A time-symmetric generalization of quantum mechanics

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I propose a time-symmetric generalization of quantum mechanics that is inspired by scattering theory. The model postulates two interacting quantum states, one traveling forward in time and one backward in time. The interaction is modeled by a unitary scattering operator. I show that this model is equivalent to pseudo-unitary quantum mechanics.

After a century of research, the principles of quantum mechanics remain the same as those proposed by Schrödinger and Heisenberg: a vector in a Hilbert space represents the state of a physical system, and a unitary operator models the dynamics. Early attempts at relaxing the unitarity requirement \cite{1,2} were soon abandoned in favor of a theory of quantized fields that retains it \cite{3}. In view of the mired progress in unifying quantum mechanics with general relativity, however, non-unitary generalizations of quantum mechanics have received renewed interest in recent years. One interesting idea that goes back to Dirac and Pauli \cite{1,2} is to replace the unitary operator with a pseudo-unitary operator that conserves a scalar product with an indefinite metric \cite{4,5}.

Here I propose an alternative approach to generalizing quantum mechanics that is arguably more intuitive and is inspired by scattering theory. In scattering theory, one considers incoming and outgoing waves that travel in the spatial dimensions and are coupled through scatterers. As quantum mechanics is a generalization of wave mechanics, it is tempting to apply scattering theory to quantum states in the time dimension as well, in which case one needs to postulate two states, one traveling forward in time and one backward in time. Coupling the two states via a scattering operator leads to a new time-evolution operator, which is given by the so-called Potapov transform \cite{6,7} of the scattering operator. If the scattering operator is unitary, the time-evolution operator turns out to be pseudo-unitary, thereby establishing a correspondence between the time-symmetric model set forth and the pseudo-unitary model in the literature.

With the interaction of the two states moving in opposite time directions, the model permits time travel, which is, of course, another interesting but controversial topic in physics. Given the model’s equivalence with the pseudo-unitary model, the possibility of time travel may explain why the latter can violate certain principles of standard physics, such as the no-signaling and no-cloning laws \cite{8,9}. One should not take the time travel and the violation of conventional principles as a failure of the models, however, as the models allow time travel only in a rigid mechanistic manner and there is no reason to believe that those conventional principles are fundamental and can survive new physics.

I stress that the mathematics here is elementary and well established in scattering theory, transmission-line theory, and optics when it comes to waves traveling in the spatial dimensions; see, in particular, the seminal works of Potapov \cite{7} and Redheffer \cite{10}. The transfer-matrix method in optics is perhaps the simplest example of the scattering theory \cite{11}. It is also known that the scattering problem with the time-independent Schrödinger equation in one spatial dimension has a (2-by-2) pseudo-unitary transfer matrix (see footnote on p. 33 of Ref. \cite{12}). The key new insight of this paper is that the formalism, though mathematically simple, offers a principled way to generalize time evolution in quantum mechanics for Hilbert spaces with arbitrary dimensions, putting time on a more equal footing with space in this fundamental law of physics. I also note that the approach here seems to share some conceptual similarities with the time-symmetric treatment of classical electrodynamics \cite{13} and the Dirac equation \cite{14} by Wheeler and Feynman, although their approach was later subsumed by the unitary quantum field theory \cite{3}.

Some works on quantum measurement theory also consider the combination of two quantum states \cite{15–18}, but those merely offer alternative interpretations or applications of standard quantum mechanics and do not modify it.

To set the stage, I first review standard quantum mechanics. Let $\mathcal{H}$ be a complex Hilbert space with an inner product denoted by $\langle u, v \rangle \in \mathbb{C}$ with $u, v \in \mathcal{H}$. Let $\psi_n \in \mathcal{H}$ be a Hilbert-space vector that models the quantum state of a physical system in the Schrödinger picture at time $t_n$. If the time evolution from $t_n$ to $t_{n+1}$ is modeled by a linear operator $U$ on $\mathcal{H}$, such that

$$\psi_{n+1} = U \psi_n, \quad (1)$$

and the norm is required to be conserved in time, viz.,

$$\langle \psi_{n+1}, \psi_{n+1} \rangle = \langle \psi_n, \psi_n \rangle, \quad (2)$$

then $U$ must be unitary (I do not consider antilinear operators). $\langle \psi, \psi \rangle$ is commonly regarded as the total probability, although I do not prescribe any physical meaning to the conservation law in the following to avoid premature interpretations. Figure 1(a) illustrates this standard quantum model by a block diagram.

The proposed time-symmetric model assumes instead that there are two worlds, one traveling forward in time and one backward in time. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces for the forward and backward worlds, respectively, and let the total
FIG. 1. (a) The standard quantum model in the Schrödinger picture, where the quantum state $\psi$ evolves forward in time from $\psi_n$ to $\psi_{n+1}$ via a unitary operator $U$. (b) The time-symmetric model, which involves two states $\psi^1$ and $\psi^2$ traveling forward and backward in time. Their interaction is modeled as an input-output relation in terms of a unitary scattering operator $S$, or equivalently a forward-time relation in terms of a pseudo-unitary transfer operator $T$.

Hilbert space for the two worlds be the direct sum $\mathcal{H}^1 \oplus \mathcal{H}^2$. Let

$$\psi_n = \psi^1_n \oplus \psi^2_n \in \mathcal{H}^1 \oplus \mathcal{H}^2$$

be the total state at time $t_n$. Frame the problem as a scattering problem, where $\psi^1_n$ and $\psi^2_{n+1}$ are the incoming waves while $\psi^1_{n+1}$ and $\psi^2_n$ are the outgoing waves, as illustrated by Figure 1(b). If the scattering is linear, it can be modeled by a scattering operator $S$ on $\mathcal{H}^1 \oplus \mathcal{H}^2$, such that

$$\begin{pmatrix} \psi^1_{n+1} \\ \psi^2_n \end{pmatrix} = \begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix} \begin{pmatrix} \psi^1_n \\ \psi^2_{n+1} \end{pmatrix},$$

where $S$ is expressed as a matrix of four operators $S^{jk}: \mathcal{H}^k \rightarrow \mathcal{H}^j$. In particular, the $S^{12}$ and $S^{21}$ operators model the interactions between the two worlds. A reasonable conservation law that can be borrowed from scattering theory is

$$\langle \psi^1_{n+1}, \psi^1_{n+1} \rangle + \langle \psi^2_n, \psi^2_n \rangle = \langle \psi^1_n, \psi^1_n \rangle + \langle \psi^2_{n+1}, \psi^2_{n+1} \rangle,$$

which implies that $S$ must be unitary. Note that this unitarity is applied to the total dynamics of the two worlds and is different from the one-world unitarity of standard quantum mechanics.

The conservation law can be rewritten as

$$\langle \psi^1_{n+1}, \psi^1_{n+1} \rangle - \langle \psi^2_{n+1}, \psi^2_{n+1} \rangle = \langle \psi^1_n, \psi^1_n \rangle - \langle \psi^2_n, \psi^2_n \rangle,$$

such that it can be interpreted as a conservation of the single-time scalar product

$$\langle \psi_{n+1}, J\psi_{n+1} \rangle = \langle \psi_n, J\psi_n \rangle$$

with the indefinite metric

$$J \equiv \begin{pmatrix} I^1 & 0 \\ 0 & -I^2 \end{pmatrix},$$

where $I^j$ is the identity operator on $\mathcal{H}^j$. Let $T$ be the transfer operator on $\mathcal{H}^1 \otimes \mathcal{H}^2$ that relates $\psi_n$ at one time to $\psi_{n+1}$ at a forward time, viz.,

$$\psi_{n+1} = T\psi_n.$$

To be more explicit,

$$\begin{pmatrix} \psi^1_{n+1} \\ \psi^2_{n+1} \end{pmatrix} = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix} \begin{pmatrix} \psi^1_n \\ \psi^2_{n+1} \end{pmatrix},$$

which is similar to Eq. (4) in that $T$ is also partitioned into four operators $T^{jk}: \mathcal{H}^k \rightarrow \mathcal{H}^j$, although $\psi^1_n$ and $\psi^2_{n+1}$ have switched places here. As $T$ is linear and conserves the $J$-weighted scalar product, it must be pseudo-unitary, or more precisely $J$-unitary, viz.,

$$T^\dagger J T = J,$$

where $\dagger$ denotes the adjoint in the usual sense. General forward-time evolution is then described by a sequence of $J$-unitary operators $T_1, T_2, \ldots$ in the form

$$\psi_N = T_{N-1} \ldots T_2 T_1 \psi_1,$$

as depicted by Fig. 2. This is precisely the model of pseudo-unitary quantum mechanics [4].

FIG. 2. A sequence of unitary interactions between the forward and backward states can be modeled by a sequence of pseudo-unitary transfer operators $T_1, T_2, \ldots$.

If $S^{22}$ is invertible, the transfer operator is related to the scattering operator through the Potapov transform [6, 7]

$$T = \Pi(S) \equiv \begin{pmatrix} S^{11} - S^{12}(S^{22})^{-1} S^{21} & S^{12}(S^{22})^{-1} \\ -(S^{22})^{-1} S^{21} & (S^{22})^{-1} \end{pmatrix}.$$

The transform is straightforward to derive: start from Eq. (4) and express $\psi^1_{n+1}$ and $\psi^2_{n+1}$ in terms of $\psi^1_n$ and $\psi^2_n$, thus switching the places of $\psi^1_n$ and $\psi^2_n$ in the matrix relation.
It is then obvious that applying the Potapov transform again to $T$ should give back the scattering operator, viz.,

$$\Pi(T) = \Pi[\Pi(S)] = S.$$  \hfill (14)

In other words, the pseudo-unitary model can be transformed back to the unitary time-symmetric model, and the two models are equivalent, as long as $T^{22}$ and $S^{22}$ are invertible.

To arrive at a definite solution, one also needs to specify the boundary conditions at the initial time $t_1$ and the final time $t_N$. In standard quantum mechanics, one may assume an initial state $\psi_1$ or a final $\psi_N$ as a boundary condition, or use the periodic boundary condition $\psi_1 = \psi_N$ to restrict the set of solutions. With the two worlds in the time-symmetric model, there are now more possible types of boundary conditions, as illustrated by Fig. 3. The first type is the open condition, which is analogous to the usual scattering problem. The inputs $\psi^1_1$ and $\psi^2_N$ can be used as the boundary conditions, although one may also pick any two states from $\{\psi^1_1, \psi^2_N, \psi^N_N, \psi^1_1\}$ as long as the reverse problem remains well posed. The second type, the half-closed condition, involves a reflection that sets $\psi^2_N = \psi^1_N$, and only $\psi^1_1$ or $\psi^2_1$ remains to be specified. The third type, where the system is closed at both ends, implies that the total world behaves like a Fabry-Pérot cavity, and a nonzero solution can exist only as a superposition of its eigen-modes. The fourth type, which introduces a periodic condition to the backward world, turns the backward world into a ring cavity and restores unitarity to the relation between the initial and final states of the forward world, although the intermediate interaction between the two worlds still makes the model different from the standard one-world model. The final type, the fully periodic condition, is similar to the closed condition but in a ring geometry.

Given the correspondence with scattering theory, the time-symmetric model can be experimentally simulated by a photonic circuit with a recirculating mesh [19] if one of the spatial dimensions is used to represent time. It is well known that a forward-only mesh can simulate a unitary system [20]; a recirculating one offers new possibilities in quantum simulation.

As a model of the universe, the time-symmetric model is consistent with standard quantum mechanics and existing experimental evidence if the two worlds interact so weakly that the transfer operator is close to unitary and current experiments cannot detect the interaction. The unitary operator of the standard model is expected to be an approximation of $S^{11}$ and $T^{11}$ for the forward world; the next questions are how one should model the rest of the operators and how the deviation from standard unitarity may be tested by experiments.

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