STABILITY ESTIMATES FOR TIME-DEPENDENT COEFFICIENTS APPEARING IN THE MAGNETIC SCHRÖDINGER EQUATION FROM ARBITRARY BOUNDARY MEASUREMENTS

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Abstract. In this work, we study the stable determination of time-dependent coefficients appearing in the Schrödinger equation from partial Dirichlet-to-Neumann map measured on an arbitrary part of the boundary. Specifically, we establish stability estimates up to the natural gauge for the magnetic potential.

1. Introduction and main results. In the present work, we deal with an inverse problem for a Schrödinger equation with a time-dependent electromagnetic potential. For this, we consider \( \Omega \subset \mathbb{R}^n \) a bounded and simply connected domain with smooth boundary \( \Gamma = \partial \Omega \), \( n \geq 2 \). We set \( Q = (0,T) \times \Omega \) and \( \Sigma = (0,T) \times \Gamma \) where \( T > 0 \) is fixed and sufficiently large. We introduce the following initial-boundary value problem (IBVP in short) for the Schrödinger equation

\[
\begin{aligned}
(i \partial_t + \mathcal{H}_{A,q})u := (i \partial_t + \Delta_A + q)u &= 0 \quad \text{in } Q, \\
u(0, \cdot) &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \Sigma,
\end{aligned}
\]

where \( q \in L^\infty(Q, \mathbb{R}) \) is a real-valued electric potential and \( \Delta_A \) the magnetic Laplacian associated to the magnetic potential \( A = (a_j)_{1 \leq j \leq n} \in W^{1,\infty}(Q, \mathbb{R}^n) \) and given by

\[
\Delta_A := \sum_{k=1}^n (\partial_{x_k} + ia_k(t,x))^2 = \Delta + 2iA(t,x) \cdot \nabla + i \text{div} A(t,x) - |A(t,x)|^2.
\]

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To study the well-posedness of the problem of interest, we start by introducing the following functional spaces. For $p, q > 0$, we set

$$H^{p,q}(Q) = H^p(0, T; L^2(\Omega)) \cap L^p(0, T; H^q(\Omega)),$$

$$H^{p,q}(\Sigma) = H^p(0, T; L^2(\Gamma)) \cap L^p(0, T; H^q(\Gamma)),$$

equipped with the norm

$$\|u\|_{H^{p,q}(0, T) \times Z}^2 = \|u\|_{H^p(0, T; L^2(Z))}^2 + \|u\|_{L^p(0, T; H^q(Z))}^2,$$

where $Z$ being either $\Omega$ or $\Gamma$. Therefore, for any Dirichlet condition disturbing the system

$$f \in \mathcal{H}_0(\Sigma) := \{ f \in H^{0,2}(\Sigma); f(0, \cdot) = \partial_t f(0, \cdot) = 0 \text{ on } \Gamma \},$$

we get the following existence and uniqueness result which is proved in [29]: Let $M > 0$, $A \in W^{2,\infty}(Q, \mathbb{R}^n)$ and $q \in W^{1,\infty}(Q, \mathbb{R})$ such that

$$\|A\|_{W^{2,\infty}(Q)} + \|q\|_{W^{1,\infty}(Q)} \leq M.$$

Then, for any $f \in \mathcal{H}_0(\Sigma)$, the IBVP (1) admits a unique solution $u \in H^{1,2}(Q)$. In addition, there exists a positive constant $C$ such that

$$\|u\|_{H^{1,2}(Q)} \leq C \|f\|_{\mathcal{H}_0(\Sigma)}.$$

In this paper, we focus on measuring arbitrarily, so that we consider an arbitrary non-empty relatively open subset $\Gamma$ of $\Gamma$. Then we set $\Sigma = (0, T) \times \Gamma$ and we introduce the partial Dirichlet-to-Neumann map (D-to-N map in short) associated to the IBVP (1)

$$\Lambda_{A,q}^\sharp : \mathcal{H}_0(\Sigma) \rightarrow L^2(\Sigma),$$

$$f \mapsto (\partial_\nu + iA \cdot \nu)u|_{\Sigma},$$

which is bounded and where $\nu$ is the outward unit normal vector to $\Gamma$.

The goal of this work is to treat the inverse problem of stably recuperating the electromagnetic potential $(A, q)$ from the knowledge of the partial operator $\Lambda_{A,q}^\sharp$.

Yet, we find an obstruction for the uniqueness of the present problem in terms that the D-to-N map $\Lambda_{A,q}^\sharp$ is not injective. As a matter of fact, and as it was mentioned in [22], the D-to-N map is invariant under the gauge transformation of the magnetic potential. Precisely, for any $\Phi \in W^{3,\infty}(Q)$ such that $\Phi = 0$ on $\Sigma$, one has

$$e^{-i\Phi} H_{A,q} e^{i\Phi} = H_{A+\nabla\Phi,q-\partial_\nu} = e^{-i\Phi} A_{A,q}^\sharp e^{i\Phi} = \Lambda_{A,q}^\sharp.$$

Hence, the magnetic potential cannot be uniquely determined by the D-to-N map $\Lambda_{A,q}^\sharp$. In geometric terms, the vector field $A$ defines the connection given by the one-form $A = \sum_{j=1}^n a_j \, dx_j$, and the non-uniqueness illustrated in (3), leads us to wish to restore the magnetic field $\text{curl}(A)$ from the D-to-N map $\Lambda_{A,q}^\sharp$, which is defined by

$$\text{curl}(A) = \sum_{j,k=1}^n \left( \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j} \right) dx_j \wedge dx_k.$$

That’s the case in [6], where Bellassoued and Choulli treated the dynamical Schrödinger equation in the presence of a time-independent magnetic potential, and stably recovered the magnetic field $\text{curl}(A)$ induced by the magnetic potential $A$. 
from the knowledge of the global Dirichlet-to-Neumann map on the whole boundary. Besides, and by imposing some geometric conditions on the domain, Chung showed in [15] that the knowledge of the Dirichlet-to-Neumann map on certain subsets of the boundary uniquely determines the magnetic Schrödinger operator.

Hence, and when working on a Riemannian manifold, Bellassoued-Dos Santos Ferreira [7] and Bellassoued [4] stably recovered the electric potential or the magnetic field from the knowledge of the dynamical D-to-N map associated to the Schrödinger equation. We mention also the recent work of Bellassoued, Kian and Soccorsi [9] where such results have been extended to unbounded cylindrical domain. Further, we can refer to several papers which similarly treated such inverse problems such as [23, 26].

In the case of a finite number of boundary obervations of the solutions, Bellassoued and Choulli derived in [5] a logarithmic stability estimate for the inverse problem of determining the potential appearing in the dynamical Schrödinger equation which is assumed to be known in a neighborhood of the boundary, with a single measurement on an arbitrary given subboundary. In addition, we cite [16] where Cristofol and Soccorsi restored the magnetic potential in the Coulomb gauge class by finitely measuring the solution of the Schrödinger equation. Recently, and by a similar type of boundary data, Ben Aïcha and Mejri recovered simultaneously in [3] the electric potential and the divergence free magnetic potential.

For stationary Schrödinger equation and from local observations as shown by Dos Santos Ferreira, Sjöstrand, Kenig and Uhlmann in [17], the magnetic field curl$(A)$ and the electric potential $q$ are uniquely determined even if the measurements are taken only on a small part of the boundary. Further, in [34], Tzou established a log-log type stability estimate for curl$(A)$ and $q$ when measuring only on a chosen subset of $\partial \Omega$ which is slightly larger than half of the boundary, and if one has full data measurements, the result can be improved to a log-type estimate.

Clearly, all the above cited results are specific with time-independent coefficients and the publications concerned with the study of the inverse problem of determining the time-dependent potentials in a Schrödinger equation are very limited. This is not the case for hyperbolic and parabolic equations where similar inverse problems have been extensively treated such as [10, 1, 12, 13, 21, 24, 25, 27, 28, 31, 33]. Further, we cite [14] in which Choulli, Kian and Soccorsi proved a logarithmic stability estimate of determining a time-dependent electric potential from boundary measurements in a one-periodic quantum cylindrical waveguide. This result was extended by Ben Aïcha [2] to the full electromagnetic potential, where the knowledge of the D-to-N map stably determines the magnetic field and the time-dependent electric potential.

In addition, it was proved by Eskin [22] that the time-dependent electric and magnetic potentials are uniquely determined by the D-to-N map in domains with obstacles. This paper is an extension of the author’s works [18, 19, 20] to the case of time-dependent potentials. Yet, Kian and Tetlow considered in [30] the inverse problem of Hölder stably determining the time- and space-dependent coefficients of the dynamical Schrödinger equation on a simple Riemannian manifold from the knowledge of the Dirichlet-to-Neumann map on the whole boundary. Similar results have been obtained by Kian and Soccorsi in [29] where the authors treated the case of a bounded subset of $\mathbb{R}^n$ with the Euclidean metric.
Furthermore, and as it was shown in [29], the time-dependent magnetic potential $A$ itself and the electric potential $q$ are Hölder stably determined from the global Dirichlet-to-Neumann map: $\Lambda_{A,q} : f \mapsto (\partial_t + iA \cdot \nu)u|_{\Sigma}$, provided that the divergence $\nabla \cdot A$ is known in $Q$. We prove in this paper an estimate, which shows that the electromagnetic potential $(A, q)$ depends stably on the partial D-to-N map $\Lambda_{A,q}^t$. More precisely, we generalize the previous result by observing on an arbitrary subboundary $\Sigma = (0, T) \times \Gamma_2$ and assuming that the magnetic potential $A$ and the electric potential $q$ are known in a neighborhood of the boundary. Further, and by removing the condition on $\nabla \cdot A$, we only recover the magnetic field curl$(A)$ induced by the magnetic potential $A$, provided that the electromagnetic potential $(A, q)$ vanishes near the boundary $\Gamma$.

First of all, we consider $O \subset \bar{O}$ an arbitrary neighborhood of $\Gamma$ and we introduce the admissible sets of coefficients $A$ and $q$: For $M > 0$, we define

$$A(M, O) := \{ A \in W^{0,\infty}(Q, \mathbb{R}^n), \| A \|_{W^{0,\infty}(Q)} \leq M, A = 0 \text{ on } (0, T) \times \partial O \},$$

$$Q(M, O) := \{ q \in W^{4,\infty}(Q, \mathbb{R}), \| q \|_{W^{4,\infty}(Q)} \leq M, q = 0 \text{ on } (0, T) \times O \}.$$ Then, we establish a stability estimate for the electromagnetic potential $(A, q)$ appearing in the Schrödinger equation (1), from observations given by the partial D-to-N map $\Lambda_{A,q}^t$.

**Theorem 1.1.** Let $M > 0$, $T > 0$, $A_k \in A(M, O)$ and $q_k \in Q(M, O), k = 1, 2$. Then, for any $T^* < T$, there exist $C > 0$ depending on $\Omega, T, T^*, M$ and $\beta \in (0, 1)$ such that we obtain

$$\| \text{curl}(A_1) - \text{curl}(A_2) \|_{L^2(0, T^*; H^s(\Omega))} \leq C \| \log \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \| \|^{-\beta},$$

provided that $\| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \|$ is small.

Further, if we assume the following condition

$$\nabla \cdot A_1(t, x) = \nabla \cdot A_2(t, x), \ (t, x) \in Q,$$

then, there exist $C > 0$ and $\beta_1, \beta_2 \in (0, 1)$ such that

$$\| A_1 - A_2 \|_{L^2(0, T^*; H^s(\Omega))} \leq C \| \log \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \| \|^{-\beta_1},$$

and

$$\| q_1 - q_2 \|_{L^2(\Omega^*)} \leq C \| \log \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \| \|^{-\beta_2},$$

provided that $\| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \|$ is small.

This article is organized as follows. In Section 2, we introduce some preliminaries and estimates for the Schrödinger equation. In Section 3, we construct geometric optics solutions used for the proof of the main results in Section 4 and 5. For the last Section, we establish a special unique continuation estimate by applying the specific Fourier-Bros-Iagolnitzer (F.B.I) transformation.

2. **Well-posedness and unique continuation.** This section will be devoted to determine some results which are strongly needed to prove our Theorem. First, let's introduce the following notations which will be used in the whole coming parts. We consider three open subsets $O_j, j = 1, 2, 3$ of $O$ neighborhood of the boundary $\Gamma$ such that

$$O_{j+1} \subset O_j, \ O_j \subset O.$$
Further, we set
\[ \Omega_j = \Omega \setminus \mathcal{O}_j, \quad Q_j = (0, T) \times \Omega_j. \]

2.1. The magnetic Schrödinger equation. Let’s start by the following Lemma for the existence of unique solutions proved in [29].

**Lemma 2.1.** Let \( T > 0, A, q \) and \( M \) defined as in Theorem 1.1 and suppose that \( g \in H^1(0, T; L^2(\Omega)) \) is such that \( g(0, \cdot) = 0 \). Then the following IBVP for the Schrödinger equation

\[
\begin{aligned}
(i \partial_t + \mathcal{H}_{A,q})u &= g & \text{in } Q, \\
u(0, \cdot) &= 0 & \text{in } \Omega, \\
u = 0 & \text{on } \Sigma,
\end{aligned}
\]

(8)

admits a unique solution
\[
u \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)).
\]

Furthermore, there exists \( C = C(T, \Omega, M) \) such that we obtain for any \( t \in (0, T) \),
\[
\|u(t, \cdot)\|_{H^2(\Omega)} + \|\partial_t u(t, \cdot)\|_{L^2(\Omega)} \leq C \|g\|_{H^1(0, T; L^2(\Omega))}.
\]

Moreover, we can prove, by multiplying the first equation of (8) by \( \pi \), integrating by parts and using the Grönwall’s inequality, that
\[
\|u(t, \cdot)\|_{L^2(\Omega)} \leq C \|g\|_{L^2(Q)}, \quad t \in (0, T).
\]

2.2. A unique continuation estimate. With the following Lemma, we will establish a stability estimate in the unique continuation of the solutions for the magnetic Schrödinger equation on an arbitrary open subset \( \Gamma_2 \) of \( \Gamma \) from lateral boundary data.

**Lemma 2.2.** Let \( q_2 \in \mathcal{Q}(M, \mathcal{O}) \) and \( A_2 \in \mathcal{A}(M, \mathcal{O}) \). We consider \( \tilde{w} \in H^{1,2}(Q) \) a solution of the following system

\[
\begin{aligned}
(i \partial_t + \mathcal{H}_{A_2, q_2})\tilde{w} &= g_0 & \text{in } Q, \\
\tilde{w}(0, \cdot) &= 0 & \text{in } \Omega, \\
\tilde{w} &= 0 & \text{on } \Sigma,
\end{aligned}
\]

(11)

where \( g_0 \in L^2(Q) \) and \( \text{Supp}(g_0) \subset (0, T) \times (\Omega \setminus \mathcal{O}) \). Then for any \( T^* \in (0, T) \), there exist \( C > 0, \gamma_+ > 0, m_1 > 0 \) and \( \mu_1 < 1 \) such that the following estimate holds
\[
\|\tilde{w}\|_{L^2((0, T^*) \times (\Omega_3 \setminus \Omega_2))} \leq C \left( \gamma^{-\mu_1} \|\tilde{w}\|_{H^{1,2}(Q)} + e^{m_1 \gamma} \|\partial_t \tilde{w}\|_{L^2(\Sigma_1)} \right),
\]

for any \( \gamma > \gamma_. \) Here the constants \( C, m_1 \) and \( \mu_1 \) depend on \( \Omega, \mathcal{O}, T^* \) and \( T \).

From Lemma 2.2, we can derive the following unique continuation property.

**Corollary 1.** Let \( q_2 \in \mathcal{Q}(M, \mathcal{O}), A_2 \in \mathcal{A}(M, \mathcal{O}) \) and \( \tilde{w} \in H^{1,2}(Q) \) a solution of (11) where \( g_0 \in L^2(Q) \) and \( \text{Supp}(g_0) \subset (0, T) \times (\Omega \setminus \mathcal{O}) \) such that \( \partial_n \tilde{w} = 0 \) on \( \Sigma_2 \). Then \( \tilde{w} = 0 \) in \((0, T) \times (\Omega_3 \setminus \Omega_2)\).

The proof of Lemma 2.2 is given in Section 6.
3. Geometric optics solutions. In this section, we construct special solutions for the electromagnetic Schrödinger equation. These solutions called geometric optics solutions are the main key for the proof of our main results.

Let’s first introduce some notations and definitions needed for all the following parts. The structure of our geometric optics solutions is similar to the one used in [29].

Given $A_j \in \mathcal{A}(M, \mathcal{O})$ and $q_j \in Q(M, \mathcal{O})$, $j = 1, 2$, and put $A(t, \cdot) = (A_1 - A_2)(t, \cdot)$, $q(t, \cdot) = (q_1 - q_2)(t, \cdot)$ extended by 0 outside $\Omega$, we begin by introducing two functions defined as follows.

Let $T^* < T$ and let $h \in (0, T^*/4)$, we set the first function $\theta = \theta_h \in C_{0}^{\infty}(\mathbb{R})$ verifying $0 \leq \theta \leq 1$ and supported in $(h, T^* - h)$, such that

$$\theta(t) = 1, \quad \text{if } t \in [2h, T^* - 2h],$$

and satisfies

$$\forall j \in \mathbb{N}, \exists C_j > 0, \|\theta\|_{W^{1, \infty}(\mathbb{R})} \leq C_j h^{-j}.$$

Further, for $\omega \in \mathbb{S}^{n-1}$, $\tau \in \mathbb{R}$, $\xi \in \omega^\perp := \{ x \in \mathbb{R}^n; x \cdot \omega = 0 \}$ and $\zeta \in \mathbb{S}^{n-1} \cap \omega^\perp$, we define the second function $\eta \in W^{5, \infty}(Q)$ by

$$\eta(t, x) = \zeta \cdot \nabla\left(e^{-i(\tau + \tau x) \omega}\exp\left(-i \int_{\mathbb{R}} \omega \cdot A(t, x + s\omega) \, ds\right)\right), \quad (t, x) \in (0, T) \times \mathbb{R}^n,$$

where $A = A_1 - A_2$ (so that $A = A_1 - A_2$ in $Q$ and $A_j = 0$ in $(0, T) \times (\mathbb{R}^n \setminus \Omega)$), and that’s how we get

$$\omega \cdot \nabla \eta(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \quad \|\eta\|_{W^{5, \infty}(Q)} \leq C\langle\tau, \xi\rangle^6,$$

where the constant $C$ depends only on $M$, and $\langle\tau, \xi\rangle$ denotes for $(1 + \tau^2 + \xi^2)^{1/2}$.

At this stage, we introduce the two following functions in $W^{5, \infty}(Q)$ defined by

$$a_{1,1}(t, x) = \theta(t) \eta(t, x) \exp\left(i \int_{0}^{\infty} \omega \cdot A_1(t, x + s\omega) \, ds\right),$$

$$a_{2,1}(t, x) = \theta(t) \exp\left(i \int_{0}^{\infty} \omega \cdot A_2(t, x + s\omega) \, ds\right).$$

Then, we get that these functions satisfy

$$\omega \cdot \nabla A_j a_{j,1} = 0, \quad \text{in } Q,$$

and

$$a_{j,1}(t, x) = 0, \quad x \in \Omega, \quad t \in (0, h) \cup (T^* - h, T^*), \quad j = 1, 2.$$

Further, there exists a constant $C > 0$ such that

$$\|a_{1,1}\|_{H^3(Q)} \leq C\langle\tau, \xi\rangle^4 h^{-3} \quad \text{and} \quad \|a_{2,1}\|_{H^3(Q)} \leq C h^{-3},$$

where $C$ depends only on $\Omega$, $T$ and $M$. In addition, by decomposing $x \in \mathbb{R}^n$ into $x = y + s\omega$ where $s = x \cdot \omega$ and $y = x - s\omega \in \omega^\perp$, for $j = 1, 2$, we set

$$a_{j,2}(t, x) = -\frac{1}{2i} \int_{0}^{s} \exp\left(-i \int_{s_1}^{s} \omega \cdot A_j(t, y + s_2\omega) \, ds_2\right) (i \partial_t + \Delta_{A_j} + q_j) a_{j,1}(t, y + s_1\omega) \, ds_1.$$

Then, a direct calculation shows that $a_{j,2}$, $j = 1, 2$, solves

$$2i \omega \cdot \nabla A_j a_{j,2} + (i \partial_t + \Delta_{A_j} + q_j) a_{j,1} = 0, \quad \text{in } Q,$$

and satisfies

$$a_{j,2} \in W^{3, \infty}(Q), \quad a_{j,2}(t, x) = 0, \quad x \in \Omega, \quad t \in (0, h) \cup (T^* - h, T^*), \quad j = 1, 2.$$
Thus, we get the following estimates
\[ \|a_{1,2}\|_{H^3(Q)} \leq C(\tau, \xi)^5 h^{-4} \] and
\[ \|a_{2,2}\|_{H^3(Q)} \leq C h^{-4}, \]
and
\[ \|(i\partial_t + \mathcal{H}_{A_1, q_1})a_{1,2}\|_{L^2(Q)} \leq C(\tau, \xi)^5 h^{-2} \] and
\[ \|(i\partial_t + \mathcal{H}_{A_2, q_2})a_{2,2}\|_{L^2(Q)} \leq C h^{-2}. \]
We are now in position to present the following Lemma in which we construct special solutions to our electromagnetic Schrödinger equation and which are the main tool strongly needed for the proof of the basic results.

**Lemma 3.1.** We consider \( \omega \in \mathbb{S}^{n-1}, \sigma > 0, A_1 \in \mathcal{A}(M, \mathcal{O}), q_1 \in \mathcal{Q}(M, \mathcal{O}). \) The following equation
\[ (i\partial_t + \mathcal{H}_{A_1, q_1})u(t, x) = 0, \quad (t, x) \in Q := (0, T) \times \Omega, \quad u(0, x) = 0, \quad x \in \Omega, \]
has a solution of the form
\[ u_{1,\sigma}(t, x) = (a_{1,1}(t, x) + \sigma^{-1}a_{1,2}(t, x))e^{i\sigma(x \cdot \omega - \sigma t)} + r_{1,\sigma}(t, x), \]
where \( r_{1,\sigma} \) satisfies
\[ r_{1,\sigma}(t, x) = 0, \quad (t, x) \in \Sigma := (0, T) \times \Gamma, \]
and
\[ r_{1,\sigma}(0, x) = 0, \quad x \in \Omega. \]
Moreover, there exists \( C > 0 \) such that
\[ \sigma\|r_{1,\sigma}(t, \cdot)\|_{L^2(\Omega)} + \|r_{1,\sigma}(t, \cdot)\|_{H^1(\Omega)} \leq C(\tau, \xi)^6 h^{-3}, \quad t \in (0, T), \]
and
\[ \|\partial_t r_{1,\sigma}(t, \cdot)\|_{L^2(\Omega)} \leq C \sigma(\tau, \xi)^6 h^{-4}, \quad t \in (0, T). \]

**Proof.** We give an idea of the proof. By considering (17), we see that \( r_{1,\sigma} \) must be the solution of the following IBVP
\[ (i\partial_t + \mathcal{H}_{A_1, q_1})r_{1,\sigma}(t, x) = g_{\sigma}(t, x) \]
\[ r_{1,\sigma}(0, x) = 0 \]
\[ r_{1,\sigma}(t, x) = 0 \]
in \( Q \),
in \( \Omega \),
on \( \Sigma \),
where
\[ g_{\sigma}(t, x) := -(i\partial_t + \mathcal{H}_{A_1, q_1})(a_{\sigma}(t, x) \psi_{\sigma}(t, x)). \]
Here \( a_{\sigma}(t, x) := a_{1,1}(t, x) + \sigma^{-1}a_{1,2}(t, x) \) and \( \psi_{\sigma}(t, x) := e^{i\sigma(x \cdot \omega - \sigma t)} \). Now, an elementary calculation gives that
\[ [i\partial_t + \mathcal{H}_{A_1, q_1}, \psi_{\sigma}] := [i\partial_t + \Delta_{A_1 + q_1}, \psi_{\sigma}] = i\partial_t \psi_{\sigma} + \Delta \psi_{\sigma} + 2\nabla \psi_{\sigma} \cdot \nabla A_1 + q_1 \psi_{\sigma} = 2i\sigma \psi_{\sigma} \omega \cdot \nabla A_1 + q_1 \psi_{\sigma}, \]
which implies, by recalling (12) and (14), that
\[ g_{\sigma} := -(i\partial_t + \mathcal{H}_{A_1, q_1})(a_{\sigma} \psi_{\sigma}) = -(i\partial_t + \Delta_{A_1 + q_1})(a_{\sigma} \psi_{\sigma}) \]
\[ = -\psi_{\sigma}(2i\sigma \psi_{\sigma} \omega \cdot \nabla A_1 a_{1,1} + (i\partial_t + \mathcal{H}_{A_1, q_1})a_{1,1} + 2i\omega \cdot \nabla A_1 a_{1,2} + \sigma^{-1}(i\partial_t + \mathcal{H}_{A_1, q_1})a_{1,2}) \]
\[ = -\sigma^{-1} \psi_{\sigma}(i\partial_t + \mathcal{H}_{A_1, q_1})a_{1,2}. \]
Now, and by referring to Lemma 2.1, we show that $r_{1,\sigma}$ is a well defined solution of the equation (20), and it satisfies
\[ r_{1,\sigma} \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega) \cap H^1_0(\Omega)). \]
Further, (10) and (16) entail
\[ (21) \quad \|r_{1,\sigma}(t,\cdot)\|_{L^2(\Omega)} \leq C \sigma^{-1} \|\left(i\partial_t + \mathcal{H}_{A_1,q_1}\right) q_{1,2}\|_{L^2(Q)} \leq C \sigma^{-1} \langle r,\xi \rangle^5 h^{-2}, \quad t \in (0,T). \]
In addition, we get from (9) and (15) that
\[ (22) \quad \|\partial_t r_{1,\sigma}(t,\cdot)\|_{L^2(\Omega)} + \|r_{1,\sigma}(t,\cdot)\|_{H^2(\Omega)} \leq C \|g\|_{H^2([0,T]; L^2(\Omega))} \leq C \sigma \langle r,\xi \rangle^6 h^{-4}, \quad t \in (0,T). \]
Then, by interpolating with (21), we obtain
\[ (23) \quad \|r_{1,\sigma}(t,\cdot)\|_{H^1(\Omega)} \leq C \sigma \langle r,\xi \rangle^6 h^{-3}, \quad t \in (0,T). \]
So that we conclude (18) from (21)-(23) and we obtain (19) from (22).

By a similar way and strategy for the proof, we get a second result for the backward equation.

**Lemma 3.2.** Let $\omega \in S^{n-1}$, $\sigma > 0$, $A_2 \in \mathcal{A}(M,\mathcal{O})$, $q_2 \in \mathcal{Q}(M,\mathcal{O})$. The following equation
\[ (i\partial_t + \mathcal{H}_{A_2,q_2})u(t,x) = 0, \quad u(T^*,x) = 0, \quad (t,x) \in Q^* := (0,T^*) \times \Omega, \]
has a solution of the form
\[ u_{2,\sigma}(t,x) = (a_{2,1}(t,x) + \sigma^{-1} a_{2,2}(t,x)) e^{i\sigma(x \cdot \omega - \sigma t)} + r_{2,\sigma}(t,x), \]
where $r_{2,\sigma}$ satisfies
\[ r_{2,\sigma}(t,x) = 0, \quad (t,x) \in \Sigma^* := (0,T^*) \times \Gamma, \]
and
\[ r_{2,\sigma}(T^*,x) = 0, \quad x \in \Omega. \]
Moreover, there exists $C > 0$ such that
\[ (24) \quad \sigma \|r_{2,\sigma}(t,\cdot)\|_{L^2(\Omega)} + \|r_{2,\sigma}(t,\cdot)\|_{H^1(\Omega)} \leq C \sigma h^{-3}, \quad \forall t \in (0,T^*). \]

4. **Stable recovery of the magnetic potential.** At the first stage, we establish stability estimates for the magnetic potential $A$. The key ingredients for the proof will be based on using the geometric optics solutions already constructed in Section 3 and on applying the special continuation estimate established in Lemma 2.2.

4.1. **Recovery of the magnetic potential without Coulomb gauge.** In the beginning, we will deal with any magnetic potentials $A_j$ where $\nabla \cdot (A_1 - A_2)$ is not fixed.

**Lemma 4.1.** There exist $C > 0$, $m_1 > 0$ and $\mu < 1$ such that the following estimate
\[ (25) \quad \left| \int_{\mathbb{R}^{n+1}} \omega \cdot A(t,x) \theta^2(t) \eta(t,x) \exp\left(i \int_0^\infty \omega \cdot A(t,x + s\omega) ds\right) dx dt \right| \leq C \left(e^{m_1^2 \gamma(\tau,\xi)} h^{-8} \|A^4_{A_1,q_1} - A^4_{A_2,q_2}\| + \gamma^{-\mu(\tau,\xi)} h^{-8}\right), \]
holds true uniformly in $\xi \in \omega^\perp$, where $C$ depends only on $\Omega$, $T$, $T^*$ and $M$. 

**Inverse Problems and Imaging** Volume 14, No. 5 (2020), 841–865
Proof. Let $a_{1,j}$, $j = 1, 2$, as above. Then, by Lemma 3.1, there exists a geometric optics solutions for the Schrödinger equation
\begin{equation}
(i\partial_t + \mathcal{H}_{A_1,q_1})u_{1,\sigma} = 0, \quad \text{in } Q,
\end{equation}
of the form
\begin{equation}
u_{1,\sigma}(t, x) = (a_{1,1}(t, x) + \sigma^{-1}a_{1,2}(t, x)) e^{i\sigma(x\omega - \sigma t)} + r_{1,\sigma}(t, x),
\end{equation}
with initial condition
\begin{equation}
u_{1,\sigma}(0, \cdot) = 0, \quad \text{in } \Omega,
\end{equation}
and where $r_{1,\sigma}$ satisfies
\begin{equation}
r_{1,\sigma}(t, x) = 0, \quad (t, x) \in \Sigma,
\end{equation}
and
\begin{equation}
\sigma \|r_{1,\sigma}(t, \cdot)\|_{L^2(\Omega)} + \|r_{1,\sigma}(t, \cdot)\|_{H^1(\Omega)} \leq C (\tau, \xi^6 h^{-3}), \quad t \in (0, T).
\end{equation}

Further, we set $f_\sigma = u_{1,\sigma}\mid_\Sigma = (a_{1,1}(t, x) + \sigma^{-1}a_{1,2}(t, x)) e^{i\sigma(x\omega - \sigma t)}$, and let $v_2$ be the $H^{1,2}(Q)$ solution of the IBVP
\begin{equation}
\begin{cases}
(i\partial_t + \mathcal{H}_{A_2,q_2})v_2 = 0 & \text{in } Q, \\
v_2(0, \cdot) = 0 & \text{in } \Omega, \\
v_2 = f_\sigma & \text{on } \Sigma.
\end{cases}
\end{equation}

Then, by defining $w = v_2 - u_{1,\sigma}$, we get
\begin{equation}
\begin{cases}
(i\partial_t + \mathcal{H}_{A_2,q_2})w = 2iA \cdot \nabla u_{1,\sigma} + G_{A,q}u_{1,\sigma} & \text{in } Q, \\
w(0, \cdot) = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \Sigma,
\end{cases}
\end{equation}
where $A = A_1 - A_2$, $q = q_1 - q_2$ and $G_{A,q} = i\text{div}(A) + |A_2|^2 - |A_1|^2 + q$. On the other side, we introduce a cut-off function $\chi \in C^\infty(\overline{\Omega})$ satisfying $0 \leq \chi \leq 1$ and
\begin{equation}
\chi(x) = \begin{cases}
0, & \text{if } x \in \mathcal{O}_3, \\
1, & \text{if } x \in \overline{\Omega} \setminus \mathcal{O}_2,
\end{cases}
\end{equation}
and if we put $\tilde{w}(t, x) = \chi(x)w(t, x)$, then we prove that $\tilde{w}$ is the solution of the following IBVP
\begin{equation}
\begin{cases}
(i\partial_t + \mathcal{H}_{A_2,q_2})\tilde{w} = 2iA \cdot \nabla u_{1,\sigma} + G_{A,q}u_{1,\sigma} + [\Delta, \chi] w & \text{in } Q, \\
\tilde{w}(0, \cdot) = 0 & \text{in } \Omega, \\
\tilde{w} = 0 & \text{on } \Sigma,
\end{cases}
\end{equation}
where we have used $\chi(x)q(t, x) = q(t, x)$ and $\chi(x)A(t, x) = A(t, x)$ in $Q$, since $A = 0$ and $q = 0$ in $(0, T) \times \mathcal{O}$. Similarly, by using Lemma 3.2, there exists a geometric optics solutions for the Schrödinger equation
\begin{equation}
(i\partial_t + \mathcal{H}_{A_2,q_2})u_{2,\sigma} = 0, \quad \text{in } Q^*, \quad u_{2,\sigma}(T^*, \cdot) = 0, \quad \text{in } \Omega,
\end{equation}
of the form
\begin{equation}
u_{2,\sigma}(t, x) = (a_{2,1}(t, x) + \sigma^{-1}a_{2,2}(t, x)) e^{i\sigma(x\omega - \sigma t)} + r_{2,\sigma}(t, x),
\end{equation}
and where \( r_{2,\sigma} \) satisfies

\[
    r_{2,\sigma}(t, x) = 0, \quad \forall (t, x) \in \Sigma^*,
\]

and

\[
    \sigma \| r_{2,\sigma}(t, \cdot) \|_{L^2(\Omega)} + \| r_{2,\sigma}(t, \cdot) \|_{H^1(\Omega)} \leq C h^{-3}, \quad t \in (0, T^*).
\]

Next, by multiplying the first equation of (30) by \( \bar{u}_{2,\sigma} \) and integrating by parts over \( Q^* \), we obtain

\[
    \int_{Q^*} (2i A \cdot \nabla u_{1,\sigma} + G_{A_q} u_{1,\sigma}) \bar{u}_{2,\sigma} \, dx \, dt + \int_{Q^*} [\Delta, \chi] w \bar{u}_{2,\sigma} \, dx \, dt
    = \int_{Q^*} \bar{w} (i \partial_t + H_{A_{2,q}}) u_{2,\sigma} \, dx \, dt = 0.
\]

Furthermore, if we replace \( u_{1,\sigma} \) and \( u_{2,\sigma} \) by their expressions given by (27) and (31), we get

\[
    \int_{Q^*} (2i A \cdot \nabla u_{1,\sigma}(t, x) + G_{A_q} u_{1,\sigma}(t, x)) \bar{u}_{2,\sigma}(t, x) \, dx \, dt
    = -2\sigma \int_{Q^*} \omega \cdot A(t, x) (a_{1,1} \bar{u}_{2,1})(t, x) \, dx \, dt + R_{\sigma}
\]

\[
    = -2\sigma \int_{Q^*} \omega \cdot A(t, x) \theta^2(t) \eta(t, x) \exp \left( i \int_0^\infty \omega \cdot A(t, x + s\omega) \, ds \right) \, dx \, dt + R_{\sigma},
\]

where

\[
    R_{\sigma} = -2\sigma \int_{Q^*} \omega \cdot A(t, x) (\sigma^{-1} \bar{u}_{2,2} + \bar{\tau}_{2,\sigma} e^{i\sigma(x \cdot \omega - \sigma t)}) \bar{u}_{1,1}(t, x) \, dx \, dt
\]

\[
    - 2 \int_{Q^*} A(t, x) \cdot (\nabla r_{1,\sigma} + (\sigma^{-1} \nabla a_{1,2} + \nabla a_{1,1}) e^{i\sigma(x \cdot \omega - \sigma t)}) \bar{u}_{2,\sigma}(t, x) \, dx \, dt
\]

\[
    + \int_{Q^*} G_{A_q} (u_{1,\sigma} \bar{u}_{2,\sigma})(t, x) \, dx \, dt.
\]

Thus, by (28) and (32), we get the estimate

\[
    |R_{\sigma}| \leq C(\tau, \xi)^6 h^{-8}.
\]

Further, by (16) and (13), we obtain

\[
    \int_{Q^*} [\Delta, \chi] w \bar{u}_{2,\sigma} \, dx \, dt \leq C\| [\Delta, \chi] w \|_{L^2(0, T^*; H^{-1}(\Omega_1 \setminus \Omega_2))} \| u_{2,\sigma} \|_{L^2(0, T^*; H^1(\Omega))}
\]

\[
    \leq C\| w \|_{L^2((0, T^*) \times (\Omega_1 \setminus \Omega_2))} \| u_{2,\sigma} \|_{L^2(0, T^*; H^1(\Omega))}
\]

\[
    \leq C \left[ \gamma^{-\mu_1} \| w \|_{H^{1,1}(Q)} + e^{m_1 \gamma} \| \partial_{\nu} w \|_{L^2(\Sigma_2)} \right] \sigma h^{-4},
\]

where we have applied Lemma 2.2 and we have used the fact that \( g_{0,\sigma} = 2i A \cdot \nabla u_{1,\sigma} + G_{A_q} u_{1,\sigma} \equiv 0 \) on \((0, T) \times \Omega\). Also, we have

\[
    \| w \|_{H^{1,1}(Q)} \leq C (\| v_2 \|_{H^{1,1}(Q)} + \| u_{1,\sigma} \|_{H^{1,1}(Q)}).
\]

We apply the inequality (2) on \( v_2 \) solution of (29) and \( u_{1,\sigma} \) solution of (26) and we recall the estimates (13) and (15) to obtain

\[
    \| w \|_{H^{1,1}(Q)} \leq C \| f_{\sigma} \|_{H_0(\Sigma)} \leq C \sigma^6 (\tau, \xi)^6 h^{-4}.
\]
In addition
\[ \|\partial_{\nu} w\|_{L^2(\Sigma)} = \|\partial_{\nu} u_2 - \partial_{\nu} u_1, \sigma\|_{L^2(\Sigma)} \]
\[ \leq \| (\Lambda_{A_1,q_1}^4 - \Lambda_{A_2,q_2}^4)(f_\sigma)\|_{L^2(\Sigma)} \]
\[ \leq C\|\Lambda_{A_1,q_1}^4 - \Lambda_{A_2,q_2}^4\|_{H_0(\Sigma)} \]
\[ \leq C\sigma^6(\tau,\xi) h^{-8}\|\Lambda_{A_1,q_1}^4 - \Lambda_{A_2,q_2}^4\|. \]
(38)

Finally, collecting (33), (34), (35), (36), (37) and (38), we obtain
\[ \sigma \left| \int_{Q^*} \omega \cdot A(t,x) \theta^2(t) \eta(t,x) \exp(i \int_0^\infty \omega \cdot A(t,x + s\omega) ds) \, dx \, dt \right| \]
\[ \leq C \left( e^{m_1} \gamma (\tau,\xi)^6 h^{-8} + \gamma^* \sigma^6(\tau,\xi) h^{-8}\|\Lambda_{A_1,q_1}^4 - \Lambda_{A_2,q_2}^4\| + (\tau,\xi)^6 h^{-8} \right) \]

Choosing \( \sigma = \gamma^{m_1/7} \) and extending \( A \) by 0 outside \((0,T^*) \times \Omega\), we obtain
\[ \left| \int_{R^{n+1}} \omega \cdot A(t,x) \theta^2(t) \eta(t,x) \exp(i \int_0^\infty \omega \cdot A(t,x + s\omega) ds) \, dx \, dt \right| \]
\[ \leq C \left( e^{m_1} \gamma (\tau,\xi)^6 h^{-8}\|\Lambda_{A_1,q_1}^4 - \Lambda_{A_2,q_2}^4\| + \gamma^{m_1} (\tau,\xi)^6 h^{-8} \right) \]

where \( \mu = \mu_1/7 \). So, we complete the proof of the Lemma.

At this stage, we define the Fourier transform of a function \( f \in L^1(R^{n+1}) \) by
\[ \hat{f}(\tau,\xi) = (2\pi)^{-\frac{n+1}{2}} \int_{R^{n+1}} e^{-i(\tau \cdot x + \xi \cdot t)} f(t,x) \, dx \, dt. \]

Further, we set
\[ \rho_{jk}(t,x) = (\partial_j(\theta^2 A_k) - \partial_k(\theta^2 A_j))(t,x), \, j,k \in \{1,\ldots,n\}. \]

Then, we get an estimate on the Fourier transform of \( \rho_{jk}, \, j,k = 1,\ldots,n \), given by the following Lemma.

**Lemma 4.2.** There exist positive constant \( C, \gamma^*, m_2 \) and \( \mu_1 \) such that
\[ |\hat{\rho}_{jk}(\tau,\xi)| \leq C \left( e^{m_2} \gamma (\tau,\xi)^6 h^{-8}\|\Lambda_{A_1,q_1}^4 - \Lambda_{A_2,q_2}^4\| + \gamma^{-\mu_1} (\tau,\xi)^6 h^{-8} \right) , \]
(39)
holds true for every \( (\tau,\xi) \in R^{n+1} \) and any \( \gamma > \gamma^* \).
Proof. We will treat the case for \( \xi \neq 0 \) (the result is evident for \( \xi = 0 \)). Let \( \omega \perp \xi \), by decomposing \( x = y + r\omega \) where \( r = x \cdot \omega \) and \( y = x - r\omega \in \omega^\perp \), we set

\[
\int_{\mathbb{R}^n+1} \omega \cdot A(t, x) \theta^2(t) \eta(t, x) \exp(i \int_0^\infty \omega \cdot A(t, x + s\omega) \, ds) \, dx \, dt
= \int \int \omega \cdot A(t, y + r\omega) \theta^2(t) \eta(t, y) \exp(i \int_r^\infty \omega \cdot A(t, y + s\omega) \, ds) \, dy \, dr \, dt

= i \int \int \theta^2(t) \eta(t, y) \left( \int \partial_r \exp(i \int_r^\infty \omega \cdot A(t, y + s\omega) \, ds) \, dr \right) \, dy \, dt

= i \int \theta^2(t) \left( \int \eta(t, y) \left( 1 - \exp(i \int R \cdot A(t, y + s\omega) \, ds) \right) \, dy \right) \, dt

= i \int \theta^2(t) e^{-i\tau} \left( \int \omega^\perp \nabla \left( e^{-i\xi \cdot y} \exp(-i \int \omega \cdot A(t, y + s\omega) \, ds) \right) \right)

\left( 1 - \exp(i \int R \cdot A(t, y + s\omega) \, ds) \right) \, dy \, dt.

\]

Here, we note that, for \( R > 0 \) such that \( \text{Supp}(A(t, \cdot)) \subset B(0, R) \), we have

\[
1 - \exp(i \int R \cdot A(t, y + s\omega) \, ds) = 0, \quad \forall y \in \omega^\perp \cap B(0, R).
\]

Moreover, if we decompose \( \nabla w \), where \( w \in H^1(\mathbb{R}^n) \), into its orthogonal part \( \omega^\perp \) and \( \omega \)-part, we find

\[
\nabla \nabla w = \nabla \left( \nabla_y w + (\omega \cdot \nabla w) \omega \right) = \nabla \nabla w.
\]

Consequently, by integrating by parts, we obtain

\[
\int \omega^\perp \nabla \left( e^{-i\xi \cdot y} \exp(-i \int \omega \cdot A(t, y + s\omega) \, ds) \right) \right)
\left( 1 - \exp(i \int \omega \cdot A(t, y + s\omega) \, ds) \right) \, dy

= -i \int \omega^\perp \nabla_y \left( \int R \cdot A(t, y + s\omega) \, ds \right) \, dy.
\]

Therefore, by (40) and using the Fubini theorem, one’s get

\[
\int_{\mathbb{R}^n+1} \omega \cdot A(t, x) \theta^2(t) \eta(t, x) \exp(i \int_0^\infty \omega \cdot A(t, x + s\omega) \, ds) \, dx \, dt

= \int \theta^2(t) e^{-i\tau} \left( \int R \cdot A(t, y + s\omega) \, ds \right) \, dy \, dt

= \int \theta^2(t) e^{-i\tau} \left( \int R \cdot A(t, y + s\omega) \, ds \right) \, dy \, dt

= \int e^{-i(t \tau + x \cdot \xi)} \theta^2(t) \nabla \left( \omega \cdot A(t, x) \right) \, dx \, dt.
\]
Next, we apply the Stokes formula for the above integral and we take $\zeta = \frac{\xi}{|\xi|}$ to find

\[
i \int_{\mathbb{R}^{n+1}} \omega \cdot A(t, x) \theta^2(t) \eta(t, x) \exp\left(i \int_0^t \omega \cdot A(t, x + s\omega) \, ds\right) \, dx \, dt
\]

\[
= -|\xi| \int_{\mathbb{R}^{n+1}} \omega \cdot A(t, x) e^{-i(\tau + x \cdot \xi)} \theta^2(t) \, dx \, dt
\]

\[
= -(2\pi)^{\frac{n+1}{2}} |\xi| \omega \cdot \hat{\theta}^2 A(\tau, \xi),
\]

that's why it implies, recalling (25), to

\[
|\xi| |\omega \cdot \hat{\theta}^2 A(\tau, \xi)| \leq C\left(e^{m_1 \gamma (\tau, \xi) h^{-8} \|A_{\Lambda_1,q_1}^4 - A_{\Lambda_2,q_2}^4\| + \gamma^{-\mu} (\tau, \xi)^6 h^{-8}}\right).
\]

Let now

\[
\beta_j(t, x) = D_j (\omega \cdot \hat{\theta}^2 A(t, x)), \quad D_j = \frac{1}{i} \partial_j.
\]

Then, we get $\hat{\beta}_j(\tau, \xi) = \xi_j \omega \cdot \hat{\theta}^2 A(\tau, \xi)$ and by recalling (41), we obtain for each $j \in \{1, \ldots, n\}$

\[
|\hat{\beta}_j(\tau, \xi)| \leq |\xi| |\omega \cdot \hat{\theta}^2 A(\tau, \xi)| \leq C\left(e^{m_1 \gamma (\tau, \xi) h^{-8} \|A_{\Lambda_1,q_1}^4 - A_{\Lambda_2,q_2}^4\| + \gamma^{-\mu} (\tau, \xi)^6 h^{-8}}\right).
\]

Yet,

\[
\hat{\beta}_j(\tau, \xi) = \sum_{k=1}^{n} \xi_j \omega_k \theta^2 A_k(\tau, \xi) - \sum_{k=1}^{n} \xi_k \omega_k \theta^2 A_j(\tau, \xi)
\]

\[
= \sum_{k=1}^{n} \omega_k \left(\xi_j \hat{\theta}^2 A_k(\tau, \xi) - \xi_k \hat{\theta}^2 A_j(\tau, \xi)\right)
\]

\[
= i \sum_{k=1}^{n} \omega_k \left(\partial_j (\theta^2 A_k) - \partial_k (\theta^2 A_j)\right)(\tau, \xi)
\]

\[
= i \sum_{k=1}^{n} \omega_k \hat{\rho}_{jk}(\tau, \xi).
\]

Thus, we collect this with (42) to find that

\[
\sum_{k=1}^{n} \omega_k \hat{\rho}_{jk}(\tau, \xi) \leq C\left(e^{m_1 \gamma (\tau, \xi) h^{-8} \|A_{\Lambda_1,q_1}^4 - A_{\Lambda_2,q_2}^4\| + \gamma^{-\mu} (\tau, \xi)^6 h^{-8}}\right), \forall \xi \in \omega^\perp.
\]

We consider now $\xi \in \mathbb{R}^n, \ell, m \in \{1, \ldots, n\}$ with $\ell \neq m$ and let $\omega = \frac{\xi_\ell e_m - \xi_m e_\ell}{|\xi_\ell e_m - \xi_m e_\ell|}$, where $\omega \cdot \xi = 0$. Then, we get

\[
\sum_{k=1}^{n} \omega_k \hat{\rho}_{jk}(\tau, \xi) = \frac{1}{|\xi_\ell e_m - \xi_m e_\ell|} \left(\xi_\ell \hat{\rho}_{mj}(\tau, \xi) - \xi_m \hat{\rho}_{\ell j}(\tau, \xi)\right).
\]
So, if $|\xi_m| \leq |\xi_\ell|$, we take $j = \ell$ in (44) to find by (43) that
\[
|\xi_\ell \hat{\rho}_{m\ell}(\tau, \xi)| \leq C (|\xi_\ell| + |\xi_m|) \left( e^{m_1 \gamma} (\tau, \xi)^6 h^{-8} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \| + \gamma^{-\mu} (\tau, \xi)^8 h^{-8} \right)
\leq C |\xi_\ell| \left( e^{m_1 \gamma} (\tau, \xi)^6 h^{-8} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \| + \gamma^{-\mu} (\tau, \xi)^8 h^{-8} \right).
\]
That’s how
\[
|\hat{\rho}_{m\ell}(\tau, \xi)| \leq C \left( e^{m_1 \gamma} (\tau, \xi)^6 h^{-8} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \| + \gamma^{-\mu} (\tau, \xi)^8 h^{-8} \right).
\]
In the other hand and if $|\xi_m| \geq |\xi_\ell|$, we take $j = m$ in (44) and we obtain similarly that
\[
|\hat{\rho}_{m\ell}(\tau, \xi)| \leq C \left( e^{m_1 \gamma} (\tau, \xi)^6 h^{-8} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \| + \gamma^{-\mu} (\tau, \xi)^8 h^{-8} \right).
\]
Collecting (45) and (46), we get the desired estimate.

Now, we are arranged to establish the first magnetic stability estimate (4) for our problem. We start by estimating the $H^{-1}(\mathbb{R}^{n+1})$ norm of $\rho_{j,k}$, to get
\[
\|\rho_{j,k}\|_{H^{-1}(\mathbb{R}^{n+1})}^2 = \left( \int_{|\tau, \xi| < R} |(\tau, \xi)|^{-2} |\hat{\rho}_{j,k}(\tau, \xi)|^2 \, d\tau \, d\xi \right.
+ \left. \int_{|\tau, \xi| \geq R} |(\tau, \xi)|^{-2} |\hat{\rho}_{j,k}(\tau, \xi)|^2 \, d\tau \, d\xi \right)
\leq C \left( R^{n+1} \|\hat{\rho}_{j,k}\|^2_{L^2(\mathbb{R}^{n+1})} + R^{-2} \|\rho_{j,k}\|^2_{L^2(\mathbb{R}^{n+1})} \right),
\]
for fixed $R > 0$. Then, by combining (47) with (39), we obtain
\[
\|\rho_{j,k}\|^2_{H^{-1}(\mathbb{R}^{n+1})} \leq C \left( R^{n+13} h^{-16} e^{2 m_1 \gamma} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \|^2
+ R^{n+13} \gamma^{-\mu} h^{-16} + R^{-2} \right).\]
Further, by interpolating we get
\[
\|\rho_{j,k}\|^2_{L^2(Q^*)} \leq C \|\rho_{j,k}\|_{H^{-1}(Q^*)} \|\rho_{j,k}\|_{H^{-1}(Q^*)} \leq C \|\rho_{j,k}\|_{H^{-1}(\mathbb{R}^{n+1})} h^{-1}
\leq C \left( R^{n+13} h^{-9} e^{m_1 \gamma} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \|^2
+ R^{n+13} \gamma^{-\mu} h^{-9} + (R h)^{-1} \right).\]
Then, we get
\[
\|\theta^2 \text{curl}(A)\|^2_{L^2(Q^*)} \leq C \left( R^{n+13} h^{-9} e^{m_1 \gamma} \| \Lambda_{A_1, q_1}^z - \Lambda_{A_2, q_2}^z \|^2
+ R^{n+13} \gamma^{-\mu} h^{-9} + (R h)^{-1} \right).\]
On the other hand, we use the fact that
\[
\|\text{curl}(A) - \theta^2 \text{curl}(A)\|^2_{L^2(Q^*)} \leq \|1 - \theta^2\|^2_{L^2(0, T^*)} \|\text{curl}(A)\|^2_{L^\infty(Q^*)},
\]
and we keep in mind that the function $(1 - \theta)$ is valued in $[0, 1]$ and verifies $1 - \theta(t) = 0$ if $t \in [2h, T^* - 2h]$, so that
\[
\|1 - \theta^2\|^2_{L^2(0, T^*)} = \int_0^{2h} (1 - \theta^2(t))^2 \, dt + \int_{T^* - 2h}^{T^*} (1 - \theta^2(t))^2 \, dt \leq 4h,
\]
which leads to
\[
\|\text{curl}(A) - \theta^2 \text{curl}(A)\|^2_{L^2(Q^*)} \leq 4M h.
\]
Thus, by (48) and (49), we obtain
\[
\|\text{curl}(A)\|_{L^2(Q^*)}^2 \leq C \left( R^{n_1 + 1} h^{-9} e^{m_1 \gamma} \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}^2\| + R^{n_2 + 1} \gamma^{-9} h^{-1} + h \right).
\]

Now, we choose \( h \) such that \( R^{-1} = h^2 \) to get
\[
\|\text{curl}(A)\|_{L^2(Q^*)}^2 \leq C \left( R^{n_1} e^{m_1 \gamma} \|\Lambda_{A_1,q_1}^2 - \Lambda_{A_2,q_2}^2\| + R^{n_2} \gamma^{-\mu} + R^{-1/2} \right),
\]
where \( N_1 \) and \( N_2 \) are positive constants depending on \( n \), and by a similar way, we pick \( R \) that is to satisfy \( R^{n_2} \gamma^{-\mu} = R^{-1/2} \) in such a way that if \( \|\Lambda_{A_1,q_1}^2 - \Lambda_{A_2,q_2}^2\| \) is sufficiently small, we end up getting that
\[
\|\text{curl}(A)\|_{L^2(Q^*)} \leq C \left( e^{m' \gamma} \|\Lambda_{A_1,q_1}^2 - \Lambda_{A_2,q_2}^2\|^{1/2 + \gamma^{-N_3}} \right),
\]
where \( N_3 < 1 \) and \( m' > 0 \) depending on \( n \) and \( \mu \).

Thus, we pick \( \gamma \) such that
\[
\gamma = \frac{1}{2m'} \left| \log \|\Lambda_{A_1,q_1}^2 - \Lambda_{A_2,q_2}^2\| \right|,
\]
to find a constant \( \beta < 1 \) such that
\[
\|\text{curl}(A)\|_{L^2(Q^*)} \leq C \left| \log \|\Lambda_{A_1,q_1}^2 - \Lambda_{A_2,q_2}^2\| \right|^{-\beta}.
\]

Therefore, we get by interpolating
\[
\|\text{curl}(A)\|_{L^2((0,T^*,H^1(\Omega))} \leq \|\text{curl}(A)\|_{L^2(Q^*)}^p \|\text{curl}(A)\|_{L^2((0,T^*,H^1(\Omega))}^{1-p}
\leq M \|\text{curl}(A)\|_{L^2(Q^*)}^p
\leq C \left| \log \|\Lambda_{A_1,q_1}^2 - \Lambda_{A_2,q_2}^2\| \right|^{-\beta_1},
\]
for some constant \( p \in (0,1) \).

That’s how we complete the proof of the first stability estimate (4) dealing with the magnetic potential.

4.2. Recovery of the magnetic potential with Coulomb gauge. This part will be devoted to treat the case where the magnetic potential \( A = A_1 - A_2 \) satisfies the Coulomb gauge (5), that is
\[
\text{div}(A) = \text{div}(A_1) - \text{div}(A_2) = 0.
\]

By using the following Lemma, the result will be based on the previous part as we will see.

**Lemma 4.3.** Let \( t \in (0,T) \) be fixed. Assuming that the magnetic potential \( A \in \mathcal{A}(M,\mathcal{O}) \) satisfies \( \text{div}(A)(t,\cdot) = 0 \), in \( \Omega \), we have the following estimate
\[
(50) \quad \|A(t,\cdot)\|_{H^1(\Omega)} \leq C \|\text{curl}(A)(t,\cdot)\|_{L^2(\Omega)},
\]
for some constant \( C > 0 \).

**Proof.** In fact, and by referring to [32], if we contradict (50), we get the existence of a sequence \( (A_k)_k \) such that \( \|A_k(t,\cdot)\|_{H^1(\Omega)} = 1 \) and \( \text{div}(A_k(t,\cdot)) = 0 \), for each \( k \in \mathbb{N} \), and which verifies
\[
1 = \|A_k(t,\cdot)\|_{H^1(\Omega)} \geq k \|\text{curl}(A_k)(t,\cdot)\|_{L^2(\Omega)}.
\]
Thus, \( \lim_{k \to +\infty} \| \text{curl}(A_k(t, \cdot)) \|_{L^2(\Omega)} = 0 \) and so \( \text{curl}(A_k(t, \cdot)) \to 0 \) in \( L^2(\Omega) \). Now, and as long as \( \| A_k(t, \cdot) \|_{H^1(\Omega)} = 1 \), we can extract a subsequence noted also \( (A_k)_k \) such that \( A_k(t, \cdot) \to A(t, \cdot) \) in \( H^1(\Omega) \) and \( A_k(t, \cdot) \to A(t, \cdot) \) in \( L^2(\Omega) \). That’s how we get

\[
\text{curl}(A_k)(t, \cdot) \rightharpoonup \text{curl}(A)(t, \cdot) = 0,
\]

\[
\text{div}(A_k)(t, \cdot) \rightharpoonup \text{div}(A)(t, \cdot) = 0.
\]

On the other hand, and as the domain \( \Omega \) is supposed to be simply connected, we can see that \( A(t, \cdot) = \nabla \Phi(t, \cdot) \) for some \( \Phi(t, \cdot) \in H^1(\Omega) \) such that \( \Phi(t, \cdot)|_{\partial \Omega} = 0 \). Then, by using (51), we find that \( \Phi \) solves \( \Delta \Phi = 0 \) in \( \Omega \) and we obtain that \( \Phi(t, \cdot) = 0 \) in \( \Omega \) and then \( A(t, \cdot) = 0 \) in \( \Omega \) which is not true. So, we end up getting (50).

We return now to prove the second stability result (6). We recall (50) and (4) to obtain

\[
\| A \|_{L^2(0, T^*; H^1(\Omega))} \leq C \| \text{curl}(A) \|_{L^2(0, T^*; L^2(\Omega))} \leq C \left| \log \| A_{A_1, q_1}^\sharp - A_{A_2, q_2}^\sharp \| \right|^{-\beta_1}.
\]

Then, by interpolating, we get from this the desired estimate

\[
\| A \|_{L^2(0, T^*; H^s(\Omega))} \leq C \| A \|_{L^2(0, T^*; H^1(\Omega))} \| A \|_{L^2(0, T^*; H^s(\Omega))} \leq C \left| \log \| A_{A_1, q_1}^\sharp - A_{A_2, q_2}^\sharp \| \right|^{-s\beta_1},
\]

for some \( s \in (0, 1) \).

We complete then the proof of the second stability estimate for the magnetic potential (6).

5. Stable recovery of the electric potential. At this stage, we are interested to prove a stability estimate for the electric potential \( q \). The work will be based on the previous stability estimate proved for the magnetic field, besides, we keep the same previous notations and definitions, unless the function \( \eta \) which will be given by the new definition as follows

\[
\eta(t, x) = e^{-i(t\tau + x\xi)}, \quad (t, x) \in (0, T) \times \mathbb{R}^n.
\]

Here we assume that \( \text{div}(A) = 0 \). Then, we recall the equality (33) but we focus now on the electric potential term to get using the estimates (13), (24), (36)-(38) that

\[
\left| \int_{Q^*} G_{A,q} (a_{1,1} \overline{\eta}_{2,1}) (t, x) \, dx \, dt \right| \leq C \left( \sigma \| A \|_{L^\infty(Q^*)} \langle \tau, \xi \rangle^{6} h^{-8} + \sigma^{6} \gamma^{-\mu_1} \langle \tau, \xi \rangle^{6} h^{-8} + e^{m_1 \gamma \sigma^6} \langle \tau, \xi \rangle^{6} h^{-8} \| A_{A_1, q_1}^\sharp - A_{A_2, q_2}^\sharp \| + \sigma^{-1} \langle \tau, \xi \rangle^{6} h^{-8} \right).
\]

We pick \( \gamma \) such that \( \gamma^{\mu_1} = \sigma^7 \), to get

\[
\left| \int_{Q^*} G_{A,q} (a_{1,1} \overline{\eta}_{2,1}) (t, x) \, dx \, dt \right| \leq C \left( \gamma^\alpha \| A \|_{L^\infty(Q^*)} \langle \tau, \xi \rangle^{6} h^{-8} + e^{m_2 \gamma} \langle \tau, \xi \rangle^{6} h^{-8} \| A_{A_1, q_1}^\sharp - A_{A_2, q_2}^\sharp \| + \gamma^{-\alpha} \langle \tau, \xi \rangle^{6} h^{-8} \right).
\]
for some $\alpha > 0$. Now, we recall the definition of $G_{\alpha,q}$ to get

$$\int_{\Omega^*} G_{\alpha,q} (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt = \int_{\Omega^*} q \, (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt - \int_{\Omega^*} A \cdot (A_1 + A_2) (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt.$$  

This, (13), (24) and (52) imply

$$\begin{align*}
\int_{\Omega^*} q (t,x) (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt &\leq C \left( \gamma^\alpha \| A \|_{L^\infty (\Omega^*)} \| \tau, \xi \|^{6} h^{-8} \\
&\quad + e^{m_2 \gamma \langle \tau, \xi \rangle^{6} h^{-8}} \| \Lambda_{A_{1,q_1}}^{2} - \Lambda_{A_{2,q_2}}^{2} \| + \gamma^{-\alpha} \langle \tau, \xi \rangle^{6} h^{-8} \right).
\end{align*}$$

Further, we have

$$\begin{align*}
\int_{\Omega^*} q (t,x) (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt &\leq \int_{\Omega^*} q (t,x) \theta^2 (t) e^{-i(\tau r + x \xi)} \exp \left( i \int_{0}^{\infty} \omega \cdot A (t,x + s \omega) \, ds \right) \, dx \, dt,
\end{align*}$$

also

$$\begin{align*}
&\left| \int_{\Omega^*} q (t,x) \theta^2 (t) e^{-i(\tau r + x \xi)} \, dx \, dt \right| \leq \left| \int_{\Omega^*} q (t,x) (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt \right|
\end{align*}$$

where

$$\left| \int_{\Omega^*} q (t,x) \theta^2 (t) e^{-i(\tau r + x \xi)} \, dx \, dt \right| \leq \left| \int_{\Omega^*} q (t,x) (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt \right| + C \| A \|_{L^\infty (\Omega^*)},$$

by applying the mean value theorem. That’s how we find, using (53) and (54), that

$$\begin{align*}
\left| \int_{\Omega^*} q (t,x) \theta^2 (t) e^{-i(\tau r + x \xi)} \, dx \, dt \right| &\leq \left| \int_{\Omega^*} q (t,x) (a_{11,\mathfrak{p}_{2,1}}) (t,x) \, dx \, dt \right| + C \| A \|_{L^\infty (\Omega^*)},
\end{align*}$$

Now, we need to estimate the $L^\infty$-norm of $A$ by the norm of the associated partial D-to-N map. To do this, we start by choosing $n_0 > n + 1$ and applying the Sobolev embedding theorem (see e.g. [11]), getting

$$\| A \|_{L^\infty (\Omega^*)} \leq C \| A \|_{W^{1,n_0} (\Omega^*)}.$$

Now, by interpolating, we obtain

$$\| A \|_{L^\infty (\Omega^*)} \leq C \| A \|_{W^{1,2,n_0} (\Omega^*)}^{1/2} \| A \|_{L^{2,0} (\Omega^*)}^{1/2} \leq C \| A \|_{L^{2,0} (\Omega^*)}^{1/2} \leq C \| A \|_{L^{2,0} (\Omega^*)}^{1/2} \leq C \| A \|_{L^{2,0} (\Omega^*)}^{1/2},$$

and then, we get by recalling (6) that

$$\| A \|_{L^\infty (\Omega^*)} \leq C \| A \|_{W^{1,2,n_0} (\Omega^*)}^{1/2} \log \| \Lambda_{A_{1,q_1}}^{2} - \Lambda_{A_{2,q_2}}^{2} \|^{-\beta_1/n_0}.$$

Next, we inject the estimate (56) in (55) to end up getting

$$|\theta^2 q (\tau, \xi)| \leq C \left( e^{m_2 \gamma \langle \tau, \xi \rangle^{6} h^{-8}} \log \| \Lambda_{A_{1,q_1}}^{2} - \Lambda_{A_{2,q_2}}^{2} \|^{-\beta_1/n_0} + \gamma^{-\alpha} \langle \tau, \xi \rangle^{6} h^{-8} \right).$$
The next step will be similar to the one achieved in the previous section, and that’s how it leads to
\[ \|q\|_{L^2(Q^\ast)} \leq C \left( e^{mz\gamma} \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \|^{-\beta' + \gamma - \alpha_1} \right), \]
where \( \alpha_1, \beta' < 1 \) depending on \( n, n_0, \alpha \) and \( \beta_1 \). Thus, we pick \( \gamma \) in such a way that
\[ \gamma = \frac{1}{2m_2} \left| \log \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \| \right|, \]
to find a constant \( \beta_2 < 1 \) such that
\[ \|q\|_{L^2(Q^\ast)} \leq C \left| \log \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \| \right|^{-\beta_2}, \]
provided that \( \| \Lambda_{A_1,q_1}^t - \Lambda_{A_2,q_2}^t \| \) is small.

This completes the proof of the estimate (7).

6. **Unique continuation estimate.** We prove now the unique continuation estimate which presented an important ingredient in the proof of our main results.

6.1. **A parabolic Carleman estimate.** We begin our section with a special inequality for PDE solutions which is the Carleman estimate. To do this, we start, by introducing the set \( \Gamma_0 \) defined by
\[ \partial O = \Gamma \cup \Gamma_0, \quad \Gamma \cap \Gamma_0 = \emptyset, \]
and we may assume, without loss of generality, that \( \Gamma_0 \) is \( C^2 \)-smooth. Also, we choose a function \( \psi \in C^2(O) \) such that
\[ \psi(x) > 0, \quad x \in O, \quad |\nabla \psi(x)| > 0, \quad x \in \overline{O}, \]
\[ \psi(x) = 0, \quad x \in \Gamma_0, \quad \partial_n \psi \leq 0, \quad x \in \partial O \setminus \Gamma_0. \]
Then, for \( \beta > 0 \), we define the functions \( \vartheta_0 \) and \( \vartheta \) by
\[ \vartheta_0(s,x) = \frac{e^{\beta(\psi(x)+b_1)}}{k(s)}, \quad x \in O, \quad s \in (-1,1), \]
and
\[ \vartheta(s,x) = \frac{e^{\beta(\psi(x)+b_1)} - e^{\beta\|\psi\|_{L^\infty(O)} + b_2}}{k(s)}, \quad x \in O, \quad s \in (-1,1), \]
where \( k(s) = (1-s)(1+s) \) and \( \psi \) is defined by (57) and (58), and
\[ \|\psi\|_{L^\infty(O)} < b_1 < b_2 < 2b_1 - \|\psi\|_{L^\infty(O)}. \]
Further, in connection with the Schrödinger operator \((i\partial_t + \mathcal{H}_{A_2,q_2})\), we define a parabolic operator, associated with some fixed parameter \( h \in (0,1) \), by
\[ \mathcal{L}_h = h^{-1}\partial_t - \Delta_{A_2} - q_2. \]
Now, we are ready to express the following Carleman estimate for the operator \( \mathcal{L}_h \). The proof of a similar result can be found in [8].
Lemma 6.1. We consider \( \partial_0 \) and \( \partial \) defined by (59) and (60) and the operator \( \mathcal{L}_h \) defined by (61). Then, there exist three constants \( \beta_0 > 0, \lambda_0 > 0 \) and \( C_1 > 0 \) such that for any \( \beta \geq \beta_0 \) and \( \lambda \geq \lambda_0 \), the following estimate

\[
\lambda \| e^{\lambda \partial} \nabla w \|_{L^2((-1,1) \times \partial)}^2 + \lambda^3 \| e^{\lambda \partial} w \|_{L^2((-1,1) \times \partial)}^2 \\
\leq C_1 \left( \| e^{\lambda \partial} \mathcal{L}_h w \|_{L^2((-1,1) \times \partial)}^2 + \lambda \| \gamma_0^{1/2} e^{\lambda \partial} \partial_0 w \|_{L^2((-1,1) \times \partial)}^2 \right),
\]

holds true for \( w \in L^2(-1,1; H^1(O)) \) satisfying \( \mathcal{L}_h w \in L^2(-1,1; L^2(O)) \) and \( \partial_0 w \in L^2(-1,1; L^2(\Gamma_\varepsilon)) \).

Next, by recalling the first part of (57) and taking into account that \( O_1 \subset O \), we deduce that there exists a constant \( m_0 > 0 \) such that

\[
\psi(x) \geq 2m_0, \quad x \in O_2 \setminus O_3.
\]

Moreover, since \( \psi |_{\Gamma_0} = 0 \), there exists a small neighborhood \( W_0 \) of \( \Gamma_0 \) such that

\[
\psi(x) \leq m_0, \quad x \in W_0, \quad W_0 \cap \overline{O_1} = \emptyset.
\]

Let’s now choose \( W_1 \subset W_0 \) a neighborhood of \( \Gamma_0 \) and introduce \( \chi_1 \in C^\infty(O) \) a cut-off function defined by

\[
\chi_1(x) = \begin{cases} 1, & \text{if } x \in O \setminus W_0, \\ 0, & \text{if } x \in W_1. \end{cases}
\]

Hence, bearing in mind that \( A_2 = q_2 = g_0 = 0 \) in \( (0,T) \times O \) and that \( \hat{w} \) is the solution of (11), we easily see that \( w(t,x) = \chi_1(x) \hat{w}(t,x) \) satisfies

\[
\begin{cases}
(i\partial_t + \Delta) w = [\Delta, \chi_1] \hat{w} := g_1(t,x) & \text{in } Q, \\
w(0,\cdot) = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \Sigma.
\end{cases}
\]

We extend \( w \) and \( g_1 \) to \( (-T,0) \times \Omega \) by the formula

\[
w(t,x) = \overline{w}(-t,x), \quad g_1(t,x) = \overline{g}_1(-t,x), \quad \forall (t,x) \in (-T,0) \times \Omega.
\]

Then, we get \( i\partial_t w = -\Delta w + g_1 \in L^2(-T,T; L^2(\Omega)) \), that is \( w \in H^{1,2}(Q) \), with \( Q = (-T,T) \times \Omega \). Thus, we will continue the proof with

\[
Q = (-T,T) \times \Omega, \quad \Sigma = (-T,T) \times \Gamma.
\]

6.2. A connection between Schrödinger and parabolic equations. We fix \( \mu \in (0,1) \) and choose \( m \in \mathbb{N}^* \) so large that

\[
\alpha := 1 - \frac{1}{2m} > \mu.
\]

Then, for any \( \gamma > 1 \), the function defined by

\[
F_\gamma(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} e^{-\left(\frac{\tau}{\mu}\right)^2} d\tau, \quad z \in \mathbb{C},
\]

is holomorphic in \( \mathbb{C} \). Moreover, there exist four positive constants \( C_j, j = 1,2,3,4 \), independent of \( \gamma \), such that

\[
|F_\gamma(z)| \leq C_1 \gamma^\alpha e^{C_2 |z|^{1/\alpha}}, \quad z \in \mathbb{C},
\]

and

\[
|F_\gamma(z)| \leq C_1 \gamma^\alpha e^{-C_3 |Rez|^{1/\alpha}}, \quad z \in \{ z \in \mathbb{C}, |Imz| \leq C_4 |Rez| \}.
\]
In addition, given \( \kappa \in (0, \frac{1}{4}) \) and \( T_0 > T \), we consider a cut-off function \( \theta_0 \in C_0^\infty(\mathbb{R}) \) defined by

\[
\theta_0(t) = \begin{cases} 
1, & \text{if } |t| \leq (1 - 2\kappa)T_0 := T_{0,2\kappa}, \\
0, & \text{if } |t| \geq (1 - \kappa)T_0 := T_{0,\kappa},
\end{cases}
\]

which satisfies

\[
\forall j \in \mathbb{N}, \exists C_j > 0, \|\theta_0\|_{W_1,\infty(\mathbb{R})} \leq C_j \kappa^{-j},
\]

and we introduce the partial Fourier-Bros-Iagolnitzer (F.B.I) transformation of \( w \in H^{1,2}(Q) \) solution of (65) as follows

\[
w_{\gamma,t}(s,x) = \mathcal{F}_\gamma w(z,x) := \int_{\mathbb{R}} F_\gamma(z-y) \theta_0(y) w(hy,x) \, dy,
\]

for all \( t \in (-T_{0,\kappa}, T_{0,\kappa}) \), where \( T_{0,\kappa} := (1 - \kappa)T_0, s \in (-1,1), \gamma > 1 \) and \( x \in \mathbb{R}^n \).

Here \( h := T/T_0 \). Since

\[
h^{-1} \partial_s w_{\gamma,t}(s,x) = -i \mathcal{F}_\gamma(\partial_t w)(z,x) - ih^{-1} \int_{\mathbb{R}} F_\gamma(z-y) \theta_0(y) w(hy,x) \, dy,
\]

and

\[
\Delta w_{\gamma,t}(s,x) = \mathcal{F}_\gamma(\Delta w)(z,x),
\]

we get by applying the FBI transform \( \mathcal{F}_\gamma \) to (65) that

\[
\begin{cases} 
\mathcal{L}_h w_{\gamma,t}(s,x) = A_{\gamma,t}(s,x) + B_{\gamma,t}(s,x), & (s,x) \in (-1,1) \times \mathcal{O}, \\
w_{\gamma,t}(s,x) = 0, & (s,x) \in (-1,1) \times \partial\mathcal{O},
\end{cases}
\]

where

\[
A_{\gamma,t}(s,x) := -\int \mathbb{R} F_\gamma(z-y) \theta_0(y)[\Delta, \chi_1]\hat{w}(hy,x) \, dy,
\]

and

\[
B_{\gamma,t}(s,x) := -ih^{-1} \int_{\mathbb{R}} F_\gamma(z-y) \theta_0(y) w(hy,x) \, dy, \quad z = t - is.
\]

We continue our proof by applying the parabolic Carleman estimate of Lemma 6.1 to the solution \( w_{\gamma,t} \) of (70) in purpose to get the next result.

**Lemma 6.2.** There exists \( \epsilon > 0 \) such that for any \( \kappa \in (0, \frac{1}{4}) \), we can find \( C > 0, T_0 > 0, \alpha_1 > 0, \alpha_2 > 0 \) and \( \gamma_* > 0 \) such that any solution \( w_{\gamma,t} \) of (70) satisfies

\[
\|w_{\gamma,t}\|_{L^2((0,T;H^1(\Omega)))}^2 + \|\nabla w_{\gamma,t}\|_{L^2((0,T;H^1(\Omega)))}^2 \\
\leq C \left( e^{-\alpha_1 \gamma} \|w\|_{L^2((0,T;H^1(\Omega)))}^2 + e^{\alpha_2 \gamma} \|\partial_t w_{\gamma,t}\|_{L^2((-1,1) \times T_\gamma)}^2 \right),
\]

uniformly in \( t \in (-T_{0,3\kappa}, T_{0,3\kappa}) \) and \( \gamma \geq \gamma_* \).

**Proof.** For \( \gamma \in (1,\infty) \) and \( t \in (-T_{0,\kappa}, T_{0,\kappa}) \), we have in view of (70) and by applying the Carleman estimate of Lemma 6.1 to \( w_{\gamma,t} \),

\[
\lambda \|e^{\lambda t} \nabla w_{\gamma,t}\|_{L^2((-1,1) \times \mathcal{O})}^2 + \lambda^3 \|e^{\lambda t} w_{\gamma,t}\|_{L^2((-1,1) \times \mathcal{O})}^2 \\
\leq C_1 \left( \|e^{\lambda t} A_{\gamma,t}\|_{L^2((-1,1) \times \mathcal{O})}^2 + \|e^{\lambda t} B_{\gamma,t}\|_{L^2((-1,1) \times \mathcal{O})}^2 + \lambda \|\theta_0 e^{\lambda t} \partial_t w_{\gamma,t}\|_{L^2((-1,1) \times \mathcal{O})}^2 \right),
\]

for every \( \lambda \geq \lambda_* \). Moreover, we notice by referring to (64) that \( A_{\gamma,t}(s,.) \) is supported in \( W_0 \) for any \( s \in (-1,1) \), and from (60) and (63) that \( \theta(s,x) \leq (-\alpha_1) \), for all
(s, x) ∈ (-1, 1) × W₀, with a₁ = e^β(∥ψ∥_{∞ + b₂}) - e^β(m₀ + b₁) > 0. Consequently, we obtain
\begin{align}
T(74) \quad e^\lambda \|A_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2 & \leq e^{-2a_1 \lambda} \|A_{γ, t}\|_{L^2((-1, 1) \times W₀)}^2 \leq e^{-2a_1 \lambda} \|A_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2.
\end{align}
The coming step consists at choosing \( ε \in (0, 1) \) so small that \( a_2 = (e^β(∥ψ∥_{∞ + b₂}) - e^β(2m₀ + b₁))/k(ε) < a_1 \). Then, keeping in mind that \( k(s) \geq k(ε) > 0 \) for any \( s \in (-ε, ε) \), we get from (62) that \( \vartheta(s, x) ≥ -a_2 \) for every \( (s, x) ∈ (-ε, ε) \times (Ω_3 \setminus Ω_2) \).
So that we obtain
\begin{align}
e^{-2a_2 \lambda} \left( \|∇w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 + \|w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 \right) \
& \quad \leq \lambda \|e^\lambda \nabla w_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2 + \lambda^3 \|e^\lambda w_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2.
\end{align}
For \( a = a_1 - a_2 \), it follows from this, (72) and (71) that
\begin{align}
(73) \quad \|w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 + \|∇w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 & \leq C \left( e^{-2a_1 \lambda} \|A_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2 + e^{2a_2 \lambda} \|B_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2 \right) + \lambda e^{2a_2 \lambda} \|∂_ε^1 e^\lambda w_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2,
\end{align}
for every \( \lambda ≥ λ_* \). On the other hand, and using the fact that \( θ_0 \) vanishes in the interval \(-T_{₀, 2κ}, T_{₀, 2κ}\) by (68), we find two positive constants \( μ_1 \) and \( μ_2 \) both independent of \( T \) and \( γ \) such that
\begin{align}
(74) \quad \|A_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2 & \leq C e^{μ_1 γ} \|w\|_{L^2(0, T; H^1(Ω))}^2, \quad t ∈ (-T_{₀, 3κ}, T_{₀, 3κ}),
\end{align}
and
\begin{align}
(75) \quad \|B_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2 & \leq C \frac{γ^{α} e^{-μ_2 γ(κT₀)^1/α}}{\kappa T} \|w\|_{L^2(Q)}^2, \quad t ∈ (-T_{₀, 3κ}, T_{₀, 3κ}).
\end{align}
Finally, we remark from (58) and (60) that \( θ_0^{1/2} e^\lambda \) is bounded on \(-1, 1) \times Γ_t \), then, from (73), (74) and (75), we deduce that
\begin{align}
(76) \quad \|w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 + \|∇w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 & \leq C \left( e^{-2a_1 \lambda + μ_1 γ} \|w\|_{L^2(0, T; H^1(Ω))}^2 + (κT₀)^{-1} γ^{α} e^{2a_2 \lambda - μ_2 γ(κT₀)^{1/α}} \|w\|_{L^2(Q)}^2 \right) + \lambda e^{2a_2 \lambda} \|∂_ε w_{γ, t}\|_{L^2((-1, 1) \times Ω)}^2, \quad \lambda ≥ λ_*.
\end{align}
As a final step, we set \( γ_* = \max(1, λ_*/κT) \) and for \( γ \in [γ_*, ∞) \), we take \( λ = κγT₀ ≥ λ_*/h \) in (76). Then, for \( T₀ \) sufficiently large and since \( 1/α > 1 \), we majorize the sum of the two first terms in the right hand side of (76) by \( e^{(-2a_1 κT₀ + μ_1 γ)(κT₀)^{-1} γ^{α} e^{2a_2 κT₀ - μ_2 (κT₀)^{1/α}})} \) \( ≤ C e^{-α_1 γ} \) and we finish up getting for all \( t ∈ (-T_{₀, 3κ}, T_{₀, 3κ}) \) that
\begin{align}
\|w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 + \|∇w_{γ, t}\|_{L^2((-ε, x) \times (Ω_3 \setminus Ω_2))}^2 & \leq C \left( e^{-α_1 γ} \|w\|_{L^2(0, T; H^1(Ω))}^2 + e^{α_2 γ} \|∂_ε w_{γ, t}\|_{L^2((-1, 1) × Γ_t)}^2 \right), \quad γ ≥ γ_*,
\end{align}
so that the desired result is obtained.
6.3. Completion of the proof of lemma 2.2. We set \( w_\gamma(t, x) = w_{\gamma,t}(0, x) \). Then we have by recalling \((69)\) that

\[
0 = (F_\gamma * \theta_0 w^\mathcal{H})(t, x),
\]

for all \( t \in (-T_{0,3^\kappa}, T_{0,3^\kappa}) \), \( \gamma > \gamma_* \) and \( x \in \mathbb{R}^n \), where \( F_\gamma \) is defined in \((67)\) and \( w^\mathcal{H}(t, x) = w(\mathcal{H}t, x) \).

Let us begin by an estimate on \( w_\gamma \).

**Lemma 6.3.** For any \( \kappa \in (0, \frac{1}{4}) \), there exist positive constants \( C, T_0 > 0, \alpha_3 > 0, \alpha_4 > 0 \) and \( \gamma_* > 0 \) such that

\[
\|w_\gamma\|^2_{L^2((-T_{0,4^\kappa}, T_{0,4^\kappa}) \times (\Omega_3 \setminus \Omega_2))} \leq C \left( e^{-\alpha_3 \gamma} \|w\|_{L^2(0,T;H^1(\Omega))} + e^{\alpha_4 \gamma} \|\partial_\nu w^\mathcal{H}\|_{L^2((-T_{0,\kappa}, T_{0,\kappa}) \times \Gamma)} \right),
\]

for any \( \gamma > \gamma_* \).

**Proof.** For \( \kappa \in (0, \frac{1}{4}) \), let \( T_0 \) given by Lemma 6.2. Then, we consider a time \( t' \in [-T_{0,4^\kappa}, T_{0,4^\kappa}] \) and \( \epsilon < \kappa T_0 \). As long as \( w_\gamma(z, x) = w_{\gamma,\mathcal{R}z}(\mathcal{R}z, x) \) is analytic in \( \{z \in \mathbb{C}, \mathcal{R}z \in (-T_0, T_0) : \mathcal{R}z \in (-1, 1)\} \) for every fixed \( x \in \Omega_3 \setminus \Omega_2 \), the Cauchy formula implies

\[
w_\gamma(t', x) = \frac{1}{2\pi i} \int_{|z| = r} \frac{w_\gamma(z, x)}{z - t'} \, dz = \frac{1}{2\pi} \int_0^{2\pi} w_\gamma(t' + r e^{i\phi}, x) \, d\phi, \quad r \in (0, \epsilon).
\]

Therefore, by the Cauchy-Schwartz inequality, we obtain

\[
|w_\gamma(t', x)|^2 \leq (2\pi)^{-1} \int_0^{2\pi} |w_\gamma(t' + r e^{i\phi}, x)|^2 \, d\phi.
\]

As the last estimate is valid uniformly for \( r \in (0, \epsilon) \), we find that

\[
|w_\gamma(t', x)|^2 \leq \frac{1}{2\pi \epsilon} \int_0^{2\pi} \int_0^{2\pi} |w_\gamma(t' + r e^{i\phi}, x)|^2 \, d\phi \, dr
\]

\[
\leq \frac{1}{2\pi \epsilon} \int_{|s| \leq \epsilon} \int_{|t - t'| \leq \epsilon} |w_\gamma(t + is, x)|^2 \, dt \, ds
\]

\[
\leq \frac{1}{2\pi \epsilon} \int_{T_{0,3^\kappa}} \|w_{\gamma,t}(\cdot, x)\|^2_{L^2(-\epsilon, \epsilon)} \, dt.
\]

Consequently, by integrating with respect to \((t', x)\) over \((-T_{0,4^\kappa}, T_{0,4^\kappa}) \times (\Omega_3 \setminus \Omega_2)\), we obtain

\[
\|w_\gamma\|^2_{L^2((-T_{0,4^\kappa}, T_{0,4^\kappa}) \times (\Omega_3 \setminus \Omega_2))} \leq C \int_{-T_{0,3^\kappa}}^{T_{0,3^\kappa}} \|w_{\gamma,t}\|^2_{L^2((-\epsilon, \epsilon) \times (\Omega_3 \setminus \Omega_2))} \, dt.
\]

Then, applying Lemma 6.2 and recalling \((68)\) and \((69)\), we end up getting

\[
\|w_\gamma\|^2_{L^2((-T_{0,4^\kappa}, T_{0,4^\kappa}) \times (\Omega_3 \setminus \Omega_2))} \leq C \left( e^{-\alpha_3 \gamma} \|w\|_{L^2(0,T;H^1(\Omega))} + e^{\alpha_4 \gamma} \|\partial_\nu w^\mathcal{H}\|_{L^2((-T_{0,\kappa}, T_{0,\kappa}) \times \Gamma)} \right).
\]

This ends the proof. \( \square \)
We are now able to complete the proof of Lemma 2.2. Let $T^* < T$ and select $\kappa \in (0, \frac{1}{4})$ such that $T^* < (1 - 4\kappa)T$. We denote by $\hat{w}(\tau, x)$ the partial Fourier transform of $w(t, x)$ with respect to $t$. Then, in view of (77), we obtain

$$
\hat{\theta}_0 \hat{w}^\delta(\tau, x) - \hat{w}_\gamma(\tau, x) = (1 - \tilde{F}_\gamma) \hat{\theta}_0 \hat{w}^\delta(\tau, x).
$$

(78)

Therefore, taking into consideration that $\tilde{F}_\gamma(\tau) = e^{-((\tau/\gamma)^2)^m}$, using the fact that $1 - e^{-\tau^2} \leq C \tau$ for all $\tau \geq 0$ (since $2m > 1$) and remembering that $\alpha > \mu$ and $\gamma > 1$ by (66), we extract from (78) that

$$
\|\hat{\theta}_0 \hat{w}^\delta(\cdot, x) - \hat{w}_\gamma(\cdot, x)\|_{L^2(\mathbb{R})} \leq \frac{1}{\gamma^\mu} \|\tau \hat{\theta}_0 \hat{w}^\delta(\cdot, x)\|_{L^2(\mathbb{R})}, \quad x \in \Omega_3 \setminus \Omega_2.
$$

(79)

Then, as long as the function $\theta_0$ is supported in $( - T_{0, \kappa}, T_{0, \kappa} )$ by (68), it follows from (79) that

$$
\|\theta_0 w^\delta(\cdot, x) - w_\gamma(\cdot, x)\|_{L^2(\mathbb{R})} \leq \frac{1}{\gamma^\mu} \|\partial_t(\theta_0 w^\delta)(\cdot, x)\|_{L^2(\mathbb{R})}
$$

$$
\leq \frac{C \kappa^{-1}}{\gamma^\mu} \|w^\delta(\cdot, x)\|_{H^1( - T_{0, \kappa}, T_{0, \kappa} )}, \quad x \in \Omega_3 \setminus \Omega_2,
$$

for some constant $C$ non depending on $x$ and $\gamma$. Further, by recalling that $\theta_0(t) = 1$ for each $t \in [-T_{0, 4\kappa}, T_{0, 4\kappa}]$, we get upon integrating the both sides of (80) with respect to $x$ over $\Omega_3 \setminus \Omega_2$, that

$$
\|w^\delta - w_\gamma\|_{L^2( - T_{0, 4\kappa}, T_{0, 4\kappa} ; (\Omega_3 \setminus \Omega_2))} \leq \frac{C \kappa^{-1}}{\gamma^\mu} \|w^\delta\|_{H^1( - T_{0, \kappa}, T_{0, \kappa} ; L^2(\Omega))}, \quad \gamma \geq \gamma_*.
$$

(81)

Hence, from this, (81) and Lemma 6.3, we get for all $\gamma > \gamma_*$ and $T_0$ sufficiently large that

$$
\|w^\delta\|_{L^2( - T_{0, 4\kappa}, T_{0, 4\kappa} ; (\Omega_3 \setminus \Omega_2))} \leq C \left( \|w^\delta - w_\gamma\|_{L^2( - T_{0, 4\kappa}, T_{0, 4\kappa} ; (\Omega_3 \setminus \Omega_2))} + \|w_\nu\|_{L^2( - T_{0, 4\kappa}, T_{0, 4\kappa} ; (\Omega_3 \setminus \Omega_2))} \right)
$$

$$
\leq C \left( \frac{\kappa^{-1}}{\gamma^\mu} \|w^\delta\|_{H^1( - T_{0, \kappa}, T_{0, \kappa} ; L^2(\Omega))} + e^{-\alpha \gamma} \|w\|_{L^2(0, T ; H^1(\Omega))} \right)
$$

$$
+ e^{\alpha \gamma} \|\partial_\nu w^\delta\|_{L^2( - T_{0, \kappa}, T_{0, \kappa} )} \times \Gamma^\circ).
$$

Since $w^\delta(t, x) = w(ht, x)$ with $h = T/T_0$, we get

$$
\|w\|_{L^2( - hT_{0, 4\kappa}, hT_{0, 4\kappa} ; (\Omega_3 \setminus \Omega_2))} \leq C \left( \frac{\kappa^{-1}}{\gamma^\mu} \|w\|_{H^1( - hT_{0, \kappa}, hT_{0, \kappa} ; L^2(\Omega))} \right)
$$

$$
+ e^{-\alpha \gamma} \|w\|_{L^2(0, T ; H^1(\Omega))} + e^{\alpha \gamma} \|\partial_\nu w\|_{L^2(\Omega_3 \setminus \Omega_2)}\).
$$

Since $hT_{0, 4\kappa} = (1 - 4\kappa)T > T^*$ and $hT_{0, \kappa} = (1 - \kappa)T > T$, we complete the proof of Lemma 2.2.

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