ACYCLIC COLOURINGS OF GRAPHS WITH OBSTRUCTIONS

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Abstract. Given a graph $G$, a colouring of $G$ is acyclic if it is a proper colouring of $G$ and every cycle contains at least three colours. Its acyclic chromatic number $\chi_a(G)$ is the minimum $k$ such that there exists a proper $k$-colouring of $G$ with no bicoloured cycle. In general, when $G$ has maximum degree $\Delta$, it is known that $\chi_a(G) = \Theta(\Delta^{4/3})$ as $\Delta \to \infty$. We study the effect on this bound of further requiring that $G$ does not contain some fixed subgraph $F$ on $t$ vertices. We establish that the bound is constant if $F$ is a subdivided tree, $O(t^{6/3}\Delta^{2/3})$ if $F$ is a forest, $O(\sqrt{7}\Delta)$ if $F$ is bipartite and 1-acyclic, $2\Delta + o(\Delta)$ if $F$ is an even cycle of length at least 6, and $O(t^{1/4}\Delta^{5/4})$ if $F = K_{3,t}$.

1. Introduction

Given a graph $G$ and an integer $k \geq 1$, an acyclic $k$-colouring $\phi$ of $G$ is a proper $k$-colouring (a partition of $V(G)$ into $k$ independent sets which we call the colour classes of $\phi$), and each pair of colour classes induces a forest (every even cycle contains at least three colours). We denote $\chi_a(G)$ the acyclic chromatic number of $G$, that is the minimum integer $k$ such that there exists a $k$-acyclic colouring of $G$.

Grunbaum [8] introduced the concept of acyclic colourings, and A. V. Kostochka [9] proved that deciding whether $G$ has an acyclic $k$-colouring is NP-complete. Most of the works in the state-of-the-art are either to compute exactly the chromatic number for certain graphs [2] or to find upper bounds for its value.

Fertin et al. [6] established a simple lower bound on the acyclic chromatic number: if $G$ has average degree $d$, then $\chi_a(G) > d/2 + 1$. Alon et al. [11] showed that there are graphs $G$ with maximum degree $\Delta(G)$ for which $\chi_a(G) = \Omega\left(\frac{\Delta(G)^{4/3}}{\log \Delta(G)^{1/t}}\right)$ as $\Delta \to \infty$. This follows from an analysis of random graphs in the Erdős–Rényi model $G(n,p)$, where $n$ vertices are fixed, and each edge has a probability $p = p(n)$ of existing.

Upper bounds over $\chi_a(G)$ in terms of the maximum degree $\Delta(G)$ have been studied for a few decades. It has been established that, for any graph $G$ of maximum degree $\Delta$, one has $\chi_a(G) = O(\Delta^{4/3})$ [1] [12] [13]. To our knowledge, the best known upper bound is given by Gonçalves, Montassier, and Pinlou: for every graph $G$ of maximum degree $\Delta$, one has $\chi_a(G) \leq \frac{3}{2}\Delta^{4/3} + O(\Delta)$ as $\Delta \to \infty$. Their proof relies on the so-called entropy compression method, which has been developed following the algorithmic proof of the Lovász Local Lemma (LLL) due to Moser and Tardos [10]. This bound is tight up to a polylogarithmic factor, as certified by the lower bound described above [1]. If moreover $G$ is a line-graph, then $\chi_a(G) = \Theta(\Delta)$ [5], and more generally for any $K_{2,t}$-free graph $G$ of maximum degree $\Delta$, $\chi_a(G) = O\left(\sqrt{t}\Delta\right)$ [1] [4]. The bound in [7] is moreover constructive, and in particular there is a random algorithm that returns in expected polynomial time an acyclic $k$-colouring of any $d$-regular $K_{2,t}$-free graph $G$, where $k$ is within a factor $O\left(\sqrt{t}\right)$ of the optimal.

Motivated by the above observation, we have sought for obstructions that let us have a constant approximation algorithm for the acyclic colouring problem over $d$-regular graphs. We show that even cycles of length at most $d^{1/3}$ are such obstructions. More generally, our work studies the impact of forbidding one or several fixed subgraphs $F$ in a graph $G$ on the value of
χa(G). Given a fixed set of graphs \( \mathcal{F} \), a graph \( G \) is called \( \mathcal{F} \)-free if no graph \( F \in \mathcal{F} \) appears as a (not necessarily induced) subgraph of \( G \). When \( \mathcal{F} = \{ F \} \), we say that \( G \) is \( F \)-free. We let \( a(d, \mathcal{F}) := \max \{ \chi_a(G) : G \text{ is } \mathcal{F} \text{-free} \} \).

The document is organised as follows. To prove our results, we need two main ingredients that we present in Section 2: structural results that bound the density of \( \mathcal{F} \)-free graphs (Section 2.1), and a technical theorem (Theorem 2) that certifies the existence of \( K \)-colourings with specific constraints (Section 2.2). In order to use Theorem 2, we need to derive upper bounds on the number of cycles in specific classes of graphs. Our work mainly consists in finding upper bounds for \( a(d, \mathcal{F}) \), with a specific focus on the case where \( \mathcal{F} = \{ F \} \) is a single-graph obstruction. We summarise those bounds in Table 1. In Section 3, we argue that we may only consider obstructions that are connected, bipartite, and sufficiently sparse. In Section 4, we characterise the single-graph obstructions that force a constant upper bound. In Section 5, we extend the previous result by characterising the single-graph obstructions that yield an upper bound that is sublinear as a function of \( d \). In Section 6, we explicit a sufficient condition for a single-graph obstructions to yield a linear upper bound. Finally, in Section 8, we describe a large family of obstructions that force an upper bound that is asymptotically smaller than the general one (which we recall is \( O\left(d^{4/3}\right) \)).

### 1.1. Notations and terminology

All graphs considered in this paper are finite and simple. Given an integer \( k \geq 0 \), we say that a graph \( G \) is \( k \)-acyclic if \( G \) has a feedback vertex set \( X \) of size \( k \) (so that \( G \setminus X \) is a forest).

Given an integer \( n \geq 1 \), \( P_n \) and \( C_n \) respectively denote the path and cycle graphs on \( n \) vertices. Given an integer \( \ell \geq 1 \), an \( \ell \)-path is a path of length \( \ell \) within a fixed graph. We denote by \( \{n\} \) the set of integers \( \{0, \ldots, n-1\} \).

Given a set of elements \( S \), and a family \( \mathcal{F} \) of subsets of \( S \), we denote by \( \Delta_q(\mathcal{F}) \) the maximum number of members of \( \mathcal{F} \) of cardinality \( q \) that intersect in a fixed element. For example, given a set of cycles \( \Pi \) in a graph \( G \), \( \Delta_2(\Pi) \) corresponds to the maximum number of cycles of length \( 2\ell \) in \( \Pi \) that contain a fixed vertex in \( V(G) \).

| \( \mathcal{F} \) | Upper bound on \( a(d, \mathcal{F}) \) | Reference |
|---|---|---|
| \( \emptyset \) | \( \frac{3}{4} \Delta^{4/3} + O(\Delta) \) | [7] |
| \( F: \) 1-subdivided tree | constant \( C_F \) | Theorem [6] |
| \( F: \) tree, size \( t \) | \( O\left(t^{8/3}d^{2/3}\right) \) | Corollary [2] |
| \( \{C_4\} \) | \( 2.763d \) | Theorem [9] |
| \( \{C_2t\} \) for some \( 3 \leq t \leq d^{1/3} \) | \( 2d + O\left(td^{2/3}\right) \) | Theorem [10] |
| \( \{C_3,C_4,C_6\} \) | \( 1.7633d + O\left(\sqrt{d}\right) \) | Theorem [11] |
| \( F: \) 1-acyclic, bipartite, size \( t \) | \( O\left(\sqrt{td}\right) \) | Theorem [8] |
| \( F: \) 2-acyclic, bipartite, size \( t \) | \( O\left(t^{1/4}d^{5/4}\right) \) | Theorem [12] |

**Table 1.** Our upper bounds with various obstructions.

### 2. The general framework

In this section, we list some technical results that are at the heart of our proofs.
2.1. Structural properties of graphs with obstructions. We focus on the average degree of $F$-free graphs.

**Lemma 1** (Folklore). Every multigraph of average degree $d$ contains a bipartite sub-multigraph of average degree at least $d/2$.

**Lemma 2** (Folklore). Every multigraph of average degree $d$ contains a sub-multigraph of minimum degree at least $\lfloor d/2 \rfloor + 1$.

**Theorem 1** (Erdős, Gallai, 1959 [1]). For every integer $t \geq 2$, if a given graph $G$ is $P_t$-free, then its average degree is at most $t - 2$.

Theorem 1 has been conjectured to hold more generally when $P_t$ is replaced with any tree on $t$ vertices.

**Conjecture 1** (Erdős, Sós, 1963). For every fixed integer $t \geq 2$, and every tree $T$ on $t$ vertices, if a graph $G$ is $T$-free, then its average degree is at most $t - 2$.

Ajtai, Komlós, Simonovits and Szemerédi have reportedly proved the Erdős-Sós conjecture when $t$ is large enough. Their work is still in preparation for publication. In order to avoid having to rely on a private result, we can use the following easy result instead.

**Lemma 3.** Let $T$ be any fixed rooted tree on $t$ vertices. If a graph $G$ has minimum degree $t - 1$, then there is a copy of $T$ rooted in each vertex $v \in V(G)$ in $G$.

Proof. The proof is standard: we include it for completeness. We construct a copy of $T$ rooted in $v$ greedily, by following a DFS ordering $v_1, \ldots, v_t$ of $V(T)$. We first fix $T_1 := \{v\}$, then for every $2 \leq i \leq t$ we construct $T_i$ by adding $v_i$ to the tree $T_{i-1}$ already constructed, so that $T_i$ is isomorphic to $T[[v_1, \ldots, v_i]]$. To do so, we need to find a neighbour of the parent node $v_j$ of $v_i$ in $T$ (by assumption we have $j < i$) that is distinct from $V(T_j)$. The existence is due to the fact that $v_j$ has at most $i - 2 < t - 1$ neighbours in $T_{i-1}$ and at least $t - 1$ neighbours in $G$. At the end of that process, we obtain a tree $T_i$ rooted in $v$ and isomorphic to $T$. □

By combining Lemma 2 and Lemma 3 we obtain a 2-approximation of the Erdős-Sós conjecture as a corollary, which will be sufficient for our applications in this paper.

**Corollary 1.** For every fixed forest $T$ on $t$ vertices, if a given graph $G$ is $T$-free, then its average degree is at most $2(t - 2)$.

**Lemma 4.** Let $F'$ be any properly 2-coloured forest on $t - 1$ vertices, and let $F$ be the bipartite graph obtained from $F'$ by adding a new vertex adjacent to one colour class. Then every $F$-free graph $G$ on $n$ vertices has average degree $O(\sqrt{t}n)$.

Proof. Let $F_0$ be the colour class of $F'$ that is linked to the new vertex in $F$.

Let $G$ be an $F$-free bipartite graph, and let $d$ be its average degree. By combining Lemmas 1 and 2 there exists a bipartite subgraph $G_0$ of $G$ of minimum degree at least $\lfloor d/4 \rfloor + 1$. Let $v_0 \in V(G_0)$ be any fixed vertex, and let $G_1 = G[N(v_0), N^2(v_0)]$ be the bipartite subgraph induced by the first and second neighbourhoods of $v_0$.

On the one hand, we have $|E(G_1)| \geq |N(v_0)|(d/4 - 1) \geq d/4(d/4 - 1)$, hence the average degree of $G_1$ is at least $\frac{2|E(G_1)|}{n-1} \geq \frac{d(d/4 - 1)}{2(n-1)}$. We infer that there exists a sub graph $G_2$ of $G_1$ of minimum degree at least $\frac{d}{4n-4}$.
On the other hand, if \( G_2 \) had minimum degree at least \( t - 3 \), then by Lemma \( \text{III} \) one could find a copy of \( F' \) in \( G_2 \) such that \( F_0 \subseteq N(v_0) \), which together with \( v_0 \) would yield a copy of \( F \) in \( G \). This is a contradiction, so the minimum degree of \( G_2 \) is at most \( t - 4 \). We conclude that \( d \leq 4\sqrt{t}n \), as desired. \( \square \)

2.2. Main technical theorem. The proof of our main theorem is an abstract generalisation of that of [I] Theorem 2, where we have replaced the compression entropy machinery with an inductive counting approach. We have made that choice because the resulting proof is shorter, less technical, and self-contained. Moreover, as a byproduct we obtain exponential lower bounds for the number of colourings satisfying the hypothesis of the theorem. We note however that it would be possible to do the same using entropy compression, and obtain a random algorithm that returns such a colouring in expected polynomial time.

We begin with the introduction of a few notions that will be needed in order to formulate the theorem, which needs a high level of generality so that we can apply it in the variety of applications appearing in the forthcoming sections.

Given a graph \( G \), a set of constraints \( \Gamma \) consists in a set of pairs of vertices of \( G \). For a given vertex \( v \in V(G) \), we let \( N_\Gamma(v) \) consist of all vertices that form a pair in \( \Gamma \) together with \( v \). The constraint-degree of \( v \) is \( \deg_\Gamma(v) := |N_\Gamma(v)| \), and the maximum degree of \( \Gamma \) is \( \Delta(\Gamma) := \max_{v \in V(G)} \deg_\Gamma(v) \). Given an integer \( k \geq 1 \), a \( \Gamma \)-proper \( k \)-colouring of \( G \) is a mapping \( \phi : V(G) \to [k] \) such that \( \phi(u) \neq \phi(v) \) for every \( \{u, v\} \in \Gamma \). For instance, an \( E(G) \)-proper \( k \)-colouring of \( G \) coincides with the usual notion of proper \( k \)-colouring. More generally, letting \( H \) be the graph on vertex set \( V(G) \) and edge set \( \Gamma \), a \( \Gamma \)-proper \( k \)-colouring of \( G \) is a proper \( k \)-colouring of \( H \).

Given an even cycle \( C = (v_0, \ldots, v_{2\ell - 1}) \) in \( G \) and a \( k \)-colouring \( \phi \) of \( G \) (which may be improper), we say that \( C \) is bicoloured in \( \phi \) if \( \phi(v_i) = \phi(v_{i+2 \mod 2\ell}) \) for every \( i \in [2\ell] \). We say that \( C \) is \( \Gamma \)-free if \( \{v_i, v_{i+2 \mod 2\ell}\} \notin \Gamma \) for every \( i \in [2\ell] \). Given a set \( \Pi \) of cycles in \( G \), we say that \( \phi \) is \( \Pi \)-acyclic if no cycle in \( \Pi \) is bicoloured in \( \phi \). Note that, if \( E(G) \subseteq \Gamma \), and if \( \Pi \) is the set of \( \Gamma \)-free cycles of \( G \), then any \( \Gamma \)-proper \( \Pi \)-acyclic colouring of \( G \) is in particular a proper acyclic colouring of \( G \).

**Theorem 2.** Let \( G \) be a graph, \( \Gamma \) a set of constraints, and \( \Pi \) a set of cycles of \( G \). Fix \( \tau > 1 \), and

\[
K := \Delta(\Gamma) + \tau + \sum_{\ell \geq 2} \frac{\Delta_2(\Pi)}{\tau^{2\ell - 3}}.
\]

Then there exist at least \( \tau^{|V(G)|} \) \( \Gamma \)-proper \( \Pi \)-acyclic \( K \)-colourings of \( G \).

**Proof.** For any subgraph \( H \subseteq G \), we denote \( X[H] := X \cap 2^V(H) \) for any family of subsets of vertices \( X \) of \( G \). We let \( \mathcal{A}(H) \) be the set of \( \Gamma[H] \)-proper \( \Pi[H] \)-acyclic \( K \)-colourings of \( H \). By convention, if \( H \) is the empty graph, \( \mathcal{A}(H) \) contains a single trivial colouring. We show the following property for any subgraph \( H \subseteq G \) by strong induction:

\[
(\text{IH 2}) \quad \forall v_0 \in V(H), \quad |\mathcal{A}(H)| \geq \tau |\mathcal{A}(H - v_0)|.
\]

If \( V(H) \) is empty, (IH 2) is trivially true. Suppose \( V(H) \neq \emptyset \) and let \( v_0 \in V(H) \). By induction, assume (IH 2) is true for all strict subgraph \( H' \subset H \). We define \( \mathcal{F} \), the set of flawed extensions, as the subset of \( K \)-colourings of \( H \) such that every colouring \( \phi \in \mathcal{F} \) is \( \Gamma[H - v_0] \)-proper \( \Pi[H - v_0] \)-acyclic, but contains a conflict due to the colour given to \( v_0 \). In symbols, \( \phi \in \mathcal{F} \) if and only if \( \phi|_{H - v_0} \in \mathcal{A}(H - v_0) \) and \( \phi \notin \mathcal{A}(H) \). We distinguish two types of conflict.

(a) There is a vertex \( u \in V(H) \) such that \( \{u, v_0\} \in \Gamma \) and \( \phi(u) = \phi(v_0) \).

(b) \( v_0 \) is contained in a bicoloured cycle of \( \Pi \).
We let \( \mathcal{F}_a \) and \( \mathcal{F}_b \) be the subsets of colourings that respectively contain a conflict of type (a) or of type (b). These two subsets are not necessarily disjoint.

By definition of \( \mathcal{F} \), we have have \( |A(H)| = K \, |A(H - v_0)| - |\mathcal{F}| \), and \( |\mathcal{F}| \leq |\mathcal{F}_a| + |\mathcal{F}_b| \).

To complete the proof of the induction, we show these two following points:

(i) \( |\mathcal{F}_a| \leq \Delta(\Gamma) \, |A(H - v_0)| \).

(ii) \( |\mathcal{F}_b| \leq |A(H - v_0)| \sum_{\ell \geq 2} \frac{\Delta_{2\ell}(\Pi)}{\tau^{2\ell-3}} \).

**Proof of point (i).** Let \( c \in A(H - v_0) \). By definition, \( v_0 \) belongs to at most \( \Delta(\Gamma) \) constraints in \( \Gamma \), hence there at most \( \Delta(\Gamma) \) flawed extensions of \( c \) that induce a conflict of type (a). Therefore, we have \( |\mathcal{F}_a| \leq \Delta(\Gamma) \, |A(H - v_0)| \).

**Proof of point (ii).** For \( \ell \geq 2 \), let \( \Omega_\ell \) be the set of cycles in \( \Pi \) of length \( 2\ell \) which contain \( v_0 \). By assumption, \( |\Omega_\ell| \leq \Delta_{2\ell}(\Pi) \). For a cycle \( C \in \Omega_\ell \), we define \( \mathcal{F}_C \subseteq \mathcal{F}_b \) the subset of colourings for which \( C \) is effectively bicoloured. Clearly, we have \( \mathcal{F}_b = \bigcup_{\ell \geq 2} \bigcup_{C \in \Omega_\ell} \mathcal{F}_C \).

Let \( C = (v_2, \ldots, v_{2\ell-1}) \in \Omega_\ell \). We construct an injection from \( \mathcal{F}_C \) to \( A(H \setminus \{v_0, \ldots, v_{2\ell-3}\}) \). Let \( \phi: \mathcal{F}_C \to A(H \setminus \{v_0, \ldots, v_{2\ell-3}\}) \) be the application that simply uncolours \( v_0, \ldots, v_{2\ell-3} \) (uncolouring \( v_0 \) ensures that all conflicts are resolved). The inverse application \( \phi^{-1} \) colours \( v_0, \ldots, v_{2\ell-3} \) by alternating the colours of \( v_{2\ell-1} \) and \( v_{2\ell-2} \), which by definition is the only way to ensure that \( C \) is bicoloured. Therefore, \( |\mathcal{F}_C| \leq |A(H \setminus \{v_0, \ldots, v_{2\ell-3}\})| \).

We apply the induction hypothesis (IH 2) iteratively on the vertices \( v_{2\ell-3}, \ldots, v_1 \) in that order, and obtain

\[
|\mathcal{F}_C| \leq |A(H \setminus \{v_0, \ldots, v_{2\ell-3}\})| \leq \frac{1}{\tau} |A(H \setminus \{v_0, \ldots, v_{2\ell-4}\})| \leq \cdots \leq \frac{1}{\tau^{2\ell-3}} |A(H - v_0)|.
\]

Finally, since \( \Delta_{2\ell}(\Pi) \) corresponds to the number of cycles \( C \in \Pi \) of length \( 2\ell \) that contain a fixed vertex \( v_0 \in V(G) \), we conclude that

\[
|\mathcal{F}_b| \leq \sum_{\ell \geq 2} \sum_{C \in \Omega_\ell} |\mathcal{F}_C| \leq \sum_{\ell \geq 2} \sum_{C \in \Omega_\ell} \frac{1}{\tau^{2\ell-3}} |A(H - v_0)| \leq \sum_{\ell \geq 2} \frac{\Delta_{2\ell}(\Pi)}{\tau^{2\ell-3}} |A(H - v_0)|.
\]

A straightforward application of (i) and (ii) completes the proof of the induction:

\[
|A(H)| \geq K \, |A(H - v_0)| - |\mathcal{F}_a| - |\mathcal{F}_b| \\
\geq \left( K - \Delta(\Gamma) - \sum_{\ell \geq 2} \frac{\Delta_{2\ell}(\Pi)}{\tau^{2\ell-3}} \right) |A(H - v_0)| \\
\geq \tau |A(H - v_0)|.
\]

An iterative application of (IH 2) to every \( v \in V(G) \) implies that \( |A(G)| \geq \tau^{|V(G)|} > 1 \).

### 3. Trimming the space of obstructions

The following theorem states that if a graph \( F \) in the family of obstructions \( \mathcal{F} \) has several connected components, then only one of them sensibly affects the value of \( a(d, \mathcal{F}) \). Therefore, it suffices to study the asymptotic behaviour of this function in the case where \( F \) is connected.
Theorem 3. Suppose $F$ has connected components $(F_i)_{i \in [r]}$. There exists a constant $c_F$ such that for any $d \geq 0$,

$$\max_{i \in [r]} a(d, F_i) \leq a(d, F) \leq \max_{i \in [r]} a(d, F_i) + c_F.$$ 

Proof. First, we prove the lower bound. Let $d \geq 0$. Let $j \in [r]$ be such that $a(d, F_j) = \max_{i \in [r]} a(d, F_i)$. Let $G$ be an $F_j$-free graph with $\Delta(G) \leq d$ such that $\chi_a(G) = a(d, F_j)$. Since $G$ must also be $F$-free, we have $\chi_a(G) \leq a(d, F)$.

Now, we focus on the upper bound. We explicitly show that for any $d \geq 0$, we have

$$(1) \quad a(d, F) \leq \max_{i \in [r]} a(d, F_i) + |V(F)|.$$ 

We proceed by induction on $r$, the number of connected components of $F$.

The base case $r = 1$ is trivial (applying the same argument for the lower bound). Suppose now that $r \geq 2$. Let $d \geq 0$. Let $G$ be an $F$-free graph with $\Delta(G) \leq d$ such that $\chi_a(G) = a(d, F)$.

Consider whether $G$ is $F_{r-1}$-free or not:

- If $G$ is $F_{r-1}$-free, then $\chi_a(G) \leq a(d, F_{r-1})$, which clearly proves inequality (1).

- Otherwise, $G$ contains a subgraph $G_0$ isomorphic to $F_{r-1}$, thus $G \setminus G_0$ must be $(F \setminus F_{r-1})$-free, therefore, $\chi_a(G \setminus G_0) \leq a(d, F \setminus F_{r-1})$. Since $F \setminus F_{r-1}$ has $r-1$ connected components, namely $F_0, \ldots, F_{r-2}$, we apply the induction hypothesis and obtain

$$a(d, F \setminus F_{r-1}) \leq \max_{i \in [r-1]} a(d, F_i) + |V(F \setminus F_{r-1})|.$$ 

Given a $k$-acyclic colouring of $G \setminus G_0$, we can construct a $(k + |V(G_0)|)$-acyclic colouring of $G$ by assigning a new unique colour to each vertex of $G_0$. Since $|V(G_0)| = |V(F_{r-1})|$, and $|V(F \setminus F_{r-1})| + |V(F_{r-1})| = |V(F)|$, we have

$$\chi_a(G) \leq a(d, F \setminus F_{r-1}) + |V(F_{r-1})| \leq \max_{i \in [r-1]} a(d, F_i) + |V(F)|.$$ 

Clearly, $\max_{i \in [r-1]} a(d, F_i) \leq \max_{i \in [r]} a(d, F_i)$, therefore inequality (1) is proven. 

The next theorem indicates that we only need to consider the case where is $F$ bipartite and has average degree less than 8, otherwise $a(d, F)$ is only within a polylog factor of the general upper bound $O(d^{1/3})$.

Theorem 4. If $F$ is not bipartite, or if $|E(F)| > 4|V(F)|$, then

$$a(d, F) = \Omega \left( \frac{d^{4/3}}{(\ln d)^{1/3}} \right).$$ 

Proof. We adapt the proof in (1) to work on the bipartite random graph model $G = G(n, n, p)$. Let $A$ and $B$ be the parts of $V(G)$ (each of size $n$). Let $p := 4(\log n/n)^{1/4}$. If $F$ is not bipartite, then $G$ is obviously $F$-free. The probability that $G$ contains a subgraph isomorphic to $F$ is bounded by $\frac{2\binom{n}{|V(F)|}\binom{|E(F)|}{|E(F)|}}{p^{|E(F)|}} = O\left( n^{|V(F)|-|E(F)|}/(\log n)^{|E(F)|/4} \right)$. If $|E(F)| > 4|V(F)|$, then that previous quantity is $o(1)$ as $n \to \infty$, implying that $G$ is $F$-free almost surely.

Let $r \leq \frac{n}{2}$, and let $V_1, \ldots, V_r$ be any partition of $V(G)$ into $r$ parts; we derive an upper bound on the probability that this partition is an acyclic colouring of $G$. Let us write $A_i := V_i \cap A$ and $B_i := V_i \cap B$. By omitting one vertex from each $A_i$ of odd cardinality, we obtain sub-parts $A'_1, \ldots, A'_r$ of even cardinality, which cover at least $n - r \geq \frac{n}{2}$ vertices in $A$. We split every non-empty set $A'_i$ into subsets of size 2. We obtain $k_A \geq \frac{n}{4}$ monochromatic pairs of vertices in $A$, which we label $A''_1, \ldots, A''_{k_A}$. Similarly, we construct $k_B \geq \frac{n}{4}$ monochromatic pairs $B''_1, \ldots, B''_{k_B}$.
of vertices in $B$. Observe that, for every $(i, j) \in [k_A] \times [k_B]$, if $A''_i \cup B''_j$ induces a $C_4$ in $G$, then this is a bicoloured cycle. We let $E_{i,j}$ be that event, which occurs with probability $p^4$. The events $E_{i,j}$ are mutually independent, therefore the probability that none of them happen is $(1 - p^4)^{k_A k_B} \leq (1 - p^4)^{n^2/16}$.

There are at most $(2n)^{2n}$ partitions of $V(G)$. By a union bound, the probability that at least one of them is an acyclic colouring of $G$ is at most

$$(2n)^{2n}(1 - p^4)^{n^2/16} \leq \exp \left(2n \log(2n) - \frac{n^2}{16} p^4 \right) \leq \exp \left(2n \log(2n) - 16n \log(n) \right) = o(1).$$

Therefore, $\chi_a(G) > \frac{n}{2}$ almost surely. The degree of each vertex follows the binomial distribution $\mathcal{B}(n, p)$, so by a tail estimate of that distribution together with a union bound over all the vertices, we have that $\Delta(G) \leq 2np = 8n^{3/4}(\log n)^{1/4}$ almost surely. Applying the function $f: x \mapsto \frac{x}{(\log x)^{1/4}}$ to that inequality (with $f$ being positive and increasing for $x \geq e^{1/4}$) yields

$$\frac{\Delta(G)^{4/3}}{(\log \Delta(G))^{1/3}} \leq 16n.$$ 

Therefore, $\chi_a(G) > \frac{\Delta(G)^{4/3}}{32(\log \Delta(G))^{1/3}}$ almost surely as $n \to \infty$. \hfill \Box

In particular $a(d, F) = \tilde{O}(d^{4/3})$ if $F$ is one of $K_{5,21}$, $K_{6,13}$, $K_{7,10}$, or $K_{8,9}$, because these bipartite graphs have average degree strictly greater than 8. Furthermore, we note that almost surely, $G$ contains no copy of $K_{2,\alpha}$, $K_{3,\beta}$ and $K_{4,\gamma}$, where $\alpha = \tilde{O}(\Delta(G)^{2/3})$, $\beta = \tilde{O}(\Delta(G)^{1/3})$ and $\gamma = \tilde{O}(1)$.

4. Constant upper bound

In this section, we want to characterise obstructions that force to have a bounded acyclic chromatic number. The 1-subdivision of a graph $G$, denoted by $G^{(1)}$, is obtained by replacing each edge in $G$ by a path of length 2. Any graph $G$ with $\Delta(G) \geq 2$ has $\Delta(G) = \Delta(G^{(1)})$. In 2008, Dvořák [3] gave a complete characterisation of the graphs of bounded acyclic chromatic number through the analysis of 1-subdivisions.

**Theorem 5 (Dvořák).** Let $\mathcal{G}$ be a class of graphs of bounded chromatic number. $\mathcal{G}$ has bounded acyclic chromatic number if and only if there exists a constant $c$ such that for any graph $H$, if $H^{(1)}$ is a subgraph of a graph in $\mathcal{G}$ then $\chi(H) \leq c$.

As a consequence of this theorem, we obtain a constant upper bound on the acyclic chromatic number of F-free graphs when $F$ is a subgraph of a 1-subdivided tree (said otherwise, $F$ is a forest in which no pair of vertices of degree at least 3 are linked by a path of odd length).

**Theorem 6.** Let $F \subseteq T^{(1)}$ where $T$ is a tree. There exists a constant $c_F$ such that for all $d \geq 0$, $a(d, F) \leq c_F$.

**Proof.** Let $\mathcal{G}$ be the non-empty class of $F$-free graphs, which have chromatic number at most $|V(F)| - 1$. Let $H$ be a graph and suppose there exists $G \in \mathcal{G}$ such that $H^{(1)} \subseteq G$. So, by assumption, $G$ is $F$-free graph, and $H^{(1)}$ is $F$-free, which in turn implies that $H^{(1)}$ is $T^{(1)}$-free. $H$ must be $T$-free, otherwise $T^{(1)} \subseteq H^{(1)}$, thus $F \subseteq G$, a contradiction. Therefore, $\chi(H) \leq |V(T)| - 1$. We conclude the proof using Theorem 5. \hfill \Box

Theorem 6 completely characterises the graphs $F$ for which $a(d, F)$ is bounded by a constant. Wood [13] determined bounds on the acyclic chromatic number of graphs subdivisions, and in particular showed that $\sqrt{n/2} < \chi_a(K_n^{(1)}) < \sqrt{n/2} + \frac{5}{2}$.

Observe that $K_n^{(1)}$ is $F$-free for every graph $F$ that is not a subgraph of a 1-subdivided tree. So by definition, we have $a(d, F) \geq \chi_a(K_{d+1}^{(1)})$, and we can derive the following.
Proposition 1. If $F$ is not a subgraph of any 1-subdivided tree, then

$$a(d, F) > \sqrt{\frac{d+1}{2}}$$

5. Sublinear upper bound

In this section, we show that for any fixed forest $F$, $a(d, F) = O(d^{2/3})$.

Theorem 7. Let $G$ be a $t$-degenerate graph of maximum degree $\Delta$. Then

$$\chi_d(G) = O\left(t^{8/3}\Delta^{2/3}\right).$$

Proof. Since $G$ is $t$-degenerate, we can find an acyclic orientation $\vec{G}$ of $G$ where each vertex has out-degree at most $t$. In what follows, an *antidirected path/cycle* is a path/cycle of $G$ which contains no directed subpath of length 2 in $\vec{G}$. Note that an antidirected cycle is necessarily even. Let $\Gamma_0$ be the set of pairs of vertices linked by a directed path of length at most 2 in $\vec{G}$, and let $\Pi$ be the set of antidirected cycles.

The graph $(V(G), \Gamma_0)$ is $(t^2 + t)$-degenerate, so we can greedily construct a $\Gamma_0$-proper $(t^2 + t + 1)$-colouring $\phi_0$ of $G$. Now, we will construct an improper $\Pi$-acyclic $O\left((t\Delta)^{2/3}\right)$-colouring $\phi_1$ of $G$. Observe that if there exists such a colouring, the Cartesian product of $\phi_0$ and $\phi_1$ yields a proper acyclic $O\left(t^{8/3}\Delta^{2/3}\right)$-colouring of $G$, and the theorem holds.

Let $\Gamma_1$ consist of all pairs of vertices $\{u, v\}$ such that $\deg_{\vec{G}}^\ell(u, v) \geq (t\Delta)^{1/3}$.

Claim 1. The maximum constraint-degree is $\Delta(\Gamma_1) \leq (t\Delta)^{2/3}$.

Proof. Let $v_0 \in V(G)$, and let $A_0$ be the set of out-going arcs from $N^-(v_0)$.

On the one hand, each vertex $w \in N^-(v_0)$ is incident to at most $t$ arcs from $A_0$ by assumption on the maximum out-degree of $\vec{G}$. Thus $t|N^-(v_0)| \geq |A_0|$. On the other hand, each vertex $u \in N_{\Gamma_1}(v)$ is incident to at least $(t\Delta)^{1/3}$ arcs from $A_0$ by definition of $\Gamma_1$. Hence we have $t|N^-(v_0)| \geq |A_0| \geq (t\Delta)^{1/3}|N_{\Gamma_1}(v_0)|$.

Since $\Delta \geq |N^-(v_0)|$ and $\deg_{\Gamma_1}(v_0) = |N_{\Gamma_1}(v_0)|$, we have $\deg_{\Gamma_1}(v_0) \leq (d\Delta)^{2/3}$.

We now let $\Pi_1 \subseteq \Pi$ be the set of $\Gamma_1$-free antidirected cycles in $\vec{G}$.

Claim 2. $\Delta_{2\ell}(\Pi_1) \leq \frac{1}{2}(t\Delta)^{\ell-2/3}$, for every integer $\ell \geq 2$.

Proof. Let $v_0 \in V(G)$.

The number of antidirected paths of length $2\ell - 2$ that begin with an in-going arc from $v_0$ is at most $\deg^-(v_0)t^{\ell-1}\Delta^{\ell-2}$, since there are at most $t$ choices for any out-going arc in $\vec{G}$. Given such a path $(v_0, \ldots, v_{2\ell-2})$, if we assume that $\{v_0, v_{2\ell-2}\} \notin \Gamma_1$, there are at most $(t\Delta)^{1/3}$ choices in order to close it into a cycle of length $2\ell$. So the number of $\Gamma_1$-free antidirected $2\ell$-cycles that contain an in-going edge from $v_0$ is at most $\frac{1}{2} \deg^-(v_0)t^{\ell-2/3}\Delta^{\ell-5/3}$, where we have divided the bound by 2 because each cycle is counted twice in the previous process, depending on the direction in which we construct it. On the other hand, the number of antidirected paths of length $2\ell - 3$ that begin with an out-going arc of $v_0$ is at most $\deg^+(v_0)(t\Delta)^{\ell-2}$. Given such a path $(v_0, \ldots, v_{2\ell-4})$, we may close it into a cycle of length $2\ell$ by first picking a vertex $v_{2\ell-1} \in N^+(v)$, then a vertex $v_{2\ell-2} \in N^-(v_{2\ell-3}, v_{2\ell-1})$. The number of choices is at most $\deg^+(v_0)(t\Delta)^{1/3}$ if we assume that $\{v_{2\ell-3}, v_{2\ell-1}\} \notin \Gamma_1$. The number of $\Gamma_1$-free antidirected $2\ell$-cycles that contain an out-going edge
from \(v_0\) is therefore at most \(\frac{1}{3} \deg^+(v_0)t^{\ell-2/3}\Delta^{\ell-5/3}\). Overall, the number of \(\Gamma_1\)-free antidirected \(2\ell\)-cycles that contain \(v_0\) is at most \(\frac{1}{3} (\deg^+(v_0) + \deg^-(v_0))t^{\ell-2/3}\Delta^{\ell-5/3} \leq \frac{1}{2} (t\Delta)^{\ell-2/3}\). \(\diamondsuit\)

We may now apply Theorem 2 on the graph \(G\), with the set of constraints \(\Gamma_1\) and the set of cycles \(\Pi_1\), after fixing \(\tau := \frac{\sqrt{2}}{2} (t\Delta)^{2/3}\). We obtain that there are at least \(\tau^{|V(G)|}\) \(\Gamma_1\)-proper \(\Pi_1\)-acyclic \(K\)-colourings of \(G\), where

\[
K = \Delta(\Gamma_1) + \tau + \sum_{\ell \geq 2} \frac{\Delta_2(\Pi_1))}{\tau^{2\ell-3}}
\leq 2 + \frac{\sqrt{2}}{2} (t\Delta)^{2/3} + \frac{\sqrt{2}}{2} \sum_{\ell \geq 2} (t\Delta)^{4/3-\ell/3}
\leq 2 + \frac{\sqrt{2}}{2} (t\Delta)^{2/3} + \frac{\sqrt{2}}{2} \cdot \frac{(t\Delta)^{2/3}}{1 - (t\Delta)^{-1/3}} = \left(1 + \frac{\sqrt{2}}{2}\right) (t\Delta)^{2/3} + O((t\Delta)^{1/3}),
\]
as \((t\Delta) \to \infty\).

We finish the proof by observing that a \(\Gamma_1\)-proper \(\Pi_1\)-acyclic \(K\)-colouring of \(G\) is in particular an improper \(\Pi\)-acyclic \(K\)-colouring of \(G\), as desired for \(\phi_1\). \(\square\)

Let \(F\) be a forest on \(t\) vertices, and let \(T\) be a tree obtained by adding edges between the connected components of \(F\). If a graph \(G\) is not \((t-2)\)-degenerate, then it contains a subgraph \(H\) of minimum degree at least \(t - 1\). By Lemma 2, \(H\) contains \(T\), and therefore also \(F\), as a subgraph. We conclude that any \(F\)-free graph \(G\) is \((t-2)\)-degenerate, hence we have the following result as a corollary.

**Corollary 2.** For every forest \(F\) on \(t\) vertices,
\[
a(d, F) = O\left(t^{8/3}d^{2/3}\right).
\]

**6. Obstructions for the Linear Upper Bound**

Having established that \(a(d, F) = O\left(d^{2/3}\right)\) whenever \(F\) is a forest, we now consider obstructions that contain a cycle.

**Proposition 2.** Let \(d \geq 2\), and let \(F\) be a graph that contains a cycle. Then
\[
a(d, F) > \frac{d}{2} + 1.
\]

**Proof.** For \(d \geq 2\) and \(g \geq 3\), the existence of \(d\)-regular graphs of girth \(g\) is well known (see [13] for example). By taking \(g\) large enough, we obtain a graph \(G\) that is \(F\)-free and \(d\)-regular, and therefore satisfies \(\chi_a(G) > \frac{d}{2} + 1\). \(\square\)

It has already been established in [11] that \(K_{2, t}\)-free graphs of maximum degree \(\Delta\) have acyclic chromatic number at most \(O\left(\sqrt{t\Delta}\right)\). When \(t\) is fixed, this bound is tight up to a multiplicative constant as a result of the previous theorem.

We recall that a graph \(F\) is 1-acyclic if there exists \(v_0 \in V(F)\) such that \(F \setminus v_0\) is a forest.

**Lemma 5.** Let \(F\) be a fixed 1-acyclic bipartite graph with \(t \geq 4\) vertices. Given an \(F\)-free graph \(G\) of maximum degree \(\Delta\), let \(\Omega_G\) denote its set of cycles. Then
\[
\Delta_{2\ell}(\Omega_G) \leq 2(t - 3)\Delta^{2\ell - 2},
\]
for any integer \(\ell \geq 2\).
Proof. Let \( u \in V(G) \). There are at most \( \Delta^{2\ell-3} (2\ell - 3) \)-paths starting from \( u \). Let \( v \) be the other endpoint of such a path; we claim that there are at most \( 4(\ell - 3)\Delta 3 \)-paths between \( u \) and \( v \). Every cycle is counted twice, therefore there are at most \( 2(\ell - 3)\Delta^{2\ell-2} \) cycles of length \( 2\ell \) in \( G \) containing \( u \). We now prove the claim.

We assume that \( \deg(u) \geq t \) and \( \deg(v) \geq t \), otherwise there are at most \( t\Delta 3 \)-paths between \( u \) and \( v \), thus proving the claim. Let \( X := N(u) \cap N(v) \), \( U := N(u) \setminus X \) and \( V := N(v) \setminus X \). Each edge in \( G[U, V], G[U, X] \) and \( G[V, X] \) defines a unique 3-path from \( u \) to \( v \), while each edge in \( G[X] \) corresponds to two such paths, depending on the direction in which it is traversed. These are the only 3-paths from \( u \) to \( v \). Let \( w \in V(F) \) be such that \( F - w \) is acyclic. Since \( G \) is \( H \)-free, \( G[X] \) must be \((F - w)\)-free, \( u \) is adjacent to every vertex in \( X \) and essentially plays the role of \( w \). \( F - w \) is a forest on \( t - 1 \) vertices, therefore by Corollary 1 there are at most \( (t-3)|X| \) edges in \( G[X] \). We now prove a similar result for \( G[U, V], G[U, X] \) and \( G[V, X] \).

We say that a neighbour of \( w \) is trivial if it is isolated in \( F - w \). Let \( (F_i)_{i \in [r]} \) be the nontrivial connected components of \( F - w \). Define \( T \) by adding an edge between \( F_i \) and \( F_{i+1} \) for \( i \in [r-1] \) in such a way that each vertex of \( T \) lies in the same part as it does in \( F \): this construction is guaranteed to exist by assumption that each component contains at least \( 2 \) vertices. Since each component \( F_i \) is acyclic, \( T \) is a tree with at most \( t - 1 \) vertices.

By contradiction, suppose \( G[U, V] \) contains \( T \) as a subgraph. By the bipartiteness of \( F \) and \( G[U, V] \), and by definition of \( T \), the nontrivial neighbours of \( w \) in \( F \) all lie in a single part of \( G[U, V] \): suppose without loss of generality that they lie in \( U \), and are therefore adjacent to \( u \). We use the assumption that \( d(u) \geq t \) to identify the trivial neighbours of \( w \) among \( N(v) \). We have therefore found a copy of \( F \) in \( G \), a contradiction. Therefore \( G[U, V] \) is \( T \)-free, and by Corollary 1 \( G[U, V] \) contains at most \( (t-3)(|U| + |V|) \) edges. In the same way, we show that \( G[U, X] \) and \( G[V, X] \) respectively contain at most \( (t-3)(|U| + |X|) \) and \( (t-3)(|V| + |X|) \) edges.

Using that \( |U| \leq \Delta - |X| \) and \( |V| \leq \Delta - |X| \), we conclude that there are at most \( (t-3)(2|U| + 2|V| + 4|X|) \leq 4(t-3)\Delta \) paths of length 3 between \( u \) and \( v \), which ends the proof.

So we can deduce the following result.

**Theorem 8.** Let \( F \) be a fixed 1-acyclic bipartite graph with \( t \geq 4 \) vertices. Then for every \( F \)-free graph \( G \) of maximum degree \( \Delta \),

\[
\chi_a(G) = \mathcal{O}\left(\sqrt{\Delta}\right)
\]

as \( \Delta \to \infty \).

**Proof.** Let \( G \) be a \( H \)-free graph of maximum degree \( \Delta \). We prove Theorem 8 through an application of Theorem 2 with \( \Gamma = E(G) \) and \( \Pi = \Omega_G \). We fix \( \alpha := \sqrt{2t - 3} \) and \( \tau := \alpha \Delta \), so that there exist \( \tau^{|V(G)|} \) proper acyclic \( K \)-colourings of \( G \), with

\[
K = \Delta + \tau + \sum_{\ell \geq 2} \frac{\Delta^2(\Omega_G)}{\tau^{2\ell-3}}
\]

\[
\leq \Delta + \alpha \Delta + (2t - 6)\Delta \sum_{\ell \geq 2} \frac{1}{\alpha^{2\ell-3}}
\]

by Lemma 5

\[
= \Delta + \alpha \Delta + (2t - 6)\Delta \frac{\alpha}{\alpha^2 - 1} = \Delta(1 + 2\alpha)
\]

\[
= \Delta\left(1 + \sqrt{8t - 20}\right).
\]

\(\square\)
We note that the condition given on $F$ could be loosened: Lemma 5 remains true for a bipartite graph $F$ of parts $X$ and $Y$ with two non-adjacent vertices $x \in X$, $y \in Y$ such that $F \setminus \{x, y\}$ is acyclic.

7. The special case of cycle obstructions

The linear bound defined of Theorem 8 depends on the size of the obstructions. For cycle obstructions, we show that this dependency is only of second order. For technical reasons, we have to treat $C_4$-free graphs separately.

7.1. $C_4$-free graphs. We begin with a bound of the number of cycles that contain a fixed vertex in $C_4$-free graphs.

**Lemma 6.** Given a $C_4$-free graph $G$, let $\Omega_G$ denote its set of cycles. Then

$$\Delta_{2\ell}(\Omega_G) \leq \frac{\Delta}{2}(\Delta - 1)^{2\ell-3},$$

for any integer $\ell \geq 3$.

**Proof.** Let $v_0 \in V(G)$ be a fixed vertex, and $\ell \geq 3$ a fixed integer. There are at most $\Delta(\Delta - 1)^{2\ell-3}$ paths of length $2\ell - 2$ starting from $v$, which can be computed by performing a BFS of depth $2\ell - 2$ from $v_0$. For each of these paths, there is a unique way to close it into a $2\ell$-cycle with a common neighbour of its extremities, since $G$ is $C_4$-free (if two vertices have two common neighbours, then they form a $C_4$). With this enumeration, each cycle is counted twice (clockwise and anticlockwise), hence we divide the total by 2. \qed

We note that the bound provided by Lemma 6 is asymptotically tight, since in the incidence graph $G$ of a projective plane, of maximum degree $\Delta$ (which can be any prime power plus one), it is straightforward to show that $\Delta_{2\ell}(\Omega_G) \geq \frac{\Delta}{2}(\Delta - \ell)^{2\ell-3}$.

We may now prove the upper bound on $a(d, C_4)$.

**Theorem 9.** If $G$ is $C_4$-free, then $\chi_a(G) < 2.763\Delta(G) - 1.457$.

**Proof.** Let $G$ be a $C_4$-free graph of maximum degree $\Delta$. We prove Theorem 9 through an application of Theorem 2 with $\Gamma = E(G)$ and $\Pi = \Omega_G$. We fix $\alpha := \arg\min_{x > 1} \left( x + \frac{1}{2(x^3 + 1)} \right) \approx 1.4576$ and $\tau := \alpha(\Delta - 1)$, so that there exist $\tau^{V(G)}$ proper acyclic $K$-colourings of $G$, with

$$K = \Delta + \tau + \sum_{\ell \geq 2} \frac{\Delta_{2\ell}(\Omega_G)}{\tau^{2\ell-3}} \leq \Delta + \alpha(\Delta - 1) + \frac{\Delta}{2} \sum_{\ell \geq 3} \frac{1}{\alpha^{2\ell-3}} \leq \Delta \left( 1 + \alpha + \frac{1}{2(\alpha^3 + \alpha)} \right) - \alpha < 2.763\Delta - 1.457.$$ 

\qed

Now, we extend this result to larger cycle obstructions.
7.2. Larger cycle obstructions. Let us fix an integer $t \geq 3$. Let $G$ be a $C_{2t}$-free graph of maximum degree $\Delta$. Let $v_0 \in V(G)$ be an arbitrary vertex, and let $X_i$ be the set of vertices at distance $i$ from $v_0$, for every $i \in \{0, \ldots, t\}$. We denote $H_i := G[X_{i-1}, X_i]$ for every $1 \leq i \leq t$.

**Claim 3** (Pikhurko, 2012 [11]). The maximum average degree of $H_i$ is at most $2t$, and that of $G[X_1]$ is at most $4t$, for every $1 \leq i \leq t$.

We note that the maximum average degree of $G[X_1]$ is actually at most $2t - 3$ by Theorem [11] because $G[X_1]$ is $P_{2t-1}$-free.

**Lemma 7.** Given a bipartite graph $H = (X, Y, E)$ with $|X| \leq |Y|$ and an integer $d \geq 1$, we say that an edge $e = xy \in X \times Y$ is $d$-branching if $\deg(y) \geq d$. If the maximum average degree of $H$ is at least $d$, then the number of $d$-branching edges in $H$ is at most $d|X|$.

**Proof.** Let $Y_d \subseteq Y$ be the set of vertices with at least $d$ neighbours in $X$, and let $H_d = H[Y_d, N(Y_d)]$ be the bipartite subgraph of $H$ induced by the $d$-branching edges. Since the average degree of $H_d$ is at most $d$, we infer that
\[
d \geq \frac{2|E(H_d)|}{|Y| + |N(Y)|} \geq \frac{2|E(H_d)|}{|E(H_d)|/d + |X|},
\]
and hence $|E(H_d)| \leq d|X|$, as desired. \qed

We fix a real parameter $\gamma \in (0, 1)$ that satisfies $\Delta^\gamma \geq 2t$. A $\gamma$-special pair in $G$ is a pair of vertices $(u, v)$ whose codegree is $\deg(u, v) \geq \Delta^\gamma$. We let $\Gamma_0$ be the set of $\gamma$-special pairs in $G$.

**Claim 4.** $\Delta(\Gamma_0) \leq 4t\Delta^{1-\gamma}$.

**Proof.** Let $E$ be the set of edges between $N_{G_0}(v_0)$ and $N_{G}(v_0)$. We let $E_1 := E \cap E(G[X_1])$, and $E_2 := E \cap E(H_2)$. We have $|E_1| \geq \Delta^\gamma |N_{G_0}(v_0) \cap X_1|/2$ and $|E_2| \geq \Delta^\gamma |N_{G_0}(v_0) \cap X_2|$. On the other hand, the average degree of $G[X_1]$ is at most $2t - 3$, hence $|E_1| \leq t\Delta$. Moreover, the average degree of $H_2$ is at most $2t$. If $|X_2| \leq |X_1|$, this directly implies that $|E_2| \leq 2t\Delta$. If on the other hand $|X_2| \geq |X_1|$, then since $\Delta^\gamma \geq 2t$, the edges in $E_2$ are $2t$-branching in $H_2$. By Lemma [7], we infer that $|E_2| \leq 2t\Delta$. We conclude that we have $|N_{G_0}(v_0) \cap X_1| \leq 2|E_1|\Delta^{-\gamma} \leq 2t\Delta^{1-\gamma}$, and $|N_{G_0}(v_0) \cap X_2| \leq |E_2|\Delta^{-\gamma} \leq 2t\Delta^{1-\gamma}$. The result follows. \qed

**Claim 5.** Let $\Pi$ be the set of $\Gamma_0$-free even cycles in $G$. Then $\Delta_{2\ell}(\Pi) \leq O(t\Delta^{2\ell-3+\gamma})$, for every $\ell \geq 3$.

**Proof.** Let $C = (v_0, v_1, \ldots, v_{2\ell-1})$ be a $\Gamma_0$-free cycle of length $2\ell$. Since the maximum degree in $H_2$ and in $H_3$ is at most $2t$, and that in $G[X_2]$ is at most $4t$, we may orient the edges of $G$ so that the maximum out-degree is at most $2t$ in $H_2$ and in $H_3$, and at most $4t$ in $G[X_2]$. If $v_2 \rightarrow v_3$ in the orientation of $G$, then there are at most $\Delta^2$ choices for the path $v_0, v_1, v_2, v_3$, and at most $8t$ choices for the out-going arc $v_2 \rightarrow v_3$, and finally at most $\Delta^{2\ell-5+\gamma}$ choices for the path $v_3, v_4, \ldots, v_0$. If on the other hand $v_3 \rightarrow v_2$, then there are at most $\Delta^{2\ell-4}$ choices for the path $v_0, v_2, v_4, \ldots, v_3$, at most $8t$ choices for the arc $v_2 \rightarrow v_3$, and finally at most $\Delta^\gamma$ choices for $v_1$. Overall, the total number of choices for a $\Gamma$-free cycle of length $2\ell$ that contains $v_0$ is at most $16t\Delta^{2\ell-3+\gamma}$. \qed

**Claim 6.** $\Delta_4(\Pi) \leq 2t\Delta^{1+\gamma}$
Proof. Let \( v_0 \in V(G) \), and let \( U \) be the set of vertices \( u \) such that \( 2 \leq \deg(u, v_0) < \Delta^\gamma \). Every 4-cycle in \( \Pi \) that contains \( v_0 \) must go through a vertex in \( U \).

Let \( H := G[N(v_0) \cup U] \), and let \( p(v) \) denote the number of \( \Gamma_0 \)-free 2-paths in \( H \) from any given vertex \( v \in N(v_0) \) to another vertex \( v' \in N(v_0) \) (this means that \( vv' \notin \Gamma_0 \)). Assume for the sake of contradiction that \( p(v) \geq 2\Delta^\gamma \) for every \( v \in N(v_0) \). We claim that it is possible to construct greedily a path \( P = u_0, u_1, \ldots, u_{2t-2} \) such that \( u_{2i} \in N(v_0) \) for every \( i < t \). Indeed, if we have constructed the \( 2i \)-subpath \( P_i \) of \( P \) for some integer \( 0 \leq i < t \), then we may extend it to a path of length \( 2i + 2 \) with one of the \( p(v_i) \) \( \Gamma_0 \)-free 2-paths starting in \( v_i \), which we choose to be disjoint from \( V(P_i) \). There are at most \( i\Delta^\gamma < t\Delta^\gamma \) of them that go through the set \( \{v_{2j+1}\}_{j<i} \) (because there are \( i \) choices for \( j \), and less than \( \Delta^\gamma \) choices for a neighbour of \( v_{2j+1} \) in \( N(v_0) \) given \( j \)), and there are at most \( i\Delta^\gamma < t\Delta^\gamma \) of them that go through the set \( \{v_{2j}\}_{j<i} \) (because given \( v_j \) such that \( v_{2i}v_{2j} \notin \Gamma_0 \), there are at most \( \Delta^\gamma \) common neighbours of \( v_{2i} \) and \( V_j \)), so this is always possible. This yields a contradiction since \( P + v_0 \) forms a 2-t-cycle in \( G \). We conclude that the minimum \( p(v) \) is at most \( 2t\Delta^\gamma \), and so by Lemma 4 applied on the multigraph induced by the \( \Gamma_0 \) 2-path in \( H \) between pairs of vertices in \( N(v_0) \), the average \( p(v) \) is at most \( 4t\Delta^\gamma \). Therefore, we infer that the number of \( \Gamma_0 \)-free 2-paths between pairs of vertices of \( N(v_0) \) is at most \( 2t\Delta^1+\gamma \). This is precisely the number of \( \Gamma_0 \)-free 4-cycles that contain \( v_0 \), so the result follows. \( \square \)

We are now ready to prove a uniform linear upper bound for \( a(d, C_{2t}) \) when \( 3 \leq t \leq \Delta^{1/3} \).

**Theorem 10.** Let \( t \geq 3 \) be a fixed integer. Let \( G \) be a \( C_{2t} \)-free graph of maximum degree \( \Delta \geq t^3 \). Then

\[
\chi_a(G) \leq 2\Delta + \mathcal{O}\left(t\Delta^{2/3}\right).
\]

**Proof.** Let \( \gamma = 1/3 \), and let \( \Gamma_0 \) be the set of \( \gamma \)-special pairs in \( G \). We apply Theorem 2 with \( \Gamma = E(G) \cup \Gamma_0 \) and \( \Pi \) the set of \( \Gamma_0 \)-free even cycles in \( G \). By fixing \( \tau := \Delta + \Delta^{2/3} \), this yields a \( \Gamma \)-proper \( \Pi \)-acyclic \( K \)-colouring of \( G \), with

\[
K \leq \Delta(\Gamma) + \tau + \sum_{\ell \geq 2} \frac{\Delta_2(\Pi)}{\tau^{2\ell-3}}
\]

\[
\leq 2\Delta + (4t + 1)\Delta^{2/3} + \mathcal{O}\left(t\Delta^{1/3}\right) \sum_{\ell \geq 2} \left(\frac{\Delta}{\tau}\right)^{2\ell-3} \quad \text{by Claims 4, 5, and 6}
\]

\[
\leq 2\Delta + (4t + 1)\Delta^{2/3} + \mathcal{O}\left(t\Delta^{1/3}\right) \frac{\tau\Delta}{\tau^2 - \Delta^2}
\]

\[
\leq 2\Delta + \mathcal{O}\left(t\Delta^{2/3}\right).
\]

The result follows by noting that a \( \Gamma \)-proper \( \Pi \)-acyclic \( K \)-colouring of \( G \) is in particular a proper acyclic \( K \)-colouring of \( G \), since a cycle that is not \( \Gamma_0 \)-free cannot be bicoloured in a \( \Gamma_0 \)-proper colouring. \( \square \)

### 7.3. Going below the 2\( \Delta \) threshold.

When \( \mathcal{F} \) is a collection of cycles, it seems that, given an \( \mathcal{F} \)-free graph \( G \), there is no better general upper bound on \( \Delta_2(\Omega_G) \) than \( \Delta(G)^{2t-C_{\mathcal{F}}} \) for some constant \( C_{\mathcal{F}} \). In particular, if we wish to apply Theorem 2 in order to obtain an upper bound on \( a(d, \mathcal{F}) \), then we need to have \( \tau > d \). Since we have \( \Delta(\Gamma) = d \) in that setting, there is no hope to obtain an upper bound below \( 2d \). We show, with a more involved technique that uses properties of proper colouring in sparse graphs, that it is possible to obtain an upper bound below that threshold for a small family of cycles \( \mathcal{F} \).
Theorem 11. For every graph $G$ of maximum degree $\Delta$ and of girth (at least) 7,

$$\chi_a(G) \leq \frac{\Delta}{W(1)} + O\left(\sqrt{\Delta}\right) < 1.7633 \Delta + O\left(\sqrt{\Delta}\right),$$

as $\Delta \rightarrow \infty$.

Proof of Theorem 11. Let $G$ be a graph of maximum degree $\Delta \geq 3$ and girth at least 7. We fix $\alpha := 1 + \frac{1}{\sqrt{\Delta}}$, $\tau := \alpha \Delta$, and $\sigma := \frac{1}{2(\alpha^2 - \alpha)} + \frac{3}{4} \left(\frac{1}{\alpha - \alpha}\right)^2$. Let $K := \left[\frac{\alpha^2 + \alpha}{\pi W(1)} + \sqrt{\Delta}\right]$. Using the fact that $\alpha^{p+2} - \alpha^p \geq 2(\alpha - 1) = \frac{2}{\sqrt{\Delta}}$ for all $p \geq 0$, we have that $K \leq W(\Delta) + O\left(\sqrt{\Delta}\right)$.

For any subgraph $H \subseteq G$ and subset of vertices $U \subseteq V(H)$, let $\Omega_H(U)$ be the set of even cycles of $H$ which contain every vertex of $U$. By definition, $\Omega_H := \Omega_H(\emptyset)$ is the set of all cycles of $H$. For any subset of cycles $\Pi \subseteq \Omega_H$, let $\mathcal{A}(H, \Pi)$ the set of proper acyclic $K$-colourings of $H$. The set of acyclic $K$-colouring of $H$ is $\mathcal{A}(H, \Omega_H)$.

The more cycles we consider, the more constraints we impose to the colouring of $H$, hence we have:

Fact 1. Let $\Pi_1 \subseteq \Pi_2 \subseteq \Omega_H$. Then $\mathcal{A}(H, \Pi_2) \subseteq \mathcal{A}(H, \Pi_1)$.

For a given vertex $v$, let $\Upsilon_v := \bigcup_{u_1, u_2 \in \mathcal{N}(v)} \Omega_G(\{u_1, u_2\})$ be the set of cycles of $G$ that contain at least two neighbours of $v$. To alleviate the notations, we write for any $\Pi \subseteq \Omega_G$:

- $\Pi \ominus v := \Pi \setminus \{v\}$ the cycles of $\Pi$ which do not contain $v$.
- $\Pi \boxtimes v := \Pi \cap \Upsilon_v$ the cycles of $\Pi$ which contain at most one neighbour of $v$.

Since any cycle containing $v$ necessarily contains at least two neighbours of $v$, we have:

Fact 2. $\Pi \boxtimes v \subseteq \Pi \ominus v$.

We show the following property for any subgraph $H \subseteq G$ by strong induction:

(IH 11) $\forall v \in V(H), \forall \Pi \subseteq \Omega_H, \ |\mathcal{A}(H, \Pi)| \geq \tau \ |\mathcal{A}(H - v, \Pi \boxtimes v)|$.

If $V(H)$ is empty, (IH 11) is trivially true. Suppose $V(H) \neq \emptyset$ and let $v \in V(H), \Pi \subseteq \Omega_H$.

By induction, assume (IH 11) is true for all strict subgraph $H' \subset H$.

Consider the set $\mathcal{A}(H, \Pi \boxtimes v)$, which contains the proper colourings of $H$ such that any cycle of $\Pi$ which happens to be bicoloured must contain at least two neighbours of $v$, i.e. belongs to $\Pi \cap \Upsilon_v$. Let $\mathcal{F} := \mathcal{A}(H, \Pi \boxtimes v) \setminus \mathcal{A}(H, \Pi)$ be the set of flawed colourings, for which at least one cycle of $\Pi \cap \Upsilon_v$ is indeed bicoloured. By definition, we have $|\mathcal{A}(H, \Pi)| = |\mathcal{A}(H, \Pi \boxtimes v)| - |\mathcal{F}|$.

To complete the proof of the induction, we show the following inequalities.

(i) $|\mathcal{A}(H, \Pi \boxtimes v)| \geq (\tau + \sigma) |\mathcal{A}(H - v, \Pi \boxtimes v)|$.

(ii) $|\mathcal{F}| \leq \sigma |\mathcal{A}(H - v, \Pi \boxtimes v)|$.

Proof of Inequality (i). The proof is inspired by [12]. Let $c$ be a random variable with uniform distribution on $\mathcal{A}(H - v, \Pi \boxtimes v)$.

For a colouring $c \in \mathcal{A}(H - v, \Pi \boxtimes v)$, let $L_c := |K| \setminus c(N(v))$ be the list of colours that are not present in the neighbourhood of $v$, and let $\ell_c := |L_c|$ be the size of this list. When extending $c$ to $v$ by giving it a colour of $L_c$, if a bicoloured cycle is created, it must be contain $v$ and is
therefore in $\Upsilon_v$, so these extensions of $c$ belong to $A(H, \Pi \boxplus v)$. Therefore, the _extensions_ of $A(H - v, \Pi \boxplus v)$ thus obtained are exactly the colourings of $A(H, \Pi \boxplus v)$. It follows that

$$|A(H, \Pi \boxplus v)| = \sum_{c \in A(H - v, \Pi \boxplus v)} \ell_c$$

$$= |A(H - v, \Pi \boxplus v)| \sum_{c \in A(H - v, \Pi \boxplus v)} \frac{\ell_c}{|A(H - v, \Pi \boxplus v)|}$$

$$= |A(H - v, \Pi \boxplus v)| \mathbb{E}[\ell_c].$$

Inequality (i) is therefore equivalent to $\mathbb{E}[\ell_c] \geq \tau + \sigma$.

For a vertex $u \in N(v)$ and a colouring $c \in A(H - v, \Pi \boxplus v)$, let $L_c^A(u)$ be the set of colours such that assigning them to $u$ when extending $c_{(H \setminus \{u, v\})}$ to $u$ does not induce a bicoloured cycle among those of $\Pi \boxplus v$. Let $\ell_c^A(u) := |L_c^A(u)|$.

We fix $t := \frac{\alpha}{\alpha - 1}$, and we say that a list $L_c^A$ is _short_ if its size is at most $t$. We have $\frac{1}{\ell_c^A(u)} = \alpha$.

For a colouring $c \in A(H - v, \Pi \boxplus v)$, let $L_c^0 := [K] \setminus \left\{ c(u) : u \in N(v) \land \ell_c^A(u) \leq t \right\}$ be the list of colours that do not appear among the neighbours of $v$ which have short lists. Let $k := |L_c^0|$. We have $\mathbb{E}[k] \geq K - \sum_{u \in N(v)} \mathbb{P}[\ell_c^A(u) \leq t]$ because each neighbour with a short list removes at most one colour. Let $u \in N(v)$. Using that $c$ is uniformly distributed on $A(H - v, \Pi \boxplus v)$, and that the number of extensions of a colouring $c \in A(H \setminus \{u, v\}, (\Pi \boxplus v) \circ u)$ to $u$ that creates no new bicoloured cycle is precisely $\ell_c^A(u)$, we have

$$\mathbb{P}\left[\ell_c^A(u) \leq t\right] = \frac{\left|\{c \in A(H - v, \Pi \boxplus v) : \ell_c^A(u) \leq t\}\right|}{|A(H - v, \Pi \boxplus v)|} \leq \frac{|A(H \setminus \{u, v\}, (\Pi \boxplus v) \circ u)|}{\tau |A(H \setminus \{u, v\}, (\Pi \boxplus v) \circ u)|}$$

by (IH 11)

Hence

$$(2) \quad \mathbb{E}[k] \geq K - \frac{\Delta t}{\tau} = \frac{\tau + \sigma}{W(1)} + \sqrt{\Delta} - \sqrt{\Delta} = \frac{\tau + \sigma}{W(1)}.$$

For a colouring $c \in A(H - v, \Pi \boxplus v)$ and a colour $x \in [K]$, let $N_c(x)$ be the set of neighbours $u \in N(v)$ such that $x \in L_c^A(u)$ and $\ell_c^A(u) \geq t$. For every $u \in N_c(x)$, we have $\frac{\ell_c^A(u)}{\ell_c^A(u) - 1} \leq \frac{1}{\ell_c^A(u)} \leq \frac{1}{\ell_c^A(u) - 1} = \alpha$, hence

$$\sum_{x \in [K]} \sum_{u \in N_c(x)} \frac{1}{\ell_c^A(u) - 1} \leq \sum_{x \in [K]} \sum_{u \in N_c(x)} \frac{\ell_c^A(u)}{\ell_c^A(u) - 1} \leq \alpha \sum_{x \in [K]} \sum_{u \in N_c(x)} \frac{1}{\ell_c^A(u)} \leq \alpha |N_c(x)| \leq \alpha \Delta = \tau.$$

**Claim 7.** Under the condition of event $\{x \in L_c^0\}$, we have $\mathbb{P}[x \in L_c^A] \geq \mathbb{E}\left[\prod_{u \in N_c(x)} \left(1 - \frac{1}{\ell_c^A(u)}\right)\right]$.

**Proof of the Claim.** We set $H_0 := H \setminus [n]$. Let $c_0$ be a possible realisation of $c_{H_0}$. We define $X_0 = \left\{ u \in N(v) : \ell_c^A(u) \geq t \right\}$, and $H_1 := (H - v) \setminus X_0$. Let $c_1$ be a possible realisation of $c_{H_1}$ under the assumption that $c_{H_0} = c_0$. By performing a random uniform recolouring of
$N_c(x)$ with the list-assignment $L^A_c$, the colouring remains proper because $G$ is triangle-free, and no cycle among $\Pi \sqsupset v$ becomes bicoloured. Assuming $c_{|H_1} = c_1$, we have $N_c(x) = N_{c_1}(x)$, therefore:

$$\mathbb{P}[x \in L_c \mid c_{|H_1} = c_1] = \mathbb{P}[c(u) \neq x, \forall u \in N_c(x) \mid c_{|H_1} = c_1] \geq \prod_{u \in N_c(x)} \left(1 - \frac{1}{\ell^A_c(u)}\right) \geq \mathbb{E}\left[\prod_{u \in N_c(x)} \left(1 - \frac{1}{\ell^A_c(x)}\right) \mid c_{|H_1} = c_1\right].$$

We conclude with the laws of total probability and total expectation.

We finish the proof with a sequence of inequalities, where we use the convexity of $x \mapsto x e^{-ax}$ on $[0, +\infty)$ for any constant $a \geq 0$.

$$\mathbb{E}[\ell_c] = \sum_{x \in [K]} \mathbb{P}[x \in L_c] \geq \mathbb{E}\left[\sum_{x \in L^0_c} \mathbb{P}[x \in L_c]\right] \quad \text{by Lemma 7 and linearity of the expectation,}$$

$$\geq \mathbb{E}\left[\sum_{x \in L^0_c} \prod_{u \in N_c(x)} \left(1 - \frac{1}{\ell^A_c(u)}\right)\right]^{\geq} \quad \text{using } 1 - \frac{1}{x} \geq e^{-1/(x-1)} \text{ for } x > 1,$$

$$\geq \mathbb{E}\left[k \exp\left(\frac{1}{k} \sum_{x \in L^0_c} \sum_{u \in N_c(x)} -\frac{1}{\ell^A_c(u)}\right)\right]^{\geq} \quad \text{by convexity,}$$

$$\geq \mathbb{E}[k \exp(-\tau/k)]^{\geq} \quad \text{by (3),}$$

$$\geq \mathbb{E}[k] \exp(-\tau/\mathbb{E}[k])^{\geq} \quad \text{by Jensen’s inequality,}$$

$$\geq \frac{\tau + \sigma}{W(1)} \exp\left(-\frac{\tau}{(\tau + \sigma)/W(1)}\right) \quad \text{by (2),}$$

$$\geq \tau + \sigma \quad \text{using } \frac{1}{W(1)} = \exp(W(1)).$$

We thus conclude the proof of Inequality (i).

**Proof of Inequality (ii).** If a colouring belongs to $\mathcal{F}$, it induces a bicoloured cycle among $\Pi \cap \Upsilon_v$. For a cycle $C \in \Pi \cap \Upsilon_v$, let $\mathcal{F}_c \subseteq \mathcal{F}$ be the subset of colourings in which the cycle $C$ is effectively bicoloured. We partition $\Pi \cap \Upsilon_v$ into two subsets:

- $\Lambda$ the cycles of $\Pi \cap \Upsilon_v$ which contain $v$.
- $\Phi$ the cycles of $\Pi \cap \Upsilon_v$ which do not contain $v$.

We have $|\mathcal{F}| \leq \sum_{C \in \Lambda} |\mathcal{F}_C| + \sum_{C \in \Phi} |\mathcal{F}_C|$. Using a similar argument as that of point (ii) in the proof of Theorem 9, we have $\sum_{C \in \Lambda} |\mathcal{F}_C| \leq \frac{1}{2(a^\alpha - a^\beta)} |\mathcal{A}(H - v, \Pi \sqsupset v)|$. There remains to establish an upper bound of $\sum_{C \in \Phi} |\mathcal{F}_C|$.

For $\ell \geq 4$, let $\Phi_{\ell}$ be the subset of cycles of $\Phi$ that have length $2\ell$. Recall that these cycles contain at least two neighbours of $v$, but not $v$ itself. There are at most $\binom{\Delta}{2}$ pairs of such neighbours. For such a pair $(u_1, u_2)$, we count the number of $2\ell$-cycles containing $u_1$ and $u_2$ but not $v$, using the fact that $G$ has girth at least 7. For $5 \leq i \leq \ell$, there are at most $\Delta_i^{i-3}$
paths of length \(i\) from \(u_1\) to \(u_2\), and at most \(\Delta^{2\ell-i-3}\) paths of length \(i-3\) from \(u_2\) back to \(u_1\). Therefore,

\[
|\Phi_\ell| \leq \left(\frac{\Delta}{2}\right) \sum_{i=5}^{\ell} \Delta^{i-3} \Delta^{2\ell-i-3} \leq \frac{\ell - 4}{2} \Delta^{2\ell-4}.
\]

Let \(C = (x_1, \ldots, x_{2\ell}) \in \Phi_\ell\) for some \(\ell \geq 4\), where \(v \notin C\). We have an injection from \(\mathcal{F}_C\) to \(\mathcal{A}(H \setminus \{x_1, \ldots, x_{2\ell-2}\}, (\Pi \boxdot v) \cup x_1 \cup \cdots \cup x_{2\ell-2})\). For any graph \(H' \subseteq H\) and set of cycles \(\Pi' \subseteq \Omega_{H'}\), we trivially have the inequality \(|\mathcal{A}(H', \Pi')| \leq K|\mathcal{A}(H' - v, \Pi')|\). We can easily check that \(K \leq 4\Delta\). We therefore have:

\[
|\mathcal{F}_C| \leq |\mathcal{A}(H \setminus \{x_1, \ldots, x_{2\ell-2}\}, (\Pi \boxdot v) \cup x_1 \cup \cdots \cup x_{2\ell-2})|
\]

\[
\leq 4\Delta |\mathcal{A}(H \setminus \{x_1, \ldots, x_{2\ell-2}\} - v, (\Pi \boxdot v) \cup x_1 \cup \cdots \cup x_{2\ell-2})| \quad \text{by the previous remark.}
\]

\[
\leq 4\Delta |\mathcal{A}(H \setminus \{x_1, \ldots, x_{2\ell-2}\} - v, (\Pi \boxdot v) \cup x_1 \cup \cdots \cup x_{2\ell-2})| \quad \text{using Facts 2 and 1}
\]

\[
\leq \frac{4\Delta}{\tau^{2\ell-2}} |\mathcal{A}(H - v, \Pi \boxdot v)| \quad \text{by applying (IH) 2 times.}
\]

Hence, by applying the previous inequality and Eq. (11), we have:

\[
\sum_{C \in \Phi} |\mathcal{F}_C| \leq \sum_{\ell \geq 2} \sum_{C \in \Phi_\ell} |\mathcal{F}_C| \leq \sum_{\ell \geq 4} 4\Delta \frac{|\Phi_\ell|}{\tau^{2\ell-2}} |\mathcal{A}(H - v, \Pi \boxdot v)|
\]

\[
\leq \sum_{\ell \geq 4} (\ell - 4) \frac{2}{\Delta} \frac{1}{\alpha^{2\ell-2}} |\mathcal{A}(H - v, \Pi \boxdot v)| = \frac{2}{\Delta} \sum_{\ell \geq 0} \frac{\ell}{\alpha^{2\ell}} |\mathcal{A}(H - v, \Pi \boxdot v)|
\]

\[
\leq \frac{2}{\Delta} \left(\frac{1}{\alpha^3 - \alpha}\right)^2 |\mathcal{A}(H - v, \Pi \boxdot v)|.
\]

We conclude the proof of Inequality (ii).

\[
|\mathcal{F}| \leq \sum_{C \in \Lambda} |\mathcal{F}_C| + \sum_{C \in \Phi} |\mathcal{F}_C|
\]

\[
\leq \frac{1}{2(\alpha^3 - \alpha)} |\mathcal{A}(H - v, \Pi \boxdot v)| + \frac{2}{\Delta} \left(\frac{1}{\alpha^3 - \alpha}\right)^2 |\mathcal{A}(H - v, \Pi \boxdot v)|
\]

\[
\leq \sigma |\mathcal{A}(H - v, \Pi \boxdot v)|.
\]

\(\diamondsuit\)

Using inequalities (i) and (ii), we have \(|\mathcal{A}(H, \Pi)| \geq \tau |\mathcal{A}(H - v, \Pi \boxdot v)|\), which ends the proof of the induction.

An iterative application of (IH) to all the vertices of \(V(G)\) implies that \(|\mathcal{A}(G, \Omega_G)| \geq \tau^{|V(G)|}\), therefore a \(K\)-acyclic colouring of \(G\) exists.

Observe that in the above proof, we have never used the fact that \(G\) is \(C_5\)-free. Hence we have the following result as a corollary.

**Corollary 3.** For every integer \(d\) large enough,

\[
a(d, \{C_3, C_4, C_6\}) < 1.7633d.
\]

**8. Non-trivial superlinear bounds**

Now, we determine a loose condition for a single-graph obstruction \(F\) to yield an upper bound on \(a(H, F)\) that is significantly smaller than the general one \(O\left(d^{4/3}\right)\).

For a fixed 2-acyclic graph \(F\) on \(t\) vertices, we determine that \(a(d, F) = O\left(t^{1/4}d^{5/4}\right)\). The proof is similar to that of Theorem 8, it differs only in the upper bound on the density of a
specific subgraph that does not contain a fixed 2-acyclic obstruction, which is weaker than the corresponding one for 1-acyclic obstructions.

Lemma 8. Let $F$ be a fixed 2-acyclic bipartite graph with $t \geq 2$ vertices. Given an $F$-free graph $G$ of maximum degree $\Delta > t$, let $\Omega_G$ denote its set of cycles. Then

$$\Delta_{2\ell}(\Omega_G) \leq (2 + 2\sqrt{2})\sqrt{t-1} \Delta^{2\ell - \frac{3}{2}},$$

for any integer $\ell \geq 2$.

Proof. Let $u \in V(G)$. There are at most $\Delta^{2\ell - 3}$ $(2\ell - 3)$-paths starting from $u$. Let $v$ be the other endpoint of such a path; we claim that there are at most $(4 + 4\sqrt{2})\sqrt{t-1} \Delta^{3/2}$ 3-paths between $u$ and $v$. By combining these two bounds we count every cycle twice, therefore there are at most $2(2 + 2\sqrt{2})\sqrt{t-1} \Delta^{2\ell - 3/2}$ cycles of length $2\ell$ in $G$ containing $u$. We now prove the claim.

We may assume that $\deg(u) \geq t$ and $\deg(v) \geq t$, otherwise there are at most $(t-1)\Delta$ 3-paths between $u$ and $v$, thus proving the claim by assumption that $\Delta > t$. Let $X := N(u) \cap N(v)$, $U := N(u) \setminus X$ and $V := N(v) \setminus X$. Each edge in $G[U, V]$, $G[U, X]$ and $G[V, X]$ defines a unique 3-path from $u$ to $v$, while each edge in $G[X]$ corresponds to two such paths, depending on the direction in which it is traversed. These are the only 3-paths from $u$ to $v$. Let $w \in V(F)$ be such that $F - w$ is 1-acyclic. Since $G$ is $F$-free, $G[X]$ must be $(F - w)$-free. By Lemma 4 there are at most $2\sqrt{t-1} |X|^{3/2}$ edges in $G[X]$. We now prove a similar result for $G[U, V]$ and $G[V, X]$.

We say that a neighbour of $w$ is trivial if it is isolated in $F - w$. Let $(F_i)_{i \in [r]}$ be the nontrivial connected components of $F - w$. Define $F'$ by adding an edge between $F_i$ and $F_{i+1}$ for $i \in [r-1]$ in such a way that each vertex of $F'$ lies in the same part as it does in $F$. $F'$ is connected, and since $F - w$ is 1-acyclic and the previous construction does not create additional cycles, $F'$ is also 1-acyclic.

By contradiction, suppose $G[U, V]$ contains $F'$ as a subgraph. By the bipartiteness of $F$ and $G[U, V]$, and by definition of $H'$, the nontrivial neighbours of $w$ in $F$ all lie in a single part of $G[U, V]$. Suppose without loss of generality that they lie in $U$, and are therefore adjacent to $u$. We use the assumption that $d(u) \geq t$ to identify the trivial neighbours of $w$ among $N(v)$. This yields a copy of $F$ in $G$, a contradiction. Therefore $G[U, V]$ is $F'$-free, and by Lemma 4, $G[U, V]$ contains at most $2\sqrt{t-1} |U| + |V|^{3/2}$ edges. In the same way, we show that $G[U, X]$ and $G[V, X]$ respectively contain at most $2\sqrt{t-1}(|U| + |X|)^{3/2}$ and $2\sqrt{t-1}(|V| + |X|)^{3/2}$ edges.

Using that $|U| + |X| \leq \Delta$, $|V| + |X| \leq \Delta$, and so $|U| + |V| \leq 2\Delta - 2|X|$, we conclude that there are at most $(4 + 4\sqrt{2})\sqrt{t-1} \Delta^{3/2}$ paths of length 3 between $u$ and $v$, which proves the claim. $\blacksquare$

Theorem 12. Let $F$ be a fixed 2-acyclic bipartite graph on $t \geq 2$ vertices. Then for every graph $G$ of maximum degree $\Delta > t$,

$$\chi_\alpha(G) = O\left(t^{1/4} \Delta^{\frac{5}{4}}\right).$$

Proof. Let $G$ be an $F$-free graph of maximum degree $\Delta > t$. We prove Theorem 12 through an application of Theorem 2 with $\Gamma = E(G)$ and $\Pi = \Omega_G$. Let $c > 1$ be a constant to be chosen later. We fix $\alpha := c(t - 1)^{1/4}$ and $\tau := \alpha \Delta^{5/4}$, so that there exist $\tau|V(G)|$ proper acyclic $K$-colourings of $G$, with
\[ K = \left[ \Delta + \tau + \sum_{\ell \geq 2} \frac{\Delta 2^\ell (\Omega_G)}{\tau^{2\ell-3}} \right] \]

\[ \leq \Delta + \alpha \Delta^{5/4} + (2 + 2\sqrt{2}) \sqrt{t-1} - 1 \sum_{\ell \geq 2} \frac{\Delta^{\ell - \frac{5}{4}}}{\alpha^{2\ell-3}} + 1 \] by Lemma \[ \frac{\alpha}{4} \]

\[ \leq \Delta + \alpha \Delta^{5/4} + \frac{2 + 2\sqrt{2}}{c} (t - 1)^{1/4} \Delta^{5/4} + o(\Delta) \]

\[ \leq (t - 1)^{1/4} \Delta^{5/4} \left( c + \frac{2 + 2\sqrt{2}}{c} \right) + \Delta + o(\Delta). \]

The first term is minimal when \( c = \sqrt{2 + 2\sqrt{2}} \), which yields

\[ K < 4.3948 t^{1/4} \Delta^{5/4} + \Delta + o(\Delta). \]

\[ \square \]

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