Stratified-algebraic vector bundles of small rank

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Abstract. We investigate vector bundles on real algebraic varieties. Our goal is to construct rank 2 real and complex stratified-algebraic vector bundles with prescribed Stiefel–Whitney and Chern classes, respectively. We obtain a partial solution of this problem and present two applications.

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1. Introduction. In the recent joint paper with Kurdyka [14], we introduced and investigated stratified-algebraic vector bundles on real algebraic varieties. They occupy an intermediate position between algebraic and topological vector bundles. A hard problem is to find a characterization of topological vector bundles admitting a stratified-algebraic structure, cf. [12,15]. In the present paper we study rank 2 real and complex stratified-algebraic vector bundles with prescribed Stiefel–Whitney and Chern classes, respectively. As an application, we obtain a criterion for a topological complex vector bundle of rank 2 and a topological quaternionic line bundle to admit a stratified-algebraic structure. This paper fits into the new direction of research in real algebraic geometry developed by several authors [1,3,4,8–15].

We use the term real algebraic variety to mean a locally ringed space isomorphic to an algebraic subset of \( \mathbb{R}^n \), for some \( n \), endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [2]). The class of real algebraic varieties is identical with the class of quasiprojective real algebraic varieties, cf. [2, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called...
regular maps. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on \( \mathbb{R} \). Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let \( X \) be a real algebraic variety. By a stratification of \( X \) we mean a finite collection \( S \) of pairwise disjoint Zariski locally closed subvarieties whose union is \( X \). Each subvariety in \( S \) is called a stratum.

Let \( F \) stand for \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{H} \) (the quaternions). When convenient, \( F \) will be identified with \( \mathbb{R}^{d(F)} \), where \( d(F) = \text{dim}_\mathbb{R} F \).

For any nonnegative integer \( n \), let \( \varepsilon^n_X(F) \) denote the standard trivial \( F \)-vector bundle on \( X \) with total space \( X \times F^n \), where \( X \times F^n \) is regarded as a real algebraic variety.

An algebraic \( F \)-vector bundle on \( X \) is an algebraic \( F \)-vector subbundle of \( \varepsilon^n_X(F) \) for some \( n \) (cf. \([2, \text{Chapters 12 and 13}]\) for various characterizations of algebraic \( F \)-vector bundles).

We now recall the fundamental notion introduced in \([14]\). A stratified-algebraic \( F \)-vector bundle on \( X \) is a topological \( F \)-vector subbundle \( \xi \) of \( \varepsilon^n_X(F) \), for some \( n \), such that for some stratification \( S \) of \( X \), the restriction \( \xi|_S \) of \( \xi \) to each stratum \( S \) in \( S \) is an algebraic \( F \)-vector subbundle of \( \varepsilon^n_S(F) \).

A topological \( F \)-vector bundle on \( X \) is said to admit a stratified-algebraic structure if it is isomorphic to a stratified-algebraic \( F \)-vector bundle on \( X \).

As usual, we denote the \( i \)th Stiefel–Whitney class of an \( \mathbb{R} \)-vector bundle \( \eta \) by \( w_i(\eta) \).

**Problem 1.1.** Given a positive integer \( k \), characterize the cohomology classes \( u_i \) in \( H^i(X; \mathbb{Z}/2) \) for which there exists a rank \( k \) stratified-algebraic \( \mathbb{R} \)-vector bundle \( \xi \) on \( X \) with \( w_i(\xi) = u_i \) for \( 1 \leq i \leq k \).

For a \( \mathbb{C} \)-vector bundle \( \eta \), let \( c_i(\eta) \) denote its \( i \)th Chern class.

**Problem 1.2.** Given a positive integer \( k \), characterize the cohomology classes \( u_i \) in \( H^{2i}(X; \mathbb{Z}) \) for which there exists a rank \( k \) stratified-algebraic \( \mathbb{C} \)-vector bundle \( \xi \) on \( X \) with \( c_i(\xi) = u_i \) for \( 1 \leq i \leq k \).

In \([14]\) we solved Problems 1.1 and 1.2 for \( k = 1 \). Presently, we address these problems for \( k \geq 2 \) (Theorems 2.3 and 2.6). The case \( k \geq 3 \) seems to be out of reach. Our partial solution of Problem 1.2 leads to a criterion for a topological \( \mathbb{C} \)-vector bundle of rank 2 and a topological \( \mathbb{H} \)-line bundle to admit a stratified-algebraic structure (Theorems 2.7 and 2.8).

It would be interesting to obtain at least a partial solution of the counterparts of Problems 1.1 and 1.2 for \( k = 2 \), where stratified-algebraic vector bundles are replaced by algebraic vector bundles. However, this would require some new approach, different from the methods used here.

2. **Constructions of rank 2 vector bundles.** For the convenience of the reader, we first review some classical topological constructions.

*Convention* Working with smooth (of class \( C^\infty \)) manifolds, we always assume that submanifolds are closed subsets of the ambient manifold.
Let $X$ be a smooth manifold, and let $M$ be a smooth codimension $k$ submanifold of $X$. Suppose that the normal bundle to $M$ in $X$ is oriented, and denote by $\tau^X_M$ the Thom class of $M$ in the cohomology group $H^k(X, X \setminus M; \mathbb{Z})$, cf. [16, p. 118]. The image of $\tau^X_M$ by the restriction homomorphism $H^k(X, X \setminus M; \mathbb{Z}) \to H^k(X; \mathbb{Z})$, induced by the inclusion map $X \hookrightarrow (X, X \setminus M)$, will be denoted by $[M]^X$ and called the cohomology class represented by $M$. If $X$ is compact and oriented, and $M$ is endowed with the compatible orientation, then $[M]^X$ is up to sign Poincaré dual to the homology class in $H_* (X; \mathbb{Z})$ represented by $M$, cf. [16, p. 136]. Similarly, without any orientability assumption, we define the cohomology class $[M]^X$ in $H^k(X; \mathbb{Z}/2)$ represented by $M$. The cohomology class $[M]^X$ is Poincaré dual to the homology class in $H_*(X; \mathbb{Z}/2)$ represented by $M$.

Let $Y$ be a smooth manifold, and let $N$ be a smooth submanifold of $Y$. Let $f: X \to Y$ be a smooth map transverse to $N$. If the normal bundle to $N$ in $Y$ is oriented and the normal bundle to the smooth submanifold $M := f^{-1}(N)$ of $X$ is endowed with the orientation induced by $f$, then $\tau^X_M = f^* (\tau^Y_N)$, where $f$ is regarded as a map from $(X, X \setminus M)$ into $(Y, Y \setminus N)$ (this follows from [5, p. 117, Theorem 6.7]). In particular,

$$[M]^X = f^* ([N]^Y).$$

Without any orientability assumption,

$$[M]^X = f^* ([N]^Y).$$

Let $\xi$ be a rank $k$ smooth $\mathbb{R}$-vector bundle on $X$. A smooth section $s: X \to \xi$ is said to be transverse regular if it is transverse to the zero section of $\xi$. In that case, the zero locus of $s$,

$$Z(s) := \{ x \in X \mid s(x) = 0 \},$$

is a smooth codimension $k$ submanifold of $X$. We identify the normal bundle to $Z(s)$ in $X$ with $\xi|_{Z(s)}$ via the isomorphism induced by $s$. In particular, if the vector bundle $\xi$ is oriented, then so is the normal bundle to $Z(s)$ in $X$ and

$$e(\xi) = [Z(s)]^X,$$

where $e(\xi)$ stands for the Euler class of $\xi$. Indeed, let $E$ be the total space of $\xi$ and $p: E \to X$ the bundle projection. Identify $X$ with the image of the zero section of $\xi$. The section $s$ is transverse to $X$ and $Z(s) = s^{-1}(X)$. Consequently, $[Z(s)]^X = s^* ([X]^E)$. Hence

$$p^* ([Z(s)]^X) = p^* (s^* ([X]^E)) = (s \circ p)^* ([X]^E) = [X]^E,$$

where the last equality holds since $s \circ p: E \to X$ is homotopic to the identity map. On the other hand, $p^* (e(\xi)) = [X]^E$, cf. [16, p. 98]. It follows that $e(\xi) = [Z(s)]^X$ since $p^*$ is an isomorphism. Similarly, if $\xi$ is not necessarily orientable, we get

$$w_k(\xi) = [Z(s)]^X.$$

Recall that on a smooth manifold each topological vector bundle is isomorphic to a smooth vector bundle, which is uniquely determined up to smooth isomorphism, cf. [5, p. 101].
For any rank $k$ $\mathbb{F}$-vector bundle $\eta$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, let $\det \eta$ denote the $k$th exterior power of $\eta$. Thus $\det \eta$ is an $\mathbb{F}$-line bundle. Furthermore,

$$w_1(\det \eta) = w_1(\eta) \quad \text{if } \mathbb{F} = \mathbb{R},$$

$$c_1(\det \eta) = c_1(\eta) \quad \text{if } \mathbb{F} = \mathbb{C},$$

cf. [6, p. 64].

Given a subset $A$ of $X$ and a cohomology class $u$ in $H^i(X; G)$, where $G = \mathbb{Z}/2$ or $G = \mathbb{Z}$, we denote by $u|_A$ the image of $u$ by the homomorphism $H^i(X; G) \to H^i(A; G)$ induced by the inclusion map $A \hookrightarrow X$.

**Proposition 2.1.** Let $X$ be a smooth manifold and let $\theta$ be a rank 2 topological $\mathbb{R}$-vector bundle on $X$. Then there exist smooth submanifolds $M_i$ of $X$ such that $\text{codim}_X M_i = i$,

$$w_1(\det \nu) = [M_1]^X|_{M_2} \quad \text{in } H^1(M_2; \mathbb{Z}/2),$$

where $\nu$ is the normal bundle to $M_2$ in $X$, and

$$w_i(\theta) = [M_i]^X \quad \text{for } i = 1, 2.$$

**Proof.** We may assume without loss of generality that the vector bundle $\theta$ is smooth. Let

$$s_1: X \to \text{det } \theta, \quad s_2: X \to \theta$$

be smooth transverse regular sections. Setting $M_i = Z(s_i)$, we get

$$w_1(\theta) = w_1(\text{det } \theta) = [M_1]^X, \quad w_2(\theta) = [M_2]^X.$$

Furthermore, since we identify $\nu$ with $\theta|_{M_2}$, we obtain $\det \nu = (\det \theta)|_{M_2}$ and

$$w_1(\det \nu) = w_1((\det \theta)|_{M_2}) = [M_1]^X|_{M_2},$$

as required. \qed

Proposition 2.1 provides motivation for the following result.

**Proposition 2.2.** Let $X$ be a smooth manifold and let $M_i$ be a smooth submanifold of $X$ such that $\text{codim}_X M_i = i$ for $i = 1, 2$, and

$$w_1(\det \nu) = [M_1]^X|_{M_2} \quad \text{in } H^1(M_2; \mathbb{Z}/2),$$

where $\nu$ is the normal bundle to $M_2$ in $X$. Then there exists a rank 2 smooth $\mathbb{R}$-vector bundle $\theta$ on $X$ with

$$w_i(\theta) = [M_i]^X \quad \text{for } i = 1, 2.$$

Furthermore, the vector bundle $\theta$ can be chosen so that there exist smooth transverse regular sections

$$s_1: X \to \text{det } \theta, \quad s_2: X \to \theta$$

satisfying $Z(s_i) = M_i$ for $i = 1, 2$. 
Proof. Let \( \lambda \) be a smooth \( \mathbb{R} \)-line bundle on \( X \) with
\[
      w_1(\lambda) = [M_1]^X. \tag{1}
\]
It is well known that there exists a smooth transverse regular section \( u : X \to \lambda \) satisfying
\[
      Z(u) = M_1. \tag{2}
\]
Let \( \rho : T \to M_2 \) be a tubular neighborhood of \( M_2 \) in \( X \). There exists a smooth transverse regular section \( \sigma : T \to \rho^* \nu \) such that \( Z(\sigma) = M_2 \). In particular,
\[
      (\rho^* \nu)|_{T \setminus M_2} = \mu \oplus \varepsilon_{\sigma}, \tag{3}
\]
where \( \varepsilon_{\sigma} \) is the trivial \( \mathbb{R} \)-line subbundle of \( (\rho^* \nu)|_{T \setminus M_2} \) generated by \( \sigma|_{T \setminus M_2} \), and \( \mu \) is a smooth \( \mathbb{R} \)-line bundle on \( T \setminus M_2 \).

We assert that the \( \mathbb{R} \)-line bundles \( \mu \) and \( \lambda|_{T \setminus M_2} \) are isomorphic. Indeed, we have
\[
      w_1(\nu) = w_1(\det \nu) = [M_1]^X|_{M_2} = w_1(\lambda)|_{M_2}. \tag{4}
\]
Consequently,
\[
      w_1(\rho^* \nu) = w_1(\lambda|_T),
\]
the map \( \rho : T \to M_2 \) being a homotopy inverse of the inclusion map \( M_2 \hookrightarrow T \). Hence
\[
      w_1((\rho^* \nu)|_{T \setminus M_2}) = w_1(\lambda|_{T \setminus M_2}). \tag{4}
\]
On the other hand, in view of (3),
\[
      w_1(\mu) = w_1((\rho^* \nu)|_{T \setminus M_2}). \tag{5}
\]
By combining (4) and (5), we get
\[
      w_1(\mu) = w_1(\lambda|_{T \setminus M_2}),
\]
which implies the assertion, cf. [7, p. 234].

Let \( \varepsilon \) be the standard trivial \( \mathbb{R} \)-line bundle on \( X \) with total space \( X \times \mathbb{R} \) and let \( \tau : X \to \lambda \oplus \varepsilon \) be the smooth section defined by \( \tau(x) = (0, (x, 1)) \) for all \( x \) in \( X \). By the assertion, there exists a smooth isomorphism
\[
      \varphi : (\rho^* \nu)|_{T \setminus M_2} \to (\lambda \oplus \varepsilon)|_{T \setminus M_2}
\]
such that \( \varphi \circ \sigma = \tau \) on \( T \setminus M_2 \).

Let \( \theta \) be the smooth \( \mathbb{R} \)-vector bundle on \( X \) obtained by gluing \( \rho^* \nu \) and \( (\lambda \oplus \varepsilon)|_{X \setminus M_2} \) over \( T \setminus M_2 \) using \( \varphi \). Similarly, let \( s_2 : X \to \theta \) be the smooth section obtained by gluing \( \sigma \) and \( \tau|_{X \setminus M_2} \) over \( T \setminus M_2 \) using \( \varphi \). By construction, \( \theta \) is a rank 2 smooth \( \mathbb{R} \)-vector bundle on \( X \), the section \( s_2 \) is transverse regular, and
\[
      Z(s_2) = M_2. \tag{6}
\]
Consequently,
\[
      w_2(\theta) = [M_2]^X. \tag{7}
\]
In view of (1), (2), (6), (7), it remains to prove that the \( \mathbb{R} \)-line bundles \( \det \theta \) and \( \lambda \) are isomorphic. To this end it suffices to show the equality
\[
      w_1(\det \theta) = w_1(\lambda), \tag{8}
\]
Note that
\[ w_1((\det \theta)|_{X\setminus M_2}) = w_1(\theta|_{X\setminus M_2}) = w_1((\lambda \oplus \varepsilon)|_{X\setminus M_2}) = w_1(\lambda|_{X\setminus M_2}) \]
and hence
\[ w_1(\det \theta)|_{X\setminus M_2} = w_1(\lambda)|_{X\setminus M_2}. \]
As a portion of the long exact cohomology sequence of the pair \((X, X \setminus M_2)\), we get
\[ H^1(X, X \setminus M_2; \mathbb{Z}/2) \rightarrow H^1(X; \mathbb{Z}/2) \xrightarrow{e^*} H^1(X \setminus M_2; \mathbb{Z}/2), \]
where \(e : X \setminus M_2 \hookrightarrow X\) is the inclusion map. Since \(H^1(X, X \setminus M_2; \mathbb{Z}/2) = 0\) (cf. [16, pp. 106, 117], it follows that \(e^*\) is a monomorphism. Hence (8) holds, as required.

If \(X\) is a smooth manifold, by combining Propositions 2.1 and 2.2, we obtain a characterization of the cohomology classes \(u_i\) in \(H^i(X; \mathbb{Z}/2)\) for which there exists a rank 2 topological \(\mathbb{R}\)-vector bundle \(\theta\) on \(X\) with \(w_i(\theta) = u_i\) for \(i = 1, 2\).

Our partial solution of Problem 1.1 is of a similar nature. We first recall a well-known phenomenon specific to real algebraic geometry. Namely, if \(X\) is a nonsingular real algebraic variety, it can happen that a nonsingular Zariski locally closed subvariety of \(X\) is Euclidean closed but not Zariski closed.

Theorem 2.3. Let \(X\) be a compact nonsingular real algebraic variety, and let \(M_i\) be a smooth submanifold of \(X\) such that \(\text{codim}_X M_i = i\) for \(i = 1, 2\), and
\[ w_1(\det \nu) = [M_1]^X_{M_2} \quad \text{in} \quad H^1(M_2; \mathbb{Z}/2), \]
where \(\nu\) is the normal bundle to \(M_2\) in \(X\). Assume that \(M_i\) is a nonsingular Zariski locally closed subvariety of \(X\) for \(i = 1, 2\). Then there exists a rank 2 stratified-algebraic \(\mathbb{R}\)-vector bundle \(\xi\) on \(X\) with
\[ w_i(\xi) = [M_i]^X \quad \text{for} \quad i = 1, 2. \]

Proof. It suffices to make use of Proposition 2.2 and [14, Theorem 1.9].

Let \(K\) be a subfield of \(F\), where \(K\) (as \(F\)) stands for \(\mathbb{R}, \mathbb{C}, \) or \(\mathbb{H}\). Any \(F\)-vector bundle \(\eta\) can be regarded as a \(K\)-vector bundle, which is indicated by \(\eta_K\). In particular, \(\eta_K = \eta\) if \(K = F\).

Suppose now that \(\xi\) is a rank \(k\) smooth \(C\)-vector bundle on a smooth manifold \(X\). Recall that
\[ c_k(\xi) = e(\xi_K), \]
where \(\xi_K\) is endowed with the orientation induced by the complex structure, cf. [16, p. 158]. If \(s : X \rightarrow \xi\) is a smooth transverse regular section, then we regard the normal bundle to \(Z(s)\) in \(X\) as a \(C\)-vector bundle, identifying it with \(\xi|_{Z(s)}\). In particular, the normal bundle to \(Z(s)\) in \(X\) is canonically oriented, the cohomology class \([Z(s)]^X\) in \(H^{2k}(X; \mathbb{Z})\) is defined, and
\[ c_k(\xi) = [Z(s)]^X. \]

If \(M\) is a smooth codimension \(2k\) submanifold of \(X\) and the normal bundle \(\nu\) to \(M\) in \(X\) is endowed with a complex structure, then the \(R\)-vector bundle
\(\nu_R\) is canonically oriented and the cohomology class \([M]_X^2\) in \(H^{2k}(X;\mathbb{Z})\) is defined.

For \(\mathbb{C}\)-vector bundles, Proposition 2.1 takes the following form.

**Proposition 2.4.** Let \(X\) be a smooth manifold, and let \(\theta\) be a rank 2 topological \(\mathbb{C}\)-vector bundle on \(X\). Then there exist smooth submanifolds \(M_i\) of \(X\) such that \(\text{codim}_X M_i = 2i\), the normal bundle \(\nu_i\) to \(M_i\) in \(X\) is endowed with a complex structure,

\[
c_1(\det \nu_2) = [M_1]^X_{|M_2} \text{ in } H^2(M_2;\mathbb{Z}),
\]

and \(c_i(\theta) = [M_i]^X\) for \(i = 1, 2\).

**Proof.** We may assume without loss of generality that the vector bundle \(\theta\) is smooth. Let \(s_1: X \to \det \theta, \ s_2: X \to \theta\) be smooth transverse regular sections. Setting \(M_i = Z(s_i)\), we get

\[
c_1(\theta) = c_1(\det \theta) = [M_1]^X, \quad c_2(\theta) = [M_2]^X.
\]

Furthermore, since we identify \(\nu_2\) with \(\theta|_{M_2}\), we obtain \(\det \nu_2 = (\det \theta)|_{M_2}\) and

\[
c_1(\det \nu_2) = c_1((\det \theta)|_{M_2}) = [M_1]^X_{|M_2},
\]

as required. \(\Box\)

The following is a counterpart of Proposition 2.2 for \(\mathbb{C}\)-vector bundles.

**Proposition 2.5.** Let \(X\) be a smooth manifold, and let \(M_i\) be a smooth submanifold of \(X\) such that \(\text{codim}_X M_i = 2i\), the normal bundle \(\nu_i\) to \(M_i\) in \(X\) is endowed with a complex structure for \(i = 1, 2\), and

\[
c_1(\det \nu_2) = [M_1]^X_{|M_2} \text{ in } H^2(M_2;\mathbb{Z}).
\]

Then there exists a rank 2 smooth \(\mathbb{C}\)-vector bundle \(\theta\) on \(X\) with

\[
c_i(\theta) = [M_i]^X \text{ for } i = 1, 2.
\]

Furthermore, the vector bundle \(\theta\) can be chosen so that there exist smooth transverse regular sections

\[
s_1: X \to \det \theta, \ s_2: X \to \theta
\]

satisfying \(Z(s_i) = M_i\) for \(i = 1, 2\).

**Proof.** The argument is analogous to that in the proof of Proposition 2.2. Let \(\lambda\) be a smooth \(\mathbb{C}\)-line bundle on \(X\) with

\[
c_1(\lambda) = [M_1]^X. \quad (1)
\]

By [14, Lemma 8.20], there exists a smooth transverse regular section \(u: X \to \lambda\) satisfying

\[
Z(u) = M_1. \quad (2)
\]

Let \(\rho: T \to M_2\) be a tubular neighborhood of \(M_2\) in \(X\). There exists a smooth transverse regular section \(\sigma: T \to \rho^*\nu_2\) such that \(Z(\sigma) = M_2\). In particular,

\[
(\rho^*\nu_2)|_{T \setminus M_2} = \mu \oplus \varepsilon_\sigma, \quad (3)
\]
where $\varepsilon$ is the trivial $\mathbb{C}$-line subbundle of $(\rho^*\nu_2)|_{T\setminus M_2}$ generated by $\sigma|_{T\setminus M_2}$, and $\mu$ is a smooth $\mathbb{C}$-line bundle on $T \setminus M_2$.

We assert that the $\mathbb{C}$-line bundles $\mu$ and $\lambda|_{T\setminus M_2}$ are isomorphic. Indeed, we have

$\text{c}_1(\nu_2) = \text{c}_1(\text{det} \nu_2) = \left[X \right]|_{M_2} = \text{c}_1(\lambda)|_{M_2}$.

Consequently,

$\text{c}_1(\rho^*\nu_2) = \text{c}_1(\lambda|_T)$,

the map $\rho: T \to M_2$ being a homotopy inverse of the inclusion map $M_2 \hookrightarrow T$. Hence

$\text{c}_1((\rho^*\nu_2)|_{T\setminus M_2}) = \text{c}_1(\lambda|_{T\setminus M_2}). \quad (4)$

On the other hand, in view of (3),

$\text{c}_1(\mu) = \text{c}_1((\rho^*\nu_2)|_{T\setminus M_2}). \quad (5)$

By combining (4) and (5), we get

$\text{c}_1(\mu) = \text{c}_1(\lambda|_{T\setminus M_2})$,

which implies the assertion, cf. [7, p. 234].

Let $\varepsilon$ be the standard trivial $\mathbb{C}$-line bundle on $X$ with total space $X \times \mathbb{C}$, and let $\tau: X \to \lambda \oplus \varepsilon$ be the smooth section defined by $\tau(x) = (0, (x, 1))$ for all $x$ in $X$. By the assertion, there exists a smooth isomorphism

$\varphi: (\rho^*\nu_2)|_{T\setminus M_2} \to (\lambda \oplus \varepsilon)|_{T\setminus M_2}$

such that $\varphi \circ \sigma = \tau$ on $T \setminus M_2$.

Let $\theta$ be the smooth $\mathbb{C}$-vector bundle on $X$ obtained by gluing $\rho^*\nu_2$ and $(\lambda \oplus \varepsilon)|_{X \setminus M_2}$ over $T \setminus M_2$ using $\varphi$. Similarly, let $s_2: X \to \theta$ be the smooth section obtained by gluing $\sigma$ and $\tau|_{X \setminus M_2}$ over $T \setminus M_2$ using $\varphi$. By construction, $\theta$ is a rank 2 smooth $\mathbb{C}$-vector bundle on $X$, the section $s_2$ is transverse regular, and

$Z(s_2) = M_2. \quad (6)$

Consequently,

$\text{c}_2(\theta) = \left[X \right]. \quad (7)$

In view of (1), (2), (6), (7), it remains to prove that the $\mathbb{C}$-line bundles $\text{det} \theta$ and $\lambda$ are isomorphic. To this end, it suffices to show the equality

$\text{c}_1(\text{det} \theta) = \text{c}_1(\lambda), \quad (8)$

cf. [7, p. 234]. Note that

$\text{c}_1((\text{det} \theta)|_{X \setminus M_2}) = \text{c}_1(\theta|_{X \setminus M_2}) = \text{c}_1((\lambda \oplus \varepsilon)|_{X \setminus M_2}) = \text{c}_1(\lambda|_{X \setminus M_2})$

and hence

$\text{c}_1(\text{det} \theta)|_{X \setminus M_2} = \text{c}_1(\lambda)|_{X \setminus M_2}.$

As a portion of the long exact cohomology sequence of the pair $(X, X \setminus M_2)$, we get

$H^2(X, X \setminus M_2; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \xrightarrow{\text{e}} H^2(X \setminus M_2; \mathbb{Z})$,
where \( e : X \setminus M_2 \hookrightarrow X \) is the inclusion map. Since \( H^2(X, X \setminus M_2; \mathbb{Z}) = 0 \) (cf. [16, pp. 110, 117]), it follows that \( e^* \) is a monomorphism. Hence (8) holds, as required.

If \( X \) is a smooth manifold, then Propositions 2.4 and 2.5 yield a characterization of the cohomology classes \( u_i \) on \( H^{2i}(X; \mathbb{Z}) \) for which there exists a rank 2 topological \( \mathbb{C} \)-vector bundle \( \theta \) on \( X \) with \( c_i(\theta) = u_i \) for \( i = 1, 2 \).

Our partial solution of Problem 1.2 is the following.

**Theorem 2.6.** Let \( X \) be a compact nonsingular real algebraic variety, and let \( M_i \) be a smooth submanifold of \( X \) such that \( \text{codim}_X M_i = 2i \), the normal bundle \( \nu_i \) to \( M_i \) in \( X \) is endowed with a complex structure for \( i = 1, 2 \), and

\[
c_1(\det \nu_2) = [M_1]^X|_{M_2} \quad \text{in } H^2(M_2; \mathbb{Z}).
\]

Assume that \( M_i \) is a nonsingular Zariski locally closed subvariety of \( X \) for \( i = 1, 2 \). Then there exists a rank 2 stratified-algebraic \( \mathbb{C} \)-vector bundle \( \xi \) on \( X \) with

\[
c_i(\xi) = [M_i]^X \quad \text{for } i = 1, 2.
\]

**Proof.** It suffices to make use of Proposition 2.5 and [14, Theorem 1.9]. \( \square \)

We conclude this paper by giving two applications of Theorem 2.6.

**Theorem 2.7.** Let \( X \) be a compact nonsingular real algebraic variety, and let \( \theta \) be a rank 2 topological \( \mathbb{C} \)-vector bundle on \( X \). Let \( M_i \) be smooth submanifolds of \( X \) such that \( \text{codim}_X M_i = 2i \), the normal bundle \( \nu_i \) to \( M_i \) in \( X \) is endowed with a complex structure,

\[
c_1(\det \nu_2) = [M_1]^X|_{M_2} \quad \text{in } H^2(M_2; \mathbb{Z}),
\]

and \( c_i(\theta) = [M_i]^X \) for \( i = 1, 2 \). Assume that \( M_i \) is a nonsingular Zariski locally closed subvariety of \( X \) for \( i = 1, 2 \). If for each integer \( k \geq 3 \) the only torsion in the cohomology group \( H^{2k}(X; \mathbb{Z}) \) is relatively prime to \( (k-1)! \), then \( \theta \) admits a stratified-algebraic structure.

**Proof.** According to Theorem 2.6, there exists a rank 2 stratified-algebraic \( \mathbb{C} \)-vector bundle \( \xi \) on \( X \) with \( c_i(\xi) = c_i(\theta) \) for \( i = 1, 2 \). Consequently,

\[
c_k(\xi) = c_k(\theta) \quad \text{for all } k \geq 0.
\]

Hence, if the condition on the torsion in the cohomology groups \( H^{2k}(X; \mathbb{Z}) \) is satisfied, then the \( \mathbb{C} \)-vector bundles \( \xi \) and \( \theta \) are stably equivalent, cf. [17, Theorem 3.2]. This implies, in view of [14, Corollary 3.14], that \( \theta \) admits a stratified-algebraic structure. \( \square \)

The second application concerns \( \mathbb{H} \)-line bundles.

**Theorem 2.8.** Let \( X \) be a compact nonsingular real algebraic variety, and let \( \lambda \) be a topological \( \mathbb{H} \)-line bundle on \( X \). Let \( M \) be a smooth submanifold of \( X \) such that \( \text{codim}_X M = 4 \), the normal bundle \( \nu \) to \( M \) in \( X \) is endowed with a complex structure, \( c_1(\det \nu) = 0 \) in \( H^2(M; \mathbb{Z}) \), and \( c_2(\lambda_\mathbb{C}) = [M]^X \). Assume that \( M \) is a nonsingular Zariski locally closed subvariety of \( X \). If for each
integer $k \geq 3$ the only torsion in the cohomology group $H^{2k}(X; \mathbb{Z})$ is relatively prime to $(k - 1)!$, then $\lambda$ admits a stratified-algebraic structure.

Proof. Suppose that the condition on the torsion in the cohomology groups $H^{2k}(X; \mathbb{Z})$ is satisfied. By Theorem 2.7, the $\mathbb{C}$-vector bundle $\lambda_C$ admits a stratified-algebraic structure. Hence, in view of [14, Theorem 1.7], the $\mathbb{H}$-line bundle $\lambda$ admits a stratified-algebraic structure. □

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