CONTROL DESIGN OF LINEAR SYSTEMS WITH SATURATING ACTUATORS: A SURVEY

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Abstract. In this paper, a survey of the extensive research investigation performed on linear systems subject to saturation including actuator, output and state types is presented. The survey takes into consideration several technical views on the analysis and design procedures leading to global or semi-global stability results and outlines basic assumptions. Research works on the design of linear feedback laws, decentralized controllers are equally emphasized. Results related stability with enlarging the domain of attraction and systems subject to multi-layered nested saturations are provided. Some typical examples are given to illustrate relevant issues.

1. Introduction. The behavior of linear, time-invariant (LTI) systems subject to actuator saturation has been extensively studied for several decades. It is known that saturation usually degrades the performance of a system and leads to instability. Over the last years systems subject to saturation has attracted a lot of researchers and a considerable amount of work has been done. Most of the studies have been done on systems subject to actuator saturation, which involves problems as global, semi-global stabilization and local stabilization, anti-windup compensation, null controllable regions, to mention a few.

More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various frameworks, see \[35\] for a nice overview of application cases requiring a formal treatment of the saturation constraints. Most of the research effort geared toward constructive linear or nonlinear control for saturated plants can be divided into two main strands. In the first one, called anti-windup design, an initial controller is designed so that its closed-loop with the plant without input saturation is well behaved (at least asymptotically stable but possibly inducing desirable unconstrained closed-loop performance). The analysis and synthesis of controllers for dynamic systems subject to actuator saturation

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have been attracting increasingly more attention, see [4, 28, 36] and the references therein. It turns out that there are mainly two approaches to dealing with actuator saturation. One approach is to take control constraints into account at the outset of control design. A low-and-high gain method was presented in [36] to design linear semi-globally stabilizing controllers.

It becomes increasingly apparent that saturation is an open topic of research in control systems and many researchers have done lot of work in this field of study. Available results can be broadly classified into

- Systems with actuator saturation, see [15, 42, 5, 20, 43, 40, 11, 21, 25, 36] and their references,
- Systems with input saturation, see [14, 10, 31, 22, 6, 44, 39, 38, 27, 34] and their references,
- Systems with output saturation, see [18, 26, 29] and their references,
- Systems subject to state saturation, see [16, 12] and their references

Researchers also considered the study on large-scale systems subject to multi-layer nested saturations [31, 41, 32]. When saturation occurs global stability becomes an issue and it can never be ensured. Usually semi-global stability was the target [26, 29]. Also the study was done on estimating the large domain of attraction in [32, 21, 2].

In some studies the set invariance coupled with LMI based optimization was established [14, 5] and feedback designs for stabilizing the system using different designs were developed [15, 43, 10, 40, 31, 7, 8, 6, 44, 39, 38, 32, 27, 34]. The anti-windup design was done for the case of actuator saturation on systems [14, 42, 5] also resulting in a lot of interest in this part of the study.

A dynamic output feedback approach was developed in [15] for the controller design using the cone complementary linearization procedure. The underlying work dealt with the estimation of domain of attraction and then a method was described for the controller design of a LTI system in the presence of actuator saturation. The feasibility problem was solved using the cone complementary linearization method. The condition for set invariance with actuator saturation was also presented. The design of controller for the system with saturation was also considered such that the estimated domain of attraction was maximized with respect to prescribed bounded convex set. This presented a solution for the state feedback case, where all states are measurable, but when the states were not measurable then only the outputs were measurable for designing the output feedback controller.

In [5], an anti-windup technique was presented to enlarge the domain of attraction for systems subject to actuator saturation. They assumed a linear anti-windup compensator which stabilizes the system in the absence of actuator saturation and then used LMI techniques to enlarge the domain of attraction. A method for estimating the domain of attraction of the origin for the system under saturated linear feedback was discussed in [20]. A set invariance condition was derived and conditions for enlarging this invariant set was done. Using these conditions analysis and design was done for both closed loop stability and disturbance rejection. The condition they developed for the determination of invariant set was less conservative than that based on the circle criterion or the vertex analysis.

In [10], the goal of study was to design controllers for saturating decentralized systems that achieve not only stabilization but also achieve high performance. Their
The contribution to the work was to provide a broad and sufficient conditions for decentralized stabilization under saturation and they have shown that stabilization is possible whenever,

- the open loop eigenvalues are in closed unit disc,
- the eigenvalues on the closed unit circle disc are not in decentralized fixed modes, and
- these eigenvalues on unit circle have algebraic multiplicity of 1.

They tried to solve the problem of developing semi-global stabilization via decentralized control, which was not achieved, but semi-global stabilization was shown.

In [31], a study on the decentralized controllers for large-scale linear systems subjected to saturation control was attained with $L_2$ disturbance rejection. For a closed loop system under a saturating decentralized feedback law conditions were identified for which an ellipsoid is contractive invariant and also within the domain of attraction. A numerical algorithm was developed to solve the optimization BMI problems. The extension of the work was done for systems subjected to nested saturation case. Various methods employed for weakly coupled and strongly coupled subsystems were discussed and in the case of strongly coupled subsystems. The first phase of [45] dealt with the design of a decentralized controller using the homotopy method in the absence of actuator saturation. In the second phase of the work, the authors considered the actuator saturation. They used the decentralized feedback law as an initial controller a path following algorithm was developed which searches a new decentralized feedback law that could achieve a larger domain of attraction or stronger disturbance rejection capability.

In [39], the authors summarized observations about the control of decentralized systems with input saturation. They were able to show that time-invariant non-linear controllers cannot be used to move the fixed modes to zero. Using the singular perturbation method for model reduction was very promising for the first step in design of stabilizing controllers for decentralized systems with input saturation.

In [18], an LTI MIMO system was considered which is controllable and observable with each output component saturated. At first the output is brought out of saturation and the states are identified which then on the application of Deadbeat control drive the states to the origin.

In [26], the output is first brought out of saturation, using a method which relies on the sign of the output. When the output comes out of the saturation, the state of the system is identified using the deadbeat control strategy. Since it was found to be difficult to bring all the outputs out of saturation at the same time in case of a MIMO system, it was found that it is better to use one output at a time out of the saturation regions, even if some others are in the saturation zone. After getting all the data for all the outputs at certain different times they merged the results and found the states of the systems. These states were brought to the origin using the deadbeat control.

In the foregoing papers, it was basically emphasized that global asymptotic stabilization is possible by output feedback and also it was illustrated that information from multiple output components at different points can be combined to identify the states of the system. On the other hand for the origin of a LTI MIMO system with saturated outputs to be globally asymptotically stabilized, it is necessary for the system to be controllable and observable.

In [29], the authors considered the problem of semi-globally stabilizing linear system using linear feedback of the saturated output measurement. They established
that a SISO linear stabilizable and detectable system subject to output saturation can be semi-globally stabilized by linear output feedback if all the invariant zeros are in the closed left-half plane, no matter where open loop poles are. The linear feedback laws were designed in such a way that the saturated output was steered to cause the system output to oscillate into the linear region of output saturation function and remain there in a finite time.

In [16], the study was focused on the problem of stability analysis and controller design for continuous time linear systems with the consideration of full state saturation as well as partial state saturation was done. A new and tractable system was constructed showing that this system is with same domain of attraction as the original system. An LMI method is used for estimating the attraction domain of the origin for new constructed system with state saturation. An algorithm was developed for the designing of of output feedback controllers guaranteeing that the attraction domain of the origin for the closed-loop system is as large as possible.

In [12], they discussed the problem of stability analysis for linear systems under state constraints and some conditions were devised for global asymptotic stability of such systems. The authors achieved certain conditions under which linear systems defined on closed hypercube and linear systems with partial saturation are globally asymptotically stable at origin. Iterative LMI formulation was proposed for verifying the asymptotically stable system.

The problem of synthesizing fixed order anti-windup compensator which meet an $L_2$ performance bound was addressed in [14]. They used the linear anti-windup augmentation method to develop a controller. It was also shown that if and only if the plant is asymptotically stable this plant order anti-windup compensation is always feasible for large $L_2$ gain. They have demonstrated that the Lyapunov formulation of this problem can be taken as a non convex optimization problem.

A new saturation control technique is developed in [42] for anti-windup design for the case of exponentially unstable LTI system. The algorithm developed guaranteed regional stability in the presence of input saturation and improves performance too. It was also commented that systems with input non-linearities such as deadzone and hysteresis can also be treated using this approach.

In [43], a method for output feedback with saturation for stabilizing the system was presented. Also the enlargement of domain of attraction was done. They have used a non-linear output feedback controller expressed in quasi-LPV system form, establishing conditions for closed loop stability.

In [9], a piece-wise quadratic Lyapunov function was developed for the analysis of global and regional performances for systems with saturation with an algebraic loop. The function incorporated the structure of the saturation non-linearity. Sector conditions were considered which are shown in the following section and also an introduction to three sector like conditions that were useful in this paper. A comparison with the non-quadratic Lyapunov function in [24] was done. They have addressed the problem of stability and performance analysis for linear systems which involve saturation/deadzone. They have compared the results from [15] with the ones produced by these new methods. In this paper they showed that the Lyapunov based approach on piece-wise quadratic function was developed to analyze the global, regional stability and performances for systems subject to saturation.

In [8], They presented the LMI based synthesis approach on output feedback design for input saturated linear systems using deadzone loops. The proposed approach will lead to regional stabilizing controllers if the plant is exponentially
unstable, to semi-global stability if the plant is non-exponentially unstable, and to
global stability if the plant is already exponentially stable, the requirement of the
plant being detectable and stabilizable.

The design of decentralized controllers for interconnected linear systems subject
to multi-layer nested saturations were considered in [32]. They formulated the de-
centralized state feedback laws that resulted in large domain of attractions. Their
study was divided into two phases, the first phase they assumed no saturation and
using the homotopy idea [45]. This decentralized control law, when subject to sat-
uration still achieves local stabilization with guaranteed domain of attraction. The
second phase, they consider the actuator saturation and designed the algorithms
for larger domain of attraction.

In [41], they showed the problem of stability for systems presenting nested sat-
urations. The generalized sector conditions were used for the stability analysis.
This work proposed that it allows a more general nested saturation structure and
addressed the global stability and global stabilization problems solved by LMI meth-
ods.

This paper addresses the contemporary research topics and theoretical advances
on linear dynamical systems with saturation. It also sheds light on open topics
for future research. The paper is organized as follows: Section II provides the
problem of actuator saturation where the problem statement and theorems as well
as assumptions used in the literature addressed are presented. Section III deals
with output saturation and Section IV focuses on state saturation. The topic of
nested saturation is the subject of Section V whereas Section VI includes the topic
of linear systems with dead-zone. Finally the paper is concluded with some open
topics for future research.

Notations and facts: In the sequel, the Euclidean norm is used for vectors. We
use $W'$ and $W^{-1}$ to denote the transpose and the inverse of any square matrix $W$,
respectively. We use $W > 0$ ($\geq, <, \leq 0$) to denote a symmetric positive definite
(positive semidefinite, negative, negative semidefinite matrix $W$ and $I$ to denote
the $n \times n$ identity matrix. Let $\mathbb{R}^+$ and $\mathbb{N}$ denote, respectively, the non-negative real
numbers and the finite set of integers $\{1, ..., N\}$. Matrices, if their dimensions are
not explicitly stated, are assumed to be compatible for algebraic ope-
rations. For a positive definite matrix $P \in \mathbb{R}^{m \times n}$ and a positive scalar $\rho$, we define the following
ellipsoid,

$$\Xi(P, \rho) = x \in \mathbb{R}^n : x'Px \leq \rho$$

For a matrix $F \in \mathbb{R}^{m \times n}$, let

$$L(F) = x \in \mathbb{R}^n : |f_i x| \leq 1, i \in I_m$$

where $f_i$ represents the ith row of the matrix $F$. We note that $L(F)$ represents the
region in $\mathbb{R}^n$ where $Fx$ does not saturate. Also let $V$ be the set of $m \times m$ diagonal
matrices whose diagonal elements are either 1 or 0. There are $2^m$ elements in $V$.
Suppose these elements of $V$ are labeled as $D_s$, $s \in I_{2^m}$. Denote $D_s^+ = I - D_s$.
Clearly, $D_s^- \in V$ if $D_s \in V$.

Let $F, H \in \mathbb{R}^{m \times n}$. Then for any $x \in L(H)$

$$\text{Sat}(Fx) \in \text{conv}D_sFx + D_s^-Hx, s \in I_{2^m}$$

where $\text{conv}$ stands for convex hull. In symmetric block matrices or complex matrix
expressions, we use the symbol $\bullet$ to represent a term that is induced by symmetry.
2. Actuator saturation. In this section we consider the important case of systems subject to actuator saturation. Other cases of systems subject to state saturation, systems subject to output saturation and systems presenting nested saturation will be dealt with in the sequel.

2.1. Problem statement. Consider the LTI plant described by,

\[ \begin{align*}
\dot{x}_p(t) &= A_p x_p(t) + B_p \text{Sat}(u) + E w(t) \\
y(t) &= C_p x_p(t) + D_p \text{Sat}(u)
\end{align*} \tag{1} \]

where \( x_p(t) \in \mathbb{R}^{n_p} \) is the plant state, \( u(t) \in \mathbb{R}^{n_u} \) is the control input, \( y(t) \in \mathbb{R}^{n_y} \) is the plant output available for measurement and \( w(t) \in \mathbb{R}^{n_w} \) is the input disturbance.

In most of the works, the following assumptions were considered:

Assumption 1. The following conditions hold

- The triple \((A_p, B_p, C_p)\) is stabilizable and detectable,
- The matrices \(B_p^t\) and \(C_p\) have full row rank,
- \(D_p = 0\)

The common objective is to address the stability analysis and control design problems for system (1) under Assumption 1. In this paper, we survey available results pertaining to both problems using alternative approaches.

2.2. Main results. Initially, assume that the controller is given for the system with actuator saturation, the problem of interest is that finding the estimate of domain of attraction. The following theorem developed in [15] provides a basic result:

**Theorem 2.1.** Given an ellipsoid \( \Xi(P, \rho) \), \( P \in \mathbb{R}^{n \times n} \), if there exists an \( H \in \mathbb{R}^{n \times n} \), such that,

\[ (A_{cl} + B_{cl} (V_i C_{cl} + V_i^{-1} H))^t P + P (A_{cl} + B_{cl} (V_i C_{cl} + V_i^{-1} H)) < 0 \tag{2} \]

for all \( V_i \in V \) and \( \Xi(P, \rho) \subset P(H) \), that is,

\[ |H_i x| \leq 1 \ \forall \ x \in \Xi(P, \rho) \]

is contractively invariant set.

**Remark 1.** System (1) was considered in [5] under Assumption 1. Introducing a typical anti-windup compensator involving a correction term of the form \( E_c (\sigma(u) - u) \) leads the closed-loop system

\[ \begin{align*}
\dot{x}_c &= A_c x_c + B_c + E_c (\sigma(u) - u), \ x_c(0) = 0 \\
u &= C_c x_c + D_c y, \ x_c \in \mathbb{R}^{n_c}\n\end{align*} \tag{3} \]

Based on Theorem (2.1), the problem estimating the domain of attraction is addressed. By using the matrix \( E \) as a free design parameter, it is shown that the domain of attraction can be enlarged via optimization procedure.

The following theorem gives a condition of set invariance as examined in [20]:

**Theorem 2.2.** Given an ellipsoid \( \Xi(P, \rho) \), if there exists an \( H \in \mathbb{R}^{m \times n} \) such that,

\[ (A + BM(v, F, H))^t P + P (A + BM(v, F, H)) < 0 \tag{4} \]
where,
\[
\{M(v, F, H) : v \in \nu\} = \{H, \begin{bmatrix} h_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ h_2 \end{bmatrix}, F\}
\]
\[
\nu = v \in \mathbb{R}^m : v_i = 1 \text{ or } 0
\]
for all \(v \in \nu\) and \(\Xi(P, \rho) \subset \ell(H)\), that is,
\[
|h_i| \leq 1, \quad \forall x \in \Xi(P, \rho), \quad i \in [1, m]
\]
then \(\Xi(P, \rho)\) is a contractively invariant set.

The class of disturbances treated in the literature is characterized below

**Assumption 2.** The input disturbance \(w(t) \in \mathbb{R}^{n_w}\) belongs to the bounded set \(W\) defined by
\[
W := \{w(t) : w(t)^t w(t) \leq 1 \forall t \geq 0\}
\]
An efficient method of determining disturbance rejection with guaranteed domain of attraction is summarized by the following theorem [20]:

**Theorem 2.3.** Given two ellipsoids
\(\Xi(P, \rho_1), \Xi(P, \rho_2), \rho_2 > \rho_1 > 0\)
If there exist matrices \(H_1, H_2 \in \mathbb{R}^{m \times n}\) and a positive number \(\eta\) such that
\[
(\begin{bmatrix} A + BM(v, F, H_1) \end{bmatrix}^t P + P(A + BM(v, F, H_1)) + \frac{1}{\eta} PEE^t P + \frac{\eta}{\rho_1} P < 0, \forall v \in \nu
\]
\[
(\begin{bmatrix} A + BM(v, F, H_2) \end{bmatrix}^t P + P(A + BM(v, F, H_2)) + \frac{1}{\eta} PEE^t P + \frac{\eta}{\rho_2} P < 0, \forall v \in \nu
\]
and \(\Xi(P, \rho_1) \subset L(H_1), \Xi(P, \rho_2) \subset L(H_2)\), then for every \(\rho \in [\rho_1, \rho_2]\), there exists a matrix \(H \in \mathbb{R}^{m \times n}\) such that
\[
(\begin{bmatrix} A + BM(v, F, H) \end{bmatrix}^t P + P(A + BM(v, F, H)) + \frac{1}{\eta} PEE^t P + \frac{\eta}{\rho} P < 0, \forall v \in \nu
\]
and \(\Xi(P, \rho) \in L(H)\). This implies that \(\Xi(P, \rho)\) is also strictly invariant.

Next, in [10] the following system is considered
\[
x(k + 1) = Ax(k) + \sum_{i=1}^{v} B_i Sat(u_i(k))
\]
\[
y_i(k) = C_i x(k), i = 1, \ldots v
\]
where \(x \in \mathbb{R}^n\) is the state, \(u_i \in \mathbb{R}^{m_i}, i = 1 \ldots v\) are control inputs, \(y_i \in \mathbb{R}^{p_i}\) are measured outputs. For system (9) an improved controller stabilization result is developed and summarized by the following theorem:

**Theorem 2.4.** For system (9), there exists non-negative integers \(s_1, s_2, \ldots, s_v\) such that for any given collection of compact sets \(W \in \mathbb{R}^n\) and \(S_i \in \mathbb{R}^{p_i}\) there exists \(v\) controllers of the form,
\[
z_i(k + 1) = K_i z_i(k) + L_i y_i(k)
\]
\[
u_i(k + 1) = M_i z_i(k) + N_i y_i(k)
\]
such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes \( W \times S_1 \times \ldots \times S_v \) only if,

- All fixed modes are in the open unit disc
- All eigenvalues of \( A \) are in the closed loop unit disc.

An additional result on the conditions of semi-global stabilizability of system Theorem 2.4 using the set of controllers (10) is established by the theorem below:

**Theorem 2.5.** Consider system (9), there exists non-negative integers \( s_1, s_2, \ldots, s_v \) such that for any given collection of compact sets \( W \in \mathbb{R}^n \) and \( S_i \in \mathbb{R}^{s_i} \), there exists \( v \) controllers of the form (10), such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes \( W \times S_1 \times \ldots \times S_v \) if,

- All fixed modes are in the open unit disc
- All eigenvalues of \( A \) are in the closed loop unit disc with those eigenvalues on the unit disc having algebraic multiplicity equal to one.

**Remark 2.** Based on the condition for set invariance developed in [15], a result on the determination of disturbance tolerance capability of the closed loop system under state feedback law is reported in [31]. This result is stated in the theorem below.

**Theorem 2.6.** Consider system (1) under the state feedback law \( u = Fx \). For a given positive definite matrix \( P \), if

\[
(A + B(D_s F + D_s^{-1} H))^t P + P(A + B(D_s F + D_s^{-1} H)) + \frac{1}{\eta} P EE^t P \leq 0, \quad s \in I_2^m
\]

and \( \Xi(P, 1 + \alpha \eta) \subset L(H) \), then every trajectory of the closed loop system that starts from inside of \( \Xi(P, 1) \) will remain inside of \( \Xi(P, 1 + \alpha \eta) \) for every \( w \in W_2^\alpha \). Additionally, there exists a matrix \( H \in \mathbb{R}^{m \times n} \) and a positive number \( \eta \) such that (11) is satisfied and \( \Xi(P, \alpha \eta) \) for every \( w \in W_2^\alpha \).

The following theorem characterizes the conditions under which the linear system under actuator saturation (1) has \( L_2 \) gain less than or equal to \( \gamma \).

**Theorem 2.7.** Let \( \alpha_{\max} \) be the maximal tolerable disturbance level. Consider an \( \alpha \in (0, \alpha_{\max}] \). For a given constant \( \gamma > 0 \), if there exists a matrix \( H \in \mathbb{R}^{m \times n} \) such that,

\[
(A + B(D_s F + D_s^{-1} H))^t P + P(A + B(D_s F + D_s^{-1} H)) + PE E^t P + \frac{1}{\gamma^2} C^t C \leq 0, \quad s \in I_2^m
\]

and \( \Xi(P, \alpha) \subset L(H) \), then the restricted \( L_2 \) gain from \( w \) to \( z \) over \( W_2^\alpha \) is less than or equal to \( \gamma \).

In establishing Theorem 2.7, the procedure for stabilization of systems subject to nested saturations developed in [1] was employed.

2.3. **Example 1.** Consider the system with

\[
A_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]
Using a standard dynamic output feedback controller, it was shown in [15] that the gains are described by

\[
A_k = -30, \quad B_k = -22, \quad C_k = -20, \quad D_k = -30
\]

Letting \( R = I_{3 \times 3} \), and solving an optimization problem, the feasible solution was attained at \( \gamma^* = 118.0139 \) and \( \Pi^* = \begin{bmatrix} 109.1588 & -0.4927 & 29.8610 \\ -0.4927 & 1.3420 & -2.6067 \\ 29.8610 & 2.6067 & 20.4395 \end{bmatrix} \)

\( H^* = \begin{bmatrix} -7.7212 & -0.7368 & -0.4017 \end{bmatrix} \)

Considering the same system and setting, \( \Xi = 2, \quad N = 5, \quad T = 5, \quad \tau = 5 \), algorithm 1 in [15], and the values obtained above, the efficiency of the ensuing design can be seen from the following results,

\[
K = \begin{bmatrix} -33.4815 & -133.8522 \\ -2.1163 & 11.8956 \end{bmatrix},
\]

\[
H_1^* = \begin{bmatrix} -7.0427 & -0.3701 & -0.3873 \end{bmatrix},
\]

\[
\Pi_1^* = \begin{bmatrix} 56.5258 & -0.0227 & 5.4276 \\ -0.0227 & 2.1547 & -0.5192 \\ 5.4276 & -0.5192 & 1.5738 \end{bmatrix}
\]

3. Output saturation. In what follows, we will examine the case of systems subject to output saturation.

3.1. Problem statement. Consider the LTI system with saturated outputs as,

\[
\dot{x} = Ax + Bu \\
y = \text{Sat}(C(x))
\]

where, \( x \in \mathbb{R}^n \) is the state of the system \( u \in \mathbb{R}^m \) is the controller input and \( y \in \mathbb{R}^p \) is the output measurement.

3.2. Main results. Basic results are established in [18] and [29] and summarized by the following theorems:

Theorem 3.1. The origin of system (13) is globally asymptotically stable, that is

\[
x(0) = 0 \Rightarrow x(t) = 0 \forall t \geq 0 \quad \text{(equilibrium)}
\]

such that \( ||x(0)|| \leq \sigma \quad \text{(stability)} \)

and \( \forall x(0), \quad \lim_{t \to \infty} x(t) = 0 \quad \text{(global attractivity)} \).

Theorem 3.2. System (13) is semi-globally asymptotically stabilizable by linear feedback of the saturated output if

- The pair \( (A,B) \) is stabilizable,
- The pair \( (A,C) \) is detectable,
- All invariant zeros of the triple \( (A,B,C) \) are in the closed left-half plane.
More specifically, for any a priori given bounded set $H_0 \subset \mathbb{R}^{2n}$, there exists a linear dynamic output feedback law of the form,

$$
\dot{z} = Fz + Gy, \quad z \in \mathbb{R}^n
$$

$$
u = Hz + H_0 y
$$
such that the equilibrium $(x,z) = (0,0)$ of the closed loop system is asymptotically stable with $H_0$ contained in its domain of attraction.

### 3.3. Example 2.

Consider the system,

$$
\dot{x}_1(t) = x_2(t) + x_3(t)
$$

$$\dot{x}_2(t) = u_1(t)
$$

$$\dot{x}_3(t) = u_2(t)
$$

$$y_1(t) = x_1(t)
$$

$$y_2(t) = x_1(t) + x_2(t)
$$

The system has an eigenvalue with multiplicity one and an eigenvalue with multiplicity 2, both at the origin, thus the system is open-loop unstable. In [18], the controller algorithm is implemented using $T = 0.5$, $\rho = 1.1$, $h_1 = C_1^t$ and $h_2 = C_2^t$ with initial conditions were $x_1(0) = 2$, $x_2(0) = -4$ and $x_3(0) = 1$. The results can be seen in the plots of Fig 1.

### 4. State saturation.

We now direct attention to the case of systems subject to state saturation.

#### 4.1. Problem statement.

Consider a class system with state saturation in the form

$$
\dot{x}_p = \text{sat}(A_p x_p) + B_p u
$$

$$y = C_p x_p
$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the plant output available for measurement.

#### 4.2. Main results.

For system (14), sufficient conditions were derived in [12] to guarantee global asymptotic stability. The theoretical results are summarized by the following two theorems. The first theorem concerns the stability analysis:

**Theorem 4.1.** If there exists a symmetric positive-definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $G \in \mathbb{R}^{n \times n}$ such that,

$$
(D_i A + D_i^{-1} G)^t P + P(D_i A + D_i^{-1} G) < 0
$$

where $G$ is (row) diagonally dominant and the diagonal is composed of negative elements as specified below

$$h(Ax + K) \in \text{co}D_i(Ax + K) + D_i^{-1} G x, \quad i \in [1,2^n] \forall x \in D^n$$

for any matrix $K \in \mathbb{R}^n$ independent of $x$.

The next theorem is for determining the globally stabilizing feedback gain $F$:...
Theorem 4.2. If there exists a symmetric positive-definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $G \in \mathbb{R}^{n \times n}$ such that,

$$
\begin{bmatrix}
A & B \\
D_i C & D_i E + D_i^{-1} G
\end{bmatrix}^t P +
P
\begin{bmatrix}
A & B \\
D_i C & D_i E + D_i^{-1} G
\end{bmatrix} < 0 \quad i \in [1, 2^n]
$$

(16)

where $D_i \in D_m$ and and $G$ is (row) diagonally dominant and the diagonal is composed of negative elements, then the system is globally asymptotically stable at origin.

4.3. Example 3. Consider the following system with state saturation

$$
A = \begin{bmatrix}-9.9 & 8 \\ 10 & 5 \end{bmatrix},
B = \begin{bmatrix} 1 \\ -9 \end{bmatrix},
C = \begin{bmatrix} 1 & 2 \end{bmatrix}
$$

Without considering the state saturation a controller is designed with gain parameters as,

$$
A_k = \begin{bmatrix}-1 & 2.5 \\ 30 & -9 \end{bmatrix},
B_k = \begin{bmatrix} -0.9 \\ -0.5 \end{bmatrix},
C_k = \begin{bmatrix} 0.1 & -2 \end{bmatrix}
$$

$$
D_k = 1
$$

$X_R = col[1100]^t, [1 - 100]^t, [-1100]^t, [-1 - 100]^t$.

By Algorithm 1 in [16], the domain of attraction was estimated and the results were as follows,

$$
\gamma^* = 1.6971
$$

$$
Q^* = \begin{bmatrix}
3.1511 & 0.9275 & 0.3591 & -0.4926 \\
0.9275 & 2.0033 & 0.4194 & -0.0770 \\
0.3591 & 0.4194 & 0.1521 & -0.0429 \\
-0.4926 & -0.0770 & -0.0429 & 1.3383
\end{bmatrix},
$$

$$
U^* = \begin{bmatrix} 0.1114 & 0 \\ 0 & 0.1111 \end{bmatrix}
$$

To design the controller, Algorithm 2 was implemented to obtain the largest domain of attraction. The controller gains were

$$
A^*_k = \begin{bmatrix} -1.6255 & 4.4283 \\ 31.3529 & -22.3786 \end{bmatrix},
B^*_k = \begin{bmatrix} -2.8123 \\ 4.5233 \end{bmatrix},
C^*_k = \begin{bmatrix} 0.3996 & -3.1210 \end{bmatrix},
D^*_k = 2.4213
$$
and the result of the domain of attraction was found to be,
\[
\gamma^* = 0.4913
\]
\[
Q^* = \begin{bmatrix}
12.2710 & -1.8511 & 0.4432 & 3.2522 \\
-1.8511 & 16.8187 & 2.9340 & -9.4093 \\
0.4432 & 2.9340 & 0.7476 & -1.5786 \\
3.2522 & -9.4093 & -1.5786 & 12.7334
\end{bmatrix},
\]
\[
U^* = \begin{bmatrix}
0.7090 & 0 \\
0 & 0.1629
\end{bmatrix}
\]

It is demonstrated in [16] that by using Algorithm 2, the index of domain of attraction is improved than that which was obtained by Algorithm 1.

5. Systems presenting nested saturation. Proceeding further, we deal with the case of systems presenting nested saturation.

5.1. Problem statement. Consider a linear system consisting of N interconnected subsystems, each subject to multi-layer nested saturation in its inputs,
\[
\dot{x}_i = A_i x_i + \Sigma_{j \neq 1} A_{ij} x_j + B_i Sat(F_{1i}x + K_{2i} Sat(F_{2i}x + \ldots + K_{pi} Sat(F_{pi}x))), \ i \in I_N
\]
where \(x_i \in \mathbb{R}^{n_i}\) is the state of the ith subsystem. For a vector \(u_i \in \mathbb{R}^{m_i}\) where
\[
Sat: \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i}
\]
is the vector valued standard saturation function defined as
\[
Sat(u_i) = \begin{bmatrix}
Sat(u_{i1}) \\
Sat(u_{i2}) \\
\vdots \\
Sat(u_{im_i})
\end{bmatrix}
\]
and
\[
Sat(u_{il}) = \text{sign}(u_{il}) \min |u_{il}|, \ l \in I_{m_i}
\]

5.2. Main results. Set invariance conditions were provided in [32] as presented by the following theorem:

**Theorem 5.1.** Given an ellipsoid \(\Xi(P, \rho)\). If there exists an \(H \in \mathbb{R}^{m \times n}\) such that,
\[
(A + B(D_s F + D_s^{-1} H))P + P(A + B(D_s F + D_s^{-1} H)) \leq 0, \ \ s \in [1, 2^m]
\]
and \(\Xi(P, \rho) \subset L(H)\). Then \(\Xi(P, \rho)\) is an invariant set. If "'\(<'\) holds for the aforementioned inequalities, then \(\Xi(P, \rho)\) is a contractively invariant set towards the origin.

For the multi-layered nested saturations, the following theorem present a useful result [31]:

**Theorem 5.2.** Consider the interconnected linear system (17). For a given ellipsoid \(\Xi(P, 1)\), if there exists some matrices \(H_\ell, \ F_\ell \in \mathbb{R}^{m \times n}, \ \ell \in I_p\), such that,
for all diagonal matrices $D_1$, $D_2$, ..., $D_{p+1} \in \mathbb{R}^{m \times m}$ whose diagonal elements can only be 0 or 1, with $\Sigma_{i=1}^{p+1} D_i = 1$ such that

\[
P(A + B(D_1 H_1 + D_2(F_1 + K_2 H_2) + \ldots +
D_p(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p H_p) +
D_{p+1}(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p F_p)) +
P(A + B(D_1 H_1 + D_2(F_1 + K_2 H_2) + \ldots +
D_p(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p H_p) +
D_{p+1}(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p F_p)))^t P < 0
\]

and $\Xi(P, 1) \subset \cap_{i=1}^{p} L(H_i)$, then $\Xi(P, 1)$ is contractively invariant set.

Extending to the situation of disturbance rejection, the following theorem establishes a workable result

**Theorem 5.3.** Consider the interconnected linear system (17). Let $P$ be a positive definite matrix.

- If there exists a positive number $\eta$ some matrices $H_\ell$, $F_\ell \in \mathbb{R}^{m \times n}$, $\ell \in I_p$, such that, for all diagonal matrices $D_1$, $D_2$, ..., $D_{p+1} \in \mathbb{R}^{m \times m}$ whose diagonal elements can only be 0 or 1, with $\Sigma_{i=1}^{p+1} D_i = 1$ such that

\[
P(A + B(D_1 H_1 + D_2(F_1 + K_2 H_2) + \ldots +
D_p(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p H_p) +
D_{p+1}(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p H_p) +
D_{p+1}(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p F_p)))^t P < 0
\]

and $\Xi(P, 1 + \alpha \eta) \subset \cap_{i=1}^{p} L(H_i)$, then every trajectory of the closed-loop system that starts from inside of $\Xi(P, 1)$ will remain inside of $\Xi(P, 1 + \alpha \eta)$ for every $w \in W_\alpha^2$.

- If there exists a positive number $\eta$ some matrices $H_\ell$, $F_\ell \in \mathbb{R}^{m \times n}$, $\ell \in I_p$, such that, for all diagonal matrices $D_1$, $D_2$, ..., $D_{p+1} \in \mathbb{R}^{m \times m}$ whose diagonal elements can only be 0 or 1, with $\Sigma_{i=1}^{p+1} D_i = 1$, inequality (20) is satisfied and $\Xi(P, \alpha \eta) \subset \cap_{i=1}^{p} L(H_i)$, then every trajectory of the closed-loop system that starts from origin will remain inside of $\Xi(P, \alpha \eta)$ for every $w \in W_\alpha^2$.

5.3. **Example 4.** For the case of multi layered nested saturation, let us consider a system with $w = 0$ and the following data

\[
A = \begin{bmatrix}
0 & 4 & 3 & 2 \\
2 & 3 & 3 & 4 \\
1 & 5 & 0 & 1 \\
1 & 3 & 4
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix},
\]

\[
p = 2, \quad K_2 = \text{blockdiag}[0.5, \ 0.5]
\]

\[
X_R = \begin{bmatrix}
1 & 0.5 & 0.6 & 0.9
\end{bmatrix}
\]
Consider the design algorithm for decentralized control for multi layered nested saturation in [32]. It is found that the feasible solution of the corresponding optimization problems was as follows:

$$\mu^* = 80.0127$$

$$P^* = \begin{bmatrix}
8.7974 & 17.0317 & -1.3232 & 2.0154 \\
17.0317 & 49.1759 & 3.6958 & 0.8127 \\
-1.3232 & 3.6958 & 29.3195 & 28.3716 \\
2.0154 & 0.8127 & 28.3716 & 45.0384
\end{bmatrix}$$

$$H_1^* = \begin{bmatrix}
-2.1505 & -5.5006 & -3.3058 & -3.6132 \\
-1.8211 & -4.6616 & -3.8394 & -4.9424
\end{bmatrix}$$

$$H_2^* = \begin{bmatrix}
-0.4850 & -1.9498 & -4.0923 & -4.4794 \\
-1.9933 & -3.4248 & -0.0002 & -1.4339
\end{bmatrix}$$

For the purpose of illustration, phase plots are presented in Figs. (1) and (2). Next, the simulation for the outer and inner input trajectories is depicted in Figs. (3)-(4).

The state trajectories of the closed loop system under a pulse disturbance with energy $\alpha = 3.3$ are plotted in Fig. 5 and the output trajectory of the closed loop system under a pulse disturbance with energy $\alpha = 3.3$ is given in Fig. 6.

6. Linear systems with deadzone. Finally, we deal with the case of systems containing deadzone.

Figure 1. Projections onto $x_1$ plane
Figure 2. Projections onto $x_2$ plane

Figure 3. Trajectories of outer input

6.1. Problem statement. Generally a system with saturation or deadzone is described as,

\[
\begin{align*}
\dot{x} &= Ax + B_yq + B_ww \\
y &= C_yx + D_{yw}q + D_{yw}w \\
z &= C_zx + D_{zw}q + D_{zw}w \\
q &= dz(y)
\end{align*}
\]  

(21)

where $x \in \mathbb{R}^n$, $q, y \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, and $z \in \mathbb{R}^p$. The deadzone function $dz(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

\[
dz(y) := y - \text{Sat}(y), \ \forall y \in \mathbb{R}^m
\]
where \( \text{Sat}(.) \) is a vector saturation function with the saturation levels given by a vector

\[
\bar{u} \in \mathbb{R}^m, \quad \bar{u}_i > 0, \quad i = 1, 2, \ldots, m
\]

In [9], they considered an algebraic loop, when \( D_{yy} \neq 0 \) and a nonlinear algebraic loop imposed by

\[
y = C_y x + D_{yy} dz(y) + D_{yw} w
\]  

Further analysis is based on the following facts:

**Fact 1:** For every diagonal matrix \( \delta \in \mathbb{R}^{s \times s}, \delta > 0 \), the deadzone function \( dz(.) \) satisfies

\[
dz(v)^T \delta \{ v - dz(v) \} \geq 0, \forall v \in \mathbb{R}^s
\]  

**Figure 4.** Trajectories of inner input

**Figure 5.** Closed-loop state trajectories
Fact 2: Given $r \in \mathbb{R}^m$ such that 

$$-\bar{u}_i \leq r_i \leq \bar{u}_i, \forall i = 1, \ldots, m$$

the following inequality holds for any diagonal matrix $\delta \in \mathbb{R}^{m \times m}$, $\delta > 0$:

$$dz(v)^t \delta \{v - dz(v) - r\} \geq 0, \forall v \in \mathbb{R}^m$$

(24)

6.2. Main results. The following results were the sector-like conditions introduced to describe the properties of the algebraic loop with deadzone:

Result 1: In view of the non-decreasing properties of saturation and deadzone functions, the following inequality holds for every diagonal matrix $\delta \in \mathbb{R}^{m \times m}$, $\delta > 0$:

$$\{dz(v_1) - dz(v_2)\}^t \delta \{sat(v_1) - sat(v_2)\} \geq 0, \forall v_1, v_2 \in \mathbb{R}^m.$$  

(25)

Result 2: For every diagonal matrix $\delta \in \mathbb{R}^{m \times m}$, the following equalities hold almost everywhere:

$$\phi(x, w)^t \delta \{\dot{u} - \phi(x, w)\} \equiv 0$$

$$dz(u)^t \delta \{\dot{u} - \phi(x, w)\} \equiv 0$$

(26)

where,

$$\dot{u} = C_y Ax + C_y B_q dz(y) + C_y B_w w + D_y \phi(x, w)$$

The conditions characterizing global analysis and regional analysis are mentioned in the following theorems:

**Theorem 6.1.** Considering system 21, the following results hold true:

1. (Exponential Stability): If there exists a matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$, $P = P^t > 0$, and diagonal matrices $\Delta_i \in \mathbb{R}^{m \times m}$, $i=1, \ldots, 5$, $\Delta_{i=1,2,3} > 0$, satisfying the LMI,

$$[I_{n+3m} \ 0_{(n+3m) \times 2}] \Psi \begin{bmatrix} I_{n+3m} \\ 0_{r \times (n+3m)} \end{bmatrix} < 0$$

(27)
then for the Lyapunov function $V(x) = \xi(x)^TP\xi$, there exists an $\Xi > 0$ such that $\ddot{V} < -\Xi|\xi|^2$ for almost all $x \in \mathbb{R}^n$ and $w = 0$. This guarantees the origin of the system is globally exponentially stable.

2. (Reachable Region): If there exists a matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$, $P = P^t > 0$, and diagonal matrices $\Delta_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, 5$, $\Delta_i = 1, 2, 3 > 0$, satisfying the LMI,

$$
\bar{\Psi} - \bar{\Psi}_6\bar{\Psi}^t_6 < 0
$$

then $\ddot{V} < w^4w$ for almost all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $\xi(x(t)) \in \Xi(P/s^2)$ for all $t \geq 0$.

3. (Regional Gain): If there exists a matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$, $P = P^t > 0$, and diagonal matrices $\Delta_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, 5$, $\Delta_i = 1, 2, 3 > 0$, satisfying the LMI,

$$
\begin{bmatrix}
\bar{\Psi} - \bar{\Psi}_6\bar{\Psi}^t_6 & \bullet \\
I_{n+3m} & 0
\end{bmatrix} < 0
$$

then $\ddot{V} + \frac{1}{\gamma}z^t < \gamma w^4w$ for almost all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^r$. If $x(0) = 0$, then $\|z\|_2 \leq \gamma\|w\|_2$, that is, the global $L_2$ gain is bounded by $\gamma$.

The theorem for regional analysis [9] is as follows:

**Theorem 6.2.** Considering system 21, the following results hold true:

1. (Exponential Stability): If there exists a matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$, $P = P^t > 0$, $H_1, H_2 \in \mathbb{R}^{n \times (n+m)}$, satisfying and diagonal matrices $\delta_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, 5$, $\delta_i = 1, 2, 3 > 0$, satisfying the LMI,

$$
[I_{n+3m} \ 0_{(n+3m) \times r}](\bar{\Psi} - \bar{\Omega} - \bar{\Omega}^t)
$$

$$
\bullet
$$

$$
[I_{n+3m} \ 0_{r \times (n+3m)}] < 0
$$

then for the Lyapunov function $V(x) = \xi(x)^TP\xi$, there exists an $\Xi > 0$ such that $\ddot{V} < -\Xi|\xi|^2$ for almost all $x \in \mathbb{R}^n$ and $w = 0$. Thus the origin of the system is globally exponentially stable. If $\xi(x(0)) \in \Xi(P)$, then $\xi(x(t)) \in \Xi(P)$ for all $t > 0$ and $\lim_{t \to \infty}x(t) = 0$.

2. (Reachable Region): Let $s > 0$. If there exists a matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$, $P = P^t > 0$, $H_1, H_2 \in \mathbb{R}^{(n+m) \times (n+m)}$ and diagonal matrices $\delta_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, 5$, $\delta_i = 1, 2, 3 > 0$, satisfying the LMI,

$$
\ddot{V} < w^4w
$$

then $\ddot{V} < w^4w$ for almost all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $\xi(x(t)) \in \Xi(P/s^2)$ for all $t \geq 0$.

3. (Regional Gain): Let $s > 0$. If there exists a matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$, $P = P^t > 0$, $H_1, H_2 \in \mathbb{R}^{(n+m) \times (n+m)}$ and diagonal matrices $\delta_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, 5$, $\delta_i = 1, 2, 3 > 0$, satisfying the LMI,

$$
\begin{bmatrix}
\bar{\Psi} - \bar{\Omega} - \bar{\Omega}^t - \bar{\Psi}_6\bar{\Psi}^t_6 & \bullet \\
C_z & 0
\end{bmatrix} < 0
$$

then $\ddot{V} + \frac{1}{\gamma}z^t < \gamma w^4w$ for almost all $x \in \Xi(P/s^2)$ and all $w \in \mathbb{R}^r$. If $\xi(x(0)) = 0$, and $\|w\|_2 \leq s$, then $\|z\|_2 \leq \gamma\|w\|_2$.
Theorem 6.3. Given $s > 0$. Consider the linear plant subject to saturation/deadzone with $||w||_2 \leq s$. Let $[N_1 \ N_2]^T$ span the null space of $[C_{py} \ D_{pyw}]$. If the following LMIs in the variables $Q_{11}, P_{11} \in \mathbb{R}^{n_u \times n_p}, Q_{11} = Q_{11}^T > 0, P_{11} = P_{11}^T > 0, Y_p \in \mathbb{R}^{n_u \times n_p}, \gamma^2 > 0, \Xi \leq \frac{1}{\gamma}$ are feasible:

$$
\begin{bmatrix}
  A_pQ_{11} + B_puY_p & B_{pw} & 0 \\
  0 & -\frac{I}{2} & 0 \\
  C_{pz}Q_{11} + D_{p,zw}Y_p & D_{p,zw} & -\frac{\gamma^2 I}{2}
\end{bmatrix} < 0
\begin{bmatrix}
  N_1 \ P_{11} \\
  I \ P_{11}
\end{bmatrix}
\begin{bmatrix}
  N_1 Y_p & Y_p^T Q_{11} \\
  Q_{11} & Y_p^T Y_p
\end{bmatrix} \geq 0
$$

then there exists an output feedback controller in the form of

$$
\dot{x}_c = A_c x_c + B_c y + E_1 dz(y_c)\quad \quad y_c = C_c x_c + D_c y + E_2 dz(y_c)
$$

of the order of $n_p$, which guaranteed the following three properties of the closed loop system:

1. the regional $L_2$ gain is bounded by $\gamma$,
2. $\Xi(\xi Q_{11}^{-1}) \times 0$ inside the domain of attraction,
3. the reachable set of the plant bounded by $\Xi((s^2 Q_{11})^{-1})$.

where $Y_{pi}$ denotes the $i$th row of $Y_p$

6.3. Example 5. Consider system (21) with the following system parameters [9]:

$$
\begin{bmatrix}
  A & B_q & B_w \\
  C_y & D_{yq} & D_{yw} \\
  C_z & D_{zq} & D_{zw}
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & -1 : 1 & 0 : 0 & 1 \\
  1 & 0 & -2 : 0 & 1 : 1 & 0 \\
  0 & 1 & -3 : 1 & -1 : 1 & 1 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  1 & 0 & 1 : -3 & -1 : 1 & -1 \\
  0 & 1 & 0 : -2 & -4 : 0 & 1 \\
  0 & 1 & 0 : 1 & 0 : -1 & 0 \\
  0 & 0 & 1 : 0 & 1 : 0 & -1
\end{bmatrix}
$$

The results of using various Lyapunov type of functions are summarized in the following table.

The associated behavior of Lyapunov functions are depicted in Figs. (7)-(9).
| Lyapunov Function                              | $-3$ | $-1$ | $-2$ | $-4$ | $-3$ | $-1.3$ | $-2.3$ | $-4$ | $-3$ | $-2$ |
|-----------------------------------------------|------|------|------|------|------|--------|--------|------|------|------|
| Piecewise; Quadratic; [9]                     | 15.13| 17.19| 25.86|
| Convex; Hull; Quadratic; [24]                 | 17.06| 19.33| 31.67|
| Max; Quadratic; [24]                          | 17.37| 20.78| 42.34|
| Quadratic; via; PDI; [24]                     | 38.96| 170.15| $\infty$|
| Lure-Postnikov; [9]                           | 46.96| $\infty$| $\infty$|
| Quadratic; via; NDI; [14], [24]               | 46.96| $\infty$| $\infty$|

**Figure 7.** Behavior of Lyapunov function 1

**Figure 8.** Behavior of Lyapunov function 2
7. **Conclusions and topics for future research.** In this paper, we have been presented a survey of the main results pertaining to linear dynamical systems subject to saturation including actuator, output and state types. The survey has outlined basic assumptions and has taken into considerations several technical views on the analysis and design procedures leading to global or semi-global stability results. A key feature has been the equal emphasis on the design of linear feedback laws, decentralized controllers. Results related stability with enlarging the domain of attraction and systems subject to multi-layered nested saturations have been provided. Some typical examples have been given to illustrate relevant issues.

Some of the topics for future research directions that can be gained from this study are as follows:

- Global stabilization of decentralized controllers subject to saturation.
- Determination of dynamic output feedback controllers using the Popov criterion.
- Design of control systems subject to output saturation to drive the output out of saturation and to steer state to origin with improved performance.
- The problem of control synthesis via dynamic output feedback for systems with state saturation has not been considered much because of its complexity.
- Decentralized control of linear systems subject to saturation is another attractive field of study.

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