Arbitrary high-order structure-preserving schemes for the generalized Rosenau-type equation

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Abstract

In this paper, we are concerned with arbitrarily high-order momentum-preserving and energy-preserving schemes for solving the generalized Rosenau-type equation, respectively. The derivation of the momentum-preserving schemes is made within the symplectic Runge-Kutta method, coupled with the standard Fourier pseudo-spectral method in space. Then, combined with the quadratic auxiliary variable approach and the symplectic Runge-Kutta method, together with the standard Fourier pseudo-spectral method, we present a class of high-order mass- and energy-preserving schemes for the Rosenau equation. Finally, extensive numerical tests and comparisons are also addressed to illustrate the performance of the proposed schemes.

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1 Introduction

In this paper, we consider the following generalized Rosenau-type equation

\[
\begin{aligned}
\partial_t u(x,t) + \kappa \partial_x u(x,t) - \delta \partial_{xxx} u(x,t) + b \partial_{xxx} u(x,t) \\
+ \alpha \partial_{xxxx} u(x,t) + \beta \partial_x (u(x,t)^p) = 0, \quad x \in \Omega \subset \mathbb{R}, \quad t > 0, \tag{1.1}
\end{aligned}
\]

where \( t \) is the time variable, \( x \) is the spatial variable, \( u := u(x,t) \) is the real-valued wave function, \( \kappa, \delta > 0, b, \alpha > 0 \) and \( \beta \) are given real constants, \( p \) is a given positive integer, \( u_0(x) \) is a given initial condition and \( \Omega = [x_l, x_r] \) is a bounded domain. Here, the Rosenau equation (1.1) will be augmented with periodic boundary condition.

The Rosenau equation was originally introduced to describe the dynamics of dense discrete systems\textsuperscript{34}. Nowadays, it has played an important role in fluid mechanics as well as atmosphere and ocean. Moreover, when \( u \) is assumed to be smooth, the equation (1.1) satisfies the following Hamiltonian formulation

\[
u_t = J \frac{\delta H}{\delta u}, \tag{1.2}
\]

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where \( J = -(1 - \delta \partial_{xx} + \alpha \partial_{xxxx})^{-1} \partial_x \) is a Hamiltonian operator and \( H \) is the Hamiltonian functional, namely

\[
H(t) = \int_{\Omega} \left( \frac{\kappa}{2} u^2 - \frac{b}{2} u_x^2 + \frac{\beta}{p+1} u^{p+1} \right) dx, \quad t \geq 0, \tag{1.3}
\]

Besides the Hamiltonian energy (1.3), the equation (1.1) also conserves the mass

\[
M(t) = \int_{\Omega} u dx \equiv M(0), \quad t \geq 0, \tag{1.4}
\]

and the momentum

\[
I(t) = \int_{\Omega} \left( \frac{1}{2} u^2 + \delta u_x^2 + \frac{\alpha}{2} u_{xx}^2 \right) dx \equiv I(0), \quad t \geq 0. \tag{1.5}
\]

In the numerical computation of such Hamiltonian partial differential equations, it is often preferable to design some special numerical schemes that inherit one or more intrinsic properties of the original system exactly in a discrete sense; they are called structure-preserving schemes [13, 14, 21]. In [10], Chung proposed an implicit finite difference (IFD) scheme, which can satisfy the discrete analogue of momentum (1.5). Furthermore, they prove rigorously in mathematics that the scheme is second-order accurate in time and space. Subsequently, Omrani et al. [30] developed a linearly implicit momentum-preserving finite difference scheme for the classical Rosenau equation, in which a linear system is to be solved at every time step. Thus it is computationally much cheaper than that of the IFD scheme. Over the years, various momentum-preserving schemes for the equation (1.1) have been proposed and analyzed (e.g., see Refs [1, 2, 22, 28, 31, 42, 43, 45, 46, 47]). However, to the best of our knowledge, all of existing momentum-preserving schemes are only second-order accurate in time at most. It has been shown in [18, 20] that, compared with the second-order schemes, the high-order ones not only provide smaller numerical errors as a large time step chosen, but also will be more advantages in the robustness. Consequently, the first aim of this paper is to present a novel paradigm for developing arbitrary high-order momentum-preserving schemes for the equation (1.1).

Besides the momentum conservation law (1.5), the equation (1.1) also satisfies the Hamiltonian energy (1.3), which is one of the most important first integrals of the Hamiltonian system. Based on the averaged vector field method [32], Cai et al. [3] proposed a second-order energy-preserving scheme, and two fourth-order energy-preserving schemes are then proposed by using the composition ideas [21]. Nevertheless, it is shown in [21] that the high-order schemes obtained by the composition method will be at the price of a terrible zig-zag of the step points (see Fig. 4.2 in Ref. [21]), which may be tedious and time consuming. Thus, the construction of high-order energy-preserving schemes for the Rosenau equation (1.1) seems to be still at its beginning stage. In this paper, the second aim is to present a new strategy for proposing arbitrary high-order energy-preserving schemes for the Rosenau equation (1.1) based on the quadratic auxiliary variable (QAV) approach [6, 16, 41], which is inspired by the idea of the energy quadratization (EQ) approach [39, 40, 48].

The rest of this paper is organized as follows. In Section 2, the high-order momentum-preserving and energy-preserving schemes for the equation (1.1) are proposed, respectively and their structure-preserving properties are analysed in details. Extensive numerical examples and some comparisons are addressed to illustrate the performance of the proposed schemes in Section 4. We draw some conclusions in Section 5.
2 High-order structure-preserving schemes

In this section, we will propose high-order momentum-preserving schemes and energy-preserving schemes for the equation \([1,1]\), respectively.

To be the start, let the spatial step size \(h = \frac{x_2 - x_1}{N}\) with an even positive integer \(N\), and denote the grid points by \(x_j = jh\) for \(j = 0, 1, 2, \ldots, N\); set \(u_j\) be the numerical approximation of \(u(x_j, t)\) for \(j = 0, 1, \ldots, N\), and \(U := (u_0, u_1, \cdots, u_{N-1})^T\) be the solution vector; we also define the discrete inner product, \(l^2\)-norm and \(l^\infty\)-norm as, respectively,
\[
(U, V)_h = h \sum_{j=0}^{N-1} u_j v_j, \quad \|U\|_h^2 = (U, U)_h, \quad \|U\|_{h, \infty} = \max_{0 \leq j < N-1} |u_j|.
\]

In addition, we denote ‘.’ as the element product of vectors \(U\) and \(V\), that is
\[
U \cdot V = (u_0v_0, u_1v_1, \cdots, u_{N-1}v_{N-1})^T.
\]

For brevity, we denote \(U \cdot U\) as \(U^2\).

As achieving high order accuracy in time, the spatial accuracy shall be comparable to that of the time-discrete discretization. Actually, we consider the periodic boundary condition in this paper, so that the Fourier pseudo-spectral method is a very good choice because of the high order accuracy and the fast Fourier transform (FFT) algorithm. Thus, we first expounded the Fourier pseudo-spectral method, as follows.

Let the interpolation space as
\[
S_h = \text{span}\{l_j(x), \ 0 \leq j \leq N - 1\}
\]
where \(l_j(x)\) is trigonometric polynomials of degree \(N/2\) given by
\[
l_j(x) = \frac{1}{N} \sum_{k=-N/2}^{N/2} \frac{1}{c_k} e^{ik\mu(x-x_j)}, \quad c_k = \begin{cases} 1, & |k| < \frac{N}{2}, \\
2, & |k| = \frac{N}{2}, \end{cases}
\]
with \(\mu = \frac{2\pi}{x_2 - x_1}\). We then define the interpolation operator \(I_N : C(\Omega) \rightarrow S_h\) as \([5,8]\)
\[
I_N u(x, t) = \sum_{k=0}^{N-1} u_k(t) l_k(x),
\]
where \(u_k(t) = u(x_k, t), \ k = 0, 1, 2, \cdots, N - 1\).

Taking the partial derivative with respect to \(x\) at the collocation points \(x_j\), we have
\[
\frac{\partial^r I_N u(x_j, t)}{\partial x^r} = \sum_{k=0}^{N-1} u_k(t) \frac{\partial^r l_k(x_j)}{\partial x^r} = \sum_{k=0}^{N-1} (D_r)_{j,k} u_k(t), \ j = 0, 1, \cdots, N - 1,
\]  \(2.1\)
where \(D_r\) represents the spectral differential matrix with elements given by
\[
(D_r)_{j,k} = \frac{\partial^r l_k(x_j)}{\partial x^r}, \ j, k = 0, 1, \cdots, N - 1.
\]

In particular, we have\([23,37]\)
\[
(D_1)_{j,k} = \begin{cases} \frac{1}{2} \mu (-1)^{j+k} \cot(\mu \frac{x_j - x_k}{2}), & j \neq k, \\
0, & j = k, \end{cases}
\]

\[
(D_2)_{j,k} = \begin{cases} \frac{1}{2} \mu (-1)^{j+k} \cot(\mu \frac{x_j - x_k}{2}) \left(1 - \cot^2(\mu \frac{x_j - x_k}{2}) \right), & j \neq k, \\
0, & j = k. \end{cases}
\]

\[
(D_3)_{j,k} = \begin{cases} \frac{1}{2} \mu (-1)^{j+k} \cot(\mu \frac{x_j - x_k}{2}) \left(1 - \cot^2(\mu \frac{x_j - x_k}{2}) \right) \left(1 - \cot^2(\mu \frac{x_j - x_k}{2}) \right), & j \neq k, \\
0, & j = k. \end{cases}
\]

\[
(D_4)_{j,k} = \begin{cases} \frac{1}{2} \mu (-1)^{j+k} \cot(\mu \frac{x_j - x_k}{2}) \left(1 - \cot^2(\mu \frac{x_j - x_k}{2}) \right) \left(1 - \cot^2(\mu \frac{x_j - x_k}{2}) \right) \left(1 - \cot^2(\mu \frac{x_j - x_k}{2}) \right), & j \neq k, \\
0, & j = k. \end{cases}
\]
Remark 2.1. We should note that [13, 37]

\[(D_2)_{j,k} = \begin{cases} \frac{1}{2} \mu^2 (-1)^{j+k+1} \csc^2\left(\frac{x_j - x_k}{2}\right), & j \neq k, \\ -\mu \frac{N^2 + 2}{12}, & j = k, \end{cases} \]

\[(D_3)_{j,k} = \begin{cases} \frac{3\mu^3}{4} (-1)^{j+k} \cos\left(\frac{x_j - x_k}{2}\right) \csc^3\left(\frac{x_j - x_k}{2}\right), & j \neq k, \\ \frac{\mu^3 N^2}{8} (-1)^{j+k+1} \cot\left(\frac{x_j - x_k}{2}\right), & j = k, \end{cases} \]

and

\[(D_4)_{j,k} = \begin{cases} \mu^4 (-1)^{j+k} \csc^2\left(\frac{x_j - x_k}{2}\right) \left(\frac{N^2}{4} - \frac{1}{2} - \frac{3}{2} \cot^2\left(\frac{x_j - x_k}{2}\right)\right), & j \neq k, \\ \mu^4 \left(\frac{N^4}{80} + \frac{N^2}{12} - \frac{1}{30}\right), & j = k. \end{cases} \]

**Remark 2.1.** We should note that [13, 37]

\[D_r = \begin{cases} F_N^H \Lambda^r F_N, & r \text{ is an odd integer}, \\ F_N^H \tilde{\Lambda}^r F_N, & r \text{ is an even integer}, \end{cases} \]

where \(\Lambda\) and \(\tilde{\Lambda}\) are

\[\Lambda = \text{diag}\left(i \mu\{0, 1, \cdots, \frac{N}{2} - 1, 0, -\frac{N}{2} + 1, \cdots, -2, -1\}\right),\]

\[\tilde{\Lambda} = \text{diag}\left(i \mu\{0, 1, \cdots, \frac{N}{2} - 1, \frac{N}{2}, -\frac{N}{2} + 1, \cdots, -2, -1\}\right),\]

and \(F_N\) is the discrete Fourier transform (DFT) and \(F_N^H\) represents the conjugate transpose of \(F_N\).

We set \(t_n = n\tau\), and \(t_{ni} = t_n + c_i \tau, \ i = 1, 2, \cdots, s, \ n = 0, 1, 2, \cdots\), where \(\tau\) is the time step size and \(c_1, c_2, \cdots, c_s\) are distinct real numbers (usually \(0 \leq c_i \leq 1\)). The approximations of the function \(u(x, t)\) at points \((x_j, t_n)\) and \((x_j, t_{ni})\) are denoted by \(u^n_j\) and \(u^{ni}_j\), respectively.

### 2.1 High-order momentum-preserving scheme

In this section, we propose a class of high-order momentum-preserving schemes for the equation \((1.1)\). To be the start, we rewrite \((1.1)\) into

\[A u_t = \mathcal{F}(u) u, \quad \mathcal{F}(u) = -\left[\kappa \partial_x + b \partial_{xxx} + \frac{p\beta}{p+1} (u^{p-1} \partial_x + \partial_x(u^p)\right], \quad (2.2)\]

where \(A = 1 - \delta \partial_{xx} + \alpha \partial_{xxxx}\) is a self-adjoint operator, \(\mathcal{F}(u)\) is an anti-adjoint operator and \(\mathcal{F}(u) u\) is defined by

\[\mathcal{F}(u) u = -\left[\kappa \partial_x u + b \partial_{xxx} u + \frac{p\beta}{p+1} (u^{p-1} \partial_x u + \partial_x(u^p)\right].\]

The Fourier pseudo-spectral method is then employed to solve \((2.2)\) in space and we obtain

\[\begin{cases} \mathcal{A}_h \frac{d}{dt} U = \mathcal{F}_h(U) U, \\ \mathcal{F}_h(U) = -\left[\kappa D_1 + b D_3 + \frac{p\beta}{p+1} (\text{diag}(U^{p-1}) D_1 + D_1 \text{diag}(U^{p-1})\right], \quad (2.3) \end{cases}\]
where $\mathcal{A}_h = I - \delta D_2 + \alpha D_4$ is a symmetric matrix, $\mathcal{F}_h(U)$ is anti-symmetric for $U$, and $F_h(U)\bar{U}$ is defined by

$$F_h(U)\bar{U} = \left[ \lambda_{D1} U + b D_3 U + \frac{p \beta}{p+1} (U^{p-1} \cdot D_1 U + D_1 (U^p)) \right].$$

(2.4)

**Theorem 2.1.** The semi-discrete system (2.3) preserves the following semi-discrete momentum conservation law

$$\frac{d}{dt} I_h(t) = 0, \quad I_h(t) = \frac{1}{2} \langle \mathcal{A}_h U, U \rangle_h, \quad t \geq 0.$$  

(2.5)

**Proof.** With noting the symmetric property of $\mathcal{A}_h$ and anti-symmetric property of $F_h(U)$ for $U$, we have

$$\frac{d}{dt} I_h(t) = \langle \mathcal{A}_h \frac{d}{dt} U, U \rangle_h = \langle F_h(U) U, U \rangle_h = 0,$$

which completes the proof. \hfill \Box

**Theorem 2.2.** If $p = 2$, the semi-discrete system (2.3) preserves the following semi-discrete mass

$$\frac{d}{dt} M_h(t) = 0, \quad M_h(t) = \langle U, 1 \rangle_h, \quad t \geq 0.$$  

(2.6)

**Proof.** It follows from (2.3) that

$$\frac{d}{dt} M_h(t) = \langle \mathcal{A}_h^{-1} \mathcal{F}_h(U) U, 1 \rangle_h = \langle F_h(U) U, \mathcal{A}_h^{-1} 1 \rangle_h = \langle F_h(U) U, 1 \rangle_h.$$  

(2.7)

With (2.4), we have

$$\langle F_h(U) U, 1 \rangle_h = -\langle \lambda_{D1} U + b D_3 U + \frac{p \beta}{p+1} (U^{p-1} \cdot D_1 U + D_1 (U^p)), 1 \rangle_h$$

$$= -\frac{p \beta}{p+1} \langle D_1 U, U^{p-1} \rangle_h.$$  

(2.8)

As $p = 2$, we obtain from the above equation

$$\langle F_h(U) U, 1 \rangle_h = 0,$$

which implies that

$$\frac{d}{dt} M_h(t) = 0.$$

This completes the proof. \hfill \Box

**Remark 2.2.** If $p > 2$ and is a positive integer, we can deduce that the Fourier spectral differential matrix $D_1$ cannot satisfy the discrete equation

$$\langle D_1 U, U^{p-1} \rangle_h = 0, \quad \text{for } \forall U.$$  

(2.9)

Thus, the system (2.3) cannot conserve the semi-discrete mass (2.6), as $p > 2$.

We then apply an RK method to the system (2.3) in time to give a class of fully discrete schemes for the equation (1.1), as follows:
Scheme 2.1. Let $b_i, a_{ij}(i, j = 1, \cdots, s)$ be real numbers and let $c_i = \sum_{j=1}^{s} a_{ij}$. For the given $U^n$, an $s$-stage Runge-Kutta method is given by

$$
\begin{align*}
A_h K^n_i &= F_h(U^{ni}) U^{ni}; \\ U^{ni} &= U^n + \tau \sum_{j=1}^{s} a_{ij} K^n_j, \ i = 1, 2, \cdots, s, \\
U^{n+1} &= U^n + \tau \sum_{i=1}^{s} b_i K^n_i.
\end{align*}
$$

(2.10)

Theorem 2.3. If the coefficients of the RK method satisfy

$$
b_i a_{ij} + b_j a_{ji} = b_i b_j, \ \forall \ i, j = 1, \cdots, s,
$$

(2.11)

Scheme 2.1 conserves the following discrete momentum

$$
I^{n+1}_h = I^n_h, \ I^n_h = \frac{1}{2} \langle U^n, A_h U^n \rangle_h, \ n = 0, 1, 2, \cdots.
$$

(2.12)

Proof. It follows from the second equality of (2.10) that

$$
I^{n+1}_h - I^n_h
= \frac{1}{2} \langle U^{n+1}, A_h U^{n+1} \rangle_h - \frac{1}{2} \langle U^n, A_h U^n \rangle_h
= \frac{1}{2} \langle U^n + \tau \sum_{i=1}^{s} b_i K^n_i, A_h (U^n + \tau \sum_{j=1}^{s} b_j K^n_j) \rangle_h - \frac{1}{2} \langle U^n, A_h U^n \rangle_h
= \frac{\tau}{2} \sum_{i=1}^{s} b_i \langle U^n, A_h K^n_i \rangle_h + \frac{\tau}{2} \sum_{i=1}^{s} b_i \langle K^n_i, A_h U^n \rangle_h + \frac{\tau^2}{2} \sum_{i,j=1}^{s} b_i b_j \langle K^n_i, A_h K^n_j \rangle_h.
$$

(2.13)

With noting

$$
\frac{\tau}{2} \sum_{i=1}^{s} b_i \langle U^n, A_h K^n_i \rangle_h = \frac{\tau}{2} \sum_{i=1}^{s} b_i \langle U^{ni} - \tau \sum_{j=1}^{s} a_{ij} K^n_j, A_h K^n_i \rangle_h
= \frac{\tau}{2} \sum_{i=1}^{s} b_i \langle U^{ni}, A_h K^n_i \rangle_h - \frac{\tau^2}{2} \sum_{i,j=1}^{s} b_j a_{ji} \langle K^n_i, A_h K^n_j \rangle_h.
$$

(2.14)

Similarly, we have

$$
\frac{\tau}{2} \sum_{i=1}^{s} b_i \langle K^n_i, A_h U^n \rangle_h = \frac{\tau}{2} \sum_{i=1}^{s} b_i \langle K^n_i, A_h U^{ni} \rangle_h - \frac{\tau^2}{2} \sum_{i,j=1}^{s} b_i a_{ij} \langle K^n_i, A_h K^n_j \rangle_h.
$$

(2.15)

We insert (2.14) and (2.15) into (2.13) and then use the symmetry of $A_h$ to obtain

$$
I^{n+1}_h - I^n_h = \frac{\tau}{2} \sum_{i=1}^{s} b_i \langle U^{ni}, A_h K^n_i \rangle_h + \langle K^n_i, A_h U^{ni} \rangle_h
+ \frac{\tau^2}{2} \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle K^n_i, A_h K^n_j \rangle_h
= \tau \sum_{i=1}^{s} b_i \langle U^{ni}, A_h K^n_i \rangle_h + \frac{\tau^2}{2} \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle K^n_i, A_h K^n_j \rangle_h.
$$

The condition (2.11) together with the equality

$$
\langle U^{ni}, A_h K^n_i \rangle_h = \langle U^{ni}, F_h(U^{ni}) U^{ni} \rangle_h = 0
$$

implies $I^{n+1}_h = I^n_h$. This completes the proof. \qed
Theorem 2.4. As \( p = 2 \), Scheme 2.1 conserves the discrete mass, as follows:

\begin{equation}
M_{n+1}^n = M_n^n, \quad M_n^n = (U^n_1)_h, \quad n = 0, 1, 2, \ldots.
\end{equation}

Proof. It follows from (2.10) that

\begin{align*}
M_{n+1}^n - M_n^n &= \tau \sum_{i=1}^{s} b_i \langle K_i^n, 1 \rangle_h \\
&= \tau \sum_{i=1}^{s} b_i \langle A_h^{-1} F_h(U^{ni}) U^{ni}, 1 \rangle_h \\
&= \tau \sum_{i=1}^{s} b_i \langle F_h(U^{ni}), 1 \rangle_h \\
&= -\tau \sum_{i=1}^{s} b_i (\kappa D_1 U^{ni} + b D_3 U^{ni} + \frac{p\beta}{p+1} ((U^{ni})^{p-1} \cdot D_1 U^{ni} + D_1 ((U^{ni})^p)), 1 \rangle_h \\
&= -\frac{p\beta}{p+1} \langle D_1 U^{ni}, (U^{ni})^{p-1} \rangle_h.
\end{align*}

With noting \( p = 2 \) and the antisymmetry of \( D_1 \), we have

\[ \langle D_1 U^{ni}, (U^{ni})^{p-1} \rangle_h = 0, \]

which implies (2.16). This completes the proof. \( \square \)

Remark 2.3. Assume that the initial condition \( u_0(x) \) is sufficiently smooth, then it follows from (2.12) that the numerical solution of Scheme 2.1 satisfies

\[ \sqrt{\|U^n\|^2_h + \delta \langle -D_2 U^n, U^n \rangle_h + \alpha \langle D_4 U^n, U^n \rangle_h} = \sqrt{\|U^0\|^2_h + \delta \langle -D_2 U^0, U^0 \rangle_h + \alpha \langle D_4 U^0, U^0 \rangle_h} \leq C, \]

which implies that (noting \( \alpha > 0 \) and \( \delta > 0 \))

\[ \|U^n\|_h \leq C, \quad \langle -D_2 U^n, U^n \rangle_h \leq C, \quad \langle D_4 U^n, U^n \rangle_h \leq C; \]

is uniformly bounded. Thus, Scheme 2.1 is unconditionally stable.

Remark 2.4. If we take \( c_1, c_2, \ldots, c_s \) as the zeros of the \( s \)th shifted Legendre polynomial

\[ d^s \left( x^s (x - 1)^s \right), \]

the RK (or collocation) method based on these nodes has the order \( 2s \) and satisfies the condition (2.11) (see Refs. [21, 35, 36] and references therein). In particular, the RK coefficients for \( s = 2 \) and \( s = 3 \) (denoted by 4th-Gauss method and 6th-Gauss method, respectively) are given in Table 1 (e.g., see Ref. [21]). In addition, we introduce two notations, as follows:

- 4th-MPS: using the 4th-Gauss method to Scheme 2.1;
- 6th-MPS: using the 6th-Gauss method to Scheme 2.1.
Table. 1: Gauss methods of order 4 and 6.

2.2 High-order energy-preserving scheme

In this section, we propose a class of high-order energy-preserving schemes for the equation (1.1). Inspired by [16, 27, 41], we first shall introduce appropriate quadratic auxiliary variables to reformulate the Hamiltonian energy into a quadratic form. For clarity, we take \( p = 2, 3 \) and 5 as examples to expound this procedure, as follows:

- **Case I**: when \( p = 2 \), we set
  \[ q := q(x, t) = u^2. \]  
  Then, the Hamiltonian energy (1.3) is rewritten into
  \[ H(t) = \int_\Omega \left( \kappa u^2 - \frac{b}{2} u^2_x + \beta \frac{3}{2} uq \right) dx, \quad t \geq 0, \]  
  and according to the energy variational principle, we obtain the following reformulated system from (1.2)
  \[ \begin{cases}
    \partial_t u = \mathcal{F}(\kappa u + b u_{xx} + \beta \frac{3}{2} uq + 2\beta \frac{3}{2} u^2), \\
    \partial_t q = 2u \cdot \partial_t u,
  \end{cases} \]  
  with the consistent initial conditions
  \[ u(x, 0) = u_0(x), \quad q(x, 0) = (u_0(x))^2. \]

- **Case II**: for \( p = 3 \), the quadratic auxiliary variable is introduced as (2.17), and the quadratic energy is given by
  \[ H(t) = \int_\Omega \left( \kappa u^2 - \frac{b}{2} u^2_x + \frac{\beta}{3} q^2 + \frac{2\beta}{3} u^2 \right) dx, \quad t \geq 0, \]  
  which implies that the reformulated system is given by
  \[ \begin{cases}
    \partial_t u = \mathcal{F}(\kappa u + b u_{xx} + \beta q), \\
    \partial_t q = 2u \cdot \partial_t u,
  \end{cases} \]  
  with the consistent initial conditions (2.20).

- **Case III**: As \( p = 5 \), we start with introducing auxiliary variables
  \[ q_1 := q_1(x, t) = u^2, \quad q_2 := q_2(x, t) = uq, \]  
  which leads to the reformulated system given by
  \[ \begin{cases}
    \partial_t u = \mathcal{F}(\kappa u + b u_{xx} + \beta uq), \\
    \partial_t q_1 = 2u \cdot \partial_t u, \\
    \partial_t q_2 = \cdots,
  \end{cases} \]  
  with the consistent initial conditions (2.20).
and then rewrite the Hamiltonian energy (1.3) as
\[ H(t) = \int_{\Omega} \left( \frac{\kappa}{2} u^2 - \frac{b}{2} u_x^2 + \frac{\beta}{6} q^2 \right) dx, \quad t \geq 0. \] (2.24)

Similarly, we obtain reformulated system from (1.2), as follows:
\[
\begin{aligned}
\partial_t u &= J \left( \kappa u + bu_{xx} + \frac{\beta}{3} q_2 (q_1 + 2u^2) \right), \\
\partial_t q_1 &= 2u \cdot \partial_t u, \\
\partial_t q_2 &= \partial_t u \cdot q_1 + u \cdot \partial_t q_1 = \partial_t u \cdot q_1 + 2u^2 \cdot \partial_t u,
\end{aligned}
\] (2.25)

with the consistent initial conditions
\[ u(x, 0) = u_0(x), \quad q_1(x, 0) = (u_0(x))^2, \quad q_2(x, 0) = u_0(x, 0)q_1(x, 0). \] (2.26)

For simplicity, in the following construction of the energy-preserving schemes, we consider the parameter \( p = 2 \) for the equation (1.1). Note that the extensions to the parameter \( p > 2 \) are straightforward.

**Theorem 2.5.** Under the periodic boundary condition, the reformulated system (2.19) conserves the mass (1.4).

**Proof.** It follows from the first equation of (2.19), together with the periodic boundary condition that
\[
\frac{d}{dt} M(t) = (\partial_t u, 1)
= \left( J(\kappa u + bu_{xx} + \frac{\beta}{3} q + \frac{2\beta}{3} u^2), 1 \right)
= -\left( \partial_x(\kappa u + bu_{xx} + \frac{\beta}{3} q + \frac{2\beta}{3} u^2), (1 - \delta \partial_{xx} + \alpha \partial_{xxx})^{-1} 1 \right)
= -\left( \partial_x(\kappa u + bu_{xx} + \frac{\beta}{3} q + \frac{2\beta}{3} u^2), 1 \right)
= 0.
\]

This completes the proof. \( \square \)

**Theorem 2.6.** Under the periodic boundary condition, the reformulation (2.19) preserves the following invariants
\[ H_{1,1} = q - u^2 = 0, \] (2.27)
\[ H(t) = \int_{\Omega} \left( \frac{\kappa}{2} u^2 - \frac{b}{2} u_x^2 + \frac{\beta}{3} q u \right) dx, \quad t \geq 0. \] (2.28)

**Proof.** It follows from the second equation of (2.19) that
\[ \partial_t H_{1,1} = \partial_t q - 2u \cdot \partial_t u = 0. \] (2.29)

According to (2.17) and (2.19) together with the anti-adjoint property of \( J \), we have
\[
\frac{d}{dt} H(t) = \int_{\Omega} \left( \kappa u \partial_t u - b \partial_x u \partial_{xx} u + \frac{\beta}{3} (q \partial_t u + u \partial_t q_u) \right) dx
= \int_{\Omega} \left( \kappa u \partial_t u + b \partial_x u \partial_t u + \frac{\beta}{3} (q + 2u^2) \partial_t u \right) dx
= \int_{\Omega} \left( \kappa u + b \partial_x u + \frac{\beta}{3} (q + 2u^2) \right) J \left( \kappa u + bu_{xx} + \frac{\beta}{3} (q + 2u^2) \right) dx
= 0.
\]

This completes the proof. \( \square \)
Next, we apply an RK method to the system (2.19) in time, together with the Fourier pseudo-spectral method in space to give a class of fully discrete schemes for (2.19), as follows:

**Scheme 2.2.** Let $b_i, a_{ij} (i, j = 1, \ldots, s)$ be real numbers and let $c_i = \sum_{j=1}^{s} a_{ij}$. For the given $(U^n, Q^n)$, an $s$-stage Runge-Kutta method is given by

\[
\begin{align*}
K_1^n &= J_h \left( \kappa U^{ni} + b D_2 U^{ni} + \frac{\beta}{3} (Q^{ni} + 2(U^{ni})^2) \right), \\
U^{ni} &= U^n + \tau \sum_{j=1}^{s} a_{ij} K_j^n, \quad J_h = -(I - \delta D_2 + \alpha D_4)^{-1} D_1, \\
Q^{ni} &= Q^n + \tau \sum_{j=1}^{s} a_{ij} L_j^n, \quad L_i^n = 2U^{ni} \cdot K_i^n, \quad i = 1, 2, \ldots, s,
\end{align*}
\]  

(2.30)

and $(U^{n+1}, Q^{n+1})$ is then updated by

\[
U^{n+1} = U^n + \tau \sum_{i=1}^{s} b_i K_i^n, \quad Q^{n+1} = Q^n + \tau \sum_{i=1}^{s} b_i L_i^n.
\]  

(2.31)

**Theorem 2.7.** If the coefficients of (2.30) and (2.31) satisfy (2.11), Scheme 2.2 preserves the following discrete invariants

\[
H_{1,1}^{n+1} = H_{1,1}^n, \quad H_{1,1}^n = Q^n - (U^n)^2,
\]  

(2.32)

\[
\mathcal{E}_h^{n+1} = \mathcal{E}_h^n, \quad \mathcal{E}_h^n = \frac{\kappa}{2} \|U^n\|^2 + \frac{b}{2} (D_2 U^n, U^n)_h + \frac{\beta}{3} (U^n, Q^n)_h, \quad n = 0, 1, \ldots.
\]  

(2.33)

**Proof.** It follows from (2.31) that

\[
H_{1,1}^{n+1} - H_{1,1}^n = (Q^{n+1} - Q^n) - (U^{n+1})^2 + (U^n)^2
\]

\[
= \tau \sum_{j=1}^{s} b_i L_i^n - \tau \sum_{i=1}^{s} b_i K_i^n \cdot U^n - \tau \sum_{i=1}^{s} b_i U^n \cdot K_i^n - \tau^2 \sum_{i,j=1}^{s} b_i b_j K_i^n \cdot K_j^n.
\]

Using the equality $U^n = U^{ni} - \tau \sum_{j=1}^{s} a_{ij} K_j^n$ and $L_i^n = 2U^{ni} \cdot K_i^n$, together with (2.11), we can obtain from the above equation

\[
H_{1,1}^{n+1} - H_{1,1}^n = \tau \sum_{j=1}^{s} b_i L_i^n - 2\tau \sum_{i=1}^{s} b_i K_i^n \cdot U^{ni} - \tau^2 \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ij}) K_i^n \cdot K_j^n
\]

\[
= 0.
\]  

(2.34)

With noting the initial condition $Q^0 - (U^0)^2 = 0$, we obtain (2.32) from (2.34).

According to (2.30) and (2.31), we have

\[
\mathcal{E}_h^{n+1} - \mathcal{E}_h^n = \frac{\kappa}{2} (U^{n+1}, U^{n+1})_h + \frac{b}{2} (D_2 U^{n+1}, U^{n+1})_h + \frac{\beta}{3} (U^{n+1}, Q^{n+1})_h
\]

\[
- \frac{\kappa}{2} (U^n, U^n)_h - \frac{b}{2} (D_2 U^n, U^n)_h - \frac{\beta}{3} (U^n, Q^n)_h
\]

\[
= \tau \sum_{i=1}^{s} b_i \left( (U^n, K_i^n)_h + (K_i^n, U^n)_h \right) + \frac{\tau^2 \kappa}{2} \sum_{i,j=1}^{s} b_i b_j (K_i^n, K_j^n)_h
\]

\[
+ \frac{\tau b}{2} \sum_{i=1}^{s} b_i \left( (D_2 U^n, K_i^n)_h + (D_2 K_i^n, U^n)_h \right) + \frac{\tau^2 b}{2} \sum_{i,j=1}^{s} b_i b_j (D_2 K_i^n, K_j^n)_h
\]
Then, we can deduce from the first equation of (2.30)

Remark 2.6. It follows from the first equation of Proof. Theorem 2.8. For any RK method, Scheme 2.2 conserves the following discrete Hamiltonian energy

\[ H^n_h = \frac{1}{2} \|U^n\|^2 + \frac{b}{2} \langle D_2 U^n, U^n\rangle_h + \frac{\beta}{3} \langle (U^n)^3, 1\rangle_h, \ n = 0, 1, 2, \cdots. \] (2.35)

Theorem 2.8. For any RK method, Scheme 2.2 conserves the following discrete mass

\[ M^{n+1}_h = M^n_h, \ M^n_h = \langle U^n, 1\rangle_h, \ n = 0, 1, 2, \cdots. \] (2.36)

Proof. It follows from the first equation of (2.31) that

\[ M^{n+1}_h = M^n_h + \tau \sum_{i=1}^s b_i \langle K^n_i, 1\rangle_h. \] (2.37)

Then, we can deduce from the first equation of (2.30) that

\[ \langle K^n_i, 1\rangle_h = -\langle D_1 (\kappa U^{ni} + b D_2 U^{ni} + \frac{\beta}{3} (Q^{ni} + 2 (U^{ni})^2)), A^{-1}_h 1\rangle_h \]

\[ = 0. \]

This completes the proof.

Remark 2.6. Here, we introduce two notations, as follows:

- 4th-EPS: using the 4th-Gauss method to Scheme 2.2;
- 6th-EPS: using the 6th-Gauss method to Scheme 2.2.
3 An efficient implementation for the proposed schemes

As it turns out, there is no existing explicit RK methods that satisfy the condition (2.11) (see Proposition 7.1.1 in [13]). Thus, in this section, motivated by [11, 50], we propose an efficient fixed-point iteration solver for the nonlinear equations of the proposed schemes. For simplicity, we consider the 4th-MP scheme where the RK coefficient is given in Table 1. Note that the extensions to $s > 2$ are straightforward.

For a given $U^n$, according to Scheme 2.1, the 4th-MP scheme is equivalent to

\[
A_hK_1^n = -\left[\kappa D_1 U^{n+1} + b D_3 U^{n+1} + \frac{p\beta}{p+1}((U^{n+1})^{p-1} \cdot D_1 U^{n+1} + D_1((U^{n+1})^p))\right],
\]

\[
A_hK_2^n = -\left[\kappa D_1 U^{n+2} + b D_3 U^{n+2} + \frac{p\beta}{p+1}((U^{n+2})^{p-1} \cdot D_1 U^{n+2} + D_1((U^{n+2})^p))\right],
\]

\[
U^{n+1} = U^n + \tau a_{11} K_1^n + \tau a_{12} K_2^n,
\]

\[
U^{n+2} = U^n + \tau a_{21} K_1^n + \tau a_{22} K_2^n,
\]

It follows from (3.1)-(3.3) that

\[
A_hK_1^n = -\left[\kappa a_{11} D_1 K_1^n + \kappa a_{12} D_1 K_2^n + \tau a_{11} D_3 K_1^n + \tau a_{12} D_3 K_2^n + F_1\right],
\]

\[
A_hK_2^n = -\left[\kappa a_{21} D_1 K_1^n + \kappa a_{22} D_1 K_2^n + \tau a_{21} D_3 K_1^n + \tau a_{22} D_3 K_2^n + F_2\right],
\]

where

\[F_1 = \kappa D_1 U^{n+1} + b D_3 U^{n+1} + \frac{p\beta}{p+1}((U^{n+1})^{p-1} \cdot D_1 U^{n+1} + D_1((U^{n+1})^p)),\]

\[F_2 = \kappa D_1 U^{n+2} + b D_3 U^{n+2} + \frac{p\beta}{p+1}((U^{n+2})^{p-1} \cdot D_1 U^{n+2} + D_1((U^{n+2})^p)).\]

Using Remark 2.1, we can obtain from (3.5)-(3.6)

\[A_h\mathbb{K}_1^n = -\left[\kappa a_{11} \Lambda K_1^n + \kappa a_{12} \Lambda K_2^n + \tau a_{11} \Lambda^3 K_1^n + \tau a_{12} \Lambda^3 K_2^n + \mathbb{F}_1\right],\]

\[A_h\mathbb{K}_2^n = -\left[\kappa a_{21} \Lambda K_1^n + \kappa a_{22} \Lambda K_2^n + \tau a_{21} \Lambda^3 K_1^n + \tau a_{22} \Lambda^3 K_2^n + \mathbb{F}_2\right],\]

where $A_h = I - \delta \Lambda^2 + \alpha \Lambda^4$ and $\mathbb{W} = \mathcal{F}_N \mathbb{W}$.

For the nonlinear algebraic equations as above, we apply the following fixed-point iteration strategy, for $l = 0, 1, 2, \cdots, M$

\[
\begin{bmatrix}
A_h + \tau a_{11} \Lambda + \tau a_{12} \Lambda^3 & \tau a_{12} \Lambda + \tau a_{12} \Lambda^3 \\
\tau a_{21} \Lambda + \tau a_{21} \Lambda^3 & A_h + \tau a_{22} \Lambda + \tau a_{22} \Lambda^3
\end{bmatrix}
\begin{bmatrix}
\mathbb{K}_{1,l+1}^n \\
\mathbb{K}_{2,l+1}^n
\end{bmatrix}
= \begin{bmatrix}
-\mathbb{F}_1^n \\
-\mathbb{F}_2^n
\end{bmatrix},
\]

which implies that

\[
\begin{bmatrix}
(A_h)_{ij} + \tau a_{11} \Lambda_j + \tau a_{12} \Lambda^3_j & \tau a_{12} \Lambda_j + \tau a_{12} \Lambda^3_j \\
\tau a_{21} \Lambda_j + \tau a_{21} \Lambda^3_j & (A_h)_{ij} + \tau a_{22} \Lambda_j + \tau a_{22} \Lambda^3_j
\end{bmatrix}
\begin{bmatrix}
(\mathbb{K}_{1,l+1}^n)_{ij} \\
(\mathbb{K}_{2,l+1}^n)_{ij}
\end{bmatrix}
= \begin{bmatrix}
-(\mathbb{F}_{1}^n)_{ij} \\
-(\mathbb{F}_{2}^n)_{ij}
\end{bmatrix},
\]

where $j = 0, 1, \cdots, N - 1$.

Solving the above equations, we obtain $\mathbb{K}_{1,l+1}^n$ and $\mathbb{K}_{2,l+1}^n$. Then $K_{1,l+1}^n$ and $K_{2,l+1}^n$ are given by $K_{1,l+1}^n = \mathcal{F}_N \mathbb{K}_{1,l+1}^n$ and $K_{2,l+1}^n = \mathcal{F}_N \mathbb{K}_{2,l+1}^n$, respectively. In our computations, we take the iterative initial value $K_{1,0}^n = U^n$ and $K_{2,0}^n = U^n$. The iteration
terminates when the number of maximum iterative step $M = 30$ is reached or the infinity norm of the error between two adjacent iterative steps is less than $10^{-14}$, that is

$$\max_{i \leqslant 2} \{ \| K_{i}^{n,l+1} - K_{i}^{n,l} \|_{\infty,h} \} < 10^{-14}.$$ 

Finally, we have $U^{n+1} = U^n + \tau b_1 K_1^{n,M} + \tau b_2 K_2^{n,M}$.

**Remark 3.1.** Similarly, the efficient iteration solver for the resulting nonlinear equations of the 4th-EP scheme (see Scheme 2.2) is given by, as follows:

$$\begin{bmatrix} I - \tau \kappa a_{11} J_h - \tau b a_{11} J_h \Lambda^2 & - \tau \kappa a_{12} J_h - \tau b a_{12} J_h \Lambda^2 \\ - \tau \kappa a_{21} J_h - \tau b a_{21} J_h \Lambda^2 & I - \tau \kappa a_{22} J_h - \tau b a_{22} J_h \Lambda^2 \end{bmatrix} \begin{bmatrix} \mathbf{K}_1^{n,l+1} \\ \mathbf{K}_2^{n,l+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix},$$

where

$$F_1 = J_h \left( \kappa U^n + bD_2 U^n + \frac{\beta}{3} (Q^n + 2(U^{n+1})^2) \right),$$

$$F_2 = J_h \left( \kappa U^n + bD_2 U^n + \frac{\beta}{3} (Q^n + 2(U^n)^2) \right),$$

$$U^{n+1} = U^n + \tau a_{11} K_1^n + \tau a_{12} K_2^n, \quad J_h = -(I - \delta \Lambda^2 + \alpha \Lambda^4)^{-1} \tilde{\Lambda},$$

$$Q^{n+1} = (U^n)^2 + \tau \sum_{j=1}^{s} a_{ij} L_i^n, \quad L_i^n = 2U^{n+1} \cdot K_i^n, \quad i = 1, 2,$$

In particular, the iteration equations as stated above can be rewritten into the following subsystems

$$\begin{bmatrix} 1 - \tau \kappa a_{11} (J_h)_{j} - \tau b a_{11} (J_h)_{j} \Lambda^2 & - \tau \kappa a_{12} (J_h)_{j} - \tau b a_{12} (J_h)_{j} \Lambda^2 \\ - \tau \kappa a_{21} (J_h)_{j} - \tau b a_{21} (J_h)_{j} \Lambda^2 & 1 - \tau \kappa a_{22} (J_h)_{j} - \tau b a_{22} (J_h)_{j} \Lambda^2 \end{bmatrix} \begin{bmatrix} (\mathbf{K}_1^{n,l+1})_{j} \\ (\mathbf{K}_2^{n,l+1})_{j} \end{bmatrix} = \begin{bmatrix} (\mathbf{F}_1)_{j} \\ (\mathbf{F}_2)_{j} \end{bmatrix},$$

where $j = 0, 1, 2, \cdots, N - 1$. Then, by the similar procedure as the 4th-MP scheme, we have $U^{n+1}$.

### 4 Numerical results

In this section, we devote to the numerical performances in the accuracy and invariants-preservation of the proposed schemes for the equation (1.1). For brevity, in the rest of this paper, 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS are only used for demonstration purposes. In order to quantify the numerical solution, we introduce the $l^2$-and $l^\infty$-norm error functions, respectively,

$$e_\infty(t^n) = \| u(\cdot, t^n) - u^n \|_{h, \infty}, \quad e_2(t^n) = \| u(\cdot, t^n) - u^n \|_{h}.$$ 

Furthermore, we also investigate the residuals on the mass, momentum and Hamiltonian energy, defined respectively, as

$$Error^m_n = |M^n_h - M^n_h|, \quad Error^p_n = |P^n_h - P^n_h|, \quad Error^H_n = |H^n_h - H^n_h|.$$ 

All simulations are performed on a Win10 machine with Intel Core i7 and 32GB using MATLAB R2015b.
4.1 Rosenau-RLW equation

As the parameters $\kappa = \delta = \alpha = \beta = 1$ and $b = 0$ are chosen, the equation (1.1) reduces to the generalized Rosenau-RLW equation, which has an exact solution [4, 31]

$$u(x, t) = \text{A} \text{sech}^\frac{4}{p-1} \left( B(x - Ct - x_0) \right),$$  \hspace{1cm} (4.1)

where

$$A = \exp \left( \frac{\ln \frac{(p+3)(3p+1)(p+1)}{4(p^2+3)(p^2+4p+7)}}{p-1} \right), \quad B = \frac{p - 1}{\sqrt{4p^2 + 8p + 20}}, \quad C = \frac{p^4 + 4p^3 + 14p^2 + 20p + 25}{p^2 + 4p^3 + 10p^2 + 12p + 21},$$

and $x_0$ represents the initial phase of the solition and the periodic boundary condition is considered.

We first verify the convergence order in time for the proposed schemes. Let us set the computational domain $\Omega = [-200, 200]$ and we take the initial value

$$u(x, 0) = \text{A} \text{sech}^\frac{4}{p-1} \left( B(x - x_0) \right), \ x \in \Omega,$$

where the initial phase $x_0 = 0$.

The temporal convergence tests are investigated by fixing the Fourier node 2048. We compute the numerical solutions at $t = 10$ using 4th-MPS and 4th-EPS with various time step sizes $\tau = 2^{-k}$, $k = 2, 3, 4, 5, 6$ as well as 6th-MPS and 6th-EPS with various time step sizes $\tau = 2^{-k}$, $k = 0, 1, 2, 3, 4$, respectively. The relation between the $l^2$ and $l^\infty$-norm errors and the time step size is summarized in Figure 1, where the upper picture corresponds to the parameter $p = 2$, the middle one corresponds to the parameter $p = 3$ and the down one is for the parameter $p = 5$. The fourth-order temporal accuracy for 4th-MPS and 4th-EPS and the sixth-order for 6th-MPS and 6th-EPS are observed, respectively, for different parameters $p$ as desired.

Then, we choose the initial value, as follows:

$$u(x, 0) = \exp(-0.05(x - 40)^2), \ x \in \Omega,$$

where the computational domain $\Omega = [-50, 250]$ with the periodic boundary condition. We take the uniform spatial size $h = 1$ and time step size $\tau = 0.1$, respectively, and compute the profile of $u$ at $T = 100$ by choosing different parameters $p$ as well as the residuals on the discrete mass, momentum and Hamiltonian energy by using 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS, respectively. Figure 2 shows the profile of $u$ provided by 4th-MPS at $t = 100$, where the left picture corresponds to the parameter $p = 2$, the middle one corresponds to the parameter $p = 3$ and the right one is for the parameter $p = 5$. The influence of the parameter $p$ on the dispersion wave propagation is observed as shown in [12]. We should note that the profiles of $u$ at $t = 100$ computed by using other schemes are similar to Figure 2, thus for brevity, we omit them. Figure 3 shows the evolutions of the residuals on the discrete mass, momentum and Hamiltonian energy of numerical solutions computed by 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS, respectively. We can draw the following observations: (1) for the parameter $p = 2$, all of proposed schemes preserve the discrete mass exactly, while the proposed momentum-preserving schemes cannot conserve the discrete mass for the parameters $p = 3$ and $p = 5$; (2) 4th-MPS and 6th-MPS can preserve discrete momentum up to the machine precision; (3) 4th-EPS and 6th-EPS conserve the discrete both mass and Hamiltonian energy for all parameters $p$. 

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Fig. 1: The $l^2$ and $l^\infty$-norm errors vs. the time step sizes provided by the proposed 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS for the Rosenau-RLW equation with different parameters $p$, respectively.

Fig. 2: The profile of $u$ provided by 4th-MPS with different parameters $p$ at $t = 100$, where the uniform spatial and time step size is chosen as $\tau = h = 0.1$ for the Rosenau-RLW equation.
Fig. 3: The residuals on the discrete mass, momentum and Hamiltonian energy from $t = 0$ to $t = 1000$ provided by the proposed 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS for the Rosenau-RLW equation with different parameters $p$, respectively.

4.2 Rosenau-KdV equation

As the parameters $\kappa = b = \alpha = 1$ and $\delta = 0$ are chosen, the equation (1.1) reduces to the generalized Rosenau-KdV equation\[51\]. For simplicity, we also consider the parameters $p = 2$, $p = 3$ and $p = 5$, respectively, where the exact solution can be given by

- **Case I**: As $\beta = \frac{1}{2}$ and $p = 2$, the Rosenau-KdV equation has an exact solution\[23\]

$$u(x,t) = k_{11}\text{sech}^4(k_{12}(x - k_{13}t)).$$

where $k_{11} = \frac{-35}{21} + \frac{35}{312}\sqrt{313}$, $k_{12} = \frac{1}{21}\sqrt{-26 + 2\sqrt{313}}$, $k_{13} = \frac{1}{2} + \frac{\sqrt{313}}{20}$.

- **Case II**: If $\beta = 1$ and $p = 3$, the Rosenau-KdV equation admits an exact solution\[33\]

$$u(x,t) = k_{21}\text{sech}^2(k_{22}(x - k_{23}t)),$$

where $k_{21} = \frac{1}{4}\sqrt{-15 + 3\sqrt{41}}$, $k_{22} = \frac{1}{4}\sqrt{-5 + \sqrt{41}}$, $k_{23} = \frac{1}{10}(5 + \sqrt{41})$.

- **Case III**: When $\beta = 1$ and $p = 5$, the exact solution of the Rosenau-KdV equation is given by\[51\]

$$u(x,t) = k_{31}\text{sech}^2(k_{32}(x - k_{33}t)),$$

where $k_{31} = \frac{4}{15}(-5 + \sqrt{34})$, $k_{32} = \frac{1}{3}\sqrt{-5 + \sqrt{34}}$, $k_{33} = \frac{1}{10}(5 + \sqrt{34})$. 

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First of all, we verify the convergence order in time for the selected four structure-preserving schemes. Let us set the computational domain $\Omega = [-200, 200]$ and we take the initial value as the exact solution at $t = 0$ for the parameters $p = 2$, $p = 3$ and $p = 5$, respectively. The temporal convergence tests are conducted by fixing the Fourier node 2048. Figure 4 shows the $l^2$ and $l^\infty$-norm errors with various time step sizes at $t = 10$, where we take time step sizes $\tau = 2^{-k}$, $k = 2, 3, 4, 5, 6$ for 4th-MPS and 4th-EPS, while for 6th-MPS and 6th-EPS, we choose time step sizes $\tau = 2^{-k}$, $k = 0, 1, 2, 3, 4$. It is clear to see that 4th-MPS and 4th-EPS are fourth-order temporal accuracy, and 6th-MPS and 6th-EPS can achieve sixth-order accuracy in time.

Then, we set the computational domain $\Omega = [-100, 100]$ and take the uniform spatial step size $h = \frac{200}{512}$ and time step size $\tau = 0.1$, respectively. Figure 5 shows the residuals on the discrete mass, momentum and Hamiltonian energy computed by using 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS, respectively, which is similar to Figure 3. We should note that the residuals on the discrete mass provided by 4th-MPS and 6th-MPS is also up to the machine precision because of the fine spatial mesh.

Fig. 4: The $l^2$ and $l^\infty$-norm errors vs. the time step size provided by the proposed 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS for the Rosenau-KdV equation with different parameters $p$, respectively.
Fig. 5: The residuals on the mass, momentum and Hamiltonian energy from \( t = 0 \) to \( t = 10000 \) provided by the proposed 4th-MPS, 4th-EPS, 6th-MPS and 6th-EPS for the Rosenau-RdV equation with different parameters \( p \), respectively.

4.3 Some comparisons

In the following numerical experiments, we compare the proposed structure-preserving schemes with the linearized Crank-Nicolson momentum-preserving scheme (LCN-MPS) \cite{25} and fourth-order energy-preserving schemes (i.e., YC-EPS and SC-EPS) \cite{3} by focusing on the numerical errors in time and the computational efficiency, respectively.

Let us still consider the Rosenau-KdW equation as above. For the sake of simplicity, we only consider the parameter \( p = 2 \) (see Case I in Section 4.2). We set the computational domain \( \Omega = [-100, 100] \) and the uniform spatial mesh \( h = \frac{200}{1024} \). Table 2 shows the numerical errors and convergence order in time for the different schemes with various time steps at \( t = 1 \). It is clear to observe that (i) LCN-MPS is second-order accurate in time and its numerical errors are largest; (ii) 4th-MPS, 4th-EPS, YC-EPS and SC-EPS are all fourth-order accurate in time, and the numerical error provided by SC-EPS is smallest, while the ones provided by YC-EPS are much larger than other fourth-order schemes.

Finally, we set the computational domain \( \Omega = [-300, 300] \) and the Fourier node \( 2^{13} \), and we then investigate the global \( L^2 \)-and \( L^\infty \)- errors of \( u \) versus the CPU time using the five selected structure-preserving schemes with various time steps at \( t = 30 \). The results are summarized in Figure 6. For a given global error, we observe that (i) the cost of the LCN-MPS is the most expensive because of the low-order accuracy in time; (ii) the cost of 4th-EPS is the cheapest; (iii) the cost of SC-EPS is much cheaper than the one provided by the 4th-MPS, and the cost of 4th-MPS is much cheaper than the one provided by the YC-EPS. We should note that as the parameter \( p \) is enlarged, more QAV variables shall be introduced, thus the computational cost of the high-order energy-preserving schemes will increase.
Table. 2: Numerical errors and convergence order for the different schemes with various time steps at $t = 1$.

| Scheme   | $\tau$ | $e_2(t_n = 1)$ | order | $e_\infty(t_n = 1)$ | order |
|----------|--------|----------------|-------|---------------------|-------|
| 4th-EPS  | $\frac{1}{10}$ | 1.585e-09 | -    | 6.292e-10 | -     |
|          | $\frac{1}{20}$ | 9.905e-11 | 4.000 | 3.933e-11 | 4.000 |
|          | $\frac{1}{40}$ | 6.191e-12 | 4.000 | 2.458e-12 | 4.000 |
|          | $\frac{1}{80}$ | 3.867e-13 | 4.001 | 1.540e-13 | 3.997 |
|          | $\frac{1}{160}$ | 1.586e-09 | -    | 5.984e-10 | -     |
| 4th-MPS  | $\frac{1}{10}$ | 9.490e-11 | 4.000 | 3.740e-11 | 4.000 |
|          | $\frac{1}{20}$ | 5.932e-12 | 4.000 | 2.338e-12 | 4.000 |
|          | $\frac{1}{40}$ | 3.706e-13 | 4.000 | 1.462e-13 | 3.999 |
|          | $\frac{1}{80}$ | 7.299e-05 | -    | 2.800e-05 | -     |
| LCN-MPS  | $\frac{1}{10}$ | 1.813e-05 | 2.009 | 6.950e-06 | 2.010 |
|          | $\frac{1}{20}$ | 4.518e-06 | 2.005 | 1.731e-06 | 2.006 |
|          | $\frac{1}{40}$ | 1.128e-06 | 2.002 | 4.319e-07 | 2.003 |
|          | $\frac{1}{80}$ | 8.024e-08 | -     | 3.400e-08 | -     |
| YC-EPS   | $\frac{1}{10}$ | 5.024e-09 | 3.997 | 2.129e-09 | 3.997 |
|          | $\frac{1}{20}$ | 3.142e-10 | 3.999 | 1.331e-10 | 3.999 |
|          | $\frac{1}{40}$ | 1.964e-11 | 4.000 | 8.326e-12 | 4.000 |
|          | $\frac{1}{80}$ | 1.081e-09 | -    | 4.355e-10 | -     |
| SC-EPS   | $\frac{1}{10}$ | 6.761e-11 | 4.000 | 2.723e-11 | 3.999 |
|          | $\frac{1}{20}$ | 4.229e-12 | 3.999 | 1.708e-12 | 3.995 |
|          | $\frac{1}{40}$ | 2.896e-13 | 3.868 | 1.186e-13 | 3.848 |

Fig. 6: The $l^2$ and $l^\infty$-norm errors vs. the CPU time provided by LCN-MPS, 4th-MPS, 4th-EPS, YC-EPS and SC-EPS for the Rosenau-KdV equation.

5 Conclusions

In this paper, we propose two classes of high-order structure-preserving schemes for the generalized Rosenau-type equation (1.1). One of the schemes can conserve the discrete momentum conservation law, which is based on the use of the symplectic RK method in time and the standard Fourier pseudo-spectral method in space, respectively. Another one conserves the discrete Hamiltonian energy and mass, where the main idea is based on the combination of the QAV approach with the the symplectic RK method in time, together with the standard Fourier pseudo-spectral method in space. Extensive
numerical tests and comparisons are also addressed to verify the performance of the proposed schemes.

We conclude this paper with two main remarks. First, we note that the construction of the energy-preserving schemes should be discussed case by case, and the proposed momentum-preserving schemes are not mass-conserving as the parameter $p > 2$. Thus in practical computations, such trade-offs between two classes of schemes shall be treated carefully. Second, as far as we know, there are some works on optimal error estimates of EQ schemes [9, 29, 38, 44] and Fourier pseudo-spectral methods [7, 17, 19, 49], but to our best knowledge, the error estimate of high-order structure-preserving Fourier pseudo-spectral schemes is still not available. Therefore, how to establish optimal error estimates for the proposed schemes will be an interesting topic for future studies. In fact, the uniformly bounded of numerical solutions in $l^\infty$-norm can be obtained by using the discrete momentum (2.12) and Sobolev imbedding theorems [20], thus our first attempt will focus on the high-order momentum-preserving schemes, followed by the energy-preserving schemes.

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References

[1] M. Ahmat and J. Qiu. SSP IMEX Runge-Kutta WENO scheme for generalized Rosenau-KdV-RLW equation. *J. Math. Study.*, 55:1–21, 2022.

[2] N. Atouani and K. Omrani. A new conservative high-order accurate difference scheme for the Rosenau equation. *Appl. Anal.*, 94:2435–2455, 2015.

[3] J. Cai, H. Liang, and C. Zhang. Efficient high-order structure-preserving methods for the generalized Rosenau-type equation with power law nonlinearity. *Commun. Nonlinear Sci. Numer. Simulat.*, 59:122–131, 2018.

[4] W. Cai, Y. Sun, and Y. Wang. Variational discretizations for the generalized Rosenau-type equations. *Appl. Math. Comput.*, 271:860–873, 2015.

[5] J. Chen and M. Qin. Multi-symplectic Fourier pseudospectral method for the nonlinear Schrödinger equation. *Electr. Trans. Numer. Anal.*, 12:193–204, 2001.

[6] Y. Chen, Y. Gong, Q. Hong, and C. Wang. A novel class of energy-preserving Runge-Kutta methods for the Korteweg-de Vries equation. *Numer. Math. Theor. Meth. Appl.*, 15:768–792, 2022.

[7] K. Cheng, W. Feng, S. Gottlieb, and C. Wang. A Fourier pseudospectral method for the “good” Boussinesq equation with second-order temporal accuracy. *Numer. Methods Partial Differ. Equ.*, 31:202–224, 2015.

[8] K. Cheng and C. Wang. Long time stability of high order multi-step numerical schemes for two-dimensional incompressible Navier-Stokes equations. *SIAM J. Numer. Anal.*, 54:3123–3144, 2016.
[9] Q. Cheng and C. Wang. Error estimate of a second order accurate scalar auxiliary variable (SAV) scheme for the thin film epitaxial equation. *Adv. Appl. Math. Mech.*, 13:1318–1354, 2021.

[10] S. K. Chung. Finite difference approximate solutions for the Rosenau equation. *Appl. Anal.*, 69:149–156, 1998.

[11] J. Cui, Y. Wang, and C. Jiang. Arbitrarily high-order structure-preserving schemes for the Gross-Pitaevskii equation with angular momentum rotation. *Comput. Phys. Commun.*, 261:107767, 2021.

[12] Y. I. Dimitrienko, S. Li, and Y. Niu. Study on the dynamics of a nonlinear dispersion model in both 1D and 2D based on the fourth-order compact conservative difference scheme. *Math. Comput. Simulat.*, 182:661–689, 2021.

[13] K. Feng and M. Qin. *Symplectic Geometric Algorithms for Hamiltonian Systems*. Springer and Zhejiang Science and Technology Publishing House, Heidelberg, Hangzhou, 2010.

[14] D. Furihata and T. Matsuo. *Discrete Variational Derivative Method: A Structure-Preserving Numerical Method for Partial Differential Equations*. Chapman & Hall/CRC, Boca Raton, 2011.

[15] Y. Gong, J. Cai, and Y. Wang. Multi-symplectic Fourier pseudospectral method for the Kawahara equation. *Commun. Comput. Phys.*, 16:35–55, 2014.

[16] Y. Gong, Q. Hong, C. Wang, and Y. Wang. Quadratic auxiliary variable Runge-Kutta methods for the Camassa-Holm equation. *Adv. Appl. Math. Mech.*, 2023, 10.4208/aamm.OA-2022-0188.

[17] Y. Gong, Q. Wang, Y. Wang, and J. Cai. A conservative Fourier pseudo-spectral method for the nonlinear Schrödinger equation. *J. Comput. Phys.*, 328:354–370, 2017.

[18] Y. Gong, J. Zhao, and Q. Wang. Arbitrarily high-order linear energy stable schemes for gradient flow models. *J. Comput. Phys.*, 419:109610, 2020.

[19] S. Gottlieb, F. Tone, C. Wang, X. Wang, and D. Wirossetisno. Long time stability of a classical efficient scheme for two-dimensional Navier-Stokes equations. *SIAM J. Numer. Anal.*, 50:126–150, 2012.

[20] S. Gottlieb and C. Wang. Stability and convergence analysis of fully discrete Fourier collocation spectral method for 3-D viscous Burgers’ equation. *J. Sci. Comput.*, 53:102–128, 2012.

[21] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*. Springer-Verlag, Berlin, 2nd edition, 2006.

[22] D. He. Exact solitary solution and a three-level linearly implicit conservative finite difference method for the generalized Rosenau-Kawahara-RLW equation with generalized Novikov type perturbation. *Nonlinear Dyn.*, 85:479–498, 2016.

[23] J. Hu, Y. Xu, and B. Hu. Conservative linear difference scheme for Rosenau-KdV equation. *Adv. Math. Phys.*, pages 1–7, 2013.
[24] C. Jiang, W. Cai, Y. Wang, and H. Li. A novel sixth order energy-conserved method for three-dimensional time-domain Maxwell’s equations. *arXiv:1705.08125*, 2017.

[25] C. Jiang, J. Cui, W. Cai, and Y. Wang. A novel linearized and momentum-preserving fourier pseudo-spectral scheme for the Rosenau-Korteweg de Vries equation. *Numer Methods Partial Differential Eq.*, 39:1558–1582, 2023.

[26] C. Jiang, J. Cui, X. Qian, and S. Song. High-order linearly implicit structure-preserving exponential integrators for the nonlinear Schrödinger equation. *J. Sci. Comput.*, 90, 2022, doi.org/10.1007/s10915-021-01739-x.

[27] C. Jiang, Y. Wang, and Y. Gong. Arbitrarily high-order energy-preserving schemes for the Camassa-Holm equation. *Appl. Numer. Math.*, 151:85–97, 2020.

[28] S. Li. Numerical analysis for fourth-order compact conservative difference scheme to solve the 3D Rosenau-RLW equation. *Comput. Math. Appl.*, 72:2388–2407, 2016.

[29] X. Li, J. Shen, and H. Rui. Energy stability and convergence of SAV block-centered finite difference method for gradient flows. *Math. Comp.*, 88:2047–2068, 2019.

[30] K. Omrani, F. Abidi, T. Achouri, and N. Khiari. A new conservative finite difference scheme for the Rosenau equation. *Appl. Math. Comput.*, 201:35–43, 2008.

[31] X. Pan and L. Zhang. On the convergence of a conservative numerical scheme for the usual Rosenau-RLW equation. *Appl. Mathe. Model.*, 36:3371–3378, 2012.

[32] G. R. W. Quispel and D. I. McLaren. A new class of energy-preserving numerical integration methods. *J. Phys. A: Math. Theor.*, 41:045206, 2008.

[33] P. Razborova, L. Moraru, and A. Biswas. Perturbation of dispersive shallow water waves with Rosenau-KdV-RLW equation and power law nonlinearity. *Rom. J. Phys.*, 59:658–676, 2014.

[34] P. Rosenau. Dynamics of dense discrete systems: high order effects. *Progr. Theor. Phys.*, 79:1028–1042, 1988.

[35] J. M. Sanz-Serna. Runge-Kutta schemes for Hamiltonian systems. *BIT*, 28:877–883, 1988.

[36] J. M. Sanz-Serna and M. Calvo. *Numerical Hamiltonian Problems*. Chapman & Hall, London, 1994.

[37] J. Shen and T. Tang. *Spectral and High-Order Methods with Applications*. Science Press, Beijing, 2006.

[38] J. Shen and J. Xu. Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows. *SIAM J. Numer. Anal.*, 56:2895–2912, 2018.

[39] J. Shen, J. Xu, and J. Yang. The scalar auxiliary variable (SAV) approach for gradient. *J. Comput. Phys.*, 353:407–416, 2018.

[40] J. Shen, J. Xu, and J. Yang. A new class of efficient and robust energy stable schemes for gradient flows. *SIAM Rev.*, 61:474–506, 2019.

[41] B. K. Tapley. Geometric integration of ODEs using multiple quadratic auxiliary variables. *SIAM J. Sci. Comput.*, 44:A2651–A2668, 2022.
[42] H. Wang, S. Li, and J. Wang. A conservative weighted finite difference scheme for the generalized Rosenau-RLW equation. *Comput. Math. Appl.*, 36:63–78, 2017.

[43] J. Wang and Q. Zeng. A fourth-order compact and conservative difference scheme for the generalized Rosenau-Korteweg de Vries equation in two dimensions. *J. Comput. Math.*, 37:541–555, 2019.

[44] M. Wang, Q. Huang, and C. Wang. A second order accurate scalar auxiliary variable (SAV) numerical method for the square phase field crystal equation. *J. Sci. Comput.*, 8, 2021, https://doi.org/10.1007/s10915-021-01487-y.

[45] X. Wang and W. Dai. A new implicit energy conservative difference scheme with fourth-order accuracy for the generalized Rosenau-Kawahara-RLW equation. *Comput. Appl. Math.*, 37:6560–6581, 2018.

[46] X. Wang, W. Dai, and Y. Yan. Numerical analysis of a new conservative scheme for the 2D generalized Rosenau-RLW equation. *Appl. Anal.*, 100:2564–2580, 2021.

[47] B. Wongsaijai, P. Charoensawan, T. Chaobankoh, and K. Poochinapan. Advance in compact structure-preserving manner to the Rosenau-Kawahara model of shallow-water wave. *Math. Meth. Appl. Sci.*, 44:7048–7064, 2021.

[48] X. Yang, J. Zhao, and Q. Wang. Numerical approximations for the molecular beam epitaxial growth model based on the invariant energy quadratization method. *J. Comput. Phys.*, 333:104–127, 2017.

[49] C. Zhang, H. Wang, J. Huang, C. Wang, and X. Yue. A second order operator splitting numerical scheme for the “good” Boussinesq equation. *Appl. Numer. Math.*, 119:179–193, 2017.

[50] G. Zhang and C. Jiang. Arbitrary high-order structure-preserving methods for the quantum Zakharov system. *arXiv:2202.13052*, 2022.

[51] M. Zheng and J. Zhou. An average linear difference scheme for the generalized Rosenau-KdV equation. *J. Appl. Math.*, pages 1–9, 2014.