Algebra

A characterization of generalized quaternion 2-groups

Une caractérisation des groupes de quaternions généralisés

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ABSTRACT

The goal of this Note is to give a characterization of generalized quaternion 2-groups by using their posets of cyclic subgroups.

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RÉSUMÉ

Le but de cette Note est de donner une caractérisation des 2-groupes de quaternions généralisés en utilisant leur ensembles partiellement ordonnés de sous-groupes cycliques.

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1. Introduction

Let $G$ be a finite group and $L(G)$ be the subgroup lattice of $G$. The starting point for our discussion is given by the paper [1], where the proper nontrivial subgroups $H$ of $G$ with the property that

for every $X \in L(G)$ we have $X \leq H$ or $H \leq X$

have been studied. Such a subgroup is called a breaking point for the lattice $L(G)$. Clearly, if $L(G)$ is a chain (i.e. $G$ is a cyclic $p$-group), then all proper nontrivial subgroups $H$ of $G$ are breaking points. On the other hand, we remark that the above concept can naturally be extended to other remarkable posets of subgroups of $G$ (and also to arbitrary posets). One of them is the poset of cyclic subgroups of $G$, denoted usually by $C(G)$. The study of the existence and of the uniqueness of breaking points in $C(G)$ constitutes the purpose of this paper.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on group theory can be found in [2] and [4]. For subgroup lattice notions we refer the reader to [3] and [5].

We mention that by a generalized quaternion 2-group we mean a group of order $2^n$ for some natural number $n \geq 3$, defined by the presentation

$$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1} \rangle.$$ 

We also recall that these groups are the unique finite noncyclic $p$-groups all of whose abelian subgroups are cyclic, or equivalently the unique finite noncyclic $p$-groups possessing exactly one subgroup of order $p$ (see (4.4) of [4], II). Obviously, this result shows that the subgroup of order 2 of $Q_{2^n}$, namely $(a^{2^{n-2}})$, is the unique breaking point of $C(Q_{2^n})$.

Our main theorem proves that generalized quaternion 2-groups exhaust all finite noncyclic groups whose posets of cyclic subgroups have breaking points.

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Theorem 1.1. Let $G$ be a finite group. Then $C(G)$ possesses breaking points if and only if $G$ is either a cyclic $p$-group of order at least $p^2$ or a generalized quaternion 2-group.

2. The proof of Theorem 1.1

We observe first that the above theorem can be easily proved in the particular case of $p$-groups.

Lemma 2.1. Let $G$ be a finite $p$-group. Then $C(G)$ possesses breaking points if and only if $G$ is either a cyclic $p$-group of order at least $p^2$ or a generalized quaternion 2-group.

Proof. Suppose that $G$ is not cyclic and let $H$ be a breaking point of $C(G)$. Then all minimal subgroups $M_1, M_2, \ldots, M_k$ of $G$ are contained in $H$. If $k \geq 2$, then we infer that $H$ is not cyclic, a contradiction. So, we have $k = 1$, that is $G$ has a unique subgroup of order $p$. This implies that $G$ is a generalized quaternion 2-group, according to the result mentioned in Section 1.

The converse implication is obvious, completing the proof. □

We are now able to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that the poset $C(G)$ of cyclic subgroups of a finite group $G$ possesses a breaking point, say $H$.

In the following we shall focus on proving that $G$ must necessarily be a $p$-group. By the way of contradiction, assume that the order of $G$ has at least two distinct prime divisors. Clearly, the same thing can be also said about the order of $H$. Let $p \in \pi(G)$ and $K$ be a cyclic $p$-subgroup of $G$. Since $H$ is not a $p$-subgroup, we infer that $K \subseteq H$. In other words, $H$ contains any cyclic $p$-subgroup of $G$ and consequently any $p$-element of $G$. This implies that all Sylow $p$-subgroups of $G$ are contained in $H$. Then $H = G$, a contradiction.

Hence $G$ is a $p$-group, for some prime $p$, and now the conclusion follows from Lemma 2.1. □

By the above results we also infer that, given a finite group $G$, the poset $C(G)$ possesses a unique breaking point if and only if $G$ is either a cyclic $p$-group of order $p^2$ or a generalized quaternion 2-group. In other words, the following corollary holds.

Corollary 2.2. The generalized quaternion 2-groups are the unique finite noncyclic groups whose posets of cyclic subgroups have exactly one breaking point.

Finally, we indicate a natural generalization of our study, suggested by the reviewers of the paper. Let $G$ be a finite group and denote by

$$C(G) = \{[H] \mid H \in C(G)\}$$

the set of conjugacy classes of cyclic subgroups of $G$. Mention that $C(G)$ is also a poset under the ordering relation

$$[H_1] \leq [H_2] \quad \text{if and only if} \quad H_1 \subseteq H_2^g, \quad \text{for some } g \in G.$$ 

Take a breaking point $[H]$ of $C(G)$. Then $H \in C(G)$ satisfies the following condition: for any cyclic subgroup $C$ of $G$, some conjugate of $C$ in $G$ contains or is contained in $H$. Clearly, this is weaker than the condition that $H$ be a breaking point of $C(G)$. We remark that for a finite $p$-group $G$ it is sufficient to guarantee the uniqueness of a subgroup of order $p$ in $G$. In other words, Lemma 2.1 also holds if we replace $C(G)$ with $C(G)$. In the general case, that is for arbitrary finite groups $G$, the problem of characterizing the existence and the uniqueness of breaking points of $C(G)$ remains still open.

3. Conclusions and further research

All previous results show that the concept of breaking point in some posets of subgroups of a (finite) group $G$ can constitute an important aspect of subgroup lattice theory. Clearly, its study started in [1] for lattices of subgroups and continued in the present paper for posets of cyclic subgroups can successfully be extended to other significant lattices/posets associated to $G$ (as the lattice of normal/subnormal/characteristic/solitary subgroups of $G$ or the poset of centralizers/conjugacy classes of elements (subgroups) of $G$). Studying the breaking points of arbitrary posets (not necessarily connected with a group $G$) seems to be also very interesting. These will surely constitute the subject of some further research.

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