Using an old method of Jacobi to derive Lagrangians: a nonlinear dynamical system with variable coefficients

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Abstract

We present a method devised by Jacobi to derive Lagrangians of any second-order differential equation: it consists in finding a Jacobi Last Multiplier. We illustrate the easiness and the power of Jacobi’s method by applying it to the same equation studied by Musielak et al. with their own method [Musielak ZE, Roy D and Swift LD. Method to derive Lagrangian and Hamiltonian for a nonlinear dynamical system with variable coefficients. Chaos, Solitons & Fractals, 2008;58:894-902]. While they were able to find one particular Lagrangian after lengthy calculations, Jacobi Last Multiplier method yields two different Lagrangians (and many others), of which one is that found by Musielak et al, and the other(s) is(are) quite new.

1 Introduction

It should be well-known that the knowledge of a Jacobi Last Multiplier always yields a Lagrangian of any second-order ordinary differential equation [9, 27]. Yet many distinguished scientists seem to be unaware of this classical result. In this paper we present again the method of the Jacobi Last Multiplier in order to compare the easiness and the power of Jacobi’s method with that proposed by Musielak et al [12] for the same purpose. We have already presented Jacobi’s method in [17]. The references in [17] and the papers [15-26] may give an idea of the many fields of applications yielded by Jacobi Last Multiplier.

In [12] the authors searched for a Lagrangian of the following second-order ordinary differential equation

\[ \ddot{x} + b(x)\dot{x}^2 + c(x)x = 0 \]  

(1)

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with \( b(x), c(x) \) arbitrary functions of the dependent variable \( x = x(t) \). After some lengthy calculations they found one Lagrangian. In the present paper we apply Jacobi’s method to equation (1) and show that many (an infinite number of) Lagrangians can be easily derived.

This paper is organized in the following way. In section 2, we illustrate the Jacobi last multiplier and its properties [5]-[9], its connection to Lie symmetries [10], [11], and its link to the Lagrangian of any second-order differential equations [9], [27]. Moreover we exemplify Jacobi’s method by determining the Lagrangians of two equations studied by Euler [4] and Jacobi himself [8] for the purpose of finding their multipliers. In section 3, we apply Jacobi’s method to equation (1) and determine some of its many Lagrangians. In section 4, we conclude with some final remarks.

Here we employ ad hoc interactive programs [14] written in REDUCE language to calculate the Lie symmetry algebra of the equations we study.

2 The method by Jacobi

The method of the Jacobi last multiplier [6]-[9]) provides a means to determine all the solutions of the partial differential equation

\[
Af = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0
\]  

or its equivalent associated Lagrange’s system

\[
\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}.
\]  

In fact, if one knows the Jacobi last multiplier and all but one of the solutions, then the last solution can be obtained by a quadrature. The Jacobi last multiplier \( M \) is given by

\[
\frac{\partial (f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial (x_1, x_2, \ldots, x_n)} = MAf,
\]  

where

\[
\frac{\partial (f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial (x_1, x_2, \ldots, x_n)} = \text{det}
\begin{bmatrix}
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\
\frac{\partial f}{\partial \omega_1} & \cdots & \frac{\partial f}{\partial \omega_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial f}{\partial \omega_{n-1}} & \cdots & \frac{\partial f}{\partial \omega_{n-1}}
\end{bmatrix} = 0
\]  

2
and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (2) or, equivalently, first integrals of (3) independent of each other. This means that $M$ is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi last multiplier are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of equation (2), then the corresponding last multiplier $N$ is linked to $M$ by the relationship:

$$N = M \frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$ 

(b) Given a non-singular transformation of variables

$$\tau: (x_1, x_2, \ldots, x_n) \longrightarrow (x'_1, x'_2, \ldots, x'_n),$$

then the last multiplier $M'$ of $A'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

where $M$ obviously comes from the $n-1$ solutions of $AF = 0$ which correspond to those chosen for $A'F = 0$ through the inverse transformation $\tau^{-1}$.

(c) One can prove that each multiplier $M$ is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0;$$

viceversa every solution $M$ of this equation is a Jacobi last multiplier.

(d) If one knows two Jacobi last multipliers $M_1$ and $M_2$ of equation (2), then their ratio is a solution $\omega$ of (2), or, equivalently, a first integral of (3). Naturally the ratio may be quite trivial, namely a constant. Viceversa the product of a multiplier $M_1$ times any solution $\omega$ yields another last multiplier $M_2 = M_1 \omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternate formulation in terms of symmetries was provided by Lie [11]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [3]. If we know $n-1$ symmetries of (2)/(3), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n)\partial_{x_j}, \quad i = 1, n - 1,$$
Jacobi’s last multiplier is given by \( M = \Delta^{-1} \), provided that \( \Delta \neq 0 \), where

\[
\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & \xi_{1,n} \\ \vdots & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}.
\]

(8)

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing the variable associated with that component, i.e., \( \partial a_i / \partial x_i = 0 \), the last multiplier is a constant, and any other Jacobi Last Multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its (almost forgotten) relationship with the Lagrangian, \( L = L(t, x, \dot{x}) \), for any second-order equation

\[
\ddot{x} = F(t, x, \dot{x})
\]

is [9], [27]

\[
M = \frac{\partial^2 L}{\partial \dot{x}^2}
\]

(10)

where \( M = M(t, x, \dot{x}) \) satisfies the following equation

\[
\frac{d}{dt}(\log M) + \frac{\partial F}{\partial \dot{x}} = 0.
\]

(11)

Then equation (9) becomes the Euler-Lagrangian equation:

\[
-\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + \frac{\partial L}{\partial x} = 0.
\]

(12)

The proof is given by taking the derivative of (12) by \( \dot{x} \) and showing that this yields (11). If one knows a Jacobi last multiplier, then \( L \) can be easily obtained by a double integration, i.e.:

\[
L = \int \left( \int M \, d\dot{x} \right) \, d\dot{x} + f_1(t, x)\dot{x} + f_2(t, x),
\]

(13)

where \( f_1 \) and \( f_2 \) are functions of \( t \) and \( x \) which have to satisfy a single partial differential equation related to [21] [22]. As it was shown in [22], \( f_1, f_2 \) are related to the gauge function \( g = g(t, x) \). In fact, we may assume

\[
f_1 = \frac{\partial g}{\partial x},
\]

\[
f_2 = \frac{\partial g}{\partial t} + f_3(t, x)
\]

(14)
where \( f_3 \) has to satisfy the mentioned partial differential equation and \( g \) is obviously arbitrary.

In [22] it was shown that if one knows several (at least two) Lie symmetries of the second-order differential equation (9), i.e.

\[
\Gamma_j = V_j(t, x) \partial_t + G_j(t, x) \partial_x, \quad j = 1, r,
\]

then many Jacobi Last Multipliers could be derived by means of (8), i.e.

\[
\frac{1}{M_{nm}} = \Delta_{nm} = \det \begin{bmatrix}
1 & \dot{x} & F(t, x, \dot{x}) \\
V_n & G_n & \frac{\partial \dot{V}_n}{\partial t} - \dot{x} \frac{\partial V_n}{\partial t} \\
V_m & G_m & \frac{\partial \dot{V}_m}{\partial t} - \dot{x} \frac{\partial V_m}{\partial t}
\end{bmatrix},
\]

with \((n, m = 1, r)\), and therefore many Lagrangians can be obtained by means of (13).

In [22] fourteen different Lagrangians \(^{1}\) were derived even for an equation as controversial as the damped linear harmonic oscillator, i.e.:

\[
\ddot{x} + cx + kx = 0,
\]

which about 80 years ago was thought not to be derivable from a variational principle by Bauer \[2\] and to be “a physically incomplete system”, namely in need of additional equations \(^2\) by Bateman \[1\].

2.1 Two examples by Euler in [8]

In [8], Jacobi found his “new multiplier” for the following class of second-order ordinary differential equations \(^3\) studied by Euler \[4\] [Sect. I, Ch. VI, §§915 ff.]:

\[
\ddot{x} + \frac{1}{2} \frac{\partial \varphi}{\partial x} \dot{x}^2 + \frac{\partial \varphi}{\partial t} \dot{x} + B = 0
\]

with \( \varphi, B \) arbitrary functions of \( t \) and \( x \). Indeed Jacobi derived that the multiplier of equation (18) is given by:

\[
M = e^{\varphi(t, x)},
\]

as it is obvious from (11). Then, Jacobi presented two examples of the class of equations (18), also studied by Euler, to illustrate the use of his multiplier, namely that the knowledge of

\(^1\)The fourteen Lagrangians are independent from each other and not related by any gauge function. They are derived from the eight-dimensional Lie symmetry algebra which is admitted by any linear second-order ordinary differential equations [11].

\(^2\)Bateman called them “a complementary sets of equations”.

\(^3\)This is not Jacobi’s original notation.
one first integral and one multiplier yields integration by quadrature. Here we report those
equations for the reader’s convenience. We also find the corresponding Lagrangian and search
for Lie symmetries of Euler’s equation in order to describe the link between Lie symmetries,
Jacobi Last Multiplier, first integrals, and of course Lagrangians. In fact it easy to derive a
Lagrangian of equation (18) by means by means of (10), i.e.
\[
L = \frac{1}{2} e^{\phi(t,x)} \dot{x}^2 + f_1(t,x) \dot{x} + f_2(t,x)
\]  
(20)
with \(f_1, f_2\) functions of \(t\) and \(x\) satisfying the following equation:
\[
\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{\phi(t,x)} B(t,x).
\]  
(21)

2.1.1 Example I

The first example by Euler that Jacobi presented is the following equation:
\[
x^2 \ddot{x} + x \dot{x}^2 + \beta x - \gamma t = 0
\]  
(22)
with \(\beta\) and \(\gamma\) arbitrary constants. Jacobi obtained the multiplier
\[
M_1 = x^2,
\]  
(23)
by means of (19) and showed how to integrate equation (22).

We can use Jacobi’s multiplier (23) to derive a Lagrangian of equation (22) by means of
(10), i.e.
\[
L_1 = \frac{1}{2} x^2 \dot{x}^2 + f_1(t,x) \dot{x} + f_2(t,x)
\]  
(24)
with \(f_1, f_2\) functions of \(t\) and \(x\) satisfying the following equation:
\[
\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} - \beta x + \gamma t = 0.
\]  
(25)
If we consider the transformation (14) between \(f_1, f_2\) and the gauge function \(g\), then equation
(25) becomes:
\[
- \frac{\partial f_3}{\partial x} - \beta x + \gamma t = 0,
\]  
(26)
which can be easily integrated, i.e.:
\[
f_3 = \frac{1}{2} (-\beta x^2 + 2\gamma tx),
\]  
(27)
\footnote{Jacobi’s original paper is in Latin, but the mathematical formulas could be understood by any mathematician.}
Then $L_1$ can be written as follows:

$$L_1 = \frac{1}{2}x^2\ddot{x} + \frac{1}{2}(-\beta x^2 + 2\gamma tx) + \frac{dg}{dt}. \quad (28)$$

We remark that equation (22) can be transformed into an autonomous equation by considering the canonical variables $\tilde{t}, \tilde{x}$ of the only Lie point symmetry admitted for any $\gamma \neq 0$, i.e.

$$\Gamma = t\partial_t + x\partial_x, \quad \Rightarrow \quad \tilde{t} = \log(t), \quad \tilde{x} = \frac{x}{t}. \quad (29)$$

Then equation (22) becomes

$$\frac{d\tilde{x}}{dt} \frac{d^2\tilde{x}}{dt^2} = -\beta \tilde{x} + \gamma - \tilde{x}^3 - 3\tilde{x}^2 \frac{d\tilde{x}}{dt} - \tilde{x} \left( \frac{d\tilde{x}}{dt} \right)^2. \quad (30)$$

with Jacobi last multiplier and consequently Lagrangian

$$\tilde{M}_1 = e^{3\tilde{t}}\tilde{x}^2, \quad \Rightarrow \quad \tilde{L}_1 = \frac{1}{2}e^{3\tilde{t}}\tilde{x}^2 \left( \frac{d\tilde{x}}{dt} \right)^2 + \tilde{f}_1(\tilde{t}, \tilde{x}) \frac{d\tilde{x}}{dt} + \tilde{f}_2(\tilde{t}, \tilde{x}). \quad (31)$$

with $\tilde{f}_1, \tilde{f}_2$ satisfying the following equation:

$$\frac{\partial \tilde{f}_1}{\partial \tilde{t}} - \frac{\partial \tilde{f}_2}{\partial \tilde{x}} - e^{3\tilde{t}}\beta \tilde{x} + e^{3\tilde{t}}\gamma - e^{3\tilde{t}}\tilde{x}^3 = 0. \quad (32)$$

We find that the Lagrangian $L_1$ (24) does not admit the Lie point symmetry $\Gamma$ (29) as a Noether symmetry [13].

If in equation (22) we assume $\gamma = 0$, i.e.:

$$x^2\ddot{x} + x\dot{x}^2 + \beta x = 0 \quad (33)$$

then we find that it admits an eight-dimensional Lie symmetry algebra generated by the following eight operators

$$
\begin{align*}
\Gamma_1 &= \frac{1}{2x}(\beta t^2 + x^2) \left( 2tx\partial_t + (x^2 - \beta t^2)\partial_x \right) \\
\Gamma_2 &= \frac{1}{x^2} \left( (3\beta t^2 x + x^3)\partial_t - 2\beta^2 t^3 \partial_x \right) \\
\Gamma_3 &= \frac{t}{x} \left( (2tx\partial_t + (x^2 - \beta t^2)\partial_x \right) \\
\Gamma_4 &= \frac{1}{x} (\beta t^2 + x^2)\partial_x \\
\Gamma_5 &= \partial_t \\
\Gamma_6 &= \frac{t}{x} (x\partial_t - \beta t\partial_x) \\
\Gamma_7 &= \frac{1}{x} \partial_x \\
\Gamma_8 &= \frac{t}{x} \partial_x
\end{align*}
$$

(34)
which means that equation (33) is linearizable by means of a point transformation [11]. In order to find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra [11], and, following Lie’s classification of two-dimensional algebras in the real plane [11], we have to transform it into the canonical form (Type II)

\[ \partial_{\tilde{t}}, \quad \tilde{t}\partial_{\tilde{x}} \]  

with \( \tilde{t} \) and \( \tilde{x} \) the new independent and dependent variables, respectively. We found that one such subalgebra is that generated by \( \Gamma_7 \) and \( \Gamma_8 \). Then, it is easy to derive that

\[ \tilde{t} = t, \quad \tilde{x} = \frac{1}{2} x^2 \]  

and equation (33) becomes

\[ \frac{d^2 \tilde{x}}{d\tilde{t}^2} = 0. \]  

Now we can use the eight Lie point symmetries (34) to generate fourteen different Jacobi last multipliers of equation (33) by means of (16), and therefore fourteen different Lagrangians of equation (33) by means of (13) as it was shown in [22]. Here we report just three.

The Jacobi last multiplier \( J_{35} \) which is derived from \( \Gamma_3 \) and \( \Gamma_5 \), i.e.

\[ J_{35} = -\frac{x^2}{(x \dot{x} + \beta t)(2tx \dot{x} + \beta t^2 - x^2)}, \]  

yields the Lagrangian

\[ L_{35} = \frac{1}{2t(\beta t^2 + x^2)}(-\beta t^2 - 2tx \dot{x} + x^2) \log(\beta t^2 + 2tx \dot{x} - x^2) - \frac{1}{2t} \]

\[ + \frac{1}{\beta t^2 + x^2} \log(\beta t + x \dot{x})(\beta t + x \dot{x}) + f_1 \dot{x} + f_2, \]

with \( \frac{\partial f_1}{\partial \dot{t}} - \frac{\partial f_2}{\partial x} = \frac{x}{\tilde{t}(\beta \tilde{t}^2 + x^2)}. \)  

The Jacobi last multiplier \( J_{58} \) which is derived from \( \Gamma_5 \) and \( \Gamma_8 \), i.e.

\[ J_{58} = -\frac{x^2}{\beta t + x \dot{x}}, \]  

yields the Lagrangian

\[ L_{58} = -(\beta t + x \dot{x}) \log(\beta t + x \dot{x}) + \beta t + x \dot{x} + f_1 \dot{x} + f_2, \]

with \( \frac{\partial f_1}{\partial \dot{t}} - \frac{\partial f_2}{\partial x} = 0. \)  

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The Jacobi last multiplier \( J_{78} \) which is derived from \( \Gamma_7 \) and \( \Gamma_8 \), i.e.

\[
J_{78} = x^2
\]  
(42)
yields the Lagrangian

\[
L_{78} = \frac{1}{2} x^2 \dot{x}^2 + f_1 \dot{x} + f_2, \quad \text{with } \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = \beta x.
\]  
(43)

We note that \( J_{78} \) corresponds to the multiplier (23) found by Jacobi.

We remind the reader that any ratio of two multipliers gives a first integral of equation (33), e.g.

\[
I_1 = \frac{J_{58}}{J_{35}} = -x^2 + 2tx \dot{x} + \beta t^2,
\]  
(44)

\[
I_2 = \frac{J_{78}}{J_{58}} = -\dot{x} - \beta t,
\]  
(45)

\[
I_3 = \frac{J_{78}}{J_{35}} = (x \dot{x} + \beta t)(x^2 - 2tx \dot{x} - \beta t^2).
\]  
(46)

### 2.1.2 Example II

The second example by Euler that Jacobi presented is the following equation:

\[
2x^3 \dddot{x} + x^2 \ddot{x}^2 - \alpha x^2 + \beta t^2 - \gamma = 0
\]  
(47)

with \( \alpha, \beta \) and \( \gamma \) arbitrary constants. Jacobi obtained the multiplier

\[
M_1 = x,
\]  
(48)

by means of (19) and showed how to integrate equation (47).

We can use Jacobi’s multiplier (18) to derive a Lagrangian of equation (47) by means of (10), i.e.

\[
L_2 = \frac{1}{2} x \dddot{x}^2 + f_1(t, x) \dot{x} + f_2(t, x)
\]  
(49)

with \( f_1, f_2 \) functions of \( t \) and \( x \) satisfying the following equation:

\[
2x^2 \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} \right) + \alpha x^2 - \beta t^2 + \gamma = 0.
\]  
(50)

If we consider the transformation (14) between \( f_1, f_2 \) and the gauge function \( g \), then:

\[
f_3 = \frac{1}{2x^2}(\alpha x^2 + \beta t^2 - \gamma),
\]  
(51)
and $L_2$ can be written as follows:

$$L_2 = \frac{1}{2} x\ddot{x}^2 + \frac{1}{2x^2}(\alpha x^2 + \beta t^2 - \gamma) + \frac{dg}{dt}. \quad (52)$$

We note that equation (47) does not possess any Lie point symmetry.

If we assume $\beta = \gamma = 0$, then equation (47), i.e.

$$2x^3\ddot{x} + x^2\dot{x}^2 - \alpha x^2 = 0 \quad (53)$$

admits a two-dimensional Lie symmetry algebra generated by the following two operators

$$\Omega_1 = t\partial_t + x\partial_x, \quad \Omega_2 = \partial_t. \quad (54)$$

Then we can use these two symmetries to generate another Jacobi last multiplier of equation (53) by means of (16), and therefore a different Lagrangian by means of (13). The Jacobi last multiplier $J_{12}$ which is derived from $\Omega_1$ and $\Omega_2$, i.e.

$$J_{12} = \frac{2}{\alpha + \dot{x}^2}, \quad (55)$$

yields the Lagrangian

$$L_{12} = -2\frac{\dot{x}}{\sqrt{\alpha}} \text{arctanh} \left(\frac{\dot{x}}{\sqrt{\alpha}}\right) - \log \left(1 - \frac{\dot{x}^2}{\alpha}\right) + f_1\dot{x} + f_2,$$

with

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = -\frac{1}{x}. \quad (56)$$

Finally the ratio of the two multipliers (48) and (55) is a first integral of equation (53), i.e.

$$I = \frac{M_1}{J_{12}} = \frac{1}{2} x(-\alpha + \dot{x}^2). \quad (57)$$

If we assume $\alpha = \beta = \gamma = 0$, then equation (47), i.e.

$$2x^3\ddot{x} + x^2\dot{x}^2 = 0 \quad (58)$$

admits an eight-dimensional Lie symmetry algebra, and fourteen different Lagrangians can be derived.

### 3 Application of the method by Jacobi to equation (11)

It is obvious that equation (11), for any $b(x)$ and $c(x)$, is a particular case of the class of equations (18) studied by Jacobi. Therefore a Jacobi Last Multiplier (19) is already known, i.e.

$$M_1 = e^{2P_b(x)}, \quad \text{with} \quad P_b(x) = \int b(x) \, dx, \quad (59)$$
and the corresponding Lagrangian (20) is

\[ L_1 = \frac{1}{2} e^{2P_b(x)} \dot{x}^2 + f_1(t, x) \dot{x} + f_2(t, x) \]  

(60)

with \( f_1, f_2 \) functions of \( t \) and \( x \) satisfying the following equation:

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{2P_b(x)} c(x) x. \]  

(61)

The Lagrangian derived by Museliak et al. [12], after lengthy calculations, i.e.

\[ L = \frac{1}{2} e^{2P_b(x)} \dot{x}^2 - \int e^{2P_b(x)} c(x) x \, dx, \]  

(62)

is a subcase of the Lagrangian (60), with \( f_1 = 0 \) and \( f_2 = f_2(x) = \int e^{2P_b(x)} c(x) x \, dx \), which is an obvious particular solution of (61). Actually we can derive another Lagrangian of (1). In fact we note that (1) admits one trivial Lie point symmetry for any \( b(x) \) and \( c(x) \), i.e. \( \Gamma = \partial_t \), which is also a Noether’s symmetry [13] for the Lagrangian (60). Therefore a first integral can be easily obtained from Noether’s theorem [13], i.e.:

\[ I_1 = \frac{1}{2} e^{2P_b} \dot{x}^2 + \int e^{2P_b} c(x) x \, dx. \]  

(63)

Let us use the property (d) of the Jacobi last multiplier. If one knows a Jacobi last multiplier \( M_1 \) in (59) and a first integral \( I_1 \) in (63) of equation (1), then their product is another Jacobi last multiplier, i.e.

\[ M_2 = M_1 I_1 = \frac{1}{2} e^{2P_b} \left( e^{2P_b} \dot{x}^2 + \int e^{2P_b} c(x) x \, dx \right) \]  

(64)

Consequently we are able to obtain a second Lagrangian of equation (1) for any \( b(x) \) and \( c(x) \), i.e.

\[ L_2 = \frac{1}{24} e^{2P_b} \dot{x}^2 \left( e^{2P_b} \dot{x}^2 + 12 \int e^{2P_b} c(x) x \, dx \right) + f_1(t, x) \dot{x} + f_2(t, x) \]  

(65)

with \( f_1, f_2 \) functions of \( t \) and \( x \) satisfying the following equation:

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{2P_b(x)} c(x) \int e^{2P_b} c(x) x \, dx. \]  

(66)

This Lagrangian admits \( \Gamma = \partial_t \) as a Noether’s symmetry and the corresponding first integral is just the square of \( I_1 \) in (63).

We can keep using property (d) to derive more and more Jacobi last multipliers and therefore

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5This energy-type first integral was obtained in [12].
Lagrangians of equation (1). In fact other Jacobi last multipliers can be obtained by simply taking any function of the first integral $I_1$ in (63) and multiplying it for either $M_1$ in (59) or $M_2$ in (64), and so on ad libitum. For example, we may take $I_2^1$, and then obtain another Jacobi last multiplier by taking the product of $M_1$ in (59) and $I_2^1$, i.e.:

$$M_3 = M_1 I_1^2 = \frac{1}{4} e^{2P_b} \left( e^{2P_b} \dot{x}^2 + \int e^{2P_b} c(x) x \, dx \right)^2$$

which yields the following third Lagrangian of equation (1)

$$L_3 = \frac{1}{120} e^{2P_b} \dot{x}^2 \left( e^{4P_b} \dot{x}^4 + 10 e^{2P_b} \dot{x}^2 \int e^{2P_b} c(x) x \, dx + 60 \left( \int e^{2P_b} c(x) x \, dx \right)^2 \right)$$

$$+ f_1(t, x) \dot{x} + f_2(t, x)$$

with $f_1, f_2$ functions of $t$ and $x$ satisfying the following equation:

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = e^{2P_b(x)} c(x) x \left( \int e^{2P_b} c(x) x \, dx \right)^2.$$  

(69)

4 Final remarks

The purpose of the present paper is to exemplify once again the method of Jacobi for finding Lagrangians, and to stress its strong connection with Lie symmetries and Noether symmetries. We advocate Jacobi Last Multiplier as an essential tool for studying nonlinear dynamical systems. As (apparently) stated by Henry S. Truman:

*There is nothing new in the world, except the history one does not know.*

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