The stability of Einstein static universe in the DGP braneworld

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Abstract

The stability of an Einstein static universe in the DGP braneworld scenario is studied in this letter. Two separate branches denoted by $\epsilon = \pm 1$ of the DGP model are analyzed. Assuming the existence of a perfect fluid with a constant equation of state, $w$, in the universe, we find that, for the branch with $\epsilon = 1$, there is no a stable Einstein static solution, while, for the case with $\epsilon = -1$, the Einstein static universe exists and it is stable when $-1 < w < -\frac{1}{3}$. Thus, the universe can stay at this stable state past-eternally and may undergo a series of infinite, non-singular oscillations. Therefore, the big bang singularity problem in the standard cosmological model can be resolved.

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I. INTRODUCTION

Although most of the problems in the standard cosmological model can be resolved by the inflation theory, the resolution of the existence of a big bang singularity in the early universe is still elusive. Based upon the string/M-theory, the pre-big bang [1] and ekpyrotic/cyclic [2] scenarios have been proposed to address the issue. Recently, Ellis et al. proposed, in the context of general relativity, a new scenario, called an emergent universe [3, 4] to avoid this singularity. In this scenario, the space curvature is positive, which is supported by the recent observation from WMAP7 [5] where it was found that a closed universe is favored at the 68% confidence level, and the universe stays, past-eternally, in an Einstein static state and then evolves to a subsequent inflationary phase. So, in an emergent theory, the universe originates from an Einstein static state rather than from a big bang singularity. However, the Einstein static universe in the classical general relativity is unstable, which means that it is extremely difficult for the universe to remain in such an initial static state in a long time due to the existence of perturbations, such as the quantum fluctuations. Therefore, the original emergent model does not seem to resolve the big bang singularity problem successfully as expected.

However, in the early epoch, the universe is presumably under extreme physical conditions, the realization of the initial state may be affected by novel physical effects, such as those stemming from quantization of gravity, or a modification of general relativity or even other new physics. As a result, the stability of the Einstein static state has been examined in various cases [6–20], from loop quantum gravity [7–9] to modified gravity (for a review see Ref. [17]), from Horava-Lifshitz gravity [19, 20] to Shtanov-Sahni braneworld scenario [10].

In this paper, we plan to examine the stability of the Einstein static universe in the DGP braneworld model [21]. In this braneworld, the whole energy-momentum is confined on a three dimensional brane embedded in a five-dimensional, infinite-volume Minkowski bulk. Since there are two different ways to embed the 4-dimensional brane into the 5-dimensional spacetime, the DGP model has two separate branches denoted by $\epsilon$ with distinct features. The $\epsilon = +1$ branch can explain the present accelerating cosmic ex-
pansion without the introduction of dark energy [22], while for the $\epsilon = -1$ branch, dark energy is needed in order to yield an accelerating expansion, as is the case in the LDGP model [25] and the QDGP model [26]. Using the $H(z)$, CMB shift and Sne Ia observational data, Lazkoz and Majerotto [27] found that the LDGP and QDGP are slightly more favored than the self-accelerating DGP model. Let us also note that a crossing of a phantom divide line, which is favored by the recent various observational data [28], is possible with a single scalar field [26, 29] in the $\epsilon = -1$ branch. In addition, inflation in the DGP model displays some new characteristics. It should be noted, however, that only in the $\epsilon = -1$ case can inflation exit spontaneously [30–35]. Also, in contrast to the Randall-Sundrum [23] and Shtanov-Sahni [24] braneworld scenarios with high energy modifications to general relativity, the DGP brane produces a low energy modification (for a review of the phenomenology of the DGP model, see Ref. [36]).

II. THE FRIEDMANN EQUATION IN DGP BRANEWORLD

For a homogeneous and isotropic universe which is described by the Friedmann-Robertson-Walker (FRW) metric.

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega \right),$$

(1)

the Friedmann equation on the warped DGP brane can be written as [37]

$$H^2 + \frac{k}{a^2} = \frac{1}{3\mu^2} \left[ \rho + \rho_0 (1 + \epsilon A(\rho, a)) \right],$$

(2)

where $H$ is the Hubble parameter, $k$ is the constant curvature of the three-space of the FRW metric, $\rho$ is the total energy density and $\mu$ is a parameter denoting the strength of the induced gravity on the brane. For $\epsilon = -1$, the brane tension can be assumed to be positive, while for $\epsilon = +1$, it is negative. $A$ is given by

$$A = \left[ A_0^2 + \frac{2\eta}{\rho_0} \left( \rho - \mu^2 \frac{E_0}{a^4} \right) \right]^{1/2},$$

(3)

where

$$A_0 = \sqrt{1 - 2\eta \frac{\mu^2 \Lambda}{\rho_0}}, \quad \eta = \frac{6m_0^5}{\rho_0 \mu^2} \quad (0 < \eta \leq 1), \quad \rho_0 = m_\Lambda^4 + \frac{6m_0^5}{\mu^2},$$

(4)
with $\Lambda$ defined as

$$\Lambda = \frac{1}{2} \left( ^{(5)} \Lambda + \frac{1}{6} \kappa_5 \lambda^2 \right).$$

Here $\kappa_5$ is the 5-dimensional Newton constant, $^{(5)} \Lambda$ the 5-dimensional cosmological constant in the bulk, $\lambda$ the brane tension, and $E_0$ a constant related to Weyl radiation. For simplicity, we will neglect the dark radiation term and restrict ourselves to the Randall-Sundrum critical case, i.e. $\Lambda = 0$, then Eq.(2) simplifies to

$$H^2 + \frac{k}{a^2} = \frac{1}{3\mu^2} \left( \rho + \rho_0 + \epsilon \rho_0 \sqrt{1 + 2 \eta \rho_0} \right).$$

Since in the very early era of the universe, the total energy density should be very high. Thus, we will, in the following, only consider the ultra high energy limit, $\rho \gg \rho_0$. In addition, we are interested in a closed universe, so we set the constant curvature $k$ to be $+1$. As a result, the Friedmann equation reduces to

$$H^2 = \frac{1}{3\mu^2} (\rho + \epsilon \sqrt{2\rho_0}) - \frac{1}{a^2}.$$  \hspace{1cm} (7)

This describes a 4-dimensional gravity with minor corrections, which implies that $\mu$ must have an energy scale as the Planck scale in the DGP model.

The energy density $\rho$ of a perfect fluid in the universe satisfies the conservation equation

$$\dot{\rho} = -3H(1 + w)\rho,$$  \hspace{1cm} (8)

where $w = \frac{p}{\rho}$ is the equation of state of the perfect fluid. A constant $w$ is considered in the present paper, which is a good approximation if the perfect fluid is a scalar field and the variation of the potential of scalar field is very slow with time.

Differentiating Eq. (7) with respect to cosmic time, one gets

$$\dot{H} = -\frac{1}{2\mu^2} (\rho + p) \left( 1 + \epsilon \sqrt{\frac{\rho_0}{2\rho}} \right) + \frac{1}{a^2},$$  \hspace{1cm} (9)

Combining this equation with the Friedmann equation given in Eq. (7), we have

$$\frac{\dot{a}}{a} = -\frac{1}{2\mu^2} (\rho + p) \left( 1 + \epsilon \sqrt{\frac{\rho_0}{2\rho}} \right) + \frac{1}{3a^2} (\rho + \epsilon \sqrt{2\rho_0}).$$  \hspace{1cm} (10)
III. THE EINSTEIN STATIC SOLUTION

The Einstein static solution is given by the conditions $\dot{a} = 0$ and $\ddot{a} = 0$, which imply

$$a = a_{Es}, \quad H(a_{Es}) = 0.$$  \hfill (11)

At the critical point determined by above conditions, we find, using Eq. (10)

$$\sqrt{\rho_{Es}} = \frac{\epsilon \sqrt{2 \rho_0 (1 - 3 \omega)}}{2(1 + 3 \omega)},$$  \hfill (12)

which means that in this dynamical system, there is only one Einstein static state solution. In order to guarantee the physical meaning of $\rho_{Es}$, it is necessary that

$$\frac{\epsilon (1 - 3 \omega)}{1 + 3 \omega} \geq 0.$$  \hfill (13)

Substituting Eq. (12) into the Friedmann equation, we obtain at the critical point

$$\frac{1}{a_{Es}^2} = \frac{\rho_0 (1 - 3 \omega) (1 + \omega)}{2 \mu^2 (1 + 3 \omega)^2},$$  \hfill (14)

with the requirement $(1 - 3 \omega)(1 + \omega) > 0$.

Before analyzing the stability of the critical point, we want to express Eq. (10) in terms of $a$ and $H$. To do so, we put the Friedmann equation in a different way

$$\sqrt{\rho} = \frac{\sqrt{2}}{2} \left( - \epsilon \sqrt{\rho_0} + \sqrt{\rho_0 + 6 \mu^2 \left( H^2 + \frac{1}{a^2} \right)} \right).$$  \hfill (15)

Thus Eq. (10) can be re-written as

$$\frac{\ddot{a}}{a} = -\frac{1}{4 \mu^2} (1 + \omega) \rho_0 + \frac{1}{4 \mu^2} \epsilon (1 + \omega) \sqrt{\rho_0^2 + 6 \mu^2 \rho_0 \left( H^2 + \frac{1}{a^2} \right) - \frac{1}{2} (1 + 3 \omega) \left( H^2 + \frac{1}{a^2} \right)}.$$  \hfill (16)

Now we study the stability of the critical point. For convenience, we introduce two variables

$$x_1 = a, \quad x_2 = \dot{a}.$$  \hfill (17)

It is then easy to obtain the following equations

$$\dot{x}_1 = x_2,$$  \hfill (18)
\[ x_2 = -\frac{1}{4\mu^2} \rho_0 (1 + w) x_1 + \frac{1}{4\mu^2} \epsilon(1 + \omega) \sqrt{\rho_0^2 x_1^2 + 6\mu^2 \rho_0 (1 + x_2^2)} - \frac{1}{2} (1 + 3\omega) \frac{x_2^2 + 1}{x_1}. \]  \hspace{1cm} (19)

In these variables, the Einstein static solution corresponds to the fixed point, \( x_1 = a_{Es}, \ x_2 = 0 \). The stability of the critical point is determined by the eigenvalue of the coefficient matrix resulting from linearizing the system described by above two equations near the critical point. Using \( \lambda^2 \) to denote the eigenvalue, we have

\[ \lambda^2 = \frac{\rho_0}{8\mu^2} \epsilon(1 + \omega)|1 + 3\omega| - \frac{3\rho_0 \omega(1 + \omega)}{2\mu^2} \frac{1}{1 + 3\omega}. \]  \hspace{1cm} (20)

If \( \lambda^2 < 0 \), the corresponding equilibrium point is a center point otherwise it is a saddle one. In order to analyze the stability of the critical point in detail, we now divide our discussions into two cases, i.e., \( \epsilon = -1 \) and \( \epsilon = 1 \).

**A. \( \epsilon = 1 \)**

In this case, \(-\frac{1}{3} < \omega < \frac{1}{3}\) is required to ensure that the critical point is physically meaningful. It then follows that \( \lambda^2 > 0 \), which means that this critical point is a saddle point. Thus, there is no stable Einstein static solution, and an emergent universe is not realistic in this case.

**B. \( \epsilon = -1 \)**

Now the requirement for the critical point to be physically meaningful is \(-1 < \omega < -\frac{1}{3}\). This exactly agrees with the condition of stability \( (\lambda^2 < 0) \). Hence, as long as the critical point exists, it is always stable. So, if the scale factor satisfies the condition given in Eq. (14) initially and \( w \) is within the region of stability, the universe can stay at this stable state past-eternally, and may undergo a series of infinite, non-singular oscillations, as shown in Fig. (1). As a result, the big bang singularity can be avoided successfully.

**IV. LEAVING THE EINSTEIN STATIC STATE**

Now, we have shown that an stable Einstein static state exists in the \( \epsilon = -1 \) branch. However, in order to have a successful cosmological scenario, a graceful exit to an inflationary epoch is needed. This is possible in the following sense. In the analysis carried
out in the present paper, the equation of state $w$ of the perfect fluid in the universe is assumed to be a constant, and this is a good approximation if the energy component in the early universe is only that of a minimally coupled scalar field with a self-interaction potential. One can show that the kinetic energy and potential energy of this scalar field should be both non-zero constants for an Einstein static solution \[3, 4, 7, 10\]. That is to say, the scalar field rolls along a plateau potential. However a realistic inflationary model clearly requires the potential to vary as the scalar field evolves. Thus, the constant potential is merely a past-asymptotic limit of a smoothly varying one, as pointed out in Refs. \[3, 4, 6, 7, 10\]. So, the essentially slowly varying potential will eventually break the equilibrium of the Einstein static state and lead to an exit from the initial Einstein phase to an inflationary one. Some specific forms of such a potential that implements these features have been constructed in Refs \[3, 4, 7\].

V. CONCLUSIONS

In this paper, we have studied the existence and stability of the Einstein static universe in the DGP braneworld scenario. By assuming the existence of a perfect fluid with a constant equation of state, which is a good approximation if the perfect fluid is a scalar field and the variation of the potential of scalar field is very slow with time, we have shown...
that for the branch with $\epsilon = 1$, there is no stable Einstein static universe, whereas, for
the branch with $\epsilon = -1$, the Einstein static universe exists and it is stable if the equation
of state $w$ satisfies $-1 < \omega < -\frac{1}{3}$. Thus, the universe can stay at this stable state past-
eternally, and may undergo a series of infinite, non-singular oscillations. Hence, in the
$\epsilon = -1$ branch of the DGP model, the universe can originate from an Einstein static state
and then enter an inflation era. Furthermore, the universe can exit, spontaneously, this
inflation phase to a radiation dominated era, as shown in previous studies [30–35]. As
a result, the big bang singularity problem in the standard cosmological scenario can be
resolved successfully.

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