Disordering effects of colour in a system of coupled Brownian motors: phase diagram and anomalous-to-normal hysteresis transition

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Abstract

A system of periodically coupled nonlinear phase oscillators—submitted to both additive and multiplicative white noises—has been recently shown to exhibit ratchetlike transport, negative zero-bias conductance, and anomalous hysteresis. These features stem from the asymmetry of the stationary probability distribution function, arising through a noise-induced nonequilibrium phase transition which is reentrant as a function of the multiplicative noise intensity. Using an explicit mean-field approximation we analyze the effect of the multiplicative noises being coloured, finding a contraction of the ordered phase (and a reentrance as a function of the coupling) on one hand, and a shift of the transition from anomalous to normal hysteresis inside this phase on the other.

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I. INTRODUCTION

One of the distinguishing features of the last decade has been a spectacular advancement of knowledge in the field of nanotechnology. Within a subfield that we might call “nanomechanics”, the topic of noise-induced transport—which concerns the mechanisms whereby one can extract useful work out of (nonequilibrium) fluctuations—has increasingly captivated researchers.

In the early works it seemed to be a requisite for the operation of microscopic rectifying devices (usually known as “molecular motors”, “Brownian motors”, or “ratchets”) that—besides a built-in ratchetlike bias—the fluctuations be correlated. That requirement was relaxed when “pulsating” ratchets, in which the rectifying effect arises from the random switching between uncorrelated noise sources, were discovered. A recent twist has been to relax also the requirement of a built-in bias: a system of periodically coupled nonlinear phase oscillators in a symmetric “pulsating” environment has been shown to undergo a noise-induced nonequilibrium phase transition, wherein the spontaneous symmetry breakdown of the stationary probability distribution function (SPDF) gives rise to an effective ratchetlike potential. Some of the striking consequences of this fact are the appearance of negative (absolute) zero-bias conductance in the disordered phase, but near the phase-transition line (for small values of the bias force \( F \), the particle current \( \langle \dot{X} \rangle \) opposes \( F \)), and anomalous hysteresis in the strong-coupling region of the ordered phase (the \( \langle \dot{X} \rangle \) vs \( F \) cycle runs clockwise, as opposed for instance to the \( B \) vs \( H \) cycle of a ferromagnet).

Exploiting our previous experience, in Ref. we addressed the model using an explicit mean-field approach (see e.g. Ref. ). We undertook a thorough exploration of the ordered phase—including the characterization of its subregions and the transition from anomalous to normal hysteresis in the behaviour of \( \langle \dot{X} \rangle \) as a function of \( F \)—and found a close relationship between the shape of the SPDF and the number of “homogeneous” mean-field solutions on one hand, and the character of the hysteresis loop on the other.

As discussed in Ref. , the multiplicative noises are expected to exhibit some degree of
time-correlation or “colour”. Hence in this work, and as a natural continuation to Ref. [6], we study (in mean-field approximation) the consequences of a finite correlation time $\tau$ of the multiplicative noises in the model of Refs. [4,6]. In the following sections we briefly describe the model and its mean-field treatment, and discuss our numerical results regarding the influence of colour on the phase diagram and on the transition from anomalous to normal hysteresis inside the ordered phase.

II. THE MODEL AND ITS MEAN-FIELD ANALYSIS

The model: Let us consider the following set of $N$ globally coupled stochastic equations of motion, in the overdamped regime:

$$\dot{X}_i = -\frac{\partial U_i}{\partial X_i} + \sqrt{2T} \xi_i(t) - \frac{1}{N} \sum_{j=1}^{N} K(X_i - X_j).$$  

The stochastic variables $X_i(t)$ are phaselike $[-L/2 \leq X_i(t) \leq L/2$, where $L$ is the period of the $U_i$] and the equation is meant to be interpreted in the sense of Stratonovich. The second term in Eq. (1) models, as usual, the effect of thermal fluctuations [the $\xi_i(t)$ are additive Gaussian white noises with zero mean and variance one: $\langle \xi_i(t) \rangle = 0$, $\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t - t')$, and $T$ represents the temperature of the environment].

The “pulsating” potentials $U_i(x, t)$ are one of the key ingredients in the model [6]. Including a “load force” $F$ as a tool for the analysis of the noise-induced ratchet effect, their form is

$$U_i(x, t) = V(x) + W(x)\sqrt{2Q} \eta_i(t) - Fx,$$

namely they consist of a static part $V(x)$ and a fluctuating one: Gaussian noises $\eta_i(t)$ with zero mean couple multiplicatively (with intensity $Q$) through a function $W(X_i)$. Whereas in Refs. [4,6] the $\eta_i(t)$ were taken as white with variance one, we now regard them instead as Ornstein-Uhlenbeck, with self-correlation time $\tau$: $\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} (\sigma^2/2\tau) \exp(-|t - t'|/\tau)$.

Besides being periodic, $V(x)$ and $W(x)$ are assumed to be symmetric $V(-x) = V(x)$ and $W(-x) = W(x)$ (there is no built-in ratchet effect). In Refs. [4,6] it is $V(x) = W(x) =$
−cos x − A cos 2x, hence L = 2π. With the choice A > 0, the direction of the particle current \( \langle \dot{X} \rangle \) turns out to be opposite to that of symmetry breaking in the SPDF \( P^{st}(x) \): it is this effect which leads in turn to such oddities as negative zero-bias conductance and anomalous hysteresis \(^4\). The interaction force \( K(x − y) = −K(y − x) \) between oscillators is a periodic function of \( x − y \) (also with period \( L \)) and in Refs. \(^{4,6} \) is chosen as \( K(x − y) = K_0 \sin(x − y) \), with \( K_0 > 0 \). We shall fix \( T = 2.0 \) and \( A = 0.15 \) as in Refs. \(^{4,7} \), so the important parameters in the model are \( K_0 \) and \( Q \). The model just set up can be visualized (at least for \( A \to 0 \)) as a set of overdamped pendula (only their phases matter, not their locations) interacting with one another through a force proportional to the sin of their phase difference (this force is always attractive in the reduced interval \( −\pi \leq x − y \leq \pi \)).

**Mean-field approximation (MFA)** With the above choice, the interparticle interaction term in Eq. \(^1\) can be cast in the form

\[
\frac{1}{N} \sum_{j=1}^{N} K(X_i − X_j) = K_0 [C_i(t) \sin X_i − S_i(t) \cos X_i].
\]

(3)

For \( N \to \infty \) we may approximate Eq. \(^3\) à la Curie-Weiss, replacing \( C_i(t) = N^{-1} \sum_j \cos x_j(t) \) by \( C_m \equiv \langle \cos x_j \rangle \) and \( S_i(t) = N^{-1} \sum_j \sin x_j(t) \) by \( S_m \equiv \langle \sin x_j \rangle \), to be determined as usual by self-consistency. This decouples the system of stochastic differential equations (SDE) in Eq. \(^3\), which reduces to essentially one SDE for the single stochastic process \( X(t) \). If \( X(t) \) happens to be Markovian, then it is a straightforward matter to write up an associated Fokker-Planck equation (FPE) whose stationary solution is the SPDF, from which all the transport properties can be readily obtained. As discussed in Ref. \(^3\), the Gaussian character of the \( \eta_i(t) \) allows in the \( \tau = 0 \) case to consider them as being coupled through an effective function \( S(X_i) \equiv \sqrt{2}[T + Q(W')^2]^{1/2} \). Hence in the white-noise case, the SDE obtained after performing the MFA is

\[
\dot{X} = R(X) + S(X) \eta(t),
\]

(4)

with \( R(x) = −V'(x) + F − K_0 [C_m \sin x − S_m \cos x] \). The associated FPE is

\[
\partial_t P(x, t) = \partial_x \{ −[R(x) + \frac{1}{2} S(x) S'(x)] P(x, t) \} + \frac{1}{2} \partial_{xx} [S^2(x) P(x, t)]
\]

(5)
and its normalised stationary solution with periodic boundary conditions and current density $J \neq 0$ is

$$P^{st}(x) = \frac{e^{-\phi(x)} H(x)}{\mathcal{N} S(x)},$$  \hspace{1cm} (6)$$

where $\phi(x) = -2 \int_0^x dy [R(y)/S^2(y)]$, $H(x) = \int_x^{x+L} dy S(y) \exp[\phi(y)]$, and $\mathcal{N} = \int_{-L/2}^{L/2} dx P^{st}(x)$. The positivity of $S(x)$ and the exponentials implies that of $H(x)$ and hence that of $P^{st}(x)$ and $\mathcal{N}$, as it should be.

**The particle current:** The appearance of a ratchet effect amounts to the existence of a nonvanishing drift term $\langle \dot{X} \rangle$ in the stationary state, in the absence of any forcing ($F = 0$); in other words, the pendula become rotators in an average sense. As it was shown in Ref. [4], the cause of this spontaneous particle current is the noise-induced asymmetry in $P^{st}(x)$.

Being the current density

$$J = [1 - e^{\phi(L)}]/2\mathcal{N},$$  \hspace{1cm} (7)$$

the sign of $J$ is that of $1 - e^{\phi(L)}$. The “holonomy” condition $e^{\phi(L)} = 1$ implies $J = 0$ and $H(x) = \text{const}$. As shown in Ref. [3] it is

$$\langle \dot{X} \rangle = \int_{-L/2}^{L/2} dx \left[ R(x) + \frac{1}{2} S(x)S'(x) \right] P^{st}(x, C_m, S_m),$$  \hspace{1cm} (8)$$

with the result

$$\langle \dot{X} \rangle = J L = \left\{ \frac{1 - e^{\phi(L)}}{2\mathcal{N}} \right\} L,$$  \hspace{1cm} (9)$$

hence $\langle \dot{X} \rangle$ has the sign of $J$ and can be also regarded as an order parameter.

Equation (8) is a self-consistency relation since both $\mathcal{N}$ and $\phi(L)$ carry information on the shape of $P^{st}(x)$ (in the latter case through $C_m$ and $S_m$). A nonzero $J$ is always associated with a symmetry breakdown in $P^{st}(x)$ (namely, $P^{st}(-x) \neq P^{st}(x)$). This may be either spontaneous (our main concern here) or induced by a nonzero $F$.

**Effective Markovian approximation for coloured noise:** The results in Eqs. (6) and (9) rely on the fact that we have been able to write the FPE Eq. (5). When the
η_i(t) in Eq. (2) are coloured, the process X(t) in Eq. (3) is in principle not Markovian and for non-Markovian processes, a FPE can at most result from some (non-systematic) approximation, like the truncation of some short correlation-time expansion. Fortunately, a consistent Markovian approximation (called “unified coloured-noise approximation” or UCNA) can be performed under certain conditions. By resorting to it one can obtain expressions for R(x) and S(x) in Eq. (5) which account for the effect of τ. Their functional forms will be published elsewhere [8].

**The self-consistency equations:** The stationary probability distribution P^{st}(x) also depends on S_m and C_m, since R(x) contains these parameters. Their values arise from requiring self-consistency, which amounts to solving the following system of nonlinear integral equations:

\[ F_{cm} = C_m, \quad \text{with} \quad F_{cm} \equiv \langle \cos x \rangle = \int_{-L/2}^{L/2} dx \cos x P^{st}(x, C_m, S_m), \quad (10) \]

\[ F_{sm} = S_m, \quad \text{with} \quad F_{sm} \equiv \langle \sin x \rangle = \int_{-L/2}^{L/2} dx \sin x P^{st}(x, C_m, S_m). \quad (11) \]

These equations give C_m and S_m for each set of the parameters (Q, K_0) that define the state of the system, assumed T, A and F fixed.

For F = 0, the choice S_m = 0 makes R(x) an odd function of x; this in turn makes φ(x) even, and then the periodicity of P^{st}(x) in Eq. (1) [in the form P^{st}(−x) = P^{st}(−x − L)] implies that the stationary probability distribution is also an even function of x. So the problem of self-consistency reduces to the numerical search of solutions to Eq. (10), with S_m = 0. Although plausibility arguments, detailed in Ref. [6], allow to have an intuition on the existence of some solutions to this integral equation (and their stability) in this symmetric case, the stability of the true solutions must be explicitly checked. Since cos x in Eq. (10) is an even function of x, it suffices to use the Curie-Weiss one-parameter criterion, namely to check whether the slope at S_m of the integral in Eq. (11) is less or greater than one. As a complementary check, a small-x expansion of φ(x) [6] confirms that P^{st}(x) is indeed Gaussian at x = 0. For small F ≠ 0, P^{st}(x) gets multiplied (in this approximation) by exp[Fx/T](≈ 1 + Fx/T) which leads to a nonzero value of S_m = kF, with k > 0. By
the mechanism discussed in Ref. \[6\], for large enough $Q$ it is $\phi(L) > 0$ and by Eq. \(7\) it is $J < 0$. This effect manifests itself in a negative zero-bias conductance since according to Eq. \(9\), $\langle \dot{X} \rangle = LJ$.

As a consequence, for $F = 0$ there are always one or more solutions to Eqs. \(10\) and \(11\) with $S_m = 0$ and one of these is the stable one in the “disordered” phase. As argued in Ref. \[4\], for $N \to \infty$ a noise-induced nonequilibrium phase transition takes place generically towards an “ordered” phase where $P^{st}(-x) \neq P^{st}(x)$. In the present scheme this asymmetry should be evidenced by the fact that the solution with $S_m = 0$ becomes unstable in favor of two other solutions such that $P^{st}_2(x) = P^{st}_1(-x)$, characterised by nonzero values $\pm |S_m|$. This fact confers on $S_m$ the rank of an order parameter.

**The phase boundary:** Since $\sin x$ is an antisymmetric function, Eq. \(11\) results impractical for the task of finding the curve that separates the ordered phase from the disordered one, given that on that curve $S_m$ is still zero. For that goal (exclusively) we solve, instead of Eqs. \(10\) and \(11\), the following system:

$$
\int_{-L/2}^{L/2} dx \cos x P^{st}(x, C_m, 0) = C_m, \quad \int_{-L/2}^{L/2} dx \sin x \left. \frac{\partial P^{st}}{\partial S_m} \right|_{S_m=0} = 1. \tag{12}
$$

**III. NUMERICAL RESULTS**

Figure \[1\] displays (in the same scale as in Ref. \[6\]) the phase diagram obtained by solving Eqs. \(12\) by the Newton-Raphson method. In the region enclosed by the thick lines (“ordered region”) the stable solution to Eqs. \(10\) and \(11\) has $S_m \neq 0$. For $\tau$ not too large this noise-induced phase transition is reentrant as a function of $Q$, for $K_0 = \text{const}$ (a fact already known for $\tau = 0$ \[4,6\]). The novelty is that for any $\tau \neq 0$, the phase transition is also reentrant as a function of $K_0$ for $Q = \text{const}$ (a feature found in Ref. \[5\] for a different system).

The multiplicity of mean-field solutions in the ordered region, together with the fact that some of them may suddenly disappear as either $K_0$ or $Q$ are varied (a fact that is closely related to the occurrence of anomalous hysteresis) hinder picking out the right solution in
this region. A more systematic characterization of the aforementioned multiple solutions is achieved when the branch to which they belong is traced from its corresponding “homogeneous” \((S_m = 0)\) solution. Accordingly, the thin lines in Fig. 1 separate two sectors within the ordered region with regard to the homogeneous solutions. Below them (“noise-driven regime” or NDR) there is a single solution with \(S_m = 0\) and \(C_m < 0\) (as already suggested, in this regime a solution with \(C_m < 0\) can be stable since it corresponds to shaking violently the pendula). Above them (“interaction-driven regime” or IDR) there are three solutions: two of them have opposite signs and (for \(K_0/Q\) large enough) \(|C_m| \simeq 0.9\); the remaining one has \(C_m \approx 0\). Note that this line presents a cusp whose meaning was discussed in Ref. [6], in relation with the character of the hysteresis loop. We have studied the shape of \(P^{st}(x)\) and the behaviour of \(\langle \dot{X} \rangle\) as a function of \(F\) for different locations in this \((Q, K_0)\) diagram. The square in Fig. 1 indicates a position inside the ordered zone for which the shape of the SPDF and the hysteresis cycle are followed as functions of \(\tau\). This point lies in the NDR for \(\tau = 0\) and in the IDR for \(\tau \neq 0\) (marginally so for \(\tau = .1\)).

Figure 2 shows (for the true solution, namely the stable \(S_m \neq 0\) one) the evolution of \(P^{st}(x)\) as a function of \(\tau\), for the state indicated by the square in Fig. 1 \((Q = 10, K_0 = 10.2)\). The SPDF is always an asymmetric function of \(x\), indicating a spontaneous breakdown of parity (since \(V\) and \(W\) remain symmetric): the system has to choose between two possible asymmetric solutions, of which just one is shown. As discussed in Ref. [6], \(P^{st}(x)\) is bimodal in the NDR. As \(\tau\) increases, it becomes unimodal (this is similar to what one achieves by decreasing \(Q\) at \(\tau = 0\)).

Figures 3(a) to 3(c) present, for the state indicated with a square in Fig. 1, a sequence of \(\langle \dot{X} \rangle\) vs \(F\) plots obtained as \(\tau\) increases. All the solutions to Eqs. (10) and (11), except the one belonging to the branch starting at \(C_m \approx 0\) for \(S_m = 0\), have been included in these figures. The sequence is analogous to the one depicted in Fig. 9 of Ref. [6]. In both cases we see a crossing from the NDR to the IDR (increasing \(\tau\) at fixed \(K_0\) has similar effects as increasing \(K_0\) at \(\tau = 0\), for fixed \(Q\)).
IV. CONCLUSIONS

From the analysis of the foregoing results, we draw the following conclusions:

1. The increase of $\tau$ tends to destroy order, and between $\tau = 0.1$ and $\tau = 0.5$ there exists a new reentrance with respect to $K_0$. In this range of $\tau$, the existence region of the ordered state is strongly shrunk. The qualitative similarity with the result arrived at in Ref. [5]—in spite of the fact that both systems are different—shows the robustness of this result.

2. Regarding the $\langle \dot{X} \rangle$ vs $F$ plots one sees that as $\tau$ increases the hysteresis cycle becomes more complex, and the range of values of $\langle \dot{X} \rangle$ corresponding to $F \neq 0$ is severely limited.

Although all of our results stem from a MFA, we see that this approximation is able to reveal the richness of the phase diagram of this model. Moreover, the mean-field results coincide with the numerical simulations we are undertaking and that will be published elsewhere [8].

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FIGURES

FIG. 1. Phase diagram of the model for $T = 2.0$, $A = 0.15$, and $F = 0.0$. Full lines: $\tau = 0.0$; dashed lines: $\tau = 0.1$; dotted lines: $\tau = 0.3$. For each value of $\tau$, the ordered region lies above and to the right of the corresponding thick line. Above the thin lines there may exist several solutions when $S_m \neq 0$, whereas below them there may exist at most one. The square corresponds to $Q = 10.0$, $K_0 = 10.2$.

FIG. 2. Shape of $P^*(x)$ at the point marked with a square in Fig. 1 ($Q = 10.0$, $K_0 = 10.2$), for $\tau = 0.0$ (full line), 0.1 (dashed line), and 0.3 (dotted line). The other parameters as in Fig. 1.

FIG. 3. The order parameter $V_m = \langle \dot{X} \rangle$ (particle current) as a function of $F$ for $Q = 10.0$ and $K_0 = 10.2$ (the square in Fig. 1). The sequence illustrates the change in the character of the hysteresis cycle as $\tau$ varies: (a) $\tau = 0.0$, (b) $\tau = 0.1$, (c) $\tau = 0.3$. Remaining parameters as in Fig. 1.
