Lyapunov exponents in $\mathcal{N} = 2$ supersymmetric Jackiw-Teitelboim gravity

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Abstract

We study $\mathcal{N} = 2$ supersymmetric Jackiw-Teitelboim (JT) gravity at finite temperature coupled to matter. The matter fields are related to superconformal primaries by AdS/CFT duality. Due to broken super reparametrisation invariance in the SCFT dual, there are corrections to superconformal correlators. These are generated by the exchange of super-Schwarzian modes which is dual to the exchange of 2D supergravity modes. We compute corrections to four-point functions for superconformal primaries and analyse the behaviour of out-of-time-ordered correlators. In particular, four-point functions of two pairs of primaries with mutually vanishing two-point functions are considered. By decomposing the corresponding supermultiplet into its components, we find different Lyapunov exponents. The value of the Lyapunov exponents depends on whether the correction is due to graviton, gravitini or graviphoton exchange. If mutual two-point functions do not vanish all components grow with maximal Lyapunov exponent.
1 Introduction

Lower dimensional instances of the AdS$_D$/CFT$_{D-1}$ correspondence [1] have attracted a wide interest in recent years. The case $D = 2$ is important from the point of view of black hole physics because AdS$_2$ is the near-horizon geometry of extremal (e.g. supersymmetric) black holes with non-zero entropy. A known problem is that pure gravity in AdS$_2$ is inconsistent with the existence of finite energy excitations above the AdS$_2$ vacuum [2–4]. The problem can be overcome by considering a deformed version of the AdS$_2$ space. In Euclidean signature, AdS$_2$ is the hyperbolic disk. The idea is to slightly deform the boundary of the disk by means of a UV brane, i.e. a curve that reduces to the old boundary as soon as a small parameter is sent to zero, without changing the AdS$_2$ geometry. In this nearly AdS$_2$ space (nAdS$_2$) it is possible to define a consistent 2-dimensional theory of gravity, the Jackiw-Teitelboim (JT) theory [5, 6], coupled to a dilaton field. The dilaton field enters as a Lagrange multiplier enforcing negative constant curvature. After imposing the constraint, JT gravity reduces to an effective action involving the Schwarzian derivative of a reparametrisation of the coordinate describing the nAdS$_2$ boundary [7–10]:

$$S_{\text{sch}} = -C \int du \text{Sch} [t(u); u],$$

(1)

where $C$ is inversely proportional to Newton’s constant $G_N$ and

$$\text{Sch} [t(u); u] = \frac{2\dot{t}^{(3)} - 3\dot{t}^2}{2t^2}.$$  

(2)

The JT theory has surprisingly some features in common with a completely different model, originally studied in condensed matter physics, the SYK model [11, 12]. It describes $N$ fermions interacting via random couplings. The model is solvable in the large $N$ limit, is chaotic and is conformally invariant in the low energy regime [12–15]. Moving away from the IR region breaks the conformal symmetry, resulting in an effective action for time reparametrisations given again by the Schwarzian. This corroborates JT gravity as the holographic dual of the SYK low-energy region. To be precise, this is an example of a nearly AdS$_2$/nearly CFT$_1$ correspondence. In the same spirit of the usual “holographic dictionary” of the canonical AdS/CFT correspondence [16], matter fields in nAdS$_2$ correspond to conformal primaries in the conformal limit of the SYK model. More precisely, correlation functions of conformal primaries are obtained by taking variational derivatives of the gravity partition function with respect to boundary conditions of matter fields. The breaking of conformal symmetry is characterised by a Schwarzian action for reparametrisations of the boundary. These are responsible for corrections to correlation functions signaling the occurrence of quantum chaos. In a chaotic system, small differences in the initial conditions lead to large and unpredictable differences in the dynamics. In a quantum theory, a diagnosis of chaos is offered by the behaviour of a special class of correlators, the out-of-time-ordered correlators (OTOC). If the system is chaotic they show an exponential behaviour with rate given by a real number $\lambda$ called Lyapunov exponent. For a causal
and unitary QFT it was argued that the Lyapunov exponent is bounded by a maximal value \[17\]. Calculations carried in JT gravity \[9\] and in the SYK model \[12,18\] gave the maximal value of Lyapunov exponent for both theories, thereby supporting the duality between them.

Both the JT theory and the SYK model can be generalised by adding supersymmetry. In \[19\] it was shown that the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) supersymmetric versions of the SYK model lead to super-Schwarzian actions. Similarly to the purely bosonic theory, the super-Schwarzian also arises as an effective action for a UV regulator brane in nAdS\(_2\) supergravity, for both the \( \mathcal{N} = 1 \) \[20\] and \( \mathcal{N} = 2 \) \[21\] supersymmetric JT theories. This suggests that the JT/SYK duality is still preserved after adding supersymmetry.

Lyapunov exponents were considered for \( \mathcal{N} = 1 \) JT super gravity in \[22,23\]. For \( \mathcal{N} = 2 \) they were computed in \[24\] within the SYK model. Here, we carry out the holographic computation for \( \mathcal{N} = 2 \) and obtain results similar to \[22\]. Whenever the leading correction to the four-point functions contains graviton exchange the Lyapunov exponent saturates the bound \( 2\pi/\beta \). There are also components of the four-point functions for which the leading correction does not contain graviton exchange. If gravitino exchange contributes, their Lyapunov exponent will be \( \pi/\beta \) whereas for combinations with graviphoton contributions only there will be just linear growth with time.

We briefly outline the procedure that we are going to follow. Firstly, we put the theory at finite temperature and compute the corresponding super-Schwarzian action. Secondly, we expand the reparametrisations of the super-boundary in terms of small fluctuations. Subsequently, we put matter fields into the nAdS\(_2\) space and compute the four-point functions of the dual operators, contracting the appearances of the fluctuations using their propagators. Using its definition, we are eventually able to extract the Lyapunov exponents from the four-point functions. The same procedure was carried out for the purely bosonic JT theory in \[9\] and reviewed in \[25\]. The paper is organised as follows. In Section 2 we review the \( \mathcal{N} = 2 \) superspace formalism and compute the super-Schwarzian at finite temperature. In Section 3 we derive the propagators of the fluctuations. In Section 4 we couple the theory to matter. In Section 5 we compute the thermal four-point functions for operators dual to the matter fields and determine the Lyapunov exponents.

## 2 The \( \mathcal{N} = 2 \) superspace formalism

In the purely bosonic JT theory, the field \( t(u) \) is the only dynamical variable describing reparametrisations of the boundary in the nAdS\(_2\) space. In a supersymmetric theory, we have to add one or more fermionic fields. The superspace formalism offers a simple way to keep track of all the relevant degrees of freedom and to make supersymmetry manifest. Within this framework, space is described in terms of the usual bosonic coordinates and by additional Grassmann variables. In a generic \( \mathcal{N} = m \) theory, super-reparametrisations of the boundary are described by an \( (m + 1) \)-tuple \( (t, \xi_1, \ldots, \xi_m) \) where \( t \) respectively \( \xi_i \) is a commuting respectively anti-commuting superfield satisfying additional constraints which can be viewed as one-dimensional versions of superconformal Killing equations.
2.1 $\mathcal{N} = 2$ superspace and super-Schwarzian

We briefly review definitions of $\mathcal{N} = 2$ super-diffeomorphisms following conventions in [19]. The $\mathcal{N} = 2$ one-dimensional superspace, or super-line, is parametrised by a bosonic variable $u$ and Grassmann variables $\theta$ and $\bar{\theta}$. The super-translation group consists of the transformations

$$u \mapsto \tau = u + \epsilon + \theta \eta + \bar{\theta} \bar{\eta}, \quad \theta \mapsto \xi = \theta + \eta, \quad \bar{\theta} \mapsto \bar{\xi} = \bar{\theta} + \bar{\eta}. \quad (3)$$

where $\epsilon(u)$ is a bosonic function and $\eta(u), \bar{\eta}(u)$ are fermionic functions. Super-derivative operators are defined as

$$D_{\theta} = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial u}, \quad \overline{D}_{\theta} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \frac{\partial}{\partial u}. \quad (4)$$

Their anticommutator is

$$\{D_{\theta}, \overline{D}_{\bar{\theta}}\} = 2 \partial_u. \quad (5)$$

Fermions and scalars can be packaged into a chiral superfield, i.e. a field $\Psi(u, \theta, \bar{\theta})$ constrained to satisfy

$$\overline{D}_{\theta} \Psi(u, \theta, \bar{\theta}) = 0, \quad (6)$$

which is solved by

$$\Psi(u, \theta, \bar{\theta}) = \Gamma(u + \theta \bar{\theta}) + \theta b(u) \quad (7)$$

for a fermion $\Gamma$ and boson $b$. The conjugate $\overline{\Psi}$ and $D_{\theta} \Psi$ are instead anti-chiral field, namely they are annihilated by $D_{\theta}$.

Transformations in (3) are super-reparametrisations if they satisfy the constraints

$$D_{\theta} \xi = 0, \quad \overline{D}_{\bar{\theta}} \tau = \xi D_{\theta} \xi, \quad \overline{D}_{\theta} \xi = 0, \quad \overline{D}_{\bar{\theta}} \tau = \xi \overline{D}_{\bar{\theta}} \xi. \quad (8)$$

Super-reparametrisations are described by an effective action which can be obtained from a supersymmetric SYK model [19] as well as from supersymmetric JT gravity [21]

$$S_{\text{eff}} = -\frac{1}{2 \pi G_N} \int_{\partial M} dud\theta d\overline{\theta} \phi_b \text{Sch} [\tau, \xi, \xi; u, \theta, \overline{\theta}], \quad (9)$$

where $\phi_b$ is the boundary value for the leading component of the dilaton $\Phi$. For convenience we rename

$$\kappa = -\frac{\phi_b}{2 \pi G_N} \quad (10)$$

and consider constant $\phi_b$ which can be achieved by an appropriate choice of the boundary. The functional $\text{Sch} [\tau, \xi, \xi; u, \theta, \overline{\theta}]$ is the $\mathcal{N} = 2$ super-Schwarzian derivative [19]

$$\text{Sch} [\tau, \xi, \xi; u, \theta, \overline{\theta}] = \frac{\partial_u \overline{D}_{\bar{\theta}} \xi}{D_{\theta} \xi} - \frac{\partial_u D_{\theta} \xi}{D_{\theta} \xi} - 2 \frac{\partial_u \xi \partial_u \overline{\xi}}{(D_{\theta} \xi)(D_{\bar{\theta}} \xi)}, \quad (11)$$

where $\tau, \xi, \overline{\xi}$ are super-reparametrisations of the boundary of nAdS$_2$ satisfying the constraints (8).
2.2 The $\mathcal{N} = 2$ super-Schwarzian at finite temperature

We now put the theory at finite temperature $T = 1/\beta$. This is achieved by reparametrising the boundary time to a tangent, $\tau(u) \mapsto \tan \frac{2\pi \tau(u)}{\beta}$. For simplicity we fix $\beta = 2\pi$. Assuming that $(\tau, \xi, \overline{\xi})$ are super-reparametrisations it is easy to check that

$$\tau' = \tan \frac{\tau}{2}, \quad \xi' = \frac{\xi}{\sqrt{2\cos \frac{\tau}{2}}}, \quad \overline{\xi}' = \frac{\overline{\xi}}{\sqrt{2\cos \frac{\tau}{2}}}. \tag{12}$$

satisfy (8). The super-Schwarzian transforms according to the chain rule \[19\]

$$\text{Sch} \left[ \tau', \xi', \overline{\xi}; u, \theta, \overline{\theta} \right] = (D_{\theta} \xi) \left( D_{\overline{\theta}} \overline{\xi} \right) \text{Sch} \left[ \tau', \xi', \overline{\xi}; \tau, \xi, \overline{\xi} \right] + \text{Sch} \left[ \tau, \xi, \overline{\xi}; u, \theta, \overline{\theta} \right]. \tag{13}$$

This allows us to express the effective action in terms of fluctuations around the zero temperature solution $(\tau, \xi, \overline{\xi}) = (u, \theta, \overline{\theta})$. Fluctuations are described by two bosonic fields $\epsilon(u)$ and $\sigma(u)$ and two fermionic fields $\eta(u)$ and $\overline{\eta}(u)$. Solving (8) up to quadratic order in the fluctuations yields \[\footnote{This expression is obtained by considering the infinitesimal limit of reparametrisation given in [19] in terms of independent fields.}

$$\tau(u) = u + \epsilon(u) + \theta \overline{\eta}(u) \left[ 1 + \epsilon(u) \right] + \overline{\theta} \eta(u) \left[ 1 + \epsilon(u) \right] + \theta \overline{\theta} \left[ \eta(u) \overline{\eta}(u) - \eta(u) \overline{\eta}(u) \right], \tag{14}$$

$$\xi(u) = \eta(u) \left[ 1 + i \sigma(u) + \frac{1}{2} \dot{\epsilon}(u) \right] + \theta \left\{ 1 + \overline{\eta}(u) \dot{\eta}(u) + i \sigma(u) - \frac{1}{2} \sigma^2(u) - \frac{1}{8} \dot{\epsilon}^2(u) + \frac{1}{2} \dot{\epsilon}(u) \right\} + \theta \overline{\theta} \left\{ \dot{\epsilon}(u) + \frac{1}{2} \epsilon(u) \right\}, \tag{15}$$

$$\overline{\xi}(u) = \overline{\eta}(u) \left[ 1 - i \sigma(u) + \frac{1}{2} \dot{\epsilon}(u) \right] + \overline{\theta} \left\{ 1 + \eta(u) \overline{\eta}(u) - i \sigma(u) - \frac{1}{2} \sigma^2(u) - \frac{1}{8} \dot{\epsilon}^2(u) + \frac{1}{2} \dot{\epsilon}(u) \right\} - \theta \overline{\theta} \left\{ \dot{\epsilon}(u) + \frac{1}{2} \epsilon(u) \right\}. \tag{16}$$

The field $\epsilon(u)$, which also appears in the purely bosonic JT gravity, is interpreted as a boundary graviton \[9\]; the fermionic fields $\eta(u)$ and $\overline{\eta}(u)$ are boundary gravitinos; finally, $\sigma(u)$ represents the boundary degree of freedom of a graviphoton \[26\]. We can now express the effective action for finite temperature super-reparametrisations in terms of this gravity supermultiplet. Neglecting total derivatives and contributions of higher than quadratic order in the fluctuations, we obtain

$$S_{\text{sch}, \beta = 2\pi}^{\mathcal{N} = 2} = \kappa \int dud\theta d\overline{\theta} \text{Sch} \left[ \tau', \xi', \overline{\xi}; u, \theta, \overline{\theta} \right] \tag{17}$$

$$= C' \int_0^{2\pi} du \left\{ \frac{1}{2} \left[ \dot{\epsilon}^2(u) - \dot{\epsilon}^2(u) \right] + 2 \sigma^2(u) + \left[ \eta(u) \overline{\eta}(u) - 4 \dot{\eta}(u) \overline{\eta}(u) \right] \right\}. \tag{17}$$

Here, we have used (12), (13) and (14), (15), (16). This action has been given already in [27]. The form of the $\epsilon$ dependent contribution coincides with the one obtained in the purely bosonic setting [9].
3 Propagators of the fluctuations

The propagator for $\epsilon$ was already obtained in [9]. Here, we closely follow the derivation detailed in [25]. In a thermal field theory, time $u$ is periodic. In our case, the period is $\beta = 2\pi$. We start by expanding fluctuations into Fourier series,

$$
\epsilon(u) = \sum_{n \in \mathbb{Z}} \epsilon_n e^{inu}, \quad \sigma(u) = \sum_{n \in \mathbb{Z}} \sigma_n e^{inu},
$$

$$
\eta(u) = \sum_{m \in \mathbb{Z} + \frac{1}{2}} \eta_m e^{imu}, \quad \overline{\eta}(u) = \sum_{m \in \mathbb{Z} + \frac{1}{2}} \overline{\eta}_m e^{-imu},
$$

(18)

where we have imposed periodic and anti-periodic boundary conditions on bosonic and fermionic fields, respectively [28]. Plugging (18) into (17) gives

$$
S_{\text{sch}, \beta = 2\pi}^{N=2} = 2\chi C \left\{ \sum_{n \in \mathbb{Z}} [(n^4 - n^2)\epsilon_n \epsilon_{-n} + 2n^2 \sigma_n \sigma_{-n}] + \sum_{m \in \mathbb{Z} + 1/2} m \left( m^2 - \frac{1}{4} \right) \eta_m \overline{\eta}_m \right\}. \quad (19)
$$

In Fourier space, the propagators are

$$
\langle \epsilon_n \epsilon_k \rangle = \frac{\delta_{n+k}}{\pi C} \frac{1}{n^2 (n^2 - 1)}, \quad \langle \sigma_n \sigma_k \rangle = \frac{\delta_{n+k}}{4\pi C} \frac{1}{n^2}, \quad \langle \eta_n \overline{\eta}_k \rangle = \frac{\delta_{m-k}}{2\pi i C} \frac{1}{m (4m^2 - 1)}. \quad (20)
$$

Divergencies due to zeros in the denominators are due to zero modes. These are removed by gauge fixing the supersymmetric extension of SL(2, $\mathbb{R}$) called SU(1, 1 | 1). This is achieved by just not summing over the associated Fourier modes when transforming back to position space. The corresponding sums can be expressed by contour integrals,

$$
\langle \epsilon(u) \epsilon(0) \rangle = \frac{1}{\pi C} \oint_{C_n} dz \frac{1}{e^{2\pi iz} - 1} \frac{e^{izu}}{z^2 (z^2 - 1)} ,
$$

$$
\langle \sigma(u) \sigma(0) \rangle = \frac{1}{4\pi C} \oint_{C_n} dz \frac{1}{e^{2\pi iz} - 1} \frac{e^{izu}}{z^2},
$$

$$
\langle \eta(u) \overline{\eta}(0) \rangle = \frac{1}{4\pi i C} \oint_{C_n} dz \frac{1}{e^{2\pi iz} - 1} \frac{e^{izu}}{z (z^2 - 1/4)},
$$

(21)

where the integration contours are sets of small circles running counter clockwise around integer $z$’s excluding zeros in the denominators in the second factors of the corresponding integrand. By contour deformation the integrals can be related to integrals in which the contour runs around the previously excluded poles. These can be evaluated by means of the residue formula. One obtains

$$
\langle \epsilon(u) \epsilon(u') \rangle = \frac{1}{2\pi C} \left[ \frac{(|u - u'| - \pi)^2}{2} + (|u - u' - \pi| \sin |u - u'| + 1 + \frac{\pi^2}{6} + \frac{5}{2} \cos |u - u'| \right], \quad (22)
$$

$$
\langle \sigma(u) \sigma(u') \rangle = \frac{1}{48\pi C} \left[ 3|u - u'|^2 - 6\pi |u - u'| + 2\pi^2 \right], \quad (23)
$$

6
\( \langle \eta(u) \bar{\eta}(u') \rangle = \frac{1}{4\pi C} \left[ (2\pi - |u - u'|) \cos \left( \frac{|u - u'|}{2} \right) + 3 \sin \left( \frac{|u - u'|}{2} \right) \right], \quad (24) \)

where we have used shift symmetry in \( u \) to obtain the dependence on the second argument \( u' \).

## 4 Coupling to matter

Superconformal invariance fixes the form of correlators. For instance, the two-point function for a superconformal primary \( O \) of dimension \( \Delta \) has the form

\[ \langle O(u_1) \bar{O}(u_2) \rangle \sim \frac{1}{\Xi^{2\Delta}}, \quad (25) \]

where \( \Xi \) is invariant under supertranslations. Furthermore, the two-point function of chiral (anti-chiral) superfields is also chiral (anti-chiral). Thus, if we are interested in the correlators for chiral and anti-chiral matter fields in \( \mathcal{N} = 2 \) supersymmetry, \( \Xi \) does not only have to be invariant under the super-translations \( (3) \) but also has to be chiral in one coordinate and anti-chiral in the other. The correct denominator that satisfies these requirements is \( (24) \)

\[ \Xi = u_1 - u_2 - 2\bar{\theta}_1 \theta_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2. \quad (26) \]

Following the general rules of AdS/CFT duality, superconformal symmetry fixes the on-shell value of the bulk matter action \( [10] \)

\[ S_{\text{matter}} = \mu \int d\tau'_1 d\tau'_2 d\xi'_1 d\xi'_2 \frac{\Psi(\tau'_2, \xi'_2) \bar{\Psi}(\tau'_1, \xi'_1)}{|\tau'_1 - \tau'_2 - 2\bar{\xi}'_1 \xi'_2 - \xi'_1 \bar{\xi}'_2 - \xi'_2 \bar{\xi}'_1|^{2\Delta}}, \quad (27) \]

where \( \mu \) is some constant and \( \Psi(\tau'_2, \xi'_2), \bar{\Psi}(\tau'_1, \xi'_1) \) are respectively chiral and anti-chiral matter multiplets corresponding to the boundary values of fields living in the bulk of nAdS\(_2\). On the CFT side, these are sources coupling to superconformal primaries of dimension \( \Delta \). The zero temperature correlator is obtained by taking variational derivatives of the exponential of \( S_{\text{matter}} \) with respect to \( \Psi \left( \tau'_i, \xi'_i, \bar{\xi}'_i \right) \) and \( \bar{\Psi} \left( \tau'_j, \xi'_j, \bar{\xi}'_j \right) \) and viewing \( \tau', \xi', \bar{\xi} \) as superspace coordinates. The finite temperature correlator follows from variational derivatives with respect to superfields depending on the unprimed coordinates \( \tau, \xi, \bar{\xi} \) where primed and unprimed quantities are related as in \( (12) \). Finally, corrections due to broken superconformal invariance can be taken into account when considering variational derivatives with respect to superfields depending on \( u, \theta, \bar{\theta} \) and the relation to \( \tau, \xi, \bar{\xi} \) is given in \( (14), (15), (16) \). In order to take the variational derivatives with respect to fields depending on \( u, \theta, \bar{\theta} \) we use the fact that they are related by super-reparametrisations to \( \tau', \theta', \bar{\theta}' \). There will be a Berezian from the superspace measure \( [19] \) whereas the transformation properties of the superfields \( \Psi \) and \( \bar{\Psi} \) follow from the AdS/CFT dictionary equating them with currents coupling to superconformal primaries. Taking all these contributions
into account we obtain

$$S_{\text{matter}}^{\mathcal{N}=2} = \mu \int du_{ac1}d\bar{\theta}_1du_{c2}d\theta_2 \left( \frac{D_{\xi_2}(\xi_2')}{D_{\xi_1}(\xi_1')^{2\Delta}} \right) \Psi(u_{c2}, \theta_2) \overline{\Psi}(u_{ac1}, \bar{\theta}_1) \frac{|\tau' - \tau'' - 2\xi_1'\xi_2' - \xi_1'\xi_2' - \xi_1''\xi_2'|^{2\Delta}}{\xi_1'\xi_2'},$$

(28)

where the primed quantities are viewed as functions of $u, \theta, \bar{\theta}$ with the corresponding label. Further, $u_c = u - \theta$ and $u_{ac} = u + \theta$ are the chiral and anti-chiral combinations on which associated superfields depend.

5 Four-point functions

In this section we compute the four-point functions at finite temperature for operators $V$ and $\overline{V}$ of conformal dimension $\Delta$, dual to the matter fields $\Psi$ and $\overline{\Psi}$. Four-point functions at zero temperature were already computed in [26]. It is useful to introduce a second pair of superconformal primaries $W$ and $\overline{W}$ which have vanishing two-point functions with $V$ and $\overline{V}$. On the supergravity side this corresponds to adding $\tilde{S}_{\text{matter}}$ which is equal to $S_{\text{matter}}$ but where the matter multiplets are labelled with an additional tilde. The generating functional of correlators for superconformal operators of the boundary theory coincides with the partition function of the gravity theory in which the currents are identified with boundary values of matter superfields

$$Z[\Psi, \overline{\Psi}, \tilde{\Psi}, \tilde{\overline{\Psi}}] = \left\langle \exp \left[ \int dud\theta d\bar{\theta} \left( \Psi \overline{V} + \overline{\Psi} V + \tilde{\Psi} \overline{W} + \tilde{\overline{\Psi}} W \right) \right] \right\rangle_{\text{SCFT}}$$

AdS/CFT $\equiv \int \mathcal{D}'\epsilon D'\sigma D'\eta D'\overline{\eta} e^{-\mathcal{S}_{\mathcal{N}=2}^\text{sch, \beta=2n} - \mathcal{S}_{\mathcal{N}=2}^{\text{matter}} - \tilde{\mathcal{S}}_{\mathcal{N}=2}^{\text{matter}}}.$

From the generating functional one can e.g. obtain four-point functions by taking variational derivaties,

$$\langle T V \overline{V} W \overline{W} \rangle = \frac{\delta^4 Z}{\delta \Psi \delta \overline{\Psi} \delta \tilde{\Psi} \delta \tilde{\overline{\Psi}}} \bigg|_{\Psi, \overline{\Psi}, \tilde{\Psi}, \tilde{\overline{\Psi}} = 0} = \left( \frac{\delta^2 Z}{\delta \Psi \delta \overline{\Psi}} \right) \left( \frac{\delta^2 Z}{\delta \tilde{\Psi} \delta \tilde{\overline{\Psi}}} \right) \bigg|_{\Psi, \overline{\Psi}, \tilde{\Psi}, \tilde{\overline{\Psi}} = 0}.$$  

(30)

where $T$ denotes time ordering.
One finds
\[
\langle \mathcal{T} \mathcal{V} (u_{ac1}, \theta_1) V (u_{ac2}, \theta_2) \mathcal{W} (u_{ac3}, \theta_3) W (u_{ac4}, \theta_4) \rangle = \int D' \epsilon D' \sigma D' \eta D' \eta N J (u_{ac1}, u_{ac2}) J (u_{ac3}, u_{ac4}) e^{-S_{Schw}},
\]
where
\[
N^{-1} = 4^{2\Delta} \sin 2\Delta \frac{u_{ac1} - u_{ac2}}{2} \sin 2\Delta \frac{u_{ac3} - u_{ac4}}{2}
\]
and
\[
J (u_{ac1}, \theta_1, u_{ac2}, \theta_2) = A (u_{ac1}, u_{ac2}) + B (u_{ac1}, u_{ac2}) \theta_1 + C (u_{ac1}, u_{ac2}) \theta_2 + D (u_{ac1}, u_{ac2}) \theta_1 \theta_2
\]
and the coefficients can be read off (35). Dropping the $a$ and $ac$ labels and introducing the convenient notation $u_{ij} = u_i - u_j$, we find
\[
A (u_1, u_2) = 1 + \Delta \left[ \frac{-\epsilon (u_1) + \epsilon (u_2)}{\tan \frac{u_{12}}{2}} + \dot{e} (u_1) + \dot{e} (u_2) + 2i (\sigma (u_1) - \sigma (u_2)) \right],
\]
\[
B (u_1, u_2) = 2\Delta \left[ \frac{\eta (u_1)}{\tan \frac{u_{12}}{2}} - \frac{\eta (u_2)}{\sin \frac{u_{12}}{2}} - 2 \dot{\eta} (u_1) \right],
\]
\[
C (u_1, u_2) = 2\Delta \left[ \frac{\bar{\eta} (u_1)}{\sin \frac{u_{12}}{2}} - \frac{\bar{\eta} (u_2)}{\tan \frac{u_{12}}{2}} - 2 \bar{\eta} (u_2) \right],
\]
\[
D (u_1, u_2) = \frac{2\Delta}{\sin \frac{u_{12}}{2}} \left[ 1 - (1 + 2\Delta) \frac{\epsilon (u_1) - \epsilon (u_2)}{2 \tan \frac{u_{12}}{2}} + (1 + 2\Delta) (\dot{e} (u_1) + \dot{e} (u_2)) \right.
\]
\[
+ 2i (-1 + 2\Delta) (\sigma (u_1) - \sigma (u_2)) \right].
\]
We decompose the superconformal primaries as
\[
V (u_2, \theta_2) = V_\phi (u_{ac2}) \phi_2 + V_\phi (u_{ac2}) \phi_2 \equiv V_{\phi_1} \phi_2 + V_{\phi_2}
\]
\[
\mathcal{V} (u_1, \theta_1) = \mathcal{V}_{\bar{\sigma}} (u_{ac1}) \bar{\sigma}_1 + \mathcal{V}_{\bar{\sigma}} (u_{ac1}) \bar{\sigma}_1 \equiv \mathcal{V}_{\bar{\sigma}_1} \bar{\sigma}_1 + \mathcal{V}_{\bar{\sigma}_2},
\]
with similar expressions holding for $W, \mathcal{W}$. By comparing both sides of (30) we are then led to the following non-vanishing connected four-point functions,
\[
\langle \mathcal{T} \mathcal{V}_{\bar{\sigma}_1} V_{\phi_2} \mathcal{W}_{\bar{\sigma}_3} W_{\phi_4} \rangle = NA (u_1, u_2) A (u_3, u_4),
\]
\[
\langle \mathcal{T} \mathcal{V}_{\bar{\sigma}_1} V_{\phi_2} \mathcal{W}_{\bar{\sigma}_3} W_{\phi_4} \rangle = ND (u_1, u_2) A (u_3, u_4),
\]
\[
\langle \mathcal{T} \mathcal{V}_{\bar{\sigma}_1} V_{\phi_2} \mathcal{W}_{\bar{\sigma}_3} W_{\phi_4} \rangle = ND (u_1, u_2) D (u_3, u_4),
\]
\[
\langle \mathcal{T} \mathcal{V}_{\bar{\sigma}_1} V_{\phi_2} \mathcal{W}_{\bar{\sigma}_3} W_{\phi_4} \rangle = -NB (u_1, u_2) C (u_3, u_4),
\]
\[
\langle \mathcal{T} \mathcal{V}_{\bar{\sigma}_1} V_{\phi_2} \mathcal{W}_{\bar{\sigma}_3} W_{\phi_4} \rangle = -NC (u_1, u_2) B (u_3, u_4).
\]
Horizontal brackets denote Wick contractions of super-Schwarzian fluctuations: only contributions quadratic in fluctuations are contained in the connected four-point function. The pair of fluctuations is replaced by its propagator. Since the propagators depend on absolute values of their arguments the results depend on the chosen ordering. The arrangement which, upon canonical continuation to Minkowski time, leads to out of time order correlators (OTOCs) is $u_1 > u_3 > u_2 > u_4$. Before continuation to Minkowski time one obtains

$$
\langle \nabla_{\phi_1} \nabla_{\phi_3} V_{\phi_2} W_{\phi_4} \rangle = \frac{N \Delta^2}{2 \pi C} \left[ \left( \frac{u_{12}}{\tan \frac{u_{23}}{2}} - 2 \right) \left( \frac{u_{34}}{\tan \frac{u_{34}}{2}} - 2 \right) + 2 \pi \frac{\sin \frac{u_{12}+u_{34}}{2}}{\sin \frac{u_{23}}{2}} - \sin \frac{u_{23}}{2} \right],
$$

$$
\langle \nabla_{\phi_1} \nabla_{\phi_3} V_{\phi_2} W_{\phi_4} \rangle = - \frac{N \Delta^2}{2 \pi C} \left[ \frac{1}{\sin \frac{u_{23}}{2}} \left\{ (2 \Delta + 1) \left[ \left( \frac{u_{12}}{\tan \frac{u_{23}}{2}} - 2 \right) \left( \frac{u_{34}}{\tan \frac{u_{34}}{2}} - 2 \right) - \sin \frac{u_{12}+u_{34}}{2} - \sin \frac{u_{23}}{2} \right] \right\} + (2 \Delta - 1) [u_{12} u_{34} + 2 \pi u_{23}] \right],
$$

$$
\langle \nabla_{\phi_1} \nabla_{\phi_3} V_{\phi_2} W_{\phi_4} \rangle = \frac{N \Delta^2}{2 \pi C} \left[ \frac{1}{\sin \frac{u_{23}}{2} \sin \frac{u_{34}}{2}} \left\{ (2 \Delta + 1) \left[ \left( \frac{u_{12}}{\tan \frac{u_{23}}{2}} - 2 \right) \left( \frac{u_{34}}{\tan \frac{u_{34}}{2}} - 2 \right) - \sin \frac{u_{12}+u_{34}}{2} - \sin \frac{u_{23}}{2} \right] \right\} + (2 \Delta - 1)^2 [u_{12} u_{34} + 2 \pi u_{23}] \right],
$$

$$
\langle \nabla_{\phi_1} \nabla_{\phi_3} V_{\phi_2} W_{\phi_4} \rangle = - \frac{2 N \Delta^2}{\pi C} \left[ \frac{1}{\sin \frac{u_{23}}{2} \sin \frac{u_{34}}{2}} \left[ u_{23} \cos \frac{u_{23}}{2} - \sin \frac{u_{23}}{2} \right],
$$

$$
\langle \nabla_{\phi_1} \nabla_{\phi_3} V_{\phi_2} W_{\phi_4} \rangle = - \frac{2 N \Delta^2}{\pi C} \left[ \frac{1}{\sin \frac{u_{23}}{2} \sin \frac{u_{34}}{2}} \left[ u_{32} \cos \frac{u_{14}}{2} + \sin \frac{u_{14}}{2} - \sin \frac{u_{12} + u_{42}}{2} + \sin \frac{u_{13} + u_{43}}{2} \right]
\right].
$$

To obtain the real time OTOCs we proceed by analytic continuation [9]. Reinstalling also temperature dependence we replace

$$
u_1 = \frac{2 \pi i}{\beta} a, \quad \nu_2 = 0, \quad \nu_3 = \frac{2 \pi i}{\beta} (b + \hat{u}) \quad \nu_4 = \frac{2 \pi i}{\beta} \hat{u}
$$

in equations (44)-(50). Furthermore, one is interested in the region with $a, b$ of the order of the dissipation time $\beta$ whereas $\hat{u}$ is much larger than the dissipation time while still much shorter than the scrambling time, i.e. $\frac{a}{\beta} \ll \hat{u} \ll \frac{2 \pi}{\beta} \log C$. Keeping only the leading contributions (growing exponentially with $\hat{u}$) one finds

$$
\langle \nabla_{\phi_1}(0) \nabla_{\phi_3}(\hat{u}) V_{\phi}(0) W_{\phi}(\hat{u}) \rangle \sim \frac{\Delta^2}{C} e^{2 \pi \hat{u}},
$$

$$
\langle \nabla_{\phi_1}(0) \nabla_{\phi_3}(\hat{u}) V_{\phi}(0) W_{\phi}(\hat{u}) \rangle \sim \frac{\Delta^2}{C} (1 + 2 \Delta)^2 e^{2 \pi \hat{u}},
$$
\begin{align}
\langle \nabla_\phi(0) \nabla_\theta(\hat{u}) \rangle_\beta & \sim \frac{\Delta^2}{C} (1 + 2\Delta) e^{\frac{2\pi}{\beta} \hat{u}}, \\
\langle \nabla_\theta(0) \nabla_\phi(\hat{u}) \rangle_\beta & \sim \frac{\Delta^2}{C} (1 + 2\Delta) e^{\frac{2\pi}{\beta} \hat{u}}.
\end{align}

The exponential growth of the above correlators originates from graviton (\(\epsilon\)) exchange. The corresponding Lyapunov exponent is

\[ \lambda^{(2)} = \frac{2\pi}{\beta}, \]

i.e. it saturates the bound of \([17]\). The superscript two indicates that this behaviour is due to graviton (spin two particle) exchange. Graviton exchange does not contribute to the four-point functions (42) and (43) at the given order in \(\frac{1}{C}\). For those one finds from gravitini exchange

\[ \langle \nabla_\phi(0) \nabla_\theta(\hat{u}) \rangle_\beta \sim \langle \nabla_\theta(0) \nabla_\phi(\hat{u}) \rangle_\beta \sim \Delta C \hat{u} e^{\frac{2\pi}{\beta} \hat{u}}, \]

implying a Lyapunov exponent

\[ \lambda^{(3/2)} = \frac{\pi}{\beta}. \]

This agrees with the pattern found in the context of \(\mathcal{N} = 1\) supersymmetry \([22, 29]\).

\[ \lambda^{(s)} = \frac{2\pi}{\beta} (s - 1), \]

where \(s\) is the spin of the gravitational mode being exchanged. From (23) we observe that combinations to which only the exchange of a graviphoton \(\sigma\) (spin one) contributes do not show exponential behaviour in agreement with (57). Notice also that for the result (55) the introduction of the second primary \(W\) with vanishing two-point function, \(\langle W\nabla \rangle = 0\), is crucial. If e.g. we had considered the 4-point functions with \(V, \nabla\)'s only, there would be additional terms with

\[ J(u_{ac1}, u_{c4}) J(u_{ac3}, u_{c2}) \]

in (31). These would result in graviton exchange contributing to (55) with \(W, \nabla\) replaced by \(V, \nabla\). Then also these components would show exponential growth with maximal Lyapunov exponent.

### 6 Summary and Outlook

In this paper we studied the \(\mathcal{N} = 2\) supersymmetric Jackiw-Teitelboim theory at finite temperature and computed the Lyapunov exponents. We found that their value depends on the particular component of the four-point function within the supermultiplet. There are components with maximal Lyapunov exponent \(2\pi/\beta\). For them the leading correction contains graviton exchange. For other components the graviton does not couple to
corresponding bi-locals. Due to gravitini exchange the leading contribution yields a Lyapunov exponent $\pi/\beta$. One could also consider superpositions with graviphoton exchange only. They do not show exponential time dependence (chaotic behaviour). Finally, we argued that for non vanishing two-point function, $\langle VW \rangle \neq 0$, all components have maximal Lyapunov exponent.

The $\mathcal{N} = 2$ case is interesting. For instance, it is related to four dimensional $1/4$ BPS black holes [26,30,31], whereas, to our knowledge, a higher dimensional solution leading to $\mathcal{N} = 1$ JT gravity in its near horizon limit has not been identified so far.

Several generalisations of the present calculation are possible. One could add flavour symmetry as in [22], or consider higher amounts of supersymmetry, e.g. $N = 3, 4$ super-Schwarzian actions. Furthermore, subleading corrections in $1/C$ might be interesting. For the bosonic case these have been analysed in [32], for $\mathcal{N} = 1$ some contributions are given in [23].

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