Two Irrational Numbers That Give the Last Non-Zero Digits of \( n! \) and \( n^n \).

Gregory P. Dresden
Washington & Lee University
Lexington, VA 24450

Author’s Note: This is a slightly revised version of the article that appeared in print in Mathematics Magazine in October of 2001. The original proof of Theorem 2 was incorrect; I’ve fixed that mistake here. My thanks to Antonio M. Oller-Marcén and José Mara Grau for pointing out to me the error.

Also, Stan Wagon pointed out in a letter to Mathematics Magazine (February 2002) that the question of the periodicity of the last non-zero digit of \( n! \) (our Theorem 1) appeared several times in Crux Mathematicorum in the 1990’s: see v. 18 n. 7 (Sep 1992) page 196 for the statement of the problem, v. 19 n. 8 (Oct 1993) page 228 for an incorrect solution, v. 19 n. 9 (Nov 1993) page 260 for Stan Wagon’s correct solution, and v. 20 n. 2 (Feb 1994) page 44 for another reference.

I wrote a sequel to this paper, called “Three Transcendental Numbers From the Last Non-Zero Digits of \( n^n \), \( F_n \), and \( n! \)”. It appeared in Mathematics Magazine, April 2008.

We begin by looking at the pattern formed from the last (i.e. unit) digit of \( n^n \). Since \( 1^1 = 1 \), \( 2^2 = 4 \), \( 3^3 = 27 \), \( 4^4 = 256 \), and so on, we can easily calculate the first few numbers in our pattern to be 1, 4, 7, 6, 5, 6, 3, 6 . . . . We construct a decimal number \( N = 0.d_1d_2d_3\ldots d_n\ldots \) such that the \( n^{\text{th}} \) digit \( d_n \) of \( N \) is the last (i.e. unit) digit of \( n^n \); that is, \( N = 0.14765636\ldots \). In a recent paper [1], R. Euler and J. Sadek showed that this \( N \) is a rational number with a period of twenty digits:

\[
N = 0.14765636901636567490.
\]

This is a nice result, and we might well wonder if it can be extended. Indeed, Euler and Sadek in [1] recommend looking at the last non-zero digit of \( n! \) (If we just looked at the last digit of \( n! \), we would get a very dull pattern of all 0’s, as \( n! \) ends in 0 for every \( n \geq 5 \).

With this in mind, let’s define \( \lnzd(A) \) to be the last nonzero digit of the positive integer \( A \); it is easy to see that \( \lnzd(A) = A/10^i \mod 10 \), where \( 10^i \) is the largest
power of 10 that divides \( A \). We wish to investigate not only the pattern formed by \( \lnzd(n!) \), but also the pattern formed by \( \lnzd(n^n) \). In accordance with [1], we define the “factorial” number \( F = 0.d_1d_2d_3 \ldots d_n \ldots \) to be the infinite decimal such that each digit \( d_n = \lnzd(n!) \), and we define the “power” number \( P = 0.d_1d_2d_3 \ldots d_n \ldots \) to be the infinite decimal such that each digit \( d_n = \lnzd(n^n) \), and we ask whether these numbers are rational (i.e. are eventually-repeating decimals) or irrational.

Although the title of this article gives away the secret, we’d like to point out that at first glance, our “factorial” number \( F \) exhibits a suprisingly high degree of regularity, and a fascinating pattern occurs. The first few digits of \( F \) are easy to calculate:

\[
\begin{align*}
1! &= 1 & 5! &= 120 & 10! &= 3628800 \\
2! &= 2 & 6! &= 720 & 11! &= 39916800 \\
3! &= 6 & 7! &= 5040 & 12! &= 479001600 \\
4! &= 24 & 8! &= 40320 & 13! &= 6227020800 \\
9! &= 362880 & 9! &= 3628800 \ldots & 14! &= 87178291200
\end{align*}
\]

Reading the underlined digits, we have:

\[
F = 0.12642242888682\ldots
\]

Continuing along this path, we have (to forty-nine decimal places):

\[
F = 0.1264224288868244846448468868224282242866264\ldots
\]

It is not hard to show that (after the first four digits) \( F \) breaks up into five-digit blocks of the form \( x x 2x x 4x \), where \( x \in \{2, 4, 6, 8\} \), and the \( 2x \) and \( 4x \) are taken mod 10. Furthermore, if we represent these five-digit blocks by symbols (\( \hat{2} \) for 22428, \( \hat{4} \) for 44846, \( \hat{6} \) for 66264, \( \hat{8} \) for 88682, and \( \hat{1} \) for the initial four-digit block of 1264), we have:

\[
F = 0.\hat{1}2\hat{8}4\hat{8}\hat{4}\hat{2}\hat{2}\hat{6}\ldots
\]

Grouping these symbols into blocks of five and then performing more calculations (with the aid of Maple) give us \( F \) to 249 decimal places:

\[
F = 0.12884482262466848226482268644224668628842466824668\ldots
\]

The reader will notice additional patterns in these blocks of five symbols (twenty-five digits). In fact, such patterns exist for any block of size \( 5^i \). However, a pattern is
different from a period, and doesn’t imply that our decimal $F$ is rational. Consider the classic example of $0.1 \ 01 \ 001 \ 0001 \ 00001 \ \ldots$, which has an obvious pattern but is obviously irrational. It turns out that our decimal $F$ is also irrational, as the following theorem indicates:

**Theorem 1.** Let $F = 0.d_1d_2d_3 \ldots d_n \ldots$ be the infinite decimal such that each digit $d_n = \lnzd(n!)$. Then, $F$ is irrational.

As for our “power” number $P$, it too might seem to be rational at first glance. $P$ is only slightly different from Euler and Sadek’s rational number $N$, as seen here:

\[
N = 0.14765\ 6369\ 16365\ 6749\ 14765\ 6369\ 16365\ 6749\ \ldots
\]
\[
\text{and } P = 0.14765\ 6369\ 16365\ 6749\ 14765\ 6369\ 16365\ 6749\ \ldots
\]

(Again, calculations were performed by Maple.) Despite this striking similarity between $P$ and $N$, it turns out that $P$, like $F$, is irrational:

**Theorem 2.** Let $P = 0.d_1d_2d_3 \ldots d_n \ldots$ be the infinite decimal such that each digit $d_n = \lnzd(n^n)$. Then, $P$ is irrational.

Before we begin with the (slightly technical) proofs, let us pause and see if we can get a feel for why these two numbers must be irrational. There is no doubt that both $F$ and $P$ are highly “regular”, in that both exhibit a lot of repetition. The problem is that there are too many patterns in the digits, acting on different scales. Taking $P$, for example, we note that there is an obvious pattern (as shown by Euler and Sadek in [1]) repeating every 20 digits with $1^1, 2^2, 3^3, \ldots, 9^9$ and $11^{11}, 12^{12}, \ldots, 19^{19}$, but this is broken by a similar pattern for $10^{10}, 20^{20}, \ldots, 90^{90}$ and $110^{110} \ldots 190^{190}$, which repeats every 200 digits. This, in turn, is broken by another pattern repeating every 2000, and so on. A similar behaviour is found for $F$, but in blocks of 5, 25, 125, and so on, as mentioned above. So, in vague terms, there are always “new patterns” starting up in the digits of $P$ and of $F$, and this is what makes them irrational.

Are there some simple observations that we can make about $P$ and $F$ which might help us to prove our theorems? To start with, we might notice that every digit of $F$ (except for the first one) is even. Can we prove this? Yes, and without much difficulty:

**Lemma 1.** For $n \geq 2$, then $\lnzd(n!)$ is in $\{2, 4, 6, 8\}$.

**Proof:** The lemma is certainly true for $n = 2, 3, 4$. For $n \geq 5$, we note that the prime
factorization of $n!$ contains more 2’s than 5’s, and thus even after taking out all the 10’s in $n!$, the quotient will still be even. To be precise, the number of 5’s in $n!$ (and thus the number of trailing zeros in its base-10 representation) is $e_5 = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor$, which is strictly less than the number of 2’s, $e_2 = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor$ (here, $\lfloor \cdot \rfloor$ represents the greatest integer function). Hence, $n!/10^{e_5}$ is an even integer not divisible by 10, and so $\lnzd(n!) = n!/10^{e_5} \mod 10$, which must be in $\{2, 4, 6, 8\}$. This completes the proof.

Another helpful observation is to note that the $\lnzd$ function appears to be multiplicative. For example,

$$\lnzd(12) \cdot \lnzd(53) = 2 \cdot 3 = 6,$$

and

$$\lnzd(12 \cdot 53) = \lnzd(636) = 6.$$

However, we note that at times this “rule” fails:

$$\lnzd(15) \cdot \lnzd(22) = 5 \cdot 2 = 10,$$

yet

$$\lnzd(15 \cdot 22) = \lnzd(330) = 3.$$

So, we can only prove a limited form of multiplicativity, but it is useful none the less:

**Lemma 2.** Suppose $a, b$ are integers such that $\lnzd(a) \neq 5$, $\lnzd(b) \neq 5$. Then, $\lnzd$ is multiplicative; that is, $\lnzd(a \cdot b) = \lnzd(a) \cdot \lnzd(b) \mod 10$.

**Proof:** Let $x'$ denote the integer $x$ without its trailing zeros; that is, $x' = x/10^i$, where $10^i$ is the largest power of 10 dividing $x$. (Note that $\lnzd(x) = x' \mod 10$.) By hypothesis, $a'$ and $b'$ are both $\neq 0 \mod 5$, and so $(a \cdot b)' \neq 0 \mod 5$ and so $(a \cdot b)' = a' \cdot b'$. Thus,

$$\lnzd(a \cdot b) = \lnzd((a \cdot b)') = \lnzd(a' \cdot b') = a' \cdot b' \mod 10$$

$$= (a' \mod 10) \cdot (b' \mod 10) = \lnzd(a') \cdot \lnzd(b') = \lnzd(a) \cdot \lnzd(b).$$

This completes the proof.

We are now ready to supply the proof of Theorem 1, in which we show that $F$ is irrational. The proof is a little technical, but it relies first on assuming that $F$ has a repeating decimal expansion, then on choosing an appropriate multiple of the period $\lambda_0$ and choosing an appropriate digit $d$, in order to arrive at a contradiction.

**Proof of Theorem 1:** We argue by contradiction. Suppose $F$ is rational. Then $F$ is eventually periodic; let $\lambda_0$ be the period (i.e. for every $n$ sufficiently large, then
$d_n = d_{n+\lambda_0}$. Write $\lambda_0 = 5^i \cdot K$ such that $5/\lambda$ (we acknowledge that $K$ could be 1) and let $\lambda = 2^i \cdot \lambda_0 = 10^i \cdot K$. Then, $\lnzd(\lambda) = \lnzd(K)$, and since $5/\lambda$, then $10/\lambda$ and so $\lnzd(K) = K \mod 10$. Note also that $\lnzd(2\lambda) = 2K \mod 10$. Choose $M$ sufficiently large so that both of the following are true: $\lnzd(10^M + \lambda) = \lnzd(\lambda)$ (this can easily be done by demanding that $10^M > \lambda$), and for all $n \geq M$, then $d_n = d_{n+\lambda_0}$, which of course would then equal $d_{n+\lambda}$. Finally, let $d = \lnzd((10^M - 1)!)$. By Lemma 1, $d \in \{2, 4, 6, 8\}$, and since $10^M! = (10^M - 1)! \cdot 10^M$, then $d$ also equals $\lnzd(10^M!)$.

Since $\lambda$ is a multiple of the period $\lambda_0$, then if we let $A = 10^M - 1 + \lambda$ and $B = 10^M - 1 + 2\lambda$, then:

$$d = \lnzd((10^M - 1)!) = \lnzd(A!) = \lnzd(B!)$$

and

$$d = \lnzd(10^M!) = \lnzd((A + 1)!) = \lnzd((B + 1)!)$$

Let’s now look at the last two terms in the above equation; it is here we will find our contradiction. Note that since $\lnzd(A!) = d$, then $\lnzd(A!) \neq 5$. Also, since $\lnzd(A + 1) = \lnzd(10^M + \lambda) = \lnzd(\lambda) = K \mod 10$, we know that $\lnzd(A + 1) \neq 5$. Thus, we can apply Lemma 2 to $\lnzd(A! \cdot (A + 1))$ to get:

$$d = \lnzd((A + 1)!) = \lnzd(A!) \cdot \lnzd(A + 1) = d \cdot K \mod 10.$$

Likewise, working with $B$, we find:

$$d = \lnzd((B + 1)!) = \lnzd(B!) \cdot \lnzd(B + 1) = d \cdot 2K \mod 10.$$

Combining these two equations, we get:

$$d = dK \mod 10 \quad \text{and} \quad d = 2dK \mod 10,$$

and this becomes $d(1 - K) = 0 = d(1 - 2K)$ mod 10. Since $d$ is even, this implies that $1 - K = 0 \mod 5$ and $1 - 2K = 0 \mod 5$, which is a contradiction. Thus, there can be no period $\lambda_0$ and so $F$ is irrational. This completes the proof.

We now turn our attention to the “power” number $P$ derived from the last non-zero digits of $n^a$. This part was more difficult, but a major step was the discovery that the sequence $\lnzd(100^{100})$, $\lnzd(200^{200})$, $\lnzd(300^{300})$ . . . was the same as the sequence $\lnzd(100^4)$, $\lnzd(200^4)$, $\lnzd(300^4)$ . . . This relies not only on the fact that $4|100$ but also on the fact that $a^b = a^{b+4} \mod 10$ for $b > 0$, as used in the following lemma:

**Lemma 3.** Suppose $100 \mid x$. Then, $\lnzd(x^x) = (\lnzd x)^4 \mod 10$. 


Proof: As in Lemma 2, let $x'$ denote the integer $x$ without its trailing zeros; that is, $x' = x/10^i$, where $10^i$ is the largest power of 10 dividing $x$. Now,

$$\lnzd(x^x) = \lnzd((10^i x')^{10^i x'})$$
$$= \lnzd((10^i x')(x')^{10^i x'})$$
$$= \lnzd((x')^{10^i x'}).$$

Since $10 \nmid x'$, then $10 \nmid (x')^{10^i x'}$, and so:

$$\lnzd(x^x) = (x')^{10^i x'} \mod 10.$$  

Since $100 \mid x$, then $4 \mid 10^i \cdot x'$, and since $(x')^n = (x')^{n+4} \mod 10$ for every positive $n$, we have:

$$\lnzd(x^x) = (x')^4 \mod 10$$
$$= (\lnzd x)^4 \mod 10.$$  

This completes the proof.

With Lemma 3 at our disposal, the proof of Theorem 2 is now fairly easy.

Proof of Theorem 2: Again, we argue by contradiction. Suppose $P$ is rational. Let $\lambda_0$ be the period, and choose $j$ sufficiently large such that $10^j > 100(\lambda_0 + 1)!$ and such that $\lnzd((10^j + n\lambda_0)^{10^j + n\lambda_0}) = \lnzd((10^j)^{10^j})$ for every positive $n$. Choosing $n = 100(\lambda_0 + 1)(\lambda_0 - 1)!$, we get:

$$\lnzd((10^j + 100(\lambda_0 + 1)!)^{10^j + 100(\lambda_0 + 1)!}) = \lnzd((10^j)^{10^j}).$$

We reduce the left side of the above equation by Lemma 3 and the right side is obviously 1, so we have:

$$(\lnzd(10^j + 100(\lambda_0 + 1)!))^4 \mod 10 = 1,$$

but since $10^j > 100(\lambda_0 + 1)!$ and $\lnzd(100(\lambda_0 + 1)!) = \lnzd((\lambda_0 + 1)!)$, we can rewrite the above equation as:

$$(\lnzd(\lambda_0 + 1)!)^4 \mod 10 = 1.$$  

Note that by Lemma 1, the only values of $\lnzd((\lambda_0 + 1)!)$ are 2, 4, 6, and 8, and raising these to the fourth power mod 10 gives us:

$$6 = 1,$$

which is a contradiction. Thus, $P$ is irrational. This completes the proof.

We close by asking the obvious (and very difficult) question: Are $F$ and $P$ algebraic or transcendental? I suspect the latter, but it is only a hunch, and I hope some curious reader will continue along this interesting line of study.
References

[1] R. Euler and J. Sadek, A number that gives the unit digit of $n^n$, *Journal of Recreational Mathematics*, 29 (1998) No. 3, pp. 203–4.