Energy for $N$–Body Motion in Two Dimensional Gravity

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Abstract

A general definition of energy is given, via the Nöther theorem, for the $N$–body problem in $(1+1)$ dimensional gravity. Within a first–order Lagrangian framework, the density of energy of a solution relative to a background is identified with the superpotential of the theory. For specific applications we reproduce the expected Hamiltonian for the motion of $N$ particles in a curved spacetime. This Hamiltonian agrees with that found through an ADM–like prescription for the energy when the latter is applicable but it also extends to a wider class of solutions provided a suitable background is chosen.

1 Introduction

The study of 2–dimensional gravity has received much attention in recent years motivated from string–inspired theories and from the necessity to study gravitational effects in a simple mathematical framework. It is well–known, for example, that the problem of $N$–body motion interacting by means of their mutual gravitational forces has no exact solution in General Relativity. This is basically due to the dissipation of energy through gravitational
radiation. The problem considerably simplified in two dimensions where gravitational radiation is absent but the main features of General Relativity are maintained. Recently, a number of exact solutions for $N = 2$ were found in [1, 2, 3, 4]. These solutions have qualitatively different features compared to their non-relativistic counterparts, and provide a rich and interesting laboratory for the study of relativistic gravitational effects.

The $N$–body problem may be formulated by taking the matter action to be that of $N$ point particles minimally coupled to gravity. Because in $(1+1)$–dimensional gravity the gravitational field has no real dynamical variables, the Lagrangian of the theory must include some dynamics through an auxiliary scalar field, referred to as the dilaton in string theories. Depending on the way the dilaton field enters into the Lagrangian, different models arise. While we develop this formalism for an arbitrary dilaton theory of gravity, we shall be mainly interested in three particular kinds of dilatonic gravity theories, classified according the value of the equations of motion impart to the Ricci scalar $R$. They are the Jackiw–Teitelboim theory $\mathbb{R}$ (JT theory) with $R = \Lambda$, the $R = T$ theory $\mathbb{R}$ where T denotes the trace of the matter stress–energy, and a more general class of theories $\mathbb{R}$ (encompassing the previous two and called GT theories from now on) with $R = \Lambda + T$. In each of these theories the evolution of the gravitational field is governed only by the matter stress–tensor (and vice versa) so that they mimic the features of General Relativity.

A general framework for deriving the Hamiltonian for the $N$–particle system was developed in [1, 3]. It was based on the $(1 + 1)$ counterpart of the ADM formalism; see e.g. [6]. A canonical reduction of the action was carried out by eliminating the Hamiltonian constraints and by imposing coordinate conditions. The reduced Hamiltonian was defined as the spatial integral of the second spatial derivatives of the dilaton field. By solving the constraints, the dilaton was given in terms of the coordinates and momenta of the particles so that the reduced Hamiltonian is consequently a function only of the parameters of the particles. Moreover, the consistency of the canonical reduction was proved in [1] and it was there shown that the reduced Hamiltonian gives rise to equations of motion which are equal to the original geodesic equations for the particles.

Nevertheless, the key point in the definition of the reduced Hamiltonian is the choice of the coordinate and boundary conditions. Roughly speaking, these choices allow one to discard all the boundary terms during the canonical reduction of the action functional. If a solution does not satisfy the required
conditions the reduced Hamiltonian can not be used to define the energy of the system. As noted in [1], the definition of energy in this situation becomes quite problematic and new surface terms have to be added “ad hoc”; see [7].

Motivated by this kind of problem we seek a definition of energy for the \( N \)-body problem not constrained by boundary and coordinate conditions. The definition we shall present in this paper is based on the Nöther theorem and the theory of the superpotential for relativistic field theories; see [8, 9, 10, 11, 12] and references quoted therein. The approach we shall describe is essentially based on the Lagrangian formulation of a field theory and conserved quantities are defined with respect to infinitesimal Lagrangian generators of symmetries on spacetime, namely, with respect to spacetime vector fields. For each one of these symmetries a superpotential (and consequently an associated conserved quantity) may be found regardless of the topology of spacetime and of the solution considered. Superpotentials play a fundamental role in the definition of conserved quantities since they enclose the energetic content of the theory. For this reason we think that the specialization of Nöther theorem to the \( N \)-body problem in 2-dimensional gravity is well-suited to define successfully a generally valid expression for the energy of the system.

The starting point for the construction of the superpotential is the definition of a covariant action functional for the theory which is first–order in the dynamical fields (see [12, 13, 14, 15] and references quoted therein). It is obtained by adding a pure divergence term to the \( N \)-body Lagrangian. Covariance is achieved by means of the introduction of a background solution chosen as a reference point (or zero level) for conserved quantities. The action functional so obtained is suitable for taking field variations where only the dynamical fields (and not their derivatives) are kept fixed on the boundary. In this way we have a Lagrangian which furnishes a definition, via Nöther theorem, of a conserved quantity which can be truly considered as the energy of the system [12]. Moreover, by inserting the background from the very beginning into the Lagrangian, we naturally obtain the energy of a region relative to the background.

We remark that the fixing of a background is a universally valid procedure because it does not imply any restrictive hypothesis on the solution: the background is fixed depending on the solution under examination and for this reason it is suited to handle solutions of widely varying asymptotic behaviour on the same footing; see [13, 14, 15, 16, 17, 18]. Another advantage is that the energy of a region of finite spatial extent can be calculated through a
suitable choice of the background.

The outline of our paper is as follows. In Section 2 the Lagrangian formulation, the ADM canonical reduction of the theory and the reduced Hamiltonian for the $N$–body system are briefly summarized. In Section 3 the Lagrangian of first–order in the dynamical fields is introduced and analysed. In Section 4 we devote considerable attention to the construction of the superpotential starting from the first–order Lagrangian. The superpotential then furnishes a definition of the energy of a solution relative to a background contained in a spacelike region, i.e. a real line interval.

In the rest of the paper we apply the formalism to explicit solutions. As far as we know the solutions here analysed cover all the known exact (i.e., non–perturbative) relativistic solutions in this dimensionality. In Section 5 we consider the $N = 2$ solution for the JT theory. Inside our framework this is the easiest theory because a Minkowski–like background can be fixed for the metric. This is not the case for the GT solutions treated in Section 6. The background here cannot be Ricci flat and, in order to avoid divergence problems, it has to be matched with the solution at the boundary of the region of integration. We shall also investigate how the problem can be simplified by defining the variation of energy along a family of solutions.

Section 7 is then devoted to the JT theory. Here the dynamical metric does not depend on the particle coordinates (they are instead contained in the dilaton field) so that the dynamical metric and the background metric are the same. While the ADM recipe does not work here for lack of the necessary hypotheses, the expected Hamiltonian is reproduced via the Nöther theorem.

In the last Section concluding remarks and perspectives are presented.

2 Lagrangian Formulation and ADM Hamiltonian

In order to describe the motion of $N$ particles in 2D gravity we consider the action functional on the spacetime $M$:

$$A = \int_M (\mathcal{L}_0 + \mathcal{L}_p) d^2x$$  \hfill (1)
with
\[ \mathcal{L}_0 = \frac{\sqrt{g}}{2\kappa} \left[ \psi R + \frac{1}{2} g^{\mu\nu} H(\psi) \nabla_\mu \psi \nabla_\nu \psi + F(\psi) \right] \] (2)

\[ \mathcal{L}_P = \sum_a \int d\tau_a \left\{ -m_a \left( -g_{\mu\nu}(x) \frac{dz_\alpha^\mu}{d\tau_a} \frac{dz_\alpha^\nu}{d\tau_a} \right)^{1/2} \delta^2(x - z_a(\tau_a)) \right\} \] (3)

where \( R \) is the Ricci scalar of the metric \( g_{\mu\nu} \), \( g \) denotes the absolute value of the metric determinant, \( \psi \) is a scalar field (the dilaton) and \( H(\psi), F(\psi) \) are arbitrary functions of the dilaton field not containing the derivative \( s \) of the dilaton itself. Here \( \tau_a \) is the proper time of the \( a \)-th particle and \( \kappa = 8\pi \) (in geometric units with \( G = c = 1 \)).

The variation of \([2]\) is given by\footnote{Here and in the sequel we use the notation \( f(j^k \phi) \) to denote that the function \( f \) depends on the field \( \phi \) together with its derivatives up to order \( k \).}:

\[ \delta \mathcal{L}_0 = \frac{\sqrt{g}}{2\kappa} \left[ G_{\mu\nu}(j^2 \psi, j^1 g) \delta g^{\mu\nu} + G(j^2 g, j^2 \psi) \delta \psi \right] + \\
+ d_\mu F^\mu(j^1 g, j^1 \psi, \delta(j^1 g), \delta(\psi)) \] (4)

where:

\[ G_{\mu\nu} = g_{\mu\nu} \nabla^\sigma \nabla_\sigma \psi - \nabla_\mu \nabla_\nu \psi + \frac{H}{2} \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \psi \nabla_\sigma \psi \right) - \frac{1}{2} g_{\mu\nu} F \]

\[ G = R - H \nabla^\sigma \nabla_\sigma \psi - \frac{1}{2} H' \nabla^\sigma \psi \nabla_\sigma \psi + F' \] (5)

\( H' \) and \( F' \) denoting the derivatives with respect to the functional argument.

The explicit expression of the divergence terms in \([4]\) will be given in Section \([7]\) – see equation \([26]\).

The field equations derived from the action \([4]\) are

\[ G = 0 \] (6)

\[ G_{\mu\nu} = \kappa T_{\mu\nu} \] (7)

\[ \frac{d}{d\tau_a} \left\{ g_{\mu\nu}(z_a) \frac{dz_\alpha^\nu}{d\tau_a} \right\} - \frac{1}{2} g_{\nu\lambda}(z_a) \frac{dz_\alpha^\nu}{d\tau_a} \frac{dz_\alpha^\lambda}{d\tau_a} = 0 \] (8)
where $G$ and $G_{\mu\nu}$ have been defined in (5) and

$$T_{\mu\nu} = \sum_a m_a \int d\tau_a \frac{1}{\sqrt{g}} g_{\mu\sigma} g_{\nu\rho} \frac{dz_a^\sigma}{d\tau_a} \frac{dz_a^\rho}{d\tau_a} \delta(x - z_a(\tau_a))$$  \hspace{1cm} (9)

Since $\nabla^\mu G_{\mu\nu} = -1/2 \nabla^\nu \psi G$, equations (8) and (7) together guarantee the conservation law $\nabla_\mu T^{\mu\nu} = 0$ when the equations of motion are satisfied.

We next consider how the three particular theories of interest we noted above arise.

1– If we set $H = 0$ and $F = -\psi \Lambda$ into the Lagrangian $L_0$ we recover the Jackiw–Teitelboim theory (JT theory) \cite{5} for the gravitational field coupled to $N$ point masses. In that case, equation (6) reduces to:

$$R = \Lambda$$  \hspace{1cm} (10)

The Ricci scalar is a constant and the other dynamical fields evolve in this spacetime of constant curvature.

2– for $H = 1$ and $F = 0$ we obtain the so–called $R = T$ theory; see \cite{1}. From (6) and (7) we have

$$R = \kappa T^{\mu\mu}_\mu$$  \hspace{1cm} (11)

In this theory the matter affects the evolution of spacetime through the trace of the stress energy $T_{\mu\nu}$.

3– if we set $H = 1$ and $F = \Lambda$ we obtain the generalised theory (GT theory) described in \cite{3}. In this case combining the trace of equation (7) with (6) we obtain:

$$R = \Lambda + \kappa T^{\mu\mu}_\mu$$  \hspace{1cm} (12)

When all bodies are massless the GT theory reduces to the JT theory whereas when the cosmological constant vanishes we recover the $R = T$ theory.

Other choices of the functions $H$ and $F$ allow one to recover the dilaton gravitational theories studied in \cite{19}. As yet there are no exact solutions to the $N$-body problem in any of these other theories.

We shall now derive the canonical form for the action (1). For simplicity, we assume $H = \text{const}$, a choice compatible with both the JT theory ($H = 0$) and the $R = T$ and GT theories ($H = 1$).
Our goal will be to obtain the definition of the Hamiltonian in the ADM formalism (see [1, 6]) and to compare it with the definition of energy we shall obtain via Nöther theorem and the theory of superpotentials.

Let us then consider the ADM splitting of the metric:

\[ g = -N_0^2 \, dt^2 + \gamma \left( dx + \frac{N_1}{\gamma} \, dt \right)^2 \]  

(13)

From now on, we shall use the symbols (\(\dot{}\)) and (\(\prime\)) to denote the derivatives \(\partial_t\) and \(\partial_x\), respectively. The action (1) transforms to:

\[ A = \int_M d^2x \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a(t)) + \pi \dot{\gamma} + \Pi \dot{\psi} + N_0 R^0 + N_1 R^1 \right\} \]  

(14)

+ boundary terms

where \(\pi\) and \(\Pi\) are the momenta conjugated to \(\gamma\) and \(\psi\), respectively

\[ \pi = \frac{1}{2\kappa\sqrt{\gamma}N_0} \left( -\dot{\psi} + \frac{N_1}{\gamma} \psi' \right) \]

\[ \Pi = \frac{1}{2\kappa N_0} \left\{ -\frac{1}{\sqrt{\gamma}} \left( \dot{\gamma} + N_1 \frac{\dot{\gamma}}{\gamma} - 2N_1' \right) + \sqrt{\gamma} H (-\dot{\psi} + \frac{N_1}{\gamma} \psi') \right\} \]

and \(R^0\) and \(R^1\) are the Hamiltonian and the momentum constraints, respectively:

\[ R^0 = -\kappa H \sqrt{\gamma} \pi^2 + 2\kappa \sqrt{\gamma} \pi \Pi + \frac{H}{4\kappa \sqrt{\gamma}} (\psi')^2 - \frac{1}{\kappa} \left( \frac{\psi'}{\sqrt{\gamma}} \right)' \]

\[ + \frac{\sqrt{\gamma}}{2\kappa} F - \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2} \delta(x - z_a(t)) \]  

(15)

\[ R^1 = \frac{\gamma'}{\gamma} \pi - \frac{1}{\gamma} \Pi \psi' + 2\pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(t)) \]  

(16)

The technique developed in [1] for defining the Hamiltonian of the system was to consider the “total generator” of the action (14). This procedure allowed one to identify the dynamic and the gauge character of the variables and, subsequently, to impose the coordinate conditions

\[ \gamma = 1 \quad \Pi = 0 \]  

(17)
Eliminating then the constraints (15) and (16) and adopting the choice (17), the action (14) then simplifies to the reduced form:

\[ A_R = \int_M d^2x \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \mathcal{H} \right\} \quad \mathcal{H} = -\frac{1}{\kappa} \psi'' \]  

which is very similar to the situation in classical mechanics. The reduced Hamiltonian for the system of particles is then identified with:

\[ H = \int dx \mathcal{H} = -\frac{1}{\kappa} \int dx \psi'' \]  

where \( \psi'' \) is understood to be a function of \( z_a \) and \( p_a \) by solving the constraints (15) and (16). Despite the very simple form of the Hamiltonian, appropriate boundary conditions have to be imposed in order to guarantee the vanishing of extra boundary terms in the action (18). These conditions will be analysed case by case in the next sections and we shall see that they play a fundamental role in establishing the equivalence between the reduced Hamiltonian (19) and the Nöther definition of energy.

3 First–Order Covariant Lagrangian

In order to calculate the energy via Nöther theorem of a system of \( N \) particles coupled to gravity we have to slightly modify the action functional (9) by adding a boundary term. As it is well known, the addition of boundary terms into the action does not affect the equations of motion provided the correct boundary conditions are taken into account in order to cancel the boundary terms which arise from the bulk action in the variation. Hence, variations with prescribed boundary conditions select suitable boundary terms to append to the action and vice versa \([12, 20, 21]\). When dealing with conserved quantities, additional boundary terms lead to different values for the Nöther currents and for the associated Nöther charges and each one has a different physical meaning. We shall here consider a modified action functional leading to the definition of a conserved quantity which can be “truly” considered as the energy of the system. It is constructed by requiring that, in the variational principle, the dynamical fields are kept fixed on the boundary while the variation of their derivatives are free, i.e. they are not constrained to vanish.
Let us consider again the variation $\delta L_0$ in (4). It is easy to verify (see (26) below) that the divergence terms depend not only on $j^1g$, $j^1\psi$, $\delta g^{\mu\nu}$, $\delta \psi$ but also on $\delta u^\alpha_{\beta\mu}$ where

$$u^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} - \delta^\alpha_{(\beta} \Gamma^\nu_{\mu)\nu}$$  \hspace{1cm} (20)

From (4) we then see that the action functional $A^0_M = \int_M L_0 d^2x$ is stationary, i.e. $\delta A^0_M = 0$, if the Euler–Lagrange equations of $L_0$, i.e. $G_{\mu\nu} = 0$ and $G = 0$, are satisfied and if $g^{\mu\nu}$ and $\psi$ are kept fixed on the boundary $\partial M$ together with certain derivatives of $g_{\mu\nu}$ (namely $\delta u^\alpha_{\beta\mu}|_{\partial M} = 0$). In order to have a variational principle where only the dilaton field and the metric are fixed on the boundary of the region of integration (while the metric derivatives are not fixed), we have to build an action functional of first order in the dynamical fields. As already explained, this may be done by adding a divergence term to the Lagrangian (2). The ultimate motivation resides in the necessity of having a well–posed definition for the energy of the system.

Moreover, when dealing with the theory of conserved quantities, it is well known that it absolute conserved quantities (for example the absolute energy of a solution) do not have a precise meaning in General Relativity since it is preferable to consider relative conserved quantities; see [15, 11, 16, 17, 18, 12, 21]. By relative we mean that the conserved quantities of a dynamical field are calculated with respect to a background solution which is chosen as a zero level or reference point.

In order to have a first order covariant Lagrangian with the background initially included in the action functional we consider a background metric $\bar{g}$. By denoting with $\bar{\Gamma}^\alpha_{\beta\mu}$ the Levi–Civita connection of the background we may define the first–order covariant Lagrangian:

$$L_1 = L_0 + L_{\text{Div}}$$

$$= \frac{\sqrt{g}}{2\kappa} \left( \psi R + \frac{1}{2} g^{\mu\nu} H \nabla_\mu \psi \nabla_\nu \psi + F \right)$$

$$- \frac{1}{2\kappa} d_{\alpha} (\sqrt{g} \psi g^{\mu\nu} w^\alpha_{\mu\nu})$$  \hspace{1cm} (21)

where

$$w^\alpha_{\beta\mu} = Q^\alpha_{\beta\mu} - \delta^\alpha_{(\beta} Q^\nu_{\mu)\nu}$$

$$Q^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} - \bar{\Gamma}^\alpha_{\beta\mu}$$  \hspace{1cm} (22)

A first glance at (21) shows that the background $\bar{g}$ does not affect the equations of motion since it is contained in the divergence term $L_{\text{Div}}$. Moreover (21) is manifestly covariant since $w^\alpha_{\beta\mu}$ is a tensor. We also observe that
\( \mathcal{L}_1 \) is first–order in the dynamical fields \( g \) and \( \psi \) (and second–order in the background \( \bar{g} \)). Indeed, the terms \( \sqrt{g} R - d_\alpha (\sqrt{g} g^{\mu \nu} \psi w^\alpha_{\beta \mu}) \) may be rewritten as

\[
\sqrt{g} \left\{ \psi g^{\mu \nu} \bar{R}_{\mu \nu} - g^{\mu \nu} \nabla_\alpha \psi w^\alpha_{\mu \nu} + g^{\mu \nu} \psi (Q^\alpha_{\sigma \nu} Q^\sigma_{\mu \alpha} - Q^\alpha_{\alpha \sigma} Q^\sigma_{\mu \nu}) \right\}
\]

Being first–order, the action functional \( \int_M \mathcal{L}_1 d^2x \) is extremized by the dynamical fields \( g \) and \( \psi \) which satisfy the equations of motion (5) provided that \( \delta g|_{\partial M} = 0 \) and \( \delta \psi|_{\partial M} = 0 \). Notice that, until now, the background has been introduced only to provide covariance for the Lagrangian \( \mathcal{L}_1 \). It also seems reasonable to require that the background \( \bar{g} \) is a solution of field equations without particles, i.e. a vacuum solution. For this reason, in describing the motion of \( N \) particles in 2–dimensional gravity one could replace the action functional (11) with:

\[
A_{\text{Tot}}^M = \int_M (\mathcal{L}_0 + \mathcal{L}_{\text{Div}} + \mathcal{L}_P - \bar{\mathcal{L}}_0) d^2x \quad (23)
\]

where \( \mathcal{L}_0 \) is the Lagrangian obtained from (2) by substituting all the fields (the metric \( g \) and also the dilaton \( \psi \)) with the corresponding background fields \( (\bar{g}, \bar{\psi}) \). In this way, the field equations for \( (\bar{g}, \bar{\psi}) \) are the same as those in (4) and (7) with \( T_{\mu \nu} = 0 \).

We remark that a similar technique was employed in General Relativity in order to calculate the corrected relative conserved quantities for a large class of solutions; see e.g. [14, 13, 18, 15, 12].

### 4 Superpotential

In this Section we shall briefly review the theory of conserved quantities via Nöther theorem. The formulation we shall give here is based on the geometric formulation of a field theory and it basically relies on covariance requirements for the Lagrangian describing the physical model. The detailed and rigorous construction of Nöther currents, superpotentials and conserved quantities requires the geometric formulation of a relativistic field theory in terms of fiber bundles and their jet prolongations. We refer the interested reader to [8, 9, 10, 11] and references quoted therein for details. Here we

\footnote{Henceforth all quantities with bars refer to background objects so that, for example, \( \bar{R}_{\mu \nu} \) denotes the Ricci tensor of the background metric \( \bar{g} \).}
specialize the formalism developed in those papers in order to construct the superpotential associated with the action (23).

First of all let us consider the Lagrangian \( L_0 \) in (23). It is a covariant Lagrangian density, i.e. it is invariant under coordinate transformation. This means that, for any vector field \( \xi = \xi^\mu \partial/\partial x^\mu \) on the spacetime \( M \), the following identity holds (see [8]):

\[
d_\lambda (\xi^\lambda L_0) = G_{\mu\nu} L_\xi(g^{\mu\nu}) + R^\mu\nu L_\xi(R_{\mu\nu}) + H\xi L_\xi(\nabla_\mu \psi) + H^\mu L_\xi(\nabla_\mu \psi) \tag{24}
\]

where:

\[
G_{\mu\nu} = \frac{\partial L_0}{\partial g_{\mu\nu}} = \frac{\sqrt{g}}{2k} \psi (R_{\mu\nu} - 1/2 g_{\mu\nu} R) = 0
\]

\[
R^\mu\nu = \frac{\partial L_0}{\partial R_{\mu\nu}} = \frac{\sqrt{g}}{2k} \psi g^{\mu\nu}
\]

\[
H = \frac{\partial L_0}{\partial \psi} = \frac{\sqrt{g}}{2k} \left( R + \frac{1}{2} g^{\mu\nu} H'(\psi) \nabla_\mu \psi \nabla_\nu \psi + F'(\psi) \right)
\]

\[
H^\mu = \frac{\partial L_0}{\partial (\nabla_\mu \psi)} = \frac{\sqrt{g}}{2k} g^{\mu\nu} H(\psi) \nabla_\nu \psi
\]

and \( L_\xi \) denotes the Lie derivative with respect to the vector field \( \xi \).

Taking into account the relation \( L_\xi R_{\mu\nu} = \nabla_\alpha (L_\xi u^\alpha_{\mu\nu}) \) (where \( u^\alpha_{\mu\nu} \) has been defined in (20)), through a covariant integration by parts the identity (24) can be rewritten as (compare with (4))

\[
d_\sigma \{ F^\sigma - \xi^\sigma L_0 \} = -\frac{\sqrt{g}}{2k} \left[ G_{\mu\nu}(j^2 \psi, j^1 g) L_\xi g^{\mu\nu} + G(j^2 g, j^2 \psi) L_\xi \psi \right] \tag{25}
\]

where

\[
F^\sigma(j^1 g, j^1 \psi, L_\xi(j^1 g), L_\xi(\psi)) = \frac{\sqrt{g}}{2k} \left\{ g^{\mu\nu} \psi L_\xi u^\sigma_{\mu\nu} + \nabla_\alpha \psi (g^{\alpha\sigma} g_{\mu\nu} - \delta^\alpha_\sigma \delta_\nu^\mu) L_\xi g^{\mu\nu} + H(\psi) g^\sigma\nu \nabla_\nu \psi L_\xi \psi \right\} \tag{26}
\]

The Nöther current \( \mathcal{E}(L_0, \xi) \) associated with the generator \( \xi \) of Lagrangian symmetries is defined as

\[
\mathcal{E}^\sigma(L_0, \xi) = F^\sigma - \xi^\sigma L_0 \tag{27}
\]

It is a \((n-1)\)-differential form on the spacetime \( M \), i.e. a 1–form.
From (25) it is clear that, whenever $g$ and $\psi$ are solution of the field equations obtained from the Lagrangian $L_0$, namely: $G_{\mu \nu} = 0$ and $G = 0$, the Nöther current $E^\sigma(L_0, \xi)$ obeys the continuity equation:

$$d_\sigma E^\sigma(L_0, \xi) = 0$$

(28)

Moreover, since the mapping $\xi \mapsto \mathcal{L}_\xi(\cdot)$ is a linear partial differential operator, the Nöther current (27) can be expanded as a linear combination of the symmetrized covariant derivatives of $\xi$ up to second order:

$$E^\alpha(L_0, \xi) = T^\alpha_\mu \xi^\mu + T^\alpha_\beta \nabla_\beta \xi^\mu + T^\alpha_\beta \gamma \nabla_(\beta \nabla_\gamma) \xi^\mu$$

(29)

where the canonical tensors $T$ are:

$$T^\alpha_\mu = \frac{\sqrt{g}}{2k} \left\{ H g^{\alpha \nu} \nabla_\nu \psi \nabla_\mu \psi + \frac{3}{2} \psi R^\alpha_\mu - \delta^\alpha_\mu L_0 \right\}$$

$$T^\alpha_\beta = \frac{\sqrt{g}}{k} h^\lambda_\mu \beta \nabla_\lambda \psi$$

$$T^\alpha_\beta \gamma = - \frac{\sqrt{g}}{k} \psi h^\alpha_(\beta \gamma)$$

(30)

with

$$h^\sigma_\nu = g^\sigma_\rho \delta^\rho_\sigma - g^{\sigma (\sigma \delta^\rho_\nu)}$$

Whenever we have a linear combination of the form (29) we can perform a covariant integration by parts to obtain for the same quantity an equivalent linear expansion whose coefficients are all symmetric with respect to upper indices, while the integrated terms are all pushed into a formal divergence [9]. Doing so, we obtain

$$E^\alpha(L_0, \xi) = \tilde{E}^\alpha(L_0, \xi) + d_\beta U^\alpha_\beta(L_0, \xi)$$

(31)

with

$$\tilde{E}^\alpha(L_0, \xi) = \tilde{E}^\alpha_\mu \xi^\mu + \tilde{E}^\alpha_\beta \nabla_\beta \xi^\mu + \tilde{E}^\alpha_\beta \gamma \nabla_(\beta \nabla_\gamma) \xi^\mu$$

(32)

$$\tilde{E}_\rho^\alpha = T^\alpha_\rho + t^\alpha_\rho - \nabla_\beta \left( T^\alpha_\beta + t^\alpha_\beta \right)$$

$$\tilde{E}_\mu^\alpha \beta = T^\alpha_\mu (\alpha \beta) + t^\alpha_\beta$$

$$\tilde{E}_\mu^\alpha \beta \gamma = T^\alpha_\mu (\beta \gamma)$$

(33)

$$t^\alpha_\rho = 1/3 T^\alpha_\beta [\sigma \beta] R^\beta_\sigma \rho$$

$$t^\alpha_\beta = -4/3 \nabla_\sigma T^\alpha_\rho [\sigma \beta]$$

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\[ U^{\alpha\beta} = \left\{ T^{[\alpha\beta]}_{\rho} - \frac{2}{3} \nabla_\sigma T^{[\alpha\beta]}_{\rho} \right\} \xi^\sigma + \frac{4}{3} T^{[\alpha\beta]}_{\rho} \nabla_\sigma \xi^\rho \]  \tag{34}

The \(1\)-form \( \tilde{\mathcal{E}}(\mathcal{L}_0, \xi) \) is called the reduced current while the \(0\)-form \( U(\mathcal{L}_0, \xi) = U^{\alpha\beta} \epsilon_{\alpha\beta} \) is called the superpotential associated with the Lagrangian \( \mathcal{L}_0 \) and relative to the vector field \( \xi \). It is easy to demonstrate from (25), (27) and (29) (see [9, 11]) that the reduced current is always vanishing on–shell (i.e. when it is evaluated on the solutions of the field equations). In the present case, from (30) and (33) we obtain:

\[
\begin{align*}
\tilde{\varepsilon}_\rho^\alpha &= \frac{\sqrt{g}}{k} G_\rho^\alpha \\
\tilde{\varepsilon}_\mu^{\alpha\beta} &= 0 \\
\tilde{\varepsilon}_\mu^{\alpha\beta\gamma} &= 0
\end{align*}
\tag{35}
\]

and

\[
U(\mathcal{L}_0, \xi) = \frac{\sqrt{g}}{2k} \left\{ 2 \psi g^{\beta\alpha} \xi^\alpha - \psi g^{\mu\nu} \nabla_\mu \xi^\nu \right\} \epsilon_{\alpha\beta}
\tag{36}
\]

A similar expression can be found for the Lagrangian \( \bar{\mathcal{L}}_0 \) in (23) by replacing all the quantities involved in (35) and (36) with the corresponding barred ones.

The same algorithmic technique can be applied to the Lagrangian \( \mathcal{L}_{\text{Div}} \) in (23) in order to compute the relative Nöther current \( \mathcal{E}(\mathcal{L}_{\text{Div}}, \xi) \), the reduced current \( \tilde{\mathcal{E}}(\mathcal{L}_{\text{Div}}, \xi) \) and the superpotential \( U(\mathcal{L}_{\text{Div}}, \xi) \). Because \( \mathcal{L}_{\text{Div}} \) is a pure divergence, it gives no contribution to the reduced current and we have:

\[
\mathcal{E}(\mathcal{L}_{\text{Div}}, \xi) = dU(\mathcal{L}_{\text{Div}}, \xi)
\tag{37}
\]

where

\[
U(\mathcal{L}_{\text{Div}}, \xi) = \frac{\sqrt{g}}{2k} \psi g^{\mu\nu} w^\beta_{\mu\nu} \xi^\alpha \epsilon_{\alpha\beta}
\tag{38}
\]

Since our goal is to construct the Nöther current \( \mathcal{E}(\mathcal{L}, \xi) \) associated with the total Lagrangian (23)

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{Div}} + \mathcal{L}_P - \bar{\mathcal{L}}_0
\tag{39}
\]

we still have to consider the \(N\)–particle matter Lagrangian \( \mathcal{L}_P \). It gives a contribution \(-\sqrt{g} T^\alpha_\rho\) to the reduced current while it gives no contribution to the superpotential. Hence, for the total Lagrangian \( \mathcal{L} \), we finally have

\[
\mathcal{E}^\alpha(\mathcal{L}, \xi) = \tilde{\mathcal{E}}^\alpha(\mathcal{L}, \xi) + d_\beta U^{\alpha\beta}(\mathcal{L}, \xi)
\tag{40}
\]
where
\[ \tilde{\mathcal{E}}^\alpha(\mathcal{L}, \xi) = \frac{\sqrt{g}}{\kappa} \left\{ \varepsilon^\alpha_{\rho} T^\rho_{\alpha} \right\} \xi^\rho - \frac{\sqrt{\bar{g}}}{\kappa} \bar{\varepsilon}^\alpha_{\rho} \xi^\rho \] (41)

and
\[ U(\mathcal{L}, \xi) = U(\mathcal{L}_0, \xi) + U(\mathcal{L}_{\text{Div}}, \xi) - U(\bar{\mathcal{L}}, \xi) \] (42)
\[ = \frac{\sqrt{g}}{2\kappa} \left\{ 2 \nabla_\beta \psi g^{\beta\alpha} \xi^\sigma - \psi g^{\alpha\mu} \nabla_\mu \xi^\sigma \right\} \epsilon_{\alpha\sigma} \] (43)
\[ + \frac{\sqrt{g}}{2\kappa} \left\{ \psi g^{\mu\nu} w^{\rho}_{\mu\nu} \xi^\alpha \right\} \epsilon_{\alpha\rho} \] (44)
\[ - \frac{\sqrt{g}}{2\kappa} \left\{ 2 \nabla_\beta \bar{\psi} \bar{g}^{\beta\alpha} \xi^\sigma - \bar{\psi} \bar{g}^{\alpha\mu} \overline{\nabla}_\mu \xi^\sigma \right\} \epsilon_{\alpha\sigma} \] (45)

From (40) it is clear that the reduced current (41) vanishes on–shell, meaning that the Noether current (40) is not only a closed 1–form on spacetime but it is also exact on–shell and this is true independent of the topology of the spacetime. Moreover, while the current \( \mathcal{E}(\mathcal{L}, \xi) \) is conserved just along solutions the quantity \( \mathcal{E}(\mathcal{L}, \xi) - \tilde{\mathcal{E}}(\mathcal{L}, \xi) \) is conserved along every field configuration, including those which are not solutions of the field equations (see (40)). We express this fact by saying that \( \mathcal{E}(\mathcal{L}, \xi) \) is weakly conserved while the difference \( \mathcal{E}(\mathcal{L}, \xi) - \tilde{\mathcal{E}}(\mathcal{L}, \xi) \) is strongly conserved. Notice that these results hold true for every vector field \( \xi \) on the spacetime \( M \), not necessarily a Killing vector field.

Now let \( D \) be a 1–dimensional space–like region of the spacetime \( M \) and let the boundary \( \partial D \) of \( D \) be formed by two points \( P_1 \) and \( P_2 \). Let \( \xi \) be a time–like vector field and let us denote by the pair \( g(x), \psi(x) \) (and \( \bar{g}(x), \bar{\psi}(x) \)) a solution of the field equations (3) and (4). According to [8, 22] we define the energy \( E^\text{Tot}_D(\mathcal{L}, \xi) \) of the solution relative to the background, contained in the region \( D \) and relative to the vector field \( \xi \) as
\[ E^\text{Tot}_D(\mathcal{L}, \xi) = U(\mathcal{L}, \xi)|_{P_2} - U(\mathcal{L}, \xi)|_{P_1} \] (46)

In other words, the superpotential (46) corresponds to the density of energy. The definition (46) of energy relies entirely on the covariant formulation of the theory and can be considered as the covariant counterpart of the ADM formulation, see [22, 23]. In this context, rather then starting from a preferred ADM foliation of the spacetime into hypersurfaces, the starting point is an arbitrary surface \( D \) and a non–vanishing vector field \( \xi \), the flow of which defines the local time.
Note that the energy (46) is already normalized in such a way it is zero when computed on the background. We also stress that the region $D$ in (46) is not required to extend out to “spatial infinity”. As we shall see below, we can compute, via the Nöther theorem, also the “quasilocal” energy, i.e. the energy contained in a region of finite spatial extent.

We turn now to specific applications of this formalism to the $N$-body problem.

5 Applications: $R = T$ Theory

We shall here present the exact solution to the problem of the relativistic motion of two point masses found in [1] and we shall see that the definition of energy (46) coincides with the ADM Hamiltonian (19).

If $z_1$ and $z_2$ (with $z_2 < z_1$) denote the positions of the two particles, the 1-dimensional slices $\{t = \text{const}\}$ of the ADM foliation of spacetime can be divided in the three regions $x < z_2$, $z_2 < x < z_1$ and $z_1 < x$. As in [1] we call them the ($-$) region, the (0) region and the (+) region, respectively.

First of all it is important to stress that in the $R = T$ theory it is necessary to impose the boundary conditions
\[
\psi = \pm 2 \kappa \chi \quad \chi' = \pi
\] (47)
which must hold in the ($-$) region as well in the (+) region. That condition, together with the coordinate condition (17), allows one to pass from the expression (14) to the final expression (18) where all the boundary terms are discarded.

The solution in the (+) and ($-$) regions is
\[
\gamma = 1 \quad \Pi = 0
\] (48)
\[
N_0 = A\phi^2 = \begin{cases} 
A\phi_+^2 & \text{(+) region} \\
A\phi_-^2 & \text{(-) region}
\end{cases}
\] (49)
\[
N_{1(+)} = N_{0(+)} - 1 \quad N_{1(-)} = -N_{0(-)} + 1
\] (50)
\[
\psi = -4 \ln |\phi| = \begin{cases} 
-4 \ln |\phi_+| & \text{(+) region} \\
-4 \ln |\phi_-| & \text{(-) region}
\end{cases}
\] (51)

15
\[ \pi = -\frac{N'_1}{\kappa N_0} \]  

(52)

where \( \phi_+ \) and \( \phi_- \) are rather complicated functions of \( x, z_1(t), z_2(t), p_1(t), p_2(t) \) (see ref. [1] for details). Notice that the boundary condition (47) is satisfied if we choose the plus sign in the (+) region and the minus sign in the (−) region, respectively.

The reduced Hamiltonian (19) becomes:

\[ H = -\frac{1}{\kappa} \int dx \psi'' = -\frac{1}{\kappa} \left[ \psi'_+ - \psi'_- \right] \]  

(53)

A detailed analysis of the result (53) was carried out in ref. [1] where it was shown that the canonical equations of motion

\[ \dot{z}_a = \frac{\partial H}{\partial p_a} \]

\[ \dot{p}_a = -\frac{\partial H}{\partial z_a} \]

derived from (53) give rise to the original geodesic equations (8). It was also shown that for sufficiently small values of the Hamiltonian the trajectories of the particles can be considered as a relativistic perturbation of the Newtonian case.

Let us now compute the energy of the two–particle system through the definitions (42) and (46). First of all we have to select a suitable background adapted to the theory under examination, i.e., a zero level for the energy. We select as a background the vacuum solution obtained from the equation of motion (11) by setting \( T_{\mu\nu} = 0 \) (no–particles). Hence, a natural choice is

\[ \bar{g} = \eta \rightarrow ds^2 = -dt^2 + dx^2 \quad \bar{\psi} = 0 \]  

(54)

Taking into account the ADM splitting of the metric (13), the background (54) and setting \( \xi = \partial / \partial t \), the superpotential (42) becomes:

\[ \mathcal{U} = -\frac{1}{\kappa \sqrt{\gamma}} N_0 \psi' + 2 N_1 \pi \]  

(55)

Employing the boundary condition (17) and the equations (48) and (50) it can be rewritten as:

\[ \mathcal{U}_+(+) = -\frac{1}{\kappa} \psi'_+ \quad \mathcal{U}_(-) = -\frac{1}{\kappa} \psi'_- \]  

(56)
in the (+) and in the (−) regions, respectively. Hence the expression (46) for the energy

\[ E^{\text{Tot}} = U(+) - U(−) = -\frac{1}{\kappa} \left[ \psi'(+)-\psi'(-) \right] \]  

perfectly agrees with the Hamiltonian (53).

### 6 Application: GT Theory

We shall here consider two classes of solutions for the GT theory (12), a single particle solution and a two–particle solution. When computing the energy via the Nöther theorem a careful analysis of the background has to be done in both cases. We stress that in GT theories the vacuum solution (i.e. the solution in absence of particles which is a good representative for the background) can not be identified with the flat metric \( \bar{g} = \eta \) together with \( \bar{\psi} = 0 \), as in the previous Section. Indeed, in the GT theory the field equations (6), (7), in absence of particles, reduce to

\[ R = \Lambda \]  
\[ g^{\mu\nu} \nabla_\mu \nabla_\nu \psi = \Lambda \]

Hence we have to choose as a background metric a constant curvature solution satisfying (58). Moreover, in order to avoid divergence problems which are commonly encountered in the computation of conserved quantities in non–Ricci flat spacetime (see, e.g. [16, 17, 18, 13, 12]), the dynamical fields and the backgrounds have to be matched on the boundary of the region \( I \) whose energy content we want to calculate. If the region is described by the expression \( I = \{-x_0 \leq x \leq x_0, t = \text{constant}\} \) we then require \( g(t, \pm x_0) = \bar{g}(t, \pm x_0) \) and \( \psi(t, \pm x_0) = \bar{\psi}(t, \pm x_0) \). The ADM reduction of the superpotential (42) then simplifies as follows:

\[ U = -\frac{N_0}{k\sqrt{\gamma}} (\psi' - \bar{\psi}') + 2 N_1 (\pi - \bar{\pi}) \]  

### 6.1 One–Particle Solution

We shall now describe a solution of the GT theory with a point particle of mass \( M \) at the origin. In this case the equations of motion (4) and (7) reduce
to

\[ R - g^{\mu\nu} \nabla_\mu \nabla_\nu \psi = 0 \]  \hspace{1cm} (61)
\[ R = \Lambda - 4\pi M\delta(x) \]  \hspace{1cm} (62)

If we look for a solution of the form:

\[ g = -\alpha(x) dt^2 + \frac{1}{\alpha(x)} dx^2 \]  \hspace{1cm} (63)

equations (62) becomes:

\[ -\frac{d^2 \alpha}{dx^2} = \Lambda - 4\pi M\delta(x) \]  \hspace{1cm} (64)

It is satisfied by

\[ \alpha(x) = 1 + 4\pi M|x| - \frac{1}{2}\Lambda x^2 \]  \hspace{1cm} (65)

Accordingly, from (61) we obtain that

\[ \psi(x) = -\ln(\alpha(x)) + Ct \]  \hspace{1cm} (66)

is a solution for every choice of the constant \( C \). We now want to compute the energy of the solution relative to a background. We can choose as a background the vacuum solution \( M = 0 \) or a solution with a different mass \( m \) as well:

\[ \bar{g} = -\bar{\alpha}(x) dt^2 + \frac{1}{\bar{\alpha}(x)} dx^2 \]  \hspace{1cm} (67)
\[ \bar{\psi} = -\ln(\bar{\alpha}(x)) + Ct \]  \hspace{1cm} (68)

where

\[ \bar{\alpha}(x) = 1 + 4\pi (M - m)x_0 + 4\pi m|x| - \frac{1}{2}\Lambda x^2 \quad x_0 = \text{const} > 0 \]  \hspace{1cm} (69)

Notice that the background solution \( \{\bar{g}, \bar{\psi}\} \) satisfies the same equations (61) and (62) (with mass \( m \)) and it is matched with the solution \( \{g, \psi\} \) on the boundary of the real line interval \( I = \{ -x_0 \leq x \leq x_0, t = \text{constant} \} \). Hence, from (60) we easily obtain

\[ U(\pm x_0) = \pm \frac{1}{2} (M - m) \]  \hspace{1cm} (70)
and from (46) we deduce that the relative energy $E_{I}^{\text{Tot}}$ contained in the region $I$, as expected, is equal to

$$E_{I}^{\text{Tot}} = U(x_0) - U(-x_0) = M - m$$ (71)

We outline that this result holds for all finite real intervals (and, of course, also asymptotically).

We also stress that the definition (19) of the ADM Hamiltonian, instead, does not lead to the correct value if applied to this solution (when $x_0$ goes to infinity it gives $H = 0$). This is basically due to the fact that the solution under examination has been obtained without imposing a priori coordinate and boundary conditions. Nevertheless, with the coordinate transformation:

$$
\begin{align*}
\tau &= t + \frac{1}{AB} \ln(1 + A|x|) - \frac{1}{2AB} \ln(1 + 2A|x| - \frac{1}{2} \Lambda x^2) \\
|y| &= \frac{1}{A} \ln(1 + A|x|) \\
A &= \frac{kM}{4}, \quad B = \sqrt{1 + (8\Lambda/k^2M^2)}
\end{align*}
$$ (72)

the solution (63), (65), (66) transforms into the solution found in [3], Appendix B. In the new coordinate system the coordinate conditions (17) are satisfied together with the suitable boundary conditions (see (73) below). Hence the definition (19) may be applied.

### 6.2 Two–Particle Solution

The exact solution for the two–particle GT theory was found in [3] where the motion of the particles was deeply analysed starting from the definition (19) of the Hamiltonian. In this case the relevant formula (19) was obtained by imposing boundary conditions which are the generalization to the GT theory of the conditions (17). In the present case they read:

$$
\psi^2 - 4\kappa \chi^2 + 2\Lambda x^2 = C_{\pm} x \quad \chi' = \pi
$$ (73)

for the (+) and (−) regions. Here $C_{\pm}$ are constants. We refer the reader to [3, 4] for the details. The solution in the (+) and (−) regions is given by

$$
\gamma = 1 \quad \Pi = 0
$$ (74)

$$
N_0 = A\phi^2 = \begin{cases} 
A\phi_+^2 & \phi_+(x) = B \exp(K_+x/2) \\
A\phi_-^2 & \phi_-(x) = C \exp(-K_-x/2)
\end{cases}
$$ (75)
\[ N_{1(+)} = \frac{Y_+}{K_+} (N_{0(+)} - 1) \quad N_{1(-)} = -\frac{Y_-}{K_-} (N_{0(-)} - 1) \quad (76) \]

\[ \psi = -4 \ln |\phi| = \begin{cases} 
-4 \ln |\phi_+| \\
-4 \ln |\phi_-| 
\end{cases} \quad (77) \]

where

\[ Y_\pm = \kappa \left[ X \pm 1/4(p_1 + p_2) \right] \]

\[ K_\pm = \sqrt{Y_\pm^2 - \Lambda/2} \quad (78) \]

\[ \pi_\pm = -X \mp 1/4(p_1 + p_2) \]

and \( A, B, C \) and \( X \) do not depend on the \( x \) coordinate but implicitly depend on the time through the positions \( z_a \) and the momenta \( p_a \) of the particles. Hence the ADM Hamiltonian (19) becomes

\[ H(z_a, p_a) = -\frac{1}{\kappa} \int_{-x_0}^{x_0} dx \psi'' = \frac{2(K_+ + K_-)}{\kappa} \quad (79) \]

where \(-x_0 < z_2 < z_1 < x_0\).

We shall see that the same expression can be obtained as well as a Nöther charge starting from the definition (12). As in the \( R = T \) theory of Section 5 the link between the ADM Hamiltonian and the Nöther energy is established via the boundary conditions (73). We stress again that only if these conditions are satisfied the expression (19) can be considered as the true Hamiltonian of the system (in fact, the one particle solution of section 5.1 is an example where (19) can not be applied). Instead, the formula based on the Nöther approach holds in any case, provided we choose the suitable background. When dealing with the explicit solution (74)–(77) it is very cumbersome to find a suitable background matching correctly the solution on the boundary. Hence we prefer to consider the infinitesimal version of equation (60). We consider a whole family of solutions (74)–(77) and we take the variation \( \delta U|_{x_0} \) of the density of energy \( U \) at the boundary \( \{-x_0, x_0\} \) along this family (see [22, 11]). Moreover, we demand that every solution of the
family has the same boundary value, i.e. \( \delta g|_{x_0} = 0 \) and \( \delta \psi|_{x_0} = 0 \). Hence, from (80) we obtain
\[
\delta U = -\frac{N_0}{\kappa \sqrt{\gamma}} \delta \psi' + 2 N_1 \delta \pi
\] (80)
and from the expressions (77) and (78) we have
\[
\begin{align*}
\delta \psi'_{\pm} &= \mp 2(Y_{\pm}/K_{\pm}) \delta Y_{\pm} \\
\delta \pi_{\pm} &= -(1/\kappa) \delta Y_{\pm}
\end{align*}
\] (81)
Then \( \delta U_{\pm} = \pm 2/\kappa \delta K_{\pm} \) and the variation of the energy becomes:
\[
\delta E^{\text{Tot}} = \delta U_{+} - \delta U_{-} = \frac{2\delta(K_{+} + K_{-})}{\kappa}
\] (82)
The latter expression obviously leads to the result
\[
E^{\text{Tot}} = \frac{2(K_{+} + K_{-})}{\kappa} + \text{const}
\] (83)
in agreement with (79). The constant of integration can be fixed arbitrarily inside the family of solutions depending on the choice for the zero of energy.

We remark that the same result (83) may be achieved if the GT theory is extended in order to include charged bodies, see [4]. The action integral for gravitational and electric fields coupled with \( N \) charged point masses leads to the same expression (42) for the superpotential. [This does not mean that the superpotential and consequently the energy are not affected by the electric field. On the contrary, the superpotential is evaluated on a solution and the latter depends on the electric field through the electric stress energy \( T_{\mu\nu} \) appearing in the equations of motion.] Moreover, since the solution [4] has the same structure as (74)–(78), the same calculation of this section may be repeated step by step yielding the same result (83).

7 Application: JT Theory
We shall here describe a two–particle solution for the Jackiw–Teitelboim theory. It is obtained, we remind, by setting \( H = 0 \) and \( F = -\psi \Lambda \) in (2) so
that the equations of motion (3)–(8) read as follows:

\[ R - \Lambda = 0 \]

\[ g_{\mu\nu} \nabla_{\sigma} \psi - \nabla_{\mu} \nabla_{\nu} \psi + \frac{1}{2} g_{\mu\nu} \psi \Lambda = \kappa T_{\mu\nu} \]  
(84)

\[ \frac{d}{d\tau_a} \left\{ g_{\mu\nu}(z_a) \frac{dz_{\nu}}{d\tau_a} \right\} - \frac{1}{2} g_{\nu\lambda,\mu}(z_a) \frac{dz_{\nu}^{\lambda}}{d\tau_a} \frac{dz_{\lambda}}{d\tau_a} = 0 \]

The solution we shall describe is obtained by considering \( \Lambda = -\frac{2}{l^2} \) (with \( l \) a real constant).

In order to find the solution, the coordinate conditions (17) were imposed together with the choice \( N_1 = 0 \). Hence (13) simplifies to \( g = -N_0^2 dt^2 + dx^2 \) and the field equations become

\[ \dot{\pi} + N_0 \left[ \frac{\psi}{2\kappa l^2} - \frac{p_1^2 \delta(x - z_1(t))}{2\sqrt{p_1^2 + m_1^2}} - \frac{p_2^2 \delta(x - z_2(t))}{2\sqrt{p_2^2 + m_2^2}} \right] \right] + \frac{N_0' \psi'}{2\kappa} = 0 \]  
(85)

\[ \psi'' - \frac{\psi}{l^2} + \kappa \left( \sqrt{p_1^2 + m_1^2 \delta(x - z_1)} + \sqrt{p_2^2 + m_2^2 \delta(x - z_2)} \right) = 0 \]  
(86)

\[ 2\pi' + p_1 \delta(x - z_1) + p_2 \delta(x - z_2) = 0 \]  
(87)

\[ N_0'' - \frac{1}{l^2} N_0 = 0 \]  
(88)

\[ \dot{\psi} + 2\kappa N_0 \pi = 0 \]  
(89)

\[ \dot{p}_1 + \frac{\partial N_0}{\partial z_1} \sqrt{p_1^2 + m_1^2} = 0 \]  
(90)

\[ \dot{p}_2 + \frac{\partial N_0}{\partial z_2} \sqrt{p_2^2 + m_2^2} = 0 \]  
(91)

\[ \dot{z}_1 - N_0 \frac{p_1}{\sqrt{p_1^2 + m_1^2}} = 0 \]  
(92)

\[ \dot{z}_2 - N_0 \frac{p_2}{\sqrt{p_2^2 + m_2^2}} = 0 \]  
(93)

with \( \pi = -\dot{\psi}/(2\kappa N_0) \) and \( \partial N_0/\partial z_a = \partial N_0(x)/\partial x|_{x=z_a} \). The solution \((g, \psi)\) is then given by:

\[ \gamma = 1 \quad N_1 = 0 \quad N_0 = \cosh \left( \frac{x}{l} \right) \]  
(94)

\[ \psi = A(t) \left[ \sqrt{p_1^2 + m_1^2 \cosh \left( \frac{x - z_1}{l} \right)} + \sqrt{p_2^2 + m_2^2 \cosh \left( \frac{x - z_2}{l} \right)} \right] \]
\[ B \left[ \sinh \left( \frac{x - z_1}{l} \right) + \sinh \left( \frac{x - z_2}{l} \right) \right] \]
\[ - \frac{\kappa l}{2} \left[ \sqrt{p_1^2 + m_1^2} \sinh \left( \frac{|x - z_1|}{l} \right) + \sqrt{p_2^2 + m_2^2} \sinh \left( \frac{|x - z_2|}{l} \right) \right] \]

(95)

with \( A(t) = a_1 \cos (t/l) + a_2 \sin (t/l) \) and \( B, a_1 \) and \( a_2 \) constants. The reduced Hamiltonian (19) now becomes

\[ H = -\frac{1}{\kappa} \int_{\infty}^{\infty} \psi'' = 0 \]

(96)

and it clearly gives rise to an unphysical result. Indeed from equations (90)–(93) the Hamiltonian is expected to be of the form:

\[ H(z_1, z_2, p_1, p_2) = N_0(z_1) \sqrt{p_1^2 + m_1^2} + N_0(z_2) \sqrt{p_2^2 + m_2^2} \]

(97)

Without entering into the details we only remark that the wrong result (96) is basically due to the “bad” asymptotic behaviour of the solution. Instead, as already observed, the definition of energy as a Noether charge is not constrained by asymptotic conditions and then it is a good candidate for defining the Hamiltonian of the system. In order to apply the definitions (42) and (46), let us then fix the background solution \((\bar{g}, \bar{\psi})\) by setting \(z_a = 0\) and \(p_a = 0\) into the solution (94), (95). We observe that the metric \(g\) does not depend on the particles coordinates so that \(\bar{g} = g\). Hence the second term in (14) is equal to zero (because \(w^\alpha_{\beta\mu} = 0\)) and the superpotential \(U\) reads as follows:

\[ U = \frac{1}{\kappa} [\psi'N_0' - \psi N_0] - \frac{1}{\kappa} [\bar{\psi}'N_0' - \bar{\psi} N_0] \]

(98)

Denoting by \(I\) the region \(I = \{-x_0 \leq x \leq x_0, t = \text{constant}\}\) the energy \(E_{I}^{\text{Tot}}\) contained in the region \(I\) becomes:

\[ E_{I}^{\text{Tot}} = \int_{-x_0}^{x_0} \frac{dU}{dx} dx \]
\[ = \frac{1}{\kappa} \int_{-x_0}^{x_0} (\psi N_0'' - \psi' N_0) dx - \frac{1}{\kappa} \int_{-x_0}^{x_0} (\bar{\psi} N_0'' - \bar{\psi}' N_0) dx \]
\[ = \int_{-x_0}^{x_0} N_0 \left( \sqrt{p_1^2 + m_1^2} \delta(x - z_1) + \sqrt{p_2^2 + m_2^2} \delta(x - z_2) \right) dx \]

(99)
where in the last equality we have taken into account the equation of motions (86) and (88). Then, if the region $I$ is “large” enough, i.e. $-x^0 < z_2 < z_1 < x^0$ the energy $E_{I}^{\text{Tot}}$ is given by

$$E_{I}^{\text{Tot}} = N_0(z_1) \sqrt{p_1^2 + m_1^2} + N_0(z_2) \sqrt{p_2^2 + m_2^2}$$

(100)

in agreement with the expected value (97).

More generally, note that the energy (99) is instead given by:

$$
\begin{align*}
E &= N_0(z_2) \sqrt{p_2^2 + m_2^2} & \text{if } & -x^0 < z_2 < x^0 < z_1 \\
E &= N_0(z_1) \sqrt{p_1^2 + m_1^2} & \text{if } & z_2 < -x^0 < z_1 < x^0 \\
E &= 0 & \text{if } & z_2 < -x^0 < x^0 < z_1
\end{align*}
$$

(101)

Finally, we point out that the above solution is easily extendable to $N$ particles, since the spacetime is of constant curvature, and the metric is given by (94). The solution for the $a$-th particle is

$$N_0(z_a) \sqrt{p_a^2 + m_a^2} = E_a$$

(102)

where $E_a$ is a constant and

$$z_a(\tau_a) = l \cosh^{-1} \left[ E_a \sqrt{\frac{2}{m_a^2 + E_a^2 + (m_a^2 - E_a^2) \sin(\tau_a/l)}} \right]$$

(103)

The total energy is

$$E = \sum_{a=1}^{N} E_a$$

(104)

provided the region of interest surrounds all of the particles.

8 Conclusions and Perspectives

We have analysed and compared two definitions of energy for the $N$–body problem in 2–dimensional gravity. They look very different from a theoretical as well from a practical point of view. Although the formula (19) is very simple and easy to deal with applications, we have shown in two examples (see
Section 6.1 and Section 7) that it is not generally valid. We have pointed out how this is related to coordinates choices and boundary conditions. Instead, the definition (19) of energy via Nöther theorem leads, in all the situations so far analysed, to the expected value for the Hamiltonian and gives rise to the same results of the ADM prescription (19) when the latter is applicable. The definition (16) seems to be more difficult in applications mainly because its (1 + 1) ADM splitting is not always the same, compare e.g. (55), (60) and (98). This is basically due to the fact that the background has to be chosen in a different way every time. In our opinion, it is exactly the presence of the background inside its expression which renders the definition (16) more general, at least at a theoretical level. It is suitable for dealing with theories admitting Minkowski–like backgrounds (such as \( R = T \) theories) as well as more general contexts (GT and JT theories).

Divergence problems in the energy are here cured not imposing a priori boundary conditions (as it is commonly done in the ADM canonical reduction) and hence fixing the asymptotic behaviour of the solution. Instead, they are avoided, a posteriori, through the choice of the background which suitably matches the solution under examination.

We expect that the definition of energy based on the superpotential could be applied also in extensions of the formalism with additional matter fields and to \( N \)–body problem in (2 + 1) dimensions, see e.g. [24, 25].

Finally, we stress that 2 dimensional gravity, far from being a pure mathematical toy model, has deep relationships with dimensionally reduced four–dimensional spherically symmetric gravity and 2D string–theoretical black hole gravity, see [26], [13] and references quoted therein. 2D gravity provides an important arena for examining the notion of gravitational energy, and we anticipate it has more to teach us about this and other interesting subjects.

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References

[1] T. Ohta, R.B. Mann, *Class. Quantum Grav.*, **13**, 2585, (1996); R.B. Mann, T. Ohta, *Class. Quantum Grav.*, **14**, 1259, (1997); R.B. Mann, T. Ohta, *Phys. Rev. D*, **57**, 4723, (1997);

[2] R.B. Mann, D. Robbins, T. Ohta, *Phys. Rev. Lett.*, **82**, 3738 (1999).

[3] R.B. Mann, D. Robbins, T. Ohta, *Phys. Rev. D* **60**, 104048, (1999)

[4] R.B. Mann, D. Robbins, T. Ohta, M. Trott, gr-qc/0005082, *Nucl. Phys. B* (to be published).

[5] R. Jackiw, *Nucl. Phys.*, **B252**, 343, (1985); C. Teitelboim, *Phys. Lett. B* **126**, 41, (1983);

[6] T. Kimura, *Prog. Theor. Phys.*, **26**, 157, (1961); T.Ohta, H. Okamura, T. Kimura, K. Hiida,*Prog. Theor. Phys.*, **51**, 1598, (1974).

[7] J.Gegenberg, G. Kunstatter, D. Louis–Martinez, *Phys. Rev. D*, **51**, 1781, (1995).

[8] M.Ferraris, in: *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, edited by M. Francaviglia (Elsevier Science Publishers B. V., 1991), pp. 451 – 488.

[9] M. Ferraris, M. Francaviglia and O. Robutti, in: *Géométrie et Physique*, Proceedings of the *Journées Relativistes 1985* (Marseille, 1985), 112 – 125; Y.Choquet-Bruhat, B. Coll, R. Kerner, A. Lichnerowicz eds. (Hermann, Paris, 1987).

[10] A. Trautman, in: *Gravitation, An Introduction to Current Research*, L. Witten ed., Wiley (New–York, 1962), pp. 169–198; A. Trautman, *Commun. Math. Phys.*, **6**, 248, (1967).

[11] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Annals of Phys.*, **275**, 27, (1999).

[12] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, gr-qc/0003019.
[13] M. Ferraris, M. Francaviglia, *8th Italian Conference on General Relativity and Gravitational Physics*, Cavalese (Trento), August 30 – September 3 (World Scientific, Singapore, 1988).

[14] J. Katz, J. Bicak, D. Lynden–Bell, *Phys. Rev. D* 55 (10), 5957, (1997); J. Katz, D. Lerer, gr-qc/9612025.

[15] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, gr-qc/9906114, to appear in Annals of Phys.

[16] S. W. Hawking, G.T. Horowitz, *Class. Quantum Grav.*, 13, 1487, (1996); S. W. Hawking, C. J. Hunter, *Class. Quantum Grav.*, 13, 2735, (1996).

[17] S. W. Hawking, C. J. Hunter, D. N. Page, hep-th/9809035; S. W. Hawking, C. J. Hunter, hep-th/9808085; C. J. Hunter, hep-th/9807010.

[18] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Phys. Rev. D* 60, 124012, (1999).

[19] R.B. Mann, *Phys. Rev. D* 47, 10, 4438, (1993); J.D.E. Creighton, R.B. Mann, *Phys. Rev. D* 54, 4569, (1996).

[20] J.W. York, *Found. of Phys.*, 16, (3), 249, (1986).

[21] J. D. Brown, J. W. York, *Phys. Rev. D*, 47, (4), 1407, (1993); J. D. Brown, J. W. York, *Phys. Rev. D*, 47, (4), 1420, (1993).

[22] M. Ferraris, M. Francaviglia, *Atti Sem. Mat. Fis. Univ. Modena*, 37, 61, (1989); M. Ferraris, M. Francaviglia, *7th Italian Conference on General Relativity and Gravitational Physics*, Rapallo (Genoa), September 3–6, 1986; M. Ferraris and M. Francaviglia, *Gen. Rel. Grav.*, 22, (9), 965 (1990);

[23] M. Ferraris, M. Francaviglia, I. Sinicco, *Il Nuovo Cimento*, 107B, N. 11, 1303, (1992).

[24] J. Louko and H.J. Matschull, *Class. Quant. Grav.*, 17, 1847 (2000); H.J. Matschull, *Class. Quant. Grav.*, 16, 1069 (1999).

[25] A. Bellini, M. Ciafaloni, P. Valtancoli, *Nuclear Phys. B* 462, 453, (1996).

[26] Kummer, Lau, *Ann. Phys.* 258 37 (1997).