SOBOLEV INEQUALITIES ON MANIFOLDS WITH NONNEGATIVE BAKRY-ÉMERY RICCI CURVATURE

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Abstract. In this note we extend a recent result of S. Brendle [3] to Riemannian manifolds with densities and nonnegative Bakry-Émery Ricci curvature.

1. Introduction

Suppose \((M, g)\) is a smooth complete noncompact Riemannian manifold of dimension \(\dim M = m\). Given a positive smooth function \(w\) (called the density) and \(\alpha > 0\) we may consider the Bakry-Émery Ricci curvature of the Riemannian metric measure space \((M, g, w d\mu)\) given by

\[
\text{Ric}_w^\alpha := \text{Ric} - D^2(\log w) - \frac{1}{\alpha} D\log w \otimes D\log w,
\]

where \(\text{Ric}\) denotes the Ricci curvature of the Riemannian metric \(g\), \(d\mu\) denotes the volume form associated of the Riemannian metric \(g\), \(Dv\) denotes the differential of a function \(v\) and \(D^2v\) denotes the Hessian of a function \(v\). If the Bakry-Émery Ricci curvature is nonnegative, the triple \((M, g, w d\mu)\) is a \(\text{CD}(0, m + \alpha)\) space.

We define the \(\alpha\)-asymptotic volume ratio \(V_{\alpha}\) of \((M, g, w d\mu)\) by

\[
V_{\alpha} = \lim_{r \to \infty} \frac{1}{r^{m+\alpha}} \int_{B_r(q)} w, \quad \text{where} \quad B_r(q) = \{p \in M | d_g(p, q) < r\},
\]

the function \(d_g\) is the distance function induced by the Riemannian metric \(g\) and \(q \in M\) is some given point. The analogue of the Bishop–Gromov volume comparison for manifolds with densities due to K.-T. Sturm [10, Theorem 2.3] implies that the limit exists and is independent of \(q \in M\). In the appendix we provide a direct proof for this result in the smooth setting.

We extend a result (Theorem 1.1 in [3]) of S. Brendle to our setting:

Theorem 1.1.

Let \((M, g)\) be a smooth complete noncompact Riemannian manifold of dimension \(\dim M = m\) with smooth positive density \(w\) and nonnegative Bakry-Émery Ricci curvature \(\text{Ric}_w^\alpha\). Suppose \(K \subset M\) is a compact domain with smooth boundary \(\partial K\) and let \(f\) be a smooth positive function on \(K\). Then we have the estimate

\[
\int_K w|Df| + \int_{\partial K} wf \geq (m + \alpha) V_{\alpha}^{\frac{1}{m+\alpha}} \left( \int_K wf \frac{1}{m+\alpha} \right)^{\frac{m+\alpha-1}{m+\alpha}},
\]

where \(V_{\alpha}\) denotes the \(\alpha\)-asymptotic volume ratio defined above.
By setting $f = 1$ in Theorem 1.1 we obtain the following sharp isoperimetric inequality:

**Corollary 1.2.**

Suppose $(M, g)$ is a smooth complete noncompact Riemannian manifold with positive smooth density $w$ and nonnegative Bakry-Émery Ricci curvature $\text{Ric}^w_\alpha$. Let $K \subset M$ be a compact subdomain in $M$ with smooth boundary $\partial K$. Then we have

\[
\int_{\partial K} w \geq (m + \alpha)^\frac{1}{m+\alpha} \left( \int_K w \right)^\frac{m+\alpha-1}{m+\alpha}.
\]

In the case of nonnegative Ricci curvature (corresponding to a constant density $w = 1$) the above inequality was obtained by S. Brendle [3, Theorem 1.1]. Previously V. Agostiniani, M. Fogagnolo and L. Mazzieri [1] proved the inequality in the three-dimensional case by proving a Willmore type inequality. Their proof builds on an argument by G. Huisken [9]. Very recently M. Fogagnolo and L. Mazzieri [7] extended the argument up to dimension seven.

Our proof will use the Alexandrov-Bakelman-Pucci (ABP) maximum principle as in [3]. A proof of the isoperimetric inequality in $\mathbb{R}^n$ using this method was first observed by X. Cabre [4, 5], see also work of X. Cabre, X. Ros-Oton and J. Serra [6] for various extensions. In a recent breakthrough S. Brendle [2] used this method to prove a sharp Michael–Simon–Sobolev inequality (and hence a sharp isoperimetric inequality) for submanifolds.

In Section 2 we discuss the proof of Theorem 1.1. In the appendix we provide a direct proof of the Bishop–Gromov volume comparison theorem for smooth manifolds with densities.

### 2. Proof of Theorem 1.1

We may assume by scaling

\[
\int_K w|Df| + \int_{\partial K} wf = (m + \alpha) \int_K w^\frac{m+\alpha}{m+\alpha-1}.
\]

Suppose $K$ is a compact domain with smooth boundary $\partial K$. We consider the linear Neumann problem given by

\[
\begin{cases}
\text{div} (wfDu) = (m + \alpha) wf^\frac{m+\alpha}{m+\alpha-1} - w|Df| & \text{in } K, \\
\langle Du, \nu \rangle = 1 & \text{on } \partial K,
\end{cases}
\]

where $\nu$ denotes the outward pointing unit normal vector field of $K$.

The scaling assumption provides the integrability condition for this Neumann problem. Since $f$ is smooth, we have $|Df| \in C^{0,1}$ and hence by standard elliptic theory (see for example Theorem 6.31 in [8]) we conclude $u \in C^{2,\gamma}$ for any $0 < \gamma < 1$.

As in [3] we define the subset $U \subset K$ by

\[
U = \{ x \in K \setminus \partial K \mid |Du(x)| < 1 \}.
\]

For any $r > 0$ we define a subset $A_r$ by

\[
A_r = \left\{ \bar{x} \in U \mid \forall x \in K : ru(x) + \frac{1}{2} d(x, \exp_\bar{x}(rDu(\bar{x})))^2 \geq ru(\bar{x}) + \frac{1}{2} r^2 |Du(\bar{x})|^2 \right\}.
\]
Let us denote the exponential map at \( x \) by \( \exp_x : T_x M \to M \). For any \( r > 0 \) we define the transport map \( \Phi_r : K \to M \) by
\[
\Phi_r(x) = \exp_x (r Du(x)).
\]
The transport map is of class \( C^{1,\gamma} \) for any \( 0 < \gamma < 1 \) by the above considerations on the regularity of the solution \( u \) of the Neumann problem.

We observe the following inequality (compare with Lemma 2.1 in [3]):

**Lemma 2.1.**

Assume \( x \in U \). Then we have the inequality
\[
w \Delta u + \langle Dw, Du \rangle \leq (m + \alpha) w f^{\frac{1}{m+\alpha-1}}.
\]

**Proof.**

If we apply the identity
\[
\text{div}(wf Du) = w \Delta u + f \langle Dw, Du \rangle + w \langle Df, Du \rangle
\]
to the PDE
\[
\text{div}(wf Du) = (m + \alpha) w f^{\frac{m+\alpha}{m+\alpha-1}} - w |Df|,
\]
we deduce by the Cauchy–Schwarz inequality
\[
f(w \Delta u + \langle Dw, Du \rangle) = (m + \alpha) w f^{\frac{m+\alpha}{m+\alpha-1}} - w (|Df| + \langle Df, Du \rangle) \leq (m + \alpha) w f^{\frac{m+\alpha}{m+\alpha-1}}.
\]
Dividing by \( f > 0 \) yields the claim.

We have the following observation on the image of the transport set:

**Lemma 2.2** (cf. S.Brendle, Lemma 2.2 in [3]).

The set
\[
\{ p \in M \mid d_g(x, p) < r \text{ for all } x \in K \}
\]
is contained in the image \( \Phi_r(A_r) \) of the transport map.

**Proof.**

The proof is identical to Lemma 2.2 in [3]. Hence we omit the details.

For a point \( \bar{x} \in A_r \) we define the path \( \bar{\gamma} : [0, r] \to M \) by \( \bar{\gamma}(t) := \exp_{\bar{x}} (t Du(\bar{x})) \). We have the following formula for the second variation of energy along \( \bar{\gamma} \):

**Lemma 2.3** (cf. S.Brendle, Lemma 2.3 in [3]).

Suppose \( Z \) is a smooth vector field along \( \bar{\gamma} \) vanishing at the end point (i.e. \( Z(r) = 0 \)). Then we have the second variation formula
\[
(D^2 u)(Z(0), Z(0)) + \int_0^r \left( |D_t Z(t)|^2 - \text{Rm}(\bar{\gamma}'(t), Z(t), Z(t), \bar{\gamma}'(t)) \right) dt \geq 0.
\]

**Proof.**

The proof is identical to Lemma 2.3 in [3]. Hence we omit the details.

The next step is to prove a vanishing result for Jacobi fields along the geodesic \( \bar{\gamma} \):
Lemma 2.4 (cf. S.Brendle, Lemma 2.4 in [3]).
Choose an orthonormal basis of the tangent space $T_x M$. Suppose $W$ is a Jacobi field along $\gamma$ satisfying
$$\langle D_t W(0), e_j \rangle = (D^2 u)(W(0), e_j)$$
for $1 \leq j \leq m$. If there exists $\tau \in (0, r)$, such that $W(\tau) = 0$, then $W$ vanishes identically.

Proof. The proof is identical to Lemma 2.4 in [3]. Hence we omit the details. □

The next proposition describes the volume expansion along the transport map $\Phi_t$:

Proposition 2.5. Assume that $x \in A_r$. Then the map
$$t \mapsto \left(1 + tf(x)\frac{1}{m+\alpha-1}\right) w(\Phi_t(x)) \det D\Phi_t(x)$$
is monotone decreasing for $t \in (0, r)$.

Proof. Let $x \in A_r$, define $\bar{\gamma} : [0, r] \rightarrow M$ by $\bar{\gamma}(t) = \exp_x(tDu(\bar{x}))$. Choose an orthonormal basis $\{e_1, \ldots, e_m\}$ of the tangent space $T_x M$, and construct geodesic normal coordinates $(x^1, \ldots, x^n)$ around $\bar{x}$, such that we have $\partial_1 = e_1$ at $\bar{x}$.

We construct for $1 \leq i \leq m$ vector fields $E_i$ along $\bar{\gamma}$ by parallel transport of the vector fields $e_i$. Moreover, we solve the Jacobi equation to obtain the unique Jacobi fields $X_i$ along $\bar{\gamma}$ satisfying $X_i(0) = e_i$ and
$$\langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j),$$
where $D_t$ denotes the covariant derivative along $\bar{\gamma}$.

Let us define a matrix-valued function $P : [0, \tau] \rightarrow \text{Mat}(m; \mathbb{R})$ by
$$[P(t)]_{ij} = \langle X_i(t), E_j(t) \rangle.$$
We observe by the above properties:
$$[P(0)]_{ij} = \delta_{ij} \text{ and } [P'(0)]_{ij} = (D^2 u)(e_i, e_j).$$

Additionally, we define a matrix-valued function $S : [0, \tau] \rightarrow \text{Mat}(m; \mathbb{R})$ by
$$[S(t)]_{ij} = R_{m} (\gamma'(t), E_i(t), E_j(t), \gamma'(t)),$$
where $R_m$ denotes the Riemann curvature tensor. For each $t \in [0, \tau]$ the matrix $S(t)$ is symmetric due to the symmetries of the Riemann curvature tensor. We have
$$\text{tr} S(t) = \text{Ric} (\gamma'(t), \gamma'(t)).$$

By the Jacobi equation for the Jacobi vector fields $X_1, \ldots, X_n$ we obtain
$$P''(t) = -P(t)S(t).$$
Moreover, $P(t)P(t)^T$ is symmetric for each $t \in [0, r]$, and $P(t)$ is invertible for each $t \in [0, r]$. We define a matrix-valued function $Q : [0, \tau] \rightarrow \text{Mat}(m; \mathbb{R})$ by
$$Q(t) = P(t)^{-1}P'(t).$$
Then $Q(t)$ is symmetric for each $t \in [0, r]$ and it satisfies the Ricci equation
$$\frac{d}{dt} Q(t) = -S(t) - Q(t)^2.$$
We compute
\[
\frac{d}{dt} \text{tr} Q(t) = \text{tr}(-S(t) - Q(t)^2) = -\text{Ric}(\bar{\gamma}'(t), \bar{\gamma}'(t)) - \text{tr}[Q(t)^2].
\]

We recall the definition of the Bakry-Émery Ricci curvature:
\[
\text{Ric}_\alpha^w = \text{Ric} - D^2(\log w) - \frac{1}{\alpha} D(\log w) \otimes D(\log w).
\]

Thus
\[
\frac{d}{dt} \text{tr} Q(t) = -(D^2 \log w)(\bar{\gamma}'(t), \bar{\gamma}'(t)) - \frac{1}{\alpha} \langle D(\log w)(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle^2 - \text{Ric}_\alpha^w(\bar{\gamma}'(t), \bar{\gamma}'(t)) - \text{tr}[Q(t)^2].
\]

We observe the differential inequality
\[
\frac{d}{dt} \left[ \text{tr} Q(t) + \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \right]
= -\frac{1}{\alpha} \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle^2 - \text{Ric}_\alpha^w(\bar{\gamma}'(t), \bar{\gamma}'(t)) - \text{tr}[Q(t)^2]
\leq -\frac{1}{m} \left[ \text{tr} Q(t) \right]^2 - \frac{1}{\alpha} \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle^2
= -\frac{1}{m + \alpha} \left( \text{tr} Q(t) + \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \right)^2 - \frac{m}{\alpha(m + \alpha)} \left( \frac{\alpha}{m} \text{tr} Q(t) - \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \right)^2
\leq -\frac{1}{m + \alpha} \left[ \text{tr} Q(t) + \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \right]^2,
\]

where we used our assumption of nonnegative Bakry-Émery Ricci curvature (ie. \( \text{Ric}_\alpha^w \geq 0 \)) and the trace inequality for symmetric matrices.

With the help of Lemma 2.1 we observe that the initial value to the above differential inequality is given by
\[
\lim_{t \to 0} \left[ \text{tr} Q(t) + \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \right] = \Delta u(\bar{x}) + \langle D \log w(\bar{x}), \log u(\bar{x}) \rangle \leq (m + \alpha) f(\bar{x})^{-\frac{1}{m+\alpha-1}}.
\]

If we apply the comparison principle for ODEs we deduce the bound
\[
\text{tr} Q(t) + \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \leq \frac{(m + \alpha) f(\bar{x})^{-\frac{1}{m+\alpha-1}}}{1 + tf(\bar{x})^{-\frac{1}{m+\alpha-1}}}.
\]

Then we have
\[
\frac{d}{dt} \log [w(\bar{\gamma}(t)) \det P(t)] = \text{tr} Q(t) + \langle D \log w(\bar{\gamma}(t)), \bar{\gamma}'(t) \rangle \leq \frac{(m + \alpha) f^{-\frac{1}{m+\alpha-1}}}{1 + tf^{-\frac{1}{m+\alpha-1}}}.
\]

This implies that the function
\[
t \mapsto \left( 1 + tf(\bar{x})^{-\frac{1}{m+\alpha-1}} \right)^{-\frac{m+\alpha}{2}} w(\bar{\gamma}(t)) \det P(t)
\]
is monotone decreasing for \( t \in (0, r) \). The proposition follows by observing that \( \det P(t) = |\det D\Phi(t, \bar{x})| \) for any \( t \in (0, r) \). \( \square \)
Corollary 2.6. 
We have for any \( x \in A_r \) the relation
\[
w(\Phi_r(x)) | \det D\Phi_r(x) | \leq \left( 1 + rf(x)^{\frac{1}{m+\alpha-1}} \right)^{m+\alpha} w(x).
\]

We complete the proof of Theorem 1.1: Indeed by Lemma 2.2, the change of variables formula and Corollary 2.6 we have
\[
\int_{\{p \in M|d_g(x,p) < r \text{ for all } x \in D\}} w \ d\mu(x) \\
\leq \int_{A_r} \left| \det D\Phi_r(x) \right| w(\Phi_r(x)) \ d\mu(x) \\
\leq \int_{U} \left( 1 + rf(x)^{\frac{1}{m+\alpha-1}} \right)^{m+\alpha} w(x) \ d\mu(x)
\]
If we divide by \( r^{m+\alpha} \) and send \( r \to \infty \) we obtain
\[
\mathcal{V}_\alpha = \lim_{r \to \infty} \frac{1}{r^{m+\alpha}} \int_{\{p \in M|d_g(x,p) < r \text{ for all } x \in K\}} w \ d\mu(x) \\
\leq \int_{U} f(x)^{\frac{m+\alpha}{m+\alpha-1}} w(x) \ d\mu(x) \\
\leq \int_{D} f(x)^{\frac{m+\alpha}{m+\alpha-1}} w(x) \ d\mu(x)
\]
If we combine the previous estimate with our initial scaling assumption, we deduce
\[
\int_{K} w |Df| + \int_{\partial K} wf = (m + \alpha) \int_{K} w f^{\frac{m+\alpha}{m+\alpha-1}} \\
\geq (m + \alpha) \left( \int_{K} w f^{\frac{m+\alpha}{m+\alpha-1}} \right)^{\frac{1}{m+\alpha}} \left( \int_{K} w f^{\frac{m+\alpha}{m+\alpha-1}} \right)^{\frac{m+\alpha-1}{m+\alpha}} \\
\geq (m + \alpha) \mathcal{V}_\alpha^{\frac{m+\alpha}{m+\alpha-1}} \left( \int_{K} w f^{\frac{m+\alpha}{m+\alpha-1}} \right)^{\frac{m+\alpha-1}{m+\alpha}}.
\]
This is the desired estimate.

Appendix A. A proof of a Bishop–Gromov volume comparison theorem for smooth manifolds with densities

We provide a short proof of the Bishop–Gromov volume comparison theorem for smooth noncompact manifolds with \( \text{Ric}^w_{\alpha} \geq 0 \).

Theorem A.1 (K.-T. Sturm, Theorem 2.3 in [10]).
Assume \((M, g)\) is a smooth complete noncompact \( m \)-dimensional smooth manifold, \( \alpha > 0 \), and \( w \) is a smooth positive function. Suppose \((M, g, w)\) has nonnegative Bakry-Émery Ricci curvature with respect to the density \( w \) and \( \alpha > 0 \), ie.
\[
\text{Ric}^w_{\alpha} = \text{Ric} - D^2(\log w) - \frac{1}{\alpha} D \log w \otimes D \log w \geq 0.
\]
Then the function

\[ r \mapsto \frac{1}{r^{m+\alpha}} \int_{B_r(q)} w, \text{ where } B_r(q) = \{ p \in M | d_g(p, q) < r \} \]

is monotone decreasing for any \( q \in M \).

**Proof.**

We define \( \Sigma_t \subset M \) by

\[ \Sigma_t = \{ \exp_q(v) | v \in \text{seg}^0(q) \text{ and } |v| = t \}. \]

The set \( \Sigma_t \) is the distance set avoiding the cut locus, here \( \text{seg}^0(p) \) denotes the interior segment domain. We denote the second fundamental form of the hypersurface \( \Sigma_t \) by \( h \) and its trace, the mean curvature of the hypersurface \( \Sigma_t \), by \( H \).

Let \( v \in T_yM, |v| = 1 \) and consider the radial geodesic \( \gamma(t) = \exp_q(tv) \) such that \( \gamma(t) \in \Sigma_t \). Then we have the Jacobi equation

\[ \left( \frac{d}{dt} H \right)(\gamma(t)) = -|h|^2(\gamma(t)) - \text{Ric}(\gamma'(t), \gamma'(t)) \]

provided \( tv \in \text{seg}^0(p) \). This implies

\[ \frac{d}{dt} \left[ H(\gamma(t)) + \langle \gamma'(t), D \log w(\gamma(t)) \rangle \right] \]

\[ = \left( \frac{d}{dt} H \right)(\gamma(t)) + (D^2 \log w)(\gamma'(t), \gamma'(t)) \]

\[ = -|h|^2(\gamma(t)) - \text{Ric}(\gamma'(t), \gamma'(t)) + (D^2 \log w)(\gamma'(t), \gamma'(t)) \]

\[ = -|h|^2(\gamma(t)) - \frac{1}{\alpha} \langle D \log w(\gamma'(t)), \gamma'(t) \rangle - \text{Ric}_w(\gamma'(t), \gamma'(t)) \]

\[ \leq -\frac{1}{m-1} H^2(\gamma(t)) - \frac{1}{\alpha} \langle D \log w(\gamma'(t)), \gamma'(t) \rangle \]

\[ = -\frac{1}{m-1+\alpha} \left[ H(\gamma(t)) + \langle \gamma'(t), D \log w(\gamma(t)) \rangle \right]^2 \]

\[ - \frac{m-1}{\alpha(m-1+\alpha)} \left[ \frac{\alpha}{m-1} H(\gamma(t)) - \langle \gamma'(t), D \log w(\gamma(t)) \rangle \right]^2 \]

\[ \leq -\frac{1}{m-1+\alpha} \left[ H(\gamma(t)) + \langle \gamma'(t), D \log w(\gamma(t)) \rangle \right]^2, \]

where we have used the trace inequality for the second fundamental form and the nonnegativity of the Bakry–Émery Ricci curvature.

Thus we established the differential inequality

\[ \frac{d}{dt} \left[ H(\gamma(t)) + \langle \gamma'(t), D \log w(\gamma(t)) \rangle \right] \leq -\frac{1}{m-1+\alpha} \left[ H(\gamma(t)) + \langle \gamma'(t), D \log w(\gamma(t)) \rangle \right]^2. \]

Integrating the above differential inequality implies

\[ H(\gamma(t)) + \langle \gamma'(t), D \log w(\gamma(t)) \rangle \leq \frac{m-1+\alpha}{t}. \]

This implies

\[ \frac{d}{dt} \left( \int_{\Sigma_t} w \right) \leq \int_{\Sigma_t} \left( H + \langle v, D \log w \rangle \right) w \leq \frac{m-1+\alpha}{t} \int_{\Sigma_t} w, \]
where the first inequality holds because the interior segment domain \( \text{seg}^0(p) \) is star-shaped and the second inequality is our differential inequality.

Integration of this differential inequality implies that the map
\[
t \mapsto t^{-(m-1+\alpha)} \int_{\Sigma_t} w
\]
is decreasing. We observe for any \( r > 0 \) by the coarea formula
\[
r^{-(m+\alpha)} \int_{B_r(q)} w = r^{-(m+\alpha)} \int_0^r \left( \int_{\Sigma_t} w \right) dt = \int_0^1 \left( r^{-(m-1+\alpha)} \int_{\Sigma_{tr}} w \right) d\tau,
\]
where we use that the cut locus is a set of measure zero. The inner bracket on the right-hand side is nonincreasing in the radius \( r \) by the previous observation, hence the left-hand side is nonincreasing in the radius \( r \) as an average of nonincreasing functions.

\[\blacksquare\]

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