On the Black Hole Acceleration in the C-metric Space-time

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Abstract

We consider the C-metric as a gravitational field configuration that describes an accelerating black hole in the presence of a semi-infinite cosmic string, along the accelerating direction. We adopt the expression for the gravitational energy-momentum developed in the teleparallel equivalent of general relativity (TEGR) and obtain an explanation for the acceleration of the black hole. The gravitational energy enclosed by surfaces of constant radius around the black hole is evaluated, and in particular the energy contained within the gravitational horizon is obtained. This energy turns out to be proportional to the square root of the area of the horizon. We find that the gravitational energy of the semi-infinite cosmic string is negative and dominant for large values of the radius of integration. This negative energy explains the acceleration of the black hole, that moves towards regions of lower gravitational energy along the string.

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1 Introduction

The C-metric is a very curious gravitational field configuration. It was first understood as a solution of Einstein’s equations that describes an accelerating black hole. Nowadays, it is clear that the line element describes not only a pair of black holes accelerated in opposite directions, but a sequence of pairs of black holes. A semi-infinite cosmic string is assumed to be attached to each one of the black holes, along which the black holes are accelerated. The acceleration of the black holes is generally supposed to be due to these semi-infinite cosmic strings, but how exactly these strings act on the black hole is not clear.

In this article we present an explanation for this acceleration. The explanation is based on energy considerations. We consider the expression for the gravitational energy-momentum established in the teleparallel equivalent of general relativity (TEGR), and evaluate the gravitational energy enclosed by surfaces of constant radius $R$, such that $R$ lies between the gravitational and acceleration horizons. This energy expresses both the energy of the black hole and of the cosmic string. The gravitational energy density due to the semi-infinite cosmic string only is negative (the semi-infinite cosmic string is characterised by both the mass parameter and the acceleration parameter), and is dominant for large values of the radius of integration $R$. The total gravitational energy (black hole plus the cosmic string) is negative and de-
creases with increasing values of the radius $R$, as we will show. Assuming
that the physical systems in nature move towards states of lower energy, the
results obtained here suggest that the black hole is dragged to regions of
lower energy state along the semi-infinite cosmic string. The absolute value
of the (negative) gravitational energy density of the cosmic string increases
along the negative $z$ axis. We also obtain the energy contained within the
gravitational horizon. This is the energy that cannot escape from the hori-
zon, and is related to the irreducible mass of the black hole. It turns out
that this energy is proportional to the square root of the area of the black
hole horizon.

We will establish a set of tetrad fields adapted to observers that accelerate
together with the black hole. The dynamical features of this frame are investi-
gated by means of the acceleration tensor. This tensor will be reviewed in
Section 3, and applied to the C-metric black hole in Section 4. The outcome
of the analysis of the acceleration tensor will help to understand that the
semi-infinite cosmic string is indeed the configuration that accelerates the
C-metric black hole, as demonstrated in section 6.

2 Description of the C-metric

The C-metric is an exact vacuum solution of Einstein’s equations that de-
pends on three parameters: $m$, $\alpha$ and $C$. When $\alpha = 0 = C$, the metric
describes the Schwarzschild solution. The parameter $\alpha$ is related to accelera-
tion, and $C$ to a deficit angle, or conical singularity. The exposition below is
based on the presentations of Refs. [1, 2]. In the latter references, one finds
the history of this solution, starting with Levi-Civita in 1918 [3], and ending
up with the work of Ehlers and Kundt [4] (see also Ref. [5]). The metric is
interpreted as describing an infinite sequence of alternating black holes and
asymptotically flat regions [1], and each asymptotically flat region is related
to a pair of causally disconnected black holes. The black holes in each pair
are supposed to accelerate away from each other along an axis of symmetry
of the space-time. This axis of symmetry contains a conical singularity that
may be physically interpreted as a cosmic string, that is related to the accel-
eration of the black hole. One does not expect this multitude of pairs of black
holes to be realized in nature. The very concept of acceleration of a black
hole is not of straightforward comprehension. Nevertheless, this metric will
be used here to model the acceleration of a single astrophysical black hole.
within a physical region that can be identified with the surroundings of an ideal observer. Such model could eventually describe features of the outcome of the merger of two black holes that are presently considered in the generation of the recently observed gravitational waves. However, here we will be mostly interested in the conceptual issues related to the characteristics of an accelerated black hole.

In spherical type coordinates $(t, r, \theta, \Phi)$, the line element of the C-metric space-time reads

$$
\frac{1}{(1 + \alpha r \cos \theta)^2} \left( -f dt^2 + \frac{1}{f} dr^2 + \frac{1}{g} r^2 d\theta^2 + g r^2 \sin^2 \theta d\Phi^2 \right),
$$

where $f = f(r) = (1 - \alpha^2 r^2)(1 - 2m/r)$, $g = g(\theta) = 1 + 2\alpha m \cos \theta$, and $(m, \alpha) > 0$. The physical solutions that we will consider satisfy $0 < 2\alpha m < 1$. It is easy to see that when $\alpha = 0$, we arrive at the line element of the Schwarzschild space-time. The angular coordinate $\Phi$ varies in the interval $-C\pi < \Phi < C\pi$.

By drawing a small circle around the half-axis $\theta = \pi$, with $(t, r)$ constant, we obtain

$$
\frac{\text{circumference}}{\text{radius}} = 2\pi C(1 - 2\alpha m),
$$

which implies the existence of a conical singularity, and doing the same around the half-axis $\theta = 0$, we find

$$
\frac{\text{circumference}}{\text{radius}} = 2\pi C(1 + 2\alpha m),
$$

which also implies the existence of a conical singularity, but with a different conicity. We choose to eliminate the excess of angular variation around the upper half-axis $\theta = 0$ by fixing the constant $C$ to satisfy $C = (1 + 2\alpha m)^{-1}$. In this way, the deficit angles at the half-axis $\theta = 0$ and $\theta = \pi$ are

$$
\delta_{\theta=0} = 0, \quad \delta_{\theta=\pi} = \frac{8\pi \alpha m}{1 + 2\alpha m},
$$

respectively. The negative $z$ axis is then identified with the semi-infinite cosmic string. This semi-infinite cosmic string makes sense only if $m \neq 0$ and $\alpha \neq 0$. Finally, we define the coordinate $\phi$ such that $\Phi = C\phi$, where $-\pi < \phi < \pi$, and arrive at the final form of the line element,
\[ ds^2 = \frac{1}{(1 + \alpha r \cos \theta)^2} \left( -f dt^2 + \frac{1}{f} dr^2 + \frac{1}{g} r^2 d\theta^2 + \frac{gr^2 \sin^2 \theta}{(1 + 2\alpha m)^2} d\phi^2 \right). \] (5)

The functions \( f \) and \( g \) are the same as in Eq. (1).

The C-metric space-time has a curvature singularity at \( r = 0 \), and two coordinate singularities: at \( r = 2m \), that yields the event horizon \( H_e \), and at \( r = 1/\alpha \), that yields the acceleration horizon \( H_a \). Thus, ignoring analytic extensions, the space-time may be divided in three regions \([1]\): I) \( 0 < r < 2m \), which is the interior of the black hole (non-static region); II) \( 2m < r < 1/\alpha \) (static region); III) \( 1/\alpha < r < \infty \) (non-static region). The coordinates in Eqs. (1) and (5) are suitable to Region II, which is the region of interest to the present analysis. The maximal analytic extension of these coordinates yields the description of a pair of accelerating black holes in opposite directions, each of them being in space-time regions that are causally disconnected.

The limit \( m \to 0 \) of the C-metric is taken by first considering Eqs. (2) and (3), and by noting that the coordinates \( \Phi \) in Eq. (1) and \( \phi \) in Eq. (5) are related by \( \Phi = C\phi \). The positive and negative half \( z \) axes have now the same angular deficit, and the line element reduces to

\[ ds^2 = \frac{1}{(1 + \alpha r \cos \theta)^2} \left[ -(1 - \alpha^2 r^2)dt^2 + \frac{1}{1 - \alpha^2 r^2} dr^2 + r^2 d\theta^2 + C^2 r^2 \sin^2 \theta d\phi^2 \right]. \] (6)

If the acceleration parameter \( \alpha \) vanishes, the line element above can be transformed into the standard form \( ds^2 = -dt^2 + d\rho^2 + \beta^2 \rho^2 d\phi^2 + dz^2 \) of a conical defect in cylindrical coordinates, provided we identify \( \beta = C \).

By means of a suitable coordinate transformation, the space-time described by the line element (6) can be transformed into the uniformly accelerated Rindler space-time in cylindrical coordinates. However, this coordinate transformation cannot be carried out globally, because the topological defect on the \( z \) axis is eliminated by such a global transformation.

In summary, we see that the C-metric space-time described by Eq. (1) is a non-linear superposition of a static black hole space-time and of a semi-infinite cosmic string along the negative \( z \)-axis.
3 The acceleration tensor

In this section we will make a brief presentation of the acceleration tensor, in order to characterise the acceleration of frames adapted to observers in the C-metric space-time. The tetrad field and the inverse frame field are denoted by \(e^a_{\mu}\) and \(e_a^{\mu}\), respectively. [Notation: \(a\) and \(\mu\) are SO(3,1) and space-time indices, respectively. The time and space components are denoted as \(a = ((0),(i))\) and \(\mu = (0,i)\). The metric tensor \(g_{\mu\nu}\) and the flat, tangent space metric tensor \(\eta_{ab} = (-1,+1,+1,+1)\) are related by \(e^a_{\mu} e^b_{\nu} \eta_{ab} = g_{\mu\nu}\).]

Along an arbitrary timelike worldline \(C\), the velocity of an observer is denoted by \(U^\mu\). We identify this velocity with the timelike component of the frame field, \(U^\mu = e^{(0)}_{\mu}\). The acceleration of the observer along this worldline is defined by the covariant derivative of \(U^\mu\) along \(C\),

\[
a^\mu = \frac{DU^\mu}{d\tau} = \frac{De^{(0)}_{\mu}}{d\tau} = U^\alpha \nabla_\alpha e^{(0)}_{\mu},
\]

where \(\tau\) is the proper time of the observer along \(C\), and the covariant derivative is constructed out of the Christoffel symbols. We have considered \(U^\alpha = dx^\alpha/d\tau\) along \(C\). Thus, \(e^a_{\mu}\) yields the velocity and acceleration of an observer along the worldline. Therefore, a given set of tetrad fields, for which \(e^{(0)}_{\mu}\) describes a congruence of timelike curves, is adapted to a particular class of observers, namely, to observers characterized by the velocity field \(U^\mu = e^{(0)}_{\mu}\), endowed with acceleration \(a^\mu\). If \(e^a_{\mu} \rightarrow \delta^a_{\mu}\) in the limit \(r \rightarrow \infty\), in an asymptotically flat space-time, then \(e^a_{\mu}\) is adapted to static observers at spacelike infinity.

An alternative characterization of tetrad fields as an observer’s frame may be given by considering the acceleration of the whole frame along an arbitrary path \(x^\mu(\tau)\) of the observer. The acceleration of the whole frame is determined by the absolute derivative (constructed out of the Levi-Civita connection) of \(e^a_{\mu}\) along \(x^\mu(\tau)\). Thus, assuming that the observer carries an orthonormal tetrad frame \(e^a_{\mu}\), the acceleration of the frame along the path is given by [6, 7, 8, 9]

\[
\frac{De^a_{\mu}}{d\tau} = \phi^{a b}_{\mu} e^b_{\mu},
\]

where \(\phi_{ab}\) is the antisymmetric acceleration tensor. As discussed in Refs. [6, 7], in analogy with the Faraday tensor we may identify \(\phi_{ab} \leftrightarrow (\mathbf{a}, \Omega)\), where \(\mathbf{a}\) is the translational acceleration \((\phi^{(0)}_{(i)} = a_{(i)})\) and \(\Omega\) is the frequency
of rotation of the local spatial frame with respect to a non-rotating, Fermi-Walker transported frame. It follows from Eq. (8) that

$$\phi_a^b = e^b_\mu \frac{De_a^\mu}{d\tau} = e^b_\mu U^\lambda \nabla_\lambda e_a^\mu.$$  

(9)

The acceleration vector $a^\mu$ may be projected on a frame in order to yield

$$a^b = e^b_\mu a^\mu = e^b_\mu U^\alpha \nabla_\alpha e^(0)(^\mu) = \phi(0)^b.$$  

(10)

Thus, $a^\mu$ and $\phi(0)(i)$ are not different translational accelerations of the frame. The expression of $a^\mu$ given by Eq. (7) may be rewritten as

$$a^\mu = U^\alpha \nabla_\alpha e(0)^\mu = \frac{dx^\alpha}{d\tau} \left( \frac{\partial U^\mu}{\partial x^\alpha} + ^0\Gamma^\mu_{\alpha\beta} U^\beta \right)$$

$$= \frac{d^2 x^\mu}{d\tau^2} + ^0\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau},$$  

(11)

where $^0\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. We see that if $U^\mu = e(0)^\mu$ represents a geodesic trajectory, then the frame is in free fall and $a^\mu = \phi(0)(i) = 0$. Therefore we conclude that non-vanishing values of the latter quantities represent inertial accelerations of the frame.

An alternative expression of the acceleration tensor is given by [8, 9]

$$\phi_{ab} = \frac{1}{2} \left[ T_{(0)ab} + T_{a(0)b} - T_{b(0)a} \right].$$  

(12)

where

$$T_{abc} = e^\mu_ke^\nu_lT_{a\mu\nu} = e^\mu_ke^\nu_l(\partial_\mu e_{\alpha\nu} - \partial_\nu e_{\alpha\mu}).$$  

(13)

The tensor $\phi_{ab}$ is invariant under coordinate transformations and covariant under global SO(3,1) transformations, but not under local SO(3,1) transformations. Because of this property, $\phi_{ab}$ may be used to characterise the inertial state of the frame. If the frame is maintained static in space-time, then the six components of the tensor $\phi_{ab}$ must cancel the six components of the gravitational acceleration on the frame.
4 Inertial accelerations in the C-metric spacetime

We will establish a set of tetrad fields constructed in terms of the coordinates $(t, r, \theta, \phi)$, whose origin coincide with the centre of the accelerating black hole. Therefore, the tetrad field below is adapted to observers that see the accelerating black hole at rest, and yields the line element (5). It reads

\[
e_{a\mu} = \Delta \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & \frac{\cos \phi \sin \theta}{A} & \frac{r \cos \theta \cos \phi}{B} & -\frac{r B \sin \theta \sin \phi}{1+2\alpha m} \\ 0 & \frac{\sin \theta \sin \phi}{A} & \frac{r \cos \theta \sin \phi}{B} & \frac{r B \cos \phi \sin \theta}{1+2\alpha m} \\ 0 & \frac{\cos \theta}{A} & -\frac{r \sin \theta}{B} & 0 \end{pmatrix}, \tag{14}
\]

where

\[
A = \sqrt{(1-2m/r)(1-\alpha^2 r^2)}, \tag{15}
\]

\[
B = \sqrt{1+2\alpha m \cos \theta}, \tag{16}
\]

\[
\Delta = \frac{1}{1+\alpha r \cos \theta}. \tag{17}
\]

Recall that we are assuming $0 < 2\alpha m < 1$. Since the black hole is accelerated in the negative $z$ direction, the observer is likewise accelerated together with the black hole. From the point of view of the frame established by Eq. (14), the observer verifies that the black hole is at rest, i.e., the 4-velocity of the observer is of the type $U^\mu = e_{(0)}^\mu = (U^0, 0, 0, 0)$.

In order to calculate the acceleration tensor out of the frame given by Eq. (14), we need the expressions of the torsion tensor $T_{\lambda\mu\nu} = e^a_\lambda T_{a\mu\nu}$. They are given by

\[
T_{001} = A \Delta \partial_r (\Delta A),
\]

\[
T_{002} = A^2 \Delta \partial_\theta \Delta,
\]

\[
T_{112} = -\frac{1}{A^2} \Delta \partial_\theta \Delta,
\]

\[
T_{212} = r \frac{\Delta}{AB^2} [A \partial_r (r \Delta) - B \Delta],
\]

\[
T_{313} = r \sin^2 \theta \frac{B \Delta}{A(1+2m\alpha)^2} [r A B \partial_r \Delta + (AB - 1 - 2m\alpha) \Delta],
\]

7
\[ T_{323} = r \sin^2 \theta \left( \frac{\Delta}{(1 + 2m\alpha)} \right)^2 \left( [B^2 \cos \theta + B \sin \theta \partial_\theta B \right. \]
\[ \left. - (1 + 2m\alpha) \cos \theta ] \Delta + B^2 \sin \theta \partial_\theta \Delta \right) . \]

The non-vanishing components of the acceleration tensor are then easily calculated, and read

\[ \phi_{0(1)} = F(r; \alpha) \cos \phi \sin \theta , \]
\[ \phi_{0(2)} = F(r; \alpha) \sin \phi \sin \theta , \]
\[ \phi_{0(3)} = F(r; \alpha) \cos \theta - \alpha B , \]

where

\[ F(r; \alpha) = \alpha \cos \theta B \]
\[ + \frac{m(1 + r^2 \alpha^2) - r^3 \alpha^2 - r \alpha [r + m (r^2 \alpha^2 - 3)] \cos \theta}{r^2 A} . \]

By combining these quantities, we have

\[ \vec{a} = \phi_{0(1)} \hat{x} + \phi_{0(2)} \hat{y} + \phi_{0(3)} \hat{z} = F(r; \alpha) \hat{r} - \alpha B \hat{z} . \]

The expression above represents the inertial accelerations that are necessary to impart to the frame (14) in order to satisfy the properties that Eq. (14) must satisfy. For instance: (i) by making \( \alpha = 0 \), we obtain

\[ \vec{a} = \frac{m}{r^2 A} \hat{r} , \]

which is the outward radial acceleration necessary to compensate the attractive radial acceleration due to the black hole (the function \( F(r; \alpha) \) generalises the expression above, for non-vanishing values of \( \alpha \)); (ii) the term \(-\alpha B \hat{z}\) represents the component of the acceleration on the frame along the negative direction of the \( z \) axis, since the frame is accelerated together with the black hole.

In the absence of the black hole, i.e., in the case \( m = 0 \), the set of tetrad fields obtained from Eq. (6), that represents a frame adapted to observers
accelerated along the negative $z$ direction, reads
\[
e_{a\mu} = \Delta \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & \frac{\cos \phi \sin \theta}{A} & r \cos \theta \cos \phi & -r \cos \theta \sin \phi \\ 0 & \frac{\sin \theta \sin \phi}{A} & r \cos \theta \sin \phi & r C \sin \theta \sin \phi \\ \cos \theta & \frac{\sin \theta \sin \phi}{A} & -r \sin \theta & 0 \end{pmatrix}.
\] (25)

It follows from the expression above that
\[
F(0; \alpha) = \alpha \left[ \cos \theta - \frac{\alpha r + \cos \theta}{\sqrt{1 - \alpha^2 r^2}} \right].
\] (26)
Therefore,
\[
\vec{a} = \alpha \left[ \cos \theta - \frac{\alpha r + \cos \theta}{\sqrt{1 - \alpha^2 r^2}} \right] \hat{r} - \alpha \hat{z}.
\] (27)

At the centre of the coordinate system, $r = 0$, we have $\vec{a}_i = -\alpha \hat{z}$. The constant $C$, that appears in Eq. (25) and that characterises the cosmic string, does not affect expressions (26) and (27) above. In fact, these expressions can be obtained directly from (22) and (23) by making $m = 0$ in the latter equations.

Finally, we mention that for both sets of tetrad fields, Eqs. (14) and (25), the frequency of rotation, given by the $\phi_{(i)(j)}$ components of the acceleration tensor, vanish. Both frames are Fermi-Walker transported.

5 A brief review of the TEGR

The gravitational energy of the C-metric space-time will be investigated in the context of the TEGR. This issue is somehow intricate, because we have, in fact, an accelerated black hole in the presence of a semi-infinite cosmic string. Up to a certain extent, we will manage to disentangle these two gravitational field configurations.

As in previous presentations, we assume that the space-time geometry is established by the tetrad fields $e^a_\mu$ only. Thus, the only possible non-trivial definition for the torsion tensor is given by $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$, as in Eq. (13). This quantity is trivially related to the torsion of the Weitzenböck connection $\Gamma_{\mu\nu}^{\lambda} = e^a_\lambda \partial_\mu e_{a\nu}$. A geometry defined solely by the tetrad field is more general than the pure Riemannian geometry, since one can make use
of both the curvature and torsion tensors, and of the Weitzenböck and Levi-Civita connections. Of course, the Riemann-Christoffel and Ricci tensors must exist in order to establish the equivalence between the TEGR and the ordinary metric formulation of general relativity.

In the TEGR, it is possible to rewrite Einstein’s equations in terms of $e^a_\mu$ and $T_{a\mu\nu}$. The Lagrangian density of the theory is defined by $\[10, 11\]

$$L = -k e^a_\mu T_{a\mu\nu} - \frac{1}{c} L_M$$

where $k = \frac{c^3}{16\pi G}$, $T_a = T^b_\cdot ba$, $T_{abc} = e^b_\mu e^c_\nu T_{a\mu\nu}$ and

$$\Sigma_{abc} = \frac{1}{4}(T_{abc} + T_{bac} - T_{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c) .$$

$L_M$ stands for the Lagrangian density for the matter fields. The Lagrangian density $L$ is invariant under the global SO(3,1) group. Invariance under the local SO(3,1) group is verified as long as we take into account the total divergence that arises in the identity

$$e R(e) \equiv - e \left( \frac{1}{4} T_{abc} T_{abc} + \frac{1}{2} T_{bac} T_{bac} - T^a T_a \right) + 2 \partial_\mu(eT^\mu),$$

where $R(e)$ is the scalar Riemannian curvature. However, the field equations derived from Eq. $\[28\]$ are covariant under local SO(3,1) transformations, and are equivalent to Einstein’s equations. They read

$$e_{a\lambda} e_{b\mu} \partial_\nu(e^{b\lambda\nu}) - e^{b\lambda\nu} a T_{b\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd} = \frac{1}{4kc} e T_{a\mu} ,$$

where $\delta L_M/\delta e^{a\mu} = e T_{a\mu}$.

Although the definition of the gravitational energy-momentum is established in the Hamiltonian framework, it may also be obtained in the framework of the Lagrangian formulation defined by $\[28\]$, according to the procedure of Ref. $\[11\]$ (we are now assuming $c = 1 = G$). Equation $\[31\]$ may be rewritten as

$$\partial_\nu(e^{a\lambda\nu}) = \frac{1}{4k} e e^a_\mu (T^{\lambda\mu} + t^{\lambda\mu}) ,$$

where $T^{\lambda\mu} = e_a^\lambda T_{a\mu}$ and $t^{\lambda\mu}$ is defined by
\[ t^{\lambda \mu} = k (4 \Sigma^{bc \lambda} T_{bc}^{\mu} - g^{\lambda \mu} \Sigma^{bcd} T_{bcd} ). \] (33)

In view of the antisymmetry property \( \Sigma^{a \mu \nu} = - \Sigma^{a \nu \mu} \), it follows that

\[ \partial_\lambda \left[ e e^a_\mu (t^{\lambda \mu} + T^{\lambda \mu}) \right] = 0. \] (34)

The equation above yields the continuity (or balance) equation,

\[ \frac{d}{dt} \int_V d^3 x e e^a_\mu (t^0_\mu + T^0_\mu) = - \oint_S dS_j \left[ e e^a_\mu (t^j_\mu + T^j_\mu) \right], \] (35)

where \( S \) is the boundary of an arbitrary 3-dimensional volume \( V \). Therefore we identify \( t^{\lambda \mu} \) as the gravitational energy-momentum tensor [11],

\[ P^a = \int_V d^3 x e e^a_\mu (t^0_\mu + T^0_\mu), \] (36)

as the total energy-momentum contained within the volume \( V \),

\[ \Phi^a_g = \oint_S dS_j \left( e e^a_\mu t^j_\mu \right), \] (37)

as the gravitational energy-momentum flux [11, 12], and

\[ \Phi^a_m = \oint_S dS_j \left( e e^a_\mu T^j_\mu \right), \] (38)

as the energy-momentum flux of matter [12, 13]. In view of (32), Eq. (36) may be written as \( P^a = - \int_V d^3 x \partial_j \Pi^a_j \), from what follows

\[ P^a = - \oint_S dS_j \Pi^a_j, \] (39)

where \( \Pi^a_j = - 4ke \Sigma^{a0j} \). A summary of all issues discussed above may be found in Ref. [14].

The passage from a volume integral to a surface integral such as Eq. (39) cannot be carried out in the presence of singularities (admitting that the space-time has singularities), and for this reason we consider Eq. (39) as the definition of the gravitational energy-momentum. It must be noted, however, that the same feature takes place in the definition of the ADM gravitational energy-momentum, where the integrals of total divergences are transformed into surface integrals. The surface integral is superior with respect to the volume integral, because the gravitational field on the surface of integration \( S \) carries information about the interior region, and the integral can be carried
out more easily. In addition, definition (39) represents the total energy of the space-time within the surface $S$.

Equation (39) is the definition for the gravitational energy-momentum presented in Ref. [15], obtained in the framework of the vacuum field equations in Hamiltonian form. It is invariant under coordinate transformations of the three-dimensional space and under time reparametrizations. Note that (34) is a true energy-momentum conservation equation. In the ordinary formulation of arbitrary field theories, energy, momentum, angular momentum and the centre of mass moment are frame dependent field quantities, that transform under the global SO(3,1) group. In particular, the energy transforms as the zero component of the energy-momentum four-vector. These features of special relativity must also hold in general relativity, since the latter yields the former in the limit of weak (or vanishing) gravitational fields.

The problem of defining the gravitational energy-momentum has a long history, and is probably as old as general relativity itself. It is well known the existence of several expressions of pseudo-tensors, including one by proposed by Einstein, and all these expressions have an obvious limitation since they are not tensors. Nowadays, the majority of the objections (if not all) against the existence of a localized expression for the gravitational energy-momentum is justified by invoking the principle of equivalence. The idea is that the affine connection in general relativity can be made to vanish at a point in space-time, or even along an arbitrary worldline (timelike or spacelike). However, as argued before [16], the vanishing of the affine connection is a feature of differential geometry, and not a principle of nature. The problem regarding the definition of the gravitational energy-momentum has to do with transformation of frames, not transformation of coordinates.

This whole issue has been thoroughly discussed in section 5 of Ref. [14]. In subsection 5.3 of the latter reference, we have shown that the definitions of energy-momentum and 4-angular momentum that arise in the TEGR satisfy the Poincaré algebra in the phase space of the theory. This result, together with the calculation of the gravitational energy contained within the external event horizon of a Kerr black hole [15], distinguishes our definition from all other existing definitions. However, the gravitational energy-momentum and 4-angular momentum must be frame dependent, as we argued above. The tetrad frame may be freely chosen, since every observer in space-time, along arbitrary timelike worldlines, carries his/her own tetrad frame.

The local SO(3,1) symmetry is not present in expressions (36) and (39) for the gravitational energy-momentum but, in practice, the latter can be
evaluated in any frame in space-time, static (with respect to the spacelike infinity), stationary, free-fall, etc.

6 Gravitational energy in the C-metric space-time

Expression (39) for the total gravitational energy takes into account altogether the contributions of the black hole and of the infinite cosmic string. Both gravitational field configurations are formally given by the integrand in Eq. (36). As we already mentioned, the analytical expression of the semi-infinite cosmic string is not yet known. Therefore, expression (39) is better suited to the analysis of the total gravitational energy of the C-metric space-time, because it incorporates the features of the non-linear superposition of the two geometrical field configurations.

We will evaluate the gravitational energy of the C-metric space-time in a region not close to the acceleration horizon $H_a$ determined by $r = 1/\alpha$. We are interested in situations of present astrophysical interest, and thus we will ignore the acceleration horizon and possible maximal extensions of the C-metric space-time. In order to evaluate the surface integral in Eq. (39), we need the quantities $\Pi^{o_j} = -4k \Sigma^{a0j}$. We find

$$\Sigma^{(0)01} = \frac{(1 + 2m\alpha - 2AB + B^2)\Delta - 2rAB\partial_r\Delta}{2rB\Delta^4}, \quad (40)$$

$$\Sigma^{(0)02} = \frac{\Delta[(1 + 2m\alpha - B^2)\cot \theta - B\partial_\theta B] - 2B^2\partial_\theta \Delta}{2r^2A\Delta^4}, \quad (41)$$

The gravitational energy $P^{(0)}$ contained within a surface of constant radius $r$ is determined by

$$P^{(0)} = 4k \oint_S dS_1 e\Sigma^{(0)01}, \quad (42)$$

where $dS_1 = d\theta d\phi$. The surface of constant radius $R$ is depicted in Figure 3 of Ref. [1], but as noted in this reference, there is no sharp vertex in the negative $z$ axis, at $\theta = \pi$, in spite of the presence of the semi-infinite cosmic string. The surface is regular at this point, so that Eq. (48) below (where $r = 2m$) may be easily obtained. By carrying out the integral above on the surface of constant radius $r = R$, we obtain
In view of the relation \( \tan^{-1} z = -\frac{i}{2} \ln \left( \frac{i - z}{i + z} \right) \), we have

\[
\left( \frac{1}{\sqrt{2m - R}} \right) \left( \tan^{-1} \sqrt{\frac{R + 2m\alpha R}{2m - R}} - \tan^{-1} \sqrt{\frac{R - 2m\alpha R}{2m - R}} \right)
\]

\[
= -\frac{1}{2} \sqrt{2m - R} \ln \left[ \left( \sqrt{R - 2m} + \sqrt{R - 2m\alpha R} \right) \left( \sqrt{R - 2m} - \sqrt{R - 2m\alpha R} \right) \right].
\] (44)

After a number of simplifications, we finally arrive at

\[
P^{(0)} = \frac{1}{2\alpha R^{1/2}(1 + 2m\alpha)} \left\{ -\frac{1}{1 - \alpha^2 R^2} \left[ (\sqrt{R - 2m\alpha R} - \sqrt{R + 2m\alpha R}) \right. \right.
\]

\[
+ 2\alpha R^{3/2} A - \alpha^2 R^2 \left( \sqrt{R - 2m\alpha R} - \sqrt{R + 2m\alpha R} \right) \right. \right.
\]

\[
\left. \left. + \frac{[R - m(1 - R\alpha)]}{\sqrt{R - 2m}} \times \ln \left[ \left( \sqrt{R - 2m} + \sqrt{R - 2m\alpha R} \right) \left( \sqrt{R - 2m} - \sqrt{R - 2m\alpha R} \right) \right] \right) \right\} .
\] (45)

The gravitational energy \( P_h^{(0)} \) contained within the event horizon \( H_g \) can be evaluated by taking the limit \( r \to 2m \) in the expression above. We find

\[
P_h^{(0)} = 4k \int dS_1 \lim_{r \to 2m} (c\Sigma^{(0)1}) = \frac{2m}{(1 + 2m\alpha)\sqrt{1 - 2m\alpha}}.
\] (46)
When $\alpha = 0$, Eq. (45) simplifies to

$$P_{\text{Schw}}^{(0)} = R \left( 1 - \sqrt{1 - \frac{2m}{R}} \right),$$

(47)

which is a well known result (obtained previously in the TEGR and by means of quasilocal expressions for the gravitational energy) that yields $P^{(0)} = m$ in the limit $R \to \infty$.

It is very interesting to note that the energy contained within the event horizon $H_g$ given by Eq. (46) is related to the area $A_h$ of the event horizon calculated in Ref. [1]. In the latter reference, the area $A_h$ is shown to be

$$A_h = \frac{16\pi C m^2}{1 - 4\alpha^2 m^2}.$$  (48)

Considering the value of $C$ adopted in Subsection 3.1 (as well as in Ref. [1]), $C = (1 + 2\alpha m)^{-1}$, it is straightforward to obtain the relation

$$P_h^{(0)} = \frac{\sqrt{A_h}}{2\pi^{1/2}}.$$  (49)

This relation may be useful in the study of the thermodynamics of the C-metric black hole.

In the limit $m \to 0$, $C$ is no longer given by $C = (1 + 2\alpha m)^{-1}$, but it can acquire arbitrary values (see Eq. (6)) [1]. The expression of $P^{(0)}$ in this limit is obtained directly from the tetrad fields (25), and represents the energy of an infinite cosmic string only, evaluated in an accelerated frame along the negative z axis. It is given by

$$P_{\text{cs}}^{(0)} = -\frac{RC}{\sqrt{1 - \alpha^2 R^2}} + \frac{1 + C}{4\alpha} \ln \left( \frac{1 + \alpha R}{1 - \alpha R} \right).$$

(50)

In the Figures below, we consider expression (45) for $P^{(0)}$ and display the total gravitational energy enclosed by a surface of constant radius $R$, considering $m = 1$ in natural units. In Figure [1] we display altogether: (i) $m = 1$, $\alpha = 0.01$; (ii) $m = 1$, $\alpha = 0$ (Schwarzschild); (iii) $m = 0$, $\alpha = 0.01$.

\footnote{We remark that when $C = 1$, the line element (6), and consequently the set of tetrad fields (25), does not represent the ordinary Minkowski space-time, but just a partition of the latter, delimited by the acceleration horizons. In section 5 of ref. [1], it is very clearly stated that test particles at the origin $r = 0$ acquire acceleration along the ±z directions, which certainly is not a feature of the ordinary, full Minkowski space-time.}
In the latter case (iii), we have considered Eq. (50) and have chosen \( C = [1 + 2(0.01)]^{-1} \) in order to make a consistent comparison with the first two cases, i.e., the value of \( C \) is the same in the three cases. In Figures 2 and 3 we consider \( \alpha = 0.02 \) and \( \alpha = 0.03 \), respectively, and the corresponding values of \( C \). In all cases, we see that for higher values of the radial coordinate \( R \), the energy of the infinite cosmic string dominates.

![Graph showing gravitational energy](image)

**Figure 1:** Gravitational energy \( P(0) \) given by Eq. (45), for various values of \( R \) of the surface of integration, considering \( R \geq 2m \), \( \alpha = 0.01 \) and \( m = 1 \) in natural units (continuous thick line). The curve for which \( \alpha = 0 \) represents the Schwarzschild black hole (Eq. (47)), and the one with \( m = 0 \) represents the infinite cosmic string only (Eq. (50)).

In Figures 1, 2 and 3 we see that the energy in the space-time of a pure infinite cosmic string is negative. As we mentioned above, this energy dominates when we consider higher volumes of integration. Thus, the energy density in regions for higher values of the radius of surface integration \( R \) (i.e., \( R \) approaching \( 1/\alpha \)) is negative. It is likely that this negative energy density is responsible for the acceleration of the black hole, since the black hole is moving towards the region of negative energy density. One argument in support of this conclusion is the following. Let us consider the gravitational
Figure 2: Gravitational energy $P^{(0)}$ given by Eq. (45), for various values of $R$ of the surface of integration, considering $R \geq 2m$, $\alpha = 0.02$ and $m = 1$ in natural units (continuous thick line). The curve for which $\alpha = 0$ represents the Schwarzschild black hole (Eq. (47)), and the one with $m = 0$ represents the infinite cosmic string only (Eq. (50)).
Figure 3: Gravitational energy $P^{(0)}$ given by Eq. (45), for various values of $R$ of the surface of integration, considering $R \geq 2m$, $\alpha = 0.03$ and $m = 1$ in natural units (continuous thick line). The curve for which $\alpha = 0$ represents the Schwarzschild black hole (Eq. (47)), and the one with $m = 0$ represents the infinite cosmic string only (Eq. (50)).
energy contained within a surface of constant radius $r$ in a Schwarzschild space-time. It is given by Eq. (47). By making $r = 2m$ and $r \to \infty$ in the latter equation, we obtain $P^{(0)} = 2m$ and $P^{(0)} = m$, respectively. The case $r = 2m$ is in agreement with Eq. (46). These results can also be obtained by means of the the quasi-local expression for the gravitational energy given by Brown and York [20]. Thus, in the region between $r = 2m$ and $r \to \infty$, the gravitational energy density is negative. The negative gravitational energy outside the event horizon may be identified with the negative Newtonian binding energy, which is attractive, as noted by Brown and York (see Eq. (6.16) of Ref. [20]). The same feature may be occurring here: a region of negative gravitational energy density exerts gravitational attraction, but in this case the black hole is being attracted, or accelerated along the semi-infinite cosmic string in the negative $z$ axis. One may think that the black hole is approaching a state of lower energy, as do ordinary bodies in classical physics.

The energy of space-time (topological) defects may be positive or negative, according to an “addition” or “removal” of a continuum medium to the space-time. Cosmic strings are disclination-type defects, and are highly energetic defects compared to dislocations. This issue is discussed in Refs. [17] [18]. (See also Eq. (32) of Ref. [19], which presents the energy per unit length of a cosmic string. For a parameter $\beta_0 > 1$, this energy is negative.) When a substantial fraction of a space-time is “removed”, as in the case of the infinite cosmic string (according to Eq. (4)), the total energy of the space-time may be negative in the frame accelerated along the negative $z$ axis.

In view of Eqs. (45) and (50), we may identify the energy of the black hole only as

$$P_{bh}^{(0)} = P^{(0)} - P_{cs}^{(0)}.$$  \hspace{1cm} (51)

In the evaluation of $P_{cs}^{(0)}$, the constant $C$ is numerically chosen to be $C = (1 + 2\alpha m)^{-1}$, where $m$ and $\alpha$ are the values that yield $P^{(0)}$. Thus, both $P^{(0)}$ and $P_{cs}^{(0)}$ are endowed with the same constant $C$.

The identification above turns out to be consistent, as we see in Figure 4, because the difference between this energy and the energy obtained from Eq. (47), that represents the energy enclosed by surfaces of constant radius $R$ in the Schwarzschild space-time, is not too much significant. Although the surfaces of constant radius $R$ are not strictly the same in the space-times
with and without the acceleration parameter \(\alpha\) (i.e., there is no covariant relation between the radius \(R\) in the two situations), the result displayed by Figure 4 is qualitatively relevant to indicate the consistency of our analysis.

\[
\begin{align*}
P_{bh}(0)(R=2m) &= 1.96121 \\
P_{Sch}(0)(R=2m) &= 2
\end{align*}
\]

Figure 4: Gravitational energy \(P^{(0)}_{bh} = P^{(0)} - P^{(0)}_{cs}\) (continuous line), for various values of the surface of integration \(R\), considering \(m = 1\) and \(\alpha = 0.01\) in natural units. The dashed line represents the Schwarzschild black hole (\(\alpha = 0\)).

As a final remark, we note that \(P^{(0)}_{cs}\) given by Eq. (50) vanishes in the flat space-time limit, which is obtained by requiring simultaneously \(\alpha = 0\) and \(C = 1\). In addition, the following limits are verified: (i) when \(\alpha \to 0\), Eq. (45) reduces to the energy of the Schwarzschild space-time, as given by Eq. (47); (ii) when both \(m\) and \(\alpha\) vanish, the gravitational energy vanishes; (iii) when only the mass parameter \(m\) vanishes, Eq. (45) reduces to Eq. (50), with \(C = 1\). With respect to the latter equation, we note that the gravitational energy of the ordinary Minkowski space-time vanishes, either in the ordinary Cartesian or Rindler coordinates. However, the space-time represented by Eq. (6) is a partition of the full Minkowski space-time, delimited by the acceleration horizons. Obviously, such a partition does not represent the
complete, ordinary Minkowski space-time, in the same way that a partition of a sphere does not represent a sphere.

7 Conclusions

In this article we have addressed the C-metric space-time and have presented an explanation for the acceleration of the black hole. We recall that the C-metric space-time is a gravitational field configuration that describes an accelerated black hole along a semi-infinite cosmic string. The black hole is characterised by the mass parameter $m$, and the acceleration $\alpha$ yields the angular deficit $C$ in the negative part of the $z$ axis ($\theta = \pi$), characterised by $C = (1 + 2\alpha m)^{-1}$.

We obtained the expression for the gravitational energy contained within a surface of constant radius $R$, around the centre of the accelerated black hole. In the limit $r \to 2m$, we found the energy contained within the gravitational horizon, given by Eq. (46). This is the energy that cannot escape from the black hole. This energy may be identified with $2M_{irr}$, where $M_{irr}$ is sometimes defined as the irreducible mass of the black hole, in analogy with the definition of the irreducible mass of the Kerr black hole. For large values of the radius of integration $R$, the total gravitational energy (black hole plus the infinite cosmic string) is negative, according to Figures 1, 2 and 3. It is clear that this negative energy is dominated by the energy of the infinite cosmic string. As we argued at the end of Section 6, we may interpret the black hole as being dragged (accelerated) towards a state of lower energy, along the infinite cosmic string. The larger is the value of the radius of integration $R$, the more negative is the gravitational energy density of the cosmic string. Therefore, the black hole moves towards regions of lower gravitational energy density.

The accelerated black hole, as described by the C-metric space-time, is not physically equivalent to the situation where the black hole is a rest, and the observer undergoes an acceleration $-\alpha$. In particular, by means of a local Lorentz transformation, we cannot remove the acceleration of the black hole in the C-metric space-time.

We mention finally that we carried out a local Lorentz transformation on the set of tetrad fields (14) such that the new frame is accelerated in the positive $z$ direction with acceleration $+\alpha$. This new frame represents a static (or nearly static) frame in space-time, where the observer is no longer
attached to the black hole. We calculated the gravitational energy in this new frame and found that the resulting relation between $P^{(0)}$ and $R$ is extremely similar to Figures 1, 2 and 3, i.e., there is not a single qualitative difference between the relation of $P^{(0)}$ and $R$ in the two situations. This result ensures the frame independence of our main conclusion, in spite of the quantitative differences for the gravitational energy arising in the consideration of the nearly static frame, as compared to Eq. (45) (i.e., the latter equation is not frame independent). The quantitative differences are due to the emergence of the quantity $\gamma(t)$ in some terms in the expression of the gravitational energy in the nearly static frame, where $\gamma(t) = (1 - v(t)^2/c^2)^{-1/2}$.

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