

\[
\nabla^\text{Ch} e_i = \omega^i_j \otimes e_j.
\]
Lemma 1 ([1, 2, 4]). The connection matrix \( \omega = (\omega^i_j) \) of \( \nabla^{\text{Ch}} \) is determined by the following structure equations,

\[
\begin{cases}
    d\vartheta = -\omega \wedge \vartheta, \\
    \omega + \omega' = -2H,
\end{cases}
\]

where \( \vartheta = (\omega^1, \ldots, \omega^n)^t \). Furthermore,

\[
\omega^\alpha_\alpha = -\omega^\alpha_n = \omega^\alpha \xi, \quad \text{and} \quad \omega^\alpha_n = 0.
\]

In [4], the authors proved that the Chern connection is just the Bott connection on \( H(SM) \) in the theory of foliation (c.f. [3]).

Let \( R^{\text{Ch}} = (\nabla^{\text{Ch}})^2 \) be the curvature of \( \nabla^{\text{Ch}} \). Let \( \Omega = (\Omega^i_{jk}) \) be the curvature forms of \( R^{\text{Ch}} \). From the torsion freeness, the curvature form has no pure vertical differential form

\[
\Omega^i_j := d\omega^i_j - \omega^i_k \wedge \omega^k_j =: \frac{1}{2} R^i_{jk} \omega^k \wedge \omega^j + P^i_{jk} \omega^k \wedge \omega^j.
\]

The Sasaki metric (1.3) induces a Euclidean metric \( \hat{g} \) on the vertical bundle \( \mathcal{F} \). As the restriction the Levi-Civita connection \( \nabla^{T(SM)} \) on \( \mathcal{F} \), \( \nabla^\mathcal{F} := p\nabla^{T(SM)}p \) is a Euclidean connection of the bundle \( (\mathcal{F}, \hat{g}) \). It is clear that along each fibre of \( S_xM, x \in M \), \( \nabla^\mathcal{F} \) is just the Levi-Civita connection \( \tilde{\nabla} \) of the Riemannian manifold \( (S_xM, \hat{g}|_{T(S_xM)}) \).

The following lemma can be found in [7]. We present here a new proof.

**Lemma 2.** Let \( \Theta = (\Theta^\gamma_\beta) \) be the connection form of the Levi-Civita connection \( \nabla^{T(SM)} \) of \( g^{T(SM)} \) with respect to the adapted frame \( \{e_A\} \). Then the connection forms of \( \nabla^\mathcal{F} \) are determined by

\[
\nabla^\mathcal{F} e_\beta = \Theta^\gamma_\beta \otimes e_\gamma,
\]

and

\[
\Theta^\gamma_\beta = \omega^\gamma_\beta + H^\beta_{\gamma\alpha} \omega^\alpha,
\]

where \( \omega^\gamma_\beta \) are the connection forms of the Chern connection and \( H^\beta_{\gamma\alpha} \omega^\alpha \) are the coefficients of the Cartan endomorphism.

**Proof.** By the definition of the Levi-Civita connection, Lemma 1 and (1.2), we obtain

\[\begin{align*}
\langle \nabla^T_{e_\alpha} e_\beta, e_\gamma \rangle &= \frac{1}{2} \langle \mathcal{L}_{e_\beta} g^{T(SM)}(e_\alpha, e_\gamma) + \frac{1}{2} d\omega^\beta(e_\alpha, e_\gamma) \\
&= -\frac{1}{2} \left( \langle [e_\beta, e_\alpha], e_\gamma \rangle + \langle [e_\beta, e_\gamma], e_\alpha \rangle + \langle [e_\alpha, e_\gamma], e_\beta \rangle \right) \\
&= -\frac{1}{2} \left( (\Omega^\gamma_\alpha + \omega^\gamma_\alpha \wedge \omega^\alpha)(e_\beta, e_\alpha) + (\Omega^\alpha_\gamma + \omega^\alpha_\gamma \wedge \omega^\gamma)(e_\beta, e_\gamma) \\
&\quad + (\Omega^\alpha_\beta + \omega^\alpha_\beta \wedge \omega^\beta)(e_\alpha, e_\gamma) \right) \\
&= -\frac{1}{2} \left( -\omega^\gamma_\beta(e_\alpha) + \omega^\gamma_\alpha(e_\beta) - \omega^\alpha_\beta(e_\gamma) + \omega^\alpha_\gamma(e_\beta) - \omega^\alpha_\gamma(e_\gamma) + \omega^\beta_\gamma(e_\alpha) \right) \\
&= \omega^\gamma_\beta(e_\alpha) + H^\beta_{\gamma\alpha} e_\gamma.
\end{align*}\]
In the following lemma, we calculate the curvature forms of $\mathring{e}$ along (1.6) $\nabla \bar{\Omega}$ that the horizontal derivatives of $\mathring{\dot{e}}$ are omitted here. Furthermore, the covariant differential of $\mathring{H}$ by the connection $\nabla ^{\mathring{F}}$ can be expressed as

$$
\nabla ^{\mathring{F}} \mathring{H} = (dH_{\alpha \beta \gamma} - H_{\mu \beta \gamma} \Theta _{\alpha}^{\mu} - H_{\alpha \nu \gamma} \Theta _{\beta}^{\nu} - H_{\alpha \beta \mu} \Theta _{\gamma}^{\mu}) \otimes \omega ^{\alpha} \otimes \omega ^{\beta} \otimes \omega ^{\gamma}.
$$

(1.5)

We will call that $H_{\alpha \beta \gamma} |_{\mu}$ the horizontal derivatives of $\mathring{H}$ by $\nabla ^{\mathring{F}}$ along $e_{\mu}$. One finds that the horizontal derivatives of $\mathring{H}$ by $\nabla ^{\mathring{F}}$ and the horizontal derivatives of $\mathring{H}$ by $\nabla ^{\text{Ch}}$ coincide. Similarly, $H_{\alpha \beta \gamma} |_{\mu}$ will be called the vertical derivatives of $\mathring{H}$ by $\nabla ^{\mathring{F}}$ along $e_{\mu}$.

Let $(\nabla ^{\mathring{F}})^{2}$ be the curvature of $\nabla ^{\mathring{F}}$. Set

$$
(\nabla ^{\mathring{F}})^{2} e_{\beta} = \Omega _{\beta}^{\gamma} \otimes e_{\gamma}.
$$

(1.6)

In the following lemma, we calculate the curvature forms of $(\nabla ^{\mathring{F}})^{2}$.

**Lemma 3.** The curvature forms $\Omega _{\beta}^{\gamma}$ of the connection $\nabla ^{\mathring{F}}$ are given by

$$
\Omega _{\beta}^{\alpha} = \frac{1}{2}(\Omega _{\beta}^{\alpha} - \Omega _{\alpha}^{\beta}) + \frac{1}{2}(H_{\beta \alpha \nu} - H_{\alpha \beta \nu})\omega ^{\beta} \wedge \omega ^{\nu} + \frac{1}{2}(H_{\beta \gamma} - H_{\gamma \beta})\omega _{\alpha} \wedge \omega ^{\beta} - \frac{1}{2}(\delta _{\beta \mu} \delta _{\alpha \nu} - \delta _{\beta \nu} \delta _{\alpha \mu})\omega ^{\beta} \wedge \omega ^{\nu},
$$

(1.7)

where $\Omega _{\beta}^{\alpha}$ are the curvature forms of the Chern connection.

**Proof.** By the definition (1.6), the curvature forms of $\nabla ^{\mathring{F}}$ are given by

$$
\Omega _{\beta}^{\gamma} = d\Theta _{\beta}^{\alpha} - \Theta _{\beta}^{\gamma} \wedge \Theta _{\gamma}^{\alpha}.
$$

(1.8)
Substituting the connection forms (1.3) into (1.8), by Lemma 1.1 and (1.2), one obtains

\[
\Omega^\alpha_\beta = d(\omega^\alpha_\beta + H_{\beta\alpha\mu}\omega^\mu) - (\omega^\alpha_\beta + H_{\beta\gamma\mu}\omega^\mu) \wedge (\omega^\gamma_\alpha + H_{\gamma\alpha\nu}\omega^\nu) \\
= d\omega^\alpha_\beta - \omega^\gamma_\alpha \wedge \omega^\alpha_\beta + \omega^\alpha_\beta \wedge \omega^\mu_\gamma - (H_{\beta\gamma\mu}\omega^\mu) \wedge (H_{\gamma\alpha\nu}\omega^\nu) \\
+ dH_{\beta\alpha\mu} \wedge \omega^\mu - H_{\beta\alpha\nu}(\Omega^\gamma_\alpha + \omega^\mu_\gamma \wedge \omega^\nu_\gamma) - \omega^\gamma_\beta \wedge (H_{\gamma\alpha\nu}\omega^\nu) - (H_{\beta\gamma\mu}\omega^\mu) \wedge \omega^\alpha_\gamma
\]

(1.9)

Hence the equation (1.7) is proved. \(\square\)

The fibres of the sphere bundle \(SM\) of a Finsler manifold \((M, F)\) are integral submanifolds of the distribution \(\mathcal{F}\), which have the induced Riemannian metrics form the Sasaki metric \(g^T(SM)\).

For any \(x \in M\), \((S_xM, \hat{g}|_{T(S_xM)})\) is a Riemannian manifold. The connection \(\nabla^{\mathcal{F}}\) along \(S_xM\) gives the corresponding Levi-Civita connection \(\nabla^{\mathcal{F}}\). Let \(i_x : S_xM \to SM\) be the inclusion map. Set

\[
\hat{\omega}^\beta := i_x^*\omega^\beta, \quad \alpha = 1, \ldots, n - 1.
\]

Then \(\{\hat{\omega}^{n+1}, \ldots, \hat{\omega}^{2n-1}\}\) is the induced orthonormal dual frame of \((S_xM, \hat{g}|_{T(S_xM)})\). Let \(\hat{\Omega}^\alpha_\beta\) and \(\hat{\Omega}^\alpha_\beta\) be the connection forms and curvature forms of the Levi-Civita connection \(\nabla^{\mathcal{F}}\). The coefficients of the curvature forms are defined as

\[
\hat{\Omega}^\alpha_\beta = \frac{1}{2}\hat{R}_{\beta\alpha\mu\nu}\omega^\mu \wedge \omega^\nu.
\]
Lemma 4. The structure equations of each fibre of the sphere bundle of a Finsler manifold are given by

\[ H_{\beta\alpha;\mu} - H_{\beta\alpha;\mu} = 0, \]  

and

\[ \dot{R}_{\beta\alpha\mu\nu} = H_{\beta\gamma\mu} H_{\gamma\alpha\nu} - H_{\beta\gamma\nu} H_{\gamma\alpha\mu} - (\delta_{\beta\mu} \delta_{\alpha\nu} - \delta_{\beta\nu} \delta_{\alpha\mu}). \]

Proof. It is clear that \( \dot{\Theta}^\alpha_{\beta} = i^* \Theta^\alpha_{\beta}. \) For any \( x \in M \), the curvature forms of \( (S_x M, \dot{g}|_{T(S_x M)}) \) are

\[ \dot{\Omega}^\alpha_{\beta} = d(i^* \Theta^\alpha_{\beta}) - (i^* \Theta^\beta_{\gamma}) \wedge (i^* \Theta^\alpha_{\gamma}) = i^* (d \Theta^\alpha_{\beta}) - i^* (\Theta^\beta_{\gamma} \wedge \Theta^\alpha_{\gamma}) = i^* \Omega^\alpha_{\beta}. \]

By \( (1.2), (1.3) \) and \( (1.13) \), one obtains

\[ \dot{\Omega}^\alpha_{\beta} = \frac{1}{2} (H_{\beta\gamma\mu} H_{\gamma\alpha\nu} - H_{\beta\gamma\nu} H_{\gamma\alpha\mu}) \omega^\mu \wedge \omega^\nu - \frac{1}{2} (\delta_{\beta\mu} \delta_{\alpha\nu} - \delta_{\beta\nu} \delta_{\alpha\mu}) \omega^\mu \wedge \omega^\nu + \frac{1}{2} (H_{\beta\alpha\nu;\mu} - H_{\beta\alpha;\nu;\mu}) \omega^\mu \wedge \omega^\nu. \]

By \( (1.10) \) and \( (1.14) \), the following equation holds

\[ \dot{R}_{\beta\alpha\mu\nu} = H_{\beta\gamma\mu} H_{\gamma\alpha\nu} - H_{\beta\gamma\nu} H_{\gamma\alpha\mu} - (\delta_{\beta\mu} \delta_{\alpha\nu} - \delta_{\beta\nu} \delta_{\alpha\mu}) + (H_{\beta\alpha;\nu;\mu} - H_{\beta\alpha;\nu;\mu}). \]

The structure equations \( (1.11) \) and \( (1.12) \) can be derived from \( (1.15) \) directly. \( \Box \)

2. The proof of the Schur type lemma

Let \( dV_M = \sigma(x) dx^1 \wedge \cdots \wedge dx^n \) be any volume form of \( M \). The distortion of \( (M, F) \) related to \( dV_M \) is defined by

\[ \tau := \ln \frac{\sqrt{\det (g_{ij}(x, y))}}{\sigma(x)}. \]

The derivative of \( \tau \) along the vector field \( G \) will be denoted by \( S := G(\tau) \) and called the S-curvature. In the previous studies \([5, 6]\), a vertical differential equation involved the S-curvature and the mean Berwald curvature plays an important role. Using notations in this paper, We restate the mentioned equation for convenience

\[ S_{\alpha;\beta} + H_{\alpha\beta\mu} S_{\mu} + S_{\delta\alpha\beta} = E_{\alpha\beta}. \]

In the following lemma, we present the Codazzi equation of \( \dot{E} \) along the fibres of the sphere bundle of a Finsler manifold.

Lemma 5. The Codazzi equation of the mean Berwald curvature is given by

\[ E_{\alpha\beta;\gamma} - E_{\alpha\gamma;\beta} = H_{\alpha\beta\mu} E_{\mu\gamma} - H_{\alpha\gamma\mu} E_{\mu\beta}. \]

Proof. The covariant derivative of \( (2.1) \) along fibres by the Levi-Civita connection \( \nabla^g \) shows that

\[ E_{\alpha\beta;\gamma} = S_{\alpha;\beta;\gamma} + H_{\alpha\beta\mu} S_{\mu} + H_{\alpha\beta\mu} S_{\mu;\gamma} + S_{\gamma\delta_{\alpha\beta}}. \]
By (2.1), (1.11) and (2.3), one obtains
\[
E_{\alpha\beta\gamma} - E_{\alpha\gamma\beta} = S_{\alpha;\beta\gamma} - S_{\alpha;\gamma\beta} + (H_{\alpha\beta\mu}S_{\mu\gamma} - H_{\alpha\gamma\mu}S_{\mu\beta}) + (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma})
\]
(2.4)
\[
= S_{\alpha;\beta\gamma} - S_{\alpha;\gamma\beta} + (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + H_{\alpha\beta\mu}(E_{\mu\gamma} - H_{\mu\gamma\nu}S_{\nu\beta} - S\delta_{\mu\beta}) - H_{\alpha\gamma\mu}(E_{\mu\beta} - H_{\mu\beta\nu}S_{\nu\gamma} - S\delta_{\mu\beta})]
\]
By (1.12), the Ricci identity holds
\[
S_{\alpha;\beta\gamma} - S_{\alpha;\gamma\beta} = S_{\nu}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma} + (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + H_{\alpha\beta\mu}E_{\mu\nu} - H_{\alpha\gamma\mu}E_{\mu\beta}.
\]
(2.5)
\[
E_{\alpha\beta\gamma} - E_{\alpha\gamma\beta} = S_{\nu}(H_{\alpha\beta\mu}H_{\mu\nu\gamma} - H_{\alpha\gamma\mu}H_{\mu\nu\beta}) - (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + S_{\nu}(H_{\alpha\gamma\mu}H_{\mu\nu\beta} - H_{\alpha\beta\mu}H_{\mu\nu\gamma} - (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + S_{\nu}(H_{\alpha\beta\mu}E_{\mu\nu} - H_{\alpha\gamma\mu}E_{\mu\beta}).
\]
Plugging (2.5) into (2.4), one has
\[
E_{\alpha\beta\gamma} - E_{\alpha\gamma\beta} = S_{\nu}(H_{\alpha\beta\mu}H_{\mu\nu\gamma} - H_{\alpha\gamma\mu}H_{\mu\nu\beta}) - (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + S_{\nu}(H_{\alpha\gamma\mu}H_{\mu\nu\beta} - H_{\alpha\beta\mu}H_{\mu\nu\gamma} - (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + (S_{\gamma}\delta_{\alpha\beta} - S_{\beta}\delta_{\alpha\gamma}) + S_{\nu}(H_{\alpha\beta\mu}E_{\mu\nu} - H_{\alpha\gamma\mu}E_{\mu\beta}).
\]
The proof is complete. □

Now we present a proof of Theorem 1.

The proof of Theorem 1. Assume that the mean Berwald curvature satisfies (0.7), or equivalently
\[
E_{\alpha\beta} = \frac{1}{n - 1}e_{\delta_{\alpha\beta}}.
\]
(2.6)
Thus
\[
E_{\alpha\beta\gamma} = \frac{1}{n - 1}e_{\gamma\delta_{\alpha\beta}}.
\]
(2.7)
On the other hand, by (2.2) and (2.6), one directly has
\[
E_{\alpha\beta\gamma} - E_{\alpha\gamma\beta} = e_{\gamma\delta_{\alpha\beta}} - e_{\beta\delta_{\alpha\gamma}} = 0
\]
(2.8)
Plugging (2.7) into (2.8) yields
\[
0 = e_{\gamma\delta_{\alpha\beta}} - e_{\beta\delta_{\alpha\gamma}}.
\]
Hence
\[
0 = (n - 2)e_{\gamma}.
\]
Assume that \( n > 2 \), we obtain \( e_{\gamma} = 0 \). Therefore the Berwald scalar curvature \( e \) is locally constant along fibres. By the connectedness of the fibres of the sphere bundle, we conclude that \( e \) is constant along fibres and \( e \in C^\infty(M) \). □
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