Cartan-Weyl 3-algebras and the BLG Theory I: Classification of Cartan-Weyl 3-algebras

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ABSTRACT: As Lie algebras of compact connected Lie groups, semisimple Lie algebras have wide applications in the description of continuous symmetries of physical systems. Mathematically, semisimple Lie algebra admits a Cartan-Weyl basis of generators which consists of a Cartan subalgebra of mutually commuting generators $H_I$ and a number of step generators $E^\alpha$ that are characterized by a root space of non-degenerate one-forms $\alpha$. This simple decomposition in terms of the root space allows for a complete classification of semisimple Lie algebras. In this paper, we introduce the analogous concept of a Cartan-Weyl Lie 3-algebra. We analyze their structure and obtain a complete classification of them. Many known examples of metric Lie 3-algebras (e.g. the Lorentzian 3-algebras) are special cases of the Cartan-Weyl 3-algebras. Due to their elegant and simple structure, we speculate that Cartan-Weyl 3-algebras may be useful for describing some kinds of generalized symmetries. As an application, we consider their use in the Bagger-Lambert-Gustavsson (BLG) theory.

KEYWORDS: D-Branes, M-Theory, Gauge symmetry, Lie $n$-algebra.
The analysis of the gravitational thermodynamics has suggested that the entropy of a large number of $N$ coincident branes should obey a power law scaling $N^2$, $N^{3/2}$, $N^3$ for D-branes, M2-branes and M5-branes respectively [1]. For coincident D-branes, the $U(1)$ gauge symmetry for individual D-brane is enhanced to a $U(N)$ gauge symmetry, and the $N^2$ dependence of the entropy is nicely accounted for by the $N^2$ degrees of freedom present in the supersymmetric $U(N)$ Yang-Mills description of the coincident D-branes. The situation is much less clear for multiple M2 or M5-branes. Some of the outstanding questions are, for example: How does the gauge symmetry get enhanced for coincident M-branes? What is the appropriate mathematical description?

Recently a new class of (2+1)-dimensional superconformal field theories with maximal $\mathcal{N} = 8$ supersymmetry has been constructed by Bagger and Lambert [2–4], and
separately by Gustavsson [5]. These field theories have been proposed as the low energy effective field theories for coincident M2-branes and thus provide a nonperturbative description of M-theory on $AdS_4 \times S^7$ according to the AdS/CFT correspondence [6]. Another proposal is due to Aharony, Bergman, Jafferis and Maldacena [7] which proposed a certain $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory as the theory describing the worldvolume of multiple M2-branes at low energies [8]. In this description, a total of $\mathcal{N} = 6$ supersymmetry is explicitly realized [9].

One of the most exciting features of the BLG theory is the use of a new mathematical structure, a metric Lie 3-algebra, in its description of the gauge symmetries of the multiple M2-branes. It has also been suggested that Lie 3-algebras also play a crucial role in the description of multiple M5-branes [10–12].

In the case of D-branes, the worldvolume theory carries a gauge symmetry that is described by a semisimple Lie algebra. Indeed the class of semisimple Lie algebras is distinguished in the theory of Lie algebras. Physically, semisimple Lie algebras are the Lie algebras of compact connected Lie groups, one that are universally used to describe the continuous symmetries of physical systems. Mathematically, semisimple Lie algebras are completely classified and are fully understood. In particular, semisimple Lie algebras admit an elegant and very simple basis of generators called the Cartan-Weyl basis. See equations (2.7) and (2.8) below. Motivated by the simplicity of the structure of a Cartan-Weyl basis of generators for a semisimple Lie algebra, one can introduce the notion of a Cartan-Weyl basis for a metric Lie 3-algebra. A Cartan-Weyl basis consists of the generators $H_I, E^\alpha$ where $H_I$ ($I = 1, \cdots, N$ for some $N$) form a Cartan subalgebra $\mathcal{H}$ of the Lie 3-algebra and $E^\alpha$ are labelled by roots $\alpha$, which are linear functions on $\mathcal{H}^{\wedge 2}$. The generators satisfy the Lie 3-brackets

$$[H_I, H_J, H_K] = 0$$

and

$$[H_I, H_J, E^\alpha] = \alpha_{IJ}E^\alpha,$$

$$[H_I, E^\alpha, E^\beta] = \begin{cases} \alpha_{IK}g^{KL}H_L, & \text{if } \alpha + \beta = 0, \\ g_I(\alpha, \beta)E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root}, \\ 0, & \text{if } \alpha + \beta \text{ is not a root}, \end{cases}$$

$$[E^\alpha, E^\beta, E^\gamma] = \begin{cases} -g_K(\alpha, \beta)g^{KL}H_L, & \text{if } \alpha + \beta + \gamma = 0, \\ c(\alpha, \beta, \gamma)E^{\alpha+\beta+\gamma}, & \text{if } \alpha + \beta + \gamma \neq 0 \text{ a root}, \\ 0, & \text{if } \alpha + \beta + \gamma \text{ is not a root}; \end{cases}$$

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and they are normalized such that
\[
\langle E^\alpha, E^\beta \rangle = \delta^{\alpha + \beta}, \quad \langle H_I, E^\alpha \rangle = 0, \quad \langle H_I, H_J \rangle = g_{IJ}.
\]

(1.3)

Here \(\langle \cdot, \cdot \rangle\) is the metric and the part \(g_{IJ}\) is assumed to be invertible with the inverse \(g^{IJ}\).

We will call a metric Lie 3-algebra a Cartan-Weyl 3-algebra if it admits such a Cartan-Weyl basis. It is natural to speculate that a classification theorem may be obtained for the Cartan-Weyl 3-algebras.

Due to their similarity in structure to semisimple Lie algebras, it is natural to speculate that Cartan-Weyl 3-algebras may also play a role in the description of certain yet to be discovered generalized symmetries of Nature. The understanding of the structure and the classification of Cartan-Weyl 3-algebras are therefore potentially important. To achieve these goals is the main motivation of this paper. Once this is achieved, it is natural to try to see whether and how this kind of Lie 3-algebras is useful for the BLG theory. This is another motivation of this work.

The organisation of the paper is as follows. In section 2.1, we review some basic facts about Lie algebras. In particular we recall the definition of a Cartan-Weyl basis. In section 2.2, we motivate and introduce the definition of a Cartan-Weyl basis for a metric Lie 3-algebra. The consistency conditions arising from the fundamental identity are analysed in details in the appendix. The resulting conditions that the two-form roots \(\alpha_{IJ}\) and the coefficients \(g_I(\alpha, \beta)\) and \(c(\alpha, \beta, \gamma)\) have to satisfy are summarized in section 2.3. In section 3, these conditions are solved fully and a complete classification of Cartan-Weyl 3-algebras is obtained. In section 4, we consider the embedding of \(A_4\) and show that Cartan-Weyl 3-algebras do not contain \(A_4\) as a subalgebra in general. This implies that a BLG theory that is based on a Cartan-Weyl 3-algebra cannot contain fuzzy \(S^3\) in its description, at least not semiclassically. Section 5 contains some further discussions. The detailed analysis of the consistency conditions is performed in the appendix A.

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1The Lie \(n\)-algebra \(A_{p,q}\) is a metric Lie \(n\)-algebra with signature \((p, q)\), \(p + q = n + 1\). It has \(n + 1\) generators \(e_i, i = 1, \cdots, n + 1\) and is defined by the metric
\[
\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij}
\]

(1.4)

and the \(n\)-bracket relations
\[
[e_1, \cdots, e_i, \cdots, e_{n+1}] = (-1)^i \varepsilon_i e_i.
\]

(1.5)

The signs \(\varepsilon_i\) are given by \((+ \cdots +)\) for \(A_0, n+1 := A_{n+1}\), \((- + \cdots +)\) for \(A_{1,n}\), \((- - + \cdots +)\) for \(A_{2,n-1}\) etc.
2. Cartan-Weyl 3-algebras and Consistency Conditions

2.1 Cartan-Weyl basis for a semisimple Lie algebra

We start with some basic definitions about Lie algebras. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{C}$. A subspace $\mathfrak{a}$ is a subalgebra if $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$. A subspace $I$ is an ideal if $[I, \mathfrak{g}] \subset I$. Suppose that $I_1, I_2$ are ideals of a Lie algebra $\mathfrak{g}$, then $[I_1, I_2]$ is also an ideal of $\mathfrak{g}$. In particular, $\mathfrak{g}$ has the following two series of ideals:

$$\mathfrak{g} \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \cdots ,$$

(2.1)

which is called the *descending central series* of $\mathfrak{g}$; and

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \cdots ,$$

(2.2)

which is called the *derived series* of $\mathfrak{g}$. Here

$$\mathfrak{g}^0 := \mathfrak{g}, \quad \mathfrak{g}^n := [\mathfrak{g}^{n-1}, \mathfrak{g}]$$

(2.3)

and

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(n)} := [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$$

(2.4)

for $n \in \mathbb{N}$. A Lie algebra $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some $n$. It is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n$. Obviously every nilpotent Lie algebra is also solvable.

A Lie algebra is *simple* if it has no proper ideals other than itself or 0. The union of all solvable ideal of $\mathfrak{g}$ is also a solvable ideal of $\mathfrak{g}$, called the radical $\mathfrak{R}(\mathfrak{g})$ of $\mathfrak{g}$. A Lie algebra is *semisimple* if its radical is 0. As a result, a semisimple Lie algebra is given by a direct sum of simple Lie algebras and Abelian ones.

In the theory of Lie algebra, a very useful device is the *Killing metric* which can be defined using the adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{g}$ by

$$\kappa(x, y) = \text{tr}((\text{ad} x)(\text{ad} y)),$$

(2.5)

for $x, y \in \mathfrak{g}$. The Killing metric is invariant:

$$\kappa([x, z], y) + \kappa(x, [y, z]) = 0, \quad \text{for all } x, y, z \in \mathfrak{g}.$$ 

(2.6)

Killing metric plays an important role in characterizing Lie algebra. In fact,

**Theorem 2.1.** *(Cartan criteria of semisimplicity)* A Lie algebra $\mathfrak{g}$ is semisimple iff the Killing metric $\kappa$ is non-degenerate.

**Theorem 2.2.** *(Cartan criteria of solvability)* A Lie algebra $\mathfrak{g}$ is solvable iff the Killing metric $\kappa([x, y], z) = 0$ for all $x, y, z \in \mathfrak{g}$. 

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It is instructive to recall the reason why semisimple Lie algebras play a fundamental role in physics. In fact the classification theorem of Lie groups states that any compact, connected Lie group is a product of a finite abelian group and simple Lie groups. By modelling the continuous symmetries that appears in nature by compact connected Lie groups, one thus arrive at the semisimple Lie algebras naturally.

An important result in the theory of Lie algebra is that a semisimple Lie algebra admits a very nice basis of generators called the Cartan-Weyl basis. A Cartan-Weyl basis consists of a set of generators $H_I$ from the Cartan subalgebra and a set of step generators $E^\alpha$ that are labelled by a vector $\alpha = (\alpha_I)$ called the root. In general any two Cartan subalgebras are conjugate relative to the group of special automorphisms generated by the exponents of nilpotent inner derivations. A special feature of the Cartan-Weyl basis is that the Cartan subalgebra is Abelian and the roots are non-degenerate. In this basis, the Killing metric reads

$$\langle E^\alpha, E^\beta \rangle = \delta_{\alpha+\beta}, \quad \langle E^\alpha, H^I \rangle = 0, \quad g_{IJ} := \langle H_I, H_J \rangle, \quad \text{(2.7)}$$

where $g_{IJ}$, the restriction of the Killing metric on the Cartan subalgebra, is non-degenerate for a semisimple Lie algebra. The Lie brackets take the form

$$[H_I, H_J] = 0, \quad [H_I, E^\alpha] = \alpha_I E^\alpha, \quad [E^\alpha, E^\beta] = \begin{cases} 0, & \text{if } \alpha + \beta \neq 0 \text{ not a root,} \\ c(\alpha, \beta) E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root,} \\ -\alpha \cdot H & \text{if } \alpha + \beta = 0, \end{cases} \quad \text{(2.8)}$$

where $\alpha \cdot H = \alpha_I g^{IJ} H_J$ and $g^{IJ}$ is the inverse of $g_{IJ}$. Here we have used the invariance of the metric in deriving the relation for $[E^\alpha, E^{-\alpha}]$. The Cartan-Weyl basis is specified by the system of roots and the coefficient $c(\alpha, \beta)$. They can be solved and gives a complete classification of semisimple Lie algebras.

The proof of the existence of a Cartan-Weyl basis for a semisimple Lie algebra rests on the theory of root space decomposition for Lie algebras, see for example, [13,14]. We have also included a proof in the companion paper [15], highlighting the most important ideas involved so as to explain the conditions that are needed to establish the existence of a generalized Cartan-Weyl basis for a Lie 3-algebra. Here let us take an elementary and more direct approach to explain the assumptions involved that lead to the existence of a generalized Cartan-Weyl basis. This exercise will also be useful for a heuristic

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\textsuperscript{2}We also recall that a simple Lie group is isomorphic to exactly one of the $SU(n), n \geq 3; Sp(n), n \geq 1; Spin(n), n \geq 7; G_2, F_4, E_6, E_7, E_8$. 

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understanding the conditions that lead to the existence of a similar Cartan-Weyl basis for a Lie 3-algebra.

Given a Lie algebra, let us start by looking for a maximal set of commuting generators $H_I, I = 1, \cdots, N$. Denote the rest of the generators by $E^\alpha$ where $\alpha$ is just a label at this point. Denote also the set of all generators by $Y^a = \{H_I, E^\alpha\}$. Most generally we have

$$[H_I, Y^a] = \psi_I^a b Y^b,$$

where $\psi_I = (\psi_I)_b^a$ is a constant matrix. Jacobi identity $[H_I, [H_J, Y^a]] + \cdots = 0$ implies that

$$[\psi_I, \psi_J] = 0$$

and so we can diagonalize $\psi_I$'s simultaneously. If one assumes that all the eigenvalues are non-degenerate, one obtains $\psi_I^a b = \delta^a_b \psi_I^b$. Obviously $\psi_I^a = 0$ for $a = K$. Therefore

$$[H_I, E^\alpha] = \alpha_I E^\alpha,$$

where we have denoted $\psi_I^a = \alpha_I$. Next using the invariance of the metric $\delta_{H_I} \langle E^\alpha, E^\beta \rangle = 0$ and $\delta_{H_I} \langle H_I, E^\alpha \rangle = 0$, one obtains immediately

$$\langle E^\alpha, E^\beta \rangle \propto \delta^{\alpha+\beta}, \quad \langle H_I, E^\alpha \rangle = 0.$$  

Normalizing the generators suitably and assuming that the metric is non-degenerate when restricted to the $H$'s, we obtain the metric (2.7). Next we note that the Jacobi identity gives $[H_I, [E^\alpha, E^\beta]] = (\alpha + \beta)_I [E^\alpha, E^\beta]$. Therefore, using also the invariance of the metric, one sees that $[E^\alpha, E^\beta]$ takes the form as in (2.8). This is precisely the Cartan-Weyl basis.

What we see from this analysis is that the existence of a Cartan-Weyl basis is equivalent to the assumption that the solutions of the “eigenvalue equation” (2.9) are non-degenerate; and that the restriction of the metric to the $H$'s is non-degenerate. This is also equivalent to the assumption of semisimplicity. For Lie 3-algebras, we will now show that a Cartan-Weyl basis exists with similar assumptions of non-degeneracy.

2.2 Cartan-Weyl 3-algebras and the Cartan-Weyl basis

A Lie $n$-algebra $\mathcal{A}$ [16,17] is a linear space over a field $\mathbb{F}$ on which defined is a multilinear $n$-bracket operation which is skew-symmetric and satisfies the fundamental identity:

$$[[b_1, \cdots, b_n], a_1, \cdots, a_{n-1}] = \sum_{i=1}^{n} [b_1, \cdots, [b_i, a_1, \cdots, a_{n-1}], \cdots, b_n]$$  

(2.13)
for all $a_i, b_j \in A$. For $n = 2$, we get back to the usual Lie algebra and the fundamental identity coincides with the Jacobi identity. We will be concerned with real Lie $n$-algebras in this paper.

The fundamental identity guarantees that the transformation $\delta : A \to A$ defined by

$$\delta(a_1, \cdots, a_{n-1})f := [f, a_1, \cdots, a_{n-1}]$$

is a derivation of the Lie $n$-algebra:

$$\delta[b_1, \cdots, b_n] = [\delta b_1, \cdots, b_n] + \cdots + [b_1, \cdots, \delta b_n].$$

The map $\delta$ is parametrized by a skew-symmetric collection of $n - 1$ elements $(a) = (a_1, \cdots, a_{n-1})$ and is a natural generalization of the usual gauge transformation $\delta_a f := [f, a]$ defined for Lie algebra.

A *metric* $\langle \cdot, \cdot \rangle : A \otimes A \to \mathbb{R}$ is a symmetric bilinear form on $A$ which is also invariant in the sense that

$$\delta(a) \langle f, g \rangle = 0,$$

i.e.

$$\langle [f, (a)], g \rangle + \langle f, [g, (a)] \rangle = 0,$$

for any $f, g \in A$ and $(a) \in A^{\wedge (n-1)}$. In addition to a metric, it is also natural to introduce the notion of an *invariant form*. An invariant form on a Lie $n$-algebra $A$ is a multilinear function $F$ on $A^{\wedge (n-1)}$ which satisfies the invariance condition:

$$\sum_{i=1}^{n-1} F(a_1, \cdots, a_iR(c), \cdots, a_{n-1}; b_1, \cdots, b_{n-1}) + F(a_1, \cdots, a_{n-1}; b_1, \cdots, b_iR(c), \cdots, b_{n-1}) = 0,$$

where $R(c)$ is the right multiplication $aR(c) := [a, c_1, \cdots, c_{n-1}]$. We say an invariant form is non-degenerate if $F(a_1, \cdots, a_n; b_1, \cdots, b_{n-1}, x) = 0$ for all $a_1, \cdots, a_n, b_1, \cdots, b_{n-1} \in A$ implies that $x = 0$. In this paper we will be mainly interested in Lie 3-algebra with an invariant non-degenerate metric. We will refer to this as a metric Lie 3-algebra.

In analogy to the Lie algebra case, one may introduce the notion of a Cartan-Weyl basis for a Lie 3-algebra as follows. Let $A$ be a metric Lie 3-algebra, we call $A$ a *Cartan-Weyl 3-algebra* if the algebra admits a basis of generators $H_I$, $I = 1, \cdots, N \geq 2$ and $E^\alpha$, $\alpha = (\alpha_{IJ})$, with the metric

$$\langle E^\alpha, E^\beta \rangle = \delta_{\alpha+\beta}, \quad \langle E^\alpha, H_I \rangle = 0, \quad g_{IJ} := \langle H_I, H_J \rangle \text{ non-degenerate,}$$
and the 3-brackets:

\[
[H_I, H_J, H_K] = 0,
\]

\[
[H_I, H_J, E^\alpha] = \alpha_{IJ} E^\alpha,
\]

\[
[H_I, E^\alpha, E^\beta] = \begin{cases}
\alpha_{IK} g^{KL} H_L, & \text{if } \alpha + \beta = 0, \\
g_I(\alpha, \beta) E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root}, \\
0, & \text{if } \alpha + \beta \text{ is not a root},
\end{cases}
\]

\[
[E^\alpha, E^\beta, E^\gamma] = \begin{cases}
- g_K(\alpha, \beta) g^{KL} H_L, & \text{if } \alpha + \beta + \gamma = 0, \\
c(\alpha, \beta, \gamma) E^{\alpha+\beta+\gamma}, & \text{if } \alpha + \beta + \gamma \neq 0 \text{ a root}, \\
0, & \text{if } \alpha + \beta + \gamma \text{ is not a root}.
\end{cases}
\]

In writing these 3-bracket relations, we have used the invariance of the metric to relate (2.22a) with (2.21) and (2.23a) with (2.22b). The number of independent \(H\)'s will be called the rank of the Cartan-Weyl 3-algebra and the set of generators \(\{H_I, E^\alpha\}\) will be called a Cartan-Weyl basis. Similarly one can define a Cartan-Weyl Lie \(n\)-algebra.

It is natural to ask if semisimplicity is again the right condition to guarantee the existence of a Cartan-Weyl basis for a Lie 3-algebra (or Lie \(n\)-algebra in general). It turns out that there is a number of ways one can generalize the concept of semisimplicity of Lie algebra to the higher Lie \(n\)-algebra, and so the answer is not obvious. In the paper [15], we will investigate this question and find out what are the precise mathematical conditions for a Lie 3-algebra to be a (generalized) Cartan-Weyl 3-algebra.

Without entering into these more abstract discussions, it is possible to state and explain quite explicitly what are the conditions that lead to the existence of a Cartan-Weyl basis. In analogy to the discussion for Lie algebras, let us start by saying we have a maximal set of “commuting” generators in the sense that

\[
[H_I, H_J, H_K] = 0, \quad I, J, K = 1, \ldots, N.
\]

Denote the rest of the generators by \(E^\alpha\) and the set of all generators by \(Y^a = \{H_I, E^\alpha\}\).

Most generally we have

\[
[H_I, H_J, Y^a] = \psi_{IJ}^a Y^b,
\]

where \(\psi_{IJ} = (\psi_{IJ})^a_b\) is a constant matrix. Fundamental identity \([H_K, H_L, [H_I, H_J, Y^a]] = \cdots\) implies that

\[
[\psi_{IJ}, \psi_{KL}] = 0
\]

and so we can diagonalize \(\psi_{IJ}\)'s simultaneously. If one assume that all the eigenvalues are non-degenerate, then we have \(\psi_{IJ}^a_b = \delta^a_b \psi_{IJ}^a\). Obviously \(\psi_{IJ}^a_a = 0\) for \(a = K\). Denoting \(\psi_{IJ}^a = \alpha_{IJ}^a\), we have

\[
[H_I, H_J, E^\alpha] = \alpha_{IJ}^a E^\alpha
\]
and so the $E$-type generators are parametrized by a two-form $\alpha_{IJ}$. Next using the invariance of the metric $\delta_{(H_I,E^\alpha)}\langle E^\beta, H_J \rangle = 0$ and $\delta_{(H_I,H_J)}\langle E^\alpha, H_K \rangle = 0$, we obtain immediately

$$\langle E^\alpha, E^\beta \rangle \propto \delta^{\alpha+\beta}, \quad \langle H_I, E^\alpha \rangle = 0. \quad (2.28)$$

Normalizing suitably the generators and assuming that the metric is non-degenerate when restricted to the $H_I$’s, we get precisely the metric (2.19). Next use the fundamental identity $[H_J, H_K, [H_I, E^\alpha, E^\beta]] = \cdots$, we obtain $[H_J, H_K, [H_I, E^\alpha, E^\beta]] = (\alpha + \beta)_{JK}[H_I, E^\alpha, E^\beta]$. Together with the assumption that the $H_I$’s forms a maximal set of commuting generators, we obtain $[H_I, E^\alpha, E^\beta] = c_I(\alpha) H_J$ for $\alpha + \beta = 0$. Invariance of the metric then implies that it is precisely of the form (2.22a). For $\alpha + \beta \neq 0$, if we denote

$$[H_I, E^\alpha, E^\beta] = \lambda_{I}^{\alpha\beta\gamma a} Y^\alpha, \quad (2.29)$$

then the fundamental identity $[[H_I, H_J, H_K], E^\alpha, E^\beta] = \cdots$ gives

$$\sum_{(I, J, K) \text{ cyclic}} \lambda_{I}^{\alpha\beta\gamma \gamma_{JK}} = 0. \quad (2.30)$$

On the other hand, the invariance of the metric $\delta_{(H_I,E^\alpha)}\langle E^\beta, E^\gamma \rangle = 0$ implies that

$$\lambda_{I}^{\alpha\beta\gamma} = \lambda_{I}^{[\alpha\beta\gamma]} \quad (2.31)$$

is completely antisymmetric in $\alpha, \beta, \gamma$. Here we have used the metric $\langle E^\alpha, E^\beta \rangle = \delta^{\alpha+\beta}$ to raise an indices of $\lambda$. Using (2.30) and (2.31) and the fundamental identity $[[H_I, H_J, E^\alpha], H_K, E^\beta] = \cdots$, one can easily deduce that $\lambda_{I}^{\alpha\beta\gamma} = 0$ unless $\alpha + \beta + \gamma = 0$. Therefore $\lambda_{I}^{\alpha\beta\gamma}$ takes the form

$$\lambda_{I}^{\alpha\beta\gamma} = g_I(\alpha, \beta), \quad \text{for } \alpha + \beta + \gamma = 0 \quad (2.32)$$

and so we obtain the 3-bracket (2.22b). Finally, assume that

$$[E^\alpha, E^\beta, E^\gamma] = \omega^{\alpha\beta\gamma a} Y^\alpha. \quad (2.33)$$

By considering the fundamental identity $[[E^\alpha, E^\beta, E^\gamma], H_J, H_K] = \cdots$, one can show that

$$(\alpha + \beta + \gamma + \delta)\omega^{\alpha\beta\gamma\delta} = 0, \quad (\alpha + \beta + \gamma)\omega^{\alpha\beta\gamma L} = 0. \quad (2.34)$$

The first condition implies that $\omega^{\alpha\beta\gamma\delta} = 0$ unless $\alpha + \beta + \gamma + \delta = 0$. In this case, by denoting $\omega^{\alpha\beta\gamma\delta} = c(\alpha, \beta, \gamma)$, we arrive at the 3-bracket (2.23b) after using the second condition of (2.34). The second condition of (2.34) gives precisely the 3-brackets (2.23a) after using the invariance of the metric.

All in all, this analysis demonstrates that the existence of a Cartan-Weyl basis for a Lie 3-algebra is equivalent to the requirement that “eigenvalue equation” (2.25) has non-degenerate solutions; and that the restriction of the metric to the $H$’s is non-degenerate.
2.3 Consequences of the consistency conditions

A Cartan-Weyl 3-algebra is specified by the data: the roots $\alpha_{IJ}$ and the structural constants $g_I(\alpha, \beta)$ and $c(\alpha, \beta, \gamma)$. Remarkably, the consistency conditions arising from the fundamental identities are so strong that one can solve them exactly, giving a complete classification of Cartan-Weyl 3-algebras.

The consistency conditions that follow from the fundamental identities are listed and analyzed in details in the appendix. It turns out that we have generally

$$c(\alpha, \beta, \gamma) = 0$$  

As for the roots, they generally have to satisfy the condition (A.1). For a Cartan-Weyl 3-algebra with a single pair of roots $\pm \alpha$, (A.1) is the only condition to be satisfied. For Cartan-Weyl 3-algebra with more than one pair of roots, the root space can generally be decomposed into a number of (say $M$) components: $\Delta_{A}(H) = \bigoplus_{\ell=1}^{M} \Omega_{\ell}$, where there is a null vector $\hat{p}(\ell)$ associated with each $\Omega_{\ell}$. The roots in each $\Omega_{\ell}$ can be decomposed in the form

$$\alpha = \hat{p}(\ell) \wedge \hat{\alpha}(\ell), \quad \alpha \in \Omega_{\ell}.$$  

The corresponding one-form parts $\hat{\alpha}(\ell)$ form the root system of a semisimple Lie algebra $g^{(\ell)}$. Moreover $\hat{p}(\ell)$ and $\hat{\alpha}(\ell)$ satisfy the conditions

$$\hat{p}(\ell) \cdot \hat{p}(\ell') = 0, \quad \text{for all } \ell, \ell',$$  

$$\hat{p}(\ell) \cdot \hat{\alpha}(\ell') = 0, \quad \text{for all } \ell, \ell',$$  

$$\hat{\alpha}(\ell) \cdot \hat{\beta}(\ell') = 0, \quad \text{for } \ell \neq \ell',$$

where the dot product is taken with respect to the metric $g_{IJ}$ of the Lie 3-algebra. As for $g_I(\alpha, \beta)$, we find

$$g_I(\alpha, \beta) = \begin{cases}  
\hat{p}(\ell) c^{(\ell)}(\hat{\alpha}(\ell), \hat{\beta}(\ell)), & \text{for } \alpha, \beta \in \Omega_{\ell}, \\
0, & \text{otherwise.} 
\end{cases}$$  

where the coefficients $c^{(\ell)}$ specifies the $\left[E, E\right] = E$ type brackets of the semisimple Lie algebra $g^{(\ell)}$ as in (A.39).

The construction and classification of Cartan-Weyl 3-algebras has thus been reduced to the problem of constructing the null vectors $\hat{p}(\ell)$ and roots $\hat{\alpha}(\ell)$ that satisfy the conditions (2.37)-(2.39). This can be done fully and explicitly. This will be our task in the next section.
3. Explicit Classification of Cartan-Weyl 3-algebras

Let us first consider the case of a rank $N$ Cartan-Weyl 3-algebra with only a single pair of roots $\pm \alpha$. In this case, the condition $(A.1)$ is the only solution to be satisfied. The general solution of it is

$$\alpha = \alpha_1 \wedge \alpha_2,$$  \hspace{1cm} (3.1)

where $\alpha_1, \alpha_2$ are linearly independent. Given $\alpha_1, \alpha_2$, one can find a basis of $N$ vectors $p_1, p_2, q_3, \cdots q_N$ such that $q_l \cdot \alpha_1 = q_l \cdot \alpha_2 = 0$ and $p_1, p_2$ lies in the plane spanned by $\alpha_1$ and $\alpha_2$. It follows immediately that

$$[q_l \cdot H, H_1, E^{\pm \alpha}] = 0,$$

$$[q_l \cdot H, E^\alpha, E^{-\alpha}] = 0, \hspace{1cm} l = 3, \cdots, N.$$  \hspace{1cm} (3.2)

Thus the $N - 2$ generators $q_l \cdot H$ are central in the algebra and the nontrivial part of the Lie 3-algebra has only 4 generators $\{p_1 \cdot H, p_2 \cdot H, E^\alpha, E^{-\alpha}\}$. Depending on the signature of the vector space spanned by the vectors $p_1, p_2$, this 3-algebra is isomorphic to $A_{0,4}, A_{1,3}$ or $A_{2,2}$.

Therefore let us consider the general case where the Cartan-Weyl 3-algebra has more than a single pair of root. Since our results given in the section 2.2 relies on the existence of the null vectors $\hat{p}^{(l)}$, therefore it is natural to classify the Cartan-Weyl 3-algebra according to the number of negative eigenvalues of the metric $g_{IJ} := \langle H_I, H_J \rangle$.

With a Hermitian structure

$$(H_I)^\dagger = H_I, \hspace{1cm} (E^\alpha)^\dagger = E^{-\alpha},$$  \hspace{1cm} (3.3)

this is also the same as the index of the metric over the whole Lie 3-algebra $A$.

3.1 Index 0 and 1

For index 0, Cartan-Weyl 3-algebra is possible only if the 3-algebra has a single pair of root. In this case the 3-algebra is isomorphic to $A_{0,4}$ plus a number of central elements. This is decomposable. Obviously it is sufficient to classify indecomposable Cartan-Weyl 3-algebra. In the following we will consider only indecomposable Cartan-Weyl 3-algebra.

Next consider the case of index 1. Let us choose a basis of the Cartan subalgebra as $\{H_I\} = \{H_\hat{I}, H_\alpha\}$, $\hat{I} = 1, \cdots, \mathfrak{N}, a = 1, 2$, such that the metric takes the form

$$g_{IJ} = \begin{pmatrix} g_{\hat{I}\hat{J}} & 1 \\ 1 & -1 \end{pmatrix},$$  \hspace{1cm} (3.4)

where $g_{\hat{I}\hat{J}}$ is Euclidean. Here we have the rank $N = \mathfrak{N} + 2$. 

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To construct a Cartan-Weyl 3-algebra, we need to pick $\hat{\alpha}(^l)$ and null vectors $\hat{p}(^l)$ such that (2.37)-(2.39) are satisfied. Knowing already that $\hat{\alpha}(^l)$ are roots of a Lie algebra, the most general solution is to take $(\hat{\alpha}_I(^l)) = (\hat{\alpha}_I(0),0,0)$ and to take the null vectors in the orthogonal complements $\hat{p}(^l) = (0,\cdot,\cdot)$. There can be (up to overall normalization) two choices of such null vectors. But since they cannot be orthogonal to each other, it means we can only use one null vector in the index 1 case. Without loss of generality, let us take

$$\hat{p} = (0,1,1)/\sqrt{2},$$

(3.5)

and the roots of the Cartan-Weyl 3-algebra are of the form

$$\alpha = \hat{p} \wedge \hat{\alpha}, \quad \text{where} \quad (\hat{\alpha}_I) = (\hat{\alpha}_I,0,0).$$

(3.6)

Explicitly, it has the nonvanishing components $\alpha_{a\bar{i}} = \hat{p}_a \hat{\alpha}_{\bar{i}}$.

3-brackets

It is convenient to reorganize the basis of Cartan generators as

$$H_I = \{H_I, v, u\},$$

(3.7)

where

$$v := \hat{p} \cdot H, \quad u := \hat{p} \cdot H$$

(3.8)

with

$$\hat{p} := (0,1,-1)/\sqrt{2}.$$  

(3.9)

It is

$$\langle v, v \rangle = \langle u, u \rangle = 0 \quad \text{and} \quad \langle v, u \rangle = \hat{p} \cdot \hat{p} = 1.$$  

(3.10)

Since the “step” generators $E^\alpha$ are characterized by the one-form part $\hat{\alpha}$, it can be written as $E^\alpha = E^{\hat{\alpha}}$. As a vector space, the Cartan-Weyl 3-algebra of index 1 is given by

$$\mathcal{A} = \mathfrak{g} \oplus \mathbb{C}(u,v).$$  

(3.11)

The Cartan-Weyl 3-algebra relations read

$$[v, \cdot, \cdot] = 0,$$

(3.12)

$$[u, H_I, E^{\hat{\alpha}}] = \hat{\alpha}_I E^{\hat{\alpha}},$$

(3.13)

$$[u, E^{\hat{\alpha}}, E^{\hat{\beta}}] = \begin{cases} 
\hat{\alpha}_I g^{ij} H_j, & \hat{\alpha} + \hat{\beta} = 0, \\
c(\hat{\alpha}, \hat{\beta}) E^{\hat{\alpha} + \hat{\beta}}, & \hat{\alpha} + \hat{\beta} \text{ nonzero roots}, 
\end{cases}$$

(3.14)

$$[E^{\hat{\alpha}}, E^{\hat{\beta}}, E^{\hat{\gamma}}] = \begin{cases} 
-c(\hat{\alpha}, \hat{\beta}) v, & \hat{\alpha} + \hat{\beta} + \hat{\gamma} = 0, \\
0, & \text{otherwise}. 
\end{cases}$$

(3.15)
The relations (3.12) - (3.15) can be summarized as
\[
\begin{align*}
[v, \cdot, \cdot] &= 0, \\
[u, g_1, g_2] &= [g_1, g_2]_g, \\
[g_1, g_2, g_3] &= -(\langle g_1, g_2 \rangle, g_3)_g, 
\end{align*}
\]
where \( g_i \in \mathbb{C}(H_I, E^{\hat{\alpha}}) := \mathfrak{g} \) is a semisimple Lie algebra with the brackets \([\cdot, \cdot]_g\) and the metric \(\langle \cdot, \cdot \rangle_g\) as defined by (2.7).

We note that the Cartan-Weyl 3-algebra of index 1 is precisely the same as the Lorentzian 3-algebra [18].

3.2 Index 2

For index 2, let us choose a basis of the Cartan generators as \(\{H_I\} = \{H_{\hat{\alpha}}, H_a\}, \hat{I} = 1, \cdots, \mathfrak{N}, a = 1, \cdots, 4\), such that the metric takes the form with nonzero entries
\[
\begin{pmatrix}
g_{\hat{I}\hat{J}} \\
1 & -1 & 1 & -1 \\
\end{pmatrix},
\]
where \(g_{\hat{I}\hat{J}}\) is Euclidean. In this case, the rank \(N = \mathfrak{N} + 4\).

We can solve the conditions (2.38) by taking \(\alpha_I = (\alpha_{\hat{I}}; 0, 0, 0, 0)\) and the null vectors in the complementary part of the vector space. There are four such linearly independent null vectors. One can choose the basis of null vectors \(\{\hat{p}^{(1)}, \hat{p}^{(2)}, \check{p}^{(1)}, \check{p}^{(2)}\}\) such that \(\hat{p}^{(i)} \cdot \hat{p}^{(j)} = 0, \check{p}^{(i)} \cdot \check{p}^{(j)} = 0, \hat{p}^{(i)} \cdot \check{p}^{(j)} = \delta_{ij}\). Moreover since any linear combinations of the \(\hat{p}\)-vectors (or of the \(\check{p}\)-vectors) are still null, the most general set (up to trivial overall normalization factor) of null vectors which satisfy (2.37) is thus given by
\[
\hat{p}^{(\lambda)} := \hat{p}^{(1)} + \lambda \check{p}^{(2)},
\]
where \(\lambda\) is an arbitrary nonvanishing finite constant. One may use as many choice of \(\lambda\) as one like. Denote such a set as \(\Lambda\). The set of usable null vectors is thus given by \(\{\hat{p}^{(i)}\} = \{\hat{p}^{(i)}, \hat{p}^{(\lambda)}\}\) with \(i = 1, 2\) and \(\lambda \in \Lambda\).

To proceed, let us further partition \((\hat{I})\) into
\[
(\hat{I}) = (\hat{I}_1, \hat{I}_2, (\hat{I}_{\lambda})_{\lambda \in \Lambda}),
\]
where
\[\hat{p}^{(1)} = (0, 1, 1, 0)/\sqrt{2}, \hat{p}^{(2)} = (0, 0, 0, 1)/\sqrt{2}, \check{p}^{(1)} = (0, 1, -1, 0)/\sqrt{2}, \check{p}^{(2)} = (0, 0, 0, 1)/\sqrt{2}.
\]
However we will not need these explicit expressions.
such that $\hat{I}_i = 1, 2, \ldots, N_i$, $\hat{I}_\lambda = 1, 2, \ldots, N_\lambda$ and $\sum_i N_i + \sum_\lambda N_\lambda = \mathfrak{h}$. The set of Cartan generators $H_I$ can be relabelled as

$$\{H_I\} = \{H_{\hat{I}_1}, H_{\hat{I}_2}, \{H_{\hat{I}_\lambda}\}_{\lambda \in \Lambda}\}. \quad (3.20)$$

Corresponds to each subset $\{H_{\hat{I}_i}\}$ or $\{H_{\hat{I}_\lambda}\}$, one can associate a set of 1-form roots such that they have nonzero components only when their indices are in $\hat{I}_i$ or $\hat{I}_\lambda$:

$$\hat{\alpha}_I^{(i)} = (0, \cdots, 0, \hat{\alpha}_I^{(i)}, 0, \cdots, 0; 0, 0, 0, 0), \quad i = 1, 2, \quad (3.21)$$

$$\hat{\alpha}_I^{(\lambda)} = (0, \cdots, 0, \hat{\alpha}_I^{(\lambda)}, 0, \cdots, 0; 0, 0, 0, 0), \quad \lambda \in \Lambda \quad (3.22)$$

The last four entries refer to the “internal” indices $a = 1, 2, 3, 4$. The 1-forms (3.21), (3.22) satisfy the conditions (2.38) and (2.39) by construction and lead to the following set of two-form roots:

$$\alpha^{(i)} = \hat{p}^{(i)} \wedge \hat{\alpha}_I^{(i)}, \quad i = 1, 2, \quad (3.23)$$

$$\alpha^{(\lambda)} = \hat{p}^{(\lambda)} \wedge \hat{\alpha}_I^{(\lambda)}, \quad \lambda \in \Lambda. \quad (3.24)$$

Or, in terms of the nonzero components, we have

$$\alpha^{(i)}_{a\hat{I}_i} = \hat{p}^{(i)}_{a \hat{I}_i} \hat{\alpha}_I^{(i)}, \quad i = 1, 2, \quad (3.25)$$

$$\alpha^{(\lambda)}_{a\hat{I}_\lambda} = \hat{p}^{(\lambda)}_{a \hat{I}_\lambda} \hat{\alpha}_I^{(\lambda)}, \quad \lambda \in \Lambda. \quad (3.26)$$

In addition to these roots which have a “mixed” indices structure, one can also consider roots with “internal” indices. In particular to satisfy (2.38) and (2.39), the only possibility is have a root which is a linear combination of $\hat{p}^{(1)}$ and $\hat{p}^{(2)}$. Wedging it with (3.18) leads to the 2-form roots

$$r^{(n)} = c_n \hat{p}^{(1)} \wedge \hat{p}^{(2)}, \quad (3.27)$$

where $c_n$ is a constant and $n$ is taken from an arbitrary set $X$. It is easy to show that $c(r^{(n)}, r^{(m)}) = 0$ for all $n, m \in X$.

All in all, we obtain the roots (3.23), (3.24) and (3.27). This give rises to the step generators

$$E^{\alpha^{(i)}}, E^{\alpha^{(\lambda)}}, E^{\pm r^{(n)}}. \quad (3.28)$$

Next let us regroup the generators of the Cartan-Weyl 3-algebra in a way which will be convenient for our analysis. Let us start with the set of Cartan generators

$$\{H_I\} = \{H_{\hat{I}_1}, H_{\hat{I}_2}, \{H_{\hat{I}_\lambda}\}_{\lambda \in \Lambda}, H_a\}. \quad (3.29)$$
To express the 3-brackets, it is convenient to use the following 4 generators instead of the $H_a$’s:

$$v^{(i)} := \hat{p}^{(i)} \cdot H, \quad u^{(i)} := \bar{\hat{p}}^{(i)} \cdot H, \quad i = 1, 2. \quad (3.30)$$

It is $\langle v^{(i)}, v^{(j)} \rangle = 0$, $\langle u^{(i)}, u^{(j)} \rangle = 0$, $\langle v^{(i)}, u^{(j)} \rangle = \delta_{ij}$. Next we denote the generators

$$x^{(n)}_{\pm} := E^{\pm r(n)}, \quad (3.31)$$

where the metric is $\langle x_{+}^{(n)}, x_{-}^{(m)} \rangle = \delta_{nm}$. It is also convenient to group the step generators and the Cartan generators in the following manners and introduce the vector spaces

$$g^{(i)} := C(H\hat{t}_i, E^{\hat{a}(i)}), \quad g^{(\lambda)} := C(H\hat{t}_\lambda, E^{\hat{a}(\lambda)}). \quad (3.32)$$

The Cartan-Weyl 3-algebra of index 2 is thus given by

$$\mathcal{A} = \bigoplus_{i=1}^{2} g^{(i)} \oplus \bigoplus_{\lambda \in \Lambda} g^{(\lambda)} \oplus \mathbb{C}(u^{(1)}, v^{(1)}) \oplus \mathbb{C}(u^{(2)}, v^{(2)}) \oplus E \quad (3.33)$$

as vector space. Here $E$ is an even dimensional vector space spanned by the elements $x_{\pm}^{(n)}$, $n \in X$. Different components of the direct sum in (3.33) are orthogonal to each other. The presence of the vector space $E$ is a new feature when the index is higher than 1.

**3-brackets**

Now let us express the 3-algebra relations (2.20)-(2.23c) in terms of these generators. Using our results (A.13) and the result that $c^{(i)}$ are those coefficients for a semisimple Lie algebra, we obtain immediately the following nonvanishing 3-brackets:

$$[u^{(i)}, g_1, g_2] = \begin{cases} [g_1, g_2] g^{(i)}, & g_1, g_2 \in g^{(i)}, \\ \lambda_1 [g_1, g_2] g^{(\lambda)}, & g_1, g_2 \in g^{(\lambda)}, \end{cases} \quad (3.34)$$

where $\lambda_1 = 1$ and $\lambda_2 = \lambda$, and

$$[g_1, g_2, g_3] = \begin{cases} -\langle [g_1, g_2], g_3 \rangle_{g^{(i)}} v^{(i)}, & g_1, g_2, g_3 \in g^{(i)}, \\ -\langle [g_1, g_2], g_3 \rangle_{g^{(\lambda)}} (v^{(1)} + \lambda v^{(2)}), & g_1, g_2, g_3 \in g^{(\lambda)}. \end{cases} \quad (3.35)$$

Also we have

$$[u^{(1)}, u^{(2)}, x_{\pm}^{(n)}] = c_n x_{\pm}^{(n)}, \quad [u^{(1)}, u^{(2)}, x_{\pm}^{(n)}] = -c_n x_{\pm}^{(n)}, \quad (3.36)$$

$$[u^{(1)}, x_{\pm}^{(n)}, x_{\mp}^{(n)}] = c_n v^{(2)}, \quad [u^{(2)}, x_{\pm}^{(n)}, x_{\mp}^{(n)}] = -c_n v^{(1)}. \quad (3.37)$$
One may also rewrite (3.36) and (3.37) in terms of a different basis. Introducing $x_\pm^{(n)} = (x^{(n)} \pm iy^{(n)})/\sqrt{2}$. The metric is $\langle x^{(n)}, x^{(m)} \rangle = \langle y^{(n)}, y^{(m)} \rangle = \delta_{nm}$. $\langle x^{(n)}, y^{(m)} \rangle = 0$. The relations (3.36) and (3.37) can be rewritten in the form

\[
[u^{(1)}, u^{(2)}, x] = Jx, \\
[u^{(1)}, x, y] = \langle Jx, y \rangle v^{(2)}, \\
[u^{(2)}, x, y] = -\langle Jx, y \rangle v^{(1)},
\]

where $x, y \in E$ in general and $J$ is an $so(E)$ matrix such that $Jx^{(n)} = ic_n y^{(n)}, Jy^{(n)} = -ic_n x^{(n)}$.

Quite amazingly, the relations (3.34), (3.35) and (3.38) are precisely the same as the 3-algebras obtained in [19] for the index 2 case. There the 3-algebra was constructed by requiring it to have a maximally isotropic center. Here we have obtained the same Lie 3-algebra from a different requirement as a Cartan-Weyl 3-algebra.

Finally we remark that in addition to the way (3.17) in splitting the metric, we can also split it differently

\[
g_{IJ} = \begin{pmatrix}
g_{ij} & \mathbf{1} \\
\mathbf{1} & -1
\end{pmatrix},
\]

such that a (1,1) subspace is singled out separately and the metric $g_{ij}$ is Lorentzian. Associated with the (1,1) subspace is a null vector which one can use to construct the two-form roots. The construction is exactly the same as for the the index 1 case except that $g_{ij}$ is now non-Euclidean. One obtain immediately the Lorentzian 3-algebra $\mathcal{A} = g \oplus \mathbb{C}(u, v)$ with $g$ being a semisimple Lie algebra of index 1. This is nothing new.

### 3.3 Higher index $m \geq 3$

The above construction can be generalized to index 3 or higher easily. First let us choose a basis of the Cartan generators such that the metric takes the form with nonzero entries

\[
g_{IJ} = \begin{pmatrix}
g_{ij} \\
l_{ab}
\end{pmatrix},
\]

where $g_{ij}$ is Euclidean and $l_{ab}$ is a $2m$-dimensional metric with signature $(m, m)$. The set of Cartan generators is $\{H_I\} = \{H_{\tilde{I}}; H_a\}$ with $\tilde{I} = 1, \ldots, \mathfrak{N}; a, = 1, \ldots, 2m$. The rank is $N = \mathfrak{N} + 2m$. 

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l_{ab}
\end{pmatrix},
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### Root system
As before, to solve (2.38), we can take the 1-form roots $\alpha_I = (\alpha_I^a; 0)$ and the null vectors in the orthogonal space: $\hat{p} = (0; \cdot)$. There are $2m$ such linearly independent null vectors. One can choose a basis of null vectors $\{\hat{p}^{(i)}, \tilde{p}^{(i)}\}$, $i = 1, \cdots, m$ such that
\begin{equation}
\hat{p}^{(i)} \cdot \hat{p}^{(j)} = 0, \quad \tilde{p}^{(i)} \cdot \tilde{p}^{(j)} = 0, \quad \hat{p}^{(i)} \cdot \tilde{p}^{(j)} = \delta_{ij}, \quad i, j = 1, \cdots, m.
\end{equation}
In addition to the $\hat{p}^{(i)}$'s, one can also solve (A.16) and (A.17) with the following null vectors
\begin{equation}
\hat{p}^{(\lambda)} := \sum_i \lambda_i \hat{p}^{(i)},
\end{equation}
where $\lambda_i$'s are arbitrary constants. One can use as many choice of $\lambda_i$ as one like. Denotes such a set as $\Lambda$. The set of usable null vectors is given by $\{\hat{p}^{(i)}\} = \{\hat{p}^{(i)}, \hat{p}^{(\lambda)}\}$ with $i = 1, \cdots, m$ and $(\lambda_i) \in \Lambda$.

Using these null vectors, one can construct solution to (2.38) and (2.39) by first partitioning $(\hat{I})$ into many parts
\begin{equation}
(\hat{I}) = (\hat{I}_i = 1, \cdots, m, \hat{I}_\lambda = 1, 2, \cdots, N_\lambda) \quad (\lambda \in \Lambda),
\end{equation}
such that $\hat{I}_i = 1, 2, \cdots, N_i$ for $i = 1, \cdots, m$, $\hat{I}_\lambda = 1, 2, \cdots, N_\lambda$ and $\sum_i N_i + \sum_\lambda N_\lambda = \mathfrak{n}$. The set of Cartan generators $H_{\hat{I}}$ can be relabelled as
\begin{equation}
\{H_{\hat{I}}\} = \{\{H_{\hat{I}_i}\}_{i=1, \cdots, m}, \{H_{\hat{I}_\lambda}\}_{\lambda \in \Lambda}\}.
\end{equation}
Corresponds to each subset $\{H_{\hat{I}_i}\}$, one can associate a set of 1-form roots such that they have nonzero components only when their indices are in $\hat{I}_i$ or $\hat{I}_\lambda$
\begin{align}
\hat{\alpha}^{(i)}_I &= (0, \cdots, 0, \hat{\alpha}^{(i)}_{\hat{I}_i}, 0, \cdots, 0; 0), \quad (i = 1, \cdots, m), \\
\hat{\alpha}^{(\lambda)}_I &= (0, \cdots, 0, \hat{\alpha}^{(\lambda)}_{\hat{I}_\lambda}, 0, \cdots, 0; 0).
\end{align}
The last entries refers to the “internal” indices $a = 1, \cdots, 2m$. The set (3.45), (3.46) lead to the following set of two-form roots:
\begin{align}
\alpha^{(i)} &= \hat{p}^{(i)} \wedge \hat{\alpha}^{(i)}, \quad (i = 1, \cdots, m), \\
\alpha^{(\lambda)} &= \hat{p}^{(\lambda)} \wedge \hat{\alpha}^{(\lambda)}, \quad (\lambda \in \Lambda).
\end{align}

In addition to these roots which have a “mixed” indices structure, one can also consider roots $\alpha_{ab}$ with only “internal” indices. In particular to satisfy (2.38) and (2.39), the only possibility is to use 1-form roots which are linear combination of $\hat{p}^{(i)}$. Wedging it with $\hat{p}^{(i)}$ or $\hat{p}^{(\lambda)}$ leads to the 2-form roots of the form
\begin{equation}
r^{(\mu)} = \sum_{ij} \mu_{ij} \hat{p}^{(i)} \wedge \hat{p}^{(j)},
\end{equation}
where \( \mu_{ij} \) are constants and not all equal to zero. There is an internal root (3.49)
corresponds to each choice of \((\mu_{ij})\). Denote the set of \((\mu)\) by \(X\). It easy to see that one
should take \(c(r^{(\mu)}, r^{(\mu')}) = 0\) for all \((\mu), (\mu') \in X\).

The roots (3.47), (3.48) and (3.49) give rises to the step generators
\[
E^{\alpha(i)}, E^{\alpha(\lambda)}, E^{\pm r^{(\mu)}}. \tag{3.50}
\]

As before, let us regroup the generators of the 3-algebra in a way which will be conven-
ient for our analysis. Let us start with the set of Cartan generators. It is convenient
to use the following \(2m\) generators instead of the \(H_a\)'s:
\[
v^{(i)} := \hat{p}^{(i)} \cdot H, \quad u^{(i)} := \tilde{p}^{(i)} \cdot H, \quad i = 1, \cdots, m. \tag{3.51}
\]

It is \(\langle v^{(i)}, v^{(j)} \rangle = 0, \langle u^{(i)}, u^{(j)} \rangle = 0, \langle v^{(i)}, u^{(j)} \rangle = \delta_{ij}\). Next we introduce the notations
\[
x^{(\mu)}_{\pm} := E^{\pm r^{(\mu)}}, \tag{3.52}
\]
where the metric is \(\langle x^{(\mu)}_{+}, x^{(\mu')}_{-} \rangle = \delta_{\mu \mu'}\). It is also convenient to group the step generators
and the Cartan generators in the following manners and introduce the vector spaces
\[
g^{(i)} := \mathbb{C}(H_{\hat{i}}, E^{\hat{i}(i)}), \quad g^{(\lambda)} := \mathbb{C}(H_{\hat{\lambda}}, E^{\hat{\lambda}(\lambda)}). \tag{3.53}
\]

The Cartan-Weyl 3-algebra of index \(m \geq 3\) is thus given by
\[
\mathcal{A} = \left( \bigoplus_{i=1}^{m} g^{(i)} \right) \oplus \left( \bigoplus_{\lambda \in \Lambda} g^{(\lambda)} \right) \oplus \left( \bigoplus_{i=1}^{m} \mathbb{C}(u^{(i)}, v^{(i)}) \right) \oplus E \tag{3.54}
\]
as vector space. Here \(E\) is a even dimensional vector space spanned by the elements
\(x^{(\mu)}_{\pm}, (\mu) \in X\). Different components of the direct sum in (3.54) are orthogonal to each
other.

3-brackets

The 3-brackets can be worked out similarly and we obtain the following nonvanishing
relations for the Cartan-Weyl 3-algebra:
\[
\begin{align*}
[u^{(i)}, g_1, g_2] &= \begin{cases} 
[g_1, g_2]_{g^{(i)}}, & g_1, g_2 \in g^{(i)}, \\
\lambda_i [g_1, g_2]_{g^{(\lambda)}}, & g_1, g_2 \in g^{(\lambda)},
\end{cases} \\
[g_1, g_2, g_3] &= \begin{cases} 
-\langle [g_1, g_2], g_3 \rangle_{g^{(i)}} v^{(i)}, & g_1, g_2, g_3 \in g^{(i)}, \\
-\langle [g_1, g_2], g_3 \rangle_{g^{(\lambda)}} (\sum \lambda_i v^{(i)}), & g_1, g_2, g_3 \in g^{(\lambda)},
\end{cases}
\end{align*} \tag{3.55}
\]
\[
-18-
\]
\[ [u^{(i)}, u^{(j)}, x^{(\mu)}] = \pm \mu_{ij} x^{(\mu)}_\pm, \quad (3.57) \]
\[ [u^{(i)}, x^{(\mu)}_+, x^{(\mu)}_-] = \sum_j \mu_{ij} v^{(j)}. \quad (3.58) \]

It is quite remarkable that, starting out as a natural generalization of a Cartan-Weyl algebras, Cartan-Weyl 3-algebras as defined by (2.20)-(2.23c) can be constructed and classified completely.

4. Remarks on Fuzzy $S^3$ in BLG Theory

In the original construction of BLG, the Lie 3-algebra $A_4$ was employed. The metric on $A_4$ is positive definite and the BLG model defines a unitary quantum field theory. The use of $A_4$ was motivated by the studies of Basu and Harvey [20] whose main objective was to construct a generalization of the Nahm equation for describing intersecting M-branes. Employing a Lie 3-algebraic structure of $A_4$, the Basu-Harvey equation admits a solution whose cross section is given by a fuzzy $S^3$ and describes the puffing up of a system of multiple M2-branes into a M5-brane.

In general, one would like to employ more general metric Lie 3-algebras with arbitrary higher ranks in the BLG description of multiple M2-branes. The choice of the Lie 3-algebras should be such that the BLG theory or generalization of the original construction, gives a unitary Quantum Field Theory. The Lie 3-algebra should also allows different embedding of $A_4$ in it in much the same way as one can find fuzzy $S^2$ of different sizes in $SU(N)$. Presumably the different $A_4$’s would then be characterized by some kinds of Casimir of the $A_4$ algebra which corresponds to having different number of M2-branes puffing up to a single M5-brane in the Basu-Harvey’s fuzzy funnel solution of the multiple M2-branes system. However achieving these objectives turns out to be highly nontrivial. For discussions about unitary BLG theory, see [18, 19, 22–26]. At present, manifest unitary remains a major obstacle of the BLG theory. To resolve it will require the use of a different kind of metric Lie 3-algebras together with a novel ghost decoupling mechanism. This is beyond the scope of this work and we will have nothing more to say on this problem.

The problem of finding a fuzzy $S^3$ solution has been considered recently for the ABJM theory [28]. Surprisingly to the best of our knowledge, the question of finding a fuzzy $S^3$ for the BLG theory in general has not been considered before. We will examine this issue in details now.

\[ ^4 \text{If one is interested only in the supersymmetric equations of motion, it is possible to do without a metric [21].} \]
It is instructive to first recall the case of multiple D1-branes. There the Nahm equation, the BPS equation for the $U(N)$ non-abelian Born-Infeld theory of the D1-strings, admits a fuzzy funnel whose transverse cross section is a fuzzy $S^2$ and describes a bunch of D1-strings puffing up into a D3-brane [27], corresponding to a D1-D3 brane intersection in string theory. Similarly, one would like to be able to describe the M2-M5 branes intersection as a fuzzy funnel solution of a BPS equation of the multiple M2-branes theory. In fact this was the original motivation leading Basu and Harvey [20] to write down such a BPS equation. It has been demonstrated in [4] that the BLG Lagrangian based on the algebra $A_4$ admits a fuzzy funnel solution whose energy density scales as expected of a M2-M5 intersection (both the massless and massive cases). However the problem has not been discussed for BLG theory based on more general Lie 3-algebras, e.g. the Lorentzian 3-algebra.

In general a fuzzy $S^3$ solution in the BLG theory is described by having $X^P$, $P = 1, 2, 3, 4$ satisfying the $A_4$ algebra

$$[X^P, X^Q, X^R] = i\epsilon^{PQR}S X^S.$$  \hspace{1cm} (4.1)

This describes a $SO(4)$ invariant distribution in the theory. In addition one needs also a certain “size” condition so that one can be sure one is describing a single fuzzy $S^3$. In the case of fuzzy $S^2$ for D1-D3 intersection, the size condition is written down with respect to the representation taken by the $X^P$’s which specifies the radius of the fuzzy $S^2$. For fuzzy $S^3$, one can expect that one will need to generalize the concept of a representation of Lie 3-algebras in order to specify a suitable “size” condition. It is not known how to write down arbitrary representations for a general Lie 3-algebra. See [29] for some discussions. However it is possible that for certain specific kinds of Lie 3-algebras, for example, such as the Cartan-Weyl 3-algebras due to the Lie algebraic structure they inherited. In fact, because of (2.7) and (2.19), the invariant metric of the Cartan-Weyl 3-algebra is precisely the same as the invariant metric of the underlying Lie algebra. Since the later can be immediately generalized to the Killing metric which is defined for an arbitrary representation of the Lie algebra, this can be extended to the Cartan-Weyl 3-algebras immediately. In the following, however, we will focus on the algebraic condition (4.1) only which is independent of any representation issues.

Let us first consider the Lorentzian 3-algebra (3.16). Let $X^P = X^P u u + X^P v v + X^P \tilde{g}^\alpha$, where the Lie algebra generators $g^\alpha$ obeys $[g^\alpha, g^\beta] = f^{\alpha\beta\gamma} g^\gamma$. The equation (4.1) gives

$$X^P = 0,$$

$$i\epsilon^{PQR}S X^Q v = -X^P \tilde{g} X^Q \beta \gamma X^R \gamma f^{\alpha\beta\gamma},$$

$$i\epsilon^{PQR}S X^R = X^P \tilde{g} X^Q \beta \gamma X^R \gamma u f^{\alpha\beta\gamma} + (P, Q, R \text{ cyclic}).$$

\hspace{1cm} (4.2)
Hence we obtain $X^S_u = X^S_\delta = X^S_v = 0$ and the Lorentzian BLG theory does not admit any fuzzy $S^3$ solution.

The above consideration can be generalized immediately to the general Cartan-Weyl 3-algebra (2.20)-(2.23c). Let $X^P = X^P_I H_I + X^P_\alpha E^\alpha$. The equation (4.1) gives

$$i\epsilon^{PQR} S^L g_{LI} = - \sum_{\alpha + \beta + \gamma = 0} X^P_\alpha X^Q_\beta X^R_\gamma g_I(\alpha, \beta)$$

$$+ \sum_\beta X^{PL} X^Q_\beta X^R_{-\beta} g_I + (P, Q, R \text{ cyclic}),$$

(4.3)

$$i\epsilon^{PQR} S^I X^\alpha_\alpha = X^{PI} X^{QJ} X^R_\alpha g_I(\beta, \alpha - \beta) + (P, Q, R \text{ cyclic}).$$

(4.4)

Now given the general solution (2.36), (2.40), we can multiply the equation (4.3) with $\hat{p}^{(\ell)I}$ and using the properties (2.37), (2.38) to obtain that

$$X^S I \hat{p}^{(\ell)I} = 0,$$

for all $\ell$ and $S$. (4.5)

Substituting this into (4.4), we then obtain $X^S_\alpha = 0$. This, substituting back into (4.3), then implies that $X^S I = 0$. Therefore the Cartan-Weyl 3-algebra does not admit $A_4$ as a subalgebra. We remark that this is quite different from the case of Lie algebra where any Lie algebra admits $SU(2)$ as a subalgebra.

The problem of finding a fuzzy $S^3$ solution has also been considered in [28] for the ABJM theory. For large value of the level $k$ where one can trust the semi-classical analysis, it was found that the ABJM theory admits only a fuzzy $S^2$ structure, rather than a fuzzy $S^3$. This is the expected result since for generic $k$, the geometry is $S^3/Z_k$ and the M-theory circle $S^1/Z_k$ becomes zero in the $k \to \infty$ limit, reducing the system to the D2-D4 branes intersection in IIA string theory. Since the large $k$ limit is needed in order to have a perturbative formulation of the ABJM theory, finding a fuzzy $S^3$ as the (semi-)classical geometry in the ABJM theory is impossible.

For the BLG theory, there is a similar problem. Presumably the level $k$ corresponds to some order of orbifolding [30] and a theory of multiple M2-branes in flat space is given by a level one BLG theory with a certain choice of Lie 3-algebras. It is not clear whether the fuzzy $S^3$ should emerge as a classical solution as in the original BLG theory [4] or only as a solution of the full quantum system as in the ABJM theory. Since the former situation is what happened in the original BLG theory, it might be natural to expect that this to be also the case for the BLG theories based on more general Lie 3-algebras. Assuming that this is the case, this no-go theorem of finding a fuzzy $S^3$ solution in the Cartan-Weyl 3-algebras then means that one need to consider more general class of Lie 3-algebras for use in the BLG theory. In the companion paper [15], a certain generalization
of the Cartan-Weyl 3-algebras is suggested and we will show that the no-go theorem can be bypassed easily in this class of Lie 3-algebras, giving the hope of having fuzzy $S^3$ solution.

5. Discussions

In this paper we have generalized the notion of a Cartan-Weyl basis for a Lie 3-algebra. We have also shown that the consistency conditions defining a Cartan-Weyl 3-algebra can be solved exactly, leading to a complete classification of Cartan-Weyl 3-algebras. This is the main result of this paper. It is natural to speculate that Cartan-Weyl 3-algebras may be useful for describing some kinds of generalized symmetry. It will be interesting to discover more of this.

We have mostly worked with metric Lie 3-algebras in this paper. It is clear that the concept of a Cartan-Weyl basis can be similarly defined for higher metric Lie $n$-algebras and the existence of a Cartan-Weyl basis is equivalent to the requirement that the solutions of the “eigenvalue equation”

$$[H_{I_1}, \cdots, H_{I_{n-1}}, Y^a] = \psi_{I_1 \cdots I_{n-1}}^a b Y^b$$

are non-degenerate, and that the restriction of the metric to the $H$’s is non-degenerate. As a result, a Cartan-Weyl $n$-algebra is equipped with a number of mutually commuting Cartan generators $H_I$ together with a number of step generators $E^\alpha$ parametrized by a root space of non-degenerate $(n-1)$-forms $\alpha$. As we have seen above, Cartan-Weyl 3-algebras have curiously built in them a structure of semisimple Lie algebras. One can expect that Cartan-Weyl $n$-algebras will also have built in them a certain special kind of Lie $(n-1)$-algebras. It will be interesting to solve the corresponding consistency conditions and construct the Cartan-Weyl $n$-algebras explicitly and find this out. We speculate that these kinds of Lie $n$-algebras may have a good chance to be of use in physics.

In order to understand which kind of Lie 3-algebras (or Lie $n$-algebras in general) might appear in physical descriptions, it will be very helpful to understand how to “integrate” the infinitesimal transformations described by Lie 3-algebras to finite transformations since it is usually much more clear how the finite transformations should be constrained. One may call these “Lie 3-group” transformations. There are many related interesting mathematical questions one may ask. For example, Lie 3-group is certainly not a Lie group. How is a Lie 3-group defined? It seems natural that Lie 3-bracket may have its origin in a tertiary, perhaps nonassociative, product structure of a Lie 3-group. Is it true? Is there a Baker-Hausdorff formula? While fascinating, there is nothing
known in the literature about how to think about a Lie 3-group. Therefore we will have to look for other traits in order to identify the desired type of Lie 3-algebras that could be relevant for physical applications.

For applications in the BLG models, the prospect of Cartan-Weyl 3-algebras is not good since the corresponding BLG models are equivalent to ordinary Yang-Mills theories and so, not surprisingly, they do not admit any fuzzy $S^3$ structure. Such BLG models describe D-branes rather than multiple M2-branes. We will argue in the paper [15] that the Lie 3-algebras of interest should satisfy a certain reduction condition. We will show that this reduction condition leads to a class of metric Lie 3-algebras which naturally generalizes the Cartan-Weyl 3-algebras introduced in this paper. These generalized Cartan-Weyl 3-algebras have the same form (1.2) of the Lie 3-brackets. However their Cartan subalgebra can be non-abelian in general, i.e. the 3-brackets $[H_I, H_J, H_K] \neq 0$. This modification is indeed a welcome one. We will explain in [15] how this modification may help with getting fuzzy $S^3$ solutions in the corresponding BLG models.

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A. Consistency Conditions of Cartan-Weyl 3-algebras

To construct a Cartan-Weyl 3-algebra, one need to specify the system of two-form roots $\alpha_{IJ}$ and the coefficients $g_I(\alpha, \beta)$, $c(\alpha, \beta, \gamma)$. These are constrained by the consistency of (2.20)-(2.23c) with the fundamental identity (FI). Let us examine this in details now.

Consistency of (2.20) with the FI gives the condition

$$\alpha_{IJ}\alpha_{KL} + \alpha_{JK}\alpha_{IL} + \alpha_{KI}\alpha_{JL} = 0.$$  \hspace{1cm} (A.1)

This condition is nontrivial only if the rank is at least four. For this general case, let us introduce auxiliary differentials $dx^K$, then the condition can be written as

$$\alpha \wedge i_\xi \alpha = 0,$$  \hspace{1cm} (A.2)

where $\alpha := \alpha_{IJ}dx^I dx^J$ is a two form and $i_\xi$ is the Cartan inner product with $\xi$ being an arbitrary vector. It is then clear that the general solution of (A.1) is given by

$$\alpha = \alpha_1 \wedge \alpha_2.$$  \hspace{1cm} (A.3)
where $\alpha_1, \alpha_2$ are arbitrary one-forms. In terms of components, it is $\alpha_{IJ} = \alpha_{1I} \alpha_{2J} - \alpha_{2I} \alpha_{1J}$. We remark that although the condition (A.1) is trivial for the case of $N = 3$, the 2-form root (dual to a vector) can still be written in the form of (A.3). The same is true for $N = 2$. So we conclude that the 2-form roots of a Cartan-Weyl 3-algebra always take the form (A.3).

\[
[[H_I, H_J, H_K], E^\alpha, E^\beta] = 0:
\]

Consistency with the FI gives the condition

\[
(\alpha + \beta)_{JK} g_I(\alpha, \beta) + (\alpha + \beta)_{KI} g_J(\alpha, \beta) + (\alpha + \beta)_{IJ} g_K(\alpha, \beta) = 0. \quad (A.4)
\]

\[
[[H_I, H_J, E^K], H_K, H_L] = 0:
\]

Consistency of (2.21) with the FI is satisfied without the need of any new condition.

\[
[[H_I, H_J, E^K], H_K, E^\beta] = 0:
\]

There are two cases to consider. For the case $\alpha + \beta = 0$, consistency with the FI is satisfied due to (A.1). For $\alpha + \beta \neq 0$, we get the new condition

\[
\beta_{JK} g_I(\alpha, \beta) + \beta_{KI} g_J(\alpha, \beta) + \beta_{IJ} g_K(\alpha, \beta) = 0. \quad (A.5)
\]

The can be written as

\[
\beta \wedge g(\alpha, \beta) = 0 \quad (A.6)
\]

with $g := g_I dx^I$. By symmetry we also have $\alpha \wedge g(\alpha, \beta) = 0$ and the condition (A.4) is a consequence of (A.5). Note that for a Cartan-Weyl 3-algebra with only one linearly independent root i.e. with a pair of roots $\alpha, -\alpha$, the condition (A.4) is satisfied automatically and there is no (A.5).

**Lemma A.1.** Denote by $\Omega$ the set of roots such that $g_I(\alpha, \beta) \neq 0$ for $\alpha, \beta \in \Omega$. The condition (A.5) implies that in general the roots $\alpha$ and $g_I$ take the following factorized form on $\Omega$

\[
\alpha = \hat{\alpha} \wedge \hat{\alpha}, \quad (A.7)
\]

\[
g_I(\alpha, \beta) = \hat{\alpha} c(\alpha, \beta), \quad (A.8)
\]

where $\hat{\alpha}_I$ is a fixed one-form and $c(\alpha, \beta)$ is a scalar function of the roots.

**Proof.** Let $\alpha = \alpha_1 \wedge \alpha_2$, $\beta = \beta_1 \wedge \beta_2$, $\gamma = \gamma_1 \wedge \gamma_2$ be any 3 roots such that $g_I$ is nonvanishing when evaluated on any two of them. The conditions $\alpha \wedge g(\alpha, \beta) = \beta \wedge g(\alpha, \beta) = 0$ with $g := g_I dx^I$, is solved by

\[
g(\alpha, \beta) = \alpha_1 + \mu \alpha_2, \quad g(\alpha, \beta) = \beta_1 + \lambda \beta_2, \quad (A.9)
\]
for some numbers $\mu, \lambda$. Some of the coefficients have been chosen to 1 by a proper normalization of $\alpha_1, \alpha_2$ etc. Similarly since $g(\alpha, \gamma) \neq 0$, then we obtain
\[ g(\alpha, \gamma) = \mu' \alpha_1 + \mu'' \alpha_2, \quad g(\alpha, \gamma) = \gamma_1 + \lambda' \gamma_2. \] (A.10)
And since $g(\beta, \gamma) \neq 0$, we have
\[ g(\beta, \gamma) = c_1 \beta_1 + c_2 \beta_2, \quad g(\beta, \gamma) = c_3 \gamma_1 + c_4 \gamma_2. \] (A.11)
It follows immediately from the consistency of these equations that $g(\alpha, \beta), g(\beta, \gamma), g(\alpha, \gamma)$ are all proportional to each other. Hence for any four roots $\alpha, \beta, \gamma, \delta$, we have
\[ c(\alpha, \beta, \gamma, \delta) = 0 \] provided that $g(\cdot, \cdot)$ is not equal to zero when evaluated on any two of the four roots. This can be satisfied only if (A.8) holds. Finally the condition $\alpha \land g = 0$ implies (A.7).

Note that here the choice of $\hat{p}$ and $c$ depends on $\Omega$. Most generally, the set $\Delta_A$ of all roots can be divided into a number of disjoint subsets $\Omega_\ell$ characterized by an orthogonality condition: $g(\alpha, \beta) = 0$ if $\alpha \in \Omega_\ell$ and $\beta \in \Omega_{\ell'}$ with $\ell \neq \ell'$. In this case, we have
\[
\boxed{\alpha = \hat{p}(\ell) \land \hat{\alpha}(\ell), \quad \alpha \in \Omega_\ell,}
\] (A.12)
and
\[
g_{I}(\alpha, \beta) = \begin{cases} 
\hat{p}_I(\ell) c(\ell)(\hat{\alpha}(\ell), \hat{\beta}(\ell)), & \text{for } \alpha, \beta \in \Omega_\ell, \\
0, & \text{otherwise}. 
\end{cases}
\] (A.13)
Thus associated with each orthogonal component $\Omega_\ell$, there is a function $c(\ell)$ and a fixed one form $\hat{p}(\ell)$.
\[
[[H_I, H_J, E^\alpha], E^\beta, E^\gamma] = 0:
\]
There are three cases to consider. The case where $\alpha + \beta + \gamma = 0$ does not give rise to any new condition. The case $\alpha + \beta + \gamma \neq 0$ and $\beta + \gamma \neq 0$ gives the condition
\[
c(\alpha, \beta, \gamma) = 0. \] (A.14)
The case $\alpha + \beta + \gamma \neq 0$ and $\beta + \gamma = 0$ gives the condition
\[
\alpha_{IL} \beta_{JK} g^{KL} = \alpha_{IL} \beta_{IK} g^{KL} \quad \text{for all } \alpha, \beta. \] (A.15)
More explicitly, for $\alpha = \hat{p}(\ell) \land \hat{\alpha}(\ell), \beta = \hat{p}(\ell') \land \hat{\beta}(\ell')$, this condition reads
\[
\hat{p}(\ell) \cdot \hat{p}(\ell') = 0, \quad \text{for all } \ell, \ell'. \] (A.16)
\[ \hat{\alpha}(\ell) \cdot \hat{\beta}(\ell') = 0, \quad \text{for all } \ell, \ell', \]  
\hspace{1cm} (A.17)

\[ \hat{\alpha}(\ell) \cdot \hat{\beta}(\ell') = 0, \quad \text{for } \ell \neq \ell'. \]  
\hspace{1cm} (A.18)

where the dot product is taken with respect to the metric \( g_{IJ} \). As consequences, it is easy to deduce that

\[ \alpha_{IK} g^{KL} g_L(\gamma, \delta) = 0, \]  
\hspace{1cm} (A.19)

\[ g_K(\alpha, \beta) g^{KL} g_L(\gamma, \delta) = 0, \]  
\hspace{1cm} (A.20)

for all roots \( \alpha, \beta, \gamma, \delta \). We note that, again, if the algebra has only a single pair of roots \( \pm \alpha \), then (A.15) is trivially satisfied and (A.17), (A.18) are not needed.

\[ [[H_I, E_\alpha, E_\beta], H_K, H_L] = 0 \ (\text{with } \alpha + \beta = 0): \]

Consistency of (2.22a) with the FI is satisfied without the need of any new condition.

\[ [[H_I, E_\alpha, E_\beta], H_J, E_\gamma] = 0 \ (\text{with } \alpha + \beta = 0): \]

There are two cases to consider. For the case where \( \alpha + \gamma = 0 \) and \( \beta + \gamma \neq 0 \), consistency is satisfied automatically. For the case where \( \alpha + \gamma \neq 0 \) and \( \beta + \gamma \neq 0 \), we obtain the condition

\[ c(\alpha, \gamma)c(-\alpha, \gamma + \alpha) - c(-\alpha, \gamma)c(\alpha, \gamma - \alpha) = -\hat{\alpha} \cdot \hat{\gamma}. \]  
\hspace{1cm} (A.21)

When \( \alpha \) and \( \gamma \) belongs to different components \( \Omega_i \) of the root space, this equation is trivially satisfied if

\[ \hat{\alpha}(\ell) \cdot \hat{\gamma}(\ell') = 0, \quad \text{if } \ell \neq \ell'. \]  
\hspace{1cm} (A.22)

When they belongs to the same component of the root space, the condition reads

\[ c(\hat{\alpha}, \hat{\gamma})c(-\hat{\alpha}, \hat{\gamma} + \hat{\alpha}) - c(-\hat{\alpha}, \hat{\gamma})c(\hat{\alpha}, \hat{\gamma} - \hat{\alpha}) = -\hat{\alpha} \cdot \hat{\gamma}. \]  
\hspace{1cm} (A.23)

Again this condition does not apply if there is only one linearly independent root.

\[ [[H_I, E_\alpha, E_\beta], E_\gamma, E_\delta] = 0 \ (\text{with } \alpha + \beta = 0): \]

There are two cases to consider. For the case where \( \gamma + \delta = 0 \), consistency with FI is satisfied due to (A.15). For the case where \( \gamma + \delta \neq 0 \), consistency with FI is satisfied due to (A.19).

\[ [[H_I, E_\alpha, E_\beta], H_J, H_K] = 0 \ (\text{with } \alpha + \beta \neq 0): \]

Consistency with FI is satisfied without the need of any new condition.

\[ [[H_I, E_\alpha, E_\beta], H_J, E_\gamma] = 0 \ (\text{with } \alpha + \beta \neq 0): \]

There are three cases to consider. First the case \( \alpha + \beta + \gamma = 0 \). Since \( \alpha + \gamma \neq 0 \) and \( \beta + \gamma \neq 0 \), one can easily see that consistency with the FI is satisfied due to (A.5). Next consider \( \alpha + \beta + \gamma \neq 0 \), we have two subcases. In the first case where \( \alpha + \gamma = 0 \)
and $\beta + \gamma \neq 0$, we require the condition (A.23). For the second case where $\alpha + \gamma \neq 0$ and $\beta + \gamma \neq 0$, we obtain the condition

$$g_I(\gamma, \beta)g_J(\gamma, \alpha) = g_I(\gamma, \alpha)g_J(\gamma, \beta). \quad (A.24)$$

This is satisfied due to (A.8).

$$[[H_I, E^\alpha, E^\beta], E^\gamma, E^\delta] = 0 \text{ (with } \alpha + \beta \neq 0):$$

One can check that FI is satisfied without the need of any new condition.

$$[[H_I, E^\alpha, E^\beta], \cdot, \cdot] = 0 \text{ (} \alpha + \beta \text{ not a root):}$$

Similarly one can check that FI is satisfied without the need of any new condition.

$$[[E^\alpha, E^\beta, E^\gamma], \cdot, \cdot] = 0 \text{ (with } \alpha + \beta + \gamma = 0):$$

One can check that FI is satisfied without the need of any new condition.

$$[[E^\alpha, E^\beta, E^\gamma], H_I, H_J] = 0 \text{ (with } \alpha + \beta + \gamma \neq 0):$$

One can check that FI is satisfied without the need of any new condition.

$$[[E^\alpha, E^\beta, E^\gamma], E^\delta, E^\epsilon] = 0 \text{ (with } \alpha + \beta + \gamma \neq 0):$$

There is only one nontrivial case to consider, that is when $\alpha + \delta + \epsilon \neq 0$, $\beta + \delta + \epsilon \neq 0$ and $\gamma + \delta + \epsilon \neq 0$. In this case, it is easy to check that FI is satisfied due to the condition (A.20).

$$[[E^\alpha, E^\beta, E^\gamma], H_K, E^\beta] = 0 \text{ (with } \alpha + \beta + \gamma \neq 0):$$

The check here is more involved. One can check that the FI is satisfied for most of the combination of $\alpha, \beta, \gamma$ and $\delta$ except for two cases:

(i) when $\alpha + \delta = 0$, $\beta + \gamma = 0$, $\beta + \delta = 0$ and $\gamma + \delta = 0$, one needs the condition

$$c(\alpha, \beta)c(\alpha, \beta) - c(\alpha, \beta)c(\alpha, \beta) = \hat{\alpha} \cdot \hat{\beta}. \quad (A.25)$$

The condition is trivially satisfied when $\alpha, \beta$ belong to different components of the root space. When they belong to the same component, the condition reads

$$c(\hat{\alpha}, \hat{\beta})c(\hat{\alpha}, \hat{\beta}) - c(\hat{\alpha}, \hat{\beta})c(\hat{\alpha}, \hat{\beta}) = \hat{\alpha} \cdot \hat{\beta}. \quad (A.26)$$

(ii) when $\alpha + \delta = -\beta - \gamma \neq 0$, $\beta + \delta \neq 0$ and $\gamma + \delta \neq 0$, one needs the condition

$$c(\alpha, -\alpha - \beta - \gamma)c(\beta, \gamma) + c(\beta, -\alpha - \beta - \gamma)c(\gamma, \alpha) + c(\gamma, -\alpha - \beta - \gamma)c(\alpha, \beta) = 0. \quad (A.27)$$

Note that since

$$c(\alpha, \beta) = c(\beta, \gamma) = c(\gamma, \alpha) \quad \text{for } \alpha + \beta + \gamma = 0, \quad (A.28)$$

the condition (A.27) can be rewritten as

$$c(\alpha, \beta)c(\gamma, \alpha + \beta) + c(\beta, \gamma)c(\alpha, \beta + \gamma) + c(\gamma, \alpha)c(\beta, \gamma + \alpha) = 0. \quad (A.29)$$
Again, the condition is trivially satisfied when $\alpha, \beta, \gamma$ belong to different components of the root space. When they belong to the same component, the condition reads

$$c(\hat{\alpha}, \hat{\beta})c(\hat{\gamma}, \hat{\alpha} + \hat{\beta}) + c(\hat{\beta}, \hat{\gamma})c(\hat{\alpha}, \hat{\beta} + \hat{\gamma}) + c(\hat{\gamma}, \hat{\alpha})c(\hat{\beta}, \hat{\gamma} + \hat{\alpha}) = 0. \quad (A.30)$$

$$[[E^\alpha, E^\beta, E^\gamma], \ldots] = 0 \ (\alpha + \beta + \gamma \text{ not a root}).$$

One can check that FI is satisfied without the need of any new condition.

Now we proceed to solve the conditions (A.23), (A.26) and (A.30). We first note that (A.26) is indeed equivalent to (A.23) due to (A.28). Next we note that the condition (A.23) is indeed precisely the same condition as one would impose for a semisimple Lie algebra. To see this, consider a semisimple Lie algebra with generators $\{E^{\hat{\alpha}}, H_I\}$, metric

$$\langle H_I, H_J \rangle := g_{IJ}, \quad \langle E^{\hat{\alpha}}, E^{\hat{\beta}} \rangle = \delta_{\alpha + \hat{\beta}}, \quad \langle E^{\hat{\alpha}}, H_I \rangle = 0, \quad (A.31)$$

and the Lie brackets

$$[H_I, H_J] = 0,$$

$$[H_I, E^{\hat{\alpha}}] = \hat{\alpha}_I E^{\hat{\alpha}}, \quad (A.32)$$

$$[E^{\hat{\alpha}}, E^{\hat{\beta}}] = \begin{cases} \hat{\alpha} \cdot H & \text{for } \hat{\alpha} + \hat{\beta} = 0, \\ c(\hat{\alpha}, \hat{\beta})E^{\hat{\alpha} + \hat{\beta}} & \text{for } \hat{\alpha} + \hat{\beta} \text{ being a root}. \end{cases}$$

One sees immediately that the condition (A.23) is precisely the same condition obtained from the Jacobi identity $[[E^{\hat{\alpha}}, E^{\hat{\beta}}], E^{-\hat{\alpha}}] + \cdots = 0$ with $\hat{\alpha} + \hat{\beta} \neq 0$; and the condition (A.30) is precisely the same condition obtained from the Jacobi identity $[E^{\hat{\alpha}}, [E^{\hat{\beta}}, E^{\hat{\gamma}}]] + \cdots = 0$ with $\hat{\alpha} + \hat{\beta} \neq 0$, $\hat{\beta} + \hat{\gamma} \neq 0$, $\hat{\alpha} + \hat{\gamma} \neq 0$. Therefore the conditions (A.23) and (A.30) can be solved if $\hat{\alpha}$ and $c(\hat{\alpha}, \hat{\beta})$ are given by those of a semisimple Lie algebra.

We remark on passing that the conditions (A.23), (A.30) do not apply if there is only a single pair of roots.

This concludes our analysis of the consistency conditions for Cartan-Weyl 3-algebras.

**** ****

Summarizing, a Cartan-Weyl 3-algebra is given by the relations

$$[H_I, H_J, H_K] = 0, \quad (A.33)$$

$$[H_I, H_J, E^\alpha] = \alpha_{IJ} E^\alpha, \quad (A.34)$$

$$[H_I, E^\alpha, E^\beta] = \begin{cases} \alpha_{IK} g^{KL} H_L, & \text{if } \alpha + \beta = 0, \\ g_l(\alpha, \beta) E^{\alpha + \beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root}, \\ 0, & \text{if } \alpha + \beta \text{ is not a root}. \end{cases} \quad (A.35)$$

- 28 -
\[ [E^\alpha, E^\beta, E^\gamma] = \begin{cases} -g_K(\alpha, \beta)g^{KL}H_L, & \text{if } \alpha + \beta + \gamma = 0, \\ 0, & \text{if } \alpha + \beta + \gamma \neq 0 \text{ a root}, \\ 0, & \text{if } \alpha + \beta + \gamma \text{ is not a root}. \end{cases} \quad (A.36a) \]

\[ g_I(\alpha, \beta) = \begin{cases} \hat{p}_I(\alpha, \beta), & \text{for } \alpha, \beta \in \Omega^\ell, \\ 0, & \text{otherwise}, \end{cases} \quad (A.38) \]

If the algebra has only a single pair of roots $\pm\alpha$, then (A.1) is the only condition to be satisfied. Otherwise, in general the root space can be decomposed into a number of (say $M$) components: \[ \Delta_\mathcal{A}(\mathcal{H}) = \bigoplus_{\ell=1}^M \Omega^\ell, \]
where there is a null vector $\hat{p}^{(\ell)}$ associated with each $\Omega^\ell$. The roots in each $\Omega^\ell$ can be decomposed in the form
\[ \alpha = \hat{p}^{(\ell)} \wedge \hat{\alpha}^{(\ell)}, \quad \alpha \in \Omega^\ell \quad (A.37) \]

where the one-form parts $\hat{\alpha}^{(\ell)}$ form the root system of a semisimple Lie algebra $g^{(\ell)}$.

The one-forms $\hat{p}^{(\ell)}$ and $\hat{\alpha}^{(\ell)}$ satisfy the conditions (A.16)-(A.18) and the coefficient $c^{(\ell)}$ specifies the $\{[E, E] = E\}$ type brackets of the semisimple Lie algebra $g^{(\ell)}$:
\[ \begin{align*}
[H_{\hat{p}^{(\ell)}}, H_{\hat{\beta}^{(\ell)}}] &= 0, \\
[H_{\hat{p}^{(\ell)}}, E^{\hat{\alpha}^{(\ell)}}] &= \hat{\alpha}^{(\ell)} E^{\hat{\alpha}^{(\ell)}}, \\
[E^{\hat{\alpha}^{(\ell)}}, E^{\hat{\beta}^{(\ell)}}] &= \begin{cases} \hat{\alpha}^{(\ell)} \cdot H^{(\ell)} & \text{for } \hat{\alpha}^{(\ell)} + \hat{\beta}^{(\ell)} = 0, \\
0 & \text{for } \hat{\alpha}^{(\ell)} + \hat{\beta}^{(\ell)} \text{ being a root.} \end{cases} \quad (A.39) \end{align*} \]

Therefore we have reduced the problem of classifying Cartan-Weyl 3-algebra to the problem of constructing the null vectors $\hat{p}^{(\ell)}$ and roots $\hat{\alpha}^{(\ell)}$ such that (A.16)-(A.18) are satisfied. This final step will depend on the signature of the metric of the 3-algebra and will be carried out in the main text of the paper.

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