IRREDUCIBILITY OF SOME REPRESENTATIONS OF THE GROUPS OF SYMPLECTOMORPHISMS AND CONTACTOMORPHISMS

ŁUKASZ GARCAREK

Abstract. We show the irreducibility of some unitary representations of the group of symplectomorphisms and the group of contactomorphisms.

1. Introduction

Let $M$ be a smooth second-countable manifold. There exists a natural diffeomorphism-invariant measure class on $M$, consisting of measures having positive density with respect to the Lebesgue measure in every coordinate chart. We will refer to them simply as Lebesgue measures.

Let $\mu$ be a Lebesgue measure on $M$. For a group $G$ acting on $M$ by diffeomorphisms we may consider a series $\Pi_\theta^\mu$ of unitary representations on $L^2(M, \mu)$ given by

$$\Pi_\theta^\mu(\gamma)f = f(\gamma^{-1}(d\gamma \cdot \mu \frac{d\gamma}{d\mu})^{1/2+i\theta}),$$

where $\theta \in \mathbb{R}$.

If a measure $\nu$ is equivalent to $\mu$, then the operator $T: L^2(M, \mu) \to L^2(M, \nu)$ defined by

$$Tf = f\left(\frac{d\mu}{d\nu}\right)^{1/2+i\theta}$$

gives an isomorphism of representations $\Pi_\theta^\mu$ and $\Pi_\theta^\nu$. In particular, if $\mu$ is equivalent to a $G$-invariant measure, the representations $\Pi_\theta^\mu$ are equivalent for all $\theta \in \mathbb{R}$.

For a diffeomorphism $\phi: M \to M$ we define its support $\text{supp} f$ as the closure of the set $\{p \in M : \phi(p) \neq p\}$. Compactly supported diffeomorphisms of $M$ form a group $\text{Diff}_c(M)$. In [6] it was proved that for an infinite measure $\mu$ the representation $\Pi_\theta^\mu$ of the group $\text{Diff}_c(M, \mu)$

---

2010 Mathematics Subject Classification. 57S20, 22A25.
of compactly supported, measure-preserving diffeomorphisms of $M$ is irreducible. It follows that the representations $\Pi^\theta_\mu$ of the groups $\text{Diff}_c(M, \mu)$ and $\text{Diff}(M)$ are irreducible for any $\theta \in \mathbb{R}$. The idea of the proof is to take two functions $f, g \in L^2(M, \mu)$ and explicitly find a diffeomorphism $\phi$ such that $\langle f, \Pi^\theta_\mu(\phi)g \rangle \neq 0$, thus showing that $f$ and $g$ cannot lie in two distinct orthogonal invariant subspaces.

Representations of various subgroups of the group of diffeomorphisms are also studied in [4].

The purpose of this note is to present an enhancement of the argument from [6], and apply it to classical groups of diffeomorphisms: the group of symplectomorphisms and the group of contactomorphisms.

2. Convolution on the Heisenberg group

On $\mathbb{R}^n$ the following theorem holds (see Theorem 4.3.3 in [3] for a proof of a more general result):

**Theorem 2.1.** If $f, g \in L^1(\mathbb{R}^n)$ are compactly supported and nonzero, then $f \ast g$ is nonzero.

**Proof.** Let $\hat{h}(\xi) = \int h(x)e^{-i\xi x} \, dx$ denote the Fourier transform of $h \in L^1(\mathbb{R}^n)$. Suppose that $f \ast g = 0$. As $f$ and $g$ are compactly supported, their Fourier transforms extend to entire functions. Since $\hat{f} \hat{g} = \hat{f} \ast \hat{g} = 0$ on $\mathbb{C}^n$, it follows by holomorphicity that $\hat{f} \hat{g} = 0$ on $\mathbb{C}^n$, and either $\hat{f}$ or $\hat{g}$ must vanish. This contradicts the assumption that $f$ and $g$ are nonzero. \qed

In this section we will prove an analogue of this theorem for square-integrable functions on the Heisenberg group.

2.1. The Heisenberg group. Let $n$ be a positive integer. The multiplicative group of all matrices of the form

\[
\begin{pmatrix}
1 & \tilde{x}^T & z \\
0 & I_n & \tilde{y} \\
0 & 0 & 1
\end{pmatrix},
\]

where $z \in \mathbb{R}$, $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, and $I_n$ denotes the $n \times n$ identity matrix, is called the Heisenberg group $H_n$. It is a unimodular Lie group diffeomorphic with $\mathbb{R}^{2n+1}$, and its Haar measure is the $(2n+1)$-dimensional Lebesgue measure. We will identify $H_n$ with $\mathbb{R}^{2n+1}$ as manifolds. The convolution of functions $f, g \in L^1(H_n)$ will be denoted $f \ast_{H} g$. 

2.2. Convolution of compactly supported functions on $H_n$. Let $f \in L^1(\mathbb{R})$. Define

$$Tf(x) = \int_{-\infty}^{x} f(t) \, dt.$$  

If $f \in L^2(\mathbb{R})$ is supported in $[a, b]$, then it is integrable; furthermore, if $\int f(t) \, dt = 0$, then $\text{supp } Tf \subseteq [a, b]$ and we may write

$$Tf(x) = \int f(t)K_{[a,b]}(t,x) \, dt,$$

where

$$K_{[a,b]}(t,x) = \begin{cases} 1 & \text{for } a \leq t \leq x \leq b, \\ 0 & \text{otherwise}. \end{cases}$$

Hence, $Tf \in L^2(\mathbb{R})$ and $\|Tf\|_2 \leq \|K_{[a,b]}\|_2 \|f\|_2$, where $\|K_{[a,b]}\|_2$ stands for the $L^2$-norm of $K_{[a,b]} \in L^2(\mathbb{R}^2)$. We may iterate the process of applying $T$ to $f$ as long as it yields a function integrating to 0. The next lemma shows that unless $f = 0$, this process terminates.

**Lemma 2.2.** If $f \in L^2(\mathbb{R})$ is nonzero and compactly supported, then there exists $k \geq 0$ such that $T^k f \in L^2(\mathbb{R})$ and $\int T^k f(x) \, dx \neq 0$.

**Proof.** If there is no such $k$, then $T^k f \in L^2(\mathbb{R})$ and $\int T^k f(x) \, dx = 0$ for all $k$. Suppose this is the case. We may assume that $\text{supp } f \subseteq [0, 1]$, and replace $T$ with a bounded operator of the form (2.3) with kernel $K_{[0,1]}$.

Since $f$ is compactly supported, $\hat{f}$ extends to an entire function on $\mathbb{C}$. We now have

$$\widehat{T^k f}(\xi) = (i\xi)^{-k} \hat{f}(\xi),$$

and by the Plancherel theorem

$$4\pi^2 \|T^k f\|_2^2 = \|\hat{T^k f}\|_2^2 \geq \int_{-1}^{1} |\hat{f}(\xi)|^2 \, d\xi$$

But $\|T\| \leq \|K_{[0,1]}\|_2 < 1$, so the left-hand side of the above inequality can be made arbitrarily small. Therefore $\hat{f} = 0$, as it is an entire function vanishing on $[-1, 1]$. This contradicts the assumption that $f$ is nonzero. \hfill \Box

Let $f \in L^2(H_n)$ be compactly supported. Define $Sf \in L^1(\mathbb{R}^{2n})$ by

$$Sf(\bar{x}, \bar{y}) = \int_{\mathbb{R}} f(\bar{x}, \bar{y}, z) \, dz.$$
If \( Sf = 0 \), we may also define \( T \) by

\[
(2.8) \quad Tf(\bar{x}, \bar{y}, z) = \int_{-\infty}^{z} f(\bar{x}, \bar{y}, t) \, dt
\]

The proof of the next lemma consists of a straightforward application of the Fubini theorem:

**Lemma 2.3.** If \( f, g \in L^2(H_n) \) are compactly supported, then

1. \( S(f \ast_H g) = Sf \ast Sg \)
2. if \( Sf = 0 \), then \( (Tf) \ast_H g = T(f \ast_H g) \)
3. if \( Sg = 0 \), then \( f \ast_H (Tg) = T(f \ast_H g) \).

**Theorem 2.4.** If \( f, g \in L^2(H_n) \) are compactly supported and nonzero, then \( f \ast_H g \neq 0 \).

**Proof.** By Lemma 2.2 there exist minimal \( k \) and \( l \) such that \( ST^k f, ST^l g \in L^1(\mathbb{R}^{2n}) \) are nonzero and compactly supported. From Lemma 2.3 and Theorem 2.1 we obtain

\[
(2.9) \quad ST^k+l(f \ast_H g) = S(T^k f \ast_H T^l g) = ST^k f \ast ST^l g \neq 0,
\]

which implies that \( f \ast_H g \neq 0 \). □

3. **Symplectic manifolds**

3.1. **Symplectic manifolds.** Let \( M \) be a symplectic manifold, that is a \( 2n \)-dimensional manifold equipped with a nondegenerate closed 2-form. A symplectomorphism of \((M, \omega)\) is a diffeomorphism \( \phi \in \text{Diff}(M) \) satisfying \( \phi^* \omega = \omega \). The group of all compactly supported symplectomorphisms will be denoted by \( \text{Sympl}_c(M, \omega) \). Since \( \omega \) is nondegenerate, \( \omega^n \) defines a positive measure \( \mu \) on \( M \), invariant under the action of \( \text{Sympl}_c(M, \omega) \).

A standard example of a symplectic manifold is \( \mathbb{R}^{2n} \) endowed with the symplectic form \( \omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i \). It is a theorem of Darboux that any symplectic manifold is locally symplectomorphic to \((\mathbb{R}^{2n}, \omega_0)\):

**Theorem 3.1.** For every \( p \in M \) there exists a chart \( \phi: U \to \mathbb{R}^{2n} \) centered at \( p \), such that \( \omega|_U = \phi^* \omega_0 \).

**Proof.** See [1], Theorem 8.1. □

The chart satisfying the conditions of Theorem 3.1 is called a Darboux chart. The pushforward of \( \mu \) through a Darboux chart is the standard Lebesgue measure, up to a constant factor.

The flow \( F^X_t \) of a complete vector field \( X \in \mathfrak{X}(M) \) consists of symplectomorphisms if and only if

\[
(3.1) \quad \mathcal{L}_X \omega = 0.
\]
There is an easy way to produce such vector fields. Namely, consider a compactly supported smooth function \( f \in C^\infty(M) \). Since \( \omega \) is non-degenerate, there exists a unique vector field \( X_f \in \mathfrak{X}(M) \) such that \( X_f \cdot \omega = df \), and it is not hard to show that this field satisfies (3.1).

For more information on symplectic manifolds see [1] and [5].

3.2. The representation \( \Pi^0_\mu \) of \( \text{Sympl}_c(M, \omega) \). As \( \mu \) is a \( \text{Sympl}_c(M, \omega) \)-invariant measure, the only interesting representation is \( \Pi^0_\mu \), taking the form

\[
\Pi^0_\mu(\gamma)f = f \circ \gamma^{-1}.
\]

Notice that the space of constant square-integrable functions is \( \Pi^0_\mu \)-invariant. It is nontrivial when \( \mu(M) < \infty \). Let us denote its orthogonal complement by \( \mathcal{H} \).

**Theorem 3.2.** The representation \( \Pi^0_\mu \) of the group \( \text{Sympl}_c(M, \omega) \) on the space \( \mathcal{H} \) is irreducible.

**Lemma 3.3.** Let \( p \in M \) and let \( \phi: U \to \mathbb{R}^{2n} \) be a Darboux chart centered at \( p \). Then there exist \( r > 0 \) and for every \( x \in B(0, 2r) \) a symplectomorphism \( \tau_x \in \text{Sympl}_c(U, \omega|_U) \subseteq \text{Sympl}_c(M, \omega) \) such that

1. \( B(0, 3r) \subseteq \phi[U] \),
2. \( \phi \tau_x \phi^{-1}(y) = y + x \) for all \( y \in B(0, r) \).

**Proof.** Take \( r > 0 \) satisfying (1) and a bump function \( h \in C^\infty(\mathbb{R}^{2n}) \) supported in \( \phi[U] \) and equal to 1 on \( B(0, 3r) \). On \( \mathbb{R}^{2n} \) there exists a linear function \( f \) such that \( X_f = x \) is a constant field. Then \( X_{fh} = x \) on \( B(0, 3r) \) and \( \text{supp } X_{fh} \subseteq \phi[U] \). The desired symplectomorphism is \( \tau_x = \phi^{-1} \Pi^0_1 X_{fh} \phi \).

By using a standard argument we obtain the following well-known corollary:

**Corollary 3.4.** The action of \( \text{Sympl}_c(M, \omega) \) on \( M \) is \( k \)-transitive for all \( k \geq 1 \).

**Lemma 3.5.** Let \( \phi: U \to \mathbb{R}^{2n} \) be a Darboux chart. Then for every nontrivial \( \Pi^0_\mu \)-invariant subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \), there exists \( f \in \mathcal{H}_0 \) such that \( f \neq 0 \) and \( \text{supp } f \subseteq U \).

**Proof.** We may assume that \( 0 \in U = \phi[U] \subseteq \mathbb{R}^{2n} \). Let \( r > 0 \) be as in Lemma 3.3. Take a nonzero \( g \in \mathcal{H}_0 \). The 2-transitivity of \( \text{Sympl}_c(M, \omega) \) allows us to assume without loss of generality that there exists \( c \in \mathbb{R} \) such that the sets \( A = \{ p \in B(0, r) : \text{Re } g(p) < c \} \)
\[ \{ p \in B(0, r) : \text{Re } g(p) > c \} \]

By the Lebesgue density theorem there exist \( a \in A \) and \( b \in B \) with the property that \( A \) (resp. \( B \)) has Lebesgue density 1 at \( a \) (resp. \( b \)). Lemma 3.3 asserts the existence of a symplectomorphism \( \tau = \tau_{b-a} \) that takes \( a \) onto \( b \) and preserves the Lebesgue density on \( B(0, 3r) \).

The function \( f = g - \Pi^0_p(\tau)g \in \mathcal{H}_0 \) then satisfies the conclusion of the lemma. \( \square \)

Proof of Theorem 3.2. Suppose that \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp \) is a nontrivial decomposition into \( \Pi^0_p \)-invariant subspaces. Let \( \phi : U \to \mathbb{R}^{2n} \) be a Darboux chart, and let \( r > 0 \) and \( \tau_x \in \text{Symp}_c(M, \omega) \) be as in Lemma 3.3. Without loss of generality assume that \( U = \phi[U] \subseteq \mathbb{R}^{2n} \). By Lemma 3.5 we may choose nonzero \( f \in \mathcal{H}_0 \) and \( g \in \mathcal{H}_0^\perp \) supported in \( B(0, r) \). We have

\[
\langle f, \Pi^0_p(\tau_x)g \rangle = \int_{B(0,r)} f(y)g(\tau^{-1}_x(y)) \, dy = f \ast g^*(x),
\]

where \( g^*(y) = \overline{g(-y)} \). But from Theorem 2.1 we know that this is nonzero for some \( x \in \text{supp } f \ast g^* \subseteq B(0, 2r) \). We obtain a contradiction, since \( \Pi^0_p(\tau_x)g \in \mathcal{H}_0^\perp \).

\( \square \)

4. Contact manifolds

4.1. Contact manifolds. Let \( \dim M = 2n + 1 \). A contact form on \( M \) is a 1-form \( \alpha \in \Omega^1(M) \) such that \( \alpha \wedge (d\alpha)^n \) is a volume form. Consider a \( 2n \)-dimensional distribution \( \xi \leq TM \). There exists an open cover \( \mathcal{U} = \{ U_i \} \) of \( M \), such that for every \( U \in \mathcal{U} \) the restriction \( \xi|_U \) is the kernel of a 1-form \( \alpha|_U \in \Omega^1(U) \). If the forms \( \alpha|_U \) are contact forms, we call \( (M, \xi) \) a contact manifold. Unless \( \xi \) is the kernel of a globally defined contact form, there is no distinguished measure on \( M \).

Assume for the rest of this section that \( (M, \xi) \) is a contact manifold. A contactomorphism of \( (M, \xi) \) is a diffeomorphism \( \phi \in \text{Diff}(M) \), such that \( \phi_* \xi = \xi \). The group of compactly supported contactomorphisms will be denoted by \( \text{Cont}_c(M, \xi) \).

An example of a contact manifold is the Heisenberg group \( H_n \) with the distribution \( \xi = \ker \alpha_0 \), where \( \alpha_0 = dz - \sum_i y^i dx^i \) is a right-invariant form on \( H_n \).

There is an analogue of Darboux theorem for contact manifolds:

**Theorem 4.1.** For every \( p \in M \) there exists a chart \( \phi : U \to H_n \) centered at \( p \), such that \( \xi|_U = \ker \phi^* \alpha_0 \).

**Proof.** See [2], Theorem 2.5.1. \( \square \)
Let $U \subseteq M$ be such that $\xi|_{U} = \ker \alpha$ for some $\alpha \in \Omega^{1}(U)$. There exists a unique vector field $R \in \mathfrak{X}(U)$ such that $\alpha(R) = 1$ and $R \cdot d\alpha = 0$, called the Reeb vector field. If $X \in \mathfrak{X}(U)$ is a complete vector field, then its flow $\text{Fl}^{X}$ consists of contactomorphisms if and only if
\begin{equation}
\label{L41}
L_{X} \alpha = u \alpha
\end{equation}
for some $u \in C^{\infty}(U)$. If we take any $f \in C^{\infty}(U)$, by nondegeneracy of $d\alpha$ there exists $X_{f} \in \mathfrak{X}(U)$ satisfying $\alpha(X_{f}) = f$ and $X_{f} \cdot d\alpha = df(R)\alpha - df$. These conditions imply equality (4.1). On the other hand, if $X$ satisfies (4.1), then it is of the form $X_{f}$ for $f = \alpha(X)$.

For more information on contact manifolds see [2].

4.2. Representations of $\text{Cont}_{\tau}(M, \xi)$.

**Lemma 4.2.** Let $p \in M$ and let $\phi: U \to H_{n}$ be a Darboux chart centered at $p$. Then there exist an open set $V \subseteq H_{n}$, a convex open neighborhood $W$ of $0$ in the Lie algebra of $H_{n}$, and for every $x \in \exp[W]$ a contactomorphism $\rho_{x} \in \text{Cont}_{\tau}(U, \xi|_{U}) \subseteq \text{Cont}_{\tau}(M, \xi)$ such that

1. $0 \in V \subseteq V V \subseteq \exp[W] \subseteq V \exp[W] \subseteq \phi[U],$
2. $\phi \rho_{x} \phi^{-1}(y) = y x$ for all $y \in V$.

**Proof.** Existence of $V$ and $W$ satisfying (1) is obvious. Let $x = \exp v$, where $v \in W$. Then $v$ extends to a left-invariant vector field $X \in \mathfrak{X}(H_{n})$, and $\text{Fl}_{\xi}^{X} = R_{\exp tv}$, where $R_{y}$ is the right multiplication by $y$. If $f = h\alpha_{0}(X)$, where $h|_{V \exp[W]} = 1$ and $\sup h \subseteq \phi[U]$, then $X_{f} = X$ on $V \exp[W]$. The contactomorphism $\rho_{x} = \phi^{-1}\text{Fl}_{\xi}^{X}\phi$ satisfies condition (2). \hfill $\square$

**Corollary 4.3.** The action of $\text{Cont}_{\tau}(M, \xi)$ on $M$ is $k$-transitive for all $k \geq 1$.

**Lemma 4.4.** Let $\phi: U \to H_{n}$ be a Darboux chart. Then for every nontrivial $\Pi^{\mu}_{\mu}$-invariant $\mathcal{H}_{0} \leq L^{2}(M, \mu)$, there exists $f \in \mathcal{H}_{0}$ such that $f \neq 0$ and $\sup f \subseteq U$.

**Proof.** Without loss of generality assume that $0 \in U \subseteq H_{n}$ and $\xi|_{U} = \ker \alpha_{0}$. Let $\delta_{t}(x, y, z) = (e^{t}x, e^{t}y, e^{2t}z)$ be the flow of the field $X = (x, y, 2z)$. We have $\delta_{t}^{\prime}\alpha_{0} = e^{2t}\alpha_{0}$, so $X = X_{g}$ for some function $g \in C^{\infty}(H_{n})$.

There exist $V = B(0, r) \subseteq \overline{V} \subseteq U$ and a function $h$ supported in $U$, such that $h|_{V} = g|_{V}$. Let $\psi_{t} = \text{Fl}_{\xi}^{X_{h}}$. Then $\psi_{t}|_{V} = \delta_{t}|_{V}$ for $t < 0$. Now, by transitivity of $\text{Cont}_{\tau}(M, \xi)$, we may take a nonzero $f \in \mathcal{H}_{0}$ such that $\sup f \cap V \neq \emptyset$. Since
\begin{equation}
\label{L42}
\int_{V} \left| \Pi^{\mu}_{\mu}(\psi_{t}) f \right|^{2} d\mu = \int_{\psi_{-1}[V]} \left| f \right|^{2} d\mu \longrightarrow 0,
\end{equation}

for $t \to -\infty$, we have that $\psi_{t}$ is a contactomorphism for $t < 0$.
there exists \( t > 0 \) such that \( f - \Pi_{\mu}^{\theta}(\psi) f \) satisfies the conclusion of the lemma.

Now, fix a Darboux chart \( \phi : U \to H_n \) and a Lebesgue measure \( \mu \) on \( M \), such that \( 0 \in \phi[U] \) and \( \phi_{*}\mu \) is the standard Lebesgue measure on \( \phi[U] \subseteq \mathbb{R}^{2n+1} \).

**Theorem 4.5.** For every \( \theta \in \mathbb{R} \) the representation \( \Pi_{\mu}^{\theta} \) of \( \text{Cont}_c(M, \xi) \) on the space \( L^2(M, \mu) \) is irreducible.

**Proof.** The proof is analogous to the proof of Theorem 3.2. Lemma 4.2 gives us \( V \subseteq U \) and contactomorphisms \( \rho_x \), such that for \( f \) and \( g \) supported in \( V \) the matrix coefficient \( \langle f, \Pi_{\mu}^{\theta}(\rho_x)g \rangle \) is nonzero for some \( \rho_x \) because of Theorem 2.4.

**References**

[1] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Springer, 2001.

[2] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge University Press, 2008.

[3] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag, 1990.

[4] R. S. Ismagilov. *Representations of Infinite-Dimensional Groups*. American Mathematical Society, 1996.

[5] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology*. Oxford University Press, 1998.

[6] A. M. Vershik, I. M. Gel’fand, and M. I. Graev. Representations of the group of diffeomorphisms. In *Representation theory: selected papers*. Cambridge University Press, 1982.