A DIRECT TEST OF PERTURBATIVE QCD AT SMALL $x$

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Abstract
We show that recent data from HERA on the proton structure function $F_2$ at small $x$ and large $Q^2$ provide a direct confirmation of the double asymptotic scaling prediction of perturbative QCD. A linear rise of $\ln F_2$ with the scaling variable $\sigma$ is observed throughout the kinematic region probed at HERA, and the measured slope is in excellent agreement with the QCD prediction. This provides a direct determination of the leading coefficient of the beta function. At large values of the scaling variable $\rho$ the data display a small but statistically significant scaling violation.

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Perturbative QCD predicts that at sufficiently large \( t \equiv \ln Q^2/\Lambda^2 \) and small \( x \) the nucleon structure function \( F_2 \) should exhibit double scaling in the two variables

\[
\sigma \equiv \sqrt{\ln \frac{x_0}{x} \ln \frac{t}{t_0}}, \quad \rho \equiv \sqrt{\ln \frac{x_0}{x} / \ln \frac{t}{t_0}},
\]

provided only that the nonperturbative input to the perturbative evolution is sufficiently soft. We have shown [1] that this prediction is indeed confirmed by the first measurements of \( F_2^p \) performed at HERA [2,3]. In fact, it turns out that not only most of the HERA data, but even some of the older data from the NMC [4], lie well inside the asymptotic regime, suggesting that the starting scale \( t_0 \equiv \ln Q^2_0/\Lambda^2 \) for the perturbative evolution should be little more than \( Q_0^2 \sim 1 \text{ GeV}^2 \). A significantly enlarged set of measurements of \( F_2^p \) has now become available [5,6], which makes it possible to test double scaling more quantitatively. Specifically, the slope of the linear rise of \( \ln F_2 \) in the scaling variable \( \sigma \) can be reliably measured, and turns out to be in excellent agreement with the QCD prediction, thus giving a direct empirical determination of the leading coefficient \( \beta_0 \) of the QCD beta–function. We also find that there is now evidence for scaling violation at large \( \rho \).

Double asymptotic scaling follows from a computation [7] of the asymptotic form of the structure function \( F_2^p(x; t) \) at small \( x \) based on the use of the operator product expansion and renormalization group at leading perturbative order. It thus relies only on the assumption that any increase in \( F_2^p(x; t) \) at small \( x \) is generated by perturbative QCD evolution, rather than being due to some other (nonperturbative) mechanism manifested by an increase in the starting distribution \( F_2^p(x; t_0) \). The resulting asymptotic behaviour takes the form

\[
F_2^p(\sigma, \rho) \sim N f(\frac{2}{\rho} \sqrt{\frac{1}{\gamma} \exp \left[ 2\gamma \sigma - \delta \left( \frac{\sigma}{\rho} \right) \right]} \left[ 1 + O\left( \frac{1}{\sigma} \right) \right]),
\]

where \( \gamma \equiv 2\sqrt{N_c/\beta_0}, \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f, \delta \equiv (1 + \frac{2n_f}{11N_c})/(1 - \frac{2n_f}{11N_c}) \), and the unknown function \( f \), which depends on the details of the starting distribution, tends to one for sufficiently small values of its argument. \( N \) is an a priori undetermined normalization factor.

In [1] we derived (2) by noting that at small-\( x \) the one loop QCD evolution equations reduce to wave equations, which propagate the parton distribution functions from their boundary values at \( t = t_0 \) and \( x = x_0 \) to larger values of \( t \) and smaller values of \( x \). Since the propagation is unstable, away from the boundaries an exponential increase with \( \sigma \) of the form (2) inevitably arises, provided only that the small-\( x \) behaviour of the starting
distributions at $t_0$ is sufficiently soft (which in practice means that if $f_s(x; t)$ is a singlet parton distribution function, $x^{1+\lambda}f_s(x; t_0) \to 0$ as $x \to 0$ for any $\lambda \leq 0.2$). The behaviour (2) is thus a rather clean prediction of perturbative QCD, in so far as it is independent of the details of the (soft) nonperturbative parton distributions which are input at $t_0$, provided that at small $x$ these conform to expectations based on Regge theory. The asymptotic behaviour can be shown [1] to set in rather rapidly as $\sigma$ increases in a region not too close to the boundaries, i.e. when $\rho$ is neither too large nor too small.

In order to compare the data for $F_2^p$ with the prediction (2) we rescale the measured values of $F_2$ by a factor

$$R'_F(\sigma, \rho) = R \exp \left( \delta(\sigma/\rho) + \frac{1}{2} \ln \sigma + \ln(\rho/\gamma) \right),$$

(3)

to remove the part of the leading subasymptotic behaviour which can be calculated in a model independent way. Then $\ln [R'_F F_2]$ is predicted to rise linearly with $\sigma$, independently of $\rho$ (when $\rho$ is large), with slope

$$2\gamma = 12/\sqrt{33 - 6n_f/N_c} = 2.4$$

(4)

if $n_f = 4$ as in the HERA kinematic range. The model-dependent subasymptotic behaviour due to the function $f$ can be eliminated by cutting all points with subasymptotically small $\rho$; the scaling analysis of Ref.[1] (see fig. 2 below) suggests that we place the cut at $\rho^2 = 2$.

All the available experimental data [4,5,6] for $F_2^p$ which pass this cut are plotted in fig. 1. The predicted linear rise in $\sigma$ is spectacularly confirmed, providing clear evidence that in the region $\sigma^2 > 1$, $\rho^2 > 2$ the asymptotic behaviour (2) has set in. Indeed, the scaling actually sets in rather precociously: even the NMC data down to $\sigma \sim 0.7$ seem to be rising linearly, with possibly an indication of a systematic normalization mismatch of around 10% between the NMC and the HERA determinations of $F_2$.

Fitting a straight line to all 80 HERA points in the plot yields a $\chi^2$ of 66, and a gradient $2\gamma_{exp} = 2.37 \pm 0.16$, in perfect agreement with the QCD prediction eq.(4). Turning this into a measurement of the leading coefficient of the beta–function gives (with $N_c = 3$) $\beta_0 = 8.6 \pm 1.1$ (to be compared with $25/3$ for $n_f = 4$). This is a direct, model independent, and highly nontrivial test of the perturbative dynamics of asymptotically free nonabelian gauge theory.

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1 The constant rescaling factor $R$ may of course be chosen arbitrarily; here we choose $R = 8.1$, so that the normalization of the figures is the same as in [1].
We next consider scaling violations, both in the subasymptotic region of small $\sigma$ and small $\rho$, and in the post-asymptotic region of large $\rho$. This is best done by rescaling $F_2^p$ by a factor

$$R_F(\sigma, \rho) = R \exp \left( -2\gamma \sigma + \delta(\sigma/\rho) + \frac{1}{2} \ln \sigma + \ln(\rho/\gamma) \right)$$

(5)

to remove all the leading behaviour in (2). The rescaled structure function should thus scale in both $\sigma$ and $\rho$ when both are sufficiently large to lie in the asymptotic region: $R_F F_2^p = N + O(1/\sigma) + O(1/\rho)$. This double asymptotic scaling behaviour is tested in the two scaling plots fig. 2, where we also display the predictions obtained [1] by applying the leading small-$x$ form of the evolution equations to a typical soft starting gluon distribution. Specifically, fig. 2a) shows that the scaling in $\sigma$ sets in very rapidly, as all the points on the plot lie in the asymptotic regime; fig. 2b) shows that the scaling in $\rho$ only sets in for $\rho^2 \gtrsim 2$. However even if $\rho$ is as low as $\rho \sim \frac{1}{2}$ the subasymptotic corrections due to $f(\gamma/\rho)$ seem fairly well accounted for by the scaling violation displayed by the curves of fig. 2.

More interestingly, at large $\rho$ there now appears to be a statistically significant rise above the scaling prediction. To test the significance of this rise, we fitted to the data a linear combination of the behaviour discussed above and displayed by the curves of fig. 2, and a “hard pomeron” behaviour, which violates scaling by rising with $\rho$ (see ref.[1] for a more detailed discussion). Including in the fit the 103 HERA points with both $\sigma^2$ and $\rho^2$ greater than one half gives the results displayed in the table. The data seem to prefer a $4 \pm 1\%$ admixture of the hard pomeron solution. One should be very cautious about taking this as evidence for the hard pomeron per se, however, since higher loop corrections should give a similar rise[8]. It should be possible to settle this issue decisively when a more detailed set of data and more accurate theoretical calculations become available.

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|     | $N_s$       | $N_h$     | $\chi^2$ |
|-----|-------------|-----------|-----------|
| a)  | $0.341 \pm 0.005$ | 0         | 96        |
| b)  | 0           | $0.156 \pm 0.002$ | 878       |
| a)+b)| $0.319 \pm 0.012$ | $0.012 \pm 0.002$ | 91        |

Table: The fitted normalizations $N_s$ and $N_h$ and the associated $\chi^2$s (103 data points). The different cases considered are a) soft pomeron b) hard pomeron, and the linear combination a) + b).
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Figure Captions

Fig. 1. Values of $R'_F F_2^p$ plotted against $\sigma$: diamonds are ZEUS data [5], squares H1 data [6], and crosses are NMC data. The best fit straight line is also shown.

Fig. 2. $R'_F F_2^p$ plotted against a) $\sigma$ and b) $\rho$. Included in the plots are all the HERA data with $\rho > 1.2$, $\sigma > 0.7$, respectively. The curves show the prediction obtained [1] evolving a typical soft starting gluon distribution: a) dot-dash curve, $\rho = 1.4$; solid curve, $\rho = 2.2$; dotted curve, $\rho = 3.2$. b) dot-dash curve, $\sigma = 1.1$; solid curve, $\sigma = 1.8$; dotted curve, $\sigma = 2.1$. 
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