Superconformal Indices, Seiberg Dualities and Special Functions

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Abstract—This is a brief account of relations between the theory of special functions, on the one side, and superconformal indices and Seiberg dualities of four-dimensional $N = 1$ supersymmetric gauge field theories, on the other side.

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PLAIN HYPERGEOMETRIC FUNCTIONS

Since this is a memorial meeting, I have decided to give a partially historical presentation. Evidently, history of the subject indicated in the talk title starts from the theory of special functions. Namely, it is necessary to go back as far as to the times of Isaak Newton, when physics and mathematics formed one science and they were not separated, as we see it nowadays. Among his numerous great achievements, in 1665 Newton proved the binomial theorem

$$ _1 F_0 (a; x) := \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n = (1 - x)^{-a}, \quad a, x \in \mathbb{C}, \quad |x| < 1, $$  \hspace{1cm} (1)

where $(a)_n = a(a+1)\cdots(a+n-1)$ is called at present the Pochhammer symbol. Actually, he established this simplest hypergeometric functions identity for fractional values of $a$ and his main achievement consisted in the treatment of infinite series.

The major development of the theory of special functions of hypergeometric type took place in the hands of Leonhard Euler [1] (I am using this textbook as a key source of historical data). Among his tremendous list of glorious discoveries one can distinguish the following ones: from 1729 he sequentially introduced the gamma function $\Gamma(x)$,

$$ \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad \text{Re}(x) > 0, $$

the beta function (integral) $B(x, y)$,

$$ B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \text{Re}(x), \text{Re}(y) > 0, $$  \hspace{1cm} (3)

and the key hypergeometric $\, _2 F_1$-function,

$$ _2 F_1 (a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} x^n, \quad |x| < 1. $$  \hspace{1cm} (4)

The Euler integral representation

$$ _2 F_1 (a, b; c; x) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1} (1-t)^{c-1} (1-xt)^{-a} dt, $$  \hspace{1cm} (5)

where $\text{Re}(c) > \text{Re}(b) > 0$ and $x \notin [1, \infty)$, follows directly from expanding integrand’s $(1-xt)^{-a}$ factor into the Taylor series according to formula (1) and using the exact integration formula (3).

Gauss (1812), Kummer (1836), Riemann (1857), Barnes (1908) investigated in detail properties of the $\, _2 F_1$-function [1]. In particular, the hypergeometric equation (considered by Euler already in 1769) satisfied by it:

$$ x(1-x)y''(x) + (c -(a+b+1)x)y'(x) - aby(x) = 0. $$

The enormous popularity of this equation is explained by the fact that it represents a very geometric object— the general differential equation of the second order with three regular singular points (fixed as 0, 1 and $\infty$). All series expansions of solutions at these singular points are expressed in terms of the $\, _2 F_1$-function ($y(x) = _2 F_1 (a, b; c; x)$ is the solution analytical near the point $x = 0$).
As to the special functions of many variables, I will mention only the multiple beta integral evaluated by Atle Selberg in 1944 [1]:

\[
\int_0^1 \prod_{0 \leq i < j \leq n} |x_i - x_j|^\gamma \prod_{j=1}^n x_j^{\alpha_j-1} (1 - x_j)^{\beta_j-1} \, dx_j
\]

\[
= \prod_{j=1}^n \frac{\Gamma(\alpha + (j - 1)\gamma)\Gamma(\beta + (j - 1)\gamma)\Gamma(1 + j\gamma)}{\Gamma(\alpha + \beta + (N + j - 2)\gamma)\Gamma(1 + \gamma)}, \quad \text{(6)}
\]

Re(\alpha), Re(\beta) > 0, \quad Re(\gamma) > -\min \left( \frac{1}{n}, \frac{\text{Re}(\alpha)}{n-1}, \frac{\text{Re}(\beta)}{n-1} \right).

It was introduced for some number-theoretical needs, but its most important applications have been found in mathematical physics: random matrices, integrable body problems, multiple orthogonal polynomials, etc.

\[ q \text{-HYPERGEOMETRIC FUNCTIONS} \]

Again, a systematic consideration of the second level of hypergeometric functions was launched by Euler, who constructed in 1748 the \( q \)-exponential functions

\[
\sum_{n=0}^{\infty} \frac{(q; q)_n}{(x; q)_n} x^n = \frac{1}{(x; q)_\infty}, \quad |x| < 1,
\]

\[
\sum_{n=0}^{\infty} \frac{n!}{(q; q)_n} (-x)^n = (x; q)_\infty, \quad |x| < 1,
\]

where the product \( (x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k) \) is known today as the \( q \)-Pochhammer symbol. Many mathematicians considered generalizations of these exact summation formulas. In particular, Rothe, Cauchy, Heine, Gauss have established the following identity

\[
\phi_0(t; q, x) = \sum_{n=0}^{\infty} \frac{(t; q)_n}{(q; q)_n} x^n = \frac{(tx; q)_\infty}{(x; q)_\infty}, \quad |x|, |q| < 1,
\]

which is referred to as the \( q \)-binomial theorem. Heine in 1847 constructed a \( q \)-analogue of the \( \, F_1 \)-function

\[
\phi_1(s; t; w; q, x) = \sum_{n=0}^{\infty} \frac{(s; q)_n(t; q)_n}{(q; q)_n} x^n,
\]

such that

\[
\phi_1(q^a; q^b; q^c; q, x) \to \, F_1(a, b; c; x) \quad \text{for} \quad q \to 1. \quad \text{(7)}
\]

The theory of hypergeometric functions was developing for more than 300 years in these two instances:

\[
\, F_r \left( \begin{array}{c} u_1, \ldots, u_{r+1} \\ v_1, \ldots, v_r \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(u_1)_n \cdots (u_{r+1})_n}{(v_1)_n \cdots (v_r)_n} x^n
\]

and

\[
\, F_r \left( \begin{array}{c} t_1, \ldots, t_{r+1} \\ w_1, \ldots, w_r \end{array} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(t_1; q)_n \cdots (t_{r+1}; q)_n}{(w_1; q)_n \cdots (w_r; q)_n} x^n,
\]

together with their multivariable extensions both in the series and integral forms. In 1980s some people expressed an opinion that no good special functions of hypergeometric type exist beyond them. Therefore it was a very big surprise when around the turn of Millennium the following functions have been discovered.

\[ \text{ELLiptic hypergeometric functions [2, 3]} \]

What is the top presently known extension of the binomial theorem, Euler beta integral, etc. and where that has been found? I was lucky to make one of the major contributions to answering this question during the work at the Bogolyubov Laboratory of Theoretical Physics, JINR. \(^1\) Namely, in 2000 the elliptic beta integral has been discovered and its exact evaluation was established as the following theorem [4].

\[
\text{Theorem.} \quad \text{Let} \quad p, q, t_j \in \mathbb{C}, \quad |p|, |q|, |t_j| < 1 \quad \text{and} \quad \prod_{j=1}^6 t_j = pq. \quad \text{Then}
\]

\[
\frac{(p^i p^j q; q)_\infty}{4\pi i} \int_0^\infty \frac{\prod_{j=1}^6 \Gamma(t_j z^{1}; p, q)}{\Gamma(z^{1}; p, q)} \frac{dz}{z} = \prod_{1 \leq j < k \leq 6} \Gamma(t_j t_k ; p, q),
\]

where \( \mathbb{T} \) is the unit circle and both sides of the equality are composed out of the elliptic gamma function

\[
\Gamma(z; p, q) := \prod_{j,k=0}^{m} \frac{1 - z^{-1} \, p^{j+1} q^{k+1}}{1 - z p^j q^k}, \quad |p|, |q| < 1,
\]

according to the conventions

\[
\Gamma(t_1, \ldots, t_6 ; p, q) := \Gamma(t_1; p, q) \cdots \Gamma(t_6; p, q),
\]

\[
\Gamma(z^{1}; p, q) := \Gamma(z; p, q) \Gamma(z^{-1}; p, q).
\]

The integrand function in (8) satisfies a linear \( q \)-difference equation of the first order with the coefficient given by a particular elliptic function, which follows from the generating equation

\[
\Gamma(qz; p, q) = \theta(z; p) \Gamma(z; p, q),
\]

\[
\theta(z; p) = (z; p)_\infty (pz^{-1}; p)_\infty,
\]

where \( \theta(z; p) \) is a Jacobi theta function with a specific normalization

\[
\theta(z; p) = -\frac{1}{(p; p)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k p^{(k-1)/2} z^k.
\]

\(^1\) Here it should be mentioned that I am a member of the N.N. Bogolyubov school from the times of my 1982 post-graduate studies at the Chair of Quantum Statistics and Field Theory of the Physics Department of the Moscow State University. My PhD thesis supervisors were V.A. Matveev and K.G. Chetyrkin—anecdotable generation members of this school.
Relation (8) is unique and pretends to be the most important exact evaluation formula of a univariate integral found so far. This statement is justified by the following facts:

- Identity (8) is an elliptic analogue of the binomial theorem.
- It is the top known univariate extension of the Euler beta integral (which includes also the Gaussian integral).
- It defines the measure for two-index biorthogonal functions—the most general univariate special functions with classical properties [5].
- It serves as a germ for all elliptic hypergeometric integrals admitting exact evaluation (including an elliptic analogue of the Selberg integral) and of the whole theory of transcendental elliptic hypergeometric functions [2].
- It proves the confinement phenomenon in a special sector of states of the simplest 4d supersymmetric field theory (as an equality of superconformal indices of dual theories) [6].
- It defines the elliptic Fourier transformation [7] with nice inversion property [8]. The key algebraic identity emerging in the corresponding Bailey lemma is nothing but the star-triangle relation in the operator form, which coincides with the braiding relation for generators of the permutation group [9]. The functional form of this relation serves as a master identity for exactly solvable 2d spin lattice systems of the Ising type [10].

**CLASSICAL SPECIAL FUNCTIONS**

Let us unfold some of the arguments given above. First we give explicit definition of the very-well-poised elliptic hypergeometric series [2]:

\[
_{r,s}V_r(t_0, t_1, \ldots, t_{r-s}; q, p) = \sum_{n=0}^{\infty} \frac{\theta(t_0 q^{2n}; p) \prod_{m=0}^{r-s-1} \theta(q t_{r-m}^n)}{\prod_{k=0}^{r-4} \theta(zq^k; p)} q^n, \tag{9}
\]

where \(\theta(z)_n = \prod_{k=0}^{n-1} \theta(z q^k; p)\) is the elliptic Pochhammer symbol. The word combination “elliptic series” in this context means that the parameters in (9) satisfy the balancing condition \(\prod_{k=1}^{r-4} t_k = t_0^{(r-5)/2} q^{(r-7)/2}\), which guarantees that each term of this series is an elliptic function (a meromorphic double periodic function) of all its parameters. However, the infinite sum of elliptic functions (9) in general does not converge. Therefore one has to terminate it by imposing the constraint \(t_j = q^{-N}, N = 0, 1, \ldots,\) for some fixed \(j\). Similar to the limit (7), for fixed parameters \(t_m\)

\[
\lim_{p \to 0} \_{r-s}V_r = \text{very-well poised, balanced}_{r-1}\varphi_{r-2}\text{-series}. \tag{10}
\]

The Frenkel–Turaev sum [11] provides a closed form expression for the terminating \(_{10}V_9\)-series and it is a special limiting case of the elliptic beta integral (8). The terminating \(_{12}V_{11}\)-series emerges in solutions of the IRF type Yang–Baxter equation [11] and of the Lax pair equations for a 2d discrete-time chain generalizing the Toda lattice [12]. The set of classical (bi)orthogonal functions is described in the enclosed Table—an extension of the Askey scheme for orthogonal polynomials. Its top left corner belongs to the Jacobi polynomials defined by the terminating \(_3F_1\)-series (4), which are orthogonal with respect to the measure (3). And that is the end of differential equations in this context. The most general classical orthogonal polynomials were found by Askey and Wilson [13] and they are defined by a terminating \(_q\varphi_2\)-series. Self-duality means that these polynomials satisfy finite-difference equations of the second order both in the degree of polynomials and in their argument with the same coefficients (i.e. there is a permutational symmetry between the corresponding variables).

Elliptic analogues of the plain and \(q\)-hypergeometric classical special functions naturally emerge only at the top \(q\)-hypergeometric level. Namely, two particular \(_{12}V_{11}\) terminating series form a pair of biorthogonal rational functions representing an elliptic extension of the \(q\)-Racah polynomials, as described in [12]. Proper generalization of the Askey–Wilson polynomials was established in [5]. Moreover, a principally new phenomenon of two-index biorthogonality of (non-rational!) functions emerges at this level, as indicated in the bottom right corner of Table 1. The latter top classical functions were also discovered at BLTP JINR [5]. The presented scheme is far from complete since not all possible limiting transitions and potential generalizations are depicted in it.

**SUPERCONFORMAL INDEX**

Four-dimensional minimal superconformal quantum field theories are based on the full symmetry group \(G_\mu \times G \times F\), where \(G_\mu = SU(2,2 | 1)\) is the flat space-time symmetry group, \(G\) is the local gauge invariance group, and \(F\) is the flavor group of global internal symmetries. The \(SU(2,2 | 1)\) supergroup is generated by \(J_i, J_i^\dagger, (SL(2,\mathbb{C})\) group generators, or Lorentz rotations), \(P_{\mu}, Q_{\alpha}, \bar{Q}_{\alpha}\) (supertranslations), \(K_{\mu}, S_{\mu}, \bar{S}_{\mu}\) (special superconformal transformations), \(H\) (dilations) and \(R\) (\(U(1)_8\)-rotations). Pick up a distinguished pair of supercharges, e.g., \(Q \sim \bar{Q}_1, Q^\dagger \sim \bar{S}_1\) and the maximal Cartan subalgebra generators comprising...
muting with them, \(H - R/2, J_3\), and \(F_k\) (maximal torus generators of \(F\)). Then one has

\[
Q^2 = (Q^1)^2 = 0, \quad \{Q, Q^1\} = 2\mathcal{H}, \quad \mathcal{H} = H - 2J_3 - 3R/2,
\]

and the superconformal index is defined as the trace of the following operator \([14, 15]\)

\[
I(p, q, y_k) = \text{Tr} \left( (-1)^F p^{3/2 + J_3} q^{3/2 - J_3} e^{-\beta \mathcal{H}} \prod y_k \right),
\]

where \(\mathcal{R} = H - R/2, (-1)^F\) is the \(\mathbb{Z}_2\)-grading operator, and \(p, q, y_k\) are fugacities (group parameters). The trace may get non-zero contributions only from the BPS states \(Q | \psi \rangle = 0\), \(\mathcal{H} | \psi \rangle = 0\), so that the \(\beta\)-dependence is cancelled out. A heuristic computation yields the matrix integral

\[
I(y; p, q) = \int_G d\mu(z) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \text{ind}(p^n, q^n, z^n, y^n) \right),
\]

where \(\mu(z)\) is the Haar measure for group \(G\) and

\[
\text{ind}(p, q, z, y) = \frac{2pq - p - q}{(1 - p)(1 - q)} \chi_{\text{adj}}(z) + \sum_f \frac{(pq)^f \chi_{R_j, f}(y) \chi_{R_0, f}(z) - (pq)^{-f} \chi_{R_{-j}, f}(y) \chi_{R_{-0, f}}(z)}{(1 - p)(1 - q)}.
\]

Here \(\chi_{R_j, f}(y), \chi_{R_{-j}, f}(y),\) and \(\chi_{R_0, f}(z)\) are the group characters of respective field representations and \(2r_f \in \mathbb{Q}\) are their \(R\)-charges.

For the unitary group \(SU(N), z = (z_1, \ldots, z_N)\),

\[
\prod_{a=1}^N z_a = 1,
\]

\[
\Delta(z) = \prod_{1 \leq a < b \leq N} (z_a - z_b).
\]

**SEIBERG DUALITY**

Consider electromagnetic duality of the following \(4d\) \(\mathcal{N} = 1\) supersymmetric field theories conjectured by Seiberg in 1994 \([16]\).

**Electric theory** (weak coupling regime): \(G = SU(2), F = SU(6)\), the field/representation content

1. vector superfield: \((\text{adj}, 1)\),

\[
\chi_{SU(2), \text{adj}}(z) = z^2 + z^{-2} + 1,
\]

2. chiral superfield: \((f, f)\),

\[
\chi_{SU(2), f}(z) = z + z^{-1}, \quad r_f = 1/6,
\]

\[
\chi_{SU(6), f}(y) = \sum_{k=1}^6 y_k, \quad \chi_{SU(6), f}(y) = \sum_{k=1}^6 y_k, \quad \prod_{k=1}^6 y_k = 1.
\]
Magnetic theory (strong coupling): \( G = 1, F = SU(6) \) with the single field/representation \( T_A \cdot \Phi_{ij} = -\Phi_{ij} \).

\[
\chi_{SU(6),T_A}(y) = \sum_{1 \leq i < j \leq 6} y_i y_j, \quad r_A = 1/3.
\]

A relation of these theories to the elliptic beta integral was discovered by Dolan and Osborn in 2008 [6]. Namely, after explicit computation of superconformal indices in these theories, \( I_E \) and \( I_M \), and equating them according to the Seiberg duality conjecture, there emerges precisely the elliptic beta integral evaluation formula (8) in the form

\[
I_E (\text{the l.h.s.}) = I_M (\text{the r.h.s.})
\]

with the identification of parameters \( t_k = (pq)^{1/6} y_k, \ k = 1, \ldots, 6 \).

In general, the explicit computability of elliptic hypergeometric integrals serves as the confinement criterion in 4d \( \mathcal{N} = 1 \) supersymmetric gauge theories. The process of integrals’ evaluation has the physical meaning of a transition from the weak to strong coupling regimes in quantum field theories. Among conceptual interpretations of the exact mathematical formulas of the type “\( A = B \)” this example is perhaps the brightest one.

General Seiberg duality [16] deals with much more complicated field theories described in the tables below, where adj, \( f \) and \( \bar{f} \) mean adjoint, fundamental and antifundamental representations, and the last two columns contain corresponding \( U(1) \)-groups charge values.

Electric theory \((G = SU(N), F = SU(M)) \times SU(M)_r \times U(1)_B, \mathcal{N} = M - N)\):

\[
\begin{array}{cccc}
SU(N) & SU(M)_f & SU(M)_r & U(1)_B \\
\hline
f & f & 1 & 1 \\
\bar{f} & 1 & \bar{f} & -1 \\
adj & f & 1 & 0 \\
\end{array}
\]

Magnetic theory \((G = SU(\mathcal{N}), F = \text{the same})\):

\[
\begin{array}{cccc}
SU(\mathcal{N}) & SU(M)_f & SU(M)_r & U(1)_B \\
\hline
f & \bar{f} & 1 & N/\mathcal{N} \\
\bar{f} & 1 & f & -N/\mathcal{N} \\
1 & f & \bar{f} & 0 \\
adj & 1 & 1 & 0 \\
\end{array}
\]

Seiberg conjectured that these two non-abelian gauge field theories are equivalent at the infrared fixed points for \( 3N/2 < M < 3N \) (the conformal window). This conjecture was proved by Dolan and Osborn [6] in the sector of BPS states using the mathematical theorems on symmetry properties of particular elliptic hypergeometric integrals. So, in proper parametrization, the electric index takes the form

\[
I_E = \kappa_N \prod_{i,j=1}^{M} \Gamma(s z_j, t^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2 \pi i z_j},
\]

where \( \prod_{j=1}^{N} z_j = 1, \kappa_N = (p, p)^{N-1}(q, q)^{N-1}/N! \). The magnetic index takes the form

\[
I_M = \kappa_M \prod_{i,j=1}^{M} \Gamma(s f_j^{-1}, p, q)
\]

\[
\times \prod_{i,j=1}^{N} \Gamma(t z_j, t^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2 \pi i z_j},
\]

where \( \prod_{j=1}^{N} x_j = 1, S = \prod_{i=1}^{M} s_i, T = \prod_{i=1}^{M} t_i, ST^{-1} = (pq)^{M-N} \).

The equality \( I_E = I_M \) in some particular cases has been established or conjectured in my papers [4, 5] and it was proved in full generality by Rains in [17].

The elliptic analogue of the Seiberg integral has been introduced in [18] and its relation to the Seiberg type dualities was described in [19]. For \( \|p\|, |q|, |t|, |m| < 1 \) and \( t^{2n-2} \prod_{m=1}^{6} t_m = pq \),

\[
\frac{(p; p)^{\sum}_{m=1} (q; q)^{\sum}_{m=1}}{(4\pi i)^n} \prod_{1 \leq j < k \leq n} \Gamma(t z_j z_k^{-1}; p, q) \prod_{j=1}^{n} \Gamma(t z_j; p, q) \prod_{m=1}^{6} \frac{dz_j}{z_j} (15)
\]

A systematic analysis [19] has shown that the physics of Seiberg like dualities yields many new complicated mathematical conjectures (special function identities), while the mathematics of elliptic hypergeometric integrals produces many new electromagnetic dualities. In a sense, on this ground physics and mathematics work again hand-to-hand as one science. As another important physical outcome, I would like to mention that superconformal indices of quiver gauge theories describe partition functions of 2d spin systems of Ising type, where Seiberg duality serves as the integrability condition [20].

Nowadays, computations of supersymmetric partition functions became an industry in producing special function identities with many contributors [21].
Let me mention some other results: the work [22] deals with applications to 2d topological field theories, the work [23] deals with a physical interpretation of $W(E_7)$ symmetry of the elliptic extension of Euler–Gauss hypergeometric function [2], in [24] an intriguing hypothesis was put forward on the relation to Nekrasov instanton sums, [25] contains an exact computation of the 4d supersymmetric partition functions by the localization technique, in [26] a deeper picture of relations to 2d solvable statistical mechanics systems is given, some more recent developments of the subject are reflected in [27, 28].

Right after giving this talk I have written my last message to Dick as a part of the Liber Amicorum collection prepared by his numerous friends and shortly handed to him, which was a very timely step.

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ADDITIONAL INFORMATION

Talk given at the conference Problems of Theoretical and Mathematical Physics dedicated to 110th anniversary of the birth of N.N. Bogolyubov (Moscow and Dubna, 09-13.09.2019), http://thproxy.jinr.ru/video/bog2019/mp4/13_Spiridonov.mp4.

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