Quark Masses
from the Linear Meson Model

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Abstract

Quark mass ratios are expressed within the linear meson model by universal relations involving only the masses and decay constants of the flavored pseudoscalars as well as their wave function renormalization. Quantitative results are in agreement with those obtained from chiral perturbation theory, with a tendency to a somewhat higher strange quark mass.
Chiral perturbation theory can predict the ratios of (current) quark masses with a very satisfactory accuracy [1]-[3]. Nevertheless, some assumptions about the convergence of the quark mass expansion have to be made since corrections quadratic in the quark masses are usually neglected. Even though these assumptions are quite reasonable it would be valuable to have an independent check of their validity. This can be provided within a linear meson model. In this model a complex $3 \times 3$ matrix $\Phi$ describes simultaneously the pseudoscalar ($O^-+\bar{\xi}$) octet and singlet as well as the scalar ($O^{++}$) octet and singlet. The meson decay constants are related to the expectation values of the unflavored scalars which constitute the real diagonal part of $\Phi$. In presence of quark masses these expectation values also determine the flavored meson masses [4]. One therefore expects relations between the masses and decay constants of the flavored mesons and the quark masses. We essentially exploit only symmetry properties and work within the framework of an effective action. This generates the 1PI-Green functions and all quantum fluctuations are supposed to be included in the effective coupling constants.

We start with the most general form of the effective action consistent with the flavor symmetry $SU_L(3) \times SU_R(3)$ as well as parity and charge conjugation

$$\Gamma[\Phi] = \int d^4x \left\{ L_{\text{kin}} + U - \frac{1}{2} \text{Tr}(\Phi^\dagger j + j^\dagger \Phi) \right\}.$$  \hspace{1cm} (1)

Here the kinetic term $L_{\text{kin}}$ contains all terms involving derivatives of $\Phi$, $U$ is the effective potential and the source term describes the response to non-vanishing quark masses. We work here within a formalism where $\Phi$ represents a composite field for quark–antiquark states and all flavor symmetry breaking is cast in a linear coupling of $\Phi$ to the source term. We consider real and diagonal sources

$$j = \text{diag}(j_u, j_d, j_s)$$  \hspace{1cm} (2)

$$j_q = 2Cm_q$$

where the current quark masses $m_q$ are evaluated at a convenient scale (say in the $\overline{MS}$ scheme at $\mu = 1$ GeV). The minimum of $U - \frac{1}{2} \text{Tr}(\Phi + \Phi^\dagger)j$ occurs for real and diagonal $\Phi$,

$$\langle \Phi \rangle = \text{diag}(\langle \phi_u \rangle, \langle \phi_d \rangle, \langle \phi_s \rangle)$$  \hspace{1cm} (3)

such that the discrete symmetries $C$ and $P$ remain conserved. For a given form of the effective potential the expectation values are determined by the field equations.
for the real diagonal components of $\Phi$

$$\frac{\partial U}{\partial \varphi_q |_{\Phi}} = j_q.$$  

(4)

In turn, the unrenormalized mass matrix can now be inferred from the second derivatives of $U$ evaluated for $\Phi = \langle \Phi \rangle$. We will denote the eigenvalues for the flavored pseudoscalars by $M^2_{\pi \pm}$, $M^2_{K \pm}$ and $M^2_{K^0}$.

In order to connect the (zero momentum) mass terms $M^2_i$ to the physical pole masses $M^2_i$ one also needs information contained in the kinetic terms. For the flavored pseudoscalars there is no mixing and we can always write the most general momentum dependence of their inverse propagators as

$$G_i^{-1}(q) = M^2_i + Z_i q^2 + h_i(q^2).$$  

(5)

Here the normalization conditions for $M^2_i$ and $Z_i$ are formulated as $h_i(-M^2_i) = h_i(0) = 0$ implying the simple relation

$$M^2_i = M^2_i / Z_i.$$  

(6)

The decay constants of the flavored pseudoscalars are defined by their leptonic decays and can again be expressed in terms of $\langle \varphi_q \rangle$ and $Z_i$

$$f_\pi = Z^\frac{1}{2}_\pi (\langle \varphi_u \rangle + \langle \varphi_d \rangle) = Z^\frac{1}{2}_\pi \bar{f}_\pi$$

$$f_{K \pm} = Z^\frac{1}{2}_{K \pm} (\langle \varphi_u \rangle + \langle \varphi_s \rangle) = Z^\frac{1}{2}_{K \pm} \bar{f}_{K \pm}$$

$$f_{K^0} = Z^\frac{1}{2}_{K^0} (\langle \varphi_d \rangle + \langle \varphi_s \rangle) = Z^\frac{1}{2}_{K^0} \bar{f}_{K^0}.$$  

(7)

For a given effective potential, say, for example,

$$U = m^2_g (\rho - 3\sigma_0^2) - \frac{1}{2} \bar{\nu}(\xi - \sigma_0 \rho + \sigma_0^2) + \frac{1}{2} \lambda_1 (\rho - 3\sigma_0^2)^2 + \frac{1}{2} \lambda_2 \tau_2 + \frac{1}{2} \lambda_3 \tau_3$$

$$\rho = \text{Tr} \Phi^+ \Phi, \quad \tau_2 = \frac{3}{2} \text{Tr} (\Phi^+ \Phi - \frac{1}{3} \rho)^2$$

$$\tau_3 = \text{Tr} (\Phi^+ \Phi - \frac{1}{3} \rho)^3, \quad \xi = \text{det} \Phi + \text{det} \Phi^+$$  

(8)

one can now relate $M^2_i$ and $\bar{f}_i$ to the quark masses. Hereby the field equations (4) and the expressions for $M^2_i$ become rather lengthy expressions involving the parameters $m^2_g, \sigma_0, \bar{\nu}, \bar{\lambda}_1, \bar{\lambda}_2$ and $\bar{\lambda}_3$. For most of the mesons described by $\Phi$ the exact relations...
between the meson and quark masses become quite involved and need a solution of
the field equation expressing \( \langle \varphi_q \rangle \) in terms of \( m_q \).

The case of the flavored pseudoscalars, however, turns out to be special. For
arbitrary values of the parameters \( m_q^2, \sigma_0, \bar{\nu}, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3 \) and arbitrary strength of the
sources \( j_q \) we find the simple exact relations

\[
\begin{align*}
M_{\pi^\pm}^2 \bar{f}_\pi &= \frac{1}{2} (j_u + j_d) = C(m_u + m_d) \\
M_{K^\pm}^2 \bar{f}_{K^\pm} &= \frac{1}{2} (j_u + j_s) = C(m_u + m_s) \\
M_{K^0}^2 \bar{f}_{K^0} &= \frac{1}{2} (j_d + j_s) = C(m_d + m_s).
\end{align*}
\]  

(9)

These relations are well known in the leading order in chiral perturbation theory for
\( m_q \to 0 \) but it may perhaps surprise that there are no corrections in higher orders
in the quark masses. In fact, a simple exercise in group theory shows that the
relations (9) are exact for an arbitrary form of the effective potential \( U \). Consider
first an \( SO(N) \) symmetric theory where the potential depends on only one vector
\( \vec{\sigma} = (\sigma_1 \ldots \sigma_N) \) and an arbitrary number of singlets \( s_k, U = U(\rho, s_k) \). We assume
that \( U \) is analytic in \( \rho = \frac{1}{2} \vec{\sigma}^2 \) for arbitrary values of \( s_k \) and denote \( U' = \partial U / \partial \rho \) etc.
The \( SO(N) \) breaking source is taken in the one-direction such that the source term
reads \( \sigma_1 j_1 + \sum_k s_k j_k \). The field equations for \( \sigma_a \)

\[
U' \sigma_a = j_1 \delta_{a1}
\]  

(10)

admit for \( j_1 \neq 0 \) only the solution

\[
\begin{align*}
\langle \sigma_a \rangle &= 0 \quad \text{for} \quad a \neq 1 \\
\langle \sigma_1 \rangle &= j_1 / U'
\end{align*}
\]  

(11)

where \( U' \) is evaluated at the expectation value for \( s_k \) and \( \sigma_a \). Because of the re-
mainning symmetry \( \langle \sigma_a \to -\sigma_a \rangle \) for \( a \neq 1 \) and \( SO(N-1) \) symmetry for \( N \geq 3 \) the
mass matrix involves no mixing of the “Goldstone modes” \( \sigma_{a \neq 1} \) with \( \sigma_1 \) or \( s_k \). We
can therefore consider the restricted matrix

\[
\begin{align*}
M_{ab}^2 &= U' \delta_{ab} \quad \text{for} \quad a, b \neq 1.
\end{align*}
\]  

(12)

Comparison with (11) yields for the eigenvalues the simple relation

\[
\begin{align*}
\langle \sigma_1 \rangle &= \frac{j_1}{\langle \sigma_1 \rangle}
\end{align*}
\]  

(13)
In order to exploit this fact for our case of a $SU_L(3) \times SU_R(3)$ invariant potential we consider first the subgroup $SU_L(2) \times SU_R(2) \simeq SO(4)$ acting on the $u$- and $d$-components. Decomposing $\Phi$ one finds two vectors $(\pi_1, \pi_2, \pi_3, \sigma_\pi)$ and $(a_1, a_2, a_3, \eta_a)$ where $\pi$ corresponds to the isospin-triplet of the pseudoscalar pions and $\eta$ denotes the isotriplet contained in the scalar $(O^+)$ octet. The scalar $\sigma_\pi = \text{Re}(\Phi_{uu} + \Phi_{dd}) = \varphi_u + \varphi_d$ and the pseudoscalar $\eta_a = \text{Im}(\Phi_{uu} + \Phi_{dd})$ are isospin singlets. Furthermore, the strange mesons belong to doublets and the rest are singlets with respect to $SO(4)$. This $SO(4)$ group is not yet sufficient for our purpose since both $j_u + j_d$ and $j_u - j_d$ act as symmetry breaking terms. We will therefore concentrate on the $SO(3)$ subgroup under which $(\sigma_\pi, \pi_1, \pi_2)$ and $(\eta_a, a_1, a_2)$ transform as vectors.

With respect to this subgroup the sources $j_s$ and $j_u - j_d$ are singlets and the only symmetry breaking term is $j_u + j_d$. Omitting for a moment the other triplet and the strange mesons we find precisely the situation described above and the relation (13) becomes equivalent to the first relation in (9). It remains only to be shown that the strange mesons which belong to two-component spinor representations of $SO(3)$ and the vector $(\eta_a, a_1, a_2)$ do not disturb this setting. First we note that for arbitrary $\langle \varphi_u \rangle, \langle \varphi_d \rangle, \langle \varphi_s \rangle$ the expectation values of these fields vanish due to symmetries (strangeness conservation for $K$, electric charge conservation for $a_1, a_2$, parity for $\eta_a$). They do therefore not affect the field equations for $(\sigma_\pi, \pi_1, \pi_2)$. Furthermore, the symmetries forbid any mixing of these fields with $(\sigma_\pi, \pi_1, \pi_2)$. Therefore the mass matrix for $(\sigma, \pi_1, \pi_2)$ is not modified by the presence of these fields either. This establishes the first relation in (9) as an exact relation independent of the specific form of $U$ and the strength of $j_u + j_d$. The two other relations follow immediately by considering appropriately rotated subgroups which are obtained from the one discussed above by the substitutions $(u \leftrightarrow s)$ or $(d \leftrightarrow s)$.

Using (9), (6) and (7) the ratios of current quark masses can now be inferred from the exact relations

\[ \frac{m_u + m_s}{m_u + m_d} = \frac{M_{K^\pm}^2 f_{K^\pm}}{M_{\pi^\pm}^2 f_\pi} \left( \frac{Z_{K^\pm}}{Z_{\pi^\pm}} \right)^{\frac{1}{2}} \]
\[ \frac{m_u + m_s}{m_d + m_s} = \frac{M_{K^\pm}^2 f_{K^\pm}}{M_{K^0}^2 f_{K^0}} \left( \frac{Z_{K^\pm}}{Z_{K^0}} \right)^{\frac{1}{2}}. \]

(14)

Beyond the electromagnetically corrected meson masses $M_{\pi^\pm} = 135.1\text{MeV}$, $M_{K^0} = 497.7\text{MeV}$, $M_{K^\pm} = (491.7 \pm 0.4)\text{MeV}$ (corresponding to $Q = 22.7 \pm 0.8$ in [3]) these
relations involve the decay constants, $f_\pi = 92.4\text{MeV}$, $f_{K^\pm} = 113\text{MeV}$, and ratios of wave function renormalization constants. Within the linear meson model the isospin violating ratios $f_{K^\pm}/f_{K^0}$ and $Z_{K^\pm}/Z_{K^0}$ can be computed as functions of $M_i^2, f_i$ and $Z_{K^\pm}/Z_{\pi^\pm}$. The ratio $Z_{K^\pm}/Z_{\pi^\pm}$ may then be related to the mixing in the $\eta - \eta'$-sector and therefore to the decay constants $f_\eta$ and $f_{\eta'}$. We use from ref. [4] the range of values

$$
\frac{Z_{K^\pm}}{Z_{\pi^\pm}} = 0.7085 - 0.7527
$$

and

$$
\frac{Z_{K^\pm}}{Z_{K^0}} = (1.00775 \pm 0.00054) - (1.00657 \pm 0.00046)
$$

$$
\frac{f_{K^\pm}}{f_{K^0}} = (0.99779 \pm 0.00015) - (0.99725 \pm 0.00019)
$$

Here, the errors in parenthesis corresponds to the uncertainty in the electromagnetically corrected mass $M_{K^\pm} = (491.7 \pm 0.4)\text{MeV}$. One finds

$$
\frac{m_u}{m_d} = (0.526 \pm 0.025) - (0.497 \pm 0.026), \quad [0.533 \pm 0.043]
$$

$$
\frac{m_s}{m_d} = (20.29 \pm 0.35) - (20.55 \pm 0.37), \quad [18.9 \pm 0.8]
$$

$$
\frac{m_s}{m_u} = (38.60 \pm 1.17) - (41.4 \pm 1.4), \quad [34.4 \pm 3.7]
$$

The first two values correspond to the two values of $Z_{K^\pm}/Z_{\pi^\pm}$ given in [5] whereas the error of each value (given in parenthesis) indicates again the uncertainty arising from the electromagnetic corrections to the mass difference $M_{K^0} - M_{K^\pm}$ (same notation as in [4]), In square brackets we have also quoted the results of a recent analysis from chiral perturbation theory [3]. The agreement is satisfactory, with a somewhat lower value of $m_s$ in chiral perturbation theory. We also note that the combinations

$$
\frac{f_{K^\pm}}{f_\pi} \left( \frac{Z_{K^\pm}}{Z_{\pi^\pm}} \right)^{\frac{1}{2}} = 1.03 - 1.06
$$

$$
\frac{f_{K^\pm}}{f_{K^0}} \left( \frac{Z_{K^\pm}}{Z_{K^0}} \right)^{\frac{1}{2}} = (1.00165 \pm 0.00012) - (1.00053 \pm 0.00004)
$$

are very close to one and corrections to the leading order relation $(m_u + m_s)/(m_u + m_d) = M_{K^\pm}/M_{\pi^\pm}$ turn therefore out to be small.
For an estimate of the error and for comparison with the results from chiral
perturbation theory it is useful to investigate the ratio

\[
\frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2} \frac{M_{K^0}^2 - M_\pi^2}{M_{K^0}^2 - M_{K^\pm}^2} (1 + \delta_Q) = Q^2 (1 + \delta_Q)
\]

(18)

where \(\hat{m} = (m_u + m_d)/2\), \(M_K^2 = (M_{K^0}^2 + M_{K^\pm}^2)/2\), \(f_K = (f_{K^0} + f_{K^\pm})/2\), \(Z_K = (Z_{K^\pm} + Z_{K^\pm})/2\), \(M_\pi^2 = M_{\pi^\pm}^2\) and (omitting negligible higher order isospin breaking
effects)

\[
\delta_Q = \frac{f_K}{f_\pi} \left( \frac{Z_K}{Z_\pi} \right)^{1/2} \left[ 1 + \frac{2(m_s + \hat{m})}{m_d - m_u} \left( 1 - \frac{f_K}{f_{K^\pm}} \left( \frac{Z_K}{Z_{K^\pm}} \right)^{1/2} \right) \right]^{-1} - 1.
\]

(19)

To first order in the quark mass expansion one has the relations

\[
\frac{Z_{K^\pm} - Z_K}{Z_K - Z_\pi} = \frac{f_{K^\pm} - f_K}{f_K - f_\pi} = -\frac{1}{2} \frac{m_d - m_u}{m_s - \hat{m}}
\]

(20)

and \(\delta_Q\) vanishes, consistent with the result from chiral perturbation theory. Using
the values \([3]\) quoted from ref. \([4]\) one finds numerically \(\delta_Q \approx 0.11 - 0.09\). Even
though formally of second order in the quark mass expansion this is a sizeable
correction. It can be explained by the relatively large deviation of \(\bar{f}_K/\bar{f}_\pi = 1.45 -
1.41\) from the lowest order value one. The convergence of the expansion in the strange
quark mass for the coefficients of the isospin violating contributions is particularly
slow \([4]\). For fixed \(m_u/m_d\) the positive value of \(\delta_Q\) enhances \(m_s/\hat{m}\) as compared to
first order chiral perturbation theory, thus explaining the tendency in eq. (16).

For the second independent ratio we choose (with \(R = (m_s - \hat{m})/(m_d - m_u)\))

\[
\frac{m_s + \hat{m}}{\hat{m}} = 2 \frac{M_K^2}{M_\pi^2} \frac{f_K}{f_\pi} \left( \frac{Z_K}{Z_\pi} \right)^{1/2} = 32.9 \left( \frac{Z_K}{Z_\pi} \right)^{1/2} = \frac{2Q^2 (1 + \delta_Q)}{R}.
\]

(21)

The error in this ratio is dominated by the uncertainty in \(Z_K/Z_\pi\). With a rather
conservative error of 15% for \(Z_K/Z_\pi\) we find

\[
\frac{m_s}{\hat{m}} = 27.0 \pm 2.0.
\]

(22)

This value turns out slightly higher than the estimate 24.4 \(\pm\) 1.5 from chiral per-
turbation theory \([3]\). Our central value corresponds to \(R \approx 43\). We observe that
in contrast to chiral perturbation theory our estimate does not need any additional assumptions beyond the extraction of the ratio $Z_K/Z_\pi$ from the two photon decays of $\eta$ and $\eta'$. Since this determination is entirely different from the one used in [3] the agreement of the two estimates is rather encouraging!

The absolute value of the quark masses needs the constant $C$ in eq. (11). Since the current quark masses are normalized at a given scale (say $\mu = 1$ GeV in the $\overline{MS}$ scheme) the same holds for $C$. Equating the flavor symmetry breaking term in the quark - and meson - language leads to a relation for the quark condensate $\langle \bar{q}q \rangle$

$$\langle \bar{q}q \rangle m_q = - (\langle \varphi_q \rangle - m_q) j_q.$$  \hspace{1cm} (23)

We use this relation for the up and down quarks and neglect isospin violation

$$C = -\frac{1}{2} (\langle \bar{u}u \rangle + \langle \bar{d}d \rangle) \frac{Z_\pi^1/2}{f_\pi - 2 \hat{m}Z_\pi^1/2} = (340 - 410)^2 \text{MeV}^2 Z_\pi^1/2.$$  \hspace{1cm} (24)

For the last equation we have taken a standard estimate from sum rules $\frac{1}{2} (< \bar{u}u > + < \bar{d}d >) = -(225 \pm 25)^3 \text{MeV}^3$ and neglected the correction $\sim \hat{m} Z_\pi^1/2$. Combining this with eq. (9) yields

$$m_s(1 \text{ GeV}) = (136 - 198) \text{ MeV}$$  \hspace{1cm} (25)

The error is dominated by the uncertainty in the value of the quark condensate. Conversely, any other independent estimate of $m_u + m_d$ or $m_s$ can be used to fix $C$ and predict the value of the quark condensate. Recent lattice estimates [5] seem to favor a value $\hat{m} = (2.9 \pm 0.5)$MeV. This would imply

$$C = (545 \pm 47)^2 \text{MeV}^2 Z_\pi^1/2$$  \hspace{1cm} (26)

$$\frac{1}{2} \left< \bar{u}u + \bar{d}d \right> = -(295 \pm 19)^3 \text{MeV}^3$$

$$m_s = (78 \pm 15) \text{MeV}.$$  

In summary, the quark mass ratios are related in the linear meson model to the masses and decay constants of the flavored mesons and their respective wave function renormalization. These relations are independent of all other parameters of the effective linear meson model. We use an earlier estimate of the different wave function renormalizations for $\pi^\pm, K^\pm$ and $K^0$ based on the two photon decay
width of the \( \eta \) and \( \eta' \). This yields quark mass ratios that resemble very closely the ones predicted from chiral perturbation theory. The two estimates are based on entirely independent experimental observations. We also compute the size of the higher order corrections which are omitted in present first order estimates from chiral perturbation theory. They amount typically to an enhancement of around 10 % for \( m_s/m_u \) and \( m_s/m_d \).

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