Composition Operators on Sobolev Spaces and Neumann Eigenvalues

V. Gol’dshtein · A. Ukhlov

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Abstract
In this paper we discuss applications of the geometric theory of composition operators on Sobolev spaces to the spectral theory of non-linear elliptic operators. Lower estimates of the first non-trivial Neumann eigenvalues of the $p$-Laplace operator in cusp domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, are given.

Keywords Sobolev spaces · Neumann eigenvalues · Quasiconformal mappings

Mathematics Subject Classification 35P15 · 46E35 · 30C65

1 Introduction
In this article we discuss a general method of applications of the geometric theory of composition operators on Sobolev spaces to spectral estimates of the non-linear Neumann-Laplace operators. It is a modification of methods proposed in our recent works [14,15].

Suppose that in a bounded domain $\Omega \subset \mathbb{R}^n$ the following Sobolev–Poincaré inequality

$$\inf_{c \in \mathbb{R}} \| f - c \|_{L^p(\Omega)} \leq B_{p,p}(\Omega) \| \nabla f \|_{L^p(\Omega)}, \quad f \in W^{1,p}_0(\Omega),$$

holds with the Poincaré constant $B_{p,p}(\Omega)$. Then the first nontrivial Neumann eigenvalue $\mu_p(\Omega)$ of the $p$-Laplace operator

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\[-\Delta_p u = - \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad p > 1, \quad (1.1)\]
can be characterized as \( \mu_p(\Omega) = B_{p,p}^{-p}(\Omega) \) (see, for example, [5]).

Exact calculations of Neumann eigenvalues are known in very limited number of domains and so estimates of \( \mu_p(\Omega) \) represent an important part of the modern spectral theory (see, for example, [1,5,10,23,26,32]). We suggest spectral estimates of the first nontrivial Neumann eigenvalue \( \mu_p(\Omega) \) of the \( p \)-Laplace operator in terms of the (weak) quasiconformal geometry of domains.

The classical upper estimate for the first nontrivial Neumann eigenvalue of the Laplace operator
\[
\mu_2(\Omega) \leq \mu_2(\Omega^*) = \frac{p_{n/2}^2}{R_*^2}
\]
was proved by Szegö [43] for simply connected planar domains via a conformal mappings technique (“the method of conformal normalization”) and by Weinberger [55] for domains in \( \mathbb{R}^n \). In this inequality \( p_{n/2} \) denotes the first positive zero of the function \( (r^{1-n/2} J_{n/2}(r))' \), and \( \Omega^* \) is an \( n \)-ball of the same \( n \)-volume as \( \Omega \) with \( R_* \) as its radius. In particular, if \( n = 2 \), we have \( p_1 = j'_{1,1} \approx 1.84118 \) where \( j'_{1,1} \) denotes the first positive zero of the derivative of the Bessel function \( J_1 \).

In convex domains \( \Omega \subset \mathbb{R}^n, n \geq 2 \), the classical lower estimates of the linear Neumann-Laplace operator (\( p=2 \)) [39] states that
\[
\mu_2(\Omega) \geq \frac{\pi^2}{d(\Omega)^2},
\]
where \( d(\Omega) \) is a diameter of a convex domain \( \Omega \). Similar estimates for \( p \neq 2 \) were obtained much later [26]:
\[
\mu_p(\Omega) \geq \left( \frac{\pi_p}{d(\Omega)} \right)^p,
\]
where
\[
\pi_p = 2 \int_0^{(p-1)^{1/p}} \frac{dt}{(1-t^p/(p-1))^{1/p}} = \frac{2\pi (p-1)^{1/p}}{p \sin(\pi/p)}.
\]

Unfortunately in non-convex domains \( \mu_p(\Omega) \) can not be characterized in the terms of its Euclidean diameters. It can be seen by considering a domain consisting of two identical squares connected by a thin corridor [2]. In our previous works [20,21] we returned to a conformal mappings techniques (that was used in [43]) in a framework of composition operators on Sobolev spaces. It permitted us to obtain lower estimates of the first nontrivial Neumann eigenvalue in the terms of the hyperbolic (conformal) radius of \( \Omega \) for a large class of non convex domains \( \Omega \subset \mathbb{R}^2 \).

In the case of space domains conformal mappings does not allow to obtain spectral estimates of the Neumann-Laplace operator and we used a generalization of conformal
mappings such as weak $p$-quasiconformal mappings [22]. The aim of the present work is to obtain lower estimates of the Neumann eigenvalues of the $p$-Laplace operator in the terms of a general weak $(p, q)$-quasiconformal geometry of domains. We use one more time the geometric theory of composition operators on Sobolev spaces [13,44] with applications to the (weighted) Poincaré–Sobolev inequalities [12,18].

The paper is organized as follows: The capacitary description of Sobolev spaces and a short survey of the geometric theory of composition operators on Sobolev spaces are presented in Sect. 2. In Sect. 3 we present a general method of applications of composition operators to the Sobolev–Poincaré inequalities and apply this method to lower estimates of the first nontrivial Neumann eigenvalue for the $p$-Laplace operator.

2 Composition Operators on Sobolev Spaces

2.1 Sobolev Spaces

Let $E$ be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. The Lebesgue space $L_p(E)$, $1 \leq p \leq \infty$, is defined as a Banach space of $p$-summable functions $f : E \to \mathbb{R}$ equipped with the following norm:

$$
\| f | L_p(E) \| = \left( \int_E |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

and

$$
\| f | L_\infty(E) \| = \operatorname{ess \sup}_{x \in E} |f(x)|, \quad p = \infty.
$$

If $\Omega$ is an open subset of $\mathbb{R}^n$, the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$
\| f | W^1_p(\Omega) \| = \| f | L_p(\Omega) \| + \| \nabla f | L_p(\Omega) \|,
$$

where $\nabla f$ is the weak gradient of the function $f$, i.e. $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$,

$$
\int_\Omega f \frac{\partial g}{\partial x_i} \, dx = -\int_\Omega \frac{\partial f}{\partial x_i} g \, dx, \quad \forall g \in C_0^\infty(\Omega), \quad i = 1, \ldots, n.
$$

As usual, $C_0^\infty(\Omega)$ is the space of infinitely smooth functions with a compact support.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$
\| f | L^1_p(\Omega) \| = \| \nabla f | L_p(\Omega) \|.
$$
Sobolev spaces are Banach spaces of equivalence classes [36]. To clarify the notion of equivalence classes we use a corresponding type of capacity.

Recall the notion of a $p$-capacity of a set $E \subset \Omega$ [36]. Let $\Omega$ be a domain in $\mathbb{R}^n$ and a compact $F \subset \Omega$. The $p$-capacity of the compact $F$ is defined by

$$\text{cap}_p(F; \Omega) = \inf \left\{ \| f |L^1_p(\Omega)|^p, \ f \geq 1 \text{ on } F, \ f \in C_0(\Omega) \cap L^1_p(\Omega) \right\},$$

where $C_0(\Omega)$ is the space of continuous functions with a compact support.

By the similar way we can define the $p$-capacity of open sets.

For arbitrary set $E \subset \Omega$ we define an inner $p$-capacity as

$$\text{cap}_p(E; \Omega) = \sup \{ \text{cap}_p(e; \Omega), \ e \subset E \subset \Omega, \ e \text{ is a compact} \},$$

and an outer $p$-capacity as

$$\overline{\text{cap}}_p(E; \Omega) = \inf \{ \text{cap}_p(U; \Omega), \ E \subset U \subset \Omega, \ U \text{ is an open set} \}.$$

A set $E \subset \Omega$ is called $p$-capacity measurable, if $\text{cap}_p(E; \Omega) = \overline{\text{cap}}_p(E; \Omega)$. The value

$$\text{cap}_p(E; \Omega) = \overline{\text{cap}}_p(E; \Omega) = \text{cap}_p(E; \Omega)$$

is called the $p$-capacity of the set $E \subset \Omega$.

The notion of $p$-capacity permits us to refine the notion of Sobolev functions. Let a function $f \in L^1_p(\Omega)$. Then refined function

$$\tilde{f}(x) = \lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy$$

is defined quasieverywhere i. e. up to a set of $p$-capacity zero and it is absolutely continuous on almost all lines [36] (7.2.4, pp. 358–359). This refined function $\tilde{f} \in L^1_p(\Omega)$ is called a unique quasicontinuous representation (a canonical representation) of function $f \in L^1_p(\Omega)$. Recall that a function $\tilde{f}$ is termed quasicontinuous if for any $\varepsilon > 0$ there is an open set $U_\varepsilon$ such that the $p$-capacity of $U_\varepsilon$ is less than $\varepsilon$ and on the set $\Omega \setminus U_\varepsilon$ the function $\tilde{f}$ is continuous (see, for example [24,36]). In what follows we will use the quasicontinuous (refined) functions only.

Note that the first weak derivatives of the function $f$ coincide almost everywhere with the usual partial derivatives (see, e.g., [36]).

### 2.2 Composition Operators

Let $\Omega$ and $\tilde{\Omega}$ be domains in $\mathbb{R}^n$, $n \geq 2$. We say that a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ induces a bounded composition operator
φ∗ : $L^1_p(\tilde{\Omega}) \rightarrow L^1_q(\Omega)$, $1 \leq q \leq p \leq \infty$,

by the composition rule $φ^*(f) = f \circ φ$, if for any function $f \in L^1_p(\tilde{\Omega})$, the composition $φ^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in $\Omega$ and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|φ^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\Omega)\|f \mid L^1_p(\tilde{\Omega})\|.$$ 

Composition operators on Sobolev spaces arise in [35] as operators generated by the class of sub-areal mappings. In [50] in the frameworks of Reshetnyak’s problem (1968) was proved that a homeomorphism $φ : \Omega \rightarrow \tilde{\Omega}$ generates an isomorphism of Sobolev spaces $L^1_n(\Omega)$ and $L^1_n(\tilde{\Omega})$ if and only if $φ$ is a quasiconformal mapping. These works motivated the study of geometric properties of mappings which generate isomorphisms (bounded operators) of Sobolev type spaces. In [51] it was proved that a homeomorphism $φ : \Omega \rightarrow \tilde{\Omega}$ generates an isomorphism of Sobolev spaces $L^1_p(\Omega)$ and $L^1_p(\tilde{\Omega})$, $p > n$, if and only if $φ$ is a bi-Lipschitz mapping. This result was extended to the cases $n - 1 < p < n$ in [17] and the case $1 \leq p < n$ in [33].

**Remark 2.1** Recall that a homeomorphism $φ$ is called a bi-Lipschitz homeomorphism if $φ$ and $φ^{-1}$ are locally Lipschitz mappings with uniformly bounded local Lipschitz constants.

Mappings that generate isomorphisms of Besov spaces were considered in [47], Nikolskii–Besov spaces and Lizorkin–Triebel spaces were considered in [48]. In [37] the theory of multipliers was applied to the change of variable problem in Sobolev spaces.

The composition operators are not necessary isomorphisms even if they are induced by diffeomorphisms or homeomorphisms of Euclidean domains. It means that the composition problem can be reformulated by a more flexible way: describe classes of homeomorphisms that induce bounded composition operators on Sobolev spaces. In the case of Sobolev spaces with the first weak derivatives it can be formulated as a characterization of homeomorphism $φ : \Omega \rightarrow \tilde{\Omega}$ that generate composition operators

$$φ^* : L^1_p(\tilde{\Omega}) \rightarrow L^1_q(\Omega), \ 1 \leq q \leq p \leq \infty,$$ 

by the standard composition rule $φ^*(f) = f \circ φ$.

Analytical characteristics of mappings that generate bounded compositions operators on Sobolev spaces are given in terms of weak derivatives of the mappings. Let a mapping $φ : \Omega \rightarrow \mathbb{R}^n$ be weakly differentiable in $\Omega$. Then the formal Jacobi matrix $Dφ(x)$ and its determinant (Jacobian) $J(x, φ)$ are well defined at almost all points $x \in \Omega$. The norm $|Dφ(x)|$ of the matrix $Dφ(x)$ is the norm of the corresponding linear operator. We will use the same notation for this matrix and the corresponding linear operator.

Recall that a mapping $φ : \Omega \rightarrow \mathbb{R}^n$ possesses the Luzin $N$-property if an image of any set of measure zero has measure zero. Lipschitz mapping possess the Luzin $N$-property.
In the case \( p = q \neq n \) an analytic description was obtained in [46] using a notion of mappings of finite distortion introduced in [52]: a weakly differentiable mapping is called a mapping of finite distortion if \(|D\varphi(x)| = 0\) a.e. on the set \( Z = \{ x \in \Omega : J(x, \varphi) = 0 \} \). This property becomes useful especially for the case \( p \neq q \).

In [13] it was obtained the geometric description of composition operators for \( n - 1 < p = q < \infty \) without using the finite distortion property. In the case \( 1 < p < n - 1 \) the geometric description requires additional apriori properties of homeomorphisms [13].

Let us reformulate the main result of [13].

**Theorem 2.2** [13] A homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) between two domains \( \Omega \) and \( \tilde{\Omega} \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_p(\Omega), \quad n - 1 < p < \infty,
\]

if and only if

\[
M_p^{(\lambda)}(\varphi; \Omega) = \sup_{x \in \Omega} \limsup_{r \to 0} \frac{L_p^p(x, r)^{n-p}}{|\varphi(B(x, \lambda r))|} < \infty,
\]

for some \( \lambda > 1 \) if \( n - 1 < p < n \) and \( \lambda = 1 \) if \( n \leq p < \infty \), where \( L_p(x, r) = \max_{|x-y|=r} |\varphi(x) - \varphi(y)| \).

On the base of this result the notion of \( p \)-quasiconformal mappings was introduced for \( n - 1 < p < \infty \). In the case \( 1 < p < n - 1 \) this result is correct under an additional apriori property of weak differentiability of \( \varphi \) [13].

Recall that a weakly differentiable homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) is called a weak \( p \)-quasiconformal mapping [13] if there exists a constant \( K_p < \infty \) such that

\[
|D\varphi(x)|^p \leq K_p^n |J(x, \varphi)| \quad \text{for almost all } x \in \Omega.
\]

The class of weak \( p \)-quasiconformal mappings is a natural generalization of quasiconformal mappings and for \( p = n \) these classes coincide.

Note that planar conformal mapping preserve the Dirichlet energy integral

\[
\int_\Omega |\nabla f(x)|^2 dx
\]

because \( |D\varphi(x)|^2 = |J(x, \varphi)| \) in \( \Omega \).

In the space \( \mathbb{R}^3 \) conformal mappings are not connected to the Dirichlet energy integral, but the class of weak 2-conformal mappings (co-conformal mappings [22]) generates bounded composition operators on the Dirichlet spaces. It is a reason for detailed study of weak 2-quasiconformal mappings that looks as an appropriate class for possible applications to elliptic equations as was mentioned in [22].

The case \( p \neq q \) is more complicated and in this case the composition operators theory is based on the countable-additive set functions, connected with norms of
composition operators [44] (see also [54]). The main result of [44] gives an analytic
description of composition operators on Sobolev spaces (see, also [54]) and asserts
that

**Theorem 2.3** [44] A homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) between two domains \( \Omega \) and \( \tilde{\Omega} \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q < p < \infty,
\]

if and only if \( \varphi \in W^1_{1,\text{loc}}(\Omega) \), has finite distortion, and

\[
K_{p,q}(\varphi; \Omega) = \left( \int_{\Omega} \left( \frac{|D\varphi(x)|^p}{|J(x, \varphi)|} \right)^{\frac{q}{p-q}} \, dx \right)^{\frac{p-q}{pq}} < \infty.
\]

Let us remark one more time that in the framework of two previous theorems function
of spaces \( L^1_p(\tilde{\Omega}) \) are quasicontinuous.

By an analogy with a notion of weak \( p \)-quasiconformal case homeomorphisms
which satisfy conditions of Theorem 2.3 are called (weak) \((p, q)\)-quasiconformal
mappings [53] or mappings of bounded \((p, q)\)-distortion [45]. A geometric description
(similar to (2.2)) of weak \((p, q)\)-quasiconformal mappings in the case \( q > n - 1 \) was
obtained in [53].

The concept of quasicontinuity wasn’t used in [27], where by an analogy with
Lebesgue spaces, Sobolev functions were defined up to set a measure zero (not up
to sets of \( p \)-capacity zero). In [27] it was stated that \((p, q)\)-quasiconformal mappings generates a bounded composition operators on Sobolev spaces under additional
assumptions continuity of Sobolev functions of \( L^1_p(\tilde{\Omega}) \), \( p > n \). Let us recall that by
the classical Sobolev embedding theorems (see, for example, [36,42]) any function
\( f \in W^1_p(\tilde{\Omega}) \), \( p > n \), has an unique continuous representation [36].

Composition operators on Sobolev spaces have applications to spectral problems
of elliptic equations. These applications are based on the Sobolev type embedding
theorems [12,18]. The following diagram illustrate the main idea of these applications:

\[
\begin{array}{ccc}
W^1_p(\tilde{\Omega}) & \xrightarrow{\varphi^*} & W^1_q(\Omega) \\
\downarrow & & \downarrow \\
L^s(\tilde{\Omega}) & \xrightarrow{(\varphi^{-1})^*} & L^r(\Omega)
\end{array}
\]

Here the operator \( \varphi^* \) defined by the composition rule \( \varphi^*(f) = f \circ \varphi \) is a bounded
composition operator on Sobolev spaces induced by a homeomorphism \( \varphi \) of a “good”
domain \( \Omega \) (for example the unit ball) to a “bad” domain \( \tilde{\Omega} \) and the operator \((\varphi^{-1})^*\)
defined by the composition rule \( (\varphi^{-1})^*(f) = f \circ \varphi^{-1} \) is a bounded composition
operator on Lebesgue spaces induced by the inverse homeomorphism.

This approach allows us to obtain stability estimates of constants in Sobolev–
Poincaré inequalities, that in its turn give us an opportunity to estimate stability of
Neumann eigenvalues of the \( p \)-Laplace operators.
In our recent works [6,7,14,15,20,21] the spectral stability problem and the lower estimates of Neumann eigenvalues in planar domains were studied. In space domains our results are more modest. Some spectral estimates in space non convex domains were obtained [11,22] with the help of weak $p$-quasiconformal mappings theory.

The geometric theory of composition operators on Sobolev spaces is closely connected with the $A_{p,q}$-Ball’s classes [3,4] and has applications in non-linear elasticity (see, for example, [9,19,49]).

In the last decade the composition operators theory on some generalizations of Sobolev spaces, like Besov spaces and Triebel–Lizorkin spaces, [25,28–30,38] was under consideration. These types of composition operators have applications to the Calderón inverse conductivity problem [8].

2.3 Composition Operators and Capacitary Inequalities

Composition operators on Sobolev spaces allow an alternative capacitory description. Recall the notion of a variational $p$-capacity [16].

A condenser in the domain $\Omega \subset \mathbb{R}^n$ is the pair $(F_0, F_1)$ of connected closed relatively to $\Omega$ sets $F_0, F_1 \subset \Omega$. A continuous function $f \in L^1_p(\Omega)$ is called an admissible function for the condenser $(F_0, F_1)$, if the set $F_i \cap \Omega$ is contained in some connected component of the set $\text{Int}\{x | u(x) = i\}$, $i = 0, 1$. We call as the $p$-capacity of the condenser $(F_0, F_1)$ (relatively to the domain) $\Omega$ the following quantity:

$$\text{cap}_p(F_0, F_1; \Omega) = \inf \left\| f \right\|_{L^1_p(\Omega)}^p.$$

Here the greatest lower bound is taken over all functions admissible for the condenser $(F_0, F_1) \subset \Omega$. If the condenser has no admissible functions we put the capacity equal to infinity.

The following theorems give a capacitary description of the composition operators on Sobolev spaces.

**Theorem 2.4** [13] Let $1 < p < \infty$. A homeomorphism $\varphi : \Omega \rightarrow \tilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L^1_p(\tilde{\Omega}) \rightarrow L^1_p(\Omega)$$

if and only if for every condenser $(F_0, F_1) \subset \tilde{\Omega}$ the inequality

$$\text{cap}_p^{1/p}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leq K \text{cap}_p^{1/p}(F_0, F_1; \tilde{\Omega})$$

holds.

**Theorem 2.5** [44,53] Let $1 < q < p < \infty$. A homeomorphism $\varphi : \Omega \rightarrow \tilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L^1_p(\tilde{\Omega}) \rightarrow L^1_q(\Omega)$$
if and only if there exists a bounded monotone countable-additive set function $\Phi$ defined on open subsets of $\tilde{\Omega}$ such that for every condenser $(F_0, F_1) \subset \tilde{\Omega}$ the inequality

$$
cap^{1/q}_q (\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leq \Phi(\tilde{\Omega} \setminus (F_0 \cup F_1))^{\frac{p-q}{pq}} \cap^{1/p}_p (F_0, F_1; \tilde{\Omega})
$$

holds.

These capacitary inequalities demonstrate a close connection of mappings that generate bounded composition operators on Sobolev spaces and so-called $Q$-homeomorphisms [34]. Descriptions of $Q$-homeomorphisms are based on the capacitary (moduli) distortion property are these classes were intensively studied at last decades (see, for example, [31,40,41]).

3 Spectral Estimates of the $p$-Laplace Operator

In this section we give spectral estimates of the $p$-Laplace operator on the base Sobolev–Poincaré inequalities.

3.1 The General Case

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is called an $(r, q)$-Sobolev–Poincaré domain, $1 \leq r, q \leq \infty$, if for any function $f \in L^1_q(\Omega)$, the $(r, q)$-Sobolev–Poincaré inequality

$$
\inf_{c \in \mathbb{R}} \| f - c \|_{L^r(\Omega)} \leq B_{r,q}(\Omega) \| \nabla f \|_{L^q(\Omega)}
$$

holds.

Weak $(p, q)$-quasiconformal mappings permits us to “transfer” this embedding property from one domain to another.

Theorem 3.1 Let a bounded domain $\Omega \subset \mathbb{R}^n$ be a $(r, q)$-Sobolev–Poincaré domain, $1 < q \leq r < \infty$, and there exists a weak $(p, q)$-quasiconformal mapping $\varphi : \Omega \to \tilde{\Omega}$ of a domain $\Omega$ onto a bounded domain $\tilde{\Omega}$, possesses the Luzin N-property and such that

$$
M_r(\Omega) = \text{ess sup}_{x \in \Omega} | J(x, \varphi) |^{\frac{1}{r}} < \infty.
$$

Then in the domain $\tilde{\Omega}$ the $(r, p)$-Sobolev–Poincaré inequality

$$
\inf_{c \in \mathbb{R}} \left( \int_{\tilde{\Omega}} \left| f(x) - c \right|^r dx \right)^{\frac{1}{r}} \leq B_{r,p}(\tilde{\Omega}) \left( \int_{\tilde{\Omega}} | \nabla f(x) |^p dx \right)^{\frac{1}{p}}, \quad f \in W^1_p(\tilde{\Omega}), \quad (3.1)
$$

holds and

$$
B_{r,p}(\tilde{\Omega}) \leq K_{p,q}(\Omega) M_r(\Omega) B_{r,q}(\Omega).
$$
Here $B_{r,q}(\Omega)$ is the best constant in the $(r,q)$-Sobolev–Poincaré inequality in the domain $\Omega$.

**Proof** Let $f \in L^1_p(\Omega)$. By the conditions of the theorem there exists a weak $(p,q)$-quasiconformal homeomorphism $\varphi : \Omega \to \tilde{\Omega}$. Hence, the composition operator

$$ \varphi^* : L^1_p(\Omega) \to L^1_q(\Omega) $$

is bounded. Because $\Omega$ is a bounded $(r,q)$-Sobolev–Poincaré domain $g = \varphi^*(f) \in W^1_q(\Omega)$.

Using the change of variable formula we obtain:

$$ \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r \, dy \right)^{\frac{1}{r}} = \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(\varphi(x)) - c|^r |J(x, \varphi)| \, dx \right)^{\frac{1}{r}} $$

$$ \leq \operatorname{ess sup}_{x \in \Omega} |J(x, \varphi)|^{\frac{1}{r}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(\varphi(x)) - c|^r \, dx \right)^{\frac{1}{r}} $$

$$ = M_r(\Omega) \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r \, dx \right)^{\frac{1}{r}}. $$

Because the domain $\Omega$ is a $(r,q)$-Sobolev–Poincaré domain we have

$$ \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r \, dx \right)^{\frac{1}{r}} \leq B_{r,q}(\Omega) \left( \int_{\Omega} |\nabla g(x)|^q \, dx \right)^{\frac{1}{q}}. $$

Hence

$$ \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r \, dy \right)^{\frac{1}{r}} \leq M_r(\Omega) B_{r,q}(\Omega) \|g| L^1_q(\Omega)\|. $$

By Theorem 2.3

$$ \|g| L^1_q(\Omega)\| \leq K_{p,q}(\Omega) \|f| L^1_p(\tilde{\Omega})\|. $$

Therefore

$$ \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r \, dy \right)^{\frac{1}{r}} \leq K_{p,q}(\Omega) M_r(\Omega) B_{r,q}(\Omega) \left( \int_{\Omega} |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}. $$

$\square$

The Theorem 3.1 immediately implies the following lower estimate for $\mu_p(\tilde{\Omega})$: 
Theorem 3.2 Suppose that there exists a weak \((p,q)\)-quasiconformal homeomorphism \(\varphi: \Omega \rightarrow \tilde{\Omega}\), of a \((r,q)\)-Sobolev–Poincaré domain \(\Omega \subset \mathbb{R}^n\) onto \(\tilde{\Omega}\), possesses the Luzin N-property and such that
\[ M_p(\Omega) = \text{ess sup}_{x \in \Omega} |J(x, \varphi)|^{\frac{1}{p}} < \infty. \]

Then
\[ \mu_p(\tilde{\Omega}) \geq \left( K_{p,q}(\Omega) M_p^p(\Omega) B_{p,q}(\Omega) \right)^{-1}. \]

Boundedness of a Jacobian of a weak \((p,q)\)-quasiconformal mapping is a sufficient but restrictive assumption. If \(\varphi\) is a Lipschitz mapping, this condition holds but an image of a Lipschitz domain can not be a domain with external singularities.

We shall use a more flexible class of weak \((p,q)\)-quasiconformal mappings with an integrable Jacobian, which allows us to map Lipschitz domains onto cusp domains.

Theorem 3.3 Let a bounded domain \(\Omega \subset \mathbb{R}^n\) be a \((r,q)\)-Sobolev–Poincaré domain, \(1 < q \leq r < \infty\), and there exists a weak \((p,q)\)-quasiconformal homeomorphism \(\varphi: \Omega \rightarrow \tilde{\Omega}\) of a domain \(\Omega\) onto a bounded domain \(\tilde{\Omega}\), possesses the Luzin N-property and such that
\[ M_{r,s}(\Omega) = \left( \int_{\Omega} |J(x, \varphi)|^{\frac{r}{r-s}} \ dx \right)^{\frac{r-s}{r}} < \infty \]
for some \(1 \leq s < r\). Then in the domain \(\tilde{\Omega}\) the \((s,p)\)-Sobolev–Poincaré inequality
\[ \left( \int_{\tilde{\Omega}} |f(x) - f_{\tilde{\Omega}}|^s \ dx \right)^{\frac{1}{s}} \leq B_{s,p}(\tilde{\Omega}) \left( \int_{\Omega} |\nabla f(x)|^p \ dx \right)^{\frac{1}{p}}, \quad f \in W^1_p(\tilde{\Omega}), \quad (3.2) \]
holds and
\[ B_{s,p}(\tilde{\Omega}) \leq K_{p,q}(\Omega) M_{r,s}(\Omega) B_{r,q}(\Omega). \]

Here \(B_{r,q}(\Omega)\) is the best constant in the \((r,q)\)-Sobolev–Poincaré inequality in the domain \(\Omega\).

Proof Let \(f \in L^1_p(\tilde{\Omega})\). By the conditions of the theorem there exists a \((p,q)\)-quasiconformal homeomorphism \(\varphi: \Omega \rightarrow \tilde{\Omega}\). By Theorem 2.3 the composition operator
\[ \varphi^*: L^1_p(\tilde{\Omega}) \rightarrow L^1_q(\Omega) \]
is bounded. Because the bounded domain \(\Omega\) is a \((r,q)\)-Sobolev–Poincaré domain then \(g = \varphi^*(f) \in W^1_q(\Omega)\).
Using the change of variable formula and the Hölder inequality we obtain:

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^s \, dy \right)^{\frac{1}{s}} = \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(\varphi(x)) - c|^s |J(x, \varphi)| \, dx \right)^{\frac{1}{s}} \\
\leq \left( \int_{\Omega} |J(x, \varphi)|^{\frac{r}{r-1}} \, dx \right)^{\frac{r-1}{r}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(\varphi(x)) - c|^r \, dx \right)^{\frac{1}{r}} \\
= M_{r,s}(\Omega) \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r \, dx \right)^{\frac{1}{r}}.
\]

Because the domain \( \Omega \) is a \((r, q)\)-Sobolev–Poincaré domain the following inequality holds:

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r \, dx \right)^{\frac{1}{r}} \leq B_{r,q}(\Omega) \left( \int_{\Omega} |\nabla g(x)|^q \, dx \right)^{\frac{1}{q}}.
\]

Combining two previous inequalities we have

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^s \, dy \right)^{\frac{1}{s}} \leq M_{r,s}(\Omega) B_{r,q}(\Omega) \| g \|_{L^1_q(\Omega)}.
\]

By Theorem 2.3

\[
\| g \|_{L^1_q(\Omega)} \leq K_{p,q}(\Omega) \| f \|_{L^1_p(\tilde{\Omega})}.
\]

Finally we obtain

\[
\inf_{c \in \mathbb{R}} \left( \int_{\tilde{\Omega}} |f(y) - c|^s \, dy \right)^{\frac{1}{s}} \leq K_{p,q}(\Omega) M_{r,s}(\Omega) B_{r,q}(\Omega) \left( \int_{\tilde{\Omega}} |\nabla f|^p \, dy \right)^{\frac{1}{p}}.
\]

It means that

\[
B_{s,p}(\tilde{\Omega}) \leq K_{p,q}(\Omega) M_{r,s}(\Omega) B_{r,q}(\Omega).
\]

\[\square\]

We are ready to establish the main lower estimate:

**Theorem 3.4** Let a domain \( \Omega \subset \mathbb{R}^n \) be a \((r, q)\)-Sobolev–Poincaré domain, \( 1 < q < p < r \), and there exists a weak \((p, q)\)-quasiconformal homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \).
of a domain \( \Omega \) onto a bounded domain \( \tilde{\Omega} \), possesses the Luzin N-property and such that

\[
M_{r,p} (\Omega) = \left( \int_\Omega |J(x, \varphi)|^{r/p} \, dx \right)^{r/p} < \infty.
\]

Then

\[
\mu_p (\tilde{\Omega}) \geq (K_{p,q} (\Omega) M_{r,p} (\Omega) B_{r,q} (\Omega))^{-1}.
\]

This theorem follows from Theorem 3.3 and gives the lower estimate of the first non-trivial Neumann eigenvalue of the \( p \)-Laplace operator in the terms of \((p, q)\)-quasiconformal geometry of domains.

### 3.2 The Anisotropic Hölder Singularities

Define domains \( H_g \) with anisotropic Hölder \( \gamma \)-singularities (introduced in [12]):

\[
H_g = \{ x \in \mathbb{R}^n : 0 < x_n < 1, 0 < x_i < g_i(x_n), i = 1, 2, \ldots, n-1 \}.
\]

Here \( g_i(\tau) = \tau^{\gamma_i}, \gamma_i \geq 1, 0 \leq \tau \leq 1 \) are Hölder functions and for the function \( G = \prod_{i=1}^{n-1} g_i \) denote by

\[
\gamma = \frac{\log G(\tau)}{\log \tau} + 1.
\]

It is evident that \( \gamma \geq n \). In the case \( g_1 = g_2 = \cdots = g_{n-1} \) we will say that domain \( H_g \) is a domain with \( \sigma \)-Hölder singularity, \( \sigma = (\gamma - 1)/(n - 1) \). For \( g_1(\tau) = g_2(\tau) = \cdots = g_{n-1}(\tau) = \tau \) we will use notation \( H_1 \) instead of \( H_g \).

Define the mapping \( \varphi_a : H_1 \to H_g, a > 0 \), by

\[
\varphi_a(x) = \left( \frac{x_1}{x_n} g_1^a(x_n), \ldots, \frac{x_{n-1}}{x_n} g_{n-1}^a(x_n), x_n^a \right).
\]

**Theorem 3.5** Let \((n - p)/(\gamma - p) < a < p(n - q)/q(\gamma - p)\). Then the mapping \( \varphi_a : H_1 \to H_g \), be a weak \((p, q)\)-quasiconformal mapping, \( 1 < q < p < \gamma \), from the Lipschitz convex domain \( H_1 \) onto the “cusp” domain \( H_g \) with

\[
K_{p,q} (H_1) \leq a^{-\frac{1}{p}} \left[ a^2 (\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i \right].
\]

**Proof** By simple calculations

\[
\frac{\partial (\varphi_a)}{\partial x_i} = g_i^a(x_n), \quad \frac{\partial (\varphi_a)}{\partial x_n} = \frac{-x_i g_i^a(x_n)}{x_n^2} + \frac{a x_i g_i^{a-1}(x_n)}{x_n} g_i'(x_n) \quad \text{and} \quad \frac{\partial (\varphi_a)}{\partial x_n} = a x_n^{a-1}
\]
for any \( i = 1, \ldots, n - 1 \). Hence \( J(x, \varphi_a) = ax_n^{a_n}G(x_n) = ax_n^{a_n}G(x_n) \leq a \) for \( a > 1 \) and

\[
D\varphi_a(x) = \begin{pmatrix}
x_n^{a_n} & 0 & \cdots & x_n^{a_n-1} \cdot (a_n - 1)x_n^{a_n-2} \\
0 & x_n^{a_n-1} & \cdots & x_n^{a_n-2} \cdot (a_n - 1)x_n^{a_n-3} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_n^{a_n-1}
\end{pmatrix}
\]

\[
= x_n^{a_n-1} \begin{pmatrix}
x_n^{a_n-a} & 0 & \cdots & x_n^{a_n-a} \cdot (a_n - 1)x_n^{a_n-a-1} \\
0 & x_n^{a_n-a} & \cdots & x_n^{a_n-a} \cdot (a_n - 1)x_n^{a_n-a-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{pmatrix}.
\] (3.3)

Because \( 0 < x_n < 1 \) and \( x_1/x_n < 1 \) we have the following estimate

\[
|D\varphi_a(x)| \leq x_n^{a_n-1} \sqrt{\sum_{i=1}^{n-1} (a_n - 1)^2 + n - 1 + a^2}
\]

\[
= x_n^{a_n-1} \sqrt{a^2 (\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i}.
\]

Then

\[
K_{p,q}(H_1) = \left( \int_{H_1} \left( \frac{|D\varphi_a(x)|^p}{J(x, \varphi_a)} \right)^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{pq}}
\]

\[
\leq \sqrt{a^2 (\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i} \left( \int_{H_1} \frac{(p(a_n-1) - (a_n - n))q}{p-q} dx \right)^{\frac{p-q}{pq}}
\]

\[
= \sqrt{a^2 (\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i} \left( \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} x_n^{(p(a_n-1) - (a_n - n))q/p-q} dx_1 \cdots dx_n \right)^{\frac{p-q}{pq}}
\]

\[
= \sqrt{a^2 (\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i} \times \left( \int_0^1 x_n^{(p(a_n-1) - (a_n - n))q/p-q} + n-1 dx_n \right)^{\frac{p-q}{pq}}
\]
\[
\frac{1}{\mu_p(H_g)} \leq \inf_{a \in \Lambda_a} \left( a^2(\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i \right)^{\frac{p}{2}} \leq B^p_{r,q}(H_1),
\]

if \((p + a\gamma - pa)q < np\) or that equivalent \(a < p(n - q)/q(\gamma - p)\).

Now we check that \(1 < q < np/(p + a\gamma - pa) < p\). The inequality \(1 < np/(p + a\gamma - pa)\) implies \(a < (np - p)/\gamma - p\), but

\[
\frac{np - p}{\gamma - p} < \frac{p(n - q)}{q(\gamma - p)}, \text{ if } q > 1.
\]

The inequality \(np/(p + a\gamma - pa) < p\) implies \(a > (n - p)/(\gamma - p)\). So, we have that \(a \in ((n - p)/(\gamma - p), p(n - q)/q(\gamma - p))\). \(\square\)

We are ready to prove spectral estimates in cusp domains.

**Theorem 3.6** Let

\[
H_g := \{ x \in \mathbb{R}^n : n \geq 3, 0 < x_n < 1, 0 < x_i < x_n^{\gamma_i}, i = 1, 2, \ldots, n - 1 \}
\]

\(\gamma_i \geq 1, \gamma := 1 + \sum_{i=1}^{n-1} \gamma_i, g := (\gamma_1, \ldots, \gamma_{n-1})\) be domains with anisotropic Hölder \(\gamma\)-singularities.

Then for \(1 < p < \gamma\)

\[
\frac{1}{\mu_p(H_g)} \leq \inf_{a \in \Lambda_a} \left( a^2(\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i \right)^{\frac{p}{2}} \leq B^p_{r,q}(H_1),
\]

where \(I_a = (\max\{(n - p)/(\gamma - p), p(n - q)/\gamma q\}, p(n - q)/q(\gamma - p))\) and \(B^p_{r,q}(H_1)\) is the best constant in the \((r, q)\)-Sobolev–Poincaré inequality in the domain \(H_1, q \leq r < \frac{nq}{n-q}\).

**Proof** By Theorem 3.5 the mapping \(\varphi_a : H_1 \to H_g, (n - p)/(\gamma - p) < a < p(n - q)/q(\gamma - p)\),

\[
\varphi_a(x) = \left( \frac{x_1}{x_n}, \frac{x_2}{g^a_1(x_n)}, \ldots, \frac{x_{n-1}}{g^a_{n-1}(x_n)}, x_n^{a} \right),
\]

maps the convex Lipschitz domain \(H_1\) onto the cusp domain \(H_g\) and it is a weak \((p, q)\)-quasiconformal mapping, \(1 < q < p < \gamma\).

Let us check conditions of Theorem 3.4. Because \(\varphi\) is a weak \((p, q)\)-quasiconformal mapping then \(K_{p,q}(H_1)\) is finite. The basic domain \(H_1\) is a Lipschitz domains and so is a \((r, q)\)-Sobolev–Poincaré domain, i.e. \(B^p_{r,q}(H_1) < \infty\).

Now we estimate the constant \(M_{r,p}(H_1)\):

\[
M_{r,p}(H_1) = \left( \int_{H_1} |J(x, \varphi_a)|^{\frac{r}{r-p}} \, dx \right)^{\frac{r-p}{r}} = a^\frac{1}{p} \left( \int_{H_1} \left( x_n^{a\gamma-n} \right)^{\frac{r}{r-p}} \, dx \right)^{\frac{r-p}{r}}
\]
\[ a^{\frac{1}{p}} \left( \int_0^1 (x_n^{a\gamma - n}) \left( \int_0^1 dx_1 \cdots \int_0^1 dx_{n-1} \right) dx_n \right)^{\frac{r-p}{rp}} \]

\[ = a^{\frac{1}{p}} \left( \int_0^1 (x_n^{a\gamma - n}) \left( \int_0^1 \cdots \int_0^1 dx_n \right)^{\frac{r-p}{rp}} \right) < \infty, \]

if

\[ \frac{(a\gamma - n)r}{r - p} + n - 1 > -1, \text{ i.e. } a > \frac{np}{\gamma r}. \]

Because \( 0 < x_n < 1 \), we have \( M_{r,p}(H_1) \leq a^{\frac{1}{p}} \) if \( a > np/\gamma r \). If we take \( r < \frac{nq}{n-q} \) we obtain that

\[ M_{\frac{nq}{n-q},p}(H_1) \leq a^{\frac{1}{p}}, \text{ if } a > \frac{p(n-q)}{\gamma q}. \]

The conditions of Theorem 3.4 is fulfilled. Therefore

\[
\frac{1}{\mu_p(H_g)} \leq K_{p,q}^r(H_1)M_{\frac{nq}{n-q},p}(H_1)B_{\frac{nq}{n-q},q}(H_1) \\
\leq \left( a^2(\gamma_1^2 + \cdots + \gamma_{n-1}^2 + 1) - 2a \sum_{i=1}^{n-1} \gamma_i \right)^{\frac{q}{r}} B_{r,q}(H_1),
\]

where \( \max\{(n-p)/(\gamma - p), p(n-q)/\gamma q\} < a < p(n-q)/q(\gamma - p) \) and \( B_{r,q}(H_1) \) is the best constant in the \((r, q)\)-Sobolev–Poincaré inequality in the domain \( H_1 \) for some \( q \in [r, \frac{nq}{n-q}) \).

Note that in [22] we proved the estimate of the Poincaré constant in the \((r, q)\)-Sobolev–Poincaré inequality in the domain \( H_1 \):

\[
B_{r,q}(H_1) \leq n \left( \frac{1 - \delta}{1/n - \delta} \right)^{1-\delta} \omega_1^{1-\frac{1}{n}} \left( \frac{1}{(n+1)!} \right)^{\frac{1}{n-\delta}}, \quad \delta = \frac{1}{q} - \frac{1}{r} \geq 0. \quad (3.4)
\]

The problem of exact values of constants in the \((r, q)\)-Sobolev–Poincaré inequalities in the case \( p \neq r \) is a complicated open problem even in the case of the unit disc \( \mathbb{D} \subset \mathbb{R}^2 \).

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