DEFORMATION THEORY OF HOLOMORPHIC VECTOR BUNDLES
EXTENDED CONFORMAL SYMMETRY
AND EXTENSIONS OF 2D GRAVITY

by

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Abstract

Developing on the ideas of R. Stora and coworkers, a formulation of two dimensional field theory endowed with extended conformal symmetry is given, which is based on deformation theory of holomorphic and Hermitian spaces. The geometric background consists of a vector bundle $E$ over a closed surface $\Sigma$ endowed with a holomorphic structure and a Hermitian structure subordinated to it. The symmetry group is the semidirect product of the automorphism group $\text{Aut}(E)$ of $E$ and the extended Weyl group $\text{Weyl}(E)$ of $E$ and acts on the holomorphic and Hermitian structures. The extended Weyl anomaly can be shifted into an automorphism chirally split anomaly by adding to the action a local counterterm, as in ordinary conformal field theory. The dependence on the scale of the metric on the fiber of $E$ is encoded in the Donaldson action, a vector bundle generalization of the Liouville action. The Weyl and automorphism anomaly split into two contributions corresponding respectively to the determinant and projectivization of $E$. The determinant part induces an effective ordinary Weyl or diffeomorphism anomaly and the induced central charge can be computed. As an application, it is shown that to any $A_1$ embedding $t$ into a simple Lie algebra $\mathfrak{g}$ and any representation $R$ of $\mathfrak{g}$ one can naturally associate a flat vector bundle $DS(t, R)$ on $\Sigma$. It is further shown that there is a deformation of the holomorphic structure of such bundle whose parameter fields are generalized Beltrami differentials of the type appearing in light cone $W$ geometry and that the projective part of the automorphism anomaly reduces to the standard $W$ anomaly in the large central charge limit. A connection between the Donaldson action and Toda field theory is also observed.
1. Introduction.

During the last few years a large body of literature has been devoted to the study of non critical string theory and 2D quantum gravity. Soon after the original discovery of a SL(2, R) symmetry in ordinary induced 2D gravity [1-2], it was found that 2D gravity emerged from a constrained gauged SL(2, R) WZWN model [3-4]. These results suggested that other models of induced 2D gravity may arise from 2D field theories endowed with an extended conformal symmetry (see ref. [5] for a comprehensive and updated review). Such trend has resulted in a wide study of $W$ strings [6-8] and $W$ gravity [9-13].

A sound formulation of $W$ gravity requires a prior understanding of the underlying geometry and its symmetries and a systematic study of anomalies [14-16]. While considerable progress has been toward a satisfactory formulation of $W$ geometry both in the light cone gauge [17-20] and in the covariant gauge [21-23], there still are a number of field theoretic issues which call for further investigation, such as locality, the proper generalization of the holomorphic factorization theorem [24-26] and quantization and gauge fixing of the geometrical fields.

In this paper, I shall attempt a formulation of a two dimensional field theory endowed with an extended conformal symmetry having in mind applications to extended 2D gravity. The approach adopted, initiated by R. Stora and coworkers [26], is based on deformation theory of holomorphic and Hermitian structures of complex spaces and is summarized below.

In ordinary conformal field theory, the geometric background consists of a world sheet $\Sigma$ endowed with a holomorphic structure $a$. Conformal fields are sections of tensor powers of the $a$–holomorphic canonical 1–cocycle $k$ of $\Sigma$. Conformal invariance consists precisely in the independence of physics from coordinate choices in $a$. The holomorphic structure $a$ may be ‘deformed’ yielding a new holomorphic structure $a'$. Deformations are parametrized in one-to-one fashion by the Beltrami field $\mu$. The classical action $S_{cl}$ thus depends on $\mu$ and $\bar{\mu}$. The response of the system under deformation is given by the holomorphic energy-momentum tensor $T_{cl} = \delta S_{cl}/\delta \mu$ and its complex conjugate.

The consistent quantization of a conformal field theory uses the $\zeta$ function scheme, which requires the introduction of a metric $h$ on $\Sigma$ that is Hermitian with respect the holomorphic structure $a$. Metrics can be parametrized in terms of the Liouville field $\phi$. This introduces a geometric degree of freedom which is extraneous to the conformal geometry of the classical theory. In the quantum theory, one may deform both $a$ and $h$. and so the effective action $I$ will depend explicitly on $\mu$, $\bar{\mu}$ and $\phi$. $\mu$, $\bar{\mu}$ and $\phi$ couple to the quantum holomorphic energy momentum tensor $T_{qu}$ and its complex conjugate and to the trace of the energy momentum tensor $T_{trace} = \delta I/\delta \phi$, respectively.
The symmetry group of ordinary conformal field theory consists in the semidirect product of the diffeomorphism group $\text{Diff}(\Sigma)$ of $\Sigma$ and the Weyl group $\text{Weyl}(\Sigma)$ of $\Sigma$. The group acts naturally on the holomorphic structures and on the Hermitian metrics subordinated or, equivalently, on $\mu$ and $\phi$. The $\zeta$ function scheme provides an diffeomorphism invariant but Weyl anomalous renormalization of the effective action $I$. The Weyl anomaly can be shifted into a diffeomorphism chirally split anomaly by adding to $I$ a local counterterm depending on $\mu$, $\bar{\mu}$ and $\phi$ [27-28]. The modified effective action is independent from $\phi$ and thus depends only on the background conformal geometry as in the classical case. See sects. 2, 3 and 4 for a brief review of the above topics.

The formulation proposed of extended conformal field theory parallels as much as possible that of ordinary conformal field theory just outlined. $\Sigma$ is extended by attaching at each point an internal vector space. Hence, the geometric background consists of a vector bundle $E$ endowed with a holomorphic structure $\mathbf{A}$. Since $\mathbf{A}$ induces a holomorphic structure $\mathfrak{a}$ on the base $\Sigma$, the holomorphic geometry of $\Sigma$ is contained in that of $E$. The extended conformal fields are conformal fields carrying internal degrees of freedom corresponding to the fiber of $E$. Such fields are sections of an $\mathfrak{a}$–holomorphic 1–cocycle of the form $k^{\otimes m} \otimes L$, where $L$ is the $\mathfrak{a}$–holomorphic matrix 1–cocycle associated to trivialization changes in $\mathbf{A}$. Extended conformal invariance consists in the independence from trivializations choices in $\mathbf{A}$. The holomorphic structure $\mathbf{A}$ can be deformed into a new holomorphic structure $\mathbf{A}'$. The deformations are parametrized in one-to-one fashion by the Beltrami field $\mu$ and a $(0, 1)$ gauge field $A^*$. The classical action $S_{\text{cl}}$ thus depends on $\mu$, $\bar{\mu}$, $A^*$ and $A^{*\dagger}$. The response of the system under deformation is given by the holomorphic energy momentum tensor $T_{\text{cl}}$ and its complex conjugate and by a holomorphic current $J_{\text{cl}} = \delta S_{\text{cl}}/\delta A^*$ and its Hermitian conjugate.

The consistent quantization of an extended conformal field theory uses the $\zeta$ function scheme, as in the standard case. This requires the introduction of a Hermitian structure $(h^\circ, H)$ on $E$ subordinated to $\mathbf{A}$, where $h^\circ$ is a Hermitian metric of $\Sigma$ with respect to $\mathfrak{a}$ and $h$ is a Hermitian metric on the fibers of $E$. Metrics can be parametrized in terms of the Liouville field $\phi^\circ$ and the Donaldson field $\Phi$. This introduces geometric degrees of freedom extraneous to the extended conformal geometry of the classical theory. In the quantum theory, one may deform both $\mathbf{A}$ and $(h^\circ, H)$ and so $I$ will depend explicitly on both $\mu$, $A^*$, $\phi^\circ$ and $\Phi$. $\mu$, $\bar{\mu}$ and $\phi^\circ$ couple to the quantum holomorphic energy momentum tensor $T_{\text{qu}}$ and its complex conjugate and to the trace energy momentum tensor $T^\circ_{\text{trace}}$. $A^*$, $A^{*\dagger}$ and $\Phi$ couple similarly to the quantum current $J_{\text{qu}}$ and its Hermitian conjugate and to a further current $J_{\text{an}}$.

In extended conformal field theory, the symmetry group is properly the semidirect product of the automorphism group $\text{Aut}(E)$ of $E$ and the extended Weyl group $\text{Weyl}(E)$
of $E$. The symmetry group acts on the holomorphic and Hermitian structures and hence on the geometrical fields $\mu$, $A^*$, $\phi^o$ and $\Phi$. The $\zeta$ function scheme provides an automorphism invariant but extended Weyl anomalous renormalization of the effective action of extended conformal field theory. The extended Weyl anomaly can be shifted into an automorphism chirally split anomaly by adding to the action a local counterterm depending on $\mu$, $\bar{\mu}$, $A^*$, $A^*$, $\phi^o$ and $\Phi$ [26]. It can be seen that both the Weyl and automorphism anomaly split into two contributions corresponding respectively to the determinant and projectivization of $E$. The determinant part induces an effective ordinary Weyl or diffeomorphism anomaly and the induced central charge can be computed. See sects. 5, 6, 7 and 8 for a discussion of these topics.

The dependence on the Donaldson field $\Phi$ is encoded in the Donaldson action [29], a vector bundle generalization of the Liouville action formally similar to a non compact gauged WZWN action in which the WZ field is $\exp \Phi$ and the gauge fields are $A^*$ and $A^*\dagger$. I believe that the Donaldson action may play an important role in the formulation of models of 2D quantum gravity a la David-Distler-Kaway [30-31]. Integrating the gauge fields may also yield effective actions for the Donaldson field $\Phi$ with black hole properties [32].

To make contact with extended gravity, it is necessary to consider a particular choice of the bundle $E$ and to restrict oneself to special deformations of its holomorphic structure and special Hermitian structures. The construction relies on $A_1$ embeddings into simple Lie algebras, in the same spirit as refs. [33-34]. To any $A_1$ embedding $t$ into a simple Lie algebra $g$ and any representation $R$ of $g$, there is naturally associated a flat unstable holomorphic vector bundle $DS(t, R)$ on any given Riemann surface. $E$ is the tensor product of a tensor power of the canonical line bundle and $DS(t, R)$. There are deformations of the holomorphic structure of $E$ whose parameter fields are the Beltrami differential $\mu$ and generalized Beltrami differentials $\nu_\eta$ of the type as those appearing in light cone $W$ geometry, one for each of the representations of $A_1$ contained in the adjoint representation of $g$. The field space formed by the Beltrami differential $\mu$ and the generalized Beltrami differentials $\nu_\eta$ possesses a $W$ symmetry. The symmetry is non local, but its action on the parameter fields translates into an action on the geometrical fields $\mu$ and $A^*$ via automorphisms. In this sense, the non local $W$ symmetry is embedded into a local one. Further, the projective part of the automorphism anomaly reduces to the standard $W$ anomaly in the large central charge limit. There is also a class of Hermitian structures of $E$ parametrized by the Liouville field $\phi^o$ and by Toda fields $\phi_\alpha$, one for each of the simple roots of $g$. When expressed in terms of such fields, the Donaldson action assumes a form closely related to Toda field theory. See sect. 9, for an illustration of these applications.
2. Holomorphic and Hermitian geometry of higher genus surfaces.

In this section, I shall provide a brief account of the basic notions of holomorphic and Hermitian geometry of a surface (see refs. [35-36] for background material). The geometrical framework illustrated below will be the starting point for the generalizations of sect. 5, and the results summarized here will be repeatedly invoked in later sections.

Let \( \Sigma \) be a connected compact smooth oriented surface of genus \( \ell \geq 2 \). A holomorphic structure \( a \) on \( \Sigma \) is a maximal atlas of local coordinates \( \{ z_a \} \) with holomorphic coordinate changes contained in the oriented differentiable structure of \( \Sigma \). The \( a \)-holomorphic canonical 1–cocycle \( k \) of \( \Sigma \) is defined by \( k_{ab} = \partial_a z_b \) for \( \text{Dom} \, z_a \cap \text{Dom} \, z_b \neq \emptyset \), where \( \partial_a = \partial/\partial z_a \).

In conformal field theory, one compares several holomorphic structures \( a, a', a'' \) etc. 2. Chosen a reference holomorphic structure \( a \), any other holomorphic structure \( a' \) can be viewed as obtained by a continuous deformation of \( a \). Since \( a \) and \( a' \) belong to the same oriented differentiable structure of \( \Sigma \)

\[
\partial_a z_a' \partial \bar{a}_a z_a' - \partial_a \bar{z}_a' \partial \bar{a}_a \bar{z}_a' > 0. \quad (2.1)
\]

Each deformation \( a' \) is characterized by a field \( \lambda = \lambda(a') \) locally given by

\[
\lambda_{aa'} = \partial_a z_a', \quad (2.2)
\]

for any two overlapping coordinates \( z_a \) and \( z_a' \) from \( a \) and \( a' \) [37]. \( \lambda \) belongs to \( S(\Sigma, k \otimes k') \). It follows from (2.1) that \( \lambda \) is nowhere vanishing. As will appear in due course, \( \lambda \) is a deformation intertwiner.

To each deformation \( a' \), one can canonically associate a field \( \mu = \mu(a') \) locally defined by [37]

\[
\mu_a = \partial_a z_a' \lambda_{aa'}^{-1}. \quad (2.3)
\]

It is easily verified that \( \mu_a \) is independent from the coordinate \( z_a' \) of \( a' \) chosen, as suggested by the notation. Further, \( \mu \) belongs to \( S(\Sigma, k^{-1} \otimes \bar{k}) \) and satisfies the bound

\[
\sup_{\Sigma} |\mu| < 1, \quad (2.4)
\]

1 In this paper, I shall denote coordinate patches and bundle trivializations by early lower case Latin subscripts. When no confusion is possible, I shall suppress such labels to simplify the notation.

2 I shall adopt the convention of attaching a corresponding number of primes to all objects referring to each of them.

3 I shall denote by \( S(\Sigma, Z) \) the space of smooth section of a smooth 1–cocycle \( Z \).
which follows directly from (2.1). A field \( \mu \) with such properties is called a Beltrami differential. The map \( a' \rightarrow \mu(a') \) can be inverted. To any Beltrami differential \( \mu \), one can associate canonically a deformation \( a' = a(\mu) \) of the holomorphic structure whose generic coordinate \( z'_{a'} \) is a local solution of the Beltrami equation

\[
(\bar{\partial}_a - \mu_a \partial_a)z'_{a'} = 0
\]  

subject to the local invertibility condition (2.1). The Beltrami differential \( \mu \) associated to \( a' \) is precisely the field \( \mu \) appearing in (2.5), as is evident from comparing (2.3) and (2.5). Hence, the set Beltr(\( \Sigma \)) of all Beltrami differentials \( \mu \) parametrizes in one-to-one fashion the family of all deformations \( a' \). Note that \( a' = a \) for \( \mu = 0 \). Beltr(\( \Sigma \)) is an infinite dimensional holomorphic manifold.

By using (2.2) and (2.5), one can easily verify that [37]

\[
dz'_{a'} = (dz_a + \mu_a dz_a)\lambda_{aa'}, \quad (2.6)
\]

\[
\bar{\partial}'_{a'} = \frac{1}{(1 - \mu_a \mu_a)\lambda_{aa'}}(\bar{\partial}_a - \mu_a \partial_a). \quad (2.7)
\]

These formulae are often employed in calculations.

For any holomorphic structure \( a \), there exists an \( a \)-holomorphic 1-cocycle \( k \otimes \frac{1}{2} \) such that \((k \otimes \frac{1}{2}) \otimes 2 = k\) [35]. In fact, there are \( 2^{2\ell} \) choices of \( k \otimes \frac{1}{2} \), each of which corresponds to a spinor structure of \( \Sigma \). Below, I shall work with a fixed but otherwise arbitrary choice.

Given a deformation \( a' \) of \( a \), define

\[
\lambda \otimes \frac{1}{2} \circ a' = \lambda_{aa'} \frac{1}{2}, \quad (2.8)
\]

with some choice of the branch of the square root for which the right hand side is smooth as a local function in \( \Sigma \) and holomorphic as a functional of \( \mu \). Under a change of coordinates in \( a \), one has \( \lambda \otimes \frac{1}{2} \circ a' = \eta_a \circ k \otimes \frac{1}{2} \circ \lambda_b \otimes \frac{1}{2} \circ a't \), where \( \eta_a \circ ab = \pm 1 \). \( \eta \circ a' \) is a trivial \( \mathbb{Z}_2 \)-valued 1-cocycle over \( \text{Dom} z'_{a'} \), since the latter can be assumed simply connected without any loss of generality. Thus, the branch of the square root in (2.8) can be chosen in such a way that \( \eta_a \circ ab = 1 \) identically. The square root is fixed by demanding that \( k \otimes \frac{1}{2} \circ a' = k \otimes \frac{1}{2} \circ ab \) for \( \mu = 0 \). It follows from here that \( k' \otimes \frac{1}{2} \circ a' b' = \lambda \otimes -\frac{1}{2} \circ aa' k \otimes \frac{1}{2} \circ ab \lambda \otimes \frac{1}{2} \circ bb' \) is a tensor square root of \( k' \) depending holomorphically on the Beltrami field \( \mu \).

A conformal field \( \psi \) of weights \( m, \bar{m} \in \mathbb{Z}/2 \) with respect to the holomorphic structure \( a \) is any element of \( S(\Sigma, k \otimes m \otimes \bar{k} \otimes \bar{m}) \). The intertwiner \( \lambda \otimes \frac{1}{2} \) induces a linear isomorphism of the spaces \( S(\Sigma, k \otimes m \otimes \bar{k} \otimes \bar{m}) \) and \( S(\Sigma, k' \otimes m \otimes \bar{k'} \otimes \bar{m}) \) of conformal fields of weights \( m, \bar{m} \)
associated to the holomorphic structures $a$ and $a'$, respectively [37]. Such isomorphism is given by

$$\psi_{a'} = \lambda^{\otimes -m}_{aa'} \bar{\lambda}^{\otimes -\bar{m}}_{aa'} \psi_a,$$  \hspace{1cm} (2.9)

for $\psi$ in $S(\Sigma, k \otimes \bar{k} \otimes \bar{k})$.

A Hermitian metric $h$ on $\Sigma$ subordinated to a given holomorphic structure $a$ is an element of $S(\Sigma, k \otimes \bar{k})$ such that $h_a$ takes only strictly positive values in its domain.

Every Hermitian metric $h$ can be represented as

$$h_a = \exp \phi_a g_a,$$  \hspace{1cm} (2.10)

where $g$ is a fixed fiducial Hermitian metric and $\phi$ is a real valued field of $S(\Sigma, 1)$, called the Liouville field. One may view $h$ as a deformation of $g$ and $\phi$ as a field parametrizing such deformation.

To a metric $h$, there is associated canonically a $(1,0)$ affine connection $\gamma_h$ compatible with $h$ given locally by

$$\gamma_h = \partial_a \ln h_a.$$  \hspace{1cm} (2.11)

The Ricci scalar $R_h$ of $h$ is given locally by

$$R_h = -2h_a^{-1} \bar{\partial}_a \partial_a \ln h_a.$$  \hspace{1cm} (2.12)

The covariant derivative associated to $\gamma_h$ will be denoted by $\partial_h$.

A deformation $a'$ of the holomorphic structure $a$ induces a deformation $h'$ of any given Hermitian metric $h$ via (2.9). Namely, $h'_{a'} = h_a |\lambda_{aa'}|^{-2}$. In the Liouville parametrization, one has $h' = \exp \phi g'$. Hence, the Liouville field is invariant under deformations.

3. The diffeomorphism and Weyl groups and the Slavnov operator.

The next step is naturally the analysis of the underlying symmetry of the geometrical setting described in the previous section. As well-known, the symmetry group is the diffeomorphism group of the surface $\Sigma$ extended by the Weyl group of $\Sigma$. Its properties, which will be recalled repeatedly in later sections, are summarized below.

I shall restrict to the group $\text{Diff}_c(\Sigma)$ of orientation preserving diffeomorphisms of $\Sigma$ homotopically connected to the identity $\text{id}_\Sigma$ in the $C^\infty$ topology. $\text{Diff}_c(\Sigma)$ acts on the family of deformations $a'$ of a reference holomorphic structure $a$ of $\Sigma$. In fact, for any diffeomorphism $f \in \text{Diff}_c(\Sigma)$, the maps

$$z''_{a''} = z'_{b'} \circ f$$  \hspace{1cm} (3.1)
form a new holomorphic structure $a''$, the pull-back $f^*a'$ of $a'$ by $f$ [37]. The pull-back action of $\text{Diff}_c(\Sigma)$ on the deformations $a'$ induces a right action of $\text{Diff}_c(\Sigma)$ on the corresponding intertwiner fields $\lambda$ and Beltrami differentials $\mu$ by the relations $f^*\lambda(a') = \lambda(f^*a')$ and $f^*\mu(a') = \mu(f^*a')$. From (3.1) and (2.2) and (2.3), one finds easily that [37]

$$(f^*\lambda)_{aa''} = (\partial_a f_b + \mu_b \circ f \partial_a \bar{f}_b)\lambda_{bb'} \circ f,$$

$$(f^*\mu)_a = \frac{\partial_a f_b + \mu_b \circ f \partial_a \bar{f}_b}{\partial_a f_b + \mu_b \circ f \partial_a \bar{f}_b}.$$

$\text{Diff}_c(\Sigma)$ acts on the space $S(\Sigma, k^\otimes m \otimes \bar{k}^\otimes \bar{m})$ of conformal fields $\psi$ of weights $m, \bar{m}$. The action is such that one has

$$(f^*\psi)''_{aa''} = \psi'_{bb'} \circ f,$$

where $a'' = f^*a'$ [37]. This is consistent since, as can easily be shown, $k''_{\otimes \frac{1}{2} a'' c''} = k''_{\otimes \frac{1}{2} b' d'} \circ f$, where $a'' = f^*a'$ and $z''_{a''}, z''_{c''}$ are related to $z'_{b'}, z'_{d'}$ by (3.1), respectively.

From (2.8), (2.9) and (3.2), one sees that

$$f^*\psi_a = \omega_{ab}(f; \mu)^{2m} \omega_{ab}(f; \mu)^{2\bar{m}} \psi_b \circ f,$$

where

$$\omega_{ab}(f; \mu) = (\partial_a f_b + \mu_b \circ f \partial_a \bar{f}_b)^{\frac{1}{2}}.$$

The branch of the square root is defined uniquely by taking the square root of both sides of (3.2) and demanding that $(f^*\lambda_{\otimes \frac{1}{2}})_{aa''} = \omega_{ab}(f; \mu)\lambda_{\otimes \frac{1}{2} bb'} \circ f$, where the square root $\lambda_{\otimes \frac{1}{2}}$ has been defined in sect. 2. Using the remark at the beginning of this paragraph, it is easy to show that the branch cannot depend on the coordinates $z''_{a''}$ and $z'_{b'}$.

When Hermitian structures are envisaged, the symmetry group is enlarged to the semidirect product $\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)$, where $\text{Weyl}(\Sigma) \cong \exp S(\Sigma, 1)$ is the Weyl group of $\Sigma$, the group multiplication being defined by

$$(f_1, v_1)(f_2, v_2) = (f_1 \circ f_2, v_1 f_1^{-1} v_2).$$

The action of a combined diffeomorphism and Weyl rescaling $(f, v)$ on the geometric fields $\lambda$ and $\mu$ and on conformal fields reduces to that of $f$ given above. For a Hermitian metric $h$, one has instead

$$f^*v_h = f^* (v^{-1} h \bar{v}^{-1}).$$

This can easily be transcribed into an action on the Liouville field $\phi$ (cf. eq. (2.10)).
In field-theoretic analyses, one is mostly interested in the infinitesimal form of the symmetry. One is thus led to considering the Lie algebra \( \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \) of \( \text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma) \). Writing formally \( f = \text{id}_\Sigma + \xi \) and \( \nu = 1 - \theta/2 \) in (3.7), it is easy to verify \( \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \) is isomorphic to the semidirect sum \( S(\Sigma, k^{-1}) \oplus S(\Sigma, 1) \) with the Lie brackets

\[
[\xi_1 \oplus \theta_1, \xi_2 \oplus \theta_2] = \left( (\xi_1 \partial + \bar{\xi}_1 \bar{\partial})\xi_2 - (\xi_2 \partial + \bar{\xi}_2 \bar{\partial})\xi_1 \right) \oplus \left( (\xi_1 \partial + \bar{\xi}_1 \bar{\partial})\theta_2 - (\xi_2 \partial + \bar{\xi}_2 \bar{\partial})\theta_1 \right). \tag{3.9}
\]

To express the infinitesimal action of the symmetry group \( \text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma) \) on field functionals, one introduces the exterior algebra \( \bigwedge^* \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \vee \) of \( \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \). The algebra is generated by the diffeomorphism ghost \( c \) and the Weyl ghost \( w \). These are the sections of \( k^{-1} \otimes \bigwedge^1 \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \vee \) and \( 1 \otimes \bigwedge^1 \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \vee \) defined by \( \langle c(p), \xi \oplus \theta \rangle = \xi(p) \) and \( \langle w(p), \xi \oplus \theta \rangle = \theta(p) \) for \( p \in \Sigma \), respectively. By a standard construction \cite{37}, the linearization of the action of the symmetry group on the relevant space \( F \) of field functionals at the identity, defines a coboundary operator \( s \) on \( \bigwedge^* \text{Lie}(\text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma)) \vee \otimes F \), called the Slavnov operator. \( s^2 = 0 \), that is \( s \) is nilpotent. From (3.9), it is not difficult to read off the structure equations

\[
sc = (c\partial + \bar{c}\bar{\partial})c, \tag{3.10}
\]
\[
sw = (c\partial + \bar{c}\bar{\partial})w. \tag{3.11}
\]

For a given Beltrami differential \( \mu \) in \( \text{Beltr}(\Sigma) \), the relevant combination of ghost fields is \( C = c + \mu\bar{c} \). \( C \) exhibits a distinguished complex structure of \( \text{LieDiff}_c(\Sigma) \). The expressions of \( s\lambda \) and \( s\mu \) can be easily obtained from (3.2), using (2.5), and (3.3):

\[
s\lambda_{a'} = \partial(C\lambda_{a'}), \tag{3.13}
\]
\[
s\mu = (\bar{\partial} - \mu\partial + (\partial\mu))C. \tag{3.14}
\]

By combining (3.10) and (3.14), one finds further that \( C \) obeys the structure equation

\[
sC = C\partial C. \tag{3.15}
\]

For any conformal field \( \psi \) of weights \( m, \bar{m} \), one has

\[
s\psi' = (c\partial + \bar{c}\bar{\partial})\psi', \tag{3.16}
\]
\[ s\psi = (c\partial + \bar{c}\bar{\partial})\psi + (m(\partial c + \mu\partial\bar{c}) + \bar{m}(\partial\bar{c} + \bar{\mu}\partial c))\psi, \quad (3.17) \]

as follows easily from (3.4)–(3.5) upon linearization. As to the Liouville field, one has

\[ s\phi = (c\partial + \bar{c}\bar{\partial})\phi + (\partial_g + \bar{\mu}\bar{\partial})c + w/2 + (\bar{\partial}_g + \mu\partial)\bar{c} + \bar{w}/2, \quad sg = 0. \quad (3.18) \]

Note that the Liouville field \( \phi \) does not transform as a conformal field of weights 0, 0.

4. Conformal field theory: Weyl versus diffeomorphism anomaly.

Consider a classical conformally invariant local field theory on a surface \( \Sigma \) endowed with a holomorphic structure \( a \). Its quantization is carried out by means of the \( \zeta \) function renormalization scheme. Such method requires the introduction of a Hermitian metric \( h \) with respect to \( a \) to properly define the adjoint of the relevant differential operators. The effective action \( I_{\text{diff}}(h) \) will thus depend on \( h \). One may fix a reference holomorphic structure \( a \) and a Hermitian metric \( h \) subordinated to \( a \) and consider any deformation \( a' \) of \( a \) and the associated deformation \( h' \) of the metric \( h \). To this there corresponds an effective action \( I'_{\text{diff}}(h') \). The symmetry group \( \text{Diff}_c(\Sigma) \times \text{Weyl}(\Sigma) \) acts on \( a' \) and \( h' \) as described in sect. 3 and thus on \( I'_{\text{diff}}(h') \). The \( \zeta \) function renormalization scheme is diffeomorphism invariant, that is \( I'_{\text{diff}}(h') \) is invariant under the action of \( \text{Diff}_c(\Sigma) \). But \( I'_{\text{diff}}(h') \) will depend in general on the scale of \( h' \) and not simply on the holomorphic structure \( a' \). A Weyl anomaly is produced in this way. The form of such anomaly is universal:

\[ sI'_{\text{diff}}(h') = \kappa A'_{\text{conf}}(w_1; h'), \quad (4.1) \]

where \( \kappa \) is the central charge of the model under consideration and

\[ A'_{\text{conf}}(w_1; h') = A_{\text{conf}}(w_1, \phi, \mu, \bar{\mu}; g) = \frac{-1}{12\pi} \int_{\Sigma} \frac{dz' \wedge dz'}{2i} w_1 \partial'\partial' \ln h' = \frac{-1}{12\pi} \int_{\Sigma} \frac{dz \wedge dz}{2i} w_1 \]

\[ \left\{ \partial\bar{\partial} \ln g + \partial\bar{\partial} \phi - (\partial - \bar{\mu}\bar{\partial} + \bar{\partial}\mu) \frac{\partial g\mu + \partial\phi\bar{\mu}}{1 - \mu\bar{\mu}} - (\bar{\partial} - \mu\partial - (\partial\mu)) \frac{\bar{\partial}g\bar{\mu} + \bar{\partial}\phi\mu}{1 - \mu\bar{\mu}} \right\}, \quad (4.2) \]

\[ w_1 = (w + \bar{w})/2. \quad (4.3) \]

Here, the dependence on the deformation \( a' \) and the scale of the metric is given explicitly in terms of the Beltrami field \( \mu \) and the Liouville field \( \phi \) relative to a reference metric \( g \) to make manifest the locality of \( A'_{\text{conf}} \).

The Weyl anomaly can be eliminated by either i) constraining the field content of the model so that the total central charge vanishes, as in the case of string theory [38], or ii) subtracting from the effective action a suitable local counterterm that absorbs the
Weyl anomaly at the cost of creating a diffeomorphism anomaly [27-28]. The form of such counterterm is also universal. Up to an overall factor $\kappa$, the counterterm is a sum of two contributions. The first contribution is just the Liouville action $S'_L(\phi; g')$ (with no cosmological term) expressing the dependence on the scale of the metric $h$

$$S'_L(\phi; g') = S_L(\phi, \mu, \bar{\mu}; g) = -\frac{1}{12\pi} \int_{\Sigma} \frac{dz' \wedge dz'}{2i} \left( \frac{1}{2} \partial' \phi \partial' \phi - \partial' \partial' \ln g' \phi \right)$$

$$= -\frac{1}{12\pi} \int_{\Sigma} \frac{dz \wedge dz}{2i} \left\{ \frac{1}{2(1 - \mu \bar{\mu})} (\partial - \bar{\mu} \partial) \phi (\bar{\partial} - \mu \partial) \phi - \left[ \partial \bar{\partial} \ln g - (\partial - \bar{\mu} \partial - (\bar{\partial} \bar{\mu})) \right] \frac{\partial g}{1 - \mu \bar{\mu}} \right\}. \tag{4.4}$$

The second contribution is the Verlinde-Knecht-Lazzarini-Thuillier action, worked out successively in refs. [27-28], and is given by

$$S_{VKLT}(\mu, \bar{\mu}; g, \mathcal{R}, \bar{\mathcal{R}}) = \frac{1}{12\pi} \int_{\Sigma} \frac{dz \wedge dz}{2i} \left\{ \mu (\mathcal{R} - r_g) + \bar{\mu} (\bar{\mathcal{R}} - \bar{r}_g) - \frac{1}{1 - \mu \bar{\mu}} \left[ \partial g (\bar{\partial} g) \right] \right\}, \tag{4.5a}$$

where $r_g$ is the projective connection

$$r_g = \partial \gamma_g - \frac{1}{2} \gamma_g^2 \tag{4.5b}$$

and $\mathcal{R}$ is a holomorphic projective connection in the reference holomorphic structure $\mathcal{R}$ satisfying

$$s \mathcal{R} = 0. \tag{4.6}$$

$\mathcal{R}$ ensures the correct conformal covariance of the integrand in the right hand side of eq. (4.5a). The counterterm is thus

$$\Delta I_{cft}(\phi, \mu, \bar{\mu}; \mathcal{R}, \bar{\mathcal{R}}) = S_L(\phi, \mu, \bar{\mu}; g) + S_{VKLT}(\mu, \bar{\mu}; g, \mathcal{R}, \bar{\mathcal{R}}). \tag{4.7}$$

The Weyl invariant effective action $I_{conf}$ is obtained by adding the counterterm to $I_{diff}$ [28]:

$$I_{conf}(\mu, \bar{\mu}; \mathcal{R}, \bar{\mathcal{R}}) = I'_{diff}(h') + \kappa \Delta I_{cft}(\phi, \mu, \bar{\mu}; \mathcal{R}, \bar{\mathcal{R}}). \tag{4.8}$$

---

4 A projective connection $\mathcal{R}$ is a collection $\{R_a\}$ of local smooth maps gluing as $R_b = k_{ba}(R_a - \{z_b, z_a\})$, where $\{f, \zeta\} = -2(\partial_\zeta f)^{\frac{1}{2}} \partial_\zeta^2 (\partial_\zeta f)^{-\frac{1}{2}}$ is the Schwarzian derivative. A holomorphic projective connections are known to exist [35].
$I_{\text{conf}}$ depends on the Beltrami fields $\mu$, $\bar{\mu}$ and the background projective connections $\mathcal{R}$ and $\bar{\mathcal{R}}$. Crucially, Weyl invariance ensures that no dependence on $g$ occurs. $I_{\text{conf}}$ obeys the Ward identity

$$sI_{\text{conf}}(\mu, \bar{\mu}; \mathcal{R}, \bar{\mathcal{R}}) = \kappa A_{\text{diff}}(C, \mu; \mathcal{R}) + \kappa\overline{A_{\text{diff}}(C, \mu; \mathcal{R})},$$

where

$$A_{\text{diff}}(C, \mu; \mathcal{R}) = \frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} C(\partial^3 + 2\mathcal{R}\partial + (\partial\mathcal{R}))\mu$$

[28]. The Weyl anomaly has been traded for a diffeomorphism anomaly whose strength is again measured by the central charge $\kappa$. The salient feature of the diffeomorphism anomaly is that it is chirally split: it is the sum of two contributions each of which depends on only one of the pairs $(C, \mu)$ and $(\bar{C}, \bar{\mu})$. This fact is intimately related to the holomorphic factorization property of $I_{\text{conf}}$, that is the chiral splitting of $I_{\text{conf}}$ itself: $I_{\text{conf}}(\mu, \bar{\mu}; \mathcal{R}, \bar{\mathcal{R}}) = I_P(\mu; \mathcal{R}) + \overline{I_P(\mu; \mathcal{R})}$ [24-25]. The functional $I_P$ is called the Polyakov action of the model under consideration [1,39].

It is important to have in mind a standard example, the bosonic spin $j b$–$c$ system, $j \in \mathbb{Z}/2$. For any holomorphic structure $a$, the classical action reads

$$S(\psi^\vee, \psi) = \frac{1}{\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \psi^\vee \bar{\partial}\psi,$$

where $\psi$ and $\psi^\vee$ vary, respectively, in $S(\Sigma, k^j)$ and $S(\Sigma, k^{1-j})$. The central charge of this model can be computed in a variety of ways. The well-known result is

$$\kappa = 2(6j^2 - 6j + 1).$$

5. Holomorphic and Hermitian geometry of complex vector bundles over higher genus Riemann surfaces.

In sect. 2, I have given a brief account of the basic notions and results of holomorphic geometry of a surface $\Sigma$ from the point of view of deformation theory. In this section, I shall show that such geometrical framework has a natural generalization where $\Sigma$ is replaced by smooth complex vector bundle $E$ over $\Sigma$. The section is divided in four subsections. In subsect. $a$, I shall indicate the basic topological properties of $E$. Subsect. $b$ describes the holomorphic and Hermitian geometry of $E$ in the language of deformation theory. Subsect. $c$ analyzes the relation between the holomorphic and Hermitian geometry of $E$ and that of its determinant and projectivization. Finally, subsect. $d$ introduces the notion
of special holomorphic and Hermitian geometry which emerges naturally in field theo
eric applications.

Before entering into the details of the discussion, it is necessary to define the notation
used and recall a few basic concepts (see refs. [36,40-41] for background material). For any
smooth fiber bundle $F$ over a smooth manifold $M$, I shall denote the bundle projection
by $\pi_F$, or simply by $\pi$ when no confusion is possible, and the fiber of $F$ at a point $p$
of $M$ by $F_p$. An isomorphism $T : F \to F'$ of fiber bundles is a diffeomorphism of $F$
onto $F'$ with the following properties. There exists a diffeomorphism $f_T : M \to M'$ of
their bases satisfying $\pi_{F'} \circ T = f_T \circ \pi_F$. $T\big|_{F_p}$ is a diffeomorphism of $F_p$
onto $F_{f_T(p)}$.

In case $F$ and $F'$ are either vector or projective bundles, it is further required that $T\big|_{F_p}$
is either a linear or a projective isomorphism, respectively. I shall denote by det the
covariant functor from the category of smooth complex vector bundles equipped with
the isomorphisms to the category of rank 1 smooth vector bundles equipped with the
isomorphisms. By definition, for any complex vector bundle $X$ over $M$, $\text{det} X$ is the rank 1
complex vector bundle over $M$ whose fiber $\text{det} X_p$ at any point $p$ of $M$ is the determinant
space of the fiber $X_p$ of $X$ at $p$. For any isomorphism $T : X \to Y$ of complex vector
bundles, $\text{det} T$ is the isomorphism $\text{det} X \to \text{det} Y$ such that $\text{det} T\big|_{\text{det} X_p}$ is the ordinary
determinant of the linear isomorphism $T\big|_{X_p}$. I shall denote by proj the covariant functor
from the category of smooth complex vector bundles equipped with the isomorphisms to
the category of smooth complex projective vector bundles equipped with the isomorphisms.
By definition, for any complex vector bundle $X$ over $M$, $\text{proj} X$ is the complex projective
bundle over $M$ whose fiber $\text{proj} X_p$ at any point $p$ of $M$ is the projective space of the fiber
$X_p$ of $X$ at $p$. For any isomorphism $T : X \to Y$ of complex vector bundles, $\text{proj} T$
is the isomorphism $\text{proj} X \to \text{proj} Y$ such that $\text{proj} T\big|_{\text{proj} X_p}$ is the projective isomorphism
canonically associated to the linear isomorphism $T\big|_{X_p}$.

a) Topological properties of $E$.

The basic topological structure which will be considered here is a smooth rank $r$
complex vector bundle $E$ over a connected compact smooth oriented surface $\Sigma$ of genus $
\ell \geq 2$.

The Chern number $c_1(\text{det} E)$ of $\text{det} E$ is by definition the degree $d$ of $E$. Now, it is
well-known that the smooth line bundles are classified topologically by their Chern number.

$^5$ Given a linear isomorphism $L : V \to W$ of vector spaces, the associated projective
isomorphism $\hat{L} : P(V) \to P(W)$ of the corresponding projective spaces is uniquely defined
by $\hat{L} \circ \varphi_V = \varphi_W \circ L$, $\varphi_X$ being the natural projection of a vector space $X$
onto its projective space.
Let $\epsilon$ be the smooth line bundle of $\Sigma$ such that $c_1(\epsilon) = 2(\ell - 1)$. ($\epsilon$ corresponds to the holomorphic canonical line bundle upon choosing a holomorphic structure on $\Sigma$). There is an integer $p > 0$ and a halfinteger $q$ such that $pc_1(\det E) = qc_1(\epsilon)$, i.e., $c_1((\det E)^\otimes p) = c_1(\epsilon^\otimes q)$. One thus has a relation of the form

$$(\det E)^\otimes p = \epsilon^\otimes q. \quad (5.1)$$

Of course, $p$ and $q$ are defined only up to multiplication by a common positive integer number. One can however fix them by requiring that $p$ and $2|q|$ are relative prime if $q \neq 0$ and that $p = 1$ if $q = 0$. The ratio

$$j = q/pr \quad (5.2)$$

is conversely an intrinsic property of $E$. It will be called the normalized weight of $E$. $j$ is related to the slope $s$ of the vector bundle $E$:

$$s = d/r = 2j(\ell - 1). \quad (5.3)$$

b) Holomorphic and Hermitian geometry of $E$.

When viewed as a real manifold, $E$ is oriented. A holomorphic structure $A$ of $E$ is a maximal collection $\{(z_a, u_a)\}$ of local trivializations contained in the oriented differentiable structure of $E$ with the following properties [40-41]. i) $a = \{z_a\}$ is a holomorphic structure of the base $\Sigma$; ii) for any two overlapping trivializations, one has

$$u_b = L_{ba} \circ \pi u_a, \quad (5.4)$$

where $L = \{L_{ba}\}$ is an $a$–holomorphic $\text{GL}(r, \mathbb{C})$–valued 1–cocycle. The holomorphic structure $a$ of $\Sigma$ and the $a$–holomorphic $\text{GL}(r, \mathbb{C})$–valued 1–cocycle $L$ are canonically associated to $A$.

In extended conformal field theory, one compares several holomorphic structures $A$, $A'$, $A''$ etc. of $E$. The following analysis parallels closely that of sect. 2. Chosen a reference holomorphic structure $A$, any other holomorphic structure $A'$ can be viewed as obtained from $A$ by means of a continuous deformation. Associated to $A'$ is a deformation $a'$ of the underlying holomorphic structure of $a$.

Each deformation $A'$ is characterized by the intertwiner field $\lambda$ of the underlying deformation $a'$ defined in (2.2) and by a further intertwiner field $V = V(A')$ defined as follows. The relation between the trivializing mappings $u_a$ and $u'_a$ of two overlapping trivializations of $A$ and $A'$ must be linear and non singular. Thus, it is of the form [26]

$$u_a = V_{aa'} \circ \pi u'_{a'}, \quad (5.5)$$
where \( V_{aa'} \) is some smooth \( \text{GL}(r, \mathbb{C}) \)-valued function. \( V \) is by definition the collection \( \{ V_{aa'} \} \). From (5.4) and (5.5), it appears that \( V \) belongs to \( S(\Sigma, L \otimes L^\vee) \).

To any deformation \( A' \) of the holomorphic structure of \( E \), one can canonically associate two geometrical fields. The first one is the Beltrami differential \( \mu = \mu(a') \) of the underlying deformation \( a' \) of the holomorphic structure of \( \Sigma \) defined by (2.3). The second one is defined as follows. Let \( A \) be a fixed fiducial \((1, 0)\) connection of the \( a \)-holomorphic \( \text{GL}(r, \mathbb{C}) \)-valued 1-cocycle \( L \) so that \( A_b = k_{ba}(L_{ba}A_aL_{ba}^{-1} + \partial_aL_{ba}L_{ba}^{-1}) \). Consider the field \( A^* = A^*(A') \) locally defined by [26]

\[
A^*_{a} = (\bar{\partial}_a - \mu_a\partial_a)V_{aa'}V_{aa'}^{-1} + \mu_aA_a. \tag{5.6}
\]

It is easily verified that \( A^*_{a} \) does not depend on the choice of the trivialization \((z'_{a'}, u'_{a'})\) of \( A' \), as suggested by the notation, since \( 0 = \bar{\partial}_{b'}L'_{a'b'} \propto (\bar{\partial}_b - \mu_b\partial_b)L'_{a'b'} \) by the \( a' \)-holomorphy of \( L' \) and by (2.7). It is also checked readily that \( A^* \) belongs to \( S(\Sigma, \bar{k}\otimes \text{End} L) \). In this way, the deformation \( A' \) is characterized by a pair \((\mu, A^*)\) formed by a Beltrami differential \( \mu \) and an element \( A^* \) of \( S(\Sigma, \bar{k} \otimes \text{End} L) \). Conversely, to any such pair \((\mu, A^*)\) one can associate a deformation \( A' = A(\mu, A^*) \) of the holomorphic structure. Let us show this briefly. As explained in sect. 2, \( \mu \) determines a holomorphic structure \( a' = \{ z'_{a'} \} \) of \( \Sigma \) by solving the Beltrami equation (2.5) subject to the local invertibility condition (2.1). Now, consider the equation

\[
(\bar{\partial}_a - \mu_a\partial_a - (A^*_a - \mu_aA_a))V_{aa'} = 0, \tag{5.7}
\]

with \( V_{aa'} \) a smooth \( \text{GL}(r, \mathbb{C}) \)-valued function. The solution of (5.7) is determined up to right multiplication by a local smooth \( \text{GL}(r, \mathbb{C}) \)-valued function \( W_{aa'} \) such that \((\bar{\partial}_a - \mu_a\partial_a)W_{aa'} = 0 \). By using this remark, it is not difficult to see that, if the domains of all trivializations involved intersect, one has \( V_{aa'} = L_{ab}V_{bb'}W_{aba'b'}^{-1} \) for some local smooth \( \text{GL}(r, \mathbb{C}) \)-valued function \( W_{aba'b'} \) satisfying \((\bar{\partial}_a - \mu_a\partial_a)W_{aba'b'} = 0 \). From here, recalling (2.7), it can be verified that, for fixed \( a' \), the matrix function \( W_{ab} = W_{aba'a'} \) defines an \( a' \)-holomorphic \( \text{GL}(r, \mathbb{C}) \)-valued 1-cocycle \( W_{a'} \) on the domain of \( z'_{a'} \). Since the latter is an open Riemann surface, \( W_{a'} \) is holomorphically trivial [36]. It then follows that the \( V_{aa'} \)’s can be chosen so that, if the domains of all trivializations involved intersect, one has \( V_{aa'} = L_{ab}V_{bb'}L'_{a'b'}^{-1} \). Recalling (2.7) once more, it is easily checked that the matrix functions \( L'_{a'b'} \) define an \( a' \)-holomorphic \( \text{GL}(r, \mathbb{C}) \)-valued 1-cocycle. Introducing trivializing maps \( u'_{a'} \) by means of (5.5), it is apparent that the collection \( \{(z'_{a'}, u'_{a'})\} \)

\[\text{For any smooth } \text{GL}(r, \mathbb{C}) \text{-valued 1-cocycle } Z, \text{ I shall denote by } Z^\vee \text{ the dual cocycle } \{Z_{ab}^{-1}\} \text{ of } Z \text{ and by } \text{End } Z \text{ the cocycle } Z \otimes Z^\vee.\]
constitutes a holomorphic structure of $E$ and that the fields $\mu$ and $A^*$ associated to it via (2.3) and (5.6) are precisely the ones which one started with. In conclusion, it has been shown that the pairs $(\mu, A^*)$ formed by a Beltrami differential $\mu$ of $\text{Beltr}(\Sigma)$ and an element $A^*$ of $S(\Sigma, \bar{k} \otimes \text{End} L)$ parametrize in one-to-one fashion the family of deformations $A'$ of the holomorphic structure of $E$, with $A' = A$ for $\mu = 0$ and $A^* = 0$. Such parametrization, however, is not unique depending on the choice of the reference connection $A$ entering in the definition of $A^*$. Note that $S(\Sigma, \bar{k} \otimes \text{End} L)$, like $\text{Beltr}(\Sigma)$, is an infinite dimensional holomorphic manifold. $V$ is non local holomorphic functional of $\mu$ and $A^*$.

For a given holomorphic structure $A$, the relevant field spaces are of the form $S(\Sigma, k \otimes \bar{k} \otimes m \otimes \bar{m} \otimes L \otimes \bar{L} \otimes Q \otimes \bar{Q})$ where $m, \bar{m} \in \mathbb{Z}/2$ and $P, \bar{P}, Q, \bar{Q} \in \mathbb{N} \cup \{0\}$. Fields of such type provide a natural generalization of ordinary conformal fields in extended conformal field theory.

The intertwiners $\lambda$ and $V$ associated to a deformation $A'$ of the holomorphic structure of $E$ induce a linear isomorphism of $S(\Sigma, k \otimes m \otimes \bar{k} \otimes \bar{m} \otimes L \otimes \bar{L} \otimes Q \otimes \bar{Q})$ and $S(\Sigma, k' \otimes m \otimes \bar{k}' \otimes \bar{m} \otimes L' \otimes \bar{L}' \otimes Q \otimes \bar{Q})$, explicitly given by

$$\Psi'_{a'} = (V \otimes -P \otimes \bar{V} \otimes -\bar{P})_{aa'} \lambda \otimes -m_{aa'} \bar{\lambda} \otimes -\bar{m}_{aa'} \Psi_a (V \otimes Q \otimes \bar{V} \otimes \bar{Q})_{aa'}.$$  \hspace{1cm} (5.8)

This relation generalizes (2.9).

The notion of Hermitian metric on a surface has a suitable generalization to the category of complex vector bundles. A Hermitian structure on $E$ subordinated to the holomorphic structure $A$ is a pair $(h^\circ, H)$, where $h^\circ$ is a Hermitian metric of $\Sigma$ subordinated to $a$ and $H$ is a Hermitian metric of $L$, that is an element of $S(\Sigma, L \otimes \bar{L})$ such that $H_a = H_a^\dagger > 0$ for any trivialization of $A$.

Upon choosing a reference Hermitian structure $(g^\circ, G^\circ)$ subordinated to $A$, one can express any other Hermitian structure $(h^\circ, H)$ in terms of a Liouville field $\phi^\circ$ and a further field $\Phi$, called the Donaldson field. The metric $h^\circ$ is given by (2.10) as a functional of $\phi^\circ$ and $g^\circ$. Similarly, $H$ can be represented locally in the form

$$H_a = \exp \Phi_a G_a.$$  \hspace{1cm} (5.9)

$\Phi = \Phi(h^\circ, H)$ is an element of $S(\Sigma, \text{End} L)$ satisfying the $G$–Hermiticity condition

$$G_a \Phi_a^\dagger G_a^{-1} = \Phi_a.$$  \hspace{1cm} (5.10)

This ensures the Hermiticity of $H$. The pair $(\phi^\circ, \Phi)$ parametrize the family of Hermitian structures $(h^\circ, H)$ viewed as deformations of the reference Hermitian structure $(g^\circ, G)$.
To any Hermitian structure \((h^\circ, H)\), there is canonically associated the \((1, 0)\) affine connection \(\gamma_{h^\circ}\) given by (2.11) and a \((1, 0)\) connection \(\Gamma_H\) of \(L\) compatible with \(H\) given locally by

\[
\Gamma_{Ha} = \partial_a H_a H_a^{-1}. \tag{5.11}
\]

The curvature of \(\gamma_{h^\circ}\) is given by (2.12) while the curvature of \(\Gamma_{Ha}\) is

\[
F_{Ha} = \tilde{\partial}_a \Gamma_{Ha}. \tag{5.12}
\]

\(F_H\) belongs to \(S(\Sigma, k \otimes \bar{k} \otimes \text{End}L)\). I shall denote by \(\partial_{h^\circ}\) and \(\partial_H\) the covariant derivatives associated respectively to \(\gamma_{h^\circ}\) and \(\Gamma_{Ha}\).

Any deformation \(A'\) of the holomorphic structure \(A\) of \(E\) induces a deformation \((h^{\circ'}, H')\) of a Hermitian structure \((h^\circ, H)\), via (2.9) and (5.8): \(h^{\circ'}_{a'a'} = h^\circ_a |\lambda_{aa'}|^{-2}\) and \(H'_{a'a'} = V_{aa'}^{-1} H_a V_{aa'}^{-1}\). The Liouville field \(\phi^\circ\) is invariant under deformations, the Donaldson field is not: \(\Phi'_{a'a'} = V_{aa'}^{-1} \Phi_a V_{aa'}\).

c) The relation between the holomorphic and Hermitian geometry of \(E\) and those of its determinant and projectivization.

To the vector bundle \(E\), there are canonically associated its determinant line bundle \(\det E\) and its projectivization \(\text{proj} E\). Such bundles will play a role in the geometrical interpretation of the field theoretic constructions of sects. 7, 8 and 9. In fact, the anomalies and the relevant counterterms of extended conformal field theory consistently appear as sums of two terms corresponding to each of them. The determinant part of the anomaly contributes to the ordinary conformal anomaly. The projective part gives rise to the anomalies of extended gravity. The relevance of projective geometry of vector bundles in extended gravity is evident in the formulations of light cone \(W\) geometry based on the parametrization of the generalized projective structures of a surface \(\Sigma\) [18-20]. The changes of trivialization of a projective bundle are indeed expressible by means of linear fractional transformations as for projective structures.

It is therefore important to understand the relation between the holomorphic and Hermitian geometry of \(E\) and those of \(\det E\) and \(\text{proj} E\). The analysis is based on the observation that the covariant functors \(\det\) and \(\text{proj}\) map any structure defined in \(E\) into an analogous structures of \(\det E\) and \(\text{proj} E\).

To each holomorphic structure \(A\) of \(E\), one can associate canonically a holomorphic structure of \(\det E\) which will be denoted by \(\det A\). The generic trivialization of \(\det A\) is the image through the functor \(\det\) of some trivialization \((z_a, u_a)\) of \(A\) and explicitly reads as \((z_a, \det u_a)\), where \(\det u_a\) is the ordinary determinant of \(u_a\) viewed as a linear isomorphism.
from each fiber onto \( C' \). The holomorphic structure of \( \Sigma \) underlying \( \det A \) is precisely \( a \). The associated \( a \)-holomorphic \( C^* \) 1–cocycle is det \( L = \{ \det L_{ab} \} \). When \( r > 1 \), the map \( A \to \det A \) is many-to-one. Two holomorphic structures \( A \) and \( A' \) of \( E \) induce the same holomorphic structure of \( \det E \) if and only if there is a bijection \( a \to a' = \varsigma(a) \) and a collection \( \{ Y_a \} \) of \( \text{SL}(r, C) \)–valued local functions such that \( z_a = z'_a \) and \( u'_a = Y_a \circ \pi u_a \).

Any deformation \( A' \) of the holomorphic structure of \( E \) induces a deformation \( \det A' \) of the corresponding holomorphic structure of \( \det E \). The intertwiners associated to the latter are \( \lambda \) and \( \det V = \{ \det V_{aa'} \} \). For any two deformations \( A' \) and \( A'' \), \( \det A' = \det A'' \) if and only if \( \mu(a') = \mu(a'') \) and \( \text{tr} A^*(A') = \text{tr} A^*(A'') \). This follows easily from the remarks at the end of the previous paragraph and from (5.5) and (5.6). It is easy to show from here that the pairs \( (\mu, a^*) \) formed by a Beltrami differential \( \mu \) and an element \( a^* \) of \( S(\Sigma, \mathbb{k}) \) parametrize in one-to-one fashion the family of induced deformations \( \det A' \). \( a^* \) is precisely the trace of the element \( A^* \) of \( S(\Sigma, \mathbb{k} \otimes \text{End} L) \) corresponding any chosen holomorphic structure \( A' \) inducing \( \det A' \) with a conventional 1/r factor: \( a^* = \frac{1}{r} \text{tr} A^*_b \).

A parallel treatment can be made for the projective bundle \( \text{proj} E \). To each holomorphic structure \( A \) of \( E \) there is canonically associated also a holomorphic structure of \( \text{proj} E \), which will be denoted by \( \text{proj} A \). The notion of holomorphic structure of a complex projective bundle over \( \Sigma \) is defined analogously to that of holomorphic structure of a complex vector bundle but the associated holomorphic 1–cocycle is valued in the group of projective isomorphisms of \( \mathbb{C}P^{r-1} \). The generic trivialization of \( \text{proj} A \) is the image through the functor \( \text{proj} \) of some trivialization \( (z_a, u_a) \) of \( A \) and explicitly reads \( (z_a, \text{proj} u_a) \), where \( \text{proj} u_a \) is the projective isomorphism associated canonically to \( u_a \) viewed as a linear isomorphism from each fiber onto \( C' \). As for \( \det E \), the holomorphic structure of \( \Sigma \) underlying \( \text{proj} A \) is precisely \( a \). The associated \( a \)-holomorphic \( C^* \) 1–cocycle is proj \( L = \{ \text{proj} L_{ab} \} \). The map \( A \to \text{proj} A \) is many-to-one. Two holomorphic structures \( A \) and \( A' \) of \( E \) induce the same holomorphic structure of \( \text{proj} E \) if and only if there is a bijection \( a \to a' = \varsigma(a) \) and a collection \( \{ y_a \} \) of \( C^* \)–valued local functions such that \( z_a = z'_a \) and \( u'_a = y_a \circ \pi u_a \). and deformations \( \det A' \) and \( \text{proj} A' \) of \( \det A \) and \( \text{proj} A \), respectively.

A deformation \( A' \) of the holomorphic structure of \( E \) induces a deformation \( \text{proj} A' \) of the corresponding holomorphic structure of \( \text{proj} E \). The intertwiners associated to the latter are \( \lambda \) and \( \text{proj} V = \{ \text{proj} V_{aa'} \} \). For any two deformations \( A' \) and \( A'' \), \( \text{proj} A' = \text{proj} A'' \) if and only if \( \mu(a') = \mu(a'') \) and \( A^*(A') - (1/r)\text{tr} A^*(A')1 = A^*(A'') - (1/r)\text{tr} A^*(A'')1 \). This follows easily from the remarks at the end of the previous paragraph and from (5.5) and (5.6). It is easy to show from here that the pairs \( (\mu, \check{A}^*) \) formed by a Beltrami differential \( \mu \) and a traceless element \( \check{A}^* \) of \( S(\Sigma, \mathbb{k} \otimes \text{End} L) \) parametrize in one-to-one fashion the family of induced deformations. \( \check{A}^* \) is precisely the traceless part of the section.
A^* of $S(\Sigma, \bar{k} \otimes \text{End} L)$ pertaining any chosen holomorphic structure $A'$ inducing $\text{proj} A'$:

$$A^*_b = A^* b - (1/r) \text{tr} A^* b 1.$$  

From the above, it appears that the deformations $\text{det} A'$ and $\text{proj} A'$ induced by a deformation $A'$ of the holomorphic structure of $E$ completely characterize it, since, from the knowledge of the former two, it is possible to uniquely reconstruct the latter.

The results of the last five paragraphs have a straightforward extension to the Hermitian geometry of $\text{det} E$ and $\text{proj} E$, as now I shall show.

To any Hermitian structure $(h^\circ, H)$ subordinated to the holomorphic structure $A$, there is associated a Hermitian structure $(h^\circ, \text{det} H)$ of $\text{det} E$ subordinated to the induced holomorphic structure $\text{det} A$. The map $(h^\circ, H) \rightarrow (h^\circ, \text{det} H)$ is clearly many-to-one. Two Hermitian structures $(h^\circ_1, H_1)$ and $(h^\circ_2, H_2)$ of $E$ induce the same Hermitian structure on $\text{det} E$ if and only if $\phi^\circ(h^\circ_1, H_1) = \phi^\circ(h^\circ_2, H_2)$ and $\text{tr} \Phi(h^\circ_1, H_1) = \text{tr} \Phi(h^\circ_2, H_2)$. Hence, the Hermitian structures $(h^\circ, H)$ of $E$ with assigned induced Hermitian structure $(h^\circ, \text{det} H)$ are parametrized by the pairs $(\phi^\circ, \phi)$ of real valued elements $S(\Sigma, 1)$: $\phi_a = (1/r) \text{tr} \Phi_a$ for any structure $(h^\circ, H)$ inducing $(h^\circ, \text{det} H)$.

Similarly, to any Hermitian structure $(h^\circ, H)$ subordinated to the holomorphic structure $A$, there is associated a Hermitian structure $(h^\circ, \text{proj} H)$ of $\text{proj} E$ subordinated to the induced holomorphic structure $\text{proj} A$, where $\text{proj} H$ is the projective isomorphism associated to $H$ when the latter is viewed as an isomorphism of the antidual $\bar{E}^\vee$ of $E$ onto $E$. The map $(h^\circ, H) \rightarrow (h^\circ, \text{proj} H)$ is also many-to-one. Two Hermitian structures $(h^\circ_1, H_1)$ and $(h^\circ_2, H_2)$ of $E$ induce the same Hermitian structure on $\text{proj} E$ if and only if $\phi^\circ(h^\circ_1, H_1) = \phi^\circ(h^\circ_2, H_2)$ and $\Phi(h^\circ_1, H_1) - (1/r) \text{tr} \Phi(h^\circ_1, H_1) 1 = \Phi(h^\circ_2, H_2) - (1/r) \text{tr} \Phi(h^\circ_2, H_2) 1$. The Hermitian structures $(h^\circ, H)$ of $E$ with assigned induced Hermitian structure $(h^\circ, \text{proj} H)$ are parametrized by the pairs $(\phi^\circ, \Phi)$, where $\phi^\circ$ is a real valued element of $S(\Sigma, 1)$ and $\Phi$ is a traceless element of $S(\Sigma, \text{End} L)$ satisfying the Hermiticity condition (5.10): $\Phi_a = \Phi_a - (1/r) \text{tr} \Phi_a 1$ for any structure $(h^\circ, H)$ inducing $(h^\circ, \text{proj} H)$.

It follows that the Hermitian structures $(h^\circ, \text{det} H)$ and $(h^\circ, \text{proj} H)$ of $\text{det} E$ and $\text{proj} E$ induced by a Hermitian structure $(h^\circ, H)$ of $E$ completely characterize the latter.

\textit{d) Special holomorphic and Hermitian geometry of $E$.}

The vector bundles relevant in ordinary conformal field theory are of the form $E = e^{\otimes m}$ with $m \in \mathbb{Z}/2$. The deformations of the holomorphic structure of such bundles are parametrized in one-to-one fashion by the Beltrami differential $\mu$ and by the field $A^*$. Such geometrical setting is not suitable for conformal field theory, since, in addition to $\mu$, it contains $A^*$, an extra undesired geometrical field. Similarly, for a general Hermitian structure $(h^\circ, H)$ of such bundles, there is no a priori relation between $h^\circ$ and $H$, while
in conformal field theory one assumes that $H = h^\otimes m$. The problem just highlighted presents itself also for more general vector bundles $E$ in extended conformal field theory, as will be shown in sect. 7. From these simple remarks, it appears that, in field theoretic applications, it is necessary to impose restrictions on the class of allowed deformations and Hermitian structures. Such restrictions are described in this subsection.

For a generic holomorphic structure $A$, the topological relation (5.1) may be written as

$$\det L_{ab}^p = k^\otimes q_{ab} e_{ab}, \quad (5.13)$$

where $e$ is a degree zero $a$–holomorphic $\mathbb{C}^*$–valued 1–cocycle. The appearance of the cocycle $e$ is unavoidable as $e$, albeit smoothly trivial, is in general not holomorphically so. However, as well-known, any such 1–cocycle $e$ is holomorphically equivalent to a flat $U(1)$–valued cocycle, i. e. one whose elements are constant phases [36]. From here, it is easy to see that any holomorphic structure $A$ of $E$ admits a reduction $A_0$ for which the $a$–holomorphic function $e_{ab}$ appearing in (5.13) is a constant of unit modulus.

Consider a deformation $A'$ of a reference holomorphic structure $A$. Using the analog of (5.13) for $A'$ and the relations $\lambda_{aa'} = k_{ab} \lambda_{bb'} k'_{a'b'}^{-1}$ and $V_{aa'} = L_{ab} V_{bb'} L'_{a'b'}^{-1}$, it is straightforward to show that

$$\det V_{aa'}^p = \lambda^\otimes q_{aa'} \exp(-pr\sigma_{aa'}), \quad (5.14)$$

where $\exp(-pr\sigma)$ is a smooth section of the smooth $\mathbb{C}^*$–valued 1–cocycle $e \otimes e'^{-1}$. $\sigma_{aa'}$ is defined up to integer multiples of $2\pi i/pr$. I have extracted a factor $pr$ from $\sigma_{aa'}$ in order to render its definition independent from the conventional choice of $p$ and $q$ made above. $\sigma$ is simply related to the normalized trace $a^*$ of $A^*$

$$a^*_{ab} = (j\partial_b + a_b) \mu_b - (\bar{\partial}_b - \mu_b \partial_b) \sigma_{bb'}, \quad a_b = \frac{1}{r} \tr A_b, \quad (5.15)$$

as follows directly from (5.6) and (5.14).

It is possible to choose the reference holomorphic structure $A$ so that $e_{ab} = 1$ for the trivializations of the reduction $A_0$, for the 1–cocycle $e$ is smoothly trivial. This conventional choice will be assumed below. A deformation $A'$ is said special if

$$\sigma_{aa'} = 0 \mod 2\pi i / pr \mathbb{Z}, \quad (\text{special deformations}) \quad (5.16)$$

for the trivializations of the reductions $A_0$ and $A'_0$. In that case, $e'_{a'b'} = 1$, for the trivializations of $A'_0$. By (5.15), a deformation $A'$ is special if and only if $a^*_{ab} = (j\partial_b + a_b) \mu_b$. The necessity follows from the fact that a flat 1–cocycle has a holomorphic section if and only if the cocycle is trivial [35]. Thus, for a special deformation, $a^*$ is a given functional of $\mu$.  

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Equivalently, \( \det A' \) is completely determined by \( a' \). Thus, the class of special deformations is parametrized in one-to-one fashion by the pairs \((\mu, \hat{A}^*)\) formed by a Beltrami field \( \mu \) and a traceless element \( \hat{A}^* \) of \( S(\Sigma, \bar{k} \otimes \text{End } L) \). This means that there is a one-to-one correspondence between the family of special deformations \( A' \) and the family of induced deformations \( \text{proj } A' \). Indeed, there is precisely one special deformations \( A' \) inducing \( \text{proj } A' \).

For bundles \( E \) of the form \( E = \epsilon^\otimes m \) with \( m \in \mathbb{Z}/2 \), the special deformations are parametrized by \( \mu \) alone, since \( \hat{A}^* = 0 \) automatically for the case considered. Thus, restricting to special deformations cures the disease mentioned at the beginning of this subsection and renders the geometric framework suitable for conformal field theory. The need for special deformations also in extended conformal field theory will appear in due course in sect. 7.

The above discussion has a close counterpart for Hermitian structures. It follows from (5.13) that, for any Hermitian structure \((h^\circ, H)\) subordinated to a holomorphic structure \( A \),

\[
\det H_a^p = h^\circ \otimes q_a \exp(-pr\tau_a),
\]

where \( \exp(-pr\tau) \) is a real valued element of \( S(\Sigma, e \otimes \bar{e}) \). This relation entails the identity

\[
\phi_a = j\phi^\circ_a - \tau_a(h^\circ, H) + \tau_a(g^\circ, G),
\]

where the definition of the notation should be obvious.

A Hermitian structure \((h^\circ, H)\) of \( E \) subordinated to a holomorphic structure \( A \) is said special if

\[
\tau_a = 0, \quad \text{(special Hermitian structures)}
\]

for the trivializations of \( A_0 \). Special Hermitian structures clearly do exist. It is convenient to choose the reference Hermitian structure \((g^\circ, G)\) to be of such type and this choice will be assumed in what follows. By (5.18), a Hermitian structure \((h^\circ, H)\) is special if and only if \( \phi_a = j\phi^\circ_a \). Thus, for a special Hermitian structure, \( \phi \) is a given functional of \( \phi^\circ \). Equivalently, the induced structure \((h^\circ, \det H)\) is completely determined by \( h^\circ \). So, the family of special Hermitian structures is parametrized in one-to-one fashion by the pairs \((\phi^\circ, \hat{\Phi})\), where \( \hat{\Phi} \) is a traceless element of \( S(\Sigma, \text{End } L) \) satisfying (5.10). This means that there is a one-to-one correspondence between the family of special Hermitian structures \((h^\circ, H)\) and the family of induced structures \((h^\circ, \text{proj } H)\). Further, there is precisely one special Hermitian structures \((h^\circ, H)\) inducing \((h^\circ, \text{proj } H)\).

For bundles \( E \) of the form \( E = \epsilon^\otimes m \) with \( m \in \mathbb{Z}/2 \), the special Hermitian structures \((h^\circ, H)\) of \( E \) subordinated to a holomorphic structure \( A \) are such that \( H_a = h^\circ \otimes m_a \) for the trivializations of \( A_0 \). This is precisely what is required in conformal field theory. In
sect. 7, it will be shown that it is natural to restrict to special Hermitian structures also for more general bundles \( E \) in extended conformal field theory.

If \( (h^\circ, H) \) is a special Hermitian structure of \( E \) with respect to a holomorphic structure \( A \), then, for an arbitrary deformation \( A' \) of \( A \), the deformed structure \( (h'^\circ, H') \) needs not be special. However, if \( A' \) is a special deformation, then \( (h'^\circ, H') \) is special.

6. The automorphism and extended Weyl groups and the Slavnov operator.

The next step is naturally the analysis of the underlying symmetry of the geometrical setting described in the previous section. As will be shown below, the symmetry group is the automorphism group of the bundle \( E \) extended by the generalized Weyl group of \( E \). This section is devoted to its study. The structure of the section parallels to some extent that of the previous one. The section is divided in four subsections. Subsect. \( a \) provides basic generalities about the automorphism group of \( E \) and describes the action of the automorphism and Weyl group on the holomorphic and Hermitian structures of \( E \). Subsect. \( b \) analyzes the relation between the symmetry group of \( E \) and those of \( \text{det} \ E \) and \( \text{proj} \ E \). Subsect. \( c \) is concerned with the special symmetry subgroup preserving special holomorphic and Hermitian structures of \( E \). Finally, subsect. \( d \) introduces the infinitesimal formulation and the Slavnov operator relevant in field theoretic applications.

\( a) \) The automorphism and generalized Weyl groups of \( E \) and their action on the holomorphic and Hermitian structures.

An automorphism \( \alpha \) of \( E \) is an orientation preserving isomorphism of \( E \) onto itself \([40-41]\). Here, I shall restrict to automorphisms homotopically connected to the identity \( \text{id}_E \) in the \( C^\infty \) topology. The set \( \text{Aut}_c(E) \) of all automorphisms of \( E \) of such type is clearly a group under composition of maps. As explained at the beginning of sect. 5, to any automorphism \( \alpha \), there is associated a diffeomorphism \( f_\alpha \) of \( \Sigma \), which is also orientation preserving. The map \( \alpha \to f_\alpha \) is a group homomorphism of \( \text{Aut}_c(E) \) onto \( \text{Diff}_c(\Sigma) \). Its kernel \( \text{Gau}_c(E) \) is the group of fiber preserving automorphisms, usually called (small) gauge transformations. \( \text{Gau}_c(E) \) is a closed normal subgroup of \( \text{Aut}_c(E) \) and one has the isomorphism \( \text{Diff}_c(\Sigma) \cong \text{Aut}_c(E)/\text{Gau}_c(E) \).

If \( \alpha \) is an automorphism of \( E \), then, for any point \( p \) of \( \Sigma \), \( \alpha|_{E_p} \) is a linear isomorphism of \( E_p \) onto \( E_{f_\alpha(p)} \). So, for any pair of trivialization \( \{(z_a, u_a)\} \) and \( \{(z_b, u_b)\} \) of a reference holomorphic structure \( A \) such that \( f_\alpha(\text{Dom} z_a) \cap \text{Dom} z_b \neq \emptyset \), there exists a smooth \( \text{GL}(r, \mathbb{C}) \)-valued function \( \alpha_{ba} \) such that

\[
\begin{align*}
  u_b \circ \alpha &= \alpha_{ba} \circ \pi u_a.
\end{align*}
\]  

(6.1)
Under changes of trivialization in $A$, one has
\[ \alpha_{dc} = L_{db} \circ f_\alpha \alpha_{ba} L_{ca}^{-1}. \]  
(6.2)

If $\alpha$ belongs to $\text{Gau}_c(E)$, $f_\alpha = \text{id}_\Sigma$. Thus, $\alpha$ can be identified with an element of $S(\Sigma, \text{End} L)$.

The automorphism group $\text{Aut}_c(E)$ acts on the set of deformations $A'$ of the holomorphic structure $A$. Given an automorphism $\alpha$ of $E$, to any trivialization $(z'_b, u'_b)$ of $A'$ one associates another trivialization $(z''_a, u''_a)$, where $z''_a$ is given by (3.1) with $f = f_\alpha$ and [26]
\[ u''_a = u'_b \circ \alpha. \]  
(6.3)

The collection $A''$ of all trivializations $(z''_a, u''_a)$ of the above form is easily checked to be a holomorphic structure of $E$. $A''$ is the pull-back of $A'$ by $\alpha$ and may be denoted by $\alpha^* A'$. The holomorphic structure $a''$ of $\Sigma$ associated to $A''$ is precisely the pull-back $f_\alpha^* a'$ of the holomorphic structure $a'$ associated to $A'$, in the sense defined in sect. 3. The $a''$–holomorphic $\text{GL}(r, \mathbb{C})$–valued 1–cocycle $L'' = \alpha^* L'$ associated to $A''$ is related to $L'$ by
\[ (\alpha^* L')_{a''b'} = L'_{c'd'} \circ f_\alpha, \]  
(6.4)

$(z''_a, u''_a)$ and $(z'_b, u'_b)$ being the images of $(z'_c, u'_c)$ and $(z'_{d'}, u'_{d'})$ under the pull-back action of $\alpha$, respectively.

The pull-back action of $\text{Aut}_c(E)$ on the deformations $A'$ induces a right action of $\text{Aut}_c(E)$ on the intertwiner fields $\lambda$ and $V$ (cf. sect. 5). For any automorphism $\alpha \in \text{Aut}_c(E)$, this is defined by $\alpha^* \lambda(A') = \lambda(\alpha^* A')$ and $\alpha^* V(A') = V(\alpha^* A')$. $\alpha^* \lambda$ is given by (3.2) with $f$ substituted by $f_\alpha$. From (5.5) and the identity $\pi \circ \alpha = f_\alpha \circ \pi$, one obtains
\[ (\alpha^* V)_{aa''} = \alpha_{ba}^{-1} V_{bb'} \circ f_\alpha. \]  
(6.5)

In similar fashion, a right action of $\text{Aut}_c(E)$ is induced on the pairs $(\mu, A^*)$ formed by a Beltrami differential $\mu$ and a field $A^*$ of $S(\Sigma, k \otimes \text{End} L)$ parametrizing the deformations of the holomorphic structure of $E$ (cf. sect. 5), by the relations $\alpha^* \mu(A') = \mu(\alpha^* A')$ and $\alpha^* A^*(A') = A^*(\alpha^* A')$. $\alpha^* \mu$ is given by (3.3) with $f$ substituted by $f_\alpha$. $\alpha^* A^*$ can be computed from (3.3), (5.6) and (6.5). The local expression of the result is
\[ (\alpha^* A^*)_a = \frac{1}{\partial_a f_{ab} + \mu_b \circ f_a \partial_a f_{ab}} \left[ (\partial_a f_{ab} \tilde{\partial}_a f_{ab} - \tilde{\partial}_a f_{ab} \partial_a f_{ab}) \alpha_{ba}^{-1} (A^*_b - \mu_b A_b) \circ f_a \alpha_{ba} 
+ (\tilde{\partial}_a f_{ab} + \mu_b \circ f_a \tilde{\partial}_a f_{ab}) (\alpha_{ba}^{-1} \partial_a \alpha_{ba} + A_a) - (\partial_a f_{ab} + \mu_b \circ f_a \tilde{\partial}_a f_{ab}) \alpha_{ba}^{-1} \partial_a \alpha_{ba} \right]. \]  
(6.6)
\text{Aut}_c(E)$ acts on the space $S(\Sigma, k^{\otimes m} \otimes \bar{k}^{\otimes \bar{m}} \otimes L^{\otimes P} \otimes \bar{L}^{\otimes P} \otimes L^{\otimes Q} \otimes \bar{L}^{\otimes Q})$. The action is such that

$$(\alpha^* \Psi)'_{\alpha^*} = \Psi' \circ f_{\alpha}, \quad (6.7)$$

where $A'' = \alpha^* A'$. From (2.8), (3.6), (5.8) and (6.5), one finds

$$\alpha^* \Psi_a = (\alpha^{\otimes -P} \otimes \bar{\alpha}^{\otimes -\bar{P}})_{ba} \bar{w}_{ab}(f_{\alpha}; \mu)^{2m} \bar{w}_{ab}(f_{\alpha}; \mu)^{2\bar{m}} \Psi_b \circ f_{\alpha}(\alpha^{\otimes Q} \otimes \bar{\alpha}^{\otimes \bar{Q}})_{ba}. \quad (6.8)$$

When Hermitian structures on $E$ are envisaged, the symmetry group must be enlarged to the semidirect product $\text{Aut}_c(E) \times \text{Weyl}(E)$. Weyl($E$) is the generalized Weyl group of $E$. Weyl($E$) is the direct product group $\exp S(\Sigma, 1) \times \exp S(\Sigma, \text{End} L)$ and contains the ordinary Weyl group Weyl($\Sigma$) as a subgroup. The group multiplication is defined by

$$(\alpha_1, \nu^1, \Upsilon_1)(\alpha_2, \nu^2, \Upsilon_2) = (\alpha_1 \circ \alpha_2, \nu^1 f_{\alpha_1}^{-1} \nu^2, \Upsilon_1 \alpha_1^{-1} \Upsilon_2). \quad (6.9)$$

The action of a combined automorphism and extended Weyl transformation $(\alpha, \nu^\circ, \Upsilon)$ on the fields $\lambda, V, \mu$ and $A^*$ and on the generalized conformal fields reduces to that of $\alpha$ given above. On a Hermitian structure $(h^\circ, H)$, it is given by (3.8) with $h, f$ and $\nu$ replaced by $h^\circ, f_\alpha$ and $\nu^\circ$ and by

$$\alpha^* \nu^\circ \Upsilon H = \alpha^*(\Upsilon^{-1} H \Upsilon^{-1}). \quad (6.10)$$

This may be translated into an action on the Donaldson field $\Phi$, though I have found no simple explicit expression.

**b) The relation between the symmetry group of $E$ and those of its determinant and projectivization.**

To each automorphism $\alpha$ of $E$, there is associated an automorphism $\text{det} \alpha$ of $\text{det} E$ by the functor $\text{det}$. From the covariance of $\text{det}$, relations analogous to (6.1)–(6.2) are found to hold for $\text{det} u_a$ and $\text{det} \alpha_{ba}$. The map $\alpha \rightarrow \text{det} \alpha$ is a many-to-one homomorphism of $\text{Aut}_c(E)$ onto $\text{Aut}_c(\text{det} E)$. Two automorphisms $\alpha$ and $\beta$ of $E$ induce the same automorphism of $\text{det} E$ if and only if $f_\alpha = f_\beta$ and $\text{det} \alpha_{ba} = \text{det} \beta_{ba}$ for all $a, b$'s. Similarly, to each automorphism $\alpha$ of $E$, there is associated an automorphism $\text{proj} \alpha$ of $\text{proj} E$ by the functor $\text{proj}$ with completely analogous properties. It is not difficult to show that the automorphisms $\text{det} \alpha$ and $\text{proj} \alpha$ completely characterize $\alpha$. The proof of this property uses crucially the fact that the automorphisms considered are homotopically connected to the identity.
For any automorphism $\alpha$ and any deformation $A'$ of the holomorphic structure of $E$, the pull-back by the automorphism $\det \alpha$ of the induced deformations $\det A'$ of $\det E$ is such that $(\det \alpha)^* \det A' = \det (\alpha^* A')$. By the covariance of the functor $\det$, the trivializations of $(\det \alpha)^* \det A'$ and its associated cocycle $\det \alpha^* \det L'$ obey relations analogous to (6.3)–(6.4). Further, (6.5) implies an analogous relation for the deformation intertwiners $\det V$. The action of $\det \alpha$ on the induced deformations $\det A'$ translates into an action on the fields $(\mu, a^*)$ parametrizing the latter (cf. sect. 5). In particular, $\det \alpha^* a^*$ is given by (6.6) with $A^*$, $A$ and $\alpha_{ba}$ replaced by $a^*_b$, $a_b = (1/r)\text{tr} A_b$ and $\det \alpha_{ba}$, respectively. Totally analogous properties hold for the pull-back action of $\text{proj} \alpha$ on the induced deformations $\text{proj} A'$. In particular, $\text{proj} \alpha^* \hat{A}^*$ is given by (6.6) with $A^*_\alpha$, $A_\alpha$ and $\alpha_{ba}$ replaced by $\hat{A}^*_\alpha$, $\hat{A}_\alpha = A_\alpha - (1/r)\text{tr} A_\alpha 1$ and $(\det \alpha_{ba})^{-\frac{1}{2}}$, respectively.

One may similarly define the extended symmetry groups $\text{Aut}_e(\det E) \times \text{Weyl}(\det E)$ and $\text{Aut}_e(\text{proj} E) \times \text{Weyl}(\text{proj} E)$ of $\det E$ and $\text{proj} E$. The covariant functors $\det$ and $\text{proj}$ yield group homomorphisms from the extended symmetry group of $E$ onto those of $\det E$ and $\text{proj} E$, namely $(\alpha, \nu^e, \Upsilon) \rightarrow (\det \alpha, \nu^e, \det \Upsilon)$ and $(\alpha, \nu^e, \det \Upsilon) \rightarrow (\text{proj} \alpha, \nu^e, \text{proj} \Upsilon)$, respectively. These homomorphisms are many-to-one. However, $(\det \alpha, \nu^e, \det \Upsilon)$ and $(\text{proj} \alpha, \nu^e, \text{proj} \Upsilon)$ completely characterize $(\alpha, \nu^e, \Upsilon)$. $(\det \alpha, \nu^e, \det \Upsilon)$ and $(\text{proj} \alpha, \nu^e, \text{proj} \Upsilon)$ act on $(h^e, \det H)$ and $(h^e, \text{proj} H)$, respectively, and thus on $\Phi$ and $\hat{\Phi}$.

c) The special symmetry group of $E$.

In sect. 5, I have introduced the notion of special deformation and special Hermitian structure. It is now necessary to find out which subgroup of the symmetry group preserves the special character of such structures. This is the topic of this subsection.

It has been shown in sect. 5 that there is a $\mathbb{C}^*$–valued 1–cocycle $e'$ associated to any deformation $A'$ (cf. eq. (5.13)) and that $e'$ has the property of being flat unitary when restricting to a certain reduction $A'_0$ of $A'$. The question arises whether such property is preserved by the action of the automorphism group. Denoting by $\alpha^* e'$ the $f^*_\alpha a'$–holomorphic $\mathbb{C}^*$–valued 1 cocycle corresponding to $\alpha^* L'$, it is not difficult to show from (5.13) and (6.4) that

$$(\alpha^* e')^*_{a'^{c'}d'} = e'^*_{c'd'} \circ f_\alpha,$$

(6.11)

where $A'' = \alpha^* A'$. This relation shows that the reduction $A'_0$ of $A'$ with respect to whose trivializations $e'_{a'^{c'}d'}$ is a constant of unit modulus is mapped by $\alpha^*$ into its counterpart $A''_0$ in $A''$. This property is crucial for the consistency of what follows.

From (5.14), (6.5) and (3.2), one finds

$$\exp \left( pr^* \sigma_{aa'} - pr^* \sigma_{bb'} \circ f_\alpha \right) = \varrho_{ab}(f_\alpha; \mu)^{2q} \det \alpha_{ba}^{-p},$$

(6.12)
where $\alpha^*\sigma$ be the intertwiner field associated to $\alpha^*V$ via (5.14). From this relation and (5.16), it appears that a general automorphism $\alpha$ does not map the subclass of special deformations into itself, since $\alpha^*\sigma_{aa'}$ may not vanish for any reduction of $A''$. From (6.12), the automorphisms $\alpha$ for which this happens are those satisfying the relation

$$\det \alpha_{ba}^{-p} = \mathcal{w}_{ab}(f\alpha; \mu)^{2q}$$

(special automorphisms) (6.13)

for the trivializations of $A_0$. I shall call such automorphisms special. They form a subgroup $\text{Aut}(E; \mu)$ of $\text{Aut}(E)$ depending on $\mu$. For any special automorphism $\alpha$, $\det \alpha$ is completely determined by $f\alpha$ and $\mu$. Since $\det \alpha$ and $\text{proj} \alpha$ characterize $\alpha$, the restriction of the group homomorphism $\alpha \rightarrow \text{proj} \alpha$ to the subgroup of special automorphisms of $E$ is a group isomorphism.

Proceeding in similar fashion for Hermitian structures, one finds the relation

$$\exp \left( pr \alpha^* \upsilon^\circ \Upsilon_a - pr \Upsilon_b \circ f\alpha \right) = \left| \mathcal{w}_{ab}(f\alpha; \mu)^{2q} \det \alpha_{ba}^{-p} \left( \upsilon^\circ_{ab} \det \Upsilon_b^p \right) \circ f\alpha \right|^2,$$

from (5.17), (3.8) and (6.10). From this relation and (5.19), one sees that the action of the symmetry group on the Hermitian structures of $E$ does not map the subclass of special Hermitian structures into itself. As appears from (6.14), in order a combined automorphism and extended Weyl transformation $(\alpha, \upsilon^\circ, \Upsilon)$ to have such a property, it is sufficient that $\alpha$ is special and that

$$\det \Upsilon_a^p = \upsilon^\circ_{\otimes q}$$

(special Weyl transformations) (6.15)

for the trivializations of $A_0$. The transformations of such type form a special subgroup $\text{Aut}_c(E; \mu) \times \text{Weyl}_0(E)$ of the full extended symmetry group $\text{Aut}_c(E) \times \text{Weyl}(E)$, depending on $\mu$. For a special combined automorphism and Weyl transformation $(\alpha, \upsilon^\circ, \Upsilon)$, $(\det \alpha, \upsilon^\circ, \det \Upsilon)$ is completely determined by $f\alpha$, $\mu$ and $\upsilon^\circ$. Since $(\det \alpha, \upsilon^\circ, \det \Upsilon)$ and $(\text{proj} \alpha, \upsilon^\circ, \text{proj} \Upsilon)$ completely characterize $(\alpha, \upsilon, \Upsilon)$, the restriction of the group automorphism $(\alpha, \upsilon^\circ, \Upsilon) \rightarrow (\text{proj} \alpha, \upsilon^\circ, \text{proj} \Upsilon)$ to the subgroup of special symmetry transformations is a group isomorphism. Note how the results of the previous paragraph generalize directly to Hermitian geometry.

\textit{d) Infinitesimal formulation and the Slavnov operator.}

As for the symmetry of surfaces, one is mainly interested in the infinitesimal action of $\text{Aut}_c(E) \times \text{Weyl}(E)$ and is thus lead to considering its Lie algebra $\text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)$ [26]. Writing formally $f\alpha = \text{id}_\Sigma + \xi$, $\alpha = \text{id}_E + M$, $\upsilon^\circ = 1 - \theta^\circ/2$ and $\Upsilon = 1 - \Theta/2$ in (6.9), one finds that $\text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)$ is isomorphic to the semidirect sum
(\bigcup_{\xi \in S(\Sigma, k^{-1})} \{ \xi \} \times (\xi A + S(\Sigma, \text{End} L))) \oplus S(\Sigma, 1) \oplus S(\Sigma, \text{End} L), \text{ where } A \text{ is a reference (1,0) connection of } L, \text{ with Lie brackets }

\left[ [\xi_1, \Omega_1] \oplus \theta^0_1 \oplus \Theta_1, (\xi_2, \Omega_2) \oplus \theta^0_2 \oplus \Theta_2 \right] = ((\xi_1 \partial + \bar{\xi}_1 \bar{\partial})\xi_2 - (\xi_2 \partial + \bar{\xi}_2 \bar{\partial})\xi_1, \\
(\xi_1 \partial + \bar{\xi}_1 \bar{\partial})\Omega_2 - (\xi_2 \partial + \bar{\xi}_2 \bar{\partial})\Omega_1 - [\Omega_1, \Omega_2]) \oplus ((\xi_1 \partial + \bar{\xi}_1 \bar{\partial})\theta^0_2 - (\xi_2 \partial + \bar{\xi}_2 \bar{\partial})\theta^0_1) \\
\oplus ((\xi_1 \partial + \bar{\xi}_1 \bar{\partial})\Theta_2 - (\xi_2 \partial + \bar{\xi}_2 \bar{\partial})\Theta_1 - [\Omega_1, \Theta_2] + [\Omega_2, \Theta_1] + (1/2)[\Theta_1, \Theta_2]). 

(6.16)

To express the infinitesimal action of the symmetry group \( \text{Aut}_c(E) \times \text{Weyl}(E) \) on the relevant space \( F \) of field functionals, one introduces the exterior algebra \( \Lambda^* \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)^\vee \) of \( \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right) \). The standard generators of the algebra are the automorphism ghosts \( c \) and \( M \) and the Weyl ghost \( w^\circ \) and \( W^7 \). \( c \) is a section of \( k^{-1} \otimes \Lambda^1 \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)^\vee \), \( w^\circ \) is a section of \( 1 \otimes \Lambda^1 \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)^\vee \), \( M - cA \) and \( W \) are sections of \( \text{End} L \otimes \Lambda^1 \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)^\vee \). \( c, M, w^\circ \) and \( W \) are defined by \( \langle (c(p), M(p)), (\xi, \Omega) \oplus \theta^0 \oplus \Theta \rangle = (\xi(p), \Omega(p)), \langle w^\circ(p), (\xi, \Omega) \oplus \theta^0 \oplus \Theta \rangle = \theta^0(p) \) and \( \langle W(p), (\xi, \Omega) \oplus \theta^0 \oplus \Theta \rangle = \Theta(p) \) for \( p \in \Sigma \), respectively.

Linearizing the action of the symmetry group on the relevant space \( F \) of field functionals, defines a nilpotent coboundary operator \( s \) on the space \( \Lambda^* \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)^\vee \otimes F \) called, as in sect. 2, the Slavnov operator. From (6.16), one can easily read off the structure equations obeyed by \( c, M, w^\circ \) and \( W \). For \( c \) and \( w^\circ \), these are (3.10) and (3.11), respectively. For \( M \) and \( W \), they read [26]

\[ sM = (c\partial + \bar{c}\bar{\partial})M - M^2, \]  
\[ sW = (c\partial + \bar{c}\bar{\partial})W - [M, W] + (1/2)W^2. \]

(6.17)  
(6.18)

For a given Beltrami differential \( \mu \) in \( \text{Beltr}(\Sigma) \) and a given element \( A^* \) of \( S(\Sigma, \bar{k} \otimes \text{End} L) \), the relevant combinations of ghost fields are \( C \), given by (3.12), and [26]

\[ X = M - cA - \bar{c}A^*, \]  
(6.19)

where \( A \) is the same reference connection of \( L \) entering the definition of \( A^* \), eq. (5.6). \( X \) is a section of \( \text{End} L \otimes \Lambda^1 \text{Lie} \left( \text{Aut}_c(E) \times \text{Weyl}(E) \right)^\vee \). (3.13), (3.14) and (3.15) hold unchanged in the present more general context. From (6.5) and (6.6), one has further

\[ sV_{a^*} = C(\partial - A)V_{a^*} - XV_{a^*}, \]  
(6.20)

7 In ref. [26], the ghost \( M \) is written as a sum \( M_{\text{pt}} + G \), where \( M_{\text{pt}} \) corresponds to parallel transport along the vector field \( c\partial + \bar{c}\bar{\partial} \) and \( G \) is the gauge ghost. Though interesting for other reasons, this separation is not relevant in the following analysis.
\[ sA^* = -(\bar{\partial} - \mu \partial_A - \text{ad} A^*)X + C(\partial A^* - \bar{\partial}A + [A^*, A]), \quad sA = 0, \quad (6.21) \]

where \( \partial_A = \partial - \text{ad} A \) is the covariant derivative of the connection \( A \). From (3.10), (6.17) and (6.21), one finds that

\[ sX = C\partial_A X - X^2. \quad (6.22) \]

From (6.7)–(6.8), it follows that, for any field \( \Psi \) in \( S(\Sigma, k^\otimes m \otimes \bar{k}^\otimes m \otimes L^\otimes P \otimes \bar{L}^\otimes P \otimes L^\otimes Q \otimes \bar{L}^\otimes Q) \), one has

\[ s\Psi' = (c\partial + c\bar{\partial})\Psi', \quad (6.23) \]

\[ s\Psi = (c\partial + c\bar{\partial})\Psi + (m(\partial c + \mu \partial c) + m(\bar{\partial}c + \bar{\mu} \partial c))\Psi - [\mathcal{R}_P(M) \otimes \bar{1}^\otimes \bar{P} + 1^\otimes P \otimes \mathcal{R}_{\bar{P}}(\bar{M})] \Psi + \Psi[\mathcal{R}_Q(M) \otimes \bar{1}^\otimes \bar{Q} + 1^\otimes Q \otimes \mathcal{R}_{\bar{Q}}(\bar{M})], \quad (6.24) \]

where \( \mathcal{R}_R(T) = \sum_{s=0}^{R-1} 1^\otimes s \otimes T \otimes 1^\otimes R-s-1 \) and \( \mathcal{R}_R(T) = \sum_{s=0}^{R-1} \bar{1}^\otimes s \otimes \bar{T} \otimes \bar{1}^\otimes R-s-1 \). These relations generalize (3.16)–(3.17).

\( s\phi^o \) is given by (3.18) in terms of \( \phi^o \) and \( w^o \). A simple calculation using (5.9) and (6.10) yields

\[ s\Phi = (c\partial_G + c\bar{\partial})\Phi + \frac{\text{ad} \Phi}{\exp \text{ad} \Phi - 1} (c\Gamma_G - M + (1/2)W) \]

\[ + G \left( \frac{\text{ad} \Phi}{\exp \text{ad} \Phi - 1} (c\Gamma_G - M + (1/2)W) \right)^\dagger G^{-1}, \quad sG = 0, \quad (6.25) \]

where \( \partial_G = \partial - \text{ad} \Gamma_G \) and \( \text{ad} \Phi/(\exp \text{ad} \Phi - 1) \) is understood as the power series of the function \( t \to t/(\exp t - 1) \) with \( t \) replaced by \( \text{ad} \Phi \).

It is simple to show that \( \text{Lie}(\text{Aut}_c(\text{det} E) \times \text{Weyl}(\text{det} E)) \) and \( \text{Lie}(\text{Aut}_c(\text{proj} E) \times \text{Weyl}(\text{proj} E)) \) are isomorphic to the semidirect sums \( \left( \bigcup_{\xi \in S(\Sigma, k-1)} \{\xi\} \times \{a + S(\Sigma, 1)\} \right) \oplus S(\Sigma, 1) \oplus S(\Sigma, 1) \) and \( \left( \bigcup_{\xi \in S(\Sigma, k-1)} \{\xi\} \times \{\xi\hat{A} \hat{S}(\Sigma, \text{End} L)\} \right) \oplus S(\Sigma, 1) \oplus \hat{S}(\Sigma, \text{End} L) \), respectively, where \( a \) and \( \hat{A} \) are the normalized trace and traceless part of the connection \( A \) and \( \hat{S}(\Sigma, \text{End} L) \) is the space of traceless sections of \( \text{End} L \). The corresponding Lie brackets are given by (6.16) upon setting formally \( \Omega_i = \omega_i \) and \( \Theta_i = \theta_i \) and \( \hat{\Omega}_i = \hat{\Omega}_i \) respectively, with \( \omega_i, \theta_i \in S(\Sigma, 1) \) and \( \hat{\Omega}_i, \hat{\Theta}_i \in \hat{S}(\Sigma, \text{End} L) \). The covariant functors det and proj induce the Lie algebras homomorphisms \( (\xi, \Omega) \oplus \theta^o \oplus \Theta \to (\xi, \omega) \oplus \theta^o \oplus \theta \) and \( (\xi, \Omega) \oplus \theta^o \oplus \Theta \to (\xi, \hat{\Omega}) \oplus \theta^o \oplus \hat{\Theta} \) of \( \text{Lie}(\text{Aut}_c(E) \times \text{Weyl}(E)) \) onto \( \text{Lie}(\text{Aut}_c(\text{det} E) \times \text{Weyl}(\text{det} E)) \) and \( \text{Lie}(\text{Aut}_c(\text{proj} E) \times \text{Weyl}(\text{proj} E)) \), respectively, where \( \omega = (1/r)\text{tr} \Omega \) and \( \hat{\Omega} = \Omega - (1/r)\text{tr} \Omega \) and similarly for \( \Theta \).

\footnote{For any two matrices \( S \) and \( T \) \( \text{ad} S \cdot T = [S, T] \).}
To the Lie algebra homomorphisms defined in the previous paragraph, there correspond a decomposition of the ghost fields $M$ and $W$ in their normalized trace and traceless parts $m, \bar{M}, w$ and $\bar{W}$, respectively. $sm, s\bar{M}, sw$ and $s\bar{W}$, can then be easily found from (6.17) and (6.18). The relevant combinations of ghost fields are the normalized trace $x$ and the trace part $\hat{X}$ of the ghost field $X$. $sa^*$ can be easily obtained by tracing (6.21) and expressing everything in terms of $x, a$ and $a^*$. $sA^*$ is given by (6.21) with $X A$ and $A^*$ replaced by $\hat{X} \hat{A}$ and $\hat{A}^*$, respectively. From (6.22), it is easy to see that $sx = C\partial x$ and that $s\hat{X}$ is given by (6.22) with $X$ and $A$ substituted by $\hat{X}$ and $\hat{A}$. From (6.25), one can also easily obtain expressions for $s\phi$ and $s\Phi$.

From (6.12), after some simple rearrangements using (5.15), one gets

$$s\sigma_{a'} = (j\partial + a + \partial\sigma_{a'})C + x. \quad (6.26)$$

From (6.14), one can easily find also a formula for $s\tau$.

The special symmetry transformations correspond to a subalgebra of the symmetry algebra defined by the constraints $\omega = -j(\partial \xi + \mu \partial \bar{\xi})$ and $\theta = j\theta^\circ$. Hence, when restricting to the special symmetry algebra, the relations

$$m = -j(\partial c + \mu \partial \bar{c}), \quad \text{(special automorphisms)} \quad (6.27)$$

$$w = jw^\circ, \quad \text{(special Weyl transformations)} \quad (6.28)$$

hold, as follows from (6.13) and (6.15). For special deformations and restricting to the special symmetry algebra one has that

$$x = -(j\partial + a)C, \quad (6.29)$$

by (5.15), (5.16), (6.27) and (6.19).

7. Extended conformal field theory: automorphism versus extended Weyl anomaly.

I shall now describe a generalization of conformal field theory, trying to highlight as much as possible the parallelism with ordinary conformal field theory. The geometric setting is that described in sects. 5 and 6. Consider a holomorphic structure $A$ of $E$ and the associated holomorphic structure $a$ of $\Sigma$. The basic fields of a generalized conformal field theory are smooth sections of $a$–holomorphic 1–cocycles of the form $k^\otimes m \otimes L$ with $m \in \mathbb{Z}/2$. The classical action is given by the integral of a $(1, 1)$ density depending locally on the fields. In this sense, one is dealing with a classical field theory with an extended conformal
symmetry. The quantization is carried out by means of the $\zeta$ function renormalization scheme. This requires the introduction of a Hermitian structure $(h^o, H)$ of $E$ to properly define the adjoint of the relevant differential operators. The resulting effective action $I_{\text{aut}}(h^o, H)$ will thus depend on this structure. One may fix a reference holomorphic structure $A$ of $E$ and the Hermitian metrics $H$ and consider any deformation $A'$ of $A$ and the deformation $(h^{o'}, H')$ of $(h^o, H)$ associated to $A'$. Correspondingly, one has the effective action $I'_{\text{aut}}(h^{o'}, H')$. In the present context, the symmetry group is $\text{Aut}(E) \times \text{Weyl}(E)$. The $\zeta$ function scheme is automorphism invariant but not Weyl invariant, for in general $I'_{\text{aut}}(h^{o'}, H')$ depends on the metric $H'$ and not simply on the holomorphic structure $A'$. As for ordinary conformal field theory, the form of the extended Weyl anomaly is universal:

$$sI'_{\text{aut}}(h^{o'}, H') = \kappa^o A'_{\text{conf}}(w^{o_1}, h^{o'}) + K A'_{\text{ext}}(W'_{H'}, H'),$$

(7.1)

where $A'_{\text{conf}}$ and $w^{o_1}$ are given by (4.2) and (4.3) and

$$A'_{\text{ext}}(W'_{H'}, H') = \frac{-1}{12\pi} \int_{\Sigma} \frac{d\bar{z}' \wedge dz'}{2i} \text{tr} \left( W'_{H'} F^i_{H'} \right),$$

(7.2)

$$W'_{H'} = (W' + H' W'^{\dagger} H'^{-1})/2,$$

(7.3)

$W'$ being the deformation of the Weyl ghost $W$. The anomaly contains a contribution which is of the form of an ordinary conformal anomaly with central charge $\kappa^o$. The other contribution reflects the dependence on the scale of $H'$. $K$ is a further coefficient measuring the anomaly strength and may be viewed as a generalized central charge. Note that $A'_{\text{ext}}$ reduces to $A'_{\text{conf}}$ when the rank $r$ of $E$ equals 1. To check the consistency of the above, one must verify that the anomaly $A'_{\text{ext}}$ is a local Slavnov cohomology class. This can be seen as follows.

The structure of the extended Weyl anomaly (7.2)–(7.3) suggests considering the following differential form on the space $\mathcal{H}'$ of Hermitian structures of $E$ subordinated to the holomorphic structure $A'$:

$$D' = \frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z}' \wedge dz'}{2i} \text{tr} \left( \delta H' H'^{-1} F^i_{H'} \right).$$

(7.4)

Using that $\delta^2 = 0$, it is simple to verify that $\delta D' = 0$, i.e. $D'$ is closed.

For any field $\Psi$, define

$$s_{\text{diff}} \Psi' = (c \partial + \bar{c} \partial) \Psi',$$

(7.5)

$$s_{\text{diff}} c = sc = (c \partial + \bar{c} \partial) c,$$

(7.6)

(cf. eq. (3.10)). $s_{\text{diff}}$ is a nilpotent operator, as is easy to verify. $s_{\text{diff}}$ expresses the infinitesimal pull-back action of the diffeomorphism group on deformations of fields. Indeed,
$s_{\text{diff}}$ is the same as $s$ when acting on all those fields for which the action of the symmetry group is just the automorphism pull-back action, but it obviously differs from $s$ when acting on fields such as $H'$ and $W'$ which do not enjoy such property. In fact, recalling (6.7)–(6.8) and (6.10), it is easy to see that

$$sH' = s_{\text{diff}}H' + W'H', \quad (7.7)$$

$$sW' = s_{\text{diff}}W' + (1/2)W'^2. \quad (7.8)$$

Since integration is invariant under the pull-back action of diffeomorphisms, one has

$$s_{\text{diff}}A'_{\text{ext}}(W'H', H') = 0. \quad (7.9)$$

Now, from (3.1), it appears that $(s - s_{\text{diff}})' = 0$. So, the computation of any expression of the form $(s - s_{\text{diff}}) \int_{\Sigma} \frac{dz' \wedge dz'}{2\pi} \tau'$ can be carried out by varying the fields but keeping the coordinates $z'$ fixed. From this remark, (7.7) and (7.9) and the closedness of $D$, one finds then

$$sA'_{\text{ext}}(W'H', H') = (s - s_{\text{diff}})A'_{\text{ext}}(W'H', H')$$

$$= (s - s_{\text{diff}}) \frac{-1}{12\pi} \int_{\Sigma} \frac{dz' \wedge dz'}{2\pi} \text{tr}
\{(s - s_{\text{diff}})H'H^{-1}F'H'\} = 0. \quad (7.10)$$

From (7.10), it follows then that $A'_{\text{ext}}$ is $s$–closed. The locality of $A'_{\text{ext}}$ can be verified directly by expressing it in terms of the fields $\mu$ and $A^*$ parametrizing the deformations $A'$ and the Donaldson field $\Phi$ parametrizing the metrics $H$ with respect to a fixed reference metric $G$ (cf. sect. 5). The reference connection $(1, 0) A$ of $L$ entering into the definition of $A^*$ (cf. eq. (5.6)) is conveniently chosen to be the metric connection $\Gamma_G$ of $G$. Using (2.6), (2.7), (5.6) and (5.8), one finds

$$A'_{\text{ext}}(W'H', H') = A_{\text{ext}}(W, W^\dagger, \Phi, \mu, \bar{\mu}, A^*, A^{*\dagger}; G) = \frac{-1}{12\pi} \int_{\Sigma} \frac{dz \wedge dz}{2\pi} \text{tr}
\{W_H \left[ F_H + \left( \partial_G - \bar{\mu} \partial - (\bar{\partial} \mu) \right) \frac{A^*_H}{1 - \mu \bar{\mu}} - \left( \bar{\partial} - \mu \partial_G - (\partial \mu) \right) \frac{G A^*_H H_1}{1 - \mu \bar{\mu}} \right] + \left[ A^*_H, G A^*_H H_1 G^{-1} \right] \}, \quad (7.11a)$$

where $W_H$ is given by an expression analogous to (7.3) and

$$A^*_H = A^* + \mu (\Gamma_H - \Gamma_G). \quad (7.11b)$$

Recall that $H = \exp \Phi G$ (cf. eq. (5.9)), $\Gamma_H = \Gamma_G + \partial_G \exp \Phi \exp - \Phi$ and $F_H = F_G + \bar{\partial} (\partial_G \exp \Phi \exp - \Phi)$.

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In analogy to ordinary conformal field theory, the extended Weyl anomaly can be absorbed by adding to \( I_{\text{aut}}(h', H') \) a local counterterm which in turn creates a chirally split automorphism anomaly. I shall now expound in some detail the calculation of such counterterm.

I have shown above that the differential form \( D' \) defined by (7.4) is closed. Since \( \mathcal{H}' \) is clearly contractible, \( D' \) is also exact. To compute a primitive of \( D' \), one can integrate \( D' \) along any functional path in \( \mathcal{H}' \). A convenient choice is given by the path \( H'_t = \exp(t\Phi')G' \) with \( 0 \leq t \leq 1 \). The result reads

\[
S'_{D}(\Phi', G') = -\frac{1}{12\pi} \int_{\Sigma} \frac{dz' \wedge dz'}{2i} \text{tr} \left[ \bar{\partial}' \Phi' \exp \text{ad} \Phi' - 1 - \text{ad} \Phi' \right] \partial'_{\bar{G}} \Phi' - \bar{F}'_{G'}, \Phi'],
\]

where \( (\exp \text{ad} \Phi' - 1 - \text{ad} \Phi')/(\text{ad} \Phi')^2 \) is the power series of the function \( \exp(t - 1 - t)/t^2 \) with \( t \) substituted by \( \text{ad} \Phi' \). \( S'_{D}(\Phi', G') \) is the Donaldson action (with no extended cosmological term) [29] and generalizes the Liouville action in the present context. Indeed, it is easy to verify that when \( E \) has rank \( r = 1 \), \( S'_{D}(\Phi', G') \) reduces to the Liouville action \( S_{L}(\phi, g') \) (cf. eq. (4.4)). It is interesting to express \( S'_{D}(\Phi', G') \) directly in terms of the fields \( \mu A^* \) and \( \Phi \). Using (2.6), (2.7), (5.6) and (5.8), one finds

\[
S'_{D}(\Phi', G') = \int_{\Sigma} \frac{dz' \wedge dz'}{2i} \text{tr} \left\{ \frac{1}{1 - \mu \bar{\mu}} \left( \bar{\partial} - \mu \partial_{G} - \text{ad} A^* \right) \Phi \\
\exp \text{ad} \Phi - 1 - \text{ad} \Phi \frac{\left( \partial_{G} - \bar{\mu} \partial + \text{ad} (GA^* G^{-1}) \right) \Phi - \left[ F_{G} - \left( \partial_{G} - \bar{\mu} \partial - \bar{\partial} \partial \right) \frac{A^*}{1 - \mu \bar{\mu}} \right] \Phi \right\}.
\]

From here, the locality of \( S'_{D}(\Phi', G') \) is apparent.

Let us compute \( sS'_{D}(\Phi', G') \). The calculation can be performed in several ways, the simplest of which is the following. Proceeding as done earlier for \( A_{\text{ext}} \), one finds that

\[
s_{\text{diff}}S'_{D}(\Phi', G') = 0,
\]

so that, recalling that \( s\zeta' = s_{\text{diff}}\zeta' \),

\[
sS'_{D}(\Phi', G') = (s - s_{\text{diff}})S'_{D}(\Phi', G') = \frac{1}{12\pi} \int_{\Sigma} \frac{dz' \wedge dz'}{2i} \text{tr} \left[ (s - s_{\text{diff}}) \exp \Phi' \exp - \Phi' + \exp \text{ad} \Phi' \left( sG' G^{-1} \right) F'_{\exp \Phi' G'} - (s - s_{\text{diff}})G' G^{-1} F'_{G'} \right].
\]

Now, from (7.7), using (5.9), one has \( sH' H'^{-1} = s \exp \Phi' \exp - \Phi' + \exp \text{ad} \Phi' (sG' G'^{-1}) = s_{\text{diff}} H' H'^{-1} + W_{H'}, = s_{\text{diff}} \exp \Phi' \exp - \Phi' + \exp \text{ad} \Phi' (s_{\text{diff}} G' G'^{-1}) + W_{H'}, \) By comparing
the two expressions, one finds that \((s - s_{\text{diff}}) \exp \Phi' \exp -\Phi + \exp \text{ad} \Phi' ((s - s_{\text{diff}}) G' G'^{-1}) = W_{H'}^t\). By using this relation in (7.15), recalling that \(H' = \exp \Phi' G\) and using (7.2), one finds

\[
sS_D'(\Phi', G') = -\mathcal{A}_{\text{ext}}(W_{H'}, H') - \frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{tr} \left[ (s - s_{\text{diff}}) G' G'^{-1} F_{G'} \right]. \tag{7.16}
\]

From (6.5), one finds easily that \(sV = s_{\text{diff}} V - MV\). Recalling that \(G' = V^{-1} G V^{-1}^{\dagger}\) and \(sG = 0\), it is straightforward to verify that \((s - s_{\text{diff}}) G' G'^{-1} = V^{-1} [M - c\Gamma_G + G(M - c\Gamma_G)^{\dagger} G^{-1}] V\). From this relation, using (2.6), (2.7), (5.6) and (5.8), (7.16) can be written as

\[
\begin{align*}
sS_D(\Phi, \mu, \bar{\mu}, A^*, A^{\dagger}; G) &= -\frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{tr} \left\{ \left[ M - c\Gamma_G + G(M - c\Gamma_G)^{\dagger} G^{-1} \right] F_G \right. \\
&\quad - \left( \partial_G - \bar{\mu} \partial - (\bar{\partial} \mu) \right) \frac{A^*}{1 - \mu \bar{\mu}} - \left( \bar{\partial} - \bar{\mu} \partial_G - (\partial \mu) \right) \frac{G A^{\dagger} G^{-1}}{1 - \mu \bar{\mu}} + \left[ A^*, G A^{\dagger} G^{-1} \right] \left[ A^{\dagger}, G A^{\dagger} G^{-1} \right] \right\} \\
&\quad - \mathcal{A}_{\text{ext}}(W, W^{\dagger}, \Phi, \mu, \bar{\mu}, A^*, A^{\dagger}; G). \tag{7.17}
\end{align*}
\]

Both (7.16) and (7.17) show that the Weyl anomaly can be absorbed by adding to the effective action \(I'_{\text{aut}}(H')\) a counterterm \(K S_D(\Phi, \mu, \bar{\mu}, A^*, A^{\dagger}; G)\) at the price of creating a chirally non-split automorphism anomaly.

To obtain a chirally split automorphism anomaly, one has to find another counterterm analogous to \(S_{V_{KLT}}\). This is the Knecht-Lazzarini-Stora action [26]. Its expression is

\[
\begin{align*}
S_{KLS}(\mu, \bar{\mu}, A^*, A^{\dagger}; G, A, A^{\dagger}) &= -\frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{tr} \left\{ \left( A - \Gamma_G \right) \left( A^* + \frac{1}{2} \mu (A - \Gamma_G) \right) \right. \\
&\quad + G \left[ \left( A - \Gamma_G \right) \left( A^* + \frac{1}{2} \mu (A - \Gamma_G) \right) \right]^{\dagger} G^{-1} + \frac{1}{1 - \mu \bar{\mu}} \left[ A^* G A^{\dagger} G^{-1} - \frac{1}{2} \mu A^2 \right. \\
&\quad \left. \left. - \frac{1}{2} \mu G A^{\dagger 2} G^{-1} \right] \right\}, \tag{7.18}
\end{align*}
\]

where \(A\) is an arbitrary \((1, 0)\) connection of \(L\) such that

\[
sA = 0. \tag{7.19}
\]

By explicit calculation, using (3.14) and (6.21), one finds

\[
\begin{align*}
sS_{KLS}(\mu, \bar{\mu}, A^*, A^{\dagger}; G, A, A^{\dagger}) &= \frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{tr} \left\{ \left[ M - c\Gamma_G + G(M - c\Gamma_G)^{\dagger} G^{-1} \right] \right. \\
&\quad \left[ F_G - \left( \partial_G - \bar{\mu} \partial - (\bar{\partial} \mu) \right) \frac{A^*}{1 - \mu \bar{\mu}} - \left( \bar{\partial} - \bar{\mu} \partial_G - (\partial \mu) \right) \frac{G A^{\dagger} G^{-1}}{1 - \mu \bar{\mu}} + \left[ A^*, G A^{\dagger} G^{-1} \right] \right\} \\
&\quad + \mathcal{A}_{\text{aut}}(X_A, A^* A; A) + \overline{\mathcal{A}}_{\text{aut}}(X_A, A^* A; A), \tag{7.20}
\end{align*}
\]
where
\[ A_{\text{aut}}(X_A, A^* A; A) = \frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{tr} \left[ X_A(\partial A^* A - \bar{\partial} A + [A^* A, A]) \right], \quad (7.21) \]

with
\[ A^* A = A^* + \mu (A - \Gamma_G), \]
\[ X_A = X - C(A - \Gamma_G). \]

\( A^* A \) and \( X_A \) differ from \( A^* \) and \( X \) by a redefinition of the reference \((1, 0)\) connection entering in the definition of the latter from \( \Gamma_G \) to \( A \).

The required counterterm contains two contributions. The first contribution is given by \( \kappa^\circ \) times the counterterm \( \Delta I_{\text{cft}} \) (cf. eq. (4.7)) and shifts the ordinary Weyl anomaly into the chirally split diffeomorphism anomaly, as explained in sect. 4. The second contribution is given by \( K \) times the counterterm
\[ \Delta I(\Phi, \mu, \bar{\mu}, A^*, A^* \dagger; G, A, A \dagger) = S_D(\Phi, \mu, \bar{\mu}, A^*, A^* \dagger; G) + S_{KLS}(\mu, \bar{\mu}, A^*, A^* \dagger; G, A, A \dagger), \quad (7.24) \]
as follows from (7.17) and (7.20). Thus the extended Weyl invariant effective action is
\[ I_{\text{ext}}(\mu, \bar{\mu}, A^*, A^* \dagger; \mathcal{R}, \bar{\mathcal{R}}, A, A \dagger) = I'_{\text{aut}}(h^\circ t, H') + \kappa^\circ \Delta I_{\text{cft}}(\phi^\circ, \mu, \bar{\mu}; g^\circ, \mathcal{R}, \bar{\mathcal{R}}) + K \Delta I(\Phi, \mu, \bar{\mu}, A^*, A^* \dagger; G, A, A \dagger), \quad (7.25) \]
which is easily verified to satisfy the Ward identity
\[ sI_{\text{ext}}(\mu, \bar{\mu}, A^*, A^* \dagger; \mathcal{R}, \bar{\mathcal{R}}, A, A \dagger) = \kappa^\circ A_{\text{diff}}(C, \mu; \mathcal{R}) + \kappa^\circ \bar{A}_{\text{diff}}(C, \mu; \bar{\mathcal{R}}) + K A_{\text{aut}}(X_A, A^* A; A) + K \bar{A}_{\text{aut}}(X_A, A^* A; A). \quad (7.26) \]

\( I_{\text{ext}} \) is manifestly independent from \( g^\circ \) and \( G \) by Weyl invariance. Further, \( I_{\text{ext}} \) is holomorphically factorized [26], that is one has the chiral splitting \( I_{\text{ext}}(\mu, \bar{\mu}, A^*, A^* \dagger; \mathcal{R}, \bar{\mathcal{R}}, A, A \dagger) = I_P(\mu, A^*; \mathcal{R}, A) + \bar{I}_P(\mu, A^*; \bar{\mathcal{R}}, A), \) where \( I_P \) is a generalized Polyakov action.

Before proceeding further, a few remarks about the above results are in order. The Donaldson action was originally introduced by Donaldson [29] to study the problem of the existence of Hermitian structures \((h^\circ t, H')\) in a holomorphic vector bundle \( E \) such that \( F_H \sim h^\circ 1. \) His method generalizes the classical analysis of Liouville reducing the problem of finding a constant curvature metric \( h \) on a Riemann surface \( \Sigma \) to a variational problem. The Donaldson action is formally similar to a non compact gauged WZWN model. Indeed, setting \( \mu = 0 \) and \( \bar{\mu} = 0 \) for simplicity, (7.13) can be written as
\[ S_D(\Phi, 0, \bar{0}, A^*, A^* \dagger; G) = S_D(\Phi, 0, 0, 0, 0; G) - \frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left[ -A^* \partial_G \exp \Phi \exp -\Phi + GA^* G^{-1} \partial \exp -\Phi \exp \Phi + A^* \exp \text{ad} \Phi (GA^* G^{-1}) - A^* G A^* G^{-1} \right]. \quad (7.27) \]
The ‘WZ field’ is the Hermitian metric \( H = \exp \Phi G \) and takes values in the subset of Hermitian non-singular elements of \( S(\Sigma, L \otimes \bar{L}) \). The ‘gauge fields’ are \( A^* \) and \( GA^*\dagger G^{-1} \). As the field \( \Phi \) is determined uniquely by \( H, G \) and the Hermiticity condition (5.10), the action is manifestly singlevalued and no level quantization occurs. Beyond the similarities, there are crucial differences. There is no underlying group theoretic structure. The gauge fields are not necessarily traceless and are related by Hermitian conjugation. They are not yielded by gauging but by deformation of the holomorphic structure of the bundle \( E \).

It is interesting to find the anomalous Ward identity obeyed by the generalized Polyakov action \( I_P \) in differential form. This consists of two parts:

\[
(\bar{\partial} - \mu \partial_A - (\partial \mu) - \text{ad} A^*) \frac{\delta I_P}{\delta A^*}(\mu, A^*; \mathcal{R}, A) = \frac{K}{12\pi} \partial_A A^*, \tag{7.28a}
\]

\[
\left[ -(\bar{\partial} - \mu \partial - 2(\partial \mu)) \frac{\delta I_P}{\delta \mu} + \text{tr} \left( \frac{\delta I_P}{\delta A^*} \frac{\delta I_P}{\delta A^*} \right) \right](\mu, A^*; \mathcal{R}, A) = \frac{\kappa^\circ}{12\pi} (\partial^3 + 2\mathcal{R} \partial + (\partial \mathcal{R})) \mu. \tag{7.28b}
\]

(7.28b) is the counterpart of the conformal Ward identity, but it differs from the latter by the second term within square brackets [39].

As shown extensively in sects. 5 and 6, the holomorphic and Hermitian geometry of \( E \) reduce essentially to those of \( \det E \) and \( \text{proj} E \). This is reflected in the splitting of the fields \( A^* \) and \( \Phi \) in their trace parts \( a^* \) and \( \phi \) and traceless parts \( \hat{A}^* \) and \( \hat{\Phi} \). Correspondingly, the action of the symmetry group factorizes into an action on \( \det E \) and one on \( \text{proj} E \). This again is reflected in the splitting of the ghost fields \( M \) and \( W \) in their trace parts \( m \) and \( w \) and traceless parts \( \hat{M} \) and \( \hat{W} \). One may express all the anomalies and the counterterm given above in terms of such component fields. As turns out, each of such functionals results to be a sum of two terms, the first depending only on the trace part fields and the second depending only on the traceless part fields, and thus referring to \( \det E \) and \( \text{proj} E \), respectively. The trace part term of each functional appears to have the same form as that of the corresponding functional in ordinary conformal field theory. Thus, it will contribute to the ordinary Weyl or diffeomorphism anomaly. Let us illustrate this in greater detail.

Assume first that the normalized weight \( j \) of \( E \) is non-zero. Then, for any Hermitian structure \( (h^{o'}, H') \) of \( E \) subordinated to \( A \) the field locally defined by

\[
h'_{\text{ind}} = \left( \det H' \right)^{1/j^r} \tag{7.29}
\]

is an induced Hermitian metric of \( \Sigma \) with respect to \( a \) provided one restricts oneself to the reduction \( A'_0 \) of \( A \) (cf. sect. 5). I shall stick to such reduction for the rest of this section. The Donaldson parametrization of the Hermitian structures of \( E \) yields directly a Liouville
parametrization of the metrics $h'_\text{ind}$ with respect to the reference metric $g_{\text{ind}} = (\det G)^{\frac{1}{j'}}$. From (5.9) and (5.14), it follows that the Liouville field $\phi_{\text{eff}}$ of $h'_\text{ind}$ is

$$
\phi_{\text{eff}} = \phi/j + \varphi_{\sigma},
$$

(7.30a)

where

$$
\varphi_{\sigma} = 2\text{Re}\sigma/j.
$$

(7.30b)

$\varphi_{\sigma}$ is a real valued element of $S(\Sigma, 1)$, since $\exp(-pr\sigma)$ is a section of the 1-cocycle $e \otimes e'^{-1}$ and the latter is flat unitary when restricted to the reductions $A_0$ and $A'_0$. Note that the indetermination of $\sigma$ modulo $2\pi i/pr$ times integers does not affect $\varphi_{\sigma}$. From (7.30) and (6.12), one can obtain easily an expression for $s\varphi_{\sigma}$. Introducing the ghost field

$$
w_{\text{1ind}} = \text{Re}w/j,
$$

(7.31)

one finds that $s\phi_{\text{eff}}$ is given by (3.18) with $\phi$ and $w_1$ replaced by $\phi_{\text{eff}}$ and by $w_{\text{1ind}}$. Thus, $\phi_{\text{eff}}$ rather than $\phi$ appears to be the proper counterpart of the Liouville field in the present context.

By a straightforward calculation, one finds

$$
A_{\text{ext}}(W, W^\dagger, \Phi, \mu, \bar{\mu}, A^*, A'^\dagger; G) = j^2rA_{\text{conf}}(w_{\text{1ind}}, \phi_{\text{eff}}, \mu, \bar{\mu}; g_{\text{ind}})
$$

+ $A_{\text{ext}}(\hat{W}, \hat{W}^\dagger, \hat{\Phi}, \mu, \bar{\mu}, \hat{A}^*, \hat{A}'^\dagger; G),
$$

(7.32)

where the first term is given explicitly by (4.2). Thus, the Ward identity (7.1) may be written as

$$
sI'_{\text{aut}}(h_{\sigma'}, H') = \kappa^\circ A_{\text{conf}}(w_1, \phi_{\sigma'}, \mu, \bar{\mu}; g^\circ) + Kj^2rA_{\text{conf}}(w_{\text{1ind}}, \phi_{\text{eff}}, \mu, \bar{\mu}; g_{\text{ind}})
$$

+ $KA_{\text{ext}}(\hat{W}, \hat{W}^\dagger, \hat{\Phi}, \mu, \bar{\mu}, \hat{A}^*, \hat{A}'^\dagger; G).
$$

(7.33)

Note further that the effective Liouville field $\phi_{\text{eff}}$ is a non local functional of $\mu, \bar{\mu}, A^*, A'^\dagger$, for $\sigma$ is a non local functional of $\mu, A^*$. However, the induced Weyl anomaly itself is local.

The Donaldson action and the Knecht-Lazzarini-Stora action undergo a similar decomposition.

$$
S_D(\Phi, \mu, \bar{\mu}, A^*, A'^\dagger; G) = j^2r[S_L(\phi_{\text{eff}}, \mu, \bar{\mu}; g_{\text{ind}}) - S_L(\phi_{\sigma}, \mu, \bar{\mu}; g_{\text{ind}})]
$$

+ $S_D(\hat{\Phi}, \mu, \bar{\mu}, \hat{A}^*, \hat{A}'^\dagger; G),
$$

(7.34)

where the Liouville action is given by (4.4).

$$
S_{KLS}(\mu, \bar{\mu}, A^*, A'^\dagger; G, A, A^\dagger) = j^2r[S_{\text{VKT}}(\mu, \bar{\mu}; g_{\text{ind}}, \mathcal{R}, \bar{\mathcal{R}}) + S_L(\phi_{\sigma}, \mu, \bar{\mu}; g_{\text{ind}})
$$

- $\hat{\Delta}I(\mu, \bar{\mu}, A^*, A'^\dagger; g_{\text{ind}}, a, \bar{a}, \mathcal{R}, \bar{\mathcal{R}}) + S_{KLS}(\mu, \bar{\mu}, \hat{A}^*, \hat{A}'^\dagger; G, \hat{A} + j\gamma_{\text{ind}}1, \hat{A}' + j\bar{\gamma}_{\text{ind}}1),
$$

(7.35)
where \( \mathcal{R} \) is an a–holomorphic projective connection, \( S_{V_{KLT}} \) is given by (4.5) and

\[
\begin{align*}
\hat{\Delta}I(\mu, \bar{\mu}, A^*, A^{*\dagger}; g, a, \bar{a}, \mathcal{R}, \bar{\mathcal{R}}) &= -\frac{1}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left\{ \mu \left[ \partial (\rho_a + \chi_\sigma) - \frac{1}{2} (\rho_a + \chi_\sigma)^2 - \mathcal{R} \right] \right. \\
& \quad + (\rho_a - \gamma_g)\chi_\sigma^* + \bar{\mu} \left[ \partial (\bar{\rho}_a + \bar{\chi}_\sigma) - \frac{1}{2} (\bar{\rho}_a + \bar{\chi}_\sigma)^2 - \bar{\mathcal{R}} \right] + (\bar{\rho}_a - \bar{\gamma}_g)\bar{\chi}_\sigma^* \left. \right\} \bigg|_{\frac{1}{2}}
\end{align*}
\]

(7.36a)

with

\[
\begin{align*}
\chi_\sigma &= \partial \sigma/j, \quad \chi_\sigma^* = \bar{\partial} \sigma/j, \\
\rho_a &= a/j.
\end{align*}
\]

(7.36b, 7.36c)

Because of the special choice of trivializations made, \( \chi_\sigma \) and \( \chi_\sigma^* \) are conformal fields of weights 1, 0 and 0, 1, respectively. Further, the indetermination of \( \sigma \) modulo \( 2\pi i/\nu \) times integers does not affect them. Hence, the above functional is well-defined, the integrand being a globally defined conformal field of weights 1, 1. Note that \( \hat{\Delta}I \) is a non local functional of \( \mu, \bar{\mu}, A^*, A^{*\dagger} \), for \( \sigma \) is a non local functional of \( \mu, A^* \). It is however chirally split, since \( \sigma \) is holomorphic.

By combining (7.24), (7.34) and (7.35), one finds

\[
\begin{align*}
\Delta I(\Phi, \mu, \bar{\mu}, A^*, A^{*\dagger}; G, A, A^{\dagger}) &= j^2 r \hat{\Delta}I(\mu, \bar{\mu}, A^*, A^{*\dagger}; g_{\text{ind}}, a, \bar{a}, \mathcal{R}, \bar{\mathcal{R}}) \\
&= j^2 r \Delta I_{\text{cft}}(\phi_{\text{eff}}, \mu, \bar{\mu}; g_{\text{ind}}, \mathcal{R}, \bar{\mathcal{R}}) + I(\Phi, \mu, \bar{\mu}, A^*, A^{*\dagger}; G, A, A^{\dagger} + j\gamma_{\text{ind}}) + j\gamma_{\text{ind}} A^{\dagger}. \\
\end{align*}
\]

(7.37)

Note that the right hand side is the sum of the counterterm \( \Delta I_{\text{cft}} \) encountered in ordinary conformal field theory plus the projective part of the counterterm \( \Delta I \). The strange looking connection \( A + j\gamma_{\text{ind}} \) is in fact what is needed since \( A \) appears always in the combination \( A - \Gamma_G \) (cf. eq. (7.18)) while the projective part of \( \Delta I \) should be independent from \( \det G \).

A straightforward calculation using (6.26) shows that

\[
\begin{align*}
 j^2 r s \hat{\Delta}I(\mu, \bar{\mu}, A^*, A^{*\dagger}; g_{\text{ind}}, a, \bar{a}, \mathcal{R}, \bar{\mathcal{R}}) &= j^2 r \left[ A_{\text{diff}}(C, \mu; \mathcal{R}) + A_{\text{diff}}(C, \mu; \bar{\mathcal{R}}) \right] \\
&- \left[ A_{\text{aut}}(x1, a^*1; a1) + A_{\text{aut}}(x1, a^*1; a1) \right].
\end{align*}
\]

(7.38)

Thus, defining

\[
\begin{align*}
\hat{I}_{\text{ext}}(\mu, \bar{\mu}, A^*, A^{*\dagger}; \mathcal{R}, \bar{\mathcal{R}}, A, A^{\dagger}) &= I_{\text{ext}}(\mu, \bar{\mu}, A^*, A^{*\dagger}; \mathcal{R}, \bar{\mathcal{R}}, A, A^{\dagger}) \\
+ K j^2 r \hat{\Delta}I(\mu, \bar{\mu}, A^*, A^{*\dagger}; g_{\text{ind}}, a, \bar{a}, \mathcal{R}, \bar{\mathcal{R}}),
\end{align*}
\]

(7.39)

\[37\]
one has

\[
\begin{align*}
\hat{s}_{\text{ext}}(\mu, \bar{\mu}, A^*, A^{\dagger}; R, \bar{R}, A, A^{\dagger}) &= (\kappa^o + Kj^2\rho) [\mathcal{A}_{\text{diff}}(C, \mu; R) + \overline{\mathcal{A}_{\text{diff}}(C, \mu; \bar{R})}]
+ K [\mathcal{A}_{\text{aut}}(\hat{X}, \hat{A}^{\ast}; \hat{A}) + \overline{\mathcal{A}_{\text{aut}}(\hat{X}, \hat{A}; \hat{A})}].
\end{align*}
\]  

(7.40)

As \( I_{\text{ext}} \), \( \hat{I}_{\text{ext}} \) does not depend on \( g^0 \) and \( G \) by extended Weyl invariance. In fact, it is easy to check from (7.36a) that \( \hat{\Delta}I \) does not depend on the scale of the metric \( g \).

The above results need to be adequately discussed and interpreted. From the Ward identity (7.33), it appears that the extended Weyl anomaly induces an ordinary Weyl anomaly with induced central charge \( Kj^2\rho \). The anomaly expresses the non trivial dependence on the scale of the metric \( h^{\prime\text{ind}} \) in the effective action and is thus distinguished from the ordinary Weyl anomaly referring to the metric \( h^o \) with central charge \( \kappa^o \). However, if one restricts oneself to special deformations and Hermitian structures of \( E \) (cf. sect. 5) and reduces the symmetry to the special symmetry (cf. sect. 6), then \( g^{\text{ind}} = g^o \), \( \phi_{\text{eff}} = (1/j)\phi = \phi^o \) and \( w_{1\text{ind}} = w_1 \), as follows from combining (5.16), (5.18), (5.19) and (6.28) with (7.30) and (7.31). Thus, the two Weyl anomalies coalesce into a single one with central charge \( \kappa^o + Kj^2\rho \).

As shown above, upon modifying the effective action \( I_{\text{ext}} \) by adding the functional \( K\hat{\Delta}I \), one obtains a new effective action \( \hat{I}_{\text{ext}} \) obeying the anomalous Ward identity (7.40). The automorphism anomaly is still chirally split. However, \( \hat{\Delta}I \) is a non local functional. So, the anomalies (7.26) and (7.40) are not equivalent as classes of the local s cohomology. However, if one restricts oneself once more to special deformations and Hermitian structures and reduces the symmetry to the special symmetry, then \( \phi_{\text{eff}} = \phi^o \), as shown in the previous paragraph, and further \( \varphi^o = 0, \chi^o = 0 \) and \( \chi^o_\ast = 0 \) by (5.16), (7.30) and (7.36b). The functional \( \hat{\Delta}I \) becomes in this way local, as appears from (7.36a) by inspection. The anomalies (7.26) and (7.40) are then equivalent in the reduced local s cohomology associated to the special symmetry. The central charge is again \( \kappa^o + Kj^2\rho \).

To summarize, I have shown that restricting to special holomorphic and Hermitian structures and their deformations and reducing the symmetry to the special symmetry shifts the ordinary central charge. The effective central charge is given by

\[
\kappa_{\text{eff}} = \kappa^o + \kappa_{\text{ind}} = \kappa^o + Kj^2\rho.
\]  

(7.41)

In the above calculations, I assumed that \( j \neq 0 \) since the definition of various fields involved a factor \( 1/j \). The analysis would seem to break down when \( j = 0 \). Happily, this is not so, provided one restricts oneself from the beginning to special deformations and Hermitian structures and to the special symmetry group. This follows from a straightforward computation based on the relations (5.15), (5.16), (5.18), (5.19), (6.27) and (6.28).
The upshot is that relations (7.32), (7.33), (7.34), (7.35), (7.37) and (7.40) hold with the terms proportional to $j^2r$ formally set to zero. In the case considered, there is no induced central charge so that $\kappa_{\text{eff}} = \kappa^\circ$.

The above discussion provides shows convincingly the naturality of restricting to special structures and special symmetry. Henceforth, this will be assumed tacitly unless otherwise stated.

Because only special deformations are admitted, $\widehat{I}_{\text{ext}}$ is effectively a functional of $\mu, \tilde{\mu}, \hat{A}^*$ and $\hat{A}^\dagger$ (cf. sect. 5). Since $\Delta I$ is chirally split, $\widehat{I}_{\text{ext}}$ satisfies the holomorphic factorization theorem if $I_{\text{ext}}$ does. Thus, one has a relation of the form $\widehat{I}_{\text{ext}}(\mu, \tilde{\mu}, \hat{A}^*; \mathcal{R}, \tilde{\mathcal{R}}, A, A^\dagger) = \widehat{I}_{P}(\mu, \tilde{\mu}, \hat{A}^*; \mathcal{R}, A) + \widehat{I}_{P}(\mu, \tilde{\mu}, \hat{A}^*; \mathcal{R}, A)$, where $\widehat{I}_{P}$ is a generalized Polyakov action. $\widehat{I}_{P}$ satisfies a Ward identity of the same type as (7.28) with $A^*$ replaced by $\hat{A}^*$ and $\kappa^\circ$ substituted by $\kappa_{\text{eff}}$.

Being a functional of $\mu$ and $\hat{A}^*$, $\widehat{I}_{P}$ depends on the holomorphic geometry of $\text{proj } E$ rather than that of $E$ (cf sect. 5). This is a rather non-trivial geometric property of the whole construction.

At this point, it is worthy considering an example. The following model is in fact a variant of the one originally envisaged in ref. [26] and may be considered as a generalization of the bosonic $b$–$c$ system. For any vector bundle $E$ and any holomorphic structure $A$, the classical action is

$$S(\Psi^\vee, \Psi) = \frac{1}{\pi} \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} \Psi^\vee \bar{\partial} \Psi,$$

where $\Psi$ and $\Psi^\vee$ vary, respectively, in $S(\Sigma, k \otimes \frac{1}{2} \otimes L)$ and $S(\Sigma, k \otimes \frac{1}{2} \otimes L^\vee)$. For such model, it is possible to compute the central charges $\kappa^\circ$ and $K$:

$$\kappa^\circ = -r, \quad K = 12,$$

[26]. From here, one can compute $\kappa_{\text{eff}}$ by using (7.41). The result can be written as

$$\kappa_{\text{eff}} = (12j^2 - 1)r = 2r(6\tilde{j}^2 - 6\tilde{j} + 1), \quad \tilde{j} = 1/2 + j.$$  

(7.44)

$\tilde{j}$ is the effective conformal weight of the field $\Psi$. The two contributions to $\tilde{j}$ correspond respectively to the first and second tensor factor of the $1$–cocycle $k \otimes \frac{1}{2} \otimes L$. Note that the second form of the expression of $\kappa_{\text{eff}}$ is that of $r$ copies of a spin $j$ bosonic $b$–$c$ system.

8. **Flat vector bundles and their flat structures and extended 2D gravity.**

Within the category of complex vector bundle, the flat ones have a considerable salience in applications. By definition, flat vector bundles admit flat structures. Recall that a flat structure $F$ subordinated to a holomorphic structure $A$ of a flat vector
bundle $E$ is a maximal reduction of $A$ such that $L_{ab}$ is a constant matrix for all pairs $a,b$ of overlapping trivializations contained in $F$. I shall write $(A,F)$ to emphasize that $F$ is subordinated to $A$. One may consider several flat structures $(A',F')$, $(A'',F'')$ viewed as deformations of a reference structure $(A,F)$. The problem arises of finding a suitable parametrization of the family of such deformations. Proceeding as in sect. 5, one finds that the deformations $(A',F')$ are parametrized in one-to-one fashion by the triples $(\mu,A^*,B)$, where $(\mu,A^*)$ specify the deformation $A'$ and $B$ is given by

$$B_a = (\partial_a - A_a)V_{aa'}V_{aa'}^{-1}, \quad a' \in F'.$$

(8.1)

$B$ is an element of $S(\Sigma, k \otimes \text{End} L)$. $\mu$, $A^*$ and $B$ satisfy the identity

$$\bar{\partial}(A + B) - \partial(A^* + \mu B) + [A + B, A^* + \mu B] = 0,$$

(8.2)

which is the zero curvature condition for the connection $(A + B)dz + (A^* + \mu B)d\bar{z}$. This relation can be cast also as

$$(\bar{\partial} - \mu \partial_A - (\partial \mu) \circ \text{ad} A^*)B = \partial A A^*.$$

(8.3)

From here, it can be seen that the flat structures $F'$ subordinated to the same holomorphic structure $A'$ are in one-to-one correspondence the $\alpha'$–holomorphic elements of $S(\Sigma, k' \otimes \text{End} L')$ (the 0–th sheaf cohomology of the $\alpha'$–holomorphic 1–cocycle $k' \otimes \text{End} L'$). It follows from (6.4) that the automorphism group of $E$ maps flat structures into flat structures. From (3.14), (3.15), (6.20), (6.21) and (6.22) one finds the relations

$$s(A + B) = \partial(CB - X) - [A + B, CB - X],$$

(8.4)

$$s(A^* + \mu B) = \bar{\partial}(CB - X) - [A^* + \mu B, CB - X],$$

(8.5)

$$s(CB - X) = (CB - X)^2.$$

(8.6)

One may consider also special deformations of a flat structure for which $\det V_{aa'} = 1$. Such deformations are parametrized in one-to-one fashion by the triples $(\mu, A^*, B)$ such that $A^*$ and $B$ are traceless (assuming that $\text{tr} A = 0$). In that instance, I shall revert to the familiar notation $\hat{A}^*$ and $\hat{B}$. For special automorphisms, one has similarly that $X$ is traceless, so that $X = \hat{X}$ (cf. sect. 6).

In concrete applications to extended 2D gravity, the vector bundle $E$ is of the form $\epsilon^j \otimes \hat{E}$ with $j \in \mathbb{Z}/2$, where $\hat{E}$ is a flat vector bundle. To any holomorphic structure hat$A$ of $\hat{E}$, there is associated a holomorphic structure $A$ of $E$ such that $a = \hat{a}$ and $L_{ab} = k \otimes m_{ab} L_{ab}$. The trivializations $(z_a,u_a)$ of $A$ are of the form $v_a \otimes \hat{u}_a$ for sum trivialization $(\hat{z}_a, \hat{u}_a)$, where
\( z_a = \hat{z}_a \) and \( v_a \) is the trivialization of \( \epsilon^m \) corresponding to \( \hat{z}_a \). Since \( S(\Sigma, \bar{k} \otimes \text{End} L) = S(\Sigma, \bar{k} \otimes \text{End} \hat{L}) \), there is a one-to-one correspondence between the special deformations of \( \hat{E} \) and \( E \), the corresponding deformations being described by the same pair of \((\mu, \hat{A}^*)\) of geometrical fields. Similarly, there is an isomorphism of the special automorphism group of \( \hat{E} \) onto that of \( E \) for a given \( \mu \), defined by \( \alpha_{ba} = \omega_{ab}(f_\alpha; \mu)^{-2m}\hat{\alpha}_{ba} \).

Restricting to special deformations and special symmetry, the generalized Polyakov action \( \hat{I}_P \) is a functional of \( \mu \) and \( \hat{A}^* \) obeying a Ward identity of the form (7.28) with \( A^* \) replaced by \( \hat{A}^* \). If one sets \( B = (12\pi/K)\delta \hat{I}_P/\delta \hat{A}^* \), (7.28a) has the same form as (8.3). The Ward identity assumes then the form of a zero curvature condition, a recurrent fact in all extensions of 2D gravity. In more geometrical terms, \( \hat{I}_P \) defines a section of the family of flat structures of \( \hat{E} \) viewed as a space fibered over the set of holomorphic structures. By substituting (7.28a) in (7.28b), one finds that \( P = (12\pi/K)\left[ -\delta \hat{I}_P/\delta \mu + (6\pi/K)\text{tr}(\delta \hat{I}_P/\delta A^*)^2 \right] + \mathcal{R} \) satisfies \( (\partial - \mu \partial - 2(\partial \mu))P = \partial^3 \mu \), which characterizes \( P \) as an \( a' \)-holomorphic projective connection [37,39]. So \( \hat{I}_P \) yields a section of the space of projective coordinate structures of \( \Sigma \) viewed as a fibered space over the set of holomorphic structures as in ordinary conformal field theory [39].

9. Applications to extended 2D gravity.

In this final section, I shall describe briefly a few examples of the general framework expounded in the previous sections. I shall consider vector bundles \( E \) of the form \( \epsilon^j \otimes \hat{E} \) with \( j \in \mathbb{Z}/2 \), where \( \hat{E} \) is a flat vector bundle. There is a choice of \( \hat{E} \) which relevant for applications which now I shall introduce. The construction exploits \( A_1 \) embeddings into simple Lie algebras and is in line with recent work in extended 2D gravity in which \( A_1 \) embeddings play a prominent role [33-34].

a) The Drinfeld-Sokolov vector bundle.

Let \( \mathfrak{g} \) be a simple complex Lie algebra. Consider a \( A_1 \) embedding into \( \mathfrak{g} \). This is given by a triple of Lie algebra elements \( t_{-1}, t_0, t_{+1} \) with Lie brackets

\[
[t_{+1}, t_{-1}] = 2t_0, \quad [t_0, t_{\pm1}] = \pm t_{\pm1},
\]

the other brackets being zero. Chosen any representation \( R \) of \( \mathfrak{g} \), one may define

\[
\hat{L}_{ab} = k_{ab}^{-t_0} \exp(\partial_a k_{ab}^{-1} t_{-1})
\]
on a Riemann surface $\Sigma$ with holomorphic structure $a$. The collection $\{k_{ab}^{-t_0}\}$ defines a SL($r$, $\mathbb{C}$)–valued 1–cocycle, where $r = \dim R$, since $R$ induces a representation of the $A_1$ embedding and this is equivalent to a unitary one in which $t_0$ is represented by a diagonal matrix with half-integer entries. By using this remark, it is straightforward to verify that $\hat{L}$ is a holomorphic SL($r$, $\mathbb{C}$) 1–cocycle. One may also easily verify that $\hat{L}$ is unstable [36]. Indeed, the fields of the form $Z = \psi t_{-1}$, where $\psi$ is a holomorphic differential of $S(\Sigma, k)$, are non trivial holomorphic elements of $S(\Sigma, \text{End} \hat{L})$. $\hat{L}$ possesses a distinguished $(1, 0)$ holomorphic connection

$$\hat{A} = (1/2)t_{+1} - \mathcal{R}t_{-1}, \quad (9.3)$$

where $\mathcal{R}$ is a holomorphic projective connection. This in turn shows the flatness of $\hat{L}$ [36].

The 1–cocycle $\hat{L}$ defines a smooth flat vector bundle $DS(t, R)$, which will be called the Drinfeld–Sokolov bundle of the $A_1$ embedding $t$ into $g$ in the representation $R$, because of the special form of the connection $\hat{A}$ [42]. It further provides a holomorphic structure $A$ naturally associated to the underlying holomorphic structure $a$ of $\Sigma$, by (9.2). Hence, any deformation $a'$ of $a$ induces a deformation $A'$ of $A$. The associated deformation intertwiners are the intertwiner $\lambda$ of $a'$ and

$$\hat{V}_{aa'} = \lambda_{aa'}^{-t_0} \exp(\partial_a \lambda_{aa'}^{-1} t_{-1}). \quad (9.4)$$

Hence, the deformation considered is special. The geometric field describing the deformation are the Beltrami differential $\mu$ and the gauge field

$$\hat{A}^* = (\mu/2)t_{+1} - \partial\mu t_0 - (\partial^2 + \mathcal{R})\mu t_{-1}, \quad (9.5)$$

where the chosen reference connection $\hat{A}$ entering into the definition of $\hat{A}^*$ according to (5.6) is given by (9.3).

The action of the diffeomorphism group on the deformations $a'$ of $a$ can be lifted to an action onto the induced deformations $A'$ of $A$ constructed above. This leads to a homomorphism of the diffeomorphism group of $\Sigma$ into the special automorphism group of $DS(t, R)$. The action of the Slavnov operator on $\mu$ and $\hat{A}^*$ is given by (3.14) and (6.21) with

$$\hat{X} = -(C/2)t_{+1} + \partial C t_0 + (\partial^2 + \mathcal{R})C t_{-1}, \quad (9.6)$$

whose validity is restricted to the image of the lift in the symmetry Lie algebra.

---

9 Since I will work with an arbitrary but fixed representation $R$, I shall not distinguish notationally between Lie algebra elements and their representatives.
It is interesting to compute the projective part of the automorphism anomaly $A_{\text{aut}}$ for the lift of the diffeomorphism group. By a straightforward calculation, one finds

$$A_{\text{aut}}(\hat{X}, \hat{A}^*; \hat{A}) = \text{tr}_R(t_0^2)A_{\text{diff}}(C, \mu; \mathcal{R}), \quad (9.7)$$

where $\text{tr}_R$ is the trace in the representation $R$ of the Lie algebra $\mathfrak{g}$. This yields a further contribution $\kappa_{\text{lift}} = \text{tr}_R(t_0^2)$ to the central charge of the form of an improvement term. On account of (7.41), the total central charge is

$$\kappa_{\text{tot}} = \kappa_{\text{eff}} + \kappa_{\text{lift}} = \kappa^o + K (j^2 \dim R + \text{tr}_R(t_0^2)). \quad (9.8)$$

As an application, consider the system (7.42) in the case where $E = \epsilon^j \otimes DS(t, \mathfrak{g})$. By combining (7.44) and (9.8), one finds

$$\kappa_{\text{tot}} = \text{tr}_R[(2(6\tilde{j}^2 - 6\tilde{j} + 1)1 + t_0^2], \quad \tilde{j} = 1/2 + j. \quad (9.9)$$

b) $W$ gravity in the light cone gauge.

Next, one would like to see whether the above framework can accommodate well-known models of extended 2D gravity, in particular $W$ gravity. To this end, I shall consider a class of flat structures of the bundle $DS(t, \mathfrak{g})$ defined by the constraint

$$\text{ad}_{t-1} B = 0, \quad (9.10)$$

where $B$ is given by (8.1) with $\hat{A}$ given by (9.3). By combining (8.2) and (9.10), one obtains then

$$\text{ad}_{t-1} \left[ \partial (A^* + \mu \hat{B}) - [\hat{A} + \hat{B}, \hat{A}^* + \mu \hat{B}] \right] = 0. \quad (9.11)$$

When restricted to the residual gauge symmetry left over by the constraint, the ghost field $\hat{X}$ obeys the following relation

$$\text{ad}_{t-1} \left[ \partial (C\hat{B} - \hat{X}) - [\hat{A} + \hat{B}, C\hat{B} - \hat{X}] \right] = 0, \quad (9.12)$$

as follows from (8.4) and (9.10). Applying the Slavnov operator $s$ on the expressions defined by the left hand sides of eqs. (9.11)–(9.12) and using (8.2), (8.4)–(8.6) leads to linear combinations of those very same expressions and their derivatives. Thus there are no more constraints besides the ones already indicated. It is not difficult to verify that (9.10)–(9.12) are compatible with changes of trivializations. The solution (9.11)–(9.12) can be found as follows.
The adjoint representation of the $A_1$ embedding $t_{-1}, t_0, t_{+1}$ in $g$ is reducible. Let us denote by $\Pi$ the set of the representations of $A_1$ appearing in the decomposition each counted with its multiplicity. To the representation $\eta$ of spin $l_\eta$, there is associated a distinguished set of generators $t_{\eta,m_\eta}$, $m_\eta = -l_\eta, -l_\eta + 1, \cdots, l_\eta - 1, l_\eta$ of $g$ such that

$$[t_d, t_{\eta,m_\eta}] = G^{d}_{l_\eta,m_\eta} t_{\eta,m_\eta + d}, \quad d = -1, 0, +1,$$

(9.13a)

where

$$G^{\pm 1}_{l,m} = [l(l+1) - m(m+1)]^{\frac{1}{2}}, \quad G^0_{l,m} = m.$$

(9.13b)

The Lie brackets of the $t_{\eta,m_\eta}$'s have the following form

$$[t_{\eta,m_\eta}, t_{\zeta,m_\zeta}] = \sum_{\xi \in \Pi} \sum_{m_\xi = -l_\xi}^{l_\xi} F_{\eta,\zeta,\xi}(l_\eta, m_\eta; l_\zeta, m_\zeta; l_\xi, m_\xi) t_{\xi,m_\xi},$$

(9.14)

where $(l_\eta, m_\eta; l_\zeta, m_\zeta; l_\xi, m_\xi)$ is a Clebsch-Gordan coefficient and the $F_{\eta,\zeta,\xi}$'s are imaginary constants depending only on the $A_1$ embedding $t$. There always is an element of $o \in \Pi$ such that $l_o = 1$, since the $t_{-1}, t_0, t_{+1}$ themselves span a representation of $A_1$ into $g$ with $t_{o,\pm 1} = \mp 2^{-\frac{1}{2}} t_{\pm 1}$ and $t_{o,0} = t_0$. Below, I shall restrict to $A_1$ embeddings for which $\Pi$ contains only integers spins. The simpleness of $g$ and the non degeneracy of the Cartan form imply that $\Pi$ cannot contain spin 0 representations. One has further

$$\text{tr}_R(t_{\eta,m_\eta} t_{\zeta,-m_\zeta}) = N_\eta (-1)^{l_\eta-m_\eta} \delta_{\eta,\zeta} \delta_{m_\eta,m_\zeta},$$

(9.15)

where $N_\eta$ is a normalization constant. In practice, the most used $A_1$ embedding is the so-called principal $A_1$, for which $t_0$ is the dual Weyl vector $\rho^\vee$. In that case, each element of $\Pi$ is non degenerate except for $l = 2k-1$ of the algebra $D_{2k}$ which is doubly degenerate.

The solution of the constraint (9.10) can be written in the form

$$\hat{B} = \sum_{\eta \in \Pi} \psi_\eta t_{\eta,-l_\eta},$$

(9.16)

where the $\psi_\eta$ are elements of $S(\Sigma, k^{\otimes l_\eta+1})$. The solution of the constraint (9.11) can be found by means of techniques analogous to those worked out in the second reference [23]. Let $Q$ be the operator defined by $Qt_{\eta,m_\eta} = 2(G^{+1}_{l_\eta,m_\eta})^{-2} t_{\eta,m_\eta}$ for $m_\eta < l_\eta$ and $Qt_{\eta,l_\eta} = 0$ and let $l_{\text{max}}$ be the largest spin in $\Pi$. Then, the result can be written as

$$\hat{A}^* = -\mu \hat{B} + \left[1 + Q\text{ad} t_{-1}(\partial_{\hat{A}} - \text{ad} \hat{B})\right]^{2l_{\text{max}}+1} \sum_{\eta \in \Pi} \nu_\eta t_{\eta,l_\eta},$$

(9.17)
where the $\nu_\eta$ are fields from $S(\Sigma, k \otimes k^{-l_\eta})$ and may be viewed as generalized Beltrami fields. Similarly, solving (9.12) yields

$$\hat{X} = C\hat{B} - \left[1 + \text{Qad} t^{-1}(\partial A - \text{ad} \hat{B})\right]^{2l_{\text{max}}+1} \sum_{\eta \in \Pi} y_\eta t_{\eta,l_\eta}, \quad (9.18)$$

where $y_\eta = -\hat{X}_{\eta,l_\eta}$ is a section of $k^{\otimes-s_\eta} \otimes \text{Lie} (\text{Aut}_c(DS(t,R)) \times \text{Weyl}(DS(t,R)))^\vee$.

The $\psi_\eta$’s and $\nu_\eta$’s are not independent, since they must satisfy a set of differential equations following from (8.3). Such equations are analogous to those found in refs. [18,20] and are rather messy.

The action of the Slavnov operator $s$ on the components fields $\psi_\eta, \nu_\eta$ and $y_\eta$ can be found by substituting (9.16)–(9.18) into (8.4)–(8.6) and then using (9.14) to compute the Lie brackets. The resulting expressions are non local, since they contain the $\psi_\eta$’s which, being solutions of (8.3), are non local functionals of the $\nu_\eta$’s. Further, they depend on the $A_1$ embedding used. However, one observes the following general structure:

$$s\tilde{\nu}_o = (\tilde{\partial} - \tilde{\nu}_o \partial + (\partial \tilde{\nu}_o))\tilde{y}_o + \ldots, \quad (9.19a)$$

$$s\nu_\eta = (\tilde{\partial} - \tilde{\nu}_o \partial + l_\eta(\partial \tilde{\nu}_o))y_\eta + \tilde{y}_o \partial \nu_\eta - l_\eta \partial \tilde{y}_o \nu_\eta + \ldots, \quad \eta \neq o, \quad (9.19b)$$

$$s\tilde{\nu}_o = \tilde{y}_o \partial \tilde{y}_o + \ldots, \quad (9.20a)$$

$$s\nu_\eta = \tilde{y}_o \partial \nu_\eta - l_\eta \partial \tilde{y}_o \nu_\eta + \ldots, \quad \eta \neq o, \quad (9.20b)$$

where $\tilde{\nu}_o = -2^{-\frac{2}{3}} \nu_o$ and $\tilde{y}_o = -2^{-\frac{2}{3}} y_o$ and the ellipses denote contributions of order at least one in both the $\nu_\eta$’s and the $y_\eta$’s with $\eta \neq o$. These relations are manifestly of the same form as those encountered in standard $W$ gravity.

The field $\tilde{\nu}_o$ has many of the formal properties of a Beltrami differential $\mu$. Indeed, this is the name usually adopted in the literature. However, such terminology is unwarranted, for $\tilde{\nu}_o$ does not represent any deformation of the holomorphic structure of $\Sigma$ as appears from the fact that it does not necessarily satisfy the bound (2.4). Similarly, $\tilde{y}_o$ shares many of the formal properties of the diffeomorphism ghost $C$ but is distinct from it.

When $\hat{A}^*$ and $\hat{X}$ have the form (9.17) and (9.18), the projective part of the automorphism anomaly

$$A_{\text{aut}}(\hat{X}, \hat{A}^*; \hat{A}) = A_{\text{aut}}(\hat{X}_0, \hat{A}^*_0; \hat{A}) + \frac{1}{12\pi} \int_{\Sigma} \text{tr}_R([\hat{A}^*, \hat{X} - \hat{X}_0]B), \quad (9.21)$$

where $\hat{B}, \hat{A}^*$ and $\hat{X}$ are given by (9.16)–(9.18) and $\hat{A}^*_0$ and $\hat{X}_0$ are given by (9.17)–(9.18) with $\hat{B}$ set to zero. To show the above identity, one has to use (8.3), (9.15), and (9.16)–(9.18) and the fact the components of $\hat{A}^* - \hat{A}^*_0$ and $\hat{X} - \hat{X}_0$ with respect to the basis
\( \{t_{\eta,m}\} \) with \( m_\eta = l_\eta, l_\eta - 1 \) vanish. The first term in the right hand side of (9.21) can be computed explicitly by using (9.1), (9.17) and (9.18). The following expression is found:

\[
A_{\text{aut}}(\hat{X}_0, \hat{A}^*_0; \hat{A}) = -\frac{1}{12\pi} \sum_{\eta \in \Pi} N_\eta \left[ \prod_{m=1}^{2l_\eta} \frac{2}{G + 1_{l_\eta, l_\eta - m}} \right] \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \gamma_\eta D_{l_\eta}(\partial, \mathcal{R}) \nu_\eta,
\]

where \( D_l(\partial, \mathcal{R}) \) is the \( l \)–th Bol operator [43]:

\[
D_1(\partial, \mathcal{R}) = \partial^3 + 2\mathcal{R}\partial + (\partial \mathcal{R}),
\]

\[
D_2(\partial, \mathcal{R}) = \partial^5 + 10\mathcal{R}\partial^3 + 15(\partial \mathcal{R})\partial^2 + [9(\partial^2 \mathcal{R}) + 16\mathcal{R}^2] + [(\partial^3 \mathcal{R}) + \mathcal{R}(\partial \mathcal{R})],
\]

etc. (9.23)

In general, \( D_l(\partial, \mathcal{R}) \) is of the form \( \partial^{2l+1} + O(\partial^{2l-1}) \). It can be shown that there is a projective coordinate structure \( \mathfrak{p} \) subordinated to the holomorphic structure \( \mathfrak{a} \) such that \( \mathcal{R}_a = 0 \) for \( a \in \mathfrak{p} \) [35]. In such coordinates, \( D_l(\partial, \mathcal{R})_a = \partial_a^{2l+1} \) exactly. It is remarkable that (9.23) is precisely of the form of the standard minimal \( W \) anomaly in the large central charge limit. The term of the anomaly corresponding to the representation \( \eta = 0 \) is formally identical to an improvement term of the diffeomorphism anomaly, though the fields appearing are \( \nu_0 \) and \( y_0 \) rather than \( \mu \) and \( C \). However, upon passing form the Ward identity to the operator product expansion formulation, this distinction is lost. The second term in the right hand side of (9.21) is a non minimal contribution to the anomaly [12].

\[ c) \text{ The Donaldson action and Toda field theory.} \]

The Drinfeld-Sokolov \( DS(t, R) \) vector bundle admits a natural special Hermitian structure \( (g^\circ, \hat{G}) \) subordinated to \( A \), where \( g^\circ \) is any Hermitian metric of \( \Sigma \) subordinated to \( \mathfrak{a} \) and

\[
\hat{G} = \exp(-\gamma g^\circ t_{-1}) \exp(-\ln g^\circ t_0) \exp(-\gamma g^\circ t_{+1}),
\]

where the connection \( \gamma g^\circ \) is given by (2.11). Here, it is assumed that the \( A_1 \) embedding is such that \( t_{d^+a} = t_{-da} \) in order to ensure the Hermiticity of \( \hat{G} \). This structure is just an element of a distinguished class of special Hermitian structures. These are of the form \((h^\circ, \hat{H})\), where \( h^\circ = \exp \phi^\circ g^\circ \) is another Hermitian metric of \( \Sigma \) subordinated to \( \mathfrak{a} \) and

\[
\hat{H} = \exp(-\gamma g^\circ t_{-1}) \exp(-\phi - \ln g^\circ t_0) \exp(-\gamma g^\circ t_{+1}),
\]

with \( \phi \) is Hermitian field of \( S(\Sigma, 1 \otimes \mathfrak{g}) \) such that \( \text{ad} t_0 \phi = 0 \). \( \hat{H} \) can be expressed in terms of the traceless Donaldson field \( \hat{\Phi} \) and of \( \hat{G} \).

It is interesting to compute the projective part of the Donaldson action \( S_D(\hat{\Phi}, 0, 0, 0; \hat{G}) \). I shall present the computation for the principal \( A_1 \) embedding. Let \( \Delta \) be the set
of simple roots of $\mathfrak{g}$, $e_{\pm \alpha}$ and $r_\alpha$, $\alpha \in \Delta$ be the generators of a Cartan-Weyl basis of $\mathfrak{g}$ and $C_{\alpha,\beta}$ be the Cartan matrix of $\mathfrak{g}$. One has

$$[r_\alpha, r_\beta] = 0, \quad [r_\alpha, e_{\pm \beta}] = \pm C_{\alpha,\beta} e_{\pm \beta}, \quad [e_\alpha, e_{-\beta}] = \delta_{\alpha,\beta} r_\alpha,$$  \hspace{1cm} (9.26)

$$n_R \text{tr}_R (r_\alpha r_\beta) = 2C_{\alpha,\beta}/\beta^2, \quad n_R \text{tr}_R (r_\alpha e_{\pm \beta}) = 0, \quad n_R \text{tr}_R (e_\alpha e_{-\beta}) = 2\delta_{\alpha,\beta}/\beta^2,$$  \hspace{1cm} (9.27)

for $\alpha, \beta \in \Delta$, where $n_R$ is a normalization such that $n_R (2/\alpha^2) \text{tr}_R (r_\alpha^2) = 2$ for the long roots $\alpha$. The generators of the principal $A_1$ are

$$t_{\pm 1} = \sum_{\alpha \in \Delta} \left[ \sum_{\beta \in \Delta} 2 C^{-1}_{\beta \alpha} \right]^{\frac{1}{2}} e_{\pm \alpha}, \quad t_0 = \sum_{\alpha, \beta \in \Delta} C^{-1}_{\beta,\alpha} r_\alpha.$$  \hspace{1cm} (9.28)

$\tilde{\phi}$ can be expanded in terms of the coroots $r_\alpha$:

$$\tilde{\phi} = \sum_{\alpha \in \Delta} \phi_\alpha r_\alpha,$$  \hspace{1cm} (9.29)

where the $\phi_\alpha$’s are realvalued elements of $S(\Sigma, 1)$. The calculation of the Donaldson action (7.13) is based on (9.26)–(9.27) and is totally straightforward though somewhat lengthy. The result found is the following:

$$S_D(\tilde{\Phi}, 0, 0, 0; \hat{G}) = -\frac{1}{12 \pi n_R} \int_\Sigma d\bar{z} \wedge dz \left\{ \frac{1}{2} \sum_{\alpha, \beta \in \Delta} \frac{2C_{\alpha,\beta}}{\beta^2} \partial \phi_\alpha \partial \phi_\beta - \partial \partial \ln g^\circ \sum_{\alpha \in \Delta} \frac{2}{\alpha^2} \phi_\alpha \right\} + \frac{1}{2} g^\circ R_{g^\circ} \sum_{\beta, \gamma \in \Delta} \frac{2C^{-1}_{\beta,\gamma}}{\gamma^2} \left[ \exp \left( - \sum_{\alpha \in \Delta} \phi_\alpha C_{\alpha,\gamma} \right) - 1 \right],$$  \hspace{1cm} (9.30)

where $R_{g^\circ}$ is given by (2.12). The action (9.30) is that of a Toda field theory with no exponential term. The last term cannot be identified with the customary exponential term of Toda field theory, since it vanishes in flat space because of the $R_{g^\circ}$ prefactor. The reason why the exponential interaction does not appear can be traced back to the absence of the generalized cosmological term in the Donaldson action.

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