MEASURES AND REGULARITY OF GROUP CANTOR ACTIONS

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Abstract. A minimal equicontinuous action of a discrete finitely generated group $G$ on a Cantor set $X$ is locally quasi-analytic (LQA) if all elements of $G$ have unique extensions on sufficiently small sets. Examples of LQA actions include topologically free actions. Examples which are not LQA include actions where stabilizers of points are pairwise distinct, as, for instance, the action of the famous Grigorchuk group on the boundary of a binary tree.

We show that an action is LQA if and only if the set of stabilizers of points with trivial holonomy in $X$ is finite. Furthermore, we give sufficient conditions under which the set of points with non-trivial holonomy in $X$ has measure zero. As a corollary, we obtain sufficient conditions under which a topologically free action is essentially free. We apply our results to study the properties of invariant random subgroups induced by minimal equicontinuous actions on Cantor sets and also provide many new examples of mean equicontinuous group actions.

1. Introduction

Let $X$ be a Cantor set, that is, a compact totally disconnected metrizable space with no isolated points. Suppose $G$ is a finitely generated discrete group and $\Phi : G \to \text{Homeo}(X)$ is an action of $G$ on $X$ by homeomorphisms. We write $(X,G,\Phi)$ to denote the dynamical system given by the action, and use the short notation $g \cdot x = \Phi(g)(x)$ for the action of $g \in G$ on $x \in X$ throughout the paper.

We assume that $(X,G,\Phi)$ is minimal and equicontinuous, and so has a unique invariant measure $\mu$, see Section 2 for definitions and details.

Let $G_x = \{ g \in G \mid g \cdot x = x \}$ be the subgroup of elements in $G$ which fix a point $x \in X$. Such a subgroup is called the stabilizer, or the isotropy subgroup of the action at $x$. A stabilizer $G_x$ is trivial if it is the trivial group, that is, $G_x = \{ e \}$ where $e$ is the identity in $G$.

We say that an action $(X,G,\Phi)$ is free if the stabilizer of every point in $X$ is trivial, and it is topologically free if the set of points of $X$ with non-trivial stabilizer is meager. For minimal continuous actions this is equivalent to $X$ containing a dense subset of points with trivial stabilizer. The action $(X,G,\Phi)$ is essentially free if the subset of points with non-trivial stabilizer has measure zero in $X$. An essentially free minimal action is always topologically free, but the converse need not hold, see the discussion below.

There are examples of actions such that no point in $X$ has a trivial stabilizer. At the extreme are actions where the stabilizers of any two points are pairwise distinct, such as actions of weakly branch groups, see Section 5.2 for the corresponding definitions.

Properties of stabilizers for measure-preserving group actions has been a topic of active research for many years, and it is not possible to give a reasonable overview of the existing literature in a few sentences. Classes of actions which have been drawing a lot of attention are, for instance, measure-preserving actions of lattices in Lie groups [10, 21, 22], and actions of self-similar groups on the boundaries of spherically homogeneous trees [11, 23]. Among the latter class, topologically free actions and actions of weakly branch groups are the most extensively studied, see [23] for a relatively recent survey. The ‘in-between’ case, when the action is not topologically free, but stabilizers of

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different points are not necessarily distinct, has received less attention, and these are the actions in which we are most interested in this paper.

More precisely, in this article we study the properties of stabilizers of points of a minimal equicontinuous Cantor action depending on its regularity. Thereby, in this work, regularity is quantified by the property of \textit{local quasi-analyticity}, which is defined further below. We determine how the presence or the absence of this property affects the number of distinct stabilizers of topologically generic points in $X$. Further, we study the genericity of points with different types of stabilizers in the measure-theoretical sense. As applications, we investigate the properties of invariant random subgroups, induced by Cantor actions, and provide novel examples of mean equicontinuous group actions. In our approach, we combine methods of topological and measurable dynamics together with concepts of foliation theory. We use examples from geometric group theory to illustrate essential aspects and key definitions.

Before we can state our main results, let us recall some basic definitions.

Let $(X,G,\Phi)$ be a minimal equicontinuous action of a discrete finitely generated group $G$ on a Cantor set $X$ by homeomorphisms. If a homeomorphism fixes a given point $x \in X$, then either it also fixes an open neighborhood of $x \in X$, or it induces a non-trivial map on any open neighborhood of $x \in X$. We define the \textit{neighborhood stabilizer} of $x \in X$ by

$$[G]_x = \{g \in G_x \mid g|_{V_x} = id \text{ for some open } V_x \ni x\}.$$  

An open set $V_g$ in (1) may depend on $g \in G$. The term neighborhood stabilizer comes from [43].

\begin{definition}
(1) We say that $x \in X$ is a point with trivial holonomy, or a point without holonomy if $G_x = [G]_x$.
(2) We say that $x \in X$ is a point with non-trivial holonomy, or a point with holonomy if $[G]_x \subset G_x$ is a proper inclusion.
\end{definition}

The terminology in Definition[14] is standard in foliation theory, where it was developed in a more general context of actions of pseudogroups of local homeomorphisms on compact spaces. We recall a basic result by Epstein, Millett and Tischler [18] for foliated spaces, which reads for the case of actions of groups on compact spaces as follows.

\begin{theorem}
Let $(X,G,\Phi)$ be an action of a discrete finitely generated group on a compact Hausdorff space. The set of points without holonomy is a residual (dense $G_\delta$) subset of $X$.
\end{theorem}

Thus points with trivial holonomy are topologically generic for minimal equicontinuous group actions on Cantor sets.

The concept of a \textit{quasi-analytic action} was first introduced by Haefliger [26] for actions on connected spaces, and it was generalized to include totally disconnected spaces by Alvarez López and Candel, see [3]. We give a localized version of this definition as formulated in [15]. Characteristics of actions with and without this property were studied in [14, 15, 30, 31, 29], and certain geometric properties of generalized suspensions of such actions were studied in [15].

\begin{definition}[3, Definition 9.4]
Let $X$ be a Cantor set with a metric, and let $G$ be a discrete finitely generated group. The action $(X,G,\Phi)$ is \textit{locally quasi-analytic}, or simply \textit{LQA}, if there exists $\epsilon > 0$ such that for any set $U \subseteq X$ with $\text{diam}(U) < \epsilon$, and for any subset $V \subseteq U$, and elements $g_1,g_2 \in G$

$$\text{if } \Phi(g_1)|V = \Phi(g_2)|V, \text{ then } \Phi(g_1)|U = \Phi(g_2)|U.$$  

An action $(X,G,\Phi)$ is quasi-analytic if one can choose $U = X$.
\end{definition}

That is, if an action is LQA, then for all $g \in G$ the homeomorphisms $\Phi(g)$ have unique extensions on subsets of uniform diameter in $X$. Composing with the inverse of $\Phi(g_i^{-1})$ on the left in (2) in
Definition 1.3, one deduces that an action is LQA if and only if every element which is the identity on $V$ is also the identity on $U$, for any $U$ with $\text{diam}(U) < \epsilon$ and any open subset $V \subset U$.

Free actions are quasi-analytic, indeed, the only element in $G$ which fixes open sets is the identity. Topologically free actions are quasi-analytic as well [30, Proposition 2.2]. Examples of actions which are locally quasi-analytic but not quasi-analytic are easy to construct: take any quasi-analytic action, and add a finite number of elements which fix an open subset of $X$ but act non-trivially on its complement, see [30, Example A.4]. A minimal equicontinuous action of a nilpotent group is always locally quasi-analytic, in fact, a similar property holds for a more general class of groups with Noetherian property [30, Theorem 1.6]. Examples of actions of finite index torsion-free subgroups of $SL(n, \mathbb{Z})$, for $n \geq 3$, which are not LQA are given in [31]. As shown in Lemma 3.5 below, actions of weakly branch groups are not LQA as well.

Our first result shows that there are minimal equicontinuous actions which are not LQA and which have points with equal stabilizers. Therefore, the class of actions which are not LQA is larger than the class of actions where points have pairwise distinct stabilizers.

**Theorem 1.4.** There exist minimal equicontinuous actions which are not LQA, and which contain distinct points with equal stabilizers. More precisely, given any minimal equicontinuous action $(Y, G, \Phi)$, which is not LQA, there exists a minimal equicontinuous action $(X, \tilde{G}, \tilde{\Phi})$, where $X = Y \times \{0, 1\}$, such that $(X, \tilde{G}, \tilde{\Phi})$ is not LQA, and for any $y \in Y$ there are at least two points $x_0, x_1 \in X$ such that $G_y = \tilde{G}_{x_0} = \tilde{G}_{x_1}$, where $G_y, \tilde{G}_{x_0}, \tilde{G}_{x_1}$ are stabilizers of the respective actions.

We proceed to study the number of distinct stabilizers of points for actions which are LQA and which are not LQA. Denote by

$$X_0 = \{x \in X \mid G_x = [G]_x\}$$

the subset of points without holonomy in $X$. Here is our next result.

**Theorem 1.5.** Let $(X, G, \Phi)$ be minimal and equicontinuous. Then the set $\{G_x \mid x \in X_0\}$ is finite if and only if the action $(X, G, \Phi)$ is locally quasi-analytic.

The result in Theorem 1.5 is optimal in the sense that it cannot be extended to include points with non-trivial holonomy. For instance, Example 2.9 is an example of a topologically free action with a single orbit of points with non-trivial holonomy. All points in $X_0$ have trivial stabilizers. The stabilizers of points with non-trivial holonomy are pairwise distinct, and so the number of distinct stabilizers of the points of $X$ is infinite.

The famous Grigorchuk group, which acts on the boundary of a binary tree and is described in [7, 23, 33] and other papers, is weakly branch (in fact, the Grigorchuk group is branch, which is a stronger property) and so the stabilizers of all points in $X$ are pairwise distinct. The action of the Grigorchuk group on the boundary of a binary tree has a single orbit of points with non-trivial holonomy [43], implying that the points in $X_0$ have an uncountable number of distinct stabilizers.

We now study measure-theoretical properties of $(X, G, \Phi)$, and include the unique invariant measure $\mu$ into our notation of the action and write $(X, G, \Phi, \mu)$. We ask the following question.

**Problem 1.6.** Let $(X, G, \Phi, \mu)$ be a minimal equicontinuous group action on a Cantor set $X$. Find necessary and sufficient conditions on the action, under which the set of points $X - X_0$ with non-trivial holonomy has measure zero.

We note that if an action $(X, G, \Phi, \mu)$ is topologically free, then every point $x \in X$ which has non-trivial stabilizer also has non-trivial holonomy. Indeed, for such actions the set of points with trivial stabilizers is dense in $X$, and so every element which fixes $x \in X$ acts non-trivially on any open neighborhood of $x$. For topologically free actions, Problem 1.6 reads as follows.

**Problem 1.7.** Let $(X, G, \Phi, \mu)$ be a minimal equicontinuous group action on a Cantor set. Suppose $(X, G, \Phi, \mu)$ is topologically free. Find necessary and sufficient conditions on the action, under which $(X, G, \Phi, \mu)$ is essentially free.
Problem [1.7] was solved for a special case of actions on trees, generated by finite automata, by Kambites, Silva and Steinberg [32], see also the discussion below. Examples of actions which are topologically free but not essentially free were given by Bergeron and Gaboriau [8], and also Abért and Elek [2]. In these examples the sets of points with non-trivial holonomy have positive measure. However, their constructions are quite involved technically, and at the end of Section 5 we give a simpler example of a class of actions where the set of points with non-trivial holonomy has positive measure. Our next theorem gives a partial answer to Problem [1.6] namely, we formulate two sufficient conditions which imply that the set of points with non-trivial holonomy has zero measure in $X$. We then recover the result of [32] as a consequence.

As we discuss in more detail in Section 6, every minimal equicontinuous action $(X,G,\Phi,\mu)$ on a Cantor set $X$ has a representation as an action on the boundary $\partial T$ of a rooted spherically homogeneous tree $T$. Recall that a tree $T$ with a vertex set $V = \bigcup_{\ell \geq 0} V_\ell$ is spherically homogeneous with spherical index $n = (n_1,n_2,\ldots)$ if for any $\ell \geq 1$ each vertex $v \in V_{\ell-1}$ is joined by edges to $n_\ell$ vertices in $V_\ell$. A tree $T$ is $d$-ary if $n_\ell = d$ for all $\ell \geq 1$. The spherical index $n$ of $T$ is bounded if there exists an integer $m > 0$ such that for all $\ell \geq 1$ we have $n_\ell \leq m$.

A representation of an action onto the boundary of a tree is given by the pair of maps

$$(\phi,\phi_*): (X,\Phi(G)) \to (\partial T, \Homeo(\partial T)),$$

where $\phi: X \rightarrow \partial T$ is a homeomorphism, and $\phi_*: \Phi(G) \to \Homeo(\partial T)$ is a monomorphism induced by $\phi$. Under $\phi$ the ergodic measure $\mu$ pushes forward to a uniform Bernoulli measure on $\partial T$ which we also denote by $\mu$, see Section 2.4 for details. We discuss the uniqueness of such representation in Sections 5 and 6. The first hypothesis of Theorem 1.9 is the existence of a representation of $(X,G,\Phi,\mu)$ as an action on the boundary of a tree with bounded spherical index.

The second condition is as follows. For a vertex $v_\ell \in V_\ell$, we denote by $T_{v_\ell}$ the subtree through the vertex $v_\ell$, and by $\partial T_{v_\ell}$ the subset of $\partial T$ consisting of paths which contain $v_\ell$. The set $\partial T_{v_\ell}$ is a clopen subset of $\partial T$.

**Definition 1.8.** Let $T$ be a spherically homogeneous tree, and let $G \subset \Homeo(\partial T)$ be a group. The action of $g \in G$ on $\partial T$ is uniformly non-constant if the following holds: there are positive integers $n_g$ and $K_g$ such that for any $\ell \geq n_g$ and any vertex $v_\ell \in V_\ell$ if $g$ fixes $v_\ell$ and $g|\partial T_{v_\ell} \neq \id$, then there exists $m \geq \ell$ and a vertex $w_m \in V_m \cap T_{v_\ell}$ such that $g \cdot w_m \neq w_m$ and $m - \ell \leq K_g$.

The action of the group $G$ is uniformly non-constant if and only if the action of every $g \in G$ is uniformly non-constant.

We show that actions on $d$-ary trees generated by finite automata are uniformly non-constant, see Example 6.3. We give an example of an action which is not uniformly non-constant in Section 6. Intuitively, Definition 1.8 ensures that, given $g \in G$, for every point $x \in X$ and for every clopen set $U \subset X$ containing $x$ such that $g \cdot U = U$ and $g|U \neq \id$, there exists a sufficiently large subset $W \subset U$ such that every point in $W$ is moved by the action of $g$. In some cases this condition can also be expressed in a more elegant form using a metric on $\partial T$, see Section 6. To avoid making more choices, we formulate Definition 1.8 in the language of trees rather than using a metric.

Here is our next main theorem.

**Theorem 1.9.** Let $(X,G,\Phi,\mu)$ be a minimal equicontinuous group action. Suppose $(X,G,\Phi,\mu)$ admits a representation $(\phi,\phi_*): (X,\Phi(G)) \to (\partial T, \Homeo(\partial T))$ on the boundary of a spherically homogeneous tree $T$ which satisfies the following two conditions:

1. The tree $T$ has bounded spherical index.
2. The action of $\phi_*(\Phi(G))$ on $\partial T$ is uniformly non-constant.

Then the set $X - X_0$ of points with non-trivial holonomy has zero measure.

As a consequence we obtain sufficient conditions for a topologically free action to be essentially free.
COROLLARY 1.10. If an action \((X,G,\Phi,\mu)\) is topologically free and satisfies the conditions of Theorem 1.9, then it is essentially free.

Kambites, Silva and Steinberg [32] showed that an action on the boundary of a rooted \(d\)-ary tree generated by finite automata is topologically free if and only if it is essentially free. Since actions generated by finite automata are uniformly non-constant, we recover this result of [32] as a consequence of Theorem 1.9.

The idea of the proof of Theorem 1.9 is inspired by a result of Hurder and Katok [28, Proposition 7.1], where it was shown that in a \(C^1\) foliation of codimenion \(n\) of a smooth manifold \(M\), the set of leaves with non-trivial linear holonomy has measure zero. Our setting is very different from the setting of [28], as for actions on Cantor sets, there are no differentials and the notion of linear holonomy does not make sense. This is the reason we must impose additional assumptions on the action to obtain our result.

REMARK 1.11. The condition that an action admits a representation onto the boundary of a tree with a bounded spherical index is a natural one. Boundedness of the spherical index ensures that the tree \(T\), with an ultrametric defined in Section 2, has finite Assouad dimension, and therefore can be embedded into a Euclidean space by a bi-Lipschitz homeomorphism [4, 9, 36, 35]. If \(G\) is finitely generated, then the action \((X,G,\Phi,\mu)\) can be suspended to obtain a lamination [15]. A lamination is a compact connected metric space where each point has a neighborhood homeomorphic to the product of a Euclidean disk and a Cantor set, and so it naturally decomposes into path-connected components, called the leaves. A natural question is when a given lamination admits an embedding as a foliated subset of a foliated manifold. Such an embedding must necessarily be bi-Lipschitz, so as to preserve the dynamical properties of \((X,G,\Phi,\mu)\) [27]. Therefore, a necessary condition for the existence of a bi-Lipschitz embedding of a lamination into a foliation of a smooth manifold is that the transverse dynamics of the lamination is conjugate to an action on the boundary of a tree with bounded spherical index.

To finish, we provide applications of our results concerning the existence of non-atomic invariant random subgroups and a rich class of novel mean equicontinuous group actions

Let \(\text{Sub}(G)\) be the set of all closed subgrops of a finitely generated discrete group \(G\). The set \(\text{Sub}(G)\) is equipped with the Chaubaty-Fell topology, and becomes a compact totally disconnected space. For the case when \(G\) is a countable group, basic open sets on \(\text{Sub}(G)\) are of the form [43, 7]

\[
O_{A,B} = \{ H \subset \text{Sub}(G) \mid A \subset H, B \cap H = \emptyset \},
\]

where \(A\) and \(B\) are finite sets. The group \(G\) acts on \(\text{Sub}(G)\) by conjugation.

DEFINITION 1.12. An invariant random subgroup (IRS) \(\nu\) is a Borel probability measure on \(\text{Sub}(G)\) invariant under the action of \(G\) on \(\text{Sub}(G)\).

For an action \((X,G,\Phi,\mu)\) consider the mapping

\[
\text{St} : X \to \text{Sub}(G) : x \mapsto G_x,
\]

which assigns to each \(x \in X\) its stabilizer. Stabilizers of points in an orbit of \(x \in X\) are conjugate, so the mapping \(\text{St}\) maps the orbit of \(x \in X\) onto the orbits of \(G_x\) in \(\text{Sub}(G)\). Unless the stabilizers of points in \(X\) are pairwise distinct, the mapping \(\text{St}\) is not injective. The pushforward \(\nu = \text{St}_*\mu\) of the invariant measure \(\mu\) is an IRS [1].

The term invariant random subgroup was first introduced in Abért, Glasner and Virag [1], and the study of IRSs has been an active topic of research in the recent years, see, for instance, surveys by Gelander [21, 22]. For discrete groups, acting on trees, the dynamics-related questions considered in the literature are of two types. First, one can study the properties of IRSs induced via the mapping [5] for a particular group. Second, for a group \(G\) of homeomorphisms of a tree \(T\), one can choose an IRS on \(\text{Sub}(G)\), which is not necessarily induced via [5], and use the properties of the action of \(G\) to study the properties of the IRS. Examples of the studies of the first kind are [12, 7], and of the
second kind are [10, 44], where the properties of IRSs for weakly branch and branch groups were considered. See also references in [21, 22, 12, 7, 10, 44] for other work on IRS.

In this paper, we study the relation between the regularity of a minimal equicontinuous action \((X, G, \Phi, \mu)\), and the properties of the invariant random subgroup \(\nu = \text{St}_{\ast} \mu\) induced on \(\text{Sub}(G)\) via the map \([5]\). The subtle aspect is that the map \([5]\) is usually not continuous, with points of discontinuity being precisely the points with non-trivial holonomy \([43]\). For actions where the points with non-trivial holonomy have measure zero, Theorems 1.5 and 1.9 lead to the following result. For the set \(X_0\) of points without holonomy, defined by \([4]\), consider the topological closure

\[
Z = \{G_x \mid x \in X_0\} \subset \text{Sub}(G).
\]

**Theorem 1.13.** Let \((X, G, \Phi, \mu)\) be a minimal equicontinuous action on a Cantor set \(X\) which satisfies the hypothesis of Theorem 1.1. Then the IRS \(\nu = \text{St}_{\ast} \mu\) is supported on \(Z\), and \((X, G, \Phi)\) is not locally quasi-analytic if and only if \(\nu\) is non-atomic. If, in addition, the restriction \(\text{St} : X_0 \to \text{Sub}(G)\) is injective, then \(\text{St}\) provides a measure-theoretical isomorphism between \((X, G, \Phi, \mu)\) and \((Z, G, \nu)\).

The question when the IRS \(\nu\) is supported on the set \(Z\) was raised in [23, p.123], and Theorem 1.13 provides a partial answer to that question. Points without holonomy are called \(G\)-typical points in [23]. The second part of the previous statement is a partial answer to [23, Problem 8.2].

As already mentioned, there are examples of actions where the sets of points with non-trivial holonomy has positive measure. The following problem is then natural.

**Problem 1.14.** Let \((X, G, \Phi, \mu)\) be a minimal equicontinuous action on a Cantor set \(X\), and suppose the set of points with non-trivial holonomy in \(X\) has positive measure. Describe the properties of the IRS \(\nu = \text{St}_{\ast} \mu\) induced on \(\text{Sub}(G)\) via the map \([5]\).

Finally, let us indicate the consequences of our results with respect to mean equicontinuous group actions. For this recall that an action \((X, G, \Phi)\) is a (topological) factor of another action \((Y, G, \Psi)\) if there is a continuous surjection \(\psi : Y \to X\) with \(\psi(\Psi(g)(y)) = \Phi(g)(\psi(y))\) for all \(g \in G\) and \(y \in Y\). In this case, \(\psi\) is called a factor map and \((Y, G, \Psi)\) is also referred to as a (topological) extension of \((X, G, \Phi)\). If \(\psi\) is further injective (and hence, a homeomorphism), we say \((X, G, \Phi)\) and \((Y, G, \Psi)\) are conjugate and call \(\psi\) a conjugacy.

To proceed, let

\[
\tilde{X} = \{(x, G_x) \mid x \in X_0\} \subset X \times Z
\]

and denote by \(\tilde{\Phi}\) the natural action of \(G\) on \(\tilde{X}\) which acts in the first coordinate by \(\Phi\) and in the second by conjugation. Then \((\tilde{X}, G, \tilde{\Phi})\) is an extension of \((X, G, \Phi)\) via the factor map \(\eta : \tilde{X} \to X\) which is the projection to the first coordinate. By [25, Proposition 1.2], we know that \((\tilde{X}, G, \tilde{\Phi})\) is minimal and \(\eta\) is almost one-to-one, meaning the set \(\{\tilde{x} \in \tilde{X} \mid \eta^{-1}(\eta(\tilde{x})) = \{\tilde{x}\}\}\) is dense in \(\tilde{X}\).

First, we obtain an alternative characterization of LQA actions using Theorem 1.15.

**Theorem 1.15.** Let \((X, G, \Phi)\) be a minimal equicontinuous action. Then \((X, G, \Phi)\) is locally quasi-analytic if and only if the almost one-to-one extension \(\eta : \tilde{X} \to X\) is a conjugacy.

To state the definition of mean equicontinuity, recall that a countable discrete group \(G\) is amenable if there is a sequence \(\{F_\ell\}_{\ell \geq 0}\), called a (left) Følner sequence, of non-empty finite sets in \(G\) such that

\[
\lim_{\ell \to \infty} \frac{|g \cdot F_\ell \triangle F_\ell|}{|F_\ell|} = 0 \quad \text{for all } g \in G,
\]

where \(\triangle\) denotes the symmetric difference. Now, given an action \((Y, G, \Psi)\) where \((Y, d)\) is a compact metric space and \(G\) is amenable, we say \((Y, G, \Psi)\) is (Weyl-)mean equicontinuous if for all \(\epsilon > 0\) there is \(\delta > 0\) such that for all \(x, y \in Y\) with \(d(x, y) < \delta\) we have

\[
\sup_{\{F_\ell\}_{\ell \geq 0}} \limsup_{\ell \to \infty} \frac{1}{|F_\ell|} \sum_{g \in F_\ell} d(g \cdot x, g \cdot y) < \epsilon,
\]

where

\[
\sum_{g \in F_\ell} d(g \cdot x, g \cdot y)
\]
where the supremum is taken over all Følner sequences \( \{F_t\}_{t \geq 0} \). For more information about mean equicontinuity, see for instance [20] and references therein.

**COROLLARY 1.16.** Suppose a minimal equicontinuous action \((X, G, \Phi)\) satisfies the conditions of Theorem 1.9 and that \(G\) is amenable. Then the action \((\tilde{X}, G, \tilde{\Phi})\) is mean equicontinuous.

Note that in the previous corollary the action \((\tilde{X}, G, \tilde{\Phi})\) could be a trivial extension, in the sense that it is equicontinuous as well (e.g., for odometer actions). However, since \(\eta\) is almost one-to-one, this can only happen when \(\eta\) is a conjugacy, see Section 7.2. This in turn implies that if \((X, G, \Phi)\) is not LQA, then \((\tilde{X}, G, \tilde{\Phi})\) is a non-trivial mean equicontinuous system, by Theorem 1.15. For instance, any group generated by bounded finite automata (see Example 6.3 for definitions) is amenable [6]. If in addition the group is weakly branch, then the extension \((X, G, \tilde{\Phi})\) is non-trivial.

The rest of the paper is organized as follows. In Section 2 we recall the necessary background on the properties of equicontinuous actions. In Section 3 we show that every minimal equicontinuous action admits a representation as an action on the boundary of a rooted spherically homogeneous tree. We discuss the uniqueness of such representation, and prove that actions of weakly branch groups on the boundaries of \(d\)-ary trees are not locally quasi-analytic. In Section 4 we recall the definition of invariant random subgroup. We prove Theorems 1.13 and 1.15 in Section 5 and Theorem 1.2 in Section 6. In Section 7 we discuss applications of our results proving Theorems 1.13 and 1.15 and their corollaries.

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## 2. Equicontinuous actions

We recall some basic properties of group actions on Cantor sets.

An action \(\Phi : G \to \text{Homeo}(X)\) of a discrete finitely generated group \(G\) on a Cantor set \(X\) is **equicontinuous** if \(X\) admits a metric \(d\) such that for all \(\epsilon > 0\) there is \(\delta > 0\) such that for any \(g \in G\) and any \(x, y \in X\) with \(d(x, y) < \delta\) we have \(d(g \cdot x, g \cdot y) < \epsilon\). An action \((X, G, \Phi)\) is **minimal** if the orbit \(G(x) = \{g \cdot x \mid g \in G\}\) of every \(x \in X\) is dense in \(X\). In this paper, \((X, G, \Phi)\) is always minimal and equicontinuous.

Let \(A \subset X\) and suppose for every \(g \in G\) we have \(g \cdot A = A\). In this case we say that \(A\) is **invariant under the action of** \(G\). Additionally, assuming that \(A\) is closed, \((A, G, \Phi_A)\) will denote the restriction of \((X, G, \Phi)\) to \(A\) and we call \(A\) a **minimal** subset if \((A, G, \Phi_A)\) is minimal.

### 2.1. Group chains

Our main tool in working with minimal equicontinuous actions are **group chains**. In this section we explain how a group chain defines a minimal equicontinuous action on a Cantor set.

**DEFINITION 2.1.** Let \(G\) be a finitely generated discrete group. A group chain is a descending infinite sequence \(\{G_t\}_{t \geq 0} : G = G_0 \supset G_1 \supset \cdots \) of finite index subgroups of \(G\).

For each \(t \geq 0\) the coset space \(X_t = G/G_t\) is finite, and there are inclusion maps
\[
X_{t+1} \to X_t : gG_{t+1} \mapsto gG_t,
\]
induced by the inclusion of groups \(G_{t+1} \subset G_t\). The inverse limit space
\[
X_\infty = \lim_{\longleftarrow} \{X_{t+1} \to X_t\} = \{(g_0G_0, g_1G_1, \ldots) \mid g_{t+1}G_{t+1} \subset g_tG_t\} \subset \prod_{t \geq 0} X_t
\]
is a Cantor set in the relative topology, induced from the product (Tychonoff) topology on \(\prod_{t \geq 0} X_t\). We denote by \((g_tG_t)_{t \geq 0}\) sequences of cosets in \(X_\infty\). Basic sets in \(X_\infty\) are given by
\[
U_{g,m} = \{(h_tG_t)_{t \geq 0} \in X_\infty \mid h_mG_m = gG_m\}.
\]
The group $G$ acts on the left on every coset space $X_\ell$, permuting the cosets in $X_\ell$, and there is an induced action by homeomorphisms on the inverse limit
\begin{equation}
G \times X_\infty \to X_\infty : (g, (g_0G_0, g_1G_1, g_2G_2, \ldots)) \mapsto (gg_0G_0, gg_1G_1, gg_2G_2, \ldots).
\end{equation}
Since the action of $G$ on every coset space $X_\ell$, $\ell \geq 0$, is transitive, the action \(10\) on $X_\infty$ is minimal. The space $X_\infty$ can be given an ultrametric $D$, for example, by
\begin{equation}
D((g_\ell G_\ell)_{\ell \geq 0}, (h_\ell G_\ell)_{\ell \geq 0}) = \frac{1}{2^m}, \text{ where } m = \min\{\ell \geq 0 \mid g_\ell G_\ell \neq h_\ell G_\ell\},
\end{equation}
that is, this metric measures for how long the sequences of cosets $(g_\ell G_\ell)_{\ell \geq 0}$ and $(h_\ell G_\ell)_{\ell \geq 0}$ coincide. Since $G$ acts on each coset space $X_\ell$, $\ell \geq 0$, by permutations, it acts on $X_\infty$ by isometries relative to the metric $D$. In particular, this action of $G$ is equicontinuous.

**Remark 2.2.** The choice of metric in \(11\) is not unique. For example, we can define
\begin{equation}
D_1((g_\ell G_\ell)_{\ell \geq 0}, (h_\ell G_\ell)_{\ell \geq 0}) = (n_1 \cdots n_m)^{-1}, \text{ where } m = \min\{\ell \geq 0 \mid g_\ell G_\ell \neq h_\ell G_\ell\},
\end{equation}
and $n_\ell = |G_\ell : G_{\ell+1}|$ is the index of $G_{\ell+1}$ in $G_\ell$. The action \(10\) is equicontinuous with respect to the metric \(12\) as well. The metrics \(11\) and \(12\) are equivalent.

### 2.2. Adapted clopen sets and group chain models

In this section, we show how to associate a group chain to a minimal equicontinuous action $(X, G, \Phi)$, where $X$ is a Cantor set, and $G$ is a finitely generated discrete group.

Let $CO(X)$ denote the collection of all clopen (closed and open) subsets of $X$, which form a basis for the topology of $X$. For $\phi \in \text{Homeo}(X)$ and $U \in CO(X)$, the image $\phi(U) \in CO(X)$. The following result is folklore, and a proof is given in [30, Proposition 3.1].

**Proposition 2.3.** A minimal Cantor action $(X, G, \Phi)$ is equicontinuous if and only if for the induced action $\Phi_* : G \times CO(X) \to CO(X)$ the orbit of every $U \in CO(X)$ is finite.

**Definition 2.4.** Let $(X, G, \Phi)$ be a minimal equicontinuous action. A clopen set $U \in CO(X)$ is adapted to the action $\Phi$ if $U$ is non-empty and for any $g \in G$ if $\Phi(g)(U) \cap U \neq \emptyset$, then $\Phi(g)(U) = U$.

That is, the translates of an adapted set by the action of $G$ form a partition of the Cantor set $X$.

**Definition 2.5.** Let $(X, G, \Phi)$ be a minimal equicontinuous action and $x \in X$. A properly descending chain of clopen sets $U_\ell = \{U_\ell \subset X \mid \ell \geq 0\}$ is said to be an adapted neighborhood basis at $x$ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 0$ with $\cap U_\ell = \{x\}$, and each $U_\ell$ is adapted to $\Phi$.

The following result is folklore, a proof can be found in [14, Appendix A].

**Proposition 2.6.** Let $(X, G, \Phi)$ be a minimal equicontinuous action. Given $x \in X$, there exists an adapted neighborhood basis $U_x$ at $x$ for the action $\Phi$.

For an adapted set $U \subset X$, consider the collection of elements in $G$ which preserve $U$, that is,
\begin{equation}
G_U = \{g \in G \mid \Phi(g)(U) = U\}.
\end{equation}
Then $G_U$ is a subgroup of $G$, called the stabilizer of $U$, or the isotropy subgroup of the action of $G$ at $U$. Since the orbit of $U$ under the action of $G$ is finite, $G_U$ has finite index in $G$.

Fix $x \in X$, and let $U_x = \{U_\ell\}_{\ell \geq 0}$ be an adapted neighborhood basis at $x \in X$. Denote by $G_\ell = G_{U_\ell}$ the stabilizer of $U_\ell$. Since $U_{\ell+1} \subset U_\ell$, every element in $G$ which stabilizers $U_{\ell+1}$ also stabilizers $U_\ell$, and so there are inclusions $G_{\ell+1} \subset G_\ell$. Thus associated to an adapted neighborhood basis $U_x$ at $x \in X$, there is an infinite group chain $G = \{G_\ell\}_{\ell \geq 0}$ of isotropy subgroups of $G$ at the sets of $U_x$.

For each $\ell \geq 0$, the translates $\{\Phi(g)(U_\ell)\}_{g \in G}$ of $U_\ell \in U_x$ form a finite partition of $X$. Define a coding map $\kappa_\ell : X \to X_\ell$, where $X_\ell = G/G_\ell$, by
\begin{equation}
\kappa_\ell(y) = gg_\ell \text{ if and only if } y \in \Phi(g)(U_\ell).
\end{equation}
The maps $\kappa_\ell$ are equivariant with respect to the action of $G$ on $X$ and $X_\ell$. Taking the inverse limit with respect to the coset inclusions $X_{\ell+1} \to X_\ell$, we obtain a homeomorphism

$$
\kappa_\infty : X \to X_\infty = \lim_{\longleftarrow} \{X_{\ell+1} \to X_\ell\}
$$

which conjugates the actions of $G$ on $X$ and $X_\infty$, the latter being given by \(10\). By construction, we have $\kappa_\infty(x) = (eG_\ell)$, where $eG_\ell$ is the coset of the identity $e$ in $G/G_\ell$.

For a given action $(X,G,\Phi)$ the choice of an adapted neighborhood system $\mathcal{U}_x$ is not unique, and so the choice of a group chain $G^\ell$ associated to the action is not unique. The relations between the group chains representing the same conjugacy class of minimal equicontinuous actions were studied in detail in \[14\], where the following result was proved.

**PROPOSITION 2.7.** \[14\] Let $G$ be a discrete finitely generated group, and let $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ and $\mathcal{G}' = \{G'_\ell\}_{\ell \geq 0}$ be group chains with $G = G_0 = G'_0$. Let $(X_\infty,G)$ and $(X'_\infty,G')$ be minimal equicontinuous actions associated to $\mathcal{G}$ and $\mathcal{G}'$ by \[13\] and \[14\]. Then $(X_\infty,G)$ and $(X'_\infty,G')$ are conjugate by a homeomorphism $\phi : X_\infty \to X'_\infty$ if and only if there exists a sequence of elements $\{g_\ell\}_{\ell \geq 0} \in G$ such that for any $n \geq 0$ and any $\ell > n$ $g_\ell G_n = g_n G_\ell$, and, possibly after passing to a subsequence, there are inclusions

$$
G = G_0 \supset G'_1 \supset g_1 G_1 g_1^{-1} \subset G'_2 \supset g_2 G_2 g_2^{-1} \supset \cdots.
$$

In addition, $\phi$ is pointed, that is, $\phi(eG_\ell) = (eG'_\ell)$, if and only if one can choose $g_\ell = e$ in \(15\) for all $\ell \geq 0$.

The condition $g_\ell G_n = g_n G_\ell$ for $\ell > n$ in Proposition \[2.7\] ensures that the chain $\{g_\ell G_\ell g_\ell^{-1}\}_{\ell \geq 0}$ is a descending group chain as in Definition \[2.1\].

**REMARK 2.8.** The inclusions \[15\] impose restrictions on the indices of subgroups in the chains $\mathcal{G}$ and $\mathcal{G}'$. Indeed, if in \[15\] we have $G'_\ell \supset g_\ell G_\ell g_\ell^{-1}$, then the index $|G : G'_\ell|$ must be a divisor of the index $|G : g_\ell G_\ell g_\ell^{-1}|$. In particular, if $|G : G'_\ell| = p_1 p_2 \cdots p_\ell$, where $p_\ell = |G_{\ell-1} : G_\ell|$, $\ell \geq 1$, are distinct primes, and $|G : G_\ell| = d^\ell$ for some integer $d \geq 2$, then the actions defined by the group chains $\mathcal{G}$ and $\mathcal{G}'$ are not conjugate, since the set of primes which divide $d$ is finite.

### 2.3. Stabilizers of points and group chains

Let $(X,G,\Phi)$ be a minimal equicontinuous system, let $\mathcal{U} = \{U_\ell\}_{\ell \geq 0}$ be an adapted neighborhood basis at $x \in X$, and let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be an associated group chain. The *kernel* of the group chain $\mathcal{G}^x$ is the subgroup

$$
\mathcal{K}(\mathcal{G}^x) = \bigcap_{\ell \geq 0} G_\ell
$$

of elements in $G$ which fix $x$, so the kernel of the group chain at $x$ is its stabilizer $G_x = \mathcal{K}(\mathcal{G}^x)$.

If $y \in X$ is another point, then for every $\ell \geq 0$ there is $g_\ell \in G$ such that $y \in \Phi(g_\ell)(U_\ell)$. It follows that $\mathcal{G}^y = \{g_\ell G_\ell g_\ell^{-1}\}_{\ell \geq 0}$ is a group chain at $y$, and we can compute the stabilizer at $y$

$$
G_y = \mathcal{K}(\mathcal{G}^y) = \bigcap_{\ell \geq 0} g_\ell G_\ell g_\ell^{-1}.
$$

If $y$ is in the orbit of $x$, that is, $y = h \cdot x$ for some $h \in G$, then we can choose $g_\ell = h$, and in this case the stabilizers $G_x$ and $G_y$ are conjugate subgroups of $G$. If $y$ is not in the orbit of $x$, then the stabilizers $G_x$ and $G_y$ need not be isomorphic, as the following example shows.

**EXAMPLE 2.9.** Let $G = \langle a,b \mid bab = a^{-1}, b^2 = e \rangle$ be the dihedral group, where $e$ denotes the identity in $G$, let $G_\ell = \langle a^{2^\ell}, b \rangle$ for $\ell > 0$, and $G_0 = G$. This example is very well-studied. For instance, the action defined by this group chain is conjugate to the action of the iterated monodromy group associated to the quadratic Chebyshev polynomial, see \[38\] and \[34\]. The group chains representing this action were studied in \[19\] and \[14\] Example 7.5].

Consider the dynamical system associated to this group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ by the construction in Section \[2.4\]. One can show that for all $\ell \geq 1$ the cosets in $X_\ell = G/G^x_\ell$ are represented by the powers of $a$, and so for each $x \in X$, the orbit of $x$ is given by $G(x) = \{a^n \cdot x \mid n \in \mathbb{Z}\}$.
So let $x = (eG_\ell)_{\ell \geq 0} \in X_\infty$, then $G_x = K(G^x) = \langle b \rangle$. The isotropy groups of the points in $G(x)$ are conjugate to $G_x$, more precisely,

$$G_{a^n x} = \langle a^n b a^{-n} \rangle.$$  

If $y \in X_\infty$ is not in the orbit of $x$, then by [19] [14] the stabilizer of $y$ is trivial, $G_y = \{ e \}$. In particular, this means that the action of $G$ on $X_\infty$ defined by $G^x$ is topologically free.

We show that the stabilizers of the points in $G(x)$ are pairwise distinct. To obtain a contradiction, suppose that $G_x = G_{a^n x}$. Then $aba^{-1} = b$, which implies that

$$a^{-2} = baba^{-1} = b^2 = e,$$

which is not true since $G$ is infinite. The claim follows by applying a similar argument to the stabilizers of arbitrary points in $G(x)$.

2.4. Uniform Bernoulli measure on $X$. Let $(X, G, \Phi)$ be a minimal equicontinuous action, then the closure $E = \Phi(G) \subset \text{Homeo}(G)$ in the uniform topology is a profinite compact group, called the Ellis, or enveloping group $[3] [17]$. The group $E$ acts on $X$, the isotropy group $E_x = \{ \tilde{g} \in E \mid \tilde{g}(x) = x \}$ of its action at $x$ is a closed subgroup of $E$, and we have $X = E/E_x$. The group $G$ is identified with a dense subgroup of $E$, and so there is an action of $G$ on $E$. The Haar measure $\tilde{\mu}$ on $E$ is invariant with respect to this action and ergodic. The measure $\tilde{\mu}$ on $E$ pushes down to a measure $\mu'$ on $X$, and with this measure $(X, G, \Phi)$ is uniquely ergodic $[13]$. Given a group chain $G^x = \{ G_\ell \}_{\ell \geq 0}$ associated to the action $(X, G, \Phi)$ at a point $x \in X$, one can define a uniform Bernoulli measure $\mu$ on the space $X_\infty$ in $(5)$ by setting for every basic set $U_{g,m}$, defined by $(9)$,

$$\mu(U_{g,m}) = \frac{1}{|G : G_m|},$$

where $|G : G_m|$ is the index of $G_m$ in $G$. This measure is easily seen to be invariant under the action of $G$, and hence coincides with $\mu'$. A consequence of this statement is that the measure $\mu$ in (16) is independent of the choice of a group chain $G^x$.

3. Actions on trees

In this section, we represent a minimal equicontinuous action $(X, G, \Phi)$ as an action on the boundary of a spherically homogeneous tree $T$. Actions on trees, especially self-similar actions on $d$-ary trees, $d \geq 2$, are an active topic in Geometric Group Theory, see [38] for an overview of developments up to 2005, and [23] [24] for more recent surveys. In the following we review certain topics about actions on trees.

3.1. Trees and group actions. A tree $T$ consists of a set of vertices $V = \bigcup_{\ell \geq 0} V_\ell$, where $V_\ell$ is a finite vertex set at level $\ell$, and of edges joining vertices in $V_{\ell+1}$ and $V_\ell$, for all $\ell \geq 0$, such that every vertex in $V_{\ell+1}$ is joined by an edge to a single vertex in $V_\ell$. A tree is rooted if $|V_0| = 1$. A tree $T$ is spherically homogeneous if there is a sequence $n = (n_1, n_2, \ldots)$, called the spherical index of $T$, such that for every $\ell \geq 1$ a vertex in $V_{\ell-1}$ is joined by edges to $n_\ell$ vertices in $V_\ell$. We assume that $n_\ell \geq 2$ for $\ell \geq 1$. A spherically homogeneous tree $T$ is $d$-ary if its spherical index $n = (n_1, n_2, \ldots)$ is constant, that is, $n_\ell = d$ for some positive integer $d$. If $d = 2$, then the tree $T$ is called a binary tree.

Let $(X, G, \Phi)$ be a minimal equicontinuous action, $U_a = \{ U_\ell \}_{\ell \geq 0}$ be a neighborhood basis at $x \in X$, and let $G^x = \{ G_\ell \}_{\ell \geq 0}$ be an associated group chain, see Section [2] for the definitions of these objects. For $\ell \geq 0$, identify the vertex set $V_\ell$ with the coset space $X_\ell = G/G_\ell$. Join $v_\ell \in V_\ell$ and $v_{\ell+1} \in V_{\ell+1}$ by an edge if and only if $v_{\ell+1} \subset v_\ell$ as cosets. The obtained tree is spherically homogeneous, with entries $n_\ell = |G_{\ell-1} : G_\ell|$ in the spherical index $n$, for $\ell \geq 1$.

An infinite path in $T$ consists of a sequence of vertices $(v_\ell)_{\ell \geq 0}$ such that $v_{\ell+1}$ and $v_\ell$ are joined by an edge, for $\ell \geq 0$. The boundary $\partial T$ of $T$ is the collection of all infinite paths in $T$, and so it is the
With the relative topology from the (Tychonoff) product topology on $\prod_{\ell \geq 0} V_\ell$, $\partial T$ is a Cantor set. It is immediate that the identification of vertex sets $V_\ell$ with coset spaces $X_\ell$ induces an identification of $\partial T'$ with the inverse limit space $X_\infty$ defined by $[S]$, with points in $X_\infty$ corresponding to infinite paths in $\partial T$.

The group $G$ acts on the vertex levels $V_\ell = X_\ell$, $\ell \geq 0$ by permutations. Since the action of $G$ preserves the containment of cosets, the action preserves the connectedness of the tree $T$, that is, the vertices $v_\ell \in V_\ell$ and $v_{\ell+1} \in V_{\ell+1}$ are joined by an edge if and only if for any $g \in G$ the images $g \cdot v_\ell \in V_\ell$ and $g \cdot v_{\ell+1} \in V_{\ell+1}$ are joined by an edge. Thus every $g \in G$ defines an automorphism of the tree $T$, and we can consider $G$ as a subgroup of the group of tree automorphisms $Aut(T)$.

The composition of the map $[13]$ with the identification $X_\infty \to \partial T$ is a homeomorphism $\phi : X \to \partial T$. The action of $G$ on vertex levels $V_\ell$, $\ell \geq 0$ induces an action of $G$ on $\partial T$ by left translations, defined by $[10]$. Thus there is the induced map $\phi_* : \Phi(G) \to Homeo(\partial T)$, and $\Phi(G)$ is identified with a subgroup of $Homeo(\partial T)$. The action $(\partial T, \Phi(G))$ is minimal and equicontinuous since $(X, G, \Phi)$ is minimal and equicontinuous. A pair of maps

$$
(\phi, \phi_*) : (X, \Phi(G)) \to (\partial T, Homeo(\partial T))
$$

is called a tree representation of $(X, G, \Phi)$. If a group $G$ is defined as a subgroup of $Homeo(\partial T)$, then we often omit the trivial homomorphism $\Phi$ from our notation.

**Remark 3.1.** The choice of a group chain, associated to an action $(X, G, \Phi)$ is not unique, and, consequently, the choice of a tree representation $[17]$ is not unique. Let $U_\ell$ and $U_\ell'$ be two adapted neighborhood bases for $(X, G, \Phi)$ with corresponding group chains $\{G_\ell\}_{\ell \geq 0}$ and $\{G'_\ell\}_{\ell \geq 0}$. Let $T$ and $T'$ be trees with representations $[17]$ constructed as above. By Proposition $[27]$ up to passing to a subsequence we have a sequence of group inclusions

$$G = G_0 \supset G'_0 \supset G'_1 \supset G_2 \supset \cdots,$$

and so there are finite-to-one surjective maps $\alpha_\ell : G/G'_\ell \to G/G_\ell$ and $\beta_\ell : G/G_{\ell+1} \to G/G_\ell$, for $\ell \geq 0$, such that, for a given $\ell \geq 0$, the cardinality of preimage sets under $\alpha_\ell$ is constant for cosets in $G/G_\ell$, and a similar statement holds for $\beta_\ell$. It follows that there are finite-to-one maps $\alpha_\ell : V'_\ell \to V_\ell$ and $\beta_\ell : V_{\ell+1} \to V'_\ell$ which preserve the paths in $T$ and $T'$, and are equivariant with respect to the action of $G$. Taking the inverse limit of such maps, we obtain homeomorphisms $\alpha_\infty : \partial T \to \partial T'$ and $\beta_\infty = \alpha_\infty^{-1} : \partial T' \to \partial T$ which conjugate the action of $G$ on $\partial T$ and $\partial T'$.

From the above discussion and Remark $[28]$ we deduce a necessary condition for actions on two trees $T$ and $T'$ to be conjugate. Namely, let $n = (n_1, n_2, \ldots)$ and $n' = (n'_1, n'_2, \ldots)$ be spherical indices for $T$ and $T'$ respectively. Then there must exist subsequences $\{i_\ell\}_{\ell \geq 0}$ and $\{i'_\ell\}_{\ell \geq 0}$ such that for all $\ell \geq 0$ the product $n_1 n_2 \cdots n_{i_\ell}$ divides the product $n'_1 n'_2 \cdots n'_{i'_{\ell}}$, and $n'_1 n'_2 \cdots n'_{i'_{\ell}}$ divides $n_1 n_2 \cdots n_{i_{\ell+1}}$.

**Remark 3.2.** There exist actions which do not admit a representation on the boundary of a spherically homogeneous tree with bounded spherical index. For instance, suppose $\infty$ has spherical index $\infty = (p_1, p_2, \ldots)$ where $\{p_1, p_2, \ldots\}$ are distinct primes. Let $T$ be another tree, with bounded spherical homogeneity index $n$. Since for $\ell \geq 1$ we have $n_{i_\ell} \leq m$ for some $m \geq 0$, the set of divisors $\{ k \in \mathbb{N} \mid k | n_{i} \text{ for some } \ell \geq 0 \}$ is bounded, and it follows by the criterion obtained in Remark 3.1 that there is no homeomorphism between the path spaces $\partial T$ and $\partial T'$.

### 3.2. Weakly branch groups

We define a class of weakly branch groups, and show that if a subgroup $G \subset Homeo(\partial T)$ for a spherically homogeneous tree $T$ is weakly branch, then the action $(\partial T, G)$ is not LQA.

Let $v \in V_\ell$, $\ell \geq 1$, be a vertex in a tree $T$. Denote by $T_v$ the subtree of $T$ which contains all paths passing through $v$. These paths form a clopen subset of $\partial T$, which we denote by $\partial T_v$.
For an element \( g \in G \) let \( \text{supp}(g) \subset \partial T \) be the set of paths in \( T \) not fixed by \( g \). The rigid stabilizer of a vertex \( v \in V_\ell \) is the subgroup

\[
\text{Rist}(v) = \{ g \in G \mid \text{supp}(g) \subset \partial T_v \},
\]

and the rigid stabilizer of the \( \ell \)-th level is the subgroup

\[
\text{Rist}(\ell) = \langle \text{Rist}(v) \mid v \in V_\ell \rangle \cong \prod_{v \in V_\ell} \text{Rist}(v),
\]

where the last isomorphism holds since the rigid stabilizers of vertices at the same level commute.

Recall that an action \((X, G, \Phi)\) is effective if for any \( g \in G, g \neq e \), there is \( x \in X \) such that \( g \cdot x \neq x \). If \( G \) is defined as a subgroup of \( \text{Homeo}(X) \), then its action is automatically effective, as the only element in \( \text{Homeo}(X) \) which fixes all points in \( X \) is the identity map.

**Definition 3.3.** Let \( G \) be a discrete finitely generated group acting minimally and effectively on the boundary \( \partial T \) of a spherically homogeneous tree \( T \). Then \( G \) is weakly branch if for all \( \ell \geq 1 \), the rigid stabilizer \( \text{Rist}(\ell) \) of the level \( \ell \) is non-trivial.

We note the following property of rigid stabilizers.

**Lemma 3.4.** Let \( G \) be a discrete finitely generated group acting minimally and effectively on the boundary \( \partial T \) of a spherically homogeneous tree \( T \). Then for \( \ell \geq 1 \) the rigid stabilizer \( \text{Rist}(\ell) \) is non-trivial if and only if \( \text{Rist}(z) \) is non-trivial for any vertex \( z \in V_\ell \).

**Proof.** Clearly \( \text{Rist}(\ell) \) is non-trivial if and only if \( \text{Rist}(z) \) is non-trivial for some \( z \in V_\ell \). We show that if \( \text{Rist}(z) \) is non-trivial for some \( z \in V_\ell \), then \( \text{Rist}(v) \) is non-trivial for any \( v \in V_\ell \), which proves the statement. Indeed, since \( G \) acts minimally on \( \partial T \), it acts transitively on \( V_\ell \), and so there exists \( h \in G \) such that \( h \cdot z = v \). Then \( h \text{Rist}(z) h^{-1} \subset \text{Rist}(v) \), and \( \text{Rist}(v) \) is non-trivial.

We now show that a weakly branch action is not LQA.

**Lemma 3.5.** Let \( G \) be a discrete finitely generated group acting minimally and effectively on the boundary \( \partial T \) of a spherically homogeneous tree \( T \). Suppose \( G \) is weakly branch. Then the action \((\partial T, G)\) is not LQA.

**Proof.** Let \( \hat{v} = (v_\ell)_{\ell \geq 0} \in \partial T \) be a path. To show that \((\partial T, G)\) is not LQA we prove that there exists a sequence of neighborhoods \( \{U_\ell\}_{\ell \geq 0} \) of \( \hat{v} \) and, for each \( \ell \geq 0 \), an element \( g_\ell \in G \) such that the restriction \( g_\ell |_{U_{\ell+1}} = id \), while the restriction \( g_\ell |_{U_\ell} \) is not the identity.

Let \( U_\ell = \partial T_{v_\ell} \), that is, \( U_\ell \) contains all paths in the subtree \( T_{v_\ell} \). Let \( z_{\ell+1} \neq v_{\ell+1} \) be a vertex in \( V_{\ell+1} \) joined to \( v_\ell \) by an edge. Then \( \partial T_{v_{\ell+1}} \cap \partial T_{z_{\ell+1}} = \emptyset \), and \( \partial T_{v_{\ell+1}} \cup \partial T_{z_{\ell+1}} \subset \partial T_{v_\ell} \). By Lemma 3.4 we can choose a non-trivial \( g_{\ell+1} \in \text{Rist}(z_{\ell+1}) \). Its support is in \( \partial T_{z_{\ell+1}} \), so \( g_{\ell+1} |_{U_{\ell+1}} = id \), while \( g_{\ell+1} |_{U_\ell} \neq id \).

We say that an action \((X, G, \Phi)\) is weakly branch if \( \Phi(G) \subset \text{Homeo}(X) \) is a weakly branch group.

Examples of actions which are not LQA and not weakly branch are constructed in the proof of Theorem 1.4 in Section 5.

### 3.3. Special properties of \( d \)-ary trees.

Let \( T \) be a \( d \)-ary tree, with the boundary \( \partial T \) consisting of all infinite connected paths in \( T \). As is common in the literature, we label the vertices in \( T \) by finite words in the alphabet \( \{0, 1, \ldots, d-1\} \) as follows. The root in \( V_0 \) is not labelled; \( d \) vertices in \( V_1 \) are labelled by integers from 0 to \( d-1 \). Every vertex in the set \( V_\ell \) is labelled by a unique word of length \( \ell \). If \( v \in V_{\ell+1} \) and \( w \in V_\ell \) are joined by an edge, and \( v = s_1 s_2 \ldots s_{\ell+1} \), where \( s_i \in \{0, 1, \ldots, d-1\} \) for \( 1 \leq i \leq \ell+1 \), then \( w = s_1 s_2 \ldots s_\ell \). It follows that every element of the boundary \( \partial T \) can be uniquely represented by an infinite sequence \( s_1 s_2 \ldots \), where \( s_\ell \in \{0, 1, \ldots, d-1\} \), \( \ell \geq 1 \). More precisely, a path \( s = s_1 s_2 \ldots \) passes through a vertex labelled by \( s_1 \) in \( V_1 \), by \( s_1 s_2 \) in \( V_2 \) and, inductively, \( s \) passes through a vertex labelled by \( s_1 s_2 \ldots s_\ell \) in \( V_\ell \) for \( \ell \geq 1 \).
Let \( w = w_1 w_2 \cdots w_k \) be a word of length \( \ell \) with \( w_k \in \{0, 1, \ldots, d-1\} \), then \( w \) determines a vertex in \( V_\ell \). For simplicity, from now on we say that \( w \) is a vertex in \( V_\ell \). As in Section 3.2, denote by \( T_w \) a subtree of \( T \) which contains all infinite paths which start with the word \( w \), and by \( \partial T_w \) the corresponding clopen subset of \( \partial T \), that is,

\[
\partial T_w = \{ s_1 s_2 \cdots s_m \cdots \in \partial T \mid s_i = w_i \text{ for } 1 \leq i \leq \ell \}.
\]

Since for all \( i \geq 0 \) the labels \( w_i \) take values in the same set \( \{0, 1, \ldots, d-1\} \), for each \( w \in V_\ell \) there is a homeomorphism

\[
\psi_w : \partial T_w \rightarrow \partial T : w_1 w_2 \cdots w_\ell s_{\ell+1} s_{\ell+2} \cdots \mapsto s_{\ell+1} s_{\ell+2} \cdots.
\]

Note that \( \partial T = \partial T_0 \cup \partial T_1 \cup \cdots \cup \partial T_d-1 \). Let \( h \in \text{Aut}(T) \), and let \( \sigma_{h,1} \) be the non-trivial permutation of \( \{0, 1, \ldots, d-1\} \) induced by the action of \( h \) on \( T \). For \( 0 \leq k \leq d-1 \), define the section \( h|k \in \text{Aut}(T) \) of \( h \) by setting, for every infinite path \( s \in \partial T \),

\[
h|k(s) = \psi_k \circ h \circ \psi_k^{-1}(s).
\]

Then we can write the element \( h \) as the composition of maps

\[
h = (h|\sigma_{h,1}^{-1}(1)) \cdots (h|\sigma_{h,1}^{-1}(d-1)) \sigma_{h,1}.
\]

Intuitively, (22) splits the action of \( h \) on \( s \) into two stages: first we apply the permutation \( \sigma_{h,1} \) to \( V_1 \), and then a suitable automorphism to each subtree \( T_k \), for \( 0 \leq k \leq d-1 \).

**REMARK 3.6.** Let \( T \) be a \( d \)-ary tree, and let \( g \in \text{Aut}(T) \). For \( \ell \geq 0 \) we have \( |V_\ell| = d^\ell \), and we can use a similar procedure to the one described above to write \( g \) as the composition

\[
g = (g|\sigma_{g,1}^{-1}(1)) \cdots (g|\sigma_{g,1}^{-1}(d-1)) \sigma_{g,1},
\]

where \( \sigma_{g,\ell} \) is the permutation of vertices in \( V_\ell \) induced by the action of \( g \), and \( g|k \in \text{Aut}(T) \), \( 0 \leq k \leq d^\ell - 1 \) are the sections of \( g \) at vertices of \( V_\ell \).

**REMARK 3.7.** The representation of an automorphism of a tree \( T \) in (22) and in Remark 3.6 uses the fact that the tree \( T \) is \( d \)-ary, and so for any finite word \( w \) in the alphabet \( \{0, 1, \ldots, d-1\} \) there is a homeomorphism (20). This construction can be generalized to the case when \( T \) is not \( d \)-ary, but its spherical index \( n = (n_1, n_2, n_3, \ldots) \) is periodic, that is, there is \( m \geq 1 \) such that \( n_{i+m} = n_i \) for all \( i \geq 1 \). In that case the map (20) is defined for finite words \( w \) of length \( km \), and formula (23) is defined for \( \ell = km \), for integer \( k \geq 1 \). On the other hand, if, for instance, the spherical index \( n \) is strictly increasing, then (20) is never defined, and we cannot define elements of \( \text{Aut}(T) \) similarly to (23).

**REMARK 3.8.** The spherical index \( n = (n_1, n_2, \ldots) \) of a tree \( T \) is bounded, if there exists \( m \geq 0 \) such that \( n_i \leq m \) for all \( i \geq 1 \). A periodic spherical index as in Remark 3.7 is bounded, but a bounded spherical index need not be periodic. Indeed, let \( p, q \) be distinct primes, and define the spherical index \( n \) by \( n_2 = p \), and \( n_k = q \) for \( k \neq 2^i, i \geq 0 \). Then \( n \) is bounded but not periodic. Moreover, for a tree \( T \) with such spherical index, the map (20) is never defined, and it is not possible to define elements of \( \text{Aut}(T) \) similarly to (23).

## 4. Invariant random subgroups

In this section we recall the necessary background on invariant random subgroups (IRS). One can also consult, for instance, references [1, 21, 22].

Recall from the Introduction that we denote by \( \text{Sub}(G) \) the space of closed subgroups of a discrete finitely generated group \( G \) equipped with the Chabauty-Fell topology. Open sets in \( \text{Sub}(G) \) are given by (4). The space \( \text{Sub}(G) \) is a compact totally disconnected space, and \( G \) acts on \( \text{Sub}(G) \) by conjugation. By Definition 1.12 an invariant random subgroup \( \mu \) is a Borel probability measure on \( \text{Sub}(G) \).

Now, let \((X,G,\Phi)\) be an equicontinuous minimal action, and consider the mapping

\[
\text{St} : X \rightarrow \text{Sub}(G) : x \mapsto G_x,
\]
which assigns to each $x \in X$ its stabilizer. Stabilizers of points in the same orbit in $(X, G, \Phi)$ are conjugate, so (24) maps the orbit of $x$ in $X$ onto the orbit of $G_x$ in $\text{Sub}(G)$. The map (24) need not be injective. For instance, if $(X, G, \Phi)$ is a free action, then for any $x \in X$ we have $G_x = \{e\}$, where $e$ is the identity in $G$. In this case the image of (24) is the trivial subgroup.

The properties of the mapping (24) were studied in many works. We recall the following result.

**Lemma 4.1.** [33] Lemma 5.4] Let $G$ act on a Hausdorff topological space $X$ by homeomorphisms. Then

1. The mapping (24) is Borel measurable.
2. The mapping (24) is continuous at $x \in X$ if and only if $[G]_x = G_x$, that is, $x$ is a point without holonomy.
3. If a sequence of points $\{x_\ell\}_{\ell \geq 0}$ converges to $x \in X$, and the sequence of stabilizers $\{G_{x_\ell}\}_{\ell \geq 0}$ converges to a closed subgroup $H \in \text{Sub}(G)$, then $[G]_x \subset H \subset G_x$.

Since a minimal equicontinuous action $(X, G, \Phi)$ is uniquely ergodic with invariant measure $\mu$, it follows that the pushforward $\nu = \text{St}_x \mu$ is an IRS. For instance, if $(X, G, \Phi)$ is a free action, then $\nu$ is an atomic measure supported on a single point in $\text{Sub}(G)$. At the other extreme, if $X = \partial T$ is a boundary of a $d$-ary tree $T$, and $G$ is weakly branch, then the stabilizers of all points in $X$ are pairwise distinct [7] Proposition 8], and it follows that $\nu$ is non-atomic.

By Lemma 4.1 the set $X_0 = \{x \in X \mid G_x = [G]_x\}$, earlier defined in [3], contains all points at which the mapping (24) is continuous. Further, recall that $Z = \{G_x \mid x \in X_0\}$, see [3].

**Lemma 4.2.** [25] Proposition 1.2] For $(X, G, \Phi)$ minimal, $Z$ is the unique minimal subset in

$$\{G_x \mid x \in X\} \subset \text{Sub}(G).$$

We now study metric properties of the action of $G$ on $\text{Sub}(G)$. For that we define a metric on $\text{Sub}(G)$ as in [1] Section 3].

Let $S$ be a finite symmetric generating set for $G$. Given a subgroup $H \subset G$, construct the Schreier graph $\Gamma_{G/H}$ as follows: the cosets in $G/H$ are the vertices of $\Gamma_{G/H}$, and two vertices $hH$ and $gH$ are joined by an edge, directed from $hH$ to $gH$ and labeled by $s \in S$ if and only if $gH = shH$. We define a length structure on $\Gamma_{G/H}$ by assigning length 1 to each edge. The length structure defines a length metric $D_{\Gamma_{G/H}}$ on $\Gamma_{G/H}$, such that the distance between any two points in $\Gamma_{G/H}$ is equal to the length of the shortest path between these points. The graph $\Gamma_{G/H}$ has a distinguished vertex which is the coset of the identity $eH$, so $\Gamma_{G/H}$ is a pointed metric space. Denote by $B_{\Gamma_{G/H}}(r)$ a metric ball of radius $r$ in $\Gamma_{G/H}$ centered at $eH$. Given $H_1, H_2 \subset \text{Sub}(G)$, the metric balls $B_{\Gamma_{G/H_1}}(r)$ and $B_{\Gamma_{G/H_2}}(r)$ are isomorphic if and only if there exists an isometry $f : B_{\Gamma_{G/H_1}}(r) \to B_{\Gamma_{G/H_2}}(r)$ which preserves the labelling of edges and such that $f(eH_1) = eH_2$.

Denote the set of all Schreier graphs associated to closed subgroups of $G$ by

$$\text{Sch}(G, S) = \{\Gamma_{G/H} \mid H \in \text{Sub}(G)\}.$$ 

We define a metric on $\text{Sch}(G, S)$ by setting

$$D_{\text{Sch}}(\Gamma_{G/H_1}, \Gamma_{G/H_2}) = \frac{1}{2^k}, \quad k = \max\{r \geq 0 \mid B_{\Gamma_{G/H_1}}(r) \text{ and } B_{\Gamma_{G/H_2}}(r) \text{ are isomorphic}\}.$$ 

The metric space $\text{Sch}(G, S)$ is compact. The action of $G$ on $\text{Sch}(G, S)$ is defined as follows: for $g \in G$ we set $g \cdot \Gamma_{G/H} = \Gamma_{G/gh^{-1}}$. It is fairly straightforward to see that if distinguished vertices are not taken into account, then the graphs $\Gamma_{G/H}$ and $\Gamma_{G/gh^{-1}}$ are isomorphic, so we can think about the action of $G$ on $\Gamma_{G/H}$ as moving the distinguished vertex to the coset of $gH$. It is immediate that the map

$$\text{Sub}(G) \to \text{Sch}(G, S) : H \mapsto \Gamma_{G/H}$$

is a homeomorphism which commutes with the action of $G$ on $\text{Sub}(G)$ and $\text{Sch}(G, S)$. 
THEOREM 5.1. We restate Theorem 1.4 for the convenience of the reader. We prove Theorems 1.4 and 1.5.

A metric on the space Sub(G) can be defined as a pullback of the metric (25) along the map (26). Such a metric depends on the choice of the generating set S. Yet, the conclusion of Lemma 4.3 is true for any finite generating set, i.e., it is independent of the particular choice.

5. STABILIZERS OF ACTIONS

For a minimal equicontinuous action \((X,G,\Phi)\), we study how the number of distinct stabilizers of points in \(X\) depends on whether the action is locally quasi-analytic (LQA), see Definition 1.3. We prove Theorems 1.4 and 1.5.

We restate Theorem 1.4 for the convenience of the reader.

THEOREM 5.1. There exist minimal equicontinuous actions which are not LQA, and which contain distinct points with equal stabilizers. More precisely, given any minimal equicontinuous action \(\Phi\), which is not LQA, there exists a minimal equicontinuous action \((Y,G,\Phi)\), where \(Y = X \times \{0,1\}\), such that \((Y,G,\Phi)\) is not LQA, and for any \(y \in Y\) there is at least two points \(x_0,x_1 \in X\) such that \(G_y = \tilde{G}_{x_0} = \tilde{G}_{x_1}\), where \(G_y,\tilde{G}_{x_0},\tilde{G}_{x_1}\) are stabilizers of the respective actions.

Proof. The space \(X = Y \times \{0,1\}\) is a Cantor set. For each \(g \in G\) define a homeomorphism \(\tilde{g} : X \to X\) by

\[
\tilde{g}(y,k) = (g(y),k),
\]

that is, the action of \(\tilde{g}\) preserves the second component. Define the homeomorphism \(z : X \to X\) by

\[
z(y,k) = (y,k+1 \mod 2),
\]

so \(z\) preserves the first component of the product space \(X\), and interchanges 0 and 1 in the second component. Then \(z\) commutes with \(\tilde{g}\), for any \(g \in G\). Let

\[
\tilde{G} = \langle z, \{\tilde{g} \mid g \in G\} \rangle,
\]

that is, \(\tilde{G}\) is generated by elements defined by (27) for \(g \in G\), and by \(z\). In particular, if \(\{g_0,\ldots, g_n\}\) is a generating set of \(G\), then \(\{\tilde{g}_0,\ldots, \tilde{g}_n, z\}\) is a generating set of \(\tilde{G}\).

By the above, we have defined an action \(\tilde{\Phi} : \tilde{G} \to \text{Homeo}(X)\). The action \((X,\tilde{G},\tilde{\Phi})\) is minimal, since the action \((Y,G,\Phi)\) is minimal. It is easy to see that since \((Y,G,\Phi)\) is equicontinuous, then \((X,\tilde{G},\tilde{\Phi})\) is also equicontinuous, using, for instance, Proposition 2.3 and finiteness of \(\{0,1\}\).

Since \((Y,G,\Phi)\) is not LQA, for any \(y \in Y\) there exists a sequence of open neighborhoods \(\{U_t\}_{t \geq 0} \subset Y\) and a sequence of elements \(E = \{g_t\}_{t \geq 0} \subset G\) such that \(g_t|U_t \neq id\) and \(g_t|U_{t+1} = id\). For \((y,k) \in X\), consider a collection \(\{U_t \times \{k\}\}_{t \geq 0}\) of clopen neighborhoods of \((y,k)\), and a collection of elements \(\{\tilde{g}_t\}_{t \geq 0}\), where \(\tilde{g}_t\) is defined by (27) with \(g_t \in E\). Then \(\tilde{g}_t|U_t \times \{k\} \neq id\), and \(\tilde{g}_t|(U_{t+1} \times \{k\}) = id\), so the action \((X,\tilde{G},\tilde{\Phi})\) is not LQA.
Fix a generating set \( \{ \tilde{g}_0, \ldots, \tilde{g}_n, z \} \) of \( \tilde{G} \), and note that \( z^{-1} = z \). Let \( y \in Y \), \( x_0 = (y, 0) \) and \( x_1 = (y, 1) \). Consider the stabilizers \( \tilde{G}_{x_0} \) and \( \tilde{G}_{x_1} \). We can represent \( \tilde{h} \in \tilde{G}_{x_0} \) as a finite word

\[
\tilde{h} = z^{\alpha_0} \tilde{g}_{i_0} z^{\alpha_1} \tilde{g}_{i_1} \cdots z^{\alpha_m} \tilde{g}_{i_m},
\]

where \( m \) is a natural number, \( \alpha_s \in \{0,1\} \) and \( \tilde{g}_{i_s} \) is a generator for \( 0 \leq s \leq m \). Since \( \tilde{h} \) fixes \( x_0 \) and \( z \) commutes with other generators of \( \tilde{G} \), the representation (29) reduces to

\[
\tilde{h} = \tilde{g}_{i_0} \tilde{g}_{i_1} \cdots \tilde{g}_{i_m},
\]

and by (27) this means that \( g_{i_0} g_{i_1} \cdots g_{i_m}(y) = y \). Then

\[
\tilde{h}(x) = \tilde{h}(y) = (g_{i_0} g_{i_1} \cdots g_{i_m}(y), 1) = (y, 1) = x_1,
\]

and so \( \tilde{h} \in \tilde{G}_{x_1} \). Thus \( \tilde{G}_{x_0} \subset \tilde{G}_{x_1} \). The reverse inclusion follows by a similar argument, and we have \( \tilde{G}_{x_0} = \tilde{G}_{x_1} \).

Recall from the Introduction that a point \( x \in X \) has trivial holonomy, or is without holonomy, if \( [G]_x = G_x \), where \( [G]_x \) denotes the neighborhood stabilizer of \( x \). If \( [G]_x \subset G_x \) is a proper inclusion, then \( x \) has non-trivial holonomy, or is a point with holonomy. Recall from the Introduction that we denote the subset of points with trivial holonomy by

\[ X_0 = \{ x \in X \mid [G]_x = G_x \}. \]

We restate Theorem 1.5 for the convenience of the reader.

**THEOREM 5.2.** Let \((X, G, \Phi)\) be a minimal equicontinuous system. Then the set \( \{G_x \mid x \in X_0\} \) is finite if and only if the action \((X, G, \Phi)\) is locally quasi-analytic.

**Proof.** In the following, let \( x \in X_0 \), and let \( U_\ell = \{ U_\ell \}_{\ell \geq 0}, U_0 = X, \) be an adapted neighborhood system at \( x \), and recall from Section 2.2 that the set \( G_\ell = \{ g \in G \mid g \cdot U_\ell = U_\ell \} \) is a group. Denote by \( \Phi_\ell : G_\ell \to \text{Homeo}(U_\ell) \), or by \( (U_\ell, G_\ell, \Phi_\ell) \) the action of \( G \) restricted to \( U_\ell \), for \( \ell \geq 0 \).

Suppose \((X, G, \Phi)\) is LQA with \( \epsilon \geq 0 \), and let \( \ell \geq 0 \) be such that \( \text{diam}(U_\ell) < \epsilon \). Then for any open set \( W \subset U_\ell \), if \( g|W = id|W \), then \( g|U_\ell = id|U_\ell \). In particular, if \( y \in U_\ell \) is without holonomy, then every element which fixes every point in an open neighborhood of \( y \) in \( U_\ell \) must fix every point in \( U_\ell \). So \( G_y = \ker(\Phi_\ell : G_\ell \to \text{Homeo}(U_\ell)) \), for all \( y \in X_0 \cap U_\ell \). The group \( \ker(\Phi_\ell) \) is a normal subgroup of \( G_\ell \), but it need not be normal in \( G \).

Let \( g \notin G_\ell \), then \( \hat{x} = g \cdot x \in g \cdot U_\ell \), with \( g \cdot U_\ell \cap U_\ell = \emptyset \). Moreover, \( G_{\hat{x}} = gG_xg^{-1} \), and \( h \in G \) fixes an open neighborhood of \( x \) if and only if \( ghg^{-1} \) fixes an open neighborhood of \( \hat{x} \), so \( x \) is a point without holonomy in \( g \cdot U_\ell \).

Let \( \hat{y} \in g \cdot U_\ell \cap X_0 \) be another point without holonomy. Then there exists a unique \( y \in U_\ell \cap X_0 \) such that \( \hat{y} = g \cdot y \). Then we have

\[ G_{\hat{y}} = gG_yg^{-1} = gG_xg^{-1} = G_{\hat{x}}, \]

so for all \( \hat{y} \in g \cdot U_\ell \cap X_0 \) the stabilizers are equal, \( G_{\hat{y}} = g \ker(\Phi_\ell)g^{-1} \). Since the orbit of \( U_\ell \) under the action of \( G \) is finite, \( \ker(\Phi_\ell) \) has a finite number of distinct conjugates in \( G \), and the set \( \{G_x \mid x \in X_0\} \) is finite.

Now suppose that the action \((X, G, \Phi)\) is not LQA. Given \( x \in X \), we show that the set \( \{G_xg \mid g \in G\} \) is infinite by induction. Namely, for each \( \ell \geq 1 \) we find a subset of the orbit \( G(x) = \{ z \in X \mid z = g \cdot x, g \in G \} \) of cardinality \( 2^\ell \) such that all points in this subset have pairwise distinct stabilizers.

Since \((X, G, \Phi)\) is not LQA, there exists an element \( g_0 \in G \) which satisfies \( g_0|U_1 = id \), and such that \( g_0|(X - U_1) \) is not the identity. Choose \( z \in X - U_1 \) such that \( g_0 \cdot z \neq z \). By continuity there is an open neighborhood \( O \ni z \) such that \( g_0 \) fixes no point in \( O \). Choose an index \( s_1 \geq 1 \) large enough so that for some \( h_{s_1} \in G \) we have \( z \in h_{s_1} \cdot U_{s_1} \subset O \), then for any \( z' \in h_{s_1} \cdot U_{s_1} \) we have \( g_0 \cdot z' \neq z' \).
Set $W_0 = U_{s_1}$, and $W_1 = h_{s_1} \cdot U_{s_1}$. Then for any $z \in W_0$ we have $g_1 \in G_z$, in particular, for $y_0 = x$. For any $z \in W_1$ we have $g_1 \notin G_z$. Since the action is minimal and the set $W_1$ is open, we can choose $y_1 \in G(x) \cap W_1$.

Now suppose we are given a finite collection of clopen sets $W_{k_1}, W_{k_2}, \ldots, W_{k_{j_k}}$, where $k_i \in \{0, 1\}$ for $1 \leq i \leq n$. That is, we have a collection of clopen sets labelled by finite words of length $1 \leq i \leq n$, consisting of 0’s and 1’s. We assume that this collection has the following properties:

1. For every $1 \leq i < n$ and every word $k_1 \cdots k_{j_1+1}$ we have an inclusion $W_{k_1} \cdots k_{j_1+1} \subset W_{k_1} \cdots k_i$. That is, every set $W_{k_1} \cdots k_i$ labelled by a word of length $i$ contains precisely two clopen sets labelled by words of length $i + 1$, $W_{k_1} \cdots k_i$ and $W_{k_1} \cdots k_i \cdot 0$.
2. For every $1 \leq i < n$ and every word $k_1 k_2 \cdots k_i$ there is an element $g_{k_1 k_2} \cdots k_0 \in G$ such that the restriction $g_{k_1 k_2} \cdots k_0 | W_{k_1 k_2} \cdots k_0$ is the identity, and for every $z \in W_{k_1 k_2} \cdots k_1$ we have $g_{k_1 k_2} \cdots k_0 \cdot z = z$.
3. For every $W_{k_1 k_2} \cdots k_n$, we choose a point $y_{k_1 k_2} \cdots k_n \in W_{k_1 k_2} \cdots k_n \cap G(y_0)$, where $y_0 = x$ by the choice made at the first step of the construction.

We claim that for any two distinct points $y_{j_1 j_2 \cdots j_n}$ and $y_{k_1 k_2} \cdots k_n$ in the collection given by property (3), the stabilizers $G_{y_{j_1 j_2 \cdots j_n}}$ and $G_{y_{k_1 k_2} \cdots k_n}$ are distinct. Indeed, consider the words $j_1 j_2 \cdots j_n$ and $k_1 k_2 \cdots k_n$, and let $s$ be the first digit such that $j_s \neq k_s$. Without loss of generality, we can assume that $j_s = 0$ and $k_s = 1$. Then by (1) and (2) we have $y_{j_1 j_2 \cdots j_{s-1} 0} \in G_{y_{j_1 j_2 \cdots j_n}}$ and $y_{j_1 j_2 \cdots j_{s-1} 0} \notin G_{y_{k_1 k_2} \cdots k_n}$, so $G_{y_{j_1 j_2 \cdots j_n}} \neq G_{y_{k_1 k_2} \cdots k_n}$.

We now enlarge the collection of points in condition (3) by implementing the inductive step.

For each $k_1 k_2 \cdots k_n$ we have $y_{k_1 k_2} \cdots k_n \in W_{k_1 k_2} \cdots k_n$, where $y_{k_1 k_2} \cdots k_n \in G(x)$. Since the action is not LQA, there exists a clopen neighborhood $W$ of $y_{k_1 k_2} \cdots k_n$ properly contained in $W_{k_1 k_2} \cdots k_n$, and an element $g_{k_1 k_2} \cdots k_0 \in G$ such that the restriction $g_{k_1 k_2} \cdots k_0 | W$ is the identity, while the restriction $g_{k_1 k_2} \cdots k_0 | (W_{k_1 k_2} \cdots k_n - W)$ is not. Let $z \in W_{k_1 k_2} \cdots k_n - W$, so that $g_{k_1 k_2} \cdots k_0 (z) \neq z$. By continuity there is a neighborhood $W' \supset z$ such that for every $z' \in W'$ we have $g_{k_1 k_2} \cdots k_0 (z') \neq z'$.

Choose an index $s_{n+1} \geq 0$ and $h_0, h_1 \in G$ such that $y_{k_1 k_2} \cdots k_n \in h_0 \cdot U_{s_{n+1}} \subset W$ and $h_1 \cdot U_{s_{n+1}} \subset W'$, for an adapted neighborhood $U_{s_{n+1}} \subset U_2$. Set $W_{k_1 k_2} \cdots k_n 0 = h_0 \cdot U_{s_{n+1}}$, and $W_{k_1 k_2} \cdots k_n 0 \cdot 1 = h_1 \cdot U_{s_{n+1}}$. Then for every point $y' \in W_{k_1 k_2} \cdots k_n 0$ we have $g_{k_1 k_2} \cdots k_0 \in G_{y'}$, and for every $z' \in W_{k_1 k_2} \cdots k_n 0$ we have $g_{k_1 k_2} \cdots k_0 \notin G_{y'}$. We set $y_{k_1 k_2} \cdots k_n 0 = y_{k_1 k_2} \cdots k_n$. Since the action is minimal, we can choose $z' \in G(x) \cap W_{k_1 k_2} \cdots k_n 0$, and then set $y_{k_1 k_2} \cdots k_n 0 = z'$. The collections of neighborhoods $W_{k_1}, W_{k_1 k_2}, \ldots, W_{k_1 k_2 \cdots k_{n+1}}$, of elements $g_0, g_1, \ldots, g_{k_1 k_2} \cdots k_{n+1}$, and of points $y_{k_1 k_2} \cdots k_{n+1}$, we have constructed satisfies (1)-(3).

For $n \geq 1$, we have $\# \{ y_{k_1 k_2} \cdots k_{n+1} | k_i \in \{0, 1\}, 1 \leq i \leq n \} = 2^n$, and all points in this set are contained in the orbit $G(x)$ and have distinct stabilizers. It follows $\{ G_x | x \in X_0 \}$ is infinite.

6. Points with non-trivial holonomy

In this section we prove Theorem 1.13. Before we start we recall the conditions in the hypothesis of Theorem 1.13 and a few results needed in the proof.

6.1. Bounded spherical index. Recall from Section 4.4 that every minimal equicontinuous group action $(X, G, \Phi)$ admits a representation as an action on the boundary of a tree. More precisely, there exists a spherically homogeneous tree $T$ with spherical index $n = (n_1, n_2, \ldots)$, and a homeomorphism $\phi : X \to \partial T$, where $\partial T$ denotes the boundary of $T$, which induces an identification of every $g \in G$ with an automorphism of $T$. Thus $\phi$ induces an injective homomorphism $\phi_* : \Phi(G) \to \text{Homeo}(\partial T)$, and so a tree representation is given by a pair of maps $(\phi, \phi_* : (X, \Phi(G)) \to (\partial T, \text{Homeo}(\partial T))$.

The spherical index $n = (n_1, n_2, \ldots)$ is bounded, if there exists $m \geq 2$ such that for any $\ell \geq 1$ we have $m \geq n_\ell$. The first hypothesis of Theorem 1.13 is that the action $(X, G, \Phi)$ admits a representation on a tree with bounded spherical index.
6.2. Uniformly non-constant actions. Let $T$ be a spherically homogeneous tree. For a vertex $v_\ell \in V_\ell$, we denote by $T_{v_\ell}$ the subtree through the vertex $v_\ell$, and by $\partial T_{v_\ell}$ the set of paths in $\partial T$ which contain $v_\ell$. The set $\partial T_{v_\ell}$ is a clopen subset of $\partial T$. We recall Definition 4.3 from the Introduction, which we restate now for the convenience of the reader.

**DEFINITION 6.1.** Let $T$ be a spherically homogeneous tree, and let $G \subset \text{Homeo}(\partial T)$ be a group. The action of $g \in G$ on $\partial T$ is uniformly non-constant if the following holds: there are positive integers $n_g$ and $K_g$ such that for any $\ell \geq n_g$ and any vertex $v_\ell \in V_\ell$, if $g$ fixes $v_\ell$ and $g|\partial T_{v_\ell} \neq \text{id}$, then there exists $m \geq \ell$ and a vertex $w_m \in V_m \cap T_{v_\ell}$ such that $g \cdot w_m \neq w_m$ and $m - \ell \leq K_g$.

The action of the group $G$ is uniformly non-constant if and only if the action of every $g \in G$ is uniformly non-constant.

**REMARK 6.2.** Intuitively, Definition 6.1 means that, given $g \in G$, for each clopen set $W \subset \partial T$ such that $g|W \neq \text{id}$, the size of the set $U \subset W$ where every point is moved by the action of $g \in G$ is proportional to the size of $W$. For example, let $\partial T$ be given a metric induced from (11), that is, for two infinite sequences $s = s_1s_2 \cdots$ and $t = t_1t_2 \cdots$

$$D((s_\ell)_{\ell \geq 0}, (t_\ell)_{\ell \geq 0}) = \frac{1}{2^m},$$

where $m = \min \{ \ell \geq 0 \mid s_\ell \neq t_\ell \}$.

Note that if $g \cdot w_m \neq w_m$, then $g$ does not fix any point in the clopen set $\partial T_{w_m}$. Then Definition 6.1 can be reformulated as follows: the action of $G$ on $\partial T$ is uniformly non-constant if and only if for every $g \in G$ there are integers $K_g > 0$ and $n_g > 0$ such that the following holds: for any $\ell \geq n_g$ and any vertex $v_\ell \in V_\ell$ if $g$ fixes $v_\ell$ and $g|\partial T_{v_\ell} \neq \text{id}$, then there exists a vertex $w_m \in V_m \cap T_{v_\ell}$ such that $g \cdot w_m \neq w_m$ and $\text{diam}(\partial T_{w_m})/\text{diam}(\partial T_{v_\ell}) \geq 2^{-K_g}$.

**EXAMPLE 6.3.** We give an example of a uniformly non-constant action. Let $T$ be a $d$-ary tree, and let $G \subset \text{Aut}(T)$ be a group acting on $T$, and suppose that the action of any $g \in G$ can be computed by a finite automaton [38]. This is equivalent to saying that for every $g \in G$ the set of sections $S_g = \{ g|_v \mid v \in V_\ell \text{ for all } \ell \geq 0 \}$ is finite [38, Section 1.3].

Number the elements in $S_g$, that is, $S_g = \{ h_1, \ldots, h_k \}$ for some $k \geq 0$. For each $1 \leq i \leq k$, there exists $\ell_i \geq 0$ and a vertex $w_{\ell_i} \in V_{\ell_i}$ such that $h_i(w_{\ell_i}) \neq w_{\ell_i}$. Then $h_i$ does not fix any point in the subtree $\partial T_{w_{\ell_i}}$. Let

$$K_g = \max \{ \ell_i \mid 1 \leq i \leq k \}.$$ 

Now, suppose $g \in G$ fixes a vertex $v \in V_\ell$. Then $g|\partial T_v = g|_v = h_i$ for some $1 \leq i \leq k$, and Definition 6.1 is satisfied with constant $K_g$ defined by (32), and $n_g = 1$. We conclude that the action $(\partial T, G)$ is uniformly non-constant.

An automaton is bounded if the set of sections $\{ g|_v \mid g|_v \neq \text{id}, v \in V_\ell \}$ has uniformly bounded cardinalities over all $\ell \geq 1$. Actions generated by bounded finite automata is a subclass of the action generated by finite automata, and so the action of $G$ is uniformly non-constant.

Example 6.4 describes an action which is not uniformly non-constant.

**EXAMPLE 6.4.** Let $T$ be a $d$-ary tree, for $d \geq 3$, and let $H \subset \text{Aut}(T)$ be a group which acts minimally and equicontinuously on $\partial T$. We define an element $c \in \text{Aut}(T)$ whose action on $\partial T$ is not uniformly non-constant. Then the action of $G = \langle c, H \rangle$ on $\partial T$ is not uniformly non-constant.

We define the action of $c$ inductively on each vertex set $V_\ell$, $\ell \geq 1$. On $V_1$, $c$ permutes the vertices labelled by 0 and 1, and fixes the remaining vertices. Then for all $k \geq 1$ and all $d^k < i \leq d^{k+1} - 1$, define the action of $c$ as follows. Let $v \in V_1$, and let $w \in V_{i-1}$ be the unique vertex joined to $v$ by an edge, that is, $v = ws$, where $0 \leq s < d - 1$. Then define $c \cdot ws = (c \cdot w)s$, that is, the action preserves the $i$-th digit, and it may or may not move one of the preceding digits. For $i = d^k$, if $c \cdot w \neq w$, define $c \cdot ws = (c \cdot w)s$. If $c \cdot w = w$, then set

$$c \cdot w0 = w1, c \cdot w1 = w0, c \cdot ws = ws$$

for $s \in \{2, \ldots, d - 1\}$,

so the action fixes all vertices connected to $w$ except $w0$ and $w1$. Since the action of $H$ is minimal, the action of $G$ is minimal. The elements of $G$ act by permutation on every $V_\ell$, $\ell \geq 0$, so the action
of $G$ on $\partial T$ is equicontinuous. For every $k \geq 0$, there exists $v_k \in V_d$ such that $g \cdot v_k = v_k$, and for every $d^k + 1 \leq i \leq d^{k+1} - 1$ every vertex in $T_{v_k} \cap V_\ell$ is fixed by the action of $c$. Since $d^k \to \infty$, this implies that the action of $c$ on $\partial T$ is not uniformly non-constant. It follows that the action of $G$ on $\partial T$ is not uniformly non-constant.

6.3. Lebesgue density. Let $(X, G, \Phi, \mu)$ be a minimal equicontinuous action, where $\mu$ is an invariant probability measure. As we discussed in Section 2.4 a choice of an adapted neighborhood system for the action allows us to give $X$ an ultrametric $D$, given by (11) or (12). A Cantor set $X$ is a Polish space, and so the following result of Miller [37] applies.

**THEOREM 6.5.** [37] Proposition 2.10] Let $X$ be a Polish space, and suppose $X$ has an ultrametric $D$ compatible with its topology. Let $\mu$ be a probability measure on $X$, and let $A$ be a Borel set of positive measure. Then the Lebesgue density of $x$ in $A$, given by

$$(34) \quad \lim_{\epsilon \to 0} \frac{\mu(A \cap B(x, \epsilon))}{\mu(B(x, \epsilon))}$$

exists and is equal to 1 for $\mu$-almost every $x \in A$.

6.4. Proof of Theorem [1.9] For the convenience of the reader we restate Theorem 1.9.

**THEOREM 6.6.** Let $(X, G, \Phi, \mu)$ be a minimal equicontinuous group action. Suppose $(X, G, \Phi, \mu)$ admits a representation $(\phi, \phi_\ast) : (X, \Phi(G)) \to (\partial T, \text{Homeo}(\partial T))$ on the boundary of a spherically homogeneous tree $T$ which satisfies the following two conditions:

1. The tree $T$ has bounded spherical index.
2. The action of $\phi_\ast(\Phi(G))$ on $\partial T$ is uniformly non-constant.

Then the set $X - X_0$ of points with non-trivial holonomy has zero measure.

Proof. Given $g \in G$, let $\text{Fix}(g) = \{x \in X \mid g \cdot x = x\}$ be the set of fixed points, containing both points with and without holonomy. By continuity, if $g \cdot z \neq z$, then there is an open neighborhood $W$ of $z$ such that for every $z' \in W$ we have $g \cdot z' \neq z'$. Therefore, the complement of $\text{Fix}(g)$ is open, and $\text{Fix}(g)$ is Borel. The set $\text{Fix}(g)$ may have positive measure, or measure zero.

Suppose $g \cdot x = x$. We say that $g$ has trivial holonomy at $x \in X$ if $g$ fixes every point in an open neighborhood of $x$. If $g$ acts non-trivially on every open neighborhood of $x$, then $g$ has non-trivial holonomy at $x$. We will show that, under the hypothesis of the theorem, the subset

$$\{x \in \text{Fix}(g) \mid g \text{ has trivial holonomy at } x \in X\}$$

of $\text{Fix}(g)$ has positive measure in $X$, while the subset

$$\{x \in \text{Fix}(g) \mid g \text{ has non-trivial holonomy at } x \in X\}$$

of $\text{Fix}(g)$ has zero measure in $X$. In particular, $\text{Fix}(g)$ can have positive measure if and only if the element $g$ fixes an open set in $X$.

First, suppose $g$ has trivial holonomy at $x \in \text{Fix}(g)$, and so fixes every point in $B(x, \epsilon)$ for some $\epsilon > 0$. Then the Lebesgue density of $x$ in $\text{Fix}(g)$, given by (34), exists and is equal to 1.

Next, suppose $g$ has non-trivial holonomy at $x \in \text{Fix}(g)$. We will show that the Lebesgue density of $\text{Fix}(g)$ at $x$ is bounded away from 1.

Let $n = (n_1, n_2, \ldots)$ be the spherical index of the tree $T$, given by the hypothesis of the theorem. Since $n$ is bounded, there exists $M > 0$ such that $n_\ell \leq M$ for all $\ell \geq 0$. Let $(v_\ell)_{\ell \geq 0} = \phi(x)$, and for $\ell \geq 0$, let $U_\ell = \partial T_{v_\ell}$ be the clopen neighborhood of $(v_\ell)_{\ell \geq 0}$, consisting of all paths which contain the vertex $v_\ell$. Assume that $\ell \geq n_g$, where $n_g$ is the integer given by Definition 6.1.

By Section 2.4 the measure of $U_\ell$ is given by

$$\mu(U_\ell) = \frac{1}{n_1 n_2 \cdots n_\ell}. $$
Since the action is uniformly non-constant, there exists a vertex \( w_{m_\ell} \in U_\ell \cap V_{m_\ell} \), for some \( m_\ell > \ell \), such that \( g \cdot w_{m_\ell} \neq w_{m_\ell} \) and \( m_\ell - \ell < K_g \), where \( K_g \) is the constant given by Definition 6.1. Then
\[
\mu(\partial T_{w_{m_\ell}}) = \frac{1}{n_1 n_2 \cdots n_{\ell+1} \cdots n_{m_\ell}} > \frac{1}{n_1 \cdots n_{\ell+K_g}}.
\]
Since the neighborhood \( \partial T_{w_{m_\ell}} \) contains no points fixed by the action of \( g \), we have the estimate
\[
\mu(\text{Fix}(g) \cap U_\ell) < \mu(U_\ell) - \mu(\partial T_{w_{m_\ell}}) < \frac{1}{n_1 \cdots n_\ell} - \frac{1}{n_1 \cdots n_{\ell+K_g}}.
\]
Using the bound on the spherical index of \( T \) given by \( n_\ell \leq M \) for \( \ell \geq 0 \), we obtain
\[
\frac{\mu(\text{Fix}(g) \cap U_\ell)}{\mu(U_\ell)} < 1 - \frac{1}{n_{\ell+1} \cdots n_{\ell+K_g}} < 1 - \frac{1}{MK_g}.
\]
It follows that the Lebesgue density of \( \text{Fix}(g) \) at the point \( x \) is bounded away from 1. Then by Theorem 6.3 either \( \text{Fix}(g) \) has positive measure, and the subset of \( \text{Fix}(g) \) of points with non-trivial holonomy has measure zero in \( X \), or \( \text{Fix}(g) \) only contains points with non-trivial holonomy, and has measure zero. So in any case \( \text{Fix}(g) \) has measure zero. Since the group \( G \) is countable, the union
\[
\bigcup_{g \in G} \{ x \in \text{Fix}(g) \mid g \text{ has non-trivial holonomy at } x \in X \}
\]
is a countable union of zero measure sets, and so has zero measure. \( \square \)

Corollary 1.10 is a reformulation of Theorem 1.9 for the case of topologically free actions.

Kambites, Silva and Steinberg [32] studied topologically and essentially free actions on \( d \)-ary trees generated by finite automata. Since \( d \)-ary trees have bounded spherical index and an action generated by finite automata is uniformly non-constant, we recover Theorem 4.3 of [32] below as a consequence of Theorem 1.9.

**COROLLARY 6.7.** [32 Theorem 4.3] Let \( G \) be a group acting on a \( d \)-ary tree \( T \), such that the action of \( g \in G \) are computed by finite state automata. Then the action \( (\partial T, G) \) is topologically free if and only if it is essentially free.

### 6.5. Examples and counterexamples

We give examples illustrating Theorem 1.9.

Let \( T \) be a tree with a spherical index \( n = (n_1, n_2, \ldots) \). Similarly to the case when \( T \) is a \( d \)-ary tree in Section 3.3 we label vertices in \( V_\ell \), \( \ell \geq 1 \), by finite words of length \( \ell \). More precisely, for \( v \in V_\ell \) we have \( v = w_1 w_2 \cdots w_\ell \), where \( w_i \in \{0,1,\ldots,n_\ell - 1\} \). Then infinite paths in \( \partial T \) are in bijective correspondence with infinite sequences \( w_1 w_2 \cdots \), where \( w_\ell \in \{0,1,\ldots,n_\ell - 1\} \). Such a path passes through the vertices \( v_\ell = w_1 \cdots w_\ell \), \( \ell \geq 1 \), so we also denote it by \((v_\ell)_{\ell \geq 1}\).

**EXAMPLE 6.8.** Let \( T \) be a tree with unbounded spherical index, for example, take \( n = (p_1, p_2, \ldots) \), where \( p_\ell \) for \( \ell \geq 1 \) are distinct primes. We define the generator of the adding machine action on \( T \) in the standard way by
\[
a(w_1 w_2 \cdots) = \begin{cases} 
(w_1 + 1) w_2 \cdots & \text{if } w_1 \neq p_1 - 1, \\
00 \cdots 0 (w_{k+1} + 1) w_{k+2} \cdots & \text{if } w_i = p_1 - 1 \text{ for } 1 \leq i \leq k, \text{ and } w_{k+1} \neq p_{k+1} - 1, \\
00 \cdots & \text{if } w_k = p_k - 1 \text{ for all } k \geq 1.
\end{cases}
\]
The infinite cyclic group \( G = \langle a \rangle \) acts transitively on every level \( V_\ell \) of \( T \) for \( \ell \geq 0 \). This action has no fixed points, so it is free and, in particular, essentially free. This example shows that the condition that \( T \) has unbounded spherical index alone is not sufficient to produce an action where the set of points with non-trivial holonomy has full measure.

Next, we define a homeomorphism \( b \) of \( \partial T \). Suppose that in the spherical index \( n = (n_1, n_2, \ldots) \) we have \( n_\ell \geq 3 \) for \( \ell \geq 1 \). Recall that \( T_{v_\ell} \) denotes the subtree of \( T \) containing all paths through a given vertex \( v_\ell = w_1 \cdots w_\ell \). Then the boundary \( \partial T_{v_\ell} \) of \( T_{v_\ell} \) consists of all infinite sequences starting with the finite word \( v_\ell \).
The root $v_0$ is joined by edges to $n_1 \geq 3$ vertices in $V_1$, labelled by $0, 1, \ldots, n_1 - 1$. Let $b$ fix the vertices $0, 1, \ldots, n_1 - 3$, and interchange the vertices $n_1 - 2$ and $n_1 - 1$. We define the action of $b$ on the rest of the tree by induction.

So suppose the action of $b$ on $V_\ell$ is defined. Let $v_\ell \in V_\ell$, then $v_\ell$ is joined by edges to $n_{\ell+1}$ vertices in $V_{\ell+1}$ which are labelled by words of length $\ell + 1$, namely $v_0, v_1, \ldots, v_{(n_{\ell+1} - 1)}$. If $b(v_\ell) \neq v_\ell$, then for $0 \leq k \leq n_{\ell+1} - 1$ we set $b(v_\ell k) = b(v_\ell) k$, that is, the action of $b$ fixes the $(\ell + 1)$-st entry in the sequence and only changes some of the preceding entries. If $b(v_\ell) = v_\ell$, then we define $b(v_\ell k) = v_\ell k$ for $0 \leq k \leq n_{\ell+1} - 3$, and we set

$$b \cdot v_\ell(n_{\ell+1} - 2) = v_\ell(n_{\ell+1} - 1) \quad \text{and} \quad b \cdot v_\ell(n_{\ell+1} - 1) = v_\ell(n_{\ell+1} - 2).$$

The set $\text{Fix}(b) \subset \partial T$ is non-empty. For instance, every infinite sequence $s = s_1 s_2 \cdots$ with $s_j \leq n_j - 3$ for $\ell \geq 1$ is fixed by $b$. The homeomorphism $b$ does not fix any open sets. Indeed, if $U \subset \partial T$ is open, it must contain a basic clopen set $\partial T_{v_\ell}$, for some vertex $v_\ell = v_1 \cdots v_\ell$. The set $\partial T_{v_\ell}$ in its turn contains a clopen set $\partial T_{v_\ell(n_{\ell+1} - 2)} \cup \partial T_{v_\ell(n_{\ell+1} - 1)}$ on which $b$ acts non-trivially by (36). Since $b$ does not fix any open set, it has non-trivial holonomy at every point in $\text{Fix}(b)$. Note that $b$ satisfies Definition 6.1 with $K_0 = 1$ and $K_1 = 1$, so the action of $b$ is uniformly non-constant.

**THEOREM 6.9.** Let $T$ be a spherically homogeneous tree with spherical index $n = (n_1, n_2, \ldots)$ such that $n_{\ell+1} > 2n_\ell$. Let $G = \langle a, b \rangle \subset \text{Aut}(T)$ where $a$ is defined by (15) and $b$ by (16). Let $\mu$ be the uniform Bernoulli measure on $\partial T$. Then the following is true:

1. The action $(\partial T, G, \mu)$ is minimal and equicontinuous.
2. The set of points in $\partial T$ with non-trivial holonomy has full measure.

**Proof.** The orbits of the points in $\partial T$ under the action of the cyclic group generated by $a$ are dense in $\partial T$, so the action of $G = \langle a, b \rangle$ on $\partial T$ is minimal. Since $a$ and $b$ act by permutations on each level $V_\ell$, $\ell \geq 0$, the action of $G$ on $\partial T$ is equicontinuous.

Let $\partial T$ be given ultrametric (111) or (112). We show that the Lebesgue density of $\text{Fix}(b)$ at every point in $\text{Fix}(b)$ is 1, and so $\text{Fix}(b)$ must have positive measure. It follows that the set $\{ x \in \partial T \mid |G|_x \neq G_x \}$ of points with non-trivial holonomy has full measure, since it is invariant under the action of $G$.

Let $(v_\ell)_{\ell \geq 0} \in \text{Fix}(b)$ be an infinite path, and denote by $U_\ell = \partial T_{v_\ell}$ the subset of sequences in $\partial T$ which start with the finite word $v_\ell = v_1 \cdots v_\ell$ for $\ell \geq 1$, and set $U_0 = \partial T$. Then

$$\mu(U_\ell) = \frac{1}{n_1 n_2 \cdots n_\ell}.$$  

We first obtain an upper estimate on the measure of the complement $U_\ell - \text{Fix}(b)$.

For each $\ell \geq 0$ the element $b$ permutes two cylinder subsets of $U_\ell$, and we have for these subsets

$$\mu \left( \partial T_{v_\ell(n_{\ell+1} - 2)} \cup \partial T_{v_\ell(n_{\ell+1} - 1)} \right) = \frac{2}{n_1 n_2 \cdots n_{\ell+1}}.$$  

Next, each clopen set $\partial T_{v_0}, \partial T_{v_1}, \ldots, \partial T_{v_{(n_{\ell+1} - 3)}}$ contains two cylinder sets of measure $\frac{1}{n_1 n_2 \cdots n_{\ell+2}}$ permuted by $b$. The measure of the union of these sets is equal to

$$\frac{2(n_{\ell+1} - 2)}{n_1 n_2 \cdots n_{\ell+2}} < \frac{2n_{\ell+1}}{n_1 \cdots n_{\ell+1}} = \frac{1}{n_1 n_2 \cdots n_\ell} \cdot \frac{2}{n_{\ell+2}}.$$  

Inductively, we obtain that

$$\mu(U_\ell - \text{Fix}(b)) = \frac{1}{n_1 \cdots n_\ell} \left( \frac{2}{n_{\ell+1}} + \frac{2(n_{\ell+1} - 2)}{n_{\ell+1} n_{\ell+2}} + \frac{2(n_{\ell+1} - 2)(n_{\ell+2} - 2)}{n_{\ell+1} n_{\ell+2} n_{\ell+3}} + \cdots \right) < \frac{1}{n_1 \cdots n_\ell} \sum_{k \geq 1} \frac{2}{n_{\ell+k}}.$$  

By assumption $n_{\ell+k} > 2^{\ell-k} n_{\ell+1}$ for $k \geq 2$, therefore,

$$\sum_{k \geq 1} \frac{2}{n_{\ell+k}} < \frac{2}{n_{\ell+1}} \left( 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) = \frac{4}{n_{\ell+1}}.$$
and so
\[
\mu(U_\ell - \text{Fix}(b)) < \frac{4}{n_1 n_2 \ldots n_{\ell+1}}.
\]
It follows that
\[
\mu(\text{Fix}(b) \cap U_\ell) > \frac{1}{n_1 \ldots n_\ell} - \frac{4}{n_1 \ldots n_{\ell+1}},
\]
which implies that
\[
1 - \frac{4}{n_{\ell+1}} < \frac{\mu(\text{Fix}(b) \cap U_\ell)}{\mu(U_\ell)} \leq 1.
\]
Since \(n_\ell \to \infty\) as \(\ell \to \infty\), we obtain that the Lebesgue density of the set \(\text{Fix}(b)\) at \((v_\ell)_{\ell \geq 0}\) is 1. \(\square\)

**Example 6.10.** Let \(G = \langle a, b \rangle\) be an action on \(\partial T\) as in Theorem 6.9 and suppose in addition that the entries of the spherical index \(n = (n_1, n_2, \ldots)\) are distinct primes. Then by Remark 3.2 \((\partial T, G, \mu)\) does not admit a representation on the boundary of a tree with bounded spherical index. Thus we obtain an example of an action for which every tree representation is on the boundary of a tree with unbounded spherical index, and where the set of points with non-trivial holonomy has positive measure.

**Remark 6.11.** We note that in the examples by Bergeron and Gaboriau [5] and Abért and Elek [2], similarly to Example 6.10, the spherical index of the associated tree representation (equivalently, the index \(|\Gamma_{\ell+1} : \Gamma_\ell|\) of subgroups in the associated group chain) is not bounded.

### 7. Applications

#### 7.1. Invariant random subgroups

We now prove Theorem 1.13. Again, we restate it first for the convenience of the reader.

Recall that \(\text{Sub}(G)\) denotes the space of all closed subgroups of \(G\) with the Chaubaty-Fell topology, and there is a mapping \(\text{St} : X \to \text{Sub}(G) : x \mapsto G_x\), where \(G_x\) denotes the stabilizer of \(x\). This mapping is described in more detail in Section 4.

Recall that \(X_0 = \{x \in X \mid G_x = [G]_x\}\) is the set of points with trivial holonomy in \(X\), and that
\[
Z = \{G_x \in \text{Sub}(G) \mid x \in X_0\}.
\]

Further, recall that two measure-preserving systems \((X, G, \Phi, \mu)\) and \((Y, G, \Psi, \nu)\) are isomorphic in the measure-theoretical sense if there are sets of full measure \(X_0 \subset X \) and \(Y_0 \subset Y\) invariant under the action of \(G\), and a bi-measurable bijection \(\psi : X_0 \to Y_0\) such that \(\mu(\psi^{-1}(A)) = \nu(A)\) for all measurable \(A \subset Y_0\) and \(\psi\) conjugates the action of \(G\) on \(X_0\) and \(Y_0\), that is,
\[
\psi(\Phi(g) \cdot x) = \Psi(g) \cdot \psi(x) \text{ for all } x \in X_0.
\]

In this case we call \(\psi\) a measure-theoretical isomorphism. We also refer to an everywhere defined measurable map \(\eta : X \to Y\) in this way if \(\eta(x) = \psi(x)\) with \(x \in X_0\) for some \(\psi\) and \(X_0\) as before.

**Theorem 7.1.** Let \((X, G, \Phi, \mu)\) be a minimal equicontinuous action on a Cantor set \(X\) which satisfies the hypothesis of Theorem 1.12. Then the IRS \(\nu = \text{St}_* \mu\) is supported on \(Z\), and \((X, G, \Phi)\) is not locally quasi-analytic (not LQA) if and only if \(\nu\) is non-atomic. If, in addition, \(\text{St} : X_0 \to \text{Sub}(G)\) is injective, then \(\text{St}\) provides a measure-theoretical isomorphism between \((X, G, \Phi, \mu)\) and \((Z, G, \nu)\).

**Proof.** Since \((X, G, \Phi, \mu)\) satisfies the hypothesis of Theorem 1.12, the set of points \(X - X_0\) has measure zero in \(X\). It follows that the pushforward measure \(\nu = \text{St}_* \mu\) is supported on \(Z\). By Theorem 1.12 \(Z\) is infinite if and only if the action \((X, G, \Phi, \mu)\) is not LQA. Therefore, \(\nu\) is non-atomic if and only if the action \((X, G, \Phi, \mu)\) is not LQA.

For the second statement, let \(Y_0 = \{G_x \in \text{Sub}(G) \mid x \in X_0\}\). Then \(Z = Y_0\). By assumption, the map \(\text{St} : X_0 \to Y_0\) is injective and, by the above, \(\nu\) is supported on \(Z\). By Theorem 1.12 \(X_0\) has full measure, and so \(Y_0\) has full measure. It follows that \((X, G, \Phi, \mu)\) and \((Z, G, \nu)\) are measure-theoretically isomorphic. \(\square\)
7.2. Mean equicontinuous actions. Recall from the Introduction the definitions of the sets $Z$ and $\tilde{X}$, see \cite{5} and \cite{7}, as well as of the almost one-to-one factor map $\eta : \tilde{X} \to X$. Moreover, recall that any continuous action has a unique (up to conjugacy) \textit{maximal equicontinuous factor}, in the sense that any other equicontinuous factor is also a factor of this maximal one, see for instance \cite{5}.

**Proposition 7.2** \cite[V(6.1)5, page 480]{16}. If $(Y, G, \Psi)$ is a minimal extension of an equicontinuous system $(X, G, \Phi)$ via a corresponding factor map which is almost one-to-one, then $(X, G, \Phi)$ is the maximal equicontinuous factor of $(Y, G, \Psi)$.

We first show Theorem 1.15 which provides the following alternative characterization of LQA actions.

**Theorem 7.3.** Let $(X, G, \Phi)$ be a minimal and equicontinuous action. We have that $(X, G, \Phi)$ is locally quasi-analytic if and only if the almost one-to-one extension $\eta : X \to \tilde{X}$ is equicontinuous if and only if $\eta$ is locally quasi-analytic.

**Proof.** First, assume that $(X, G, \Phi)$ is LQA. By Theorem 1.14 $Z$ is finite, and the action of $G$ on $Z$ by conjugation is equicontinuous. Then the product action of $G$ on $X \times Z$ is also equicontinuous, see \cite[Lemma 4 in Chapter 2]{5}. This in turn implies that the orbit closure of every point in $X \times Z$ is minimal, see \cite[Lemma 3 in Chapter 2]{5}. Now, using that $\tilde{X}$ is the unique minimal subset in $(\{(x, G_x) \mid x \in X\} \subset X \times \text{Sub}(G)$ by \cite[Proposition 1.2 (3)]{25}, we get that $\tilde{X} = X \times Z$. Finally, since $\eta : \tilde{X} \to X$ is almost one-to-one, we have that $(X, G, \Phi)$ is the maximal equicontinuous factor of $(\tilde{X}, G, \Phi)$, by Proposition 7.2, but this immediately implies that $\eta$ must be a conjugacy.

For the opposite direction, assume that $\eta$ is a conjugacy. Then the action of $G$ on $\tilde{X}$ is equicontinuous. Moreover, since $Z$ is a factor system of $\tilde{X}$ (simply by projecting in the second coordinate), we get that the action of $G$ on $Z$ by conjugation is equicontinuous as well, see \cite[Corollary 6 in Chapter 2]{5}. Hence by Lemma 1.13 $Z$ is finite and so, by Theorem 1.14 the action $(X, G, \Phi)$ is LQA. \hfill $\square$

In closing, let us now deduce Corollary 1.16 which is a direct consequence of the following equivalent characterization of mean equicontinuity for general minimal actions by amenable groups.

**Theorem 7.4** \cite[Corollary 1.5]{20}. Let $(Y, G, \Psi)$ be minimal, $G$ amenable and denote by $(X, G, \Phi)$ the maximal equicontinuous factor of $(Y, G, \Psi)$. Furthermore, let $\eta : Y \to X$ be the corresponding factor map and $\mu$ the unique invariant measure of $(X, G, \Phi)$. Then $(Y, G, \Psi)$ is mean equicontinuous if and only if $(Y, G, \Psi)$ has a unique invariant measure $\nu = \eta_* \mu$ and the factor map $\eta$ provides a measure-theoretical isomorphism between $(Y, G, \Psi, \nu)$ and $(X, G, \Phi, \mu)$.

**Proof.** (of Corollary 1.16). By Proposition 7.2 the action $(X, G, \Phi)$ is the maximal equicontinuous factor of $(\tilde{X}, G, \Phi)$. Further, under the hypothesis of Theorem 1.14 we know that $\mu(\tilde{X}_0) = 1$. This implies immediately that $(\tilde{X}, G, \Phi)$ has a unique invariant measure which is the pushforward $\nu = \eta_* \mu$ and $\eta$ provides a measure-theoretical isomorphism between $(\tilde{X}, G, \Phi, \nu)$ and $(X, G, \Phi, \mu)$. By using the previous theorem, we obtain Corollary 1.16. \hfill $\square$

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