Horizontal Monotonicity of the Modulus of the Riemann Zeta Function and Related Functions

Yuri Matiyasevich, Filip Saidak, Peter Zvengrowski

Abstract

As usual let \( s = \sigma + it. \) For any fixed value \( t = t_0 \) with \( |t_0| \geq 8, \) and for \( \sigma \leq 0, \) we show that \( |\zeta(s)| \) is strictly monotone decreasing in \( \sigma, \) with the same result also holding for the related functions \( \xi \) of Riemann and \( \eta \) of Euler. The following inequality relating the monotonicity of all three functions is proved:

\[
\Re \left( \frac{\eta'(s)}{\eta(s)} \right) < \Re \left( \frac{\zeta'(s)}{\zeta(s)} \right) < \Re \left( \frac{\xi'(s)}{\xi(s)} \right).
\]

It is also shown that extending the above monotonicity result from \( \sigma \leq 0 \) to \( \sigma \leq 1/2, \) for any of \( \zeta, \xi, \eta, \) is equivalent to the Riemann hypothesis.

1 Introduction

Starting from the work of Riemann \[13\], the zeta function \( \zeta(s) \) (with \( s = \sigma + it \)) has been primarily investigated in the vertical sense, especially in the critical strip \( 0 \leq \sigma \leq 1 \) and on the critical line \( \sigma = 1/2. \) Questions related to the horizontal behaviour of \( |\zeta(s)| \) (as usual we write \( s = \sigma + it \)) have been considered by Saidak and Zvengrowski in \[14\], and earlier by Spira \[16\]. Indeed, the opening page of the article on the Riemann zeta function in the Wolfram MathWorld \[21\] has a plot showing horizontal “ridges” of \( |\zeta(\sigma + it)| \) for \( 0 < \sigma < 1 \) and \( 0 < t < 100. \) To quote from this article, “the fact that the ridges decrease monotonically for \( 0 < \sigma < 1/2 \) is not a
coincidence since it turns out that monotone decrease implies the Riemann hypothesis,” cf. [14] and [2]. In this note, among other things, we shall not only prove the converse of this assertion, but also the fact that $|ζ(s)|$ is monotone decreasing in $σ$ in the region $σ < 0$, subject to the (minor) additional condition $|t| ≥ 8$.

Recently a paper by Sondow and Dumitrescu [15] has appeared exploring this question for the related Riemann $ξ$ function (defined by $ξ(s) := (s − 1)Γ(1 + s/2)π^{−s/2}ζ(s)$). Here we shall consider this question for $ζ(s)$ (as mentioned above), as well as for $ξ(s)$ and Euler’s function $η$ (cf. [4]) (also known as the Dedekind $η$ function) defined by $η(s) := (1 − 2^{1−s})ζ(s)$ or, for $σ > 0$, by the alternating Dirichlet series $η(s) = ∑_{n≥1}(-1)^{n+1}/n^s$. A recent paper of Srinivasan and Zvengrowski [17] also examines this question, for the $Γ$ function, and another recent paper of Alzer [1], titled “Monotonicity Properties of the Riemann Zeta Function,” concerns itself with monotonicity of a function related to the zeta function, but only along the real line. For completeness, let us quote the results in [15] and [17].

**Theorem 1.1** (Sondow–Dumitrescu) : The $ξ$ function is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no zeros of $ξ$. Similarly, the modulus decreases along each horizontal half-line lying in any zero-free, open, left half-plane.

**Theorem 1.2** (Srinivasan–Zvengrowski) : For any fixed $t$ with $|t| > 5/4$, $|Γ(s)|$ is monotone increasing in $σ$.

Section 2 starts by quoting an elementary lemma from [17] that relates the horizontal increase or decrease of $|f(s)|$, for any holomorphic function $f$, to $ℜ(f′(s)/f(s))$, the real part of the logarithmic derivative of $f$. Using this lemma we give a very short proof of the Sondow–Dumitrescu theorem. We also show how a portion of this theorem was implicitly anticipated in a paper of Pólya [12] written in 1927. It is also related to work of Lagarias [9], Haglund [6], and others, again this is briefly discussed in Section 2.

In Section 3 we prove our first main result, namely

**Theorem 1.3** : For $|t| ≥ 8$, $σ < 1/2$, one has

$$ℜ\left(\frac{η′(s)}{η(s)}\right) < ℜ\left(\frac{ζ′(s)}{ζ(s)}\right) < ℜ\left(\frac{ξ′(s)}{ξ(s)}\right),$$

which relates the horizontal growth rates of all three functions under consideration. The second main result, which follows as a corollary of this inequality together with the results in Section 2, is now stated.

**Theorem 1.4** : The moduli of all three functions $η(s), ζ(s)$, and $ξ(s)$ are monotone decreasing with respect to $σ$ in the region $σ ≤ 0$, $|t| ≥ 8$. 

2
Extending this region to $\sigma \leq 1/2$, for any of the three functions, is equivalent to the Riemann hypothesis.

The inequality given in Theorem 1.3 seems to indicate that in order to seek further results on monotonicity, for $\sigma < 1/2$, the most promising of the three functions is $\eta$, and the least promising $\xi$. On the other hand, combining the monotonicity results for $\zeta$ together with the Voronin Universality Theorem [20] for $\zeta$ (or for $\log \zeta$) seems to offer an approach to possibly showing that the Riemann hypothesis is false. We also note that the inequality $|t| \geq 8$ is essential. Slightly smaller numbers than 8 will also work but, for $|t| < 6.2897$, the conclusion of Theorem 3.4 is false for both $\zeta$ and $\eta$. Also, for $\sigma > 1/2$, neither $|\zeta|$ nor $|\eta|$ are monotone (by “monotone” we always mean monotone with respect to $\sigma$).

2 Monotonicity of $|\xi|$

To measure the rate of change of $|f(s)|$ with respect to $\sigma$, the following elementary lemma is useful. For a proof cf. [17].

**Lemma 2.1:** For any holomorphic function $f$, with $f(s) \neq 0$ in some open domain $D$,

$$\Re \left( \frac{f'(s)}{f(s)} \right) = \frac{1}{|f(s)|} \cdot \frac{\partial |f(s)|}{\partial \sigma}, \quad s \in D.$$ 

**Corollary 2.2:** For $s \in D$, 

$$\text{sgn} \left( \frac{\partial |f(s)|}{\partial \sigma} \right) = \text{sgn} \left( \Re \left( \frac{f'(s)}{f(s)} \right) \right).$$

The fact that Lemma 2.1 does not apply at a zero of $f$ is not a problem towards our main objectives, as the next lemma shows.

**Lemma 2.3:** (a) Let $f$ be holomorphic in an open domain $D$ and not identically zero. Let us also suppose $\Re(f'(s)/f(s)) < 0$ for all $s \in D$ such that $f(s) \neq 0$. Then $|f(s)|$ is strictly decreasing with respect to $\sigma$ in $D$, i.e. for each $s_0 \in D$ there exists a $\delta > 0$ such that $|f(s)|$ is strictly monotonically decreasing with respect to $\sigma$ on the horizontal interval from $s_0 - \delta$ to $s_0 + \delta$.

(b) Conversely, if $|f(s)|$ is decreasing with respect to $\sigma$ in $D$, then $\Re(f'(s)/f(s)) \leq 0$ for all $s \in D$ such that $f(s) \neq 0$.

Proof of (a): From Lemma 2.1 and Corollary 2.2 it clearly suffices to show this for those $s_0 = \sigma_0 + it_0 \in D$, where $f(s_0) = 0$. Thanks to $f$ being holomorphic and not identically 0 there exists $\delta > 0$ with $\{s : |s - s_0| < \delta\} \subset D$ and with no further zeros of $f$ in this open disc. Then using the next part of the hypothesis and Corollary 2.2, $|f(s)|$ is strictly decreasing.
with respect to \( \sigma \) on the two horizontal open intervals from \( \sigma_0 - \delta + it_0 \) to \( \sigma_0 + it_0 \), and from \( \sigma_0 + it_0 \) to \( \sigma_0 + \delta + it_0 \). Since \( |f| \) is continuous in \( \mathcal{D} \), a simple continuity argument shows that it must be strictly decreasing on the entire horizontal interval from \( \sigma_0 - \delta + it_0 \) to \( \sigma_0 + \delta + it_0 \).

Proof of (b): Conversely, we are assuming \( \partial |f(s)|/\partial \sigma \leq 0 \) in \( \mathcal{D} \), so Lemma 2.1 implies that \( \Re(f'(s)/f(s)) \leq 0 \) at any \( s \in \mathcal{D} \) for which \( f(s) \neq 0 \). \( \square \)

Of course the analogous results hold for monotone increasing and \( \Re(f'(s)/f(s)) > 0 \). Combining Lemma 2.3 with the fact that a function can have no zeros in an open domain in which its modulus is strictly monotone decreasing (increasing) with respect to \( \sigma \) gives the next result.

**Corollary 2.4:** With the same hypotheses as in Lemma 2.3 (a), \( f \) has no zeros in \( \mathcal{D} \).

Let us now apply the above to the Riemann \( \xi \) function and thereby give a short proof of Theorem 1.1. It is well known that \( \xi(1 - s) = \xi(s) \) and that \( \xi(\overline{s}) = \overline{\xi(s)} \). Hence \( |\xi(1/2 - \sigma + it)| = |\xi(1/2 + \sigma - it)| = |\xi(1/2 + \sigma + it)| \), which shows that \( |\xi| \) is symmetrical about the critical line \( \sigma = 1/2 \). So showing that \( |\xi| \) is monotone decreasing in a domain to the left of the critical line is equivalent to showing it is monotone increasing in the reflection of the same domain about the point \( s = 1/2 \), and this is what we shall show.

**Theorem 1.1:** (Sondow–Dumitrescu) Let \( \sigma_0 \) be greater than or equal to the real part of any zero of \( \xi \). Then \( |\xi(s)| \) is strictly monotone increasing in the half plane \( \sigma > \sigma_0 \).

Proof: We start with the formula due to Hadamard [5] and von Mangoldt [10] (also cf. [8], (36), or simply take the logarithmic derivative of the final formula given in [3], §2.8)

\[
\frac{\xi'(s)}{\xi(s)} = \sum_{\rho} \frac{1}{s - \rho},
\]

where the summation is taken over all zeros \( \rho \) of \( \xi \) (which, as is well known, lie in the critical strip \( 0 < \Re(\rho) < 1 \)), in conjugate pairs and in order of increasing \( \Im(\rho) \). If any such zero be written as \( \rho = \alpha + i\beta \), then by hypothesis \( \sigma > \alpha \). It is then trivial to check that \( \Re(1/(s - \rho)) = (\sigma - \alpha)/[(\sigma - \alpha)^2 + (t - \beta)^2] > 0 \), hence \( \Re(\xi'(s)/\xi(s)) > 0 \) and by Corollary 2.4 \( |\xi(s)| \) is monotone increasing in \( \sigma \), in the given half plane \( \sigma > \sigma_0 \). \( \square \)

Combining this theorem with well known facts about the zeros of \( \xi \), and the fact that a function can have no zeros in an open domain where its modulus is monotone increasing (decreasing), gives the next result.

**Corollary 2.5:** (Sondow–Dumitrescu) In the right (left) half plane \( \sigma \geq 1 \) (\( \sigma \leq 0 \)), \( |\xi| \) is monotone increasing (decreasing). The same is true for
the right (left) half plane \( \sigma \geq 1/2 \) (\( \sigma \leq 1/2 \)) if and only if the Riemann hypothesis is true.

The second part of this corollary, which is the same as Corollary 1 in [15], is actually implicit in a paper written by Pólya in 1927 [12] which discusses the “Nachlass” of J.L.W.V. Jensen, after suitable interpretation. Namely, following I’ on p. 18 of [12], and using \( z = x + iy \) as in this reference, we consider the holomorphic function \( F(z) = \xi(1/2 - iz) = \xi(1/2 + y - ix) \). Note that \( |F(z)| = |\xi(1/2 + y - ix)| = |\xi(1/2 + y + ix)| \), since \( \xi(\overline{s}) = \overline{\xi(s)} \). The condition that all zeros of \( F \) are real is precisely the Riemann hypothesis, indeed this was Riemann’s original formulation. According to condition I’, this is equivalent to \( \frac{\partial^2|\xi(1/2 + y + ix)|^2}{\partial^2 y} \geq 0 \). This implies that \( |\xi(1/2 + y + ix)|^2 \) is a convex function of \( y \). By symmetry it has zero derivative at \( y = 0 \), hence it is monotone increasing for \( y \geq 0 \) and monotone decreasing for \( y \leq 0 \). The same is then also true for \( |\xi(1/2 + y + ix)| \). And conversely, as already remarked before Corollary 2.5, these monotonicity properties imply the Riemann hypothesis.

The fact that \( \Re(\xi'(s)/\xi(s)) > 0 \) when \( \sigma > 1 \), and that the Riemann hypothesis is equivalent to the same statement for \( \sigma > 1/2 \) (for which we gave a short proof above) also appears in the 1999 paper of Lagarias [9] and the 1997 paper of Hinkkanen [7]. Combining this with Lemma 2.3 gives an immediate proof of Theorem 1.1. Another version the Sondow-Dumitrescu result appears as a “known result” at the beginning of Section 6 of [6], this time for the related \( \Xi \) function (the horizontal monotonicity of \( \xi \) being equivalent to vertical monotonicity of \( \Xi \)), but no reference or proof is given.

3 Proof of Theorems 1.3 and 1.4

For convenience we label the first inequality of Theorem 1.3 as (A), and the second (B). To prove either of these we shall take the logarithmic derivatives of the formulae given for \( \xi, \eta \) in the Introduction, and then look at the real part of these logarithmic derivatives. Again, for convenience, we will divide the proof into corresponding parts (A), (B), and separately give two lemmas that will be of use.

Lemma 3.1: For \( \sigma < 1 \), one has \( \Re\left(\frac{1}{2^{s-1} - 1}\right) < 0 \).

Proof: First note that \( 2^{s-1} - 1 = 0 \) if and only if \( s = 1 + 2n\pi i/\log 2, \ n \in \mathbb{Z} \).
Z. In particular \(2^{s-1} - 1 \neq 0\) for \(\sigma < 1\). Now

\[
\Re \left( \frac{1}{2^{s-1} - 1} \right) = \frac{2^{\sigma-1} \cos(t \log 2) - 1}{|2^{s-1} - 1|^2}.
\]

The denominator of this expression is strictly positive since \(\sigma < 1\). As for the numerator, one has \(|2^{\sigma-1} \cos(t \log 2)| < |\cos(t \log 2)| \leq 1\), so the numerator is strictly negative.

Proof of (A): From the formula given at the beginning of the Introduction for \(\eta(s)\), it follows that 

\[
\log(\eta(s)) = \log(1 - 2^{s-1}) + \log(\zeta(s)).
\]

Differentiating,

\[
\eta'(s) = \frac{2^{1-s} \log 2}{1 - 2^{1-s}} + \frac{\zeta'(s)}{\zeta(s)} = \frac{\log 2}{2^{s-1} - 1} + \frac{\zeta'(s)}{\zeta(s)}.
\]

Taking the real parts, and using Lemma 3.1 as well as \(\log 2 > 0\), completes the proof (indeed for \(\sigma < 1\)).

For the second inequality it will be necessary to recall the digamma function \(\Psi(s) := \frac{\Gamma'}{\Gamma}(s)\). We list a few of its properties as the next lemma.

**Lemma 3.2:**

(i) \(\Psi(s) - \Psi(1 - s) = -\pi \cot(\pi s)\),

(ii) \(|\Re(\Psi(s)) - \Re(\Psi(1 - s))| < 3\pi e^{-2\pi t}, \quad t \geq 0,\)

(iii) in the sector of the complex plane \(-\theta < \arg(s) < \theta\), one has

\[
\Psi(s) = \log(s) - \frac{1}{2s} + R'_0(s), \quad \text{where} \quad |R'_0(s)| \leq \sec^3(\theta/2) \cdot \frac{B_2}{2|s|^2},
\]

with \(B_2 = 1/6\) being the second Bernoulli number,

(iv) \(|x/(x^2 + t^2)| \leq 1/(2|t|), \quad \text{for any} \quad x, t \in \mathbb{R}, \quad t \neq 0,\)

(v) for any \(\sigma \geq 1/2, \text{ and } |t| \geq 8, \quad \Re(\Psi(s)) > 2.0096.\)

Formula (i) is a simple consequence of Euler’s reflexion formula for the \(\Gamma\) function, it can be found e.g. in [15], p.14. Formula (ii) follows from (i) and doing an elementary estimate of \(\Re(\cot(z))\), since (i) implies

\[
|\Re(\Psi(s)) - \Re(\Psi(1 - s))| \leq |\Psi(s) - \Psi(1 - s)| = \pi|\cot(z)|,
\]

where for convenience we set \(\pi s = z = x + iy\). We outline the remaining steps towards proving (ii), which are essentially an exercise in calculus. First recall that

\[
\cot(z) = \frac{\cos x \cosh y - i \sin x \sinh y}{\sin x \cosh y + i \cos x \sinh y}.
\]

From this it is easy to derive

\[
\Re(\cot(z)) = \frac{\sin(2x)}{b - \cos(2x) + 1} =: g_b(x), \quad \text{where} \quad b = 2 \sinh^2(y) > 0.
\]
We claim that \(|g_b(x)| < 3e^{-2y}\), when \(y > \log(3)/4\). Indeed, using elementary calculus one shows that \(|g_b(x)|_{\text{max}} = \frac{1}{\sqrt{b^2 + 2b}}\), hence proving the claim reduces to showing \(\frac{1}{\sqrt{b^2 + 2b}} < 3e^{-2y}\). Using the definition of \(b\), this inequality reduces to \(y > \log(3)/4\) and the claim is thus proved. Finally, substituting \(z = \pi s\), we obtain (ii) with \(y = \pi t > \log(3)/4\), i.e. \(t > \log(3)/(4\pi) = .08742\).

Formula (iii) is a special case \((n = 0)\) of the Stirling series

\[
\Psi(s) = \log(s) - \frac{1}{2s} - \sum_{k=1}^{n} \frac{B_{2k}}{2k s^{2k}} + R'_{2n}
\]

for digamma, together with the Stieltjes estimate for the error term (cf. [3], p.114, or the original manuscript of Stieltjes [19])

\[
|R'_{2n}| \leq (\sec(\theta/2))^{2n+3} \left| \frac{B_{2n+2}}{(2n+2)s^{2n+2}} \right|
\]

Formula (iv) is equivalent to \(0 \leq (|x| - |t|)^2\). Finally, from (iii) applied to the sector \(-\pi/2 < \theta < \pi/2\), we have

\[
\Re(\Psi(s)) = \log |s| - \frac{\sigma}{2|s|^2} + \Re(R'_0(s)),
\]

where \(|\Re(R'_0(s))| \leq |R'_0(s)| < 2\sqrt{2}/(6|s|^2)\). Now assume as in (v) that \(\sigma \geq 1/2\) and \(|t| \geq 8\), then using this estimate for the remainder term as well as \(|s| > 8\), we obtain

\[
\Re(\Psi(s)) \geq \log 8 - 1/16 - \sqrt{2}/(3 \cdot 64) = 2.0096,
\]

where (iv) was used to give the 1/16 estimate for the second term. This completes the proof of Lemma 3.2.

Proof of (B): Taking the logarithmic derivative of the formula given at the beginning of the Introduction for \(\xi(s)\) gives

\[
\frac{\xi'(s)}{\xi(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s - 1} + \frac{1}{2} \Psi\left(\frac{s}{2} + 1\right) - \frac{1}{2} \log \pi.
\]

Hence, to complete the proof of (B), it suffices to show that

\[
\Re\left(\frac{1}{s - 1} + \frac{1}{2} \Psi\left(\frac{s}{2} + 1\right)\right) - \frac{1}{2} \log \pi > 0, \quad \sigma < \frac{1}{2}, \quad 8 \leq |t|.
\]
Now, by Lemma 3.2 (iv), the first term is greater than or equal to $-1/16$. By Lemma 3.2 (v) \((1/2)\Re(\Psi(z)) > 1.0048\), at least when \(\sigma \geq 1/2\) and \(|t| \geq 8\). However, applying Lemma 3.2 (ii), we see that the same holds for any \(z\) with \(|t| \geq 8\), at least to within \(3\pi e^{-16\pi}\) which is negligible here. Thus the sum in question is greater than \(-1/16 + 1.0048 - \log \pi/2 > 0\).

Theorem 3.4: The moduli of all three functions \(\eta(s), \zeta(s),\) and \(\xi(s)\) are monotone decreasing with respect to \(\sigma\) in the region \(\sigma \leq 0, \ |t| \geq 8\). Extending this region to \(\sigma \leq 1/2\), for any of the three functions, is equivalent to the Riemann hypothesis.

Proof: For \(\sigma \leq 0\), we have seen in the proof of Theorem 2.5 that \(\Re(\xi'(s)/\xi(s)) < 0\). Combining this with the inequalities in Theorem 3.1 shows that the same is true for \(\zeta\) and \(\eta\), thus all three are monotone decreasing in modulus for \(\sigma \leq 0, \ |t| \geq 8\). And the same argument used in Corollary 2.5 shows that extending this to the larger region \(\sigma \leq 1/2, \ |t| \geq 8\), is equivalent to the Riemann hypothesis.

References

[1] Alzer, H. Monotonicity properties of the Riemann zeta function, Mediterranean J. Math (2011)

[2] Borwein, J., Bailey, D., Mathematics by Experiment – Plausible Reasoning in the Twenty-first Century, A.K.Peters, Wellesley, Massachusetts (2003).

[3] Edwards, H.M., Riemann’s Zeta Function, Academic Press, New York (1974). Reprinted by Dover Publications, Mineola, N.Y. (2001).

[4] Euler, L., Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques (lu en 1749) (presented 1761), Hist. Acad. Roy. Sci. Belles-Lettres Berlin 17 (1768), 83–106. Also in “Opera Omni,” Ser. 1, Vol. 15, 70–90. Original and English translation available via [http://www.math.dartmouth.edu/~euler/pages/E352.html](http://www.math.dartmouth.edu/~euler/pages/E352.html)

[5] Hadamard, J., Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann, J. Math. Pures Appl. (4) 9 (1893), 171–215.

[6] Haglund, J., Some conjectures on the zeros of approximates to the Riemann \(\Xi\) function and incomplete gamma functions, Central European J. Math. 9 (2) (2011), 302–318.
[7] Hinkkanen, A. *On functions of bounded type*, Complex Variables 34 (1997), 119–139.

[8] Kudryavtseva, E., Saidak, F., Zvengrowski, P., *Riemann and his zeta function*, Morfismos 9, No. 2 (2005), 1–48.

[9] Lagarias, J.C., *On a positivity property of the Riemann $\xi$ function*, Acta Arith. 89 (3) (1999), 217–234.

[10] von Mangoldt, H., *Zu Riemann’s Abhandlung ‘Über die Anzahl der Primzahlen unter einer gegebenen Größe’*, J. Reine Angew. Math. 114 (1895), 255–305.

[11] Murty, M. R., *Problems in Analytic Number Theory*, Graduate Texts in Math. 206, Springer-Verlag, N.Y., Berlin, Heidelberg (2001).

[12] Pólya, G., *Über die algebraisch-funktionstheoretischen Untersuchungen von J.L.W.V. Jensen*, Kgl. Danske Vid. Sel. Math.-Fys. Medd. 7 (1927), 3–33. Also available in Pólya, George, Collected Papers Vol. II, Location of Zeros, edited by R. P. Boas, Mathematicians of Our Time, Vol. 8, The MIT Press, Cambridge, Mass., London (1974).

[13] Riemann, B., *Über die Anzahl der Primzahlen unter einer gegebenen Größe* (1859, Monatsberichter der Berliner Akademie, November 1859. Included in : Riemann, B., Gesammelte Werke, Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953. Original and English translation available via http://www.claymath.org/millenium/Riemann_Hypothesis/1859_manuscript/)

[14] Saidak, F., Zvengrowski, P., *On the modulus of the Riemann zeta function in the critical strip*, Math. Slovaca 53 No. 2 (2003), 145–172.

[15] Sondow, J., Dumitrescu, C., *A monotonicity property of Riemann’s xi function and a reformulation of the Riemann hypothesis*, Periodica Math. Hung. 60 I (2010), 37–40. Also available at http://arxiv.org/abs/1005.1104

[16] Spira, R., *An inequality for the Riemann zeta-function*, Duke Math. J. 32 (1965), 247–250.

[17] Srinavasan, G.K., Zvengrowski, P., *On the horizontal monotonicity of $|\Gamma(s)|$*, Can. Math. Bull. 54 (3) (2011), 538–543.
[18] Srivastava, H.M., Choi, J., Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht (2001).

[19] Stieltjes, T. J., *Sur le developpement de logΓ(a)*, J. Math. Pures Appl. (9) 5 (1889), 425-444.

[20] Воронин, С. М. [Voronin, S. M.], *Теорема об “универсальности” дзета-функции Римана* [Theorem on the universality of the Riemann zeta function], Изв. Акад. Наук СССР, сер. матем. 39:3 (1975), 475-486. Translated in Math. USSR Izv. 9 (1975), 443-445.

[21] Wolfram MathWorld [http://mathworld.wolfram.com/RiemannZetaFunction.html](http://mathworld.wolfram.com/RiemannZetaFunction.html)

Yuri Matiyasevich
St. Petersburg Department
of Steklov Institute of Mathematics
do Russian Academy of Sciences
(POMI RAN)
27, Fontanka
St. Petersburg, 191023, Russia
e-mail: yumat@pdmi.ras.ru

Filip Saidak
Department of Mathematics
University of North Carolina
Greensboro, NC 27402, U.S.A.
e-mail: saidak@nz11.com

Peter Zvengrowski
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada T2N 1N4
e-mail: zvengrow@ucalgary.ca