Gaiotto’s Lagrangian Subvarieties via Loop Groups

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April 19, 2018

Abstract

The purpose of this note is to give a simple proof of the fact that a certain substack, defined in [2], of the moduli stack \( T^* \text{Bun}_G(\Sigma) \) of Higgs bundles over a curve \( \Sigma \), for a connected, simply connected semisimple group \( G \), possesses a Lagrangian structure. The substack, roughly speaking, consists of images under the moment map of global sections of principal \( G \)-bundles over \( \Sigma \) twisted by a smooth symplectic variety with a Hamiltonian \( G \)-action.

1 Introduction

In this note we work over the complex numbers \( \mathbb{C} \). Throughout, let \( G \) be a fixed connected, simply connected semisimple algebraic group and \( \Sigma \) a fixed smooth irreducible projective curve, with a choice \( K_1^{1/2} \) of square root of its canonical bundle.

In the differential-geometric setting, Gaiotto constructed in [2], for each symplectic vector space with a linear symplectic \( G \)-action, a subspace of the moduli space of stable Higgs bundles over \( \Sigma \) and proved that this subspace is Lagrangian (see Appendix A of [2]). A more detailed proof in this setting, as well as explanations of the word ‘Lagrangian’, can be found in Section 2 of [6]. In the algebro-geometric setting, a proof has recently been given in [4] (Theorem 1.3), based on the powerful theory of derived symplectic geometry. The purpose of this note is to give an algebro-geometric proof of Gaiotto’s statement in a down-to-earth manner. In particular, our proof does not make use of any notions from derived algebraic geometry. One interesting feature of our approach is that it allows us to see the picture very concretely - Gaiotto’s statement boils down to the basic fact that the sum of residues of a rational 1-form on \( \Sigma \) equals zero.

Before stating the main theorem, let us fix some notations. Let \( \text{Bun}_G(\Sigma) \), or simply \( \text{Bun}_G \), be the moduli stack of \( G \)-torsors over \( \Sigma \). It is well-known that the cotangent stack \( T^* \text{Bun}_G \) is the moduli stack of Higgs bundles over \( \Sigma \). The stack \( T^* \text{Bun}_G \), as a cotangent stack, has a canonical symplectic form \( \Omega \) (see Definition 2 below), defined as the differential of the Liouville 1-form \( \Lambda \) (see Definition 1 below) on \( T^* \text{Bun}_G \). In the stacky context, there have been attempts to generalize the usual notions of isotropic/Lagrangian submanifolds to the notions of isotropic/Lagrangian substacks. The interested readers are referred to page 1 of [3] for more details. In the derived context, such generalizations have been proposed in [8] (Definitions 2.7, 2.8 in [8]). See also the definitions in Sections 2 and 3, as well as Section 5.2, of [4] for more discussions about these notions. The main theorem of this note will be to the effect that a certain substack of \( T^* \text{Bun}_G \) is Lagrangian.

Remark. If it helps psychologically, one may replace \( T^* \text{Bun}_G \) with \( T^* \text{Bun}^\text{reg}_G \), the open substack of Higgs bundles without non-scalar automorphisms, so that all the usual notions
of symplectic geometry make perfect sense. Essentially nothing is lost by doing this.

Let \((X, \omega)\) be a smooth symplectic variety with a Hamiltonian \(G\)-action preserving the symplectic form \(\omega\) whose moment map is \(\mu : X \rightarrow \mathfrak{g}^*\). Assume moreover that there is a \(\mathbb{G}_m\)-action on \(X\) such that the symplectic form \(\omega\) has weight 2 and the \(\mathbb{G}_m\)-action commutes with the \(G\)-action. Given a \(G\)-torsor \(P\), we write \(X_P\) for \(P \times_G X\). If \((K^{1/2})^\times\) stands for the total space of the bundle \(K^{1/2}\) without the zero section, then we write, by slightly abusing notation, \(X_P \otimes K^{1/2}\) for \((X_P \times \Sigma (K^{1/2})^\times)/\mathbb{G}_m\), where \(\mathbb{G}_m\) acts on \(X_P\) via its action on \(X\), on \((K^{1/2})^\times\) by dilation, and on \(X_P \times \Sigma (K^{1/2})^\times\) by the diagonal action.

For the ease of future references, let us write \(Y\) for the stack that parametrizes pairs \((P, s)\), where \(P\) is a \(G\)-torsor over \(\Sigma\) and \(s\) is a global section of \(X_P \otimes K^{1/2}\). More precisely, given a \(\mathbb{C}\)-scheme \(S\), the set of \(S\)-points of the stack \(Y\) is defined as

\[
Y(S) := \{(P, s) : P \text{ is a } G\text{-torsor over } \Sigma \times S, s \text{ is a global section of } X_P \otimes \text{pr}_1^* K^{1/2}\},
\]

where \(\text{pr}_1 : \Sigma \times S \rightarrow \Sigma\) is the natural projection to the first factor and \(X_P\), as well as \(X_P \otimes \text{pr}_1^* K^{1/2}\), are constructed in an analogous way as in the previous paragraph.

Note that the moment map \(\mu : X \rightarrow \mathfrak{g}^*\), being a homogeneous polynomial of degree 2, induces, for each \(\mathbb{C}\)-scheme \(S\) and each \(G\)-torsor \(P\) over \(\Sigma \times S\), a morphism

\[
X_P \otimes \text{pr}_1^* K^{1/2} \rightarrow \mathfrak{g}_P^* \otimes \text{pr}_1^* K, \tag{1}
\]

where \(\mathfrak{g}_P^*\) is the coadjoint bundle associated to \(P\) and \(K\) is the canonical bundle of \(\Sigma\). Since \(T^* \text{Bun}_G\) parametrizes pairs \((P, \phi)\), where \(P\) is a \(G\)-torsor over \(\Sigma\) and \(\phi\) is a global section of \(\mathfrak{g}_P^* \otimes K\), the morphism \(1\) above induces a morphism \(\bar{\mu} : Y \rightarrow T^* \text{Bun}_G\). The main theorem of this note is

**Theorem 1.1.** The 2-form \(\bar{\mu}^*(\Omega)\) vanishes on \(Y\).

**Remak 1.** If we write \(Z\) for the substack of \(T^* \text{Bun}_G\) whose set of \(S\)-valued points, for a \(\mathbb{C}\)-scheme \(S\), is

\[
Z(S) := \{(P, \phi) : P \text{ is a } G\text{-torsor over } \Sigma \times S, \phi \in \Gamma(\Sigma \times S, \mathfrak{g}_P^* \otimes \text{pr}_1^* K) \text{ factors through the morphism } 1\text{ above}\},
\]

then this theorem can be viewed as saying that the substack \(Z\) of \(T^* \text{Bun}_G\) is isotropic.

**Remark 2.** Our theorem is more flexible than Gaiotto's in [2] in that we allow \(X\) to be a symplectic variety with a Hamiltonian \(G\)-action, not just a symplectic vector space equipped with a linear symplectic \(G\)-action.

## 2 The Proof

In this section, we will try to be as humble and concrete as possible - whenever it helps with geometric intuition, we will pretend that \(\text{Bun}_G\) and \(T^* \text{Bun}_G\) are actual varieties, rather than stacks, and regard, for example, \(\text{Bun}_G\) as consisting of (\(\mathbb{C}\)-valued) points given by transition morphisms with respect to some (fixed) cover of \(\Sigma\). The reason for this choice of perspective is that it is not hard to formalize what we are about to say rigorously in the stacky language (for a general \(\mathbb{C}\)-scheme \(S\) and a \(G\)-torsor \(P\) over \(S\), apply Theorem 3 of [1] to trivialize \(P\), then one can define the forms \(\Lambda\) and \(\Omega\) in an analogous manner to Definitions 1 and 2 below),
and that, probably more importantly, the readers will benefit from seeing the argument in concrete geometric terms.

Throughout, when we speak of tangent vectors to stacks, we mean vecteur tangent universel in the sense of [7] (see page 169). For the meaning of cotangent vectors we refer the readers to Théorème 17.16 of [7].

The plan of our proof is as follows. We will first give a formula for the Liouville 1-form $\Lambda$ on $T^*Bu_{\mathbb{A}}$; then differentiate it to obtain a formula for the canonical symplectic form $\Omega$. We then pull the form $\Omega$ back to the stack $Y$ and argue that the pull-back equals zero.

## 2.1 Trivializations

Let $P$ be a given $G$-torsor over $\Sigma$. We first consider the special case where $P$, as well as $K^{1/2}$, can be trivialized over $\Sigma^0 := \Sigma - \{pt\}$ for some point $pt \in \Sigma$ (see Satz 3.3 of [5]), although this is in general not always the case. The general case will follow easily as long as we fully understand this special case.

Let $D := \text{Spec } \mathbb{C}[[z]]$ be a formal disk centered at $pt$. Since any bundle over $D$ is trivial, $P$ and $K^{1/2}$ are determined by transition morphisms. Define $D^\times := \text{Spec } \mathbb{C}((z))$ to be the punctured formal disk. Then we write $g^{-1} \in G(D^\times)$ (resp. $T : D^\times \to \mathbb{G}_m$) for the transition morphism from $\Sigma^0$ to $D$ for $P$ (resp. $K^{1/2}$). More precisely, for closed points, $(z, h) \in D^\times \times G$ over $\Sigma^0$ is identified with $(z, g(z)^{-1}h) \in D^\times \times G$ over $D$. Similarly, $(z, t) \in D^\times \times \mathbb{A}^1$ over $\Sigma^0$ is identified with $(z, T(z)t) \in D^\times \times \mathbb{A}^1$ over $D$.

With these notations, the bundle $X_P \otimes K^{1/2}$ has a simple description as follows. Over $\Sigma^0$, since $P$ is $G$-equivariantly isomorphic to the trivial $G$-torsor, $X_P$ is isomorphic to $\Sigma^0 \times X$. Since $K^{1/2}$ is $\mathbb{G}_m$-equivariantly isomorphic to $\Sigma^0 \times \mathbb{G}_m$ over the same chart, we have the isomorphism $X_P \otimes K^{1/2} \simeq \Sigma^0 \times X$ over $\Sigma^0$. In a similar way, we see that $X_P \otimes K^{1/2} \simeq D \times X$ over $D$. Using the transition morphisms from the previous paragraph, we see that the transition morphism for $X_P \otimes K^{1/2}$ from $\Sigma^0$ to $D$ is $T^{-1}g^{-1}$.

## 2.2 The ‘Source Stack’ $Y$

Let us describe the source stack $Y$ in this section. In concrete words, $Y$ parametrizes equivalence classes of triples $(g, s^o, s') \in G(D^\times) \times \text{Map}(\Sigma^0, X) \times \text{Map}(D, X)$ satisfying the equation

$$T^{-1}g^{-1}s^o = s', \quad (2)$$

where $(g, s^o, s')$ is equivalent to $(h, t^o, t')$ if there exist $g_1 \in G(\Sigma^0)$ and $g_2 \in G(D)$ such that $g_1g_2^{-1} = h$, $g_1s^o = t^o$, and $g_2s' = t'$. In the sequel, for simplicity, we will suppress the words ‘equivalence classes of’ and write only ‘triples’ or ‘tuples’ in all similar situations.

Given a point $(g, s^o, s') \in Y$, $T_{(g, s^o, s')}$ consists of 1st order infinitesimal deformations of $(g, s^o, s')$. Concretely, a tangent vector to $Y$ at $(g, s^o, s')$ consists of a triple $(\dot{g}, \dot{s}^o, \dot{s}') \in g(D^\times) \times \text{Map}(\Sigma^0, TX) \times \text{Map}(D, TX)$ so that $\pi \circ \dot{s}^o = s^o$, $\pi \circ \dot{s}' = s'$, where $\pi : TX \to X$ is the projection, and

$$\dot{s}' = T^{-1}g^{-1}\dot{s}^o - \rho_{s'}(\dot{g}), \quad (3)$$

where $\rho_{s'}(\dot{g})$ stands for the infinitesimal action of $\dot{g}$ at $s'$. Note that equation (3) is obtained from equation (2) by considering 1st order infinitesimal deformations.

Heuristically, one can view tangent vectors as infinitesimal curves and write $s^o$ (resp. $s'$) for the infinitesimal curve representing $s^o$ (resp. $s'$). Using the suggestive notation
Definition 1. The Liouville 1-form \( \Lambda \) is given by
\[
\Lambda(\dot{g}, \dot{\phi}, \dot{\phi}') := (g, \phi^o, \phi')(\dot{g}) = \text{Res}_{s=0}(\langle \phi', \dot{g} \rangle dz),
\]
where we used \( dz \) to trivialize \( K \over D \).

2.3 The Liouville 1-form \( \Lambda \) on \( T^* \text{Bun}_G \)

Recall that, given a \( G \)-torsor \( P \), one has \( T^*_P \text{Bun}_G = H^0(\Sigma, \mathfrak{g}_P^* \otimes K) \), where \( \mathfrak{g}_P^* \) stands for the coadjoint bundle associated to \( P \) and \( K \) stands for the canonical bundle of \( \Sigma \). Since \( T \) is the transition morphism for \( K^{1/2} \), the transition morphism for \( K \) is given by \( T^2 \). The same analysis as in Section 2.2 then tells us that a global section of \( \mathfrak{g}_P^* \otimes K \) consists of a pair \( (\phi^o, \phi') \in \mathfrak{g}^*(\Sigma^o) \times \mathfrak{g}^*(\Sigma) \) satisfying \( \phi' = T^{-2}g \phi^o \), where \( g \in G(D^x) \) corresponds to \( P \) and acts on \( \phi^o \) by the coadjoint action.

Definition 2. The symplectic 2-form \( \Omega \) on \( T^* \text{Bun}_G \) is given by
\[
\Omega((\dot{g}_1, \dot{\phi}_1^o, \dot{\phi}_1'), (\dot{g}_2, \dot{\phi}_2^o, \dot{\phi}_2')) := \text{Res}_{z=0}(\langle \phi', \dot{g}_2 \rangle dz) - \text{Res}_{z=0}(\langle \phi'_2, \dot{g}_1 \rangle dz) - \text{Res}_{z=0}(\langle \phi'_1, [\dot{g}_1, \dot{g}_2] \rangle dz). \tag{4}
\]

This section is devoted to the justification of this definition. We will interpret \( \Omega \) as obtained from \( \Lambda \) by Cartan’s formula for exterior derivatives.

Notice first that the 1-form \( \Lambda \) is actually defined on the space \( g(D^x) \times g^*(\Sigma^o) \times g^*(\Sigma) \), not just the subspace \( T^* \text{Bun}_G \) of \( g(D^x) \times g^*(\Sigma^o) \times g^*(\Sigma) \). So, if we write \( \Lambda \) for this 1-form on the ambient space \( g(D^x) \times g^*(\Sigma^o) \times g^*(\Sigma) \) and \( i \) for the inclusion of \( T^* \text{Bun}_G \) into the ambient space, we have
\[
d\Lambda = d^* \lambda = i^* d\tilde{\Lambda}.
\]

For \( i = 1, 2 \), extend the tangent vectors \((\dot{g}_i, \dot{\phi}_i^o, \dot{\phi}_i')\) to vector fields \((\dot{G}_i, \dot{\Phi}_i^o, \dot{\Phi}_i')\) on \( g(D^x) \times g^*(\Sigma^o) \times g^*(\Sigma) \) near \((g, \phi^o, \phi')\). Since in the computation of \( d\Lambda((\dot{g}_1, \dot{\phi}_1^o, \dot{\phi}_1'), (\dot{g}_2, \dot{\phi}_2^o, \dot{\phi}_2')) \), it does not matter which extension we use, we will choose the one where \( G_i(h, \psi^o, \psi') = \dot{g}_i \) for \( i = 1, 2 \), where \((h, \psi^o, \psi')\) is a point in \( g(D^x) \times g^*(\Sigma^o) \times g^*(\Sigma) \) near \((g, \phi^o, \phi')\).
Observe that the function $\tilde{\Lambda}(\dot{G}_z, \dot{F}_z, \dot{F}_z')$ sends a point $(h, \psi^o, \psi')$ near the point $(g, \phi^o, \phi')$ to $\text{Res}_{z=0}(\langle \psi', \dot{g}_2 \rangle dz)$. Applying the vector field $(\dot{G}_z, \dot{F}_z, \dot{F}_z')$ to the function $\Lambda(\dot{G}_z, \dot{F}_z, \dot{F}_z')$ and evaluating at $(g, \phi^o, \phi')$, we get

$$\text{Res}_{z=0}(\langle \phi'_1, \dot{g}_2 \rangle dz).$$

Similarly, applying the vector field $(\dot{G}_z, \dot{F}_z, \dot{F}_z')$ to the function $\tilde{\Lambda}(\dot{G}_z, \dot{F}_z', \dot{F}_z')$ and evaluating at $(g, \phi^o, \phi')$, we get

$$\text{Res}_{z=0}(\langle \phi'_2, \dot{g}_1 \rangle dz).$$

Finally, observe that the push-forward of $\left[ \dot{\Phi}_z, \Phi_z, \dot{\Phi}_z' \right]$ at the point $(\alpha, \rho, \dot{\rho})$, using Cartan’s formula for exterior derivatives, we have

$$d\Lambda((\dot{g}_1, \dot{\Phi}_z, \dot{\Phi}_z'), (\dot{g}_2, \dot{\Phi}_z, \dot{\Phi}_z'))$$

$$= \text{Res}_{z=0}(\langle \phi'_1, \dot{g}_2 \rangle dz) - \text{Res}_{z=0}(\langle \phi'_2, \dot{g}_1 \rangle dz) - \text{Res}_{z=0}(\langle \phi', [\dot{g}_1, \dot{g}_2] \rangle dz),$$

(5)

thus justifying Definition 2.

2.5 Pull-back of $\Omega$ to $Y$

We now put together everything we have seen so far. Let $(g, s^o, s')$ be a $C$-valued point in the source stack $Y$ and let $(\dot{g}_i, s^o_i, s'_i)$ be tangent vectors to $Y$ at $(g, s^o, s')$ $(i = 1, 2)$. These two vectors are pushed forward by $\tilde{\mu}$ to $(\dot{\mu}_i, d\mu(s^o_i), d\mu(s'_i))$ $(i = 1, 2)$. Using formula (5) we have seen in Section 2.4, we see that

$$\Omega((\dot{g}_1, d\mu(s^o_1), d\mu(s'_1)), (\dot{g}_2, d\mu(s^o_2), d\mu(s'_2)))$$

$$= \text{Res}_{z=0}(\langle d\mu(s^o_1), \dot{g}_2 \rangle dz - \langle d\mu(s'_2), \dot{g}_1 \rangle dz - \langle \mu \circ s', [\dot{g}_1, \dot{g}_2] \rangle dz)$$

$$= \text{Res}_{z=0}(\langle -\omega(s^o_1, \rho' \circ (\dot{g}_2)) dz + \omega(s'_2, \rho' \circ (\dot{g}_1)) dz - \langle \mu \circ s', [\dot{g}_1, \dot{g}_2] \rangle dz).$$

Now let $\alpha$ be the nowhere vanishing 1-form on $\Sigma^o$ used in the trivialization of $K$. So we have $\alpha = T^{-2} dz$ on $D^c$. Using equation (3) from Section 2.2, we have

$$\omega(g^{-1} s^o_1, g^{-1} s^o_2) \alpha$$

$$= \omega(g^{-1} s^o_1, g^{-1} s^o_2) T^{-2} dz$$

$$= \omega(T^{-1} g^{-1} s^o_1, T^{-1} g^{-1} s^o_2) dz$$

$$= \omega(s'_1 + \rho' \circ (\dot{g}_1), s'_2 + \rho' \circ (\dot{g}_2)) dz$$

$$= \omega(s'_1, s'_2) dz + \omega(s'_1, \rho' \circ (\dot{g}_2)) dz - \omega(s'_2, \rho' \circ (\dot{g}_1)) dz + \omega(\rho' \circ (\dot{g}_1), \rho' \circ (\dot{g}_2)) dz$$

$$= \omega(s'_1, s'_2) dz + \omega(s'_1, \rho' \circ (\dot{g}_2)) dz - \omega(s'_2, \rho' \circ (\dot{g}_1)) dz + \langle \mu \circ s', [\dot{g}_1, \dot{g}_2] \rangle dz$$

(6)

The very first line of (6), by $G$-invariance of $\omega$, is equal to $\omega(s^o_1, s^o_2) \alpha$, which is a 1-form defined on $\Sigma$ with a pole at pt. Since the sum of residues of a rational 1-form on $\Sigma$ is zero, we see that the residue at $z = 0$ of the bottom line of (6) is zero. Since $\omega(s'_1, s'_2) dz$ is a 1-form defined on the disk $D$, its residue at $z = 0$ is also zero. This implies that the residue at $z = 0$ of the last three terms in the bottom line of (6) is zero, thus proving that $\tilde{\mu}^* \Omega$ vanishes.
2.6 The General Case

Finally we remove the assumption that $P$ and $K^{1/2}$ can be trivialized over $\Sigma - \{pt\}$ as follows. In this general case we remove finitely many points $pt_1, \ldots, pt_n$ so that $P$ and $K^{1/2}$ are can be trivialized over $\Sigma^o := \Sigma - \{pt_1, \ldots, pt_n\}$ (see Satz 3.3 of [5] and Theorem 3 of [1] for more details). As before, we write $D_i := \text{Spec } \mathbb{C}[[z]]$ (i = 1, \ldots, n) for formal disks centered at $pt_i$ and $D_i^S := \text{Spec } \mathbb{C}(z)$ for punctured formal disks centered at $pt_i$. We write $g_i^{-1} \in G(D_i^S)$ (i = 1, \ldots, n) for the transition morphisms of $P$ from $\Sigma^o$ to $D_i$ and $T_i : D_i^S \rightarrow \mathbb{G}_m$ for the transition morphisms of $K^{1/2}$ from $\Sigma^o$ to $D_i$. Using these transitions, the bundle $X_P \otimes K^{1/2}$ can be described by the transition morphisms $T_i^{-1}g_i^{-1}$ for $i = 1, \ldots, n$, and, hence, the source stack $Y$ consists of tuples $(g_i, s^o, s'_i) \in \prod_{i=1}^n G(D_i^S) \times \text{Map}(\Sigma^o, X) \times \prod_{i=1}^n \text{Map}(D_i, X)$ satisfying the equations $s'_i = T_i^{-1}g_i^{-1}s^o$ for all $i$.

Given a tuple $(g_i, s^o, s'_i)$ as in the previous paragraph, a tangent vector to $Y$ at $(g_i, s^o, s'_i)$ is a tuple $(\dot{g}_i, \dot{s}^o, \dot{s}'_i) \in \prod_{i=1}^n g(D_i^S) \times \text{Map}(\Sigma^o, TX) \times \prod_{i=1}^n \text{Map}(D_i, TX)$ such that $\pi \circ \dot{s}^o = s^o$, $\pi \circ \dot{s}'_i = \dot{s}_i$ and

$$s'_i = T_i^{-1}g_i^{-1}s^o - \rho_{\alpha}(\dot{g}_i)$$

for all $i$, where $\pi : TX \rightarrow X$ is the natural projection.

For a point $(\dot{g}_i, \dot{s}^o, \dot{s}'_i) \in T^*Bun_G$, and a tangent vector $(\dot{g}_i, \dot{s}^o, \dot{s}'_i)$ to $T^*Bun_G$ at $(g_i, \dot{s}^o, \dot{s}'_i)$, we define $\Omega(\dot{g}_i, \dot{s}^o, \dot{s}'_i) := \sum_{i=1}^n \text{Res}_{z=0}(\langle \dot{s}'_i, \dot{g}_i \rangle \ dz)$. The exact same computation as in Section 2.4 justifies the following definition:

$$\Omega((\dot{g}_{i,1}, \dot{s}^o_{i,1}, \dot{s}'_{i,1}),(\dot{g}_{i,2}, \dot{s}^o_{i,2}, \dot{s}'_{i,2})) := \sum_{i=1}^n \text{Res}_{z=0}(\langle \dot{s}'_i, \dot{g}_i \rangle \ dz - \langle \dot{s}'_i, \dot{g}_{i,1} \rangle \ dz - \langle \dot{s}'_i, \dot{g}_{i,2} \rangle \ dz)$$

for tangent vectors $(\dot{g}_{i,j}, \dot{s}^o_j, \dot{s}'_{i,j})$ (j = 1, 2) to $T^*Bun_G$ at $(g_i, \dot{s}^o, \dot{s}'_i)$. This tells us that, if $(\dot{g}_i, s^o, s'_i)$ is a point in $Y$ and $(\dot{g}_{i,j}, \dot{s}^o_j, \dot{s}'_{i,j})$ are tangential to $Y$ at $(\dot{g}_i, s^o, s'_i)$ (j = 1, 2), then

$$\Omega((\dot{g}_{i,1}, d\mu(s^o_{i,1}), d\mu(s'_{i,1})), (\dot{g}_{i,2}, d\mu(s^o_{i,2}), d\mu(s'_{i,2}))) = \sum_{i=1}^n \text{Res}_{z=0}(\omega(s^o_{i,1}, \rho_{\alpha}(\dot{g}_{i,2})) \ dz + \omega(s'_{i,2}, \rho_{\alpha}(\dot{g}_{i,1})) \ dz - \langle \mu \circ s'_i, [\dot{g}_{i,1}, \dot{g}_{i,2}] \rangle \ dz).$$  \hspace{1cm} (7)

We conclude that the bottom line of (7) is zero. Recall that $\alpha$ is a nowhere vanishing 1-form on $\Sigma^o$ used in the trivialization of $K$. A computation similar to the one in the previous section tells us that

$$\omega(s'_{i,1}, s'_{i,2}) \ dz - \omega(s^o_{i,1}, s^o_{i,2}) \ dz$$

$$= \omega(s^o_{i,1}, \rho_{\alpha}(\dot{g}_{i,2})) \ dz + \omega(s^o_{i,2}, \rho_{\alpha}(\dot{g}_{i,1})) \ dz - \langle \mu \circ s'_i, [\dot{g}_{i,1}, \dot{g}_{i,2}] \rangle \ dz$$

for all $i$. Since $\omega(s^o_{i,1}, s^o_{i,2}) \ dz$ is a 1-form defined on $D_i$, its residue at $z = 0$ is zero. From this we see that the bottom line of (7) equals $-\sum_{i=1}^n \text{Res}_{z=0}(\omega(s^o_{i,1}, s^o_{i,2}) \ dz)$. This is zero, again because the sum of residues of a rational 1-form on $\Sigma$ is equal to zero.

Acknowledgments. The author would like to thank V. Ginzburg for introducing this problem to the author, for his continual support and for many invaluable discussions during the preparation of this note. The author is also grateful to A. Beilinson for answering several questions about trivializations of principal $G$-bundles and to M. Nori for discussions about a preliminary version of this note.
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