Visualization of Rabi oscillations in a magnetic resonance

E. A. Ivanchenko

Institute for Theoretical Physics, National Science Center “Institute of Physics and Technology”,
1, Akademicheskaya str., 61108 Kharkov, Ukraine

(Dated: 12 November 2013)

A visualization scheme for dynamics of a qudit polarization vector in a time-dependent magnetic field is presented by solving equations for a density matrix in Hermitian basis. This is realized by means of mapping solution for the polarization vector on the three-dimensional spherical curve (vector hodograph). The obtained results obviously display the interference of precessional and nutational effects on the polarization vector in a magnetic resonance. The study can find the practical applications in a magnetic resonance and 3D visualization of computational data.

PACS numbers: 87.63.L, 82.56.-b, 03.67.-a

Keywords: 3D Visualization, Spin magnetic resonance, Quantum information

a)Electronic mail: yevgeny@kipt.kharkov.ua, eaivanchenko1@gmail.com
I. INTRODUCTION

Imaging and visualization are among the most dynamic and innovative areas of research of the past decades. This activity arises from the requirements of important practical applications such as the visualization of computational data, the processing of medical images for assisting medical diagnosis and intervention, and the 3D geometry reconstruction and processing for computer simulations. Due to the development of more powerful hardware resources, mathematical and physical methods, investigators have been incorporating advanced computational techniques to derive methodologies that can better enable the solution of the problems encountered.

Realization of magnetic resonance, as it is known, depends on the kind of the magnetic field modulations. Let us consider the spin dynamics in an alternating field of the form

\[ \vec{h}(t) = (h_1 \text{cn}(\omega t|k), h_2 \text{sn}(\omega t|k), H \text{dn}(\omega t|k)), \]  

where \( \text{cn}, \text{sn}, \text{dn} \) are the Jacobi elliptic functions, \( \omega \) is the field frequency. Such field modulation under the changing of the elliptic modulus \( k \) from 0 to 1 describes the whole class of field forms from trigonometric \( (\text{cn}(\omega t|0) = \cos \omega t, \text{sn}(\omega t|0) = \sin \omega t, \text{dn}(\omega t|0) = 1) \) to the exponentially impulse ones \( (\text{cn}(\omega t|1) = \frac{1}{\cosh \omega t}, \text{sn}(\omega t|1) = \tanh \omega t, \text{dn}(\omega t|1) = \frac{1}{\cosh \omega t}) \). The elliptic functions \( \text{cn}(\omega t|k) \) and \( \text{sn}(\omega t|k) \) have the real period \( 4K/\omega \), while the function \( \text{dn}(\omega t|k) \) has a period of half the duration. Here \( K \) is the full elliptic integral of the first kind. In other words, even though the field is periodic with a common real period \( 4K/\omega \), but as we can see, the frequency of the longitudinal field amplitude modulation is twice as high as that of the transverse field. We call such field consistent.

At \( k = 0 \) and \( h_1 = h_2 = h \) it is a circularly polarized magnetic field, where the scalar of the vector of the magnetic field does not depend on time (a rotating magnetic field), and at \( k = 0 \) and \( h_1 = h, h_2 = 0 \) it is a linearly polarized field.

The solution of the Schrodinger equation for a wave function does not yield directly to the physical observed values. In paper real functions have been constructed from the Schrodinger equation solutions, which have direct physical sense and whose temporary evolution supposes visualization. The objective of this work is to present the scheme of visualization of the dynamics polarization vector of the qudit in a magnetic field on the base of both analytical and numerical solutions for the density matrix in the Hermitian base. The solution represents a real generalized Bloch vector and a mapping of the solution for
polarization vector on the tridimensional geometrical object-oriented spherical curve, which
for qubit, as it will be shown further, will present a precession, similar to the precession
of the symmetrical top in the field of gravity. The 3D sphere is used for visualization of
results of numerical simulation. Creation of such a representation and its numerical approval
is an important step in clarification of extremely complicated interrelation between classical
and quantum randomness. It opens new possibilities for application of the mathematical
formalism of quantum mechanics in other domains of science. The 3D visualization brings
additional information for interpretations of the characteristic features of dynamics.

The paper has the following content. In section II, in the presentation of the real Bloch
vector, a set of equations for qubit dynamics is derived, taking into account its environment
as well as its solution in a rotating magnetic field. The scheme of visualization of the polar-
ization vector and the analytical formulas characterizing visualization in a resonance case,
are presented in section III. Section IV contains the analytical solution for qutrit, taking into
account the anisotropy in a rotating magnetic field in case of a resonance, as an example of
visualization of the polarization vector of a qudit. In Section V, the results are presented
pictorially at concrete parameters. Deductions are presented in the short conclusion. In the
Appendix the subsidiary analytical results are numbered.

II. HAMILTONIAN AND MASTER EQUATION IN LINDBLAD FORM

The Hamiltonian of a magnetic qubit (spin 1/2 particle) which is in the external variable
magnetic field $\overrightarrow{H}(t) = (h_1(t), h_2(t), h_3(t))$ is equal to

$$\hat{H} = h_i(t)s_i,$$

(2)

where $h_i(t)$ are the Cartesian components of the external magnetic field in frequency units
(we suppose $\hbar = 1$); $s_i = \frac{1}{2}\sigma_i$, is the matrix representation of the spin of the qubit operators;
Pauli matrices are equal to $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The Liouville - Neumann equation for the density matrix $\rho$, describing the qubit dynamics
and taking into account the environment in the form of Lindblad, takes the form

$$\partial_t \rho = -i[\hat{H}, \rho] + \frac{1}{2} \sum_{\alpha, \beta = 1}^{3} a_{\alpha\beta} (2\sigma_\alpha \rho \sigma_\beta - \sigma_\beta \sigma_\alpha \rho - \rho \sigma_\beta \sigma_\alpha), \rho(t = 0) = \rho_0,$$

(3)

where $a_{11} = a_{22} = \gamma_1/4, a_{33} = \gamma_2/2 - \gamma_1/4, a_{12} = a_{12}^* = -i\gamma_1 R_3(t = 0)/4, a_{23} = a_{32} = a_{31} = a_{13} = 0$. The constants $\gamma_1, \gamma_2$, and $R_{eqr}$ are identified as the longitudinal and transverse
lifetimes, and the equilibrium value of the population difference respectively.

We present the equation solution \( \rho = \frac{1}{2} R \sigma \alpha \), \( \rho^+ = \rho \), \( \text{Tr} \rho = 1 \), \( R_0 = 1 \), in which, here and further, we imply the summation on repetitive Greek coefficients from zero to three and on Latin from one to three. The coherence vector (the Bloch vector) which is widely used in the theory of the magnetic resonance,

\[
R_i = \text{Tr} (\rho \sigma_i),
\]

characterizes the behavior of the qubit. At unitary evolution the length of the Bloch vector \( b \) is conserved

\[
b = \sqrt{R_i^2}.
\]

In the terms of functions \( R_i \) the Liouville - Neumann equation takes a real shape in the form of a closed system of 3 differential equations of first order for the components of the Bloch vector

\[
\begin{align*}
\partial_t R_1 &= h_2 R_3 - h_3 R_2 - \gamma_2 R_1, \\
\partial_t R_2 &= h_3 R_1 - h_1 R_3 - \gamma_2 R_2, \\
\partial_t R_3 &= h_1 R_2 - h_2 R_1 - \gamma_1 (R_3 - R_{eqr})
\end{align*}
\]

at the initially given conditions.

### III. RABI MODULATION

Taking into account an environment with parameters \( \gamma_2 = \gamma_1 = \gamma \), \( R_{eqr} = 0 \), in terms of the Bloch vector the Rabi model is described by a matrix equation with a initial state \( R(t = 0) = (R_1(t = 0), R_2(t = 0), R_3(t = 0))^T \) (here \( T \) indicates transposition)

\[
\partial_t R = MR,
\]

where the matrix looks as

\[
M = \begin{pmatrix}
\gamma & -h & h \sin \omega t \\
H & \gamma & -h \cos \omega t \\
-h \sin \omega t & h \cos \omega t & -\gamma
\end{pmatrix}.
\]

Let’s introduce a vector \( r = aR \) in which the orthogonal matrix is equal to \( a = \begin{pmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{pmatrix}. \)

The equation for \( r \) takes the form

\[
\partial_t r = M r
\]
with a constant matrix \( \tilde{M} = \begin{pmatrix} \gamma & -\delta & 0 \\ \delta & \gamma & -h \\ 0 & h & \gamma \end{pmatrix} \). Now the solution can be written as: \( R = a^{-1}e^{\tilde{M}t} R(t = 0) \). For the pure initial state \( R(t = 0) = (\cos \varphi_0 \sin \theta_0, \sin \varphi_0 \sin \theta_0, \cos \theta_0)^T \) the solution is given in the Appendix (A.1)-(A.3). For the initial state \( \varphi_0 = \theta_0 = 0 \) the solution becomes

\[
R_1 = \frac{e^{-\gamma t}h}{\Omega^2} (\Omega \sin \Omega t \sin \omega t + \delta (1 - \cos \Omega t) \cos \omega t),
\]

\( \text{Eq. 10a} \)

\[
R_2 = \frac{e^{-\gamma t}h}{\Omega^2} (\delta (1 - \cos \Omega t) \sin \omega t - \Omega \sin \Omega t \cos \omega t),
\]

\( \text{Eq. 10b} \)

\[
R_3 = \frac{e^{-\gamma t}t}{\Omega^2} (\delta^2 + h^2 \cos \Omega t),
\]

\( \text{Eq. 10c} \)

where \( \delta = H - \omega \) and \( \Omega = \sqrt{\delta^2 + h^2} \) is the nonresonance Rabi frequency.

The probability of the spin-flip is equal to

\[
P = \frac{1 - R_3}{2}.
\]

\( \text{Eq. 11} \)

This probability at a resonance \( \delta = H - \omega = 0 \) has an oscillating behavior and also reaches a unity. At big detuning \( \delta \gg 1 \) the probability of spin-fip aims at zero. If the longitudinal field \( H = 0 \), then the peak probability is equal to 1/2.

In the case of an elliptic field \( k \neq 0 \) and \( h_1 = h_2 = h \), as an expansion of the Rabi model, the fulfilled rotational displacement of the matrix is \( \tilde{a} = \begin{pmatrix} \cos(\omega t | k) & \sin(\omega t | k) & 0 \\ -\sin(\omega t | k) & \cos(\omega t | k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Thus in the matrix \( \tilde{M}(t) = \begin{pmatrix} \gamma & -\delta \sin(\omega t | k) & 0 \\ \delta \sin(\omega t | k) & \gamma & -h \\ 0 & h & \gamma \end{pmatrix} \) the time dependence at a resonance \( \delta = 0 \) vanishes and the solution in an elliptic field is the following:

\[
R_1 = e^{-\gamma t} \sin(\omega t | k) \sin ht, \quad R_2 = -e^{-\gamma t} \cos(\omega t | k) \sin ht, \quad R_3 = e^{-\gamma t} \cos ht.
\]

\( \text{Eq. 12} \)

In the model under consideration, the components of the Bloch vector decay in due course, but the unit polarization vector \( \vec{p} = (p_1, p_2, p_3) \) does not depend on \( \gamma \).

**IV. VISUALIZATION SCHEME OF DYNAMICS**

One of the major stages of investigations is the visualization of the data obtained. As known, the difficulty arising in quantum mechanics is not only to find the solutions but also to understand their meaning\(^{11}\). In this paper it will be useful to map the solution for the polarization vector on geometrical model\(^7\). To this end, it is convenient to parameterize the unit polarization vector by spherical angles \( (p_1, p_2, p_3) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \).
Thus, the parameter $\varphi (0 \leq \varphi \leq 2\pi)$ becomes a precession angle of the end of the vector $\vec{p}$ on a sphere, referred to as apex, and the angle $\theta (0 \leq \theta \leq \pi)$ characterizes the nutation. $\theta = 0$ corresponds to the north pole on a sphere. These angles are expressed through the components of polarization vector $\vec{p}$ according to the following formulae:

$$
\theta(t) = \arccos p_3, \quad \sin \varphi(t) = \frac{p_2}{\sqrt{p_1^2 + p_2^2}}, \quad \cos \varphi(t) = \frac{p_1}{\sqrt{p_1^2 + p_2^2}}. \quad (13)
$$

The angular velocities are given by

$$
\theta'(t) = \frac{h_2p_1 - h_1p_2}{\sqrt{1 - p_3^2}}, \quad \varphi'(t) = (\arctan \frac{p_2}{p_1})' = h_3 - \frac{(h_1p_1 + h_2p_2)p_3}{p_1^2 + p_2^2}. \quad (14)
$$

The oriented spherical curve $\Gamma$ is characterized by the curvature $k$, torsion of the curve $\kappa$, modulus of velocity of an apex $v$, and length of a path $s$. All these quantities can be easily calculated in terms of the components of polarization vector using the following formulae$^{12}$:

$$
k = \frac{||\vec{p}', \vec{p}''||}{||\vec{p}'||^3}, \quad \kappa = \frac{(\vec{p}', \vec{p}'', \vec{p}''')}{||\vec{p}', \vec{p}'', \vec{p}'''||^2}, \quad v = \sqrt{p_1^2 + p_2^2 + p_3^2}, \quad s = \int_0^t |\vec{p}'| d\tau, \quad (15)
$$

where the prime is used to denote the derivative with respect to time, $' \equiv \partial_t$. One can show$^{12}$ that the squared radius of adjoining sphere (in our case, a unit sphere) is related to a curvature, torsion, and module of velocity of an apex by the formula

$$
b^2 = 1 = \frac{1}{k^2} + \left(\frac{k'}{\sqrt{k^2} \kappa}\right)^2. \quad (16)
$$

A. Visualization of dynamics in a circularly polarized field

Let us give results for the explicit solution of the Rabi model for the initial state $\varphi_0 = \theta_0 = 0$. In a circularly polarized field $\vec{h}(t) = (h \cos \omega t, h \sin \omega t, H)$ the angular velocities of a nutation and precession have the form

$$
\theta' = \frac{h^2\Omega \sin \Omega t}{\sqrt{\Omega^4 - (\delta^2 + h^2 \cos \Omega t)^2}}, \quad \varphi' = \frac{\omega \delta^2 + \Omega^2 \delta + \omega \Omega^2 - \omega (\delta^2 - \Omega^2) \cos \Omega t}{\delta^2 + \Omega^2 + (\Omega^2 - \delta^2) \cos \Omega t}. \quad (17)
$$

But in the case of a resonance, the module of velocity of nutation is constant and the velocity of the precession is constant, too

$$
\theta'(\delta = 0) = h \text{sgn}(\sin \omega t), \quad \varphi'(\delta = 0) = \omega \quad (18)
$$
with a period \( T = 2\pi/h \).

The curvature, torsion, velocity and path length at an exact resonance are expressed by the formulas

\[
k_{\text{res}} = \sqrt{(h^2 + 3\omega^2)(8h^4 + 4\omega^2h^2 + \omega^4) - g(t)},
\]

\[(19a)\]

\[
\kappa_{\text{res}} = -\frac{4h\omega (4h^4 + 7\omega^2h^2 + \omega^4 (h^2 - \omega^2) \cos 2ht) \sin ht}{(h^2 + 3\omega^2)(8h^4 + 4\omega^2h^2 + \omega^4) - g(t)},
\]

\[(19b)\]

\[
v_{\text{res}} = \frac{1}{\sqrt{2}} \sqrt{2h^2 + \omega^2 - \omega^2 \cos 2ht},
\]

\[(19c)\]

\[
s_{\text{res}} = F(ht) - \frac{\omega^2}{h^2},
\]

\[(19d)\]

where \( g(t) = 4(\omega^4 - h^4 + 3\omega^2h^2)\omega^2 \cos 2ht + (\omega^2 - h^2) \omega^4 \cos 4ht, \)

\( F(\phi|m) = \int_0^\phi (1 - m \sin^2 \vartheta)^{1/2} d\vartheta \) is the elliptic integral of the second kind.

In the consistent Rabi field \( (1) (h_1 = h_2 = h, k \neq 0) \) at a resonance, the angular velocities depend on time and are equal to

\[
\theta'_{d\text{Rabi}}(\delta = 0) = h \text{sgn}(\sin ht), \quad \varphi'_{d\text{Rabi}}(\delta = 0) = \omega \text{dn}(\omega t|k).
\]

\[(20)\]

V. SPIN \( S > 1/2 \)

The qudit is characterized by the generalized Bloch vector, the dimension of which is equal to \((2S + 1)^2 - 1\). The so developed scheme for a spin 1/2 is completely applicable for qudits, for the visualization of the polarization vector and the anisotropy influence on polarization. For the qutrit, the 2D parametric time dependence of the components of the Bloch vector, was studied\(^{13,14}\).

The Hamiltonian of a spin 1 (qutrit) in a magnetic field \( \overrightarrow{h}(t) = (h_1(t), h_2(t), h_3(t)) \) taking into account anisotropy has the form\(^{15}\)

\[
\hat{H} = h_i(t)S_i + Q(S_3^2 - \frac{2}{3} E_{3\times3}) + d(S_1^2 - S_2^2),
\]

\[(21)\]

where \( S_i \) - spin matrices and \( Q, d \) - the anisotropy constants, take account of the contributions of the one-quantum and double-quantum transitions in qutrit\(^{15}\). The explicit form of the matrices of the spin and the set of equations, determining the qutrit dynamics are given in paper\(^{15}\). The exact solution in a circularly polarized field at a resonance without the influence of constant \( d \) is given in the Appendix (A.4)-(A.11). The components of
the qutrit polarization vector are equal to \( \pi_1 = q_1/N, \ \pi_2 = q_2/N, \ \pi_3 = q_3/N \), where \( N = \sqrt{q_1^2 + q_2^2 + q_3^2} \).

For closed paths at \( \omega = 0 \) in the solution (A.4)-(A.11) the requirement of a commensurability of frequencies at \( d = 0 \) looks as \( x\sqrt{4h^2 + Q^2} = yQ \), from which it follows that

\[
h = \pm \sqrt{y^2 - x^2 \frac{Q}{2x}},
\]

(22)

where \( y, x \) are integer numbers. It is possible to be shown that the condition for the closed trajectory at \( d \neq 0 \) has the form \( h + \sqrt{2}d = \pm \sqrt{y^2 - x^2 \frac{Q}{2x}} \).

VI. NUMERICAL RESULTS

Qubit. In a circularly polarized field in Fig. 1 the curvature, torsion and velocities are presented, which for descriptive reasons are combined with the help of introducing a scale factor (for the curvature). It becomes clear that when the torsion \( \kappa \) changes its sign from plus to minus, modulus of velocity of an apex \( v \equiv |\vec{p}'| \) decreases, the velocity of precession is minimum, the velocity of nutation changes its sign from minus to plus, and the curvature \( k \) increases.

The interaction of the effects of precession and nutation is geometrically described in Fig. 2. The occurrence of loops (self-superposition) follows from the fact that velocity of precession on the upper parallel is opposite to the velocity on the lower parallel.

At the cusps, as it is obvious in Fig. 3 there is a sharp transition from some geometrical characteristics of the hodograph to others (that is, linking trajectories with different \( k, \kappa \)). In the transition vicinity the velocity \( v \) sharply decreases, the curvature and torsion sharply increase and the precession and nutation velocities converge to zero.

If motion begins from \( \theta = 0 \) or \( \theta = \pi \), then the upper or accordingly the lower parallel of the sphere degenerates in a point. Fig. 4 presents dynamics, where in the vicinity of the cusps the velocity \( v \) sharply decreases, the curvature and torsion sharply increase, and the precession and nutation velocities actually transit through zero.

In all the cases in the cusps the energy of qubit \( E_{\text{Rabi}} = \text{Tr}(\rho \hat{H}) \) has its local maximum.

At coincidence of the frequency of the external field and the frequency of eigen oscillations there is a resonance in the system. On the sphere in Fig. 5 it is visible that at very small frequencies which correspond to a small velocity of precession (22), on the closed
apex trajectories (with beginning on the northern pole towards the southern one and termination on the northern pole for one period), which is a purely periodic driving with a period \( T = 2\pi/h \), there are no self-intersections. With increasing frequency \( \omega \), and at a constant Rabi frequency \( h \), (see Fig. 6), the bigger is the frequency \( \omega \), the more the path trajectory on the sphere coils, like the so called loxodrome curve. Let us emphasize that the trajectory is not flat, but three-dimensional. This is visible along the path, which is more than \( 2\pi \), and by means of another criteria it is possible to show that at a plane curve torsion is equal to zero in each point. The whole class of closed trajectories satisfies the requirement for closed curve which for the solution (A.1)-(A.3) is determined by the formula

\[
h = \pm \sqrt{y^2 \omega^2/x^2 - (H - \omega)^2}.
\]

As it is known, in a linearly polarized field \( \vec{h}(t) = (h \cos \omega t, 0, H) \), the resonance happens at \( \omega \neq H \), (Bloch-Siegert resonance frequency shift). The visualization for such modulation was partially considered in paper. We would like only to add that in a linearly polarized field instead of loops or cusps, the hodograph derived from the numerical solution, at parameters such as in Fig. 3 is more smooth (see Fig. 7) as the velocity of precession does not change its sign and the curvature and torsion in such field are much less than in a rotating field.

**Qutrit.** We present some numerical results describing qutrit dynamics in the field \( \vec{h}(t) = (h, 0, 0) \) for \( d = 0 \). A smooth closed trajectory goes through both of the poles that means that there is a two-photon transition. In detail, a flip of the third component of polarization vector \( \pi_3 \) occurs and the population of the lowest level \( P_{-1} \) is equal to unity over the time \( (5(2\pi/f)) \). In the vicinity of the southern pole, the precession velocity takes its minimum value and the nutation velocity changes its sign from plus to minus. Fig. 8 shows both dynamics of level populations with spin projections \((1,0,-1)\) and third projection of polarization vector \( \pi_3 \). Fig. 9 presents the dynamics of velocities. A comparison with the hodograph, given in Fig. 10 shows that precession corresponds to small oscillations of \( \pi_3 \) (two orbits on the sphere over the time \( 2(2\pi/f) \)). Then a sharp nutational change of \( \pi_3 \) (the smooth curve) occurs during the time \( (2\pi/f) \). Next four precessional orbits go through the south pole of the sphere over the time \( 4(2\pi/f) \)). Further sharp nutational change of \( \pi_3 \) (the smooth curve) takes place during \( (2\pi/f) \). The full period \( 10(2\pi/f) \) is completed by two precessional orbits on the northern pole of the sphere over the time \( 2(2\pi/f) \)). In other words, there is a quasitrapping \( \pi_3 \) on the poles. All the velocities \( v, \varphi', \theta' \) grow with
increasing anisotropy and the periods of "hodograph components" decrease. However, the shape of hodograph does not depend on anisotropy parameter at positive $Q$. The torsion $\kappa$ is small and repeatedly changes its sign during the full period. In fact, such a hodograph is the Lissajous figure on the three-dimensional sphere.

The influence of anisotropy constant $d$ caused the double-quantum transitions leads to an increase of transition frequency. When $d = 0.1$, the frequency of double-quantum transitions is increased approximately twice, due to reduction of the number of precessional orbits on the poles.

VII. CONCLUSION

A system of closed equations describing dynamics of the Bloch vector components for a qubit in an arbitrary time-dependent external magnetic field is derived. On the basis of analytical solutions we propose a scheme of visualization of a qubit dynamics. This is realized by mapping the solution on a three-dimensional parametric hodograph of an apex of polarization vector on a sphere. The numerical results for the qubit dynamics in a circularly polarized field are presented for different initial states.

The offered scheme of visualization is also applicable for 3D mapping the analytical or numerical solution of a qudit.

We believe that this study will be useful since 3D visualization of polarization vectors can be applicable to a double magnetic resonance and manipulations of quantum bits. The visualization may have potential applications in magnetometry and quantum information processing.

Appendix

The exact solution for the qubit for a pure initial general state in a circularly polarized field looks:

$$e^{\gamma t} \Omega^2 R_1 = -\Omega \left( \sin \theta_0 (\delta \cos \varphi_0 \sin \Omega t + \Omega \cos \Omega t \cos \varphi_0) - h \cos \theta_0 \sin \Omega t \right) \sin \omega t$$

$$+ (h\delta \cos \theta_0 (1 - \cos \Omega t) + \sin \theta_0 (\cos \varphi_0 (h^2 + \delta^2 \cos \Omega t) - \delta \Omega \sin \varphi_0 \sin \Omega t)) \cos \omega t.$$  (A.1)
\[ e^{\gamma t} \Omega^2 R_2 = (\sin \theta_0 \cos \varphi_0 (h^2 + \delta^2 \cos \Omega t) - \delta h \cos \theta_0 (\cos \Omega t - 1) - \delta \Omega \sin \theta_0 \sin \varphi_0 \sin \Omega t) \sin \omega t \]

\[ (h \Omega \cos \theta_0 \sin \Omega t - \delta \Omega \sin \theta_0 \cos \varphi_0 \sin \Omega t - \Omega^2 \sin \theta_0 \varphi_0 \cos \Omega t) \cos \omega t, \quad (A.2) \]

\[ e^{\gamma t} \Omega^2 R_3 = \cos \theta_0 \left( \delta^2 + h^2 \cos \Omega t \right) + h \sin \theta_0 \left( \delta \cos \varphi_0 (1 - \cos \Omega t) + \Omega \sin \varphi_0 \sin \Omega t \right). \quad (A.3) \]

The exact solution for the qutrit, taking into account only the anisotropy \( Q \) at the start from the northern pole \( \rho_{\text{qutrit}}(t = 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) in a circularly polarized magnetic field\( ^{\text{c}} \) at a resonance \( \omega = H \) has the form

\[ q_1 = \frac{\sqrt{6} h \sin \frac{f t}{2}}{f^2} (f \cos \frac{Q t}{2} \sin \omega t + Q \sin \frac{f t}{2} \cos \omega t), \quad (A.4) \]

\[ q_2 = \frac{\sqrt{6} h \sin \frac{f t}{2}}{f^2} (Q \sin \frac{f t}{2} \sin \omega t - f \cos \frac{Q t}{2} \cos \omega t), \quad (A.5) \]

\[ q_3 = \frac{\sqrt{3}}{2} (Q \sin \frac{f t}{2} \cos \frac{Q t}{2} + \cos \frac{f t}{2} \cos \frac{Q t}{2}), \quad (A.6) \]

\[ q_4 = \frac{1}{f^2} \sqrt{\frac{3}{2}} (-2h^2 \sin^2 \frac{f t}{2} \sin 2\omega t + \left( f^2 \cos \frac{f t}{2} \sin \frac{Q t}{2} - fQ \sin \frac{f t}{2} \cos \frac{Q t}{2} \right) \cos 2\omega t), \quad (A.7) \]

\[ q_5 = \frac{\sqrt{6} h \sin \frac{f t}{2}}{f} (\sin \frac{Q t}{2} \sin \omega t - \cos \frac{f t}{2} \cos \omega t), \quad (A.8) \]

\[ q_6 = \frac{h^2 + Q^2 + 3h^2 \cos ft}{\sqrt{2} f^2}, \quad (A.9) \]

\[ q_7 = \frac{\sqrt{6} h \sin \frac{f t}{2}}{f} (\cos \frac{f t}{2} \sin \omega t + \sin \frac{Q t}{2} \cos \omega t), \quad (A.10) \]

\[ q_8 = \frac{1}{f^2} \sqrt{\frac{3}{2}} \left( \frac{f(Q \sin \frac{f t}{2} \cos \frac{Q t}{2} - f \cos \frac{f t}{2} \sin \frac{Q t}{2})}{\sin 2\omega t - 2h^2 \sin^2 \frac{f t}{2} \cos 2\omega t}, \quad (A.11) \]

where \( q_1, q_2, q_3 \) are the spin components, \( f = \sqrt{4h^2 + Q^2} \).

The population levels in the qutrit are equal to

\[ P_{+1} = \frac{1}{6} (2 + \sqrt{6} q_3 + \sqrt{2} q_6), \quad P_0 = \frac{1}{3} (1 - \sqrt{2} q_6), \quad P_{-1} = \frac{1}{6} (2 - \sqrt{6} q_3 + \sqrt{2} q_6). \quad (A.12) \]

REFERENCES

1. E. A. Ivanchenko, Physica B 358, 308 (2005).
2. E. A. Ivanchenko, Low Temp. Phys. 31, 577 (2005).
3. M. Abramovitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1968).
Fig. 1. Time dependencies of the curvature $k$, torsion $\kappa$, modulus of velocity of an apex $v$, velocities of the precession $\varphi'$ and mutation $\theta'$, for the initial state $\theta_0 = \arccos \frac{1}{\sqrt{3}}$, $\varphi_0 = 0$ with parameters of the circularly polarized field $\omega = 3$; $H = 0.45$; $h = -0.6$; $k = k/20$ during $2\pi/\Omega$. 
Fig. 2. Mapping of the dynamics of the qubit on the hodograph of the apex (interference of precession and nutation) with parameters as in Fig. 1 for total time $T_{\text{all}} = 7(2\pi/\Omega)$. The probability of spin-flip oscillates in the limits of $0.06 \leq P \leq 0.211$. The path length equals $s = 10.44$.

Fig. 3. Cusps at $\theta_0 = \arccos \frac{1}{\sqrt{3}}$, $\varphi_0 = \pi/4$ with field parameters $\omega = 3$, $H = 0.5$, $h = 0.6$ during $6(2\pi/\Omega)$; $0.19 \leq P \leq 0.4$; $0.015 \leq v \leq 0.78$, $-0.02 \leq \varphi_t \leq 0.67$, $-0.6 \leq \theta_t \leq 0.6$, $1 \leq k \leq 8500$, $-1550 \leq \kappa \leq 1550$, $s = 8.6$.

Fig. 4. 3D parametric plot of the polarization vector of qubit $p_1(t), p_2(t), p_3(t)$ vs. time. $\theta_0 = \pi$, $\varphi_0 = 3\pi/4$; $\omega = 0.5$, $H = 0.05$, $h = 0.5$ during $4(2\pi/\Omega)$; $0.005 \leq v \leq 0.5$, $1.5 \leq k \leq 25000$, $-20 \leq \kappa \leq 20$, $-0.003 \leq \varphi_t \leq 0.28$, $-0.5 \leq \theta_t \leq 0.5$, $0.45 \leq P \leq 1$, $s = 14$. 
Fig. 5. Apex hodograph at a resonance for initial state $\theta_0 = 0, \varphi_0 = \pi/4; \omega = 0.2, H = 0.2, h = 0.5$ for a period $T = 2\pi/h; 0 \leq P \leq 1, 0.5 \leq v \leq 0.54, 1 \leq k \leq 1.28, -0.64 \leq \kappa \leq 0.64, s = 6.53$.

Fig. 6. Hodograph at a resonance for initial state $\theta_0 = 0, \varphi_0 = \pi/4; \omega = 5, H = 5, h = 0.5, T = 2\pi/h; 0 \leq P \leq 1, 0.47 \leq v \leq 5.02, 1 \leq k \leq 20, -0.55 \leq \kappa \leq 0.55, s = 40, 84$.

Fig. 7. Apex dynamics of qubit in a linearly polarized field with parameters as in Fig. 3 during $14.66; 0.14 \leq v \leq 0.77, 1 \leq k \leq 22, -200 \leq \kappa \leq 350, 0.08 \leq \varphi_t \leq 0.87, -0.58 \leq \theta_t \leq 0.6, 0.14 \leq P \leq 0.3, s = 6.44$. 
Fig. 8. Dynamics of populations $\{A, 12\}$ and of the third component of the qutrit polarization vector $\pi_3$ with model parameters (for $x = 4$, $y = 5$ in the formula (??)) $\omega = H = 0$, $Q = 1$, $h = 3Q/8$ for the full time period $10(2\pi/f)$.

Fig. 9. Time dependence of the modulus of velocity of a qutrit apex $v$, angular velocities $\varphi'$ and $\theta'$ with model parameters as in a Fig. 8.

Fig. 10. 3D parametric plot of the qutrit polarization vector $(\pi_1(t), \pi_2(t), \pi_3(t))$ with parameters as in a Fig. 8. The torsion is small $-10^{-26} < \kappa < 10^{-26}$ and also changes its sign 28 times for the full period, while the curvature varies within $0.005 \leq k \leq 29$; $s = 22.13$. 

