JACOB’S LADDERS AND $|\zeta|^{-2}$-REPRESENTATION OF SOME FUNCTIONALS GENERATED BY A NEW CLASS OF TRANSCENDENTAL INTEGRALS

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Abstract. In this paper we introduce a new infinite set of transcendental integrals. Each of them is expressed by corresponding value of the function $|\zeta\left(\frac{1}{2} + it\right)|^{-2}$. Such a property is another argument about universality of the Riemann zeta-function $\zeta\left(\frac{1}{2} + it\right)$ in the field of pure mathematics.

1. Introduction

1.1. In our paper [7] we have postulated the radius of the Universe $R(t)$ as

\[ R(t) = h\left(\left|\zeta\left(\frac{1}{2} + it\right)\right|\right), \quad t \to \infty, \tag{1.1} \]

(see [7], (3.5), (4.1)). By making use this together with the Riemann hypothesis we have constructed:

(a) a new infinite set of mathematical universes (of the Einstein’s type),
(b) an infinite subset of microscopic universes (on the Planck’s scale) of the Einstein’s type in the early period (just after inflation) of the evolution of the Universe.

Remark 1. Above mentioned results can be understood as an argument for the universality (comp. [2], p. 135) of the modulus of Riemann zeta-function on the critical line.

1.2. In this paper we will study the property of universality of the function

\[ \left|\zeta\left(\frac{1}{2} + it\right)\right| \]

in another direction. Namely, we will study a new class of transcendental integrals of the following type

\[ \int_a^b |H[\varphi_1(t); \tau]|^\alpha H'_\varphi_1[\varphi_1(t); \tau]|dt, \quad a < \tau < b, \quad \alpha > 0, \]

where

\[ \varphi_1(t) \sim t - (1 - c)\pi(t), \quad \pi(t) \sim \frac{t}{\ln t}, \quad t \to \infty, \tag{1.2} \]

and $\pi(t)$ is the prime-counting function and $c$ is the Euler constant. We will prove that there are numbers

\[ \bar{a}, \bar{b}, \bar{\alpha}, \bar{t}_H, \]

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and a function

\[ h_1(u) \]

such that

\[
\int_{\tilde{a}}^{\tilde{b}} |H[\varphi_1(t); \tau]|^{\tilde{\alpha}} |H'[\varphi_1(t); \tau]| \, dt =
\]

\[
= h_1 \left( \left| \zeta \left( \frac{1}{2} + it_H \right) \right| \right), \quad t_H = t(H; \tilde{\alpha}), \quad \tilde{a} \to \infty
\]

(comp. with (1.1)), for every fixed Jacob’s ladder \( \varphi_1(t) \).

1.3. The function \( \varphi_1(t) \) (see (1.1) – (1.3)) that we call Jacob’s ladder (see [5]) according to the Jacob’s dream in Chumash, Bereishis, 28:12 has the following properties:

(a) \[
\varphi_1(t) = \frac{1}{2} \varphi(t),
\]

(b) the function \( \varphi(t) \) is the solution of nonlinear integral equation (see [5], [6])

\[
\int_{\mu(x(T))}^{\mu(T)} Z^2(t)e^{-\frac{\pi t^2}{4}} \, dt = \int_{\mu(y)}^{T} Z^2(t) \, dt,
\]

where

\[
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right),
\]

\[
\vartheta(t) = -\frac{t}{2} \ln \pi + \ln \Gamma \left( \frac{1}{4} + \frac{i t}{2} \right),
\]

and each advisable function \( \mu(y) \) generates a solution

\[
y = \varphi_\mu(T) = \varphi(T); \quad \mu(y) \geq 7y \ln y.
\]

Remark 2. The main reason for the introduction of the Jacob’s ladders in the paper [5] lies in the proof of the following theorem:

the Hardy-Littlewood integral (1918)

\[
\int_{0}^{T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt
\]

has – in addition to the Hardy-Littlewood (and other similar) expression possessing an unbounded error term as \( T \to \infty \) – the following infinite set of almost exact expressions

\[
\int_{0}^{T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi) \varphi_1(T) + c_0 +
\]

\[
+ \mathcal{O} \left( \frac{\ln T}{T} \right), \quad T \to \infty,
\]

where \( c_0 \) is the constant from the Titchmarsh-Kober-Atkinson formula (see [8], p. 141).
Remark 3. Simultaneously with (1.5) we have proved that the following transcendental equation
\[ \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = V(T) \ln V(T) + (c - \ln 2\pi) V(T) + c_0 \]
has an infinite set of asymptotic solutions
\[ V(T) = \varphi_1(T), \; T \to \infty. \]

2. The result

The following theorem holds true in the direction (1.3).

Theorem. Let
(a) \( G(t) \in C_1[T_0, \infty), \; T_0 > 0, \)
(b) the symbols
\[ \{\gamma\}, \{t_0\}; \; t_0 \neq \gamma; \; \gamma = \gamma[G], t_0 = t_0[G] \]
denote the sequence of the roots of equations
\[ G(t) = 0, \; G'(t) = 0 \]
respectively,
(c) the points of the sequences \( \{\gamma\}, \{t_0\} \) are separated each from other, i.e.
\[ \gamma' < t_0 < \gamma''; \; \gamma'' - \gamma' \in \left(0, \frac{\gamma'}{\ln \gamma'}\right), \]
where \( \gamma', \gamma'' \) are neighbouring points of the sequence \( \{\gamma\} \), (i.e. \( G(t_0) \) is the local extreme of the function \( G(t), t \in [\gamma', \gamma''] \)),
(d) \( \varphi_1 ([\gamma', \gamma'']) = [\gamma', \gamma''] \),
(e) \( H[\varphi_1(t); t_0] = \frac{G[\varphi_1(t)]}{G[\varphi_1(t_0)]} = \frac{G[\varphi_1(t)]}{G(t_0)}; \; t_0 = \varphi_1(t_0). \)

Then there is the function
(2.1) \[ \omega(x) = \omega(x; H) = 1 + \mathcal{O} \left( \frac{\ln \ln x}{\ln x} \right), \; x \to \infty \]
and the point
\[ t_H = t(H; \omega) \in (\gamma', \gamma'') \]
such that
(2.2) \[ \int_{\gamma'}^{\gamma''} |H[\varphi_1(t); t_0]|^{2\omega(\gamma')} \ln \gamma' - 1 \ln |H'_{\varphi_1}[\varphi_1(t); t_0]| dt = \frac{1}{\left| \zeta \left( \frac{1}{2} + it_H \right) \right|^2}, \; \gamma' \to \infty, \]
where, of course,
\[ 2\omega(\gamma') \ln \gamma' - 1 \to \infty, \; \gamma' \to \infty. \]

Remark 4. The formula (2.2):
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Jacob's ladder . . .

(a) introduces a new infinite set of transcendental integrals each of them is expressed by the corresponding value of the function

\[ \left| \zeta \left( \frac{1}{2} + it \right) \right|^{-2}, \]

i. e. \( h_1(u) = u^{-2} \), comp. \[1.3\].

(b) supports the argument about the universality of the Riemann zeta-function \( |\zeta \left( \frac{1}{2} + it \right)| \) in the field of pure mathematics (comp. Remark 1),

(c) is not reachable by recent methods in the theory of the Riemann zeta-function.

3. Examples concerning Jacobi elliptical functions

3.1. Let us consider the continuum set of functions

\[ \text{sn} \, x = \text{sn}(x; k), \; x > 0, \; k^2 \in (0, 1), \]

where (comp. \[9\], pp. 35-40)

\[ K = K(k) = \int_{0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \]

Then we have

\[ \gamma = \gamma[\text{sn}]; \; \gamma_l = 2lK, \; l \in \mathbb{N}_0, \; \gamma'' - \gamma' = 2K = O(1); \; \gamma_l \to \infty, \]

\[ t_0 = t_0[\text{sn}]; \; t_0(l) = (2l + 1)K, \]

\[ \text{sn}' x = \text{cn} x \; \text{dn} x; \; \text{dn} x > 0, \]

\[ |\text{sn} t_0| = |\text{sn}(2l + 1)K| = 1 \Rightarrow |G(t_0)| = 1 \Rightarrow \]

\[ \Rightarrow |H[\varphi_1(t); t_0]| = |G[\varphi_1(t)]| = |\text{sn}[\varphi_1(t)]|. \]

Consequently we obtain (see \(2.2\)) the following

**Example 1.** There is

\[ t_s \in (\gamma'[\text{sn}], \gamma''[\text{sn}]) \]

such that

\[ \int_{\gamma'}^{\gamma''} |\text{sn}[\varphi_1(t)]|^{2\omega(\gamma')} \ln |\text{cn}[\varphi_1(t)]| \; \text{dn}[\varphi_1(t)] \; dt = \]

\[ = \frac{1}{\left| \zeta \left( \frac{1}{2} + it_s \right) \right|^2}, \; \gamma' \to \infty. \]

3.2. Next, for continuum set of functions

\[ \text{cn} \, x = \text{cn}(x; k), \; x > 0 \]

we have by the similar way that

\[ \gamma = \gamma[\text{cn}]; \; \gamma_l = (2l + 1)K, \; l \in \mathbb{N}_0, \]

\[ t_0 = t_0[\text{cn}]; \; t_0(l) = (2l + 2)K, \]

\[ \text{cn}' x = - \text{sn} x \; \text{dn} x, \]

\[ |\text{cn} t_0| = |\text{cn}(2l + 2)K| = 1 \Rightarrow |G(t_0)| = 1 \Rightarrow \]

\[ \Rightarrow |H[\varphi_1(t); t_0]| = |G[\varphi_1(t)]| = |\text{cn}[\varphi_1(t)]|. \]

Consequently, we have the following

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Example 2. There is
\[ t_c \in (\gamma''[\text{cn}], \gamma''[\text{cn}]) \]
such that
\[ \int_{\gamma'}^{\gamma''} \left| \text{cn}[\varphi_1(t)] \right|^{2\omega(\gamma')} \ln \gamma' - 1 \left| \text{sn}[\varphi_1(t)] \right| \text{dn}[\varphi_1(t)] \, dt = \frac{1}{\zeta \left( \frac{1}{2} + it_c \right)^2}, \quad \gamma' \to \infty. \]

4. Example concerning the Bessel’s functions
Let
\[ \gamma = \gamma \left[ \frac{J_\nu(x)}{x^\nu} \right], \quad t_0 = t_0 \left[ \frac{J_\nu(x)}{x^\nu} \right]. \]
For our purpose it is sufficient to consider the case
\[ \nu > -1, \quad x \to \infty. \]
We have from the theory of the Bessel’s functions that
\[ \gamma' < t < \gamma'', \quad \gamma'' - \gamma' \sim \pi, \quad \gamma' \to \infty, \]
(comp. [11], p. 361; [10], p. 91) where, of course,
\[ \frac{J_{\nu+1}(x)}{x^\nu} = -\frac{d}{dx} \left[ \frac{J_\nu(x)}{x^\nu} \right]. \]
Next we have
\[ G[\varphi_1(t)] = \frac{J_\nu[\varphi_1(t)]}{[\varphi_1(t)]^\nu}, \quad G(t_0) = \frac{J_\nu(t_0)}{t_0^\nu}; \quad \varphi_1(t_0) = t_0, \]
\[ H[\varphi_1(t); t_0] = \frac{J_\nu[\varphi_1(t)]}{[\varphi_1(t)]^\nu} \frac{t_0}{J_\nu(t_0)}. \]
Consequently, we obtain (see (2.2)) the following

Example 3. There is
\[ t_J \in \left( \gamma' \left[ \frac{J_\nu(x)}{x^\nu} \right], \gamma'' \left[ \frac{J_\nu(x)}{x^\nu} \right] \right) \]
such that
\[ \int_{\gamma'}^{\gamma''} \left| \frac{J_\nu[\varphi_1(t)]}{[\varphi_1(t)]^\nu} \right| \frac{t_0^\nu}{J_\nu(t_0)} \left| 2^{2\omega(\gamma')} \ln \gamma' - 1 \right| \frac{J_{\nu+1}[\varphi_1(t)]}{[\varphi_1(t)]^\nu} \frac{t_0^\nu}{J_\nu(t_0)} \, dt = \frac{1}{\zeta \left( \frac{1}{2} + it_J \right)^2}, \quad \gamma' \to \infty. \]
5. Example concerning the Riemann zeta-function

In the case of the function $Z(t)$ (see (1.4)) we have, on the Riemann hypothesis, that

$$\gamma' < t_0 < \gamma'', \gamma' \to \infty,$$

where

$$\gamma = \gamma[Z], \quad t_0 = t_0[Z],$$

i.e. the points of the sequences $\{\gamma\}, \{t_0\}$ are separated each from other (see [4], Corollary 3). Next, on the Riemann hypothesis also the Littlewood’s estimate

$$\gamma'' - \gamma' < A \ln \ln \gamma', \quad \gamma' \to \infty$$

holds true (see [3]). Since

$$H[\varphi_1(t); t_0] = \frac{Z[\varphi_1(t)]}{Z(t_0)},$$

then we obtain the following

**Example 4.** On the Riemann hypothesis there is

$$t_Z \in (\check{\gamma}'[Z], \check{\gamma}''[Z])$$

such that

$$\int_{\check{\gamma}'}^{\check{\gamma}''} \left| \frac{Z[\varphi_1(t)]}{Z(t_0)} \right|^2 \left| \frac{Z'[\varphi_1(t)]}{Z(t_0)} \right| \, dt = \frac{1}{|\zeta \left( \frac{1}{2} + it \right)|^2}, \quad \gamma' \to \infty. \tag{5.1}$$

**Remark 5.** Consequently, we obtain by (5.1) by the first formula in (1.4) the following sufficiently complicated integral

$$\int_{\check{\gamma}'}^{\check{\gamma}''} \left| \frac{\zeta \left( \frac{1}{2} + it \varphi_1(t) \right) \zeta' \left( \frac{1}{2} + it \varphi_1(t) \right)}{\zeta' \left( \frac{1}{2} + it_0 \right) \zeta \left( \frac{1}{2} + it_0 \right)} \right|^2 \left| \frac{\zeta' \left( \frac{1}{2} + it \varphi_1(t) \right) + \zeta \left( \frac{1}{2} + it \varphi_1(t) \right)}{\zeta \left( \frac{1}{2} + it \right) \zeta' \left( \frac{1}{2} + it_0 \right)} \right| \, dt = \frac{1}{|\zeta \left( \frac{1}{2} + it \right)|^2}, \quad \gamma' \to \infty.$$

6. Topological deformations of some element and $|\zeta|^{-2}$-representation of a corresponding functionals connected with (2.2)

Let some element

$$\tilde{G}(t) = C_1[\gamma', \gamma'']$$

fulfill the assumptions (b) and (c) of Theorem. Let

$$\{G_T(t)\}$$

denote the continuum set of all topological deformations $G_T(t)$ of the graph of the function $\tilde{G}(t)$ such that every

$$G_T(t), \quad t \in [\gamma', \gamma'']$$
fulfils (b) and (c) of Theorem, and

\[ G_T(t) \in C_1[\gamma', \gamma'']. \]

Let, finally,

\[ H_T[\varphi_1(t); t_0] \in D\{[\gamma', \gamma'']\} \iff\]

\[ H_T[\varphi_1(t); t_0] = \frac{G_T[\varphi_1(t)]}{G_T[\varphi_1(t_0)]}, \quad t \in [\gamma', \gamma''], \tag{6.1} \]

where

\[ \varphi_1\{[\gamma', \gamma'']\} = [\gamma', \gamma'']. \]

Then we have that by the integral

\[ \mathcal{F}[H_T] = \frac{1}{\zeta(\frac{1}{2} + it_H)} \int_{\gamma'}^{\gamma''} |H_T[\varphi_1(t); t_0]|^{2\omega(\gamma')\ln \gamma' - 1} \times \]

\[ \times |(H_T)'_{\varphi_1}[\varphi_1(t); t_0]|dt, \quad H_T \in \mathbb{D} \tag{6.2} \]

is defined the functional on \( \mathbb{D} \).

**Corollary.** The formula (see (2.2), (6.2))

\[ \mathcal{F}[H_T] = \frac{1}{\zeta(\frac{1}{2} + it_H)} \int_{\gamma'}^{\gamma''} |H_T[\varphi_1(t); t_0]|^{2\omega(\gamma')\ln \gamma' - 1} \times \]

\[ \times |(H_T)'_{\varphi_1}[\varphi_1(t); t_0]|dt, \quad H_T \in \mathbb{D} \]

expresses the representation of the functional \( \mathcal{F}[H_T] \) by the corresponding set of values of the function \( |\zeta(\frac{1}{2} + it)|^{-2} \).

### 7. Proof of Theorem

Let us remind that we have proved the following theorem (see [6], (9.7)): for every Lebesgue integrable function

\[ f(x), \quad x \in [T, T + U], \quad f(x) \geq 0 \quad (\leq 0), \quad x \in [T, T + U]; \]

\[ \varphi_1\{[\hat{T}, \hat{T} + \bar{U}]\} = [T, T + U] \]

we have

\[ \int_{\hat{T}}^{\hat{T} + \bar{U}} f[\varphi_1(t)] \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \]

\[ = \omega(T) \ln T \int_T^{T + U} f(x)dx, \quad U \in \left( 0, \frac{T}{\ln T} \right], \tag{7.1} \]

where

\[ \omega(T) = \omega(T; f) = 1 + \mathcal{O} \left( \frac{\ln \ln T}{\ln T} \right), \quad T \to \infty. \]

Next, from (a) – (c) of Theorem we obtain that

\[ \int_{\gamma'}^{\gamma''} |G(t)|^\alpha |G'(t)|dt = \frac{2}{\alpha + 1} |G(t_0)|^{\alpha + 1}, \quad \alpha > 0, \]

\[ G(t) \geq 0 \quad (\leq 0), \quad t \in [\gamma', \gamma'']. \tag{7.2} \]
Now, in the case $f(t) = |G(t)||G'(t)|, \ t \in \left[\gamma', \gamma''\right]$ we have (see (6.1), (6.2) and (b) of Theorem)

$$\int_{\gamma'}^{\gamma''} [G(\varphi_1(t))]^\alpha |G'_{\varphi_1}(\varphi_1(t))| \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \frac{2\omega(\gamma') \ln \gamma'}{\alpha + 1} |G(t_0)|^{\alpha + 1}.$$ (7.3)

Hence, putting

$$\alpha = 2\omega(\gamma') \ln \gamma' - 1,$$

we obtain by (6.3) and (e) of Theorem that

$$\int_{\gamma'}^{\gamma''} |H[\varphi_1(t); t_0]|^{2\omega(\gamma') \ln \gamma' - 1} |H'_{\varphi_1}(\varphi_1(t); t_0)| \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = 1.$$ (7.4)

Finally, if we use the mean-value theorem in (7.3), we obtain

$$\left| \zeta \left( \frac{1}{2} + it_H \right) \right|^2 \int_{\gamma'}^{\gamma''} |H[\varphi_1(t); t_0]|^{2\omega(\gamma') \ln \gamma' - 1} |H'_{\varphi_1}(\varphi_1(t); t_0)| dt = 1,$$

i.e. the formula (2.2) is verified.

8. **Concluding remarks: the Dirac property of the topological deformation $H_T$ at the point $\hat{t}_0$**

Since

$$\varphi_1(\hat{t}_0) = t_0 \Rightarrow H_T[\varphi_1(\hat{t}_0); t_0] = 1$$

(see (6.1)), then we have (see (7.4))

$$\int_{\gamma'}^{\gamma''} |H[\varphi_1(t); t_0]|^{2\omega(\gamma') \ln \gamma' - 1} |H'_{\varphi_1}(\varphi_1(t); t_0)| \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = |H_T[\varphi_1(\hat{t}_0); t_0]|^{2\omega(\gamma') \ln \gamma' - 1}.$$ (8.1)

Hence, by using the Dirac $\delta$-function (see [1], pp. 58-61) we obtain

$$\int_{\gamma'}^{\gamma''} |H[\varphi_1(t); t_0]|^{2\omega(\gamma') \ln \gamma' - 1} \delta(t - \hat{t}_0) dt = |H_T[\varphi_1(\hat{t}_0); t_0]|^{2\omega(\gamma') \ln \gamma' - 1}.$$ (8.2)

The set

$$(-\infty, \hat{\gamma}') \cup (\hat{\gamma}'', \infty)$$

is irrelevant, comp. [1], p. 59.

**Remark 6.** By (8.1), (8.2) we see that the continuous function (*proper* function in terminology of Dirac)

$$|H'_{\varphi_1}(\varphi_1(t); t_0)| \left| \zeta \left( \frac{1}{2} + it \right) \right|^2$$

acts at the point $\hat{t}_0$ as the Dirac $\delta$-function (*improper* function in terminology of Dirac, see [1], p. 58).

**Remark 7.** It is sufficient to quote Dirac as the discoverer of the generalized functions.
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REFERENCES

[1] P.A.M. Dirac, ‘The principles of quantum mechanics’, Oxford, Clarendon Press, 1958.
[2] A.A. Karatsuba, ‘Complex analysis in number theory’, CRC Press, Boca Raton, Ann Arbor, London, Tokyo, 1995.
[3] J.E. Littlewood, ‘Two notes on the Riemann zeta-function’, Proc. Cambr. Phil. Soc., 22 (1924), 234-242.
[4] J. Moser, ‘Some properties of the Riemann zeta-function on the critical line’, Acta Arith., 26 (1974), 33-39, (in Russian); arXiv: 0710.0943.
[5] J. Moser, ‘Jacob’s ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral’, Math. Notes 88, 414-422 (2010), arXiv: 0901.3937.
[6] J. Moser, ‘Jacob’s ladders, the structure of the Hardy-Littlewood integral and some new class of nonlinear integral equations’, Proc. Stek. Inst. 276, 208-221 (2011), arXiv: 1103.0359.
[7] J. Moser, ‘Riemann hypothesis and some infinite set of microscopic universes of the Einstein’s type in the early period of the evolution of the Universe’, arXiv: 1307.1095.
[8] E.C. Titchmarsh, ‘The theory of the Riemann zeta-function’ Clarendon Press, Oxford, 1951.
[9] F.G. Tricomi, ‘Differential equations’, Blackie and Son Limited, 1961 (in Russian).
[10] F.G. Tricomi, ‘Lezioni sulle equazioni a derivate parziali’, Gheroni, Torino, 1954 (in Russian).
[11] E.T. Whittaker, G.N. Watson, ‘A course of modern analysis’, Cambridge, Cambridge Univ. Press, 1927.

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