Hamiltonian Zoo for systems with one degree of freedom

Saksilpa Srisukson\textsuperscript{1,a}, Kittapat Ratanaphupha\textsuperscript{2,b} and Sikarin Yoo-Kong\textsuperscript{3,c,d}

\textit{a} Department of Physics, Faculty of Science, Chulalongkorn University,
254 Phayathai Road, Patumwan, Bangkok 10330, Thailand.
\textit{b} Darunisikkhalai Science School (Engineering-Science Classroom),
King Mongkut’s University of Technology Thonburi (KMUTT),
126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
\textit{c} The institute for fundamental study (IF), Naresuan University,
Phitsamulok, 65000, Thailand
\textit{d} Theoretical and Computational Science Centre (TaCs), Faculty of Science,
King Mongkut’s University of Technology Thonburi, Bangkok 10140, Thailand.

\textsuperscript{1}saksilpa@gmail.com, \textsuperscript{2}bosskittapat@hotmail.com, \textsuperscript{3}sikariny@nu.ac.th

Abstract

We present alternative \textit{explicit} forms of the standard Hamiltonian for systems with one degree of freedom. This new class of infinite Hamiltonians is called Newton-equivalent Hamiltonian zoo, producing the same equation of motion. These Hamiltonians are directly solved from the Hamilton’s equations and come with extra-parameters, which are interpreted as scaling factors for the time evolution on phase space. Moreover, each Hamiltonian in the zoo can be used as a generating function for a Hamiltonian hierarchy.

Keywords: Lagrangian, Hamiltonian, Calculus of Variations, Newton-equivalent Hamiltonian, Non-uniqueness.

1 Introduction

In classical mechanics, Lagrangian is commonly used to study the dynamics of the physical system. For the system with one degree of freedom, the Lagrangian is given by $L_N(\dot{x}, x) = T(\dot{x}) - V(x)$, where $T(\dot{x}) = m\dot{x}^2/2$ is the kinetic energy and $V(x)$ is the potential energy. The Euler-Lagrange equation

$$\frac{\partial L_N}{\partial x} - \frac{d}{dt} \frac{\partial L_N}{\partial \dot{x}} = 0$$

(1.1)

gives us the equation of motion:

$$\ddot{x} = -\frac{1}{m}dV(x)/dx \equiv F.$$  

(1.2)
Using Legendre transformation, \( L_N(\dot{x}, x) = p\dot{x} - H_N(p, x) \), we obtain the Hamiltonian \( H_N(p, x) \) given by \( H_N(p, x) = T(p) + V(x) \), where \( T(p) = p^2/2m \). The Hamilton’s equations

\[
\dot{x} = \frac{\partial H_N}{\partial p}, \quad \text{and} \quad \dot{p} = -\frac{\partial H_N}{\partial x}
\]

(1.3)
give us again the equation of motion.

It is commonly known that the Lagrangian possesses non-uniqueness property. Simply speaking, the equation of motion (1.2) does not change under the modification of the Lagrangian: \( L'_N(\dot{x}, x) = \alpha L_N(x, \dot{x}) + \beta + df(x, t)/dt \), where \( \alpha \) and \( \beta \) are constants. The last term is the total derivative of the function \( f(x, t) \) which makes no contribution to the variation of the action.

Recently, a new type of Lagrangian, called the multiplicative Lagrangian, was successfully constructed by Surawutinack et al., [1], see also [2], for the system with one degree of freedom. This Lagrangian came with an extra variable namely \( \lambda \) which is in velocity unit. Then it is called the \( \lambda \)-extended class or the 1-parameter extended class of the standard Lagrangian \( L_N \) given by

\[
L_\lambda(\dot{x}, x) = m\lambda^2 \left[ e^{-E(x, \dot{x})/m\lambda^2} + \frac{\dot{x}}{\lambda^2} \int_0^\dot{x} e^{-E(q, \dot{q})/m\lambda^2} dq \right],
\]

(1.4)

where \( E(\dot{x}, x) = m\dot{x}^2/2 + V(x) \) is the energy function. What has been found is that if the value of \( \lambda \) is infinitely large: \( \lim_{\lambda \to \infty} (L_\lambda(\dot{x}, x) - m\lambda^2) = L_N(\dot{x}, x) \), the standard Lagrangian is recovered. The interesting point is that the multiplicative Lagrangian can be considered as a generating function for Lagrangian hierarchy:

\[ \{ L_1, L_2, L_3, \ldots \} \]

and the first three Lagrangians are given by

\[
L_1(\dot{x}, x) = L_N(\dot{x}, x) = T(\dot{x}) - V(x),
\]

\[
L_2(\dot{x}, x) = \frac{T^2(\dot{x})}{3} + 2T(\dot{x})V(x) - V^2(x),
\]

\[
L_3(\dot{x}, x) = \frac{T^3(\dot{x})}{5} + T^2(\dot{x})V(x) + 3T(\dot{x})V^2(x) - V^3(x).
\]

Intriguingly, all Lagrangians in the hierarchy produce the same equation of motion. Then the construction for Lagrangian (1.4) represents a non-trivial way, but systematic, to produce infinitely different Lagrangians describing the same physics.

The motivation of [1] is directly related to the inverse Lagrangian problem. The question is that for a given equation of motion “Does there exist a corresponding Lagrangian?” Sonin [3] (see also [4] and review note please see [5]) proved that every ordinary second-order differential equation admits a Lagrangian.

**Theorem 1.1.** (Sonin): For every function \( \mathcal{F} \) there exists a solution \((g, L)\) of equation

\[
g(\mathcal{F} - \ddot{x}) = \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}, \quad \text{where} \quad g = \frac{\partial^2 L}{\partial \dot{x}^2} \neq 0.
\]

(1.5)

While what we did in [1], we went further to show that actually the ordinary second order differential equation (1.5) actually admits infinite solutions. The explicit
forms of the 1-paramenter extended class of Lagrangians were obtained both in non-relativistic and relativistic cases.

One can ask “Does it exist the multiplicative Hamiltonian?” The answer is yes and it is given by
\[ H_\lambda(p, x) = -m\lambda^2 e^{-\frac{H_N(p, x)}{m\lambda^2}}. \]

This Hamiltonian is also called the \(\lambda\)-extended class of the standard Hamiltonian \(H_N\). It was also found that if the value of \(\lambda\) is infinitely large
\[ \lim_{\lambda \to \infty} (H_\lambda(p, x) + m\lambda^2) = H_N(p, x), \]
the standard Hamiltonian is retrieved. The multiplicative Hamiltonian is also the generating function for Hamiltonian hierarchy: \(\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots\}\) and the first three Hamiltonians are given by
\[
\begin{align*}
\mathcal{H}_1(p, x) &= H_N = T(p) + V(x), \\
\mathcal{H}_2(p, x) &= H_N^2 = (T(p) + V(x))^2 = T^2(p) + 2T(p)V(x) + V^2(x), \\
\mathcal{H}_3(p, x) &= H_N^3 = (T(p) + V(x))^3 = T^3(p) + 3T^2(p)V(x) + 3T(p)V^2(x) + V^2(x),
\end{align*}
\]
where \(T(p) = p^2/2m\) is the kinetic energy in terms of momentum variable. These Hamiltonians form an infinite Newton-equivalent set of Hamiltonians producing the same equation of motion. The parameter \(\lambda\) in this construction was interpreted as a time scaling factor, which would be discussed further in the text.

An alternative form of the Hamiltonian for a one-dimensional harmonic oscillator was studied in [6] and the one-parameter extended class of the Hamiltonian was obtained, but in different form with (1.6). In this context, we may call these alternative Hamiltonians as the non-standard Hamiltonians. For the case of the non-standard Lagrangians, a number of works have been done in the area of non-linear dynamics and dissipative systems [7]-[9]. Some applications in physics have been investigated in Cosmology [10], quantum dynamics [11] and general relativity [12].

In the present paper, we will explore more families of the Hamiltonian, many-parameter extended classes, for systems with one degree of freedom by directly solving the Hamilton’s equations which are modified to a second order partial differential equation (PDE). This method can be considered as the inverse engineering of the Hamiltonian problem. In section 2, the exponential family of Hamiltonians, namely Cabbatonian, is derived. A non-trivial equation is found as an extra relation for solving for the Hamiltonian family. Interestingly, this extra relation turns out to be the conservation of energy. We then construct the zoo of Newton-equivalent Hamiltonians. In section 3, the physical meaning of the extra parameter will be examined through a concrete example namely the harmonic oscillator. Finally, the summary and remarks will be given in the last section.

## 2 Hamiltonian Zoo

With a given equation
\[ \dot{p} = -\frac{dV(x)}{dx}, \]
the key equation to solve for the Hamiltonian is the combination of the Hamilton’s equations

\[ 0 = \frac{1}{m} \frac{\partial H}{\partial x} + p \frac{\partial^2 H}{\partial p^2} + \frac{p}{m} \frac{\partial^2 H}{\partial p \partial x}, \quad (2.2) \]

where \( H(p, x) \) is the Hamiltonian. Obviously, if we take the Hamiltonian in the form \( H(p, x) = T(p) + V(x) \) \((2.2)\) gives the kinetic energy in the standard form \( T(p) = p^2 / 2m \). If we take the Hamiltonian in the form \( H(p, x) = K(p)W(x) \), where \( K(p) \) and \( W(x) \) are to be determined, \((2.2)\) gives the multiplicative Hamiltonian \((1.6)\).

In this section, we will go further in the inverse engineering of \((2.2)\) subject to \((2.1)\). We introduce an ansatz exponential form of the Hamiltonian given by

\[ H_{\alpha_1, \alpha_2}(p, x) = \alpha_2 e^{\alpha_1 F(p, x)}, \quad (2.3) \]

where \( F(p, x) \) is a function defined on the phase space and needs to be determined. The parameters \( \alpha_1 \) and \( \alpha_2 \) are constants to be also determined. Substituting the Hamiltonian \((2.3)\) into \((2.2)\), we obtain

\[ -\frac{\alpha_1 \alpha_2 e^{\alpha_1 F} \frac{\partial F}{\partial x}}{m} \dot{p} \alpha_1 \alpha_2 e^{\alpha_1 F} \left( \frac{\partial^2 F}{\partial p^2} + \alpha_1 \left( \frac{\partial F}{\partial p} \right)^2 \right) + \frac{p}{m} \left[ \alpha_1 \alpha_2 e^{\alpha_1 F} \left( \frac{\partial^2 F}{\partial p \partial x} \right) \right], \quad (2.4) \]

which can be rewritten as

\[ 0 = \frac{1}{m} \frac{\partial F}{\partial x} + \frac{\partial^2 F}{\partial p^2} + \frac{p}{m} \frac{\partial^2 F}{\partial p \partial x} + \alpha_1 \left[ \dot{p} \left( \frac{\partial F}{\partial p} \right)^2 + \frac{p}{m} \frac{\partial F}{\partial p} \frac{\partial F}{\partial x} \right]. \quad (2.5) \]

We find that if \( F(p, x) = H(p, x) \) is the Hamiltonian the first three terms in \((2.5)\) give \((2.2)\). Then the last bracket must vanish and gives an extra-relation

\[ 0 = \dot{p} \left( \frac{\partial F}{\partial p} \right)^2 + \frac{p}{m} \left( \frac{\partial F}{\partial p} \frac{\partial F}{\partial x} \right), \quad (2.6) \]

in order to preserve the structure of \((2.2)\). The equation \((2.6)\) can be further simplified in the form

\[ 0 = \dot{p} \left( \frac{\partial F}{\partial p} \right)^2 + \frac{p}{m} \left( \frac{\partial F}{\partial p} \frac{\partial F}{\partial x} \right). \quad (2.7) \]

Equation \((2.7)\) and together with

\[ 0 = \frac{1}{m} \frac{\partial F}{\partial x} + \frac{\partial^2 F}{\partial p^2} + \frac{p}{m} \frac{\partial^2 F}{\partial p \partial x}, \quad (2.8) \]

can be used to solve for the function \( F(p, x) \) in \((2.3)\). However, obviously, \((2.7)\) is much more simpler to handle with.

**Remark 1:** If the function \( F(p, x) \) equals Hamiltonian \( H(p, x) \) \((2.7)\) is actually a consequence of the Hamilton’s equations as well as \((2.8)\). This can be transparently seen by substituting the Hamilton’s equations in \((2.7)\), resulting in

\[ 0 = -\frac{\partial H}{\partial p} \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \frac{\partial H}{\partial x}, \quad (2.9) \]
which holds. If the Hamiltonian implicitly depends on time (2.7) can be expressed in the form:

\[ 0 = \frac{\partial H}{\partial p} \frac{dp}{dt} + \frac{\partial H}{\partial x} \frac{dx}{dt} = \frac{dH}{dt}, \]  

which is the conservation of energy. However, (2.8) can be directly obtained from performing partial derivative (2.7) with respect to \( p \) variable

\[ \frac{\partial}{\partial p} \left( \frac{\dot{p} \partial F}{\partial p} + \frac{p}{m} \frac{\partial F}{\partial x} \right) = \frac{1}{m} \frac{\partial F}{\partial x} + \frac{\dot{p}}{m} \frac{\partial^2 F}{\partial p^2} + \frac{p}{m} \frac{\partial^2 F}{\partial p \partial x} = 0. \]  

Next, we are going to use (2.7) to solve the function \( F(p, x) \) with the technique of separation of variables.

### 2.1 Case I: Additive case

If we take the function, \( F(p, x) \) in the form: \( F(p, x) = T(p) + V(x) \) and substitute in (2.7) we obtain

\[ 0 = \dot{p} \frac{dT}{dp} + \frac{p}{m} \frac{dV}{dx}. \]  

Using the equation of motion \( \dot{p} = -\frac{dV}{dx} \), we obtain

\[ 0 = \dot{p} \left( \frac{dT}{dp} - \frac{p}{m} \right). \]  

Since \( \dot{p} \neq 0 \), the terms inside the bracket must be zero, resulting in

\[ \int dT = \int \frac{p}{m} dp \Rightarrow T(p) = \frac{p^2}{2m} + C, \]  

where \( C \) is a constant of integration which can be set to be zero. Then the function \( F(p, x) \) is nothing but standard Hamiltonian \( H_N(p, x) = \frac{p^2}{2m} + V(x) \).

### 2.2 Case II: Multiplicative case

If we take the function \( F(p, x) \) in the multiplicative form \( F(p, x) = K(p)B(x) \) and substitute in (2.7) we obtain

\[ 0 = \frac{m}{K \dot{p}} \frac{dK}{dp} + \frac{p}{m} \left( K \frac{dB}{dx} \right) \]

\[ \Rightarrow \frac{m}{K \dot{p}} \frac{dK}{dp} = \frac{1}{B} \frac{dB}{dV}. \]  

We see that both sides of (2.15) are independent to each other. Then equation holds if both sides equal to a constant \( \beta \). We consider first the left-hand-side of (2.15) and obtain

\[ \frac{m}{K \dot{p}} \frac{dK}{dp} = \beta \]

\[ \int \frac{dK}{K} = \int \beta \frac{p}{m} dp \Rightarrow K(p) = A_1 e^{\beta T(p)}, \]  

where \( A_1 \) is a constant of integration.
where $A_1$ is a constant to be determined. Next, we consider the right-hand-side of (2.15)
\[
\frac{1}{B} \frac{dB}{dV} = \beta \\
\int \frac{dB}{B} = \int \beta dV \Rightarrow B(x) = A_2 e^{\beta V(x)},
\]
where $A_2$ is a constant to be determined. Then the function $F(p, x)$ becomes
\[
F(p, x) = A_1 A_2 e^{\beta H_N(p, x)},
\]
where $H_N(p, x) = T(p) + V(x)$ is the standard Hamiltonian. In fact, the function $F(p, x)$ is a multiplicative Hamiltonian $H_\lambda(p, x)$ given in (1.6) with the choices $A_1 A_2 = \pm \frac{m\lambda^2}{1 - q}$ and $\beta = \pm \frac{1}{m\lambda^2}$. Then, we now define
\[
H_2(p, x) \equiv H_{\lambda_1, \lambda_2}(p, x) = \pm \frac{m\lambda^2_1 e^{\lambda_2 - m\lambda^2_2}}{m\lambda^2_1}.
\]
which the $\lambda$ is replaced by $\lambda_1$ for the later use.

Remark 2: We now define $\lambda_1 = 1/(1 - q)$. Then $q$ approaches 1 is the same as $\lambda$ approaches infinity, see (1.7). In this sense, we find that the Hamiltonian (2.19) can be considered as the $q$-deformation of the standard Hamiltonian. The same mathematical analogy can be found in the context of Renyi entropy [13].

2.3 Case III: Cabbatonian
We now insert (2.19) into (2.3) and obtain
\[
H_{\alpha_1, \alpha_2}(p, x) = \alpha_2 e^{\alpha_1 H_1(p, x)} = \alpha_2 e^{\pm \alpha_1 m\lambda^2_1 e^{\pm H_N(p, x) / m\lambda^2_1}}.
\]
The remaining task is to determine the constants $\alpha_1$ and $\alpha_2$. We find that the appropriate choices are $\alpha_1 = \pm 1/m\lambda^2_1$ and $\alpha_2 = \pm m\lambda^2_2$, where $\lambda_1$ and $\lambda_2$ are in the velocity unit. Then we have
\[
H_2(p, x) \equiv H_{\lambda_1, \lambda_2}(p, x) = \pm m\lambda^2_2 e^{\lambda_2 - m\lambda^2_1}.
\]
The question why do we need two different $\lambda$-parameters in the Hamiltonian (2.21) can be answered by considering the limit on the $\lambda$-parameters. When $\lambda_2$ in $H_2$ is under the limit approaching to infinity, $H_2$ reduces to $H_1$ or the multiplicative Hamiltonian:
\[
\lim_{\lambda_2 \to \infty} (H_2(p, x) \mp m\lambda^2_2) = H_1(p, x).
\]
and $H_2$ can be further reduced to standard Hamiltonian by taking the limit on $\lambda_1$ in
\[
\lim_{\lambda_1 \to \infty} \lim_{\lambda_2 \to \infty} (H_2(p, x) \mp m\lambda^2_2 \mp m\lambda^2_1) = H_N(p, x).
\]
What we have now in (2.21) is the two $\lambda$-parameters extended class of the standard Hamiltonian or the one $\lambda$-parameter extended class of the multiplicative Hamiltonian, and it is not difficult to see that this new Hamiltonian also gives the same
equation of motion with the standard Hamiltonian as well as the multiplicative Hamiltonian.

We now extend the Hamiltonian (2.3) to the super-exponential function given by

$$H_j(p, x) \equiv H_{\alpha_1, \alpha_2, \ldots, \alpha_j}(p, x) = \sum_{j} \alpha_j e^{\alpha_j-1 e^{\alpha_j-2 e^{\alpha_1 F(p, x)}}}$$

where \((\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_j)\) are constants to be determined. Inserting (2.24) into (2.8), we obtain

$$0 = \frac{1}{m} \frac{\partial F}{\partial x} + \dot{p} \frac{\partial^2 F}{\partial p^2} + \frac{p}{m} \frac{\partial}{\partial p} \frac{\partial^2 F}{\partial p \partial x} + \sum_{j=1}^{j-1} \alpha_{h+1} \left( \prod_{i=1}^{h} \alpha_i g(i) \right) \left[ \dot{p} \left( \frac{\partial F}{\partial p} \right)^2 + \frac{p}{m} \frac{\partial F}{\partial p} \frac{\partial F}{\partial x} \right]$$

where the function \(g(i) = e^{\alpha_i g(i-1)}\) for \(i \geq 2\), and \(g(1) = e^{\alpha_1 F}\). Again, it requires that the last term must vanish resulting in the equations (2.7) and (2.8). Then the function \(F(p, x)\) takes exactly the same form as in the previous cases.

With the appropriate choices of \(\alpha_j = \pm 1/m \lambda_j^2\), where \(\lambda_j\) is in the velocity unit, the standard Hamiltonian and the Hamiltonian (2.24) form an infinite hierarchy given below

$$H_0(p, x) = H_N(p, x) = T(p) + V(x), \quad (2.26a)$$

$$H_1(p, x) = H_{\lambda_1}(p, x) = \pm m \lambda_1^2 e^{\pm \frac{H_0(p, x)}{m \lambda_1^2}}, \quad (2.26b)$$

$$H_2(p, x) = H_{\lambda_1, \lambda_2}(p, x) = \pm m \lambda_2^2 e^{\pm \frac{H_1(p, x)}{m \lambda_2^2}} = \pm m \lambda_2^2 e^{\pm \frac{H_0(p, x)}{m \lambda_2^2}}, \quad (2.26c)$$

$$\vdots$$

$$H_j(p, x) = H_{\lambda_1, \lambda_2, \ldots, \lambda_j}(p, x) = \pm m \lambda_j^2 e^{\pm \frac{H_{j-1}(p, x)}{m \lambda_j^2}} = \pm m \lambda_j^2 e^{\pm \frac{H_0(p, x)}{m \lambda_j^2}} \quad (2.26d)$$

which is called the Cabbatonian\(^1\) To recover the standard Hamiltonian \(H_0(p, x)\) from the Cabbatonian \(H_j(p, x)\), a series of the limit on \(\lambda\)-parameters is considered

$$\prod_{j=1}^{j} \lim_{\lambda_j \to \infty} \left( H_j(p, x) \pm \sum_{k=1}^{j} m \lambda_k^2 \right) = H_0(p, x), \quad (2.27)$$

where

$$\prod_{j=1}^{j} \lim_{\lambda_j \to \infty} = \lim_{\lambda_1 \to \infty} \lim_{\lambda_2 \to \infty} \lim_{\lambda_3 \to \infty} \ldots \lim_{\lambda_{j-1} \to \infty} \lim_{\lambda_j \to \infty}. \quad (2.28)$$

\(^1\)The word Cabbatonian results from the combination between Cabbage (a multi-layered vegetable) and Hamiltonian.
Furthermore, if the Cabbatonian $H_j(p, x)$ is expanded with respect to the parameter $\lambda_j$ the infinite series is obtained

$$H_j(p, x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \pm \frac{1}{m\lambda_j^2} \right)^{k-1} H_{j-1}^k(p, x), \quad \text{where } j = 1, 2, \ldots$$

(2.29)

which gives another Hamiltonian hierarchy $\{H_{j-1}, H_j^1, \ldots, H_j^k, \ldots\}$, see also [1]. Again it is not difficult to check that the Cabbatonian (2.29) give the same equation of motion with the standard Hamiltonian.

**Remark 3:** If we now define new parameters $\Omega_{j+1} \equiv \pm m\lambda_{j+1}^2$, the Cabbatonian can be written in a form,

$$H_{j+1}(p, x) = \Omega_{j+1} e^{H_j(p, x)} \Omega_{j+1}, \quad \text{where } j = 0, 1, 2, \ldots,$$

(2.30)

which forms an iterated exponential map with the given initial value $H_0(p, x)$. Then the Hamiltonians $\{H_0, H_1, \ldots, H_j, \ldots\}$ in (2.26) can be treated as variables for the dynamical system (2.30).

**Remark 4:** In [1], the Hamiltonian (2.26) was shown to be the generating function

$$H_1 = \sum_{k=0}^{\infty} a_k^1(\lambda_1) H_0^k, \quad \text{where } a_k^1(\lambda_1) = \frac{1}{k!} \left( -\frac{1}{m\lambda_1^2} \right)^{k-1} \text{ and } H_0^k = (T + V)^k,$$

(2.31)

for Hamiltonian hierarchy $\{H_0^0, H_0^1, H_0^2, H_0^3, \ldots\}$. The first six Hamiltonians are given by

$$
\begin{align*}
H_0^0 &= (T + V)^0 = 1 \\
H_0^1 &= (T + V)^1 = T + V \\
H_0^2 &= (T + V)^2 = T^2 + 2TV + V^2 \\
H_0^3 &= (T + V)^3 = T^3 + 3T^2V + 3TV^2 + V^3 \\
H_0^4 &= (T + V)^4 = T^4 + 4T^3V + 6T^2V^2 + 4TV^3 + V^4 \\
H_0^5 &= (T + V)^5 = T^5 + 5T^4V + 10T^3V^2 + 10T^2V^3 + 5TV^4 + V^5
\end{align*}
$$

We observe that the coefficients of $T$ and $V$ form the well known structure called the Pascal triangle given in figure 1(a). Furthermore, it is also well known that the Pascal triangle processes the fractal structure. It is automatically known that the Pascal triangle possesses the fractal structure by shading all odd numbers resulting in the Sierpinski’s triangle given in figure 1(b). Next the Cabbatonian (2.26) can also be written in the form

$$H_j = \sum_{k=0}^{\infty} a_j^k(\lambda_1, \lambda_2, \ldots, \lambda_j) H_0^k,$$

(2.32)

where the coefficient $a_j^k$ depends on $\{\lambda_1, \lambda_2, \ldots, \lambda_j\}$, e.g. $j = 1$, we get (2.31). Then obviously the Cabbatonian (2.32) processes the Pascal triangle structure and possibly the hidden fractal structure through the Sierpinski’s triangle given in figure 1.
Figure 1: (a) The Pascal triangle obtained by expanding \((T + V)^k\), where \(k = 0, 1, 2, \ldots\).
(b) The Sierpinski’s triangle obtained from the Pascal triangle by shading all odd numbers.

### 2.4 Lagrangian hierarchy

The Lagrangian hierarchy associated with the Cabbatonian can be obtained by using the Legendre transformation.

\[
L_j(x, \dot{x}) = p_j \dot{x} - H_j(p, x), \quad \text{where} \quad p_j = \frac{\partial L_j}{\partial \dot{x}}. \quad (2.33)
\]

We find that for \(j = 0\) we obtain the standard Lagrangian is \(L_0(\dot{x}, x) = T(\dot{x}) - V(x)\) with the momentum variable \(p_0 = m \dot{x}\). Here the kinetic energy \(T(\dot{x}) = m \dot{x}^2 / 2\). For \(j = 1\), we obtain the Lagrangian

\[
L_1(\dot{x}, x) = m \lambda^2_1 \left[ e^{-\frac{E(\dot{x}, x)}{m \lambda^2_1}} + \frac{\dot{x}}{\lambda^2_1} \int_0^{\dot{x}} e^{-\frac{E(q, x)}{m \lambda^2_1}} dq \right], \quad (2.34)
\]

which was explicitly derived in [1].

Consider for \(j = 2\), the momentum variable \(p_2\) can be derived by Hamilton’s equations, \(\partial H_2 / \partial x = -\dot{p}_2\) and it gives

\[
\dot{p}_2 = \frac{\partial H_2}{\partial \dot{x}} = -m \lambda^2_2 e^{\frac{\lambda^2_1}{\lambda^2_2}} \left( \frac{\lambda^2_1}{\lambda^2_2} \right) e^{-\frac{H_0}{m \lambda^2_1}} \left( -\frac{1}{m \lambda^2_1} \frac{dV}{dx} \right)
\]

\[
\Rightarrow \dot{p}_2 = \dot{p}_0 e^{-\frac{H_0}{m \lambda^1_1}} e^{\frac{\lambda^2_1}{\lambda^2_2}} e^{-\frac{H_0}{m \lambda^2_1}}. \quad (2.35)
\]

Using \(p_0 = m \dot{x}\), the momentum variable \(p_2\) can be directly from (2.35)

\[
p_2 = m \int_0^{\dot{x}} e^{-\frac{E(q, x)}{m \lambda^2_1}} e^{\frac{\lambda^2_1}{\lambda^2_2}} e^{-\frac{E(q, x)}{m \lambda^2_1}} dq, \quad (2.36)
\]

Substituting (2.36) into the Legendre transformation, we obtain

\[
L_2(\dot{x}, x) = m \lambda^2_2 \left[ e^{\frac{\lambda^2_1}{\lambda^2_2}} e^{-\frac{E(q, x)}{m \lambda^2_1}} + \frac{\dot{x}}{\lambda^2_2} \int_0^{\dot{x}} e^{-\frac{E(q, x)}{m \lambda^2_1}} e^{\frac{\lambda^2_1}{\lambda^2_2}} dq \right]. \quad (2.37)
\]
Applying the same steps, we obtain the rest of Lagrangians. Then we now have a Lagrangian hierarchy

\[ L_0(\dot{x}, x) = L_N(\dot{x}, x) = T(\dot{x}) - V(x), \]

\[ L_1(\dot{x}, x) = m\lambda_1^2 \left[ e^{\frac{E(\dot{x}, x)}{m\lambda_1^2}} \frac{\dot{x}}{\lambda_1^2} - \frac{E(\dot{x}, x)}{m\lambda_1^2} \int_0^{\dot{x}} e^{-\frac{E(q)}{m\lambda_1^2}} dq \right], \]

\[ L_2(\dot{x}, x) = m\lambda_2^2 \left[ e^{\frac{E(\dot{x}, x)}{m\lambda_2^2}} \frac{\dot{x}}{\lambda_2^2} - \frac{E(\dot{x}, x)}{m\lambda_2^2} \int_0^{\dot{x}} e^{-\frac{E(q)}{m\lambda_2^2}} \frac{\dot{x}^2}{\lambda_2^2} e^{\frac{E(q)}{m\lambda_2^2}} dq \right], \]

\[ L_3(\dot{x}, x) = m\lambda_3^2 \left[ e^{\frac{E(\dot{x}, x)}{m\lambda_3^2}} \frac{\dot{x}}{\lambda_3^2} - \frac{E(\dot{x}, x)}{m\lambda_3^2} \int_0^{\dot{x}} e^{-\frac{E(q)}{m\lambda_3^2}} \frac{\dot{x}^2}{\lambda_3^2} e^{\frac{E(q)}{m\lambda_3^2}} dq \right], \]

\[ \vdots \]

\[ L_j(\dot{x}, x) = m\lambda_j^2 \left[ e^{\frac{E(\dot{x}, x)}{m\lambda_j^2}} \frac{\dot{x}}{\lambda_j^2} - \frac{E(\dot{x}, x)}{m\lambda_j^2} \int_0^{\dot{x}} e^{-\frac{E(q)}{m\lambda_j^2}} \frac{\dot{x}^2}{\lambda_j^2} e^{\frac{E(q)}{m\lambda_j^2}} dq \right], \]

\[ \vdots \]

where

\[ E_j(\dot{x}, x) = e^{\frac{E_j-1(\dot{x}, x)}{m\lambda_j^2}} = e^{\frac{\lambda_j^2 - 2}{\lambda_j^2} e^{\frac{E(q)}{m\lambda_j^2}} \int_0^{\dot{x}} e^{-\frac{E(q)}{m\lambda_j^2}} \frac{\dot{x}^2}{\lambda_j^2} e^{\frac{E(q)}{m\lambda_j^2}} dq}. \]

Furthermore, we find that

\[ \lim_{\lambda_j \to \infty} \left( L_j(\dot{x}, x) - m\lambda_j^2 \right) = L_{j-1}(\dot{x}, x), \]

and to recover the standard Lagrangian \( L_0(\dot{x}, x) \), we consider

\[ \prod_{j=1}^{\infty} \lim_{\lambda_j \to \infty} \left( L_j(\dot{x}, x) - \sum_{k=1}^{j} m\lambda_k^2 \right) = L_0(\dot{x}, x). \]

We now successfully derive the Lagrangian hierarchy associated with the Cabbatonian \( (2.26) \).

### 2.5 More Hamiltonians

We are still wondering that “Are there more families of the Hamiltonian, producing the same equation of motion, to be explored systematically?” Then we set out to find more families of the Hamiltonian using the key equations \( (2.2) \). We now take an ansatz form of the Hamiltonian as

\[ H(p, x) = F(p, x)e^{aF(p, x)}, \]

where

\[ a = \frac{\lambda_1^2}{2} e^{\frac{E(q)}{m\lambda_1^2}} \int_0^{\dot{x}} e^{-\frac{E(q)}{m\lambda_1^2}} \frac{\dot{x}^2}{\lambda_1^2} e^{\frac{E(q)}{m\lambda_1^2}} dq. \]
where $a$ is a constant to be determined. Substituting (2.42) into (2.8), we obtain
\[
0 = \frac{1}{m} \frac{\partial F}{\partial x} + \dot{p} \left( \frac{\partial^2 F}{\partial p^2} + \left( \frac{\partial F}{\partial p} \right)^2 a \frac{2 + aF}{1 + aF} \right) + \frac{p}{m} \left( \frac{\partial^2 F}{\partial p \partial x} + \frac{\partial F}{\partial p} \frac{\partial F}{\partial x} a \frac{2 + aF}{1 + aF} \right)
\]
\[
= \frac{1}{m} \frac{\partial F}{\partial x} + \dot{p} \frac{\partial^2 F}{\partial p^2} + \frac{p}{m} \frac{\partial^2 F}{\partial p \partial x} + a \frac{2 + aF}{1 + aF} \left( \dot{p} \frac{\partial F}{\partial p} + \frac{\partial F}{\partial x} \right)
\]
(2.43)

What we have in (2.43) is similar to (2.5), again resulting in (2.7) and (2.8). If we now choose $F(p, x) = H_1(p, x)$ we have
\[
H_{\sigma_1}(p, x) = H_1(p, x) e^{-\frac{H_0(p, x)}{m\sigma_1^2}} e^{\frac{\lambda_1}{\sigma_1^2}},
\]
(2.44)
where the constant $a$ is chosen to be $-1/m\sigma_1^2$. The parameter $\sigma_1$ has the same unit with the parameter $\lambda_1$. We can choose to express the Hamiltonian (2.44) in terms of the Hamiltonian $H_0$ as
\[
H_{\sigma_1}(p, x) = -m\lambda_1^2 e^{-\frac{H_0(p, x)}{m\lambda_1^2}}.
\]
(2.45)

Next if we choose $F(p, x) = H_2(p, x)$ we have
\[
H_{\sigma_2}(p, x) = H_2(p, x) e^{-\frac{H_0(p, x)}{m\sigma_2^2}},
\]
(2.46)
where the constant $a$ is chosen to be $-1/m\sigma_2^2$. Again the parameter $\sigma_2$ is also in the velocity unit. We can write the Hamiltonian (2.46) in terms of the Hamiltonian $H_0$ as
\[
H_{\sigma_2}(p, x) = -m\lambda_2^2 e^{-\frac{H_0(p, x)}{m\lambda_2^2}} e^{\frac{\lambda_2}{\sigma_2^2}},
\]
(2.47)

Then if we choose $F(p, x) = H_j(p, x)$ we have
\[
H_{\sigma_j}(p, x) = H_j(p, x) e^{-\frac{H_0(p, x)}{m\sigma_j^2}},
\]
(2.48)
where $\sigma_j$ is in velocity unit. The structure of (2.44), (2.46) and (2.48) forms a new hierarchy of the Hamiltonians given by
\[
H_{\sigma_0}(p, x) = H_0 e^{-\frac{H_0(p, x)}{m\sigma_0^2}},
\]
(2.49a)
\[
H_{\sigma_1}(p, x) = H_1 e^{-\frac{H_1(p, x)}{m\sigma_1^2}} = -m\lambda_1^2 e^{-\frac{H_0(p, x)}{m\lambda_1^2}} e^{\frac{\lambda_1}{\sigma_1^2}},
\]
(2.49b)
\[
H_{\sigma_2}(p, x) = H_2(p, x) e^{-\frac{H_2(p, x)}{m\sigma_2^2}} = -m\lambda_2^2 e^{-\frac{\lambda_2}{\sigma_2^2}} e^{\frac{\lambda_2}{\sigma_2^2}},
\]
(2.49c)
\[
\vdots
\]
\[
H_{\sigma_j}(p, x) = H_j(p, x) e^{-\frac{H_j(p, x)}{m\sigma_j^2}} = -m\lambda_j^2 e^{-\frac{\lambda_{j-1}}{\sigma_{j-1}^2}} e^{\frac{\lambda_{j-1}}{\sigma_{j-1}^2}} e^{\frac{\lambda_j}{\sigma_j^2}},
\]
(2.49d)
\[
\vdots
\]
Here for this new hierarchy of the Hamiltonians, we have two limits to consider. The first one is that if the \( \sigma_j \) approaches to infinity: 

\[
\lim_{\sigma_j \to \infty} H_{\sigma_j}(p, x) = H_j(p, x)
\]

we recover the Cabbatonian (2.26). The second limit is on parameter \( \lambda_j \) such that 

\[
\lim_{\lambda_j \to \infty} H_{\sigma_j}(p, x) = -m\lambda_j^2 + e^{\lambda_j^2 - 1} H_{\sigma_j-1}(p, x),
\] (2.50)

where \( \sigma_j \) on the right-hand-side is replaced by \( \sigma_j - 1 \) in order to obtain \( H_{\sigma_j-1}(p, x) \).

What we have in (2.49) is a bigger family of the Hamiltonians involving many extra-parameters \( \{\sigma_j, \lambda_j\} \) and obviously the Cabbatonian is a special case in this family.

3 The harmonic oscillator

In this section, we would like to examine the role of the parameter \( \lambda \) presenting in the multiplicative Hamiltonian (1.6) and Lagrangian (1.4) through a concrete example, namely the simple harmonic oscillator. We knew that the standard Hamiltonian is given by 

\[
H_0(p, x) = \frac{p^2}{2m} + \frac{kx^2}{2}.
\] (3.1)

Then multiplicative Hamiltonian reads 

\[
H_\lambda(p, x) = m\lambda^2 e^{\frac{1}{m\lambda^2}} \left( \frac{p^2}{2m} + \frac{kx^2}{2} \right).
\] (3.2)

The infinite hierarchy is given by 

\[
H_\lambda(p, x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{m\lambda^2} \right)^{k-1} H^k_0, \quad \text{where} \quad H^k_0 = \left( \frac{p^2}{2m} + \frac{kx^2}{2} \right)^k.
\] (3.3)

Now let \( \eta = (x, p) \) and then we consider 

\[
\frac{d\eta}{dt_\lambda} = J \frac{\partial H_\lambda}{\partial \eta},
\] (3.4)

where \( t_\lambda \) is a time variable associated with the Hamiltonian \( H_\lambda \) and \( \partial / \partial \eta = (\partial / \partial x, \partial / \partial p) \). The \( J \) is the symplectic matrix given by 

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\] (3.5)

Inserting (3.3) into (3.4), we obtain 

\[
\frac{d\eta}{dt_\lambda} = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{m\lambda^2} \right)^{k-1} J \frac{\partial H^k_0}{\partial \eta} = \sum_{k=1}^{\infty} \frac{d\eta}{dt_k},
\] (3.6)

where 

\[
\frac{d}{dt_k} = \frac{E^{k-1}}{(k-1)!(m\lambda^2)^{k-1}} \frac{d}{dt_0}, \quad k = 1, 2, 3, \ldots,
\] (3.7)

\(^2\)For simplicity, we work with the case of one parameter.
where $E = T + V$ is the total energy of the system. Here $t_0 (= t_1)$ is the standard time variable associated with the standard Hamiltonian $H_0$. Equation (3.6) tells us that the $\lambda$-flow is constituted of infinite different flows on the same trajectory on phase space, see figure 2. This means that we can choose any Hamiltonian in the hierarchy to study the system of harmonic oscillator resulting in the same physics, but with different time scale. Then we may say that the parameter $\lambda$ plays a role of scaling in the Hamiltonian flow on phase space. We also observe that if $m\lambda^2 \to \infty$, all extra-flows will be terminated and only the standard flow $d/dt_0$ survives. This is consistent with the condition $\lim_{\lambda \to \infty} H_\lambda + m\lambda^2 = H_0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{The $\lambda$-flow is constituted of infinite different flows on the same trajectory on phase space.}
\end{figure}

Now we need to explore more a little bit on Lagrangian point of view. The standard Lagrangian for the harmonic oscillator is given by

$$L_0(\dot{x}, x) = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}. \quad (3.8)$$

The multiplicative Lagrangian (1.4) becomes

$$L_\lambda(\dot{x}, x) = m\lambda^2 \left[ \frac{E(\dot{x}, x)}{m\lambda^2} + \frac{\dot{x}}{\lambda^2} \int_0^\dot{x} \frac{E(q, x)}{m\lambda^2} dq \right], \quad (3.9)$$

where $E(\dot{x}, x) = m\dot{x}^2/2 + kx^2/2$ is the total energy of the system. We know that the Lagrangian (3.9) can be rewritten in the form

$$L_\lambda(\dot{x}, x) = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{1}{m\lambda^2} \right)^{k-1} L_k(\dot{x}, x), \quad (3.10)$$

where

$$L_k(\dot{x}, x) = \sum_{j=0}^{k} \left( \frac{k!(m\dot{x}^2/2)^{k-j}(kx^2/2)^j}{(k-j)!(2j-(2j+1))} \right). \quad (3.11)$$

The action of the system is given by

$$S[\dot{x}] = \int_0^T dt L_\lambda(\dot{x}, x) = \sum_{k=1}^{\infty} \int_0^T dt_k L_k(\dot{x}, x), \quad (3.12)$$
where
\[ dt_k = \frac{1}{k!} \left( \frac{1}{m\lambda^2} \right)^{k-1} dt_0. \]  
(3.13)

We then perform the variation: \( x \to x + \delta x \) with end-point conditions: \( \delta x(0) = \delta x(T) = 0 \), resulting in
\[ \delta S = \sum_{k=1}^{\infty} \int_0^T dt_k \left( \frac{\partial L_k}{\partial x} - \frac{d}{dt_k} \left( \frac{\partial L_k}{\partial x_k} \right) \right), \]
(3.14)
where \( x_k = dx/dt_k \). Least action principle, \( \delta S = 0 \), gives infinite copies of the Euler-Lagrangian equation
\[ \frac{\partial L_k}{\partial x} - \frac{d}{dt_k} \left( \frac{\partial L_k}{\partial x_k} \right) = 0 \Rightarrow \frac{d^2}{dt_k^2} x = -\frac{k}{m} x, \quad k = 1, 2, 3, \ldots, \]
(3.15)
associated with different time variables. Again in this case, we have the same structure of equation of motion for each Lagrangian, but with different time scale and of course, if \( \lambda \to \infty \), only the standard flow survives resulting in
\[ \frac{d^2}{dt_0^2} x = -\frac{k}{m} x. \]
(3.16)

From structure of (3.12) and (3.15), the parameter \( \lambda \) plays also a role of time scaling parameter in the Lagrangian context.

**Remark 5:** We see that there are many forms of the Hamiltonian that you can work with. One may start with the assumption that any new Hamiltonian can be written as a function of the standard Hamiltonian \( H_N : H = f(H_0) \). Inserting this new Hamiltonian into the Hamilton’s equations, we obtain
\[ f'(E) \frac{\partial H}{\partial x} = -\frac{\partial p}{\partial \tau}, \quad f'(E) \frac{\partial H}{\partial p} = -\frac{\partial x}{\partial \tau} \]
(3.17)
where \( f'(E) \equiv df(H_0)/H_0 \) with fixing \( H_0 = E \) and \( t = f'(E)\tau \) is the rescaling of time parameter. This result agrees with what we have in this section, rescaling the time evolution of the system. However, there are some major different features as follows. The first thing is that our new Hamiltonians contain a parameter \( \lambda \), since the explicit forms of the Hamiltonian are obtained. With this parameter, it makes our rescaling much more interesting with the fact that the rescaling time variables depends on also the parameter \( \lambda \), see (3.7). Then it means that we know how to move from one scale to another scale and of course we know how to obtain the standard time evolution by playing with the limit of the parameter \( \lambda \). Without explicit form of the new Hamiltonian, which contains a parameter, we cannot see this fine detail of family of rescaling time variables, since there is only a fixed parameter \( E \) in (3.17). The second thing is that actually the new Hamiltonian (3.2), which is a function of the standard Hamiltonian, can be obtained from the Lagrangian (3.9) by means of Legendre transformation. What we have seen is that Lagrangian (3.9) is nontrivial and is not a function of the standard Lagrangian. Again this new Lagrangian contains a parameter \( \lambda \), the same with the one in the new Hamiltonian. With this parameter, the Lagrangian hierarchy (3.11) is obtained. What we have here is a family of nontrivial Lagrangians to work with, producing the same equation
of motion, as a consequence of non-uniqueness property. An important thing is that there is no way you can guess the form of this family of Lagrangian without the mechanism in [1]. This means that with the Hamiltonian in the form $H = f(H_0)$ cannot deliver all these fine details. The explicit form of the Hamiltonian (3.2) allows us to study deep in more details and is definitely richer than the standard one.

4 Concluding summary

We have found that there are infinite ways to express the Hamiltonian for the systems with one degree of freedom without altering the equation of motion. These alternative Hamiltonians or newton-equivalent Hamiltonians are obtained from (2.2) by using technique of separation of variables and they come with extra-parameters. What we have here in the present paper is a zoo of the Hamiltonians and obviously points out that Hamiltonian is also not unique. Then we come to the conclusion that for every function $F' = -dV(x)/dx$ there exist infinite Hamiltonians of equation

$$F' - \dot{p} = \dot{p} \frac{\partial H}{\partial p} + \frac{p}{m} \frac{\partial H}{\partial x}.$$ (4.1)

However, we believe that there are more families of the Hamiltonian to be systematically solved from (2.8). The most important question right now is that why does nature provide us such a huge variety of Hamiltonians even for just the system with one degree of freedom? We do not have a good answer for this at the moment, but we hope that we could come up with the resolution soon. One may also ask why do we need the Hamiltonian Zoo, since the standard Hamiltonian is much more simpler to work with. We totally agree with that because these new Hamiltonians are indeed complicated, but at least they provide us a new interpretation for the time evolution of the system arising from existing of extra-parameters as time scaling parameters, see also [2]. Moreover, these Hamiltonian is richer than the standard one in the sense that they contain more parameters to play with.

In the case of many degrees of freedom, the problem turns out to be very difficult. Even in the case of 2 degrees of freedom, the problem is already hard to solve from scratch. We may start with an anzast form of the Lagrangian: $L(\dot{x}, \dot{y}, x, y) = F(\dot{x}, \dot{y})G(x, y)$. This difficulty can be seen from the fact that we have to solve a non-separable coupled equation. A mathematical trig or further assumptions might be needed for solving $F(\dot{x}, \dot{y})$ and $G(x, y)$. The investigation is now monitored.

Furthermore, promoting the Hamiltonian (1.6) to be a quantum operator in the context of Schordingers equation is also an interesting problem. This seems to suggest that alternative form of the wave function for a considering system is possibly obtained. This can be seen as a result from that fact that with new Hamiltonian operator we need to solve different eigenvalue equation and of course a new appropriate eigenstate is needed. From the Lagrangian point of view, extension to the quantum realm in the context of Feynman path integrals is quite natural to address. However, this problem is not easy to deal with since the multiplication Lagrangian is not in the quadratic form. Then a common procedure for deriving the propagator is no longer applicable. Further study is on our program of investigation.
Acknowledgements

The authors would like to thank Dr. Ekapong Hirirunsirisawat for pointing out the structure of the Cabbatonian at the early state of this work. Sikarin Yoo-Kong was supported by Computational and Applied Science for Smart Innovation Cluster (CLASSIC), Faculty of Science, King Mongkut's University of Technology Thonburi. Saksilpa Srisukson was supported by National Science and Technology Development Agency (NSTDA) under the Junior Science Talent Project (JSTP).

References

[1] Surawuttinack, K., Yoo-Kong, S., Tanasittikosol, M. (2016), On the multiplicative form of the Lagrangian, Mathematical and Theoretical Physics, 189, pp. 1693-1711
[2] Srisukson, S., Surawuttinack, K., Yoo-Kong, S. (2017), The multiplicative Hamiltonian and its hierarchy, Journal of Physics: Conference Series (Siam Physics Congress 2017), 901, 012167
[3] Sonin N J. (1886), About determining maximal and minimal properties of plane curves (in Russian), Warsawskye Universitetskye Izvestiya, 12, 168; English translation, Lepage Inst. Archive, No. 1 (2012).
[4] Douglas J. (1939), Solution of the inverse problem of the calculus of variations, Douglas, J. Solution of the Inverse Problem of the Calculus of Variations Proceedings of the National Academy of Sciences of the United States of America vol. 25(12), pp. 631-637.
[5] Krupka D. (2015), The Sonin-Douglas problem, The inverse problem of the calculus of variations, Springer, pp. 31-73.
[6] Degasperis A. and Ruijsenaars S. (2001), Newton-Equivalent Hamiltonians for the Harmonic Oscillator, Annals of Physics, 293, pp. 92-109.
[7] Nucci C M. and Tamaizhmani M K. (2010), Lagrangian for dissipative nonlinear oscillators: the method of Jacobi last multiplier, Journal of nonlinear mathematical physics, 17(02), pp. 167-178.
[8] Saha A. and Talukdar B. (2013), On the non-standard Lagrangian equations, arXiv:1301.2667
[9] Musielak E Z. (2008), Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients, Journal of Physics A: Mathematical and Theoretical, 41, 055205.
[10] El-Nabulsi A R. (2013), Nonstandard Lagrangian cosmology, Journal of Theoretical and Applied Physics, 7, 58.
[11] El-Nabulsi A R. (2015), Non-standard power-law Lagrangians in classical and quantum dynamics, Applied Mathematics Letters, 43, pp.120-127.
[12] El-Nabulsi A R. (2013), Nonstandard fractional exponential Lagrangians, fractional geodesic equation, complex general relativity and discrete gravity, Canadian Journal of Physics, 91(8), pp. 618-622.
[13] Renyi A. (1960), On measure of information and entropy, in Proceedings of the 4th Berkeley Symposium on Mathematics, Statistic and Probability, pp. 547-561.