Late time acceleration in a non-commutative model of modified cosmology

B. Malekolkalami\textsuperscript{1*}, K. Atazadeh\textsuperscript{2†} and B. Vakili\textsuperscript{3‡}

\textsuperscript{1}Department of Physics, University of Kurdistan, P Pandora St., Sanandaj, Iran
\textsuperscript{2}Department of Physics, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran
\textsuperscript{3}Department of Physics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

Abstract

We investigate the effects of noncommutativity between the position-position, position-momentum and momentum-momentum of a phase space corresponding to a modified cosmological model. We show that the existence of such noncommutativity results in a Moyal Poisson algebra between the phase space variables in which the product law between the functions is of the kind of an $\alpha$-deformed product. We then transform the variables in such a way that the Poisson brackets between the dynamical variables take the form of a usual Poisson bracket but this time with a noncommutative structure. For a power law expression for the function of the Ricci scalar with which the action of the gravity model is modified, the exact solutions in the commutative and noncommutative cases are presented and compared. In terms of these solutions we address the issue of the late time acceleration in cosmic evolution.

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1 Introduction

According to the astronomical evidences and corresponding observations, we live in an accelerated expanding universe. To explain this acceleration, many models have been proposed which in general may be classified in two categories. The first is the models which deal with ordinary (or dark) matter and energy including scalar fields in the context of the Einstein’s general theory of relativity \cite{1}. The second one is the so-called modified theories of gravity for which there are good theoretical reasons to consider the possibility that gravity is not described precisely by Einstein’s theory (frame) but rather by some alternative theories \cite{2}.

In the past decade, investigations of the noncommutative (NC) structures in the field of the first category have caused a renewed interest on noncommutativity in the classical and quantum levels. In these studies, the influence of noncommutativity (in the configuration space or even in momentum sector) has been explored by formulation of a version of NC cosmology in which noncommutativity shows itself as a deformation parameter in the classical or quantum commutation relations between minisuperspace \cite{3-6} or phase-space \cite{7-9} variables while the geometrical structure of the underlying space-time remains unchanged.

However, as far as the authors are aware, studying the subject in the level of the second category (modified theories of gravity) has seldom been studied in the literature. As an excellent review in

\*e-mail: b.malakolkalami@uok.ac.ir
\†e-mail: atazadeh@azaruniv.ac.ir
\‡e-mail: b.vakili@iauctb.ac.ir
which many aspects of gravitational issues have been studied in the context of extended theories of gravity, we can refer to the work of Capozziello and De Laurentis [10]. Apart from the conceptual problems in astrophysics and cosmology, this work has dealt with the recently emerged problems in the subject of gravity such as inflation, dark energy, dark matter, large scale structure, quantum gravity and so on. In particular, it is shown in [10] that the space-time metric deformations can be related to the generation of gravity theory. To do this, they considered the deformations as extended conformal transformations to address the relations between the Jordan and the Einstein frames. Although our main motivation in this work is the idea of the space-time deformation such as what mentioned in [10], we do not do this at the level of the space-time geometry. Instead, our purpose in this work is to build an NC scenario for a classical modified gravity model. To do this, we will investigate the effects of noncommutativity either in the configuration space with the NC parameter $\theta$ or in the momentum sector whose NC parameter is $\beta$. Specially, we will show that the latter case causes more ability in tuning and problem debugging of time solutions of interested variables.

The work is organized as follows: Section 2 is devoted to introduce the modified gravity model we are going to deal with in the rest of the paper. In the first subsection of the third section we have presented the time evolution of model and in its second subsection we will solve the corresponding cosmology but this time in the NC framework. We finally summarize the work in section 4 by a discussion about how the cosmological picture is changed when one takes into account the NC considerations.

## 2 The model

Let us start with an $f(\mathcal{R})$ theory whose action in the Jordan frame may be written as

$$A = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{R}),$$

(1)

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{R}$ is the Ricci scalar corresponding to $g_{\mu\nu}$ and we let $f$ to be an arbitrary function which its form will be specified later. Under the conformal transformation

$$\tilde{g}_{\mu\nu} = e^{\phi} g_{\mu\nu},$$

(2)

where

$$\phi = \ln f'(\mathcal{R}),$$

(3)

the above action can be expressed in the Einstein frame as

$$A = \frac{3}{2\kappa} \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\mathcal{R}}{3} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right],$$

(4)

in which tilde-quantities should be evaluated with respect to the metric $\tilde{g}_{\mu\nu}$ and

$$V(\phi) = \frac{\mathcal{R} f'(\mathcal{R}) - f(\mathcal{R})}{f'(\mathcal{R})^2},$$

(5)

where a prime represents differentiation with respect to $\mathcal{R}$. As the background geometry we consider a spatially flat FRW space-time where following [11] its metric is given by

$$ds^2 = -\frac{\tilde{N}^2(t)}{\tilde{a}^2(t)} dt^2 + \tilde{a}^2(t) \delta_{ij} dx^i dx^j,$$

(6)

where $\tilde{N}(t)$ and $\tilde{a}(t)$ being the lapse function and the scale factor, respectively. The choice of the square of the scale factor dividing the lapse function as the 00 component of the metric seems to be a little unusual. However, as we will see it simplifies the calculations and gives a Hamiltonian
function with a quadratic form. By substituting (6) into (4) it is easy to obtain an effective point-like Lagrangian for the model as

$$L = \frac{1}{2} \tilde{N} \left( -\dot{a}^2 \dot{a}^2 + \frac{1}{4} \dot{a}^4 \dot{\phi}^2 \right) - \frac{\tilde{N}}{6} \dot{a}^2 V(\phi),$$

(7)

where we have re-scaled the scale factor as $\tilde{a} \rightarrow \tilde{a}/\sqrt{2}$. We may write the Lagrangian in a simpler form by introducing a new set of variables [12]

$$x = \frac{\tilde{a}^2}{2} \cosh \phi, \quad y = \frac{\tilde{a}^2}{2} \sinh \phi,$$

(8)

in terms of which Lagrangian (7) takes the form

$$L = \frac{1}{2} \tilde{N} (\dot{y}^2 - \dot{x}^2) - \frac{\tilde{N}}{3} (x - y) e^\phi V(\phi).$$

(9)

So far, we did not consider a specific form for the function $f(R)$. From now on, let us suppose that this function has the form $f(R) = f_0 R^m$, for which according to (5) the potential function will be

$$V(\phi) = \frac{m-1}{m^2 f_0} R^{2-m}.$$

(10)

Also equation (3) gives

$$R = (m f_0)^{\frac{1}{1-m}} e^{\frac{\phi}{m-1}},$$

(11)

and finally, in terms of $\phi$ the potential becomes

$$V(\phi) = V_0 e^{-\alpha \phi},$$

(12)

where we have introduced the abbreviations

$$V_0 = 3 \frac{m-1}{m} (m f_0)^{\frac{1}{1-m}} \quad \text{and} \quad \alpha = \frac{m-2}{m-1}.$$

For the numeric value $m = 3/2$ ($\alpha = 1$) the last term in the Lagrangian (9) will be simplified as

$$L = \frac{1}{2} \tilde{N} (\dot{y}^2 - \dot{x}^2) - \tilde{N} V_0 (x - y),$$

(13)

with the corresponding Hamiltonian

$$\mathcal{H} = \tilde{N} \left( -\frac{1}{2} \dot{p}_x^2 + \frac{1}{2} \dot{p}_y^2 + V_0 (x - y) \right),$$

(14)

which satisfies the constraint

$$\mathcal{H} = 0.$$

(15)

Therefore, we are led to a two dimensional minisuperspace model whose coordinates vary in the intervals $0 < a < \infty$, $-\infty < \phi < +\infty$. The boundaries of this two dimensional manifold may be divided into the nonsingular and singular types. By definition, at the nonsingular boundary we have $a = 0$ with $|\phi| < +\infty$, while at the singular boundary, at least one of the two variables is infinite [13]. When we use the variables $x$ and $y$, introduced in (8), to recover the above mentioned minisuperspace, the range of them should be restricted as $x > 0$, $x > |y|$, and therefore the nonsingular boundary may be represented by $x = y = 0$. In what follows we are going to deal with the resulting cosmological solution based on the Hamiltonian (14) which seems to have more suitable form for study the subject in the NC framework.
3 Cosmological dynamics

In the context of the Hamiltonian formalism the classical solutions of the model described by Hamiltonian \( \mathcal{H} \) can be easily obtained. Since, we are interested to know how NC considerations affect the usual commutative solutions, in what follows we will discuss these two cases separately and compare the results with each other.

3.1 Commutative case

In this case the Poisson brackets for the classical phase space variables will be

\[
\{x_i, x_j\} = \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij},
\]

where \( x_i(i = 1, 2) = x, y \) and \( p_i(i = 1, 2) = p_x, p_y \). Assuming \( \tilde{N} = 1 \), the equations of motion become

\[
\dot{x} = \{x, \mathcal{H}\} = -p_x, \quad \dot{p}_x = \{p_x, \mathcal{H}\} = -V_0,
\]

\[
\dot{y} = \{y, \mathcal{H}\} = p_y, \quad \dot{p}_y = \{p_y, \mathcal{H}\} = V_0.
\]

These equations can be immediately integrated to yield the solutions

\[
x(t) = \frac{1}{2} V_0 t^2 - p_{0x} t + x_0, \quad p_x(t) = -V_0 t + p_{0x}, \tag{19}
\]

\[
y(t) = \frac{1}{2} V_0 t^2 + p_{0y} t + y_0, \quad p_y(t) = V_0 t + p_{0y}, \tag{20}
\]

where \( x_0, p_{0x}, y_0 \) and \( p_{0y} \) are some integration constants whose values, due to the zero energy condition \( \mathcal{H} = 0 \), are restricted by the following relation

\[
p^2_{0x} - p^2_{0y} = 2V_0(y_0 - x_0). \tag{21}
\]

We notice that the equations (19) and (21) have the same form of ones for a particle moving in a plane with a constant acceleration \( \mathbf{a} = (V_0, V_0) \) while \( -p_x(t) \) and \( p_y(t) \) play the role of its velocity components. Note that the condition \( x > 0 \) implies that \( p^2_{0x} - 2V_0 x_0 < 0 \), thus, equation (21) results in \( p^2_{0y} - 2V_0 y_0 < 0 \), which means that \( y > 0 \). Therefore, in classical cosmology only half of the minisuperspace: \( x > y > 0 \) or \( (\tilde{a} > 0, \phi < 0) \) is recovered by the dynamical variables \( x(t) \) and \( y(t) \).

Now, going back to the relation (8) we can find the scale factor and scalar field as

\[
\tilde{a}(t) = [8|p_{0x}|(V_0 t^3 + 2x_0 t)]^{1/4}, \tag{22}
\]

\[
\phi(t) = \frac{1}{2} \ln \left( \frac{V_0 t^2 + 2x_0}{2|p_{0x}| t} \right), \tag{23}
\]

in which we have set \( x_0 = y_0 \) and \( p_{0x} = p_{0y} \) compatible to (21). Now, let us assume the following form for the metric tensor appeared in action (11)

\[
g_{\mu\nu} = \text{diag}(-N^2, a^2, a^2, a^2). \tag{24}
\]

Then, from (2), after some simple algebra, we get the Gordan frame counterparts of the lapse function and the scale factor as

\[
N^2(t) = \frac{e^{-\phi}}{a^2} = \frac{1}{2(V_0 t^2 + 2x_0)^2}, \tag{25}
\]

\[
a^2(t) = \tilde{a}^2 e^{-\phi} = 4|p_{0x}| t. \tag{26}
\]

As is clear, the scale factor describes a deceleration universe, more accurately a pure radiation universe \( (\tilde{a}(t) \propto \sqrt{t}) \) for \( t > 0 \). Also, the lapse function has a singularity at \( t = 0 \) for initial condition \( x_0 = 0 \). In the next section, we will see how these problems may be solved in the NC frame.
3.2 Noncommutative case

Let us now go forward by considering the noncommutativity concepts in classical cosmology obtained above. Classically, noncommutativity can be described by a deformed product known as the star product law between two arbitrary functions on a $2N$-dimensional phase space as

$$(f * \alpha g)(q, p) = \exp \left[ \frac{1}{2} \alpha^{ab} \partial_{a}^{(1)} \partial_{b}^{(2)} \right] f(q_1, p_1) g(q_2, p_2) |_{q_1 = q_2 = q, p_1 = p_2 = p},$$

such that

$$\alpha_{ab} = \left( \begin{array}{cc} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{array} \right),$$

where $\theta$ and $\beta$ are the $N \times N$ antisymmetric matrices which represent the noncommutativity in coordinates and momenta, respectively. The relation between the above star-product of phase space functions and the Poisson brackets becomes more clear if the formula (27) is expressed as follows [14]

$$f * g = fg + \frac{1}{2} \{f, g\} + \sum_{k=2}^{\infty} \left( \frac{1}{2} \right)^{k} \frac{1}{k!} D_k(f, g),$$

where the bidifferential operator $D_k$ is defined as

$$D_k(f, g)(q, p) = \left[ \left( \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_2} \frac{\partial}{\partial p_1} \right)^{\delta_{ij}} f(q_1, p_1) g(q_2, p_2) \right]_{q_1 = q_2 = q, p_1 = p_2 = p}.$$ (30)

According to (27)-(30) one is led to the following definition for the Moyal bracket as a deformed Poisson bracket

$$\{f, g\}_\alpha = f * \alpha g - g * \alpha f = \{f, g\} + \sum_{k=2}^{\infty} \left( \frac{1}{2} \right)^{k} \frac{1}{k!} \left[ D_k(f, g) - D_k(g, f) \right],$$

which looks like an $\alpha$-commutation relation between two function $f$ and $g$ (a more detailed analysis of Moyal and Poisson brackets is given in [14]). Now, a simple calculation shows that the Poisson brackets between the phase space coordinates will be deformed as

$$\{x_i, x_j\}_\alpha = \theta_{ij}, \quad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij}, \quad \{p_i, p_j\}_\alpha = \beta_{ij}.$$ (32)

However, we may find a transformations on the classical phase space, as [15]

$$x_i' = x_i - \frac{1}{2} \theta_{ij} p_j, \quad p_i' = p_i + \frac{1}{2} \beta_{ij} x_j,$$

such that the canonical pair $(x_i, p_j)$ obeys the usual Poisson algebra (16) and then

$$\{x_i', x_j'\} = \theta_{ij}, \quad \{x_i', p_j'\} = \delta_{ij} + \sigma_{ij}, \quad \{p_i', p_j'\} = \beta_{ij},$$ (34)

where $\sigma_{ij} = -\frac{1}{8} (\theta^k_i \beta_{kj} + \beta^k_i \theta_{kj})$. These commutative relations have the same form as (32) with this difference that here the brackets are the ordinary Poisson brackets. Consequently, for introducing noncommutativity, it is more convenient to work with Poisson brackets \([31]\) than $\alpha$-star deformed Poisson brackets \([32]\). It is important to note that the relations represented by equations (32) are defined in the spirit of the Moyal product given above. However, in the relations defined by \([31]\), the variables $(x_i, p_j)$ obey the usual Poisson bracket relations so that the two sets of deformed and ordinary Poisson brackets represented by relations (32) and (34) should be considered as distinct.

In this work, we consider a noncommutative phase space in which $\theta_{12} = \theta$ and $\beta_{12} = \beta$, such that the non-vanishing Poisson brackets of the phase space variables are

$$\{x'_i, y'_j\} = \theta, \quad \{x'_i, p'_x\} = \{y'_j, p'_y\} = 1 - \theta \beta / 4, \quad \{p'_x, p'_y\} = \beta.$$ (35)
Also, transformations \[33\] take the form:

\[
\begin{align*}
x' &= x - \frac{1}{2} \beta p_y, \quad p'_x = p_x + \frac{1}{2} \beta y, \\
y' &= y + \frac{1}{2} \beta p_x, \quad p'_y = p_y - \frac{1}{2} \beta x.
\end{align*}
\]

On the other hand, to construct Hamilton’s equations of motion for the noncommutative phase space variables, we consider the Hamiltonian of the noncommutative model as having the same functional form as equation \[14\], but in which the dynamical variables satisfy the deformed Poisson brackets, that is

\[
\mathcal{H}_{nc} = -\frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + V_0 (x' - y'),
\]

where by using of \[36\] takes the form

\[
\mathcal{H}_{nc} = \frac{p_y^2 - p_x^2}{2} + \frac{\beta^2}{8} (x^2 - y^2) - \frac{\beta}{2} (xp_y + yp_x) - \frac{V_0}{2} (p_x + p_y) + V_0 (x - y).
\]

Therefore, the equations of motion read

\[
\begin{align*}
\dot{x} &= \{x, \mathcal{H}_{nc}\} = -p_x - \frac{\beta}{2} y - \frac{\theta V_0}{2}, \quad \dot{p}_x = \{p_x, \mathcal{H}_{nc}\} = -\frac{\beta^2}{4} x + \frac{\beta}{2} p_y - V_0, \\
\dot{y} &= \{y, \mathcal{H}_{nc}\} = p_y - \frac{\beta}{2} x - \frac{\theta V_0}{2}, \quad \dot{p}_y = \{p_y, \mathcal{H}_{nc}\} = \frac{\beta^2}{4} y + \frac{\beta}{2} p_x + V_0.
\end{align*}
\]

Upon integration of these equations we get the following expressions for the functions \(x(t)\) and \(y(t)\),

\[
\begin{align*}
x(t) &= Ae^{\beta t} + Be^{-\beta t} + ht + C_1, \\
y(t) &= -Ae^{\beta t} + Be^{-\beta t} + ht + C_2,
\end{align*}
\]

where \(h \equiv (1 - \theta \beta/4)V_0/\beta\) and \(A, B, C_1\) and \(C_2\) are integration constants which their values are restricted to satisfy the constraint equation \(\mathcal{H}_{nc} = 0\), that is

\[
2\beta^2 AB = h(C_1 - C_2).
\]

A calculation similar to what we have done in commutative case yields

\[
N^2(t) = e^{-\phi/\alpha^2} = \frac{1}{4(B(1 + \frac{\beta \theta}{4})e^{-\beta t} + \frac{\beta^2}{V_0}t + D)},
\]

\[
a^2(t) = a^2 e^{-\phi} = 4A((1 + \frac{\beta \theta}{4})e^{\beta t} + \frac{B \beta^3}{V_0}).
\]

The form of the scale factor may be simplified if we impose the initial condition \(a(0) = 0\), for which we obtain

\[
a^2(t) = 4A(1 + \frac{\beta \theta}{4})(e^{\beta t} - 1).
\]

Now, it can be seen that while for the early times of cosmic evolution the scale factor behaves as \(a(t) \propto \sqrt{t}\), i.e., is consistent with the behavior of the commutative scale factor, its late time behavior is like as an accelerated de-Sitter universe, that is \(a(t) \propto e^{\beta t/2}\). This may be interpreted such that the NC parameter \(\beta\) can play the role of cosmological constant and in this way may be considered as a candidate for dark energy. This means that the noncommutative effects derive the universe toward an accelerating phase without the need for any additional fields. As a matter of fact, we should note that the noncommutativity in the momentum sector of the phase space (with noncommutative

\[1\]It is important to point out that, transformations \[33\] can be inverted if \(1 - \beta \theta/4 \neq 0\).
parameter $\beta$) has its roots in the string theory corrections to the Einstein gravity \cite{16}. In this sense, such noncommutative models may be considered as an effective theory to describe the quantum effects in cosmology. Here, the introduction of noncommutativity causes additional terms to appear in Hamiltonian (38) as compared to (14) which may be interpreted as the effects of high energy corrections of a full theory \cite{17}. Now, since our model can address the issue of dark energy in a simple cosmological setting, it is reasonable to consider the noncommutative relations (35) as an alternative to the model theories with dark energy and the accelerating universe. Also, we see that unlike the commutative case\cite{2} in the NC frame the lapse function (44) is free of singularity. It is important to note that the NC parameter $\beta$ is responsible for removing this singularity and from this point of view the role of the NC parameter $\theta$ is only to shift when the singularity occurs. To illustrate this issue we may set $\beta = 0$ in the equations (43) and (44) and solve them once again to obtain

$$x(t) = \frac{1}{2}V_0t^2 - (p_{0x} + \theta V_0/2)t + x_0, \quad p_x(t) = -V_0t + p_{0x},$$  \hspace{1cm} (47)$$

$$y(t) = \frac{1}{2}V_0t^2 + (p_{0y} - \theta V_0/2)t + y_0, \quad p_y(t) = V_0t + p_{0y},$$  \hspace{1cm} (48)$$

in which the Hamiltonian constraint impose the following relation between the constants of integration

$$p_{0x}^2 - p_{0y}^2 - 2V_0(y_0 - x_0) + \theta(p_{0x} + p_{0y})V_0 = 0.$$ 

If as the commutative case we choose $x_0 = y_0$, unlike what we saw in the commutative case this constraint prevents the relation $p_{0x} = p_{0y}$ between the initial values of the momenta. Indeed, the constraint leads to $p_{0y} - p_{0x} = \theta V_0$, and hence the lapse function and the scale factor corresponding to the solutions (47) and (48) take the forms

$$N^2(t) = \frac{1}{2\left(V_0t^2 - \theta V_0 t + 2x_0 - \frac{\theta^2 V_0^2}{2}\right)} = \frac{1}{2V_0\left[(t - \frac{\theta}{2})^2 - \frac{3\theta^2}{4}\right]} + 4x_0,$$  \hspace{1cm} (49)$$

$$a^2(t) = 4|p_{0x} + \theta V_0/2|(t + \frac{\theta}{2}).$$  \hspace{1cm} (50)$$

As functions of time, these have the same time dependence as the commutative case, i.e. equations (25) and (26), but with a constant time-shift due to the existence of the NC parameter $\theta$. This means that the singularity has not removed and in this sense role of the parameter $\theta$ is only to change the initial conditions from which the cosmological functions began their evolution.

4 Summary

In this letter we have studied the classical evolution of a modified $f(R)$ cosmology with a noncommutative phase space. Motivation of such a study is that introduction of a deformed phase space can be interpreted as an effective theory which may bring some signals from a quantum theory. In this work after considering the scale factor and a scalar field and their momenta as the phase space variable, we introduced a set of new variables in terms of which the corresponding minisuperspace takes the form of a Minkowskian space. In this space we have dealt with the commutation relationships between variables. In the case where the phase space variables obey the usual Poisson algebra, i.e., they Poisson-commute with each other, and for a power law expression for the $f(R)$ function, the evolution of the universe is like the motion of a particle (universe) moving on a plane with a constant acceleration. We saw that in this case the universe decelerates its expansion both in early and late times of cosmic evolution, in contrast to the current observation data. We then looked for an accelerated phase of the universe, suggested by recent supernova observations, in the context of a deformed phase space of our $f(R)$ model. We showed that a late time acceleration will occur in the

\footnote{For $x_0 = 0$, the lapse function (25) had a singularity at $t = 0$.}
history of the universe due to introduction of noncommutativity in phase space. We have found that while the noncommutative parameter between the momenta is responsible for this acceleration and also removing a singularity from the classical model, the noncommutative parameter between the configuration variables only causes a time shift in the commutative solutions. All in all, our study showed that while the usual classical model cannot give an accelerated universe, a classical model with noncommutative phase space variables can derive a late time acceleration compatible with the current observations. In particular, we showed that the existence of the deformation $\beta$ parameter in the momentum sector can play the role of a cosmological constant at late times which means that such a noncommutativity can be related to the issue of the dark energy. However, note that in spite of the nature of the cosmological constant in general relativity as a parameter which is introduced by hand into the action to fix the problem of the static universe, here its appearance at late times is a result of some noncommutative structures which in turn, may be considered as a signal from the quantum effects in gravity.

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