On the Algebraic Approach to Solvable Lattice Models.

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ABSTRACT

We develop an algebraic approach to solvable lattice models based on a chain of algebras obeyed by the models. In each subalgebra we use a unit, giving a chain of ideals. Thus, we divide the models into distinct sectors which do not mix. This method gives the usual Bethe ansatz results in cases it is known, but generalizes it to non-integrable models. We exemplify the method on the Temperley–Lieb and Fuss–Catalan algebras. For the Fuss–Catalan algebra we show that the ground state energy is zero and there is a mass gap of one for $\alpha > \sqrt{2}$, and that for $\alpha = 1$ we seem to get an RCFT as the scaling limit.
1. Introduction.

Solvable lattice models in two dimensions have attracted considerable attention (for a review see [1]). This due to a large extent owing to their solvability along with the fascinating interplay with quantum field theory close to the critical points. Here we use the algebraic structure of such models to study their spectrum. Our main results are about the Fuss–Catalan models where we find a mass gap of 1 and a ground state energy zero in the principal regime. For $\alpha = 1$ we conjecture to get a series of conformal field theories and no mass gap.

2. The algebras.

In the process of solving various lattice models one encounters different algebras. It is an old idea of Baxter ref. [2] that the algebraic structure itself can be used to solve the models completely. This is the line of thought that we will follow here. We develop further the interesting ideas of Levy on the subject [3].

The best known example is the Temperley–Lieb algebra [4]. It is defined on the $n$ points of the model by,

$$
e_{i}^{2} = \alpha e_{i}$$
$$e_{i}e_{i+1} = e_{i}$$
$$e_{i}e_{j} = e_{j}e_{i} \quad \text{for } |i - j| \geq 2,$$

(1)

where the $e_{i}$ are the generators of the algebra, and $i = 1, 2, \ldots, n$. Originally, this algebra was used to solve the $q$ states Potts model where $\alpha = q^{1/2}$. Another interesting model is the Heisenberg spin chain where one takes,

$$\mathcal{H} = \sum_{i=1}^{n} e_{i}.$$  

(2)

Another algebra of this kind is the Fuss–Catalan algebra (FC), [5, 6], which is a generalization of the Temperley Lieb algebra (TL) to more than one color. The
generators here are $e_i^{(a)}$ where $a$ labels the colors, $a = 1, 2, \ldots, f$ where $f$ is the number of colors. $f = 1$ is just the usual Temperley–Lieb algebra.

The two colors Fuss–Catalan algebra is defined by,

\begin{align*}
(e_i^{(1)})^2 &= \alpha e_i^{(1)} \\
(e_i^{(2)})^2 &= \alpha^2 e_i^{(2)} \\
e_i^{(1)} e_i^{(2)} &= \alpha e_i^{(1)} \\
e_i^{(1)} e_{i+1}^{(1)} &= e_{i+1}^{(1)} e_i^{(1)} \\
e_i^{(2)} e_{i+1}^{(1)} &= e_{i+1}^{(1)} e_i^{(2)} \\
e_i^{(1)} e_{i+1}^{(2)} &= e_i^{(1)} e_{i+1}^{(2)} \\
e_i^{(2)} e_{i+1}^{(2)} &= e_i^{(2)} e_{i+1}^{(2)} \\
e_i^{(1)} e_i^{(2)} e_{i+1}^{(2)} &= e_i^{(1)} e_i^{(2)} e_{i+1}^{(2)} \\
e_i^{(2)} e_i^{(1)} e_{i+1}^{(2)} &= e_i^{(2)} e_i^{(1)} e_{i+1}^{(2)} \quad (3)
\end{align*}

Here $\alpha$ is a free parameter defining the algebra, and we assume that $i = 1, 2, \ldots, n$ is the number of points. We take the Hamiltonian to be,

$$
H = \sum_{i=1}^{n} \frac{e_i^{(1)}}{\alpha} + \frac{e_i^{(2)}}{\alpha^2 - 2} \quad (4)
$$

We take this Hamiltonian so as to become an integrable model, where the transfer matrix obeys the Yang–Baxter relation, ref. [6].

For more than two colors the algebra is given by the relations,

\begin{align*}
e_i^{(m)} e_j^{(p)} &= \alpha^{\min(m,p)} e_i^{(\max(m,p))} \\
e_i^{(m)} e_j^{(p)} &= e_j^{(p)} e_i^{(m)} \quad \text{if } |i - j| > 1 \text{ or } j = i \pm 1 \text{ and } m + p \leq f. \\
e_i^{(m)} e_{i+1}^{(p)} e_{i+1}^{(m)} &= \alpha^{(f-p)} e_i^{(m)} e_{i+1}^{(f-m)} e_{i+1}^{(m)} \quad \text{for } m + p > f, \\
e_i^{(m)} e_{i+1}^{(p)} e_{i+1}^{(q)} &= \alpha^{(f-p)} e_i^{(m)} e_{i+1}^{(f-q)} e_{i+1}^{(q)} \quad \text{if } m \geq q \text{,}
\end{align*}

\begin{align*}
e_i^{(m)} e_{i+1}^{(p)} e_{i+1}^{(q)} &= \alpha^{(f-p)} e_i^{(m)} e_{i+1}^{(f-q)} e_{i+1}^{(q)} \quad \text{if } m \leq q. \quad (5)
\end{align*}

Again, the algebra depends only on one parameter, $\alpha$ and the number of points $n$. (Actually, the algebra is defined with $f - 1$ parameters, and, we specialized all to
be equal so as to give a solution of the Yang–Baxter equation [6]. As in the two color case, the Hamiltonian is taken to be,

\[ \mathcal{H} = \sum_{i=1}^{n} \sum_{a=1}^{f} e_i^{(a)} k_a, \]  

where the \( k_a \) are some function of the color which depends only on \( \alpha \), and which guarantees that the model solves the Yang–Baxter relation, ref. [6]. The Boltzman weights are all positive in the principal regime of the model which is defined by \( \alpha > \sqrt{2} \).

Note that all of these algebras are nested inside each other. Namely, the algebra for \( r \) points is a subalgebra of the algebra for \( r + 1 \) points. This, very nontrivial fact, is going to be the basis for the subsequent discussion.

3. The projectors

Consider thus any of the algebras discussed above, \( \mathcal{A} \). To be specific we can take \( \mathcal{A} \) to be any of the Temperley–Lieb or Fuss–Catalan algebras for some \( f \), the number of colors, and some \( n \), the number of points. Importantly, we do not include the unit in this algebra, but just the generators \( e_i^{a} \) obeying eq. (5).

Now, denote by \( V_n \) a left and right unit of the algebra,

\[ V_n e_i^{(a)} = e_i^{(a)} \]
\[ e_i^{(a)} V_n = e_i^{(a)}, \]  

for all \( i = 1, 2, \ldots, n \) and for all \( a = 1, 2, \ldots, f \), where we suppose that the unit indeed exists. Below we will give the explicit expression for \( V_n \) for the algebras we discuss.

Now, a standard result is that the unit, if it exists, is unique. Suppose that \( b \)
is another unit (left or right; it does not matter). Then, for a right unit, say,

$$b = bV_n = V_n,$$  \hspace{1cm} (8)

which shows that the unit is always unique. From the unit, we can form a projection operator, $Z_n$, by taking,

$$Z_n = 1 - V_n.$$  \hspace{1cm} (9)

Since $V_n$ does not include a 1, the projector $Z_n$ is nontrivial. For the projector $Z_n$ we have,

$$e_i^{(a)} Z_n = Z_n e_i^{(a)} = 0,$$  \hspace{1cm} (10)

for all $i = 1, 2, \ldots, n$ and for all the $a = 1, 2, \ldots, f$.

Before proceeding, let us give the explicit expression for the projectors $Z_n$ for the algebras on which we will concentrate here, namely, the Temperley–Lieb and the Fuss–Catalan algebras.

For the TL algebra the projectors are given recursively by [7] and reported by [8],

$$Z_n = Z_{n-1}(1 - a_ne_n)Z_{n-1},$$  \hspace{1cm} (11)

where

$$a_n = \frac{\sinh n\theta}{\sinh(n+1)\theta},$$  \hspace{1cm} (12)

and $\theta$ obeys,

$$e^\theta + e^{-\theta} = \alpha,$$  \hspace{1cm} (13)

and we define $Z_0 = 1$. For this projector, it can be shown that eq. (10) is indeed obeyed. The projectors are not defined for

$$\alpha = 2\cos(\pi/N), \hspace{1cm} \text{for } N = 2, 3, \ldots,$$  \hspace{1cm} (14)

where the $a_n$ are ill defined, for some $n$. These are exactly, the ‘minimal’ models of this algebra, which are the only unitary models with $\alpha < 2$, and they correspond
to conformal field theories. To avoid this problem, we can take $\alpha$ to be close, but not identical, to any of the minimal points. However, we shall not consider here these points in detail, and in the sequel we shall assume that $\alpha$ is different from the values in eq. (14), and that the projectors $Z_n$ are, thus, well defined.

For the Fuss–Catalan algebra the expression for the projectors is even simpler,

$$Z_n = \prod_{i=1}^{n} \left( 1 - \frac{c_i^{(1)}}{\alpha} \right).$$

(15)

Here, it is entirely trivial that eq. (10) is obeyed, and that $Z_n$ is the unique projector for the Fuss–Catalan algebra. Here the projector is well defined for any $\alpha$, such that $\alpha \neq 0$, and we shall assume this to be the case.

Now, we wish to diagonalize the Hamiltonian of the system. I.e., to find solutions to the equation,

$$\mathcal{H}\psi_n = E_n\psi_n,$$

(16)

where the $E_n$ are the energy levels and $\psi_n$ are the wave functions (which we can take to be any of the operators in the algebra, acting on some state, which is not of significance). What we will describe below is a method of diagonalizing the Hamiltonian, by dividing the Hilbert space into $n + 1$ sectors which do not mix.

The discussion here is general for any algebra. So let us assume that we have a descending chain of algebras $\mathcal{C}_r$, such that each is a subalgebra of the other,

$$\mathcal{C}_1 \supset \mathcal{C}_2 \supset \ldots \supset \mathcal{C}_n.$$ 

(17)

Assume also that for each algebra, we have, a unique projector, $Z_r$, which is the projector of the $\mathcal{C}_r$ algebra.

Now, assume that we have the full algebra $\mathcal{C}_1$ for some number of points $n$. We can consider the chain of subalgebras $\mathcal{C}_r$ where $r = 1, 2, \ldots, n$. For concreteness
we take $C_r$ to be the algebra of the points $e_i^{(a)}$, where $i = r, r+1, \ldots, n$. Thus, we have also the chain of projectors, $Z_r$ which obey, as above,

$$e_i^{(a)} Z_r = Z_r e_i^{(a)} = 0,$$

for all $n \geq i \geq r$, and any $a$, and $Z_0 = 0$ by convention.

It follows, that we can define a left ideal, $I_r$, as the ideal of all the elements in $C_1$, which annihilate $Z_r$ from the left,

$$I_r = \{ \omega \mid \omega Z_r = 0 \}.$$

Evidently, $I_r$ is a left ideal. I.e., multiplying any element of $I_r$ from the left, by any element of the algebra, gives a result which is still in the ideal,

$$\omega \in I_r \text{ and } r \in C_1 \implies r \omega \in I_r.$$

Because the algebras $C_r$ are subalgebras of each other, it follows that the ideals also form a chain,

$$I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_{n+1} = (0),$$

where $I_0$ is the entire algebra, $I_0 = C_1$.

It follows that we can define sectors of the algebra by taking

$$U_r = I_r \setminus I_{r+1},$$

that is, the elements of the $r$'th sector ($r = 0, 1, \ldots, n$), are defined as all the elements obeying,

$$\omega Z_r = 0, \text{ and } \omega Z_{r+1} \neq 0.$$

In this way we divide the algebra $C_1$ into $n + 1$ distinct sectors, $U_r$, the union of which is the entire algebra, $C_1$. 

7
Now, our aim is to solve Schrödinger equation for some Hamiltonian, which we denoted by $\mathcal{H}$, e.g., eq. (2) or eq. (4). Alternatively, we may solve the equation in the algebra itself, without applying it to any state,

$$\mathcal{H}\psi = E\psi,$$  \hspace{1cm} \text{(24)}

where $\psi$ is any element of the algebra. We can assume now that $\psi$ is given by,

$$\psi = \sum C_s \beta_s,$$  \hspace{1cm} \text{(25)}

where $\beta_s$ is in the $s$th ideal, $\beta_s \in I_s$, and $C_s$ are some numerical coefficients to be determined. Since $I_s$ is a left ideal, acting by $\mathcal{H}$ from the left leaves us within the ideal. So, we can ignore the lower ideals, $I_m$ with $m < s$. Further, we can multiply from the right by the projector $Z_{r+1}$. It follows that the Schrödinger equation becomes,

$$\mathcal{H}\psi Z_{r+1} = E\psi Z_{r+1},$$  \hspace{1cm} \text{(26)}

and the only non-trivial words that appear are the ones such that $\psi Z_r = 0$ and $\psi Z_{r+1} \neq 0$. It follows that we can diagonalize independently only the words that are in the $r$th sector, $\psi_r \in U_r$, and in eq. (25) we can assume that $C_s = 0$ for $s \neq r$ where $r$ labels the sector.

The prescription to do this is thus very simple. We take only words from the $r$’th sector. Then we act with $\mathcal{H}$ and eliminate from the result any words which are in any higher sector, i.e., those in $I_{r+1}$.

This allows us to diagonalize the Hamiltonian sector by sector, without any mixing with other sectors. Note, that this method does not require integrability at all, and we can take any algebra and any Hamiltonian.

Consider then the first example of the Temperly–Lieb algebra. It is not hard
to see that a basis for the $r$th sector is given by,

$$C^{m_1,m_2,\ldots,m_r} = K_1^{m_1}K_2^{m_2}\ldots K_r^{m_r},$$

where the word $K_n^m$ is defined as,

$$K_n^m = e_ne_{n-1}\ldots e_m,$$

and $n \geq m$. We impose also

$$0 \leq m_1 < m_2 \ldots < m_r \leq n,$$

and $m_r \geq r$. It is immediate that $C^{m_1,m_2,\ldots,m_r}$ lies in the ideal $I_r$. It is less immediate, but correct, that it forms a basis for the $r$th sector, $U_r$, and that all the words of this type are, in fact, independent of each other.

For the Fuss–Catalan algebra the situation is more complicated, because some of the $e_i^{(a)}$ commute with $e_{i+1}^{(b)}$, for some $a$ and $b$, eqs. (3,5). Here we take for the basis elements, of the $r$th ‘prime sector’, $P_r$

$$C^{m_1,m_2,\ldots,m_r} = K_1^{m_1}K_2^{m_2}\ldots K_r^{m_r},$$

where again the $m_i$ obey eq. (29) and the $K_n^m$ are defined by

$$K_n^m = e_n^{(a_n^m)} e_{n-1}^{(a_{n-1}^m)} \ldots e_m^{(a_m^m)},$$

where the $a_n^m$'s can take any value of the color, $a_n^m = 1, 2, \ldots, f$. Here the sector $U_r$ is defined as the union of all the prime sectors, $P_1, \ldots, P_r$ which obey,

$$\omega Z_r = 0 \quad \text{and} \quad \omega Z_{r+1} \neq 0.$$ 

An additional complication is that now not all the words defined by eq. (30) are linearly independent, and some words are needed to be discarded.
In all the models, the \( k = 0 \) sector contains just one element, the unit element, 
\( U_0 = \{1\} \). Applying the Hamiltonian we see that,

\[
H \psi = 0, \tag{33}
\]

up to elements that are in higher sectors. We conclude that there is just one state 
in this sector, and its energy is zero, \( E = 0 \). This state can be the ground state, or 
not, depending on the algebra and the range of its parameters, as will be discussed 
below.

Consider now the \( k = 1 \) sector for the Temperley–Lieb model. We take the 
Hamiltonian as in eq. (2). A basis for this sector is given by the elements, eq. 
(27),

\[
A_r = e_r e_{r-1} \ldots e_1, \tag{34}
\]

where \( r = 1, 2, \ldots, n \). We take, then, for the wave function,

\[
\psi = \sum_{i=1}^{n} a_i A_i, \tag{35}
\]

where the \( a_i \) are some numerical coefficients to be determined. Applying \( H \) on \( \psi \) 
and discarding any elements in higher sectors, we arrive at the equation,

\[
a_{i+1} + \alpha a_i + a_{i-1} = E a_i, \tag{36}
\]

along with the boundary condition, \( a_0 = a_{n+1} = 0 \). It is very easy to solve this 
equation exactly, by taking the ansatz,

\[
a_r = a q_1^r + b q_2^r, \tag{37}
\]

where \( a \) and \( b \) are some coefficients determined by the boundary condition, and \( q_1 \)
and $q_2$ are the two solutions of the second order equation,

$$q^2 + (\alpha - E)q + 1 = 0. \quad (38)$$

This equation (first derived by Levy, ref. [3]) is in fact identical to the Bethe Ansatz solution of the Heisenberg spin chain, with one spin flipped, as derived by Alcaraz et al. (ref. [9]). Showing that in this case, our method reproduces the Bethe Ansatz.

Actually, it is quite easy, for the Temperley–Lieb case, to write the general solution for any sector $k$, using the basis, eq. (27). Again, it can be seen to reproduce the results of the Bethe Ansatz. Since the Bethe ansatz for this model is fully known, we will omit the details. Actually, unlike the Bethe ansatz, we do not require integrability and we can solve the model for any Hamiltonian we wish, by exactly the same method.

4. Numerical work.

Let us turn now to the Fuss–Catalan model. We take here two colors, $f = 2$, and the Hamiltonian as in eq. (4). Again, for the $k = 0$ sector, there is only the unit field and the energy of it is $E = 0$. For the $k = 1$ sector, and for five points, $n = 5$, we find 20 independent elements, which are all in the form eq. (30), and which obey, $\omega Z_1 = 0$ and $\omega Z_2 \neq 0$, where $\omega$ is the operator (as otherwise they would be in an higher sector, $k > 1$). We diagonalized the Hamiltonian for these 20 elements, and the results are listed in the table for $\alpha = 1, 1.5, 1.8, 2, 2.5, 3$.

The results are as follows. For $\alpha > \sqrt{2}$ the ground state energy is 0 and the next state is at energy $E = 1$. That is, in this regime, which is the principal regime, we find that there is a mass gap of $E = 1$ and the model is massive. Thus, we expect the theory to be at the scaling limit, that of a massive integrable model. It is possible to check also the $k = 2$ sector. The results persist also in this sector, and we find only states with the energy $E \geq 1$. Note that all the eigenvalues in
this regime are positive. We do not enclose the calculations for the sector \( k = 2 \) for the sake of brevity. We also run the program on other number of points, up to \( n = 11 \), and the results above, indeed, persist in these calculations.

For \( 0 < \alpha < \sqrt{2} \) regime the situation is different. Here, for the \( k = 1 \) sector, we always find complex eigenvalues, indicating that the model is not unitary. There is one important exception to this which is for \( \alpha = 1 \). For this point we get for \( k = 1 \) only non-negative integers (see the table). However, for \( k = 2 \) we find some (non-integral) negative eigenvalues. For example, for \( n = 4 \) points, in the \( k = 2 \) sector, the lowest energy state, i.e., the ground state of this sector, we find,

\[
E = \frac{1 - \sqrt{5}}{2} = -0.618034..., \tag{39}
\]

and it is the only negative energy state in this sector. We conclude that for \( \alpha = 1 \) the ground state energy is negative, and that there is probably no mass gap. This theory is analogous to the \( \alpha < 2 \) unitary minimal model for the TL algebra (where \( \alpha \) is given by eq. (14)). We conjecture then that the \( \alpha = 1 \) model is some conformal field theory in the scaling limit. However, more work is needed to determine precisely the RCFT.

We expect these results to persist for more than two colors. Again in the principal regime \( \alpha > \sqrt{2} \) we expect to have a zero energy ground state and a mass gap of 1. Again, \( \alpha = 1 \) is the only unitary point for \( \alpha < \sqrt{2} \), and it is conjectured to be some conformal field theory in the scaling limit. We thus have a series of RCFT’s or \( \alpha = 1 \) and some number of colors, \( f \), where \( f = 2, 3, 4, \ldots \). It is left to future work to determine exactly the RCFT’s.
5. Conclusions.

We described here an algebraic method for diagonalizing the Hilbert space of solvable lattice models. We exemplified the method on the Temperley–Lieb and Fuss–Catalan models. We expect this method to be of interest for other models obeying such a chain of algebras. We hope that this will be beneficial to the study of lattice models in two dimensions along with their interplay with quantum field theory.

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Table.
The spectrum for $n = 5$ and $k = 1$.

| $\alpha = 1$ | $\alpha = 1.5$ | $\alpha = 1.8$ | $\alpha = 2$ | $\alpha = 2.5$ | $\alpha = 3$ |
|--------------|----------------|----------------|--------------|---------------|-------------|
| 2            | 17.8301        | 6.4410         | 5.3364       | 4.4705        | 4.2857      |
| 2            | 17.4643        | 5.9635         | 5.            | 4.3268        | 3.9383      |
| 2            | 15.6332        | 5.7745         | 5.            | 4.1244        | 3.7901      |
| 2            | 14.7850        | 5.6129         | 4.5647        | 3.7744        | 3.5469      |
| 2            | 12.            | 5.0815         | 4.3660        | 3.4705        | 3.2857      |
| 2            | 11.7821        | 4.0134         | 3.1037        | 3.4577        | 3.1428      |
| 2            | 11.            | 4.6129         | 4.            | 3.2352        | 3.0943      |
| 1            | 10.6190        | 4.1216         | 3.5000        | 3.2049        | 3.0312      |
| 1            | 7.0000         | 3.8064         | 3.4595        | 3.1544        | 2.8565      |
| 1            | 6.4306         | 3.5165         | 3.3240        | 2.8456        | 2.7387      |
| 1            | 5.4417         | 3.1236         | 2.7772        | 2.6961        | 2.6384      |
| 1            | 3.2737         | 2.5994         | 2.6339        | 2.5814        | 2.6050      |
| 1            | 2.5278         | 2.4557         | 2.5000        | 2.4598        | 2.5387      |
| 1            | 2.3667         | 2.2941         | 2.3395        | 2.4520        | 2.3233      |
| 1            | 2.0939         | 2.2682         | 2.1876        | 2.0910        | 2.0559      |
| 1            | 1.6871         | 1.7383         | 1.7623        | 1.8047        | 1.8332      |
| 1            | 1.2134         | 1.4311         | 1.5000        | 1.60438       | 1.6678      |
| 1            | 1.0972         | 1.2822         | 1.3567        | 1.4794        | 1.5582      |
| 1            | 1.0636         | 1.2171         | 1.2881        | 1.4129        | 1.4971      |
| 0            | 1.             | 1.             | 1.            | 1.            | 1.          |