From quantum disorder to quantum chaos

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We study the statistics of wave functions in a ballistic chaotic system. The statistical ensemble is generated by adding weak smooth random potential, which allows us to apply the ballistic $\sigma$-model approach. We analyze conditions of applicability of the $\sigma$-model, emphasizing the role played by the single-particle mean free path and the Lyapunov exponent due to the random potential. In particular, we present a resolution of the puzzle of repetitions of periodic orbits counted differently by the $\sigma$-model and by the trace formula.

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1. INTRODUCTION

The central problem in the field of quantum chaos is understanding of statistics of eigenfunctions and energy levels of a quantum system whose classical counterpart is chaotic and their relation to the underlying classical dynamics. The Bohigas-Giannoni-Schmit\textsuperscript{1} conjecture states that, generically, statistical properties of levels in such a system are described (in the leading approximation) by universal results of the random matrix theory (RMT). A related hypothesis concerning statistics of wave functions has been put forward by Berry\textsuperscript{2} (see also\textsuperscript{3}) who conjectured that an eigenfunction of a chaotic billiard can be represented as a random superposition of plane waves with a fixed absolute value $k$ of the wave vector (determined by the energy $k^2/2m = E$, where $m$ is the mass and we set $\hbar = 1$). This implies Gaussian statistics of the eigenfunction amplitude $\psi(\mathbf{r})$,

$$\mathcal{P}\{\psi\} \propto \exp \left[ -\frac{\beta}{2} \int d^2 r d^2 r' \psi^*(\mathbf{r}) C^{-1}(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \right] , \quad (1)$$

*dedicated to Peter Wölle on the occasion of his 60th birthday
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determined solely by the correlation function (we consider a two-dimensional system) $C(\mathbf{r}_1, \mathbf{r}_2) \equiv \langle \psi^* (\mathbf{r}_1) \psi (\mathbf{r}_2) \rangle = J_0(k|\mathbf{r}_1 - \mathbf{r}_2|)/V$. Here $\beta = 1$ (or 2) for a system with preserved (respectively broken) time reversal symmetry, $V$ is the system area, and $J_0$ the Bessel function. Note that, when taken literally, Eq. (1) contradicts the wave function normalization,
\[ \int d^2 r \left[ \langle |\psi^2 (\mathbf{r})\psi^2 (\mathbf{r}')| \rangle - \langle |\psi^2 (\mathbf{r})| \rangle \langle |\psi^2 (\mathbf{r}')| \rangle \right] = 0, \] since the integrand is equal to $C^2(\mathbf{r}, \mathbf{r}') > 0$ according to (1). Therefore, the limits of validity of this conjecture have to be understood.

Despite much effort spent in this direction, no proof of these conjectures has been obtained via semiclassical methods, and deviations from universality have not been calculated in a controlled way. This is because the standard semiclassical tool – representation of a Green’s function in terms of a sum over classical trajectories – is only justified for times much shorter than the Heisenberg time $t_H = 2\pi\hbar/\Delta$ (where $\Delta$ is the mean level spacing), which is not sufficient for the problems considered.

On the other hand, a considerable progress has been achieved in investigation of the statistical properties of energy levels and wave functions of diffusive disordered systems. In this case the supersymmetry method serves as a tool for a systematic analytical description of the level and eigenfunction statistics. After averaging over an ensemble of realizations of the random potential the problem is mapped onto a supermatrix $\sigma$-model, which is further studied by various analytical means. This has allowed one not only to prove the relevance of the RMT results, but also to calculate system-specific deviations from universality determined by the diffusive classical dynamics, see [5] for a review.

This success of the diffusive $\sigma$-model gave rise to an expectation that the supersymmetry method may be also useful in the context of quantum chaos. In a seminal paper [6] Muzykantskii and Khmelnitskii conjectured that a chaotic ballistic system can be described by a ballistic $\sigma$-model, with the Liouville operator replacing the diffusion operator in the action. They presented a derivation of this model using the averaging over a white-noise disorder and conjectured that it remains valid in the limit of vanishing disorder. Subsequently, Andreev, Agam, Simons and Altshuler [7] proposed another derivation of the model, by considering a clean system and employing the energy averaging only. This led them to the conclusion that the statistical properties of a chaotic system can be obtained from the results found for diffusive systems [5] by replacing eigenvalues and eigenfunctions of the diffusion operators by those of the (properly regularized) Liouville operator.

However, soon after publication of [6, 7] these conclusions were criticized from several points of view. Prange [8] showed that the energy averaging is
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insufficient to detect non-universal corrections to the level statistics, in view of the statistical noise. This indicates that one cannot study an individual system but should rather consider some ensemble of systems. The necessity of an ensemble averaging has been also pointed out by Zirnbauer, who concentrated on rigorous formulation and justification of the Bohigas-Giannoni-Schmit conjecture of universality. Bogomolny and Keating emphasized the discrepancy between the prediction of the ballistic $\sigma$-model for the smooth part of the level correlation function and the result of the diagonal approximation to the semiclassical trace formula. Specifically, repetitions of periodic orbits are counted differently within the two approaches. These two issues (ensemble averaging and repetitions) point to serious problems with the ballistic $\sigma$-model approach and require clarification. This will be done below.

2. Ballistic $\sigma$-Model from Averaging over Smooth Disorder

Let us come back to the derivation of the $\sigma$-model. It turns out that the derivations proposed in both Refs. and are, in fact, not quite correct. Specifically, if averaging over a white-noise disorder with a mean free path $l$ is used (as in Ref.7), the ballistic $\sigma$-model is only valid for momenta $q < l^{-1}$, i.e. in the region of validity of the diffusive $\sigma$-model, but not in the ballistic range of larger momenta. On the other hand, the energy averaging of Ref.7 leaves a continuum of zero modes allowing for arbitrary fluctuations of the supermatrix field transverse to the energy shell and spoiling the derivation of the ballistic $\sigma$-model, which requires that these fluctuations be frozen.

We note that the methods of and may be considered as opposite extremes: while in Ref.7 the averaging over a random potential with zero correlation length $d = 0$ was proposed, the energy averaging of Ref.7 corresponds to a random potential with $d = \infty$. As often happens, the truth lies in between: the ballistic $\sigma$-model can be obtained if one averages over a smooth random potential with a finite correlation length $d$. In fact, this type of derivation had been performed for the first time by Wölfle and Bhatt ten years before the notion of the ballistic $\sigma$-model was introduced. However, the aim of Ref.7 was demonstration of the applicability of the diffusive $\sigma$-model to the problem of a smooth random potential, and the ballistic action appeared only implicitly as an intermediate step of the calculation. This derivation was generalized to the case of a random magnetic field in yet again the main point of that work was the diffusive action on distances exceeding the transport mean free path. More recently, it was emphasized that averaging over a smooth disorder is exactly the proper way of derivation of the ballistic $\sigma$-model. This established the connection between Refs.
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on the one hand, and Refs. on the other hand.

Starting from a system with a Hamiltonian \( \hat{H} \), we generate a statistical ensemble by adding a random potential \( U(r) \) characterized by a correlation function \( W(r - r') = \langle U(r)U(r') \rangle \) with a correlation length \( d \). Parameters of this random potential are assumed to satisfy \( k^{-1} \ll d \ll l_s \), where \( l_s = v\tau_s \), \( l_t = v\tau_t \), \( v \) is the velocity, and \( \tau_s (\tau_t) \) is the single-particle (respectively transport) relaxation time. Note that the potential is smooth, \( k d \gg 1 \), so that \( l_t / l_s \sim (kd)^2 \gg 1 \). In order to apply this idea to a ballistic system of a characteristic size \( L \), we require \( l_s \ll L \ll l_t \). The condition \( l_t \gg L \) preserves the ballistic nature of the system, while the inequality \( l_s \ll L \) guarantees that the ensemble of quantum systems is large enough to produce meaningful result.

After the ensemble averaging, the problem can be reduced to a “non-local ballistic \( \sigma \)-model” of a supermatrix field \( Q(r, n) \) with the action (for definiteness, we consider the case \( \beta = 2 \))

\[
S[Q] = \text{Str} \ln \left[ E - \hat{H} + \omega \Lambda - \frac{i}{2} \int d n' Q(r, n')w(n, n') \right] - \frac{\pi \nu}{4} \int d^2 r d n d n' \text{Str} Q(r, n)w(n, n')Q(r, n'),
\]

where \( \nu \) is the density of states, \( w(n, n') = 2\pi \nu W(|n - n'|) \) is the scattering cross-section for the random potential, and \( n \) is a unit vector characterizing the direction of velocity on the energy surface. Note that despite the non-local \( \text{Str} \ln \) form, the action is not an exact representation of the original problem, but rather a low-energy theory with only soft modes kept: the momentum variable of \( Q \) is constrained to the energy surface, and \( Q \) satisfies the usual \( \sigma \)-model condition \( Q^2 = 1 \). To obtain the ballistic \( \sigma \)-model in the local form (as proposed in Ref. 6, 7), one has to perform a gradient- and frequency-expansion, which is justified provided \( ql_s \ll 1, \omega \tau_s \ll 1 \). The result is

\[
S[Q] = \pi \nu \int d^2 r d n \text{Str} \left[ \Lambda T^{-1}(r, n)\hat{\mathcal{L}}T(r, n) - \frac{i\omega}{2} \Lambda Q(r, n) \right] + \frac{\pi \nu}{4} \int d^2 r d n d n' \text{Str} Q(r, n)w(n, n')Q(r, n'),
\]

where \( Q(r, n) = T(r, n)\Lambda T^{-1}(r, n) \). The symbol \( \hat{\mathcal{L}} \) denotes the Liouville operator characterizing the classical motion; for a billiard \( \hat{\mathcal{L}} = vn \nabla \) supplemented by appropriate boundary conditions. The local ballistic \( \sigma \)-model is thus only applicable on length scales \( \gg l_s \). From the point of view of the semiclassical (periodic-orbit) theory, this corresponds to the condition of applicability of the diagonal approximation. On shorter scales, one has to use the more general, non-local, form.
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3. EIGENFUNCTION STATISTICS IN A BALLISTIC SYSTEM

The two-point correlation function of the wave function intensities is expressed in this approach as

\[
\langle |\psi^2(r)\psi^2(r')| \rangle = \lim_{\eta \to 0} \frac{\eta \Delta}{\pi} \langle [G_{11}(r, r)G_{22}(r', r') + G_{12}(r, r')G_{21}(r', r)] \rangle_{S[Q]},
\]

where \( \hat{G} \) is the Green’s function in the field \( Q \),

\[
\hat{G} = \left[ E - \hat{H} + i\eta \Delta - \frac{i}{2} \int d\mathbf{n} Q(\mathbf{r}, \mathbf{n}') w(\mathbf{n}, \mathbf{n}') \right]^{-1},
\]

and the subscripts 1, 2 refer to the advanced-retarded decomposition (the boson-boson components being implied). Here \( \eta \) is an infinitesimal positive, and \( \omega \) in the action \( S[Q] \) is given by \( \omega = i\eta \).

We first evaluate Eq. (5) in the zero-mode approximation, \( Q(\mathbf{r}) = Q_0 \). The Green’s function \( \hat{G} \) is given in the leading order by

\[
G_0(r, r') = i\text{Im} G_R(r, r') Q_0 + \text{Re} G_R(r, r'),
\]

\[
G_R(r, r') = \langle r | (E - \hat{H} + i\tau_\eta)^{-1} | r' \rangle.
\]

Substituting Eq. (7) in (5) and expanding the action (3) up to the linear-in-\( \eta \) term, \( S[Q] \approx \pi \eta V \text{Str} Q_0 \Delta \), one finds, in a full analogy with the case of diffusive systems,

\[
V^2 \langle |\psi^2(r_1)\psi^2(r_2)| \rangle \simeq 1 + k_q(r_1, r_2);
\]

\[
k_q(r, r') = \text{Im} G_R(r, r') \text{Im} G_R(r', r) / (\pi \nu)^2,
\]

with the two contributions on the r.h.s. of (3) originating from the terms \( \langle G_{11} G_{22} \rangle \) and \( \langle G_{12} G_{21} \rangle \) in (3), respectively. The result (5), corresponding exactly to the conjecture (7) of the Gaussian statistics, is in conflict with the wave function normalization, as explained above.

To resolve this problem, we evaluate the term \( \langle G_{11} G_{22} \rangle \) more accurately by expanding the Green’s function (5) to the order \( \eta \) and the action (3) to the order \( \eta^2 \). While these terms (usually neglected in the \( \sigma \)-model calculations) are of the next order in \( \eta \) and may be naively thought to vanish in the limit \( \eta \to 0 \) performed in (3), this is not so, since \( Q_0 \propto \eta^{-1} \). As a result, we get in the zero-mode approximation

\[
V^2 \langle |\psi^2(r_1)\psi^2(r_2)| \rangle_{\text{ZM}} - 1 = k_q(r_1, r_2) - \tilde{k}_q(r_1) - \tilde{k}_q(r_2) + \tilde{k}_q
\]

(terms of still higher orders in \( \eta \) produce corrections small in the parameter \( \Delta \tau_\eta \ll 1 \)), where \( \tilde{k}_q(r) = V^{-1} \int d^2 \mathbf{r}' k_q(\mathbf{r}, \mathbf{r}') \), \( \tilde{k}_q = V^{-2} \int d^2 \mathbf{r} d^2 \mathbf{r}' k_q(\mathbf{r}, \mathbf{r}') \).
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The contribution of non-zero modes is expressed in terms of the $\sigma$-model propagator
\[ V^2 \langle \psi(r_1) \psi(r_2) \rangle_{\text{NZM}} = \tilde{\Pi}_B(r_1, r_2), \] (12)
where $\tilde{\Pi}_B(r_1, r_2) = \Pi_B(r_1, r_2) - \Pi_B^{(0)}(r_1, r_2)$ describes the (integrated over direction of velocity) probability of classical propagation from $r_1$ to $r_2$.

We analyze now the total result given by the sum of (11) and (12). First of all, we stress that it satisfies exactly the condition of wave function normalization. Next, we consider sufficiently short distances, $|r_1 - r_2| \ll l_s$. In this case the correlation function is dominated by the first term in the r.h.s. of Eq. (11), returning us to the result (9). Furthermore, we can generalize this result to higher correlation functions,
\[ \langle \psi^*(r_1) \psi(r_1') \ldots \psi^*(r_n) \psi(r_n') \rangle = -\frac{1}{2V(n-1)!} \times \lim_{\eta \to 0} (2\pi\nu\eta)^{n-1} \left\{ \sum \prod_{i=1}^n \frac{1}{\pi\nu} G_{p_i p_{\sigma(i)}}(r_i, r_{\sigma(i)}) \right\}, \] (15)
where the summation goes over all permutations $\sigma$ of the set $\{1, 2, \ldots, n\}$, $p_i = 1$ for $i = 1, \ldots, n - 1$, and $p_n = 2$. If all the points $r_i, r_i'$ are within a distance $\ll l_s$ from each other, the leading contribution to this correlation function is given by the zero-mode approximation with higher-order terms in $\eta$ neglected [i.e. by the same approximation which leads to Eq. (3)], yielding
\[ V^n \langle \psi^*(r_1) \psi(r_1') \ldots \psi^*(r_n) \psi(r_n') \rangle = \sum_{\sigma} \prod_{i=1}^n f_F(r_i, r'_{\sigma(i)}), \]
\[ f_F(r, r') = -\text{Im} G_R(r, r') / (\pi\nu). \] (15)
This result is identical to the statement of the Gaussian statistics of eigenfunctions conjectured in [3]. We have thus proven that Eq. (1) holds within a spatial region of an extension $\ll l_s$, with the kernel $C(r_1, r_2) = f_F(r_1, r_2) / V$ given by Eq. (13).

We turn now to the behavior of the correlator $\langle |\psi^2(r_1)\psi^2(r_2)| \rangle$ at larger separations $|r_1 - r_2| \gg l_s$. In this situation, the correlations are dominated by the contribution (12) of non-zero modes. Let us further note that the smooth
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part of the zero-mode contribution $\Pi^{(0)}_B$ (i.e. with Friedel-type oscillations neglected) is exactly equal to $\Pi_B^{(0)}$. Therefore, the smoothed correlation function is given by the classical propagator,

$$V^2(|\psi^2(r_1)\psi^2(r_2)|)_{\text{smooth}} - 1 = \Pi_B(r_1, r_2),$$

independent of the relation between $|r_1 - r_2|$ and $l_s$. The mean free path $l_s$ manifests itself only in setting the scale on which the oscillatory part of $\langle |\psi^2(r_1)\psi^2(r_2)| \rangle$ gets damped.

To summarize, the disorder averaging generates the scale $l_s$ which separates the regions of applicability of the Gaussian statistics and of the quasiclassical theory $\Pi^{(0)}_B$. However, this is not the full story yet. As we demonstrate below, the disorder averaging induces one more scale which plays a crucial role for the problem considered.

4. PROBLEM OF REPETITIONS

As was mentioned in Sec. 1 of Ref. 10 emphasized discrepancy between the $\sigma$-model and the trace formula as concerned the counting of repetitions in the expression for the level correlation function. As we demonstrate in this Section, this problem, while not affecting (except for a non-generic situation when the points $r_1$ and $r_2$ lie close to one and the same short periodic orbit) the lowest-order correlation function $\langle |\psi^2(r_1)\psi^2(r_2)|^2 \rangle$ considered in Sec. 3 is also relevant to higher-order correlators of wave function amplitudes. We will clarify the origin of this puzzling discrepancy and formulate the conditions under which each of the two results apply. We assume $\beta = 1$ in this section (so that eigenfunctions are real) and start our consideration from the correlation function

$$\Gamma^{(4)}(r - r') = \langle \psi_\mu(r)\psi_\nu(r)\psi_\rho(r')\psi_\sigma(r')\psi_\mu'(r)\psi_\nu'(r')\psi_\rho'(r)\psi_\sigma'(r') \rangle,$$

where $\psi_{\mu,\nu,\rho,\sigma}$ are different eigenfunctions with sufficiently close energies. We will further assume that $l_s \ll |r - r'| \ll l_{tr}$, i.e. the distance $|r - r'|$ is in the expected range of applicability of the ballistic $\sigma$-model. The objects of the type (17) naturally arise when one studies fluctuations of matrix elements of the electron-electron interaction, which are important for statistics of electron transport through quantum dots, see [18].

We first consider the situation of a sufficiently large system size $L$ (e.g., we can assume a diffusive system, $L \gg l_{tr}$); in this case the value of $L$ will be irrelevant for the results (except for normalization of wave functions). In the end of the section we will generalize our conclusions to the case of a ballistic system. We start by performing an averaging over an auxiliary
random potential with a single-particle mean free path \( \tilde{l}_s = v\tau_s \) satisfying \(|r - r'| \ll \tilde{l}_s \ll L \) (which is much weaker than our “main” random potential and will thus not enter the final result). This auxiliary averaging allows us to present (17) in the form

\[
\Gamma^{(4)}(r - r') = (\pi \nu)^{-4} \langle \text{Im} G_R(r, r') \rangle^4 = \frac{3}{8} (\pi \nu)^{-4} \langle G^2_R(r, r')G^2_A(r, r') \rangle, \tag{18}
\]

where \( G_{R,A} = (E - \hat{H} \pm i/2\tilde{\tau}_s)^{-1} \) is a Green’s function in a given realization of the “main” random potential, the averaging over which remains to be performed in the r.h.s. of (18), as indicated by \( \langle \ldots \rangle \). The cumulant \( \langle \langle \psi^4(r)\psi^4(r') \rangle \rangle \) can be also reduced to the form (18), if one uses the \( \sigma \)-model approach to average over the auxiliary random potential, in full analogy with the derivation of Eqs. (15), (16) (in view of \(|r - r'| \ll \tilde{l}_s \) the zero-mode approximation is appropriate). An analogous trick of an auxiliary averaging was used in [12] in order to study ballistic wave function correlations in a random magnetic field.

Let us first evaluate the r.h.s. of (18) using the ballistic \( \sigma \)-model. The calculation is trivial and yields

\[
(2\pi^2 \nu^2)^{-2}\langle G^2_R(r, r')G^2_A(r, r') \rangle = 2\Pi^2_B(r, r') \simeq \frac{2}{(\pi k|r - r'|)^2}. \tag{19}
\]

The result (19) is fully transparent from the point of view of diagrammatics: there are two possibilities to couple the retarded and advanced Green’s functions in two “ballistic-diffuson” ladders, yielding the factor of 2 in front of the squared propagator \( \Pi^2_B(r, r') \).

We are going to show now that the result (19) of the ballistic \( \sigma \)-model is only correct for sufficiently large distances, \(|r - r'| \gg l_L \) (the scale \( l_L \) will be specified below), while in the opposite limit, \(|r - r'| \ll l_L \), it is wrong by factor of 2. To evaluate the r.h.s. of (18), we use the path integral approach. The product of the four Green’s functions in (18) can be written as (we set \( r = 0 \) and \( R = r' - r \))

\[
\langle G^2_R(0, R)G^2_A(0, R) \rangle = \prod_{i=1}^4 \int_0^\infty dt_i \int_{r_i(0)=0}^{r_i(t_i)=R} \mathcal{D}r_i \exp[iS_{\text{kin}} - S_W]
\times \exp[iE(t_1 + t_2 - t_3 - t_4) - (t_1 + t_2 + t_3 + t_4)/\tilde{\tau}_s]; \tag{20}
\]

\[
S_{\text{kin}} = \frac{m}{2} \left( \int_0^{t_1} dt_1 \tilde{r}_1^2 + \int_0^{t_2} dt_2 \tilde{r}_2^2 - \int_0^{t_3} dt_3 \tilde{r}_3^2 - \int_0^{t_4} dt_4 \tilde{r}_4^2 \right); \nonumber
\]

\[
S_W = \frac{1}{2}(S_{11} + S_{22} + S_{33} + S_{44}) + S_{12} + S_{34} - S_{13} - S_{14} - S_{23} - S_{24}; \nonumber
\]

\[
S_{ij} = \int_0^{t_i} \int_0^{t_j} W(r_i(t) - r_j(t'))dtdt', \tag{21}
\]
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where the paths $r_1(t_1), r_2(t_2)$ correspond to the retarded, and $r_3(t_3), r_4(t_4)$ to the advanced Green’s functions. It is useful to perform the change of variables, introducing $T = (t_1 + t_2 + t_3 + t_4)/4$, $t_- = (t_1 + t_2 - t_3 - t_4)/2$, $\tau_{12} = t_1 - t_2$, $\tau_{34} = t_3 - t_4$, and $R_+ = (r_1 + r_2 + r_3 + r_4)/4$, $R_- = (r_1 + r_2 - r_3 - r_4)/2$, $r_{12} = r_1 - r_2$, $r_{34} = r_3 - r_4$. The kinetic part of the action then reads

$$S_{\text{kin}} \simeq m \int_0^T dt \left\{ \left( -\frac{t_-}{T} + \frac{\tau_{12}^2 - \tau_{34}^2}{4T^2} \right) \ddot{R}_+^2 + 2\dot{R}_+ \dot{R}_- + \frac{\dot{r}_{12}^2 - \dot{r}_{34}^2}{4} - \frac{\ddot{R}_+ \tau_{12} \dot{r}_{12} - \tau_{34} \dot{r}_{34}}{2T} \right\}. \quad (22)$$

Since we are interested in the ballistic scales ($\ll l_t$) it is convenient to split $r, r_{12}, r_{34}$ into components parallel ($\| \$ and perpendicular ($\perp$) to $R$. As shown below, $S_W$ depends only on the $\perp$-components. Shifting the parallel components of $r_{12}, r_{34}$ via $\rho_{12} = r_{12}^\| - P_+ \tau_{12}/T$, $\rho_{34} = r_{34}^\| - P_+ \tau_{34}/T$, we can perform integration over $\rho_{12}, \rho_{34}, \tau_{12}, \tau_{34}$, which yields the factor $(T/R)^2$. Furthermore, integration over $P_+, P_+, t_-$, and $T$ produces the factor $m/2E$ and sets $T = R/v$ with $v = (2E/m)^{1/2}$. Finally, introducing

$$\rho_\pm = (r_{12}^\| \pm r_{34}^\|)/2, \quad r_\pm = R_\pm^\|, \quad (23)$$

we simplify the action to the form

$$\tilde{S}_{\text{kin}} = m \int_0^T dt \{ 2\dot{r}_+ \dot{r}_- + \dot{\rho}_+ \dot{\rho}_- \}. \quad (24)$$

Now we turn to the disorder-induced part of the action, $S_W$ (where all $t_i$ can be approximated by $T$, see [21]). Expanding the correlator $W$ up to the second order in $r$, we obtain

$$S_W \simeq \int_0^T U(\rho_- (t), \rho_+ (t)) dt - 2 \int_0^T \{ G(\rho_- (t)) + G(\rho_+ (t)) \} r_\perp^2 (t) dt, \quad (25)$$

where

$$U(y, y') = 2 \{ F(y) + F(y') \} - F(y + y') - F(y-y'). \quad (26)$$

Here the functions $F(y)$ and $G(y)$ are given by

$$F(y) \equiv \int_0^\infty \frac{dx}{v} [W(x, 0) - W(x, y)], \quad G(y) \equiv \int_0^\infty \frac{dx}{v} \frac{\partial^2}{\partial y^2} W(x, y), \quad (27)$$

with the following asymptotic behavior $F(y \ll d) \simeq -G(0)y^2/2$, $F(y \gg d) \simeq \tau_s^{-1}$, $G(0) = -m^2v^2/\tau_t$, and $G(y \gg d) \to 0$.

We examine first the contribution of the region $\rho_+, \rho_- \gg d$, where all four paths are uncorrelated. Then the action [23] is large, $S_W \simeq 2TF(y \gg d)$.
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d) = 2T/τs, since the phases acquired by the waves traveling along these paths do not cancel each other. Thus the contribution of this region decays exponentially at \( R \gg l_s \) (which is the scale of our interest), in the same way as the single Green’s function does. We conclude that the path integral (20) is dominated by paths with at least one of \( \rho_+ \) much less than \( d \), which corresponds to correlations between “retarded” and “advanced” paths.

Let us now consider the contribution of \( \rho_+ \gg d, \rho_- \ll d \), when the two “ballistic diffusons” (formed by the pairs of paths 1, 3 and 2, 4) are not correlated. Then \( S_W \) acquires a form

\[
S_W \simeq \int_0^T \left[ 2\frac{w_0}{v} r_+^2(t) + \frac{w_0}{v} \rho_+^2(t) \right] dt, \tag{28}
\]

where \( w_0 = -vG(0) \sim v(d^2\tau_s)^{-1} \). Combining Eqs. (24) and (28) we see that the variables \( r_\pm \) and \( \rho_\pm \) separate, each of the two pairs being characterized by the diffusive action (24) (describing the dynamics of the “center of mass” \( r_+ \) and the distance between the two “diffusons” \( \rho_\pm \), respectively). This gives squared propagator \( \Pi_B^2(r, r') \), as in Eq. (19). The region \( \rho_- \gg d, \rho_+ \ll d \) (diffusons formed by paths 1, 4 and 2, 3) yields an identical contribution, reproducing the factor 2 in the \( \sigma \)-model result (19).

However, the approximation (28) is only justified at sufficiently large \( T \). Indeed, the action (24), (28) implies a diffusion of the velocity \( \dot{\rho}_+ / v \) with the diffusion coefficient \( 1/\tau_{tr} \). Taking into account the boundary conditions \( \rho_+(0) = \rho_+(T) = 0 \), it requires a time of the order of

\[
\tau_L \sim (d/l_{tr})^{2/3} \tag{29}
\]

for \( \rho_+ \) to reach a value \( \sim d \), which implies \( T \gg \tau_L \) as a condition of validity of Eq. (28) and thus of the result (19). As we discuss below, the scale \( \tau_L \) has a meaning of the inverse Lyapunov exponent; hence the subscript “L”.

At shorter distances \( R \ll l_L = v\tau_L \), both \( \rho_+ \) and \( \rho_- \) are small compared to \( d \). Expanding (28) up to the second order in both \( \rho_+ \) and \( \rho_- \) we have

\[
S_W \simeq \int_0^T \left[ \frac{4w_0}{v} r_+^2(t) + \frac{w_2}{v} \rho_+^2(t) \rho_-^2(t) \right] dt, \tag{30}
\]

where \( w_2 = -vG''(0)/2 \sim w_0/d^2 \) and we neglected cross-terms of the type \( r_+^2 \rho_\pm^2 \), which are small compared to the first term in the integrand in (30). We note that the variables \( r_\pm \) and \( \rho_\pm \) again separate. The first pair corresponds to the diffusion of \( \dot{r}_\pm \) [with a diffusion coefficient two times smaller than in Eq. (28)], while the second one now describes the divergence of the two pairs of paths. The differential equation for a Green’s function which corresponds
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to the \( \rho \)-part of the action reads

\[
\left( \frac{\partial}{\partial t} - \frac{i}{m} \frac{\partial^2}{\partial \rho_+ \partial \rho_-} + \frac{w_2}{v} \rho_+^2 \rho_-^2 \right) g(\rho_+, \rho_-, t) = \delta(t) \delta(\rho_+) \delta(\rho_-),
\]

and we are interested in the quantity \( g(0, 0, t) \), the correlation function \( g \)

being given by

\[
(2\pi^2 v^2)^{-2} \langle G_R^2(r, r') G_A^2(r, r') \rangle = \frac{2}{\pi m^2 v^3 R} g(0, 0, R/v).
\]

Now we analyze the solution of \((31)\). Clearly, \( \tau_L \equiv (m^2 v/w_2)^{1/3} \) sets a characteristic time scale for this equation. It is remarkable that upon the Fourier transformation, \( \rho_+ \rightarrow i(mv)^{-1} \partial/\partial \phi \) and \( \partial/\partial \rho_+ \rightarrow -i w_2 \), the operator in the l.h.s. of \((31)\) coincides with the one obtained by Aleiner and Larkin in Ref. [23] from the analysis of divergence of classical paths in a weak smooth random potential. As shown in [22], the scale \( \tau_L \) has the meaning of a corresponding inverse Lyapunov exponent. We have thus demonstrated that the same scale arises in the treatment of a quantum \((l_s \gg d)\) random potential, despite the diffractive nature of the scattering [22].

For \( t \ll \tau_L \) one can neglect the term \( \rho_+^2 \rho_-^2 \) in the l.h.s of \((31)\), yielding

\[
g(\rho_+, \rho_-, t) \simeq \frac{m}{2\pi t} \exp \frac{i \rho_+ \rho_-}{mt}, \quad t \ll \tau_L.
\]

In fact, the solution remains a function of the product \( \rho_+ \rho_- \) for all \( t \), and Eq. \((31)\) can be reduced, in dimensionless variables \( \tau = -i^{5/3} t/\tau_L, r = (i/4)^{-1/6} (m \rho_+ \rho_- / \tau_L)^{1/2} \), to an equation for a 2D anharmonic oscillator,

\[
(-i \partial_r - r^{-1} \partial_r r \partial_r + r^4) \tilde{g}(r, \tau) = \delta^2(r) \delta(\tau).
\]

Therefore, at \( \tau \gg 1 \) (corresponding to \( t \gg \tau_L \)) its solution decays exponentially, \( g \propto \exp \{-i^{2/3} \epsilon_0 t/\tau_L \} \), where \( \epsilon_0 \) is the (dimensionless) oscillator ground-state energy. However, as explained above, at this time scales the approximation \( \rho_+ \rho_- \ll d \) leading to Eq. \((31)\) loses its validity, and the result is dominated by the contributions with either \( \rho_+ \gg d \) or \( \rho_- \gg d \) described by the action \((28)\) and yielding the result \((33)\).

We thus conclude that

\[
(2\pi^2 v^2)^{-2} \langle G_R^2(r, r') G_A^2(r, r') \rangle = \frac{1}{(\pi k R)^2} \mathcal{F}(R/\ell_L),
\]

where \( \mathcal{F}(x) \) is a parameterless function with asymptotics \( \mathcal{F}(x \ll 1) = 1 \) and \( \mathcal{F}(x \gg 1) = 2 \). To calculate the crossover function \( \mathcal{F}(x) \), one has to solve a differential equation

\[
\left( \frac{\partial}{\partial t} - \frac{i}{m} \frac{\partial^2}{\partial \rho_+ \partial \rho_-} + \mathcal{U}(\rho_+, \rho_-) \right) g(\rho_+, \rho_-, t) = \delta(t) \delta(\rho_+) \delta(\rho_-),
\]
with $U(\rho_+, \rho_-)$ defined in (24); the desired correlation function is then given by Eq. (32). The solution in the crossover region depends on the specific form of the random potential correlation function $W(r)$ and, in general, can only be found numerically; however, the explicit form of the function $F(x)$ is not important for our analysis. A similar analysis can be performed for higher-order correlation function $\langle G^n_R(r, r')G^n_A(r, r') \rangle$, yielding the crossover from $F_n(x \ll 1) = 1$ to $F_n(x \gg 1) = n!$.

Therefore, the ballistic $\sigma$-model result is only valid at distances $R \gg l_L$. At shorter scales, the four paths do not split into two pairs yielding each a ballistic diffusion (as the $\sigma$-model approach assumes) but rather propagate together, remaining all strongly correlated. In the diagrammatic language this means that all the four Green’s functions are coupled by impurity lines. This is closely related to the non-Markovian memory effects showing up in transport in a smooth random potential.

For ballistic systems this implies that the $\sigma$-model approach is only valid provided the random potential over which the averaging is performed is sufficiently strong, so that $l_L \ll L$. Note that such an averaging can be characterized as “strongly invasive”: the Lyapunov exponent due to the random potential $\tau^{-1}_L$ is much larger than that of the clean system itself (which is $\sim v/L$ for a generic system of a size $L$). Therefore, although such a random potential does not essentially influence (in view of $l_t \gg L$) the positions of Ruelle resonances governing the relaxation rate in the chaotic system, it strongly affects the Lyapunov exponents.

In the opposite case $l_L \gg L$ (corresponding to a “non-invasive” averaging, keeping intact all the classical features of the system) the ballistic $\sigma$-model does not give correct results for non-universal level and eigenfunction correlations. More precisely, it is still valid when applied to the lowest-order correlation function $\langle |\psi^2(r)\psi^2(r')| \rangle$, see Eq. (16). However, when one considers higher-order correlation functions (or, in fact, $\langle |\psi^2(r)\psi^2(r')| \rangle$ with $r$ and $r'$ located close to a short periodic orbit), the approach breaks down, since it does not take into account correlations between four or more close paths. This is precisely the problem which shows up in the counting of repetitions in the formula for the level correlation function. In that case, relevant paths for both $G_R$ and $G_A$ wind $n$ times in the vicinity of a periodic orbit. If $L \ll l_L$, all these paths are correlated (in the same way as four Green’s functions in the example considered above). The $\sigma$-model neglects this fact, combining the two paths (“R” and “A”) in a ladder, which can be done in $n$ possible ways. This leads to an overestimate of the contribution by a factor of $n$ (similarly to the overestimate of the correlation function (14) at $R \ll l_L$ by a factor of 2). Therefore, the results of the $\sigma$-model and of the trace formula (14) for the smooth part of the level correlation function correspond to
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different types of averaging: the former to \( l_L \ll L \), while the latter to \( l_L \gg L \)
(in both the cases the condition \( l_s \ll L \) should be assumed, see Sec. 3).

It is worth noting that we assumed the clean system to be characterized
by a single scale \( L \), so that the corresponding Lyapunov exponent is \( \sim L^{-1} \).
An example of a system which does not belong to this class is given by a
billiard with a surface roughness producing a diffuse boundary scattering.\(^{11}\)
In this case repetitions give a negligibly small contribution, and the \( \sigma \)-model
approach as applied in \(^{11}\) is justified with the only assumption \( l_s \ll L \).

Finally, let us note a close similarity between the condition \( l_L \gg L \) of
the failure of the \( \sigma \)-model and the condition \( l_L \gg R_c \) (with \( R_c \) the cyclotron
radius) under which memory effects develop into adiabaticity which affects
strongly magnetoresistivity in a smooth disorder.\(^{26}\) In both cases, correlations
between multiple traversals of a periodic (respectively cyclotron) orbit
lead to breakdown of the Boltzmann kinetic equation.

5. CONCLUSIONS

We have studied the statistics of wave function in a chaotic system on
the ballistic scale. To define the statistical ensemble, we have added to
the system a smooth quantum random potential with a correlation length
\( d \gg k^{-1} \). We required the corresponding quantum (single-particle) mean
free path to satisfy \( l_s \ll L \), which ensures that the ensemble of quantum sys-
tems is sufficiently large and provides meaningful statistics. Using the ballis-
tic \( \sigma \)-model approach, we have shown that on scales \( \ll l_s \) the wave function
statistics is Gaussian, Eq. (15), proving the Berry’s conjecture.\(^{2,3}\) On larger
scales Friedel-type oscillations in the correlation function \( \langle |\psi^2(\mathbf{r})\psi^2(\mathbf{r'})| \rangle \)
are exponentially damped, and the latter is given by the classical ballistic propagator, Eq. (12). However, an attempt to use the \( \sigma \)-model approach for
calculation of higher-order correlation functions in this regime makes us to
face a problem: the \( \sigma \)-model does not take into account strong correlations
between 4 (or more) Green’s functions dominated by paths which are all
close to each other. Considering the correlation function (18) as an exam-
ple and performing its detailed analysis, we demonstrated that the ballistic
\( \sigma \)-model result is only correct on distances \( R \gg l_L = v\tau L \), where \( \tau_L^{-1} \) is
the classical Lyapunov exponent associated with the random potential, see
Eq. (29). It is remarkable that the Lyapunov scale \( l_L \) arises despite the fact
that the random potential is assumed to be quantum, with \( l_s \gg d \). On
shorter distances, \( R \ll l_L \), the \( \sigma \)-model result for (18) is wrong by factor of
two. In this regime the path integral approach is more appropriate; it allows
one also to describe (at least, in principle) the crossover between the two
regimes (\( R \sim l_L \)).
A completely analogous situation is encountered when one studies the level correlation function. Our results thus resolve the discrepancy between the predictions of the $\sigma$-model and of the trace formula for the smooth part of the two-level correlation function. The $\sigma$-model result is generically valid provided $l_L \ll L$, where $L$ is the system size. This condition means that the averaging is “invasive”, since it strongly affects the Lyapunov exponent (although it does not affect, in view of the assumption $l_{tr} \gg L$, positions of the Ruelle resonances). In the opposite case of a “non-invasive” (preserving all classical parameters of the system) averaging, $l_L \gg L$, the $\sigma$-model approach neglecting correlations between multiple traversals of a periodic orbit loses its validity, and the smooth part of the level correlation function is given by the trace formula. Calculation of the system-specific oscillatory contribution in this situation remains an open problem. Such contributions are related to the behavior of the spectral form-factor at times close to the Heisenberg time $t_H$, which is beyond the region of validity of the trace formula. Bogomolny and Keating proposed a procedure allowing to obtain the oscillatory contribution from the trace formula. However, in view of the ad hoc nature of their suggestion, the status of the result is unclear. On the other hand, the $\sigma$-model approach, which would be more appropriate for an analysis of contributions of this kind (non-perturbative from the field-theoretical point of view), does not treat properly correlations of multiple close paths. A natural idea would be to combine the two approaches, i.e. to formulate a field theory which would contain not only conventional “2-diffusons” (i.e. those generated by a product $G_RG_A$ of two Green’s functions) $D(rn, r'n')$, but also “4-diffusons”, “6-diffusons”, etc. This remains a challenge for future research.

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