Product Rigidity in Von Neumann and C*-Algebras Via S-Malleable Deformations

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Abstract: We provide a new large class of countable icc groups $\mathcal{A}$ for which the product rigidity result from Chifan et al. (Geom Funct Anal 26(1): 136–159, 2016) holds: if $\Gamma_1, \ldots, \Gamma_n \in \mathcal{A}$ and $\Lambda$ is any group such that $L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda)$, then there exists a product decomposition $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ such that $L(\Lambda_i)$ is stably isomorphic to $L(\Gamma_i)$, for any $1 \leq i \leq n$. Class $\mathcal{A}$ consists of groups $\Gamma$ for which $L(\Gamma)$ admits an s-malleable deformation in the sense of Sorin Popa and it includes all non-amenable groups $\Gamma$ such that either (a) $\Gamma$ admits an unbounded 1-cocycle into its left regular representation, or (b) $\Gamma$ is an arbitrary wreath product group with amenable base. As a byproduct of these results, we obtain new examples of W*-superrigid groups and new rigidity results in the C*-algebra theory.

1. Introduction

Every countable group $\Gamma$ gives rise to the group von Neumann algebra $L(\Gamma)$ by considering the weak operator closure of the complex group algebra $\mathbb{C}[\Gamma]$ acting on the Hilbert space $l^2(\Gamma)$ by left convolution [MvN43]. A main theme in operator algebras is the classification of group von Neumann algebras which is centered around the following question: what properties of the group $\Gamma$ are remembered by $L(\Gamma)$? This problem is the most interesting when $\Gamma$ is icc (i.e., all non-trivial conjugacy classes of $\Gamma$ are infinite), which corresponds to $L(\Gamma)$ being a II$_1$ factor. In the amenable case, the classification is completed by the work of Connes [Co76] which asserts that any two icc amenable groups give rise to isomorphic von Neumann algebras. Therefore, besides the amenability of the group, no information can be recovered from $L(\Gamma)$ when $\Gamma$ is icc amenable.

In the non-amenable case the situation is radically different and far more complex. An outstanding progress has been achieved since the invention of Popa’s deformation/rigidity theory [Po07] and there have been discovered many instances when various
algebraic and analytic properties of a group $\Gamma$ can be recovered from $L(\Gamma)$, see the surveys [Va10a,Io12b,Io17]. Remarkably, Ioana, Popa and Vaes found in [IPV10] the first class of countable groups $\Gamma$ that are $W^*$-superrigid. Roughly speaking, this means that the group $\Gamma$ is completely recovered by its von Neumann algebra $L(\Gamma)$. Subsequently, several other classes of $W^*$-superrigid groups have been found [BV12,Be14,Ci17,CD-AD20].

However, in general, one can only expect to recover certain aspects of a group $\Gamma$ from its von Neumann algebra $L(\Gamma)$. We only highlight the following developments. Ozawa showed in [Oz03] that the group von Neumann algebra of a non-amenable bi-exact icc group is prime, in particular implying that $L(\Gamma) \not\cong L(\Gamma_1 \times \Gamma_2)$, for all infinite groups $\Gamma_1$ and $\Gamma_2$. Ozawa and Popa then proved that any tensor product of $\text{II}_1$ factors of non-amenable hyperbolic icc groups admits a unique prime decomposition into prime factors [OP03]. As a corollary, their result shows that if $L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda_1 \times \cdots \times \Lambda_m)$ for some icc hyperbolic groups $\Gamma_i$’s and infinite groups $\Lambda_j$’s, then $m = n$ and after a permutation of indices we have $L(\Gamma_i)$ is stably isomorphic to $L(\Lambda_i)$, for all $1 \leq i \leq n$. More recently, Chifan, de Santiago and Sinclair strengthened the previous corollary of [OP03] by discovering the following product rigidity phenomenon: if the groups $\Gamma_i$’s are icc hyperbolic, then any group $\Lambda$ such that $L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda)$ admits a decomposition $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ satisfying $L(\Gamma_i)$ is stably isomorphic to $L(\Lambda_i)$, for any $1 \leq i \leq n$ [CdSS15]. While a large number of other unique prime factorization results have been obtained since [OP03] (see, e.g., the introduction of [DH16]), the above product rigidity result has been extended to the class of non-amenable bi-exact groups only very recent [CD-AD20]. Some of the methods from [CD-AD20], including an augmentation technique, will play an important role in our work as well.

The goal of this paper is to provide a new class of countable groups, denoted Class $\mathcal{A}$, for which the above product rigidity holds, see Theorem A. A common feature of these groups is that their von Neumann algebras admit an s-malleable deformation in the sense of Popa [Po01,Po03] (see Definition 3.1) and a key ingredient of the proof of our first main result is the use of Popa’s spectral gap principle that was developed in [Po06a,Po06b]. In fact, Theorem A follows from a more general and conceptual result, see Theorem 5.2.

**Class $\mathcal{A}$**. We say that a countable non-amenable icc group $\Gamma$ belongs to Class $\mathcal{A}$ if $\Gamma$ satisfies one of the following conditions:

1. $\Gamma$ admits an unbounded cocycle for some mixing representation $\pi : \Gamma \to \mathcal{O}(H_\mathbb{R})$ such that $\pi$ is weakly contained in the left regular representation of $\Gamma$.
2. $\Gamma = \Gamma_1 \ast\Sigma \Gamma_2$ is an amalgamated free product group satisfying $[\Gamma_1 : \Sigma] \geq 2$ and $[\Gamma_2 : \Sigma] \geq 3$, where $\Sigma < \Gamma$ is an amenable almost malnormal\(^1\) subgroup.
3. $\Gamma = \Sigma \rtimes_{G/H} G$ is a generalized wreath product group with $\Sigma$ amenable, $G$ non-amenable and $H < G$ is an amenable almost malnormal subgroup.

For any $i \in \{1, 3\}$, if $\Gamma$ belongs to $\mathcal{A}$ satisfying condition (i), then we say that $\Gamma$ belongs to $\mathcal{A}_i$. Note that $\mathcal{A}_i$ contains all non-amenable icc groups $\Gamma$ with $\beta_1^{(2)}(\Gamma) > 0$.

**Theorem A.** Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 1$ countable groups that belong to $\mathcal{A}$ and denote $M = L(\Gamma)$. Let $\Lambda$ be any countable group such that $M^t = L(\Lambda)$ for some $t > 0$.

Then there exist a product decomposition $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$, a unitary $u \in \mathcal{U}(M^t)$ and some positive numbers $t_1, \ldots, t_n$ with $t_1 \cdots t_n = t$ such that $uL(\Lambda_i)u^* = L(\Gamma_i)^{t_i}$, for any $i \in \{1, n\}$.

\(^1\) A subgroup $H < G$ is called almost malnormal if $gHg^{-1} \cap H$ is finite for any $g \in G \setminus H$. 

Remark that [CdSS15, Theorem A] shows that if the groups $\Gamma_i$’s are non-amenable free groups and $L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda)$ for some group $\Lambda$, then $\Lambda$ admits a product decomposition into $n$ infinite groups. Theorem A strengthens this fact by replacing the groups $\Gamma_i$’s with the more general class of arbitrary free product groups.

To put our result into a better perspective, we note that Popa’s deformation/rigidity theory led to striking rigidity results for group von Neumann algebras of wreath product groups. To recall this results, fix a non-trivial abelian group $A$. Popa showed in [Po03, Po04] that the von Neumann algebras $L(A \wr \Gamma_1)$ are pairwise non-isomorphic for different icc property (T) groups $\Gamma_1$. This result was strengthened by Ioana, Popa and Vaes in [IPV10] by showing that for any icc property (T) group $\Gamma_1$, the isomorphism $L(A \wr \Gamma_1) \cong L(\Lambda)$ implies that there exists a semi-direct product decomposition $\Lambda = B \rtimes \Lambda_1$ such that $\Gamma_1 \cong \Lambda_1$. Several other rigidity results have been obtained for group von Neumann algebras of (generalized) wreath product groups, including primeness, unique prime factorization and relative solidity, see [Io06,Po06a,CI08,IPV10,SW11,BV12,IM19]. Theorem A provides on the other hand a new general rigidity result by showing that the von Neumann algebra of a product of wreath product groups with amenable base completely remembers the product structure.

**Remark 1.1.** We also emphasize the following new rigidity phenomenon that Theorem A (more precisely, Theorem 5.2) leads to. First, we consider the class $\mathcal{M}_0$ of all non-amenable II_1 factors $M$ that admit an $s$-malleable deformation $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}})$ with the properties that $M \subset \tilde{M}$ is mixing, $_MM^2(M) \otimes L^2(M)_M$ is weakly contained in the coarse bimodule $\mathcal{M}^2(M) \otimes L^2(M)_M$ and $\alpha_t$ does not converge uniformly on $(M)_1$ (see also Sect. 3.2). In Theorem 5.2 we classify all tensor product decompositions of any group von Neumann algebra for which the tensor factors belong to $\mathcal{M}_0$. In contrast to the product rigidity result from [CdSS15], the II_1 factors from $\mathcal{M}_0$ are not necessarily group von Neumann algebras. For instance, any tracial non-amenable free product $M_1 \ast M_2$ belongs to $\mathcal{M}_0$, see Remark 3.6.

Next, we show that Theorem A can be used together with [IPV10] to derive new examples of W*-superrigid product groups. First, recall that a countable group $\Gamma$ is W*-superrigid if for any group $\Lambda$ and any *-isomorphism $\theta : L(\Gamma)^t \to L(\Lambda)$ for some $t > 0$, we have $t = 1$ and there exist a group isomorphism $\delta : \Gamma \to \Lambda$, a character $\omega : \Gamma \to \mathbb{T}$ and a unitary $w \in L(\Lambda)$ such that $\theta(u_\delta) = \omega(\delta)wv_\delta(w)^*$. Here, we denoted by $\{u_\delta\}_{\delta\in\Gamma}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ the canonical generating unitaries of $L(\Gamma)$ and $L(\Lambda)$, respectively.

The first class of W*-superrigid product groups has recently been found in [CD-AD20] by considering products of W*-superrigid groups from [IPV10] that are bi-exact. As a consequence of Theorem A we can actually drop the bi-exactness assumption and therefore obtain that all products of W*-superrigid groups from [IPV10] are again W*-superrigid. To illustrate our result, we introduce the following class of groups that was considered in [IPV10].

**Class $\mathcal{I}\mathcal{P}\mathcal{V}$.** We say that a countable group $\Gamma$ belongs to Class $\mathcal{I}\mathcal{P}\mathcal{V}$ if $\Gamma = (\mathbb{Z}/n\mathbb{Z}) \wr I$ $G$ is a generalized wreath product group that satisfies:

- $n \in \{2, 3\}$ and $I = G/H$, where $H < G$ is an infinite amenable almost malnormal subgroup.
- $G$ admits an infinite normal subgroup that either has relative property (T) or its centralizer is non-amenable.

**Corollary B.** If $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of W*-superrigid groups that belong to $\mathcal{A}$ (e.g., $\Gamma_i \in \mathcal{I}\mathcal{P}\mathcal{V}$ for any $i$), then $\Gamma$ is W*-superrigid.
The problem of proving that the W*-superrigid property is closed with respect to direct products is notoriously hard and remains open. In Corollary B (see also Theorem 5.2) we make some progress on this problem, by showing that within the class of generalized wreath product groups with almost malnormal stabilizers, the W*-superrigidity property is preserved by taking direct products.

Finally, we will discuss some applications of Theorem A to the C*-algebra theory. In contrast to the von Neumann algebra setting, the classification of reduced C*-algebras is not governed by an amenable/non-amenable dichotomy in the sense that the reduced group C*-algebra $C_r^*(\Gamma)$ of an amenable group $\Gamma$ does not provide the same striking lack of rigidity as its von Neumann algebra $L(\Gamma)$. In fact, any torsion free abelian group $\Gamma$ is C*-superrigid [Sc74, Theorem 1], which roughly means that $C_r^*(\Gamma)$ completely remembers the group $\Gamma$. By building upon the result of [Sc74], several other classes of amenable C*-superrigid groups have been found, see the introduction of [ER18]. In the non-amenable case, the only examples of C*-superrigid groups are obtained in [CI17, CD-AD20] via Popa’s deformation/rigidity theory using their von Neumann algebraic superrigid behavior combined with the unique trace property [BKKO14] of their reduced C*-algebra. In a similar way, the von Neumann rigidity result from Theorem A can be transferred to a product rigidity in C*-algebras.

Corollary C. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 1$ countable non-amenable icc groups such that $\beta_1^{(2)}(\Gamma_i) > 0$ or $\Gamma_i \in \mathcal{A}_2$, for any $i \in \overline{1,n}$. Let $\Lambda$ be any countable group satisfying $C_r^*(\Gamma) = C_r^*(\Lambda)$.

Then $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ admits a product decomposition into infinite groups such that $L(\Lambda_i)$ is stably isomorphic to $L(\Gamma_i)$, for any $i \in \overline{1,n}$.

We note that the above product rigidity has already been obtained for icc hyperbolic groups [CdSS15] and extended to infinite direct sums of icc hyperbolic property (T) groups [CU18]. Finally, it would be interesting to show that in certain cases the conclusion of Corollary C can be strengthened to derive that the C*-algebras of the factors $C_r^*(\Gamma)$ and $C_r^*(\Lambda)$ are isomorphic for any $i \in \overline{1,n}$.

Outline of the proof of Theorem A. We outline briefly and informally the proof of our main result, which is Theorem A. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 1$ countable groups that belong to Class $\mathcal{A}$; to better illustrate the proof, we assume that $\beta_1^{(2)}(\Gamma_i) > 0$, for any $i \in \overline{1,n}$. Denote $M_i = L(\Gamma_i)$, for any $i \in \overline{1,n}$ and let $M = L(\Gamma)$.

Our goal is to show that for any group von Neumann algebra decomposition $M^t = L(\Lambda)$, where $t > 0$ and $\Lambda$ is a countable group, the underlying group is a product group $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$. To simplify notation, we assume $t = 1$. In order to attain this goal, we use a number of techniques from Popa’s deformation/rigidity theory. Following [PV09], we define the *-homomorphism $\Delta : M \to M \bar{\otimes} M$ by letting $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$, for any $\lambda \in \Lambda$.

Throughout the proof we repeatedly use Popa’s spectral gap rigidity principle together with the s-malleable deformation constructed by Sinclair [Si10] for all $L(\Gamma_i)$’s. Their first combined application shows that
\[ \forall i \in \overline{1,n}, \exists \ell_i \in \overline{1,n} \text{ such that } \Delta(M_{\ell_i}) \prec M \bar{\otimes} M_{\ell_i}. \tag{1.1} \]

Here, $P < Q$ denotes that a corner of $P$ embeds into $Q$ inside the ambient algebra, in the sense of Popa [Po03] and $M_{\ell} = \bar{\otimes}_{j \neq i} M_j$. Next, by combining Ioana’s ultrapower technique [Io11] with some recent techniques in the framework of von Neumann algebras without property Gamma [BMO19,IM19], we are able to upgrade (1.1) to
\[ \Delta(M_I) \prec M \bar{\otimes} M_I, \text{ for any } I \subset \overline{1,n}, \tag{1.2} \]
We also derive that for any subset $I \subset \overline{1, n}$ there exists a subgroup $\Sigma_I < \Lambda$ with non-amenable centralizer $C_\Lambda(\Sigma)$ such that

$$L(\Sigma_I) < M_I \text{ and } M_I < L(\Sigma_I).$$

(1.3)

Here, we denoted by $M_I = \bigotimes_{i \in I} M_i$. This is achieved in Sect. 4. Next, by using some general recent results on s-malleable deformations [dSHHS20] and an augmentation technique developed in [CD-AD20] we are able to use (1.2) and (1.3) to show that there exists a non-zero projection $e \in L(\Sigma_n)'^\perp \cap M$ such that

$$L(\Sigma_n)e \prec^e M_n \text{ and } M_n < L(\Sigma_n)e.$$  

(1.4)

Here, we denoted by $P \prec^e Q$ to indicate that $Pp' \prec Q$ for any non-zero projection in the relative commutant of $P$. Relation (1.3) shows in particular that

$$L(\Sigma_n) < M_n \text{ and } M_n < L(\Sigma_n).$$

(1.5)

Finally, by building upon some results from [DHI16], we show that (1.4) and (1.5) imply that there exists a product decomposition $\Lambda = \Lambda_n \times \Lambda_n$ such that $M_n = L(\Lambda_n)$ and $M_n = L(\Lambda_n)$, modulo unitary conjugacy and amplifications. The result follows now by induction.

**Organization of the paper.** Besides the introduction there are four other sections in the paper. In Sect. 2 we review Popa’s intertwining-by-bimodules techniques and some other tools. In Sect. 3 we recall some general properties of s-malleable deformations and some structural results for $L(\Gamma)$, when $\Gamma \in \mathcal{A}$. Next, in Sect. 4 we review Ioana’s ultrapower technique and present an important corollary which, besides being of independent interest, will be used in the proof of Theorem A. Finally, in Sect. 5 we prove our main results.

2. Preliminaries

2.1. Terminology. In this paper we consider *tracial von Neumann algebras* $(M, \tau)$, i.e. von Neumann algebras $M$ equipped with a faithful normal tracial state $\tau : M \to \mathbb{C}$. This induces a norm on $M$ by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$, for all $x \in M$. We will always assume that $M$ is a separable von Neumann algebra, i.e. the $\|\cdot\|_2$-completion of $M$ denoted by $L^2(M)$ is separable as a Hilbert space. We denote by $\mathcal{Z}(M)$ the *center of $M$* and by $\mathcal{U}(M)$ its *unitary group*. For two von Neumann subalgebras $P_1, P_2 \subset M$, we denote by $P_1 \vee P_2 = W^*(P_1 \cup P_2)$ the von Neumann algebra generated by $P_1$ and $P_2$.

All inclusions $P \subset M$ of von Neumann algebras are assumed unital. We denote by $E_P : M \to P$ the unique $\tau$-preserving *conditional expectation* from $M$ onto $P$, by $e_P : L^2(M) \to L^2(P)$ the orthogonal projection onto $L^2(P)$ and by $(M, e_P)$ the Jones’ basic construction of $P \subset M$. We also denote by $P' \cap M = \{ x \in M | x y = y x, \text{ for all } y \in P \}$ the relative commutant of $P$ in $M$ and by $\mathcal{N}_M(P) = \{ u \in \mathcal{U}(M) | u P u^* = P \}$ the *normalizer of $P$ in $M$*. We say that $P$ is *regular* in $M$ if the von Neumann algebra generated by $\mathcal{N}_M(P)$ equals $M$. Next, we define the quasi-normalizer $\mathcal{Q} \mathcal{N}_M(P)$ as the subgroup of all elements $x \in M$ for which there exist $x_1, \ldots, x_n \in M$ such that $P x \subset \bigcup_i x_i P$ and $x P \subset \bigcup_i P x_i$ (see [Po99, Definition 4.8]).

The *amplification* of a II$_1$ factor $(M, \tau)$ by a number $t > 0$ is defined to be $M^t = p(\mathbb{B}(L^2(\mathbb{Z})) \bar{\otimes} M)p$, for a projection $p \in \mathbb{B}(L^2(\mathbb{Z})) \bar{\otimes} M$ satisfying $(\text{Tr} \otimes \tau)(p) = t$. Here $\text{Tr}$ denotes the usual trace on $\mathbb{B}(L^2(\mathbb{Z}))$. Since $M$ is a II$_1$ factor, $M^t$ is well defined.
Note that if $M = P_1 \bar{\otimes} P_2$, for some $\II_1$ factors $P_1$ and $P_2$, then there exists a natural identification $M = P_1^t \bar{\otimes} P_2^{1/t}$, for every $t > 0$.

For a countable group $\Gamma$ and for two subsets $S, T \subset \Gamma$, we denote by $C_S(T) = \{g \in S | gh = hg, \text{ for all } h \in T\}$ the centralizer of $T$ in $S$. Given a group inclusion $\Sigma < \Gamma$, the quasi-normalizer $QN_\Gamma(\Sigma)$ is the group of all $g \in \Gamma$ for which exists a finite set $F \subset \Gamma$ such that $\Sigma g \subset F \Sigma$ and $g \Sigma \subset \Sigma F$; equivalently, $g \in QN_\Gamma(\Sigma)$ if and only if $[\Sigma : g\Sigma g^{-1} \cap \Sigma] < \infty$ and $[\Sigma : g^{-1}\Sigma g \cap \Sigma] < \infty$.

Finally, for a positive integer $n$, we denote by $1, n$ the set $\{1, \ldots, n\}$. For any subset $S \subset 1, n$ we denote its complement by $\hat{S} = 1, n \setminus S$. If $S = \{i\}$, we will simply write $\hat{i}$ instead of $\{i\}$. Also, given any product group $G = G_1 \times \cdots \times G_n$ and any tensor product $M_1 \bar{\otimes} \cdots \bar{\otimes} M_n$, we will denote their subproduct supported on $S$ by $G_S = \times_{i \in S} G_i$ and $M_S = \bar{\otimes}_{i \in S} M_i$, respectively.

2.2. Intertwining-by-Bimodules. We next recall from [Po03, Theorem 2.1 and Corollary 2.3] the powerful intertwining-by-bimodules technique of S. Popa.

**Theorem 2.1** ([Po03]). Let $(M, \tau)$ be a tracial von Neumann algebra and $P \subset pMp$, $Q \subset qMq$ be von Neumann subalgebras. Let $U \subset U(P)$ be a subgroup such that $U'' = P$.

Then the following are equivalent:

1. There exist projections $p_0 \in P$, $q_0 \in Q$, a $*$-homomorphism $\theta : p_0 P p_0 \rightarrow q_0 Q q_0$ and a non-zero partial isometry $v \in q_0 M p_0$ such that $\theta(x)v = vx$, for all $x \in p_0 P p_0$.
2. There is no sequence $(u_n) \subset U$ satisfying $\|E_Q(xu_n y)\|_2 \rightarrow 0$, for all $x, y \in M$.

If one of the equivalent conditions of Theorem 2.1 holds true, we write $P \prec_M Q$, and say that a corner of $P$ embeds into $Q$ inside $M$.

**Notation 2.2.** Throughout the paper we will use the following notation.

- If $Pp' \prec_M Q$ for any non-zero projection $p' \in P' \cap pMp$, then we write $P \prec_M^p Q$.
- If $P \prec_M Q q'$ for any non-zero projection $q' \in Q' \cap qMq$, then we write $P \prec_M^q Q$.
- If both $P \prec_M^p Q$ and $P \prec_M^q Q$ hold, then we write $P \prec_M^{s,s'} Q$.

2.3. From Tensor Decompositions to Group Product Decompositions. In this subsection we review how some intertwining relations and a tensor product decomposition in a group von Neumann algebra can be used to deduce a direct decomposition of the underlying group, see Theorem 2.3. The result is actually a mild generalization of [DHI16, Theorem 6.1]. The proof follows the same idea, but for the convenience of the reader we include all the details.

**Theorem 2.3** ([DHI16]). Let $M = L(\Gamma)$ be the $\II_1$ factor of an icc group $\Gamma$ and assume $M = P_1 \bar{\otimes} P_2$. Let $\Sigma_1, \Sigma_2 \subset \Gamma$ be subgroups and $e \in L(\Sigma_1)' \cap M$ a non-zero projection such that

1. $L(\Sigma_1)e \prec_M^p P_1$ and $P_1 \prec_M L(\Sigma_1)e$,
2. $L(\Sigma_2) \prec_M P_2$ and $P_2 \prec_M L(\Sigma_2)$.

Then there exist a decomposition $\Gamma \cong \Gamma_1 \times \Gamma_2$, a decomposition $M = P_1^{t_1} \bar{\otimes} P_2^{t_2}$ for some $t_1, t_2 > 0$ with $t_1 t_2 = 1$ and a unitary $u \in M$ such that $P_i^{t_i} = u L(\Gamma_i) u^*$, for any $i \in \overline{1, 2}$.
We first need the following two lemmas inspired by [DHI16, Section 6]. Lemma 2.5 below can also be seen as an extension of [Dri19, Lemma 2.6].

**Lemma 2.4.** Let $M = L(\Gamma)$ be the II$_1$ factor of an icc group $\Gamma$ and assume $M = P_1 \bar{\otimes} P_2$. Let $\Sigma < \Gamma$ be a subgroup and $e \in L(\Sigma)' \cap M$ a non-zero projection such that $L(\Sigma)e \prec_M P_1$ and $P_1 \prec_M L(\Sigma)e$.

Then $QN_{\Gamma}(\Sigma) < \Gamma$ has finite index.

**Proof.** We first use [DHI16, Lemma 2.4] and derive that there exists a non-zero projection $f \in Z(L(\Sigma)' \cap M)$ satisfying $f \preceq e$ and

$$L(\Sigma)f \prec_M P_1 \text{ and } P_1 \prec_M L(\Sigma)f.$$  \hfill (2.1)

Note that $N_M(L(\Sigma)' \cap M) \cap M \subset L(\Sigma)' \cap M$. Hence, by passing to relative commutants in (2.1) it follows from [Va08, Lemma 3.5] and [DHI16, Lemma 2.4] that

$$P_2 \prec_M (L(\Sigma)' \cap M)f \text{ and } (L(\Sigma)' \cap M)f \prec_M P_2.$$  \hfill (2.2)

By repeating the same argument, we deduce that $((L(\Sigma)' \cap M)' \cap M)f \prec_M P_1$. In combination with (2.2) we derive that $Z((L(\Sigma)' \cap M)f \prec_M P_1$ and $Z((L(\Sigma)' \cap M)f \prec_M P_2$. Therefore, by [DHI16, Lemma 2.8(2)] we further obtain that $Z((L(\Sigma)' \cap M)f \prec_M C_1$, and hence, there exists a non-zero projection $f_1 \in Z((L(\Sigma)' \cap M))$ with $f_1 \preceq f$ such that $Z((L(\Sigma)' \cap M)f_1 = C_1$.

By letting $Q = L(\Sigma) \vee (L(\Sigma)' \cap M)$, it follows that $Q' \cap M \subset Z(L(\Sigma)' \cap M)$, and therefore, $(Qf_1)' \cap f_1 Mf_1 = C_1$. Using relations (2.1) and (2.2) we have $P_1 \prec_M Qf_1$ and $P_2 \prec_M Qf_1$. Therefore, we can apply [Dri19, Lemma 2.6] and derive that $M \prec_M Q$. Let $\Omega = \{g \in \Gamma| O(\Sigma)(g) \text{ is finite}\}$, where $O(\Sigma)(g) = \{hgh^{-1}| h \in \Sigma\}$ is the orbit of $g$ under conjugation by $\Sigma$. Note that $\Omega$ is normalized by $\Sigma$ and $L(\Sigma)' \cap M$ is a subgroup in $L(\Omega)$. Since $\Omega \Sigma < QN_{\Gamma}(\Sigma)$, it follows that $Q \subset L(QN_{\Gamma}(\Sigma))$. Altogether, it follows that $M \prec_M L(QN_{\Gamma}(\Sigma))$, which proves the lemma by using [DHI16, Lemma 2.5(1)].

\[\Box\]

**Lemma 2.5.** Let $\Gamma \cong B$ be a trace preserving action and denote $M = B \times \Gamma$. Let $\Sigma_1 < \Sigma_2 < \Gamma$ be subgroups such that $L(QN_{\Gamma}(\Sigma_1))' \cap M = C_1$

Let $P_1, P_2 \subset pMa$ be von Neumann subalgebras such that there exist commuting subalgebras $P_0, P_1, P_2 \subset M$ satisfying $P_1 \subset P_1, P_2 \subset P_2$ and $P_0 \vee P_1 \vee P_2 = M$.

If $P_1 \prec_M B \times \Sigma_1$, for any i $\in \{1, 2\}$, then $P_1 \prec_P \vee P_2 \prec_M B \times \Sigma_2$.

**Proof.** Since $P_1 \prec_M B \times \Sigma_1$ and $P_2 \prec_M B \times \Sigma_2$, it follows that there exist projections $p_1 \in P_1, p_2 \in P_2$ and non-zero elements $v_1 \in p_1 M, v_2 \in M p_2$ such that

$$(p_1 p_1 p_1) v_1 \subset v_1 (B \times \Sigma_1)_1 \text{ and } v_2 (p_2 p_2 p_2) \subset (B \times \Sigma_2)_1 v_2.$$  \hfill (2.3)

We continue by showing that there exists $g \in QN_{\Gamma}(\Sigma_1)$ such that $v_1 u_g v_2 \neq 0$. If this is not the case, we derive that $v_1 z v_2 = 0$, for all $z \in U(L(QN_{\Gamma}(\Sigma_1)))$ and hence, $v_1 z v_2 z^\ast v_1 = 0$. By letting $a = v_1 z u_z (L(QN_{\Gamma}(\Sigma_1))) v_2 v_2^\ast$, it follows that $v_1 a = 0$. Since $a \in L(QN_{\Gamma}(\Sigma_1))' \cap M = C_1$, we derive that $v_1 = 0$ or $v_2 = 0$, contradiction.

Next, we consider $g \in QN_{\Gamma}(\Sigma_1)$ such that $v_1 u_g v_2 \neq 0$. This implies that there exist some elements $g_1, \ldots, g_n \in \Gamma$ such that $(B \times \Sigma_1)_1 u_g \subset \sum_{j=1}^n u_g (B \times \Sigma_1)_1$. Using (2.3), we derive that

$$U(p_1 p_1 p_1) (v_1 u_g v_2) U(p_2 p_2 p_2) \subset \sum_{i=1}^n v_1 u_g (B \times \Sigma_2)_1 v_2.$$  \hfill (2.4)
Now, we assume by contradiction that \( p_1 P_1 p_1 \lor p_2 P_2 p_2 \not\preceq_M B \rtimes \Sigma_2 \). Hence, there exist two sequences of unitaries \( (a_n)_n \subset \mathcal{U}(p_1 P_1 p_1) \) and \( (b_n)_n \subset \mathcal{U}(p_2 P_2 p_2) \) such that \( \| E_{B \rtimes \Sigma_2} (x a_n b_n y) \|_2 \to 0 \), for all \( x, y \in M \). Since \( P_1 \subset \tilde{P}_1, P_2 \subset \tilde{P}_2 \) and \( P_0 \lor \tilde{P}_1 \lor \tilde{P}_2 = M \), we further derive that \( \| E_{B \rtimes \Sigma_2} (x a_n (v_1 z v_2) b_n y) \|_2 \to 0 \), for all \( x, y \in M \).

For any subset \( F \subset \Gamma \), we denote by \( P_F \) the orthogonal projection onto the closed linear span of \( \{ Bu_g \mid g \in F \} \) and notice that \( \| P_{G \Sigma_2 H} (x) \|_2^2 \leq \sum_{g \in G, h \in H} \| E_{B \rtimes \Sigma_2} (u_g^* x u_h^*) \|_2^2 \), for all finite subsets \( G, H \subset \Gamma \) and \( x \in M \). Hence, the previous paragraph implies that \( \| P_{G \Sigma_2 H} (a_n (v_1 z v_2) b_n) \|_2 \to 0 \), for all finite subsets \( G, H \subset \Gamma \). By using Kaplansky’s density theorem, this contradicts (2.4) and therefore ends the proof of the lemma.

**Proof of Theorem 2.3.** From assumption we have \( L(\Sigma_1) <_M L(\Sigma_2) \cap M \). By applying [DHI16, Theorem 6.2] we denote that there exist finite index subgroups \( \Theta_1 < k \Sigma_1 k^{-1} \) and \( \Theta_2 < \Sigma_2 \) for some \( k \in \Gamma \) such that the commutator subgroup \( [\Theta_1, \Theta_2] \) is finite. Denote by \( \Theta < \Gamma \) the subgroup generated by \( \Theta_1 \) and \( \Theta_2 \).

By Lemma 2.4 we have \( [\Gamma : QN_F(\Sigma_1)] < \infty \). Since \( \Gamma \) is icc, we derive that \( L(Q \Sigma_1) \cap M = \mathbb{C}1 \). Therefore, by Lemma 2.5 we derive that \( M <_M L(\Theta) \), which shows that \( [\Gamma : \Theta] < \infty \). In particular, \( \Theta \) is icc. Since \( [\Theta_1, \Theta_2] \) is a finite normal subgroup of \( \Theta \), it must be trivial. Therefore, \( \Theta_1 \) and \( \Theta_2 \) are commuting subgroups of \( \Gamma \) such that \( \Theta_1 \cap \Theta_2 = 1 \) and \( [\Gamma : \Theta_1 \Theta_2] < \infty \).

Finally, by applying verbatim the second part of the proof of [DHI16, Theorem 6.1] (or the second part of the proof of [CdSS15, Theorem 4.14]), we obtain the conclusion of the theorem. \( \Box \)

### 2.4. An Augmentation Technique for Intertwining

An essential tool for the proof of our main result, Theorem A, is the augmentation technique developed in [CD-AD20]; this was also crucial for the recent work [CD-AD21]. One place where the augmentation technique is used in our paper is in the following result.

**Proposition 2.6.** Let \( \Gamma \) be a countable icc group and let \( M = L(\Gamma) \). Let \( \Sigma, \Theta < \Gamma \) be subgroups and \( M = P \hat{\otimes} Q \) a tensor product decomposition. Let \( G_0 \subset G \) be countable groups and let \( G \rtimes A \) be a trace preserving action such that \( Q = A \rtimes G_0 \). Assume that the following two conditions hold:

- \( P <_M L(\Sigma) \) and \( L(\Sigma) <_M P \).
- \( Q <_M L(\Theta) \) and \( L(\Theta) <_M P \hat{\otimes} (A \rtimes G_0) \).

Then \( [G : G_0] < \infty \).

**Proof.** Following the augmentation technique from [CD-AD20, Section 3], we consider a Bernoulli action \( \Gamma \rtimes D \) with abelian base. Denote \( M = D \rtimes \Gamma \) and let \( \Psi : M \to M \hat{\otimes} M \) be the \( * \)-homomorphism given by \( \Psi (d u_g) = d u_g \otimes u_g \), for all \( d \in D \) and \( g \in \Gamma \).

From [DHI16, Lemma 2.4(2)] we get that \( P <^s_M L(\Sigma) \), and hence, \( P <^{s,s'}_M D \rtimes \Sigma \). By using [DHI16, Remark 2.2] and [Dr19, Lemma 2.3] we further obtain that \( \Psi(P) <^{s,s'}_M \hat{\otimes} M \hat{\otimes} L(\Sigma) \). Using \( L(\Sigma) <_M P \) and [Dr19, Lemma 2.4(2)], we get \( \Psi(P) <^{s,s'}_M \hat{\otimes} M \hat{\otimes} P \). In the same way, we derive that \( \Psi(Q) <^{s,s'}_M \hat{\otimes} P \hat{\otimes} (A \rtimes G_0) \). Using [BV12, Lemma 2.3] we get that \( \Psi(M) <^{s,s'}_M \hat{\otimes} P \hat{\otimes} (A \rtimes G_0) \). Since \( \Psi(x) = x \otimes 1 \), for all \( x \in D \), we deduce that \( \Psi(M) <_M \hat{\otimes} P \hat{\otimes} (A \rtimes G_0) \). By [IPV10, Lemma 10.2] we obtain that \( M <_M P \hat{\otimes} (A \rtimes G_0) \). Since \( QN^{(1)}_M(M) = M \), we derive that \( M <_M P \hat{\otimes} (A \rtimes G_0) \), which implies from [DHI16, Lemma 2.5(1)] that \( [G : G_0] < \infty \). \( \Box \)
2.5. Relative Amenability and Weak Containment of Bimodules. A tracial von Neumann algebra \((M, \tau)\) is amenable if there exists a positive linear functional \(\Phi : \mathbb{B}(L^2(M)) \to \mathbb{C}\) such that \(\Phi|_M = \tau\) and \(\Phi\) is \(M\)-central, meaning \(\Phi(xT) = \Phi(Tx)\), for all \(x \in M\) and \(T \in \mathbb{B}(L^2(M))\). By Connes’ breakthrough classification of amenable factors [Co76], it follows that \(M\) is amenable if and only if \(M\) is approximately finite dimensional.

Next, we recall the notion of relative amenability introduced by Ozawa and Popa in [OP07]. Let \((M, \tau)\) be a tracial von Neumann algebra. Let \(p \in M\) be a projection and \(P \subseteq pMp\), \(Q \subseteq M\) be von Neumann subalgebras. Following [OP07, Definition 2.2], we say that \(P\) is amenable relative to \(Q\) inside \(M\) if there exists a positive linear functional \(\Phi : p\langle M, e_Q\rangle p \to \mathbb{C}\) such that \(\Phi|_{pMp} = \tau\) and \(\Phi\) is \(P\)-central. Note that \(P\) is amenable relative to \(Q\) inside \(M\) if and only if \(P\) is amenable. We say that \(P\) is strongly non-amenable relative to \(Q\) if \(Pp'\) is non-amenable relative to \(Q\) for any non-zero projection \(p' \in P' \cap pMp\).

Let \(M, N\) be tracial von Neumann algebras. An \(M\text{-}N\) bimodule \(M\mathcal{H}_N\) is a Hilbert space \(\mathcal{H}\) equipped with a \(\ast\)-homomorphism \(\pi_\mathcal{H} : M \otimes N^{\text{op}} \to \mathbb{B}(\mathcal{H})\) that is normal on \(M\) and \(N^{\text{op}}\), where \(M \otimes N^{\text{op}}\) is the algebraic tensor product between \(M\) and the opposite von Neumann algebra \(N^{\text{op}}\) of \(N\). For two \(M\text{-}N\) bimodules \(M\mathcal{H}_N\) and \(M\mathcal{K}_N\), we say that \(M\mathcal{H}_N\) is weakly contained in \(M\mathcal{K}_N\) if \(\|\pi_\mathcal{H}(x)\| \leq \|\pi_\mathcal{K}(x)\|\), for any \(x \in M \otimes N^{\text{op}}\). Examples of bimodules include the trivial bimodule \(M L^2(M)_M\) and the coarse bimodule \(M L^2(M) \otimes L^2(N)_N\).

Finally, if \(Q \subseteq M\) are tracial von Neumann algebras, then \(M L^2(\langle M, e_Q\rangle)_M \cong_M (L^2(M) \otimes_Q L^2(M))_M\). Also, a subalgebra \(P \subseteq pMp\) is amenable relative to \(Q\) if and only if \(p L^2(\langle M, e_Q\rangle)_M\) is weakly contained in \(p L^2(\langle M, e_Q\rangle)_M\).

2.6. Mixing Inclusion of Von Neumann Algebras. An important tool in the intertwining-by-bimodules techniques is the mixing notion relative to a subalgebra (see [Po05, Definition 2.9], [PS09, Definition 2.3] and [Bo14, Definition A.4.2]).

**Definition 2.7.** Let \(A \subseteq M \subseteq \hat{M}\) be an inclusion of tracial von Neumann algebras. We say that \(M \subseteq \hat{M}\) is mixing relative to \(A\) if for any sequence \((u_n)_n \subseteq (M)_{1}\) satisfying \(\|E_A(xu_n y)\|_2 \to 0\), for all \(x, y \in M\), we have \(\|E_M(\tilde{x} u_n \tilde{y})\|_2 \to 0\), for all \(\tilde{x}, \tilde{y} \in \hat{M} \ominus M\).

In Definition 2.7, if \(A = \mathbb{C} 1\) we simply say that \(M \subseteq \hat{M}\) is mixing. We record the following known lemmas and include the proof of the first one only for the convenience of the reader; the second one can be proven in a similar way.

**Lemma 2.8.** Let \(\Sigma < \Gamma\) be countable groups and denote \(I = \Gamma/\Sigma\). Let \(A_0 \subseteq B_0\) be tracial von Neumann algebras and denote \(M = A_0^I \times \Gamma\) and \(\hat{M} = B_0^I \times \Gamma\). Note that \(M \subseteq \hat{M}\).

Then \(M \subseteq \hat{M}\) is mixing relative to \(A_0^I \times \Sigma\).

**Proof.** Let \((w_n)_n \subseteq \mathcal{U}(M)\) be a sequence of unitaries such that \(\|E_{A_0^I \times \Sigma}(x w_n y)\|_2 \to 0\), for all \(x, y \in M\). We have to show that \(\|E_M(\tilde{x} w_n \tilde{y})\|_2 \to 0\), for all \(\tilde{x}, \tilde{y} \in \hat{M} \ominus M\). By Kaplansky’s density theorem, it is enough to assume \(\tilde{x} = b = \{b_0 \otimes A_0\}_h^k \Sigma^c\) for some \(h, k \in I\). If we let \(w_n = \sum_{g \in \Gamma} (w_n u_g) \in A_0^I \times \Gamma\), note that \(E_M(b w_n^g u_g(c)) = 0\) if \(g \notin h \Sigma^k - 1\) and hence

\[
\|E_M(\tilde{x} w_n \tilde{y})\|_2 \leq \|b\| \|c\| \| \sum_{g \in h \Sigma^k - 1} w_n^g u_g\|_2 = \|b\| \|c\| \|E_{A_0^I \times \Sigma}(u_{h^{-1} w_n u_k})\|_2 \to 0,
\]
Lemma 2.9. Let $\tilde{M} = M_1 \ast_A M_2$ be an amalgamated free product of tracial von Neumann algebras. Then $M_1 \subset \tilde{M}$ is mixing relative to $A$.

3. Two Classes of Von Neumann Algebras that Admit S-Malleable Deformations

3.1. Malleable Deformations. In [Po01, Po03] Popa introduced the notion of an s-malleable deformation of a von Neumann algebra. In the framework of his powerful deformation/rigidity techniques, this notion has led to a remarkable progress in the theory of von Neumann algebras, see the surveys [Po07, Va10a, Io12b, Io17]. See also [dSHHS20] for a comprehensive overview on s-malleable deformations and for recent developments.

Definition 3.1. Let $(M, \tau)$ be a tracial von Neumann algebra. A pair $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ is called an s-malleable deformation of $M$ if the following conditions hold:
- $(\tilde{M}, \tilde{\tau})$ is a tracial von Neumann algebra such that $M \subset \tilde{M}$ and $\tau = \tilde{\tau}|_M$.
- $(\alpha_t)_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M}, \tilde{\tau})$ is a 1-parameter group with $\lim_{t \to 0} \|\alpha_t(x) - x\|_2 = 0$, for any $x \in \tilde{M}$.
- There exists $\beta \in \text{Aut}(\tilde{M}, \tilde{\tau})$ that satisfies $\beta|_M = \text{Id}_M$, $\beta^2 = \text{Id}_{\tilde{M}}$ and $\beta \alpha_t = \alpha_t \beta$, for any $t \in \mathbb{R}$.
- $\alpha_t$ does not converge uniformly to the identity on $(M)_1$ as $t \to 0$.

Theorem 3.2. ([dSHHS20]) Let $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ be an s-malleable deformation of a tracial von Neumann algebra $M$. Let $A \subset M$ and $Q \subset qMq$ be some von Neumann subalgebras such that $M \subset \tilde{M}$ is mixing relative to $A$, $\alpha_t \to \text{Id}$ uniformly on $(Q)_1$ and $Q \not\subset M A$. Then the following hold:
1. $\alpha_t \to \text{Id}$ uniformly on $(Q \lor (Q' \cap qMq))_1$.
2. Let $Q = Q_1 \subset Q_2 \subset \cdots \subset qMq$ be an ascending sequence of von Neumann subalgebras with $\alpha_t \to \text{Id}$ uniformly on $(Q_i)_1$, for all $i \geq 1$. Then $\alpha_t \to \text{Id}$ uniformly on $(\bigvee_{i \geq 1} Q_i)_1$.

Proof. Part (1) follows from [dSHHS20, Corollary 6.7(ii)]. For part (2), by using [dSHHS20, Lemma 6.3] it follows that $Q' \cap q\tilde{M}q \subset qMq$ and the conclusion follows from [dSHHS20, Theorem 3.5].

We will also need the following result from [dSHHS20, Proposition 5.6].

Proposition 3.3. ([dSHHS20]) Let $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ be an s-malleable deformation of a tracial von Neumann algebra $M$. Let $Q \subset qMq$ be a von Neumann subalgebra and let $q_0 \in Q$ be a non-zero projection such that $\alpha_t \to \text{Id}$ uniformly on $(q_0Qq_0)_1$.

Then $\alpha_t \to \text{Id}$ uniformly on $(Qz)_1$, where $z$ is the central support of $q_0$ in $Q$.

3.2. Class $\mathcal{M}$. We say that a non-amenable II$_1$ factor $M$ is in Class $\mathcal{M}$ if there exists an s-malleable deformation $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ of $M$ and an amenable subalgebra $A \subset M$ satisfying:
1. The inclusion $M \subset \tilde{M}$ is mixing relative to $A$. 
(2) For any tracial von Neumann algebra $N$ and for any subalgebra $P \subset p(M \bar{\otimes} N) \rho$ such that $P' \cap p(M \bar{\otimes} N) \rho$ is strongly non-amenable relative to $1 \otimes N$, it follows that:

(i) $\alpha_t \otimes \text{id} \to \text{id}$ uniformly on $(P)_1$.
(ii) If $P \prec_{M \bar{\otimes} N} A \otimes N$, then $P \prec_{M \bar{\otimes} N} 1 \otimes N$.

While this class of II$_1$ factors seems somewhat technical, it actually contains all group von Neumann algebras $L(\Gamma)$ with $\Gamma \in A$ and all non-trivial tracial free products $M_1 \ast M_2$, see Proposition 3.4 bellow and its proof. Note also that if $A = \mathbb{C}1$, then condition (2) is simply reflecting Popa’s spectral gap principle.

**Proposition 3.4.** If $\Gamma \in A$, then $L(\Gamma) \in \mathcal{M}$.

**Proof.** If $\Gamma \in A_1$ we recall that Sinclair constructed in [Si10, Section 3] an $s$-malleable deformation $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ in the sense of Definition 3.1; see also [Va10b, Section 3.1] and [Io11, Section 2]. We will prove that $M \in \mathcal{M}$ with $A = \mathbb{C}1$. Note first that $M \subset \tilde{M}$ is mixing since $\pi$ is a mixing representation. Next, Lemma 3.5 bellow shows that condition (2) of Class $\mathcal{M}$ is satisfied.

Next, if $\Gamma = \Gamma_1 \ast \Sigma \Gamma_2 \in A_2$, we recall that [IP05, Section 2.2] shows that $M = L(\Gamma)$ admits an $s$-malleable deformation $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ in the sense of Definition 3.1 with $\tilde{M} = M \ast L(\Sigma)(L(\Gamma_1) \bar{\otimes} L(\mathbb{F}_2))$. It follows that $M \in \mathcal{M}$ with $A = L(\Sigma)$. Indeed, we note first that $M \subset \tilde{M}$ is mixing relative to $A$ by Lemma 2.9. Next, since we have the decomposition $M \bar{\otimes} N = (L(\Gamma_1) \bar{\otimes} N) \ast L(\Sigma) \bar{\otimes} N (L(\Gamma_2) \bar{\otimes} N)$, condition (2.i) follows from [Io12a, Lemma 6.5]. To show condition (2.ii), let $N$ be a tracial von Neumann algebra and $P \subset p(M \bar{\otimes} N) \rho$ a subalgebra such that $P' \cap p(M \bar{\otimes} N) \rho$ is strongly non-amenable relative to $1 \otimes N$. If $P \prec_{M \bar{\otimes} N} A \otimes N$ and $P \not\prec_{M \bar{\otimes} N} 1 \otimes N$, then we derive (see, e.g., [Dr17, Proposition 3.7]) that $P' \cap p(M \bar{\otimes} N) \rho \prec_{M \bar{\otimes} N} A \otimes N$. It implies by [DH16, Lemma 2.6(3)] that $P' \cap p(M \bar{\otimes} N) \rho$ is not strongly non-amenable relative to $1 \otimes N$, contradiction.

Finally, if $\Gamma \in A_3$, we recall that Ioana constructed in [Io06, Section 2] an $s$-malleable deformation $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ in the sense of Definition 3.1, see Remark 3.8(1). Next, we note that $M \subset \tilde{M}$ is mixing relative to $A := L(\Sigma)^{G/H} \rtimes H$ by Lemma 2.8. Next, parts (i) and (ii) from condition (2) of Class $\mathcal{M}$ follow from [IPV10, Corollary 4.3] and its proof. □

The following lemma is a standard application of Popa’s spectral gap rigidity principle [Po06b] and it is essentially contained in the proof of [Ho15, Theorem 6.4] (see also [Io11, Lemma 2.2]). For completeness, we include a proof.

**Lemma 3.5.** Let $\Gamma$ be a countable non-amenable group that admits an unbounded cocycle for some mixing representation $\pi : \Gamma \to O(H_\mathbb{R})$ such that $\pi$ is weakly contained in the left regular representation of $\Gamma$. Let $\Gamma \curvearrowleft N$ be a trace preserving action and denote $M = N \rtimes \Gamma$.

If $P \subset pMp$ is a von Neumann subalgebra that is strongly non-amenable relative to $N$, then $\alpha_t \to \text{id}$ uniformly on $(P' \cap pMp)_1$.

**Proof.** Since $\pi$ is contained in the left regular representation, it follows by [Va10b, Lemma 3.5] that the $M-M$ bimodule $L^2(\tilde{M} \ominus M)$ is weakly contained in the $M-M$ bimodule $L^2(M) \otimes_N L^2(M)$. Since $P$ is strongly non-amenable relative to $N$, it follows that for any non-zero projection $z \in \mathcal{Z}(P' \cap pMp)$ we have that $M L^2(M)_{Pz}$ is not weakly contained in $M L^2(M) \otimes_N L^2(M)_{Pz}$. Hence, for any non-zero projection $z \in \mathcal{Z}(P' \cap pMp)$ we derive that $M L^2(M)_{Pz}$ is not weakly contained in $M L^2(M \ominus M)_{Pz}$. 

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Let $\epsilon > 0$. Therefore, by [IPV10, Lemma 2.3] we obtain that there exist $a_1, \ldots, a_n \in P$ and $\delta > 0$ such that if $x \in (p\tilde{M}p)_1$ satisfies $\|xa_i - a_i x\|_2 \leq \delta$, for any $i \in \overline{1,n}$, then $\|x - E_M(x)\|_2 \leq \epsilon$. We choose $t_0 > 0$ such that $\|\alpha_t(a_i) - a_i\|_2 \leq \delta/2$, for all $|t| \leq t_0$ and $i \in \overline{1,n}$. Let $x \in (P' \cap pMp)_1$ and $t \in \mathbb{R}$ such that $|t| \leq t_0$. By using the triangle inequality we derive that for all $i \in \overline{1,n}$ and $|t| \leq t_0$ we have

$$\|a_i\alpha_t(x) - \alpha_t(x)a_i\|_2 = \|\alpha_{-t}(a_i)x - x\alpha_{-t}(a_i)\|_2 \leq 2\|\alpha_{-t}(a_i) - a_i\|_2 \leq \delta.$$ 

As a consequence, we deduce that $\|\alpha_t(x) - E_M(\alpha_t(x))\|_2 \leq \epsilon$, for any $|t| \leq t_0$. Using Popa’s transversality property [Va10b, Lemma 3.1] we obtain that

$$\|\alpha_t(x) - x\|_2 \leq \sqrt{2}\epsilon, \text{ for all } x \in (P' \cap pMp)_1 \text{ and } |t| \leq t_0.$$ 

This concludes the proof. \qed

**Remark 3.6.** The proof of Lemma 3.5 shows that the class $\mathcal{M}_0$ defined in Remark 1.1 is contained in $\mathcal{M}$ and that any non-amenable tracial free product $M_1 \ast M_2$ belongs to $\mathcal{M}_0$ (see also [Io12a, Lemma 2.10]).

We end this subsection by showing that any von Neumann algebra $M \in \mathcal{M}$ does not have property Gamma, i.e. for any uniformly bounded sequence $(x_n)_n \subset M$ with $\|x_n y - y x_n\|_2 \to 0$, for any $y \in M$, must satisfy $\|x_n - \tau (x_n)\|_2 \to 0$.

**Lemma 3.7.** Let $M$ be a tracial von Neumann algebra that belongs to Class $\mathcal{M}$. Then $M$ does not have property Gamma.

**Proof.** We assume the contrary. By using [HU15, Theorem 3.1] it follows that there exists a decreasing sequence of diffuse abelian von Neumann subalgebras $A_n \subset M$ with $n \geq 1$ such that $M = \bigvee_{n \geq 1} (A_n' \cap M)$. Let $(\bar{M}, (\alpha_t)_{t \in \mathbb{R}})$ be an s-malleable deformation of $M$ and $A \subset M$ a subalgebra as given by the definition of Class $\mathcal{M}$. Since $M$ is non-amenable, it follows that there exists $n \geq 1$ such that $A_n' \cap M$ is non-amenable. Let $z \in Z(A_n' \cap M)$ such that $(A_n' \cap M)z$ is strongly non-amenable relative to $C^*1$. By using the fact that $M$ belongs to Class $\mathcal{M}$, it follows that $\alpha_t \to \text{id}$ uniformly on $(A_nz)_1$. In particular,

$$\alpha_t \to \text{id} \text{ uniformly on } (A_mz)_1, \text{ for any } m \geq n. \quad (3.1)$$

Note here that $z \in A_m' \cap M$, for any $m \geq n$. If $A_mz \nless_M A$, condition (2.ii) of Class $\mathcal{M}$ shows that there exists a non-zero projection $z_1 \in (A_m' \cap M)' \cap M \subset (A_n' \cap M)' \cap M$ with $z_1 \leq z$ such that $(A_m' \cap M)z_1$ is amenable. This proves that $(A_n' \cap M)z_1$ is amenable, contradiction.

Hence, $A_mz \nless_M A$. Since $M \subset \bar{M}$ is mixing relative to $A$, it follows from (3.1) and Theorem 3.2(1) that $\alpha_t \to \text{id}$ uniformly on $(z(A_m' \cap M)z)_1$, for any $m \geq n$. Since $z(A_m' \cap M)z \nless_M A$ and $zMz = \bigvee_{m \geq n} z(A_m' \cap M)z$, we apply Theorem 3.2(2) and derive that $\alpha_t \to \text{id}$ uniformly on $(zMz)_1$. Since $M$ is a factor, we apply Proposition 3.3 and deduce that $\alpha_t \to \text{id}$ uniformly on $(M)_1$, contradiction. Therefore, $M$ does not have property Gamma. \qed
3.3. Class $\mathcal{M}_{wr}$. We say that a von Neumann algebra $M$ is in Class $\mathcal{M}_{wr}$ if there exists a decomposition $M = B_0^I \rtimes \Lambda$ satisfying the following properties:

- $B_0$ is a tracial amenable von Neumann algebra and $\Lambda$ is a non-amenable group.
- There exists $k \geq 1$ such that $\text{Stab} J$ is finite whenever $J \subset I$ with $|J| \geq k$.

Remark 3.8. We record the following properties of von Neumann algebras that belong to $\mathcal{M}_{wr}$.

1. Throughout the proofs of the main results, we will use the fact that any $M \in \mathcal{M}_{wr}$ admits an s-malleable deformation $(\hat{M}, (\alpha_t))_{t \in \mathbb{R}}$ in the sense of Definition 3.1 by using the free product deformation, see [Io06, Section 2]. To recall this construction, we define a self-adjoint unitary $h \in L(\mathbb{Z})$ with spectrum $[-\pi, \pi]$ such that $\exp(ih)$ equals the canonical generating unitary $u \in L(\mathbb{Z})$. For any $t \in \mathbb{R}$, define $u_t = \exp(i \pi h) \in L(\mathbb{Z})$. We let $\hat{M} = (B_0 \ast L(\mathbb{Z}))^I \rtimes \Lambda \subset M$ and $\alpha_t = \otimes_{i \in I} \text{Ad}(u_t) \in \text{Aut}(\hat{M})$.

2. If $M \in \mathcal{M}_{wr}$, then $M$ is a II$_1$ factor without property Gamma (see, e.g., [Dr20, Proposition 4.3]).

4. From Commuting Subalgebras to Commuting Subgroups

One of the crucial ingredients of the proof of Theorem A is an ultrapower technique due to Adrian Ioana [Io11], which we recall in the following form. This result is essentially contained in the proof of [Io11, Theorem 3.1] (see also [CdSS15, Theorem 3.3]). The statement that we will use is a particular case of [DHI16, Theorem 4.1].

**Theorem 4.1.** ([Io11]) Let $\Gamma$ be a countable icc group and denote by $M = L(\Gamma)$. Let $\Delta : M \to M \bar{\otimes} M$ be the *-homomorphism given by $\Delta(u_g) = u_g \bar{\otimes} u_g$, for all $g \in \Gamma$.

Let $B, Q \subset M$ be von Neumann subalgebras such that $\Delta(B) \prec_M M \bar{\otimes} Q$.

Then there exists a decreasing sequence of subgroups $\Sigma_k < \Gamma$ such that $B \prec_M L(\Sigma_k)$, for every $k \geq 1$, and $Q \cap M \prec_M L(\bigcup_{k \geq 1} C(\Sigma_k))$.

Note that the ultrapower technique [Io11] has been of crucial use in several other works in order to obtain structural and rigidity results for certain classes of group and group measure space von Neumann algebras [CdSS15, KV15, DHI16, CI17, CU18, Dr19, CDK19, CDHK20, CD-AD20, CD-AD21].

By combining Theorem 4.1 with a recent characterization of von Neumann algebras that do not have property Gamma [BMO19, IM19] we obtain the following useful consequence.

**Theorem 4.2.** We consider the context of Theorem 4.1. In addition, assume that $M = P \bar{\otimes} Q$ where $P$ does not have property Gamma.

Then there exists a subgroup $\Sigma < \Gamma$ with non-amenable centralizer $C(\Sigma)$ such that $B \prec_M L(\Sigma)$ and $L(\Sigma) \prec_M Q$.

**Proof.** The proof is inspired by [IM19, Lemma 5.2]. We start the proof by recalling two general facts from [BMO19]. First, [BMO19, Proposition 3.2] is stating that if a tracial von Neumann algebra $N$ does not have property Gamma, then any $N$-$N$ bimodule $\mathcal{K}$ that is weakly equivalent to $L^2(N)$ (i.e., $\mathcal{K}_N$ is weakly contained in $L^2(N)$ and $L^2(N)$ is weakly contained in $\mathcal{K}_N$) must contain $L^2(N)$.

Second, by using [BMO19, Lemma 3.4] we derive that for any subalgebra $Q_0 \subset M$ we have

$$L^2((M, e_{Q_0 \cap M}))$$

is weakly contained in $L^2(M)$ as $M$-$M$ bimodules. (4.1)
Next, we consider a decreasing sequence of subgroups $\Sigma_k < \Gamma$ as in the conclusion of Theorem 4.1. Since $P$ is non-amenable, we can assume without loss of generality that $C^*_\Gamma(\Sigma_k)$ is non-amenable for any $k \geq 1$. Denote the $M$–$M$ bimodule $\mathcal{H} = \bigoplus_{k \geq 1} L^2((M, e_{L(\Sigma_k) \cap M}))$ and notice that (4.1) implies that $\mathcal{H}$ is weakly contained in $L^2(M)$ as $M$–$M$ bimodules. (4.2)

On the other hand, since $P <_M L(\bigcup_{k \geq 1} C^*_\Gamma(\Sigma_k))$, it follows by Lemma 2.4(2) and Lemma 2.6(3) from [DHI16] that $P$ is amenable relative to $\bigvee_{k \geq 1} (L(\Sigma_k) \cap M)$ inside $M$. Hence, $L^2(M)$ is weakly contained in $L^2((M, e_{\bigvee_{k \geq 1} (L(\Sigma_k) \cap M)}))$ as $M$–$P$ bimodules. By using the moreover part of [IM19, Proposition 2.5], we deduce that $L^2((M, e_{\bigvee_{k \geq 1} (L(\Sigma_k) \cap M)}))$ is weakly contained in $\mathcal{H}$ as $M$–$M$ bimodules. Therefore,

$$L^2(M) \text{ is weakly contained in } \mathcal{H} \text{ as } M$–$P \text{ bimodules.}$$

(4.3)

Since $M = P \bar{\otimes} Q$, it follows that $L^2(M)$ is a multiple of $L^2(P)$ as $P$–$P$ bimodules. In combination with (4.2) and (4.3), we deduce that $L^2(P)$ is weakly equivalent to $\mathcal{H}$ as $P$–$P$ bimodules. Finally, since $P$ does not have property Gamma, we use the first paragraph of the proof and derive that $L^2(P)$ is contained in $p\mathcal{H}P$. Hence, there exists $k_0 \geq 1$ such that $L^2(P)$ is contained in $pL^2(M, e_{L(\Sigma_{k_0}) \cap M})P$. This implies that $P <_M L(\Sigma_{k_0}) \cap M$. By passing to relative commutants and using [Va08, Lemma 3.5], the conclusion of the corollary is obtained by taking $\Sigma = \Sigma_{k_0}$. $\square$

As an application of Theorem 4.2, we can use the augmentation technique from [CD-AD20, Section 3] and derive the following result which is an important ingredient of the proof of Theorem A.

**Theorem 4.3.** Let $\Gamma$ be a countable icc group such that $L(\Gamma)$ does not have property Gamma. Denote $M = L(\Gamma)^{1/t}$ for some $t > 0$ and let $\Delta : M^t \to M^t \bar{\otimes} M^t$ be the $*$-homomorphism given by $\Delta(u_g) = u_g \otimes u_g$, for any $g \in \Gamma$.

Assume $M = M_1 \bar{\otimes} \ldots \bar{\otimes} M_n$ is the tensor product of $n \geq 1$ II$_1$ factors with the property that for any $i \in \overline{1, n}$, there exists $f(i) \in \overline{1, n}$ such that $\Delta(M^t_i) <_{M^t \bar{\otimes} M^t} M^t_i \bar{\otimes} M^t_{f(i)}$.

Then $\Delta(M^t_i) <_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} M^t_{f(i)}$, for any subset $I \subset \overline{1, n}$.

**Proof.** We may assume that $t = 1$ since this simplification does not hide any essential part of the argument. Indeed, note that for any $i \in \overline{1, n}$ there is a natural identification $M_i = M^t_i \bar{\otimes} M^t_i$. Assume by contradiction that there exists $i$ such that $f(i) \neq i$. Using the assumption we can apply Theorem 4.2 and derive that for any $j \in \overline{1, n}$ there exists a subgroup $\Sigma_j < \Gamma$ such that

$$M^t_j <_{M^t \bar{\otimes} M^t} L(\Sigma_j) \text{ and } L(\Sigma_j) <_{M^t} M^t_{f(j)}.$$  

(4.4)

Here, we also used [DHI16, Lemma 2.4(2)]. Note that (4.4) shows that $L(\Sigma_i) <_{M^t} M^t_{f(i)}$ and $M^t_{f(i)} <_{M^t} L(\Sigma_{f(i)})$. Using [Va08, Lemma 3.7] we deduce that $L(\Sigma_i) <_{M^t} L(\Sigma_{f(i)})$. We can apply [CI17, Lemma 2.2] and derive that there exist $g \in \Gamma$ and a finite index subgroup $\Sigma^0_i < \Sigma_i$ such that $g \Sigma^0_i g^{-1} \subset \Sigma_{f(i)}$. Using (4.4) for $j = f(i)$ we get that $L(\Sigma^0_i) <_{M^t} M^t_{f^2(i)}$. Since $[\Sigma_i : \Sigma^0_i] < \infty$, it follows that

$$L(\Sigma_i) <_{M^t} M^t_{f^2(i)}.$$  

(4.5)
Following an idea from [CD-AD20, Section 3], we consider a Bernoulli action $\Gamma \actson D$ with abelian base and let $M = D \rtimes \Gamma$. Let $\Psi: \mathcal{M} \to \mathcal{M} \bar{\otimes} M$ be the $\ast$-homomorphism given by $\Psi((d_{t} g) = d_{t} g \otimes u_{g}$, for all $d \in D$ and $g \in \Gamma$. From (4.4) we have that $M_{t} \prec_{M}^{s} L(\Sigma_{i})$, which gives $M_{t} \prec_{\mathcal{M}}^{s', s'} D \rtimes \Sigma_{i}$. By applying [Dr19, Lemma 2.3], we further derive that $\Psi(M_{t}) \prec_{\mathcal{M} \bar{\otimes} M}^{s, s'} \mathcal{M} \bar{\otimes} M L(\Sigma_{i})$. Next, by using [Dr19, Lemma 2.4] and the relation (4.5) we obtain that
\[
\Psi(M_{t}) \prec_{\mathcal{M} \bar{\otimes} M}^{s} \mathcal{M} \bar{\otimes} M_{f_{2}(t)}^{s}. \tag{4.6}
\]

Note that $i \in f(t)$. Using our assumption, we obtain that $\Delta(M_{i}) \prec_{\mathcal{M} \bar{\otimes} M}^{s} \mathcal{M} \bar{\otimes} M_{f_{2}(i)}^{s}$. Hence, by [DHI16, Remark 2.2] we get that
\[
\Psi(M_{i}) \prec_{\mathcal{M} \bar{\otimes} M}^{s} \mathcal{M} \bar{\otimes} M_{f_{2}(i)}^{s}. \tag{4.7}
\]

Next, by using [Is19, Lemma 2.6], (4.6) and (4.7), we derive that $\Psi(M) \prec_{\mathcal{M} \bar{\otimes} M}^{s} \mathcal{M} \bar{\otimes} M_{f_{2}(i)}^{s}$. Since $\Psi(x) = x \otimes 1$, for any $x \in D$, we deduce that $\Psi(M) \prec_{\mathcal{M} \bar{\otimes} M}^{s} \mathcal{M} \bar{\otimes} M_{f_{2}(i)}^{s}$. By [PV11, Lemma 10.2] we obtain that $M \prec_{\mathcal{M}} M_{f_{2}(i)}^{s}$. Since $\mathcal{Q}_{\mathcal{M}}^{(1)}(M) = M$, we derive that $M \prec_{M} M_{f_{2}(i)}^{s}$, which implies that $M_{f_{2}(i)}^{s}$ is not diffuse, contradiction.

Hence, $\Delta(M_{j}) \prec_{M \bar{\otimes} M}^{s} M \bar{\otimes} M_{j}$, for any $j \in \overline{1, n}$. By applying [DHI16, Lemma 2.6(2)], the conclusion of the theorem follows. $\square$

5. Proofs of the Main Results

5.1. Proof of Theorem A. Before proceeding to the proof of Theorem A, we need the following result.

**Theorem 5.1.** Let $M = M_{1} \bar{\otimes} \ldots \bar{\otimes} M_{n}$ be the tensor product of $n \geq 1$ factors from $\mathcal{M} \cup \mathcal{M}_{ur}$. Assume $\Gamma$ is a countable icc group such that $M_{t} = L(\Gamma)$ for some $t > 0$. Denote by $\Delta: M^{t} \to M^{t} \bar{\otimes} M^{t}$ the $\ast$-homomorphism given by $\Delta(u_{g}) = u_{g} \otimes u_{g}$, for any $g \in \Gamma$.

Then for any $i \in \overline{1, n}$, there exists $f(i) \in \overline{1, n}$ such that $\Delta(M_{t}^{f(i)}) \prec_{M^{t} \bar{\otimes} M^{t}}^{s} M_{t}^{f(i)}.$

**Proof.** Let $(\tilde{M}_{i}, (\alpha_{i}^{t})_{i \in \mathbb{R}})$ be an s-malleable deformation of $M_{i}$ given by the fact that $M_{i}$ belongs to $\mathcal{M} \cup \mathcal{M}_{ur}$, see Remark 3.8(1). If $M_{t} \in \mathcal{M}$, we let $A_{t}$ be given as in the definition of Class $\mathcal{M}$. We naturally extend $\alpha_{i}^{t}$ to an automorphism $\alpha_{i}^{t} \in \text{Aut}(\tilde{M}_{i} \bar{\otimes} \tilde{M}_{i})$. We may assume without loss of generality that $t = 1$ since this simplification does not hide any essential part of the argument; note that for any $i \in \overline{1, n}$ there is a natural identification $M^{t} = M_{t}^{t} \bar{\otimes} M_{t}^{t}$.

We first prove that there exists a function $f \in \overline{1, n} \to \overline{1, n}$ such that
\[
\Delta(M_{j}) \text{ is non-amenable relative to } M \bar{\otimes} M_{f(j)} \text{ for any } j \in \overline{1, n}. \tag{5.1}
\]

If (5.1) does not hold, then there exists $j \in \overline{1, n}$ such that $\Delta(M_{j})$ is amenable relative to $M \bar{\otimes} M_{k}$, for any $k \in \overline{1, n}$. By [PV11, Proposition 2.7], it follows that $\Delta(M_{j})$ is amenable relative to $M \bar{\otimes} 1$. This further implies by [PV10, Proposition 7.2(4)] that
$M_j$ is amenable, contradiction. Therefore, there exists a function $f$ that satisfies (5.1). Next, we show that

$$\Delta(M_j) \prec_{M \hat{\otimes} M_f(j)} M \hat{\otimes} M_f(j), \text{ for any } j \in 1, n. \tag{5.2}$$

To show this, fix $j \in 1, n$ and notice first that $N_{M \hat{\otimes} M}(\Delta(M_j)) \cap M \hat{\otimes} M = \varnothing$ since $\Gamma$ is icc. Thus, using [DHI16, Lemma 2.6(2)], we obtain from (5.1) that

$$\Delta(M_j) \text{ is strongly non-amenable relative to } M \hat{\otimes} M_f(j). \tag{5.3}$$

**Case 1.** $M_f(j) \in M$.

In this case it follows directly from (5.3) that $\id \otimes \alpha_t^{f(j)} \rightarrow \id$ uniformly on $(\Delta(M_j))_1$. Assume by contradiction that $\Delta(M_j) \not\prec_{M \hat{\otimes} M} M \hat{\otimes} M_f(j)$. If $\Delta(M_j) \not\prec_{M \hat{\otimes} M} M \hat{\otimes} (M_{f(j)} \hat{\otimes} A_f(j))$, we obtain from Theorem 3.2(1) that $\id \otimes \alpha_t^{f(j)} \rightarrow \id$ uniformly on $(\Delta(M))_1$. This shows that $\alpha_t^{f(j)} \rightarrow \id$ uniformly on $(M)_1$, and hence on $M_f(j)$, contradiction. Therefore, $\Delta(M_j) \prec_{M \hat{\otimes} M} M \hat{\otimes} (M_{f(j)} \hat{\otimes} A_f(j))$. Using condition (ii) from Class $M$, it follows that (5.2) holds in this case.

**Case 2.** $M_f(j) \in M_{wr}$.

In this case we can write $M_f(j) = B_f(j) = \Lambda_f(j)$ as the von Neumann algebra of a generalized Bernoulli action as in Class $M_{wr}$. By (5.3) and [BV12, Theorem 3.1] we have that $\id \otimes \alpha_t^{f(j)} \rightarrow \id$ uniformly on $(\Delta(M_j))_1$. Using [IPV10, Theorem 4.2] we obtain that $\Delta(M_f(j)) \prec_{M \hat{\otimes} M} M \hat{\otimes} M_f(j)$ or $\Delta(M) \prec_{M \hat{\otimes} M} M \hat{\otimes} (M_{f(j)} \hat{\otimes} \Lambda_f(j))$ or $\Delta(M) \prec_{M \hat{\otimes} M} M \hat{\otimes} (M_{f(j)} \hat{\otimes} (B_f(j) \hat{\otimes} \theta_f(j))))$, where $\theta_f(j) < \Lambda_f(j)$ is an infinite index subgroup. From [IPV10, Proposition 7.2] the last two possibilities give a contradiction. This shows that (5.2) holds.

Finally, by using [DHI16, Lemma 2.4(2)], we end the proof of the theorem. \qed

We are now ready to prove the following result which is a generalization of Theorem A.

**Theorem 5.2.** Let $M = M_1 \otimes \ldots \otimes M_n$ be the tensor product of $n \geq 1$ II$_1$ factors from $M \cup M_{wr}$. Assume $\Gamma$ is a countable icc group such that $M^t = L(\Gamma)$ for some $t > 0$.

Then there exists a product decomposition $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$, a unitary $u \in M$ and some positive numbers $t_1, \ldots, t_n$ with $t_1 \cdots t_n = t$ such that $M_i^t = uL(\Gamma_i)u^*$, for any $i \in 1, n$.

**Proof.** Let $(\tilde{M}_i, (\alpha_i^u)_{t \in \mathbb{R}})$ be an s-malleable deformation of $M_i$ given by the fact that $M_i$ belongs to Class $M \cup M_{wr}$, see Remark 3.8(1). If $M_i \in M$, let $A_i$ be given as in the definition of Class $M$. We naturally extend $\alpha_i^t$ to an automorphism $\alpha_i^t \in \text{Aut}(M_i \hat{\otimes} \tilde{M}_i)$. Next, we may assume that $t = 1$ since this simplification does not hide any essential part of the argument.

Following [PV09], we denote by $\Delta : M \rightarrow M \hat{\otimes} M$ the $*$-homomorphism given by $\Delta(u_g) = u_g \otimes u_g$, for any $g \in \Gamma$. By Theorem 5.1 and Theorem 4.3, we have that

$$\Delta(M_I) \prec_{M \hat{\otimes} M} M \hat{\otimes} M_I, \text{ for any subset } I \subset 1, n. \tag{5.4}$$

Our goal is to prove the following claim.

**Claim.** There exist a subgroup $\Sigma < \Gamma$ and a non-zero projection $f \in L(\Sigma)' \cap M$ such that $M_{\Sigma} \prec_{M} L(\Sigma)f$ and $L(\Sigma)f \prec_{M} M_{\Sigma}$.
Proof of Claim. Using Theorem 4.2, we can find a subgroup $\Sigma < \Gamma$ with non-amenable centralizer $C_{\Gamma}(\Sigma)$ such that $M_\tilde{n} \prec_{M} L(\Sigma)$ and $L(\Sigma) \prec_{M} M_\tilde{n}$. By using [DHI16, Lemma 2.4(4)] there exists a non-zero projection $f \in L(\Sigma C_{\Gamma}(\Sigma))' \cap M$ such that

$$M_\tilde{n} \prec_{M}^f L(\Sigma) f \text{ and } L(\Sigma) \prec_{M} M_\tilde{n}. \quad (5.5)$$

Next, by using [Va08, Lemma 3.5] and [DHI16, Lemma 2.6(3)] we deduce from (5.5) that $L(C_{\Gamma}(\Sigma)) f$ is amenable relative to $M_\tilde{n}$. Since $C_{\Gamma}(\Sigma)$ is non-amenable, we derive from [PV11, Proposition 2.7] that $L(C_{\Gamma}(\Sigma)) f$ is non-amenable relative to $M_\tilde{n}$. Using [DHI16, Lemma 2.4] there exists a non-zero projection $f_1 \in L(\Sigma C_{\Gamma}(\Sigma))' \cap M$ with $f_1 \leq f$ such that

$$L(C_{\Gamma}(\Sigma)) f_1 \text{ is strongly non-amenable relative to } M_\tilde{n}. \quad (5.6)$$

Next, we note that

$$\alpha_i^n \to \text{id uniformly on } (L(\Sigma) f_1)_1. \quad (5.7)$$

Indeed, if $M_\Lambda$ belongs to $M$, this follows immediately, while if $M$ belongs to $M_{w}M_{\Lambda}$, this follows from [BV12, Theorem 3.1]. Next, remark that the claim would follow if $L(\Sigma) f_1 \prec_{M} M_\tilde{n}$ because we could use [DHI16, Lemma 2.4] and derive that there exists a non-zero projection $f_0 \in L(\Sigma)' \cap M$ with $f_0 \leq f_1$ such that $L(\Sigma) f_0 \prec_{M} M_\tilde{n}$.

Hence, we assume by contradiction that $L(\Sigma) f_1 \not\prec_{M} M_\tilde{n}$.

Following [CdSS15, Section 4], we let $\Omega = \{g \in \Gamma | O_\Sigma(g) \text{ is finite}\},$ where $O_\Sigma(g) = \{hgh^{-1} | h \in \Sigma\}$, and notice that $L(\Sigma)' \cap M \subset L(\Omega)$. We continue by proving the following. \hfill \Box

Subclaim. There exists a non-zero projection $f_2 \in L(\Omega)' \cap M$ such that

$$\alpha_i^n \to \text{id uniformly on } (L(\Omega) f_2)_1. \quad (5.7')$$

Proof of subclaim. We split the proof of the subclaim in two parts.

Case 1. $M_\tilde{n}$ belongs to Class $M$.

Note that there exists a sequence of increasing subgroups $\Omega_i < \Omega$ such that $\Omega = \bigvee_{i \geq 1} \Omega_i$ and by letting $\Sigma_i = C_\Sigma(\Omega_i)$, we have $[\Sigma : \Sigma_i] < \infty$, for any $i \geq 1$. Indeed, let $\{O_i\}_{i \geq 1}$ be a countable enumeration of all the finite orbits of the action by conjugation of $\Sigma$ on $\Gamma$. Notice that $\Omega_i := (\bigcup_{j=1}^i O_j) < \Omega$, $i \geq 1$, is an ascending sequence of subgroups with $\Omega = \bigvee_{i \geq 1} \Omega_i$. Since $\bigcup_{j=1}^i O_j \subset \Omega$ is a finite set, it follows that $\Sigma_i := \bigcap_{g \in \bigcup_{j=1}^i O_j} C_\Sigma(g) = C_\Sigma(\Omega_i)$ is a decreasing sequence of finite index subgroups of $\Sigma$.

Next, note that $f_1 \in L(\Sigma_i)' \cap M$ and $L(\Sigma_i) f_1 \not\prec_{M} M_\tilde{n}$ since $[\Sigma : \Sigma_i] < \infty$, for any $i \geq 1$. Moreover, we note that $L(\Sigma_i) f_1 \not\prec_{M} M_\tilde{n} \otimes A_n$. Otherwise, by using condition (2) in Class $M$, we get that there exists a non-zero projection $f_2 \in L(\Sigma_i)' \cap M \cap L(C_{\Gamma}(\Sigma))' \cap M$ with $f_2 \leq f_1$ such that $(L(\Sigma_i)' \cap M) f_2$ is amenable relative to $M_\tilde{n}$, which contradicts (5.6).

Therefore, since $M \subset M_\tilde{n} \otimes \tilde{M}_n$ is mixing relative to $M_\tilde{n} \otimes A_n$ we obtain from (5.7) and Theorem 3.2(1) that

$$\alpha_i^n \to \text{id uniformly on } (f_1(L(\Sigma_i) \vee (L(\Sigma_i)' \cap M)) f_1)_1, \text{ for any } i \geq 1. \quad (5.8)$$
Next, since $A_n$ is amenable, we notice that (5.6) together with [DHI16, Theorem 2.6(3)] imply that $f_1(L(\Sigma_i)' \cap M)) f_1 \not\sim_M M_\bar{\otimes} A_n$ for any $i \geq 1$. We can therefore combine (5.8) with Theorem 3.2(2) and deduce that

$$\alpha^n_i \to \text{id} \text{ uniformly on } (f_1(\bigvee_{i \geq 1} (L(\Sigma_i)' \cap M)))_1.$$ 

We can apply Proposition 3.3 and derive that if we denote by $f_2$ the central support of $f_1$ in $\bigvee_{i \geq 1} (L(\Sigma_i)' \cap M)$, we further obtain from that

$$\alpha^n_i \to \text{id} \text{ uniformly on } (\bigvee_{i \geq 1} (L(\Sigma_i)' \cap M)) f_2_1.$$ 

Since $L(\Omega) \subset \bigvee_{i \geq 1} L(\Sigma_i)' \cap M$ and $f_2 \in \mathcal{Z}(\bigvee_{i \geq 1} (L(\Sigma_i)' \cap M))$, it follows that the subclaim is proven in this case.

**Case 2.** $M_n$ belongs to Class $\mathcal{M}_{wr}$. We can write $M_n = B_n \times \Lambda_n$ where $\Lambda_n \subset B_n$ is a generalized Bernoulli action with amenable base as given by Class $\mathcal{M}_{wr}$. For proving the subclaim, we follow a slightly different approach than the one used in Case 1. Using Popa’s compression formulas for quasi-normalizers [Po03], we have

$$\mathcal{QN}_{f_1Mf_1}(L(\Sigma)f_1) = f_1\mathcal{QN}_M(L(\Sigma)f_1).$$

By applying [IPV10, Theorem 4.2] to (5.7), we derive that (i) $L(\Sigma)f_1 \not\sim_M M_\bar{\otimes}$ or (ii) $L(\mathcal{QN}_\Gamma(\Sigma)) f_1 \not\sim_M M_\bar{\otimes}(B_n \rtimes \theta_n)$, where $\theta_n \subset \Lambda_n$ is an infinite index subgroup, or (iii) there exists a partial isometry $w \in M$ with $ww^* = f_1$ and $w^*\mathcal{QN}_{f_1Mf_1}(L(\Sigma)f_1)''w \subset M_\bar{\otimes}L(\Lambda_n)$. Option (i) is not possible since we assumed by contradiction that $L(\Sigma)f_1 \not\sim_M M_\bar{\otimes}$.

We now show that option (ii) leads to a contradiction as well. Note that by passing to relative commutants in (5.5), we derive that $M_n \not\sim_M L(\Omega)$. Since $\Omega \subset \mathcal{QN}_\Gamma(\Sigma)$, option (ii) implies that $L(\Omega) \not\sim_M M_\bar{\otimes}(B_n \rtimes \theta_n)$. Combining all these with (5.5), we can apply Proposition 2.6 and derive that $[\Lambda_n : \theta_n] < \infty$, contradiction.

Next, note that option (iii) combined with (5.9) implies that

$$\alpha^n_i \to \text{id} \text{ uniformly on } (f_1\mathcal{QN}_M(L(\Sigma)f_1))_1.$$ 

Since $\Omega \subset \mathcal{QN}_\Gamma(\Sigma)$, the subclaim follows by using Proposition 3.3. □

Following the augmentation technique from [CD-AD20, Section 3], we consider a Bernoulli action $\Gamma \subset D$ with abelian base. Denote $\mathcal{M} = D \rtimes \Gamma$ and let $\Psi : \mathcal{M} \to \mathcal{M}\bar{\otimes}L(\Omega)$ be the $*$-homomorphism given by $\Psi(du_g) = du_g \bar{\otimes} u_g$, for all $d \in D$ and $g \in \Gamma$. Next, by passing to relative commutants in (5.5) we obtain that $M_n \not\sim_M L(\Omega)$, which implies that $M_n \not\sim_M D \rtimes \Omega$. This shows using [Dr19, Lemma 2.3] that $\Psi(M_n) \not\sim_{\mathcal{M}\bar{\otimes}L(\Omega)} \mathcal{M}\bar{\otimes}L(\Omega)$. In particular, $\Psi(M_n) \not\sim_{\mathcal{M}\bar{\otimes}M} \mathcal{M}\bar{\otimes}L(\Omega) f_2$.

Hence, there exist some projections $p \in M_n, q \in \mathcal{M}\bar{\otimes}L(\Omega)f_2$, a non-zero partial isometry $w \in \mathcal{QN}_\Gamma(\Sigma)\Psi(p)$ and a $*$-homomorphism $\theta : (\mathcal{M}\bar{\otimes}M)\Psi(pM_n p) \to q(\mathcal{M}\bar{\otimes}L(\Omega)f_2)q$ such that $\theta(x)w = wx$, for any $x \in \Psi(pM_n p)$. Let $\tilde{p} = \Psi(p)$ and note that $w^*w \in \Psi(pM_n p)' \cap \tilde{p}(\mathcal{M}\bar{\otimes}M)\tilde{p}$. Therefore, by using the multiplicativity of $\text{id} \otimes \alpha^n_i$ and its pointwise $\|\cdot\|_2$-convergence to the identity, the subclaim implies that

$$\text{id} \otimes \alpha^n_i \to \text{id} \text{ uniformly on } (\Psi(pM_n p)w^*w)_1.$$ 

(5.10)
The claim will be proven by considering again two separate cases.

**Case 1.** $M_n$ belongs to Class $\mathcal{M}$.

If $\Psi(pM_n p) w^* w \not\prec_{\mathcal{M} \otimes_M} M \otimes (M_n \otimes A_n)$, then Theorem 3.2(1) and (5.10) give $id \otimes \alpha^n \to id$ uniformly on $(w^* w (\Psi(pM_n p) \lor \Psi(pM_n p))' \cap \tilde{p}(M \otimes \tilde{M}) \tilde{p}) w^* w 1)$. Let $\tilde{p}_1$ be the central support of $w^* w$ in $\Psi(M_n) \lor \Psi(M_n)' \cap (M \otimes M)$ and note that $\tilde{p}_1 \in \Psi(M)' \cap (M \otimes M) = 1$ since $\Gamma$ is icc. By using Proposition 3.3, we further obtain that $id \otimes \alpha^n \to id$ uniformly on $(\Psi(M))_1$, which shows that $\alpha^n \to id$ uniformly on $(M)_1$, contradiction.

Hence, $\Psi(pM_n p) w^* w \prec_{\mathcal{M} \otimes_M} M \otimes (M_n \otimes A_n)$, and therefore $\Psi(M_n) \prec_{\mathcal{M} \otimes_M} M \otimes (M_n \otimes A_n)$ by [DHI16, Lemma 2.4(2)], Relation (5.4) implies in particular that $\Psi(M_n) \prec_{\mathcal{M} \otimes_M} M \otimes M_n$. By using [DHI16, Lemma 2.8(2)], we derive that $\Psi(M_n) \prec_{\mathcal{M} \otimes_M} M \otimes A_n$, which implies by [IPV10, Lemma 10.2] that $M_n$ is amenable, contradiction. Hence, the claim is proven in this case.

**Case 2.** $M_n$ belongs to Class $\mathcal{M}_{ur}$. As before, we assume $M_n = B_n \rtimes \Lambda_n$ where $\Lambda_n \sim C^*_n =: B_n$ is a generalized Bernoulli action with amenable base as given by Class $\mathcal{M}_{ur}$. If $\Psi(pM_n p) w^* w \not\prec_{\mathcal{M} \otimes_M} M \otimes M_n$, we derive from [IPV10, Theorem 4.2] and (5.10) that $\Psi(M) \prec_{\mathcal{M} \otimes_M} M \otimes (B_n \rtimes \theta_n)$, where $\theta_n \prec \Lambda_n$ is an infinite index subgroup, or $\Psi(M) \prec_{\mathcal{M} \otimes_M} M \otimes L(\Lambda_n)$, and therefore, $\Psi(M) \prec_{\mathcal{M} \otimes_M} M \otimes L(\Lambda_n)$, respectively. By using [IPV10, Lemma 10.2] it is easy to see that we would get a contradiction. Hence, $\Psi(pM_n p) w^* w \prec_{\mathcal{M} \otimes_M} M \otimes M_n$, and by proceeding as in the last paragraph of Case 1 above, we obtain that $\Psi(M_n) \prec_{\mathcal{M} \otimes_M} M \otimes 1$, which is again a contradiction. \qed

Finally, note that (5.4) implies in particular that $\Delta(M_1) \prec_{\mathcal{M} \otimes_M} M \otimes M_1$. Using Theorem 4.2, we obtain a subgroup $\Theta < \Gamma$ such that $M_1 \prec_{\mathcal{M}} L(\Theta)$ and $L(\Theta) \prec_{\mathcal{M}} M_1$. In combination with the claim, it follows from Theorem 2.3 that there exist a product decomposition $\Gamma = \Gamma^{n-1}_1 \times \Gamma_n$, a decomposition $M = M_n^{\otimes_M} M_1^{1/s}$ for some $s > 0$ and a unitary $u \in U(M)$ such that $uM_n^{\otimes_M} u^* = L(\Gamma^{n-1}_1)$ and $uM_1^{1/s} u^* = L(\Gamma_n)$. Therefore, we obtain the conclusion of the theorem by a simple induction argument. \qed

**5.2. Proof of Corollary B.** Let $\theta : L(\Gamma)^t \to L(\Lambda)$ be a $*$-isomorphism where $\Lambda$ is any countable group and $t > 0$. By Theorem A, there exist a product decomposition $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$, some positive numbers $t_1, \ldots, t_n > 0$ with $t_1 \cdots t_n = t$ and a unitary $w \in L(\Lambda)$ such that $\theta(L(\Gamma)^t) = wL(\Lambda_i)w^*$, for any $i \in \overline{1, n}$. Since $\Gamma_i$ is $W^*$-superrigid, it follows that $t_1 = 1$ and there exist a group isomorphism $\delta_i : \Gamma_i \to \Lambda_i$, a unitary $w_i \in L(\Lambda_i)$ and a character $\omega_i : \Gamma_i \to \mathbb{T}$ such that $\theta(u_{g_i}) = \omega_i(g)w_i^* v_{\delta(g)}w_i^*$, for all $i \in \overline{1, n}$ and $g \in \Gamma_i$. Hence, $t = 1$ and by letting $\omega = \prod_{i=1}^n \omega_i$, $\delta = \prod_{i=1}^n \delta_i$ and $u = \prod_{i=1}^n w_i$, we get the desired conclusion. Finally, we notice that any group from $\mathcal{IPV}$ is $W^*$-superrigid by [IPV10, Theorem 8.3]. \qed

**5.3. Proof of Corollary C.** On one hand, note that if $\Gamma_0$ is an icc non-amenable countable group with $\beta^{(2)}_0(\Gamma_0) > 0$, then its amenable radical is trivial, i.e. any normal amenable subgroup of $\Gamma_0$ is trivial. On the other hand, note that any group from $\mathcal{A}_2$ has trivial amenable radical by using [CD-AD20, Proposition 6.3].

Therefore, $\Gamma$ has trivial amenable radical, so by using [BKKO14, Theorem 1.3], it follows that $C^*_r(\Gamma)$ has a unique trace. This implies that any $*$-isomorphism $\theta : C^*_r(\Gamma) \to$
$C^*_r(\Lambda)$ extends to a $\ast$-isomorphism $\theta : L(\Gamma) \to L(\Lambda)$. The conclusion now follows from Theorem A. □

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