STOCHASTIC HOMOGENIZATION OF CERTAIN NONCONVEX
HAMILTON-JACOBI EQUATIONS

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ABSTRACT. In this paper, we prove the stochastic homogenization of certain nonconvex Hamilton-Jacobi equations. The nonconvex Hamiltonians, which are generally uneven and inseparable, are generated by a sequence of quasiconvex Hamiltonians and a sequence of quasiconcave Hamiltonians through the min-max formula. We provide a monotonicity assumption on the contact values between those stably paired Hamiltonians so as to guarantee the stochastic homogenization.

1. INTRODUCTION

1.1. The problem. Let us consider the following Hamilton-Jacobi equation (HJ) in any space dimension $d \geq 1$.

\[
\begin{align*}
(HJ) \quad \begin{cases}
    u_t + H(Du, x, \omega) = 0 & (x, t) \in \mathbb{R}^d \times (0, \infty) \\
    u(x, 0, \omega) = u_0(x) & x \in \mathbb{R}^d
\end{cases}
\]

where the Hamiltonian $H(p, x, \omega) : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is coercive in $p$, uniformly in $(x, \omega) \in \mathbb{R}^d \times \Omega$. Here $x \in \mathbb{R}^d$ represents the space variable and $\omega \in \Omega$ is the sample point from an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Usually, an assumption of stationary ergodicity is imposed on $(x, \omega)$. We set the initial condition $u_0(x) \in \text{BUC}(\mathbb{R}^d)$, the space of bounded uniformly continuous functions, so that (HJ) has a unique viscosity solution (c.f. [9, 14]). For any $\epsilon > 0$, let $u^\epsilon(x, t, \omega)$ be the viscosity solution of the equation (HJ$^\epsilon$) as follows.

\[
\begin{align*}
(HJ^\epsilon) \quad \begin{cases}
    u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}, \omega) = 0 & (x, t) \in \mathbb{R}^d \times (0, \infty) \\
    u^\epsilon(x, 0, \omega) = u_0(x) & x \in \mathbb{R}^d
\end{cases}
\]

Stochastic homogenization means this: there exists $\Omega_0 \in \mathcal{F}$, with $\mathbb{P}(\Omega_0) = 1$, such that for any $\omega \in \Omega_0$, $\lim_{\epsilon \to 0} u^\epsilon(x, t, \omega) = \pi(x, t)$ locally uniformly in $\mathbb{R}^d \times (0, \infty)$. Moreover, $\pi(x, t)$ is the unique viscosity solution of the homogenized Hamilton-Jacobi equation (HJ), in which $\overline{H}(p)$ is called the effective Hamiltonian.

\[
\begin{align*}
(HJ) \quad \begin{cases}
    \pi_t + \overline{H}(D\pi) = 0 & (x, t) \in \mathbb{R}^d \times (0, \infty) \\
    \pi(x, 0) = u_0(x) & x \in \mathbb{R}^d
\end{cases}
\]

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1.2. Overview.

1.2.1. The literature. The stochastic homogenization for convex Hamilton-Jacobi equations was first established by Souganidis [33] and by Rezakhanlou and Tarver [31], independently. It was extended to spatio-temporal case by Schwab [32] if the Hamiltonian has a super-linear growth in the the gradient variable and by Jing, Souganidis and Tran [20] if \( H(p,x,t,\omega) = a(x,t,\omega)|p| \). When the Hamiltonian is quasiconvex (or level-set convex) in \( p \), the homogenization results were due to Davini and Siconolfi [12] if \( d = 1 \) and Armstrong and Souganidis [4] for general \( d \). Based on a finite range of dependence structure imposed on the random media, the quantitative results were obtained by Armstrong, Cardaliaguet and Souganidis [2]. For the second-order Hamilton-Jacobi equations, the homogenization results were established by Lions and Souganidis [27, 28], by Kosygina, Rezakhanlou and Varadhan [22], by Kosygina and Varadhan [23], by Armstrong and Souganidis [3], and by Armstrong and Tran [5]. Except the (quasi)convexity, the general fine properties of the effective Hamiltonian \( \overline{H}(p) \) is not well-known. In the periodic case, the inverse problem has been investigated by Luo, Tran and Yu [29], by Jing, Tran and Yu [21] and by Tran and Yu [34].

One of the main open questions in this field is whether or not the stochastic homogenization of a genuinely nonconvex Hamilton-Jacobi equation holds. The first positive result has been obtained by Armstrong, Tran and Yu [6], where \( H(p,x,\omega) = (|p|^2 - 1)^2 + V(x,\omega) \). And later, the same authors proved in [7] the homogenization results for the coercive Hamiltonians of the type \( H(p) + V(x,\omega) \) if \( d = 1 \). The author of this paper then justified the general homogenization result for inseparable coercive Hamiltonians \( H(p,x,\omega) \) in one dimension (see [18]). In the random media with a finite range of dependence, Armstrong and Cardaliaguet [1] confirmed the homogenization of Hamiltonians that are positive homogeneous in the gradient variable. This result was extended by Feldman and Souganidis [16] to Hamiltonians with star-shaped sub-level sets. Recently, Qian, Tran and Yu [30] provided a new decomposition method to prove homogenization for some general classes of even separable nonconvex Hamiltonians in multi dimensions, which include the result in [6] as a special case. In the case of second-order nonconvex Hamilton-Jacobi equations, the homogenization result was proved if \( d = 1 \) and the Hamiltonian takes certain special forms. The first result was contributed by Davini and Kosygina [11] when the Hamiltonian is piecewise level-set convex and pinned at junctions. With the help of a probabilistic approach, Kosygina, Yilmaz and Zeitouni [24] (see also Yilmaz and Zeitouni [35]) established the homogenization for a Hamiltonian that takes a ‘W’ shape and the potential function satisfies certain valley-hill assumption.

The failing of homogenization indeed exists for nonconvex Hamiltonians. The first counter example was discovered by Ziliotto [36], in which the distribution of \( H(p,x,\omega) \) correlates distant regions of space. It is not clear if homogenization is still valid when the correlation vanishes at infinity. Later, it was shown by Feldman and Souganidis [16] that the existence of a strict saddle point of the Hamiltonian in gradient variable may result in non-homogenization.
Therefore, in order to prove the homogenization result in a general random media, one should consider a Hamiltonian that is free of strict saddle point. Generally, such Hamiltonians could still be extremely complicated. One typical class of such Hamiltonians are of the rotational type, i.e., \( H(p, x, \omega) = h(|p|, x, \omega) \). It has been conjectured in Qian, Tran and Yu [30] that the homogenization holds for any coercive Hamiltonian of the type \( H(p, x, \omega) = \varphi(|p|) + V(x, \omega) \). On the other hand, the approaches in Armstrong, Tran and Yu [6] and Qian, Tran and Yu [30] only apply to special even and separable Hamiltonians. An existing open question is if the evenness and separability are necessary in stochastic homogenization. Combining all these issues, we conjecture (see the Conjecture 1) that a strict-saddle-point-free Hamiltonian of the form (1.1) has the stochastic homogenization. In particular, this indicates that neither the evenness nor the separability is necessary. This conjecture is clearly more general since an arbitrary rotational Hamiltonian \( H(p, x, \omega) = h(|p|, x, \omega) \) can be approximated by such kind of Hamiltonians. Note that although a special min-max formula has appeared in Qian, Tran and Yu [30], it is still worth investigating the general Hamiltonians of the type (1.1), which has not been considered before. The goal of this article is to prove the Conjecture 1 under a very weak monotonicity condition (M). Moreover, we provide an explicit expression of the effective Hamiltonian.

1.2.2. The difficulties and the key ideas. The classical periodic homogenization was based on the well-posedness of the cell problem (see Lions, Papanicolaou and Varadhan [25], Evans [15] and Ishii [19], etc.). However, in a general stationary ergodic media, Lions and Souganidis [26] showed that the corresponding cell problem may not exist. Instead, one considers a convergence property, i.e., the regularly homogenizability (see Definition 2), of the auxiliary macroscopic problem (2.1). This has been established for the aforementioned Hamiltonians (see [28, 4, 1, 6, 7, 18, 30], etc.). Recently, Cardaliaguet and Souganidis [8] proved the existence of the cell problem for all extreme points of the convex hull of the sublevel sets of the effective Hamiltonian, if it exists. One natural idea in the homogenization of a nonconvex Hamiltonian is to decompose the nonconvex structure into its convex/concave components (see [6, 7, 18, 30]). Since the viscosity solution is in general not classical, we can only talk about its subdifferentials and superdifferentials, which cannot be estimated properly in general. This becomes a major difficulty in operating the decomposition.

To achieve our goal of decomposition, we approximate the auxiliary macroscopic problem of convex/concave Hamiltonians from two sides. More precisely, to the equation (2.1), we find a subsolution (resp. supersolution), such that its superdifferential (resp. subdifferential) has certain lower (resp. upper) bound. This idea enables us to compare the nonconvex Hamiltonian with its convex/concave components effectively. Note that part of this idea has been illustrated in [6, 30] but it does not apply to our situation. The second new ingredient in our setting is the inseparability. Unlike the separable case [6, 7], one can no longer deal with the kinetic energy and the potential energy separately. Instead, we introduce a family of auxiliary potential energy functions associated to each junction level (see...
the section 4) as a replacement. It turns out that the existence of flat regions of the effective Hamiltonian does not depend on their horizontal oscillations (see the Lemma 14). Thus, an appropriate adjustment of certain vertical oscillation suffices to fulfill our needs (see the Lemma 15). Lastly, another difficulty that the previous methods cannot overcome is the unevenness of the Hamiltonian in the gradient variable. Essentially, this issue is related to the non symmetry between the subdifferential and the superdifferential of the viscosity solution. However, the well-known inf-sup formula (c.f. [12, 4], etc.) of the effective Hamiltonian for any quasiconvex Hamilton-Jacobi equation retains a strong symmetry property (see the Lemmas 5, 6). This softly hints the homogenization of an uneven Hamiltonian. The connection between the symmetry of the inf-sup formula and the aforementioned machinery can be built by considering the Hamiltonian

\[ H(p, x, \omega) \]

and its even duality \( H^{de}(p, x, \omega) := H(-p, x, \omega) \), simultaneously. It turns out that in (2.1), the subdifferential (resp. superdifferential) of the solution associated to \( H \) and \( p \) is symmetric to the superdifferential (resp. subdifferential) of the solution associated to \( H^{de} \) and \(-p\). This helps to settle the issue of the unevenness. The assumption (M) plays its role in the inductive step (see the Section 4.2). Basically, it helps to provide certain lower/upper bound of the effective Hamiltonian (see the Lemma 16 and the Lemma 18). We point out that the corresponding results are not valid once the assumption (M) breaks down. These ideas together prove the regularly homogenizability of our Hamiltonian, which brings about the stochastic homogenization (see the Proposition 7) based on a variant of the perturbed test function method (c.f. [15]).

1.3. The assumptions and main results. From now on, we fix an integer \( \ell > 0 \) and let \( \hat{H}_i(p, x, \omega), i = 1, \cdots, \ell, \) be \( \ell \) quasiconvex Hamiltonians. Meanwhile, let \( \hat{H}_i(p, x, \omega), i = 1, \cdots, \ell, \) be \( \ell \) quasiconcave Hamiltonians. We consider a nonconvex Hamiltonian \( H_\ell(p, x, \omega) \) generated through a min-max formula as follows.

\[
H_\ell := \max \left\{ \hat{H}_\ell, \min \left\{ \hat{H}_2, \cdots, \max \left\{ \hat{H}_1, \hat{H}_1 \right\} \right\} \right\} \cdots \]

Based on a min-max identity established in the Lemma 1 and the Corollary 1, and that the max (resp. min) of quasiconvex (resp. quasiconcave) functions is still quasiconvex (resp. quasiconcave), we can assume without loss of generality that

\[
\hat{H}_1 \geq \hat{H}_2 \geq \cdots \geq \hat{H}_\ell \quad \text{and} \quad \hat{H}_1 \leq \hat{H}_2 \leq \cdots \leq \hat{H}_\ell
\]

\[
\begin{tikzpicture}
  \node at (0,0) {Figure 1. An illustration of \( H_2(p, x, \omega) \) under (A4)};
\end{tikzpicture}
\]
The following assumptions (A1) - (A4) are in force throughout the article. Let us denote $\tilde{H}_k$ or $\mathring{H}_k$ by $\hat{H}_k$.

(A1) Stationary Ergodicity. For any $p \in \mathbb{R}^d$ and $1 \leq k \leq \ell$, $\hat{H}_k(p, x, \omega)$ is stationary ergodic in $(x, \omega)$. To be more precise, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a group $\{\tau_z\}_{z \in \mathbb{R}^d}$ of $\mathcal{F}$-measurable, measure-preserving transformations $\tau_z : \Omega \to \Omega$, i.e., for any $y, z \in \mathbb{R}^d$ and any $A \in \mathcal{F}$, we have $\tau_{y+z} = \tau_y \circ \tau_z$ and $\mathbb{P}[\tau_y(A)] = \mathbb{P}[A]$.

Stationary: $\hat{H}_k(p, y, \tau_z \omega) = \hat{H}_k(p, y + z, \omega)$ for all $y, z \in \mathbb{R}^d$ and $\omega \in \Omega$;

Ergodic: if $A \in \mathcal{F}$, $\tau_z(A) = A$ for all $z \in \mathbb{R}^d$, then $\mathbb{P}[A] \in \{0, 1\}$.

(A2) Coercivity. Fix any $1 \leq k \leq \ell$, then
\[
\liminf_{|p| \to \infty} \text{ess inf}_{(x, \omega) \in \mathbb{R}^d \times \Omega} \left| \hat{H}_k(p, x, \omega) \right| = \infty
\]

(A3) Continuity and Boundedness. Fix any $\omega \in \Omega$ and any compact set $K \subset \mathbb{R}^d$, there exists a modulus of continuity $\rho(\cdot) = \rho_{\omega, K}(\cdot)$, such that for $1 \leq k \leq \ell$,
\[
\left| \hat{H}_k(p, x, \omega) - \hat{H}_k(q, y, \omega) \right| \leq \rho(|p - q| + |x - y|), \quad (p, x), (q, y) \in K \times \mathbb{R}^d
\]

And $\hat{H}_k(p, x, \omega)$ is bounded on $K \times \mathbb{R}^d \times \Omega$.

(A4) Stably pairing. For each $(x, \omega)$, $(\hat{H}_i, \hat{H}_j)$, $1 \leq i \leq \ell$, and $(\hat{H}_{j+1}, \hat{H}_j)$, $1 \leq j \leq \ell - 1$, are all stable pairs (c.f. Definition 1).

**Remark 1.** The assumptions (A1) - (A3) are standard setup in stochastic homogenization of Hamilton-Jacobi equations. Meanwhile, as mentioned before, the assumption (A4) is a natural way to exclude any strict saddle point, which could lead to non-homogenization (c.f. [16]).

**Conjecture 1.** Let us fix a positive integer $\ell$ and assume (A1) - (A4), where $H_\ell(p, x, \omega)$ is the Hamiltonian defined in (1.1) - (1.2), then the stochastic homogenization of $H_\ell(p, x, \omega)$ holds.

In this article, our goal is to prove the above conjecture under the following monotonicity condition (M).

**Definition 1** (the stable pair and the contact value). Let $V(p), \Lambda(p) : \mathbb{R}^d \to \mathbb{R}$, such that $V(\cdot)$ is quasiconvex and $\Lambda(\cdot)$ is quasiconcave. Let us denote $\Delta := \{p \in \mathbb{R}^d | \Lambda(p) \geq V(p)\}$. We call both $(V, \Lambda)$ and $(\Lambda, V)$ stable pairs if either (i) or (ii) of the following holds.

(i) $\Delta = \emptyset$;  
(ii) $\Delta \neq \emptyset$, $V|_{\partial \Delta}$ is a constant, and $V|_{\partial \Delta} > V|_{\partial \Delta}$.

If $(V, \Lambda)$ (resp. $(\Lambda, V)$) is a stable pairs, we call $\mathcal{C}_V(\Lambda)$ (resp. $\mathcal{C}_\Lambda(V)$), which is defined as below, the contact value between $V(\cdot)$ and $\Lambda(\cdot)$ (resp. between $\Lambda(\cdot)$ and $V(\cdot)$).

\[
\mathcal{C}_V(\Lambda) := \begin{cases} 
\min V & \text{if } \Delta = \emptyset \\
V|_{\partial \Delta} & \text{if } \Delta \neq \emptyset
\end{cases}, \quad 
\mathcal{C}_\Lambda(V) := \begin{cases} 
\max \Lambda & \text{if } \Delta = \emptyset \\
\Lambda|_{\partial \Delta} & \text{if } \Delta \neq \emptyset
\end{cases}
\]
Notation 1. Based on the Definition 1 and the assumption (A4), let us denote
\[ m_k(x, \omega) := \sup_{(x, \omega) \in \mathbb{R}^d \times \Omega} H_k(x, \omega) \quad \text{and} \quad M_k(x, \omega) := \inf_{(x, \omega) \in \mathbb{R}^d \times \Omega} M_k(x, \omega) \quad \text{for} \quad 1 \leq k \leq \ell \]

Next, for the simplicity of the notations, we set \( \hat{H}_0 := \infty \) and denote
\[ M_k(x, \omega) := \frac{\hat{H}_k(x, \omega)}{\bar{H}_k(x, \omega)} - 1 \quad \text{and} \quad M_k(x, \omega) \quad \text{for} \quad 1 \leq k \leq \ell \]

Let us denote by \((M)\) the monotonicity condition as follows.
\[ m_1 \geq m_2 \geq \cdots \geq m_\ell \quad \text{and} \quad M_1 \leq M_2 \leq \cdots \leq M_\ell \]

Theorem 1. Assume (A1) - (A4) and \((M)\), let \( H_\ell(p, x, \omega) \) be the Hamiltonian defined in (1.1) - (1.2) and let \( u_0 \in \text{BUC}(\mathbb{R}^d) \). For any \( \epsilon > 0 \) and \( \omega \in \Omega \), let \( u_\epsilon(x, t, \omega) \) be the unique viscosity solution of the Hamilton-Jacobi equation.

\[
\begin{cases}
  u_\epsilon' + H_\ell(Du_\epsilon', \frac{x}{\epsilon}, \omega) = 0 & (x, t) \in \mathbb{R}^d \times (0, \infty) \\
  u_\epsilon(x, 0, \omega) = u_0(x) & x \in \mathbb{R}^d 
\end{cases}
\]

Then there exists an effective Hamiltonian \( \overline{H}_\ell(p) \in C(\mathbb{R}^d) \) with \( \lim_{|p| \to \infty} \overline{H}_\ell(p) = \infty \), such that for a.e. \( \omega \in \Omega \), \( \lim_{\epsilon \to 0} u_\epsilon(x, t, \omega) = \overline{u}(x, t) \) locally uniformly in \( \mathbb{R}^d \times (0, \infty) \), where \( \overline{u}(x, t) \) is the unique viscosity solution of the homogenized Hamilton-Jacobi equation.

\[
\begin{cases}
  \overline{u}' + \overline{H}_\ell(D\overline{u}) = 0 & (x, t) \in \mathbb{R}^d \times (0, \infty) \\
  \overline{u}(x, 0) = u_0(x) & x \in \mathbb{R}^d 
\end{cases}
\]

Furthermore, the effective Hamiltonian \( \overline{H}(p) \) is expressed as below.
\[
\overline{H}_\ell = \max \left\{ \frac{\overline{H}_\ell}{\bar{H}_\ell}, \min \left\{ \frac{\overline{H}_\ell}{\bar{H}_\ell}, M_1, \cdots, \max \left\{ \frac{\overline{H}_2}{\bar{H}_2}, m_2, \min \left\{ \frac{\overline{H}_1}{\bar{H}_1}, M_1, \frac{\overline{H}_1}{\bar{H}_1} \right\} \right\} \right\} \cdots \}
\]

where \( \overline{H}_i(p) \) is the effective Hamiltonian of \( \bar{H}_i(p, x, \omega) \).
2. Preliminaries

Let us start with a min-max identity that justifies the generality of the ordering assumption in (1.2).

**Lemma 1.** Fix a positive integer $N$ and real numbers $a_i$, $b_i$, $1 \leq i \leq N$. Denote
\[ \alpha_k := \max_{1 \leq j \leq k} a_j \quad \text{and} \quad \beta_k := \min_{1 \leq j \leq k} b_j \quad 1 \leq k \leq N \]
then $I_N = \Pi_N$, where
\[ I_N := \max \{a_1, \min \{b_1, \max \{b_2, \ldots, \max \{a_N, b_N\}\}\}\} \]
\[ \Pi_N := \max \{\alpha_1, \min \{\beta_1, \max \{\alpha_2, \min \{\beta_2, \ldots, \max \{a_N, \beta_N\}\}\}\}\} \]

**Proof.** Let us first discuss two extreme cases.

**Case 1:** $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{N-1} \geq \beta_N \geq \alpha_N \geq \alpha_{N-1} \geq \cdots \geq \alpha_2 \geq \alpha_1$. This means that $a_i \leq b_j$ for any $i, j$. Then
\[ \Pi_N = \beta_N = \min_{1 \leq i \leq N} b_i = I_N \]

**Case 2:** $\alpha_N \geq \alpha_{N-1} \geq \cdots \geq \alpha_2 \geq \alpha_1 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{N-1} \geq \beta_N$. This means that $a_i > b_1$. Then
\[ \Pi_N = \alpha_1 = a_1 = I_N \]

Next, let us consider other intermediate cases. We denote
\[ k^* := \min \{1 \leq k \leq N \mid \alpha_k > \beta_k\} \]
From the previous discussions, it suffices to consider $2 \leq k^* \leq N$. By the above definition, we have either (i): $a_{k^*} = \alpha_{k^*} > \alpha_{k^*-1}$; or (ii): $a_{k^*} < \alpha_{k^*-1}$ and $b_{k^*} = \beta_{k^*} < \beta_{k^*-1}$.

**Case A:** $\beta_{k^*} < \alpha_{k^*} < \beta_{k^*-1}$. In this case, we have $\Pi_N = a_{k^*}$.
If (i) holds, then, $\Pi_N = a_{k^*}$. On the other hand, we have $a_{k^*} > b_{k^*}$, then
\[ \max \{a_{k^*}, \min \{b_{k^*}, \ldots, \max \{a_N, b_N\}\}\} = a_{k^*} \]
therefore, we have the following relation and get the desired conclusion.
\[ I_N = \max \{a_1, \min \{b_1, \cdots, \max \{a_{k^*-1}, b_{k^*-1}, \max \{a_{k^*-2}, b_{k^*-2}, \max \{a_{k^*-3}, b_{k^*-3}, \cdots, a_1, b_1\}\}\}\} \cdots \} \]
\[ = \max \{a_1, \min \{b_1, \cdots, \max \{a_{k^*-2}, b_{k^*-2}, \max \{a_{k^*-3}, b_{k^*-3}, \cdots, a_1, b_1\}\}\}\} \cdots \}
\[ = \max \{a_1, \min \{b_1, \cdots, \max \{b_{k^*-2}, a_{k^*-2}, \cdots, a_1, b_1\}\}\} \cdots \]
\[ = \cdots = a_{k^*} = \Pi_N \]
If (ii) holds, let us denote
\[ X := \min \{b_{k^*}, \max \{a_{k^*+1}, \cdots, \min \{b_{N-1}, \max \{a_N, b_N\}\}\}\} \leq b_{k^*} = \min_{1 \leq i \leq k^*} b_i \]
By the definition of $k^*$, we have that
\[ a_j \leq \alpha_j \leq a_{k^*+1} \leq \beta_{k^*+1} \leq a_1 \leq b_i, \quad 1 \leq i \leq k^* - 1, \quad 1 \leq j \leq k^* \]
Then
\[
I_N = \max \{ a_1, \min \{ b_1, \cdots \max \{ a_{k^* - 1}, \min \{ b_{k^* - 1}, \max \{ a_{k^*}, X \} \} \} \} \cdots \}
\]
\[
= \max \{ a_1, \min \{ b_1, \cdots \max \{ a_{k^* - 1}, a_{k^*}, X \} \} \cdots \}
\]
\[
= \cdots
\]
\[
= \max \{ a_1, \cdots, a_{k^* - 1}, a_{k^*}, X \}
\]
\[
= \max \{ \alpha_{k^*}, X \} = \alpha_{k^*} = I_N
\]

Case B: \( \alpha_{k^*} > \beta_{k^* - 1} \). In this case, we have \( I_N = \beta_{k^* - 1} \). By the definition of \( k^* \), the case (ii) is vacuum, so we must have (i) holds. Let us denote

\[
Y := \max \{ a_{k^*}, \min \{ b_{k^*}, \cdots, \max \{ a_N, b_N \} \} \} \geq a_{k^*} \geq \max_{1 \leq i \leq k^*} a_i
\]

By the definition of \( k^* \), we have that

\[
b_i > \beta_i > \beta_{k^* - 1} \geq \alpha_{k^* - 1} \geq \alpha_j \geq a_j, \quad 1 \leq i, j \leq k^* - 1
\]

Then

\[
I_N = \max \{ a_1, \min \{ b_1, \cdots, \min \{ b_{k^* - 2}, \max \{ a_{k^* - 1}, \min \{ b_{k^* - 1}, Y \} \} \} \} \cdots \}
\]
\[
= \max \{ a_1, \min \{ b_1, \cdots, \min \{ b_{k^* - 2}, b_{k^* - 1}, Y \} \} \cdots \}
\]
\[
= \cdots
\]
\[
= \min \{ b_1, \cdots, b_{k^* - 2}, b_{k^* - 1}, Y \}
\]
\[
= \min \{ \beta_{k^* - 1}, Y \} = \beta_{k^* - 1} = I_N
\]

\[\square\]

**Corollary 1.** Fix a positive integer \( \ell \), quasiconvex Hamiltonians \( \tilde{H}_i(p, x, \omega) \), \( i = 1, \cdots, \ell \) and quasiconvex Hamiltonians \( \tilde{H}_j(p, x, \omega) \), \( j = 1, \cdots, \ell \). Let us denote

\[\tilde{H}^*_k(p, x, \omega) := \max_{k \leq j \leq \ell} \tilde{H}_j(p, x, \omega) \quad \text{and} \quad \tilde{H}^*_k(p, x, \omega) := \min_{k \leq j \leq \ell} \tilde{H}_j(p, x, \omega)
\]

then

\[
\max \left\{ \tilde{H}_\ell, \min \left\{ \tilde{H}_\ell, \cdots, \max \left\{ \tilde{H}_2, \min \left\{ \tilde{H}_1, \tilde{H}^*_1 \right\} \right\} \cdots \right\} \right\}
\]
\[
= \max \left\{ \tilde{H}^*_\ell, \min \left\{ \tilde{H}^*_\ell, \cdots, \max \left\{ \tilde{H}^*_2, \min \left\{ \tilde{H}^*_1, \tilde{H}^*_1 \right\} \right\} \cdots \right\} \right\}
\]

**Proof.** Fix any \((p, x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega \), let us set the following \( a_i, b_i, 1 \leq i \leq \ell \).

\[
a_i := \tilde{H}_{\ell - i + 1}(p, x, \omega) \quad \text{and} \quad b_i := \tilde{H}_{\ell - i + 1}(p, x, \omega)
\]

The conclusion follows directly from the Lemma 1. \(\square\)

**Definition 2** (Definition 1.1 in [7]). A Hamiltonian \( H(p, x, \omega) \) is called **regularly homogenizable** at \( p_0 \in \mathbb{R}^d \) if there exists a number \( \overline{H}(p_0) \), such that

\[
P \left[ \omega \in \Omega \mid \limsup_{\lambda \to 0} \max_{|x| \leq R} \left| \lambda \nu_{\lambda}(x, p_0, \omega) + \overline{H}(p_0) \right| = 0, \text{ for any } R > 0 \right] = 1
\]

where \( \lambda > 0, \omega \in \Omega \), and \( \nu_{\lambda}(x, p_0, \omega) \) is the unique viscosity solution of the equation:

\[
\frac{\lambda \nu_{\lambda} + H(p_0 + D\nu_{\lambda}, x, \omega)}{\lambda} = 0, \quad x \in \mathbb{R}^d
\]
If \( H(p, x, \omega) \) is regularly homogenizable for all \( p \in \mathbb{R}^d \), then the function \( \overline{H}(p) : \mathbb{R}^d \to \mathbb{R} \) is called the effective Hamiltonian.

**Proposition 1** (Lemma 5.1 in [4]). Under the assumption of (A1), a Hamiltonian \( H(p, x, \omega) \) is regularly homogenizable at \( p_0 \in \mathbb{R}^d \) if and only if there exists a number \( \overline{H}(p_0) \), such that

\[
P \left[ \omega \in \Omega \left| \lim_{\lambda \to 0} \left| \lambda v_\lambda(0, p_0, \omega) + \overline{H}(p_0) \right| = 0 \right. \right] = 1
\]

**Lemma 2.** Fix \( p_0, p_1 \in \mathbb{R}^d \), let \( H^{(0)}(p, x, \omega) \) be a Hamiltonian that is regularly homogenizable at \( p_0 \), then the Hamiltonian \( H^{(i)}(p, x, \omega) := H^{(0)}(p - p_1 + p_0, x, \omega) \) is regularly homogenizable at \( p_1 \). Moreover, \( \overline{H^{(i)}}(p_1) = \overline{H^{(0)}}(p_0) \).

**Proof.** For any \( \lambda > 0, \omega \in \Omega \) and \( i \in \{0, 1\} \), let \( v_\lambda^{(i)} = v_\lambda^{(i)}(x, p_i, \omega) \) be the unique viscosity solution of the following equation.

\[
\lambda v_\lambda^{(i)} + H^{(i)}(p_i + Dv_\lambda^{(i)}, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

The uniqueness of the solution for the above equation implies that \( v_\lambda^{(0)}(x, p_0, \omega) \equiv v_\lambda^{(1)}(x, p_1, \omega) \), therefore,

\[
P \left[ \omega \in \Omega \left| \lim_{\lambda \to 0} \left| \lambda v_\lambda^{(0)}(0, p_0, \omega) + \overline{H^{(0)}}(p_0) \right| = 0 \right. \right] = 1
\]

is equivalent to

\[
P \left[ \omega \in \Omega \left| \lim_{\lambda \to 0} \left| \lambda v_\lambda^{(1)}(0, p_1, \omega) + \overline{H^{(1)}}(p_1) \right| = 0 \right. \right] = 1
\]

with \( \overline{H^{(1)}}(p_1) = \overline{H^{(0)}}(p_0) \). \( \square \)

**Lemma 3.** Let \( H^{(n)}(p, x, \omega), n = 0, 1, \cdots \) be a sequence Hamiltonians, such that

(i) Uniformly coercive.

\[
\lim_{|p| \to \infty} \inf_{(x, \omega) \in \mathbb{R}^d \times \Omega} \min_{n \geq 0} H^{(n)}(p, x, \omega) = +\infty
\]

(ii) Convergence. Fix any \( \omega \in \Omega \) and any \( K \subset \mathbb{R}^d \), then

\[
\lim_{n \to \infty} \|H^{(n)}(p, x, \omega) - H^{(0)}(p, x, \omega)\|_{L^\infty(K \times \mathbb{R}^d)} = 0
\]

(iii) For each \( n \geq 1 \), \( H^{(n)}(p, x, \omega) \) is regularly homogenizable at \( p_n \in \mathbb{R}^d \) with the corresponding effective Hamiltonian \( \overline{H^{(n)}}(p_n) \), and \( \lim_{n \to \infty} p_n = p_0 \).

then \( H^{(0)}(p, x, \omega) \) is regularly homogenizable at \( p_0 \), and \( \overline{H^{(0)}}(p_0) = \lim_{n \to \infty} \overline{H^{(n)}}(p_n) \).

**Proof.** It is a direct imitation of the Lemma 2 in [18]. \( \square \)

**Corollary 2.** Let \( H(p, x, \omega) \) be a Hamiltonian that satisfies (A1) - (A3), then

(i) The set \( \mathcal{R}_H := \{ q \in \mathbb{R}^d | H \text{ is regularly homogenizable at } q \} \) is closed.
(ii) $\overline{H}(p)$ is continuous on $\mathcal{R}_H$.

Proof. Fix any sequence $\{p_n\}_{n \geq 1} \subset \mathcal{R}_H$, such that $\lim_{n \to \infty} p_n = p_0$. An application of the Lemma 3 to $H^{(n)} := H$ and $\{p_n\}_{n \geq 0}$ gives (i) and (ii).

Lemma 4. Fix $\ell \in \mathbb{N}$ and let $H_\ell(p,x,\omega)$ be the Hamiltonian defined in (1.1) - (1.2), we assume (A1) - (A4) and (M), then for any $\epsilon > 0$, there exists a Hamiltonian $H^{(\epsilon)}_\ell(p,x,\omega)$ that satisfies (A1) - (A4) and (M+) as follows. Moreover, $\|H_\ell(p,x,\omega) - H^{(\epsilon)}_\ell(p,x,\omega)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R} \times \Omega)} < \epsilon$.

Proposition 2. Fix a Hamiltonian $H(p,x,\omega)$ that satisfies (A1) - (A3), and let $0 < R, 0 < \lambda < 1, \omega \in \Omega$ and $p_0 \in \mathbb{R}^d$. Suppose $u(x,p_0,\omega)$ and $v(x,p_0,\omega)$ are (viscosity) subsolution and supersolution of the following equation, respectively.

$$\lambda \gamma + H(p_0 + D\gamma, x, \omega) = 0, \quad x \in B_R(0)$$

Moreover, assume that $\max_{x \in B_R(0)} (|\lambda u| + |\lambda v|) \leq C_1$, for some constant $C_1$ which only depends on $p_0$. Then there is a constant $C_2$ which only depends on $p_0$, such that

$$\lambda u(x,p_0,\omega) - \lambda v(x,p_0,\omega) \leq \frac{C_1}{R} \sqrt{|x|^2 + 1} + \frac{C_1 C_2}{R}, \quad x \in B_R(0)$$

Proof. It is a trivial extension of the Lemma 3 in [18].

3. Some properties of the quasiconvex/quasiconcave Hamiltonians

In this section, we present some new propositions regarding stochastic homogenization of Hamiltonian that is either quasiconvex or quasiconcave. As mentioned in the introduction, the stochastic homogenization of such Hamiltonians has been proved by Davini and Siconolfi [12] and by Armstrong and Souganidis [4] (see also [17] a remark by the author). However, a deeper understanding of the (sub- or super-) differentials in the auxiliary macroscopic problem (2.1) is not available. We approximate the solution of the auxiliary macroscopic problem (2.1) by a subsolution/supersolution, with additional constraints on their super/sub-differentials. It turns out that such construction helps in later decomposition of nonconvex Hamiltonians.

Definition 3 (c.f. [9]). Let $f(x) : \mathbb{R}^d \to \mathbb{R}$ be a continuous function, for any $x_0 \in \mathbb{R}^d$, we denote the superdifferential of $f$ at $x_0$ by $D^+ f(x_0)$, to be more precise,

$$D^+ f(x_0) := \left\{ p \in \mathbb{R}^d \mid \limsup_{x \to x_0} \frac{f(x) - f(x_0) - p \cdot (x - x_0)}{|x - x_0|} \leq 0 \right\}$$

Similarly, the subdifferential of $f$ at $x_0$, which is denote as $D^- f(x_0)$, is defined by

$$D^- f(x_0) := \left\{ p \in \mathbb{R}^d \mid \liminf_{x \to x_0} \frac{f(x) - f(x_0) - p \cdot (x - x_0)}{|x - x_0|} \geq 0 \right\}$$
Lemma 5. Let $H^{(1)}(p, x, \omega)$ be a coercive Hamiltonian that is regularly homogenizable for all $p \in \mathbb{R}^d$ with effective Hamiltonian $\overline{H}^{(1)}(p)$. Then $H^{(2)}(p, x, \omega) := -H^{(1)}(-p, x, \omega)$ is also regularly homogenizable for all $p \in \mathbb{R}^d$ with the effective Hamiltonian $\overline{H}^{(2)}(p) = -\overline{H}^{(1)}(-p)$.

**Proof.** Fix any $p \in \mathbb{R}^d$, for any $\lambda > 0$, $\omega \in \Omega$ and $i \in \{1, 2\}$, let $v^{(i)}_\lambda = v^{(i)}_\lambda(x, p, \omega)$ be the unique viscosity solution of the equations as follows.

$$\lambda v^{(i)}_\lambda + H^{(i)}(p + Dv^{(i)}_\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d$$

Let us denote $w_\lambda(x, p, \omega) := -v^{(1)}_\lambda(x, -p, \omega)$, then for any $x \in \mathbb{R}^d$, we have that

$$q^+ \in D^+ w_\lambda(x, p, \omega) \iff -q^+ \in D^- v^{(1)}_\lambda(x, -p, \omega)$$

it means that,

$$\lambda v^{(1)}_\lambda(x, -p, \omega) + H^{(1)}(-p - q^+, x, \omega) \geq 0$$

which is equivalent to that,

$$\lambda w_\lambda(x, p, \omega) + H^{(2)}(p + q^+, x, \omega) \leq 0$$

Similarly, for any $q^- \in D^- w_\lambda(x, p, \omega)$, we have that,

$$\lambda w_\lambda(x, p, \omega) + H^{(2)}(p - q^-, x, \omega) \geq 0$$

Therefore, the following equation holds in the viscosity sense (c.f. [9, 14]).

$$\lambda w_\lambda(x, p, \omega) + H^{(2)}(p + Dw_\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d$$

By the uniqueness of the solution to the above equation, $w_\lambda(x, p, \omega) = v^{(2)}_\lambda(x, p, \omega)$, thus, $\overline{H}^{(2)}(p) = -\overline{H}^{(1)}(-p)$ is an immediate consequence of the following equalities.

$$\lim_{\lambda \to 0} -\lambda v^{(2)}_\lambda(0, p, \omega) = \lim_{\lambda \to 0} -\lambda w_\lambda(0, p, \omega) = -\lim_{\lambda \to 0} -\lambda v^{(1)}_\lambda(0, -p, \omega)$$

□

Lemma 6. Let $H^{(1)}(p, x, \omega)$ be a quasiconvex Hamiltonian that satisfies the assumptions of (A1) - (A3) and let $H^{(2)}(p, x, \omega) := H^{(1)}(-p, x, \omega)$, then $\overline{H}^{(2)}(p) = \overline{H}^{(1)}(-p)$.

**Proof.** Let us denote by $\mathcal{L}$ the set of all globally Lipschitz functions on $\mathbb{R}^d$, and

$$S^+ := \left\{ \phi(x) \in \mathcal{L} \mid \liminf_{|x| \to \infty} \frac{\phi(x)}{|x|} \geq 0 \right\}, \quad S^- := \left\{ \phi(x) \in \mathcal{L} \mid \limsup_{|x| \to \infty} \frac{\phi(x)}{|x|} \leq 0 \right\}$$

The inf-sup formula established in [12, 4] states that,

$$\overline{H}^{(2)}(p) = \inf_{\phi(x) \in S^+} \operatorname{ess sup}_{(x, \omega) \in \mathbb{R}^d \times \Omega} H^{(2)}(p + D\phi(x), x, \omega)$$

$$= \inf_{\phi(x) \in S^-} \operatorname{ess sup}_{(x, \omega) \in \mathbb{R}^d \times \Omega} H^{(2)}(p + D\phi(x), x, \omega)$$

Therefore,
\[
H^{(2)}(p) = \inf_{\phi(x) \in \mathcal{S}^+} \sup_{(x,\omega) \in \mathbb{R}^d \times \Omega} H^{(2)}(p + D\phi(x), x, \omega)
\]
\[
= \inf_{\phi(x) \in \mathcal{S}^+} \sup_{(x,\omega) \in \mathbb{R}^d \times \Omega} H^{(1)}(-p - D\phi(x), x, \omega)
\]
\[
= \inf_{\psi(x) \in \mathcal{S}^+} \sup_{(x,\omega) \in \mathbb{R}^d \times \Omega} H^{(1)}(-p + D\psi(x), x, \omega)
\]
\[
= \overline{H^{(1)}}(-p)
\]
□

3.2. Subsolutions with constraint superdifferentials.

Lemma 7 (quasiconvex Hamiltonian). Under the assumptions of (A1) - (A3), let \( H(p,x,\omega) \) be a quasiconvex Hamiltonian, whose effective Hamiltonian is \( \overline{H}(p) \), and set \( \mu := \min_{p \in \mathbb{R}^d} \overline{H}(p) \). Fix any \( p_0 \in \mathbb{R}^d \), with \( \overline{H}(p_0) > \mu \), then there exists an event \( \tilde{\Omega} \subseteq \Omega \) with \( P(\tilde{\Omega}) = 1 \), a number \( \epsilon_0 = \epsilon_0(p_0) > 0 \), and a statement as follows.

Fix any \( (\epsilon, R, \omega) \in (0,\epsilon_0) \times (0,\infty) \times \tilde{\Omega} \), there exists \( \lambda_0 = \lambda_0(p_0,\epsilon, R, \omega) > 0 \), such that if \( 0 < \lambda < \lambda_0 \), then the following (i) - (iii) hold.

(i) There is a subsolution \( w^{x,R}_\lambda(p_0,\omega) \) of the following equation.
\[
\lambda u + H(p_0 + Du, x, \omega) = 0, \quad x \in B_R(0)
\]

(ii) \( -\lambda w^{x,R}_\lambda(p_0,\omega) \) stays in the \( \epsilon \)-neighborhood of \( \overline{H}(p_0) \), uniformly in \( B_R(0) \).

\[
\max_{|x| \leq R} \left| \lambda w^{x,R}_\lambda(p_0,\omega) + \overline{H}(p_0) \right| < \epsilon
\]

(iii) Fix any \( x \in B_R(0) \), then
\[
p_0 + D^+ w^{x,R}_\lambda(p_0,\omega) \subseteq \left\{ q \in \mathbb{R}^d \mid H(q, x, \omega) > \overline{H}(p_0) - \frac{\epsilon}{3} \right\}
\]

Proof. Let us denote by \( H^{de}(p,x,\omega) \) the dual Hamiltonian of \( H(p,x,\omega) \) in the sense of evenness. To be more precise, \( H^{de}(p,x,\omega) := H(-p,x,\omega) \). The Lemma 6 states that \( H^{de}(-p_0) = \overline{H}(p_0) \). For any \( (\lambda, \omega) \in (0,\infty) \times \Omega \), let \( v^\lambda_\omega(x,-p_0,\omega) \) be the unique viscosity solution of the following equation.
\[
\lambda v^\lambda_\omega(x,-p_0,\omega) + H^{de}(-p_0 + Dv^\lambda_\omega(x,-p_0,\omega), x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

By the regularly homogenizability of \( H^{de}(p,x,\omega) \) at \( -p_0 \), then
\[
P \left[ \omega \in \Omega \mid \limsup_{\lambda \to 0} \max_{|x| \leq R} \left| \lambda v^\lambda_\omega(x,-p_0,\omega) + H^{de}(-p_0) \right| = 0, \text{ for any } R > 0 \right] = 1
\]

Let us first set
\[
\tilde{\Omega} := \left\{ \omega \in \Omega \mid \limsup_{\lambda \to 0} \max_{|x| \leq R} \left| \lambda v^\lambda_\omega(x,-p_0,\omega) + H^{de}(-p_0) \right| = 0, \text{ for any } R > 0 \right\}
\]
Let us also set \( \epsilon_0 := 3 \left( \overline{H}^{de}(-p_0) - \mu \right) > 0 \) and fix \((\epsilon, R, \omega) \in (0, \epsilon_0) \times (0, \infty) \times \tilde{\Omega} \).

Then let us define the constant \( \lambda_0 = \lambda_0(p_0, \epsilon, R, \omega) > 0 \) as below.

\[
\lambda_0 := \sup \left\{ \lambda \in (0, 1) \mid \max_{|x| \leq \frac{3}{2}} |v_{de}^{\lambda}(x, -p_0, \omega) + H^{de}(-p_0)| < \frac{\epsilon}{3} \right\}
\]

Now, for any \( 0 < \lambda < \lambda_0 \), let us pick the function \( w_{\lambda}^{R}(x, p_0, \omega) \) as follows.

\[
w_{\lambda}^{R}(x, p_0, \omega) := -v_{de}^{\lambda}(x, -p_0, \omega) - \frac{6H^{de}(-p_0) + 2\epsilon}{3\lambda}, \quad x \in B_{\frac{3}{2}}(0)
\]

Step 1: (i). By the Lipschitz continuity of \( v_{de}^{\lambda}(\cdot, -p_0, \omega) \) and \( w_{\lambda}^{R}(\cdot, p_0, \omega) \), the relations below hold for a.e. \( x \in B_{\frac{3}{2}}(0) \).

\[
\begin{align*}
\lambda w_{\lambda}^{R}(x, p_0, \omega) + H(p_0) + Dw_{\lambda}^{R}(x, p_0, \omega), x, x), \\
&= \lambda w_{\lambda}^{R}(x, p_0, \omega) + H^{de}(-p_0) + Dw_{de}^{\lambda}(x, -p_0, \omega), x, x), \\
&\leq -2\lambda v_{de}^{\lambda}(x, -p_0, \omega) - 2H^{de}(-p_0) + \frac{2\epsilon}{3}, \\
&\leq 0
\end{align*}
\]

Because \( H(\cdot, x, \omega) \) is quasiconvex, the inequality below holds in the viscosity sense.

\[
\lambda w_{\lambda}^{R}(x, p_0, \omega) + H(p_0) + Dw_{\lambda}^{R}(x, p_0, \omega), x, x), \quad x \in B_{\frac{3}{2}}(0)
\]

Step 2: (ii). Fix any \( x \in B_{\frac{3}{2}}(0) \), then

\[
\left| \lambda w_{\lambda}^{R}(x, p_0, \omega) + \overline{H}(p_0) \right| = \left| v_{de}^{\lambda}(x, -p_0, \omega) + \overline{H}^{de}(-p_0) + \frac{2\epsilon}{3} \right| < \epsilon
\]

Step 3: (iii). Fix any \( x \in B_{\frac{3}{2}}(0) \) and let \( q_0 \in D^{+}w_{\lambda}^{R}(x, p_0, \omega) \), then \( -q_0 \in D^{-}v_{de}^{\lambda}(x, -p_0, \omega) \). Equivalently,

\[
\lambda v_{de}^{\lambda}(x, -p_0, \omega) + H^{de}(-p_0 - q_0, x, x), \quad \lambda v_{de}^{\lambda}(x, -p_0, \omega) + H^{de}(-p_0 - q_0, x, x) \geq 0
\]

Owing to \( 0 < \lambda < \lambda_0 \), (iii) follows naturally from below (recall the Lemma 6).

\[
H(p_0 + q_0, x, x) = H^{de}(-p_0 - q_0, x, x), \quad \geq -\lambda v_{de}^{\lambda}(x, -p_0, \omega) > \overline{H}^{de}(-p_0) - \frac{\epsilon}{3} = \overline{H}(p_0) - \frac{\epsilon}{3}
\]

\[\square\]

**Lemma 8** (quasiconcave Hamiltonian). Under the assumptions of (A1) - (A3), let \( H(p, x, \omega) \) be a quasiconcave Hamiltonian, whose effective Hamiltonian is \( \overline{H}(p) \).

Fix any \( p_0 \in \mathbb{R}^d \), then there exists an event \( \tilde{\Omega} \subseteq \Omega \) with \( \mathbb{P}\left[\tilde{\Omega}\right] = 1 \). Such that for any \( (\epsilon, R, \omega) \in (0, 1) \times (0, \infty) \times \tilde{\Omega} \), there exists \( \lambda_0 = \lambda_0(p_0, \epsilon, R, \omega) > 0 \), and the following holds.

\[
p_0 + D^{+}v_{\lambda}(x, p_0, \omega) \subseteq \left\{ q \in \mathbb{R}^d \mid H(q, x, \omega) > \overline{H}(p_0) - \epsilon \right\}, \quad x \in B_{\frac{3}{2}}(0)
\]
where $0 < \lambda < \lambda_0$, and $v_\lambda(x, p_0, \omega)$ is the unique solution to the following equation.

\begin{equation}
\lambda v_\lambda(x, p_0, \omega) + H(p_0 + Dw_\lambda(x, p_0, \omega), x, \omega) = 0, \quad x \in \mathbb{R}^d
\end{equation}

Proof. Let us fix $p_0 \in \mathbb{R}^d$ and denote that,

\[ G(p, x, \omega) := -H(-p, x, \omega) \quad \text{and} \quad u_\lambda(x, -p_0, \omega) := -v_\lambda(x, p_0, \omega) \]

The Lemma 5 indicates the regularly homogenizability of $H(p, x, \omega)$, moreover,

\[ \lambda w_\lambda(x, -p_0, \omega) + G(-p_0 + Du_\lambda(x, -p_0, \omega), x, \omega) = 0, \quad x \in \mathbb{R}^d \]

Let us select $\tilde{\Omega}$ as follows,

\[ \tilde{\Omega} := \left\{ \omega \in \Omega \left| \limsup_{\lambda \to 0} \max_{|x| \leq \frac{R}{\lambda}} |\lambda v_\lambda(x, p_0, \omega) + \mathcal{P}(p_0)| = 0, \text{ for any } R > 0 \right. \right\} \]

Then for any $(\epsilon, R, \omega) \in (0, 1) \times (0, \infty) \times \tilde{\Omega}$, we define the constant $\lambda_0 = \lambda_0(p_0, \epsilon, R, \omega) > 0$ as below

\[ \lambda_0 := \sup \left\{ \lambda \in (0, 1) \left| \max_{|x| \leq \frac{R}{\lambda}} |\lambda v_\lambda(x, p_0, \omega) + \mathcal{P}(p_0)| < \frac{\epsilon}{3} \right. \right\} \]

Let us also denote that,

\[ G^{de}(p, x, \omega) := G(-p, x, \omega) = -H(p, x, \omega) \]

\[ w_\lambda(x, p_0, \omega) := v_\lambda(x, p_0, \omega) + \frac{6\mathcal{P}(p_0) - 2\epsilon}{3\lambda} \]

Owing to the Lipschitz continuity of $w_\lambda(\cdot, p_0, \omega)$ and $u_\lambda(\cdot, -p_0, \omega)$, for any $0 < \lambda < \lambda_0$ and for a.e. $x \in B_\frac{R}{\lambda}(0)$, we have that

\[ \lambda w_\lambda(x, p_0, \omega) + G^{de}(p_0 + Dw_\lambda(x, p_0, \omega), x, \omega) = \lambda w_\lambda(x, p_0, \omega) + G(-p_0 + Du_\lambda(x, -p_0, \omega), x, \omega) = \lambda w_\lambda(x, p_0, \omega) - \lambda u_\lambda(x, -p_0, \omega) = 2\lambda w_\lambda(x, p_0, \omega) + 2\mathcal{P}(p_0) - \frac{2\epsilon}{3} \leq 0 \]

Quasiconvexity of $G^{de}(\cdot, x, \omega)$ induces the following inequality in viscosity sense.

\[ \lambda w_\lambda(x, p_0, \omega) + G^{de}(p_0 + Dw_\lambda(x, p_0, \omega), x, \omega) \leq 0, \quad x \in B_\frac{R}{\lambda}(0) \]

Then for any $q_0 \in D^+u_\lambda(x, p_0, \omega) = D^+w_\lambda(x, p_0, \omega)$, where $0 < \lambda < \lambda_0$, the desired conclusion can be drawn from the following relations.

\[ H(p_0 + q_0, x, \omega) = -G^{de}(p_0 + q_0, x, \omega) \geq \lambda w_\lambda(x, p_0, \omega) = \lambda v_\lambda(x, p_0, \omega) + 2\mathcal{P}(p_0) - \frac{2\epsilon}{3} > \mathcal{P}(p_0) - \epsilon \]

\[ \Box \]
3.3. Supersolutions with constraint subdifferentials.

**Lemma 9** (quasiconvex Hamiltonian). Under the assumptions of (A1) - (A3), let $H(p, x, \omega)$ be a quasiconvex Hamiltonian, whose effective Hamiltonian is $\overline{H}(p)$. Fix any $p_0 \in \mathbb{R}^d$, then there exists an event $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$. Such that for any $(\epsilon, R, \omega) \in (0, 1) \times (0, \infty) \times \tilde{\Omega}$, there exists $\gamma_0 = \gamma_0(p_0, \epsilon, R, \omega) > 0$, and the following holds.

$$p_0 + D^- v_\lambda(x, p_0, \omega) \subseteq \left\{ q \in \mathbb{R}^d | H(q, x, \omega) < \overline{H}(p_0) + \epsilon \right\}, \quad x \in B_\mathbb{R}(0)$$

where $0 < \lambda < \gamma_0$, and $v_\lambda(x, p_0, \omega)$ is the unique solution to the following equation.

(3.2) $$\lambda v_\lambda + H(p_0 + Dv_\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d$$

**Proof.** Let us fix $p_0 \in \mathbb{R}^d$ and denote

$$G(p, x, \omega) := -H(-p, x, \omega) \quad \text{and} \quad w_\lambda(x, -p_0, \omega) = -v_\lambda(x, p_0, \omega)$$

then as the proof of the Lemma 5,

$$\lambda w_\lambda(x, -p_0, \omega) + G(-p_0 + Dw_\lambda(x, -p_0, \omega), x, \omega) = 0, \quad x \in \mathbb{R}^d$$

Let $q_0 := -p_0$, an application of the Lemma 8 to the Hamiltonian $G(p, x, \omega)$ and the solution $w_\lambda(x, q_0, \omega)$ at $q_0$ implies the statement as follows. There exists an event $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$. Such that for any $(\epsilon, R, \omega) \in (0, 1) \times (0, \infty) \times \tilde{\Omega}$, there exists $\lambda_0 = \lambda_0(q_0, \epsilon, R, \omega) > 0$, such that the following holds for any $0 < \lambda < \lambda_0$.

$$q_0 + D^+ w_\lambda(x, q_0, \omega) \subseteq \left\{ q \in \mathbb{R}^d | G(q, x, \omega) > \overline{G}(q_0) - \epsilon \right\}, \quad x \in B_\mathbb{R}(0)$$

which can be equivalently stated as follows (recall the Lemma 5).

$$p_0 + D^- v_\lambda(x, p_0, \omega) \subseteq \left\{ q \in \mathbb{R}^d | H(q, x, \omega) < \overline{H}(p_0) + \epsilon \right\}$$

The Lemma follows if we keep the same $\tilde{\Omega}$ and pick $\gamma_0(p_0, \epsilon, R, \omega) := \lambda_0(q_0, \epsilon, R, \omega)$. □

**Lemma 10** (quasiconcave Hamiltonian). Under the assumptions of (A1) - (A3), let $H(p, x, \omega)$ be a quasiconcave Hamiltonian, whose effective Hamiltonian is $\overline{H}(p)$, and set $M := \max_{p \in \mathbb{R}^d} \overline{H}(p)$. Fix any $p_0 \in \mathbb{R}^d$, with $\overline{H}(p_0) < M$, then there exists an event $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$, a number $\epsilon_0 = \epsilon_0(p_0) > 0$, and a statement as follows.

Fix any $(\epsilon, R, \omega) \in (0, \epsilon_0) \times (0, \infty) \times \tilde{\Omega}$, there exists $\gamma_0 = \gamma_0(p_0, \epsilon, R, \omega) > 0$, such that if $0 < \lambda < \gamma_0$, then the following (1) - (3) hold.

1. There is a supersolution $m_\lambda^{\epsilon, R}(x, p_0, \omega)$ of the following equation.

$$\lambda u + H(p_0 + Du, x, \omega) = 0, \quad x \in B_\mathbb{R}(0)$$

2. $-\lambda m_\lambda^{\epsilon, R}(x, p_0, \omega)$ stays in the $\epsilon$-neighborhood of $\overline{H}(p_0)$, uniformly in $B_\mathbb{R}(0)$.

$$\max_{|x| \leq \epsilon} \left| \lambda m_\lambda^{\epsilon, R}(x, p_0, \omega) + \overline{H}(p_0) \right| < \epsilon$$
(3) Fix any \( x \in B_{\mathbb{R}}(0) \), then

\[
p_0 + D^- m^c_\lambda(x, p_0, \omega) \subseteq \left\{ q \in \mathbb{R}^d \mid H(q, x, \omega) < \mathcal{P}(p_0) + \frac{\epsilon}{3} \right\}
\]

Proof. Let us denote \( G(p, x, \omega) := -H(-p, x, \omega) \) and \( q_0 := -p_0 \). The Lemma 5 demonstrates that \( \mathcal{G}(q_0) = -\mathcal{P}(p_0) \). An application of the Lemma 7 to \( G(p, x, \omega) \) at \( q_0 \) indicates the existence of \( \Omega \subseteq \Omega \) with \( \mathbb{P}[\Omega] = 1 \), a number \( \epsilon_0 = \epsilon_0(p_0) > 0 \) and a statement as follows. For any \((\epsilon, R, \omega) \in (0, \epsilon_0) \times (0, \infty) \times \Omega\), there exists \( \lambda_0 = \lambda_0(q_0, \epsilon, R, \omega) > 0 \) such that if \( 0 < \lambda < \lambda_0 \), then the following (i) - (iii) hold.

(i) There is a subsolution \( w^{c, R}_\lambda(x, q_0, \omega) \) of the following equation.

\[
\lambda u + G(q_0 + Du, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

(ii) \( -\lambda w^{c, R}_\lambda(x, q_0, \omega) \) stays in the \( \epsilon \)-neighborhood of \( G(q_0) \), uniformly in \( B_{\mathbb{R}}(0) \).

\[
\max_{|x| \leq R} |\lambda w^{c, R}_\lambda(x, q_0, \omega) + G(q_0)| < \epsilon
\]

(iii) Fix any \( x \in B_{\mathbb{R}}(0) \), then

\[
q_0 + D^- w^{c, R}_\lambda(x, q_0, \omega) \subseteq \left\{ q \in \mathbb{R}^d \mid G(q, x, \omega) > \mathcal{G}(q_0) - \frac{\epsilon}{3} \right\}
\]

Let us set \( m^{c, R}_\lambda(x, p_0, \omega) := -w^{c, R}_\lambda(x, q_0, \omega) \) and \( \gamma_0(p_0, \epsilon, R, \omega) := \lambda_0(q_0, \epsilon, R, \omega) \). Then the Lemma 5 shows the equivalence between (i) - (iii) and (1) - (3).\( \square \)

4. The proof of the regularly homogenizability

4.1. The base case. If \( \ell = 1 \), then the monotonicity condition \((M)\) is vacuum. Throughout this subsection, let the Hamiltonian \( H_1(p, x, \omega) \) be defined through (1.1) (1.2). For any \((p, \lambda, \omega) \in \mathbb{R}^d \times (0, \infty) \times \Omega\), let \( v_\lambda(x, p, \omega) \) be the unique solution of the equation as follows.

\[
\lambda v_\lambda + H_1(p + Dv_\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

Similarly, let \( \tilde{v}_\lambda(x, p, \omega) \) be either “or”, be the unique solution to the equation below.

\[
\lambda \tilde{v}_\lambda + \tilde{H}_1(p + D\tilde{v}_\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

Proposition 3. Let \( \ell = 1 \) and the assumptions of \((A1) - (A4)\) be in force, then the Hamiltonian \( H_1(p, x, \omega) \) is regularly homogenizable for any \( p \in \mathbb{R}^d \). Moreover, the effective Hamiltonian \( \overline{H}_1(p) \) is characterized as follows.

\[
\overline{H}_1(p) = \max \left\{ \overline{H}_1(p), \overline{m}_1, \overline{H}_1(p) \right\}
\]

Proof. It follows from the Remark 2 and the Lemmas 11, 12, 13, 14, 15.\( \square \)

Remark 2. By the Lemma 3, the homogenization is stable. We can perturb the Hamiltonians and assume without loss of generality that for any \((x, \omega) \in \mathbb{R}^d \times \Omega\),
the sets \( \{ p \in \mathbb{R}^d | \tilde{H}_1(p, x, \omega) = m_1(x, \omega) \} \), \( ^- \) is either \( ^- \) or \( ^- \), have no interior point. i.e.,

\[
\text{(E)} \quad \text{int}\ \{ p \in \mathbb{R}^d | \tilde{H}_1(p, x, \omega) = m_1(x, \omega) \} = \emptyset
\]

**Lemma 11.** Let \( \ell = 1 \) and the assumptions of (A1) - (A4) be in force, then

\[
\lim_{\lambda \to 0} \inf -\lambda v_\lambda(0, p, \omega) \geq \max \left\{ \overline{H}_1(p), \tilde{H}_1(p), m_1 \right\}
\]

**Proof.** For any \( (p, \lambda, \omega, i) \in \mathbb{R}^d \times (0, \infty) \times \Omega \times \{ 1, 2 \} \), let \( \bar{v}_\lambda(x, p, \omega) \) be from (4.2). Since \( H_1(p, x, \omega) \geq \tilde{H}_1(p, x, \omega) \), the comparison principle indicates that

\[
\lim_{\lambda \to 0} \inf -\lambda v_\lambda(0, p, \omega) \geq \lim_{\lambda \to 0} \inf -\lambda \bar{v}_\lambda(0, p, \omega) = \overline{H}_1(p)
\]

Finally, it is well-known that (see the Lemma 25 in [18] for a similar proof)

\[
\lim_{\lambda \to 0} \inf -\lambda v_\lambda(0, p, \omega) \geq \min_{q \in \mathbb{R}^d} \sup_{(y, \omega) \in \mathbb{R}^d \times \Omega} H_1(q, y, \omega) = m_1, \quad x \in \mathbb{R}^d
\]

**Lemma 12.** Let \( \ell = 1 \) and the assumptions of (A1) - (A4) be in force, then the Hamiltonian \( H_1(p, x, \omega) \) is regularly homogenizable at any \( p \in \left\{ q \in \mathbb{R}^d | \overline{H}_1(q) > m_1 \right\} \).

**Proof.** Fix any \( p_0 \in \mathbb{R}^d \), such that \( \overline{H}_1(p_0) > m_1 \). Let us set \( \epsilon_0 = 3 \left( \overline{H}_1(p_0) - m_1 \right) \) and apply the Lemma 7 to the Hamiltonian \( H_1(p, x, \omega) \) at \( p_0 \) with the above choice of \( \epsilon_0 > 0 \). There exists an event \( \tilde{\Omega} \subseteq \Omega \) with \( \mathbb{P} \left[ \tilde{\Omega} \right] = 1 \). Such that for any \( (\epsilon, R, \omega) \in (0, \epsilon_0) \times (0, \infty) \times \tilde{\Omega} \), there exists \( \lambda_0 = \lambda_0(p_0, \epsilon, R, \omega) > 0 \), such that if \( 0 < \lambda < \lambda_0 \), then the following (i) - (iii) hold.

(i) There is a subsolution \( u^R_\lambda(x, p_0, \omega) \) of the following equation.

\[
\lambda u + H_1(p_0 + Du, x, \omega) = 0, \quad x \in B_{\frac{\epsilon}{3}}(0)
\]

(ii) \( -\lambda u^R_\lambda(x, p_0, \omega) \) stays in the \( \epsilon \)-neighborhood of \( \overline{H}_1(p_0) \), uniformly in \( B_{\frac{\epsilon}{2}}(0) \).

\[
\max_{|x| \leq \frac{\epsilon}{3}} \left| \lambda u^R_\lambda(x, p_0, \omega) + \overline{H}_1(p_0) \right| < \epsilon
\]

(iii) Fix any \( x \in B_{\frac{\epsilon}{2}}(0) \), then

\[
p_0 + D^+ u^R_\lambda(x, p_0, \omega) \leq \left\{ q \in \mathbb{R}^d | \overline{H}_1(q, x, \omega) > \overline{H}_1(p_0) - \frac{\epsilon}{3} \right\}
\]

The choice of \( \lambda_0 \) in the Lemma 7, combined with above (i) - (iii), implies that

\[
\lambda u^R_\lambda + H_1(p_0 + Du^R_\lambda, x, \omega) \leq 0, \quad x \in B_{\frac{\epsilon}{2}}(0)
\]
The Proposition 2 applied to above \( w^{c,R}_\lambda(x, p_0, \omega) \) and \( v_\lambda(x, p_0, \omega) \) in (4.1) shows the existence of a constant \( C = C(p_0) > 0 \), such that

\[
\lambda w^{c,R}_\lambda(0, p_0, \omega) - \lambda v_\lambda(0, p_0, \omega) \leq \frac{C}{R}
\]

then (by recalling (ii))

\[
\limsup_{\lambda \to 0} -\lambda v_\lambda(0, p_0, \omega) \leq \frac{C}{R} + \limsup_{\lambda \to 0} -\lambda w^{c,R}_\lambda(0, p_0, \omega) \leq \overline{H}_1(p_0) + \frac{C}{R} + \epsilon
\]

Let \((\epsilon, R) \to (0, \infty)\), we get \(\limsup_{\lambda \to 0} -\lambda v_\lambda(0, p_0, \omega) \leq \overline{H}_1(p_0)\). The opposite inequality follows from the Lemma 11. Therefore, \(H_1(p, x, \omega)\) is regularly homogenizable at \(p_0\) and \(\overline{H}_1(p_0) = \overline{H}_1(p_0)\). Lastly, the closedness is from the Corollary 2. \(\square\)

**Lemma 13.** Let \(\ell = 1\) and the assumptions of (A1) - (A4) be in force, then the Hamiltonian \(H_1(p, x, \omega)\) is regularly homogenizable at any \(p \in \{ q \in \mathbb{R}^d \mid \overline{H}_1(q) > \overline{m}_1 \}\).

Moreover, \(\overline{H}_1(p) = \overline{H}_1(p)\) for such \(p\).

**Proof.** Fix any \(p_0 \in \mathbb{R}^d\), such that \(\overline{H}_1(p_0) > \overline{m}_1\). Let us apply the Lemma 8 to \(H_1(p, x, \omega)\) and \(\tilde{v}_\lambda(x, p_0, \omega)\) (defined in (4.2)) at \(p_0\). Then there exists an event \(\tilde{\Omega} \subseteq \Omega\) with \(P[\tilde{\Omega}] = 1\). Such that for any \((\epsilon, R, \omega) \in (0, c_0) \times (0, \infty) \times \tilde{\Omega}\), where \(c_0 := \min \{ \overline{H}_1(p_0) - \overline{m}_1, 1 \}\), there exists \(\lambda_0 = \lambda_0(p_0, \epsilon, R, \omega) > 0\), and the following holds (for \(0 < \lambda < \lambda_0\)).

\[
p_0 + D^+ \tilde{v}_\lambda(x, p_0, \omega) \subseteq \{ q \in \mathbb{R}^d \mid \overline{H}_1(q) > \overline{H}_1(p_0) - \epsilon \}, \quad x \in B_{\frac{1}{\lambda}}(0)
\]

which implies that

\[
\lambda \tilde{v}_\lambda(x, p_0, \omega) + H_1(p_0 + D\tilde{v}_\lambda(x, p_0, \omega), x, \omega) \leq 0, \quad x \in B_{\frac{1}{\lambda}}(0)
\]

By applying the Proposition 2 to \(v_\lambda(x, p_0, \omega)\) and \(\tilde{v}_\lambda(x, p_0, \omega)\), we can find some constant \(C = C(p_0) > 0\), such that

\[
\lambda \tilde{v}_\lambda(0, p_0, \omega) - \lambda v_\lambda(0, p_0, \omega) \leq \frac{C}{R}
\]

then

\[
\limsup_{\lambda \to 0} -\lambda v_\lambda(0, p_0, \omega) \leq \frac{C}{R} + \limsup_{\lambda \to 0} -\lambda \tilde{v}_\lambda(0, p_0, \omega) = \frac{C}{R} + \overline{H}_1(p_0)
\]

By sending \(R \to \infty\), we see that \(\limsup_{\lambda \to 0} -\lambda v_\lambda(0, p_0, \omega) \leq \overline{H}_1(p_0)\). The other direction of the inequality comes from the Lemma 11. Hence, we proved the regular homogenizability of \(H_1(p, x, \omega)\) at \(p_0\) and \(\overline{H}_1(p_0) = \overline{H}_1(p_0)\). The Corollary 2 establishes the closedness and then the Lemma is justified. \(\square\)

Next, let us introduce a family of auxiliary Hamiltonians indexed by \(\kappa \in [0, 1]\).

\[
H_1^\kappa(p, x, \omega) := H_1(p, x, \omega) + \kappa (\overline{m}_1 - m_1(x, \omega))
\]
\[ \tilde{H}_1^\kappa(p, x, \omega) := \tilde{H}_1(p, x, \omega) + \kappa (\mathfrak{m}_1 - m_1(x, \omega)), \] 

is either \textquotedblleft or \textquotedblright

For any \((p, \lambda, \omega) \in \mathbb{R}^d \times (0, \infty) \times \Omega\), let \(v_\kappa^\lambda(x, p, \omega)\) be the unique solution of the equation as follows.

\[
(4.3) \quad \lambda v_\kappa^\lambda + H_1^\kappa(p + Dv_\kappa^\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

Similarly, let \(\bar{v}_\kappa^\lambda(x, p, \omega)\) be the unique solution of the equation below.

\[
(4.4) \quad \lambda \bar{v}_\kappa^\lambda + \bar{H}_1^\kappa(p + D\bar{v}_\kappa^\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

**Lemma 14.** Let \(\ell = 1\), \((E)\) and the assumptions of \((A1)\) - \((A3)\) be in force, then

\[
\partial \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) > m_1 \right\} = \partial \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) = m_1 \right\}
\]

**Proof.** The statement \((E)\) is equivalent to that

\[
\text{int} \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p, x, \omega) = m_1 \right\} = \emptyset, \quad \text{is either \textquotedblleft or \textquotedblright}
\]

The Lemma 5 and the Lemma 11 imply that \(\left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) > m_1 \right\} \neq \emptyset\). Moreover, the continuity of \(\bar{H}_1^\kappa(p)\) justifies the equivalence as below.

\[
\partial \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) > m_1 \right\} = \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) = m_1 \right\}, \quad \text{is either \textquotedblleft or \textquotedblright}
\]

To prove the Lemma, we only need to show that

\[
\left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) = m_1 \right\} = \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) = m_1 \right\}
\]

By the Lemma 5, it suffices to prove the following, where \(G_1^\kappa(p, x, \omega) := -\bar{H}_1^\kappa(-p, x, \omega)\).

\[
A := \left\{ p \in \mathbb{R}^d | \bar{H}_1^\kappa(p) = m_1 \right\} = B := \left\{ p \in \mathbb{R}^d | \bar{G}_1^\kappa(-p) = -m_1 \right\}
\]

Based on \([12, 13, 4]\), the set \(A\) is characterized by the maximum subsolutions \(U(\cdot, x, \omega), (x, \omega) \in \mathbb{R}^d \times \Omega\), of the following metric problems.

\[
\begin{align*}
\bar{H}_1^\kappa(Du(y, x, \omega), y, \omega) \leq m_1, & \quad y \in \mathbb{R}^d \setminus \{x\} \\
u(x, x, \omega) = 0
\end{align*}
\]

Similarly, the set \(B\) is fully determined by the maximum subsolutions \(V(\cdot, x, \omega), (x, \omega) \in \mathbb{R}^d \times \Omega\), of the equations below.

\[
\begin{align*}
\bar{G}_1^\kappa(-Dv(y, x, \omega), y, \omega) \leq -m_1, & \quad y \in \mathbb{R}^d \setminus \{x\} \\
v(x, x, \omega) = 0
\end{align*}
\]

It is clear that \(U \equiv V\), as a result of this, \(A = B\). \hfill \Box

**Lemma 15.** Let \(\ell = 1\), \((E)\) and the assumptions of \((A1)\) - \((A3)\) be in force, then

\[
\limsup_{\lambda \to 0} -\lambda v_\kappa^\lambda(0, p, \omega) \leq m_1, \quad p \in \left( \bigcup_{\text{is either \textquotedblleft or \textquotedblright}} \left\{ q \in \mathbb{R}^d | \bar{H}_1^\kappa(q) > m_1 \right\} \right)^c
\]
Proof. Since \( \hat{H}^\kappa_t(\cdot, \cdot, \cdot) \), \( \cdot \) is either \( - \) or \( \cdot \), is continuous and increasing in \( \kappa \in [0, 1] \), so is \( \hat{H}^\kappa_t(\cdot) \). By taking the Lemma 14 into account, it gives that
\[
\mathcal{L} := \bigcup_{\kappa \in [0, 1]} \left\{ q \in \mathbb{R}^d | \hat{H}^\kappa_t(q) > \mathfrak{m}_1 \right\} = \bigcup_{\kappa \in [0, 1]} \left\{ q \in \mathbb{R}^d | \hat{H}^\kappa_t(q) = \mathfrak{m}_1 \right\}
\]
Therefore, for any \( p \in \mathcal{L} \), there exists \( \cdot \), which is either \( - \) or \( \cdot \), and \( \kappa(p) \in [0, 1] \), such that \( \hat{H}^\kappa_t(p) = \mathfrak{m}_1 \), then for \( v_{\lambda}(x, p, \omega) \) and \( \hat{v}_{\lambda}(p, x, \omega) \), \( \lambda > 0 \), from (4.1) and (4.4), respectively, we have
\[
\limsup_{\lambda \to 0} -\lambda v_{\lambda}(0, p, \omega) \leq \limsup_{\lambda \to 0} -\lambda \hat{v}_{\lambda}(0, p, \omega) = \hat{H}^\kappa_t(p) = \mathfrak{m}_1
\]
where we have applied the Lemma 12 and the Lemma 13 to those Hamiltonians with index \( \kappa(p) \).

4.2. The inductive steps. In this section, we prove that if for a particular integer \( \ell_0 \geq 1 \), \( H_{\ell_0}(p, x, \omega) \) defined through (1.1) - (1.2), under the assumptions (A1) - (A4), is regularly homogenizable for all \( p \in \mathbb{R}^d \), then so is \( H_{\ell_0+1}(p, x, \omega) \).

4.2.1. Some preparations. For any \( 0 \leq \ell \leq \ell_0 \), let us denote
\[
H_{\ell+\frac{1}{2}}(p, x, \omega) := \begin{cases} H_1(p, x, \omega) & \ell = 0 \\ \min \left\{ H_{\ell+1}(p, x, \omega), H_{\ell}(p, x, \omega) \right\} & 1 \leq \ell \leq \ell_0 \end{cases}
\]
For any \( (p, \lambda, \omega) \in \mathbb{R}^d \times (0, \infty) \times \Omega \), let \( v_{\lambda,i}(x, p, \omega) \), where \( s \in \{1, \frac{3}{2}, 2, \frac{5}{2}, \cdots, \ell_0, \ell_0 + \frac{1}{2}, \ell_0 + 1\} \), be the unique solution of the equation as follows.
\[
\lambda v_{\lambda,i} + H_s(p + Dv_{\lambda,s}, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]
Similarly, let \( \hat{v}_{\lambda,i}(x, p, \omega) \), where \( \cdot \) is either \( - \) or \( \cdot \) and \( i \in \{1, 2, \cdots, \ell_0, \ell_0 + 1\} \), be the unique solution to the equation below.
\[
\lambda \hat{v}_{\lambda,i} + \hat{H}_s(p + D\hat{v}_{\lambda,s}, x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

4.2.2. The inductive step: from \( \ell_0 \) to \( \ell_0 + \frac{1}{2} \). In this part, let us fix an integer \( \ell_0 > 0 \) and denote that
\[
(I_{\ell_0}^1) \text{ Assumptions of (A1) - (A4) and (M^*)}, \ 1 \leq \ell \leq \ell_0 + 1, \text{ are in force.}
\]
\[
(I_{\ell_0}^2) \text{ } H_{s}(p, x, \omega), \text{ defined through (1.1) - (1.2), with } 1 \leq \ell \leq \ell_0, \text{ is regularly homogenizable for all } p \in \mathbb{R}^d, \text{ such that}
\]
\[
\mathcal{H}_t = \max \left\{ H_t, \mathfrak{m}_1, \min \left\{ H_t, M_{t+\frac{1}{2}}, \cdots, \max \left\{ H_t, \mathfrak{m}_1, \mathcal{H}_1, \right\} \right\} \right\}
\]
Let us combine those into \( (I_{\ell_0}) \) as follows.
\[
(I_{\ell_0}) \quad \text{both } (I_{\ell_0}^1) \text{ and } (I_{\ell_0}^2) \text{ hold}
\]

**Proposition 4.** Assume \( (I_{\ell_0}) \), then \( H_{\ell_0+\frac{1}{2}}(p, x, \omega) \) is also regularly homogenizable for all \( p \in \mathbb{R}^d \). Furthermore, the following equality holds.
\[
H_{\ell_0+\frac{1}{2}}(p) = \min \left\{ H_{\ell_0+1}(p), M_{\ell_0+1}, \mathcal{H}_{\ell_0}(p) \right\}
\]
Lemma 17. Assume (I\(_{\ell_0}\)) with \(\ell_0 = 1\) is basically the result of the Proposition 3. Based on the Definition 2 and the Proposition 1, this proposition follows from the Lemma 16 and the Lemma 17.

**Proof.** Note that the assumption (I\(_{\ell_0}\)) with \(\ell_0 = 1\) is basically the result of the Proposition 3. Based on the Definition 2 and the Proposition 1, this proposition follows from the Lemma 16 and the Lemma 17.

**Lemma 16.** Assume (I\(_{\ell_0}\)), let \(v_{\lambda,\ell_0+\frac{1}{2}}(x, p_0, \omega)\) be from (4.6), we have that

\[
\limsup_{\lambda \to 0} -\lambda v_{\lambda,\ell_0+\frac{1}{2}}(0, p, \omega) \leq \min \left\{ H_{\ell_0+1}(p), M_{\ell_0+1}, \overline{H}_{\ell_0}(p) \right\}
\]

**Proof.** For any \((p, \lambda, \omega) \in \mathbb{R}^d \times (0, \infty) \times \Omega\), let \(v_{\lambda,\ell_0}(x, p, \omega)\) and \(\hat{v}_{\lambda,\ell_0+1}(x, p, \omega)\) be from (4.6) and (4.7), respectively. By (4.5), the comparison principle indicates that

\[
\limsup_{\lambda \to 0} -\lambda v_{\lambda,\ell_0+\frac{1}{2}}(0, p, \omega) \leq \limsup_{\lambda \to 0} -\lambda v_{\lambda,\ell_0}(0, p, \omega) = \overline{H}_{\ell_0}(p)
\]

Finally, by a proof similar to that of the Lemma 25 in [18], we get that

\[
\limsup_{\lambda \to 0} -\lambda v_{\lambda,\ell_0+\frac{1}{2}}(0, p, \omega) \leq \max_{p \in \mathbb{R}^d} \inf_{(y, \omega) \in \mathbb{R}^d \times \Omega} H_{\ell_0+\frac{1}{2}}(p, y, \omega) = M_{\ell_0+1}
\]

**Lemma 17.** Assume (I\(_{\ell_0}\)), let \(v_{\lambda,\ell_0+\frac{1}{2}}(x, p_0, \omega)\) be from (4.6), we have that

\[
\liminf_{\lambda \to 0} -\lambda v_{\lambda,\ell_0+\frac{1}{2}}(0, p, \omega) \geq \min \left\{ H_{\ell_0+1}(p), M_{\ell_0+1}, \overline{H}_{\ell_0}(p) \right\}
\]

**Proof.** Let us denote

\[
\mathcal{H}_{\ell_0+\frac{1}{2}} := \min \left\{ \mathcal{H}_{\ell_0+1}, \max \left\{ \mathcal{H}_{\ell_0}, \ldots, \min \left\{ \mathcal{H}_2, \mathcal{H}_1 \right\} \ldots \right\} \right\}
\]

Let us apply (I\(_{\ell_0}\)) to \(\ell_0\) quasiconcave Hamiltonians \(\{-\mathcal{H}_i(-p, x, \omega)\}_{i=1}^{\ell_0}\) and to \(\ell_0\) quasiconvex Hamiltonians \(\{-\mathcal{H}_j(-p, x, \omega)\}_{j=2}^{\ell_0+1}\). In view of the Lemma 5, we get that

\[
\overline{\mathcal{H}}_{\ell_0+\frac{1}{2}} = \min \left\{ \overline{\mathcal{H}}_{\ell_0+1}, M_{\ell_0+1}, \max \left\{ \overline{\mathcal{H}}_{\ell_0}, m_{\ell_0}, \ldots, \min \left\{ \overline{H}_2, M_2, H_1 \right\} \ldots \right\} \right\}
\]

On the other hand, the following relation

\[
\mathcal{H}_{\ell_0+\frac{1}{2}}(p, x, \omega) = \max \left\{ \mathcal{H}_{\ell_0+\frac{1}{2}}(p, x, \omega), H_1(p, x, \omega) \right\}
\]

implies that

\[
\liminf_{\lambda \to 0} -\lambda v_{\lambda,\ell_0+\frac{1}{2}}(0, p, \omega) \geq \max \left\{ \overline{\mathcal{H}}_{\ell_0+\frac{1}{2}}(p), H_1(p) \right\}
\]

We finish the proof by recalling (M) and by observing the following equation.

\[
\max \left\{ \overline{\mathcal{H}}_{\ell_0+\frac{1}{2}}(p), H_1(p) \right\} = \min \left\{ \overline{\mathcal{H}}_{\ell_0+1}(p), M_{\ell_0+1}, \overline{H}_{\ell_0}(p) \right\}
\]
4.2.3. The inductive step: from \( \ell_0 + \frac{1}{2} \) to \( \ell_0 + 1 \).

**Proposition 5.** Assume (I\(_{\ell_0} \)), then \( H_{\ell_0+1}(p, x, \omega) \) is also regularly homogenizable for all \( p \in \mathbb{R}^d \). Furthermore, the following equality holds.

\[
\overline{H}_{\ell_0+1}(p) = \max \left\{ \overline{H}_{\ell_0+1}(p), \overline{m}_{\ell_0+1}, \overline{H}_{\ell_0+\frac{1}{2}}(p) \right\}
\]

**Proof.** Based on the Definition 2 and the Proposition 1, this proposition follows from the Lemma 18 and the Lemma 19.

\[\square\]

**Lemma 18.** Assume (I\(_{\ell_0} \)), then for \( v_{\lambda, \ell_0+1}(x, p_0, \omega) \) be from (4.6), we have that

\[
\liminf_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+1}(0, p, \omega) \geq \max \left\{ \overline{H}_{\ell_0+1}(p), \overline{m}_{\ell_0+1}, \overline{H}_{\ell_0+\frac{1}{2}}(p) \right\}
\]

**Proof.** For any \((p, \lambda, \omega) \in \mathbb{R}^d \times (0, \lambda) \times \Omega\), let \( v_{\lambda, \ell_0+\frac{1}{2}}(x, p, \omega) \) and \( \hat{v}_{\lambda, \ell_0+1}(x, p, \omega) \) be from (4.6) and (4.7), respectively. Because of the ordering relations \( H_{\ell_0+1}(p, x, \omega) \geq H_{\ell_0+\frac{1}{2}}(p, x, \omega) \) and \( H_{\ell_0+1}(p, x, \omega) \geq H_{\ell_0+1}(p, x, \omega) \), we have that

\[
\lambda v_{\lambda, \ell_0+\frac{1}{2}} + H_{\ell_0+1}(p + Dv_{\lambda, \ell_0+\frac{1}{2}}, x, \omega) \geq 0, \quad x \in \mathbb{R}^d
\]

and

\[
\lambda \hat{v}_{\lambda, \ell_0+1} + H_{\ell_0+1}(p + D\hat{v}_{\lambda, \ell_0+1}, x, \omega) \geq 0, \quad x \in \mathbb{R}^d
\]

By applying comparison principle to the above supersolutions and the solution \( v_{\lambda, \ell_0+1}(x, p, \omega) \), we get that

\[
\liminf_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+1}(0, p, \omega) \geq \liminf_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+\frac{1}{2}}(0, p, \omega) = \overline{H}_{\ell_0+\frac{1}{2}}(p)
\]

\[
\liminf_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+1}(0, p, \omega) \geq \liminf_{\lambda \to 0} -\lambda \hat{v}_{\lambda, \ell_0+1}(0, p, \omega) = \overline{H}_{\ell_0+1}(p)
\]

Finally, we can employ a proof similar to that of the Lemma 25 in [18] to get that

\[
\liminf_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+1}(0, p, \omega) \geq \min_{q \in \mathbb{R}^d} \text{ess sup}_{(y, \omega) \in \mathbb{R}^d \times \Omega} H_{\ell_0+1}(q, y, \omega) = \overline{m}_{\ell_0+1}
\]

\[\square\]

**Lemma 19.** Assume (I\(_{\ell_0} \)), then for \( v_{\lambda, \ell_0+1}(x, p_0, \omega) \) be from (4.6), we have that

\[
\limsup_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+1}(0, p, \omega) \leq \max \left\{ \overline{H}_{\ell_0+1}(p), \overline{m}_{\ell_0+1}, \overline{H}_{\ell_0+\frac{1}{2}}(p) \right\}
\]

**Proof.** Let us denote

\[
\mathcal{H}_{\ell_0+1} := \max \left\{ \hat{H}_{\ell_0+1}, \min \left\{ \hat{H}_{\ell_0+1}, \cdots \max \left\{ \hat{H}_2, \hat{H}_2 \right\} \right\} \right\}
\]

Then, we can apply the inductive assumption (I\(_{\ell_0} \)) to \( \ell_0 \) quasiconcave Hamiltonians \( \left\{ \hat{H}_i(p, x, \omega) \right\}_{i=2}^{\ell_0+1} \) and \( \ell_0 \) quasiconvex Hamiltonian \( \left\{ \hat{H}_j(p, x, \omega) \right\}_{j=2}^{\ell_0+1} \). This induces that

\[
\overline{H}_{\ell_0+1} = \max \left\{ \overline{H}_{\ell_0+1}, \overline{m}_{\ell_0+1}, \min \left\{ \overline{H}_{\ell_0+1}, \overline{M}_{\ell_0+1}, \cdots, \max \left\{ \overline{H}_2, \overline{m}_2, \overline{H}_2 \right\} \right\} \right\}
\]

Let us also denote

\[
\overline{\mathcal{H}}_{\ell_0+1}(p, x, \omega) := \max \left\{ \hat{H}_1(p, x, \omega), \hat{H}_1(p, x, \omega) \right\}
\]
From the Proposition 3, we get that

\[ \overline{H}_{\ell_0+1}(p) = \max \left\{ \overline{H}_1(p), \overline{m}_1, \overline{H}_1(p) \right\} \]

Since we have the ordering relations

\[ H_{\ell_0+1}(p, x, \omega) \leq H_{\ell_0+1}(p, x, \omega) \quad \text{and} \quad H_{\ell_0+1}(p, x, \omega) \leq \overline{H}_{\ell_0+1}(p, x, \omega) \]

By recalling (M), we have that

\[ \lim \sup_{\lambda \to 0} -\lambda v_{\lambda, \ell_0+1}(0, p, \omega) \leq \min \left\{ \overline{H}_{\ell_0+1}(p), \overline{m}_{\ell_0+1}, \overline{\lambda v_{\ell_0+1}}(p) \right\} = \max \left\{ \overline{H}_{\ell_0+1}(p), \overline{m}_{\ell_0+1}, \overline{H}_{\ell_0+1}(p) \right\} \]

\[ \square \]

4.3. A summary.

**Proposition 6.** Fix a positive integer \( \ell \), assume (A1) - (A4) and (M), let \( H_\ell(p, x, \omega) \) be the Hamiltonian defined through (1.1) - (1.2), then \( H_\ell(p, x, \omega) \) is regularly homogenizable for all \( p \in \mathbb{R}^d \). Moreover, the effective Hamiltonian \( \overline{H}_\ell(p) \) has the following expression.

\[ \overline{H}_\ell = \max \left\{ \overline{H}_0, \overline{m}_0, \min \left\{ \overline{H}_1, \overline{M}_1, \cdots, \max \left\{ \overline{H}_2, \overline{M}_2, \min \left\{ \overline{H}_3, \overline{M}_3, \cdots, \right\} \right\} \right\} \]

**Proof.** If \( \ell = 1 \), the result follows from the Proposition 3. Suppose the result holds for a fixed positive integer \( \ell = \ell_0 \), then according to the Proposition 4 and the Proposition 5, the result also holds for \( \ell = \ell_0 + 1 \). By the inductive argument, the result holds for any positive integer \( \ell \). \( \square \)

5. Homogenization

The goal of this section is to establish that if a Hamiltonian \( H(p, x, \omega) \), under the assumptions of (A1) - (A3), is regularly homogenizable for all \( p \in \mathbb{R}^d \), then the stochastic homogenization holds. The proof is based on certain regularity result of Hamilton-Jacobi equation and a variant of the perturbed test function method [15] (see also [12, 3] for similar arguments).

**Lemma 20.** Let \( H(p, x, \omega) : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) be a coercive Hamiltonian that satisfies (A3) and let \( u_0(x) : \mathbb{R}^d \to \mathbb{R} \) be a bounded Lipschitz function. For any \( (\epsilon, \omega) \in (0, 1) \times \Omega \), let \( u^\epsilon(x, t, \omega) \) be the viscosity solution of the equation (HJ) with \( u_0(x) \) as its initial condition. Then for any \( \omega \in \Omega \) and any sequence \( J = \{ \epsilon_j \}_{j=1}^\infty \) with \( \lim_{j \to \infty} \epsilon_j = 0 \), there exists a subsequence \( \{ \epsilon_{j_k} \}_{k=1}^\infty \), such that \( u^{\epsilon_{k}}(x, t, \omega) \), as \( k \to \infty \), converges locally uniformly in \( \mathbb{R}^d \times (0, \infty) \).

**Proof.** Let \( L \) be the Lipschitz constant of \( u_0(x) \), then by the Definition 3, both of \( D^+ u_0(x) \) and \( D^- u_0(x) \), \( x \in \mathbb{R}^d \) are bounded by \( B_L(0) \). According to the assumption (A3), let us denote

\[ K := \text{ess sup}_{(p, x, \omega) \in B_L(0) \times \mathbb{R}^d \times \Omega} |H(p, x, \omega)| \quad \text{and} \quad \tilde{u}^\pm(x, t, \omega) := u_0(x) \pm Kt \]

\[ \square \]
Then $u^+$ (resp. $u^-$) is a supersolution (resp. subsolution) of the equation (HJ'). The usual comparison principle (c.f. [10]) implies that

$$|u^+(x, t, \omega) - u_0(x)| \leq Kt, \quad (x, t, \omega) \in \mathbb{R}^d \times [0, \infty) \times \Omega$$

For any $0 < t_1 < t_2 < \infty$, the usual comparison principle applied to $u^+(x, t + t_2 - t_1, \omega)$ and $u^+(x, t, \omega)$ implies that

$$\sup_{x \in \mathbb{R}^d} |u^+(x, t + t_2 - t_1, \omega) - u^+(x, t, \omega)| \leq K|t_2 - t_1|$$

Therefore, $|u^+(x, t, \omega)| \leq K$. Then, by the coercivity of $H(p, x, \omega)$, let us denote

$$R := \max \left\{ 0 < r < \infty \mid p \in B_r(0), \quad \text{ess inf}_{(x, \omega) \in \mathbb{R}^d \times \Omega} H(p, x, \omega) \leq K \right\}$$

Then we get that $|Du^+(x, t, \omega)| \leq R$, independent of $\epsilon$. So for any $\omega \in \Omega$ and any $T > 0$, $\{u^+(x, t, \omega)\}_{0 \leq t \leq T}$ is uniformly bounded and equicontinuous on $\mathbb{R}^d \times [0, T]$. The conclusion follows from the Arzelà-Ascoli theorem.

**Lemma 21.** Let $H(p, x, \omega) : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a coercive continuous Hamiltonian that is regularly homogenizable for all $p \in \mathbb{R}^d$ with the effective Hamiltonian $\overline{H}(p)$. For any $(\epsilon, \omega) \in (0, 1) \times \Omega$, let $u^\epsilon(x, t, \omega)$ be the viscosity solution of the equation (HJ'). Suppose for a sequence $J = \{\epsilon_j = \epsilon_j(\omega)\}_{j=1}^\infty$ with $\lim_{j \to \infty} \epsilon_j = 0$, such that $\lim_{j \to \infty} u^\epsilon_j(x, t, \omega) = \overline{\pi}(x, t, \omega)$, locally uniformly in $\mathbb{R}^d \times (0, \infty)$. Then $\overline{\pi}(x, t, \omega)$ is the solution of the homogenized Hamilton-Jacobi equation (HJ). In particular, $\overline{\pi}$ is independent of $\omega$.

**Proof.** Let us only prove that $\overline{\pi}$ is a subsolution of (HJ) since the property of being a supersolution can be shown similarly. Suppose on the contrary that $\overline{\pi}$ is not a subsolution at $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$, then there exists a smooth test function $\varphi(x, t)$ in a neighborhood of $(x_0, t_0)$, such that $\overline{\pi}(x, t, \omega) - \varphi(x, t)$ obtains a strict local maximum at $(x_0, t_0)$, but we have that

$$\theta := \varphi_t(x_0, t_0) + \overline{H}(D\varphi(x_0, t_0)) > 0$$

Let us denote $p_0 := \varphi(x_0, t_0)$, for any $\lambda > 0$, let $v_\lambda(x, p_0, \omega)$ be the unique solution of the following equation.

$$\lambda v_\lambda + H(p_0 + Dv_\lambda, x, \omega) = 0, \quad x \in \mathbb{R}^d$$

Based on the regularly homogenizability of $H(p, x, \omega)$ at $p_0$, we denote that

$$\Omega_{p_0} := \left\{ \omega \in \Omega \mid \lim_{\lambda \to 0} \max_{x \in \mathbb{R}^d} |\lambda v_\lambda(x, p_0, \omega) + \overline{H}(p_0)| = 0, \text{ for any } R > 0 \right\}$$

Then $P[\Omega_{p_0}] = 1$. Fix any $\omega \in \Omega_{p_0}$, let us denote by $\phi^\epsilon_j(x, t, \omega)$ the following perturbed test function.

$$\phi^\epsilon_j(x, t, \omega) := \varphi(x, t) + \epsilon_j v_\epsilon_j \left( \frac{x}{\epsilon_j} , p_0, \omega \right)$$
Let us send the effective Hamiltonian has a minimum at $y \in \mathcal{H}(A1) - (A3)$. Assume Proposition 7.

Recall that $\epsilon_j = \epsilon_j(x_0, t_0)$. By the continuity of $\epsilon_j$, there exists a small number $r_0 > 0$ such that for any $(x_1, \epsilon_j) \in B_{r_0}(x_0) \times (t_0 - r_0, t_0 + r_0)$, we have

$$H(p_0 + D\psi(\epsilon_j y_1, t_1) - D\varphi(\epsilon_j y_1, t_1), y_1, \omega) > H(0)$$

and

$$|\epsilon_j v_{\epsilon_j}(x_1, \epsilon_j, p_0, \omega) + \overline{\mathcal{H}}(p_0)| < \frac{\theta}{6}, \quad |\varphi_t(x_0, t_0) - \varphi_t(x_1, t_1)| < \frac{\theta}{6}$$

Recall that $\overline{\mathcal{H}}(p_0) = \theta - \varphi_t(x_0, t_0)$ and $\varphi_t(x_1, t_1) = \phi^{e_j}(x_1, t_1) = \psi_t(x_1, t_1)$

Then (Q) implies (P). Then the comparison principle applied to $u^{e_j}(x, t, \omega)$ and $\phi^{e_j}(x, t, \omega)$ on $B_{r_0}(x_0) \times (t_0 - r_0, t_0 + r_0)$ indicates that

$$\max_{B_{\epsilon_1}(x_0) \times [t_0 - r_0, t_0 + r_0]} (u^{e_j} - \phi^{e_j})(\cdot, \cdot, \omega) = \max_{\partial(B_{\epsilon_1}(x_0) \times [t_0 - r_0, t_0 + r_0])} (u^{e_j} - \phi^{e_j})(\cdot, \cdot, \omega)$$

Let us send $j \to \infty$ and recall the regularly homogenizable, then

$$\max_{B_{\epsilon_1}(x_0) \times [t_0 - r_0, t_0 + r_0]} (\overline{\mathcal{H}}(\cdot, \cdot, \omega) - \varphi(\cdot, \cdot)) = \max_{\partial(B_{\epsilon_1}(x_0) \times [t_0 - r_0, t_0 + r_0])} (\overline{\mathcal{H}}(\cdot, \cdot, \omega) - \varphi(\cdot, \cdot))$$

which contradicts the fact that $\overline{\mathcal{H}}(x, t, \omega) - \varphi(x, t)$ has a strict maximum at $(x_0, t_0)$. Hence,

$$\varphi_t(x_0, t_0) + \overline{\mathcal{H}}(D\varphi(x_0, t_0)) \leq 0$$

Proposition 7. Let $H(p, x, \omega) : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a Hamiltonian that satisfies (A1) - (A3). Assume $H(p, x, \omega)$ is regularly homogenizable for all $p \in \mathbb{R}^d$ with the effective Hamiltonian $\overline{\mathcal{H}}(p)$ (see the Definition 2). Let $u^e(x, t, \omega)$ be the unique viscosity solution of the equation (H.1), then there exists an event $\Omega \subseteq \Omega$, with $P(\Omega) = 1$ and a deterministic function $\overline{\mathcal{H}}(x, t) : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, such that for any $\omega \in \Omega$, $\lim_{t \to 0} u^e(x, t, \omega) = \overline{\mathcal{H}}(x, t)$, locally uniformly in $\mathbb{R}^d \times (0, \infty)$. Moreover, $\overline{\mathcal{H}}(x, t)$ is the unique solution of the homogenized Hamilton-Jacobi equation (H.1).
Proof. For any \((p, \lambda, \omega) \in \mathbb{R}^d \times (0, 1) \times \Omega\), let \(v_\lambda(x, p, \omega)\) be the unique viscosity solution of the following equation.

\[
\lambda v_\lambda(x, p, \omega) + H(p + Dv_\lambda(x, p, \omega), x, \omega) = 0, \quad x \in \mathbb{R}^d
\]

As in the Lemma 21, let us denote

\[
\Omega_p := \left\{ \omega \in \Omega \mid \limsup_{\lambda \to 0} \max_{|z| \leq \frac{R}{\lambda}} |\lambda v_\lambda(x, p, \omega) + \overline{H}(p)| = 0, \text{ for any } R > 0 \right\}
\]

We set \(\tilde{\Omega} := \bigcap_{p \in Q^d} \Omega_p\). Then, by the continuous dependence of \(v_\lambda(x, p, \omega)\) and \(\overline{H}(p)\) on \(p\), we have that for any \((p, \omega) \in \mathbb{R}^d \times \tilde{\Omega}\) that

\[
\limsup_{\lambda \to 0} \max_{|z| \leq \frac{R}{\lambda}} |\lambda v_\lambda(x, p, \omega) + \overline{H}(p)| = 0, \text{ for any } R > 0
\]

Because \(u_0(x) : \mathbb{R}^d \to \mathbb{R}\) is bounded uniformly continuous, there exists a family of bounded Lipschitz functions \(\{u^n_0(x)\}_{n=1}^{\infty}\), such that

\[
\lim_{n \to \infty} u^n_0(x) = u_0(x), \text{ uniformly in } \mathbb{R}^d
\]

For each \(n\), let us denote by \(u^{\epsilon, n}(x, t)\) the solution of the following equation.

\[
(HJ^{\epsilon, n}) \begin{cases} 
u_\epsilon^{\epsilon, n} + H(Du^{\epsilon, n}, \frac{x}{\epsilon}, \omega) = 0 & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u^{\epsilon, n}(x, 0, \omega) = u^n_0(x) & x \in \mathbb{R}^d \end{cases}
\]

By the comparison principle, we get that

\[
\lim_{n \to \infty} u^{\epsilon, n}(x, t, \omega) = u^{\epsilon}(x, t, \omega), \text{ uniformly in } \mathbb{R}^d \times (0, \infty)
\]

According to the Lemma 20, for any sequence \(\{\epsilon_j\}_{j=1}^{\infty}\) with \(\lim_{j \to \infty} \epsilon_j = 0\), there exists a subsequence \(\{\epsilon_{j_k}\}_{k=1}^{\infty}\), such that

\[
\lim_{k \to \infty} u^{\epsilon_{j_k}, n}(x, t, \omega) = \overline{u}^n(x, t, \omega), \text{ locally uniformly in } \mathbb{R}^d \times (0, \infty)
\]

By the Lemma 21, for any \(\omega \in \tilde{\Omega}\), \(\overline{u}^n(x, t, \omega)\) is the unique viscosity solution of the equation \((HJ)\) with the initial condition replaced by \(u^n_0\). i.e., \(\overline{u}^n(x, t, \omega)\) is independent of \(\omega \in \tilde{\Omega}\), let us denote it by \(\overline{u}^n(x, t)\). By the uniqueness of \(\overline{u}^n(x, t)\),

\[
\lim_{\epsilon \to 0} u^{\epsilon, n}(x, t, \omega) = \overline{u}^n(x, t), \text{ locally uniformly in } \mathbb{R}^d \times (0, \infty)
\]

Next, we apply the comparison principle to \(\overline{u}^n(x, t)\) and \(\overline{u}(x, t)\) and get that

\[
\lim_{n \to \infty} \overline{u}^n(x, t) = \overline{u}(x, t), \text{ uniformly in } \mathbb{R}^d \times (0, \infty)
\]
Finally, for any $\omega \in \tilde{\Omega}$, the following locally uniformly convergence holds in $\mathbb{R}^d \times \mathbb{R}^+$. 

$$\lim_{\epsilon \to 0} |u^\epsilon(x, t, \omega) - \overline{u}(x, t)| \leq \lim_{n \to \infty} \lim_{\epsilon \to 0} (|u^\epsilon(x, t, \omega) - u^n(x, t, \omega)| + |u^n(x, t, \omega) - \overline{u}(x, t)| + |\overline{u}(x, t) - \overline{u}(x, t)|)$$

$$\leq \lim_{n \to \infty} \lim_{\epsilon \to 0} (|u_0(x) - u_0^n(x)| + |u^n(x, t, \omega) - \overline{u}(x, t)| + |u_0(x) - u_0^n(x)|)$$

$$= 0$$

$\square$

Proof of the Theorem 1. It follows from the Proposition 6 and the Proposition 7. $\square$

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