SPECTRUM OF THE LICHNEROWICZ LAPLACIAN ON ASYMPTOTICALLY HYPERBOLIC SURFACES

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Abstract. We show that, on any asymptotically hyperbolic surface, the essential spectrum of the Lichnerowicz Laplacian $\Delta_L$ contains the ray $[\frac{1}{4}, +\infty]$. If moreover the scalar curvature is constant then $-2$ and $0$ are infinite dimensional eigenvalues. If, in addition, the inequality $\langle \Delta u, u \rangle_{L^2} \geq \frac{1}{4} ||u||_{L^2}^2$ holds for all smooth compactly supported function $u$, then there is no other value in the spectrum.

Keywords: Asymptotically hyperbolic surfaces, Lichnerowicz Laplacian, symmetric 2-tensor, essential spectrum, asymptotic behavior.

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1. Introduction

This article is a complement of the papers [7], [8] where the study of the Lichnerowicz Laplacian $\Delta_L$ is given in dimension $n$ greater than 2. We refer the reader to those papers for all the motivations. In the preceding papers, the spectrum was only given for $n \geq 3$ because of the natural relation to the prescribed Ricci curvature problem. In dimension 2 this study does not appear because the corresponding problem is conform. The present paper, firstly given for completeness, appears to be particularly interesting because of the quite big differences with the other dimensions.

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For instance on the hyperbolic space, when \( n \geq 3 \) the spectrum of \( \Delta_L \) on trace free symmetric two tensors is the ray

\[
\left[ \frac{(n-1)(n-9)}{4}, +\infty \right].
\]

This spectrum is essentially characterized by non trivial trace free tensors on the boundary at infinity. In dimension 2 (so 1 at infinity) those tensors do not exist, and the situation is very different.

Also, in dimension two, the cohomology of the manifold appears naturally in the spectrum. This situation was already noticed by Avez [3] [2] and Buzanca [5] [6].

The principal result is the following

**Theorem 1.1.** Let \((M, g)\) be an asymptotically hyperbolic surface. The essential spectrum of \(\Delta_L\) on trace free symmetric two tensors contains the ray \([1/4, +\infty]\). If moreover \(g\) has constant scalar curvature \(R = -2\) then \(-2\) and 0 are also in the essential spectrum. Moreover their eigenspaces are in one to one correspondence with the space of harmonic one forms respectively in \(L^4\) and in \(L^2\) (in particular they are infinite dimensional). Finally, if in addition, as for the hyperbolic plane, for all smooth compactly supported function \(u\), \(\langle \Delta u, u \rangle_{L^2} \geq \frac{1}{4}||u||_{L^2}^2\), then the spectrum of \(\Delta_L\) is

\[
\{-2\} \cup \{0\} \cup \left[ \frac{1}{4}, +\infty \right].
\]

Along the paper we also obtain some relative results on more general surfaces, with or without constant scalar curvature.

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## 2. Definitions, notations and conventions

Let \(\overline{M}\) be a smooth, compact surface with boundary \(\partial M\). Let \(M := \overline{M} \setminus \partial M\) be a non-compact surface without boundary. In our context the boundary \(\partial M\) will play the role of a conformal boundary at infinity of \(M\). Let \(g\) be a Riemannian metric on \(M\). The manifold \((M, g)\) is **conformally compact** if there exists on \(\overline{M}\) a smooth defining function \(\rho\) for \(\partial M\) (that is \(\rho \in C^\infty(\overline{M})\), \(\rho > 0\) on \(M\), \(\rho = 0\) on \(\partial M\) and \(d\rho\) is nowhere vanishing on \(\partial M\)) such that \(\overline{g} := \rho^2 g\) is a \(C^{2,\alpha}(\overline{M}) \cap C^{\infty}(M)\) Riemannian metric on \(\overline{M}\). We will denote by \(\hat{g}\) the metric induced on \(\partial M\). Now if \(|d\rho|_{\bar{g}} = 1\) on \(\partial M\), it is well known (see [11] for instance) that \(g\) has asymptotically sectional curvature \(-1\) near its boundary at infinity. In this case we say that \((M, g)\) is **asymptotically hyperbolic**. Along the paper, it will be assumed sometimes than \((M, g)\) has constant scalar curvature : then the asymptotic hyperbolicity enforces the normalisation

\[
R(g) = -2,
\]

where \(R(g)\) is the scalar curvature of \(g\).

The basic asymptotically hyperbolic surface is the real hyperbolic Poincaré disc. In this case \(M\) is the unit disc of \(\mathbb{R}^2\), with the hyperbolic metric

\[
g_0 = \omega^{-2} \delta,
\]
\( \delta \) is the Euclidean metric, \( \omega(x) = \frac{1}{2}(1 - |x|^2) \).

We denote by \( T^p_q \) the set of rank \( p \) covariant and rank \( q \) contravariant tensors. When \( p = 2 \) and \( q = 0 \), we denote by \( S_2 \) the subset of symmetric tensors, and by \( \dot{S}_2 \) the subset of \( S_2 \) of trace free symmetric tensors. We use the summation convention, indices are lowered with \( g_{ij} \) and raised with its inverse \( g^{ij} \).

The Laplacian is defined as
\[
\Delta = -\text{tr} \nabla^2 = \nabla^* \nabla,
\]
where \( \nabla^* \) is the \( L^2 \) formal adjoint of \( \nabla \). In dimension 2, the Lichnerowicz Laplacian acting on trace free symmetric covariant 2-tensors is
\[
\Delta_L = \Delta + 2R,
\]
where \( R \) is the scalar curvature of \( g \).

For \( u \) a covariant 2-tensor field on \( M \) we define the divergence of \( u \) by
\[
(\text{div } u)_j = -\nabla^j u_{ji}.
\]
If \( u \) is a symmetric covariant 2-tensor field on \( M \), it can be seen as a one form with values in the cotangent bundle. Thus we can define its exterior differential with
\[
(d^\nabla u)_{ijk} := \nabla_i u_{jk} - \nabla_j u_{ik},
\]
which is a two form with values the cotangent bundle.

For \( \omega \), a one form on \( M \), we define its divergence
\[
d^\nabla \omega = -\nabla^i \omega_i,
\]
the symmetric part of its covariant derivative : 
\[
(L\omega)_{ij} = \frac{1}{2}(\nabla_i \omega_j + \nabla_j \omega_i),
\]
(note that \( L^* = \text{div} \)) and the trace free part of that last tensor :
\[
(\dot{L}\omega)_{ij} = \frac{1}{2}(\nabla_i \omega_j + \nabla_j \omega_i) + \frac{1}{2}d^\nabla \omega g_{ij}.
\]

The well known \([4]\) Weitzenböck formula for the Hodge-De Rham Laplacian on 1-forms, in dimension 2, reads
\[
\Delta_H \omega_i = \nabla^* \nabla \omega_i + \text{Ric}(g)_{ik} \omega^k = \nabla^* \nabla \omega_i + \frac{R}{2} \omega_i.
\]
We recall also the Weitzenböck formula
\[
\Delta_K := (d^\nabla)^* d^\nabla + \text{div}^* \text{div} = \Delta + R = \Delta_L - R.
\]

For a one form \( \omega \), we will consider the trace free symmetric covariant two tensor defined by
\[
(\dot{S} \omega)_{ij} = \omega_i \omega_j - \frac{|\omega|^2}{2} g_{ij}.
\]
A \( TT \)-tensor (Transverse Traceless tensor) is by definition a symmetric divergence free and trace free covariant 2-tensor.

\( L^2 \) denotes the usual Hilbert space of functions or tensors with the product (resp. norm)
\[
\langle u, v \rangle_{L^2} = \int_M \langle u, v \rangle d\mu_g \quad \text{resp.} \quad |u|_{L^2} = (\int_M |u|^2 d\mu_g)^{\frac{1}{2}},
\]
where $\langle u, v \rangle$ (resp. $|u|$) is the usual product (resp. norm) of functions or tensors relative to $g$, and the measure $d\mu_g$ is the usual measure relative to $g$ (we will omit the term $d\mu_g$). For $k \in \mathbb{N}$, $H^k$ will denote the Hilbert space of functions or tensors with $k$-covariant derivative in $L^2$, endowed with its standard product and norm.

We will first work near the infinity of $M$, so it is convenient to define for small $\varepsilon > 0$, the manifold

$$M_\varepsilon = \{ x \in M, \rho(x) < \varepsilon \}.$$ 

It is well known that near infinity, we can choose the defining function $\rho$ to be the $g$-distance to the boundary. Thus, if $\varepsilon$ is small enough, $M_\varepsilon$ can be identified with $(0, \varepsilon) \times \partial M$ equipped with the metric

$$g = \rho^{-2}(d\rho^2 + \hat{g}(\rho)d\theta^2),$$

where $\{\hat{g}(\rho)\}_{\rho \in (0,\varepsilon)}$ is a family of smooth, positive functions on $\partial M$, with $\hat{g}(0) = \hat{g}$.

Let $P$ be an uniformly degenerate elliptic operator of order 2 on some tensor bundle over $M$ (see [9] for more details). We recall here a criterion for $P$ to be semi-Fredholm. We first need the

**Definition 2.1.** We say that $P$ satisfies the asymptotic estimate

$$\langle Pu, u \rangle_{L^2} \geq \infty C\|u\|_{L^2}^2$$

(resp. $\|Pu\|_{L^2} \geq \infty C\|u\|_{L^2}$)

if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all smooth $u$ with compact support in $M_\delta$, we have

$$\langle Pu, u \rangle_{L^2} \geq (C - \varepsilon)\|u\|_{L^2}^2$$

(resp. $\|Pu\|_{L^2} \geq (C - \varepsilon)\|u\|_{L^2}$).

Proposition 2.2 below is standard in the context of non-compact manifolds (see [7] for instance). It shows that the essential spectrum is characterized near infinity.

**Proposition 2.2.** Let $P : H^2 \longrightarrow L^2$. Then $P$ is semi-Fredholm (ie. has finite dimensional kernel and closed range) if and only if $P$ satisfies an asymptotic estimate

$$\|Pu\|_{L^2} \geq \infty C\|u\|_{L^2}$$

for some $c > 0$.

This proposition will be used to compute the essential spectrum of $\Delta_L$ which is, by definition, the closed set

$$\sigma_e(\Delta_L) = \{ \lambda \in \mathbb{R}, \Delta_L - \lambda I d \text{ is not semi-Fredholm} \}.$$

**3. Commutators of some natural operators**

**Lemma 3.1.** On one forms, we have

$$\text{div} \circ \mathcal{L} = \frac{1}{2}(\Delta - \frac{R}{2}) = \frac{1}{2}(\Delta_H - R).$$
Proof. In local coordinates, \(2 \text{div} \circ \tilde{L}(\omega)\) is equal to:
\[
\begin{align*}
-\nabla^i (\nabla_i \omega_j + \nabla_j \omega_i - \nabla^k \omega_k g_{ij}) &= \Delta \omega_j - \nabla^k \nabla_j \omega_k + \nabla_j \nabla^k \omega_k \\
&= (\Delta - \text{Ric}) \omega_j \\
&= (\Delta - \frac{R}{2}) \omega_j.
\end{align*}
\]
\[\square\]

Recall that in dimension 2 (see Corollary 3.2 of [8] for instance) we have:

**Lemma 3.2.** Let \((M, g)\) be a Riemannian surface with Levi-Civita connexion \(\nabla\). Then the following equality holds for trace free symmetric covariants two tensors:
\[
\text{div} \circ \Delta_L = \Delta_H \circ \text{div}.
\]

So we obtain

**Corollary 3.3.** If \(h\) is a trace free symmetric covariant two tensor with \(\Delta_L h = \lambda h\) then \(\Delta_H \text{div} h = \lambda \text{div} h\).

**Lemma 3.4.** On a Riemannian surface with Levi-Civita connexion \(\nabla\), we have
\[
\Delta_L \circ \tilde{L} = \tilde{L} \circ \Delta_H - \hat{S}(dR, \cdot),
\]
where \(\hat{S}(dR, \xi)_{ij} = \frac{1}{2}(\nabla_j R \xi_i + \nabla_i R \xi_j - \nabla_p R \xi_p g_{ij})\). In particular
\[
\Delta_L \circ \tilde{L} = \tilde{L} \circ \Delta_H - \hat{S}(dR, \cdot).
\]
Moreover \(R\) is constant iff
\[
\Delta_L \circ \tilde{L} = \tilde{L} \circ \Delta_H \text{ so } \Delta_L \circ \tilde{L} = \tilde{L} \circ \Delta_H.
\]

**Proof.** The first part comes from [8] lemma 3.3 where here \(\text{Ric}(g) = \frac{R(g)}{2} g\).

Now, if \(\Delta_L \circ \tilde{L} = \tilde{L} \circ \Delta_H\) then for any one form \(\xi\), \(\hat{S}(dR, \xi) = 0\). At any point \(x \in M\), we take an orthonormal basis \((e_1, e_2)\) on \(T_x^* M\), and choose \(\xi = e_1\). We then see that the matrix of \(\hat{S}(dR, \xi)\) has the form \(\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\), where \((a, b)\) are the coordinates of \(dR\). We finally deduce that \(dR = 0\). \[\square\]

4. Some decompositions of trace free symmetric two tensors

In this section, we recall two well known natural decompositions. We give their simple proofs for completeness.

**Lemma 4.1.** For all \(k \in \mathbb{N}\),
\[
H^{k+1}(M, \tilde{S}_2) = \ker \text{div} \oplus \text{Im} \tilde{L},
\]
where the decomposition is orthogonal in \(L^2\).

**Proof.** For \(\omega \in C^\infty_c(M)\), we have
\[
\int_M < \tilde{L}(\omega), h > = \int_M < \omega, \text{div} h >.
\]
Thus \(\tilde{L}^* = \text{div}\), which gives \((\text{Im} \tilde{L})^\perp = \text{Ker} \text{div}\). \[\square\]
Lemma 4.2. For all $k \in \mathbb{N}$, 
\[ H^{k+1}(M, T_1) = \ker \Delta_H \oplus \text{Im} d \oplus \text{Im}(d^*), \]
where the decomposition is orthogonal in $L^2$.

Remark 4.3. Recall that, from the definition of $\Delta_H$, we have: \[ \ker \Delta_H = \ker d \cap \ker d^*. \]

Proof. First, from the definition of $d^*$, it is clear that $(\text{Im} d)^\perp = \ker d^*$, and so $H^{k+2}(M, T_1) = \ker d^* \oplus \text{Im} d$. For all $H^1$ function $u$ and all $H^1$ one forms $\omega$, we have
\[ \int_M \langle *du, \omega \rangle = \int_M \langle d^* u, \omega \rangle = \int_M \langle u, d\omega \rangle. \]
As a consequence, if $\langle *du, \omega \rangle_{L^2} = 0$ for all $u \in C_c^\infty(M)$, then $d \omega = 0$, and if in addition $d^* \omega = 0$ then $\Delta_H \omega = 0$. This shows that $\ker d^* = \text{Im}(d^*) \oplus \ker \Delta_H$. \hfill $\square$

From Lemma 4.2, any one form $\omega$ in $H^1$ can be decomposed in a unique way with
\[ (4.1) \quad \omega = \eta + du + *dv, \]
where $\Delta_H \eta = 0$.

5. The spectrum on TT-tensors

Lemma 5.1. Let $M$ be any Riemannian surface. If $h \in C^2(M, \tilde{S}_2)$, then the following properties are equivalent:
\begin{enumerate}
  \item [(i)] $\text{div} h = 0$,
  \item [(ii)] $d^\nabla h = 0$,
  \item [(iii)] $h = \tilde{S}(\omega)$, where $\omega$ is a harmonic one form.
\end{enumerate}

They imply
\begin{enumerate}
  \item [(iv)] $\Delta_L h = Rh$.
\end{enumerate}

Moreover, if $h \in L^2$, then (iv) implies (i), (ii) and (iii).

Proof. The first part is due to Avez ([3] Lemma A and Lemma C). The second part is simply due to the following Weitzenböck formula [10]:
\[ (d^\nabla)^* d^\nabla + \text{div}^* \text{div} = \Delta_K = \Delta_L - R, \]
and the fact that if $h \in L^2$ solves (iv) weakly, then elliptic regularity gives $h \in H^\infty \subset C^\infty$. \hfill $\square$

Corollary 5.2. There exists a non trivial eigen-TT-tensor of $\Delta_L$ iff $R$ is constant. In this case any TT-tensor is an eigentensor with eigenvalue $R$.

Proof. The "if" part is clear. For the "only if" direction, assume that $h$ is a non trivial eigen-TT-tensor of $\Delta_L$, so that $\text{div} h = 0$ and $\Delta_L h = \lambda h$ hold for some $\lambda \in \mathbb{R}$. From Lemma 5.1, $(\Delta_L - R)h = 0$ and then $(R - \lambda)h = 0$. If $R \neq \lambda$ near a point, then $h$ has to be trivial near this point, so from the unique continuation property, $h$ is trivial. This contradicts the assumption on $h$ and proves the result. \hfill $\square$
6. Spectrum on \( \text{Im} \hat{L} \)

If \( R \) is constant then from Lemma 3.4 and the fact that \( \Delta_H \) preserves the decomposition 4.1, it suffices to study the spectrum of \( \Delta_L \) on \( \text{Im} \hat{L} \), restricted successively to \( \text{Ker} \Delta_H \), \( \text{Im} d \) and \( \text{Im}(d^*) \).

**Lemma 6.1.** When \( R \) is constant then \( \hat{L}(\text{Ker} \Delta_H) \) is in the kernel of \( \Delta_L \). If \( R \) is moreover negative, \( \hat{L}(\text{Ker} \Delta_H) \) is in one to one correspondence with \( \text{Ker} \Delta_H \).

**Proof.** If \( h = \hat{L}\eta \), with \( \eta \in \text{Ker} \Delta_H \), then from Lemma 3.4 \( \Delta_L h = 0 \). Now from Lemma 3.1 we have

\[
2 \text{div}\circ\hat{L}(\eta) = (\Delta - \frac{R}{2})\eta.
\]

Thus if \( R < \text{cte} < 0 \), then \( \hat{L} \) is injective on \( H^2 \).

We are now interested in the spectrum on \( \text{Im} \hat{L} \circ d \). We begin with a lemma.

**Lemma 6.2.** If \( h = \hat{L}\omega \), with \( \omega \in H^1 \) then:

\[
||h||^2_{L^2} = \frac{1}{2} ||\omega||^2_{H^1} - \int_M (\frac{R}{2} + 1)||\omega||^2.
\]

In particular, if \( R = -2 \) we obtain

\[
2 ||h||^2_{L^2} = ||\omega||^2_{H^1}.
\]

**Proof.** Using Lemma 3.4 we compute:

\[
\int_M |h|^2 = \langle \hat{L}\omega, \hat{L}\omega \rangle_{L^2} = \langle \text{div} \hat{L}\omega, \omega \rangle_{L^2} = \frac{1}{2} \int_M (||\nabla \omega||^2 - R/2 ||\omega||^2) = \frac{1}{2} (||\omega||^2_{H^1} - \int_M (R/2 + 1)||\omega||^2).
\]

\( \square \)

**Corollary 6.3.** On an A.H. surface, for all \( \varepsilon > 0 \), there exists \( \delta_0 > 0 \) small such that, for all \( \delta \in (0, \delta_0) \) and all one forms \( \omega \) with compact support in \( M_\delta \), if \( h = \hat{L}\omega \) then

\[
||\omega||^2_{H^1} \geq 2(1 - \varepsilon)||h||^2_{L^2}.
\]

**Proof.**

\[
||\omega||^2_{H^1} - \int_M (R/2 + 1)||\omega||^2 = ||\omega||^2_{H^1} - \int_{M_\delta} O(\rho)||\omega||^2 \leq ||\omega||^2_{H^1} + C\delta ||\omega||^2_{L^2} \leq (1 + C\delta) ||\omega||^2_{H^1},
\]

where \( C \) is a positive constant. Lemma 6.2 concludes the proof.

\( \square \)

Let us recall a well known lemma.
Lemma 6.4. Let \( u \) be a smooth compactly supported function. If
\[
\langle \Delta u, u \rangle_{L^2} \geq c||u||_{L^2}^2,
\]
then
\[
\langle \Delta_H du, du \rangle_{L^2} \geq c||du||_{L^2}^2
\]
and
\[
\langle \Delta_H (*du), (*du) \rangle_{L^2} \geq c||*du||_{L^2}^2.
\]

Proof.
\[
\langle \Delta_H du, du \rangle = \langle dd^* du, du \rangle = \langle d^* du, d^* du \rangle = ||\Delta u||^2 \geq c||u|| ||\Delta u|| \geq c\langle u, \Delta u \rangle = c||du||^2.
\]
\[
\langle \Delta_H (*du), (*du) \rangle = \langle d^* d(*du), *du \rangle = \langle d * du, d * du \rangle = ||\Delta u||^2 \geq c||u|| ||\Delta u|| \geq c\langle u, \Delta u \rangle = c||du||^2 = c||*du||^2.
\]
\(\square\)

We would like an equivalent to this lemma when substituting one forms to functions. This is achieved by the following lemma and its corollary.

Lemma 6.5. Let \( \omega \) be a smooth compactly supported one form. If
\[
\langle \Delta_H \omega, \omega \rangle_{L^2} \geq c||\omega||_{L^2}^2
\]
then
\[
\langle \Delta_L \mathcal{L} \omega, \mathcal{L} \omega \rangle \geq c \frac{\omega}{2} ||\omega||_{H^1}^2 + \frac{1}{2} \int_M \left( \frac{R}{2} + 1 \right) ||\omega||^2 - \int_M \left( \frac{R}{2} + 1 \right) \langle \Delta_H \omega, \omega \rangle - \langle S(dR, \omega), \mathcal{L} \omega \rangle_{L^2}.
\]
Proof.

\[ \langle \Delta L \hat{\omega}, \hat{\omega} \rangle_{L^2} = \langle \hat{L} \Delta_H \omega, \hat{\omega} \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ = \langle \Delta_H \omega, \text{div} \hat{\omega} \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ = \frac{1}{2} \langle \Delta_H \omega, (\Delta_H - R) \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ = \frac{1}{2} ||\Delta_H \omega||_{L^2}^2 - \frac{1}{2} \langle R \Delta_H \omega, \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ \geq \frac{1}{2} c ||\Delta_H \omega||_{L^2}^2 ||\omega||_{L^2}^2 - \frac{1}{2} \langle R \Delta_H \omega, \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ \geq \frac{1}{2} c \langle \Delta_H \omega, \omega \rangle_{L^2}^2 - \frac{1}{2} \langle R \Delta_H \omega, \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ \geq \frac{1}{2} c ||\nabla \omega||_{L^2}^2 + \frac{1}{2} c \int_M R |\omega|^2 - \frac{1}{2} \langle R \Delta_H \omega, \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ \geq \frac{1}{2} c ||\nabla \omega||_{L^2}^2 + \frac{1}{2} c \int_M R |\omega|^2 + c ||\omega||_{L^2}^2 \]
\[ \geq \frac{1}{2} \langle (R + 2) \Delta_H \omega, \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]
\[ \geq \frac{1}{2} c ||\omega||_{H^1}^2 + \frac{1}{2} c \int_M (\frac{R}{2} + 1) |\omega|^2 \]
\[ \geq \frac{1}{2} \langle (R + 2) \Delta_H \omega, \omega \rangle_{L^2} - \langle \hat{S}(dR, \omega), \hat{\omega} \rangle_{L^2} \]

\[ \square \]

Remark 6.6. Under the assumptions of Lemma 6.4, the assumptions of lemma 6.5 are satisfied by \( \omega = \text{d}u \) or \( \omega = *\text{d}u \).

Proposition 6.5 together with Lemma 6.2 give:

**Corollary 6.7.** If \( R = -2 \) and \( \langle \Delta_H \omega, \omega \rangle_{L^2} \geq c ||\omega||_{L^2}^2 \) then
\[ \langle \Delta L \hat{\omega}, \hat{\omega} \rangle_{L^2} \geq c ||\hat{\omega}||_{L^2}^2. \]

In the A.H. setting we have

**Corollary 6.8.** On an A.H. surface, for \( \omega \in \text{Im} d \) or \( \omega \in \text{Im} *d \),
\[ \langle \Delta L \hat{\omega}, \hat{\omega} \rangle_{L^2} \geq \frac{1}{4} ||\hat{\omega}||_{L^2}^2. \]

**Proof.** It is well know that on A.H. surfaces, \( \langle \Delta u, u \rangle \geq \frac{1}{4} ||u||_{L^2}^2 \) holds. Then (see lemma 6.4 for instance) \( \langle \Delta_H \omega, \omega \rangle_{L^2} \geq \frac{1}{4} ||\omega||_{L^2}^2 \). We will show that the three terms in the right-hand side of Lemma 6.5 do not contribute at infinity. We work with a one form \( \omega \) compactly supported in \( M_3 \) with small \( \delta \). We recall that \( R + 2 = O(\rho) \) and \( ||d(R + 2)|| = O(\rho) \).
Let us begin with
\[
\int_M \left( \frac{R}{2} + 1 \right) \langle \Delta_H \omega, \omega \rangle = \int_M \left( \frac{R}{2} + 1 \right) \langle \Delta \omega, \omega \rangle + \int_M \frac{R}{2} ||\omega||^2
\]
\[
= \int_M \omega^i \nabla^i \left( \frac{R}{2} + 1 \right) \nabla_j \omega_j + \int_M \left( \frac{R}{2} + 1 \right) ||\nabla \omega||^2
\]
\[
+ \int_M \frac{R}{2} ||\omega||^2.
\]

The two last terms are clearly bounded in absolute value by \(C_1 \delta ||\omega||^2_{H^1}\). Let
\[
A(\omega) = \int_M \omega^j \nabla^i \left( \frac{R}{2} + 2 \right) \nabla_i \omega_j.
\]
Then:
\[
|A(\omega)| \leq ||\nabla \omega||_{L^2} \left( \int_M ||d(R + 2)\omega||^2 \right)^{1/2}
\]
\[
\leq C_2 \delta ||\nabla \omega||_{L^2} \left( \int_M ||\omega||^2 \right)^{1/2}
\]
\[
\leq \frac{C_2}{2} \delta ||\omega||^2_{H^1}.
\]

We so get:
\[
\left| \int_M \left( \frac{R}{2} + 1 \right) \langle \Delta_H \omega, \omega \rangle \right| \leq C_3 \delta ||\omega||^2_{H^1}.
\]

The term \(\langle \hat{S}(dR, \omega), \hat{L} \omega \rangle_{L^2}\) proceed in a manner similar to \(A(\omega)\) to obtain the same estimate, perhaps with a different constant. Finally, the term \(\int_M \left( \frac{R}{2} + 1 \right) ||\omega||^2\) is clearly bounded in absolute value by \(C_4 \delta ||\omega||^2_{H^1}\). The conclusion follows from Lemma 6.5, the triangular inequality and Corollary 6.3.

\[
\square
\]

**Proposition 6.9.** Let \(\lambda \geq \frac{1}{4}\) and \(C > 0\). Let \(P\) be the operator \(\Delta_L - \lambda \text{Id}: H^2 \rightarrow L^2\). There is no asymptotic estimate
\[
|Pu|_{L^2} \geq C ||u||_{L^2},
\]
for \(P\) on \(\text{Im}(\hat{L} \circ d)\).

**Proof.** Let \(\lambda \geq \frac{1}{4}\), and \(\mu := \sqrt{\lambda - \frac{1}{4}}\). The idea of the proof is to construct a family of tensors \(\{h_R\} = \{\hat{L}(df_R)\} = \{\text{Hess} f_R\}\) with compact support in \(M_{e^{-R/2}}\) such that \(|Ph_R|_{L^2(M)}\) goes to zero when \(R\) goes to infinity but \(|h_R|_{L^2(M)}\) goes to infinity when \(R\) goes to infinity.

It is well known (see \([L1, \text{lemma 5.1}]\) for example) that we can change the defining function \(\rho\) into a defining function \(r\) such that the metric takes the form
\[
g = r^{-2} \bar{g} = r^{-2} (dr^2 + \bar{g}(r)),
\]
on \(M_\delta = \{0, \delta \times \partial_\infty M\}\) (reducing \(\delta\) if necessary), where \(\bar{g}(r)\) is a metric on \(\{r\} \times \partial_\infty M\).

The non trivial Christoffel symbols of \(g = r^{-2} [dr^2 + \bar{g}(r)d\theta^2]\) are
\[
\Gamma^r_{rr} = -r^{-1},
\]
\[
\Gamma^r_{rr} = \frac{\partial\bar{g}}{\partial r},
\]
\[
\Gamma^\theta_{r\theta} = \frac{\partial\bar{g}}{\partial \theta},
\]
\[
\Gamma^\theta_{\theta r} = -\frac{\partial\bar{g}}{\partial \theta},
\]
\[
\Gamma^\theta_{\theta \theta} = \frac{\partial^2 \bar{g}}{\partial \theta^2}.
\]
where the primes denote $r$-derivatives. If $f$ is a "radial" function, i.e., $f = f(r)$, we compute:

$$\text{Hess } f = (f'' + r^{-1} f')dr^2 + (-r^{-1} \tilde{g} + \frac{1}{2} \tilde{g}'')f'd\theta^2.$$ 

We deduce

$$\Delta f = -r^2(f'' + \frac{1}{2} \tilde{g} - \tilde{g}' f').$$

We also have

$$\text{Hess } f = (\frac{1}{2}f'' + r^{-1} f' - \frac{1}{4} \tilde{g}^{-1} \tilde{g}' f')(dr^2 - \tilde{g}d\theta^2) =: F_f(r)(dr^2 - \tilde{g}d\theta^2).$$

This tensor is in the set $V_2$ of [9] page 201: substitute $f$ there by $F_f$ here, $	ilde{q}$ there by $dr^2 - \tilde{g}d\theta^2$ here, $\rho$ there by $r$ here and the dimension $n+1$ there by 2 here. Recall that the Lichnerowicz Laplacian in our context is $\Delta_L = \Delta + \mathcal{K}$, where $\mathcal{K} = -4 + O(r)$. Thus, from [9] Lemma 2.9 page 202, we obtain

$$(\Delta_L - \lambda)(F(r)\tilde{q}) = I_2(F(r))\tilde{q} + rX(F),$$

where

$$I_2(F) = -r^2F'' - 4rF' - 2FF$$

has for characteristic exponents

$$s_1, s_2 = \frac{1}{2}(-3 \pm \sqrt{1 - 4\lambda}),$$

and $X = \tilde{q}r^2\frac{d^2}{dr^2} + \tilde{b}r\frac{d}{dr} + \tilde{c}$ is a second order operator polynomial in $r\frac{d}{dr}$ with $\tilde{g}$-bounded coefficients depending on $\tilde{q}$ and $\tilde{q}$.

In particular, if $\lambda \geq \frac{3}{4}$ and $f(r) = \sqrt{r}(a \cos(\mu \ln(r)) + b \sin(\mu \ln(r)))$, where $\mu = \sqrt{\lambda - \frac{1}{4}}$, then $F_f(r) = r^{-3/2}(A \cos(\mu \ln(r)) + B \sin(\mu \ln(r)) + O(r^{-1/2})$ and $(A, B) \neq 0$ if $(a, b) \neq 0$. Thus we obtain

$$I_2(F_f(r)) = O(r^{-1/2}).$$

Let us now define the function

$$f_R(r) = f(r)\Psi_R(r),$$

where $\Psi_R$ is as in Lemma 8.1. A simple calculation shows that

$$F_{f_R}(r) = \Psi_R(r)F_f(r) + O(R^{-1})O(r^{-3/2}).$$

Therefore:

$$I_2(F_{f_R}(r)) = O(r^{-1/2}) + O(R^{-1})O(r^{-3/2}),$$

and

$$rX(F_{f_R}(r)) = O(r^{-1/2}).$$

Then:

$$(\Delta_L - \lambda)(F_{f_R}(r)\tilde{q}) = O(r^{-1/2}) + O(R^{-1})O(r^{-3/2}).$$

We deduce that

$$\| (\Delta_L - \lambda)(F_{f_R}(r)\tilde{q}) \|_{L^2}^2 = O(R^{-1}).$$
On the other hand, we have
\[ \| (F_{fr}(r)\mathcal{I}) \|_{L^2}^2 \geq cR, \]
where \( c \) is a positive constant. Letting \( R \) going to infinity, this concludes the proof of the proposition.

\[ \square \]

7. Conclusion

From Proposition 6.9 and Corollary 6.8, the essential spectrum of \( \Delta_L \) restricted to \( \text{Im}(\mathcal{L} \circ d) \) is
\[ \left[ \frac{1}{4}, +\infty \right]. \]
In particular this ray is in the essential spectrum of \( \Delta_L \).

If \( R \) is constant then the A. H. condition forces \( R = -2 \). Lemma 6.1 shows that any tensor in \( \mathcal{L}(\ker \Delta_H) \) is in the kernel of \( \Delta_L \). The eigenspace for 0 is then infinite dimensionnal as \( \ker \Delta_H \) (recall that \( \mathcal{L} \) is injective if \( R < 0 \)).

From Lemma 5.1, any TT-tensor \( h \) is an eigentensor for the eigenvalue \(-2\) and there is a harmonic one form \( \omega \) such that \( h = \mathcal{S}(\omega) \). Moreover \( \omega \) is in \( L^4 \) iff \( h \) is in \( L^2 \).

Assume now that
\[ \langle \Delta u, u \rangle_{L^2} \geq \frac{1}{4} \| u \|_{L^2}, \]
holds for all smooth compactly supported functions \( u \). Then Lemma 6.4 and Corollary 6.7 give, when \( \omega = df \) or \( \omega = *df \),
\[ \langle \Delta_L \mathcal{L} \omega, \mathcal{L} \omega \rangle_{L^2} \geq \frac{1}{4} \| \mathcal{L} \omega \|_{L^2}. \]
This proves that there are no eigentensors with eigenvalue less than \( \frac{1}{4} \) in \( \text{Im}(\mathcal{L} \circ d) \) nor in \( \text{Im}(\mathcal{L} \circ (\ast d)) \). Recall that, when \( R \) is a constant, the Lichnerowicz Laplacian commute with the Hodge Laplacian, and also that the Hodge Laplacian preserves the decomposition 4.1. We so get that on \( \text{Im} \mathcal{L} \), the essential spectrum of \( \Delta_L \) is
\[ \{ 0 \} \cup \left[ \frac{1}{4}, +\infty \right]. \]
This concludes the proof of the main theorem 1.1.

8. Appendix : A family of cutoff functions

In this appendix, we give a family of cutoff functions. Standard in the A.H. context, they can be found in [1], Definition 2.1 p.1362 for instance.

**Lemma 8.1.** Let \((M, g, \rho)\) be an asymptotically hyperbolic manifold. For \( R \in \mathbb{R} \) large enough, there exits a cutoff function \( \Psi_R : M \to [0, 1] \) depending only on \( \rho \), supported in the annulus \( \{e^{-8R} < \rho < e^{-R}\} \), equal to 1 in \( \{e^{-4R} < \rho < e^{-2R}\} \) and which satisfies for \( R \) large :
\[ \left| \frac{d^k \Psi_R}{d\rho^k}(\rho) \right| \leq \frac{C_k}{R^k \rho^k}, \]
for all \( k \in \mathbb{N} \setminus \{0\} \), where \( C_k \) is independent of \( R \).
Proof. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function equal to 1 on $]-\infty, 1]$ and 0 on $[2, +\infty[$. We define

$$\chi_R(x) := \chi\left(\frac{\ln(\rho(x))}{-R}\right),$$

we then have $\chi_R : M \rightarrow [0, 1]$ is equal to 1 on $\rho \geq e^{-R}$ and 0 on $\rho \leq e^{-2R}$. Now we define

$$\Psi_R := \chi_4 R (1 - \chi_R)$$

which satisfies the announced properties. \qed

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