Generating function matrix for random walks on a simple ladder

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Abstract. We consider some basic properties of random walks on a simple ladder including the first and the second moments, the probability of returning to the starting site, the probability of ever reaching a given site, the conditional mean first-passage time to a given site and the expected number of distinct sites visited. These basic properties provide us a great deal of information about mobility, diffusivity and exploration of the random walker. We study these properties by using two different approaches, i.e., the Roerdink and Shuler’s approach and the direct generating function approach. Most of the results are identical to those for one-dimensional lattice except for a renormalization of coefficients.

1. Introduction
Transport phenomena in many physical, chemical and biological processes have been studied by random walk (RW) models on various kinds of structures [1,2]. The effect of dimensionality on the recurrence of periodic space lattice was considered by Polya [3] and the probabilities of returning to the starting site of regular lattices were accurately calculated by Montroll E W and Weiss G H [4]. RWs on hexagonal lattice with trapping centers were studied by Montroll E W [5] and some properties of RWs on the generalized Sierpinski gasket were discussed by Balakrishnan V [6]. More recently, the effect of the absence of loops or RWs on complex trees was studied by Baronchelli A, Catanzaro M, and Romualdo P S [7]. In addition, non-Markovian RWs on the so-called comb structures were considered by Méndez V, Iomin A, Campos D, and Horsthemke W [8]. It is thus evident that the problems of RWs on structures of different topologies are still open and challenging.

Even though the problems of random walks on different kinds of lattices have been constantly studied since 1905, there are not many works discussing RWs on a simple ladder. The study of these models may be relevant to many transport processes in nature. To a certain extent, it is related to the study of diffusion of two-state particles, such as a protein recognizing its DNA target through a combination of one- and three-dimensional diffusion [9], and a molecule diffusing through the chromatographic bed with the mobile and stationary phases [10]. In addition, it is possible to apply works on these models to other models on generalized ladders, e.g., ladders with more sites on rungs between the rails. This generalization has some intimate connections with comb structures [8].

In this paper, we consider some basic properties of RWs on a simple ladder including the first and the second moments, the probability of returning to the starting site, the probability of ever reaching a given site, the conditional mean first-passage time to a given site and the expected number of distinct sites visited. These basic properties provide us a great deal of information about mobility, diffusivity and exploration of the random walker. We study these properties by two different approaches, i.e., the...
Roerdink and Shuler’s approach [11] and the direct generating function approach (see [12,13] for an overview).

The organization of this paper is as follows. Section 2 is devoted to give readers the definition of a simple ladder and the RW properties which we are interested in. A brief introduction of the probability generating function formalism is also included. Section 3 shows the application of the Roerdink and Shuler’s approach to the problem of RWs on the ladder. In Section 4, the direct generating function approach is employed. In Section 5, we present a summary and discussion of this paper.

2. The model and the generating function formalism

As mentioned in the introduction, we are interested in RWs on a simple ladder. A simple ladder is two-dimensional structure as shown in figure 1. The structure has two-side rails defined as + rail (lower) and – rail (upper). Let \((x, \pm)\) denote the position of the walker being at site \(x\) of \(\pm\) rail. The walker located at site \((x, +)\) may jump to the right, left, or up neighbouring sites with probabilities \(\alpha_+, \gamma_+, \beta_+\) respectively. Similarly, the walker located at site \((x, –)\) may jump to the right, left, or down neighbouring sites with probabilities \(\alpha_-, \gamma_-, \beta_-\) respectively.

![Figure 1. A simple ladder structure consisting of two parallel rails connected to each other with the arrows showing possible transitions.](image)

We devote this section to discuss a general formalism which relates our considered properties to a single function, the probability generating function vector \(P(\xi)\) defined below. A basic quantity of interest is \(P_{ab}(x)\), the probability that the walker is at site \(x\) after \(n\) steps, having started from \((0, b)\). The evolution of the walker is described by the Chapman-Kolmogorov equation for Markovian RWs,

\[
P_{ab}(x) = \delta_{a,b,0} \delta_{a,b,0} + \sum_{x'} T_{ac}(x-x') P_{cb}(x')_{a-1}
\]

where \(T_{ac}(x-x')\) is the transition probability from site \((x', c)\) to site \((x, a)\). We have assumed here that the walk is translationally invariant. For convenience, we may write this relation in the matrix form

\[
P_b(x) = \delta_{a,b,0} U + \sum_{x'} T(x-x') P_b(x')_{a-1}
\]

where \(P_b(x, \xi)[P_{ab}(x) P_{ba}(x)]\), \(T(x-x')\equiv \begin{bmatrix} T_{+}(x-x') & T_{-}(x-x') \\ T_{-}(x-x') & T_{+}(x-x') \end{bmatrix}\), \(U\equiv[u_+ u_-]^T\) corresponds to the initial state. For this model, we can write the transition probability matrix in the form

\[
T(x) = \begin{bmatrix}
\alpha_+ \delta_{1,x} + \gamma_+ \delta_{-1,x} & \beta_+ \delta_{0,x} \\
\beta_- \delta_{0,x} & \alpha_- \delta_{1,x} + \gamma_- \delta_{-1,x}
\end{bmatrix}.
\]
The derivation of the considered properties is facilitated by introducing the probability generating function vector $P_b(x; \xi) = \sum_{n=0}^{\infty} P_n(x) \xi^n$ where $|\xi| < 1$. It can be shown that

$$P_b(x; \xi) = \frac{1}{2\pi} \int \tilde{P}_b(k; \xi) e^{-ikx} dk$$

(3)

where the moment generating function vector $\tilde{P}_b(k; \xi) = [\tilde{P}_{xb}(k; \xi) \quad \tilde{P}_{xb}(k; \xi)]^T$ is given by

$$\tilde{P}_b(k; \xi) = \sum_{n} e^{\xi k} P_n(x; \xi) = \Lambda(k; \xi) U$$

(4)

with

$$\Lambda(k; \xi) = \left(1 - \xi \tilde{T}(k) \right)^{-1}$$

(5)

and $\tilde{T}(k) = \sum_{x} T(x) e^{ikx}$.

The generating functions for the probability of returning to the starting site after $n$ steps, $P_{ab}(0; a)$, the first moment, $\langle x \rangle_{b,a} = \sum_{a,b} x P_{ab}(x)$, and the second moment, $\langle x^2 \rangle_{b,a} = \sum_{a,b} x^2 P_{ab}(x)$ may be written in terms of the moment generating functions $\tilde{P}_{ab}(k; \xi)$, as follows [12,13]: the generating function of the probability of returning to the starting site after $n$ steps is

$$P_{ab}(0; \xi) = \sum_{n=0}^{\infty} P_{ab}(0) \xi^n = \frac{1}{2\pi} \int \tilde{P}_{xb}(k; \xi) dk ;$$

(6)

The generating function of the first moment is

$$\langle x \rangle_{b} = \sum_{n=0}^{\infty} \langle x \rangle_{b,a} \xi^n = \frac{1}{i} \frac{\partial}{\partial k} \sum_{n=0}^{\infty} \tilde{P}_{xb}(k; \xi) \bigg|_{k=0} \right.$$

(7)

The generating function of second moment is

$$\langle x^2 \rangle_{b} = \sum_{n=0}^{\infty} \langle x^2 \rangle_{b,a} \xi^n = \frac{1}{i^2} \frac{\partial^2}{\partial k^2} \sum_{n=0}^{\infty} \tilde{P}_{xb}(k; \xi) \bigg|_{k=0} \right.$$ \n
(8)

To discuss the remaining properties, we need the concept of the probability $F_{ab}(x)$ of visiting for the first time at $(x,a)$ after $n$ steps, having commenced from $(0,b)$. The probability $P_{ab}(x)$ and $F_{ab}(x)$ are connected by the following relation,

$$F_{ab}(x) = \delta_{a,0} \delta_{b,0} \delta_{x,0} + \sum_{m=1}^{n} P_{ab}(0) \cdot F_{ab}(x) \bigg|_{m=0} \right.$$

(9)

The idea is: the probability of the walker being at site $(x,a)$ after $n$ steps is equal to the summation of all probabilities that the walker being at that site for the first time after walking $m$ steps and, the rest of its walk, it can go anywhere on the structure but eventually it must be at that site. We turn equation (9) into the equation for the generating function, $P_{ab}(x; \xi) = \delta_{a,0} \delta_{b,0} \delta_{x,0} + P_{ab}(0; \xi) F_{ab}(x; \xi)$ which gives

$$F_{ab}(x; \xi) = P_{ab}^{-1}(0; \xi) \left( P_{ab}(x; \xi) - \delta_{a,0} \delta_{b,0} \delta_{x,0} \right).$$

(10)

If we define $(A)_{b,m}$ as the number of new sites visited at $m^{th}$ step given the walker started at site $(0,b)$, then the number of distinct sites visited in the first $n$ steps is just the summation of new sites visited at all steps, $(S)_{b,n} = \sum_{j=0}^{n} (A)_{b,j}$. It is easier to analyse $\langle S \rangle_{b,n}$, the expected number of distinct sites visited
after $n$ steps. From the fact that $\langle \Delta \rangle_{b,n} = \sum_{x,y} F_{ab}(x)_{n}$ [13] and the average of the sum of random variables is the sum of the average of random variables, we can show that

$$\langle S' \rangle_b (\xi) = \sum_{n=0}^{\infty} \langle S' \rangle_{b,n} \xi^n = \frac{1}{1-\xi} \left( \sum_{x,a} P_{ab}(x;\xi) \right)$$

(11)

where $|\xi|<1$. From equation (3) and equation (10), we can obtain the conditional mean first-passage time and the probability of ever reaching a given site (see Section 4).

Just before we move on, it is important to remark that asymptotic behaviours of statistical properties of RWs are governed by the singular part $\tilde{P}_{ab}(k;\xi)$ of $P_{ab}(k;\xi)$. In this work, we often implicitly use this fact.

3. The Roerdink and Shuler’s approach

The problem of RWs on the ladder may be considered as the problem of two-state RWs on one-dimensional lattice in which the walker may jump to the neighbouring sites or may change its current state. If the walker is on the upper rail, we can say that it is in “the upper state”. If the walker is on the lower rail, it is in “the lower state”. In 1985, Roerdink and Shuler proposed a general approach to study the asymptotic properties of multistate RWs. Their approach is based on determining the embedded Markov chain of the model which is resulted from the projection of the RWs on the set of possible states of the model. They found that if the embedded Markov chain has equilibrium occupation probabilities, the asymptotic behaviours of certain properties of RWs on the whole structure are determined by the properties of the embedded Markov chain.

In this part, we roughly sketch Roerdink and Shuler’s procedure to derive the expression for their most central quantity $[\Lambda(k;\xi)]_{1,1}$, from which other quantities may be derived. They first introduced a similarity transformation $H$, which diagonalizes $\tilde{T} = \sum_{x=0}^{\infty} T(x)$, the transition probability matrix of the embedded Markov chain. With this transformation, equation (5) becomes

$$\Lambda(k;\xi) = \left(1 - \xi \tilde{T}(k)\right)^{-1}$$

and $\tilde{\Lambda}(k) = \tilde{T} + ikM - \frac{1}{2}k^2S + O(k^3)$ where $M = \sum_{x=0}^{\infty} xT(x)$ is the mean displacement per step matrix and $S = \sum_{x=0}^{\infty} x^2T(x)$ is the mean-square displacement per step matrix. Here the notation $\Lambda$ denotes a matrix similar to the matrix $A$ via the matrix $H$. They found that if the embedded Markov chain has equilibrium occupation probabilities, the singular part $\tilde{P}_{ab}(k;\xi)$ of $P_{ab}(k;\xi)$ comes solely from the entry $[\Lambda(k;\xi)]_{1,1}$, i.e., $\tilde{P}_{ab}(k;\xi) = \pi_a \Lambda(k;\xi)_{1,1}$ where $\pi_a = [H]_{1,1}$. From this, they obtained their central relation

$$\left[\Lambda(k;\xi)\right]_{1,1} - \left[1 - \xi \Lambda(k;\xi)_{1,1} - \frac{1}{2}k^2S\right]^{-1}$$

(12)

where $m = [M]_{1,1}$ and $s = [S]_{1,1}$.

Applying Roerdink and Shuler’s procedure to our model, we obtain the following expressions:

$$\tilde{T} = \begin{bmatrix} \alpha_s + \gamma & \beta_s \\ \beta_s & \alpha_s + \gamma \end{bmatrix}; \quad M = \begin{bmatrix} \alpha_s - \gamma & 0 \\ 0 & \alpha_s - \gamma \end{bmatrix}; \quad S = \begin{bmatrix} \alpha_s + \gamma & 0 \\ 0 & \alpha_s + \gamma \end{bmatrix};$$
\[
H = \begin{bmatrix}
\pi_+ & -\pi_-
\pi_- & \pi_+
\end{bmatrix};
H^{-1} = \begin{bmatrix}
1 & 1
-1 & \beta
\end{bmatrix}
\]
where \( \pi_s = \frac{\beta_+}{\beta_+ + \beta_-} \) is the equilibrium occupation probabilities of the embedded Markov chain. From equations (7), (8), (12) and Tauberian theorem [14], it can be shown that \( \langle x \rangle_{h,n} - mn \) and \( \langle x^2 \rangle_{h,n} - (\langle x \rangle_{h,n}^2 + sn) \). With this, the asymptotic behaviours of the first and the second moments are given by \( \langle x \rangle_{h,n} - (\pi_+ (\alpha_+ - \gamma_+) + \pi_- (\alpha_- - \gamma_-))n \) and \( \langle x^2 \rangle_{h,n} - (\langle x \rangle_{h,n}^2 + 2Dn) \) where \( D = \alpha_+ \pi_+ + \alpha_- \pi_- \) respectively.

For unbiased RWs defined by the conditions, \( \alpha_+ = \gamma_+ \) and \( \alpha_- = \gamma_- \), using equations (3), (11), (12) and Tauberian theorem, we can show that the asymptotic behaviours of the probability of returning to the starting point after \( n \) steps and the expected number of distinct sites visited are \( P_{ab}(0)_n \sim \pi_b(\sqrt{\pi Dn})^{-1} \) and \( \langle S \rangle_{h,n} - 8\sqrt{Dn\pi^{-1}} \) respectively.

4. The direct generating function approach
In this section, we straightforwardly calculate the generating functions of the model. From equations (2) and (4), we can express the moment generating function vector as
\[
\tilde{P}_s(k; \xi) = \frac{1}{(1 - \xi L_s(k))(1 - \xi L_s(k)) - \xi^2 \beta_+ \beta_-} \begin{bmatrix}
1 - \xi L_s(k) \\
1 - \xi L_s(k)
\end{bmatrix} \begin{bmatrix}
\xi \beta_- \\
\xi \beta_+
\end{bmatrix} u_s + \sum_{k=0}^{\infty} \frac{dL_s(k)}{dk} \left(1 - \xi \right)^{-1} u_s
\]
where \( L_s(k) \equiv \alpha_+ e^{ik} + \gamma_+ e^{-ik} \) and \( L_s(k) \equiv \alpha_- e^{ik} + \gamma_- e^{-ik} \). Using equations (7), (8), (13), we obtain
\[
\langle x \rangle_b(\xi) - \left( \sum_{u=0}^{\infty} \pi_u \frac{dL_u(k)}{dk} \right) \left(1 - \xi \right)^{-1} \text{ and } \langle x^2 \rangle_b(\xi) - \left( \sum_{u=0}^{\infty} \pi_u \frac{d^2L_u(k)}{dk^2} \right) \left(1 - \xi \right)^{-1} \text{.}
\]
The asymptotic behaviours of the first and the second moments can then be obtained by applying Tauberian theorem, \( \langle x \rangle_{h,n} - (\pi_+ (\alpha_+ - \gamma_+) + \pi_- (\alpha_- - \gamma_-))n \) and \( \langle x^2 \rangle_{h,n} - (\langle x \rangle_{h,n}^2 + 2Dn) \) respectively. The remaining properties are directly related to the probability generating function \( P_{ab}(x; \xi) \). Thus, if equation (3) can be integrated analytically, we can straightforwardly obtain the remaining properties. Unfortunately, only some specific cases are possible, i.e., the case of unbiased RWs and the case of symmetric RWs defined by the conditions, \( \alpha_+ = \alpha_- \) and \( \gamma_+ = \gamma_- \). and \( \beta_+ = \beta_- \).

The explicit forms of the probability generating functions for unbiased RWs are
\[
P_{\pm+}(x; \xi) = \frac{\alpha_+}{K} \left( -\left(2b_+ - d_+ \right) + \frac{A_{\pm+}}{\sqrt{b_+^2 - 1}} \right),
\]
\[
P_{\pm-}(x; \xi) = \frac{\beta_-}{K} \left( \frac{A_{\pm-}}{\sqrt{b_-^2 - 1}} \right)
\]
where \( A_\pm = b_\pm - \sqrt{b_\pm^2 - 1} \), \( b_\pm = ((\alpha_+ + \alpha_-) \pm K)(4\xi \alpha_+ \alpha_-)^{-1} \), \( K = \sqrt{\left(\alpha_+ + \alpha_-\right)^2 - 4(1 - \xi^2 \beta_+ \beta_-) \alpha_+ \alpha_-} \) and \( d_\pm = \frac{1}{\xi \alpha_\pm} \). The asymptotic behaviours of generating functions of probability of returning to the starting site and the expected number of distinct sites visited can be obtained from equations (14) and
(11), \[ P_{\pm}(0;\xi) = \pi_{\pm}\left(\sqrt{D(1-\xi)}\right)^{-1} \] and \[ \langle S \rangle_{\pm}(\xi) = \frac{4\sqrt{D}}{(1-\xi)^{3/2}} \] respectively. Again, we can determine the asymptotic behaviours of the corresponding sequences from applying Taubarian theorem to them, \[ P_{ab}(0;\xi) = \pi_{a}\left(\sqrt{Dn}\right)^{-1} \] and \[ \langle S \rangle_{ab}(\xi) = 8\sqrt{Dn\pi^{-1}} \] respectively. With explicit form of the probability generating function, we can derive some other properties, not obtained by the Roerdink and Shuler’s approach. The probability that site \((x,a)\) is ever reached by a walker started at \((0,b)\), \[ R_{ab}(x) \], is equal to the summation over all steps of the probability of visiting to site for the first time, \[ R_{ab}(x) = \sum_{n=0}^{\infty} F_{ab}(x)n = \lim_{\xi^{\pm\rightarrow 0}} P_{ab}^{-1}(0;\xi) \left( P_{ab}(x;\xi) - \delta_{a,b}\delta_{x,0} \right). \] The point \(\xi = 1\) is the singular point of \( P_{\pm}(0;\xi) \) so from equation (16), \( R_{\pm}(0) = 1 \). From the recurrence theorem [12] and accessibility of every site on the ladder, \( R_{\pm}(0) \) is unity implies that \( R_{ab}(x) \) is also unity for every site on the structure. In other words, for unbiased RWs on the ladder, the event that every site is ever reached by a walker is certain. Even though \( R_{ab}(x) \) is very important to all RW problems, it gives us merely a crude picture of the behaviour of the RWs [12]. The more informative statistical property is the statistics of the step number on which site \((x,a)\) is first visited, that is, the first-passage time. When \( R_{ab}(x) < 1 \), the mean first-passage time is infinite in some cases. To avoid this difficulty, we ought to consider here the conditional mean first-passage time, \[ \tau_{ab}(x) \equiv \sum_{n=0}^{\infty} n F_{ab}(x)n_\xi^{\pm\rightarrow 0} = \lim_{\xi^{\pm\rightarrow 0}} \frac{\partial}{\partial \xi} \left( P_{ab}^{-1}(0;\xi) \left( P_{ab}(x;\xi) - \delta_{a,b}\delta_{x,0} \right) \right). \] From equations (14), (15), (16) and (17), we can show that \( \tau_{ab}(x) \) is infinite for every site \((x,a)\) on the structure. Together with \( R_{\pm}(0) = 1 \), this means that the walker is certain to be at \((x,a)\) but we must wait for long time generally.

Now, let us move to symmetric RWs. The explicit forms of the probability generating functions are
\begin{align*}
P_{\pm}(x;\xi) = \frac{1}{2\xi} \left\{ \frac{(B_+)^{\gamma}}{\sqrt{\xi^{-1} - \beta}} - \frac{(B_-)^{\gamma}}{\sqrt{\xi^{-1} + \beta}} \right\} \times \begin{cases} \alpha^{\left|\gamma\right|} & x \leq 0 \\ \gamma^{\left|\gamma\right|} & x > 0 \end{cases} \end{align*}
where \( B_{\pm} = \frac{1}{2} \left( \xi^{-1} \pm \beta \right) - \sqrt{\left(\xi^{-1} \pm \beta\right)^2 - 4\gamma} \). From the symmetry, it does not matter which site the walker started so let us omit the second subscript (the starting rail). Since the case of symmetric unbiased RWs is included in the case of unbiased RWs, for the remaining of consideration in this section, let us consider only symmetric biased RWs (\( \alpha \neq \gamma \)). From equation (18), we can show that the asymptotic limit of the generating function probability of returning to the starting site does not follow a power law and approaches a specific constant governed by the probabilities of jumping, \[ \lim_{\xi^{\gamma\rightarrow 0}} P_{\pm}(0;\xi) = \frac{1}{2} \left\{ \left|\alpha - \gamma\right| + \left(\left|\alpha - \gamma\right| + 4\beta\right)^{1/2} \right\} ^{1/2}. \] From equation (19), we cannot obtain the asymptotic behaviour of the corresponding sequence by using Tauberian theorem. Anyway, equation (19) still plays a significant role in the model as follows. From equation (11), the generating function of the expected number of distinct sites visited for symmetric
biased RWs is \( \langle S \rangle_b(\xi) = \frac{2}{(1-\xi)^2} P_{\xi}(0;\xi) \) and, after using Tauberian theorem, the corresponding sequence is \( \langle S \rangle_{b,n} - 4 \left( |\alpha - \gamma|^{-1} + \left( |\alpha - \gamma|^2 + 4\beta \right)^{1/2} \right)^n \). From equations (16) and (18), we can show that

\[
R_x(0) = 1 - \left( \frac{1}{2} \right) \left( |\alpha - \gamma|^{-1} + \left( |\alpha - \gamma|^2 + 4\beta \right)^{1/2} \right)^n \quad \text{and} \quad R_x(x \neq 0) = 1 + |\alpha - \gamma| \left( |\alpha - \gamma|^2 + 4\beta \right)^{1/2} \]

when the direction of \( x \) is the same as the direction of bias otherwise, \( R_x(x \neq 0) \equiv 0 \) where \( |x| \gg 0 \) and \( \beta \neq 0 \). Similarly, from equations (17) and (18), we can show that \( \tau_x(x) \approx |x| |\alpha - \gamma|^{-1} \) when the direction of \( x \) is the same as the direction of bias otherwise, \( \tau_x(x) \approx \infty \) when the direction of \( x \) is the same as the direction of bias otherwise, \( \beta \neq 0 \).

5. Discussion and conclusion

We have considered some basic properties of RWs on a simple ladder. Most of the properties are identical to those for one-dimensional lattice except for a renormalization of coefficients and the probability of ever reaching a given site for symmetric biased RWs. For unbiased RWs on the ladder, the diffusion coefficient can be varied by the weighted average, \( D = \alpha, \pi + \alpha, \pi \), while, for RWs on one-dimensional lattice, the diffusion coefficient is fixed to one-half. The asymptotic behaviours of various other properties of RWs are governed by this diffusion coefficient. For instance, the probability of returning to the starting site after \( n \) steps, \( P_{\pi}(0; \pi) \approx 2 \pi \left( \sqrt{4\pi Dn} \right)^{-1} \) and the expected number of distinct sites visited after \( n \) steps, \( \langle S \rangle_{b,n} \approx 8\sqrt{Dn\pi} \). From these results, roughly speaking, if the diffusion coefficient of the walker increases, it is more likely to find new sites but less likely to returning to the starting site. Consideration for the probability of ever reaching a given site \( (x, a) \) and the conditional mean first-passage time leads us to the paradox, which also occurs in RWs on one-dimensional lattice, that the walker is certain to be at every site on the structure but we must wait for long time generally. For symmetric biased RWs, we found that the walker moves with “the constant speed”, \( |\alpha - \gamma| \), in the direction of bias.

On one hand, even though the Roerdink and K. E. Shuler’s approach provides us a general approach to study the asymptotic properties of RWs on a lattice built up by periodically repeated unit cells, its application is limited to certain properties. On the other hand, although the direct generating function approach provides us the more informative tools to obtain various statistical properties of RWs, the derivation of the probability generating functions is more involved. For RWs on the ladder, we can analytically express them only in the cases of unbiased RWs and symmetric RWs because the full analysis is involved with the factorization of fourth degree complex polynomial function.

Last but not least, it is possible to apply works on RWs on the ladder to RWs on generalized ladders, e.g., ladders with more sites on rungs between the rails. This generalization has some intimate connections with widely used comb structures.

Acknowledgements

We acknowledge support from the 100th Anniversary Chulalongkorn University Fund for Doctoral Scholarship.
References

[1] Montroll E W and M F Shlesinger 1984 Nonequilibrium Phenomena II : From Stochastics to Hydrodynamics (Amsterdam: Elsevier Science)

[2] Oliver C IBE 2013 Elements of Random Walk and Diffusion Processes (New Jersey: John Wiley & Sons)

[3] Polya G 1921 Math. Ann. 84 149

[4] Montroll E W and Weiss G H 1965 J. Math. Phys. 6 167

[5] Montroll E W 1969 J. Math. Phys. 10 753

[6] Balakrishnan V 1995 Mater. Sci. Eng. B 32 201

[7] Baronchelli A, Catanzaro M and Romualdo P S 2008 Phys. Rev. E 78 011114

[8] Méndez V, Iomin A, Campos D and Horsthemke W 2015 Phys. Rev. E 92 062112

[9] Krepel D and Levy Y 2016 J. Phys. A: Math. Theor. 49 494003

[10] Felinger A 2008 J. Chromatogr. A 1184 20

[11] Roerdink J B T M and Shuler K E 1985 J. Stat. Phys. 40 205

[12] Hughes B D 1995 Random Walks and Random Environments vol 1 (Oxford: Clarendon Press)

[13] Klafter J and Sokolov I M 2011 First Steps in Random Walks: From Tools to Applications (New York: Oxford University Press)

[14] Lawler G F and Limic V 2010 Random Walk: A Modern Introduction vol 123 (Cambridge: Cambridge University Press)