The impact of two-dimensional elastic disk

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The impact of macroscopic materials is characterized by the coefficient of restitution (COR) defined by

\[ e = -\frac{v_r}{v_i}, \]  

(1)

where \( v_i \) and \( v_r \) are the relative velocities of incoming and outgoing particles, respectively. COR \( e \) had been believed to be a material constant. In general, however, experiments show that COR for three dimensional materials is not a constant even in approximate sense but depends strongly on the impact velocity.\(^1-4\)

The origin of the dissipation in inelastic collisions is the transfer of the kinetic energy of the center of mass into the internal degrees of freedom during the impacts. Gerl and Zippelius\(^5\) performed the microscopic simulation of two-dimensional collision of an elastic disk with a wall. Their simulation is mainly based on the mode expansion of an elastic disk under the force free boundary condition. Then, they analyze Hamilton’s equation;

\[ \dot{P}_{n,l} = -\frac{\partial H}{\partial Q_{n,l}}; \quad \dot{Q}_{n,l} = \frac{\partial H}{\partial P_{n,l}}, \]  

(2)

under the Hamiltonian

\[ H = \frac{p_0^2}{2M} + \sum_{n,l} \left( \frac{P_{n,l}^2}{2M} + \frac{1}{2}M\omega_{n,l}^2Q_{n,l}^2 \right) + V_0 \int_{\pi/2}^{\pi/2} d\phi e^{-a|y(\phi,t)|}. \]  

(3)

Here \( Q_{n,l} \) and \( P_{n,l} \) are respectively the expansion coefficient of the elastic deformation and the canonical momentum, where \( n \) and \( l \) are the mode indices. \( y(\phi,t) \) is the shape of the elastic disk in the polar coordinate.\(^5,6\) \( M \) is the mass of the disk, and \( p_0 \) satisfies \( \dot{p}_0 = M\ddot{y}_0 = -(\partial H/\partial y_0) \) with the position of the center of mass \( y_0 \). \( V_0 \) and \( a \) are parameters to express the strength of the wall potential and \( \omega_{n,l} \) is the angular frequency of the \((n,l)\) mode.\(^5,6\) Their results indicate that COR

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decreases with the impact velocity, which strongly depends on Poisson’s ratio. Since the relation between quasi-static theory of impact\(^7\,9\) and their microscopic simulation\(^5\) is not clear, we have to clarify the relation between two typical approaches.

In this short note, we will perform the microscopic simulation of the impact of a two-dimensional elastic disk with a wall. We introduce a continuum model which is identical to that by Gerl and Zippelius.\(^5\) Through our simulation, we found that this model does not recover the quasi-static theories.\(^7\,9\) Details of this short note have been published as a longer conference report.\(^6\)

Let us explain our model. In this model, the wall exists at \(y = 0\), and the center of mass keeps the position at \(x = 0\). The disk approaches from the region \(y > 0\) and is rebounded by the wall. As in the simulation by Gerl and Zippelius\(^5\) we introduce the wall whose potential is given by \(V_0 e^{-ay}\) in this model, \(V_0 = aMc^2/2\) and \(a = 500/R\) with the radius of the disk \(R\). This choice of \(a\) is aimed to simulate the collision between two identical disks, though we have not extrapolated our results to the limit of \(a \to \infty\). Such the extrapolation has been checked by Gerl and Zippelius for this model.\(^5\) We only simulate the case of Poisson’s ratio \(\sigma = 1/3\). The numerical scheme of the integration of this model is the fourth order symplectic integral method with \(\Delta t = 5.0 \times 10^{-3} R/c\) with \(c = \sqrt{Y/\rho}\) where \(Y\) is Young’s modulus and \(\rho\) is the density.

At first, we carry out the simulation of this model with the initial condition at \(T = 0\) (i.e. no internal motion). Figure 1 is the plot of the COR against the impact velocity. We show the results of 437 modes and 1189 modes\(^6\) which clearly demonstrates the convergence of the result for the number of modes. Each line decreases smoothly as impact velocity increases.

Second, we investigate the force acting on the center of mass of the disk caused by the interaction with the wall in this model. In the limit of \(v_i \to 0\), we expect that the Hertzian contact theory can be used.\(^5,10,11\) The small amount of transfer from the translational motion to the internal motion is the macroscopic dissipation. Thus, we can check the validity of quasi-static approaches\(^7\,9\) from our simulation by the difference between the observed force acting on the center of mass and the Hertzian contact force. The two dimensional Hertzian contact law\(^5,11\) is given by the relation between the macroscopic deformation of the center of mass \(h\) and the elastic force \(F_{el}\) as

\[
h \simeq -\frac{F_{el}}{\pi Y} \{ \ln \left( \frac{4\pi Y R}{F_{el} (1 - \sigma^2)} \right) - 1 \},
\]

where \(Y\), \(\sigma\) and \(R\) are the Young modulus, Poisson’s ratio and the radius of the disk without deformation, respectively. If \(h\) is given, we can calculate the elastic force by solving eq.\((4)\) numerically. Since in the limit of \(v_i \to 0\) we may replace eq.\((4)\) by \(F_{el} \simeq -\pi Y h/\ln(4R/h)\).\(^5\) Thus, the dissipative force \(F_{dis}\) in the two-dimensional quasi-static theory are expected to be \(F_{dis} \propto -\pi Y h/\ln(4R/h)\). The total force \(F_{tot}\) in the two-dimensional quasi-static theory are expected to be

\[
F_{tot} \simeq -\frac{\pi Y h}{\ln(4R/h)} - A\frac{\pi Y h}{\ln(4R/h)},
\]

\(\)
where $A$ is an undetermined constant. Figure 2 is the comparison of our simulation in this model (1189 modes) with the Hertzian contact theory (4). The result of our simulation at the impact velocity $v_i = 0.01c$ with $c = \sqrt{Y/\rho}$ shows the beautiful hysteresis as suggested in the simulation at $v_i = 0.1c$.\(^5\) This means the compression and rebound are not symmetric. The hysteresis curve is still self-similar even at $v_i = 0.04c$ but the loop becomes noisy at $v_i = 0.1c$.

For the low impact velocity $v_i = 0.001c$, the hysteresis loop almost disappears but the total force observed in our simulation is almost a linear function of $\dot{h}$ which is a deviation from Hertzian contact theory and quasi-static theory. In particular, the turning point at $\dot{F} = 0$ is a deviation from the Hertzian curve. This deviation is in clear contrast to the quasi-static theory, because the dissipative force in the quasi-static theory must be zero at the turning point which $\dot{h} = 0$ should satisfy. In this point, our model seems not to be appropriate to represent the nature of slow impact. This deviation may be originated from the defect of our model. Actually, our model cannot describe the equilibrium compression of a disk on the wall. We may need thermal deformations which cannot be described by elastic deformation of disks. The linearity of the total repulsion force is not surprising, because $e^{-ay(\phi,t)}$ in the potential term in eq.(3) can be expanded in a series of $Q_{n,l}$ for very slow impact.\(^5,6\)

We have numerically studied the impact of a two dimensional elastic disk with a wall. The results can be summarized as (i) The coefficient of restitution (COR) decreases with the impact velocity. (ii) The result of our simulation is not consistent with the result of the two-dimensional quasi-static theory. For small velocity, there remains the inelastic force even at $\dot{h} = 0$.

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Fig.1.: Coefficient of restitution for normal collision of our model as a function of impact velocity, where \( c = \sqrt{\frac{Y}{\rho}} \) with the Young’s modulus \( Y \) and the density \( \rho \). 437 and 1189 modes are chosen for this model.

Fig.2.: The comparison of the Hertzian force in eq.(4) with our simulation at \( v_i = 0.01c \) (a) and \( v_i = 0.001c \) (b) at \( T = 0 \) in this model. \( F_{tot} \) is the total force originated from the wall potential.
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