LADDER REPRESENTATIONS OF $GL(n, \mathbb{Q}_p)$

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To David with admiration

Abstract. In this paper, we recover certain known results about the ladder representations of $GL(n, \mathbb{Q}_p)$ defined and studied by Lapid, Mínguez, and Tadić. We work in the equivalent setting of graded Hecke algebra modules. Using the Arakawa-Suzuki functor from category $O$ to graded Hecke algebra modules, we show that the determinantal formula proved by Lapid-Mínguez and Tadić is a direct consequence of the BGG resolution of finite dimensional simple $\mathfrak{gl}(n)$-modules. We make a connection between the semisimplicity of Hecke algebra modules, unitarity with respect to a certain hermitian form, and ladder representations.

Contents

1. Introduction 1
2. The • star operation 2
3. Ladder representations: definitions 4
4. Ladder representations: functors from category $O$ to $H$-modules 7
5. Ladder representations: pairs of commuting nilpotent elements 11
References 13

1. Introduction

In this paper we study a class of representations of the graded affine Hecke algebra which are unitary for a star operation which we call •. The •-unitary dual for type $A$ is determined completely. In this case, the unitary dual corresponds via the Borel-Casselmann equivalence of categories [Bo] composed with the reduction to the affine graded Hecke algebra of [LM] to the ladder representations defined and studied in [LM] and [Ta] for $GL(n, \mathbb{Q}_p)$.

The classification of the unitary dual of real and $p$-adic reductive groups is one of the central problems of representation theory. Typically, by results of Harish-Chandra, this problem is reduced to an algebraic one, the study of admissible representations of an algebra endowed with a star operation. In the case of real groups, this algebra is the enveloping algebra, in the case of $p$-adic groups, an Iwahori-Hecke type algebra with parameters. In both cases, the star operation is derived from the antiautomorphism $g \mapsto g^{-1}$. In the real case, David Vogan and his collaborators [ALTV] make a deep study of signatures of hermitian forms of admissible modules by exploiting the relationship between two different star operations, one related to the real form of the reductive group, the other related
to the compact form of the group. Motivated by this, we study the analogues of
these star operations for the graded affine Hecke algebras. The star operation
coming from the $p$-adic group is made explicit in [BM2]. In [BC1], we introduce
and study another star operation which we denote by $\bullet$, the analogue of the star
operation for a compact form. The problem of the unitarity of representations
for $\bullet$ seemed an artificial one. However, the results of Opdam [Op], and more
recently Oda [Od], show that spherical representations of graded affine Iwahori-
Hecke algebras play an important role in harmonic analysis of symmetric spaces
of compact type. Motivated by this result, we initiated a systematic study of
$\bullet$—unitary representations. This is the topic of this paper.

The first set of results is a connection between $\bullet$—unitary representations, and
representations which are $A$—semisimple. This is the content of Propositions
$2.3.2$ and $2.3.3$. This provides a connection to the work of [Ch], [KR], and [Ra].

In ongoing research we are planning to determine the entire $\bullet$—unitary dual for
graded affine Hecke algebras of arbitrary type. The most complete results to date
are for type $A$. In the process we found the links to the ladder representations in
the title, and the results [LM], [CR], and [Ta].

A seminal idea, pioneered by D. Vogan, was to try to make a connection between
the unitary dual of real and $p$-adic groups via intertwining operators, via petite
$K$—types and $W$—types. This was developed systematically by the authors of this
paper, jointly and separately, in particular to determine the full spherical unitary
dual of split $p$-adic (and split real classical) groups. We follow this approach in this
paper. We relate the $\bullet$—unitary (star for the compact form of the Lie algebra) dual
of Verma modules to the $\bullet$—unitary dual of the graded affine Hecke algebra using
the functors introduced by Arakawa and Suzuki, [AS], [Su]. The advantage of this
method is that it provides interesting connections between the Bernstein-Gelfand-
Gelfand resolution and results about character formulas of ladder representations.

Some time ago, motivated by conjectures of Arthur concerning unipotent rep-
resentations, D. Barbasch, S. Evens, and A. Moy conjectured the existence of an
action of the Iwahori-Hecke algebra on the cohomology of the incidence variety of
a pair of nilpotent elements whose $sl(2)$—triples commute (the conjectures actually
referred to more general pairs). In section 5 we provide evidence for this conjecture,
by establishing connections to the work of [Gi] and [EP].

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## 2. The $\bullet$ star operation

### 2.1. Graded affine Hecke algebra

Let $\Phi = (V, R, V^\vee, R^\vee, \Pi)$ be a reduced based $\mathbb{R}$-root system. Let $W \subset GL(V)$ be the Weyl group generated by the simple
reflections $\{s_\alpha : \alpha \in \Pi\}$. The positive roots are $R^+$ and the positive coroots are
$R^\vee, +$. The complexifications of $V$ and $V^\vee$ are denoted by $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$, respectively,
and we denote by $\bar{}$ the complex conjugations of $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$ induced by $V$ and $V^\vee$,
respectively.
Let \( k : \Pi \to \mathbb{R}_{>0} \) be a function such that \( k_{\alpha} = k_{\alpha'} \) whenever \( \alpha, \alpha' \in \Pi \) are \( W \)-conjugate. Let \( \mathbb{C}[W] \) denote the group algebra of \( W \) and \( S(V_c) \) the symmetric algebra over \( V_c \). The group \( W \) acts on \( S(V_c) \) by extending the action on \( V \). For every \( \alpha \in \Pi \), denote the difference operator by

\[
\Delta : S(V_c) \to S(V_c), \quad \Delta_{\alpha} (a) = \frac{a - s_{\alpha}(a)}{\alpha}, \text{ for all } a \in S(V_c). \tag{2.1.1}
\]

If \( a \in V_c \), then \( \Delta_{\alpha} (a) = \langle a, \alpha \rangle^\vee \).

**Definition 2.2.1** ([Lu]). The graded affine Hecke algebra \( \mathbb{H} = \mathbb{H}(\Phi, k) \) is the unique associative unital algebra generated by \( \{ a : a \in S(V_c) \} \) and \( \{ t_w : w \in W \} \) such that

(i) the assignment \( t_w a \mapsto w \otimes a \) gives an isomorphism of \( (\mathbb{C}[W], S(V_c)) \)-bimodules between \( \mathbb{H} \) and \( \mathbb{C}[W] \otimes S(V_c) \);

(ii) \( at_{s_{\alpha}} = t_{s_{\alpha s_{\alpha}}} + k_{\alpha} \Delta_{\alpha} (a) \), for all \( \alpha \in \Pi, a \in S(V_c) \).

The center of \( \mathbb{H} \) is \( S(V_c)^W \) ([Lu]). By Schur’s Lemma, the center of \( \mathbb{H} \) acts by scalars on each irreducible \( \mathbb{H} \)-module. The central characters are parameterized by \( W \)-orbits in \( V_c^\vee \). If \( X \) is an irreducible \( \mathbb{H} \)-module, denote by \( \operatorname{cc}(X) \in V_c^\vee \) (actually in \( W \backslash V_c^\vee \)) its central character.

### 2.2. Star operations

Let \( w_0 \) denote the long Weyl group element, and let \( \delta \) be the involutive automorphism of \( \mathbb{H} \) determined by

\[
\delta(t_w) = t_{w_0 w w_0}, \quad w \in W, \quad \delta(\omega) = -w_0(\omega), \quad \omega \in V_c. \tag{2.2.1}
\]

When \( w_0 \) is central in \( W \), \( \delta = \text{Id} \).

**Definition 2.2.1.** Let \( \kappa : \mathbb{H} \to \mathbb{H} \) be a conjugate linear involutive algebra anti-automorphism. An \( \mathbb{H} \)-module \( (\pi, X) \) is said to be \( \kappa \)-hermitian if \( X \) has a hermitian form \( (\ , \ ) \) which is \( \kappa \)-invariant, i.e.,

\[
(\pi(h)x, y) = (x, \pi(\kappa(h))y), \quad x, y \in X, \quad h \in \mathbb{H}.
\]

A hermitian module \( X \) is \( \kappa \)-unitary if the \( \kappa \)-hermitian form is positive definite.

**Definition 2.2.2.** Define a conjugate linear algebra anti-involution \( * \) of \( \mathbb{H} \) by

\[
t_w^* = t_{w^{-1}}, \quad w \in W, \quad \omega^* = \overline{\operatorname{Ad} t_{w_0} (\delta(a))} = -t_{w_0} \cdot w_0(\omega) \cdot t_{w_0}, \quad \omega \in V_c. \tag{2.2.2}
\]

Similarly define \( \bullet \) by

\[
t_w^\bullet = t_{w^{-1}}, \quad w \in W, \quad \omega^\bullet = \overline{\omega}, \quad \omega \in V_c. \tag{2.2.3}
\]

The operations \( * \) and \( \bullet \) are related by

\[
* = \operatorname{Ad} t_{w_0} \circ \bullet \circ \delta, \quad \text{for all } h \in \mathbb{H}. \tag{2.2.4}
\]

**Remark 2.2.3.** In [BCI], it is proved that \( * \) and \( \bullet \) are the only star operations of \( \mathbb{H} \) that satisfy certain natural conditions. When \( \mathbb{H} \) is obtained by grading the Iwahori-Hecke algebra of a reductive \( p \)-adic group, \( * \) corresponds to the natural star operation of the \( p \)-adic group. The operation \( \bullet \) is the analogue of the compact star operation defined for real reductive groups in [ALT].
2.3. Semisimplicity. In this section, suppose the parameters $k_\alpha$ are positive, but arbitrary. Let $(\pi, X)$ be a finite dimensional $H$-module. For every $\lambda \in V_C^*$, define

$$
X_\lambda = \{ x \in X : \pi(\omega)x = (\omega, \lambda)x, \text{ for all } \omega \in V_C \},
$$

$$
X_\lambda^{\text{gen}} = \{ x \in X : (\pi(\omega) - (\omega, \lambda))^n x = 0 \text{ for some } n \in \mathbb{N}, \text{ for all } \omega \in V_C \}. \tag{2.3.1}
$$

A functional $\lambda \in V_C^*$ is called a weight of $X$ if $X_\lambda \neq 0$. Let $\text{Wt}(X)$ denote the set of weights of $X$. It is straightforward that $\text{Wt}(X) \subset W \cdot \text{cc}(X)$.

Definition 2.3.1. The module $(\pi, X)$ is called $\mathcal{A}$-semisimple if $X_\lambda = X_\lambda^{\text{gen}}$ for all $\lambda$.

Proposition 2.3.2. Assume $(\pi, X)$ is a $\bullet$-unitary $H$-module. Then $X$ is $\mathcal{A}$-semisimple.

Proof. Let $(x, y)_X$ be the positive definite $\bullet$-form on $X$. Let $\lambda$ be a weight of $X$ and $x_\lambda \neq 0$ a weight vector. Define

$$
\{x_\lambda\}^\perp = \{ y \in X : (x_\lambda, y)_X = 0 \}.
$$

Let $y \in \{x_\lambda\}^\perp$ be given. Since

$$
0 = (\alpha, \lambda)(x, y)_X = (\pi(\omega)x_\lambda, y)_X = (x_\lambda, \pi(\omega^*)y)_X = (x_\lambda, \pi(\mathcal{W})y), \quad \omega \in V_C,
$$

it follows that $\{x_\lambda\}^\perp$ is $\mathcal{A}$-invariant. Since the form $(x, y)_X$ is positive definite, we have $X = \mathbb{C}x_\lambda \oplus \{x_\lambda\}^\perp$ as $\mathcal{A}$-modules. By induction, it follows that $X$ is a direct sum of one-dimensional $\mathcal{A}$-modules, thus $\mathcal{A}$-semisimple. \qed

Remark 2.3.3. The above proposition can also be interpreted as the following linear algebra statement: if $J$ is a hermitian matrix, and $N$ a nonzero nilpotent matrix such that

$$
JN = N^*J,
$$

then $J$ is not positive definite.

Remark 2.3.4. The proof and statement of Proposition 2.3.2 can be easily generalized by replacing $\mathcal{A}$ with any parabolic subalgebra of $H$.

Proposition 2.3.5 ([BC\text{\textsection}2]). Assume the central character $\chi$ of $\pi$ satisfies

$$
|\langle \chi, \tilde{\alpha} \rangle| \geq 1,
$$

for all simple roots $\alpha$. If $\pi$ is $\mathcal{A}$-semisimple, it is $\bullet$-unitary.

This is a converse to Proposition 2.3.2 and more difficult. We refer to [BC\text{\textsection}2] for a proof. Notice that, in particular, Propositions 2.3.2 and 2.3.5 imply that at integral central character $\chi$, i.e. $\langle \chi, \alpha^\vee \rangle \in \mathbb{Z}$ for all roots $\alpha$, a simple $H$-module is $\bullet$-unitary if and only if it is $\mathcal{A}$-semisimple.

3. Ladder representations: definitions

We consider the graded Hecke algebra of type $A$. In this case, we can classify the $\bullet$-unitary dual. We begin by recalling Zelevinsky’s classification [Ze1] of the simple modules. We will phrase the classification “with quotients” rather than “submodules”, cf. [Ze1] §10.
3.1. Multisegments. We restrict to \( \mathbb{H} \)-modules with real central character. By [BC2] Corollary 4.3.2 or Corollary 5.1.3, every simple \( \mathbb{H} \)-module with real central character admits a nondegenerate \( \bullet \)-invariant hermitian form.

A segment is a set \( \Delta = \{ a, a + 1, a + 2, \ldots, b \} \), where \( a, b \in \mathbb{R} \) and \( a \equiv b \) (mod \( \mathbb{Z} \)). We will write \( \Delta = [a, b] \) and \( |\Delta| = b - a + 1 \) for the length. A multisegment is an ordered collection \( (\Delta_1, \Delta_2, \ldots, \Delta_r) \) of segments. Following [Ze1] §4.1, two segments \( \Delta_1 \) and \( \Delta_2 \) are called

(a) linked, if \( \Delta_1 \not\subseteq \Delta_2, \Delta_2 \not\subseteq \Delta_1, \) and \( \Delta_1 \cup \Delta_2 \) is a segment;
(b) juxtaposed, if \( \Delta_1, \Delta_2 \) are linked and \( \Delta_1 \cap \Delta_2 = \emptyset \);

One says that

(c) \( \Delta_1 \) precedes \( \Delta_2 \) if \( \Delta_1, \Delta_2 \) are linked and \( a_1 < a_2 \).

For every segment \( \Delta \) with \( m = b - a + 1 \), let \( \langle \Delta \rangle \) denote the one-dimensional \( \mathbb{H}_m \)-module which extends the sign \( W \)-representation and on which \( \mathbb{A} \) acts by the character \( \mathbb{C}_{[a,b]} \). If \( (\Delta_1, \Delta_2, \ldots, \Delta_r) \) is a multisegment, denote by

\[
\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_r \rangle
\]

the induced module \( \mathbb{H}_n \otimes \mathbb{H}_{m_1} \otimes \mathbb{H}_{m_2} \otimes \cdots \otimes \mathbb{H}_{m_r} \) \( (\langle \Delta_1 \rangle \boxtimes \langle \Delta_2 \rangle \boxtimes \cdots \boxtimes \langle \Delta_r \rangle) \), where \( m_i = b_i - a_i + 1 \) and \( n = \sum m_i \).

We need two of the main results from [Ze1].

**Theorem 3.1.1 ([Ze1] Theorem 4.2).** The following conditions are equivalent:

1. The module \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle \) is irreducible.
2. For each \( i, j = 1, \ldots, r \), the segments \( \Delta_i \) and \( \Delta_j \) are not linked.

**Theorem 3.1.2 ([Ze1] Theorem 6.1).** (a) Let \( (\Delta_1, \ldots, \Delta_r) \) be a multisegment. Suppose that for each \( i < j \), \( \Delta_i \) does not precede \( \Delta_j \). Then the representation \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle \) has a unique irreducible quotient denoted by \( \langle \Delta_1, \ldots, \Delta_r \rangle \).

(b) The modules \( \langle \Delta_1, \ldots, \Delta_r \rangle \) and \( \langle \Delta'_1, \ldots, \Delta'_r \rangle \) are isomorphic if and only if the corresponding multisegments are equal up to a rearrangement.

(c) Every simple \( \mathbb{H}_n \)-module is isomorphic to one of the form \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle \).

**Remark 3.1.3.** For the most part, the above results are instances of the Langlands classification. A multisegment corresponds to data \( (M, \sigma, \nu) \) where

\[
M = GL(b_1 - a_1 + 1) \times \cdots \times GL(b_r - a_r + 1)
\]

is a Levi component, the tempered representation \( \sigma \) is the Steinberg, and the \( (a_i, b_i) \) determine the \( \nu \). The fact that \( \Delta_i \) precedes \( \Delta_j \) is the usual dominance condition for \( \nu \). The remaining results are sharpenings of the Langlands classification in the case of \( GL(n) \).

**Definition 3.1.4** (Ladder representations [LM]). Let \( \Delta_i = [a_i, b_i] \) \( 1 \leq i \leq r \) be Zelevinsky segments. If \( a_1 > a_2 > \cdots > a_r \) and \( b_1 > b_2 > \cdots > b_r \), call the irreducible representation \( \langle \Delta_1, \Delta_2, \ldots, \Delta_r \rangle \) a ladder representation.

**Example 3.1.5** (Speh representations [BM]). Let \( \Delta_i, 1 \leq i \leq r \) be segments as in Definition 3.1.4 such that \( b_i - a_i + 1 = d \) for a fixed \( d \) and \( a_i - a_{i+1} = 1 \) for all \( i \). Then \( \langle \Delta_1, \ldots, \Delta_r \rangle \) is irreducible as an \( S_n \)-representation, isomorphic to the \( S_n \)-representation parameterized by the rectangular Young diagram with \( r \) rows and \( d \) columns. These modules are both \( \bullet \)-unitary and \( \bullet \)-unitary ([BM] [CM]) and correspond to the (\( I \)-fixed vectors) of Speh representations.
3.2. Cherednik’s construction. As in Definition [3.1.4] let \((\Delta_1, \ldots, \Delta_r), a_1 > a_2 > \cdots > a_r, b_1 > b_2 > \cdots > b_r\) be a ladder representation. The interesting case is when \(\Delta_i\) is linked to \(\Delta_{i+1}\) for all \(i\). In fact, since tensoring with a character of the center of \(H\) does not change \(A\)-semisimplicity, we may even assume that \(a_i, b_i \in \mathbb{Z}\) for all \(i\). From now on, this type of ladder representations will be called integral.

Following [Ch], we give a combinatorial construction of integral ladder representations. Let \((\Delta_1, \ldots, \Delta_r)\) be an integral ladder representation with the notation as above. Set

\[
\lambda = (a_1, \ldots, b_1, a_2, \ldots, b_2, \ldots, a_r, \ldots, b_r) \in \mathbb{Z}^n
\]

(viewed as an element of \(V_C^\vee \cong \mathbb{C}^n\). The underlying multisegment \((\Delta_1, \ldots, \Delta_r)\) gives a skew-Young diagram, where each box in the Young diagram corresponds to an integer in one of the multisegments. More precisely, the underlying skew diagram is formed as follows. The first segment \(\Delta_1\) gives the top row with \(|\Delta_1|\) boxes, each box for one of the integers in \(\Delta_1\) in order. The segment \(\Delta_2\) gives the second row with \(|\Delta_2|\) boxes, immediately below the first, etc. The rows are aligned so that

(1) the two boxes are in the same column if and only if they correspond to the same integer in the multisegment, and

(2) two boxes is two adjacent columns correspond to two consecutive integers in the multisegment.

For example, if the multisegment is \(([2, 4], [0, 2], [-2, -1])\), the resulting skew Young diagram is

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \\
\text{ } & \text{ } & \text{ } & \\
\text{ } & \text{ } & \text{ } & \\
\text{ } & \text{ } & \text{ } & \\
\text{ } & \text{ } & \text{ } & \\
\end{array}
\]

(3.2.2)

Notice that the skew Young diagram does not recover the integral ladder representation uniquely, only up to tensoring with a central character. However, if we specify an integer \(a\) such that the first segment starts with \(a\), then the multisegment is determined.

We fix a skew Young diagram as above and we will form skew Young tableaux with that shape. Let \([1 \ldots n]\) be the set of integers \(1, 2, \ldots, n\). Let \(Y_1\) be the skew Young tableau with entries in \([1 \ldots n]\) such that in the first row, the entries are, in order, \(1, 2, \ldots, b_1 - a_1 + 1\), in the second row, \(b_1 - a_1 + 2, b_1 - a_1 + 3, \ldots, b_1 + b_2 - a_1 - a_2 + 2\), etc. In our example,

\[
Y_1 = \begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & \\
\end{array}
\]

(3.2.3)

Consider all skew Young tableaux with entries in \([1 \ldots n]\) subject to the requirements:

(1) the entries are increasing left-right on each row;

(2) the entries are increasing up-down on each 45°-diagonal.

Denote every such tableau by \(Y_w\), where \(w \in S_n\) is the permutation transforming \((1, 2, \ldots, n)\) to the entries of the tableau read in order from the top row to the bottom row and on each row from left to right. Let

\[
W(\Delta_1, \ldots, \Delta_r) = \text{ the set of } w \in S_n \text{ parameterizing the tableaux } Y_w \text{ for } (\Delta_1, \ldots, \Delta_r).
\]

(3.2.4)
Theorem 3.2.1 (Cherednik [Ch] Theorem 4, see also Ram [Ra]). The set \( \{ Y_w \} \) defined above is a basis of a simple \( \mathbb{H} \)-module \( C(\Delta_1, \ldots, \Delta_r) \) such that

1. \( Y_w \) is an \( \mathcal{A} \)-weight vector with weight \( w(\lambda) \).
2. The action of \( W \) on \( \{ Y_w \} \) is as follows:

\[
\pi(t_s)Y_w = \begin{cases} 
\frac{1}{(\alpha, w(\lambda))}Y_w, & \text{if } s_\alpha w \notin W(\Delta_1, \ldots, \Delta_r) \\
\frac{1}{(\alpha, w(\lambda))}Y_w + (1 + \frac{1}{(\alpha, w(\lambda))})Y_s, & \text{if } s_\alpha w \in W(\Delta_1, \ldots, \Delta_r),
\end{cases}
\]

for every \( \alpha \in \Pi \).

Since the weight \( \lambda \) is also the Langlands weight of the irreducible ladder representation \( (\Delta_1, \ldots, \Delta_r) \), we clearly have

\[
(\Delta_1, \ldots, \Delta_r) \cong C(\Delta_1, \ldots, \Delta_r),
\]

for every integral ladder representation.

4. LADDER REPRESENTATIONS: FUNCTORS FROM CATEGORY \( \mathcal{O} \) TO \( \mathbb{H} \)-MODULES

In this section, we apply some constructions of Zelevinsky [Ze2] and Arakawa and Suzuki [AS, Su] to the study of \( \mathfrak{b} \)-unitary representations.

4.1. Category \( \mathcal{O} \). Let \( \mathfrak{g} \) be a complex reductive Lie algebra with universal enveloping algebra \( U(\mathfrak{g}) \). Fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \), and a Borel subalgebra \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \). Let \( R \subset \mathfrak{h}^* \) denote the roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \), and let \( R^+ \) be the positive roots with respect to \( \mathfrak{b} \). Let \( W = N_G(\mathfrak{h})/Z_G(\mathfrak{h}) \) be the Weyl group with length function \( \ell \).

Let \( \mathfrak{n}^- \) be the nilradical of the opposite Borel subalgebra. Let \( \Pi \) be the simple roots defined by \( R^+ \), and for every root \( \alpha \), let \( \alpha^\vee \in \mathfrak{h} \) be the coroot. Let \( \alpha_i, i = 1, |\Pi| \) denote the simple roots, and \( \omega_i \) the corresponding fundamental coweights. Set \( \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \). We denote by \( \langle , \rangle \) the pairing between \( \mathfrak{h}^* \) and \( \mathfrak{h} \).

Define

\[
\Lambda = \{ \lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \text{ for all } \alpha \in R \};
\]

\[
\Lambda^+ = \{ \lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}, \text{ for all } \alpha \in R^+ \}.
\]

Let \( \mathcal{O} \) denote the category of finitely generated \( U(\mathfrak{g}) \)-modules, which are \( \mathfrak{n} \)-locally finite and \( \mathfrak{h} \)-semisimple. If \( X \) is a module in \( \mathcal{O} \), let \( \Omega(X) \) denote the set of \( \mathfrak{h} \)-weights of \( X \).

For every \( \mu \in \mathfrak{h}^* \), let \( M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\mu \) denote the Verma module with highest weight \( \mu \) and infinitesimal character \( \mu + \rho \). Then \( M(\mu) \in \mathcal{O} \) has a unique simple quotient, the highest weight module \( L(\mu) \). As it is well known, \( L(\mu) \) is a simple finite dimensional module if and only if \( \mu \in \Lambda^+ \).

For every \( w \in W \), \( \lambda \in \mathfrak{h}^* \), define

\[
w \circ \lambda = w(\lambda + \rho) - \rho.
\]

For \( X \in \mathcal{O} \), let

\[
H_0(\mathfrak{n}^-, X) = X/\mathfrak{n}^- X
\]

denote the 0-th \( \mathfrak{n}^- \)-homology space, viewed as an \( \mathfrak{h} \)-module. For every \( \lambda \in \Lambda^+ \) and every finite dimensional \( \mathfrak{g} \)-module \( \mathcal{V} \), define the functor

\[
F_{\lambda, \mathcal{V}} : \mathcal{O} \to \text{Vect}, \quad F_{\lambda, \mathcal{V}}(X) = H_0(\mathfrak{n}^-, X \otimes \mathcal{V})_{\lambda},
\]

(4.1.2)
where the subscript stands for the $\lambda$-weight space, and Vect denotes the category of $\mathbb{C}$-vector spaces.

**Remark 4.1.1.** In general, $F_{\lambda,V}$ need not be an exact functor. However, if one assumes that $\lambda + \rho \in \Lambda^+$, then $F_{\lambda,V}$ is exact. See for example [AS Proposition 1.4.2].

Let $L(\mu)$ be a simple finite dimensional module. Recall that there exists a resolution of $L(\mu)$ in $O$ defined by Bernstein-Gelfand-Gelfand:

$$0 \to C_N \to C_{N-1} \to \cdots \to C_1 \to C_0 \to L(\mu) \to 0,$$

where $C_i = \bigoplus_{w \in W, \ell(w) = i} M_{w_\mu}$. \hfill (4.1.3)

In particular, applying the Euler-Poincaré principle, the identity

$$L(\mu) = \sum_{w \in W} \text{sgn}(w) M_{w_\mu}$$ \hfill (4.1.4)

holds in the Grothendieck group of $O$.

**Proposition 4.1.2** ([Ze2, Proposition 1]). Fix $\lambda \in \Lambda^+$, $\mu \in \Lambda^+$, $\chi \in \Lambda$, and a finite dimensional representation $V$.

1. The functor $F_{\lambda,V}$ transforms the BGG resolution (4.1.3) into an exact sequence.
2. There are natural $\mathbb{C}$-linear isomorphisms

$$F_{\lambda,V}(M(\chi)) = V_{\lambda-\chi} \text{ and } F_{\lambda,V}(L(\mu)) = V_{\lambda-\mu}[\mu],$$ \hfill (4.1.5)

where $V_{\lambda-\chi}$ denotes the $(\lambda-\chi)$-weight space of $V$, and

$$V_{\lambda-\mu}[\mu] = \{ v \in V_{\lambda-\mu} : e^\mu_\alpha v = 0, \text{ for all } \alpha \in \Pi \}.$$

Here $e_\alpha \in n$ denotes a fixed root vector for $\alpha \in \Pi$.

As a corollary, one can transfer formula (4.1.4) via $F_{\lambda}$. This is particularly interesting when the images of modules in $O$ under $F_{\lambda}$ admit actions by a different group (such as in the classical Schur-Weyl duality) or other algebras.

4.2. **The Arakawa-Suzuki functor.** We specialize to $g = gl(n, \mathbb{C})$. Let $E_{i,j}$ denote the matrix with 1 in the $(i, j)$-position and 0 elsewhere. Let $s_{ij} \in S_n$ denote the transposition. Fix a positive integer $\ell$, and set

$$V_\ell = (\mathbb{C}^n)^{\otimes \ell},$$ \hfill (4.2.1)

with the diagonal $g$-action.

**Remark 4.2.1.** If $\ell = n$, the finite dimensional $g$-module $V_n$ has the property that its 0-weight space is naturally isomorphic to the standard representation of $S_n$.

For every $0 \leq i < j \leq \ell$, consider the operator

$$\Omega_{i,j} = \sum_{1 \leq k,m \leq n} (E_{k,m})_i \otimes (E_{m,k})_j \in \text{End}(X \otimes V_\ell),$$ \hfill (4.2.2)

where $()_i$ means that the corresponding element acts on the $i$-th factor of the tensor product. It is well known that $\Omega_{i,j}$, $1 \leq i < j \leq n$ flips the $i,j$ factors of tensor product, i.e., $\Omega_{i,j}(x \otimes v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_\ell) = x \otimes v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_\ell.$
extends to an action of the graded Hecke algebra $\mathbb{H}_\ell$ of $\mathfrak{gl}(\ell)$ on $X \otimes \mathcal{V}_\ell$.

Notice the presence of the minus sign in the action of $s_{i,i+1}$ which is not the convention in [AS]. We make this adjustment so that the results fit with the previous sections. This is because the standard modules for $H$ are induced from Steinberg modules to conform with the Langlands classification, not the trivial modules as in [AS] and [Su] (Lemma 4.2.2).

In this way, the functor $F_{\lambda,\mu}$ from (4.1.2) maps to $\mathbb{H}_\ell$-modules. Since we will consider $\lambda$ such that $\lambda + \rho \in \Lambda^+$, this will be an exact functor.

In [AS] and [Su], the images of Verma modules and highest weight modules are computed. We recall their results now.

Let $P(\mathcal{V}_\ell) \subset \mathfrak{h}^*$ denote the set of weights of $\mathcal{V}_\ell$. If we identify $\mathfrak{h}^*$ with $\mathbb{C}^n$, then these weights are of the form $(\ell_1, \ldots, \ell_n)$ where $\sum \ell_i = \ell$ and $\ell_i \geq 0$.

Assume that $\lambda + \rho \in \Lambda^+$ and let $\mu \in \mathfrak{h}^*$ be such that $\lambda - \mu \in P(\mathcal{V}_\ell)$. Define the multisegment $\Phi_{\lambda,\mu} = (\Delta_1, \ldots, \Delta_n)$, $\Delta_i = [(\mu + \rho, \epsilon_i^\vee), (\lambda + \rho, \epsilon_i^\vee) - 1]$, and the standard $\mathbb{H}_\ell$-module

$$\mathcal{M}(\lambda, \mu) = \langle \Delta_1, \ldots, \Delta_n \rangle = \mathbb{H}_\ell \otimes \mathbb{H}_{\ell_1} \otimes \cdots \otimes \mathbb{H}_{\ell_n} (\mathfrak{st} \otimes \mathbb{C}_{\Delta_1}) \otimes \cdots \otimes (\mathfrak{st} \otimes \mathbb{C}_{\Delta_n}).$$

Let $\mathcal{L}(\lambda, \mu)$ denote the unique simple quotient of $\mathcal{M}(\lambda, \mu)$. The lowest $S_\ell$-type of $\mathcal{L}(\lambda, \mu)$ is parameterized by the partition of $\ell$ obtained by ordering $\lambda - \mu = (f_1, \ldots, f_n)$ in decreasing order.

**Theorem 4.2.3** ([Su] Theorems 3.2.1, 3.2.2). Assume $\lambda + \rho \in \Lambda^+$ and $\mu \in \lambda - P(\mathcal{V}_\ell)$.

1. $F_{\lambda,\mu}(\mathcal{M}(\mu)) = \mathcal{M}(\lambda, \mu)$ as $\mathbb{H}_\ell$-modules.
2. If $\mu$ satisfies the condition

$$\langle \mu + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0} \text{ for all } \alpha \in R^+ \text{ satisfying } \langle \lambda + \rho, \alpha^\vee \rangle = 0,$$

then

$$F_{\lambda,\mu}(\mathcal{L}(\mu)) = \mathcal{L}(\lambda, \mu).$$

3. If $\mu$ does not satisfy condition (4.2.5), then

$$F_{\lambda,\mu}(\mathcal{L}(\mu)) = 0.$$

Notice that if $\lambda$ in the theorem is such that $\langle \lambda + \rho, \alpha^\vee \rangle \geq 1$ for all simple roots $\alpha$, then condition (4.2.5) is vacuously true.

4.3. We apply the previous results to the ladder representations. Consider segments $\Delta_i = [a_i, b_i]$, $i = 1, n$, such that $a_1 > a_2 > \ldots$ and $b_1 > b_2 > \ldots$. Let $C(\Delta_1, \ldots, \Delta_n)$ denote the ladder representation for $\mathbb{H}_\ell$. Identify $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ with elements of $\mathfrak{h}^* \cong \mathbb{C}^n$. Set

$$\mu = (a_1, \ldots, a_n) - \rho, \quad \lambda = (b_1 + 1, \ldots, b_n + 1) - \rho.$$  

(4.3.1)
In coordinates $\rho = \left( \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2} \right)$.

Assume from now on that $(a_1, \ldots, a_n) \equiv \rho \mod \mathbb{Z}$. Then $\lambda$ and $\mu$ just defined satisfy the conditions of Theorem 4.2.3 and, in fact, $\langle \lambda + \rho, \alpha \rangle \geq 1$ for all simple roots $\alpha$.

**Proposition 4.3.1.** With the notation as above, we have:

1. $F_{\lambda, V_\ell}(L(\mu)) = C(\Delta_1, \ldots, \Delta_n)$;
2. $C(\Delta_1, \ldots, \Delta_n) = \sum_{w \in S_n} \text{sgn}(w) \langle w \cdot \Delta_1, \ldots, w \cdot \Delta_n \rangle$, where $w \cdot \Delta_i := [a_{w(i)}, b_i]$, and the standard representation $\langle w \cdot \Delta_1, \ldots, w \cdot \Delta_n \rangle$ is understood to be 0 if $a_{w(i)} > b_i$ for some $i$.

**Proof.** Part (1) follows immediately from Theorem 4.2.3(2). For (2), we first apply the functor $F_{\lambda, V_\ell}$ to the BGG formula (4.1.4), and then identify the images of the Verma modules as in Theorem 4.2.3(1). \(\square\)

**Remark 4.3.2.** Proposition 4.3.1(2) recovers the known “determinantal” character formula for ladder representations of Tadić [Ta], and Lapid-Minguez [LM, Theorem 1], see also [CR]. This approach also provides a resolution of the ladder representations which is the image of the BGG resolution under the functor.

### 4.4. The functor $F_{\lambda, V_\ell}$ behaves well with respect to invariant hermitian (or symmetric bilinear) forms, and in fact, this is an ingredient in the proof of Theorem 4.2.3(2). We recall the results in the setting of hermitian rather than symmetric forms, with the obvious modifications.

Recall that $g = gl(n, \mathbb{C})$ viewed as a Lie algebra admits a complex conjugate linear anti-automorphism $* : A \mapsto A^T$. A module $X \in O$ is called hermitian if it admits an invariant form $(\ , \ )_X$ satisfying

$$(Ax, y)_X = (x, A^* y)_X, \quad \text{for all } A \in g = gl(n). \quad (4.4.1)$$

The standard representation $\mathbb{C}^n$ is hermitian, the usual inner product

$$(x, y)_{\mathbb{C}^n} = \sum_i x_i \overline{y}_i$$

has property (4.4.1).

If $X$ admits an invariant hermitian form, then $X \otimes V_\ell = X \otimes (\mathbb{C}^n)^{\otimes \ell}$ can be endowed with the product form. The following lemma is straightforward.

**Lemma 4.4.1 ([Su, Lemma 4.1.4]).** Suppose $X$ admits a $g$-invariant form as in (4.4.1). Then the form on $X \otimes V_\ell$ is $H_\ell$-invariant with respect to the $\bullet$ star operation of $H_\ell$.

If the form on $X$ is nondegenerate (positive definite), then the form obtained on $F_{\lambda, V_\ell}(X)$ is nondegenerate (positive definite).

Combining Lemma 4.4.1 with Proposition 4.3.1, we obtain as a consequence the known semisimplicity result for ladder representations [LM].

**Proposition 4.4.2.** Every ladder representation $C(\Delta_1, \ldots, \Delta_n)$ is $\bullet$-unitary, and therefore $H_M$-semisimple for every parabolic Hecke subalgebra $H_M$.

**Proof.** Apply Lemma 4.4.1 with $X = L(\mu)$, where $\mu$ and $\lambda$ are as in Proposition 4.3.1(1). \(\square\)
**Remark 4.4.3.** The $\mathbb{H}_{M}$-semisimplicity of ladder representations from Proposition 4.4.2 is the Hecke algebra equivalent of the semisimplicity of the Jacquet modules of ladder representations proved in [LM].

5. LADDER REPRESENTATIONS: PAIRS OF COMMUTING NILPOTENT ELEMENTS

We relate the $A$-semisimple $H$-modules to the geometry of pairs of commuting nilpotent elements considered by [Gi]. Let $g$ be a complex semisimple Lie algebra and $G = \text{Ad}(g)$.

**Definition 5.0.4** ([Gi], [EP]). A pair $\underline{e} = (e_1, e_2) \in g \times g$ is called a nilpotent pair if $[e_1, e_2] = 0$ and for all $(t_1, t_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$, there exists $g \in G$ such that $\text{Ad}(g)(e_1, e_2) = (t_1 e_1, t_2 e_2)$. In addition:

1. $\underline{e}$ is called principal if $\text{dim} \ z_g(e) = \text{rank} \ g$;
2. $\underline{e}$ is called distinguished if
   a. $z_g(e)$ contains no semisimple elements, and
   b. there exists a semisimple pair $\underline{h} = (h_1, h_2) \in g \times g$ such that $\text{ad}(h_1), \text{ad}(h_2)$ have rational eigenvalues,

\[
[h_1, h_2] = 0, \quad [h_i, e_j] = \delta_{ij} e_j, \quad \text{and} \quad z_g(h) \text{ is a Cartan subalgebra.}
\]

3. $\underline{e}$ is called rectangular if $e_1, e_2$ can be embedded in commuting $sl(2)$ triples.

By [Gi, Theorem 1.2], every principal nilpotent pair $\underline{e}$ is distinguished, and in fact, the associated semisimple pair $\underline{h}$ has the property that the eigenvalues of $\text{ad} \ h_i$ are integral.

5.1. We summarize some of the results from [Gi].

**Theorem 5.1.1** ([Gi] Theorem 3.7, Theorem 3.9, Corollary 3.6).

1. Any two principal nilpotent pairs $\underline{e}$ and $\underline{e}'$ with the same associated semisimple pair $\underline{h}$ are conjugate to each other by the maximal torus $T = Z_G(h)$.
2. There are finitely many adjoint $G$-orbits of principal nilpotent pairs.
3. For every principal nilpotent pair $\underline{e}$, the centralizer $Z_G(e)$ is connected.

The construction of principal pairs is as follows. Let $p = l \oplus u$ be a parabolic subalgebra and $e_1 \in l$ a principal nilpotent element. Assume that $e_2 \in z_u(e_1)$ is a Richardson element for $p$. Set $\underline{e} = (e_1, e_2)$. The following are equivalent:

1. $\underline{e}$ is a principal nilpotent pair.
2. The orbit $\text{Ad} \ Z_p(e_1) \cdot e_2$ is Zariski open dense in $z_u(e_1)$.

Every principal nilpotent pair is of this form. More precisely, for a given principal pair $\underline{e}$, let $\underline{h} = (h_1, h_2)$ be the associated semisimple pair. Let $g = \oplus_{p,q} g_{p,q}$ be the bigradation of $g$ defined by the adjoint action of $\underline{h}$. Define

\[
g_{p, *} = \oplus_q g_{p,q} \quad \text{and} \quad g_{*, q} = \oplus_p g_{p,q},
\]

and the parabolic subalgebras

\[
p^\text{east} = \oplus_{p \geq 0} g_{p,*} \quad \text{and} \quad p^\text{south} = \oplus_{q \geq 0} g_{*, q},
\]

with Levi subalgebras $g^1 = g_{*, 0}$ and $g^2 = g_{0,*}$, respectively. Then $(e_1, e_2)$ are given by the above construction for $p = p^\text{south}$ and $l = g^1$. 
5.2. The notation is motivated by the example $g = sl(n)$. In this case let $\sigma$ be a Young diagram visualized as in the following example:

```
\begin{array}{ccc}
0 & 1 & 2 \\
\end{array}
```

Enumerate the boxes $1, 2, \ldots, n$ in some order and label the basis of $\mathbb{C}^n$ by the box with the corresponding number. Let $e_1, e_2 \in \text{End}(\mathbb{C}^n)$ be defined as follows:

- $e_1$: sends a basis vector corresponding to a box to the vector corresponding to the next box on the row (to the east) or 0 if it’s the last row box;
- $e_2$: same as $e_1$ except the direction is down (south) on the columns.

**Theorem 5.2.1 ([Gi]).** Suppose $g = sl(n)$. Every adjoint $G$-orbit of principal nilpotent pairs has a representative obtained from a Young diagram by the procedure described above.

The classification of the larger class of distinguished nilpotent pairs has a similar flavor. Consider $\sigma$ to be a skew Young diagram, i.e. the set difference of two Young diagrams as before with the same corner. Moreover, assume that $\sigma$ is connected. Define $e = (e_1, e_2)$ as in the Young diagram case, but for the skew diagram $\sigma$.

**Theorem 5.2.2 ([Gi] Theorem 5.6).** The adjoint $G$-orbits of distinguished nilpotent pairs are in one to one correspondence, via the construction above, with connected skew diagrams $\sigma$.

The rectangular distinguished nilpotent pairs (in the sense of Definition 5.0.4(3)) correspond to rectangular Young diagrams, i.e. usual Young diagrams in the shape of rectangles (Example 3.1.5).

5.3. We make the connection with ladder representations. Given an $a \in \mathbb{Z}$ and $\sigma$ a connected skew diagram, we associate an integral ladder representation $C(\sigma, a)$ as follows.

Form a skew Young tableau as follows: the leftmost box of the first row of $\sigma$ gets content (the number in the box) $a$, then the contents increase to the right and decrease to the left on rows, and stay constant on the columns. In the following example, $a = 2$:

```
\begin{array}{ccc}
0 & 1 & 2 \\
\end{array}
```

(5.3.1)

Suppose $a'_i$ is the leftmost content in row $i$, while $b'_i$ is the rightmost content. Define the segments:

$$\Delta_i = [a_i, b_i], \text{ where } a_i = -(i - 1) + a'_i \text{ and } b_i = -(i - 1) + b'_i.$$ (5.3.2)

In other words, move the $i$-th row $(i - 1)$-units to the left, for every $i$. In our example,

```
\begin{array}{ccc}
2 & 3 & 4 \\
0 & 1 \\
\end{array}
```

(5.3.3)

**Definition 5.3.1.** The integral integral ladder representation defined above will be called

$$C(\sigma, a) := \langle \Delta_1, \ldots, \Delta_r \rangle.$$ (5.3.4)
Consider the variety $B(e, h)$ of Borel subalgebras of $g$ containing the elements $(e_1, e_2, h_1, h_2)$. When $e$ is distinguished, $B(e, h)$ is 0-dimensional. More precisely, suppose $b \in B(e, h)$. Since $h_1, h_2 \in h$, also $\mathfrak{B}(h) \subset b$. As $e$ is distinguished, $h := \mathfrak{B}(h)$ is a Cartan subalgebra. This means that every $b \in B(e, h)$ contains the Cartan subalgebra $h$. Let $W$ be the Weyl group of $h$. If $b_0$ is a Borel subalgebra containing $h$ such that $e_1, e_2 \in b_0$, then

$$B(e, h) = \{wb_0 : w \in Wb_0\}$$

where

$$Wb_0 = \{w \in W : w^{-1}e_1 \in b_0, w^{-1}e_2 \in b_0\}.$$  \hspace{1cm} (5.3.5)

Clearly, if $b'_0 = ub_0$ is another Borel subalgebra in $B(e, h)$, with $u \in W$, then

$$W(e, b'_0) = W(e, b_0)u^{-1}.$$ 

**Proposition 5.3.2.** Suppose $g = sl(n, \mathbb{C})$. Let $\sigma$ be a connected skew diagram. Let $e$ be a distinguished nilpotent pair with associated semisimple pair $h$, such that $e$ is attached to $\sigma$ by Theorem 5.2.2. Let $(\Delta_1, \ldots, \Delta_r)$ be the multisegment constructed from $\sigma$ by procedure [5.3.2]. Then, for every Borel subalgebra $b_0 \in B(e, h)$, we have

$$W(e, b_0) = W(\Delta_1, \ldots, \Delta_r)u^{-1},$$  \hspace{1cm} (5.3.6)

where $W(e, b_0)$, $W(\Delta_1, \ldots, \Delta_r)$ are defined in [5.3.3] and [3.2.4], respectively.

**Proof.** It is sufficient to prove that if $b_0$ is the lower triangular Borel subalgebra and $h$ is the diagonal Cartan subalgebra, then $W(e, b_0) = W(\Delta_1, \ldots, \Delta_r)$. If we assign to the boxes of $\sigma$ the standard basis elements of $\mathbb{C}^n$ in row order, e.g. the boxes of the first row correspond to $x_1, x_2, \ldots, x_{m_1}$, where $m_1 = |\Delta_1|$, etc., then the nilpotent element $e_1 \in b_0$ is a sum

$$e_1 = \sum_{i=1}^r X_i,$$

where $X_1 = E_{21} + E_{32} + \cdots + E_{m_1, m_1-1}$, etc., \hspace{1cm} (5.3.7)

Since $w \cdot E_{ij} = E_{w(i), w(j)}$, for $w \in S_n$, it is clear that the condition $w^{-1} \cdot e_1 \in b_0$ translates to the same rule as the “row rule” (1) used in defining $W(\Delta_1, \ldots, \Delta_r)$.

Similarly, $e_2$ is defined using the columns of $\sigma$. Then the restrictions imposed by the condition $w^{-1} \cdot e_2 \in b_0$ are the same as the “45°-diagonal rule” (2) used in the definition of $W(\Delta_1, \ldots, \Delta_r)$. Recall that $(\Delta_1, \ldots, \Delta_r)$ is obtained from $\sigma$ by shifting each row to the left and therefore the column relations become diagonal relations.

Proposition 5.3.2 has the following immediate corollary.

**Corollary 5.3.3.** Keep the notation from Proposition 5.3.2. For every $a \in \mathbb{Z}$, the $\lambda$-weights of the ladder representation $C(\sigma, a)$ defined in [5.3.4] are in one-to-one correspondence with the points of the variety $B(e, h)$.

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