Flatness of families induced
by hypersurfaces on flag varieties

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Abstract

We answer a question posed by S. Kleiman concerning flatness of the family
of complete quadrics. We also show that any flat family of hypersurfaces on
Grassmann varieties induces a flat family of intersections with the corresponding
flag variety.

Introduction

Let \( S \) be the variety of complete quadrics, \( S^{nd} \) the open subset of nondegenerate
quadrics and \( F \) the scheme of complete flags in \( \mathbb{P}^n \). Let \( \varphi_0 : S^{nd} \to \text{Hilb}(F) \) be
the morphism that assigns to each nondegenerate quadric the locus of its tangent flags.
We prove the following.

**Theorem.** \( \varphi_0 \) extends to a morphism \( \varphi : S \to \text{Hilb}(F) \).

This answers affirmatively a question S. Kleiman asked in ([K], p.362).

We first show that \( S \) parametrizes a flat family that restricts, over \( S^{nd} \), to the family
of the graphs of the Gauss map (point \( \mapsto \) tangent hyperplane) of nondegenerate quadric
hypersurfaces. The family pertinent to Kleiman’s question is obtained by composing
the family of graphs with the appropriate flag bundle (point \( \in \) line \( \subset \ldots \subset \) hyperplane).

Our proof of flatness for the completed family of graphs relies on Laksov’s description
[L] of Semple–Tyrrell’s “standard” affine open cover of \( S \).

The space of complete conics has recently reappeared as a simple instance of Kontsevich’s spaces of stable maps (cf. [P]). It is also instrumental for the counting of rational
curves on a K3 surface double cover of the plane (cf. [V]). Complete quadric surfaces
play a role in Narasimhan–Trautmann [NT] study of a compactification of a space of
instanton bundles.

\( ^1 \) Partially supported by Brasil’s CNPq. Thanks are due to the UFMG for the warm atmosphere
and hospitality during the preparation of this manuscript.
We also show that any flat family of hypersurfaces on Grassmann varieties induces a flat family of subschemes of the corresponding flag variety (cf. (7.2)).

This statement was first obtained as an earlier attempt to answer Kleiman’s question. We observe that for the case of quadric hypersurfaces the family described in the proposition does not induce the family of tangent flags. In fact, for conics it yields a double structure on the graph of the Gauss map. (cf. §7 for details).

1 The tangent flag to a smooth quadric

Write \( x = (x_1, \ldots, x_{n+1}) \) (resp. \( y = (y_1, \ldots, y_{n+1}) \)) for the vector of homogeneous coordinates in \( \mathbb{P}^n \) (resp. \( \mathbb{P}^n \)). Let \( F_{0,n-1} \subset \mathbb{P}^n \times \mathbb{P}^n \) be the incidence correspondence “point \( \in \) hyperplane”. It is the zeros of the incidence section \( x \cdot y \) of \( \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \).

Let \( \kappa \subset \mathbb{P}^n \) denote a smooth quadric represented by a symmetric matrix \( a \). The Gauss map \( \gamma : \kappa \to \mathbb{P}^n \) is given by \( x \mapsto y = x \cdot a \). Hence we have

\[
\gamma^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^n}(1)|_\kappa.
\]

The tangent flag \( \tilde{\kappa} \subset F_n \) of \( \kappa \) is equal to the restriction of the flag bundle

\[
F_n \to F_{0,n-1} \subset \mathbb{P}^n \times \mathbb{P}^n
\]

over the graph \( \Gamma_\kappa \) of \( \gamma \). Consequently, flatness of the family \( \{\tilde{\kappa}\} \) of tangent flags is equivalent to flatness of the family of graphs \( \{\Gamma_\kappa\} \). The latter will be handled in §4.

We proceed to compute the Hilbert polynomial of the graph \( \Gamma \) of the Gauss map of a general quadric hypersurface \( \kappa \subset \mathbb{P}^n \).

1.1 Lemma. Notation as above, the Hilbert polynomial \( \chi(\mathcal{O}_\Gamma(\mathcal{L}^\otimes t)) \) with respect to

\[
\mathcal{L} = (\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(1))|_\Gamma
\]

is equal to

\[
\binom{2t + n}{n} - \binom{2(t - 1) + n}{n}.
\]
Proof. We have $L \cong \mathcal{O}_{\mathbb{P}^n}(2)|_K$ under the identification $\Gamma \cong \kappa$. Thus we may compute

$$
\chi(L^\otimes t) = \chi(\mathcal{O}_{\mathbb{P}^n}(2t))|_K
= \chi(\mathcal{O}_{\mathbb{P}^n}(2t)) - \chi(\mathcal{O}_{\mathbb{P}^n}(2t - 2))
= \binom{2t+n}{n} - \binom{2(t-1)+n}{n}.
$$

\[\square\]

2 Hilbert polynomial of loci of rank 1 matrices

The image of the Segre imbedding $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^N$ is the variety of matrices of rank one. The image $\Delta$ of the diagonal $\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^N$ is the subvariety of symmetric matrices of rank one. It’s Hilbert polynomial is easily found to be

$$
\dim (H^0(\Delta, \mathcal{O}_{\mathbb{P}^N}(t))) = \binom{2t+n}{n}.
$$

The bi-homogeneous ideal $I_{\Delta}$ of the diagonal is generated by the $2 \times 2$ minors of the matrix

(1)

$$
\begin{bmatrix}
x_1 & x_2 & \ldots & x_{n+1} \\
y_1 & y_2 & \ldots & y_{n+1}
\end{bmatrix}.
$$

Write

$$
S = k[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]
$$

for the polynomial ring in $2n + 2$ variables, and let $S_{i,j}$ denote the space of bi-homogeneous polynomials of bi-degree $(i,j)$. We have

$$
\dim_k S_{t,t}/(I_{\Delta})_{t,t} = \binom{2t+n}{n}.
$$

Quite generally, for a closed subscheme $X \subseteq \mathbb{P}^m \times \mathbb{P}^n$ defined by a bi-homogeneous ideal $I \subseteq S$ we have (cf. [KTB], p. 189)

$$
H^0(X, \mathcal{O}_{\mathbb{P}^m}(t) \otimes \mathcal{O}_{\mathbb{P}^n}(t)|_X) = S_{t,t}/(I)_{t,t} \text{ for all } t >> 0.
$$

Indeed, the homomorphism

(2)

$$
R = k[z_{i,j}]/\langle z_{i,j} z_{r,s} - z_{r,j} z_{i,s} \rangle \twoheadrightarrow S \quad \mapsto \quad x_i y_j
$$
maps $R_t$ isomorphically onto $S_{t,t}$. Let the bi-homogeneous ideal $I \subseteq S$ be generated by polynomials of some fixed bidegree $(d,d)$. Its inverse image via (2) generates a homogeneous ideal $I^\# \subseteq R$. We have

$$(R/I^\#)_t \xrightarrow{\sim} S_{t,t}/I_{t,t}.$$ 

Now for $t \gg 0$ we may write

$$(R/I^\#)_t = H^0(X, \mathcal{O}_{\mathbb{P}^n}(t)|_X) = H^0(X, \mathcal{O}_{\mathbb{P}^m}(t) \otimes \mathcal{O}_{\mathbb{P}^n}(t)|_X).$$

Let $L(I)$ denote the monomial ideal of initial terms of $I$ with respect to some bi-graded monomial order. Then we have the equality of Hilbert functions,

$$\varphi_I(i,j) = \varphi_{L(I)}(i,j).$$

This is rather standard: let $f_1, \ldots, f_k$ be linearly independent forms of bidegree $(i,j)$ in $I$. Replacing if needed each $f_\mu$ by $f_\mu - cf_\nu$ for suitable $c \in k$, we may assume their initial terms $L f_\mu \neq L f_\nu$. Hence the initial terms $L f_1, \ldots, L f_k$ are linearly independent monomials in $L(I)_{i,j}$. This shows that $\varphi_I(i,j) \leq \varphi_{L(I)}(i,j)$. Conversely, pick monomials $g_1 > \ldots > g_k$ in $L(I)_{i,j}$. We have each $g_\mu = L f_\mu$ for some $f_\mu \in I_{i,j}$. It follows that $f_1, \ldots, f_k$ are linearly independent.

2.1 Lemma. Let $\Gamma_0$ be the subscheme of $\mathbb{P}^n \times \mathbb{P}^n$ defined by the ideal

$$\langle x_iy_j \mid 1 \leq i < j \leq n + 1 \rangle + \langle \sum x_iy_i \rangle.$$

Then we have

$$\varphi_{\Gamma_0}(t) = \binom{2t+n}{n} - \binom{2(t-1)+n}{n}.$$ 

Proof. The whole point is to notice that the $x_iy_j$ span the ideal of initial terms of $I_\Delta$ with respect to a suitable order. In fact, the set of $2 \times 2$ minors of (1) is known to be a (universal) Gröbner basis for $I_\Delta$ (see Sturmfel [BS]). By the above discussion, we may write

$$\varphi_{L(I_\Delta)}(t) = \varphi_{I_\Delta}(t) = \binom{2t+n}{n}.$$ 

I’m indebted to P. Gimenez for his precious help on this matter.
One checks at once that $\sum x_iy_i$ is a nonzero divisor mod $L(I_\Delta)$ (see (7.1) (i)). Therefore
\[ \varphi_{\Gamma_0}(t) = \varphi_{L(I_\Delta)}(t) - \varphi_{L(I_\Delta)}(t-1). \]
\[ \square \]

We will deduce flatness for the “completed” family of Gauss maps from the fact that the above Hilbert polynomial at the special point $\Gamma_0$ coincides with the generic one (1.1).

3 Semple-Tyrrell-Laksov cover

Let $U_n$ denote the group of lower triangular unipotent $(n+1)$-matrices. Thus, $U_n$ is isomorphic to the affine space $\mathbb{A}^{n(n+1)/2}$ with coordinate functions $u_{i,j}$, $1 \leq j \leq i-1$, $i = 2 \ldots n+1$. These are thought of as entries of the matrix,

\[ u = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
u_{2,1} & 0 & 1 & \cdots & 0 \\
u_{3,1} & u_{3,2} & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
u_{n+1,1} & u_{n+1,2} & u_{n+1,3} & \cdots & 1
\end{bmatrix}. \]

Let $d_1, \ldots, d_n$ be coordinate functions in $\mathbb{A}^n$. Put

(3) \[ d^{(1)} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & d_1 & 0 & \cdots & 0 \\
0 & 0 & d_1d_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & d_1d_2 \cdots d_n
\end{bmatrix}. \]

For a matrix $A$ let it’s $i$th adjugate be the matrix $\hat{\Lambda}A$ of all $i \times i$ minors. We denote by $d^{(i)}$ the matrix obtained from $\hat{\Lambda}d^{(1)}$ by removing the common factor $d_1^{i-1}d_2^{i-2} \cdots d_{i-1}$. E.g., for $n = 3$ we have
\[ d^{(1)} = \text{diag}(1, d_1, d_1d_2, d_1d_2d_3) \]
\[ d^{(2)} = \frac{\text{diag}(d_1, d_1d_2, d_1d_2d_3, d_1^2d_2d_3, d_1^2d_2^2d_3, d_1^2d_2^2d_3d_1)}{(d_1)} = \text{diag}(1, d_2, d_2d_3, d_1d_2, d_1d_2d_3, d_1d_2^2d_3) \]
\[ d^{(3)} = \text{diag}(1, d_3, d_2d_3, d_1d_2d_3). \]
The map $U_n \times \mathbb{A}^n \to S \subset \prod_{i=1}^{i=n} \mathbb{P}(S_2(\wedge^2 k^{n+1}))$ defined by sending $(u, d)$ to 

$$(u \cdot d^{(1)} u^t, (\wedge u) \cdot d^{(2)} u^t, \ldots, (\wedge u) \cdot d^{(n)} u^t)$$

is an isomorphism onto an affine open subset $S^0$ of $S$. The variety of complete quadrics may be covered by translates of $S^0$ (cf. Laksov [L]).

Let $S^0_d \cong U_n \times \mathbb{A}^n$ be the principal open piece defined by $d_1 d_2 \cdots d_n \neq 0$. It maps isomorphically onto an open subvariety of $S^0$.

4 Graph of the Gauss map

The variety $S^0$ of nondegenerate quadrics parametrizes a flat family of graphs of Gauss maps. For a nondegenerate quadric represented by a symmetric matrix $a \in S^0$ the Gauss map is given by $x \mapsto \gamma = x \cdot \gamma$. We define $\mathcal{K}^{nd} \subset S^0 \times \mathbb{P}^n \times \mathbb{P}^n$ by the bi-homogeneous ideal generated by the incidence relation $x \cdot y$ together with the $2 \times 2$ minors of the $2 \times (n + 1)$ matrix with rows $y, x \cdot z$, where $z$ denotes the generic symmetric matrix. Clearly $\mathcal{K}^{nd} \to S^0$ is a map of $\text{GL}_{n+1}$-homogeneous spaces.

Now write $a = v^{(1)} c^{(1)} v^t$ with $v \in U_n, c \in \mathbb{A}^n$ (as in (3)), and put $x' = x v$, $y' = y (v^{-1})^t$. We have $y = x a$ iff $y' = x' c^{(1)}$. Let

$$(4) \qquad \mathcal{K}^0_d \subset S^0_d \times \mathbb{P}^n \times \mathbb{P}^n.$$ 

be defined by $x \cdot y$ together with the $2 \times 2$ minors of the $2 \times (n + 1)$ matrix

$$(5) \qquad \begin{bmatrix} x_1' & d_1 x_2' & d_1 d_2 x_3' & \ldots & d_1 \cdots d_n x_{n+1}' \\ y_1' & y_2' & y_3' & \ldots & y_{n+1}' \end{bmatrix}$$

where we put $x_j' = \sum_i u_{ij} x_i$ and likewise $y_j$ denotes the $j$th entry of $y (u^{-1})^t$. Thus $\mathcal{K}^0_d$ is the total space of the family of Gauss maps parametrized by $S^0_d$. Note that $\mathcal{K}^0_d \to S^0_d$ is a smooth quadric bundle. Its fibre over $(I, (1, \ldots, 1)) \in U_n \times \mathbb{A}^n$ is equal to the quadric given by $\sum x_i^2$ inside the “diagonal” $y_1 = x_1, \ldots, y_{n+1} = x_{n+1}$ of $\mathbb{P}^n \times \mathbb{P}^n$.

Let

$$(6) \qquad \mathcal{K}^0 \subset S^0 \times \mathbb{P}^n \times \mathbb{P}^n$$

be defined by $x \cdot y$ together with the ideal

$$(7) \qquad J = \langle x_1' y_2' - d_1 y_1' x_2', \ldots, x_1' y_{n+1}' - d_1 \cdots d_n y_1' x_{n+1}', x_2' y_3' - d_2 y_2' x_3', \ldots, x_n' y_{n+1}' - d_n y_n' x_{n+1}' \rangle$$
obtained by cancelling all $d_i$ factors occurring in the above $2 \times 2$ minors. We obviously have $\mathbb{K}_d^0 |_{S_d^0} = \mathbb{K}_d^0$.

We will show that $\mathbb{K}^0$ is the scheme theoretic closure of $\mathbb{K}_d^0$ in $S^0 \times \mathbb{P}^n \times \mathbb{P}^n$ (cf. (6.3)).

5 A torus action

Notation as in (3), imbed $\mathfrak{G}^\times_m$ in $\mathfrak{GL}_{n+1}$ by sending $c = (c_1, \ldots, c_n) \in \mathfrak{G}^\times_m$ to $c^{(1)} = \text{diag}(1, c_1, c_1c_2, \ldots)$. We let $\mathfrak{G}^\times_m$ act on $S^0$ by

$$c \cdot (v, b) = (c^{(1)} v (c^{(1)})^{-1}, (c_1^2 b_1, \ldots, c_n^2 b_n)).$$

This action is compatible with the natural action of $\mathfrak{GL}_{n+1}$ on the space $\mathbb{P}(S_2(k^{n+1*}))$ of quadrics, i.e., for a symmetric matrix $a(v, b) := v b^{(1)} v^t$ as above, we have

$$c^{(1)} \cdot a(v, b) = c^{(1)} a(v, b) (c^{(1)})^t = c^{(1)} v b^{(1)} v^t (c^{(1)})^t$$

$$= c^{(1)} v (c^{(1)})^{-1} c^{(1)} b^{(1)} c^{(1)} ((c^{(1)})^t)^{-1} v^t (c^{(1)})^t$$

$$= c^{(1)} v (c^{(1)})^{-1} (c^{(1)})^2 b^{(1)} ((c^{(1)})^t)^{-1} v^t (c^{(1)})^t$$

$$= a(c \cdot (v, b)).$$

It can be also easily checked that $\mathfrak{G}^\times_m$ acts compatibly on $S^0 \times \mathbb{P}^n \times \mathbb{P}^n$ and $\mathbb{K}^0$ is invariant. Indeed, let $((v, b), x, y) \in \mathbb{K}^0$. Pick $c \in \mathfrak{G}^\times_m$. We have

$$c \cdot ((v, b), x, y) = ((c^{(1)} v (c^{(1)})^{-1}, (c_1^2 b_1, \ldots, c_n^2 b_n)), x (c^{(1)})^{-1}, y (c^{(1)})^t).$$

Now $x' = xv$ changes to

$$x'' = (x (c^{(1)})^{-1}) (c^{(1)} v (c^{(1)})^{-1}) = x v (c^{(1)})^{-1} = x' (c^{(1)})^{-1}$$

so that the first row $x' b^{(1)}$ in (5) (evaluated at $((v, b), x, y)$) changes to

$$x'' (b^{(1)} (c^{(1)})^2) = x' (c^{(1)})^{-1} (b^{(1)} (c^{(1)})^2) = x' (b^{(1)} c^{(1)}).$$

Similarly, $y' = y (v^{-1})^t$ changes to

$$y'' = (y (c^{(1)})^t) ((c^{(1)} v (c^{(1)})^{-1})^{-1})^t = y (v^{-1})^t (c^{(1)})^t = y' c^{(1)}.$$

Therefore (5) changes to the matrix with rows $x' (b^{(1)} c^{(1)})$ and $y' c^{(1)}$. Thus evaluation of (7) at $c \cdot ((v, b), x, y)$ and at $((v, b), x, y)$ differ only by nonzero multiples.
5.1 Lemma. The orbit of $(I, 0) \in S^0$ is the unique closed orbit.

Proof. Conjugation of $v \in U_n$ by the diagonal matrix $c^{(1)}$ replaces each entry $v_{ij}, j < i$ by

\[(c^{(1)} v (c^{(1)})^{-1})_{ij} = c^{(1)}_{ii} (v (c^{(1)})^{-1})_{ij} = c^{(1)}_{ii} v_{ij} ((c^{(1)})^{-1})_{jj} = v_{ij} c^{(1)}_{ii} / c^{(1)}_{jj} = v_{ij} c_{i-1} \cdots c_j.\]

Thus, letting $c \to 0$, we see that $(I, 0)$ is in the orbit closure $\mathcal{G}_m^{\times n} \cdot (v, b)$.

\[\square\]

6 Proof of the theorem

6.1 Lemma. Notation as in (5), the family $K^0 \to S^0$ is flat.

Proof. Since $K^0 \to S^0$ is equivariant for the $\mathcal{G}_m^{\times n}$-action, it suffices to check that the Hilbert polynomial of the fiber over the representative $(I, 0)$ of the unique closed orbit is right, i.e., coincides with the generic one (cf. Hartshorne [H], thm.9.9, p.261). Evaluating (7) at $(I, 0)$ yields the monomial ideal (1). We are done by virtue of (1.1) and (2.1).

\[\square\]

6.2 Lemma. Let $f : X \to Y$ be a flat, surjective morphism of schemes. If $U \subseteq Y$ is open and schematically dense in $Y$ then $f^{-1}U$ is open and schematically dense in $X$.

Proof. We may assume $X, Y$ affine. Let $A \subseteq B$ be a flat ring extension and let $a \in A$ be such that $\text{Spec} A_a$ is schematically dense in $\text{Spec} A$. This means that every element in $\ker (A \to A_a)$ is nilpotent. Flatness implies $\ker (B \to B_a)=\ker (A \to A_a) \otimes B$. Hence $\text{Spec} B_a$ is schematically dense in $\text{Spec} B$.

\[\square\]
6.3 Lemma. Notation as in (3) and (1), we have that $\mathcal{I}_K^0$ is equal to the scheme theoretic closure of $\mathcal{I}_d^0$.

Proof. In view of (6.1), we may apply the previous lemma to $\mathcal{I}_K^0 \rightarrow \mathcal{S}^0 \supset \mathcal{S}_d^0$. □

6.4 Lemma. Let $G$ be an algebraic group and let

\[
\begin{align*}
X^0 \subset X \\
\downarrow \quad \downarrow \\
Y^0 \subset Y
\end{align*}
\]

be a commutative diagram of maps of $G$–varieties. Let $\overline{X}$, $\overline{Y}$ denote the closures of $X^0$, $Y^0$. If $\overline{X} \rightarrow \overline{Y}$ is flat over a neighborhood of a point in each closed orbit then $\overline{X} \rightarrow \overline{Y}$ is flat.

Proof. Immediate. □

We may now finish the proof of the theorem. Let $\mathcal{I}_K \subset \mathcal{S} \times \mathbb{P}^n \times \tilde{\mathbb{P}}^n$ be the scheme theoretic closure of $\mathcal{I}_K^0$. We have $\mathcal{I}_K \cap (\mathcal{S} \times \mathbb{P}^n \times \tilde{\mathbb{P}}^n) = \mathcal{I}_K^0$ flat over $\mathcal{S}^0$ by (6.4). The latter is a neighborhood of a point in the unique closed orbit of $\mathcal{S}$. Now apply the previous lemma to $G = \text{GL}_{n+1}$, $X = \mathcal{S} \times \mathbb{P}^n \times \tilde{\mathbb{P}}^n$, $Y = \mathcal{S}$, $Y^0 = \mathcal{S}^n$, $X^0 = \mathcal{I}_K^{\text{nd}}$. Finally, since the family of tangent flags is defined by the fibre square,

\[
\begin{align*}
\overline{\mathcal{I}_K} & \rightarrow \mathcal{F}_n \times \mathcal{S} \\
\downarrow \quad \downarrow \\
\mathcal{I}_K & \rightarrow \mathcal{F}_{0,n-1} \times \mathcal{S}
\end{align*}
\]

the composition $\overline{\mathcal{I}_K} \rightarrow \mathcal{I}_K \rightarrow \mathcal{S}$ is flat.

7 Final remarks

7.1 (i) The primary decomposition of the monomial ideal in (2.1) can be checked to be given by

\[
\langle x_1, x_2, \ldots, x_n \rangle \cap \cdots \cap \langle x_1, \ldots, x_i, y_{i+2}, \ldots, y_{n+1} \rangle \cap \cdots \cap \langle y_2, y_3, \ldots, y_{n+1} \rangle.
\]
Thus enlarging it to include the nonzero divisor $x \cdot y$ we see that the special fiber $\Gamma_0$ presents no imbedded component.

(ii) The example of $\mathbb{P}^n$ acted on by the stabilizer of a point, blown up at that point might clarify why we were not able to show directly that the closure of $\mathbb{K}^{nd}$ is flat over $S$.

(iii) For $n = 1$ we may write the following global equations for $\mathbb{K}$. Let $z, w$ be a pair of symmetric $3 \times 3$ matrices of independent indeterminates. Then $\mathbb{K} \subset \mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ is given by the $2 \times 2$ minors of the $2 \times 3$ matrices with rows $x \cdot z, y$ and $x, y \cdot w$, in addition to the incidence relation $x \cdot y$ together with the equation $3z \cdot w = \text{trace}(z \cdot w)I$ for $S \subset \mathbb{P}^5 \times \mathbb{P}^5$. It would be nice to give a similar description for higher dimension.

(iv) Still assuming $n = 1$, put

$$\Gamma = \{(P, \ell, \kappa, \kappa') \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^5 \times \mathbb{P}^5 \mid P \in \kappa \cap \ell, \ell \in \kappa'\}.$$ 

It is easy to check that $\Gamma |_{S} = \mathbb{K}$ as sets. Furthermore, $\Gamma$ may be endowed with a natural scheme structure such that $\Gamma \to \mathbb{P}^5 \times \mathbb{P}^5$ is flat and with Hilbert polynomial of its fibers equal to $4t$. Thus, $\Gamma |_{S} \to S$ is a family of double structures of genus one on the fibers of $\mathbb{K}$.

In fact, we have the following.

7.2 Proposition. Any flat family of hypersurfaces on Grassmann varieties induces a flat family of subschemes of the corresponding flag variety.

Before considering the general case, we describe the situation in the projective plane. Thus, let $F_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the incidence correspondence "point $\in$ line". Let $f_0$ (resp. $f_1$) denote a curve in $\mathbb{P}^2$ (resp. $\mathbb{P}^2$). Set

$$\Gamma_{\text{F}} := (f_0 \times f_1) \cap F_2.$$ 

Then $\Gamma_{\text{F}}$ is easily seen to be regularly imbedded of codimension 2 in $F_2$ (cf. (7.3)). Moreover, its Hilbert polynomial with respect to the ample sheaf $O_{\mathbb{P}^2}(1) \otimes O_{\mathbb{P}^2}(1)$ restricted to $F_2$ depends only on the degrees, say $d_0, d_1$ of $f_0, f_1$. In fact, the Koszul
complex that resolves the ideal of \( f_0 \times f_1 \) in \( \mathbb{P}^2 \times \mathbb{P}^2 \) restricts to a resolution of \( \Gamma_f \) in \( F_2 \). One finds the Hilbert polynomial,

\[
\chi_f(t) = (d_0 + d_1)t - d_0d_1(d_0 + d_1 - 4)/2.
\]

(8)

Therefore, as in the final argument for the proof of (6.1), the parameter space of pairs \((f_0, f_1)\), call it \( T = \mathbb{P}^{n_0} \times \mathbb{P}^{n_1} \) for suitable \( n_0, n_1 \), carries a flat family of curves on \( F_2 \). Precisely, let

\[
W_0 \subset \mathbb{P}^2 \times \mathbb{P}^{n_0} \text{ and } W_1 \subset \mathbb{P}^2 \times \mathbb{P}^{n_1}
\]

denote the total spaces of the universal plane curve parametrized by \( \mathbb{P}^{n_i} \). Then

\[
\Gamma := (W_0 \times W_1) \cap F_2 \rightarrow T
\]

is a flat family of curves in \( F_2 \), with fiber \( \Gamma_f \).

For the proof of (7.2) we let \( G_{r,n} \) denote the grassmannian of projective subspaces of dimension \( r \) of \( \mathbb{P}^n \).

Recall that the dimension of the variety of complete flags \( F_n \subset \prod G_{i,n} \) is

\[
\dim F_n = 1 + \cdots + n.
\]

The proposition is an easy consequence of the following.

7.3 Lemma. Let \( f_0, f_1, \ldots, f_n \) be arbitrary hypersurfaces of points, lines, \ldots, hyperplanes in the appropriate grassmannians of subspaces of \( \mathbb{P}^{n+1} \). Then the intersection

\[
\Gamma_f := (f_0 \times \cdots \times f_n) \cap F_{n+1}
\]

is of codimension \( n + 1 \) in \( F_{n+1} \).

Proof. We shall argue by induction on \( n \).

We may assume all \( f_i \) irreducible. Let \( n = 1 \). Pick a line \( h \in f_1 \). Set

\[
h^{(0)} = \{ P \in \mathbb{P}^2 \mid P \in h \}.
\]

The fiber \( (\Gamma_f)_h \simeq h^{(0)} \cap f_0 \) is zero dimensional unless \( h^{(0)} = f_0 \). This occurs for at most one \( h \in f_1 \), hence \( \Gamma_f \) is 1–dimensional (otherwise most of its fibres over \( f_1 \) would be at least 1–dimensional).
For the inductive step, we set for $h \in \mathbb{P}^{n+1}$,

$$(9)\quad h^{(r)} = \{ g \in G_{r,n+1} | g \subseteq h \} \simeq G_{r,n}.$$  

If the intersection

$$f'_r = h^{(r)} \cap f_r$$

were proper for all $r$ and $h \in f_n$ then we would be done by induction. Indeed, we have

$$(\Gamma_f)_h \simeq (f'_0 \times \cdots \times f'_{n-1}) \cap F_n.$$  

By the induction hypothesis, this is of the right dimension

$$1 + \cdots + n - n = 1 + \cdots + (n - 1).$$

Since $h$ varies in the $n$–dimensional hypersurface $f_n$ of $G_{n,n+1} = \mathbb{P}^{n+1}$, we would have

$$\dim \Gamma_f = (1 + \cdots + (n - 1)) + n = (1 + \cdots + (n + 1)) - (n + 1)$$

as desired.

However, just as in the case $n = 1$, it may well happen that the intersection $h^{(r)} \cap f_r$ be not proper for some $h, r$. Thus it remains to be shown that, whenever $\dim (\Gamma_f)_h$ exceeds the right dimension, say by $\delta$, the hyperplane $h$ is restricted to vary in a locus of codimension at least $\delta$ in $f_n$. This is taken care of by the lemma below.

7.4 Lemma. Notation as in (9), for $r = 0, \ldots, n$ we have

$$\dim \{ h \in \mathbb{P}^{n+1} | h^{(r)} \subseteq f_r \} \leq r.$$  

Proof. Let $F_{r,n} \subset \mathbb{P}^{n+1} \times G_{r,n+1}$ be the partial flag variety. Form the diagram with natural projections,

$$\begin{array}{ccc}
\mathbb{P}^{n+1} & \xrightarrow{\pi_n} & F_{r,n} \\
F_{r,n} & \xrightarrow{\pi_r} & G_{r,n+1}
\end{array}$$

For $g_r \in G_{r,n+1}$, set

$$g_r^{(n)} = \{ h \in \mathbb{P}^{n+1} | g_r \subseteq h \}.$$
We have $g^{(n)} \simeq \mathbb{P}^{n-r}$ whence it hits any subvariety of $\mathbb{P}^{n+1}$ of dimension $\geq r + 1$. In other words, for any subvariety $Z \subseteq \mathbb{P}^{n+1}$ such that $\dim Z \geq r + 1$, we have

$$\pi_r \pi_n^{-1} Z = \{ g_r | \exists h \in Z \text{ s.t. } h \supseteq g_r \}$$

$$= \{ g_r | g^{(n)} \cap Z \neq \emptyset \}$$

$$= G_{r,n+1}.$$

The lemma follows by taking $Z = \{ h \in \mathbb{P}^{n+1} | h^{(r)} \subseteq f_r \}$. Indeed, if $\dim Z \geq r + 1$, then for all $g_r \in G_{r,n+1}$ there exists $h \in Z$ s.t. $h \supseteq g_r$, so $g_r \in h^{(r)} \subseteq f_r$, contradicting that $f_r$ is a hypersurface of $G_{r,n+1}$. $\blacksquare$ (for (7.4))

Continuing the proof of (7.3) we consider the stratification of $f_n$ by the condition of improper intersection of $f_r$ with $h^{(r)}$, namely,

$$f_{n,0} = \{ h \in f_n | h^{(0)} \subseteq f_0 \},$$

$$f_{n,1} = \{ h \in f_n | h^{(1)} \subseteq f_1 \} \setminus f_{n,0},$$

$$\vdots$$

$$f_{n,n} = \{ h \in f_n | h^{(n)} \subseteq f_n \} \setminus \bigcup_{j < n} f_{n,j}.$$

We will be done if we show

$$\dim (\Gamma_f) h \leq 1 + \cdots + n - r \quad \forall h \in f_{n,r}.$$

We have already seen that $\dim (\Gamma_f) h = 1 + \cdots + n - 1$ for $h$ in $f_{n,n}$. Also, for $r = 0$, the desired estimate holds because we have $(\Gamma_f) h \subseteq (F_{n+1}) h \simeq F_n$ and $\dim F_n = 1 + \cdots + n$. Let $r > 0$ and pick a hyperplane $h \in f_{n,r}$. Then the intersections,

$$f_i' = h^{(i)} \cap f_i,$$

are proper for $i = 0, \ldots, r - 1$, whereas for the subsequent index, we have

$$h^{(r)} \cap f_r = h^{(r)} \simeq G_{r,n}.$$

Thus, we may write,

$$(\Gamma_f) h \hookrightarrow (f_0' \times \cdots \times f_{r-1}' \times G_{r,n} \times \cdots \times G_{n-1,n}) \cap F_n.$$

By the induction hypothesis the intersection above is of dimension $\dim F_n - r$ in view of the following easy

**Remark.** The validity of (7.3) for a given $n$ implies properness of the “partial” intersection

$$(f_0 \times \cdots \times G_{r,n+1} \times \cdots \times f_n) \cap F_{n+1},$$

where one (or more) of the hypersurfaces $f_r \subset G_{r,n+1}$ is replaced by the corresponding full Grassmannian. $\blacksquare$ (for (7.3)) (Feb.2'96)

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