HAHN ANALYTIFICATION AND CONNECTIVITY
OF HIGHER RANK TROPICAL VARIETIES

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ABSTRACT. We show that the tropicalization of a connected variety over a higher rank valued field is a path connected topological space. This establishes an affirmative answer to a question posed by Banerjee. Higher rank tropical varieties are studied as the images of “Hahn analytifications”, introduced in this paper. A Hahn analytification is a space of valuations on a scheme over a higher rank valued field. We prove that the Hahn analytification is related to higher rank tropicalization by means of an inverse limit theorem, extending well-known results in the non-Archimedean case. We also establish comparison results between the Hahn analytification and the Huber and Berkovich analytifications, as well as the Hrushovski-Loeser stable completion.

1. INTRODUCTION

In recent years, numerous authors have studied the relationship between tropical and analytic geometry over rank-1 valued fields [2, 4, 5, 8, 23, 27]. The purpose of the present text is to extend this study to varieties over valued fields of finite rank greater than 1. We introduce a new form of tropicalization over higher rank valued fields and prove that the tropicalization of a connected subvariety of a torus gives rise to a path connected topological space. This resolves a question posed by Banerjee in [3]. The proof of connectivity relies on a new theory of analytification over higher rank valued fields, which reduces to Berkovich analytification in rank-1. We describe in detail the relationship between this theory and the Berkovich and Huber approaches to geometry over non-Archimedean fields [4, 19], as well as Hrushovski and Loeser’s theory of stable completions [17].

In a subsequent paper, we will present a systematic study of the combinatorial geometry of skeletons arising from degenerations of varieties over higher rank valuation rings.

1.1. Hahn analytification. Let \( K \) be a field equipped with a valuation

\[ \nu : K^\times \rightarrow \Gamma, \]

where \( \Gamma \) is a totally ordered abelian group. Fix a positive integer \( k \) and denote by \( \mathbb{R}^{(k)} \) the \( k \)-fold product of \( \mathbb{R} \), equipped with the lexicographic ordering. Choose an order preserving homomorphism \( \rho : \Gamma \rightarrow \mathbb{R}^{(k)} \).

Let \( X = \text{Spec} \ A \) be an affine \( K \)-variety. The Hahn analytification of \( X \) is the set of ring valuations

\[ |X^0| := \left\{ A \xrightarrow{\text{val}} \mathbb{R}^{(k)} \sqcup \{\infty\} : \text{val}(z) = \rho \circ \nu(z), \text{ for all } z \in K \right\}. \]

By definition, a ring valuation \( \text{val} : A \rightarrow \mathbb{R}^{(k)} \sqcup \{\infty\} \) is a map satisfying \( \text{val}(0) = \infty \), \( \text{val}(ab) = \text{val}(a) + \text{val}(b) \), and \( \text{val}(a+b) \geq \min\{\text{val}(a), \text{val}(b)\} \).

We put two distinct topologies on the set \( |X^0| \):

\[ \text{Date: April 28, 2015.} \]
(The Extended Order Topology.) Give $\mathbb{R}^{(k)} \sqcup \{\infty\}$ the extended order topology by declaring that $r < \infty$ for all $r \in \mathbb{R}^{(k)}$. Equip the set $|X^{0}|$ with the weakest topology making the evaluation functions

$$
ev_{f} : |X^{0}| \longrightarrow \mathbb{R}^{(k)} \sqcup \{\infty\}$$

$$\val_{x} \longrightarrow \val_{x}(f)$$

continuous with respect to this extended order topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$, for all $f \in A$. Denote the resulting topological space by $X^{0}$. 

(The Extended Euclidean Topology.) Extend the Euclidean topology on $\mathbb{R}$ to the topology on $\mathbb{R}_{\infty} = \mathbb{R} \sqcup \{\infty\}$ for which the completed rays $(a, \infty)$, $a \in \mathbb{R}$, form a basis of open neighborhoods at $\infty$. The extended Euclidean topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$ is the subspace topology obtained by identifying $\mathbb{R}^{(k)} \sqcup \{\infty\}$ with the subspace $\mathbb{R}^{k} \sqcup \{(\infty, \ldots, \infty)\}$ of $(\mathbb{R}_{\infty})^{k}$. Equip $|X^{0}|$ with the weakest topology making the evaluation functions $\ev_{f}$ continuous with respect to the resulting subspace topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$. Denote the resulting topological space by $X_{#}^{0}$. We think of the point $\infty$ as being a “sharp” corner that partially compactifies $\mathbb{R}^{k}$.

Remark 1.1.1. (Basic topological properties). When $k = 1$, the extended Euclidean topology and extended order topology on $\mathbb{R} \sqcup \{\infty\}$ coincide. For $k \geq 1$, the space $\mathbb{R}^{(k)} \sqcup \{\infty\}$ is Hausdorff and non-compact in both topologies. When $k \geq 2$, the order topology on $\mathbb{R}^{(k)}$ is strictly finer than the Euclidean topology on $\mathbb{R}^{(k)}$, while the extended order and Euclidean topologies on $\mathbb{R}^{(k)} \sqcup \{\infty\}$ are incomparable.

Remark 1.1.2. (Alternative topologies). Note that for $k \geq 2$, there are other ways in which one might extend the Euclidean topology on $\mathbb{R}^{(k)}$ to a topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$. For instance, if one gives $\infty$ an open neighborhood basis consisting of sets of the form $U_{a} := \{\infty\} \cup \{(r_{1}, \ldots, r_{k}) \in \mathbb{R}^{k} : r_{1} > a\}$ for all $a \in \mathbb{R}$, then the resulting topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$ is strictly coarser than the extended order topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$. The results of this paper that make use of the extended Euclidean topology are unaffected if we replace the extended Euclidean topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$ by any other topology that is path connected.

1.2. Hahn tropicalization. Let $T$ be a split algebraic torus of dimension $d$ with character lattice $M$, and let $X = \text{Spec } A$ be a closed subvariety of $T$. For each point $x \in X^{0}$, i.e., for each valuation $\val_{x} : A \longrightarrow \mathbb{R}^{(k)} \sqcup \{\infty\}$, consider the composite

$$(1) \quad M \longrightarrow K[M] \longrightarrow A \longrightarrow \mathbb{R}^{(k)} \sqcup \{\infty\}. $$

For characters $\chi^{u}, \chi^{v} \in M$, the valuation axioms imply $\val_{x}(\chi^{u+v}) = \val_{x}(\chi^{u}) + \val_{x}(\chi^{v})$. Thus, (1) can be taken to be a homomorphism of abelian groups, which we denote

$$\text{trop}(x) : M \longrightarrow \mathbb{R}^{(k)}.$$ 

In this way, we obtain a tropicalization map

$$(2) \quad |X^{0}| \longrightarrow \text{Hom}_{Z}(M, \mathbb{R}^{(k)}).$$

The Hahn tropicalization of $X$, denoted $|\text{trop}(X)|$, is the image of $X^{0}$ under (2). When no confusion can arise, we simply refer to $|\text{trop}(X)|$ as the tropicalization of $X$. The order and Euclidean topologies on $\mathbb{R}^{(k)}$ determine two distinct topologies on the set $\text{Hom}_{Z}(M, \mathbb{R}^{(k)})$. We let $\text{trop}(X)$ denote the Hahn tropicalization with the subspace topology for the order topology on $\mathbb{R}^{(k)}$, and we let $\text{trop}_{#}(X)$ denote the Hahn tropicalization with the subspace topology for the Euclidean topology. Observe that the topological space $\text{trop}(X)$ (resp. $\text{trop}_{#}(X)$) is the continuous image of $X^{0}$ (resp. $X_{#}^{0}$) under the map trop.
Remark 1.2.1. When $k = 1$, $\text{trop}_q(X)$ and $\text{trop}(X)$ coincide with the image of the Berkovich analytification of $X$ under the standard tropicalization map. Aroca [1] previously studied tropicalizations of hypersurfaces over higher rank valued fields by extending the theory of Newton polygons. When $K$ is a field with value group equal to $\mathbb{R}^{(k)}$, his definition coincides with ours. Note that Aroca’s tropicalizations do not carry a topology. The higher rank tropicalization studied by Banerjee [3] is the closure of $\text{trop}_q(X)$ in the Euclidean topology. We discuss the relationship with Banerjee’s work in more detail in Section 2.3.

1.3. Main results. If $X$ is a connected, closed subvariety of a torus over a rank-1 non-Archimedean field, then the usual tropicalization of $X$ is connected. This result was first proved by Einsiedler, Kapranov, and Lind [8] using rigid analytic techniques, but it can also be obtained as an elementary consequence of connectivity of the Berkovich analytification $X^an$. Our main results in the present text are extensions of this connectivity result to the higher rank setting.

Theorem A. Let $X$ be a connected subvariety of an algebraic torus over $K$. Then $\text{trop}_q(X)$ is a path connected topological space.

From Theorem A together with the results of Section 2.3 we deduce connectivity for Banerjee’s tropicalization.

Theorem B. Let $X$ be a connected subvariety of an algebraic torus over $K$. Then $\text{trop}(X)$ is a definable and definably path connected space.

The connectivity results of Theorems A and B are consequences of basic properties of the Hahn analytification itself.

Theorem C. If $X$ is a geometrically connected $K$-variety, then $X^\diamond$ is path connected. Furthermore, for any $\mathbb{R}^{(k)}$-valued field $F$ extending $K$, each pair of $F$-rational points $x$ and $y$ in $X_F := X \times_K F$ are connected by a definable path in $X^\diamond_F$.

If $K$ is algebraically closed, and if $X$ is a $\mathbb{K}$-variety that can be realized as a closed subvariety of a toric $\mathbb{K}$-variety, then $X^\diamond$ is a prodefinable set.

The Euclidean connectivity, definable connectivity, and prodefinability results in Theorem C are restated and proved in Theorems 2.2.1, 2.5.7, and 4.2.2 respectively.

Remark 1.3.1. Definability and prodefinability are basic concepts in logic and the theory of o-minimal structures [29]. The relevant ideas are reviewed in Section 2.4. One can understand the definability of $\text{trop}(X)$ as a reflection of the fact that $\text{trop}(X)$ is a finite union of polyhedra in $\left(\mathbb{R}^{(k)}\right)^d$. The definable path connectivity of $\text{trop}(X)$ is the statement that any two points in $\text{trop}(X)$ may be connected by an interval in the ordered abelian group $\mathbb{R}^{(k)}$. Loosely speaking, this interval is embedded in $\left(\mathbb{R}^{(k)}\right)^d$ as a rational 1-dimensional polyhedral complex. Note that the naive connectivity statement for the order topology is false, as $\mathbb{R}^{(k)}$ is disconnected for $k \geq 2$.

In Section 3 we discuss the relationship between the Hahn analytification and the Huber adic space, and the stable completion appearing in recent work of Hrushovski and Loeser. In Section 4, we prove that the Hahn analytification and tropicalizations are related by an inverse limit theorem (Theorem 4.2.1) in the spirit of [11, 27]. The prodefinability of $X^\diamond$ is an immediate consequence.
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2. HAHN ANALYTIFICATION AND CONNECTIVITY THEOREMS

Throughout the present section, fix the following notation. Let $K$ be a field equipped with a valuation $\nu : K^\times \to \Gamma$. Fix an order preserving homomorphism $\rho : \Gamma \to \mathbb{R}(k)$. We refer to the data of $K$ together with the maps $\nu$ and $\rho$ as a Hahn valued field. Let $R$ denote the valuation ring associated to the valuation $\rho \circ \nu$. A Hahn field extension of the triple $(K, \nu, \rho)$ is a field extension $L$ of $K$ equipped with a valuation $\nu_L : L^\times \to \mathbb{R}(k)$ such that $\nu_L(z) = \rho \circ \nu(z)$ for all $z \in K$.

Remark 2.0.2. It is often desirable to choose $\rho$ to be an embedding into $\mathbb{R}(k)$. Such an embedding always exists by a theorem of Hahn [15]. If the rank$^1$ of $\Gamma$ is finite, $k$ can be taken to be equal to the rank of $\Gamma$. Nonetheless, it is convenient to allow $\rho$ to not be non-injective in general.

Let $X$ be a separated finite type $K$-scheme. Consider pairs consisting of a Hahn valued field extension $L$ of $K$ and a point $x \in X(L)$. There is an equivalence relation generated by declaring that $(L, x) \sim (L', x')$ whenever there is an embedding $L \hookrightarrow L'$, such that $\nu_L$ restricts to $\nu_{L'}$ on the subfield $L$, and that $x \mapsto x'$ under the induced inclusion of point sets $X(L) \hookrightarrow X(L')$. Set theoretically, we define the Hahn analytification $|X^0|$ to be the resulting collection of equivalence classes of points of $X$ over valued extensions of $K$. That is, $|X^0| := \{(L, x)\}/\sim$.

Suppose $X = \text{Spec}(A)$. Given a valuation $\text{val} : A \to \mathbb{R}(k) \sqcup \{\infty\}$, the kernel $\text{val}^{-1}(\infty)$ is a prime ideal. In turn this gives rise to a Hahn valuation fraction field $\text{Frac}(A/p)$. Thus, the above definition of analytification is equivalent for affine schemes to the one given in Section 1.1. As $X$ is covered by affine opens, $|X^0|$ is the union of the Hahn analytifications of these affine opens. As in the Berkovich setting [4, Section 3.4], the topologies on these affines agree on their overlaps and determine a global topology on the set $|X^0|$. We let $X^0$ denote $|X^0|$ with its order topology, and we let $X^0_\#$ denote $|X^0|$ with its Euclidean topology.

The construction of $X^0$ and $X^0_\#$ is covariantly functorial. A morphism $f : X \to Y$ of finite-type $K$-schemes induces a natural a map $f^0 : |X^0| \to |Y^0|$ that is continuous in both the order and Euclidean topologies.

$^1$Recall that the rank of a totally ordered abelian group is the number of proper convex subgroups contained in it.
Given a point \( x \in X(K) \), the composition of evaluation at \( x \) with the valuation on \( K \) defines a valuation on any affine open neighborhood of \( x \). In this way, \( X(K) \) becomes a subset of \( |X^0| \).

If \( K \) is Henselian [9, Chapter 4], then the valuation on \( K \) extends uniquely to the algebraic closure, giving an inclusion of the set of closed scheme theoretic points of \( X \) into \( |X^0| \).

### 2.1. Tower of projections and relation to Berkovich analytification

For any \( 0 \leq j \leq k \), there exists a unique continuous order-preserving projection \( \pi^k_j : \mathbb{R}^{(k)} \to \mathbb{R}^{(j)} \), namely projection to the first \( j \) factors. Composing the map \( \rho \) with this projection \( \pi^k_j \), we obtain a new Hahn valued field with the same underlying field \( K \). Correspondingly, one obtains a tower of Hahn analytifications with continuous maps between them:

\[
X^0_k \overset{\pi^k_{k-1}}{\longrightarrow} X^0_{k-1} \overset{\cdots}{\longrightarrow} X^0_2 \overset{\pi^k_1}{\longrightarrow} X^0_1 \overset{\pi^k_0}{\longrightarrow} X^0_0.
\]

Here \( X^0_j \) denotes the Hahn analytification of \( X \) with respect to \( K \) with its valuation \( K \times \nu \xrightarrow{\rho} \Gamma \xrightarrow{\pi^k_j} \mathbb{R}^{(j)} \).

**Warning 2.1.1.** Some care is required in order to handle the case of \( j = 0 \). The set \( \{0\} \cup \{\infty\} \) is topologized as a connected doubleton, where the open sets are \( \emptyset \), \( \{0\} \) and \( \{0, \infty\} \). Observe that the projection \( \mathbb{R} \sqcup \{\infty\} \to \{0\} \sqcup \{\infty\} \) taking \( \mathbb{R} \to \{0\} \) is continuous.

We point out two special cases.

**Example 2.1.2.** The space \( X^0_0 \) coincides with the set of scheme theoretic points of \( X \) in the Zariski topology. For \( X \) affine, the map \( X \to X^0_0 \) taking a prime \( p \) to the trivial valuation on \( K[X]/p \) yields a set theoretic bijection \( |X| \cong |X^0| \). To see that \( X^0_0 \) and \( X \) coincide as topological spaces, note that for any regular function \( f \) on an affine scheme, \( \text{ev}^{-1}_f(\infty) \) consists of exactly those prime ideals \( p \) that contain \( f \). In other words, \( \text{ev}^{-1}_f(\infty) = \text{V}(f) \). These generate the closed sets of the Zariski topology on \( X \).

**Example 2.1.3.** The space \( X^0_1 \) is homeomorphic to the Berkovich analytification\(^2\) of \( X \) with respect to the composite rank-1 valuation

\[
\mathbb{K}^X \xrightarrow{\nu} \Gamma \xrightarrow{\rho} \mathbb{R}^{(k)} \xrightarrow{\pi^k_1} \mathbb{R}.
\]

By Example 2.1.2, the map \( X^0_1 \to X^0_0 \) is the continuous map, to the underlying scheme, that realizes the universal property of Berkovich analytification [4, Section 3.5].

### 2.2. Euclidean path connectivity

In this subsection we prove Theorem A. The result is deduced from the following result about the structure of the Hahn analytification.

**Theorem 2.2.1.** If \( X \) is a geometrically connected \( K \)-variety, then the topological space \( X^0_\# \) is path connected.

Our proof of Theorem 2.2.1 requires the following auxiliary construction.

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\(^2\)Note that for \( K \) a non-complete rank-1 field, Berkovich analytification is redefined by Hrushovski and Loeser as a space of types, recovering the usual definition when \( K \) is complete. Employing this definition, we can ignore questions about whether or not \( K \) is complete with the rank-1 valuation described. See [18, Section 5] for a nice discussion.
Construction 2.2.2. Let $G$ be a finite graph with edges $e_1, \ldots, e_r$. Fix an $r$-tuple

$$\ell = (\ell(e_1), \ldots, \ell(e_r)) \in (\mathbb{R}^{(k)}_{\geq 0} \sqcup \{\infty\})^r.$$ 

For each $i$, consider the interval

$$[0, \ell(e_i)] = \{ \gamma \in \mathbb{R}^{(k)} \sqcup \{\infty\} : 0 \leq \gamma \leq \ell(e_i) \}.$$ 

Let $|e_i|_{\#}$ be the interval $[0, \ell(e_i)]$ equipped with the subspace topology for the extended Euclidean topology on $\mathbb{R}^{(k)} \sqcup \{\infty\}$. Choose a bijection between the endpoints of $[0, \ell(e_i)]$ and the vertices of $G$ incident to the edge $e_i$. We define a topological space

$$|G(\ell)|_{\#} := \left( \bigsqcup_i |e_i|_{\#} \right)/\sim,$$

where the relation “$\sim$” identifies the endpoints of $|e_i|_{\#}$ and $|e_j|_{\#}$ whenever the corresponding vertices are identified in the graph $G$.

Each interval in $\mathbb{R}^{(k)} \sqcup \{\infty\}$ is path connected in the extended Euclidean topology. Hence if $G$ is a connected graph, then $|G(\ell)|_{\#}$ is path connected.

Remark 2.2.3. Suppose $\mathcal{C}$ is a proper, regular, marked semistable $\mathbb{R}$-curve with generic fiber $C$ and special fiber $\mathcal{C}_0$. Assume that the fibers of $\mathcal{C}$ have no self intersections, and let $G$ be the marked dual graph of $\mathcal{C}_0$. The graph $G$ has one vertex for each irreducible component of $\mathcal{C}_0$ and each horizontal mark in $\mathcal{C}$, and $G$ has an edge between two vertices whenever the two corresponding components of $\mathcal{C}_0$ share a node, or whenever the horizontal mark corresponding to one of the vertices intersects the component of $\mathcal{C}_0$ corresponding to the other vertex. If $e_i$ is an edge corresponding to a node $p_i$ where two components of $\mathcal{C}_0$ intersect, then the local equation for $\mathcal{C}$ near $p_i$ is given by $x_iy_i = f_i$ for $f_i \in \mathbb{R}$. Set $\ell(e_i) = \nu(f_i)$. If $e_i$ is an edge corresponding to the intersection of a horizontal mark with a component of $\mathcal{C}_0$, then set $\ell(e_i) = \infty$.

As we now show, the dual graph of a model $\mathcal{C}$ above embeds naturally into the Hahn analytification of the generic fiber. The construction is a variant of the standard construction over rank-1 fields, see [2, 25].

Proposition 2.2.4. There is a continuous embedding $|G(\ell)|_{\#} \hookrightarrow C_0^\mathbb{R}$. If $e$ is an edge corresponding to a marked point $p$ of the generic fiber $C$, the infinite point of $|e|_{\#}$ is mapped to the image of $p$ under the inclusion $C(K) \hookrightarrow C_0^\mathbb{R}$.

Proof. Let $\omega \in |G(\ell)|_{\#}$ be a point lying in a subspace $|e|_{\#} \subset |G(\ell)|_{\#}$ that corresponds to a node $p$ between two components of the special fiber $\mathcal{C}_0$. We build a point of $C_0^\mathbb{R}$ as follows. Since the fibers of $\mathcal{C}$ have no self-intersections, Zariski locally near $p$ the $R$-model $\mathcal{C}$ is of the form $\text{Spec } R[x,y]/(xy - f)$ where $\ell(e) = \nu(f)$. For each $g = \sum a_j x^j y^k$, define

$$\text{val}_\omega(g) = \min \left\{ \nu(a_j) + j\omega + k(\ell(e) - \omega) \right\}.$$ 

Analogously to the rank-1 case, $\text{val}_\omega$ defines a valuation on $R[x,y]/(xy - f)$ that extends to the field of rational functions $K(C)$. Similarly, if $e$ corresponds to a node where a marked section of $\mathcal{C}$ intersects $\mathcal{C}_0$, then one may choose local coordinates to describe $\mathcal{C}$ as $\text{Spec } R[x]$, where the marked section is cut out by $x$. Again, given $\omega \in |e|_{\#}$, one may construct a monomial valuation $\text{val}_\omega$ that takes each $g = \sum a_j x^j$ to

$$\text{val}_\omega(g) = \min \left\{ \nu(a_j) + j\omega \right\}.$$ 

The resulting map $\iota : |G(\ell)|_{\#} \rightarrow C_0^\mathbb{R}$ is a continuous inclusion. To obtain the continuous inverse on the image of $\iota$, observe that each edge $|e|_{\#}$ corresponding to a node (resp. a
marked point) of \( \mathcal{K}_0 \) the set \(|e|_\# \) as a subspace of \((\mathbb{R}^{(k)})^2 \) (resp. \( \mathbb{R}^{(k)} \cup \{\infty\} \)). The inverse map to \( \iota \) on the image is given by evaluating a valuation at the coordinates in \( \mathbb{R}[x,y]/(xy - f) \) (resp. \( \mathbb{R}[x] \)). □

Let \( F \) be any Hahn valued extension of \( K \).

**Proposition 2.2.5.** Let \( C \) be a geometrically irreducible \( F \)-curve. Then the space \( C^\delta_\# \) is path connected.

**Proof.** An arbitrary point \( z \) of \( C^\delta_\# \) is represented by a valuation

\[
\text{val}_z : \mathcal{O}_C(U) \rightarrow \mathbb{R}^{(k)} \cup \{\infty\}
\]

for some Zariski open \( U \subset C \). Let \( E \) be the fraction field of \( \mathcal{O}_C(U)/\text{val}_z^{-1}(\infty) \), equipped with the valuation \( \text{val}_z \). By base changing to \( E \), we obtain a continuous map \( C^\delta_\# := (C \times_F E)^\delta \rightarrow C^\delta \) and an \( E \)-valued point \( z' \) in \( C \) mapping to \( z \). Since \( E \) is a valued field extension of \( F \), the \( F \)-rational points of \( C \) include into \( C^\delta_\# \). Thus it suffices to connect the \( E \)-rational points in \( C^\delta_\# \), for an arbitrary \( \mathbb{R}^{(k)} \)-valued field extension \( E \) of \( F \).

Now consider two \( E \)-valued points \( z \) and \( w \) in \( C \). As the normalization map is surjective, we reduce to the case that \( C \) is smooth. Choose a compactification \( \hat{C}_E \) of \( C_E \), and a collection \( \mathcal{P} = \{|p_i|\} \) of distinct points of \( \hat{C}_E \) such that \( x, y \in \mathcal{P} \) and the marked curve \( (\hat{C}, \mathcal{P}) \) is stable. By the valuative criterion for properness of \( \overline{M}_{g,n} \) (possibly after a finite base change) we may take the stable marked model \( \mathcal{E} \) of \( (\hat{C}_E, \mathcal{P}) \) over the valuation ring \( R_E \) of \( E \). After blowup, we may assume that the components of the special fiber of \( \mathcal{E} \) do not have self-intersection. Now Proposition 2.2.4 above furnishes a path connected subspace of \( C^\delta_\# \) that contains the points \( z \) and \( w \) and the result follows. □

**Proof of Theorem 2.2.1.** As in the proof of the proposition above, we may reduce the claim to a proof that if \( E \) is any Hahn valued field extending \( K \), then for any two \( \mathbb{R} \)-rational points \( z, w \in X_E \), there exists a path in \( X^\delta_\# \) connecting \( z \) and \( w \). Furthermore, we may assume that \( E \) is algebraically closed. Then by Bertini’s theorem, as stated in [24, p. 53], the points \( z \) and \( w \) lie on an irreducible \( E \)-curve \( C \) in \( X \). We may apply the above proposition to find a path connecting \( z \) and \( w \). Projecting to \( X^\delta_\# \), the result follows. □

**Proof of Theorem A.** By definition, \( \text{trop}_\#(X) \) comes with a surjective continuous map \( \text{trop}_\# : X^\delta_\# \rightarrow \text{trop}_\#(X) \). Thus path connectivity of \( \text{trop}_\#(X) \) follows immediately from Theorem 2.2.1. □

### 2.3. Connections to Banerjee’s tropicalizations.

Recall [10] that a higher local field \( K \) has value group isomorphic to \( \mathbb{Z}^{(k)} \), and that its algebraic closure \( K^{\text{alg}} \) has value group isomorphic \( \mathbb{Q}^{(k)} \). Given a subvariety \( X \) of a split, \( d \)-dimensional algebraic torus over \( K \), Banerjee [3] defines the tropicalization of \( X \) to be the subset of \( \mathbb{R}^{k \times d} \) obtained by taking Euclidean closure of the image of the coordinatewise valuation map \( X(K^{\text{alg}}) \rightarrow \mathbb{R}^{k \times d} \). In the present section, we refer to the resulting topological space as Banerjee’s tropicalization, and denote it \( \text{trop}_{\text{Ban}}(X) \).

The spaces \( \text{trop}_\#(X) \) and \( \text{trop}_{\text{Ban}}(X) \) do not coincide. The essential reason for this is the following. A **half-space** in \((\mathbb{R}^{(k)})^d\) is any subset

\[
H = \{ \mathbf{r} = (x_1, \ldots, x_d) \in (\mathbb{R}^{(k)})^d : u_1 x_1 + \cdots + u_d x_d + \gamma \geq 0 \},
\]
for \( u = (u_1, \ldots, u_d) \in \mathbb{Z}^d \) and \( \gamma \in \mathbb{R}^k \). A halfspace is closed in the product-order topology on \((\mathbb{R}^k)^d\), but is not closed in the Euclidean topology on \( \mathbb{R}^{k \times d} \). For instance, consider the halfspace

\[
H' = \{ r \in \mathbb{R}^{(2)} : r \geq (0,0) \}.
\]

It coincides with the subset of \( \mathbb{R}^{(2)} \) given by the right-half plane, minus the (open) vertical negative axis, and thus it is not closed in \( \mathbb{R}^2 \) (see Figure 1).

![Figure 1](image)

**FIGURE 1.** The half-space \( H' = \{ r \in \mathbb{R}^{(2)} : r > (0,0) \} \) has a larger closure in the Euclidean topology on \( \mathbb{R}^2 \) than it does in the order topology on \( \mathbb{R}^2 \).

For an explicit example of how the tropicalizations differ, let \( X = V(x + y + 1) \) in the 2-dimensional torus. Then \( \text{trop}(X) \) consists of 3 copies of the halfspace \( H' \) glued at their origins, whereas \( \text{trop}_{\text{Ban}}(X) \) consists of three copies of the set \( \{ (r_1, r_2) \in \mathbb{R}^2 : r_1 \geq 0 \} \) glued together at their origins.

Despite the fact that they do not coincide, the two tropicalizations are related by Proposition 2.3.1 below. Together with Theorem A, this resolves a question posed by Banerjee in [3, p. 2].

**Proposition 2.3.1.** Let \( K \) be a higher local field of rank \( k \), and let \( X \) be a subvariety of a \( d \)-dimensional split algebraic torus \( T \) over \( K \). Then \( \text{trop}_{\text{Ban}}(X) \) is the closure of \( \text{trop}_\nu(X) \) in the Euclidean topology.

**Proof.** Choose a splitting of the torus \( T \), inducing an isomorphism of \( \text{Hom}(M, \mathbb{R}^k) \) with \((\mathbb{R}^k)^d\). Let \( P \) be a \( \mathbb{Q}^k \)-rational polyhedron in \((\mathbb{R}^k)^d\). Identify \((\mathbb{R}^k)^d\) with the (unordered) abelian group \( \mathbb{R}^{kd} \), and consider \( P \) as a Euclidean subset of \( \mathbb{R}^{kd} \). The set of points of \( P \) with coordinates in \( \mathbb{Q} \) is dense in \( P \). The value group of the algebraic closure of \( K \) is \( \mathbb{Q}^k \), and thus \( |\text{trop}(X)| \) is a union of \( \mathbb{Q}^k \)-rational polyhedra [26, Proposition 1.2, Remark 1.3]. This implies that the set \( |\text{trop}(X)(\mathbb{Q})| \) of points of \( \text{trop}_\nu(X) \) with coordinates in \( \mathbb{Q} \) is dense in \( \text{trop}_\nu(X) \). It follows from [26, Proposition 1.1] that \( |\text{trop}(X)(\mathbb{Q})| \) is precisely the image of \( X(K_{\text{alg}}) \). The result follows. \( \square \)

**Corollary 2.3.2.** If \( X \) is a geometrically connected subvariety of a split algebraic \( K \)-torus, then \( \text{trop}_{\text{Ban}}(X) \) is path connected.

2.4. **Definability.** We require certain rudimentary notions from model theory. We give a brief and self-contained treatment, sufficient for our purposes. For a more detailed introduction and an overview of the work of Hrushovski and Loeser [17], see [7]. For background on \( o \)-minimal and definable structures, see [29, Chapter 3].

The composition \( \rho \circ \nu : K \times \rightarrow \mathbb{R}^k \) is a valuation on \( K \). Let \( K_{\text{alg}} \) denote an algebraically closed field extending \( K \), together with a valuation \( \nu_{K_{\text{alg}}} : (K_{\text{alg}})^\times \rightarrow \mathbb{R}^k \) extending \( \rho \circ \nu \).
Such an extension always exists since \( \mathbb{R}^{(k)} \) is divisible [9, Theorem 3.2.4]. Denote by \( K \) the pair\(^3\) \((K^{\text{alg}}, \mathbb{R}^{(k)})\).

Let \( \mathbf{ACVF} \) be the category in which an object is any algebraically closed field \( F \) extending \( K^{\text{alg}} \) and equipped with a valuation \( \nu : F^\times \to \mathbb{R}^{(k)} \) extending the valuation on \( K^{\text{alg}} \). Morphisms are given by isometric embeddings over \( K \).

Let \( X \) and \( Y \) be finite-type \( K \)-schemes. A subfunctor \( D \) of \( X \) is called \( K \)-definable if it can be defined Zariski locally in \( X \) by a (finite) boolean combination of inequalities of the form \( \nu(f) \bowtie \lambda + \nu(g) \) in \( \mathbb{R}^{(k)} \), where \( f \) and \( g \) are regular functions, \( \lambda \in \mathbb{R}^{(k)} \sqcup \{ \infty \} \), and \( \bowtie \in \{ =, \geq, <, > \} \).

A subset of \( X(F) \) is said to be \( K \)-definable if it can be defined Zariski locally in \( X \) by a (finite) boolean combination of inequalities of the form \( \nu(f) \bowtie \lambda + \nu(g) \). When \( k = 1 \), this reduces to the familiar notion of a semi-algebraic subset of \( X(F) \). Similarly, one defines \( K \)-definable maps \( X(F) \to Y(F) \) by requiring the graph to be a definable subset of the product.

**Definition 2.4.1.** A functor \( D : \mathbf{ACVF} \to \mathbf{Sets} \) is \( K \)-definable if it is isomorphic to a quotient \( E/R \) of a \( K \)-definable subfunctor \( E \subset X \) of the functor of points \( X \) of some \( K \)-scheme by an equivalence relation \( R \subset E \times E \) that is itself a \( K \)-definable subfunctor of \( X \times X \).

A natural transformation \( \varphi : D \to E \) between \( K \)-definable functors is said to be \( K \)-definable if the graph of \( \varphi \) is a \( K \)-definable functor.

If \( \Delta \) is a \( K \)-definable functor, for every \( F \in \mathbf{ACVF} \), a subset of \( \Delta(F) \) is a \( K \)-definable subset if it can be written as \( D(F) \) for some \( K \)-definable subfunctor of \( \Delta \).

### 2.5. Model theoretic connectivity.
In this subsection, the choice of \( K \) will be implicit in the discussion, and we will use the term definable in place of \( K \)-definable.

There is a natural pairing
\[
\langle -, - \rangle : (\mathbb{R}^{(k)} \sqcup \{ \infty \})^d \times \mathbb{Z}^d \to \mathbb{R}^{(k)} \sqcup \{ \infty \}
\]
\[
(\gamma_1, \ldots, \gamma_d, (u_1, \ldots, u_d)) \mapsto \sum u_i \gamma_i.
\]

A rational halfspace in \((\mathbb{R}^{(k)} \sqcup \{ \infty \})^d\) is any set of the form \( H_{\Sigma, \delta} := \{ \gamma \in (\mathbb{R}^{(k)} \sqcup \{ \infty \})^d : \langle \gamma, u \rangle \geq \delta \} \), for fixed slope \( u \in \mathbb{Z}^d \) and affine constraint \( \delta \in \mathbb{R}^{(k)} \sqcup \{ \infty \} \). Its boundary is \( \{ \gamma \in (\mathbb{R}^{(k)} \sqcup \{ \infty \})^d : \langle \gamma, u \rangle = \delta \} \). A rational polyhedron \( P \in (\mathbb{R}^{(k)} \sqcup \{ \infty \})^d \) is any intersection of finitely many rational halfspaces
\[
(4) \quad P = H_{\Sigma_1, \delta_1} \cap \cdots \cap H_{\Sigma_m, \delta_m}.
\]

If \( P \) is a rational polyhedron given by an intersection (4), then a face of \( P \) is any intersection of \( P \) with any number of the boundaries of the rational halfspaces in (4). A rational polyhedral complex in \((\mathbb{R}^{(k)} \sqcup \{ \infty \})^d\) is a finite a collection \( \{ P_j \}_{j \in J} \) of polyhedra in \((\mathbb{R}^{(k)} \sqcup \{ \infty \})^d\) such that every face of every \( P_j \) also lies in the collection, and such that the intersection of any two polyhedra in the collection again lies in the collection.

The following proposition follows from the discussion in [7, Section 1.4].

**Proposition 2.5.1.** A subset of \((\mathbb{R}^{(k)} \sqcup \{ \infty \})^d\) is definable if and only if it is isomorphic to a finite union of rational polyhedra. In particular, the set underlying any rational polyhedral complex in \((\mathbb{R}^{(k)} \sqcup \{ \infty \})^d\) is a definable subset of \((\mathbb{R}^{(k)} \sqcup \{ \infty \})^d\).

\(^3\)As in [17, Secion 14], we view \( K \) as a substructure of the theory of algebraically closed valued fields, in valued field and value group sorts.
The analogous statements hold upon replacing \( \mathbb{R}^k \sqcup \{\infty\} \) with \( \mathbb{R}^k \).

**Theorem 2.5.2.** If \( X \) is a subvariety of a \( d \)-dimensional torus over \( K \), then \( \trop(X) \) carries the structure of a rational polyhedral complex. In particular, \( \trop(X) \) is a definable subset of \( (\mathbb{R}^k)^d \cong \text{Hom}_\mathbb{Z}(M, \mathbb{R}^k) \).

**Proof.** This is proved for valued fields of arbitrary rank in [26, Proposition 1.2, Remark 1.3] by choosing an embedding of the torus into \( \mathbb{P}^d \) and then using the Gröbner complex to give \( \trop(X) \) a polyhedral complex structure. \( \square \)

We now give a more precise restatement of the connectivity part of Theorem B.

A **generalized interval** is any definable space obtained as follows: Given an interval \( [\gamma_a, \gamma_b] \subset \mathbb{R}^k \sqcup \{\infty\} \), we may consider either \( [\gamma_a, \gamma_b] \) with its induced lexicographic order, or with its opposite lexicographic order. The choice of order is called an **orientation** of the interval. A generalized interval is any definable space obtained from a collection of oriented intervals

\[
[\gamma_1a, \gamma_1b], [\gamma_2a, \gamma_2b], \ldots, [\gamma_na, \gamma_nb] \subset \mathbb{R}^k \sqcup \{\infty\}
\]

by identifying the consecutive endpoints, respecting the orientations. That is, by identifying the largest endpoint of \( [\gamma_ma, \gamma_mb] \) with the smallest endpoint of \( [\gamma_{(m+1)a}, \gamma_{(m+1)b}] \).

**Theorem 2.5.3.** (Restatement of Theorem B). Let \( X \) be a geometrically connected subvariety of a split algebraic torus over \( K \). Given two points \( x, y \in \trop(X) \), there exists a generalized interval \( I \), together with a definable, continuous morphism \( \mathbb{P} : I \rightarrow \trop(X) \) whose endpoints map to \( x \) and \( y \).

We begin the proof with an analogue of Construction 2.2.2 for the extended order topology.

**Construction 2.5.4.** Let \( G \) be a finite graph with edges \( e_1, \ldots, e_r \). Fix a tuple \( \| \in (\mathbb{R}^k \sqcup \{\infty\})^r \).

Denote by \( |e_i| \) the interval \( [0, \ell(e_i)] \subset \mathbb{R}^k \sqcup \{\infty\} \), considered as a subspace in the extended order topology. Denote by \( |G(\|)| \) the topological space obtained by gluing endpoints of \( |e_i| \) and \( |e_j| \) when the corresponding endpoints \( e_i \) and \( e_j \) are identified.

Let \( M \) be a proper, regular, marked semistable \( \mathbb{R} \)-model with generic fiber \( C \). Let \( G \) be the marked dual graph of associated to \( M \) as in Remark 2.2.3, let \( \ell(e_i) \in \mathbb{R}^k \) denote the length assigned to each edge \( e_i \) in \( G \), and let \( |G(\|)| \) the resulting topological space described in Construction 2.5.4.

**Proposition 2.5.5.** There is a continuous embedding \( |G(\|)| \rightarrow C^\mathbb{R} \). Moreover, \( e \) is an edge corresponding to a marked point \( p \) of the generic fiber \( C \), the infinite point of \( |e| \) is mapped to the image of \( p \) under the inclusion \( C(K) \rightarrow C^\mathbb{R} \).

**Proof.** The proof is identical to that of Proposition 2.2.4, replacing the Euclidean topology with the order topology throughout. \( \square \)

The rank-1 analogue of the following proposition was proved in [2, Theorem 6.22].

**Proposition 2.5.6 (Faithful tropicalization for curves).** Let \( M \) be a marked model as above. There exists a rational map \( \varphi : C \rightarrow \mathbb{C}^n_m \) such that the tropicalization of \( C \) with respect to \( \varphi \) is injective on the subspace \( |G(\|)| \subset C^\mathbb{R} \).
Proof. The result follows from arguments similar to those given for the rank-1 case in the proof of [14, Theorem 9.5]. We explain how this proof may be adapted to the present context.

If \( U \) is an affine open in \( \mathcal{C} \), define \( U := \mathcal{V}_k \), and let \( U^\delta \geq 0 \) denote the subspace of \( U^\delta \) consisting of those points represented by morphisms \( \text{Spec} \ L \rightarrow U \) that extend to a morphism \( \text{Spec} \ R \rightarrow U \), where \( R \) denotes the valuation ring of \( L \). There is a reduction map

\[
\text{red}_U : U^\delta \geq 0 \rightarrow \mathcal{V}_0
\]

given by sending a point \( \text{Spec} \ R \rightarrow U \) to the image of the closed point of \( \text{Spec} \ R \). Since the model \( \mathcal{C} \) is proper, \( C^\delta \) is covered by the sets \( U^\delta \geq 0 \), and one can check, as in the rank-1 case [25, Section 2], that the local reduction maps \( (5) \) fit together to give a well defined reduction map

\[
\text{red}_C : C^\delta \rightarrow C_0.
\]

Let \( \mathcal{I} \) be the set consisting of the marks on \( \mathcal{C} \), the generic points of the irreducible components of \( C_0 \), and the nodes of \( C_0 \). Projectivity and quasicompactness of \( \mathcal{C} \) implies that for each \( \zeta \in \mathcal{I} \), there exists a finite cover of \( \mathcal{C} \) by affine opens \( \mathcal{V}_{\zeta,i} \) containing \( \zeta \). On each open \( \mathcal{V}_{\zeta,i} \) there are finitely many regular functions \( f_{\zeta,i,j} \) whose reduction to the special fiber have zero set \( \zeta \cap \mathcal{V}_{\zeta,i} \). For every \( p \in U^\delta_{\zeta,i,j} \), we have that \( \text{val}_p(f_{\zeta,i,j}) = 0 \) if the reduction of \( p \) is not contained in the closure of \( \zeta \), and \( \text{val}_p(f_{\zeta,i,j}) = 0 \) is strictly positive otherwise.

Near each node \( q_i \) of \( C_0 \), we have rational functions \( x_i \) and \( y_i \) cutting out the components meeting at \( q_i \). Similarly, for each marked section \( s_r \), we obtain a rational function \( z_r \). The collection of functions \( \{x_i, y_i, z_r, f_{\zeta,i,j} \} \) determine a rational map \( C \rightarrow \mathbb{G}_m^n \). If \( p, q \in |G(\ell)| \subset C^\delta \) are points whose reductions \( \text{red}_C(p) \) and \( \text{red}_C(q) \) lie in adjacent strata of \( C_0 \), then some pair of functions \( x_i \) and \( y_i \) separate \( p \) and \( q \). If the reductions \( \text{red}_C(p) \) and \( \text{red}_C(q) \) lie in non-adjacent strata of \( C_0 \), there is a rational function \( f_{\zeta,i,j} \) in our collection such that \( \text{val}_p(f_{\zeta,i,j}) \neq \text{val}_q(f_{\zeta,i,j}) \). One checks as in the concluding paragraph of the proof of [14, Theorem 9.5] that there exists a \( \zeta \in \mathcal{I} \) and a function \( f_{\zeta,i,j} \) such that \( \text{val}_p(f_{\zeta,i,j}) \) is strictly positive, but \( \text{val}_q(f_{\zeta,i,j}) \) is 0.

Proof of Theorem 2.5.3. Let \( X \) be a geometrically connected subvariety of a d-dimensional torus over \( K \). Choose \( z, w \in \text{trop}(X) \). After base changing \( X \) to a sufficiently large field \( F \) over \( K \), we can assume that there exist points \( z', w' \in X_F(F) \subset X^\delta_F \) mapping to \( z \) and \( w \) under tropicalization. As before, Bertini’s theorem produces an irreducible \( F \)-curve \( C \) in \( X_F \) connecting \( z' \) and \( w' \). Let \( \hat{C} \) be an \( F \)-curve compactifying the normalization of \( C \). Choose a semistable \( R \)-model for \( \hat{C} \) marked at \( z' \), \( w' \). Denote by \( |G| \subset C^\delta \) the graph constructed as in the proposition above. Since \( C \) is connected, \( G \) is also connected, and one may find a generalized interval \( I \subset |G| \subset C^\delta \). Compositing with the tropicalization map, we obtain a map \( P : I \rightarrow \text{trop}(X) \) connecting \( z \) and \( w \). Continuity is clear, and so it remains only to show the definability of this path. The curve \( C \) is embedded in the torus \( \mathbb{G}_m^d \). Using Proposition 2.5.6 above, one may enlarge the embedding set of functions for \( C \) to obtain a rational morphism \( \psi : C \rightarrow \mathbb{G}_m^{d+m} \), such that the tropicalization of \( C \) with respect \( \psi \) is injective on \( |G| \) and hence on \( I \). The path \( P \) factors as

\[
\begin{array}{ccc}
I & \xrightarrow{P} & (\mathbb{R}(k))^{d+m} \\
\downarrow & & \downarrow \text{(}\mathbb{R}(k)\text{) } ^d \\
\end{array}
\]
The first map is an embedding of a 1-dimensional polyhedral complex and is consequently definable. The second map is projection onto the first d factors, hence also definable. Thus P is definable and continuous.

The arguments above imply the following connectivity property of \( X^0 \).

**Theorem 2.5.7.** Let \( X \) be a geometrically connected \( K \)-variety. Given any two points \( z, w \in X^0 \), there exists a Hahn valued field \( F \) extending \( K \), points \( \tilde{z}, \tilde{w} \in (X \times_K F)^0 \) mapping to \( z \) and \( w \), and a continuous definable path \( P : I \to (X \times_K F)^0 \) connecting \( \tilde{z} \) and \( \tilde{w} \).

### 3. Comparison results

We briefly explain how the Hahn analytification relates to the Huber analytification and the Hrushovski-Loeser spaces.

#### 3.1. Comparison results I: Huber adic space. Let \( K \) be a field, complete with respect to a nontrivial rank-1 valuation. For any affine \( K \)-scheme \( X = \text{Spec} \ K[\mathbf{X}] \), the adic space \( X^\text{ad} \) associated to \( X \) is the set

\[
X^\text{ad} = \left\{ \text{valuations } K[\mathbf{X}] \to \Upsilon \sqcup \{\infty\} \text{ factoring through } K \to v_K(K^\times) \sqcup \{\infty\} \hookrightarrow \Upsilon \sqcup \{\infty\} \right\}/\sim,
\]

where \( \sim \) is the equivalence relation generated by the requirement that two valuations \( K[\mathbf{X}] \to \Upsilon_1 \sqcup \{\infty\} \) and \( K[\mathbf{X}] \to \Upsilon_2 \sqcup \{\infty\} \) be equivalent if there exists an inclusion \( \Upsilon_1 \hookrightarrow \Upsilon_2 \) of totally ordered abelian groups such that the diagram

\[
\begin{array}{ccc}
K[\mathbf{X}] & 
\Upsilon_2 \sqcup \{\infty\} \\
\Upsilon_1 \sqcup \{\infty\} & 
\end{array}
\]

commutes. The space \( X^\text{ad} \) is equipped with the topology generated by sets of the form

\[
U(f, g) := \{ x \in X^\text{ad} : \text{val}_x(g) < \text{val}_x(f) \},
\]

where \( f \) and \( g \) are rational functions on \( X^\text{ad} \).

If we fix an embedding of totally ordered abelian groups \( \Gamma \hookrightarrow \mathbb{R}^{(k)} \), then every valuation \( \text{val}_x : K[X] \to \mathbb{R}^{(k)} \) that describes a point \( x \) of the resulting Hahn analytification \( X^0 \) satisfies \( \text{val}_x \mid_K = v_K \). Thus we obtain a map

\[
\eta : X^0 \to X^\text{ad}.
\]

**Theorem 3.1.1.** If we choose our Hahn embedding \( \Gamma \hookrightarrow \mathbb{R}^{(k)} \) to be the inclusion \( r \mapsto (r, 0, \ldots, 0) \) into the 1st factor of \( \mathbb{R}^{(k)} \), and if

\[
k \geq 1 + \dim_{K_{\text{nr}}} X,
\]

then the map \( \eta : X^0 \to X^\text{ad} \) is surjective.

**Proof.** We may assume that \( X \) is affine with coordinate ring \( K[\mathbf{X}] \). Consider a point \( x \in X^\text{ad} \) represented by the valuation \( \nu : K[\mathbf{X}] \to \Upsilon \sqcup \{\infty\} \), and note that this valuation induces an inclusion \( \Gamma \hookrightarrow \Upsilon \).

The chain of convex subgroups of \( \Upsilon \) is in bijection with a chain of prime ideals of \( K[\mathbf{X}] \) (see [9]). Since \( X \) is finite dimensional, this chain of convex subgroups is finite. Thus we may assume that \( \Upsilon \) has finite rank bounded by \( \dim(X) + 1 \). Since every convex subgroup
is a union of archimedean equivalence classes of \( \Upsilon \), we deduce that \( \Upsilon \) admits an order-preserving embedding \( \Upsilon \rightarrow \mathbb{R}^{(k)} \). It is straightforward to check that this embedding can be chosen to be compatible with the embedding \( \Gamma \hookrightarrow \mathbb{R}^{(k)} \). In this way, we obtain a valuation

\[
K[X] \rightarrow \Upsilon \sqcup \{\infty\} \overset{\sim}{\rightarrow} \mathbb{R}^{(k)} \sqcup \{\infty\},
\]

and thus a point \( x_h \) of \( X^\partial \). It is clear that \( x_h \mapsto x \).

**Remark 3.1.2.** The map \( \eta : X^\partial \rightarrow X^{ad} \) will not be continuous in general. For instance, let \( \Gamma = \mathbb{R} \) and \( k = 1 \). Then \( X^\partial = X^{an} \), and \( \eta \) becomes the section

\[(7)\]

\[
X^{an} \rightarrow X^{ad}
\]

of the maximal Hausdorff quotient map \( X^{ad} \longrightarrow X^{an} \), sending a higher rank valuation \( \text{val} : \mathcal{O}(U) \rightarrow \Upsilon \sqcup \{\infty\} \) to the valuation \( \mathcal{O}(U) \rightarrow \Upsilon \sqcup \{\infty\} \rightarrow (\Upsilon/\Upsilon_1) \sqcup \{\infty\} \), where \( \Upsilon_1 \) is the largest proper convex subgroup of \( \Upsilon \). This section \((7)\) is not continuous in general \([19,\text{ Proposition 8.3.1, Lemma 8.1.8}]\).

### 3.2. Comparison results II: Stable completion.

In \([17]\), Hrushovski and Loeser associate to any variety \( X \) over an algebraically closed valued field \( K \) a space \( \hat{X} \) called stable completion of \( X \). The relationship between \( \hat{X} \) and \( X^\partial \) is similar to the relationship between \( \hat{X} \) and the Berkovich analytification \( X^{an} \) when \( K \) has a nontrivial rank-1 valuation, as we now explain. The surveys by Ducros \([6,7]\) provide an essentially self-contained introduction to the model theoretic background to this section. The reader may safely skip this section, as the rest of the paper does not logically depend upon it.

**The stable completion functor.** Let \( K \) be an algebraically closed valued field with value group \( \Gamma \). Let \( M \) denote the category of algebraically closed valued extensions of \( K \), where morphisms are taken to be isometric \( K \)-embeddings. For \( F \in M \), let \( M_F \) be the category of valued extensions of \( F \) that belong to \( M \).

Let \( X = \text{Spec}(A) \) be an affine \( F \)-scheme. A type \( t \) on \( X \) is an element of the valuative spectrum of \( X \). That is, \( t \) is the data of a scheme theoretic point \( x \) of \( X \), together with a valuation \( \nu_x \) on the residue field of \( X \) at \( x \). In particular, a type \( t \) thus gives rise to a valuation \( \varphi_t \) on \( A \).

A type \( t \) is said to be \( F \)-definable if and only if for every finite-dimensional \( F \)-subspace subspace \( E \) of \( A \), the following subsets are \( F \)-definable in the sense of Definition 2.4.1:

- The set of elements \( e \in E \) such that \( \varphi_t(e) = \infty \);
- The set of elements \( e \in E \) such that \( \varphi_t(e) \geq 0 \).

A type \( t \) is said to be orthogonal to \( \Gamma \) if and only if it is \( F \)-definable and \( \varphi_t \) takes values in \( \nu(F) \).

**Definition 3.2.1.** Let \( X \) be an affine \( K \)-scheme of finite type and let \( F \in M \). The stable completion of \( X \) at \( F \) is the set \( \hat{X}(F) \) of types on \( X \times_K F \) that are orthogonal to \( \Gamma \).

As with the Berkovich and Hahn analytifications, the set \( \hat{X}(F) \) is given the weak topology for the evaluation functions \( ev_f : \hat{X}(F) \rightarrow \Gamma \sqcup \{\infty\} \) for all \( f \in A \). The construction extends to arbitrary finite-type \( K \)-schemes in the natural way.
The comparison map. Suppose \( K \) is complete with respect to a rank-1 valuation and that \( F \in M \) also has rank-1. An element of \( \hat{X}(F) \) can be interpreted as a valuation with values in \( \nu(F) \subset \mathbb{R} \sqcup \{\infty\} \). This gives rise to a continuous map
\[
\pi_F : \hat{X}(F) \rightarrow X^{an}
\]

Hrushovski and Loeser prove the following result about the comparison map \( \pi_F \).

**Proposition 3.2.2** ([17, Lemma 14.1.1]). If the valuation \( F^x \rightarrow \mathbb{R} \) is surjective, and if \( F \) is maximally complete, then the map (8) is a proper surjection.

The map (8) plays a crucial role in [17, Section 14] for deducing tameness results about the Berkovich space from results about the spaces of types. The following extension of Proposition 3.2.2 gives further evidence that the Hahn analytification \( X^0 \) is an analogue of the Berkovich analytification in the higher rank setting.

Let \( K \) be an arbitrary algebraically closed valued field with Hahn valuation
\[
K^x \xrightarrow{\nu} \Gamma \xrightarrow{\rho} \mathbb{R}^k.
\]
Assume that \( \rho \) is an embedding of ordered abelian groups. Note that this is not a serious restriction since we may always replace \( \Gamma \) with the image of \( \Gamma \) under \( \rho \).

**Proposition 3.2.3.** If \( F \) is a maximally complete, algebraically closed field extending \( K \), with surjective valuation \( F^x \rightarrow \mathbb{R}^k \) extending the valuation on \( K \), then there is a natural continuous surjection \( \pi_F : \hat{X}(F) \rightarrow X^0 \).

**Proof.** Consider a point \( x \in X^0 \) associated to a valuation \( \text{val}_x : \mathcal{A} \rightarrow \mathbb{R}^k \). The prime ideal \( \text{val}_x^{-1}(\infty) \) gives rise to a scheme theoretic point of \( X \), and hence a map \( \text{Spec}(L) \rightarrow X \). The valuation \( \text{val}_x \) gives rise to a valuation on the field \( L \). We may assume that \( L \) is algebraically closed. Let \( L^{\text{max}} \) denote the field (unique by Kaplansky’s theorem [21]) having value group \( \mathbb{R}^k \) and residue field equal to the residue field of \( L \). We may represent the point \( x \) by a map \( \text{Spec}(L^{\text{max}}) \rightarrow X \). The field \( F \) includes into \( L^{\text{max}} \). Thus we obtain a type \( t \) on \( \text{Spec}(A \otimes_K F) \). Since \( F \) is maximally complete, and since the value group \( \mathbb{R}^k \) has no Archimedean extensions, we may apply the result of Haskell, Hrushovski, and Macpherson [16, Theorem 12.18] to conclude that the type is orthogonal\(^4\). It is clear that this type \( t \) maps to the point \( x \) under \( \pi_F \), and surjectivity follows. \( \square \)

**Remark 3.2.4.** Note that fields \( F \) extending a Hahn valued field \( K \) as in Proposition 3.2.3 above always exist. Indeed, the group ring
\[
K[\mathbb{R}^k] := \left\{ \left. \text{all sums } f(t) = \sum_{r \in \mathbb{R}^k} a_t^r \mathbb{R} \right| \text{with finite support } \text{supp}(f(t)) \right\}
\]
comes with a surjective map
\[
\nu_{\text{mon}} : K[\mathbb{R}^k] \rightarrow \mathbb{R}^k \sqcup \{\infty\},
\]
given by \( \nu_{\text{mon}}(0) := \infty \) and
\[
\nu_{\text{mon}}(f(t)) := \inf_{r \in \text{supp}(f(t))} (\nu_K(a_{t^r}) + r)
\]
for nonzero \( f(t) = \sum_{r \in \mathbb{R}^k} a_t^r \mathbb{R} \). One checks, as in [22, Proposition 2.1.2], that this defines a valuation. Passing to a maximally complete algebraic closure of this field yields the desired extension.

\(^4\)See also the statement and remarks following [17, Theorem 2.9.2]
4. Extended tropicalization and an inverse limit theorem

4.1. Tropicalization in toric varieties. Let $K$ be a field with valuation $v : K \to \mathbb{R}^{(k)} \sqcup \{\infty\}$.

The Hahn tropicalization of $K$-tori and their subvarieties extends naturally to subvarieties of toric varieties, generalizing the construction of Kajiwara and Payne \[20, 27\]. Let $M$ be a lattice and $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ the dual lattice. Denote by $U(\sigma) = \text{Spec} \left( K[S\sigma] \right)$ the affine toric variety associated to a cone $\sigma$ in $N_{\mathbb{R}}$. Consider the set $N(\sigma)$ of semigroup homomorphisms from the commutative semigroup $S\sigma$ into the commutative semigroup underlying $\mathbb{R}^{(k)} \sqcup \{\infty\}$, i.e.,

$$N(\sigma) := \text{Hom}_{\text{Grp}} \left( S\sigma, \mathbb{R}^{(k)} \sqcup \{\infty\} \right).$$

Give $N(\sigma)$ the subspace topology for its inclusion into $(\mathbb{R}^{(k)} \sqcup \{\infty\})^{S\sigma}$, where $\mathbb{R}^{(k)} \sqcup \{\infty\}$ is given the extended order topology. Let $x = \text{val}_x : K[S\sigma] \to \mathbb{R}^{(k)} \sqcup \{\infty\}$ be a point of $U(\sigma)^\circ$. Restricting to $S\sigma$, we obtain a point $\text{trop}(X)$ of $N(\sigma)$. This furnishes a well defined, continuous tropicalization map

$$\text{trop} : U(\sigma)^\circ \to N(\sigma).$$

Let $\Delta$ be a fan in $N_{\mathbb{R}}$ with associated toric variety $Y(\Delta)$. If $\tau$ is a face of some cone $\sigma$ in $\Delta$, then one may restrict functions from $U(\sigma)$ to $U(\tau)$ to obtain a map $S\tau \hookrightarrow S\sigma$ and consequently an embedding of $N(\tau)$ into $N(\sigma)$. Gluing along these inclusions, we obtain a topological space $N(\Delta)$. The tropicalizations on affine invariant patches can be glued to form a continuous map

$$\text{trop} : Y(\Delta)^\circ \to N(\Delta).$$

The map above may be seen as a projection of $Y(\Delta)^\circ$ onto a closed subspace.

**Proposition 4.1.1.** The tropicalization map (9) admits a continuous section $s : N(\Delta) \to Y(\Delta)^\circ$.

*Proof.* It suffices to consider the affine case $Y(\Delta) = \text{Spec} \ K[S\sigma]$ for a single cone $\sigma$ in $\Delta$. Given a point $\omega \in N(\sigma)$, one constructs a monomial valuation with weight $\omega$, as in \[28, Proposition 2.9\], and these monomial valuations provide the desired section. \hfill \Box

Given a torus-equivariant map of toric varieties $\varphi : Y(\Delta) \to Y(\Delta')$, one may pullback characters and, following the construction above, obtain a map $\varphi^{\text{trop}} : N(\Delta) \to N(\Delta')$. In particular, tropicalization is covariantly funcitorial.

A closed embedding $\iota : X \hookrightarrow Y(\Delta)$ induces a continuous embedding $X^\circ \hookrightarrow Y(\Delta)^\circ$. Composing with (9) yields a subspace $\text{trop}(X, \iota)$ of $N(\Delta)$. If $U$ is an open subscheme of $X$, then we let $\text{trop}(U, \iota)$ denote the image of $U^\circ$ under trop : $X^\circ \to \text{trop}(X, \iota)$.

4.2. Inverse limit theorem and prodefinability of $X^\circ$. For this section, we assume that $K$ is algebraically closed. Let $X$ be a $K$-variety admitting a closed embedding into at least one toric variety. Covariant functoriality for torus-equivariant morphisms between toric varieties gives us a nonempty, inverse system $\tilde{S}$ of tropicalizations of $X$ under closed embeddings into toric varieties. This, in turn, gives rise to a continuous map

$$\tau : X^\circ \to \varprojlim \text{trop}(X, \iota).$$

**Theorem 4.2.1.** If $X$ is a $K$-variety admitting a closed embedding into at least one toric variety, then the map (10) is a homeomorphism.
Proof. The proof is closely modeled after the proofs of [27, Theorem 4.2] and [11, Theorem 1.2]. Fix, at the outset, a single closed embedding \( t_0 : X \hookrightarrow Y(\Delta_0) \) into a toric variety. Let \( S_0 \) denote the subcategory of \( S \) consisting of all those closed embeddings that factor through \( t_0 \). Then it suffices to prove that the map

\[
\tau_0 : X^0 \longrightarrow \lim_{\underset{\mathfrak{t} \in S_0}{\leftarrow}} \trop(X, \mathfrak{t})
\]

is a homeomorphism.

The arguments of [11, Section 4] make no use of the valuation on \( K \). They require only that \( K \) be algebraically closed. This means that there exists a finite open affine cover \( \{U_1, \ldots, U_r\} \) of \( X \) with the following property:

\((*)\) For any \( 1 \leq i \leq r \) and any nonzero regular function \( f \in K[\mathcal{U}_i] \), there exists a closed embedding \( \iota_f : X \hookrightarrow Y(\Delta_f) \) in \( S_0 \) such that \( \mathcal{U}_i \) is the \( \iota_f \)-preimage of a torus-invariant open subset of \( Y(\Delta_f) \), and such that \( f \) is the pullback of a monomial on \( Y(\Delta_f) \).

We claim that for each open set \( \mathcal{U}_j \) in this cover, the map (11) restricts to a homeomorphism of \( \mathcal{U}_j^0 \) onto the preimage of \( \trop(\mathcal{U}_j, t_0) \) in \( \lim_{\underset{\mathfrak{t} \in S_0}{\leftarrow}} \trop(X, \mathfrak{t}) \).

To verify injectivity, note that if \( x \neq y \in \mathcal{U}_j^0 \), then there exists a regular function \( f \in K[\mathcal{U}_j] \) with \( \val_x(f) \neq \val_y(f) \). Thus (\( \ast \)) provides us with a closed embedding \( \mathfrak{t} \in S_0 \) such that the images of \( x \) and \( y \) in \( \trop(X, \mathfrak{t}) \) are distinct.

To verify surjectivity, fix a point inside the preimage of \( \trop(\mathcal{U}_j, t_0) \) in \( \lim_{\underset{\mathfrak{t} \in S_0}{\leftarrow}} \trop(X, \mathfrak{t}) \). Such a point is given by a compatible system \( (\mathcal{V}_i)_{\mathfrak{t} \in S_0} \) of points \( \nu_i \in \trop(X, \mathfrak{t}) \), with \( \nu_{t_0} \in \trop(\mathcal{U}_j, t_0) \). For each \( f \in K[\mathcal{U}_j] \), use (\( \ast \)) to produce a closed embedding \( \mathfrak{t} \in S_0 \) where \( f \) becomes the pullback of a monomial \( a_{X,f}^u \), with \( a \in K \). We claim that the assignment

\[
\val_x : K[\mathcal{U}_j] \longrightarrow \mathbb{R}^k \cup \{\infty\} \quad f \longmapsto v(a) + \nu_i(u)
\]

is a well defined multiplicative valuation on \( K[\mathcal{U}_j] \). Well definedness of \( \val_x \) follows from compatibility of the system \( (\mathcal{V}_i)_{\mathfrak{t} \in S_0} \) combined with the fact that \( S_0 \) is closed under products. To verify multiplicativity and subadditivity, observe that the use of Włodarczyk’s algorithm [30, Proof of Lemma 4.2] in the construction of the closed embedding in (\( \ast \)) gives us enough flexibility to construct closed embeddings \( \mathfrak{t} \in S_0 \) in which any triple of functions, \( f, g \), and \( f + g \) in \( K[\mathcal{U}_j] \) become pullbacks of monomials.

The definition of the topology on \( \mathcal{U}_j^0 \) along with the existence of the closed embedding \( \iota_f \) for each \( f \in K[\mathcal{U}_j] \) implies that the topology on \( \mathcal{U}_j^0 \) coincides with the inverse limit topology on the preimage of \( \trop(\mathcal{U}_j, t_0) \) in \( \lim_{\underset{\mathfrak{t} \in S_0}{\leftarrow}} \trop(X, \mathfrak{t}) \). The local result at each \( \mathcal{U}_j \) in the open cover of \( X \) implies the global result, i.e., that (11) is a homeomorphism.

The inverse limit statement of Theorem 4.2.1 immediately implies the following model theoretic consequence. See [17, Section 3.1] for the corresponding statement for the stable completion.

**Theorem 4.2.2.** Let \( X \) be a variety admitting at least one closed embedding into a toric variety. Then the space \( X^0 \) is a \( K \)-prodefinable set in the sense of [17, Section 2.2].

**Remark 4.2.3.** Scheme theoretic and universal tropicalization. The constructions presented in this paper are compatible with recent work in which Giansiracusa and Giansiracusa develop a “scheme theoretic” framework for tropicalization [12, 13]. Given a subvariety \( X \) of a toric variety over a Hahn valued field \( K \), one may consider the associated tropical
scheme $\mathcal{T}rop(X)\), as constructed in [12]. The homomorphism $\rho : \Gamma \to \mathbb{R}^{(k)}$ allows one to consider the $\mathbb{R}^{(k)} \setminus \{\infty\}$-valued points of $\mathcal{T}rop(X)$. This underlying set of points coincides with the extended Hahn tropicalization above. Similarly, the $\mathbb{R}^{(k)} \setminus \{\infty\}$-valued points of the universal embedding of $X$ constructed in [13] coincide with the underlying point set of the Hahn analytification. Together with Theorem 3.1.1, this makes precise the observation [13, pg.3] that the tropicalization of the universal embedding contains information about the Huber adic space.

REFERENCES

[1] F. AROCA, Krall-tropical hypersurfaces, Annales de la faculté des sciences de Toulouse Mathématiques, 19 (2010), pp. 525–538. 1.2.1
[2] M. BAKER, S. PAYNE, and J. RABINOFF, Nonarchimedean geometry, tropicalization, and metrics on curves, arXiv:1104.0320, (2011). 1, 2.2, 2.5
[3] S. D. BANERJEE, Tropical geometry over higher dimensional local fields, Journal für die reine und angewandte Mathematik (Crelles Journal), 2015 (2015), pp. 71–87. 1, 1.2.1, 2.3, 2.3
[4] V. G. BERKOVICH, Spectral theory and analytic geometry over non-Archimedean fields, vol. 33, American Mathematical Society, 1990. 1, 2, 2.1.3
[5] R. BIERI and J. R. GROVES, The geometry of the set of characters induced by valuations., J. Reine Angew. Math. (Crelle’s Journal), 347 (1984), pp. 168–195. 1
[6] A. DUCROS, Les espaces de Berkovich sont modérés, d’après E. Hrushovski et F. Loeser, arXiv:1210.4336, (2012). 3.2
[7] A. DUCROS, About Hrushovski and Loeser’s work on the homotopy type of Berkovich spaces, arXiv:1309.0340, (2013). 2.4, 2.5, 3.2
[8] M. EINSIEDLER, M. KAPRANOV, and D. LIND, Non-archimedean amoebas and tropical varieties, J. Reine Angew. Math. (Crelle’s Journal), 2006 (2006), pp. 139–157. 1, 1.3
[9] A. J. ENGELER and A. PRESTEL, Valued fields, Springer, 2005. 2, 2.4, 3.1
[10] I. FESENKO and M. KURIHARA, eds., Invitation to higher local fields, vol. 3 of Geometry & Topology Monographs, Geometry & Topology Publications, Coventry, 2000. Papers from the conference held in Münster, August 29–September 5, 1999. 2.3
[11] T. FOSTER, P. GROSS, and S. PAYNE, Limits of tropicalizations, Israel Journal of Mathematics, 201 (2014), pp. 835–846. 1.3, 4.2, 4.2
[12] J. GIANSIRACUSA and N. GIANSIRACUSA, Equations of tropical varieties, arXiv:1308.0042, (2013). 4.2.3
[13] ________, The universal tropicalization and the Berkovich analytification, arXiv:1410.4348, (2014). 4.2.3
[14] W. GUBLER, J. RABINOFF, and A. WERNER, Skeletons and tropicalizations, arXiv:1404.7044, (2014). 2.5, 2.5
[15] H. HAHN, Über die nichtarchimedischen größensysteme, in Hans Hahn Gesammelte Abhandlungen Band 1/Hans Hahn Collected Works Volume 1, Springer, 1995, pp. 445–499. 2.0.2
[16] D. HASKELL, E. Hrushovski, and D. Macpherson, Stable domination and independence in algebraically closed valued fields, volume 30 of Lecture Notes in Logic, (2008). 3.2
[17] E. Hrushovski and F. Loeser, Non-archimedean tame topology and stably dominated types, Annals of Mathematics Studies, To Appear. 1, 2.4, 3, 3.2, 3.2.2, 3.2, 4, 4.2, 4.2.2
[18] E. Hrushovski, F. Loeser, and B. POONEN, Berkovich spaces embed in euclidean spaces, Enseign. Math., (To appear). 2
[19] R. Huber, Étale cohomology of rigid analytic varieties and adic spaces, aspects of math, E30, Friedr. Vieweg & Sohn, Braunschweig, (1996). 1, 3.1.2
[20] T. KAJIWARA, Tropical toric geometry, Contemporary Mathematics, 460 (2008), pp. 197–208. 4.1
[21] I. KAPLANSKY, Maximal fields with valuations, Duke Math. J., 9 (1942), pp. 303–321. 3.2
[22] K. S. KEDLAYA, p-adic differential equations, vol. 125 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2010. 3.2.4
[23] D. MACLAUGAN and B. STURMFELS, Introduction to Tropical Geometry, vol. 161 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2015. 1
[24] D. MUMFORD, C. P. RAMANUJAM, and J. I. MANIN, Abelian varieties, vol. 108, Oxford Univ Press, 1974. 2.2
[25] M. MUSTATA and J. NICAISE, Weight functions on non-archimedean analytic spaces and the kontsevich-soibelman skeleton, Algebraic Geometry, (To appear). 2.2, 2.5
[26] M. NISSE and F. SOTTILE, Non-archimedean coamoebae, Tropical and Non-Archimedean Geometry (Hole-town, 2013), (2011), pp. 73–91. 2.3, 2.5
[27] S. Payne, Analytification is the limit of all tropicalizations, Math. Res. Lett., 16 (2009), pp. 543–556. 1, 1.3, 4.1, 4.2

[28] A. Thuillier, Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d’homotopie de certains schémas formels, Manuscripta Math., 123 (2007), pp. 381–451. 4.1

[29] L. Van den Dries, Tame topology and o-minimal structures, vol. 248, Cambridge university press, 1998. 1.3.1, 2.4

[30] J. Włodarczyk, Embeddings in toric varieties and prevarieties, J. Algebraic Geom., 2 (1993), pp. 705–726. 4.2

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