Trees in the Real Field

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Abstract. This paper proposes an algebraic view of trees which opens the doors to an alternative computational scheme with respect to classic algorithms. In particular, it is shown that this view is very well-suited for machine learning and computational linguistics.

1 Introduction

In the last few years models of deep learning have been successfully applied to computational linguistics. Amongst others, the translation problem has benefited significantly from simple approaches based on recurrent neural networks. In particular, because of the classic problem of capturing long-term dependencies [1], LSTM [3] architectures have been mostly used which can better deal with this classic problem.

In this paper we go beyond this approach and assume to characterize linguistic production by means of generative trees by relying on the principle that the complexity of the problem of long-term dependencies is dramatically reduced because of the exponential growth of nodes of the trees with respect to their height. In general the relations between trees and their corresponding linear encoding is not easy to grasp. For example, when restricting to binary trees, it can be proven that we need a pair of traversals to fully characterize a given tree, one of which may be the symmetric one [4]. However, whenever a sequence presents a certain degree of regularity, the ambition arises to establish a bijection with a corresponding tree (e.g. the parsing tree).

While encoding mechanisms are quite straightforward to design every time that it is possible to assign to each sequence a tree-like structure; it is sufficient to propagate the information (for example with a linear scheme) through the nodes up to the root of the tree ([2]), it is much harder to came up with a decoding scheme that generates the translated sequence. Here we prove that we can construct a decoding scheme that naturally extend those used nowadays in recurrent neural nets that can be potentially very interesting in computational linguistics.
2 Uniform real-valued tree representations

A binary tree is recursively defined as

\[ T = \begin{cases} \emptyset \quad \text{basis} \\ (L, y, R) \quad \text{induction} \end{cases} \tag{1} \]

where \( \emptyset \) is the empty tree, \( y \in \Sigma \) is the labeled root, which takes on values from the alphabet \( \Sigma \), \( L \) (Left), and \( R \) (Right) are trees. We assume that we are given a coding function \( \ell : \Sigma \to \mathcal{Y} \subset \mathbb{R}^p \), so as the nodes of the tree are related to an associated point \( \gamma \) of \( \mathcal{Y} \). Now, let us consider the the pair

\[ i. \quad T := (L, x, R) \]
\[ ii. \quad \gamma : \mathcal{X} \subset \mathbb{R}^n \to \mathcal{Y}, \quad \gamma(x) := Cx \tag{2} \]

which consists of the triple \((L, x, R)\) and of the linear labeling function \( \gamma \), which returns points, that will be related to the labels of \( T \). In the triple, we have \( x \in \mathbb{R}^n \), \( L, R \in \mathbb{R}^{n \times n} \). Basically, we introduce a computational scheme on the embedding space \( \mathcal{X} \). If \( x = 0 \) then we assume that \((L, 0, R) \sim (0, 0, 0) := T_\emptyset \).

We want to explore the relations between the tree definition \((1)\) and the related real-valued representation given by equations \((2)\). To this end, we start noticing that the void tree \( T_\emptyset \) can be associated with \( T_\emptyset \). The idea is that we can specify a tree \( T \) once the triple \((L, x, R)\) and \( C \) are given. Beginning from \( Cx = \text{root}(T) \), we process the children of the root by applying \( L \) and \( R \) to \( x \), so that \( CRx \) is the right child and \( CLx \) is the left child. Then the left child of the left child of the root is obtained as \( CLLx \), and the right child of the left child of the root as \( CRLx \), and so on and so forth, until we find, for each branch of the tree, a node \( l \in \mathbb{R}^n \) for which \( Ll = Rl = 0 \). This will be the leaf of that particular path, and we will say that the children of the leaves are buds; more generally every null node will be denoted as a bud.

We say that \((L, x, R)\) is an \( n \)-dimensional real representation of the obtained tree \( T \).

In order to get an insight on this construction let us consider the following examples.

**Example 1.** The first non-trivial example is the tree that consists of the root only. In our representation this tree is obtained by picking up any two matrices \( L \) and \( R \), such that \( x \) in their kernel, that is \( Lx = Rx = 0 \). The simplest next example is given by

\[
(L, x, R) = \begin{cases} \mathcal{X} \\
1 \quad y(R) \\
2 \\
3 \\
\end{cases}
\]

\[^3\text{In the following we will often regard the elements of } T \text{ as elements of } \mathbb{R}^p, \text{ without mentioning function } \ell \text{ explicitly.}\]
The decoding equations that defines this tree are

\[
\begin{cases}
Cx = \text{root}(T); & CLx = 0, \quad \text{bud 1}; \\
CRx = y(R), & CLRx = 0, \quad \text{bud 2}; \\
& CR^2x = 0, \quad \text{bud 3}.
\end{cases}
\]

They are conveniently separated into the “node conditions” and “bud conditions”. In order to be even more explicit consider the case \( C = I, x = (1, 0)' \) and \( y(R) = (0, 1)' \), then it is easy to check that

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

We can easily see that in this special case, this representation is unique in \( \mathbb{R}^2 \).

As soon as we think about the next example with two nodes \( \bullet \), a symmetry property of the decoding scheme becomes evident. Given \( T \) let us define the symmetric left-right \( T' \) as the tree that one obtains from \( T \) by recursively exchanging the left with the right subtrees. For example

\[
T = \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\text{D}
\end{array}, \quad T' = \begin{array}{c}
\text{C} \\
\text{D} \\
\text{A} \\
\text{B}
\end{array}.
\]

are related by the defined symmetry operation. Clearly, for those trees we can state an immediate property on their representation.

**Proposition 1.** Let \( T \) and \( T' \) be related by left-right symmetry and let \( (L, x, R) \) be a real representation of \( T \). Then \( (R, x, L) \) is the representation of \( T' \).

**Proof.** Straightforward. \( \square \)

This result immediately shows us when looking at the tree given by (3), that we have

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

**Example 2.** In this case we show the role of the embedding space \( \mathcal{X} \subset \mathbb{R}^n \). In particular, we will see that the decoding might not be solvable at certain
dimensions and that there could be also infinite solutions. Let us consider the following tree with the associated decoding equations

\[
(L, x, R) = \begin{pmatrix} y(L) \\ y(RL) \end{pmatrix}, \quad \begin{cases} Cx = \text{root}(T); \\ CLx = y(L); \\ CRLx = y(RL), \end{cases} \begin{cases} CL^2x = 0, & \text{bud 1;} \\ CLRLx = 0, & \text{bud 2;} \\ CR^2Lx = 0, & \text{bud 3;} \\ CRx = 0, & \text{bud 4.} \end{cases}
\]

We consider two different cases \( n = 2 \) and \( n = 3 \).

- **Case** \( n = 2 \). Let us consider \( n = 2 \) and assume \( C = I \). In addition, let us assume that the nodes of \( T \) are coded by \( \text{root}(T) = (1, 0)' \), \( y(L) = (0, 1)' \), \( y(RL) = (1, 1)' \).

From \( CL^2x = 0 \) and from \( CLx = y(L) \) we get \( L(Lx) = 0 \), that is \( Ly(L) = 0 \).

This yields a constraint on the structure of \( L \); we have

\[
\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow L = \begin{pmatrix} l_{11} \\ l_{21} \end{pmatrix}.
\]

Likewise from \( CRLx = y(RL) \) we get

\[
\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow R = \begin{pmatrix} r_{11} \\ r_{21} \end{pmatrix}
\]

From \( CLRLx = 0 \) we get

\[
\begin{align*}
l_{11}(r_{11}l_{11} + l_{21}) &= 0 \\
l_{21}(r_{21}l_{11} + l_{21}) &= 0
\end{align*}
\]

Now, let \( x = (x_{1}, x_{2})' \) be. From \( Lx = y(L) \) we get \( l_{11}x_{1} = 0 \) and \( l_{21}x_{1} = 1 \). Then \( l_{11} = 0 \), which, in turn, satisfies (4). Then, from (5) we get \( l_{21} = 0 \). Then, we end up into an impossible satisfaction of \( l_{21}x_{1} = 1 \).

- **Case** \( n = 3 \). Let us consider \( n = 3 \) and still assume \( C = I \). In addition, let us assume that the nodes of \( T \) are coded by \( \text{root}(T) = (1, 0, 0)' \), \( y(L) = (0, 1, 0)' \), \( y(RL) = (0, 0, 1)' \).

From \( Cx = \text{root}(T) \) we get \( x = (1, 0, 0)' \). From \( CL^2x = 0 \) and from \( CLx = y(L) \) we get \( L(Lx) = 0 \), that is \( Ly(L) = 0 \). This yields a constraint on the structure of \( L \); we have

\[
\begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow L = \begin{pmatrix} l_{11} & 0 & l_{13} \\ l_{21} & 0 & l_{23} \\ l_{31} & 0 & l_{33} \end{pmatrix}.
\]
From $Lx = y(L)$ we get
\[
\begin{pmatrix}
 l_{11} & 0 & l_{13} \\
 l_{21} & 0 & l_{23} \\
 l_{31} & 0 & l_{33}
\end{pmatrix}
\begin{pmatrix}
 1 \\
 0 \\
 0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 1 \\
 0
\end{pmatrix}
\]
that is $l_{21} = 1$ and $l_{11} = l_{31} = 0$. Likewise from $CRLx = y(RL)$ we get
\[
\begin{pmatrix}
 r_{11} & r_{12} & r_{13} \\
 r_{21} & r_{22} & r_{23} \\
 r_{31} & r_{32} & r_{33}
\end{pmatrix}
\begin{pmatrix}
 0 \\
 1 \\
 0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 0 \\
 1
\end{pmatrix}
\rightarrow R = 
\begin{pmatrix}
 r_{11} & 0 & r_{13} \\
 r_{21} & 0 & r_{23} \\
 r_{31} & 1 & r_{33}
\end{pmatrix}
.
\]
From $CLRLx = 0$ we get
\[
\begin{pmatrix}
 0 & 0 & l_{13} \\
 1 & 0 & l_{23} \\
 0 & 0 & l_{33}
\end{pmatrix}
\begin{pmatrix}
 r_{11} & 0 & r_{13} \\
 r_{21} & 0 & r_{23} \\
 r_{31} & 1 & r_{33}
\end{pmatrix}
\begin{pmatrix}
 0 \\
 1 \\
 0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 0 \\
 0
\end{pmatrix}
,
\]
which is satisfied if $l_{13} = l_{23} = l_{33} = 0$.

From $CR^2Lx = 0$ we get
\[
\begin{pmatrix}
 r_{11} & 0 & r_{13} \\
 r_{21} & 0 & r_{23} \\
 r_{31} & 1 & r_{33}
\end{pmatrix}
\begin{pmatrix}
 r_{11} & 0 & r_{13} \\
 r_{21} & 0 & r_{23} \\
 r_{31} & 1 & r_{33}
\end{pmatrix}
\begin{pmatrix}
 0 \\
 1 \\
 0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 0 \\
 0
\end{pmatrix}
\rightarrow R = 
\begin{pmatrix}
 r_{11} & 0 & r_{13} \\
 r_{21} & 0 & r_{23} \\
 r_{31} & 1 & r_{33}
\end{pmatrix}
\begin{pmatrix}
 0 \\
 1 \\
 0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 0 \\
 0
\end{pmatrix}
.
\]
Finally, from $Rx = 0$ we need $r_{11} = 0$. Then we conclude that $L$ and $R$ are solutions whenever they have the structure
\[
L = 
\begin{pmatrix}
 0 & 0 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix}
R = 
\begin{pmatrix}
 0 & 0 & 0 \\
 r_{21} & 0 & 0 \\
 r_{31} & 1 & 0
\end{pmatrix}
.
\]
Notice that in this case we discover infinite solutions. In addition, it is worth mentioning that this solution originates from the required labeling, since it immediately requires to choose $x = (1,0,0)$. This makes it possible to satisfy the matrix monomial equations without requiring strong nilpotent conditions on the matrices. In addition, in this case, there is no solution for any $x$, since otherwise we need to require $R = 0$. As a consequence, the other labelling conditions would not be met. If we assume to keep a representation based on the above matrices $L, R$ then a different choice of $x$ may lead to a completely different tree. For example, we can easily see that the choices $x = (0,1,0)^t, (0,0,1)^t$ yield infinite trees.
Interestingly, the generation of infinite trees is not an exception, but quite a common property of the introduced generative scheme.

Let us consider a simple example that clearly shows the possible explosion of the introduced generation scheme. Let us consider a tree whose elements are two dimensional vectors, and consider a two dimensional representation; in addition, for the sake of simplicity, let us assume that \( C = I \) and root(T) = (1, 0)’. Then let us assume that \( R \) is a \( \pi \) rotation and \( L \) is a projection onto the \( y \) axis:

\[
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

An infinite tree with flipping labels is generated that is shown in the side figure.

As shown in the previous examples, we are interested in solving equations involving monomials of matrices. Let us focus on the algebraic side and consider the following example.

**Example 3.** Let us consider the monomial equation

\[
LR = 0.
\]

What are the non-null matrices \( L \) and \( R \) which satisfy this equation? Clearly, equations like \( L^2 = 0 \) and \( R^2 = 0 \) define nilpotent matrices of order 2. Equation (6) can be regarded as a sort of generalization of the notion of nilpotent matrix to the case in which the property involves two matrices.

This problem has generally infinite solutions. Any pair of matrices \( L, R \) such that the image space of \( R \) is in the kernel of \( L \) is a solution. The pair \( L = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \) and \( R = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix} \) is an example. The image space of \( R \) is in the kernel of \( L \). Of course, matrix \( R \) must be singular, otherwise its image space would invade the whole \( \mathbb{R}^2 \) and \( \text{Ker}(A) = \{0\} \), which would require matrix \( L = 0 \).

As discussed in Example 2, in general we need the satisfaction of monomial equations that also involve \( x \in \mathbb{R}^n \).

**Example 4.** Suppose we are given \( T = (x, L, R) \) where \( L = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \) and \( R = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). We can promptly see that \( L^2 = R^2 = 0 \), and \( [L, R] = LR - RL = 0 \). The last one comes out in any case in which \( R = \alpha L \), with \( \alpha \in \mathbb{R} \) (here \( \alpha = 1/2 \)). We can immediately conclude that any pair \( (L, R) \), where \( L^2 = 0 \) and \( R = \alpha L \) corresponds with a balanced tree composed of three nodes.

\[
\begin{align*}
&\quad Cx \\
\downarrow & y(L) \\
\downarrow & 1 \\
\downarrow & 2 \\
\downarrow & 3 \\
\downarrow & 4 \\
\downarrow & y(R)
\end{align*}
\]

\[
L^2 = R^2 = LR = RL = 0.
\]

Notice that in order to define the formal correspondence with this non-void balanced tree we need to restrict to the condition \( x \notin \text{Ker} L \). On the opposite,
if we choose $x = \beta(1, 1)'$ with $\beta \in \mathbb{R} \setminus \{0\}$ then the triple represents a tree composed of the root only. If $x = 0$ then the triple degenerates to one of the infinite representations of the void tree.

Now, let us consider the problem of mapping the above tree in the representation $(L, x, R)$. We need to match the labels $\text{root}(T)$, $y(L)$ and $y(R)$. Hence we must impose:

$$Cx = \text{root}(T), \quad CLx = y(L), \quad CRx = y(R).$$

Since $R = \alpha L$ we have $y(R) = \alpha CLx = \alpha y(L)$. This clearly indicates that while the representation $(L, x, \alpha L)$ is a balanced tree, there is a strong restriction on the label that it can produce.

Paths and monomial correspondence. The discussion on the representation of trees in the real field given in the previous examples enlightens on a nice connection between paths and monomials. In order to decode a certain node we generally need to associate nodes with monomials like $L$, $R$, $L^2$, $LR$, $RL$, $R^2$, $L^3$, $L^2R$, $RL^2$, $R^3$, $LRL$, $RLR$,... composed with the two variables $L$ and $R$. This kind of monomials turn out to be just another way of expressing a path in a tree. The above monomial are of degree 3, but we are interested in monomials of any order, which can be represented by the language generated with symbols $L$ and $R$. For instance, the sequence

$$LRLRLLLLRRLRLRLR = (LR) \cdot (L^2) \cdot (R^3) \cdot (L^2) \cdot (R^2) \cdot (LR)^3$$

is a way of constructing a monomial with $L$ and $R$, that could also be regarded as an element of the language generated by $S_1 = R$, $S_2 = L^2$, $S_3 = LR$. This monomials can be described as follows. Let $\ell_\nu$ and $r_\nu$ be the integer vectors that count the repetitions of $L$ and $R$ is the sequence, respectively. In the above sequence we have

$$\ell_\nu = (1, 2, 4, 1, 1, 1)$$
$$r_\nu = (1, 1, 2, 1, 1, 1).$$

This notation makes is possible to express the sequence as

$$\pi_\nu = LRLRLLLLRRLRLRLR := L^{(1,2,4,1,1,1)} R^{(1,1,2,1,1,1)} = L^{\ell_\nu} R^{r_\nu},$$

where we assume that the above path characterizes node $\nu$. Consistently with what we have done so far will indicate the label on the node $\nu$ with the notation $y(\pi_\nu) \in \mathcal{Y}$. Here, the notations $L^{\ell_\nu} R^{r_\nu}$ reminds us of a generalized notion of matrix power for the matrices $L$ and $R$. The notation used for $\pi_\nu$ reminds the characterization of the node $\nu$, while the generic arc of the path $\pi_\nu$ is simply an element $\pi_\nu^k$ of vector $\pi_\nu$. Moreover, we also use the notation $|\ell_\nu| = \sum_\nu \ell_\nu$ and $|r_\nu| = \sum_\nu r_\nu$. Clearly $|\pi_\nu| = |\ell_\nu| + |r_\nu|$. 

Example [3] gives an insight to draw the following general conclusion
Proposition 2. Let $\alpha \in \mathbb{R}$ and $R = \alpha L$ be. Moreover, let us assume that $h \in \mathbb{N}$ and $h \geq 1$ is the first integer such $L^h = 0$. If $x \notin \ker L^{h-1}$ then the decoding of the triple $T = (L, x, R)$ is a balanced tree $T$ with height $h$.

Proof. The proof can be given straightforwardly by induction on $h$. □

The possible generation of infinite trees raises the question on which conditions we need to impose in order to gain the guarantee that a given representation yields finiteness. In addition to the condition stated in Proposition 2, in the next section we will present another class of representations which gives rise to finite tree. The following proposition states a general property on the generation of “vanishing trees”.

Proposition 3. Given $T = (L, x, R)$ let us assume that $\|L\| < 1$, $\|R\| < 1$. Then if $\nu$ is a leaf of path $\pi^\nu$

$$\lim_{|\pi| \to \infty} L^\nu R^\nu x = 0.$$  

Proof. We have

$$y(\pi^\nu) = CL^\nu R^\nu x.$$  

When taking the norm on both sides

$$\|y(\pi^\nu)\| \leq \|C\| \cdot \|L^\nu R^\nu\| \cdot \|x\|.$$  

Now, let $\theta < 1$ be an upper bound of $\|L\|$ and $\|R\|$. Then

$$\|y_\nu\| \leq \|C\| \cdot \|x\| \cdot \theta^{|\pi|}.$$  

Finally, the proof follows when computing $\lim_{|\pi| \to \infty}$. □

We are now ready to formulate the decoding problem in its general form.

Decoding Problem. Given the tree $T$ with $m$ nodes we consider the equations

$$\begin{cases} CL^\nu R^\nu x = y(\pi^\nu) & \text{for all nodes } \nu; \\ CL^\beta R^\beta x = 0 & \text{for all buds } \beta, \end{cases}$$

which refers to the nodes and to the buds, respectively (remember that a binary tree with $m$ nodes has $m+1$ buds). When using the vectorial form, we can rewrite
this conditions in the form \( Mx = y \) where

\[
M := \begin{pmatrix}
CL^{l_1}R^{r_1} \\
CL^{l_2}R^{r_2} \\
\vdots \\
CL^{l_m}R^{r_m} \\
CL^{m+1}R^{r_{m+1}} \\
\vdots \\
CL^{2m+1}R^{r_{2m+1}}
\end{pmatrix}
\quad \text{and} \quad
y := \begin{pmatrix}
y(\pi^1) \\
y(\pi^2) \\
\vdots \\
y(\pi^m) \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

**Definition 1.** The representation \((L, x, R)\) of \(T\) is completely reachable if and only if \(\text{rank } M = \min\{n, p \cdot (2m + 1)\}\).

**Proposition 4.** Let us consider any completely reachable pair \((L, R)\) of \(T\). If \(n \geq p \cdot (2m + 1)\) then the decoding problem of \(T\) admits the solution

\[
x = M^+y,
\]

where \(M^+\) is Penrose pseudo-inverse of \(M\).

### 3 Non-commutative left-right matrices

As we have already seen, \(T\) can yield an infinite tree. Here is another example.

**Example 5.** Let us consider the triple \(T = (L, x, R)\) where

\[
L = \begin{pmatrix}
0 & 0 & 0 \\
b_l & 0 & 0 \\
a_l & c_l & 0
\end{pmatrix}
\quad \text{and} \quad
R = \begin{pmatrix}
0 & 0 & 0 \\
b_r & 0 & 0 \\
a_r & c_r & 0
\end{pmatrix}
\]

We can easily check that the recursive propagation yields an infinite tree.

No matter whether a finite or an infinite tree is generate, a uniform representation \((T, \gamma)\) is especially interesting whenever \([L, R] \neq 0\). In the opposite case, as already seen, the representation is dramatically limited. The following example suggests to consider a nice class of uniform non-commutative representations. The following example shows a representation \((L, x, R)\) which yields finite trees.

**Example 6.** Let us consider the triple \(T = (L, x, R)\) where

\[
[L, R] = \begin{pmatrix}
2 & -1 \\
4 & -2
\end{pmatrix} \cdot \begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix} - \begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix} \cdot \begin{pmatrix}
2 & -1 \\
4 & -2
\end{pmatrix} = \begin{pmatrix}
3 & -2 \\
4 & -3
\end{pmatrix}
\]

We can easily check that the recursive propagation yields an infinite tree.
Fig. 1. Balanced tree on the left in the case of non-null coefficients. If $b_r = 0$ then the asymmetry yields the unbalanced tree on the right.

Let $a_l, b_l, c_l, a_r, b_r, c_r$ be non-null reals and associate any non-null real with symbol $\odot$. Then we have

$$|\pi^\nu| = 2 \rightarrow \pi^\nu = \begin{pmatrix} 0 & 0 & 0 \\ \odot & 0 & 0 \\ \odot & \odot & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \odot & 0 \\ \odot & \odot \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \odot & 0 & 0 \\ \odot & \odot & 0 \end{pmatrix}$$

$$|\pi^\nu| = 3 \rightarrow \pi^\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \odot & \odot & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \odot & 0 \\ \odot & \odot \end{pmatrix} = 0$$

This corresponds with the balanced tree in Fig. 1.

Now, we can exploit the non-commutativity $[L, R] \neq 0$ to generate other trees with missing nodes. We easily see that if $b_r = 0$ then $RL = R^2 = 0$ (see Fig. 1).

4 Conclusions

The encoding-decoding scheme presented in this paper opens the doors to new learning algorithms that seem to be adequate in computational linguistics. A different path that may be followed is the one of restricting to commuting matrices where different matrices are used for any layer.

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