Quantum decoherence due to imperfect manipulation of quantum devices is a key issue in the noisy intermediate-scale quantum (NISQ) era. Standard analyses in quantum information and quantum computation use error rates to parameterize quantum noise channels. However, there is no explicit relation between the decoherence effect induced by a noise channel and its error rate. In this work, we propose to characterize the decoherence effect of a noise channel by the physical implementability of its inverse, which is a universal parameter quantifying the difficulty to simulate the noise inverse with accessible quantum channels. We establish two concise inequalities connecting the decrease of the state purity and logarithmic negativity after a noise channel to the physical implementability of the noise inverse. Our results are numerically demonstrated on several commonly adopted two-qubit noise models.

**Introduction.**—Quantum entanglement is an important resource in quantum computers [1, 2], empowering the establishment of quantum supremacy [3–5]. The characterization and detection of quantum entanglement [6] in physical systems have been the primary concern of quantum information and computation for decades. On the other hand, imperfect control of quantum systems in the noisy intermediate-scale quantum (NISQ) era will induce errors [6] into quantum circuits composed of unitary gates, which can be described by general quantum channels.

An interesting problem arises: how does the quantum entanglement vary after the implementation of a quantum channel? Roughly speaking, entangled unitary gates generate entanglement, while noise channels destroy entanglement. The balance between these two parts gives the critical point of whether highly entangled states can be generated in quantum computers, which is a necessary condition to achieve universal quantum computation [7–9].

In this manuscript, we try to study the decoherence effects of noise channels on quantum states. Standard analyses on noises in quantum computation often use error rates to describe the strength of noise effects [10]. Nevertheless, for different noise models, there is no universal relation between their error rates and decoherence effects. Our purpose is to use a universal parameter of the noise channel to characterize how much entanglement it destroys. We believe that the physical implementability of the noise inverse, which represents the sampling cost to implement a linear map [11], is a prime candidate. Intuitively, the harder to implement the noise inverse, the more destructive the noise itself to quantum entanglement.

In this Letter, we establish two concise and universal inequalities which connect the destructive effects of noise channels to the change of entanglement between the input and output states. More specifically, we bound the decrease of the state purity and logarithmic negativity under noise channels with the physical implementability of the noise inverse. We believe that these relations contribute to the theoretical research on the entanglement properties of noise channels and provide guiding principles for quantum circuit design.

**Choi operator for linear maps.**—To begin with, we briefly introduce the notations to be used. We denote the Hilbert space of a quantum system as $\mathcal{H}$, where the space of linear operators is labeled as $\mathcal{L}(\mathcal{H})$. A quantum state is represented by a density operator $\rho \in \mathcal{L}(\mathcal{H})$, which is a Hermitian and positive semidefinite operator with unit trace.

A quantum channel $\mathcal{T}$ is a linear map between two operator spaces $\mathcal{T} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ with the completely positive (CP) and trace-preserving (TP) conditions. The most commonly adopted representation for a quantum channel $\mathcal{T}$ is the Choi operator [10, 14], defined as

$$
\Lambda_{\mathcal{T}} := (\mathcal{I}_\sigma \otimes \mathcal{T}) (d |\Phi\rangle\langle \Phi|_{\sigma\tau}),
$$

where $d$ is the dimension of the Hilbert space, while $\tau$ and $\sigma$ are the indices of the physical system and the ancillary system (which is isomorphic to the physical system) respectively. $|\Phi\rangle$ is the maximally entangled state $|\Phi\rangle_{\sigma\tau} = \sum_{i=0}^{d-1} |i\sigma i\tau\rangle / \sqrt{d}$ with $\{|i\rangle\}$ being a set of orthonormal basis. Then the implementation of $\mathcal{T}$ can be calculated in terms of the Choi operator as

$$
\mathcal{T} (\rho) = \text{Tr}_\sigma [(\rho^\tau \otimes I_\tau) \Lambda_{\mathcal{T}}].
$$

For an invertible CPTP map $\mathcal{T}$, there is no guarantee that its inverse $\mathcal{T}^{-1}$ is CPTP again. It has been further proved that the inverse of a CPTP map $\mathcal{T}$ is Hermitian-preserving (HP) and TP [11, 15]. Additionally, $\mathcal{T}^{-1}$ is CPTP if and only if $\mathcal{T}$ is unitary [16]. To represent these non-CP maps, we generalize the Choi operator to any HPTP map with the same definition as Eq. (1).

**Physical implementability.**—The quasi-probability method [17, 18] for quantum error mitigation and its variants [19, 20, 21] involve the simulation of the
inverse of noise channels with physically implementable quantum channels. The sampling cost for implementing an HPTP map \( \mathcal{N} \) is characterized by its physical implementability \([11]\), defined as

\[
\nu(\mathcal{N}) := \log_2 \min_{T_i \text{ is CPTP}} \left\{ \sum_i |q_i| \left| \mathcal{N} = \sum_i q_i T_i, q_i \in \mathbb{R} \right. \right\}.
\]

(3)

Consider a mixed unitary map \( \mathcal{N} \) on a \( d \)-dimensional Hilbert space decomposed by a set of mutually orthogonal unitaries, i.e.,

\[
\mathcal{N}(\cdot) = \sum_i q_i U_i(\cdot) U_i^\dagger,
\]

(4)

where \( q_i \in \mathbb{R} \) and \( \{U_i\} \) being a set of orthogonal unitaries with \( \text{Tr}[U_i U_j^\dagger] = d \delta_{ij} \). The physical implementability of \( \mathcal{N} \) is related to the trace norm of its Choi operator \([11]\)

\[
2^\nu(\mathcal{N}) = \frac{\|A_{\mathcal{N}}\|_1}{d},
\]

(5)

where the trace norm \( \|X\|_1 := \text{Tr} \sqrt{X^\dagger X} \) is equal to the summation of all singular values of \( X \).

In Supplemental Material, we also discuss the upper and lower bounds for the physical implementability in terms of the maximum and minimum eigenvalues of the Choi matrix. \([23]\).

**Physical implementability and purity.**—In this section, we try to connect the variation of purity to the physical implementability of the noise inverse. First, we consider a single-qubit system described by a Bloch sphere. The purity of a quantum state \( \rho \) is defined as \([10]\)

\[
\mathcal{P}(\rho) := \text{Tr}[\rho^2] = \text{Tr} \left[ \rho \left( I + \sum_{i=1}^3 \frac{1}{2} \sigma_i r_i \right) \right] = \frac{1 + |\vec{r}|^2}{2},
\]

(6)

where \( \sigma_i \) is the Pauli matrix and \( r_i = \text{Tr}[\rho \sigma_i] \) for \( i = 1, 2, 3 \). We note that \( \vec{r} \) is just the Bloch vector, the length of which is invariant under unitary transformations.

The above discussion can be directly generalized to quantum systems with a \( d \)-dimensional Hilbert space, with Eq. (6) becoming \( \mathcal{P}(\rho) = (1 + |\vec{r}|^2)/d \), where \( r_\alpha \) is defined by the expectation values of a set of complete and orthonormal basis \( \{O_\alpha\} \) for the operator space except the identity \( I \), i.e., \( r_\alpha = \text{Tr}[\rho O_\alpha] \) \([10]\). This relation enables us to calculate the purity change via the transformation of the Bloch vector.

**Theorem 1.** For a mixed unitary map \( \mathcal{N} \) decomposed by a set of mutually orthogonal unitaries, the purity of the input state \( \rho_0 \) and the output state \( \rho \) satisfy

\[
\log_2 \left( \frac{\mathcal{P}(\rho)}{\mathcal{P}(\rho_0)} \frac{d-1}{d-1} \right) \leq 2\nu(\mathcal{N}),
\]

(7)

where \( d \) is the dimension of the Hilbert space, and \( \nu(\mathcal{N}) \) is the physical implementability of \( \mathcal{N} \).

**Proof.** The corresponding transformation of the Bloch vector is

\[
\vec{r}(\rho) = \vec{r} \left( \sum_i q_i U_i \rho_0 U_i^\dagger \right) = \sum_i q_i \vec{r} \left( U_i \rho_0 U_i^\dagger \right),
\]

(8)

where \( \vec{r}(\rho) \) is the Bloch vector of the state \( \rho \), which is a linear map by definition. Since unitary transformations leave the Bloch vector length unchanged, we reach

\[
|\vec{r}(\rho)| = \sum_i |q_i| \left| \vec{r} \left( U_i \rho_0 U_i^\dagger \right) \right| \leq \sum_i |q_i| \left| \vec{r} \left( U_i \rho_0 U_i^\dagger \right) \right| = \sum_i |q_i| \left| \vec{r}(\rho_0) \right| = 2^\nu(\mathcal{N}) |\vec{r}(\rho_0)|.
\]

(9)

Finally, we build the relation between the physical implementability of \( \mathcal{N} \) and the ratio of purity

\[
\log_2 \left( \frac{\mathcal{P}(\rho) d-1}{\mathcal{P}(\rho_0) d-1} \right) = \log_2 \left( \frac{|\vec{r}(\rho)|^2}{|\vec{r}(\rho_0)|^2} \right) \leq 2\nu(\mathcal{N}).
\]

(10)

We now define a special class of noises.

**Definition 1.** Let \( \mathcal{E} \) be an invertible noise. We say that \( \mathcal{E} \) is an orthogonally mixed unitary noise if both \( \mathcal{E} \) and \( \mathcal{E}^{-1} \) are mixed unitary maps decomposed by a set of mutually orthogonal unitaries.

We note that several commonly used noise models, such as the (multiquit) Pauli noise, depolarizing noise, and dephasing noise, all belong to this category. We further prove that the composition of orthogonally mixed unitary noises with unitary channels or the tensor product of orthogonally mixed unitary noises is also an orthogonally mixed unitary noise \([23]\). However, some other noise models, such as the phase damping and amplitude damping noises, are not mixed unitary channels themselves, and thus are not orthogonally mixed unitary noises.

From Theorem 1 and the definition of orthogonally mixed unitary noises, we can directly derive the following corollary, which is one of the main results of this work.

**Corollary.** For an orthogonally mixed unitary noise \( \mathcal{E} \), the purity of the input state \( \rho_0 \) and the output state \( \rho \) satisfy

\[
-2\nu(\mathcal{E}^{-1}) \leq \log_2 \left( \frac{\mathcal{P}(\rho) d-1}{\mathcal{P}(\rho_0) d-1} \right) \leq 2\nu(\mathcal{E}) = 0,
\]

(12)

where \( d \) is the dimension of the Hilbert space.

The last equality in Eq. (12) follows from the fact that \( \nu(T) = 0 \) for any CPTP map \( T \) by definition. It
can be easily verified that both sides of this inequality can be reached for the bit-flip and phase-flip noises. As for a more generic multiqubit noise, such as the $n$-qubit dephasing noise, we discuss the bounds in Supplemental Material \[23\], where we conclude that the equality may hold when $n = 1$, while the lower bounds can be further tightened for $n \geq 2$.

**Physical implementability and negativity.**—Now we turn to consider the noise effects on quantum entanglement, which limits the potential power of quantum computers \[8\]. There are many entanglement measures \[5\] for bipartite mixed states, such as concurrence \[24\], entanglement of formation \[25\], entanglement of assistance \[26\], localizable entanglement \[27\], entanglement cost \[28\], etc. Here we choose the logarithmic negativity \[12, 13\] to measure the state entanglement, which characterizes the violation of the well-known positive partial transpose (PPT) criterion \[29\]. Consider a quantum state $\rho$ on a bipartite system $A \otimes B$, its logarithmic negativity is defined as

$$E_N(\rho) := \log_2 \| \rho^{TB} \|_1,$$

where $TB$ denotes the partial transpose of subsystem $B$.

To fully understand this issue, we need to analyze the entanglement property of a quantum channel itself. The Choi-Jamiolkowski isomorphism \[11, 30\] between linear maps and density operators motivates us to define entangled quantum channels, which is a generalization of entangled quantum states \[15\].

**Definition 2.** Let $T$ be a quantum channel on a bipartite system $A \otimes B$. We say that $T$ is separable, if there exist $q_i > 0$ and product channels $T_i^A \otimes T_i^B$ such that

$$T(\cdot) = \sum_i q_i (T_i^A \otimes T_i^B)(\cdot).$$

(14)

Otherwise, we call it an entangled channel.

This definition can also be applied to HPTP maps.

**Definition 3.** Let $\mathcal{N}$ be an HPTP map on a bipartite system $A \otimes B$. We say that $\mathcal{N}$ is separable, if there exist $q_i \in \mathbb{R}$ and product channels $T_i^A \otimes T_i^B$ such that

$$\mathcal{N}(\cdot) = \sum_i q_i (T_i^A \otimes T_i^B)(\cdot).$$

(15)

Otherwise, we call it an entangled map.

To study the influence of noises on the state negativity, we try to connect the partial transpose of output and input states. It can be proved that $\rho^{TB} = \mathcal{N}^{TB}(\rho^T_0)$ \[23\], where the (partial) transpose of a linear map $\mathcal{N}$ is defined by the (partial) transpose of its Choi operator, i.e., $\Lambda_{\mathcal{N}^{TB}}(\rho) = \Lambda_{\mathcal{N}^T}(\rho^T)$. Therefore, we need to take the property of the partial transpose of an HPTP map into consideration. In the following, we try to generalize the PPT concept to quantum channels.

**Definition 4.** A quantum channel $T$ is CPPT if it has a completely positive partial transpose.

**Corollary.** A quantum channel $T$ is CPPT $\iff$ Its Choi operator $\Lambda_T$ is PPT.

The well-known PPT criterion can also be generalized to quantum channels.

**Theorem 2** (CPPT Criterion for quantum channels). Let $T$ be a separable quantum channel on a bipartite system, then $T$ is CPPT.

**Proof.** For a separable $T(\cdot) = \sum_i q_i (T_i^A \otimes T_i^B)(\cdot)$ with $q_i > 0$, its partially transposed Choi operator is written as

$$\Lambda_T^{TB} = \sum_i q_i \Lambda_{T_i^A \otimes T_i^B} = \sum_i q_i \Lambda_{T_i^A \otimes (T_i^B)^T}.$$  

(16)

Since $T_i^B$ is HP and $\Lambda_{TB}^T$ is Hermitian, $\Lambda_{TB}^T$ and $\Lambda_{TB}^A$ share the same eigenvalue spectra. Thus we have

$$T_i^B \text{ is CP } \iff \Lambda_{TB}^T \geq 0$$

$$\iff \Lambda_{TB}^T = \Lambda_{TB}^T \iff (T_i^B)^T \text{ is CP } \iff \Lambda_{TB}^A \otimes (T_i^B)^T \geq 0.$$  

Finally we obtain

$$\Lambda_T^{TB} = \sum_i q_i \Lambda_{T_i^A \otimes (T_i^B)^T} \geq 0.$$  

(18)

The following theorem provides a general bound for the output state negativity concerning the partially transposed Choi operator.

**Theorem 3.** For an HPTP map $\mathcal{N}$ on a bipartite system $A \otimes B$, the logarithmic negativity of the input state $\rho_0$ and the output state $\rho$ satisfy

$$E_N(\rho) - E_N(\rho_0) \leq \log_2 d + \log_2 \| \Lambda_{\mathcal{N}^T}^{TB} \|_1,$$

(19)

where $d$ is the dimension of the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.

**Proof.** We can bound the trace norm of $\rho^{TB}$ as

$$\| \rho^{TB} \|_1 = \| \mathcal{N}^{TB}(\rho_0^T) \|_1 = \| \text{Tr}_B \left( \left( \rho_0^T \otimes I_T \right) \Lambda_{\mathcal{N}^T}^{TB} \right) \|_1$$

$$\leq \left\| \left( \rho_0^T \otimes I_T \right) \Lambda_{\mathcal{N}^T}^{TB} \right\|_1 \leq \left\| \left( \rho_0^T \otimes I_T \right) \right\|_1 \left\| \Lambda_{\mathcal{N}^T}^{TB} \right\|_1$$

$$= d \left\| \rho_0^T \right\|_1 \left\| \Lambda_{\mathcal{N}^T}^{TB} \right\|_1 = d \left\| \rho_0^{TB} \right\|_1 \left\| \Lambda_{\mathcal{N}^T}^{TB} \right\|_1.$$  

(20)

The first inequality comes from $\text{Tr}[X] \leq \| X \|_1$, while the second inequality is a general property of any kind of the norm.
Theorem 4. The state negativity is non-increasing under separable quantum channels.

Proof. Consider a separable quantum channel $\mathcal{T} = \sum_i q_i T_i^A \otimes T_i^B$, we have
\[
\left\| \rho_{T_i}^{B} \right\|_1 = \left\| \sum_i q_i \left[ T_i^A \otimes T_i^B \left( \rho_0 \right) \right]^{T_i} \right\|_1 \\
\leq \sum_i |q_i| \left\| \left[ T_i^A \otimes T_i^B \left( \rho_0 \right) \right]^{T_i} \right\|_1 \\
\leq \sum_i |q_i| \left\| \rho_0^{T_i} \right\|_1 = \left\| \rho_0^{T_i} \right\|_1,
\]
where the second inequality follows from a general property of any entanglement measure $E(\rho)$, saying that $E(\rho)$ does not increase under local operations and classical communication (LOCC) \[23\] [31].

However, for those separable non-CP maps, such as the inverse of a noise channel, we believe that they can counteract the noise effect and recover coherence. It helps us to characterize how much entanglement the noise channel destroys.

Theorem 5. For a separable HPTP map $\mathcal{N}$, the logarithmic negativity of the input state $\rho_0$ and the output state $\rho$ satisfy
\[
E_N(\rho) - E_N(\rho_0) \leq \nu(\mathcal{N}),
\]
where $\nu(\mathcal{N})$ is the physical implementability of $\mathcal{N}$.

Proof. Suppose that the optimized decomposition of $\mathcal{N}$
\[
\mathcal{N} = \sum_i q_i T_i
\]
gives the physical implementability $2^{\nu(\mathcal{N})} = \sum_i |q_i|$, where $T_i$ are separable quantum channels. Then the output state negativity reads
\[
\left\| \rho_{T_i}^{B} \right\|_1 = \left\| \sum_i q_i \left[ T_i \left( \rho_0 \right) \right]^{T_i} \right\|_1 \leq \sum_i |q_i| \left\| T_i \left( \rho_0 \right) \right\|_1 \\
\leq \sum_i |q_i| \left\| \rho_0^{T_i} \right\|_1 = 2^{\nu(\mathcal{N})} \left\| \rho_0^{T_i} \right\|_1,
\]
where we have used Theorem [4] \[12\].

With the above theorem, we can derive another important conclusion of our study.

Corollary. For a noise channel $\mathcal{E}$, if both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are separable, the logarithmic negativity of the input state $\rho_0$ and the output state $\rho$ will satisfy
\[
-\nu(\mathcal{E}^{-1}) \leq E_N(\rho) - E_N(\rho_0) \leq \nu(\mathcal{E}) = 0.
\]
than the commonly used error rate. We believe that our work is a big step forward and opens a new avenue towards the theoretical research of noise channels from the entanglement perspective.

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Supplemental material

In this Supplemental Material, we provide more details on the Choi operator representation, mixed unitary maps, noise models and physical implementability, an example of purity change, partial transpose of linear maps, and estimation for physical implementability.
Choi operator representation

We denote the Hilbert space of a quantum system as $\mathcal{H}$, where the space of linear operators is labeled as $\mathcal{L}(\mathcal{H})$. A quantum state is represented by a density operator $\rho \in \mathcal{L}(\mathcal{H})$, which is a Hermitian ($\rho = \rho^\dagger$) and positive semidefinite ($\rho \geq 0$) operator with unit trace ($\text{Tr}[\rho] = 1$).

A quantum channel $\mathcal{T}$ is a linear map between two operator spaces $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ with the completely positive (CP) and trace-preserving (TP) conditions. We say that a linear map $N$ is

- trace-preserving (TP), if $\text{Tr}[N(O)] = \text{Tr}[O]$ for $\forall O \in \mathcal{L}(\mathcal{H})$,
- Hermitian-preserving (HP), if $N(O)^\dagger = N(O)$ for $\forall O \in \mathcal{L}(\mathcal{H})$ with $O^\dagger = O$,
- positive, if $N(O) \geq 0$ for $\forall O \in \mathcal{L}(\mathcal{H})$ with $O \geq 0$,
- completely positive (CP), if $I_{\mathcal{A}'} \otimes N_A$ is positive for an arbitrary ancillary system $\mathcal{A}'$.

The most commonly adopted representation of a quantum channel $\mathcal{T}$ is the Choi operator \cite{Choi1975}, which is defined on a joint system as

$$\Lambda_T := (I_{\sigma} \otimes \mathcal{T}_\tau) (d |\Phi\rangle\langle\Phi|_{\sigma\tau}) ,$$

(S1)

where $d$ is the dimension of the Hilbert space, $\tau$ and $\sigma$ are the indices of the physical system and the ancillary system (which is isomorphic to the physical system) respectively. $|\Phi\rangle$ is the maximally entangled state

$$|\Phi\rangle_{\sigma\tau} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_{\sigma} |i\rangle_{\tau}$$  

(S2)

with $\{|i\rangle_{\tau}\}$ and $\{|i\rangle_{\sigma}\}$ being the orthonormal basis of the physical system and the ancillary system respectively. On the same basis, the matrix elements of the Choi operator are

$$\langle i_{\sigma}k_{\tau} | \Lambda_T | j_{\sigma}l_{\tau} \rangle = \langle k | \mathcal{T} (|i\rangle\langle j|) |l\rangle ,$$

(S3)

which is just the matrix representation of map $\mathcal{T}$ in an orthonormal basis of $\mathcal{L}(\mathcal{H})$ with reshuffled physical indices. In this sense, we can consider $\sigma$ and $\tau$ as the input and output indices respectively. Then the implementation of $\mathcal{T}$ can be calculated in terms of the Choi operator as

$$\mathcal{T} (\rho) = \text{Tr}_\tau [(\rho^T \otimes I_{\sigma}) \Lambda_T] .$$

(S4)

The Choi representation can be directly generalized to any linear map $\mathcal{N}$ beyond CPTP ones with the same definition as Eq. (S1). Due to the Choi-Jamiołkowski isomorphism, the conditions mentioned above for a linear map $\mathcal{N}$ are guaranteed by the following properties of $\Lambda_N$ \cite{Choi1975}.

- $\mathcal{N}$ is TP $\iff$ $\text{Tr}[\Lambda_N] = I_{\sigma}$.
- $\mathcal{N}$ is HP $\iff$ $\Lambda_N^\dagger = \Lambda_N$.
- $\mathcal{N}$ is CP $\iff$ $\Lambda_N \geq 0$.

Mixed unitary maps

The convex combination of unitary channels is called a mixed unitary channel \cite{Braunstein1994}

$$\mathcal{T} (\cdot) = \sum_{i} q_i U_i (\cdot) U_i^\dagger$$

with $q_i > 0$,  

(S5)

which can be naturally generalized to real coefficients and denoted as a mixed unitary map \cite{Caves1994}.

$$\mathcal{N} (\cdot) = \sum_{i} q_i U_i (\cdot) U_i^\dagger$$

(S6)

with $q_i \in \mathbb{R}$.
We define the multiqubit Pauli noises as

\[ \mathcal{E} \rho = (1 - \varepsilon) \rho + \varepsilon \sigma \rho \sigma, \quad \sigma \in \{1, 2, 3\}. \] (S8)

The inverse of this form can be easily obtained

\[ \mathcal{E}^{-1} \rho = \frac{1 - \varepsilon}{1 - 2\varepsilon} \rho - \frac{\varepsilon}{1 - 2\varepsilon} \sigma \rho \sigma, \] (S9)

which is a mixed unitary map. We further note that the two decomposed unitaries in \( \mathcal{E} \) (and \( \mathcal{E}^{-1} \)) are orthogonal \( \text{Tr}[I \sigma] = 0 \), hence we can directly calculate the physical implementability of \( \mathcal{E}^{-1} \) as \( \nu(\mathcal{E}^{-1}) = \log_2 \left( \frac{1}{1 - 2\varepsilon} \right) \).

**The multiqubit Pauli noises.** We define the multiqubit Pauli noises as

\[ \mathcal{E} \rho = (1 - \varepsilon) \rho[n] + \varepsilon \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right) \rho[n] \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right)^\dagger, \] (S10)

where \( \sigma_{\alpha_i}^i \) represents the Pauli matrix \( \sigma_{\alpha} \) applied on the \( i \)-th site. The inverse of this noise is similar to the single-qubit case

\[ \mathcal{E}^{-1} \rho = \frac{1 - \varepsilon}{1 - 2\varepsilon} \rho[n] - \frac{\varepsilon}{1 - 2\varepsilon} \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right) \rho[n] \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right)^\dagger, \] (S11)

which is also a mixed unitary map, with the physical implementability being \( \nu(\mathcal{E}^{-1}) = \log_2 \left( \frac{1}{1 - 2\varepsilon} \right) \). The two-qubit phase flip noise in Fig. 1(c) corresponds to taking \( \alpha_1 = \alpha_2 = 3 \) here, i.e., \( \mathcal{E}[2] \left( \rho[2] \right) = (1 - \varepsilon) \rho[2] + \varepsilon (\sigma_2^z \otimes \sigma_2^z) \rho[2] (\sigma_2^z \otimes \sigma_2^z)^\dagger \).

**The multiqubit depolarizing noise.** The \( n \)-qubit depolarizing noise is defined as

\[ \mathcal{E}[n] \rho[n] = (1 - \varepsilon) \rho[n] + \frac{\varepsilon}{2^n} \mathbb{I}[n] \]

\[ = (1 - \varepsilon) \rho[n] + \frac{\varepsilon}{4^n} \sum_{\alpha} \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right) \rho[n] \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right)^\dagger, \] (S12)

where the indices \( \alpha_i \) are summed from 0 to 3. Its inverse can be directly calculated with the first equality

\[ \mathcal{E}[n]^{-1} \rho[n] = \frac{1}{1 - \varepsilon} \rho[n] - \frac{\varepsilon}{(1 - \varepsilon)2^n} \mathbb{I}[n] \]

\[ = \frac{1}{1 - \varepsilon} \rho[n] - \frac{\varepsilon}{(1 - \varepsilon)4^n} \sum_{\alpha} \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right) \rho[n] \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^i \right)^\dagger. \] (S13)
Therefore, $\mathcal{E}^{-1}$ is a mixed unitary map. The unitaries in the decomposition of $\mathcal{E}$ and $\mathcal{E}^{-1}$ are also mutually orthogonal

$$
\text{Tr} \left[ \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right)^{\dagger} \left( \bigotimes_{i=1}^{n} \sigma_{\beta_i}^{i} \right) \right] = 2^n \prod_{i=1}^{n} \delta_{\alpha_i \beta_i},
$$

allowing us to analytically calculate the physical implementability as

$$
\nu \left( \mathcal{E}^{-1} \right) = \log_{2} \left[ \left( \frac{1}{1-\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)2^n} \right) + (2^n - 1) \frac{\varepsilon}{(1-\varepsilon)2^n} \right] = \log_{2} \left[ \frac{1 + \left( 1 - \frac{2^n}{2^n} \right) \varepsilon}{1 - \varepsilon} \right].
$$

**The multiqubit dephasing noise.** The n-qubit dephasing noise is defined as

$$
\mathcal{E}^{[n]} \left( \rho^{[n]} \right) = (1-\varepsilon) \rho^{[n]} + \frac{\varepsilon}{2^n} \sum_{\{\alpha_i \in \{0,3\}\}} \left[ \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right) \rho^{[n]} \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right)^{\dagger} \right],
$$

where the summation only contains $\sigma_0$ and $\sigma_3$. We assume that the inverse of $\mathcal{E}^{[n]}$ takes a similar form

$$
\mathcal{E}^{-1} \left( \rho^{[n]} \right) = A \rho^{[n]} - B \sum_{\{\alpha_i \in \{0,3\}\}} \left[ \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right) \rho^{[n]} \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right)^{\dagger} \right]
$$

and solve the undetermined coefficients $A$ and $B$. Finally we obtain

$$
\mathcal{E}^{-1} \left( \rho^{[n]} \right) = \frac{1}{1-\varepsilon} \rho^{[n]} - \frac{\varepsilon}{(1-\varepsilon)2^n} \sum_{\{\alpha_i \in \{0,3\}\}} \left[ \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right) \rho^{[n]} \left( \bigotimes_{i=1}^{n} \sigma_{\alpha_i}^{i} \right)^{\dagger} \right],
$$

implying that $\mathcal{E}^{[n]}$ is an orthogonally mixed unitary noise. Similarly, we can derive the physical implementability of the noise inverse

$$
\nu \left( \mathcal{E}^{-1} \right) = \log_{2} \left[ \left( \frac{1}{1-\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)2^n} \right) + (2^n - 1) \frac{\varepsilon}{(1-\varepsilon)2^n} \right] = \log_{2} \left[ \frac{1 + \left( 1 - \frac{2^n}{2^n} \right) \varepsilon}{1 - \varepsilon} \right].
$$

**The amplitude damping noise.** The amplitude damping noise is defined with the Kraus operator $E_0 = |0\rangle\langle 0| + \sqrt{1-\varepsilon} |1\rangle\langle 1|$ and $E_1 = \sqrt{\varepsilon} |0\rangle\langle 1|$ with the operator-sum representation

$$
\mathcal{E} (\rho) = E_0 \rho E_0^{\dagger} + E_1 \rho E_1^{\dagger}.
$$

The physical implementability of $\mathcal{E}^{-1}$ was analytically studied in [11], given by $\nu \left( \mathcal{E}^{-1} \right) = \log_{2} \left[ \frac{1+\varepsilon}{1-\varepsilon} \right]$. The multiqubit amplitude damping noise is just defined as the tensor product of single-qubit noise channels, i.e.,

$$
\mathcal{E}^{[n]} = \bigotimes_{i=1}^{n} \mathcal{E}^{i}
$$

with the physical implementability of the noise inverse being $\nu \left( \mathcal{E}^{-1} \right) = n \log_{2} \left[ \frac{1+\varepsilon}{1-\varepsilon} \right]$.

In summary, all of the above noise models are separable noises for any bipartite quantum system. Therefore, the relation in Eq. [25] holds for these noises.

In the following lemmas, We prove that the composition of orthogonally mixed unitary noises with unitary channels or the tensor product of orthogonally mixed unitary noises is also an orthogonally mixed unitary noise.

**Lemma.** For an orthogonally mixed unitary noise $\mathcal{E}$, $\mathcal{U} \circ \mathcal{E} \circ \mathcal{V}$ is also an orthogonally mixed unitary noise for any unitary channels $\mathcal{U} (\cdot) = U (\cdot) U^{\dagger}$ and $\mathcal{V} (\cdot) = V (\cdot) V^{\dagger}$. 

Proof. Consider the orthogonal decomposition $\mathcal{E}(\cdot) = \sum_i q_i U_i(\cdot) U_i^\dagger$, we can decompose
\[
\mathcal{U} \circ \mathcal{E} \circ \mathcal{V}(\cdot) = \sum_i q_i \mathcal{U} U_i V(\cdot) (U_i V)^\dagger
\]
with orthogonal unitaries $\{UU_i V\}$ satisfying
\[
\text{Tr}
\left[
(U_i U_i V)^\dagger (U_i U_i V)\right] = \text{Tr}
\left[
V^\dagger U_i^\dagger U_i U_i V\right] = \text{Tr}
\left[
U_i^\dagger U_i\right] = d \delta_{ij},
\]
where $d$ is the dimension of the Hilbert space. Similarly we can prove that the inverse $(\mathcal{U} \circ \mathcal{E} \circ \mathcal{V})^{-1} = \mathcal{V}^{-1} \circ \mathcal{E}^{-1} \circ \mathcal{U}^{-1}$ is also a mixed unitary map decomposed by a set of mutually orthogonal unitaries. \hfill \Box

**Lemma.** For two orthogonally mixed unitary noises $\mathcal{E}^A$ and $\mathcal{E}^B$, $\mathcal{E}^A \otimes \mathcal{E}^B$ is also an orthogonally mixed unitary noise.

**Proof.** Consider the orthogonal decomposition $\mathcal{E}^X(\cdot) = \sum_{ij} q_i q_j U_i^A \otimes U_i^B (\cdot) U_i^{A\dagger} \otimes U_i^{B\dagger}$.

Obviously this decomposition also satisfies the orthogonal condition
\[
\text{Tr}
\left[
(U_i^{A\dagger} \otimes U_i^{B\dagger}) (U_i^A \otimes U_i^B)\right] = \text{Tr}
\left[
U_i^{A\dagger} U_i^A\right] \text{Tr}
\left[
U_j^{B\dagger} U_j^B\right] = d^A d^B \delta_{ik} \delta_{jl},
\]
where $d^A$ and $d^B$ are the dimensions of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. It can also be easily verified that $\mathcal{E}^{A^{-1}} \otimes \mathcal{E}^{B^{-1}}$ is a mixed unitary map decomposed by a set of mutually orthogonal unitaries. \hfill \Box

**An example of purity change**

We consider a quantum state $\rho_0^{[n]}$ undergoing an $n$-qubit dephasing noise defined in Eq. (S16). We measure the expectation values of $O_{\{\alpha\}} = \otimes_{i=1}^n \sigma^i_{\alpha_i}$, which form a set of orthogonal basis of the operator space.

\[
\langle O_{\{\alpha\}} \rangle = \text{Tr}
\left[
\rho_0^{[n]} O_{\{\alpha\}}\right] = \text{Tr}
\left[
\mathcal{E}^{[n]}(\rho_0^{[n]}) O_{\{\alpha\}}\right]
\]

\[
= (1 - \varepsilon) \text{Tr}
\left[
\rho_0^{[n]} O_{\{\alpha\}}\right] + \frac{\varepsilon}{2^n} \sum_{\{\beta_i \in \{0,3\}\}} \text{Tr}
\left[
\left(\bigotimes_{i=1}^n \sigma^i_{\beta_i}\right) \rho_0^{[n]} \left(\bigotimes_{i=1}^n \sigma^i_{\beta_i}\right) \dagger\left(\bigotimes_{i=1}^n \sigma^i_{\alpha_i}\right)\right]
\]

\[
= (1 - \varepsilon) \text{Tr}
\left[
\rho_0^{[n]} O_{\{\alpha\}}\right] + \frac{\varepsilon}{2^n} \sum_{\{\beta_i \in \{0,3\}\}} \text{Tr}
\left[
\rho_0^{[n]} \left(\bigotimes_{i=1}^n \sigma^i_{\beta_i}\right) \dagger\left(\bigotimes_{i=1}^n \sigma^i_{\alpha_i}\right)\right]
\]

\[
= (1 - \varepsilon) \text{Tr}
\left[
\rho_0^{[n]} O_{\{\alpha\}}\right] + \frac{\varepsilon}{2^n} \text{Tr}
\left[
\rho_0^{[n]} \left(\bigotimes_{i=1}^n \sigma^i_{\beta_i}\right) \dagger\left(\bigotimes_{i=1}^n \sigma^i_{\alpha_i}\right)\right]
\]

We note that
\[
\sum_{\beta_i \in \{0,3\}} \sigma^i_{\beta_i} \sigma^i_{\alpha_i} \sigma^i_{\alpha_i} = \sigma^i_{\alpha_i} + \sigma^i_{\alpha_i} \sigma^i_{\alpha_i} = \begin{cases}
2 \sigma^i_{\alpha_i}, & \text{if } \alpha_i \in \{0,3\}, \\
0, & \text{if } \alpha_i \in \{1,2\},
\end{cases}
\]
from which we can derive
\[
\langle O_{\{\alpha\}} \rangle = \begin{cases}
\langle O_{\{\alpha_i\}} \rangle (0), & \text{if } \alpha_i \in \{0,3\} \text{ for } \forall i, \\
(1 - \varepsilon) \langle O_{\{\alpha_i\}} \rangle (0), & \text{otherwise},
\end{cases}
\]

Therefore, the length of the Bloch vector satisfies
\[
(1 - \varepsilon) \leq \frac{|\vec{r}(\rho)|}{|\vec{r}(\rho_0)|} \leq 1.
\]

Compared with Eq. (S19), we conclude that the bounds in Eq. (12) can be reached for $n = 1$, while for a big $n$, the physical implementability still serves as a good estimation.
Partial transpose of linear maps

In the following lemma, we prove that the partial transpose of the output state can be calculated in terms of the partial transpose of the linear map.

Lemma. For a linear map \(N\) on a bipartite system \(A \otimes B\), the partial transpose of the output operator satisfies

\[
\rho_B^{\text{T}} = N_B^{\text{T}} \left( \rho_0^{B^{\text{T}}} \right), \tag{S30}
\]

where the (partial) transpose of a linear map \(N\) is defined by the (partial) transpose of its Choi operator, i.e.,

\[
\Lambda_{N^{\text{T}}(B)} = \Lambda_N^{B^{\text{T}}}. \tag{S31}
\]

Proof. We first take the Schmidt decomposition of operators

\[
\rho_0 = \sum_k \alpha_k \rho_k^A \otimes \rho_k^B, \tag{S32}
\]

\[
\Lambda_N = \sum_k \beta_k \Lambda_k^A \otimes \Lambda_k^B, \tag{S33}
\]

then the partially transposed output operator is calculated in terms of the Choi operator as

\[
\rho_B^{\text{T}} = \text{Tr}_{\sigma} \left[ \left( \rho_0^{B^{\text{T}}} \otimes \mathbb{I}^{\tau} \right) \Lambda_N \right]^{B^{\text{T}}}
\]

\[
= \text{Tr}_{\sigma} \left[ \sum_{i,j} \alpha_i \beta_j \left( \left( \rho_i^{A^{\text{T}}} \otimes \mathbb{I}^{\tau}_A \right) \otimes \left( \rho_i^{B^{\text{T}}} \otimes \mathbb{I}^{\tau}_B \right) \right) \left( \Lambda_j^A \otimes \Lambda_j^B \right) \right]^{B^{\text{T}}}
\]

\[
= \text{Tr}_{\sigma} \left[ \sum_{i,j} \alpha_i \beta_j \left( \left( \rho_i^{A^{\text{T}}} \otimes \mathbb{I}^{\tau}_A \right) \otimes \left( \Lambda_j^B \right) \right) \left( \Lambda_j^A \otimes \left( \rho_i^{B^{\text{T}}} \otimes \mathbb{I}^{\tau}_B \right) \right) \right]
\]

\[
= \text{Tr}_{\sigma} \left[ \rho_0^{A^{\text{T}}} \otimes \mathbb{I}^{\tau} \right] \Lambda_N^{B^{\text{T}}}
\]

\[
= \text{Tr}_{\sigma} \left[ \left( \rho_0^{B^{\text{T}}} \otimes \mathbb{I}^{\tau} \right) \Lambda_N^{B^{\text{T}}} \right]
\]

\[
= \Lambda_N^{B^{\text{T}}} \left( \rho_0^{B^{\text{T}}} \right).
\]

For the third equality, the partial transpose will introduce a permutation between \(\rho_i^{B^{\text{T}}} \otimes \mathbb{I}^{\tau}_B\) and \(\Lambda_j^B\), while the permutation in the fourth equality comes from the partial trace of \(\sigma\) part and the identity of \(\tau\) part.

Bounds for physical implementability

It was proved in \([11]\) that the physical implementability for a general HPTP map is bounded by the trace norm of its Choi operator

\[
\frac{\| \Lambda_N \|_1}{d} \leq 2^{\nu(N)} \leq \| \Lambda_N \|_1, \tag{S35}
\]

where \(d\) is the dimension of the Hilbert space. In the following, we provide two lower bounds for the trace norm in terms of maximum and minimum eigenvalues.

Lemma. For a Choi operator \(\Lambda\), its trace norm satisfies

\[
\| \Lambda \|_1 \geq 2\lambda_{\text{max}} - d, \tag{S36}
\]

\[
\| \Lambda \|_1 \geq d - 2\lambda_{\text{min}}, \tag{S37}
\]

where \(\lambda_{\text{min}}\) and \(\lambda_{\text{max}}\) are the minimum and maximum eigenvalues of \(\Lambda\) respectively, and \(d\) is the dimension of the Hilbert space.
Proof. Suppose the eigenvalues of Λ are \( \lambda_{\min} = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{d^2-1} = \lambda_{\max} \) with \( \text{Tr}[\Lambda] = \sum_{i=0}^{d^2-1} \lambda_i = d \).

\[
\|\Lambda\|_1 = \sum_{i=0}^{d^2-1} |\lambda_i| \geq |\lambda_{\max}| + \sum_{i=0}^{d^2-2} \lambda_i = |\lambda_{\max}| + |d - \lambda_{\max}| \geq 2\lambda_{\max} - d. \tag{S38}
\]

\[
\|\Lambda\|_2 = \sum_{i=0}^{d^2-1} |\lambda_i| \geq |\lambda_{\min}| + \sum_{i=1}^{d^2-1} \lambda_i = |\lambda_{\min}| + |d - \lambda_{\min}| \geq d - 2\lambda_{\min}. \tag{S39}
\]

The equality in Eq. \(\text{[S36]}\) holds when \(\lambda_{d^2-1} \geq 0 \geq \lambda_{d^2-2}, \) while the equality in Eq. \(\text{[S37]}\) holds when \(\lambda_0 \leq 0 \leq \lambda_1.\) With Eq. \(\text{[S35]}\), we can provide two lower bounds for the physical implementability.

Corollary. For an HPTP map \(\mathcal{N}\), its physical implementability is bounded by

\[
2^{\nu(\mathcal{N})} \geq \frac{2\lambda_{\max}}{d} - 1, \tag{S40}
\]

\[
2^{\nu(\mathcal{N})} \geq 1 - \frac{2\lambda_{\min}}{d}. \tag{S41}
\]

where \(\lambda_{\min}\) and \(\lambda_{\max}\) are the minimum and maximum eigenvalues of \(\Lambda_{\mathcal{N}}\) respectively, and \(d\) is the dimension of the Hilbert space.

It can be verified that the bound in Eq. \(\text{[S40]}\) can be reached for the inverse of the multiqubit Pauli, dephasing, and depolarizing noises.

As for the possible upper bound, we note that there exists a trivial decomposition for the Choi operator of any HPTP map \(\mathcal{N}\) which is not CP,

\[
\Lambda_{\mathcal{N}} = \Lambda_{\mathcal{N}} - \lambda_{\min}I_\sigma \otimes \tau + \lambda_{\min}I_\sigma \otimes \tau = \Lambda_{\mathcal{N}} - \lambda_{\min}d \frac{I_\sigma \otimes \tau}{d} - (-\lambda_{\min})d \frac{I_\sigma \otimes \tau}{d}, \tag{S42}
\]

where \(I_\sigma \otimes \tau\) is the identity operator on the extended \(d^2\)-dimensional Hilbert space with \(\text{Tr}_\sigma[I_\sigma \otimes \tau] = dI_\tau\), Therefore, \(I_\sigma \otimes \tau/d\) is a proper Choi operator for a CPTP quantum channel, denoted as \(\mathcal{T}_I\). Actually it can be verified that \(\mathcal{T}_I(\rho) = I\) for any density operator \(\rho\), i.e., it projects all states to the maximally mixed state. Consequentially, we can decompose the map \(\mathcal{N}\) with two CPTP maps as

\[
\mathcal{N} = (1 - \lambda_{\min} d) \frac{\mathcal{N} - \lambda_{\min}d \mathcal{I}_T}{1 - \lambda_{\min}d} + \lambda_{\min}d \mathcal{I}_T, \tag{S43}
\]

from which we obtain the upper bound of the physical implementability

\[
2^{\nu(\mathcal{N})} \leq 1 - 2\lambda_{\min}d. \tag{S44}
\]

For example, for the \(n\)-qubit depolarizing noise \(\mathcal{E}^{[n]}\), the maximum and minimum eigenvalues of \(\Lambda_{\mathcal{E}^{[n]}-1}\) are

\[
\lambda_{\max} = \frac{4^n - \varepsilon}{(1 - \varepsilon)2^n}, \tag{S45}
\]

\[
\lambda_{\min} = -\frac{\varepsilon}{(1 - \varepsilon)2^n}, \tag{S46}
\]

which can be directly derived from Eq. \(\text{[S13]}\). Then Eq. \(\text{[S40]}\) and \(\text{[S44]}\) give

\[
\log_2 \left[ 1 + \frac{1 + \left( \frac{\varepsilon}{2^n} \right)}{1 - \varepsilon} \right] \leq \nu \left( \mathcal{E}^{[n]} - 1 \right) \leq \log_2 \left[ 1 + \frac{\varepsilon}{1 - \varepsilon} \right]. \tag{S47}
\]

It is implied that if we use these two bounds to estimate the physical implementability, the precision will grow exponentially with the number of qubits \(n\) in this case and the estimation is exact in the thermodynamic limit.