I. INTRODUCTION

Network theory has made a significant contribution to understanding the complexity of different systems, such as communication networks [1], transportation systems [2] and even neuronal networks [3, 4]. The resilience of such networks is often studied under a percolation process [5-8] where a fraction $\frac{1}{p}$ of nodes is removed from the network and the size of the largest connected component, $P_\infty$ is evaluated. It has been found that these networks experience a second-order phase transition at a critical point $p_c$ [9]. Below this transition point, the size of the giant component is zero and non-zero above it. The giant component is used to describe the network functionality where nodes that are connected to it are considered functional, while isolated clusters are considered non-functional.

Many real-world networks are spatially embedded [10-14], and it has been shown that the spatiality of a network affects the phase transition of the percolation process [15]. Examples of such spatial networks are infrastructure networks [16], brain network [10] and transportation networks [17-19]. Percolation processes have been studied thoroughly for complex networks [9, 20-22], but the robustness have not been studied for spatial networks in the presence of reinforced nodes.

Recently, the question of centralization vs. decentralization of infrastructures has been emerged, which led to the idea of reinforced nodes. These reinforced nodes can function and support other nodes in their cluster even if they are not connected to the giant component [23, 24] and can be significant for the robustness of real-life networks. For example, in the case of the internet, satellites [24] can also be used in order to exchange information, meaning they are functional without being directly connected to the giant component. Another example are power-grids, where reinforced nodes represent generators, each having their own source of energy and being able to support themselves and their clusters.

In this paper, we will study the effects of reinforced nodes on spatial networks and how to optimally distribute them.

II. MODEL

Our model consists of a 2D lattice of size $N = L \times L$ initially without links. Next, we generate the length of the links, $r$, between the nodes which follow an exponential distribution [14, 15, 26-28].

\[ P(r) \sim e^{-r/\zeta}. \]  

(1)

The process of creating a spatial network requires 3 successive steps. The first step is to randomly choose a node. The second step is to randomly draw a link length from the distribution in Eq. (1). The third step is to randomly choose an angle and identify the closest target node for the link and create an edge. For our specific model, we chose the average degree (number of links) of the nodes to be $\langle k \rangle = 4$. In order to achieve a given $\langle k \rangle$ we repeat these steps $N \cdot \langle k \rangle / 2$ times.

FIG. 1: Illustration. The nodes are placed as the sites of a 2D lattice, while the links are added according to Eq. (1). The characteristic length of the links is $\zeta$ (left link in both boxes) while a small fraction $\rho$ of the nodes are randomly reinforced (red nodes). (left box) The network at a certain point of the percolation process, contains the giant component and finite clusters. (right box) $P_\infty$ represents the functioning nodes in the network, i.e., it includes all nodes that are part of the giant component or connected to a finite cluster with at least one reinforced node, total 13 nodes.

It is important to note, that for values of $\zeta \rightarrow 0$ our model generates a 2D lattice, where each node is connected only to its nearest neighbors, and has a known
value of $p_c \simeq 0.59$ [13]. While for values of $\zeta \to \infty$ our model generates an $ER$ network, where all links have a pre-determined probability of being cast, and has a known value of $p_c = 1/(k) = 1/4 = 0.25$ [13].

After generating the spatial network, we chose a fraction $\rho$ of the nodes of the network to be reinforced nodes (see Fig. 1). The reinforced nodes are chosen randomly, with the only condition that the nodes are part of the original giant component of the network, i.e., at $p = 1$. We now use the notation $P_{\infty}$ as the fraction of functioning nodes in the network and we analyze $P_{\infty}$ as a function of $p$, where $1 - p$ is the fraction of non-removed nodes from the network.

III. RESULTS

The resilience of spatial networks can be studied using percolation process, as shown in Fig. 2 for the model above, for different values of $\zeta$ and different values of $\rho$. We included only 2 values of $\zeta$ in the figure, other values of $\zeta$ show similar results and can be seen in the appendix. As we can see, the existence of reinforced nodes makes the spatial network more resilient to random failures, and the phase transition is removed. It can also be seen that the higher the fraction of reinforced nodes, the more resilient is the network. And lastly, the shorter is $\zeta$, i.e., the stronger is spatiality, the higher is the impact of the reinforced nodes on the network.

Once we establish that reinforced nodes increase significantly the resilience of spatial networks, we can address the question of identifying better strategies, i.e., we can ask what is the best strategy to distribute the reinforced nodes to maximize the network resilience? To do so, we repeated the percolation process for different values of $\zeta$ and different values $\rho$ testing 6 different distribution strategies:

- Random distribution, as shown in Fig. 2.
- Nodes with the highest degree $k$.
- Nodes with the lowest degree $k$.
- Nodes with the longest average link length.
- Nodes with the shortest average link length.
- Node with highest weighted degree, defined as:

$$w_i = \sum_{j=1}^{N} A_{ij} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

where $A_{ij}$ is the adjacency matrix. The weighted degree strategy is a combination of the four previous strategies since it takes into consideration both the average length of the links and the degree $k$ of the node.

![FIG. 2: Randomly distributed reinforced nodes in spatial networks. (a) The giant component, $P_{\infty}$, as a function of $p$ for $\zeta = 0.2$ (similar to 2D lattice). The phase transition at $p_c \simeq 0.59$ is being removed by even a small fraction of reinforced nodes. The network becomes more resilient as the fraction of reinforced nodes increases. (b) The giant component, $P_{\infty}$, as a function of $p$ for $\zeta = 50$. As $\zeta$ increases, we can see similar results but a significantly weaker effect compared to (a). Note, due to large $\zeta$, the phase transition is close to the $ER$ limit of $p_c = 0.25$ (here $\langle k \rangle = 4$).](image)

![FIG. 3: Prioritize strategies for reinforced nodes distribution. (a) We tested $P_{\infty}$ as a function of $p$ for 6 different strategies of reinforced nodes distribution: random distribution, high and low degree node preference, longest and shortest average length preference, and weighted degree preference. Here $\zeta = 2$. (b) Zoom in of the lower values $p$ (below $p_c$). We find that for all values of $\zeta$ (like here for $\zeta = 2$) and for low values of $p$ (close to the percolation threshold), the best nodes to reinforce are the nodes with the highest degree, and the worst nodes to reinforce are the nodes with the lowest degree. (c) Zoom in for the high values $p$ (above $p_c$). For high values of $p$ the statement in (b) is reversed, but the difference becomes very small and almost does not impact the network robustness for the same $\zeta$ value compared to the case of no reinforced nodes. Here $\rho = 0.1$. (d) Shows the degree distribution of the nodes having degree $k$ in the giant component. Here $\zeta = 2$ and $p = 0.1$. As can be seen, the distribution shifts to the right for low values of $p$, i.e., towards larger degree values. Thus, at early stages of percolation, it is more useful to reinforce low degree nodes since they fail more frequently, while at small values of $p$ it is better to reinforce the high degree nodes. We can see that our simulation results (circles) and the analytic MF solution (the continuous lines), are in good agreement, indicating that although the analytical solution is based on MF theory, it catches also quite well the behaviour of spatial networks.](image)
As we can see in Fig. 3(a) - (c), the functionality of the network changes depending on the distribution strategy of the reinforced nodes. For high values of $\rho$, the best distribution strategy of reinforced nodes is low degree nodes. For low values of $\rho$, the best distribution strategy is to reinforce high degree nodes. These results are shown for $\zeta = 2$, but valid also for other values of $\zeta$ (see appendix). In order to understand why the percolation process has different best distribution strategies for reinforced nodes at different stages, we study the distribution of the degree $k$ in the giant component for different stages in the percolation (different values of $p$). We find, as shown in Fig. 3(d), that the distribution is shifting for lower values of $p$ towards large degrees, which means that at the earlier stages of the percolation, low degree nodes have a much higher probability to be disconnected from the giant component. Thus, at early stages of percolation, making low degree nodes that have high probability of failing as reinforced nodes improve the robustness, while at later stages, high degree nodes are preferred to be chosen as reinforced nodes.

**IV. TEST ON A REAL NETWORK: THE EU POWER GRID**

To further validate our results, we studied the impact of reinforced nodes on the EU power grid network. The edge distribution of this network was found to follow Eq. (4) [15]. The number of nodes in the network is $N = 1254$, while the average degree is $\langle k \rangle = 1.44$ [15]. The size here is much smaller compared to our simulations before, thus, we expect the results to be noisy and, due to the lower degree, higher value of $p_c$. In Fig. 4(a) we demonstrate the impact of randomly distributed reinforced nodes in a real world network, the EU power grid, for different values of $\rho$. As one can see, we get similar results as those found in our model. As shown in Fig. 2, the presence of a small fraction of reinforced nodes makes the network significantly more resilient. In Fig. 4(b) we show the percolation process of the power grid of Europe network with 2 strategies according to Fig. 3, low degree distribution and high degree distribution. We can clearly see similar results to our model. For early stages of the percolation process, it is better to reinforce the nodes with the lower degree, while for later stages of the percolation, it is better to reinforce the nodes with the higher degree. In Fig. 4(c) - (d) we can see the power grid network of Europe with low degree distribution and high degree distribution strategies.

**V. EXTERNAL FIELD ANALOGY**

Here we argue that the concentration $\rho$ of reinforced nodes is analogous to external field in percolation [6, 8]. The key critical exponents $\beta$, $\delta$, and $\gamma$ that describe the behavior of the system near criticality will be derived below [26 - 30]. These key critical exponents also fulfill Widom’s identity $\delta - 1 = \gamma / \beta$ [6, 8, 23, 29, 30].

The critical exponent $\beta$ describes the behavior of the order parameter $P_\infty$ near the critical point with zero-field ($\rho = 0$, i.e., no reinforced nodes) and is given by

$$P_\infty(0, p) \sim (p - p_c)^\beta. \quad (3)$$

At the critical point, $p = p_c$, the increase of the order parameter with the magnitude of the field, i.e., the concentration of reinforced nodes $\rho$, is expected to yield the critical exponent $\delta$ as

$$P_\infty(\rho, p_c) \sim \rho^{1/\delta}. \quad (4)$$

The susceptibility of the system, $\chi$, is given by the partial derivative of the order parameter with respect to the field, $\rho$, and scales near the critical point with the exponent $\gamma$ as

$$\chi \equiv \left( \frac{\partial P_\infty(\rho, p)}{\partial \rho} \right)_{p \to 0} \sim |p - p_c|^{-\gamma}. \quad (5)$$
VI. SUMMERY

We showed that reinforced nodes have a significant effect on the resilience of spatial networks. Even a small fraction of reinforced nodes destroy the phase transition of the percolation process and increase the functional component. We showed that the higher the fraction of reinforced nodes, the more resilient is the network, i.e., increase the functional component. We also found that the lower the value of $\zeta$, the stronger is the effect for the same fraction of reinforced nodes on the resilience of the network. We showed that the best reinforced node distribution strategy highly depends on the stage of the percolation process. For earlier stages of the percolation process, low $1 - p$, it is better to reinforce the low degree nodes, while for later stages of the percolation process, it is better to reinforce the high degree nodes. We also showed the effect of reinforced nodes in a real-world spatial embedded network. Finally, we showed that reinforced nodes are analogous to external field also in the presence of spatial constraints with a crossover phenomenon for intermediate values of $\zeta$ and that the critical exponents that we found satisfy the Widom’s identity $\delta - 1 = \gamma/\beta$.

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Appendix A: The effect of reinforced nodes for different values of $\zeta$

In this appendix section, we will show random distributed reinforced nodes for other values of $\zeta$ (i.e. $\zeta = 10$ and $\zeta = 1000$). As can be seen in Fig. A1, we get similar results as in Fig. 2 in the main text, and the existence of reinforced nodes makes the spatial network more resilient to random failures, and the phase transition is removed. The meaning is that those results apply for all values of $\zeta$ since we cover the entire range from $\zeta = 0.2$ (Fig. 2(a) in the main text) to $\zeta = 1000$ (Fig. A1(b)).

Appendix B: Prioritized distribution of reinforced nodes for different values of $\zeta$

In this appendix section, we will identify preferred strategies for reinforced distribution for other values of $\zeta$. As we can see in Fig. B1, for other values of $\zeta$ we obtain similar results to those seen in Fig. 3 in the main text. We see also here that different reinforced node distributions are preferred for different stages of the percolation process. While the low degree nodes are better to reinforce at the earlier stages of the percolation process.
(large $p$), it is better to reinforce the high degree nodes at later stages of the percolation process (low $p$). These similarities suggest that our results are valid for the entire $\zeta$ range. Notice that the lowest value of $\zeta$ that we included here is $\zeta = 2$ instead of $\zeta = 0.2$ in Fig. 2. This is since for $\zeta = 0.2$ the degree and edge distributions are similar to those of 2D lattice i.e. $\delta_{k,4}$ for the degree distribution and all four nearest neighbors are edges. In this case all methods will show similar behaviour.

In Fig. C1 we can see the degree distribution of nodes in the giant component at different stages of the percolation process and for different values of $\zeta$. Here we can see the simulation results (circles) and the analytic solution for ER (continuous lines) from Eq. (C6). As seen already in Fig. 3 in the main text, we see here also similar behaviour and good agreement between the simulation results and the analytic solution (of mean-field ER) for all values of $\zeta$ indicating that although the analytical solution is mean-field it catches the behaviour of spatial networks.

Appendix C: Analytical results for the degree distribution of the nodes in the giant component during percolation process

In this section, we will solve analytically the node degree distribution in the giant component for ER networks.

We start by defining two events: $A$ is the event of a node being of degree $k$ and $B$ is the event of a node being in the giant component after the random removal of a fraction of $1-p$ of nodes. Thus, the degree distribution of nodes in the giant component will be obtained by the conditional probability $P(A|B)$ which according to the Bayes’ theorem is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \quad (C1)$$

The probability for a node to have degree $k$ is

$$P(A) = p_k. \quad (C2)$$

The probability of a node being a part of the giant component is

$$P(B) = P_\infty. \quad (C3)$$

The conditional probability is:

$$P(B|A) = p(1 - u^k). \quad (C4)$$

where $u$ is the probability that a node at the end of an edge of a randomly chosen node is not in the giant component \[31\]. Thus,

$$P(A|B) = \frac{p(1 - u^k)p_k}{P_\infty}. \quad (C5)$$

In the case of ER, $P_\infty = 1 - u$ \[31\] and $p_k = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}$. Thus,

$$P(A|B) = \frac{p(1 - (1 - P_\infty)k^k e^{-\langle k \rangle})}{P_\infty k!}. \quad (C6)$$

where \[32\],

$$P_\infty = p(1 - e^{-\langle k \rangle}P_\infty). \quad (C7)$$

In Fig. C1 we can see the degree distribution of nodes in the giant component at different stages of the percolation process and for different values of $\zeta$. Here we can see the simulation results (circles) and the analytic solution for ER (continuous lines) from Eq. (C6). As seen already in Fig. 3 in the main text, we see here also similar behaviour and good agreement between the simulation results and the analytic solution (of mean-field ER) for all values of $\zeta$ indicating that although the analytical solution is mean-field it catches the behaviour of spatial networks.

Appendix D: The effect of different values of $\langle k \rangle$ on the model

We also tested our model for a different value of average degree, $\langle k \rangle = 3$.

In Fig. D1 we can see that for all values of $\zeta$ we get similar results to the results shown in Fig. 2 in the main text, meaning the effects of randomly distributed reinforced nodes do not change for different values of average degree $\langle k \rangle$. However, there is one exception to these
FIG. C.1: Degree distribution of nodes in the giant component (for $\langle k \rangle = 4$). (a) The degree distribution of nodes, $P(k)$, as a function of $k$ for $\zeta = 2$. (b) The degree distribution of nodes, $P(k)$, as a function of $k$ for $\zeta = 10$. (c) The degree distribution of nodes, $P(k)$, as a function of $k$ for $\zeta = 50$. (d) The degree distribution of nodes, $P(k)$, as a function of $k$ for $\zeta = 1000$. We can see that for all values of $\zeta$ we get similar results to those shown in Fig. 3(d) in the main text. We can see that our simulation results (circles) and the analytic solution (continuous lines) overlap for $\zeta = 1000$ and are close enough to lower values of $\zeta$.

FIG. D.1: Randomly distributed reinforced nodes in spatial networks with a different average degree, $\langle k \rangle = 3$. (a) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 2$. The phase transition is at $p_c \approx 0.72$. (b) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 10$. (c) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 50$. (d) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 1000$, here $p_c = 0.33$ (since $\langle k \rangle = 3$). We can see that for all values of $\zeta$ we get similar results to those shown in Fig. 2 in the main text expect for the values of $p_c$.

FIG. D.2: Preferred strategies for reinforced distribution in spatial networks with a different average degree, $\langle k \rangle = 3$. (a) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 2$. (b) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 10$. (c) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 50$. (d) The giant component, $P_\infty$, as a function of $p$ for $\zeta = 1000$. We can see that for all values of $\zeta$ we get similar results to those shown in Fig. 3 in the main text, meaning the prioritize strategies for reinforced distribution do not change for different values of average degree $\langle k \rangle$ and it still depends on the stages of the percolation process.

In Fig. D2 we can see that for all values of $\zeta$ we get similar results to those in Fig. 3 in the main text, meaning the prioritize strategies for reinforced distribution do not change for different values of average degree $\langle k \rangle$ and it still depends on the stages of the percolation process.
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