Spin Mixing in Spinor Fermi Gases

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We study a spinor fermionic system under the effect of spin-exchange interaction. We focus on the interplay between the spin-exchange interaction and the effective quadratic Zeeman shift. We examine the static and the dynamic properties of both two- and many-body system. We find that the spin-exchange interaction induces coherent Rabi oscillation in the two-body system, but the oscillation is quickly damped when the system is extended to the many-body case.

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I. INTRODUCTION

Spinor quantum gas has received tremendous attention ever since the first creation of a spin-1 Bose-Einstein condensate in an optical trap [1]. Most of these studies deal with spinor Bose gases. The most salient feature of spinor condensates is the presence of spin-exchange interaction which drives coherent spin-mixing dynamics. A natural question can be asked is: can similar behavior be observed in a quantum degenerate fermionic system? The current work is an attempt to address this question.

Quantum degenerate Fermi gases have been realized in many laboratories [2-5]. Different from the normal electronic system, many fermionic atoms have spins higher than 1/2 in their lowest hyperfine manifold. These large-spin ultracold Fermi gases provide us a unique opportunity to investigate exotic many-body physics [6-8], and have stimulated a great deal of theoretical interest [9-12]. Considerable experimental progress has also been made recently in the system of \textsuperscript{87}Sr (f = 9/2, where f is total hyperfine spin) [13] and \textsuperscript{173}Yb (f = 5/2) [14]. Both \textsuperscript{87}Sr and \textsuperscript{173}Yb have an alkaline-earth-like atomic structure with all electron shells filled, thus their hyperfine spins completely come from nuclear spins. This will lead to a spin-independent atom-atom interaction since the nuclear spins are deep within the atom. In these alkaline-earth-like fermionic system, the spin-independent interaction can give rise to the so-called SU(N) symmetry, with N = 2f + 1 [15,17]. However, for a more general cold Fermi system including non-alkaline-earth atoms with large spins, the SU(N) symmetry may not be reserved. As a result, one can find a more rich phases diagram in its ground state [18]. In a seminal work published very recently [19], a spinor Fermi gas of \textsuperscript{40}K was realized. By taking advantage of the spin conservation, the effective spin of the system can be tuned from 1/2 to 9/2.

In this article, we focus on the simplest large-hyperfine-spin systems with f = 3/2 with four internal components. This can be either true hyperfine spin (e.g., \textsuperscript{133}Cs, \textsuperscript{28}Be and \textsuperscript{201}Hg) or effective spin such as realized in Ref. [19]. If we consider atoms with nonzero electron spins due to partially filled electron shells, then the interaction among atoms will be spin-dependent. One of the key features in this kind of systems is that there will be spin-exchange interactions which constantly mix different spin components. For example, two atoms with respective hyperfine spins −1/2 and +1/2 interact and become two atoms with hyperfine spins −3/2 and +3/2, as schematically depicted in Fig. 1. A similar case of spin mixing has been well studied for spinor condensates [22,23].

The paper is organized as follows: We present the model Hamiltonian in Sec. II. The ground state properties and the spin-mixing dynamics are discussed in Secs. III and IV, respectively. Finally, we conclude in Sec. V.

II. MODEL HAMILTONIAN

To begin we consider a homogeneous dilute gas of fermionic atoms with hyperfine spin f = 3/2 in a box with volume V. The second quantized Hamiltonian of the system is given by

\[
H = \int d\mathbf{r} \sum_{\lambda} \psi_{\lambda}^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + p_\lambda\right)\psi_{\lambda}(\mathbf{r}) + \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} U_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \psi_{\lambda_1}^\dagger(\mathbf{r})\psi_{\lambda_2}^\dagger(\mathbf{r})\psi_{\lambda_3}(\mathbf{r})\psi_{\lambda_4}(\mathbf{r}),
\]

where \(\psi_{\lambda}(\lambda = -3/2,-1/2,1/2,3/2)\) is the atomic field annihilation operator associated with atoms in the hyperfine spin state \(f = 3/2, m_f = \lambda\). The summation indices in (1) run through the values −3/2, −1/2, 1/2, 3/2. \(p_\lambda\) is the bare atomic energy for spin state \(\lambda\). We will consider an effective quadratic Zeeman shift such that


\[ p_{-1/2} = p_{1/2} \text{ and } p_{-3/2} = p_{3/2} \] as the linear Zeeman shift can be gauged away. If we consider the s-wave scattering only, then the interaction between atoms can be characterized by the coefficients \( U_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \) which are obtained from the two-body interaction model \( \hat{U} = g_0 \hat{P}_0 + g_2 \hat{P}_2 \). Here, \( \hat{P}_F \) is the projection operator which projects the pair into a total hyperfine spin \( F \) state, \( g_F \) is the interaction strength in the total spin \( F \) channel, which, in the calculation, will be replaced in favor of the s-wave scattering length \( a_F \) via the regularization procedure:

\[
\frac{1}{g_F} \to \frac{m}{4\pi \hbar^2 a_F} - \frac{1}{V} \sum_k \frac{m}{\hbar^2 k^2}. \tag{2}
\]

\[ H = \sum_{k, \lambda} \mathcal{E}_{\lambda k} \lambda_k^\dagger \lambda_k + \sum_{k, k', p} [c_2 (a_{k' + p}^\dagger b_{k - p}^\dagger - b_{k + p}^\dagger a_{k'}^\dagger) + \mu_{k' + p}^\dagger a_{k'} + \mu_{k - p}^\dagger b_{k} + a_{k'}^\dagger \mu_{k'}^\dagger b_{k'} - b_{k'}^\dagger \lambda'_{k'} \lambda_k] + \frac{g_2}{V} (a_{k' + p}^\dagger b_{k - p}^\dagger - b_{k + p}^\dagger a_{k'}^\dagger + \beta_{k' + p}^\dagger \beta_{k} + \beta_{k'}^\dagger \beta_{k'} + \alpha_{k'}^\dagger \mu_{k'}^\dagger b_{k'} - b_{k'}^\dagger \lambda'_{k'} \lambda_k)] \tag{3}
\]

For Fermi gases, there is no s-wave interaction in the odd total spin \((F = 1, 3)\) channels, since these channels are forbidden by Pauli’s exclusion principle.

We expand the field operators with plane wave function \( \psi_\lambda = \sum_k \lambda_k e^{i k \cdot r} / \sqrt{V} \) and complete the spatial integral to the Hamiltonian into momentum space:

\[
H = \sum_{k, \lambda} (\mathcal{E}_{\lambda k} + h_\lambda) \lambda_k^\dagger \lambda_k + \sum_k [\Delta^* \beta_{-k} \alpha_k - \Delta^\ast \nu_{-k} \mu_k + h.c]. \tag{4}
\]

where \( h_\lambda = g_0 \sum_k (\lambda_k^\dagger \lambda_k) / 2V \) and \( \Delta = \frac{g_0}{2V} (C_1 - C_3) \). To diagonalize this Hamiltonian, we can perform the Bogoliubov transformation

\[
\begin{align*}
\alpha_k &= u_k a_k + v_k b_k^\dagger, & \beta_{-k} &= u_k b_{-k} - v_k a_k^\dagger, \\
\mu_k &= s_k u_k + t_k v_k^\dagger, & \nu_{-k} &= s_k v_{-k} - t_k a_k^\dagger
\end{align*} \tag{5}
\]

Here, the new coefficients should meet the following conditions to ensure the anticommutativity of new operators.

\[
|s_k|^2 + |t_k|^2 = 1, \quad \text{and} \quad |u_k|^2 + |v_k|^2 = 1. \tag{6}
\]

Then we can write down the Bogoliubov-de Gennes (BdG) equations as

\[
\begin{pmatrix}
\mathcal{E}_{\alpha k} + h_\alpha - \mu \\
\Delta^* \beta_{-k} - \Delta^\ast \nu_{-k} + \mu
\end{pmatrix}
\begin{pmatrix}
u_k \\
\dot{v}_k
\end{pmatrix}
= E_1
\begin{pmatrix}
u_k \\
\dot{v}_k
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathcal{E}_{\mu k} + h_\mu - \tilde{\mu} \\
\Delta^* \nu_{-k} - \Delta^\ast \alpha_{-k} + \tilde{\mu}
\end{pmatrix}
\begin{pmatrix}s_k \\
\dot{t}_k
\end{pmatrix}
= E_3
\begin{pmatrix}s_k \\
\dot{t}_k
\end{pmatrix} \tag{7}
\]

where we have introduced the chemical potential \( \tilde{\mu} \) into the equations. We assume that there are \( 2N \) particles in total, and the population in opposite spin states are equal, i.e., \( N_\alpha = N_\beta \) and \( N_\mu = N_\nu. \) Then we have \( N = \sum_k (|s_k|^2 + |t_k|^2) \) and the order parameter is given by

\[
\Delta = \frac{g_0}{2V} \sum_k (u_k v_k^* - s_k t_k^*). \tag{8}
\]

III. GROUND STATE PROPERTIES

Throughout this work, we take temperature to be zero. We will first consider the mean-field ground state property of the system. In the case of \( g_2 = 0 \), it is obvious from Hamiltonian \( \mathcal{H} \) that superfluid pairing can occur between spin components \((\alpha, \beta)\) and \((\beta, \alpha)\). If we denote \( C_1 = \sum_k (\beta_{-k} \alpha_k) \) and \( C_3 = \sum_k (\nu_{-k} \mu_k) \), the Hamiltonian under the mean-field approximation can be

As our focus is on the effect of spin-mixing interaction, we will consider the case with \( g_0 \neq g_2 \). Great simplification can be further achieved by assuming \( g_2 = 0 \) and \( g_0 \neq 0 \), in which case the second line of Eq. \( \mathcal{H} \) vanishes. This is the case we will consider in this work. We note that the essential physics does not change qualitatively if \( g_2 \neq 0 \).
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gas, which should exhibits a superfluid ground state at

the system reduces into a two-component spin-1/2 F
er
sea. Indeed, Fig. 2(c) shows that the magnitude of the

the so-called Cooper problem where two extra particles

Order parameter decreases as

p
a
order parameter decreases as

p
a

As
f
= ±
1/2

increases, the population in the spin components

3

increases. Intuitively , one

Without loss of generality, we take the effective

quadratic Zeeman shift as
p
= 16a0kF/π with kF being the Fermi wavenumber. (c)

Order parameter Δ as a function of

p
are given by

E\psi_{\alpha\beta}(k) = E_{\alpha\beta}\psi_{\alpha\beta}(k) + \frac{g_0}{2V} \sum_{k'} (\psi_{\alpha\beta}(k') - \psi_{\mu\nu}(k'))

E\psi_{\mu\nu}(k) = E_{\mu\nu}\psi_{\mu\nu}(k) + \frac{g_0}{2V} \sum_{k'} (\psi_{\mu\nu}(k') - \psi_{\alpha\beta}(k'))\]

where

E_{\alpha\beta} = \frac{\hbar^2k^2}{2m} - 2E_F and

E_{\mu\nu} = \frac{\hbar^2k^2}{2m} - 2E_F + 2p, and

\psi_{\alpha\beta} and \psi_{\mu\nu} are pairing amplitudes between

m_f = ±1/2, m_f = ±3/2, respectively. The presence of the Fermi

sea imposes the restriction that k and k' lie outside of the Fermi sea, i.e., k, k' ≥ k_F. A negative value of the
eigenenergy E means that the two particles form a bound
pairing with binding energy |E|. The larger |E| is, the
more strongly the pair binds.

After some mathematical manipulation, the two eigen-
value equations (9) leads to the following single equation
for E:

1 = \frac{g_0}{2V} \sum_{k>k_F} \left( \frac{1}{E - E_{\alpha\beta}} + \frac{1}{E - E_{\mu\nu}} \right) , \quad (10)

which can be solved numerically. The solution of E as a
function of p is displayed in Fig. 3 from which one can see
that, as p increases, |E| tends to zero. In other words,
the pairs becomes less and less bound. This result is
consistent with the many-body result illustrated in Fig. 2.

We therefore reach the conclusion that the energy mis-
mismatch induced by the quadratic Zeeman shift p, together
with the spin-exchange interaction, tends to break the
pair apart. This phenomenon is reminiscent of the ef-
f
fect by a magnetic impurity on a spin-1/2 superconduc-
tor [21, 21]. In this latter system, the magnetic impurity
induces an energy difference between the two pairing
particles and has the tendency of destroying the pairing.

IV. SPIN-MIXING DYNAMICS

So far we have focused on the ground state of the sys-
tem. Now, let us turn to the spin-mixing dynamics. Be-
fore dealing with the many-body situation, it may be
helpful to investigate the Cooper problem first. We take
the initial state to be the ground state of the Cooper

FIG. 2: (Color Online) (a), (b) The momentum distribution in spin components ±1/2 and ±3/2 for quadratic Zeeman
shift p = 0 (solid blue), p = 1 (dot-dashed orange) and p = 10
(dashed red), in units of the Fermi energy E_F. In the calculation
we set \( \tilde{a}_0 = -4 \) for (a1) and (a2), and \( \tilde{a}_0 = 1 \) for (b1) and
(b2), where the dimensionless interaction strength is defined as \( \tilde{a}_0 = 16a_0k_F/\pi \) with k_F being the Fermi wavenumber. (c)

The upper panels of Fig. 2 display the momentum distri-
bution in different spin components at various values of
p. As p increases, the population in the spin components
m_f = ±3/2 decreases, and that in m_f = ±1/2 increases.
This can be easily understood from the energetic point of
view. However, somewhat surprisingly, for large p, the
momentum distribution in m_f = ±1/2 components ap-
proaches a step function, exemplifying a normal Fermi
sea. Indeed, Fig. 2(c) shows that the magnitude of the
order parameter decreases as p increases. Intuitively, one
might have thought that for very large p, the population
in the m_f = ±3/2 components becomes negligible and
the system reduces into a two-component spin-1/2 Fermi
gas, which should exhibits a superfluid ground state at
zero temperature. Results presented in Fig. 2 apparently
contradicts this intuitive picture.

To gain some insights into this problem, let us consider
the so-called Cooper problem where two extra particles
with attractive interaction lie on the surface of a filled
Fermi sea noninteracting atoms. The eigenvalue equa-
tions describing the two interacting particles in k-space

FIG. 3: (Color Online) The bound state energy of the Cooper
pair as a function of p for dimensionless interaction strength
(defined in Fig. 2) \( \tilde{a}_0 = -4 \) (solid blue) and \( \tilde{a}_0 = 1 \) (dashed red), respectively.
system under an effective quadratic Zeeman field with $p = 10E_F$. At $t = 0$, we suddenly turn this field off so that $p = 0$ and the system starts to evolve. The dynamics of system is governed by Eq. (9) after replacing $E$ on the left hand side with $i\hbar \partial / \partial t$. The equations can be simplified if we redefine two quantities as follows:

$$\psi_\pm(k, t) = \frac{1}{\sqrt{2}} [\psi_{\alpha\beta}(k, t) \pm \psi_{\mu\nu}(k, t)],$$

and the dynamical equations for $p = 0$ can be rewritten as

$$i\hbar \frac{\partial}{\partial t} \psi_+(k, t) = \frac{\hbar^2 k^2}{m} \psi_+(k, t),$$

$$i\hbar \frac{\partial}{\partial t} \psi_-(k, t) = \frac{\hbar^2 k^2}{m} \psi_-(k, t) + \frac{g_0}{V} \sum_{k'} \psi_-(k', t) \tag{12}$$

Hence the two equations for $\psi_\pm$ are decoupled and the interaction term only appears in the equation for $\psi_-$. The above equations can be easily solved and results are presented in Fig. 4. Initially, due to the presence of the large quadratic Zeeman shift, almost all the populations are in spin states $m_f = \pm 1/2$. Spin mixing dynamics is initiated by quenching this Zeeman shift at $t = 0$. We compare the cases for two different values of the interaction strength: a weak interaction with $a_0 = -4$ and a strong interaction with $a_0 = 1$. In both scenarios, damping is observed in spin mixing, and stronger interaction gives rise to a much faster damping. This can be intuitively understood as follows: the initial state can be regarded as a superposition of different eigenstates of the quenched Hamiltonian. The stronger the interaction, the larger the number of the eigenstates contained in the initial state. For $t > 0$, different eigenstates oscillate at different frequencies which result in the damping of the population dynamics. The more eigenstates are involved, the faster the damping. Therefore, such damping is a result of the intrinsic multi-mode nature of the Fermi gas. For a system of spinor condensate near zero temperature, as all the atoms occupy the same lowest-energy orbitals, nearly undamped spin-mixing oscillations can be observed [23, 24].

We now turn our attention to the dynamics in the many-body setting. As in the Cooper problem, we prepare the system in the ground state with $p = 10E_F$ and at $t = 0$ quench the quadratic Zeeman field to zero. The ensuing mean-field dynamics can be simulated by solving the time-dependent BdG equations, obtained by replacing the eigenenergies at the right hand side of Eqs. (7) with $i\hbar \partial / \partial t$ [26]. At each time step, the order parameter will be updated as

$$\Delta(t) = -\frac{g_0}{2V} \sum_k \langle u_k(t) \psi_k^*(t) - \psi_k(t) \psi_k^*(t) \rangle .$$

Representative results are shown in Fig. 5. The dynamics exhibits qualitatively similar properties as in the Cooper problem: damping is observed in both the dynamics of the population and that of the order parameter, and the stronger the interaction, the faster the damping. This can be understood using a similar intuitive argument we presented for the Cooper problem. We note that such damping was observed in the experiment reported in Ref. [19]. Finally we remark that the ground state corresponding to $p = 0$ should have equal population in all four spin states and an order parameter value indicated by the horizontal line in the lower panels of Fig. 4. In the absence of any dissipation, as in our simulation, the system remains far away from the ground state.

V. CONCLUSION

To conclude, we have examined the spin mixing interaction of a degenerate Fermi gas with four internal spin components. It is quite remarkable that in the presence of an effective quadratic Zeeman field that shifts relatively the bare energies of spin-(\(\pm 1/2\)) and spin-(\(\pm 3/2\)) states, the system becomes almost normal. This is analogous to
the effect of the Zeeman field that breaks the symmetry of the two spin components in a two-component (spin-1/2) Fermi gas. We have also investigated the spin-mixing dynamics initiated by quenching the quadratic Zeeman field and show that, unlike in the case of a spinor condensate, damping will necessarily occur in a many-body Fermi gas due to the intrinsic multi-mode nature of fermions.

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