Killing tensors in pp-wave spacetimes

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Abstract

The formal solution of the second-order Killing tensor equations for the general pp-wave spacetime is given. The Killing tensor equations are integrated fully for some specific pp-wave spacetimes. In particular, the complete solution is given for the conformally flat plane wave spacetimes and we find that irreducible Killing tensors arise for specific classes. The maximum number of independent irreducible Killing tensors admitted by a conformally flat plane wave spacetime is shown to be six. It is shown that every pp-wave spacetime that admits an homothety will admit a Killing tensor of Koutras type and, with the exception of the singular scale-invariant plane wave spacetimes, this Killing tensor is irreducible.

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1. Introduction

Let \( \mathcal{M} \) denote a four-dimensional spacetime manifold with Lorentzian metric \( g_{ab} \) and metric connection \( \Gamma^c_{ab} \), \( R_{abcd} \), \( R_{ab} \) and \( C_{abcd} \) denote the Riemann curvature tensor, Ricci tensor and Weyl tensor respectively. \( \mathcal{L}_X \) denotes the Lie derivative operator with respect to a vector field \( X \) on \( \mathcal{M} \) and a semicolon denotes the covariant derivative arising from \( g_{ab} \) in the usual way. Symmetrization of index pairs of a tensor field on \( \mathcal{M} \) is indicated by round brackets, i.e. \( T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \). For Lie algebras \( A \) and \( B \), the notation \( A \supset B \) means \( B \) is a subalgebra of \( A \).

A vector field \( X \) on \( \mathcal{M} \) which satisfies

\[
(\mathcal{L}_X g)_{ab} = 0 \iff X_{(a;b)} = 0
\]

is referred to as a Killing vector field (KV). The concept of a KV can be generalized in a variety of ways. A totally symmetric tensor field \( K \) of order \( r \) on \( \mathcal{M} \) satisfying

\[
K_{(a_1 \ldots a_r ; a_{r+1})} = 0
\]

is referred to as an irreducible Killing tensor (IKT). The concept of an IKT can be generalized in a variety of ways.
is referred to as a Killing tensor field (KT). A KT with $r = 1$ is a KV. KTs are of interest principally because of their association with polynomial first integrals of the geodesic equation: if $t$ is the geodesic tangent vector, then the quantity $K_{a_1\ldots a_r} t^{a_1} \cdots t^{a_r}$ is a first integral of the geodesic motion. The set of all KVs on $\mathcal{M}$ form a Lie algebra under the bracket operation. Similarly, the set of all KTs on $\mathcal{M}$ form a graded algebra under the Schouten–Nijenhuis bracket operation [1]. In this work we shall restrict attention to the second-order KTs, i.e. those which satisfy
\[ K_{(a|b;c)} = 0, \quad K_{ab} = K_{ba}. \tag{1.3} \]
The metric tensor itself and all symmetrized products of KVs are KTs, as are all linear combinations with constant coefficients, i.e.
\[ K_{ab} = c_0 g_{ab} + \sum_{I=1}^{n} \sum_{J=1}^{n} c_{IJ} X_I(a) X_J(b) \tag{1.4} \]
is a KT, where $J \geq 1$ and $n$ is the dimension of the Lie algebra of KVs. KTs which can be written in the form (1.4) are known as reducible, otherwise they are irreducible.

The maximum number of independent KTs admitted by a four-dimensional spacetime is 50 and this maximum number is attained if and only if the spacetime is of constant curvature [2], and in which case the 50 KTs are reducible [3]. The Kerr metric [4] is probably the most well known and interesting example of a spacetime admitting an irreducible KT [5, 6]. KTs are admitted by other Petrov type $D$ spacetimes [6–8]. Kimura [9–11] investigated KTs in static spherically symmetric spacetimes having spatial parts of non-constant curvature: of particular note are spacetimes that admit 8 and 11 independent irreducible KTs [11]. Hauser and Malhiot made a similar study under the assumption that the KTs are independent of time [12]. The Taub-NUT spacetime admits four KTs [13] and the Euclidean Taub-NUT spacetime admits three KTs [14].

It is of interest to investigate the existence of KTs in other spacetimes and in this paper we investigate KTs, in particular irreducible KTs, in pp-wave spacetimes. There has been some previous work in this area: Cosgrove [15] considered stationary axisymmetric vacuum spacetimes and has found some examples of KTs in pp-wave spacetimes. Our motivation for considering KTs in pp-wave spacetimes is twofold. (i) A particular class of pp-wave spacetimes, the plane wave spacetimes, exhibit a high degree of symmetry/conformal symmetry being either of Petrov type $O$ or $N$. The type $O$ spacetimes admit the maximum conformal symmetry and the type $N$ plane wave spacetimes admit the highest degree of conformal symmetry below the type $O$ [16, 17]. Thus, it is natural to consider this class of spacetime since it is likely to admit further symmetries. (ii) Physically, pp-wave spacetimes represent radiation moving at the speed of light, and so are of particular interest in the theory of gravitational radiation. Further, gravitational plane wave spacetimes have applications in string theory [18] and arise naturally as the Penrose limit [19] of any spacetime, see also [18].

We shall now state precisely what we mean by a pp-wave spacetime. We define a pp-wave spacetime to be a non-flat spacetime which admits a covariantly constant, nowhere zero, null bivector. The line element for such a spacetime can be written [20]
\[ ds^2 = -2 du dv - 2 H(u, y, z) du^2 + dy^2 + dz^2. \tag{1.5} \]
A pp-wave spacetime admits a covariantly constant, nowhere zero, null vector field $k$, which is necessarily a KV, and has the form
\[ k^a = \delta_y^a, \quad k_a = -\delta_y^a. \tag{1.6} \]
Sippel and Goenner [21] give the form of the Riemann, Weyl and Ricci tensors for the spacetime with line element (1.5), the latter being
\[ R_{ab} = (H_{yy} + H_{zz}) k_ak_b. \]
It is clear from this that vacuum and pure radiation fields (with the possibility for null electromagnetic fields as a special case) can occur. The weak and dominant energy conditions [22] are satisfied if

\[ H_{yy} + H_{zz} \geq 0. \] (1.7)

The spacetime (1.5) is vacuum if \( H_{yy} + H_{zz} = 0 \) and conformally flat if \( H_{yy} = H_{zz} \) and \( H_{yz} = 0 \). The pp-wave spacetime is of Petrov type \( N \) or \( O \), which can be deduced from the form of the Weyl tensor [21], and the vacuum case cannot occur for type \( O \). The Riemann curvature tensor satisfies \( R_{abcd}k^d = 0 \) and if the Weyl tensor is nowhere zero then \( R_{abcd}k^d = C_{abcd}k^d = 0 \) and \( k \) is a repeated principal null direction of the Weyl tensor. The similarity of the Weyl tensor for the type \( N \) pp-wave spacetime and the electromagnetic field tensor for electromagnetic plane waves permits the interpretation as gravitational waves. The vanishing of the expansion and twist of the rays justifies the term plane-fronted, and the constancy of \( k \) implies parallel rays: hence the designation of the fields as plane-fronted waves with parallel rays, or pp-waves. We note that in [23] a pp-wave is defined to be a spacetime admitting only the covariantly constant, nowhere zero, null vector field \( k \) and that the imposition of the conditions on the energy–momentum tensor of the types above are required in order to obtain the line element of the form (1.5). Ehlers and Kundt [20] defined a pp-wave to be a vacuum spacetime. See [23, 24] for an overview.

The wave interpretation permits one to define an amplitude and polarization for the type \( N \) pp-wave spacetimes [20]. (Type \( O \) spacetimes have vanishing Weyl tensor and hence zero amplitude.) A pp-wave spacetime is said to be a plane wave spacetime if the amplitude is constant in every wavefront and in this case the metric function can be written

\[ 2H = A(u)y^2 + 2B(u)yz + C(u)z^2 \] (1.8)

where \( A, B \) and \( C \) are arbitrary functions. A conformally flat pp-wave spacetime is necessarily a plane wave spacetime and has a metric function given by

\[ 2H = A(u)(y^2 + z^2) \] (1.9)

where \( A \) is an arbitrary function.

We now state some preliminary geometrical results. A vector field \( X \) is said to be a conformal Killing vector field (CKV) if and only if

\[ (\mathcal{L}_X g)_{ab} = 2\phi g_{ab} \] (1.10)

where \( \phi \) is some function of the coordinates (conformal scalar). When \( \phi \) is not constant the CKV is said to be proper and if \( \phi_{ab} = 0 \), the CKV is a special CKV (SCKV). When \( \phi \) is a constant, \( X \) is a homothetic vector field (HKV) and when the constant \( \phi \) is nonzero \( X \) is a proper HKV. When \( \phi = 0 \), \( X \) is a KV as mentioned above. The set of all CKV (respectively, SCKV, HKV and KV) form a finite-dimensional Lie algebra denoted by \( \mathcal{C} \) (respectively, \( \mathcal{S}, \mathcal{H} \) and \( \mathcal{G} \)). Koutras [25] devised an algorithm to find KTs using CKVs and this algorithm was generalized by Rani, Edgar and Barnes [26, 27]. Given a pair of CKVs \( X, Y \) satisfying

\[ (\mathcal{L}_X g)_{ab} = 2\phi g_{ab}, \quad (\mathcal{L}_Y g)_{ab} = 2\psi g_{ab} \]

then if the quantity \( \phi Y_a + \psi X_a \) is a gradient, i.e. \( \phi Y_a + \psi X_a = \Phi_a \) for some scalar \( \Phi \), it follows that the tensor field

\[ L_{ab} = X_a Y_b - \Phi g_{ab} \]

is a KT. Special cases are dealt with in the theorems and corollaries given in [26]. This algorithm does not, nor claims to, produce irreducible KTs in general. We note that for a KV \( X \) and a KT \( K \) it is straightforward to show that the tensor \( (\mathcal{L}_X K)_{ab} \) is a KT. This can
be regarded as following from the more general result given in [1] where it is shown that the graded algebra of KTs has the same structure constants as the corresponding Poisson bracket Lie algebra of first integrals. We note that for a pp-wave spacetime, it follows from (1.3) and (1.6) that

\[ \mathcal{L}_k K_{ab} = -\mathcal{L}^k g_{ab} \quad \text{where} \quad \dot{k}_a = k^b K_{ab}. \]

For details of the isometry and conformal algebras of the pp-wave spacetimes see [20, 21, 28, 29]. Here we note that the general plane wave spacetime (1.8) admits an \( \mathcal{H}_6 \supset \mathcal{G}_5 \) and the general conformally flat spacetime (1.9) admits a \( \mathcal{C}_{15} \supset \mathcal{H}_7 \supset \mathcal{G}_6 \). In the conformally flat case the underlying conformal algebra is the \( so(4, 2) \) conformal algebra of Minkowski spacetime. Of particular relevance for this work are the plane wave spacetimes which admit additional symmetries, i.e.

\[ 2H = u^{-2}(ay^2 + 2byz + cz^2) \]  

and

\[ 2H = ay^2 + 2byz + cz^2 \]  

where \( a, b \) and \( c \) are arbitrary constants (not all zero). Note that a transformation of the \( y \) and \( z \) coordinates allows us to set \( b = 0 \) in both (1.11) and (1.12). For arbitrary values of \( a, b \) and \( c \), the metric functions (1.11) and (1.12) are Sippel and Goenner classes 11 and 13 respectively and both admit an \( \mathcal{H}_7 \supset \mathcal{G}_6 \). When \( a = c \) and \( b = 0 \), (1.11) and (1.12) are Sippel and Goenner classes 16 and 17 respectively, are conformally flat and admit a \( \mathcal{C}_{15} \supset \mathcal{H}_8 \supset \mathcal{G}_7 \). Plane wave spacetimes with metric function (1.11) are \textit{singular scale-invariant} plane waves and those with metric function (1.12) are \textit{symmetric} plane waves [18].

The Koutras algorithm is an indirect method to construct KTs and, as we have pointed out, does not produce irreducible KTs in general. We are interested in irreducible KTs and particularly those which cannot be obtained from the Koutras algorithm. However, we shall now see that if a pp-wave spacetime admits a homothety, then the Koutras KT arises naturally.

**Theorem 1.** A pp-wave spacetime which admits an HKV \( Y \) will admit a KT, which will be irreducible in general. This KT is obtained from the \( Y \) and KV \( k \) via the Koutras algorithm. The only pp-wave spacetime for which this KT is reducible is the plane wave spacetime with metric function (1.11).

**Proof.** Given the pairs \((k, \phi = 0), (Y, \psi = \text{constant})\) then \( \phi Y_a + \psi k_a = -\psi \delta^u_a = \Phi_{,a} \), i.e. \( \Phi = -\psi u \) and the Koutras algorithm generates the KT

\[ L_{ab} = k_{(a} Y_{b)} + \psi u g_{ab}. \]  

(1.13)

The expressions for the most general HKV and KV in a pp-wave spacetime are given in [29]. The general HKV is given by \( \alpha Z + X \) where \( \alpha \) is a constant and \( Z \) and \( X \) are given by equations (15) and (16) respectively in [29]; the general KV is given by \( X \). Using these expressions one can write down the most general reducible KT formed from a sum of the metric tensor \( g_{ab} \) and the symmetrized product of the general KV \( X \), and comparison with (1.13) leads one to the conclusion that the metric function \( H \) must have the form of a plane wave (1.8). Subsequent application of the CKV equation (1.10) leads to a metric function of the form (1.11).

**Corollary 1.** All plane wave spacetimes admit a KT of Koutras type (1.13) and, with the exception of those with metric function (1.11), this KT is irreducible.
The general form of the KT components for a general pp-wave spacetime is given in the appendix. Despite the large number and apparent complexity of the general equations we are able to obtain explicit solutions for a selection of specific pp-wave spacetimes and these are presented in section 2. Some examples of plane wave spacetimes are given in section 3. In section 4 we solve for the KTs explicitly for the conformally flat plane wave spacetimes. The maximum number of independent irreducible KTs admitted by a conformally flat plane wave spacetime is shown to be six but some admit none at all.

2. pp-wave spacetimes

The full KT equations are given in the appendix. We present two examples.

Example 1. The type Biv pp-wave spacetime of [29] has the metric
\[
d s^2 = -2\, du\, dv - 2l(\alpha z - \beta y)^{-2}\, du^2 + dy^2 + dz^2
\]
where \(l, \alpha\) and \(\beta\) are constants such that \(\alpha^2 + \beta^2 \neq 0\). A coordinate transformation puts this metric in the form
\[
d s^2 = -2\, du\, dv - 2z^{-2}\, du^2 + dy^2 + dz^2
\]
which is a 1+3 spacetime [30]. This spacetime admits a conformal algebra \(S_6 \supset H_5 \supset G_4\) with the basis
\[
X_1 = \partial_v, \quad X_2 = \partial_u, \quad X_3 = \partial_y, \quad X_4 = y\partial_v + u\partial_y, \quad X_5 = 2u\partial_u + y\partial_y + z\partial_z, \quad X_6 = u^2\partial_u + \frac{1}{2}(y^2 + z^2)\partial_v + u(y\partial_y + z\partial_z).
\]
Solving the KT equations (A.1)–(A.19) we find that there are 16 independent KTs corresponding to the 16 arbitrary constants of which 5 are irreducible KTs. The irreducible KTs are
\[
(K_1)_{ab} = -2yz^2(\delta_{(a}^v\delta_{b)}^v - \delta_{(a}^u\delta_{b)}^u) + 2y^2\delta_{(a}^v\delta_{b)}^v - y^2\delta_{(a}^u\delta_{b)}^u,

(K_2)_{ab} = 2yz^2(\delta_{(a}^v\delta_{b)}^v - \delta_{(a}^u\delta_{b)}^u) + y\delta_{(a}^v\delta_{b)}^v - y\delta_{(a}^u\delta_{b)}^u + u\delta_{(a}^v\delta_{b)}^v + u\delta_{(a}^u\delta_{b)}^u,

(K_3)_{ab} = 2uz^2(\delta_{(a}^v\delta_{b)}^v - \delta_{(a}^u\delta_{b)}^u) - 2uz\delta_{(a}^v\delta_{b)}^v - 2uz\delta_{(a}^u\delta_{b)}^u + u\delta_{(a}^v\delta_{b)}^v - u\delta_{(a}^u\delta_{b)}^u,

(K_4)_{ab} = (z^2 + 2u^2z^2)\delta_{(a}^v\delta_{b)}^v - 2uz\delta_{(a}^v\delta_{b)}^v + u^2\delta_{(a}^v\delta_{b)}^v.
\]
We note that \(K_2, K_4\) and \(K_5\) can be derived from the Koutras algorithm (in combination with reducible KTs) whereas \(K_1\) and \(K_3\) cannot.

Example 2. The metric of the isometry class 9 pp-wave spacetime of [21] can be written
\[
d s^2 = -2\, du\, dv - 2\exp(y)\, du^2 + dy^2 + dz^2.
\]
This spacetime admits a \(G_5\) and no irreducible KTs.

Thus, there exist pp-wave spacetimes which admit no irreducible KTs.

3. Plane wave spacetimes

Let us consider the special case of the plane wave spacetime (1.8). For arbitrary \(A, B\) and \(C\) this spacetime admits a \(H_6 \supset G_5\) with the basis (see [29])
\[
X_1 = \partial_v, \quad X_2 = f(u)v\partial_v + f\partial_y, \quad X_3 = g(u)z\partial_v + g\partial_y, \quad X_4, X_5 = g(u)z\partial_v + g\partial_y, \quad X_6 = 2v\partial_v + y\partial_y + z\partial_z.
\]
where the functions \( f \) and \( g \) satisfy
\[
 f_{uu} + Af + Bg = 0, \\ g_{uu} + Bf + Cg = 0. \tag{3.2}
\]
\( X_1, \ldots, X_5 \) are KVs and \( X_6 \) is a proper HKV with \( \phi = 1 \). The KT components for the general plane wave spacetimes are obtained from the equations in the appendix with \( H \) given by (1.8) and, as a consequence, \( \sigma = \zeta = \mu = \epsilon = 0 \). The Koutras KT arising from \( X_1 \) and \( X_6 \) is given by
\[
 L_{ab} = 2(v - uH)\delta^u_{(a} \delta^u_{b)} - y\delta^y_{(a} \delta^y_{b)} - z\delta^z_{(a} \delta^z_{b)} - 2u\delta^u_{(a} \delta^v_{b)} + u(\delta^y_{(a} \delta^v_{b)} + \delta^z_{(a} \delta^v_{b)}). \tag{3.3}
\]
When \( A = -C \) the spacetime is vacuum and when \( A = C \) and \( B = 0 \) the spacetime is conformally flat, the latter being dealt with in section 4.

**Theorem 2.** The singular scale-invariant plane wave spacetime given by (1.11) in general admits no irreducible KTs. The only exception occurs in the case of the conformally flat plane wave spacetime with the metric function
\[
 2H = \frac{3}{16}u^{-2}(y^2 + z^2). \tag{3.4}
\]
in which case there are six independent irreducible KTs.

**Proof.** A straightforward but lengthy calculation using (1.11) in the KT equations gives the general result. The plane wave with metric function (3.4) arises in the analysis of the general conformally flat plane wave spacetimes in section 4 where the second part of the theorem is proved. We note that the metric function \( 2H = -\frac{3}{2}u^{-2}(y^2 + z^2) \) will admit an irreducible KT; however, we discard this solution because it does not satisfy the energy conditions (1.7).

**Corollary 2.** The singular scale-invariant vacuum plane wave spacetimes, i.e. those with the metric function
\[
 2H = \kappa u^{-2}(y^2 - z^2) \tag{3.5}
\]
where \( \kappa \) is a constant, admit no irreducible KTs.

**Example 3.** The vacuum plane wave spacetime with \( 2H = y^2 - z^2 \), i.e.
\[
 ds^2 = -2du dv - (y^2 - z^2) du^2 + dy^2 + dz^2 \tag{3.6}
\]
admits an \( \mathcal{H}_7 \supset G_6 \) composed of (3.1) and the extra KV
\[
 X_7 = \partial_u.
\]
In this case the independent solutions of (3.2) are
\[
 f_1 = \sin u, \quad f_2 = \cos u, \quad g_1 = \sinh u, \quad g_2 = \cosh u.
\]
The solution of the KT equations involves 22 independent arbitrary constants. However, there are only 21 independent reducible KTs. There are 21 symmetrized products of the KVs and the metric tensor but the metric tensor is a linear combination of 5 of the symmetrized products of KVs, i.e.
\[
 g_{ab} = -2X_{1(a}X_{7b)} + X_{3(a}X_{3b)} + X_{2(a}X_{2b)} + X_{4(a}X_{4b)} - X_{5(a}X_{5b)}.
\]
The irreducible KT is the Koutras KT arising from the KV \( X_1 = k = \partial_u \) and the HKV \( X_6 \), i.e.
\[
 L_{ab} = [2v - u(y^2 - z^2)]\delta^u_{(a} \delta^u_{b)} - y\delta^y_{(a} \delta^y_{b)} - z\delta^z_{(a} \delta^z_{b)} - 2u\delta^u_{(a} \delta^v_{b)} + u(\delta^y_{(a} \delta^v_{b)} + \delta^z_{(a} \delta^v_{b)}). 
\]
4. Conformally flat plane wave spacetimes

These spacetimes have a metric function given by (1.9). Since we are primarily interested in irreducible KTs and whether the KTs can be obtained from the Koutras algorithm, we begin by writing down a basis for the CKV of this spacetime and, where appropriate, the nonzero conformal scalars \( \phi \). The covariant components of the CKVs are given in terms of the functions \( f_1 \) and \( f_2 \) which are two independent solutions of

\[
f_{u,u} + Af = 0. \tag{4.1}
\]

The components are as follows:

\[
\begin{align*}
X_{1a} &= -\delta_u^a, & X_{2a} &= z\delta_u^a - y\delta_u^b, \\
X_{3a} &= -f_{1,u}y\delta_u^a + f_1\delta_u^b, & X_{4a} &= -f_{2,u}y\delta_u^a + f_2\delta_u^b, \\
X_{5a} &= -f_{1,u}z\delta_u^a + f_1\delta_u^b, & X_{6a} &= -f_{2,u}z\delta_u^a + f_2\delta_u^b, \\
X_{7a} &= -2\nu\delta_u^a + y\delta_u^b + z\delta_u^c, & \phi &= 1 \\
X_{8a} &= -\frac{1}{2}A(y^2 + z^2)^2 - u^2\delta_u^a - \frac{1}{2}(y^2 + z^2)\delta_u^b + v(y\delta_u^a + z\delta_u^a), & \phi &= \nu \\
X_{9a} &= -\frac{1}{2}Af_{y}(y^2 + z^2) - f_{1,u}vy\delta_u^a - f_{1,y}\delta_u^b - \left[ \frac{1}{2}f_{2,u}(y^2 - z^2) + f_{1,v}\right]b^a + f_{1,u}y\delta_u^a, \\
\phi &= f_{1,u}y, \\
X_{10a} &= -\frac{1}{2}Af_{2}(y^2 + z^2) - f_{2,u}vy\delta_u^a - f_{2,y}\delta_u^b - \left[ \frac{1}{2}f_{2,u}(y^2 - z^2) + f_{2,v}\right]b^a + f_{2,u}y\delta_u^a, \\
\phi &= f_{2,u}y, \\
X_{11a} &= -\frac{1}{2}Af_{1}(y^2 + z^2) - f_{1,u}vz\delta_u^a - f_{1,z}\delta_u^b - \left[ \frac{1}{2}f_{1,u}(y^2 - z^2) + f_{1,v}\right]b^a + f_{1,u}z\delta_u^a, \\
\phi &= f_{1,u}z, \\
X_{12a} &= -\frac{1}{2}Af_{2}(y^2 + z^2) - f_{2,u}vz\delta_u^a - f_{2,z}\delta_u^b - \left[ \frac{1}{2}f_{2,u}(y^2 - z^2) + f_{2,v}\right]b^a + f_{2,u}z\delta_u^a, \\
\phi &= f_{2,u}z, \\
X_{13a} &= -\frac{1}{2}(Af_{1}^2 + f_{1,u}^2)(y^2 + z^2)\delta_u^a - f_{1,u}\delta_u^a + f_{1,u}(y\delta_u^a + z\delta_u^a), & \phi &= f_{1,u}, \\
X_{14a} &= -\frac{1}{2}(Af_{2}^2 + f_{2,u}^2)(y^2 + z^2)\delta_u^a - f_{2,u}\delta_u^a + f_{2,u}(y\delta_u^a + z\delta_u^a), & \phi &= f_{2,u}, \\
X_{15a} &= -\frac{1}{2}(Af_{1}^2 + f_{1,u}^2)(y^2 + z^2)\delta_u^a - f_{1,u}\delta_u^a + f_{1,u}(y\delta_u^a + z\delta_u^a), & \phi &= f_{1,u}. 
\end{align*}
\]

Thus, for a conformally flat plane wave spacetime, the conformal symmetries depend only on the independent solutions of the differential equation (4.1). \( X_1, \ldots, X_6 \) are KVs, \( X_7 \) is a proper HKV and, in general, \( X_8, \ldots, X_{15} \) are proper CKV.

There are some special cases of note as identified by Sippel and Goenner [21]. When \( A \) is constant, \( X_{15} \) is a KV, which we shall denote as \( Z = \delta_u \) and

\[
Z_u = -A(y^2 + z^2)\delta_u^a - \delta_u^a.
\]

When \( A = \kappa u^{-2} \), there is also an extra KV: for \( \kappa < 1/4 \), \( X_{15} \) is an HKV; for \( \kappa = 1/4 \), \( X_{13} \) is an HKV; for \( \kappa > 1/4 \), \( X_{13} + X_{14} \) is an HKV. In each case taking a linear combination of the HKV with \( X_7 \) allows us to replace the HKV with the KV \( Y = u\delta_u - v\delta_u \). The covariant components of \( Y \) are

\[
Y_a = [v - \kappa u^{-1}(y^2 + z^2)]\delta_u^a - u\delta_u^a.
\]

The KT components for the conformally flat plane wave spacetimes are obtained from the equations in the appendix with \( H \) given by (1.9) and, as a consequence, \( \sigma = \zeta = \mu = \epsilon = 0 \). In the case of the conformally flat plane wave spacetimes the KT equations separate into independent groups. Most of the groups lead only to reducible KTs but the following five groups lead to irreducible KTs.
(i) Equations involving $\rho$, $\Psi$ and $\Lambda$ only

\[ \rho + \Psi + \Lambda = 0 \]  
(4.2)

\[ \rho_{,uu} = \Psi_{,uu} \]  
(4.3)

\[ \rho_{,uuu} = -A_{,u}(\rho + \Psi) - 2A(\rho_{,u} + \Psi_{,u}) \]  
(4.4)

\[ 3\rho_{,uu} + 2A(\rho + \Psi) = 0 \]  
(4.5)

\[ 8A\rho_{,u} + 3A_{,u}\rho - A_{,u}\Psi = 0 \]  
(4.6)

\[ 8A\Psi_{,u} + 3A_{,u}\Psi - A_{,u}\rho = 0 \]  
(4.7)

\[ 10A\rho_{,uu} + A_{,u}\rho_{,u} + 4A_{,u}\Psi_{,u} + A_{,uu}\Psi + 4A^2(\rho + \Psi) = 0 \]  
(4.8)

\[ 2A\rho_{,uu} - 7A_{,u}\rho_{,u} + 2A_{,u}\Psi_{,u} + A_{,uu}(\Psi - 2\rho) + 4A^2(\rho + \Psi) = 0. \]  
(4.9)

(ii) Equations involving $\tau$, $\omega$, $\theta$ and $\pi$ only

\[ \theta_{,uuu} = 0 \]  
(4.10)

\[ 3\pi_{,u} = 2A(\tau + \omega) \]  
(4.11)

\[ \tau_{,uuu} = -2A_{,u}\tau - 4A\tau_{,u} - A_{,u}\theta_{,u} - 2A\theta_{,uu} \]  
(4.12)

\[ \omega_{,uuu} = -2A_{,u}\omega - 4A\omega_{,u} - A_{,u}\theta_{,u} - 2A\theta_{,uu} \]  
(4.13)

\[ 3\tau_{,uu} + 4A\tau = 2A_{,u}\theta + 2A\theta_{,u} \]  
(4.14)

\[ 3\omega_{,uu} + 4A\omega = 2A_{,u}\theta + 2A\theta_{,u} \]  
(4.15)

\[ 24A(A\theta_{,u} + A_{,u}\theta) = 16A\tau_{,uu} + 5A_{,u}\tau_{,u} + A_{,uu}\tau + 16A^2\tau \]  
(4.16)

\[ 8A(A\theta_{,u} + A_{,u}\theta) = 2A(\tau + \omega)_{,uu} - \frac{1}{2}A_{,u}(\tau + \omega)_{,u} - \frac{1}{2}A_{,uu}(\tau + \omega) + \pi_{,uuu} + 4A\pi_{,uu} \]  
(4.17)

\[ 4A(\tau - \omega)_{,u} + A_{,u}(\tau - \omega) = 0 \]  
(4.18)

\[ 24A(A\theta_{,u} + A_{,u}\theta) = 16A\omega_{,uu} + 5A_{,u}\omega_{,u} + A_{,uu}\omega + 16A^2\omega. \]  
(4.19)

(iii) Equations involving $\xi$, $\Sigma$, $q$ and $s$ only

\[ \xi_{,uuu} + 2A_{,u}\xi + 4A\xi_{,u} + 2A_{,u}q + 4Aq_{,u} = 0 \]  
(4.20)

\[ \Sigma_{,uuu} + 2A_{,u}\Sigma + 4A\Sigma_{,u} + 2A_{,u}q + 4Aq_{,u} = 0 \]  
(4.21)

\[ q = -\frac{1}{2}\alpha u + \beta \]  
(4.22)

\[ s = \alpha = \text{constant}. \]  
(4.23)

(iv) Equations involving $l$, $\Gamma$ and $\Omega$ only

\[ l_{,uu} + A l = 0 \]  
(4.24)

\[ 3\Omega_{,uu} + A(3\Omega - 4l) + 2A_{,u}\Gamma = 0 \]  
(4.25)

\[ \Omega_{,uuu} + A_{,u}\Omega + A_{,uu}\Omega + 2A_{,u}\Gamma_{,uu} + 2A_{,u}\Gamma_{,u} + A_{,uu}\Gamma + 2A^2\Gamma - A_{,u}l = 0 \]  
(4.26)

\[ 2l_{,u} + A\Gamma_{,uu} + A\Gamma = 0 \]  
(4.27)

\[ A_{,u}l + 4A\Gamma_{,u} + 6A\Gamma_{,uu} + 4A_{,u}\Gamma_{,u} + A_{,uu}\Gamma + 6A^2\Gamma = 0. \]  
(4.28)
Equations involving $h$, $v$ and $\chi$ only. These equations have the same form as those in the previous group.

We consider each group in turn as follows.

(i) Equations (4.6) and (4.7) give

$$\rho = \rho_0 A^{-1/4} + \Psi_0 A^{-1/2}, \quad \Psi = \rho_0 A^{-1/4} - \Psi_0 A^{-1/2}$$

where $\rho_0$ and $\Psi_0$ are constants, and equation (4.3) becomes

$$\Psi_0 (A^{-1/2})_{, uu} = 0$$

so that either $\Psi_0 = 0$ and $A$ is an arbitrary nonzero function, or $\Psi_0$ is an arbitrary constant and $A = \kappa$ or $\kappa u^{-2}$, where $\kappa$ is an arbitrary nonzero constant. Equation (4.5) becomes

$$\rho_0 A^{-9/4}(15A,u^2 - 12AA_{, uu} + 64A^3) + 12\Psi_0 A^{-5/2}(3A,u^2 - 2AA_{, uu}) = 0$$

and, on account of the above conditions on $A$ and $\Psi_0$, the second term vanishes and we have either $\rho_0 = 0$ or

$$12AA_{, uu} - 15A,u^2 - 64A^3 = 0.$$

This equation integrates to give

$$A^{-5/2}A,u^2 = \frac{64}{3}A^{1/2} + \eta$$

where $\eta$ is an arbitrary constant and integration of this equation gives

$$A = \begin{cases} \frac{4}{3}u^2 & \text{for } \eta = 0 \\ \left(u^2 - \frac{4}{3}\right)^2 & \text{for } \eta \neq 0. \end{cases}$$

There are only two cases satisfying equations (4.2)–(4.9) and leading to an irreducible KT.

(a) $A = \frac{4}{3}u^2, \Psi_0 = 0$ and $\rho_0$ arbitrary.

(b) $A = \left(u^2 - \frac{4}{3}\right)^2, \Psi_0 = 0$ and $\rho_0$ arbitrary.

In both cases the irreducible KT is given by

$$(K_1)_{, ab} = \left[-\frac{4}{3}A^{1/2} yzv + \frac{4}{3}A A^{-1/2} yv (y^2 + z^2) \right] \delta_{(a, b)}^u \delta_{(a, b)}^u + A^{-5/4} A_{, u} yz \delta_{(a, b)}^u \delta_{(a, b)}^v$$

$$+ \left[-\frac{2}{3} A^{-5/4} A_{, u} zv + \frac{4}{3} A^{1/2} (3y^2 + z^2) \right] \delta_{(a, b)}^u \delta_{(a, b)}^y$$

$$+ \left[-\frac{2}{3} A^{-5/4} A_{, u} yv + \frac{2}{3} A^{1/2} (y^2 + 3z^2) \right] \delta_{(a, b)}^u \delta_{(a, b)}^z$$

$$+ 2A^{-1/4} \delta_{(a, b)}^u \delta_{(a, b)}^v + 2A^{-1/4} y \delta_{(a, b)}^u \delta_{(a, b)}^y - 4A^{-1/4} v \delta_{(a, b)}^u \delta_{(a, b)}^z.$$

(ii) Equation (4.10) gives

$$\theta = \theta_1 u^2 + \theta_2 u + \theta_3$$

where $\theta_1, \theta_2$ and $\theta_3$ are constants. From equation (4.18) we obtain

$$\tau - \omega = a_0 A^{-1/4}$$

where $a_0$ is a constant. Equations (4.14), (4.15) and (4.33) give

$$a_0 (12AA_{, uu} - 15A,u^2 - 64A^3) = 0.$$
and an identical equation can be derived for \( \omega \). Using combinations of equations (4.12), (4.14), (4.16) and their derivatives we find

\[
\begin{align*}
\theta_1 (32AA,uu^2 + 64A^2u + 2A,uuuu^2 + 18A,uuu^2 + 30A,uu) + \theta_2 (32AA,uu^2 + 32A^2 + 2A,uuuu^2 + 9A,uu) + \theta_3 (32AA,uu + 2A,uuuu) &= 0. \quad (4.36)
\end{align*}
\]

These equations can also be used to obtain

\[
\frac{1}{2} (12AA,uu - 15A_a^2 - 64A^4) \tau
= \theta_1 (6AA,uu^2 - 4AA,uuuu^2 + 42A,uuuu^2 - 36AA,uuu^2 - 12AA,uu)
+ \theta_2 (6AA,uuuu^2 - 4AA,uuuuu - 21A,uu^2 - 18AA,uuu)
+ \theta_3 (6AA,uu - 4AA,uuuu)
\]

(4.37)

and an identical equation can be derived for \( \omega \). We note that each of \( \theta_1 \), \( \theta_2 \) and \( \theta_3 \) are either zero or arbitrary constants. There are two cases to consider: \( a_0 \neq 0 \) and \( a_0 = 0 \). If \( a_0 \neq 0 \), then (4.34) reduces to equation (4.31) which has solutions (4.32). In the case \( A = \frac{1}{16} u^{-2} \), equations (4.10)–(4.19) and (4.33) give \( \theta_3 = 0 \) and

\[
\tau = \frac{1}{2} \omega = \omega_1 u^2 - \frac{1}{2} \theta_2, \quad \pi = -\frac{1}{4} \left( \tau_1 + \omega_1 \right) u^{-1/2} + \frac{1}{8} \theta_2 u^{-1} + \pi_0
\]

where \( \theta_1, \theta_2, \tau_1, \omega_1 \) and \( \pi_0 \) are arbitrary constants and \( \omega_1 = \tau_1 - 2a_0/\sqrt{3} \). The KTs associated with the constants \( \pi_0 \) and \( \theta_1 \) are reducible. However, the KTs associated with the constants \( \tau_1 \), \( \omega_1 \) and \( \pi_0 \) are irreducible and given by, respectively,

\[
(K_2)_{ab} = \left[ -\frac{1}{3} u^{-3/2} v^2 - \frac{3}{12} u^{-5/2} y^2 (y^2 + z^2) \right] \delta^v (a^v b^v)
-\frac{1}{2} u^{-1/2} y^2 \delta^v (a^v b^v) + \frac{1}{8} u^{-3/2} z^2 \delta^v (a^v b^v)
+ \left( u^{-1/2} v y + \frac{1}{2} u^{-3/2} y^3 + \frac{1}{2} u^{-3/2} y^2 z \right) \delta^v (a^v b^v)
+ 2u^{-1/2} y \delta^v (a^v b^v) + (-2u^{-1/2} v + \frac{1}{4} u^{-1/2} z^2) \delta^v (a^v b^v)
- \frac{1}{2} u^{-1/2} y z \delta^v (a^v b^v) + \frac{1}{2} u^{-1/2} y^2 \delta^v (a^v b^v)
\]

(4.38)

We will now consider the case \( A = \left( u^2 - \frac{4}{3} \right)^{-2} \). Inserting the function \( A \) into (4.36) we find that \( \theta_1 = \theta_2 = \theta_3 = 0 \) and equations (4.11)–(4.19) give

\[
\tau = \frac{1}{2} \omega = \omega_1 \left( u^2 - \frac{4}{3} \right)^{1/2}, \quad \pi = \pi_0 - \frac{1}{2} (\tau_1 + \omega_1) u \left( u^2 - \frac{4}{3} \right)^{-1/2}.
\]

The KT associated with the constant \( \pi_0 \) is reducible and the KTs associated with the constants \( \tau_1 \) and \( \omega_1 \) are irreducible and given by, respectively,

\[
(K_3)_{ab} = \left[ -\frac{2}{3} (u^2 - \frac{4}{3})^2 + \frac{1}{3} \left( \frac{1}{2} u^2 + \frac{2}{3} u \left( u^2 - \frac{4}{3} \right) (y^2 - 2z^2) \right) \right] \left( u^2 - \frac{4}{3} \right)^{-7/2} y^2 \delta^v (a^v b^v)
- 2u \left( u^2 - \frac{4}{3} \right)^{-1/2} y^2 \delta^v (a^v b^v) + \frac{1}{2} (u^2 - \frac{4}{3})^{-3/2} y z \delta^v (a^v b^v)
\]
In this case the only irreducible KT is the Koutras KT

Equation (4.37) and equations (4.10)–(4.19) must be checked for consistency. The function \( \pi \) can then be solved for using equation (4.11), i.e.

\[
\pi_u = \frac{2}{3} A \tau \tag{4.39}
\]

We now consider the case \( a_0 = 0 \), i.e. \( \tau = \omega \). We note that \( K_4 \) corresponds to this case. For a given \( A(u) \), it is necessary to first solve (4.35), then (4.36), insert the \( \tau \) and \( \theta \) into (4.37) and equations (4.10)–(4.19) and the KT associated with the constant

and the general solution of this is

\[
\Omega_{uu} + A \Omega = \frac{2}{3} A(l_1 f_1 + l_2 f_2) - \frac{2}{3} A \omega(l_1 f_1 + l_2 f_2 - l_1 u f_1 - l_2 u f_2) \tag{4.37}
\]

where \( \Gamma_1 \) and \( \Gamma_2 \) are arbitrary constants. As a result, equation (4.25) gives

\[
\Omega_{uu} + A \Omega = \frac{2}{3} A(l_1 f_1 + l_2 f_2 - l_1 u f_1 - l_2 u f_2) \tag{4.38}
\]

and the general solution of this equation is

\[
\Omega = \Omega_1 f_1 + \Omega_2 f_2 + \frac{2}{3} \Gamma_1 f_1, u + \frac{2}{3} \Gamma_2 f_2, u - \frac{2}{3} l_1 u f_1, u - \frac{2}{3} l_2 u f_2, u \tag{4.39}
\]

where \( \Omega_1 \) and \( \Omega_2 \) are arbitrary constants. As a result equations (4.27) and (4.28) yield

\[
4(l_1 f_1 + l_2 f_2, u)(2A + u A_u, u) + (l_1 f_1 + l_2 f_2)(2A + u A_u, u, u)
\]

\[
- 4(\Gamma_1 f_1 + \Gamma_2 f_2, u) A_u, u - (\Gamma_1 f_1 + \Gamma_2 f_2) A_{uu} = 0. \tag{4.40}
\]

We have been unable to find the general solution to this equation. However, we note that if \( \Gamma_1 = \Gamma_2 = 0 \), the equation yields either \( A = \kappa u^{-2} \) where \( \kappa \) is an arbitrary constant, resulting in only reducible KTs, or \( l = l_1 f_1 + l_2 f_2 = \alpha A + u A_u \)–1/\( A \), where \( \alpha \) is an arbitrary constant. In this case equation (4.24) gives

\[
5X, u^2 - 4XX, uu + 16AX^2 = 0 \quad \text{where} \quad X = 2A + u A_u. \tag{4.41}
\]

If a function \( A(u) \) satisfying this differential equation can be found, then the corresponding KT is irreducible and is given by

\[
\begin{align*}
(K_4)_{ab} &= \left[ -2l_1 u \right] v + \left[ \frac{1}{2} (2A + u A_u) l - 2u A l \right] y(y^2 + z^2) \delta_{(a} u \delta_{b)} \\
&\quad + 2u A A l \gamma_{(a} \delta_{b)} + 2l_1 u (2A l (3y^2 + 2z^2) - 1u y z \delta_{(a} \delta_{b)} \\
&\quad + \frac{1}{2} (l_u - u A l) \delta_{(a} \delta_{b)} - 2u A A l \delta_{(a} \delta_{b)} + \frac{1}{2} u A l y \delta_{(a} \delta_{b)} - \frac{1}{2} u A l y \delta_{(a} \delta_{b)}. \tag{4.42}
\end{align*}
\]
If instead, \( l_1 = l_2 = 0 \), equation (4.40) yields either \( A_u = 0 \), which leads only to reducible KTs, or \( \Gamma = \Gamma_1 f_1 + \Gamma_2 f_2 = a |A_u|^{-1/4} \). Equation (4.27) then gives

\[
5A_{uu}^2 - 4A_u A_{uuu} + 16A A_u^2 = 0. \tag{4.42}
\]

Any function \( A(u) \) satisfying this equation will have a corresponding irreducible KT given by

\[
(K_8)_{ab} = -\left(2A\Gamma_u + \frac{1}{4} A_u \Gamma \right) y(2z^2) \delta_{(a} u \delta_{b)} - 2\Gamma_u y \delta_{(a} \delta_{b)}^u u + \frac{2}{3} A \Gamma (3y^2 + 2z^2) \delta_{(a} \delta_{b)}^u + \frac{2}{3} A \Gamma y \delta_{(a} \delta_{b)}^u + 2\Gamma \delta_{(a} \delta_{b)}^u
\]

\[-\frac{2}{3} \Gamma_u \delta_{(a} \delta_{b)}^u y + \frac{2}{3} \Gamma_u \delta_{(a} \delta_{b)}^u z.
\]

We have found two solutions to (4.42), namely

(a) \( A(u) = \frac{3}{16} u - \frac{2}{3} \). This is the only function of the form \( A(u) = \kappa u^{-2} \) that satisfies equation (4.42). The corresponding irreducible KT is \( K_0 \) with \( f_1 = u^{3/4}, f_2 = u^{1/4} \), \( \Gamma_2 = 0 \).

(b) \( A(u) = \left[\frac{1}{4} + W(-e^{-u})\right][1 + W(-e^{-u})]^{-1} \), where \( W(x) \) is the Lambert \( W \)-function [31]. The corresponding irreducible KT is \( K_0 \) with \( f_1, f_2 \) given by

\[
f_1^2 = -(1 + W) W^{-1}, \quad f_2^2 = -(1 + W) W.
\]

Now \( A(u) > 0 \) for the energy conditions to hold and \( f_1^2, f_2^2 \) are each positive so it follows that the solution is valid on the principle branch of the \( W \)-function in the interval \(-1 < W(-e^{-u}) < -1/4 \) so that \( u \) is confined to the interval \( 1 > u < 1.615 \), approximately.

(v) By symmetry with the case above, the functions \( h, v \) and \( \chi \) lead to the same solutions \( A(u) \) but with new irreducible KTs, \( K_9 \) and \( K_{10} \), that can be derived from \( K_7 \) and \( K_8 \) by interchanging the coordinates \( y \) and \( z \).

The results of this section can be summarized in the following theorem.

**Theorem 3.** An arbitrary conformally flat plane wave spacetime will admit only one irreducible KT, and that KT is the Koutras KT Lab. Special subcases exist:

(i) \( A(u) = \kappa u^{-2}, \kappa = \text{constant}, \kappa \neq 3/16 \). In this case the Koutras KT Lab is reducible and the spacetime admits no irreducible KTs.

(ii) \( A(u) = (3/16) u^{-2} \). In this case the Koutras KT Lab is reducible and there exist six independent irreducible KTs: \( K_1, K_2, K_3, K_4, K_8 \) and \( K_{10} \).

(iii) \( A(u) = (u^2 - \frac{1}{3})^{-2} \). In this case there exist four independent irreducible KTs: the Koutras KT Lab, \( K_1, K_5 \) and \( K_6 \).

(iv) \( A(u) \) is not given by any of the above and satisfies (4.36) and there can be at most three independent KTs, one of which is the irreducible Koutras KT Lab. The other KTs may or may not be irreducible.

(v) \( A(u) \) is not given by any of the above and satisfies (4.41) and there are three irreducible KTs: the Koutras KT Lab, \( K_7 \) and \( K_9 \).

(vi) \( A(u) \) is not given by any of the above and satisfies (4.42) and there are three irreducible KTs: the Koutras KT Lab, \( K_8 \) and \( K_{10} \).

**Corollary 3.** The maximum number of independent irreducible KTs in a conformally flat plane wave spacetime is six.

We remark that the above results are similar to those found by Kimura [9–11] in that only a very few specific metrics admit irreducible KTs and some of those which do, admit many. It is worth noting that even the class with \( A = \text{constant} \) admits only one irreducible KT, i.e.
the Koutras KT $L_{ab}$. Further, we have been unable to find the general solutions corresponding to cases (iv), (v) and (vi) of theorem 3.

The conformally flat plane wave spacetimes correspond to the Sippel and Goenner spacetimes of classes 15, 16 and 17, which are homogeneous pure radiation solutions and can also be interpreted as Einstein–Maxwell solutions, see [21] for details. The Sippel and Goenner [21] class 16 spacetime, i.e. $A = \text{constant}$, can also be interpreted as an Einstein–Klein–Gordon solution, see [28] for details.

5. Discussion

We have given the formal solution of the second-order KT equations for the general pp-wave spacetime, and have presented some noteworthy examples. We note that all physically meaningful pp-wave spacetimes are subject to the energy condition (1.7). The complete solution is given for the conformally flat plane wave spacetimes and we have found that irreducible KTs arise for specific classes. A number of theorems are given regarding the number of irreducible KTs admitted by pp-wave spacetimes. It is worth noting that the technique used in the proof of theorem 1 can be applied to any gradient CKV and the condition for reducibility determined. Further, one could in principle apply the reducibility condition to the general KT components given in section 2. So far we have been unable to explain geometrically why the value $\kappa = 3/16$ is singled out amongst the singular scale-invariant plane wave spacetimes. To this end we investigated the geodesic equations: as we have stated already, KTs are of interest principally because of their association with quadratic first integrals of the geodesic equation. However, this did not provide any further illumination. The geodesic equations can be integrated without recourse to the use of the irreducible KTs which is in contrast to the situation in [5] where complete integration of the geodesic equations required the first integral arising from the irreducible KT. The equation of geodesic deviation is important in the analysis of gravitational waves and may provide some insight into the problem but we only touch upon geodesic deviation briefly in what follows. The general results for KTs in vacuum pp-wave spacetimes will be presented elsewhere.

The existence of KTs is an interesting topic in its own right; however, we shall now mention some applications of the results obtained in this paper.

Equation of geodesic deviation

Consider a family of geodesics $x^a = x^a(\lambda)$ in an arbitrary spacetime, where $\lambda$ is an affine parameter. Let $t^a = dx^a/d\lambda$ be the tangent vector to a geodesic. The equation of geodesic deviation is

$$t^b \nabla_b \nabla_c \xi^a = R^a_{bcd} t^b \xi^d$$

where $\xi^a$ is the vector field representing the geodesic separation. Let $X_A$, $A = 1, \ldots, r$, be a basis for the isometry algebra $G_r$. Further, let $K_{a \ldots}^{a \ldots}$ be a KT of order $p + 1$ and define the field $W_B = K_{a \ldots}^{a \ldots} t^a \ldots t^a$. We shall denote the set of such fields as $W_B$, $B = 1, \ldots, s$. The general solution of the equation of geodesic deviation involves eight independent solutions and the set consisting of $t$, $\lambda t$, $X_A$ and $W_B$ are the solutions of the equation of geodesic deviation [32]. For example, in the case of the general plane wave spacetime (1.8) we have $r = 5$ and, on account of the existence of the Koutras KT (3.3), $s = 1$. These together with $t$ and $\lambda t$ provide eight solutions; however, their independence would have to be verified on a case-by-case basis.
**Penrose limits**

We will now investigate the existence of KTs in the Penrose limits of two important spacetimes. These Penrose limits are derived in [18]. First consider the Schwarzschild spacetime $ds^2 = -f(r)\, dt^2 + f(r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta\, d\phi^2), \quad f(r) = 1 - 2M/r.$

For radial null geodesics at constant $r$ (i.e., the unstable circular orbits at $r = 3M$) is the type $N$ vacuum plane wave spacetime (3.6) which admits one irreducible KT. For the non-radial, non-circular null geodesics, the Penrose limit of the singularity is given by the type $N$ vacuum plane wave spacetime

$$ds^2 = -2\, du\, dv - \kappa(y^2 - z^2)u^{-2} du^2 + dy^2 + dz^2,$$

which, from corollary 2, admits no irreducible KTs. Now consider the FRW spacetime

$$ds^2 = -dt^2 + a(t)^2[dr^2 + f_r(r)^2(d\theta^2 + \sin^2 \theta\, d\phi^2)]$$

where $f_r(r) = r, \sin r, \sinh r$ for $\epsilon = 0, +1, -1$, respectively, with the equation of state $p(t) = \omega_0 a(t)$ where $\omega > -1$ is constant. In the case $\omega = -1$ the Penrose limit is Minkowski spacetime. In what follows, the constant $h$ is defined by $h = 2/[3(1 + \omega)]$. For $\omega > -1$, the Penrose limit for the singularity of the FRW spacetime is given by the type $O$ plane wave spacetime

$$ds^2 = -2\, du\, dv - \kappa(y^2 + z^2)u^{-2} du^2 + dy^2 + dz^2.$$

For the case $\epsilon = 0$, and the case $\epsilon = \pm 1$ with $0 < h < 1$, the constant $\kappa = h(1 + h)^{-2}$ and it follows that $0 < \kappa < 1/4$. Theorem 3 states that in general this type $O$ plane wave spacetime will admit no irreducible KTs, with the exception of the case $\kappa = 3/16$ which admits the maximum number of six. For $\epsilon = 0$ this value of $\kappa$ corresponds to $h = 3$. The value $\kappa = 3/16$ can also arise in the regime $\epsilon = \pm 1$, $h \geq 1$, see [18] for details.

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**Appendix**

The nonzero connection coefficients for (1.5) are

$$\Gamma^v_{uu} = H_{,u}, \quad \Gamma^v_{uv} = \Gamma^v_{vv} = H_{,v}, \quad \Gamma^z_{uu} = \Gamma^z_{uv} = \Gamma^z_{vv} = H_{,z}.$$

For the pp-wave spacetime, (1.3) gives 20 independent differential equations. 11 of these equations are independent of the metric function $H$ and its derivatives, and 9 equations are dependent upon the metric function $H$. Direct integration of these equations gives the following expressions for the components $K_{ab}$:

- $K_{uu} = [\mu H_{,y} + \epsilon H_{,z} - \mu_{,uu}y - \epsilon_{,uu}z + \theta_{,uu}]v^2 + Mv + N$
- $K_{uv} = (\mu_{,u}y + \epsilon_{,u}z - \theta_{,u})v + D$
- $K_{vy} = -\mu_{,u}v^2 + Pu + R$
- $K_{yz} = -\epsilon_{,u}v^2 + Sv + Q$
- $K_{yu} = W_1$
- $K_{yy} = \mu v + W_2$
- $K_{xz} = \epsilon v + W_3$
- $K_{yv} = -2(\sigma z + \tau)v - \pi z^2 + \chi z + \xi$
- $K_{yz} = (\sigma y + \xi z + \Lambda)v + \pi yz - \frac{1}{2}(\chi y + \Omega z) + \Phi$
- $K_{xz} = -2(\xi y + \omega)v - \pi y^2 + \Omega y + \Sigma$
where $M, N, D, P, Q, R, S, W_1, W_2$ and $W_3$ are functions of $u, y$ and $z$; $\mu, \epsilon, \theta, \sigma, \tau, \xi, \Psi, \Gamma, \omega, \rho, v, \pi, \chi, \xi, \Lambda, \Omega, \Phi$ and $\Sigma$ are functions of $u$ only. The functions $W_1, W_2$ and $W_3$ are given by

$$W_1 = -2\mu y - 2\epsilon z + 2\theta, \quad W_2 = \sigma y z + \tau y - \xi z^2 + \Psi z + \Gamma$$

$$W_3 = \epsilon y z + \omega z - \sigma y^2 + \rho y + v.$$

The functions and constants are governed by the following differential equations:

$$M_u = 2H_u(\mu_u y + \epsilon_u z - \theta_u) + 2H_y P + 2H_z S \quad (A.1)$$

$$N_u = 2H_u D + 2H_y R + 2H_z Q \quad (A.2)$$

$$M + 2D_u - 2H_u W_1 - 2H_y W_2 - 2H_z W_3 = 0 \quad (A.3)$$

$$P_{,y} = \sigma_u z + \tau_u + 2\mu H_y \quad (A.4)$$

$$\chi_{,u} z - \pi_{,u} z^2 + \xi_u + 2R_{,y} - 4H_y W_2 = 0 \quad (A.5)$$

$$S_z = \xi_u y + \omega_u + 2\epsilon H_z \quad (A.6)$$

$$\Omega_{,u} y - \pi_{,u} y^2 + \Sigma_u + 2Q_{,z} - 4H_{,z} W_3 = 0 \quad (A.7)$$

$$\Psi_{,u} + D_y + P + \sigma_{,u} y z + \tau_{,u} y - \xi_{,u} z^2 + \Gamma_u - 2H_y W_1 = 0 \quad (A.8)$$

$$\rho_{,u} y + D_z + S + \xi_{,u} y z + \omega_{,u} z - \sigma_{,u} y^2 + \nu_u - 2H_{,z} W_1 = 0 \quad (A.9)$$

$$Q_y + R_z + \pi_{,u} y z - \frac{1}{2}\chi_{,u} y - \frac{1}{2}\Omega_{,u} z + \Phi_u - 2H_{,y} W_3 - 2H_{,z} W_2 = 0 \quad (A.10)$$

$$\rho + \Lambda + \Psi = 0 \quad (A.11)$$

$$\mu H_{,uu} y + 3\mu_{,u} H_y y + \epsilon H_{,uu} z + 3\epsilon_{,u} H_z - \mu_{,uu} y - \epsilon_{,uu} z + \theta_{,uu} = 0 \quad (A.12)$$

$$- 3\mu_{,uu} + \mu H_{,yy} y + \epsilon H_{,yy} z = 0 \quad (A.13)$$

$$- 3\epsilon_{,uu} + \epsilon H_{,zz} z + \mu H_{,zz} z = 0 \quad (A.14)$$

$$S_{,y} + P_{,u} y + \xi_{,u} z + \Lambda_{,u} - 2\epsilon H_{,y} - 2\mu H_{,z} = 0 \quad (A.15)$$

$$M_{,y} + 2P_{,u} - 4H_{,y}(\mu_{,u} y + \epsilon_{,u} z - \theta_{,u}) - 2\mu H_{,u} + 4H_{,y}(\sigma y z + \tau) - 2H_{,z}(\sigma y + \xi z + \Lambda) = 0 \quad (A.16)$$

$$N_{,y} + 2R_{,u} - 4H_{,y} D - 2H_u W_2 - 2H_y(-\pi z^2 + \chi z + \xi)$$

$$- 2H_{,z}(\pi y z - \frac{1}{2}\chi z + \frac{1}{2}\Omega z + \Phi) = 0 \quad (A.17)$$

$$M_{,z} + 2S_{,u} - 4H_{,z}(\mu_{,u} y + \epsilon_{,u} z - \theta_{,u}) - 2\epsilon H_{,u} - 2H_{,y}(\sigma y z + \xi z + \Lambda) + 4H_{,z}(\chi y + \omega) = 0 \quad (A.18)$$

$$N_{,z} + 2Q_{,u} - 4H_{,z} D - 2H_u W_3 - 2H_y(-\pi y^2 + \Omega y + \Sigma)$$

$$- 2H_{,y}(\pi y z - \frac{1}{2}\chi y - \frac{1}{2}\Omega z + \Phi) = 0. \quad (A.19)$$

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