A Hint on the External Field Problem for Matrix Models

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Abstract

We reexamine the external field problem for $N \times N$ hermitian one-matrix models. We prove an equivalence of the models with the potentials $\text{tr} \left( \frac{1}{2N} X^2 + \log X - \Lambda X \right)$ and $\sum_{k=1}^{\infty} t_k \text{tr} X^k$ providing the matrix $\Lambda$ is related to $\{t_k\}$ by $t_k = \frac{1}{k} \text{tr} \Lambda^{-k} - \frac{N}{2} \delta_{k2}$. Based on this equivalence we formulate a method for calculating the partition function by solving the Schwinger–Dyson equations order by order of genus expansion. Explicit calculations of the partition function and of correlators of conformal operators with the puncture operator are presented in genus one. These results support the conjecture that our models are associated with the $c = 1$ case in the same sense as the Kontsevich model describes $c = 0$.

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1 Introduction

The external field problem for matrix models has received recently much attention due to the work by Kontsevich [1]. Generically, the \((N \times N)\) hermitian one-matrix model in an external field is defined by the partition function

\[
Z[\Lambda; N] = \int DX \, e^{N \text{tr}(\Lambda X - V_0(X))}
\]  

(1.1)

where \(V_0(X)\) is some potential. As follows from the results by Kontsevich, this model with a cubic potential \(V_0(X) \propto X^3\) describes Witten’s formulation [2] of 2D topological gravity.

The partition function (1.1) can be calculated by the standard methods of solving matrix models. While the orthogonal polynomial technique can not be used due to the presence of the matrix \(\Lambda\), the method [3] of Schwinger–Dyson equations written in terms of eigenvalues of \(\Lambda\) has been applied recently [4, 5] to explicitly solve the Kontsevich model by genus expansion. An analogous solution in genus zero has been obtained [3] for the case of the Kostov–Metha potential [7]

\[
V_0(X) = \frac{1}{2}X^2 - \alpha \log X,
\]  

(1.2)

when it is associated with an external field problem for the Penner model [8].

The external field \(\Lambda\) in Eq.(1.1) plays a role of a source for correlators of \(\text{tr} \, X^k\) which can be obtained by differentiating w.r.t. \(\Lambda\) and putting then \(\Lambda = 0\). An alternative way to calculate these correlators is to introduce the most general potential

\[
V(X) = \sum_{k=0}^{\infty} t_k X^k
\]  

(1.3)

and to consider the partition function

\[
Z[t; N] = \int DX \, e^{-\text{tr} \, V(X)}
\]  

(1.4)

whose derivatives w.r.t. \(t_k\)’s reproduce the same correlators of \(\text{tr} \, X^k\)’s provided one puts \(V(X) = NV_0(X)\) after the differentiation. In this approach, the role of an external field is played by the set of couplings \(\{t_k\}\). It is evident that the two ways of introducing an external field are equivalent. As far as we know, this equivalence has not been elaborated, however, in the literature.

In the present paper we prove the exact relation between the partition functions (1.1) and (1.4):

\[
Z[\Lambda; N] = e^{\frac{N}{2} \text{tr} \, \Lambda^2} Z[t; \alpha N],
\]  

(1.5)

which is valid provided \(\Lambda\) and \(\{t_k\}\) are related by the Miwa–type transformation

\[
t_k = \frac{1}{k} \text{tr} \, \Lambda^{-k} - \frac{N}{2} \delta_{k2} \quad \text{for} \quad n \geq 1, \quad t_0 = \text{tr} \, \log \Lambda^{-1}
\]  

(1.6)
and \( N \to \infty \) which makes \( t_k \)'s to be independent variables. The proof is based on the fact that the Schwinger–Dyson equations for both models are equivalent. Based on this equivalence, we formulate then a method for calculating the partition functions order by order of genus expansion which is closed in spirit to that proposed by Gross and Newman \([4]\) for the unitary matrix model and for the hermitian one with a cubic potential. We calculate explicitly the partition function in genus one and apply the result to calculate genus one contribution to correlators of conformal operators with the puncture operator. Our genus one results support the conjecture \([6]\) that the model \((1.1)\) with the potential \((1.2)\) (and therefore the model \((1.4)\) with the potential \((1.3)\)) is associated with the \( c = 1 \) case, if \( \alpha \) is identified with the cosmological constant, in the same sense as the Kontsevich model is associated with \( c = 0 \).

2 The Virasoro constraints

A simplest way to prove the equivalence between the partition functions \((1.1)\) and \((1.4)\) is to show that the Schwinger–Dyson equation for the model \((1.1)\) with the potential \((1.2)\),

\[
\left\{ \frac{\partial^2}{\partial \Lambda^2} + N \Lambda \frac{\partial}{\partial \Lambda} - \alpha N^2 \right\} e^{-\frac{1}{2} \text{tr} \Lambda^2} Z[\Lambda] = 0,
\]

(2.1)
is equivalent to the set of the Virasoro constraints,

\[
L_n Z[t.] = 0 \quad \text{for} \quad n \geq -1
\]

(2.2)

where

\[
L_n = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}} + \sum_{k=0}^{n} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k}},
\]

(2.3)

imposed on the partition function \((1.4)\).

Eq.\((2.1)\) has been discussed in Ref.\([6]\). The extra gaussian factor in front of \( Z \) is introduced to cancel one arising from the integral (one can easily see this is the case of \( \alpha = 0 \)). The commutation of this gaussian factor with the derivatives makes the sign of the second term in parentheses opposite to what would be naively expected. As for the Virasoro constraints \((2.2), (2.3)\), they were obtained in Refs.\([9]\) using the invariance of the integral \((1.4)\) under the shift \( X \to X + \epsilon_n X^{n+1} \).

Now our purpose is to make a change of variables from (the eigenvalues of) \( \Lambda \) to \( \{ t \} \) in Eq.\((2.1)\) similarly to what was done in Refs.\([10, 11]\) to derive the continuum Virasoro constraints from the Kontsevich matrix model (in that case only odd \( k \)'s entered the Kontsevich–Miwa transformation \((1.6)\)). The only what we need is the chain rule

\[
\frac{\partial}{\partial \Lambda} = -\sum_{k=0}^{\infty} \Lambda^{-k-1} \frac{\partial}{\partial t_k}
\]

(2.4)

which is a consequence of Eq.\((1.6)\).
Making this change of variables, one obtains from Eq.(2.1)
\[ \sum_{n=-1}^{\infty} \Lambda^{-n-2} L_n e^{-\frac{N}{2} \text{tr} \Lambda^2} Z = 0 \] (2.5)
with \( L_n \) given by Eq.(2.3) provided that
\[ \frac{\partial}{\partial t_0} Z = -\alpha NZ \] (2.6)
which emerges formally for \( n = -2 \). The last equation coincides with the normalization condition for \( \alpha N \times \alpha N \) hermitian one-matrix model. We have reproduced, therefore, exactly the Virasoro constraints (2.2), providing the partition functions are related by Eq.(1.5).

The fact that the Virasoro constraints (2.2), (2.3) can be obtained from the Schwinger–Dyson equation for the Kontsevich model making the transformation (1.6) with both even and odd \( k \)’s was noticed by Marshakov, Mironov and Morozov [10]. It was not recognized, however, that the proper external field problem is given by the partition function (1.1) with the potential (1.2).

3 Loop equations

Loop equations are successfully applied for studying the matrix models since the work by Kazakov [12] (for a review of recent results, see e.g. Ref.[13]).

An insertion of the one-loop operator (with \( \lambda \) being the Laplace transformed momentum which corresponds to a loop of the length \( l \)) is given by
\[ \frac{\delta}{\delta V(\lambda)} = -\sum_{k=0}^{\infty} \lambda^{-k-1} \frac{\partial}{\partial t_k}. \] (3.1)
This operator can be rewritten in terms of \( \partial/\partial \Lambda \). Due to Eq.(1.6), one gets
\[ \frac{\partial}{\partial \lambda_i} = -\sum_{k=0}^{\infty} \lambda_i^{-k-1} \frac{\partial}{\partial t_k} \] (3.2)
where \( \lambda_i \) stands for an eigenvalue of \( \Lambda \). As \( N \to \infty \) when Eq.(3.2) becomes exact, one can define the density of eigenvalues of \( \Lambda \):
\[ \rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \] (3.3)
so that
\[ \frac{\delta}{\delta V(\lambda)} = \frac{1}{N} \frac{\partial}{\partial \lambda} \frac{\delta}{\delta \rho(\lambda)}. \] (3.4)
Similarly, one can relate (the derivative of) the loop source \( V(\lambda) \), which is defined by Eq.\((1.3)\), to \( \rho \). Using Eq.(1.6), one gets
\[
V'(\lambda) = \sum_{k=1}^{\infty} kt_k \lambda^{k-1} = \sum_{j=1}^{N} \frac{1}{\lambda_j - \lambda} - N\lambda, \tag{3.5}
\]
where the sum over \( k \) is convergent if \( \lambda < \min |\lambda_i| \). As \( N \to \infty \) Eq.(3.5) determines \( V' \) to be the Hilbert transform of \( \rho \):
\[
\frac{1}{N} V'(\lambda) = \int dx \frac{\rho(x)}{x - \lambda} - \lambda. \tag{3.6}
\]
This equation expresses unambiguously each term of the \( \lambda \)-expansion of \( V'(\lambda) \) via \( \rho \) provided the support of \( \rho(x) \) vanishes at some finite interval which includes the point \( x = 0 \) (this is an analog of the above condition \( \lambda < \min |\lambda_i| \)). All the integrals below are rigorously defined providing \( \rho(\lambda) \) possesses this property. The discussion of general properties of Eq.(3.6) is beyond the scope of the present publication.

We compare now two equations. The first one is the Schwinger–Dyson equation for the model \((1.1)\) written in terms of \( \lambda_i \) \[6\]
\[
\left\{ \frac{\partial^2}{\partial \lambda_i^2} + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \left( \frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j} \right) + N\lambda_i \frac{\partial}{\partial \lambda_i} - \alpha N^2 \right\} e^{\frac{N}{2} \sum_k \lambda_k^2} Z[\lambda] = 0. \tag{3.7}
\]
The second one is the loop equation written for the one-loop average \( W(\lambda) = \frac{\delta}{\delta V(\lambda)} \log Z[t] \)
\( \tag{3.8} \)
which reads (see, e.g. \[13\])
\[
\int_{C_1} \frac{d\omega}{2\pi i (\lambda - \omega)} V'(\omega) W(\omega) = (W(\lambda))^2 + \frac{\delta}{\delta V(\lambda)} W(\lambda). \tag{3.9}
\]
This equation is equivalent to the Virasoro constraints \((2.2), (2.3)\) which can be obtained by expanding both sides of Eq.(3.9) in \( 1/\lambda \).

It follows immediately from Eqs.(3.1), (3.4), (3.8) that the r.h.s. of Eq.(3.9) coincides with the first term (with the double derivative) on the l.h.s. of Eq.(3.7). In order to compare the remaining terms, one inserts Eq.(3.6) into the l.h.s. of Eq.(3.9) and calculates the contour integral over \( \omega \) by taking residuals at the two poles. Applying again Eqs.(3.1), (3.4), (3.8), one gets exactly the remaining terms on the l.h.s. of Eq.(3.7) providing the normalization condition \((2.6)\) is satisfied. This completes the proof of the equivalence between Eqs.(3.7) and (3.9).

4 Comparison of genus zero solutions

It is instructive to compare the known (one-cut) genus zero solution of loop equation (3.9) with that of Eq.(3.7) which has been obtained in Ref\([6\).
The genus zero solution to Eq. (3.9), which was first obtained for a general $V(\lambda)$ that involves both even and odd powers of $\lambda$ by Migdal [14], reads

$$W_0(\lambda) = \int_{C_1} \frac{d\omega}{4\pi i (\lambda - \omega)} \frac{V'(\omega)}{\sqrt{\lambda^2 + 4b\lambda + 4c}} \sqrt{\omega^2 + 4b\omega + 4c}$$

(4.1)

with $b$ and $c$ given by

$$\int_{C_1} \frac{2\pi i}{d\omega} V'(\omega) \sqrt{\omega^2 + 4b\omega + 4c} = 0, \quad \int_{C_1} \frac{2\pi i}{d\omega} \omega V'(\omega) \sqrt{\omega^2 + 4b\omega + 4c} = 2\alpha N.$$ (4.2)

Inserting Eq. (3.6) and doing the contour integral as before, one gets from Eq. (4.1)

$$\frac{1}{N} W_0(\lambda) = \frac{1}{2} \left[ \sqrt{\lambda^2 + 4b\lambda + 4c} - \lambda - \int dx \frac{\rho(x)}{x - \lambda} \left( \frac{\sqrt{\lambda^2 + 4b\lambda + 4c}}{\sqrt{x^2 + 4bx + 4c}} - 1 \right) \right]$$

(4.3)

while Eq. (4.2) yields

$$\frac{1}{2} \int dx \frac{\rho(x)}{\sqrt{x^2 + 4bx + 4c}} + b = 0, \quad \frac{1}{2} \int dx \frac{x \rho(x)}{\sqrt{x^2 + 4bx + 4c}} + c - 3b^2 = \alpha + \frac{1}{2}.$$ (4.4)

One sees that (4.3), (4.4) coincide with the corresponding formulas of Ref. [6]. The $W(x)$ which was introduced there is related to our $W_0$ by $W(x) = \frac{1}{N} W_0(\lambda) + \frac{1}{2}$ while the definitions of $b$ and $c$ are the same. One more formula which would be useful for applications can be obtained by differentiating Eq. (4.3) (or (4.1)) w.r.t. $\alpha$:

$$\frac{1}{N} \frac{dW_0(\lambda)}{d\alpha} = \frac{1}{\sqrt{\lambda^2 + 4b\lambda + 4c}}.$$ (4.5)

Being expanded in $\frac{1}{N}$, this expression reproduces Eq. (5.20) of Ref. [6]. An advantage of the approach of Ref. [6] is that it allowed to calculate $\log Z$ itself.

For an iterative solution of Eq. (5.7) which is considered below, we shall need an explicit expression for the irreducible two-loop correlator

$$W(\lambda, \mu) = \delta \frac{\delta}{\delta V(\mu)} \frac{\delta}{\delta V(\lambda)} \log Z.$$ (4.6)

Applying Eq. (3.4) to $W_0$ given by Eq. (4.3), one gets in genus zero

$$W_0(\lambda, \mu) = \frac{1}{2(\lambda - \mu)^2} \left( \frac{\lambda \mu - 2b(\lambda + \mu) + 4c}{\sqrt{\lambda^2 + 4b\lambda + 4c}} \right)^{\frac{1}{2}}, \quad \frac{b^2 - c}{(\lambda^2 + 4b\lambda + 4c)^2}.$$ (4.7)

which coincides with the known result by Ambjørn et al. [9].

The solution (4.3), (4.4) is simplified if $\rho(\lambda)$ is a symmetric function $\rho(\lambda) = \rho(-\lambda)$. The first equation in (4.4) yields then $b = 0$ while only even powers of $x$ enter the second one (as well as Eq. (4.3)). This case corresponds to the so-called reduced hermitian one-matrix model, i.e. to vanishing odd times $t_{2m+1}$. It implies for the Miwa transformation (1.6) that only even $k$’s are present. This situation is complementary to the original Kontsevich
model when only odd $k$’s appear. The loop equations for the reduced hermitian one-matrix model can not be formulated, however, entirely in terms of the even times. The odd times inevitably appear for this model to higher orders. This does not happen for the case of the complex matrix model (see, e.g. [13]) which is formulated entirely via the even times. It is the model which is complementary in this sense to the Kontsevich model. A study of the external field problem for the complex matrix model will be published elsewhere.

5 The genus one solution

Our idea of how to solve Eq.(3.7) (or (3.9)) iteratively is similar to one which has been proposed by Gross and Newman [4] for the unitary matrix model and for the hermitian one with a cubic potential.

Let us introduce the new variables

$$B_p = 2^{p-1}(2p-1)!! \int d\lambda \rho(\lambda) \frac{1}{(\lambda^2 + 4b\lambda + 4c)^{p+1/2}},$$

$$C_p = 2^{p-1}(2p-1)!! \int d\lambda \rho(\lambda) \frac{\lambda}{(\lambda^2 + 4b\lambda + 4c)^{p+1/2}} - (\alpha + \frac{1}{2})\delta_p^0$$

where $b$ and $c$ are determined by Eq.(4.4) which reads now

$$B_0 + b = 0, \quad C_0 + c - 3b^2 = 0$$

where $(-1)!! = 1$ by definition. Let us define $F_g$ — genus $g$ contribution to log $Z$ — by the formula

$$\log Z = \sum_{g=0}^{\infty} N^{2-2g} F_g.$$  

Now our conjecture is that $F_g$ would depend at $1 \leq g < \infty$ only on $B_p$ and $C_p$ for $p \leq P$ where $P$ is some finite number (the larger $g$ the larger $P$). This is in contrast to the $t$-dependence of $F_g$ which always depends on the whole set $\{t_k\}$. Such a behavior of $F_g$ for the Kontsevich model has been advocated recently by Itzykson and Zuber [15].

Let us find explicitly the genus one correction to the genus zero results which are described in the previous section. Defining

$$W_1(\lambda) = \frac{\delta}{\delta V(\lambda)} F_1$$

with $\delta/\delta V(\lambda)$ given by Eq.(3.4) and substituting into Eq.(3.7), one gets the following linear equation for $W_1(\lambda)$:

$$\left(\frac{2}{N} W_0(\lambda) + \lambda\right)W_1(\lambda) + \int dx \rho(x) \frac{W_1(\lambda) - W_1(x)}{\lambda - x} + \frac{1}{N} W_0(\lambda, \lambda) = 0,$$

where $W_0(\lambda)$ and $W_0(\lambda, \lambda)$ are given by Eqs.(1.3) and (1.7), respectively.
For an arbitrary $\rho$, the solution of Eq.(5.3) requires tedious calculations. Some simplifications occur if $\rho$ is a symmetric function $\rho(\lambda) = \rho(-\lambda)$. As is discussed above, this corresponds to the reduced hermitian matrix model (i.e. to an even potential $V$). For this case, one gets $b = 0$ and $B_p = 0$ in Eqs.(5.1) and (5.2) which simplifies calculations. Using the property $\rho(\lambda) = \rho(-\lambda)$, Eq.(4.3) can be rewritten as

$$2\frac{N}{N}W_0(\lambda) + \lambda = \sqrt{\lambda^2 + 4c} - \int dx \rho(x) \frac{x}{x^2 - \lambda^2} \sqrt{\lambda^2 + 4c}.$$  (5.6)

By a direct differentiation of this formula one gets

$$\frac{\partial}{\partial c} (2\frac{N}{N}W_0(\lambda) + \lambda) = \frac{2(1 - C_1)}{\sqrt{\lambda^2 + 4c}}.$$  (5.7)

which will be extensively used below. We shall need as well the following rules of differentiation:

$$\frac{\partial C_p}{\partial c} = -C_{p+1}, \quad \frac{\delta c}{\delta V(\lambda)} = \frac{2c}{(\lambda^2 + 4c)^{\frac{3}{2}}(C_1 - 1)},$$

$$\frac{\delta C_1}{\delta V(\lambda)} = \left[ \frac{cC_2}{1 - C_1} - 1 \right] \frac{2}{(\lambda^2 + 4c)^{\frac{3}{2}}} + \frac{12c}{(\lambda^2 + 4c)^{\frac{5}{2}}}.$$  (5.8)

which are easy to derive from Eqs.(5.1), (5.2).

Using the property $\rho(\lambda) = \rho(-\lambda)$, let us represent the second term on the l.h.s. of Eq.(5.5) as the linear operator $K$:

$$KW_1(\lambda) = \int dx \rho(x) K(\lambda, x) W_1(x) \equiv \int dx \rho(x) \frac{\lambda W_1(\lambda) - x W_1(x)}{\lambda^2 - x^2}.$$  (5.9)

In matrix notations, when $\lambda$ and $x$ are replaced by the eigenvalues $\lambda_i$ and $\lambda_j$, this operator becomes an $N \times N$ matrix $K_{kl}$. Such an operator was considered in Ref.[4]. Eq.(5.5) can now be rewritten as

$$(\frac{2N}{N}W_0(\lambda) + \lambda)W_1(\lambda) + \int dx \rho(x) K(\lambda, x) W_1(x) = \frac{1}{N} \frac{c}{(\lambda^2 + 4c)^{\frac{1}{2}}}.$$  (5.10)

It is the equation which is solved below.

The form of the operator $K$ and the fact that $W_1(\lambda) = -W_1(-\lambda)$ suggest the following ansatz

$$W_1(\lambda) = \sum_{n=0}^{\infty} \frac{A_n}{(\lambda^2 + 4c)^{n+\frac{1}{2}}}.$$  (5.11)

where the coefficients $\{A_n\}$ are functions of $\{C_p\}$. It is easy to calculate of how the operator $K$ acts on the ‘basis vectors’ $1/(\lambda^2 + 4c)^{n+\frac{1}{2}}$. One first calculates for $n = 0$ and then obtains a general formula by applying $(\partial/\partial c)^n$. The result reads

$$\int dx \rho(x) K(\lambda, x) \frac{1}{(x^2 + 4c)^{n+\frac{1}{2}}} = \frac{(-1)^n}{2^n(2n - 1)!!} \left( \frac{\partial}{\partial c} \right)^n \left[ 1 - \frac{2W_0(\lambda) + \lambda}{\sqrt{\lambda^2 + 4c}} \right].$$  (5.12)
The term arising from the action of \( \frac{\partial}{\partial c} \) on \( \frac{1}{\sqrt{\lambda^2 + 4c}} \) equals to the first term on the l.h.s. of Eq.(5.10) with the minus sign and cancels it when inserted into Eq.(5.10). The other terms can be easily calculated using Eq.(5.7). Calculating the derivatives, one gets finally

\[
\begin{align*}
(2W_0(\lambda) + \lambda + K) \frac{1}{(\lambda^2 + 4c)^{n+\frac{1}{2}}} &= \frac{1 - C_1}{(\lambda^2 + 4c)^n} - \sum_{p=2}^{n} \frac{C_p}{2^{p-1}(2p-1)!!} \frac{1}{(\lambda^2 + 4c)^{n-p+1}}, \\
(2W_0(\lambda) + \lambda + K) \frac{1}{\sqrt{\lambda^2 + 4c}} &= 1 \quad \text{for} \; n = 0
\end{align*}
\]

(5.13)

where the operator notation for the l.h.s. of Eq.(5.10) has been used. This formula is similar to that by Gross and Newman [4] while the definition of the ‘moments’ \( C_p \) is different for our model.

One sees now that the r.h.s. of Eq.(5.10) is reproduced by the \( n = 2 \) term so that Eq.(5.10) is satisfied if all \( A_n = 0 \) except

\[
A_1 = \frac{1}{N} \frac{C_2c}{6(1 - C_1)^2}, \quad A_2 = \frac{1}{N} \frac{c}{1 - C_1}.
\]

(5.14)

Therefore, we have found the genus one solution to be

\[
W_1(\lambda) = \frac{1}{N} \left\{ \frac{C_2c}{6(1 - C_1)^2 (\lambda^2 + 4c)^{\frac{3}{2}}} + \frac{c}{1 - C_1 (\lambda^2 + 4c)^{\frac{3}{2}}} \right\}.
\]

(5.15)

It is worth noticing that we have obtained the explicit genus one solution to the reduced model with an arbitrary potential \( V(\lambda) = V(-\lambda) \). Previously it was calculated [4] by solving loop equations only for the case of a polynomial potential with the highest power \( \lambda^6 \). We have verified that the result for the quartic potential can be reproduced by our formula (5.15) when \( \rho(\lambda) \) is such that only \( t_2 \) and \( t_4 \) are nonvanishing so that

\[
C_1 = 1 + 2t_2 - 24t_4c, \quad C_2 = 24t_4, \quad C_p = 0 \quad \text{for} \; p \geq 3
\]

and Eq.(5.2) for \( c \) reads

\[
-2t_2c + 12t_4c^2 = \alpha.
\]

(5.17)

The general formulas which express the ‘moments’ \( C_p \) via \( \{t_{2k}\} \) for a polynomial \( V(\lambda) \) look similar.

Using the collection of formulas from Eq.(5.8), one integrates Eq.(5.15) to obtain

\[
F_1 = -\frac{1}{12} \log \{c(1 - C_1)\}.
\]

(5.18)

While our genus one result looks similar to that of Ref.[4, 15], the coefficient in front of the logarithm is now \( \frac{1}{12} \) instead of \( \frac{1}{24} \). As is discussed in the next section this is related to the fact that our matrix model is associated with \( c = 1 \) and not with \( c = 0 \).
6 The relation to c=1

We discuss in this section of how our genus one results support the conjecture of Ref. [6] (which was based on the genus zero calculations) that the model (1.1) with the potential (1.2) is associated with $c = 1$ providing $\alpha$ is identified with the cosmological constant. This model has been called the Kontsevich–Penner model.

Let us start with the case of $\Lambda \to \infty$ when the Kontsevich–Penner model is reduced to the standard Penner model [8] which corresponds [6] to $b = 0$, $c = \alpha$, $B_p = 0$ and $C_p = 0$ for $p \geq 1$ in our notations. This case can be easily recovered by the genus one solution from the previous section. One gets from Eq.(5.18)

$$F_1 = -\frac{1}{12} \log \alpha$$

(6.1)

which exactly reproduces the corresponding result of Distler and Vafa [17] for the Penner model. Notice that we did not take the ‘double scaling limit’ to obtain this result since our model is associated with the continuum case.

The correlators of conformal operators $\mathcal{O}_n$ with the puncture operator $\mathcal{P}$ can be obtained for our model in genus one by applying $d/d\alpha$ to Eq.(5.15) and expanding in $1/\lambda$. The only $C_0$ depends explicitly on $\alpha$ so that our collection of formulas (5.8) should be supplemented by

$$\frac{dc}{d\alpha} = \frac{1}{1-C_1}.$$ (6.2)

By a direct differentiation of Eq.(5.18) one gets

$$\frac{d^2}{d\alpha^2} F_1 = \frac{1}{12(1-C_1)^2} \left\{ \frac{1}{c^2} + \frac{C_2}{c(1-C_1)} + \frac{C_3}{(1-C_1)} + \frac{2C_2^2}{(1-C_1)^2} \right\}$$

(6.3)

which corresponds to genus one contribution to the correlator of the two puncture operators $\mathcal{P}$. One can combine this result with the genus zero calculation of Ref.[6] to give

$$\frac{d^2}{d\alpha^2} F = \log (-C)$$

(6.4)

where $C$ is defined by

$$C = c + \frac{1}{12N^2(1-C_1)^2} \left\{ \frac{1}{c} + \frac{C_2}{(1-C_1)} + \frac{cC_3}{(1-C_1)} + \frac{2cC_2^2}{(1-C_1)^2} \right\}.$$ (6.5)

The meaning of this formula is discussed below.

To calculate the next correlators, one differentiates Eq.(5.15) w.r.t. $\alpha$. The result reads

$$N \frac{d}{d\alpha} W_1(\lambda) = \left\{ \frac{C_2 - cC_3}{(1-C_1)^2} - \frac{2cC_2^2}{(1-C_1)^3} \right\} \frac{1}{6(\lambda^2 + 4c)^2}$$

$$+ \left\{ \frac{1}{(1-C_1)^2} - \frac{2cC_2}{(1-C_1)^3} \right\} \frac{1}{(\lambda^2 + 4c)^2} - \frac{10c}{(1-C_1)^2 (\lambda^2 + 4c)^2}.$$ (6.6)
The explicit expressions for the correlators of $O_{2k}$ with $P$, which can be obtained by expanding the r.h.s. of Eq.(6.6) in $1/\lambda^2$, are rather involved while some simplifications occur after the shift (6.5). As an example we present the sum of the genus zero and genus one results for the correlator of $P$ and $O_2$ (the dilaton operator in the case of the Penner model) which reads

$$\langle O_2 P \rangle \equiv \frac{\partial^2}{\partial t_2 \partial \alpha} F = -2C + \frac{1}{6N^2(1-C_1)^2} \left[ \frac{2C_2}{(1-C_1)} + \frac{1}{c} \right].$$

(6.7)

When expressed in terms of $\{t_{2k}\}$, this formula is to be compared with the corresponding correlator for $c = 1$ CFT.

Let us speculate now on a nonperturbative essence of our results. Eq.(6.4) looks like the equation $\partial^2 F/\partial t^2_0 = u$ which expresses the susceptibility $u$ via $F$ for the Kontsevich model while Eq.(6.3) is an analog of the string equation (to the given order of the genus expansion) which relates $u$ to the ‘times’ $\{t_k\}$. The transformation (1.6) looks like the transformation which has been introduced by Kharchev et al. [18] for the generalized Kontsevich models to relate their partition functions to the (reduced) KP $\tau$-function while differs from it by the fact that the role of the cosmological constant is played now not by $t_0$ but by the ‘conjugate’ (in the sense of Eq.(2.6)) variable $\alpha$. As is shown in Ref.[18], the whole description of $c < 1$ CFT’s can be obtained in this framework and the set of Virasoro and W– constraints can be constructed. Our conjecture indicates, therefore, that the set of the Virasoro constraints (2.2), (2.3) which is constructed with the use of the variable $t_0$ — ‘conjugate’ to the cosmological constant $\alpha$ — can be used alternatively to describe a $c = 1$ CFT.

One more argument in favour of our conjecture is a similarity between the Virasoro constrains (2.2), (2.3) and those for the case of the Witten’s generalization [19] of the intersection theory on moduli space analytically continued to $k = -3$ (or to $p = -1$ in the notations of Ref.[20]) when it is equivalent in a certain sense to the Penner model. It would be very interesting to study the relation between all these approaches.

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