Entanglement evolution for quantum trajectories

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Abstract. Entanglement is a key resource in quantum information. It can be destroyed or sometimes created by interactions with a reservoir. In recent years, much attention has been devoted to the phenomena of entanglement sudden death and sudden birth, i.e., the sudden disappearance or revival of entanglement at finite times resulting from a coupling of the quantum system to its environment [1, 2, 3]. We investigate the evolution of the entanglement of noninteracting qubits coupled to reservoirs under monitoring of the reservoirs by means of continuous measurements. Because of these measurements, the qubits remain at all times in a pure state \( |\psi(t)\rangle \), which evolves randomly. To each measurement result (or “realization”) corresponds a quantum trajectory in the Hilbert space of the qubits. We show that for two qubits coupled to independent baths subjected to local measurements, the average of the qubits’ concurrence over all quantum trajectories is either constant or decays exponentially. The corresponding decay rate depends on the measurement scheme only. This result contrasts with the entanglement sudden death phenomenon exhibited by the qubits’ density matrix in the absence of measurements. Our analysis applies to arbitrary quantum jump dynamics (photon counting) as well as to quantum state diffusion (homodyne or heterodyne detections) in the Markov limit. We discuss the best measurement schemes to protect the entanglement of the qubits. We also analyze the case of two qubits coupled to a common bath. Then, the average concurrence can vanish at discrete times and may coincide with the concurrence of the density matrix. The results explained in this article have been presented during the “Fifth International Workshop DICE2010” by the first author and have been the subject of a prior publication [4].

1. Introduction
As already noted by Erwin Schrödinger in 1935 ”Entanglement is not one but rather the characteristic trait of Quantum Mechanics” [5]. Nowadays, entanglement is considered as a key resource in quantum information. However, it can be destroyed by interactions with reservoirs. When the two parts of a bipartite system are coupled to independent baths, entanglement typically disappears after a finite time [1, 3, 6]. This phenomenon is called “entanglement sudden death” (ESD). When the two parts of the system are coupled to a common bath, sudden revivals of entanglement may also take place after the state has become separable [7, 8, 9].

In this article, we investigate the evolution of the entanglement of two qubits coupled to two or one bath when the baths are monitored by continuous measurements. Because of these measurements, the qubits remain at all times in a pure state \( |\psi(t)\rangle \), which evolves randomly. To each measurement result (or “realization”) corresponds a quantum trajectory \( t \in \mathbb{R}_+ \mapsto |\psi(t)\rangle \) in the Hilbert space \( \mathbb{C}^4 \) of the qubits. In the Markovian regime, the dynamics is given by the
quantum jump (QJ) model [10, 11] or, in the case of homodyne and heterodyne detections, by the so-called quantum state diffusion (QSD) models [11, 12, 13]. We study how the entanglement of the qubits evolves in time by calculating the average $C_{\psi(t)}$ of the Wootters concurrence over all quantum trajectories (QTs). $C_{\psi(t)}$ differs in general from the concurrence $C_{\rho(t)}$ of the density matrix $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ (here and in what follows the overline denotes the average over all QTs) [14, 15]. For two qubits coupled to independent baths subjected to local measurements, we find that

$$C_{\psi(t)} = C_0 e^{-\kappa t} \quad (1)$$

where $C_0 = C_{\psi(0)}$ is the initial concurrence and $\kappa \geq 0$ depends on the measurement scheme but not on the initial state $|\psi(0)\rangle$. In particular, if $C_0 > 0$ and $t_{\text{ESD}} \in [0, \infty)$ is the time at which entanglement disappears in the density matrix (assuming that this time is finite), then $C_{\rho(t)} = 0$ at times $t \geq t_{\text{ESD}}$ whereas $C_{\psi(t)}$ can only vanish asymptotically. The continuous measurements on the two baths thus protect on average the qubits from ESD. Of course, this does not mean that all random wavefunctions $|\psi(t)\rangle$ remain entangled at all times. But in some cases, such as for pure dephasing and for infinite temperature baths, one can find measurement schemes such that $\kappa = 0$; then, for all QTs, if the qubits are maximally entangled at $t = 0$ (i.e., $C_0 = 1$) they remain maximally entangled at all times.

We investigate the best measurement schemes to protect entanglement. Related strategies using quantum Zeno effect [16], entanglement distillation [17], quantum feedback [18], and encoding in qutrits [19] have been proposed. It is assumed in this work that the measurements on the baths are performed by perfect detectors. The impact of detection errors has been studied in [19]. When the qubits are coupled to a common bath, we find that $C_{\psi(t)}$ has a more complex time behavior than in (1) and may vanish at finite discrete times and be equal to the density matrix concurrence for some initial states. In [18, 20], the authors study the problem to find measurement schemes depending on the initial state such that $C_{\psi(t)} = C_{\rho(t)}$.

This article is organized as follows. We begin to recall the definition of the concurrence and explain why it is a good measure of entanglement (Sec. 2). We review the quantum jump model in Sec. 3. In Sec. 4, we prove the formula (1) in the case of two two-level atoms coupled to the electromagnetic field initially in the vacuum for photon counting detection. We generalize this result in Sec. 5 to general quantum jump dynamics. In Secs. 6 and 7 we treat the case of homodyne and heterodyne detections and we compare the decay of $C_{\psi(t)}$ with that found for photon counting. Finally, Sec. 8 is devoted to the evolution of the entanglement of two qubits coupled to a common bath at zero temperature monitored by continuous measurements.

2. Entanglement measures

The entanglement of formation [21] of a bipartite quantum system $S$ in a pure state $|\psi\rangle$ is defined by means of the von Neumann entropy $E_\psi = -\text{tr}(\rho_A \ln \rho_A) = -\text{tr}(\rho_B \ln \rho_B)$ of the density matrices $\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \text{tr}_A(|\psi\rangle\langle\psi|)$ of the two subsystems $A$ and $B$ composing $S$. If $S$ is in a mixed state, $E_\rho$ is the minimum of $\sum p_k E_{\psi_k}$ over all convex decompositions $\rho = \sum p_k |\psi_k\rangle\langle\psi_k|$ of its density matrix (with $p_k \geq 0$ and $\|\psi_k\| = 1$). When $A$ and $B$ have two-dimensional Hilbert spaces, $E_\rho = f(C_\rho)$ is related to the concurrence [22] $C_\rho$ by a convex increasing function $f : [0, 1] \rightarrow [0, \ln(2)]$. A state $\rho$ is separable if and only if $C_\rho = 0$, i.e., $E_\rho = 0$. For a pure state

$$C_\psi = |\langle \sigma_y \otimes \sigma_y T \rangle_{\psi}| \quad (2)$$

where $\sigma_y = i(|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|)$ is the $y$-Pauli matrix, $T : |\psi\rangle = \sum s,s' c_{ss'} |s,s'\rangle \rightarrow \sum s,s' c_{ss'}^* |s,s'\rangle$ the anti-unitary operator of complex conjugation in the canonical basis $\{|s,s'\rangle = |s\rangle \otimes |s'\rangle; s, s' = \uparrow, \downarrow\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $\langle \psi \rangle = \langle \psi | \cdot |\psi\rangle$ the quantum expectation in state $|\psi\rangle$.
For quantum trajectories, one has always \( \overline{E_{\psi(t)}} \geq E_{\rho(t)} \), this inequality being strict excepted if the decomposition

\[
\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|} = \int dp(\psi) |\psi(t)\rangle\langle\psi(t)|
\]

realizes the infinum defining \( E_{\rho(t)} \). Thanks to the convexity of \( f \), \( \overline{E_{\psi(t)}} \geq f(\overline{\mathcal{C}(\psi(t))}) \). Thus Eq. (1) shows that for independent baths and if \( C_0 > 0 \) then \( \overline{E_{\psi(t)}} \geq f(C_0e^{-\kappa t}) > 0 \) whatever the measurement scheme.

It is legitimate to ask which entanglement measure should be averaged. The concurrence is a natural candidate as it corresponds for pure states to the maximum over all self-adjoint local observables \( J_A \) and \( J_B \) with norms \( \|J_A\|, \|J_B\| \leq 1 \) of the covariance \( \langle J_A \otimes J_B \rangle_{\psi(t)} - \langle J_A \otimes 1_B \rangle_{\psi(t)} \langle 1_A \otimes J_B \rangle_{\psi(t)} \). Moreover, as we will see below, \( \overline{\mathcal{C}(\psi(t))} \) is easy to calculate in the Markovian regime and it gives a lower bound on \( \overline{E_{\psi(t)}} \).

3. Quantum jump model

Let us briefly recall the QJ dynamics \([10, 23]\). As a result of a measurement on a particle (e.g. a photon) of the bath scattered by the qubits, the qubits’ wavefunction suffers a quantum jump

\[
|\psi(t)\rangle \rightarrow |\psi_{\text{jump}}^{(m,i)}\rangle = \frac{J^i_m |\psi(t)\rangle}{\|J^i_m |\psi(t)\rangle\|}
\]

where the jump operator \( J^i_m \) is related to the particle-qubits coupling and the indices \( m, i \) label all possible measurement results save for the most likely one, which we refer as a “no detection”. In the small coupling limit, the probability that a measurement in the small time interval \([t, t+dt] \) gives the result \((m, i)\) is very small and equal to \( dp^i_m(t) = \gamma^i_m \|J^i_m |\psi(t)\rangle\|^2 dt \). The jump rate \( \gamma^i_m \) does not depend on \( |\psi(t)\rangle \) and is proportional to the square of the particle-qubits coupling constant. In the no-detection case the wavefunction of the qubits evolves as

\[
|\psi(t + dt)\rangle = \frac{e^{-iH_{\text{eff}}dt}|\psi(t)\rangle}{\|e^{-iH_{\text{eff}}dt}|\psi(t)\rangle\|}, \quad H_{\text{eff}} = H_0 - \frac{i}{2} \sum_{m,i} \gamma^i_m J^i_m J^i_m
\]

where \( H_0 \) is the Hamiltonian of the qubits. The probability that no jump occurs in the time interval \([t_0, t] \) is \( p_{\text{noj}}(t_0, t) = \|e^{-iH_{\text{eff}}(t-t_0)} |\psi(t_0)\rangle\|^2 \). It is not difficult to show \([10]\) that the density matrix \( \rho(t) = |\psi(t)\rangle\langle\psi(t)| \) satisfies the Lindblad equation

\[
\frac{d\rho}{dt} = -i[H_0, \rho] + \sum_{m,i} \gamma^i_m \{J^i_m \rho J^i_m, \rho\} - \frac{1}{2} \sum_{m,i} \{J^i_m J^i_m, \rho\}
\]

where \( \{., .\} \) denotes the anticommutator. It is known that many distinct QJ dynamics unravel the same master equation (6).

For two qubits coupled to independent reservoirs \( R_A \) and \( R_B \), the jump operators are \( local \), i.e., they have the form

\[
J^A_m \otimes 1_B, \quad 1_A \otimes J^B_m
\]

depending on whether the measurements are performed on \( R_A \) or \( R_B \). Here \( J^\text{\_m}_m \) are \( 2 \times 2 \) matrices.

The absence of ESD, mentioned in the introduction, for the mean concurrence of two qubits coupled to independent baths can be traced back to the existence of trajectories for which \( |\psi(t)\rangle \) remains entangled at all times. Actually, for a trajectory without jump, \( |\psi_{\text{noj}}(t)\rangle \propto e^{-iH_{\text{eff}}t} |\psi(0)\rangle \) up to a normalization factor, see (5). By (7) and for noninteracting qubits, the effective Hamiltonian \( H_{\text{eff}} \) is the sum of two (commuting) local operators acting on each qubit. If \( |\psi_{\text{noj}}(t)\rangle \) would be separable at a given time \( t \) then, by reversing the dynamics (i.e., by applying \( e^{iH_{\text{eff}}t} \))
to $|\psi_{ij}(t)\rangle$ one would deduce that $|\psi(0)\rangle$ is separable. Hence, $C_{\psi_{ij}(t)}>0$ if $C_0>0$. But the no-detection probability between times 0 and $t$ is nonzero and thus $C_{\psi(t)}>0$ at all times. Note that this argument does not apply if nonlocal observables of the two baths are measured or if the two qubits are coupled to a common bath, due to the non-locality of $J_m$.

4. Photon counting

Let us illustrate the random dynamics described previously on a simple and experimentally relevant example [24]. Each qubit is a two-level atom coupled resonantly to the electromagnetic field initially in the vacuum (zero-temperature photon bath). The two atoms are far from each other and thus interact with independent baths. Two perfect photon counters $D_i$ make a click when a photon is emitted by qubit $i$ ($i=A,B$), whatever the direction of the emitted photon (see figure 1). Doing the rotating wave approximation, the jump operators are $J_i = \sigma_i^+ - \sigma_i^- = |\downarrow\rangle\langle\uparrow|$. For simplicity we take $H_0 = 0$.

If no photon is detected in the time interval $[0, t]$, the qubits wavefunction is, by (5),

$$|\psi(t)\rangle = \mathcal{N}(t)^{-1} \sum_{s,s'=\uparrow,\downarrow} c_{ss'} e^{-\gamma_{ss'}t/2} |s, s'\rangle$$

with :

(i) $c_{ss'} = \langle s, s'|\psi(0)\rangle$

(ii) $\mathcal{N}(t)^2 = \sum_{s,s'} |c_{ss'}|^2 e^{-\gamma_{ss'}t}$

(iii) $\gamma_{\uparrow\uparrow} = \gamma_A + \gamma_B$, $\gamma_{\uparrow\downarrow} = \gamma_A$, $\gamma_{\downarrow\uparrow} = \gamma_B$ and $\gamma_{\downarrow\downarrow} = 0$ ($\gamma_i$ being the jump rate for detector $D_i$).

The concurrence (2) of the state $|\psi(t)\rangle$ is $C(t) = C_0 \mathcal{N}(t)^{-2} e^{-(\gamma_A + \gamma_B)t/2}$ with $C_0 = 2|c_{\uparrow\uparrow}c_{\downarrow\downarrow} - c_{\downarrow\uparrow}c_{\uparrow\downarrow}|$.

If a photon is detected at time $t_j$ by, say, the photon counter $D_A$, the qubits are just after the jump (4) in the separable state $|\psi(t_j)\rangle \propto |\downarrow\rangle \otimes (c_{\uparrow\uparrow} e^{-\gamma_{\uparrow\uparrow}t_j/2}|\uparrow\rangle + c_{\downarrow\uparrow} e^{-\gamma_{\downarrow\uparrow}t_j/2}|\downarrow\rangle)$. Since neither a jump nor the inter-jump dynamics can create entanglement (the jump operators (7)}
Figure 2. Concurrences of two qubits coupled to independent baths at positive temperature as a function of $\gamma t$ for $\gamma_+ = \gamma_-/2 = \gamma$ with $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - i|\downarrow\downarrow\rangle)$.

(2a) $C_{\rho(t)}$ for the density matrix (blue dashed lines).
(2b) $C_{\psi(t)}$ for a single trajectory (black dotted line).
(2c) $C_{\psi(t)}$ averaged over 1500 trajectories (red plain line) and from Eq. (10) (red solid lines).
(2d) $C_{\psi(t)}$ for the best measurement scheme (green dotted-dashed line).

being local), the qubits state remains separable at all times $t \geq t_j$, even if more photons are subsequently detected. Thus $C(t) = 0$ if at least one photon is detected in the time interval $[0, t]$.

Averaging over all realizations of the QTs and using the probability $p_{nj}(0, t) = N(t)^2$ that no photon is detected in $[0, t]$, one finds

$$C(t) = C_0 e^{-\left(\gamma_A + \gamma_B\right)t/2}. \quad (9)$$

This argument is easily extended to baths at positive temperatures $T_i$. One must then add two jump operators $J_i^+ = \sigma_i^+$ with rates $\gamma_i^+ = e^{-\hbar \omega_i/kT_i} \gamma_i^-$ such that $\hbar \omega_i$ is the energy difference between the two levels of qubit $i$ and $k$ is the Boltzmann constant. Then the mean concurrence is given by

$$C(t) = C_0 e^{-\left(\gamma_A^+ + \gamma_B^+ + \gamma_A^- + \gamma_B^-\right)t/2}. \quad (10)$$

It is compared in Fig. 2 with the concurrence for the density matrix obtained by solving the master equation (6) which shows ESD for all initial states.
5. General quantum jump dynamics

We now consider a general QJ dynamics with jump operators given by (7). The Hamiltonian of the qubits has the form \( H_0 = H_A \otimes 1_B + 1_A \otimes H_B \). Let \( K = K_A \otimes 1_B + 1_A \otimes K_B \) with

\[
K_i = \frac{1}{2} \sum_m \gamma^i_m J^i_m J^i_m^\dagger,
\]

\( \gamma^i_m \) being the jump rate for the detector \( D_i \) \((i = A, B)\). We first assume that no jump occurs between \( t \) and \( t + dt \). By expanding the exponential in (5) one gets

\[
C(t + dt) = p_{nj}(t, t + dt)^{-1} \langle \sigma_y \otimes \sigma_y T \rangle_{\psi(t)} + idt \langle H^\dagger_{\text{eff}} \sigma_y \otimes \sigma_y T + \sigma_y \otimes \sigma_y T H_{\text{eff}} \rangle_{\psi(t)} + \mathcal{O}(dt)^2
\]

where \( p_{nj}(t, t + dt) = (1 - 2K dt + \mathcal{O}(dt)^2)_{\psi(t)} \) is the probability that no jump occurs between \( t \) and \( t + dt \). Now, for any local operator \( O_i \) acting on qubit \( i \), one has

\[
\langle O_i \sigma_y \otimes \sigma_y T \rangle_{\psi(t)} = \frac{C(t)}{2} \text{tr}_{C^2}(O_i)
\]

where we have introduced \( \gamma_{ij}^m \) for the jump rates for the detector \( D_i \). The solution of this differential equation is given by (1). Note that \( \kappa_{QJ} = 0 \) if all matrices \( J^i_m \) are self-adjoint and traceless. This is the case e.g. for pure dephasing (jump operators \( J^i = v_i \cdot \vec{\sigma} \) with \( v_i \in \mathbb{R}^3 \) and \( \vec{\sigma} \) the vector formed by the three Pauli matrices \( \sigma_x, \sigma_y \) and \( \sigma_z \)).

We can now give the optimal measurement scheme to protect the entanglement of two qubits coupled to independent baths at positive temperatures. Let us replace the photon-counting jump

\[
\int H_{\text{eff}} dt = \mathcal{O}(dt)^2.
\]
operators $J_{i}^{\pm} = \sigma_{i}^{\pm}$ by $I_{i} = \sum_{m=\pm} (\gamma_{i}^{m}/\gamma_{i}^{m+})^{1/2} u_{i}^{m} \sigma_{i}^{m}$ where $U_{i} = (u_{i}^{m,n})_{m=\pm, n=\pm}$. This corresponds to a rotation of the measurement basis and gives another unraveling of the master equation (6). Let us stress that the new jump operators $I_{i}^{\pm}$ still act locally on each qubits. Using (18), $\sum_{m} |u_{i}^{m,\pm}| = 1$, $\sum_{m} u_{i}^{m,+} u_{i}^{m,-} = 0$ and optimizing over all unitaries $U_{i}$, one finds that the smallest disentanglement rate arises for e.g. $u_{1,\pm} = \pm u_{2,\pm} = 1/\sqrt{2}$ ($N = 2$) and is given by

$$\kappa_{\text{QJ}}^{\text{opt}} = \frac{1}{2} \sum_{i=A,B} \left( \sqrt{\gamma_{i}^{+}} - \sqrt{\gamma_{i}^{-}} \right)^{2}.$$  \hspace{1cm} (19)

The decay of $\overline{C_{\psi}(t)}$ for this optimal measurement scheme is shown in Fig. 2. Note that $\kappa_{\text{QJ}}^{\text{opt}} = \kappa_{\text{QJ}}$ at zero temperature and $\kappa_{\text{QJ}}^{\text{opt}} = 0$ (perfect protection) at infinite temperature.

6. Homodyne detection

Let us come back to our example of two atoms coupled to the electromagnetic field initially in the vacuum. If homodyne photo-detection is used instead of photon counting, the jump operators become $J_{i \pm}^{\pm} = \sigma_{i}^{\pm} \pm \alpha_{i}$ with the rates $\gamma_{i \pm} = \gamma_{i}/2$, $\alpha_{i}$ being the amplitudes of the classical laser fields (there are now four jump operators since each homodyne detection involves two photon counters) [25]. By applying (18), we find that the disentanglement rate for homodyne detection is the same as for photon counting, $\kappa_{\text{QJ}}(\alpha) = (\gamma_{A} + \gamma_{B})/2$.

Let us now consider a general QJ model with jump operators $J_{i}^{\pm}$. A new unraveling of the master equation (6) is obtained from the new QJ dynamics with jump operators $J_{i \pm}^{\pm} = J_{i}^{\pm} \pm \alpha_{i}$ and rates $\gamma_{i \pm} = \gamma_{i}/2$. For large positive laser amplitudes $\alpha_{i} \gg 1$, this dynamics converges after an appropriate coarse graining in time to the quantum state diffusion (QSD) model described by the stochastic differential equation [12]

$$|d\psi\rangle = \left[ (-iH_{0} - K)dt + \sum_{m,i} \gamma_{i}^{m} \left( \Re \langle J_{i}^{m}\rangle_{\psi} J_{i}^{m} - \frac{\left( \Re \langle J_{i}^{m}\rangle_{\psi} \right)^{2}}{2} \right) dt + \sum_{m,i} \sqrt{\gamma_{i}^{m}} (J_{i}^{m} - \Re \langle J_{i}^{m}\rangle_{\psi}) dw_{i}^{m} \right] |\psi\rangle$$  \hspace{1cm} (20)

where $dw_{i}^{m}$ are the Itô differentials for independent real Wiener processes satisfying the Itô rules $dw_{i}^{m} dw_{i}^{n} = \delta_{ij} \delta_{mn} dt$.

It is shown in [4] by taking the limit $\alpha_{i} \gg 1$ of the mean concurrence for the quantum jump dynamics with jump operators $J_{i \pm}^{\pm}$ that the concurrence for the QSD model is still given by Eq. (1) but with a new rate

$$\kappa_{\text{ho}} = \frac{\text{tr}_{\text{C1}}(K)}{2} - \sum_{m,i} \gamma_{i}^{m} \left( \text{Re} \text{det}_{\text{C2}}(J_{i}^{m}) + \frac{3}{4} \text{tr}_{\text{C2}}(J_{i}^{m}) \right)^{2}. \hspace{1cm} (21)$$

Unlike $\kappa_{\text{QJ}}$, $\kappa_{\text{ho}}$ changes when the operators $J_{i}^{m}$ in (20) acquire a phase factor $e^{-i\theta_{i}^{m}}$. This arises for homodyne detection with complex laser amplitudes $\alpha_{i}^{m} = |\alpha_{i}^{m}| e^{i\theta_{i}^{m}}$ as $|\alpha_{i}^{m}| \gg 1$. Minimizing over all laser phases $\theta_{i}^{m}$ yields

$$\kappa_{\text{ho}}^{\text{opt}} = \frac{1}{2} \text{tr}_{\text{C1}}(K) - \sum_{m,i} \gamma_{i}^{m} \left( \frac{\text{det}_{\text{C2}}(J_{i}^{m})}{4} - \frac{1}{4} \text{tr}_{\text{C2}}(J_{i}^{m}) \right)^{2} + \frac{1}{4} \text{tr}_{\text{C2}}(J_{i}^{m}) \right)^{2}. \hspace{1cm} (22)$$
Using the inequality $|a - b^2| + |b|^2 \geq |a|$ for $a, b \in \mathbb{C}$, we have
\[ \kappa_{\text{het}}^{\text{opt}} \leq \kappa_{\text{QJ}}. \] (23)

Thus optimal homodyne detection protects entanglement better than or as well as photon counting.

7. Heterodyne detection

This case is very similar to the previous one. The corresponding jump operators $J^i_{m,\pm\alpha}(t_q) = J^i_m \pm \alpha^i_m e^{\Omega^i_m t_q}$ depend on the time $t_q$ of occurrence of the $q$th jump [25] and the associated rates are $\gamma^i_{m,\pm\alpha} = \gamma^i_m/2$.

For $\alpha^i_m > 0$, in the limit $(\alpha^i_m)^2 \gg \Omega^i_m/\gamma^i_m \gg 1$ of large laser intensities and rapidly oscillating laser amplitudes, and with an appropriate coarse graining in time, the QJ dynamics with jump operators $J^i_{m,\pm\alpha}(t_q)$ converges to the QSD model given by the stochastic differential equation [12, 23]

\[ \dot{|d\psi}\rangle = \left[ (-iH_0 - K)dt + \frac{1}{2} \sum_{m,i} \gamma^i_m \left( (J^i_m)^* \psi J^i_m - \frac{|(J^i_m)\psi|^2}{2} \right) dt \right. \]
\[ \left. + \sum_{m,i} \sqrt{\gamma^i_m} \left( (J^i_m - \frac{1}{2}(J^i_m)^*)d\xi^i_m - \frac{1}{2}(J^i_m)^*d\xi^i_m \right) \right]|\psi\rangle \] (24)

where $d\xi^i_m$ are the Itô differential of independent complex Wiener processes satisfying the Itô rules $d\xi^i_m d\xi^j_n = \delta_{ij}\delta_{mn}dt$.

It is proven in [4] that the mean concurrence is still given by Eq. (1) with the rate

\[ \kappa_{\text{het}} = \frac{\text{tr}C_4(K)}{2} - \frac{1}{4} \sum_{m,i} \gamma_m^i |\text{tr}C_2(J^i_m)|^2. \] (25)

We note that $\kappa_{\text{het}} \geq \kappa_{\text{ho}}^{\text{opt}}$, so that the best measurement scheme to protect the qubits against disentanglement is given by homodyne detection with optimally chosen laser phases.

8. Qubits coupled to a common bath

If the qubits are closed from each other then the situation is different from that studied in Sec. 4: the two qubits with equal frequencies are now coupled resonantly to the same modes of the electromagnetic field. We assume that the field is initially in the vacuum. A photon counter $D$ makes a click when a photon is emitted by qubit $A$ or $B$. The jump operator $\sigma_- \otimes 1_B + 1_A \otimes \sigma_-$ is now nonlocal. We take $H_0 = 0$. Proceeding as for independent baths, the contribution to the mean concurrence of QTs with no jump between 0 and $t$ is $p_{\text{nj}}(0, t) C_{\text{nj}}(t) = |(e^{-Kt} \Psi \otimes \sigma_y T e^{-Kt})|_{\psi(0)}$ and can be calculated with the help of (14).

QTs having one jump between 0 and $t$ gives a contribution which can be found by multiplying $C_{\text{nj}}(t) = 2N_{\text{i},j}^2(t) e^{-2\gamma t}|C_{\uparrow\uparrow}|^2$ by the probability density that the jump occurs at time $t_j \in [0, t]$, given by $p_{\text{nj}}(t_j, t) ||\langle|J\psi(t_j-\rangle||^2 p_{\text{nj}}(0, t_j) = \gamma N_{\text{i},j}(t)^2 = \gamma \|e^{-K(t-t_j)}J e^{-Kt_j}\psi(0)\|^2$, and integrating over $t_j$ from 0 to $t$.

After two clicks, $|\psi(t)e\rangle = |\downarrow\downarrow\rangle$ is in an invariant separable state. Therefore, trajectories with more than one jump do not contribute to the mean concurrence.

Setting $c_{\pm} = c_{\downarrow\downarrow} \pm c_{\uparrow\downarrow}$, one gets [4]

\[ C(t) = \frac{1}{2}c_+^2 - c_+^2 e^{-2\gamma t} + 4c_+ c_{\downarrow\downarrow} e^{-\gamma t} + 2|c_{\downarrow\downarrow}|^2 \gamma t e^{-2\gamma t}. \] (26)
Figure 3. Concurrency of two qubits coupled to a common bath at zero temperature as a function of $\gamma t$ with $|\Psi(0)\rangle = \frac{2}{\sqrt{5}}|↑↑\rangle + \frac{1}{\sqrt{5}}|↓↓\rangle$.

(1a) $C_{\rho(t)}$ (blue dashed lines).

(1b) $C_{\psi(t)}$ for a single trajectory (black dotted line).

(1c) $C_{\psi(t)}$ given by (26) (red plain line superposed on the blue line).

Inset (2) is the same for $|\Psi(0)\rangle = \frac{7i}{\sqrt{30}}|↑↑\rangle + \frac{2i}{\sqrt{30}}|↓↓\rangle$.

The time behavior of the concurrency (26) depends strongly on the initial state. For $c_{↑↑} = 0$ and $\arg(c_{↑↓}) = \arg(c_{↓↑})$, $C(t) = 0$ if and only if $t = \gamma^{-1}\ln(c_+/c_-)$. Thus entanglement is lost after a finite time but this loss is immediately followed by a revival.

For any initial state such that $c_{↑↑} = 0$, $C(t) = \frac{1}{2}|c_+^2 - c_-^2 e^{-2\gamma t}|$ and it can be shown that $C(t)$ coincides with $C_{\rho(t)}$. In contrast, if $c_{↑↑} \neq 0$ then $C_{\rho(t)}$ and $C(t)$ have very different behaviors as we can see in Fig. 3. Finally, $C(t)$ converges at large times to the same limit $C_{\infty} = |c_-|^2/2$ as the concurrence $C_{\rho(t)}$ [16, 26].
9. Conclusion
We have found explicit formulas for the mean concurrence of quantum trajectories and have shown that the measurements on the baths may be used to protect the entanglement of two qubits. These results shed new light on the phenomenon of ESD. It should be possible to check our findings experimentally by employing similar optical devices as in Ref. [6].

Note: after the completion of this work we learned that related results have been obtained in [27].

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