Taut-distance-regular graphs and the subconstituent algebra

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Abstract

We consider a bipartite distance-regular graph $\Gamma$ with diameter $D \geq 4$ and valency $k \geq 3$. Let $X$ denote the vertex set of $\Gamma$ and fix $x \in X$. Let $\Gamma^2_x$ denote the graph with vertex set $\hat{X} = \{y \in X \mid \partial(x, y) = 2\}$, and edge set $\hat{R} = \{yz \mid y, z \in \hat{X}, \partial(y, z) = 2\}$, where $\partial$ is the path-length distance function for $\Gamma$. The graph $\Gamma^2_x$ has exactly $k_2$ vertices, where $k_2$ is the second valency of $\Gamma$. Let $\eta_1, \eta_2, \ldots, \eta_{k_2}$ denote the eigenvalues of the adjacency matrix of $\Gamma^2_x$; we call these the local eigenvalues of $\Gamma$. Let $A$ denote the adjacency matrix of $\Gamma$. We obtain upper and lower bounds for the local eigenvalues in terms of the intersection numbers of $\Gamma$ and the eigenvalues of $A$. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A, E^*_0, E^*_1, \ldots, E^*_D$, where for $0 \leq i \leq D$, $E^*_i$ represents the projection onto the $i$th subconstituent of $\Gamma$ with respect to $x$. We refer to $T$ as the subalgebra (or Terwilliger algebra) of $\Gamma$ with respect to $x$. An irreducible $T$-module $W$ is said to be thin whenever $\dim E^*_i W \leq 1$ for $0 \leq i \leq D$. By the endpoint of $W$ we mean $\min \{ i | E^*_i W \neq 0 \}$. We give a detailed description of the thin irreducible $T$-modules that have endpoint 2 and dimension $D - 3$. In [Discrete Math., 225(2000), 193–216] MacLean defined what it means for $\Gamma$ to be taut. We obtain three characterizations of the taut condition, each of which involves the local eigenvalues or the above $T$-modules.

Keywords: Distance-regular graph, association scheme, Terwilliger algebra, subconstituent algebra.

1 Introduction

Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and intersection numbers $a_i, b_i, c_i$ (see Section 2 for formal definitions). In this paper we obtain some results on the subconstituent algebra $\mathcal{S}_t$ of $\Gamma$, and some related results concerning the taut condition [22]. We will state our results shortly, but first we motivate things with a brief discussion of the subconstituent algebra. Let $X$ denote the vertex set of $\Gamma$ and fix $x \in X$. We view $x$ as a “base vertex.” Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A, E^*_0, E^*_1, \ldots, E^*_D$, where $A$ denotes the adjacency matrix of $\Gamma$ and $E^*_i$ represents the projection onto the $i$th subconstituent of $\Gamma$ with respect to $x$. The algebra $T$ is called the subconstituent algebra (or Terwilliger algebra) of $\Gamma$ with respect to $x$ [35]. Observe $T$ has finite dimension. Moreover $T$ is semi-simple; the reason is each of $A, E^*_0, E^*_1, \ldots, E^*_D$ is symmetric with real entries, so $T$ is closed under the conjugate-transpose map [10] p. 157]. Since $T$ is semi-simple, each $T$-module is a direct sum of irreducible $T$-modules. Describing the irreducible $T$-modules is an active area of research [44, 49, 51–54, 56, 59, 63, 65, 68, 71].

In this paper we are concerned with the irreducible $T$-modules that possess a certain property. In order to define this property we make a few observations. Let $W$ denote an irreducible $T$-module. Then $W$ is the direct sum of the nonzero spaces among $E^*_0 W, E^*_1 W, \ldots, E^*_D W$. There is a second decomposition of interest. To obtain it we make a definition. Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of $A$, and for $0 \leq i \leq D$ let $E_i$ denote the primitive idempotent of $A$ associated with $\theta_i$. Then $W$ is the direct sum of the nonzero spaces among $E_0 W, E_1 W, \ldots, E_D W$. If the dimension of $E^*_i W$ is at most 1 for $0 \leq i \leq D$ then the dimension of $E_i W$ is at most 1 for $0 \leq i \leq D$ [35] Lemma 3.9; in this case we say $W$ is thin. Let $W$ denote an irreducible $T$-module. By the endpoint of $W$ we mean $\min \{ i | 0 \leq i \leq D, E^*_i W \neq 0 \}$. There exists a unique irreducible $T$-module with endpoint 0 [12] Proposition 8.4. We call this module $V_0$. The module $V_0$ is thin; in fact $E^*_i V_0$ and $E_i V_0$ have dimension 1 for $0 \leq i \leq D$ [35] Lemma 3.6. For a detailed description of $V_0$ see [6, 12].
For the rest of this section assume $\Gamma$ is bipartite. There exists, up to isomorphism, a unique irreducible $T$-module with endpoint 1 [6 Corollary 7.7]. We call this module $V_1$. The module $V_1$ is thin; in fact each of $E_i^1 V_1$, $E_i V_1$ has dimension 1 for $1 \leq i \leq D -1$ and $E_i^2 V_1 = 0$, $E_0 V_1 = 0$, $E_D V_1 = 0$. For a detailed description of $V_1$ see [6]. In this paper we are concerned with the thin irreducible $T$-modules with endpoint 2.

We now state our results. In order to state our first result we define some parameters. Let $\Gamma_2^2 = \Gamma_2^2(x)$ denote the graph with vertex set $\bar{X}$ and edge set $\bar{R}$, where

$$\bar{X} = \{ y \in X \mid \partial(x,y) = 2 \},$$

$$\bar{R} = \{ yz \mid y, z \in \bar{X}, \partial(y,z) = 2 \},$$

and where $\partial$ is the path-length distance function for $\Gamma$. The graph $\Gamma_2^2$ has exactly $k_2$ vertices, where $k_2$ is the second valency of $\Gamma$. Also, $\Gamma_2^2$ is regular with valency $p_{22}$. We let $\eta_1, \eta_2, \ldots, \eta_{k_2}$ denote the eigenvalues of the adjacency matrix of $\Gamma_2^2$. By [22] Theorem 11.7, these eigenvalues may be ordered such that $\eta_1 = \rho_{22}^2$ and $\eta_i = b_i - 1$ ($2 \leq i \leq k$).

Abbreviate $d = \lfloor D/2 \rfloor$. Our first main result is that $\tilde{\theta}_1 \leq \theta_i \leq \tilde{\theta}_d$ for $1 \leq i \leq k_2$, where $\tilde{\theta}_1 = -1 - b_3 b_4 (\theta_{22}^2 - b_2)^{-1}$ and $\tilde{\theta}_d = -1 - b_3 b_4 (\theta_{22}^2 - b_2)^{-1}$. We remark $\theta_{22}^2 > b_2 > \theta_{22}^2$ by [22] Lemma 2.6, so $\tilde{\theta}_1 < -1$ and $\tilde{\theta}_d \geq 0$.

In order to state our next set of results we make a definition. Let $W$ denote a thin irreducible $T$-module with endpoint 2. Observe $E_2^2 W$ is a 1-dimensional eigenspace for $E_2^2 A_2 E_2^2$; let $\eta$ denote the corresponding eigenvalue. It turns out $\eta$ is one of $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$ so $\tilde{\eta}_1 \leq \eta \leq \tilde{\eta}_d$. We call $\eta$ the local eigenvalue of $W$. To describe the structure of $W$ we distinguish four cases: (i) $D$ is odd, and $\eta = \theta_1$ or $\eta = \theta_d$; (ii) $D$ is even and $\eta = \theta_1$; (iii) $D$ is even and $\eta = \theta_d$; (iv) $\tilde{\eta}_1 < \eta < \tilde{\eta}_d$. We investigate cases (i), (ii) in the present paper. We will investigate the remaining cases in a future paper.

Our results concerning the $T$-modules are as follows. Choose $n \in \{1, d\}$ if $D$ is odd, and let $n = 1$ if $D$ is even. Define $\eta = \tilde{\theta}_n$. Let $W$ denote a thin irreducible $T$-module with endpoint 2 and local eigenvalue $\eta$. Let $v$ denote a nonzero vector in $E_{2n}^2 W$. We show $W$ has a basis $E_i v$ ($1 \leq i \leq D - 1$, $i \neq n$, $i \neq D - n$). We show this basis is orthogonal (with respect to the Hermitian dot product) and we compute the square norm of each basis vector. We show $W$ has a basis $E_i A_i v$ ($0 \leq i \leq D - 4$), where $A_i$ denotes the $i$th distance matrix for $\Gamma$. We find the matrix representing $A$ with respect to this basis. We show this basis is orthogonal and we compute the square norm of each basis vector. We find the transition matrix relating our two bases for $W$. We show the following scalars are equal: (i) The multiplicity with which $\eta$ appears in the standard module $\mathbb{C}^X$; (ii) The number of times $\eta$ appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$.

In order to state our remaining results we recall the taut condition. In [23] Theorem 18] Curtin showed that $b_2 (k - 2) \geq (c_2 - 1) \theta_{22}^2$ with equality if and only if $\Gamma$ is 2-homogeneous in the sense of Nomura [25]. In [8] Theorem 12] Curtin showed $\Delta \geq 0$, where $\Delta = (k - 2) (c_3 - 1) - (c_2 - 1) p_{22}^2$. In [22] Lemma 3.8] MacLean proved that

$$b_3 (b_2 (k - 2) - (c_2 - 1) \theta_{22}^2) (b_2 (k - 2) - (c_2 - 1) \theta_{22}^2) \geq b_1 \Delta (\theta_{22}^2 - b_2) (\theta_{22}^2 - \theta_{22}^2). \quad (1)$$

We mentioned earlier that $\theta_{22}^2 > b_2 > \theta_{22}^2$, so the last two factors on the right in (1) are positive. Observe each factor in (1) is nonnegative. From these comments we find that $\Gamma$ is 2-homogeneous if and only if $\Delta = 0$ and equality holds in (1). MacLean defined $\Gamma$ to be taut [22] whenever $\Delta \neq 0$ and equality holds in (1).

Assume for the moment that $\Gamma$ is taut. It turns out that the structure of $\Gamma$ depends to a large extent on the parity of $D$. We investigated this structure for $D$ even in [23], and for $D$ odd in [24]. In this paper we obtain three characterizations of the taut condition, two of which require the assumption that $D$ is odd.

In order to state our first characterization of the taut condition we make a definition. We say $\Gamma$ is spectrally taut with respect to $x$ whenever $\eta_i$ is one of $\theta_1, \theta_d$ for $1 \leq i \leq k_2$. We show the following are equivalent:
(i) $\Gamma$ is taut; (ii) $\Delta \neq 0$ and $\Gamma$ is spectrally taut with respect to each vertex; (iii) $\Delta \neq 0$ and $\Gamma$ is spectrally taut with respect to at least one vertex.

For the rest of this section assume $D$ is odd. We obtain two additional characterizations of the taut condition. In order to state the first one we make a definition. We say $\Gamma$ is taut with respect to $x$ whenever every irreducible $T$-module with endpoint 2 is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_d$. We show the following are equivalent: (i) $\Gamma$ is taut; (ii) $\Delta \neq 0$ and $\Gamma$ is taut with respect to each vertex; (iii) $\Delta \neq 0$ and $\Gamma$ is taut with respect to at least one vertex.

Before we state our last characterization we recall two concepts. Recall that $\Gamma$ is an antipodal 2-cover and 2-thin with respect to at least one vertex.

For more information on the taut condition and related topics we refer the reader to \[1\], \[3\], \[15\] or \[35\].

\section{Preliminaries concerning distance-regular graphs}

In this section we review some definitions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to \[1\], \[3\], \[15\] or \[35\].

Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We endow $V$ with the Hermitian inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle u, v \rangle = u^T \overline{v} \quad (u, v \in V),$$

where $t$ denotes transpose and $-$ denotes complex conjugation. As usual, we abbreviate $\|u\|^2 = \langle u, u \rangle$ for all $u \in V$. For all $y \in X$, let $\hat{y}$ denote the element of $V$ with a 1 in the $y$ coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for $V$. The following formula will be useful. For all $B \in \text{Mat}_X(\mathbb{C})$ and for all $u, v \in V$,

$$\langle Bu, v \rangle = \langle u, B^T v \rangle.$$  \hspace{1cm} (3)

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $R$. Let $\partial$ denote the path-length distance function for $\Gamma$, and set $D = \max \{\partial(x, y) \mid x, y \in X\}$. We refer to $D$ as the diameter of $\Gamma$. We write $\lfloor D/2 \rfloor$ to denote the greatest integer at most $D/2$. Let $x, y$ denote vertices of $\Gamma$. We say $x, y$ are adjacent whenever $xy$ is an edge. Let $k$ denote a nonnegative integer. We say $\Gamma$ is regular with valency $k$ whenever each vertex of $\Gamma$ is adjacent to exactly $k$ distinct vertices of $\Gamma$.

We say $\Gamma$ is distance-regular whenever for all integers $h, i, j$ ($0 \leq h, i, j \leq D$) and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of $x$ and $y$. The integers $p^h_{ij}$ are called the intersection numbers of $\Gamma$. We abbreviate $c_i = p^1_{i, i}$, $a_i = p^i_{i, i}$ ($0 \leq i \leq D$), and $b_i = p^i_{i, i+1}$ ($0 \leq i \leq D - 1$). For notational convenience we define $c_0 = 0$ and $b_D = 0$. We note $a_0 = 0$ and $c_1 = 1$.

For the rest of this paper we assume $\Gamma$ is distance-regular with diameter $D \geq 3$.

By \[1\] and the triangle inequality, $p^h_{ij} = 0$ if one of $h, i, j$ is bigger than the sum of the other two ($0 \leq h, i, j \leq D$). Observe $\Gamma$ is regular with valency $k = b_0$, and that $c_i + a_i + b_i = k$ ($0 \leq i \leq D$). Moreover $b_i > 0$ ($0 \leq i \leq D - 1$) and $c_i > 0$ ($1 \leq i \leq D$).
For $0 \leq i \leq D$ we abbreviate $k_i = p_i^0$, and observe

$$k_i = |\{z \in X \mid \partial(x, z) = i\}|,$$

where $x$ is any vertex in $X$. Apparently $k_0 = 1$ and $k_1 = k$. By [3, p. 128] we have

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D).$$

We refer to $k_i$ as the $i$th valency of $\Gamma$.

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $xy$ entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call $A_i$ the $i$th distance matrix of $\Gamma$. For convenience we define $A_i = 0$ for $i < 0$ and $i > D$. We abbreviate $A = A_1$ and call this the adjacency matrix of $\Gamma$. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^D A_i = J$; (aiii) $\overline{A}_i = A_i$ ($0 \leq i \leq D$); (av) $A_i A_j = \sum_{k=0}^D p_k^0 A_k$ ($0 \leq i, j \leq D$), where $I$ denotes the identity matrix and $J$ denotes the all $1$'s matrix. Let $M$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$. Using (ai) and (av) one can readily show $A_0, A_1, \ldots, A_D$ form a basis for $M$. We refer to $M$ as the Bose-Mesner algebra of $\Gamma$. By [3, p. 45] $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $\sum_{i=0}^D E_i = I$; (eiii) $E_i E_j = E_i$ ($0 \leq i \leq D$); (ev) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$). We refer to $E_0, E_1, \ldots, E_D$ as the primitive idempotents of $\Gamma$. We call $E_0$ the trivial idempotent of $\Gamma$.

We recall the eigenvalues of $\Gamma$. Since $E_0, E_1, \ldots, E_D$ form a basis for $M$, there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^D \theta_i E_i$. Combining this with (ev) we find $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. Using (aiii) and (eii) we find $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe $\theta_0, \theta_1, \ldots, \theta_D$ are distinct since $A$ generates $M$. By [21 Proposition 3.1] we have $\theta_0 = k$ and $-k \leq \theta_i \leq k$ for $0 \leq i \leq D$. Throughout this paper we assume $E_0, E_1, \ldots, E_D$ are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_D$. We refer to $\theta_i$ as the eigenvalue of $\Gamma$ associated with $E_i$. For $0 \leq i \leq D$ let $m_i$ denote the rank of $E_i$. We refer to $m_i$ as the multiplicity of $E_i$ (or $\theta_i$). From (ei) we find $m_0 = 1$. Using (eii)–(ev) we find

$$V = E_0 V + E_1 V + \cdots + E_D V$$

(orthogonal direct sum). (7)

For $0 \leq i \leq D$ the space $E_i V$ is the eigenspace of $A$ associated with $\theta_i$. We observe the dimension of $E_i V$ is $m_i$.

We now recall the dual eigenvalues of $\Gamma$. Let $\theta$ denote an eigenvalue of $\Gamma$ and let $E$ denote the associated primitive idempotent. Since $A_0, A_1, \ldots, A_D$ is a basis for $M$, there exist complex scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ such that

$$E = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i.$$

Evaluating [3] using (aiii) and (eii) we see $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are in $\mathbb{R}$. We refer to $\theta_i^*$ as the $i$th dual eigenvalue of $\Gamma$ with respect to $E$ (or $\theta$). We call $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ the dual eigenvalue sequence associated with $E$ (or $\theta$). By [3, p. 128] we have

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta^* \theta_i^* \quad (0 \leq i \leq D),$$

where $\theta_{-1}^*, \theta_{D+1}^*$ are indeterminates. We remark by [11, p. 62] that $\theta_0^* = m_i$ where $\theta = \theta_i$.

The following lemma will be useful.

**Lemma 2.1** [21 Lemma 2.6] Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Then
(i) \(-1 < \theta_1 < k\).
(ii) \(a_i - k \leq \theta_D < -1\).

Later in this paper we will discuss polynomials in one or two variables. We will use the following notation. We let \(\lambda\) denote an indeterminate. We let \(\mathbb{R}[\lambda]\) denote the \(\mathbb{R}\)-algebra consisting of all polynomials in \(\lambda\) that have coefficients in \(\mathbb{R}\). We let \(\mu\) denote an indeterminate that commutes with \(\lambda\). We let \(\mathbb{R}[\lambda, \mu]\) denote the \(\mathbb{R}\)-algebra consisting of all polynomials in \(\lambda\) and \(\mu\) that have coefficients in \(\mathbb{R}\).

3 Bipartite distance-regular graphs

We now consider the case in which \(\Gamma\) is bipartite. We say \(\Gamma\) is bipartite whenever the vertex set \(X\) can be partitioned into two subsets, neither of which contains an edge. In the next few lemmas, we recall some routine facts concerning the case in which \(\Gamma\) is bipartite. To avoid trivialities, we will generally assume \(D \geq 4\).

Lemma 3.1 Let \(\Gamma\) denote a distance-regular graph with diameter \(D \geq 4\), valency \(k\), and eigenvalues \(\theta_0 > \theta_1 > \cdots > \theta_D\). The following are equivalent:

(i) \(\Gamma\) is bipartite.
(ii) \(p_{ij}^h = 0\) if \(h + i + j\) is odd \((0 \leq h, i, j \leq D)\).
(iii) \(a_i = 0\) \((0 \leq i \leq D)\).
(iv) \(c_i + b_i = k\) \((0 \leq i \leq D)\).
(v) \(\theta_{D-i} = -\theta_i\) \((0 \leq i \leq D)\).

Lemma 3.2 Let \(\Gamma\) denote a bipartite distance-regular graph with diameter \(D \geq 4\) and eigenvalues \(k = \theta_0 > \theta_1 > \cdots > \theta_D\).

(i) Assume \(D\) is even and let \(d = D/2\). Then \(\theta_d = 0\).
(ii) Assume \(D\) is odd and let \(d = (D - 1)/2\). Then \(\theta_d > 0\) and \(\theta_{d+1} = -\theta_d\).

Proof. Immediate from Lemma 3.1(v). \(\Box\)

Lemma 3.3 Let \(\Gamma\) denote a bipartite distance-regular graph with diameter \(D \geq 4\). Let \(\theta\) denote an eigenvalue of \(\Gamma\) and let \(\theta_0^*\), \(\theta_1^*\), \ldots, \(\theta_k^*\) denote the corresponding dual eigenvalue sequence. Then the dual eigenvalue sequence associated with \(-\theta\) is \(\theta_0^*, -\theta_1^*, \theta_2^*, \ldots, (1)^D \theta_D^*\).

Lemma 3.4 Let \(\Gamma = (X, R)\) denote a bipartite distance-regular graph with diameter \(D \geq 4\), and eigenvalues \(\theta_0 > \theta_1 > \cdots > \theta_D\). Then \(E_D = |X|^{-1} J'\), where

\[
J' = \sum_{i=0}^{D} (-1)^i A_i. \tag{10}
\]

Proof. By (a(ii), e(i)), we have \(E_0 = |X|^{-1} \sum_{i=0}^{D} A_i\). Combining this with \(\Box\) and Lemma 3.3 we find \(E_D = |X|^{-1} \sum_{i=0}^{D} (-1)^i A_i\). The result follows. \(\Box\)

Lemma 3.5 Let \(\Gamma\) denote a bipartite distance-regular graph with diameter \(D \geq 4\) and eigenvalues \(\theta_0 > \theta_1 > \cdots > \theta_D\). Then \(\theta_1^2 > b_2 > \theta_2^*\), where \(d = \lfloor D/2 \rfloor\).

Proof. Apply Lemma 3.1 to the halved graph of \(\Gamma\), and use \(\Box\) Proposition 4.2.3]. \(\Box\)

Lemma 3.6 Let \(\Gamma\) denote a bipartite distance-regular graph with diameter \(D \geq 4\). Then

\[
p_{22}^2 = (b_2(c_3 - 1) + c_2(k - 2))c_2^{-1}. \tag{11}
\]
4 Two families of polynomials

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$. In this section we recall two types of polynomials associated with $\Gamma$. To motivate things, we recall by (av) and the triangle inequality that

$$AA_i = b_{i-1}A_{i-1} + c_{i+1}A_{i+1} \quad (0 \leq i \leq D),$$

where $b_{-1} = 0$ and $c_{D+1} = 0$. Let $f_0, f_1, \ldots, f_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying $f_0 = 1$ and

$$\lambda f_i = b_{i-1}f_{i-1} + c_{i+1}f_{i+1} \quad (0 \leq i \leq D - 1),$$

where $f_{-1} = 0$. Let $i$ denote an integer $(0 \leq i \leq D)$. The polynomial $f_i$ has degree $i$, and the coefficient of $\lambda^i$ is $(c_1c_2 \cdots c_i)^{-1}$. Comparing (12) and (13) we find $f_i(A) = A_i$. By [1, p. 63] the polynomials $f_0, f_1, \ldots, f_D$ satisfy the orthogonality relation

$$\sum_{h=0}^D f_i(\theta_h)f_j(\theta_h)m_h = \delta_{ij}|X|k_i \quad (0 \leq i, j \leq D).$$

Let $\theta$ denote an eigenvalue of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenvalue sequence. Comparing (12) and (13) using (10) we routinely obtain

$$f_i(\theta) = k_i\theta_i^* / \theta_0^* \quad (0 \leq i \leq D).$$

We remark on two special cases. Setting $i = 0$, $i = 1$ in (13) we routinely find $f_1 = \lambda$, $f_2 = (\lambda^2 - k)/c_2$. Now setting $i = 1$, $i = 2$ in (15) we find

$$\theta_1^* / \theta_0^* = \theta / k; \quad \theta_2^* / \theta_0^* = (\theta^2 - k)/(kb_1).$$

We now recall some polynomials related to the $f_i$. Let $p_0, p_1, \ldots, p_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying

$$p_i = \begin{cases} f_0 + f_2 + f_4 + \cdots + f_i, & \text{if } i \text{ is even} \\ f_1 + f_3 + f_5 + \cdots + f_i, & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq D).$$

Let $i$ denote an integer $(0 \leq i \leq D)$. The polynomial $p_i$ has degree $i$, and the coefficient of $\lambda^i$ is $(c_1c_2 \cdots c_i)^{-1}$. Recalling $f_0(A) = A_0$ $(0 \leq j \leq D)$, we observe

$$p_D(A) + p_{D-1}(A) = J, \quad p_D(A) - p_{D-1}(A) = (-1)^D J',$$

where $J'$ is from (18).

A bit later we find an orthogonality relation satisfied by the polynomials $p_i$. To obtain it we use the following result.

**Lemma 4.1** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials $f_0, f_1, \ldots, f_D$ be from (15), and let the polynomials $p_0, p_1, \ldots, p_D$ be from (17). Then

(i) \( p_i - p_{i-2} = f_i \) \quad $(2 \leq i \leq D)$,

(ii) \( (k^2 - \lambda^2)p_i = b_{i+1}f_{i+1} - c_{i+1}c_{i+2}f_{i+2} \) \quad $(0 \leq i \leq D - 2)$.

**Proof.** (i) Immediate from (17).

(ii) To see that the two sides are equal, in the expression on the left eliminate $p_i$ using (17), and evaluate the result by repeatedly applying (13) and Lemma 3.1(iv). \qed
Theorem 4.2 Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials $p_0, p_1, \ldots, p_D$ be as in (14). Then $p_0 = 1$ and

$$\lambda p_i = c_{i+1} p_{i+1} + b_{i+1} p_{i-1} \quad (0 \leq i \leq D-1),$$

where $p_{-1} = 0$.

Proof. Evaluate each side of (19) using (17) and (15), and simplify using Lemma 3.1(iv). $\square$

The polynomials $p_i$ satisfy the following orthogonality relation.

Lemma 4.3 Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $p_0, p_1, \ldots, p_D$ be as in (14). Then $p_D(\theta_h) = 0$ and $p_{D-1}(\theta_h) = 0$ for $1 \leq h \leq D - 1$. Moreover,

$$\sum_{h=0}^{D} p_i(\theta_h)(k^2 - \theta_h^2)m_h = \delta_{ij}|X|k_ib_i b_{i+1} \quad (0 \leq i, j \leq D - 2).$$

(We recall $m_h$ denotes the multiplicity of $\theta_h$ for $0 \leq h \leq D$.)

Proof. We first show $p_D(\theta_h) = 0$ and $p_{D-1}(\theta_h) = 0$ for $1 \leq h \leq D - 1$. Let $h$ be given. Multiplying both sides of the equations in (15) by $E_h$ and recalling $J, J'$ are scalar multiples of $E_0, E_D$, respectively, we find

$$p_D(A)E_h + p_{D-1}(A)E_h = 0, \quad p_D(A)E_h - p_{D-1}(A)E_h = 0$$

in view of (ev). Solving the equations in (21), we find $p_D(A)E_h = 0$, $p_{D-1}(A)E_h = 0$. Observe $p_D(A)E_h = p_D(\theta_h)E_h$, $p_{D-1}(A)E_h = p_{D-1}(\theta_h)E_h$, and thus $p_D(D_h) = 0$, $p_{D-1}(\theta_h) = 0$. Concerning (20), let the integers $i, j$ be given. Without loss of generality, we may assume $i \leq j$. We first assume $i$ is even. By (17) and Lemma 3.1(ii), the left-hand side of (20) is equal to

$$\sum_{h=0}^{D} (f_i(\theta_h) + f_{i-2}(\theta_h) + \cdots + f_0(\theta_h)) \left( b_j b_{j+1} f_j(\theta_h) - c_{j+1} c_{j+2} f_{j+2}(\theta_h) \right) m_h.$$

Evaluating (22) using (14) and recalling $i \leq j$, we find (22) is equal to the right-hand side of (20). The result follows. The case in which $i$ is odd is similar. $\square$

The following fact will be useful.

Lemma 4.4 Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let $\theta$ denote one of $\theta_1, \theta_2, \ldots, \theta_{D-1}$ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ denote the corresponding dual eigenvalue sequence. Then $\theta^*_0 \neq \theta^*_2$ and

$$p_i(\theta) = \frac{b_{2b_3} b_{i+1}}{c_1 c_2 \cdots c_i} \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_0 - \theta^*_2} \quad (0 \leq i \leq D - 2),$$

where the polynomials $p_i(\theta)$ are from (14).

Proof. Using the equation on the right in (16), we routinely verify that $\theta^*_0 \neq \theta^*_2$. To obtain (23), set $\lambda = \theta$ in Lemma 3.1(ii), and simplify the result using (6), (15), and (16). $\square$

Corollary 4.5 Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let $\theta$ denote one of $\theta_1, \theta_2, \ldots, \theta_{D-1}$ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ denote the corresponding dual eigenvalue sequence. Then

$$\frac{\theta^2 - b_2}{b_2 b_3} = \frac{\theta^*_0 - \theta^*_2}{\theta_0 - \theta_2}.$$
5 The polynomials $\Psi_i$

Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. In the previous section we used $\Gamma$ to define two families of polynomials in one variable. We called these polynomials the $f_i$ and the $p_i$. Later in this paper we will use $\Gamma$ to define a third family of polynomials in one variable. We will call these polynomials the $g_i$. To define and study the $g_i$, it is convenient to first consider some polynomials $\Psi_i$ in two variables.

**Definition 5.1** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. For $0 \leq i \leq D - 2$ let $\Psi_i$ denote the polynomial in $\mathbb{R}[\lambda, \mu]$ given by

$$\Psi_i = \sum_{h=0}^{i} \frac{p_h(\lambda)p_i(\mu)}{k_h b_{h+1}} b_{h} b_{h+1},$$

(25)

where the polynomials $p_0, p_1, \ldots, p_{D-2}$ are from $[17]$. We observe $\Psi_0 = 1$ and $\Psi_1 = \lambda \mu$.

**Lemma 5.2** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials $p_i, \Psi_i$ be as in $[17], [25]$, respectively. Then

$$p_i(\lambda)p_i(\mu) = \Psi_i - \frac{b_i b_{i+1}}{c_i c_{i-1}} \Psi_{i-2} \quad (2 \leq i \leq D - 2).$$

(26)

**Proof.** This is immediate from Definition 5.1. \qed

The following equation is a variation of the Christoffel-Darboux Formula.

**Lemma 5.3** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials $p_i, \Psi_i$ be as in $[17], [25]$, respectively. Then for $1 \leq i \leq D - 1$,

$$p_{i+1}(\lambda)p_{i-1}(\mu) - p_{i-1}(\lambda)p_{i+1}(\mu) = c_i^{-1} c_{i+1}^{-1} (\lambda^2 - \mu^2) \Psi_{i-1}.$$  

(27)

**Proof.** Repeatedly applying $[19]$, we find that for $0 \leq h \leq D - 2$,

$$\lambda^2 p_h(\lambda) = c_{h+1} c_{h+2} p_{h+2}(\lambda) + (c_{h+1} b_{h+2} + b_{h+1} c_h) p_h(\lambda) + b_h b_{h+1} p_{h-2}(\lambda),$$

(28)

where we define $p_{-1} := 0, p_{-2} := 0$. Similarly,

$$\mu^2 p_h(\mu) = c_{h+1} c_{h+2} p_{h+2}(\mu) + (c_{h+1} b_{h+2} + b_{h+1} c_h) p_h(\mu) + b_h b_{h+1} p_{h-2}(\mu).$$

(29)

Subtracting $(k_h b_h b_{h+1})^{-1} p_h(\lambda)$ times $[20]$ from $(k_h b_h b_{h+1})^{-1} p_h(\mu)$ times $[28]$ and using $[10]$, we find

$$\frac{(\lambda^2 - \mu^2) p_h(\lambda)p_h(\mu)}{k_h b_h b_{h+1}} = \frac{p_{h+2}(\lambda)p_h(\mu) - p_h(\lambda)p_{h+2}(\mu)}{k_{h+2}} - \frac{p_h(\lambda)p_{h-2}(\mu) - p_{h-2}(\lambda)p_h(\mu)}{k_h},$$

(30)

for $0 \leq h \leq D - 2$. Fix an integer $i$ $(1 \leq i \leq D - 1)$. Summing $[30]$ over all $h$ such that $0 \leq h \leq i - 1$ and such that $i - 1 - h$ is even, and using $[10], [25]$, we obtain $[27]$.

**Lemma 5.4** Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $p_i, \Psi_i$ be as in $[17], [25]$, respectively. Then for $0 \leq i, j \leq D - 2$,

$$\sum_{h=0}^{D} \Psi_i(\theta_h, \mu) \Psi_j(\theta_h, \mu) (k^2 - \theta_h^2) (\mu^2 - \theta_h^2) m_h$$

(31)

$$= \delta_{ij} |X| p_i(\mu)p_{i+2}(\mu) k_h b_{h+1} c_{i+1} c_{i+2}.$$  

(32)

(We recall $m_h$ denotes the multiplicity of $\theta_h$ for $0 \leq h \leq D$.)
Proof. Let the integers $i, j$ be given. Without loss of generality we may assume $i \leq j$. By Lemma 5.3 we have

$$p_{j+2}(\theta_h)p_j(\mu) - p_j(\theta_h)p_{j+2}(\mu) = c^{-1}_{j+1}c_{j+2}(\theta^2_h - \mu^2)\Psi_j(\theta_h, \mu)$$

for $0 \leq h \leq D$, so the sum is equal to

$$c_{j+1}c_{j+2}\sum_{h=0}^{D}\Psi_j(\theta_h, \mu)(p_{j+2}(\mu)p_j(\theta_h) - p_j(\mu)p_{j+2}(\theta_h))(k^2 - \theta^2_h)m_h.$$  \hspace{1cm} (33)

Eliminating $\Psi_i(\theta_h, \mu)$ in (33) using (24), and evaluating the result using Lemma 5.3 and $i \leq j$, we obtain (22). The result follows. \hfill \Box

Lemma 5.5 Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $p_i$ be as in (17). Then the following (i), (ii) hold.

(i) Abbreviate $\theta = \theta_1$. Then $p_i(\theta) > 0$ for $0 \leq i \leq D - 2$, and $p_{D-1}(\theta) = 0$, $p_D(\theta) = 0$.

(ii) Abbreviate $\theta = \theta_{D-1}$. Then $(-1)^i p_i(\theta) > 0$ for $0 \leq i \leq D - 2$, and $p_{D-1}(\theta) = 0$, $p_D(\theta) = 0$.

Proof. (i) Observe $p_{D-1}(\theta_1) = 0$, $p_D(\theta_1) = 0$ by Lemma 5.3. Suppose there exists an integer $i$ ($0 \leq i \leq D - 2$) such that $p_i(\theta) \leq 0$. Let us pick the minimal such $i$. Observe $i \geq 2$ since $p_0(\theta) = 1$, $p_1(\theta) = \theta$. Apparently $p_{i-2}(\theta) > 0$. We claim there exists an integer $h$ ($2 \leq h \leq D - 2$) such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. To see this, observe by Definition 5.1 that $\Psi_{i-2}(\lambda, \theta)$ is a polynomial in $\lambda$ with degree $i - 2$. In this polynomial the coefficient of $\lambda^{i-2}$ is $p_{i-2}(\theta)(c_1c_2 \cdots c_{i-2})^{-1}$. Apparently this polynomial is not identically 0 so there exist at most $i - 2$ integers $h$ ($2 \leq h \leq D - 2$) such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. By this and since $i \leq D - 2$, there exists at least one integer $h$ ($2 \leq h \leq D - 2$) such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. We have now proved our claim. We may now argue

$$0 < \sum_{h=2}^{D-2} \Psi^2_{i-2}(\theta_h, \theta)(k^2 - \theta^2_h)(\theta^2 - \theta^2_h)m_h$$

$$= \sum_{h=0}^{D} \Psi^2_{i-2}(\theta_h, \theta)(k^2 - \theta^2_h)(\theta^2 - \theta^2_h)m_h$$

$$= |X| p_{i-2}(\theta)p_i(\theta)k_{i-2}b_{i-2}b_{i-1}c_{i-1}c_i \hspace{1cm} \text{(by Lemma 5.4)}$$

$$\leq 0.$$  \hspace{1cm} (34)

We now have a contradiction and the result follows. (ii) Similar to the proof of (i) above. \hfill \Box

Lemma 5.6 Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Assume $D$ is odd and abbreviate $d = (D - 1)/2$. Let the polynomials $p_i$ be as in (17). Then the following (i), (ii) hold.

(i) Abbreviate $\theta = \theta_d$. Then $(-1)^{i/d} p_i(\theta) > 0$ for $0 \leq i \leq D - 2$, and $p_{D-1}(\theta) = 0$, $p_D(\theta) = 0$.

(ii) Abbreviate $\theta = \theta_{d+1}$. Then $(-1)^{(i+d)/2} p_i(\theta) > 0$ for $0 \leq i \leq D - 2$, and $p_{D-1}(\theta) = 0$, $p_D(\theta) = 0$.

Proof. (i) Observe $p_{D-1}(\theta_d) = 0$, $p_D(\theta_d) = 0$ by Lemma 5.3. Suppose there exists an integer $i$ ($0 \leq i \leq D - 2$) such that $(-1)^{i/d}p_i(\theta) \leq 0$. Let us pick the minimal such $i$. Observe $i \geq 2$ since $p_0(\theta) = 1$, $p_1(\theta) = \theta$. Apparently $(-1)^{i/d}p_{i-2}(\theta) > 0$, so $p_i(\theta)$, $p_{i-2}(\theta)$ do not have opposite signs. We claim there exists an integer $h$ ($1 \leq h \leq D - 1$), $h \neq d$, $h \neq d + 1$ such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. To see this, observe by Definition 5.1 that $\Psi_{i-2}(\lambda, \theta)$ is a polynomial in $\lambda$ with degree $i - 2$. In this polynomial the coefficient of $\lambda^{i-2}$ is $p_{i-2}(\theta)(c_1c_2 \cdots c_{i-2})^{-1}$. Apparently this polynomial is not identically 0 so there exist at most $i - 2$ integers $h$ ($1 \leq h \leq D - 1$) such that $\Psi_{i-2}(\theta_h, \theta) = 0$. By this and since $i \leq D - 2$, there exists at least one integer $h$
(1 ≤ h ≤ D − 1), h ̸= d, h ̸= d + 1 such that \( \Psi_{i-2}(\theta_h, \theta) \neq 0 \). We have now proved our claim. We may now argue

\[
0 > \sum_{h=1, h \neq d, h \neq d+1}^{D-1} \Psi_{i-2}^2(\theta_h, \theta)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2)m_h
\]

\[
= \sum_{h=0}^{D} \Psi_{i-2}^2(\theta_h, \theta)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2)m_h
\]

\[
= |X|p_{i-2}(\theta)p_i(\theta)k_{i-2}b_{i-2}b_{i-1}c_{i-1}c_i \quad \text{(by Lemma 5.4)}
\]

\[
\geq 0.
\]

We now have a contradiction and the result follows. 

(ii) Similar to the proof of (i) above.

\[\square\]

6 A third family of polynomials

In this section we will use the following notation.

**Notation 6.1** Throughout this section, \( \Gamma \) will denote a bipartite distance-regular graph with diameter \( D \geq 4 \) and eigenvalues \( k = \theta_0 > \theta_1 > \cdots > \theta_D \). Let the polynomials \( p_i \) be as in \( 17 \). If \( D \) is odd, we let \( \theta \) denote one of \( \theta_1, \theta_d, \theta_{d+1}, \theta_{D-1} \), where \( d = (D-1)/2 \). If \( D \) is even, we let \( \theta \) denote one of \( \theta_1, \theta_{D-1} \). We remark \( p_i(\theta) \neq 0 \) for \( 0 \leq i \leq D - 2 \) by Lemma 5.5 and Lemma 5.6.

In this section we use \( \Gamma \) to define a family of polynomials in one variable. We call these polynomials the \( g_i \).

**Definition 6.2** With reference to Notation 6.1 for \( 0 \leq i \leq D - 2 \) we define the polynomial \( g_i \in \mathbb{R}[\lambda] \) by

\[
g_i = \sum_{h=0}^{i} \sum_{k:h \text{ even}} \frac{p_h(\theta)k_hb_hb_{h+1}}{p_i(\theta)k_hb_hb_{h+1}}p_h.
\]

We observe

\[
\Psi_i(\lambda, \theta) = p_i(\theta)g_i,
\]

where \( \Psi_i \) is from Definition 5.1. We emphasize \( g_i \) depends on \( \theta \) as well as the intersection numbers of \( \Gamma \).

**Lemma 6.3** With reference to Notation 6.1 let \( g_0, g_1, \ldots, g_{D-2} \) denote the associated polynomials from Definition 6.2 Then

\[
p_i = g_i - \frac{b_ib_{i+1}p_{i-2}(\theta)c_{i-1}c_i}{p_i(\theta)g_{i-2}} \quad (2 \leq i \leq D - 2).
\]

**Proof.** Set \( \mu = \theta \) in \( 26 \) and simplify the result using \( 34 \). \[\square\]

**Lemma 6.4** With reference to Notation 6.1 let \( g_0, g_1, \ldots, g_{D-2} \) denote the associated polynomials from Definition 6.2 Then (i) and (ii) hold below for \( 0 \leq i \leq D - 2 \):

(i) The polynomial \( g_i \) has degree exactly \( i \).

(ii) The coefficient of \( \lambda^i \) in \( g_i \) is \((c_1c_2 \cdots c_i)^{-1}\).
Proof. Routine.

We now present a three-term recurrence satisfied by the polynomials $g_i$.

**Theorem 6.5** With reference to Notation 6.1 let $g_0, g_1, \ldots, g_{D-2}$ denote the associated polynomials from Definition 6.2. Then $g_0 = 1$ and

$$\lambda g_i = c_{i+1} g_{i+1} + \omega_i g_{i-1}$$

(36)

for $0 \leq i \leq D - 3$, where $\omega = \omega_0 = 0$, and

$$\omega_i = \frac{b_{i+1} c_{i+2} p_{i-1}(\theta) p_{i+2}(\theta)}{p_i(\theta) p_{i+1}(\theta)}$$

(1 \leq i \leq D - 3).

(37)

**Proof.** We find $g_0 = 1$ by Definition 6.2. To prove (36), we proceed by induction. It is routine to show equality holds in (36) for $i = 0, 1$ using Definition 6.2 and (19). Now suppose equality holds in (36) for $i = j - 2$, where $2 \leq j \leq D - 3$. We show equality holds in (36) for $i = j$. By the inductive hypothesis, we have

$$\lambda g_{j-2} = c_{j-1} g_{j-1} + \omega_{j-2} g_{j-3}.$$  

(38)

We must prove $\lambda g_j = c_{j+1} g_{j+1} + \omega_j g_{j-1}$. First consider the expression $c_{j+1} g_{j+1} + \omega_j g_{j-1}$. Eliminating $g_{j+1}, \omega_j$ in this expression using (36), (37), respectively, and then simplifying the result using (19), we find

$$c_{j+1} g_{j+1} + \omega_j g_{j-1} = c_{j+1} p_{j+1} + \frac{b_{j+1} \theta p_{j-1}(\theta)}{c_j p_j(\theta)} g_{j-1}.$$

(39)

Now consider the expression $\lambda g_j$. Replacing $g_j$ in this expression using (36), and eliminating $\lambda p_j$, $\lambda g_{j-2}$ in the result using (19), (38), respectively, we find

$$\lambda g_j = c_{j+1} p_{j+1} + b_{j+1} p_{j-1} + \frac{b_{j+1} b_{j+1} p_{j-2}(\theta)}{c_j c_{j+1} c_j p_j(\theta)} (c_{j-1} g_{j-1} + \omega_{j-2} g_{j-3}).$$

(40)

Eliminating $\omega_{j-2}$ in (40) using (37) and eliminating $b_{j-1} b_j p_{j-3}(\theta)(c_{j-2} c_{j-1} p_{j-1}(\theta))^{-1} g_{j-3}$ in the result using (36), we find

$$\lambda g_j = c_{j+1} p_{j+1} + b_{j+1} \frac{c_j p_j(\theta) + b_j p_{j-2}(\theta)}{c_j p_j(\theta)} g_{j-1}.$$  

(41)

Observe the right-hand sides of (39), (41) are equal in view of (19), and thus the left-hand sides are equal. We find equality holds in (36) for $i = j$, as desired.

**Theorem 6.6** With reference to Notation 6.1 let $g_0, g_1, \ldots, g_{D-2}$ denote the associated polynomials from Definition 6.2. Then

$$\sum_{h=0}^{D} g_i(\theta_h) g_j(\theta_h)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2)m_h = \delta_{ij} |X| k_i b_i c_{i+1} c_{i+2} \frac{p_{i+2}(\theta)}{p_i(\theta)}$$

(42)

for $0 \leq i, j \leq D - 2$.

**Proof.** To verify (42), on the left-hand side first eliminate $g_i(\theta_h)$ and $g_j(\theta_h)$ using (36), and then evaluate the result using Lemma 5.4.
7 The subconstituent algebra and its modules

In this section we recall some definitions and basic concepts concerning the subconstituent algebra and its modules. For more information we refer the reader to \[4, 6, 7, 13, 16, 35.\]

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. We recall the dual Bose-Mesner algebra of $\Gamma$. For the rest of this section, fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $yy$ entry

\[(E^*_i)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).\]  

(43)

We call $E^*_i$ the $i^{th}$ dual idempotent of $\Gamma$ with respect to $x$. We observe (di) $\sum_{i=0}^{D} E^*_i = I$; (dii) $E^*_i = E^*_i (0 \leq i \leq D)$; (diii) $E^*_i E^*_j = \delta_{ij} E^*_i (0 \leq i, j \leq D)$. Using (di) and (dii) we find $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$. We call $M^*$ the dual Bose-Mesner algebra of $\Gamma$ with respect to $x$. We recall the subconstituents of $\Gamma$. Using \[43\] we find

\[E^*_i V = \text{Span} \{ \hat{y} \mid y \in X, \quad \partial(x, y) = i \} \quad (0 \leq i \leq D).\]  

(44)

By \[43\] and since $\{ \hat{y} \mid y \in X \}$ is an orthonormal basis for $V$ we find

\[V = E^*_0 V + E^*_1 V + \cdots + E^*_D V \quad \text{(orthogonal direct sum)}.\]  

(45)

Combining \[43\] and \[45\] we find the dimension of $E^*_i V$ is $k_i$ for $0 \leq i \leq D$. We call $E^*_i V$ the $i^{th}$ subconstituent of $\Gamma$ with respect to $x$.

We recall how $M$ and $M^*$ are related. By \[35\] Lemma 3.2,

\[E^*_i A_i E^*_j = 0 \quad \text{if and only if} \quad p^h_{ij} = 0 \quad (0 \leq h, i, j \leq D).\]  

(46)

Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the subconstituent algebra of $\Gamma$ with respect to $x$ \[35\]. We observe $T$ has finite dimension. Moreover $T$ is semi-simple; the reason is that $T$ is closed under the conjugate-transpose map \[10\] p. 157).

We now consider the modules for $T$. By a $T$-module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. We refer to $V$ itself as the standard module for $T$. Let $W$ denote a $T$-module. Then $W$ is said to be irreducible whenever $W$ is nonzero and $W$ contains no $T$-modules other than $0$ and $W$. Let $W, W'$ denote $T$-modules. By an isomorphism of $T$-modules from $W$ to $W'$ we mean an isomorphism of vector spaces $\sigma : W \rightarrow W'$ such that

\[(\sigma B - B \sigma)W = 0 \quad \text{for all } B \in T.\]

The modules $W, W'$ are said to be isomorphic as $T$-modules whenever there exists an isomorphism of $T$-modules from $W$ to $W'$.

Let $W$ denote a $T$-module and let $W'$ denote a $T$-module contained in $W$. Using \[35\] we find the orthogonal complement of $W'$ in $W$ is a $T$-module. It follows that each $T$-module is an orthogonal direct sum of irreducible $T$-modules. We mention any two nonisomorphic irreducible $T$-modules are orthogonal \[10\] Chapter IV.

Let $W$ denote an irreducible $T$-module. Using (di)--(div) above we find $W$ is the direct sum of the nonzero spaces among $E^*_0 W, E^*_1 W, \ldots, E^*_D W$. Similarly using (dii)--(ev) we find $W$ is the direct sum of the nonzero spaces among $E^*_0 W, E^*_1 W, \ldots, E^*_D W$. If the dimension of $E^*_i W$ is at most 1 for $0 \leq i \leq D$ then the dimension of $E^*_i W$ is at most 1 for $0 \leq i \leq D$ \[35\] Lemma 3.9; in this case we say $W$ is thin. Let $W$ denote an irreducible $T$-module. By the endpoint of $W$ we mean

\[\min \{ i \mid 0 \leq i \leq D, \quad E^*_i W \neq 0 \}.\]

In the rest of the paper we will assume $\Gamma$ is bipartite. We adopt the following notational convention.
Definition 7.1 For the rest of this paper we let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers $b_i, c_i$, adjacency matrix $A$, Bose-Mesner algebra $M$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. For $0 \leq i \leq D$ we let $E_i$ denote the primitive idempotent of $\Gamma$ associated with $\theta_i$. We define $d = \lfloor D/2 \rfloor$. We let $V$ denote the standard module for $\Gamma$. We fix $x \in X$ and abbreviate $E_i = E_i(x)$ $(0 \leq i \leq D)$, $M^* = M^*(x)$, $T = T(x)$. We define

$$s_i = \sum_{y \in X} \hat{y}$$

$(0 \leq i \leq D)$. (47)

8 The $T$-module of endpoint 0

With reference to Definition 7.1 there exists a unique irreducible $T$-module with endpoint 0 $[12]$ Proposition 8.4]. We call this module $V_0$. The module $V_0$ is described in $[6]$, $[12]$. We summarize some details below in order to motivate the results that follow.

The module $V_0$ is thin. In fact each of $E_i V_0$, $E_i^* V_0$ has dimension 1 for $0 \leq i \leq D$. We give two bases for $V_0$. The vectors $E_0 \hat{x}, E_1 \hat{x}, \ldots, E_D \hat{x}$ form a basis for $V_0$. These vectors are mutually orthogonal and $\|E_i \hat{x}\|^2 = m_i |X|^{-1}$ for $0 \leq i \leq D$. To motivate the second basis we make some comments. For $0 \leq i \leq D$ we have $s_i = A_i \hat{x}$. Moreover $s_i = E_i^* \delta$, where $\delta = \sum_{y \in X} \hat{y}$. The vectors $s_0, s_1, \ldots, s_D$ form a basis for $V_0$. These vectors are mutually orthogonal and $\|s_i\|^2 = k_i$ for $0 \leq i \leq D$. With respect to the basis $s_0, s_1, \ldots, s_D$ the matrix representing $A$ is

$$
\begin{pmatrix}
0 & b_0 & 0 & 0 \\
c_1 & 0 & b_1 & 0 \\
c_2 & \cdots & \cdots & b_{D-1} \\
0 & \cdots & c_D & 0
\end{pmatrix}.
$$

The two bases for $V_0$ given above are related as follows. For $0 \leq i \leq D$ we have

$$s_i = \sum_{h=0}^{D} f_i(\theta_h) E_h \hat{x},$$

where the polynomial $f_i$ is from $[13]$.

9 $T$-modules of endpoint 1

With reference to Definition 7.1 there exists, up to isomorphism, a unique irreducible $T$-module with endpoint 1 $[6]$ Corollary 7.7]. We call this module $V_1$. The module $V_1$ is described in $[6]$, $[13]$. We summarize some details below in order to motivate the results that follow.

The module $V_1$ is thin with dimension $D - 1$. We give two bases for $V_1$. Let $v$ denote a nonzero vector in $E_1^* V_1$. Then the sequence $E_1^* v, E_2^* v, \ldots, E_{D-1}^* v$ is a basis for $V_1$. These vectors are mutually orthogonal and

$$\|E_i^* v\|^2 = \frac{m_i (k^2 - \theta_i^2)}{|X| (k - 1)} |v|^2 \quad (1 \leq i \leq D - 1).$$

To motivate the second basis we make some comments. We have $E_{i+1}^* A_i v = p_i(A) v$ for $0 \leq i \leq D - 2$ and $E_D^* A_{D-1} v = 0$, where the polynomial $p_i$ is from $[17]$. The vectors $E_1^* A_0 v, E_2^* A_1 v, \ldots, E_{D-1}^* A_{D-2} v$ form a basis for $V_1$. These vectors are mutually orthogonal and

$$\|E_{i+1}^* A_i v\|^2 = \frac{b_2 \cdots b_{i+1}}{c_1 \cdots c_i} |v|^2 \quad (0 \leq i \leq D - 2).$$
With respect to the basis $E_1^*A_0 v, E_2^* A_1 v, \ldots, E_{D-1}^* A_{D-2} v$, the matrix representing $A$ is

$$
\begin{pmatrix}
0 & b_2 & & & \\
c_1 & 0 & b_3 & & \\
c_2 & & \ddots & \ddots & \\
0 & \cdots & & b_{D-1} & \\
& & & c_{D-2} & 0
\end{pmatrix}.
$$

The two bases for $V_1$ given above are related as follows. For $0 \leq i \leq D - 2$ we have

$$E_i^* A_i v = \sum_{h=1}^{D-1} p_i(\theta_h) E_h v.$$

We comment that $V_1$ appears in $V$ with multiplicity $k - 1$. We will need the following result.

**Corollary 9.1** With reference to Definition 7.1, let $W$ denote an irreducible $T$-module with endpoint 1. Observe $E_2^* W$ is an eigenspace for $E_2^* A_2 E_2^*$. The corresponding eigenvalue is $b_3 - 1$.

**Proof.** The desired eigenvalue is the entry in the second row and second column of the matrix representing $A_2$ with respect to the basis $E_1^* A_0 v, E_2^* A_1 v, \ldots, E_{D-1}^* A_{D-2} v$. To compute this entry, first set $i = 1$ in (12) and observe that $c_2 A_2 = A^2 - k I$. Using this fact and the above matrix display of $A$, we verify the specified matrix entry is $b_3 - 1$. \hfill $\square$

## 10 The local eigenvalues

Later in the paper we will consider the thin irreducible $T$-modules with endpoint 2. In order to discuss these we introduce some parameters we call the local eigenvalues.

**Definition 10.1** With reference to Definition 7.1 we let $\bar{\Gamma}_2^2 = \bar{\Gamma}_2^2(x)$ denote the graph $(\bar{X}, \bar{R})$, where

$$
\bar{X} = \{y \in X \mid \partial(x, y) = 2\},
\bar{R} = \{yz \mid y, z \in \bar{X}, \partial(y, z) = 2\},
$$

where we recall $\partial$ denotes the path-length distance function for $\Gamma$. The graph $\bar{\Gamma}_2^2$ has exactly $k_2$ vertices, where $k_2$ is the second valency of $\Gamma$. Also, $\bar{\Gamma}_2^2$ is regular with valency $p_{22}^2$. We let $\bar{A}$ denote the adjacency matrix of $\bar{\Gamma}_2^2$. The matrix $\bar{A}$ is symmetric with real entries; therefore $\bar{A}$ is diagonalizable with all eigenvalues real. We let $\eta_1, \eta_2, \ldots, \eta_{k_2}$ denote the eigenvalues of $\bar{A}$. We call $\eta_1, \eta_2, \ldots, \eta_{k_2}$ the **local eigenvalues** of $\Gamma$ with respect to $x$.

With reference to Definition 10.1 we consider the second subconstituent $E_2^* V$. We recall the dimension of $E_2^* V$ is $k_2$. Observe $E_2^* V$ is invariant under the action of $E_2^* A_2 E_2^*$. To illuminate this action we make an observation. For an appropriate ordering of the vertices of $\Gamma$ we have

$$E_2^* A_2 E_2^* = \begin{pmatrix}
\bar{A} & 0 \\
0 & 0
\end{pmatrix},$$

where $\bar{A}$ is from Definition 10.1. Apparently the action of $E_2^* A_2 E_2^*$ on $E_2^* V$ is essentially the adjacency map for $\bar{\Gamma}_2^2$. In particular the action of $E_2^* A_2 E_2^*$ on $E_2^* V$ is diagonalizable with eigenvalues $\eta_1, \eta_2, \ldots, \eta_{k_2}$. We observe the vector $s_2$ from (47) is contained in $E_2^* V$. One may easily show that $s_2$ is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $p_{22}^2$. Let $v$ denote a vector in $E_2^* V$. We observe the following are equivalent: (i) $v$ is orthogonal to $s_2$; (ii) $E_0 v = 0$; (iii) $J v = 0$; (iv) $E_{D'} v = 0$; (v) $J' v = 0$. Let $V_1$ denote an irreducible $T$-module of endpoint 1, and let $v$ denote a vector in $E_2^* V_1$. By Corollary 9.1, $v$ is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $b_3 - 1$. Reordering the eigenvalues if necessary, we have $\eta_i = p_{22}^2$ and $\eta_i = b_3 - 1$ ($2 \leq i \leq k$).

For the rest of this paper we assume the local eigenvalues of $\Gamma$ are ordered in this way.

We now need some notation.
**Definition 10.2** With reference to Definition 7.1 let $Y$ denote the subspace of $V$ spanned by the irreducible $T$-modules with endpoint 1. We define the set $U$ to be the orthogonal complement of $E_2^1V_0 + E_2^1Y$ in $E_2^1V$.

**Definition 10.3** With reference to Definition 7.1 let $\Phi$ denote the set of distinct scalars among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$, where the $\eta_i$ are from Definition 10.1. For $\eta \in \mathbb{R}$ we let mult$_\eta$ denote the number of times $\eta$ appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. We observe mult$_\eta \neq 0$ if and only if $\eta \in \Phi$.

Using (8) we find $U$ is invariant under $E_2^1A_2E_2^*$ and $E_2^1R_2$. Apparently the restriction of $E_2^1A_2E_2^*$ to $U$ is diagonalizable with eigenvalues $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. For $\eta \in \mathbb{R}$ let $U_\eta$ denote the set consisting of those vectors in $U$ that are eigenvectors for $E_2^1A_2E_2^*$ with eigenvalue $\eta$. We observe $U_\eta$ is a subspace of $U$ with dimension mult$_\eta$.

We emphasize the following are equivalent: (i) mult$_\eta \neq 0$; (ii) $U_\eta \neq 0$; (iii) $\eta \in \Phi$. By (8) and since $E_2^1A_2E_2^*$ is symmetric with real entries we find

$$U = \sum_{\eta \in \Phi} U_\eta \quad \text{(orthogonal direct sum).}$$  \hspace{1cm} (48)

The following result will be useful.

**Lemma 10.4** With reference to Definition 7.1 let $E$ denote a primitive idempotent of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the corresponding dual eigenvalue sequence. Then

$$|X|E_2^1EE_2^* = (\theta_0^* - \theta_1^*)E_2^* + (\theta_2^* - \theta_3^*)E_2^1A_2E_2^* + \theta_4^*E_2^1JE_2^*.$$  \hspace{1cm} (49)

**Proof.** By (aii),

$$E_2^1JE_2^* = E_2^1 \left( \sum_{i=0}^{D} A_i \right) E_2^* = E_2^* + E_2^1A_2E_2^* + E_2^1A_1E_2^*$$  \hspace{1cm} (50)

in view of (ai), (8). Using (8) we similarly find

$$|X|E_2^1EE_2^* = E_2^1 \left( \sum_{i=0}^{D} \theta_i^* A_i \right) E_2^* = \theta_0^*E_2^* + \theta_2^*E_2^1A_2E_2^* + \theta_4^*E_2^1A_1E_2^*.$$  \hspace{1cm} (51)

Eliminating $E_2^1A_1E_2^*$ in (91) using (50) we get (90). \hfill \Box

### 11 Some inner products

In this section we are concerned with the following basis.

**Lemma 11.1** With reference to Definition 7.1 let $v$ denote a nonzero vector in $E_2^1V$ which is orthogonal to $s_2$. Then the nonvanishing vectors among

$$E_1v, E_2v, \ldots, E_{D-1}v$$  \hspace{1cm} (52)

form an orthogonal basis for $Mv$.

**Proof.** Recall $E_0, E_1, \ldots, E_D$ form a basis for $M$. We assume $v$ is orthogonal to $s_2$ so $E_0v = 0$, $E_2v = 0$. Now apparently the vectors in (52) span $Mv$. The vectors in (52) are mutually orthogonal by (7) and the result follows. \hfill \Box

Referring to Lemma 11.1 in this section we determine which of $E_1v, E_2v, \ldots, E_{D-1}v$ are zero. To do this, we compute the square norms of these vectors.
Thus the equations in (53), (55) hold since both sides of the equations are zero. Now assume

First assume \( \eta \neq -1 \). Then

\[
\| E_i v \|^2 = \frac{m_i (\theta_i - k)(\theta_i + k)(\theta_i^2 - \psi)}{|X| k b_1 (\psi - b_2)} \|v\|^2 \quad (0 \leq i \leq D),
\]

where

\[
\psi = b_2 \left( 1 - \frac{b_3}{1 + \eta} \right).
\]

We remark the denominator in (53) is nonzero by the form of (54).

(ii) Assume \( \eta = -1 \). Then

\[
\| E_i v \|^2 = \frac{m_i (k - \theta_i)(k + \theta_i)}{|X| k b_1} \|v\|^2 \quad (0 \leq i \leq D).
\]

Proof. First assume \( i \in \{0, D\} \). Then \( \theta_i \in \{-k, k\} \). We assume \( v \) is orthogonal to \( s_2 \) so \( E_0 v = 0; E_D v = 0 \).

Now assume \( 1 \leq i \leq D - 1 \). Observe \( E = E^2_{2} v \) by construction so \( E_i v = E_i E^2_{2} v \). We may now argue

\[
\| E_i v \|^2 = (E_i E^2_{2} v)^t E_i E^2_{2} v \quad \text{by (2)}
\]

\[
= v^t E^2_{2} E_i E^2_2 v \quad \text{by (11)}
\]

\[
= v^t E^2_{2} E_i E^2_2 v \quad \text{by (ev), (dii), (diii)}
\]

To evaluate (56) we apply Lemma 10.2. To do this we make some comments. Let \( \theta_0, \theta_1, \ldots, \theta_D \) denote the dual eigenvalue sequence for \( E_i \). We assume \( v \) is orthogonal to \( s_2 \) so \( J v = 0 \). We already mentioned \( E^2_{2} v = v \). By assumption \( E^2_{2} A_2 E^2_{2} v = \eta v \). Since each of \( J, E^2_{2}, A_2 \) has real entries and since \( \eta \) is real, we see \( J v = 0 \), \( E^2_{2} v = v \), and \( E^2_{2} A_2 E^2_{2} v = \eta v \). Evaluating (56) using Lemma 10.2 and our above comments, we find \( |X| \|E_i v\|^2 \) is equal to \( \|v\|^2 \) times

\[
\theta_0 - \theta_i + \eta (\theta_2 - \theta_i^2).
\]

Observe (57) is equal to \( \theta_0 - \theta_2 \) times

\[
1 + (1 + \eta) \frac{\theta_2 - \theta_i}{\theta_0 - \theta_i}.
\]

Using (10) and recalling \( \theta_0 = m_i \), we find

\[
\theta_0 - \theta_2 = m_i (k - \theta_i)(k + \theta_i)/(k b_1).
\]

As before, we may now argue

\[
\| E_i v \|^2 = \frac{m_i (k - \theta_i)(k + \theta_i)}{|X| k b_1} \|v\|^2 \quad (0 \leq i \leq D).
\]

With reference to Definition 11.1, our next goal is to get upper and lower bounds for the local eigenvalues \( \eta_i \) for all \( z \in \mathbb{C} \cup \{\infty\} \).

\[
\tilde{z} = \begin{cases} 
-1 - \frac{b_1 b_2}{z - b_2}, & \text{if } z \neq \infty, \ z^2 \neq b_2 \\
\infty, & \text{if } z^2 = b_2 \\
-1, & \text{if } z = \infty.
\end{cases}
\]

By Lemma 3.3, neither \( \theta_1^2, \theta_d^2 \) is equal to \( b_2 \), so

\[
\hat{\theta}_1 = -1 - b_2 b_3 (\theta_1^2 - b_2)^{-1}, \quad \hat{\theta}_d = -1 - b_2 b_3 (\theta_d^2 - b_2)^{-1}.
\]

By the data in Lemma 3.5 we have \( \hat{\theta}_1 < -1 \). Moreover \( \hat{\theta}_d > b_3 - 1 \) if \( D \) is odd and \( \hat{\theta}_d = b_3 - 1 \) if \( D \) is even.

In either case \( \hat{\theta}_d \geq 0 \).
Theorem 11.4 With reference to Definitions \[\text{7.1} \] and \[\text{10.1} \] we have \( \tilde{\theta}_1 \leq \eta_i \leq \theta_d \) for \( k + 1 \leq i \leq k_2 \).

Proof. Let the integer \( i \) be given, and abbreviate \( \eta = \eta_i \). Let \( v \) denote a nonzero vector in \( U_\eta \). First suppose \( \eta < \tilde{\theta}_1 \). By Definition \[\text{11.3} \] and Lemma \[\text{3.6} \] we find
\[
\eta + 1 < \frac{b_2 b_3}{b_2 - \theta_1^2} < 0.
\] (60)

Using \[\text{5.1} \] and \[\text{6.1} \], we find \( \psi > b_2 \) and \( \psi < \theta_1^2 \). Now \( \| E_1 v \|^2 < 0 \) by Theorem \[\text{11.2} \] a contradiction. The inequality \( \eta_i \leq \theta_d \) is proven similarly. \( \square \)

Referring to Lemma \[\text{11.3} \] we now determine which of \( E_1 v, E_2 v, \ldots, E_{D-1} v \) are zero.

Lemma 11.5 With reference to Definition \[\text{7.1} \] let \( v \) denote a nonzero vector in \( U \). Then (i)–(vi) hold below.

(i) \( E_0 v = 0 \) and \( E_D v = 0 \).

(ii) For \( 1 \leq i \leq D - 1 \), \( E_i v \neq 0 \) provided \( i \) is not among \( 1, d, D - d, D - 1 \).

(iii) \( E_1 v = 0 \) if and only if \( v \in U_{\tilde{\theta}_1} \).

(iv) \( E_{D-1} v = 0 \) if and only if \( v \in U_{\tilde{\theta}_1} \).

(v) \( E_d v = 0 \) if and only if \( v \in U_{\tilde{\theta}_d} \).

(vi) \( E_{D-d} v = 0 \) if and only if \( v \in U_{\tilde{\theta}_d} \).

Proof. By Definition \[\text{11.2} \] \( U \) is orthogonal to \( V_0 \); in particular \( v \) is orthogonal to \( s_2 \). Thus \( E_0 v, E_D v \) are zero. Now suppose there exists an integer \( n \ (1 \leq n \leq D - 1) \) such that \( E_n v = 0 \). We show \( n \) is among \( 1, d, D - d, D - 1 \), and that \( v \in U_{\tilde{\theta}_n} \). We claim \( v \) is an eigenvector for \( E_2^* A_2 E_2^* \). To see this, in \[\text{10.1} \] set \( E = E_n \) and apply both sides to \( v \). Using \( v = E_2^* v \) and \( J v = 0 \) we find
\[
0 = (\theta_0^* - \theta_1^*) v + (\theta_2^* - \theta_4^*) E_2^* A_2 E_2^* v,
\]
where \( \theta_0^*, \theta_2^*, \theta_4^* \) are dual eigenvalues for \( E_n \). Observe \( \theta_2^* \neq \theta_4^* \); otherwise \( \theta_0^* = \theta_4^* \) by the above line, forcing \( \theta_0^* = \theta_2^* \) and contradicting Lemma \[\text{11.4} \]. By assumption \( E_n v = 0 \) so \( \| E_n v \|^2 = 0 \). Applying Theorem \[\text{11.2} \] we find \( \theta_2^* = \psi \) and thus \( \eta = \eta_i \). Thus \( n \) is among \( 1, d, D - d, D - 1 \) by Theorem \[\text{11.2} \] Lemma \[\text{3.6} \] (v), and the fact that \( \theta_0 > \theta_1 > \cdots > \theta_d \). Apparently \( v \in U_{\tilde{\theta}_n} \). To finish the proof, suppose \( n = 1 \) or \( n = d \) and assume \( v \in U_{\tilde{\theta}_n} \). We show \( E_n v = 0, E_{D-n} v = 0 \). Observe \( v \) is an eigenvector for \( E_2^* A_2 E_2^* \) with eigenvalue \( \tilde{\theta}_n \). By Theorem \[\text{11.2} \] we find \( \| E_n v \|^2 = 0 \) so \( E_n v = 0 \), as desired. Observe \( E_{D-n} v = 0 \) since \( \| E_{D-n} v \|^2 = \| E_n v \|^2 \) by \[\text{5.3} \]. The result follows. \( \square \)

Corollary 11.6 With reference to Definition \[\text{7.1} \] let \( v \) denote a nonzero vector in \( U \). Then (i)–(iv) hold below.

(i) If \( v \in U_{\tilde{\theta}_1} \) then \( M v \) has dimension \( D - 3 \).

(ii) If \( v \in U_{\tilde{\theta}_d} \) and \( D \) is odd, then \( M v \) has dimension \( D - 3 \).

(iii) If \( v \in U_{\tilde{\theta}_d} \) and \( D \) is even, then \( M v \) has dimension \( D - 2 \).

(iv) If \( v \notin U_{\tilde{\theta}_1} \) and \( v \notin U_{\tilde{\theta}_d} \) then \( M v \) has dimension \( D - 1 \).

Proof. Combine Lemmas \[\text{11.3} \] and \[\text{11.5} \] and observe the integers \( d, D - d \) are distinct precisely when \( D \) is odd. \( \square \)

The following equations will be useful.
Lemma 11.7 With reference to Definitions 7.1 and 11.8 the following (i)–(iii) hold.

(i) \( k + \sum_{\eta \in \Phi} \text{mult}_\eta = k_2. \)

(ii) \( p_{22}^2 + (k - 1)(b_3 - 1) + \sum_{\eta \in \Phi} \eta \text{mult}_\eta = 0. \)

(iii) \( (p_{22}^2)^2 + (k - 1)(b_3 - 1)^2 + \sum_{\eta \in \Phi} \eta^2 \text{mult}_\eta = k_2 p_{22}^2. \)

Proof. (i) There are \( k_2 - k \) elements in the sequence \( \eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}. \)

(ii) Recall the matrix \( \tilde{A} \) from Definition 11.8. Each diagonal entry of \( \tilde{A} \) is zero, so the trace of \( \tilde{A} \) is zero. Recall \( \eta_1, \eta_2, \ldots, \eta_{k_2} \) are the eigenvalues of \( \tilde{A} \), so \( \sum_{i=1}^{k_2} \eta_i = 0. \) By this and since \( \eta_1 = p_{22}^2, \eta_i = b_3 - 1 \) \( (2 \leq i \leq k) \), we have the desired result.

(iii) Recall \( \Gamma_2^3 \) is regular with valency \( p_{22}^2 \), so each diagonal entry of \( \tilde{A}^2 \) is \( p_{22}^2 \). Apparently the trace of \( \tilde{A}^2 \) is \( k_2 p_{22}^2 \), so \( \sum_{i=1}^{k_2} \eta_i^2 = k_2 p_{22}^2 \). By this and since \( \eta_1 = p_{22}^2, \eta_i = b_3 - 1 \) \( (2 \leq i \leq k) \), we have the desired result. □

Definition 11.8 With reference to Definition 11.8 let \( W \) denote a thin irreducible \( T \)-module with endpoint 2. Observe \( E_2^3 W \) is a 1-dimensional eigenspace for \( E_2^3 A_2 E_2^3 \); let \( \eta \) denote the corresponding eigenvalue. We observe \( E_2^3 W \) is contained in \( E_2^3 V \) and is orthogonal to any irreducible \( T \)-module with endpoint 0 or 1, so \( E_2^3 W \subseteq U_\eta \). Apparently \( U_\eta \neq 0 \) so \( \eta \) is one of \( \eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}. \) We have \( \theta_1 \leq \eta \leq \theta_d \) by Theorem 11.3. We refer to \( \eta \) as the \emph{local eigenvalue} of \( W \).

With reference to Definition 11.8 let \( W \) denote a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \eta. \) In order to describe \( W \) we distinguish four cases: (i) \( D \) is odd, and \( \eta = \theta_1 \) or \( \eta = \theta_d; \) (ii) \( D \) is even and \( \eta = \theta_1 \); (iii) \( D \) is even and \( \eta = \theta_d; \) (iv) \( \theta_1 < \eta < \theta_d. \) We investigate cases (i), (ii) in the present paper. We will investigate the remaining cases in a future paper.

12 The spaces \( U_{\tilde{\theta}_1} \) and \( U_{\tilde{\theta}_d} \)

We state our main goal for this section. With reference to Definition 11.8 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \theta_n. \) We show that for all nonzero \( v \in U_\eta \) the space \( Mv \) is a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \eta. \)

Lemma 12.1 With reference to Definition 7.1 let \( v \) denote a vector in \( E_2^3 V. \) Then

\[ E_i^* A_j v = 0 \quad \text{if} \quad |i - j| > 2 \quad (0 \leq i, j \leq D), \]

and

\[ E_i^* A_j v = 0 \quad \text{if} \quad i + j \text{ is odd} \quad (0 \leq i, j \leq D). \]

Proof. Let \( i, j \) be given and observe \( E_2^3 v = v \) so \( E_i^* A_j v = E_i^* A_j E_2^3 v. \) The result now follows from (46). □

Lemma 12.2 With reference to Definition 7.1 let \( v \) denote a vector in \( E_2^3 V \) which is orthogonal to \( s_2. \)

Then

\[ \sum_{j=0}^{D} E_i^* A_j v = 0 \quad (0 \leq i \leq D). \]

Proof. Observe \( Jv = 0 \) so \( E_i^* Jv = 0. \) Eliminate \( J \) in this expression using (a(ii)) to get the result. □
Lemma 12.3 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \tilde{\eta}_n \). Then for all \( v \in U_\eta \) we have

\[
\sum_{j=0}^{D} \theta_j^* A_j v = 0 \quad (0 \leq i \leq D),
\]

where \( \theta_0^*, \theta_1^*, \ldots, \theta_D^* \) denotes the dual eigenvalue sequence for \( \eta_n \).

Proof. Observe \( E_n v = 0 \) by Lemma 12.3 so \( E_i^* E_n v = 0 \). Eliminate \( E_n \) in this expression using 12.1 to get the result. \( \square \)

Lemma 12.4 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \tilde{\eta}_n \). Then for all \( v \in U_\eta \) we have

\[
E_i^* A_i v = \frac{\theta_{i-2}^* - \theta_{i+2}^*}{\theta_{i+2}^* - \theta_i^*} E_i^* A_i v \quad (2 \leq i \leq D - 2), \tag{61}
\]

\[
E_i^* A_{i+2} v = \frac{\theta_{i-2}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+2}^*} E_i^* A_{i+2} v \quad (2 \leq i \leq D - 2), \tag{62}
\]

where \( \theta_0^*, \theta_1^*, \ldots, \theta_D^* \) denotes the dual eigenvalue sequence for \( \eta_n \). Moreover

\[
E_i^* A_i v = 0, \quad E_i^* A_0 v = 0, \quad E_{D-1}^* A_i v = 0, \quad E_D^* A_i v = 0 \quad (0 \leq i \leq D). \tag{63}
\]

We note the denominators in (61), (63) are nonzero by Lemmas 12.4, 5.5, and 5.6.

Proof. By Lemmas 12.3 and 7.6 we find \( \theta_{i-2}^* \neq \theta_i^* \) for \( 2 \leq i \leq D \). Solving the equations in Lemma 12.2 and Lemma 12.3 using Lemma 12.1 we routinely obtain (61)–(63). \( \square \)

Lemma 12.5 With reference to Definition 7.1 let \( v \) denote a vector in \( E_i^* V \) which is orthogonal to \( s_2 \). Let the polynomials \( p_0, p_1, \ldots, p_D \) be from (7). Then

\[
p_i(A) v = E_{i+2}^* A_i v - E_i^* A_{i+2} v \quad (0 \leq i \leq D - 2). \tag{64}
\]

Moreover \( p_{D-1}(A) v = 0 \), \( p_D(A) v = 0 \).

Proof. For \( 0 \leq i \leq D - 2 \) we have

\[
p_i(A) v = \sum_{0 \leq s \leq i, i-s \text{ even}} A_s v
\]

\[
= (E_0^* + E_1^* + \cdots + E_D^*) \sum_{0 \leq s \leq i, i-s \text{ even}} A_s v
\]

\[
= \sum_{0 \leq s \leq i, i-s \text{ even}} E_s^* A_s v, \tag{65}
\]

where the final sum is over all integers \( r, s \) such that \( 0 \leq r \leq D, 0 \leq s \leq i \), and \( i-s \) is even. Cancelling terms in (65) using Lemmas 12.1 and 12.4 we obtain (64). Using (64), we may solve for each of \( p_{D-1}(A), p_D(A) \) as a linear combination of \( J \) and \( J' \). Recall \( Jv = 0 \), \( J'v = 0 \), and thus \( p_{D-1}(A) v = 0, p_D(A) v = 0 \). \( \square \)

Theorem 12.6 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \tilde{\eta}_n \). Then for all \( v \in U_\eta \) we have

\[
E_{i+2}^* A_i v = \sum_{0 \leq h \leq i, i-h \text{ even}} \frac{\theta_h^* - \theta_{h+2}^*}{\theta_i^* - \theta_{i+2}^*} p_h(A) v \quad (0 \leq i \leq D - 2), \tag{66}
\]

where \( \theta_0^*, \theta_1^*, \ldots, \theta_D^* \) denotes the dual eigenvalue sequence for \( \eta_n \). Moreover each side of (66) is zero for \( i = D - 3, i = D - 2 \). We note the denominators in (66) are nonzero by Lemmas 12.4, 5.5, and 5.6.
Proof. To verify (65), in the expression on the right eliminate \( p_h(A)v \) using (63), and simplify the result using (62), (64). We now have (66). For \( i = D - 3, i = D - 2 \), each side of (66) is zero by (65). \( \square \)

**Theorem 12.7** With reference to Definition 7.1 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \bar{\theta}_n \). Let \( \theta = \bar{\theta}_n \), and let \( g_0, g_1, \ldots, g_{D - 2} \) denote the associated polynomials from Definition 6.2. Then for all \( v \in U_\eta \) and for \( 0 \leq i \leq D - 2 \),

\[
E_{i+2}^* A_i v = g_i(A)v.
\] (67)

Moreover, each side of (67) is zero for \( i = D - 3, i = D - 2 \).

**Proof.** Choose an integer \( h \) \((0 \leq h \leq D - 2)\). Using Lemma 12.5 and (6), we find

\[
\frac{k_i b_i b_{i+1}}{k_i b_{i+1}} \frac{p_h(\theta)}{p_i(\theta)} = \frac{\theta^*_h - \theta^*_{h+2}}{\theta^*_i - \theta^*_i+2},
\] where denominators are nonzero by Lemmas 5.5, 5.6. The result now follows by Definition 6.2 and Theorem 12.7. \( \square \)

**Lemma 12.8** With reference to Definition 7.1 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \bar{\theta}_n \). Then for all nonzero \( v \in U_\eta \), the vectors \( E_{i+2}^* A_i v \) \((0 \leq i \leq D - 4)\) form a basis for \( Mv \).

**Proof.** By Corollary 11.6 the dimension of \( Mv \) is \( D - 3 \). By this and since \( A \) generates \( M \) we find \( v, Av, A^2 v, \ldots, A^{D-4} v \) form a basis for \( Mv \). For \( 0 \leq i \leq D - 2 \) let the polynomial \( g_i \) be as in Definition 6.2 (with \( \theta = \bar{\theta}_n \)). Recall \( g_i \) have degree \( i \). Apparently the vectors \( g_i(A)v \) \((0 \leq i \leq D - 4)\) form a basis for \( Mv \). By Theorem 12.7 we have \( g_i(A)v = E_{i+2}^* A_i v \) \((0 \leq i \leq D - 4)\). The result follows. \( \square \)

**Theorem 12.9** With reference to Definition 7.1 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \bar{\theta}_n \). Then for all nonzero \( v \in U_\eta \) the space \( Mv \) is a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \eta \).

**Proof.** We first show \( Mv \) is a \( T \)-module. It is clear \( Mv \) is closed under \( M \). By Lemma 12.8 and (div) we find \( Mv \) is closed under \( M^* \). Recall \( M \) and \( M^* \) generate \( T \) so \( Mv \) is a \( T \)-module. We show \( Mv \) is irreducible. From Lemma 12.8 we find \( v \) is a basis for \( E_2^* Mv \). In particular \( E_2^* Mv \) has dimension 1. Since \( Mv \) is a \( T \)-module it is a direct sum of irreducible \( T \)-modules. It follows there exists an irreducible \( T \)-module \( W' \) such that \( W' \subseteq Mv \) and such that \( E_2^* W' \neq 0 \). We show \( W' = Mv \). Observe \( E_2^* Mv \subseteq E_2^* Mv \), and we mentioned \( E_2^* Mv \) has dimension 1, so \( E_2^* W' = E_2^* Mv \). Now apparently \( v \in E_2^* W' \). Observe \( W' \) is \( M \)-invariant, so \( Mv \subseteq W' \), and it follows \( W' = Mv \). In particular \( Mv \) is irreducible. From Lemma 12.8 we find \( E_i^* Mv \) is 0 for \( i \in \{0, 1, D - 1, D\} \) and has dimension 1 for \( 2 \leq i \leq D - 2 \). Apparently \( Mv \) is thin with endpoint 2. We mentioned \( v \) is a basis for \( E_2^* Mv \). From the construction \( v \in U_\eta \) so \( Mv \) has local eigenvalue \( \eta \). \( \square \)

13 The thin irreducible \( T \)-modules with endpoint 2 and local eigenvalue \( \bar{\theta}_1 \) or \( \bar{\theta}_d \)

With reference to Definition 7.1 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. We now describe the thin irreducible \( T \)-modules with endpoint 2 and local eigenvalue \( \bar{\theta}_n \).

**Theorem 13.1** With reference to Definition 7.1 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Let \( W \) denote a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \bar{\theta}_n \). Let \( v \) denote a nonzero vector in \( E^*_2 W \). Then \( W = Mv \). The vectors

\[
E_i v \quad (1 \leq i \leq D - 1, \ i \neq n, \ i \neq D - n)
\] (68)

form a basis for \( W \) and \( E_0 v = 0, E_n v = 0, E_{D-n} v = 0, E_D v = 0 \).
Proof. Observe $W$ is $M$-invariant and $v \in W$ so $Mv \subseteq W$. Observe $v \in U_{\theta_n}$ by Definition 11.8 combining this with Theorem 12.8 we find $Mv$ is a $T$-module. Now $W = Mv$ by the irreducibility of $W$. We mentioned $v \in U_{\theta_n}$ by this and Lemma 11.9 we find each of the vectors in (68) are nonzero. Moreover $E_0v = 0$, $E_nv = 0$, $E_{D-n}v = 0$, and $E_Dv = 0$. Applying Lemma 11.1 we find the vectors in (68) form a basis for $Mv$. \[\square\]

Theorem 13.2 With reference to Definition 7.1 choose $n \in \{1, d\}$ if $D$ is odd, and let $n = 1$ if $D$ is even. Let $W$ denote a thin irreducible $T$-module with endpoint 2 and local eigenvalue $\theta_n$. The vectors in (68) are mutually orthogonal and
\[
\|E_i v\|^2 = \frac{m_i(\theta_n^2 - k^2)(\theta_n^2 - \theta_{n-1}^2)}{|X|kb_1(\theta_n^2 - b_2)}\|v\|^2 \quad (1 \leq i \leq D - 1, \; i \neq n, \; i \neq D - n), \tag{69}
\]
where the scalar $m_i$ denotes the multiplicity of $\theta_i$. We remark the denominator in (69) is nonzero by Lemma 7.3.

Proof. The vectors in (68) are mutually orthogonal by (7). To obtain (69) we apply Theorem 11.2. Set $\eta = \theta_n$ and observe $\eta \neq -1$ by Definition 11.3. Now (69) holds and (68) follows. \[\square\]

Theorem 13.3 With reference to Definition 7.1 choose $n \in \{1, d\}$ if $D$ is odd, and let $n = 1$ if $D$ is even. Let $W$ denote a thin irreducible $T$-module with endpoint 2 and local eigenvalue $\tilde{\theta}_n$. Let $v$ denote a nonzero vector in $E^*_2W$. Then the vectors $E_{i+2}^*Av (0 \leq i \leq D - 4)$ form a basis for $W$.

Proof. Observe $v \in U_{\tilde{\theta}_n}$ by Definition 11.8 and $W = Mv$ by Theorem 13.1. The result now follows in view of Lemma 12.8. \[\square\]

Theorem 13.4 With reference to Definition 7.1 choose $n \in \{1, d\}$ if $D$ is odd, and let $n = 1$ if $D$ is even. Let $W$ denote a thin irreducible $T$-module with endpoint 2 and local eigenvalue $\tilde{\theta}_n$. Abbreviate $\theta = \theta_n$, and let $g_0, g_1, \ldots, g_{D-2}$ denote the associated polynomials from Definition 6.2. Let $v$ denote a nonzero vector in $E^*_2W$. Then for $0 \leq i \leq D - 4$ we have
\[
E^*_i A_i v = \sum_{1 \leq j \leq D - 1, \; i \neq j \neq D - n} g_i(\theta_j)E_j v. \tag{70}
\]

Proof. By Theorem 12.7 we have $E^*_i A_i v = g_i(A)v$. In this equation, multiply $g_i(A)v$ on the left by $I$, expand using (eii), and simplify the result using $AE_j = \theta_j E_j (0 \leq j \leq D)$. Observe $E_0v = 0$, $E_nv = 0$, $E_{D-n}v = 0$, $E_Dv = 0$ by Theorem 13.1. \[\square\]

Theorem 13.5 With reference to Definition 7.1 choose $n \in \{1, d\}$ if $D$ is odd, and let $n = 1$ if $D$ is even. Let $W$ denote a thin irreducible $T$-module with endpoint 2 and local eigenvalue $\tilde{\theta}_n$. Let $v$ denote a nonzero vector in $E^*_2W$. The vectors $E^*_i A_i v (0 \leq i \leq D - 4)$ are mutually orthogonal and
\[
\|E^*_i A_i v\|^2 = \frac{k_i b_1 b_{i+1} c_{i+1} c_{i+2} p_i(\theta_n)}{kb_1(\theta_n^2 - b_2)}\|v\|^2 \quad (0 \leq i \leq D - 4), \tag{71}
\]
where the polynomials $p_i$ are as in (17). We remark the denominators in (71) are nonzero by Lemmas 6.3 and 5.2.

Proof. The vectors $E^*_i A_i v (0 \leq i \leq D - 4)$ are mutually orthogonal by (69). To verify (71), in the left-hand side eliminate $E^*_i A_i v$ using (70), and evaluate the result using Theorem 6.3 and Theorem 13.2. \[\square\]

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Theorem 13.6 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Let \( W \) denote a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \tilde{\theta}_n \). With respect to the basis for \( W \) given in Theorem 13.3 the matrix representing \( A \) is

\[
\begin{pmatrix}
0 & \omega_1 & 0 \\
c_1 & 0 & \omega_2 \\
c_2 & \ddots & \ddots \\
0 & \ddots & \ddots & \omega_{D-4} \\
0 & & c_{D-4} & 0
\end{pmatrix},
\]

where

\[
\omega_i = \frac{b_{i+1}c_{i+2}p_{i-1}(\theta_n)p_{i+2}(\theta_n)}{c_i p_i(\theta_n)p_{i+1}(\theta_n)} \quad (1 \leq i \leq D - 4).
\]

We remark the denominator in (72) is nonzero by Lemmas 5.5 and 5.6.

Proof. Let \( \theta = \theta_n \), and let \( g_0, g_1, \ldots, g_{D-2} \) denote the associated polynomials from Definition 6.2. Setting \( \lambda = A \) in (50) and applying the result to \( v \), we find

\[
Ag_i(A)v = c_{i+1}g_{i+1}(A)v + \omega_i g_{i-1}(A)v \quad (0 \leq i \leq D - 4),
\]

where \( g_{-1} = 0, \omega_0 = 0 \). The result follows in view of Theorem 12.7. \( \square \)

In summary we have the following theorem.

Theorem 13.7 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Let \( W \) denote a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \tilde{\theta}_n \). Then \( W \) has dimension \( D - 3 \). For \( 0 \leq i \leq D \), \( E_i^\ast W \) is zero if \( i \in \{0, 1, D - 1, D\} \) and has dimension 1 if \( i \notin \{0, 1, D - 1, D\} \). Moreover \( E_iW \) is zero if \( i \in \{0, n, D - n, D\} \) and has dimension 1 if \( i \notin \{0, n, D - n, D\} \).

Proof. The dimension of \( W \) is equal to \( D - 3 \) by Theorem 13.3. Fix an integer \( i \) (\( 0 \leq i \leq D \)). From Theorem 13.3 we find \( E_i^\ast W \) is zero if \( i \in \{0, 1, D - 1, D\} \) and has dimension 1 if \( i \notin \{0, 1, D - 1, D\} \). From Theorem 13.1 we find \( E_iW \) is zero if \( i \in \{0, n, D - n, D\} \) and has dimension 1 if \( i \notin \{0, n, D - n, D\} \). \( \square \)

14 Some multiplicities

With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Let \( W \) denote a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \tilde{\theta}_n \). In this section we consider the multiplicity with which \( W \) appears in the standard module \( V \).

Theorem 14.1 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Let \( W \) denote a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \tilde{\theta}_n \). Let \( W' \) denote an irreducible \( T \)-module. Then the following (i), (ii) are equivalent.

(i) \( W \) and \( W' \) are isomorphic as \( T \)-modules.

(ii) \( W' \) is thin with endpoint 2 and local eigenvalue \( \tilde{\theta}_n \).

Proof. (i) \( \Rightarrow \) (ii) Clear.

(ii) \( \Rightarrow \) (i) We display an isomorphism of \( T \)-modules from \( W \) to \( W' \). Observe \( E_i^\ast W \) and \( E_i^\ast W' \) are both nonzero. Let \( v \) (resp. \( v' \)) denote a nonzero vector in \( E_i^\ast W \) (resp. in \( E_i^\ast W' \)). By Theorem 13.3 the vectors

\[
E_{i+2}^\ast Ay \quad (0 \leq i \leq D - 4)
\]

form a basis for \( W \). Similarly the vectors

\[
E_{i+2}^\ast A_i v' \quad (0 \leq i \leq D - 4)
\]
Lemma 14.2 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \tilde{\theta}_n \). Then

\[
U_\eta = E^*_2 H_\eta,
\]

where \( H_\eta \) denotes the subspace of \( V \) spanned by all the thin irreducible \( T \)-modules with endpoint 2 and local eigenvalue \( \eta \).

Proof. We first show \( U_\eta \subseteq E^*_2 H_\eta \). Assume \( U_\eta \neq 0 \); otherwise the result is trivial. Let \( v \) denote a nonzero vector in \( U_\eta \). By Theorem 12.6 we find \( Mv \) is a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \eta \), so \( Mv \subseteq H_\eta \). Of course \( v \in Mv \) so \( v \in H_\eta \). By the construction \( v \in E^*_2 V \) so \( v = E^*_2 v \). It follows \( v \in E^*_2 H_\eta \). We have now shown \( U_\eta \subseteq E^*_2 H_\eta \). Next we show \( U_\eta \supseteq E^*_2 H_\eta \). To see this observe \( E^*_2 H_\eta \) is spanned by the \( E^*_2 W \), where \( W \) ranges over all thin irreducible \( T \)-modules with endpoint 2 and local eigenvalue \( \eta \). For all such \( W \) the space \( E^*_2 W \) is contained in \( U_\eta \) by Definition 11.8. It follows \( U_\eta \supseteq E^*_2 H_\eta \). \( \Box \)

Definition 14.3 With reference to Definition 7.1 and from our discussion in Section 7, the standard module \( V \) can be decomposed into an orthogonal direct sum of irreducible \( T \)-modules. Let \( W \) denote an irreducible \( T \)-module. By the multiplicity with which \( W \) appears in \( V \), we mean the number of irreducible \( T \)-modules in the above decomposition which are isomorphic to \( W \).

Definition 14.4 With reference to Definition 7.1 choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \tilde{\theta}_n \). We let \( \mu_\eta \) denote the multiplicity with which \( W \) appears in \( V \), where \( W \) is a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \eta \). If no such \( W \) exists we set \( \mu_\eta = 0 \).

Theorem 14.5 With reference to Definition 7.1, choose \( n \in \{1, d\} \) if \( D \) is odd, and let \( n = 1 \) if \( D \) is even. Define \( \eta = \tilde{\theta}_n \). Then the following scalars (i)–(iii) are equal:

(i) The scalar \( \mu_\eta \) from Definition 14.4.

(ii) The dimension of \( U_\eta \).

(iii) The scalar \( \text{mult}_\eta \) from Definition 10.8.

Proof. We mentioned below Definition 10.8 that the above scalars (ii), (iii) are equal. We now show the scalars (i), (ii) are equal. In view of Lemma 13.8 it suffices to show the dimension of \( E^*_2 H_\eta \) is \( \mu_\eta \). Observe \( H_\eta \) is a \( T \)-module so it is an orthogonal direct sum of irreducible \( T \)-modules. More precisely

\[
H_\eta = W_1 + W_2 + \cdots + W_m \quad \text{(orthogonal direct sum),}
\]

where \( m \) is a nonnegative integer, and where \( W_1, W_2, \ldots, W_m \) are thin irreducible \( T \)-modules with endpoint 2 and local eigenvalue \( \eta \). Apparently \( m \) is equal to \( \mu_\eta \). We show \( m \) is equal to the dimension of \( E^*_2 H_\eta \). Applying \( E^*_2 \) to \( (75) \) we find

\[
E^*_2 H_\eta = E^*_2 W_1 + E^*_2 W_2 + \cdots + E^*_2 W_m \quad \text{(orthogonal direct sum).}
\]

Observe each summand on the right in \( (76) \) has dimension 1. These summands are mutually orthogonal so \( m \) is equal to the dimension of \( E^*_2 H_\eta \). Now apparently \( \mu_\eta \) is equal to the dimension of \( E^*_2 H_\eta \), as desired. It follows the scalars (i), (ii) above are equal. \( \Box \)
15 Taut graphs and the local eigenvalues

In this section we assume $\Gamma$ is bipartite with diameter $D \geq 4$, valency $k \geq 3$, and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let $d = \lceil D/2 \rceil$. We now recall the taut condition. In [8, Theorem 18] Curtin showed that $b_2(k - 2) \geq (c_2 - 1)\theta_1^2$ with equality if and only if $\Gamma$ is 2-homogeneous in the sense of Nomura [25]. In [8, Theorem 12] Curtin showed $\Delta \geq 0$, where

$$\Delta = (k - 2)(c_3 - 1) - (c_2 - 1)p_{22}^2.$$  

In [22] Lemma 3.8] MacLean proved that

$$b_3\left(b_2(k - 2) - (c_2 - 1)\theta_1^2\right)(b_2(k - 2) - (c_2 - 1)\theta_2^2) \geq b_1(\theta_2^2 - b_2)(b_2 - \theta_2^2).$$  

(77)

We remark that the inequality [77] looks different from the inequality presented in [22], but it is straightforward to show that these two inequalities are equivalent. Recall from Lemma 8.3 that $\theta_1^2 > b_2 > \theta_2^2$, so the last two factors on the right in (77) are positive. Observe each factor in (77) is nonnegative. From these comments we find that $\Gamma$ is 2-homogeneous if and only if $\Delta = 0$ and equality holds in (77). MacLean defined $\Gamma$ to be taut whenever $\Delta \neq 0$ and equality holds in (77).

Fix $x \in X$. In this section we give a characterization of the taut condition in terms of the local eigenvalues of $\Gamma$ with respect to $x$. In order to motivate our results we sketch a proof of (77).

**Proof of (77).** Fix $x \in X$ and let $\eta_1, \eta_2, \ldots, \eta_k$ denote the corresponding local eigenvalues of $\Gamma$. Consider the sum

$$\sum_{i=1}^{k_2}(\eta_i - \hat{\theta}_1)(\eta_i - \hat{\theta}_d).$$

(78)

We evaluate (78) in two ways. First, by Theorem 11.4 and since $\eta_i = p_{22}^2$, $\eta_i = b_3 - 1$ (2 $\leq i \leq k$) we find (78) is at most $(p_{22}^2 - \hat{\theta}_1)(p_{22}^2 - \hat{\theta}_d) + (k - 1)(b_3 - 1 - \hat{\theta}_1)(b_3 - 1 - \hat{\theta}_d)$. Second, we determine (78) by computing $\sum_{i=1}^{k_2} \eta_i$ and $\sum_{i=1}^{k_2} \eta_i^2$. By Lemma 11.7(ii),(iii) we find $\sum_{i=1}^{k_2} \eta_i = 0$ and $\sum_{i=1}^{k_2} \eta_i^2 = k_2p_{22}^2$. From these comments the expression (78) is equal to $k_2(p_{22}^2 + \hat{\theta}_1\hat{\theta}_d)$. We now have an inequality involving $k, k_2, p_{22}^2, b_3, \hat{\theta}_1, \hat{\theta}_d$; eliminating $\hat{\theta}_1, \hat{\theta}_d$ using 55 and simplifying using 10 and 11, we get (77). \[\square\]

In order to gain some insight into the case in which $\Gamma$ is taut, we examine the above proof. We begin with a definition.

**Definition 15.1** With reference to Definition 14.1 we say $\Gamma$ is spectrally taut with respect to $x$ whenever $\eta_i$ is one of $\hat{\theta}_1, \hat{\theta}_d$ for $k + 1 \leq i \leq k_2$. (The scalars $\eta_i$ and $\hat{\theta}_i$ are from Definition 10.1 and Definition 11.3 respectively.)

From the above proof of (77) we routinely obtain the following.

**Theorem 15.2** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Then the following (i)–(iii) are equivalent.

(i) $\Gamma$ is taut.

(ii) $\Delta \neq 0$ and $\Gamma$ is spectrally taut with respect to each vertex.

(iii) $\Delta \neq 0$ and $\Gamma$ is spectrally taut with respect to at least one vertex.

**Corollary 15.3** With reference to Definition 12.1 assume that $\Gamma$ is taut. Then

$$\text{mult}_{\hat{\theta}_1} = \frac{k(\theta_1^2 - b_2)(b_2(k - 2) - (c_2 - 1)\theta_2^2)}{(\theta_1^2 - \theta_d^2)b_2c_2},$$

(79)

$$\text{mult}_{\hat{\theta}_d} = \frac{k(\theta_d^2 - b_2)(b_2(k - 2) - (c_2 - 1)\theta_1^2)}{(\theta_d^2 - \theta_1^2)b_2c_2}.$$  

(80)

Moreover each of $\text{mult}_{\hat{\theta}_1}, \text{mult}_{\hat{\theta}_d}$ is nonzero.
Proof. By Theorem 15.2 \( \Phi \subseteq \{ \bar{\theta}_1, \bar{\theta}_d \} \), where \( \Phi \) is from Definition 10.3. Applying Lemma 11.1 (i), (ii), we find

\[
k + \operatorname{mult}_{\bar{\theta}_1} + \operatorname{mult}_{\bar{\theta}_d} = k_2, \quad p_{22}^2 + (k - 1)(b_3 - 1) + \bar{\theta}_1 \operatorname{mult}_{\bar{\theta}_1} + \bar{\theta}_d \operatorname{mult}_{\bar{\theta}_d} = 0.\]

Solving these two equations for \( \operatorname{mult}_{\bar{\theta}_1} \), \( \operatorname{mult}_{\bar{\theta}_d} \) and simplifying using (9), (11) and (52), we obtain (79), (80). Now we show \( \operatorname{mult}_{\bar{\theta}_1} \), \( \operatorname{mult}_{\bar{\theta}_d} \) are nonzero. Suppose \( \operatorname{mult}_{\bar{\theta}_1} = 0 \). Since \( \theta_1^2 - b_2 \neq 0 \) by Lemma 3.5 by the form of (79) we conclude that \( b_2(k - 2) - (c_2 - 1)\theta_2^2 = 0 \). Since equality holds in (77), we have \( \Delta = 0 \), contradicting Theorem 15.2. The proof that \( \operatorname{mult}_{\bar{\theta}_d} \) is nonzero is similar.

\[\square\]

16 Taut graphs and the subconstituent algebra

With reference to Definition 7.1, assume \( D \) is odd. In this section we give two characterizations of the taut condition in terms of the subconstituent algebra \( T \). We begin with the following definition.

Definition 16.1 With reference to Definition 7.1, assume \( D \) is odd. We say \( \Gamma \) is algebraically taut with respect to \( x \) whenever every irreducible \( T \)-module with endpoint 2 is thin with local eigenvalue \( \bar{\theta}_1 \) or \( \bar{\theta}_d \).

The notions of spectrally taut and algebraically taut are related as follows.

Lemma 16.2 With reference to Definition 7.1, assume \( D \) is odd. Then the following (i), (ii) are equivalent.

(i) \( \Gamma \) is spectrally taut with respect to \( x \).

(ii) \( \Gamma \) is algebraically taut with respect to \( x \).

Proof. (i) \( \Rightarrow \) (ii) Let \( W \) denote an irreducible \( T \)-module with endpoint 2. We show \( W \) is thin with local eigenvalue \( \bar{\theta}_1 \) or \( \bar{\theta}_d \). Observe \( E_2W \) is nonzero and invariant under \( E_2^*A_2E_2 \). Therefore there exists a nonzero vector \( v \in E_2^*W \) which is an eigenvector for \( E_2^*A_2E_2 \). Let \( \eta \) denote the corresponding eigenvalue. Let \( Y \) denote the subspace of \( V \) spanned by all the irreducible \( T \)-modules with endpoint 1. Observe \( E_2^*W \) is orthogonal to \( V_0 + Y \) so \( v \in U \) by Definition 10.2. Now \( \eta \) is one of \( \eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2} \). By this and Definition 15.1 we find \( \eta \) is one of \( \bar{\theta}_1, \bar{\theta}_d \). By Theorem 15.2 we find \( Mv \) is a thin irreducible \( T \)-module with endpoint 2 and local eigenvalue \( \eta \). Observe \( Mv \subseteq W \) so \( Mv = W \) by the irreducibility of \( W \). Apparently \( W \) is thin with local eigenvalue \( \eta \) and the result follows.

(ii) \( \Rightarrow \) (i) Let \( S \) denote the subspace of \( V \) spanned by all the irreducible \( T \)-modules with endpoint 2. Then

\[
S = H_{\bar{\theta}_1} + H_{\bar{\theta}_d} \quad \text{(orthogonal direct sum)},
\]

where \( H_{\bar{\theta}_1} \) and \( H_{\bar{\theta}_d} \) are from Lemma 14.2. Let \( Y \) denote the subspace of \( V \) spanned by all the irreducible \( T \)-modules with endpoint 1. Recall \( U \) denotes the orthogonal complement of \( E_2^*V_0 + E_2^*Y \) in \( E_2^*V \). Applying \( E_2^* \) to each term in (81), and evaluating the result using Lemma 14.2 and \( E_2^*S = U \), we obtain

\[
U = U_{\bar{\theta}_1} + U_{\bar{\theta}_d} \quad \text{(orthogonal direct sum)}.
\]

Comparing (82) and (50) we find \( \eta_i \) is one of \( \bar{\theta}_1, \bar{\theta}_d \) for \( k + 1 \leq i \leq k_2 \). Now \( \Gamma \) is spectrally taut with respect to \( x \) by Definition 15.1. \( \square \)

Definition 16.3 With reference to Definition 7.1, assume \( D \) is odd. We say \( \Gamma \) is taut with respect to \( x \) whenever the equivalent conditions (i), (ii) hold in Lemma 10.2.

Combining Theorem 15.2 and Definition 16.3, we immediately obtain the following theorem.

Theorem 16.4 Let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 4 \) and valency \( k \geq 3 \). Assume \( D \) is odd. Then the following (i)–(iii) are equivalent.

(i) \( \Gamma \) is taut.
(ii) \( \Delta \neq 0 \) and \( \Gamma \) is taut with respect to each vertex.

(iii) \( \Delta \neq 0 \) and \( \Gamma \) is taut with respect to at least one vertex.

For our final characterization of the taut condition, we will need the following two definitions.

**Definition 16.5** Let \( \Gamma \) denote a distance-regular graph with vertex set \( X \) and diameter \( D \geq 3 \). We say \( \Gamma \) is an antipodal 2-cover whenever for all \( x \in X \), there exists a unique vertex \( y \in X \) such that \( \partial(x, y) = D \). In other words, \( \Gamma \) is an antipodal 2-cover if and only if \( k_D = 1 \).

**Definition 16.6** With reference to Definition 7.1, we say \( \Gamma \) is 2-thin with respect to \( x \) whenever every irreducible \( T \)-module with endpoint 2 is thin.

**Theorem 16.7** Let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 4 \) and valency \( k \geq 3 \). Assume \( D \) is odd. Then the following (i)–(iii) are equivalent:

(i) \( \Gamma \) is taut or 2-homogeneous.

(ii) \( \Gamma \) is an antipodal 2-cover and 2-thin with respect to each vertex.

(iii) \( \Gamma \) is an antipodal 2-cover and 2-thin with respect to at least one vertex.

*Proof.* (i) \( \Rightarrow \) (ii) Let \( x \) denote a vertex of \( \Gamma \). First suppose \( \Gamma \) is taut. Then \( \Gamma \) is an antipodal 2-cover by [24, Theorem 6.4], and \( \Gamma \) is 2-thin with respect to \( x \) by Theorem 16.4. Now suppose \( \Gamma \) is 2-homogeneous. By [8, Theorem 42], \( \Gamma \) is an antipodal 2-cover, and \( \Delta = 0 \) and equality holds in (77). Then \( \Gamma \) is spectrally taut with respect to \( x \) by the proof of (77). Then \( \Gamma \) is 2-thin with respect to \( x \) by Definition 16.1 and Lemma 16.2.

(ii) \( \Rightarrow \) (iii) Clear.

(iii) \( \Rightarrow \) (i) By assumption there exists a vertex with respect to which \( \Gamma \) is 2-thin. Denote this vertex by \( x \) and write \( T = T(x) \). Let \( W \) denote an irreducible \( T \)-module of \( \Gamma \) with endpoint 2. Then the dimension of \( W \) is \( D - 3 \) by [5, Lemma 14.1]. Now by Corollary 11.6, the local eigenvalue of \( W \) is either \( \hat{\theta}_1 \) or \( \hat{\theta}_d \). Then \( \Gamma \) is algebraically taut with respect to \( x \) by Definition 16.1. Thus \( \Gamma \) is spectrally taut with respect to \( x \) by Lemma 16.2, so equality holds in (77). Thus \( \Gamma \) is taut or 2-homogeneous. \( \square \)

17 Directions for further research

In this section we give some suggestions for further research.

We start with a problem that we admit is quite general but we believe is important.

**Problem 17.1** With reference to Definition 7.1, assume that up to isomorphism there exist at most two irreducible \( T \)-modules with endpoint 2, and they are both thin. Investigate the combinatorial and algebraic implications of this assumption.

**Remark 17.2** With reference to Definition 7.1, assume \( \Gamma \) is taut and \( D \) is odd. Then \( \Gamma \) satisfies the assumptions of Problem 17.1 by Theorem 16.4.

**Remark 17.3** With reference to Definition 7.1, assume \( \Gamma \) is \( Q \)-polynomial. Then \( \Gamma \) satisfies the assumptions of Problem 17.1 [3, Section 14].

**Remark 17.4** Assume \( \Gamma \) satisfies the assumptions of Problem 17.1. Then using [27, Theorem 13.1] we can recursively obtain the intersection numbers of \( \Gamma \) in terms of the diameter \( D \), the local eigenvalues of the \( T \)-modules mentioned in Problem 17.1, and the multiplicities with which these modules appear in \( V \). The resulting formulae are not attractive however.

**Problem 17.5** Assume \( \Gamma \) satisfies the assumptions of Problem 17.1. Obtain the intersection numbers of \( \Gamma \) in closed form as attractive rational expressions involving \( D \) and at most four complex parameters.

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Problem 17.6 With reference to Definition 17.4 let $W$ denote a thin irreducible $T$-module with endpoint 2. Let $\eta$ denote the local eigenvalue of $W$ and assume $\vartheta_1 < \eta < \vartheta_d$. Describe the structure of $W$ along the lines of Section 13 in the present paper.

The following problem is of interest in view of Theorem 15.2.

**Problem 17.7** With reference to Definition 17.4 and Definition 19.2 assume $D$ is even and there exists a nonzero vector $v \in U$ that is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $b_3 = 1$. Show the $T$-module $Tv$ is irreducible with basis $\{E_i^* A_i v \mid 0 \leq i \leq D - 3\} \cup \{E_i^* A_i v \mid 3 \leq i \leq D - 3, \; i \text{ odd}\}$.

Compute the matrix representing $A$ with respect to this basis.

**Problem 17.8** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $E, F$ denote pseudo primitive idempotents of $\Gamma$ [32]. We say the pair $E, F$ is *taut* whenever $E \circ F$ is a linear combination of at most two pseudo primitive idempotents of $\Gamma$. Find all the taut pairs of pseudo primitive idempotents of $\Gamma$. See [32] for related results concerning tight pairs of pseudo primitive idempotents.

**Conjecture 17.9** With reference to Definition 17.4 assume that up to isomorphism there exist exactly two irreducible $T$-modules with endpoint 2, and they are both thin. Pick $\xi, \chi \in \mathbb{C} \cup \infty$ such that $\xi$ and $\chi$ are the local eigenvalues for these $T$-modules. Let $E$ (resp. $F$) denote a pseudo primitive idempotent [39] of $\Gamma$ for $\xi$ (resp. $\chi$). Then the pair $E, F$ is taut in the sense of Problem 17.8.

**Conjecture 17.10** Let $\Gamma$ denote a taut distance-regular graph with odd diameter $D \geq 5$. Recall $\Gamma$ is an antipodal 2-cover by Theorem 16.7, we conjecture that the antipodal quotient of $\Gamma$ is $Q$-polynomial.

**Problem 17.11** Let $\Gamma = (X, \mathcal{R})$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Fix $x, y \in X$ at distance $\partial(x, y) = 2$. For $0 \leq i, j \leq D$ define $w_{ij} = \sum \hat{z}$, where the sum is over all $z \in X$ such that $\partial(x, z) = i$ and $\partial(y, z) = j$. Show that the following are equivalent: (i) $\Gamma$ is taut; (ii) The vectors $E_2 \hat{x}, E_1 \hat{y}, E_1 w_{11}, E_1 w_{22}$ are linearly dependent and the vectors $E_d \hat{x}, E_d \hat{y}, E_d w_{11}, E_d w_{22}$ are linearly dependent, where $d = \lfloor D/2 \rfloor$.

**Problem 17.12** Let $\Gamma = (X, \mathcal{R})$ denote a taut distance-regular graph with odd diameter $D \geq 5$. Fix $x, y \in X$ at distance $\partial(x, y) = 2$, and let the vectors $w_{ij}$ be as in Problem 17.11 For $2 \leq i \leq D - 2$ define $w_{ii}^+ = \sum | \{r \in X \mid \partial(r, x) = 1, \partial(r, y) = 1, \partial(r, z) = i - 1\} \cdot \hat{z} |$, where the sum is over all $z \in X$ such that $\partial(x, z) = i$ and $\partial(y, z) = i$. Show that the vectors

$$\{w_{ij} \mid 0 \leq i, j \leq D, \; |i - j| = 2\} \cup \{w_{ii} \mid 1 \leq i \leq D - 1\} \cup \{w_{ii}^+ \mid 2 \leq i \leq D - 2\}$$

form a basis for an $A$-invariant subspace of the standard module. Find the matrix representing $A$ with respect to this basis.

**References**

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London, 1984.

[2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, London, 1994.

[3] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[4] J. S. Caughman IV, The Terwilliger algebras of bipartite P- and Q-polynomial schemes, *Discrete Math.*, 196(1999), 65-95.

[5] B. Collins, The Terwilliger algebra of an almost-bipartite distance-regular graph and its antipodal 2-cover, *Discrete Math.*, 216(2000), 35-69.
[6] B. Curtin, Bipartite distance-regular graphs I, Graphs Comb., 15(1999), 143-158.
[7] B. Curtin, Bipartite distance-regular graphs II, Graphs Comb., 15(1999), 377-391.
[8] B. Curtin, 2-homogeneous bipartite distance-regular graphs, Discrete Math., 187(1998), 39–70.
[9] B. Curtin and K. Nomura, Distance-regular graphs related to the quantum enveloping algebra of $sl(2)$, J. Algebr. Comb., 12(2000), 25–36.
[10] C. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, 1962.
[11] G. Dickie, Twice $Q$-polynomial distance-regular graphs are thin, Europ. J. Combinatorics, 16(1995), 555–560.
[12] E. Egge, A generalization of the Terwilliger algebra, J. Algebra, 233(2000), 213-252.
[13] J. T. Go, The Terwilliger algebra of the hypercube, Europ. J. Combinatorics, 23(2002), 399–429.
[14] J. T. Go and P. Terwilliger, Tight distance-regular graphs and the subconstituent algebra, Europ. J. Combinatorics, 23(2002), 793–816.
[15] C. D. Godsil, Algebraic Combinatorics, Chapman and Hall Inc., New York, 1993.
[16] S. A. Hobart and T. Ito, The structure of nonthin irreducible $T$-modules: Ladder bases and classical parameters, J. Algebr. Comb., 7(1998), 53-75.
[17] A. Jurišić and J. Koolen, A local approach to 1-homogeneous graphs, Des. Codes Cryptogr., 21(2000), 127–147.
[18] A. Jurišić and J. Koolen, Nonexistence of some antipodal distance-regular graphs of diameter four, Europ. J. Combinatorics, 21(2000), 1039–1046.
[19] A. Jurišić and J. Koolen, 1-homogeneous graphs with cocktail party $\mu$-graphs, J. Algebr. Comb., 18(2003), 79-98.
[20] A. Jurišić and J. Koolen, Krein parameters and antipodal distance-regular graphs with diameter 3 and 4, Discrete Math., 244(2002), 181-202.
[21] A. Jurišić, J. Koolen, and P. Terwilliger, Tight distance-regular graphs, J. Algebr. Comb., 12(2000), 163-197.
[22] M. MacLean, An inequality involving two eigenvalues of a bipartite distance-regular graph, Discrete Math., 225(2000), 193–216.
[23] M. MacLean, Taut distance-regular graphs of even diameter, J. Comb. Theory Ser. B, 91(2004), 127-142.
[24] M. MacLean, Taut distance-regular graphs of odd diameter, J. Algebr. Comb., 17(2003), 125-147.
[25] K. Nomura, Homogeneous graphs and regular near polygons, J. Comb. Theory Ser. B, 60(1994), 63–71.
[26] K. Nomura, Spin models on bipartite distance-regular graphs, J. Comb. Theory Ser. B, 64(1995), 300–313.
[27] K. Nomura, Spin models and almost bipartite 2-homogeneous graphs, Adv. Stud. Pure Math., 24 Mathematical Society Japan, Tokyo, 1996, 285–308.
[28] A. A. Pascasio, Tight graphs and their primitive idempotents, J. Algebr. Comb., 10(1999), 47-59.
[29] A. A. Pascasio, Tight distance-regular graphs and $Q$-polynomial property, Graphs Comb., 17(2001), 149–169.
[30] A. A. Pascasio, An inequality on the cosines of a tight distance-regular graph, *Linear Algebr. Appl.*, 325(2001), 147–159.

[31] A. A. Pascasio, An inequality in character algebras, *Discrete Math.*, 264(2003), 201–210.

[32] A. A. Pascasio and P. Terwilliger, The pseudo cosine sequences of a distance-regular graph, *J. Algebr. Comb.*, submitted.

[33] K. Tanabe, The irreducible modules of the Terwilliger algebras of Doob schemes, *J. Algebr. Comb.*, 6(1997), 173–195.

[34] P. Terwilliger, A new feasibility condition for distance-regular graphs, *Discrete Math.*, 61(1986), 311–315.

[35] P. Terwilliger, The subconstituent algebra of an association scheme I, *J. Algebr. Comb.*, 1(1992), 363–388.

[36] P. Terwilliger, The subconstituent algebra of an association scheme II, *J. Algebr. Comb.*, 2(1993), 73–103.

[37] P. Terwilliger, The subconstituent algebra of an association scheme III, *J. Algebr. Comb.*, 2(1993), 177–210.

[38] P. Terwilliger, The subconstituent algebra of a distance-regular graph; thin modules with endpoint one, *Linear Algebr. Appl.*, 356(2002), 157–187.

[39] P. Terwilliger and Chih-wen Weng, Distance-regular graphs, pseudo primitive idempotents, and the Terwilliger algebra, *Europ. J. Combinatorics*, 25(2004), 287–298.

[40] M. Tomiyama, On the primitive idempotents of distance-regular graphs, *Discrete Math.*, 240(2001), 281–294.

[41] M. Tomiyama and N. Yamazaki, The subconstituent algebra of a strongly regular graph, *Kyushu J. Math.*, 48(1998), 323–334.