GLOBAL KATO SMOOTHING AND STRICHARTZ ESTIMATES FOR
SCHRÖDINGER TYPE EQUATIONS WITH ROUGH DECAY
POTENTIALS

HARUYA MIZUTANI AND XIAOHUA YAO†

Abstract. Let $H = (-\Delta)^m + V$ be a higher-order elliptic operator on $L^2(\mathbb{R}^n)$, where $V$ is a general bounded decaying potential. This paper focuses on the global Kato smoothing and Strichartz estimates for solutions to Schrödinger-type equation associated with $H$. In particular, we first establish sharp global Kato smoothing estimates for $e^{itH}$, based on uniform resolvent estimates of Kato-Yajima type for the absolutely continuous part of $H$. As a consequence, we also obtain optimal local decay estimates. Using these local decay estimates, we then prove the full set of Strichartz estimates, including the endpoint case. Notably, we derive Strichartz estimates with sharp smoothing effects for higher-order cases with rough potentials, which are applicable to the study of nonlinear higher-order Schrödinger equations. Finally, we introduce new uniform Sobolev estimates of the Kenig-Ruiz-Sogge type, incorporating an additional derivative term, which are crucial for establishing the sharp Kato smoothing estimates.

1. Introduction

1.1. Main results. Consider the following higher-order elliptic operator with a potential on $L^2(\mathbb{R}^n)$:

$$H = H_0 + V(x), \quad H_0 = (-\Delta)^m,$$

where $n > 2m$, $m \in \mathbb{N}$, and $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian. The potential $V(x)$ is a real-valued measurable function that satisfies $|V(x)| \leq C \langle x \rangle^{-s}$ for some $s > 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

It is well-known that higher-order elliptic operators (including more general forms $P(D) + V$) have been extensively studied as general Hamiltonian operators in various contexts. For instance, Schechter [57] has explored their spectral theory, while Kuroda [45], Agmon [1] and Hörmander [30] have contributed to their scattering theory. Ben-Artzi and Devinatz [8] investigated the limiting absorption principle, and Davies [14], along with Davies and Hinz [15] and Deng et al. [13], have examined their semigroup theory. Additionally, Herbst and Skibsted [28, 29] studied the eigenfunctions of pseudodifferential operators (PDOs), and further interesting topics can be found in these works [7, 59].
For any bounded real potential $V$, it follows directly from Kato-Rellich theorem (see, e.g., Simon [60, Chapter 7]) that $H$ is a self-adjoint operator with the same Sobolev domain $H^{2m}(\mathbb{R}^n)$ of order $2m$ as $H_0$. In particular, Stone’s theorem asserts that the Schrödinger unitary group $e^{itH}$ provides the solution to Schrödinger equation:

$$(i\partial_t + H)\psi(t, x) = 0, \quad \psi(0, x) = \psi_0(x), \quad (t, x) \in \mathbb{R}^{1+n},$$

by $\psi(t) = e^{itH}\psi_0$ for $t \in \mathbb{R}$.

In this paper, we aim to establish global regularity and decay estimates for the solution of the Cauchy problem (1.1): global Kato smoothing estimates and Strichartz estimates. These estimates are not only interesting in themselves, but also crucial for investigating the local and global behavior of nonlinear dispersive equations with potentials (see, e.g., [51]). This work extends our previous study [50], where we considered the case with Hardy potential $V(x) = a|x|^{-2m}$ and more generally, critically decaying potentials $V(x) = O(\langle x \rangle^{-2m})$ under several repulsive conditions. It is well known that under such repulsive conditions, $H$ is purely absolutely continuous and has no eigenvalues.

Here using a different approach based on the limiting absorption principle, we establish these estimates for a general bounded potential $V(x)$ that decays slightly faster than $|x|^{-2m}$ as $|x| \to \infty$. Under this setting, $H$ may possibly have negative eigenvalues. Such potentials naturally arise in many physically relevant models, such as one-body higher-order Schrödinger operators or linearized operators associated with soliton solutions to higher-order nonlinear Schrödinger equations.

The first results are stated as follows:

**Theorem 1.1.** Let $m \in \mathbb{N}$, $m \geq 2$, $n > 2m$, $H = (-\Delta)^m + V$ and $|V(x)| \leq C\langle x \rangle^{-s}$ for $s > 2m$. Assume that $H$ has no positive eigenvalues and no zero resonance/eigenvalue (see Definition 3.3 in Section 3.2). Let $P_{ac}(H)$ denote the projection onto the absolutely continuous subspace of $H$. Then the following statements (i.e. the global kato smoothing estimates) are satisfied:

(i) If $m - n/2 < \gamma < m - 1/2$, then

$$\| |x|^{-m+\gamma}D^\gamma e^{itH}P_{ac}(H)\psi_0 \|_{L^2_t L_x^2} \lesssim \| \psi_0 \|_{L^2_x}. \tag{1.2}$$

where $D = -(i\partial_{x_1}, i\partial_{x_2}, \ldots, i\partial_{x_n})$, $|D| = \sqrt{-\Delta}$. In particular, as $\gamma = 0$, the following local decay estimate holds:

$$\| |x|^{-m}e^{itH}P_{ac}(H)\psi_0 \|_{L^2_t L_x^2} \lesssim \| \psi_0 \|_{L^2_x}. \tag{1.3}$$

(ii) If $\gamma = m - 1/2$, then for any $\epsilon > 0$,

$$\| \langle x \rangle^{-1/2-\epsilon}D^{m-1/2}e^{itH}P_{ac}(H)\psi_0 \|_{L^2_t L_x^2} \lesssim \| \psi_0 \|_{L^2_x}. \tag{1.4}$$
In particular, $e^{itH}\psi_0$ belongs to $\mathcal{H}^{m-1/2}_{L \text{oc}}(\mathbb{R}^n)$ for a.e. $t \in \mathbb{R}$ and initial data $\psi_0 = P_{ac}(H)\psi_0 \in L^2(\mathbb{R}^n)$, and satisfies the following local smoothing estimate:

$$\int_{-\infty}^{\infty} \int_{|x| \leq R} |D|^{m-1/2} e^{itH} \psi_0|^2 \, dx \, dt \leq C(R) \|\psi_0\|_{L^2_x}^2. \quad (1.5)$$

**Remark 1.2.** Several remarks are given as follows:

- We first note that for $m = 1$, Kato [37] showed the absence of positive eigenvalues for the operator $H = -\Delta + V$ with potentials decaying as $V = o(\langle x \rangle^{-1})$ as $|x| \to \infty$. However, for $m \geq 2$, the higher-order Schrödinger operators $H = (-\Delta)^m + V$ may have discrete eigenvalues embedded in the positive real line, even for smooth potentials $V \in C^\infty_0(\mathbb{R}^n)$ (see, e.g., Remark [3.2] below for the construction of counterexamples). This observation demonstrates that the decay and regularity of potentials do not always prevent the appearance of positive eigenvalues in higher-order cases.

- On the other hand, it is known from the work of Feng et al. [20] that if a bounded potential satisfies the repulsive condition (i.e., $(x \cdot \nabla)V \leq 0$), then for each $m \geq 1$, the operator $H = (-\Delta)^m + V$ has no eigenvalues. This result is particularly useful for further studies of higher-order dispersive problems. Additionally, for this and other assumptions given in Theorem [1.1], more detailed comments can be found in Subsection [1.2] and Remark [3.2] below.

- Furthermore, we would like to provide additional remarks on the Kato smoothing estimates discussed above. When $V = 0$ (i.e., $H = (-\Delta)^m$), local smoothing estimates like [(1.5)], originally traced back to Kato [38] for the KdV equation, were first proved by Constantin and Saut [10] for general dispersive equations. These estimates were further studied by many other authors, including Ben-Artzi and Devinatz [4], and Kenig-Ponce-Vega [41].

  The global-in-time smoothing estimates [(1.2)] and [(1.4)] were first established by Kato and Yajima [39] for $m = 1$ (i.e., the Laplacian operator $-\Delta$) using uniform resolvent estimates, based on the smooth perturbation method [36]. These results were also reproved by Ben-Artzi and Klainerman [5] using the spectral measure integral.

  For higher-order operators $(-\Delta)^m$ with $m \geq 2$, and even for fractional operators $(-\Delta)^\alpha$ with $\alpha > 0$, optimal global smoothing estimates such as [(1.2)] and [(1.4)] have been studied in the works of Ruzhansky and Sugimoto [55]. References therein provide further insights into these results.

- In the case where $V \neq 0$, local smoothing estimates have been considered by Constantin and Saut [11] for (higher-order) Schrödinger equations, even under general perturbations of the form $V(x, D)$. In particular, they proved the following local-in-time and space smoothing estimate (see Corollary 2.4 of [11]):

$$\int_{-T}^{T} \int_{|x| \leq R} |D|^{m-1/2} \psi(t, x)|^2 \, dx \, dt \leq C(T, R) \|\psi_0\|_{L^2_x}^2, \quad (1.6)$$
which is weaker than the estimate (1.5) since $C(T, R)$ depends on $T$.

For global-in-time smoothing estimates with potentials, the situation is more delicate than in the local-in-time case (1.6), as it particularly depends on the global time and space behavior of the solution $e^{itH}\psi_0$ and the spectral properties of $H$ at the threshold (i.e., the critical values of the symbol $P(\xi)$). For $H = -\Delta + V$ (i.e., $m = 1$) under the same conditions as in Theorem 1.1, Ben-Artzi and Klainerman [5] obtained the following (similar but not identical) global estimate:

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\langle x \rangle^{-1/2-\epsilon} (1 + H)^{1/4} e^{itH} \psi_0|^2 \, dx \, dt \leq C \|\psi_0\|_{L^2}^2,$$

(1.7)

(see also [48, Theorem 1.10], where the same estimate as (1.4) was obtained for $m = 1$).

To the best of our knowledge, global smoothing estimates like (1.2) and (1.4) are less known for $m \geq 2$. In particular, we remark that the estimates (1.2) and (1.4) in Theorem 1.1 are optimal when compared to the free case (see, e.g., Ruzhansky and Sugimoto [55]). Moreover, the local decay estimates (1.3) will play a crucial role in establishing the endpoint Strichartz estimates for equation (1.2) (see Theorem 1.4 below).

Besides, in the following Theorem 1.3, by an abstract argument due to [36] and [12], we are also able to establish Kato smoothing effect for the solution $\psi$ of the inhomogeneous equation

$$(i\partial_t + H)\psi(t, x) = F(t, x), \quad \psi(0, x) = \psi_0(x),$$

(1.8)

with data $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $F \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ (the Schwartz function space). The solution $\psi$ is given by the Duhamel formula:

$$\psi = e^{itH}\psi_0 + i \int_0^t e^{i(t-s)H} F(s) \, ds.$$

(1.9)

**Theorem 1.3.** Under the same conditions as in Theorem 1.1, the solution $\psi$ to (1.8) given by (1.9) satisfies the following statements:

(i) If $m - n/2 < \gamma < m - 1/2$, then

$$\| |x|^{-m+\gamma} D^\gamma P_{ac}(H)\psi \|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2} + \| |x|^{m-\gamma} D^{-\gamma} F \|_{L^2_t L^2_x}.$$  

Furthermore, if $\gamma = 0$, then the following local decay estimate holds:

$$\| |x|^{-m} P_{ac}(H)\psi \|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2} + \| |x|^m F \|_{L^2_t L^2_x}.$$  

(ii) If $\gamma = m - 1/2$, then for any $\epsilon > 0$,

$$\| |x|^{-1/2-\epsilon} D^{m-1/2} P_{ac}(H)\psi \|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2} + \| |x|^{1/2+\epsilon} D^{-m+1/2} F \|_{L^2_t L^2_x}.$$  

Finally, we come to state the results on the Strichartz estimates for the equation (1.8) above under the same conditions as in Theorem 1.1 and will see that the $L^p$-type of global smoothing estimates can happen for higher-order dispersive equations. To the end, recall that $(1/p, 1/q) \in [0, 1]^2$ is said to be a (sharp) $\alpha$-admissible pair if

$$2 \leq p, q \leq \infty, \quad 1/p = \alpha/(2 - 1/q), \quad (p, q, \alpha) \neq (2, \infty, 1).$$ (1.10)

**Theorem 1.4.** Under the same conditions as in Theorem 1.1, the solution $\psi$ to (1.8) satisfies the following statements:

(i) If $(p_1, q_1)$ and $(p_2, q_2)$ satisfy (1.10) with $\alpha = n/(2m)$, then $\psi$ satisfies the following standard Strichartz estimates:

$$\left\| P_{\alpha c}(H)\psi \right\|_{L^p_t L^q_x} \lesssim \left\| \psi_0 \right\|_{L^2_x} + \left\| F \right\|_{L^p_t L^q_x}. \tag{1.11}$$

In particular, the following endpoint Strichartz estimates hold:

$$\left\| e^{itH} P_{\alpha c}(H)\psi_0 \right\|_{L^2_t L^{\frac{2n}{m}}_x} \lesssim \left\| \psi_0 \right\|_{L^2_x}, \tag{1.12}$$

where

$$\left\| \int_0^t e^{i(t-s)H} P_{\alpha c}(H)F(s)ds \right\|_{L^2_t L^{\frac{2n}{m}}_x} \lesssim \left\| F \right\|_{L^p_t L^q_x}. \tag{1.13}$$

(ii) Let $(p_1, q_1)$ and $(p_2, q_2)$ satisfy (1.10) with $\alpha = n/2$. Then $\psi$ satisfies the following improved Strichartz estimates with gain of regularity:

$$\left\| |D|^{2(m-1)/p_1} P_{\alpha c}(H)\psi \right\|_{L^p_t L^q_x} \lesssim \left\| \psi_0 \right\|_{L^2_x} + \left\| |D|^{2(1-m)/p_2} F \right\|_{L^p_t L^q_x}. \tag{1.14}$$

(iii) Let $\psi_0 = P_{\alpha c}(H)\psi_0 \in L^2(\mathbb{R}^n)$, $F \equiv 0$ and $(p_1, q_1) = (2, 2n/(n-2))$. Then the estimate (1.14) implies that $\psi = e^{itH}\psi_0$ belongs to $F^m_{-\infty} L^{p_1}_{\text{loc}}$ for a.e. $t \in \mathbb{R}$, and satisfies the following global $L^p$-smoothing estimate (comparing with the estimate (1.5)):

$$\int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^n} |D|^{m-1} e^{iH} \psi_0 \frac{2n}{m-2} dx \right)^{\frac{m-2}{2}} dt \leq C \left\| \psi_0 \right\|_{L^2_x}^2. \tag{1.15}$$

Note that when $m = 1$, the inequalities (1.11) and (1.14) are the same and have been proved by Rodnianski-Schlag 54. However, for $m > 1$, the situations are different. In fact, if $(p, q)$ is $n/2$-admissible and $(p, q_1)$ is $n/(2m)$-admissible, then $p, q, q_1$ must satisfy $1/q - 1/q_1 = 2(m-1)/(np)$. Sobolev’s inequality then implies

$$\left\| f \right\|_{L^q} \lesssim \left\| |D|^{2(m-1)/p} f \right\|_{L^p_x}. \tag{1.16}$$

Hence (1.11) immediately follows from (1.14). In this sense, (1.14) has an additional smoothing effect compared with (1.11) in the higher-order case $m > 1$.

Moreover, we notice that such Strichartz estimates with gain of regularity above have been extensively studied for the free case $H = (-\Delta)^m$ for all $m > 1$ and played an important
role in the study of higher-order and fractional NLS equations (see e.g. [11], [52] and references therein). Therefore, we believe that Theorems 1.4 have many potential applications to higher-order NLS equations with potentials satisfying $V(x) = O(|x|^{-2m-\varepsilon})$.

Finally, we should mention that, besides of Kato smoothing and Strichartz estimates above, there are some recent interesting works on the pointwise time-decay estimates of $e^{itH}$ for higher-order Schrödinger operators with fast decay potentials. In case of $H = (-\Delta)^m + V$, for instance, we can refer to [21], [25] and [16] for Kato-Jensen estimates or $L^1-L^\infty$ estimates of $e^{itH}$ in the case $m = 2$, and [20] for general $m \geq 2$. However, comparing with the abundant point-wise decay results on Schrödinger operators $-\Delta + V$ (see e.g. Journé et al [35], Yajima [64], Schlag’s survey [58] and references therein), the $L^1-L^\infty$ decay estimates for higher-order Schrödinger operators are still far from completion and deserve to be further investigated.

1.2. Further remarks. We will make more remarks on all theorems above. In particular, the optimality of the conditions for the operator $(-\Delta)^m + V(x)$ is discussed.

- **(The restriction of dimension $n > 2m$)** The dimensional restriction $n > 2m$ plays a critical role in the validity of several important estimates, including the local decay estimates with $\gamma = 0$ (as in equation (1.2)), uniform Sobolev estimates in Theorem 2.4 and the endpoint Strichartz estimates in Theorem 1.4 when either $p_1 = 2$ or $p_2 = 2$. This restriction ensures the necessary spatial and decay conditions for these estimates to hold.

  For instance, without the condition $n > 2m$, the estimates do not hold even for the free operator $H = H_0$. The failure of these estimates for the free operator implies that the methods used to establish the estimates for more general operators would also break down based on the perturbation argument used in Sections below.

- **(The decay rate of potential $V$)** The decay index $s > 2m$ is indeed critical for the validity of several estimates, particularly when dealing with Schrödinger operators (i.e., $m = 1$). For example, in the case of Schrödinger operators ($m = 1$), the work of Goldberg, Vega, and Visciglia [23] demonstrated that Strichartz estimates fail to hold for certain classes of repulsive potentials that decay slower than $|x|^{-2}$. This shows that for potentials with slower decay, even fundamental estimates as Strichartz estimates collapse, leaving only the trivial $L^\infty_t L^2_x$ estimate. This highlights the delicate balance between the decay of the potential and the validity of key dispersive and smoothing estimates.

  In contrast, in our previous paper [50], we considered higher-order and fractional Schrödinger operators with scaling-critical potentials of the form $V(x) = O(|x|^{-2m})$, alongside certain conditions on $\nabla V$. In that work, we successfully established Kato smoothing and Strichartz estimates for higher-order Schrödinger operators with critical-decay Hardy-type potentials of the form $a|x|^{-2m}$. However, the approach in [50] was based on Mourre theory, which is fundamentally different from the method used in the current paper. It’s important to note that the potential $V(x)$ considered in this
work generally does not satisfy the assumptions in [50], which is why the two results do not overlap.

• (The absence of positive eigenvalues of $H$) The assumption regarding the absence of positive eigenvalues is indeed crucial for the dispersive analysis of the time-evolution operator $e^{itH}$. For the case of $m = 1$, corresponding to the Schrödinger operator $H = -\Delta + V$, Kato [37] established that if the potential decays as $|V(x)| = o(|x|^{-1})$ as $|x| \to \infty$, then the operator has no positive eigenvalues embedded in the continuous spectrum. This foundational result was later generalized by Agmon [2], Simon [61], and Froese et al. [22], Ionescu and Jerison [33] and Koch and Tataru [44]. However, for higher-order operators of the form $H = (-\Delta)^m + V$ with $m \geq 2$, the situation is more complicated. In fact, for any even $m = 2k$ (with $k \in \mathbb{N}$), it has been shown that there exist compactly supported smooth potentials $V$ such that $H = (-\Delta)^m + V$ can have positive eigenvalues.

Despite this, there are general virial criteria that provide conditions for the absence of positive eigenvalues for a wider class of higher-order (and even fractional) operators. These criteria often apply to repulsive potentials that satisfy conditions $x \cdot \nabla V \leq 0$.

1.3. Organization of the paper. In this subsection, we will describe the organization of paper and outline some specific results and ideas.

In Section 2, the limiting absorption principle of resolvent $R_0(z)$ of $H_0 = (-\Delta)^m$ was first considered, which says that the resolvent operator $R_0(z)$ can continue into the spectrum point $\lambda \geq 0$ if one considers $R_0(z)$ as an operator function with value in $\mathcal{B}(L^2_s, L^2_{-s})$ for some $s > 0$. For any $\lambda > 0$ (regular values), we need only choose $s > 1/2$ due to Agmon [1]. For $\lambda = 0$ (threshold value), the limit is more complex than regular values since the zero is the critical point (i.e. $\nabla|\xi|^{2m} = 0$ when $\xi = 0$), we need to take $s > m$ for $\lambda = 0$.

Secondly, we also show the following uniform Sobolev estimates in the sense of Kenig-Ruiz-Sogge type:

$$\|D^\alpha u\|_{L^2} \lesssim \|(-\Delta)^m - z\|_{L^p} u \|_{L^p}, \quad z \in \mathbb{C}, \quad u \in C^\infty_0(\mathbb{R}^n),$$

where $2m - n < \alpha \leq 2m - 2n/(n+1)$, $1/p - 1/q = (2m - \alpha)/n$, $1 < p < 2n/(n+1)$ and $2n/(n-1) < q < \infty$.

Note that when $2m - n < 0$ and $\alpha = 0$, the uniform estimates have been proved by Kenig-Ruiz-Sogge [12] for $m = 1$ and Huang-Yao-Zheng [31] for $m \geq 2$. When $\alpha \neq 0$, these estimates with derivatives are new for all $m \geq 2$, which are indispensable to show the sharp Kato smoothing estimates of $e^{itH}$ with potentials shown in Section 3.2 below.

In Section 3, we will prove Theorems 1.1 and 1.3. The main point of the proofs is to establish the following uniform resolvent estimates of Kato-Yajima type (i.e. $H$-supersmoothing estimates):

$$\sup_{\lambda \in \mathbb{R}, \theta \in (0,1]} \|x|^{-m+\gamma}|D|^\gamma P_{ac}(H)(H - \lambda \mp i\theta)^{-1}|D|^\gamma |x|^{-m+\gamma}\|_{L^2 \to L^2} < \infty,$$
and
\[ \sup_{\lambda \in \mathbb{R}, \theta \in (0,1]} \| \langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} P_{ac}(H)(H - \lambda \mp i\theta)^{-1} |D|^{m-1/2} \langle x \rangle^{-1/2-\epsilon} \|_{L^2 \to L^2} < \infty, \]
which in turn is based on the following resolvent formula:
\[ R_V(z) = R_0(z) - R_0(z)v[M(z)]^{-1} v R_0(z). \]

Here \( v = \sqrt{|V|}, U = \text{sgn} V(x) \) and \( M(z) = U + v R_0(z) v \) will be proved to have a uniformly bounded inverse on \( L^2(\mathbb{R}^n) \) at the absolutely continuous spectra regime of \( H \).

Section 4 is devoted to proving Strichartz estimates in Theorem 1.4. The proof of Theorem 1.4 relies on a perturbation method by Rodnianski-Schlag [54] (see also [9], [8]). Specifically, let \( U_H, \Gamma_H \) be homogeneous and inhomogeneous Schrödinger evolutions defined by
\[ U_H f = e^{itH} f, \Gamma_H F = \int_0^t e^{i(t-s)H} F(s) ds. \]

Then they satisfy the following Duhamel formulas
\[ U_H = U_{H_0} + i\Gamma_{H_0} V U_H, \quad \Gamma_H = \Gamma_{H_0} + i\Gamma_{H_0} V \Gamma_H = \Gamma_{H_0} + i\Gamma_{H_0} V \Gamma_{H_0}. \]

Using these formulas, Sobolev’s inequality (1.16), (1.3) in Theorem 1.1 and the same Strichartz estimates as (4.1) for \( U_{H_0} \) and \( \Gamma_{H_0} \), we can finish the proof of Theorem 1.4.

In Section 5, we will give the proofs of Theorems 2.1 and 2.4 based on many specific presentations of the free resolvent \( R_0(z) = ((-\Delta)^m - z)^{-1} \). For instance, in order to deal with limiting absorption principle at the zero energy \( z = 0 \) in Theorems 2.1, the following formula
\[ R_0(z) = \frac{1}{mz} \sum_{\ell=0}^{m-1} z_\ell (-\Delta - z_\ell)^{-1}, \quad z_\ell = z^\frac{1}{m} e^{\frac{2\pi i}{m}}, \quad z \in \mathbb{C} \setminus [0, \infty), \]
will be used. Moreover, in the proof of Theorem 2.4, we will use oscillatory integral techniques from Harmonic analysis based on the asymptotic properties of the Fourier transform of the kernel of \( R_0(z) \). These details will be left to the last section.

1.4. Notations. In order to state our main results, we will use the following notations.

- \( \langle x \rangle \) stands for \( \sqrt{1 + |x|^2} \).
- Let \( \mathbb{C}^\pm = \{ z \in \mathbb{C}; \pm \text{Im} z > 0 \} \) denote the upper and lower complex planes, respectively, and \( \overline{\mathbb{C}^\pm} \) be the closures of \( \mathbb{C}^\pm \).
- Let \( \mathcal{B}(X,Y) \) be the Banach space of bounded operators from \( X \) to \( Y \), \( \mathcal{B}(X) = \mathcal{B}(X,X) \) and \( \| \cdot \|_{X \to Y} := \| \cdot \|_{\mathcal{B}(X,Y)} \). Let \( \mathcal{B}_\infty(X) \) be the set of compact operators on \( X \).
- For \( p \in [1, \infty], p' := p/(p-1) \) is its Hölder conjugate exponent.
- For each \( s \in \mathbb{R}, \mathcal{H}^s(\mathbb{R}^n) \) denotes the \( L^2 \)-based Sobolev space of order \( s \) and \( L^2_s(\mathbb{R}^n) \) denotes the usual weighted space consisting of the function \( f \) with \( \langle x \rangle^s f \in L^2(\mathbb{R}^n) \).
Theorem 2.1. Let $H_0 = (-\Delta)^m$ be the polyharmonic operator on $L^2(\mathbb{R}^n)$, where $m \in \mathbb{N}$, $m \geq 2$, $n > 2m$ and $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ is the Laplacian. It is well-known that $H_0$ is self-adjoint on $L^2(\mathbb{R}^n)$ with the domain $\mathcal{D}(H_0) = \mathcal{H}^{2m}(\mathbb{R}^n)$ and the spectrum of $H_0$ is $[0, \infty)$ by the Fourier transform presentation $\hat{H}_0 f(\xi) = |\xi|^{2m} \hat{f}(\xi)$. Moreover, the resolvent $R_0(z)$ is analytic on $z \in \mathbb{C} \setminus [0, \infty)$ in the uniform operator topology of $\mathcal{B}(L^2(\mathbb{R}^n))$. As $z$ closes to $\lambda \geq 0$, it is clear that $R_0(z)$ can not continue into the spectrum point $\lambda \geq 0$ in the uniform operator topology of $\mathcal{B}(L^2(\mathbb{R}^n))$ (or any weak $L^2$ topology). The celebrated limiting absorption principle, however, shows that such limits do exist if one considers $R_0(z)$ as an operator function with value in $\mathcal{B}(L^2_s, L^2_{-s})$ for some $s > 0$. Indeed, for any $\lambda > 0$ (regular values), we need only choose $s > 1/2$.

For $\lambda = 0$ (the threshold value), the limit is more complex than the case with regular values since the zero is the critical point (i.e. $\nabla|\xi|^{2m} = 0$ when $\xi = 0$). The studying situation depends on the specific operator. In the present case (i.e. $H_0 = (-\Delta)^m$), we need to take $s > m$ for $\lambda = 0$.

In the following two theorems, we will present the limiting absorption principle of $R_0(z)$ and collect several interesting uniform resolvent estimates. The proofs of Theorems 2.1 and 2.4 will be given in Section 5 below.

2. Free resolvent estimates for $H_0 = (-\Delta)^m$

2.1. The limiting absorption principle for $R_0(z)$. Let $H_0 = (-\Delta)^m$ be the polyharmonic operator on $L^2(\mathbb{R}^n)$, where $m \in \mathbb{N}$, $m \geq 2$, $n > 2m$ and $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ is the Laplacian. It is well-known that $H_0$ is self-adjoint on $L^2(\mathbb{R}^n)$ with the domain $\mathcal{D}(H_0) = \mathcal{H}^{2m}(\mathbb{R}^n)$ and the spectrum of $H_0$ is $[0, \infty)$ by the Fourier transform presentation $\hat{H}_0 f(\xi) = |\xi|^{2m} \hat{f}(\xi)$.

For any $z \in \mathbb{C} \setminus [0, \infty)$, the resolvent $R_0(z) = (H_0 - z)^{-1}$ is well-defined as a bounded operator on $L^2(\mathbb{R}^n)$ and its operator norm is

$$|| (H_0 - z)^{-1} ||_{L^2 - L^2} = d(z, [0, \infty))^{-1}.$$ 

Moreover, the resolvent $R_0(z)$ is analytic on $z \in \mathbb{C} \setminus [0, \infty)$ in the uniform operator topology of $\mathcal{B}(L^2(\mathbb{R}^n)).$ As $z$ closes to $\lambda \geq 0$, it is clear that $R_0(z)$ can not continue into the spectrum point $\lambda \geq 0$ in the uniform operator topology of $\mathcal{B}(L^2(\mathbb{R}^n))$ (or any weak $L^2$ topology). The celebrated limiting absorption principle, however, shows that such limits do exist if one considers $R_0(z)$ as an operator function with value in $\mathcal{B}(L^2_s, L^2_{-s})$ for some $s > 0$. Indeed, for any $\lambda > 0$ (regular values), we need only choose $s > 1/2$.

For $\lambda = 0$ (the threshold value), the limit is more complex than the case with regular values since the zero is the critical point (i.e. $\nabla|\xi|^{2m} = 0$ when $\xi = 0$). The studying situation depends on the specific operator. In the present case (i.e. $H_0 = (-\Delta)^m$), we need to take $s > m$ for $\lambda = 0$.

In the following two theorems, we will present the limiting absorption principle of $R_0(z)$ and collect several interesting uniform resolvent estimates. The proofs of Theorems 2.1 and 2.4 will be given in Section 5 below.

Theorem 2.1. Let $n > 2m$, $m \geq 2$ and $H_0 = (-\Delta)^m$. Consider $R_0(z) = (H_0 - z)^{-1}$ as an analytic operator function on $\mathbb{C} \setminus [0, \infty)$ with values in $\mathcal{B}(L^2_s, L^2_{-s})$ for some $s > 0$. Then the following conclusions hold.

(i) If $s > 1/2$, then the following two limits exist for any $\lambda > 0$ in the uniform operator topology of $\mathcal{B}(L^2_s, L^2_{-s})$:

$$\lim_{z \rightarrow \lambda} R_0(z) = R_0^\pm (\lambda),$$

where $R_0^\pm (\lambda)$ can be written as follows:

$$\langle R_0^\pm (\lambda) f, g \rangle = \text{p.v.} \int_{\mathbb{R}^n} \frac{\hat{f}(\xi) \overline{\hat{g}(\xi)}}{|\xi|^{2m} - \lambda} d\xi \pm \frac{1}{2m} \lambda^{\frac{1}{2m}} \pi i \int_{|\xi| = \lambda^{\frac{1}{2m}}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\sigma(\xi),$$

where $\sigma(\xi)$ is the Radon-Nikodym derivative of the surface measure of $|\xi| = \lambda^{\frac{1}{2m}}$.
for \( f, g \in C^\infty_0(\mathbb{R}^n) \). In other words, \( R_0(z) \) is a uniform-operator-topology continuous function on \( \mathbb{C}^\pm \setminus \{0\} \) with values in \( \mathbb{B}(L^2_s, L^2_{-s}) \) for \( s > 1/2 \).

(ii) If \( s > m \), then \( R_0(z) \) is an uniform-operator-topology continuous function on \( \mathbb{C}^\pm \) (also on \( \mathbb{C}^{-} \)) with values in \( \mathbb{B}(L^2_s, L^2_{-s}) \), where we take \( R_0(0) = (-\Delta)^{-2m} \) and \( R_0(z) = R^\pm_0(\lambda) \) for \( z = \lambda \in \mathbb{C}^\pm \) and \( \lambda > 0 \).

(iii) The following uniform estimates for \( R_0(z) \) hold:

\[
\sup_{z \in \mathbb{C} \setminus (0, \infty)} \| R_0(z) \|_{L^2_{-s} \to L^2_{-s}} \leq C_{s,m} < \infty, \quad s > m; \tag{2.3}
\]

\[
\sup_{z \in \mathbb{C} \setminus (0, \infty)} \| |D|^{m-\frac{1}{2}} R_0(z) |D|^{m-\frac{1}{2}} \|_{L^2_{-s} \to L^2_{-s}} \leq C'_{s,m} < \infty, \quad s > 1/2. \tag{2.4}
\]

where \( |D| = \sqrt{-\Delta} \), \( C_{s,m} \) and \( C'_{s,m} \) are the two positive constants independent of \( z \).

Remark 2.2. For Theorem 2.1 above, we make several comments as follows:

- The statement (i) is actually well-known due to Agmon [1], where general higher-order elliptic operators \( P(D) \) or the differential operators of principal type were considered. Note that by Stone’s formula, we have

\[
\langle E'_{H_0}(\lambda) f, g \rangle = \frac{1}{2\pi i} \langle (R^+_{0}(\lambda) - R^-_{0}(\lambda)) f, g \rangle, \quad \lambda > 0, \tag{2.5}
\]

where \( E'(\lambda) \) is the spectral measure density of \( H_0 \). Hence for any \( s > 1/2 \), it follows by the trace lemma that (see e.g. [5]):

\[
|\langle E'_{H_0}(\lambda) f, f \rangle| = \frac{1}{2m} \lambda^{1-2m} \int_{|\xi| = \lambda^{1/2m}} |\hat{f}(\xi)|^2 \ d\sigma \lesssim \min(\lambda^{\frac{2s-2m}{2m}}, \lambda^{\frac{1-2m}{2m}}) \| f \|_{L^2}^2. \tag{2.6}
\]

- In the statement (ii), the main point is to prove that \( R_0(z) \) is continuous at \( z = 0 \) in the cost of higher weight index. We will adopt the arguments from Agmon [1] to give the proof of (ii) in Section 5 where we use the following resolvent decomposition formula:

\[
R_0(z) = ((-\Delta)^m - z)^{-1} = \frac{1}{mz} \sum_{\ell=0}^{m-1} z_{\ell} (-\Delta - z_{\ell})^{-1}, \tag{2.7}
\]

where \( z_{\ell} = z^{\frac{1}{m}} e^{i \frac{2\pi \ell}{m}}, z \in \mathbb{C} \setminus [0, \infty) \). This can make us to use the resolvent kernel of \( -\Delta \) to present the kernel of \( R_0(z) \) for \( z \neq 0 \). Moreover, note that \( \|(-\Delta - z)\|_{L^2_{-s} \to L^2_{-s}} = O(|z|^{-1/2}) \) for any \( s > 1/2 \) as \( |z| \to \infty \) (see e.g. [43, p. 59]), we then immediately deduce the following high energy decay estimates from the formula (2.7):

\[
\| R_0(z) \|_{L^2_{-s} \to L^2_{-s}} \leq C_{s,m,\delta} |z|^{(1-2m)/2m}, \quad |z| \geq \delta > 0, \quad s > 1/2, \tag{2.8}
\]

where \( C_{s,m,\delta} \) is a positive constant depending on \( s, m, \delta \).
Remark 2.3. The further remarks about the estimates (2.3) and (2.4) are given as follows:

- The inequality (2.3) can be deduced from the following uniform Sobolev estimate:
  \[ \| (H_0 - z) u \|_{L^{2q/2m}} \leq \| (H_0 - z) u \|_{L^{2q/m}}, \quad z \in \mathbb{C}, \quad u \in C_0^\infty(\mathbb{R}^n), \]
  (2.11)
  which was proved by Kenig-Ruiz-Sogge [42] for \( m = 1 \) and [31] for \( m \geq 2 \). In fact, if \( s > m \), then it follows by Hölder’s inequality that
  \[ \| (H_0 - z) u \|_{L^{2q/2m}} \leq \| (H_0 - z) u \|_{L^{2q/m}} \]
  (2.12)
  uniformly in \( z \in \mathbb{C} \setminus [0, \infty) \), which gives the desired estimate (2.3). Moreover, by means of the real interpolation theory, the estimate (2.11) can be refined to the estimate
  \[ \| u \|_{L^{2q/2m}} \leq \| (H_0 - z) u \|_{L^{2q/m}} \]
where \( L^{p,q}(\mathbb{R}^n) \) is the Lorentz space (see [24]). Since \( |x|^{-m} \in L^{n/m,\infty} \), by the weak Hölder inequality (see [6]) and (2.11) we have the following sharp uniform estimates:
  \[ \sup_{z \in \mathbb{C} \setminus [0, \infty)} \| (H_0 - z) u \|_{L^{2q/m}} < \infty. \]
  (2.13)

- The inequality (2.4) is due to Agmon [1] Lemma A.2], where he actually showed that for any \( s > 1/2 \) and \( u \in C_0^\infty(\mathbb{R}^n), \)
  \[ \int (1 + |x|^2)^{-s} |P(D) u|^2 dx \leq C_{m,s} \int (1 + |x|^2)^s (|P(D) u|^2 dx, \]
  (2.14)
  where \( P(D) \) is any differential operator of order \( 2m \) and \( P^j(\xi) = \frac{\partial^j}{\partial \xi^j} P(\xi) \) for \( j = 1, 2, \ldots, n \). Taking \( P(D) = (-\Delta)^m - z \), then the desired inequality (2.4) immediately follows from the (2.14) above.

Besides of the special uniform Sobolev estimates (2.11) we actually can show the following uniform Sobolev estimates with derivatives, which will play key roles in sharp Kato smoothing estimates of \( e^{itH} \) with potentials shown in Section 3.2.
Theorem 2.4. Let $H_0 = (-\Delta)^m$ with $n > 2m$ and $R_0(z) = (H_0 - z)^{-1}$ for $z \in \mathbb{C} \setminus [0, \infty)$. Then for any $2m - n < \alpha \leq 2m - 2n/(n+1)$, the following uniform estimates hold:

$$
\|D|^{\alpha}u\|_{L^p} \lesssim \|(H_0 - z)u\|_{L^p}, \ z \in \mathbb{C}, \ u \in C_0^\infty(\mathbb{R}^n), \ (2.15)
$$

where $1/p - 1/q = (2m - \alpha)/n$, $1 < p < 2n/(n+1)$ and $2n/(n-1) < q < \infty$. Moreover, the following uniform $L^p$-$L^q$ resolvent estimates hold:

$$
\sup_{z \in \mathbb{C} \setminus [0, \infty)} \|D|^{\alpha/2}R_0(z)|D|^{\alpha/2}\|_{L^p-L^q} = \|D|^{\alpha}R_0(z)\|_{L^p-L^q} < \infty. \ (2.16)
$$

Remark 2.5. For any $\alpha, p, q$ satisfying the conditions in Theorem 2.4 one can find $p_0, p_1, q_0, q_1 \in (1, \infty)$ satisfying $p_0 < p < p_1$, $q_0 < q < q_1$ and the conditions in Theorem 2.4. The real interpolation theory (see [6]) implies that the estimate (2.16) still holds, where $L^p, L^q$ replaced by the Lorentz spaces $L^{p,2}, L^{q,2}$, respectively.

By virtue of the continuous embedding $L^p \subset L^{p,2}, L^{q,2} \subset L^q$, this gives a slightly stronger estimate than (2.16) which will be used in proving Theorem 3.4 below. We refer to [50, Appendix C] where the real interpolation theorems and basic properties of Lorentz spaces used in the present paper have been summarized.

Remark 2.6. Some further comments on Theorem 2.4 are given as follows

- Besides of the estimate (2.15) above, there exist actually more pairs $(p, q)$ such that the following Sobolev type estimates hold:

$$
\|D|^{\alpha}u\|_{L^p} \lesssim |z| \left(\frac{n}{2m} \left(\frac{1}{p} - \frac{1}{q}\right) \right)^{\frac{2m - \alpha}{2m}} \|((\Delta)^m - z)u\|_{L^p}, \ z \in \mathbb{C} \setminus \{0\}, \ (2.17)
$$

for $2m - n < \alpha \leq 2m - 2n/(n+1)$ and

$$
\frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2m - \alpha}{n}, \ 1 < p < \frac{2n}{n+1}, \ \frac{2n}{n-1} < q < \infty. \ (2.18)
$$

Clearly, the range of $\alpha$ is exactly from the conditions (2.18). When $\alpha = 0$, the estimates (2.17) have been proved by [42] for $m = 1$ and [31] for $m \geq 2$.

When $\alpha \neq 0$, these estimates (2.17) are new for all $m \geq 1$, which are indispensable to show the sharp Kato smoothing estimates of $e^{itH}$ with potentials shown in Section 3.2 below. In particular, note that uniform Sobolev estimates with $\alpha = 0$ above have played a crucial role in the unique continuity of elliptic equations and the $L^p$-multiplier estimates (see e.g. [42], [31]), as well as in recent developments on Lieb-Thirring type inequalities (see e.g. [17], [18], [47] and reference therein). Hence, the authors believe that the estimates (2.15) and (2.17) with $\alpha \neq 0$ would potentially have more further applications.

- The range $2m - n < \alpha \leq 2m - 2n/(n+1)$ in Theorem 2.4 is optimal. Indeed, on the one hand, if the inequality (2.15) hold for $\alpha = 2m - n$, then we only have the pair $(p, q) = (1, \infty)$ satisfying the estimate (2.15) which is impossible since the classical embedding estimate $\|u\|_{L^\infty} \lesssim \|D|^\alpha u\|_{L^1}$ does not hold (if taking $z = 0$ in (2.15)).
One the other hand, if the inequality \[ (2.15) \] hold for \( \alpha > 2m - 2n/(n + 1) \), then we can choose some \( p_0 > 2(n + 1)/(n + 3) \) such that

\[
\sup_{z \in \mathbb{C} \setminus [0, \infty)} \| D^\alpha R_0(z) \|_{L^{p_0} \to L^{p_0'}} < \infty,
\]

which leads to the following boundary resolvent estimate as \( z \to \lambda \),

\[
\left| \langle D^\alpha R_0^0(\lambda)f, g \rangle \right| \lesssim \| f \|_{L^{p_0}} \| g \|_{L^{p_0}}, \quad f, \ g \in C_0^\infty(\mathbb{R}^n).
\] (2.19)

In view of the formula \[(2.2)\] of \( R_0^\pm(\lambda) \), if \( \lambda = 1 \), then the \[(2.19)\] gives the Fourier restriction estimate:

\[
\int_{S^{n-1}} |\hat{f}(\xi)|^2 d\sigma \lesssim \| \langle D \rangle^\alpha (R^+_0(1) - R_-^0(1)) f \| \lesssim \| f \|^2_{L^{p_0}},
\] (2.20)

for \( p_0 > 2(n + 1)/(n + 3) \). This contradicts with the famous Stein-Tomas theorem (see e.g. [24]), which says that Fourier restriction operator

\[
\mathfrak{R} : f \in L^p(\mathbb{R}^n) \mapsto \hat{f} |_{S^{n-1}} \in L^2(S^{n-1}),
\]

is bounded if only if \( 1 \leq p \leq 2(n + 1)/(n + 3) \).

- If \( q = p' \), since \( |x|^{-m+\gamma} \in L^{n/(m-\gamma),\infty} \), we learn by weak Hölder’s inequality (see [6]) that

\[
\| |x|^{-m+\gamma} |D^\gamma R_0(z) D^\gamma |x|^{-m+\gamma} \|_{L^2 \to L^2} \lesssim \| |D| \gamma R_0(z) D^\gamma \|_{L^{p,2} \to L^{p',2}},
\] (2.21)

if \( 1/p - 1/p' = 2(m - \gamma)/n \). Hence for any \( m - n/2 < \gamma \leq m - n/(n + 1) \), it follows immediately from the estimate \[(2.16)\] and Remark \[(2.5)\] that the uniform estimates hold:

\[
\sup_{z \in \mathbb{C} \setminus [0, \infty)} \| |x|^{-m+\gamma} |D^\gamma R_0(z) D^\gamma |x|^{-m+\gamma} \|_{L^2 \to L^2} < \infty.
\] (2.22)

However, we remark that the range of \( \gamma \) above is not sharp. Actually, the optimal range of \( \gamma \) such that the \[(2.22)\] holds is \( (m - n/2, m - 1/2) \) (see e.g. Kato-Yajima [39], Ruzhansky and Sugimoto [55]).

### 2.2. The compactness of \( \langle x \rangle^{-s} R_0(z) \langle x \rangle^{-s} \).

In the subsection, we first prove that operators \( \langle x \rangle^{-s} R_0(z) \langle x \rangle^{-s} \) is compact on \( L^2(\mathbb{R}^n) \) for any \( z \in \mathbb{C} \setminus [0, \infty) \) for any \( s > 0 \), then by the continuity in Theorem 2.1 we can extend the compactness to the boundary value operators \( \langle x \rangle^{-s} R_0^\pm(\lambda) \langle x \rangle^{-s} \) for sufficiently large \( s \).

Let \( s > 0 \), since \( \langle x \rangle^{-s} \) is bounded, it is enough to prove that \( \langle x \rangle^{-s} R_0(z) \) is compact for any \( s > 0 \). Moreover, by virtue of the resolvent formula

\[
R_0(z) = R_0(z') - (z - z') R_0(z') R_0(z), \quad z, z' \in \mathbb{C} \setminus [0, \infty),
\]

it hence suffices to show that \( \langle x \rangle^{-s} (1 + H_0)^{-1} \) is compact on \( L^2(\mathbb{R}^n) \) for any \( s > 0 \). Indeed, since the bounded functions \( f(x) = \langle x \rangle^{-s} \) and \( g(x) = (1 + |x|^{2m})^{-1} \) both decay to 0 as \( |x| \to \infty \), it hence follows that the operator \( f(X) g(D) = \langle x \rangle^{-s} (H_0 + 1)^{-1} \) is a compact operator of \( \mathcal{B}(L^2) \) (see e.g. Simon [60], p. 160]). Thus, by Theorem 2.1 (ii) and the closeness
of the family of compact operators $\mathcal{B}_\infty(L^2)$ in $\mathcal{B}(L^2)$, we immediately conclude the following result:

**Theorem 2.7.** If $s > m$, then $z \mapsto \langle x \rangle^{-s} R_0(z) \langle x \rangle^{-s}$ is an uniform-operator-topology continuous function on $\mathbb{C}^+$ (also on $\mathbb{C}^-$) with compact operator values in $\mathcal{B}_\infty(L^2)$, where we set $R_0(0) = (-\Delta)^{-2m}$ and $R_0(z) = R_0^\pm(\lambda)$ as $z = \lambda \in \mathbb{C}^\pm$ and $\lambda > 0$.

Note that, by Theorem 2.4(i), if $\lambda > 0$ then we only need $s > 1/2$ to show that the operators $\langle x \rangle^{-s} R_0^\pm(\lambda) \langle x \rangle^{-s}$ are compact. For $\lambda = 0$, the restriction $s > m$ is essentially optimal in the sense that $\langle x \rangle^{-m} R_0(0) \langle x \rangle^{-m}$ is not compact, although it is bounded on $L^2(\mathbb{R}^n)$.

3. Kato smoothing estimates of $e^{itH}$

In this section, we will show sharp Kato smoothing estimates of $e^{itH}$ with a general bounded decaying potential $V$. Let us first discuss the spectrum of $H = H_0 + V$ in Subsection 3.1 and then give our main results in Subsection 3.2.

3.1. The spectrum of $H_0 + V$. Let $n > 2m$, $H = H_0 + V$, $H_0 = (-\Delta)^m$ and $V(x)$ be a real valued measurable function satisfying $|V(x)| \lesssim \langle x \rangle^{-s}$ for some $s > 0$.

It is well known that $V$ is a relatively compact perturbation of $H_0$ (by using the compactness of $\langle x \rangle^{-s}(1 + H_0)^{-1}$ mentioned above). Hence, $H$ is a self-adjoint operator with the same domain as $\mathcal{D}(H_0) = \mathcal{H}^{2m}(\mathbb{R}^n)$ by Kato-Rellich’s theorem and its essential spectrum is the same set $[0, \infty)$ as $H_0$ by Weyl’s theorem. Moreover, the spectrum located at $(-\infty, 0)$ is only discrete eigenvalues of finite multiplicity with a possible limiting point at zero.

In particular, if $s > 1$ (i.e. the potential $V$ is short-range), then the positive eigenvalues embedding in the essential spectrum $(0, \infty)$ of $H$ are also discrete and of finite multiplicity, as well as their only possible limiting point is zero point (see e.g. Agmon [1]). Now let us sum up some spectral results of $H$ as follows:

**Lemma 3.1.** Let $n > 2m$, $H = (-\Delta)^m + V$ and $V(x)$ be a real valued measurable function satisfying $|V(x)| \leq C \langle x \rangle^{-s}$ for $s > 2m$. Then the following conclusions hold:

(i) The spectrum $\sigma(H) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N_0} < 0\} \cup [0, \infty)$, where each $\lambda_j$ is the negative discrete eigenvalue of $H$ and $N_0$ denotes the number of negative eigenvalues by counting its finite multiplicity, which satisfies with the following bound:

$$N_0 = |\{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N_0} < 0\}| \leq C_{n,m} \int_{\mathbb{R}^n} |V(x)|^{n/2m} dx < \infty. \quad (3.1)$$

Moreover, the positive eigenvalues embedding in $(0, \infty)$ are also discrete and of finite multiplicity, as well their only possible limiting point is zero point. Finally, the singular continuous spectrum is absent.

(ii) If $m = 2k$ ($k \in \mathbb{N}$), then there exists an even $C_0^\infty(\mathbb{R}^n)$-potential $V$ such that $H = H_0 + V$ has at least one positive eigenvalue.
(iii) If the potential $V$ further satisfies with the repulsive condition $(x \cdot \nabla)V(x) \leq 0$, then the point spectrum $\sigma_p(H) = \emptyset$. In particular, $H$ has no any embedding positive eigenvalues.

**Remark 3.2.** Some remarks are given as follows:

- In the case of $m = 1$ the number estimate of negative eigenvalues (3.1) is well-known and usually called by Cwikel-Lieb-Rozenbljum bound. see e.g. Simon [60, p. 674]. For the cases $m \geq 2$, it is due to Birman and Solomyak [7]. The statements (ii) and (iii) have been proved in [20, Section 7]. In particular, the conclusion (iii) also works for much general elliptic operator $P(D) + V$ and fractional operator $(-\Delta)^s + V$ with $s > 0$.

- For even $m \geq 2$, there exist higher-order Schrödinger type operators $H = (-\Delta)^m + V$ with a positive eigenvalue embedded in the continuous spectrum, even for $C_0^\infty$-potentials. Indeed, if we have a strictly positive smoothing function $\phi$ such that $\phi(x) = (-\Delta)^m \phi(x)$ for $|x| > \delta > 0$, then the potential

$\quad V(x) = \phi^{-1}(x) \left( \phi(x) - (-\Delta)^m \phi(x) \right)$. \quad (3.2)$

has compact support in $B(0, \delta)$ and satisfies $(-\Delta)^m \phi + V \phi = \phi$ (i.e. 1 is an eigenvalue of $H$).

For instance, in $n = 3$, since

$\quad \Delta(r^{-1}e^{-r}) = \left( \frac{d^2}{dr^2} + 2 \frac{d}{dr} \right) (r^{-1}e^{-r}) = r^{-1}e^{-r}, \quad r = |x| > \delta > 0$. \quad (3.3)$

so we can define a radial function $\phi(x) > 0$ such that $\phi(x) = |x|^{-1}e^{-|x|}$ for $|x| > \delta > 0$, which clearly satisfies $\phi(x) = (-\Delta)^m \phi(x)$ for $|x| > \delta > 0$. Thus by (3.2) we construct a potential $V(x) \in C_0^\infty(\mathbb{R}^3)$ such that $((-\Delta)^m + V)\phi = \phi$ for each even integer $m \geq 2$. For any other dimension $n$, we also obtain the same results if choosing the Bessel kernel function $G(|x|)$ of $(1 - \Delta)^{-1}$, instead of $|x|^{-1}e^{-|x|}$ (see e.g. [20, Section 7] for more details and references therein).

### 3.2. Uniform resolvent estimates and Kato smoothing estimates

In the subsection, we will show the following local $H$-supersmoothing estimates, which imply Kato smoothing estimates of $e^{itH}$. For the end, let us give the definition of zero resonance of $H$.

**Definition 3.3.** Zero is said to be a resonance of $H$ if there exists some

$\quad 0 \neq \psi \in \bigcap_{\sigma > m} L^2_\sigma(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$

such that $(-\Delta)^m \psi + V \psi = 0$ in the distributional sense. In particular, we say that zero is an eigenvalue of $H$ if $\psi \in L^2(\mathbb{R}^n)$.

Note that under the assumptions on $V = O(|x|^{-s})$ for $s > 2m$ with $n > 2m$, the solution $\psi \in \bigcap_{\sigma > m} L^2_\sigma(\mathbb{R}^n)$ which satisfies with $(-\Delta)^m \psi + V \psi = 0$, must belong to the smaller space $\bigcap_{\sigma > 2m - \frac{n}{2}} L^2_\sigma(\mathbb{R}^n)$ if $2m < n \leq 4m$ and be an eigenfunction in $L^2(\mathbb{R}^n)$ if $n > 4m$. This means that zero is not a resonance and only is a possible eigenvalue if $n > 4m$. In particular, as
$m = 1$, these situations exactly return to the case of Schrödinger operator $-\Delta + V$. For more details and comments, one may see Jensen and Kato [34] for $m = 1$ and Feng et al. [20] for $m \geq 2$.

Let $\mathcal{E}_\nu = \{ \lambda \in \mathbb{R} \mid \text{dist}_\mathbb{R}(\lambda, \sigma_p(H)) < \nu \}$ be the $\nu$-neighborhood of $\sigma_p(H)$ in $\mathbb{R}$. If $H$ has no eigenvalues, we set $\mathcal{E}_\nu = \emptyset$. Note that if $H$ has no nonnegative eigenvalues nor a zero resonance, then $\sigma_p(H)$ consists of finitely many negative discrete eigenvalues due to Lemma 3.1 (i).

**Theorem 3.4.** Let $n > 2m$, $H = (-\Delta)^m + V$ and $|V(x)| \leq C\langle x \rangle^{-s}$ for some $s > 2m$. Assume that $H$ has no positive eigenvalues and no zero resonance/eigenvalue. Then, for any $\nu > 0$,

$$\sup_{\lambda \in \mathbb{R} \setminus \mathcal{E}_\nu, 0 < \theta \leq 1} \left\| x^{-m+\gamma} |D|^\gamma (H - \lambda \mp i\theta)^{-1} |D|^\gamma |x|^{-m+\gamma} \right\|_{L^2 - L^2} \leq C_\nu < \infty, \quad (3.4)$$

for any $m - \frac{n}{2} < \gamma < m - \frac{1}{2}$. Moreover, if $\gamma = m - \frac{1}{2}$, then for any $\epsilon > 0$

$$\sup_{\lambda \in \mathbb{R} \setminus \mathcal{E}_\nu, 0 < \theta \leq 1} \left\| \langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} (H - \lambda \mp i\theta)^{-1} |D|^{m-1/2} \langle x \rangle^{-1/2-\epsilon} \right\|_{L^2 - L^2} \leq C_\nu < \infty, \quad (3.5)$$

**Proof.** The proof are divided into the following two steps.

**Step 1.** Let $M(z) = I + wR_0(z)v$ for $z \in \mathbb{C} \setminus [0, \infty)$, where $v(x) = \sqrt{|V|}$ and $w(x) = v(x)\text{sgn}V(x)$. In the sequel, we will show that the inverse $M^{-1}(\lambda \pm i\theta)$ exists on $L^2(\mathbb{R}^n)$ for $(\lambda, \theta) \in (\mathbb{R} \setminus \mathcal{E}_\nu) \times [0, 1]$ and that $M^{-1}(\lambda \pm i\theta)$ is continuous on $(\mathbb{R} \setminus \mathcal{E}_\nu) \times [0, 1]$, satisfying

$$\sup_{\lambda \in \mathbb{R} \setminus \mathcal{E}_\nu, 0 < \theta \leq 1} \left\| M^{-1}(\lambda \pm i\theta) \right\|_{L^2 - L^2} < \infty. \quad (3.6)$$

Clearly $V(x) = v(x)w(x)$ and $|v(x)| = |w(x)| \leq C\langle x \rangle^{-s/2}$. In view of the compactness and continuity of $\langle x \rangle^{-s/2}R_0(z)\langle x \rangle^{-s/2}$ on $\mathbb{C}^+$ (also on $\mathbb{C}^-$) from Theorem 2.7, it follows that the operator $wR_0(z)\text{v}$ is also compact on $L^2(\mathbb{R}^n)$ for each $z \in \mathbb{C} \setminus [0, \infty)$. In particular, as an operator valued function, $z \mapsto wR_0(z)v$ is analytic in $\mathbb{C}^\pm$ and extends continuously to the boundary set $\mathbb{R}$ in the uniform topology of $\mathbb{B}(L^2)$. Note that by the high energy estimate (2.8) we have that $\|wR_0(z)v\|_{L^2 - L^2} \leq 1/2$ if $z \in \mathbb{C} \setminus [0, \infty)$ and $|z| \geq r$ for some $r > 0$. Hence by Neumann series expansion, it follows that the series

$$M^{-1}(z) = \sum_{j=0}^{\infty} (-1)^j (wR_0(z)v)^j$$

converges uniformly in the operator norm for $|z| \geq r$ and $z \in \mathbb{C} \setminus [0, \infty)$. Thus by the continuity in $z$ of $wR_0(z)v$, we have that $M^{-1}(\lambda \pm i\theta)$ is continuous on $[r, \infty) \times [0, 1]$ and

$$\sup_{\lambda \geq r, 0 \leq \theta \leq 1} \left\| M^{-1}(\lambda \pm i\theta) \right\|_{L^2 - L^2} \leq 2. \quad (3.7)$$

It remains to deal with the invertibility and continuity of $M(z)$ on the domain $\Omega_\pm := ((-\infty, r] \setminus \mathcal{E}_\nu) \pm [0, 1]$. Let $z \in \Omega_\pm$. Since $wR_0(z)v$ is compact on $L^2(\mathbb{R}^n)$, it follows by
We may consider the case $z > 0$, which means $\lambda > 0$ on $M$. Fredholm-Riesz theory that the inverse $M^{-1}(z)$ exists in $\mathcal{B}(L^2)$ if and only if $\text{Ker}_{L^2}(M(z)) = \{0\}$. To show $\text{Ker}_{L^2}(M(z)) = \{0\}$, we suppose there exists $\psi \in L^2$ such that
\[
M(z)\psi = \psi + wR_0(z)v\psi = 0.
\] (3.8)
Set $f = v\psi \in L^2_{s/2}(\mathbb{R}^n)$ with some $s > 2m$ and $g = R_0(z)f$. Then we have $(H_0 + V - z)g = 0$. In the sequel, we will divide the three cases to show $\psi = 0$.

Case (i). If $z \in \Omega_+$ and $\text{Im} z > 0$ or $z = \lambda < 0$, then $g = R_0(z)f \in L^2(\mathbb{R}^n)$ and $g$ must be 0 due to $z \notin \sigma(H)$. So $f = v\psi = 0$, which implies that $\psi = 0$ from the equation (3.8).

Case (ii). If $\Omega_+ \ni z = \lambda > 0$, then $g = R_0^+(\lambda)f \in L^{2-\beta}(\mathbb{R}^n)$ for $\beta > s/2$ by Theorem 2.1(i) and $(H_0 + V - \lambda)g = 0$. In fact, by Agmon-Hörmander scattering theory (see e.g. Hörmander [30, Theorem 14.5.2]), we can obtain that $g$ is a rapidly decreasing eigenfunction, i.e.
\[
\int_{\mathbb{R}^n} (1 + |x|^2)^N |(D^n g)(x)|^2 dx < \infty \quad \text{for all } N \in \mathbb{N} \text{ and } |\alpha| \leq 2m.
\] (3.9)
Note that (3.9) also holds for all eigenfunctions associated with negative eigenvalues of $H$. This means $\lambda > 0$ must be an eigenvalues of $H$, which will contract our assumption unless $g = 0$. Thus as shown in the case (i), we can deduce $\psi = 0$.

Case (iii). If $z = 0$, then $g = R_0(0)f \in \bigcap_{\sigma > m} L^{2-\sigma}(\mathbb{R}^n)$ by Theorem 2.1(ii) and $(H_0 + V)g = 0$. Since zero is not neither resonance nor zero eigenvalue of $H$ from the assumption, so $g = 0$, which again deduce $\psi = 0$.

Now we need to prove the inverse operator function $M^{-1}(z)$ is continuous for $z \in \Omega_+$. We may consider the case $z \in \Omega_+$ only. In fact, let $z_0 \in \Omega_+$, since $wR_0(z)v$ is continuous on $z$ in $\mathcal{B}(L^2)$, hence for any $\epsilon > 0$, there exists a $\delta > 0$ depending on $z_0$ such that when $|z - z_0| < \delta$,
\[
\|M^{-1}(z) - M^{-1}(z_0)\|_{L^2-L^2} = \left\| \left( (I + (wR_0(z)v - wR_0(z_0)v)M^{-1}(z_0))^{-1} - I \right) M^{-1}(z_0) \right\|_{L^2-L^2} < \epsilon.
\]
Thus the continuity of $M^{-1}(z)$ can give $\sup_{(\lambda, \theta) \in \Omega_+} \|M^{-1}(\lambda + i\theta)\|_{L^2-L^2} < \infty$, which combining with the (3.7) leads to the desired (3.6).

Step 2. Let $z = \lambda + i\theta \in (\mathbb{R} \setminus \mathcal{E}_\nu) \pm i(0,1]$. Firstly, recall that the following resolvent formula
\[
R(z) = (H_0 + V - z)^{-1} = R_0(z) - R_0(z)wM^{-1}(z)vR_0(z)
\] (3.10)
holds, where $M(z) = I + wR_0(z)v$ and $V(x) = v(x)w(x)$ defined as in Step 1 above. To prove (3.5) in view of the inequality (2.9) of $R_0(z)$, it suffices to show the following estimate:
\[
\sup_{\lambda \in \mathbb{R} \setminus \mathcal{E}_\nu, 0 < \theta \leq 1} \left\| \langle x \rangle^{-1/2-\epsilon} |D|^{-m/2} R_0(z)wM^{-1}(z)vR_0(z) |D|^{-m/2} \langle x \rangle^{-1/2-\epsilon} \right\|_{L^2-L^2} < \infty,
\] (3.11)
for any $\epsilon > 0$. Since $\langle x \rangle^{-1/2-\epsilon} |D|^{-m/2} R_0(z)w(x)$ is essentially dual each other with the operator $v(x)R_0(z)|D|^{-m/2} \langle x \rangle^{-1/2-\epsilon}$, using of the uniform estimate (3.6) for $M^{-1}(z)$, it
also its absolutely continuous part

\[ e^{-z^2} |D|^p \mathcal{R}_0(z) w \]  

suffices to show

\[ \sup_{z \in \mathbb{C} \setminus [0, \infty)} \left\| \langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} R_0(z) w \right\|_{L^2 - L^2} < \infty. \]  (3.12)

We now apply Theorem 2.3 to show (3.12) as follows. Choose \((1/p_0, 1/q_0) = ((n+2m)^{-1}/2n), (n-1)^+/2n)\) be such that the inequality (2.16) holds, where \(a^+\) (resp. \(a^-\)) denotes some number which is arbitrarily close but larger (resp. less) than \(a\). Then by Hölder inequality we have

\[ \left\| \langle x \rangle^{-\frac{1}{2}-\epsilon} |D|^{m-\frac{1}{2}} R_0(z) w \right\|_{L^2 - L^2} \leq \left\| \langle x \rangle^{-\frac{1}{2}-\epsilon} \right\|_{L^{(2n)\text{-}}} \left\| |D|^{m-\frac{1}{2}} R_0(z) \right\|_{L^{p_0} - L^{q_0}} \left\| w \right\|_{L^{(n/m)-}}, \]  (3.13)

where we have used the estimate \(|w(x)| \lesssim \langle x \rangle^{-m^-} \) and \(\langle x \rangle^{-m^-} \in L^{(n/m)-}\). Hence we get (3.12) and then the desired estimate (3.5).

To prove the (3.4) recall that the same uniform estimate for the free resolvent

\[ \sup_{z \in \mathbb{C} \setminus [0, \infty)} \left\| |x|^{-m+2\gamma} |D|^{\gamma} R_0(z) \right\|_{L^2 - L^2} < \infty \]  \hspace{1cm} (3.14)

holds for any \(m - n/2 < \gamma < m - 1/2\) (see [55]). Then in view of the resolvent formula (3.10) and a similar argument above, it suffices to prove

\[ \sup_{z \in \mathbb{C} \setminus [0, \infty)} \left\| |x|^{-m+\gamma} |D|^{\gamma} R_0(z) w \right\|_{L^2 - L^2} < \infty. \]  \hspace{1cm} (3.15)

To this end, we note that, as mentioned in Remark 2.3 (2.16) together with the real interpolation theory implies

\[ \sup_{z \in \mathbb{C} \setminus [0, \infty)} \left\| |D|^{\gamma} R_0(z) \right\|_{L^{p_0,2} - L^{q_0,2}} < \infty, \]  \hspace{1cm} (3.16)

where \((1/p_0, 1/q_0) = ((n+2m)/2n, (n-2m+2\gamma)/2n)\). Hence combining with weak Hölder’s inequality, we obtain from (3.16) that

\[ \left\| |x|^{-m+\gamma} |D|^{\gamma} R_0(z) w \right\|_{L^2 - L^2} \leq \left\| |x|^{-m+\gamma} \right\|_{L^{\infty} - L^{\infty}} \left\| |D|^{\gamma} R_0(z) \right\|_{L^{p_0,2} - L^{q_0,2}} \left\| w \right\|_{L^{\infty} - L^{\infty}} \lesssim 1 \]  \hspace{1cm} \text{(3.17)}

uniformly in \(z \in \mathbb{C} \setminus [0, \infty)\), which immediately deduce the desired bound (3.15) \(\square\)

If \(H\) has no eigenvalues (in which case \(\mathcal{E}_\nu = \emptyset\)), then Theorem 3.4 means that \(|x|^{-m+\gamma} |D|^\gamma\) and \(\langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2}\) are \(H\)-supersmooth in the sense of Kato-Yajima [39], which implies Kato smoothing estimates, i.e. Theorem 1.1. However, if \(H\) has eigenvalues, then Theorem 3.4 is not sufficient to achieve Theorem 1.1 since we have to deal with not only \(e^{itH}\), but also its absolutely continuous part \(e^{itH} P_{ac}(H)\). To this end, we need to replace the resolvent \((H - z)^{-1}\) by its absolutely continuous part in Theorem 3.4 which is the main point of the following corollary.

**Corollary 3.5.** Let \(P_{ac}(H)\) denote the projection onto the absolutely continuous spectral space of \(H = (-\Delta)^m + V\). Then, under the conditions in Theorem 3.4, \(|x|^{-m+\gamma} |D|^\gamma P_{ac}(H)\) and \(\langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} P_{ac}(H)\) are \(H\)-supersmooth, i.e. the following uniform estimates hold:
and
\[ \sup_{\lambda \in \mathbb{R}, \theta \in (0,1]} \| x^{-m+\gamma} |D|^{\gamma} P_{ac}(H)(H - \lambda \mp i\theta)^{-1} P_{ac}(H) |D|^{m} x^{-m+\gamma} \|_{L^2 - L^2} < \infty, \]
and
\[ \sup_{\lambda \in \mathbb{R}, \theta \in (0,1]} \| \langle \tau \rangle^{-1/2 - \epsilon} |D|^{m-1/2} P_{ac}(H)(H - \lambda \mp i\theta)^{-1} P_{ac}(H) |D|^{m-1/2} \langle \tau \rangle^{-1/2 - \epsilon} \|_{L^2 - L^2} < \infty. \]

To prove this corollary, we prepare two lemmas. The first one, which is a special case of [63, Theorem B*], concerns with the weighted \( L^2 \)-boundedness of the fractional integral operator.

**Lemma 3.6 (63 Theorem B*).** Let \( 0 < \lambda < n, \alpha, \beta < n/2, \alpha + \beta \geq 0 \) and \( \lambda + \alpha + \beta = n \). Then \( |x|^{-\beta} |D|^{-n+\lambda} |x|^{-\alpha} \) extends to a bounded operator on \( L^2(\mathbb{R}^n) \).

The second one is concerned with several mapping properties of the projection \( P_{ac}(H) \). Note that the estimates (3.18) and (3.19) are sufficient to obtain Corollary 3.5, while (3.20) will be used to prove Strichartz estimates (i.e. Theorem 1.4) in the next section.

**Lemma 3.7.** Let \( n > 2m, m \in \mathbb{N}, 0 \leq \gamma < m - 1/2 \) and \( H = (-\Delta)^m + V \) be as in Theorem 3.4. Then
\[ \| |x|^{-m+\gamma} |D|^{\gamma} P_{ac}(H)f \|_{L^2} \lesssim \| |x|^{-m+\gamma} |D|^{\gamma} f \|_{L^2}; \tag{3.18} \]
\[ \| \langle \tau \rangle^{-1/2 - \epsilon} |D|^{m-1/2} P_{ac}(H)f \|_{L^2} \lesssim \| \langle \tau \rangle^{-1/2 - \epsilon} |D|^{m-1/2} f \|_{L^2}, \quad \epsilon > 0. \tag{3.19} \]
Moreover, for any admissible pair \( (p,q) \) satisfying (1.10) with \( \alpha = n/2 \),
\[ \| |D|^{-2(m-1)/p} P_{ac}(H)f \|_{L^q'} \lesssim \| |D|^{-2(m-1)/p} f \|_{L^q'}. \tag{3.20} \]

**Proof.** Under the assumption of \( H \), we know that \( H \) has at most finitely many negative eigenvalues and there are neither embedded eigenvalues nor singular continuous spectrum (see Lemma 3.1). Hence, with some finite integer \( N_0 \geq 0 \), \( P_{ac}(H) \) is of the form
\[ P_{ac}(H)f = f - \sum_{j=1}^{N_0} \langle \psi_j, f \rangle \psi_j, \]
where \( \psi_1, ..., \psi_{N_0} \) are eigenfunctions associated with the negative eigenvalues of \( H \).

Let \( G := |x|^{-m+\gamma} |D|^{\gamma} \) and \( (G^{-1})^* = |x|^{m-\gamma} |D|^{-\gamma} \). Then one has
\[ \| GP_{ac}(H)f \|_{L^2} \leq \| Gf \|_{L^2} + \sum_{j=1}^{N_0} \| G\psi_j \|_{L^2} \| (G^{-1})^* \psi_j \|_{L^2} \| Gf \|_{L^2}. \tag{3.21} \]
Note that Lemma 3.6 with \( (\lambda, \alpha, \beta) = (n - m + \gamma, 0, m - \gamma) \) and (3.9) imply
\[ \| G\psi_j \|_{L^2} = \| |x|^{-m+\gamma} |D|^{-m+\gamma} \cdot |D|^{m} \psi_j \|_{L^2} \lesssim \| |D|^{m} \psi_j \|_{L^2} < \infty. \]
Again, using Lemma 3.6 with \( (\lambda, \alpha, \beta) = (n - \gamma, m, -m + \gamma) \) and (3.9) we also have
\[ \| (G^{-1})^* \psi_j \|_{L^2} = \| |x|^{m-\gamma} |D|^{-\gamma} |x|^{-m} \cdot |x|^{m} \psi_j \|_{L^2} \lesssim \| |x|^{m} \psi_j \|_{L^2} < \infty. \]
Therefore, the desired estimate (3.18) follows from (3.21). The proof of (3.19) is essentially the same as (3.18) by using the variant of Lemma 3.6 and we thus omit it.

Next, let us consider the last estimate (3.20). By the same argument, it suffices to show that $|D|^{2(m-1)/p} \psi_j \in L^q$ and $|D|^{-2(m-1)/p} \psi_j \in L^{q'}$, which can be also deduced from (3.9) as follows. Firstly, since $p, q$ satisfy $2/p = n(1/2 - 1/q)$, the Sobolev inequality implies

$$
\|D|^{2(m-1)/p} \psi_j\|_{L^q} \leq \|D|^{2(m-1)/p} \psi_j\|_{L^2} = \|D|^{2m/p} \psi_j\|_{L^2} \leq \|(D)^m \psi_j\|_{L^2} \leq \infty,
$$

where we have used the fact $p \geq 2$ and (3.9). Secondly, by (3.9) with $N > n/2$ and Hölder’s inequality, we have $\psi_j \in L^1 \cap L^2$ and hence $\psi_j \in L^r$ for any $1 \leq r \leq 2$. This fact, combined with Sobolev’s inequality, shows that

$$
\|D|^{-2(m-1)/p} \psi_j\|_{L^{q'}} \lesssim \|\psi_j\|_{L^r} \leq \infty,
$$

where $1/r = 1/q' + 2(m-1)/(np) = 1/2 + 2m/(np) > 1/2$. Therefore, we have proved the desired estimate (3.20).

\[ \square \]

**Proof of Corollary 3.5** We only prove the result for $G := |x|^{-m+\gamma}|D|^\gamma$, since the proof for the operator $\langle x \rangle^{-1/2+\epsilon}|D|^{-m+1/2}$ is analogous. Note that $P_{ac}(H) = P_{ac}(H)^2$, it is enough to show

$$
\sup_{\lambda \in \mathbb{R}, \theta \in [0,1]} \|GP_{ac}(H)(H - \lambda \mp i\theta)^{-1}G^*\|_{L^2} < \infty. \tag{3.22}
$$

The proof of (3.22) is divided into two cases $0 \leq \gamma < m - 1/2$ and $m - n/2 < \gamma < 0$.

We first let $0 \leq \gamma < m - 1/2$ and observe from Lemma 3.7 that $P_{ac}(H)$ satisfies

$$
\|GP_{ac}(H)f\|_{L^2} \lesssim \|Gf\|_{L^2}, \quad f \in D(G). \tag{3.23}
$$

Let $\nu > 0$ be so small that $\text{dist}_\mathbb{R}(\mathcal{E}_\nu, [0, \infty)) \geq \nu$ which is possible since $\sigma_p(H)$ consists of finitely many negative eigenvalues under our assumption (see Lemma 3.1 above). We consider two cases $\lambda \in \mathbb{R} \setminus \mathcal{E}_\nu$ or $\lambda \in \mathcal{E}_\nu$ separately. If $\lambda \in \mathbb{R} \setminus \mathcal{E}_\nu$, (3.22) follows from (3.4) and (3.23). If $\lambda \in \mathcal{E}_\nu$, then since $\|Gf\|_{L^2} \lesssim \|D^m f\|_{L^2}$ by Hardy’s inequality (also as seen in the proof of Lemma 3.7), we have

$$
\|GP_{ac}(H)(H - \lambda \mp i\theta)^{-1}G^*\|_{L^2} \lesssim \|D^m P_{ac}(H)(H - \lambda \mp i\theta)^{-1}|D^m|\|_{L^2}.
$$

Since $\|D^m \langle t \rangle^{-1/2}\|_{L^2} \lesssim 1$ by the fact $D(H) = \mathcal{H}^{2m}(\mathbb{R}^n)$, the spectral theorem implies

$$
\|D^m P_{ac}(H)(H - \lambda \mp i\theta)^{-1}|D^m|\|_{L^2} \lesssim \|P_{ac}(H)\langle t \rangle^{-1}\|_{L^2} \leq \sup_{t \in [0, \infty)} (\langle t \rangle |t - \lambda \mp i\theta|^{-1}) \lesssim \nu^{-1},
$$

where we have used the fact $\text{dist}_\mathbb{R}(\lambda, [0, \infty)) \geq \nu$. This proves (3.23) for $0 \leq \gamma < m - 1/2$.

Next, the case $m - n/2 < \gamma < 0$ follows easily from the previous case. Indeed, by letting $\lambda = n + \gamma$, $\alpha = -m$ and $\beta = m - \gamma$ in Lemma 3.6, one has $|x|^{-m+\gamma}|D|^\gamma|x|^m \in \mathcal{B}(L^2)$.
We also have $|x|^m |D|^{\gamma} |x|^{-m+\gamma} \in \mathcal{B}(L^2)$ by taking the adjoint. Therefore, (3.22) with $\gamma = 0$ implies that
\[
\| |x|^{-m+\gamma} |D|^{\gamma} P_{ac}(H)(H - \lambda \mp i\theta)^{-1} |D|^{\gamma} |x|^{-m+\gamma} f \|_{L^2} \\
\lesssim \| |x|^{-m} P_{ac}(H)(H - \lambda \mp i\theta)^{-1} |x|^{-m} \|_{L^2-L^2} \| |x|^m |D|^{\gamma} |x|^{-m+\gamma} f \|_{L^2} \\
\lesssim \| f \|_{L^2}
\]
for $m - n/2 < \gamma < 0$ uniformly in $\lambda \in \mathbb{R}$ and $\theta \in (0,1]$. This completes the proof of (3.22).

Now we are in a position to give the proof of Theorems 1.1 and 1.3, which actually are the direct consequences of the following theorem.

**Theorem 3.8.** Let $n > 2m$, $H = (-\Delta)^m + V$ and $|V(x)| \leq C\langle x \rangle^{-s}$ for $s > 2m$. Assume that $H$ has no positive eigenvalues and no zero resonance/eigenvalue. Then the following statements (i.e. Kato smoothing estimates) were proved:

(i) If $m - n/2 < \gamma < m - 1/2$, then
\[
\| |x|^{-m+\gamma} |D|^{\gamma} e^{itH} P_{ac}(H)\psi_0 \|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2_x},
\]
\[
\| |x|^{-m+\gamma} |D|^{\gamma} \int_0^t e^{-i(t-s)H} P_{ac}(H) F(s) ds \|_{L^2_t L^2_x} \lesssim \| |x|^{-m-\gamma} |D|^{-\gamma} F \|_{L^2_t L^2_x}.
\]

In particular, as $\gamma = 0$, the following local decay estimate holds:
\[
\| |x|^{-m} e^{itH} P_{ac}(H)\psi_0 \|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2_x},
\]
\[
\| |x|^{-m} \int_0^t e^{-i(t-s)H} P_{ac}(H) F(s) ds \|_{L^2_t L^2_x} \lesssim \| |x|^m F \|_{L^2_t L^2_x}.
\]

(ii) If $\gamma = m - 1/2$, then for any $\epsilon > 0$,
\[
\| \langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} e^{itH} P_{ac}(H)\psi_0 \|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2_x},
\]
\[
\| \langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} \int_0^t e^{-i(t-s)H} P_{ac}(H) F(s) ds \|_{L^2_t L^2_x} \lesssim \| \langle x \rangle^{1/2+\epsilon} |D|^{-m+1/2} F \|_{L^2_t L^2_x}.
\]

**Proof.** These estimates are direct consequences of Corollary 3.5 (i.e. the $H$-supersmoothness of $|x|^{-m+\gamma} |D|^{-\gamma} P_{ac}(H)$ and $\langle x \rangle^{-1/2-\epsilon} |D|^{m-1/2} P_{ac}(H)$) and Kato’s smooth perturbation theory [36] (see also [12] for the inhomogeneous estimates).

Finally, note that if $(x \cdot \nabla)V(x) \leq 0$ and $\lim_{x \to \infty} V(x) = 0$, then $V(x) \geq 0$. In fact, it can be easily concluded by the following integral
\[
V(x) = -\int_1^\infty \frac{d}{ds} (V(sx)) ds \geq 0, \quad x \neq 0.
\]
where \( \frac{d^2}{ds^2}(V(sx)) = \frac{1}{2}(sx \cdot \nabla)V(sx) \leq 0 \). Thus, we obtain under these conditions that \( H = H_0 + V \) is a nonnegative self-adjoint operator and that \( P_{ac}(H) = \text{Id} \) in the previous results.

4. Strichartz estimates of \( e^{itH} \)

In this section, we prove Theorem 1.4. The proof relies on a method by Rodnianski-Schlag [54] (see also [9] for the homogenous endpoint estimate and [8] for the double endpoint estimate). This method requires the corresponding estimates for the free evolutions which are summarized in the following lemma.

**Lemma 4.1.** Let \( H_0 = (-\Delta)^m \) with \( m \in \mathbb{N} \), \((p_1, q_1), (p_2, q_2)\) satisfy (1.10) with \( \alpha = n/2 \). Then,

\[
\left\| D \right\|_{L^p_1L^{q_1}_2}^{2(m-1)/p} \lesssim \left\| \psi_0 \right\|_{L^2_2} \quad \text{and} \quad \left\| D \right\|_{L^p_1L^{q_1}_2}^{2(1-m)/p} \lesssim \left\| D \right\|_{L^{p'_2}_1L^{q'_2}_2}^{2(1-m)/p}. \tag{4.2}
\]

**Proof.** The lemma follows from dispersive estimates for \( \left\| D \right\|_{L^p_1L^{q_1}_2}^{2(m-1)/p} \) and Keel-Tao’s theorem [40]. We refer to [52] Section 3 and [50] Appendix A for details.

**Proof of Theorem 1.4.** Note that (1.11) follows from (1.14) and Sobolev’s inequality since

\[
\frac{1}{q_1} - \frac{1}{q} = \left( \frac{2m}{n} - \frac{2}{n} \right) \frac{1}{p_1} = \frac{2(m-1)}{p_1 n}
\]

as long as \((p_1, q_1)\) is \( n/2 \)-admissible and \((p_1, q)\) is \( n/(2m) \)-admissible. It is thus enough to show (1.14). Let \( \Lambda_p = \left\| D \right\|_{L^{p_1}_1L^{q_1}_2}^{2(m-1)/p} \) for short. For a given self-adjoint operator \( A \), we set

\[
\Gamma_A F(t, x) = \int_0^t e^{i(t-s)A}F(s, x)ds.
\]

Then (1.14) can be decomposed into the following two estimates

\[
\| \Lambda_{p_1}e^{itH}P_{ac}(H)\psi_0 \|_{L^{p_1}_1L^{q_1}_2} \lesssim \| \psi_0 \|_{L^2_2}, \quad \text{(4.3)}
\]

\[
\| \Lambda_{p_1}\Gamma_H P_{ac}(H)F \|_{L^{p_1}_1L^{q_1}_2} \lesssim \| \Lambda_{p_2}^{-1}F \|_{L^{p'_2}_1L^{q'_2}_2}. \quad \text{(4.4)}
\]

We first consider (4.4) whose proof relies on the following Duhamel formulas:

\[
\Gamma_H = \Gamma_{H_0} - i\Gamma_{H_0}VT_H = \Gamma_{H_0} - i\Gamma_H VT_{H_0}
\]

(see e.g. [8] Section 4 for the proof of these formulas). By (4.2), one first has

\[
\| \Lambda_{p_1}\Gamma_H P_{ac}(H)F \|_{L^{p_1}_1L^{q_1}_2} \lesssim \| \Lambda_{p_1}\Gamma_{H_0}P_{ac}(H)F \|_{L^{p_1}_1L^{q_1}_2} + \| \Lambda_{p_1}\Gamma_{H_0}VT_H P_{ac}(H)F \|_{L^{p_1}_1L^{q_1}_2} \lesssim \| \Lambda_{p_2}^{-1}P_{ac}(H)F \|_{L^{p'_2}_1L^{q'_2}_2} + \| \Lambda_{p_1}\Gamma_{H_0}V P_{ac}(H)\Gamma_H F \|_{L^{p_1}_1L^{q_1}_2}. \tag{4.5}
\]

Since Lemma 3.7 implies

\[
\| \Lambda_{p_2}^{-1}P_{ac}(H)F \|_{L^{p'_2}_1L^{q'_2}_2} \lesssim \| \Lambda_{p_2}^{-1}F \|_{L^{p'_2}_2},
\]
hence the term \( \Lambda^{-1}_{p_2} P_{ac}(H) F \) in (4.5) satisfies the desired estimate
\[
\| \Lambda^{-1}_{p_2} P_{ac}(H) F \|_{L^p \to L^q} \lesssim \| \Lambda^{-1}_{p_2} F \|_{L^{p_2} \to L^{q_2}}. \tag{4.6}
\]
For the operator \( \Lambda_{p_1} \Gamma H_0 V P_{ac}(H) \Gamma_H F \), we first apply (4.2) with \( (p_2, q_2) = (2, \frac{2n}{n-2}) \) to obtain
\[
\| \Lambda_{p_1} \Gamma H_0 V P_{ac}(H) \Gamma_H F \|_{L^{p_1} \to L^{q_1}} \lesssim \| \Lambda^{-1}_2 V P_{ac}(H) \Gamma_H F \|_{L^{p_2} \to L^{q_2}}. \tag{4.7}
\]
Recall here that \( V = vw \) with \( v, w \in L^{n/m} \). With the equalities
\[
\frac{n + 2m}{2n} - \frac{n + 2}{2n} = \frac{m - 1}{n}, \quad \frac{n + 2m}{2n} = \frac{m + 1}{2}
\]
at hand, we see from Sobolev’s and Hölder’s inequalities that
\[
\| \Lambda^{-1}_2 V P_{ac}(H) \Gamma_H F \|_{L^{p_2} \to L^{q_2}} \lesssim \| v \|_{L^{p_3}} \| w P_{ac}(H) \Gamma_H F \|_{L^{p_4} \to L^{q_4}}. \tag{4.8}
\]
Now we use again the Duhamel formula to estimate the right hand side of (4.8) as
\[
\| w P_{ac}(H) \Gamma_H F \|_{L^{p_4} \to L^{q_4}} \lesssim \| w P_{ac}(H) \Gamma H_0 F \|_{L^{p_5} \to L^{q_5}} + \| w P_{ac}(H) \Gamma H V \Gamma H_0 F \|_{L^{p_6} \to L^{q_6}}. \tag{4.9}
\]
Since \( \| w P_{ac}(H) f \|_{L^2} \lesssim \|w f\|_{L^2} \) (which can be verified by the same proof as that of (3.18)), the first term of the right hand side of (4.9) is dominated by \( \|w \Gamma H_0 F\|_{L^2} \). Moreover, since \( |x|^{-m} P_{ac}(H) \) is \( H \)-supersmooth by Corollary 3.5, so is \( w P_{ac}(H) \) and hence \( w P_{ac}(H) \Gamma H w \in \mathcal{B}(L^2) \) by the same argument as in the proof of Theorem 3.8. Therefore, the second term of the right hand side of (4.9) is dominated by \( \|v \Gamma H_0 F\|_{L^2} \). Since \( |v| = |w| \), we conclude that
\[
\| w P_{ac}(H) \Gamma_H F \|_{L^{p_5} \to L^{q_5}} \lesssim \| w \Gamma H_0 F \|_{L^{p_5} \to L^{q_5}} \lesssim \| w \|_{L^{p_3}} \| \Lambda^{-1}_2 \Gamma H_0 F \|_{L^{p_4} \to L^{q_4}} \lesssim \| F \|_{L^{p_1} \to L^{q_1}}. \tag{4.10}
\]
where we have used Hölder’s and Sobolev’s inequalities in the second line and (4.2) with \( (p_1, q_1) = (2, \frac{2n}{n-2}) \) in the last line. Finally, (4.5)–(4.10) yield the desired bound (4.4).

Next, the estimate (4.3) can be obtained similarly by using (1.3) (4.1) (4.2) and the usual Duhamel formula \( U_H = U_{H_0} - i \Gamma H_0 V U_H \), where \( U_{H_0} = e^{itH_0} \) and \( U_H = e^{itH} \). Since the proof is essentially same as (or even simpler than) that of (4.4), we omit these details. \( \square \)

5. The proofs of Theorems 2.1 and 2.4

5.1. The proof of Theorem 2.1 The statements (i) and (iii) in Theorems 2.1 are due to Agmon [1] (also see Remark 2.2 for more comments). Hence, it remains to show the statement (ii) of Theorems 2.1. Actually, comparing with the conclusion (i) of Theorems 2.1, it suffices to add the continuity of the resolvent \( R_0(z) \) at \( z = 0 \) in the uniform operator topology of \( \mathcal{B}(L^2) \) if \( s > m \). Following the arguments of Agmon [1, Theorem 4.1], we first prove the weak continuity of \( R_0(z) \) on \( \mathbb{C}^\pm \), and then lift to the uniform-topology continuity of \( R_0(z) \) by the compact arguments.
The proof of Theorem 2.1(ii): The proof is divided into the following three steps:

Step 1. Let \( s > m \) and \( f, g \in L^2_s \). We first prove that \( \langle R_0(z) f, g \rangle \), which is an analytic function on \( \mathbb{C} \setminus [0, \infty) \), has a continuous boundary value on the two sides of \([0, \infty)\). Clearly, for any \( \lambda > 0 \), by the statement (i) of Theorem 2.1, we have

\[
\lim_{z \to 0} \langle R_0(0^+) f, g \rangle = \langle R_0(\lambda) f, g \rangle, \tag{5.1}
\]

where \( R_0(\lambda) = R_0(\lambda \pm i0) \) are defined in [2.2]. For \( \lambda = 0 \), we need to show the limit

\[
\lim_{z \to 0} \langle R_0(z) f, g \rangle = \langle R_0(0) f, g \rangle \tag{5.2}
\]

holds, where \( R_0(0) = (-\Delta)^{-m} \). Note that \(|\langle R_0(z) f, g \rangle| \leq C \| f \|_{L^2_s} \| g \|_{L^2_s} \) holds uniformly in \( z \in \mathbb{C} \setminus [0, \infty) \). So we may assume \( f, g \in S(\mathbb{R}^n) \) (the Schwartz function class) by a density argument. By the kernel function expansion of \((-\Delta - z)^{-1}\) and the decomposition formula

\[
R_0(z) = \frac{1}{mz} \sum_{k=0}^{m-1} z_k (-\Delta - z_k)^{-1}, \quad z_k = z e^{i \frac{2\pi k}{m}}, \quad z \in \mathbb{C} \setminus [0, \infty), \tag{5.3}
\]

we can establish the kernel expansion of \( R_0(z) \) as \( z \) close to 0:

\[
R_0(z)(x, y) = c_n |x - y|^{2m-n} + E(z, x, y),
\]

where \( c_n |x - y|^{2m-n} \) is the kernel of \( R_0(0) \) and the integral operator \( E(z) \) with the kernel \( E(z, x, y) \) belongs to \( \mathbb{B}(L^2_{\sigma}, L^2_{-\sigma}) \) for any \( \sigma > n/2 + 2 \) and satisfies \( \| E(z) \|_{L^2_{\sigma} \to L^2_{-\sigma}} = O(|z|^{\epsilon}) \) for some \( \epsilon > 0 \) depending on \( n, m \) (see e.g. [20 Proposition 2.4]). Hence it follows that

\[
|\langle (R_0(z) - R_0(0)) f, g \rangle| \leq \| E(z) \|_{L^2_{\sigma} \to L^2_{-\sigma}} \| f \|_{L^2_s} \| g \|_{L^2_s} \to 0,
\]

as \( z \to 0 \) for any \( f, g \in S(\mathbb{R}^n) \), from which we thus conclude [5.2] by a density argument.

Step 2. Let \( \mathcal{H}^{m}(\mathbb{R}^n) \) denote the weighted Sobolev space defined by the norm

\[
\| f \|_{\mathcal{H}^{m}(\mathbb{R}^n)} = \| (1 - \Delta)^{k/2} f \|_{L^2(\mathbb{R}^n)}.
\]

Also define \( R_0^\pm(z) = R^\pm(\lambda) \) if \( z = \lambda > 0 \), and \( R_0^\pm(z) = R(z) \) if \( z \in \mathbb{C}^\pm \setminus (0, \infty) \). By Step 1, we know that the operator function \( R^\pm(z) \) is weak continuous on \( \mathbb{C}^\pm \). In this step, we will show that \( R^\pm(z) \) is continuous on \( \mathbb{C}^\pm \) in the strong operator topology of \( \mathcal{B}(L^2_s, L^2_{-s}) \) for \( s > m \). First, note that the equality

\[
(1 + H_0)R_0(z)f = f + (z + 1)R_0(z)f
\]

hold for \( z \in \mathbb{C} \setminus [0, \infty) \), which leads to the following uniform bounds for any \( M > 0 \).

\[
\| R_0(z)f \|_{\mathcal{H}^{m}(\mathbb{R}^n)} \leq C_M \| f \|_{L^2(\mathbb{R}^n)}, \quad z \in \mathbb{C} \setminus [0, \infty), \quad |z| \leq M. \tag{5.4}
\]

By the weak continuity of \( R^\pm_0(z) \) on \( \mathcal{B}(L^2_s, L^2_{-s}) \) and [5.4] we can actually obtain for any \( f \in L^2_s \) and any \( z_0 \in \mathbb{C}^\pm \) that

\[
\text{w-lim}_{z \to 0} R_0^\pm(z)f = R^\pm_0(z_0)f, \quad \text{in} \ \mathcal{H}^{2m}(\mathbb{R}^n), \tag{5.5}
\]
and $R^+_\pm(z) \in B(L^2_s, \mathcal{H}^{2m}_{s',z})$ for any $z \in \mathbb{C}^\pm$ and $s > m$. Since $R^+_\pm(z)f \in \mathcal{H}^{2m}_{s',z}$ for any $f \in L^2_s(\mathbb{R}^n)$ and the space embedding relation $\mathcal{H}^{2m}_{s',z} \hookrightarrow L^2_{-s}(\mathbb{R}^n)$ is compact for any $0 < s' < s$, hence by the compactness, we can lift the weak convergence of (5.5) up to the following strong convergence:

$$s\lim_{z \to z_0} R^+_\pm(z)f = R^+_\pm(z_0)f, \text{ in } L^2_{-s}(\mathbb{R}^n),$$

(5.6)

where $f \in L^2_s(\mathbb{R}^n)$. Thus we conclude that $R^+_\pm(z)$ is continuous on $\mathbb{C}^\pm$ in the strong operator topology of $\mathbb{B}(L^2_s, L^2_{-s})$ for $s > m$.

**Step 3.** Finally, we come to prove that $R^+_\pm(z)$ is continuous on $\mathbb{C}^\pm$ in the uniform operator topology of $\mathbb{B}(L^2_s, L^2_{-s})$ for $s > m$. To the end, suppose that it is not true by contradiction. Then there exist sequences $\{z_j\} \subset \mathbb{C}^\pm$ with $z_j \to z_0 \in \mathbb{C}^\pm$, and $\{f_j\} \subset L^2_s(\mathbb{R}^n)$ with $\|f_j\|_{L^2_s} = 1$, such that

$$\lim_{j \to \infty} \| (R^+_\pm(z_j) - R^+_\pm(z_0)) f_j \|_{L^2_{-s}} > 0.$$  

(5.7)

Note that $\{f_j\}$ always has a weak convergent subsequence, so we may assume that $f_j \to f$ in $L^2_s(\mathbb{R}^n)$ (in weak sense). Now by using a similar argument in Step 2, we actually can prove that the following convergence holds:

$$s\lim_{j \to \infty} R^+_\pm(z_j)f_j = R^+_\pm(z_0)f, \text{ in } L^2_{-s}(\mathbb{R}^n),$$

(5.8)

which clearly gives a contradiction to (5.7). Thus summing up three Steps above, we have finished the proof of Theorem 2.1 (ii).

\[ \square \]

### 5.2. The proof of Theorem 2.4

Theorem 2.4 is actually the special case (i.e. $1/p - 1/q = (2m - \alpha)/n$) of the estimates (5.11) in the following lemma, which will be proved based on the Fourier method involved with the famous Carleson-Sjölin oscillatory integral argument (see [62, p.69]). Note that the argument of the proof have been used similarly in Sikora-Yan-Yao [59]. Here we emphasize that the following results are new as $\alpha \neq 0$ even for the second order case, and crucial to Kato smoothing estimates studies in this paper.

**Lemma 5.1.** Let $n > 2m$, $H_0 = (-\Delta)^m$ and $z \in \mathbb{C}$. Consider arbitrary auxiliary cutoff function $\psi$ such that $\psi \in C^\infty_0(\mathbb{R})$, $\psi(s) \equiv 1$ if $s \in [1/2, 2]$ and $\psi$ is supported in the interval $[1/4, 4]$. Suppose also that exponent $(1/p, 1/q) \in (0, 1)^2$ satisfy the following conditions:

$$\min \left( \frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q} \right) > \frac{1}{2n}, \quad \frac{2}{n+1} \leq \left( \frac{1}{p} - \frac{1}{q} \right) \leq 1.$$  

(5.9)

Then there exists positive constants $C_{p,q,\alpha}$ independent of $|z|$ such that

$$\| |D|^{\alpha}(H_0 - z)^{-1}\psi(H_0/|z|)\|_{p \to q} \leq C_{p,q,\alpha} |z|^\frac{\alpha}{2m} \left( \frac{1}{p} - \frac{1}{q} \right)^{\frac{2m-\alpha}{2m}},$$  

(5.10)

for all $z \in \mathbb{C}^\pm \setminus \{0\}$ and $\alpha \in \mathbb{R}$, where $|D| = \sqrt{-\Delta}$. Furthermore, besides of the conditions (5.9), assume that $\frac{1}{p} - \frac{1}{q} \leq \frac{2m-\alpha}{n}$ and $2m - n < \alpha \leq 2m - \frac{2m}{n+1}$. Then, for all $z \in \mathbb{C}^\pm \setminus \{0\}$, \n
$$\| |D|^{\alpha}(H_0 - z)^{-1}\|_{p \to q} \leq C_{p,q,\alpha} |z|^\frac{\alpha}{2m} \left( \frac{1}{p} - \frac{1}{q} \right)^{\frac{2m-\alpha}{2m}}.$$  

(5.11)
Proof. By a scaling argument, we may assume $z = e^{i\theta}$ with $0 < |\theta| \leq \pi$. If $\delta < |\theta| \leq \pi$ for any small $\delta > 0$, then $|D|^\alpha(H_0 - e^{i\theta})^{-1}$ is a standard constant coefficient Fourier multiplier operator of order $-2m + \alpha$ with the symbol $|\xi|^\alpha(|\xi|^{2m} - e^{i\theta})^{-1}$. Hence the estimate (5.11) follows from the standard Sobolev estimates. A similar argument shows that for any $p \leq q$ the multiplier $|D|^\alpha(H_0 - e^{i\theta})^{-1}(\psi)(H_0)$ is bounded as an operator from $L^p$ to $L^q$. Thus we may assume that $0 < |\theta| \leq \delta$ (that is, $z$ belongs to one neighborhood containing positive real line), and by symmetry it is enough to consider only the case $\text{Im} z > 0$.

Using the reduction above, we may set $z = (\lambda + i\lambda\varepsilon)^{2m}$ for some $\lambda \sim 1$ and $0 < \varepsilon \ll 1$. Since $|z| \sim \lambda^{2m} \sim 1$, by a scaling argument again in $\lambda$, it suffices to estimate $|D|^\alpha(H_0 - (1 + i\varepsilon)^{2m})^{-1}$ and $|D|^\alpha(H_0 - (1 + i\varepsilon)^{2m})^{-1}(\psi)(H_0)$ uniformly for $0 < \varepsilon \ll 1$. Let $K^\varepsilon$ be the convolution kernel of $|D|^\alpha(H_0 - (1 + i\varepsilon)^{2m})^{-1}$. Then Fourier transform gives that

$$K^\varepsilon = \mathcal{F}^{-1}\left(\frac{|\xi|^\alpha(|\xi|^{2m} - (1 + i\varepsilon)^{2m})^{-1}}{|\xi|^{2m} - (1 + i\varepsilon)^{2m}}\right).$$

Decompose $K^\varepsilon = K_1 + K_2$, where

$$K_1 = \mathcal{F}^{-1}\left(\frac{|\xi|^\alpha\psi(|\xi|^{2m})}{|\xi|^{2m} - (1 + i\varepsilon)^{2m}}\right), \quad K_2 = \mathcal{F}^{-1}\left(\frac{|\xi|^\alpha(1 - \psi(|\xi|^{2m}))}{|\xi|^{2m} - (1 + i\varepsilon)^{2m}}\right).$$

To show (5.10) and (5.11), it is crucial to verify that the operator $K_1 * f$ satisfies (5.10) since (5.11) can immediately follows by combining $K_1 * f$ with the simpler part $K_2 * f$.

Estimate for $K_2 * f$. By the support property of $\psi$ the symbol of $K_2$ satisfies that

$$|D^\beta\left(\frac{|\xi|^\alpha(1 - \psi(|\xi|^{2m}))}{|\xi|^{2m} - (1 + i\varepsilon)^{2m}}\right)| \leq C_{\alpha\beta}(1 + |\xi|)^{-2m + \alpha - |\beta|}, \ \xi \neq 0,$$

for any $\beta \in \mathbb{N}_0^\alpha$. Hence by the Fourier transform we obtain that $|K_2(x)| \leq C_N|x|^{2m - \alpha - n - N}$ for any $N \in \mathbb{N}_0$. Then Young’s inequality and interpolation (note that $2m - \alpha < n$) give that

$$\|K_2 * f\|_{L^q} \leq C_{p,q}\|f\|_{L^p}$$

for all $(p, q)$ satisfying $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2m - \alpha}{n}$ and $1 < p \leq q < \infty$.

Estimate for $K_1 * f$. In order to apply the stationary phase method to $K_1$, we first write

$$K_1(x) = \int_{\mathbb{R}_n} \frac{e^{ix\xi} \tilde{\psi}(\xi)}{|\xi|^{2m} - (1 + i\varepsilon)^{2m}} d\xi = \int_0^{\infty} s^{n-1}\tilde{\psi}(s)\int_{s^{-1}}^\infty e^{is\omega}dw\omega ds,$$

where $x = s\omega$, $\tilde{\psi}(s) = s^\alpha\psi(s^{2m})(s^{2m-2} + s^{2m-2}(1 + i\varepsilon) + \ldots + (1 + i\varepsilon)^{2m-1})^{-1}$.

Note that $K_1$ is the Fourier transform of a compactly supported distribution including taking limits with $\varepsilon$ goes to $\pm 0$ so $|K_1(x)| \leq C$ for all $|x| \leq 1$. To handle the remaining case $|x| > 1$, we recall the following stationary phase formula for the Fourier transform of a smooth measure on hypersurface $S^{n-1}$ (see e.g. [62 p.51]):

$$\int_{S^{n-1}} e^{iy\omega}d\omega = |y|^{-\frac{n-1}{2}}c_+(y)e^{iy} + |y|^{-\frac{n-1}{2}}c_-(y)e^{-iy},$$

(5.14)
where, for $|y| \geq 1/4$, the coefficients satisfy
\[ | \frac{\partial^\beta}{\partial y^\beta} c_+(y) | + | \frac{\partial^\beta}{\partial y^\beta} c_-(y) | \leq C_\alpha |y|^{-|\beta|}, \quad \beta \in \mathbb{N}_0. \] (5.15)

Thus combining (5.13) with (5.14) one has
\[ K_1(x) = \sum_{\pm} \int_0^\infty \frac{s^{n-1} \tilde{\psi}(s)}{s - 1 - i\varepsilon} \left( |sx|^{-\frac{n-1}{2}} c_\pm(sx)e^{\pm is|x|} \right) ds \]
\[ = \sum_{\pm} |x|^{-\frac{n-1}{2}} b_\varepsilon^\pm(x)e^{\pm i|x|}, \quad |x| > 1/4, \] (5.16)

where
\[ b_\varepsilon^\pm(x) = \int_{-\infty}^\infty \frac{(s + 1)^{\frac{n-1}{2}} \tilde{\psi}(s + 1)}{s - i\varepsilon} c_\pm((s + 1)x)e^{\pm is|x|} ds. \]

Note that the function $s \mapsto (s + 1)^{\frac{n-1}{2}} \tilde{\psi}(s + 1)c_\pm((s + 1)x)$ is smooth and compactly supported near $s = 0$. So one can obtain uniformly in $\varepsilon > 0$ that
\[ |\partial^\beta b_\varepsilon^\pm(x)| \leq C_\beta |x|^{-|\beta|}, \quad |x| > 1/4. \]

Hence in view of (5.16) we can further smoothly decompose $K_1 = K' + K''$ in such a way that $\text{supp } K' \subset B(0,1)$ (the unit ball of $\mathbb{R}^n$), $K'$ is bounded and $K''$ can be expressed as
\[ K''(x) = \sum_{\pm} |x|^{-\frac{n-1}{2}} a_\pm(x)e^{\pm i|x|}, \] (5.17)

where $a_\pm \in C^\infty(\mathbb{R}^n)$ satisfy $a_\pm(x) = 0$ for $|x| \leq 1/2$ and $|\partial^\beta a_\pm(x)| \leq C_\beta |x|^{-|\beta|}$ for any $\beta \in \mathbb{N}_0$. By Young’s inequality, we have for all $1 \leq p < q \leq \infty$ that
\[ \|K' * f\|_{L^q} \leq C \|f\|_{L^p} \] (5.18)

To estimate $K''$, we first note that $|K''(x)| \leq (1 + |x|)^{-(n-1)/2}$ from the expression (5.17). Hence, for all $1 < p \leq q < \infty$ satisfying $\frac{n+1}{2n} \leq \frac{1}{p} - \frac{1}{q} \leq 1$, one has
\[ \|K'' * f\|_{L^q} \leq C \|f\|_{L^p}. \] (5.19)

However, this argument does not give the whole range of pairs $(p, q)$ for which (5.19) holds. It is possible to extend it by using the oscillatory factor $e^{\pm i|x|}$ of $K''(x)$ in the formula (5.17), i.e.
\[ K'' * f(x) = \sum_{\pm} \int_{\mathbb{R}^n} |x - y|^{-\frac{n-1}{2}} a_\pm(x - y)e^{\pm i|x-y|} f(y) dy. \] (5.20)

Indeed, the phase function $|x - y|$ satisfies the so-called $n \times n$-Carleson-Sjölin conditions, see [52] p.69. Hence the celebrated Carleson-Sjölin argument can be used to estimate $K'' * f$.

Let $\phi(s) \in C_0^\infty(\mathbb{R})$ be a such function that supp $\phi \in [\frac{1}{2}, 2]$ and $\sum_{\ell=0}^\infty \phi(2^{-\ell}s) = 1$ for $s \geq 1/2$. Set $K''_\ell(x) = \phi(2^{-\ell}|x|)K''(x)$ for all $\ell = 0, 1, 2, \ldots$, so
\[ K'' * f(x) = \sum_{\ell=0}^\infty (K''_\ell * f)(x), \] (5.21)
where
\[
K'' \ast f(x) := \int_{\mathbb{R}^n} |x-y|^{-\frac{n-1}{2}} \phi(2^{-\ell}|x-y|) a_{\pm}((x-y)) e^{\pm i|x-y|} f(y) dy.
\] (5.22)

Put \(\lambda = 2^\ell\). Then the scaling gives
\[
(K'' \ast f)(\lambda x) = \lambda^{\frac{n+1}{2}} \int_{\mathbb{R}^n} w(x-y) e^{\pm \lambda|x-y|} f(\lambda y) dy,
\] (5.23)

where \(w(x) = |x|^{-\frac{n-1}{2}} \phi(|x|) a_{\pm}(\lambda x)) \in C^\infty_0(\mathbb{R}^n \setminus 0)\) satisfying \(|\partial^\beta w(x)| \leq C_\beta\) for any \(\beta \in \mathbb{N}^n_0\).

Now we can apply Carleson-Sjölin argument (see [62, p.69]) to (5.23), obtaining that
\[
\|K'' \ast f\|_q \leq C \lambda^{-n/p+(n+1)/2} \|f\|_{L^p}, \quad \lambda = 2^\ell, \ \ell = 0, 1, \ldots,
\] and hence
\[
\|K'' \ast f\|_{L^q} \leq C \|f\|_{L^p},
\] (5.24)

for all \(q = \frac{n+1}{n} p', 1 \leq p < 2n/(n+1)\) as \(n \geq 3\). Furthermore, by interpolating between the estimates (5.19) and (5.24), we can conclude that
\[
\|K'' \ast f\|_{L^q} \leq C \|f\|_{L^p},
\] (5.25)

for all \((p, q)\) such that \(\frac{2}{n+1} < \frac{1}{p} - \frac{1}{q} \leq 1\) and
\[
\min \left( \frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q} \right) > \frac{1}{2n}.
\]

Therefore the estimates [5.12], [5.18] together with [5.25] yield the estimate (5.11) besides of the boundary line \(2/(n+1) = 1/p - 1/q\). But based on the oscillatory integral presentations (5.20) and (5.23) of \(K''\), this can be proved by showing the weak estimates of the endpoint case, and using duality and real interpolation method. We refer reader to see [27] or [32] for such technical details of the remained cases. Thus we have finished the proof of Lemma 5.1.

\[\square\]

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Email address: haruya@math.sci.osaka-u.ac.jp

School of Mathematics and Statistics, Key Laboratory of Nonlinear Analysis and Applications (Ministry of Education), Central China Normal University, Wuhan, 430079, P.R. China

Email address: yaoxiaohua@ccnu.edu.cn