On the smoothness of multi-M2 brane horizons

Chethan N Gowdigere, Siddharth Satpathy and Yogesh K Srivastava

National Institute of Science Education and Research. Sachivalaya Marg, PO Sainik School, Bhubaneswar 751005, India

E-mail: chethan.gowdigere@niser.ac.in, yogeshs@niser.ac.in and siddharthsatpathy.ss@gmail.com

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Abstract
We calculate the degree of horizon smoothness of a multi-M2 brane solution with branes along a common axis. We find that the metric is generically only thrice continuously differentiable at any of the horizons. The 4-form field strength is found to be only twice continuously differentiable. We work with Gaussian null-like co-ordinates which are obtained by solving geodesic equations for multi-M2 brane geometry. We also find different, exact co-ordinate transformations which take the metric from isotropic co-ordinates to co-ordinates in which the metric is thrice differentiable at the horizon. Both methods give the same result that the multi-M2 brane metric is only thrice differentiable at the horizon.

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1. Introduction

Multi-black-hole solutions in the four-dimensional Einstein–Maxwell theory have been known and analyzed extensively [5] in the literature. In [1], smoothness of multi-black-hole spacetimes in higher dimensional Einstein–Maxwell theory was analyzed using Gaussian null co-ordinates, building on earlier work by [4]. In [2], a similar analysis was performed for multi-BMPV black holes. In this work, we analyze the issue of horizon smoothness for the case of M2 brane of M-theory. Among the fundamental objects in string and M-theories, M2 branes [6] are special in having an analytic horizon. In [3], a multi-M2 brane metric was given and the possibility that horizon may not be analytic was mentioned. In [8], authors considered the case of infinite array of M2 branes and found that the horizon is analytic in this case. This case is parallel to the higher dimensional black holes in compactified spacetimes, considered in [9]. In this work, we explore the problem of degree of smoothness of a multi-M2 brane horizon in detail. We use Gaussian null co-ordinates to construct a co-ordinate chart that covers the horizon. The metric near the horizon is constructed as a series in an affine parameter along a geodesic and the resulting metric coefficients are found to be thrice differentiable.
only. We also construct exact $C^3$ extension of the metric by means of suitable co-ordinate transformations. Both approaches give the same result. Because of finite differentiability at the horizon, the extension and hence the interior metric are not unique.

2. Single-centered M2 brane

First, we analyze the case of single-centered M2 brane and find co-ordinates in which the metric is analytic at the horizon and can be continued to the interior. Here, the term analytic refers to real analyticity, i.e being infinitely differentiable at the horizon as real functions and the existence of a Taylor series expansion around the horizon which converges to the function. As a matter of notation, we will call a function $f(x)$ the $C^k$ function if the function $f(x)$ and its derivatives $f', f'', \ldots , f^k$ exist and are continuous at the point under consideration. The single or coincident M2 brane metric is

$$\text{ds}^2 = H^{-2/3}(-\text{d}t^2 + \text{d}x_1^2 + \text{d}x_2^2) + H^{1/3} (\text{d}r^2 + r^2 \text{d}\Omega_3^2), \quad (2.1)$$

$$C_{012} = H^{-1}, \quad H = 1 + \frac{\mu_1}{\rho}. \quad (2.2)$$

To go to non-singular co-ordinates, we define the following [7]:

$$r = (\mu_1)^{1/6}(\rho^{-3} - 1)^{1/6} = (\mu_1)^{1/6}\sqrt{\rho}(1 - \rho^{-3})^{-1/6}. \quad (2.3)$$

With this, we have $H = \frac{1}{\rho}$ and the metric becomes

$$\text{ds}^2 = \rho^2(-\text{d}t^2 + \text{d}x_1^2 + \text{d}x_2^2) + (\mu_1)^{1/3}\left(\frac{\text{d}\rho^2}{4\rho^2} + \text{d}\Omega_7^2\right) + (\mu_1)^{1/3} \frac{\text{d}\rho^2}{4\rho^2}[(1-\rho^{-3})^{-7/3} - 1] + (\mu_1)^{1/3}[(1 - \rho^{-3})^{-1/3} - 1] \text{d}\Omega_3^2. \quad (2.4)$$

First line above is the metric for $\text{AdS}_4 \times S^7$ in the Poincaré co-ordinates (locally), while the second line is regular at $\rho \rightarrow 0$. By performing the expansion in terms of $\rho$ around the horizon, we see that only the positive integer powers of $\rho$ occur in the metric coefficients, and hence, the single-centered case has an analytic horizon. Later in this section, we show that, in spite of appearance, the AdS$_4$ part metric is also analytic and we can find the co-ordinate transformation which express this. In terms of $\rho$, the horizon (which was at $r = 0$ in previous isotropic co-ordinates) is at $\rho = 0$, while asymptotic infinity corresponds to $\rho \rightarrow 1^-$. Now, we derive explicit regular co-ordinates for AdS$_4$ which will also be useful later on:

$$\text{ds}^2_{\text{AdS}_4} = \rho^2(-\text{d}t^2 + \text{d}x_1^2 + \text{d}x_2^2) + (\mu_1)^{1/3}\left(\frac{\text{d}\rho^2}{4\rho^2} = \rho^2(\text{d}u \text{d}v + \text{d}x_2^2) + (\mu_1)^{1/3}\left(\frac{\text{d}\rho^2}{4\rho^2}\right), \quad (2.5)$$

where we have defined light-cone co-ordinates $u, v = x_1 \pm t$. First thing to note is that the horizon is not just $\rho = 0$ but also $t \rightarrow \infty$. So to go to the horizon we need the limits

$$\rho \rightarrow 0, \quad u, v \rightarrow \infty. \quad (2.6)$$

Now, we define new co-ordinates

$$v = -\frac{1}{V}, \quad u = U + \frac{X_2^2}{V}, \quad \rho = \frac{\mu_1^{1/6}VW}{2}, \quad x_2 = \frac{X_2}{V}. \quad (2.7)$$

In these co-ordinates, horizon is at $V = 0$, while $W$ is finite at the horizon. Using these, we obtain

$$\text{ds}^2_{\text{AdS}_4} = (\mu_1)^{1/3}\frac{1}{4}\left(W^2(\text{d}U \text{d}V + \text{d}X_2^2) + \frac{\text{d}W^2}{W^2}\right). \quad (2.8)$$

One can see that in these co-ordinates the metric for $\text{AdS}_4$ is regular. Rest of the metric (2.4) is easily seen to be analytic at the horizon in these co-ordinates.
3. Differentiability of the two-centered M2 brane horizon using axial null geodesics

In this section, we follow [1] to perform a quick calculation to determine the differentiability of one particular component of the metric. This will prepare us for the use of Gaussian null co-ordinates for the multi-centered M2 brane metric which is done in next section. Here, we use the argument given in [1] which proves the following: if the metric admits a $C^k$ extension through one of the horizons, with $k \geq 2$, and that this extension admits a Killing vector field $V$, then $V$ must be $C^k$. We will apply this result to the isometries of the solution corresponding to spacetime translations along the brane world volume, i.e. $V = \partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{\Omega_1}$. The norm of $V$ is $\propto H^{-2/3}$. Since both the metric and $V$ are $C^k$, it follows that $H^{-2/3}$ must also be $C^k$ through the horizon. We can use this to determine an upper bound on $k$ by considering $H^{-2/3}$ along an axial null geodesic.

Axial geodesics are best studied in a co-ordinate system that uses cylindrical polar co-ordinates $z, \sigma, \Omega_6$ in the transverse space. The metric for the two-centered M2 brane solutions in cylindrical polar co-ordinates is

$$\text{d} s^2 = H^{-2/3}(-\text{d}x^2 + \text{d}x_1^2 + \text{d}x_2^2) + H^{1/3}(\text{d}z^2 + \text{d}\sigma^2 + \sigma^2 \text{d}\Omega_6^2)$$  \hspace{0.5cm} (3.9)

with

$$H = 1 + \frac{\mu_1}{(\sigma^2 + z^2)^{3}} + \frac{\mu_2}{(\sigma^2 + (z - a)^2)^3}.$$  \hspace{0.5cm} (3.10)

Consider a future-directed null geodesic approaching the origin along the positive $z$-axis. This geodesic has a non-trivial dependence on the affine parameter for only four co-ordinates, i.e. $t(\lambda), \ x_1(\lambda), \ x_2(\lambda)$ and $z(\lambda)$; all other co-ordinates take constant values except $\sigma(\lambda) = 0$. Non-trivial geodesic equations are

$$\frac{d}{d\lambda}(-H^{-\frac{2}{3}} t) = 0, \quad \frac{d}{d\lambda}(H^{-\frac{1}{3}} x_1) = 0, \quad \frac{d}{d\lambda}(H^{-\frac{1}{3}} x_2) = 0,$$

$$H^{-\frac{2}{3}} (-\dot{t}^2 + \dot{x}_1^2 + \dot{x}_2^2) + H^{\frac{1}{3}}(\dot{z}^2) = 0.$$  \hspace{0.5cm} (3.11)

Here, $\lambda$ derivatives are denoted by dot over the respective variables. In this special case, these geodesic equations can be integrated once$^1$ to give

$$-H^{2/3} + H^{1/3} \dot{z}^2 = 0,$$  \hspace{0.5cm} (3.13)

which result in

$$\frac{dz}{d\lambda} = -H^{1/6} = -\left(1 + \frac{\mu_1}{z(\lambda)^6} + \frac{\mu_2}{(z(\lambda) - a)^6}\right)^{\frac{1}{3}}.$$  \hspace{0.5cm} (3.14)

We know that the horizon is located at $z = 0$. We can choose that the affine parameter takes the value zero at the horizon so that $z(\lambda)$ is small near $\lambda = 0$. A small $z$ expansion of $H^{1/6}$ will have $1/z$ as the leading order term. Hence, $\frac{dz}{d\lambda} \sim -\frac{1}{z}$, and this implies that $z(\lambda)$ can be expanded in terms of powers of $\sqrt{-\lambda}$, the minus sign because $\lambda$ is negative outside the horizon and becomes zero at the horizon, i.e. $z \rightarrow 0^+$ as $\lambda \rightarrow 0^-$. Hence, we make the ansatz

$$z(\lambda) = \sum_n c_n (\sqrt{-\lambda})^n.$$  \hspace{0.5cm} (3.15)

Now, we use (3.14) to determine the coefficients $c_n$:

$^1$ See the next section for more details about this.
The norm of the Killing vector fields is then computed:

\[ H^{-2/3} = \frac{4}{\mu_1^{1/3}} + \frac{56}{3 a^6 \mu_1^{5/6}} \kappa^6 - \frac{1024}{9 a^7 \mu_1^{3/4}} \sqrt{2} \mu_1^2 (\sqrt{-\kappa})^{11} + O(\lambda^6). \]  

We thus see that the norm is a \( C^5 \) function and not a \( C^6 \) function; the addition of an extra center has decreased the horizon smoothness. Also, the single-centered case can be obtained by substituting \( \mu_2 = 0 \) into the above and we see that the metric is analytic (at least to the order we have displayed) but of course we know this from the construction of exact co-ordinates in the previous section.

One can try to consider the norm of Killing vectors corresponding to the \( SO(7) \) symmetry of the metric to determine the degree of differentiability of corresponding metric components. But the six-sphere along which these Killing vector fields are supported becomes of zero size everywhere on the axial geodesic, and consequently, the norms of the Killing vector fields vanish. We would need to consider radial geodesics which we do in the next section.

4. Gaussian null-like co-ordinates

In this section, we will construct a co-ordinate system which will provide a good co-ordinate system in the neighborhood of the event horizon. The co-ordinate system is obtained from the family of non-axial null geodesics. In this sense, it is similar to the Gaussian null co-ordinate system for the neighborhood of an event horizon. But the co-ordinate system that we construct is not exactly the Gaussian null co-ordinate system; on the horizon hypersurface, one of the Gaussian null co-ordinates is the affine parameter along the null geodesic generators of the horizon but the co-ordinates that we construct below do not have this feature. Nevertheless, it does provide a good co-ordinate system for the neighborhood of the horizon which is what is needed to address the smoothness of the metric at the horizon and to extend it into the interior. We will refer to the co-ordinate system that we construct as Gaussian null-like co-ordinates.

The family of non-axial null geodesics is best studied in a spherical co-ordinate system in the neighborhood of the event horizon. But the co-ordinate system that we construct is the affine parameter along the null geodesic generators of the horizon but the co-ordinates that we construct below do not have this feature. Nevertheless, it does provide a good co-ordinate system for the neighborhood of the horizon which is what is needed to address the smoothness of the metric at the horizon and to extend it into the interior. We will refer to the co-ordinate system that we construct as Gaussian null-like co-ordinates.

The horizon of the first center is at \( r = 0 \) and we will investigate the smoothness there. It is convenient to expand in terms of the relevant spherical harmonics

\[ H = \frac{\mu_1}{r^6} + \sum_{n=0}^{\infty} h_n r^n Y_n(\cos \theta), \]  

with

\[ \mu_1 = \frac{1}{\mu_1^{1/3}} + \sum_{j=0}^{N} \frac{\mu_j}{(r^2 - 2 a_j r \cos \theta + a_j^2)^{3/2}}. \]
where \( Y_n(\cos \theta) \) are certain Gegenbauer polynomials:

\[
\begin{align*}
Y_0(\cos \theta) &= 1, \\
Y_1(\cos \theta) &= 6 \cos \theta, \\
Y_2(\cos \theta) &= 24 \cos^2 \theta - 3, \\
Y_3(\cos \theta) &= 80 \cos^3 \theta - 24 \cos \theta, \\
Y_4(\cos \theta) &= 240 \cos^4 \theta - 120 \cos^2 \theta + 6, \\
Y_5(\cos \theta) &= 672 \cos^5 \theta - 480 \cos^3 \theta + 60 \cos \theta, \\
Y_6(\cos \theta) &= 1792 \cos^6 \theta - 1680 \cos^4 \theta + 360 \cos^2 \theta - 10
\end{align*}
\]

and the coefficients \( h_n \) are

\[
h_n = \delta_{n,0} + \sum_{i=2}^{N} \frac{\mu_i}{\delta_{i,n}}.
\] (4.21)

4.1. Constructing Gaussian null-like co-ordinates

We expect that the \( SO(7) \) symmetry of the solution outside the horizon continues to be a symmetry of the extension of the metric. Hence, we will only need to consider geodesics that have a constant angular momentum along \( S^5 \), i.e. angular co-ordinates along \( S^5 \) will not change along the geodesic. The co-ordinates that are the non-trivial functions of the affine parameter are \( t, x_1, x_2, r, \theta \). The geodesic equations in isotropic co-ordinates then are five coupled second-order differential equations, four of which can be integrated once:

\[
\begin{align*}
\frac{d}{d\lambda}(-H^{-\frac{1}{2}} i) &= 0, \\
\frac{d}{d\lambda}(H^{-\frac{1}{2}} i \dot{x}_1) &= 0, \\
\frac{d}{d\lambda}(H^{-\frac{1}{2}} i \dot{x}_2) &= 0,
\end{align*}
\] (4.22)

\[
H^{-\frac{1}{2}} (-\dot{r}^2 + \dot{x}_1^2 + \dot{x}_2^2) + H^\frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) = 0,
\] (4.23)

\[
\dot{r} = \frac{\partial_\lambda H}{3 H^\frac{1}{2}} + \frac{\partial_\lambda H}{6H} (\dot{r}^2 - r^2 \dot{\theta}^2) - r \dot{\theta}^2 + \frac{\partial_\lambda H}{3H} \dot{\theta} = 0.
\] (4.24)

The angles on \( S^5 \) do not change along the geodesic; they continue to be co-ordinates in the Gaussian null-like co-ordinate system. The angle \( \theta \) changes along the geodesic; its value on the horizon, denoted by \( \Theta \), is taken to be one of Gaussian null-like co-ordinates. The other four co-ordinates in the Gaussian null-like co-ordinate system are the affine parameter \( \lambda \) and the integration constants that result by integrating the \( t, x_1, x_2 \) geodesic equations. We first solve equations (3.11) in the following manner:

\[
\begin{align*}
t &= v - f(v, w, y_2) T(\lambda, \Theta), \\
x_1 &= w - g(v, w, y_2) T(\lambda, \Theta), \\
x_2 &= y_2 - h(v, w, y_2) T(\lambda, \Theta),
\end{align*}
\] (4.25)

where

\[
T(\lambda, \Theta) = \int d\lambda H^{-\frac{1}{2}} (\lambda, \Theta)
\] (4.26)

will be determined below; the integration constants \( v, w, y_2 \) will form three of the Gaussian null-like co-ordinates. Isotropic co-ordinates \( t, x_1, x_2 \) go bad as we approach the horizon. This can be seen from (4.25), where \( T(\lambda, \Theta) \to \infty \) as \( \lambda \to 0 \). Gaussian null-like co-ordinates \( v, w, y_2 \) are well defined and finite at the horizon.

We choose to introduce the functions \( f, g, h \) of integration constants in the above because of a simpler choice, such as constants will not make the metric non-singular at the horizon;
then, we would not have obtained a good co-ordinate system for the neighborhood of the horizon. A completely arbitrary choice of the functions \( f, g, h \) does not make the metric in these co-ordinates non-singular. We will encounter various conditions along the way, one of them is that they need to satisfy the constraint

\[
S \equiv f^2 - g^2 - h^2 - 1 = 0.
\]  

(4.27)

Although we do not have a solution to all the constraints\(^2\) that \( f, g, h \) would need to satisfy by the end of the analysis, we do have many examples; thus, we do have multiple examples of Gaussian null-like co-ordinate systems. Constraint (4.27) means that we now need to solve the equations

\[
-H \dot{r} + H (\dot{r}^2 + r^2 \dot{\theta}^2) = 0,
\]  

(4.28)

\[
\dot{r} - \frac{\partial r}{\partial H} + \frac{\partial H}{6H} (\dot{r}^2 - r^2 \dot{\theta}^2) - r \dot{\theta}^2 + \frac{\partial h}{3H} \dot{\theta} = 0.
\]  

(4.29)

Let us assign the affine parameter \( \lambda = 0 \) for the event horizon; here, \( \lambda \) takes positive values outside the horizon (note that this convention is different from the previous section). \( r(\lambda, \Theta) \) and \( \theta(\lambda, \Theta) \) are the solutions to (4.28) and (4.29) with the initial conditions

\[
r(0, \Theta) = 0, \quad \theta(0, \Theta) = \Theta, \quad \dot{\theta}(0, \Theta) = 0.
\]  

(4.30)

We assume series expansions of the form

\[
r(\lambda, \Theta) = \sum_{n=1}^{\infty} c_n(\Theta) \lambda^{\frac{n}{2}}, \quad \theta(\lambda, \Theta) = \Theta + \sum_{n=1}^{\infty} b_n(\Theta) \Lambda^\frac{n}{2},
\]  

(4.31)

the \( \lambda^{\frac{1}{2}} \) expansion parameter again resulting from considering the leading order behavior of the geodesic equations at \( \lambda = 0 \). The result of solving (4.28) and (4.29) order by order is

\[
r(\lambda, \Theta) = \sqrt{2} \mu_1^{1/12} \lambda^{1/2} + \frac{h_0}{3 \sqrt{2} \mu_1^{5/12}} \lambda^{7/2} + \frac{16 h_1}{9 \mu_1} \cot \lambda^4 + \frac{4 \sqrt{2} h_2}{3 \mu_1^{1/2}} (3 + 4 c_{20}) \lambda^{9/2} + O(\lambda^5),
\]  

(4.32)

\[
\theta(\lambda, \Theta) = \Theta - \frac{24 \sqrt{2} h_1}{35 \mu_1^{5/12}} s_{\Theta} \lambda^{7/2} - \frac{4 h_2}{\mu_1^{1/2} s_{20}} \lambda^4 - \frac{32 \sqrt{2} h_3}{21 \mu_1^{1/4}} (3 s_{\Theta} + 5 s_{30}) \lambda^{9/2} + O(\lambda^5),
\]  

(4.33)

where \( c_\alpha = \cos \alpha, s_\alpha = \sin \alpha \). Appendices provides more details of obtaining these expansions as well as more terms all of which are needed to obtain the results in this section. With these expansions, the computation in (4.26) can be performed. We thus have, in (4.25), (4.32) and (4.33), the co-ordinate transformations between the Gaussian null-like co-ordinates \( v, w, \gamma_2, \lambda, \Theta, \Omega_\Theta \) and the isotropic co-ordinates \( t, x_1, x_2, r, \theta, \Omega_\Theta \). The functions \( f, g, h \) thus enter in the definition of the Gaussian null-like co-ordinates; they only need to satisfy the constraint (4.27) and any other conditions that ensures the metric on the horizon in the Gaussian null-like co-ordinates is non-singular. We will see below that we have many choices for \( f, g, h \).

\(^2\) There are only two of them: (4.27) and (4.36).
4.2. Metric is $C^3$

We use the co-ordinate transformations (4.25), (4.32) and (4.33) to compute the metric in the Gaussian null-like co-ordinates:

$$\text{d}s^2 = H^{-\frac{1}{2}}(\lambda, \Theta) \left[ \left( -1 + 2T(\lambda, \Theta) \partial_\lambda f + T^2(\lambda, \Theta) z_1(v, w, y_2) \right) \text{d}v^2 + \left( -1 + 2T(\lambda, \Theta) \partial_v g + T^2(\lambda, \Theta) z_2(v, w, y_2) \right) \text{d}w^2 \right. \right.$$  
$$\left. + \left( -1 + 2T(\lambda, \Theta) \partial_w h + T^2(\lambda, \Theta) z_3(v, w, y_2) \right) \text{d}y_2^2 \right]$$  
$$+ 2H^{-\frac{1}{2}}(\lambda, \Theta) T(\lambda, \Theta) \left[ q_1(v, w, y_2) + T(\lambda, \Theta) q_2(v, w, y_2) \right] \text{d}v \text{d}w + \left[ q_3(v, w, y_2) + T(\lambda, \Theta) q_4(v, w, y_2) \right] \text{d}v \text{d}y_2$$  
$$+ 2f \text{d}v \text{d}\lambda - 2g \text{d}w \text{d}\lambda - 2h \text{d}y_2 \text{d}\lambda + 2H^{-\frac{1}{2}}(\lambda, \Theta) \partial_\Theta T(\lambda, \Theta) [f \text{d}v \text{d}\Theta - g \text{d}w \text{d}\Theta - h \text{d}y_2 \text{d}\Theta]$$  
$$+ O(\lambda^2) \partial_\lambda^2 + H^{-\frac{1}{2}}(\lambda, \Theta)^2 \partial_\Theta^2 + O(\lambda^2) \partial_\lambda \partial_\Theta + H^{-\frac{1}{2}}(\lambda, \Theta)^2 \text{sin}^2(\theta(\lambda, \Theta)) \partial_\Omega^2.$$

Each of the unknown functions appearing above is defined in (B.1) and (B.2), respectively. The condition $S = 0$ ensures that the $g_{\lambda\lambda}$ component is well behaved at the horizon. Derived conditions $\partial_\Theta S = 0$, $\partial_v S = 0$, $\partial_w S = 0$ ensure that $g_{v\lambda}$, $g_{w\lambda}$, $g_{y_2 \lambda}$, respectively, are well behaved at the horizon.

As we can see from the explicit components of the metric, the least differentiable components of the metric are $g_{\Theta\Theta}$ and $g_{\Theta_0 \Theta_0}$, which are $C^3$ functions. Hence, we conclude that a multi-centered membrane solution is only $C^3$ at any of its horizons.

The metric (4.34) has the usual co-ordinate singularities of spherical co-ordinates at $\Theta = 0, \pi$. Apart from these, we also require that the metric is non-singular at $\lambda = 0$. We can compute the determinant of the metric on the horizon, i.e. at $\lambda = 0$:

$$g = \frac{\mu^4}{16} \sin^{12} \Theta \ g_{\Theta\Theta} \left[ f^2(q_6^2 - z_2z_3) + g^2(q_4^2 - z_3z_1) + h^2(q_2^2 - z_1z_2) \right.$$
$$\left. + 2fq_4q_6 - q_2z_3) + 2gh(q_4q_6 - q_2z_3) + 2f \text{d}q_4 \text{d}q_6 - 2gh \text{d}q_4 \text{d}q_6 \right] + 2f \text{d}h \text{d}q_4 \text{d}q_6 - 2gh \text{d}q_4 \text{d}q_6,$$

where $g_{\Theta\Theta}$ is the determinant of the round metric on the unit six-sphere and the $q_i$’s and the $z_i$’s are defined in (B.2). Requiring that the determinant does not vanish on the horizon gives us the following condition that our choice of $f, g, h$ functions must satisfy:

$$f^2(q_6^2 - z_2z_3) + g^2(q_4^2 - z_3z_1) + h^2(q_2^2 - z_1z_2) + 2fq_4q_6 - 2gh(q_4q_6 - q_2z_3)$$
$$- 2gh(q_4q_6 - q_2z_3) + 2f \text{d}q_4 \text{d}q_6 - 2gh \text{d}q_4 \text{d}q_6 - 2f \text{d}h \text{d}q_4 \text{d}q_6 - 2gh \text{d}q_4 \text{d}q_6 \neq 0.$$

Simple algebraic manipulations give that this is equivalent to

$$\left( f \sqrt{-q_6^2 + z_2z_3} - g \sqrt{-q_4^2 + z_3z_1} - h \sqrt{-q_2^2 + z_1z_2} \right)^2 \neq 0.
\quad (4.37)$$

We do not analyze this condition in detail here. But we do have many examples for the $f, g, h$ functions that satisfy (4.36) and (4.27), two of which are

$$f(v, w, y_2) = \frac{1}{2} \left( w + \frac{1}{w} + \frac{y_2^2}{w} \right), \quad g(v, w, y_2) = \frac{1}{2} \left( -w + \frac{1}{w} + \frac{y_2^2}{w} \right), \quad h(v, w, y_2) = y_2,$$
$$f(v, w, y_2) = \sqrt{1 + y_2^2 \cosh w}, \quad g(v, w, y_2) = \sqrt{1 + y_2^2 \sinh w}, \quad h(v, w, y_2) = y_2. \quad (4.38)$$
4.3. Volume of $S^6$

In the previous section, we deduced that the metric is only $C^5$ from the differentiability of the norm of the Killing vector field corresponding to time translations. This was achieved by examining the norm of the Killing vector field along the axial geodesic. This strategy does not work for computing the norm of the $SO(7)$ Killing vector fields because the six-sphere on which these Killing vector fields are supported becomes of zero size everywhere on the axial geodesic, and consequently, the norms of the Killing vector fields vanish. By considering radial geodesics, we have obtained the metric in the neighborhood of the event horizon. Using this metric, one can determine volume of $S^6$:

$$A_6 = \mu_1 \sin^6 \Theta + 8h_0\mu_1^2 \sin^6 \Theta \lambda^3 + \frac{1536\sqrt{2}}{35}h_1\mu_1^{7/12}\cos \Theta \sin^6 \Theta \lambda^{7/2} + O(\lambda^4).$$  \hspace{1cm} (4.39)

We thus see that the volume is a $C^3$ function. From the relation of the norm of the $SO(7)$ Killing fields to the volume of $S^6$, we can conclude that the norm is also only $C^3$.

4.4. Maxwell field strength is $C^2$

We now consider the degree of differentiability of 4-form field strength in the Gaussian null-like co-ordinates. The 3-form gauge potential in the Gaussian-null co-ordinates is obtained using the co-ordinate transformations (4.25), (4.32), (4.33):

$$A_{[3]} = H^{-1} \, dt \wedge dx_1 \wedge dx_2 \wedge (dv - f \, dt - T \, df)$$
$$= H^{-1} \, (dw - g \, dt - T \, dg) \wedge (dy_2 - h \, dt - T \, dh).$$ \hspace{1cm} (4.40)

The full expressions can be found in (C.2), (C.3) and (C.4). The 4-form field strength in the Gaussian null-like co-ordinates is

$$F_{[4]} = \delta(H^{-1}) \wedge dt \wedge dx_1 \wedge dx_2$$
$$= (\partial_{\lambda} H^{-1} \, d\lambda + \partial_{\Theta} H^{-1} \, d\Theta) \wedge (dv - f \, dt - T \, df)$$
$$\wedge (dw - g \, dt - T \, dg) \wedge (dy_2 - h \, dt - T \, dh).$$ \hspace{1cm} (4.41)

The nonzero components are the following:

$$F_{\lambda w y_2} = \partial_{\lambda} H^{-1} \{1 - u_1(v, w, y_2) T + u_2(v, w, y_2) T^2\},$$
$$F_{\Theta vw y} = \partial_{\Theta} H^{-1} \{1 - u_1(v, w, y_2) T + u_2(v, w, y_2) T^2\},$$
$$F_{\Theta w v y} = (\partial_{\Theta} H^{-1} \partial_{\omega} T - \partial_{\omega} T \partial_{\Theta} H^{-1})[h(v, w, y_2) - u_3(v, w, y_2) T + u_4(v, w, y_2) T^2],$$
$$F_{\lambda w y_2} = (\partial_{\lambda} H^{-1} \partial_{\omega} T - \partial_{\omega} T \partial_{\lambda} H^{-1})[g(v, w, y_2) - u_5(v, w, y_2) T + u_6(v, w, y_2) T^2],$$
$$F_{\Theta v y_2} = (\partial_{\Theta} H^{-1} \partial_{\omega} T - \partial_{\omega} T \partial_{\Theta} H^{-1})[f(v, w, y_2) - u_7(v, w, y_2) T + u_8(v, w, y_2) T^2].$$ \hspace{1cm} (4.42)

The various undefined terms above are gathered in (C.3), (C.4) and (C.1). Conditions $\partial_{\alpha} S = 0$, $\partial_{\mu} S = 0$, $\partial_{\gamma} S = 0$ require

$$\begin{pmatrix}
\partial_{\lambda} f \\
\partial_{\mu} f \\
\partial_{\gamma} f
\end{pmatrix} =
\begin{pmatrix}
-\partial_{\lambda} g \\
-\partial_{\mu} g \\
-\partial_{\gamma} g
\end{pmatrix} =
\begin{pmatrix}
-\partial_{\lambda} h \\
-\partial_{\mu} h \\
-\partial_{\gamma} h
\end{pmatrix} = 0$$ \hspace{1cm} (4.43)

for having a non-trivial solution for $f$, $g$, $h$. This condition ensures that components $F_{\lambda w y_2}$ and $F_{\Theta w y_2}$ are regular at the horizon.

From the formulae in appendices, particularly (C.3) and (C.4), we can see that the $F_{\lambda w y_2}$ component is a $C^3$ function, $F_{\Theta w y_2}$ is a $C^4$ function and $F_{\lambda w y_2}$, $F_{\Theta w y_2}$, $F_{\Theta v y_2}$ are $C^2$ functions. We thus conclude that $F_{[4]}$ is only $C^2$ at the horizon.
4.5. Equation of motion

\[ R_{\mu\nu} = \frac{1}{12} \left( F_{\mu\alpha\beta\gamma} F^\alpha F^\beta - \frac{1}{12} F^2 g_{\mu\nu} \right). \]  \hspace{1cm} (4.44)

We have shown that the metric is \( C^3 \). Roughly, one then expects that the curvature components are \( C^1 \) functions, i.e. the left-hand side of the equation of motion (4.44) is \( C^1 \). On the other hand, we have shown that the Maxwell field strength is \( C^2 \) which means roughly that the right-hand side of (4.44) have \( C^2 \) components. We are thus left with a puzzle about the mismatch of differentiability of the right- and left-hand sides of the equation of motion. To settle this, we simply perform the computations of the left- and right-hand sides of (4.44). We choose a specific choice of Gaussian null-like co-ordinates (D.1) and the results of the computations are collected in (D.4) and (D.3). From an examination of the Ricci tensor components (D.4), it is clear that there are no components which are only \( C^1 \), thus solving our puzzle. The components on the right-hand side of (4.44) have also been computed and they match with the Ricci tensor components as they should.

4.6. Extending through the horizon and the interior metric

Since the metric at the horizon is only of finite differentiability, infinitely many interior solutions can be matched to the exterior solution given by (4.18). We will assume that the interior metric has the same Killing vectors as those of the exterior metric. As done for the black hole case in [1], we assume that the interior metric takes the same form as the exterior metric in isotropic co-ordinates but with a different harmonic function \( \hat{H}(r, \theta) \). The range of the co-ordinates (except \( r \)) remains the same as the exterior metric:

\[ ds^2 = \hat{H}^{-2/3} (dr^2 + dx^2_1 + dx^2_2) + \hat{H}^{1/3} (d\theta^2 + r^2 \sin^2 \theta \, d\Omega^2_2), \]  \hspace{1cm} (4.45)

where \( \hat{H} \) is chosen to agree with \( H \) (for exterior metric) at the leading order:

\[ \hat{H} = \frac{\mu_1}{r^6} + \sum_{n=0}^{\infty} \hat{h}_n r^n Y_n(\cos \theta). \]  \hspace{1cm} (4.46)

In the exterior region, we have \( \lambda > 0 \) (this is consistent with section 4 but not with section 3), which is the affine parameter along a past-directed geodesic. For the interior region, we define the parameter \( \hat{\lambda} \) to be the affine parameter along a future-directed null geodesic. We construct (nearly) Gaussian null-like co-ordinates as before. The geodesic equation for \( r \) and \( \theta \) remains the same except for \( \lambda \to \hat{\lambda} \) and \( h_n \to \hat{h}_n \):

\[ t = v + f(v, w, y_2) \hat{T}(\hat{\lambda}, \Theta), \quad \hat{T}(\hat{\lambda}, \Theta) \equiv \int \hat{H}(\hat{\lambda}, \Theta)^{2/3} \, d\hat{\lambda}, \]  \hspace{1cm} (4.47)

and similarly for \( x_1, x_2 \) with \( \hat{T}(\hat{\lambda}, \Theta) \) defined as above:

\[ x_1 = w + g(v, w, y_2) \hat{T}(\hat{\lambda}, \Theta), \quad x_2 = y_2 + h(v, w, y_2) \hat{T}(\hat{\lambda}, \Theta). \]  \hspace{1cm} (4.48)

The interior metric that we obtain is

\[ ds^2 = \hat{H}^{-2} (\hat{\lambda}, \Theta) \left[ (1 - 2\hat{T}(\hat{\lambda}, \Theta) \hat{\partial}_z f + \hat{T}^2(\hat{\lambda}, \Theta) z_1(v, w, y_2)) dv^2 \right. \]

\[ + [1 + 2\hat{T}(\hat{\lambda}, \Theta) \hat{\partial}_z g + \hat{T}^2(\hat{\lambda}, \Theta) z_2(v, w, y_2)] dw^2 \]

\[ + [1 + 2\hat{T}(\hat{\lambda}, \Theta) \hat{\partial}_z h + \hat{T}^2(\hat{\lambda}, \Theta) z_3(v, w, y_2)] dy_2^2 \]

\[ + \hat{H}^{-2} (\hat{\lambda}, \Theta) \hat{T}(\hat{\lambda}, \Theta) \left[ - q_1(v, w, y_2) + \hat{T}(\hat{\lambda}, \Theta) q_2(v, w, y_2) \right] dv \, dw \]

\[ + [- q_1(v, w, y_2) + \hat{T}(\hat{\lambda}, \Theta) q_2(v, w, y_2)] dv \, dy_2 \]

\[ + [- q_1(v, w, y_2) + \hat{T}(\hat{\lambda}, \Theta) q_2(v, w, y_2)] dv \, dy_2 \]
Defining new co-ordinates at a distance not very illuminating though. Here, $d$ metric. This sets $C_{\text{H}} = 0$, we have to choose $\hat{\lambda} = -\lambda$ for the interior region. We match interior and exterior metrics up to order $\lambda^3$ since we only have a $C^3$ metric. This sets $h_0 = -\dot{h}_0$ and the rest of the coefficients are unconstrained. Hence, we have an infinite family of interior metrics parametrized by $h_n$ for $n \geq 1$. Similarly, the field strength can be matched up to order $\lambda^2$ terms without any further constraints on $h_n$.

5. Two-centered M2-brane solution: exact $C^3$ metric

The metric for two-centered M2 branes written in nearly Gaussian null co-ordinates provides a $C^3$ extension of the metric, written as a series in the affine parameter $\lambda$. In this section, we give an exact $C^3$ metric for the two-centered M2 brane case. The final form of the metric is not very illuminating though.

As we wrote earlier, for two M2 branes, one at the origin and other one along the $z$-axis, at a distance $a$, we have

$$H = 1 + \frac{\mu_1}{\rho^2} + \frac{\mu_2}{(r^2 + a^2 - 2ar \cos \theta)^{1/2}}.$$ \hspace{1cm} (5.50)

In section 2, we had given co-ordinate transformations which make the metric of single-centered M2 brane analytic at the horizon. For the two-centered M2-brane case, we will not obtain an analytic metric but only a $C^3$ metric. We start by making the same co-ordinate transformation as we use for a single-centered case: $r = \mu_1^{1/6}(\rho^{-3} - 1)^{-1/6}$. Using this, we obtain $H = \frac{f(\rho, \theta)}{\rho^3}$ and metric becomes

$$ds^2 = \frac{\rho^2}{f(\rho, \theta)} (dv^2 + dy^2) + f(\rho, \theta)^{1/3} \left( 1 - \rho^3 \right)^{-7/3} \frac{d\rho^2}{4}\left(1 - \rho^3\right)^{-1/3} d\Omega_3^2.$$ \hspace{1cm} (5.51)

Here,

$$f(\rho, \theta) = 1 + \rho^3 \sum_{n=0}^{\infty} h_n Y_n(\cos \theta)(1 - \rho^3)^{-n/6} \rho^{n/2}$$ \hspace{1cm} (5.52)

and $Y_n(\cos \theta)$ are the Gegenbauer polynomials and $h_n$ are constants given by

$$h_n = \frac{\mu_2 \mu_1^{n/6}}{a^6 a^n}.$$ \hspace{1cm} (5.53)

Defining new co-ordinates $U, V, W, X_3$ using the co-ordinate transformations given earlier in (2.7), we obtain the metric of the following form:

$$ds^2 = \frac{\mu_1^2}{4} (A(V, W, \theta) dV^2 + B(V, W, \theta) dV dU + 2C(V, W, \theta) dV dW + D(V, W, \theta) dX_3^2 + E(V, W, \theta) dW^2 + F(V, W, \theta) d\Omega_3^2).$$ \hspace{1cm} (5.54)

Here, the metric coefficients are given by following expressions:

$$A(V, W, \theta) = \frac{f(1 - \rho^3)^{-7/3} - 1}{V^2 f^{2/3}}, \quad B(V, W, \theta) = \frac{W^2}{f^{2/3}},$$ \hspace{1cm} (5.55)

$$C(V, W, \theta) = \frac{f(1 - \rho^3)^{-7/3} - 1}{W f^{2/3}}, \quad D(V, W, \theta) = \frac{W^2}{f^{2/3}}.$$ \hspace{1cm} (5.56)
we are interested in checking how differentiable are the metric coefficients at the horizon. The integer powers of \( V \) do not create any problems and so we concentrate on non-integer powers.

Knowing the expression for \( f \), we can see that the metric is \( C^1 \) as \( V \to 0 \). But we can obtain a \( C^2 \) metric by noting that the lowest non-integer power of \( V \) occurs in \( dV^2 \) term (\( V^{3/2} \)) and that can be canceled by making further co-ordinate transformation \( U \to U - G(V, W, \theta) \), tailored to cancel the term proportional to \( V^{3/2} \). In fact, by choosing

\[
G(V, \theta, W) = V^{5/2} L_1(W, \theta) + V^{7/2} L_2(W, \theta) + \cdots,
\]

we can cancel the half-integral powers of \( V \) in the coefficient of \( dV^2 \) term. Explicitly, the co-ordinate transformation is

\[
dV dU \to dU dV = \frac{\partial G(V, \theta, W)}{\partial V} dV^2 - \frac{\partial G(V, \theta, W)}{\partial W} dV dW - \frac{\partial G(V, \theta, W)}{\partial \theta} dV d\theta.
\]

This co-ordinate transformation makes the metric \( C^2 \) but it generates terms proportional to \( V^{5/2} \) in coefficients of \( dV dW \) and \( dV d\theta \). Note that since \( dV^2 \) does not contain any non-integer powers of \( V \), only terms which contain terms proportional to \( V^{5/2} \) are coefficients of \( dV dW \) and \( dV d\theta \). To obtain a \( C^2 \) metric at the horizon, we need to get rid of these terms proportional to \( V^{5/2} \). To do that, we make further co-ordinate transformations. First, we write \( W = e^P \) so that \( \frac{dw}{w} = dP \). Then, we make the following co-ordinate transformations:

\[
P \to P - K_1(P, \theta)V^{7/2}, \quad \theta \to \theta - K_2(P, \theta)V^{3/2}.
\]

Choosing \( K_1 \) and \( K_2 \) carefully, we can get rid of terms proportional to \( V^{5/2} \). Rest of the terms only have \( V^{3/2} \) or higher powers of \( V \), and hence, the metric at the horizon \( V = 0 \) is \( C^3 \) in these co-ordinates.

6. Conclusions and Outlook

In this work, we analyzed in detail the question of degree of smoothness of a multi-M2 brane metric. By solving for null geodesics in this geometry, we constructed ‘nearly Gaussian null co-ordinates’ and found that in terms of these co-ordinates, the metric can be extended across the horizon but the extended metric is only \( C^1 \) at the horizon. We also found an exact set of co-ordinate transformations which take from multiple M2-brane metric in isotropic co-ordinates to co-ordinates (different from Gaussian null co-ordinates) in which metric is \( C^3 \) at the horizon.

Finite differentiability of the metric at the horizon means that an observer falling through the horizon can detect the presence of horizon through local measurement [10]. Finite differentiability means that some derivatives of the Riemann tensor will blow up at the horizon and these can in principle be observed by an infalling observer. This singularity is very mild but this is unlike the case of single-centered M2 branes and black holes, which have an analytic horizon. It would be interesting to consider the case of a probe M2-brane in this spacetime to see if such kind of singularities have some effects. It is not clear if the fact that the interior metric of multiple M2-branes is not unique has any significance for the world-volume theory.

One may wonder whether higher derivative corrections have any effect on such singularity. In [2], a particular four-derivative term was considered for the case of multi-centered black holes in five-dimensional supergravity and found not to change the degree of smoothness of the metric at the horizon. It is not clear what happens if other higher derivative terms are included. For our M-theory case, higher derivative terms are either not well understood or very difficult to analyze. But the fact that corrections to classical two-derivative theory can happen at the scale of the horizon is quite significant in itself.
This work also leads to the question of the degree of smoothness of horizons when the multiple membranes are not confined to a common axis in the transverse space. One could for example consider multiple membranes confined to be only on a plane in the transverse space. The exterior metric would then have only an $SO(6)$ symmetry. The author of [2] argues that the degree of smoothness should decrease. The more symmetries of the single-centered solution that the multi-centered solution breaks, the less is the degree of horizon smoothness. Although the above statement is not a theorem, it seems reasonable. We are currently analyzing this issue in the context of multi-black-holes and multi-membrane solutions. We plan to report on this issue in a forthcoming work [11].

Appendix A. Series expansions for $r(\lambda, \Theta)$ and $\theta(\lambda, \Theta)$

In this appendix, we give a brief account of the computations leading to (4.32) and (4.33). We plug expansions (4.31) into equations (4.28) and (4.29) and solve it order by order in $\lambda$ [1]. Instead of imposing the boundary conditions (4.30), we impose the modified boundary conditions:

$$r(0, \Theta) = 0, \quad \theta(0, \Theta) = \Theta, \quad \dot{\theta}(0, \Theta) = b. \quad (A.1)$$

Doing this ensures that every $c_i$ and $b_j$ coefficient at whatever order it appears first appears linearly, thus giving a unique solution. They do occur at higher orders with higher powers but by then they have been determined and become constants in terms of which the solutions to whatever coefficients that occur linearly at that order are determined. We do this order by order and then take the limit $b \to 0$ to obtain the following series expansions. Note that $c_\alpha \equiv \cos \alpha, s_\alpha \equiv \sin \alpha$

$$r(\lambda, \Theta) = \sqrt{2} \mu_1^{1/12} \lambda^{1/2} + \frac{h_0}{3\sqrt{2} \mu_1^{5/12}} \lambda^{7/2} + \frac{16h_1}{9 \mu_1^{1/3}} c_{13} \lambda^4 + \frac{4\sqrt{2}h_2}{5 \mu_1^{1/4}} (3 + 4c_{26}) \lambda^{9/2}$$
$$+ \frac{64h_3}{33 \mu_1^{1/6}} (9c_{13} + 5c_{26}) \lambda^5 + \frac{8\sqrt{2}h_4}{3 \mu_1^{1/12}} (6 + 10c_{26} + 5c_{26}) \lambda^{11/2}$$
$$+ \frac{64h_5}{13} (20c_{13} + 15c_{26} + 7c_{26}) \lambda^6 \quad (A.2)$$
\[\begin{align*}
-635 & 250h_0\hbar_4 + c_{260}(1984 752h_2^2 - 3103 360h_1h_3 \\
-1058 & 750h_0h_4 + 25 872 000h_{10\mu_1}) + c_{460}(569 184h_2^2 - 977 600h_1h_3 \\
-529 & 375h_0h_4 + 22 176 000h_{10\mu_1} + 16 632 000h_{10\mu_1}c_{800} \\
+ & 10 164 000h_{10\mu_1}c_{880} + 406 600h_{10\mu_1}c_{1060})^{1/2} \\
\frac{64}{513 513\mu_1} c_{60}(-14 774 760h_2h_3 & - 8500 349h_1h_4 \\
-2411 & 640h_0h_5 + 42 378 336h_{11\mu_1}) + c_{360}(-8162 154h_2h_3 \\
-5646 & 560h_0h_4 - 1808 730h_0h_5 + 38 918 880h_{11\mu_1}) \\
+ c_{560}(-2104 830h_2h_3 & - 1661 803h_1h_4 - 844 074h_0h_5 + 32 432 400h_{11\mu_1}) \\
+ & 23 783 760h_{11\mu_1}c_{700} + 14 270 256h_{11\mu_1}c_{900} + 562 616h_{11\mu_1}c_{1100})^{1/2} \\
\frac{1}{249 729 480 \sqrt{2\mu_1^{1/12}}} & [(306 997 691h_0^3 - 115 430 515 200h_6\mu_1h_0 \\
+ & 1670 723 518 464h_{12\mu_1^2} - 569 874 898 944h_{2\mu_1^2} \\
- & 895 030 456 320h_2\mu_1h_4 - 446 778 630 144h_1h_5\mu_1) \\
+ & c_{260}(-881 051 738 112h_{2\mu_1^2} + 322 109 642 752h_{12\mu_1^2} \\
- & 1476 800 252 928h_2h_4\mu_1 - 776 493 527 040h_1h_5\mu_1 - 207 774 927 360h_0h_6\mu_1) \\
+ c_{460}(-417 291 264 000h_{2\mu_1^2} + 2876 883 609 600h_{12\mu_1^2} \\
- & 738 400 126 464h_2h_4\mu_1 - 474 275 450 880h_1h_5\mu_1 - 145 442 449 152h_0h_6\mu_1) \\
+ c_{660}(-9706 441 600h_{2\mu_1^2} + 2344 127 385 600h_{12\mu_1^2} - 179 006 091 264h_2h_4\mu_1 \\
- & 133 822 402 560h_1h_5\mu_1 - 6464 088 512h_0h_6\mu_1) + 1687 771 717 632h_{12\mu_1^2}c_{800} \\
+ & 997 319 651 328h_{12\mu_1^2}c_{1000} + 387 846 531 072h_{12\mu_1^2}c_{1200})^{1/2} \\
\frac{32}{14 189 175\mu_1^{1/12}} & [c_{60}(9214 205h_1h_2^2 - 567 567 000h_7\mu_1h_0 + 7264 857 600h_{13\mu_1^2} \\
- & 7311 447 000h_3h_4\mu_1 - 5050 854 900h_2h_5\mu_1 - 2327 222 040h_1h_6\mu_1) \\
+ & c_{360}(6810 804 000h_{13\mu_1^2} - 4918 914 900h_3h_4\mu_1 - 3736 223 820h_2h_5\mu_1 \\
- & 1855 812 816h_1h_6\mu_1 - 476 756 280h_2h_7\mu_1) + c_{560}(5945 940 000h_{13\mu_1^2} \\
- & 2134 875 600h_3h_4\mu_1 - 1756 305 180h_2h_5\mu_1 - 1072 119 048h_1h_6\mu_1 \\
- & 317 837 520h_1h_7\mu_1) + c_{760}(4756 752 000h_{13\mu_1^2} - 467 181 000h_3h_4\mu_1 \\
- & 409 786 020h_2h_5\mu_1 - 294 390 096h_1h_6\mu_1 - 136 216 080h_0h_7\mu_1) \\
+ & 3372 969 600h_{13\mu_1^2}c_{900} + 1967 565 600h_{13\mu_1^2}c_{1100} \\
+ & 756 756 000h_{13\mu_1^2}c_{1300})^{1/2} + O(\lambda^{10}),
\end{align*}\]
In this appendix, the components of the 3-form potential (4.40) are explicitly given in (C.2), (C.3) and (C.4). The various terms defined in the 4-form field strength (4.42) are in (C.3).
Various combinations which occur in (4.42) have the following series expansions in $\lambda$:

$$\partial_{\lambda}H^{-1} = \frac{24}{\sqrt{\mu_1}} \lambda^2 - \frac{336h_0}{\mu_1} \lambda^5 - \frac{6656 \sqrt{2} \cos \Theta h_1}{3 \mu_1^{11/12}} \lambda^{11/2} + O(\lambda^6),$$

$$T \partial_{\lambda}H^{-1} = -\frac{6}{\mu_1} \lambda^4 + \frac{98h_0}{\mu_1} \lambda^4 + \frac{9344 \sqrt{2} \cos \Theta h_1}{15 \mu_1^{11/12}} \lambda^{9/2} + O(\lambda^5),$$

$$T^2 \partial_{\lambda}H^{-1} = \frac{3}{2} \mu_1^{1/6} - \frac{28h_0}{\mu_1^{1/3}} \lambda^3 + \frac{864 \sqrt{2} \cos \Theta h_1}{5 \mu_1^{11/4}} \lambda^{7/2} + O(\lambda^4),$$

$$\partial_0 H^{-1} = \frac{1024 \sqrt{2} h_1 \sin \Theta}{3 \mu_1^{11/12}} \lambda^{13/2} + O(\lambda^7),$$

$$T \partial_0 H^{-1} = -\frac{256 \sqrt{2} h_1 \sin \Theta}{3 \mu_1^{7/12}} \lambda^{11/2} + O(\lambda^6),$$

$$T^2 \partial_0 H^{-1} = \frac{64 \sqrt{2} h_1 \sin \Theta}{3 \mu_1^{7/4}} \lambda^{9/2} + O(\lambda^5).$$

(C.4) and (C.1):

$$u_1(v, w, y_2) \equiv \partial_y h + \partial_v f + \partial_w g,$$

$$u_2(v, w, y_2) \equiv \partial_v f - f \partial_y h + h \partial_w g - g \partial_v h,$$

$$u_3(v, w, y_2) \equiv f \partial_v g - g \partial_v f + \partial_v g h - \partial_v h g,$$

$$u_4(v, w, y_2) \equiv \partial_v h \partial_w g - \partial_w h \partial_v g + g \partial_v f \partial_w h - \partial_v h \partial_w f,$$

$$u_5(v, w, y_2) \equiv f \partial_v h - h \partial_v f + f \partial_v g h - f \partial_v h g,$$

$$u_6(v, w, y_2) \equiv \partial_v h \partial_w g - \partial_w h \partial_v g + g \partial_v f \partial_w h - \partial_v h \partial_w f.$$
\[
\begin{align*}
\partial_\lambda H^{-1} \partial_\Theta T - \partial_\Theta T \partial_\lambda H^{-1} &= \frac{768 \sqrt{2} h_1 \sin \Theta}{5 \mu_1^{1/12}} \lambda^{9/2} + \mathcal{O}(\lambda^5), \\
T (\partial_\lambda H^{-1} \partial_\Theta T - \partial_\Theta T \partial_\lambda H^{-1}) &= -\frac{192 \sqrt{2} h_1 \sin \Theta}{5 \mu_1^{1/4}} \lambda^{7/2} + \mathcal{O}(\lambda^4), \\
T^2 (\partial_\lambda H^{-1} \partial_\Theta T - \partial_\Theta T \partial_\lambda H^{-1}) &= \frac{48}{5} \sqrt{2} \mu_1^{1/12} h_1 \sin \Theta \lambda^{5/2} + \mathcal{O}(\lambda^3). (\text{C.4})
\end{align*}
\]

**Appendix D. Equation of motion**

In this appendix, we gather all the formulae that goes into the computations of section 4.5. We work with a specific choice of Gaussian null-like co-ordinates:

\[
\begin{align*}
t &= v - f(w, y_2) T(\lambda, \Theta), \\
x_1 &= v - g(w, y_2) \frac{T(\lambda, \Theta)}{\Theta}, \\
x_2 &= y_2 T(\lambda, \Theta),
\end{align*}
\]

where

\[
\begin{align*}
f(w, y_2) &= \frac{1}{2} \left( w + \frac{1}{w} + \frac{y_2}{w} \right), \\
g(w, y_2) &= \frac{1}{2} \left( -w + \frac{1}{w} + \frac{y_2}{w} \right). (\text{D.2})
\end{align*}
\]

In this specific Gaussian null-like co-ordinate system, the 4-form field strength is

\[
\begin{align*}
F_{[4]} &= (\partial_\lambda H^{-1} \partial_\Theta T + \partial_\Theta T \partial_\lambda H^{-1} \sin \Theta) \left[ ((f - g)T \partial_\lambda T \partial_\Theta + \partial_\lambda (f - g)T \partial_\Theta T \partial_\lambda + \partial_\Theta (f - g)T \partial_\lambda T \partial_\Theta) \\
&\quad - T^2 \partial_\lambda \partial_\Theta T \partial_\lambda T \partial_\Theta + f T^2 \partial_\lambda \partial_\Theta T \partial_\lambda T \partial_\Theta + g T^2 \partial_\lambda \lambda \partial_\Theta T \partial_\lambda T \partial_\Theta + g T^2 \partial_\Theta \partial_\lambda T \partial_\Theta T \partial_\lambda \right].
\end{align*}
\]

\[
\begin{align*}
F_{[4]} &= \left( \frac{48 \sqrt{2} \mu_1^{1/12} h_1 \sin \Theta}{5 w} \lambda^{5/2} + \frac{72 \mu_1^{1/6} h_2 \sin(2\Theta)}{w} \lambda^3 + \mathcal{O}(\lambda^{7/2}) \right) \partial_\lambda \partial_\Theta \partial_\lambda T \partial_\Theta T + \mathcal{O}(\lambda^5/2) \\
&\quad + \left( \frac{3 \mu_1^{1/6}}{2} \frac{28 h_0}{\mu_1^{1/3}} \lambda^3 - \frac{864 \sqrt{2} \cos \Theta h_1}{5 \mu_1^{1/4}} \lambda^{7/2} + \mathcal{O}(\lambda^4) \right) \partial_\lambda \partial_\Theta \partial_\lambda T \partial_\Theta T + \mathcal{O}(\lambda^{7/2}) \\
&\quad + \left( \frac{64 \sqrt{2} h_1 \sin \Theta}{3 \mu_1^{1/4}} \lambda^9/2 + \mathcal{O}(\lambda^5) \right) \partial_\Theta \partial_\lambda \partial_\Theta T \partial_\lambda T + \mathcal{O}(\lambda^{7/2}) \\
&\quad + \left( \frac{192 \sqrt{2} y_2 h_1 \sin \Theta}{5 \mu_1^{1/4}} \lambda^{7/2} - \frac{288 y_2 h_2 \sin(2\Theta)}{\mu_1^{1/6}} \lambda^4 + \mathcal{O}(\lambda^{7/2}) \right) \partial_\Theta \partial_\Theta \partial_\lambda T \partial_\Theta T + \mathcal{O}(\lambda^5/2).
\end{align*}
\]

The nonzero components of the Ricci tensor (up to and including \(\mathcal{O}(\lambda^3)\)) are listed here:

\[
\begin{align*}
R_{uv} &= \frac{12}{\mu_1^{1/3}} \lambda + \mathcal{O}(\lambda^{7/2}), \\
R_{uw} &= -\frac{3 \left( 1 + \frac{y_2}{w} \right)}{w^2} + \frac{84 \left( 1 + \frac{y_2}{w} \right) h_0}{w^2 \sqrt{\mu_1}} \lambda^3 + \mathcal{O}(\lambda^{7/2}), \\
R_{uw2} &= \frac{3 y_2}{w} - \frac{84 y_2 h_0}{w \sqrt{\mu_1}} \lambda^3 + \mathcal{O}(\lambda^{7/2}).
\end{align*}
\]
\[ R_{ij3} = -3 + \frac{84h_0}{\sqrt{\mu_1}} \lambda^3 + O(\lambda^{7/2}), \]
\[ R_{i\lambda} = -\frac{12w}{\mu_1^{7/3}} + \frac{224w h_0}{\mu_1^{1/6}} \lambda^3 + O(\lambda^{7/2}), \]
\[ R_{i\Theta} = -\frac{576\sqrt{2}h_1 \sin \Theta}{5 \mu_1^{5/12}} \lambda^{5/2} = \frac{1728h_2 \cos \Theta \sin \Theta}{\mu_1^{1/3}} \lambda^3 + O(\lambda^{7/2}), \]
\[ R_{\Theta\Theta} = 6 - \frac{96h_0}{\sqrt{\mu_1}} \lambda^3 + O(\lambda^{7/2}), \]
\[ R_{ab} = 6 \left( 1 - \frac{16h_0}{\sqrt{\mu_1}} \lambda^3 + O(\lambda^{7/2}) \right) \sin^2 \Theta g_{ab}, \tag{D.4} \]

where in the last equation \( R_{ab} \) and \( g_{ab} \) are the components of the Ricci tensor and the metric tensor along the six-sphere. Other components may be nonzero but they all start after \( O(\lambda^3) \).

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