The mixed affine quermassintegrals

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Abstract In this paper, we introduce first the mixed affine quermassintegrals of \( j \) convex bodies. The Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of \( j \) convex bodies is established. As a application, the Minkowski’s, Brunn’s Minkowski’s inequalities for the mixed affine quermassintegrals are also derived.

Keywords convex body, affine quermassintegrals, Minkowski inequality, Aleksandrov-Fenchel inequality.

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1. Introduction

Lutwak [1] proposed to define the affine quermassintegrals for a convex body \( K \), \( \Phi_0(K) \), \( \Phi_1(K) \), \ldots, \( \Phi_n(K) \), by taking \( \Phi_0(K) := V(K) \), \( \Phi_n(K) := \omega_n \) and for \( 0 < j < n \),

\[
\Phi_{n-j}(K) := \omega_n \left[ \int_{G_{n,j}} \left( \frac{\text{vol}_j(K|\xi)}{\omega_j} \right)^{-n} \mu_j(\xi) \right]^{-1/n},
\]

where \( G_{n,j} \) denotes the Grassman manifold of \( j \)-dimensional subspaces in \( \mathbb{R}^n \), and \( \mu_j \) denotes the gauge Haar measure on \( G_{n,j} \), and \( \text{vol}_j(K|\xi) \) denotes the \( j \)-dimensional volume of the positive projection of \( K \) on \( j \)-dimensional subspace \( \xi \subset \mathbb{R}^n \) and \( \omega_j \) denotes the volume of \( j \)-dimensional unit ball (see [2]). Lutwak showed the Brunn-Minkowski inequality for the affine quermassintegrals. If \( K \) and \( L \) are convex bodies and \( 0 < j < n \), then

\[
\Phi_{n-j}(K + L)^{1/j} \geq \Phi_{n-j}(K)^{1/j} + \Phi_{n-j}(L)^{1/j}.
\]

In this paper, we introduce the mixed affine quermassintegrals of \( j \) convex bodies. The Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of \( j \) convex bodies is established. As a application, and the Minkowski inequality is also derived.

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2. The mixed affine quermassintegrals

In the section, we introduce first the following concept and list its properties and related inequalities.

**Definition 2.1** (The mixed affine quermassintegrals of \( j \) convex bodies) The mixed affine quermassintegral of \( j \) convex bodies \( K_1, \ldots, K_j \), denoted by \( \Phi_{n-j}(K_1, \ldots, K_j) \), defined by

\[
\Phi_{n-j}(K_1, \ldots, K_j) := \omega_n \left[ \int_{G_{n,j}} \left( \frac{\text{vol}_j((K_1, \ldots, K_j) \mid \xi)}{\omega_j} \right)^{-n} \, d\mu_j(\xi) \right]^{-1/n},
\]

(2.1)

where \( 0 \leq j \leq n \).

When \( K_1 = \cdots = K_j = K \), \( \Phi_{n-j}(K_1, \ldots, K_j) \) becomes Lutwak’s affine quermassintegral \( \Phi_{n-j}(K) \).

When \( K_1 = \cdots = K_{j-1} = K \) and \( K_j = L \), \( \Phi_{n-j}(K_1, \ldots, K_j) \) becomes a new affine geometric quantity, denoted by \( \Phi_{n-j}(K, L) \) and call it mixed affine quermassintegral of \( K \) and \( L \). When \( K_1 = \cdots = K_{j-i-1} = K \), \( K_{j-i+1} = \cdots = K_{j-1} = B \) and \( K_{j-i} = L \), \( \Phi_{n-j}(K_1, \ldots, K_j) \) becomes another new affine geometric quantity, denoted by \( \Phi_{n-j,i}(K, L) \) and call it \( i \)-th mixed affine quermassintegral of \( K \) and \( L \), where \( 0 \leq i < j \leq n \).

Obviously, the mixed affine quermassintegrals of \( j \) convex bodies is invariant under simultaneous unimodular centro-affine transformation.

**Lemma 2.1** If \( K_1, \ldots, K_j \in K_n^0 \) and \( 0 \leq j \leq n \), then for any \( g \in SL(n) \)

\[
\Phi_{n-j}(gK_1, \ldots, gK_j) = \Phi_{n-j}(K_1, \ldots, K_j).
\]

As we all know, according to the Brunn-Minkowski theory, a very natural question is raised: are there some isoperimetric inequalities about the mixed affine quermassintegrals of \( j \) convex bodies? The following perfectly answers the question and establish Minkowski’s, and Aleksandrov-Fenchel’s and Brunn-Minkowski’s inequalities for the mixed affine quermassintegrals.

**Theorem 2.1** (The Minkowski inequality for mixed affine quermassintegrals) If \( K, L \in K_n^0 \) and \( 0 \leq j \leq n \), then

\[
\Phi_{n-j}(K, L)^j \geq \Phi_{n-j}(K)^{j-1}\Phi_{n-j}(L),
\]

(2.2)

with equality if and only if \( K \) and \( L \) are homothetic.

**Proof** This follows immediately from the Minkowski’s, and Hölder’s inequalities. \( \square \)

Next, we establish an Aleksandrov-Fenchel inequality for the mixed affine quermassintegral of \( j \) convex bodies \( K_1, \ldots, K_j \).
Theorem 2.3 (The Aleksandrov-Fenchel inequality for mixed affine quermassintegrals of $j$ convex bodies) If $K_1, \ldots, K_j \in K_n^o$, $0 \leq j \leq n$ and $0 < r \leq j$, then

$$
\Phi_{n-j}(K_1, \ldots, K_j) \geq \prod_{i=1}^{r} \Phi_{n-j}(K_i, \ldots, K_j, K_{r+1}, \ldots, K_j)^{1/r}.
$$

(2.3)

Proof This follows immediately from the Aleksandrov-Fenchel inequality and Hölder's inequality. □

Unfortunately, the equality conditions of the Aleksandrov-Fenchel inequality are, in general, unknown.

Corollary 2.1 If $K_1, \ldots, K_j \in K_n^o$ and $0 \leq j \leq n$, then

$$
\Phi_{n-j}(K_1, \ldots, K_j)^j \geq \Phi_{n-j}(K_1) \cdots \Phi_{n-j}(K_j),
$$

(2.4)

with equality if and only if $K_1, \ldots, K_j$ are homothetic.

Proof The special case $r = j - 1$, of inequality (4.3), is

$$
\Phi_{n-j}(K_1, \ldots, K_j)^{j-1} \geq \Phi_{n-j}(K_1, K_j) \cdots \Phi_{n-j}(K_{j-1}, K_j).
$$

When above inequality is combined with the Minkowski inequality (4.2), the result is

$$
\Phi_{n-j}(K_1, \ldots, K_j)^j \geq \Phi_{n-j}(K_1) \cdots \Phi_{n-j}(K_j),
$$

with equality if and only if $K_1, \ldots, K_j$ are homothetic. □

Finally, we simply prove the Brunn-Minkowski inequality for the affine quermassintegrals by using the mixed affine quermassintegrals theory introduced in this section.

Theorem 2.3 (The Brunn-Minkowski inequality for affine quermassintegrals) If $K, L \in K_n^o$ and $0 \leq j \leq n$, then for $\varepsilon > 0$

$$
\Phi_{n-j}(K + \varepsilon \cdot L)^{1/j} \geq \Phi_{n-j}(K)^{1/j} + \varepsilon \Phi_{n-j}(L)^{1/j}.
$$

(2.5)

Proof This follows immediately from (2.1) and (2.2). □

References

[1] E. Lutwak, A general isepiphanic inequality, *Proc. Amer. Math. Soc.*, 90 (1984), 451-421.

[2] E. Lutwak, Inequalities for Hadwiger's harmonic quermassintegrals, *Math. Ann.*, 280 (1988), 165-175.