Asymptotic behaviors of linear advanced systems of differential equations

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Abstract

In this paper, we use the fundamental matrix solution of the system $y'(t) = D(t) y(t)$, and the technique of the fixed point theorem to obtain sufficient conditions satisfying the convergence and exponential convergence of solutions for the linear system of advanced differential equations. The considered system with multiple variable advanced arguments is discussed as well. The obtained theorems generalize previous results of Dung [8], from the one dimension to the n dimension.

Key words: Fixed points, asymptotic behaviors, advanced systems, exponential stability, fundamental matrix solution.

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1 Introduction and Preliminaries

The study of advanced differential equations began in the middle of the last century by Myschkis [10] and Bellman & Cooke [1], this type of equations has been studied considerably by many authors see for example [3, 4, 9, 13, 14, 15]. However the advanced systems of differential equations has not been studied before, for this reason, in this paper we have studied the asymptotic behaviors of linear advanced systems of differential equations.

The important techniques used in the literature to investigate the qualitative behaviors of paths of linear and non-linear differential equations, without finding
the explicit solutions, are known as the second Lyapunov function(al) method, perturbation theory, fixed point method, the variation of constants formula and so on (see [5, 6] and the references therein).

Dung N T in [8], studied the asymptotic behaviors of the following linear advanced differential equations

\[ x' (t) + a (t) x (t + h (t)) + b (t) x (t + r (t)) = 0, \quad t \geq t_0, \]  

where \( a (t) \) and \( b (t) \) are continuous on \([t_0, +\infty)\), and \( h (t), \ r (t) \) are continuous functions with \( h (t) \geq 0 \) and \( r (t) \geq 0 \).

In this paper, we consider the linear system of advanced differential equations

\[ x' (t) + A (t) x (t + h (t)) + B (t) x (t + r (t)) = 0, \quad t \geq t_0 \geq 0, \]  

in which the functions \( h (t) \geq 0 \) and \( r (t) \geq 0 \) are continuous on \([t_0, +\infty)\), \( A, B : [t_0, +\infty) \to \mathbb{R}^{2n} \) are \( n \times n \) matrices with continuous real-valued functions as its elements.

Motivated by [8] and some previous works ([11, 12]), we use in the analysis the fundamental matrix solution of

\[ y' (t) = D (t) y (t), \]  

to invert the system (2) into an integral system which we derive a fixed point mapping. After then, we define prudently a suitable complete space, depending on the initial condition, so that the mapping is a contraction.

We recall now some definitions and results for fundamental matrix, see also [7].

**Definition 1.** An \( n \times n \) matrix function \( t \to \Phi (t) \), defined on an open interval \( J \), is called a matrix solution of the homogeneous linear system (3) if each of its columns is a (vector) solution.

**Definition 2.** The state transition matrix for the homogeneous linear system (3) on the open interval \( J \) is the family of fundamental matrix solutions \( t \to \Phi (t, r) \) parametrized by \( r \in J \) such that \( \Phi (r, r) = I \).

**Proposition 1** ([7 Proposition 2.14]). If \( t \to \Phi (t) \) is a fundamental matrix solution for the system (3) on \( J \), then \( \Phi (t, r) := \Phi (t) \Phi^{-1} (r) \) is the state transition matrix. Also, the state transition matrix satisfies the Chapman–Kolmogorov identities

\[ \Phi (r, r) = I, \quad \Phi (t, s) \Phi (s, r) = \Phi (t, r), \]

and the identities

\[ \Phi (t, s)^{-1} = \Phi (s, t), \quad \frac{\partial \Phi (t, s)}{\partial s} = -\Phi (t, s) A (s). \]

**Remark 1.** Notice that, \( \Phi (t, t_0) \) will be \( e^{(t-t_0)D} \) if \( D \) is a constant matrix.
Throughout this paper, \( \Phi (t, t_0) \) will denote a fundamental matrix solution of the homogeneous (unperturbed) linear problem \( [3] \).

**Lemma 1.** Let \( x (t) : [t_0, +\infty) \to \mathbb{R}^n \) be the solution of \( [2] \). Then, the system \( [3] \) is equivalent to

\[
x (t) = \Phi (t, t_0) x (t_0) + \int_{t_0}^{t} \Phi (t, s) A (s) \int_{s}^{t + h(s)} E_x (u) duds + \int_{t_0}^{t} \Phi (t, s) B (s) \int_{s}^{t + r(s)} E_x (u) duds.
\]

where \( E_x (u) := A (u) x (u + h (u)) + B (u) x (u + r (u)) \).

**Proof.** First we can write

\[
x (t + h (t)) = x (t) + \int_{t}^{t + h(t)} x' (u) du \quad \text{and} \quad x (t + r (t)) = x (t) + \int_{t}^{t + r(t)} x' (u) du,
\]

substituting these relations into \( [2] \), we obtain

\[
x' (t) + A (t) x (t) + A (t) \int_{t}^{t + h(t)} x' (u) du + B (t) x (t) + B (t) \int_{t}^{t + r(t)} x' (u) du = 0,
\]

then

\[
x' (t) = - (A (t) + B (t)) x (t) - A (t) \int_{t}^{t + h(t)} x' (u) du - B (t) \int_{t}^{t + r(t)} x' (u) du, \quad (5)
\]

Second, we put \( D (t) := - (A (t) + B (t)) \) and we put \( E_x (u) := A (u) x (u + h (u)) + B (u) x (u + r (u)) \), then, the substitution of \( [2] \) in \( [5] \) yields

\[
x' (t) = D (t) x (t) + A (t) \int_{t}^{t + h(t)} E_x (u) du + B (t) \int_{t}^{t + r(t)} E_x (u) du.
\]

Now, if we assume that a solution in the interval \([t_0, \infty )\) is given by

\[
x (t) = \Phi (t, t_0) \lambda (t), \quad (6)
\]

where \( \nu (t) \) is a differentiable vector valued function to be determined in the following fashion.

By the product rule for differentiation we have that

\[
x' (t) = \Phi' (t, t_0) \nu (t) + \Phi (t, t_0) \nu' (t)
\]

\[
= D (t) \Phi (t, t_0) \nu (t) + \Phi (t, t_0) \nu' (t).
\]

By the differential equation that \( x (t) \) satisfies on \([t_0, \infty )\), this implies

\[
D (t) \Phi (t, t_0) \nu (t) + \Phi (t, t_0) \nu' (t)
\]

\[
= D (t) x (t) + A (t) \int_{t}^{t + h(t)} E_x (u) du + B (t) \int_{t}^{t + r(t)} E_x (u) du,
\]
then
\[ D(t) \Phi(t, t_0) \nu(t) + \Phi(t, t_0) \nu'(t) \]
\[ = D(t) \Phi(t, t_0) \nu(t) \]
\[ + A(t) \int_t^{t+h(t)} E_x(u) \, du + B(t) \int_t^{t+r(t)} E_x(u) \, du. \]

Thus

\[ \nu'(t) = \Phi(t_0, t) A(t) \int_t^{t+h(t)} E_x(u) \, du + \Phi(t, t_0) B(t) \int_t^{t+r(t)} E_x(u) \, du. \]

The previous expression, after integrating from \( t_0 \) to \( t \) and using the fact that \( \nu(t_0) = x(t_0) \), implies

\[ \nu(t) = x(t_0) + \int_{t_0}^{t} \Phi(t_0, s) A(s) \int_s^{s+h(s)} E_x(u) \, duds \]
\[ + \int_{t_0}^{t} \Phi(t_0, s) B(s) \int_s^{s+r(s)} E_x(u) \, duds, \]

substituting into (6) we get (4).

The converse implication is easily obtained and the proof is complete. \( \square \)

2 Asymptotic behaviors

Let \( C([t_0, +\infty), \mathbb{R}^n) \) is the space of all \( n \)-vector continuous functions \( x(t) \) on \([t_0, +\infty)\) such that \( x(t_0) = x_0 \).

It is seen that \( C([t_0, +\infty), \mathbb{R}^n) \) is a complete metric space, endowed with the supremum norm
\[ \| x(\cdot) \| = \sup_{t \in [t_0, +\infty)} |x(t)|, \]
where \( |\cdot| \) denotes the infinity norm for \( x \in \mathbb{R}^n \). Also, if \( A \) is an \( n \times n \) real matrix, then we define the norm of \( A \) by
\[ |A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \sup_{t \in [t_0, +\infty)} |a_{ij}(t)|. \]

Let \( \{x^*(t), t \geq t_0\} \) be an arbitrary solution of (2). Then we can define \( x_0 = x^*(t_0) \). Thanks to Lemma 1, we know that \( \{x^*(t), t \geq t_0\} \) is a solution of equation (4) with an initial condition \( x(t_0) = x_0 \). So, we define for all \( t \geq t_0 \) the mapping \( \mathcal{H} \) by
\[
\begin{align*}
(\mathcal{H}\varphi)(t) &= \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} \Phi(t, s) A(s) \int_s^{s+h(s)} E_x(u) \, duds \\
&\quad + \int_{t_0}^{t} \Phi(t, s) B(s) \int_s^{s+r(s)} E_x(u) \, duds. \quad (7)
\end{align*}
\]
In this paper we assume that, for, we assume that for all \( s_2 \geq s_1 \in [t_0, \infty) \), let us have the uniform bound, in other words let
\[
\sup_{s_2 \geq s_1 \geq t_0} \| \Phi (s_2, s_1) \| \leq K < \infty. \tag{8}
\]

**Theorem 1.** Assume that (8) the following conditions hold,
\[
\lim_{t \to +\infty} \Phi (t, t_0) = 0 \tag{9}
\]
\[
\int_{t_0}^{t} |\Phi (t, s)| |A (s)| \int_{s}^{s+h(s)} (|A (u)| + |B (u)|) \, du \\
+ \int_{t_0}^{t} |\Phi (t, s)| |B (s)| \int_{s}^{s+r(s)} (|A (u)| + |B (u)|) \, du := \alpha < 1, \tag{10}
\]
Then every solution \( \{x (t), t \geq t_0\} \) of (7) with initial condition \( x (t_0) \) converges to zero.

**Proof.** In order to obtain the desired result, it is enough to show that the mapping (7) has an unique solution and this solution converges to zero as \( t \) tends to \( \infty \). So, we consider a closed subspace \( S \) of \( C ([t_0, +\infty), \mathbb{R}^n) \)
\[
S = \left\{ x \in C ([t_0, +\infty), \mathbb{R}^n) : \|x\| \leq L \text{ and } \lim_{t \to \infty} x (t) = 0 \right\}.
\]

Firstly, we must prove that \( \mathcal{H} \) maps \( S \) into itself.

**Step 1.** By definition of \( S \), we must show for \( x \in S \) that \( \|\mathcal{H}x\| \leq L, t \geq t_0 \). We have that, noticing that \( \|x\| \leq L \) by definition of \( S \), since (8), (9) and (10) hold, so that
\[
\|\mathcal{H}x\| \leq \sup_{t \in [t_0, +\infty)} \|\Phi (t, t_0) x (t_0)\| \\
+ \sup_{t \in [t_0, +\infty)} \left| \int_{t_0}^{t} \Phi (t, s) A (s) \int_{s}^{s+h(s)} E_x (u) \, du \right| \\
+ \sup_{t \in [t_0, +\infty)} \left| \int_{t_0}^{t} \Phi (t, s) B (s) \int_{s}^{s+r(s)} E_x (u) \, du \right| \\
\leq K \|x_0\| + \alpha L \leq L.
\]
we can choose \( \|x_0\| \leq \frac{(1-\alpha)L}{K} \) to obtain \( \|\mathcal{H}x\| \leq L \) for every \( t \geq t_0 \).

**Step 2.** We show that for \( x \in S \), \( \mathcal{H}x(t) \to 0 \) as \( t \to \infty \). By definition of \( S \), \( x (t) \to 0 \) as \( t \to \infty \). Thus we have
\[
|\mathcal{H} \varphi (t)| \leq |\Phi (t, t_0) x (t_0)| + \int_{t_0}^{t} |\Phi (t, s)| A (s) \int_{s}^{s+h(s)} |E_x (u)| \, du \\
+ \int_{t_0}^{t} |\Phi (t, s)| B (s) \int_{s}^{s+r(s)} |E_x (u)| \, du \\
:= I_1 + I_2 + I_3. \tag{11}
\]

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By (9) \[ I_1 = |\Phi(t, t_0) x(t_0)| \to 0 \text{ as } t \to \infty. \]

Moreover, it follows from the fact \( x \in \mathcal{S} \) that for any \( \epsilon > 0 \), there exists \( T \geq t_0 \) such that \( |x(t)| < \frac{\epsilon}{2} \) for all \( t \geq T \). Hence, we have

\[
I_2 = \int_{t_0}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} |E_x(u)| \, duds \\
= \int_{t_0}^{T} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} |E_x(u)| \, duds \\
+ \int_{T}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} |E_x(u)| \, duds \\
< \int_{t_0}^{T} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} (|A(u)| + |B(u)|) \, duds \\
+ \frac{\epsilon}{2} \int_{T}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} (|A(u)| + |B(u)|) \, duds.
\]

We observe that \( \int_{T}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} |E_x(u)| \, duds \) converges to zero as \( t \to \infty \) due to condition (9). Thus, there exists \( T_1 \geq T \), such that

\[
I_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} \int_{T}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} (|A(u)| + |B(u)|) \, duds.
\]

Using (10) we get \( I_2 < \epsilon \) for all \( t \geq T_1 \). In other words, we have \( I_2 \to 0 \) as \( t \to \infty \).

Similarly, we also have \( I_3 \to 0 \) as \( t \to \infty \), this proves that \( (\mathcal{H} \varphi)(t) \to 0 \) as \( t \to \infty \).

We now prove that \( \mathcal{H} \) is a contraction.

**Step 3.** Clearly, for each \( x \in \mathcal{S} \), we have that \( \mathcal{H}x \) is continuous. Let \( x, y \in \mathcal{S} \). For \( t \geq t_0 \) we get by the condition (10), we get

\[
|\mathcal{H}(x)(t) - \mathcal{H}(y)(t)| \\
\leq \int_{t_0}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} |E_x(u) - E_y(u)| \, duds \\
+ \int_{t_0}^{t} |\Phi(t, s)||B(s)| \int_s^{s+r(s)} |E_x(u) - E_y(u)| \, duds \\
\leq \int_{t_0}^{t} |\Phi(t, s)||A(s)| \int_s^{s+h(s)} (|A(u)| + |B(u)|) \, duds \|x - y\| \\
+ \int_{t_0}^{t} |\Phi(t, s)||B(s)| \int_s^{s+r(s)} (|A(u)| + |B(u)|) \, duds \|x - y\| \\
\leq \alpha \|x - y\|.
\]

Since \( \alpha < 1 \). Thus \( \mathcal{H} \) is a contraction on \( \mathcal{S} \). This implies that there is a unique solution to (2) with initial condition \( x_0 \). \qed
Let us now recall a fundamental concept (see, for instance, [2]) that will be used in the next theorem.

**Definition 3.** The differential equation \( x(t) + D(t)x(t) = 0, \ t \geq t_0 \) is called exponentially stable, if there exist \( M_0 > 0, \lambda_0 > 0 \) such that,

\[
|\Phi(t,s)| \leq M_0 e^{-\lambda_0(t-s)}, \ t_0 \leq s \leq t < \infty,
\]

where \( M_0 \) and \( \lambda_0 \) do not depend on \( s \).

**Theorem 2.** Assume that \( A(t) \) and \( B(t) \) are bounded on \([t_0, +\infty)\) and there exist \( M_0 > 0, \lambda_0 > 0 \) such that \((12)\) holds. Then any solution \( u \) of \((2)\) is defined for \( t \geq t_0 \) and satisfies

\[
|x(t)| \leq Me^{-\lambda t}, t_0 \leq t < \infty.
\]

**Proof.** Let us define another closed subspace of \( C([t_0, +\infty), \mathbb{R}^n) \) as

\[
\mathcal{E} = \{ x \in C([t_0, +\infty), \mathbb{R}^n) : \exists M, \lambda > 0 \text{ such that } |x(t)| \leq Me^{-\lambda t} \ \forall t \geq t_0 \}.
\]

We will show that \( \mathcal{H}(\mathcal{E}) \subset \mathcal{E} \). So, we use the same notation \( I_1, I_2 \) and \( I_3 \) in (11). Then by (12), we have

\[
I_1 \leq M_0 \|x_0\| e^{-\lambda_0(t-t_0)}, t \geq t_0,
\]

without loss of generality, we may assume that \( \lambda_0 \neq \lambda \) for \( \lambda \) as in the definition of \( \mathcal{E} \).

To estimate \( I_2 \) in (11), we observe that \( h(t), r(t) \geq 0 \), and hence

\[
I_2 = \int_{t_0}^{t} |\Phi(t,s)||A(s)| \int_{s}^{s+h(s)} |E_x(u)| \, du \, ds \\
\leq M \int_{t_0}^{t} |\Phi(t,s)||A(s)| \int_{s}^{s+h(s)} (|A(u)| + |B(u)|) e^{-\lambda(u+h(u))} \, du \, ds \\
\leq M\overline{A}(\overline{A} + \overline{B}) \int_{t_0}^{t} |\Phi(t,s)| \int_{s}^{s+h(s)} e^{-\lambda u} \, du \, ds \\
\leq \frac{M\overline{A}(\overline{A} + \overline{B})}{\lambda} \int_{t_0}^{t} |\Phi(t,s)| e^{-\lambda s} (1 - e^{-\lambda h(s)}) \, ds \\
\leq \frac{M\overline{A}(\overline{A} + \overline{B}) M_0}{\lambda} \int_{t_0}^{t} e^{-\lambda_0(t-s)} e^{-\lambda s} \, ds,
\]

where \( |A(t)| \leq \overline{A} \) and \( |B(t)| \leq \overline{B} \). If \( \lambda < \lambda_0 \), then we have

\[
\int_{t_0}^{t} e^{-\lambda_0(t-s)} e^{-\lambda s} \, ds = e^{-\lambda t} \int_{t_0}^{t} e^{-(\lambda_0-\lambda)(t-s)} e^{-\lambda s} \, ds \leq \frac{e^{-\lambda t}}{\lambda_0 - \lambda},
\]

and if \( \lambda > \lambda_0 \), then we can write

\[
\int_{t_0}^{t} e^{-\lambda_0(t-s)} e^{-\lambda s} \, ds = e^{-\lambda_0 t} \int_{t_0}^{t} e^{-(\lambda-\lambda_0)(t-s)} e^{-\lambda s} \, ds \leq \frac{e^{-(\lambda-\lambda_0) t} e^{-\lambda_0 t}}{\lambda_0 - \lambda}.
\]
From this fact for $I_2$, we can write

$$I_2 \leq \begin{cases} 
\frac{M A (A - B)}{\lambda (\lambda_0 - \lambda)} e^{-\lambda t}, & \lambda < \lambda_0, \\
\frac{MA (A+B) M_0}{\lambda (\lambda_0 - \lambda)} e^{-(\lambda - \lambda_0) t} e^{-\lambda_0 t}, & \lambda > \lambda_0.
\end{cases}$$

In the same way, for $I_3$, we also obtain

$$I_3 \leq \begin{cases} 
\frac{MB (A + B)}{\lambda (\lambda_0 - \lambda)} e^{-\lambda t}, & \lambda < \lambda_0, \\
\frac{MB (A+B) M_0}{\lambda (\lambda_0 - \lambda)} e^{-(\lambda - \lambda_0) t} e^{-\lambda_0 t}, & \lambda > \lambda_0.
\end{cases}$$

As $|\mathcal{H}(x(t))| \leq I_1 + I_2 + I_3$, we infer that

$$\| \mathcal{H}(x(t)) \| = \begin{cases} 
M_0 \|x_0\| e^{\lambda_0 t} e^{-\lambda_0 t} + \frac{M (A+B)^2 M_0}{\lambda (\lambda_0 - \lambda)} e^{-\lambda t}, & \lambda < \lambda_0, \\
M_0 \|x_0\| e^{\lambda_0 t} e^{-\lambda_0 t} + \frac{M (A+B)^2 M_0}{\lambda (\lambda_0 - \lambda)} e^{-(\lambda - \lambda_0) t} e^{-\lambda_0 t}, & \lambda > \lambda_0,
\end{cases}$$

then $\mathcal{H}(\mathcal{E}) \subset \mathcal{E}$ will be hold since

$$M_0 \|x_0\| e^{\lambda_0 t} + \frac{M (A+B)^2 M_0}{\lambda (\lambda_0 - \lambda)} \leq M \text{ and } \lambda < \lambda_0.$$ 

The remainder of the proof is similar to that of Theorem 1. So, we omit it here. \hfill \Box

**Remark 2.** Note that, if $A, B$ are continuous real-valued functions on $[t_0, +\infty)$ to $\mathbb{R}$, then Theorems 1 and 2 become the [8, Theorems 2.3 and 2.5 respectively]s.

### 3 General Problem

Now, the methods in the previous section can be extended to the following system

$$x'(t) + \sum_{j=1}^{N} A_j(t) x(t+h_j(t)) = 0.$$  \hfill (14)

With a same way in Lemma 1, suppose that

$$D(t) := -\sum_{j=1}^{N} A_j(t),$$

then

$$x(t) = \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} \Phi(t, s) \sum_{j=1}^{N} A_j(s) \int_{s}^{s+h_j(s)} E_x(u) \, du,$$  \hfill (15)
where $E_x(u) := \sum_{j=1}^{N} A_j(u)x(u + h_j(u))$ and $\Phi(t, t_0)$ is the solution of

$$x'(t) = D(t)x(t).$$

The proof of the following theorem is similar to that of Theorems 1 and 2. Hence, we omit it.

**Theorem 3.** Suppose that the following conditions hold,

$$\sup_{s_2 \geq s_1 \geq t_0} \|\Phi(s_2, s_1)\| \leq K < \infty,$$

(i)

$$\int_{t_0}^{t} |\Phi(t, s)| \sum_{j=1}^{N} |A_j(s)| \int_{s}^{s + h_j(s)} \sum_{k=1}^{N} |A_k(u)| \, du \, duds := \bar{\alpha} < 1,$$

Then every solution $\{x(t), t \geq t_0\}$ of (14) with initial condition $x(t_0)$ converges to zero.

(ii) If $A_j(t), j = 1, N$ are bounded and the equation $x(t) + D(t)x(t) = 0$ is exponentially stable, then any solution $\{x(t), t \geq t_0\}$ of (14) with initial condition $x(t_0)$ exponentially converges to zero.

**Remark 3.** Note that, if $A_j(t), j = 1, N$ are continuous real-valued functions on $[t_0, +\infty)$ to $\mathbb{R}$, then Theorem 3 become the [8, Theorem 3.1].

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