On Planar Polynomial Geometric Interpolation

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Abstract
In the paper, the planar polynomial geometric interpolation of data points is revisited. Simple sufficient geometric conditions that imply the existence of the interpolant are derived in general. They require data points to be convex in a certain discrete sense. Since the geometric interpolation is based precisely on the known data only, one may consider it as the parametric counterpart to the polynomial function interpolation. The established result confirms the Höllig-Koch conjecture on the existence and the approximation order in the planar case for parametric polynomial curves of any degree stated quite a while ago.

Keywords: polynomial curve, geometric interpolation, existence, approximation order

2010 MSC: 41A10, 65D05, 65D17

1. Introduction

The study of geometric polynomial interpolation problems was initiated in [1] where Hermite cubic interpolation of two points, tangent directions, and curvatures was analyzed. It soon became apparent that geometric interpolation brings two important practical advantages compared to linear polynomial interpolation schemes adapted to the curve case. First, the interpolating curve depends on geometric quantities only: data points, tangent directions, curvatures, etc. As to the second one, we avoid the choice of parametrization in advance. This allows us to stick to lower degree polynomials, but use the implicit parametrization to keep the

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1 The author acknowledges the financial support from the Slovenian Research Agency (research core funding No.P1-0291)
approximation power of the higher degree component-wise linear interpolants. In practical applications, the shape of an interpolating curve is often an issue. One would require that the interpolant at least roughly preserves the shape of the data it is based upon. However, a priori choice of parametrization combined with higher degree polynomials may make this goal hard to achieve. As an example, consider the points, sampled from smooth convex curves (Fig. 1). With three commonly used parametrizations applied, none of the interpolants is satisfying. But the same data interpolated geometrically, determine quite differently a satisfactory curve approximation (Fig. 2). So understandably a lot of work that followed [1] investigated the geometric interpolation quite thoroughly. The problems considered included Lagrange interpolation of data points only, and its limit case, Hermite, where also data directions, data curvatures, etc. are prescribed. If all data appear at one point, the Hermite case reduces to a Taylor one. But the problems involved are generally nonlinear, so the existence, the approximation order, and the construction are the burning issues. The asymptotic approach was often the tool to tackle the first two goals. There, the data is assumed to be taken from a sufficiently small part of a convex curve that could be locally parametrized by one of the components.

In [2], a natural conjecture on the geometric polynomial interpolation was made, based upon the authors’ previous work and results referenced therein. This conjecture in the special, Lagrange planar case, predicts that a polynomial curve of degree \( \leq n \) can if some appropriate assumptions are met interpolate \( 2n \) points,
with optimal approximation order $2n$. The conjecture was confirmed in some particular cases, mostly for low degree polynomials, and in the asymptotic sense only \cite{1, 2, 3, 4, 5, 6, 7, et al.}.

It is somewhat surprising how long the result for the general degree curves was slipping one’s hands. In this paper, we close this gap for the planar Lagrange interpolation problem, and we present quite general geometric conditions that assure both the existence and the optimal approximation order for arbitrary degree cases. Recently, a similar, positive result for the Taylor case polynomial curve expansion was given in \cite{8} for quite a large infinite subset of $\mathbb{N}$.

The result of the paper is based on an old but far-reaching achievement developed to the Bolzano–Poincaré–Miranda theorem for simplices in \cite{9, Theorem 2.1}: if a set of continuous functions satisfies certain sign conditions at the simplex boundary, it must vanish at some interior point of the simplex.

The outline of the paper is as follows. In Section 2, the Lagrange interpolation problem is defined, and the main theorems are listed. In Section 3, the functions and the corresponding equations to be solved are derived. Section 4, the most technical part, contains the analysis of the functions involved. Based upon Section 4, the last two sections provide the proofs of the theorems given in Section 2.

2. Lagrange interpolation problem

The Lagrange interpolation problem is formulated as follows. Suppose that a sequence of $2n$ distinct points

$$
P_\ell = \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} \in \mathbb{R}^2, \quad \ell = 0, 1, \ldots, 2n - 1, \quad a := (a_\ell)_{\ell=0}^{2n-1}, \quad b := (b_\ell)_{\ell=0}^{2n-1}, \quad (1)
$$

is given. Find a parametric polynomial curve

$$
p_n : [0, 1] \to \mathbb{R}^2
$$

of degree $\leq n$ that interpolates the given data points \eqref{P_ell} at some parameter values $t_\ell \in [0, 1]$ in increasing order, i.e.,

$$
p_n(t_\ell) = P_\ell, \quad \ell = 0, 1, \ldots, 2n - 1. \quad (2)
$$

Since a linear reparametrization preserves the degree of a parametric polynomial curve, one can assume $t_0 := 0$ and $t_{2n-1} := 1$, but the remaining parameters

$$
t := (t_\ell)_{\ell=1}^{2n-2}
$$
are unknown, ordered as
\[ t_0 = 0 < t_1 < \cdots < t_{2n-2} < t_{2n-1} = 1. \]

Let \( \Delta \) denote the forward difference, i.e.,
\[
\Delta w_\ell := w_{\ell+1} - w_\ell, \quad \Delta P_\ell := P_{\ell+1} - P_\ell.
\]

We have the following assertion.

**Theorem 1.** If, possibly after a linear transformation, the differences
\[
\Delta a_\ell, \Delta b_\ell, \quad \ell = 0, 1, \ldots, 2n - 2,
\]
(satisfy the assumptions (3) and (4), are all of the same sign, the interpolation problem (2) has at least one solution.

Some obvious remarks are right on the spot. Even if the data (1) don’t satisfy the assumptions (3) and (4), they may be satisfied after a proper linear data transformation. In particular, the role of both data components may be reversed, any data rotation or reflection is allowed, etc. All of the asymptotic approaches to the interpolation problem are covered here too. Generally, Theorem 1 gives the positive answer for all data that could be transformed to a sampled monotone convex function curve by an affine transformation, as is the data case on Fig. 2 (left). But the right side of the same figure requires more general assumptions.

**Theorem 2.** If the determinants
\[
\|\Delta P_{\ell-1} \Delta P_\ell\| = \Delta a_{\ell-1} \Delta b_\ell - \Delta a_\ell \Delta b_{\ell-1}, \quad \ell = 1, 2, \ldots, 2n - 2,
\]
are all of the same sign, the interpolation problem (2) has at least one solution.

Note that the assumptions (5) are affinely invariant. The proof of Theorem 1 will carry most of the burden, and Theorem 2 will be confirmed with its help. There is little hope that simple necessary and sufficient conditions that guarantee existence could be found in general. In the case \( n = 2 \) the geometric interpolation problem has a solution iff (5) are satisfied as observed already in [10] and further elaborated in [11]. This case was generalized, together with necessary and sufficient conditions for the existence of the interpolant, to any dimension in [6] where a
Figure 2: Data that satisfy Theorem 1 interpolated by a cubic parametric curve (left), and data that satisfy Theorem 2 and corresponding polynomial interpolant of degree 5 (right). Data curves are plotted with a small offset to the right since otherwise, any distinction between the original curve and its interpolant would not be visible.

particular form of (5) already appeared. But if the degree $n$ is larger, i.e., $n \geq 3$ in the planar case, the analysis of all possibilities becomes very complicated even for planar cubic interpolation [12, 13]. If the data are not convex in the discrete sense of either one of the theorems, the existence of the solution depends heavily on the particular data set. One would expect that the results carry over to the Hermite interpolation problems, but we leave the confirmation to the future work.

3. Equations of the problem

As a vector in $\mathbb{R}^{2n-2}$, the unknowns $t$ should belong to the interior $\text{int}(S)$ of a $(2n-2)$-simplex $S$,

$$S := \{ t \in \mathbb{R}^{2n-2} | 0 \leq t_1 \leq \cdots \leq t_{2n-2} \leq 1 \}.$$ 

The system of equations (2) has to determine the unknown $p_n$ as well as the parameters $t$. But the two tasks can be separated if one can provide enough linearly independent functionals, depending on $t$ only, that map $p_n$ to zero. Divided differences, based upon $\geq n + 2$ values, are a natural choice. Let us apply the divided differences

$$[t_{j-1}, t_j, \ldots, t_{n+j}], \quad j = 1, 2, \ldots, n-1,$$

(6)
to both sides of (2). Since \( \deg p_n \leq n \), the left side vanishes, and so should the right one. But the parameters \( t_\ell \) are distinct and this condition becomes

\[
\sum_{\ell=j-1}^{n+j} \frac{1}{\prod_{m=j-1}^{n+j} (t_\ell - t_m)} P_\ell = 0, \quad j = 1, 2, \ldots, n - 1.
\]

This nonlinear system depends on the data \( P_\ell \) and the unknowns \( t \) only. If the solution \( t \in \text{int}(S) \) is found, one can apply any algorithm inherited from the polynomial function interpolation to construct the interpolatory curve \( p_n \) component-wise. For each \( j \) the system (7) provides two equations based upon the first and the second component of the data. For our purpose, it will be more convenient to rewrite (7) in a polynomial form. However, from a computational point of view, the equations (7) are less sensitive to numerical cancellation, and they produced the examples of Fig. 2 in a couple of Newton steps starting with an equidistant initial guess. Let \( V(t_{j_1}, \ldots, t_{j_r}) \) be a Vandermonde determinant based upon \( t_{j_1}, \ldots, t_{j_r} \),

\[
V(t_{j_1}, \ldots, t_{j_r}) := \prod_{i=1}^{r} \prod_{\ell=1}^{r-1} (t_{j_\ell} - t_{j_i}),
\]

and let

\[
V(t_{j_1}, \ldots, t_{j_r}; c_{j_1}, \ldots, c_{j_r})
\]

denote a determinant, obtained from (8) by replacing the last determinant row \((t_{j_1}^{r-1}, \ldots, t_{j_r}^{r-1})\) by the values \((c_{j_1}, \ldots, c_{j_r})\),

\[
V(t_{j_1}, \ldots, t_{j_r}; c_{j_1}, \ldots, c_{j_r}) = \sum_{\ell=1}^{r} (-1)^{r+\ell} c_{j_\ell} V(t_{j_1}, \ldots, t_{j_{\ell-1}}, t_{j_{\ell+1}}, \ldots, t_{j_r}).
\]

The divided difference (6) applied to a sequence \((c_\ell)_{\ell=j-1}^{\ell=n+j}\) as a quotient of the Vandermonde determinants reads

\[
\frac{V(t_{j-1}, \ldots, t_{n+j}; c_{j-1}, \ldots, c_{n+j})}{V(t_{j-1}, \ldots, t_{n+j})}.
\]

Thus, if we multiply (7) by \( V(t_{j-1}, \ldots, t_{n+j}) \) we obtain the polynomial form of the system

\[
f_{a,j} (t_{j-1}, \ldots, t_{n+j}) = 0, \quad j = 1, 2, \ldots, n - 1,
\]

\[
f_{b,j} (t_{j-1}, \ldots, t_{n+j}) = 0, \quad j = 1, 2, \ldots, n - 1.
\]
where

\[ f_{c,j}(t_{j-1}, \ldots, t_{n+j}) := V(t_{j-1}, \ldots, t_{n+j}; c_{j-1}, \ldots, c_{n+j}). \]

Here and throughout the paper, coefficients \( c \) will be used as a placeholder for either \( a \) or \( b \) meaning that any relation or text involving \( c \) holds for either of them. Note that \( f_{c,j} \) depends linearly on the constants \( c_{j-1}, c_j, \ldots, c_{n+j} \) only. It is also invariant under data translation since divided differences map constants to zero. So \( f_{c,j} \) depends linearly on data differences \( \Delta c_{\ell} \),

\[
\begin{align*}
  f_{c,j}(t_{j-1}, \ldots, t_{n+j}) &= \\
  &= \sum_{k=j}^{n+j} (-1)^{n+j-k} (c_k - c_{j-1}) V(t_{j-1}, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n+j}) \\
  &= \sum_{r=j}^{n+j} \sum_{k=r}^{n+j} (-1)^{n+j-k} V(t_{j-1}, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n+j}).
\end{align*}
\]

This shows that the equations derived follow the affine nature of the interpolation problem, and it explains the following remark.

**Remark 1.** For each \( j \) separately, \( 1 \leq j \leq n - 1 \), we may replace the data

\[
(\Delta P_{j-1} \quad \Delta P_j \quad \cdots \quad \Delta P_{n+j-1})
\]

which in (9) determine the function pairs \( f_{a,j} \), and \( f_{b,j} \), by

\[
(\Delta P_{j-1} \quad \Delta P_j \quad \cdots \quad \Delta P_{n+j-1}),
\]

where \( A_j : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is any nonsingular linear transformation. The system of equations generated by the modified functions is equivalent to the original one.

The following remark contributes an error term in case the data are sampled from a data curve.

**Remark 2.** Suppose that the data

\[
P_\ell = g(\xi_\ell), \quad \ell = 0, 1, \ldots, 2n - 1,
\]

are taken from a smooth parametric curve \( g : [\xi_0, \xi_{2n-1}] \rightarrow \mathbb{R}^2 \) at increasing parameter values \( \xi_0 < \xi_1 < \cdots < \xi_{2n-1} \). If the solution \( t \in \text{int}(\mathcal{S}) \) of (9) exists, the function case polynomial interpolation provides the remainder in the
parametric one too. Let \( \varphi : [0, 1] \to [\xi_0, \xi_{2n-1}] \) be any regular reparametrization of \( g \) that satisfies
\[
\varphi(t_\ell) = \xi_\ell, \quad \ell = 0, 1, \ldots, 2n - 1.
\]
Then
\[
g(\varphi(t)) = p_n(t) + \left( \prod_{\ell=0}^{2n-1} (t - t_\ell) \right) [\xi_0, \xi_1, \ldots, \xi_{2n-1}, \varphi(t)] g, \quad t \in [0, 1].
\]

As an example, the normal reparametrization \( \varphi \), introduced in [14], is often a sensible choice (Fig. 3).

![Figure 3: Normal reparametrization \( \varphi \) and corresponding parametric difference graphs \( g \circ \varphi - p \) of the examples outlined in Fig. 2.](image)

**Remark 3.** If the data curve \( g \) in Remark 2 is additionally convex on \([\xi_a, \xi_b]\), the points \( P_\ell \) satisfy the assumptions of Theorem 2 for any increasing choice of parameters \( \xi_\ell \in [\xi_a, \xi_b] \). So by [7, Theorem 4.6] the interpolating polynomial parametric curve \( p_n \) approximates \( g \) with the optimal approximation order \( 2n \). Fig. 4 shows a numerical estimate of the approximation order for the data of Fig. 2 as a function of shrinking data curve parameter interval computed as in [15, at several spots]. The numerical evidence clearly confirms the expected result, \( 2n = 6 \) (left), and \( 2n = 10 \) (right).
4. Analysis of polynomials $f_{c,j}$

Somewhat sloppy, we shall consider the polynomial $f_{c,j}$ wherever needed also as a function with the domain extended naturally,

$$f_{c,j} : \mathbb{R}^{2n-2} \to \mathbb{R} : (t_\ell)_{\ell=1}^{2n-2} \mapsto f_{c,j}(t_{j-1}, \ldots, t_{n+j}),$$

and

$$f_j := (f_{a,j}, f_{b,j})^T, \quad f := (f_{a,1}, f_{b,1}, \ldots, f_{a,n-1}, f_{b,n-1})^T.$$

This helps us to define varieties $\mathcal{V}(f_{c,j})$, $\mathcal{V}(f_j)$ and $\mathcal{V}(f)$, where the variety definition of a scalar function $g$,

$$\mathcal{V}(g) := \{ t \in \mathbb{R}^{2n-2} \mid g(t) = 0 \},$$

is naturally extended to the vector function case $\mathcal{V}(g)$. The system (9) is equivalent to (7) except for possible extraneous solutions at the simplex boundary $\partial S := S \setminus \text{int}(S)$, and there is to prove $\mathcal{V}(f) \cap \text{int}(S) \neq \emptyset$. Of particular interest will be a variety part

$$\mathcal{V}_S(g) := \text{cl}(\mathcal{V}(g) \cap \text{int}(S))$$

the restriction of $\mathcal{V}(g)$ to $S$ that drops out the variety boundary points that can’t be reached from $\text{int}(S)$. The simplex $S$ requires some additional notation. It is a convex hull of points,

$$S = \text{co}\{Q_0, \ldots, Q_{2n-2}\},$$

where

$$Q_\ell := (0, \ldots, 0, 1, \ldots, 1)^T, \quad \ell = 0, 1, \ldots, 2n-2.$$  \hspace{1cm} (12)
The simplex faces will be denoted by
\[ S_{\{\ell_1, \ldots, \ell_r\}} = \text{co}\{Q_j\}_{j \in \{0, \ldots, 2n-2\} \setminus \{\ell_1, \ldots, \ell_r\}}. \]

In this notation, the boundary of the simplex \( S \) is determined by the \((2n-3)\)-simplex faces
\[ S_{\{\ell\}}, \quad \ell = 0, 1, \ldots, 2n-2. \] (13)

The simplest case \( n = 2 \) is shown in Fig. 5. The first observation is straightforward. Since precisely the components \( \ell \) and \( \ell + 1 \) of \( Q_\ell \) are by (12) distinct, the \((2n-3)\)-simplex \( S_{\{\ell\}} \) is the maximal face of \( S \) characterized by \( t_\ell = t_{\ell+1} \).

We then deduce from (10)
\[
\begin{align*}
  f(c, j) (t_{j-1}, \ldots, t_{r_2-1}, t_{r_1}, t_{r_2+1}, \ldots, t_{n+j}) = \\
  (-1)^{n+r_2-j} (c_{r_2} - c_{r_1}) V(t_{j-1}, \ldots, t_{r_2-1}, t_{r_2+1}, \ldots, t_{n+j}).
\end{align*}
\] (14)

Let us define the signum function by
\[
\text{sign } w := \begin{cases} 
  1, & w > 0, \\
  0, & w = 0, \\
  -1, & w < 0.
\end{cases}
\]

![Figure 5: The case \( n = 2 \): the simplex \( S \subset \mathbb{R}^2 \), the points \( Q_0, Q_1, Q_2 \), and the faces \( S_{\{0\}}, S_{\{1\}}, S_{\{2\}} \).](image)
and let us extend it to a vector argument component-wise, \( \text{sign} \ (w_i) := \text{sign} \ w_i \).

The choice \( r_2 = r_1 + 1 = \ell + 1 \) in (14) proves the following lemma.

**Lemma 1.** Let \( j - 1 \leq \ell \leq n + j - 1 \). Then

\[
\text{sign} \ f_{c,j} (t) = (-1)^{n+j-1-\ell} \text{sign} \ \Delta c_{\ell}, \quad t \in \text{int} (S_{\setminus \{\ell\}}),
\]

and further,

\[
\text{sign} \ f_{c,j} (t)_{t \in S_{\setminus \{\ell,k\}}} = \begin{cases} 
0, & k = j - 1, \ldots, \ell - 1, \ell + 1, \ldots, n + j - 1, \\
\text{sign} \ f_{c,j} (t)_{t \in \text{int} (S_{\setminus \{\ell\}})}, & k = 0, \ldots, j - 2, n + j, \ldots, 2n - 2.
\end{cases}
\]

So the sign of \( f_{c,j} \) at the open faces

\[
\text{int} (S_{\setminus \{\ell\}}), \quad \ell = j - 1, \ldots, n + j - 1,
\]

is known for each \( j \), and this fact even simplifies if all differences \( \Delta c_{\ell} \) are of the same sign. This paves the way to an application of the Bolzano–Poincaré–Miranda theorem on simplices [9, Theorem 2.1]. For the reader’s convenience, we recall the theorem here.

**Theorem 3.** (9 Theorem 2.1) Assume that \( \Sigma := \text{co} \{U_0, \ldots, U_m\} \) is an \( m \)-simplex in \( \mathbb{R}^m \) with vertices \( U_\ell \in \mathbb{R}^m \). Let \( g := (g_1, \ldots, g_m)^T : \Sigma \to \mathbb{R}^m \) be a continuous function such that \( g_i (U_\ell) \neq 0, i = 1, 2, \ldots, m; \ell = 0, 1, \ldots, m \), and \( g(x) \neq 0, x \in \partial \Sigma \). Assume that the vertices \( U_\ell \) are reordered such that the following hypotheses are fulfilled:

(i) \( \text{sign} \ g_{i0} (U_\ell) \text{sign} \ g_i (x) = -1, \quad x \in \Sigma_{\setminus \{i\}}, \quad i = 1, \ldots, m \),

(ii) \( \text{sign} \ g (U_0) \neq \text{sign} \ g (x), \quad x \in \Sigma_{\setminus \{0\}} \).

Then, there is at least one \( x \in \text{int} (\Sigma) \) such that \( g(x) = 0 \).

The sign assumptions of Theorem 3 are too demanding for a straightforward application in our case since by Lemma 4 functions involved vanish at a significant part of the boundary \( \partial S \). An intermediate step is needed. We illustrate it with two simple examples shown in Fig. 6. Both functions \( g : \Sigma \to \mathbb{R}^2 \) there have an equal sign pattern at the boundary \( \partial \Sigma \), and there are almost none of the required assumptions fulfilled. So Theorem 3 can’t be applied. But there is an \( x \in \text{int} (\Sigma) \) such that \( g(x) = 0 \) in Fig. 6 (left), and no such interior point exists in the right
part of the figure. This encourages one to seek a modification \( g \to \tilde{g} \), based upon some additional information on \( g \) at or very close to the boundary, such that the theorem could be put to work on \( \tilde{g} \) where appropriate. Of course, the modification must preserve the zero set \( \{ x \in \text{int} (\Sigma) \mid g(x) = 0 \} \). The following observation offers a simple sufficient condition that helps us to verify this.

**Lemma 2.** Suppose that \( g := (g_1, \ldots, g_m)^T : \Sigma \to \mathbb{R}^m \) is a continuous function. Choose \( g_i \) that is to be replaced by a continuous \( \tilde{g}_i : \Sigma \to \mathbb{R} \), and let

\[
M := \{ x \in \text{int} (\Sigma) \mid (g_i(x) = 0 \land \tilde{g}_i(x) \neq 0) \lor (\tilde{g}_i(x) = 0 \land g_i(x) \neq 0) \}.
\]

If there exists a component \( g_{\ell} \), \( 1 \leq \ell \leq m \), such that

\[
\{ x \in \text{int} (\Sigma) \mid g_{\ell}(x) = 0 \} \cap M = \emptyset,
\]

then the replacement \( g_i \to \tilde{g}_i \) doesn’t alter the set \( \{ x \in \text{int} (\Sigma) \mid g(x) = 0 \} \).

**Proof.** The set \( M \subset \text{int} (\Sigma) \) determines the points where the zero sets of \( g_i \) and \( \tilde{g}_i \) differ. Since \( g_{\ell} \) doesn’t vanish there, so can’t \( g \), and the replacement \( g_i \to \tilde{g}_i \) does not influence the zero set of \( g \). \( \square \)

Recall Fig. 6. After several steps where each of them satisfies the assumptions of Lemma 2 the examples are modified as shown in Fig. 7. The function \( \tilde{g} \) of the left image fulfills the assumptions of Theorem 3 thoroughly, but for the right counterpart understandably this is not possible. All the other assumptions been satisfied, it still misses (ii).

To apply Theorem 3 a proper association between function \( f_{c,j} \) and simplex faces \( S_{\setminus \{\ell\}} \) has to be found. For each of the functions (9) identified by the index pair \( (c,j) \) we have to select a face \( S_{\setminus \{\sigma^*\}} \), determined by a map

\[
\{a, b\} \times \{1, \ldots, n - 1\} \to \{0, \ldots, 2n - 2\} : (c, j) \mapsto \sigma^* := \sigma^*(c, j)
\]  \( (16) \)
Figure 7: Modifications that satisfy Lemma 2 applied to the examples of Fig. 6. The assumptions of Theorem 3 are met on the left, but not all on the right part of the figure.

such that

$$\text{sign} f_{c,j}(t) \text{ sign} f_{c,j}(Q_{\sigma^*}) = -1, \quad t \in S \setminus \{\sigma^*\}. \quad (17)$$

The map (16) should be injective, and combined with (17) it represents (i) of Theorem 3. There are only $2n - 2$ functions in (9) and $2n - 1$ simplex faces in (13). One face is left, i.e., $S \setminus \{\sigma^{**}\}$, at which the final condition (ii) of Theorem 3 is imposed,

$$\text{sign} f(t) \neq \text{sign} f(Q_{\sigma^{**}}), \quad t \in S \setminus \{\sigma^{**}\}. \quad (18)$$

Quite clearly, given $j$ and $f_{c,j}$, the candidates for $\sigma^* (c,j)$ are only the indices $j - 1, j, \ldots, n + j - 1$. Note that (17) requires $\text{sign} f_{c,j}$ to be constant at the particular simplex, which partially collides with (15) established by Lemma 1. So modifications of the functions $f_{c,j}$ are needed, and Lemma 2 together with the examples of Fig. 6 will help us in the construction.

Let us elaborate (14) one step further. Note first that

$$\frac{\partial}{\partial t_{j_i}} V(t_{j_1}, \ldots, t_{j_r}) \bigg|_{t_{j_i} = t_{j_m}} =$$

$$= (-1)^{r-i} V(t_{j_1}, \ldots, t_{j_{i-1}}, t_{j_{i+1}}, \ldots, t_{j_r}) \prod_{\ell = 1}^{r} (t_{j_m} - t_{j_{\ell}}). \quad (19)$$

From (14) and (19) it is straightforward to derive first order terms of the Taylor expansion of $f_{c,j}$. Let

$$j - 1 \leq r_1, r_2, r_3, r_4 \leq n + j, \quad r_2 \neq r_1, r_4 \neq r_3, r_2 \leq r_4,$$
and
\[
\mathcal{L}_{t,c,j} (r_1, r_2, r_3, r_4) := (-1)^{r_4-r_2} (c_{r_4} - c_{r_2}, c_{r_2} - c_{r_1}) \begin{bmatrix}
-\omega'_{r_2,r_4} (t_{r_1}) & 0 \\
0 & \omega'_{r_2,r_4} (t_{r_3})
\end{bmatrix} (t_{r_2} - t_{r_1}) (t_{r_4} - t_{r_3}).
\]

with
\[
\omega_{r_2,r_4} (z) := \prod_{\ell=j-1, \ell \neq r_2,r_4}^{n+j} (z - t_{\ell}).
\]

Then \( f_{c,j} \) at \( t_{r_2} \approx t_{r_1} \) and \( t_{r_4} \approx t_{r_3} \) expands as
\[
f_{c,j} (t_{j-1}, \ldots, t_{n+j}) = V(t_{j-1}, \ldots, t_{r_2-1}, t_{r_2+1}, \ldots, t_{r_4-1}, t_{r_4+1}, \ldots, t_{n+j}) \cdot \
\mathcal{L}_{t,c,j} (r_1, r_2, r_3, r_4) + O_2 (t_{r_2} - t_{r_1}, t_{r_4} - t_{r_3}).
\]

Here, \( O_2 (u,v) \) denotes the remainder involving terms \( u^i v^{m-i}, 0 \leq i \leq m, m = 2, \ldots \), with polynomial coefficients that depend on \( t \).

With \( j \) fixed, let us choose \( k - 1, j \leq k \leq n + j \), as a candidate for \( \sigma^+ (c, j) \).

Following Lemma [1] we have to analyze \( f_{c,j} \) near the faces
\[
\frac{S \setminus \{j-1,k-1\}, \ldots, S \setminus \{k-3,k-1\}, S \setminus \{k-2,k-1\}}{k-j},
\]

and
\[
\frac{S \setminus \{k-1,k\}, S \setminus \{k-1,k+1\}, \ldots, S \setminus \{k-1,n+j-1\}}{n+j-k}
\]

since \( f_{c,j} \) vanishes at these \((2n-4)\)-simplices. Let us consider now (23) only since the inspection of (22) follows by symmetry. The faces will be analyzed with help of an intersection of \( S \) and the \((t_k,t_{k+i})\)-plane, \( 1 \leq i \leq n + j - 1 - k \), with the rest of the components \( t \in \text{int} (S) \) considered as parameters. Two different generic cases are to be examined only, \( i = 1 \) yields the intersection as a triangle
\[
\left\{ (t_k, t_{k+1})^T \in \mathbb{R}^2 \mid t_{k-1} \leq t_k \leq t_{k+1} \leq t_{k+2} \right\},
\]

and the rest of \( i \) as a rectangle
\[
\left\{ (t_k, t_{k+i})^T \in \mathbb{R}^2 \mid t_{k-1} \leq t_k \leq t_{k+1}, t_{k+i-1} \leq t_{k+i} \leq t_{k+i+1} \right\}.
\]

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As the first one, we take a look at the intersection of \( S \) and the \((t_k, t_{k+1})\)-plane. This intersection is a triangle, determined by the vertices

\[
T_0 := \left( \frac{t_{k-1}}{t_{k-1}} \right), \quad T_1 := \left( \frac{t_{k-1}}{t_{k+2}} \right), \quad T_2 := \left( \frac{t_{k+2}}{t_{k+2}} \right),
\]

(24)
i.e., particular points in

\[ S \setminus \{k-1, k\}, S \setminus \{k, k+1\}, S \setminus \{k-1, k+1\}, \]

and edges as particular line segments in

\[ S \setminus \{k-1\}, S \setminus \{k\}, S \setminus \{k+1\}. \]

If \( t \in \text{int}(S) \) moves, the functions \( f_{c,j} \) as well as the actual positioning of the points (24) in \( \mathbb{R}^{2n-2} \) change but the crucial relations that will be proved in the following lemmas remain intact.

**Lemma 3.** Suppose that \( j \leq k \leq n+j-2 \), and

\[
\Delta c_{k-1} > 0, \quad \Delta c_k > 0, \quad \Delta c_{k+1} > 0.
\]

(25)

Suppose that the components of \( t \in S \) satisfy

\[
t_{j-1} < t_j < \cdots < t_{k-1} < t_{k+2} < t_{k+3} < \cdots < t_{n+j}.
\]

(26)

At the triangle \( T_0T_1T_2 \), the variety \( V(f_{c,j}) \) has the following properties (Fig. 8):

(i) near the vertices \( T_\ell \) it could be expressed as a continuous function of \( t_k \) (or \( t_{k+1} \)). At \( T_1 \) it only touches the triangle from the outside, and at \( T_0 \) as well as at \( T_2 \) it continues in the triangles interior,

(ii) any path connecting the open line segment \( T_0T_2 \) with the open edge on the polygon \( T_0T_1T_2 \) crosses the variety odd times,

(iii) it lies entirely inside the triangle except for the vertices \( T_i \), and it continuously connects \( T_0 \) and \( T_2 \).

**Proof.** The assumptions (25), (26), and Lemma 1 imply that the function \( f_{c,j} \) is of signs \((-1)^{n+j-k}, (-1)^{n+j-k}, \) and \((-1)^{n+j-k+1} \) at the open line segments \( T_0T_1, T_1T_2, \) and \( T_2T_0 \) respectfully. The expansion (21) shows that we may
apply the implicit function theorem to \( f_{c,j} \) to obtain \( V(f_{c,j}) \) at \( T_\ell \) locally as a continuous function of \( t_k \) or \( t_k+1 \). At \( T_2 \), the significant part of the expansion reads

\[
L_{t,c,j}(k-1, k, k+2, k+1) = -\omega'_{k,k+1}(t_{k-1}) \Delta c_{k+1}(t_k - t_{k-1}) + \omega'_{k,k+1}(t_{k+2}) \Delta c_{k-1}(t_{k+2} - t_{k+1}).
\]

Since \( \text{sign} \omega'_{k,k+1}(t_{k+2}) = -\text{sign} \omega'_{k,k+1}(t_{k-1}) = (-1)^{n+j-k} \), the first assertion is verified at \( T_2 \), and similarly at \( T_0 \). At \( T_1 \), the first order expansion can vanish only if \( \text{sign} \Delta t_{k-1} = -\text{sign} \Delta t_{k+1} \), and the touch is verified too. Further, the variety \( V(f_{c,j}) \) must block any path from the open line segment \( T_0 T_2 \) to \( T_1 \), so it must continuously connect \( T_0 \) and \( T_2 \), and the rest of the assertions is confirmed.

In the second case, we consider the rectangular intersection of \( S \) and \((t_k, t_{k+i})\)-plane where \( 2 \leq i, k+i \leq n+j-1 \). Its edges correspond to particular line segments of the faces

\[
S\setminus\{k-1\}, S\setminus\{k+i\}, S\setminus\{k\}, S\setminus\{k+i-1\},
\]

and its vertices

\[
V_0 := \begin{pmatrix} t_{k-1} \\ t_{k+i-1} \end{pmatrix}, \quad V_1 := \begin{pmatrix} t_{k-1} \\ t_{k+i} \end{pmatrix}, \quad V_2 := \begin{pmatrix} t_{k+1} \\ t_{k+i+1} \end{pmatrix}, \quad V_3 := \begin{pmatrix} t_{k+1} \\ t_{k+i-1} \end{pmatrix},
\]

\[
T_1
\]

\[
\text{Figure 8: The variety } V(f_{c,j}) \text{ separates an open edge on the polygon } T_0 T_1 T_2 \text{ from the open line segment } T_0 T_2.
\]
are particular points of their intersections

\[ S_{\{k-1,k+i-1\}}, S_{\{k-1,k+i\}}, S_{\{k,k+i\}}, S_{\{k,k+i-1\}}. \]

We have the following observation, very similar to Lemma 3.

**Lemma 4.** Suppose that \( j \leq k \), \( k + 1 < k + i \leq n + j - 1 \), and

\[ \Delta c_{k-1} > 0, \; \Delta c_k > 0, \; \Delta c_{k+i-1} > 0, \; \Delta c_{k+i} > 0. \] (27)

Suppose that the components of \( t \in S \) satisfy

\[ t_{j-1} < \cdots < t_{k-1} < t_{k+1} < \cdots < t_{k+i-1} < t_{k+i+1} < \cdots < t_{n+j}. \] (28)

At the rectangle \( V_0V_1V_2V_3 \) the variety \( V(f_{c,j}) \) has the following properties (Fig. 9). Near the vertices \( V_\ell \) it could be expressed as a continuous function of \( t_k \) (or \( t_{k+i} \)), and for an even \( i \) one has

![Diagram](image_url)

Figure 9: The variety \( V(f_{c,j}) \) separates an open edge on the polygon \( V_3V_0V_1 \) from an open edge on \( V_1V_2V_3 \) (left, \( i \) even), and an open edge on \( V_0V_1V_2 \) from an open edge on \( V_2V_3V_0 \) (right, \( i \) odd).
(i) at $V_0$ and $V_2$ it only touches the rectangle, but at $V_1$ and $V_3$ it continues to its interior,

(ii) any path connecting an open edge on the polygon $V_3V_0V_1$ to an open edge on $V_1V_2V_3$ crosses the variety an odd number of times,

(iii) $V(f_{c,j})$ lies entirely inside the rectangle except for the vertices, and it continuously connects $V_1$ and $V_3$.

If $i$ is odd, the role of $V_0$ and $V_1$, as well as $V_2$ and $V_3$, is reversed.

PROOF. The assumption (28) allows us to use the expansion (20) again, and we obtain significant expansion parts at four rectangle corners as,

$L_{t,c,j}(k-1,k,k+i-1,k+i) = (-1)^i \left( -\Delta c_{k+i-1} \omega'_{k,k+i} (t_{k-1})(t_k-t_{k-1}) + \Delta c_{k-1} \omega'_{k,k+i} (t_{k+i-1})(t_{k+i} - t_{k+i-1}) \right)$, at $V_0$,

$L_{t,c,j}(k-1,k,k+i+1,k+i) = (-1)^i \left( -\Delta c_{k+i} \omega'_{k,k+i} (t_{k+i+1})(t_{k+i+1} - t_{k+i}) + \Delta c_{k+i} \omega'_{k,k+i} (t_{k-1})(t_k - t_{k-1}) \right)$, at $V_1$,

$L_{t,c,j}(k+1,k,k+i+1,k+i) = (-1)^i \left( -\Delta c_{k+i} \omega'_{k,k+i} (t_{k+i+1})(t_{k+i+1} - t_{k+i}) + \Delta c_{k+i} \omega'_{k,k+i} (t_{k-1})(t_k - t_{k-1}) \right)$, at $V_2$,

$L_{t,c,j}(k+1,k,k+i-1,k+i) = (-1)^i \left( -\Delta c_{k+i-1} \omega'_{k,k+i} (t_{k+i-1})(t_{k+i} - t_{k+i-1}) + \Delta c_{k+i} \omega'_{k,k+i} (t_{k-1})(t_k - t_{k-1}) \right)$, at $V_3$.

Note also

$\text{sign } \omega'_{k,k+i} (t_{k+1}) = -\text{sign } \omega'_{k,k+i} (t_{k-1}) = (-1)^{n+j-k}$,

$\text{sign } \omega'_{k,k+i} (t_{k+i}) = -\text{sign } \omega'_{k,k+i} (t_{k+i+1}) = (-1)^{n+j-k+i}$.

(29)

With the assumption (27) it is now straightforward to verify the assertions by arguments used already in the proof of Lemma 3. □

To investigate the system (9) further, we have to make use of the distinction between $f_{a,j}$, and $f_{b,j}$. Recall Lemma 3 and Fig. 8. The variety $V(f_{c,j})$ continuously connects $T_0$ and $T_2$. Let us denote by $\theta_{c,T_0}$ the angle between abscissa direction and $V(f_{c,j})$ at $T_0$, and let $\theta_{c,T_2}$ be the angle between $V(f_{c,j})$ and ordinate direction at $T_2$. 

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Lemma 5. Suppose assumptions of Lemma 3 are satisfied and suppose additionally
\[
\begin{bmatrix}
\Delta a_{k-1+\ell} & \Delta a_{k+\ell} \\
\Delta b_{k-1+\ell} & \Delta b_{k+\ell}
\end{bmatrix} > 0, \quad \ell = 0, 1.
\] (30)

Then (Fig. 10)
\[
1 < \tan \theta_{a,T_0} = 1 + \frac{\Delta a_k}{\Delta a_{k-1}} < \tan \theta_{b,T_0} = 1 + \frac{\Delta b_k}{\Delta b_{k-1}} < \infty,
\] (31)
\[
1 < \tan \theta_{b,T_2} = 1 + \frac{\Delta b_k}{\Delta b_{k+1}} < \tan \theta_{a,T_2} = 1 + \frac{\Delta a_k}{\Delta a_{k+1}} < \infty.
\]

The number of intersections of the varieties \(V(f_{a,j})\) and \(V(f_{b,j})\) in \(\text{int}(T_0T_1T_2)\) is odd.

PROOF. At \(T_0\), the variety \(V(f_{c,j})\) is determined from the significant part of the \(f_{c,j}\) expansion (21)
\[
L_{t,c,j}(k - 1, k, k - 1, k + 1) = 0 = \omega'_{k,k+1}(t_{k-1}) ((\Delta c_{k-1} + \Delta c_k) (t_k - t_{k-1}) - \Delta c_{k-1} (t_{k+1} - t_{k-1})),
\]
so
\[
\tan \theta_{c,T_0} = 1 + \frac{\Delta c_k}{\Delta c_{k-1}}.
\]
and similarly
\[ \tan \theta_{c,r_2} = 1 + \frac{\Delta c_k}{\Delta c_{k+1}} \]
at \( T_2 \). The inequalities (31) follow then from (25) and (30). So varieties \( \mathcal{V}(f_{a,j}) \) and \( \mathcal{V}(f_{b,j}) \) interlace, and the proof is completed. \( \square \)

Figure 11: The case \( n = 2 \) before and after modifications of \( f_{a,j} \to \tilde{f}_{a,j} \) and \( f_{b,j} \to \tilde{f}_{a,j} \) based upon Lemma 5 and Lemma 2 (the image left and right respectively).

**Remark 4.** In the particular case, i.e., \( n = 2 \), the proof of Theorem 1 is completed. If the assumptions are satisfied, Lemma 5 in hand with the implicit function theorem reveals that modifications \( f_{a,1} \to \tilde{f}_{a,1}, f_{b,1} \to \tilde{f}_{b,1} \) which satisfy Lemma 2 can be carried over at \( Q_2 \) and \( Q_0 \) in some \( \varepsilon \) neighborhood (Fig. 11). The function \( \tilde{f}_{a,1} \) is of opposite sign at \( S_{\{0\}} \) and \( Q_0 \), so is \( \tilde{f}_{b,1} \) at \( S_{\{2\}} \) and \( Q_0 \). Thus
\[ \sigma^*(a, 1) = 0, \quad \sigma^*(b, 1) = 2, \quad \text{and} \quad \sigma^{**} = 1. \]

After removing the isolated zero at \( Q_1 \) we have
\[ \text{sign} \tilde{f}_{a,1}(Q_1) = \text{sign} \tilde{f}_{b,1}(Q_1) = 1, \]
but nowhere at \( S_{\{1\}} \) both functions are positive simultaneously. So the existence of the solution follows from [9, Theorem 2.1]. Of course, there are much shorter
ways to handle $n = 2$, even the closed form solution is available,

$$v_{i,k} := \| \Delta P_i, \Delta P_k \| > 0, \quad i = 0, 1; k = 1, 2,$$

$$t_1 = t_0 + \frac{\nu_0}{\nu_0 (\nu_1 + \nu_2) + \sqrt{\nu_0 (\nu_1 + \nu_2) (\nu_1 + \nu_2) (\nu_2 + \nu_2)}},$$

$$t_2 = t_0 + \frac{\nu_0 (\nu_1 + \nu_2) + \sqrt{\nu_0 (\nu_1 + \nu_2) (\nu_1 + \nu_2) (\nu_2 + \nu_2)}}{\nu_2 (\nu_1 + \nu_2) + \sqrt{\nu_0 (\nu_1 + \nu_2) (\nu_1 + \nu_2) (\nu_2 + \nu_2)}}.$$

Nevertheless, this remark suggests how to complete the general case too.

Since the case $n = 2$ is covered, we shall assume from now on $n \geq 3$. We return now to Lemma 4, with sharpened assumptions. Let us denote by $\tau_{c, \ell}$ the angle between abscissa direction and $V(f_{c,j})$ at $V_{\ell}$.

**Lemma 6.** Suppose assumptions of Lemma 4 are satisfied and suppose additionally

$$\left| \frac{\Delta a_{k-1+\ell}}{\Delta b_{k-1+\ell}} \right| > 0, \quad \ell = 0, 1.$$

Then the angles $\tau_{c, \ell}$ at vertices $V_{\ell}$ satisfy (Fig. 12)

$$0 < \tan \tau_{a, V_1} < \tan \tau_{b, V_1} < \infty, \quad 0 < \tan \tau_{a, V_3} < \tan \tau_{b, V_3} < \infty,$$

if $i$ is even, and

$$0 < \tan \tau_{a, V_0} < \tan \tau_{b, V_0} < \infty, \quad 0 < \tan \tau_{a, V_2} < \tan \tau_{b, V_2} < \infty,$$

for $i$ that is odd. The number of intersections of the varieties $V(f_{a,j})$ and $V(f_{b,j})$ in $\text{int}(V_0 V_1 V_2 V_3)$ is odd.

**Proof.** If $i$ is even, the linearized equation

$$\mathcal{L}_{t,c,j}(k - 1, k, k + i + 1, k + i) = 0$$

at $V_1$ yields

$$\tan \tau_{c, V_1} = \frac{\omega_{k,k+1} (t_{k-1}) \Delta c_{k+i}}{\omega_{k,k+1} (t_{k+i+1}) \Delta c_{k-1}},$$

and

$$\mathcal{L}_{t,c,j}(k + 1, k, k + i - 1, k + i) = 0$$
similarly

\[ \tan \tau_{c,V_3} = \frac{\omega'_{k,k+1}(t_{k+1})}{\omega'_{k,k+i}(t_{k+i-1})} \frac{\Delta c_{k+i-1}}{\Delta c_k} \]

at \( V_3 \). The asserted inequalities follow then from (27), (29), and (33). We omit the odd case \( i \) part of the proof. \( \square \)

Let us verify now that (3) and (4) imply the determinant sign assumptions of the previous lemmas of this section.

**Lemma 7.** Let \( k, \ell, i \) be indices that satisfy \( 0 \leq k < \ell < i < 2n - 1 \). Suppose that

\[ \Delta a_k > 0, \Delta b_k > 0, \Delta a_\ell > 0, \Delta b_\ell > 0, \Delta a_i > 0, \]

and

\[ \begin{vmatrix} \Delta a_k & \Delta a_\ell \\ \Delta b_k & \Delta b_\ell \end{vmatrix} > 0, \quad \begin{vmatrix} \Delta a_\ell & \Delta a_i \\ \Delta b_\ell & \Delta b_i \end{vmatrix} > 0 \]

Then

\[ \begin{vmatrix} \Delta a_k & \Delta a_i \\ \Delta b_k & \Delta b_i \end{vmatrix} > 0. \] (34)

**Proof.** If the assumptions hold, but not (34), we observe

\[ \Delta b_k \geq \Delta b_i \Delta a_k \Delta a_i > \Delta b_\ell \Delta a_\ell \Delta a_i > \Delta b_k \Delta a_\ell \Delta a_k = \Delta b_k, \]

a contradiction. \( \square \)

Figure 12: Interlacing of the varieties \( \mathcal{V}(f_{a,j}) \) and \( \mathcal{V}(f_{b,j}) \) in \( \text{int}(V_0V_1V_2V_3) \) for even and odd \( i \), left and right respectively.
Let the assumptions of Theorem 1 be satisfied. A brief look reveals that
\[ \Delta a_k \Delta b_i - \Delta a_i \Delta b_k > 0, \quad 0 \leq k < i \leq 2n - 1, \]
clearly follows from Lemma 7, so all the assumptions in Lemma 3, Lemma 4, Lemma 5, and Lemma 6 are met too. Let us introduce further discussion by

Figure 13: The cubic case, \( S \in \mathbb{R}^4 \), and the functions \( f_1 = (f_{a,1}, f_{b,1})^T \) considered. The boundary simplex \( S_{\{0\}} \) in the \( (t_2, t_3, t_4)^T \)-subspace, with a particular choice of \( T_0T_1 \) and \( V_0V_1 \) based upon the same choice of \( t_4 \) (left), and the corresponding intersections discussed in Lemma 5 or Lemma 6 in \( (t_1, t_2, t_3)^T \)-subspace of \( \text{int}(S) \) (right). The dash-dotted curve shows \( \mathcal{V}_S(f_1) \) for the particular choice of \( t_4 \). It connects \( Q_3Q_4 \) and \( Q_0Q_4 \).

a cubic example (Fig. 13). In this case, \( S \) lives in \( \mathbb{R}^4 \), and only projections are available. The left image shows the boundary simplex \( S_{\{0\}} \) in \( (t_2, t_3, t_4)^T \)-subspace. The line segments \( T_0T_1 \) and \( V_0V_1 \), referenced in Lemma 5, Lemma 6 and Lemma 4,Lemma 6 respectively, are obtained at \( t_4 = \text{const} \). The functions \( f_{a,1}, f_{b,1} \) are negative inside \( S_{\{0\}} \), and they change the sign when crossing through the faces \( S_{\{1,0\}} := \text{co}\{Q_2Q_3Q_4\} \) and \( S_{\{0,3\}} \) to the simplices \( S_{\{1\}} \) and \( S_{\{3\}} \) respectively. The right image shows the corresponding \( S \) projection at \( t_4 = \text{const} \). Curves shown at \( T_0T_1T_2 \) or \( V_0V_1V_2V_3 \) are parts of \( \mathcal{V}(f_{a,1}) \) and \( \mathcal{V}(f_{b,1}) \). Near \( S_{\{0,1\}} \) or \( S_{\{0,1\}} \) they both could be expressed as continuous functions of variables other than \( t_1 \). The variety \( \mathcal{V}_S(f_1) \) as dash-dotted curve connects points at \( S_{\{0,1,2\}} = Q_3Q_4 \) and \( S_{\{1,2,3\}} = Q_0Q_4 \). Let us denote \( t_{\{\ell\}} := (t_1, \ldots, t_{\ell-1}, t_{\ell+1}, \ldots, t_{2n-2})^T \).

**Lemma 8.** Suppose the assumptions of Theorem 1 are satisfied. Let \( M \) stand either for \( S_{\{j-1,j-1+k\}} \) or \( S_{\{n+j-1-k,n+j-1\}} \), with \( 1 \leq k \leq n \). Then
(i) if $k$ is even, $\text{int}(\mathcal{M}) \cap \mathcal{V}_S(f_{e,j}) = \emptyset$,
(ii) if $k$ is odd, $\mathcal{M} \subset \mathcal{V}_S(f_{e,j})$. The variety could be, close enough to $\mathcal{M}$, expressed as a continuous function
\[
\mathcal{V}_S(f_{e,j}): \mathbb{R}^{2n-3} \to \mathbb{R}: t_{\setminus \{t\}} \mapsto \mathcal{V}_S(f_{e,j})\left(t_{\setminus \{t\}}\right),
\]
with $\ell = j$ for the first case of $\mathcal{M}$, and $\ell = n + j - 1$ for the second one.
(iii) $\text{int}(\mathcal{M}) \cap \mathcal{V}_S(f_j) = \emptyset$,
(iv) $S_{\setminus \{j-1\}} \cap \mathcal{V}_S(f_j) = S_{\setminus \{j-1\}} \cap \mathcal{V}_S(f_j) = S_{\setminus \{n+j-3,n+j-2,n+j-1\}}$, and $\mathcal{V}_S(f_j)$ connects $S_{\setminus \{j-1,j,j+1\}}$ with $S_{\setminus \{n+j-3,n+j-2,n+j-1\}}$.

**Proof.** Lemma 4 allows one to apply previous lemmas of this section. The assertion (ii) is confirmed by Lemma 4 and Lemma 5. So is (iii), but only on int $(\mathcal{M})$. Since $f_{e,j}$ is a polynomial, vanishing at $\partial S_{\setminus \{j-1\}}$ and $\partial S_{\setminus \{n+j-1\}}$, we may extend the conclusion to the entire $\mathcal{M}$ by continuity. For an even $k$, (iii) follows from (ii) and Lemma 5. Lemma 6 implies it for an odd one. From (iii) it follows that $\mathcal{V}_S(f_j)$ may include only a part of $\partial \mathcal{M}$, i.e., parameter vectors $t$ where at least three of the parameters
\[ t_{j-1}, t_j, \ldots, t_{n+j} \]
coincide. Here, $t_{j-1}$ and $t_{n+j}$ are assumed to be constant. Recall Lemma 5 and Lemma 6, the intersections $T_0 T_1 T_2$ or $V_0 V_1 V_2 V_3$ studied there provided the basic step of the (iii) proof. In particular, $T_0 T_1 T_2$ keeps $\mathcal{V}_S(f_j)$ trapped inside the triangle independently of the boundary as long it remains nontrivial, i.e., $t_{j-1} < t_{j+2}$ for the first case of $\mathcal{M}$, and $t_{n+j-3} < t_{n+j}$ for the second one. Thus one may apply continuity again, and the first part of (iv) is confirmed. Any intersection considered in Lemma 5 or Lemma 6 contains an odd number of points of $\mathcal{V}_S(f_j)$, so this joint variety must continuously connect $S_{\setminus \{j-1,j,j+1\}}$ and $S_{\setminus \{n+j-3,n+j-2,n+j-1\}}$. This concludes the last part of the proof. □

Let us denote $\Gamma_j^0 := \mathcal{V}_S(f_{e,j}) \cap \partial S_{\setminus \{j-1\}}$, and let $t^0 \in \Gamma_j^0$. By Lemma 7 there exists $\varepsilon(t^0) > 0$ such that the variety $\mathcal{V}_S(f_{e,j})$ could be expressed as a continuous function of variables $t_{\setminus \{t\}}, t \in \mathcal{S}$, determined by the implicit function theorem from
\[ f_{e,j}(t_{j-1}, \mathcal{V}_S(f_{e,j}), t_{j+1}, \ldots, t_{n+j}) = 0, \]
as long as they satisfy $\|t_{\setminus \{t\}} - t_{\setminus \{t\}}\| < \varepsilon(t^0)$, where $\|\cdot\|$ denotes the Euclidean norm. Since $\Gamma_j^0$ is compact, there exists a smallest, but positive $\varepsilon(t^0)$,
\[ \varepsilon_1^* := \min_{t^0 \in \Gamma_j^0} \varepsilon(t^0) > 0, \]
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that holds for both \( f_{a,j} \) and \( f_{b,j} \). So the domains of \( \mathcal{V}_S(f_{a,j}) \) and \( \mathcal{V}_S(f_{b,j}) \) as functions include \( \Gamma(j,\varepsilon) \), with

\[
\Gamma(j,\varepsilon) := \left\{ t \in S \mid \text{dist}(t_{\setminus(j)}, \Gamma(j,\varepsilon)) < \varepsilon \right\}.
\]

Let us elaborate \( \mathcal{V}_S(f_j) \) near \( \mathcal{S}_{\setminus(j-1,j+1)} \) further. Since

\[
V(t_{j-1}, \ldots, \varepsilon) = \begin{cases} 
V(t_{j-1}, \ldots, t_{k+1}, \ldots, t_{n+j}) = \\
V(t_{j-1}, \ldots, t_{j+2}) V(t_{j+3}, \ldots, t_{n+j}) j+2 \prod_{i=1}^{n+j} (t_m - t_i), \\
j \leq k \leq j+2, \\
V(t_{j-1}, \ldots, t_{j+2}) V(t_{j+3}, \ldots, t_{k+1}, \ldots, n+j) j+3 \prod_{i=1}^{n+j} (t_m - t_i), \\
j+3 \leq k \leq n+j,
\end{cases}
\]

and \( V(t_{j-1}, \ldots, t_{j+2}) \) is by the order 3 smaller than \( V(t_{j-1}, \ldots, t_{k+1}, \ldots, t_{j+2}) \) there, \( f_{c,j} \) in (10) simplifies to

\[
f_{c,j} = (c_j - c_{j-1}, c_{j+1} - c_{j-1}, c_{j+2} - c_{j-1}) v_j + \ldots,
\]

where dots denote higher order terms, and

\[
v_j := \left( V(t_{j-1}, t_{j+1}, t_{j+2}), -V(t_{j-1}, t_j, t_{j+2}), V(t_{j-1}, t_j, t_{j+1}) \right)^T.
\]

Thus the main part of the variety \( \mathcal{V}(f_j) \) is determined from

\[
B_j v_j = 0, \quad B_j := \begin{bmatrix} a_j - a_{j-1} & a_{j+1} - a_j & a_{j+2} - a_{j-1} \\
b_j - b_{j-1} & b_{j+1} - b_{j-1} & b_{j+2} - b_{j-1} \end{bmatrix}.
\] (35)

Note that (35) is just the quadratic case considered in Remark 4, with indices 0, 1, \ldots been replaced by \( j - 1, j, \ldots \). The equation (35) can be written as \( v_j \in \ker B_j \), where \( \ker B_j \) is spanned by a cofactor vector

\[
\begin{pmatrix}
| a_{j+1} - a_{j-1} & a_{j+2} - a_{j-1} \\
| b_{j+1} - b_{j-1} & b_{j+2} - b_{j-1} \\
- | a_j - a_{j-1} & a_{j+2} - a_{j-1} \\
| b_j - b_{j-1} & b_{j+2} - b_{j-1} \\
| a_j - a_{j-1} & a_{j+1} - a_{j-1} \\
| b_j - b_{j-1} & b_{j+1} - b_{j-1} |
\end{pmatrix}.
\] (36)
All the determinants in (36) are positive by Lemma 7, and the corresponding parameters
\[ t_{j-1+k} = t_{j-1} + \text{const}_{j,k} \xi, \quad \text{const}_{j,k} > 0, \quad k = 1, 2, 3, \quad \xi \geq 0, \]
could be derived similarly to (32). They depend on the data involved in (36) only. Thus we have established
\[ V_S(f_j) = t^0 + \xi h_j + \ldots, \quad t^0 \in S_{\{j-1,j,j+1\}}, \quad \xi \geq 0, \quad \xi \text{ small enough}, \]
(37)
Suppose now additionally \( j < n - 1 \), \( t^0 \in V_S(f_{j+1}) \). By Lemma 8 this is possible only if \( t^0 \in S_{\{j-1,j,j+1,j+2\}} \), and we may use (37) again, with \( j \to j + 1 \). An application of the Cauchy–Schwarz inequality
\[ 0 < h_j^T h_{j+1} = \text{const}_{j,1} \text{const}_{j+1,1} + \text{const}_{j,2} \text{const}_{j+1,2} \leq \sqrt{\text{const}_{j,2}^2 + \text{const}_{j,3}^2} \sqrt{\text{const}_{j+1,1}^2 + \text{const}_{j,3}^2} < \| h_j \| \| h_{j+1} \| \]
\[ (38) \]
verifies that directions of \( h_j \) and \( h_{j+1} \) are separated by a constant angle \( \psi_j \in (0, \frac{\pi}{2}) \) since the inequality (38) is strict. Based upon this discussion we are able now to prove the major observation of this section.

**Lemma 9.** Suppose the assumptions of Theorem 1 are satisfied. The functions \( f_{a,j}, f_{b,j} \) can be modified to \( \tilde{f}_{a,j}, \tilde{f}_{b,j} \) in such a way that
\[ \text{sign} \tilde{f}_{a,j}(t) = (-1)^n, \quad t \in S_{\{j-1\}}, \quad \text{sign} \tilde{f}_{a,j}(Q_j) = (-1)^{n+1}, \]
and
\[ \text{sign} \tilde{f}_{b,j}(t) = 1, \quad t \in S_{\{n+j-1\}}, \quad \text{sign} \tilde{f}_{b,j}(Q_{n+j-1}) = -1. \]
The substitution \( f_{a,j} \to \tilde{f}_{a,j}, \quad f_{b,j} \to \tilde{f}_{b,j} \) doesn’t bring about any additional solution of the system (9).

**Proof.** Let \( 0 < \eta < \tilde{\eta} \ll 1 \), and let
\[ \chi_W : \mathbb{R}^{2n-2} \to \mathbb{R} : t \mapsto \chi_W(t) := \chi_{W,\eta,\tilde{\eta}}(t) := \begin{cases} 1, & \text{dist}(t, W) \leq \eta, \\ 0, & \text{dist}(t, W) \geq \tilde{\eta}, \\ 0 < \cdot < 1, & \text{otherwise}, \end{cases} \]
\[ (41) \]
The parameter $\tilde{t}_{j+2}$ is chosen small enough so that the varieties $V_S(f_{a,j})$ and $V_S(f_{b,j})$ could be expressed as functions, and $V_S(f_j)$ as a curve parameterized by $t_{j+2}$.

Denote a smooth characteristic wrapper function of a set $W \subset \mathbb{R}^{2n-2}$ with $\text{dist}$ denoting Hausdorff distance based upon Euclidean norm. Let us consider (39) first. From Lemma 1 and Lemma 8 (i) we conclude that there are only three boundary parts of $S_{\{j-1\}}$ that require a particular attention:

(i) $\mathcal{M}(i) = S_{\{j-1,j+1,j+2\}}$,
(ii) $\mathcal{M}(ii) = S_{\{j-1,j,j+1\}} \setminus \mathcal{M}(i)$,
(iii) $\mathcal{M}(iii) = \text{int} \left( S_{\{j-1,j-1+k\}} \right)$, with $k$ odd.

If $t^0 \in S_{\{j-1,j+1\}}$, Lemma 8 implies that there exists

$$\tilde{t}_{j+2} = \tilde{t}_{j+2} \left( t^0_{j-1} \right), \quad 0 < \tilde{t}_{j+2} < \varepsilon^*_1,$$

such that the varieties $V_S(f_{a,j})$ and $V_S(f_{b,j})$ restricted to a tetrahedron $\mathcal{T}$,

$$\mathcal{T}: \ t^0_{j-1} \leq t_j \leq t_{j+1} \leq t_{j+2} \leq \tilde{t}_{j+2} \leq t^0_{j+3},$$

are functions of variables $t_{j+1}$ and $t_{j+2}$ (Fig. 14). By (37), we may assume that $V_S(f_j)$ is a curve parameterized by $t_{j+2} \in [t^0_{j-1}, \tilde{t}_{j+2}]$. If not, a smaller $\tilde{t}_{j+2} > 0$ can be found that allows this assumption. Let us now choose somewhat arbitrary

$$x := (x_i)^T := -t^0 + V_S(f_j)_{t_{j+2} = \tilde{t}_{j+2}}, \quad \tilde{\eta} = \frac{2}{3} \|x\|, \quad \eta = \frac{2}{3} \tilde{\eta}, \quad \varrho := x_j.$$
A modification

\[ f_{a,j} \rightarrow \tilde{f}_{a,j} := f_{a,j} + (-1)^n \varrho \chi_{S_j \cup S_{j+1}} \]

by Lemma 2 doesn’t introduce any additional solution of (9) provided the corresponding modified part of \( V_{\mathcal{S}}(\tilde{f}_j) \) doesn’t meet \( V_{\mathcal{S}}(f_{j+1}) \). This is obvious for the set \( M_{(ii)} \), and also by (37) and (38) for the case \( M_{(i)} \) if \( \tilde{t}_{j+2} \) is small enough (Fig. 15). However, \( S_{\{j-1,j,j+1\}} \) is compact, so the smallest values \( \tilde{t}_{j+2}, \tilde{\eta} \) and \( \varrho \) can be found, and the modification (40) works uniformly. Note that this modification covers also the points of \( M_{(iii)} \) that are close enough to \( S_{\{j-1\}} \). For those that are not, consider \( t^0 \in V_{\mathcal{S}}(\tilde{f}_{a,j}) \cup \partial S_{\{j-1\}} \). Lemma 8 implies that in small enough neighbourhood

\[ \{ t \in \text{int}(S) \mid \|t^0 - t\| < \varepsilon(t^0) \} \]

the varieties \( V_{\mathcal{S}}(\tilde{f}_{a,j}) \) and \( V_{\mathcal{S}}(f_{b,j}) \) are separated. Since \( V_{\mathcal{S}}(\tilde{f}_{a,j}) \cup \partial S_{\{j-1\}} \) is compact, the smallest \( \varepsilon \) exists. So one may again construct a modification of the type (41) that will by Lemma 2 introduce no additional solutions of (9). By combining both wrappers we finally obtain the admissible modification of \( f_{a,j} \) at \( S_{\{j-1\}} \)

\[ f_{a,j} \rightarrow \tilde{f}_{a,j} := f_{a,j} + (-1)^n \varrho^* \chi_{S_{\{j-1\}}}, \quad \varrho^*_a > 0, \quad (42) \]
that removes also $V(f_{a,j}) \setminus V_S(f_{a,j})$ from the simplex boundary and proves the first part of (39). The first part of (40) follows in a similar way. We have to verify only that a modification $f_{a,j} \to \tilde{f}_{a,j}$ doesn’t spoil the arguments used in the proof of (39). At the common face of $S_{\{j-1\}}$ and $S_{\{n+j-1\}}$ we observe $S_{\{j-1\}} \cup V_S(f_{c,j}) = \emptyset$

if $n$ is even since

$$\text{sign} f_{c,j}(t) = (-1)^n = 1 = \text{sign} f_{c,j}(\tilde{t}), \quad t \in S_{\{j-1\}}, \tilde{t} \in S_{\{n+j-1\}}.$$ 

If $n$ is odd, modifications $\tilde{f}_{a,j} - f_{a,j}, \tilde{f}_{b,j} - f_{b,j}$ are of opposite sign, so also the first assertion of (40) is confirmed. Suppose now that $f_{a,j} \to \tilde{f}_{a,j}, f_{b,j} \to \tilde{f}_{b,j}$ has been carried out as in (42), so $\text{sign} f_{b,j}(t) = 1, \ t \in S_{\{n+j-1\}}$. So one may $\tilde{f}_{a,j}$ by Lemma 8 further modify

$$\tilde{f}_{a,j} \to \tilde{f}_{a,j} + (-1)^{n+1} \varrho_{a,n+j-1} \chi_{Q_{n+j-1}}, \quad (43)$$

for sufficiently small $\varrho_{a,n+j-1} > 0$. The modification (43) doesn’t introduce any new intersection between $\tilde{f}_{a,j}$, and $\tilde{f}_{b,j}$. The argument works at $Q_{n+j-1}$ too. This completes the proof. \[ \square \]

5. Proof of Theorem 1

Let us apply [9, Theorem 2.1] on

$$\tilde{f} := \left( \tilde{f}_{a,1}, \tilde{f}_{b,1}, \ldots, \tilde{f}_{a,n-1}, \tilde{f}_{b,n-1} \right)^T,$$

derived from $f$ by Lemma 9. The choice of the map $\sigma^*$ defined in (16) that determines the simplex-vertex pairs at which the functions $\tilde{f}_{c,j}$ are of different sign is straightforward,

$$\sigma^*(a, j) = j - 1, \quad \sigma^*(b, j) = n + j - 1, \quad j = 1, 2, \ldots, n - 1.$$ 

By Lemma 9,

$$\text{sign} \tilde{f}_{a,j}(t) = (-1)^n \neq \text{sign} \tilde{f}_{a,j}(Q_{j-1}) = (-1)^{n+1}, \quad t \in S_{\{j-1\}},$$

$$\text{sign} \tilde{f}_{b,j}(t) = 1 \neq \text{sign} \tilde{f}_{b,j}(Q_{n+j-1}) = -1, \quad t \in S_{\{n+j-1\}},$$

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and (17) is satisfied. With \( \sigma^{**} = n - 1 \), it remains to verify (18),

\[
\text{sign} \tilde{f}(t) \neq \text{sign} \tilde{f}(Q_{n-1}), \; t \in S_{\{n-1\}}.
\]  

(44)

Since \( 1 \leq j < n \), and \( Q_{n-1} \in S_{\{j-1,n+j-1\}} \), we have

\[
\text{sign} \tilde{f}_{a,j}(Q_{n-1}) = (-1)^n, \text{ sign} \tilde{f}_{b,j}(Q_{n-1}) = 1, \; j = 1, \ldots, n - 1.
\]

Lemma further reveals

\[
\text{sign} \tilde{f}_{a,j}(t) = (-1)^j, \; t \in \text{int}(S_{\{n-1\}}).
\]

For an odd \( n - j \) then

\[
\text{sign} \tilde{f}_{a,j}(t) \neq \text{sign} \tilde{f}_{a,j}(Q_{n-1}), \; t \in S_{\{n-1\}} \setminus \text{supp} \chi_{S_{\{j-1\}}}.
\]

Note that points in \( S_{\{n-1\}} \cap \text{supp} \chi_{S_{\{j-1\}}} \) are arbitrary close to \( S_{\{j-1,n-1\}} \). Similarly, for an odd \( j \), we conclude

\[
\text{sign} \tilde{f}_{b,j}(t) \neq \text{sign} \tilde{f}_{b,j}(Q_{n-1}), \; t \in S_{\{n-1\}} \setminus \text{supp} \chi_{S_{\{n+j-1\}}}.
\]

Tab. 1 summarizes the sign distribution detected. If \( n \) is odd, the pairs \( \tilde{f}_{a,j} \) and

| \( n \) | \( j \) | \( n - j \) | \( n - j \) at \( S_{\{n-1\}} \) | \( n - j \) at \( Q_{n-1} \) | \( n - j \) at \( S_{\{n-1\}} \) | \( n - j \) at \( Q_{n-1} \) |
|-----|-----|-----|----------------|----------------|----------------|----------------|
| odd | odd | even | \(-1\) | \(-1\) | \(-1\) | \(1\) |
| odd | even | odd | \(1\) | \(-1\) | \(1\) | \(1\) |
| even | odd | odd | \(-1\) | \(1\) | \(-1\) | \(1\) |
| even | even | even | \(1\) | \(1\) | \(1\) | \(1\) |

Table 1: Sign distribution of \( f_{a,j}, \tilde{f}_{a,j}, f_{b,j}, \tilde{f}_{b,j} \) at \( S_{\{n-1\}} \) and \( Q_{n-1} \).

\( \tilde{f}_{b,j} \) demonstrate similar behavior regardless of \( j \) been even or odd. Suppose it is odd. Then there is no \( \chi_{S_{\{j-1\}}} \cap \chi_{S_{\{n-1\}}} \) except near \( S_{\{j-1,n-1\}} \cap S_{\{n-1\}} \). On the other hand, \( \text{sign} \tilde{f}_{a,j}(Q_{j-1}) = 1 \), and \( \text{sign} \tilde{f}_{a,j}(Q_{j-1}) \in \{0,1\} \) except near \( S_{\{j-1\}} \cap \chi_{S_{\{n+j-1\}}} \). So \( \tilde{f}_{j} \) confirms (44) except possibly near

\[
S_{\{j-1\}} \cap S_{\{n-1\}} \cap S_{\{n+j-1\}} = S_{\{j-1,n-1,n+j-1\}}.
\]
But
\[
\bigcap_{j=1}^{n-1} S \{j-1, n-1, n+j-1\} = \emptyset,
\]
what confirms (44) for an odd \( n \). If \( n \) is even, \( \tilde{f}_j \) confirms again (44) except possibly near

\[
S \{j-1\} \cap S \{n-1\} \cap S \{n+j-1\} = S \{j-1, n-1, n+j-1\}.
\]

To observe this note that \( \tilde{f}_{a,j} \) verifies (44) except close to \( S \{j-1\} \cap S \{n-1\} \) for an odd \( j \), and close to \( S \{n+j-1\} \cap S \{n-1\} \) for an even one. The rest of the proof is obvious. This finally completes the proof of Theorem 1. \( \square \)

A brief inspection of the proofs of Theorem 1 and the lemmas of Section 4 it is based upon reveals that there is no need for the constant \( \Delta a, \Delta b, \Delta c, \Delta d \) to be the same for distinct values of \( j \) as long as their required sign behavior is preserved. The following corollary illuminates this fact.

**Corollary 1.** Suppose that the data (11) that determine \( f_{a,j} \) and \( f_{b,j} \) depend on \( j \),

\[
\Delta a_\ell = \Delta a_{\ell,j}, \quad \Delta b_\ell = \Delta b_{\ell,j}, \quad \ell = j-1, j, \ldots, n+j-1.
\]

Suppose that the following assumptions are fulfilled for each \( j, j = 1, 2, \ldots, n-1 \), separately: the data differences

\[
\Delta a_{\ell,j}, \quad \Delta b_{\ell,j}, \quad \ell = j-1, j, \ldots, n+j-1,
\]

are all positive or all negative, and determinants

\[
\begin{vmatrix}
\Delta a_{\ell,j} & \Delta a_{\ell+1,j} \\
\Delta b_{\ell,j} & \Delta b_{\ell+1,j}
\end{vmatrix}, \quad \ell = j-1, j, \ldots, n+j-2,
\]

are all of the same sign. The system of equations (9) has at least one solution.

**6. Proof of Theorem 2**

We will prove Theorem 2 with help of Remark 1 and Corollary 1. Without losing generality we may assume that the sign of determinants (5) is positive.
Consider fixed $j$, $1 \leq j \leq n - 1$. By Remark 1, we may modify the data (11) by a matrix product

$$M \begin{pmatrix} \Delta b_{n+j-1} & -\Delta a_{n+j-1} \\ -\Delta b_{j-1} & \Delta a_{j-1} \end{pmatrix}, \quad M := \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad \varepsilon > 0.$$ 

This yields the new data differences

$$\left( \begin{array}{c} \Delta \tilde{a}_\ell \\ \Delta \tilde{b}_\ell \end{array} \right) := M \left( \begin{array}{c} \Delta b_{n+j-1} - \Delta a_{n+j-1} \\ -\Delta b_{j-1} + \Delta a_{j-1} \end{array} \right) \left( \begin{array}{c} \Delta a_\ell \\ \Delta b_\ell \end{array} \right), \quad \ell = j - 1, j, \ldots, n + j - 1.$$

But

$$\Delta \tilde{a}_\ell = \left\| \begin{array}{cc} \Delta a_\ell & \Delta a_{n+j-1} \\ \Delta b_\ell & \Delta b_{n+j-1} \end{array} \right\| + \varepsilon \left\lVert \begin{array}{cc} \Delta a_{j-1} & \Delta a_\ell \\ \Delta b_{j-1} & \Delta b_\ell \end{array} \right\rVert > 0,$$

$$\Delta \tilde{b}_\ell = \varepsilon \left\lVert \begin{array}{cc} \Delta a_\ell & \Delta a_{n+j-1} \\ \Delta b_\ell & \Delta b_{n+j-1} \end{array} \right\rVert + \left\lVert \begin{array}{cc} \Delta a_{j-1} & \Delta a_\ell \\ \Delta b_{j-1} & \Delta b_\ell \end{array} \right\rVert > 0,$$

by the assumption of Theorem 2. Also,

$$\left\| \begin{array}{cc} \Delta \tilde{a}_\ell & \Delta \tilde{a}_{\ell+1} \\ \Delta \tilde{b}_\ell & \Delta \tilde{b}_{\ell+1} \end{array} \right\| = (1 - \varepsilon^2) \left\| \begin{array}{cc} \Delta a_{j-1} & \Delta a_{n+j-1} \\ \Delta b_{j-1} & \Delta b_{n+j-1} \end{array} \right\| \left\| \begin{array}{cc} \Delta a_\ell & \Delta a_{\ell+1} \\ \Delta b_\ell & \Delta b_{\ell+1} \end{array} \right\rVert > 0,$$

$$\ell = j - 1, j, \ldots, n + j - 2,$$

for $\varepsilon$ small enough. But then the assumptions of Corollary 1 are met. This concludes the proof of the theorem. \hfill \Box
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