Mechanical Hamiltonian systems with respect to linear Poisson structures and Jacobi–Reeb dynamics

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Abstract
In this paper, we present a relation between Jacobi–Reeb dynamics and the dynamics associated with a mechanical Hamiltonian system with respect to a linear Poisson structure on a vector bundle. For this purpose, we will use the so-called Jacobi bundle metrics induced by the mechanical Hamiltonian system. These constructions extend classical results on the relation between standard mechanical Hamiltonian systems on cotangent bundles and Reeb dynamics.

Keywords Mechanical Hamiltonian systems · Linear Poisson structures · Jacobi structures · Reeb dynamics

Mathematics Subject Classification 53D17 · 53D25 · 70G45 · 70H05

1 Introduction

1.1 Jacobi metrics, contact structures and Reeb dynamics
For a mechanical system on a configuration manifold $Q$, with Hamiltonian $H_{(g,V)} : T^*Q \to \mathbb{R}$ given as the sum of the kinetic energy associated to a Riemannian metric $g$ on $Q$ and a potential energy $V : Q \to \mathbb{R}$, the Jacobi–Maupertuis Principle of Least Action states that fixing a constant energy $e$, for the points in the open subset
$U_e = \{ q \in Q : V(q) < e \}$ of $Q$, the solutions can be restricted to the energy surface $H = e$ and they minimize the action given by $\int \sqrt{2(e - V(q))} ds$ [1]. An interesting connection between the classical Jacobi–Maupertuis principle and the Routh reduction has been discussed in [2, 3].

From a geometric perspective, using the Riemannian metric $g$ and the energy $V$, fixing $e \in \mathbb{R}$, there is a Riemannian metric $g_e$ on $U_e$, the Jacobi metric, in such a way that the geodesics of $g_e$ with velocity 1 are, up to reparametrization, the trajectories of the system with fixed energy $e$. In addition, one can construct a contact structure on the spheric cotangent bundle $S_{g_e}(T^*U_e)$ of radius 1, with respect to the Jacobi metric $g_e$, and the corresponding Reeb vector field is just the restriction to $S_{g_e}(T^*U_e)$ of the Hamiltonian vector field of the kinetic energy $\kappa_{g_e}$ associated with the Jacobi metric $g_e$ on $U_e$, that is, the geodesic flow on the Riemannian space $(U_e, g_e)$ (for more details see, for instance [4]).

Why are these results relevant? Some answers for this question are the following:

- In relation with geodesic flows: The negative curvature of a compact Riemannian manifold has a strong impact on the behavior of the geodesic flow (this is related to the exponential instability of geodesics on the manifold). For instance, almost all phase trajectories are dense in $\kappa_{g_e}^{-1}(\frac{1}{2})$. This is also related with the ergodicity of the system (see Anosov’s monograph [5]; see also Arnold’s book [1]).

On the other hand, from the previous discussion one can deduce that the dynamics on the level set of the Hamiltonian function is essentially Reeb dynamics. This implies strong consequences. In particular, the Weinstein conjecture [6] claims that on a compact contact manifold, its Reeb vector field should carry at least one periodic orbit. The Weinstein conjecture has been proved for contact hypersurfaces in $\mathbb{R}^{2n}$ [7], for cotangent bundles [8] or for closed 3-dimensional manifolds [9]. For closed higher dimension manifolds is an open problem.

- In relation with integrability: First integrals for the mechanical Hamiltonian system $(T^*U_e, H(g, V))$ with configuration space $U_e$ are just first integrals for the kinetic Hamiltonian system $(T^*U_e, \kappa_{g_e})$. On the other hand, linear first integrals on the phase space for $g_e$ can be obtained using Noether Theorem for the cotangent lift of an isometric action for the Jacobi metric $g_e$ on $U_e$ (see, for instance, [10]). But, also, quadratic (or higher order) first integrals on the phase space for $g_e$ can be obtained looking for Killing tensors. See, for instance, [11] and the references therein.

We also remark that, very recently, the classical results on the relation between standard mechanical Hamiltonian systems, geodesic flows and Reeb dynamics have been extended for standard mechanical systems subjected to linear nonholonomic constraints (see [12]).

### 1.2 Poisson and Jacobi structures

As it is well-known, Poisson manifolds are a natural generalization of symplectic manifolds and play an important role in Classical Mechanics. In fact, Poisson brackets appear, in a natural way, in the study of several mechanical systems such as systems with constraints or in the reduction of systems with symmetries. A particular type
of Poisson structures is the linear one over the dual $A^*$ of a vector bundle $A \to M$, i.e. Poisson brackets closed for fiberwise linear functions (see [13]). For instance, the Poisson structure induced by the canonical symplectic 2-form on the cotangent bundle $T^*Q$ of an $n$-dimensional manifold $Q$ is an example of a linear Poisson structure.

On the other hand, Jacobi manifolds are natural extensions of Poisson manifolds and, in addition, they include contact manifolds (like the case of $\mathcal{S}_g(T^*Q)$) or locally conformal symplectic manifolds. More precisely, a Jacobi structure on a manifold $M$ is a bracket of functions on $M$ which is a local Lie algebra in the sense of Kirillov [14]. Alternatively, a Jacobi structure on $M$ is a pair $(\Lambda, E)$, where $\Lambda$ is a 2-vector and $E$ is a vector field on $M$ (the Reeb vector field of the Jacobi structure) such that $[\Lambda, \Lambda] = 2\Lambda \wedge E$ and $[E, \Lambda] = 0$, with $[\cdot, \cdot]$ the Schouten-Nijenhuis bracket (see [15]). For a standard mechanical Hamiltonian system $(Q, H(g, V))$ and a fixed energy $e$, the spheric cotangent bundle $\mathcal{S}_{ge}(T^*U_e)$ associated with the Jacobi metric $ge$, admits a Jacobi structure (in fact, a contact structure) and its Reeb vector field is, up to reparametrization, the restriction to $\mathcal{S}_{ge}(T^*U_e)$ of the Hamiltonian vector field of $H(g, V)$.

1.3 Aim of the paper

Now, we will consider a more general situation. Let $\tau : A \to Q$ be a vector bundle over $Q$ such that the dual bundle $A^*$ admits a linear Poisson structure and let $H(g, V) : A^* \to \mathbb{R}$ be a Hamiltonian function of mechanical type associated with a bundle metric $g$ on $A$ and a potential energy $V : Q \to \mathbb{R} \in C^\infty(Q)$. For a fixed energy $e$, we can define (as in the standard case) the corresponding Jacobi bundle metric $ge$ on $A_e = \tau^{-1}(U_e)$ and the spheric dual bundle $\mathcal{S}_{ge}(A^*_e)$ associated with $ge$. Then, a natural question arises: is it possible to define a Jacobi structure (not necessarily a contact one) on $\mathcal{S}_{ge}(A^*_e)$ such that the Reeb vector field is the restriction to $\mathcal{S}_{ge}(A^*_e)$ of the Hamiltonian vector field on $A^*_e$ associated with the Hamiltonian function $H(g, V)$? In this paper, we will give an affirmative answer to this question (see Theorems 3.3 and 3.5).

1.4 Structure of the paper

The paper is organized as follows. In Sect. 2, we will review the geometric description of the relation between the dynamics of mechanical Hamiltonian systems on the cotangent bundle of a Riemannian manifold, geodesic flows and Reeb dynamics and we will present some motivating examples. In Sect. 3, which contains the main results of the paper, we extend the previous relation for mechanical Hamiltonian systems with respect to fiberwise linear Poisson structures on vector bundles (see Theorems 3.3 and 3.5). In Sect. 4, we will apply our main results to mechanical Hamiltonian systems with respect to linear Poisson structures on the dual bundle of an Atiyah vector bundle and linear semi-direct Poisson structures. As particular cases of the previous situation, we discuss the coupled planar pendula and the heavy top. Finally, in Sect. 5, we point out some interesting future lines of research opened up by the results of this paper.
2 Motivation: the classical problem

First of all, we will review the relation between the dynamics of mechanical Hamiltonian systems on the cotangent bundle of a Riemannian manifold and the Reeb dynamics (for more details see, for instance [4]).

Let \((Q, g)\) be a Riemannian manifold of dimension \(n\) and denote by \(\lambda_Q\) the Liouville 1-form on \(T^*Q\) and by \(\omega_Q = -d\lambda_Q\) the corresponding symplectic 2-form on \(T^*Q\). Consider the kinetic Hamiltonian function \(\kappa_g : T^*Q \to \mathbb{R}\) induced from the metric \(g\), given by

\[
\kappa_g(\alpha_q) = \frac{1}{2}\|\alpha_q\|^2_g, \quad \text{for } \alpha_q \in T_q^*Q,
\]

where \(\|\cdot, \cdot\|_g\) is the norm induced by \(g\) on \(T^*Q\). Then, the spheric cotangent bundle of radius 1, \(S_g(T^*Q) = \kappa_g^{-1}(\frac{1}{2})\), is a regular hypersurface of \(T^*Q\) and

\[
\eta := \iota^*(\lambda_Q)
\]

defines a contact structure on \(S_g(T^*Q)\), being \(\iota : S_g(T^*Q) \to T^*Q\) the canonical inclusion. We recall that the 1-form \(\eta\) is a contact 1-form if

\[
\eta \wedge d\eta \wedge \cdots \wedge (n-1) \cdots d\eta \in \Omega^{2n-1}(S_g(T^*Q))
\]

is a volume form on \(S_g(T^*Q)\). On a contact manifold there is a distinguished vector field \(R\), the Reeb vector field, characterized by \(i_R\eta = 1\) and \(i_Rd\eta = 0\). In our particular case, the Hamiltonian vector field \(X_{\kappa_g} \in \mathfrak{X}(T^*Q)\) of \(\kappa_g\) is the geodesic flow of \(g\) in \(T^*Q\). This means that the trajectories of the Hamiltonian system \((Q, \kappa_g)\) (that is, the projection via \(\tau_Q^* : T^*Q \to Q\) of the integral curves of \(X_{\kappa_g}\)) are the geodesics of the metric \(g\). In addition, the Reeb vector field \(R\) associated with the contact manifold \((S_g(T^*Q), \eta)\) is just the restriction to \(S_g(T^*Q)\) of \(X_{\kappa_g}\), that is,

\[
R = (X_{\kappa_g})|_{S_g(T^*Q)}.
\]

Indeed, using that \(i_{X_{\kappa_g}}\lambda_Q = 2\kappa_g\) and \(i_{X_{\kappa_g}}\omega_Q = d\kappa_g\), we deduce that

\[
\begin{align*}
\iota_R(dt^*\lambda_Q) &= -t^*(i_{X_{\kappa_g}}\omega_Q) = -d(\kappa_g \circ t) = 0, \\
\iota_R(t^*\lambda_Q) &= t^*(i_{X_{\kappa_g}}\lambda_Q) = 2(\kappa_g \circ t) = 1.
\end{align*}
\]

Moreover, there is a symplectomorphism between the symplectic open subset \(T^*Q - O_Q\), where \(O_Q\) is the zero-section on \(T^*Q\), and the symplectification of \((S_g(T^*Q), \iota^*(\lambda_Q))\) i.e. the manifold \(S_g(T^*Q) \times \mathbb{R}\) endowed with the symplectic structure \(\omega_{S_g}(T^*Q) \times \mathbb{R} = -\epsilon'(dt^*\lambda_Q - t^*\lambda_Q \wedge dt)\). The symplectic isomorphism \(\varphi : S_g(T^*Q) \times \mathbb{R} \mapsto T^*Q - O_Q\) is given by \((\alpha_q, t) \mapsto \varphi(\alpha_q, t) = e^t\alpha_q\).

Now, suppose that the Hamiltonian function is defined by

\[
H_{(g, V)} := \kappa_g + V \circ \tau_Q^*,
\]
$V \in C^\infty(Q)$ being the potential energy. Let $e \in \mathbb{R}$ such that $U_e = \{q \in Q \mid V(q) < e\}$ is a non-empty set of $Q$. For instance, if $V$ is bounded above by a certain $e \in \mathbb{R}$ (which happens, in particular, when $Q$ is compact), $U_e = Q$. Denote by $T^*U_e$ the cotangent bundle of $U_e$ given by

$$T^*U_e = \bigcup_{q \in U_e} T^*_q Q,$$

and by $\tau^*_U : T^*U_e \to U_e$ the corresponding projection. Now, we consider the Riemannian metric on $U_e$

$$g_e = 2(e - V|_{U_e})g|_{U_e}$$

and the kinetic Hamiltonian function on $T^*U_e$ given by

$$\kappa_{g_e}(\alpha_q) = \frac{1}{2} \|\alpha_q\|_{g_e}^2 = \frac{1}{4(e - V(q))}\|\alpha_q\|_{g}^2 = \frac{1}{2(e - V(q))}\kappa_g(\alpha_q), \quad \text{for } \alpha_q \in T^*_q U_e.$$

$g_e$ is the Jacobi metric and the spheric cotangent bundle of radius 1 associated with $g_e$ is

$$S_{g_e}(T^*U_e) = \kappa_{g_e}^{-1}\left(\frac{1}{2}\right) = \{\alpha_q \in T^*_q U_e : \kappa_g(\alpha_q) = e - V(q)\} = H_{(g,V)}^{-1}(e)$$

is a submanifold of codimension 1 of $T^*U_e$ and a bundle over $U_e$ with fiber by the point $q \in U_e$, the sphere of center $0_q \in T^*_q Q$ and radius $\sqrt{2(e - V(q))}$, with respect to the metric $g$. Moreover,

$$X_{\kappa_{g_e}|S_{g_e}(T^*U_e)} = \frac{1}{2(e - V \circ \tau^*_U)} X_{H_{(g,V)}|S_{g_e}(T^*U_e)}.$$

Thus, the trajectories of the Hamiltonian system $(U_e, H_{(g,V)}|T^*U_e)$ associated with solutions on $S_{g_e}(T^*U_e) = H_{(g,V)}^{-1}(e)$ are, up to reparametrization, geodesics of the Riemannian metric $g_e$ on $U_e$. In fact, if $c : I \to U_e$ and $c_e : I_e \to U_e$, with $0 \in I \cap I_e$, are trajectories of the corresponding systems with the same initial condition, then there is a strictly increasing reparametrization $h : I \to I_e$ such that $c(s) = c_e(h(s))$,

$$\frac{dh}{ds} = 2(e - V \circ c) \quad \text{and} \quad h(0) = 0.$$

On the other hand, $S_{g_e}(T^*U_e)$ is a contact manifold and the corresponding Reeb vector field is just

$$R = X_{\kappa_{g_e}|S_{g_e}(T^*U_e)} = \frac{1}{2(e - V \circ \tau^*_U)} X_{H_{(g,V)}|S_{g_e}(T^*U_e)}.$$
Remark 2.1 An extension of the Maupertuis principle for a mechanical Hamiltonian function $H_{(g,V)} := \kappa g + V \circ \tau_Q^*$, where $g$ is Lorentz metric, has been considered, for instance, in [16].

2.1 The case of the harmonic oscillator

We consider the Hamiltonian function on the cotangent bundle $T^*\mathbb{R}^2$

$$H : T^*\mathbb{R}^2 \to \mathbb{R}, \quad H(q, p) = \frac{1}{2} \left( \sum_{i=1,2} p_i^2 + \sum_{i=1,2} q_i^2 \right).$$

We take an energy level $e > 0$ and we denote by $U_e$ the open subset

$$U_e = \{(q_1, q_2) \in \mathbb{R}^2 : q_1^2 + q_2^2 < e\}.$$

In this case, the level set of $H$ for $e$ is

$$H^{-1}(e) = \{(q_1, q_2, p_1, p_2) \in U_e \times \mathbb{R}^2 : p_1^2 + p_2^2 + (q_1^2 + q_2^2) = 2e\},$$

which can be described as the level set for the value $\frac{1}{2}$ of the kinetic energy for the Riemannian metric on $U_e$

$$g_e = 2(e - q_1^2 - q_2^2)(dq_1 \otimes dq_1 + dq_2 \otimes dq_2).$$

The trajectories on $H^{-1}(e)$ of the system $(U_e, H|_{U_e \times \mathbb{R}^2})$ are

$$t \mapsto c(t) = (A_1 \sin t + B_1 \cos t, A_2 \sin t + B_2 \cos t)$$

with $A_1^2 + A_2^2 + B_1^2 + B_2^2 = e$. Note that all of them are periodic. Moreover, we deduce that they are, up to reparametrization, geodesics of the Riemannian metric $g_e$ on $U_e$.

2.2 The case of the hyperbolic plane

We consider the hyperbolic plane

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\} = \mathbb{R} \times \mathbb{R}^+$$

with the metric

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

Let $\Gamma$ be the discrete group of isometries generated by the isometry $\varphi_a : \mathbb{H} \to \mathbb{H}$ given by $\varphi_a(x, y) = (a^2 x, a^2 y)$ with $a \in \mathbb{N}, a \neq 0, 1$. This group acts on $\mathbb{H}$ and $\mathbb{H}/\Gamma$
is a 2-dimensional compact Riemannian manifold of constant curvature \(-1\), with the metric \(\tilde{g}\) induced by \(g\) on \(\mathbb{H}/\Gamma\).

If \(V : \mathbb{H} \to \mathbb{R}\) is an invariant function with respect to \(\Gamma\), on the cotangent bundle \(T^*(\mathbb{H}/\Gamma)\) we can consider the Hamiltonian function

\[
H : T^*(\mathbb{H}/\Gamma) \to \mathbb{R}, \quad H(z, \alpha_z) = \frac{1}{2} \|\alpha_z\|^2_{\tilde{g}} + \tilde{V}(z),
\]

with \(\tilde{V} : \mathbb{H}/\Gamma \to \mathbb{R}\) the function characterized by

\[
\tilde{V} \circ \pi = V,
\]

\(\pi : \mathbb{H} \to \mathbb{H}/\Gamma\) being the canonical projection.

In this case, if \(e\) is a positive real number such that \(\tilde{V}\) is bounded by \(e\), then the 3-dimensional submanifold \(H^{-1}(e)\) is just a \(S^1\)-bundle on \(\mathbb{H}/\Gamma\) and it can be described as the level set for the value \(\frac{1}{2}\) of the kinetic energy for the Riemannian metric on \(\mathbb{H}/\Gamma\)

\[
\tilde{g}_e = 2(e - \tilde{V})\tilde{g},
\]

induced by the Jacobi metric \(g_e = 2(e - V)g\) on \(\mathbb{H}\).

The sectional curvature \(c_{g_e}(x, y)\) at the point \((x, y)\) of the Riemannian manifold \((\mathbb{H}, g_e)\) is just

\[
c_{g_e}(x, y) = \frac{1}{2} \left( -y^2 + y^4 \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \right) = \frac{1}{2} (-y^2 + y^4 \Delta(V)).
\]

Therefore, if \(\Delta(V) < \frac{1}{y^2}\), we have that \((\mathbb{H}/\Gamma, \tilde{g}_e)\) is a compact Riemannian manifold with negative curvature.

The trajectories of the system \((\mathbb{H}/\Gamma, H)\) on the level set \(H^{-1}(e) = S_{\tilde{g}_e}(T^*(\mathbb{H}/\Gamma))\) are, up to reparametrization, geodesics of the Riemannian manifold \((\mathbb{H}/\Gamma, g_e)\). Using the results of Anosov [5], we deduce that the periodic trajectories determine an everywhere dense set in \(H^{-1}(e) = S_{\tilde{g}_e}(T^*(\mathbb{H}/\Gamma))\) and the system is ergodic.

### 3 The more general case of a linear Poisson structure

In this section, we will extend the results in the previous section for mechanical Hamiltonian systems with respect to linear Poisson structures on a vector bundle.

Let \(\tau_A : A \to Q\) be a vector bundle of rank \(n\) endowed with a linear Poisson structure on the dual vector bundle \(\tau_{A^*} : A^* \to Q\), i.e. the corresponding Lie bracket \(\{\cdot, \cdot\}_{A^*}\) of functions on \(A^*\) satisfies that the bracket of two fiberwise linear functions is again a fiberwise linear function. Note that there exists a one-to-one correspondence between the set of fiberwise linear functions on \(A^*\) and the space \(\Gamma(A)\) of sections of
\( \tau_A : A \to Q \) which is defined by

\[
\hat{}: \Gamma(A) \mapsto \{ \text{fiberwise linear functions on } A^* \}
X \mapsto \hat{X}
\]

with

\[
\hat{X}(\alpha_q) = (\alpha_q, X(q)), \quad \text{for } \alpha_q \in A^*_q \text{ and } q \in Q.
\]

Thus, for the linear Poisson structure \( \{ \cdot, \cdot \}_A^* \) on \( A^* \), we have that

\[
\{ \hat{X}, \hat{Y} \}_A^* = -\hat{[X, Y]}, \quad \text{for } X, Y \in \Gamma(A),
\]

with \( [X, Y] \in \Gamma(A) \).

**Remark 3.1** The Poisson structure \( \{ \cdot, \cdot \}_{T^*Q} \) associated with the canonical symplectic structure \( \omega_Q \) on \( T^*Q \) is linear. In fact, if \( X, Y \in \mathfrak{X}(Q) \) then

\[
\{ \hat{X}, \hat{Y} \}_{T^*Q} = -\hat{[X, Y]},
\]

where \( [X, Y] \in \mathfrak{X}(Q) \) is the standard Lie bracket of vector fields on \( Q \).

More generally, linear Poisson structures on a vector bundle \( A^* \) are in one-to-one correspondence with Lie algebroid structures on the dual vector bundle \( A \) (see [13]).

We recall that a Lie algebroid structure \( (\{ \cdot, \cdot \}, \rho) \) on a vector bundle \( A \to M \) is a Lie algebra bracket \( \{ \cdot, \cdot \} \) on \( \Gamma(A) \) and a bundle map \( \rho : A \to TM \), called the anchor map, such that

\[
\{ X, fY \} = f\{ X, Y \} + \rho(X)(f)Y, \quad \text{for } X, Y \in \Gamma(A), f \in C^\infty(M).
\]

For a linear Poisson structure \( \{ \cdot, \cdot \}_A^* \) on \( A^* \), it follows that

\[
\{ f \circ \tau_{A^*}, \hat{X} \}_A^* = \rho(X)(f) \circ \tau_{A^*}, \quad \{ f \circ \tau_{A^*}, h \circ \tau_{A^*} \}_A^* = 0,
\]

for \( f, h \in C^\infty(Q) \) and \( X \in \Gamma(A) \) (see, for instance, [13]).

So, if \( (q^i, y_\alpha) \) are local coordinates on \( A^* \), associated with local coordinates \( (q^i) \) on \( Q \) and local coordinates \( (y_\alpha) \) coming from a local basis \( \{ e^\alpha \} \) of sections of \( A^* \), then

\[
\hat{e}_\alpha = y_\alpha
\]

where \( \{ e_\alpha \} \) is the dual basis of sections of \( A \). Moreover, if

\[
\{ e_\alpha, e_\beta \} = C^\gamma_{\alpha\beta} e_\gamma, \quad \rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i},
\]
we have that
\[ \{ y_\alpha, y_\beta \}_{A^*} = -C^\gamma_{\alpha\beta} y_\gamma, \quad \{ q^i, y_\alpha \}_{A^*} = \rho^i_\alpha, \quad \{ q^i, q^j \}_{A^*} = 0, \]
and the Poisson 2-vector \( \Pi_{A^*} \), associated with the linear Poisson bracket \( \{ \cdot, \cdot \}_{A^*} \), has local expression
\[ \Pi_{A^*}(q, y) = \rho^i_\alpha(q) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial y_\alpha} - \frac{1}{2} C^\gamma_{\alpha\beta}(q) y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}. \] (1)

The previous linearity conditions can be described in term of the homogeneity with respect to the Liouville vector field \( \Delta_{A^*} \) of \( \tau_{A^*} \) which is locally defined by
\[ \Delta_{A^*} = \sum_{\alpha} y_\alpha \frac{\partial}{\partial y_\alpha}. \] (2)

Indeed, \( F : A^* \rightarrow \mathbb{R} \in C^\infty(A^*) \) is fiberwise linear if and only if \( \Delta_{A^*}(F) = F \). Then, one can prove that the 2-vector \( \Pi_{A^*} \) on \( A^* \) associated with a Poisson bracket on \( A^* \) is linear if and only if
\[ \mathcal{L}_{\Delta_{A^*}} \Pi_{A^*} = -\Pi_{A^*}. \]

Now, suppose that we have a bundle metric \( g \) on \( A \). Consider the Hamiltonian function \( \kappa_{(A, g)} : A^* \rightarrow \mathbb{R} \) given by
\[ \kappa_{(A, g)}(\alpha_q) = \frac{1}{2} \| \alpha_q \|^2_g, \text{ for all } \alpha_q \in A^*_q \text{ and } q \in Q, \]
where \( \| \cdot \|_g \) is the norm on \( A^* \) induced by \( g \). If \( (g^\alpha_\beta) \) is the matrix of the bundle metric \( g \) on \( A^* \) with respect to a local basis \( \{ e^\gamma \} \) of \( \Gamma(A^*) \), then the local expression of \( \kappa_{(A, g)} \) is
\[ \kappa_{(A, g)}(q, y) = \frac{1}{2} g^\alpha_\beta(q) y_\alpha y_\beta. \] (3)

Like in the classical case on \( T^*Q \), we consider the spheric dual bundle of radius 1
\[ \mathbb{S}_g(A^*) = \kappa_{(A, g)}^{-1} \left( \frac{1}{2} \right), \]
which is a fiber bundle on \( Q \).

Extending the construction in Sect. 2, we will show that on \( \mathbb{S}_g(A^*) \) one can define a Jacobi structure.

We recall that a Jacobi structure on a manifold \( M \) is a pair \( (\Lambda, E) \), with \( \Lambda \in \mathcal{V}^2(M) \) a 2-vector on \( M \) and \( E \) a vector field on \( M \) such that
\[ [\Lambda, \Lambda] = 2\Lambda \wedge E \text{ and } \mathcal{L}_E \Lambda = 0. \]
Here $\langle \cdot, \cdot \rangle$ is the Schouten-Nijenhuis bracket. Every Jacobi structure $(\Lambda, E)$ induces a bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on the space of real $C^\infty$-functions on $M$ given by

$$\{f_1, f_2\} = \Lambda(df_1, df_2) + f_1 E(f_2) - f_2 E(f_1),$$

which is $\mathbb{R}$-bilinear, skew-symmetric, satisfies the Jacobi identity and it associates a first order differential operator with any function. Conversely, every bracket of functions on the manifold $M$ with these properties induces a Jacobi structure on $M$ [14, 15]. This bracket is called the Jacobi bracket on $M$.

Contact structures are particular examples of Jacobi structures. In fact, if $\eta$ is a contact structure on $M$ then the vector bundle morphism $\flat_{\eta} : TM \to T^*M$ given by

$$\flat_{\eta}(v) = i_v d\eta(\tau_M(v)) + \langle \eta(\tau_M(v)), v \rangle \eta(\tau_M(v)),$$

for $v \in TM$, with $\tau_M : TM \to M$ the canonical projection, is a vector bundle isomorphism. So, we can consider the 2-vector $\Lambda_{\eta}$ on $M$ defined by

$$\Lambda_{\eta}(\alpha, \beta) = -d\eta(\flat_{\eta}^{-1}(\alpha), \flat_{\eta}^{-1}(\beta)), \quad \text{for } \alpha, \beta \in \Omega^1(M).$$

Moreover, if $R$ is the Reeb vector field, one may see that the couple $(\Lambda_{\eta}, E_{\eta} = -R)$ is a Jacobi structure on $M$ (see, for instance [15]).

**Remark 3.2** Let $g$ be a Riemannian metric on $Q$, $S_g(T^*Q)$ the spheric cotangent bundle of radius 1 and $\eta$ the canonical contact structure on $S_g(T^*Q)$ (see Section 2). Then, the Jacobi structure $(\Lambda_{S_g(T^*Q)}, E_{S_g(T^*Q)})$ on $S_g(T^*Q)$ associated with $\eta$ is given by

$$\Lambda_{S_g(T^*Q)} = (\Pi_{T^*Q} + \Delta_{T^*Q} \wedge X_{k_g})|_{S_g(T^*Q)}, \quad E_{S_g(T^*Q)} = - (X_{k_g})|_{S_g(T^*Q)},$$

where $\Pi_{T^*Q}$ is the canonical Poisson 2-vector on $T^*Q$ and $X_{k_g} = -i(dk_g)\Pi_{T^*Q}$ is the Hamiltonian vector field of $k_g$, with $k_g : T^*Q \to \mathbb{R}$ the kinetic energy on $T^*Q$ induced by the Riemannian metric on $g$. In fact, if $X, Y \in \mathcal{X}(T^*Q)$ then

$$\omega_Q(X, Y) = \Pi_{T^*Q}(\flat_{\omega_Q}(X), \flat_{\omega_Q}(Y)),$$

with $\flat_{\omega_Q} : \mathcal{X}(T^*Q) \to \Omega^1(T^*Q)$ the isomorphism of $C^\infty(T^*Q)$-modules given by $\flat_{\omega_Q}(X) = i_X \omega_Q$. So, using that $i^*\omega_Q = -d\eta$, it follows that

$$-d\eta(X, Y) = \Pi_{T^*Q}(i_X d\eta, i_Y d\eta) = \Pi_{T^*Q}(\flat_{\eta}X, \flat_{\eta}Y) + \eta(Y) \Pi_{T^*Q}(\eta, \flat_{\eta}X) - \eta(X) \Pi_{T^*Q}(\eta, \flat_{\eta}Y).$$

Now, since $i_{\lambda_{\eta}} \Pi_{T^*Q} = \Delta_{T^*Q}$, we deduce that

$$-d\eta(X, Y) = \Pi_{T^*Q}(\flat_{\eta}X, \flat_{\eta}Y) + (\flat_{\eta}X)(\Delta_{T^*Q})\eta(Y) - (\flat_{\eta}Y)(\Delta_{T^*Q})\eta(X).$$
On the other hand, the Reeb vector field of \( S_g(T^*Q) \) is \( \mathcal{R} = (X_{\kappa_g})|_{S_g(T^*Q)} \) (see Sect. 2). This implies that

\[
(b_\eta X)(X_{\kappa_g}) = \eta(X)
\]

and, thus,

\[
-d\eta(X, Y) = (\Pi_{T^*Q} + \Delta_{T^*Q} \wedge X_{\kappa_g})|_{S_g(T^*Q)}(b_\eta X, b_\eta Y)
\]

which proves the result.

The previous remark provides a good motivation to claim the following theorem.

**Theorem 3.3** Let \( \tau_A : A \to Q \) be a vector bundle such that \( \Pi_A^* \) is a linear Poisson structure on \( A^* \) and let \( g : A \times A \to \mathbb{R} \) be a bundle metric on \( A \). Then:

(i) The pair \( (\Lambda_A^*, E_A^*) \in \mathcal{V}^2(A^*) \times \mathfrak{X}(A^*) \) given by

\[
\Lambda_A^* = \Pi_A^* + \Delta_A^* \wedge X_{\kappa(A,g)}^*, \quad E_A^* = -X_{\kappa(A,g)}^*
\]

is a Jacobi structure on \( A^* \), where \( \Delta_A^* \) is the Liouville vector field on the vector bundle \( \tau_A^* : A^* \to M \) and \( X_{\kappa(A,g)}^* \in \mathfrak{X}(A^*) \) is the Hamiltonian vector field of \( \kappa(A,g) \in C^\infty(A^*) \) with respect to the linear Poisson structure \( \Pi_A^* \), i.e.

\[
X_{\kappa(A,g)}^* = -i(d\kappa(A,g))\Pi_A^*.
\]

(ii) The Jacobi structure \( (\Lambda_A^*, E_A^*) \) induces a Jacobi structure \( (\Lambda_{S_g}(A^*), E_{S_g}(A^*)) \) on \( S_g(A^*) \) such that, if \( O_Q \) is the zero section of \( A^* \), the map

\[
\Psi : S_g(A^*) \times \mathbb{R} \to A^* - O_Q, \quad \Psi(\beta_q, t) = e^t \beta_q
\]

is a Poisson isomorphism when we consider on \( A^* - O_Q \) the induced Poisson structure by \( \Pi_A^* \) and on \( S_g(A^*) \times \mathbb{R} \) the Poissonization of the Jacobi structure on \( S_g(A^*) \), i.e.

\[
e^{-t}\left(\Lambda_{S_g}(A^*) - E_{S_g}(A^*) \wedge \frac{\partial}{\partial t}\right).
\]

(iii) The Jacobi bracket \( \{\cdot, \cdot\}_{S_g}(A^*) \) on \( S_g(A^*) \) and the Poisson bracket \( \{\cdot, \cdot\}_{A^*} \) on \( A^* \) are related as follows

\[
\{G_1 \circ t, G_2 \circ t\}_{S_g}(A^*) = \{G_1, G_2\}_{A^*} + \{\kappa(A,g), G_1\}_{A^*}(G_2 - \Delta_A^*(G_2)) - \{\kappa(A,g), G_2\}_{A^*}(G_1 - \Delta_A^*(G_1)) \circ t
\]

for all \( G_1, G_2 \in C^\infty(A^*) \), with \( t : S_g(A^*) \to A^* \) the inclusion map.
\textbf{Proof} Our theorem is a consequence of some results which where proved in [17] (see also [18, 19]). Anyway, in order to make our paper self-contained, we will give a simple proof of the theorem.

(i) It is well-known that $X_{\kappa(A,g)}$ is an infinitesimal automorphism of the Poisson structure $\Pi_{A^*}$, that is,

$$\mathcal{L}_{X_{\kappa(A,g)}} \Pi_{A^*} = 0.$$ 

Then Grabowski and Marmo [17, Cor.1] implies that $\Lambda_{A^*} = \Pi_{A^*} + \Delta_{A^*} \wedge X_{\kappa(A,g)}$, $E_{A^*} = -X_{\kappa(A,g)}$ is a Jacobi structure on $A^*$.

(ii) Using (2) and (3), we have that $\Delta_{A^*}(\kappa(A,g)) = 2\kappa(A,g)$, that is, $\kappa(A,g)$ is homogeneous of degree 2. Therefore, from Grabowski and Marmo [17, Cor.2], $\Lambda_{A^*}$ and $E_{A^*}$ restrict to $S_g(A^*)$.

On the other hand, note that the diffeomorphism $\Psi$ is just the restriction of the flow of $\Delta_{A^*}$ to $S_g(A^*)$. Thus, we have that

$$T\Psi \circ \frac{\partial}{\partial t} = \Delta_{A^*} \circ \Psi.$$ 

This, using (5), implies that $\Psi$ is a Poisson isomorphism between $S_g(A^*) \times \mathbb{R}$ and $A^* - \mathcal{O}_Q$.

(iii) It is a consequence from (4) and (5). $\square$

\textbf{Remark 3.4} From (5), we deduce that the Jacobi structure $(\Lambda_{A^*}, E_{A^*})$ on $A^*$ is polynomial of degree 3.

Now, let $g$ be a bundle metric on $A$ and $V \in C^\infty(Q)$. Consider the Hamiltonian function $H_{(A,g,V)} : A^* \rightarrow \mathbb{R}$ of mechanical type given by

$$H_{(A,g,V)}(\alpha_q) = \kappa(A,g)(\alpha_q) + V(q), \text{ for all } \alpha_q \in A^*_q \text{ and } q \in Q. \quad (6)$$

Suppose that $e \in \mathbb{R}$ is such that $U_e = \{ q \in Q : V(q) < e \}$ is a non-empty subset of $Q$. Denote by

$$A_e = \bigcup_{q \in U_e} A_q$$

the corresponding vector bundle over $U_e$ and by $A^*_e$ its dual bundle. Now, we consider the Jacobi bundle metric $g_e$ on $A_e$ defined by

$$g_e = 2(e - V|_{U_e})g|_{A_e}$$

and the corresponding Hamiltonian function $\kappa_{(A_e, g_e)} : A^*_e \rightarrow \mathbb{R}$ given by

$$\kappa_{(A_e, g_e)}(\alpha_q) = \frac{1}{2} \| \alpha_q \|^2_{g_e} = \frac{1}{2} \left( \frac{e - V(q)}{4(e - V(q))} \right) \| \alpha_q \|^2_g = \frac{1}{2} \left( \frac{e - V(q)}{2(e - V(q))} \right) \kappa_{(A_e, g)}(\alpha_q) \quad (7)$$
for $\alpha_q \in (A^*_e)_{q}$, with $q \in U_e$. On the other hand, we may introduce the spheric dual bundle on $U_e$ defined by

$$S_{ge}(A^*) = \kappa_{(A_e, ge)} \left( \frac{1}{2} \right) = \bigcup_{q \in U_e} \{ \alpha_q \in A^*_q : \|\alpha_q\|_g^2 = 2(e - V(q)) \} = H_{(A_e, g, V)}^{-1}(e).$$

(8)

Then, one has the following result

Theorem 3.5

(i) $S_{ge}(A^*)$ is a submanifold of codimension 1 in $A^*_e$ and if $\alpha_q \in S_{ge}(A^*)$

$$T_{\alpha_q}S_{ge}(A^*) = \{ w \in T_{\alpha_q}A^*_e : \langle dH_{(A_e, g, V)}(\alpha_q), w \rangle = 0 \}$$

$$= \{ w \in T_{\alpha_q}A^*_e : \langle d\kappa_{(A_e, ge)}(\alpha_q), w \rangle = 0 \}. \quad (9)$$

(ii) $S_{ge}(A^*)$ is a bundle over $U_e$ with fiber by $q \in U_e$ the sphere of center $0_q \in A^*_q$ and radius $\sqrt{2(e - V(q))}$, with respect to the bundle metric $g$.

(iii) The Hamiltonian vector fields of $H_{(A, g, V)}$ and $\kappa_{(A, ge)}$ are tangent to $S_{ge}(A^*)$ and

$$X_{H_{(A, g, V)}|S_{ge}(A^*)} = 2(e - V|U_e)X_{\kappa_{(A, ge)}|S_{ge}(A^*)}.$$

(iv) The bundle $S_{ge}(A^*)$ admits a Jacobi bundle structure $(\Lambda_{S_{ge}(A^*)}, E_{S_{ge}(A^*)})$ with

$$\Lambda_{S_{ge}(A^*)} = \Pi A^*|S_{ge}(A^*) + \frac{1}{2(e - V|U_e)} \Delta A^*|S_{ge}(A^*) \land X_{H_{(A, g, V)}|S_{ge}(A^*)},$$

$$E_{S_{ge}(A^*)} = - \frac{1}{2(e - V|U_e)} X_{H_{(A, g, V)}|S_{ge}(A^*)}.$$

(v) If the curve $c : I \to A^*_e$ (resp., $c_e : I_e \to A^*_e$), with 0 belonging to the interval $I$ (resp. $I_e$), is the trajectory of the system $(A^*_e, H_{(A, g, V)}$) (resp. $(A^*_e, \kappa_{(A, g)})$) such that $c(0) = \alpha_q$ (resp. $c_e(0) = \alpha_q$), then there is a strictly increasing reparametrization $h : I \to I_e$ such that

$$c(s) = c_e(h(s)).$$

Moreover,

$$\frac{dh}{ds} = 2(e - V \circ \tau_{A^*} \circ c), \quad h(0) = 0.$$

Proof Since

$$\langle dH_{(A_e, g, V)}(\alpha_q), \Delta A^*(\alpha_q) \rangle = \|\alpha_q\|_g^2 = 2(e - V(q)) > 0,$$
and
\[
\langle d\kappa_{(A_e, g_e)}(\alpha_q), \Delta_{A^*}(\alpha_q) \rangle = \|\alpha_q\|^2_{g_e} = 1 > 0,
\]
then
\[
dH_{(A, g, V)}(\alpha_q) \neq 0 \text{ and } d\kappa_{(A_e, g_e)}(\alpha_q) \neq 0.
\]

Thus, \( S_{g_e}(A^*) \) is a submanifold of \( A_e^* \) of codimension 1 and (9) holds. Moreover,
\[
T_{\alpha_q}A_e^* = T_{\alpha_q}S_{g_e}(A^*) \oplus \langle \Delta_{A^*}(\alpha_q) \rangle.
\]

Therefore, using that \( \Delta_{A^*} \) is vertical with respect to the projection \( \tau_{|A_e^*} : A_e^* \to U_e \), it follows that the restriction of \( \tau_{|A_e^*} \) to \( S_{g_e}(A^*) \) is a fibration
\[
\tau_{|S_{g_e}(A^*)} : S_{g_e}(A^*) \to U_e.
\]

In addition, it is easy to prove that the fiber of \( \tau_{|S_{g_e}(A^*)} \) by \( q \in U_e \) is just the sphere of center \( 0_q \in A^*_e \) and radius \( \sqrt{2(e - V(q))} \), with respect to the bundle metric \( g \). This proves (i) and (ii).

Now, using (9), it is clear that the restriction to \( S_{g_e}(A^*_e) \) of the Hamiltonian vector fields \( X_{H_{(A_e, g, V)}} \) and \( X_{\kappa_{(A_e, g_e)}} \) is tangent to \( S_{g_e}(A^*_e) \). In addition, if \( \alpha_q \in S_{g_e}(A_e^*) \), we have that
\[
\kappa_g(\alpha_q) = e - V(q),
\]
and, from (7), we deduce that
\[
d\kappa_{(A_e, g_e)}(\alpha_q) = \frac{1}{2(e - V(q))}dH_{(A_e, g, V)}(\alpha_q).
\]
This implies that
\[
(X_{H_{(A_e, g, V)}})|_{S_{g_e}(A^*_e)} = 2(e - V|_{U_e})(X_{\kappa_{(A_e, g_e)}})|_{S_{g_e}(A^*_e)},
\]
which proves (iii).

(iv) follows using (iii) and Theorem 3.3.

Finally, (v) is a consequence of (iii). \( \square \)

**Remark 3.6** If we apply Theorem 3.5 to the particular case when \( A = TQ \) and the linear Poisson structure on \( T^*Q \) is the canonical symplectic structure then, using Remark 3.2, we deduce all the results in Sect. 2.
If \((q^i, y_\alpha)\) are local coordinates on \(A^*\) then, using (1), (3) and (6), we deduce that
\[
X_{\kappa(\lambda; \mu; \nu)} = \frac{1}{2(e - V)} X_{(H, g, V)} = \frac{1}{2(e - V)} \left( \left( \frac{\partial g^{\alpha \beta}}{\partial q^i} \beta^{\mu \nu} y_\alpha y_\gamma \right. \right.
\[
- C^{\alpha \beta \gamma} y_\gamma y_\mu + \left. \frac{\partial V}{\partial q^i} \beta^{\mu \nu} \frac{\partial}{\partial y_\beta} \right) \right).
\]

**Remark 3.7** Note that if in Theorems 3.3 and 3.5 we replace the bundle metric \(g\) by a non-degenerate symmetric section of the vector bundle \(A^* \otimes A^* \to Q\) then both results remain valid. This implies that the results in this section can also be applied to the dynamics of relativistic particles. In such a case, we have a Lorentz bundle metric.

### 4 Examples

In this section we will apply the previous constructions to some Hamiltonian mechanical systems with respect to linear Poisson structures on vector bundles.

#### 4.1 Mechanical Hamiltonian systems on the dual bundle of an Atiyah Lie algebroid associated with a trivial principal bundle

Let \(G\) be a Lie group with Lie algebra \(g\) and consider the product manifold \(M = G \times Q\), with \(Q\) a smooth manifold. Then, \(M\) is the total space of a trivial principal \(G\)-bundle
\[
p: M = G \times Q \to Q,
\]
where the action of \(G\) on \(M\) is just by left-translation on the first factor.

The cotangent lift of the \(G\)-action on \(M\) induces a principal action of \(G\) on \(T^* M = T^*G \times T^*Q\). Indeed, using the left-trivialization of \(T^* G\), we can identify \(T^* M\) with the product manifold \((G \times g^*) \times T^* Q\) and, under this identification, we have that the quotient space \(T^* M / G\) is just the trivial vector bundle (over \(Q\)), \(T^* Q \times g^* \to Q\) (see, for instance [10]).

In the total space of the vector bundle \(T^* Q \times g^*\) there is a fiberwise linear Poisson structure, denoted by \(\Pi_{T^* Q \times g^*}\), which is the product of the canonical symplectic structure on \(T^* Q\) and the Lie-Poisson structure on \(g^*\), whose bracket is characterized as follows
\[
\{ f_1, f_2 \}_{g^*}(\alpha) = -\alpha([df_1(\alpha), df_2(\alpha)]_g), \quad \text{for all } \alpha \in g^* \text{ and } f_1, f_2 \in C^\infty(g^*).
\]

Here, \([\cdot, \cdot]_g\) is the Lie bracket on \(g\). If \((q^i, p_i)\) are local fibred coordinates on \(T^* Q\) and \(\{ e_a \}\) is a basis of \(g\), we have the corresponding linear coordinates \((q^i, y_\alpha) = (q^i, p_i, y_\alpha)\) on \(T^* Q \times g^*\) and, for every \(\varphi, \psi \in C^\infty(T^* Q \times g^*)\)
\[
\{ \varphi, \psi \}_{T^* Q \times g^*} = \left( \frac{\partial \varphi}{\partial q^i} \frac{\partial \psi}{\partial p_i} - \frac{\partial \varphi}{\partial p_i} \frac{\partial \psi}{\partial q^i} \right) - \frac{1}{2} c_{ab}^{\cdot \cdot}(e) \left( \frac{\partial \varphi}{\partial y_a} \frac{\partial \psi}{\partial y_b} - \frac{\partial \varphi}{\partial y_b} \frac{\partial \psi}{\partial y_a} \right).
\]
where \( c^c_{ab} \) are the structure constants of the Lie algebra \( g \) with respect to the basis \( \{ e_a \} \).

**Remark 4.1** The canonical symplectic structure on \( T^* M \) is \( G \)-invariant. Thus, it induces a Poisson structure on the quotient space \( T^* M / G \cong T^* Q \times g^* \). This Poisson structure is just \( \Pi_{T^* Q \times g^*} \) (see, for instance [20]).

Now, let \( g \) be a bundle metric on the vector bundle \( T^* Q \times g^* \to Q \) and \( V \in C^\infty(Q) \) a real \( C^\infty \)-function on \( Q \). Then, we can consider the mechanical Hamiltonian function \( H_{(g, V)} : T^* Q \times g^* \to \mathbb{R} \) given by

\[
H_{(g, V)}(\alpha_q, \gamma) = \frac{1}{2} \| (\alpha_q, \gamma) \|_g^2 + V(q), \quad \text{for } \alpha_q \in T^*_q Q \text{ and } \gamma \in g^*.
\] (11)

**Remark 4.2** Hamiltonian functions as in (11) may be obtained by reduction, from \( G \)-invariant Riemannian metrics and \( G \)-invariant potential functions on \( M = G \times Q \).

If

\[
\begin{pmatrix}
g^{ij}(q) & g^{ia}(q) \\
g^{ia}(q) & g^{ab}(q)
\end{pmatrix}
\]

is the matrix of coefficients of the scalar product \( g(q) \) on \( T^*_q Q \times g^* \) then it follows that

\[
H_{(g, V)}(q^i, p_i, y_a) = \frac{1}{2} (g^{ij}(q)p_ip_j + g^{ab}(q)y_ay_b) + g^{ia}(q)p_iy_a + V(q).
\]

Next, let \( e \) be a real number such that \( U_e = \{ q \in Q : V(q) < e \} \neq \emptyset \). Then

\[
H_{(g, V)}^{-1}(e) = \{ \alpha_q \in T^* U_e : \| \alpha_q \|_g^2 = 2(e - V(q)) \}
= \kappa_{g_e}^{-1} \left( \frac{1}{2} \right) = S_{g_e}(T^* U_e \times g^*),
\]

where \( \kappa_{g_e} : T^* U_e \times g^* \to \mathbb{R} \) is the kinetic energy associated with the Jacobi bundle metric on \( T^* U_e \times g^* \) given by

\[
g_e = 2(e - V \circ \tau_Q^* )g.
\]

Using (10), it follows that the local expression of the Hamiltonian vector field on \( T^* U_e \times g^* \) of the kinetic energy \( \kappa_{g_e} \) is

\[
X_{\kappa_{g_e}}(q, p, y) = \frac{1}{2(e - V)} \left( (g^{ia}(q)y_a) \frac{\partial}{\partial q^i} - \frac{1}{2} \frac{\partial g^{a'\gamma}}{\partial q^i} y_a y_\gamma + \frac{\partial V}{\partial q^i} \frac{\partial}{\partial p_i} ight.
\]
\[
+ \left. (c^c_{ab}g^{ia}(q)p_iy_c + c^c_{ab}g^{ad}(q)y_cy_d) \frac{\partial}{\partial y_b} \right).
\]
In addition, the Jacobi structure on $S_{ge}(T^*U_e \times g^*)$ is the restriction to $S_{ge}(T^*U_e \times g^*)$ of the Jacobi structure $(\Lambda_{T^*U_e \times g^*}, E_{T^*U_e \times g^*})$ on $T^*U_e \times g^*$ given by

$$\Lambda_{T^*U_e \times g^*} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} - \frac{1}{2} \epsilon_{abc} y^c \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b} + y^a \frac{\partial}{\partial y^a} \wedge X_{\kappa_{ge}},$$

$$E_{T^*U_e \times g^*} = -X_{\kappa_{ge}}.$$

**Remark 4.3** If $Q$ is a single point, then the Hamiltonian function is the kinetic energy

$$\kappa_{\langle \cdot, \cdot \rangle} : g^* \to \mathbb{R}, \quad \kappa_{\langle \cdot, \cdot \rangle}(\alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle^*,$$

associated with a scalar product $\langle \cdot, \cdot \rangle$ on $g$. Here, $\langle \cdot, \cdot \rangle^* : g^* \times g^* \to \mathbb{R}$ denotes the corresponding scalar product on $g^*$. In this setting, the Jacobi structure $(\Lambda_{g^*|S_{\langle \cdot, \cdot \rangle}(g^*)}, E_{g^*|S_{\langle \cdot, \cdot \rangle}(g^*)})$ on

$$S_{\langle \cdot, \cdot \rangle}(g^*) = \kappa_{\langle \cdot, \cdot \rangle}^{-1}(\frac{1}{2}) = \{ \alpha \in g^* : \langle \alpha, \alpha \rangle^* = 1 \},$$

is $p$-projectable on the conformal Jacobi structure introduced by Lichnerowicz in [21] in the quotient space $(g^* - \{0\})/(\Delta_{g^*}|_{g^* - \{0\}})$, $p : S_{\langle \cdot, \cdot \rangle}(g^*) \to (g^* - \{0\})/(\Delta_{g^*}|_{g^* - \{0\}})$ being the canonical covering map.

An interesting example of this situation is given by the rigid body, in which $g$ is the Lie algebra $so(3)$ of the orthogonal group $SO(3)$.

**Example 4.4 (The coupled planar pendula)** As a simple example of a mechanical Hamiltonian system on the dual bundle of an Atiyah Lie algebroid, we will consider the coupled planar pendular [22]. Let us consider two pendulums which are coupled moving under the influence of a potential depending on the hinge angle between them. Denote by $\theta_1$ and $\theta_2$ the angles formed by the straight lines through the joint and their centers of mass relative to an inertial coordinate system fixed in space. Thus, the configuration space is $M = T^2 = S^1 \times S^1$. The metric in this case is just

$$g = \frac{1}{2}(d\theta_1 \otimes d\theta_1 + d\theta_2 \otimes d\theta_2).$$

and the potential energy is of the form $V(\theta_1 - \theta_2)$. There is an action of $S^1$ on $M$

$$\phi : S^1 \times M \to M$$

which, considering coordinates $(\varphi, \psi)$ given by

$$\varphi = \frac{\theta_1 + \theta_2}{\sqrt{2}}, \quad \psi = \frac{\theta_1 - \theta_2}{\sqrt{2}},$$
can be written as $\phi(\theta, (\varphi, \psi)) = (\varphi + \theta, \psi)$, and we have the trivial principal $S^1$-bundle $p: S^1 \times S^1 \rightarrow S^1$, $(\varphi, \psi) \mapsto \psi$. In addition, the quotient for the lifted $S^1$-action on $T^*M$ is $T^*S^1 \times \mathbb{R}$. We will consider coordinates $(\psi, p_\psi, y)$ where $y = (p_\psi d\varphi) \cdot \varphi^{-1}$. Thus, the Hamiltonian is

$$H_{(g, V)}(\psi, p_\psi, y) = \frac{1}{2}(p_\psi^2 + y^2) + V(2\sqrt{\psi}).$$

Let $e \in \mathbb{R}$ and the open set $U_e = \{\psi \in S^1 : V(\sqrt{2}\psi) < e\}$. Then

$$S_{g_e}(T^*U_e \times \mathbb{R}) = \{(\psi, p_\psi, y) \in T^*U_e : y^2 + p_\psi^2 = 2(e - V(\sqrt{2}\psi))\}$$

and where $g_e$ is the metric on $U_e$ given by

$$g_e = 2(e - V(\sqrt{2}\psi))g.$$

The local expression of $X_{*g_e}$ is

$$X_{*g_e} = \frac{1}{2(e - V(\sqrt{2}\psi))}\left(\frac{1}{2}p_\psi \frac{\partial}{\partial \psi} + \frac{\partial V}{\partial \psi} \frac{\partial}{\partial p_\psi}\right)$$

and the Jacobi structure on $T^*U_e \times \mathbb{R}$ is given by

$$\Lambda_{T^*U_e \times \mathbb{R}} = \frac{\partial}{\partial \psi} \wedge \frac{\partial}{\partial p_\psi} + y \frac{\partial}{\partial y} \wedge X_{*g_e}$$

$$E_{T^*U_e \times \mathbb{R}} = -\frac{1}{2(e - V(\sqrt{2}\psi))}\left(\frac{1}{2}p_\psi \frac{\partial}{\partial \psi} + \frac{\partial V}{\partial \psi} \frac{\partial}{\partial p_\psi}\right).$$

### 4.2 Mechanical Hamiltonian systems with respect to linear semi-direct Poisson structures

Let $G$ be a Lie group and $\phi: G \times P \rightarrow P$ an action of $G$ on a manifold $P$. If $\mathfrak{g}$ is the Lie algebra of $G$, then we consider the trivial vector bundle $P \times \mathfrak{g} \rightarrow P$ whose space of sections may be identified with the set $C^\infty(P, \mathfrak{g})$ of $\mathfrak{g}$-valued functions on $P$. The dual bundle, $P \times \mathfrak{g}^*$, admits a linear Poisson structure $\Pi_{P \times \mathfrak{g}^*}$, called the semi-direct Poisson structure [23, 24], whose bracket of functions is given by

$$\{F_1, F_2\}_{P \times \mathfrak{g}^*}(p, \alpha) = -\alpha((D_2F_1)(p, \alpha), (D_2F_2)(p, \alpha))_{\mathfrak{g}}$$

$$+ (D_2F_1(p, \alpha))_{\mathfrak{g}}(p)(F_2) - (D_2F_2(p, \alpha))_{\mathfrak{g}}(p)(F_1),$$

for $F_1, F_2 \in C^\infty(P \times \mathfrak{g}^*)$ and $(p, \alpha) \in P \times \mathfrak{g}^*$, where $(D_2F)(p, \alpha) = (dF(p, \cdot))(\alpha) \in T^*_p \mathfrak{g}^* \cong \mathfrak{g}$ and, for every $\xi \in \mathfrak{g}$, $\xi_p \in \mathfrak{X}(P)$ is the infinitesimal generator of the action $\phi$ associated with $\xi$. Additionally, suppose that there is a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and we denote by $\langle \cdot, \cdot \rangle^*$ the corresponding scalar product on $\mathfrak{g}^*$. Let
Hamilton equations of the system $(\text{Mechanical Hamiltonian systems with respect to linear... Page 19 of 23}$

Moreover, the trajectories of the Hamilton equations for the Hamiltonian system $H_{(\cdot, \cdot), V} : P \times g^* \rightarrow \mathbb{R}$ be the Hamiltonian function given by

$$H_{(\cdot, \cdot), V}(p, \alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle^* + V(p), \quad \text{for } (p, \alpha) \in P \times g^*.$$ 

Now, let $e \in \mathbb{R}$ a real number such that $U_e = \{ p \in P : V(p) < e \} \neq \emptyset$. Then

$$\mathbb{S}_{g_e}(P \times g^*) = H_{(\cdot, \cdot), V}^{-1}(e) = (\kappa_{g_e})^{-1}
\left( \frac{1}{2} \right) = \{(p, \alpha) \in P \times g^* : \langle \alpha, \alpha \rangle^* = 2(e - V(p))\},$$

where $\kappa_{g_e}$ is the kinetic energy induced by the Jacobi metric on $P \times g^*$

$$g_e = 2(e - V)(\cdot, \cdot)^*.$$

Moreover, the trajectories of the Hamilton equations for the Hamiltonian system $(P \times g^*, H_{(\cdot, \cdot), V})$ on $\mathbb{S}_{g_e}(P \times g^*)$ are, up to reparametrization, the trajectories of the Hamilton equations of the system $(P \times g^*, \kappa_{g_e})$ on $\mathbb{S}_{g_e}(P \times g^*)$.

Now, let $\{e_\alpha\}$ be a basis of $g$ such that

$$[e_\alpha, e_\beta]_g = c_{\alpha \beta}^\gamma e_\gamma, \quad (e_\alpha)_P = -\rho_\alpha^i \frac{\partial}{\partial q^i}.$$

Then, the Jacobi structure on $\mathbb{S}_{g_e}(P \times g^*)$ is the restriction to $\mathbb{S}_{g_e}(P \times g^*)$ of the Jacobi structure $(\Lambda_{P \times g^*}, E_{P \times g^*})$ on $P \times g^*$ given by

$$\begin{align*}
\Lambda_{P \times g^*} &= \left( \rho_\alpha^i - \frac{1}{2(e - V)} \rho_\alpha^i (\cdot, \cdot)^{\gamma \beta} y_\alpha y_\beta \right) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial y_\alpha} \\
&\quad - \left( \frac{1}{2} \rho_\alpha^i y_\gamma + \frac{1}{2(e - V)} \left( \rho_\beta^\gamma y_\alpha \frac{\partial V}{\partial q^i} - c_{\gamma \beta}^{\gamma \mu} y_\gamma y_\mu y_\mu \right) \right) \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}, \\
E_{P \times g^*} &= -\frac{1}{2(e - V)} \left( \rho_\alpha^i (\cdot, \cdot)^{\cdot \cdot \beta} y_\beta \right) \frac{\partial}{\partial q^i} - \left( \rho_\beta^\gamma y_\alpha \frac{\partial V}{\partial q^i} - c_{\alpha \beta}^{\gamma \mu} y_\gamma y_\mu \right) \frac{\partial}{\partial y_\alpha} \end{align*}$$

where $\langle \cdot, \cdot \rangle^{\alpha \beta} = \langle e^\alpha, e^\beta \rangle^*$. 

So, the restriction to $\mathbb{S}_{g_e}(P \times g^*)$ of $X_{\kappa_{g_e}}$ is just $-(E_{P \times g^*})|_{\mathbb{S}_{g_e}(P \times g^*)}$.

**Example 4.5 (The heavy-top)** Let $G$ be the special orthogonal group $SO(3)$, $P$ be the sphere $S^2$ and the action $\phi : SO(3) \times S^2 \rightarrow S^2$ given by $(A, q) \mapsto Aq$. We take the basis $\{e_1, e_2, e_3\}$ of $\mathfrak{so}(3)$ such that

$$[e_1, e_2]_{\mathfrak{so}(3)} = e_3, \quad [e_2, e_3]_{\mathfrak{so}(3)} = e_1, \quad [e_3, e_1]_{\mathfrak{so}(3)} = e_2.$$ 

Using this basis, we can identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ and the infinitesimal vector fields of the action are described by $\xi_{S^2}(q) = \xi \times q$, for $\xi \in \mathbb{R}^3$ and $q \in S^2$, where $\times$ is the vector product in $\mathbb{R}^3$. 
We consider the scalar product on $\mathfrak{so}(3)$ given by

$$(x, y) = x \cdot I y,$$

$I$ being the inertia tensor of the top. For simplicity in the expressions, we will assume that $I$ is diagonal, that is,

$$\langle(x^1, x^2, x^3), (y^1, y^2, y^3)\rangle = \sum_{i=1}^{3} I_i x^i y^i,$$

for $(x^1, x^2, x^3), (y^1, y^2, y^3) \in \mathbb{R}^3 \cong \mathfrak{so}(3)$, where the positive numbers $I_1, I_2, I_3$ are the principal moments of inertia. The potential energy $V: S^2 \to \mathbb{R}$ is defined by $V(q) = mgl(q \cdot a)$, for $q \in S^2$, where $a$ is the unit vector from the fixed point to the center of mass and $m, g$ and $l$ are constants. Thus, if $e \in \mathbb{R}$ and $U_e = \{q \in S^2 : q \cdot a < \frac{e}{mgl}\}$

$$\mathbb{S}_e(S^2 \times \mathfrak{so}(3)^*) = \{(q, \alpha) \in U_e \times \mathfrak{so}(3)^* : \sqrt{\alpha \cdot I^{-1} \alpha} = 2(e - mgl(q \cdot a))\}.$$

The trajectories of the Hamilton equations of $(S^2 \times \mathfrak{so}(3)^*, H_{(\cdot, \cdot, V)})$ on $\mathbb{S}_e(S^2 \times \mathfrak{so}(3)^*)$ are just the trajectories of the Hamilton equations of the system $(S^2 \times \mathfrak{so}(3)^*, \kappa_e)$ on $\mathbb{S}_e(S^2 \times \mathfrak{so}(3)^*)$, where $g_e$ is the metric on $S^2 \times \mathfrak{so}(3)^*$. The metric $g_e(x)(\alpha, \alpha') = 2(e - mgl x \cdot a) \alpha \cdot I^{-1} \alpha'$.

If $(y_1, y_2, y_3)$ are the coordinates on $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ with respect to the base $\{e_1, e_2, e_3\}$ and $(q_1, q_2, q_3)$ are canonical coordinates on $\mathbb{R}^3$ then the linear Poisson structure $\Pi_{S^2 \times \mathfrak{so}(3)^*}$ has the expression

$$\Pi_{S^2 \times \mathfrak{so}(3)^*} = -q^3 \frac{\partial}{\partial q^1} \land \frac{\partial}{\partial y_2} + q^2 \frac{\partial}{\partial q^1} \land \frac{\partial}{\partial y_3} + q^3 \frac{\partial}{\partial q^2} \land \frac{\partial}{\partial y_1} - q^3 \frac{\partial}{\partial q^2} \land \frac{\partial}{\partial y_3} - q^2 \frac{\partial}{\partial q^3} \land \frac{\partial}{\partial y_1} + q^1 \frac{\partial}{\partial q^3} \land \frac{\partial}{\partial y_2} - \frac{1}{2} \left(y_3 \frac{\partial}{\partial y_1} \land \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial y_3} \land \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2} \land \frac{\partial}{\partial y_3}\right),$$

the Hamiltonian vector field $X_{\kappa_e}$ is given by

$$X_{\kappa_e} = -\frac{1}{2(e - V)} \left[\left(q^2 y_3 - \frac{q^3 y_2}{I_3}\right) \frac{\partial}{\partial y_1} + \left(q^3 y_1 - \frac{q^1 y_3}{I_3}\right) \frac{\partial}{\partial q^1} + \left(q^1 y_2 - \frac{q^2 y_1}{I_1}\right) \frac{\partial}{\partial q^2} + \left(q^2 y_1 - \frac{q^3 y_2}{I_2}\right) \frac{\partial}{\partial q^3} + \left(q^3 y_2 - \frac{q^1 y_3}{I_1}\right) \frac{\partial}{\partial y_1} + \left(q^1 y_3 - \frac{q^2 y_1}{I_2}\right) \frac{\partial}{\partial y_2} + \left(q^2 y_3 - \frac{q^1 y_2}{I_3}\right) \frac{\partial}{\partial y_3}\right].$$
and the Jacobi structure is given by

\[
\Lambda_{S^2 \times \mathfrak{so}(3)^*} = \Pi_{S^2 \times \mathfrak{so}(3)^*} + \Delta_{S^2 \times \mathfrak{so}(3)^*} \wedge X_{k^g},
\]

\[
E_{S^2 \times \mathfrak{so}(3)^*} = -X_{k^g},
\]

where \(\Delta_{S^2 \times \mathfrak{so}(3)^*} = \gamma \alpha \frac{\partial}{\partial y^\alpha}\) is the Liouville vector field.

## 5 Conclusions and future work

We have described the relation between the dynamics of mechanical Hamiltonian systems with respect to a linear Poisson structure on a vector bundle and the Jacobi–Reeb dynamics on the corresponding spherical bundle. In the particular case when the vector bundle is the cotangent bundle \(T^* Q\) of a manifold \(Q\) and the linear Poisson structure on \(T^* Q\) is just the canonical symplectic structure, we recover the classical case.

We remark that the classical result establishes an interesting connection between mechanical Hamiltonian flows on \(T^* Q\) and geodesic flows associated with Riemannian metrics on \(Q\) (or on open subsets \(U_e\) of \(Q\)). Indeed, for a fixed energy \(e\), the Reeb vector field of the contact structure on the spheric cotangent bundle \(S_{g^e}(T^* U_e)\), associated with the Jacobi metric \(g^e\), is just the restriction to \(S_{g^e}(T^* U_e)\) of the geodesic flow on \(T^* U_e\) for the Jacobi metric \(g^e\). This fact has important dynamical consequences. For instance, when \(Q\) is compact and the Riemannian curvature of \(Q\) is negative (see Introduction).

In the more general case when we replace the cotangent bundle \(T^* Q\) by an arbitrary vector bundle \(\tau_A: A^* \to Q\) endowed with a linear Poisson structure, some aspects stay similar. More precisely, we have that \(A\) is a Lie algebroid (see Remark 3.1) and if \(g\) is a bundle metric on \(A\), then one may introduce the Levi-Civita connection as an \(A\)-connection on \(A\) (for the theory of such connections see, for instance [25] and the references therein) and the geodesics of \(g\) as a particular class of admissible curves on \(A\) (see [26, Example 3.6]) or, equivalently, on \(A^*\) (note that the metric \(g\) induces a vector bundle isomorphism between \(A\) and \(A^*\)). In fact, the geodesics are the integral curves of the Hamiltonian vector field \(X_{k^g}\) of the kinetic energy \(k^g: A^* \to \mathbb{R}\). \(X_{k^g}\) is the geodesic flow on \(A^*\) of \(g\). In addition, using the Levi–Civita connection, one can also introduce the curvature of the bundle metric \(g\) and the sectional curvature of a 2-dimensional subspace of \(A_q\), with \(q \in Q\).

The previous comments and Theorem 3.5 imply, as in the classical case, that there also is a connection between mechanical Hamiltonian dynamics, with respect to linear Poisson structures on vector bundles, and geodesic flows on such spaces.

So, it would be interesting to develop a research program on the geometry (geodesics and curvature) of bundle metrics on vector bundles endowed with linear Poisson structures (or, equivalently, bundle metrics on Lie algebroids). The results of this research could be applied in the description of mechanical Hamiltonian dynamics with respect to such Poisson structures. In fact, following the ideas in [5] for the classical case, some possible topics such as:
- Structural stability of the trajectories,
- Periodic orbits,
- Ergodicity of the system,
- Killing tensors,

could be discussed. The case when the base space of the vector bundle is compact and the sectional curvature is negative should be remarkable.

Another interesting goal is to develop hybrid variational integrators for mechanical Hamiltonian systems with respect to linear Poisson structures based on the results in this paper and following the ideas in [27].

Finally, we remark that there exists a relevant connection between the Jacobi–Maupertuis principle and the Schrödinger variational principle of wave mechanics (for a discussion on this topic, see [28]). So, it would be interesting to extend these ideas to the more general setting of Lie algebroids or, equivalently, vector bundles endowed with linear Poisson structures.

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Declarations

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