Derivation of ARCH(1) process from market price changes based on deterministic microscopic multi-agent

Aki-Hiro Sato\textsuperscript{1}\textsuperscript{*} and Hideki Takayasu\textsuperscript{2}
Department of Applied Mathematics and Physics,
Kyoto University, Kyoto 606-8501, Japan, and
\textsuperscript{2} Sony Computer Science Lab., Takanawa Muse Bldg., 3-14-13,
Higashi-Gotanda, Shinagawa-ku, Tokyo 141-0022, Japan.

October 27, 2018

Abstract

A model of fluctuations in the market price including many deterministic dealers, who predict their buying and selling prices from the latest price change, is developed. We show that price changes of the model is approximated by ARCH(1) process. We conclude that predictions of dealers affected by the past price changes cause the fat tails of probability density function. We believe that this study bridges stochastic processes in econometrics with multi-agent simulation approaches.

Key words. ARCH(1), artificial market, microscopic, macroscopic, multi-agent simulation

1 Introduction

Market prices have been analyzed by many researchers. It is a famous result that the probability density function of difference of the market prices follows

\textsuperscript{*}Electronic mail: aki@sawada.riec.tohoku.ac.jp
the power law distribution (Mantegna et al. 1995). On the other hand the stochastic processes of ARCH type are one of the most exciting studies in econometrics. Statistical properties of ARCH processes have been clarified by many researches. Namely a probability density of a dynamical variable of the ARCH process follows the power law distribution and its characteristics is applied to explain fluctuations in the real market (Engle 1982, Bollerslev 1986, Nelson 1990). Recently economists and physicists are interested in multi-agent simulations of problems motivated economically (Takayasu et al. 1992, Bak et al. 1997, Sato et al. 1998, Johnson et al. 1998, Lux et al. 1999). This approach is to study an artificial market in which programming agents sell and buy their stocks and actually market price changes can be calculated by an interaction of many dealers.

On the viewpoint of statistical mechanics we can consider investigating statistical properties of market prices, the stochastic processes of the ARCH type and multi-agent simulations to be macroscopic, mesoscopic and microscopic, respectively. (see fig. 1). Our study will bridge the multi-agent simulation with stochastic processes of the ARCH type.

In this article we develop a simple market model with many deterministic dealers. The dealers estimate their buying and selling prices from the latest change of the market price. We show that from market price changes of the model a stochastic process of the ARCH type can be derived. Let us give a brief outline of this article. In the second chapter we describe the process of the ARCH type. In the third chapter we mention a simple model of fluctuations in market price based on dealers. In the fourth chapter we analytically derive an ARCH(1) process for price changes from the proposed model. The fifth chapter is devoted to the concluding remarks.

2 The ARCH (q) processes

The ARCH abbreviates autoregressive conditional heteroskedasticity, which has been introduced by Engle in econometrics (Engle 1982). The ARCH(q) process is formalized by

\[
\begin{align*}
\epsilon_s &= \sigma_s \cdot Z_s \\
\sigma_s^2 &= \alpha_0 + \alpha_1 \epsilon_{s-1}^2 + \cdots + \alpha_{s-q} \epsilon_{s-q}^2,
\end{align*}
\]
where $\epsilon_s$ is an interesting variable, $Z_s$ is a stochastic variable, $\sigma_s$ is called volatility. $\alpha_i$ ($i = 0 \ldots q$) is a positive parameter. Consider $q = 1$ as the most simple case of the ARCH(q) stochastic process. It seems sufficient to discuss the case of $q = 1$ in order to see statistical properties of the ARCH process. Especially when $q = 1$ eq. (1) is written by

$$
\begin{cases}
\epsilon_s = \sigma_s \cdot Z_s \\
\sigma_s^2 = \alpha_0 + \alpha_1 \epsilon_{s-1}^2.
\end{cases}
$$

Eliminating $\sigma_s$ into eq. (2) one can get an alternative expression,

$$
\epsilon_s = \sqrt{\alpha_0 + \alpha_1 \epsilon_{s-1}^2} \cdot Z_s.
$$

Moreover eq. (3) is approximated as

$$
\epsilon_s = \begin{cases}
\sqrt{\alpha_0} \cdot Z_s & (|\epsilon_{s-1}| \leq \sqrt{\alpha_0/\alpha_1}) \\
\sqrt{\alpha_1} \cdot |\epsilon_{s-1}| \cdot Z_s & (|\epsilon_{s-1}| > \sqrt{\alpha_0/\alpha_1})
\end{cases}
$$

This equation is equivalent to a random multiplicative process (Takayasu et al. 1997). Using the random multiplicative process theory (Sato et al. 2000) it is easy to prove that a probability density function of $\epsilon_s$ has power law tails,

$$
p(\epsilon) \propto |\epsilon|^{-\beta-1},
$$

where $\beta$ is given by

$$
\alpha_1^{\beta/2} \langle |Z|^\beta \rangle = 1.
$$

3 Dealer model

We show a brief explanation of the market model consisted of many simple deterministic dealers. As shown in fig. 2 this model can be separated into two parts: One is a market mechanism, which describes how to determine a market price from orders. The other is an algorithm of agents, which governs how to order a selling or buying on the market and how to modify their bid prices. In the following subsections we explain these parts, initial conditions and parameters and also results of numerical simulations.
3.1 Market mechanism

Suppose that a market of a competitive buying and selling in which $N$ dealers give their orders for a common order board. $S_i(t)$ and $B_i(t)$ represent a selling price and buying price for the $i$th dealer at time $t$, respectively. It is assumed that buying prices and selling prices individually compete in this market. Namely the maximum buying price and the minimum selling price are effective in the market. Thus the condition for a trade to occur is given by the inequality,

$$L(t) = \max\{B(t)\} - \min\{S(t)\} \geq 0,$$

where $\max\{B(t)\}$ represents the maximum buying price in all the buyers, and $\min\{S(t)\}$ the minimum selling price in all the sellers. Suppose that the market price $P(t)$ is determined as an arithmetic mean of the selling price and the buying one when the transaction occurs. Otherwise the latest market price is maintained. Namely,

$$P(t) = \begin{cases} \frac{1}{2}(\max\{B\} + \min\{S\}) & (L(t) \geq \Lambda) \\ P(t-1) & (L(t) < \Lambda) \end{cases}.$$

3.2 Dealer algorithm

We suppose that a seller goes on decreasing his expectation of selling price until he can sell a stock and that a buyer goes on increasing his expectation of buying price until he can buy a stock. A rule to modify his expecting price is given by

$$B_i(t+1) = B_i(t) + |1 + c_i\Delta P_{prev}|a_i(t),$$

where $\Delta P_{prev}$ denotes the latest change of the market price. From the above assumption for the seller and the buyer the $i$th dealer is a seller when $a_i(t) < 0$ and he is a buyer when $a_i(t) > 0$. A term of $|1 + c_i\Delta P_{prev}|$ means that modification of his expectation of price depends on the latest market price change. Here $c_i$ is a prediction coefficient dependent on the dealer.

Suppose that all the dealers have a small asset. Thus it means that a seller/buyer changes his position into a buyer/seller after a trade. Because of the assumptions that each dealer keeps his position till a trading, an evolution rule of $a(t)$ is given by

$$a_i(t+1) = \begin{cases} -a_i(t) & \text{(a seller and a buyer when a trade occurs)} \\ a_i(t) & \text{(otherwise)} \end{cases}.$$
3.3 Initial conditions and parameters

\( B_i(0), a_i(0) \) and \( c_i \) are given by random numbers of which a range is \([-\Lambda/2, \Lambda/2]\), \([-\alpha, \alpha]\) and \([-c^*, c^*]\), respectively. We put \( P(0) = 0 \) and \( \Delta P_{prev} = 0 \). The dealer’s rule is deterministic except initial conditions. This model has four parameters; the dealer number \( N \), \( \alpha \) for \( a_i(t) \), difference between selling and buying price \( \Lambda \) and \( c^* \) for \( c_i \).

3.4 Numerical Simulation

Fig. 3 show a typical example of time series of market price \( P(t) \) simulated numerically with price changes \( \Delta P(t) = P(t) - P(t - 1) \). Because an event of a change of the market price occurs in Poissonian we define a jump \( \Delta p \) as a market price change on which a trade occurs. Fig. 4 shows probability density functions (PDF) of the jumps and their corresponding cumulative distribution functions (CDF), which is defined by

\[
P(\geq |x|) = \int_{-\infty}^{-|x|} p(x')dx' + \int_{|x|}^{\infty} p(x')dx'.
\]  

(11)

From fig. 4 we can find linear part meaning that CDFs follow power law distributions of the following form

\[
P(\geq |x|) \propto |x|^{-\beta},
\]  

(12)

where \( \beta \) is a power law exponent. Moreover the power law exponent obviously depends on a value of \( c^* \).

4 From Microscopic to mesoscopic

As shown in fig. 5 \( \Delta p_s \) represents a change of the market price on the \( s \)th trade, and \( n_s \) a time interval between the \( s \)th trade and the \( s + 1 \)st. We define \( M_s \) as the buying price at the \( s \)th trade and \( m_s \) as the selling price, respectively. From the assumptions of eq. (8) \( \Delta p_s \) can be estimated as

\[
\Delta p_s = \frac{1}{2}(m_s + M_s + \Lambda) - \frac{1}{2}(m_{s-1} + M_{s-1} + \Lambda) \\
= \frac{1}{2}(m_s - m_{s-1}) + \frac{1}{2}(M_s - M_{s-1}).
\]  

(13)
The first term and the second term on the second line are approximated by

\[ m_s - m_{s-1} = |1 + c_i \Delta p_{s-1}|a_i n_{s-1}, \quad (14) \]
\[ M_s - M_{s-1} = |1 + c_j \Delta p_{s-1}|a_j n_{s-1}, \quad (15) \]

where the subscript \( i \) denotes the dealer who gives the lowest selling price, and \( j \) the dealer who gives the highest buying one. Substituting eqs. (14) and (15) into eq. (13) yields

\[ \Delta p_s = \frac{1}{2} |1 + c_i \Delta p_{s-1}|a_i n_{s-1} + \frac{1}{2} |1 + c_j \Delta p_{s-1}|a_j n_{s-1}. \quad (16) \]

When the dealer number \( N \) is large it can be assumed that dealers’ prices of expectation are distributed uniformly. By taking square of eq. (16) for all the dealers and averaging over ensemble under the condition that \( \Delta p_{s-1} \) is realized, we get the following equation,

\[ \langle \Delta p_s^2 \rangle = \frac{1}{2} \langle a^2 \rangle (1 + \langle c^2 \rangle \Delta p_{s-1}^2) \langle n_{s-1}^2 \rangle. \quad (17) \]

From the assumptions of \( c_i \) and \( a_i \) we have \( \langle c \rangle = 0, \langle c^2 \rangle = c^*^2/3, \langle a \rangle = 0 \) and \( \langle a^2 \rangle = a^2/3 \). Assuming that \( \langle \Delta p \rangle = 0 \), eq. (17) becomes,

\[ \langle \Delta p_s^2 \rangle = \alpha^2 (1 + \frac{c^*^2}{3} \Delta p_{s-1}^2) \langle n_{s-1}^2 \rangle. \quad (18) \]

Eq. (18) shows that the variance of \( \Delta p_s \) relates to a realized value on \( s - 1 \) and it has the same form with eq. (2) without the term of \( \langle n_{s-1}^2 \rangle \). Supposing that \( \sigma_s' \) is a stochastic variable with a zero-average and a normal variance we can also rewrite eq. (18) as

\[ \Delta p_s = \sqrt{1 + \frac{c^*^2}{3} \Delta p_{s-1}^2} \alpha \sqrt{\langle n_{s-1}^2 \rangle} \sigma_s'. \quad (19) \]

Eq. (19) becomes identical to eq. (3) with the relations \( \alpha_0 = 1, \alpha_1 = c^*^2/3 \) and \( \sigma_s = \alpha \sqrt{\langle n_{s-1}^2 \rangle} \sigma_s' \).
5  Conclusions

We introduced the model of an artificial market with many deterministic dealers and showed the outline of the stochastic process of ARCH type. The ARCH(1) process can be derived from the changes of the market price in this model. We conclude that the fluctuation of the market price is approximated by the ARCH type process if each dealer changes his expectation of a buying/selling price proportional to the lastest market price change. We expect that our approach will bridge the stochastic processes of the ARCH type in econometrics with dynamical market models consisted of dealers, and our understanding about the basic properties of markets will be deepened.

References

[1] Bak P., Paczuski M. and Shubik M. (1997) Price variations in a stock market with many agents. Physica A 246:430–453.

[2] Bollerslev T. (1986) Generalized autoregressive conditional heteroskedasticity. J. Econometrics 31:307–327.

[3] Deutsch J. M. (1994) Probability distributions for one component equations with multiplicative noise. Physica A 208:433–444.

[4] Engle R.F. (1982) Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. Econometrica 50:987–1007.

[5] Johnson N.F., Jarvis S., Jonson R., Cheung P. and Kwong Y.R. (1998) Volatility and agent adaptability in a self-organizing market. Physica A 258:230–236.

[6] Lux T. and Marchesi M. (1999) Scaling and criticality in a stochastic multi-agent model of a financial market. Nature (London) 397:498–500.

[7] Mantegna R. N. and Stanley H. E. (1995) Scaling behaviour in the dynamics of an economics index. Nature (London) 397:46–49.

[8] Nelson D.B. (1990) ARCH Models as diffusion approximations. J. Econometrics 45:7–38.
[9] Sato A.-H., Takayasu H. (1998) Dynamic numerical models of stock market price: from microscopic determinism to macroscopic randomness. Physica A 250:231–252.

[10] Sato A.-H., Takayasu H. and Sawada Y. (2000) Invariant power law distribution of Langevin systems with colored multiplicative noise. Phys. Rev. E, 61:1081–1087.

[11] Schenzle A. and Brand H. (1979) Multiplicative stochastic processes in statistical physics. Phys. Rev. A 20:1628–1647.

[12] Takayasu H., Miura H., Hirabayashi T. and Hamada K. (1992) Statistical properties of deterministic threshold elements—the case of market price. Physica A 184:127–134.

[13] Takayasu H., Sato A.-H. and Takayasu M. (1997) Stable infinite variance fluctuations in randomly amplified Langevin systems, Phys. Rev. Lett. 79:966–969.

Macroscopic  Mesoscopic  Microscopic

Statistical properties  ARCH type processes  Multi-agent simulations

Figure 1: The conceptual illustration of correspondence of scope to study methods.
Figure 2: The conceptual illustration of dealer model. The inputs of the market are orders from dealers. The output of the market is a market price. The input of an agent is a sequence of latest price changes. The output of an agent is a selling or a buying.
Figure 3: Time series of artificial prices (right). Time series of changes of the artificial prices (left). X axis represents a step \( t \). Parameters are \( N = 100, \alpha = 0.01, \Lambda = 1.0, c^* = 390.0 \).
Figure 4: Semi-log plots of the probability density functions of the changes (right). Log-log plots of the cumulative distributions of the changes (left). Parameters are $N = 100, \alpha = 0.01, \Lambda = 1.0$. 

11
Figure 5: A conceptual illustration of a temporal development of the market price $P(t)$. $\tau(s)$ represents a step for the $s$th trade to occur. $\Delta p_s$ is a price change on that step. $n_s$ denotes steps between the $s-1$th trade and the $s$th.