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A SPECTRAL THEOREM FOR MATRICES
OVER FIELDS OF POWER SERIES

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Abstract. Let $K = \mathbb{R}((t_1, \ldots, t_m))$ be a field of formal power series in one or several variables with real coefficients. We prove that every symmetric square matrix $A \in \text{Mat}_n(K)$ can be diagonalized by means of an orthogonal matrix $U \in \text{Mat}_n(K)$. Our proof is based on a recursive construction and prepares the way for effectively computing the transition matrix $U$ (and therefore the eigenvalues of $A$ and their multiplicities). The result carries over to certain Henselian fields of power series in infinitely many variables.

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Introduction. The most prominent result in the theory of real or complex matrices is the Spectral Theorem which says that every symmetric [resp. hermitian] square matrix can be put into diagonal form by means of an orthogonal [resp. a unitary] matrix. The far-reaching applications of this result and its generalization to bounded linear operators on infinite-dimensional Hilbert spaces have been intensively studied. However, little is known about analogous decompositions of matrices with entries in more general fields. Diarra [2] showed that symmetric matrices over fields of p-adic numbers cannot be diagonalized in general. In turn, Adkins [1] proved a theorem on diagonalization of matrices with entries in discrete hermitian rings.

In the present paper we consider fields $K = \mathbb{R}((t_1, \ldots, t_m))$ of formal power series in one or several variables with real coefficients. Our main result states that over these fields $K$ every symmetric matrix can be orthogonally diagonalized. We shall show that this result even carries over to fields of generalized power series in infinitely many variables.

In the classical case of a real symmetric matrix $A$ the diagonalization is obtained by computing the characteristic polynomial of $A$ and using the fact that the quadratic extension $\mathbb{C} = \mathbb{R}(\sqrt{-1})$ is algebraically closed. In the present case this way of reasoning
fails. For, our fields $K = \mathbb{R}((t_1, \ldots, t_m))$ are too far from being algebraically closed; in fact these fields admit finite extensions of any degree. Our method of proof combines two ideas and can be outlined as follows. First, write $K = \mathbb{R}((t_1, \ldots, t_m)) = K_0((t))$ where $t = t_m$ and $K_0 = \mathbb{R}((t_1, \ldots, t_{m-1}))$. The field $K = K_0((t))$ is complete with respect to a non-archimedean, discrete valuation. This allows us to represent a given symmetric matrix $A$ with entries in $K$ as a convergent power series $A = A_0 + A_1 \cdot t + A_2 \cdot t^2 + \cdots$ with coefficients $A_k$ in a smaller matrix ring. Secondly, we shall set up a recursive construction that produces an orthogonal transition matrix $U = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots$ such that $U^\text{tr} A U$ is decomposed into two blocks of smaller size. The proof is then finished by an easy induction.

It is a remarkable feature of this proof that it does not involve the spectrum. Indeed, the eigenvalues of $A$ are obtained at the end as a by-product. Thus the proof is potentially a tool to study arithmetical properties of fields of power series.

We should like to mention that the paper has grown out of studies in the theory of orthomodular spaces. These are, by definition, vector spaces $E$ endowed with a hermitean form $\phi$ such that the Projection Theorem holds for $(E, \phi)$: every orthogonally closed linear subspace $U \subseteq E$ is a direct summand of the whole space. Classical examples are the Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$ and for a long time there were no others. Then, in 1980, numerous non-classical, infinite-dimensional orthomodular spaces were discovered. They are constructed over certain non-archimedean, complete fields; the valuations in question are of infinite rank. These new spaces carry a natural non-archimedean norm, so there is a notion of "bounded linear operator". The central question is whether a bounded, selfadjoint linear operator $T : E \to E$ always admits an orthogonal decomposition derived from its spectrum. By using the technique of reduction modulo residual spaces the task of decomposing an infinite-dimensional operator $T : E \to E$ is seen to be closely related to the problem of decomposing finite matrices over fields of power series (or of rational functions). For details we refer to [3] and [4].

1. Fields of power series. Given any field $K_0$ with $\text{char}(K_0) \neq 2$ we let $K = K_0((t))$ be the field of formal power series in the indeterminate $t$ with coefficients in $K_0$, and we let $\varphi : K \to \mathbb{Z}$ be the usual exponential valuation. Thus for a typical $\alpha = \sum_{i \in \mathbb{Z}} a_i t^i$ in $K$ we have $\varphi(\alpha) = \min \{ i \in \mathbb{Z} \mid a_i \neq 0 \}$ if $\alpha \neq 0$, $\varphi(\alpha) = \infty$ if $\alpha = 0$. The valued field $(K, \varphi)$ is henselian (cf. [4]). To $\varphi$ there corresponds a valuation ring $R := \{ \alpha \in K \mid \varphi(\alpha) \geq 0 \}$ with maximal ideal $J = \{ \alpha \in K \mid \varphi(\alpha) > 0 \}$. The residue field $\hat{K} := R/J$ is isomorphic to $K_0$, thus there is a canonical epimorphism $\pi$ from $K$ onto $K_0$.

We shall need the following fact.

Lemma 1. $K$ is a purely transcendental extension of $K_0$.

Proof : Suppose that $\vartheta \in K$ is algebraic over $K_0$. We have to show that $\vartheta \in K_0$. Let $p(X) = \sum_{i=0}^m a_i X^i \in K_0[X]$ be the irreducible polynomial of $\vartheta$. Then

$$a_0 + a_1 \vartheta + a_2 \vartheta^2 + \cdots + a_m \vartheta^m = 0.$$
There are at least two indices $0 \leq i < j \leq m$ such that $\varphi(a_i \vartheta^i) = \varphi(a_j \vartheta^j)$, for otherwise the terms on the lefthand side couldn't cancel. Since $\varphi(a_i) = \varphi(a_j) = 0$ it follows that $\varphi(\vartheta^i) = \varphi(\vartheta^j)$, hence $\varphi(\vartheta) = 0$. Thus $\vartheta \in R$. Applying now the epimorphism $\pi : R \to K_0$ to the above equality and noticing that $\pi(a_i) = a_i$ for all $i$ we obtain $a_0 + a_1 \pi(\vartheta) + a_2 \pi(\vartheta)^2 + \ldots + a_m \pi(\vartheta)^m = 0$, i.e. $\pi(\vartheta) \in K_0$ is a root of the polynomial $p(X)$. Since $p(X)$ is irreducible this is possible only when $m = 1$. We conclude that $\vartheta \in K_0$, as claimed.

2. Matrices over fields of power series. We consider the ring $Mat_n(K)$ of all square matrices of size $n \times n$ with entries in $K$ along with the subring $Mat_n(K_0)$ consisting of all matrices with entries in the subfield $K_0 \subset K$. We shall denote the matrices in $Mat_n(K)$ by $A, B, \ldots, U \ldots$ and those in $Mat_n(K_0)$ by $A, B, \ldots, U \ldots$. The unit matrix is always denoted by $I$.

A matrix $A \in Mat_n(K)$ is called orthogonal if its transpose $A^*$ is equal to the inverse $A^{-1}$, i.e. if $A^* A = A A^* = I$. We say that $A$ is diagonal if all entries outside the main diagonal are 0, more generally, we say that $A$ is $(r, n-r)$-blockdiagonal if it has the shape

\[
A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}
\]

where $B$ and $C$ are square matrices of size $r \times r$ and $(n-r) \times (n-r)$ respectively.

Our computations later on will rely on a representation of the elements $A \in Mat_n(K)$ as a formal power series with coefficients in the (non-commutative) subring $Mat_n(K_0)$. Put $\varphi(A) := \min\{\varphi(a_{ij})|1 \leq i, j \leq n\}$ and assume, for sake of simplicity, that $\varphi(A) = 0$. Then each entry $a_{ij}$ can be expressed as

\[
a_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)} t + a_{ij}^{(2)} t^2 + \ldots + a_{ij}^{(m)} t^m + \ldots.
\]

For $m = 0, 1, \ldots$ we collect the coefficients $a_{ij}^{(m)}$ in a matrix

\[
A_m := \begin{bmatrix} a_{11}^{(m)} & \ldots & a_{1n}^{(m)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(m)} & \ldots & a_{nn}^{(m)} \end{bmatrix} \in Mat_n(K_0).
\]

Then we can express $A$ in the form

\[
A = A_0 + A_1 \cdot t + A_2 \cdot t^2 + \ldots + A_m \cdot t^m + \ldots.
\]

This is the representation mentioned above.

3. The main result. Our purpose is to prove
Theorem 1: Let $K = K_0((t))$ and $n \geq 1$. The following conditions are equivalent:
(a) Every symmetric matrix $A \in \text{Mat}_n(K_0)$ can be diagonalized by means of an orthogonal matrix $U \in \text{Mat}_n(K_0)$.
(b) Every symmetric matrix $A \in \text{Mat}_n(K)$ can be diagonalized by means of an orthogonal matrix $U \in \text{Mat}_n(K)$.

The proof will be divided into several steps. We begin with the easy part.

Proof of the implication (b) $\Rightarrow$ (a): Suppose that $A$ in $\text{Mat}_n(K_0)$ is symmetric. By hypothesis (b) there exists an orthogonal matrix $U \in \text{Mat}_n(K)$ such that

$$D := U^*AU = \begin{bmatrix}
\lambda_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_{nn}
\end{bmatrix}$$

Here the diagonal entries $\lambda_{ii}$ are the eigenvalues of the matrix $A$, that is, the roots of the characteristic polynomial $p_A(X) = \det(X \cdot I - A)$. Since the coefficients of $p_A(X)$ belong to $K_0$, the $\lambda_{ii}$'s are algebraic over $K_0$. By Lemma 1 we conclude that $\lambda_{11}, \ldots, \lambda_{nn} \in K_0$.

Consider an eigenvalue $\lambda_{ii}$ and let $m_i$ be its algebraic multiplicity. Then $\lambda_{ii}$ is repeated $m_i$ times in $D$ and consequently

$$A - \lambda_{ii}I = U(D - \lambda_{ii}I)U^*$$

has rank $n - m_i$ over $K$. But the matrix $A - \lambda_{ii} \cdot I$ is in $\text{Mat}_n(K_0)$, so its rank over $K_0$ is the same as its rank over $K$ as can be easily seen by applying the Gaussian algorithm. Consequently $\lambda_{ii}$ has geometric multiplicity $m_i$. Thus for each eigenvalue of $A$ the algebraic and the geometric multiplicity coincide. This entails that there exists an orthogonal matrix $U$ in $\text{Mat}_n(K)$ such that $U^*AU$ is diagonal, as asserted.

The substantial part is the converse implication to which we now turn.

4. Proof of the implication (a) $\Rightarrow$ (b). In this section we assume throughout that $K_0$ is a coefficient field satisfying condition (a). Let there be given a symmetric matrix $A \neq 0$ in $\text{Mat}_n(K)$.

4.1. Multiplying $A$ by a suitable power of $t$ we may assume that $\varphi(A) = 0$. Then $A$ can be expressed as a power series

$$A = A_0 + A_1 \cdot t + \cdots + A_m \cdot t^m + \cdots$$

Notice that all the $A_m$'s are symmetric. We first reduce the general case to the special one in which the initial coefficient matrix $A_0$ of $A$ satisfies the condition

$$A_0 \text{ is diagonal but not a multiple of the unit matrix } I.$$
In fact, if all the $A_m$'s ($m = 0, 1, \ldots$) are multiples of $I$ then so is $A$ and there is nothing to prove. Otherwise let $m := \min\{k \in \mathbb{N}_0 | A_k \text{ is not a multiple of } I\}$. Then $\sum_{k=0}^{m-1} A_k \cdot t^k = \lambda \cdot I$ for some $\lambda \in K$. Put

$$B := t^{-m} \cdot (A - \lambda I) = t^{-m} \cdot \left( A - \sum_{k=0}^{m-1} A_k \cdot t^k \right) = A_m + A_{m+1} \cdot t + \cdots$$

Since $A_m$ is symmetric there exists, by (a), an orthogonal matrix $V \in \text{Mat}_n(K_0)$ such that $D := V^*A_m V$ is diagonal. Now look at

$$C := V^*B V = (V^*A_m V) + (V^*A_{m+1} V) \cdot t + (V^*A_{m+2} V) \cdot t^2 + \cdots$$

The expansion of $C$ starts with a coefficient matrix $C_0 = V^*A_m V$ that is diagonal but not a multiple of $I$. Moreover, if we succeed in finding an orthogonal matrix $U \in \text{Mat}_n(K)$ which diagonalizes $C$ then $V \cdot U$ will provide a diagonalization of $B$ and therefore also of $A = t^m \cdot B + \lambda I$. Hence we may assume from the start that the initial coefficient of $A$ satisfies (1).

We should like to point out that it is only in the above preliminary step where the hypothesis (a) is actually needed. However, the condition (a) can hardly be replaced by an assumption on the initial matrix $A_0$ because it will be used repeatedly in the inductive argument at the end (see section 4.6).

4.2. The idea is to construct recursively an orthogonal transition matrix

$$U = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots + U_m \cdot t^m + \cdots$$

that diagonalizes $A$. When trying to do so it turns out that the recursive computation of $U_0, U_1, U_2, \cdots$ can be carried out provided the diagonal entries of $A_0$ are pairwise different. However, when some diagonal entries of $A_0$ are repeated there arise serious troubles. The underlying geometric reason for these obstacles in that in the second case the given matrix $A$ may have multiple eigenvalues and consequently $U$ is not uniquely determined by $A$. The way out of the difficulties is as follows: we shall not attempt to put the given matrix $A$ into diagonal form at once, but we will first decompose $A$ into blocks the sizes of which are determined by the multiplicities occurring in $A_0$. The clue is given by the following result.

**Lemma 2:** Let $A = A_0 + A_1 \cdot t + \cdots + A_m \cdot t^m + \cdots \in \text{Mat}_n(K)$ be symmetric. Assume that $A_0$ satisfies (1). Then there exist an integer $r$ with $1 \leq r \leq n - 1$ along with an orthogonal matrix $U \in \text{Mat}_n(K)$ such that $U^*AU$ is $(r, n - r)$-blockdiagonal.

The proof will be divided into several steps and will cover the next three subsections.

4.3. Write

$$A_0 = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}$$
Since $A_0$ is not a multiple of $I$ the multiplicity $r$ of $a_{11}$ is strictly less than $n$. After conjugating by some permutation matrix we may assume that

$$a_{ii} = a_{11} \quad \text{for} \quad 1 \leq i \leq r, \quad a_{ii} \neq a_{11} \quad \text{for} \quad r + 1 \leq i \leq n.$$  

The multiplicity $r$ is the number $r$ referred to in the statement of Lemma 2.

4.4. We shall construct recursively matrices $U_0, U_1, \ldots$ in $\text{Mat}_n(K)$ such that

$$U = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots + U_m \cdot t^m + \cdots$$

satisfies both

$$U^* U = I,$$

and

$$U^* A U \quad \text{is} \quad (r, n - r) \text{-blockdiagonal.}$$

The first task is to express the above two conditions in terms of the $U_k$'s. Multiplying out the series $U = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots$ and $U^* = U_0^* + U_1^* \cdot t + U_2^* \cdot t^2 + \cdots$ we find

$$U^* U = U_0^* U_0 + (U_1^* U_0 + U_0^* U_1) \cdot t + \cdots + \left( \sum_{i+j=k} U_i^* U_j \right) \cdot t^k + \cdots$$

Hence the condition that $U^* U = I$ is satisfied if and only if

$$U_0^* U_0 = I,$$

and for all $k \geq 1$ we have

$$\sum_{i+j=k} U_i^* U_j = 0.$$  

Next, multiplication of the series for $U^*, A$ and $U$ yields

$$U^* A U = V_0 + V_1 \cdot t + V_2 \cdot t^2 + \cdots + V_k \cdot t^k + \cdots$$

where

$$V_0 := U_0^* A_0 U_0, \quad V_k = \sum_{i+j+h=k} U_i^* A_j U_h.$$  

It follows that $U^* A U$ is $(r, n - r)$-blockdiagonal if and only all the matrices $V_k \ (k = 0, 1, \ldots)$ given by (4) are $(r, n - r)$-blockdiagonal.
4.5. We start the recursive construction by putting

\[ U_0 := I. \]

Then (2) is satisfied, and the matrix \( V_0 \) given by (4) is trivially block-diagonal since \( A_0 \) is diagonal.

Assume that we have already constructed \( U_0, \ldots, U_{m-1} \) such that (3) holds for \( 1 \leq k \leq m - 1 \) and \( V_k \) is block-diagonal for \( 0 \leq k \leq m - 1 \). Consider then (3) with \( k = m \). Since \( U_0 = I \) we can rewrite this condition as

\[ U_m^* + U_m + \sum_{i+j=m, i \neq m} U_i^* U_j = 0. \]

Now \( S_m := \sum U_i^* U_j \) is symmetric. Hence (3) holds if and only if \( U_m \) has the shape

\[ U_m = -\frac{1}{2} S_m + Q_m \]

where \( Q_m \) is any antisymmetric matrix. Since \( S_m \) is determined by the matrices \( U_0, \ldots, U_{m-1} \) already constructed, the task is to choose \( Q_m \) in such a way that the resulting matrix \( V_m \) given by (4) is block-diagonal.

Separating in (4) the two summands corresponding to \( (i, j, h) = (m, 0, 0) \) and \( (i, j, h) = (0, 0, m) \) we obtain

\[ V_m = U_m^* A_0 + A_0 U_m + \sum_{i+j+h=m, i \neq m, h \neq m} U_i^* A_j U_h. \]

Substituting (5) into the above expression we obtain

\[ V_m = -Q_m A_0 + A_0 Q_m + T_m \]

where

\[ T_m = -\frac{1}{2} (S_m A_0 + A_0 S_m) + \sum_{i+j+h=m, i \neq m, h \neq m} U_i^* A_j U_h. \]

Since \( S_m \) and all the \( A_k \)'s are symmetric it follows that \( T_m \) is symmetric. Notice that \( T_m \) is expressed in terms of matrices already determined.

Write

\[ V_m = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix}, \quad Q_m = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}, \quad T_m = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}. \]
Computing the matrix products in (6) and taking into account that
\[ A_0 = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} \]
is diagonal we find
\[ v_{ij} = -q_{ij} a_{jj} + a_{ii} q_{ij} + t_{ij} = -q_{ij} (a_{jj} - a_{ii}) + t_{ij} \]
for all \( 1 \leq i, j \leq n \). Consider now a pair \((i, j)\) outside the blocks, i.e. either \( i \in \{1, \ldots, r\} \) and \( j \in \{r+1, \ldots, n\} \) or \( i \in \{r+1, \ldots, n\} \) and \( j \in \{1, \ldots, r\} \). Then \( a_{ii} \neq a_{jj} \) by definition of \( r \) (cf. section 4.3.) Thus if we put
\[ q_{ij} := \frac{t_{ij}}{a_{jj} - a_{ii}} \]
then \( v_{ij} = 0 \), as required. Notice also that \( q_{ij} = -q_{ji} \) for these pairs \((i, j)\). If \( i, j \) are both in \( \{1, \ldots, r\} \) or both in \( \{r+1, \ldots, n\} \) then we choose \( q_{ij} \) arbitrarily but such that \( q_{ij} = -q_{ji} \).

The matrix \( Q_m \) thus obtained is antisymmetric. Moreover, if we put \( U_m := -\frac{1}{2} S_m + Q_m \) then the matrix \( V_m \) given by (4) is \((r, n-r)\)-block-diagonal. This completes the recursive construction.

By construction the matrix \( U = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots \) is orthogonal and \( U^* A U \) is block-diagonal. The proof of Lemma 2 is complete.

4.6. We can now finish the proof of Theorem 1 by an easy induction on the size \( n \). The case \( n = 1 \) is trivial, so assume \( n > 1 \). Let there be given a symmetric matrix \( A \) in \( Mat_n(K) \) with initial coefficient \( A_0 \) satisfying (1). By Lemma 2 there exists a natural number \( r < n \) and an orthogonal matrix \( U \) in \( Mat_n(K) \) such that
\[ U^* A U = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \]
where \( A_1 \in Mat_r(K), A_2 \in Mat_{n-r}(K) \). Clearly \( A_1 \) and \( A_2 \) are symmetric. By induction, there exist orthogonal matrices \( V_1 \in Mat_r(K) \) and \( V_2 \in Mat_{n-r}(K) \) such that \( V_1^* A_1 V_1 \) and \( V_2^* A_2 V_2 \) are diagonal. Put
\[ W := \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}. \]

Then \( U W \) is orthogonal, and
\[
(U W)^* A (U W) = \begin{bmatrix} V_1^* A_1 V_1 & 0 \\ 0 & V_2^* A_2 V_2 \end{bmatrix}
\]
is diagonal. The proof is complete.

5. Applications. The classical Spectral Theorem (for finite dimensions) states that every symmetric matrix can be orthogonally diagonalized over the field $\mathbb{R}$ of reals. Applying Theorem 1 repeatedly we deduce the following result.

**Theorem 2:** Let $m \geq 0$ and let

$$K := \mathbb{R}((t_1, \ldots, t_m)) = \mathbb{R}((t_1))((t_2)) \cdots ((t_m))$$

be the field of formal power series in $m$ indeterminates with real coefficients. Then every symmetric matrix can be orthogonally diagonalized over $K$.

**Proof:** By induction on $m$. The case $m = 0$ is the classical one, and the induction step is just the assertion "(a) \Rightarrow (b)" of Thm. 1.

**Corollary:** Let $K := \mathbb{R}((t_1, \ldots, t_m))$. If the matrix $A \in \text{Mat}_n(K)$ is symmetric then its characteristic polynomial

$$p_A(X) = \det(X \cdot I - A) \in K[X]$$

decomposes into linear factors over $K$.

The above Spectral Theorem can even be generalized to fields of power series in infinitely many variables as we shall now show. We start with a direct sum

$$\Gamma := \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \cdots$$

of infinitely many copies of the group of integers. $\Gamma$ is an abelian, additive group under componentwise operations. We order $\Gamma$ antilexicographically.

Next we form the field

$$K := \mathbb{R}((\Gamma))$$

of generalized power series with exponents in $\Gamma$ and real coefficients. $K$ can be described as the field of all functions $\xi : \Gamma \to \mathbb{R}$ for which the support

$$\text{supp}(\xi) := \min\{\gamma \in \Gamma|\xi(\gamma) \neq 0\}$$

is well-ordered. The operations in $K$ are the obvious ones: $(\xi + \eta)(\gamma) := \xi(\gamma) + \eta(\gamma)$ and $(\xi \cdot \eta)(\gamma) := \sum_{\delta + \delta' = \gamma} \xi(\delta) \cdot \eta(\delta')$. There is a natural valuation

$$\varphi : K \to \Gamma \cup \{\infty\}, \text{ given by } \varphi(\xi) := \min \text{supp}(\xi).$$

The valued field $(K, \varphi)$ is complete and henselian; for details we refer to [5] or [6].

Now we can state
Theorem 3: Over the field \( K := R((F)) \) every (finite) symmetric matrix can be orthogonally diagonalized.

Outline of the proof For \( m = 0, 1, 2, \ldots \) we define the subgroup \( \Delta_m \subseteq \Gamma \) by
\[
\Delta_m := \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \{0\} \oplus \{0\} \oplus \cdots.
\]
The \( \Delta_m \)'s are isolated (or convex) subgroups of \( \Gamma \). To each \( \Delta_m \) there corresponds a valuation ring \( R_m := \{ \xi \in K | \varphi(\xi) \geq \delta \} \) for some \( \delta \in \Delta_m \) with maximal ideal \( J_m := \{ \xi \in K | \varphi(\xi) > \delta \} \) for all \( \delta \in \Delta_m \) and residue field \( \mathbb{K}_m := R_m/J_m \). It is readily verified that \( \mathbb{K}_m \cong R((t_1, \ldots, t_m)) \). In particular, each residue field \( \mathbb{K}_m \) can be considered as a subfield of \( \mathbb{K} \), moreover there is a canonical epimorphism \( \pi_m : R_m \rightarrow R((t_1, \ldots, t_m)) \).

Now let there be given a symmetric matrix
\[
\mathcal{A} = \begin{bmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{n1} & \cdots & \alpha_{nn}
\end{bmatrix} \in \text{Mat}_n(K).
\]
We may suppose that \( \varphi(\alpha_{ij}) \geq 0 \) for all \( i, j \), thus \( \alpha_{ij} \in R_m \) for \( m = 0, 1, 2, \ldots \). For each \( m \in \mathbb{N}_0 \) we form the reduced matrix
\[
\mathcal{A}_m := \pi_m(\mathcal{A}) = \begin{bmatrix}
\pi_m(\alpha_{11}) & \cdots & \pi_m(\alpha_{1n}) \\
\vdots & \ddots & \vdots \\
\pi_m(\alpha_{n1}) & \cdots & \pi_m(\alpha_{nn})
\end{bmatrix}.
\]
Applying Theorem 2 we obtain, for each \( m \in \mathbb{N}_0 \), an orthogonal matrix \( \hat{U}_m \in \text{Mat}_n(\mathbb{K}_m) \subseteq \text{Mat}_n(K) \) such that \( \hat{U}_m^* \mathcal{A}_m \hat{U}_m \) is diagonal.

The point is to show that the orthogonal matrices \( \hat{U}_m \) can be chosen in such a way that
\[
(7) \quad \pi_m(\hat{U}_{m+1}) = \hat{U}_m.
\]
If all the eigenvalues of the given matrix \( \mathcal{A} \) are simple then (7) is automatically satisfied as is shown by a routine verification. In the case where \( \mathcal{A} \) has multiple eigenvalues then the orthogonal matrices \( \hat{U}_m \) are not unique and one has to choose a suitable basis in each eigenspace.

Since \( K \) is complete we easily deduce from (7) that the sequence \( (\hat{U}_m)_{m \in \mathbb{N}_0} \) converges in the valuation topology to some matrix \( \mathcal{U} \in \text{Mat}_n(K) \); by continuity we conclude that \( \mathcal{U} \) is orthogonal and \( \mathcal{U}^* \mathcal{A} \mathcal{U} \) is diagonal. This completes the proof.

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