Numerical studies of the optimization of the first eigenvalue for the heat diffusion in inhomogeneous media

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Abstract

In this paper, we study optimization of the first eigenvalue of \(-\nabla(\rho(x)\nabla)u = \lambda u\) on a bounded domain \(\Omega \subset \mathbb{R}^n\) under several constraints for the function \(\rho\). We consider this problem in various boundary conditions and various type of topology of domains. As a result, we numerically observe several common criteria of \(\rho\) for optimizing eigenvalues in terms of corresponding eigenfunctions, which are independent of topology of domains and boundary conditions. The geometric characterization of optimizers are also numerically observed.

Keywords: eigenvalue problem, topology optimization, dependence of optimizers on topology.

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1 Introduction

In this paper we consider the following eigenvalue problem

\[-\nabla(\rho(x)\nabla)u = \lambda u, \quad x \in \Omega, \quad \rho \in \mathcal{K}\]

(1.1)

for a bounded domain \(\Omega\) in \(\mathbb{R}^n\) with suitable boundary conditions, where

\[\mathcal{K} := \left\{ \rho \in L^\infty(\Omega) \mid \rho = 1 \text{ or } c, \text{ a.e. on } \Omega, \int_\Omega \rho dx = (cm_0 + (1 - m_0))|\Omega| \right\},\]

(1.2)

\(|\Omega|\) denotes the Lebesgue measure of \(\Omega\) in \(\mathbb{R}^n\). Let \(S := \{x \in \Omega \mid \rho(x) = c\}\), then (1.2) immediately implies

\[|S|/|\Omega| = m_0.\]

(1.3)

Under these setting, we consider the following problem:

Problem 1.1. Find \(\rho_* \in \mathcal{K}\) which attains the supremum or infimum of the first eigenvalue \(\lambda_1(\rho)\) on \(\mathcal{K}\) under suitable boundary condition. If exists, characterize the shape of the domain \(S_* = \{x \in \Omega \mid \rho_*(x) = c\}\).

We shall call the smallest positive eigenvalue “the first eigenvalue” since we would consider this problem under Dirichlet, Neumann or mixed boundary condition.

Our motivation in the above problem is as follows. Assume two different materials in a given domain \(\Omega\) with fixed volume ratio of these materials. How do we arrange such materials to optimize the heat conductivity of \(\Omega\)? Since the long-time behaviour of heat transfer of the heat equation is controlled by the first eigenvalue of \(-\nabla(\rho(x)\nabla)u\), we would consider the Problem 1.1 as one of toy models of this problem.

In this paper, we numerically study the eigenvalue optimization for (1.1) and we observe the following results for Problem 1.1:

1. The element \(\rho_* \in \mathcal{K}\) optimizing \(\lambda_1(\rho)\) can be characterized by inequalities with respect to \(\rho_*(\nabla u_*)\), where \(u_*\) is the eigenfunction associated with \(\lambda_1(\rho_*)\) of (1.1). As a consequence, the domain \(S_* = \{x \in \Omega \mid \rho_*(x) = c\}\) is given by super- or sub-level set of \(|\rho_*\nabla u_*|\). This characterization is independent of topology and geometry of \(\Omega\) and boundary conditions on \(\partial\Omega\).

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2. If $\Omega$ is symmetric, then so is $S^\ast$.

3. Optimized region $S^\ast$ depends continuously on boundary conditions on $\partial \Omega$.

The precise statements are described in Section 4 (Observation 4.2 - 4.4) with various numerical results, which lead to expectations that all these results hold with mathematical rigor.

This paper is organized as follows. In Section 2, we provide more precise setting of our problems. A numerical method for finding optimizers we apply here, the level set approach, is also derived therein. In Section 3, numerical and mathematical known results for a well-considered problem is discussed. Section 4 is where our main discussion is developed. We show several numerical observations about eigenvalue optimization criteria, geometry of the level set $S^\ast = \{ x \in \Omega \mid \rho^\ast(x) = c \}$ for the optimizer $\rho^\ast$, and continuous dependence of $S^\ast$ on boundary conditions.

2 Setting

2.1 Setting of problems

Here we provide the precise setting of Problem 1.1.

Problem 2.1 (Precise version of Problem 1.1). For $\rho \in K$ define

$$\lambda_1(\rho) := \inf_{u : \text{admissible}} \frac{\int_{\Omega} \rho(x) |\nabla u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx}. \quad (2.1)$$

Find an element $\rho^\ast \in K$ which attains

$$\sup_{\rho \in K} \lambda_1(\rho). \quad (2.2)$$

If exists, characterize $\rho^\ast$ and the shape of domain $S^\ast = \{ x \in \Omega \mid \rho^\ast(x) = c \}$.

We also consider same question for $\inf_{\rho} \lambda_1(\rho)$.

It is mathematically known that the eigenvalue of the linear operator $A_\rho u = -\nabla(\rho(\cdot) \nabla)u$ is characterized by the Rayleigh quotient (2.1) (see e.g. [3]). Here we also consider the minimization of $\lambda_1(\rho)$ as a comparison of maximization.

We consider the problem in the case of various type of domains like ones with piecewise smooth boundary, non-convex or non-simply connected ones, as well as various boundary conditions (Dirichlet, Neumann or mixed boundary condition).

Remark 2.2. A well-known mathematical theory (see e.g. [2]) tells us that, in each case, the eigenvalue problem possesses discrete eigenvalues

$$(0 <) \lambda_1(\rho) \leq \lambda_2(\rho) \leq \cdots \to \infty. \quad (2.3)$$

In the case of homogeneous Dirichlet boundary value problem, 0 is not an eigenvalue and hence the infimum in (2.1) is attained in $u$ being not identically zero. Thus, in case of Dirichlet boundary value problems, we call $u$ is admissible if and only if $u \not= 0$ in an appropriate function space.

On the other hand, in the case of homogeneous Neumann boundary value problems, 0 is admitted as an eigenvalue with a constant function as an eigenfunction. Therefore, 0 is the smallest eigenvalue. According to the general theory of eigenfunctions, eigenfunctions associated with $\lambda_1(\rho)$ have to be orthogonal to constant functions in an inner product on investigating functional space $L^2(\Omega)$. Hence we call $u$ is admissible if and only if $u$ satisfies $\int_{\Omega} u \, dx = 0$.

2.2 The level set method

Our problem is one of typical problems called topology optimization. One of well-known methods for determining topology of the optimal objective is the level set approach, originally developed in Osher and Sethian [12]. This method is a kind of topology optimization methods, namely, tracing the time evolution of shapes of the objective including the change of its topological type. This method provides an efficient way of describing time evolving curves and surfaces which may undergo topological change from the initial shape to the desiring topology. Osher and Santosa [13] improve the method so that this method can be applied to optimization problems with one or
more constraints like the volume constraint. We follow their method as a numerical approach for
determining optimizers in our problems. Here we briefly review implementations of the level set
approach discussed in [13].

We represent a subset $S \subset \Omega$ as the super-level set of a function $\phi : \Omega \to \mathbb{R}$, i.e.
$$ S = \{ x \in \Omega \mid \phi(x) > 0 \}. $$

Then the function $\rho$ in Problem 2.1 can be represented as a function of $\phi$. Consequently, $\lambda_1$ in
Problem 2.1 can be also regarded as a function of $\phi$ because $\lambda_1$ originally depends on $\rho$. Therefore,
our optimization problem results in the minimization of the following energy functional $L$ :
$$ L(\phi) = F(\phi) + \nu G(\phi), $$

where $F(\phi) = \lambda_1(\phi) := \lambda_1(\rho)$ for minimizing eigenvalues and $F(\phi) = -\lambda_1(\phi)$ for maximizing
eigenvalues, $\nu$ is the Lagrange multiplier and $G(\phi) = \int_{\phi(x) > 0} dx - m_0|\Omega|$. Remark that the equation
$G(\phi) = 0$ corresponds to the volume constraint (1.3).

The first variation of $L$ and $\nu$ can be calculated by ordinary methods. Level set approach
is reduced to solve the gradient flow associated with $L$, namely, the evolutionary equation whose
solution orbit decreases $L$. More precisely, our concern is reduced to solve the following evolutionary
equation :
$$ \frac{\partial \phi}{\partial t} = -(v_0 + \nu)|\nabla \phi| \text{ on } \partial S \quad (2.5) $$
in a certain functional space, where $v_0 = v_0(x)$ is given by
$$ v_0(x) = \frac{c - 1}{\int_{\Omega} u_\phi^2 dx} |\nabla u_\phi(x)|^2, \quad (2.6) $$

where $u_\phi$ is the associated eigenfunction of $\lambda_1(\phi)$. We omit the detail here how to obtain this
system since it is completely done by following the method in [13].

Remark 2.3. In practice, direct computation of (2.5) may cause a loss of smoothness of $\phi$ be-
cause, especially in Problem 2.1, the right hand side in (2.5) possesses a point in $\Omega$ losing $\phi$’s
smoothness. To avoid such a difficulty, we artificially add a viscosity term with sufficiently small positive coefficient $\epsilon$ to (2.5), namely, we solve
$$ \frac{\partial \phi}{\partial t} = \epsilon \Delta \phi - (v_0 + \nu)|\nabla \phi| \quad (2.7) $$

instead of (2.5) itself. Then (2.7) is just a semi-linear parabolic evolutionary equation in the
ordinary sense and hence the well-known explicit or implicit scheme enables us to solve this equation
numerically so that $\phi$ keeps its smoothness during evolutions. This technique is well-known as
viscosity vanishing method. As for Hamilton-Jacobi type equations like (2.5), it is known that
solutions of (2.7) approach to those of (2.5) as $\epsilon \to 0$ in a suitable sense (see e.g. [6]).

3 Known results

As for eigenvalue optimization problems, there are a lot of preceding works for
$$ - \Delta u = \mu \sigma(x) u, \quad x \in \Omega, \quad \sigma \in \mathcal{K}. $$

This is a generalized eigenvalue problem of Laplacian, which is, for example, well-considered for
studying a frequency of drums with spatially inhomogeneous density on $\Omega$. Osher-Sethian [12] and
Osher-Santosa [13] considered (3.1) as a problem of the level set approach (Section 2.2) and has
been well treated during a variety of improvements and extension of this approach. In [12, 13] and
many related works, the following problem is considered, which is an alternative one of Problem
2.1.

Problem 3.1. For $\sigma \in \mathcal{K}$ define
$$ \mu_1(\sigma) := \inf_{u : \text{admissible}} \frac{\int_{\Omega} |\nabla u(x)|^2 dx}{\int_{\Omega} \sigma(x)|u(x)|^2 dx}. \quad (3.2) $$
Find an element \( \sigma_* \in \mathcal{K} \) which attains
\[
\inf_{\sigma \in \mathcal{K}} \mu_1(\sigma).
\] (3.3)

If exists, characterize \( \sigma_* \) and the shape of domain \( S_* = \{ x \in \Omega \mid \sigma_*(x) = c \} \).

For Problem \ref{prob:3.1} there are several mathematical works for determining \( \sigma_* \) which optimizes \( \mu_1(\sigma) \) as well as the geometry of the domain \( S_* = \{ x \in \Omega \mid \sigma_*(x) = c \} \).

In \cite{krein1957}, Krein mathematically considered the largest or smallest \( k \)-th natural frequency of fixed endpoint strings on the interval \((0,1)\). In words of Problem \ref{prob:3.1}, he considered \( \mu_k, \ k \in \mathbb{N} \), on \( \Omega = (0,1) \) with homogeneous Dirichlet boundary condition. He completely solved this problem by detecting the optimizer \( \sigma_* \). A couple of decades later, Cox and McLaughlin extended Krein’s arguments in arbitrary dimensions \cite{cox1963}. They obtained the following optimization criteria for eigenvalues in terms of corresponding eigenfunctions.

**Theorem 3.2** (Cox and McLaughlin \cite{cox1963}). Consider Problem \ref{prob:3.1} with homogeneous Dirichlet boundary condition. If \( \sigma_{\min} \in \mathcal{K} \) is the minimizer of \( \mu_1 \) in \( \mathcal{K} \), then there is a positive constant \( \alpha > 0 \) such that, for the corresponding eigenfunction \( u_1 \), the following inequalities hold:
\[
\sigma_{\min}(x) = c \quad \Rightarrow \quad u_1(x) \geq \alpha,
\]
\[
\sigma_{\min}(x) = 1 \quad \Rightarrow \quad u_1(x) \leq \alpha.
\]

Similarly if \( \sigma_{\max} \in \mathcal{K} \) is the maximizer of \( \mu_1 \) in \( \mathcal{K} \), then there is a positive constant \( \alpha' > 0 \) such that the following inequalities hold:
\[
\sigma_{\max}(x) = c \quad \Rightarrow \quad u_1(x) \leq \alpha',
\]
\[
\sigma_{\max}(x) = 1 \quad \Rightarrow \quad u_1(x) \geq \alpha'.
\]

Moreover, they obtained results for geometries of \( S \) which attained optimization of the first eigenvalue using symmetry arguments derived from the maximum principle.

**Theorem 3.3** (Cox and McLaughlin \cite{cox1963}). Consider Problem \ref{prob:3.1} with homogeneous Dirichlet boundary condition. Assume that \( \sigma_{\min} \in \mathcal{K} \) is the minimizer of \( \mu_1 \) in \( \mathcal{K} \) and \( S_{\min} = \{ x \in \Omega \mid \sigma_{\min}(x) = c \} \). If \( \Omega \) is convex and symmetric in \( N \) orthogonal directions, then so is \( S_{\min} \). Moreover, \( S_{\min} \) is star-shaped with respect to the center of symmetry. Similarly, if \( \sigma_{\max} \in \mathcal{K} \) is the maximizer of \( \mu_1 \) in \( \mathcal{K} \) and \( S_{\max} = \{ x \in \Omega \mid \sigma_{\max}(x) = c \} \). Then, under the same assumption as minimizers, the same statements hold for \( S_{\max}^c \).

Theorem \ref{thm:3.3} also claims that geometry of \( S_{\min} \) and \( S_{\max} \) are also reduced to the analysis of eigenfunctions since \( S_{\max} \) and \( S_{\max}^c \) are characterized by corresponding eigenfunctions. However, these researches focus on Problem \ref{prob:3.1} with homogeneous Dirichlet boundary condition. On the other hand, the same problem neither with other boundary conditions nor in Problem \ref{prob:2.1} are concerned therein and there are few studies about them (even numerically) in last a few decades.

**Remark 3.4.** Lou and Yanagida \cite{lou2007} discuss an indefinite linear eigenvalue problem related to biological invasion of species. The formation of their problem is similar to Problem \ref{prob:3.1} with the homogeneous Neumann condition, but the assumption of the weight \( \sigma(x) \) is different. That is, \( \sigma \) is bounded by a positive and negative constant and the total weight is a fixed negative constant. In \cite{lou2007}, they concentrate on one-dimensional problems and prove that the global minimizer \( \sigma_* \) of the principal eigenvalue \( \mu_1 \) (i.e. the first eigenvalue in our arguments) should be the specific two-valued function, which is often called “bang-bang” type. Such bang-bang type weight \( \sigma_* \) is characterized by the super-level set of the eigenfunction associated with \( \mu_1(\sigma_* \) ), which is similar to Theorem 3.2.

### 4 Numerical Study

#### 4.1 Optimization criteria

We consider Problem \ref{prob:2.1} by the level set method. This problem is well-considered neither mathematically nor numerically so far but the same level set approach as in the case of Problem \ref{prob:3.1} can be applied. If no confusion arises, let \( S = \{ x \in \Omega \mid \rho(x) = c \} \) for \( \rho \) under consideration in this section.
Recall that optimizers of eigenvalues are determined by the stationary solution of (2.5). As shown in (2.5), stationary solutions, namely solutions of \( \partial \phi / \partial t = 0 \), are essentially characterized by \(|\nabla u|\) being constant on \( \partial S \). Thus one can expect that \(|\nabla u|\) determines the optimizer.

First we study one-dimensional problems on \( \Omega_0 := (0, 1) \subset \mathbb{R} \) shown in Figures 1 and 2. As shown in Figure 1, the differential \( u' = \nabla u = du/dx \) of the eigenfunction \( u \) has discontinuities on \( \partial S \) and there is little hope of the correspondence between \( S \) for optimizers and \( u' \). Next we consider the correspondence between \( S \) for optimizers and \( pu' \) instead of \( u' \) itself. The graph of \( pu' \) and \( S \) for optimizers are shown in Figure 2. Both in the case of Dirichlet and Neumann boundary conditions, there seems to be a correspondence between two objects similar to Theorem 3.2.

Next, we consider two-dimensional problems. We fix \( e = 1.1 \) and the constant of volume constraint \( m_0 = 0.5 \) in (1.3) unless otherwise noted and consider the following \( \Omega \):

\[
\begin{align*}
\Omega_1 & := (-\pi, \pi) \times (-\pi, \pi) \subset \mathbb{R}^2, \quad \text{(square)} \\
\Omega_2 & := (0, 3) \times (0, 3) \setminus \{(0, 1] \times (0, 1] \cup \{(0, 1] \times [2, 3]) \cup [2, 3] \times (0, 1] \cup [2, 3] \times [2, 3]) \} \subset \mathbb{R}^2, \quad \text{(disc, e.g. Figure 5)} \\
\Omega_3 & := \{(x^2 + y^2 < 1) \} \subset \mathbb{R}^2, \quad \text{(disc)} \\
\Omega_4 & := \{(x^2 + y^2 < 1) \} \setminus \{(x - 0.5)^2 + y^2 < 0.04 \} \cup \{(x + 0.5)^2 + y^2 < 0.04 \} \subset \mathbb{R}^2, \quad \text{(disc with two holes, e.g. Figure 9)} \\
\Omega_5 & := (-2\pi, 2\pi) \subset \mathbb{R}^2, \quad \text{(rectangle)} \\
\Omega_6 & := (0.5 \times (0, 3) \setminus \{(0, 2] \times (0, 1] \cup \{(0, 2] \times [2, 3]) \cup [3, 5] \times (0, 1] \cup [3, 5] \times [2, 3]) \} \subset \mathbb{R}^2, \quad \text{(crossed rectangulars, e.g. Figure 14)} \\
\Omega_7 & := \{x^2 + y^2/4 < 1 \} \subset \mathbb{R}^2 \quad \text{and} \\
\Omega_8 & := \{x^2 + y^2/4 < 1 \} \setminus \{(x - 0.5)^2 + y^2/16 < 0.04 \} \cup \{(x + 0.5)^2 + y^2/16 < 0.04 \} \subset \mathbb{R}^2. \\
\end{align*}
\]

In two-dimensional problems we used the FreeFEM++ library \( \mathbb{F} \) for whole computations.

Focus on Figure 3 \( 8 \) \( 9 \) \( 10 \) \( 11 \) \( 12 \) \( 14 \) \( 16 \) and \( 18 \). In any cases, \( S \) for optimizers corresponds to the super- or sub-level set of \( |\rho \nabla u_1| \) regardless of the topology of \( \Omega \) and boundary conditions as in one-dimensional problems, where \( u_1 \) is the associated eigenfunction of \( \lambda_1(\rho_*) \). More precisely, if \( \rho \in K \) minimizes or maximizes \( \lambda_1 \) and if \( \lambda_1 \) is simple (see Table 1), then the corresponding eigenfunction \( u_1 \) satisfies

\[
\begin{align*}
\text{Minimization} : & \quad |\rho(x)\nabla u_1(x)| \leq |\rho(y)\nabla u_1(y)| \quad \text{a.e. } x \in S \text{ and } y \in S^c. \\
\text{Maximization} : & \quad |\rho(y)\nabla u_1(y)| \leq |\rho(x)\nabla u_1(x)| \quad \text{a.e. } x \in S \text{ and } y \in S^c.
\end{align*}
\]

Inequalities (4.1) and (4.2) can be considered the analogue of those in Theorem 3.2. If (4.1) and (4.2) holds, then the simpleness of \( \lambda_1(\rho) \) holds and vice versa (Table 1). Indeed, if \( \lambda_1 \) is not simple, then the inequality (4.2) does not hold like the case shown in Figure 20 and Table 1. Nevertheless, it is natural to consider that the optimizer of the \( k \)-tuple eigenvalue \( \lambda_1(\rho) \) can be characterized by the collection of independent eigenfunctions \( \{u_i\}_{i=1}^k \) associated with \( \lambda_1(\rho) \).

**Remark 4.1.** The maximum principle guarantees the simpleness of the smallest eigenvalue for (uniformly) elliptic operators. In the case of (1.1) with homogeneous Dirichlet boundary condition, the smallest eigenvalue \( \lambda_1 \) is positive. Therefore, \( \lambda_1(\rho) \) is always simple for any \( \rho \in K \) in the case of Dirichlet boundary condition. However, in the case of homogeneous Neumann boundary conditions, the smallest eigenvalue is not \( \lambda_1(\rho) \) but 0. The maximum principle only guarantees the simpleness of 0 eigenvalue. On the other hand, the smallest positive eigenvalue \( \lambda_1(\rho) \) is actually the second smallest one, so simpleness of \( \lambda_1(\rho) \) does not guaranteed without any additional assumptions.

Assume that \( \rho_{\max} \) attains the maximum of \( \lambda_1(\rho) \) on \( K \) and that \( \lambda_1(\rho_{\max}) \) has multiplicity two as shown in Figure 20. Then the energy functional in (2.4) with eigenvalue \( \lambda_1(\rho_{\max}) = \lambda_2(\rho_{\max}) \) with multiplicity two can be also written by

\[
L(\phi) = F(\phi) + \nu_1 G_1(\phi) + \nu_2 G_2(\phi),
\]

\[
F(\phi) = -\lambda_1(\phi) = -(a_1 \lambda_1(\phi) + a_2 \lambda_2(\phi)), \quad a_1, a_2 \geq 0 \text{ with } a_1 + a_2 = 1,
\]

\[
G_1(\phi) = \int_{\phi(x) > 0} dx - m_0|\Omega|, \quad G_2(\phi) = \lambda_2(\phi) - \lambda_1(\phi), \quad \nu_1, \nu_2 \in \mathbb{R}.
\]
Corresponding \( v_0(x) \) in (2.6) for obtaining the steepest descent flow is

\[
v_0(x) = -\left( \frac{a_1(c - 1)}{\int_{\Omega} u_1^2, \phi \, dx} |\nabla u_1, \phi(x)|^2 + \frac{a_2(c - 1)}{\int_{\Omega} u_2^2, \phi \, dx} |\nabla u_2, \phi(x)|^2 \right).
\]

Since \( \lambda_1(\phi) = \lambda_2(\phi) \) holds for optimizers, then one can see that the optimizer should satisfy

\[
a_1'|\nabla u_1(x)|^2 + a_2'|\nabla u_2(x)|^2 \equiv \text{constant on } \partial S, \quad \text{where } a_i' = a_i(\int_{\Omega} u_i^2, \phi \, dx)^{-1}. \quad \text{Recalling the definition of } S \text{ via } \phi, \rho = 1 \text{ holds on } \partial S \text{ and hence the above equality on } \partial S \text{ is equivalent to}
\]

\[
\rho(x)^s(a_1'|\nabla u_1(x)|^2 + a_2'|\nabla u_2(x)|^2) \equiv \text{constant on } \partial S \quad \text{for some } s > 0. \tag{4.3}
\]

Moreover, the constraint \( G_1(\phi) = 0 \) and \( G_2(\phi) = 0 \) must be kept during evolution of \( \phi \) via (2.5). Optimizers then have to satisfy

\[
D_\phi G_i(\phi) \delta \phi = 0, \quad i = 1, 2
\]

for the first variation \( \delta \phi \) of \( \phi \). In particular, it follows that

\[
\int_{\partial S} \rho(x)^s \left( \frac{a_1|\nabla u_1, \phi(x)|^2}{\int_{\Omega} u_1^2, \phi \, dx} - \frac{a_2|\nabla u_2, \phi(x)|^2}{\int_{\Omega} \sigma(\phi) u_2^2, \phi \, dx} \right) \, ds(x) = 0
\]

should be satisfied by calculations discussed in [13]. In particular, positive constants \( a_1' \) and \( a_2' \) must be identical and hence \( a_1, a_2 > 0 \).

Since \( a_1 \) and \( a_2 \) are arbitrary, we may choose \( a_1 = a_2 = 1/2 \). If we normalize eigenfunctions \( u_1 \) and \( u_2 \) so that \( \int_{\Omega} |u_i|^2 = 1 \) \( (i = 1, 2) \), then the following inequality will be the maximization criterion for \( \lambda_1(\rho) \) with multiplicity two:

\[
\rho(y)^2(|\nabla u_1(y)|^2 + |\nabla u_2(y)|^2) \leq \rho(x)^2(|\nabla u_1(x)|^2 + |\nabla u_2(x)|^2), \quad x \in S \text{ and } y \in S^c. \tag{4.4}
\]

Here we chose \( s = 2 \) in (4.3), which seems to be natural because \( \rho \) and \( \nabla u \) have the same order in (4.1) and (4.2). We can see that (4.4) is actually satisfied (see Figure 20).

We conclude our numerical observations for optimization criteria in Problem 2.1.

**Observation 4.2.** Consider Problem 2.1 with bounded domain \( \Omega \subset \mathbb{R}^n \). If \( \rho_{\text{min}} \in \mathcal{K} \) is the minimizer of \( \lambda_1 \) in \( \mathcal{K} \), then the eigenfunction \( u_1 \) associated with \( \lambda_1 \) satisfies (4.1). Similarly, if \( \rho_{\text{max}} \in \mathcal{K} \) is the maximizer of \( \lambda_1 \) in \( \mathcal{K} \) and if \( \lambda_1(\rho_{\text{max}}) \) is simple, then the eigenfunction \( u_1 \) associated with \( \lambda_1(\rho_{\text{max}}) \) satisfies (4.2). If \( \rho_{\text{max}} \in \mathcal{K} \) is the maximizer of \( \lambda_1 \) in \( \mathcal{K} \) and if \( \lambda_1(\rho_{\text{max}}) \) has multiplicity two, then the eigenfunction \( u_1 \) associated with \( \lambda_1(\rho_{\text{max}}) \) satisfies (4.4).

This optimization criteria will be able to generalized to higher dimensional problems, in particular, the case that \( \lambda_1(\rho) \) has multiplicity \( k \geq 3 \) in the similar manner.

### 4.2 Geometry of optimizers

Next, we consider the geometry of the domain \( S_* = \{ x \in \Omega \mid \rho_*(x) = c \} \) defined by the optimizer \( \rho_* \). Inequalities (4.1), (4.2) and (4.4) imply that various properties of \( S_* \) for optimizers come from corresponding eigenfunctions. It is natural to expect that *some geometric properties inherit from \( \Omega \).* Here, for example, we would consider connectivity, convexity and star-shapedness as geometric properties.

For example, in Figure 20 (square with Dirichlet boundary), neither \( S_* \) nor \( S_*^c \) are even connected even if \( \Omega \) is convex. In the case of Neumann boundary value problems, either \( S_* \) or \( S_*^c \) is convex if \( \Omega \) is convex, according to our computation results. On the other hand, in case of the non-simply connected domain \( \Omega_1 \) (Figure 28), \( S_* \) maximizing \( \lambda_1(\rho) \) is not connected, although \( \Omega_1 \) is connected. However, according to the inequality (4.2), one can easily confirm that we can choose \( m_0 \in (0, 5, 1) \) so that \( S_* \) maximizing \( \lambda_1(\rho) \) is connected. As a consequence, there are generally no topological correspondences between \( \Omega \) and \( S_* \) (or \( S_*^c \)) which are independent of boundary conditions on \( \partial \Omega \) or \( m_0 \).

Next we focus on the symmetry of \( S_* \). Inequalities (4.1) and (4.2) imply that the symmetry of \( S \) comes from that of \( \rho \nabla u \) given by corresponding eigenfunction. Thus if \( \rho \nabla u \) is symmetric in a certain axis direction or in rotation, then, thanks to the original equation \(-\nabla(\rho \nabla u) = \lambda u \), so will
be \( u \) and hence \( \nabla u \), too. Finally the symmetry of \( \rho \) will hold from the symmetry of \( \rho \nabla u \). The key point is then whether the symmetry of \( \rho \nabla u \) associated with the optimizer \( \rho_\ast \) inherits from \( \Omega \). As far as one sees our numerical simulations (Figure 3, 5, 7, 9, 12, 14, 16 and 18), the symmetry of \( \rho \nabla u_\ast \) inherits from \( \Omega \), and so is \( \rho_\ast \). Therefore one observes the following.

**Observation 4.3.** For Problem 2.1, let \( \rho_\ast \) be the optimizer of \( \lambda_1(\rho) \), \( u_\ast \) be the associated eigenfunctions of \( \lambda_1(\rho_\ast) \) and \( S_\ast = \{ x \in \Omega \mid \rho_\ast(x) = c \} \). If \( \Omega \) is symmetric in a certain direction or in rotation, then so are \( S_\ast \) and \( S_\ast^\ast \) regardless of the topology of \( \Omega \) and boundary conditions on \( \partial \Omega \).

### 4.3 Continuous dependence of optimizers on boundary conditions

Finally we consider Problem 2.1 with mixed boundary condition to discuss the continuous dependence of optimizers on boundary conditions. Let \( \Omega_1 = (-\pi, \pi) \times (-\pi, \pi) \) and the boundary condition on \( \partial \Omega_1 \) be

\[
  u = 0 \quad \text{on} \quad x = \pm \pi \quad \text{and} \quad y = \pi, \quad \eta u + \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad y = -\pi, \quad (4.5)
\]

where \( \eta \geq 0 \) is a non-negative constant. In general, as in the case of homogeneous Dirichlet and Neumann boundary conditions, the unique existence of the boundary value problem of elliptic equation

\[
  Lu = f \quad \text{in} \quad \Omega, \quad a(x)u + \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\]

is well-known under suitable assumptions, where \( L \) is an elliptic operator, say, \( Lu = -\Delta u + b(x)u \) for a given bounded function \( b(x) \) and \( a(x) \) is a piecewise continuous function on \( \partial \Omega \). Moreover, the unique solution \( u \) depends continuously on \( a(x) \) in \( L^\infty \) (see e.g. [3]). In previous sections we observed that optimizers are dominated by eigenfunctions associated with corresponding eigenvalues. Thus it is natural to consider that optimizers also depend continuously on the boundary condition as well as eigenfunctions as solutions of elliptic equations.

**Observation 4.4.** Consider Problem 2.1 with the boundary condition (4.5). Then the region \( S_\ast \) given by the optimizer \( \rho_\ast \) of \( \lambda_1 \) depends continuously on \( \eta \geq 0 \). Sample numerical results are shown in Figure 22 and 23.

Although these are just simple examples, combining this observation with the continuous dependence of eigenfunctions on boundary conditions, there is a possibility that one can mathematically prove the continuous dependence of optimizers on boundary conditions in a suitable topology.

### 4.4 Comparative observations – Problem 3.1

As a comparison with Problem 2.1 we consider the optimization of the first eigenvalue for (3.1). This type of problems is well-considered in various papers like [13], while almost all such considerations are only with Dirichlet boundary condition on rectangular domain. If no confusion arises, let \( S = \{ x \in \Omega \mid \sigma(x) = c \} \) for \( \sigma \) under consideration and fix \( c = 2 \) in this section.

Similar calculations to Section 4.1 except

\[
v_0(x) = \frac{\mu_1(\phi)(c-1)}{\int_\Omega \sigma u_\phi^2 \, dx} |u_\phi(x)|^2
\]

in (2.6) yield numerical results listed in Figures 4, 5, 7, 9, 12, 14, 16 and 18. These results imply if \( \sigma \in K \) minimizes or maximizes \( \mu_1 \) and if \( \mu_1 \) is simple (see Table 2) then the corresponding eigenfunction \( u_1 \) associated with \( \mu_1 \) satisfies

\[
  \begin{align*}
    \text{Minimization:} & \quad |u_1(x)| \leq |u_1(y)| & \text{for} \ x \in S \text{ and } y \in S^c, \\
    \text{Maximization:} & \quad |u_1(y)| \leq |u_1(x)| & \text{for} \ x \in S \text{ and } y \in S^c.
  \end{align*}
\]

Several sample examples of mixed boundary value problem are also shown in Figure 24 and 25 which give us the same as Observation 4.4. Note that inequalities (4.6) and (4.7) are exactly the same correspondences as the optimization criteria for Dirichlet boundary value problems, Theorem 3.2. One of key points of these observations is that \( \mu_1 \) is assumed to be simple. Table 2 shows the ratio between \( \mu_1(\sigma) \) and the second eigenvalue \( \mu_2(\sigma) \). Via the similar discussions to Section 4.1 we obtain the following eigenvalue observation criteria in Problem 3.1 including the double eigenvalue cases.
Observation 4.5. For Problem 3.1 in bounded domain $\Omega \subset \mathbb{R}^n$. If $\sigma_{\min} \in \mathcal{K}$ is the minimizer of $\mu_1$ in $\mathcal{K}$, then the eigenfunction $u_1$ associated with $\mu_1$ satisfies (4.6). Similarly, if $\sigma_{\max} \in \mathcal{K}$ is the maximizer of $\mu_1$ in $\mathcal{K}$ and if $\mu_1(\sigma_{\max})$ is simple, then the eigenfunction $u_1$ associated with $\mu_1(\sigma_{\max})$ satisfies (4.7). If $\sigma_{\max} \in \mathcal{K}$ is the maximizer of $\mu_1$ in $\mathcal{K}$ and if $\mu_1(\sigma_{\max})$ has multiplicity two, then eigenfunctions $u_1$ and $u_2$ associated with $\mu_1(\sigma_{\max})$ satisfy

$$|u_1(y)|^2 + |u_2(y)|^2 \leq |u_1(x)|^2 + |u_2(x)|^2 \quad \text{for } x \in S \text{ and } y \in S^c$$

(4.8) under the normalization $\int_\Omega |u_i|^2 dx = 1 \ (i = 1, 2)$, which can be confirmed in Figure 20.

Next we consider the geometry of $S_*$. In the case of Dirichlet boundary, there is a mathematical result for describing the geometric property of $S_*$ from $\Omega$, Theorem 3.3. In the case of Neumann boundary value problems in Problem 3.1, unlike Dirichlet boundary value problems, associated eigenfunctions do not have identical sign in $\Omega$. Instead, we consider connected components of off-zero-level set of eigenfunctions (nodal domains). It is mathematically well-known that the eigenfunction associated with $\mu_1(\sigma)$ has exactly two nodal domains in the case of Neumann boundary value problems (see e.g. [2]).

Assume that $u$ is the eigenfunction associated with $\mu_1(\sigma)$ under Neumann boundary condition. Then the function $v_1$ and $v_2$ restricting $u$ on each nodal domain, say, $\Omega_1$ and $\Omega_2$, are eigenfunctions of the same equation with an identical sign in whole domain, respectively, with

$$\frac{\partial v_i}{\partial n} = 0 \quad \text{on } \partial \Omega \setminus \{u = 0\}, \quad v_i = 0 \quad \text{on } \{u = 0\}$$

In this case, the same discussion as homogeneous Dirichlet boundary can be applied to analyzing properties of $v_i$ (see e.g. [14]) including our criteria (4.6) and (4.7).

On the other hand, in Figure 13, 15, 17 and 19 $\Omega$ is symmetric with respect to the set $\{x \in \Omega \mid u(x) = 0\}$. In Figure 21 $\Omega_1$ is rotationally symmetric with respect to the origin, namely, $\{x \mid |u_1(x)|^2 + |u_2(x)|^2 = 0\}$. In such cases, we can see that $S_*$ is also symmetric with respect to the null set of $u$ (or $|u_1|^2 + |u_2|^2$). Consequently we observe the following:

Observation 4.6. For Problem 3.1 with homogeneous Neumann boundary condition. Assume that $\sigma_{\min} \in \mathcal{K}$ is the minimizer of $\mu_1$ in $\mathcal{K}$ and $S_{\min} = \{x \in \Omega \mid \sigma_{\min}(x) = c\}$. If, for each nodal domain $\Omega_i \ (i = 1, 2)$, $\Omega_i$ is convex and symmetric in $N$ orthogonal directions, then so is the set $\{x \in \Omega_i \mid \sigma_{\min}(x) = c\} = S_{\min} \mid \Omega_i$ as well as being star-shaped with respect to the center of symmetry.

Similarly, assume that $\sigma_{\max} \in \mathcal{K}$ is the maximizer of $\mu_1$ in $\mathcal{K}$ and $S_{\max} = \{x \in \Omega \mid \sigma_{\max}(x) = c\}$. Then, under the same assumption as minimizers, the same statements hold for the set $\{x \in \Omega_i \mid \sigma_{\max}(x) = 1\} = S_{\max} \mid \Omega_i$.

Finally, if $\Omega$ is symmetric with respect to the null set of eigenfunctions $\{x \in \Omega \mid u(x) = 0\}$ (or $\{x \in \Omega \mid |u_1(x)|^2 + |u_2(x)|^2 = 0\}$ in case that $\mu_1(\sigma_1)$ is not simple), then so are $S_*$ and $S_*^c$. These observations are numerically confirmed in Figure 13, 15, 17, 19 and 21.

We also discuss the geometry of $S_*$ in Problem 3.1 in case that $\Omega$ is not even star-shaped. Our numerical results partially answer the inheritance problem in this case, as shown in Figure 9, 10, 11, 18 and 19.

Preceding mathematical result, Theorem 3.3 refers to the star-shapedness of $S_*$ only in case that $\Omega$ is convex. Our numerical results newly suggest if $\Omega$ is star-shaped then so is $S_*$ like, say, Figure 6. However, this is not the case of $\Omega$ which is not even star-shaped. See Figure 11. In this figure, minimization in Problem 3.1 for $\Omega_8$ with homogeneous Dirichlet boundary condition is considered. Only the difference between two figures is the ratio of the volume constraint in (1.3). One of them is $m_0 = 0.3$ and the other is $m_0 = 0.7$. In the case of $m_0 = 0.3$, $S_{\min} = \{x \in \Omega \mid \sigma_{\min}(x) = c\}$ for the minimizer $\sigma_{\min}$ of $\mu_1(\sigma)$ is star-shaped although $\Omega_8$ is not even star-shaped. On the other hand, in the case of $m_0 = 0.7$, $S_{\min}$ is not star-shaped.

As a conclusion, we can say that there is generally no relationship about star-shapedness between $\Omega$ and optimizers if $\Omega$ is not star-shaped. Nevertheless, the symmetry inheritance of $S_*$ and $S_*^c$ from $\Omega$ still holds both in Dirichlet and Neumann boundary problems. Consequently,

Observation 4.7. For Problem 3.1 Let $\sigma_*$ be the optimizer of $\mu_1(\sigma)$, $u_*$ be the associated eigenfunctions of $\mu_1(\sigma_*)$ and $S_* = \{x \in \Omega \mid \sigma_*(x) = c\}$. If $\Omega$ is not star-shaped, there are both cases that either $S_*$ or $S_*^c$ is star-shaped and that neither $S_*$ nor $S_*^c$ are star-shaped.

Even if $\Omega$ is not star-shaped, if $\Omega$ is symmetric with respect to a certain direction or rotationally symmetric, then so are $S_*$ and $S_*^c$. 
4.5 Convergence rate
Throughout numerical studies in this paper we numerically solved (2.7) with \( \epsilon = 1.0 \times 10^{-4} \) via the following implicit scheme

\[
\int_{\Omega} \frac{\phi^n - \phi^{n-1}}{\Delta t} w_h dx = -\epsilon \int_{\Omega} \nabla \phi^n \cdot \nabla w_h dx - \int_{\Omega} (v_0(\phi^{n-1})(x) + \nu) |\nabla \phi^{n-1}| w_h dx \tag{4.9}
\]

where \( w_h \) is an arbitrary element of a finite element subspace of \( H^1(\Omega) \) with suitable boundary conditions. Fix the initial level set function by \( \phi^0(x, y) = \phi(x, y) = x \). Then we inductively solve (4.9) so that all \( \phi^n \) \((n \geq 0)\) keep the volume constraint \( G(\phi^n) = 0 \) following the discussion in [13]. Here \( c = 1.1 \) (Problem 2.1), \( c = 2 \) (Problem 3.1) and \( m_0 = 0.5 \) are fixed in all cases, and the step size \( \Delta t^n \) is defined by \( \Delta t^n = (\sup_{(x,y) \in \Omega} |\phi^n(x, y)|)^{-1} \) for each step. Several sample examples are shown in Figure 26 - 29. Although the initial shape of \( S = \sigma^{-1}(c) \) or \( \rho^{-1}(c) \) concerns, we may observe that the convergence rate is independent of the geometry of \( \Omega \) for each problem with each boundary condition.

5 Conclusion
In this paper, we study the eigenvalue optimization of spatially inhomogeneous diffusion operator \( A_{\rho} u = -\nabla (\rho(x) \nabla u) \) with a given constraint, which is motivated by the control of heat conductivity of spatially inhomogeneous media. We applied the level set approach to characterizing optimizers for Problem 2.1. Collecting our numerical observations, one knows that

The region \( S_* = \{ x \in \Omega \mid \rho_*(x) = c \} \) determined by the eigenvalue optimizer \( \rho_* \) is characterized by the super- or sub-level set of \(|\rho_* \nabla u_*|\) even if \( \lambda_1(\rho) \) has multiplicity greater than two, where \( u_* \) is the eigenfunction associated with \( \lambda_1(\rho_*) \). This characterization is independent of the topology of \( \Omega \) and boundary conditions on \( \partial \Omega \). Moreover, if \( \Omega \) has a certain symmetry, \( S_* \) inherits it from \( \Omega \).

One of key points in our numerical study here is that eigenvalue optimizers can be characterized by associating eigenfunctions including symmetry. Therefore analysis of optimizers can be reduced to corresponding eigenfunctions. We can say the similar properties for Problem 3.1.

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A Tables

In Problem 2.1 (Problem 3.1) with homogeneous Neumann boundary condition, the simpleness of the first eigenvalue $\lambda_1(\rho)$ ($\mu_1(\sigma)$) is nontrivial. $\lambda_{\min}$ and $\lambda_{\max}$ ($\sigma_{\min}$ and $\sigma_{\max}$) denote the minimizer and the maximizer of $\lambda_1(\rho)$ ($\mu_1(\sigma)$), respectively. As for Neumann boundary value problems for $\Omega = \Omega_5, \Omega_6, \Omega_7$ or $\Omega_8$, $\lambda_1(\rho)/\lambda_2(\rho) < 1$ ($\mu_1(\sigma)/\mu_2(\sigma) < 1$) holds for minimizations and maximizations, which implies that $\lambda_1(\rho)$ ($\mu_1(\sigma)$) is simple in those cases.

In the case of maximization problem for $\Omega = \Omega_1$, the ratio $\lambda_2(\rho_{\max})/\lambda_3(\rho_{\max})$ is close to 1 and hence the first eigenvalue can be considered not being simple. Indeed, Figure 20 and Figure 21 implies that (4.2) and (4.7) does not hold, respectively. On the other hand, $\lambda_2(\rho_{\max})/\lambda_3(\rho_{\max}) < 1$ and $\mu_2(\sigma_{\max})/\mu_3(\sigma_{\max}) < 1$ holds, which implies that the first eigenvalue for for $\Omega_1$ has multiplicity two.

| $\Omega$ | $\lambda_1(\rho_{\min})/\lambda_2(\rho_{\min})$ | $\lambda_1(\rho_{\max})/\lambda_2(\rho_{\max})$ | $\lambda_2(\rho_{\max})/\lambda_3(\rho_{\max})$ |
|---------|----------------------------------|----------------------------------|----------------------------------|
| $\Omega_1$ | 0.908758 | 0.99951 | 0.510619 |
| $\Omega_5$ | 0.243209 | 0.258043 | -- |
| $\Omega_6$ | 0.491202 | 0.500293 | -- |
| $\Omega_7$ | 0.289396 | 0.313473 | -- |
| $\Omega_8$ | 0.396238 | 0.417894 | -- |

Table 1: Multiplicity of $\lambda_1(\rho)$ ($c = 1.1$, $m = 0.5$).

| $\Omega$ | $\mu_1(\sigma_{\min})/\mu_2(\sigma_{\min})$ | $\mu_1(\sigma_{\max})/\mu_2(\sigma_{\max})$ | $\mu_2(\sigma_{\max})/\mu_3(\sigma_{\max})$ |
|---------|----------------------------------|----------------------------------|----------------------------------|
| $\Omega_1$ | 0.605469 | 0.999452 | 0.424123 |
| $\Omega_5$ | 0.221712 | 0.338271 | -- |
| $\Omega_6$ | 0.353771 | 0.649946 | -- |
| $\Omega_7$ | 0.236784 | 0.440369 | -- |
| $\Omega_8$ | 0.298349 | 0.746742 | -- |

Table 2: Multiplicity of $\mu_1(\sigma)$ ($c = 2$, $m = 0.5$).
B Figures

B.1 1-dimensional case

Optimization of $\lambda_1(\rho)$ in Problem 2.1 on $\Omega_0 = (0, 1)$. The optimal region $S_*$ and the graph of associated eigenfunction $u$ are drawn.

Figure 1:
(a) : Minimization of $\lambda_1(\rho)$ with homogeneous Dirichlet boundary condition. (b) : Maximization of $\lambda_1(\rho)$ with homogeneous Dirichlet boundary condition. (c) : Minimization of $\lambda_1(\rho)$ with homogeneous Neumann boundary condition. (d) : Maximization of $\lambda_1(\rho)$ with homogeneous Neumann boundary condition. The region in $\Omega_0$ where impulses are hung on is $S_*$ in each figure. Computed eigenvalues with $c = 5$ are (a) : 11.158517, (b) : 26.563359, (c) : 11.487066 and (d) : 27.073145. One can see that $\partial S$ corresponds to the discontinuity of the differential $u'$ of $u$.

Figure 2:
The graph of $\rho u'$ of associated eigenfunction $u$ of the corresponding symbol in Figure 1. For example, (a) is the graph of $\rho u'$ of $u$ in Figure 1-(a). The region in $\Omega_0$ where impulses are hung on is $S_*$. The rest of figures are drawn in the same manner. One can expect that there is a certain correspondence between $S_*$ and super- or sub-level set of $\rho u'$. 
B.2 Dirichlet boundary condition

For Problem 2.1 figures of eigenfunction stand for $|\rho \nabla u|^2$, and for Problem 3.1 figures of eigenfunction stand for $|u|^2$. Each figure shows optimizer of the first eigenvalue and associated eigenfunction. The red region in figures of optimizer stands for $\{x \in \Omega \mid \rho(x) = c\}$ or $\{x \in \Omega \mid \sigma(x) = c\}$.

Figure 3: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_1$
(a) minimizer $\rho_{\text{min}}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$, (c) maximizer $\rho_{\text{max}}$, (d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$.

Figure 4: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_1$
(a) minimizer $\sigma_{\text{min}}$, (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$, (c) maximizer $\sigma_{\text{max}}$ and (d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$.

Figure 5: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_2$
(a) minimizer $\rho_{\text{min}}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$, (c) maximizer $\rho_{\text{max}}$, (d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$. 
Figure 6: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_2$
(a) minimizer $\sigma_{\text{min}}$ (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. (c) maximizer $\sigma_{\text{max}}$ and 
(d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$.

Figure 7: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_3$
(a) minimizer $\rho_{\text{min}}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$. (c) maximizer $\rho_{\text{max}}$, 
(d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$.

Figure 8: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_3$
(a) minimizer $\sigma_{\text{min}}$ (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. (c) maximizer $\sigma_{\text{max}}$ and 
(d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$.

Figure 9: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_4$
(a) minimizer $\rho_{\text{min}}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$. (c) maximizer $\rho_{\text{max}}$, 
(d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$.

Figure 10: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_4$
(a) minimizer $\sigma_{\text{min}}$ (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. (c) maximizer $\sigma_{\text{max}}$ and 
(d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. 
Figure 11: **Geometry of \( \{ x \mid \sigma_{\text{min}}(x) = c \} \) for a non-star-shaped region \( \Omega_4 \)**

Minimization in Problem 3.1 for the non-star-shaped region \( \Omega_4 \) (cf. Figure 10) with homogeneous Dirichlet boundary condition is considered. Figure (a) is the minimizer \( \sigma_{\text{min}} \) with the volume constraint ratio \( m_0 = 0.3 \) and (b) is \( \sigma_{\text{min}} \) with the volume constraint ratio \( m_0 = 0.7 \). The super-level set \( S_{\text{min}} = \{ x \in \Omega_4 \mid \sigma_{\text{min}}(x) = c \} \) is star-shaped in case of (a), which is not the case of (b).
B.3 Neumann boundary condition

For Problem 2.1, figures of eigenfunction stand for $|\rho \nabla u|^2$ and for Problem 3.1, figures of eigenfunction stand for $|u|^2$. Each figure shows optimizer of the first eigenvalue and associated eigenfunction. The red region in figures of optimizer stands for $\{x \in \Omega \mid \rho(x) = c\}$ or $\{x \in \Omega \mid \sigma(x) = c\}$.

Figure 12: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_5$
(a) minimizer $\rho_{\min}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\min})$. (c) maximizer $\rho_{\max}$, (d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\min})$.

Figure 13: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_5$
(a) minimizer $\sigma_{\min}$, (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\min})$. (c) maximizer $\sigma_{\max}$ and (d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\min})$.

Figure 14: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_6$
(a) minimizer $\rho_{\min}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\min})$. (c) maximizer $\rho_{\max}$, (d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\min})$. 
Figure 15: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_6$
(a) minimizer $\sigma_{\text{min}}$, (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. (c) maximizer $\sigma_{\text{max}}$ and (d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$.

Figure 16: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_7$
(a) minimizer $\rho_{\text{min}}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$. (c) maximizer $\rho_{\text{max}}$, (d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$.

Figure 17: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_7$
(a) minimizer $\sigma_{\text{min}}$, (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. (c) maximizer $\sigma_{\text{max}}$ and (d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$.

Figure 18: Optimizers and eigenfunctions for Problem 2.1 on $\Omega_8$
(a) minimizer $\rho_{\text{min}}$, (b) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$. (c) maximizer $\rho_{\text{max}}$, (d) $|\rho \nabla u|^2$ of the associated eigenfunction of $\lambda_1(\rho_{\text{min}})$.

Figure 19: Optimizers and eigenfunctions for Problem 3.1 on $\Omega_8$
(a) minimizer $\sigma_{\text{min}}$, (b) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$. (c) maximizer $\sigma_{\text{max}}$ and (d) $|u|^2$ of the associated eigenfunction of $\mu_1(\sigma_{\text{min}})$.
Figure 20: **Maximizers and eigenfunctions for Problem 2.1 on \( \Omega_1 \)**

Figure (a) shows the maximizer \( \rho_{\text{max}} \) of \( \lambda_1(\rho) \). Figure (b) and (c) show corresponding \( |\rho_{\text{max}} \nabla u_{1,\text{max}}|^2 \) and \( |\rho_{\text{max}} \nabla u_{2,\text{max}}|^2 \), where \( u_{1,\text{max}} \) and \( u_{2,\text{max}} \) are associated eigenfunctions of \( \lambda_1(\rho_{\text{max}}) \) and \( \lambda_2(\rho_{\text{max}}) \) (actually equal to \( \lambda_1(\rho_{\text{max}}) \)), respectively, after the normalization so that \( \int_{\Omega} |u_{i,\text{max}}|^2 = 1 \) holds. Figure (d) shows \( |\rho_{\text{max}} \nabla u_{1,\text{max}}|^2 + |\rho_{\text{max}} \nabla u_{2,\text{max}}|^2 \) after normalizations.

Figure 21: **Maximizers and eigenfunctions for Problem 3.1 on \( \Omega_1 \)**

Figure (a) shows the maximizer \( \sigma_{\text{max}} \) of \( \mu_1(\sigma) \). Figure (b) and (c) show corresponding \( |u_{1,\text{max}}|^2 \) and \( |u_{2,\text{max}}|^2 \), where \( u_{1,\text{max}} \) and \( u_{2,\text{max}} \) are associated eigenfunctions of \( \mu_1(\sigma_{\text{max}}) \) and \( \mu_2(\sigma_{\text{max}}) \) (actually equal to \( \mu_1(\sigma_{\text{max}}) \)), respectively, after the normalization so that \( \int_{\Omega} \sigma_{\text{max}} |u_{i,\text{max}}|^2 = 1 \) holds. Figure (d) shows \( |u_{1,\text{max}}|^2 + |u_{2,\text{max}}|^2 \) after normalizations.
B.4 Continuous dependency on boundary condition

We calculate the dependency on boundary condition. The boundary condition is given by (4.5).

Figure 22: Minimizer for Problem 2.1 on $\Omega_1$
(a) shows the minimizer $\rho_{\min}$ and (b) shows $|\rho_{\min} \nabla u_{\min}|^2$ for the eigenfunction $u_{\min}$ of $\rho_1(\lambda_{\min})$ with various $\eta$. The minimization criterion (4.1) is actually satisfied in each case. The larger $\eta$ becomes, the closer $\rho_{\min}$ is to the minimizer with homogeneous Dirichlet boundary condition (Figure 3, a).

Figure 23: Maximizer for Problem 2.1 on $\Omega_1$
(a) shows the maximizer $\rho_{\max}$ and (b) shows $|\rho_{\max} \nabla u_{\max}|^2$ for the eigenfunction $u_{\max}$ of $\rho_1(\lambda_{\max})$ with various $\eta$. The maximization criterion (4.2) is actually satisfied in each case. The larger $\eta$ becomes, the closer $\rho_{\max}$ is to the maximizer with homogeneous Dirichlet boundary condition (Figure 3, c).
Figure 24: **Minimizer for Problem 3.1 on** $\Omega_1$

(a) shows the minimizer $\sigma_{\text{min}}$ and (b) shows $|u_{\text{min}}|^2$ for the eigenfunction $u_{\text{min}}$ of $\mu_1(\sigma_{\text{min}})$ with various $\eta$. The minimization criterion (4.6) is actually satisfied in each case. The larger $\eta$ becomes, the closer $\sigma_{\text{min}}$ is to the minimizer with homogeneous Dirichlet boundary condition (Figure 4(a)).

Figure 25: **Maximizer for Problem 3.1 on** $\Omega_1$

(a) shows the minimizer $\sigma_{\text{max}}$ and (b) shows $|u_{\text{max}}|^2$ for the eigenfunction $u_{\text{min}}$ of $\mu_1(\sigma_{\text{max}})$ with various $\eta$. The maximization criterion (4.7) is actually satisfied in each case. The larger $\eta$ becomes, the closer $\sigma_{\text{max}}$ is to the maximizer with homogeneous Dirichlet boundary condition (Figure 4(c)).
B.5 Convergence to optimizer

We calculate the convergence to optimizer. Each figure shows the density function $\lambda$ and $\sigma$ after $t$ steps as we solve (2.7) towards the optimizer in Problem 2.1 and Problem 3.1 respectively.

(a) $t = 0$ $t = 100$ $t = 125$ $t = 300$
(b) $t = 0$ $t = 20$ $t = 250$ $t = 450$

Figure 26: Problem 2.1 with Dirichlet boundary

In all cases $c = 1.1$ and $m_0 = 0.5$ are fixed. The rightmost graph is the evolution of $\lambda_1(\rho)$ as we solve (2.7). (a) : Maximization of $\lambda_1(\rho)$ on $\Omega_1$. (b) : Maximization of $\lambda_1(\rho)$ on $\Omega_4$.

(a) $t = 0$ $t = 5$ $t = 10$ $t = 20$
(b) $t = 0$ $t = 20$ $t = 40$ $t = 100$
(c) $t = 0$ $t = 20$ $t = 60$ $t = 100$

Figure 27: Problem 2.1 with Neumann boundary

In all cases $c = 1.1$ and $m_0 = 0.5$ are fixed. The rightmost graph is the evolution of $\lambda_1(\rho)$ as we solve (2.7). (a) : Maximization of $\lambda_1(\rho)$ on $\Omega_5$. (b) : Maximization of $\lambda_1(\rho)$ on $\Omega_8$. (c) : Maximization of $\lambda_1(\rho)$ on $\Omega_1$. In this case the optimized eigenvalue $\lambda_1(\rho_{\text{max}})$ has multiplicity two (cf. Figure 20 and Table 7) and hence the function $v_0$ in the level set evolution (2.5) is set $v_0(x) = -(c - 1)|\nabla u_1,\phi(x)|^2 + (c - 1)|\nabla u_2,\phi(x)|^2$ after the normalization $\int_\Omega |u_i,\phi(x)|^2 dx = 1$.  

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Figure 28: Problem 3.1 with Dirichlet boundary
In all cases $c = 2$ and $m_0 = 0.5$ are fixed. The rightmost graph is the evolution of $\mu_1(\sigma)$ as we solve (2.7). (a) : Minimization of $\mu_1(\sigma)$ on $\Omega_1$. (b) : Minimization of $\mu_1(\sigma)$ on $\Omega_5$. (c) : Minimization of $\mu_1(\sigma)$ on $\Omega_4$.

Figure 29: Problem 3.1 with Neumann boundary
In all cases $c = 2$ and $m_0 = 0.5$ are fixed. The rightmost graph is the evolution of $\mu_1(\sigma)$ as we solve (2.7). (a) : Minimization of $\mu_1(\sigma)$ on $\Omega_5$. (b) : Minimization of $\mu_1(\sigma)$ on $\Omega_4$. (c) : Maximization of $\mu_1(\sigma)$ on $\Omega_8$. In this case the optimized eigenvalue $\mu_1(\sigma_{max})$ has multiplicity two (cf. Figure 21 and Table 2) and hence the function $v_0$ in the level set evolution (2.5) is set $v_0(x) = \{\mu_1(\phi)(c - 1)|u_{1,\phi}(x)|^2 + \mu_2(\phi)(c - 1)|u_{2,\phi}(x)|^2\}$ after the normalization $\int_{\Omega} \sigma(x)|u_{i,\phi}(x)|^2 dx = 1.$