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A maximally-graded invertible cubic threefold that does not admit a full exceptional collection of line bundles

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Abstract
We show that there exists a cubic threefold defined by an invertible polynomial that, when quotiented by the maximal diagonal symmetry group, has a derived category that does not have a full exceptional collection consisting of line bundles. This provides a counterexample to a conjecture of Lekili and Ueda.

1. Introduction

Let $\mathbb{C}$ be the complex numbers. We say a polynomial $w \in \mathbb{C}[x_1, \ldots, x_n]$ is invertible if it is of the form

$$w = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}$$

where $A = (a_{ij})_{i,j=1}^{n}$ is a non-negative integer-valued matrix satisfying the following conditions:
A. The matrix $A$ is invertible over $\mathbb{Q}$;

B. The polynomial $w$ is quasihomogeneous: that is, there exist positive integers $q_j$ such that $d := \sum_{j=1}^{n} q_j a_{ij}$ is constant for all $i$; and

C. The polynomial $w$ is quasi-smooth: that is, the map $w : \mathbb{C}^n \to \mathbb{C}$ has a unique critical point at the origin.

Let $\mathbb{G}_m$ be the multiplicative torus. Consider the following group:

$$\Gamma_w := \{(t_1, \ldots, t_{n+1}) \in \mathbb{G}_m^{n+1} \mid w(t_1x_1, \ldots, t_{n+1}x_n) = t_{n+1}w(x_1, \ldots, x_n)\}. \quad (1)$$

This group $\Gamma_w$ acts on $\mathbb{A}^n$ by projecting onto its first $n$ coordinates and then acting diagonally. Lekili and Ueda made the following conjecture concerning the bounded derived category associated to the polynomial $w$ and the group $\Gamma_w$.

**Conjecture 1.1 (Conjecture 1.3 of [20]).** For any invertible polynomial $w$, the bounded derived category $\mathbb{D}^B(\text{coh} X_w)$ of coherent sheaves on the stack

$$X_w := \left[\text{Spec}(\mathbb{C}[x_1, \ldots, x_n]/(w)) \setminus 0/\Gamma_w\right]$$

has a tilting object, which is a direct sum of line bundles.

In this paper, we show that

$$w = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 \quad (2)$$

provides a counterexample to this conjecture. In fact, the maximal length of any exceptional collection of line bundles on $\mathbb{D}^B(\text{coh} X_w)$ is 24. On the other hand, we calculate that 54 line bundles would be required in any full exceptional collection, let alone a tilting object.

### 1.1. Relation to current literature and mirror symmetry

The result above is analogous to the case of toric varieties. It was asked by King if the derived category of a smooth projective toric variety admits a tilting object that is a direct sum of line bundles. This later became known as King’s conjecture. The first counterexamples to King’s conjecture were provided by Hille-Perling [10] and then later by Efimov [2] in the Fano case. Nevertheless, in [15], Kawamata proved that the derived category of any smooth projective toric Deligne-Mumford stack has a full exceptional collection. It just need not consist of line bundles (or sheaves, for that matter, see [16, Remark 7]).

The Landau-Ginzburg B-model analogue to $\mathbb{D}^B(\text{coh} X_w)$ given by the singularity category of $(\mathbb{C}^n, \Gamma_w, w)$ is well-studied in the context of homological mirror symmetry. At present, it is known to have a full exceptional collection [3]. It is also known to have a full strong exceptional collection in certain cases: for example, when $n \leq 3$ [18] or when $w$ can be written as the Thom-Sebastiani sum of Fermat and chain polynomials [12]. This has been desirable in order to establish homological mirror symmetry for mirror pairs of (gauged) Landau-Ginzburg models [4, 5, 8, 13, 14, 20, 21].

### 1.2. Plan of paper

In Section 2, we show that the Picard group of the stack $X_w$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$. In Section 3, we calculate that the Chen-Ruan cohomology of $X_w$ is 54-dimensional. This implies that the cardinality of any full exceptional collection for $\mathbb{D}^B(\text{coh} X_w)$ must be 54 (Corollary 3.2). On the other hand, in Section 4, we find a sharp upper bound of 24 on the cardinality of an exceptional collection for $\mathbb{D}^B(\text{coh} X_w)$ consisting of line bundles.
2. Line bundles on $X_w$

To address Conjecture 1.1, we first require an explicit description of the Picard group of $X_w$.

2.1. The group $\Gamma_w$

First, we define the group of diagonal automorphisms of the invertible polynomial $w$ to be

$$G_w := \{(t_1, \ldots, t_n) \in \mathbb{G}_m^n \mid w(t_1x_1, \ldots, t_nx_n) = w(x_1, \ldots, x_n)\}. \quad (5)$$

This sits in an exact sequence

$$0 \rightarrow G_w \rightarrow \Gamma_w \xrightarrow{\chi_{n+1}} \mathbb{G}_m \rightarrow 0 \quad (6)$$

where $\chi_{n+1}$ is the projection onto the $(n+1)$th term of $\Gamma_w$. Indeed, we know that $\chi_{n+1}$ is surjective, as, given $\lambda \in \mathbb{G}_m$, we have that $(\lambda^{q_1/d}, \ldots, \lambda^{q_n/d}, \lambda) \in \Gamma_w$.

By Lemma 1.6(B) of [17] for a loop polynomial

$$w = x_1^{d_1}x_2 + x_2^{d_2}x_3 + \ldots + x_{n-1}^{d_{n-1}}x_n + x_n^{d_n},$$

we have $G_w \cong \mathbb{Z}/(a_1 \cdots a_n + (-1)^{n+1})\mathbb{Z}$ with generator $(e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_n})$, where

$$\varphi_j := \frac{(-1)^{n+1-j}a_1 \cdots a_{j-1}}{a_1 \cdots a_n + (-1)^{n+1}}. \quad (7)$$

Recall that $w$ is quasi-homogeneous: that is, we can choose $q_i$ such that $d := \sum_{j=1}^n q_j a_{ij}$ is constant for all $i$ and such that $\text{gcd}(q_1, \ldots, q_n) = 1$. This yields a subgroup $J_w \cong \mathbb{G}_m$ defined by

$$f : J_w \rightarrow \Gamma_w; \quad f(\lambda) = (\lambda^{q_1}, \ldots, \lambda^{q_n}, \lambda^d)$$

known as the exponential grading operator in the literature.

Furthermore, the inclusion $f$ gives rise to a split short exact sequence

$$0 \rightarrow J_w \rightarrow \Gamma_w \rightarrow \overline{G_w} \rightarrow 0 \quad (8)$$

where $\overline{G_w} := G_w/(J_w \cap G_w)$ is the quotient group. Since $\text{gcd}(q_1, \ldots, q_n) = 1$, there exist $b_i$ with $\sum_{i=1}^n b_i q_i = 1$, which gives rise to the splitting of the exact sequence given by

$$g : \Gamma_w \rightarrow J_w; \quad g(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) = \prod_{i=1}^n \lambda_i^{b_i}. \quad (9)$$

Hence $\Gamma_w \cong J_w \times \overline{G_w}$.

The isomorphism $\Gamma_w \cong J_w \times \overline{G_w}$ gives rise to an intermediate quotient stack associated to $J_w$,

$$Z_w = [(\text{Spec}(\mathbb{C}[x_1, \ldots, x_n]/(w)) \setminus 0)/J_w],$$

which is a hypersurface in the weighted projective stack

$$[(\text{Spec}(\mathbb{C}[x_1, \ldots, x_n]) \setminus 0)/J_w] = \mathbb{P}(q_1 : \cdots : q_n).$$

This allows us to identify $X_w$ with the quotient $[Z_w/\overline{G_w}]$. 
Example 2.1. Let \( w = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1, \) as in (2). Then \( G_w = \mathbb{Z}/33\mathbb{Z} \) with generator
\[
g = (\xi, \xi^{-2}, \xi^4, \xi^{-8}, \xi^{16})
\]
(9)
where \( \xi \) is a primitive 33rd root of unity. Here, the intersection \( J_w \cap G_w \) is generated by \( g^{11} = (\xi^{11}, \xi^{11}, \xi^{11}, \xi^{11}, \xi^{11}) \). Hence \( \overline{G_w} \) can be identified with the symmetry group generated by \( (\xi, \xi^9, \xi^4, \xi^3, \xi^5) \), where \( \xi \) is a primitive 11th root of unity.

2.2. The Picard group of \( X_w \)

The Grothendieck–Lefschetz theorem allows us to calculate the Picard group of \( X_w \) as follows.

**Proposition 2.2.** Let \( w \) be an invertible polynomial with \( n \geq 5 \) and \( q_1 = \ldots = q_n = 1 \). The Picard group of \( X_w \) is isomorphic to \( \mathbb{Z} \times \overline{G_w} \), where \( \overline{G_w} \) is the group of characters of \( G_w \).

**Proof.** Since \( X_w = [Z_w/\overline{G_w}] \) is a global quotient stack, \( \text{Pic}(X_w) \) is nothing more than the \( \overline{G_w} \)-equivariant Picard group of \( Z_w \). Note that there is a (surjective) pullback map
\[
\text{Pic}(X_w) \xrightarrow{f} \text{Pic}(Z_w)
\]
that just forgets the equivariant structure. By the Grothendieck–Lefschetz theorem (see, for example, [9, Corollary 3.2]), \( \text{Pic}(Z_w) \cong \mathbb{Z} \); that is, any line bundle is of the form \( O(n) \). As \( O(n) \) admits an equivariant structure, the forgetful map \( f \) is surjective.

Furthermore, as any two equivariant structures differ by a character of \( \overline{G_w} \), we get a short exact sequence
\[
0 \rightarrow \overline{G_w} \rightarrow \text{Pic}(X_w) \xrightarrow{f} \mathbb{Z} \rightarrow 0.
\]
Since \( \mathbb{Z} \) is a projective \( \mathbb{Z} \)-module, this splits to give the desired isomorphism. \( \square \)

**Example 2.3.** Let \( w = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 \) so that \( \overline{G_w} = \mathbb{Z}/11\mathbb{Z} \). Then by Proposition 2.2, we have \( \text{Pic}(X_w) \cong \mathbb{Z} \times (\mathbb{Z}/11\mathbb{Z}) \).

3. Dimension of the Hochschild homology of \( D^b(\text{coh} \ X_w) \)

In this section, we compute the dimension of the Chen–Ruan cohomology of \( X_w \) to be 54. This implies that any full exceptional collection for \( D^b(\text{coh} \ X_w) \) must have 54 objects.

**Proposition 3.1.** Let \( w = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 \). Then \( \dim(H^*_{CR}(X_w; \mathbb{C})) = 54 \).

**Proof.** As vector spaces, the (ungraded) Chen–Ruan cohomology of \( X_w \) is the direct sum of ordinary cohomology groups of twisted sectors
\[
H^*_{CR}(X_w; \mathbb{C}) = \bigoplus_{\gamma \in \Gamma_w} H^*(\{w = 0\}_y/\Gamma_w; \mathbb{C})
\]
where \( \{w = 0\}_y := \{x \in \{w = 0\}_C \cap \{0\} | y \cdot x = x \} \) [1, Section 3].

First, note that if \( \gamma = (\lambda_1, \ldots, \lambda_5) \) so that \( \lambda_i \neq 1 \) for all \( i \), then \( \gamma \cdot x \neq x \) for all \( x \in C^5 \setminus \{0\} \). This implies that the twisted sector corresponding to \( \gamma \) contributes the cohomology of the empty set; that is, nothing.

First, we address the twisted sector associated to the identity element \( \gamma = e \). Note that \( H^*(\{w = 0\}/\Gamma_w; \mathbb{C}) = H^*(Z_w; \mathbb{C})^G_w \), so we must see how \( G_w \) acts on the cohomology of \( Z_w \). Recall that the
Hodge diamond of the cubic $Z_w$ is of the form

\[
\begin{array}{ccc}
1 \\
0 & 0 \\
0 & 1 & 0 \\
0 & 5 & 5 & 0 \\
0 & 1 & 0 \\
0 & 0 \\
1
\end{array}
\]

This is computed using the Griffiths’ residue map [6], which also allows us to describe the action of $\overline{G_w}$. Namely, any element $H^{2,1}(Z_w)$ can be written as the residue of a 4-form

\[
\varphi = \frac{Q}{w} \Omega_0, \quad \Omega_0 = \sum_{i=1}^{5} (-1)^i x_i \, dx_1 \wedge \ldots \wedge \overline{dx_i} \wedge \ldots \wedge dx_5
\]

where $Q$ is a degree 1 polynomial in $\mathbb{C}[x_1, \ldots, x_5]$. By looking at the action by the generator $\rho$ of $\overline{G_w}$, we can see that $w$ and $\Omega_0$ are invariant under its action; however, no degree 1 polynomial is invariant, so the $\overline{G_w}$-invariant subspace of $H^{2,1}(Z_w; \mathbb{C})$ is zero. Similarly, the $\overline{G_w}$-invariant subspace of $H^{1,2}(Z_w; \mathbb{C})$ is zero. The hyperplane classes, on the other hand, are all invariant cycles, so

\[
\dim H^*(\{w = 0\}/\Gamma_w; \mathbb{C}) = 4.
\]

Lastly, there are 50 non-identity elements

\[
S := \{ (\rho \tau^{-1})^a, (\rho \tau^{-9})^a, (\rho \tau^{-4})^a, (\rho \tau^{-3})^a, (\rho \tau^{-5})^a \mid 1 \leq a \leq 10 \} \subseteq \Gamma_w
\]

with a fixed point where $\rho := (\xi, \xi^9, \xi^4, \xi^3, \xi^5)$ is the generator of $\overline{G_w}$ and $\tau = (\xi, \xi, \xi, \xi, \xi)$. In fact, each has a single fixed point and hence contributes one dimension to the Chen-Ruan cohomology.

We conclude that $\dim(H^*_C(X_w; \mathbb{C})) = 4 + |S| = 4 + 50 = 54$. □

This proposition implies the following corollary.

**Corollary 3.2.** For $w$ as defined in (2), we have that $\dim(HH_*(D^b(\text{coh } X_w))) = 54$. In particular, any full exceptional collection for $D^b(\text{coh } X_w)$ has precisely 54 objects.

**Proof.** By an unpublished result of Toën (reproven in [7, Proposition 3.16]),

\[
\dim(HH_*(D^b(\text{coh } X_w))) = \dim(H^*_C(X_w; \mathbb{C})) = 54.
\]

The fact that any full exceptional collection must have 54 objects follows from the additivity of Hochschild homology under semi-orthogonal decomposition. □

**Remark 3.3.** In [3, Theorem 1.1], the authors prove that there is a strong exceptional collection for the singularity category $D[\mathbb{A}^3, \Gamma_w, w]$. It is of length 32, the Milnor number of its mirror LG-model. By the equivariant version of Orlov’s theorem (proven by Hirano [11, Theorem 1.3]), it follows that $D^b(\text{coh } X_w)$ has a full exceptional collection of length $32 + 2(11) = 54$. From this, it also follows that any full exceptional collection must have 54 objects.

### 4. Computations of Ext between line bundles on $X_w$

By Corollary 3.2, any full exceptional collection for $D^b(\text{coh } X_w)$ has 54 objects. However, in this section, we show that an exceptional collection consisting of line bundles on $X_w$ has at most 24 objects (and remark that this bound is achieved).
Table 1. The \((a, b)\)th entry is an \((a, b)\)-bigraded monomial in \(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]/(w)\).

| \(\mathbb{Z}\)-grading | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------------------------|---|---|---|---|---|---|---|---|---|---|----|
| \(\mathbb{Z}/11\mathbb{Z}\)-grading | \(x_1\) | \(x_2\) | \(x_4\) | \(x_3\) | \(x_5\) | \(x_1x_3\) | \(x_1x_5\) | \(x_3x_5\) | \(x_2\) | \(x_1x_2\) | \(x_1^2\) |

Lemma 4.1. For \(a \geq 0\), \(\text{Hom}(\mathcal{O}, \mathcal{O}(a, b)) \neq 0\) unless \(a = 0\) and \(b \neq 0\) or \((a, b) \in \mathcal{X} := \{(1, 0), (1, 2), (1, 6), (1, 7), (1, 8), (1, 10), (2, 0)\}\).

Proof. Observe that \(\text{Hom}(\mathcal{O}, \mathcal{O}(a, b))\) is the space of bidegree \((a, b) \in \mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}\) polynomials in \(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]/(w)\). By Example 2.1, \(\overline{G_w} = ((\xi, \xi^2, \xi^4, \xi^3, \xi^5)) \cong \mathbb{Z}/11\mathbb{Z}\), where \(\xi\) is a primitive 11th root of unity. Hence,

\[
\deg(x_1) = (1, 1), \quad \deg(x_2) = (1, 9), \quad \deg(x_3) = (1, 4), \quad \deg(x_4) = (1, 3), \quad \deg(x_5) = (1, 5).
\]

So Table 1 exhibits an element in \(\text{Hom}(\mathcal{O}, \mathcal{O}(a, b))\) for \(1 \leq a \leq 3\), unless \((a, b) \in \mathcal{X}\). We conclude that \(\text{Hom}(\mathcal{O}, \mathcal{O}(a, b))\) is non-zero for \(a \geq 3\) by multiplying any monomial in \(\text{Hom}(\mathcal{O}, \mathcal{O}(3, b - a + 3))\) by \(x_1^{a-3}\).

\[\square\]

Lemma 4.2. For \(a \geq 2\), we have that \(\text{Ext}^3(\mathcal{O}(a, b), \mathcal{O}) \neq 0\) unless \(a = 2\) and \(b \neq 0\) or \((a, b) \in \mathcal{X}' := \{(3, 0), (3, 2), (3, 6), (3, 7), (3, 8), (3, 10), (4, 0)\}\).

Proof. By adjunction, the canonical bundle is \(\mathcal{O}(-2, 0)\). Therefore by Serre duality,

\[
\text{Ext}^i(\mathcal{O}(a, b), \mathcal{O}) \overset{\text{Serre}}{\cong} \text{Ext}^{3-i}(\mathcal{O}, \mathcal{O}(a, b) \otimes_{\mathcal{O}} \mathcal{O}(-2, 0))^*.
\]

The result follows from Lemma 4.1.

\[\square\]

Proposition 4.3. An exceptional collection of line bundles in \(D^b(\text{coh } X_w)\) has at most 24 objects and hence cannot be full (by Corollary 3.2).

Proof. By Example 2.3, any line bundle on \(X_w\) is of the form \(\mathcal{O}(a, b)\) for \((a, b) \in \mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}\). Let \(\mathcal{E}\) denote an exceptional collection of line bundles, and take the minimal \(a\) such that \(\mathcal{O}(a, b) \in \mathcal{E}\) for some \(b \in \mathbb{Z}/11\mathbb{Z}\). Since \(\mathcal{E} \otimes \mathcal{O}(-a, -b)\) is an exceptional collection, we can assume \((a, b) = (0, 0)\).

Notice that \(\mathcal{E}\) cannot have an object of the form \(\mathcal{O}(a, b)\) for \(a \geq 5\), as, by Lemma 4.1, \(\mathcal{O}(a, b)\) receives a non-zero map from \(\mathcal{O}\) and, by Lemma 4.2, there is a non-trivial 3-extension of \(\mathcal{O}\) by \(\mathcal{O}(a, b)\).

By Table 1, observe that if \(b \neq b'\), then for any \(a\), one has non-zero elements

\[
f_1 \in \text{Hom}(\mathcal{O}(a, b), \mathcal{O}(a + 2, b')) \quad \text{and} \quad f_2 \in \text{Hom}(\mathcal{O}(a, b'), \mathcal{O}(a + 2, b)).
\]

Therefore, denoting by \(\mathcal{S}(a, b)\) the Serre functor applied to the identity map on \(\mathcal{O}(a, b)\), one has a loop:

\[
\mathcal{O}(a, b) \xrightarrow{f_1} \mathcal{O}(a + 2, b') \xrightarrow{\mathcal{S}(a+2, b')} \mathcal{O}(a, b) \xrightarrow{f_2} \mathcal{O}(a + 2, b) \xrightarrow{\mathcal{S}(a+2, b)} \mathcal{O}(a, b).
\]

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We conclude that $E$ cannot have a quadruple of objects
\[
\{O(a, b), O(a, b'), O(a+2, b), O(a+2, b')\}.
\]
For example, taking $a = 0$ (respectively, $a = 1$), $E$ cannot have multiple objects with $a = 0$ and $a = 2$ (respectively, $a = 1$ and $a = 3$). This forces there to be at most 12 line bundles in $E$ with $a = 0, 2$ and $a = 1, 3$, respectively.

Now, again by Lemma 4.2, $E$ cannot have an object of the form $O(a, b)$ for $a \geq 4$ except $(a, b) = (4, 0)$. Hence, we can have at most 1 more object. But if $O(4, 0) \in E$, Lemma 4.2 also forces $O(0, b) \notin E$ for $b \neq 0$. Hence, if we already have 12 line bundles in $E$ with $a = 0, 2$ then $O(2, b) \in E$ for all $b$. This gives a contradiction, as $O, O(2, 0), O(4, 0)$ also form a loop
\[
O \xrightarrow{x_1^2x_2x_3} O(4, 0) \xrightarrow{S(4, 0)} O(2, 0) \xrightarrow{S(2, 0)} O
\]
and therefore cannot be in the same exceptional collection. We conclude that this 1 additional object cannot take us beyond 24 exceptional objects. \hfill \square

**Remark 4.4.** The upper bound of 24 exceptional objects is sharp. It is achieved by the exceptional collection drawn below. This exceptional collection is not strong, however; we only draw the degree-0 maps for aesthetic simplicity. The required vanishing can be checked using Lemmas 4.1 and 4.2 and the fact that $\text{Ext}^1, \text{Ext}^2$ vanish for line bundles on a 3-fold hypersurface in projective space (for example, using the long exact sequence for the divisor).

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