GLOBAL SOLUTIONS TO THE
ELECTRODYNAMIC TWO-BODY PROBLEM
ON A STRAIGHT LINE

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ABSTRACT. The classical electrodynamic two-body problem has been a long standing open problem in mathematics. For motion constrained to the straight line, the interaction is similar to that of the two-body problem of classical gravitation. The additional complication is the presence of unbounded state-dependent delays in the Coulomb forces due to the finiteness of the speed of light. This circumstance renders the notion of local solutions meaningless, and therefore, straightforward ODE techniques cannot be applied. Here, we study the time-symmetric case, i.e., the Fokker-Schwarzschild-Tetrode (FST) equations, comprising both advanced and retarded delays. We extend the technique developed in [3], where existence of FST solutions was proven on the half-line, to ensure global existence – a result that had been obtained by Bauer [2] in 1997. Due to the novel technique, the presented proof is shorter and more transparent but also relies on the idea to employ asymptotic data to characterize solutions.

Keywords: Fokker-Schwarzschild-Tetrode electrodynamics; Wheeler-Feynman electrodynamics; delay differential equations

1. INTRODUCTION

The time-symmetric electrodynamic interaction of point-charges is described by the so-called Fokker-Schwarzschild-Tetrode (FST) equations. Historically, these equations were first discussed in the works [10, 11, 9]. Later Wheeler and Feynman took up these equations in their seminal works [12, 13] to show that the electrodynamic arrow of time is derived from the thermodynamic one; see also [4]. In our case, in which we restrict ourselves to two point-charges moving along a straight line having positions $a(t), b(t) \in \mathbb{R}$ at time $t \in \mathbb{R}$, the FST equations take the form

$$\frac{d}{dt} \left( \frac{\dot{a}(t)}{\sqrt{1 - \dot{a}(t)^2}} \right) = \frac{\kappa_a}{2} \left[ \frac{1 + \dot{b}(t_2^\pm)}{1 - \dot{b}(t_2^\pm)} \frac{1}{(a(t) - b(t_2^\pm))^2} + \frac{1 - \dot{b}(t_2^\pm)}{1 + \dot{b}(t_2^\pm)} \frac{1}{(a(t) - b(t_2^\pm))^2} \right],$$

$$\frac{d}{dt} \left( \frac{\dot{b}(t)}{\sqrt{1 - \dot{b}(t)^2}} \right) = -\frac{\kappa_b}{2} \left[ \frac{1 - \dot{a}(t_1^\pm)}{1 + \dot{a}(t_1^\pm)} \frac{1}{(b(t) - a(t_1^\pm))^2} + \frac{1 + \dot{a}(t_1^\pm)}{1 - \dot{a}(t_1^\pm)} \frac{1}{(b(t) - a(t_1^\pm))^2} \right].$$

(1)

Here, we use the dot notation, i.e., any derivative w.r.t. time parameter $t$ is denoted by an overset dot such as $\dot{a}(t) = \frac{d}{dt} a(t)$ and $\ddot{a}(t) = \frac{d^2}{dt^2} a(t)$, and furthermore, units such that speed of light equals one. Furthermore, $\kappa_a, \kappa_b > 0$ denote coupling constants and the so-called advanced and retarded times $t_1^+$ and $t_1^-$ for $i = 1, 2$ are given implicitly as solutions to the following equations

$$t_1^+ = t_1^+(a, b(t), t) = t \pm |a(t_1^+(a, b(t), t)) - b(t)|,$$

$$t_2^+ = t_2^+(a(t), b, t) = t \pm |a(t) - b(t_2^+(a(t), b, t))|.$$  

(2)

Thus, $t_1^+$ and $t_2^+$ are functionals of trajectory $a$ and $b$, respectively. To keep the notation slim we will often omit their arguments. Geometrically, equations (2) can be understood as the intersection times of the forward and backward light-cones of the respectively other trajectory. These intersection points exist as long as the trajectories $a$ and $b$ have velocities that are bounded away from one, i.e., the speed of light.

We shall establish in this paper the existence of solutions to the FST equations (1) satisfying a prescribed asymptotic behavior; see Theorem 2 below.
The presence of advanced and delayed terms is the main source of difficulty when aiming at a global existence result. Namely, it implies that the right-hand sides of the equations of motion in (1) involve not only terms evaluated at the same time instant \( t \) but also at the respectively future or past times \( t^\pm \). Since these times are potentially unbounded functionals of the entire trajectories, the notion of local solutions is meaningless as even a very small interval of the trajectory \( a \) may depend on a very large interval of the trajectory \( b \) and vice versa. Therefore, straight-forward ODE techniques based on finding local solutions first and extending them to global solutions with an additional a priori estimate can not be applied. A particularly interesting question is therefore in which sense one may hope for a well-posed initial value problem. Due to the delayed terms it is not clear if in general Cauchy data, i.e., position and velocities of both charges at one time instant, suffices to characterize solutions uniquely or if even whole strips of the trajectories have to be prescribed as initial data. In case the advanced terms are omitted, the respective equations are called Synge equations. For this case, solutions on the half-line can be found by integration, but obtaining global ones is still highly non-trivial; see \([7, 1, 5]\). The goal of this work, i.e., Theorem 2 below, is to ensure global existence of solutions to equations (1) including both the advanced and retarded terms. Before Theorem 2 can be spelled out precisely and its special formulation can be understood we need to recall some previous results and discuss the asymptotic behavior of potential solutions.

To date the only result about uniqueness of solutions to the FST equations is given in \([8]\). There, uniqueness of solutions for two charges on the straight line was obtained in the special situation of initially prescribed zero velocities and sufficiently large separation of the two charges; for a general discussion of valid initial data and uniqueness in the case of a FST toy model see \([6]\). General global existence of solutions on the straight line was later shown in \([2]\). As yet the only result towards a solution theory of the FST equations in three space-dimensions is given in \([3]\) where, for \( N \) rigid charges and a prescribed asymptotic behavior of the trajectories for times \(|t| > \tau\), existence of solutions on \([-\tau, \tau]\) for arbitrary large \( 0 < \tau < \infty \) was shown. Apart from the different setting, the crucial difference in the latter two works lies in the prescription of the initial data. In \([2]\) it was given as asymptotic data in the remote past and in \([3]\) as Cauchy data. Both choices seem to have advantages and disadvantages: In order to use asymptotic data, a priori, one must determine the asymptotic behavior of potential scattering solutions; until now this was only done successfully on the straight line; see \([2]\). This knowledge then provides sufficient global control on the charge trajectories to employ topological fixed-point methods. However, in general, such methods do not provide information about uniqueness and about how the asymptotic data relates to potential data at finite times. On the contrary, possible notions of initial data at finite times are suggested readily when recasting the FST equations into integral form to yield potential candidates for self-maps. The trouble with this approach is that one usually lacks sufficient control on the global behavior of the solutions in order to apply fixed-point methods globally. Because of this one is usually only able to prove existence of solutions on finite time intervals as in \([3]\).

In a recent work \([5]\), for two charges on the straight line, it was possible to infer from prescribed initial data existence of solutions not only in a finite time interval but on the half-line. More precisely, it was shown in \([5, \text{Theorem 1.1}]\):

**Theorem 1** (FST solutions on the half-line). Given an initial position \( a_0 \in \mathbb{R} \) and velocity \( \dot{a}_0 \in ]-1, 1[ \) of charge \( a \) at a time \( T \in \mathbb{R} \), and in addition, an initial trajectory strip \( b_0 \in C^\infty([-T, T^+], ]-\infty, a_0[) \) of charge \( b \) such that \( T^\pm = T \pm (a_0 - b_0(T^\pm)), \) there is a trajectory pair \((a, b)\) fulfilling:

(i) Trajectory \( a \) fulfills the equation of motion (1) for all \( t \geq T \) and \( b \) for all \( t \geq T^+ \);
(ii) The pair \((a, b)\) satisfies the initial conditions \( a(T) = a_0, \dot{a}(T) = \dot{a}_0, b(T) = b_0 \),
(iii) The pair \((a, b)\) fulfills \( a \in C^\infty([T, \infty[) \) and \( b \in C^\infty([T^+, \infty[) \).

Henceforth, we will refer to such solutions as “conditional solutions” corresponding to the prescribed initial data. This result is basic to this work: In order to prove the global existence result for the FST equations, Theorem 2 below, we combine Theorem 1 with the a priori results on the asymptotic behavior of solutions that were also exploited in \([2]\). The strategy of proof
consists of two steps: First, we identify the asymptotic trajectories \((x, y)\) of potential global solutions. As it turns out, the velocities \((\dot{a}, \dot{b})\) of any global solution \((a, b)\) are bounded away from one \([5, (24a)\) in Proposition 2.1], and since \(\dot{a} > 0\) and \(\dot{b} < 0\) due to (1), they converge to limiting values

\[-1 < u_{-\infty} = \lim_{t \to -\infty} \dot{a}(t) < \lim_{t \to -\infty} \dot{b}(t) = v_{-\infty} < 1.\]  

Moreover, as we discuss in Section 2, it is even possible to show that these two asymptotic velocities together with two other reals \(x_{-\infty}, y_{-\infty}\), referred to as asymptotic positions, parametrize all possible asymptotes \((x, y)\) of potential global solutions. In a second step, relying on this asymptotic information, we will then extract initial data to infer conditional solutions conditional solutions \(a_T, b_T\) satisfying

\[a_T(T) = x(T), b_T[|T^-|, T^+] = y[|T^-|, T^+],\]  

for sufficiently large negative number \(T\) (meaning large magnitude \(|T|\) but \(T < 0\)), where

\[T^\pm = t_i^\pm(x(T), y, T).\]  

For convenience, \(a_T\) and \(b_T\) will be extended to all times \(t \in \mathbb{R}\) by means of the asymptotes \((x, y)\)

\[a_T(t) = x(t) \text{ for } t \in ]-\infty, T[ \quad b_T(t) = y(t) \text{ for } t \in ]-\infty, T[-.\]  

Since their velocities are bounded away from the speed of light uniformly in \(T\) we are able to establish sufficiently strong uniform estimates that allow to prove that a certain subsequence of the family of trajectory pairs \((a_T, b_T)\) converges in a suitable sense to a global solution to (1) as \(T \to \infty\).

The paper is structured as follows: In Section 2 we discuss the asymptotic behavior of potential global solutions. In Section 3 we use the introduced notions to characterize the asymptotes in order to formulate our global existence result in Theorem 2 and provide the proof.

### 2. Asymptotic Behavior of Potential Global Solutions

As the highest derivative in (1) is on the right-hand side, one immediately observes that any global solution \((a, b)\) to (1) must be smooth. As discussed earlier (3), the velocities \(\dot{a}\) and \(\dot{b}\) of any potential global solution have a modulus which is bounded away from one, which implies that the implicitly defined functions \(t_i^\pm\) in (2) are well-defined and smooth. Moreover, because of \(\dot{a} > 0\) and \(\dot{b} < 0\), the velocities converge monotonically to asymptotic velocities (3). Without restricting generality, we will assume

\[a(t) > b(t) \quad \text{for } t \in \mathbb{R}\]  

throughout the paper. In other words, asymptotically, the velocities are constant, and to a first approximation, the difference \(a(t) - b(t^*_i(t))\) can be expected to be of order \(t\) for large \(|t|\). The other terms on the right-hand side in (1) involve only velocities, so that the acceleration

\[\ddot{a}(t) = (1 - \dot{a}(t)^2)^{\frac{1}{2}} \frac{d}{dt} \left( \frac{\dot{a}(t)}{\sqrt{1 - \dot{a}(t)^2}} \right)\]  

should be of order \(\frac{1}{t^2}\) for large \(|t|\), which leads to the guess

\[a(t) = x_{-\infty} + u_{-\infty}t - C \ln(-t) + o(\ln(-t))\]  

for \(t \to -\infty\), where we use the notation \(f(s) = o(g(s)) \iff \lim_{s \to 0} \frac{f(s)}{g(s)} = 0\) and \(f(s) \sim g(s) \iff \lim_{s \to 0} \frac{f(s)}{g(s)} = 1\). The logarithmic corrections in (8) are characteristic to Coulomb interaction in 3+1 dimensions and well-known in the case without delays (note that although we restrict the dynamics to a straight line, we use the electrodynamic interaction, which is determined by the Green’s function of the d’Alembert operator, in 3+1 dimensions).
The result (8) was obtained in [2] for the first time, but since that publication is only available in German and it is crucial for understanding the formulation of our main result, Theorem 2, we shall briefly discuss the intuition behind its proof here. This discussion will consist of a rigorous part until equation (16) which will provide formulas that will be used in the proof of our main result and, in particular, make the reader familiar with the terms (34)-(38) below. The shorter second part will then explain in a nonrigorous way how the estimate (16) leads to (8) including an identification of the constant \( C \). However, a proof of the latter will not be given in this section. Rather, it will proven as part of the main result Theorem 2 that the established solutions comply with the asymptotic behavior given here.

In order to infer more details about the asymptotic behavior, it is convenient to bring the integrated version of the equation of motion (1), i.e.,

\[
\dot{a}(t) = \dot{a}(T) + \frac{k_a}{2} \int_T^t (1 - \dot{a}(s))^2 \left( 1 + \frac{1}{1 - b(t_2^*(s))} \frac{1}{(a(s) - b(t_2^*(s)))^2} + \frac{1 - \dot{b}(t_2^*(s))}{1 + \dot{b}(t_2^*(s))} \frac{1}{(a(s) - b(t_2^*(s)))^2} \right) \, ds,
\]

for \( t, T \in \mathbb{R} \), into a form similar to (8) by partial integration such that the factors \((a(s) - b(t_2^*(s)))^{-2}\) turn into \( \ln (a(s) - b(t_2^*(s))) \). By definition of \( t_2^* \) in (2), we have

\[
t_2^*(s) = 1 \pm a(s) \mp b(t_2^*(s)) t_2^*(s) = \frac{1 \pm a(s)}{1 \pm b(t_2^*(s))},
\]

and therefore,

\[
\frac{d}{ds} (a(s) - b(t_2^*(s))) = \pm \frac{d}{ds} (t_2^*(s) - s) = \pm (t_2^*(s) - 1) = \frac{\dot{a}(s) - b(t_2^*(s))}{1 \pm b(t_2^*(s))}.
\]

We recall from [5, Lemma 2.1] that the modulus in definition (2) can be omitted since, as the speed of light equals one, one has

\[
\frac{a(t) - b(t)}{2} \leq \left\{ \frac{a(t) - b(t_2^*)}{a(t_2^*) - b(t)} \right\} \leq \frac{a(t) - b(t)}{1 - \max \{ \| \dot{a} \|_\infty, \| b \|_\infty \}},
\]

with \( \| \cdot \|_\infty \) denoting the \( L^\infty \) norm. Consequently, in the first integration by parts, we take the antiderivative of

\[
\frac{d}{ds} \frac{1}{a(s) - b(t_2^*(s))} = -\frac{\dot{a}(s) - b(t_2^*(s))}{1 \pm b(t_2^*(s))} \frac{1}{(a(s) - b(t_2^*(s)))^2}
\]

and the derivative of the rest. This yields

\[
\dot{a}(t) = \dot{a}(T) + \frac{k_a}{2} (1 - \dot{a}(T)^2)^{\frac{3}{2}} \left[ \frac{1 + \dot{b}(t_2^*(T))}{\dot{a}(T) - b(t_2^*(T))} \frac{1}{a(T) - b(t_2^*(T))} \right. \]
\[
+ \frac{1 - \dot{b}(t_2^*(T))}{\dot{a}(T) - b(t_2^*(T))} \frac{1}{a(T) - b(t_2^*(T))} \left. \right] \]
\[
- \frac{k_a}{2} (1 - \dot{a}(t)^2)^{\frac{3}{2}} \left[ \frac{1 + \dot{b}(t_2^*(t))}{\dot{a}(t) - b(t_2^*(t))} \frac{1}{a(t) - b(t_2^*(t))} \right. \]
\[
+ \frac{1 - \dot{b}(t_2^*(t))}{\dot{a}(t) - b(t_2^*(t))} \frac{1}{a(t) - b(t_2^*(t))} \left. \right] \]
\[
+ \frac{k_a}{2} \int_T^t \frac{d}{ds} \left( (1 - \dot{a}(s)^2)^{\frac{3}{2}} \frac{1 + \dot{b}(t_2^*(s))}{\dot{a}(s) - b(t_2^*(s))} \right) \frac{1}{a(s) - b(t_2^*(s))} \, ds
\]
In the second step, we integrate the latter equation for $\dot{a}(t)$ again from $T$ to $t$ in order to obtain an equation for $a(t)$. The corresponding integral over (13d) is again done by parts using $\frac{d}{ds} \ln(a(s) - b(t^+_2(s))) = \frac{\dot{a}(s) - b(t^+_2(s))}{a(s) - b(t^+_2(s))}$ as antiderivate. We find

$$a(t) = a(T) + \dot{a}(T)(t - T) + \frac{\kappa_a}{2} (1 - \dot{a}(T)^2)^{\frac{3}{2}} \left[ \frac{1 + \dot{b}(t^-_2(T))}{\dot{a}(T) - b(t^-_2(T))} \ln \left( a(T) - b(t^-_2(T)) \right) + \frac{1 - \dot{b}(t^-_2(T))}{\dot{a}(T) - b(t^-_2(T))} \right] (t - T)$$

$$- \frac{\kappa_a}{2} (1 - \dot{a}(T)^2)^{\frac{3}{2}} \left[ \frac{1 - \dot{b}(t^-_2(T))^2}{\dot{a}(T) - b(t^-_2(T))} \ln \left( a(T) - b(t^-_2(T)) \right) + \frac{1 - \dot{b}(t^-_2(T))^2}{\dot{a}(T) - b(t^-_2(T))} \right]$$

$$+ \frac{\kappa_a}{2} (1 - \dot{a}(T)^2)^{\frac{3}{2}} \left[ \frac{1 - \dot{b}(t^-_2(T))^2}{\dot{a}(T) - b(t^-_2(T))} \ln \left( a(T) - b(t^-_2(T)) \right) + \frac{1 - \dot{b}(t^-_2(T))^2}{\dot{a}(T) - b(t^-_2(T))} \right]$$

Let us turn to the asymptotic behavior where we are interested in negative $t$ and have the arbitrary parameter $T$ at our expense. Thanks to the convergence of the velocities in (3), identity (11), and the de l’Hospital rule, we observe $a(t) - b(t^+_2(t)) \sim \frac{u_{-\infty} - v_{-\infty}}{1 - v_{-\infty}} t$ for $t \to -\infty$ so that

$$\eta_1 := \lim_{t \to -\infty} \kappa_a (1 - \dot{a}(t)^2)^{\frac{3}{2}} \left[ \frac{1 - \dot{b}(t^-_2(T))^2}{\dot{a}(t) - b(t^-_2(T))} \right] \ln \left( a(t) - b(t^-_2(T)) \right) = \kappa_a (1 - u_{-\infty}^2)^{\frac{3}{2}} \left[ \frac{1 - v_{-\infty}^2}{(u_{-\infty} - v_{-\infty})^2} \right],$$

and, (14b) $\sim \eta_1 \frac{t - T}{T}$ for $T \to -\infty$. Furthermore, we have

$$\ln \left( a(t) - b(t^+_2(t)) \right) \sim \ln \left( \frac{u_{-\infty} - v_{-\infty}}{1 - v_{-\infty}} \right) + \ln(-t),$$
for $t \to -\infty$, so that

$$(14c) \sim -\frac{\kappa_a}{2} (1 - u_{-\infty}^2) \frac{1 - v_{-\infty}^2}{(u_{-\infty} - v_{-\infty})^2} \times \left[ 2 \ln(-t) + \ln \left( \frac{u_{-\infty} - v_{-\infty}}{1 - v_{-\infty}} \right) + \ln \left( \frac{u_{-\infty} - v_{-\infty}}{1 + v_{-\infty}} \right) \right]$$

for $t \to -\infty$ and, correspondingly,

$$(14d) \sim \eta_1 \ln(-T) + \frac{\eta_1}{2} \ln \left( -\frac{(u_{-\infty} - v_{-\infty})^2}{1 - v_{-\infty}^2} \right)$$

for $T \to -\infty$. When computing the derivatives in the remaining integral and double integral terms by chain rule, each summand is proportional to $\ddot{a}(t)$ or $\ddot{b}(t)$. Since $T$ is at our expense, we choose a scaling such that $T \ll t \ll 0$. In this case, thanks to the equation of motion (1), for each acceleration one gains again a $(a(t) - b(t)^2) o - (a(t^2) - b(t^2)) o^2$ term, and we find that (14e) is of order $\frac{\ln(-t)}{t}$ and (14f) of order $\frac{1}{t}$ and, as such, subleading w.r.t. the logarithmic correction. This implies

$$a(t) = a(T) + \dot{a}(T)(t - T) + \eta_1 \frac{t - T}{T} - \eta_1 \ln(-t) + \eta_1 \ln(-T) + o(\ln(-t)) \quad (16)$$

for $t, T \to -\infty$.

Next, in the second part, we discuss nonrigorously how (16) leads to (8) and how to identify the constant $C$. The proof of our main result, Theorem 2, will provide a proof of what comes next for the case of global solutions. Using ansatz (8) for $a$ and omitting the remainder terms gives

$$a(T) + \dot{a}(T)(t - T) = x_{-\infty} + u_{-\infty} T - C \ln|T| + \left( u_{-\infty} - \frac{C}{T} \right) (t - T).$$

Substituting this expression together with ansatz (8) into eq. (16), again neglecting the remainders, provide the formula

$$C \left( \ln|T| - \ln|t| + \frac{t - T}{T} \right) = \eta_1 \left( \ln|T| - \ln|t| + \frac{t - T}{T} \right)$$

which indicates $C = \eta_1$. Hence, we are led to the guess

$$a(t) = x_{-\infty} + u_{-\infty} t - \eta_1 \ln|t| + o(\ln(|t|))$$

for $t \to -\infty$. Likewise, one obtains

$$b(t) = y_{-\infty} + v_{-\infty} t + \eta_2 \ln|t| + o(\ln(|t|))$$

with

$$\eta_2 = \kappa_b \left( 1 - v_{-\infty}^2 \right) \frac{1 - u_{-\infty}^2}{(u_{-\infty} - v_{-\infty})^2}.$$  \quad (17)

3. Main result and its proof

Based on the information about the asymptotic behavior of solutions that was provided in Section 2 we can now make our main result precise. As Banach space for the potential solutions to the FST equations (1) we employ the space $B$ of pairs of trajectories $(a, b) \in C^1(\mathbb{R}, \mathbb{R}^2)$ equipped with the norm

$$\|(a, b)\| := \max \left( |a(0)|, |b(0)|, \|\dot{a}\|_{\infty}, \|\dot{b}\|_{\infty} \right). \quad (18)$$

In this notation our main result reads:
Theorem 2 (Global Existence). Let

\[ x_{-\infty}, y_{-\infty} \in \mathbb{R}, \quad -1 < u_{-\infty} < v_{-\infty} < 1 \]

and

\[
\begin{align*}
x(t) &= x_{-\infty} + u_{-\infty} t - \eta_1 \ln |t| \\
y(t) &= y_{-\infty} + v_{-\infty} t + \eta_2 \ln |t|
\end{align*}
\]

for \( t < -1 \) \hspace{1cm} (19)

with \( \eta_1 \) and \( \eta_2 \) defined in eq. (15), (17). Then, the following statements hold true:

(i) There exists a global solution \((a, b)\in C^\infty(\mathbb{R}, \mathbb{R}^2)\) to the FST equations (1) with

\[
a(t) = x(t) + O\left(\frac{\ln |t|}{t}\right), \quad b(t) = y(t) + O\left(\frac{\ln |t|}{t}\right) \quad \text{for} \ t \to -\infty.
\]

(ii) Let furthermore \( T_0 < -1 \) be a sufficiently large negative number such that \( x(t) > y(t) \) and \(|\dot{x}(t)|, |\dot{y}(t')| < 1 \) for all \( t \leq T_0 \) and \( t' \leq T_0^+ \) is fulfilled, and let \((a_T, b_T)_{t \leq T \leq T_0}\) denote the family of conditional solutions to the FST equations (1) inferred by Theorem 1 satisfying the initial conditions (4), (5). Then, there is a sequence \((T_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} T_n = -\infty \) fulfilling

\[
\lim_{n \to \infty} \| (a_{T_n}, b_{T_n}) - (a, b) \| = 0.
\]

The proof will be given at the end of this section. Before we will collect some more technical results. At the core of our proof are estimates that show a uniform closeness of the conditional solutions \((a_T, b_T)\) and their velocities \((\dot{a}_T, \dot{b}_T)\) to the asymptotes \((x, y)\) and the asymptotic velocities \((u_{-\infty}, v_{-\infty})\), respectively.

Lemma 1. Let \( T_0 \) be a sufficiently large negative number as in Theorem 2. There are \( t_0 < T_0 \) and \( C > 0 \) such that, for all \( T \leq t_0 \) and \( t \in [T, t_0] \),

\[
\begin{align*}
u_{-\infty} &< \dot{a}_T(t) \leq u_{-\infty} - \frac{C}{t}, \\
v_{-\infty} &> \dot{b}_T(t) \geq v_{-\infty} + \frac{C}{t}
\end{align*}
\]

and furthermore, for all \( t \leq t_0 \),

\[
|a_T(t) - x(t)|, |b_T(t) - y(t)| < C \frac{\ln |t|}{|t|}
\]

with asymptotes \((x, y)\) defined in eq. (19), (15), (17).

Lemma 1 will be proven with the help of two further lemmata.

Note that the pair of asymptotes \((x, y)\) scatter apart for \( t \to -\infty \). The time instant \( t_0 \), thus, has to be chosen to be a sufficiently large negative number such that the choice (4), (5) of the initial conditions on \( x \) and \( y \) ensures a sufficient decay of the Coulomb terms \( \frac{1}{(a_T(t) - b_T(t)z)} \) in the FST (1) equations for \( t \leq t_0 \). At first the estimates (20) and (21) can be more easily derived from the FST (1) equations under additional assumptions on the distance and relative velocity of \( a \) and \( b \) that ensure such a decay:

Lemma 2. Let \( T_0 \) be a sufficiently large negative number as in Theorem 2. Furthermore, let also \( t_0 < T_0 \) be sufficiently large, \( T \leq t_0 \) and

\[
\mu := \frac{u_{-\infty} - v_{-\infty}}{2} < 0.
\]

Assume that there is a \( t^* \in [T, t_0] \) such that

\[
a_T(t) - b_T(t) \geq \mu t, \quad \dot{a}_T(t) - \dot{b}_T(t) \leq \mu \quad \text{for all} \ t \in [T, t^*].
\]

Then, estimates (20) and (21) hold true for all \( t \in [T, t^*] \) and a suitable \( C > 0 \).
Nevertheless, the computations are tedious and therefore deferred to section 3.1.

At first sight, this might look circular: We can prove (20), (21) under the decay assumption (22), but in order to prove the latter, something like (20), (21) (at least in a slightly weakened form) seems to be necessary. However, by definition of \( x, y, \lim_{t \rightarrow -\infty} (\dot{x}(t) - \dot{y}(t)) = 2\mu \), so that the initial conditions (4) fulfill

\[
 a_T(T) - b_T(T) = x(T) - y(T) > \mu T
\]

and

\[
 \dot{a}_T(T) - \dot{b}_T(T) = \dot{x}(T) - \dot{y}(T) < \mu
\]

sufficiently large negative \( T \). Therefore, if \( t_0 \) is adapted conveniently, continuity of \((a_T, b_T)\) implies that at least a small interval \([T, t^*]\), potentially with \( t^* < t_0 \), on which (22) and thus, by Lemma 2, (20) and (21), hold true, exists. But if (20) and (21) hold on some interval, one can readily show that (22) is satisfied on a larger one:

**Lemma 3.** Let \( t_0 \) be a sufficiently large negative number, \( T \) as in Lemma 2, and \( t^* \in [T, t_0] \) such that estimates (20) and (21) hold true for all \( t \in [T, t^*] \). Then, there exists \( t^{**} \in [t^*, t_0] \) such that

\[
 a_T(t) - b_T(t) \geq \mu t, \quad \dot{a}_T(t) - \dot{b}_T(t) \leq \mu \quad \text{for all } t \in [T, t^{**}].
\]

By iterating Lemma 2 and 3, one then arrives at Lemma 1.

**Proof of Lemma 3.** By assumption (21) and definition (19) of \( x \) and \( y \),

\[
 a_T(t) - b_T(t) \geq \frac{2C}{|t|} = \mu t + \left( x_{-\infty} - y_{-\infty} + |\mu| |t| - (\eta_1 + \eta_2) \ln |t| \right) - \frac{2C}{|t|} \quad \text{for } t \in [T, t^*].
\]

The bracket becomes positive for sufficiently large \( |t_0| \) and \( t \leq t_0 \), so

\[
 a_T(t) - b_T(t) > \mu t
\]

in this case and, by continuity, \( \geq \) is satisfied on on some larger closed interval. Moreover, by assumption (20),

\[
 \dot{a}_T(t) - \dot{b}_T(t) \leq \left( u_{-\infty} - v_{-\infty} + \frac{C}{|t|} \right) \leq u_{-\infty} - v_{-\infty} + \frac{C}{|t_0|} \leq \mu + \left( \frac{C}{|t_0|} \right) \quad \text{for } t \in [T, t^*]
\]

and, again, \( \leq \) holds on some larger interval. This implies the existence of a \( t^{**} \in [t^*, t_0] \) such that (24) holds. \( \square \)

Now we prove Lemma 1:

**Proof of Lemma 1.** Fix a sufficiently large negative \( t_0 \) such that both Lemma 2 and 3 are applicable and inequality (23) holds true for all \( T \leq t_0 \). Iterated application of these two lemmata reveals that inequalities (22) hold true on \([T, t_0] \): Assuming the contrary, fix \( T \) as required and define

\[
 \tilde{t} := \inf \left\{ t \in [T, t_0] \mid a_T(t) - b_T(t) < \mu t \lor \dot{a}_T(t) - \dot{b}_T(t) > \mu \right\}.
\]

By continuity of \( a_T \) and \( b_T \), inequalities (22) are fulfilled on \([T, \tilde{t}] \), so Lemma 2 implies that Lemma 3 is applicable with \( t^* = \tilde{t} \), resulting in a contradiction to the definition of \( \tilde{t} \).

By definition (6), \( a_T(t) - x(t) = 0 = b_T(t) - y(t) \) for \( t \leq T \), so inequalities (21) are also valid for such times \( t \). \( \square \)

Beyond the estimates in Lemma 1 for large negative \( t \), we need a priori estimates for all \( t \in \mathbb{R} \) that prevent the terms in the FST equations (1) from becoming singular:
Lemma 4. Let $t_0$ be a sufficiently large negative number as in Lemma 1. There are $V \in [0,1]$ and $D > 0$ such that all conditional solutions for $(a_T, b_T)$ with $T \leq t_0$ corresponding to initial conditions (4) with asymptotes in (19) satisfy
\[
\|\dot{a}_T\|_{\infty}, \|\dot{b}_T\|_{\infty} \leq V
\]
and
\[
a_T(t) - b_T(t) \geq D(1 + |t|) \text{ for all } t \in \mathbb{R}.
\]

Before we discuss the proof, we remark that (27) is a corollary to Proposition 2.1 from [5], but (28) is deduced from Lemma 1. This poses no problem since (28) will only be exploited in the proof of the main result Theorem 2, whereas the estimate (27) will be used throughout the proof of Lemma 2. In the proof of Lemma 4, and whenever dealing with the advanced and retarded times, the following estimates, already mentioned before in (12) and proven in [5, Lemma 2.1], will be useful:

Lemma 5. For any $C^1$-trajectories $(a, b)$ with $\|\dot{a}\|_{\infty}, \|\dot{b}\|_{\infty} \leq C < 1$ and $a(t) > b(t)$ for all $t \in \mathbb{R}$, the advanced and retarded times $t_i^\pm$ introduced in (2) are globally well-defined and
\[
\frac{a(t) - b(t)}{2} \leq a(t) - b(t^\pm(a(t), b(t))) \leq \frac{a(t) - b(t)}{1 - \|\dot{b}\|_{\infty}},
\]
\[
\frac{a(t) - b(t)}{2} \leq a(t^\pm(a, b(t), t)) - b(t) \leq \frac{a(t) - b(t)}{1 - \|\dot{a}\|_{\infty}}.
\]

Proof of Lemma 4: According to the proof of Proposition 2.1 in [5] (the equation after 29), for any conditional solution with initial data $a_T(T) = a_0$, $a_T(T) = \dot{a}_0$, $b_T|_{T^-} = b_0$ the estimate
\[
\sup_{t \geq T} |\dot{a}_T(t)| \leq \sqrt{1 - \frac{(1 - \|\dot{b}_0\|_{\infty})^2}{\left(\frac{4}{\left(1 - (\max(a_0, \|x_0\|_\infty))x_0\right) + \frac{3v_0}{\pi_0 - \theta_0(f)}}\right)^2}}
\]
holds true. In [5], it was formulated only for the case $T = 0$, but it holds without change for different $t$. $x_0$ denotes a reference trajectory, which, in the case at hand, can be chosen in such a way that $\|x_0\|_{\infty} = u$ where $u$ is an arbitrary number between $\sup_{t \leq T_0} |x(t)|$ and 1. Substituting as initial data the corresponding segments of the asymptotes $x, y$ and recalling from their explicit form (19) and the choice of $t_0 \leq T_0$ that their velocities are bounded away from 1 and the distances $a_0 - b_0(T)$ from 0, we conclude
\[
\sup_{T \leq t_0} \sup_{t \geq T} |\dot{a}_T(t)| < 1.
\]
Since $a_T(t) = x(t)$ for $t \leq T$, we get
\[
\sup_{T \leq t_0} \|\dot{a}_T\|_{\infty} < 1
\]
and, by the analogous reasoning for $b$, the existence of $V \in [0,1]$ such that
\[
\|\dot{a}_T\|_{\infty}, \|\dot{b}_T\|_{\infty} \leq V
\]
for all $T \leq t_0$. Estimate (30) in [5] then implies the existence of a uniform lower bound for $\inf_{t \geq T} (a_T(t) - b_T(t))$ and $T \leq t_0$ which, as the distance of the asymptotes is bounded away from 0, can again be extended to a bound $\bar{D}$ valid for all $t \in \mathbb{R}$.

In order to prove (28), it remains to find a negative upper bound for $\dot{a}_T(t) - \dot{b}_T(t)$ for sufficiently large negative $t$ and a positive lower bound for sufficiently large $t$, both uniform in $T$. The first of them is immediately given by Lemma 1, inequality (20), and the fact that $u_{-\infty} - v_{-\infty} < 0$. To find the latter, we employ a proof by contradiction. We observe that, according to Lemma 1, we can find $S$, $V^\prime$ and $D^\prime$ such that $a_T(S) - b_T(S) \leq D^\prime$ and $\dot{a}_T(S) - \dot{b}_T(S) \geq V^\prime$
hold true for all $T \leq t_0$. Assuming now that $\dot{a}_T(t) - \dot{b}_T(t) \leq 0$ for all $t \in [S, \tilde{S}]$ with $\tilde{S} > S$, Lemma 5 together with the uniform velocity bound (27) implies

$$a(t) - b(t^+_2) \leq \frac{D'}{V}$$

for these $t$ and thus, due to (7) and the equation (1) of motion, yields a uniform lower bound

$$\dot{a}_T(t) \geq \frac{k_a}{2} (1 - \dot{a}_T(t)^2)^{\frac{3}{2}} \frac{1 - \dot{b}_T(t^+_2)}{1 + \dot{b}_T(t^+_2)} \geq \frac{k_a}{2} (1 - V^2)^{\frac{3}{2}} (1 - V) =: C_1,$$

on the acceleration of particle $a$. Analogously, one obtains

$$\ddot{b}_T(t) \leq -C_2$$

and, contrary to the assumption,

$$\dot{a}_T(\tilde{S}) - \dot{b}_T(\tilde{S}) \geq V' + \int_{S}^{\tilde{S}} (C_1 + C_2) ds$$

must be positive for a sufficiently large $\tilde{S}$ independent of $T$.

From time $\tilde{S}$ on, one gets a uniform upper bound on $a(t) - b(t^+_2)$ by assuming that $a$ moved to the right and $b$ to the left with the maximal possible velocity $V$ already from time $S$ on. The resulting lower bound on $\dot{a}_T(t)$ for $t \geq \tilde{S}$ and the analogous upper one on $\dot{b}_T(t)$ lead to the desired uniform positive lower bound on $\dot{a}_T(t) - \dot{b}_T(t)$ e.g. for times $t \geq \tilde{S} + 1$.

The sequence $(a_{T_n}, b_{T_n})$ as in the Theorem can now be found by a compactness argument. We formulate an appropriate generalization of the Arzela-Ascoli theorem, the proof of which can be found in [5, Lemma 2.3]:

**Lemma 6.** If a sequence $f_1, f_2, \ldots$ of bounded continuous functions on $\mathbb{R}$ is uniformly bounded and equicontinuous and

$$\lim_{S \to \infty} \sup_{t > S, n \in \mathbb{N}} \max \{|f_n(t) - f_n(S)|, |f_n(-t) - f_n(-S)|\} = 0,$$

then it has a uniformly convergent subsequence.

Finally, with these technical lemmata, we can prove our main result:

**Proof of Theorem 2.** Recall the family of solutions on the half-line $(a_T, b_T)_{T \leq T_0}$ corresponding to the initial conditions (4) that are provided by Theorem 1. The proof is divided in several steps:

1) Existence of an accumulation point: We start by observing that $(\dot{a}_T)_{T \leq t_0}$ is uniformly bounded according to Lemma 4. For $t \geq T$, according to the FST equations (1) and Lemmata 5 and 4,

$$0 < \dot{a}_T(t) \leq \frac{k_a}{2} \left(1 - V^2\right)^{\frac{3}{2}} \left[2 \cdot \frac{4}{1 - V D^2(1 + |t|)^2}\right] \leq \frac{8k_a}{(1 - V)D^2(1 + |t|)^2}.$$

For $t \leq T$, $a_T(t) = x(t)$, so, by definition (19),

$$0 < \dot{a}_T(t) = \frac{\eta}{t^2}$$

and therefore,

$$0 < \dot{a}_T(t) \leq \frac{C}{1 + t^2}$$

holds for a suitable $C > 0$ for all $t \leq T \leq t_0$. Consequently, for all $s, t \in \mathbb{R}$ with $s < t$, we have

$$|\dot{a}_T(t) - \dot{a}_T(s)| \leq \int_s^t \frac{C}{1 + u^2} du,$$

implying that $(\dot{a}_T)_{T \leq t_0}$ is equicontinuous and that condition (29) from Lemma 6 is fulfilled, so that a uniformly convergent sequence $(a_{T_n})_{n \in \mathbb{N}}$ with $T_n \to \infty$ exists.
Furthermore, choosing \( t \leq t_0 \) and using estimate (21) from Lemma 1 together with \( |a_T(t)| \leq 1 \), gives
\[
|a_T(0)| = |a_T(t) + \int_t^0 \dot{a}_T(s)ds| \leq |x(t)| + |t|.
\]
This shows us that \((a_T(0))_{T \leq t_0}\) is bounded. Therefore, for a suitable subsequence \((T_{n_k})_{k \in \mathbb{N}}\) of \((T_n)_{n \in \mathbb{N}}\), also \((a_{T_{n_k}}(0))_{k \in \mathbb{N}}\) converges. From \((T_{n_k})_{k \in \mathbb{N}}\), we can, by an analogous reasoning, extract a subsequence \((T_{n_k})_{k \in \mathbb{N}}\) such that also \((b_{T_{n_k}})_{k \in \mathbb{N}}\) and \((b_{T_n}(0))_{n \in \mathbb{N}}\) converge. In conclusion, there is a subsequence, with slight abuse of notation again denoted by \((a_n, b_n)_{n \in \mathbb{N}}\), which converges with respect to the norm (18).

2) Estimates for the accumulation point. Estimate (21) in Lemma 1, as well as Lemma 4, hold also true for
\[
(a, b) := \lim_{n \to \infty} (a_n, b_n).
\]
This can be seen by applying these estimates to \((a_n(t), b_n(t))\) for fixed \( t \) and exploiting the convergence \( a_n(t) \to a(t) \) and \( b_n(t) \to b(t) \) for \( n \to \infty \). Therefore,
\[
a(t) = x(t) + O\left(\frac{\ln|t|}{|t|}\right), \quad b(t) = y(t) + O\left(\frac{\ln|t|}{|t|}\right).
umber{30}
\]

3) Fulfillment of the FST equations. First, we observe that, according to Lemma 4 and 5, for \((a, b)\), the advanced and retarded times \( t^\pm_2 \) in (2) are well-defined, and so is the right-hand sides of the FST equations. It remains to show that \((a, b)\) solve the FST equations. The integral equation fulfilled by \( a_n \) is given by
\[
\dot{a}_n(t) = \dot{a}_n(T) + \frac{\kappa a}{2} \int_T^t (1 - \dot{a}_n(u)^2)^{\frac{1}{2}} \left[ 1 + \frac{\dot{b}_n(t)}{1 - \dot{b}_n(t)} \right] \frac{1}{(a_n(u) - b_n(t))^2} \left[ 1 + \frac{\dot{b}_n(t)}{1 - \dot{b}_n(t)} \right] du.
umber{31}
\]
Hence, it suffices to show that we may exchange the limit \( n \to \infty \) with the integration. For this it is sufficient to show that the integrand converges uniformly on compact intervals \([-S, S]\); and likewise one has to repeat the proof for \( b_n \). However, we know that \( a_n, \dot{a}_n, b_n, \dot{b}_n \) converge uniformly thanks to the definition of the norm, the denominators are bounded away from zero by Lemma 4, and furthermore, the uniform convergence of \( t^\pm_2(a(\cdot), b(\cdot), \cdot) =: t^\pm_{2,n}(\cdot) \) to \( t^\pm_2(a(\cdot), b(\cdot), \cdot) =: t^\pm_2(\cdot) \) follows from the estimate
\[
|t^\pm_{2,n}(t) - t^\pm_2(t)| = |t \pm a_n(t) \mp b_n(t^\pm_{2,n}(t)) - t \mp a(t) \pm b(t^\pm_2(t))|
\leq |a_n(t) - a(t)| + |b_n(t^\pm_{2,n}(t)) - b_n(t^\pm_2(t))| + |b_n(t^\pm_2(t)) - b(t^\pm_2(t))|
\leq |a_n(0) - a(0)| + |S||\dot{a}_n - \dot{a}_|| + V |t^\pm_{2,n}(t) - t^\pm_2(t)|
+ |b_n(0) - b(0)| + |t^\pm_2(t)||\dot{b}_n - \dot{b}| + \frac{1 + |S| + \sup_{t \in [-S, S]} |t^\pm_2(t)|}{1 - V} ||(a_n - a, b_n - b)||.
\]
Note that the supremum is finite since \( t^\pm_2 \) is continuous. Hence, we may interchange the limit \( n \to \infty \) with the integral in (31); and likewise for the corresponding integral equation for \( b_n \). By Theorem 1 and the uniform convergence, we know that \((a, b)\) is smooth. Hence, we may take the derivate of the integral equations (31) recover the FST equations (1).

In summary, the last step proves the existence of a smooth global solution to the FST equations (1) that results from the convergence of the sequence \((a_n, b_n)_{n \in \mathbb{N}}\), which furthermore obeys the asymptotic behavior (30), which concludes the proof. \(\square\)
3.1. Proof of Lemma 2. In this last section we provide the remaining proof of Lemma 2. Recall that Lemma 2 is supposed to ensure the claims of Lemma 1, i.e., estimate (20) and (21), under the stronger condition (22). In the following we prove both claims separately denoted by Part I and Part II. We only show the estimates for $a_T$, the ones for $b_T$ are obtained analogously. In our notation, $C$ will denote finite and positive constants that may vary from line to line.

Ideally we would like to proof Lemma 2 assuming only that $T < T_0$ and that $t^* \in [T, T_0]$ exists such that estimates (22) hold true for all $t \in [T, t^*]$, and consider $t \in [T, t^*]$. However, several steps in the proof of Lemma 2, including the auxiliary lemmata in this section, will only hold under finitely many additional conditions of the form that $t$ is a sufficiently large negative number, i.e., $t \leq t_n$ for finitely many $t_n$. Since we pick up these extra constraints along the way in the proof of Lemma 2, we possibly have to adjust $T_0$ each time and start over with the proof – at most finitely many times. This is unproblematic since all previous estimates hold also for larger negative values of $T_0$.

Therefore, in order to keep the presentation reasonably short we employ a slight abuse of notation to avoid repetition of the proof: Instead of keeping $T_0$ fixed, we adjust its value from $T_0$ to $T_0 \wedge t_n$ each time we pick up another constraint $t \leq t_n$, keeping in mind that in the end, the proof will only hold for $t_0 := \min_n t_n$ – exactly in the form given in Lemma 2.

**Proof of Lemma 2, Part I: Estimate (20).** We observe that $\ddot{a}_T(t) > 0$ for $t > T$ and, by definition (19), also $\ddot{a}_T(t) = \frac{\ddot{a}}{\dot{a}} > 0$ holds true for $t < T$. Hence, $\lim_{t \to -\infty} \ddot{a}_T(t) = u_{-\infty}$ implies $\ddot{a}_T(t) > a_{-\infty}$ for all $t$. Using the integrated equation of motion (9) (for $a_T$ instead of $a$), the velocity estimate (27), Lemma 5 and assumption (22), we find

\[
\dot{a}_T(t) \leq \dot{a}_T(T) + C \int_t^T \frac{1}{(a_T(s) - b_T(t_2))^2} \, ds
\]

\[
\leq \dot{a}_T(T) + C \int_t^T \frac{1}{(a_T(s) - b_T(s))^2} \, ds
\]

\[
\leq \dot{a}_T(T) + C \int_t^T \frac{1}{s^2} \, ds.
\]

Since $\dot{a}_T(T) = \ddot{x}(T) = u_{-\infty} - \frac{\eta_1}{T}$, estimate (20) follows.\hfill \square

In order to prove estimate (21) and with it provide Part II of the proof of Lemma 2, we use the fact that equation (14) also holds true for $(a_T, b_T)$ instead of $(a, b)$ as long as $t \in [T, t^*]$; in the following we refer to (14) in the sense of $(a, b)$ replaced by $(a_T, b_T)$. Our goal is to employ this formula in order to estimate the distance $|a_T(t) - x(t)|$ by observing cancellations or asymptotically vanishing terms. Term (14a) now reads

\[
a_T(T) + \dot{a}_T(T)(t - T) = x_{-\infty} + u_{-\infty}T - \eta_1 \ln |T| + \left( u_{-\infty} - \frac{\eta_1}{T} \right) (t - T)
\]

\[
= x_{-\infty} + u_{-\infty}t - \eta_1 \ln |T| - \eta_1 \frac{t - T}{T}.
\]

(32)

In order to gain some intuition about the terms, we observe that the first two summands cancel with the ones in definition (19) of $x$. Moreover, as indicated in the section 2, the factor of term (14b) in front of $(t - T)$, multiplied by $T$, converges to $\eta_1$. Thus, term (14b) should cancel the last summand in (32) asymptotically. Term (14c) approaches

\[
-\eta_1 \ln |t| - \frac{\eta_1}{2} \ln \left( \frac{(u_{-\infty} - v_{-\infty})^2}{1 - v_{-\infty}^2} \right)
\]

and, likewise, term (14d),

\[
\eta_1 \ln |T| + \frac{\eta_1}{2} \ln \left( \frac{(u_{-\infty} - v_{-\infty})^2}{1 - v_{-\infty}^2} \right).
\]
Finally, (14e) and (14f) are expected to vanish separately. Correspondingly, it is convenient to group the terms

\[ |a_T(t) - x(t)| \leq \sum_{n=1}^{5} |A_n(t)|, \]  

where

\[ A_1(t) := \frac{\kappa_a}{2} (1 - \dot{a}_T(T))^2 \left[ \frac{1 + \dot{b}_T(t_2^- (T))}{\dot{a}_T(T) - \dot{b}_T(t_2^- (T))} \frac{t}{T} + \frac{1 - \dot{b}_T(t_2^+ (T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+ (T))} \right] \frac{t - T}{T} - \eta_1 \frac{t - T}{T}, \]  

\[ A_2(t) := \frac{\kappa_a}{2} (1 - \dot{a}_T(t))^2 \left[ \frac{1 - \dot{b}_T(t_2^- (t))^2}{(\dot{a}_T(t) - \dot{b}_T(t_2^- (t)))^2} \ln \left(\frac{a_T(t) - b_T(t_2^- (t))}{\dot{a}_T(t) - \dot{b}_T(t_2^- (t))}\right) \right. \]  

\[ \left. + \frac{1 - \dot{b}_T(t_2^+ (t))^2}{(\dot{a}_T(t) - \dot{b}_T(t_2^+ (t)))^2} \ln \left(\frac{a_T(t) - b_T(t_2^+ (t))}{\dot{a}_T(t) - \dot{b}_T(t_2^+ (t))}\right) \right] - \eta_1 \ln |t| - \frac{\eta_1}{2} \ln \left(\frac{u_{-\infty} - v_{-\infty}}{1 - v_{-\infty}^2}\right), \]  

\[ A_3(t) := A_2(T), \]  

\[ A_4(t) := \frac{\kappa_a}{2} \int_T^t \left[ \frac{d}{ds} \left( (1 - \dot{a}_T(s))^2 \right)^{\frac{1}{2} \ln \left(\frac{a_T(s) - b_T(t_2^- (s))}{\dot{a}_T(s) - \dot{b}_T(t_2^- (s))}\right)} \right] \ln \left(\frac{a_T(s) - b_T(t_2^- (s))}{\dot{a}_T(s) - \dot{b}_T(t_2^- (s))}\right) ds, \]  

\[ + \left[ \frac{d}{ds} \left( (1 - \dot{a}_T(s))^2 \right)^{\frac{1}{2} \ln \left(\frac{a_T(s) - b_T(t_2^- (s))}{\dot{a}_T(s) - \dot{b}_T(t_2^- (s))}\right)} \right] \ln \left(\frac{a_T(s) - b_T(t_2^- (s))}{\dot{a}_T(s) - \dot{b}_T(t_2^- (s))}\right) ds, \]  

\[ A_5(t) := \frac{\kappa_a}{2} \int_T^t \int_T^s \left[ \frac{d}{dr} \left( (1 - \dot{a}_T(r))^2 \right)^{\frac{1}{2} \ln \left(\frac{a_T(r) - b_T(t_2^- (r))}{\dot{a}_T(r) - \dot{b}_T(t_2^- (r))}\right)} \right] \frac{1}{a_T(r) - \dot{b}_T(t_2^- (r))} \]  

\[ + \left[ \frac{d}{dr} \left( (1 - \dot{a}_T(r))^2 \right)^{\frac{1}{2} \ln \left(\frac{a_T(r) - b_T(t_2^- (r))}{\dot{a}_T(r) - \dot{b}_T(t_2^- (r))}\right)} \right] \frac{1}{a_T(r) - \dot{b}_T(t_2^- (r))} dr ds, \]  

and show, term by term, that

\[ |A_n(t)| \leq \frac{C}{\sqrt{|t|}} \]
holds. For later use, we remark that a different rearrangement (with \( A_2 \) on the other side as in estimate (33)) gives

\[
|a_T(t) - x_{-\infty} - u_{-\infty}t - \frac{\eta t}{2} \ln \left( \frac{(u_{-\infty} - v_{-\infty})^2}{1 - v_{-\infty}^2} \right) + \frac{\kappa a}{2} (1 - \dot{a}_T(t)^2)^{3/2} \left[ \frac{1 - \dot{b}_T(t_{-2})(t)^2}{(\dot{a}_T(t) - \dot{b}_T(t_{-2})(t))^2} \ln (a_T(t) - b_T(t_{-2})(t)) \right] + \frac{1 - \dot{b}_T(t_{+2})(t)^2}{(\dot{a}_T(t) - \dot{b}_T(t_{+2})(t))^2} \ln (a_T(t) - b_T(t_{+2})(t)) | \leq \sum_{n \neq 2} |A_n(t)| .
\]

The following three lemmata will provide the essential ingredients for the estimate of (33).

**Lemma 7.** For sufficiently large negative numbers \( T, t \) such that \( T \leq t \), it holds that

\[
\dot{a}_T(t^+_1) - \dot{a}_T(t) < \frac{C}{|t|}, \quad b_T(t) - b_T(t^+_2) < \frac{C}{|t|}, \tag{41a}
\]
\[
\dot{a}_T(t) - b_T(t^+_2) \leq \mu, \quad \dot{a}_T(t^-_1) - b_T(t) \leq \mu, \tag{41b}
\]
\[
\dot{a}_T(t^+_1) - b_T(t) \leq \frac{\mu}{2}, \quad \dot{a}_T(t) - b_T(t^+_2) \leq \frac{\mu}{2}. \tag{41c}
\]

**Proof.** In the integrated equation (1) of motion

\[
\dot{a}_T(t^+_1)(t) - \dot{a}_T(t) = \frac{\kappa a}{2} \int_t^{t^+_1} (1 - \dot{a}_T(s)^2)^{3/2} \left[ \frac{1 + \dot{b}_T(t^+_2)(s)}{1 - \dot{b}_T(t^+_2)(s)} \left( a_T(s) - b_T(t^+_2)(s) \right)^2 \right. \left. \frac{1}{a_T(s) - b_T(t^+_2)(s)} \right] ds , \tag{42}
\]

we shall estimate the denominators \( a_T(s) - b_T(t^+_2)(s) \) by terms of the form \( f(t) + g(t)(t - s) \) which will be possible even if the integration can extend beyond the time up to which estimates (22) are valid. Due to the mean value theorem and the fact that \( \dot{b}_T < 0 \), which implies \( \dot{b}_T(t) < \dot{b}_T(s) \) for \( t > s \), we find

\[
b_T(t_{-2})(s) \leq b_T(t) - b_T(t_{-2})(t - t_{-2})(s) .
\]

Definition (2) implies \( s - t_{-2}(s) = a_T(s) - b_T(t_{-2}(s)) \), so that we have

\[
a_T(s) - b_T(t_{-2}(s)) \geq a_T(s) - b_T(t) + \dot{b}_T(t)(t - s) + \dot{b}_T(t) (a_T(s) - b_T(t_{-2}(s))) \geq \frac{a_T(s) - b_T(t) + \dot{b}_T(t)(t - s)}{1 - \dot{b}_T(t)} .
\]

Employing the analogous estimate \( a_T(s) \geq a_T(t^+_1(t)) - \dot{a}_T(t^+_1)(t) (t^+_1(t) - t) \) we get

\[
a_T(s) - b_T(t^-_2(s)) \geq \frac{f(t) + g(t)(t - s)}{1 - \dot{b}_T(t)}
\]

for functions

\[
f(t) := a_T(t^+_1(t)) - b_T(t) - \dot{a}_T(t^+_1(t)) (t^+_1(t) - t) = (1 - \dot{a}_T(t^+_1(t))) (a_T(t^+_1(t)) - b_T(t))
\]

– where, for the last equality,

\[
t^+_1(t) - t = a_T(t^+_1(t)) = b_T(t) \tag{43}
\]

should be recalled – and

\[
g(t) := \dot{b}_T(t) - \dot{a}_T(t^+_1(t)) .
\]

Likewise, one obtains

\[
a_T(s) - b_T(t^+_2(s)) \geq \frac{f(t) + g(t)(t - s)}{1 + \dot{b}(t)} .
\]
Using both of these estimates in the integral equation \((42)\) together with the bound on the velocities \((27)\) in order to estimate \(b_T\), performing the integration, and exploiting the identity \((43)\), we find
\[
\dot{a}_T(t_1^+(t)) - \dot{a}_T(t) \leq \int_{t}^{t_1^+(t)} \frac{C}{f(t) + g(t)(t - s)^2} ds
\]
\[
= \frac{C}{g(t)[f(t) + g(t)(t - t_1^+(t))]} - \frac{C}{g(t)[f(t) + g(t)(t - t)]}
\]
\[
= \frac{C(f(t) - [f(t) + g(t)(t - t_1^+(t))])}{g(t)f(t)[f(t) + g(t)(t - t_1^+(t))])}
\]
\[
= \frac{C(t_1^+(t) - t)}{f(t)[f(t) - g(t)(t - t_1^+(t) - t)][}
\]
\[
= \frac{C}{(1 - \dot{a}_T(t_1^+(t)))(1 - \dot{b}_T(t))(a_T(t_1^+(t)) - b_T(t))}.
\]
Here, we have resubstituted the functions \(f\) and \(g\) again. Now Lemma 5 and assumption \((22)\) allow us to conclude
\[
\dot{a}_T(t_1^+(t)) - \dot{a}_T(t) \leq \frac{C}{a_T(t) - b_T(t)} \leq \frac{C}{|t|}.
\]
Since \(\dot{b}_T < 0\), we also get
\[
\dot{a}_T(t) - \dot{b}_T(t_2^-) \leq \dot{a}_T(t) - \dot{b}_T(t) \leq \mu
\]
thanks to assumption \((22)\). If \(t\) is a sufficiently large negative number, recalling that \(\mu < 0\), \((41a)\) implies
\[
\dot{a}_T(t_1^+) - \dot{b}_T(t) = \dot{a}_T(t_1^+) - \dot{a}_T(t) + \dot{a}_T(t) - \dot{b}_T(t) \leq \frac{\mu}{2}.
\]
We omit a proof of the remaining inequalities which are obtained by very similar arguments. \(\Box\)

**Lemma 8.** For sufficiently large negative numbers \(T, t\) such that \(T \leq t\), it holds that
\[
\left| \kappa_a \left(1 - \dot{a}_T(t)^2\right)^{\frac{3}{2}} \frac{1 - \dot{b}_T(t_2^+)^2}{(\dot{a}_T(t) - \dot{b}_T(t_2^+))^2} - \eta_1 \right| \leq \frac{C}{|t|}.
\]

**Proof.** By definition \((15)\) of \(\eta_1\), we find
\[
D(t) := \left| \kappa_a \left(1 - \dot{a}_T(t)^2\right)^{\frac{3}{2}} \frac{1 - \dot{b}_T(t_2^+)^2}{(\dot{a}_T(t) - \dot{b}_T(t_2^+))^2} - \eta_1 \right|
\]
\[
\leq \kappa_a \left| (1 - \dot{a}_T(t)^2)^{\frac{3}{2}} \frac{1 - \dot{b}_T(t_2^+)^2}{(\dot{a}_T(t) - \dot{b}_T(t_2^+))^2} - (1 - \dot{a}_T(t)^2)^{\frac{3}{2}} \frac{1 - v_{-\infty}^2}{(\dot{a}_T(t) - v_{-\infty})^2} \right|
\]
\[
+ \kappa_a \left| (1 - \dot{a}_T(t)^2)^{\frac{3}{2}} \frac{1 - v_{-\infty}^2}{(\dot{a}_T(t) - v_{-\infty})^2} - (1 - u_{-\infty}^2)^{\frac{3}{2}} \frac{1 - v_{-\infty}^2}{(u_{-\infty} - v_{-\infty})^2} \right|.
\]
The facts
\[
\frac{\partial}{\partial u} \left[ (1 - u^2)^{\frac{3}{2}} \frac{1 - v^2}{(u - v)^2} \right] = \frac{(1 - u^2)^{\frac{3}{2}}(1 - v^2)(-u^2 + 3uv - 2)}{(u - v)^3}
\]
and
\[
\frac{\partial}{\partial v} \left[ (1 - u^2)^{\frac{3}{2}} \frac{1 - v^2}{(u - v)^2} \right] = \frac{2(1 - u^2)^{\frac{3}{2}}(1 - uv)}{(u - v)^3},
\]

\[
(44)
\]
\[
(45)
\]
together with the mean value theorem and the bound $|\dot{a}_T|, |\dot{b}_T| \leq 1$ yield

$$D(t) \leq \frac{C}{|\dot{a}_T(t) - v_s|^3} \left| v_* - \dot{b}_T(t_s^\pm) \right| + \frac{C}{|u_* - v_*|^3} |\dot{a}_T(t) - u_*|$$

Equation (46)

for some $u^* \in [u_*-\infty, \dot{a}_T(t)]$ and $v^* \in [\dot{b}_T(t_s^\pm), v_*-\infty]$. Since $\dot{a}_T > 0, \dot{b}_T < 0$, we get

$$u_* - v_* \leq u^* - v^* \leq \dot{a}_T(t) - v^* \leq \dot{a}_T(t) - \dot{b}_T(t_s^\pm) \leq \mu < 0$$

thanks to Lemma 7. Therefore, $|\dot{a}_T(t) - v^*|, |u_* - v_*| \geq C$ holds. Employing this bound together with (20) in (46) concludes the proof.

\[ \square \]

**Lemma 9.** For all $t \in \mathbb{R}$ it holds that

$$\frac{t}{a_T(t) - b_T(t_s^\pm(t))} = \frac{1 \pm b_T(\tilde{t}^\pm)}{a_T(t) - b_T(\tilde{t}^\pm(t))}$$

for some $\tilde{t}^\pm$ between $t$ and $t_s^\pm(t) := t_s^\pm(a_T(t), b_T(t), t)$.

**Proof.** Rearranging terms and using definition (2) of $t_s^\pm$, we get

$$\frac{t}{a_T(t) - b_T(t_s^\pm(\tilde{t}))} = \frac{t}{a_T(t) - b_T(\tilde{t})} \left( 1 + \frac{b_T(t_s^\pm(\tilde{t})) - b_T(\tilde{t})}{a_T(t) - b_T(t_s^\pm(\tilde{t}))} \right)$$

$$= \frac{t}{a_T(t) - b_T(\tilde{t})} \left( 1 + \frac{b_T(t_s^\pm(\tilde{t})) - b_T(\tilde{t})}{t_s^\pm(\tilde{t}) - t} \right).$$

Applying the mean value theorem to the term in brackets, the claim follows. \[ \square \]

Finally, we have all necessary ingredient for Part II of the proof:

**Proof of Lemma 2, estimate (21).** It suffices to provide the bound (39) on terms $A_1, A_2, A_3, A_4, A_5$ given in (34)-(38).

**Term** $A_5$: Starting from the expressions

$$\frac{\partial}{\partial u} \left[ (1 - u^2)^{\frac{3}{2}} \frac{1 \mp v}{u - v} \right] = \frac{(1 - u^2)^{\frac{3}{2}} (1 \mp v) (-2u^2 + 3uv - 1)}{(u - v)^2}$$

and

$$\frac{\partial}{\partial v} \left[ (1 - u^2)^{\frac{3}{2}} \frac{1 \mp v}{u - v} \right] = \frac{(1 - u^2)^{\frac{3}{2}} (1 \mp u)}{(u - v)^2}$$

for $u \neq v \in [-1, 1]$ and employing formula (10) for $t_s^\pm$, we get

$$\frac{d}{dr} \left[ (1 - \dot{a}_T(r)^2)^{\frac{3}{2}} \frac{\dot{a}_T(r) \mp \dot{b}_T(t_s^\pm(r))}{\dot{a}_T(r) - b_T(t_s^\pm(r))} \right]$$

$$= \dot{a}_T(r) \left( 1 - a_T(r)^2 \right)^{\frac{3}{2}} \frac{(1 \mp \dot{b}_T(t_s^\pm(r)))(-2\dot{a}_T(r)^2 + 3\dot{a}_T(r)\dot{b}_T(t_s^\pm(r)) - 1)}{(\dot{a}_T(r) - b_T(t_s^\pm(r)))^2}$$

$$+ \dot{b}_T(t_s^\pm(r)) \frac{(1 - \dot{a}_T(r)^2)^{\frac{3}{2}}}{(1 \mp \dot{b}_T(t_s^\pm(r)))(\dot{a}_T(r) - b_T(t_s^\pm(r)))^2}$$

for $T \leq r \leq t^*$. The velocity estimate (27) and Lemma 7 applied to $\dot{a}_T(r) - \dot{b}_T(t_s^\pm(r))$ imply

$$\left| \frac{d}{dr} \left[ (1 - \dot{a}_T(r)^2)^{\frac{3}{2}} \frac{\dot{a}_T(r) \mp \dot{b}_T(t_s^\pm(r))}{\dot{a}_T(r) - b_T(t_s^\pm(r))} \right] \right| \leq C \left( \dot{a}_T(r) - \dot{b}_T(t_s^\pm(r)) \right).$$

Equation (47)

From the FST equation (1), eq. (7) for $\ddot{a}_T$ and Lemma 5, we obtain

$$\dot{a}_T(r) \leq \frac{C}{(a_T(r) - b_T(t_s^\pm(r))^2}$$

Equation (48)
for $T \leq r \leq t^*$ as well as

$$-\dot{b}_T(t_2^+(r)) \leq C \left[ \frac{1}{(a_T(r) - b_T(r))^2} + \frac{1}{(a_T(t_2^+(r)) - b_T(t_2^+(r)))^2} \right],$$

for $T \leq r \leq t^*$ in the “$t_2^-(r)$” case and for $T \leq r \leq t^*$, but only such that $t_2^-(r) > T^+$ holds, in the “$t_2^+(r)$” case; recall that for particle $b_T$ the FST equations (1) are only guaranteed to hold from $T^+$ on by Theorem 1. By the mean value theorem, the velocity estimate, definition (2) of $t_2^\pm$ and Lemma 5, we furthermore find

$$a_T(t_2^+(r) - b_T(t_2^+(r)) \geq a_T(r) - V(t_2^+(r) - r) - b(t_2^+(r))$$

$$= a_T(r) - V(a_T(r) - b_T(t_2^+(r))) - b_T(t_2^+(r)) \geq \frac{C}{a_T(r) - b_T(r)}$$

and

$$a_T(t_2^-(r) - b_T(t_2^-(r)) \geq \frac{C}{a_T(r) - b_T(r)}$$

holds true for the values of $r$ considered above. This is because assumption (22) implies that $a_T(r) - b_T(r)$ is monotonically decreasing in $r$. If, instead, $T \leq r \leq t^*$ but now $t_2^-(r) \leq T^+$, then, using in addition definition (19) of the asymptotes, we get

$$-\dot{b}_T(t_2^+(r)) = \frac{\eta_2}{t_2^+(r)^2} < \frac{\eta_2}{(r - t_2^-(r))^2} \leq \frac{\eta_2}{(a_T(r) - b_T(t_2^-(r)))^2} \leq \frac{C}{(a_T(r) - b_T(r))^2}.$$ 

In summary, in all cases, we find

$$-\dot{b}_T(t_2^+(r)) \leq \frac{C}{(a_T(r) - b_T(r))^2}. \quad (49)$$

Using this for $b_T$ together with estimate (48) for $\dot{a}_T$ in (47), we can provide the claimed bound (39) on $A_5$ in (38) by using once more Lemma 5 and assumption (22). Indeed, we infer the even better bound

$$|A_5(t)| \leq \int_T^t \int_T^s \frac{C}{(a_T(r) - b_T(r))^3} \, dr \, ds \leq \int_T^t \int_T^s \frac{C}{|t|^3} \, dr \, ds \leq \frac{C}{|t|}.$$ 

**Term $A_4$:** We compute the derivatives in the integrand of $A_4$ in (37) via formulas (44) and (45). Using the assumption (22) to estimate the denominator, the boundedness $|\dot{a}_T|, |\dot{b}_T| \leq 1$ to estimate the remaining velocities as well as and estimates (48) and (49) for $\dot{a}_T$ and $\dot{b}_T$, gives

$$\frac{d}{ds} \left( (1 - \dot{a}_T(t_2^+(s)))^\frac{3}{2} \left( 1 - \dot{b}_T(t_2^+(s))^2 \right) \right)$$

$$= \left| \dot{a}_T(s) \left( 1 - \dot{a}_T(s)^2 \right)^{\frac{3}{2}} \left( 1 - \dot{b}_T(t_2^+(s))^2 \right) \left( -\dot{a}_T(s)^2 + 3\dot{a}_T(s)\dot{b}_T(t_2^+(s)) - 2 \right) \right|$$

$$\leq C \left( \dot{a}_T(s) - \dot{b}_T(t_2^+(s)) \right) \leq \frac{C}{(a_T(s) - b_T(s))^2}$$

for $T \leq s \leq t^*$. For sufficiently large negative $s$, assumption (22) implies

$$\frac{a_T(s) - b_T(s)}{2} \geq \frac{\mu s}{2} \geq 1.$$  

(50)
Invoking Lemmata 5 and 4, yields
\[ 0 = \ln(1) \leq \ln \left( a_T(s) - b_T(t_2^+(s)) \right) \leq \ln \left( \frac{a_T(s) - b_T(s)}{1 - V} \right) \] (51)
and therefore,
\[ |A_4(t)| \leq \int_T^t \frac{C}{(a_T(s) - b_T(s))^2} \ln \left( \frac{a_T(s) - b_T(s)}{1 - V} \right) \, ds . \]
The integrand is increasing in $a_T(s) - b_T(s)$ for sufficiently large negative $s$, so, using (50), $A_4$ can further be bounded by
\[ |A_4(t)| \leq \int_T^t \frac{C \ln(C|s|)}{s^2} \, ds \leq C \int_T^t \ln(C|s|) - \frac{1}{s} \, ds = C \int_T^t \frac{d \ln(C|s|)}{ds} \, ds \leq C \ln \left| \frac{t}{|t|} \right| . \]
So far the terms had sufficient decay in time to asymptotically vanish. As discussed above, in the following estimates it will be important to observe certain cancellations between the term in order to provide the corresponding estimates (39).

**Term $A_1$:** Since $T \leq t < 0$ we have $|\frac{T}{t}| \leq 1$, and furthermore
\[ |A_1(t)| \leq \left| \frac{\kappa_a}{2} (1 - \dot{a}_T(T))^2 \right| \frac{1 - \dot{b}_T(t_2^+(T))^2}{\left( a_T(T) - b_T(t_2^+(T)) \right)^2} + \frac{1 - \dot{b}_T(t_2^+(T))^2}{\left( \dot{a}_T(T) - \dot{b}_T(t_2^+(T)) \right)^2} - \eta_1 \]
(53)
\[ + \frac{\kappa_a}{2} (1 - \dot{a}_T(T))^2 \frac{1 + \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \left( \frac{T}{a_T(T) - b_T(t_2^+(T))} - \frac{1 - \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right) \]
\[ + \frac{\kappa_a}{2} (1 - \dot{a}_T(T))^2 \frac{1 - \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \left( \frac{T}{a_T(T) - b_T(t_2^+(T))} - \frac{1 + \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right) \]
\[ \leq \frac{C}{|T|} + C \left| \frac{T}{a_T(T) - b_T(t_2^+(T))} - \frac{1 - \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| \]
\[ + C \left| \frac{T}{a_T(T) - b_T(t_2^+(T))} - \frac{1 + \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| , \]
where Lemma 8 has been used for the summand (53) and the usual velocity estimates for the other two summands (55),(56). With $\bar{T}^\pm$ between $T$ and $t_2^+(T)$ in order to apply Lemma 9, and in addition using Lemma 7 and $\bar{b}_T < 0$, we find
\[ \left| \frac{T}{a_T(T) - b_T(t_2^+(T))} - \frac{1 \pm \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| \]
\[ \leq \left| \frac{1 \pm \dot{b}_T(T^\pm)}{a_T(T) - b_T(t_2^+(T))} - \frac{1 \pm \dot{b}_T(T^\pm)}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| + \left| \frac{\dot{b}_T(T^\pm)}{a_T(T) - b_T(t_2^+(T))} - \frac{\dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| \]
\[ \leq 2 \left| \frac{1}{a_T(T) - b_T(t_2^+(T))} - \frac{1}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| + \frac{2}{\mu} \left| \dot{b}_T(T) - \dot{b}_T(t_2^+(T)) \right| . \]
By assumption (22) and Lemma 7, the denominators in the first modulus are at most $\frac{8}{5}$. Applying the mean value theorem, Lemma 7, the fact that $a_T(t) = x(t)$ and $b_T(t) = y(t)$ for $t \leq T$ and the definition of the asymptotes in (19), yields
\[ \left| \frac{T}{a_T(T) - b_T(t_2^+(T))} - \frac{1 \pm \dot{b}_T(t_2^+(T))}{\dot{a}_T(T) - \dot{b}_T(t_2^+(T))} \right| \]
\[
\begin{align*}
&\leq \frac{8}{\mu^2} \left| \frac{a_T(T)}{T} - \frac{b_T(T)}{t} - ( \dot{a}_T(T) - b_T(t_2^{-}(T))) \right| + \frac{2}{\mu} \left| \tilde{b}_T(T) - \tilde{b}_T(t_2^{-}(T)) \right| \\
&\leq C \left( \frac{x(T)}{T} - \dot{x}(T) \right) + C \left( \frac{y(T)}{T} - \dot{y}(T) \right) + C \left| \tilde{b}_T(T) - \tilde{b}_T(t_2^{-}(T)) \right| \\
&\leq C \left| \ln \frac{T}{|T|} \right| \leq C \ln \frac{|t|}{|t|},
\end{align*}
\]
and therefore,
\[
|A_1(t)| \leq C \ln \frac{|t|}{|t|},
\]
which complies with the claimed estimate (39).

Term $A_2$: We compute for term $A_2$ in (35) for $T \leq t$
\[
|A_2(t)| \leq \left| \frac{\kappa_a}{2} (1 - \dot{a}_T(t)^2)^{\frac{3}{2}} \frac{1 - \tilde{b}_T(t_2^{-}(t))^2}{(\dot{a}_T(t) - \tilde{b}_T(t_2^{-}(t))^2) \frac{\eta_1}{2}} \ln \left( \frac{a_T(t) - b_T(t_2^{-}(t))}{\tilde{b}_T(t)^{-2}(t)} \right) \right|
\]
\[
+ \left| \frac{\kappa_a}{2} (1 - \dot{a}_T(t)^2)^{\frac{3}{2}} \frac{1 - \tilde{b}_T(t_2^{-}(t))^2}{(\dot{a}_T(t) - \tilde{b}_T(t_2^{-}(t))^2) \frac{\eta_1}{2}} \ln \left( \frac{a_T(t) - b_T(t_2^{-}(t))}{\tilde{b}_T(t)^{-2}(t)} \right) \right|
\]
\[
+ \frac{\eta_1}{2} \ln \left( - \frac{a_T(t) - b_T(t_2^{-}(t))}{t} \right) - \ln \left( - \frac{u_{-\infty} - v_{\infty}}{1 - u_{-\infty}} \right)
\]
\[
+ \frac{\eta_2}{2} \ln \left( - \frac{a_T(t) - b_T(t_2^{-}(t))}{t} \right) - \ln \left( - \frac{u_{-\infty} - v_{\infty}}{1 + v_{-\infty}} \right)
\]
\[
\leq C \left| \ln \frac{a_T(t) - b_T(t_2^{-}(t))}{1 - V} \right|
\]
\[
+ C \left( \frac{a_T(t) - b_T(t_2^{-}(t))}{t} - \frac{u_{-\infty} - v_{\infty}}{1 - V} \right) + \frac{a_T(t) - b_T(t_2^{-}(t))}{t} - \frac{u_{-\infty} - v_{\infty}}{1 + V} \right)
\]
\[
(58)
\]
where in the first two summands Lemma 8 has been applied to the factors in the brackets involving the difference for \( \frac{\eta_1}{2} \), and the logarithmic factors were bounded by estimate (51). For the differences of the logarithms we exploited once again the mean value theorem noting that their arguments are all bounded from below by $\frac{\eta_2}{2}$ thanks to assumption (22) and $|v_{-\infty}| < 1$.

With $\tilde{t}^{\pm}$ between $t$ and $t_2^{\pm}(t)$ in order to apply Lemma 9 and noting that $\tilde{b}_T < 0$, we find
\[
\left| \frac{a_T(t) - b_T(t_2^{\pm}(t))}{t} \right| - \frac{u_{-\infty} - v_{\infty}}{1 \pm V} + \left| \frac{b_T(t)^{-2}(t) - u_{-\infty} - v_{\infty}}{1 \pm V} \right|
\]
\[
\leq C \left( \left| \frac{a_T(t) - b_T(t_2^{\pm}(t))}{t} - u_{-\infty} \right| + \left| \frac{b_T(t)^{-2}(t) - u_{-\infty} - v_{\infty}}{1 \pm V} \right| - \frac{u_{-\infty} - v_{\infty}}{1 \pm V} \right)
\]
\[
(59)
\]

Lemmata 7 and 1 and the fact that $\ddot{b}_T < 0$ implies
\[
\left| \dot{b}_T(t)^{\pm} - v_{\infty} \right| \leq \left| b_T(t_2^{\pm}(t)) - v_{\infty} \right| \leq \left| b_T(t_2^{\pm}(t)) - b_T(t) \right| + \left| \dot{b}_T(t) - v_{\infty} \right| \leq \frac{C}{|t|}.
\]
Collecting these results going back to eq. (59) and then (58), we have shown
\[
|A_2(t)| \leq \frac{C}{|t|} \left( |\ln |a_T(t) - b_T(t)|| + |a_T(t) - u_{-\infty}t| + |b_T(t) - v_{\infty}t| + 1 \right).
\]
(60)
Note that the term $A_3(t)$ in (36) is constant in $t$ and fulfills $A_3(t) = A_2(T)$. If $t = T$, then $a_T(t) = x(t)$ and $b_T(t) = y(t)$ and, for sufficiently large negative $t$, the definition of the asymptotes $(x, y)$ in (19) yields

$$|A_3(t)| = |A_2(T)| \leq C \frac{\ln |T|}{|T|} \leq C \frac{\ln |t|}{|t|}.$$ 

In order to provide a similar bound for $A_2(t)$ for general $t$, we first need an upper bound on $a_T(t) - b_T(t)$ to estimate the right-hand side of (60). This can be obtained by collecting the preceding estimates together in the form (40): First, we may omit the terms in (40) involving \( \ln (a_T(t) - b_T(t^2(t))) \), which are negative by (51), and find

$$a_T(t) \leq x_{-\infty} + u_{-\infty} t + \frac{\eta}{2} \ln \left( \frac{(u_{-\infty} - v_{-\infty})^2}{1 - v_{-\infty}^2} \right) + \frac{C}{\sqrt{|t|}} \leq C_1 + u_{-\infty} t + \frac{C}{\sqrt{|t|}},$$

and likewise

$$b_T(t) \geq C_2 + v_{-\infty} t - \frac{C}{\sqrt{|t|}},$$

where $C_1, C_2$ are not necessarily positive constants. Recalling the choice $u_{-\infty} < v_{-\infty}$, we have

$$0 < a_T(t) - b_T(t) \leq C + (u_{-\infty} - v_{-\infty}) t + \frac{C}{\sqrt{|t|}} \leq C (1 + |t|)$$

and, for sufficiently large negative $t$,

$$|\ln (a_T(t) - b_T(t))| \leq C \ln |t|. \quad (61)$$

Applying now (40) in the opposite direction and using, beside the estimate just derived, Lemma 4 and inequality (22) for the velocities, we obtain

$$a_T(t) \geq C_3 + u_{-\infty} t - C \ln |t| - \frac{C}{\sqrt{|t|}}.$$ 

Together with the upper bound, this implies

$$|a_T(t) - u_{-\infty} t| \leq C \ln |t|$$

for sufficiently large negative $t$. Going back to (60), the latter bound and the corresponding one for $b_T$ together with estimate (61), we find

$$|A_2(t)| \leq C \frac{\ln |t|}{|t|}.$$ 

which again complies with the necessary estimate (39).

In summary, we have shown that (39) holds for $T \leq t \leq t^*$ which concludes the proof of Lemma 2. \qed

**Acknowledgement.** This work was partially funded by the Elite Network of Bavaria through the Junior Research Group ‘Interaction between Light and Matter’.

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