Neutrino-axion-dilaton interconnection

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We show that a recently proposed framework that provides a simple connection between Majorana neutrinos and an invisible axion in minimal scalar extensions of the standard electroweak model can be naturally embedded in a classically scale-invariant setup. The explicit breaking of the scale invariance à la Coleman-Weinberg generates the Peccei-Quinn and electroweak scales. The spontaneous breaking of the chiral $U(1)_{PQ}$ triggers the generation of neutrino masses via Type-II seesaw and, at the same time, provides a dynamical solution to the strong CP problem as well as the axion as a dark matter candidate. The electroweak and neutrino mass scales are obtained via a technically natural ultraweak limit of the singlet scalar interactions. Accordingly, a realistic and perturbatively stable scalar spectrum, possibly in the reach of the LHC, is naturally obtained. A very light pseudodilaton characterizes such a setting. The vacuum stability of the extended setup is discussed.

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I. INTRODUCTION

In spite of the fundamental importance for the understanding of the electroweak symmetry breaking, the discovery of what appears to be the long-sought Higgs boson still leaves many issues of the Standard Model (SM) of particle interactions unaddressed. Laboratory and astrophysical observations give us an extremely detailed picture of massive neutrino and lepton mixing which clearly indicate the need for physics beyond the SM. Dark matter is required to account for more than 20% of the mass of the Universe, where antimatter is a very rare component. From a theoretical point of view, the absence of new particles at the TeV scale raises the issue of the stability of the electroweak scale in the presence of hypothetical new heavy states associated to, e.g., grand unification or other high-scale dynamics (if nothing else gravity does).

In a recent paper [1] we attempted to show that most of these issues can be organically addressed in a minimal renormalizable framework that just extends the Higgs sector of the SM while providing a structural connection between different open questions, the model naturally offers a stable spectrum of exotic scalar states at the TeV scale, thus opening the possibility of a test at the LHC.

The proposal in [1] extends the model discussed in [2,3] which provided a connection between an invisible axion à la Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) [4,5] and the one-loop generation of neutrino masses à la Zee [6,7]. As explicit and potentially realistic realizations of such a scheme we surveyed in [1] three setups where the neutrino mass arises at different loop orders, namely, at the tree level via Type-II seesaw [8–12], at one loop as in [13], and at two loops as in the Zee-Babu model [14,15], respectively.

Regardless of the presence of the large Peccei-Quinn (PQ) scale, in all variants of the scheme discussed in [1], a natural and stable electroweak setup is obtained by invoking a decoupling behaviour of the scalar singlet field responsible for the PQ symmetry breaking. All its interactions in the scalar potential (besides the self-interaction) are scaled down by powers of the electroweak over the PQ scales in such a way that all non-singlet states acquire weak-scale masses. This “ultraweak” setting of the singlet scalar interactions is in fact technically stable since the decoupling of the singlet corresponds to an extended Poincaré symmetry of the action [2,16].

The presence of the invisible axion requires at least two Higgs doublets and a complex singlet field. One or more additional scalars are then responsible for the generation of Majorana neutrino masses. The scalar spectra

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1 Comments on the role of gravity in such a setting can be found in [17].
obtained in [1] are naturally compatible with one light
SM-like Higgs (a general discussion on the decoupling
and alignment limits in a two Higgs doublet context is
found in [18]). On top of that, perturbative naturalness
implies that the new scalars should be within the reach
of the LHC. As a fringe benefit of the extension of the
scalar sector the stability of the electroweak vacuum is
expected to be improved with respect to the SM [19] [20]
(for a recent overview see [21]).

Hence, a stable renormalizable and realistic extension
of the SM is obtained that not only addresses the ori-
gen of the neutrino masses and mixings but, at the same
time, connects them to the presence of an invisible ax-
ion, a viable dark matter candidate (at variance with the
DFSZ model, in the current setup the axion entertains
tiny couplings to the neutrinos). These models, in their
minimal realizations, do not exhibit additional sources
of CP violation, thus fostering the dynamical solution
of the strong CP problem via the PQ mechanism [22].

In the present paper we investigate the embedding
of such a PQ related neutrino mass framework into a
classically scale-invariant setup, exploiting the intriguing
idea [23] that mass scales in nature may originate from
quantum effects. Classical scale symmetry is explicitly
broken by the renormalization of dimension four inter-
actions. The logarithmic dependence on the mass scales
allows for large hierarchies in terms of order one ratios of
the dimensionless Lagrangian parameters, thus naturally
protecting the fundamental mass scale of the theory from
any large scale.

The conjecture that classical scale symmetry may pro-

tect the electroweak scale from large perturbative effects
was pioneered in [24] (see [25] [26] for a recent reap-
praisal). At variance with the QCD-like strong dynamics,

scale invariance is broken by perturbative quantum loops
that contribute to the stress-tensor trace anomaly with
terms proportional to the beta-functions of the dimen-
sionless couplings (a pedagogical introduction to scale
invariance is found in [27], while its relation with confor-
mal invariance is reviewed in [28]).

Obtaining a realistic scenario in such a context re-
requires as well an extension of the standard Higgs sector.
The embedding of the DFSZ invisible axion model in
a classically scale-invariant setup has been recently dis-
cussed in [29]. As an archetypical implementation of the
scale invariant framework to the neutrino mass models
considered in [1], we focus our analysis on the PQ ex-
tended Type-II seesaw model. The PQ scale, induced
via dimensional transmutation, triggers in turn the elec-
tric weak symmetry breaking. The hierarchy between
the two scales is set and stabilized by the ultraweak limit
of the singlet scalar couplings. Attention is paid to the
analysis of the scalar spectrum, aiming at a realistic fit
of the present LHC data.

The simultaneous presence of the PQ and classical
scale symmetries sharply constrain the scalar potential
of the theory. All but two of the ultraweak couplings
are determined by the minimization conditions in terms
of the other quartic couplings. We show that a natural
(i.e., not fine-tuned) and radiatively stable decoupling
limit is feasible; in such a case the lightest Higgs boson
is fully compatible with the data (that require moderate
\tan \beta \) values), while all other physical scalars satisfy
the present collider bounds. The needed decoupling and
alignment limits of the extra doublet Higgs states are
controlled by just one of the two independent ultraweak
couplings, while the second one drives both the neutrino
mass as well as the decoupling of the scalar triplet states.
All this is achieved within a stable and a fully perturba-
tive setup.

The model exhibits an invisible axion and a very light
neutral scalar that plays the role of a pseudodilaton, both
with tiny couplings to neutrinos that bear no relevance
for today’s astrophysical and cosmological data (the

cosmology of the ultralight pseudodilaton is thoroughly
discussed in [29]). The smallness of the pseudodilaton mass
is due to quantum effects which require a tiny quartic
self interaction. This is a characteristic feature of the
scale-invariant embedding, at variance with the setups
discussed in [1], where the strength of the singlet self
interaction is unconstrained and a heavy singlet scalar
state is allowed.

In summary, a relatively simple extension of the stan-
dard Higgs sector gives rise to a renormalizable and per-
turbatively stable scenario where a number of observa-
tional and theoretical issues of the SM find a correlated
and natural explanation. Were perturbative naturalness
a “fundamental” principle rather than a theorist preju-
dice, a plethora of new scalar states could be well within
the LHC reach.

The study is organized as follows. In the first part of
the paper we introduce the model and study the mini-
mization of the one-loop effective potential. In the sec-
ond part we analyze the pattern and phenomenology of
the extended scalar sector and discuss the conditions for
vacuum stability. Detailed aspects of the analysis are
summarized in the appendices.

II. PQ EXTENDED TYPE-II SEESAW

The field content and the charge assignment of the PQ
extended Type-II seesaw model was worked out in [1]
and it is displayed for convenience in Table I. On top
of the usual SM field content, the scalar sector includes
two Higgs doublets, one isospin triplet with a unit hy-
charge and one complex SM singlet. Since the PQ
current is axial, it is proportional to the difference be-
 tween the charges of the left- and right-handed fermions.
Hence, without loss of generality, we may set the PQ
current of the quark doublet $X_q = 0$ such that the color
anomaly of the PQ current turns out to be proportional
to $X_u + X_d \neq 0$ [30].
In Eqs. (1)–(2) as follows

\[ V = \lambda_1 |H_u|^4 + \lambda_2 |H_d|^4 + \lambda_{12} |H_u|^2 |H_d|^2 
+ \lambda_4 |H_u^\dagger H_d|^2 + \lambda_{13} |\sigma|^2 |H_u|^2 + \lambda_{23} |\sigma|^2 |H_d|^2 
+ \lambda_3 |\sigma|^4 + \text{Tr}(\Delta^\dagger \Delta) [\lambda_{11} |H_u|^2 + \lambda_{22} |H_d|^2 
+ \lambda_{12} |\sigma|^2 + \lambda_{21} \text{Tr}(\Delta^\dagger \Delta)] 
+ \lambda_7 H_u^\dagger \Delta^\dagger H_u + \lambda_8 H_d^\dagger \Delta^\dagger H_d + \lambda_9 \text{Tr}(\Delta^\dagger \Delta)^2 
+ (\lambda_5 \sigma^2 H_u^\dagger H_u + \lambda_6 \sigma H_d^\dagger H_d + \text{h.c.}), \]  

(2)

where \( H_u = i \tau_2 H_u^* \).

We shall parameterize the complex scalar fields appearing in Eqs. (1)–(2) as follows

\[ H_u = \left( \frac{h_u^0 + i h_u^0}{\sqrt{2}} \right), \]  

(3)

\[ H_d = \left( \frac{h_d^0 + i h_d^0}{\sqrt{2}} \right), \]  

(4)

\[ \sigma = \frac{\sigma^0 + i \sigma^0}{\sqrt{2}}, \]  

(5)

\[ \Delta = \frac{\tau \cdot \Delta}{\sqrt{2}} = \left( \frac{\delta^+ + \delta^+}{\sqrt{2}} \frac{\delta^+ + \delta^+}{\sqrt{2}} \right). \]  

(6)

In Eq. (4), \( \tau = (\tau_1, \tau_2, \tau_3) \) are the Pauli matrices and \( \Delta = (\Delta_1, \Delta_2, \Delta_3) \) are the \( SU(2)_R \) components of the scalar triplet. In what follows, the vacuum expectation values (VEV) of the neutral components of the scalar multiplets, arising at the quantum level, will be denoted as

\[ \langle H_u \rangle = v_u, \quad \langle H_d \rangle = v_d, \quad \langle \sigma \rangle = V_\sigma, \quad \langle \Delta \rangle = v_\Delta. \]  

(7)

Note that the form of the scalar potential in Eq. (2) is uniquely fixed by the charge assignment in Table I. In particular, terms like \( H_u^\dagger H_d^\dagger \text{Tr}(\Delta \Delta^\dagger) \) or \( H_u^\dagger H_d^\dagger \text{Tr}(\Delta^\dagger \Delta) \) are not allowed since the QCD anomaly of the PQ current requires \( X_u + X_d \neq 0 \). Moreover, the identity \( H_u^\dagger_d (\Delta \Delta^\dagger) H_u^d = |H_u^d|^2 \text{Tr}(\Delta \Delta^\dagger) \) holds, so that only two out of three apparently different invariants are linearly independent.

We remind that the interaction terms \( \lambda_5 \sigma^2 H_u^\dagger H_d \) and \( \lambda_6 \sigma H_u^\dagger H_d \) are both needed in order to assign a nonvanishing PQ charge to the singlet \( \sigma \) and to generate the neutrino mass. The simultaneous presence of \( \lambda_5 \), \( \lambda_6 \) and \( V_\sigma \) ensures the explicit breaking of the lepton number. If any of these couplings is missing, either lepton number is exact and neutrinos are massless (\( \lambda_5 = 0 \)) or lepton number is spontaneously broken (\( \lambda_5 = 0 \)) and the vacuum exhibits a Majoron together with a Wilczek-Weinberg axion (in the latter case, discussed in [2], there exist a charge assignment such that \( \sigma \) carries two units of lepton number, while being PQ neutral). Both couplings \( \lambda_5 \) and \( \lambda_6 \) can be set real by two independent rephasings of the scalar fields. No spontaneous CP violation arises from such a scalar potential [13].

By normalizing the PQ charge of the scalar singlet to unity and by imposing its orthogonality to the SM hypercharge (i.e., \( X_u v_u^2 = X_d v_d^2 \)) one obtains

\[ X_u = \frac{2}{\tan^2 \beta + 1}, \quad X_d = \frac{2 \tan^2 \beta}{\tan^2 \beta + 1}, \]  

(8)

\[ X_\ell = \frac{\tan^2 \beta - 3}{2(\tan^2 \beta + 1)}, \quad X_e = \frac{5 \tan^2 \beta - 3}{2(\tan^2 \beta + 1)}, \]  

where \( \tan \beta \equiv v_u/v_d \).

B. PQ scale via dimensional transmutation

A simple but comprehensive discussion of the embedding of the DFSZ invisible axion model [13] in a classically scale-invariant setup has been recently presented in [29].

We shall now analyze the analogous embedding of our Type II invisible axion model, paying attention to the spectrum of the weak-scale scalars, in particular to the conditions for obtaining a 125 GeV SM-like Higgs. In this respect large \( \tan \beta \) values are not allowed by the current limits on the Higgs down-quark couplings. The needed
decoupling of the other scalar eigenstates then further constrains the scalar potential parameter space.

According to the discussion in [1] we assume the technically natural ultraweak limit of the singlet scalar interactions:

$$\lambda_{13, 5} \sim \mathcal{O}\left(\frac{v^2}{V^2}\right), \quad \lambda_6 \sim \mathcal{O}\left(\frac{v_\Delta}{V}\right),$$  \hspace{1cm} (9)

where $v^2 = v_u^2 + v_d^2$. The reference scaling of the singlet couplings is dictated by the requirement that the physical ultraweak limit perturbatively stable. It is convenient to introduce the rescaled couplings $c_\lambda$ as

$$\lambda_{13, 5} \equiv c_{13, 5} \frac{v^2}{V^2}, \quad \lambda_6 \equiv c_6 \frac{v_\Delta}{V}. \hspace{1cm} (10)$$

This setting is at the origin of the hierarchy between the PQ and EW scales and the stability of the Higgs mass, thus making the setup insensitive to the large PQ scale. The limit $\lambda_{13, 5}, \lambda_6 \to 0$ is associated with the emergence of an additional Poincaré symmetry of the action [2, 10] (see [17] for a recent reassessment) which makes the ultraweak limit perturbatively stable. It is readily verified that the renormalization of the couplings connecting the “light” and “heavy” sectors is, as a set, multiplicative (the relevant beta functions exhibit a fixed point for vanishing couplings, as it is verified from inspection of the one-loop beta coefficients in Appendix [3]). Note that the hierarchy among the ultraweak couplings in Eq. (9) is stable since $\lambda_6^2 \ll \lambda_{13}$. The couplings $\lambda_5$ and $\lambda_6$ are themselves multiplicatively renormalized since lepton number is restored when one of them is vanishing.

The stronger scaling pattern of $\lambda_3$

$$\lambda_3 \sim \mathcal{O}\left(\frac{v^4}{V^4}\right) \hspace{1cm} (11)$$

is required by the Coleman-Weinberg (CW) mechanism in order to be effective along the singlet direction, so that the PQ scale is obtained by dimensional transmutation [24]. This is also a renormalization safe assumption since the limit $\lambda_{13}, \lambda_5, \lambda_6, \lambda_3 \to 0$ is associated with the emergence of a shift symmetry of the noninteracting scalar singlet [24]. These considerations hold as long as we neglect gravity. For a brief discussion of gravity induced effects we refer to [11] and references therein.

C. CW potential and the vacuum configuration

The generation of the electroweak breaking vacuum via the CW mechanism [24] was shown to require a very light Higgs mass, of about 10 GeV [24, 32]. In this paper, analogously to the proposal of [29], we provide a realistic setup where quantum correction are responsible for the generation of the PQ scale. This leads, as we will see, to a very light neutral scalar acting as a pseudodilaton, still phenomenologically viable. The hierarchy between the PQ and the electroweak scale is ensured by the technically natural ultraweak limit on the singlet couplings to the other scalar fields [11, 2, 10], as stated in Eq. (9).

To this end the relevant one-loop CW potential in the $\overline{MS}$ scheme can be written as

$$V_1(\bar{\sigma}) = \frac{1}{64\pi^2} \text{Tr} M^4(\bar{\sigma}) \left(\log \frac{M^2(\bar{\sigma})}{\mu^2} - \frac{3}{2}\right), \hspace{1cm} (12)$$

where $\bar{\sigma}$ denotes a spacetime-independent classical field and the trace is over the tree-level mass matrices in the shifted $\sigma \to \bar{\sigma} + \sigma$ theory,

$$M^2_{ij}(\bar{\sigma}) = \left. \frac{\partial^2 V_0}{\partial \phi_i \partial \phi_j} \right|_{\phi \to (\bar{\sigma}, 0, \ldots, 0)} \hspace{1cm} (13)$$

where the vector $\phi$ stands for the whole set of the real fields in the model. In our case the leading contributions to the effective potential in the $\sigma$-field direction read (from now on we drop the bar symbol over $\sigma$)

$$V_1 = \frac{1}{64\pi^2} \left[ 4 \lambda_+^2 |\sigma|^4 \left(\ln \frac{\lambda_+ |\sigma|^2}{\mu^2} - \frac{3}{2}\right) + \right. \left. 4 \lambda_2^2 |\sigma|^4 \left(\ln \frac{\lambda_- |\sigma|^2}{\mu^2} - \frac{3}{2}\right) + \right. \left. 6 \lambda_{3\Delta}^2 |\sigma|^4 \left(\ln \frac{\lambda_{3\Delta} |\sigma|^2}{\mu^2} - \frac{3}{2}\right) \right], \hspace{1cm} (14)$$

where one may recognize the complex doublet and triplet contributions. The singlet quartic interaction is negligible because of Eq. (11), as detailed in the following. The functional masses of the two Higgs doublets in the singlet direction are written in terms of

$$\lambda_\pm = \frac{1}{2} \left( \lambda_{13} + \lambda_{23} \pm \sqrt{(\lambda_{13} - \lambda_{23})^2 + 4 \lambda_5^2} \right). \hspace{1cm} (15)$$

We limited the dependence of the functional masses to the $\sigma$ component, assuming that in the other field directions the tree-level quartic couplings dominate the potential. By doing so the perturbative effective potential may develop an imaginary part. This is just a consequence of having effectively chosen a nonconvex point of the one-loop potential in other neutral field directions but the singlet one. The correct minimum in the $\sigma$ field direction is nevertheless obtained by taking the real part of the effective potential [33].

According to the original CW approach we minimize the scalar potential along $\sigma$, which is, by construction, the only field direction sensitive to radiative corrections. The stationarity equation obtained from the derivative of $V \equiv V_0 + V_1$ reads
The difference amounts to a redefinition of term analogous to $\lambda$ group (RG) based discussion of [29, 34] one sees that a dependence of $\lambda$ quartic coupling, as given in Appendix B. Of the two terms in the above expression one has a linear leading dependence on $V_\sigma$ while the other is cubic. On the other hand, in view of the ultraweak limit in Eq. [9], both terms are of the same order and must be equally considered. We see that a stationary point exists either when $V_\sigma = 0$ or $V_\sigma$ arbitrary and

$$\lambda_3 = \frac{\lambda'_3 v^2}{2V_\sigma^2} - \frac{1}{16\pi^2} \left[ \lambda_+^2 \left( \ln \frac{\lambda_+ V_\sigma^2}{\mu^2} - 1 \right) + \lambda_-^2 \left( \ln \frac{\lambda_- V_\sigma^2}{\mu^2} - 1 \right) + \frac{3}{2} \lambda_{3\sigma}^2 \left( \ln \frac{\lambda_{3\sigma} V_\sigma^2}{\mu^2} - 1 \right) \right]$$

(17)

where we defined

$$\lambda_3' v^2 \equiv 2\lambda_5 v_u v_d - \lambda_1 v_u^2 - \lambda_3 v_d^2 - \lambda_{3\sigma} v_\sigma^2$$

(18)

By plugging the above expression for $\lambda_3$ into the potential one finally finds

$$V_{\text{eff}} = (V_0 - \lambda_3 |\sigma|^4) + \frac{\lambda'_3 v^2}{2V_\sigma^2} |\sigma|^4 + \frac{1}{16\pi^2} \left[ \lambda_+^2 + \lambda_-^2 + \frac{3}{2} \lambda_{3\sigma}^2 \right] \left( \ln \frac{|\sigma|^2}{V_\sigma^2} - 1/2 \right) |\sigma|^4$$

$$= (V_0 - \lambda_3 |\sigma|^4) + \frac{\lambda'_3 v^2}{2V_\sigma^2} |\sigma|^4 + \frac{\beta_{\lambda_3}}{2} \left( \ln \frac{|\sigma|^2}{V_\sigma^2} - 1/2 \right) |\sigma|^4,$$

(19)

where $\beta_{\lambda_3}$ is the one-loop beta function of the singlet quartic coupling, as given in Appendix [13].

By comparing our result with the renormalization group (RG) based discussion of [29, 34] one sees that a term analogous to $\lambda'_3$ is missing in the effective potential. The difference amounts to a redefinition of $V_\sigma$ (or a shift on the singlet quartic coupling) which does not bear any physical consequences.

The effective potential in Eq. (19) does not explicitly depend on the renormalization scale, since the explicit $\mu$ dependence of $\lambda_3$ in Eq. (17) precisely cancels the explicit $\mu$ dependence of the one-loop effective potential. It is worth remarking that the coefficient of the quartic term for the singlet coincides precisely with the beta function of $\lambda_3$ (which includes a negligible contribution of the $\sigma$ quartic self interaction, proportional to $\lambda_{3\sigma}^2$).

The minimization proceeds with the derivation of the stationarity equations in the remaining field directions, with $\lambda'_3 = O(v^2/V_\sigma^2)$. As mentioned above, since the functional masses in Eq. (14) are taken at a non convex point of the perturbative effective potential ($v_{u,d} = 0$) an imaginary part may develop [33]. Accordingly, the real part of Eq. (17) is understood.

We have traded the dimensionless parameter $\lambda_3$ for $\langle \sigma \rangle / \mu$. Obviously, for the CW mechanism to work an (almost) flat field direction is needed in the scalar potential. In the case at hand the singlet quartic coupling $\lambda_3$ is required to be of the order of the square of the other ultraweak couplings and loop suppressed (this justifies its omission from the one-loop effective potential).

By taking into account Eq. (9) and that electroweak measurements bound $v_\Delta/v$ to be less than a percent, Eq. (18) can be cast into the form

$$\lambda_3' v^2 \approx 2 \frac{v_\Delta}{V_\sigma} (\lambda_1 v_u^4 + \lambda_2 v_d^4 + \lambda_3 v_\sigma^2 v_u^2),$$

(23)
which holds up to $O(v_\Delta^2/v^2)$ corrections.

It is worth remarking that due to the ultraweak size of the singlet couplings the relevant scale for the minimization of the effective potential is not the PQ scale, but rather the electroweak scale. The latter in fact minimizes the logarithmic terms in Eq. (14), and therefore minimizes higher-order corrections to the vacuum. We shall therefore consider the stationarity equations as constraints on the couplings evaluated at the weak scale.

D. Scalar spectrum

The scalar spectrum of the model is worked out in detail in Appendix A. Considering the hierarchy among the VEVs ($v_\Delta \ll v_{u,d} \ll V_\sigma$) and the ultraweak limit we may safely neglect at the percent level accuracy the mixings of the triplet with the other scalars and perform an analytic diagonalization of the doublet-singlet mass matrix.

The results can be cast in fairly simple form. Of four neutral CP-even scalars, two are made mostly of the $h_0^0$ and $h_0^0$ doublet fields (see Eqs. (A22)–(A23) in Appendix A). The mass eigenvalues are given in Eq. (A21). In view of the results of the phenomenological discussion to follow, it is convenient to give their expression in the limit of large $c_5 \gg \lambda_{1,2,12}$, namely

$$m^2_H \approx 2 \left(\lambda_1 v_u^2 + \lambda_2 v_d^2\right) + 2 \left(\lambda_1 v_u^2 - \lambda_2 v_d^2\right) \tan \beta - \cot \beta \tan \beta + \cot \beta \left(\tan \beta + \cot \beta\right)$$

and

$$m^2_H \approx c_5 v^2 (\tan \beta + \cot \beta),$$

respectively (tan $\beta = v_u/v_d$). Eqs. (24)–(25) exhibit the large tan $\beta$ (cot $\beta$) mass dependence as well. The lighter state $h$ will be identified with the SM-like Higgs. The mixing between the two neutral scalars is parametrized by the angle $\alpha$ (defined in Eq. (A26) of Appendix A). For $\cos(\alpha - \beta) \to 0$ the couplings of the lightest eigenstate overlap with those of the SM Higgs (alignment limit [23]). The other neutral scalar ($H$) can be made parametrically heavier, with mass in the TeV range, by increasing $c_5$. Correspondingly, $\cos(\alpha - \beta)$ decreases, as we shall shortly detail.

The remaining two neutral scalars coincide with high precision with the $\sigma^0$ and $\delta^0$ fields, respectively. Their masses are given by

$$m^2_H = 2\beta \lambda_1 v^2,$$

$$m^2_{\Delta} = -c_6 v u v d,$$

where $c_6 < 0$. The coefficients $c_4$, defined in Eq. (10), are $O(1)$ parameters that gauge the degree of naturalness of the model. Since the mass eigenstates scale roughly with their square root, deviations within a factor of 10 from unity are to be considered natural [1]. The very light (mainly singlet) eigenstate can be identified with a pseudodilaton [29–34]. The ultraweak character of its couplings underlies a shift symmetry that makes in the limit the field $\sigma$ formally analogous to a dilaton, as arising from spontaneous breaking of the scale invariance. Due to the ultraweak setup in Eq. (9) its mass is bound to be below the MeV scale. Since the CW mechanism is effective only in the $\sigma$ direction, it is also the only scalar state whose mass is determined by quantum effects, as a consequence of the explicit breaking of the scale invariance.

The pseudoscalar spectrum consists of the neutral Goldstone boson (GB) “eaten” by $Z$ (spanned predominantly on $\eta^0_u$ and $\eta^0_d$), the invisible axion (mostly $\eta^0$ orthogonal to the GB above) and $\Delta_A$ (mostly $\eta^0$) with masses

$$m^2_{\Delta_A} = -c_6 v u v d,$$

where $c_5 > 0$. The leading order (LO) equality of $\Delta_\sigma$ and $\Delta_A$ masses is a consequence of the neglect of the triplet mixings with the other scalars (see Appendix A).

Among the singly-charged scalars one of the mass eigenstates is the charged GB “eaten” by $W^\pm$ and there are two massive states (mostly $h^0_u$ and $h^0_d$, respectively):

$$m^2_{H^\pm} = \lambda_4 v^2 + c_5 v^2 (\tan \beta + \cot \beta),$$

$$m^2_{\Delta^\pm} = \frac{1}{2} (\lambda_7 v_u^2 - \lambda_8 v_d^2) - c_6 v u v d.$$

Finally, the doubly-charged component of the triplet field acquires the mass

$$m^2_{\Delta^{++}} = \lambda_7 v_u^2 - \lambda_8 v_d^2 - c_6 v u v d.$$

Notice the $\frac{1}{2} (\lambda_7 v_u^2 - \lambda_8 v_d^2)$ mass isosplitting among the components of the scalar triplet.

While the masses of the (mainly) doublet and triplet states are driven by the tree-level part of the potential, the weak-scale pseudodilaton mass (the ultraweak setup is assumed) is genuinely obtained at one loop. As such Eq. (26) should be renormalized down to its characteristic scale ($\ll$ MeV), and finite momentum corrections should be included. On the other hand, the pseudodilaton couples to matter and gauge fields only through its mixings with the doublet and triplet scalars which are suppressed by the PQ scale and, thus, the running of the pseudodilaton mass is of a higher order. Analogously, finite momentum dependent one-loop corrections are suppressed by the mass of the heavier particles in the loop, and for the scope of the present analysis they can be neglected as well.

The physics of the invisible axion is analogous to that of the DFSZ model with the addition of tiny couplings to the neutrinos that, however, do not bear observable cosmological or astrophysical implications [1]. A short discussion on the light pseudodilaton phenomenology is deferred to Sect. III B.
 FIG. 1. The “hug” diagram, responsible for the neutrino mass in the PQ extended Type-II seesaw model.

E. Neutrino masses

In the PQ extended Type II seesaw model [1], the neutrino masses are generated through the tree-level diagram in Fig. 1. The corresponding (symmetric) mass matrix is readily obtained from the Yukawa Lagrangian in Eq. (1) as

$$ M_{\nu} = Y \Delta v \Delta \approx -Y \Delta \lambda \frac{v_u v_d}{M^2} \, , $$

(33)

where $M_{\Delta}$ is the common mass of the neutral triplet components. Consequently, the bound on the heaviest neutrino $m_{\nu_3} \lesssim 1$ eV translates into the constraint

$$ |\lambda| \lesssim 10^{-18} \left( \frac{10^9 \text{ GeV}}{v_\sigma} \right) \, . $$

(34)

The smallness of the absolute neutrino mass scale may have different origins and the naturalness of the setup plays a relevant role. The fact that $M_{\Delta}$ is by construction not far from the electroweak scale sharply constrains the size of the triplet Yukawa couplings $Y_{ij}$ in the tree-level lepton flavor violation processes [1]. Complementary to that, the smallness of the ultraweak coupling $\lambda \approx v_\Delta / V_\sigma$ is a crucial factor for the required neutrino mass suppression, that can be obtained even for somewhat large $Y_{\Delta}$, depending on the actual size of $v_\Delta$.

III. HIGGS PHENOMENOLOGY AND VACUUM STABILITY

In the technically natural ultraweak limit of Eq. (9) a number of new scalar states may possibly fall within the reach of the present and near future collider searches.

In this section we shall discuss the conditions under which the scalar spectrum is phenomenologically viable and the vacuum structure of the model is stable. A necessary prerequisite to that is the analysis of the constraints on the scalar couplings coming from the measured value of the Higgs mass ($125.09 \pm 0.21 \pm 0.11$ GeV [37]) and the allowed deviations of its couplings to matter and gauge fields (for a recent study of the Higgs LHC physics and the electroweak precision observables in the DFSZ ultraweak setup see [38]). We shall give a benchmark setting of the model parameters that satisfies all the present LHC constraints with the heavier scalars possibly below the TeV scale, and thus accessible to the collider searches. We also include a brief phenomenological overview of the light pseudodilaton phenomenology.

A. Experimental constraints

The Yukawa couplings of the lightest neutral scalar eigenstate $h$ can be parametrized at the LO in $\cos(\beta - \alpha) \ll 1$ as [18]

$$ L_Y \approx 1 - \tan \beta \cos(\beta - \alpha) \frac{v_D M_D h}{v} + 1 + \cot \beta \cos(\beta - \alpha) \frac{1}{v} (1 - \cos^2(\beta - \alpha)) \, . $$

(35)

where $\alpha$ is the mixing angle between the light and heavy eigenstates (see Appendix A).

Analogously, the $h$ couplings with the gauge bosons are conveniently written as

$$ g_{hVV} \approx 1 - \frac{\cos^2(\beta - \alpha)}{2} \, . $$

(36)

Based on the results in Appendix A Fig. 2 shows the constraints on $c_5$ and $\tan \beta$ coming from the present fit of the Higgs mass and couplings [39] (with benchmark values of the relevant quartic couplings). Only moderate values of $\tan \beta$ are allowed. The strong constraint on $\tan \beta$ is a consequence of the specific pattern of the scalar couplings and can be readily inferred from Eq. (35). In the large $c_5$ limit the leading $c_5$ contributions cancel in the lightest scalar mass eigenvalue (while they sum up
in the heavier one). The assumed perturbativity of the scalar couplings up to the Planck scale together with the constraints from the Higgs mass lead to the typical $10^{-1}$ scale for the relevant doublet couplings. For doublet couplings of similar size a double solution above and below $\tan \beta = 1$ appears, as expected. A value of $\tan \beta$ near unity is allowed for $\lambda_1 = \lambda_2 = \lambda_{12} \approx 0.17$ which, however, is too large for perturbativity. On the other hand, when considering lower values, a hierarchy between $\lambda_1$ and the other doublet couplings is imposed by the requirement of vacuum stability ($\lambda_1$ is affected by a large top quark negative renormalization and it is bound to larger values). This, altogether, selects the $\tan \beta > 1$ solution. For decreasing $c_5 < 1$ a solution is maintained by increasing $\tan \beta$. These features are apparent in Fig. 2 for typical values of the doublet couplings. The specific pattern there is a consequence of the hierarchy between $\lambda_1$ and the other doublet couplings which is needed to reconcile perturbativity with vacuum stability. The “lower part” of the plot (i.e., $c_5 \lesssim 0.5$) is cut out by the constraints from the $h \rightarrow b \bar{b}$ data that still allow for about 30% deviation from the SM coupling [39]. In combination, a rather sharp constraint on $\tan \beta$ emerges.

A phenomenologically acceptable mass gap between the lightest Higgs and the other physical doublet eigenstates can then be obtained, even for moderate values of $\tan \beta$ and perturbative quartic couplings, by raising $c_5$, as we shall shortly detail (no issue of perturbativity arises since $\lambda_5$ is an ultraweak coupling). In the large $c_5$ limit the doublet mixing angle $\alpha$ is simply related to $\tan \beta$. From Eq. (A26) one obtains

$$\tan 2\alpha \approx \frac{2}{\cot \beta - \tan 2\beta}.$$  

(A37)

According to the doublet eigenstates and mixings defined in Appendix A, $\sin 2\alpha < 0$ in the limit. This leads to $\cos(\beta - \alpha) \approx 0$, which corresponds to the alignment with the SM Higgs couplings discussed in [18]. We recall that, in the broken PQ phase, the $\lambda_5$ quartic coupling induces a mixing between the two Higgs doublets. The pattern of the scalar doublet spectrum for large $c_5$ then corresponds to the decoupling limit for large $m_{12}^2$ in Ref. [18].

The detailed dependence of $\cos(\beta - \alpha)$ on $\tan \beta$ is shown in Fig. 3 for the benchmark values of the couplings in Fig. 2 and few typical values of $c_5$. This allows one to readily estimate the size of the deviations of the light Higgs couplings to quarks and gauge bosons from their SM values. In particular, we understand why the present experimental uncertainties do not strongly affect the model even for moderate values of $\tan \beta$: even considering the largest allowed value of $\tan \beta$ in Fig. 2 the present experimental constraints on the $hbb$ couplings lead to $|\cos(\beta - \alpha)| < 0.09$, well above all model values for $c_5 > 1$, cf. Fig. 3. When the constraints on the Higgs couplings become stronger the mixing angle $\alpha$ must be further tuned toward its effective decoupling value $\beta - \pi/2$ by means of a larger $c_5$, with a corresponding increase in the mass of the heavier “doublet states” and a progressive loss of naturalness of the setup.

It is remarkable that the deviations of the couplings of the light Higgs boson to the top quark and to the gauge bosons are always very small (below 1% in the
allowed range of $\tan \beta$ for $c_5 > 1$). This justifies the assumption of the usual SM Higgs production rates used in our numerical analysis.

In Fig. 2 we show the contour plots for the values of the ultraweak couplings $\lambda_{13}$ and $\lambda_{23}$ in the $(c_5, \tan \beta)$ plane, in the same area of the parameter space shown in Fig. 2. We see that $c_{13}$ may be negative for small $c_5$, while $c_{23} > 1$ in the allowed region. The fact that $\lambda_{13}$ may be negative at the weak scale is not an issue for the vacuum stability since, as we shall see, the positivity of the relevant boundedness condition, which involves other quartic couplings, is always satisfied.

In Table II the chosen benchmark values of the independent scalar couplings are reported. The starred couplings are related by the vacuum to the other ones and their reference values are given for $\tan \beta = 3.1$, $c_5 = -c_6 = 1$ and $V_\sigma = 10^5$ GeV ($v_\Delta < 1$ GeV). As discussed in the text the tiny scale of the ultraweak couplings is protected by symmetry and it is therefore technically natural.

| Quartic coupling | Electroweak-scale value |
|------------------|-------------------------|
| $\lambda_1$     | 0.15                    |
| $\lambda_2 = \lambda_{12} = \lambda_4$ | 0.08                    |
| $\lambda_7 = \lambda_8 = \lambda_9$ | 0.08                    |
| $\lambda_{1\Delta 1} = \lambda_{1\Delta 2} = \lambda_{4\Delta}$ | 0.08                    |
| $\lambda_6/c_5$ | $3.0 \times 10^{-14} \left(\frac{v_\Delta}{V_\Delta}\right)^2$ |
| $\lambda_6/c_6$ | $1.0 \times 10^{-9} \left(\frac{v_\Delta}{V_\Delta}\right)^2$ |
| $\lambda_{13}(\ast)$ | $1.3 \times 10^{-15}$ |
| $\lambda_{23}(\ast)$ | $9.1 \times 10^{-14}$ |
| $\lambda_{3\Delta}(\ast)$ | $6.2 \times 10^{-15}$ |
| $\lambda_3(\ast)$ | $1.1 \times 10^{-28}$ |

TABLE II. Benchmark values of quartic couplings at the electroweak scale $v \approx 174$ GeV. The starred couplings are related to the vacuum to the other ones and their reference values are given for $\tan \beta = 3.1$, $c_5 = -c_6 = 1$ and $V_\sigma = 10^5$ GeV ($v_\Delta < 1$ GeV). As discussed in the text the tiny scale of the ultraweak couplings is protected by symmetry and it is therefore technically natural.

This analysis allows us to conclude that the model can accommodate all the present experimental constraints [40–42], while maintaining, as we shall detail, vacuum stability and absence of Landau poles up to the Planck scale. Indeed, as already mentioned, among all the Higgs doublet couplings, $\lambda_1$ takes the largest value in order to avoid the vacuum instability due to the large renormalization effect induced by the top quark. The smaller values chosen for the other couplings then ensure the absence of Landau poles below the Planck scale.

In Table III and Table IV, typical mass spectra of the scalar fields are displayed. The mass scale of the exotic doublet and triplet states is controlled by $c_5$ and $c_6$, respectively. The large value of $c_6$ which drives the triplet masses does not affect neither the stability of the $\lambda_6$ coupling (which, alike $\lambda_5$, is multiplicatively renormalized), nor perturbativity since $\lambda_6$ is ultraweak. Rather, as for $c_5$, it is a measure of the naturalness of the setup. Roughly speaking, the degree of fine tuning related to the stability of the lightest Higgs mass against radiative corrections induced by the scalar triplet states is proportional to the square root of $|c_6|$.

In terms of the rescaled coupling $c_6$ such a condition can be conservatively written as

$$|c_6| < \frac{v}{v_\Delta},$$

independently on the PQ scale, $V_\sigma$. Depending on the actual value of $v_\Delta < 1$ GeV, stability of the ultraweak setup is maintained even in the presence of very large values of $c_6$ that, indeed, may be needed by the heavy triplet states.

Finally, $\lambda_5$ and $\lambda_6$ are individually multiplicatively renormalized since lepton number is restored when one of

| $\tan \beta$ | $c_5$ | $m_5$[GeV] | $m_H$[GeV] | $m_A$[GeV] | $m_{H^+}$[GeV] |
|-------------|-------|------------|------------|------------|----------------|
| 3.1         | 1.0   | 125        | 324        | 322        | 326            |
| 3.0         | 2.0   | 125        | 451        | 449        | 452            |
| 3.0         | 5.0   | 125        | 711        | 710        | 712            |

TABLE III. Typical values of the doublet scalar masses for the values of the quartic couplings in Table II, when varying $\tan \beta$ and $c_5$ within the allowed region in Fig. 2.

| $\tan \beta$ | $c_6$ | $m_{\Delta 0}$[GeV] | $m_{\Delta^+}$[GeV] | $m_{\Delta^{++}}$[GeV] | $m_\sigma$[GeV] |
|-------------|-------|---------------------|---------------------|------------------------|----------------|
| 25          | 476   | 477                 | 478                 | 0.5 × 10^{-4}          |
| 75          | 825   | 826                 | 826                 | 1.3 × 10^{-4}          |

TABLE IV. Typical triplet and pseudodilaton masses for the benchmark values of the quartic couplings in Table II. The scalar masses are given for reference $\tan \beta$ and $c_5$ values that accommodate the present collider limits on the doubly-charged triplet component. The dependence of the pseudodilaton mass on $c_5$, in the interval 1 to 5, ranges from a factor of 2 to a factor of 1.1 as $c_6$ increases.
them vanishes. In the limit $c_5 \rightarrow 0$ there is a PQ charge assignment that leaves the singlet scalar carrying only the lepton number. We thus recover at the tree level two massless pseudoscalar states: an invisible Majoron and a weak-scale axion. In the limit $c_6 \rightarrow 0$ lepton number is restored and it remains unbroken as there is no induced triplet VEV. From the inspection of the triplet mass spectrum we observe the vanishing of the mass of the neutral triplet components, while for $\lambda_{7,8} \rightarrow 0$ all triplet-dominated fields turn out to be massless as well. These tree-level results are related to the neglect of the triplet mixings, and they can be generally understood in terms of accidental shift symmetries of the scalar potential for vanishing triplet couplings. All these features are explicitly verified by the inspection of the mass spectrum and the one-loop beta coefficients reported in Appendix B. We can therefore conclude that the pattern of the benchmark values given in Table II is technically natural, and leads (within the present experimental limits on the scalar spectrum) to a natural and stable model setup.

From Table II and Table IV we see that the heavy part of the scalar spectrum of the model may be accessible at the LHC. It goes without saying that heavier masses for the exotic scalar fields can be achieved at the expense of the strict naturalness requirement we asked for.

B. Ultralight pseudodilaton phenomenology

The smoking gun of the current scenario at low energy is the light pseudodilaton, whose basic features are rather similar to those of the analogous state discussed in 29, 34.

In particular, concerning the pseudodilaton couplings to the matter fields, they are driven by its mixing with the other neutral scalars. As one can see by inverting Eqs. (A27)–(A29), its projection onto the neutral components of the Higgs doublets amounts to $\sim v/V_\sigma$ which, in turn, yields couplings to the SM fermions of the order of $m_f/V_\sigma$. Although this, in principle, admits for a possible pseudodilaton detection in “5th force” experiments, this interaction turns out to be too weak for the existing limits to provide nontrivial constraints. Since the relevant mediator mass $m_\pi$ is of $O(v^2/V_\sigma) \approx 30 \times (10^9 \text{GeV}/V_\sigma) \text{keV}$, the current bound reads $\alpha_5/\alpha_{EM} \lesssim 10^{-8} - 10^{-16}$ for $V_\sigma$ in the $10^9 - 10^{12} \text{GeV}$ range, far above the size of the pseudodilaton interaction strength with ordinary matter. The ultraweak size of the couplings of the complex singlet scalar field, which drives both the axion and the light pseudodilaton states, does not lead to visible collider signatures either, at variance with the case of less constrained scale invariant Higgs extensions [44].

Considering the role the pseudodilaton may play in the early Universe cosmology, the main concerns have to do with the energy stored in the coherent oscillations of the pseudodilaton field after inflation. To that end, two basic scenarios, characterized basically by the hierarchy between the reheat temperature and the PQ breaking scale, emerge 29, 34. It turns out, however, that in either of these cases there is a natural way to dissipate the excess energy either by the pseudodilaton interactions with the SM thermal bath or by taming its very production in the first place. The former, in the current setup, may work even better than estimated in the large $\tan \beta$ scenario of 34 since the small $\tan \beta$ constraint implies larger up-type Yukawa couplings and, in turn, more efficient dissipation. There is no natural domain for the relevant parameters in which the pseudodilaton may contribute a significant fraction of $\Omega_\text{DM}$ (unlike the axion): either its production is negligible or it is unstable on cosmological scales.

Given all that, the main difference among our setup and that in 34, is the direct coupling of the pseudodilaton to the light neutrinos that follows from its mixing with the triplet scalar in the Type-II seesaw. On the other hand, the absolute size of the corresponding effective coupling, $g_{\nu\nu} \propto m_\nu/V_\sigma$, is utterly small and, at the present time, it yields no observable effect.

C. Vacuum stability

An added value of scalar extensions of the SM is their potential to improve on the stability of the electroweak vacuum. This issue has been discussed at length in the literature (see for instance Refs. 45, 46), and we just briefly recall the argument here. The key effect is the positive contribution of the new scalars (through, e.g., the Higgs portal couplings) to the beta-function of the
Higgs quartic coupling \( \lambda_H \). As such, they tend to stabilize the Higgs potential if they enter the running below the instability scale.\(^2\)

Nevertheless, the presence of multiple scalar field directions may potentially reintroduce the issue of instability already at tree level, especially in those cases where operators featuring an odd power of the same field are present (e.g., those associated with the couplings \( \lambda_5 \) and \( \lambda_6 \) in Eq. (2)). On the other hand, even at the tree-level potential level, a fully analytical determination of the necessary and sufficient conditions for the boundedness of Eq. (2) from below turns out to be a formidable problem.

In Fig. 6 we display the one-loop running of the “large” (i.e., \( O(10^{-1}) \)) couplings for the weak-scale benchmark values given in Table II.

Regardless of the fact that \( \lambda_4 \) and \( \lambda_7 \) both fall negative at large energies, it can be shown that the relevant boundedness conditions are always satisfied. To this end, in Appendix C we provide a compact parametrization of the scalar potential manifold based on the “invariants’ method” (see e.g. Ref. [49]), which allows us to probe the critical scalar field directions by fully exploiting the symmetries of the system. In particular, the sixteen real field variables of the potential in Eq. (2) are traded for ten real parameters (three angles and six invariants defined over a compact domain plus one radial coordinate). In this way, given a set of benchmark values for the scalar potential couplings it is possible to perform a fast numerical check of the vacuum stability by randomly scanning over the scalar potential variables.

Moreover, by selecting specific field directions (angles), certain sufficient conditions for the vacuum stability can be explicitly worked out (cf. Appendix C). This provides an analytical understanding of the reason why, e.g., the fact that \( \lambda_7 \) runs sharply negative, as shown in Fig. 6, is not necessarily a problem. As a matter of fact, the relevant boundedness condition (in the Type-II seesaw limit – cf. Appendix C) yields the relation (most restrictive for \( \lambda_9 > 0 \))

\[
\lambda_{\Delta 1} + \lambda_7 + 2 \sqrt{\lambda_1 \left( \lambda_{\Delta 4} + \frac{1}{2} \lambda_9 \right)} > 0,
\]

which, as shown in Fig. 7, is always satisfied for the benchmark point in Table II. We have verified by analytic and numerical means that for the chosen set of benchmark values this conclusion holds for all field directions.

IV. CONCLUSIONS

We have discussed a minimal extension of the SM that addresses in a structural and natural way a number of its observational and theoretical open issues. The origin of a Type-II seesaw neutrino mass is strictly related to the presence of an invisible axion à la DFSZ [1]. In spite of the large PQ scale, a technically stable ultraweak setup of the singlet scalar couplings makes the model perturbatively stable and natural, with a scalar spectrum potentially within the reach of the LHC. The presence of the axion field provides a natural solution to both the dark matter issue and the strong CP problem. At the same time, the required extension of the scalar sector improves on the stability of the SM vacuum. We do not require additional fermions (recent proposals that include right-handed neutrinos in connection with an invisible axion were presented in [50–55], while an extensive list of studies on the subject is found in [1]).

The embedding of the Type-II seesaw model, discussed in [1], in a classically scale invariant setup explicitly broken by perturbative quantum effects, allows us to further constrain the model and describe the scalar spectrum in terms of just few parameters. A setup consistent with all the LHC constraints is naturally obtained. In addition to

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\(^2\) The instability scale of the SM effective potential is a gauge dependent quantity [47]. A gauge invariant criterium to include the effects of new physics can be devised [48].
the invisible axion a very light pseudodilaton characterizes the present setup. Both fields exhibit tiny couplings to neutrinos, that, however, do not affect present-day laboratory, astrophysical and cosmological searches.

While not excluding additional ultraviolet physics, the model represents a minimal renormalizable extension of the SM that aims at answering, from a particle physics perspective, some of the yet open issues of the standard electroweak theory. With this ambition, we are currently investigating whether cold baryogenesis with axion dynamics at play \cite{56}, may find as well a structural embedding in such a context.

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**Appendix A: Scalar Mass spectrum**

The scalar spectrum is readily obtained from the quadratic part of the shifted potential. The minimization of the CW potential in Eq. \[19\] provides, together with Eq. \[17\] for \(\lambda_3\), three additional relations among the quartic scalar couplings. We choose to write the three ultraweak couplings \(\lambda_{i3} (i = 2, 3, \Delta)\) in terms of the remaining ones as

\[
\begin{align*}
\lambda_{13} &= -\left(\lambda_{\Delta 1} + c_5 \cot \beta \right) v^2_{\Delta} + 2 \lambda_1 v^2_u - c_5 v^2 \cot \beta + \lambda_{12} v^2_d, \\
\lambda_{23} &= -\left(\lambda_8 + \lambda_{\Delta 2} + c_6 \tan \beta \right) v^2_{\Delta} + 2 \lambda_2 v^2_d - c_5 v^2 \tan \beta + \lambda_{12} v^2_u, \\
\lambda_{33} &= -2 \left(\lambda_9 + \lambda_{\Delta 3} \right) v^2_{\Delta} + \left(\lambda_8 + \lambda_{\Delta 2}\right) v^2_d + \lambda_{\Delta 1} v^2_u + c_6 v_d v_u,
\end{align*}
\]

where, according to Eq. \[10\], we conveniently used the rescaled couplings \(c_5\) and \(c_6\). Since the adopted ultraweak limit for the singlet interactions leads to a scalar mass spectrum clustered below the TeV scale, the minimization of the higher-order effects in the CW one-loop potential implies that the relevant scale for the stationarity equations is the weak scale. From the known constraints on the scalar spectrum this allows us to readily derive the corresponding set of low-scale values of the scalar couplings underpinning the vacuum stability analysis.

1. Neutral scalars

With the help of Eqs. \[A1\] \- \[A3\] the mass matrix entries for the neutral scalars in the \(\{h^0_u, h^0_d, \sigma^0, \sigma^0\}\) basis can be written as

\[
\begin{align*}
M_S^2[1, 1] &= \frac{4 \lambda_1 v^3_u + \lambda_5 v d V^2_{\sigma} - \lambda_6 v_{\Delta} v_d V_{\sigma}}{v_u}, \\
M_S^2[1, 2] &= \frac{M_S^2[2, 1]}{v_u} = \lambda_6 v_{\Delta} V_{\sigma} - \lambda_5 V^2_{\sigma} + 2 \lambda_{12} v_d v_u, \\
M_S^2[3, 1] &= \frac{M_S^2[3, 1]}{V_{\sigma}^2} = -\frac{2 v^2_{\Delta} \lambda_{\Delta 1} v_u}{V^2_{\sigma}} \\
&\quad + v_{\Delta} \lambda_6 V_{\sigma} v_d + 4 \lambda_1 v^3_u + 2 \lambda_{12} v^2_d v_u, \\
M_S^2[4, 1] &= \frac{M_S^2[4, 1]}{V_{\sigma}^2} = 2 \lambda_{\Delta 1} v_{\Delta} v_u + \lambda_6 v_d V_{\sigma}, \\
M_S^2[2, 2] &= \frac{4 \lambda_2 v^3_d + \lambda_5 V^2_{\sigma} v_u - \lambda_6 v_{\Delta} v_u V_{\sigma}}{v_d}, \\
M_S^2[3, 2] &= \frac{M_S^2[3, 2]}{V_{\sigma}^2} = -\frac{2 v^2_{\Delta} \lambda_{\Delta 1} v_u}{V^2_{\sigma}} \\
&\quad + v_{\Delta} \lambda_6 V_{\sigma} v_u + 4 \lambda_2 v^3_d + 2 \lambda_{12} v^2_d v_u^2, \\
M_S^2[4, 2] &= \frac{M_S^2[4, 2]}{V_{\sigma}^2} = 2 (\lambda_8 + \lambda_{\Delta 2}) v_d v_{\Delta} + \lambda_6 V_{\sigma} v_u.
\end{align*}
\]
where \( \sigma \) is given at Eq. (A14) can be conveniently written as

\[
M^2_\sigma = \begin{pmatrix}
4 \lambda_1 v_u^2 + c_5 v^2 \cot \beta & -c_5 v^2 + 2 \lambda_2 v_d v_u & -\frac{1}{V_\sigma} (4 \lambda_1 v_u^3 + 2 \lambda_2 v_d v_u) \\
-c_5 v^2 + 2 \lambda_2 v_d v_u & 4 \lambda_2 v_d^2 + c_5 v^2 \tan \beta & -\frac{1}{V_\sigma} (4 \lambda_2 v_d^3 + 2 \lambda_2 v_d v_u) \\
-\frac{1}{V_\sigma} (4 \lambda_1 v_u^3 + 2 \lambda_2 v_d v_u) & -\frac{1}{V_\sigma} (4 \lambda_2 v_d^3 + 2 \lambda_2 v_d v_u) & 2 \lambda_3 V_d^2 + 2 \lambda_3 v^2
\end{pmatrix},
\]

(A14)

Using Eq. (A23) a straightforward computation yields

\[
V^T (M^2_\sigma)^{-1} V = \frac{4}{V_\sigma} (\lambda_1 v_u^4 + \lambda_2 v_d^4 + \lambda_2 v_u^2 v_d^2)
\]

(A19)

from where we obtain the pseudodilaton mass

\[
m^2_\sigma = 2 \beta \lambda_3 V_d^2,
\]

(A20)

which, as expected, turns out to be proportional to the explicit breaking of the scale invariance induced by the running of \( \lambda_3 \) in the stress-tensor trace anomaly. This result obviously holds for the complete mass matrix as well.

The mass eigenvalues of the light \( \tilde{h} \) and heavy \( \tilde{H} \) neutral scalars are then given by

\[
2 m^2_{\tilde{h}, \tilde{H}} \approx [4 \lambda_1 v_u^2 + 4 \lambda_2 v_d^2 + c_5 v^2 (\tan \beta + \cot \beta)] \pm \sqrt{[4 \lambda_2 v_d^2 - 4 \lambda_1 v_u^2 + c_5 v^2 (\tan \beta - \cot \beta)]^2 + 4 (2 \lambda_2 v_u v_d - c_5 v^2)^2}.
\]

(A21)

where \( \alpha \) is the mixing angle between the neutral doublet fields, given by

\[
tan 2 \alpha = \frac{2(2 \lambda_1 v_u v_d - c_5 v^2)}{4(\lambda_2 v_u^2 - \lambda_2 v_d^2) + c_5 (\tan \beta - \cot \beta) v^2}.
\]

(A26)

with \(-\pi/2 < \alpha < \pi/2\). Eqs. (A22)-(A26) are consistent with the two-doublet mixing and eigenstate definitions of Ref. [18].

Were the triplet mixings negligible with respect to those of the singlet sector \( (v_\Delta / v \ll v/V_\sigma) \) the eigenstates
would read
\[ \tilde{h} \approx \cos \alpha \left( \frac{v_u}{\sqrt{\sigma}} h^0_u - \frac{v_d}{\sqrt{\sigma}} h^0_d \right) - \sin \alpha \left( \frac{v_d}{\sqrt{\sigma}} h^0_u - \frac{v_u}{\sqrt{\sigma}} h^0_d \right), \] (A27)
\[ \tilde{H} \approx \cos \alpha \left( \frac{v_u}{\sqrt{\sigma}} h^0_u - \frac{v_d}{\sqrt{\sigma}} h^0_d \right) + \cos \alpha \left( \frac{v_d}{\sqrt{\sigma}} h^0_u - \frac{v_u}{\sqrt{\sigma}} h^0_d \right), \] (A28)
\[ \tilde{\sigma}_0 \approx \sigma_0 + \frac{v_u}{\sqrt{\sigma}} h^0_u + \frac{v_d}{\sqrt{\sigma}} h^0_d, \] (A29)
where the result
\[ (M^2_{2h})^{-1} V = -\frac{1}{\sqrt{\sigma}} \begin{pmatrix} v_u \\ v_d \end{pmatrix}. \] (A30)
is used in the seesaw-like diagonalization of \( M^2_\sigma \) in Eq. (A15).

2. Pseudoscalar fields

Analogously, for the pseudoscalar fields we obtain in the \((\eta'_u, \eta'_d, \eta'_\sigma, \eta''_l)\) basis

\[ M^2_{PS}[1, 1] = -\frac{v_d V_{\sigma}}{v_u} (v_\Delta \lambda_6 - \lambda_5 V_{\sigma}), \] (A31)
\[ M^2_{PS}[1, 2] = M^2_{PS}[2, 1] = \lambda_6 v_\Delta V_{\sigma} + \lambda_5 V_{\sigma}^2, \] (A32)
\[ M^2_{PS}[1, 3] = M^2_{PS}[3, 1] = \lambda_6 v_\Delta v_d + 2 \lambda_5 v_d V_{\sigma}, \] (A33)
\[ M^2_{PS}[1, 4] = M^2_{PS}[4, 1] = -\lambda_6 v_\Delta V_{\sigma}, \] (A34)
\[ M^2_{PS}[2, 2] = -\frac{V_{\sigma} v_d}{v_\Delta} (v_\Delta \lambda_6 - \lambda_5 V_{\sigma}), \] (A35)
\[ M^2_{PS}[2, 3] = M^2_{PS}[3, 2] = -\lambda_6 v_\Delta v_u + 2 \lambda_5 V_{\sigma} v_d, \] (A36)
\[ M^2_{PS}[2, 4] = M^2_{PS}[4, 2] = \lambda_6 V_{\sigma} v_d, \] (A37)
\[ M^2_{PS}[3, 3] = -\lambda_6 \frac{V_{\sigma} v_d v_d}{v_\Delta} + 4 \lambda_5 v_d v_d, \] (A38)
\[ M^2_{PS}[3, 4] = M^2_{PS}[4, 3] = \lambda_6 v_d v_u, \] (A39)
\[ M^2_{PS}[4, 4] = -\frac{\lambda_6 V_{\sigma} v_d v_u}{v_\Delta}. \] (A40)

After diagonalization and by neglecting the \( v_\Delta \) terms, we finally get
\[ M^2_{PS} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{5, 6} \frac{v_d}{v_\Delta} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_{5, 6} \frac{v_d}{v_\Delta} \end{pmatrix}, \] (A41)
where \( c_{5, 6} \) are \( O(1) \) numbers in the ultraweak limit. The two massless states correspond to the weak neutral GB and to the invisible axion fields, respectively. The latter acquires a tiny mass via QCD instantons.

3. Singly charged scalars

Working in the \( \{h^+_u, h^+_d, \delta^+\} \) basis we obtain
\[ M^2_{PV}[1, 1] = \lambda_7 v_\Delta^2 - v_\Delta \lambda_6 \frac{v_d}{v_u} V_\sigma + \lambda_4 v_d^2 + \lambda_5 \frac{v_d}{v_u} V_{\sigma}^2, \] (A42)
\[ M^2_{PV}[1, 2] = M^2_{PV}[2, 1] = \lambda_6 v_d v_u + \lambda_5 V_{\sigma}^2, \] (A43)
\[ M^2_{PV}[1, 3] = M^2_{PV}[3, 1] = \lambda_6 (\lambda_7 v_\Delta v_\sigma - \lambda_6 v_d V_{\sigma}), \] (A44)
\[ M^2_{PV}[2, 2] = -v_\Delta^2 \lambda_6 - v_\Delta V_{\sigma} \lambda_6 \frac{v_d}{v_u} + \lambda_4 v_d^2 + \lambda_5 V_{\sigma}^2 \frac{v_d}{v_u}, \] (A45)
\[ M^2_{PV}[2, 3] = M^2_{PV}[3, 2] = \frac{1}{\sqrt{2}} (\lambda_7 v_\Delta v_d + \lambda_6 V_{\sigma} v_u), \] (A46)
\[ M^2_{PV}[3, 3] = \frac{1}{2} (\lambda_7 v_\Delta^2 - \lambda_6 v_d^2) - \lambda_6 \frac{V_{\sigma} v_d v_u}{v_\Delta}. \] (A47)

After diagonalization and by neglecting \( v_\Delta \) terms, we finally get
\[ M^2_{PV} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_4 v_d^2 + c_{5, 6} \frac{v_d}{v_\Delta} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} (\lambda_7 v_\Delta^2 - \lambda_6 v_d^2) - c_{5, 6} \frac{v_d}{v_\Delta} \end{pmatrix}, \] (A48)
where the massless state corresponds to the weak charged GB.

4. Doubly charged scalars

The mass of the doubly charged component of the scalar triplet reads
\[ M^2_{PV} = \frac{\lambda_7 v_\Delta^2 - \lambda_6 v_d^2 - c_{5, 6} v_d v_u}{v_\Delta}. \] (A49)
Notice the \( \frac{1}{2} (\lambda_7 v_\Delta^2 - \lambda_6 v_d^2) \) mass isosplitting among the components of the scalar triplet.

Appendix B: Scalar one-loop beta functions

We consider the field content given in Table 1 with the scalar potential in Eq. (1) and the Yukawa Lagrangian in Eq. (1). For the purpose of the present discussion we neglect the \( Y_d, Y_e \), and \( Y_\Delta \) couplings and take
\[ Y_u \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y_t & 0 \end{pmatrix}. \] (B1)
As usual, we denote by \( g_3, g_2 \) and \( g_1 \) the gauge couplings associated to the three factors of the \( SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \) gauge group, respectively. In this approximation the one-loop running of the top Yukawa coupling coincides with the SM result, namely
\[ \frac{dy_t}{d\ln \mu} = \frac{y_t}{16\pi^2} (9/2g_t^2 - 8g_3^2 - 9/4g_2^2 - 17/12g_1^2). \] (B2)
On the other hand, the gauge coupling running is modified by the presence of the additional scalars: 

\[
\frac{\text{d}a_i}{\text{d} \ln \mu} = \frac{a_i}{2\pi}, \tag{B3}
\]

with \( a = \{8, -\frac{7}{3}, -7\} \) and \( \alpha_i \equiv g_i^2/(4\pi) \).

Parametrizing the \( \beta \) functions of the scalar couplings as

\[
\frac{\text{d}\lambda}{\text{d} \ln \mu} = \frac{b_\lambda}{16\pi^2} \equiv \beta_\lambda, \tag{B4}
\]

we obtain the following one-loop results \([57]\):

\[
\begin{align*}
\beta_{\lambda_1} &= 24\lambda_1^2 + 2\lambda_1^2 + 2\lambda_1\lambda_4 + \lambda_4^2 + \lambda_1^2 + 3\Delta_1^2 + 3\lambda_1\lambda_7 + 5/4\lambda_7^2 - 6y_4^4 + 12y_7^2 \lambda_1 \\
&+ 3/8g_1^4 + 3/4g_1^2g_2^2 + 9/8g_2^4 - 3\lambda_1(g_1^2 + 3g_2^2), \\
\beta_{\lambda_2} &= 24\lambda_2^2 + 2\lambda_2^2 + 2\lambda_2\lambda_4 + \lambda_4^2 + 3\Delta_2^2 + 3\lambda_1\lambda_8 + 5/4\lambda_8^2 \\
&+ 3/8g_1^4 + 3/4g_1^2g_2^2 + 9/8g_2^4 - 3\lambda_2(g_1^2 + 3g_2^2), \\
\beta_{\lambda_{12}} &= 12\lambda_1\lambda_{12} + 4\lambda_1\lambda_4 + 12\lambda_2\lambda_8 + 4\lambda_2\lambda_4 + 2\lambda_{13}\lambda_{23} + 6\lambda_{12}\lambda_{2} + 3\lambda_{12}\lambda_8 + 3\lambda_{12}\lambda_7 \\
&+ 1/2\lambda_7\lambda_8 + 4\lambda_7^2 + 2\lambda_7^2 + 2\lambda_6^2 + 6y_2^4 + 3/4g_1^4 + 3/2g_1^2g_2^2 + 9/4g_2^4 - 3\lambda_{12}(g_1^2 + 3g_2^2), \\
\beta_{\lambda_4} &= 4\lambda_4(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_4) + 2\lambda_7\lambda_8 - 4\lambda_5^2 - 6y_2^4 \lambda_4 - 3y_1^2g_2^2 - 3\lambda_4(g_1^2 + 3g_2^2), \\
\beta_{\lambda_5} &= 20\lambda_5^2 + 2\lambda_5^2 + 2\lambda_4^2 + 3\lambda_3^2, \\
\beta_{\lambda_{13}} &= 8\lambda_3\lambda_{13} + 12\lambda_1\lambda_{13} + 4\lambda_{23}\lambda_{12} + 2\lambda_{23}\lambda_4 + 6\lambda_{33}\lambda_{11} + 3\lambda_{31}\lambda_7 + 4\lambda_1^2 + 8\lambda_5^2 + 3\lambda_6^2 \\
&+ 6y_2^2\lambda_{13} - 3/2\lambda_{13}(g_1^2 + 3g_2^2), \\
\beta_{\lambda_{23}} &= 8\lambda_3\lambda_{23} + 12\lambda_2\lambda_{23} + 4\lambda_{13}\lambda_{12} + 2\lambda_{13}\lambda_4 + 6\lambda_{33}\lambda_{22} + 3\lambda_{31}\lambda_8 + 4\lambda_{23}^2 + 8\lambda_5^2 + 3\lambda_6^2 \\
&- 3/2\lambda_{23}(g_1^2 + 3g_2^2), \\
\beta_{\lambda_{14}} &= 12\lambda_1\lambda_{14} + 4\lambda_1\lambda_7 + 4\lambda_{12}\lambda_{2} + 2\lambda_{12}\lambda_8 + 2\lambda_{13}\lambda_{13} + 16\lambda_{11}\lambda_{14} \\
&+ 12\lambda_{11}\lambda_9 + 6\lambda_7\lambda_{14} + 2\lambda_7\lambda_9 + 4\lambda_1^2 + \lambda_7^2 + 2\lambda_5^2 + 6y_2^2\lambda_{11} \\
&+ 3y_1^4 + 9y_1^2g_2^2 + 6y_2^4 - 3/2\lambda_{11}(5g_1^2 + 11g_2^2), \\
\beta_{\lambda_{24}} &= 12\lambda_2\lambda_{24} + 4\lambda_2\lambda_8 + 4\lambda_{12}\lambda_{14} + 2\lambda_1\lambda_7 + 2\lambda_{23}\lambda_{13} + 16\lambda_{22}\lambda_{14} \\
&+ 12\lambda_{22}\lambda_9 + 6\lambda_7\lambda_{24} + 2\lambda_8\lambda_9 + 4\lambda_1^2 + \lambda_7^2 + 3y_1^4 - 6y_2^2g_2^2 + 6y_2^4 - 3/2\lambda_{22}(5g_1^2 + 11g_2^2), \\
\beta_{\lambda_{34}} &= 8\lambda_3\lambda_{34} + \lambda_{33}(6\lambda_{34} + 12\lambda_9) + 2\lambda_{13}(2\lambda_{11} + \lambda_7) + 2\lambda_{23}(2\lambda_{22} + \lambda_8) \\
&+ 4\lambda_1^2 + 2\lambda_5^2 - 3\lambda_{33}(2g_1^2 + 4g_2^2), \\
\beta_{\lambda_{44}} &= 28\lambda_4^2 + 24\lambda_4\lambda_9 + 6\lambda_5^2 + 2\lambda_{11}\lambda_7 + 2\lambda_2\lambda_{22} + 3\lambda_5^2 \lambda_3 \\
&+ 6y_4^4 - 12y_7^2g_2^2 + 15g_2^4 - 3\lambda_{14}(4g_1^2 + 8g_2^2)), \\
\beta_{\lambda_{54}} &= 5(4\lambda_5 + 2\lambda_{12} + 4\lambda_{13} + 4\lambda_{23} - 2\lambda_4) + 3y_1^2\lambda_5 - 3/2\lambda_{54}(g_1^2 + 3g_2^2), \\
\beta_{\lambda_6} &= 2\lambda_6(\lambda_{13} + \lambda_{23} + \lambda_{12} + \lambda_{21}) + \lambda_6(3\lambda_7 - \lambda_7) + 2\lambda_6\lambda_{33} + 2\lambda_6\lambda_{12} + 3y_1^2\lambda_6 - 3/2\lambda_6(3g_1^2 + 7g_2^2), \\
\beta_{\lambda_7} &= 4\lambda_1\lambda_7 + 4\lambda_1\lambda_7 + 8\lambda_1\lambda_7 + 2\lambda_4\lambda_8 + 8\lambda_{11}\lambda_7 + 4\lambda_7^2 - 2\lambda_5^2 + 6y_2^2\lambda_7 \\
&- 12g_1^2g_2^2 - 3/2\lambda_7(5g_1^2 + 11g_2^2), \\
\beta_{\lambda_8} &= 4\lambda_2\lambda_8 + 4\lambda_4\lambda_8 + 8\lambda_8\lambda_9 + 2\lambda_4\lambda_7 + 8\lambda_{12}\lambda_8 + 4\lambda_8^2 + 2\lambda_5^2 + 12g_1^2g_2^2 - 3/2\lambda_8(5g_1^2 + 11g_2^2), \\
\beta_{\lambda_9} &= 24\lambda_4\lambda_9 + 18\lambda_6^2 + \lambda_7^2 + \lambda_9^2 + 24g_1^2g_2^2 - 6g_2^4 - 3\lambda_9(4g_1^2 + 8g_2^2). \tag{B20}
\end{align*}
\]

Notice that \( \lambda_5 \) and \( \lambda_6 \) are individually multiplicatively renormalized and, together with the \( \lambda_{13} \) couplings, renormalize multiplicatively as a subset. These results are expected from symmetry arguments and hold at any order in perturbation theory, thus making the ultraweak limit technically natural. As expected, the quartic singlet coupling \( \lambda_3 \) scales quadratically with the ultraweak couplings which makes it natural to assume the dominance of loop corrections in the CW potential in the singlet direction.
Appendix C: Invariants’ method for the vacuum stability analysis

In order to study the vacuum stability of the potential in Eq. (2) we employ the following polar representation

\[
|H_u| = r \sin \chi \sin \theta \cos \phi, \quad (C1)
\]
\[
|H_d| = r \sin \chi \sin \theta \sin \phi, \quad (C2)
\]
\[
|\sigma| = r \sin \chi \cos \theta, \quad (C3)
\]
\[
|\Delta| = r \cos \chi, \quad (C4)
\]

with \( r \in [0, \infty) \), \( \chi \in [0, \frac{\pi}{2}] \), \( \theta \in [0, \frac{\pi}{2}] \), \( \phi \in [0, \frac{\pi}{2}] \) and define the (normalized) invariants

\[
\zeta_4 = \frac{|H_u|^4|H_d|^2}{|H_u|^2|H_d|^2}; \quad (C5)
\]
\[
\zeta_5 = \frac{\Re(\sigma^2 H_u^3 H_d)}{|\sigma|^2 |H_u| |H_d|}; \quad (C6)
\]
\[
\zeta_6 = \frac{\Re(\sigma H_u^3 \Delta^3 H_d)}{|\sigma| |H_u| |\Delta| |H_d|}; \quad (C7)
\]

\[
V_0 = r^4 \left\{ (\lambda_{D4} + \zeta_5 \lambda_9) \cos^4 \chi + \left[ \lambda_{D3} \cos^2 \theta + (\lambda_{D1} + \zeta_7 \lambda_7) \cos^2 \phi + (\lambda_{D2} + \zeta_6 \lambda_8) \sin^2 \phi \right] \sin^2 \chi \sin^2 \theta 
+ 2 \zeta_6 \lambda_6 \cos \phi \sin \phi \cos \theta \sin^2 \theta \cos \chi \sin^3 \chi + \lambda_{D3} \cos^4 \theta + (\lambda_{D1} \cos^2 \phi + \lambda_{D2} \sin^2 \phi + 2 \zeta_5 \lambda_5 \cos \phi \sin \phi) \cos^2 \theta \sin^2 \theta 
+ (\lambda_1 \cos^4 \phi + \lambda_2 \sin^4 \phi + (\lambda_{12} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi) \sin^4 \theta \right\}. \quad (C11)
\]

The (tree-level) stability problem, \( V_0 > 0 \), is hence reduced to an inequality in terms of nine real variables spanning over a compact domain. Given a benchmark set of the scalar couplings (e.g., the one in Table II), we can check that the potential is bounded from below by means of a numerical scan over the potential parameters, which consist of three angles and six invariants.

In the following, we consider few specific field directions where the (sufficient) stability conditions in some limiting cases can be determined analytically.

1. DFSZ limit: (\( \chi = \frac{\pi}{4} \))

In this case the field \( \Delta \) is decoupled from the DFSZ model and the tree-level potential is given by

\[
V_0 = r^4 \cos^4 \theta \left\{ \lambda_3 + (\lambda_{13} \cos^2 \phi + \lambda_{23} \sin^2 \phi + 2 \zeta_5 \lambda_5 \cos \phi \sin \phi) \tan^2 \theta 
+ \left[ \lambda_1 \cos^4 \phi + \lambda_2 \sin^4 \phi + (\lambda_{12} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi \right] \tan^4 \theta \right\}. \quad (C12)
\]

The expression in the curly bracket is a biquadratic function of \( \tan \theta \in [0, \infty) \); this readily gives the following positivity constraints\(^3\)

\[^3\text{In order to determine the domain of } \zeta_9 \text{ it is useful to reexpress it as } \zeta_9 = 1 - \frac{1}{2} \frac{(D^4 \Delta^4)(\Delta^4 \Delta)}{|\Delta|^4}, \text{ where we used the definition in Eq. (4) and } \text{Tr}(\tau_i \tau_j \tau_k \tau_l) = 2(\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}).\]

\[^4\text{Given the polynomial } V(\chi) = a + bx^2 + cx^4, \text{ with } x \in [0, \infty), \text{ the condition } V(x) > 0 \text{ yields } a > 0, c > 0 \text{ and } b + 2\sqrt{ac} > 0.\]
\( i) \lambda_3 > 0 , \)
\( ii) \lambda_1 \cos^4 \phi + \lambda_2 \sin^4 \phi + (\lambda_{12} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi > 0 , \)  
\( iii) \lambda_{13} \cos^2 \phi + \lambda_{23} \sin^2 \phi + 2 \zeta_5 \lambda_5 \cos \phi \sin \phi + 2 \sqrt{\lambda_3} (\lambda_1 \cos^4 \phi + \lambda_2 \sin^4 \phi + (\lambda_{12} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi) > 0 . \)  
\( \text{(C13)} \)
\( \text{(C14)} \)
\( \text{(C15)} \)

Let us consider the conditions \( ii) \) and \( iii \). By factorizing \( \cos^4 \phi \) out of \( ii \) we get again a biquadratic form in \( \tan \phi \in [0, \infty) \)
\( \lambda_1 + (\lambda_{12} + \zeta_4 \lambda_4) \tan^2 \phi + \lambda_2 \tan^4 \phi > 0 , \)  
which yields the constraints
\( iia) \lambda_1 > 0 , \)
\( iib) \lambda_2 > 0 , \)
\( iic) \lambda_{12} + \zeta_4 \lambda_4 + 2 \sqrt{\lambda_1 \lambda_2} > 0 . \)  
\( \text{(C16)} \)
\( \text{(C17)} \)
\( \text{(C18)} \)
\( \text{(C19)} \)

In particular, since \( \zeta_4 \in [0, 1] \), \( iic \) leads (by considering \( \lambda_4 \) either positive or negative) to
\( iic1) \lambda_{12} + 2 \sqrt{\lambda_1 \lambda_2} > 0 , \)
\( iic2) \lambda_{12} + \lambda_4 + 2 \sqrt{\lambda_1 \lambda_2} > 0 . \)  
\( \text{(C20)} \)
\( \text{(C21)} \)

We are hence left with discussing \( iii \). Although we were not able to solve it analytically, it is still useful to consider some specific field directions. For instance, for \( \phi = 0, \pi/2, \pi/4 \) we get, respectively:
\( iia) \lambda_{13} + 2 \sqrt{\lambda_1 \lambda_3} > 0 , \)
\( iib) \lambda_{23} + 2 \sqrt{\lambda_2 \lambda_3} > 0 , \)
\( iic) \lambda_{13} + \lambda_{23} + 2 \zeta_5 \lambda_5 + 2 \sqrt{\lambda_1 (\lambda_1 + \lambda_2 + \lambda_{12} + \zeta_4 \lambda_4)} > 0 . \)  
\( \text{(C22)} \)
\( \text{(C23)} \)
\( \text{(C24)} \)

Notice that \( \phi = \pi/4 \) maximizes the contribution to the invariant \( \zeta_5 \). However this choice does not need to yield the most restrictive condition on \( \lambda_3 \), since other couplings might be magnified in other directions. By taking into account the uncorrelated ranges of \( \zeta_4 \in [0, 1] \) and \( \zeta_5 \in [-1, 1] \) (thus leading to a sufficient condition), we can further expand \( iiiic \) into
\( iiiic1) |\lambda_5| < \frac{1}{2} (\lambda_{13} + \lambda_{23}) + \sqrt{\lambda_3 (\lambda_1 + \lambda_2 + \lambda_{12})} , \)
\( iiiic2) |\lambda_5| < \frac{1}{2} (\lambda_{13} + \lambda_{23}) + \sqrt{\lambda_3 (\lambda_1 + \lambda_2 + \lambda_{12} + \lambda_4)} . \)  
\( \text{(C25)} \)
\( \text{(C26)} \)

2. **Type-II seesaw limit:** \( (\theta, \phi) = (\frac{\pi}{2}, 0) \)

This limit corresponds to an effective Type-II seesaw model where only the field directions \( \Delta \) and \( H_d \) are considered in the potential, which reads
\( V_0 = r^4 \cos^4 \chi (\lambda_{\Delta 4} + \zeta_9 \lambda_9) + (\lambda_{\Delta 1} + \zeta_7 \lambda_7) \tan^2 \chi + \lambda_1 \tan^4 \chi . \)  
\( \text{(C27)} \)

The positivity constraints are hence
\( i) \lambda_1 > 0 , \)
\( ii) \lambda_{\Delta 4} + \zeta_9 \lambda_9 > 0 , \)
\( iii) \lambda_{\Delta 1} + \zeta_7 \lambda_7 + 2 \sqrt{\lambda_1 (\lambda_{\Delta 4} + \zeta_9 \lambda_9)} > 0 . \)  
\( \text{(C28)} \)
\( \text{(C29)} \)
\( \text{(C30)} \)

Given the (uncorrelated) domains of \( \zeta_7 \in [0, 1] \) and \( \zeta_9 \in [\frac{1}{2}, 1] \) we find the (sufficient) conditions \( [19] \)
\( iia) \lambda_{\Delta 4} + \frac{1}{2} \lambda_9 > 0 , \)
\( iib) \lambda_{\Delta 4} + \lambda_9 > 0 , \)  
\( \text{(C31)} \)
\( \text{(C32)} \)

and
\( iia) \lambda_{\Delta 1} + \lambda_7 + 2 \sqrt{\lambda_1 (\lambda_{\Delta 4} + \frac{1}{2} \lambda_9)} > 0 , \)
\( iib) \lambda_{\Delta 1} + 2 \sqrt{\lambda_1 (\lambda_{\Delta 4} + \frac{1}{2} \lambda_9)} > 0 , \)
\( iic) \lambda_{\Delta 1} + \lambda_7 + 2 \sqrt{\lambda_1 (\lambda_{\Delta 4} + \lambda_9)} > 0 , \)
\( iidd) \lambda_{\Delta 1} + 2 \sqrt{\lambda_1 (\lambda_{\Delta 4} + \lambda_9)} > 0 . \)  
\( \text{(C33)} \)
\( \text{(C34)} \)
\( \text{(C35)} \)
\( \text{(C36)} \)

Analogous results are obtained by projecting along the \( \Delta–H_d \) direction \( (\theta, \phi) = (\frac{\pi}{2}, \frac{\pi}{2}) \), which entails the replacements \( \lambda_1 \to \lambda_2, \lambda_7 \to \lambda_8 \) and \( \lambda_{\Delta 1} \to \lambda_{\Delta 2} \) in conditions \( iiiic–d \).

3. **2HDM Type-II seesaw limit:** \( (\theta = \frac{\pi}{2}) \)

In this limit the singlet is decoupled and we get
\[ V_0 = r^4 \cos^4 \chi \left\{ (\lambda_{\Delta 4} + \zeta_9 \lambda_9) + [(\lambda_{\Delta 1} + \zeta_7 \lambda_7) \cos^2 \phi + (\lambda_{\Delta 2} + \zeta_8 \lambda_8) \sin^2 \phi] \tan^2 \chi \right. \\
\left. + [\lambda_1 \cos^4 \phi + (\lambda_{\Delta 2} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi + \lambda_2 \sin^4 \phi] \tan^4 \chi \right\}, \]  
\tag{C37}
which yields the positivity conditions
\[ i) \lambda_{\Delta 4} + \zeta_9 \lambda_9 > 0, \tag{C38} \]
\[ ii) \lambda_1 \cos^4 \phi + (\lambda_{\Delta 2} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi + \lambda_2 \sin^4 \phi > 0, \tag{C39} \]
\[ iii) (\lambda_{\Delta 1} + \zeta_7 \lambda_7) \cos^2 \phi + (\lambda_{\Delta 2} + \zeta_8 \lambda_8) \sin^2 \phi \]
\[ + 2 \sqrt{(\lambda_{\Delta 4} + \zeta_9 \lambda_9) (\lambda_1 \cos^4 \phi + (\lambda_{\Delta 2} + \zeta_4 \lambda_4) \cos^2 \phi \sin^2 \phi + \lambda_2 \sin^4 \phi)} > 0. \tag{C40} \]

Note that condition i) is equivalent to
\[ iia) \lambda_{\Delta 4} + \frac{1}{2} \lambda_9 > 0, \tag{C41} \]
\[ iib) \lambda_{\Delta 4} + \lambda_9 > 0, \tag{C42} \]
while ii) yields
\[ iia) \lambda_1 > 0, \tag{C43} \]
\[ iib) \lambda_2 > 0, \tag{C44} \]
\[ iic) \lambda_{\Delta 2} + \zeta_4 \lambda_4 + 2 \sqrt{\lambda_1 \lambda_2} > 0. \tag{C45} \]
The latter can be further expanded into
\[ iic1) \lambda_{\Delta 2} + 2 \sqrt{\lambda_1 \lambda_2} > 0, \tag{C46} \]
\[ iic2) \lambda_{\Delta 2} + \lambda_4 + 4 \sqrt{\lambda_1 \lambda_2} > 0. \tag{C47} \]

Condition iii) for \( \phi = 0 \) and \( \frac{\pi}{2} \) has been already considered in the previous section.

4. Single Higgs doublet limit: (\( \phi = 0 \))

In this case \( H_d \) is decoupled from the scalar potential, which reads
\[ V_0 = r^4 \cos^4 \chi \left\{ (\lambda_{\Delta 4} + \zeta_9 \lambda_9) + [(\lambda_{\Delta 3} \cos^2 \theta + (\lambda_{\Delta 1} + \zeta_7 \lambda_7) \sin^2 \theta] \tan^2 \chi \right. \\
\left. + [\lambda_3 \cos^4 \theta + \lambda_{13} \cos^2 \theta \sin^2 \theta + \lambda_1 \sin^4 \theta] \tan^4 \chi \right\}, \]  
\tag{C48}
thus yielding the constraints
\[ i) \lambda_{\Delta 4} + \zeta_9 \lambda_9 > 0, \tag{C49} \]
\[ ii) \lambda_3 \cos^4 \theta + \lambda_{13} \cos^2 \theta \sin^2 \theta + \lambda_1 \sin^4 \theta > 0, \tag{C50} \]
\[ iii) \lambda_{\Delta 3} \cos^2 \theta + (\lambda_{\Delta 1} + \zeta_7 \lambda_7) \sin^2 \theta + 2 \sqrt{(\lambda_{\Delta 4} + \zeta_9 \lambda_9) (\lambda_3 \cos^4 \theta + \lambda_{13} \cos^2 \theta \sin^2 \theta + \lambda_1 \sin^4 \theta)} > 0. \tag{C51} \]

After further expanding i) and ii), we get
\[ iia) \lambda_3 > 0, \tag{C54} \]
\[ iib) \lambda_1 > 0, \tag{C55} \]
\[ iic) \lambda_{13} + 2 \sqrt{\lambda_1 \lambda_3} > 0. \tag{C56} \]

The case \( \phi = \frac{\pi}{2} (H_u \text{ decoupled}) \) is obtained by replacing \( \lambda_1 \rightarrow \lambda_2 \) and \( \lambda_{13} \rightarrow \lambda_{23} \) in the inequalities above.

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