ADAPTIVE TIME-STEPPING FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH NON-LIPSCHITZ DRIFT

STUART CAMPBELL AND GABRIEL LORD

Abstract. We introduce an explicit, adaptive time-stepping scheme for simulation of SPDEs with non-Lipschitz drift coefficients. Strong convergence is proven for the full space-time discretisation with multiplicative noise by considering the space and time discretisation separately. Adapting the time-step size to ensure strong convergence is shown numerically to produce more accurate solutions when compared to alternative fixed time-stepping strategies for the same computational effort.

1. Introduction

We construct an adaptive time-stepping method to ensure strong convergence of the numerical approximation to a semilinear stochastic partial differential equation (SPDE) with non-globally Lipschitz drift and multiplicative noise. We consider SPDEs of the form

\[ dX = [-AX + F(X)] \, dt + B(X) \, dW, \quad X(0) = X_0, \quad t \in [0, T], \quad T > 0, \]

on a separable Hilbert space \( H = L^2(\mathcal{D}) \), with domain \( \mathcal{D} \subset \mathbb{R}^d \). The linear operator \(-A\) is assumed to be the generator of an analytic semigroup, \( S(t) := e^{-tA} \). Under additional assumptions, given in Section 2, existence and uniqueness of the mild solution,

\[ X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s)) \, ds + \int_0^t S(t-s)B(X(s)) \, dW(s), \]

can be proved. It is known [7] that the standard Euler-Maruyama method for stochastic differential equations (SDEs) does not converge strongly if the drift does not satisfy a global Lipschitz condition. Unfortunately many applications of interest for both SDEs and SPDEs have non-Lipschitz drift functions. There are several directions that have been taken to ensure strong convergence in the case of non-globally Lipschitz drift. Firstly, implicit methods are strongly convergent for SDEs with non-Lipschitz drift, however are more computationally expensive than explicit methods. This can render them impractical, particularly for large-scale Monte-Carlo estimations or inverse problems. For this reason there has been much interest in the construction of explicit methods that ensure strong convergence under non-Lipschitz drift.

For particular SPDEs with non-Lipschitz drift, there several schemes that ensure convergence, for example in [2] and [17], strong convergence results are proven for the Allen-Cahn equation with additive space-time white noise.

Strong convergence of a spatial semi-discrete approximation is achieved for general one-sided Lipschitz drift and additive noise via random PDE equivalence in [3]. Convergence in probability was proven in [1] for the one dimensional non-linear heat equation. We make the distinction in this paper that we do not restrict to any specific drift function and consider multiplicative noise for the full spatial-temporal discretisation.

For this general case, to our knowledge, there are two existing explicit schemes that ensure strong convergence. Firstly tamed methods, introduced initially for SDEs in [8], perturb the drift to ensure boundedness of the (numerical) solution moments. Roughly speaking, the perturbation

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of the drift ensures linear growth of the new drift function and the perturbed drift function converges to the original drift function at the time-step is decreased towards zero. Convergence and stability of a tamed Euler scheme for SPDEs was proved in the variational setting in [5].

An alternative to taming the drift was introduced for SPDEs in [9], where the contributions of the nonlinear drift and diffusion are “switched off” if the nonlinearities are too large when compared to the scheme’s time-step. The switching behaviour ensures control of the numerical scheme and allows for strong convergence to be proven.

In this paper, we propose an adaptive time-stepping method for SPDEs that ensures strong convergence in the presence of non-Lipschitz drift. This is achieved by reducing the time-step of the scheme when the numerical drift response is large, ensuring at most linear growth of the numerical drift response with respect to the solution of the numerical scheme. The underlying numerical method is based on a finite element or spectral Galerkin exponential integrator, which was considered for multiplicative noise with a fixed time-step in e.g. [13], [15].

A similar adaptive time-stepping scheme for SDEs was first introduced in [10]. There, it was shown that such an adaptive method can result in more accurate simulations when compared to tamed, fixed time-step methods and therefore lead to more efficient multi-level Monte-Carlo simulations.

The content of the paper is organised as follows, in Section 2 we formalise the setting and assumptions made, as well as stating some well known properties of the semigroup $S(t)$. We define what we call an admissable time-stepping strategy in Section 3. This ensures there is no unwanted growth in the numerical solution and is key to proving our main results of strong convergence in Section 6. Due to the nature of the time-stepping strategies introduced, we require a conditional Itô Isometry and conditional regularity in time of the mild solution, which we prove in Section 4. Section 5 is a brief overview of the necessary Galerkin methods for the spatial discretisation and we conclude with some numerical comparisons to alternative methods in Section 7.

2. Setting and Assumptions

We consider the SPDE in (1.1) on a Hilbert space $H$ with norm $\|\cdot\|$. The linear operator $-A$ is assumed to be the generator of an analytic semigroup, $S(t) := e^{-tA}$ for $t \geq 0$. Both $F$ and $B$ are possibly nonlinear in $X$, with $F$ satisfying a monotone condition and a polynomial bound. The diffusion coefficient $B$ is assumed to be globally Lipschitz. We define a $Q$-Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$, taking values in a Hilbert space $U$, with covariance operator $Q \in \mathcal{L}(U)$. We write $W(t)$ as

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j} \chi_j \beta_j(t),$$

where $\{\chi_j\}_{j \in \mathbb{N}}$ are the eigenfunctions of $Q$ that form an orthonormal basis of $U$ (see [4, 12]). The $q_j$ are real eigenvalues of $Q$ such that $q_j \geq 0$ and $\sum_{j=1}^{\infty} q_j < \infty$. Finally the $\beta_j$ are i.i.d. $\mathcal{F}_t$-adapted one-dimensional Brownian motions. To complete the setting for the diffusion coefficient $B$, we define a subset of the Hilbert space $U$ as $U_0 := \{Q^{1/2}u : u \in U\}$. We take $B \in L^2_0$, the set of Hilbert-Schmidt operators from $U_0$ to $H$. That is $B : U_0 \rightarrow H$ satisfies

$$\|B\|_{L^2_0} := \left(\sum_{j=1}^{\infty} \|BQ^{1/2}\chi_j\|^2\right)^{1/2} < \infty,$$

with $\{\chi_j\}$ an orthonormal basis of $U$.

We are interested in strong convergence of the numerical scheme to the true solution, that is convergence with respect to the norm

$$\|x\|_{L^2(\Omega; H)} := \left(\mathbb{E}[\|x\|^2]\right)^{1/2}.$$
During the analysis we need to consider conditional expectations in time to ensure the adaptive scheme maintains normally distributed Weiner increments and does not introduce bias. Therefore we introduce the following notation

$$\|x\|_{L^2(\Omega|G_n;H)} := \left( \mathbb{E} \left[ \|x\|^2 \mid G_n \right] \right)^{\frac{1}{2}},$$

which is required when discussing the full space-time convergence of the adaptive scheme. We now formalise the above assumptions and state some standard properties of the semigroup $S(t)$.

**Assumption 2.1.** The operator $A : \mathcal{D}(A) \to H$ is the generator of an analytic semigroup $S(t) = e^{-tA}, t \geq 0$.

Let $\lambda_j > 0$ and $\phi_j$ be the eigenvalues and eigenfunctions of $A$ with $\lambda_j \leq \lambda_{j+1}$. For $\alpha \in \mathbb{R}$ we define the fractional power operator $A^\alpha : \mathcal{D}(A^\alpha) \to H$ as

$$A^\alpha u := \sum_{j=1}^{\infty} \lambda_j^\alpha u_j \phi_j,$$

for all $u = \sum_{j=1}^{\infty} u_j \phi_j$ such that $A^\alpha u \in H$.

**Lemma 2.2.** Under Assumption 2.1 the following bounds hold for $0 \leq \alpha \leq 1$, $\beta \geq 0$ and some $C > 0$,

$$\|(−A)^\beta S(t)\|_{\mathcal{L}(H)} \leq Ct^{-\beta} \quad \text{for } t > 0$$

$$\|((−A)^{-\alpha}(I−S(t)))\|_{\mathcal{L}(H)} \leq Ct^\alpha \quad \text{for } t \geq 0.$$

**Proof.** See [6].

For $v \in \mathcal{D}((-A)^{\alpha/2})$, $\alpha \in \mathbb{R}$, we use throughout the notation $\|v\|_\alpha := \|(−A)^{\alpha/2}v\|$. Under this norm, $\mathcal{D}((-A)^{\alpha/2})$ becomes a Hilbert space. We apply Lemma 2.2 to attain the following useful inequality

$$(2.1) \quad \|(I−S(t))v\|^2 = \|((−A)^{-\alpha/2}(I−S(t))(−A)^{\alpha/2}v\|^2 \leq Ct^\alpha \|v\|_\alpha^2,$$

for $v \in \mathcal{D}((-A)^{\alpha/2})$ and $0 \leq \alpha \leq 1$.

The usual global Lipschitz assumption on $F$ is weakened to the following one-sided Lipschitz condition and a polynomial bound on the derivative.

**Assumption 2.3.** The function $F$ satisfies a one sided Lipschitz growth condition. That is, there exists a constant $\alpha > 0$ such that for all $X, Y \in H$,

$$(F(X) − F(Y), X − Y) \leq \alpha \|X − Y\|^2.$$

**Assumption 2.4.** The derivative of $F$ is bounded polynomially, that is

$$\|DF(X)\|_{\mathcal{L}(H)} \leq c_1(1 + \|X\|^{c_2}),$$

for some $c_1, c_2 \in (0, \infty)$.

Under Assumption 2.4, the growth of $F$ can be bound polynomially by

$$(2.2) \quad \|F(X)\| \leq c_3(1 + \|X\|^{c_2+1}),$$

with $c_3 := 2c_1 + \|F(0)\|$. A global Lipschitz assumption on the diffusion term $B$ is assumed throughout this work.

**Assumption 2.5.** The diffusion coefficient $B$ satisfies a global Lipschitz growth condition, i.e. there exists a constant $L_1 > 0$ such that

$$\|B(X) − B(Y)\|_{L^2} \leq L_1 \|X − Y\|, \quad \forall X, Y \in H.$$
Assumption 2.6. For \( r \in (0, 1) \) and some constant \( L_2 > 0 \) the following bound holds
\[
\left\| \left( -A \right)^{r/2} B(X) \right\|_{L^p} \leq L_2(1 + \| X \|_r), \quad \text{for all} \, X \in \mathcal{D}(\left( -A \right)^{r/2}).
\]

The parameter \( r \) in Assumption 2.6 determines the spatial regularity of the mild solution. The classical result of \cite{4} proves existence and uniqueness for the mild solution of \eqref{eq:1.1} under global Lipschitz assumption on the drift function \( F \). We require a similar result for the more general assumptions on \( F \), as shown in \cite{9}.

Theorem 2.7. Under Assumptions 2.1 and 2.3 to 2.6 with initial data \( X_0 \in L^2(\mathbb{D}, \mathcal{D}(\left( -A \right)^{1/2})) \), there exists a mild solution, unique up to equivalence given by
\[
X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s),
\]
Further that \( p \geq 2 \) and \( \eta \in (0, 1) \),
\[
\sup_{t \in [0,T]} E \| \| X(t) \|_p^\mathcal{G} < \infty.
\]

Proof. See \cite{9}, in particular Proposition 7.3 for the moment bounds. \( \Box \)

3. Adaptive time-stepping and admissible strategies

In this section we define the adaptive time-stepping strategy and outline the conditions required for the strong convergence of the numerical approximation.

Definition 3.1. Let each member of the family \( \{ t_n \}_{n \in \mathbb{N}} \) be an \( \mathcal{F}_t \) - stopping time, that is \( \{ t_n \leq t \} \in \mathcal{F}_t \) for all \( t \geq 0 \) with \( (\mathcal{F}_t)_{t \geq 0} \) as the natural filtration of \( W \). We define the discrete-time filtration \( \{ \mathcal{G}_n \}_{n \in \mathbb{N}} \) as
\[
\mathcal{G}_n := \{ B \in \mathcal{F} : \mathcal{B} \cap \{ t_n \leq t \} \in \mathcal{F}_t \}, \quad n \in \mathbb{N}.
\]

Assumption 3.2. Suppose each \( \Delta t_n := t_n - t_{n-1} \) is \( \mathcal{G}_{n-1} \)-measurable and \( N \) is a random integer such that
\[
N := \max \{ n \in \mathbb{N} : t_{n-1} < T \} \quad \text{and} \quad t_N = T.
\]

We constrain the sizes of the minimum and maximum allowed time-steps, which ensures the numerical method terminates in a finite number of steps and to prove convergence as \( \Delta t_{\max} \to 0 \). We impose a \( \Delta t_{\min} \) and a \( \Delta t_{\max} \) such that
\[
\Delta t_{\max} = \rho \Delta t_{\min}, \quad \text{for} \quad \rho \geq 1.
\]

By construction, the random time-step \( \Delta t_{n+1} \) being \( \mathcal{G}_n \)-measurable ensures the Weiner increments \( \Delta \beta_j := \beta_j(t_{n+1}) - \beta_j(t_n) \) are \( \mathcal{G}_n \)-conditionally normally distributed with mean zero and variance \( \Delta t_{n+1} \). That is
\[
\mathbb{E} [ \Delta \beta_{j+1} \| \mathcal{G}_n ] = 0, \quad \mathbb{E} [ \| \Delta \beta_{j+1} \|^2 \| \mathcal{G}_n ] = \Delta t_{n+1}.
\]

Without this construction, we may not assume the increments \( \Delta \beta_j(t_{n+1}) \) are normally distributed as the chosen step-size \( \Delta t_{n+1} \) would depend on the solution value at \( t_n \). The final ingredient for the adaptive time-stepping strategy is the admissibility of the time-stepping strategy, which we now define as in \cite{10}.

Definition 3.3. Let \( \{ X_n \} \) be an approximate numerical solution of \eqref{eq:1.1}. The time-stepping of the numerical solution \( \{ \Delta t_n \}_{n \in \mathbb{N}} \) is called an admissible time-stepping strategy if Assumption 3.2 is satisfied and there exists non-negative constants \( R_1 \) and \( R_2 \) such that
\[
\| F(X_n) \|_2^2 \leq R_1 + R_2 \| X_n \|_2^2, \quad n = 0, 1, ..., N - 1,
\]
for \( \Delta t_{\min} < \Delta t_n \leq \Delta t_{\max} \).
The requirement to restrict the minimum time-step size implies we cannot always ensure control of the drift response via time-step selection - without a potentially infinite number of time-steps. To construct a viable scheme we require a backstop scheme to run over a time-step of size $\Delta t_{\text{min}}$ whenever the scheme selects a time-step size less than or equal to $\Delta t_{\text{min}}$. The backstop may be any method that converges strongly and for our numerical simulations in Section 7, we use the so called nonlinearities-stopped exponential Euler of [9], which we refer to as NSEE. We can relate the admissible time-stepping exponential Euler of \[3.4\], which we refer to as NSEE.

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4. Conditional Itô Isometry and Regularity of Mild Solution

In this section, we give a conditional Itô Isometry for the $Q$-Wiener process and prove a conditional regularity result for the mild solution, which is required for the analysis of the numerical method.

**Lemma 4.1.** Let $B : U_0 \rightarrow H$ be an $L^2_0$-valued predictable process such that
\[
\int_0^T \mathbb{E} \left[ \| B(s) \|_{L^2_0}^2 \right] ds < \infty,
\]
and let $\mu, \tau$ be two stopping times with $0 \leq \mu \leq \tau \leq T$. Then the following identities hold,
\[
\Phi := \mathbb{E} \left[ \left\| \int_\mu^\tau B(s) dW(s) \right\|^2 \middle| \mathcal{F}_\mu \right] = \mathbb{E} \left[ \int_\mu^\tau \| B(s) \|_{L^2_0}^2 ds \middle| \mathcal{F}_\mu \right] = 0.
\]

**Proof.** The proof is an expansion of the stochastic term $\Phi$ as an inner product in a basis of $H$, similarly to Theorem 10.16 in [12]. Then an application of Theorem 5.21 and Lemma 5.22 from [14] to the individual terms of the expansion yields the result. \qed

We now prove a conditional regularity in time result for the mild solution, similar to standard regularity results for Lipschitz drift but of lower order. The conditional expectation is required due to the nature of the adaptive time-stepping scheme.

**Lemma 4.2.** Let $X$ be the mild solution given by (2.3) and that Assumptions 2.1, 2.3, 2.4 and 2.6 hold. Let $t_n, t_{n+1} \in [0, T]$ be such that $\Delta t_{n+1} := t_{n+1} - t_n > 0$ and $\epsilon > 0$. The following conditional regularity estimates holds for an a.s. finite, $\mathcal{G}_n$-measurable random variable $K_1$:
\[
\| X(t_{n+1}) - X(t_n) \|_{L^2(\Omega; \mathcal{G}_n; H)}^2 \leq K_1 \Delta t_{n+1}^{1-\epsilon} \text{ a.s., } \mathbb{E}[K_1] < \infty.
\]

**Proof.** The solution $X(t_n) \in D((-A)^{\frac{\eta}{2}})$ for $\eta \in [0, 1)$ by (2.3). We can decompose the error terms between any two elements $u, v \in D((-A)^{\frac{\eta}{2}})$ using Lemma 2.2, in particular (2.1) and the triangle inequality as follows
\[
\| u - v \|^2 = \| u - S(\Delta t_{n+1})v + S(\Delta t_{n+1})v - v \|^2 \\
\leq 2 \| u - S(\Delta t_{n+1})v \|^2 + 2 \| (I - S(\Delta t_{n+1}))v \|^2 \\
\leq 2 \| u - S(\Delta t_{n+1})v \|^2 + 2C \Delta t_{n+1} \| v \|^2_{\eta}.
\]

Setting $u = X(t_{n+1})$ and $v = X(t_n)$ we have,
\[
\| X(t_{n+1}) - X(t_n) \|^2 \leq 2C \Delta t_{n+1}^\eta \| X(t_n) \|^2_{\eta} + \| X(t_{n+1}) - S(\Delta t_{n+1})X(t_n) \|^2 \leq 2 \left( C \Delta t_{n+1}^\eta \| X(t_n) \|^2_{\eta} + \right.
\]
\[
+ 2 \left\| \int_{t_n}^{t_{n+1}} S(t_{n+1} - s)F(X(s))ds \right\|^2 +
\]
\[
+ 2 \left\| \int_{t_n}^{t_{n+1}} S(t_{n+1} - s)B(X(s))dW(s) \right\|^2 \right)
\]
\[
:= 2 \left( \| I \|^2 + \| II \|^2 + \| III \|^2 \right).
\]

The first term is directly bound by (2.3) with $\eta = 1 - \epsilon$,
\[
\| I \|_{L^2(\Omega; \mathcal{G}_n; H)}^2 \leq C \Delta t_{n+1}^{1-\epsilon} \sup_{s \in [0, T]} \mathbb{E} \left[ \| X(s) \|^2_{1-\epsilon} \middle| \mathcal{G}_n \right].
\]
For part II, we apply Jensen’s inequality, the boundedness of the semigroup, (2.2) and Theorem 2.7
\[
\|II\|_{L^2(\Omega;\mathcal{G}_n;H)}^2 = \left\| \int_{t_n}^{t_{n+1}} S(t_{n+1} - s)F(X(s))ds \right\|_{L^2(\Omega;\mathcal{G}_n;H)}^2
\leq \Delta t_{n+1} \int_{t_n}^{t_{n+1}} \|S(t_{n+1} - s)F(X(s))\|_{L^2(\Omega;\mathcal{G}_n;H)}^2 ds
\leq C\Delta t_{n+1} \int_{t_n}^{t_{n+1}} \|F(X(s))\|_{L^2(\Omega;\mathcal{G}_n;H)}^2 ds
\leq C\Delta t_{n+1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ (1 + \|X(s)\|^{c+1})^2 \right] ds
\leq C\Delta t_{n+1}^2 \sup_{s \in [0,T]} \mathbb{E} \left[ (1 + \|X(s)\|^{2c+2}) \right] ds.
\] (4.6)

For part III we show, via Lemma 4.1, boundedness of the semigroup, Assumption 2.6 and Theorem 2.7, that
\[
\|III\|_{L^2(\Omega;\mathcal{G}_n;H)}^2 = \left\| \int_{t_n}^{t_{n+1}} S(t_{n+1} - s)B(X(s))dW(s) \right\|_{L^2(\Omega;\mathcal{G}_n;H)}^2
= \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|S(t_{n+1} - s)B(X(s))\|_{L^2}^2 ds \right] \mathcal{G}_n
\leq C \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ (1 + \|X(s)\|)^2 \right] ds
\leq C\Delta t_{n+1} \sup_{s \in [0,T]} \mathbb{E} \left[ 1 + \|X(s)\|^2 \right] ds.
\] (4.7)

We can then define the a.s. finite \(\mathcal{G}_n\)-measurable random variable \(K_1\) as
\[
K_1 := C \left( \sup_{s \in [0,T]} \mathbb{E} \left[ \|X(s)\|_1 \right] \mathcal{G}_n \right) + \sup_{s \in [0,T]} \mathbb{E} \left[ (1 + \|X(s)\|^{2c+2}) \right] \mathcal{G}_n
+ \sup_{s \in [0,T]} \mathbb{E} \left[ 1 + \|X(s)\|^2 \right] \mathcal{G}_n).
\]

Together, Equations (4.5) to (4.7) imply
\[
\|X(t_{n+1}) - X(t_n)\|_{L^2(\Omega;\mathcal{G}_n;H)} \leq K_1 \Delta t_{n+1}^{1-\epsilon} \quad a.s.,
\]
and Theorem 2.7 implies that \(\mathbb{E}[K_1] < \infty\). \(\square\)

For the analysis of the numerical scheme, we require conditional bounds on the remainder terms when we Taylor expand the drift and diffusion terms.

**Lemma 4.3.** Define the remainder terms of \(F\) and \(B\) from the Taylor expansions as follows,
\[
F(X(s)) = F(X(t_n)) + R_F(s, t_n, X(t_n))
B(X(s)) = B(X(t_n)) + R_B(s, t_n, X(t_n)),
\]
with \(R_F\) and \(R_B\) given by
\[
R_z(s, t_n, X(t_n)) := \int_0^1 Dz \left( X(t_n) + \tau(X(s) - X(t_n)) \right) (X(s) - X(t_n)) d\tau,
\]
for $z$ as either $F$ or $B$. Then the following bounds hold for a.s. finite $\mathcal{G}_n$-measurable random variables $K_2, K_3, K_4$

\[
\begin{align*}
i) & \quad \mathbb{E} \left[ \left\| \int_{t_n}^{t_{n+1}} R_F(s, t_n, X(t_n)) \, ds \right\| \mathcal{G}_n \right] \leq K_2 \Delta t_n^{\frac{3}{2} - \epsilon} \text{ a.s.}, \\
ii) & \quad \mathbb{E} \left[ \left\| \int_{t_n}^{t_{n+1}} R_F(s, t_n, X(t_n)) \, ds \right\|^2 \mathcal{G}_n \right] \leq K_3 \Delta t_n^{3 - \epsilon} \text{ a.s.}, \\
iii) & \quad \mathbb{E} \left[ \left\| \int_{t_n}^{t_{n+1}} R_B(s, t_n, X(t_n)) \, dW(s) \right\|^2 \mathcal{G}_n \right] \leq K_4 \Delta t_n^{2 - \epsilon} \text{ a.s.}, \\
iv) & \quad \mathbb{E} [K_i] < \infty \quad i = 2, 3, 4.
\end{align*}
\]

**Proof.** Using Assumption 2.4 on $F$ and the Cauchy-Schwarz inequality we have

\[
\begin{align*}
\mathbb{E} \left[ \left\| R_F(s, t_n, X(t_n)) \right\| \mathcal{G}_n \right] & \leq \mathbb{E} \left[ \int_0^1 c_1 (1 + \| X(t_n) \| + \| X(s) - X(t_n) \|^2) \| X(s) - X(t_n) \| \, d\tau \mathcal{G}_n \right] \\
& \leq c_1 \| (X(s) - X(t_n)) \|_{L^2(\Omega; \mathcal{G}_n; H)} \\
& \quad \times \sqrt{\mathbb{E} \left[ \int_0^1 c_1 (1 + \| X(t_n) \| + \| X(s) - X(t_n) \|^2) \, d\tau \mathcal{G}_n \right]}. 
\end{align*}
\]

We can then use Lemma 4.2 to bound the first part a.s. and define an a.s. finite $\mathcal{G}_n$-measurable random variable

\[
M_1 := \sup_{s \in [0, T]} \mathbb{E} \left[ 2c_1^2 + 2c_1^2 \| X(s) \|^2 \mathcal{G}_n \right],
\]

to bound the second term. Which yields

\[
\mathbb{E} \left[ \left\| R_F(s, t_n, X(t_n)) \right\| \mathcal{G}_n \right] \leq \sqrt{K_1 M_1 (s - t_n)^{\frac{3}{2} - \epsilon}} \text{ a.s.}
\]

Therefore,

\[
\begin{align*}
\mathbb{E} \left[ \left\| \int_{t_n}^{t_{n+1}} R_F(s, t_n, X(t_n)) \, ds \right\| \mathcal{G}_n \right] & \leq \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \left\| R_F(s, t_n, X(t_n)) \right\| \mathcal{G}_n \right] \, ds \\
& \leq \sqrt{K_1 M_1} \int_{t_n}^{t_{n+1}} (s - t_n)^{\frac{3}{2} - \epsilon} \, ds \text{ a.s.} \\
& = K_2 \Delta t_n^{\frac{3}{2} - \epsilon} \text{ a.s.}
\end{align*}
\]

where $K_2 := \sqrt{K_1 M_1}$. 

For part ii), we apply Jensen’s inequality

\[
\left\| \int_{t_n}^{t_{n+1}} R_F(s, t_n, X(t_n)) \, ds \right\|_{L^2(\Omega; \mathcal{G}_n; H)}^2 \leq \Delta t_n \int_{t_n}^{t_{n+1}} \left\| R_F(s, t_n, X(t_n)) \right\|_{L^2(\Omega; \mathcal{G}_n; H)}^2 \, ds,
\]

and consider $\left\| R_F(s, t_n, X(t_n)) \right\|_{L^2(\Omega; \mathcal{G}_n; H)}^2$. Define

\[
D_q(s) := c_1 (1 + 2^{2q+1} \| X(t_n) \|^2 + 2c_2^2 \| X(s) \|^2)^{2q-1} \| X(s) - X(t_n) \|,
\]

where $c_1$ and $c_2$ are defined in Assumption 2.4. We note that $D_q$ satisfies the following relation

\[
D_q^2(s) = D_{q+1}(s) \| X(s) - X(t_n) \| .
\]
Applying Young's inequality to $\mathbb{E}[D_q(s) \| X(s) - X(t_n)\|]$ we see that
\[
\mathbb{E}[D_q(s) \| X(s) - X(t_n)\|] \leq \mathbb{E}[\| X(s) \|^2 + \| X(t_n) \|^2]
\]
\[
+ \frac{c_1^2}{2} (1 + 2^{\varphi+1} \| X(t_n) \|^\varphi + 2^{\varphi} \| X(s) \|^\varphi 2^q)
\]
\[
\leq \sup_{s \in [0,T]} \mathbb{E}[\| X(s) \|^2 + \| X(t_n) \|^2]
\]
\[
+ \frac{c_1^2}{2} (1 + 2^{\varphi+1} \| X(s) \|^\varphi + 2^{\varphi} \| X(s) \|^\varphi 2^q).
\]
\[
:= \bar{D}_q < \infty,
\]
due to Theorem 2.7. Going back to $\| R_F(s, t_n, X(t_n)) \|^2_{L^2(\Omega; \mathcal{G}_n; H)}$ we can apply the Cauchy-Schwarz inequality $q$ successive times, for $q > 1 - \log_2 \epsilon$ together with Lemma 4.2 to show that
\[
\| R_F(s, t_n, X(t_n)) \|^2_{L^2(\Omega; \mathcal{G}_n; H)} \leq \mathbb{E}[ D_1(s) \| X(s) - X(t_n) \| \mathcal{G}_n]
\]
\[
\leq \mathbb{E}[ D_q(s) \| X(s) - X(t_n) \| \mathcal{G}_n]^{1/(2^{q-1})}
\]
\[
\times \| X(s) - X(t_n) \|^2 x^{1/(2^{q-1})}
\]
\[
\leq \bar{D}_q^{1/(2^{q-1})} K_1^{1-\epsilon} (s - t_n)^{1-\epsilon} \text{ a.s.}
\]
(4.10)
Therefore
\[
\left\| \int_{t_n}^{t_{n+1}} R_F(s, t_n, X(t_n))ds \right\|^2_{L^2(\Omega; \mathcal{G}_n; H)} \leq \Delta t_{n+1} \int_{t_n}^{t_{n+1}} \| R_F(s, t_n, X(t_n)) \|^2_{L^2(\Omega; \mathcal{G}_n; H)} ds
\]
\[
\leq \bar{D}_q^{1/(2^{q-1})} K_1^{1-\epsilon} \Delta t_{n+1} \int_{t_n}^{t_{n+1}} (s - t_n)^{1-\epsilon} ds \text{ a.s.}
\]
\[
\leq K_3 \Delta t_{n+1}^{1-\epsilon}.
\]
For part iii) first apply the conditional Itô isometry, (4.1) then Assumption 2.6 which implies $DB$ is globally bound by the constant $L_1$ to show that
\[
\mathbb{E}\left[ \left\| \int_{t_n}^{t_{n+1}} R_B(s, t_n, X(t_n))dW(s) \right\|^2 \mathcal{G}_n \right]
\]
\[
= \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \int_0^1 \| DB(X(t_n) + \tau (X(s) - X(t_n))) \| (X(s) - X(t_n)) d\tau \|_{L^2_0}^2 ds \mathcal{G}_n \right]
\]
\[
\leq \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \int_0^1 \| DB(X(t_n) + \tau (X(s) - X(t_n))) \|_{L^2_0}^2 d\tau \| X(s) - X(t_n) \|^2 ds \mathcal{G}_n \right]
\]
\[
\leq L_1^2 \int_{t_n}^{t_{n+1}} \| X(s) - X(t_n) \|^2_{L^2(\Omega; \mathcal{G}_n; H)} ds.
\]
Finally, application of Lemma 4.2 yields
\[
\mathbb{E}\left[ \left\| \int_{t_n}^{t_{n+1}} R_B(s, t_n, X(t_n))dW(s) \right\|^2 \mathcal{G}_n \right] \leq L_1^2 \int_{t_n}^{t_{n+1}} \| X(s) - X(t_n) \|^2_{L^2(\Omega; \mathcal{G}_n; H)} ds
\]
\[
\leq L_1^2 K_1 \int_{t_n}^{t_{n+1}} (s - t_n)^{1-\epsilon} ds \text{ a.s.}
\]
\[
\leq K_4 \Delta t_{n+1}^{2-\epsilon} \text{ a.s.}
\]
\[
\square
5. Galerkin Methods

With these preliminary results in hand, we can now turn our attention to the numerical scheme. We present a brief overview of the relevant Galerkin methods required for the spatial discretisation of the SPDE problem in this section. A detailed presentation of these methods can be found in [16]. Let \((V_h)_{h \in (0,1)} \subset V := D((-A)^s)\) be a sequence of finite dimensional subspaces of \(V\) and define the Riesz projection \(R_h : V \rightarrow V_h\) by

\[
(-AR_h v, w_h) = a(v, w_h) := (A^{1/2}v, A^{1/2}w_h) \quad \text{for all } v \in V, w_h \in V_h.
\]

We require an assumption on the order of the approximation of elements in \(V\) by elements in \(V_h\), which we later show is satisfied by the usual finite element and spectral Galerkin methods.

**Assumption 5.1.** For \((V_h)_{h \in (0,1)} \subset V\) there exists a constant \(C\) such that

\[
\|R_h v - v\| \leq Ch^s \|v\|_s \quad \text{for all } v \in D((-A)^{s/2}), \ s \in \{1, 2\}, h \in (0,1).
\]

As usual we first consider the linear deterministic problem. That is, given \(u(0) = v\) find \(u \in V\) such that

\[
\frac{du}{dt} = Au, \quad \text{for } t \in (0, T).
\]

We define \(A_h : V_h \rightarrow V_h\) as the discrete version of the linear operator \(A\) via the relation

\[
(A_h v_h, w_h) = a(v_h, w_h) \quad \text{for all } v_h, w_h \in V_h.
\]

The discrete operator \(A_h\) is also the generator of an analytic semigroup \(S_h(t) := e^{-tA_h}\) and therefore satisfies similar smoothing properties as \(S(t)\) does in Lemma 2.2.

We require two additional projection operators in order to discretise the SPDE, firstly the orthogonal projection operator \(P_h : L^2(\Omega) \rightarrow V_h\) defined by

\[
(P_h v, w) = (v, w) \quad \text{for all } v \in L^2(\Omega), w \in V_h.
\]

Note that \(\|P_h v\| \leq \|v\|\) for all \(v \in L^2(\Omega)\) and therefore the operator norm \(\|P_h\|_{\mathcal{L}(H)} \leq 1\).

The second projection operator \(P_J\), is the projection of \(u \in L^2(\Omega)\) onto the first \(J\) terms of an orthonormal basis \(\{\chi_j\}\) of \(L^2(\Omega)\),

\[
P_J u := \sum_{j=1}^{N} (\chi_j, u)\chi_j.
\]

It is easily seen that \(\|P_J\|_{\mathcal{L}(H)} \leq 1\). We finish this section with two examples that fall under the above framework, firstly the standard finite element method and secondly the spectral Galerkin method.

**5.1. Finite Element Method.** Assume that \(\Omega \subset \mathbb{R}^d\) for \(d = 1, 2, 3\) and \(\Omega\) is a bounded convex domain that either has a smooth boundary or is a convex polygon. We define two spaces, \(\mathbb{H} \subset V\) which depend on the boundary conditions of the SPDE. For Dirichlet boundary conditions we have that

\[
V = \mathbb{H} = H^1_0(\Omega)
\]

and for Robin boundary conditions

\[
V = H^1(\Omega)
\]

\[
\mathbb{H} = \{v \in H^2(\Omega) \frac{\partial v}{\partial n} + c_r v = 0 \text{ on } \partial \Omega\}, \quad c_r \in \mathbb{R}.
\]

The space \(\mathbb{H}\) encodes the boundary conditions of the SPDE and the following characterisations of \(D((-A)^{s/2})\) for \(s \in \{1, 2\}\) are well known

\[
\|v\|_{H^s(\Omega)} \equiv \|v\|_s, \quad \text{for all } v \in D((-A)^{s/2})
\]

\[
D((-A)^{s/2}) = \mathbb{H} \cap H^s(\Omega) \quad (\text{Dirichlet B.C.s})
\]

\[
D((-A)) = \mathbb{H}, \quad D((-A)^{1/2}) = H^1(\mathbb{H}) \quad (\text{Robin B.C.s}).
\]
Let $T_h$ be a $d$-dimensional triangulation of the domain $\mathbb{D}$ satisfying the usual regularity assumptions with maximal mesh size $h$. Then $V_h \subset V$ is the space of continuous functions that are piecewise linear over the mesh $T_h$.

5.2. **Spectral Galerkin Method.** For spectral Galerkin to be applicable we have to assume that for the combination of domain $\mathbb{D} \subset \mathbb{R}^d$, operator $-A$, and boundary data, that the eigenvalues and orthonormal eigenfunctions are explicitly computable. For example when $\mathbb{D} = [0, a]$ with Dirichlet or Periodic boundary conditions and $-A$ is equal to the Laplacian. In this setting we can compute the eigenvalues $\lambda_j$ and eigenfunctions $\phi_j$ of $A$. By setting $h := \lambda_{J+1}^{-1/2}$ for some $J \in \mathbb{N}$, the spaces $V_h$ are then defined as

$$V_h := \text{span}\{\phi_j : j = 1, \ldots, J\}.$$ 

6. **Numerical Solution and Analysis**

With the notation of the previous section, the semi-discrete problem can be defined as the solution to

$$dX_h = [-A_h X_h + P_h F(X_h)] \, dt + P_h B(X_h) P_J \, dW$$

$$X_h(0) = P_h X_0.$$ 

Using the mild solution and the approximations in time,

$$F(X_h(s)) \approx F(X_h(t_n)), \quad s \in [t_n, t_{n+1}],$$

$$S_h(t_{n+1} - s) P_h B(X_h(s)) \approx S_h(\Delta t_{n+1}) P_h B(X_h(t_n)), \quad s \in [t_n, t_{n+1}],$$

leads to our adaptive numerical scheme ASETD1 (for $\Delta t_{n+1} > \Delta t_{\text{min}}$),

$$X_h^{n+1} = e^{-\Delta t_{n+1} A_h} (X_h^n + P_h B(X_h^n) P_J \Delta W_{n+1})$$

$$+ \Delta t_{n+1} \varphi_1(\Delta t_{n+1} A_h) P_h F(X_h^n),$$

with the standard $\varphi_1$ function defined by $\varphi_1(z) := \frac{e^z - 1}{z}$.

We can rewrite (6.3) as

$$X_h^{n+1} = X_h^n + P_h B(X_h^n) P_J \Delta W_{n+1}$$

$$+ \Delta t_{n+1} \varphi_1(\Delta t_{n+1} A_h) (A_h(X_h^n + P_h B(X_h^n) P_J \Delta W_{n+1}) + P_h F(X_h^n)),$$

in order to compute only a single matrix exponential at each time-step, which may be considerably more efficient for finite element spatial discretisations.

If instead we use the approximations

$$S_h(t_{n+1} - s) P_h F(X_h(s)) \approx S_h(\Delta t_{n+1}) P_h F(X_h(t_n)), \quad s \in [t_n, t_{n+1}],$$

$$S_h(t_{n+1} - s) P_h B(X_h(s)) \approx S_h(\Delta t_{n+1}) P_h B(X_h(t_n)), \quad s \in [t_n, t_{n+1}],$$

we arrive at an alternative adaptive scheme ASETD0

$$X_h^{n+1} = \varphi_0(-\Delta t_{n+1} A_h) (X_h^n + P_h F(X_h^n) \Delta t_{n+1} + P_h B(X_h^n) P_J \Delta W_{n+1}),$$

where $\varphi_0(z) := e^z$. For the analysis, we recall several properties of the error in the semigroup approximation $S_h(t)$ of $S(t)$.

**Lemma 6.1.** Let $0 \leq \alpha \leq \beta \leq 2$, $0 \leq \nu \leq 1$ and $h \in (0, 1]$. For the approximation operator $T_h(t) := S(t) - S_h(t) P_h$, the following error bounds hold

$$i) \|T_h(t)x\| \leq C_T h^{\beta} t^{-\frac{\alpha - \nu}{2}} \|x\|_\alpha, \quad \text{for all } x \in \mathcal{D}((-A)\frac{\alpha}{2}).$$

$$ii) \left\| \int_0^t T_h(t)x \, ds \right\| \leq Ch^{2-\nu} \|x\|_{-\nu}, \quad \text{for all } x \in \mathcal{D}((-A)\frac{\nu}{2}).$$

**Proof.** For i) see [16]. Estimate ii) is proven in [11] lemma 4.2. □
We state a non-linear generalisation of the Gronwall inequality which we apply to estimate the error in the spatial discretisation.

**Lemma 6.2.** Let \( u(t) \) be a non-negative function which satisfies the inequality
\[
    u(t) \leq c + \int_0^t au(s) + bu^\delta(s)ds,
\]
for \( a, b, c \geq 0 \). For \( 0 \leq \delta < 1 \), \( u(t) \) can be bounded by
\[
    u(t) \leq e^{at} \left( c^\delta + \frac{b}{a} (1 - e^{-\delta at}) \right)^{1/\delta}.
\]

**Proof.** This is a specific form of the non-linear Gronwall inequality for homogeneous coefficients, which is proved in generality in [18]. \( \Box \)

We have four solutions to consider. First the (mild) solution to (1.1) is
\[
    X(T) = S(T)X_0 + \int_0^T S(T - s)F(X(s))ds + \int_0^T S(T - s)B(X(s))dW(s). \tag{6.5}
\]

Secondly the semi-discrete approximation, i.e. the mild solution to (6.1)
\[
    X_h(T) = S_h(T)P_hX_0 + \int_0^T S_h(T - s)P_hF(X_h(s))ds \tag{6.6}
    + \int_0^T S_h(T - s)P_hB(X_h(s))P_JdW(s).
\]

The fully discrete approximation for ASETD1, which uses the \( \varphi_1 \) approximation for the drift
\[
    X_h^N = S_h(T)P_hX_0 + \sum_{k=0}^{N-1} \left( \int_{t_k}^{t_{k+1}} S_h(T - s)P_hF(X_h^k)ds \right) \tag{6.7}
    + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} S_h(T - t_k)P_hB(X_h^k)P_JdW(s).
\]

Finally we have the fully discrete approximation for ASETD0,
\[
    X_h^N = S_h(T)P_hX_0 + \sum_{k=0}^{N-1} \left( \int_{t_k}^{t_{k+1}} S_h(T - s)P_hF(X_h^k)ds \right) \tag{6.8}
    + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} S_h(T - t_k)P_hB(X_h^k)P_JdW(s).
\]

We now state our main result on strong convergence of the adaptive time-stepping method when the time-stepping strategy is admissible.

**Theorem 6.3.** Let \( X(T) \) be the mild solution defined in (6.5) at time \( T \). Let \( X_h(T) \) be the semi-discrete approximation defined in (6.6). Then, for \( X_0 \in L^2(\mathbb{D}, \mathcal{D}((-A)^{1/2})) \), the following estimate holds
\[
    \| X(T) - X_h(T) \|_{L^2(\Omega; \mathcal{H})} \leq C(T) \left( h^{1+r-\epsilon} + \lambda_{J+1}^{\frac{1}{2}\epsilon} + \epsilon \right).
\]

**Theorem 6.4.** Let \( X(T) \) be the mild solution defined in (6.5) at time \( T \). Let \( X_h^N \) be the numerical approximation defined in (6.7) with \( \{ t_n \}_{n \in \mathbb{N}} \) an admissible time-stepping strategy. Then for \( X_0 \in L^2(\mathbb{D}, \mathcal{D}((-A)^{1/2})) \), the following estimate holds
\[
    \| X(T) - X_h^N \|_{L^2(\Omega; \mathcal{H})} \leq C(T) \left( h^{1+r-\epsilon} + \lambda_{J+1}^{\frac{1}{2}\epsilon} + \Delta_{\max}^{\frac{1}{2}\epsilon} \right).
\]
Corollary 6.5. Let $X(T)$ be the mild solution defined in (6.5) at time $T$. Let $X^N_h$ be the numerical approximation defined in (6.8) with $\{t_n\}_{n \in \mathbb{N}}$ an admissible time-stepping strategy. Then for $X_0 \in L^2(\mathbb{D}, \mathcal{D}(-(A)^{1/2}))$, the following estimate holds

$$
\|X(T) - X^N_h\|_{L^2(\Omega; H)} \leq C(T) \left( h^{1+r-\epsilon} + \lambda_{j+1} h^{\frac{1}{j+1} + \epsilon} + \Delta t_{\text{max}}^{\frac{1}{2} - \epsilon} \right).
$$

We decompose the total error at time $t_n$ as

$$
X(t_n) - X^N_h = X(t_n) - X_h(t_n) + X_h(t_n) - X^N_h.
$$

The total error $E_n$ can be bounded by

$$
\|E_n\|_{L^2(\Omega; H)}^2 \leq 2 \|E_h(t_n)\|_{L^2(\Omega; H)}^2 + 2 \|E_h(t_n)\|_{L^2(\Omega; H)}^2 + \|E_h(t_n)\|_{L^2(\Omega; H)}^2,
$$

and we consider the error terms in space and time separately.

Proof of Theorem 6.3. The spatial approximation error $E_h$ can be written using the error operator introduced in Lemma 6.1 as,

$$
E_h(T) = T_h(T)X_0 + \int_0^T T_h(T - s)F(X(s))\,ds + \int_0^T T_h(T - s)P_h\left(F(X(s)) - F(X_h(s))\right)\,ds + \int_0^T T_h(T - s)B(X(s)) + S_h(T - s)P_hB(X(s))(I - P_J) + S_h(T - s)P_h(B(X(s)) - B(X_h(s)))P_J \,dW(s).
$$

We estimate the error $\|E_h\|_{L^2(\Omega; H)}^2$ using (standard) Itô’s isometry

$$
\|E_h(T)\|_{L^2(\Omega; H)}^2 \leq C \|T_h(T)X_0\|_{L^2(\Omega; H)}^2 + CE \left( \int_0^T \|T_h(T - s)F(X(s))\|_{L^2(\Omega; H)}^2 \,ds \right)^2 + C \left( \int_0^T \|T_h(T - s)P_h\left(F(X(s)) - F(X_h(s))\right)\|_{L^2(\Omega; H)}^2 \,ds \right)^2 + C \left( \int_0^T \|T_h(T - s)B(X(s))\|_{L^2(\Omega; H)}^2 \,ds \right)^2 + C \left( \int_0^T \|S_h(T - s)P_h(I - P_J)B(X(s))\|_{L^2(\Omega; H)}^2 \,ds \right)^2 + C \left( \int_0^T \|S_h(T - s)P_hB(X(s)) - B(X_h(s))\|_{L^2(\Omega; H)}^2 \,ds \right)^2.
$$

(6.10)

Focusing on the third term for now, we re-write $F(X(s))$ in terms of $F(X_h(s))$ and a remainder term similar to Lemma 4.3.

$$
F(X(s)) = F(X_h(s)) + \int_0^T DF(X_h(s)) + \tau(X(s) - X_h(s)) (X(s) - X_h(s)) d\tau.
$$

Defining the constant

$$
\bar{E}_q := \sup_{s \in [0, T]} \mathbb{E}\left[ \frac{C^q}{2} (1 + 2^{c_2+1} \|X_h(s)\|^{c_2} + 2^{c_2} \|X(s)\|^{c_2})^{2q} + \|X(s)\|^2 + \|X_h(s)\|^2 \right],
$$

we have that $\bar{E}_q < \infty$ by Theorem 2.7 and the finite dimensional counterpart, e.g. Theorem 4.1 in [14]. Following the same strategy as the proof of Lemma 4.3 part ii, we can bound the third term, for any $\delta \in (0, 1)$ as
\[
\| \int_0^T S_h(T-s)P_h(F(X(s)) - F(X_h(s))) \|_{L^2(\Omega;H)}^2
= \| \int_0^T S_h(T-s)P_hR_F(s, s, X_h(s)) \|_{L^2(\Omega;H)}^2 \\
\leq T\hat{E}_q^{1/(2^\gamma - 1)} \int_0^T \| E_h(s) \|_{L^2(\Omega;H)}^{2-\delta} ds.
\]

(6.11)

Returning to (6.10), we proceed using Lemma 6.1 i) with \( \beta = 1 + r \) and \( \alpha = 1 \) for the first term. The final term is estimated by Assumption 2.5 together with the boundedness of the semigroup and the boundedness of \( \| P_j \|_{L(\mathcal{H})} \) and \( \| P_h \|_{L(\mathcal{H})} \). We are left with

\[
\| E_h(T) \|_{L^2(\Omega;H)}^2 \leq C h^{2(1+r)} |T|^{-r} \mathbb{E} \| X_0 \|_{X_{\gamma}}^2 + C T \hat{E}_q^{1/(2^\gamma - 1)} \int_0^T \| E_h(s) \|_{L^2(\Omega;H)}^{2-\delta} ds \\
+ C \int_0^T L^2 \| E_h(s) \|_{L^2(\Omega;H)}^2 ds + C E \int_0^T \| T_h(T-s)F(X(s)) \|_{L^2_{\mathcal{H}}}^2 ds \\
+ C E \int_0^T \| T_h(T-s)B(X(s)) \|_{L^2_{\mathcal{H}}}^2 ds \\
+ C E \int_0^T \| S_h(T-s)P_h(I-P_j)B(X(s)) \|_{L^2_{\mathcal{H}}}^2 ds.
\]

(6.12)

We consider each of the three remaining terms individually. Firstly, part I is estimated using a Taylor expansion of \( F(X(s)) \) in terms of \( F(X(T)) \), we have

\[
I := 2^6 \left\| \int_0^T T_h(T-s)F(X(s))ds \right\|_{L^2(\Omega;H)}^2
= C \left\| \int_0^T T_h(T-s)(F(X(T)) + R_F(s, s, X(T)))ds \right\|_{L^2(\Omega;H)}^2 \\
\leq 2C \left\| \int_0^T T_h(T-s)F(X(T))ds \right\|_{L^2(\Omega;H)}^2 + 2C \left\| \int_0^T T_h(T-s)R_F(s, s, X(T))ds \right\|_{L^2(\Omega;H)}^2 \\
= I_1 + I_2.
\]

For \( I_1 \) we apply Lemma 6.1 part ii) with \( \nu = 1 - r \), which requires \( r \leq 1 \), together with Assumption 2.4 to show,

\[
I_1 \leq C h^{2(1+r)} \left\| (-\Delta)^{-\frac{1+r}{2}} F(X(T)) \right\|_{L^2(\Omega;H)}^2 \\
\leq C h^{2(1+r)} \lambda_1^{-\frac{1+r}{2}} \| F(X(T)) \|_{L^2(\Omega;H)}^2 \\
\leq C h^{2(1+r)}.
\]
The last line uses Theorem 2.7 for the true solution $X(t)$. Term $I_2$ is bounded by Lemma 6.1 i) with $\beta = 1 + r - \epsilon$ and $\alpha = 0$, we have

$$I_2 \leq C \int_0^T \|T_h(T - s)R_F(s, T, X(T))\|^2_{L^2(\Omega; H)} ds$$

$$\leq C \int_0^T h^{2(1+r-\epsilon)}(T - s)^{-1(1+r-\epsilon)} \|R_F(s, T, X(T))\|^2_{L^2(\Omega; H)} ds.$$

We have already shown, (4.10), that

$$\|R_F(s, T, X(T))\|^2_{L^2(\Omega; H)} \leq C(T - s)^{1-\epsilon}.$$ 

Inserting into the bound for $I_2$ it follows

$$I_2 \leq C \int_0^T h^{2(1+r-\epsilon)}(T - s)^{-r} ds$$

$$\leq C h^{2(1+r-\epsilon)}.$$ 

In the last step we require $r < 1$, otherwise the integral does not converge. Overall we have

(6.13) 

$$I \leq C h^{2(1+r-\epsilon)}.$$ 

Moving onto part $II$, we apply Lemma 6.1 i) with $\beta = 1 + r - \epsilon$ and $\alpha = r$ to the individual terms inside $\|\|_{L^2_0}$. That is

$$II := \int_0^T \sum_{j=1}^T \left\|T_h(T - s)B(X(s))Q^\frac{1}{2}\right\|^2 ds$$

$$\leq C h^{2(1+r-\epsilon)} \int_0^T \sum_{j=1}^\infty (T - s)^{-1+\epsilon} \left\|A^\frac{1}{2}B(X(s))Q^\frac{1}{2}\right\|^2 ds$$

$$= C h^{2(1+r-\epsilon)} \int_0^T (T - s)^{-1+\epsilon} \left\|A^\frac{1}{2}B(X(s))\right\|^2_{L^2_0} ds.$$ 

Application of Assumption 2.6 and Theorem 2.7 completes the estimate of $II$, 

(6.14)

$$II \leq C h^{2(1+r-\epsilon)}.$$ 

The final term of (6.10), $III$, we have for $r \in [0, 1)$ and any $\epsilon \in (0, 1)$

$$III := CE \int_0^T \|S_h(T - s)P_h(I - P_J)B(X(s))\|^2_{L^2_0} ds$$

$$\leq CE \int_0^T \left\|(-A)^{\frac{1}{2}}s_h(T - s)P_h\right\|^2_{L^2(\Omega; H)} \left\|(I - P_J)(-A)^{-\frac{1}{2}}\right\|^2_{L^2(\Omega; H)} \left\|(-A)^{\frac{1}{2}}B(X(s))\right\|^2_{L^2_0} ds.$$ 

We use Lemma 2.2 to bound first term in the integral and Assumption 2.6 on the third term in the integral to show that

$$III \leq C \int_0^T (T - s)^{-1+\epsilon} ds \lambda_{J+1}^{-(1+r)+\epsilon}(1 + \sup_{s \in [0,t]} E \|X(s)\|^2_{L^2})$$

(6.15)

$$\leq C \lambda_{J+1}^{-(1+r)+\epsilon}.$$ 

Returning to (6.12), Equations (6.13) to (6.15) show that

$$\|E_h(T)\|^2_{L^2(\Omega; H)} \leq C_1 h^{2(1+r-\epsilon)} + C_2 \lambda_{J+1}^{-(1+r)+\epsilon} + C_3 \int_0^T E_h(s)\|^2_{L^2(\Omega; H)} ds$$

$$+ C_4 \int_0^T \|E_h(s)\|^2_{L^2(\Omega; H)} ds.$$
Application of Lemma 6.2 yields
\[
\|E_h(T)\|_{L^2(\Omega; H)}^2 \leq e^{C_3 L^2 T} \left[ \left( C_1 h^{2(1+r-c)} + C_2 \lambda_{J+1}^{-(1+r)+c} \right) + \frac{C_4}{C_3 L^2} \right]^{1/\delta}
\]
\[
\leq e^{C_3 L^2 T} \left[ \left( C_1 h^{2(1+r-c)} + C_2 \lambda_{J+1}^{-(1+r)+c} \right) + C_4 T \delta \right]^{1/\delta}
\]
\[
\leq e^{C_4 T} e^{1/e} e^{C_3 L^2 T} \left( C_1 h^{2(1+r-c)} + C_2 \lambda_{J+1}^{-(1+r)+c} \right).
\]

Proof of Theorem 6.4. The time discretisation error, \( E_h(t_n) \), must be considered over one step with conditional expectation. This is to ensure the time-step size selection, computed by the current solution does not bias the numerical scheme. The time discretisation error can be written as

\[
E_h(t_{n+1}) = S_h(t_{n+1} - t_n) P_h E_h(t_n)
\]
\[
+ \int_{t_n}^{t_{n+1}} S_h(t_{n+1} - s) P_h F(X_h(s)) - S_h(t_{n+1} - s) P_h F(X_h^n) ds
\]
\[
+ \int_{t_n}^{t_{n+1}} S_h(t_{n+1} - s) P_h B(X_h(s)) - S_h(t_{n+1} - t_n) P_h B(X_h^n) dW(s)
\]

Over one step, we Taylor expand the drift and diffusion coefficients around \( X_h(t_n) \) as follows,

\[
E_h(t_{n+1}) = S_h(t_{n+1} - t_n) P_h E_h(t_n)
\]
\[
+ \int_{t_n}^{t_{n+1}} S_h(t_{n+1} - s) P_h \left( F(X_h(t_n)) - F(X_h^n) \right) ds
\]
\[
:= I
\]
\[
+ \int_{t_n}^{t_{n+1}} (S_h(t_{n+1} - s) - S_h(t_{n+1} - t_n)) P_h B(X_h(t_n)) dW(s)
\]
\[
:= I_1
\]
\[
+ S_h(t_{n+1} - t_n) P_h (B(X_h(t_n)) - B(X_h^n)) \Delta W_{n+1}
\]
\[
:= I_2
\]
\[
+ \int_{t_n}^{t_{n+1}} S_h(t_{n+1} - s) P_h R_F(s, t_n, X_h(t_n)) ds
\]
\[
:= \tilde{R}_F
\]
\[
+ \int_{t_n}^{t_{n+1}} S_h(t_{n+1} - s) P_h R_B(s, t_n, X_h(t_n)) dW(s)
\]
\[
:= \tilde{R}_B
\]

(6.16)

\[:= S_h(t_{n+1} - t_n) P_h E_h(t_n) + \tilde{R}_K,\]

where we have defined \( R_K := I + II + \tilde{R}_F + \tilde{R}_B \) and \( II := I_1 + I_2 \). Applying the operator \( S_h(T - t_{n+1}) \) to (6.16) yields

\[
S_h(T - t_{n+1}) E_h(t_{n+1}) = S_h(T - t_n) P_h E_h(t_n) + S_h(T - t_{n+1}) R_K(t_{n+1}).
\]

(6.17)

\[:= E_h^3 + I_S + II_S + \tilde{R}_SF + \tilde{R}_SB.\]
The norm $\|E_{S}^{n+1}\|^2$ is expanded and re-written using the inequality $\langle A, B \rangle \leq \frac{1}{2}(\|A\|^2 + \|B\|^2)$ as

$$\|E_{S}^{n+1}\|^2 = \langle E_{S}^{n} + I_{S} + II_{S} + \hat{R}_{SF} + \hat{R}_{SB}, E_{S}^{n+1} \rangle$$

$$\leq \frac{1}{2}(\|E_{S}^{n}\|^2 + \|E_{S}^{n+1}\|^2) + \langle I_{S} + II_{S} + \hat{R}_{SF} + \hat{R}_{SB}, E_{S}^{n+1} \rangle.$$ 

Therefore the error over one time-step can be bound by

$$\|E_{S}^{n+1}\|^2 - \|E_{S}^{n}\|^2 \leq 2\langle I_{S} + II_{S} + \hat{R}_{SF} + \hat{R}_{SB}, E_{S}^{n+1} \rangle.$$ 

Expansion of both $R_{S}^{n+1}$ and $E_{S}^{n+1}$ inside the inner product then collation of the stochastic terms shows that

$$\|E_{S}^{n+1}\|^2 - \|E_{S}^{n}\|^2 \leq 2\left(\langle I_{S}, E_{S}^{n} + I_{S} + II_{S} + \hat{R}_{SF} + \hat{R}_{SB} \rangle + \langle \hat{R}_{SF}, E_{S}^{n} + I_{S} + II_{S} + \hat{R}_{SF} + \hat{R}_{SB} \rangle + \langle II_{S} + \hat{R}_{SB}, E_{S}^{n} + I_{S} + II_{S} + \hat{R}_{SF} + \hat{R}_{SB} \rangle \right).$$

Further applications of the inequality $\langle A, B \rangle \leq \frac{1}{2}(\|A\|^2 + \|B\|^2)$ and rearrangement yields,

$$\|E_{S}^{n+1}\|^2 - \|E_{S}^{n}\|^2 \leq 2\langle I_{S}, E_{S}^{n} \rangle + 2\langle \hat{R}_{SF}, E_{S}^{n} \rangle + 4\|I_{S}\|^2 + 7\|\hat{R}_{SF}\|^2 + 8\|II_{S}\|^2 + 8\|\hat{R}_{SB}\|^2$$

$$+ 2\langle 2I_{S} + E_{S}^{n}, II_{S} + \hat{R}_{SB} \rangle. \quad (6.18)$$

The expected value, conditioned on the filtration at $t_{n}$, of the final term $\langle 2I_{S} + E_{S}^{n}, II_{S} + \hat{R}_{SB} \rangle$, is zero. This is true since both $I_{S}$ and $E_{S}^{n}$ are $\mathcal{G}_{n}$ measurable and $\mathbb{E}[II_{S} + \hat{R}_{SB} | \mathcal{G}_{n}] = 0$. We bound each of the terms in (6.18) in turn. Boundedness of the semigroup and Assumption 2.3 on the drift are applied to the first term to show

$$\langle I_{S}, E_{S}^{n} \rangle = \left( \int_{t_{n}}^{t_{n+1}} S_{h}(T-s)P_{h} \left( F(X_{h}(t_{n})) - F(X_{h}^{n}) \right) ds, S_{h}(T-t_{n})P_{h}E_{k}(t_{n}) \right)$$

$$\leq C\Delta t_{n+1} \langle F(X_{h}(t_{n})) - F(X_{h}^{n}), E_{k}(t_{n}) \rangle$$

$$\leq C\alpha \Delta t_{n+1} \|E_{k}(t_{n})\|^2. \quad (6.19)$$

The second term is estimated using the fact $\|E_{S}^{n}\|$ is $\mathcal{G}_{n}$ measurable, Lemma 4.3, the boundedness of the semigroup and Young’s inequality,

$$\mathbb{E}\left[ \langle \hat{R}_{SF}, E_{S}^{n} \rangle | \mathcal{G}_{n} \right] \leq \mathbb{E}[\|\hat{R}_{F} \| \| E_{S}^{n} \| | \mathcal{G}_{n}]$$

$$\leq C \|E_{k}(t_{n})\| \mathbb{E}[\|\hat{R}_{F} \| | \mathcal{G}_{n}]$$

$$\leq C\Delta t_{n+1}^{2-\varepsilon} \|E_{k}(t_{n})\|$$

$$\leq \frac{1}{2}C\Delta t_{n+1}^{2(1-\varepsilon)} + \frac{1}{2}C\Delta t_{n+1} \|E_{k}(t_{n})\|^2. \quad (6.20)$$
For $\|I_S\|^2$ we use Jensen’s inequality, the boundedness of the semigroup and the admissibility of the time-stepping strategy,

$$
\|I_S\|^2 = \left\| \int_{t_n}^{t_{n+1}} S_h(T - s) P_h \left( F(X_h(t_n)) - F(X^n_h) \right) ds \right\|^2 \\
\leq \left( \int_{t_n}^{t_{n+1}} \| S_h(T - s) P_h \|_{L(H)} \| F(X_h(t_n)) - F(X^n_h) \| ds \right)^2 \\
\leq \Delta t_{n+1}^2 \| F(X_h(t_n)) - F(X^n_h) \|^2 \\
\leq 2\Delta t_{n+1}^2 \left( \| F(X_h(t_n)) \|^2 + \| F(X^n_h) \|^2 \right) \\
\leq 2\Delta t_{n+1}^2 \left( (1 + \| X_h(t_n) \|)^{c+1} + R_1 + 2R_2 \| X_h(t_n) \|^2 + 2R_2 \| E_k(t_n) \|^2 \right).
$$

Therefore

$$(6.21)\quad \|I_S\|^2_{L^2(\Omega; G_n; H)} \leq K_5 \Delta t_{n+1}^2 + 4R_2 \Delta t_{n+1}^2 \| E_k(t_n) \|^2 \quad a.s.,$$

for the a.s. finite $G_n$-measurable random variable

$$K_5 := 2 \left( (1 + \sup_{s \in [0,T]} \mathbb{E} \| X_h(s) \|^{c+1} \| G_n \|)^2 + R_1 + 2R_2 \sup_{s \in [0,T]} \mathbb{E} \| X_h(s) \|^2 \| G_n \| \right).$$

Lemma 4.3 and the boundedness of the semigroup implies

$$(6.22)\quad \| \tilde{R}_{SF} \|^2_{L^2(\Omega; G_n; H)} \leq K_3 \Delta t_{n+1}^{3-\epsilon} \quad a.s.,$$

$$(6.23)\quad \| \tilde{R}_{SB} \|^2_{L^2(\Omega; G_n; H)} \leq K_4 \Delta t_{n+1}^{2-\epsilon} \quad a.s.$$.

The next term to estimate in (6.18) is $\|II_S\|^2$,

$$
\|II_S\|^2_{L^2(\Omega; G_n; H)} \\
\leq 2\mathbb{E} \left[ \left\| \int_{t_n}^{t_{n+1}} (S_h(T - s) - S_h(T - t_n)) P_h B(X_h(t_n)) dW(s) \right\|^2 \ | G_n \right] \\
+ 2\mathbb{E} \left[ \| S_h(T - t_n) P_h (B(X_h(t_n)) - B(X^n_h)) \Delta W_{n+1} \|^2 \ | G_n \right] \\
:= II_{S1} + II_{S2}.
$$

Firstly using Lemma 4.1, Assumption 2.5 and Lemma 2.2 we have for $\epsilon \in (0, 1)$ that $II_{S1}$ can be bounded by

$$
II_{S1} \leq 2\mathbb{E} \int_{t_n}^{t_{n+1}} \left( -A_h \right)^{\frac{1-\epsilon}{2}} \left( -A_h \right)^{-\frac{1-\epsilon}{2}} S_h(T - s) \left\| -A_h \right\|^{\frac{1-\epsilon}{2}} \left\| S_h(T - s) \right\|_{L(H)} \\
\times \left\| (I - S_h(s - t_n)) P_h \left\| B(X_h(t_n)) \right\|_{L_2^2} ds \right\|_{L_2} \left| G_n \right| \\
\leq 2C\Delta t_{n+1}^{1-\epsilon} \int_{t_n}^{t_{n+1}} (T - s)^{-1+\epsilon} ds \mathbb{E} \left[ \| B(X_h(t_n)) \|^2_{L_2^2} \ | G_n \right] \\
\leq 2K_6 \Delta t_{n+1}^{1-\epsilon} \int_{t_n}^{t_{n+1}} (T - s)^{-1+\epsilon} ds \quad a.s.
$$

(6.24)

With the a.s. finite $G_n$-measurable random variable $K_6$ defined as

$$K_6 := 2CL_2 \sup_{t \in [0,T]} \mathbb{E} \left[ (1 + \| X_h(s) \|)^2 \ | G_n \right] .$$
To bound $II_{S2}$ we use Lemma 4.1, the boundedness of the semigroup and $P_h$ operator with assumption 2.5 to show

$$II_{S2} = 2\mathbb{E} \left[ \|S_h(T - t_n)P_h(B(X_h(t_n)) - B(X^n_h))\Delta W_{n+1}\|^2 \bigg| \mathcal{G}_n \right]$$

$$\leq 2C\Delta t_{n+1} \|B(X_h(t_n)) - B(X^n_h)\|_{L_0^2}^2$$

$$\leq C\Delta t_{n+1}L_0^2\|E_k(t_n)\|^2. \quad (6.25)$$

Combining the estimates in Equations (6.19) to (6.25) for (6.18) implies

$$\mathbb{E} \left[ \|E_{S}^{n+1}\|^2 \bigg| \mathcal{G}_n \right] - \|E_{S}^n\|^2 \leq C_1\Delta t_{n+1}^{2-\epsilon} + C_2\Delta t_{n+1}\|E_k(t_n)\|^2 + C_3\Delta t_{n+1}^{1-\epsilon} \int_{t_n}^{t_{n+1}} (T - s)^{-1+\epsilon} ds.$$  

We sum both sides from $n = 0$ to $n = N - 1$ and take expectation, noting that $E_{S}^N := S(T - T)E_k(T) = E_k(T)$, to show that

$$\mathbb{E}[\|E_k(T)\|^2] \leq C_1\Delta t_{\max}^{1-\epsilon} + \sum_{n=0}^{N-1} \Delta t_{\max} C_2\mathbb{E}[\|E_k(t_n)\|^2] + C_3\Delta t_{\max}^{1-\epsilon} \int_0^T (T - s)^{-1+\epsilon} ds$$

$$\leq C_3\Delta t_{\max}^{1-\epsilon} + \sum_{n=0}^{N-1} \Delta t_{\max} C_2 \|E_k(t_n)\|^2 \quad (6.26)$$

Applying the discrete Gronwall lemma to (6.26) yields,

$$\mathbb{E}[\|E_k(T)\|^2 | \mathcal{G}_n] \leq C\Delta t_{\max}^{1-\epsilon} \exp(C_2N\Delta t_{\max})$$

$$\leq C\Delta t_{\max}^{1-\epsilon} \exp(C_2\rho T). \quad (6.27)$$

Combining the estimate from Theorem 6.4 with (6.27) yields the result. \hfill \Box

To conclude the section we give the proof of Corollary 6.5.

Proof of Corollary 6.5. If the $\varphi_0$ approximation is used instead of $\varphi_1$ for the drift contribution, that is approximations (6.4) instead of (6.2), an additional term

$$III_S := \int_{t_n}^{t_{n+1}} (S_h(T - s) - S_h(T - t_n)) P_h F(X_h(t_n)) ds,$$

appears in (6.17). This introduces additional inner product terms against the existing terms in (6.18). All terms not involving $III$ can be dealt with as before. The term $III$ is $\mathcal{G}_n$-measurable and therefore the inner product with the stochastic terms has zero mean. The norm of $III_S$ can be bound as,

$$\mathbb{E} \left[ \|III_S\|^2 \bigg| \mathcal{G}_n \right] := \mathbb{E} \left[ \left\| \int_{t_n}^{t_{n+1}} (S_h(T - s) - S_h(T - t_n)) P_h F(X_h(t_n)) ds \right\|^2 \bigg| \mathcal{G}_n \right]$$

$$\leq \Delta t_{n+1} \int_{t_n}^{t_{n+1}} \left\| (A_h)^{1/2} S_h(T - s) \right\|_{\mathcal{L}(H)}^2 \times \left\| (A_h)^{-1/2} (I - S_h(s - t_n)) \right\|_{\mathcal{L}(H)}^2 ds \mathbb{E} \left[ \|P_h F(X_h(t_n))\|^2 \bigg| \mathcal{G}_n \right]$$

$$\leq C\Delta t_{n+1}^{2-\epsilon} \int_{t_n}^{t_{n+1}} (T - s)^{-1+\epsilon} ds.$$
The term $\mathbb{E} \left[ \langle III, E^S_2 \rangle | \mathcal{G}_n \right]$ is bounded via the boundedness of the semigroup, Jensen’s inequality and Young’s inequality

$$\mathbb{E} \left[ \langle III, E^S_2 \rangle | \mathcal{G}_n \right] \leq \| E^S_2 \| \mathbb{E} \| III \| | \mathcal{G}_n \|
$$

whenever $\Delta t_{n+1} \leq \| E_k(t_n) \| \left( \int_{t_n}^{t_{n+1}} (T-s)^{-1+\varepsilon} ds \right)^{1/2}.
$$

\leq C \Delta t_{n+1} \| E_k(t_n) \| \left( \int_{t_n}^{t_{n+1}} (T-s)^{-1+\varepsilon} ds + \Delta t_{n+1} \| E_k(t_n) \| ^2 \right).$$

The two remaining inner products of $III$ against $I_S$ and $R_{SF}$ contain higher powers of $\Delta t_{n+1}$, leaving the final order of convergence unaffected after application of the discrete Gronwall lemma.

\[ \square \]

### 7. Numerical Experiments

We continue with several numerical tests to demonstrate the effectiveness of the adaptive time-stepping scheme in comparison to alternative numerical schemes that ensure strong convergence in the presence of (general) non-globally Lipschitz drift. The numerical experiments all utilise the spectral Galerkin approach. We firstly restate our adaptive numerical scheme and introduce several additional schemes to compare against. The first adaptive numerical scheme is defined as

$$X^{n+1}_h = e^{-\Delta t_{n+1} A_h} \left( X^n_h + \Delta t_{n+1} B(X^n_h) P_h \Delta W_{n+1} \right)
$$

whenever $\Delta t_{n+1} > \Delta t_{\text{min}}$. We refer to this scheme as ASETD1. A less computationally intensive adaptive scheme can be defined by using the $\varphi_0$ approximation to the semigroup in the drift,

$$X^{n+1}_h = e^{-\Delta t_{n+1} A_h} \left( X^n_h + \Delta t_{n+1} \varphi_1(\Delta t_{n+1} A_h) P_h F(X^n_h) \right),
$$

whenever $\Delta t_{n+1} > \Delta t_{\text{min}}$, that we call ASETD0. Both adaptive schemes use the stopped scheme of [9] as a backstop if $\Delta t_{n+1} = \Delta t_{\text{min}}$, which we re-define here.

$$X^{n+1}_h = e^{-\Delta t A_h} X^n_h + e^{\Delta t A_h} \left[ P_h F(X^n_h) \Delta t + P_h B(X^n_h) P_j \Delta W_{n+1} \right] 1_{\{F, \Delta t \}},
$$

with

$$1\{F, \Delta t\} := 1\left\{ \| P_h F(X^n_h) \| \leq (1/\Delta t)^{\theta} \right\}.
$$

We take $\theta$ as large as possible (according to [9]), which corresponds to $\theta = 1/2$ and refer to this scheme as NSEE. In this method, the nonlinear contributions to the scheme are switched off if they are too large in comparison to the timestep size. This controls any unwanted growth due to the non-Lipschitz drift and ensures strong convergence of the scheme. The Lipschitz assumptions in [9] are more general than in this work, as they allow (potentially) non-globally Lipschitz drift $B$ to be controlled by the one-sided Lipschitz drift $F$ in a combined estimate such as

$$\left\| F(X) - F(Y), X - Y \right\|_{L^2} \leq L \| X - Y \|^2.
$$

As this work only considers Lipschitz drift $B$, we have relaxed the indicator bound to (7.4) instead of the stronger bound of

$$1\{F, \Delta t\} := 1\left\{ \| P_h F(X^n_h) \| + \| P_h B(X^n_h) \|_{HS(H,H)} \leq (1/\Delta t)^{\theta} \right\},
$$

as originally defined in [9], to improve the performance of the stopped scheme.

The second fixed step numerical scheme we compare against is the tamed Euler-Maryuama introduced in [5]. This work is of the variational approach and therefore satisfies different assumptions than in this paper. In [5] the linear and non-linear operators are considered as a single operator which consists of two components. These components satisfy different growth and coercivity conditions than in this work. Existence and uniqueness of solutions as well convergence
of a tamed Euler-Maryuama scheme is proven in [5]. We test a similar tamed Euler-Maryuama method but again remark the framework and assumptions of [5] is quite different to what we consider here. To that end, define $C(X) := -AX + F(X)$, where $A$ and $F$ are the linear and nonlinear parts of the SPDE. Then the tamed operator $\tilde{C}$ is defined by

$$\tilde{C} := \frac{C(X)}{1 + \sqrt{\Delta t} \|C(X)\|},$$

and the tamed scheme as,

$$X^{n+1}_h = X^n_h + \Delta t P_h \tilde{C}(X^n_h) + P_h B(X^n_h) P_j \Delta W_{n+1},$$

called TEM. The final method we compare against, named TSETD0, is a speculative method, inspired by the tamed schemes of e.g. [5] and [8]. We define the tamed exponential scheme as

$$X^{n+1} = e^{-\Delta t A_h} \left( X^n_h + P_h \tilde{F}(X^n_h) \Delta t + P_h B(X^n_h) P_j \Delta W_{n+1} \right),$$

with $\tilde{F}$ defined analogously as in (7.6) for the nonlinear operator $F$ in place of $C$.

In order to make a fair comparison between the adaptive schemes and the fixed time-step schemes we firstly compute an approximation using ASETD1, then calculate the average time-step size to use in the fixed time-step schemes. Hence the adaptive and fixed schemes take approximately the same number of time-steps for each realisation.

To ensure that the linear operator is the generator of an analytic semigroup with eigenvalues $\lambda_j < 0$, we add and subtract $c_0 X dt$ on the right hand side of (1.1) for some $c_0 \in \mathbb{R}$, taking $-c_0 X$ into the (re)definition of $A$ and $c_0 X$ into $F$ for the exponential methods. In all simulations we set $c_0 = \max\{\lambda_j, 0\} + 1$.

### 7.1. Allen-Cahn Equation

We test the adaptive scheme firstly on the Allen-Cahn SPDE with multiplicative noise, that is we solve the following SPDE

$$dX = \left[-\Delta X + X - X^3\right] dt + \sigma X dW,$$

on the 1D domain $[0, 32\pi]$. We set $\sigma = 1$ and compare the mean-square error over 1000 trials of the final solution at $T = 1$, against a reference solution. The reference solution is computed with a time-step size of $\Delta t_{ref} = 5 \cdot 10^{-6}$. The first test is to show convergence of the adaptive scheme with respect to $\Delta t_{max}$. We solve the Allen-Cahn equation over a range of $\Delta t_{max}$ values in the range $[2^{-3}, 2^{-12}]$ with a fixed number of points on the spatial mesh of $N_x = 512$. For the adaptive time-stepping, we use the time-stepping rule, $\Delta t_n \leq \frac{\rho}{\|F(X^n_h)\|}$ and set $\rho = 100$. In Figure 1a we plot the mean-square error against $\Delta t_{max}$ for a range of noise regularities and observe convergence at a rate equal to the theoretical rate of $\frac{1}{2}$.

In Figure 1b we compare the two adaptive schemes, ASETD1 and ASETD0, against the three fixed time-stepping methods NSEE, TEM and TSETD0. To compare the adaptive scheme against the fixed time-step schemes fairly, we firstly solve the SPDE using the adaptive scheme and then compute $\Delta t := T / N_{adapt}$, where $N_{adapt}$ is the number of timesteps taken by the adaptive scheme for the particular realisation. We observe significantly lower errors and a higher convergence rate for ASETD1 and ASETD0 when compared to both TEM and TSETD0. NSEE cannot produce accurate solutions for larger values of $\Delta t$, due to the switching off of the non-linear contributions. After sufficient reduction in $\Delta t$, the errors in NSEE and ASETD0 coincide, with a slight overhead for the adaptive scheme. For the parameter $r = 0$ we do not see significant difference between ASETD1 and ASETD0, however for smoother noise, such as shown in Figure 4b, the increase of accuracy of ASETD1 becomes apparent. A final point to note is the difference in accuracy between TEM and TSETD0 for $r = 0$, for smoother noise the accuracy of these two schemes coincide.

In Figure 2 we plot, for a single realisation of Allen-Cahn equation, the change in $\Delta t_n$ over the time period $[0, 5]$ in the upper plot. Below this we show the change in $\|F(X)\|$ over time. We observe close agreement between ASETD1 and the reference solution, which is not shown for
Figure 1. Plot a) convergence of ASETD1 with respect to $\Delta t_{\text{max}}$. We show convergence for a range of noise regularity parameters, $r \in [0, 1]$ with reference slopes of $\frac{1}{2}$ and $\frac{1}{4}$. Plot b) shows RMS error against average CPU time for all schemes with noise regularity $r = 0$ and $\rho = 100$.

Figure 2. Upper plot: change in $\Delta t_n$ over time for a single solve of Allen-Cahn equation. Lower plot: size of the non-linearity $\|F(X)\|$ for the same realisation. The solid blue line corresponding to ASETD1, dashed orange for NSEE and dotted yellow signifying TSETD0. The shaded grey area indicates the times the stopped method excluded the non-linear contributions.

clarity. The buffer between $\Delta t_{\text{max}}$ and $\Delta t_{\text{min}}$ provided by $\rho$ enables ASETD1 to approximate the solution without use of the backstop at any point for this realisation. Conversely, for this particular realisation, NSEE has to discard the non-linear contribution for significant portions of the overall time period as seen by the orange dashed line and grey shaded areas. In this realisation the parameters used were $r = 0$, $\Delta t_{\text{max}} = 10^{-2}$ and $\rho = 100$. The resulting mean timestep was $\bar{\Delta t} = 3 \cdot 10^{-3}$. 
7.2. Swift-Hohenberg equation. We consider a similar numerical experiment as in the previous section, with the Swift-Hohenberg SPDE. This is a $4^{th}$ order SPDE that arises in applications involving pattern formation such as fluid flow and the study of neural tissue. It is defined as

$$dX = \left[\eta X - (1 + \Delta)^2 X + cX^2 - X^3\right] dt + \sigma X dW.$$ 

We set the parameters $\eta = 0.7$, $c = 1.8$ and $\sigma = 1$, then solve to a final time of $T = 5$ with $N_x = 512$ points in the spatial domain $[0, 32\pi]$. The same time-stepping rule of $\Delta t_n \leq \frac{5}{\|F(X_n)\|}$ is used as previously and again reference solutions are computed at $\Delta t_{\text{ref}} = 5 \cdot 10^{-6}$. We maintain $\rho = 100$ for all 1000 simulations and vary $\Delta t_{\text{max}}$ between $[2^{-3}, 2^{-11}]$. We plot example solutions in Figure 3 for all schemes. We clearly see the difference between ASETD1, which faithfully recreates the patterns formed by the reference solution, and NSEE which is unable to do so for this realisation. Again this effect is due to the size of $\Delta t$ being too large when compared to $\|F(X)\|$ and after sufficient reduction in $\Delta t$ we would of course see the correct pattern appearing for the stopped method.

Figure 4b shows the RMS error against CPU time for the various methods and $r = \frac{1}{2}$. The smoother noise shows the gain in accuracy of ASETD1 over ASETD0 at larger values of $\Delta t_{\text{max}}$. It is clear that NSEE and TEM suffer at these values of $\Delta t$ and cannot accurately solve the $4^{th}$ order SPDE. ASETD1 and ASETD0 produce solutions with significantly lower errors, even when $\Delta t_{\text{max}}$ is very large. We observe the expected convergence rate for ASETD1 in Figure 4a over the entire range of $\Delta t_{\text{max}}$. In comparison, the fixed step methods are only starting to reach their asymptotic convergence rate at the smallest values of $\Delta t$. We remark that after reducing $\Delta t$ to a sufficiently small value, we would see the same behaviour as in Figure 1b, where the
error in NSEE should jump down and agree with ASETD0. However it should be noted that these experiments were carried out for 1D spectral Galerkin and thus are relatively inexpensive to run. For a large finite element discretisation in higher dimensions, it may be difficult to reach a sufficiently small $\Delta t$ to ensure NSEE always includes the (important) non-linear contributions.

8. Conclusion

We have constructed an adaptive time-stepping scheme for semilinear SPDEs with non-Lipschitz drift coefficients. We have proven the scheme strongly converges, with respect to $\Delta t_{\text{max}}$, with the same rate as for globally Lipschitz drift. It is important to note the convergence of the adaptive scheme is valid only when the time-step size stays above $\Delta t_{\text{min}}$. If the minimum time-step size is ever required then an alternative “backstop” scheme must be employed to ensure overall convergence. In practice, we have shown that for moderate choices in the fixed ratio $\Delta t_{\text{max}}/\Delta t_{\text{min}}$, we always maintain a time-step size above $\Delta t_{\text{min}}$. Two numerical experiments in Section 7 confirm the theoretical convergence rates and highlight the efficiency gains over alternative fixed time-step methods for general non-Lipschitz drift.

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