A local combinatorial formula for the Chern class of a triangulated $S^1$ bundle in terms of shellings

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Abstract

Here we will fix an output of a trivial calculation based on Konsevich’s differential 2-form for the Chern class of polygon bundle. As a result an interesting combinatorics and arithmetics jumps right out of a jukebox. The calculation gives very simple rational combinatorial characteristics (we call it "curvature") of a triangulated $S^1$ bundle over a 2-simplex, which is a local combinatorial formula for the first Chern class. The curvature is expressed in terms of cyclic word in 3-character alphabet associated to the bundle. From the point of view of simplicial combinatorics the word is a canonical shelling of the total complex. If you know a triangulation of a bundle - you can really easily compute the Chern class.

0.1 Introduction

In this article $I$ always denote an oriented interval and $S$ always denote an oriented circle $S^1$.

Elementary simplicial $S$-bundle ("elementary $S$-bundle" for short) over an ordered $n$-simplex is a map $\mathcal{E} \xrightarrow{p} [\Delta^n]$ of simplicial complexes, such that $|\mathcal{E}| \xrightarrow{|p|} \Delta^n$ is a piecewise linear $S$-fiber bundle (see Figure 1). Here $\Delta^n$ is a standard geometric simplex, $[\Delta^n]$ is a face complex of $\Delta^n$ and $[|...|]$ the geometric realization functor.

Let $SB(n)$ be the set of all elementary $S$-bundles over ordered $n$-simplices. The set $SB(n)$ is countable. Let $SB(n) \xrightarrow{\delta_i} SB(n-1), i = 0...n$ be a map assigning to the bundle $p$ its $i$-th face - the restriction on $i$-th face of $[\Delta^n]$. This boundary operators $\delta_i$ creates on elementary $S$-bundles a structure of semi-simplicial set (a simplicial set without degenerations) $SB$. Now we will define a based cochain complex $SB^*$ over $\mathbb{Q}$. Define $SB^n$ to be a vector space of all maps $SB(n) \rightarrow \mathbb{Q}$, and define the differential $SB^n \xrightarrow{d} SB^{n+1}$ by the formula $df(p) = \sum_{i=0}^{n} (-1)^i f(\delta_i p)$.

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A smooth $U(1)$-bundle $E \xrightarrow{p} B$ on a smooth manifold $B$ can be triangulated. This means following. A triangulation $t$ of a manifold $M$ is a homomorphism $|\mathcal{M}| \xrightarrow{t} M$ between the geometric realization of some simplicial complex $\mathcal{M}$ and $M$. A triangulation of $p$ is a simplicial map of simplicial complexes $\mathcal{E} \xrightarrow{p} \mathcal{B}$ and a pair of triangulations $|\mathcal{E}| \xrightarrow{e} E, |\mathcal{B}| \xrightarrow{b} B$ such the the diagram of maps (1) is commutative

$$
\begin{array}{ccc}
|\mathcal{E}| & \xrightarrow{e} & E \\
|p| & \downarrow & \downarrow p \\
|\mathcal{B}| & \xrightarrow{b} & B
\end{array}
$$

Here $|p|$ is a geometric realization of the simplicial map $p$.

Suppose that some local ordering on the vertices of the base complex $\mathcal{B}$ is fixed. Then $\mathcal{B}$ obtains a structure of semi-simplicial set and we got a Gauss map of semi-simplicial sets $\mathcal{B} \xrightarrow{C^1_p} SB$ sending an ordered simplex of $\sigma \in \mathcal{B}$ to the induced elementary bundle $p^{-1}(\sigma) \xrightarrow{p^*} \sigma$.

The bundle $p$ has a rational Chern characteristic class $c_1(p) \in H^2(B, \mathbb{Q})$. A "local combinatorial function for $c_1$" is a cocycle $C_1 \in SB^2$, such that for any triangulation $p$ of $p$ the cocycle having value $C_1(G_p(\sigma))$ on the simplex $\sigma$ in the ordered cochain complex of the triangulation $b$ represents $c_1(p)$. The existence of such functions is a standard result of geometric topology, the particular expressions - "formulas" for such functions is a problem of "local combinatorial formulas for characteristic classes" (see [2] for a survey).

Our aim is to define a cocycle $	ext{Curv} \in SB^2$ by a very simple combinatorial formula and demonstrate that it is a local combinatorial formula for the first Chern class of $U(1)$ bundles. First we will introduce a comfortable way of encoding elementary $I$-bundles by words and elementary $S$ bundles by cyclic words. This is done in §0.2. Then in §0.3 we define the cochain...
Curv in terms of combinatorics of words. Once Curv is guessed one can directly check that it is a local formula for the first Chern class. It is done in §0.4. In §0.5 we demystify the formula for Curv by showing that it naturally comes from Kontsevich’s metric polygon bundles and naturally points to the relation between combinatorics of elementary bundles and combinatorics of words.

The author is grateful to Peter Zograf for the reference on Kontsevich’s polygon bundles.

0.2 Elementary bundles as words in ordered alphabet

Let we have an alphabet made from $(n + 1)$ ordered characters numbered by 0...$n$. An $n$-word is a finite word in which all the $(n + 1)$ characters are present. (We need such a "+1 shift" in the definition. It originates in the fact that $n$-dimensional simplex has $(n + 1)$ vertices. So a 0-word is just a finite sequence of identical characters like $aaaaaa$ characterized only by a natural number - its length.) Denote by $IW(n)$ the set of all $n$-words. Consider boundary operators $IW(n) \overset{\delta_i}{\rightarrow} IW(n - 1), i = 0, ..., n$, corresponding to deletion of all entrances of $i$-th character from the word. For example: consider the word $bcabbccacb$ in the ordered alphabet $a > b > c$. Then $bcabbccacb \in IW(2), \delta_0(bcabbccacb) = bcbbcccb \in IW(1)$ in alphabet $b > c$, $\delta_1(bcabbccacb) = caccac \in IW(1)$ in alphabet $a > c$, $\delta_2(bcabbccacb) = babbab \in IW(1)$ in alphabet $a > b$.

Elementary simplicial oriented $I$-bundle ("elementary $I$-bundle") over an ordered $n$-simplex is a map $\mathcal{F} \overset{q}{\rightarrow} [\Delta^n]$ of simplicial complexes, such that $|\mathcal{F}| \overset{|q|}{\rightarrow} \Delta^n$ is a piecewise linear $I$ fiber bundle, where $I$ is ordered interval (see Figure 2). Let $IB(n)$ be the set of all elementary $I$-bundles over ordered $n$-simplices. Endowed with natural boundary operators elementary simplicial oriented $I$-bundles forms a semi-simplicial set $IB$.

We will establish a canonical isomorphism between the semi-simplicial sets $IW$ and $IB$ (See Figure 2 for the illustration of the correspondence.) Pick a bundle $q \in IB(n)$, $\mathcal{F} \overset{q}{\rightarrow} [\Delta^n]$. The total simplicial complex $\mathcal{F}$ has a dimension $(n + 1)$. Consider a simplex $\sigma \in \mathcal{F}_{n+1}$ of maximal dimension. The simplicial map $\sigma \overset{q}{\rightarrow} [\Delta^n]$ is onto by dimension reasons and hence there is exactly one vertex $V(\sigma)$ of the base simplex $[\Delta^n]$ such that its preimage is a 1-dimensional face of $\sigma$ and for all other vertices the preimage is a single vertex. We got a map $\mathcal{F}_{n+1} \overset{V}{\rightarrow} [\Delta^n]_0$. Now look at the geometric realization of the bundle $|\mathcal{F}| \overset{|q|}{\rightarrow} \Delta^n$, pick up a point $x \in \text{int} \Delta^n$ from the interior of the base and consider the fiber over this point - an oriented interval $I_x = \text{int} \Delta^n \overset{\iota}{\rightarrow} \Delta^n$.

The fiber $I_x$ intersects the interior of every $n + 1$-dimensional simplex $|\sigma| \in |\mathcal{F}|_{n+1}$ by an open interval $\iota(\sigma)$. All this intervals are disjoint and hence totally ordered by the orientation of $I_x$. Marking such an interval $\iota(\sigma)$ by the vertex $V(\sigma)$ of the base we got a $n + 1$ word $\mathcal{W}_I(p)$ in the ordered alphabet consisting of names of the vertices of the base. When we move the point $x$ to the the point in the interior of $i$-th face $\delta_i([\Delta^n])$ of base simplex all intervals in $I_x$ marked by the vertex number $i$ disappeared and other preservers order. So the map of semi-simplicial sets $IB \overset{\mathcal{W}_I}{\rightarrow} IW$ is defined. Mention that the word $\mathcal{W}_I(p)$ is a simple case of "shelling" for
combinatorial balls. Here it is a special shelling of the total complex which induces shellings on all bundle-faces of the complex.

Now let we have an n-word \( w \) we suppose to construct an \( I \)-bundle \( B_I(w) \). Call by "interval" in \( w \) any subword which is contained between two characters. The single characters are also "intervals".

Let \( a^0, \ldots a^n \) be our ordered alphabet. Let the character \( a^i \) has \( k_i \) entrances \( a^i_1 \ldots a^i_{k_i} \) ordered from right to left by appearance in the word \( w \). Now call by \( a^i \)-interval an interval of \( w \) which either stars from the beginning of \( w \) and ends at character \( a^i_1 \) or starts from \( a^i_{k_i} \) and ends at the end of \( w \), or starts from \( a^i_j \) and ends at \( a^i_{j+1} \) for \( j = 1, \ldots k_i - 1 \). Consider the set \( E^i_0(w) \) -the set of \( a^i \) intervals in \( w \). Each \( E^i_0(w) \) is a covering of the word \( w \) by intervals. Put \( E_0(w) = \bigcup_{i=0}^{n} E^i_0(w) \). Consider the Chech simplicial complex \( E(w) \) of the covering \( E_0(w) \). The \((k - 1)\)-simplex of \( E(w) \) a set of \( k \) intervals from \( E_0(w) \) having nonempty intersection. So intervals from \( E_0(w) \) are the vertices of \( E(w) \). The simplicial map \( E(w) \xrightarrow{B_I(w)} \Delta^n \) is induced by the map sending \( a^i \)-interval (the vertex of \( E(w) \)) to the vertex with number \( i \) of the base simplex. The simplicial map \( B(w) \) is an elementary oriented combinatorial \( I \) bundle. So the map of semi-simplicial sets \( IW \xrightarrow{B_I} IB \) is defined.

One may check that \( W_I B_I(w) = w, B_I W_I(q) = q \) and the correspondence \( IB \xrightarrow{W_I} IW \) is an isomorphism of semi-simplicial sets.

One can place a word from \( IW \) on an oriented circle and thus got a cyclic word. Cyclic words in the alphabet made from ordered characters forms a semi-simplicial set \( SW \) exactly by the same construction as for \( IW \). The deep process of placing on an oriented circle generates a
map of semi-simplicial sets $IW \xrightarrow{W_U} SW$. It is useful to consider simultaneously a cyclic word as an orbit of action of cyclic shift on the ordinary words$^1$.

We will formulate a false statement: If we have an elementary $I$ bundle then we can glue together the top and bottom simplices and get a $S$-elementary bundle. The statement is wrong in our definition of elementary $S$ bundle since we required that they are maps of simplicial complexes. We should have not less then four vertices in the fibers over each vertex of base to get a simplicial complex. Generally if we have less points we will get a very nice simplicial cell complex. We will ignore this trouble since everything is valid in this more general situation and if one want to stay in simplicial complexes one may everywhere change the simplicial sets $IB$ $IW$ and $SW$ to simplicial sets $IB$ $IW$ and $SW$ where the words of $IW$ and $SW$ have no less the 3 entrances of each character and bundles from $IB$ have no less then 4 vertices in each fiber over a base vertex. Anyway such a gluing produces a map $IB \xrightarrow{U_W} SB$. As in words situation it is useful to identify a $S$-bundle with an orbit of cyclic shift acting on $I$-bundles. The action is following. The simplicial sections of an elementary $I$-bundle are ordered by the orientation of fiber. The action of cyclic shift is to glue together the first and the last sections and to cut the resulting complex by the second section (see Figure 3). The cyclic shift action commute with maps $IB \xrightarrow{W_I} IW$ and so we obtain an induced canonical isomorphism of semi-simplicial sets $SB \xrightarrow{W_S} SW$. We have a commutative diagram

$$
\begin{array}{ccc}
IB & \xrightarrow{W_I} & IW \\
\downarrow U_B & \quad & \downarrow U_W \\
SB & \xrightarrow{W_S} & SW
\end{array}
$$

0.3 The formula - combinatorial curvature

We suppose to define here a cocycle $\text{Curv} \in SB^2$.

Now we can identify the cochain complex $SB^\bullet$ with a subcomplex of cochains from $IB^\bullet$ which are invariant under the cyclic shift action on basis. Analogously we define a cochain cochain complexes $IW^\bullet$ and its subcomplex $SW^\bullet$ of invariant under the cyclic shift cochains. The commutative diagram (2) produces the following commutative diagram of based cochain

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$^1$One can naturally extend the action of cyclic permutations on an individual word to the action of cyclic semi-simplicial groupoid $CW$ on $IW$. This turns the map $U_W$ into a correct discrete replacement of the universal $U(1)$-bundle over $\mathbb{C}P^\infty$ and the groupoid $CW$ into a discrete replacement of $U(1)$ as a Lie group. We will not touch this subject here.
Figure 3: Cyclic shift of $I$-bundle

First we will define a cochain $\text{Ind} \in IW^1 \approx IB^1$. The word from $w \in IW(1)$ is a word in ordered alphabet having two characters. We can correspond to the word $w$ a sort of Young diagram $Y(w)$ as follows. (See Figure [for an illustration.]) Let $a > b$ be a 2-character ordered alphabet and $w$ is a word in this alphabet. Number consequent entrances of $a$ in $w$ by $a_1...a_{k_0}$ and number consequent entrances of $b$ in $w$ by $b_1...b_{k_1}$ (thus our word became a what is called "shuffle"). Draw on white checked paper a rectangle with $k_0$ rows numbered by $a_1...a_{k_0}$ and $k_1$ columns numbered by $b_1...b_{k_1}$. Paint in black a cell $i,j$ if $a_i$ stays to the left from $b_j$ in the word $w$. We got a Young diagram $Y(w)$ drawn in $k_0 \times k_1$ rectangle. Now define a rational number $\text{Ind}(w)$

$$\text{Ind}(w) = \frac{(\# \text{ black squares}) - (\# \text{ white squares})}{2k_0k_1} = \frac{(\# \text{ black squares})}{k_0k_1} - \frac{1}{2}$$

For the elementary bundle $p \in IB(1)$ put $\text{Ind}(p) = \text{Ind}(\mathcal{W}_i(p))$. The cochain $\text{Ind} \in IW^1(\subseteq IB^1)$ is not invariant under the action of cyclic shifts on words (bundles). So it is not belong to $SW^1$ (SB$^1$). Now define

$$\text{Curv} = d(\text{Ind}) \in IW^2(\subseteq IB^2)$$
The coordinates expression is following:

For 3-character word in ordered alphabet \( w \in IW(2) \)

\[
\text{Curv}(w) = \text{Ind}(\delta_0 w) - \text{Ind}(\delta_1 w) + \text{Ind}(\delta_2 w) \in IW^2
\]  

and for \( b \in IB(2) \):

\[
\text{Curv}(p) = \text{Curv}(W_I(p)) \in IB^2
\]

**Theorem 0.1.** *The cochain Curv is invariant under cyclic shifts, so it belongs to SW^2 (SB^2), it is a cocycle in SW^\bullet (SB^\bullet), representing a local combinatorial formula for the first Chern class.*

**0.4 Proof by check**

Having guessed the formula for Curv we can prove Theorem 0.1 by direct check.

First of all we mention that cochain subcomplex SW^2 of IW^2 inherits the differential and hence, if \( \text{Curv} \in SW^2 \) then it is a cocycle there, because it is defined as a coboundary in IW^2.

Now we will check the result of cyclic shift action on Curv. We will inspect the behavior of expression (6) under a cyclic shift. Let \( w \in IW(2) \), \( a^0 > a^1 > a^2 \) be our ordered alphabet and \( a_0^0...a_k^0, a_1^0...a_k^1, a_1^2...a_k^2 \) be the entrances of characters in the word \( w \) in the order of the word \( w \). First let us see what happen when we change the order on the alphabet. The trivial observation that the only result of acting by a permutation an the alphabet - the changing the sign of Curv on parity of the permutation.

After this observation we may suppose that the first character of \( w \) is \( a_0^0 \). Consider the shifted word \( \overline{w} \), where the first character \( a_0^0 \) has jumped to the end. Compare the summands of expression (6) for \( \text{Curv}(w) \) and \( \text{Curv}(\overline{w}) \). The summand number 0 is not affected at all by the shift since the words \( \delta_0 w = \delta_0 \overline{w} \). Other two summands are affected:

\[
\text{Ind}(\delta_i w) = \text{Ind}(\delta_i w) - \frac{1}{k_0}, i = 1, 2
\]
So the adjustment term \( \frac{1}{k_0} \) comes with opposite signs into expression (6) for \( \text{Curv}(w) \) and hence \( \text{Curv}(w) = \text{Curv}(\overline{w}) \).

So \( \text{Curv} \in \text{SW}^2 \) and it is a cocycle.

Now we can make a reference to transcendental but classic results of geometric topology that tells us that the cochain complex \( \text{SW}^\bullet \) is quasiisomorphic to singular cochain complex of \( \mathbb{C}P^2 \approx BU(1) \). First one can use the result that \( BU(1) \approx BPL(S^1) \) (see [1]) for the survey. Than one can add degenerations to semi-simplicial set \( SB \) and got a simplicial set \( \widetilde{SB} \) having homotopy and which classifies PL bundles with an oriented fiber \( S^1 \). This is a form of Whitehead theorem on functorial triangulations of fiber bundles coupled as always with Zeeman’s relative simplicial approximation theorem. The semi-simplicial set \( SB \) is just the set of nongenerate simplices of \( \widetilde{SB} \) and as such those ordered cohomology complexes are quasiisomorphic.

We know the cohomology group \( H^2(BU(1), \mathbb{Q}) \) it is one-dimensional and generated by the first Chern class \( c_1 \). So same is true for \( H^2(\text{SW}^\bullet) \). To check that the cocycle \( \text{Curv} \) represents Chern class, it is sufficient to verify that it is not a coboundary and that it is correctly normalized. Both goal will be archived if using \( \text{Curv} \) we will successfully compute Chern number of, for example, Hopf fibration. This calculation is simple if some triangulation of the Hopf fibration is granted. We may take a triangulation from [4] with a base \( \partial[\Delta^3] \) - a standard face triangulation of the boundary of \( \Delta^3 \). Applying the map \( \mathcal{W}_S \) to the elementary bundles from the paper we got the coloring of \( \partial[\Delta^3] \) by cyclic words presented on Figure 5. Here a sign on an ordered simplex is a sign of the simplex in the fundamental cycle - its orientation relatively to the global orientation. Calculating the value of cocycle \( \text{Curv} \) on the fundamental cycle of \( \partial[\Delta^3] \) we
are happily obtaining a hoped for number 1 - the Chern number of the Hopf fibration. This finishes proof of the Theorem 0.1 by formal check.

0.5 The partial demystification of the formula – integration of Kontsevich’s form

The demystification is following and very simple. Take a simplicial $S^1$-bundle. Pick up a the standard geometric realization with a standard metrics on simplices. As a result the bundle got a canonical combinatorially locally defined structure of $U(1)$ bundle - Kontsevich’s metric polygon bundle (see [3, §2.2]), together with a canonical, ruled by combinatorics, $U(1)$-invariant connection [3, p.8 line 22 from the top]. Integration by base of elementary 2-subbundle of the curvature 2-form [3, p.8 line 26 from the top] of this connection, which is elementary integral, gives us exactly our local formula. If a simplicial bundle triangulates some differential $U(1)$-bundle then the two $U(1)$-bundle structures are isomorphic. The demystification is partial because the combinatorics and arithmetics which jump right out of a jukebox is a bit mystery.

0.6 Conclusion

The first remark is philosophical: For $S^1$-bundles the construction literally demonstrates a thesis for combinatorial-differential topology "triangulation is a connection". We hope to show in forecoming joint work with G. Sharygin that for general combinatorial replacements of vector bundles the a refined thesis "triangulation is a flat superconnection" is exactly the same useful for local combinatorial formulas for characteristic classes.

The second remark is that the the relation between the combinatorics of elementary $S$-bundle (cyclic word) and the value of its curvature is a mystery. The curvature takes rational value in the interval $[-\frac{1}{2}, ... \frac{1}{2}]$. All the rational numbers can show up (it is doubtful) or some of them? The values $\pm \frac{1}{2}$ are archived on boring words of type $abc, aaaaabbbbbc$ - where all 3 characters are joined in three monolith blocks. Value 0 the curvature has on "cyclic palindromes" - the words which are equal to its mirror up to cyclic shift. Is the word a cyclic palindrome if its curvature is 0?

The Theorem 0.1 with any proof - by check or by integration of Kontsevich’s form tells us that Curv is a representative of a Chern cocycle. Generally the Chern cocycle is defined as any cocycle only up to a shift on coboundary. Is Curv distinguished somehow? As a granted we have two extra axioms valid for Curv: it changes sign with odd permutation of alphabet and it changes sign on a cyclic-mirror word. How much coboundary shifts kills this two extra conditions? It kills some. We can prove that a Chern cocycle on words of length 3, which satisfies the two mentioned conditions can be only the Curv. The proof is an exercise on painting facets of tetrahedron by cyclic words of length 3 to get a tangent bundle for 2-sphere. Symmetries will give us a uniqueness for the value.

Some numeric experiments shows evidence that Curv is a harmonic cocycle relatively to a natural scalar multiplication on $SB^\bullet(SW^\bullet)$. It would be nice to get an electrodynamics understanding of such a scalar multiplications.
We supply a Maple procedure for computing the curvature

\begin{verbatim}
Curv:=proc(a,b,c,S)
# The curvature of 3-character word.
# Using: Curv(first character,second character, third character, word )
local Index, L, Lab,Lac,Lbc,i;
Index := proc(a,b, S)
local L, k_0, k_1, i,j, Ind;
L:=convert(convert(S,string),list); k_0 :=0; k_1 := 0; Ind :=0;
for i from 1 to nops(L) do
  if L[i] = convert(a,string) then k_0:= k_0 + 1;
    for j from 1 to nops(L) do
      if L[j] = convert(b,string) then
        if j < i then Ind := Ind +1 fi; if j>i then Ind:=Ind-1; fi
      fi
    od fi od; k_1 := nops(L) - k_0; Ind/(2*k_0*k_1) end proc;
L:=convert(convert(S,string),list); Lab :=""; Lac:=""; Lbc="";
for i from 1 to nops(L) do
  if L[i] = convert(a,string) then Lab:=cat(Lab,L[i]); Lac:=cat(Lac,L[i]); fi;
  if L[i] = convert(b,string) then Lab:=cat(Lab,L[i]); Lbc:=cat(Lbc,L[i]); fi;
  if L[i] = convert(c,string) then Lac:=cat(Lac,L[i]); Lbc:=cat(Lbc,L[i]); fi;
od;
Index(b,c,Lbc)-Index(a,c,Lac)+Index(a,b,Lab) end proc:

Examples

2:
Sell less eels! (“kdv2005”)
Curv(s,e,l,sellessesels);

-1
--
16

Cat tact act (“kdv2005”):
Curv(c,a,t,cattactact);

1
--
18

Dad rescued a puppy (in Russian: papa spas psa) (“rus4”):
Curv(a,p,s,papaspaspsa);

1
--
24
\end{verbatim}

\footnote{I am deeply grateful to my LiveJournal.com friends with niknames “kdv2005”, “rus4” and “tatjaana” for this examples.}
Fooling Google (in Russian: “lgu guglu” - sounds poetic and this is a cyclic palindrome) (“tatjaana”):

\[ \text{Curv}(l,g,u,lgugugu); \]

0

References

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