Bianchi V inflation in the Brans-Dicke theory?

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Abstract

It is shown some exact solutions in the Brans-Dicke (BD) theory for a Bianchi V metric having the property of inflationary expansion, graceful exit, and asymptotic evolution to a Friedmann-Robertson-Walker (FRW) open model. It is remarkable that an inflationary behaviour can occur, even without a cosmological potential or constant. However, the horizon and flatness problems cannot be solve within the standard BD theory because the inflationary period is severely restricted by the value of the BD parameter $\omega$. 
1 INTRODUCTION

It is well known that the Universe is homogeneous and isotropic on very large scales. Supporting this assertion are different measurements as the isotropy measured in the Cosmic Microwave Background Radiation (CMBR) by the Cosmic Background Explorer (COBE) satellite [1]; isotropy in x-ray backgrounds (e.g., quasars at high red shift); and in number counts in faint radio sources, cf. Ref. [2]. All these measurements give evidence for the cosmological principle, which consistently should be valid on very large scales of our Universe. However, on smaller scales there are a variety of inhomogeneities and anisotropies that have been produced by growth of matter perturbations, at least since the time of last scattering until now. Accordingly, many authors have tried to model the early Universe with non-FRW metrics allowing general initial conditions to explain, after its evolution, small and large scale properties of our Universe. Thus, the chaotic cosmology programme [3] tried, unsuccessfully, to isotropize anisotropic cosmologies in order to understand, within general footings, the present large scale isotropy. Collins and Hawking [4] proved, within General Relativity (GR), that if the dominant energy condition and positive pressure criterion are satisfied, the universe can approach isotropy only if it is one of the types I, V, VII0, and VIIh. However, at finite times other anisotropic models can be bounded from above [5] and even they are consistent with COBE measurements [6], that is, anisotropic cosmologies do not need inflationary scenarios to explain the observed $\Delta T/T \approx 10^{-5}$ in the CMBR. However, a causal origin of the perturbations is not provided by anisotropic cosmologies alone, and inflationary cosmologies do provide such an explanation, up to some fine tuning. This is one reason why inflation is still the most interesting proposal to the standard model of cosmology.

The question of the entry to an inflationary phase, now known as the cosmic no hair conjecture [7], as well as the question of its exit (graceful exit problem) are two key aspects of inflation that have been intensively discussed. No hair theorems have been proved in the context of GR within anisotropic and some inhomogeneous cosmologies [8]. The entry to inflation has been also discussed in some other contexts, including tilt matter models [9], and some anisotropic models in scalar tensor theories [10]. In general, one has realized that for some set of initial conditions there exists an inflationary attractor for the evolution. However, the exit of inflation is not necessarily guaranteed when a cosmological constant ($\Lambda$) is present. Instead of $\Lambda$ one has been considering a cosmological function $V(\phi)$ that plays the role of a constant during some time $\tau = NH^{-1}$, where $N$ is the number of e-foldings of exponential expansion and $H \approx \text{const.}$ the Hubble parameter. Adding a properly chosen constant to $V(\phi)$ one gets after inflation that $V(\phi_f) \approx 0$, then the Universe

\footnote{So fine tuned to avoid having a cosmological constant nowadays ($n$) bigger than $8\pi G\rho_n = \ldots$}
preheats, reheats, and experiences a metamorphosis to a FRW dynamics filled with a particle content characterized by the decay of the $\phi$–particles. The introduction of a constant in $V(\phi)$, to allow inflation to occur and to graceful exit, is well motivated but it is put by hand. In this sense, it would be desirable to have a model in which both features, the entry and exit of inflation, arise naturally, as a consequence of the exact, integrating dynamics. In GR, however, one needs to introduce a cosmological constant (or function) to generate inflation. In scalar-tensor theories this term appears naturally, but to obtain successful inflation it implies the fine tuning of parameters that not always are in accord with the particle physics models behind.

Recently, effort has been put on the analysis of FRW models in the Brans-Dicke (BD) theory, which is simplest gravity theory beyond GR. In Refs. a qualitative analysis of the general evolution and asymptotic behaviour is studied, as well as the conditions for inflation even without a cosmological function or constant, see also Ref. . Motived by these analyses we study in the present work Bianchi V anisotropic solutions in the BD theory, because it is one of the simplest step further in complexity and permits the analysis of the anisotropic properties of models; in particular the isotropization of some solutions has been studied in Refs. . We found that Bianchi V models can have the properties of inflation (without a potential or a cosmological constant), graceful exit of the inflationary stage, and evolution to an effective FRW open Universe, like the one we observe nowadays. The range of values for the coupling constant $\omega$ is, however, very different from what one would desire, i.e. $\omega > 500$.

In the next section we show the analytic solutions and, thereafter, in section we analyse their behaviour. In section some comments are made on fluctuations of the BD field in these models. Finally, section is devoted to conclusions.

## 2 BRANS-DICKE BIANCHI V SOLUTIONS

The BD theory has the following Einstein-Hilbert action, with signature $(+,-,-,-)$:

$$
\mathcal{L} = \left( \phi R + \frac{\omega}{\phi} \phi_{\mu} \phi^{\mu} + 16\pi L_M \right) \frac{\sqrt{-g}}{16\pi} 
$$

(1)

where $R$ is the Ricci scalar, $\phi$ the scalar field, the symbol $|$ partial derivative, $\omega$ the coupling constant of the theory, and $g$ the determinant of the metric tensor. After

$$3H_n^2 \sim 10^{-83}\text{GeV}^2.$$
varying this equation one derives the BD field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi}{\phi} T_{\mu\nu} - \frac{\omega}{\phi^2} \left[ \phi_{\mu\nu} \phi_{\nu\lambda} - \frac{1}{2} \phi_{\mu\lambda} \phi_{\nu\lambda} g_{\mu\nu} \right] - \frac{1}{\phi} \left[ \phi_{\mu||\nu} - \phi_{\lambda||\lambda} g_{\mu\nu} \right] \]

(2)

and

\[ \phi_{||\lambda} = \frac{8\pi}{3 + 2\omega} T, \]

(3)

where the symbol || stands for the covariant derivative and \( T \) is the trace of the energy-momentum tensor, \( T_{\mu\nu} \). The continuity equation (energy-momentum conservation law) reads

\[ T_{\mu}^{\nu} ||_{\nu} = 0. \]

(4)

We consider homogeneous, anisotropic Bianchi-type models to study the dynamics, and particularly a Bianchi type V spacetime symmetry in a synchronous coordinate frame given by the line element \([19]\):

\[ ds^2 = dt^2 - a_1^2 dx^2 - a_2^2 e^{-2x} dy^2 - a_3^2 e^{-2x} dz^2. \]

(5)

Following, we shall use scaled variables to find and analyse the solutions in a simple way: the scaled field \( \psi \equiv \phi a^{3(1-\nu)} \), a new cosmic time parameter \( d\eta \equiv a^{-3\nu} dt \), \( \psi' \equiv \frac{d\psi}{d\eta} \), the ‘volume’ \( a^3 \equiv a_1 a_2 a_3 \), and the Hubble parameters \( H_i \equiv \frac{a_i'}{a_i} \) corresponding to the scale factors \( a_i = a_i(\eta) \) for \( i = 1, 2, 3 \). We assume comoving coordinates and a barotropic equation of state for the perfect fluid present \( (T_{\mu\nu}) \), \( p = \nu \rho \), with \( \nu \) a constant. Using these definitions and the above given metric, one obtains the cosmological equations from Eqs. (2) and (3):

\[ (\psi H_i)' - \psi a^{6\nu} C_{iV} = \frac{8\pi}{3 + 2\omega} [1 + (1 - \nu)\omega] \rho a^{3(1+\nu)} \] for \( i = 1, 2, 3 \). \]

(6)

\[ H_1 H_2 + H_1 H_3 + H_2 H_3 + [1 + (1 - \nu)\omega] (H_1 + H_2 + H_3) \frac{\psi'}{\psi} \]

\[ -(1 - \nu)[1 + (1 - \nu)/2](H_1 + H_2 + H_3)^2 - \omega \left( \frac{\psi'}{\psi} \right)^2 - \frac{C_{1V}}{2} a^{6\nu} \]

\[ = 8\pi \frac{\rho a^{3(1+\nu)}}{\psi}, \]

(7)

\[ \psi'' + (\nu - 1)a^{6\nu} C_{V} \psi = \frac{8\pi}{3 + 2\omega} [2(2 - 3\nu) + 3(1 - \nu)^2\omega] \rho a^{3(1+\nu)} \],

(8)
and

\[ H_2 + H_3 = 2H_1 , \] (9)

where \( C_V \equiv \Sigma_i C_{iV} \) with the partial the curvature terms equal to each other, i.e., \( C_{iV} = 2/a_i^2 \) for \( i = 1, 2, 3 \). Eq. (9) implies that \( a_2 \) and \( a_3 \) are inverse proportional functions, \( a_2 a_3 = a_1^2 \); note that \( 3H \equiv H_1 + H_2 + H_3 = 3H_1 \), \( H \) being the mean Hubble parameter.

Additionally, the continuity equation yields:

\[ \rho a^{3(1+\nu)} = \text{const.} \equiv M_\nu \] (10)

\( M_\nu \) being a dimensional constant depending on the fluid present. The vacuum case is attained when \( M_\nu = 0 \).

The system of ordinary differential equations, Eqs. (6-9), can be decoupled to obtain an equation for \( \psi \) alone, by means of the relation \( a_1 = a \), which can be once integrated to get:

\[
\psi \psi'' - \frac{2}{3(1-\nu)} \psi'^2 - \frac{2(1-3\nu)}{3(1-\nu)} [m_\nu (1-3\nu)\eta + \eta_0] \psi' + \left[ 2 + (1-\nu)(1+3\nu)\omega \right] m_\nu \psi + \frac{2(1-3\nu)}{3(1-\nu)} [2(2-3\nu) + 3(1-\nu)^2 \omega] \left[ \frac{m_\nu}{2} (1-3\nu) \eta^2 + \eta_0 \eta + \eta_1^2 \right] m_\nu = K , \] (11)

where \( \eta_0 \), \( \eta_1 \), and \( K \) are integration constants; \( m_\nu \equiv \frac{8\pi M_\nu}{3+2\omega} \). Solutions to Eq. (11) are to be used to get the mean Hubble parameter through the following equation

\[ H_1 + H_2 + H_3 = \frac{1}{1-\nu} \left[ \frac{\psi'}{\psi} - \frac{(1-3\nu)m_\nu \eta + \eta_0}{\psi} \right] , \] (12)

obtained by using Eqs. (5) and (8). Additionally, each Hubble rate is written as follows (similar to the Bianchi type I model Ref. [20]):

\[ H_i = \frac{1}{3} (H_1 + H_2 + H_3) + \frac{h_i}{\psi} = \frac{\psi' - (1-3\nu)m_\nu \eta - \eta_0 + 3(1-\nu)h_i}{3(1-\nu)\psi} , \] (13)

where the \( h_i \)’s are constants that determine the anisotropic character of the solutions. If \( h_i = 0 \) for \( i = 1, 2, 3 \) simultaneously, no anisotropy is present. Furthermore, the Bianchi V model obeys the condition

\[ h_1 + h_2 + h_3 = 0, \quad \text{with} \quad h_1 = 0 , \] (14)

the last relation stems from the fact that \( H_1 = H \), see Eq. (3).

In order to analyse the anisotropic character of the solutions, we have constructed the following ‘constraint’ equation for the anisotropic shear, \( \sigma \):

\[ \sigma(\eta) \equiv -(H_1 - H_2)^2 - (H_2 - H_3)^2 - (H_3 - H_1)^2 = \]
\[
\frac{3}{2(1-\nu)} \left( \frac{\psi''}{\psi} \right) - \frac{1}{(1-\nu)^2} \left( \frac{\psi'}{\psi} \right)^2 - \frac{(1-3\nu)}{(1-\nu)^2} \left( \frac{(1-3\nu)m_\nu \eta + \eta_0}{\psi} \right) \left( \frac{\psi'}{\psi} \right) + \frac{[2-3\nu + \frac{3}{2}\omega (1-\nu)^2]}{(1-\nu)^2} \left( \frac{(1-3\nu)m_\nu \eta + \eta_0}{\psi} \right)^2 + \frac{3[2 + \omega (1-\nu)(1+3\nu)]m_\nu}{2(1-\nu)\psi}.
\]

Equation (15)

\(\sigma = 0\) is a necessary condition to obtain a FRW cosmology since it implies \(H_1 = H_2 = H_3\), cf. Ref. [21, 16]. If the sum of the squared differences of the Hubble expansion rates tends to zero, it would mean that the anisotropic scale factors tend to a single function of time which is, certainly, the scale factor of the open Friedmann model.

The anisotropic shear becomes, using Eq. (14),

\[
\sigma(\eta) = -\frac{6h^2}{\psi^2},
\]

Equation (16)

or the dimensionless shear parameter [22]

\[
\frac{\sigma}{H^2} = -\frac{54(1-\nu)^2 h^2}{[\psi' - (1-3\nu)m_\nu \eta - \eta_0]^2}.
\]

Equation (17)

These equations admit solutions such that \(\sigma \to 0\), \(\sigma/H^2 \to 0\) as \(\eta \to \infty\) (or \(t \to \infty\)), that is, one has time asymptotic isotropization solutions, similar to the solutions found for the Bianchi V model in GR, see Ref. [5]. In fact, one does not need to impose an asymptotic, infinity condition, but just that \(\eta \gg \eta_0\), where \(\eta_0\) is yet some arbitrary value to warrant that \(\sigma (\sigma/H^2)\) can be bounded from above.

The above equations are valid for any value \(\nu\) of the equation of state, but the solutions are different for \(\nu \neq \frac{1}{3}\) and \(\nu = \frac{1}{3}\). Therefore, we treat both cases separately.

### 2.1 Solutions with \(\nu \neq \frac{1}{3}\)

The explicit solutions are following given. A particular solution of Eq. (11) is [16]:

\[
\psi = A_V \eta^2 + B_V \eta + C_V,
\]

Equation (18)

with the constants

\[
A_V = -\frac{(1-3\nu)^2}{(1+3\nu)} m_\nu,
\]

\[
B_V = -2 \left( \frac{1-3\nu}{1+3\nu} \right) \eta_0,
\]

Equation (19)

being the same as for the isotropic, \(k = \pm 1\) cases [13], but now

\[
m_\nu (1+3\nu) C_V = -\frac{(1+3\nu)^2(h_1^2 + h_2^2 + h_3^2)}{18\nu+\omega(1+3\nu)^2} - \eta_0^2.
\]

Equation (20)
The constants $B_V$ and $C_V$, being proportional to the $h^i$s, encode information on the anisotropic shear, that is, on the nature of the anisotropic character of this Bianchi type model. The constants $\eta_1$ and $K$ of Eq. (14) are determined through Eq. (20) but they have no further significance.

The Hubble expansion rates, given through Eqs. (13, 18, 19), are

$$H_i(\eta) = -\frac{1}{1 + 3\nu} \frac{(1 - 3\nu)m_\nu \eta + \eta_0 - (1 + 3\nu)h_i}{(A_\nu \eta^2 + B_\nu \eta + C_\nu)} \quad \text{for} \quad i = 1, 2, 3 \quad (21)$$

which can be directly integrated to get the scale factors:

$$a_1 = \left[ \frac{-2(1 + 3\nu)}{[2 + (1 - \nu)(1 + 3\nu)\omega] m_\nu} \right]^{\frac{1}{2(1 - 3\nu)}} \left[ A_\nu \eta^2 + B_\nu \eta + C_\nu \right]^{\frac{1}{2(1 - 3\nu)}}, \quad (22)$$

$$a_2 = a_1 \exp \left[ \frac{-2h_2}{\sqrt{\Delta}} \arctanh \left( \frac{-2(1 - 3\nu)(1 - 3\nu)m_\nu \eta + \eta_0}{(1 + 3\nu)\sqrt{\Delta}} \right) \right], \quad \Delta > 0 \quad (23)$$

or

$$a_2 = a_1 \exp \left[ \frac{2h_2}{\sqrt{-\Delta}} \arctan \left( \frac{-2(1 - 3\nu)(1 - 3\nu)m_\nu \eta + \eta_0}{(1 + 3\nu)\sqrt{-\Delta}} \right) \right], \quad \Delta < 0 \quad (24)$$

and according to Eq. (9) one gets

$$a_3 = \frac{a_1^2}{a_2}, \quad (25)$$

where the discriminant is

$$\Delta = B_V^2 - 4A_\nu C_V = \frac{-8(1 - 3\nu)^2}{18\nu + (1 + 3\nu)^2\omega} h_2^2, \quad (26)$$

being proportional to the shear, see Eq. (16). Independent of the value $\Delta$ might have, $h_1 + h_2 + h_3 = 0$ is always true. The type V model with $\Delta \neq 0$ fulfills that $h_2 = -h_3$ with $h_1 = 0$ to have truly anisotropic solutions. But for $\Delta = 0$,

$$a_3 = a_2 = a_1, \quad (27)$$

since one has that $h_1 = h_2 = h_3 = 0$. In this case, the scale factors are equal to each other, given by Eq. (22), up to a constant that can be scaled away.

The BD field is

$$\phi = \frac{\psi}{a^3(1 - \nu)} = \left[ \frac{[2 + (1 - \nu)(1 + 3\nu)\omega] m_\nu}{-2(1 + 3\nu)} \right]^{\frac{3(1 - \nu)}{2(1 - 3\nu)}} \left[ A_\nu \eta^2 + B_\nu \eta + C_\nu \right]^{-\frac{1}{2(1 - 3\nu)}}. \quad (28)$$
The above-presented solutions reduce to the previously known for the isotropic, FRW model with an open space \((k = -1)\), cf. Refs. \([23, 24, 25]\), when \(\Delta = 0\), implying that \(h_1 = h_2 = h_3 = 0\), to have no shear, \(\sigma = 0\). The above solutions have no analogues in GR since the scalar field \(\phi\) cannot be set to a constant.

It has been conjectured, for the Bianchi \(V\) dust \((\nu = 0)\) model, that it is almost impossible to find solutions because of the complexity of the equation for \(\psi\), which is presented in Ref. \([26]\) as a fourth-order differential equation. Therefore, only particular solutions are expected to be found, if any. In the present work it is shown an integrated equation for \(\psi\) which is now of second-order, see Eq. \((11)\), and possesses at least one solution for \(\nu \neq 1/3\) given by Eq. \((18)\); other solutions are hoped to be found because of the relative simplicity of Eq. \((11)\). The dust-solution in Ref. \([26]\) is written in other variables, but it is a special case of our above solution when one takes \(B = \nu = 0\). Referring to this, one of the main problems of the physical interpretation in the field of exact solutions in cosmology has been that results are written in \emph{scaled} variables and the physics behind them is partially hidden. In our case, for the dust model \((\nu = 0)\) the scaled time is the same as the original cosmic time, written in Eq. \((5)\). For \(\nu \neq 0\) effort should be made to interpret physical situations.

Following we present the solutions for the radiation case.

\[ 2.2 \text{ Solutions with } \nu = 1/3 \]

In the radiation case the curvature terms in Eqs. \((6, 7, 8)\) turn out to be especially simple \((a^{6\nu}C_{iV} = 2, a^{6
nu}C_V = 6)\). Therefore, Eq. \((8)\) reduces to

\[
\psi'' - 4\psi - \frac{16\pi M_{1/3}}{3} = 0, \tag{29}
\]

that can be directly integrated to get\(^2\)

\[
\psi = c_1 e^{-2\eta} + c_2 e^{2\eta} - \frac{4\pi M_{1/3}}{3}, \quad c_1, c_2 \text{ are integration constants.} \tag{30}
\]

The general solution for the Hubble expansion rates, using Eqs. \((12, 13, 30)\), is given by

\[
H_i(\eta) = -\frac{c_1 e^{-2\eta} + c_2 e^{2\eta} - c_3 + h_i}{c_1 e^{-2\eta} + c_2 e^{2\eta} - \frac{4\pi M_{1/3}}{3}} \quad \text{for } i = 1, 2, 3 \tag{31}
\]

where \(c_3 \equiv \eta_o/2\) is an arbitrary integration constant and the constants \(h_i\), accounting for the anisotropy of the models, are determined by:

\[
h_1 = 0,
\]

\(^2\)This solution is also valid for Eq. \((11)\) with \(\nu = 1/3\).
\[ h_2 = -h_3, \]
\[ h_3 = \mp \sqrt{12\Delta - (3 + 2\omega) c_3^2} \]  
(32)

where \( \Delta \equiv (2\pi M_{1/3}/3)^2 - c_1c_2 \). The constants given in Eq. (22) are consistent with Eqs. (13, 14, 16). For if \( h_1 = h_2 = h_3 = 0 \), it implies that \( \sigma = 0 \).

From Eq. (31) one obtains the scale factors

\[ a_i = \alpha_{i0} \psi^{1/2} \exp \left[ \frac{c_3 - h_i}{2\sqrt{\Delta}} \arctan \left( \frac{1}{\sqrt{\Delta}}(c_2 e^{2\eta} - 2\pi M_{1/3}/3) \right) \right], \quad \Delta > 0, \]  
(33)

\[ a_i = \alpha_{i0} \psi^{1/2} \exp \left[ \frac{h_i - c_3}{2\sqrt{-\Delta}} \arctan \left( \frac{1}{\sqrt{-\Delta}}(c_2 e^{2\eta} - 2\pi M_{1/3}/3) \right) \right], \quad \Delta < 0, \]  
(34)

or

\[ a_i = \alpha_{i0} \psi^{1/2} \exp \left[ \frac{c_3 - h_i}{2c_2 e^{2\eta} - 4\pi M_{1/3}/3} \right], \quad \Delta = 0, \]  
(35)

where \( \alpha_{i0} \) are integration constants proportional to some initial size of the Universe. Note that Eq. (25) is valid again.

The BD field \( (\phi = \psi/a^2) \) is

\[ \phi = (\alpha_1 \alpha_2 \alpha_3)^{-2/3} \exp \left[ -\frac{c_3}{\sqrt{\Delta}} \arctan \left( \frac{1}{\sqrt{\Delta}}(c_2 e^{2\eta} - 2\pi M_{1/3}/3) \right) \right], \quad \Delta > 0, \]  
(36)

\[ \phi = (\alpha_1 \alpha_2 \alpha_3)^{-2/3} \exp \left[ \frac{c_3}{\sqrt{-\Delta}} \arctan \left( \frac{1}{\sqrt{-\Delta}}(c_2 e^{2\eta} - 2\pi M_{1/3}/3) \right) \right], \quad \Delta < 0, \]  
(37)

or

\[ \phi = (\alpha_1 \alpha_2 \alpha_3)^{-2/3} \exp \left[ -\frac{c_3}{c_2 e^{2\eta} - 2\pi M_{1/3}/3} \right], \quad \Delta = 0. \]  
(38)

From Eq. (32) one observes that if \( \Delta \leq 0 \), then \( 3 + 2\omega \leq 0 \) in order to have real solutions \( (h_2, h_3 \) being real numbers) for the scale factors Eq. (34) or Eq. (35).

Eqs.(33,38), together with Eq. (10), show the most general solution for the Bianchi model V containing a radiation field or a fluid of relativistic particles. For some particular values of our constants \( c_1, c_2, c_3, \) and \( h_i \), they were firstly presented in Ref. [26]. The solution for \( i = 1 \) is the most general solution for the FRW open Universe, cf. [27]. In the particular case that \( c_1 = -c_2 \) and \( h_i = 0 \) one gets the open FRW solution given in Ref. [28], where a misprint is found: \( \beta \rightarrow -\beta \) in Eq. (76) of that paper.

In the particular case that \( c_1 = c_2 = 0 \) one obtains the isotropic solution with \( \phi < 0 \). In this case, one has that \( H(\eta) = \text{const.}, \psi(\eta) = \text{const.} \) and since \( a(\eta)d\eta = dt \), one has that \( H_1(t) = H_2(t) = H_3(t) = 1/t \) with \( a_1 = a_1 t, \ a_2 = a_2 t, \ a_3 = a_3 t \), getting that \( \phi = -\frac{4\pi M_{1/3}}{3a_1 t^2}, \) a sign of antigravity?, a particular solution found elsewhere [29].
For if $c_3 = 0$, the solution becomes the same as in the GR theory with $\phi = \text{const} \equiv M^2 \rho_{l}$, first obtained in Ref. [30]. Further, if $h_i = c_3 = 0$, one obtains the $k = -1$ isotropic solutions of GR discussed in the late 60’s [31].

3 SOLUTIONS ANALYSIS

The solutions presented in the preceding section show various behaviors depending upon the value of the BD parameter ($\omega$), the type of fluid present ($\nu$), the amount of matter ($m$), and the anisotropic degree ($h_i$) of the models; let us consider each of them. In the BD theory measurements imply that $\omega > 500$, cf. Ref. [18]. However, in general scalar tensor theories the coupling constant can be much less than one [32, 33]. If such a theory is to be valid during some epoch in the early Universe, then it can be mimicked by an effective BD theory, but with its proper value of $\omega$. Therefore, it is interesting to consider both great and small $\omega$-values.

In the preceding section we have presented two types of solutions for models with $\nu \neq 1/3$ and $\nu = 1/3$. Following, we present the analysis of these solutions separately.

3.1 The case $\nu = 0$

Within the most interesting barotropic equations of state ($p = \nu \rho$) with $\nu \neq 1/3$ are the cases of dust ($\nu = 0$), stiff fluid or supernuclear density ($\nu = 1$) [34], particle creation ($\nu = 2/3$) [35], and vacuum energy ($\nu = -1$). For concreteness we shall consider in this subsection a gas of nonrelativistic massive bosons that could have played a dominant role (in the stress energy tensor) for the dynamics of the very early universe. This gas behaves as dust with an equation of state determined by $\nu = 0$. Accordingly, the total mass of the gas is given by the quantity $M_o = \frac{(3+2\omega)}{8\pi} m_o = \rho_o a_o^3$.

The values of $h_i$ are related to the degree of anisotropy present in the model. As pointed out already, if $h_i = 0$ then $\Delta = 0$ implying that $a_1 = a_2 = a_3$, to have the open FRW model. Anisotropic solutions are found for the case that the above constants fulfill that $\Delta \neq 0$. For if $\Delta > 0$ with $\nu = 0$, then $\omega < 0$, cf. Eq. (26). In this case the solution is given by Eqs. (22, 23, 25) implying that it is valid only in the interval $\left[\eta - \frac{\sqrt{-2/\omega |h_2|-\eta_0}}{m_o}, \frac{\sqrt{-2/\omega |h_2|}}{|h_2|}, \eta + \frac{\sqrt{-2/\omega |h_2|}}{|h_2|}\right]$. Then, for this solution to evolve long times one should demand that $\frac{\sqrt{2|h_2|}}{8\pi M_o} \gg \frac{\sqrt{-\omega}}{|3+2\omega|}$, that is, this would be favored by large anisotropies ($h_2$), small masses, or small negative $\omega$-values. This solution is for $h_2 > 0$ deflationary in the scale factor $a_2$ and inflationary in $a_3$; for

\[\text{Note that for } \nu = 0 \text{ one has that } \eta = \text{const. } t.\]
$h_2 < 0$ the roles of $a_2$ and $a_3$ are inverted. However, there is no exit of inflation because of the continuously increasing arctanh function in Eq. (23). Therefore, this behaviour can be only useful if after some inflation time some other physical source (e.g. another scalar field) begins to dominate the dynamics to stop the otherwise eternal inflationary behaviour.

The most interesting case seems to be for $\Delta < 0$, which is actually the case for $\omega > 0$. The solution is given by Eqs. (22, 24, 25) and is valid in the whole interval $-\infty < \eta < \infty$ without restrictions. For $h_2 > 0$ this solution is initially inflationary in $a_2$ and deflationary in $a_3$; again their roles can be inverted by choosing $h_2 < 0$. In contrast with the $\Delta > 0$ solution, in the present case the solution has exit, since the arctan function in Eq. (24) tends asymptotically to $\frac{\pi}{2}$. Thus, during some time interval this function behaves inflationary, after which a FRW behaviour follows.

The strong exponential expansion is possible, though no cosmological constant or function is present; this is because the non-minimal coupling in Eq. (1) implies new kinetic terms to the dynamical equations in comparison with GR [15]. In order to achieve a successful inflationary scenario one should demand that the three scale factors, or its volume, inflate simultaneously. This is apparently not possible from the form of the solution for $a_1$, Eq. (22), which is a power-law standard expansion type. An inflationary stage may occur, however, when the denominator of Eq. (13) is effectively a constant ($\psi \approx \text{const.}$), while the numerator is a linear function, so $H_i \sim \eta$. This behaviour can be attained only during some time interval, since $\psi$, given by Eq. (18), is a quadratic function and is eventually numerically larger than the numerator to give rise to $H_i \sim \frac{1}{\eta}$. To get enough e-foldings ($\frac{a_f}{a_o} = e^N$) of inflation to solve the horizon and flatness problems, one can try to adjust the model parameters $\omega, h_i, m_o$. Let us impose the necessary conditions for it. On the one hand, $\psi \approx \text{const.}$ from $\eta = 0$ until the time $\eta_*$, with $\eta_* \equiv \frac{2^{1/2}}{m_o} \left(\eta_0^2 + \frac{h_2}{\omega} \right)^{1/2} - \frac{m_0}{m_o}$. On the other hand, to last sufficient amount of e-foldings of inflation ($N \sim 68$) the accelerated expansion must run at least until the time $\eta_f$, with $\eta_f \equiv \frac{1}{m_o} \left((\eta_0 - h_i)^2 + 2N(\eta_0^2 + \frac{2h_2}{\omega}) \right)^{1/2} - \frac{m_0 - h_i}{m_o}$. Thus, the inflationary epoch is valid in the interval $0 \leq \eta < \eta_f$, provided that $\eta_f < \eta_*$. Unfortunately, the latter inequality is valid for very restricted values of $\omega$, in fact for $\omega < -\frac{2h_2}{\eta_0^2}$. Otherwise, for arbitrary $\omega$ the number of e-foldings is at most $N = 1/2$ only. For instance, one may try to fit the parameter $m_o$ (if $m_o$ is augmented, $N$ grows at a fix time) to get the desired number of e-foldings, but then the above inequality is no more valid. Analogously, whereas arbitrary large values of $\frac{h_2}{\omega}$ can make arbitrary large the inflation time [37], the time $\eta_f$ grows in the same proportion, avoiding to have

\footnote{For initial conditions with lower energy scales than Planckian, the number of e-folds can be smaller than 68 [4]. In our models we pursue to identify the asymptotic value of the scalar field with the Newtonian constant $G$. Therefore, the energy scale $G^{1/2}$ appears.}
\[ \eta_f < \eta_o. \] In BD FRW cosmologies a similar result is found, where only negative values for \( \omega \) can accomplish an accelerated, successful expansion \[15\]: A solution to the flatness problem is not possible because the current value \( \omega = 500 \) \[18\] restricts the solution types, not allowing an accelerated expansion, and not allowing to have nowadays \( n \) a value of \( \phi_0^{-1} = 16\pi G, G \) being the Newtonian constant \[38\].

In figures 1-3 analytic solutions are plotted for some cosmological parameters. Figure 1 shows the Hubble parameters and the BD field for \( \nu = 0, \omega = 500, \) and \( h_2 = \eta_o = m_0 = 1. \) Initially the Hubble parameter \( H_2 \) grows almost linearly until \( \eta = 0.42 \) and, therefore, the solution is inflationary in this direction with \( a_2 \sim e^{\kappa_1 \eta + \kappa_2 \eta^2}, \) where \( \kappa_1, \kappa_2 \) are constants. Afterwards \( H_2 \) continues growing at a lower rate until it stops at \( \eta = 1.0. \) After this time this Hubble parameter is dominated by its denominator, cf. Eq. \[15\], and the solution becomes \( H_2 \sim \frac{1}{\eta}. \) Later on, after \( \eta = 1.6 \) all three \( H_i \) are dominated by its denominator and the solution is of FRW type.

Figure 2 shows the solution for \( \omega = 500, \nu = 0, \eta_o = m_0 = 1, \) and \( h_2 = 10. \) In this case, the initial anisotropy parameter was augmented, provoking an initial contraction of \( a_2 (H_{2o} < 0), \) after which an expansion follows. If one leaves above parameters fixed, but one augments the value of \( m_o, a_2 \) goes from a contraction to an inflation period.

Finally, figure 3 is similar to figure 1, but in figure 3 a small \( \omega \) was chosen to mimic a model of induced gravity valid when its potential plays no role, see Ref. \[37\]. In figure 3 we observe an initially inflationary behaviour for the three scale factors, its exit, and its evolution to the open FRW. Again the same effect is attained by augmenting \( m_o, \) the three scale factors inflate but with a restricted number of \( N \) e-foldings, as mentioned above.

Note that during the inflationary era the solutions are still anisotropic; they tend to isotropize once \( H_i \sim 1/\eta, \) see figures 1, 2, and 3.

For small \( \omega \)'s, like in some induced gravity models \[32\], the three Hubble parameters grow, as can be observed from figure 3. For \( \omega \ll 1, \) cf. Ref. \[33\], the models isotropize during the inflationary era \[37\]; later on, they evolve asymptotically to the open FRW model.

The dimensionless shear parameter, Eq. \[17\], turns out to be

\[ \frac{\sigma}{H^2} = -\frac{6h_2^2}{[m_o \eta - \eta_o]^2}, \quad (39) \]

which for the \( \Delta < 0 \) solution, one observes an asymptotic, isotropic behaviour

---

5Our solutions are applicable in the induced gravity theory during a time interval, when its potential is not significant for the dynamics \[37\].
Figure 1:
Figure 2:
Figure 3:
\[ \sigma/H^2 \to 0, \text{ when } \eta \to \infty, \] to yield the following solution:

\[
    a_1 \to \sqrt{\frac{2}{2 + \omega}} \eta, \\
    a_2 \to e^{-\sqrt{\frac{\pi}{2}} a_1}, \\
    a_3 \to e^{\sqrt{\frac{\pi}{2}} a_1}, \\
    \phi \to \left(\frac{2 + \omega}{2}\right)^{\frac{3}{2}} m_o \eta, \quad (40)
\]

the bigger \( \omega \) is, the larger (smaller) is the ratio between \( a_1 \) and \( a_2 \) (\( a_3 \)). In our graphics we observe only small differences in the \( H_i \) \( (i = 1, 2, 3) \) for \( \eta \sim 20 \) to get, in the asymptotic limit, the solutions found in Refs. [23, 24, 25]. This asymptotic solution is different from the open FRW solution in GR, and is described by a fixed point in the autonomous phase plane \((\dot{a}/a, \dot{\phi}/\phi)\) analysis for FRW cosmologies in the BD theory [14].

### 3.2 The case \( \nu = 1/3 \)

The general solution found for \( \nu = 1/3 \) is valid for a fluid of ultra-relativistic particles and/or radiation. The isotropic, open solutions are obtained when \( h_i = 0 \), and for some particular values of our constants they are reported in Refs. [27, 28]. Solutions given by Eqs. (33)-(35) have a similar structure: they are multiplied by the power-law term \( \psi^{1/2} \), as well as by the exponential term. Initially the exponential term maybe more significant, and from it one could possibly obtain an inflationary stage in the evolution. The time asymptotic behaviour is dominated by the power-law term. The details depend on the value of the discriminant \( (\Delta) \). For the \( \Delta = 0 \) case the number of e-foldings of inflation is given by

\[
    N = \frac{c_3}{2c_2 e^{2\eta_f} - \frac{4\pi M_{1/3}}{3}} - \frac{c_3}{2c_2 - \frac{4\pi M_{1/3}}{3}} + \frac{1}{2} \ln \frac{\psi_f}{\psi_o} \quad (41)
\]

where \( \eta_f \) is the time when the power-law solution begins to dominate, i.e., when the exponential term tends to one. The number of e-foldings depends finely on the value of the constants \( c_2, c_3, \) and \( M_{1/3} \), subject to the following inequalities: \( i \) \( c_2 > 2\pi M_{1/3}/3 > 0 \), the first inequality is to avoid the point where \( a_i \) becomes infinity, and the second is to have \( a_i, \rho > 0 \). Note that \( c_1 > 0 \) since \( c_1 c_2 = \left(2\pi M_{1/3}/3\right)^2 > 0 \); \( ii \) \( c_1 + c_2 > 4\pi M_{1/3}/3 \) to have \( \psi_o, a_{io} > 0 \); and \( iii \) \( c_3 < 0 \) to have \( N > 0 \). The right choice of these parameters gives the desired number of e-folding of inflation \( (N) \). Unfortunately, the present case \( (\Delta = 0) \) is valid for \( \omega \leq -3/2 \), otherwise \( a_2 \) and \( a_3 \) become complex functions, cf. Eqs. (32) and (33). Recall that values for \( \omega < -3/2 \)
correspond to a negative scalar field density in the conformally rescaled Einstein frame \( \hat{g}_{\mu\nu} = G\phi g_{\mu\nu} \), see for instance [14]. However, if one believes that the correct physics is in the Jordan frame, negative \( \omega \)'s maybe still interesting. Recall that changing frames via conformal transformations imply physically different gravity theories since the matter couplings are different, as well as geodesic equations, see for instance [39].

Solutions with \( \Delta > 0 \) have the same problem as in the \( \nu \neq \frac{1}{3} \) case: they inflate anisotropically during the whole evolution given by the time interval

\[
\ln \left[ \left( \frac{2\pi M_{1/3}}{3} - 1 \right) \left( \frac{2\pi M_{1/3}}{3c_2^2} - \frac{c_1}{c_2} \right) \right]^{1/2} < \eta < \ln \left[ \left( \frac{2\pi M_{1/3}}{3} + 1 \right) \left( \frac{2\pi M_{1/3}}{3c_2^2} - \frac{c_1}{c_2} \right) \right]^{1/2}
\]

and the graceful exit problem remains because inflation never ends.

Solutions with \( \Delta < 0 \) imply that \( \omega < -\frac{3}{2} \) in order to have real (instead of complex) solutions for the scale factors in accordance with Eqs. (32) and (34). Figures 4 and 5 show solutions for the different values of the physical parameters. One notes that terms containing \( c_2 e^{2\eta} \) dominate the dynamics very rapid, and therefore shall account for the major contribution to the dynamics in asymptotic times. Terms with \( c_1 e^{-2\eta} \) could be important for the very beginning. In figure 4 we have chosen the parameters to be \( \omega = -2, h_2 = c_1 = c_2 = 1 \), and \( M_{1/3} = -1 \), which corresponds to \( M_{1/3} = +1/(8\pi) \) to have a positive density. In figure 5 we have augmented the value of the anisotropic parameter \( h_2 (=100) \), and the values of \( c_1 \) and \( c_2 \) \((c_1 < c_2)\) as well, to observe the influence of the \( c_1 e^{-2\eta} \) term. Otherwise, if \( c_2 > c_1 \) the solution becomes the isotropic solution yet from the very beginning.

The solutions with \( \Delta < 0 \) have the chance to achieve a number of e-foldings of inflation \( \left( \frac{2\pi}{\eta_o} = e^N \right) \) given by

\[
N = \left[ -\frac{c_3}{2\sqrt{-\Delta}} \right] \left[ \frac{1}{\sqrt{-\Delta}} \left( c_2 e^{2\eta} - \frac{2\pi M_{1/3}}{3} \right) + \frac{1}{\sqrt{-\Delta}} \left( c_2 - \frac{2\pi M_{1/3}}{3} \right) \right] + \frac{1}{2} \ln \frac{\psi_f}{\psi_o} .
\]

One can choose the integration constants to have the desired number of e-foldings, subject to following constrains (the arguments are the same as in the \( \Delta = 0 \) case): 
(i) \( c_2 > 0, M_{1/3} > 0 \). Note that \( \Delta < 0 \) implies that \( c_1 c_2 > \left( \frac{2\pi M_{1/3}}{3} \right)^2 > 0 \), hence, \( c_1 > 0 \); (ii) \( c_1 + c_2 > 4\pi M_{1/3}/3 \); and (iii) \( c_3 < 0 \). Accordingly, figure 6 shows the parameter dependence of the number of e-foldings of inflation \( (N) \). Small values of \( c_2 \) together with big \( h_i \) values are requested to achieve a sufficient number for \( N \).

The dimensionless shear parameter, Eq. (17), turns out to be

\[
\frac{\sigma}{H^2} = -\frac{6h_2^2}{\left[ -c_1 e^{-2\eta} + c_2 e^{2\eta} - c_3 \right]^2},
\]

which for the \( \Delta \leq 0 \) solutions, their asymptotic behaviour, when \( \eta \to \infty \), implies
Figure 4:
Figure 5:
\[ \sigma / H^2 \rightarrow 0, \text{ yielding the following isotropic solution:} \]

\[
H_{i}(\eta) \rightarrow 1, \quad a_{i}(\eta) \rightarrow a_{i0} \sqrt{c_{2}} e^{\eta}, \quad \phi(\eta) \rightarrow (a_{10}a_{20}a_{30})^{-2/3} \quad \Rightarrow \\
H_{i}(t) \rightarrow \frac{1}{t}, \quad a_{i}(t) \rightarrow a_{i0} t, \quad \phi(t) \rightarrow (a_{10}a_{20}a_{30})^{-2/3}.
\]

\[ (44) \]

Independent of the values all parameters may have, the solution evolves to its isotropic behaviour \[29\], as can be seen in figures \[4\] and \[5\]. The above limit solution, Eq. (44), tends to the radiation solution in GR for asymptotic times.

\section{4 FLUCTUATIONS OF THE BD FIELD}

Although we are working in a framework of a classical theory, one may also consider quantum fluctuations of the BD field during inflation. It is well known that isocurvature and curvature fluctuations are generated, and they produce perturbations in the density and in the CMBR. Fluctuations of the BD field has been analysed \[40, 41\] mostly in the Einstein frame\[6\], since there the new scalar field (\(\hat{\phi}\)) is minimally coupled and its kinetic term is canonical, therefore, the role of fluctuations is easy to interpret physically, and to import results from what is known in the Einstein theory \[42, 40\]. For example, if it were the case that we consider an additional effective constant potential (false vacuum energy) in the Jordan frame, this would turn out to behave like an exponential potential in the Einstein frame, i.e., a la extended inflation \[40\]. In this case, one gets that \(\delta \rho / \rho\) depends on \(\lambda^{4/(2\omega-1)}\) due to power law inflation, where \(\lambda\) is some perturbation scale \[43, 40, 41\]. Then, all machinery of reconstructing the inflation potential could be applied \[44\]. In our case, one does not have a potential that forces \(H = \text{const.}\), but in our models one has that \(H \sim t\) during some initial stage. Because of this fact, one expects that the spectrum is non-scale invariant too. In our inflationary models, however, if one considers the Einstein frame to analyse the perturbations, one finds that the transformation factors involve quantities with \(\sqrt{3+2\omega}\) (see footnote \[3\]), which in our solutions with inflation, it is a complex number. Then, one cannot correctly interpret our inflationary solutions in the Einstein frame. Furthermore, values of \(\omega < -3/2\) generate negative scalar-field densities in the conformally rescaled Einstein frame \[14\].

For \(\omega > -3/2\) we do not have an inflationary behaviour, therefore, the production of quantum fluctuations is out of interest because the horizon problem remains. However, if one starts the anisotropic models with Planckian initial conditions one can explain \[3\] the observed anisotropy of the CMBR measured by COBE \[1\].

\footnote{The transformation of the metric is \(g_{\mu\nu} = G\hat{g}_{\mu\nu}\) and the new field \(\hat{\phi} = \sqrt{\frac{2}{3+2\omega}} \ln(G\phi)\).}
5 CONCLUSIONS

Our analysis is based on the exact solutions reported in this work and in Ref. [16]. For a fluid with \(\nu \neq 1/3\) the particular solution Eq. (13) was used, but other solutions may be found from Eq. (11), which is considerably simpler than other equations found in the past [20]. For instance, our time asymptotic solution does not tend to the GR one, therefore, the latter must correspond to another solution of Eq. (11). For the radiation case (\(\nu = 1/3\)) the most general solution is given. The FRW solution is same as the given by our \(i = 1\) case (recall that \(h_1 = 0\)). The GR Bianchi V case is achieved when \(c_3 = 0\).

The above-presented exact solutions have a variety of possible behaviours depending on the values of the physical parameters \(\omega, \nu, m_\nu,\) and \(h_2\). Within some parameter range it is possible to encompass a cosmological model that incorporates the issues mentioned in the introduction: to have anisotropic initial conditions that leads to an inflationary stage, and, as time goes on, to isotropize towards an open FRW model. In this way, inflation represents a transient attractor, and the FRW behaviour is an asymptotic attractor. Further, these models can explain the observed anisotropy degree of the CMBR measured by COBE [1], if the model is to be started with Planckian initial conditions [3].

It is peculiar that these solutions show an inflationary behaviour, even without a cosmological constant or function. This is because the non-minimal coupling in Eq. (1) implies new kinetic terms to the dynamical equations in comparison with GR [13]. For the non-radiating (\(\nu \neq 1/3\)), anisotropic solutions with \(\Delta \neq 0\) the inflationary stage takes place when the denominator of Eq. (13) is almost a constant, whereas the numerator is linear. This is achieved during some time interval depending on the physical parameters \(\omega, \nu, h_i,\) and \(m_\nu\). Especially, large values of \(m_\nu\) or \(\frac{h_2^2}{\omega}\) favor strong exponential expanding solutions of the type \(a_i \sim e^{\kappa_1 \eta + \kappa_2 \eta^2}\), but the number of e-foldings of inflation cannot be greater than \(1/2\) for \(\omega > 0\). Solutions with \(\Delta > 0\) are inflationary, but without exit, diverging at asymptotic times. Solutions with \(\Delta < 0\), independent of a possible inflationary behaviour, asymptotically isotropize to a FRW open model. For \(\omega \gg 1\) the isotropization mechanism happens after the (short) inflationary era (see figure [4]), and for \(\omega \ll 1\) it occurs during inflation; some figures of the latter case are shown elsewhere [57]. For the radiating case (\(\nu = 1/3\)), anisotropic solutions are found for \(\Delta \neq 0\) and \(\Delta = 0\), as well. Solutions with \(\Delta > 0\) are inflationary again, but with the same properties as in the non-radiating case: inflation is only present in one scale factor, and there is no exit of it, i.e. the solution asymptotically diverges. In the cases with \(\Delta \leq 0\) inflation takes place, but again \(\omega\) must be negative (\(\omega \leq -3/2\)) to guarantee a sufficient amount of e-folds of expansion (\(N \sim 68\)). The reason behind this can be found in Eqs. (2) and (3).
For $\omega \leq -3/2$ the sign of some source terms (related to the trace $T$) and kinetic terms are reverse and play an inverse role than normally, making possible to have transient inflationary solutions.

It seems that the most interesting cosmological exact solutions in the BD theory are those with a BD parameter different from the desired $\omega > 500$, in fact for negative $\omega$'s. For the latter case there have been reported even more solutions than with $\omega > 0$, see for instance $[23, 16]$. However, values of $\omega < -3/2$ correspond to negative scalar field densities in the conformally rescaled Einstein frame ($\hat{g}_{\mu \nu} = G\phi g_{\mu \nu}$) $[14]$. Therefore, such solutions have to be taken with caution. Furthermore, within the BD theory alone it is not possible to have reheating in the Universe, since there is no potential nor any energy transfer to mass terms. However, reheating is necessary after inflation, and therefore, a more general theory than BD is in order. In this context, we have analysed the case of an induced gravity theory, where after a mild inflation period (due to non-minimal coupling) the Bianchi V model isotropizes, afterwards inflation due to the potential of the theory follows and reheating takes place $[33, 37]$. Our solutions may be also of physical interest in a scalar-tensor theory, perhaps some kind of scalar-tensor with $\omega(\phi)$, in which the solutions are to be valid during some (cosmic) time stage when the physical theory is mimiced by a BD effective action, yet with some other value for $\omega$, not restricted to be bigger than 500 at the begining of times.

Finally, we think that much work has to be done in the field of exact solutions in the context of modern cosmology. There are many solutions written in scaled variables, whose physics is hidden because of its complexity, and because less effort has been put on the analysis of its physical meaning. Nice properties as inflation, graceful exit, and isotropization has been shown in the present work, and without any cosmological constant or function, but for a restricted range of physical parameters.

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Figure captions

Figure 1: The upper graph shows the Hubble parameters as a function of the time $\eta$. As the universe evolves the three Hubble rates tend to their corresponding open FRW solution $H_1$. For these plots we have taken $\nu = 0$, $\omega = 500$, and $h_2 = \eta_o = m_0 = 1$. For these values one has that $H_{1o} > 0$, $H_{3o} > 0$, and $H_{2o} = 0$. In the figure below the evolution of the scalar field is shown. For $\eta \gg 2.4$, $\phi \sim 1/\eta$ as in the FRW cosmology.

Figure 2: The upper graph shows the Hubble parameters as a function of the time $\eta$. In these plots we have taken the parameters to be the same as in figure 1, except for the anisotropy parameter that we have augmented to be $h_2 = 10$. Because of this, $a_2$ undergoes a contraction that diminishes as time evolves. At $\eta \approx 9$ the contraction stops and an expansion follows. $a_1$ and $a_3$ always expand. The figure below shows the scalar field as a function of the time $\eta$.

Figure 3: The upper graph shows the Hubble parameters as a function of the time $\eta$. For these plots we have taken $\nu = 0$, $h_2 = \eta_o = m_o = 1$, and $\omega = 0.1$. That is, we have changed $\omega$ to be a small number to show an example of the type of induced gravity without potential. The evolution of the scalar field is shown in the figure below.

Figure 4: The upper graph shows the Hubble parameters as a function of the time $\eta$ for the parameters $\omega = -2$, $h_2 = c_1 = c_2 = -m_{1/3} = 1$. For these values the three initial Hubble functions are positive, $H_{1o} > 0$. The expansion begins super-inflationary until some turning point after which the three Hubble rates tend to the isotropic solution. In the figure below the evolution of the scalar field is shown.

Figure 5: The upper graph shows the Hubble parameters as a function of the time $\eta$ for the parameters $\omega = -2$, $h_2 = c_1 = 100$, $c_2 = 10$, $m_{1/3} = -1$. For these values $H_{1o} > 0$, $H_{2o} > 0$, and $H_{3o} < 0$. $H_1$ and $H_2$ begin expanding, and $H_3$ contracting, until some turning point after which the three Hubble rates tend to the isotropic solution. In the figure below the evolution of the scalar field is shown.

Figure 6: The number of e-foldings ($N$) of exponential expansion as a function of $h_2$ and $c_2$. This plot corresponds to an integration time from $\eta = 0$ to 5. The plane $N = 68$, representing the threshold of successful expansion, is shown for comparison.