Is de Sitter space a fermion?

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Abstract

Following up on a recent model yielding fermionic geometries, I turn to more familiar territory to address the question of statistics in purely geometric theories. Working in the gauge formulation of gravity, where geometry is characterized by a symmetry broken Cartan connection, I give strong evidence to suggest that de Sitter space itself, and a class of de Sitter-like geometries, can be consistently quantized fermionically. Surprisingly, the underlying mathematics is the same as that of the Skyrme model for strongly interacting baryons. This promotes the question "Is geometry bosonic or fermionic?" beyond the realm of the rhetorical and places it on uncomfortably familiar ground.

1 Introduction

It is generally taken on faith that the geometry underlying general relativity is bosonic. In quantum gravity, this faith is buttressed by the observation that low energy, small-amplitude excitations of the gravitational field are spin-2 gravitons, and should therefore be quantized as bosons. On the other hand, it is well known that certain constrained or otherwise non-linear field theories whose low energy excitations are bosonic, can nevertheless give rise to emergent structures in the non-perturbative regime with fermionic (or anyonic) statistics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. It is conceivable then that the phase space structure of general relativity could admit fermionic modes upon quantization.

To support this hypothesis, in a recent article we constructed a purely geometric theory with precisely this peculiar property [11, 12]. The model presented there is not general relativity, however it is a dynamical geometry theory. Specifically, we considered a propagating torsion theory in flat Minkowski space, where the torsion is constrained in such a way that the local degrees of freedom of the model are described by a non-linear sigma model with target space $\text{Spin}(3, 1)$. By formally mapping the model onto the Skyrme model for strongly interacting baryons, we showed that the system could be quantized in such a way that isolated torsional charges of even charge behave as bosons under rotations and exchanges, whereas odd charges behave as fermions.

The existence of these fermionic geometries leads one to question whether this could be a generic feature of dynamical geometries or simply a peculiarity of the particular model. One may raise the speculative objection, for example, that the possibility of more exotic non-perturbative statistics is novel to torsional theories and

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cannot necessarily be extrapolated to non-torsional geometries. In this paper I will give strong evidence to quell this objection.

Specifically I will show that a much more familiar geometry, namely de Sitter space itself and a class of generalized de Sitter-like geometries can be quantized fermionically. The geometric arena I will work in is the gauge formulation of gravity where geometry is described by a reductive Cartan connection on a Spin$(4,1)$-bundle [13] [14] [15] [16] [17] [18] [19] [20]. Surprisingly, the underlying mathematics allowing for fermionic quantization is the same as for the Skyrme model.

Bosonic geometry has long been an implicit tenet of quantum gravity theories. The possibility of fermionic geometries will likely require significant rethinking of foundational issues in quantum geometry. Furthermore, de Sitter space plays an important role in cosmology as the expected ground state of a universe with a positive cosmological constant and potentially as the future asymptote of our universe. This promotes the question “Is geometry bosonic or fermionic?” beyond the realm of the rhetorical and places it on uncomfortably familiar ground.

2 Geometry from Cartan’s perspective

The mathematical arena that I will work in is the reformulation of Einstein-Cartan theory as a symmetry broken gauge theory. I will briefly review this formalism below, but I refer the reader to my review paper [20] for a more extensive presentation.

In the gauge formulation of gravity, geometry is characterized by a Cartan connection $\mathcal{A}$ taking values in a reductive Cartan algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. The reductive split of the algebra into a stabilizer subalgebra $\mathfrak{h}$ and its complement $\mathfrak{p}$ is facilitated through the introduction [21] [22] of a new field $V$. This field acts as a symmetry breaking field analogous to the Higgs, “breaking” the $\mathcal{G}$ symmetry of the principle $\mathcal{G}$-bundle down to a subgroup $\mathcal{H}$ which stabilizes $V$ at each point. The field itself takes values in a space isomorphic to the coset space $\mathcal{G}/\mathcal{H}$. Thus, in total the geometry is characterized by a specification of the pair $\{\mathcal{A}, V\}$, modulo $\mathcal{G}$-gauge transformations. The breaking of the symmetry allows for a separation of the spin connection from the frame field corresponding to the reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, so that $\mathcal{A} = \omega \oplus \frac{1}{\ell} e$ where $\ell$ is a parameter with dimension of length. Here, $\omega$ is the $\mathfrak{h}$-valued spin-connection, and $e$ is the $\mathfrak{p}$-valued tetrad. For the case of gravity in $(3+1)$-dimensions with a positive cosmological constant, the gauge group is $\mathcal{G} = Spin(4,1)$, the double cover of the de Sitter group, and the stabilizer subgroup is $\mathcal{H} = Spin(3,1)$. The parameter $\ell$ can be related to the cosmological constant $\Lambda$ by $\ell = \sqrt{3/\Lambda}$.

To see this more explicitly, consider the adjoint representation of $\mathcal{G} = Spin(4,1)$. Let hatted-upper case Roman indices $\{\hat{I}, \hat{J}, \hat{K}, \ldots\}$ range from 0 to 4, and unhatted indices $\{I, J, K, \ldots\}$ range from 0 to 3. The symmetry breaking field $V^I$ is a vector representation of $Spin(4,1)$ whose magnitude is constrained by $\eta_{\hat{I}\hat{J}} V^I \bar{V}^J = 1$ with $\eta_{ij} = \text{diag}(-1, 1, 1, 1, 1)$. The tetrad can then be identified with $e^I \equiv \ell D_A V^I$ and the
spin connection with \( \omega^{ij} = A^{ij} - 2D_A V^i V^j \).

One can always choose a specific gauge locally where \( V^i = (0, 0, 0, 0, 1) \). For the rest of the paper we will refer to this gauge as the Einstein-Cartan (EC) gauge. In the EC gauge, the tetrad simplifies to \( e^i = \ell A^i \), and the spin connection to \( \omega^{ij} = A^{ij} \). In a Clifford algebra notation which we will employ here (see Appendix A) the symmetry breaking field is given by \( V = V^I \gamma^I \) with \( \gamma^I = (\gamma^I, \gamma_5) \). The action of a Spin(4,1) gauge transformation generated by \( g = g(x) \) is given by \( \{ A, V \} \rightarrow \{ gA g^{-1} - dgg^{-1}, gV g^{-1} \} \).

Corresponding to the split \( A = \omega + \frac{1}{\ell} \gamma_5 e \), the curvature also splits into

\[
\mathcal{F}_A = R_\omega - \frac{1}{\ell^2} e \wedge e + \frac{1}{\ell} \gamma_5 T .
\]

The constant curvature \( (R_\omega = \frac{1}{\ell^2} e \wedge e) \), zero torsion \( (T = 0) \) condition is then given succinctly by \( \mathcal{F}_A = 0 \). Thus, in the gauge framework of gravity, de Sitter space is described by a flat connection. This allows one to easily embed the de Sitter solution into a larger space, which is at the same time small enough to easily characterize topologically. For this reason, I will focus on the configurations consisting of the pair \( \{ A, V \} \) where \( A \) is restricted to be a flat connection. I will refine this phase space shortly.

### 2.1 Flat Cartan geometries

Restrict attention to manifolds with the topology of de Sitter space, namely \( M \simeq \mathbb{R} \times S^3 \). Since \( \pi_1(M) = 0 \), all flat connection are “gauge related” to the zero connection. This implies that there exists a \( g : \mathbb{R} \times S^3 \rightarrow Spin(4,1) \) such that \( A = -dgg^{-1} \). The geometry is therefore characterized by the pair \( \{ A, V \} = \{-dg^{-1}, V\} \). However, one can always transform both fields to a gauge where \( A = 0 \) and \( V' = g^{-1} V g \) thereby pushing all the geometric information into the symmetry breaking field itself. In this gauge, which I will refer to as the trivial gauge, the geometry is completely characterized by the map \( V : \mathbb{R} \times S^3 \rightarrow \mathcal{G}/\mathcal{H} \simeq \mathbb{R} \times S^3 \).

The goal is to categorize the space of flat Cartan connections topologically. To this end, consider first the set of maps that are deformable to a time independent map \( V : S^3 \rightarrow \mathcal{G}/\mathcal{H} \simeq \mathbb{R} \times S^3 \). The homotopy class of maps of this sort fall into discrete classes characterized by \( \pi_3(\mathcal{G}/\mathcal{H}) = \mathbb{Z} \). The integer labelling the sector in which the map resides is the number of times the \( S^3 \) of the range winds around the \( S^3 \) of the domain, or the winding number of the map.

As shown in [14, 20] the \( \mathbb{Z} \)-sectors of the phase space can be constructed as follows.

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\(^1\text{It is convenient in the Clifford algebra notation to define } e = \frac{1}{2} \gamma_I e^I \text{ which pulls the } \gamma_5 \text{ out of the expression in the decomposition. See Appendix A for more details.}\)
Define $m_n$ to be the time-independent map $m_n : S^3 \to Spin(4,1)$ given by

$$m_n = \begin{bmatrix} h^m & 0 \\ 0 & h^n \end{bmatrix} \quad \text{with} \quad h = X^4 1 + X^i i\sigma_i$$

(2)

where $X$ denotes the usual embedding of the three sphere in $\mathbb{R}^4$ given explicitly in three-dimensional polar coordinates by

$$
\begin{align*}
X^1 &= \sin \chi \sin \theta \cos \phi \\
X^2 &= \sin \chi \sin \theta \sin \phi \\
X^3 &= \sin \chi \cos \theta \\
X^4 &= \cos \chi
\end{align*}
$$

(3)

The field $h$ is a map $h : S^3 \to SU(2)$ with winding number one, and as such it is the generator of $\pi_3(SU(2)) = \mathbb{Z}$. Since $Spin(4) \cong SU(2) \uparrow \times SU(2) \downarrow$ and $Spin(4)$ is a subgroup of $Spin(4,1)$, $h(x) \in Spin(4,1)$ and it has winding number $m + n$. To add back in the time dependence of the de Sitter solutions, first time-translate $m_n$ to $\tilde{m}_n = h_t m_n$ where $h_t = \exp(\frac{1}{2} t \gamma^5 \gamma^0)$. A class of topologically distinct Cartan geometries is then given, in the EC gauge, by the configurations

$$\{A, V\} = \left\{ \tilde{m}_n \equiv -d\tilde{g}_n^{-1}, \gamma_5 \right\} .$$

(4)

To understand the topological structure of these configurations more thoroughly it is useful to transform to the trivial gauge where $A = 0$. In this gauge, it can be shown that $V' = \tilde{V}_n \equiv \tilde{g}_n^{-1} \gamma_5 \tilde{g}_n^{-1}$ viewed as a map $\tilde{V}_n : \mathbb{R} \times S^3 \to \mathcal{G}/\mathcal{H}$ has winding number $-q$ where $q \equiv m - n$. Thus, the integer $-q$ delineates the topological sectors of the space of flat Cartan connections corresponding to the homotopy group $\pi_3(\mathcal{G}/\mathcal{H})$.

Geometrically these configurations have a very simple interpretation. The metric induced from the pair $\{\tilde{A}, V = \gamma_5\}$ is given by

$$\tilde{g}_n = -dt^2 + \cosh^2(t/\ell) \left( |q|^2 d\chi^2 + \sin^2(|q|\chi) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right) .$$

(5)

For $|q| > 0$, this metric describes a “string of pearls” geometry consisting of $|q|$ copies of de Sitter space attached at their poles by regions where the metric becomes degenerate (see Fig. 1). The geometry of each individual 3-sphere is identical to that of de Sitter space itself aside from the poles. One can show that the volume of the three-sphere defined at the throat of de Sitter space at $t = 0$ is given by $2\pi^2 \ell^3(-q)$. 

4
It should be stressed that all these configurations are solutions to the Einstein Cartan field equations, despite being degenerate [14, 20]. This serves to generalize de Sitter space, and allows one to embed the pure de Sitter solution into a larger phase space that is still manageable to work with.

2.2 $\mathbb{Z}_2$ degeneracy of the phase space

I will now show that the phase space containing the de Sitter configuration admits one further degeneracy. But first it is necessary to refine and define the phase space more precisely. Consider the set of flat Cartan geometries that asymptotically approach the pair $\{A, V = \gamma_3\}$ in the past and future. In the trivial gauge this can be simplified to the set of maps $V : \mathbb{R} \times S^3 \to G/H$ that asymptote to $V$. In this fixed gauge, the only remaining gauge invariance is a global action of a constant $g \in Spin(4,1)$. Rather than modding out by diffeomorphism equivalence classes, we will consider the action of diffeomorphisms on the phase space that preserves the geometry on the boundaries. This allows for a classification of the action of diffeomorphisms into types categorized by topological properties. Thus, I will take the phase space $Q$ to be the space of flat Cartan connections modulo the set of identity connected, boundary isometries.

The two lowest non-zero homotopy groups of the target space of $V$ are $\pi_3(G/H) = \pi_3(S^3) = \mathbb{Z}$ and $\pi_4(G/H) = \pi_4(S^3) = \mathbb{Z}_2$. I have already shown that the former is the topological property that allowed for the construction of a class of configurations labelled by the integer $-q$. The remaining $\mathbb{Z}_2$ degeneracy I will argue allows for fermionic quantization of de Sitter space. In total, the space of flat Cartan connections $Q$ splits into topologically distinct sectors labelled by the winding number, and the $\mathbb{Z}_2$ degeneracy.

Figure 1: A visualization of the generalized de Sitter space corresponding to $|q| = 3$ on a constant time slice. On the left is the standard visualization where the radius is weighted by volume. The geometry consists of a string of $|q|$ three-spheres attached at the poles along two-spheres where the metric becomes degenerate. On the right is a more topologically accurate picture where the degenerate surfaces are pictured as extended regions denoted by dotted lines.
3 The action of diffeomorphisms on $Q$

The goal now is to construct a generator of the $Z_2$ degeneracy. In fact, the degeneracy can be related to the action of diffeomorphisms on $Q$. First, however, I will discuss generically how this degeneracy comes about.

Start with the $q = 0$ sector denoted $Q_0$. In this sector, the configuration should asymptote to $\{A^0 = -dh_t h^{-1}_t, \gamma_5\}$. As in [3], to analyze the topological properties, it is convenient to define a canonical homeomorphism between each topological sector and a space where the analytic properties of the map are more apparent. Thus, for each sector $Q_q$, we define a homeomorphism to a new sector $Q^*_0$, which is itself homotopic to $Q_0$. The homeomorphisms is defined as follows (see Fig. 2). Given a configuration in the $Q_q$ sector first transform to the EC gauge where $V = \gamma_5$. Next map the pair $\{A, \gamma_5\} \rightarrow \{h^{-1}_t A h_t - dh^{-1}_t h_t, \gamma_5\}$. This serves to remove the time dependence of the fiducial configurations $\{A^0, \gamma_5\}$. Now transform to the trivial gauge. Finally transform the configuration by $\{0, V\} \rightarrow \{0, h V h^{-1}\}$. The point behind this homeomorphism is that it transforms each of the fiducial configurations $\{A^0, \gamma_5\}$ defined in the previous section to the convenient base point $\{0, \gamma_5\}$. This will make it easier to determine the homotopy properties of the configuration.

The resulting configuration lives in a space $Q^*_0$. This space is formed by the set of maps $V : \mathbb{R} \times S^3 \rightarrow \mathcal{G}/\mathcal{H}$ that asymptote to $V = \gamma_5$ in the asymptotic past and future. Because of the latter restriction, one can add the endpoints $t = \{-\infty\}$ and $t = \{+\infty\}$ and compactify the domain to $S^4$. Thus $V : S^4 \rightarrow \mathcal{G}/\mathcal{H}$. The homotopy classes of maps of this type are characterized by $\pi_4(\mathcal{G}/\mathcal{H}) = \mathbb{Z}_2$. Thus, via this homeomorphism, all states in $Q$ can be classified into two groups: those configurations that when mapped to $Q^*_0$ are homotopic to $\{0, \gamma_5\}$ and those that are homotopic to a generator of $\pi_4(\mathcal{G}/\mathcal{H}) = \mathbb{Z}_2$ in $Q^*_0$.

3.1 Rotational diffeomorphisms and the $Z_2$ degeneracy

Now let us consider the action of diffeomorphisms on $Q$. To preserve the phase space, one should restrict attention to those that asymptote to an isomorphism in the asymptotic past and future. However, since an isomorphism is equivalent to a gauge transformation by a constant element of $Spin(4,1)$ and are therefore contained in the local gauge group, we will restrict to $Diff_0$, the set of diffeomorphisms that tend to the identity at future and past asymptotic infinity.

Consider first the $q = -1$ sector where the fiducial configuration $\{A^0, \gamma_5\}$ represents ordinary de Sitter space. Define a one-parameter rotational diffeomorphism $\varphi$ by its action on the coordinates $\varphi(t, \chi, \theta, \phi) = \{t, \chi, \theta, \phi' \equiv \phi - \phi_0(t)\}$ where $\phi_0(t)$ is a smooth function of $t$ only that varies from 0 to $2\pi$ in the interval $[t_i, t_f]$, and is constant
Figure 2: Schematic of the homeomorphism between the $Q_q$ sectors and the reference space $Q^*_0$. On the left is the $Q_{-1}$ sector containing the de Sitter solution (black point) and the twisted de Sitter solution (red point) which are mapped to different $\mathbb{Z}_2$ sectors of $Q^*_0$.

outside the interval. Assume also that the derivative $\partial_t \phi_0$ vanishes at $t_i$ and $t_f$. For definiteness, choose $t_i = -\ell$ and $t_f = \ell$. Explicitly, the diffeomorphism is given by $\varphi = \exp(-\phi_0 L_{\phi})$ where $\phi = \frac{\partial}{\partial \phi}$, and its action on de Sitter space is represented by a $2\pi$ twist. The metric transforms to (see Fig. 3 for visualization)

$$
\gamma^0_1 = -(1 - \alpha) dt^2 + 2 \beta dt d\phi' + \cosh^2(t/\ell) \left( d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\phi'^2 \right) \right)
$$

(6)

where $\alpha = \cosh^2(t/\ell) \sin^2 \chi \sin^2 \theta (\partial_t \phi'_0)^2$ and $\beta = \cosh^2(t/\ell) \sin^2 \chi \sin^2 \theta \partial_t \phi'_0$.

Consider now the action of of $\varphi$ on the configuration $\{A, \gamma_5\}$. The diffeomorphism leaves $V = \gamma_5$ fixed in this gauge, but transforms $A = -d\gamma_0^0 g^{-1}$ to $\varphi(A) = -d\gamma_0^0 g^{-1}$ where

$$
\gamma_0^0 = h_t U^{-1} h U \quad \text{with} \quad U(t) = \begin{bmatrix} \exp(\phi_0 \frac{i}{2} \sigma_3) & 0 \\ 0 & \exp(\phi_0 \frac{i}{2} \sigma_3) \end{bmatrix}
$$

(7)

Under the canonical homomorphism to $Q^*_0$ given above, this configuration maps to

$$
V^i = (u^{-1} h u h^{-1})^i \quad \text{with} \quad u = \exp \left( \phi_0 \frac{i}{2} \sigma_3 \right)
$$

(8)

where it is understood in this expression that the components $V^i$ are extracted from the expression $V = V^4 1 + V^i i \sigma_i$. I claim that this is a generator of $\pi_4(G/H) = \mathbb{Z}_2$. To see this, I must first digress to discuss more generally the generators of $\pi_4(S^3) = \mathbb{Z}_2$ (see e.g. [2, 3, 10] for a related discussion in the context of the Skyrme model).
3.2 The Hopf map and suspensions

The group $\pi_4(S^3) = \mathbb{Z}_2$ has just two elements, so it is sufficient to find a single map taking $S^4 \to S^3$ that is not deformable to the identity. Any map that is homotopic to this map will then also be a generator of $\mathbb{Z}_2$. To construct the generator, consider first the lower dimensional case of maps from $S^3$ onto $S^2$. Since $\pi_3(S^2) = \mathbb{Z}$, this group is also generated by a single map, and the canonical generator is known as the Hopf map.

This map can be thought of as a fibration of the three-sphere into a non-trivial bundle of $U(1)$ fibers over $S^2$. The projection map of the bundle $\pi : P = S^4 \to M = S^2$ is the Hopf map itself. Explicitly it can be constructed as follows. Consider a vector field $n^i$ in the tangent space of $S^3$ whose magnitude $\delta_{ij} n^i n^j = 1$ constrains it to live in a $S^2$ submanifold of the tangent space. The group $SU(2)$ acts transitively on this space via the adjoint action $n^i \to n^i + a n^i$ for any $a : S^3 \to SU(2)$. Now, take $a$ to be the standard generator, $h$, of $\pi_3(S^3)$ given in (2), and take $n^i = (0, 0, 1)$. Then, since $h$ is not deformable to the identity and has winding number one, the map $n^i' = h n^i h^{-1} = h i \sigma_3 h^{-1}$ is a map from $S^3$ to $S^2$ with winding number one. It therefore serves as the generator of $\pi_3(S^2) = \mathbb{Z}$. This is the Hopf map.

A well known result of homotopy theory is that any suspension of the Hopf map is a generator of $\pi_4(S^3) = \mathbb{Z}_2$. An example of such a suspension is any continuous map $V : S^4 \to S^3$ such that the restriction of the map to the $S^3$ equator of $S^4$ reduces to the Hopf map. For example, take $\chi$ to be the azimuthal angle on $S^4$. Then the map $V^i = (h u^{-1} h u^{-1})^i$, where as before $u = \exp(2 \chi \frac{1}{2} \sigma_3)$, is such a suspension since on the equator at $\chi = \frac{\pi}{2}$, the map reduces to $V(\pi/2) = h i \sigma_3 h^{-1}$, the Hopf map.

3.3 The $\mathbb{Z}_2$ configurations in $Q$

We now return to our configuration $V^i = (u^{-1} h u^{-1})^i$. Evaluated at $t = 0$, this map reduces to $-i \sigma_3 h \sigma_3 h^{-1}$. This is simply the Hopf map followed by a constant rotation by $-\pi$, and is therefore homotopic to the Hopf map. In turn, the full map $V^i$ is homotopic to a suspension of the Hopf map in $S^4$, and is therefore a generator of $\pi_4(G/H) = \mathbb{Z}_2$. Thus, we have constructed the generator of the $\mathbb{Z}_2$ degeneracy in the sector $Q_{-1}$. Clearly a similar construction holds in the sector $Q_1$. Moreover, it should be clear that any time dependent spatial rotation of the same generic

Figure 3: The twisted version of de Sitter space given by the metric $g_{\phi'}$. The red lines represent lines of constant $\phi'$. 
Figure 4: The exchange of two copies of de Sitter space in the $q = \pm 2$ sector. This map is homotopic to $2\pi$ twist of just one of the copies, and is therefore a generator of $\mathbb{Z}_2$ in the $q = \pm 2$ sector.

form as $\varphi$ is a generator of $\mathbb{Z}_2$. It is slightly less obvious, but still true, that a time dependent spatial translation of de Sitter space about one full revolution is also a generator of $\mathbb{Z}_2$.

To construct the generator in all sectors $\mathcal{Q}_q$ we borrow well known results from the Skyrme model \cite{2, 3, 10}. Consider the action of $\varphi$ on the base points $\{m, n, \gamma_5\}$, where we recall $q \equiv m - n$. It can be shown that in any sector $\mathcal{Q}_q$ with odd charge $q$, the action of $\varphi$ on the base point $\{m, n, \gamma_5\}$ is a generator of $\mathbb{Z}_2$. On the other hand, the map is deformable to the trivial map in any sector with even $q$. In sectors of even charge, the generator can be constructed from exchanges of the geometries (see Fig. 4).

Consider for example the $q = -2$ sector whose base point consists of two copies of de Sitter space attached along a degenerate surface. A diffeomorphism representing an exchange of these two copies is homotopic to a $2\pi$ twist of just one copy. Thus, the exchange is the generator of $\mathbb{Z}_2$ in this case. Similar results hold in all sectors with even $q$.

\footnote{The easiest way to see this is to picture de Sitter space as the hyperboloid embedded in $\mathbb{R}^{1,4}$ with coordinates $\{T, X, Y, Z, W\}$. A global $2\pi$-rotation about the $XY$ plane is clearly homotopic to a global $2\pi$-rotation about $XZ$ since any spatial plane can be continuously deformed into another. By the same token it is homotopic to a $2\pi$-rotation about the $XW$ plane. But the latter is interpreted as a translation of the spacetime about one full revolution. The same reasoning applies for time dependent rotations and translations.}
4 Fermionic Quantization

I now turn to the quantization of the generalized de Sitter spaces. Our goal here is to present the overall structure of a quantization scheme, with emphasis on the ways that it differs from a more conventional quantization. Most importantly, one must pay particularly careful attention to the role of symmetries, exact or asymptotic, and their effect on the wavefunction. As with asymptotically flat spacetimes, given our asymptotic (generalized) de Sitter past and future boundary conditions, one must make a distinction between an arbitrary diffeomorphism and a diffeomorphism representing an isometry. The latter are genuine symmetries of the system and the associated generators of the symmetries in the canonical theory are generally associated with physical quantities (e.g. energy, momentum, angular momentum, etc...). As a consequence, a diffeomorphism representing an isometry or asymptotic isometry should act non-trivially on the wavefunction in the quantum theory.

Since I have restricted attention to the space of flat connections, all our solutions are maximally symmetric solutions with ten Killing vectors. Choose two spatial hypersurfaces $\Sigma_1$ and $\Sigma_2$, for simplicity chosen to be located at finite values $t_1$ and $t_2$. Without loss of generality, assume $t_1 < 0$ and $t_2 > 0$. Fix the geometry on the hypersurfaces to be the pull-back of (4) and (5) in the canonical coordinate system $\{\chi, \theta, \phi\}$. We wish to consider the set of diffeomorphisms that preserves the embedding of the spatial hypersurfaces $\Sigma_1$ and $\Sigma_2$ and their respective geometries. Naturally, this includes the set of isometries of $mg$. So, let $\text{Diff}$ be the set of identity connected diffeomorphisms that preserve the embedding of the hypersurfaces $\Sigma_1$ and $\Sigma_2$ and the pull-back of the metric to these surfaces, and let $\text{Diff}_0 \subset \text{Diff}$ be the subgroup that restricts to the identity on the boundary. The coset $\text{Diff}/\text{Diff}_0$ generically acts non-trivially on the wavefunction in the quantum theory.

To simplify the analysis, we will restrict attention to the $|q| = 1$ de Sitter case and consider the action of $\text{Diff}/\text{Diff}_0$ on it. A typical element of the coset space is given by a pair $\{g_1, g_2\}$ representing the action of the diffeomorphism on the two endcaps $\Sigma_1$ and $\Sigma_2$, possibly subject to some as yet undefined equivalence relations. Given a constant-$t$ hypersurface of the manifold, the action of the subgroup of diffeomorphisms that preserve the hypersurface and the restriction of the metric to the hypersurface can be represented by a group $SO(4)_{\Sigma_t}$. Thus, $g_1 \in SO(4)_{\Sigma_1}$ and $g_2 \in SO(4)_{\Sigma_2}$.

Consider first a one-parameter family of diffeomorphisms $\varphi_\epsilon$ representing a smooth twist of the manifold as $\epsilon$ ranges from 0 to 1. Suppose that for each value $\epsilon$, the diffeomorphism represents a $2\pi\epsilon$ rotation in the $\frac{\partial}{\partial \phi}$ direction of the $\Sigma_2$ hypersurface while keeping $\Sigma_1$ fixed. In the interior between $t_i$ and $t_f$, the diffeomorphism smoothly interpolates between the identity on $\Sigma_1$ and the rotation on $\Sigma_2$. For example, the action of the diffeomorphism could be represented by its action on the coordinates by

$$\varphi_\epsilon \{\chi, \theta, \phi\} = \{\chi, \theta, \phi' \equiv \phi - \epsilon \phi_0(t)\}$$

where $\phi_0$ is defined as before. Viewed as an element of $\text{Diff}/\text{Diff}_0$, the twist can
be represented by the pair \( \{g_1, g_2\} = \{1, g(\epsilon)\} \) where \( g(\epsilon) \) represents a \( 2\pi\epsilon \) rotation in \( SO(4) \). It can be shown that the action of the diffeomorphism on the boundary data alone is a loop parameterized by \( \epsilon \) in the set of boundary data that is contractable if \( q \) is even, and non-contractable if \( q \) is odd. Thus, the fact that \( \varphi_{\epsilon=1} \) is a generator of \( \mathbb{Z}_2 \) only in the odd sectors is matched by the pure boundary description.

In the boundary description appropriate for canonical quantization, one first identifies a configuration space \( C \) on each boundary by choosing a polarization of the phase space of boundary data equipped with a symplectic structure. Although the configuration space will depend on the polarization and the details of the quantization we will make the assumption that a \( 2\pi \) rotation is a non-contractable loop in the configuration space. Following alongside Skyrme theory, the formal procedure for fermionic quantization is to first take the double cover \( \widetilde{C} \) of the configuration space prior to quantization. In practice this is difficult to do, so fermionic quantization usually proceeds by first identifying the key non-contractable loops and promoting them to operators on the Hilbert space. The constraints, which go by the name of Finkelstein-Rubenstein constraints \([2]\) in Skyrme theory, associate a \(-1\) phase to non-contractable loops. In other words consider a parameterized loop \( \gamma : S^1 \to C \), parameterized by \( \epsilon \in [0, 1] \), generated by a Hamiltonian flow. Promoting this to a parameterized set of operators \( \hat{O}_{\gamma(\epsilon)} \) acting on the Hilbert space, we require

\[
\hat{O}_{\gamma(1)}|\Psi\rangle = \begin{cases} 
|\Psi\rangle & \text{if } \gamma(\epsilon) \text{ is contractable} \\
-|\Psi\rangle & \text{if } \gamma(\epsilon) \text{ is non-contractable}
\end{cases} .
\] (10)

In our case, the \( \mathbb{Z}_2 \) generators are given by the action of \( \overline{Diff}/\overline{Diff}_0 \) on the two endcaps of the manifold, represented by \( \overline{C}_{\Sigma_1} \cup \overline{C}_{\Sigma_2} \). Restricting to the \( q = \pm 1 \) sector containing ordinary de Sitter space, we can easily guess the action of this group by considering the generators of the \( \mathbb{Z}_2 \) degeneracy on \( Q_{\pm 1} \). We first recall that \( \overline{Diff} \) consists of the set of identity connected diffeomorphisms that restrict to hypersurface and geometry preserving diffeomorphisms on the boundary \( \partial M = \Sigma_1 \cup \Sigma_2 \). The subgroup of the isometry group of de Sitter space that preserves the embedding of a spatial hypersurface with topology \( S^3 \) is \( SO(4) \), which consists of the spatial rotations and compact spatial translations. Taking the double cover, we expect that \( \overline{Diff}/\overline{Diff}_0 \) can be identified with some appropriate equivalence class of \( Spin(4)_{\Sigma_1} \times Spin(4)_{\Sigma_2} \). To find this equivalence class, we consider three separate cases.

First, consider the action of a diffeomorphism representing a rotational or spatial translational isometry in a fixed direction. As an element of \( \overline{Diff}/\overline{Diff}_0 \), the action of the isometry on \( \overline{C}_{\Sigma_1} \cup \overline{C}_{\Sigma_2} \) can be represented by the pair \( \{g_1, g_2\} = \{g(\epsilon), g(\epsilon)\} \) where \( \epsilon \) ranges from 0 to 1 and \( g(\epsilon) \) represents a \( 2\pi\epsilon \) rotation. Such an isometry is \textit{not} a generator of the \( \mathbb{Z}_2 \) degeneracy as can be seen as follows. The action of the

\[^3\text{If, for example, one were to consider the phase space corresponding to Einstein-Cartan theory via the EC action restricted to the set of flat Cartan connections, then this property would hold.}\]
diffeomorphism on \( Q \) can be constructed by fixing the data at \( t = 0 \) and twisting each endcap by \( \epsilon \) while keeping \( \Sigma_0 \) fixed. At \( \epsilon = 1 \), the lower and upper halves are both generators of \( \mathbb{Z}_2 \) on \( Q \), but the product of two generators of \( \mathbb{Z}_2 \) is itself deformable to the identity. As a consequence, we should expect that for an isometry, \( \{g(0), g(0)\} \approx \{g(1), g(1)\} \). Since \( g(1) = -1 \), we have \( \{1, 1\} \approx \{-1, -1\} \).

Next consider a true generator of \( \mathbb{Z}_2 \) on \( Q \). This can be constructed in two equivalent ways. One way is to fix the hypersurface \( \Sigma_1 \) and rotate or translate only the other endcap \( \Sigma_2 \). This can be represented by the pair \( \{g_1, g_2\} = \{1, g(\epsilon)\} \). The endpoint of the twist at \( \epsilon = 1 \) is therefore \( \{1, -1\} \). On the other hand, the generator can also be constructed by holding \( \Sigma_2 \) fixed and rotating or translating \( \Sigma_1 \) in the opposite direction. In this case the action is represented by the pair \( \{g_1, g_2\} = \{g(\epsilon)^{-1}, 1\} \). The endpoint of the twist in this case is given by \( \{-1, 1\} \). However, since both cases represent the same action at the endpoint \( \epsilon = 1 \), we must have \( \{1, -1\} \approx \{-1, 1\} \).

In total this implies that the action of \( \text{Diff}/\text{Diff}_0 \) on the configuration space is given by

\[
\frac{\text{Spin}(4) \times \text{Spin}(4)}{\{-1, -1\}} \rhd \overline{\Sigma}_1 \cup \overline{\Sigma}_2. \tag{11}
\]

One consequence of this is that even when the double cover is taken, the action of the set of boundary preserving isometries on \( \overline{\Sigma}_1 \cup \overline{\Sigma}_2 \) is still isomorphic to \( \text{SO}(4) \).

### 4.1 The fermionic inner product

Since we now understand the action of \( \overline{\text{Diff}}/\overline{\text{Diff}}_0 \) on \( \overline{\Sigma}_1 \cup \overline{\Sigma}_2 \) (in the \( q = \pm 1 \) sector) it will be easiest to consider the inner product between an initial state \( |\Psi_1\rangle \) on \( \Sigma_1 \) and the final state \( |\Psi_2\rangle \) on \( \Sigma_2 \).

First, consider an inner product that carries a faithful representation of the full group \( \overline{\text{Diff}}/\overline{\text{Diff}}_0 \). Since the action of the isometry subgroup is isomorphic to \( \text{SO}(4) \), the inner product can be thought of as a vector representation of \( \text{SO}(4) \), hence, it carries an index denoted \( \langle \Psi_2 | \Psi_1 \rangle^I \). The absolute-square is then given by

\[
|\langle \Psi_2 | \Psi_1 \rangle^I|^2 \equiv \eta_{ij} \langle \Psi_2 | \Psi_1 \rangle^i \langle \Psi_2 | \Psi_1 \rangle^j. \tag{12}
\]

To be more specific, let us work a particular representation. First associate 4-component Dirac spinors (or some appropriate subset therein) \( \psi_1 \) and \( \psi_2 \) with the initial and final states. Defining \( \gamma^I = \{\gamma^I, \gamma_5\} \), the intermediate inner product might be given by

\[
\langle \Psi_2 | \Psi_1 \rangle^I \equiv \bar{\psi}_2 \gamma^I \psi_1. \tag{13}
\]

This inner product is chosen because it is faithful to the action of \( \overline{\text{Diff}}/\overline{\text{Diff}}_0 \) on \( \overline{\Sigma}_1 \cup \overline{\Sigma}_2 \). Note that the action of \( \{g_1, g_2\} \) is given by

\[
\langle g_2 \Psi_2 | g_1 \Psi_1 \rangle^I \equiv \bar{\psi}_2 g_2^{-1} \gamma^I g_1 \psi_1. \tag{14}
\]
The isometry subgroup therefore acts on the inner product by

\[ \langle g \Psi_2 | g \Psi_1 \rangle^I \equiv \bar{\psi}_2 g^{-1} \gamma^I g \psi_1 = g^I_J \bar{\psi}_2 \gamma^J \psi_1 \]

(15)

where \( g^I_J \) is the adjoint representation of Spin(4), which is itself isomorphic to SO(4) as should be expected from an isometry. It follows that \( |\langle \Psi_2 | \Psi_1 \rangle|^2 \) is invariant under isomorphisms. However, under a twist corresponding to a generator of the \( \mathbb{Z}_2 \) degeneracy of \( Q \), the inner product transforms by

\[ \langle \Psi_2 | \Psi_1 \rangle^I \rightarrow \langle g_2 \Psi_2 | g_1 \Psi_1 \rangle^I = -\langle \Psi_2 | \Psi_1 \rangle^I. \]

(16)

Alternatively, we can define an inner product that is invariant under isomorphisms, but still retains information about twists by

\[ \langle \langle \Psi_2 | \Psi_1 \rangle \rangle \equiv \bar{\psi}_2 \psi_1. \]

(17)

5 Concluding Remarks

In ordinary quantum field theory, it is generally taken as given that bosons are comprised of ordinary commuting fields, and fermions are comprised of Grassman fields. However, the non-linear structure of certain field theories admits an alternative means for fermionic statistics to emerge. Here I have given strong evidence that this possibility may be realized in quantum gravity. Together with our previous work on emergent fermions in torsional systems \([11, 12]\), we now have two examples of fermionic geometries. Regarding the present work, the prospect that an entire spacetime could itself behave fermionically challenges many implicit tenets of quantum gravity, quantum cosmology, and even quantum field theory. Furthermore, the spacetime geometry in question is hardly exotic, being the simplest model spacetime beyond Minkowski space. Thus, the physical ramifications of fermionic geometries need to be explored in depth.

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A Clifford algebra conventions

Here I will review the basic conventions used in this paper regarding the Clifford algebra notation. The Clifford algebra is spanned by a set of Clifford matrices \( \{ \gamma^I \} \) satisfying \( \gamma^I \gamma^J + \gamma^J \gamma^I = 2 \eta^{IJ} \) where \( \eta^{IJ} = diag(-1, 1, 1, 1) \). The volume element
$\gamma_5$ is defined to be $\gamma_5 = i^{\frac{1}{2}} \epsilon_{IJKL} \gamma^I \gamma^J \gamma^K \gamma^L = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$, where it is understood that the alternating symbol $\epsilon_{IJKL}$ is such that $\epsilon_{0123} = -\epsilon_{0123} = 1$. The ten dimensional de Sitter algebra, $\text{spin}(4,1)$ is spanned by the basis elements $\{\frac{1}{2} \gamma^{[I} \gamma^{J]} , \frac{1}{2} \gamma_5 \gamma^K \}$. The Lorentz stabilizer subalgebra, denoted $\mathfrak{h} = \text{spin}(3,1)$ is spanned by $\{ \frac{1}{2} \gamma^{[I} \gamma^{J]} \}$ and its complement $\mathfrak{p}$ in $\text{spin}(4,1)$ is spanned by $\{ \frac{1}{2} \gamma_5 \gamma^K \}$. Given an antisymmetric Lorentz valued matrix $A^{IJ} = A^{[IJ]}$, the index free object $A$ is given by $A \equiv \frac{1}{2} \gamma_5 [\gamma_I, \gamma_J] A^{IJ}$.

The $\text{spin}(4,1)$ connection coefficient in a local trivialization denoted by $A$ splits correspondingly. However, it is convenient to represent the tetrad as simply $e \equiv \frac{1}{2} \gamma_5 e^I$ which pulls the $\gamma_5$ out of the decomposition $A = \omega \oplus \frac{1}{2} \gamma_5 e$.

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