Controlled Operator Frames in Hilbert $C^*$-modules

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Abstract

Controlled frame has been the subject of interest because of its ability to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In this paper, we introduce the notion of controlled $K$-frame in Hilbert $C^*$-modules. We establish the equivalent condition for controlled $K$-frame. We investigate some operator theoretic characterizations of controlled $K$-frames and controlled bessel sequences. Moreover we establish the relationship between the $K$-frame and controlled $K$-frame. At the end we prove a perturbation result on controlled $K$-frame.

Keywords: Hilbert $C^*$-module, Frame, K-frame, Controlled frame

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1 Introduction

Frames as more flexible substitute of bases in Hilbert spaces were first proposed by Duffin and Schaeffer [6] in 1952 while studying nonharmonic Fourier series. Daubechies, Grossmann and Meyer [5] reintroduced and developed the theory of frames in 1986. Due to their rich structure the subject draws the attention of many mathematician, physicists and engineers as it was largely applicable in signal processing [10], image processing [4], coding and communications [21], sampling [7, 8], numerical analysis, filter theory [3]. Nowadays it is used in compressive sensing, data analysis and other areas. In general frame can be viewed as a redundant representation of basis. Due to its redundancy it becomes more applicable not only in theoretic point of view but also in various kind of applications.

Hilbert $C^*$-modules are generalization of Hilbert spaces by allowing the inner product to take values in $C^*$-algebra rather than in the field of complex numbers. They were introduced and investigated by Kaplansky [16]. Frank and Larson [11] defined the concept of standard frames in finitely or countably generated Hilbert $C^*$ -modules over unital $C^*$ -algebra. For
more details of frames in Hilbert $C^*$-modules one may refer Doctoral Dissertation [15], Han et al. [14] and Han et al. [13]. In 2012, L. Gavruta [12] introduced the notions of $K$-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator $K$. Controlled frames in Hilbert spaces have been introduced by P. Balazs [2] to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Rahimi [19] defined the concept of controlled $K$-frames in Hilbert spaces and showed that controlled $K$-frames are equivalent to $K$-frames due to which the controlled operator $C$ can be used as preconditions in applications. In [18], Najati et al. introduced the concepts of atomic system for operators and $K$-frames in Hilbert $C^*$-modules. Controlled frames in Hilbert $C^*$-modules were introduced by Rashidi and Rahimi [17], and the authors showed that they share many useful properties with their corresponding notions in Hilbert space. Motivated from the above literature, we introduce the notion of controlled $K$-frame in Hilbert $C^*$-modules.

2 Preliminaries

In this section we give some basic definitions related to Hilbert $C^*$-modules, frames, $K$-frames, Controlled frames in Hilbert $C^*$-modules which we quote from the literature. Hilbert $C^*$-modules are generalization of Hilbert spaces by allowing the inner product to take values in $C^*$-algebra rather than $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let $A$ be a $C^*$-algebra. An inner product $A$-module is a complex vector space $\mathcal{H}$ such that
(i) $\mathcal{H}$ is a right $A$-module i.e there is a bilinear map
$$\mathcal{H} \times A \rightarrow A: (x, a) \mapsto x \cdot a$$
satisfying $(x \cdot a) \cdot b = x \cdot (ab)$ and $(\lambda x) \cdot a = x \cdot (\lambda a)$, and $x \cdot 1 = x$ where $A$ has a unit 1.
(ii) There is a map $\mathcal{H} \times \mathcal{H} \rightarrow A: (x, y) \mapsto \langle x, y \rangle$ satisfying
1. $\langle x, x \rangle \geq 0$
2. $\langle x, y \rangle^* = \langle y, x \rangle$
3. $\langle ax, y \rangle = a \langle x, y \rangle$
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
5. $\langle x, x \rangle = 0$ if and only if $x = 0$ (for every $x, y, z \in \mathcal{H}$, $a \in A$).

Definition 2.2. A Hilbert $C^*$-module over $A$ is an inner product $A$-module with the property that $(\mathcal{H}, \|\cdot\|_\mathcal{H})$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

Definition 2.3. ([15]) Let $A$ be a unital $C^*$-algebra and $j \in J$ be a finite or countable index set. A sequence $\{\psi_j\}_{j \in J}$ of elements in a Hilbert $A$-module $\mathcal{H}$ is said to be a frame if there
exist two constants $C, D > 0$ such that

$$C\langle f, f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \leq D\langle f, f \rangle, \forall f \in \mathcal{H}. \quad (2.1)$$

The frame $\{\psi_j\}_{j \in J}$ is said to be tight frame if $C = D$, and is said to be Parseval or a normalized tight frame if $C = D = 1$.

Suppose that $\{\psi_j\}_{j \in J}$ is a frame of a finitely or countably generated Hilbert $C^*$-module $\mathcal{H}$ over a unital $C^*$-algebra $\mathcal{A}$. The operator $T: \mathcal{H} \to l^2(\mathcal{A})$ defined by

$$Tf = \{(f, \psi_j)\}_{j \in J}$$

is called the analysis operator.

The adjoint operator $T^*: l^2(\mathcal{A}) \to \mathcal{H}$ is given by

$$T^*\{c_j\}_{j \in J} = \sum_{j \in J} c_j \psi_j$$

$T^*$ is called pre-frame operator or the synthesis operator.

By composing $T$ and $T^*$, we obtain the frame operator $S: \mathcal{H} \to \mathcal{H}$

$$Sf = T^*Tf = \sum_{j \in J} \langle f, \psi_j \rangle \psi_j. \quad (2.2)$$

**Definition 2.4.** [18] A sequence $\{\psi_j\}_{j \in J}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a $K$-frame ($K \in L(\mathcal{H})$) if there exist constants $C, D > 0$ such that

$$C\langle K^*f, K^*f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \leq D\langle f, f \rangle, \forall f \in \mathcal{H}. \quad (2.3)$$

**Definition 2.5.** [17] Let $\mathcal{H}$ be a Hilbert $C^*$-module and $C \in GL(\mathcal{H})$. A frame controlled by the operator $C$ or $C$-controlled frame in Hilbert $C^*$-module $\mathcal{H}$ is a family of vectors $\{\psi_j\}_{j \in J}$, such that there exist two constants $A, B > 0$ satisfying

$$A\langle f, f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \leq B\langle f, f \rangle, \forall f \in \mathcal{H}. \quad (2.4)$$

Likewise, $\{\psi_j\}_{j \in J}$ is called a $C$-$\mathcal{A}$-controlled Bessel sequence with bound $B$, if there exists $B > 0$ such that

$$\sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \leq B\langle f, f \rangle, \forall f \in \mathcal{H},$$

where the sum in the above inequalities converges in norm.

If $A = B$, we call $\{\psi_j\}_{j \in J}$ as $C$-$\mathcal{A}$-controlled tight frame, and if $A = B = 1$ it is called a $C$-controlled Parseval frame.
3 Controlled operator frames

For the rest of the paper we assume that $H$ is a Hilbert $C^*$-module over unital $C^*$-algebra $A$ with $A$-valued inner product $\langle.,.\rangle$ and norm $\|\|$. $L(H)$ denotes the set of all adjointable operators on Hilbert $C^*$-module $H$, and $GL^+(H)$ indicates the set of all bounded linear positive invertible operators on $H$ with bounded inverse. We define below the controlled operator frame or $C$-controlled $K$-frame on a Hilbert $C^*$-module $H$.

**Definition 3.1.** Let $H$ be a Hilbert $A$-module over a unital $C^*$-algebra, $C \in GL^+(H)$ and $K \in L(H)$. A sequence $\{\psi_j\}_{j \in J}$ in $H$ is said to be $C$-controlled $K$-frame if there exist two constants $0 < A \leq B < \infty$ such that

$$A\langle C^{\frac{1}{2}}K^*f,C^{\frac{1}{2}}Kf \rangle \leq \sum_{j \in J} \langle f,\psi_j \rangle \langle C\psi_j,f \rangle \leq B\langle f,f \rangle, \forall f \in H. \quad (3.4)$$

If $C = I$, the $C$-controlled $K$-frame $\{\psi_j\}_{j \in J}$ is simply $K$-frame in $H$ which was discussed in [18]. The sequence $\{\psi_j\}_{j \in J}$ is called a $C$-controlled Bessel sequence with bound $B$, if there exists $B > 0$ such that

$$\sum_{j \in J} \langle f,\psi_j \rangle \langle C\psi_j,f \rangle \leq B\langle f,f \rangle, \forall f \in H, \quad (3.5)$$

where the sum in the above inequalities converges in norm.

If $A = B$, we call this $C$-controlled $K$-frame a tight $C$-controlled $K$-frame, and if $A = B = 1$ it is called a Parseval $C$-controlled $K$-frame.

Let $\{\psi_j\}_{j \in J}$ be a $C$-controlled Bessel sequence for Hilbert module $H$ over $A$.

The operator $T : H \to l^2(A)$ defined by

$$Tf = \{\langle f,\psi_j \rangle\}_{j \in J}, f \in H \quad (3.6)$$

is called the **analysis operator**. The adjoint operator $T^* : l^2(A) \to H$ given by

$$T^*(\{c_j\}_{j \in J}) = \sum_{j \in J} c_j C\psi_j \quad (3.7)$$

is called **pre-frame operator or the synthesis operator**. By composing $T$ and $T^*$, we obtain the $C$-controlled frame operator $S_C : H \to H$ as

$$S_Cf = T^*Tf = \sum_{j \in J} \langle f,\psi_j \rangle C\psi_j. \quad (3.8)$$

We quote the following results from the literature that will be used in our work.

**Lemma 3.1.** [1] Let $A$ be a $C^*$-algebra. Let $U$ and $V$ be two Hilbert $A$-modules and $T \in \text{End}^*_A(U,V)$. Then the following statements are equivalent
1. $T$ is surjective.

2. $T^*$ is bounded below with respect to norm i.e there exists $m > 0$ such that $\|T^*f\| \geq m\|f\|$ for all $f \in U$.

3. $T^*$ is bounded below with respect to inner product i.e there exists $m > 0$ such that $\langle T^*f, T^*f \rangle \geq m\langle f, f \rangle$ for all $f \in U$.

**Lemma 3.2.** [20] Let $U$ and $V$ be Hilbert $A$-modules over a $C^*$-algebra $A$ and let $T : U \to V$ be a linear map. Then the following conditions are equivalent:

1. The operator $T$ is bounded and $A$-linear.

2. There exists $k \geq 0$ such that $\langle Tx, Tx \rangle \leq k\langle x, x \rangle$ holds for all $x \in U$.

**Theorem 3.1.** [9] Let $E$, $F$ and $G$ be Hilbert $A$-modules over a $C^*$-algebra $A$. Let $T \in L(E, F)$ and $T' \in L(G, F)$ with $R(T^*)$ be orthogonally complemented. Then the following statements are equivalent:

1. $T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$;

2. There exists $\mu > 0$ such that $\|T'^*z\| \leq \mu\|T^*z\|$ for all $z \in F$;

3. There exists $D \in L(G, E)$ such that $T' = TD$, that is the equation $TX = T'$ has a solution;

4. $R(T') \subseteq R(T)$.

For the rest of the paper we indicate that $S_C$ stands for the controlled frame operator as we have defined in (3.8), and $S$ stands for the classical frame operator in Hilbert $C^*$-module $H$ as defined in (2.2). In the following theorem, we establish an equivalence condition for $C$-controlled $K$-frame in a Hilbert $C^*$-module $H$.

**Theorem 3.2.** Let $H$ be a finitely or countably generated Hilbert $A$-module over a unital $C^*$ algebra $A$, $\{\psi_j\}_{j \in J} \subset H$ is a sequence, $C \in GL^+(H)$, $K \in L(H)$, $KC = CK$ and $\|K^*\| \geq 1$. Then $\{\psi_j\}_{j \in J}$ is a $C$-controlled $K$-frame in Hilbert $C^*$-module if and only if there exist constants $0 < A \leq B < \infty$ such that

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle\| \leq B\|f\|^2, \forall f \in H.$$  \hspace{1cm} (3.9)

**Proof.** ($\implies$) Obvious.

Now we assume that there exist constants $0 < A, B \leq \infty$ such that

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle\| \leq B\|f\|^2, \forall f \in H.$$
We prove that \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for Hilbert \( C^* \)-module \( \mathcal{H} \). As \( S \) and \( C \) are both positive operator, they are self adjoint. Thus we have
\[
A\|C^\frac{1}{2}K^*f\|^2 \leq \left\| \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \right\|
= \| (SCf, f) \| = \| (CSf, f) \| = \| (CS)\frac{1}{2}f, (CS)\frac{1}{2}f \|, \text{ as } SC = CS
= \| (CS)\frac{1}{2}f \|^2.
\] (3.10)

and
\[
A\|C^\frac{1}{2}K^*f\|^2 = A\|K^*C^\frac{1}{2}f\|^2 \geq A\|C^\frac{1}{2}f\|^2, \text{ as } \|K^*\| > 1.
\] (3.11)

Therefore, by using (3.10) and (3.11) we get
\[
A\|C^\frac{1}{2}f\|^2 \leq \| (CS)\frac{1}{2}f \|.
\]

Now by using Lemma 3.1 we have
\[
A\langle C^\frac{1}{2}f, C^\frac{1}{2}f \rangle \leq \langle (CS)\frac{1}{2}f, (CS)\frac{1}{2}f \rangle = \langle SCf, f \rangle.
\] (3.12)

Now using (3.12), we get
\[
\langle C^\frac{1}{2}K^*f, C^\frac{1}{2}K^*f \rangle \leq \frac{A}{\|K^*\|^2}\langle C^\frac{1}{2}f, C^\frac{1}{2}f \rangle \leq \frac{A}{\|K^*\|^2}\langle SCf, f \rangle,
\]

This implies that
\[
\frac{A}{\|K^*\|^2}\langle C^\frac{1}{2}K^*f, C^\frac{1}{2}K^*f \rangle \leq \langle SCf, f \rangle.
\] (3.13)

Since \( SC \) is positive, self adjoint and bounded \( A \)-linear map, we can write
\[
\langle S^\frac{1}{2}_C f, S^\frac{1}{2}_C f \rangle = \langle SCf, f \rangle = \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle
\]

and hence by using Lemma 3.2, there exist some \( B' \) such that
\[
\langle S^\frac{1}{2}_C f, S^\frac{1}{2}_C f \rangle \leq B' \langle f, f \rangle
\]
\[
\Rightarrow \langle SCf, f \rangle \leq B' \langle f, f \rangle, \forall f \in \mathcal{H}.
\] (3.14)

Therefore from (3.13) and (3.14), we conclude that \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled \( K \)-frame in Hilbert \( C^* \)-module \( \mathcal{H} \) with bounds \( \frac{A}{\|K^*\|^2} \) and \( B' \). \( \square \)

**Lemma 3.3.** Let \( C \in GL^+((\mathcal{H})) \), \( CS_C = SC \) and \( R(S^\frac{1}{2}_C) \subseteq R((CSR_C)^{\frac{1}{2}}) \) with \( R((CSR_C)^{\frac{1}{2}}) \) is orthogonally complemented. Then \( \|S^\frac{1}{2}_C f\|^2 \leq \lambda\|((CSR_C)^{\frac{1}{2}})\|^2 \) for some \( \lambda > 0. \)
Proof. By the assumption that \( R(\overline{S}^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}}) \) with \( R((S_C^{\frac{1}{2}})^*) \) orthogonally complemented. Then by using Theorem 3.1 there exists some \( \lambda > 0 \) such that

\[
(S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* \leq \lambda((CS_C)^{\frac{1}{2}})((CS_C)^{\frac{1}{2}})^*.
\]

This implies that

\[
\langle (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* f, f \rangle \leq \lambda \langle ((CS_C)^{\frac{1}{2}})((CS_C)^{\frac{1}{2}})^* f, f \rangle
\]

\[
\Rightarrow \| S_C^{\frac{1}{2}} f \|^2 \leq \lambda \| (CS_C)^{\frac{1}{2}} f \|^2, \forall f \in \mathcal{H}.
\]



Proposition 3.1. Let \( \{\psi_j\}_{j \in \mathbb{J}} \) be a sequence of a finitely or countably generated Hilbert \( A \)-module \( \mathcal{H} \) over a unital \( C^* \)-algebra \( A \). Suppose that \( C \) commutes with the controlled frame operator \( S_C \) and \( R(\overline{S}^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}}) \) with \( R((S_C^{\frac{1}{2}})^*) \) is orthogonally complemented. Then \( \{\psi_j\}_{j \in \mathbb{J}} \) is a \( C \)-controlled Bessel sequence with bessel bound \( B \) if and only if the operator \( U : l^2(A) \rightarrow \mathcal{H} \) defined by

\[
U \{a_j\}_{j \in \mathbb{J}} = \sum_{j \in \mathbb{J}} a_j C \psi_j
\]

is a well defined bounded operator from \( l^2(A) \) into \( \mathcal{H} \) with \( \|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\| \).

Proof. First, suppose that \( \{\psi_j\}_{j \in \mathbb{J}} \) is a \( C \)-controlled Bessel sequence with bessel bound \( B \). Therefore we have

\[
\| \sum_{j \in \mathbb{J}} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \| = \| \langle S_C f, f \rangle \| \leq B \| f \|^2, \forall f \in \mathcal{H}.
\]

We first show that \( U \) is a well-defined operator. For arbitrary \( n > m \), we have

\[
\left\| \sum_{j=1}^{n} a_j C \psi_j - \sum_{j=1}^{m} a_j C \psi_j \right\|^2 = \left\| \sum_{j=m+1}^{n} a_j C \psi_j \right\|^2
\]

\[
= \sup_{\|f\|=1} \left\| \sum_{j=m+1}^{n} \langle a_j C \psi_j, f \rangle \right\|^2
\]

\[
= \sup_{\|f\|=1} \left\| \sum_{j=m+1}^{n} a_j \langle C \psi_j, f \rangle \right\|^2
\]

\[
\leq \sup_{\|f\|=1} \left\| \sum_{j=m+1}^{n} a_j \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \right\| \left\| \sum_{j=m+1}^{n} a_j \right\|
\]
This shows that \( \sum_{j \in J} a_j C\psi_j \) is a Cauchy sequence which is convergent in \( \mathcal{H} \). Thus \( U(\{a_j\}_{j \in J}) \) is a well defined operator from \( l^2(A) \) into \( \mathcal{H} \).

For boundedness of \( U \), we consider

\[
\|U\{a_j\}_{j \in J}\|^2 = \sup_{\|f\|=1} \|\langle U\{a_j\}, f \rangle\|^2
\]

\[
= \sup_{\|f\|=1} \left\| \sum_{j \in J} a_j \langle C\psi_j, f \rangle \right\|^2
\]

\[
\leq \sup_{\|f\|=1} \left\| \sum_{j \in J} \langle f, C\psi_j \rangle \langle C\psi_j, f \rangle \right\| \left\| \sum_{j \in J} a_j^* \right\|
\]

\[
= \sup_{\|f\|=1} \left\| \sum_{j \in J} \langle f, C\psi_j \rangle C\psi_j, f \right\| \left\| \sum_{j \in J} a_j^* \right\|
\]

\[
= \sup_{\|f\|=1} \left\| (CS_C)f, (CS_C)^*f \right\| \left\| \sum_{j \in J} a_j^* \right\|
\]

\[
= \sup_{\|f\|=1} \| (CS_C)^{\frac{1}{2}} f \|^2 \|a_j\|^2
\]

\[
\leq B \|C^{\frac{1}{2}}\|^2 \|a_j\|^2 \leq B \|C^{\frac{1}{2}}\|^2 \|a_j\|^2.
\]

This implies that \( \|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\| \).

Now assume that \( U \) is well defined operator from \( l^2(A) \) into \( \mathcal{H} \) and \( \|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\| \). We now prove that \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled Bessel sequence with bessel bound \( B \).
For arbitrary \( f \in \mathcal{H} \) and \( \{a_j\} \in l^2(A) \), we have
\[
\langle f, U\{a_j\} \rangle = \langle f, \sum_{j \in J} a_j C\psi_j \rangle \\
= \langle \sum_{j \in J} a_j^* Cf, \psi_j \rangle \\
= \sum_{j \in J} \langle Cf, \psi_j \rangle a_j^*.
\]
Therefore we get
\[
\langle f, U\{a_j\} \rangle = \langle \{ \langle Cf, \psi_j \rangle \}, \{a_j\} \rangle.
\]
This implies that \( U \) is adjointable and \( U^* f = \{ \langle Cf, \psi_j \rangle \} \). Also, \( \|U\| = \|U^*\| \).

So we have
\[
\|U^* f\|^2 = \|U^* f, U^* f\| = \|UU^* f\| = \|CS_C f, f\| = \|\langle C \psi_j, f \rangle\|^2 \leq B\|C\psi\|^2. \tag{3.15}
\]
By using Lemma 3.3 we have \( \|S_C^{\frac{1}{2}} f\|^2 \leq \lambda \|\langle C \psi_j \rangle\|^2 \) for some \( \lambda > 0 \). Using (3.15) we get
\[
\|S_C^{\frac{1}{2}} f\|^2 \leq \lambda \|\langle C \psi_j \rangle\|^2 \leq \lambda B\|C\psi\|^2. \]
Therefore \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled Bessel sequence with bessel bound \( \lambda B\|C\psi\|^2 \).

**Proposition 3.2.** Let \( \{\psi_j\}_{j \in J} \) be a \( C \)-controlled \( K \)-frame in \( \mathcal{H} \). Then \( \text{ACKK}^* I \leq S_C \leq BI \).

**Proof.** Suppose \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled \( K \)-frame with bounds \( A \) and \( B \). Then
\[
A\langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \leq B\langle f, f \rangle, \forall f \in \mathcal{H}.
\]
\[
\Rightarrow A\langle CKK^* f, f \rangle \leq \langle SC f, f \rangle \leq B\langle f, f \rangle.
\]
\[
\Rightarrow \text{ACKK}^* I \leq S_C \leq BI. \hfill \Box
\]

**Proposition 3.3.** Let \( \{\psi_j\}_{j \in J} \) be a \( C \)-Controlled Bessel sequence in \( \mathcal{H} \) and \( C \in GL^+(\mathcal{H}) \). Then \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for \( \mathcal{H} \), if and only if there exists \( A > 0 \) such that \( CS \geq ACKK^* \).

**Proof.** The sequence \( \{\psi_j\}_{j \in J} \) is a Controlled \( K \)-frame for \( \mathcal{H} \) with frame bounds \( A, B \) and frame operator \( S_C \), if and only if
\[
A\langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \leq B\langle f, f \rangle, \forall f \in \mathcal{H}.
\]
\[
\Leftrightarrow A\langle CKK^* f, f \rangle \leq \langle SC f, f \rangle \leq B\langle f, f \rangle.
\]
\[
\Leftrightarrow A\langle CKK^* f, f \rangle \leq \langle CS f, f \rangle \leq B\langle f, f \rangle.
\]
\[
\Leftrightarrow ACKK^* I \leq SC. \hfill \Box
\]
Proposition 3.4. Let \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled Bessel sequence with bound \( D \). Let \( T \in L(\mathcal{H}) \) and \( CT = TC \). Then \( \{T\psi_j\}_{j \in J} \) is also \( C \)-controlled Bessel sequence with bound \( D\|T^*\|^2 \).

Proof. Suppose \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled Bessel sequence. Then we have
\[
\sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \leq B\langle f, f \rangle, \forall f \in \mathcal{H}.
\]
For every \( f \in \mathcal{H} \),
\[
\sum_{j \in J} \langle f, T\psi_j \rangle \langle CT\psi_j, f \rangle = \sum_{j \in J} \langle T^* f, \psi_j \rangle \langle TC\psi_j, f \rangle \\
= \sum_{j \in J} \langle T^* f, \psi_j \rangle \langle C\psi_j, T^* f \rangle \\
\leq D\langle T^* f, T^* f \rangle \\
\leq D\|T^*\|^2 \langle f, f \rangle.
\]
Thus \( \{T\psi_j\}_{j \in J} \) is also \( C \)-controlled Bessel sequence with bound \( D\|T^*\|^2 \).

Proposition 3.5. Let \( C \in GL^+(\mathcal{H}) \), \( K \in L(\mathcal{H}) \) with \( \|K^*\| > 1 \), and \( \{\psi_j\}_{j \in J} \) be a \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with lower and upper frame bounds \( A \) and \( B \), respectively. Then \( \{\psi_j\}_{j \in J} \) is a \( K \)-frame for \( \mathcal{H} \) with lower and upper frame bounds \( A\|C^{\frac{1}{2}}\|^2 \) and \( B\|C^{-\frac{1}{2}}\|^2 \), respectively.

Proof. Suppose \( \{\psi_j\}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with bound \( A \) and \( B \). Then by Theorem 3.2, we have
\[
A \|C^{\frac{1}{2}}K^*f\|^2 \leq \| \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \| \leq B\|f\|^2, \forall f \in \mathcal{H}.
\]
Now,
\[
A\|K^*f\|^2 = A\|C^{\frac{1}{2}}C^{\frac{1}{2}}K^*f\|^2 \\
\leq A\|C^{\frac{1}{2}}\|^2 \|C^{-\frac{1}{2}}K^*f\|^2 \\
\leq \|C^{\frac{1}{2}}\|^2 \| \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \|.
\]
This implies that \( A\|C^{\frac{1}{2}}\|^2 \|K^*f\|^2 \leq \| \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \|.\]
On the other hand for every $f \in \mathcal{H}$,

\[ \| \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \| = \| \langle Sf, f \rangle \| = \| \langle C^{-1}CSf, f \rangle \| = \| \langle (C^{-1}CS)^{\frac{1}{2}} f, (C^{-1}CS)^{\frac{1}{2}} f \rangle \| \leq \| (C^{-1}CS)^{\frac{1}{2}} f \|^2 \leq \| C^{-\frac{1}{2}} \|^2 \| (CS)^{\frac{1}{2}} f \|^2 = \| (CSf, f) \| \leq \| C^{-\frac{1}{2}} \|^2 \| BSf \|^2. \]

Therefore $\{ \psi_j \}_{j \in J}$ is a $K$-frame with lower and upper frame bounds $A\| C^{-\frac{1}{2}} \|^{-2}$ and $B\| C^{-\frac{1}{2}} \|^2$, respectively.

**Theorem 3.3.** Let $C \in GL^+(\mathcal{H})$, $\{ \psi_j \}_{j \in J}$ be a $C$-controlled $K$-frame for $\mathcal{H}$ with bounds $A$ and $B$. Let $M, K \in L(\mathcal{H})$ with $R(M) \subset R(K)$, $R(K^*)$ orthogonally complemented, and $C$ commutes with $M$ and $K$ both. Then $\{ \psi_j \}_{j \in J}$ is a $C$-controlled $M$-frame for $\mathcal{H}$.

**Proof.** Suppose $\{ \psi_j \}_{j \in J}$ is a $C$-controlled $K$-frame for $\mathcal{H}$ with bounds $A$ and $B$. Then

\[ A \langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \leq B \langle f, f \rangle, \quad \forall f \in \mathcal{H}. \tag{3.16} \]

Since $R(M) \subset R(K)$, from Theorem 3.1 there exists some $\lambda' > 0$ such that $MM^* \leq \lambda' KK^*$. So we have

\[ \langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle \leq \lambda' \langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle \]

Multiplying the above inequality by $A$, we get

\[ \frac{A}{\lambda} \langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle \leq A \langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle \]

From (3.16), we have

\[ \frac{A}{\lambda} \langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle \leq B \langle f, f \rangle, \quad \forall f \in \mathcal{H}. \]

Therefore $\{ \psi_j \}_{j \in J}$ is a $C$-controlled $M$-frame with lower and upper frame bounds $\frac{A}{\lambda}$ and $B$, respectively. \hfill \Box

**Theorem 3.4.** Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$ and $\{ \psi_j \}_{j \in J}$ be a $C$-controlled $K$-frame for $\mathcal{H}$ with lower and upper bounds $A$ and $B$, respectively. If $T \in L(\mathcal{H})$ with closed range such that $R(TK)$ is orthogonally complemented and $C, K, T$ commute with each other. Then $\{ T \psi_j \}_{j \in J}$ is a $C$-controlled $K$-frame for $R(T)$.
Proof. Suppose \( \{ \psi_j \}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with bound \( A \) and \( B \). Then
\[
A(C^{\frac{1}{2}}K^*f,C^{\frac{1}{2}}K^*f) \leq \sum_{j \in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle \leq B \langle f, f \rangle, \forall f \in \mathcal{H}.
\]
We know that if \( T \) has closed range then \( T \) has Moore-Penrose inverse \( T^\dagger \) such that \( TT^\dagger = T \) and \( T^\dagger TT^\dagger = T^\dagger \). So \( TT^\dagger|_{R(T)} = I_{R(T)} \) and \( (TT^\dagger)^* = I^* = I = TT^\dagger \).

We have
\[
\langle K^*C^{\frac{1}{2}}f,K^*C^{\frac{1}{2}}f \rangle = \langle (TT^\dagger)^*K^*C^{\frac{1}{2}}f,(TT^\dagger)^*K^*C^{\frac{1}{2}}f \rangle
\]
\[
= \langle T^\dagger T^*K^*C^{\frac{1}{2}}f,T^\dagger T^*K^*C^{\frac{1}{2}}f \rangle
\]
\[
\leq \| (T^\dagger)^* \|^2 \langle T^*K^*C^{\frac{1}{2}}f,T^*K^*C^{\frac{1}{2}}f \rangle.
\]
This implies that
\[
\| (T^\dagger)^* \|^2 \langle K^*C^{\frac{1}{2}}f,K^*C^{\frac{1}{2}}f \rangle \leq \langle T^*K^*C^{\frac{1}{2}}f,T^*K^*C^{\frac{1}{2}}f \rangle.
\] (3.17)

Since \( R(T^*K^*) \subset R(K^*T^*) \), by using Theorem 3.1 there exists some \( \lambda' > 0 \) such that
\[
\langle T^*K^*C^{\frac{1}{2}}f,T^*K^*C^{\frac{1}{2}}f \rangle \leq \lambda' \langle K^*T^*C^{\frac{1}{2}}f,K^*T^*C^{\frac{1}{2}}f \rangle.
\] (3.18)

Therefore, using (3.17) and (3.18) we get
\[
\sum_{j \in J} \langle f, T\psi_j \rangle \langle CT\psi_j, f \rangle = \sum_{j \in J} \langle T^*f, \psi_j \rangle \langle TC\psi_j, f \rangle
\]
\[
= \sum_{j \in J} \langle T^*f, \psi_j \rangle \langle C\psi_j, T^*f \rangle
\]
\[
\geq A \langle C^{\frac{1}{2}}K^*T^*f,C^{\frac{1}{2}}K^*T^*f \rangle
\]
\[
\geq \lambda' \langle T^*C^{\frac{1}{2}}K^*f,T^*C^{\frac{1}{2}}K^*f \rangle
\]
\[
\geq A \langle T^*C^{\frac{1}{2}}K^*f,T^*C^{\frac{1}{2}}K^*f \rangle.
\]

This gives the lower frame inequality for \( \{ T\psi_j \}_{j \in J} \). On the other hand by Proposition 3.1, \( \{ T\psi_j \}_{j \in J} \) is a \( C \)-controlled bessel sequence. So \( \{ T\psi_j \}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for \( R(T) \).

\[\square\]

**Proposition 3.6.** Let \( C \in GL^+(\mathcal{H}) \), \( K \in L(\mathcal{H}) \) with \( \| K^* \| > 1 \) and \( KC = CK \). Let \( \{ \psi_j \}_{j \in J} \) be a \( K \)-frame for \( \mathcal{H} \) with lower and upper frame bounds \( A \) and \( B \), respectively. Then \( \{ \psi_j \}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with lower and upper frame bounds \( A \) and \( B\|C\| \), respectively.

**Proof.** Suppose \( \{ \psi_j \}_{j \in J} \) is a \( K \)-frame with frame bounds \( A \) and \( B \). Then by equivalence condition [12] of \( K \)-frame, we have
\[
A\| K^*f \|^2 \leq \| \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, f \rangle \| \leq B\|f\|^2, \forall f \in \mathcal{H}.
\]
For any \( f \in \mathcal{H} \),
\[
A \| C^{\frac{1}{2}} K^* f \|^2 = A \| K^* C^{\frac{1}{2}} f \|^2 \\
\leq \| \sum_{j \in J} (C^{\frac{1}{2}} f, \psi_j) (\psi_j, C^{\frac{1}{2}} f) \| \\
= \| \sum_{j \in J} (C^{\frac{1}{2}} f, \psi_j) \psi_j, C^{\frac{1}{2}} f) \| \\
= \| (C^{\frac{1}{2}} S f, C^{\frac{1}{2}} f) \| \\
= \| (C S f, f) \|. \quad \tag{3.19}
\]

On the other hand for every \( f \in \mathcal{H} \),
\[
\| (C S f, f) \| = \| (S f, C f) \| \\
\leq \| S f \| \| C f \| \\
\leq B \| C \| \| f \|^2. \quad \tag{3.20}
\]

Therefore from (3.21), (3.22) and Theorem 3.2, we conclude that \( \{ \psi_j \}_{j \in J} \) is a \( C \)-controlled \( K \)-frame with bounds \( A \) and \( B \| C \| \).

**Theorem 3.5.** Let \( F = \{ f_j \}_{j \in J} \) be a \( C \)-controlled \( K \)-frame (\( \| K^* \| > 1 \)) for \( \mathcal{H} \), with controlled frame operator \( S_C \). If \( G = \{ g_j \}_{j \in J} \) is a non zero sequence in \( \mathcal{H} \), and \( E = T_F - T_G \) be a compact operator, where \( T_G(\{ c_j \}_{j \in J}) = \sum_{j \in J} c_j g_j \) for \( \{ c_j \}_{j \in J} \in l^2(A) \), then \( G = \{ g_j \}_{j \in J} \) is a \( C \)-controlled \( K \)-frame for \( \mathcal{H} \).

**Proof.** Let \( \{ f_j \}_{j \in J} \) be a \( C \)-controlled \( K \)-frame with bounds \( A \) and \( B \), then because of Theorem 3.2, we have
\[
A \| C^{\frac{1}{2}} K^* f \|^2 \leq \sum_{j \in J} \| (f, f_j) (C f_j, f) \| \leq B \| f \|^2, \forall f \in \mathcal{H}.
\]
This implies \( \| T_F \|^2 \leq B \| C^{\frac{1}{2}} \|^2 \).

Let \( V = T_F - E \) be an operator from \( l^2(A) \) into \( \mathcal{H} \). Since \( T_F \) and \( E \) are bounded, then the operator \( V \) is bounded. Therefore \( \| V \| = \| V^* \| \).

For any \( f \in \mathcal{H} \),
\[
V^* f = T_F^* f - E^* f \\
= \{ (f, f_j) \}_{j \in J} - \{ (f, f_j - g_j) \}_{j \in J} \\
= \{ (f, f_j) \}_{j \in J} - \{ (f_j - g_j, f) \}^*_{j \in J} \\
= \{ (f, f_j) \}_{j \in J} - \{ (f_j, f)^* - (g_j, f)^* \}_{j \in J} \\
= \{ (f, f_j) \}_{j \in J} - \{ (f_j, f) - (f, g_j) \}_{j \in J} \\
= \{ (f, g_j) \}_{j \in J}.
\]
We have
\[ V(\{c_j\}_{j \in J}) = \sum_{j \in J} c_j g_j, \quad \text{and} \quad S_G = VV^*. \] (3.21)

Now using (3.21), we have
\[
\|\langle f, CS_G f \rangle \| = \|\langle f, CVV^* f \rangle \|
= \|C^{1/2} V f, C^{1/2} V f \|
\leq \|C^{1/2}\|^2 \|V f \|^2
\leq \|C^{1/2}\|^2 \|(T_F - E) f \|^2
\leq \|C^{1/2}\|^2 \|T_F - E\|^2 \|f \|^2
\leq (\|T_F\|^2 + 2\|T_F\|\|E\| + \|E\|^2) \|C^{1/2}\|^2 \|f \|^2
\leq (B \|C^{1/2}\|^2 + 2\sqrt{B}\|C^{1/2}\|\|E\| + \|E\|^2) \|C^{1/2}\|^2 \|f \|^2
= B\left(\|C^{1/2}\|^2 + \frac{\|E\|}{\sqrt{B}}\right)^2 \|C^{1/2}\|^2 \|f \|^2. \] (3.22)

This inequality shows that \( \{g_j\}_{j \in J} \) is a controlled Bessel sequence with bound
\[ B\left(\|C^{1/2}\|^2 + \frac{\|E\|}{\sqrt{B}}\right)^2 \|C^{1/2}\|^2. \]

Again we have
\[
VV^* = (T_F - E)(T_F - E)^*
= (T_F - E)(T_F^* - E^*)
= T_F T_F^* - T_F E^* - E T_F^* + E E^*
= S_F - T_F E^* - E T_F^* + E E^*
\]

Since \( E, T_F \) and \( S_F \) are compact operators, then \( S_F - T_F E^* - E T_F^* + E E^* \) is a compact operator. Therefore \( S_F - T_F E^* - E T_F^* + E E^* + I \) is a bounded operator with closed range. Thus, \( VV^* = S_F - T_F E^* - E T_F^* + E E^* \) is a bounded operator with closed range. Also \( VV^* \) is injective as \( V \) is injective. Hence \( VV^* (= S_G) \) is bounded below. So there exists some constant \( A > 0 \) such that
\[ A\|C^{1/2} f \| \leq \|S_G C^{1/2} f \|. \] (3.23)

Now
\[
\|C^{1/2} K^* f \|^2 = \|K^* C^{1/2} f \|^2
\leq \|K^*\|^2 \|C^{1/2} f \|^2
\leq \frac{1}{A^2} \|K^*\|^2 \|S_G C^{1/2} f \|^2.
\]

This implies that
\[ \frac{A^2}{\|K^*\|^2} \|C^{1/2} K^* f \|^2 \leq \|S_G C^{1/2} f \|^2. \] (3.24)
Therefore from (3.22) and (3.24), we conclude that $G = \{g_k\}_{k \in J}$ is a $C$-controlled $K$-frame for $\mathcal{H}$ with frame bounds $\frac{4^2}{\|K\|^2}$ and $B(\|C^{\frac{1}{2}}\| + \frac{\|E\|}{\sqrt{B}})^2 \|C^{\frac{1}{2}}\|^2$.

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