MAPPINGS WITH MAXIMAL RANK

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1 Introduction

We study Maximal Rank Maps and Riemannian Submersions $\pi : M \to B$, where $M$ and $B$ are Riemannian manifolds.

As essential tools in this work we are interested in equivalence relations between non-compact Riemannian manifolds given by Rough Isometries, a concept first introduced by M. Kanai [6].

Motivated by O’Neill [10] we investigated the question: when does a maximal rank map differ only by a rough isometry of $M$ from the simplest type of Riemannian submersions, the projection $p_B : F \times B \to B$ of a Riemannian product manifold on one of its factors. Firstly, for Riemannian submersions $\pi : M \to B$ we show that, if the base manifold $B$ is compact and connected, then the fibers $F$ can be roughly isometrically immersed into $M$, and thus, $M$ is roughly isometric to the product $F \times B$ of any fiber and the base space [Theorem 4.1.1]. When $B$ is noncompact, connected and complete, and $\text{diam}(F)$ is uniformly bounded, the Riemannian Submersion $\pi$ is a rough isometry, and thus, if a fixed fiber $F$ is compact then $M$ is roughly isometric to the product $F \times B$ of that fiber and the base space [Theorem 4.1.2]. Secondly, for onto maximal rank maps that are not necessarily submersions, by adding control on the length of horizontal vector lifts we have the same consequences [Theorem 4.2.1 Theorem 4.2.3]. We provide Counterexamples in section 4.2 to show that the assumptions made are necessary conditions.

The paper begins with background.

2 Rough Isometries and Riemannian Submersions

In this section we define some notation and provide some definitions according to M. Kanai [6] and O’Neill [10].

We will be interested in equivalence relations given by rough isometries, a concept first introduced in [6].

Definition 2.1 A map $\varphi : M \to N$, between two metric spaces $(M, \delta)$ and $(N, d)$, not necessarily continuous, is called a rough isometry, if it satisfies the following two axioms:

1. Rough Isometry Condition: For any $p \in M$ and any $\varepsilon > 0$, there exists a ball $B(p, \delta(p, N))$ in $M$ such that $\varphi(B(p, \delta(p, N))) \subset B(\varphi(p), \varepsilon)$.
2. Rough Inverse Condition: For any $q \in N$ and any $\varepsilon > 0$, there exists a ball $B(q, d(q, M))$ in $N$ such that $\varphi^{-1}(B(q, d(q, M))) \subset B(p, \varepsilon)$ for some $p \in M$.
(RI.1) There are constants $A \geq 1, C \geq 0$, such that,

$$\frac{1}{A} \delta(p_1, p_2) - C \leq d(\varphi(p_1), \varphi(p_2)) \leq A\delta(p_1, p_2) + C, \quad \forall p_1, p_2 \in M$$

(RI.2) The set $\text{Im}\varphi := \{q = \varphi(p), \forall p \in M\}$ is full in $N$, i.e.

$$\exists \varepsilon > 0 : N = B_\varepsilon(\text{Im}\varphi) = \{q \in N : d(q, \text{Im}\varphi) < \varepsilon\}$$

In this case we say that $\text{Im}\varphi$ is $\varepsilon$-full in $N$.

One can easily show that if $\varphi : M \rightarrow N$ and $\psi : N \rightarrow M$ are rough isometries, then the composition $\psi \circ \varphi : M \rightarrow M$ is also a rough isometry.

We will denote by $\varphi^- : N \rightarrow M$ a rough inverse of $\varphi$, defined as follows: for each $q \in N$, choose $p \in M$ so that $d(\varphi(p), q) < \varepsilon$, and define $\varphi^-(q) := x$.

We point out here that such a $p$ exists because of the condition (RI.2). $\varphi^-$ is a rough isometry such that both $\delta(\varphi^-(p), p)$ and $d(\varphi \circ \varphi^-(q), q)$ are bounded in $p \in M$ and in $q \in N$, respectively.

We refer to O’Neill [10] for the properties of Riemannian submersions.

We start recalling their definition.

Let $M^m$ and $B^n$ be Riemannian manifolds with dimensions $m$ and $n$, respectively, where $m \geq n$.

**Definition 2.2** A map $\pi : M \rightarrow B$ has maximal rank $n$ if the derivative map $\pi_* : T_xM \rightarrow T_{\pi(x)}B$ is surjective.

According to [10], a tangent vector on $M$ is said to be **vertical** if it is tangent to a fiber, **horizontal** if it is orthogonal to a fiber. A vector field on $M$ is vertical if it is always tangent to fibers, horizontal if it is always orthogonal to fibers.

Since the derivative map $\pi_*x$ of $\pi$ is surjective for all $x \in M$, its rank is maximal. We can define the projections of the tangent space of $M$ onto the subspaces of vertical and horizontal vectors, which we will denote respectively by $(VT)_x$ and $(HT)_x$ for each $x \in M$. In that case, we can decompose each tangent space to $M$ into a direct orthogonal sum $T_xM = (VT)_x \oplus (HT)_x$.

**Definition 2.3** A **Riemannian submersion** $\pi : M \rightarrow B$ is an onto mapping satisfying the following two axioms:

(S.1) $\pi$ has maximal rank;

(S.2) $\pi_*$ preserves lengths of horizontal vectors.
3 Long Curves and Their Lifts

Here we begin with background from O’Neill [10] and continue with an investigation of curves and their lifts.

Let $\pi : M \to B$ denote an onto mapping with maximal rank $n$ between Riemannian manifolds $M^m$ and $B^n$ with $m \geq n$.

From the maximality of the rank of the onto mapping $\pi$ we have the unique horizontal vector property:

**Lemma 3.1** Let $b \in B$. Given any $w \in T_b B$ and $x \in M$ satisfying $\pi(x) = b$, there exists a unique horizontal vector $v \in T_x M$ which is $\pi$-related to $w$, i.e. satisfying $v \in (HT)_x$ and $(\pi_*)_x(v) = w$.

If, in addition, one has control from below over the length of horizontal vectors, then one has control from below over the distance in $M$. This is the essence of the following Lemma.

**Lemma 3.2** Assume that $M$ and $B$ are both connected and geodesically complete. Let $x, x' \in M$, $\Gamma_{\text{min}} \subset M$ be a minimal geodesic joining $x$ to $x'$, and $\gamma_{\text{min}} \subset B$ be a minimal geodesic joining $\pi(x)$ to $\pi(x')$. Suppose that for all $b \in B$ and for all $x \in F_b$ there exist constants $\alpha \geq 1$ and $\beta > 0$, both independent of $b$ and $x$, such that

$$\frac{1}{\alpha} ||w||_B - \beta \leq ||v||_M$$

(1)

for all $w \in T_b B$, where $v$ is the unique horizontal lift of $w$ through $x$ that we assume satisfies $||v||_M \leq 1$, and $|| \cdot ||_M, || \cdot ||_B$ denote the inner product on $TM$ and $TB$, respectively.

Then, $d_M(x, x') = \ell(\Gamma_{\text{min}}) \geq \frac{1}{\alpha} \ell(\gamma_{\text{min}}) - \beta = \frac{1}{\alpha} d_B(\pi(x), \pi(x')) - \beta$

**Proof.** Without loss of generality, we may assume that the horizontal lift $v \in (HT)_x$ of $w$ satisfies $||v||_M \leq 1$, and that both parametrizations of $\Gamma_{\text{min}}$ and $\gamma_{\text{min}}$ are defined in the interval $[0, 1]$.

We may write

$$\Gamma'_{\text{min}}(t) = \Gamma'_V(t) \oplus \Gamma'_H(t) \in T_x M = (VT)_x \oplus (HT)_x, \forall t \in [0, 1]$$

Notice that by Lemma 3.1 $\Gamma'_H(t) \in T_{\Gamma_{\text{min}}(t)} M$ is the unique horizontal vector which is $\pi$-related to $\frac{d}{dt} (\pi \circ \Gamma_{\text{min}})(t)$, for each $t \in [0, 1]$. Assume that $\Gamma$ is parametrized proportionally to arclength, and that $||\Gamma'_H(t)||_M \leq 1$. 

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We have,

\[ d_M(x,x') = \ell(\Gamma_{\text{min}}) = \int_0^1 ||\Gamma'_V(t) \oplus \Gamma'_H(t)||_M dt \geq \int_0^1 ||\Gamma'_H(t)||_M dt \geq \]

\[ \geq \frac{1}{\alpha} \int_0^1 \left| \frac{d}{dt} (\pi \circ \Gamma_{\text{min}})(t) \right|_M dt - \beta = \frac{1}{\alpha} \ell(\pi \circ \Gamma_{\text{min}}) - \beta \]

which concludes the Lemma.

In what follows lifts of curves are defined.

**Definition 3.3** Let \( \gamma : [t_1,t_2] \to B \) be a smooth embedded curve in \( B \) and \( \Gamma : [t_1,t_2] \to M \) be any curve in \( M \) satisfying \( \pi \circ \Gamma = \gamma \). The curve \( \Gamma \) is called a lift of \( \gamma \).

If in addition, \( \Gamma \) is horizontal, i.e., \( \Gamma'(t) \in (HT)_{\Gamma(t)}, \forall t \in [t_1,t_2], \) where \( \Gamma(t_1) = x_0 \in M \) with \( \gamma(t_1) = \pi(x_0) \), the curve \( \Gamma \) is called a horizontal lift of \( \gamma \) through \( x_0 \). Recall that the horizontal lift of a curve in \( B \), through a point \( x_0 \in M \) is unique.

Next, we define long curves.

**Definition 3.4** Let \( \beta \) be any positive constant. A smooth embedded curve \( \gamma : [t_1,t_2] \to B \) is said to be a \( \beta \)-long curve if

\[ \inf_{t_1 \leq t \leq t_2} ||\gamma'(t)|| \geq \beta. \]

In that case, \( \ell(\gamma) \geq \int_{t_1}^{t_2} ||\gamma'(t)|| dt \geq \beta(t_1 - t_2) \). We say that a curve \( \gamma \) is simply a long curve if it is a \( \beta \)-long curve for some constant \( \beta > 0 \).

Let \( \gamma : [t_1,t_2] \to B \) denote a smooth embedded curve and let \( \Gamma : [t_1,t_2] \to M \) denote a lift of \( \gamma \).

In the next two Propositions, under control from above (below) on the derivative of the maximal rank mapping \( \pi \), we have control from below (above) over the length of any lift of a curve.

For instance, in **Proposition 3.5** for a long curve \( \gamma \) in \( B \) any of its lift \( \Gamma \) in \( M \) cannot be short, and **Proposition 3.6** the length of a lift \( \Gamma \) of a long curve \( \gamma \) is bounded above by the length of \( \gamma \).

We denote by \( || \cdot ||_M \) and \( || \cdot ||_B \) the Riemannian norms in \( TM \) and \( TB \), respectively.
Proposition 3.5 Assume there are constants $\alpha \geq 1$ and $\beta > 0$ such that,
\[
\|((\pi_*)_x v)\|_B \leq \alpha \|v\|_M + \beta
\] (2)
for all $x \in M$, for all $v \in T_x M$ satisfying $\|v\|_M \leq 1$.

If $\gamma$ is any smooth $\beta$-long curve in $B$, then,
\[
\ell(\Gamma) \geq \frac{1}{\alpha} [\ell(\gamma) - \beta(t_2 - t_1)] > 0
\]
where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves $\Gamma$ and $\gamma$, respectively.

Proof. First, we choose a parametrization proportional to arc length of $\Gamma : t \in [t_1, t_2] \rightarrow \Gamma(t) \in M$, an arbitrary lift of $\gamma \subset B$. We may assume without loss of generality that $\|\Gamma'(t)\|_M \leq 1$.

If we use $v = \Gamma'(t)$ in (2) and $\pi \circ \Gamma = \gamma$, we obtain
\[
\|\gamma'(t)\|_B = \|((\pi_*)_{\Gamma(t)} \Gamma'(t))\|_B \leq \alpha \|\Gamma'(t)\|_M + \beta, \quad \forall t \in [t_1, t_2] \quad (3)
\]

Finally, if we integrate (3), we get
\[
\ell(\gamma) = \int_{t_1}^{t_2} \|\gamma'(t)\|_B dt \leq \alpha \int_{t_1}^{t_2} \|\Gamma'(t)\|_M dt + \beta \int_{t_1}^{t_2} dt = \alpha \cdot \ell(\Gamma) + \beta(t_2 - t_1) \Rightarrow
\]
\[
\ell(\Gamma) \geq \frac{1}{\alpha} [\ell(\gamma) - \beta(t_2 - t_1)] > 0
\]
which proves the proposition.

That the second hand side of the last inequality above is positive follows from the assumption that $\gamma$ is a long curve. Therefore, as it can be interpreted from the inequality shown, for a long curve $\gamma$ any of its lift $\Gamma$ cannot be short.

\[\square\]

Proposition 3.6 Let $\gamma : [t_1, t_2] \rightarrow B$ denote a smooth embedded curve and let $\Gamma : [t_1, t_2] \rightarrow M$ denote a lift of $\gamma$. For horizontal vectors $v \in TM$ only, assume that there is a universal constant $\alpha \geq 1$ such that,
\[
\|((\pi_*)_x v)\|_B \geq \frac{1}{\alpha} \|v\|_M - \beta
\] (4)
for all $x \in M$, for all $v \in T_x M \setminus (VT)_x = (HT)_x = [\ker(\pi_*)_x]^\perp$. 
If $\gamma$ is a $\beta$-long curve then,
\[ \ell(\Gamma) \leq \alpha \{\ell(\gamma) + \beta(t_2 - t_1)\} \]
where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves $\Gamma$ and $\gamma$, respectively.

Now, in the proof of Proposition 3.6 we will need the following Lemma: for a long curve in $B$, any of its lift in $M$ is non-vertical.

**Lemma 3.7** Let $\gamma : [t_1, t_2] \to B$ be a smooth embedded curve in $B$ and let $\Gamma : [t_1, t_2] \to M$ be a lift of $\gamma$. If $\gamma$ is a long curve, then, $\Gamma$ is non-vertical, i.e. there exists an interval $[t_1, t_2]$, such that,
\[ (\Gamma'(t))_H \neq 0, \quad \forall t \in [t_1, t_2] \]
where $\Gamma'(t) = (\Gamma'(t))_V \oplus (\Gamma'(t))_H \in T_{\Gamma(t)} M = (VT)_{\Gamma(t)} \oplus (HT)_{\Gamma(t)}$.

**Proof.** Since $\gamma$ is a smooth, embedded long curve, there exists an interval let us say $[t_1, t_2]$, for which,
\[ \gamma'(t) \neq 0, \quad \forall t \in [t_1, t_2] \]
Moreover, since for all $t \in [t_1, t_2]$ the restriction of the derivative map
\[ (\pi_\ast)_{\Gamma(t)} \mid_{(HT)_{\Gamma(t)}} \] is an isomorphism,
we thus obtain
\[ (\pi_\ast)_{\Gamma(t)} (\Gamma'(t))_H = (\pi_\ast)_{\Gamma(t)} \{ (\Gamma'(t))_V \oplus (\Gamma'(t))_H \} = (\pi_\ast)_{\Gamma(t)} \{ \Gamma'(t) \} = \gamma'(t) \neq 0 \]
for all $t \in [t_1, t_2]$, and thus $\Gamma$ is non-vertical.

**Proof of Proposition 3.6** We first notice that because $\gamma$ is a long curve, by Lemma 3.7 $\Gamma$ is non-vertical.

If we use the horizontal vector $v = \Gamma'(t)$ in (4), we may write
\[ 0 \neq \| (\pi_\ast)_{\Gamma(t)} \Gamma'(t) \|_B \geq \frac{1}{\alpha} \| \Gamma'(t) \|_M - \beta, \quad \forall t \in [t_1, t_2] \]
and using $\pi \circ \Gamma = \gamma$ in the above inequality, we obtain
\[ 0 \neq \| \gamma'(t) \|_B \geq \frac{1}{\alpha} \| \Gamma'(t) \|_M - \beta, \quad \forall t \in [t_1, t_2] \]
which in turn implies that
\[ ||\Gamma'(t)||_M \leq \alpha \left( ||\gamma'(t)||_B + \beta \right), \quad \forall t \in [t_1, t_2] \]

Finally, integrating the above inequality gives us
\[ \ell(\Gamma) = \int_{t_1}^{t_2} ||\Gamma'(t)||_M dt \leq \alpha \int_{t_1}^{t_2} ||\gamma'(t)||_B dt + \beta \alpha \int_{t_1}^{t_2} dt = \alpha \cdot \ell(\gamma) + \beta \alpha (t_2 - t_1) = \alpha \left[ \ell(\gamma) + \beta (t_2 - t_1) \right] > 0 \]
which proves the proposition.

Therefore, as it can be interpreted from the above inequality, the length of a lift \( \Gamma \) of a long curve \( \gamma \) is controlled by above by the length of \( \gamma \).

\[ \square \]

4 Riemannian Submersions, Maximal Rank Maps and Counterexamples

In this section we will explore Riemannian submersions and maximal rank maps \( \pi : M \to B \) between Riemannian manifolds \( M \) and \( B \).

Motivated by O’Neill [10], we will investigate this question: when does a maximal rank map \( \pi : M \to B \) differ only by a rough isometry of \( M \) from the simplest type of Riemannian submersions, the projection \( p_B : F \times B \to B \) of a Riemannian product manifold on one of its factors.

4.1 Riemannian Submersions

We first show that, if the base manifold \( B \) is compact and connected, then the fibers \( F \) can be roughly isometrically immersed into \( M \), and thus, \( M \) is roughly isometric to the product \( F \times B \) of any fiber \( F \) and the base space \( B \) [Theorem 4.1.1]. Secondly, when \( B \) is noncompact, connected and complete, and \( diam(F) \) is uniformly bounded, we show that the Riemannian submersion \( \pi : M \to B \) is a rough isometry, and thus, if a fixed fiber \( F \) is compact then \( M \) is roughly isometric to the product \( F \times B \) of that fiber \( F \) and the base space \( B \) [Theorem 4.1.2].

**Theorem 4.1.1** Let \( \pi : M \to B \) be a Riemannian submersion. Suppose \( B \) is compact and connected, and for each \( b \in B \) the fiber \( \pi^{-1}(b) \) has the induced metric from \( (M, d) \). Then for each \( b \in B \), the inclusion \( \iota : \pi^{-1}(b) \hookrightarrow M \) is a rough isometry.

In particular, since \( B \) is compact, \( M \) is roughly isometric to the product \( \pi^{-1}(b) \times B \).
Theorem 4.1.2 Let $\pi : M \to B$ be a Riemannian submersion, where $B$ is connected and complete. Suppose that, for some constant $m > 0$, all fibers satisfy the universal diameter property:

$$(UDF) \quad \text{diam} \ (\pi^{-1}(b)) \leq m < \infty, \forall b \in B.$$ 

Then, $\pi : M \to B$ is a rough isometry.

In particular, if for some $b_0$ the fiber $\pi^{-1}(b_0)$ is compact, then $M$ is roughly isometric to the product $\pi^{-1}(b_0) \times B$.

Note that Theorem 4.1.1 is a Corollary of Theorem 4.2.1 and Theorem 4.1.2 is a Corollary of Theorem 4.2.3 proven in the next section.

4.2 Non-Submersions Surjective Maximal Rank Maps

In this section, we prove that for onto smooth mappings with maximal rank $\pi : M \to B$, that are not necessarily submersions, the same results as in Theorem 4.1.1 and Theorem 4.1.2 hold, as long as we make extra assumptions on the subspaces of horizontal vectors by adding control from above on the length of horizontal vector lifts [Theorem 4.2.1 Theorem 4.2.3]. Counterexamples are provided in this section to show that if any of the assumptions are removed those results cease to follow [Counterexample 4.2.2 Counterexample 4.2.4 Counterexample 4.2.5 Counterexample 4.2.6].

Theorem 4.2.1 Assume that $B$ is compact, and for each $b \in B$, the fiber $\pi^{-1}(b) = F_b$ is endowed with the induced metric from $(M,d)$. Suppose that for all $b \in B$, there are constants $\alpha \geq 1$ and $\beta > 0$, independent of $b$ such that, the following inequality holds:

$$||v||_M \leq \alpha ||w||_B + \beta$$

(5)

for all $x \in F_b$ and $w \in T_b B$, where $v \in (HT)_x \subset T_x M$ is the horizontal lift of $w$ through $x$.

Then, for each $b \in B$, the inclusion map $\iota : F_b \hookrightarrow M$ is a rough isometry.

In particular, since $B$ is compact, $M$ is roughly isometric to the product $\pi^{-1}(b) \times B$.

Proof. We must verify axioms (RI.1) and (RI.2) for $\iota$, given any $b \in B$.

Clearly axiom (RI.1) holds since each fiber has the induced metric.

Let us denote by $(M,d_M)$ and $(B,d_B)$ the Riemannian metric spaces, and let $b \in B$ be fixed.
To verify axiom (RI.2), we need to prove that $M$ is an $\epsilon$-neighborhood of $\iota(F_b) \subseteq M$, for some $\epsilon > 0$, i.e. we must find a constant $\epsilon > 0$ for which

$$d_M(y, \iota(F_b)) < \epsilon, \quad \forall y \in M$$

Without loss of generality, we may assume that $B$ is connected, otherwise, we can repeat, on each connected component of $B$, the argument that will follow.

Now, since $B$ is compact and connected it is also complete.

Thus, for any $y \in M$ there exists a minimal geodesic $\gamma$ joining $\pi(y)$ to $b$, which we will parametrize by

$$\gamma : [0, 1] \to B, \quad \gamma(0) = \pi(y), \quad \gamma(1) = b$$

Since $\gamma$ has a unique horizontal lift $\Gamma_y : [0, 1] \to M$, through $y$, and so $\Gamma_y$ connects $y$ to the fiber $F_b$, we can write,

$$d_M(y, \iota(F_b)) \leq \ell(\Gamma_y) := \int_0^1 ||\Gamma_y'||_M dt \leq$$

$$\leq \alpha \int_0^1 ||(\pi_*)\Gamma_y(t)\Gamma_y'(t)||_B dt + \beta =$$

$$\pi \sigma \Gamma_y = \gamma \quad \alpha \int_0^1 ||\gamma'(t)||_B dt + \beta = \alpha \cdot \ell(\gamma) + \beta \quad (6)$$

Now, by the compactness of $B$,

$$\text{diam } B := \sup_{b_1, b_2 \in B} \{d_B(b_1, b_2)\} < \infty$$

Moreover, since $\gamma$ is a minimal geodesic joining $\pi(y)$ to $b$,

$$\ell(\gamma) = d_B(\pi(y), b) \leq \text{diam } B < \infty \quad (7)$$

By substituting (7) in (6), we obtain,

$$d_M(y, \iota(F_b)) \leq \alpha \cdot \ell(\gamma) + \beta \leq \alpha \cdot (\text{diam } B) + \beta \quad (8)$$

Define $\epsilon := \alpha \cdot (\text{diam } B) + \beta$, which is a positive constant independent of $y$, and also of $b \in B$.

For that choice of $\epsilon$, since $y \in M$ is arbitrary, we see that (8) is exactly axiom (RI.2) for the inclusion map $\iota : F_b \hookrightarrow M$. 

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In what follows, we provide a Counterexample to illustrate how assumption (5) is essential in Theorem 4.2.1. We show that if (5) doesn’t hold for some $b \in B$, then the inclusion map $\iota : F_b \hookrightarrow M$ ceases to be a rough isometry.

**Counterexample 4.2.2** We will exhibit $M, B, \pi$ satisfying all the conditions in Theorem 4.2.1 with the exception of (5), i.e.,

For any given constants $\alpha \geq 1$ and $\beta > 0$ there exist $b \in B$ and $x \in F_b$ satisfying:

$$||v||_M > \alpha||w||_B + \beta$$

where $v$ is the horizontal lift of $w$ in $(HT)_x \subset T_x M$.

In this case, the inclusion map $\iota : F_b \hookrightarrow M$ is not a rough isometry.

Let $M$ and $B$ be the following Riemannian manifolds,

$$M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2 + 1\}$$

and the compact unit circle,

$$B = S^1 = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$$

where the metrics on $M$ and $B$ are induced by the Euclidean metric on $\mathbb{R}^3$.

Let $\pi : M \rightarrow B$ be defined by,

$$\pi(x_1, x_2, x_3) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, 0\right)$$

Clearly $\pi : M \rightarrow B$ is an onto smooth maximal rank map.

Firstly, we remark that (9) can be verified with a series of calculations (c.f. [1]).

Lastly, we show that for each $b = (b_1, b_2, 0) \in B$ the inclusion $\iota : F_b \hookrightarrow M$ is not a rough isometry.

In that direction, we claim that (RI.2) fails, i.e.

$$\forall \epsilon > 0, \exists y_\epsilon = y_\epsilon(\epsilon, b) \in M, \text{ satisfying } d_M(y_\epsilon, \iota(F_b)) \geq \epsilon$$

Let $\gamma$ be a compact connected smooth curve in $B = S^1$, parametrized by,

$$\gamma(t) = (\cos(t), \sin(t), 0) \in B, \ \forall t \in [0, 1]$$
Figure 1: The map \( \pi : M \to B \) in Counterexample 4.2.2

with \( b = \gamma(t_b) \) for some \( t_b \in [0, 1] \).

A generic element in the fiber \( F_b \subseteq M \) can be described as,

\[
\xi_r := \left( \cos(t_b) \cdot \sqrt{r^2 + 1}, \sin(t_b) \cdot \sqrt{r^2 + 1}, r \right) \in M
\]

where \( s \in \mathbb{R} \) is constant.

Thus, the fiber \( F_b \), where \( b = \gamma(t_b) = (\cos t_b, \sin t_b, 0) \), can be described as,

\[
F_b = \{ \xi_r := (\sqrt{r^2 + 1} \cdot \cos t_b, \sqrt{r^2 + 1} \cdot \sin t_b, r), r \in \mathbb{R} \}
\]

It can be shown (see [1]) that the unique horizontal lift \( \Gamma_r \) (see Fig. 2) of \( \gamma \) through \( \xi_r \), where \( r > 0 \), can be parametrized by,

\[
\Gamma_r(t) = \left( \gamma_1(t) \cdot \sqrt{r^2 + 1}, \gamma_2(t) \cdot \sqrt{r^2 + 1}, r \right) \quad \forall t \in [0, 1]
\]
Figure 2: Curve $\gamma$ and its horizontal lift $\Gamma_r$.

Notice that $M \setminus F_b \neq \emptyset$.

Now, any element $y_r$ of $M \setminus F_b$ is of the form,

$$y_r = \left( \sqrt{r^2 + 1} \cdot \cos \bar{t}, \sqrt{r^2 + 1} \cdot \sin \bar{t}, r \right)$$

for some $r \in \mathbb{R}$ and $\bar{t} \in [0, 2\pi), \bar{t} \neq t_b$, where $\gamma(t_b) = (\cos t_b, \sin t_b, 0) = b \neq \pi(y_r) = (\cos \bar{t}, \sin \bar{t}, 0) = \gamma(\bar{t})$ (see Fig. 3).

We may choose $\bar{t} \in [0, 2\pi)$ as follows,

$$\bar{t} := \begin{cases} 
  t_b + \pi, & \text{if } 0 \leq t_b < \pi \\
  t_b - \pi, & \text{if } \pi \leq t_b < 2\pi
\end{cases} \quad (0 \leq \bar{t} < 2\pi)$$

In particular, $\bar{t} \neq t_b$ and $|\bar{t} - t_b| = \pi$. 

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Moreover,

\[
d_M(\xi_r, y_r) = \sqrt{(r^2 + 1)(\cos t_b - \cos t)^2 + (r^2 + 1)(\sin t_b - \sin t)^2} = \\
= \sqrt{r^2 + 1 - 2 \cos t_b \cos t - 2 \sin t_b \sin t + 1} = \\
= \sqrt{r^2 + 1 - 2 \cos(t_b - t)} = \sqrt{r^2 + 1 - 2 \sqrt{2}} = 2 \sqrt{r^2 + 1} \quad (10)
\]

Let \( \epsilon > 0 \) be arbitrary.

If \( \epsilon \leq 2 \), by (10) we have (see Fig. 4),

\[
d_M(\iota(F_b), y_0) = d_M(\xi_0, y_0) = 2 \geq \epsilon
\]
which shows that \([R\text{I}.2]\) fails for \(y_\epsilon := y_0 = (\cos \bar{t}, \sin \bar{t}, 0) \in M \setminus F_b\).

If \(\epsilon > 2\), consider any \(r \in \mathbb{R}\) satisfying
\[
 r > \frac{\sqrt{\epsilon^2 - 4}}{2} = \frac{\sqrt{\epsilon - 2} \sqrt{\epsilon + 2}}{2} > 0
\]
this choice of \(r\) being possible, because property \([9]\) holds for this Counterexample.

In that case, we have,
\[
2r > \sqrt{\epsilon^2 - 4} > 0 \implies 4r^2 > \epsilon^2 - 4 > 0 \implies 4(r^2 + 1) > \epsilon^2 \implies 2\sqrt{r^2 + 1} > \epsilon
\]
In what follows we will define (see Fig. 4),
\[
y_\epsilon = \left(\sqrt{r^2 + 1} \cdot \cos \bar{t}, \sqrt{r^2 + 1} \cdot \sin \bar{t}, r_\epsilon\right) \in M \setminus F_b
\]
satisfying the 2 conditions,

- \(r_\epsilon > r\); and
- the unique straight line passing through \(y_\epsilon\) and \(\xi_\epsilon\) is perpendicular to \(F_b\) at \(\xi_\epsilon\), thus giving us the realization of the distance \(d_M(y_\epsilon, \epsilon(F_b)) = d_M(y_\epsilon, \xi_\epsilon)\).

We may assume for the sake of a much simplified calculation, that, \(t_b = \frac{3\pi}{2}\) and \(\bar{t} = \frac{\pi}{2}\), since \(M\) is symmetric with respect to both axis \(e_3\) and \(e_2\).

Thus we have, \(\gamma(\frac{3\pi}{2}) = (0, -1, 0) = b, \gamma(\frac{\pi}{2}) = (0, 1, 0)\), a generic element \(\xi_\epsilon = (0, -\sqrt{r^2 + 1}, r)\) in the fiber \(F_b\), and a generic element \(y_\epsilon = (0, \sqrt{r^2 + 1}, r)\) in \(M\), but not in the fiber \(F_b\).

In this case, \(F_b\) is given by \(x_2 = -\sqrt{x_3^2 + 1}\), and the perpendicular line to \(F_b\) at \(\xi_\epsilon\) has equation,
\[
x_2 + \sqrt{r^2 + 1} = \frac{-1}{\mu}(x_3 - r) = \frac{\sqrt{r^2 + 1}}{r}(x_3 - r), \quad \forall x_3 \in \mathbb{R}
\]
where \(\mu = \frac{d}{du} - \sqrt{u^2 + 1} \mid_{u=r} = \frac{-2r}{2\sqrt{r^2 + 1}}\).
Since \( y_\epsilon = (0, \sqrt{r_\epsilon^2 + 1}, r_\epsilon) \) is on the line (12), we obtain the following equation,
\[
\sqrt{r_\epsilon^2 + 1} + \sqrt{r^2 + 1} = \frac{\sqrt{r^2 + 1}}{r}(r_\epsilon - r) > 0
\]
which defines \( r_\epsilon \).

Indeed, equation (13) has only one solution,
\[
\sqrt{r_\epsilon^2 + 1} = \frac{\sqrt{r^2 + 1}}{r}r_\epsilon - 2\sqrt{r^2 + 1} = \sqrt{r^2 + 1}\left(\frac{r_\epsilon}{r} - 2\right) \Rightarrow \\
\Rightarrow r_\epsilon^2 + 1 = (r^2 + 1)\left(\frac{r_\epsilon}{r}^2 + 4 - 4\frac{r_\epsilon}{r}\right) \Rightarrow 
\]
\[
\Rightarrow r_\varepsilon^2 + 1 = r_\varepsilon^2 + \frac{r_\varepsilon^2}{r^2} + 4(r^2 + 1) - 4\left(\frac{r^2 + 1}{r}\right) r_\varepsilon \Rightarrow \\
\Rightarrow \frac{1}{r^2} r_\varepsilon^2 - 4\left(\frac{r^2 + 1}{r}\right) r_\varepsilon + 4r^2 + 3 = 0 \Rightarrow \\
\Rightarrow r_\varepsilon = \frac{4\left(\frac{r^2 + 1}{r}\right) \pm \sqrt{\Delta}}{2 \frac{1}{r^2}}
\]

where

\[
\Delta = 16\left(\frac{r^2 + 1}{r}\right)^2 - 4\frac{1}{r^2} (4r^2 + 3) = 16\left(r^2 + 2 + \frac{1}{r^2}\right) - 16 - \frac{12}{r^2} = \\
= 16r^2 + 16 + \frac{4}{r^2} = 4\left(4r^2 + 4 + \frac{1}{r^2}\right) = 4\left(2r + \frac{1}{r}\right)^2 \Rightarrow \\
\Rightarrow \sqrt{\Delta} = 2\left(2r + \frac{1}{r}\right) = \left(4r + \frac{2}{r}\right)
\]

which implies that,

\[
r_\varepsilon = \frac{4\left(\frac{r^2 + 1}{r}\right) \pm \left(4r + \frac{2}{r}\right)}{2 \frac{1}{r^2}} = 2r^3 + 2r \pm (2r^3 + r) = \begin{cases} 
4^3 + 3r \text{ (solution)} \\
4r \text{ (not a solution)}
\end{cases}
\]

Then, the only solution of equation (13) is,

\[
r_\varepsilon = 4^3 + 3r = r(4^2 + 3) > r > 0 \quad (14)
\]

Finally, by employing the expressions,

\[
y_\varepsilon = \left(0, \sqrt{r_\varepsilon^2 + 1}, r_\varepsilon\right) \quad \text{and} \quad \xi_r = (0, -\sqrt{r^2 + 1}, r)
\]

and by using (13) and (14), we can estimate the distance from \(y_\varepsilon\) to the fiber \(F_b\), only in terms of \(r\),

\[
d_M(y_\varepsilon, \xi_r) = \sqrt{0^2 + \left(\sqrt{r_\varepsilon^2 + 1} + \sqrt{r^2 + 1}\right)^2 + (r_\varepsilon - r)^2} = \\
= \sqrt{\left(\frac{r^2 + 1}{r^2}\right)(r_\varepsilon - r)^2 + (r_\varepsilon - r)^2 = \sqrt{\left(\frac{r^2 + 1}{r^2} + 1\right)(r_\varepsilon - r)^2}}
\]
\[ \sqrt{\frac{(2r^2 + 1)(r_\epsilon - r)^2}{r^2}} = \sqrt{\frac{(2r^2 + 1)(4r^3 + 3r - r)^2}{r^2}} = \sqrt{(2r^2 + 1)(4r^2 + 2)} = 2\sqrt{2r^2 + 1} (2^2 + 1) = 2 (2r^2 + 1)^{\frac{3}{2}} \] (15)

Next, we claim that,
\[ (2r^2 + 1)^{\frac{3}{2}} > \sqrt{r^2 + 1} \] (16)

The function \( f \in C^\infty(\mathbb{R}) \) defined by,
\[ f(r) := (2r^2 + 1)^{\frac{3}{2}} - (r^2 + 1) \]
is clearly strictly increasing on \([0, \infty)\), and thus,
\[ f(r) > f(0) = 0, \ \forall r > 0 \Rightarrow (2r^2 + 1)^{\frac{3}{2}} > (r^2 + 1)^{\frac{1}{2}}, \ \forall r > 0 \]
which is claim (16).

Now, if we combine (15), (16) and (11), we get,
\[ d_M(y_\epsilon, \iota(F_b)) = d_M(y_\epsilon, \xi_r) > 2\sqrt{r^2 + 1} > \epsilon \]
which shows that [RI.2] fails for \( y_\epsilon := (0, \sqrt{r^2 + 1}, r_\epsilon) \in M \setminus F_b \).

We have thus shown that for any \( \epsilon > 0 \) there exists \( y_\epsilon \in M \setminus F_b \) for which [RI.2] fails. Consequently, the inclusion map \( \iota : F_b \to M \) is not a rough isometry.

This describes the Counterexample.

Next, including a lower bound in assumption (13), and adding an universal diameter upper bound condition on the fibers, we will show that \( \pi : M \to B \) is a rough isometry.

**Theorem 4.2.3** Let \( \pi : M \to B \) be an onto smooth map with maximal rank, where \( B \) is complete. Assume the following,

- **(UDF)** \( \exists m > 0, \) a universal constant, such that \( \text{diam} \{ \pi^{-1}(b) \} \leq m < \infty, \) for all \( b \in B \); and

- **(HLC)** \( \exists \alpha \geq 1 \) and \( \beta > 0 \) such that, for all \( b \in B \) the inequality holds:
\[ \frac{1}{\alpha}||w||_B - \beta \leq ||v||_M \leq \alpha ||w||_B + \beta \]
for all \( x \in F_b \) and \( w \in T_bB \), where \( v \in (HT)_x \subset T_xM \) is the horizontal lift of \( w \) through \( x \) and we assume that \( v \) satisfies \( ||v||_M \leq 1 \).
Then, \( \pi : M \to B \) is a rough isometry.

In particular, if the fiber \( \pi^{-1}(b_0) \) is compact for some \( b_0 \), then \( M \) is roughly isometric to the product \( \pi^{-1}(b_0) \times B \).

**Proof.** Firstly, note that in (HLC) the horizontal lift \( v \in (HT)_x \) of \( w \) is assumed to satisfy \( ||v||_M \leq 1 \). Otherwise, if \( ||v||_M > 1 \) we define \( \tilde{v} := \frac{v}{||v||_M} \), with the properties

- \( \tilde{v} := \frac{v}{||v||_M} \in (HT)_x \)
- \( ||\tilde{v}||_M = 1 \)
- \( (\pi_x)_* (\tilde{v}) = \frac{w}{||v||_M} \)

and if we use \( \frac{w}{||v||_M} \) and \( \tilde{v} \) in (HLC), we thus obtain the equivalent inequality,

\[
\frac{1}{\alpha} \left| \frac{w}{||v||_M} \right|_B - \beta \leq ||\tilde{v}||_M \leq \alpha \left| \frac{w}{||v||_M} \right|_B + \beta \Rightarrow
\]

\[
\Rightarrow \frac{1}{\alpha} \left| \frac{w}{||v||_M} \right|_B - \beta \leq \frac{||v||_M}{||v||_M} \leq \alpha \left| \frac{w}{||v||_M} \right|_B + \beta \Rightarrow
\]

\[
\Rightarrow \frac{1}{\alpha} ||w||_B - \beta ||v||_M \leq ||v||_M \leq \alpha ||w||_B + \beta ||v||_M \Rightarrow
\]

\[
\Rightarrow \frac{1}{\alpha} ||w||_B \leq (\beta + 1) ||v||_M \land (1 - \beta) ||v||_M \leq \alpha ||w||_B \Rightarrow
\]

\[
\Rightarrow \begin{cases} 
\frac{1}{\alpha (\beta + 1)} ||w||_B \leq ||v||_M \leq \frac{\alpha}{(1 - \beta)} ||w||_B, & \text{if } \beta \neq 1 \\
\frac{1}{\alpha (\beta + 1)} ||w||_B \leq ||v||_M, & \text{if } \beta = 1
\end{cases}
\]

for \( w \in T_b B \), where \( v \) is the unique horizontal lift of \( w \) through \( x \) with \( ||v||_M > 1 \).

We must verify the validity of (RI.1) and (RI.2).

Clearly, axiom (RI.2) holds since \( \pi \) is onto.

To verify (RI.1), let \( x, y \in M \).

We may assume that \( B \) is connected. Otherwise, we repeat the argument which will be utilized in this proof, on each connected component and the result will follow.
Because $B$ is complete, there exists a minimal geodesic $\gamma$ joining $\pi(x)$ to $\pi(y)$, with $\ell(\gamma) = d_B(\pi(x), \pi(y))$, which we parametrize by $\gamma : [0, 1] \to B$, where, $\gamma(0) := \pi(x), \gamma(1) := \pi(y)$.

Recall that $\gamma$ has a unique horizontal lift $\Gamma_x : [0, 1] \to M$, through $x$, so $\Gamma_x$ intersects the fiber $F_{\pi(y)}$ containing $y$.

We may assume, without loss of generality, that $\Gamma_x$ is parametrized proportionally to arc length and $||\Gamma'_x(t)||_M \leq 1$ for all $t \in [0, 1]$.

Thus we can write,

$$\ell(\Gamma_x) = \int_0^1 ||\Gamma'_x||_M dt \overset{(HLC)}{\leq} \alpha \int_0^1 ||(\pi_\ast \Gamma_x(t))\Gamma'_x(t)||_B dt + \beta \overset{\text{UDF}}{=} \ell(\gamma)$$

By the triangle inequality, by hypothesis and the above, we have,

$$d_M(x, y) \leq d_M(x, \Gamma_x(1)) + d_M(\Gamma_x(1), y) \overset{\text{dist.}}{\leq} \ell(\Gamma_x) + \beta \overset{\text{UDF}}{\leq} \ell(\Gamma_x) + m \leq \alpha \cdot d_B(\pi(x), \pi(y)) + \beta + m$$

which can be rewritten as,

$$d_B(\pi(x), \pi(y)) \geq \frac{1}{\alpha} d_M(x, y) - \frac{(\beta + m)}{\alpha}$$

Now, we claim that for $\gamma$, the minimal geodesic joining $\pi(x)$ to $\pi(y)$, its length $\ell(\gamma)$ satisfies,

$$d_B(\pi(x), \pi(y)) = \ell(\gamma) \leq \alpha \cdot \ell(\varsigma) + \alpha \cdot \beta$$

for any smooth curve $\varsigma : [0, 1] \to M$, joining $x$ to $y$.

First, observe that for any orthogonal vectors $U$ and $W$,

$$||U \oplus W||^2 = ||U||^2 + ||W||^2 \geq \max\{||U||^2, ||W||^2\}$$

Now, since each tangent vector is the direct sum of a horizontal and a vertical vector, we can write,

$$\ell(\varsigma) = \int_0^1 ||\varsigma'(t)||_M dt = \int_0^1 ||\varsigma'_H(t) \oplus \varsigma'_V(t)||_M dt \geq \int_0^1 ||\varsigma'_H(t)||_M dt$$
where we are assuming here that $\varsigma_H$ is parametrized proportional to arclength, and $||\varsigma_H'(t)||_M \leq 1, \forall t \in [0,1]$.

Since, $(VT)_x = \ker(\pi_*), \forall x \in M$, we have,

$$\ell(\pi \circ \varsigma) = \int_0^1 ||(\pi_*)_{\varsigma(t)}\varsigma'(t)||_B dt = \int_0^1 ||(\pi_*)_{\varsigma(t)}\varsigma'(t)||_B dt$$  \hspace{1cm} (21)

From the left-hand side of (HLC),

$$\frac{1}{\alpha} \cdot ||(\pi_*)_{\varsigma(t)}\varsigma'_H(t)||_B - \beta \leq ||\varsigma'_H(t)||_M \Rightarrow ||(\pi_*)_{\varsigma(t)}\varsigma'_H(t)||_B \leq \alpha \cdot ||\varsigma'_H(t)||_M + \alpha \cdot \beta$$ \hspace{1cm} (22)

for all $t \in [0,1]$.

If we combine (20), (21) and (22), we get,

$$\ell(\pi \circ \varsigma) \leq \int_0^1 ||(\pi_*)_{\varsigma(t)}\varsigma'_H(t)||_B dt \leq \alpha \int_0^1 ||\varsigma'_H(t)||_M + \alpha \beta \leq \alpha \ell(\varsigma) + \alpha \beta$$ \hspace{1cm} (23)

Inequality (23) and the fact that $\gamma$ is a minimal geodesic joining $\pi(x)$ to $\pi(y)$ imply that we can finally write,

$$d_B(\pi(x),\pi(y)) = \ell(\gamma) \leq \ell(\pi \circ \varsigma) \leq \alpha \ell(\varsigma) + \alpha \beta$$

for any smooth curve $\varsigma : [0,1] \rightarrow M$, joining $x$ to $y$, which is claim (19).

We recall that by definition of infimum, $d_M(x,y)$ is the greatest lower bound for $\{\ell(\varsigma), \text{ where } \varsigma : [0,1] \rightarrow M \text{ is any smooth curve joining } x \text{ to } y\}$, and since $\varsigma$ is arbitrary in (19), we obtain,

$$d_B(\pi(x),\pi(y)) \leq \alpha \cdot d_M(x,y) + \alpha \cdot \beta$$ \hspace{1cm} (24)

Let $A := \alpha \geq 1$ and $C := \max \left\{ \frac{\beta + m}{\alpha}, \alpha \cdot \beta \right\} > 0$.

If we now, rewrite (18) and (24) in terms of $A$ and $C$, as follows,

$$\frac{1}{A} d_M(x,y) - C \leq \frac{1}{\alpha} d_M(x,y) - \frac{\beta + m}{\alpha} \leq$$

$$\leq \frac{1}{A} d_M(x,y) - C \leq \frac{1}{\alpha} d_M(x,y) - \frac{\beta + m}{\alpha} \leq$$

$$\leq d_B(\pi(x),\pi(y)) \leq \alpha d_M(x,y) + \alpha \beta \leq Ad_M(x,y) + C$$

we obtain (RI.1) for $\pi$.  

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In the following two Counterexamples, we show that the universal di-
ameter property (UDF) of the fibers, and the control over the length of horizontal lifts (HLC) of tangent vectors are both necessary conditions in Theorem 4.2.3.

Counterexample 4.2.4 We will exhibit $M, B, \pi$, where $B$ is connected and each fiber $F_b$ is compact for all $b \in B$, satisfying all but condition (HLC) in Theorem 4.2.3 i.e.,

For any given constants $\alpha \geq 1$ and $\beta > 0$, there exist $\bar{b} \in B$, $\bar{x} \in F_{\bar{b}}$, $\bar{w} \in T_{\bar{b}}B$ such that either one of the following holds:

$$||\bar{v}||_M > \alpha ||\bar{w}||_B + \beta \quad \text{or} \quad ||\bar{v}||_M < \frac{1}{\alpha} ||\bar{w}||_B - \beta$$

(25)

where $\bar{v}$ is the unique horizontal lift of $\bar{w}$ through $\bar{x}$.

In this case, the map $\pi : M \to B$ is not a rough isometry.

Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 1, y \in \mathbb{R}\}$ and $B = \mathbb{R}$, a complete and connected Riemannian manifold.

We first define an auxiliary $C^1$-diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ by,

$$f(y) = \begin{cases} 
e^y - 1 & \in [0, \infty) \quad \text{if } y \geq 0 \\ 1 - e^{-y} & \in (-\infty, 0] \quad \text{if } y \leq 0 \end{cases}$$

(26)
Let $\pi : M \to B$ be given by $\pi(x, y, z) := f(y)$. The map $\pi$ is onto and $C^\infty$, since both the projection $(x, y, z) \mapsto y$ and $f$ have those properties. The rank of $\pi$ is maximal and no (HLC) is easily verified (see [1]).

Notice that the fibers have either form,

\[ F_b = \pi^{-1}(b) = \begin{cases} \{(x, \ln(b + 1), z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}, & \text{if } b \geq 0 \\ \{(x, -\ln(1 - b), z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}, & \text{if } b < 0 \end{cases} \]

Therefore, each fiber $F_b$ is compact and $\text{diam } F_b \leq m$, for all $b \in B$, where $m = 3 > 0$ is the universal upper bound for the fibers’ diameters.

Finally, we claim that $\pi$ does not satisfy (RI.1). It suffices to verify that (RI.1) fails for $\pi$, for particular pairs of elements in $M$. We will show that $\forall A \geq 1, \forall C > 0, \exists y_{AC} \in \mathbb{R}$, a positive number such that,

\[ d_B (\pi(x, 0, z), \pi(x, y, z)) > A \cdot d_M ((x, 0, z), (x, y, z)) + C \Leftrightarrow |f(0) - f(y)| > A \cdot y + C \Leftrightarrow e^y - 1 > A \cdot y + C \]  

(27)

for all $y > y_{AC}$, where $x, z \in \mathbb{R} : x^2 + z^2 = 1$ are arbitrary.

Fixing constants $A \geq 1$ and $C > 0$, introduce $g \in C^\infty(\mathbb{R})$, by

\[ g : y \mapsto g(y) := e^y - 1 - Ay - C \]

One can show that,

\[ \exists y_{AC} > 0 : \forall y > y_{AC} \Rightarrow g(y) > 0 \]  

(28)

using the functional behavior of $g$ (see [1]).

Therefore, (RI.1) holds and the claim follows, and consequently $\pi$ is not a rough isometry.

This describes the Counterexample.
Counterexample 4.2.5 We will exhibit $M, B, \pi$, where $B$ is connected, satisfying all the conditions in Theorem 4.2.3 with the exception of (UDF), i.e.,

The fibers’ diameters are not uniformly bounded, in other words:

$$\forall m > 0, \exists b_m \in B : \text{diam } F_{b_m} > m$$

In this case, the map $\pi : M \to B$ is not a rough isometry.

Proof. Let $M = \{(0, y, z) \in \mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^2$ and $B = \mathbb{R}$, a complete and connected Riemannian manifold.

Let $\pi : M \to B$ be the projection $\pi(x, y, z) := y$.

The map $\pi$ is onto, $\mathcal{C}^\infty$, and $\pi$ has maximal rank=1, and (HLC) is easily verified (see [1]).

Each fiber is given by,

$$F_b = \pi^{-1}(b) = \{(0, b, z) : z \in \mathbb{R}\}, \quad b \in B = \mathbb{R}$$

which is a line passing through $(0, b, 0)$, determined by the intersection of $M$ with the plane $y = b$. Hence each fiber is not compact as a subset of $\mathbb{R}^3$, they all have infinite diameter, and therefore the fibers’ diameters are not uniformly bounded.

Our goal next is to show that $\pi$ is not a rough isometry.
It suffices to verify that $\pi$ does not satisfy (RI.1) for particular pairs of elements in $M$, i.e.,
\[
\forall A \geq 1, \forall C > 0, \exists \eta_{AC} \in \mathbb{R} \setminus \{0\},
\]
\[
d_B (\pi \circ X(\mu, \eta), \pi \circ X(\mu, 0)) < \frac{1}{A} \cdot d_M (X(\mu, \eta), X(\mu, 0)) - C \iff
\]
\[
|\mu - \mu| < \frac{1}{A} \cdot d_M ((0, \mu, \eta), (0, \mu, 0)) - C \iff
\]
\[
0 < \frac{1}{A} \cdot \eta - C, \ \forall \eta \geq \eta_{AC}
\]
(29)

Let $A \geq 1, C > 0$ be arbitrary, and define the real positive number $\eta_{AC} := AC + 1 > 0$.

We see that,
\[
\eta_{AC} = AC + 1 > AC \iff \frac{1}{A} \eta_{AC} > C \iff \frac{1}{A} \eta_{AC} - C > 0
\]
(30)

and since, for all $\eta \geq \eta_{AC}$,
\[
\frac{1}{A} \eta - C \geq \frac{1}{A} \eta_{AC} - C > 0
\]

inequality (29) is verified.

Therefore, holds and (RI.1) fails for $\pi$, which shows that $\pi$ is not a rough isometry.

This describes the Counterexample.

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