The physics of stars, their workings and their evolution, is a goldmine of problems in statistical mechanics and thermodynamics. We discuss many examples that illustrate the possibility of deepening student’s knowledge of statistical mechanics by an introductory study of stars. The matter constituting the various stellar objects provides examples of equations of state for classical or quantal and relativistic or non-relativistic gases. Maximum entropy can be used to characterize thermodynamic and gravitational equilibrium which determines the structure of stars and predicts their instability above a certain mass. Contraction accompanying radiation induces either heating or cooling, which explains the formation of stars above a minimum mass. The characteristics of the emitted light are understood from black-body radiation and more precisely from the Boltzmann-Lorentz kinetic equation for photons. The luminosity is governed by the transport of heat by photons from the center to the surface. Heat production by thermonuclear fusion is determined by microscopic balance equations. The stability of the steady state of stars is controlled by the interplay of thermodynamics and gravitation.

We report on a teaching experience at the Ecole Polytechnique, a high-level non-specialized scientific institution in France. Its two academic years correspond to the middle years of universities in the United States; they are preceded by two years of preparation for the admittance examination, and most often followed by specialization. During the first year all students attend introductory courses in quantum mechanics and statistical physics. The course described here was taken by second-year students on an optional basis. One purpose of the course was to present a systematic but elementary introduction to the physics of stars, including our Sun. For this aspect, an article by Nauenberg and Weisskopf has been a major source of inspiration. At the same time the course was used as an opportunity for students to strengthen their knowledge of many important topics in statistical physics, such as maximum entropy, chemical potential, equilibrium of classical and quantum gases, and microscopic transport.

Although the course was not intended to train future astrophysicists or professional physicists, we measured its success by the enthusiasm of the students and their progress in understanding physics. The appeal of astrophysics is an ideal means of arousing interest in physics. Many basic laws of physics are brought into play in the workings of stars and have unexpected consequences. For instance, a striking fact is the omnipresence of quantum mechanics for such large objects, which is essential for explaining how a star radiates, how the nuclear plant at its center works, and why extinct stars do not collapse. A surprising feature of diffusion is also uncovered: as a result of its Brownian motion a photon emitted at the center of the Sun reaches the surface after having covered a distance of $2 \times 10^{10}$ times the Sun’s radius.

The necessity of relying on statistical physics is especially obvious in the study of stars, inaccessible objects known only via the radiation we receive from them. The conditions of temperature, pressure, or density which prevail in them cannot be reproduced in our laboratories. We have therefore to use the theoretical tools of statistical mechanics to deduce the properties of stars. The success of this approach is remarkable because we deal with exotic states of matter having no equivalent on Earth.

It is also noteworthy that simple and apparently crude models are sufficient to account, often unexpectedly, for the main phenomena, hence offering an exceptional opportunity for getting students more familiar with qualitative or semi-quantitative analyses using order of magnitude estimates. Such applications of statistical physics provide a welcome balance to the more abstract approaches which emphasize its underlying mathematical rigor. Of course, students who turn to astrophysics will have to face the full complexity of more realistic equations, which eventually lead, for instance, to a quantitative model for the Sun. At some places the course indicates how such equations can be obtained.

Another interesting feature of this subject is that it appeals to many branches of physics, thus exhibiting the unity of physics. For example, it is the only instance where the four fundamental forces of nature come into play in a characteristic and spectacular manner: the stars are formed by a collapse of matter caused by gravitational attraction, the light that they emit is generated by electromagnetic interactions, strong interactions provide their main source of energy, and weak interactions contribute in a crucial way to make their lifetime so long. The variety of the domains of physics involved in this course was reflected in the composition of the teaching team, including Marie-Noëlle Bussac, Robert Mochkovitch, Michel Spiro and the authors: our background is in astrophysics and statistical mechanics as well as nuclear, particle and plasma physics.

The course was delivered in 7 or 8 “blocks,” each one including a formal lecture plus a class devoted to teaching through problems. Most of the topics covered are outlined in the following. A more detailed account of several of them can be found in Ref. 1.
This second-year course on statistical physics is not to be regarded as an advanced course, but rather as a means for deepening the understanding of some of the concepts already acquired in the first-year course. No new techniques are introduced. On the other hand, this course is the first contact of students with astrophysics, which dictates the order in which the various topics are presented. Hence, the course proceeded in a nonlinear way, which has pedagogical value but is not fully reflected here. Approximations often are based on qualitative arguments which partly anticipate the results to be established. Students may be puzzled by such a procedure before they understand that it is fruitful and not vitiated by circularity. After some astrophysical background (Sec. II), the various topics are grouped into sections, each of which focuses on a specific aspect of statistical mechanics or thermodynamics: equations of state for the matter of the various types of stars (Sec. III), gravitational equilibrium viewed as an application of the maximum entropy criterion (Sec. IV), emission of light from the surface as an illustration of both black-body radiation and photon kinetics (Sec. V), energy transport by photons within the star (Sec. VI), and the use of balance equations to determine the production of heat in the core (Sec. VII). A synthesis is made in Sec. VIII, where further applications of statistical mechanics and thermodynamics to the physics of stars are suggested. Small type is used for material appropriate for problems. More details can be found in Refs. 3–7.

The course begins with a brief overview of stellar characteristics including mass, radius, luminosity, and surface temperature, and how these quantities can be obtained through observations. Statistical physics enters immediately, for example, in the determination of the radius $R$ of a distant star. We first deduce the luminosity $L$, which is the total radiative power emitted by the star, from the light flux that we receive from it and from the estimate of its distance. We then find $R$ from the Stefan-Boltzmann law,

$$L = 4\pi R^2 \sigma T_s^4,$$

$$\sigma = \frac{\pi^2 k^4}{605\hbar^3 c^2},$$

where $\sigma = 5.67 \times 10^{-8} \text{Wm}^{-2} \text{K}^{-4}$ is Stefan’s constant. The surface temperature $T_s$ of the star is obtained by comparing its spectrum to Planck’s law (see Sec. VA).

A. Three types of stars

It is useful to keep in mind various orders of magnitude corresponding to the three principal classes of stars. Our Sun is a typical example of the so-called main sequence. Its radius is $R_{\odot} = 7 \times 10^8 \text{m}$, its mass is $M_{\odot} = 2 \times 10^{30} \text{kg}$, its luminosity is $L_{\odot} = 3.8 \times 10^{26} \text{W}$, and its surface temperature is $T_{s,\odot} = 6000 \text{K}$. Hence, its average density is $1.4 \text{g cm}^{-3}$, comparable to that of water on earth. The masses of all the stars lie between $0.1 M_{\odot}$ and $100 M_{\odot}$. The Sun is mainly made of hydrogen, with 28% of its mass consisting of $^4\text{He}$ nuclei and 2% of other light elements. Its number of protons is the order of $10^{57}$. Heavier stars include red giants, a branch detached from the main sequence.

A second category of stars, the white dwarfs, have masses between 0.5 and $1.4 M_{\odot}$, and typical radii of 5000 km (like the Earth). Their density is of the order of $10^6 \text{g cm}^{-3}$.

The third category, the neutron stars, are even more compact objects. The density of neutron stars is the order of $3 \times 10^{14} \text{g cm}^{-3}$ and is comparable to that of the matter inside an atomic nucleus. A typical neutron star has a mass of $1.4 M_{\odot}$ and a radius of 10 km. Neutron stars are observed as pulsars: we receive from them regularly spaced pulses of electromagnetic radiation because they rotate rapidly and radiate only in particular directions. Beyond $3 M_{\odot}$, neutron stars are believed to collapse into black holes.

B. A few queries

One achievement of statistical physics is to explain the existence of these three very different and well-separated classes of stars. More precisely, we wish to understand the above orders of magnitude from simple information about the constituents of each type of star.

An obvious but most remarkable property of stars is that they shine. Furthermore, in the case of the Sun, we know that its luminosity has remained constant over a period comparable to the age of the Earth, that is, $4.5 \times 10^9$ years, a number obtained by measuring the proportions of radioactive elements in rocks. Understanding the source of energy for this radiation remained a challenge for 100 years.

At this stage, simple order of magnitude estimates can be suggested to students. Assuming the Sun is merely undergoing some chemical combustion, how long could it keep its present luminosity? Or, assuming that its only source of energy is gravitational contraction, evaluate its decrease in energy since the epoch when it was a dilute gas (see Sec. VII). If we also assume (without justification) that this energy has been radiated at the present rate, what would be the age of the Sun? Such calculations were made as far back as 1854 by H. von Helmholtz, but it became clear when radioactive dating of rocks began 50 years later that gravitational contraction was insufficient to explain why the radiation of the Sun lasts billions of years.

II. BACKGROUND
As soon as the huge heat associated with radioactivity was discovered, the idea emerged to look there for the origin of the energy radiated by the Sun. The first attempts were made in 1919 by J. Perrin and A. Eddington and a satisfactory explanation was given in 1937 by H. Bethe and F. von Weizäcker. However, a new question arises. The reactions of nuclear fusion of hydrogen into helium that take place in the central part of the Sun produce some amount of heat per unit time, which is exactly equal to the luminosity because the state of the Sun is stationary. However, such reactions are very sensitive to small changes in the temperature: they are activated by a rise, hindered by a decrease. Thus, if it happens at some instant that a little more power is produced in the core than what is evacuated by radiation from the surface, why does the internal temperature not rise, eventually resulting in an explosion of the Sun? Conversely if the opposite perturbation occurs, why does the Sun not become extinct? In short, why is the radiation from the Sun so stable? We shall answer this question in Sec. VIII.

We shall also wonder about the evolution of stars. How are they created? Why do they have a seemingly steady state in spite of their radiation? What happens after all their nuclear fuel is burnt?

III. THE MATTER OF STARS: EQUILIBRIUM STATISTICAL MECHANICS

A. Evolution of stars and models for their matter

As a first approximation, the Sun is made of hydrogen. The binding energy of the hydrogen atom is 13.6 eV, while the molecular binding energy of H₂ is 103 kcal mol⁻¹ or equivalently 4.5 eV. Because 1 eV corresponds to 11600 K, the surface temperature of 6000 K is not sufficient to dissociate the H₂ molecules. However, as will be seen, the bulk of the Sun is much hotter, typically 2 × 10⁶ K or 200 eV. It thus consists of completely ionized hydrogen plasma. Because it is globally neutral, we neglect the Coulomb interactions, and treat the Sun as a mixture of two perfect gases of protons and electrons with equal local densities.

The same model holds for any star of the main sequence. It will be shown that such a star is due to the contraction of a hydrogen cloud due to self-gravitation. As the density and the pressure increase, the temperature and hence the luminosity also increase. The protostar thus formed turns into a star if the temperature of its core becomes sufficiently high so as to initiate nuclear reactions of fusion. The study of this evolution will rely on the model of a self-gravitating perfect gas of protons and electrons (Sec. IV).

The emission of light in stars is maintained as long as nuclear reactions take place. However, when the nuclear fuel is exhausted, the production of heat in the core ceases, the internal pressure cannot be sustained, and gravitational contraction is resumed. For stars of masses between 0.1 and 0.5 M☉, fusion results in a plasma of He nuclei and electrons. As long as this gas behaves classically, it contracts rapidly, but this process stops when the electrons become a Fermi gas. The object becomes a white dwarf, an inert star which shines while cooling down slowly. Its matter can be represented by a model of a locally neutral mixture of two independent particle gases, the He nuclei and the electrons, with the electron gas quantum mechanical, while the He gas remains classical. This treatment is justified a posteriori.

Between 0.5 and 10 M☉, the stars reach a temperature which is sufficiently high so that fusion of nuclei produces elements heavier than ⁴He, such as ¹²C, ¹⁶O, ²⁰Ne, and ²⁴Mg, depending on the mass of the star. The resulting white dwarfs thus involve such nuclei instead of the He nuclei. The Sun will eventually become a C-O white dwarf.

From 10 to 100 M☉, fusion may lead to the most stable nuclei being around ⁵⁶Fe, but the temperature rises so much that most of the star explodes (supernova) while its core implodes, becoming a neutron star. Again neglecting the interactions, we can describe the constituents of such objects as a quantum neutron gas with a density comparable to that of nuclear matter.

We also have to consider photon gases. Within stars photons are emitted, absorbed, and scattered by matter. Thermal equilibrium is thus reached locally, at a temperature imposed by the medium.

B. Equations of state

All the equilibrium properties of a gas of non-interacting particles are embedded in the grand potential Ω(T, μ, V), a function of the temperature T, the chemical potential μ, and the volume V. It is given by

\[ \Omega = - \int d\varepsilon \mathcal{N}(\varepsilon) f(\varepsilon), \]

where \( \mathcal{N}(\varepsilon) \) is the number of single-particle states with energy less than \( \varepsilon \), and

\[ f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/kT} - 1}. \]
with $\eta = +1$ for Bosons and $\eta = -1$ for Fermions. The equations of state, which relate the extensive quantities, the internal energy $U$, the particle number $N$, the entropy $S$, and the volume $V$, to the intensive quantities, $T, \mu,$ and the pressure $P$, are obtained from

$$\Omega = U - TS - \mu N = -PV, \quad (4)$$

$$d\Omega = -SDT - Nd\mu - PdV. \quad (5)$$

In particular,

$$N = \int d\varepsilon D(\varepsilon), \quad U = \int d\varepsilon D(\varepsilon)\varepsilon f(\varepsilon) \quad (6)$$

are expressed in terms of the density of states $D(\varepsilon) = dN/d\varepsilon$. If a potential (here the gravitational potential) is included in the single-particle energies, it also contributes to $\mu$ for a given density.

The classical limit is attained when $\eta$ can be neglected in Eq. (3), that is, when $kT \gg \varepsilon_F$, where $\varepsilon_F$ is the function of the density defined by $N(\varepsilon_F) = N$; for fermions it is the Fermi energy. This condition is satisfied for both the electrons and the protons in the Sun and for the nuclei in white dwarfs. The classical limit yields $PV = NK_T$. However the nuclei do not always behave as a gas: they may crystallize in white dwarfs as studied in Problem 16.10 of Ref. 2 by a mean-field method. The above particles are non-relativistic because their mass $m$ satisfies $mc^2 \gg kT$.

We have for particles of spin $s = \frac{1}{2}$,

$$N(\varepsilon) = \frac{2s + 1}{h^3}V \int d^3p \theta(\varepsilon - \frac{p^2}{2m}) = \frac{V}{3\pi^2h^3}(2m\varepsilon)^{3/2}. \quad (7)$$

The uncertainty principle is reflected by the factor $\hbar^2/2mV^{2/3}$ because the momentum of each particle is order $\hbar/V^{1/3}$. The Pauli principle provides the factor $N^{2/3}$ in $\varepsilon_F$ because exclusion compels the particle number to behave as $p_{F}^{3}$. For any non-relativistic gas (classical or quantal), the combination of Eqs. (2), (4), (6) and (7), that is $N(\varepsilon) \propto \varepsilon^{3/2},$ yields a relation between the pressure and the internal energy per unit volume:

$$P = \frac{2U}{3V}, \quad (10)$$

which can also be derived from kinetic theory as an exercise.

In the ultra-relativistic limit, $\varepsilon \sim cp \gg mc^2$, which is relevant for the electron gas in heavy white dwarfs, we have

$$N(\varepsilon) = \frac{8\pi V}{3h^3c^3}\varepsilon^3, \quad (11)$$

and the relation between pressure and internal energy becomes

$$P = \frac{1}{3} \frac{U}{V}. \quad (12)$$

Protostars behave as classical non-relativistic gases. However, their elementary constituents are protons and electrons (like in stars) only when they are sufficiently hot. Otherwise we may have H-atoms or even H$_2$-molecules,
depending on the density and the temperature. Ionization equilibrium associated with the reaction $p + e \rightarrow H$ is characterized by

$$\mu_p + \mu_e = \mu_H, \quad N_p = N_e,$$  \hspace{1cm} (13)

where the chemical potentials are given by

$$\frac{\mu}{kT} = \ln \frac{N}{V} - \frac{3}{2} \ln \frac{mkT}{2\pi\hbar^2} - \ln \zeta(T)$$  \hspace{1cm} (14)

for ideal gases; $\zeta(T)$ is the internal partition function if the particle is composite. With $\zeta_p = \zeta_e = 2$ and $kT \ln \zeta_H \sim |\varepsilon_0|$, where $\varepsilon_0$ is the ground-state energy of the hydrogen atom ($-13.6 \text{ eV}$), valid for $T \ll |\varepsilon_0|/k \approx 10^8 \text{ K}$, Eqs. (13) and (14) constitute the Saha formula for the ionization ratio of the H-gas. This ratio increases with the temperature, decreases with the density, and equals $1/2$ for

$$\log \frac{n}{T^{3/2}} = 22 - \frac{6.9 \times 10^4}{T},$$ \hspace{1cm} (15)

in SI units. Chemical equilibrium for $2H \rightarrow \text{H}_2$ is treated likewise, but $\text{H}_2$ occurs only in the coldest and least dense clouds, at the very early stages of the formation of stars.

For the gas of photons in thermal equilibrium which coexists with matter in all stars, we have $\eta = +1$ and $\mu = 0$ in Eq. (13); Eqs. (11) and (12) still hold. The density of energy,

$$u_r = \frac{U_r}{V} = \frac{4}{c} \sigma T^4,$$ \hspace{1cm} (16)

is generally negligible compared to that of matter, given by $U = \frac{2}{3} N kT$ for $kT \gg \varepsilon_F$ or by Eq. (9) for $kT \ll \varepsilon_F$. However at the very high temperatures attained in massive stars and in supernovae, the pressure of radiation given by Eqs. (12) and (16) becomes essential due to its rise as $T^4$. It is responsible for stellar wind which expels matter, and for the existence of a maximum mass of about $100 M_\odot$ for stars (see Sec. VIII).

C. Radius of a neutron star

It is already possible at this stage to present the basic properties of white dwarfs and neutron stars as student problems. We focus here on neutron stars. By estimating their density from their mass (slightly larger than $M_\odot$) and from their radius (which will be found to be in the 10 km range) and by using Eq. (12), we expect that $\varepsilon_F$ defined by $N(\varepsilon_F) = N$ is of the order $100 \text{ MeV}$. Even though the internal temperature is of the order of $10^8 \text{ K}$, $kT/\varepsilon_F$ is as small as $10^{-4}$; the density is so large that the neutrons should be regarded as as if their temperature were zero. The gas is non-relativistic because $m_n c^2 \approx 900 \text{ MeV} \gg \varepsilon_F$. The internal energy $U$ is then given by Eq. (4) and the pressure by Eq. (10); they are $4000$ times larger than if the gas were classical. The specific heat per particle found from Eq. (3) is

$$\frac{1}{N} \frac{dU}{dT} = \frac{\pi^2 k^4}{60 \hbar^3 c^2} \frac{2^2 T^2}{\varepsilon_F},$$ \hspace{1cm} (17)

which is weaker than that of a perfect gas by a factor of $1/3000$.

To determine the radius of the neutron star from its mass, we note that it is in gravitational equilibrium. Because the temperature is negligible, this condition is expressed by requiring that the total energy $E = U + E_G$, the sum of the internal and the self-gravitational energies, be a minimum as a functional of the density of particles $n(r)$ at each point, under the constraint $m_n \int d^3 r n(r) = \int d^3 r \rho(r) = M$, the total mass of the star. From Eq. (10) we obtain

$$U = \frac{3(3\pi^2)^{2/3} \hbar^2}{10 m_n} \int d^3 r [n(r)]^{5/3},$$ \hspace{1cm} (18)

while the gravitational energy is

$$E_G = -\frac{G}{2} \int d^3 r d^3 r' \frac{\rho(r) \rho(r')}{|r - r'|}$$ \hspace{1cm} (19)

We use a crude model for which the density $n$ within the star is taken as a constant. Then Eqs. (13) and (17) yield

$$U = \frac{3^{7/3} \pi^{2/3} \hbar^2 M^{5/3}}{2^{7/3} 5 m_n^{8/3}} \frac{1}{R^2},$$ \hspace{1cm} (20)
the energy of neutrons although neutron stars (but not for white dwarfs). The present argument is faulty because it uses the non-relativistic approximation for where light is trapped by gravity. This mass is smaller than the mass of collapse in Eq. (25) which is therefore irrelevant for

$$E_G = -2U, \quad E = -U.$$  (22)

Gravitational equilibrium thus implies in this model a radius

$$R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{h^2}{Gm_n^{8/3}M^{1/3}}.$$  (23)

The more massive the star, the smaller its radius. A numerical estimate for $M = M_\odot$ provides $R = 12$ km, $n = 0.15$ fm$^{-3}$, $\varepsilon_F = 60$ MeV, which confirms the hypotheses on which our model is based.

A similar theory holds for white dwarfs provided that the neutrons are replaced by electrons (for the evaluation of $E$) and that the nuclei are taken into account (for the evaluation of $E_c$). Their radius and density are found as function of their mass, with numerical values in agreement with the data of Sec. IIA.

D. Stability problems for neutron stars and white dwarfs

For more massive neutron stars than considered above, the Fermi energy becomes significant compared to $m_n c^2$ and the relativistic form $\varepsilon = (m^2 c^4 + c^2 p^2)^{1/2}$ of the kinetic energy should be taken into account. This form produces a crossover from Eq. (4) to Eq. (11), for $\mathcal{N}(\varepsilon)$, and from Eqs. (13) or (24) to

$$U = \frac{3^{4/3}hc}{8\pi^{1/3}} \int d^3 r \left[n(r)\right]^{4/3} \simeq \frac{3^{5/3}hcM^{4/3}}{2^{11/3}\pi^{2/3}m_n^{4/3}} \frac{1}{R}$$  (24)

for $U$. Hence, for sufficiently small $R, E = U + E_G$ is linear in $R^{-1}$, with a coefficient which becomes negative for

$$M > \frac{3}{16\pi m_n^2} \left(\frac{5hc}{2G}\right)^{3/2} \simeq 7M_\odot.$$  (25)

Thus, neutron stars heavier than this limit would collapse because the Fermi pressure is insufficient to resist gravitational attraction.

As shown later, this calculation is irrelevant because another effect prevents the existence of such massive neutron stars. However, for white dwarfs, the corresponding calculation yields the Chandrasekhar limit mass beyond which white dwarfs cannot exist. The occurrence of this maximum mass implies an upper bound on the luminosity of white dwarfs. The Chandrasekhar mass is found to be $1.7M_\odot$ by a crude estimate analogous to Eq. (23), and to be $1.4M_\odot$ by a more precise calculation taking into account the complete form of the relativistic energy $\varepsilon$ and the variation of the density.

Returning to neutron stars, we note that Eq. (23) implies that $\varepsilon_F = \frac{1}{2}GMm_n/R$. Hence, the total energy of a neutron at the surface of the star, $\varepsilon = GMm_n/R$, is negative because $\varepsilon < \varepsilon_F$. No neutron can escape from the star because the escape velocity,

$$\sqrt{\frac{2GM}{R}} \simeq 1.5 \times 10^8 \left(\frac{M}{M_\odot}\right)^{2/3} \text{ms}^{-1},$$  (26)

is larger than the Fermi velocity. The velocity in Eq. (24) reaches $c$ for $M = 3M_\odot$, suggesting that photons themselves cannot escape if $M > 3M_\odot$. Thus there exists a maximum mass for neutron stars, beyond which such objects become black holes where light is trapped by gravity. This mass is smaller than the mass of collapse in Eq. (25) which is therefore irrelevant for neutron stars (but not for white dwarfs). The present argument is faulty because it uses the non-relativistic approximation for the energy of neutrons although $\varepsilon_F$ equals $\frac{1}{2}m_n c^2$ when the velocity in Eq. (26) reaches $c$; furthermore general relativity should be used to describe such strong gravitational fields. However, the conclusion turns out to be correct.

A neutron decays as

$$n \rightarrow p + e + \nu,$$  (27)

with an average lifetime of 15 minutes. How can a neutron star be stable against such a $\beta$-decay? Actually the electrons and protons created by the reaction (27) can in turn react as

$$p + e \rightarrow n + \nu,$$  (28)
so that a stationary regime sets up where the star contains, beside \( N_n \) neutrons, a number \( N_p = N_e \) of protons and electrons. The neutrinos practically do not interact with matter and are radiated, only carrying out a small amount of energy. If we forget about them, we determine \( N_e \) by the chemical equilibrium condition

\[
\mu_n = \mu_p + \mu_e.
\]  

The chemical potentials include contributions \( mc^2 \) which can be omitted because \( (m_n - m_p)c^2 = 1.3 \text{ MeV} \) and \( m_e c^2 = 0.5 \text{ MeV} \) are negligible compared to the Fermi energy \( \varepsilon_p^F \) of the neutrons. They also involve contributions from gravity which are almost the same on both sides of Eq. (29). Assuming that the protons and the electrons constitute "cold" Fermi gases, and assuming the electrons to be ultra-relativistic — hypotheses to be checked in the end — we note that \( N_p = N_e \) implies that the Fermi momenta \( p_p^F = p_e^F \) are equal and hence that \( \varepsilon_p^F \ll \varepsilon_e^F \). Thus \( N_e \) results from \( \varepsilon_p^F = \varepsilon_e^F \), which together with Eqs. (1) and (11) yields

\[
\frac{N_e}{N_n} = \frac{3\pi^2}{8} \left( \frac{\hbar}{m_n c} \right)^3 \frac{N_n}{V},
\]

which is \( 2 \times 10^{-3} \) for \( M = M_\odot \). The presence of this small number of electrons and protons is sufficient to prevent the \( \beta \)-decay of the neutrons.

### IV. SELF-GRAVITATING OBJECTS: MAXIMIZING THE ENTROPY

#### A. The relation between mass, radius and temperature

Like any thermodynamic equilibrium, gravitational equilibrium is expressed by looking for the maximum of the entropy, subject to constraints on conserved quantities. The existence of radiation within a star causes a negligible departure from equilibrium (except for very massive objects), so that we have to write that \( S \) is maximum for a fixed value of the total energy \( E = U + E_G \). We begin with the same crude approximation as in Sec. IIIC, neglecting the variations of density and temperature within the star or protostar. We thus have to maximize

\[
S - \beta(U + E_G)
\]

with respect to \( U \) and to the radius, where \( \beta \) is the lagrangian multiplier associated with the constraint on \( E \). The first condition implies that \( T^{-1} = \beta \). If we use Eq. (21) and vary the above with respect to \( R \), we obtain

\[
3PV = -E_G = \frac{3GM^2}{5R},
\]

where \( P = -\partial U/\partial V \) is the average thermodynamic pressure. For a given mass \( M \) and a given temperature \( T \), the radius \( R \) is determined by Eq. (22), where \( P \) is a function of \( T \) and the density.

The long-range nature of the gravitational forces implies a non-extensivity of the matter of stars. Doubling the mass of a star for a given temperature does not double its volume. In particular, the volume of a neutron star or a white dwarf is inversely proportional to its mass (see Eq. (23)).

From the microscopic viewpoint, the gravitational attraction at short interparticle separations produces another pathology. The Boltzmann-Gibbs distribution \( e^{-H/kT} \) contains in classical statistical mechanics a factor \( \exp \left[ \sum_{i>j} Gm^2/r_{ij}kT \right] \) for particles with masses \( m \) and relative distances \( r_{ij} \). This factor becomes so large at short distances that the partition function diverges, and hence, there exists no equilibrium in classical statistical mechanics. The expressions (12) or (21) for the gravitational energy implicitly rely on a mean-field approximation and thus disregard the correlations which invalidate the classical calculation. However, as seen in Secs. IIIC and IID, the Pauli principle cures this difficulty by providing a short-range cut-off \( \hbar/p_F \), and thus prevents the occurrence of the collapse, at least for not too massive stars.

The condition (22) characterizes a stable equilibrium provided \( S \) is a maximum, that is, \( d^2S \equiv d \left[ T^{-1}(dU + PdV) \right] < 0 \) for \( dE = 0 \) and \( dS = 0 \). This condition is expressed by

\[
\frac{dP}{dV} \bigg|_S < \frac{d^2E_G}{dV^2} = \frac{4}{9} \frac{E_G}{V^2} = -\frac{4}{3} \frac{P}{V}.
\]

Hence, the gravitational equilibrium is stable only if the adiabatic compressibility \( \kappa_S \equiv - (P/V)(dV/dP) \bigg|_S \) is smaller than \( 3/4 \). This condition is satisfied for non-relativistic gases, whether classical or quantal (for which \( \kappa_S = 3/5 \)), but not for ultra-relativistic Fermi gases (for which \( \kappa_S = 3/4 \)). Such a relativistic instability is responsible for the gravitational collapse of massive objects, in particular for the implosion of a supernova leading to a neutron star.
For non-relativistic gases, Eqs. (10) and (32) imply that
\[ E = U + E_G = \frac{1}{2} E_G = -U = -\frac{3}{2} P V, \]  
(34)
whereas for ultra-relativistic gases, Eq. (10) yields the total energy
\[ E = 3 P V + E_G = 0, \]  
(35)
which thus vanishes when the star becomes unstable.

If the stellar object radiates, and if the corresponding loss of energy given by Eq. (1) is not compensated for by a production of heat in the core through nuclear fusion, the decrease \( |dE| \) in energy produces a shift of the gravitational equilibrium. For non-relativistic gases, the resulting changes in the various thermodynamic quantities are obtained from Eqs. (2)–(7), by noting that the Massieu potential per particle \( \Psi(\beta, V) = \left( \frac{k - 1}{S} - \frac{\beta U}{N} \right) \) is a function of the dimensionless variable \( \frac{\beta - 1}{V^2/3 - 2/3 \hbar - 2/3 m} \). (This result can alternatively be found from Eq. (34).) Hence, using the fact that the first and second partial derivatives of \( \Psi \) with respect to \( \beta \) yield \( U \) and the constant volume specific heat \( C_V = \partial U/\partial T \mid V \), one can find as an exercise:
\[ -\frac{dE}{U} = -\frac{dR}{R} = \frac{1}{4} \frac{dP}{P} = \frac{dU}{U} = -\frac{T dS}{U} = \frac{C_V dT}{2C_V T - U}. \]  
(36)
The temperature may increase or decrease, depending on the sign of \( 2C_V T - U \). The entropy always decreases, even when the temperature rises, due to the contraction which accompanies the loss of energy. A larger increase of entropy takes place in the outside world, because an amount \( |dE| \) of energy is radiated toward regions with temperature lower than \( T \).

We recover from the above results those of Secs. IIIC and IIID on neutron stars and non-relativistic white dwarfs. In these cases, \( C_V T \) is negligible compared to \( U \), and radiation induces cooling.

B. Evolution of a hydrogen cloud

A hydrogen cloud behaves as a perfect gas, so that
\[ -E = U = \frac{3}{2} N k T = C_V T, \]  
(37)
where \( N \) is the total particle number. Hence, a loss of energy \( |dE| \) is accompanied by heating. Gravitational forces thus induce an effective “specific heat” \( dE/dT = -\frac{1}{4} N k \) which is negative. In contrast, for extensive states of matter without gravity, the concavity of entropy implies that \( \Psi \) is convex and hence that the true specific heat \( dU/dT \) is positive.

As the cloud radiates and shrinks, the temperature rises, and the heating becomes more and more efficient, especially if the cloud has a large mass. Once the ionization temperature given by Eq. (15) is reached, the particle number is \( N = 2M/m_p \) and the condition (32) expresses the radius as
\[ R = \frac{G m_p M}{10 k T}. \]  
(38)
This relation holds for protostars as well as for stars of the main sequence for which the electron gas is classical, with
\[ e^{n_e/kT} = \frac{n}{2} \left( \frac{2\pi \hbar^2}{m_e k T} \right)^{3/2} \ll 1. \]  
(39)
We denote by \( n = M/m_p V \) the common density of electrons and protons (see Figs. 1 and 2). For the Sun, Eq. (38) yields an average internal temperature of \( 2 \times 10^6 \) K.

A star of the main sequence is in a stationary regime, where the luminosity \( L \) is exactly compensated for by the power \( Q \) created in the core by nuclear reactions. Its temperature remains fixed at a value about \( 10^7 \) K or 1 KeV in the core (Secs. VII and VIII). Its radius has the value corresponding to Eq. (38) and the energy \( E \) (excluding the nuclear mass energy) remains fixed at the value \(-3MkT/m_p\).

For a protostar the radius decreases with the total energy as shown by Eq. (38), while the temperature rises until it reaches (for \( M > 0.08 M_\odot \)) the ignition value for H, where the contraction stops. Clouds with masses \( M < 0.08 M_\odot \)
do not result in stars; as seen below, their temperature never attains the ignition value. The cloud results in an aborted star, a brown dwarf, or for smaller masses an object like Jupiter (\(M = 10^{-3} M_\odot\)).

Once a sizeable proportion of the protons in the core has been transformed into helium, its fusion can no longer take place. Radiation is no longer compensated for by heat production, which entails anew a decrease of \(E\), hence shrinking and heating. If the star is sufficiently massive (\(M > 0.5 M_\odot\)), the temperature may then reach the higher ignition value for fusion of He into C and O. For still more massive and hotter stars (\(M > 8 M_\odot\)), fusion can lead to heavier elements.

During these processes, the left-hand side of Eq. (39), proportional to \(T^{3/2} M^{-2}\) because \(n \propto M R^{-3}\) and \(R \propto M T^{-1}\), increases until the condition (39) is violated. The electrons gradually enter the quantum regime, described by Eqs. (2)–(7), while the classical approximation still holds for the residual hydrogen and for the nuclei produced. According to Eq. (36), while the radius always decreases when the either aborted or extinct star radiates its energy without thermonuclear production, the temperature first rises, reaches a maximum, and then decreases: while \(dT = |dE|/C_V\) in the classical regime of Eq. (38), we find from Eq. (38) that \(dT = -|dE|/C_V\) in the Fermi gas limit, with \(C_V \propto T\).

More precisely, the relations between the density, the temperature, and the energy for a given mass can be found by using Eqs. (1), (6), and (7) and \(\rho \equiv \frac{N}{V}\) without thermonuclear production, the temperature first rises, reaches a maximum, and then decreases: while \(dT = |dE|/C_V\) in the classical regime of Eq. (38), we find from Eq. (38) that \(dT = -|dE|/C_V\) in the Fermi gas limit, with \(C_V \propto T\).

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The numerical values were estimated for an aborted star where \(N_{\text{nuc}} = N_e = M/m_p\), but this behavior is general. As expected, higher maximum temperatures are reached for more massive stars. When the star cools sufficiently, it becomes a white dwarf, described by letting \(\alpha \to \infty\) in Eqs. (10)–(12). In particular \(E\) and \(R\) end up at a minimum value given by Eqs. (10)–(12), where \(m_u^{8/3}\) should be replaced by \(m_u (m_p A/Z)^{5/3}\) and \(Z\) and \(A\) are the average charge and mass numbers of the nuclei. It is instructive to represent the above results as graphs in the density-temperature plane (Fig. 1) and in the density-energy plane (Fig. 2).

A numerical estimate of the thermal plus gravitational energy of the Sun from Eqs. (22) and (24) shows that it is 1000 times smaller than the radiative energy released during the whole life of the Sun (for \(10^{10}\) years at the rate of \(L_\odot = 3.8 \times 10^{26}\) W). Thus the energy of gravitational contraction has been negligible compared to the nuclear energy released and radiated; it has only contributed to bringing the core of the Sun to the ignition temperature for the fusion of hydrogen. However, once the nuclear reactions stop, a new contraction will produce an overheating and a huge increase of luminosity. The amount of energy released in this process can be estimated from the final energy of the Sun as a white dwarf, given by Eq. (41) for \(\alpha \to \infty\). It is found to be order of one tenth of the total radiated energy. Thus, in the stage following exhaustion of the nuclear fuel, the radiative energy fed by gravitation only will be a significant fraction of the thermonuclear energy.

C. Local density

A more precise theory requires us to account for the non-uniformity of the mass density \(\rho(r)\) and the temperature \(T(r)\) within the stellar object. We assume that the total angular momentum vanishes, so that gravitational equilibrium ensures spherical symmetry. The entropy \(S\) and the internal energy \(U\) are now sums of contributions \(s_i\) and \(u_i\) from
small volume elements $\omega_i$, and $E_G$ is given by Eq. (19). Equilibrium is expressed by maximizing Eq. (31) with respect to variations $\delta u_i$ and to deformations which displace matter from $r$ to $r + \delta r$, where $\delta r$ is a function of $r$. The volumes $\omega_i$ are changed according to $\delta \omega_i = \omega_i \nabla \cdot \delta r$, and the variation of $E_G$ is

$$\delta E_G = \int d^3r \, \rho(r) \nabla W \cdot \delta r,$$

where $W(r)$ is the gravitational potential

$$W(r) = -G \int \frac{d^3r'}{|r - r'|} \rho(r') - 4\pi G \int r' dr' \rho(r').$$

The variation of $S$ is

$$\delta S = \sum_i \left[ \frac{\delta u_i}{T_i} + \frac{P_i}{T_i} \delta \omega_i \right] = \sum_i \frac{\delta u_i}{T_i} + \int d^3r \frac{P}{T} \nabla \cdot \delta r.$$

Thermodynamic equilibrium thus implies for independent variations $\delta u_i$ that $T(r)$ is uniform and equal to $\beta^{-1}$. It also implies for arbitrary deformations $\delta r$ that

$$\nabla P = \rho \nabla W$$

at each point. We thus recover the equation of hydrostatics as a consequence of the laws of thermodynamics.

Because $P$ is a function of $T$ and the density, Eq. (48) is an integro-differential equation for the mass density, to be solved with $\rho(\infty) = 0$ and $\int d^3r \rho(r) = M$. It is equivalent to the differential equation

$$\nabla \cdot \frac{\nabla P}{\rho} = -4\pi G \rho.$$

A consequence of Eq. (48) is the virial theorem

$$3 \int d^3r P = -E_G.$$

It can be proved directly by writing that the thermodynamic potential (31) is stationary under a dilation $\delta r = r \varepsilon$, $\delta u_i = -P_i \delta \omega_i$, for which $\delta U = -3 \int d^3r P \varepsilon$, and $\delta E_G = -E_G \varepsilon$. The approach of Secs. IIIA and IIB was based on the virial theorem in the crude form of Eq. (42).

Actually, stars are not in thermal equilibrium because they radiate and because heat is produced in their core. The temperature $T(r)$ decreases from the center to the surface. The time-scales involved for thermal exchanges are large (it takes $10^7$ years for a photon to propagate through the Sun), whereas gravitational equilibrium is reached very rapidly due to the long-range forces (it takes only a few hours for sound waves to propagate across the Sun). Likewise, electric neutrality is locally ensured by the long-range Coulomb forces. In the above calculation, the variations $\delta u_i$ and $\delta \omega_i$ thus correspond to mechanical energy only, without heat transfer nor diffusion. They are thus not independent, but satisfy $\delta u_i = -P_i \delta \omega_i$. We thereby obtain no condition on $T(r)$, but Eq. (48) still holds.

In stars which have already transformed a significant part of their hydrogen into helium and heavier elements, equilibrium is also not reached regarding the variation in space of the proportion of elements. This situation occurs in particular for red giants and for supernovae. In the gravitational potential $W$, the density of nuclei with masses $m_i$ would vary in equilibrium as $\exp[-m_i W(r)/kT]$, whereas nuclei remain in stars at the place where they were formed because their diffusion is a slow process. (Convection can however raise heavier elements up to the surface.) The above reasoning, in which matter is deformed as a whole, applies here and leads again to Eq. (48).

As an example, for a not too massive white dwarf (with $M < 0.5M_\odot$), the thermodynamics is governed by the electron gas which is non-relativistic. The nuclei, which for such stars are H or He, only contribute to the gravitation. Equations (1) and (3) hold for the local density of electrons $n_e(r)$, the internal energy per electron, and the pressure $P(r)$, which are related to one another by the local Fermi energy $\varepsilon_F(r)$. The mass density equals

$$\rho(r) = \frac{m_e A}{Z} n_e(r) = \frac{m_e A}{Z} \left( \frac{2m_e}{3\pi^2 h^3} \right)^{3/2} \left[ \varepsilon_F(r) \right]^{3/2},$$
where $A$ and $Z$ denotes the average mass and charge numbers of the nuclei, because on average one electron drags $1/Z$ nuclei. On the other hand, we find from $\dot{U} = \dot{P}V$ and $dU = -PdV$, that $\dot{P}n_{e}dP$ is the differential of the energy per particle given by Eq. (48), so that

$$
\frac{1}{\rho} \nabla P = \frac{Z}{m_{p}A} \nabla \varepsilon_{F}.
$$

(52)

The differential equation (13) for $\rho(r)$ can thus be rewritten as

$$
\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \frac{d\varphi(\xi)}{d\xi} \right] + \left[ \varphi(\xi) \right]^{3/2} = 0
$$

(53)

in terms of the dimensionless variables

$$
\xi \equiv \frac{r}{a}, \quad \varphi(\xi) \equiv \frac{\varepsilon_{F}(r)}{\varepsilon_{F}(0)} = \left[ \frac{\rho(r)}{\rho(0)} \right]^{2/3}, \quad a^2 \equiv \frac{1}{4\pi G \left( \frac{Z}{m_{p}A} \right)} \frac{\varepsilon_{F}(0)}{\rho(0)}.
$$

(54)

The solution of Eq. (53), the Lane-Emden equation of index 3/2, with boundary conditions $\varphi(0) = 1$ and $\varphi'(0) = 0$, is a function which decreases from 1 to 0, a value attained for $\xi_1 = 3.65$. At this value $\varphi(\xi)$ has a finite derivative, $\xi_1^{3/2} \varphi'(\xi_1) = -2.71$, and beyond it vanishes. The parameter $a$, and consequently $\varepsilon_{F}(0)$ and $\rho(0)$, are determined by expressing the total mass as

$$
M = \int d^3r \rho(r) = 4\pi a^3 \rho(0) \int_{0}^{\xi_1} \xi^2 d\xi \left[ \varphi(\xi) \right]^{3/2}
$$

$$
= \frac{9a^2}{16} \left( \frac{Z}{m_{p}A} \right)^5 \left( \frac{\hbar^2}{2m_{e}G\mu} \right)^3 \left| \xi_1^2 \varphi'(\xi_1) \right|.
$$

(55)

where we have used successively Eqs. (54), (31), and (53).

We see that a universal function $\varphi(\xi)$ describes the density of all white dwarfs with masses $M < 0.5 M_{\odot}$. A sharp edge occurs at the radius $R_1 = a\xi_1$, which scales with the mass as $M^{1/3}$, and which equals 1.2 times the radius provided by the crude approximation of Eq. (23). Near $R_1$, the density decreases as $(R_1 - r)^{3/2}$. The central density $\rho(0)$ also scales as $R_1^{1/3}$. Its value is larger than the one found in Sec. IIIC by a factor of 3.3. For white dwarfs with masses $M > M_{\odot}$, the large value of $\rho(0)$ implies the failure of the non-relativistic approximation, and $\varepsilon = p^2/2m$ should be replaced by $(m^2 c^4 + c^2 p^2)^{1/2}$ for the electrons in the central part of the star. When $M$ approaches the Chandrasekhar mass, the ultra-relativistic approximation of Eq. (11) may be used in that region.

Because for such stars, $P$ depends only on the density through Eq. (22), the hydrostatic equation (18) can be integrated as

$$
\varepsilon_{F}(r) + \frac{m_{p}A}{Z} W(r) = \mu.
$$

(56)

The integration constant $\mu$ is interpreted as the total electronic chemical potential, including the contribution of the effective gravitational potential seen by an electron together with the nuclei that it drags. Indeed, equilibrium implies that the chemical potential is uniform, at least when the temperature is also uniform or as here negligible; otherwise, its gradient is obtained from Eq. (18) and from the Gibbs-Duhem relation as $-s \nabla T$, where $s$ is the entropy per particle.

V. THE LIGHT EMITTED BY A STAR: PHOTON KINETICS AND BLACK-BODY RADIATION

A. A star as a black body

The radiation of a body is characterized by its luminance $L_{\nu}(\theta)$, defined as follows. The power $\delta w$ radiated by a surface element $\delta S$ in a solid angle $\delta \omega$ around a direction making an angle $\theta$ with the normal to the surface and in a range of frequency $\nu, \nu + \delta \nu$ equals

$$
\delta w = L_{\nu}(\theta) \cos \theta \, \delta S \, \delta \omega \, \delta \nu.
$$

(57)

The main features of light emission by a star can be understood by regarding its surface as a black body, that is, as a material which is a perfect absorber. Standard thermodynamic reasoning shows that the luminance $L_{\nu}^0(T)$ of a black body at temperature $T$ is related to the spectral energy density of a photon gas in equilibrium in an enclosure. From Eqs. (3), (3), and (3), we can obtain Planck’s law

$$
L_{\nu}^0(T) = \frac{2\hbar \nu^3}{c^2} \frac{1}{e^{\hbar \nu/kT} - 1}.
$$

(58)
Photometry shows that the spectrum of the light that we receive from stars can be fitted to Eq. (58), the parameter $T$ being identified with the surface temperature $T_s$ of the star (5800K for the Sun). As a function of the frequency, $L_0^s$ has a maximum at $h\nu_{\text{max}} = 2.82kT$ (Wien’s law), so that the color of a star gives us direct information about its surface temperature. One classification of stars is actually based upon their color.

The total luminosity $L$ of a star is obtained by integrating Eqs. (57) and (58), which yields Eq. (1). On the Hertzsprung-Russell diagram, stars are plotted according to their color and luminosity, which exhibits regularities that will be explained in Sec. VID.

The fact that $L_0^s$ does not depend on $\theta$ is Lambert’s law. It expresses the fact that the power $\delta w$ received by an observer is proportional, not to the surface $\delta S$, but to the apparent surface $\delta S \cos \theta$ of the emitter. The angle $\theta$ is not accessible for distant stars, which are always seen as points. For the Sun, $\theta$ increases from 0 at the center of the solar disk to $\pi/2$ at its edge. We shall discuss the validity of Lambert’s law in Sec. V.C.

**B. Kinetics of photons in matter**

A microscopic theory is needed to explain the above results and to obtain corrections. We must take into account the structure of the matter of the star, and in particular, the fact that its surface is ill-defined. The light comes from a superficial shell which is partly transparent, the photosphere. In this region the temperature $T(r)$ and the density $\rho(r)$ decrease with $r$ and we need to understand the meaning of the quantity $T_s$ which enters Eqs. (58) and (1). The propagation and thermalization of particles moving in a gaseous medium, such as photons in the photosphere, is described by the Boltzmann-Lorentz kinetic equation for the particle density $f(r, p, t)$. This equation has the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = I(f).$$

(59)

For photons with energy $\varepsilon = h\nu = cp$, it is convenient to rewrite Eq. (59) in terms of

$$\Lambda_\nu(r, u, t) \equiv \frac{h^4 v^3}{c^2} f(r, p, t),$$

(60)

where $u = p/p$ is a unit vector in the direction of propagation. When $r$ lies just outside the star, $\Lambda_\nu$ can be identified with the luminance $L_\nu$ entering Eq. (57). The kinetic equation then reads

$$\frac{\partial \Lambda_\nu}{\partial t} + cu \cdot \nabla \Lambda_\nu = I_c + I_{\text{sp}} + I_{\text{st}} - I_a,$$

(61)

where the right-hand side depends on the distribution function $\Lambda_\nu$ and on the local properties of matter. The term $I_c$ is the collision term describing elastic or inelastic scattering of photons, $I_{\text{sp}}$ and $I_{\text{st}}$ are the source terms associated with spontaneous and stimulated emission, and $I_a$ is the absorption term. At each point, matter is in local thermal equilibrium at temperature $T(r)$, but the photons are far from equilibrium, especially near the surface where their distribution is extremely anisotropic as they are escaping. In the photosphere, $T$ lies below the ionization temperature and the matter mainly contains atoms which govern the dominant processes, absorption and emission. Inside the star where matter is completely ionized, the dominant process is the scattering by electrons.

We focus here on absorption and emission. The corresponding three terms in Eq. (61) are associated with the reactions $\gamma + A_1 \rightarrow A_2$, where the states $A_1$ and $A_2$ of an atom have energies $\varepsilon_1$ and $\varepsilon_2 = \varepsilon_1 + h\nu$. Because the photosphere is in thermal equilibrium, the densities $n_1$ and $n_2$ of the two species satisfy $n_2 = n_1 e^{-h\nu/kT}$. The average number $\langle N_\gamma \rangle$ of photons in a single mode with frequency $\nu$ is

$$\langle N_\gamma \rangle = \frac{1}{2} \frac{h^3 f}{2\hbar \nu^3} \Lambda_\nu.$$  

(62)

Denoting by $\sigma_a$ the cross-section for absorption of a single photon by a single atom, the flux of photons reaching this atom is $cf$ and the absorption term in Eq. (61) is therefore $I_a = c \sigma_a n_1 \Lambda_\nu$. Probabilities of opposite elementary processes are equal. Hence, the simulated emission term is $I_{\text{sp}} = c \sigma_a n_2 \Lambda_{\nu}$, and the spontaneous emission term $I_{\text{st}}$ is obtained for each photon mode by dividing the contribution of stimulated emission by the number $N_\nu$ of photons. (A similar relation between two inverse processes holds for inelastic scattering.) With these considerations and Eq. (62), Eq. (61) becomes

$$\frac{1}{c} \frac{\partial \Lambda_\nu}{\partial t} + u \cdot \nabla \Lambda_\nu = \sum \sigma_a [n_2(\Lambda_\nu + \frac{2h\nu^3}{c^2}) - n_1 \Lambda_\nu].$$

(63)
The summation is carried over the various processes which can occur due to the existence of various constituents in the photosphere. Using the relation \( n_1 = n_2 e^{h\nu/kT} \), we can rewrite Eq. (63) as

\[
\frac{1}{c} \frac{\partial \Lambda_\nu}{\partial t} + \mathbf{u} \cdot \nabla \Lambda_\nu = k_\nu (L_\nu^0 - \Lambda_\nu), \tag{64}
\]

where the coefficient \( k_\nu \) is defined by

\[
k_\nu = \sum \sigma_a (n_1 - n_2). \tag{65}
\]

The quantity \( k_\nu^{-1} \) can be interpreted as a mean free path for the photons with frequency \( \nu \). The quantity \( L_\nu^0 \) on the right-hand side of Eq. (64) is the same as in Eq. (58), evaluated at the local temperature \( T(r) \) of the medium.

As expected from equilibrium statistical mechanics, the right-hand side of the kinetic equation vanishes when the photons are thermalized with matter, their distribution then being the same as within an enclosure at temperature \( T(r) \). The temperature \( T(r) \) appears not only through \( L_\nu^0 \), but also through the factor \( k_\nu \), which moreover is proportional to the local density \( \rho \) and depends on the frequency through the cross-sections \( \sigma_a \).

Regarding the surface of a star as a plane, the quantities \( k_\nu \) and \( L_\nu^0 \) are functions of the altitude \( z \). In a stationary regime, \( \Lambda_\nu \), a function of \( z \) and of the angle \( \theta \) between the direction of propagation and the \( z \)-axis, should be obtained by solving

\[
\frac{d\Lambda_\nu}{dz} = \frac{k_\nu}{\cos \theta} (L_\nu^0 - \Lambda_\nu). \tag{66}
\]

If the surface of the star (at \( z = 0 \)) were sharp, and if the temperature and the density were uniform for \( z < 0 \), the solution of Eq. (66) would be \( L_\nu^0 = \Lambda_\nu \) everywhere for \( \cos \theta > 0 \), and would satisfy \( \Lambda_\nu = 0 \) for \( z \geq 0 \) and \( \cos \theta < 0 \) (no incoming light from outside the star), and go to \( L_\nu^0 \) for \( z \to -\infty \) and \( \cos \theta < 0 \) (light propagating inwards inside the star). The star would thus radiate like a black body as indicated in Sec. II.A.

To solve Eq. (66) for the realistic case of an inhomogeneous photosphere, we define the optical depth as

\[
\zeta \equiv \int_{z}^{\infty} d\zeta' \frac{k_\nu}{\cos \theta} (L_\nu^0 - \Lambda_\nu). \tag{67}
\]

This quantity is dimensionless and depends on the frequency and on the geometrical depth \( z \). It vanishes outside the star, increases when the altitude \( z \) decreases, and tends to infinity deep within the photosphere (\( z \to -\infty \)). The optical depth characterizes the penetration of light in the photosphere, which is governed by the cumulative effect on photons of matter lying above \( z \); an incoming beam with \( \cos \theta < 0 \) would be attenuated as \( e^{-\zeta/|\cos \theta|} \) as it propagates inward.

Inverting Eq. (67) gives \( z \) as function of \( \zeta \) for a given frequency. Thus the luminance \( L_\nu^0 \) can be regarded as a function of \( \zeta \), and the kinetic equation (66) for \( \Lambda_\nu(\zeta, \theta) \) reads

\[
\frac{d\Lambda_\nu}{d\zeta} = \frac{\Lambda_\nu - L_\nu^0}{\cos \theta}. \tag{68}
\]

Equation (68) should be supplemented with boundary conditions expressing for \( \cos \theta > 0 \), the boundedness of \( \Lambda_\nu \) deep within the photosphere (\( \zeta \to \infty \)), and for \( \cos \theta < 0 \), the absence of incoming light outside the star (\( \Lambda_\nu = 0 \) for \( \zeta = 0 \)). The solution is

\[
\Lambda_\nu(\zeta, \theta) = \int_{0}^{\infty} \frac{d\zeta'}{\cos \theta} e^{-\zeta'/|\cos \theta|} L_\nu^0(\zeta + \zeta'), \quad (0 < \theta < \pi/2), \tag{69}
\]

and

\[
\Lambda_\nu(\zeta, \theta) = \int_{0}^{\zeta} \frac{d\zeta'}{|\cos \theta|} e^{-\zeta'/|\cos \theta|} L_\nu^0(\zeta - \zeta'), \quad (\pi/2 < \theta < \pi). \tag{70}
\]

Using Eq. (68) outside the star where \( \zeta = 0 \) and where \( \Lambda_\nu = L_\nu \), we find the luminance of the star

\[
L_\nu(\theta) = \int_{0}^{\infty} \frac{d\zeta}{\cos \theta} e^{-\zeta/|\cos \theta|} L_\nu^0 \left[T(z)\right], \tag{71}
\]
where \( L_\nu^0 \) should be evaluated from Eq. (58) at an altitude \( z \) which is related to the optical depth \( \zeta \) and to the frequency by Eqs. (53) and (67).

**C. Departures from black-body radiation**

The combined effect of emission and absorption has resulted in the expression (71) for the luminance of the star, which for fixed \( \theta \) and \( \nu \) appears as a weighted superposition of black-body radiation associated with successive layers. Contributions from deeper and deeper layers involve larger and larger values of \( L_\nu^0 \), because the temperature increases with the depth, but they are weighted by \( e^{-\zeta/\cos \theta} \), a decreasing function of the optical depth. The region which contributes to the radiation is therefore the one in which \( \zeta \) rises from 0 to some number of order one.

The photosphere thus defined is thin. For the Sun, the optical depth \( \zeta \) rises from 0 to 1 in a region of only 20 km thickness. It reaches 10 when the altitude decreases by 500 km, small in comparison to \( R_\odot = 700,000 \) km. Because the photons that we receive all come from nearly the same altitude, the edge of the Sun appears optically very sharp, although the density remains significant in the chromosphere, a shell with thickness 1500 km which lies above the photosphere. Moreover, because the photosphere is thin, its temperature is nearly uniform, which explains why the black-body approximation of Sec. [3] is fairly good, both for the total luminosity and for the spectrum of the emitted light. Indeed the temperature decreases in the photosphere of the Sun by 7% for an increase by 50 km of the altitude, and the parameter \( T_\nu \), which in Eqs. (10) and (13) is fitted to the observed radiation, corresponds to an average temperature in the photosphere.

A better approximation consists of linearizing \( T(z) \) and \( L_\nu^0(T) \) in the photosphere. Moreover, we neglect the variation in the photosphere of \( k_\nu(z) \) defined by Eq. (53). Equation (71) then provides, using \( \zeta \approx -k_\nu z \),

\[
L_\nu(\theta) \approx \int_0^\infty \frac{d\zeta}{\cos \theta} e^{-\zeta/\cos \theta} \left[ L_\nu^0[T(0)] + z \frac{dL_\nu^0[T(0)]}{dz} \right] 
\approx L_\nu^0[T(0)] - \frac{\cos \theta}{k_\nu} \frac{dL_\nu^0[T(0)]}{dz} \approx L_\nu^0\left[T(-\frac{\cos \theta}{k_\nu})\right]. \tag{72}
\]

For given \( \nu \) and \( \theta \), the luminance \( L_\nu(\theta) \) of a star is therefore the same as that of a black body at the temperature which prevails at the altitude \( z = -\cos \theta/k_\nu \) such that the optical depth \( \zeta \) equals \( \cos \theta \).

The dependence of \( L_\nu(\theta) \) on the frequency arises both directly in \( L_\nu^0 \) and through the temperature \( T(z = -\cos \theta/k_\nu) \). The contribution to \( k_\nu \) of inelastic processes is fairly constant. However, we see from Eq. (53) that atomic transitions correspond to sharp resonance peaks in the absorption cross-sections and hence in \( k_\nu \), with widths inversely proportional to the lifetime of the excited level. Around such a peak, the depth \( |z| = \cos \theta/k_\nu \) has a narrow minimum, and \( L_\nu^0(T) \) should therefore be evaluated at an altitude where the temperature is significantly lower than outside the resonance. Thus, for each atomic resonance, the luminance has a sharp minimum as function of the frequency, and the spectrum displays dark lines associated with an increased absorption. The intensity of light at the resonance frequency is determined by the temperature of the layer involved.

A converse effect also exists. In the chromosphere and in the corona which extends beyond the altitude of 2000 km, the temperature strongly rises reaching 1000 K. The density is very low, which strongly reduces \( k_\nu \). However at a resonance, \( k_\nu \) has a significant value and the optical depth is no longer negligible. The very high temperatures involved in \( L_\nu^0(T) \) thus produces bright lines associated with an increased emission, compared to the average Planck spectrum associated with the temperature of 5800 K.

The dependence of \( L_\nu(\theta) \) on the angle \( \theta \) arises from the factor \( \cos \theta \) entering the altitude \( z = -\cos \theta/k_\nu \) in Eq. (72). The light that we receive from the center of the solar disk corresponds to \( L_\nu^0[T(z = -1/k_\nu)] \), whereas the edges send us light associated with \( L_\nu^0[T(0)] \). Thus the periphery of the solar disk appears both less bright and more red than its center, because the effective surface temperature which characterizes the radiation is lower there. Photographs show these features. If the Sun behaved as a black body with uniform temperature, Lambert’s law would be satisfied, and the Sun would appear to us as a uniformly bright, seemingly flat disk.

When the radiation of the photosphere is occulted by a total eclipse, the corona becomes visible. Its luminance is weak in spite of its high temperature, contrary to what happens for a black body. This can be understood as a consequence of Kirchhoff’s law, which expresses the luminance of a body as the product of the luminance \( L_\nu^0(T) \) of a black body at the same temperature by the absorption coefficient. This coefficient is here very small because the low density of the corona makes it transparent. A quantitative approach relies on the kinetic equations (61). Disregarding the scattering of photons but now taking into account the spherical geometry of the Sun, we can find as an exercise the full stationary solution of Eq. (64). Hence we derive the luminance \( L_\nu(x) \) which can be observed along a line passing at the distance \( x \) from the centre of the Sun. For \( x < R_\odot \) we recover the result of Eqs. (7) or (22) with \( x = R_\odot \sin \theta \). For the outer atmosphere \( x > R_\odot \) which is transparent, we have \( k_\nu R_\odot \ll 1 \). In this limit the luminance of the corona is found to be

\[
L_\nu(x) = 2 \int_x^\infty \frac{rdr}{\sqrt{r^2 - x^2}} k_\nu(r) L_\nu^0[T(r)], \tag{73}
\]
a quantity much smaller than the black-body radiation at the temperature of the corona, but still following Planck’s spectral
distribution at this temperature.

Quantitative studies of $L_\nu(\theta)$ as function of $\nu$ and $\theta$ provide extensive information on the elements constituting
the external shells and on the variations of the temperature and the density with the altitude.

VI. RADIATIVE TRANSFER WITHIN STARS: DIFFUSION AND BROWNIAN MOTION

The light emitted by a star originates from the heat transported from the center outward. This heat is produced
in the core either by gravitational contraction (protostars), or by thermonuclear reactions (stars in the main sequence,
see Sec. VII), or it is stored there due to earlier evolution (white dwarfs). The heat flow is mainly induced by thermal
conduction in white dwarfs and by convection in red giants. Convection is also important in the upper shell of the
Sun (1/4 of its radius). However, the dominant process of energy transport within protostars and stars of the main
sequence is radiative transfer by photons which are emitted, absorbed, and scattered by matter.

A. Transport equations

At the microscopic level, the dynamics of photons inside a star is governed by the kinetic equation (59) or (61)
as in the photosphere. However, not only the matter but also the photon gas are now in local equilibrium due to
the effects of emission, absorption and inelastic scattering processes. We therefore can use the macroscopic equations
of non-equilibrium thermodynamics. The distribution $\Lambda_\nu(r, u)$ of photons defined by Eq. (60) is close to the black-
body distribution $L_\nu^0[T(r)]$ of Eq. (58), but the current density of energy $J(r)$ for photons, which is the dominant
contribution to the heat flux in the considered regime, originates from the deviation $\Lambda_\nu - L_\nu^0$ from equilibrium. It is
given at each time by

$$J(r) = \int d^3p \, c u f(r, p) \, cp = \int d^2u \, d\nu \, u \Lambda_\nu(r, u).$$

(74)

Conservation of energy is expressed by the balance equation

$$\frac{\partial u}{\partial t} + \nabla \cdot J = q,$$

(75)

where $u$ is the total energy density, mainly carried by matter and evaluated in Sec. IIIB. The source term $q(r)$ on the
right-hand side describes the heat production by thermonuclear reactions.

Thermal conduction is characterized by the heat conductivity, the ratio between the heat current $J$ and $-\nabla T$. For photons, it is customary to replace $\nabla T$ by the gradient of the energy density of photons,

$$u_\gamma(T) = \int d^3p \, f(r, p)cp \simeq \frac{1}{c} \int d^2u \, d\nu \, L_\nu^0(r) = \frac{4}{c} \sigma T^4.$$  

(76)

This replacement defines a transport coefficient, the opacity $\kappa$, through

$$J = -\frac{c}{3 \rho \kappa} \nabla u_\gamma.$$  

(77)

The mass density $\rho$ of the medium has been introduced in Eq. (77), because for a given constitution of matter, the
flux of photons generated by a fixed thermal gradient is inversely proportional to the density of particles on which
they scatter. The opacity is measured in m$^2$ kg$^{-1}$, and the quantity

$$\ell = \frac{1}{\rho \kappa}$$

(78)

can be interpreted as the mean free path of photons between two successive collisions.

Equations (75), (76), and (77) are sufficient to determine the transport of energy in terms of the opacity $\kappa$. The value of $\kappa$ can be evaluated by solving the Boltzmann-Lorentz equation (51). Within a star, when ionization
is complete, the main radiative process is the elastic Thomson scattering of photons by electrons, for which the
cross-section,

$$\frac{d\sigma_{\text{Th}}}{d\omega} = \frac{\sigma_{\text{Th}}}{4\pi} = \frac{2}{3} \left( \frac{e^2}{4\pi\epsilon_0 m_e c^2} \right)^2 = 5.3 \times 10^{-30} \text{ m}^2.$$  

(79)

is isotropic. From the definition of a cross-section, the elastic collision term in the kinetic equation (51) is obtained as
\[ I_c = c n e \int d^2 \mathbf{u}' \frac{d \sigma_{\text{Th}}}{d \omega} [\Lambda_{\nu}(\mathbf{r}, \mathbf{u}') - \Lambda_{\nu}(\mathbf{r}, \mathbf{u})]. \] (80)

This collision term dominates the right-hand side of Eq. (61). However, the remaining absorption and emission terms given by Eq. (64) are not irrelevant: they ensure thermalization, and allow us to replace \( \Lambda_{\nu} \) by \( L_0 \) in the left hand side of Eq. (61). Multiplication by \( \mathbf{u} \) and integration over the direction of \( \mathbf{u} \) of Eq. (61) with the right hand side (80) then yields, in a stationary regime,

\[ \int d^2 \mathbf{u} \mathbf{u} (\mathbf{u} \cdot \nabla L_0^\nu) = -n_e \sigma_{\text{Th}} \int d^2 \mathbf{u} \mathbf{u} \Lambda_{\nu}. \] (81)

Using Eqs. (74) and (76) and integrating Eq. (81) over \( \nu \), we find

\[ J = -\frac{c}{3n_e \sigma_{\text{Th}}} \nabla u_\gamma. \] (82)

which proves microscopically the empirical equation (77) and provides

\[ \kappa = \frac{\sigma_{\text{Th}}}{m_p} = 4 \times 10^{-2} \text{ m}^2 \text{ kg}^{-1}. \] (83)

B. The random walk of photons

Let us first disregard the variations of density and temperature in the medium. From Eqs. (75) and (77) we find for \( u_\gamma \), the density of energy of photons, the heat diffusion equation

\[ \frac{\partial u_\gamma}{\partial t} - \frac{c}{3\rho\kappa} \nabla^2 u_\gamma = q. \] (84)

According to Eq. (84), a unit pulse of energy which is given to the photons at the point \( r = 0 \) at time \( t = 0 \) diffuses for \( t > 0 \) as

\[ u_\gamma(r, t) = \left( \frac{3\rho\kappa}{4\pi ct} \right)^{3/2} \exp \left[ -\frac{3\rho\kappa r^2}{4ct} \right]. \] (85)

The perturbation spreads as \( \langle r^2 \rangle = (2ct/\rho\kappa)t \).

Taking for \( \rho \) the average density of the Sun, \( 1.4 \times 10^3 \) kg m\(^{-3}\), we find that it takes 1500 years for \( \langle r^2 \rangle \) to reach \( R_\odot^2 \). If the matter were transparent, the photons created at the center would reach the surface after only 2 seconds. Their scattering by the electrons slows them down enormously. The opacity for the photons of stellar matter is therefore essential for preventing the energy confined in the core from escaping immediately. In contrast, matter is transparent for neutrinos, so that most of the available energy in supernovae is suddenly evacuated in the form of neutrinos.

An alternative approach is based on the fact that each collision of a photon changes its direction of propagation at random. Its motion is thus a random walk. The mean square length of each step is \( \ell \sqrt{2} \), where \( \ell \) is the mean free path (because the distance travelled between two successive collisions obeys Poisson statistics). The theory of Brownian motion implies that the mean square distance traveled after \( n \) steps is \( \langle r^2 \rangle = 2\ell^2 n = 2\ell ct \), in agreement with the solution of the diffusion equation. The mean free path given by Eq. (78) equals \( \ell = 1.8 \) cm, and the average number of collisions of a photon during its travel from the center to the surface is thus \( n = 8 \times 10^{20} \).

C. The temperature within the star

Consider a stationary regime. The power \( Q = \int d^3 \mathbf{r} q(r) \) produced in the active core of the star should exactly balance the luminosity \( L \). Equations (75)–(77) yield

\[ \frac{d}{dr} \left( \frac{\sigma v^2}{\rho \kappa} \frac{dT^4}{dr} \right) = -\frac{3v^2}{4}q. \] (86)

Taking into account the boundedness of \( T(0), \), Eq. (86) provides for a given \( \rho(r) \) and \( q(r) \) the temperature profile in terms of a single boundary condition, \( T(0) \) or \( T_s \).

In a model where \( \rho \) is regarded as a constant and where \( q(r) \) is also taken as uniform in the core with radius \( R_c \), the solution of Eq. (86) is
$$T^4(r) = T^4(0) - \frac{\rho \kappa q}{8 \sigma} r^2 \quad (r < R_c)$$ \hspace{1cm} (87)$$

$$T^4(r) = T^4(0) - \frac{\rho \kappa q}{8 \sigma} \left( 3 R_c^2 - 2 \frac{R_c^3}{r} \right) \quad (r > R_c)$$ \hspace{1cm} (88)$$

For $r = R$, Eq. (88) relates the surface temperature $T_s = T(R)$ and the central temperature $T(0)$.

For the Sun, if we take $R_c = \frac{1}{15} R_\odot$ and if we make use of $L_\odot = Q = \frac{4}{3} \pi R_\odot^3 q$, we find from Eq. (88) the central temperature

$$T(0) \approx \left[ \frac{9 \rho \kappa q}{32 \pi \sigma R_c} \right]^{1/4} \simeq 5 \times 10^6 \text{K}.$$ \hspace{1cm} (89)$$

This temperature is determined mainly by the heat production rate and by the properties of the core (opacity, density, and size). Its insensitivity to the outer regions is related to the efficiency of the photons confinement. The temperature of the core is nearly uniform as $T(R_c) = 0.9 T(0)$. The value of the temperature in Eq. (89) is not sufficient (by a factor of 2) to ensure the ignition of the thermonuclear reactions. Actually, besides their Thomson scattering off the electrons, photons can interact with partly ionized atoms such as Fe which, although scarce, have large absorption cross-sections. Moreover, the non-uniformity of the density enhances the confinement of photons near the center where $\rho(r)$ has a maximum, and hence raises the central temperature above the value of Eq. (89).

Outside the core and up to the surface, the temperature given by Eq. (88) decreases as

$$\frac{T^4(r) - T^4(0)}{T^4(0)} = \frac{2 R_c}{3 R - 2 R_c} \frac{R - r}{r}.$$ \hspace{1cm} (90)$$

D. Relation between mass and luminosity

The luminosity $L$ of a star is related to its surface temperature through the Stefan-Boltzmann law, Eq. (1), and to the internal temperature gradient, through the diffusion equation for photons, Eqs. (76) and (77). The temperature $T$ outside the core thus satisfies

$$L = -\frac{16 \pi \sigma r^2}{3 \rho \kappa} \frac{dT^4}{dr}.$$ \hspace{1cm} (91)$$

Equation (91) is consistent with Eq. (88) and $L = Q$.

We have found in Sec. IV the relation (38) between the mass, the radius, and the internal temperature of a star. By eliminating the temperature, we can therefore express the luminosity in terms of the mass and the radius. From Eq. (90) we find for $-r^2 dT^4/dr$ the estimate $\frac{2}{3} R_c T^4(0)$. The average temperature entering Eq. (88) is found from Eqs. (87) and (88) to be $(R_c/R)^{4/3} T(0)$. We thus obtain in SI units, using Eq. (88) and the expression for $\sigma$,

$$L = \frac{32 \pi \sigma R}{9 \rho \kappa} \left( \frac{G m_p M}{10 k R} \right)^4 \simeq \frac{7.7 \times 10^{-4} G^4 m_p^4}{h^3 c^2 \kappa} M^3 \approx \frac{10^{-66}}{\kappa} M^3.$$ \hspace{1cm} (92)$$

Letting $M = M_\odot$ and taking for $\kappa$ the value in Eq. (83), we find $L = 2.2 \times 10^{26}$ W, in reasonable agreement with the actual luminosity $L_\odot = 3.8 \times 10^{26}$ W of the Sun.

Remarkably, the luminosity is found to depend only on the mass of the star, with a power law $L \propto M^3$ which is consistent with the observational data for $M_\odot < M < 100 M_\odot$. The fact that the radius (or the temperature) does not enter Eq. (12) implies that the luminosity $L$ is independent of the power production $Q$. In the stationary regime $L$ and $Q$ are equal, but perturbations around this regime do not modify the luminosity although $Q$ is changed. We can recover the result $L \propto M^3$ by means of a qualitative argument, using $L \propto R T^4/\rho \kappa \propto (R T)^4 / M k$, and, from the virial equation (38), $RT \propto GM$. The coefficient is found within a numerical factor by dimensional analysis.

For smaller stars where microscopic radiative processes other than Thomson scattering occur, the opacity $\kappa$ is not a constant but varies as $\rho T^{-7/2}$. It will be left as a problem to show that $L \propto M^5 T^{1/2}$ for such stars.

A. Nuclear reactions VII. THE NUCLEAR FURNACE: BALANCE EQUATIONS
The source of heating in the core of the stars of the main sequence is thermonuclear fusion. For \( M < 0.5 \, M_\odot \), the dominant process is the sequence of reactions

\[
\begin{align*}
p + p & \rightarrow d + e^+ + \nu + 0.4 \text{ MeV}, \\
e^+ + e^- & \rightarrow \gamma + 1.0 \text{ MeV}, \\
d + p & \rightarrow ^3\text{He} + \gamma + 5.5 \text{ MeV}, \\
^3\text{He} + ^3\text{He} & \rightarrow ^4\text{He} + 2p + 12.9 \text{ MeV},
\end{align*}
\]

which altogether amount to

\[
4p + 2e^- \rightarrow ^4\text{He} + 2\nu + 26.7 \text{ MeV}.
\]

The same reactions also take place in the first stage of the life of heavier stars, with \( 0.5 \, M_\odot < M < 100 \, M_\odot \). However, in such stars after these reactions have taken place in the core and after the temperature has sufficiently risen, new fusion reactions involving He can occur. Strong interactions are involved in the reactions (93) and (95), and electromagnetic interactions in the reactions (94) and (96). These three reactions have large cross-sections compared to (93), which involves weak interactions, and which is a prerequisite for the other ones to take place. The reactions (93), (95), and (96) are moreover hindered by the Coulomb repulsion between the two nuclei, which must be overcome before they can fuse.

The reactions (93)–(96) are all exothermic, and energy is released in the form of kinetic energy of the particles produced. Collisions distribute this energy to other particles, and thermalization takes place, so that the effect is heat production. However, the neutrinos created by the reaction (93) escape the star without having interacted with matter, so that their energy, on average of 0.3 MeV per neutrino, is lost. The final balance is thus an average production of heat of \( \eta = 13 \text{ MeV} \) per fusion of a pair of protons.

The cross-section for the reaction (93) is especially small because this reaction combines weak interactions with the need for the two protons to approach each other at a distance comparable to the range \( r_0 \) of nuclear interactions, of the order of 1 fm. For this reason, the reaction (93) cannot be produced in a laboratory, and its cross-section needs to be evaluated theoretically. The value of the Coulomb repulsion at the separation \( r_0 \) has a geometrical origin, and the remaining coefficient \( S(\varepsilon) \) which accounts for the short range nuclear interactions varies slowly with energy (except near resonances). The integral runs from the range \( r_0 \) of the nuclear forces to the turning point, but the lower bound can be replaced by 0, which yields

\[
\sigma = \frac{S(\varepsilon)}{\varepsilon} \exp \left[ -\frac{2\sqrt{\pi}}{\hbar} \left( \frac{Z_1 Z_2 e^2}{4 \pi \epsilon_0 r} - \varepsilon \right)^{1/2} dr \right].
\]

The relative mass is denoted by \( \mu = m_1 m_2 / (m_1 + m_2) \), and the relative energy by \( \varepsilon = \frac{1}{2} \mu (v_1 - v_2)^2 \) in terms of the velocities of the colliding nuclei. The factor \( 1/\varepsilon \) has a geometrical origin, and the remaining coefficient \( S(\varepsilon) \) which accounts for the short range nuclear interactions varies slowly with energy (except near resonances). The integral runs from the range \( r_0 \) of the nuclear forces to the turning point, but the lower bound can be replaced by 0, which yields

\[
\sigma = \frac{S(\varepsilon)}{\varepsilon} \exp(-\sqrt{\varepsilon_B/\varepsilon}), \quad \varepsilon_B = \frac{\mu Z_1^2 Z_2^2 e^4}{8 \hbar^2 c_0^2}.
\]

The energy \( \varepsilon_B \), which characterizes the strength of the potential barrier, is large (500 keV for the p-p reaction) compared to the thermal energy \( (1 \text{ keV} \text{ at } 10^7 \text{ K}) \). The exponent in Eq. (99) is thus small, and varies rapidly with \( \varepsilon \). For the weak interaction (93), the coefficient \( S(\varepsilon) \) may be estimated by using the Fermi golden rule, the coupling constant being inferred from the \( \beta \)-decay half life of the neutron. Its value, \( S \approx 3.8 \times 10^{-20} \text{ m}^2 \text{ keV} \), is smaller than for strong interactions by more than 20 orders of magnitude: for the reaction (95), an extrapolation of data obtained by ion collision experiments at higher than thermal energies yields \( S \approx 5 \times 10^{-25} \text{ m}^2 \text{ keV} \).

**B. The Gamow peak**

The probability of a single reaction induced by the collision of two positively charged particles is characterized by its cross-section which is of the form (99). We wish to evaluate from it the average number \( w \) of such reactions which take place per unit time in a unit volume where the particles constitute a gas in thermal equilibrium. If particle 2 with mass \( m_2 \) and velocity \( v_2 \) is regarded as a target, the relative flux of particles 1 having the density \( n_1 \) and a velocity within the volume \( d^3v_1 \) around \( v_1 \) is, according to Maxwell’s distribution,
\[ |v_1 - v_2| n_1 \left( \frac{m_1}{2\pi kT} \right)^{3/2} e^{-m_1 v_1^2/2kT} d^3v_1. \]  

(100)

The corresponding average number of reactions per unit time of the particle 2 is the product of Eqs. (99) and (100). The average number \( w \) is then obtained by summing over the particles 2 as

\[
w = n_1n_2 \left( \frac{m_1}{2\pi kT} \right)^{3/2} \left( \frac{m_2}{2\pi kT} \right)^{3/2} \int d^3v_1 \int d^3v_2 |v_1 - v_2| S(\varepsilon) \exp \left( -\sqrt{\varepsilon_B/\varepsilon} - (m_1 v_1^2 + m_2 v_2^2)/2kT \right). \]

(101)

Integration over all variables except \( \varepsilon = \frac{1}{2} \mu (v_1 - v_2)^2 \) yields

\[
w = \frac{4n_1n_2}{(2\pi \mu)^{3/2}(kT)^{3/2}} \int d\varepsilon S(\varepsilon) \exp \left( -\sqrt{\varepsilon_B/\varepsilon} - \varepsilon/kT \right).
\]

(102)

Due to the rapid increase with \( \varepsilon \) of the tunneling factor in the cross-section, and to the exponential decrease of the Boltzmann factor, the integrand in Eq. (102) has a sharp peak, called the Gamow peak. The position \( \varepsilon_G \) of this peak, referred to as the Gamow energy, is the value of \( \varepsilon \) for which the exponent is largest, namely

\[
\varepsilon_G = \left( \frac{kT}{2} \right)^{2/3} \varepsilon_B^{1/3}.
\]

(103)

The Gamow energy is much larger than the thermal energy, and much smaller than the barrier energy \( \varepsilon_B \). The width of the Gamow peak is of order \( (kT\varepsilon_G)^{1/2} \). The particular shape of the integrand in Eq. (102) implies that most reactions take place at a relative energy in the range \( \varepsilon_G \pm (kT\varepsilon_G)^{1/2} \). Slower particles are not sufficiently energetic to pass through the barrier, while the Maxwell distribution cuts off the number of faster particles that would collide more efficiently. A compromise is thus needed, which results in the Gamow peak.

Neglecting the variation of \( S(\varepsilon) \) within the Gamow peak and replacing the exponential in Eq. (102) by a gaussian, we obtain finally

\[
w = \frac{4(2\varepsilon_B)^{1/6}}{(3\pi)^{1/2}(kT)^{3/2}} n_1n_2 S(\varepsilon_G) \exp \left[ -3\left( \frac{\varepsilon_B}{4kT} \right)^{1/3} \right].
\]

(104)

The exponential factor, with \( \varepsilon_B \) defined in Eq. (104), comes from both the Coulomb barrier and the Maxwell factor, because \( \varepsilon_B/(4kT)^{1/3} = (\varepsilon_B/4\varepsilon_G)^{1/2} = \varepsilon_G/kT \). It dominates \( w \), increasing rapidly with the temperature.

For the first reaction \( (13) \), we have \( Z_1 = Z_2 = 1, \mu = \frac{1}{2} m_p, \varepsilon_B = 490 \text{ keV} \), and a factor \( \frac{1}{2} \) should be included in \( (104) \) due to the indistinguishability of the two colliding protons. For the Sun, a numerical estimate of \( w \) can be obtained by assuming that the density of protons in the core is 100 times larger than the average density (see Fig. 1), that is, \( n_1 = n_2 = 10^{32} \text{ m}^{-3} \), and that the temperature is \( kT = 1 \text{ keV} \); by using \( S = 3.8 \times 10^{-30} \text{ m}^2 \text{ keV} \) we find a rate \( w = 2 \times 10^{14} \text{ m}^{-3} \text{ s}^{-1} \).

C. Power production

Whereas the major part of the energy is generated by the strong interactions \( (13) \) and \( (10) \), the time-rate of the entire process is controlled by the interaction \( (13) \) which is both weak and hindered by the Coulomb repulsion. As soon as a positron and a deuteron are produced by the reaction \( (10) \), they readily find in their neighborhood an electron and a proton to react according to \( (14) \) and \( (15) \), which have much larger cross-sections. The final reaction \( (16) \) also has a large cross-section, although the probability of tunneling is strongly reduced by the fact that \( \varepsilon_B \) in Eq. (10) is 48 times larger for this reaction than for the p-p reaction. However, due to the factor \( \left[ n(3\text{He}) \right]^2 \) which enters Eq. (10), the reaction \( (16) \) occurs less often than \( (13) \) as long as a sufficient amount of \( 3\text{He} \) has not yet been accumulated (which typically takes 10^6 years). The stationary regime is attained when \( n(3\text{He}) \) has reached a value such that the rate at which \( 3\text{He} \) is produced by the reactions \( (13) \) and \( (16) \) equals the rate at which it disappears through the reaction \( (13) \).

This stationary regime is therefore governed by the rate \( w \) evaluated above for the p-p reactions \( (13) \). Because each fusion of a pair of protons releases altogether an energy \( \eta = 13 \text{ MeV} \), the production \( q(r) \) of heat per unit time and unit volume (or unit mass) is

\[
q = \eta w \approx 500 \text{ W m}^{-3} \approx 4 \times 10^{-3} \text{ W kg}^{-1}.
\]

(105)
This value of $q$ is small, smaller for example than the heat radiated by a human body, which is of the order of 1.5 W kg$^{-1}$. This small value implies that the nuclear reactions do not significantly perturb the state of the Sun. The total heat power $Q$ is large due to the huge size of the Sun. By assuming that the active core is a sphere of radius $R_c = \frac{1}{15} R_{\odot}$, we find

$$Q = \frac{4}{3} \pi R_c^3 q \simeq 2 \times 10^{26} \text{W},$$

(106)

in reasonable agreement with the actual luminosity of the Sun, $L_{\odot} = 3.8 \times 10^{26} \text{W}$.

The small efficiency of nuclear fusion in stars is illustrated by the average time it takes for a particular proton in the core to react, $\frac{1}{2} m_p n_w^{-1} \approx 8 \times 10^{9} \text{years}$. Accordingly, the total time during which the Sun can radiate as it does presently is large; assuming that 10% of its mass can be burnt, we find

$$\frac{\eta}{2} \frac{M_{\odot}}{10 m_p} \frac{1}{L_{\odot}} = 10^{10} \text{years}.$$  

(107)

The flux of energy carried by photons is accompanied by a flux of neutrinos, produced by the reaction (93). According to Eq. (97), one neutrino should be emitted for each 13 MeV released. Thus the flux of neutrinos emitted by the Sun can be explained.

More precise calculations should account for the non-uniformity of the density and the temperature, and for the presence of elements other than H. The latter effect is especially important for the heavier stars and for the later stages of evolution, when the central part of a star becomes much hotter and much more dense than the outer shells and when elements other than H and He have already been synthesized. Indeed, while the onset of the primary cycle (93)–(96) of reactions takes place at temperatures of the order of $10^7$ K, the onset of other reactions requires higher temperatures because the Coulomb barrier in Eq. (99) increases with the charge and the mass of the colliding nuclei. In red giants with masses $M_{\odot} < M < 10 M_{\odot}$, H can thus fuse into He outside the inert helium core, but through reactions other than the cycle (93)–(96) which are catalyzed by the presence of C. Helium can in turn fuse into C provided the temperature reaches $10^8$ K (instead of $10^7$ K for H). In this case, an additional effect requires a high temperature: whereas the reactions in (93)–(96) are exothermic, the successive reactions $^4\text{He} + ^4\text{He} + 92 \text{keV} \rightarrow ^8\text{Be}^*$ and $^4\text{He} + ^8\text{Be}^* + 67 \text{keV} \rightarrow ^{12}\text{C}^*$ are endothermic, and the thermal energy of $^4\text{He}$ should overcome this threshold. Heat is thereafter retrieved through the electromagnetic deexcitation $^{12}\text{C}^* \rightarrow ^{12}\text{C} + \gamma + 7.4 \text{MeV}$.

**D. The stability of stellar equilibrium**

The stars in the main sequence reach a stationary regime in which their luminosity $L$ equals the heat production rate $Q$: the radiated energy is then exactly compensated for by the thermonuclear power. This stationary state, referred to by astrophysicists as “equilibrium,” lasts for a long time in the life of the star until the nuclear fuel is exhausted.

The stability of the stationary regime is paradoxical because the production $Q$ of heat is a rapidly increasing function of the core temperature, as shown by Eqs. (104) and (106), whereas the luminosity $L$ depends practically only on the mass, as shown in Sec. V D. To study perturbations around the stationary state, we use the dynamical balance equation:

$$\frac{dE}{dt} = Q - L,$$

(108)

which expresses energy conservation. If a small perturbation slightly raises the core temperature above its value for which $Q = L$, the right-hand side of Eq. (108) becomes positive, and the total energy $E$ increases. The very slow progression of heat from the core to the surface (typically $10^4$ years) prevents the extra heat that is produced from being radiated. If we naively accept the idea that the total energy $E$ varies in the same direction as the internal energy $U$, the initial perturbation would result in an increase of $U$, and hence in an extra increase of the temperature. The heat production $Q$ would rise again, and so on, leading eventually to a heating instability and to an explosion. If conversely the core of the star happens to cool down slightly below the temperature of the stationary regime, the thermonuclear reactions would by the same hypothesis stop progressively.

Fortunately, the total energy $E = E_G + U$ is dominated not by $U$, but by the gravitational contribution $E_G$, equal to $-2U$ when the star is in gravitational equilibrium. We noted in Sec. V C that the time-scale for the establishment of gravitational equilibrium is short, of the order of a few hours. Hence, an accidental increase of the core temperature $T_c$ which leads to an increases of $Q$ and thus of $E$, produces an immediate mechanical reaction: the star slightly expands according to Eq. (30), cooling the core of the star back to its original temperature. Conversely, an accidental
lowering of the thermonuclear activity produces a decrease of $E$, and restoration of gravitational equilibrium results in a contraction and a heating so that the rate $Q$ of heat production returns to its stationary value $L$. The stability of the stationary state of a star is therefore due to the negative sign of $dE/dT$, an unusual feature that we noted in Sec. IVB.

Thus, the regulation of the thermonuclear power plant constituted by the core of a star is ensured by the gravitational forces, which prevent $Q$ from shifting away from $L$.

VIII. SYNTHESIS, PROPERTIES AND EVOLUTION OF STARS

The state of a star is characterized by the nature and proportion of the constituent particles; the local mass density $\rho(r)$ and the radius $R$; the pressure $P(r)$ and the local energy density $u(r)$; the temperature $T(r)$ and its extreme values at the surface $T_s$ and at the center $T_c$; the total energy $E$, including the gravitation energy $E_G$ and the internal energy $U$; the local rate $q(r)$ and total rate $Q$ of heat production in the core for the active stars of the main sequence; the luminosity $L$ and the spectral constitution of the emitted light. The last quantity is the only one that we can observe by photometry of the radiation that we receive.

In Secs. III to VII we have used statistical mechanics to establish relations among all the quantities listed above. We will now summarize the foregoing results to show that a single quantity, the mass $M$ of the star, essentially determines all of the properties of the object as a function of time.

Actually, a star always begins as a cloud of hydrogen, with about 25% in mass of helium and with small quantities of heavier elements. The presence of these elements may influence some properties, in particular through their interactions with photons (Secs. III and IV). They may thus play an important role at some stages of the evolution of stars, but they do not govern the main common features of stars. On the other hand, we have considered only isotropic objects; some peculiar features can emerge for other geometries. Binary stars rotating close to each other can in particular produce accretion disks or explosive events (novae).

For a given constitution of matter, equilibrium statistical mechanics provides the equations of state which relate $P(r)$ and $u(r)$ to $\rho(r)$ and $T(r)$. The ionization ratio and the criterion for using quantum mechanics or relativity depend only on $\rho(r)$ and $T(r)$.

The constitution of the matter of stars is determined mainly by their age and by their mass which governs their evolution. Indeed, like protostars, stars of the main sequence are mainly made of hydrogen plus an increasing amount of helium and of other elements produced by nuclear fusion. White dwarfs are made of the latter elements, in proportions depending on the degree to which fusion has been effective, itself determined by the temperature of the core which is hotter for heavier stars. For neutron stars produced in the supernovae explosions, nuclear fusion is complete.

The gravitational energy $E_G$ results from the density through Eq. (14) and is related to the pressure through the virial equation (19). The hydrostatic equilibrium equation (19) provides the function $\rho(r)$, and hence the radius $R$ and the total energy $E$, in terms of the function $T(r)$ and the mass $M$.

The temperature $T(r)$ is in turn a solution of the heat transport equation (16), which relates the surface temperature $T_s$ to that of the center $T_c$ and to the rate $q(r)$ of heat production. This solution depends on a single parameter, say $T_c$.

The luminosity $L$ of the star then follows from the surface temperature. The more detailed characteristics of its radiation can be obtained by solving the Boltzmann-Lorentz equation (14) in the outer shells.

The theory of thermonuclear reactions expresses the heat production $Q$ as a function of the temperature and the density in the active core through Eq. (108).

All the above quantities are thus related by means of a set of self-consistent equations to the mass $M$ and to the only remaining parameter $T_c$. Finally, conservation of energy yields the differential equation (108), which determines the time-dependence of the parameter $T_c$ and hence of all quantities. Altogether, apart from minor differences, all stars having the same mass have the same history.

The classification and evolution of stars according to their mass can be visualized by the diagrams of Figs. 1 and 2 or the schematic diagram of Fig. 3. As an exercise, the various features of Fig. 3 can be derived from the results of Secs. III and IV. For protostars, white dwarfs and neutron stars, we have $Q = 0$ in Eq. (108) and energy is gradually lost. The evolution is represented by constant mass lines in the various figures, except for supernovae. The time-dependence is not uniform. In particular the evolution stops for a long time for the stars of the main sequence. As a general rule, more massive stars evolve faster. This behavior can be illustrated by the duration of the stationary regime of a star in the main sequence, the luminosity of which is proportional to $M^3$. Because the quantity of nuclear fuel which supplies this radiation is proportional to $M$, the lifetime of the star is proportional to $1/M^2$. Massive stars thus waste their nuclear stocks.

The simple model that we often used, neglecting the variations with $r$ of most quantities, is adequate in many situations, but some stars need to be analyzed in shells having different qualitative properties. For instance, the study of heat transfers in
white dwarfs (Ref. 2, Exer. 15f) requires that we distinguish the interior, which is a completely ionized gas of elements such as C or O, from the envelope, which is a colder gas of non-ionized, lighter atoms. Heat conduction takes place in the interior through transport and collisions of electrons on the nuclei like in a plasma or a metal, and it is so efficient that the interior temperature is uniform. In the envelope the transport takes place through radiative processes as in Sec. [V], which shows that the surface temperature is approximately proportional to the internal temperature, $T_s \approx T/2000$. Noting that the specific heat includes the contribution $\frac{1}{3}$ of the electrons and the classical contribution of the nuclei, we can then study the duration of the glare of white dwarfs. Starting from an initial temperature $T$ of $2 \times 10^5$ K, it is found that $T$ decreases very slowly, by 10% in $2 \times 10^4$ years. The statistics of the observed luminosities of such stars thus reflects the statistics of their ages. It is consistent with a constant rate of creation of white dwarfs in our galaxy — except for the occurrence of a clear cut-off toward the faintest objects: no white dwarfs are seen with luminosities below $2.5 \times 10^{-3} L_\odot$. This anomalous absence of very old, cool white dwarfs can be used to estimate the age of our galaxy, which is thus found to be of the order of $10^{10}$ years.

Another example of heterogeneous stars are the red giants, a stage reached by the stars of the main sequence with masses $M_\odot < M < 10 M_\odot$ after hydrogen in the central part has been converted into helium. These stars have a large radius, typically $100 R_\odot$, and hence a large luminosity although their surface temperature is lower than that of the Sun. These properties can be understood by the use of a two-zone model, including a very dense and hot inert core of helium, and a huge hydrogen envelope, the deepest layer of which is heated by the gravitational contraction of the core and undergoes thermonuclear reactions.

For very bright stars, the contribution of photons to the energy and pressure becomes important. The stability of the dilute envelope of such a star can be studied as a problem. According to Eqs. (12) and (77), the radiation pressure $P_\gamma$ satisfies

$$\frac{dP_\gamma}{dr} = -\frac{\rho_{\text{env}}}{\pi c r^2} L,$$

where $\rho_{\text{env}}$ is the opacity of the envelope. Typically $\rho_{\text{env}} = 5\kappa$, where $\kappa$ is the opacity in the bulk. From Eq. (118), we infer the existence of a critical luminosity $4\pi c GM/\kappa_{\text{env}}$ above which the envelope is expelled by radiation. An upper bound for the mass of stars is thus obtained from the mass-luminosity relation (92).

$$M_{\text{max}} = \frac{130}{m_\gamma^2} \left( \frac{\hbar c}{\gamma} \right)^{3/2} \left( \frac{\kappa}{\kappa_{\text{env}}} \right)^{1/2},$$

which yields $M_{\text{max}} = 100 M_\odot$.

We have considered stationary or slow regimes only, in which the evolution is due to radiation and governed by Eq. (108). More rapid dynamical processes can suggest additional student problems. The above equations should then be supplemented with conservation equations, involving the time-derivatives of the energy density $u$ and the photon density for transport of heat or of radiative energy as in Sec. VI, and time-derivatives of the mass and momentum densities for transport of matter. These time-derivatives are related to the corresponding thermodynamic forces. When local equilibrium is not reached, one should instead resort to kinetic equations as in Sec. [V]. Various phenomena can be understood by solving the resulting partial differential equations. For instance, simple models explain the solar oscillations. Thermal convection, which also exists in the Sun, governs the transfers of matter and heat occurring in the envelope of red giants. Important quantities of matter can be expelled at some stages of evolution, in particular during the transformation of a red giant into a white dwarf, and especially in heavy stars, which for $M > 40 M_\odot$ can loose half of their mass due to radiation pressure. Dynamics is essential in all the transient regimes. For instance, when the ignition temperature of helium is reached in the core of a red giant, a flash occurs, which rapidly brings the star from one stationary regime to another. When nuclear fuel is exhausted, processes are also rapid because nothing opposes the gravitational contraction. Depending on the mass, the resulting instability may produce pulsations, implosions, or explosions. The dynamics and luminosity of accretion disks and the understanding of novae provide problems of thermodynamics and statistical mechanics accessible to students.

We have successfully proposed to students a series of exercises explaining the main features of supernovae and involving only the above ideas. We now sketch what can be learned from these exercises. In its preliminary stage, a supernova is a red supergiant with typical mass $10 M_\odot$, and radius $5 \times 10^9$ km, made of a compact core of $^{56}$Fe at a temperature of $5 \times 10^{10}$ K surrounded by successive shells made of lighter and lighter elements. The mass of the core grows slowly from $1 M_\odot$ by nuclear fusion. It is shown as in Secs. [XI] and [X] that the thermodynamics and the gravitational equilibrium of the core are dominated by the ultrarelativistic Fermi gas of electrons. Hence, beyond some mass analogous to the Chandrasekhar limit mass for white dwarfs, the core no longer tends to expand, being maintained by the pressure of the envelope, but tends to shrink. This gravitational instability produces a sudden collapse when the critical mass is reached. The duration of the collapse can be estimated as the time of free fall of a particle under the effect of gravity from a height equal to the initial radius of the core, say 2000 km, which yields a fraction of a second. The collapse produces heating and fusion of the Fe nuclei into a unique gigantic single nucleus, which is a neutron star. Stability is then recovered because the equation of state becomes that of the non-relativistic Fermi gas of neutrons. Most of the energy of this implosion is released through neutrinos formed in the nuclear reaction, which can escape freely. Their number can be estimated by assuming their average energy to be 10 MeV and by comparing the energies of the initial Fe core and of the resulting neutron star (a difference of $10^{50}$ J is found). One can also explain the number of neutrinos detected on Earth during the collapse of the supernova 1987A, which is $150 \times 10^6$ light-years from us. Part of the energy produces a shock wave in the envelope which explodes, in the same way as the shock on the ground of a pair of elastic balls falling together on top of each other expels the upper one to a high altitude (an experiment easily done in class). Assuming that, after the shock wave has passed across the envelope, the kinetic and thermal energies of matter are equal, one can estimate the speed of expansion and the temperature of the envelope. As a consequence, one can show that the energy is mainly carried by the photons in thermal equilibrium, and hence that the temperature varies as $1/T$ during the subsequent rapid expansion. The
luminosity after an initial intense flash can then be evaluated as in Sec. VI. It is found to remain nearly constant in time for a few months during which an isentropic expansion takes place. Its value, of order $10^9 L_\odot$, and the duration of the flare-up can be estimated by assuming that the shock-wave carries 1% of the total energy released, that is, $10^{44}$ J. Afterwards, when the total energy radiated becomes equal to the residual thermal energy, the expansion is no longer isentropic and the luminosity rapidly decreases while the cloud formed by the envelope goes on expanding. It is found that the total energy radiated by photons is 1% of the kinetic energy, and $10^{-4}$ of the energy is carried away by the neutrinos.

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Figure Captions

Figure 1. Protostars and stars in the density-temperature plane, where the coordinates are log $n_e$ and log $T$ in units of $m^{-3}$ and K, respectively. The curves describe the evolution for a fixed mass $M$ (in units of $M_\odot$), obtained from Eqs. (40) and (41) with $N_e = N_{\text{nuc}}$. While the density increases, the temperature rises, proportionally to $n^{1/3}$ in the classical regime, and then decreases. The maximum temperature (dashed line) is given by Eq. (44). The dotted line indicates the ionization boundary of Eq. (15) in the classical regime. When the mass is sufficient ($M > 0.1 M_\odot$), nuclear ignition takes place and the evolution of $n$ and $T$ stops. The star indicates the stationary state of the Sun. Note that its coordinates represent average density and temperature; the actual temperature of the core is higher ($10^7$ K). White dwarfs are obtained after exhaustion of their nuclear fuel and nearly complete contraction in the extreme quantum regime.

Figure 2. Protostars and stars in the density-energy plane, where the coordinates are log $n_e$ and log $|E|/N_e = \log \frac{P}{n_e}$ in units of $m^{-3}$ and eV, respectively. The same conventions as in Fig. 1 are used. The thin curves are the isotherms (labelled by log $T$ in K): in the classical regime they are constants, reflecting the equipartition theorem, and they bend up in the quantum regime. The entire evolution, contraction, and loss of energy of a star with mass $M$ (in units of $M_\odot$) is represented by a straight line with slope 1/3 ending up on the dashed-dotted line which marks the end of the evolution (Jupiter-like objects and white dwarfs). The dashed line is the locus of the maximum temperatures reached during the evolution.

Figure 3. Evolution of the stars in the mass-density plane with coordinates log $M/M_\odot$ and log $\rho$ ($\rho$ in kg m$^{-3}$). The evolution takes place upward, with a loss of mass for massive stars when matter is expelled after exhaustion of hydrogen and overheating. The thin lines represent the isotherms of Eq. (38), for average temperatures corresponding to the ignition of H (core temperature $10^7$ K), He ($10^8$ K), C and O ($10^9$ K); we assume the core temperature to be 10 times the average temperature. The stars in the main sequence correspond to the isotherm $10^6$ K for $0.1 M_\odot < M < 100 M_\odot$. The isotherms bend up for large masses as an effect of the radiation pressure; for small masses they bend down (not drawn) toward the line of dwarfs due to the Fermi gas behavior of electrons. The dotted line is the ionization boundary of Eq. (15). The dashed-dotted lines mark the end of the evolution for aborted stars and white dwarfs (with $M$ less than the Chandrasekhar mass $1.4 M_\odot$), and for neutron stars (Eq. (23)). The maximum temperature (44) is attained along the dashed line. The thick dashed line is the black-hole boundary of Eq. (26).
