Schrödinger Self-adjoint Extension and Quantum Field Theory

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ABSTRACT

We argue that the results obtained using the quantum mechanical method of self-adjoint extension of the Schrödinger Hamiltonian can also be derived using Feynman perturbation theory in the investigation of the corresponding non-relativistic field theories. We show that this is indeed what happens in the study of an anyon system, and, in doing so, we establish a field theoretical description for “colliding anyons”, i.e. anyons whose quantum mechanical wave functions satisfy the non-conventional boundary conditions obtained with the method of self-adjoint extension. We also show that analogous results hold for a system of non-abelian Chern-Simons particles.

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I Introduction

The method of self-adjoint extension of the Hamiltonian has been advocated in several occasions in the literature (see for example Refs. [1-7]). In some instances, like in the case of the Dirac Hamiltonian for the spinning cone in Ref. [2], a self-adjoint extension of the Hamiltonian (in which the wave functions are allowed to diverge at a finite number of points, provided they remain square integrable) is necessary because the requirement that the wave functions be regular does not lead to a self-adjoint Hamiltonian. For other physical systems, including the ones considered in this Letter, the demand of regularity does lead to a self-adjoint Hamiltonian, but one notices that a one-parameter family of self-adjoint Hamiltonians (which includes, in correspondence with a specific choice of the self-adjoint extension parameter, the conventional Hamiltonian with regular wave functions) can be found by relaxing this requirement (see for example Refs. [1,4-6]).

In the study of quantum mechanical Schrödinger problems [1, 5, 6] the results obtained using the method of self-adjoint extension raise an interesting issue which has not yet been addressed in the literature. It has not been established whether or not there is a procedure that allows to rederive these results in the framework of the field theoretical description (which is supposed to be completely equivalent to the quantum mechanical description) of the same non-relativistic physical system.

In this Letter, we study a system of anyons [8] and a system of non-abelian Chern-Simons (NACS) particles [9, 10], and we prove that the scattering amplitude obtained by a field theoretical perturbative calculation takes the form of the corresponding result obtained using the quantum mechanical method of self-adjoint extension, provided the renormalized strength of the contact interaction that is induced by renormalization [11, 12] is chosen to be related in a specific way (for fixed renormalization scale) to the self-adjoint extension parameter.

The particular physical systems that we consider have been recently examined from different viewpoints in Refs. [11-15], and are closely related to other extensively studied problems. Indeed, anyons, which can be useful for understanding the Fractional Quantum Hall Effect [16], are particles that acquire fractional statistics through the Aharonov-Bohm effect [17], and our results are therefore related to the Aharonov-Bohm scattering problem. Moreover, in Refs. [2, 3] it was observed that for an energy eigenstate the equations for a particle in the gravitational field of a massless spinning source in two spatial dimensions, which are also relevant to the study of spinning infinite cosmic strings in three spatial dimensions, are equivalent to those of an infinitely thin flux tube in a background Aharonov-Bohm gauge field, and are therefore also related to the system here studied.

II Abelian Case

In this section, we consider anyons, which can be described as non-relativistic bosons in 2+1 dimensions interacting through an abelian Chern-Simons gauge field. The Lagrange density is

\[ \mathcal{L} = \frac{\kappa}{2} e^{\alpha \beta \gamma} A_\alpha \partial_\beta A_\gamma + i \phi^\dagger D_t \phi - \frac{1}{2} (D \phi)^\dagger \cdot D \phi, \]  

where \( \phi \) is a complex bosonic field, and \( D_t \equiv \partial_t + ieA_0 \) and \( D \equiv \nabla - ieA \) are the covariant derivatives. With the help of the number operator that is defined as the
Noether charge of the global $U(1)$ symmetry of the Lagrange density, the quantum theory may be equivalently formulated as a quantum mechanical N-body Schrödinger problem\cite{18}. In particular, we shall be interested in the time-independent Schrödinger equation describing the 2-body relative motion:

$$H\psi(r) \equiv -\left(\nabla + i\nu \nabla \times \ln r\right)^2 \psi(r) = p^2 \psi(r),$$

(2)

where $\nu = \frac{\epsilon^2}{2\pi R}$ is the statistical parameter, which we can restrict to be in the interval $[-1, 1]$ without loss of generality\cite{8}, and $p$ is the relative momentum.

If we demand that the wave functions be regular ($i.e.$ finite everywhere), the scattering problem for the system here considered is exactly the one solved by Aharonov and Bohm\cite{17}. However, in general interesting physical predictions can also be obtained if the regularity requirement is relaxed, allowing the wave functions to diverge at a finite number of points, provided they remain square integrable and the Hamiltonian is self-adjoint. Allowing the wave functions $\psi(r)$ to be non-regular at the origin (i.e. when the particle positions coincide) leads\cite{5} to the following one-parameter family of boundary conditions at the origin for the s-wave functions:

$$r^{|\nu|} \psi(r) - w R^{2|\nu|} \frac{dr^{|\nu|} \psi(r)}{dr^{2|\nu|}} \bigg|_{r=0} = 0,$$

(3)

which can be equivalently expressed as the following requirement on the form of $\psi$

for $r \sim 0$

$$\psi(r) \to a(r^{|\nu|} + w R^{2|\nu|} r^{-|\nu|}) \quad \text{for } r \sim 0.$$

(4)

Here $R$ is a reference scale with dimensions of a length, $w$ is a dimensionless real parameter, the self-adjoint extension parameter, which characterizes the type of boundary condition\cite{11} and $a$ is a constant.

Note that the conformal symmetry possessed\cite{19} by the Lagrange density in Eq. (1) is in general broken by the boundary condition (3) due to the presence of the dimensionful quantity $w R^{2|\nu|}$. Only at the critical points $w = 0$, which corresponds to the conventional Aharonov-Bohm-type scale independent boundary condition $\psi(0) = 0$, and $w \sim \infty$, which corresponds to the scale independent boundary condition $\frac{dr^{|\nu|} \psi(r)}{dr^{2|\nu|}} \bigg|_{r=0} = 0$, the scale symmetry is preserved.

One can easily see\cite{5} that for non-s-wave functions square integrability is only consistent with the $\psi(0) = 0$ boundary condition; thus, the method of self-adjoint extension only affects the s-wave part of the calculations, which are therefore the ones we shall be concerned with.

The s-wave scattering amplitude for anyons satisfying the boundary condition (3) can be evaluated exactly by using a rather straightforward generalization of the analysis given in Ref.\cite{17}, which concerned the special case $w = 0$; we find (also see Ref.\cite{5})

$$A_s(p) = -i \sqrt{\frac{2}{\pi p}} (e^{i\pi|\nu|} - 1) \frac{\int_0^{\frac{\pi}{2}} \left(\frac{2}{\pi R}\right)^{2|\nu|} \Gamma(1+|\nu|)}{1+\frac{2}{\pi} e^{i\pi|\nu|} \left(\frac{2}{\pi R}\right)^{2|\nu|} \Gamma(1+|\nu|)}$$

$$= -\frac{2\pi}{p} \left\{|\nu| \left|\frac{1-w}{1+w} - \frac{\pi}{2} \nu^2 - \nu^2 \frac{4w}{(1+w)^2} \left(\ln \frac{pR}{2} + \gamma - \frac{i\pi}{2}\right) \right. \right.$$  

$$\left. - \frac{\pi^2}{4} \nu^3 \frac{1-w}{1+w} - |\nu|^3 \frac{4(1-w)w}{(1+w)^4} \left(\ln \frac{pR}{2} + \gamma - \frac{i\pi}{2}\right)^2 + O(\nu^4) \right\},$$

(5)

\(^1\)Note that, in Ref.\cite{5}, the self-adjoint extension is parametrized in terms of a dimensionful quantity $R_0$, which is related to our $R$ and $w$ by the relation $(R_0)^{2|\nu|} = w R^{2|\nu|}$.\n
\[2]
where \( p \equiv |p| \), and \( \gamma \) denotes the Euler constant.

Note that scale invariance is in general broken by the dependence of \( A_s \) on \( R \). Consistently with our preceding observation, the scale symmetry is only preserved at the critical values of \( w \), for which the s-wave amplitude can be written as

\[
A_s = -i \sqrt{\frac{2}{\pi p}} (e^{\mp i\pi|\nu|} - 1) ,
\]

where the upper (lower) sign holds for the \( w = 0 \) \((w \sim \infty)\) critical point.

As stated in the Introduction, we intend to show that it is possible to rederive the quantum mechanical result for the scattering amplitude obtained using the method of self-adjoint extension of the Schrödinger Hamiltonian in a field theoretical perturbative calculation.

The field theoretical description of the system that we are considering has been discussed in Refs.\[11, 12\]. It was shown that renormalizability requires the addition of a contact term \(-\pi g_b (\phi^\dagger \phi)^2\) to the Lagrangian density \( L \) of Eq.(1). The two-particle scattering amplitude was calculated to one-loop order in Ref.\[12\]; its s-wave part \(^{\dagger}\) (including the appropriate kinematic factor) can be written as

\[
A_s,1-l(p) = -\sqrt{\frac{2\pi}{p}} \left\{ g_b - \frac{i\pi}{2} \nu^2 + (g_b^2 - \nu^2) \left( \ln \frac{\mu}{\mu'} - \frac{i\pi}{2} \right) - g_b^2 \epsilon \left( \left( \ln \frac{\mu}{\mu'} - \frac{i\pi}{2} \right)^2 + \frac{\pi^2}{24} \right) \right\}
\]

\[
= -\sqrt{\frac{2\pi}{p}} \left[ g - \frac{i\pi}{2} \nu^2 + (g^2 - \nu^2) \left( \ln \frac{\mu}{\mu'} - \frac{i\pi}{2} \right) \right] ,
\]

where \( \epsilon \) is the usual cut-off used in dimensional regularization, \( \mu \) is the renormalization scale, \( \mu' \), which we introduced just in order to simplify the notation, is defined by \( \ln \mu' = \ln \mu - \frac{\gamma}{2} - \frac{\ln 4\pi}{2} \), and we also introduced a one-loop renormalized coupling \( g_r \) defined in terms of the bare coupling \( g_b \) by the relation

\[
g_r = g_b - (g_b^2 - \nu^2) \left( \frac{1}{2\pi} - \frac{\gamma}{2\pi} \right) .
\]

Note that only at the critical values \( g_r = \pm |\nu| \) of the renormalized contact coupling the scale invariance of the classical theory is preserved at the quantum (one-loop) level. Moreover, it was observed in Refs.\[11, 12\] that at the repulsive critical value of the renormalized contact coupling, i.e. \( g_r = |\nu| \), the result \( ^{(3)} \) is consistent with the Aharonov-Bohm scattering amplitude (which is given by the \( w \to 0 \) limit of Eq.(3)), and it is in this sense that this field theory for \( g_r = |\nu| \) describes the conventional anyons, which satisfy the Aharonov-Bohm-type regular boundary condition.

Our objective is to establish a general connection (as we just mentioned, this connection was only understood for the special case \( w = 0, g_r = |\nu| \)) between the \( g_r \)-dependent field theoretical results and the \( w \)-dependent quantum mechanical results of the method of self-adjoint extension. We identify this connection by comparison of the results in Eqs.(3) and \( ^{(3)} \); in fact, we observe that if one uses the relations

\[
g_r = |\nu| \frac{1 - w}{1 + w} , \quad \mu = \frac{2}{Re\gamma}
\]

\(^{\dagger}\)Note that in the field theoretical calculations one can easily show (to all orders) that the non-s-wave part of the scattering amplitude is cut-off-independent (and therefore it plays no role in the renormalization procedure), contact-coupling-independent, and, besides the overall kinematic factor, scale-independent.
the one-loop field theoretical result (7) reproduces exactly the $O(\nu^2)$ approximation of the quantum mechanical result (5) obtained using the method of self-adjoint extension.

We observe that, in particular, Eq. (7) implies that (as it should be expected based on the analysis of scale invariance), like $w = 0$ corresponds to $g_r = |\nu|$, the other critical value ($w \sim \infty$) of the self-adjoint extension parameter corresponds to attractive critical strength $§(g_r = -|\nu|)$ of the contact interaction.

Further insight into the correspondence (9) between the quantum mechanical variables $w, R$ and the field theoretical variables $g_r, \mu$ can be gained from the following observations. First, we notice that using the renormalization-group equation which states that the physical scattering amplitude is independent on the choice of the renormalization scale $\mu$, one can derive, to the order $\nu^2$, the following beta function for the coupling $g_r$:

$$\beta(g_r) \equiv \frac{dg_r}{d\ln \mu} = g_r^2 - \nu^2. \quad (10)$$

Eq. (10), which indicates that $g_r$ and $\mu$ are not physically independent, can be integrated to give the relation

$$\frac{|\nu| + g_r(\mu_1)}{|\nu| - g_r(\mu_1)} \mu_1^{2|\nu|} = \frac{|\nu| + g_r(\mu_2)}{|\nu| - g_r(\mu_2)} \mu_2^{2|\nu|}. \quad (11)$$

Similarly in the exact result (5), which was obtained in the quantum mechanical framework, $R$ is only a reference scale, and obviously physics must be independent of the choice of $R$. Indeed, all physical quantities (see, for example, Eqs. (3) and (5)) depend on $w$ and $R$ only through the quantity $wR^{2|\nu|}$, and the independence of physics on the choice of $R$ is realized by the fact that if $R$ is changed from a value $R_1$ to a value $R_2$ this must be accompanied by a corresponding change of $w$ as described by the relation

$$w(R_1) R_1^{2|\nu|} = w(R_2) R_2^{2|\nu|}. \quad (12)$$

Clearly, the Eqs. (11) and (12) are perfectly consistent with the relations (9). This observation is even more remarkable considering that Eq. (12) is exact, whereas the Eqs. (3) and (11) are just based on a one-loop analysis. Evidently, the Eqs. (3) and (11) have more general validity than one would expect based on the fact that they have been derived at one-loop. In order to test whether indeed the Eqs. (3) and (11) receive vanishing higher loop contributions, we now calculate the two-loop $s$-wave scattering amplitude. The computation is simplified by the fact that it is easy to show that the only two-loop contributions to the $s$-wave scattering amplitude come from the two diagrams in Fig. 1.

§Note that, in Ref. [18] it was shown that for attractive critical value of the contact coupling the classical version of the field theory here considered admits static solutions, i.e. solitons, that satisfy a self-dual equation which is equivalent to the Liouville equation. It would be interesting to investigate whether characteristic structures also arise at the quantum mechanical level in correspondence of $w \sim \infty$.  

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We find that the contribution $A_{db}$ of the double-bubble diagram (Fig. 1a) and the contribution $A_{ic}$ of the ice-cone diagram (Fig. 1b) are given by

$$A_{db} = -\sqrt{\frac{2\pi}{p}} g_b^3 \left[ \frac{1}{4\epsilon^2} - \frac{1}{\epsilon} \left( \ln \frac{p}{\mu'} - \frac{i\pi}{2} \right) + 2 \left( \ln \frac{p}{\mu'} - \frac{i\pi}{2} \right)^2 + \frac{\pi^2}{24} \right],$$  \hspace{1cm} (13)$$

$$A_{ic} = \sqrt{\frac{2\pi}{p}} g_b \nu^2 \left[ \frac{1}{4\epsilon^2} - \frac{1}{\epsilon} \left( \ln \frac{p}{\mu'} - \frac{i\pi}{2} \right) + 2 \left( \ln \frac{p}{\mu'} - \frac{i\pi}{2} \right)^2 + \frac{5\pi^2}{24} \right].$$  \hspace{1cm} (14)$$

Adding $A_{db}$ and $A_{ic}$ to the one-loop s-wave scattering amplitude, and introducing a two-loop renormalized contact coupling $g_r$ related to the bare coupling $g_b$ by

$$g_r = g_b - (g_b^2 - \nu^2)(\frac{1}{2\epsilon} - \frac{\gamma - \ln 4\pi}{2}) + g_b(g_b^2 - \nu^2)\left(\frac{1}{2\epsilon} - \frac{\gamma - \ln 4\pi}{2}\right),$$  \hspace{1cm} (15)$$

we obtain the following final result for the renormalized s-wave scattering amplitude to two-loop order:

$$A_{s,2-l} = -\sqrt{\frac{2\pi}{p}} \{ g_r - \frac{i\pi}{2} \nu^2 - \frac{\pi^2}{12} g_r \nu^2 + (g_r^2 - \nu^2) \left( \ln \frac{p}{\mu'} - \frac{i\pi}{2} \right) + g_r (g_r^2 - \nu^2) \left( \ln \frac{p}{\mu'} - \frac{i\pi}{2} \right) \}. \hspace{1cm} (16)$$

It is easy to verify that this result reproduces the $O(\nu^3)$ approximation of the exact result (5) if one uses again the relations (9); therefore, we find that, as anticipated by our hypothesis that these relations are exact, there is no two-loop correction to the relations (9). Analogously, from Eq. (16) one can verify that there is no two-loop correction to the $\beta$-function given in Eq. (10). (Using a different formalism, in Ref. [13] it was shown that also the three-loop correction to Eq. (10) vanishes.)

### III Non-abelian Generalization

In this section, we discuss the generalization of the analysis presented in the preceding section to the case of a non-abelian gauge-symmetry group. The system considered is therefore exactly the one of Ref. [12], and we adopt the notation and conventions introduced in that paper, with the only exception that, for simplicity, we study particles of unit mass.
The non-abelian generalization of the Schrödinger problem (2) is given by\[10, 11\]
\[H\Psi(r) \equiv -(\nabla + i\Omega \nabla \times \ln r)^2\Psi(r) = p^2,\]
where \(\Omega = -\frac{g^2}{2\pi}\tau_a \otimes T_a\), and \(\Psi = \sum_{nm} \psi_{nm}(r) |nm\rangle\) is a two-NACS-particle state (the \(\psi_{nm}\) are the components of \(\Psi\) in the \(|nm\rangle\) basis, see Ref.\[12\]).

By choosing a basis which diagonalizes \(\Omega\) this non-abelian problem can be essentially reduced to the abelian one. Exploiting this simplification, it is easy to show that in the non-abelian case the method of self-adjoint extension of the Schrödinger Hamiltonian leads to the following requirement on the form of the s-wave functions:

\[\Psi(r) \rightarrow \langle r^{|\Omega|} + r^{-|\Omega|} W R^{|\Omega|} \rangle \sum_{nm} a_{nm} |nm\rangle \quad \text{for } r \sim 0,\]

where the \(a_{nm}\) are constants, \(W\) is a Hermitian matrix\[12\] whose components are dimensionless parameters that characterize the self-adjoint extension, and \(|\Omega|\) is the matrix which in the basis that diagonalizes \(\Omega\) has elements given by the absolute value of the elements of \(\Omega\).

In a basis that diagonalizes \(\Omega\), it is also easy to obtain the non-abelian generalization of Eq.(5). We find that the boundary condition (18) leads to an s-wave solution for the scattering \(n_1, m_1 \rightarrow n_2, m_2\) which is given by \(\langle n_2, m_2 | A_s | n_1, m_1 \rangle + \langle m_2, n_2 | A_s | n_1, m_1 \rangle\) with

\[A_s = -\frac{i e^{i|\Omega|}}{2\sqrt{2p}} \left(1 - e^{-\frac{i|\Omega|}{2}} \left(\frac{2}{pR}\right)^{|\Omega|} \Gamma(1 + |\Omega|) \frac{1}{W \Gamma(1-|\Omega|)} \left(\frac{2}{pR}\right)^{|\Omega|} e^{i|\Omega|} \right)\right.\]

\[\cdot \left\{1 + e^{\frac{i|\Omega|}{2}} \left(\frac{2}{pR}\right)^{|\Omega|} \Gamma(1 + |\Omega|) \frac{1}{W \Gamma(1-|\Omega|)} \left(\frac{2}{pR}\right)^{|\Omega|} e^{i|\Omega|} \right\}^{-1}.\]

As done in the preceding section for the abelian case, we now proceed to establish a prescription that allows to rederive this result, which was obtained using the method of self-adjoint extension, in the framework of the field theoretical technique of Feynman perturbation theory. The renormalized scattering amplitude was calculated in field theory to one-loop order in Ref.\[14\]; its s-wave part is

\[A_s,1-l = -\sqrt{\frac{2}{2p}} \left[ G_r - \frac{i\pi}{2} \Omega^2 + (G_r^2 - \Omega^2) \left(\ln \frac{\mu}{\mu - \frac{i\pi}{2}}\right) \right],\]

where \(G_r\) is the renormalized contact coupling matrix that appears in the contact term required for renormalizability of the non-abelian model (specifically, \(G_r\) is defined in terms of the matrix \(C_r\) introduced in Ref.\[12\] by the relation \(G_r = C_r/4\pi\)). We observe that Eq.(20) reproduces the \(O(|\Omega|^2)\) approximation of Eq.(19) if one uses the following relations

\[G_r = |\Omega|(1 - W)(1 + W)^{-1}, \quad \mu = \frac{2}{Re^{\gamma}}.\]

We observe that for a general Hermitian matrix the form \[13\] of the s-wave functions is not gauge covariant. Gauge covariance can be achieved by demanding that \(W\) satisfies the relation \([W, T_a \otimes 1 + 1 \otimes T_a] = 0\), which also implies that \(W\) commutes with \(|\Omega|\). For completeness, we present formulas valid for a general Hermitian matrix \(W\), i.e. we keep track of the ordering of the matrices \(W\) and \(|\Omega|\).
Also for this non-abelian case, in order to present some evidence that the relations (21) are stable with respect to higher order contributions, we calculated the renormalized two-loop s-wave scattering amplitude; our result is

\begin{equation}
A_{s,2-l} = -\sqrt{\frac{\pi}{4p}} (G_r - \frac{i\pi}{2} \Omega^2 - \frac{\pi^2}{12} (G_r \Omega^2 + \Omega^2 G_r) + (G_r^2 - \Omega^2) \left(\ln \frac{p}{\mu} - \frac{i\pi}{2}\right)) + \frac{i}{2} (G_r (G_r^2 - \Omega^2) + (G_r^2 - \Omega^2) G_r) \left(\ln \frac{p}{\mu} - \frac{i\pi}{2}\right)^2 ,
\end{equation}

which does indeed reproduce the $O(|\Omega|^3)$ approximation of Eq.(19) when the relations (21) are used.

\section*{IV Conclusion}

In our study of anyons and of NACS particles, we have identified relations between the quantities that appear in the (quantum mechanical) method of self-adjoint extension and the quantities that appear in the (field theoretical) Feynman perturbation theory, which allow to put in one to one correspondence the results of the two methods. In establishing this result, we have extended to two-loop (see Eqs.(16) and (22)) some of the results of Refs.[11, 12], and we have generalized to the case of NACS particles (see Eqs.(18)) the results of the method of self-adjoint extension presented for anyons in Refs.[5, 6].

Based on our analysis, we argue that in general the results obtained using the method of self-adjoint extension of the Schrödinger Hamiltonian should be equivalently derivable by a (suitably renormalized) perturbative calculation in the framework of the corresponding field theoretical problem.

A problem of theoretical physics that is related to the ones here studied, is the connection between boundary conditions for the wave functions and contact interactions, which was recently investigated in quantum mechanics[1, 14, 15]. The analysis presented in the preceding sections shows that also in field theory the introduction of contact interactions can be used to implement in the perturbative calculations a choice of boundary conditions for the wave functions.

Our investigation is also relevant to the issue of which boundary conditions at the points of overlap of particle positions are most natural in the case of anyons[6, 20] or NACS particles. The results of Sec.II (Sec.III) establish a field theoretical description of “colliding anyons”\cite{3} (“colliding NACS particles”), i.e. we identified the strength of the contact coupling $g_r$ which (at fixed renormalization scale) is to be used in the field theory calculations to describe anyons (NACS particles) whose quantum mechanical wave functions satisfy the non-conventional boundary conditions obtainable with the method of self-adjoint extension with parameter $w$ (parameter matrix $W$). In this field theoretical formalism there appears to be no reason for restricting oneself to the case of the conventional “non-colliding anyons”\cite{3}, the ones whose wave functions vanish at the points of overlap of particle positions\cite{4}.

Lastly, we want to point out that the calculations presented in this Letter give one of the rare opportunities of comparing exact results to renormalization requiring

\footnote{The only special property of the non-colliding anyons, which correspond to $w = 0$, is the preservation of the scale invariance; however, this property is shared by the case of colliding anyons with $w \sim \infty$, and, anyway, it is not clear to us whether there is any physical motivation to exclude values of $w$ that do not preserve scale invariance.}
perturbative field theoretical results, and we hope they can be used to gain some insight in the physics behind the regularization and renormalization procedure. For example, in our analysis it appears that the necessity of a cut-off is not a relict of some unknown ultraviolet physics, but rather an artifact of the perturbative methods used. This is in contrast with the conventional wisdom on renormalization; however, our results are derived in the context of non-relativistic field theory, and it is unclear to us whether similar conclusions could be reached in the case of relativistic field theories, which is the framework where renormalization is customarily used.

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References

[1] R. Jackiw, in *M.A.B. Bell memorial volume*, A. Ali and P. Hoodbhoy eds. (World Scientific, Singapore, 1991).

[2] P. de Sousa Gerbert and R. Jackiw, Commun. Math. Phys. **124** (1989) 229.

[3] P. de Sousa Gerbert, Phys. Rev. **D40** (1989) 1346

[4] B.S. Kay and U.M. Studer, Commun. Math. Phys. **139** (1991) 103.

[5] C. Manuel and R. Tarrach, Phys. Lett. **B268** (1991) 222.

[6] M. Bourdeau and R.D. Sorkin, Phys. Rev. **D45** (1992) 687.

[7] D.K. Park, HEPTH-9405009 (1994).

[8] J. M. Leinaas and J. Myrheim, Nuovo Cimento **B37** (1977) 1; F. Wilczek, Fractional Statistics and Anyon Superconductivity, (World Scientific, 1990).

[9] T. Lee and P. Oh, Phys. Lett. **B319** (1993) 497.

[10] D. Bak, R. Jackiw and S.-Y. Pi, Phys. Rev. **D49** (1994) 6778.

[11] O. Bergman and G. Lozano, Ann. Phys. (NY) **229** (1994) 416.

[12] D. Bak and O. Bergman, MIT-CTP-2283/hep-th-9403134 (1994).

[13] D. Freedman, G. Lozano and N. Rius, Phys. Rev. **D49** (1994) 1054.

[14] G. Amelino-Camelia, Phys. Lett. **B322** (1994) 286; MIT-CTP-2321 (1994).

[15] C. Manuel and R. Tarrach, Phys. Lett. **B328** (1994) 113; S. Ouvry, IPNO/TH-71 (1993).

[16] *The Quantum Hall effect*, R. Prange and S. Girvin, eds., (Springer, Berlin, 1990).

[17] Y. Aharonov and D. Bohm, Phys. Rev. **115** (1959) 485; S. Ruijsenaars, Ann. Phys. (NY) **146** (1983) 1.

[18] R. Jackiw and S.-Y. Pi, Phys. Rev. **D42** (1990) 3500.

[19] R. Jackiw, Ann. Phys. (NY) **201** (1990) 83.

[20] G. Amelino-Camelia and L. Hua, Phys. Rev. Lett. **69** (1992) 2875; K. H. Cho and C. Rim, Phys. Rev. Lett. **69** (1992) 2877.