Orbifold Symmetry Reductions of Massive Boson-Fermion Degeneracy

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Abstract

We investigate the existence of string vacua with Massive Spectrum Degeneracy Symmetry (MSDS) in Heterotic and Type II orbifold constructions. We present a classification of all possible $\mathbb{Z}_2^N$-orbifolds with MSDS symmetry that can be constructed in the formalism of the 2d free fermionic construction. We explicitly construct several two-dimensional models whose Reduced Massive Spectrum Degeneracy Symmetry (RMDS) is due to a set of $\mathbb{Z}_2$-orbifold projections induced naturally in the framework of the free fermionic construction. In all proposed models the massive boson and fermion degrees of freedom exhibit Massive Spectrum Degeneracy Symmetry while the number of massless bosons $n(b)$ and massless fermions $n(f)$ are different; $n(b) \neq n(f)$. This property distinguishes the MSDS $\mathbb{Z}_2$-twisted theories from ordinary supersymmetric ones. Some comments are stated concerning the large marginal $JJ$-deformations of the proposed models connecting them to higher-dimensional gauged-supergravity theories with non-trivial geometrical fluxes.
1 Introduction

String theory provides a consistent framework that unifies all interactions including gravity [1]. Focusing ourselves on stringy gravity and cosmology new interesting phenomena occur. Conventional notions from general relativity like geometry, topology etc., are well defined in the string framework only as low energy approximations of the stringy approach [2–4]. At small distances physics deviates drastically from naive field-theoretic intuition. Various examples of purely stringy phenomena have already been identified in the past, which in several cases imply that the physics at strong curvature scales can be quite different from what one might expect from the field theory approximation [2]. They indicate new possibilities in the context of quantum cosmology and especially in the context of the “Stringy Big-Bang” picture [2–4] versus the initial time singularity picture of the “Big-Bang” in General Relativity. Assuming for instance a compact space and sufficiently close to the singularity, the typical scale of the universe reaches at these early times the gravitational scale (string scale). Obviously at this early epoch classical gravity is no longer valid and has to be replaced by a more fundamental singularity-free theory such as (super-)string theory.

Changing our framework from field theory to strings is by far a non-straightforward task, since even (super-)string theories are marked by Hagedorn-like singularities [5–7] which have to be resolved either by stringy phase transitions [6, 8] or by choosing Hagedorn-free string vacua in the early stage of the universe [9, 10]. From our viewpoint, it is of fundamental importance to show that the space of Hagedorn-free vacua is not empty and that their existence is at least equally natural as the Hagedorn-singular ones. Recently, a noticeable progress has been made in constructing Hagedorn-free string vacua in High Temperatures in the presence of non-trivial magnetic fluxes [10], which has shown explicitly the existence of non-pathological string vacua. Furthermore, a new symmetry “Massive boson-fermion Spectrum-Degeneracy Symmetry” (MSDS) was discovered in stringy vacua [9] where at least eight of the nine space dimensions are compact with a typical compactification scale close to the string scale.

The fact that the MSDS string vacua proposed in ref. [9] were constructed in a $d \leq 2$ target space background does not at all exclude them from being the most serious candidates able
to describe the early “Stringy non-geometric era” of the universe. On the contrary, in the spirit of ref. [9], this is actually quite natural. Assuming compact transverse space, the “non-geometrical stringy era” is expected as a consequence of the stringy $T$-duality symmetry. At the $T$-self-dual points the geometric description of space breaks down. In stringy framework, however, the theory makes sense in terms of a “non-geometrical description” based on non-abelian gauge field theory. These stringy phenomena are well known in several stringy compactifications around the so called “extended gauge symmetry points” of the moduli space.

All 2d MSDS string vacua, heterotic, type II and orientifolds proposed in ref. [9] are non-geometrical in terms of the internal compactified space but are well-characterized by the non-abelian gauge group $H_L \times H_R$. In the massless spectrum there are scalar bosons $M_{I_L, J_R}$, $I_L = 1, 2, ..., d_L$, $J_R = 1, 2, ..., d_R$, parametrizing the manifold

$$\mathcal{K} = \frac{SO(d_L, d_R)}{SO(d_L) \times SO(d_R)},$$

(1.1)

where $d_L$, $d_R$ are the dimensions of the $H_L$, $H_R$ gauge groups, respectively. Because of the non-abelian structure of $H_L \times H_R$, the MSDS vacua admit marginal deformations (flat directions) associated to the Cartan sub-algebra $U(1)^{r_L} \times U(1)^{r_R}$, with $r_L$ and $r_R$ being the ranks of $H_L$ and $H_R$, respectively. Following ref. [9], the moduli space of these deformations is of current-current type $M_{I_L, J_R}$ deformation and is given by the coset:

$$\mathcal{M} = \frac{SO(r_L, r_R)}{SO(r_L) \times SO(r_R)}.$$  

(1.2)

What is of main importance is the ultimate connection of the $M_{I_L, J_R}$ deformation parameters with the “induced effective higher-dimensional space geometry” in the large $M_{I_L, J_R}$-deformation limit (i.e. when the MSDS-vacua are strongly-deformed). In this limit one recovers the geometric field theory description in terms of an effective “higher-dimensional” conventional superstring theory in which the space-time supersymmetry is spontaneously broken by “geometrical” and “thermal” fluxes [9–11]. This fundamental generic property of the deformed MSDS-vacua strongly suggests that they be considered as the most (semi-) realistic candidate vacua able to describe the “early non-singular phase of our Universe”, being free of any initial “general relativity-like” or “Hagedorn-like” stringy singularities.

The originally proposed MSDS-vacua [9] and in particular the ones with $H_L \equiv SU(2)^8$,
are far too symmetric to be phenomenologically viable. Indeed, in the extreme large-$M$ deformation limit (decompactification limit), the induced effective theory is that of non-chiral extended gauge supergravities, implemented with a well-defined set of geometrical fluxes. However, from our cosmological viewpoint, the strongly deformed $MSDS$-vacua should consistently represent our late time universe and, thus, should contain a non-trivial net number of chiral families as well as a reduced gauge group unifying in the most realistic possible manner the standard model interactions.

The main aim of this work is to show the existence of less symmetric $MSDS$-vacua which are eventually connected via large $M$-deformations to phenomenologically acceptable four-dimensional vacua. In our days there are several well-known procedures that may be utilized in order to reduce symmetries of string vacua and at the same time ensure the presence of chiral matter representations of the unified gauge group. Such well-established procedures that create chiral $N = 1$ superstring models with (spontaneously) broken supersymmetry include symmetric orbifolds [12] (≡ Calabi-Yau [1]) compactification, fermionic constructions [13, 14], covariant lattices [15] and Gepner constructions [16], asymmetric orbifolds [17] (≡ generalized CY with torsion [18]), or type II orientifold compactifications [19] with or without geometrical [20] [21] or non-geometrical [22] fluxes. In this work we apply the (Asymmetric Freely Acting) orbifold construction to $MSDS$-vacua, so that the “Strongly $M$-Deformed $MSDS$-vacua” one would obtain in late cosmological times be phenomenologically acceptable.

The paper is organized as follows. In Section 2 a brief review of the construction of maximally symmetric $MSDS$-vacua is presented. These theories are non-singular and are based on a Spectral-Flow Super-Conformal Symmetry on the world-sheet. The space-time spectrum exhibits a Massive Spectrum Degeneracy Symmetry $(MSDS)$ between massive bosons and fermions. In Sections 3 and 4 we employ fermionic and orbifold construction techniques in order to construct several less symmetric models in Type II (Section 3) and Heterotic (Section 4) theories, that are still characterized by a Reduced Massive Spectrum Degeneracy Symmetry $(RMSDS)$. In Section 5 we derive the necessary conditions that permit the construction of all possible $RMSDS$-vacua, by utilizing free fermionic construction techniques. The reduced moduli space of the models is studied in section 6; in the same section we discuss the large $M$-
deformation limit of RMSDS-vacua and their plausible connections to phenomenologically acceptable models in late cosmological times. Section 7 is devoted to our conclusions.

2 Review of the maximally symmetric \textit{MSDS}-vacua

In the maximally symmetric \textit{MSDS}-vacua all nine, or at least eight- space coordinates are compact and closed to the string scale [9]. Furthermore, all compact space coordinates are expressed in terms of free 2d world-sheet fermions rather than the conventional compact bosonic coordinates [13, 14, 23]. The advantage of this fermionization lies in the consistent separation of left- and right-moving world-sheet degrees of freedom into terms of left- and right-moving 2d fermions that permit easier manipulations of the left-right asymmetric (and even non-geometrical) constructions of vacua in string theory. In what follows we restrict ourselves to the case of two flat target-space dimensions, leaving the remaining eight dimensions compactified to the string scale.

2.1 Generalities

2.1.1 Type II degrees of freedom

In the “critical” Type II theories the left- and right- moving degrees of freedom are:

- The light-cone degrees: \((\partial X^0, \Psi^0), (\partial X^L, \Psi^L)\)
- The super-reparametrization ghosts: \((b, c), (\beta, \gamma)\)
- The transverse super-coordinates: \((\partial X^I, \Psi^I), I = 1,...,8\)

In the fermionic construction the transverse super-coordinates are replaced by a set of free fermions in the adjoint representation of a semi-simple gauge group \(H\) (refs. [13, 23]): \(\{\chi^a\}, a = 1,...,n, \ n = \dim[H] = 24\). The simplest choice of \(H\) is:

\[
H = SU(2)^8 \equiv SO(4)^4,
\]  

where the transverse super-coordinates \((\partial X^I, \Psi^I)\) are replaced by \((y^I, w^I, \Psi_I)\) so that for every \(I = 1,...,8\), the coordinate currents \(i\partial X^I = y^I w^I\) are expressed in terms of the \(y^I, w^I\) 2d world-sheet fermions. For every \(I\), \(\{y^I, w^I, \Psi_I\}\) define the adjoint representation of a \(SU(2)_{k=2}\) current algebra. The choice \(H = SU(2)^8\) of the coordinate-fermionization is by no means unique [9, 23]. Other non-trivial choices of the coordinate-fermionization are also
possible:

\[ H = SU(5), \quad H = SO(7) \times SU(2), \quad H = G_2 \times Sp(4), \quad H = SU(4) \times SU(2)^3, \quad H = SU(3)^3. \]

(2.2)

In this work we restrict our attention to \( SU(2)^8 \)-fermionization for both left- and right-moving transverse degrees of freedom.

2.1.2 Heterotic degrees of freedom

The left-moving sector is identical to that of Type II theories, whereas the right-moving degrees of freedom are [9]:

- The light-cone degrees: \( (\partial X_0, \partial X_L) \)
- The reparametrization ghosts: \( (b, c) \)
- The transverse coordinates: \( (\partial X^I, I = 1, \ldots 8) \)
- The extra 32 right-moving fermions \( (\psi^A, A = 1, 2, \ldots 32) \) necessary for the conformal anomaly cancelation.

In total there are 48 free fermions in the right moving sector \( \{ \bar{\chi}^a, a = 1, 2, \ldots 48 \} \) which can be separated into: i) the 16 right-moving fermions \( (y^I, w^I, I = 1, 2, \ldots, 8) \) from coordinate fermionization \( i\partial X^I = y^I w^I \) plus ii) the extra 32 right-moving fermions \( (\psi^A, A = 1, 2, \ldots 32) \).

2.1.3 The basic left- and right-moving chiral operators and partition functions

The fundamental operators are as usual the left- and right-moving energy-momentum tensor \( T_B \) with conformal dimension \( h_B = 2 \) and the left- and right-moving superconformal operator \( T_F \) with \( h_F = 3/2 \) in Type II, responsible for the local left- and right-moving \( \mathcal{N} = (1, 1) \) world-sheet superconformal symmetry [1]. In Heterotic theories only the left-moving \( T_F \) exists, giving rise to an \( \mathcal{N} = (1, 0) \) superconformal symmetry. In both Type II and Heterotic theories the left-moving \( T_B \) and \( T_F \) have the same form:

\[
T_B = -\frac{1}{2} (\partial X_0)^2 - \frac{1}{2} \Psi_0 \partial \Psi_0 + \frac{1}{2} (\partial X_L)^2 + \frac{1}{2} \Psi_L \partial \Psi_L + \sum_{a=1}^{24} \frac{1}{2} \chi^a \partial \chi^a
\]

\[
T_F = i \partial X_0 \Psi_0 + i \partial X_1 \Psi_1 + \sum_{a,b,c} f_{abc} \chi^a \chi^b \chi^c ,
\]

(2.3)
where \( f_{abc} \) are the structure constants of the group \( H_L = SU(2)^8 \) and \( \{ \chi^a \} \) (\( a = 1, 2, ..., 24 \)) denote the 8 fermionized super-coordinates:

\[
\{ \chi^a \}, \ a = 1, 2, ..., 24 \equiv \{ \Psi^I, y^I, w^I \}, \ I = 1, 2, ..., 8.
\] (2.4)

The heterotic right-moving \( \bar{T}_B(\bar{z}) \) becomes:

\[
\bar{T}_B = -\frac{1}{2} (\bar{\partial}X_0)^2 + \frac{1}{2} (\bar{\partial}X_L)^2 + \sum_{a=1}^{48} \frac{1}{2} \bar{\chi}^a \bar{\partial} \chi^a.
\] (2.5)

Following the rules of the fermionic construction and respecting the \( H_L \times H_R = SU(2)^8 \times SU(2)^8 \) in type II or the \( H_L \times H_R = SU(2)^8 \times SO(48) \) in the heterotic, we can construct very special tachyon free vacua, with left–right holomorphic factorization of the partition function [9]. If the choice of boundary conditions on the world-sheet respects the global existence of the \( H_L \times H_R \) symmetry, the latter is promoted to a local gauge symmetry on the target space-time, both in Type II and the Heterotic cases [9, 13, 23]. The simplest constructions are those where all left-moving fermions \( \{ \chi^a, \ a = 1, 2, ...24 \} \) are taken with the same boundary conditions. All right-moving ones \( \{ \bar{\chi}^a, \ a = 1, 2, ...n_R \} \) have the same boundary conditions as well \( (n_R = 24 \text{ in Type II and } n_R = 48 \text{ in Heterotic}) \). Both Type II and Heterotic partition functions appear in simple factorized forms. In terms of the \( SO(2n) \) characters \( (n = 12 \text{ or } n = 24) \):

\[
V_{2n} = \frac{\theta^n_3 - \theta^n_4}{2\eta^n}, \quad O_{2n} = \frac{\theta^2_4 + \theta^n_4}{2\eta^n}, \quad S_{2n} = \frac{\theta^2_4 - \theta^n_4}{2\eta^n}, \quad C_{2n} = \frac{\theta^2_4 + \theta^n_4}{2\eta^n},
\] (2.6)

the Type II and Heterotic partition functions are:

\[
Z_{II} = \int_{\mathcal{F}} \frac{d^2 \tau}{(1 \text{m} \tau)^2} \left( V_{24} - S_{24} \right) \left( \overline{V}_{24} - \overline{S}_{24} \right),
\]

\[
Z_{Het} = \int_{\mathcal{F}} \frac{d^2 \tau}{(1 \text{m} \tau)^2} \left( V_{24} - S_{24} \right) \left( \overline{O}_{48} + \overline{C}_{48} \right).
\] (2.7)

The above expression for \( Z_{II} \) remains the same for any choice of left- and right-moving \( H \)-group \( H_L, H_R \), since the dimension of each is always equal to 24. In this respect, \( Z_{II} \) is a unique tachyon-free partition function (modulo the chirality of the left- and right-spinors) \textit{that respects the } \( H_L \times H_R \text{ gauge symmetry} \). The expression of the left-moving part in \( Z_{Het} \) remains the same as well. The right-moving part, however, depends on the choice of \( H_R \) (i.e. \( SO(48), E_8 \times SO(32), E_8^3 \)).
Both $Z_{II}$ and $Z_{Het}$ show a Massive Spectrum Degeneracy Symmetry. This spectacular property reflects the relations between the characters of the $SO(24)$-affine algebra [9, 24]:

$$V_{24} - S_{24} = \text{constant} = 24.$$  \hfill (2.8)

This follows from the well-known Jacobi identities between theta functions:

$$\theta^4_3 - \theta^4_4 - \theta^4_2 = 0, \quad \theta^4_1 = 0, \quad \theta^2_2 \theta^3_3 \theta^4_4 = 2\eta^3,$$

that further imply the identity:

$$\frac{\theta^{12}_3 - \theta^{12}_4}{2\eta^{12}} - \frac{\theta^{12}_2 - \theta^{12}_1}{2\eta^{12}} = 24.$$  \hfill (2.9)

The above identity shows that the spectrum of massive bosons and massive fermions is identical to all string mass levels! This is similar to the analogous property of supersymmetric theories. In the massless level, however, the situation is radically different: although there are 24 left-moving bosonic degrees of freedom there are no massless fermionic states.

- In Type II there are 24 right-moving bosonic states as well, so in total there are $24 \times 24$ scalar bosons at the massless level transforming under the adjoint representations of $H_L$ and $H_R$.

- The integrated partition function is thus equal to $[d(H_L) \times d(H_R)] \times \mathcal{I}$, where $\mathcal{I}$ is the integral over the fundamental domain

$$\mathcal{I} = \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \tau)^2} = \frac{\pi^2}{3}, \quad Z_{II} = \frac{\pi^2}{3} d(H_L) \times d(H_R).$$

- In the Heterotic string the left-moving sector gives constant contribution as in the Type II case ($d(H_L) = 24$). The right-moving massive states are expressed in terms of the unique holomorphic modular invariant function $j(\tau)$:

$$Z_{Het} = \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \tau)^2} d(H_L) \times \{d(H_R) + [j(\bar{\tau}) - 744]\} = \frac{\pi^2}{3} d(H_L) \times d(H_R).$$  \hfill (2.10)

The final integrated expression of $Z_{Het}$ is similar to $Z_{II}$. Both are proportional to the number of massless states of the models. This is because the contribution of the anti-holomorphic function $[j(\bar{\tau}) - 744]$ vanishes when integrated over the fundamental domain.
Depending on the choice of $H_R$ in the Heterotic, the number of the massless states can be: $d[SO(48)] = 1128$, $d[E_8 \times SO(32)] = d[E_8^3] = 744$.

2.2 Chiral superconformal algebra and spectral flow in MSDS

The symmetry operators of the MSDS vacuum are the usual holomorphic (anti-holomorphic) operators $T_B$, $T_F$ ($\bar{T}_B$, $\bar{T}_F$) giving rise to the standard $\mathcal{N} = (1,1)$ world-sheet superconformal symmetry in type II and the $\mathcal{N} = (1,0)$ in the heterotic, realizing a left-moving (and right-moving in type II) Operator Product Expansion [1] (OPE) with $\hat{c} = \frac{2}{3}c = 8$. The extra symmetry operators are the currents of conformal weight $h_J = 1$, associated with the $H_L$- and $H_R$-affine algebras: $J^a \equiv f^a_{bc} \chi^b \chi^c$ and $\bar{J}^a \equiv \bar{f}^a_{bc} \bar{\chi}^b \bar{\chi}^c$. Finally, there are two $SO(24)$ spin-field operators with conformal weight $\frac{3}{2}$ and opposite chirality:

$$C = Sp\{\chi^a\}_+ \quad \text{and} \quad S = Sp\{\chi^a\}_- \quad (2.11)$$

Following ref. [9], the existence of the chiral operator $C$, of conformal weight $h_C = \frac{3}{2}$, together with $T_B, T_F, J^a, \chi^a$, form a new chiral superconformal algebra implying the massive boson-fermion degeneracy of the Vacuum. The operator $O_{3/2} \equiv (\partial \hat{\chi} + \hat{\chi} \hat{\chi} \hat{\chi})$ which appears in the rhs of Eq. (2.12) is used to define a massive bosonic vertex operator of conformal weight $h_1 = 2$ in the $(-1)$ ghost picture [1, 25]:

$$V_{(1)} \equiv e^{-\Phi} (\partial \hat{\chi} + \hat{\chi} \hat{\chi} \hat{\chi}) \quad (2.13)$$
2.2.1 Spectral flow and the MSDS operator-relations

As usual the vertex operators are dressed by the super-reparametrization ghosts [1,25]: the space-time boson vertices are expressed either in the 0 or the (−1) ghost picture. The space-time fermions are in the (−1/2) or (−3/2) pictures.

\[ V(0) = e^{-\Phi} \hat{\chi}, \quad S = e^{-\frac{1}{2}\Phi - \frac{1}{2}iH_0} S \quad \text{or} \quad S = e^{-\frac{3}{2}\Phi + \frac{1}{2}iH_0} S, \quad (2.14) \]

where the \( H_0 \) is the usual helicity field defined via bosonization of \( \Psi_0 \) and \( \Psi_L : i\partial H_0 = \Psi_0 \Psi_L \). The conformal weight \( h_q \) of the operator [1,25]:

\[ e^{q\Phi} \rightarrow h_q = -\frac{1}{2} q (q + 2) \quad (2.15) \]

is such that \( V(0), V(1) \) have conformal weight \( h_0 = 1, h_1 = 2 \), while \( S \) has weight \( h_S = 2 \) in both the (−1/2) and (−3/2) pictures. The string spectrum of bosons starts from a massless sector which is described by \( V(0) \). On the contrary, all space-time fermions are massive, starting from mass level 1. The flow of \( V(0),(1) \)-states to \( S \)-states is expressed by the action of a “Spectral-Flow Operator” \( C \):

\[ C \equiv e^{\frac{1}{2}(\Phi - iH_0)} C. \]

Here, \( C \) is written in the (−1/2) ghost picture. It has conformal dimension \( h_C = 1 \) and (−1/2) helicity charge. Thus, generically, \( C \) acting on “physical” bosonic states produces fermionic states at the same string level and vice-versa. Although the \( C \)-action looks like a space-time supersymmetry transformation, the actual situation turns out to be drastically different from that of supersymmetry. Indeed, the \( C \)-action leaves the massless bosonic states of the theory invariant, therefore the boson-to-fermion mapping does not exist for the massless states. This statement is visualized in the OPE:

\[ C(z) V_0 \sim S, \quad \text{finite as } z \to w. \quad (2.16) \]

The absence of singular terms in \( z - w \) shows clearly that the massless states are invariant under the \( C \)-transformation. On the other hand, \( C \) acts not-trivially on the massive states:

\[ C(z) V_1(w) \sim \frac{S(w)}{(z - w)} + \text{finite terms}. \quad (2.17) \]

The above equation shows that the massive bosonic states are mapped into the fermionic ones. To show the inverse map \( C(z) : S(w) \to V(1)(w) \), it is necessary to perform standard
picture-changing manipulations \citep{1, 25} and consider $S$ in the $(-\frac{3}{2})$ picture so that $V(1)$ will appear in its conventional ghost-picture:

$$C(z) S(w) \sim \frac{V(1)(w)}{(z - w)} + \text{finite terms} \quad (2.18)$$

Summarizing:

- $T_B, T_F, C_{3/2}$ and $(J^a, \chi^a)$ define via the OPEs a new super-conformal algebra.
- The closure of the algebra is guarantied when $c = 12$, so that $C_{3/2}$ is a chirality “+” spin-field of $SO(24)$ with conformal weight $h_C = 3/2$.
- The realization of the algebra divides the “physical” states in two sectors:
  i) A massless sector invariant under $C$ spectral-flow transformations.
  ii) Massive fermionic states $S$ with “−” chirality, which are in one-to-one correspondence with the massive bosonic states $C : V(1) \leftrightarrow S$ \textit{massive supersymmetry}.

3 Type II MSDS Orbifold Vacua

In this section we provide explicit examples of reduced MSDS vacua ($RMSDS$), by introducing $\mathbb{Z}_2$-twists and shifts on the internal compactified coordinates $i\partial X^I \equiv y^I w^I$ of Type II theories. Our constructions utilize the conformal field theory techniques of the free fermionic construction \citep{13}, that can be easily translated in the symmetric and asymmetric $\mathbb{Z}_2$-orbifold language. The orbifold representation will be especially useful for the study of deformed $RMSDS$ vacua via $J \times \bar{J}$ marginal deformations (see Section 6).

3.1 $\mathbb{Z}_2$-orbifolds with MSDS in Type II theories

3.1.1 $\mathbb{Z}_2$-twisted MSDS in Type II

The initial MSDS vacuum in Type II factorizes the world-sheet fermions into two basis sets $\{H_L, H_R\}$:

i) the left-moving set $\rightarrow H_L = \{\chi^{1...8}, y^{1...8}, w^{1...8}\},$

ii) the right-moving set $\rightarrow H_R = \{\chi^{1...8}, \bar{y}^{1...8}, \bar{w}^{1...8}\}$.

By introducing the additional (breaking) set: $B_t = \{\chi^{5...8} y^{5...8}, \chi^{5...8} \bar{y}^{5...8}\}$,
the four internal coordinates are $Z_2$-twisted as follows:

$$i \partial X^I = y^I w^I \rightarrow -i \partial X^I = -y^I w^I, \quad i \bar{\partial} X^{\bar{I}} = y^{\bar{I}} w^{\bar{I}} \rightarrow -i \bar{\partial} X^{\bar{I}} = -\bar{y}^{\bar{I}} \bar{w}^{\bar{I}}, \quad I = 5, 6, 7, 8. \quad (3.1)$$

The partition function of the $Z_2$-twisted model $\{H_L, H_R, B_t\}$ is:

$$Z_{II}^{B_t} = \frac{1}{2^2} \sum_{a,b,a,b} \frac{1}{2} \sum_{h,g} (-)^{a+b} \frac{\theta_{[b]}^{[a]}}{\eta^{12}} \frac{\theta_{[b+g]}^{[a+h]}}{\bar{\eta}^{12}} (-)^{\bar{a}+\bar{b}} \frac{\bar{\theta}_{[\bar{b}]}^{[\bar{a}]} \bar{\theta}_{[\bar{b}+g]}^{[\bar{a}+h]}}{\bar{\eta}^{12}}. \quad (3.2)$$

The $Z_2$-projection induced by the $B_t$ set, $Z_2^{B_t}$-reduces the initial spectrum symmetry:

$$SO(24)_L \times SO(24)_R \rightarrow [SO(16) \times SO(8)]_L \times [SO(16) \times SO(8)]_R, \quad (3.3)$$

so that the spectrum is naturally expressed in terms of the $[SO(16) \times SO(8)]_{L,R}$ characters:

$$V_{16}, O_{16}, S_{16}, C_{16}, \quad V_8, O_8, S_8, C_8, \quad V_{16}, \quad O_{16}, \quad S_{16}, \quad C_{16}, \quad V_8, \quad O_8, \quad S_8, \quad C_8. \quad (3.4)$$

The $Z_2^{B_t}$-action is non-trivial on:

$$Z_2^{B_t} : \{V_8, \quad S_8, \quad V_8, \quad S_8\} \rightarrow -\{V_8, \quad S_8, \quad V_8, \quad S_8\}. \quad (3.5)$$

The $Z_{II}^{B_t}$ partition function is naturally organized into four products of holomorphic times anti-holomorphic terms $\sum A_i \times \bar{A}_i$ transforming under the same irreducible representation of the $Z_2^{B_t}$-group. There are two terms arising from the untwisted sector $h = 0$ and two from the twisted one $h = 1$:

$$Z_{II}^{B_t} = \frac{1}{2} \sum_g Z_{+g}^{[0]} \bar{Z}_{+g}^{[0]} + \frac{1}{2} \sum_g Z_{-g}^{[0]} \bar{Z}_{-g}^{[0]} + \frac{1}{2} \sum_g Z_{+g}^{[1]} \bar{Z}_{+g}^{[1]} + \frac{1}{2} \sum_g Z_{-g}^{[1]} \bar{Z}_{-g}^{[1]} \quad (3.6)$$

The first two terms, coming from the untwisted sector, are:

(i) Untwisted $(+, +)$:

$$\frac{1}{2} \sum_g Z_{+g}^{[0]} \bar{Z}_{+g}^{[0]} = (V_{16} O_8 - S_{16} C_8) \times (\bar{V}_{16} O_8 - \bar{S}_{16} \bar{C}_8), \quad (3.7)$$

(ii) Untwisted $(-, -)$:

$$\frac{1}{2} \sum_g Z_{-g}^{[0]} \bar{Z}_{-g}^{[0]} = (O_{16} V_8 - C_{16} S_8) \times (\bar{O}_{16} \bar{V}_8 - \bar{C}_{16} \bar{S}_8). \quad (3.8)$$

The last two terms come from the twisted sector:

(iii) Twisted $(+, +)$:

$$\frac{1}{2} \sum_g Z_{+g}^{[1]} \bar{Z}_{+g}^{[1]} = (V_{16} C_8 - S_{16} O_8) \times (\bar{V}_{16} \bar{C}_8 - \bar{S}_{16} \bar{O}_8), \quad (3.9)$$

11
\[(iv) \text{Twisted } (-,-): \quad \frac{1}{2} \sum_g Z_{-\frac{1}{2}}^{[1]} \overline{Z}_{-\frac{1}{2}}^{[1]} = (O_{16}S_8 - C_{16}V_8) \times (O_{16}S_8 - C_{16}V_8). \quad (3.10)\]

It is remarkable that the \(\{H_L, H_R, B_t\}\)-twisted vacuum enjoys a massive boson-fermion degeneracy symmetry similar to the original \(\{H_L, H_R\}\) Type II model. What is even more remarkable is that the \(MSDS\) properties (reduced by \(Z_{Bt}^2\)) are not only valid sector-by-sector individually but also for each holomorphic and anti-holomorphic factor separately. The (anti-)holomorphic part of each contribution turns out to be constant:

\[
\begin{align*}
V_{16}O_8 - S_{16}C_8 &= 16, \quad (3.11) \\
O_{16}V_8 - C_{16}S_8 &= 8, \quad (3.12) \\
V_{16}C_8 - S_{16}O_8 &= 0, \quad (3.13) \\
O_{16}S_8 - C_{16}V_8 &= 8. \quad (3.14)
\end{align*}
\]

The above holomorphic twisted \(\theta^{12}\)-identities can be easily proved by using the \(\theta^4\)-abstrusa" and "triple-product" identities \([2.9]\) of Jacobi.

Adding all four contributions we obtain the partition function of the orbifolded model:

\[
Z_{II}^{B_t} = 16 \times 16 + 8 \times 8 + 0 \times 0 + 8 \times 8 = 384. \quad (3.15)
\]

The fact that the holomorphic and anti-holomorphic parts of each sector are separately equal to constants, as shown in Eqs. \((3.11) - (3.14)\), implies that the \(MSDS\)-structure originates separately from the holomorphic and anti-holomorphic part in each sector. This precise property hints at the existence of a chiral world-sheet algebra, whose spectral-flow is responsible for the massive boson-fermion degeneracy symmetry of the spectrum. In fact, the algebra in question is a \(Z_2^{B_t}\)-truncation of the original chiral superconformal algebra of the previous section. The \(Z_2^{B_t}\)-orbifold truncates the original spectral-flow operator \(C_{24}\), dual to the \(C_{24}\)-character of \(SO(24)\), down to the \(Z_2^{B_t}\)-invariant operator:

\[
Z_2^{B_t} : C_{24} = C_{16}C_8 + S_{16}S_8 \rightarrow C_{16}C_8
\]

\[
C_{B_t} = e^{\frac{i}{2}(\Phi - iH_0)} C_{16}C_8. \quad (3.17)
\]

The global existence of \(C_{B_t}\), along with the truncated chiral algebra, are sufficient to guarantee massive supersymmetry of the spectrum.
We close this subsection with a few comments on the massless spectrum of the $\mathbb{Z}_2^{B_t}$-orbifold considered above. Its massless states are still purely bosonic as a result of the left-right symmetry of the model. Specifically, there are:

- $16 \times \overline{16}$ states: $\{\chi_{\frac{1}{2}}^{1,4} \oplus y_{\frac{1}{2}}^{1,4} \oplus w_{\frac{1}{2}}^{1,8}\}|0\rangle_L \otimes \{\chi_{\frac{1}{2}}^{1,4} \oplus y_{\frac{1}{2}}^{1,4} \oplus w_{\frac{1}{2}}^{1,8}\}|0\rangle_R$ from $V_{16}O_8 \times \overline{V}_{16O_8}$
- $8 \times \overline{8}$ states: $\{\chi_{\frac{1}{2}}^{5,8} \oplus y_{\frac{1}{2}}^{5,8}\}|0\rangle_L \otimes \{\chi_{\frac{1}{2}}^{5,8} \oplus y_{\frac{1}{2}}^{5,8}\}|0\rangle_R$ from $O_{16V_8} \times \overline{O}_{16V_8}$
- $8 \times \overline{8}$ states: $Sp\{\chi_{\frac{1}{2}}^{5,8} \oplus y_{\frac{1}{2}}^{5,8}\} - |0\rangle_L \otimes Sp\{\chi_{\frac{1}{2}}^{5,8} \oplus y_{\frac{1}{2}}^{5,8}\} - |0\rangle_R$ from $O_{16S_8} \times \overline{O}_{16S_8}$

We note that each of the (anti-)holomorphic contributions to the massless states come with a positive sign, implying that they are bosonic.

3.1.2 $\mathbb{Z}_2$-shifted MSDS in Type II

A different way of reducing the initial MSDS symmetry is via $\mathbb{Z}_2$-shifted orbifolds, where some (or all) of the compactified coordinates are half-shifted:

$$Z^{B_s}_{II} = \sum_{h,g} \sum_{a,b,\bar{a},\bar{b}} (-)^{a+b} \frac{\theta[a][4] \theta[a+h][8]}{\eta^{12}} (-)^{\bar{a}+\bar{b}} \frac{\theta[\bar{a}][4] \theta[\bar{a}+\bar{h}][8]}{\bar{\eta}^{12}}.$$  (3.20)

Shifting the summation variables:

$$a \rightarrow a + h, \quad b \rightarrow b + g; \quad \bar{a} \rightarrow \bar{a} + h, \quad \bar{b} \rightarrow \bar{b} + g,$$  (3.21)

we see that $Z^{B_t}_{II}$ becomes identical to the partition function $Z^{B_t}_{II}$ of the twisted model $3.2$. The shift of variables produces an extra phase $(-)^{h+g}$ in the holomorphic sector, which is
cancelled by the corresponding extra phase of the anti-holomorphic part and the partition functions of the $\mathbb{Z}_2$-twisted and $\mathbb{Z}_2$-shifted models are algebraically equal. However, this similarity between twists and shifts is due to the fact that the theory has been written in the very special “fermionic point” of the moduli space. The fundamental difference between the two models will become apparent as soon as the theory is deformed away from that point. Indeed, the twisted model corresponds to a non-freely acting orbifold, whose deformation radii have two fixed points each, whereas the shifted model is freely-acting and effectively corresponds to a model whose radii are reduced by half.

As in the twisted case, contributions to the partition function are naturally organized into four products of holomorphic times anti-holomorphic terms $\sum A_i \times \overline{A}_i$, transforming under the same irreducible representation of the $\mathbb{Z}_B^2$-group. Two of them arise from the untwisted sector $h = 0$ and two from the twisted one $h = 1$:

(i) Untwisted $(+, +)$: $\frac{1}{2} \sum_g Z_+^{[0]} \overline{Z}_+^{[0]} = (V_8 O_{16} - S_8 C_{16}) \times (V_8 O_{16} - S_8 C_{16}) = 8 \times 8$

(ii) Untwisted $(-, -)$: $\frac{1}{2} \sum_g Z_-^{[0]} \overline{Z}_-^{[0]} = (O_8 V_{16} - C_8 S_{16}) \times (O_8 V_{16} - C_8 S_{16}) = 16 \times 16$

(iii) Twisted $(+, +)$: $\frac{1}{2} \sum_g Z_+^{[1]} \overline{Z}_+^{[1]} = (V_8 C_{16} - S_8 O_{16}) \times (V_8 C_{16} - S_8 O_{16}) = (-8) \times (-8)$

(iv) Twisted $(-, -)$: $\frac{1}{2} \sum_g Z_-^{[1]} \overline{Z}_-^{[1]} = (O_8 S_{16} - C_8 V_{16}) \times (O_8 S_{16} - C_8 V_{16}) = 0 \times 0$ (3.22)

where, as usual, the bars denote complex conjugation. We note that the chiral and anti-chiral contributions of the shifted $(+, +)$-sector $\sum_g Z_+^{[1]} \overline{Z}_+^{[1]}$ now come with a negative sign each, indicating an abundance of fermions in both the holomorphic and anti-holomorphic sides. However, due to the left-right symmetry of the model, the tensor product of the left- and right-handed spin fields $Sp\{\chi^{1...8}\}_- \otimes Sp\{\overline{\chi}^{1...8}\}_-$ from $(-S_8 O_{16}) \times (-\overline{S_8 O_{16}})$ produce space-time bosons, which accounts for the overall positive sign $(-8) \times (-8) = +64$. The overall number of massless bosonic states is the same as in the twisted case:

$$Z_{II}^{B_s} = 16 \times 16 + 8 \times 8 + (-8) \times (-8) = 384.$$ (3.23)

As in the twisted case, the spectral-flow operator responsible for the $MSDS$-symmetry is the truncation of the original operator, invariant under the $\mathbb{Z}_2^B$-shift:

$$C_{B_s} = e^{\frac{i}{2}(\Phi - iH_0)} C_{16} C_8.$$ (3.24)
Another interesting possibility in Type II is to combine together an holomorphic $\mathbb{Z}_2$-twist with an anti-holomorphic $\mathbb{Z}_2$-shift. Because of the chiral nature of the spectral-flow degeneracy, the resulting model will again have a reduced $\text{MSDS}$-structure, since the (anti-) holomorphic side does. The result will be an asymmetric model whose additional basis set $B_{ts}$ acts differently on the left- and right-moving side:

$$B_{ts} = \{ \chi^{5..8} y^{5..8} | \bar{\eta}^{1..8} \bar{w}^{1..8} \}.$$  

(3.25)

Its partition function is:

$$Z_{II}^{B_{ts}} = \frac{1}{2} \sum_{h,g} \sum_{a,b,\bar{a},\bar{b}} (-)^{a+b+hg} \frac{\theta_{[a]}^8 \theta_{[a+h]}^4}{\eta^{12}} \frac{\bar{\theta}_{[\bar{a}]}^4 \bar{\theta}_{[\bar{a}+\bar{h}]}^8}{\bar{\eta}^{12}}.$$

(3.26)

The various contributions to the partition function are organized as usual:

(i) Untwisted $(+, +)$:

$$\frac{1}{2} \sum_{g} Z_+^{[0]} \bar{Z}_+^{[0]} = (V_{16} O_8 - S_{16} C_8) \times (V_{8} O_{16} - S_{8} C_{16}) = 16 \times \overline{8}$$

(ii) Untwisted $(-, -)$:

$$\frac{1}{2} \sum_{g} Z_-^{[0]} \bar{Z}_-^{[0]} = (O_{16} V_8 - C_{16} S_8) \times (O_{8} V_{16} - C_{8} S_{16}) = 8 \times \overline{16}$$

(iii) Twisted $(+, +)$:

$$\frac{1}{2} \sum_{g} Z_+^{[1]} \bar{Z}_+^{[1]} = (O_{16} S_8 - C_{16} V_8) \times (V_{16} C_{16} - S_{16} O_{16}) = 8 \times (-8)$$

(iv) Twisted $(-, -)$:

$$\frac{1}{2} \sum_{g} Z_-^{[1]} \bar{Z}_-^{[1]} = (V_{16} C_8 - S_{16} O_8) \times (O_{8} S_{16} - C_{8} V_{16}) = 0 \times \overline{0}$$

(3.27)

The chiral and anti-chiral algebras need to be truncated by the $\mathbb{Z}_2$-twist and $\mathbb{Z}_2$-shift, respectively. Similarly, the holomorphic spectral-flow operator $C_{B_{ts}}$ will be invariant under the twist, while its anti-holomorphic counterpart $\overline{C}_{B_{ts}}$ under the shift:

$$C_{B_{ts}} = e^{\frac{i}{2}(\Phi - iH_0)} C_{16} C_8,$$

$$\overline{C}_{B_{ts}} = e^{\frac{i}{2}(\bar{\Phi} - i\bar{H}_0)} \overline{C}_8 \overline{C}_{16}.$$

(3.28)

The full partition function of the model is then:

$$Z_{II}^{B_{ts}} = 16 \times \overline{8} + 8 \times \overline{16} + 8 \times (-8) + 0 \times \overline{0} = 192$$

(3.29)

and exhibits massive supersymmetry, as expected. Note that the spectrum now contains massless fermions in the twisted sector, as a result of the left-right asymmetry of the
model. There are 64 massless states \( Sp\{\chi^{5..8} y^{5..8}\} - \otimes Sp\{\chi^{1..8}\} - \) arising from the sector \((O_{16} S_8)(-S_8 O_{16})\). Massless fermions will also appear in the heterotic extension of (3.26) which we consider in Section 4.

### 3.2 \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-orbifolds with MSDS in Type II theories

To further reduce the massive boson-fermion degeneracy of the initial Type II maximally symmetric MSDS-vacuum, we next consider examples of \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-orbifolds that exhibit Reduced Massive Spectral boson-fermion Degeneracy Symmetry (RMSDS). Here, as in the previous cases, the massive boson-fermion degeneracy follows from (anti-)holomorphic \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-twisted \(\theta^{12}\)-identities induced by an (anti-)holomorphic spectral-flow operator, invariant under \(\mathbb{Z}_2 \times \mathbb{Z}_2\).

#### 3.2.1 \(T^6 / \mathbb{Z}_2 \times \mathbb{Z}_2\) MSDS orbifold in Type II

In the language of the free fermionic construction, the \(T^6 / \mathbb{Z}_2 \times \mathbb{Z}_2\) orbifold of the original MSDS-vacuum is constructed by introducing two additional basis sets \(B_1^t, B_2^t\):

\[
B_1^t = \{\chi^{5,6,7,8}, y^{5,6,7,8}, \chi^{5,6,7,8}, \chi^{5,6,7,8}\}, \\
B_2^t = \{\chi^{3,4,7,8}, y^{3,4,7,8}, \chi^{3,4,7,8}, \chi^{3,4,7,8}\},
\]

(3.30)
giving different boundary conditions to the fermions \(y^I\) and \(w^I\), which bosonize into the six compact scalars \(i\partial X^I, I = 3, \ldots, 8\). The modular invariant partition function of the \(\{H_L, H_R, B_1^t, B_2^t\}\)-vacuum is:

\[
Z_{II}^{B_1^t, B_2^t} = \frac{1}{2^2} \sum_{h_1, g_1} \sum_{a, b, \bar{a}, \bar{b}} \frac{(-)^{a+b}}{\eta^{12}} \times \frac{(-)^{a+b}}{\bar{\eta}^{12}} \theta \left[ a \right]_b^6 \theta \left[ a+h_1 \right]_b^{h_2} \theta \left[ a-h_1-h_2 \right]_b^2 \theta \left[ a+h_2 \right]_b^2 \theta \left[ a-h_2 \right]_b^{h_1} \theta \left[ a-h_1 \right]_b^{h_2} \theta \left[ a-h_1-h_2 \right]_b^2 \theta \left[ a-h_1 \right]_b^{h_2} \theta \left[ a-h_2 \right]_b^{h_1} \theta \left[ a-h_1-h_2 \right]_b^2 \theta \left[ a-h_1 \right]_b^{h_2} \theta \left[ a-h_2 \right]_b^{h_1} \theta \left[ a-h_1-h_2 \right]_b^2 \theta \left[ a-h_1 \right]_b^{h_2} \theta \left[ a-h_2 \right]_b^{h_1} \theta \left[ a-h_1-h_2 \right]_b^2 .
\]

(3.31)
The \(\mathbb{Z}_2^{B_1^t} \times \mathbb{Z}_2^{B_2^t}\) reduce the initial spectrum symmetry:

\[
SO(24)_{L,R} \rightarrow [SO(12) \times SO(4) \times SO(4) \times SO(4)]_{L,R} ,
\]

(3.32)
so that the spectrum is naturally expressed in terms of the \([SO(12) \times SO(4) \times SO(4) \times SO(4)]_{L,R}\) characters. Taking the group elements of \(\mathbb{Z}_2^{B_1^t} \times \mathbb{Z}_2^{B_2^t}\) to be \(\{1, a, b, ab\}\), we may organize the
contributions to the partition function into terms transforming as the irreducible representations:

\[ Z_2^{B_1} \times Z_2^{B_2} : \{ \chi(\pm,\pm,\pm), \chi(\pm,\pm,\mp), \chi(\pm,\mp,\pm), \chi(\mp,\pm,\pm) \} \]

\[ \longrightarrow \{ 1 \chi(\pm,\pm,\pm), a \chi(\pm,\pm,\mp), b \chi(\pm,\mp,\pm), ab \chi(\mp,\pm,\pm) \} \] .  (3.33)

The transformation properties under \( Z_2^{B_1} \times Z_2^{B_2} \) of the characters of the remaining spectrum symmetry, \([SO(12) \times SO(4) \times SO(4) \times SO(4)]_{L,R}\), are:

\[
\begin{align*}
\{ C_{12}, C_4, C_4, C_4 \} & \longrightarrow \{ 1 C_{12}, 1 C_4, 1 C_4, 1 C_4 \}, \\
\{ O_{12}, O_4, O_4, O_4 \} & \longrightarrow \{ 1 O_{12}, 1 O_4, 1 O_4, 1 O_4 \}, \\
\{ V_{12}, V_4, V_4, V_4 \} & \longrightarrow \{ 1 V_{12}, a V_4, b V_4, a b V_4 \}, \\
\{ S_{12}, S_4, S_4, S_4 \} & \longrightarrow \{ 1 S_{12}, a S_4, b S_4, a b S_4 \}. 
\end{align*}
\]  (3.34)

The \( Z_2^{B_1} \times Z_2^{B_2} \) partition function is naturally organized into products of sixteen holomorphic times anti-holomorphic terms \( \sum A_i \times \overline{A}_i \) transforming under the same irreducible representation of the \( Z_2^{B_1} \times Z_2^{B_2} \)-group. Four of those come from the untwisted sector \((h_1, h_2) = (0, 0)\) and four from each of the three twisted planes \((h_1, h_2) = (1, 0), (h_1, h_2) = (0, 1)\) and \((h_1, h_2) = (1, 1)\).

\( \alpha \) Untwisted sector \((h_1, h_2) = (0, 0)\):

\[ \begin{align*}
i) & \quad \left| A_{(+,+,+)}^{(0,0)} \right|^2 = |V_{12}O_4O_4O_4 + O_{12}V_4V_4V_4 - S_{12}C_4C_4C_4 - C_{12}S_4S_4S_4|^2 = 12 \times \overline{T}, \\
ii) & \quad \left| A_{(+,-,+)}^{(0,0)} \right|^2 = |V_{12}O_4V_4 + O_{12}V_4O_4O_4 - S_{12}C_4S_4S_4 - C_{12}S_4C_4C_4|^2 = 4 \times T, \\
iii) & \quad \left| A_{(+,-,-)}^{(0,0)} \right|^2 = |V_{12}O_4V_4 + O_{12}V_4O_4O_4 - S_{12}S_4C_4S_4 - C_{12}C_4C_4C_4|^2 = 4 \times T, \\
iv) & \quad \left| A_{(+-,-)}^{(0,0)} \right|^2 = |V_{12}V_4O_4O_4 + O_{12}O_4V_4V_4 - S_{12}S_4S_4C_4 - C_{12}C_4S_4C_4|^2 = 4 \times T. 
\end{align*} \]  (3.35)

The last three terms are algebraically equal due to the permutation symmetry under the interchange of the three \( SO(4) \) factors. Summing the above, we obtain the untwisted contribution to the partition function:

\[ \left| A_{(+,+,+)}^{(0,0)} \right|^2 + \left| A_{(+,-,+)}^{(0,0)} \right|^2 + \left| A_{(+,-,-)}^{(0,0)} \right|^2 + \left| A_{(+-,-)}^{(0,0)} \right|^2 = 12 \times \overline{T} + (4 \times T) \times 3 = 192 . \]  (3.36)
The twisted contributions per twisted plane can again be grouped into four terms according to their transformation properties.

\(3)\) Twisted sector \((h_1, h_2) = (1, 0)\):

\[
\begin{align*}
\text{i)} & \quad \left| A^{(1,0)}_{(+,+,+,+)} \right|^2 = |V_{12}O_4S_4S_4 + O_{12}V_4C_4C_4 - S_{12}C_4V_4V_4 - C_{12}S_4O_4O_4|^2 = 0 \times 0, \\
\text{ii)} & \quad \left| A^{(1,0)}_{(+,+,-,+)} \right|^2 = |V_{12}O_4C_4C_4 + O_{12}C_4S_4S_4 - S_{12}C_4O_4O_4 - C_{12}S_4V_4V_4|^2 = 0 \times 0, \\
\text{iii)} & \quad \left| A^{(1,0)}_{(+,-,+,+)} \right|^2 = |V_{12}V_4S_4C_4 + O_{12}O_4C_4S_4 - S_{12}S_4V_4O_4 - C_{12}C_4O_4V_4|^2 = 4 \times 4, \\
\text{iv)} & \quad \left| A^{(1,0)}_{(+,-,+,+)} \right|^2 = |V_{12}V_4C_4S_4 + O_{12}O_4S_4C_4 - S_{12}O_4V_4S_4 - C_{12}C_4V_4O_4|^2 = 4 \times 4. 
\end{align*}
\]

\(\gamma\) Twisted sector \((h_1, h_2) = (0, 1)\):

The contribution of the \((h_1, h_2) = (0, 1)\) twisted sector is similar to the \((h_1, h_2) = (1, 0)\) one. It is obtained by interchanging the characters of the first \(SO(4)\) with the second \(SO(4)\) factor.

\(\delta\) Twisted sector \((h_1, h_2) = (1, 1)\):

The contribution of the \((h_1, h_2) = (1, 1)\) twisted sector is similar to the \((h_1, h_2) = (1, 0)\) one. It is obtained by interchanging the characters of the first \(SO(4)\) with the third \(SO(4)\) factor. Thus, all three orbifold planes are equivalent, giving rise to equal contributions per twisted sector, \((2 \times 4 \times 4 = 32)\). Therefore, the total partition function of the \(\{H_L, H_R, B^1_t, B^2_t\}\)-vacuum is constant:

\[
Z_{H}^{B^1_t, B^2_t} = 192 + 32 \times 3 = 288. \tag{3.38}
\]

In the \(\{H_L, H_R, B^1_t, B^2_t\}\)-vacuum there are no massless fermions. The chiral spectral-flow operator, invariant under \(\mathbb{Z}_2^{B^1_t} \times \mathbb{Z}_2^{B^2_t}\), is:

\[
C_{B^1_t, B^2_t} = e^{i(\Phi - iH_0)} \left[ C_{12}C_4C_4C_4 + S_{12}S_4S_4S_4 \right]. \tag{3.39}
\]

3.2.2 \(T^8/\mathbb{Z}_2 \times \mathbb{Z}_2\) MSDS orbifold in Type II

We next discuss another example of \(\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifold with a different factorizable embedding in the compactified eight dimensional target space \(T^4/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2\). Consider the orbifold
constructed by the following choice of basis sets:

\[ b_1^1 = \{ \chi^{5...8} y^{5...8} | \chi^{5...8} y^{5...8} \}, \]
\[ b_1^2 = \{ \chi^{1...4} y^{1...4} | \chi^{1...4} y^{1...4} \}. \] (3.40)

The partition function of the \( \{ H_L, H_R, b_1^1, b_1^2 \} \)-model is:

\[
Z_{II}^{b_1^1, b_1^2} = \frac{1}{2^2} \sum_{h_1, b_1} \frac{1}{2^2} \sum_{a, b, a, b} (-)^{a+b} \frac{\theta[a]}{\eta^{12}} \frac{\theta[a+h_1]}{\eta^{12}} \frac{\theta[a+h_2]}{\eta^{12}} \frac{\theta[a+h_2]}{\eta^{12}} (-)^{a+b} \frac{\theta[\bar{a}]}{\eta^{12}} \frac{\theta[\bar{a}+h_1]}{\eta^{12}} \frac{\theta[\bar{a}+h_2]}{\eta^{12}} \frac{\theta[\bar{a}+h_2]}{\eta^{12}} .
\] (3.41)

In this model the initial spectrum symmetry \( SO(24)_L \times SO(24)_R \) is reduced by \( \mathbb{Z}_2^{b_1^1} \times \mathbb{Z}_2^{b_1^2} \) to a product of three \( SO(8) \) factors:

\[
SO(24)_{L,R} \rightarrow \left[ SO(8) \times SO(8) \times SO(8) \right]_{L,R} ,
\] (3.42)

so that the spectrum is naturally expressed in terms of the \( \left[ SO(8) \times SO(8) \times SO(8) \right]_{L,R} \) characters. Taking the group elements of \( \mathbb{Z}_2^{b_1^1} \times \mathbb{Z}_2^{b_1^2} \) to be \( \{ 1, a, b, ab \} \), we may organize the contributions to the partition function into terms transforming as the irreducible representations of the discrete orbifold group. The transformation properties of the characters of the remaining spectrum symmetry \( \left[ SO(8) \times SO(8) \times SO(8) \right]_{L,R} \) under \( \mathbb{Z}_2^{b_1^1} \times \mathbb{Z}_2^{b_1^2} \) are:

\[
\{ C_8, C_8, C_8 \} \rightarrow \{ 1 C_8, 1 C_8, 1 C_8 \} ,
\]
\[
\{ O_8, O_8, O_8 \} \rightarrow \{ 1 O_8, 1 O_8, 1 O_8 \} ,
\]
\[
\{ V_8, V_8, V_8 \} \rightarrow \{ 1 V_8, a V_8, b V_8 \} ,
\]
\[
\{ S_8, S_8, S_8 \} \rightarrow \{ 1 S_8, a S_8, b S_8 \} .
\] (3.43)

The \( Z_{II}^{b_1^1, b_1^2} \) partition function is naturally organized into products of sixteen holomorphic times anti-holomorphic terms \( \sum A_i \times \bar{A}_i \) transforming under the same irreducible representation of the \( \mathbb{Z}_2^{b_1^1} \times \mathbb{Z}_2^{b_1^2} \)-group. Four of those come from the untwisted sector \( (h_1, h_2) = (0, 0) \) and four come from each of the three twisted planes \( (h_1, h_2) = (1, 0), (h_1, h_2) = (0, 1) \) and \( (h_1, h_2) = (1, 1) \).

a) \textit{Untwisted sector} \( (h_1, h_2) = (0, 0) \): 

i) \[
\left| A_{(++,++,+)}^{(0,0)} \right|^2 = |V_8 O_8 O_8 - S_8 C_8 C_8|^2 = 8 \times \bar{8} ,
\]
\[ ii) \left| A_{(+,-,-,-)}^{(0,0)} \right|^2 = |O_8 O_8 V_8 - C_8 C_8 S_8|^2 = 8 \times \bar{8}, \]
\[ iii) \left| A_{(+,-,-,-)}^{(0,0)} \right|^2 = |O_8 V_8 O_8 - C_8 S_8 C_8|^2 = 8 \times \bar{8}, \]
\[ iv) \left| A_{(+,-,+,-)}^{(0,0)} \right|^2 = |V_8 V_8 V_8 - S_8 S_8 S_8|^2 = 0 \times \bar{0}. \]

(3.44)

\[ \beta) \text{Twisted sector } (h_1, h_2) = (1, 0) : \]
\[ i) \left| A_{(+,+,-,+)}^{(1,0)} \right|^2 = |V_8 C_8 O_8 - S_8 O_8 C_8|^2 = 0 \times \bar{0}, \]
\[ ii) \left| A_{(+,+,-,-)}^{(1,0)} \right|^2 = |O_8 C_8 V_8 - C_8 O_8 S_8|^2 = 0 \times \bar{0}, \]
\[ iii) \left| A_{(+,-,+,-)}^{(1,0)} \right|^2 = |O_8 S_8 O_8 - C_8 V_8 C_8|^2 = 8 \times \bar{8}, \]
\[ iv) \left| A_{(+,-,+,-)}^{(1,0)} \right|^2 = |V_8 S_8 V_8 - S_8 V_8 S_8|^2 = 0 \times \bar{0}. \]

(3.45)

The two remaining twisted sectors are \((h_1, h_2) = (0, 1)\) and \((h_1, h_2) = (1, 1)\), modulo permutations of the three \(SO(8)\) factors. Summing up the contribution of the untwisted sector as well as the contribution of the three twisted sectors, the total partition function of the \(\{H_L, H_R, b_1^1, b_1^2\}\)-vacuum is:

\[ Z_{11}^{b_1^1, b_1^2} = 192 + 64 \times 3 = 384. \]

(3.46)

The \(\{H_L, H_R, B_1^1, B_1^2\}\)-vacuum contains 384 massless bosons. There are no massless fermions, as is the case in all examples of left-right symmetric orbifolds. The chiral spectral-flow operator, invariant under \(\mathbb{Z}_2^{h_1^1} \times \mathbb{Z}_2^{h_1^2}\) and responsible for the \(MSDS\)-structure, is:

\[ C_{b_1^1, b_1^2} = e^{\frac{i}{2} (\Phi - i H_0)} C_8 C_8 C_8. \]

(3.47)

### 4 Heterotic \(MSDS\) Orbifold Vacua

The fact that \(MSDS\)-structure is the result of a \textit{chiral spectral-flow} permits the construction of a large number of Type II and Heterotic \(MSDS\)-vacua. It will be sufficient to choose the holomorphic part of the partition function to be a suitable twisting and/or shifting of the original model so that the holomorphic contributions yield constants, while the anti-holomorphic part is allowed to vary, respecting only the consistency conditions of modular
invariance. The resulting models will all have \textit{MSDS}-symmetry, since any non-constant contribution coming from the anti-holomorphic side will necessarily violate level-matching conditions and will, thus, not contribute to the spectrum neither to the integrated partition function.

In Heterotic \textit{MSDS}-vacua the anti-holomorphic contributions to the partition function are no longer constant numbers and the full partition function becomes an anti-holomorphic modular invariant function \(Z_{\text{het}}(\bar{q})\), \(\bar{q} = \exp(-2i\pi \tau)\). It is well-known that any such function can be expressed as a rational function \(Q(\bar{j})\) of the Klein invariant \(\bar{j}(\tau)\). It is not difficult to see that the Klein \(j\)-function \textit{can only appear linearly} in the partition function as a result of the structure of the anti-holomorphic part. Indeed, since the full partition function will necessarily contain only simple would-be-tachyon poles \(\sim 1/\bar{q}\), the rational function \(Q(\bar{j}(\tau))\) is fixed to be at most linear. The partition function will then necessarily be of the form:

\[
Z_{\text{het}} = n + m [\bar{j}(\tau) - 744]
\]

for some constant integers \(n, m\). It is straightforward to show that \(n\) equals: \(n(b) - n(f)\), namely the number of massless bosons minus the number of massless fermions of the model, whereas \(m\) is essentially the number of the would-be-tachyon poles in \(\bar{q}\). Moreover, it is clear that the integration over \(\tau\) in the fundamental domain eliminates the spurious \([\bar{j}(\tau) - 744]-\)terms leaving only the constant contribution \(n\) of the massless spectrum, as expected. In what follows we give explicit examples of Heterotic models with reduced \textit{MSDS}-structure, all of which are constructed in this spirit.

### 4.1 Heterotic \textit{MSDS} \(\mathbb{Z}_2\)-orbifolds

The Heterotic \(\mathbb{Z}_2\)-orbifold models with \textit{MSDS}-symmetry can be constructed by coupling a holomorphic partition function \(Z_{\text{hol}}^{[h]}\) with \textit{MSDS}-structure, such as the ones studied in Section 3, to an anti-holomorphic heterotic partition function \(\bar{Z}_{\text{hol}}^{[h]}\), so that the full partition function be modular invariant:

\[
Z_{\text{het}} = \frac{1}{2} \sum_{h, g} Z_{[h]}^{[h]} \bar{Z}_{[g]}^{[h]}.
\]
4.1.1 Heterotic \( \mathbb{Z}_2 \)-twisted MSDS with \( SO(32) \times E_8 \)

As a first example we consider the \( \mathbb{Z}_2 \)-twisted holomorphic partition function:

\[
Z^{[g]}_{[a]} = \frac{1}{2} \sum_{a,b} (-)^{a+b+h_g} \frac{\theta^{[a]}_{[b]} \theta^{[a+h]}_{[b+g]}}{\eta^{12}}.
\]

For the right-moving side we choose an \( SO(32) \times E_8 \) gauge group:

\[
\tilde{Z}^{[b]}_{[g]} = \frac{1}{2} \sum_{a,b} \frac{\theta^{[a+16]}_{[b]} \theta^{[a+h]}_{[b+g]}}{\bar{\eta}^{24}}.
\]

This model is generated in the free fermionic construction by the following choice of basis elements:

\[
H_L = \{ \chi^{1...8}, y^{1...8}, w^{1...8} \}, \quad H_R = \{ \chi^{1...48} \}, \quad B = \{ \chi^{5...8} y^{5...8} \chi^{33...48} \}.
\] (4.3)

The contributions to the partition function are organized according to their transformation under the \( \mathbb{Z}_2^B \)-twist, as usual:

\[
\begin{align*}
\frac{1}{2} \sum_{g} Z_{(+,+)}^{[0]} \tilde{Z}_{(+,+)}^{[0]} & = (V_{16}O_8 - S_{16}C_8) \times (O_{32}O_{16} + C_{32}C_{16}) = 16 \times (O_{32}O_{16} + C_{32}C_{16}) \\
\frac{1}{2} \sum_{g} Z_{(+,-)}^{[0]} \tilde{Z}_{(+,-)}^{[0]} & = (O_{16}V_8 - C_{16}S_8) \times (V_{32}V_{16} + S_{32}S_{16}) = 8 \times (V_{32}V_{16} + S_{32}S_{16}) \\
\frac{1}{2} \sum_{g} Z_{(+,+)}^{[1]} \tilde{Z}_{(+,+)}^{[1]} & = (O_{16}S_8 - C_{16}V_8) \times (O_{32}C_{16} + C_{32}O_{16}) = 8 \times (O_{32}C_{16} + C_{32}O_{16}) \\
\frac{1}{2} \sum_{g} Z_{(+,-)}^{[1]} \tilde{Z}_{(+,-)}^{[1]} & = (V_{16}C_8 - S_{16}O_8) \times (V_{32}S_{16} + S_{32}V_{16}) = 0 \times (V_{32}S_{16} + S_{32}V_{16})
\end{align*}
\] (4.4)

The holomorphic part for each individual term is constant due to the holomorphic MSDS-structure, as explained in Section 3. To determine the full partition function it is sufficient to derive the number of massless and would-be-tachyonic states. It is clear that massless states can only occur from \( V_{16}O_8O_{32}O_{16}, O_{16}V_{32}V_{16} \) and \( O_{16}S_8O_{32}C_{16} \). Specifically we have:

\[
V_{16}O_8O_{32}O_{16} : 16 \times 616 \text{ states } \{ \chi^{1...4}_\frac{1}{2} \oplus y^{1...4}_\frac{1}{2} \oplus w^{1...8}_\frac{1}{2} \} \otimes \{ \chi^{a}_\frac{1}{2} \chi^{b}_\frac{1}{2} \oplus \bar{\chi}^{a}_\frac{1}{2} \bar{\chi}^{b}_\frac{1}{2} \}
\]

\[
O_{16}V_8V_{32}V_{16} : 8 \times 512 \text{ states } \{ \chi^{5...8}_\frac{1}{2} \oplus y^{5...8}_\frac{1}{2} \} \otimes \bar{\chi}^{a}_\frac{1}{2} \bar{\chi}^{I}_\frac{1}{2}
\]

\[
O_{16}S_8O_{32}C_{16} : 8 \times 128 \text{ states } Sp\{ \chi^{5...8}_- \} \otimes Sp\{ \chi^{I}_+ \}
\]
where $a, b = 1 \ldots 32$ span the fermions in $SO(32)$ while $I, J = 33 \ldots 48$ run over the fermions in $SO(16)$. There are in total $n = 14976$ massless states in the model, while there is only one anti-holomorphic tachyonic contribution from $O_{32}O_{16}$. After coupling to the holomorphic part, we obtain the coefficient $m = 16$ of the pole term. Using (4.1) we determine the partition function of the model:

$$Z^B_{het} = 14976 + 16 \left[ \bar{j}(\bar{\tau}) - 744 \right]. \quad (4.5)$$

It is instructive to see how this structure appears at the level of characters. Expanding the various contributions above in terms of $SO(8)$-characters and using the triality relations $V_8 = S_8 = C_8$ we can write the partition function in a particularly simple way:

$$Z^B_{het} = 16 \cdot \left[ O_8^6 + 12 O_8^4 V_8^2 + 21 O_8^2 V_8^4 + 30 V_8^6 \right].$$

This expression should contain a $j$-function whose character expansion is:

$$\bar{j}(\bar{\tau}) = (O_8^2 + V_8^2 + S_8^2 + C_8^2)^3 = (O_8^2 + 3V_8^2)^3. \quad (4.6)$$

Using (4.6) to eliminate the $O_8^6$-term in the partition function in terms of the $j$-function, we find:

$$Z^B_{het} = 16 \left[ \bar{j}(\bar{\tau}) + 3 V_8^2 (O_8^2 - V_8^2)^2 \right].$$

We next note that the second term is nothing but the square of $V_8 O_8 S_8 - S_8 C_8 C_8 = 8$ that was already encountered as (3.45) in the $Z_2 \times Z_2$ model with triple triality. Therefore, the partition function becomes:

$$Z^B_{het} = 3072 + 16 \bar{j}(\bar{\tau}) = 14976 + 16 \left[ \bar{j}(\bar{\tau}) - 744 \right], \quad (4.7)$$

which is, of course, the same as (4.5) found above by counting massless and tachyonic states.

### 4.1.2 Heterotic $Z_2$-twisted MSDS with $SO(16) \times SO(16) \times E_8$

An interesting variation of the previous model comes from further breaking $SO(32)$ down to $SO(16) \times SO(16)$. We consider the model generated by the following basis elements:

$$H_L = \{ \chi^{1\ldots8}, y^{1\ldots8}, w^{1\ldots8} \}, \quad H_R = \{ \bar{\chi}^{1\ldots48} \}, \quad G = \{ \bar{\chi}^{1\ldots32} \}, \quad b = \{ \chi^{5\ldots8} y^{5\ldots8} | \bar{\chi}^{16\ldots32} \}.$$
The partition function of the model is:

$$Z_{het}^{G,b} = \frac{1}{2} \sum_{a,b} \left( -1 \right)^{a+b} \frac{\theta[a+h] \theta[2a-h] \theta[2b+h] \theta[2b-h]}{\eta^{12}} \cdot \frac{1}{2^2} \sum_{a,b} \frac{\theta[a+h] \theta[2a-h] \theta[2b+h] \theta[2b-h]}{\eta^{24}}.$$  (4.8)

The partition function in terms of SO(16)-characters is organized into the following contributions:

$$\frac{1}{2} \sum_g Z_{(+,+)[g]} Z_{(+,+)[g]} = (V_{16} O_8 - S_{16} C_8) \times (O_{16} O_{16} O_{16} + O_{16} C_{16} O_{16} + C_{16} C_{16} C_{16})$$

$$\frac{1}{2} \sum_g Z_{(+,-)[g]} Z_{(+,-)[g]} = (O_{16} V_8 - C_{16} S_8) \times (V_{16} V_{16} O_{16} + V_{16} V_{16} C_{16} + S_{16} S_{16} O_{16} + S_{16} S_{16} C_{16})$$

$$\frac{1}{2} \sum_g Z_{(+,+)^{[1]}[g]} Z_{(+,+)^{[1]}[g]} = (O_{16} S_8 - C_{16} V_8) \times (O_{16} C_{16} O_{16} + O_{16} C_{16} C_{16} + C_{16} O_{16} C_{16} + C_{16} O_{16} C_{16})$$

$$\frac{1}{2} \sum_g Z_{(+,-)^{[1]}[g]} Z_{(+,-)^{[1]}[g]} = (V_{16} C_8 - S_{16} O_8) \times (V_{16} S_{16} O_{16} + V_{16} S_{16} C_{16} + S_{16} V_{16} O_{16} + S_{16} V_{16} C_{16})$$

As before, the holomorphic part for each individual term is constant (and equal to 16, 8, 8 and 0, respectively) due to the holomorphic MSDS-structure, as shown in Section 3. To determine the full partition function it is sufficient to derive the number of massless and would be tachyonic states. Specifically the massless spectrum is:

$$V_{16} O_8 O_{16} O_{16} O_{16} : 16 \times 360 \text{ states } \{ \chi^{1,4}_{-\frac{1}{2}} \oplus y^{1,4}_{-\frac{1}{2}} \oplus w^{1,8}_{-\frac{1}{2}} \} \otimes \{ \bar{\chi}^{a}_{-\frac{1}{2}} \bar{\chi}^{b}_{-\frac{1}{2}} \oplus \bar{\chi}^{C}_{-\frac{1}{2}} \bar{\chi}^{D}_{-\frac{1}{2}} \oplus \bar{\chi}^{\alpha}_{-\frac{1}{2}} \bar{\chi}^{\beta}_{-\frac{1}{2}} \}$$

$$V_{16} O_8 O_{16} C_{16} : 16 \times 128 \text{ states } \{ \chi^{1,4}_{-\frac{1}{2}} \oplus y^{1,4}_{-\frac{1}{2}} \oplus w^{1,8}_{-\frac{1}{2}} \} \otimes Sp\{\chi^a\}$$

$$O_{16} V_8 V_{16} V_{16} O_{16} : 8 \times 256 \text{ states } \{ \chi^{5,8}_{-\frac{1}{2}} \oplus y^{5,8}_{-\frac{1}{2}} \} \otimes \bar{\chi}^{a}_{-\frac{1}{2}} \bar{\chi}^{I}_{-\frac{1}{2}}$$

$$O_{16} S_8 O_{16} C_{16} O_{16} : 8 \times 256 \text{ states } Sp\{\chi^5 y^8\} \oplus Sp\{\bar{\chi}^I\}$$

The contribution of the massless states is therefore \( n = 11904 \). Moreover, there are 16 would-be-tachyonic states from \( V_{16} O_8 O_{16} O_{16} O_{16} \) giving a pole contribution \( m = 16 \). Therefore, the partition function of the model is:

$$Z_{het}^{G,b} = 16 \tilde{j}(\tau) = 11904 + 16 \left[ \tilde{j}(\tau) - 744 \right].$$  (4.9)

We see that the heterotic \( Z_2^{G,b} \)-twisted model has no massless fermions, since those can only appear from the holomorphic side. The situation changes if one considers a holomorphic \( Z_2 \)-shift, as shown below.
4.1.3 Heterotic $\mathbb{Z}_2$-shifted MSDS with $SO(32) \times E_8$

We next illustrate an example of Heterotic MSDS-vacuum with massless fermions. We consider the model that couples the $\mathbb{Z}_2$-shifted holomorphic MSDS-partition function to the anti-holomorphic side with gauge group $SO(32) \times E_8$. The breaking set of the shifted model is:

$$B_s = \{ y^{1...8} w^{1...8} | \chi^{33...48} \} \quad \text{(4.10)}$$

and the partition function becomes:

$$Z^{B_s}_{het} = \frac{1}{2} \sum_{h,g} \frac{1}{2} \sum_{a,b} (-)^{a+b} \theta[a]_4 \theta[a+h]_8 \theta[b]_3 \theta[b+g]_8 \cdot \frac{1}{2^2} \sum_{a,b} \bar{\theta}[a]_4 \bar{\theta}[a+h]_8 \bar{\theta}[b]_3 \bar{\theta}[b-g]_8. \quad \text{(4.11)}$$

The contributions to the partition function are organized in this case as:

$$\frac{1}{2} \sum_g Z_{(+,+)}[g] \tilde{Z}_{(+,+)}[g] = (V_8 O_{16} - S_8 C_{16}) \times (O_{32} O_{16} + C_{32} C_{16}) = 8 \times (O_{32} O_{16} + C_{32} C_{16})$$

$$\frac{1}{2} \sum_g Z_{(+,-)}[g] \tilde{Z}_{(+,-)}[g] = (O_8 V_{16} - C_8 S_{16}) \times (V_{32} V_{16} + S_{32} S_{16}) = 16 \times (V_{32} V_{16} + S_{32} S_{16})$$

$$\frac{1}{2} \sum_g Z_{(+,+)}[g] \tilde{Z}_{(+,+)}[g] = (V_8 C_{16} - S_8 O_{16}) \times (O_{32} C_{16} + C_{32} O_{16}) = -8 \times (O_{32} C_{16} + C_{32} O_{16})$$

$$\frac{1}{2} \sum_g Z_{(+,-)}[g] \tilde{Z}_{(+,-)}[g] = (O_8 S_{16} - C_8 V_{16}) \times (V_{32} S_{16} + S_{32} V_{16}) = 0 \times (V_{32} S_{16} + S_{32} V_{16}) \quad \text{(4.12)}$$

The massless states of the shifted model are the following:

$$V_8 O_{16} O_{32} O_{16} : 8 \times 616 \text{ states } \chi_{-\frac{1}{2}} \otimes \{ \tilde{\phi}_{-\frac{1}{2}} \tilde{\phi}_{-\frac{1}{2}} + \tilde{\phi}_{-\frac{1}{2}} \tilde{\phi}_{-\frac{1}{2}} \}$$

$$O_8 V_{16} V_{32} V_{16} : 16 \times 512 \text{ states } \{ y^{\frac{1}{2}} + w^{\frac{1}{2}} \} \otimes \tilde{\phi}_{\frac{1}{2}} \tilde{\phi}_{\frac{1}{2}}$$

$$-S_8 O_{16} O_{32} C_{16} : (-8) \times 128 \text{ states } Sp\{ \chi_{-\frac{1}{2}} \} \otimes Sp\{ \tilde{\phi}_{+}\}$$

We notice the appearance of 8 massless fermions from $-S_8 O_{16} O_{32} C_{16}$. Adding together the massless contributions we find $n = n(b) - n(f) = 12096$, $n(f) = 1024$, while the tachyonic states in $V_8 O_{16} O_{32} O_{16}$ give the pole coefficient $m = 8$. Therefore, the partition function of the $\mathbb{Z}_2$-shifted model is found to be:

$$Z^{R_s}_{het} = 12096 + 8 \left[ \bar{j}(\bar{\tau}) - 744 \right]. \quad \text{(4.13)}$$
4.2 Heterotic $\mathbb{Z}_2$-shifted MSDS with $SO(16) \times SO(16) \times E_8$

A variation of the previous model can be obtained by further breaking $SO(32)$ to $SO(16) \times SO(16)$. The breaking sets in this case are $G = \{\chi^{1\ldots32}\}$ and $B_s = \{y^{1\ldots8}w^{1\ldots8}|\chi^{33\ldots48}\}$ and the corresponding partition function is:

$$Z^G_{het} = \frac{1}{2} \sum_{h,g} \frac{1}{2} \sum_{a,b} (-)^{a+b} \frac{\theta[a]^{14} \theta[2^{a+b+h}]^8}{\eta^{12}} \cdot \frac{1}{2^2} \sum_{\bar{a},\bar{b},\gamma,\delta} \frac{\bar{\theta}[\bar{a}]^{18} \bar{\theta}[\bar{b}]^{18} \bar{\theta}[\gamma]^{18}}{\eta^{24}}. \quad (4.14)$$

A similar analysis shows the presence of massless bosons and fermions with $n = n(b) - n(f) = 5952$, $n(f) = 2048$ and a pole coefficient $m = 8$. The partition function is then:

$$Z_{heterotic} = 5952 + 8 \left[ \tilde{j}(\tau) - 744 \right]. \quad (4.15)$$

4.2 Heterotic $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbifolds

There is a plethora of reduced MSDS orbifold vacua in the heterotic framework. The classification rules will be given in the next section where the (left-moving) holomorphic MSDS constraints will be derived. Here we present a typical example that will be used as a representative paradigm in Section 6 concerning our discussion about the geometrical interpretation of the marginally deformed MSDS-vacua. In fact, in the limit of large marginal deformations an effective four-dimensional space-time will emerge. Aspiring to the construction of semi-realistic four-dimensional heterotic chiral models, with $SO(10) \times U(1)^3 \times SO(16)$ as right-moving gauge group, we choose the basis set of the representative MSDS-vacuum to be:

$$H_L = \{\chi^{1\ldots8}, y^{1\ldots8}, w^{1\ldots8}\}, \quad H_R = \{\bar{y}^{1\ldots8}, \bar{w}^{1\ldots8}, \bar{\eta}, \bar{\eta}^2, \bar{\eta}^3, \bar{\psi}^{1\ldots5}, \bar{\varphi}^{1\ldots8}\},$$

$$G = \{\bar{y}^{1\ldots8}, \bar{w}^{1\ldots8}\}, \quad z = \{\bar{\varphi}^{1\ldots8}\}$$

$$b_1 = \{\chi^{3,4,5,6}, y^{3,4,5,6}, \bar{y}^{1,\psi^{1\ldots5}}\}, \quad b_2 = \{\chi^{1,2,5,6}, y^{1,2,5,6}, \bar{y}^{1,\bar{\psi}^{1\ldots5}}\}. \quad (4.16)$$

In the above $\chi^I, y^I, w^I, \bar{y}^I, \bar{w}^I$ are considered to be real fermions while $\bar{\eta}^{1,2,3}, \bar{\psi}^{1\ldots5}$ and $\bar{\varphi}^{1\ldots8}$ are complex. The holomorphic part of the partition function is:

$$Z^{[h_1 h_2]}_{[g_1 g_2]} = \frac{1}{2} \sum_{a,b} (-)^{a+b} \frac{\theta[a]^{16} \theta[a+h_1]^{12} \theta[a+h_2]^{12} \theta[a-h_1-h_2]^{12}}{\eta^{12}}, \quad (4.17)$$
whereas the anti-holomorphic part is:
\[
\bar{Z}_{[h_1 \bar{g}_2]} = \frac{1}{2^3 \eta^{24}} \sum_{\gamma, \delta} \theta^{[\gamma]} \bar{\theta}^{[\gamma + h_1]} \bar{\theta}^{[\gamma + h_2]} \bar{\theta}^{[\gamma - h_1 - h_2]} \sum_{\epsilon, \zeta} \theta^{[\epsilon]} \bar{\theta}^{[\epsilon + h_1]} \bar{\theta}^{[\epsilon + h_2]} \bar{\theta}^{[\epsilon - h_1 - h_2]} \sum_{a, b} \bar{\theta}^{[\bar{a}]} \bar{\theta}^{[\bar{b}]} .
\]
(4.18)

The full partition function can be written in a conventional shifted and twisted “Γ_{8,8}-lattice form” [29, 30] as follows:
\[
Z_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \frac{1}{2^6 \eta^{12} \bar{\eta}^{24}} \sum_{a, b, \gamma, \delta, h_i, g_i} (-)^{a+b} \theta^{[a]} \theta^{[b+g_1]} \theta^{[b+g_2]} \theta^{[a-h_1-h_2]} \times \Gamma_{8,8} \left[ \frac{a}{b}, \frac{\gamma}{\delta}, \frac{h_1}{g_1} \right] \sum_{\epsilon, \zeta} \theta^{[\epsilon]} \bar{\theta}^{[\epsilon + h_1]} \bar{\theta}^{[\epsilon + h_2]} \bar{\theta}^{[\epsilon - h_1 - h_2]} \sum_{\bar{a}, \bar{b}} \bar{\theta}^{[\bar{a}]} \bar{\theta}^{[\bar{b}]} ,
\]
(4.19)
where Γ_{8,8} \left[ \frac{a}{b}, \frac{\gamma}{\delta}, \frac{h_1}{g_1} \right] indicates the contribution of the eight fermionized coordinates \{y', \omega' | \bar{y}', \bar{\omega}'\}.

The MSDS-structure of the holomorphic side has already been studied in Section 3, where \mathbb{Z}_2 \times \mathbb{Z}_2 Type-II models were considered. Thus, the heterotic model under consideration also possesses MSDS-structure. Specifically, the partition function in this representative example is found to be:
\[
Z = 12 \bar{j}(\bar{\tau}) = 12 \times 744 + 12 \left[ \bar{j}(\bar{\tau}) - 744 \right] .
\]
(4.20)

Inserting in the above representative model all possible discrete torsion coefficients permitted by the fermionic construction, a plethora of MSDS Heterotic models can be obtained. The resulting models will in general exhibit different bosonic and fermionic massless spectra in different representations of the chiral (right-moving) gauge group \(SO(10) \times U(1)^3 \times SO(16)\), similarly to the four-dimensional supersymmetric chiral models of ref. [26]. The generic property of all those models is that their partition function will be always of the form of Eq. (4.1).

5 Classification of fermionic MSDS Vacua

In the previous section we presented various examples of \(\mathbb{Z}_2\)-orbifolds of the original Type II and Heterotic MSDS-vacua and showed that the MSDS-structure is inherited by the “daughter” models we presented. There it was mentioned that the MSDS-structure of the spectrum results from a consistent truncation of the original chiral superconformal algebra.

In this section we elaborate on the necessary conditions for such a truncation to be consistent and obtain a simple set of rules that permit the construction of all fermionic MSDS-vacua.
As \(MSDS\)-structure has been seen to be a chiral property, it is sufficient for now to concentrate on the holomorphic side. Once \(MSDS\)-symmetry in the sense of spectral-flow has been secured on the holomorphic side, one may couple it to an arbitrary anti-holomorphic part, respecting only constraints of modular invariance. The resulting model will necessarily have \(MSDS\)-structure, since the partition function in that case can easily be shown to equal at most a constant plus a possible term proportional to the Klein \(j\)-invariant. After projecting out the unphysical states, one remains with the surviving constant contribution of the massless states only.

To motivate the consistency conditions we are about to derive, we consider the simple \(\mathbb{Z}_2\)-orbifold already examined in the previous sections. The holomorphic part of the partition function is:

\[
(−)^{a+b+hg} \frac{\theta^{[a]}_b \theta^{[a+h]}_{b+g}}{\eta^{12}}, \quad (5.1)
\]

where we explicitly added the phase \((-)^{hg}\) to make the holomorphic part modular invariant by itself. The contributions to the untwisted sector are \(O_{16}V_8 - C_{16}S_8\) and \(V_{16}O_8 - S_{16}C_8\). We first note the existence of a \(\mathbb{Z}_2\)-invariant spectral-flow operator:

\[
C = e^{\frac{i}{2}(\Phi - iH_0)} C_{16} C_8. \quad (5.2)
\]

Such an operator must map massive boson states to massive fermion states in the untwisted sector because the latter is simply a projection of the initial \(MSDS\)-vacuum. We notice that the dressed spectral-flow operator \(C(z)\) has conformal weight \(\Delta_C = 1\) and, thus, it effectively acts as a current:

\[
\hat{j}_{MSDS}(z) \equiv C(z). \quad (5.3)
\]

This spectral-flow is responsible for the isomorphism that maps the massive tower of states \(O_{16}V_8\) into \(C_{16}S_8\), leaving only the massless states invariant. For this mapping to exist, as in the case of ordinary supersymmetry, we must be able to define a BRST-invariant charge \(Q_{MSDS}\):

\[
Q_{MSDS} = \oint \frac{dz}{2\pi i} \hat{j}_{MSDS}(z) = \oint \frac{dz}{2\pi i} e^{\frac{i}{2}(\Phi - iH_0)} C_{16} C_8(z), \quad (5.4)
\]

with a well-defined action on the states of the spectrum. We will refer to this operator as the \(MSDS\) charge. The problem of classifying \(MSDS\)-vacua is, therefore, twofold. One must
first examine under what conditions the action of the MSDS-charge is well-defined on the spectrum of states and, secondly, to ensure that massless states are annihilated by its action.

The states of the vacuum representation contributing to the $O_{16}$-character can be split into those generated by the identity operator $1_{16}$ (with conformal weight $\Delta_1 = 0$), as well as by the those in the adjoint $\text{Adj}_{16}(z) = (\hat{\chi}\hat{\chi})_{16}$ (with weight $\Delta_{\text{Adj}} = 1$). In what follows, we consider the adjoint as part of the affine descendants of the identity operator. The fusion relation shows that massless states do not transform, since in this case the MSDS-charge vanishes:

$$j_{\text{MSDS}}(z) \cdot (1_{16} V_8)(w) \sim C_{16} S_8(w) . \quad (5.5)$$

On the other hand, the massive states do transform:

$$j_{\text{MSDS}}(z) \cdot (\text{Adj}_{16} V_8)(w) \sim \frac{C_{16} S_8(w)}{z - w} . \quad (5.6)$$

The same MSDS-mapping would, of course, be true for the descendant states generated by $(1_{16} V_{(1),8})$:

$$j_{\text{MSDS}}(z) \cdot (1_{16} V_{(1),8})(w) \sim \frac{C_{16} S_8(w)}{z - w} , \quad (5.7)$$

where

$$V_{(1)}(z) \equiv e^{-\Phi} \left( \partial \chi + \hat{\chi}\hat{\chi} \right) \quad (5.8)$$

is the first descendant operator of $V_{(0)}(z)$.

The spectral-flow relations (5.5), (5.6) and (5.7) are responsible for the isomorphism that maps the massive tower of states $O_{16} V_8$ into $C_{16} S_8$, leaving only the massless states invariant. This can be seen explicitly by considering the action of the MSDS-charge on the massless and massive states mentioned above. The difference of characters is, therefore, completely determined by the excess in massless states, giving:

$$O_{16} V_8 - C_{16} S_8 = 8 - 0 = 8 , \quad (5.9)$$

since only $O_{16} V_8$ contributes to the massless spectrum. Now consider the action of the orbifold. Under the $\mathbb{Z}_2$-twist the vertex operators charged under $SO(8)$ transform as $O \leftrightarrow C$ and $V \leftrightarrow S$, respecting parity. In order to preserve MSDS symmetry in the twisted sector.
we need to ensure that the twisted states have similar fusion rules:

\[ j_{\text{MSDS}}(z) \cdot (1_{16}S_8)(w) \sim C_{16}V_8(w) \]  

(5.10)

\[ j_{\text{MSDS}}(z) \cdot (\text{Adj}_{16}S_8)(w) \sim \frac{C_{16}V_8(w)}{z - w}, \]  

(5.11)

so that, again, the massless states remain invariant, whereas the massive ones transform. A similar fusion rule is responsible for the mapping of the remaining massive descendants of \( 1_{16}S_8 \) into \( C_{16}V_8 \). The spectral-flow is inherited to the twisted sector and one obtains the twisted character formula:

\[ -O_{16}S_8 + C_{16}V_8 = -8. \]  

(5.12)

On the other hand, one could also perform the inverse mapping:

\[ j_{\text{MSDS}}(z) \cdot (C_{16}S_8)(w) \sim \frac{1_{16}V_8(w)}{(z - w)^2} + \frac{\text{Adj}_{16}V_8(w) + 1_{16}V_{(1),8}(w)}{z - w}. \]  

(5.13)

This is in exact agreement with our previous results, showing the mapping of the spinorial primary states generated by \( S(z) \) into the descendant bosonic states in \( \text{Adj}_{16}V_8(z) \) and \( 1_{16}V_{(1),8}(z) \) that lie at the same mass level. This explicitly demonstrates that the \( MSDS \)-mapping:

\[ \{ S(0)|0\rangle \} \leftrightarrow \{ \text{Des}(1)[1_{16}V_8]|0\rangle \} \]  

(5.14)

is indeed bijective, as it should. For convenience, we denote as:

\[ \{ \text{Des}(1)[1_{16}V_8]|0\rangle \} \equiv \{ \text{Adj}_{16}V_8(0)|0\rangle \oplus 1_{16}V_{(1),8}(0)|0\rangle \} \]  

(5.15)

the set of states generated by the descendant operators of \( 1_{16}V_8 \) at the next mass level. These descendant operators, in turn, generate the remaining tower of massive states of the vectorial representation by the repeated action of the ‘covariant-like derivative’ \( D \equiv \partial + \hat{\chi}\hat{\chi} \).

It is now important to notice that the spectral-flow is valid in the twisted sector only for specific choices of the breaking sets. Consider introducing a breaking set \( b \) twisting \( n_L(b) \) left-moving fermions and an appropriate number of right-moving fermions in such a way that modular invariance constraints are satisfied. The relevant characters will now transform under \( SO(24 - n_L(b)) \times SO(n_L(b)) \). The spectral flow is always valid in the untwisted sector:

\[ j_{\text{MSDS}}(z) \cdot (1_{24-n_L}V_{n_L})(w) \sim C_{24-n_L}S_{n_L}(w) \]  

(5.16)
\[ j_{\text{MSDS}}(z) \cdot \text{Des}(1)[1_{24-n_L} V_{n_L}](w) \sim \frac{C_{24-n_L} S_{n_L}(w)}{z-w}, \quad (5.17) \]

where the symbol \( \text{Des}(q)[A] \) stands for the descendants of \( A \) starting from the \( q \)-th next mass level after that of \( A \).

We now see that this fusion rule is not preserved by the twist for generic \( n_L(b) \). The corresponding fusion rule in the twisted sector is:

\[ j_{\text{MSDS}}(z) \cdot \text{Des}(q)[1_{24-n_L} S_{n_L}](w) \sim \sum_{q'=0} \frac{\text{Des}(q')[C_{24-n_L} V_{n_L}(w)]}{(z-w)^{n_L/8-1+q-q'}} \], \quad (5.18)

where \( q, q' \in \{0, 1, 2, 3, \ldots\} \). The cases \( q = 0, q' = 0 \) correspond to the original (primary) operators \( 1_{24-n_L} S_{n_L} \) and \( C_{24-n_L} V_{n_L} \), respectively. Clearly the spectral-flow algebra is preserved under the twist only for choices \( b \) of breaking sets satisfying \( n_L(b) = 0 \mod 8 \), otherwise the fusion OPE contains cuts and the action of the current on the vertex operators is non-local. In the latter case, the \( \text{MSDS} \)-charge \( Q_{\text{MSDS}} \) cannot be defined, implying the absence of mapping from massive bosonic to fermionic states. Assuming now that the \( \text{MSDS} \)-charge is well-defined, the mapping can only occur if there appears a simple pole in the OPE, implying:

\[ \frac{1}{8} n_L + q - q' = 2 \quad (5.19) \]

for some \( q' \geq 0 \). It is easy to see that this condition is only violated by the massless states, which correspond precisely to \( n_L = 8 \) and \( q = 0 \) and, therefore, they do not transform.

We are now ready to formulate the analogous argument for a generic \((\mathbb{Z}_2)^N\)-twist. Generically, acting with the \( \text{MSDS} \)-charge on a state \(|A\rangle\), created by the local operator \( A(w) \) with weight \( \Delta_A \), one produces a state \(|B\rangle\) that is created by an operator \( B(w) \) with weight \( \Delta_B \). Taking into account the possible descendants that can appear, the fusion relation is:

\[ j_{\text{MSDS}}(z) \cdot \text{Des}(q)[A](w) \sim \sum_{q'=0} \frac{\text{Des}(q')[B](w)}{(z-w)^{\Delta_A - \Delta_B + q-q'+1}} \]. \quad (5.20)

Since only simple poles give a spectral-flow mapping\(^1\), the condition for the existence of the \( \text{MSDS} \)-mapping between the two states is:

\[ \Delta_A - \Delta_B = q' - q , \quad \text{where} \; q, q' \in \{0, 1, 2, \ldots\} . \quad (5.21) \]

\(^1\)This means that only states of the same mass level can map into each other, as should be expected.
The case $\Delta_A = \Delta_B$ corresponds to a ‘primary’ ($q = 0$) operator $A$ being mapped into another ‘primary’ ($q' = 0$) operator $B$. The case $\Delta_A \neq \Delta_B$ implies that the dominant contributions to the OPE come with higher-order poles and, therefore, only their descendants appearing with simple poles eventually contribute to the mapping. Supposing that the $MSDS$-charge is well-defined for a particular model, the massless states are the only ones that always violate (5.21) for any $q' \geq 0$, as they correspond to $\Delta_A = 1/2$ and $q = 0$.

The $MSDS$-charge is, therefore, well-defined on all states provided that $\Delta_A - \Delta_B \in \mathbb{Z}$ for any primary states $A, B$. While this is trivially satisfied in the untwisted sector, in the twisted sector it imposes powerful constraints on the form of permitted breaking sets. To show this, consider the most general contribution to the untwisted sector:

$$A(w) = \prod_{n_i} V_{n_i} \prod_{m_j} O_{m_j}(w) , \quad (5.22)$$

which has conformal weight $\Delta_A = n/2$. Of course, consistency requires that:

$$\sum_{n_i} n_i + \sum_{m_j} m_j = 24 . \quad (5.23)$$

Acting with the spectral flow operator $j_{MSDS}(z)$ we obtain another operator:

$$B(w) = \prod_{n_i} S_{n_i} \prod_{m_j} C_{m_j}(w) , \quad (5.24)$$

with conformal weight $\Delta_B = 3/2$. The condition (5.21) for the existence of a mapping between the two states is :

$$\frac{n - 3}{2} = q' - q . \quad (5.25)$$

Since the $GGSO$-projection of the models under consideration forces the overall parity to be negative, $n$ will always be odd so that $\Delta_A - \Delta_B$ is always an integer. It is important to note that only the massless states violate this condition for all $q' \geq 0$, since they correspond to $q = 0$ and $n = 1$. All other cases correspond to massive states and transform, in agreement to our previous considerations.

We now proceed to impose condition (5.21) on the twisted sector. The original operator becomes, under a generic $\mathbb{Z}_2$-like twist:

$$A'(z) = \prod_{n_i} V_{n_i} \prod_{m_j} S_{m_j} \prod_{a} O_{m_j} \prod_{b} C_{m_j}(w) , \quad (5.26)$$
with conformal weight:
\[ \Delta_{A'} = \frac{n-a}{2} + \sum_{n_i} \frac{n_i}{16} + \sum_{m_j} \frac{m_j}{16}. \]  
(5.27)

The action of \( j_{MSDS}(z) \) on the ‘twisted’ operator produces:
\[ B'(z) = \prod_{n_i} S_{n_i} \prod_{n_i} V_{n_i} \prod_{m_j} C_{m_j} \prod_{m_j} O_{m_j}(w), \]  
(5.28)

with conformal weight:
\[ \Delta_{B'} = \frac{a}{2} + \sum_{n_i} \frac{n_i}{16} + \sum_{m_j} \frac{m_j}{16}. \]  
(5.29)

In this case, condition (5.21) becomes simply:
\[ \frac{1}{8} \left( \sum_{n_i} n_i + \sum_{m_j} m_j \right) + \frac{n-3}{2} - a = q' - q. \]  
(5.30)

To ensure that the above is indeed an integer we require:
\[ \sum_{n_i} n_i + \sum_{m_j} m_j = 0 \pmod{8}, \]  
(5.31)

which forces the left-moving twisted fermions to appear in multiples of eight:
\[ n_L(b_i) = 0 \pmod{8} \]  
(5.32)

for any element \( b_i \) in the basis of the parity group \( \Xi \). Note that the mapping condition (5.30) is violated at precisely the massless cases, as would be expected. Indeed, massless states in the twisted sector correspond to:
\[ a = n \quad , \quad q = 0 \]  
(5.33)

and
\[ \frac{1}{16} \left( \sum_{n_i} n_i + \sum_{m_j} m_j \right) = \frac{1}{2}, \]  
(5.34)

for which (5.30) is violated for any \( q' \geq 0 \), since \( n \) is odd by the GGSO-projections. In fact, (5.32) together with overall modular invariance of the full partition function automatically imply the following two conditions on the left-moving degrees of freedom of basis elements:
\[ n_L(b_i \cap b_j) = 0 \pmod{4} \quad , \quad n_L(b_i \cap b_j \cap b_k) = 0 \pmod{2}. \]  
(5.35)
The above constraints imply that the basis elements in $\Xi$, if effectively truncated to their left-moving parts only, would still generate a holomorphic modular invariant partition function. It is clear that this holomorphic partition function can only equal a constant integer, since no term linear in the Klein $j$-invariant can be constructed by quantizing only 24 real fermions.

We finally summarize the above results into a classification theorem that permits the $\mathbb{Z}_2^N$-orbifold construction of all fermionic $MSDS$-vacua in 2 flat spacetime dimensions:

**Theorem:** For any choice of parity group $\Xi$ containing $n_L(F) = 24$ free left-moving fermions whose basis elements satisfy, in addition to the usual modular invariance constraints, the holomorphic constraint:

$$n_L(b_i) = 0 \text{ (mod 8)} ,$$

and whose basis elements $b_i \in \Xi$ preserve the global definition\footnote{Indeed, to create a Klein $j(\tau)$-function one would need 48 real fermions in 2 space-time dimensions. This can only happen in the absence of worldsheet supersymmetry, as in the heterotic case.} of the spectral-flow operator $j_{MSDS}(z)$, the resulting fermionic model has $MSDS$-structure and its partition function, depending on the anti-holomorphic structure, will at most equal a constant, plus an additional linear antiholomorphic $j$-invariant term in the Heterotic case.

In Type-II and Heterotic theories the anti-holomorphic constraint $n_R(b_i) = 0 \text{ (mod 8)}$ for the right-movers is automatically satisfied once (5.36) is imposed on the left-movers, because of modular invariance. The $MSDS$-structure, thus, automatically appears in the anti-holomorphic side of Type II theories as well and the partition function is simply equal to a constant.

On the other hand, in Heterotic models where there is no anti-holomorphic spectral-flow operator to guarantee $MSDS$-structure in the right-moving side, an additional Klein $j$-function is generated. The latter, in turn, participates in the massless contribution while its "spurious" massive terms will eventually give zero contribution to the integrated partition function.

\footnote{This requirement can be easily imposed by requiring the sum of all twist/shift indices to vanish $\sum_i h_i = 0$, similarly to the case of ordinary supersymmetry.}
function. These remarks permit the construction of all real fermionic Type II and Heterotic 

**6 Marginal deformations of RMSDS Orbifold Vacua**

The initial 2d MSDS string vacua proposed in ref. [9] are non-geometrical in terms of the internal compactified space but are rather characterized by the non-abelian gauge group $H_L \times H_R$. In the massless spectrum there are scalar bosons $M_{I_L,J_R}$, $I_L = 1, 2, ..., d_L$, $J_R = 1, 2, ..., d_R$, parametrizing the manifold given in Eq. (1.1). As was already mentioned in the introduction, because of the non-abelian structure of $H_L \times H_R$, the MSDS vacua admit marginal deformations (flat directions) associated with the Cartan sub-algebra $U(1)^{r_L} \times U(1)^{r_R}$, with $r_L$ and $r_R$ being the ranks of $H_L$ and $H_R$ respectively as in Eq. (1.2).

Ultimately, the $M_{I,J}$-deformation parameters are connected with an “induced effective higher dimensional space geometry” in the large $M_{I,J}$-deformation limit (e.g. when the MSDS-vacua are strongly-deformed). In this limit one recovers the geometrical field theory description in terms of an effective “higher dimensional” conventional superstring theory in which space-time supersymmetry is spontaneously broken by “geometrical” and “thermal” fluxes. This fundamental generic property of the deformed MSDS-vacua suggests the following Cosmological Conjecture formulated in ref. [9]:

- The MSDS-vacua, (or most likely their less symmetric orbifold reductions, such as those considered here) are potential candidates to describe the early non-singular phase of a stringy cosmological universe.

- The deformation moduli $M_{I,J} \rightarrow M_{I,J}(t)$ are subject to cosmological evolution and as such, they eventually acquire time dependence. Once the $M_{I,J}(t)$ become sufficiently large (modulo $S, T, U$-dualities), an effective field theory description emerges along with an induced “space-time geometry” of an effective higher dimensional space-time. The relevant degrees of freedom and interactions will be then described by the effective “no-scale” supergravity theories [27] of conventional superstrings [28].

- The MSDS-structure at the early cosmological times induces, in the large moduli limit, non-trivial “geometrical” fluxes [10,11,20] which, in the language of the effective
supergravity, give rise to a spontaneous breaking of supersymmetry [29, 30] and to a finite temperature description of the effective theory [8, 10, 11].

In this respect one may consider \( RMSDS \)-models as the most (semi-) realistic candidate vacua able to describe the “early non-singular phase of our Universe”, being free of any initial “general relativity-like” or “Hagedorn-like” stringy singularities.

The moduli space of the \( RMSDS \) orbifolds obviously contains a subspace of would-be geometrical \( M_{11} \)-deformations, associated with the conventional supersymmetric (freely acting) orbifolds. For instance, in the representative \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) Heterotic example defined in Eq. (4.19), the shifted lattice \( \Gamma_{8,8} \left[ a^i, \gamma^{\delta \gamma^i} h_i \right] \) is also twisted by \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and so, the initial deformation space is reduced to:

\[
\mathbb{Z}_2 \times \mathbb{Z}_2 : \quad \frac{SO(8,8)}{SO(8) \times SO(8)} \longrightarrow \frac{SO(4,4)}{SO(4) \times SO(4)} \times \frac{SO(2,2)}{SO(2) \times SO(2)} \times \frac{SO(2,2)}{SO(2) \times SO(2)}. \quad (6.1)
\]

Assuming very large deformations in the \((2,2)\) sub-space of \( SO(4,4) \), a four-dimensional flat space-time is generated, together with a six-dimensional compact space described by \( T^6_{\mathbb{Z}_2 \times \mathbb{Z}_2} \). This class of models is then connected with the semi-realistic \( N = 1 \) chiral supersymmetric models based on the \( SO(10) \) unified gauge group which were classified in ref. [26]. Furthermore, the \( RMSDS \)-deformed models in this class provide vacua with \( N = 1 \) supersymmetry, spontaneously broken by very specific geometrical fluxes! This remarkable property follows from the fact that the initial \( \Gamma_{8,8} \left[ a^i, \gamma^{\delta \gamma^i} h_i \right] \) lattice is shifted by a set of well-defined \( R \)-symmetry charges [29, 30], as dictated by the non-deformed \( RMSDS \) vacua. Generically, for large but not infinitely large deformations, the obtained vacua are those of “spontaneously broken supersymmetric vacua in the presence of geometrical fluxes” [20], studied in detail in refs [29,30]. Notice also that in the Euclidian version some of the models correspond to “thermal stringy vacua” in the presence of non-trivial left-right asymmetric “gravito-magnetic fluxes” studied recently in refs [10,11]. The would-be “initial” classical singularity of general relativity as well as the stringy Hagedorn-like singularities are both resolved by these fluxes!
The existence of a new massive boson-fermion degeneracy symmetry is shown by explicit orbifold constructions in Type II and Heterotic string theories. In all constructions, the target space-time is 2-dimensional and the spectrum consists of massless bosonic degrees of freedom as well as of massless fermionic ones with \((n(b) - n(f)) \neq 0\). All massive boson and fermion degrees of freedom exhibit Massive Spectrum Degeneracy Symmetry (MSDS). This remarkable property follows from the modular properties between the Vector (V), Spinor (S) and Anti-Spinor (C) characters of the affine \(G \subset SO(24)\) algebra, twisted by the \(\mathbb{Z}_2\)-orbifolds that are formulated algebraically in terms of twisted \(\theta^{12}\)-identities.

A new chiral \(\mathcal{N} = 1\) superconformal algebra is proposed based on the usual \(\mathcal{N} = 1\) super-Virasoro operators \(T_B (h_B = 2)\) and \(T_F (h_F = 3/2)\), together with \(C (h_C = 3/2)\) and \(J^a (h_J = 1)\), where \(J^a\) are the currents of the semi-simple gauge group \(H\) reduced by the orbifolds. The reduced massive boson-fermion degeneracy follows from a “spectral flow” relation induced by the algebra \(\{T_B, T_F, C, J^a\}\). In this work we derived the necessary conditions leading to the classification of all fermionic \(\mathbb{Z}_2^N\)-orbifold constructions of vacua with MSDS-structure. These classification rules are of main importance since the RMSDS-vacua are eventually related, via “Cosmological Large Marginal Deformations”, to some effective “four-dimensional” semi-realistic chiral superstring vacua with spacetime supersymmetry spontaneously broken by the RMSDS-induced “geometric” and “thermal” fluxes. The connection of RMSDS-vacua with “gauged supergravity theories” is by now transparent in the “strongly deformed phase” via the induced geometrical fluxes of the effective higher-dimensional theories. It is, thus, strongly suggested that the deformed orbifold RMSDS-models be considered as the most (semi-)realistic candidate vacua able to describe the “early non-singular phase of our Universe”, free of initial “general relativity-like” as well as of any “Hagedorn-like” singularities.

The observation that massless space-time fermions can appear in the twisted sectors of RMSDS-orbifold constructions hints at the possibility of constructing field theories with unbroken RMSDS and massless chiral fermions in higher than two dimensions, the case of \(d = 4\) dimensions being the most theoretically and phenomenologically appealing. Progress
in that direction may produce interesting alternatives to the conventional supersymmetry approach, possibly even bypassing some of the well-known mathematical inconsistencies related to the hierarchy and to the cosmological constant problems.

Finally, a noticeable property of 2-dimensional RMSDS-orbifold vacua is the holomorphic factorization property of their partition function. Although these theories have non-trivial massive spectra, thanks to the MSDS structure, all non-topological degrees of freedom are effectively washed out of the partition function! In this respect, RMSDS-orbifold vacua realize 2d target-space conformal field theories with holomorphic factorization properties similar to those initially proposed by Witten [31] in connection with BTZ-black holes [32]. In this context, the 2d MSDS-vacua (especially the Heterotic ones) are identified with the boundary 2d conformal field theories of $AdS_3$ [32]. Following Witten’s conjecture, the massive bosonic spectrum is identified with the mass spectrum of BTZ-black holes [31]. The MSDS-theories, however, additionally suggest the existence of a fermionic “massive supersymmetric” partner having the same mass spectrum as the bosonic one!

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