New Coupled Method for Solving Burger’s Equation

L N M Tawfiq¹, A H Khamas²

¹Department of Mathematics, College of Education for Pure Science Ibn Al-Haitham, University of Baghdad. Email: luma.n.m@ihcoedu.uobaghdad.edu.iq
²Department of Clinical Laboratory Sciences, College of Pharmacy, Mustansiriyah University. Email: dr.aseel@uomustansiriyah.edu.iq

Abstract In this paper one, two, three-dimensional Burger’s equation have been solved by using efficient method based on coupled new integral transform with homotopy perturbation method. The implementation of the suggested method demonstrates the usefulness in finding solution in wider domain. The practical implications show the effectiveness of proposed method and it is easily implemented in finding solution, since it can be solve the problems without resorting to the frequency domain and discretization.

Keywords: Partial differential equations, Burger’s equation, homotopy perturbation method.

1. Introduction

Differential Equations (DEs) are globally used to describe the behaviour of natural phenomena around us that experience changes due to a specific factor or multiple factors. These equations link between the unknown function of that behaviour/phenomenon with its derivative. While such unknown functions represent a specific physical quantity, its derivative represent its rate of change with respect to a specific factor (time, space, temperature, pressure, etc…)[1, 2].

The DEs usually come in two types: ordinary and partial differential equations (ODEs and PDEs) respectively. Models involving these two types had been appeared in many fields such as fluids, mechanical engineering, physics, earth sciences, relativity, and finance [3-5].

Many well-developed solution techniques were presented by researchers over the last two centuries. Analytical solutions were preferred for a number of given DEs. However, models described by DEs were often so complex or so large. In such cases, analytical solutions were even not so tractable or cannot be solved to the exact solution using such methods. For that reason, the numerical solution methods have to be developed to be implemented in such cases [6].

Numerical methods have been used recently to develop simulations for dynamic mathematical models and it has been the focus of an intensive research. This field exhibits an extended range of applications and for that reason continuous demands for better and more reliable method grows simultaneously [7].

Nonlinear partial differential equations (NLPDEs) became an important tool to model physical phenomena. It has a great importance in the fields of science and engineering [8]. Most of these phenomena are models of our daily life problems, an example of such problems are the solution for Burger’s equation. Over the last few years a variety of a powerful methods have been developed to offer better solution for the PDEs represent such phenomena, examples of such methods are Fourier Transform (FT) [9], Homotopy Perturbation Method (HPM) [10, 11], Homotopy Analysis Method (HAM) [12], semi analytic method [13-15], Variational Iterational Method (VIM) [16, 17], differential transform method (DTM) [18] and Adomian Decomposition Method (ADM) [19, 20].It is well known that the need to obtain analytical solutions due to the limitation of the numerical solutions that cannot tell us much about the qualitative behaviour of systems is remained. The first attempts at analytical solutions were to apply perturbation techniques to obtain approximate analytical solutions. These methods depend on the availability of a perturbation parameter where the approximate solutions are dependent on the chosen parameter which forms part of the equation and/or the boundary conditions and not the independent variable. This is placing an additional restriction on the problem.
In the recent the researchers suggest some efficient technique such as parallel processing technique such neural networks (ANNs) [21, 22], Simulink method [23] and coupled method [24, 25] and recently the coupled based transform [26] were developed.

2. The New Transform
The new integral transform, which is known as the new transform, was proposed by Luma and Alaa[27], and defined as follow:

\[ f^*(s) = \mathcal{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{s}\right) dt \quad (1) \]

The properties of this transform where introduced in [27]. We illustrate some of this properties in Table (1), the symbol \(\mathcal{T}\) denote the new transform:

| Function          | Property                                      |
|-------------------|-----------------------------------------------|
| \(t^n\)           | \(\frac{n!}{s^n} e^{-a t} f(s-a)\)           |
| \(r^a\), \(a>0\)  | \(\mathbb{F}(a+1)s^a\)                      |
| \(e^{at}\)        | \(s/s-a\) if \(a\leq 0\) \(\mathcal{T}\\{\mathcal{T}\{f(t)\}\}\) |
| \(\sin(at)\)      | \(a^2+s^2\)                                 |
| \(\cos(at)\)      | \(s^2\)                                     |
| \(t^n f\)         | \((-1)^n s^{\frac{n}{2}} \mathcal{T}\{f(t)\}\) |
| \(e^{at} f\)      | \(\frac{s}{s-a} f(s-a)\)                    |
| \((f * g)(t)\)    | \(\int_0^t f(\tau) g(t-\tau) d\tau\)       |
| \(f^{(n)}\)       | \(s^n \mathcal{T}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k} f^{(k)}(0)\) |
| \(\frac{\partial^n}{\partial x^n} f(t,x)\) | \(\frac{\partial^n}{\partial x^n} \mathcal{T}\{f(t,x)\}\) |

3. The Idea of Proposed Method
We consider the following nonlinear PDE:

\[ L[u(X,t)] + R[u(X,t)] + N[u(X,t)] = g(X,t) \quad (2) \]

Subject to initial condition (IC):

\[ u(X,0) = f(X) \]

Where \(X \in \mathbb{R}^n\), \(L\) is the linear differential operator \(L = \frac{\partial}{\partial t}\), \(R\) the remained of the linear operator, \(N\) is the nonlinear differential operator and \(g(X,t)\) is the inhomogeneous term.

Taking the new transform with respect to the variable \(t\) for the equation (2) we have:
\[ T\{L(u)\} + T\{R(u)\} + T\{N(u)\} = T\{g\} \]  

(3)

Now by using the differentiation property of new transform illustrated in Table (1), when \( n=1 \), we see that equation (3) becomes:

\[ s(T\{u\} - f(X)) + T\{R(u)\} + T\{N(u)\} = T\{g\} \]

Hence, we get:

\[ T\{u\} = f(X) + \frac{T\{g\}}{s} - \frac{T\{R(u)\}}{s} - \frac{T\{N(u)\}}{s} \]

(4)

By taking the inverse of new transform for both sides of equation (4), we have:

\[ u(X,t) = G(X,t) - T^{-1}\left\{ \frac{T\{R(u) + N(u)\}}{s} \right\} \]

(5)

Where \( G(X,t) = f(X) + T^{-1}\left\{ \frac{T\{g\}}{s} \right\} \).

Now, suppose that the solution is an infinite series as following:

\[ u(X,t) = \sum_{n=0}^{\infty} u_n(X,t) \]

(6)

We can decomposed the nonlinear part as:

\[ N(\theta) = \sum_{n=0}^{\infty} A_n \]

(7)

Such that \( A_n \) is the Adomain polynomial \([28]\)of \( u_0, u_1, u_2, \ldots, u_n \) and can be defined as:

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0} \quad , \quad n = 0, 1, 2 \ldots \]

(8)

Then substitute equations (6), (7) and (8) in equation (5) to get:

\[ \sum_{n=0}^{\infty} u_n(X,t) = G(X,t) - T^{-1}\left\{ \frac{T\{R[\sum_{n=0}^{\infty} u_n] + \sum_{n=0}^{\infty} A_n\}}{s} \right\} \]

(9)

By comparing both sides of the equation (9) we have:
Now, we illustrate the implementation of above steps by the following application for Burger's equation

4. Application
In this section we use NTHPM to solve 4D, 2nd order Burger's equation:

\[ \frac{\partial u}{\partial t} + u \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) = \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]

We see that

\[ L[u(X,t)] = \frac{\partial u(x,y,z,t)}{\partial t}, \quad R[u(X,t)] = -\alpha \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right], \]

\[ N[u(X,t)] = u \left[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right] \text{and } g(X,t) = 0 \]

Then \( G(X, t) = f(X) \) and hence \( u_0 = f(X) \).

To compute the Adomain polynomials \( A_n \) to the nonlinear term \( N[\theta] \) by using equation (8) we have:

\[ A_0 = u_0 \left[ \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial z} \right] \]

\[ A_1 = u_1 \left[ \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial z} \right] + u_0 \left[ \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} + \frac{\partial u_1}{\partial z} \right] \]

\[ A_2 = u_2 \left[ \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial z} \right] + u_1 \left[ \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} + \frac{\partial u_1}{\partial z} \right] + u_0 \left[ \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_2}{\partial z} \right] \]

and so on.

Follow three problems of Burger's equation with initial condition. Firstly, we illustrate the implementation of NTHPM in 1D then in 3D.

4.1. Problem 1
Consider the following 2nd order, nonlinear Burger's equation:
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}
\]

Subject to IC: \( f(x) = u(x, 0) = 1 - \tanh \left( \frac{x}{2\alpha} \right) \)

Now, we calculate \( A_n \) as:

\[
A_0 = -\left[ \tanh \left( \frac{x}{2\alpha} \right) - 1 \right]^2 \left[ \tanh \left( \frac{x}{2\alpha} \right) + 1 \right] \frac{1}{2\alpha}
\]

\[
A_1 = -t \left[ \tanh \left( \frac{x}{2\alpha} \right) - 1 \right]^2 \left[ \tanh \left( \frac{x}{2\alpha} \right) + 1 \right] \left[ 3 \tanh \left( \frac{x}{2\alpha} \right) + 1 \right] \frac{1}{4\alpha^2}
\]

\[
A_2 = -t^2 \left[ \tanh \left( \frac{x}{2\alpha} \right) - 1 \right]^2 \left[ 2\tanh \left( \frac{x}{2\alpha} \right) + 9 \tanh \left( \frac{x}{2\alpha} \right)^2 + 6 \left( \tanh \left( \frac{x}{2\alpha} \right) \right)^3 - 1 \right] \frac{1}{8\alpha^3}
\]

and so on.

Moreover, we calculate \( u_n \) to get the series solution as follow:

\[
u_0 = 1 - \tanh \left( \frac{x}{2\alpha} \right)
\]

\[
u_1 = -t \frac{\left( \tanh \left( \frac{x}{2\alpha} \right) \right)^2 - 1}{2\alpha}
\]

\[
u_2 = -t^2 \tanh \left( \frac{x}{2\alpha} \right) \left[ \left( \tanh \left( \frac{x}{2\alpha} \right) \right)^2 - 1 \right] \frac{1}{4\alpha^2}
\]

\[
u_3 = -t^3 \left[ 3 \left( \tanh \left( \frac{x}{2\alpha} \right) \right)^4 - 4 \left( \tanh \left( \frac{x}{2\alpha} \right) \right)^2 + 1 \right] \frac{1}{24\alpha^3}
\]

\[
u_4 = -t^4 \tanh \left( \frac{x}{2\alpha} \right) \left[ 3 \left( \tanh \left( \frac{x}{2\alpha} \right) \right)^4 - 5 \left( \tanh \left( \frac{x}{2\alpha} \right) \right)^2 + 2 \right] \frac{1}{48\alpha^4}
\]

and so on.

It is easy to verify that this series converge to the exact solution.

\[
u(x, t) = 1 - \tanh \left( \frac{x - t}{2\alpha} \right)
\]

4.2. Problem 2

Consider the following 3D, 2\(^{nd}\) order, nonlinear Burger's equation:

\[
\frac{\partial u}{\partial t} + u \left[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right] = \alpha \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]
\]

Subject to IC:
\[ f(x, y, z) = u(x, y, z, 0) = 1 - \tanh\left(\frac{x + y + z}{2\alpha}\right) \]

Now, we calculate \(A_0\) as follow:
\[
A_0 = -\frac{[\tanh\left(\frac{x+y+z}{2\alpha}\right) - 1]^2 [\tanh\left(\frac{x+y+z}{2\alpha}\right) + 1]}{2\alpha}
\]

\[
A_1 = -9t \frac{\left[\tanh\left(\frac{x+y+z}{2\alpha}\right) - 1\right]^2 \left[4 \tanh\left(\frac{x+y+z}{2\alpha}\right) + 3 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^2 + 1\right]}{4\alpha^2}
\]

\[
A_2 = -27t^2 \frac{\left[\tanh\left(\frac{x+y+z}{2\alpha}\right) - 1\right]^2 \left[2\tanh\left(\frac{x+y+z}{2\alpha}\right) + 9 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^2 + 6 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^3 - 1\right]}{8\alpha^3}
\]

and so on.

Moreover, the series solution \(u_n\) can be calculated as follow:

\[
u_0 = 1 - \tanh\left(\frac{x + y + z}{2\alpha}\right) \]

\[
u_1 = -3t \frac{\left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^2 - 1}{2\alpha}
\]

\[
u_2 = -9t^2 \frac{\tanh\left(\frac{x+y+z}{2\alpha}\right) \left[\tanh\left(\frac{x+y+z}{2\alpha}\right)\right]^2 - 1}{4\alpha^2}
\]

\[
u_3 = -9t^3 \frac{3 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^4 - 4 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^2 + 1}{8\alpha^3}
\]

\[
u_4 = -27t^4 \frac{\tanh\left(\frac{x+y+z}{2\alpha}\right) \left[3 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^4 - 5 \left(\tanh\left(\frac{x+y+z}{2\alpha}\right)\right)^2 + 2\right]}{16\alpha^4}
\]

and so on. It is easy to illustrate that the series solution converge to the exact solution.

\[ u(x, y, z, t) = 1 - \tanh\left(\frac{x + y + z - 3t}{2\alpha}\right) \]

4.3. Problem 3

Consider the equation in problem 2 with the following initial condition

\[ f(x, y, z) = \theta(x, y, z, 0) = 1 - \tanh\left(\frac{x + y + z}{6\alpha}\right) \]

\(A_n\) can be calculated as:

\[
A_0 = -\frac{[\tanh\left(\frac{x+y+z}{6\alpha}\right) - 1]^2 [\tanh\left(\frac{x+y+z}{6\alpha}\right) + 1]}{6\alpha}
\]

\[
A_1 = -t \frac{\left[\tanh\left(\frac{x+y+z}{6\alpha}\right) - 1\right]^2 \left[4 \tanh\left(\frac{x+y+z}{6\alpha}\right) + 3 \left(\tanh\left(\frac{x+y+z}{6\alpha}\right)\right)^2 + 1\right]}{36\alpha^2}
\]

6
\[ A_2 = -t^2 \left[ \frac{\tanh \left( \frac{x+y+z}{6\alpha} \right) - 1}{216\alpha^3} \right]^2 \left[ 2 \tanh \left( \frac{x+y+z}{6\alpha} \right) + 9 \left( \tanh \left( \frac{x+y+z}{6\alpha} \right)^2 + 6 \left( \tanh \left( \frac{x+y+z}{6\alpha} \right)^3 - 1 \right) \right] \]

and so on.

Moreover, the series solution \( u_n \) can be calculated as:

\[ u_0 = 1 - \tanh \left( \frac{x+y+z}{6\alpha} \right) ; \]

\[ u_1 = -t \frac{\tanh \left( \frac{x+y+z}{2\alpha} \right)^2 - 1}{6\alpha} ; \]

\[ u_2 = -t^2 \frac{\tanh \left( \frac{x+y+z}{6\alpha} \right) \left( \tanh \left( \frac{x+y+z}{6\alpha} \right)^2 - 1 \right)}{36\alpha^2} ; \]

\[ u_3 = -t^3 \frac{3 \left( \tanh \left( \frac{x+y+z}{6\alpha} \right)^4 - 4 \left( \tanh \left( \frac{x+y+z}{6\alpha} \right)^2 - 1 \right) \right)}{648\alpha^3} ; \]

\[ u_4 = t^4 \frac{\tanh \left( \frac{x+y+z}{6\alpha} \right) \left[ 3 \left( \tanh \left( \frac{x+y+z}{6\alpha} \right) \right)^4 - 5 \left( \tanh \left( \frac{x+y+z}{6\alpha} \right) \right)^2 - 2 \right]}{3888\alpha^4} ; \]

and so on.

Then the solution of the problem is close to the form:

\[ u(x, y, z, t) = 1 - \tanh \left( \frac{x + y + z - t}{6\alpha} \right) \]

This represents the exact solution of the problem.

5. Convergence of the Solution

Now, we must prove the convergence of series solution to the exact form when we used the NTHPM, the series solution is given in equation (11), where \( u_n, (\text{n= 0, 1, ...}) \), are calculated by new transform, i.e.,

\[ u(x, y, z, t) = u_0(x, y, z, t) + u_1(x, y, z, t) + \cdots + \sum_{n=0}^{\infty} u_n(x, y, z, t) \quad (11) \]

\[ u_n = -T^{-1} \left\{ \frac{\mathbb{T}\left\{R[u_{n-1}] + A_{n-1}\right\}}{\nu} \right\}, n \geq 1 \quad (12) \]

And \( A_n, (n = 0, 1, \ldots) \), are defined as

\[ A_n = u_n \frac{\partial u_0}{\partial z} + u_{n-1} \frac{\partial u_1}{\partial z} + \cdots + u_0 \frac{\partial u_n}{\partial z} = \sum_{k=0}^{n} u_k \frac{\partial u_{n-k}}{\partial z} \quad (13) \]}
Now we show the convergence in the following.

**Lemma 1**

If \( f \) be continues function then \( \frac{\partial}{\partial t} \int_0^t f(t - \tau) d\tau = f(t) \)

**Proof**

Suppose that

\[
\int f(x) dx = F(x) + c
\]

Assume that \( x = t - \tau \) then \( dx = -d\tau \) then

\[
\frac{\partial}{\partial t} \int_0^t f(t - \tau) d\tau = -\frac{\partial}{\partial t} \int_0^t f(x) dx = \frac{\partial}{\partial t} \left[ F(x) \right]_0^t = \frac{\partial}{\partial t} [F(t) - F(0)]
\]

\[
= \frac{\partial}{\partial t} F(t) - \frac{\partial}{\partial t} F(0) = f(t)
\]

so,

\[
\frac{\partial}{\partial t} \int_0^t f(t - \tau) d\tau = f(t)
\]

**Lemma 2**

Let \( \mathbb{T} \) is new transform. Then

\[
\frac{\partial}{\partial t} \left( \mathbb{T}^{-1} \left\{ \frac{1}{u} \mathbb{T}\left[ f(X,t) \right] \right\} \right) = f(X,t) , \text{ where } X = (x,y,z)
\]

**Proof**

Using property 2 and 3 of NT, and lemma (1), we have

\[
\frac{\partial}{\partial t} \left( \mathbb{T}^{-1} \left\{ \frac{1}{u} \mathbb{T}\left[ f(X,t) \right] \right\} \right) = \frac{\partial}{\partial t} \left( \mathbb{T}^{-1} \left\{ \frac{1}{u} \mathbb{T}\{1\} \mathbb{T}\{f(X,t)\} \right\} \right) = \frac{\partial}{\partial t} \left( \mathbb{T}^{-1} \{\mathbb{T}\{1*f(X,t)\}\} \right)
\]

\[
= \frac{\partial}{\partial t} (1 * f(X,t)) = \frac{\partial}{\partial t} \left( \int_0^t f(X,t - \tau) d\tau \right) = f(X,t)
\]
**Theorem 3 (Convergence Theorem)**

If the series form (11) which was calculated by NTHPM is convergent then the limit point converges to the exact solution for the equation (2). Suppose that the limit point is:

\[ w(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t) \]

Now, from left hand side of equation (2) we have:

\[ \frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} u_n(x, y, z, t) = \frac{\partial}{\partial t} \left[ u_0(x, y, z, t) + \sum_{n=1}^{\infty} u_n(x, y, z, t) \right] \]

\[ = \frac{\partial}{\partial t} \left[ T^{-1}\{f\} - \sum_{n=1}^{\infty} \frac{T[R(u_{n-1}) + A_{n-1}]}{v} \right] \]

\[ = \frac{\partial}{\partial t} \left[ T^{-1}\{f\} - \sum_{n=0}^{\infty} \frac{T[R(u_n)]}{v} \right] - \sum_{n=0}^{\infty} \frac{T[A_n]}{v} \]

\[ = \frac{\partial f}{\partial t} - \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left[ T^{-1}\left(\frac{T[R(u_n)]}{v}\right) \right] - \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left[ T^{-1}\left(\frac{T[A_n]}{v}\right) \right] \] (14)

By lemma (2.2) and equation (14) we get:

\[ \frac{\partial w}{\partial t} = 0 - \sum_{n=0}^{\infty} R[u_n] - \sum_{n=0}^{\infty} A_n \] (15)

However, from equation (13) we have
\[ \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} u_k \frac{\partial u_{n-k}}{\partial z} \]

\[ = u_0 \frac{\partial u_0}{\partial z} + u_0 \frac{\partial u_1}{\partial z} + u_1 \frac{\partial u_0}{\partial z} + u_0 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial u_0}{\partial z} + u_0 \frac{\partial u_3}{\partial z} + \ldots \]

\[ + u_1 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial u_1}{\partial z} + u_3 \frac{\partial u_0}{\partial z} + \ldots \]

\[ = u_0 \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \ldots \right) + u_1 \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \ldots \right) \]

\[ + u_2 \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \ldots \right) + u_3 \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \ldots \right) \]

\[ + \ldots \]

\[ = (u_0 + u_1 + u_2 + u_3 + \ldots) \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \ldots \right) = \left( \sum_{n=0}^{\infty} u_n \right) \left( \sum_{n=0}^{\infty} \frac{\partial u_n}{\partial z} \right) \]

\[ = \left( \sum_{n=0}^{\infty} u_n \right) \left( \frac{\partial}{\partial z} \sum_{n=0}^{\infty} u_n \right) \]

Then substitute equation (16) in equation (15) to obtain

\[ \frac{\partial w}{\partial t} = -R \left[ \sum_{n=0}^{\infty} u_n \right] - \left( \sum_{n=0}^{\infty} u_n \right) \left( \frac{\partial}{\partial z} \sum_{n=0}^{\infty} u_n \right) = -R[w] - w \frac{\partial w}{\partial z} \]

then \[ \frac{\partial w}{\partial t} = \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] - w \frac{\partial w}{\partial z} \]

Then \( w(x, y, z, t) \) is satisfy equation (2). So , its exact solution.

6. Conclusion

The combination of new transform suggested by Luma and Alaa with HPM is used to get exact solution of the 2 and 4 dimensional, 2nd order, nonlinear Burger's equation. The proposed method is free of unnecessary mathematical complexities and discretization of domain. The accuracy, efficiency, and reliability of the proposed method are guaranteed. The convergence of obtained solution to the exact form by using NTHPM is proved.
7. References

[1] Mauch, S., 2002, Advanced Mathematical Methods for Scientists and Engineers, Mauch Publishing Company, 1st Edition.

[2] Zwillinger, D., 1997, *Handbook of Differential Equations*, 3rd edition, Academic Press.
[3] Rahimi, E., Rahimifar, A., Mohammadyari, R., Rahimipetroudi, I., and Rahimi, E.M. 2016, Analytical approach for solving two-dimensional laminar viscous flow between slowly expanding and contracting walls. *ASEJ.*, 7(4): 1089-1097. doi:10.1016/j.asej.2015.07.013.

[4] Salih, H., Tawfiq, L.N.M., Yahya, Z.R.I., and Zin, S.M. 2018 Solving Modified Regularized Long Wave Equation Using Collocation Method, *Journal of Physics: Conference Series*, 1003(012062): 1-10. doi:10.1088/1742-6596/1003/1/012062.

[5] Wazwaz, A.M.2009 *Partial Differential Equations and Solitary Waves Theory*, 1st edition, Beijing and Berlin: Springer, ISBN 978-3-642-00250-2, e-ISBN 978-3-642-00251-9.

[6] Javidi, M. 2006, A numerical solution of Burger’s equation based on modified extended BDF scheme, *International Mathematical Forum*, 1(32): 1565 – 1570.

[7] Tawfiq, L.N.M., Jasim, K.A. and Abdulhmeed, E.O. 2016 Numerical Model for Estimation the Concentration of Heavy Metals in Soil and its Application in Iraq, *Global Journal of Engineering science and Researches*, 3(3):75-81.

[8] Tawfiq, L.N.M., and Jabber, A.K., 2018 Steady State Radial Flow in Anisotropic and Homogenous in Confined Aquifers. *Journal of Physics: Conference Series*, 1003(012056): 1-12.

[9] Tawfiq, L. N. M. and Jabber, A. K., 2017 Solve the groundwater model equation using Fourier transforms Method, *Int. J. Adv. Appl. Math. and Mech.*, 5(1): 75 – 80.

[10] He, J.H. 1999 Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 178 (3-4): 257-262. doi:10.1016/s0045-7825(99)00018-3.

[11] Tawfiq, LNM.; Jabber, AK. Mathematical Modeling of Groundwater Flow. Global Journal of Engineering Science and Researches. 2016, 3, 10, 15-22.

[12] Jamil, H.J., Albahri, M.R.A., Al-Noor, N.H., ...Arnetz, B., Tawfiq, L.N.M. 2020, Hookah Smoking with Health Risk Perception of Different Types of Tobacco, Journal of Physics: Conference Series, 2020, 1664(1), 012127.

[13] Tawfiq, L.N.M. and Yassien, S.M., 2013, Solution of High Order Ordinary Boundary Value Problems Using Semi-Analytic Technique, *Ibn Al-Haitham Journal for Pure & Applied Sciences*, 26(1): 281-291.
[14] Tawfiq, L.N.M. and Hussein, R.W., 2014 Efficient Semi-Analytic Technique for Solving Nonlinear Singular Initial Value Problems, Ibn Al-Haitham Journal for Pure & Applied Sciences, 27 3 pp 513-522.

[15] Tawfiq, L.N.M., and Hilal, M.M., 2014 Solution of 2nd Order Nonlinear Three-Point Boundary Value Problems By Semi-Analytic Technique, Ibn Al-Haitham Journal for Pure & Applied Sciences, 27 3 pp 501-512.

[16] Tawfiq LNM, 2005, On Training of Artificial Neural Networks, Al-Fatih journal, 1(23), 130-139.

[17] Salih H., and Tawfiq LNM. 2020. Solution of Modified Equal Width Equation Using Quartic Trigonometric-Spline Method. Journal of Physics: Conference Series. 1664, 012033: 1-10. doi:10.1088/1742-6596/1664/1/012033.

[18] Tawfiq LNM and Altaie H. 2020. Recent Modification of Homotopy Perturbation Method for Solving System of Third Order PDEs. Journal of Physics: Conference Series. 1530, 012073: 1-8.

[19] Kareem Z.H. and Tawfiq L.N.M. 2020. Solving Three-Dimensional Groundwater Recharge Based on Decomposition Method. Journal of Physics: Conference Series. 1530, 012068: 1-8.

[20] Tawfiq, L.N.M. and Ali M.H. 2020. Efficient Design of Neural Networks for Solving Third Order Partial Differential Equations. Journal of Physics: Conference Series. 1530, 012067: 1-8.

[21] Tawfiq, L.N.M. and Hussein, A.A.T., 2013 Design feed forward neural network to solve singular boundary value problems, ISRN Applied Mathematics, 2013 pp 1-8, http://dx.doi.org/10.1155/2013/650467.

[22] Tawfiq, L.N.M., and Hasan, M.A., 2019, Evaluate the Rate of Pollution in Soil using Simulink Environment, Ibn Al-Haitham Jour. for Pure & Appl. Sci., 32 1 pp 132-138.

[23] Tawfiq LNM, and Abood I N, 2018 Persons Camp Using Interpolation Method, Journal of Physics: Conference Series. 1003(012055): 1-10.

[24] Enadi, M.O., and Tawfiq, L.N.M., 2019, New Approach for Solving Three Dimensional Space Partial Differential Equation, Baghdad Science Journal, 16 3 pp 786-792.

[25] Ghazi, FF., and Tawfiq, L.N.M. 2020. New Approach for Solving Two Dimensional Spaces PDE. Journal of Physics: Conference Series. 1530, 012066: 1-10

[26] Hussein NA and Tawfiq LNM. 2020. New Approach for Solving (1+1)-Dimensional Differential Equation. Journal of Physics: Conference Series. 1530. 012098: 1-11.

[27] Tawfiq, L.N.M., and Jabber, A.K. 2018, New Transform Fundamental Properties and its Applications, Ibn Al-Haitham Journal for Pure & Applied Sciences, 31 1 pp 151-163.

[28] Tawfiq, LNM, Hassan MA. 2018 Estimate the Effect of Rainwaters in Contaminated Soil by Using Simulink Technique. Journal of Physics: Conference Series. 1003(012057):1-7.
[29] Tawfiq, L.N.M. and Salih, O. M., 2019 Design neural network based upon decomposition approach for solving reaction diffusion equation, Journal of Physics: Conference Series, 1234 012104, pp.1-8.

[30] Tawfiq, L.N.M, Al-Noor, N.H. and Al-Noor, T.H. 2019 Estimate the Rate of Contamination in Baghdad Soils By Using Numerical Method, Journal of Physics: Conference Series, 1294 (032020), doi:10.1088/1742-6596/1294/3/032020.

[31] Ali MH, Tawfiq LNM, Thirthar AA, 2019 Designing Coupled Feed Forward Neural Network to Solve Fourth Order Singular Boundary Value Problem. Reviesta Aus Vol.26(2) pp.140–6. DOI: 10.4206/aus.2019.n26.2.20.