A model of Public key cryptography using multinacci matrices

Munesh Kumari\textsuperscript{1,*}, Jagmohan Tanti\textsuperscript{2†}

\textsuperscript{1,2}Department of Mathematics, Central University of Jharkhand, India, 835205

March 20, 2020

Abstract

In this paper, we have proposed a Public Key Cryptography (PKC) using block matrices with generalized Fibonacci sequence. First, we have shown the multiplicative commutativity of generalized matrices which are constructed using generalized Fibonacci sequences and then we develop a cryptographical scheme. We also discuss the efficiency and strength of proposed scheme in context of block matrices.

\textbf{Keyword}: Public Key Cryptography, Multinacci Sequences & Matrices, Key Distribution \\
\textbf{Mathematics Subject Classifications}: 11T71, 14G50, 68P25, 68P30, 68R01, 94A60

1 Introduction

Information is one of the most valuable assets since the dawn of civilizations. The secured transmission of information is of prime importance. Cryptography is the science of study about the security, privacy, and confidentiality of information transmitted over a secured channel. Although many encryption-decryption schemes exist but the need for new nonstandard encryption algorithms raises to prevent any traditional opportunities to steal or change the information. Also reducing the time and space complexity of key generation with enhancement in security so that it is difficult for an intruder to steal or change the information becomes a need for today’s time..

\*E-mail: muneshnasir94@gmail.com, ORCID ID: https://orcid.org/0000-0002-6541-0284/ \\
†E-mail: jagmohan.t@gmail.com
The problem concerned to the topic is to develop a model of public key cryptography with the following parameters:

- Reduction in complexity for key generation.
- Enhancement of security in the transmission of information.

As an example, for the same level of security, the RSA cryptosystem uses a bigger key size than the key size of Elliptic curve cryptography. There is a need for finding one model which has the least complexity but maximum possible security.

In 1976, Diffie and Hellman [2] provided a solution to the long-standing problem of key exchange and pointed the way to digital signature. In 1978, Rivest, Shamir & Adleman [9] proposed a public key cryptosystem which famed as RSA cryptosystem. The security of RSA cryptosystem depends on the difficulty level of factoring large integers.

R. ALVAREZA, et.al [1] proposed a public key cryptosystem (PKC) based on the generalization of the discrete logarithm problem for block matrices over the field $\mathbb{Z}_p$ with reduced key length for a given level of security. Kuppuswamy, et.al [8] has given two different encryption algorithms; One is public key cryptography based on a linear block cipher and the second one is private key cryptography based on a simple symmetric algorithm. Viswanath and Kumar [12] proposed a public key cryptosystem using Hill’s cipher, in which security of the system depends on the involvement of two digital signatures.

M. Zeriouh, et.al [13], proposed the concept of key exchange between Alice and Bob using specially designed matrices. In this key exchange scheme, each of the sender and receiver first chooses a square matrix of suitable order and then both publish their corresponding set of matrices which commute with their corresponding chosen matrices.

In this paper, we propose a modified key exchange scheme and public key cryptography based on generalized Fibonacci matrices and affine-hill cipher. In this proposed scheme, the time complexity of key matrix generation is reduced to $O(1)$. The security of our scheme depends on the discrete logarithm problem. So our proposed scheme is secure against various known attacks.

This paper is organized as follows. In section 2, the basic concepts of Affine Cipher, generalized matrices and block matrices are outlined. In section 3, we have shown commutativity of the generalized Fibonacci matrices and developed public key cryptography. In section 4, we have described an example of proposed scheme. Finally in section 5, we present the crypt-analytic strength of model.
2 Preliminaries

Definition 2.1 (Affine-Hill Cipher). Affine-Hill Cipher is defined as:

**Encryption:** \[ C = (KP + B) \mod p \] (2.1)

**Decryption:** \[ P = K^{-1} (C - B) \mod p \] (2.2)

where \( P, C \) & \( K \) represents plaintext, ciphertext & key matrix respectively and \( p \geq 26 \) is a prime with \( \gcd(\det(K), \ p) = 1 \).

2.1 Generalization of Fibonacci sequence and Fibonacci matrix

Definition 2.2 (Fibonacci Sequence). Fibonacci Sequence is defined by the recurrence relation

\[ t_{k+2} = t_k + t_{k+1}, \quad k \geq 0, \text{ with } t_0 = 0, \ t_1 = 1 \] (2.3)

and the terms of this sequence are called Fibonacci numbers.

Note that, the Fibonacci sequence can also be extended in negative direction [5][7].

**Fibonacci Matrix:** Fibonacci Matrix was first used by Brenner and it’s basic properties were enumerated by King [6]. In 1985, Honsberger [4] showed that the Fibonacci matrix \( F \) is a square matrix of order \( 2 \times 2 \) of the form

\[
F_2 = \begin{bmatrix}
t_2 & t_1 \\
t_1 & t_0
\end{bmatrix} = \begin{bmatrix}1 & 1 \\1 & 0
\end{bmatrix}
\]

and it’s \( k^{th} \) power is defined as

\[
F_2^k = \begin{bmatrix}t_{k+1} & t_k \\
t_k & t_{k-1}
\end{bmatrix}
\] (2.4)

In notation, here we use \( F_2 \) instead of \( F_2^1 \) and in general \( F_n \) instead of \( F_n^1 \) for \( k = 1 \).

Generalization of Fibonacci matrix

Definition 2.3. The generalized Fibonacci sequence of order \( n \) is defined by the recurrence relation

\[ t_{k+n} = t_k + t_{k+1} + t_{k+2} + \ldots + t_{k+n-1}, \quad k \geq 0 \] (2.5)

with \( t_0 = t_1 = \ldots = t_{n-2} = 0 \) and \( t_{n-1} = 1 \).

It can also be stated as multinacci sequence [5].

In particular, for \( n = 2 \), relation (2.5) is Fibonacci sequence[A000045] and for \( n = 3 \), (2.5) is called tribonacci sequence [5][7][A000073].

The generalized Fibonacci matrix \( F_n \) of order \( n \) is given by
Lemma 2.1. Matrix $A_{j,l} = A_{l,j}$ if $G.M = M.G$ and $H.N = N.H$.
3 Main Work

Lemma 3.1. Generalized Fibonacci matrices commute with each other i.e. if \( F^k_n \) and \( F^l_n \) are two generalized matrices for some \( k, l \in \mathbb{N} \) then \( F^k_n \cdot F^l_n = F^l_n \cdot F^k_n \).

Proof. We prove this lemma by mathematical induction on \( l \) and fixing \( k \).

For, \( l = 1 \),

\[
F^k_n \cdot F^1_n = \begin{bmatrix}
  t_{k+n-1} & t_{k+n-2} + \ldots + t_k & \ldots & & t_{k+n-2} \\
  t_{k+n-2} & t_{k+n-3} + \ldots + t_{k-1} & \ldots & & t_{k+n-3} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  t_{k+1} & t_k + \ldots + t_{k-(n-2)} & \ldots & t_k \\
  t_k & t_{k-1} + \ldots + t_{k-(n-1)} & \ldots & t_{k-1}
\end{bmatrix}
\begin{bmatrix}
  1 & 1 & \ldots & 1 & 1 \\
  1 & 0 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & 0 & 0 \\
  0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  t_{k+n-1} + \ldots + t_k & t_{k+n-1} + \ldots + t_{k+1} & \ldots & t_{k+n-1} \\
  t_{k+n-2} + \ldots + t_{k-1} & t_{k+n-2} + \ldots + t_k & \ldots & t_{k+n-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  t_{k+1} + \ldots + t_{k-(n-2)} & t_{k+1} + \ldots + t_{k-(n-3)} & \ldots & t_{k+1} \\
  t_k + \ldots + t_{k-(n-1)} & t_k + \ldots + t_{k-(n-2)} & \ldots & t_k
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  t_{k+n} & t_{k+n-1} + t_{k+n-2} + \ldots + t_{k+1} & \ldots & t_{k+n-1} \\
  t_{k+n-1} & t_{k+n-2} + t_{k+n-3} + \ldots + t_k & \ldots & t_{k+n-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  t_{k+2} & t_{k+1} + t_k + \ldots + t_{k-(n-3)} & \ldots & t_{k+1} \\
  t_{k+1} & t_k + t_{k-1} + \ldots + t_{k-(n-2)} & \ldots & t_k
\end{bmatrix}
\]

\[
= F^k_{n+1}.
\]
And,

\[
F_n^1 \cdot F_n^k = \begin{bmatrix}
1 & 1 & \ldots & 1 & 1 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \ldots & 1 & 1 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix} = \begin{bmatrix}
t_k + n - 1 + t_k & t_k + n - 2 + t_k & \ldots & t_k + n - 2 & t_k + n - 1 \\
t_k + n - 2 + t_k & t_k + n - 3 + t_k & \ldots & t_k + n - 3 & t_k + n - 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t_k + 2 & t_k + 1 + t_k & \ldots & t_k + 1 & t_k \\
t_k + 1 & t_k + t_k & \ldots & t_k + t_k & t_k
\end{bmatrix}
\]

Using (2.5)

\[
= F_n^{k+1}. \tag{3.2}
\]

Thus, from equation (3.1) and (3.2),

\[
F_n^1 \cdot F_n^k = F_n^k \cdot F_n^1 = F_n^{k+1}. \tag{3.3}
\]

Therefore result holds for \( l = 1 \).

Assume that the statement is true for \( l = r \),

\[
\implies F_n^r \cdot F_n^k = F_n^k \cdot F_n^r. \tag{3.4}
\]

Then for \( l = r + 1 \),

\[
F_n^{r+1} \cdot F_n^k = F_n^r \cdot F_n^1 \cdot F_n^k = F_n^r \cdot F_n^k \cdot F_n^1 = F_n^k \cdot F_n^r \cdot F_n^1 = F_n^k \cdot F_n^{r+1}. \]

So, result is true for \( l = r + 1 \).

Hence, by the principal of mathematical induction, result is true for every \( l \in \mathbb{N} \).

Here, we see that generalized Fibonacci matrix commute with each other.

Hence satisfy lemma (2.1).

### 3.1 Key Generation Algorithm

Let the set \( S = \{ F_n^k | F_n^k \text{ is a generalized Fibonacci matrix, } k \in \mathbb{N} \} \).
Construction of public key:

i. Alice chooses a prime number $p$, $l \in \mathbb{N}$ and matrices $K, G, H \in S$.

ii. Calculate

$$K_j = \sum_{r=0}^{l-1} (G)^{l-1-r} \ast K \ast (H)^r \pmod{p}.$$

Here, $(p, K, K_l)$ is Alice’s public key and $(G, H, l)$ is her secret key.

Key Generation and Encryption:

Using Alice’s public key $(p, K, K_l)$, Bob will generate first encryption key and then encrypts his plaintext as follows:

i. Bob choose his secret key $j \in \mathbb{N}$ and $M, N \in S$.

ii. Calculate $K_j$ as

$$K_j = \sum_{s=0}^{j-1} (M)^{l-1-s} \ast K \ast (N)^s \pmod{p}.$$

iii. Further,

$$K_{l,i,j} = \sum_{s=0}^{j-1} (M)^{l-1-s} \ast K_l \ast (N)^s \pmod{p}.$$

Thus, encryption key (say $E_k$) = $K_{l,i,j}$.

iv. Calculate $B$, where $B$ is a row vector in $\mathbb{Z}_p$ whose $i^{th}$ column is the sum of elements of $i^{th}$ column of $E_K$.

v. Encryption takes place as,

$$C \equiv (PE_K + B) \pmod{p}.$$

vi. Bob sends $(K_j, C)$ to Alice.

Decryption:

After receiving $(K_j, C)$ from Bob, Alice perform following operations to recover plaintext:

i. Alice first calculate key matrix $E_k$ as,

$$E_k = K_{j,l} = \sum_{r=0}^{l-1} (G)^{l-1-r} \ast K_j \ast (H)^r \pmod{p}.$$
ii. Thus, decryption key (say $D_K$) = $(K_{j,l})^{-1}$.

iii. Decryption of ciphertext

$$P \equiv ((C - B)D_K) \pmod{p},$$

where $B$ is a row vector whose $i^{th}$ column is the sum of elements of $i^{th}$ column of $K_{j,l}$ in $\mathbb{Z}_p$.

4 Example

Example 1. Let prime $p=47$ and $S = \{F^k_3 | F_3$ is a generalized Fibonacci matrix of order 3, $k \in \mathbb{N}\}$. Encrypt the plaintext HEY.

Construction of Public Key:

(i). Suppose Alice’s secret key $l = 5$, secret matrices $G, H \in S$ and public matrix $K \in S$ are

$$G = F^9_3 = \begin{bmatrix} 8 & 31 & 34 \\ 34 & 21 & 44 \\ 44 & 37 & 24 \end{bmatrix}, H = F^{13}_3 = \begin{bmatrix} 13 & 21 & 34 \\ 34 & 26 & 34 \\ 34 & 0 & 9 \end{bmatrix} \text{ and } K = F^2_3 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (4.1)$$

(ii). Construction of public key,

$$K_5 = \sum_{r=0}^{4} (G)^{4-r} \ast K \ast (H)^r \pmod{47} \equiv \begin{bmatrix} 42 & 25 & 5 \\ 5 & 37 & 20 \\ 20 & 32 & 17 \end{bmatrix}. $$

Thus, here Alice’s public Key is

$$\begin{bmatrix} 47, & 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 42 & 25 & 5 \\ 5 & 37 & 20 \\ 20 & 32 & 17 \end{bmatrix}$$

and

Alice’s secret key is

$$\begin{bmatrix} 5, & 8 & 31 & 34 \\ 34 & 21 & 44 \\ 44 & 37 & 24 \end{bmatrix}, \begin{bmatrix} 13 & 21 & 34 \\ 34 & 26 & 34 \\ 34 & 0 & 9 \end{bmatrix}.$$

Key Generation and Encryption:

Bob generate encryption key using Alice’s public key

$$\begin{bmatrix} 47, & 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 42 & 25 & 5 \\ 5 & 37 & 20 \\ 20 & 32 & 17 \end{bmatrix}.$$
1. $P \leftarrow [07, 04, 24]$. \hspace{1cm} (Here, $[H, E, Y] = [07, 04, 24]$)

2. Bob chooses his secret number, say $m=3$ and secret matrices $M, N \in S$, which are

$$M = F_3^7 = \begin{bmatrix} 44 & 37 & 24 \\ 24 & 20 & 13 \\ 13 & 11 & 7 \end{bmatrix} \quad \text{and} \quad N = F_3^{15} = \begin{bmatrix} 34 & 0 & 34 \\ 34 & 0 & 13 \\ 13 & 21 & 34 \end{bmatrix}. \quad (4.2)$$

3. Calculation of $K_3$,

$$K_3 = \sum_{s=0}^{2} (M)^{2-s} \ast K \ast (N)^{s} \mod 47 = \begin{bmatrix} 24 & 4 & 19 \\ 19 & 5 & 32 \\ 32 & 34 & 20 \end{bmatrix}.$$ 

4. Calculation of encryption key,

$$K_{5,3} = \sum_{s=0}^{2} (M)^{2-s} \ast K_3 \ast (N)^{s} \mod 47 = \begin{bmatrix} 34 & 19 & 5 \\ 5 & 29 & 14 \\ 14 & 38 & 15 \end{bmatrix}.$$ 

So, encryption key $E_K = K_{5,3}$.

5. Row vector $B$ over $\mathbb{Z}_{47}$, $B = [6 \hspace{1cm} 39 \hspace{1cm} 34]$.

6. Encryption: $C \equiv (P E_K + B) \mod 47$.

$$C \equiv \begin{bmatrix} 7 & 4 & 24 \end{bmatrix} \ast \begin{bmatrix} 34 & 19 & 5 \\ 5 & 29 & 14 \\ 14 & 38 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 39 & 34 \end{bmatrix} \mod 47$$

$$\equiv \begin{bmatrix} 36 & 25 & 15 \end{bmatrix} \rightarrow [> Z P].$$

Here, plaintext $[H E Y]$ encrypted as $[> ZP]$.

7. Bob sends $(C, K_3) = \begin{bmatrix} [> ZP], \\ 24 & 4 & 19 \\ 19 & 5 & 32 \\ 32 & 34 & 20 \end{bmatrix}$ to Alice.

Decryption:

After receiving $(C, K_3) = \begin{bmatrix} [> ZP, \\ 24 & 4 & 19 \\ 19 & 5 & 32 \\ 32 & 34 & 20 \end{bmatrix}$, Alice recover plaintext as
1. First calculate key matrix as

\[ K_{3:5} = S_K = \sum_{r=0}^{4} (G)^{4-r} \cdot K_3 \cdot (H)^r = \begin{bmatrix} 34 & 19 & 5 \\ 5 & 29 & 14 \\ 14 & 38 & 15 \end{bmatrix} . \]

2. Decryption key, \( D_K = (K_{3:5})^{-1} = \begin{bmatrix} 43 & 30 & 36 \\ 36 & 7 & 41 \\ 41 & 42 & 13 \end{bmatrix} . \)

3. Decryption:

\[ P \equiv ((C - B)D_K) \pmod{47}, \quad (4.3) \]

4. \( C = [>, Z, P] \rightarrow [36 \ 25 \ 15] \) and row vector \( B = [6 \ 39 \ 34] \).

Now recovering the plaintext according to (4.3), we have

\[ P \equiv \left( \begin{bmatrix} 36 & 25 & 15 \\ 6 & 39 & 34 \end{bmatrix} - \begin{bmatrix} 6 & 39 & 34 \end{bmatrix} \right) \cdot \begin{bmatrix} 43 & 30 & 36 \\ 36 & 7 & 41 \\ 41 & 42 & 13 \end{bmatrix} \pmod{47} \]

\[ \equiv \begin{bmatrix} 7 & 4 & 24 \end{bmatrix} \rightarrow [H \ E \ Y]. \]

Thus, Alice recovers \([H \ E \ Y]\) from \([> \ Z \ P]\) successfully.

5. **Complexity of Key matrix**

In our work, time complexity \([\Omega]\) of generating key matrix using equation (2.6) is reduced to

\[ B_l = \sum_{r=0}^{l-1} (F_{n}^{k_1})^{l-1-r} \cdot (F_{n}^{k_2}) \cdot (F_{n}^{k_3})^r \]

\[ = \sum_{r=0}^{l-1} (F_{n})^{r(k_3-k_1)+k_1(l-1)+k_2} \quad (5.1) \]

where, \( M = F_{n}^{k_1}, C = F_{n}^{k_2} \) and \( N = F_{n}^{k_3} \).

So time complexity of construction is reduced to \( O(1) \). Security strength of proposed method depends upon the private keys of sender \((j, M, N)\) and receiver \((l, G, H)\). Since, it is almost impossible to recover \( M, N \) from \( K_{l:j} \) because there is no any deterministic algorithm (Discrete logarithm problem \([3][10]\)) even intruder knows the matrix \( K \).
6 Conclusion

Here we have proposed public key cryptography and modification in key exchange method discussed in paper [?] using generalized Fibonacci matrices. We have used commutative property of generalized Fibonacci matrices to reduce complexity of construction. In above proposed scheme, neither sender nor receiver need to publish their corresponding set of matrices which commute with chosen matrix of sender and receiver respectively. Also our proposed scheme is secured against various known attacks. So, Our method is quite robust and can be implemented easily.

7 Acknowledgment

First author would like to thank University Grant Commission, India for financial support. The authors are grateful to Central University of Jharkhand, India for kind support.

References

[1] ALVAREZ, R., MARTINEZ, F.-M., VICENT, J.-F., AND ZAMORA, A. A new public key cryptosystem based on matrices. WSEAS Information Security and Privacy (2007), 3639.

[2] DIFFIE, W., AND HELLMAN, M. New directions in cryptography. IEEE transactions on Information Theory 22, 6 (1976), 644654.

[3] HOFFSTEIN, J., PIPHER, J., SILVERMAN, J. H., AND SILVERMAN, J. H. An introduction to mathematical cryptography, vol. 1. Springer, 2008.

[4] HONSBERGER, R. Mathematical gems iii, dolciani math, 1985.

[5] JOHNSON, R. C. Fibonacci numbers and matrices. manuscript available at http://www. dur. ac. uk/bob. johnson/fibonacci (2008).

[6] KING, C. Some further properties of the fibonacci numbers. San Jose: CA (1960).

[7] KOSHY, T. Fibonacci and Lucas numbers with applications. John Wiley & Sons, 2019.

[8] KUPPUSWAMY, P., AND AL-KHALIDI, S. Q. Hybrid encryption/decryption technique using new public key and symmetric key algorithm. MIS Review: An International Journal 19, 2 (2014), 113.
[9] RIVEST, R. L., SHAMIR, A., AND ADLEMAN, L. A method for obtaining digital signatures and public-key cryptosystems. Communications of the ACM 21, 2 (1978), 120126.

[10] STINSON, D. R. Cryptography: theory and practice. Chapman and Hall/CRC, 2005.

[11] STOTHERS, A. J. On the complexity of matrix multiplication, The University of Edinburgh (2010).

[12] VISWANATH, M., AND KUMAR, M. R. A public key cryptosystem using hills cipher. Journal of Discrete Mathematical Sciences and Cryptography 18, 1-2 (2015), 129 138.

[13] ZERIOUH, M., CHILLALI, A., AND BOUA, A. Cryptography based on the matrices. Boletim da Sociedade Paranaense de Matematica 37, 3 (2019), 7583.