The spectral radius of graphs with no intersecting odd cycles

Yongtao Li, Yuejian Peng†

School of Mathematics, Hunan University
Changsha, Hunan, 410082, P.R. China

January 14, 2022

Abstract

Let \( H_{s,t_1\ldots t_k} \) be the graph with \( s \) triangles and \( k \) odd cycles of lengths \( t_1, \ldots, t_k \geq 5 \) intersecting in exactly one common vertex. Recently, Hou, Qiu and Liu [Discrete Math. 341 (2018) 126–137], and Yuan [J. Graph Theory 89 (1) (2018) 26–39] determined independently the maximum number of edges in an \( n \)-vertex graph that does not contain \( H_{s,t_1\ldots t_k} \) as a subgraph. In this paper, we determine the graphs of order \( n \) that attain the maximum spectral radius among all graphs containing no \( H_{s,t_1\ldots t_k} \) for \( n \) large enough.

Key words: Spectral radius; Intersecting odd cycles; Extremal graph; Stability method.

2010 Mathematics Subject Classification. 05C50, 15A18, 05C38.

1 Introduction

In this paper, we consider only simple and undirected graphs. Let \( G \) be a simple connected graph with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and edge set \( E(G) = \{e_1, \ldots, e_m\} \). Let \( N(v) \) or \( N_G(v) \) be the set of neighbors of \( v \), and \( d(v) \) or \( d_G(v) \) be the degree of a vertex \( v \) in \( G \). Let \( S \) be a set of vertices. We write \( d_S(v) \) for the number of neighbors of \( v \) in the set \( S \), that is, \( d_S(v) = |N(v) \cap S| \). And we denote by \( e(S) \) the number of edges contained in \( S \). Let \( K_{s,t} \) be the complete bipartite graph with parts of sizes \( s \) and \( t \). In particular, when \( s = 1 \), the graph \( K_{1,t} \) is the star graph with \( t \) edges. Let \( M_t \) be the matching of size \( t \), i.e., the union of \( t \) disjoint edges.

*This paper was firstly announced in May, 2021, and was later published on Discrete Mathematics 345 (2022) 112907. See https://doi.org/10.1016/j.disc.2022.112907 E-mail addresses: ytli0921@hnu.edu.cn (Y. Li), ypeng1@hnu.edu.cn (Y. Peng, corresponding author).
The Turán number of a graph $F$ is the maximum number of edges in an $n$-vertex graph without a subgraph isomorphic to $F$, and it is usually denoted by $\text{ex}(n, F)$. We say that a graph $G$ is $F$-free if it does not contain an isomorphic copy of $F$ as a subgraph. A graph on $n$ vertices with no subgraph $F$ and with $\text{ex}(n, F)$ edges is called an extremal graph for $F$ and we denote by $\text{Ex}(n, F)$ the set of all extremal graphs on $n$ vertices for $F$. It is a cornerstone of extremal graph theory to understand $\text{ex}(n, F)$ and $\text{Ex}(n, F)$ for various graphs $F$; see [24, 28, 40] for surveys.

In 1941, Turán [41] posed the natural question of determining $\text{ex}(n, K_{r+1})$ for $r \geq 2$. Let $T_r(n)$ denote the complete $r$-partite graph on $n$ vertices where its part sizes are as equal as possible. Turán [41] (also see [5, p. 294]) extended a result of Mantel [30] and obtained that if $G$ is an $n$-vertex graph containing no $K_{r+1}$, then $e(G) \leq e(T_r(n))$, equality holds if and only if $G = T_r(n)$. There are many extensions and generalizations on Turán’s result. The problem of determining $\text{ex}(n, F)$ is usually called the Turán-type extremal problem. The most celebrated extension always attributes to a result of Erdős, Stone and Simonovits [15, 14], which states that

$$\text{ex}(n, F) = \left(1 - \frac{1}{\chi(F)} - 1 + o(1)\right)\frac{n^2}{2},$$

where $\chi(F)$ is the vertex-chromatic number of $F$. This provides good asymptotic estimates for the extremal numbers of non-bipartite graphs. However, for bipartite graphs, where $\chi(F) = 2$, it only gives the bound $\text{ex}(n, F) = o(n^2)$. Although there have been numerous attempts on finding better bounds of $\text{ex}(n, F)$ for various bipartite graphs $F$, we know very little in this case. The history of such a case began in 1954 with the theorem of Kővari, Sós and Turán [29], which states that if $K_{s,t}$ is the complete bipartite graph with vertex classes of size $s \geq t$, then $\text{ex}(n, K_{s,t}) = O(n^{2-1/t})$; see [20, 21] for more details. In particular, we refer the interested reader to the comprehensive survey by Füredi and Simonovits [24].

### 1.1 History and background

In this section, we shall review the exact values of $\text{ex}(n, F)$ for some special graphs $F$, instead of the asymptotic estimation. A graph on $2k+1$ vertices consisting of $k$ triangles which intersect in exactly one common vertex is called a $k$-fan (also known as the friendship graph) and is denoted by $F_k$. Since $\chi(F_k) = 3$, the theorem of Erdős, Stone and Simonovits in [11] implies that $\text{ex}(n, F_k) = n^2/4 + o(n^2)$. In 1995, Erdős, Füredi, Gould and Gunderson [16] proved the following exact result.

**Theorem 1.1.** [11] For every $k \geq 1$, and for every $n \geq 50k^2$,

$$\text{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k, & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even}. \end{cases}$$

The extremal graphs of Theorem 1.1 are as follows. For odd $k$ (where $n \geq 4k-1$), the extremal graphs are constructed by taking $T_2(n)$, the balanced complete bipartite
graph, and embedding two vertex disjoint copies of $K_k$ in one side. For even $k$ (where now $n \geq 4k-3$), the extremal graphs are constructed by taking $T_2(n)$ and embedding a graph with $2k-1$ vertices, $k^2 - \frac{3}{2}k$ edges with maximum degree $k-1$ in one side.

Let $C_{k,q}$ be the graph consisting of $k$ cycles of length $q$ which intersect exactly in one common vertex. Clearly, when we set $q = 3$, then $C_{k,3}$ is just the $k$-fan graph; see Theorem 1.1. When $q$ is an odd integer, we can see that $\chi(C_{k,q}) = 3$, the theorem of Erdős, Stone and Simonovits also implies that $\text{ex}(n, C_{k,q}) = n^2/4 + o(n^2)$. In 2016, Hou, Qiu and Liu [26] determined exactly the extremal number for $C_{k,q}$ with $k \geq 1$ and odd integer $q \geq 5$.

**Theorem 1.2.** [26] For an integer $k \geq 1$ and an odd integer $q \geq 5$, there exists $n_0(k,q)$ such that for all $n \geq n_0(k,q)$, we have

$$\text{ex}(n, C_{k,q}) = \left\lfloor \frac{n^2}{4} \right\rfloor + (k-1)^2.$$ 

Moreover, an extremal graph must be a Turán graph $T_2(n)$ with a $K_{k-1,k-1}$ embedding into one class.

We remark here that when $q$ is even, then $C_{k,q}$ is a bipartite graph where every vertex in one of its parts has degree at most 2. For such a sparse bipartite graph, a classical result of Füredi [19] or Alon, Krivelevich and Sudakov [2] implies that $\text{ex}(n, C_{k,q}) = O(n^{3/2})$. Recently, a breakthrough result of Conlon, Lee and Janzer [11, 12] shows that for even $q \geq 6$ and $k \geq 1$, we have $\text{ex}(n, C_{k,q}) = O(n^{3/2-\delta})$ for some $\delta = \delta(k,q) > 0$. It is a challenging problem to determine the value $\delta(k,q)$. In particular, for the special case $k = 1$, this problem reduces to determine the extremal number for even cycle.

Next, we shall introduce a unified extension of both Theorem 1.1 and Theorem 1.2. Let $s$ be a positive integer and $t_1, \ldots, t_k \geq 5$ be odd integers. We write $H_{s,t_1,\ldots,t_k}$ for the graph consisting of $s$ triangles and $k$ odd cycles of lengths $t_1, \ldots, t_k$ in which these triangles and cycles intersect in exactly one common vertex. The graph $H_{s,t_1,\ldots,t_k}$ is also known as the flower graph with $s+k$ petals. We remark here that the $k$ odd cycles can have different lengths. Clearly, when $t_1 = \cdots = t_k = 0$, then $H_{s,0,\ldots,0} = F_s$, the $s$-fan graph; see Theorem 1.1. In addition, when $s = 0$ and $t_1 = \cdots = t_k = q$, then $H_{0,q,\ldots,q} = C_{k,q}$; see Theorem 1.2.

In 2018, Hou, Qiu and Liu [27] and Yuan [43] independently determined the extremal number of $H_{s,t_1,\ldots,t_k}$ for $s \geq 0$ and $k \geq 1$. Let $F_{n,s,k}$ be the family of graphs with each member being a Turán graph $T_2(n)$ with a graph $Q$ embedded in one partite set, where

$$Q = \begin{cases} K_{s+k-1,s+k-1}, & \text{if } (s,k) \neq (3,1), \\ K_{3,3} \text{ or } 3K_3, & \text{if } (s,k) = (3,1), \end{cases}$$

where $3K_3$ is the union of three disjoint triangles.
**Theorem 1.3.** [27, 43] For every graph $H_{s,t_1,\ldots,t_k}$ with $s \geq 0$ and $k \geq 1$, there exists $n_0$ such that for all $n \geq n_0$, we have

$$ex(n, H_{s,t_1,\ldots,t_k}) = \left\lfloor \frac{n^2}{4} \right\rfloor + (s + k - 1)^2.$$  

Moreover, the only extremal graphs for $H_{s,t_1,\ldots,t_k}$ are members of $F_{n,s,k}$.

### 1.2 Spectral extremal problem

Let $G$ be a simple graph on $n$ vertices. The adjacency matrix of $G$ is defined as $A(G) = [a_{ij}]_{n \times n}$ where $a_{ij} = 1$ if two vertices $v_i$ and $v_j$ are adjacent in $G$, and $a_{ij} = 0$ otherwise. We say that $G$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ if these values are eigenvalues of the adjacency matrix $A(G)$. Let $\lambda(G)$ be the maximum value in absolute among the eigenvalues of $G$, which is known as the spectral radius of graph $G$, that is,

$$\lambda(G) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } G\}.$$  

By the Perron–Frobenius Theorem [25, p. 534], the spectral radius of a graph $G$ is actually the largest eigenvalue of $G$ since the adjacency matrix $A(G)$ is nonnegative. We usually write $\lambda_1(G)$ for the spectral radius of $G$. The spectral radius of a graph sometimes can give some information about the structure of graphs. For example, it is well-known [4, p. 34] that the average degree of $G$ is at most $\lambda(G)$, which is at most the maximum degree of $G$.

In this paper we consider spectral analogues of Turán-type problems for graphs. That is, determining $ex_{sp}(n,F) = \max\{\lambda(G) : |G| = n, F \not\subseteq G\}$. It is well-known that

$$ex(n,F) \leq \frac{n}{2} ex_{sp}(n,F)$$  

(2)

because of the fundamental inequality $\frac{2m}{n} \leq \lambda(G)$. For most graphs, this study is again fairly complete due in large part to a longstanding work of Nikiforov [37]. For example, he extended the classical theorem of Turán, by determining the maximum spectral radius of any $K_{r+1}$-free graph $G$ on $n$ vertices.

The following problem regarding the adjacency spectral radius was proposed in [31]: What is the maximum spectral radius of a graph $G$ on $n$ vertices without a subgraph isomorphic to a given graph $F'$? Wilf [42] and Nikiforov [31] obtained spectral strengthening of Turán’s theorem when the forbidden substructure is the complete graph. Soon after, Nikiforov [32] showed that if $G$ is a $K_{r+1}$-free graph on $n$ vertices, then $\lambda(G) \leq \lambda(T_r(n))$, equality holds if and only if $G = T_r(n)$. Moreover, Nikiforov [32] (when $n$ is odd), and Zhai and Wang [44] (when $n$ is even) determined the maximum spectral radius of $K_{2,2}$-free graphs. Furthermore, Nikiforov [34], Babai and Guiduli [3] independently obtained the spectral generalization of the theorem of Kővari, Sós and Turán when the forbidden graph is the complete bipartite graph $K_{s,t}$. Finally, Nikiforov [35] characterized the spectral radius of graphs without paths.
and cycles of specified length. In addition, Fiedler and Nikiforov \[17\] obtained tight sufficient conditions for graphs to be Hamiltonian or traceable. For many other spectral analogues of results in extremal graph theory we refer the reader to the survey \[37\]. It is worth mentioning that a corresponding spectral extension \[36\] of the theorem of Erdős, Stone and Simonovits states that

\[ \text{ex}_{sp}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1) \right)n. \]

From this result, we know that \( \text{ex}_{sp}(n, F_k) = n/2 + o(n) \) where \( F_k \) is the \( k \)-fan graph. Recently, Cioabă, Feng, Tait and Zhang \[8\] generalized this bound by improving the error term \( o(n) \) to \( O(1) \), and obtained a spectral counterpart of Theorem 1.1. More precisely, they proved the following theorem.

**Theorem 1.4.** \[8\] Let \( G \) be a graph of order \( n \) that does not contain a copy of \( F_k \) where \( k \geq 2 \). For sufficiently large \( n \), if \( G \) has the maximal spectral radius, then

\[ G \in \text{Ex}(n, F_k). \]

Recall that \( H_{s,t_1,\ldots,t_k} \) is the graph consisting of \( s \) triangles and \( k \) odd cycles of lengths \( t_1, \ldots, t_k \) which intersect in exactly one common vertex. In this paper, we shall prove the following theorem.

**Theorem 1.5 (Main result).** Let \( G \) be a graph of order \( n \) that does not contain a copy of \( H_{s,t_1,\ldots,t_k} \), where \( s \geq 0 \) and \( k \geq 1 \). For sufficiently large \( n \), if \( G \) has the maximal spectral radius, then

\[ G \in \text{Ex}(n, H_{s,t_1,\ldots,t_k}). \]

It is interesting that the spectral extremal example sometimes differs from the usual extremal example. For instance, Nikiforov \[32\], and Zhai and Wang \[44\] proved that the maximum spectral radius of a \( C_4 \)-free graph on \( n \) vertices is uniquely achieved by the friendship graph. This is very different from the usual extremal problem for the maximum number of edges in a \( C_4 \)-free graphs, since Füredi \[22\] showed that for \( n \) large enough with the form \( n = q^2 + q + 1 \), the extremal number is attained by the polarity graph of a projective plane. From Theorem 1.4 and Theorem 1.5 we know that graphs attaining the maximum spectral radius among all \( F_k \)-free \( (H_{s,t_1,\ldots,t_k} \)-free) graphs also contain the maximum number of edges among all \( F_k \)-free \( (H_{s,t_1,\ldots,t_k} \)-free) graphs.

Our theorem is a spectral result of the Turán extremal problem for \( H_{s,t_1,\ldots,t_k} \), it can be viewed as an extension of Theorem 1.4 as well as a spectral analogue of Theorem 1.3. Our treatment strategy of the proof is mainly based on the stability method. To some extent, this paper could be regarded as a continuation and development of \[8\]. The heart of the proof and all key ideas lie in the proof of stability. We know that if we forbid the substructure \( F_k \), then the neighborhood of each vertex does not contain a matching of \( k \) edges. While we forbid the intersecting odd-length cycles,
the neighborhood of each vertex does not contain a long path, which can be viewed as
a key observation in our extension. In addition, the embedding method of $H_{s,t_1,...,t_k}$ is
slightly different from that of $F_k$, we need to prove the existence of a larger bipartite
subgraph. We remark here that the spectral stability method is also used in a recent
paper to deal with the extremal problem of odd-wheel graph [9].

2 Some Lemmas

In this section, we state some lemmas which are needed in our proof.

**Lemma 2.1.** [13] Let $P_t$ denote the path on $t$ vertices. If $G$ is a $P_t$-free graph on $n$
vertices, then $e(G) \leq \frac{(t-2)n}{2}$, equality holds if and only if $G$ is the disjoint union of
copies of $K_{t-1}$.

**Lemma 2.2.** [8] If $G$ has $t$ triangles, then $e(G) \geq \lambda(G)^2 - \frac{3t}{\lambda(G)}$.

The next is the famous triangle removal lemma [38], which is a direct consequence
of the Szemerédi regularity lemma; see, e.g., [10, 18] for more details.

**Lemma 2.3.** [38] For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that every $n$-vertex
graph with at most $\delta(\varepsilon) \cdot n^3$ triangles can be made triangle-free by removing at most
$\varepsilon n^2$ edges.

**Lemma 2.4** (Füredi [23]). Let $G$ be a triangle-free graph on $n$ vertices. If $a > 0$ and $e(G) = e(T_2(n)) - a$, then there exists a bipartite subgraph $H \subseteq G$ such that
$e(H) \geq e(G) - a$.

Let $G$ be a simple graph with matching number $\beta(G)$ and maximum degree $\Delta(G)$.
For given two integers $\beta$ and $\Delta$, define $f(\beta, \Delta) = \max\{e(G) : \beta(G) \leq \beta, \Delta(G) \leq \Delta\}$.

In 1976, Chvátal and Hanson [7] obtained the following result.

**Lemma 2.5** (Chvátal–Hanson [7]). For every two integers $\beta \geq 1$ and $\Delta \geq 1$, we have

$$f(\beta, \Delta) = \Delta \beta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \beta \left\lfloor \frac{\Delta}{2} \right\rfloor \right\rfloor \leq \Delta \beta + \beta.$$

We will frequently use a special case proved by Abbott, Hanson and Sauer [1]:

$$f(k - 1, k - 1) = \begin{cases} k^2 - k, & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even}. \end{cases}$$

Furthermore, the extremal graphs attaining the equality case are exactly those we
embedded into the Turán graph $T_2(n)$ to obtain the extremal $F_k$-free graph.
3 Proof of Theorem 1.5

In the sequel, we always assume that $G$ is a graph on $n$ vertices containing no $H_{s,t_1,\ldots,t_k}$ as a subgraph and attaining the maximum spectral radius. The aim of this section is to prove that $e(G) = \text{ex}(n, H_{s,t_1,\ldots,t_k})$ for $n$ large enough.

First of all, we note that $G$ must be connected since adding an edge between different components will increase the spectral radius and also keep the resulting graph being $H_{s,t_1,\ldots,t_k}$-free. Let $\lambda(G)$ be the spectral radius of $G$. By the Perron–Frobenius Theorem [25, p. 534], we know that $\lambda_1$ has an eigenvector with all entries positive. We denote such an eigenvector by $x$. For a vertex $v \in V(G)$, we will write $x_v$ for the eigenvector entry of $x$ corresponding to $v$. We may normalize $x$ so that it has maximum entry equal to 1, and let $z$ be a vertex such that $x_z = 1$. If there are multiple such vertices, we choose and fix $z$ arbitrarily among them.

In the sequel, we shall prove Theorem 1.5 iteratively, giving successively better lower bounds on both $e(G)$ and the eigenvector entries of all of the other vertices, until finally we can show that $e(G) = \text{ex}(n, H_{s,t_1,\ldots,t_k})$.

The proof of Theorem 1.5 is outlined as follows.

♠ We apply Lemma 2.2 to give a lower bound $e(G) \geq \frac{n^2}{4} - O(n)$; see Lemma 3.1. Then we use the triangle removal lemma and Füredi’s stability result, and show that $G$ has a very large bipartite subgraph on parts $S,T$ with $\frac{n}{2} - o(n) \leq |S|, |T| \leq \frac{n}{2} + o(n)$. Moreover, we also have $e(S,T) \geq \frac{n^2}{4} - o(n^2)$; see Lemma 3.2.

♥ We show that the number of vertices that have $\Omega(n)$ neighbors on its side of the partition is bounded by $o(n)$, and the number of vertices that have degree less than $(\frac{1}{2} - c_0)n$ is bounded by $O(1)$ for some small constant $c_0 > 0$; see Lemmas 3.3 and 3.4 respectively. Furthermore, we will prove that such vertices do not exist, and both $G[S]$ and $G[T]$ are $K_{1,s+k}$-free and $M_{s+k}$-free; see Lemmas 3.6, 3.7 and 3.8.

♣ Based on the previous lemmas, we shall refine the structure of $G$, and improve the lower bound of $e(G)$ to $e(G) \geq \frac{n^2}{4} - O(1)$ and refine the bisection $\frac{n}{2} - O(1) \leq |S|, |T| \leq \frac{n}{2} + O(1)$ and also $e(S,T) \geq \frac{n^2}{4} - O(1)$; see Lemma 3.9. Moreover, we shall prove that $x_u = 1 - o(1)$ for every $u \in V(G)$; see Lemma 3.10.

◊ Once we know that all vertices have eigenvector entries close to 1, we can show that the bipartition is balanced; see Lemmas 3.11, 3.12 and 3.13. This implies that $G$ can be converted to a graph in $\text{Ex}(n, H_{s,t_1,\ldots,t_k})$ by deleting small number of edges within $S,T$ and adding small number of edges between $S$ and $T$. Invoking these facts, we finally show that $e(G) = \text{ex}(n, H_{s,t_1,\ldots,t_k})$.

Let $H$ be an $H_{s,t_1,\ldots,t_k}$-free graph on $n$ vertices with maximum number of edges. Since $G$ is the graph maximizing the spectral radius over all $H_{s,t_1,\ldots,t_k}$-free graphs, in
view of Theorem 1.3, we can see by the Rayleigh quotient [25, p. 234] or [45, p. 267] that
\[
\lambda(G) \geq \lambda(H) \geq \frac{1^T A(H) 1}{1^T 1} = \frac{2(|n^2/4| + (s + k - 1)^2)}{n} > \frac{n}{2}.
\]

**Lemma 3.1.** Let \(c\) be the largest length of cycles of \(H_{s,t_1,...,t_k}\). Then
\[
e(G) \geq \frac{n^2}{4} - (s + k)c n.
\]

**Proof.** Since \(G\) is \(H_{s,t_1,...,t_k}\)-free, the neighborhood of any vertex does not contain \(P_{(s+k)c}\) (a path on \((s + k)c\) vertices) as a subgraph. Otherwise, \(G\) contains the join graph \(K_1 \vee P_{(s+k)c}\), which contains a copy of \(H_{s,t_1,...,t_k}\). Let \(t\) be the number of triangles in \(G\). By Lemma 2.1, we know that
\[
t = \sum_{v \in V(G)} e(G[N(v)]) \leq \sum_{v \in V(G)} ex(n, P_{(s+k)c}) < \sum_{v \in V(G)} (s + k)c n \leq 2(s + k)c n.\]
This gives \(t \leq \frac{(s + k)c}{6} n^2\). From Lemma 2.2 and (3), we obtain
\[
e(G) \geq \lambda^2(G) - \frac{6t}{n} \geq \frac{n^2}{4} - (s + k)c n.
\]
This completes the proof. \(\square\)

**Lemma 3.2.** Let \(\varepsilon\) be a fixed positive constant. There exists an \(n_0(\varepsilon, s, k, c)\) such that \(G\) has a partition \(V \neq S \cup T\) which gives a maximum bipartite subgraph, and
\[
e(S, T) \geq \left(\frac{1}{4} - \varepsilon\right) n^2
\]
for \(n \geq n_0(\varepsilon, s, k, c)\). Furthermore
\[
\left(\frac{1}{2} - \sqrt{\varepsilon}\right) n \leq |S|, |T| \leq \left(\frac{1}{2} + \sqrt{\varepsilon}\right) n.
\]

**Proof.** Let \(\delta(\varepsilon)\) be the parameter chosen from the Triangle Removal Lemma 2.3. In the proof of Lemma 3.1, we know that \(t \leq \frac{(s + k)c}{6} n^3 \leq \delta(\varepsilon) n^3\) for \(n \geq n_0 = \frac{(s + k)c}{6\delta(\varepsilon)/4}\). By Lemma 2.3, there exists a large integer \(n_0\) such that the graph \(G_1\) obtained from \(G\) by deleting at most \(\frac{s}{4} n^2\) edges is \(K_3\)-free for \(n \geq n_0\). Hence the size of the graph \(G_1\) of order \(n\) satisfies
\[
e(G_1) \geq e(G) - \frac{\varepsilon}{4} n^2 \geq \frac{n^2}{4} - (s + k)c n - \frac{\varepsilon}{4} n^2.
\]
Note that \(e(G_1) \leq e(T_2(n))\) by the Mantel Theorem. We define \(a := e(T_2(n)) - e(G_1)\), then \(0 \leq a \leq (s + k)c n + \frac{s}{4} n^2\). By Lemma 2.4, \(G_1\) contains a bipartite subgraph \(G_2\) such that \(e(G_2) \geq e(G_1) - a\). Hence, for \(n\) sufficiently large, we have
\[
e(G_2) \geq e(G_1) - a \geq \frac{n^2}{4} - 2(s + k)c n - \frac{s}{2} n^2 \geq \left(\frac{1}{4} - \varepsilon\right) n^2.
\]
Therefore, $G$ has a partition $V = S \cup T$ which gives a maximum cut such that

$$e(S, T) \geq e(G) \geq \left(\frac{1}{4} - \varepsilon\right) n^2. \quad (7)$$

Furthermore, without loss of generality, we may assume that $|S| \leq |T|$. If $|S| < \left(\frac{1}{2} - \sqrt{\varepsilon}\right)n$, then $|T| = n - |S| > \left(\frac{1}{2} + \sqrt{\varepsilon}\right)n$. So

$$e(S, T) \leq |S||T| < \left(\frac{1}{2} - \sqrt{\varepsilon}\right) n \left(\frac{1}{2} + \sqrt{\varepsilon}\right) n = \left(\frac{1}{4} - \varepsilon\right) n^2,$$

which contradicts to Eq. (7). Therefore it follows that

$$\left(\frac{1}{2} - \sqrt{\varepsilon}\right) n \leq |S|, \ |T| \leq \left(\frac{1}{2} + \sqrt{\varepsilon}\right) n.$$

Hence the assertion (6) holds. \hfill \Box

For a vertex $v$, let $d_S(v) = |N(v) \cap S|$ and $d_T(v) = |N(v) \cap T|$. Next, we consider the set of vertices that have many neighbors which are not in the cut.

**Lemma 3.3.** Let $\varepsilon, \delta$ be two sufficiently small constants with $\varepsilon < \delta^2 / 3$. We denote

$$W := \{v \in S : d_S(v) \geq \delta n\} \cup \{v \in T : d_T(v) \geq \delta n\}. \quad (8)$$

For sufficiently large $n$, we have

$$|W| \leq \frac{2\delta}{3} n + \frac{2(s + k - 1)^2}{\delta n} < \delta n.$$

**Proof.** Firstly, by Theorem 1.3, we know that $e(G) \leq \text{ex}(n, H_{s,t_1,\ldots,t_k}) \leq \frac{n^2}{4} + (s + k - 1)^2$. Note that $e(S, T) \geq \left(\frac{1}{4} - \varepsilon\right) n^2$ by Lemma 3.2. Hence

$$e(S) + e(T) = e(G) - e(S, T) \leq \frac{n^2}{4} + (s + k - 1)^2 - \left(\frac{1}{4} - \varepsilon\right) n^2 \leq \varepsilon n^2 + (s + k - 1)^2. \quad (9)$$

On the other hand, if we denote $W_1 = W \cap S$ and $W_2 = W \cap T$, then we get

$$2e(S) = \sum_{u \in S} d_S(u) \geq \sum_{u \in W_1} d_S(u) \geq |W_1| \delta n,$$

and similarly, we also have

$$2e(T) = \sum_{u \in T} d_T(u) \geq \sum_{u \in W_2} d_T(u) \geq |W_2| \delta n.$$
So
\[ e(S) + e(T) \geq (|W_1| + |W_2|) \frac{\delta n}{2} = \frac{\delta n}{2} |W|. \]  
(10)
Combining (9) and (10), we get
\[ \frac{\delta n}{2} |W| \leq \varepsilon n^2 + (s + k - 1)^2, \]
i.e.,
\[ |W| \leq \frac{2\varepsilon n^2 + 2(s + k - 1)^2}{\delta n}. \]
Note that \( \varepsilon < \frac{\delta^2}{3} \), we can get \( |W| < \delta n \) for sufficiently large \( n \).

\[ \square \]

**Lemma 3.4.** Let \( k \geq 2 \). We denote \( c_0 := \frac{1}{8c(s+k)} \) and
\[ L := \{ v \in V(G) : d(v) \leq (\frac{1}{2} - c_0) n \}. \]  
(11)
Then
\[ |L| \leq 16c^2(s + k)^2. \]

**Proof.** Suppose that \( |L| > 16c^2(s + k)^2 \). Then let \( L' \subseteq L \) with \( |L'| = 16c^2(s + k)^2 \). Then it follows that
\[ e(G - L') \geq e(G) - \sum_{v \in L'} d(v) \geq \frac{n^2}{4} - (s + k)cn - 16c^2(s + k)^2 \left( \frac{1}{2} - \frac{1}{8c(s+k)} \right) n = \frac{n^2}{4} - 8c^2(s + k)^2n + (s + k)cn \]
\[ > \frac{(n - 16c^2(s + k)^2)^2}{4} + (s + k - 1)^2 \]
for sufficiently large \( n \), where the second inequality is by (5). Hence by Theorem 1.3, \( G - L' \) contains \( H_{s,t_1,\ldots,t_k} \), which implies that \( G \) contains \( H_{s,t_1,\ldots,t_k} \). So the assertion holds.

\[ \square \]

Now, we have proved that \( |W| = o(n) \) and \( |L| = O(1) \) by Lemmas 3.3 and 3.4 respectively. Next we will improve the bound on \( W \) and actually show that \( W \) is a subset of \( L \), so \( |W| = O(1) \). To proceed, we first need the following lemma which can be proved by induction or double counting.

**Lemma 3.5.** Let \( A_1, A_2, \ldots, A_p \) be \( p \) finite sets. Then
\[ \left| \bigcap_{i=1}^{p} A_i \right| \geq \sum_{i=1}^{p} |A_i| - (p - 1) \left| \bigcup_{i=1}^{p} A_i \right|. \]

**Lemma 3.6.** Let \( W \) and \( L \) be sets of vertices defined in (8) and (11). Then \( W \subseteq L \).
Proof. Suppose on the contrary that there exists a vertex \( u_0 \in W \) and \( u_0 \notin L \). Recall that \( W_1 = W \cap S \) and \( W_2 = W \cap T \). Similarly, let \( L_1 = L \cap S \) and \( L_2 = L \cap T \). Without loss of generality, we may assume that \( u_0 \in S \), that is, \( u_0 \in W_1 \) and \( u_0 \notin L_1 \). Since \( S \) and \( T \) form a maximum bipartite subgraph, we have \( d_T(u_0) \geq \frac{1}{2}d(u_0) \). Indeed, otherwise, we can move the vertex \( u_0 \) into the part \( T \), it will increase strictly the number of edges between \( S \) and \( T \). On the other hand, invoking the fact \( u_0 \notin L \), we get \( d(u_0) \geq \left( \frac{1}{2} - \frac{1}{8c(s+k)} \right)n \). So
\[
d_T(u_0) \geq \frac{1}{2}d(u_0) \geq \left( \frac{1}{4} - \frac{1}{16c(s+k)} \right)n.
\]

Recall in Lemmas 3.3 and 3.4 that
\[
|W| < \delta n, \quad |L| \leq 16c^2(s + k)^2.
\]
Hence, for fixed \( \delta < \frac{1}{10(k+1)^2} \) and sufficiently large \( n \), we have
\[
|S \setminus (W \cup L)| \geq \left( \frac{1}{2} - \sqrt{\varepsilon} \right)n - \delta n - 16c^2(s + k)^2 \geq (s + k)c.
\]

Claim. \( u_0 \) is adjacent to at most \( s + k - 1 \) vertices in \( S \setminus (W \cup L) \).

Suppose that \( u_0 \) is adjacent to \( s + k \) vertices \( u_1, u_2, \ldots, u_{s+k} \) in \( S \setminus (W \cup L) \). Since \( u_i \notin L \), we have \( d(u_i) \geq \left( \frac{1}{2} - \frac{1}{8c(s+k)} \right)n \). On the other hand, we have \( d_S(u_i) \leq \delta n \) because \( u_i \notin W \). So \( d_T(u_i) = d(u_i) - d_S(u_i) \geq \left( \frac{1}{2} - \frac{1}{8c(s+k)} - \delta \right)n \). In addition, we can choose other vertices \( u_{s+k+1}, \ldots, u_{s+k+c} \) in the set \( S \setminus (W \cup L) \). Similarly, we also have \( d_S(u_i) \geq \left( \frac{1}{2} - \frac{1}{8c(s+k)} - \delta \right)n \) for each \( i \in [s + k + 1, (s + k)c] \). By Lemma 3.5, we consider the cardinality of common neighbors
\[
\left| N_T(u_0) \cap N_T(u_1) \cap \cdots \cap N_T(u_{s+k}) \right| \\
\geq \sum_{i=0}^{(s+k)c} \left| N_T(u_i) \right| - (s + k)c \left| \bigcup_{i=0}^{(s+k)c} N_T(u_i) \right| \\
\geq d_T(u_0) + d_T(u_1) + \cdots + d_T(u_{s+k}) - (s + k)c|T| \\
\geq \left( \frac{1}{4} - \frac{1}{16c(s+k)} \right)n + \left( \frac{1}{2} - \frac{1}{8c(s+k)} - \delta \right)n \cdot (s + k)c - (s + k)c \left( \frac{1}{2} + \sqrt{\varepsilon} \right)n \\
= \left( \frac{1}{8} - \frac{1}{16c(s+k)} - (s + k)c\delta - (s + k)c\sqrt{\varepsilon} \right)n > (s + k)c
\]
for sufficiently large \( n \), where the last inequality follows from the fact that \( \delta \) and \( \varepsilon \) are small enough, e.g., \( \delta < \frac{4}{100(k+1)^2} \) and \( \varepsilon < \frac{\delta^2}{3} \). So there exist \( (s + k)c \) vertices \( v_1, v_2, \ldots, v_{(s+k)c} \) in \( T \) such that the induced subgraph by two partitions \( \{u_1, \ldots, u_{(s+k)c}\} \) and \( \{v_1, \ldots, v_{(s+k)c}\} \) is complete bipartite. The subgraph of \( G \) formed by the vertex \( u_0 \) together with such a complete bipartite graph can contain many
disjoint odd-length cycles. For example, we can choose $u_0u_1v_1u_0$ to find a copy of triangle, and we can choose $u_0u_1v_1u_{s+k+1}v_2u_0$ to form a copy of pentagon and so on. Hence, it follows that $G$ contains $H_{s,t_1,...,t_k}$, this is a contradiction. Therefore $u_0$ is adjacent to at most $s + k - 1$ vertices in $S \setminus (W \cup L)$.

Hence, applying Lemmas 3.3 and 3.4 again, we have

\[
d_S(u_0) \leq |W| + |L| + s + k - 1 < \frac{2\delta}{3} n + \frac{2(s + k - 1)^2}{\delta n} + 16c^2(s + k)^2 + s + k - 1 < \delta n\]

for sufficiently large $n$. This is a contradiction to the fact that $u_0 \in W$. Similarly, there is no vertex $u$ such that $u \in W_2$ and $u \notin L_2$. Hence $W \subseteq L$. \hfill \Box

**Lemma 3.7.** There exist independent sets $I_S \subseteq S$ and $I_T \subseteq T$ such that

\[
|I_S| \geq |S| - 20c^2(s + k)^2 \quad \text{and} \quad |I_T| \geq |T| - 20c^2(s + k)^2.
\]

**Proof.** Since $S \setminus L$ is large enough by reviewing (12) in the proof of Lemma 3.6, we next prove that there exists a large complete bipartite subgraph between $S$ and $T$. Let $u_1, \ldots, u_{(s+k)c}$ be $(s+k)c$ vertices chosen arbitrarily from $S \setminus L$. Since $u_i \notin L$, we have

\[
d(u_i) \geq \left(\frac{1}{2} - \frac{1}{8c(s + k)}\right)n.
\]

Note that $W \subseteq L$ by Lemma 3.6 so $u_i \notin W$, then $d_S(u_i) \leq \delta n$. Hence

\[
d_T(u_i) = d(u_i) - d_S(u_i) \geq \left(\frac{1}{2} - \frac{1}{8c(s + k)} - \delta\right)n.
\]

Furthermore, by Lemma 3.5 we have

\[
\left|\bigcap_{i=1}^{(s+k)c} N_T(u_i)\right| \geq \left|\bigcup_{i=1}^{(s+k)c} N_T(u_i)\right| - (s + k)c - 1
\]

\[
\geq \left(\frac{1}{2} - \frac{1}{8c(s + k)} - \delta\right) n \cdot (s + k)c - ((s + k)c - 1) \left(\frac{1}{2} + \sqrt{\varepsilon}\right)n
\]

\[
= \left(\frac{3}{8} - (s + k)c\delta - ((s + k)c - 1)\sqrt{\varepsilon}\right)n > (s + k)c
\]

for sufficiently large $n$. Hence for any $(s + k)c$ vertices $u_1, u_2, \ldots, u_{(s+k)c}$ in $S \setminus L$, there exist $(s + k)c$ vertices $v_1, v_2, \ldots, v_{(s+k)c} \in T$ such that the subgraph formed by two partitions $\{u_1, \ldots, u_{(s+k)c}\}$ and $\{v_1, \ldots, v_{(s+k)c}\}$ is a complete bipartite graph.

**Claim.** $G[S \setminus L]$ is both $K_{1,s+k}$-free and $M_{s+k}$-free.
If \( G[S \setminus L] \) contains a copy of \( K_{1,s+k} \) centered at vertex \( u_0 \) with leaves \( u_1, u_2, \ldots, u_{s+k} \), then by the discussion above, there exist \( u_{s+k+1}, \ldots, u_{(s+k)c} \in S \setminus (L \cup \{u_1, \ldots, u_{s+k}\}) \) and \( v_1, \ldots, v_{(s+k)c} \in T \) such that \( \{u_1, \ldots, u_{(s+k)c}\} \) and \( \{v_1, \ldots, v_{(s+k)c}\} \) form a complete bipartite graph. Therefore, we can see that \( u_0u_1v_1, u_0u_2v_2, \ldots, u_0u_sv_s \) form \( s \) triangles centered at \( u_0 \). Moreover, there exist \( u_x, \ldots, u_z \in S \setminus (L \cup \{u_1, \ldots, u_{s+k}\}) \) and \( v_y, \ldots, v_w \in T \setminus \{v_1, \ldots, v_{s+k}\} \) such that \( u_0u_{s+1}v_{s+1}u_{x}v_{y} \cdots u_{z}v_{w}u_0 \) forms an odd cycle and in fact we can find all other odd cycles similarly. Since there are \((s+k)c\) such \( u_i \) and \((s+k)c\) such \( v_i \), it is enough to form \( s \) triangles and \( k \) odd cycles of lengths no more than \( c \). Hence there is a copy of \( H_{s,k} \) centered at \( u_0 \). Therefore, \( G[S \setminus L] \) is \( K_{1,s+k}-\text{free} \). Now, we assume that \( \{u_1u_2, u_3u_4, \ldots, u_{2(s+k)−1}u_{2(s+k)}\} \) is a matching of size \( s+k \) in \( S \setminus L \). Then \( v_1u_1u_2, v_1u_3u_4, \ldots, v_1u_{2s−1}u_{2s} \) form \( s \) triangles centered at vertex \( v_1 \), and there exist distinct vertices \( v_i, \ldots, v_x \in T \setminus \{v_1\} \) and \( u_j, \ldots, u_y \in S \setminus (L \cup \{u_1, \ldots, u_{2s}\}) \) such that \( v_1u_{2s+1}u_{2s+2}vi_j \cdots u_yv_1 \) forms an odd cycle and so on. So \( G[S \setminus L] \) is \( M_{s+k} \)-free.

Hence both the maximum degree and the maximum matching number of \( G[S \setminus L] \) are at most \( s+k−1 \), respectively. By Lemma 2.5,

\[
e(G[S \setminus L]) \leq f(s+k-1, s+k-1).
\]

The same argument gives

\[
e(G[T \setminus L]) \leq f(s+k-1, s+k-1).
\]

Since \( G[S \setminus L] \) has at most \( f(s+k-1, s+k-1) \) edges, then the subgraph obtained from \( G[S \setminus L] \) by deleting one vertex of each edge in \( G[S \setminus L] \) contains no edges, which is an independent set of \( G[S \setminus L] \). By Lemma 3.4, there exists an independent set \( I_S \subseteq S \) such that

\[
|I_S| \geq |S \setminus L| - f(s+k-1, s+k-1)
\geq |S| - 16c^2(s+k)^2 - (s+k)^2 \geq |S| - 20c^2(s+k)^2.
\]

The same argument gives that there is an independent set \( I_T \subseteq T \) with

\[
|I_T| \geq |T| - 20c^2(s+k)^2.
\]

This completes the proof. \( \square \)
In Lemma 3.7 we have showed that there are two large independent sets with 
\((\frac{1}{2} - o(1))n\) vertices, one in \(S\) and the other in \(T\). Invoking this fact, we next shall prove that \(L\) is actually an empty set.

**Lemma 3.8.** \(L\) is empty, and both \(G[S]\) and \(G[T]\) are \(K_{1,s+k}\)-free and \(M_{s+k}\)-free.

**Proof.** Recall that \(A\mathbf{x} = \lambda_1 \mathbf{x}\) and \(z\) is defined as a vertex with maximum eigenvector entry and satisfies \(x_z = 1\). So we have

\[
d(z) \geq \sum_{w \sim z} x_w = \lambda_1 x_z = \lambda_1 \geq \frac{n}{2}.
\]

Hence \(z \notin L\). Without loss of generality, we may assume that \(z \in S\). Since the maximum degree in the induced subgraph \(G[S \setminus L]\) is at most \(s + k - 1\) (containing no \(K_{1,s+k}\)), from Lemma 3.4 we have \(|L| \leq 16c^2(s + k)^2\) and

\[
d_S(z) = d_{S \setminus L}(z) + d_{S \setminus L}(z) \leq 16c^2(s + k)^2 + s + k - 1 \leq 20c^2(s + k)^2.
\]

Therefore, by Lemma 3.7 we have

\[
\lambda_1 = \lambda_1 x_z + \sum_{v \sim z} x_v = \sum_{v \sim z, v \in S} x_v + \sum_{v \sim z, v \in T} x_v \leq d_S(z) + \sum_{v \in T} x_v + \sum_{v \in T \setminus I_T} 1 \leq 20c^2(s + k)^2 + \sum_{v \in I_T} x_v + |T| - |I_T| \leq \sum_{v \in I_T} x_v + 40c^2(s + k)^2.
\]

Combining (3), we can get

\[
\sum_{v \in I_T} x_v \geq \frac{n}{2} - 40c^2(s + k)^2. \tag{13}
\]

Next we are going to prove \(L = \emptyset\).

By way of contradiction, suppose that there is a vertex \(v \in L \cap S\), so \(d_G(v) \leq (\frac{1}{2} - \frac{1}{8c(s+k)})n\). Consider the graph \(G^+\) with vertex set \(V(G)\) and edge set \(E(G^+) = (E(G) \setminus \{vu : u \in N_G(v)\}) \cup \{vw : w \in I_T\}\). In this process, the number of added edges is larger than that of deleted edges. Note that \(I_T\) is an independent set and then adding edges connecting \(v\) and vertices in \(I_T\) does not create any triangles or odd
cycles. So the new graph $G^+$ is still $H_{s,t_1,\ldots,t_k}$-free. Note that $\mathbf{x}$ is a vector such that $\lambda(G) = \frac{x^T A(G) \mathbf{x}}{x^T \mathbf{x}}$, and the Rayleigh theorem implies $\lambda(G^+) \geq \frac{x^T A(G^+) \mathbf{x}}{x^T \mathbf{x}}$. Furthermore,

$$\lambda(G^+) - \lambda(G) \geq \frac{2 \mathbf{x}_v}{\mathbf{x}^T \mathbf{x}} \left( \sum_{w \in I_T} \mathbf{x}_w - \sum_{uv \in E(G)} \mathbf{x}_u \right)$$

where the last inequality holds for $n$ large enough and $\mathbf{x}_v > 0$, which follows from the Perron–Frobenius theorem and the fact that $G$ is connected. This contradicts $G$ has the largest spectral radius over all $H_{s,t_1,\ldots,t_k}$-free graphs, so $L$ must be empty. Furthermore, the claim in the proof of Lemma 3.7 implies that both $G[S]$ and $G[T]$ are $K_{1,s+k}$-free and $M_{s+k}$-free.

Let $G$ be an $H_{s,t_1,\ldots,t_k}$-free graph on $n$ vertices with maximum spectral radius. In the previous lemmas, we have proved that $G$ contains at most $O(n^2)$ triangles and has $\frac{n^2}{4} - O(n)$ edges. In addition, $G$ contains a bipartite subgraph with parts $S$ and $T$ such that $\frac{n}{2} - o(n) \leq |S|, |T| \leq \frac{n}{2} + o(n)$. Next we shall refine the structure of $G$. We shall show that the number of triangles in $G$ is at most $O(n)$ and the number of edges in $G$ is at least $\frac{n^2}{4} - O(1)$, and the two vertex parts $S, T$ satisfies $\frac{n}{2} - O(1) \leq |S|, |T| \leq \frac{n}{2} + O(1)$. More precisely, we state these results as in the following lemma.

**Lemma 3.9.** For $n$ and $k$ defined as before, we have

$$e(G) \geq \frac{n^2}{4} - 12(s + k)^2,$$

$$e(S, T) \geq \frac{n^2}{4} - 14(s + k)^2,$$

$$\frac{n}{2} - 4(s + k) \leq |S|, |T| \leq \frac{n}{2} + 4(s + k),$$

and

$$\frac{n}{2} - 14(s + k)^2 \leq \delta(G) \leq \lambda(G) \leq \Delta(G) \leq \frac{n}{2} + 5(s + k).$$

**Proof.** From Lemma 3.8 both $G[S]$ and $G[T]$ are $K_{1,s+k}$-free and $M_{s+k}$-free. By Lemma 2.5 so we have $e(S) + e(T) \leq 2f(s + k - 1, s + k - 1) < 2(s + k)^2$. This means
that the number of triangles in $G$ is bounded above by $2(s + k)^2 n$ since any triangle contains an edge of $E(S) \cup E(T)$. By Lemma 2.2 we have

$$e(G) \geq \lambda_1^2 - \frac{6t}{n} \geq \frac{n^2}{4} - 12(s + k)^2.$$ 

Since $e(S) + e(T) \leq 2(s + k)^2$, then we have

$$e(S, T) = e(G) - e(S) - e(T) \geq \frac{n^2}{4} - 14(s + k)^2.$$ 

Suppose that $|S| \leq \frac{n}{2} - 4(s + k)$, then $|T| = n - |S| \geq \frac{n}{2} + 4(s + k)$. Hence

$$e(S, T) \leq |S||T| \leq \left(\frac{n}{2} - 4(s + k)\right) \left(\frac{n}{2} + 4(s + k)\right) = \frac{n^2}{4} - 16(s + k)^2,$$

which contradicts to $e(S, T) \geq \frac{n^2}{4} - 14(s + k)^2$. So we have

$$\frac{n}{2} - 4(s + k) \leq |S|, |T| \leq \frac{n}{2} + 4(s + k).$$

Moreover, by Lemma 3.8 the maximum degree of $G[S]$ and $G[L]$ is at most $s + k - 1$, which yields

$$\Delta(G) \leq (\frac{n}{2} + 4(s + k)) + (s + k - 1) < \frac{n}{2} + 5(s + k).$$

So

$$\lambda_1 \leq \Delta(G) < \frac{n}{2} + 5(s + k).$$

Furthermore, we claim that the minimum degree of $G$ is at least $\frac{n}{2} - 14(s + k)^2$. Otherwise, removing a vertex $v$ of minimum degree $d(v)$, we have

$$e(G - v) = e(G) - d(v) \geq \frac{n^2}{4} - 12(s + k)^2 - \left(\frac{n}{2} - 14(s + k)^2\right) = \frac{n^2}{4} - \frac{n}{2} + 2(s + k)^2 > \frac{(n-1)^2}{4} + (s + k - 1)^2,$$

which implies the induced subgraph $G - v$ contains a copy of $H_{s,t_1,...,t_k}$ by Theorem L3.8. 

\[\square\]

**Lemma 3.10.** For all $u \in V(G)$, we have that $x_u \geq 1 - \frac{120(s+k)^2}{n}$.
Proof. Without loss of generality, we may assume that \( z \in S \). We consider the following two cases.

**Step 1.** We first consider the case \( u \in S \). Since \( G[S] \) is \( K_{1,s+k} \)-free, then \( d_S(u) \leq s + k - 1 \). By Lemma 3.9, we have

\[
|N_T(u)| = d_T(u) = d(u) - d_S(u) \geq \delta(G) - d_S(u) \\
\geq \frac{n}{2} - 14(s + k)^2 - (s + k - 1) \\
\geq \frac{n}{2} - 15(s + k)^2.
\]

Similarly, we also have \( |N_T(z)| \geq \frac{n}{2} - 15(s + k)^2 \). Then

\[
|N_T(u) \cap N_T(z)| = |N_T(u)| + |N_T(z)| - |N_T(u) \cup N_T(z)| \\
\geq 2 \left( \frac{n}{2} - 15(s + k)^2 \right) - \left( \frac{n}{2} + 4(s + k) \right) \\
\geq \frac{n}{2} - 34(s + k)^2.
\]

Note that \( d_T(z) \leq |T| \). By Lemma 3.9 again, we can get

\[
d_T(z) - |N_T(u) \cap N_T(z)| \leq \frac{n}{2} + 4(s + k) - \left( \frac{n}{2} - 34(s + k)^2 \right) \leq 38(s + k)^2.
\]

Hence, we have

\[
\lambda_1 x_u - \lambda_1 x_z = \sum_{v \sim u} x_v - \sum_{v \sim z} x_v \\
= \sum_{v \sim u, v \in T, v \neq z} x_v + \sum_{v \sim u, v \in S} x_v - \sum_{v \sim z, v \in T, v \neq u} x_v - \sum_{v \sim z, v \in S} x_v \\
\geq - \sum_{v \sim z, v \in T, v \neq u} x_v - \sum_{v \sim z, v \in S} x_v \\
\geq - \sum_{v \sim z, v \in T, v \neq u} 1 - \sum_{v \sim z, v \in S} 1 \\
\geq - \left( d_T(z) - |N_T(u) \cap N_T(z)| \right) - d_S(z) \\
\geq -38(s + k)^2 - (s + k)^2 \\
= -39(s + k)^2.
\]

Recall that \( x_z = 1 \). Therefore, for any \( u \in S \), we have

\[
x_u \geq 1 - \frac{39(s + k)^2}{\lambda_1} > 1 - \frac{39(s + k)^2}{n/2} = 1 - \frac{78(s + k)^2}{n}.
\]

(14)

**Step 2.** Now we consider the case \( u \in T \). By (14), we get

\[
\lambda_1 x_u = \sum_{v \sim u} x_v \geq \sum_{v \sim u, v \in S} x_v \geq \left( 1 - \frac{78(s + k)^2}{n} \right) d_S(u).
\]
By Lemma 3.9, we can see that $d(u) \geq \delta(G) \geq \frac{n}{2} - 14(s+k)^2$. Recall that $G[T]$ is $K_{1,s+k}$-free, so we have $d_T(u) \leq s + k - 1$. Then

$$d_S(u) = d(u) - d_T(u) \geq \frac{n}{2} - 15(s+k)^2.$$  

Hence

$$x_u \geq \frac{(1 - \frac{78(s+k)^2}{n})d_S(u)}{\lambda_1} \geq \frac{(1 - \frac{78(s+k)^2}{n})\left(\frac{n}{2} - 15(s+k)^2\right)}{\frac{n}{2} + 5(s+k)}$$

$$= \frac{n}{2} - 54(s+k)^2 + \frac{1170(s+k)^4}{n}$$

$$> 1 - \frac{120(s+k)^2}{n}.$$  

From the above two cases, the result follows. \[\square\]

Using this refined bound on the eigenvector entries, we will show that the partition $V = S \cup T$ is balanced (Lemma 3.13). First of all, we fix some notation for convenience. Let $B = K_{s,t}$ be the complete bipartite graph with partite sets $S$ and $T$, and let $G_1 = G[S] \cup G[T]$ and $G_2$ be the graph on $V(G)$ with the missing edges between $S$ and $T$, that is, $E(G_2) = E(B) \setminus E(G)$. Note that $e(G) = e(G_1) + e(B) - e(G_2)$.

From Lemma 3.8, we know that both $G[S]$ and $G[T]$ are $K_{1,s+k}$-free and $M_{s+k}$-free, then $e(G_1) = e(S) + e(T) \leq 2f(s+k-1,s+k-1) \leq (s+k)^2$. Next we shall give an improvement in the sense that $(G_2)$ is close to zero.

**Lemma 3.11.** Let $G_1, G_2$ and $B$ be graphs as defined above. Then

$$e(G_1) - e(G_2) \leq (s+k-1)^2.$$  

**Proof.** Without loss of generality, we may assume that $|T| \geq |S|$ and denote

$$S' := \{v \in S : N(v) \subseteq T\},$$

$$T' := \{v \in T : N(v) \subseteq S\}.$$  

Since $e(G[S]) \leq f(s+k-1,s+k-1) \leq (s+k)^2$ by Lemma 3.8, there exist at most $2(s+k)^2$ vertices in $S$ having a neighbor in $S$. Hence

$$|S'| \geq |S| - 2(s+k)^2.$$  

Similarly,

$$|T'| \geq |T| - 2(s+k)^2.$$  

Let $C \subseteq T'$ be a set having $|T| - |S|$ vertices, which is well-defined, as we can see from Lemma 3.9 that $|T| - |S| \leq 8(s+k)$ and $|T'| \geq |T| - 2(s+k)^2 \geq \frac{n}{2} - 4(s+k) - 2(s+k)^2 > 8(s+k)$. Then $G \setminus C$ is a graph on $2|S|$ vertices such that

$$e(G) - e(C, S) = e(G \setminus C) \leq ex(2|S|, H_{s,t_1,\ldots,t_k}) \leq \frac{(2|S|)^2}{4} + (s + k - 1)^2.$$  

18
Hence
\[ e(G) \leq |S|^2 + |C||S| + (s + k - 1)^2 = |S||T| + (s + k - 1)^2. \]
Note that \( e(G_1) - e(G_2) = e(G) - e(B) \). This completes the proof. \( \square \)

**Lemma 3.12.**
\[ \frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor - \sqrt{|S||T|} \leq \frac{7200(s + k)^4}{n(n - 240(s + k)^2)}. \]

**Proof.** By Lemma 3.10 we have,
\[ x^T x \geq n \left(1 - \frac{120(s + k)^2}{n}\right)^2 > n \left(1 - \frac{240(s + k)^2}{n}\right) = n - 240(s + k)^2, \]  
and that \( \lambda(B) = \sqrt{|S||T|} \). By Lemma 3.9 we know that \( e(G_1) \leq 2(s + k)^2 \), we obtain
\[ e(S, T) = e(G) - e(G_1) \geq \frac{n^2}{4} - 12(s + k)^2 - 2(s + k)^2 = \frac{n^2}{4} - 14(s + k)^2, \]
which implies that
\[ e(G_2) = e(B) - e(S, T) \leq |S||T| - \left(\frac{n^2}{4} - 14(s + k)^2\right) \leq 14(s + k)^2. \]

Applying Lemma 3.10 again, we can obtain
\[ x^T A(G_2) x = 2 \sum_{uv \in E(G_2)} x_u x_v \geq 2e(G_2) \left(1 - \frac{120(s + k)^2}{n}\right)^2 \]
\[ \geq 2e(G_2) \left(1 - \frac{240(s + k)^2}{n}\right). \]
Combining this result together with (3) and Lemma 3.11 we can get
\[ \frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{2(s + k - 1)^2}{n} \leq \lambda(G) = \frac{x^T (A(B) + (G_1) - A(G_2)) x}{x^T x} \]
\[ = \frac{x^T A(B) x}{x^T x} + \frac{x^T A(G_1) x}{x^T x} - \frac{x^T A(G_2) x}{x^T x} \]
\[ \leq \lambda(B) + \frac{2e(G_1) - 2e(G_2) \left(1 - \frac{240(s + k)^2}{n}\right)}{x^T x} \]
\[ \leq \lambda(B) + \frac{2(e(G_1) - e(G_2)) + 2e(G_2) \frac{240(s + k)^2}{n}}{x^T x} \]
\[ \leq \lambda(B) + \frac{2(s + k - 1)^2}{x^T x} + \frac{2 \cdot 14(s + k)^2 \frac{240(s + k)^2}{n}}{x^T x}. \]
Then we have
\[
\frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor - \sqrt{|S||T|} \leq 2(s + k - 1)^2 \left( \frac{1}{x^T x} - \frac{1}{n} \right) + \frac{28(s + k)^2 240(s+k)^2}{x^T x n} \\
\leq 2(s + k)^2 \left( \frac{1}{n - 240(s+k)^2} - \frac{1}{n} \right) + \frac{6720(s + k)^4}{n(n - 240(s+k)^2)} \\
= \frac{480(s + k)^4}{n(n - 240(s+k)^2)} + \frac{6720(s + k)^4}{n(n - 240(s+k)^2)} \\
= \frac{7200(s + k)^4}{n(n - 240(s+k)^2)}.
\]

This completes the proof.

Lemma 3.13. The sets \( S \) and \( T \) have sizes as equal as possible. That is
\[
||S| - |T|| \leq 1.
\]

Proof. We assume on the contrary that \( |T| \geq |S| + 2 \). We consider two cases.

Case 1: \( n \) is even. Since \( |S| + |T| = n \), we have
\[
\frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor - \sqrt{|S||T|} \geq \frac{n}{2} - \sqrt{\left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} + 1 \right)} \\
= \frac{n}{2} - \sqrt{\frac{n^2}{4} - 1} = \frac{1}{n + \sqrt{\frac{n^2}{4} - 1}} > \frac{1}{n}.
\]

So by Lemma 3.12 we have
\[
\frac{1}{n} < \frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor - \sqrt{|S||T|} \leq \frac{7200(s + k)^4}{n(n - 240(s+k)^2)}.
\]

This is a contradiction for sufficiently large \( n \).

Case 2: \( n \) is odd. Since \( |S| + |T| = n \), we have
\[
\frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor - \sqrt{|S||T|} \geq \frac{n^2 - 1}{2n} - \sqrt{\left( \frac{n - 3}{2} \right) \left( \frac{n + 3}{2} \right)} \\
= \frac{1}{2} \left( n - \frac{1}{n} - \sqrt{n^2 - 9} \right) = \frac{(n - \frac{1}{n})^2 - (n^2 - 9)}{2(n - \frac{1}{n} + \sqrt{n^2 - 9})} \\
= \frac{7 + \frac{1}{n^2}}{2(n - \frac{1}{n} + \sqrt{n^2 - 9})} > \frac{1}{n}.
\]

So by Lemma 3.12 again, we get
\[
\frac{1}{n} < \frac{2}{n} \left\lfloor \frac{n^2}{4} \right\rfloor - \sqrt{|S||T|} \leq \frac{7200(s + k)^4}{n(n - 240(s+k)^2)}.
\]

This is a contradiction for sufficiently large \( n \). Therefore for \( n \) large enough we must have that \( ||S| - |T|| \leq 1 \).
Recall that $G$ is an $H_{s,t_1,\ldots,t_k}$-free graph with the maximum spectral radius. Finally, we will show that $e(G) = \text{ex}(n,H_{s,t_1,\ldots,t_k})$. In other words, $G$ also attains the maximum number of edges among all $H_{s,t_1,\ldots,t_k}$-free graphs.

Proof of Theorem 1.5 By way of contradiction, we may assume that $e(G) \leq \text{ex}(n,H_{s,t_1,\ldots,t_k}) - 1$. By Lemma 3.13 we know that $|S| - |T| \leq 1$. Let $H$ be an $H_{s,t_1,\ldots,t_k}$-free graph with $\text{ex}(n,H_{s,t_1,\ldots,t_k})$ edges on the same vertex set as $G$ such that the crossing edges between $S$ and $T$ span a complete bipartite graph in $H$, this is possible because every graph in $\text{Ex}(n,H_{s,t_1,\ldots,t_k})$ has a maximum cut of size $\lfloor n^2/4 \rfloor$ by Theorem 1.3. We denote $E_+ = E(H) \setminus E(G)$ and $E_- = E(G) \setminus E(H)$. Note that $E_+$ and $E_-$ are sets of edges such that $(E(G) \cup E_+) \setminus E_- = E(H)$. Thus $e(G) + |E_+| - |E_-| = e(H)$, which together with $e(H) \geq e(G) + 1$ implies that

$$|E_+| \geq |E_-| + 1.$$ 

Furthermore, we have that $|E_-| \leq e(G[S]) + e(G[T]) < 2(s + k)^2$. By Lemma 3.9 we have that $|E_+| \leq \left\lfloor \frac{n^2}{4} \right\rfloor - e(S,T) + 2f(s + k - 1,s + k - 1) \leq 16(s + k)^2$. Now, by Lemma 3.10 we have that

$$\lambda(H) \geq \frac{x^T A(H) x}{x^T x} = \lambda(G) + \frac{2}{x^T x} \sum_{ij \in E_+} x_i x_j - \frac{2}{x^T x} \sum_{ij \in E_-} x_i x_j \geq \lambda(G) + \frac{2}{x^T x} \left( |E_+| \left( 1 - \frac{120(s + k)^2}{n} \right)^2 - |E_-| \right) \geq \lambda(G) + \frac{2}{x^T x} \left( |E_+| - |E_-| - \frac{240(s + k)^2}{n} |E_+| + \frac{(120(s + k)^2)^2}{n^2} |E_+| \right) \geq \lambda(G) + \frac{2}{x^T x} \left( 1 - \frac{240(s + k)^2}{n} |E_+| + \frac{(120(s + k)^2)^2}{n^2} |E_+| \right) \geq \lambda(G)$$

for sufficiently large $n$, where the last inequality follows by $|E_+| < 16(s + k)^2$. Therefore we have that for $n$ large enough, $\lambda(H) > \lambda(G)$, a contradiction. Hence $e(G) = e(H)$. By Theorem 1.3 we know that $G \in \text{Ex}(n,H_{s,t_1,\ldots,t_k})$. The proof of Theorem 1.5 is complete.

4 Concluding remarks

To avoid unnecessary calculations, we did not attempt to get the best bound on the order of graphs in the proof. Our proof used the Triangle Removal Lemma, which means that the condition “sufficiently large $n$” is needed in our proof. It is interesting to determine how large $n$ needs to be for our result.

Recently, Cioabă, Desai and Tait [9] investigated the largest spectral radius of an $n$-vertex graph that does not contain the odd-wheel graph $W_{2k+1}$, which is the
Conjecture 4.1. Let $F$ be any graph such that the graphs in $\text{Ex}(n, F)$ are Turán graphs plus $O(1)$ edges. Then for sufficiently large $n$, a graph attaining the maximum spectral radius among all $F$-free graphs is a member of $\text{Ex}(n, F)$.

We say that $F$ is edge-color-critical if there exists an edge $e$ of $F$ such that $\chi(F - e) < \chi(F)$. Let $F$ be an edge-color-critical graph with $\chi(F) = r + 1$. By a result of Simonovits [39] and a result of Nikiforov [36], we know that $\text{Ex}(n, F) = \text{Ex}_{sp}(n, F) = \{T_e(n)\}$ for sufficiently large $n$. This shows that Conjecture 4.1 is true for all edge-color-critical graphs. As we mentioned before, Theorem 1.4 says that Conjecture 4.1 holds for the $k$-fan graph $F_k$. In addition, our main result (Theorem 1.5) tells us that Conjecture 4.1 also holds for the flower graph $H_{s,t_1,...,t_k}$. Note that both $F_k$ and $H_{s,t_1,...,t_k}$ are not edge-color-critical.

Let $S_{n,k}$ be the graph consisting of a clique on $k$ vertices and an independent set on $n - k$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. Clearly, we can see that $S_{n,k}$ does not contain $F_k$ as a subgraph. Recently, Zhao, Huang and Guo [46] proved that $S_{n,k}$ is the unique graph attaining the maximum signless Laplacian spectral radius among all graphs of order $n$ containing no $F_k$ for $n \geq 3k^2 - k - 2$. So it is a natural question to consider the maximum signless Laplacian spectral radius among all graphs containing no $C_{k,q}$, the graph defined as $k$ cycles of odd-length $q$ intersecting in a common vertex. We write $q(G)$ for the signless Laplacian spectral radius, i.e., the largest eigenvalue of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = \text{diag}(d_1, \ldots, d_n)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix. We end with the following conjecture (Clearly, when $t = 1$, our conjecture reduces to the result of Zhao et al. [46]).

Conjecture 4.2. For integers $k \geq 2, t \geq 1$ and $q = 2t + 1$, there exists an integer $n_0(k,t)$ such that if $n \geq n_0(k,t)$ and $G$ is a $C_{k,q}$-free graph on $n$ vertices, then

$$q(G) \leq q(S_{n,kt}),$$

equality holds if and only if $G = S_{n,kt}$.

Remark. After we submitted our paper, this conjecture was recently solved by Chen, Liu and Zhang [6].

Another interesting problem on this topic is to determine the Turán number of $C_{k,q}$ for even $q$. In general, it is challenging to determine the Turán number of $H_{s,t_1,...,t_k}$ where the cycles have even lengths.

Acknowledgements

The authors would like to thank anonymous reviewers for their valuable comments and suggestions to improve the presentation of the paper. The first author would like to express his sincere thanks to Prof. Lihua Feng and Lu Lu for many illuminating discussions. This work was supported by NSFC (Grant No. 11931002).
References

[1] H.L. Abbott, D. Hanson, H. Sauer, Intersection theorems for systems of sets, J. Combin. Theory Ser. A, 12 (1972) 381–389.

[2] N. Alon, M. Krivelevich, B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, Combin. Probab. Comput. 12 (2003) 477–494.

[3] L. Babai, B. Guiduli, Spectral extrema for graphs: the Zarankiewicz problem, Electronic J. Combin. 15 (2009) R123.

[4] R.B. Bapat, Graphs and matrices, (2nd), Universitext. Springer, London; Hindustan Book Agency, New Delhi, 2014.

[5] B. Bollobás, Extremal Graph Theory, Academic Press, New York, 1978.

[6] M.-Z. Chen, A.-M. Liu, X.-D. Zhang, The signless Laplacian spectral radius of graphs without intersecting odd cycles, 11 pages, 9 August, 2021, arXiv:2108.03895v1.

[7] V. Chvátal, D. Hanson, Degrees and matchings, J. Combin. Theory Ser. B, 20 (1976) 128–138.

[8] S. Cioabă, L.H. Feng, M. Tait, X.-D. Zhang, The spectral radius of graphs with no intersecting triangles, Electron. J. Combin. 27 (4) (2020) P4.22.

[9] S. Cioabă, D.N. Desai, M. Tait, The spectral radius of graphs with no odd wheels, European J. Combin. 99 (2022) 103420.

[10] D. Conlon, J. Fox, Graph removal lemmas, Surveys in Combinatorics, London Math. Soc. Lecture Note Ser., 409, Cambridge Univ. Press, Cambridge, 2013, pp. 1–49.

[11] D. Conlon, J. Lee, On the extremal number of subdivisions, Int. Math. Res. Not. IMRN 2021, no. 12, 9122–9145.

[12] D. Conlon, O. Janzer, J. Lee, More on the extremal number of subdivisions, Combinatorica 41 (4) (2021) 465–494.

[13] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Hungar. 10 (1959) 337–356.

[14] P. Erdős, M. Simonovits, A limit theorem in graph theory, Stud. Sci. Math. Hungar. 1 (1966) 51–57.

[15] P. Erdős, A.H. Stone, On the structure of linear graphs, Bull. Am. Math. Soc. 52 (1946) 1087–1091.
[16] P. Erdős, Z. Füredi, R.J. Gould, D.S. Gunderson, Extremal Graphs for Intersecting Triangles, J. Combin. Theory. Ser. B 64 (1995) 89–100.

[17] M. Fiedler, V. Nikiforov, Spectral radius and Hamiltonicity of graphs, Linear Algebra Appl., 432 (2010) 2170–2173.

[18] J. Fox, A new proof of the graph removal lemma, Ann. of Math. 174 (2011) 561–579.

[19] Z. Füredi, On a Turán type problem of Erdős, Combinatorica 11 (1991) 75–79.

[20] Z. Füredi, An upper bound on Zarankiewicz’s problem, Comb. Probab. Comput. 5 (1996) 29–33.

[21] Z. Füredi, New asymptotics for bipartite Turán numbers, J. Combin. Theory, Ser. A 75 (1996) 141–144.

[22] Z. Füredi, On the number of edges of quadrilateral-free graphs, J. Combin. Theory Ser. B 68 (1996) 1–6.

[23] Z. Füredi, A proof of the stability of extremal graphs, Simonovits’ stability from Szemerédi’s regularity, J. Combin. Theory Ser. B 115 (2015) 66–71.

[24] Z. Füredi, M. Simonovits, The history of degenerate (bipartite) extremal graph problems, in Erdős Centennial, Bolyai Soc. Math. Stud., 25, János Bolyai Math. Soc., Budapest, 2013, pp. 169–264.

[25] R.A. Horn, C.R. Johnson, Matrix Analysis, 2nd edition, Cambridge University Press, Cambridge, 2013.

[26] X. Hou, Y. Qiu, B. Liu, Extremal graph for intersecting odd cycles, Electron. J. Combin. 23 (2) (2016) P2.29.

[27] X. Hou, Y. Qiu, B. Liu, Turán number and decomposition number of intersecting odd cycles, Discrete Math. 341 (2018) 126–137.

[28] P. Keevash, Hypergraph Turán problems, in Surveys in Combinatorics, Cambridge University Press, Cambridge, 2011, pp. 83–140.

[29] T. Kővári, V.T. Sós, P. Turán, On a problem of K. Zarankiewicz, Colloq. Math. 3 (1954) 50–57.

[30] W. Mantel, Problem 28, Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff. Wiskundige Opgaven, 10 (1907) 60–61.

[31] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, Combin. Probab. Comput. 11 (2002) 179–189.
[32] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl. 427 (2007) 183–189.

[33] V. Nikiforov, Spectral saturation: inverting the spectral turán theorem, Electron. J. Combin. 16 (1) (2009) R33.

[34] V. Nikiforov, A contribution to the Zarankiewicz problem, Linear Algebra Appl. 414 (2010) 1405–1411.

[35] V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length, Linear Algebra Appl. 432 (2010) 2243–2256.

[36] V. Nikiforov, A spectral Erdős–Stone–Bollobás theorem, Combin. Probab. Comput. 18 (2009) 455–458.

[37] V. Nikiforov, Some new results in extremal graph theory, Surveys in Combinatorics, London Math. Soc. Lecture Note Ser., 392, Cambridge Univ. Press, Cambridge, 2011, pp. 141–181.

[38] I.Z. Ruzsa, E. Szemerédi, Triple systems with no six points carrying three triangles, in Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai II, 939–945.

[39] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in Theory of Graphs, Tihany, Hungary, 1966, Academic, New York, 1968, pp. 279–319.

[40] M. Simonovits, Paul Erdős' influence on Extremal graph theory, in The Mathematics of Paul Erdős II, R.L. Graham, Springer, New York, 2013, pp. 245–311.

[41] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941), pp. 436–452. (in Hungarian).

[42] H.S. Wilf, Spectral bounds for the clique and independence numbers of graphs, J. Combin. Theory Ser. B, 65 (1986) 113–117.

[43] L.T. Yuan, Extremal graphs for the $k$-flower, J. Graph Theory 89 (1) (2018) 26–39.

[44] M. Zhai, B. Wang, Proof of a conjecture on the spectral radius of $C_4$-free graphs, Linear Algebra Appl. 430 (2012) 1641–1647.

[45] F. Zhang, Matrix Theory: Basic Results and Techniques, 2nd edition, Springer, New York, 2011.

[46] Y. Zhao, X.Y. Huang, H. Guo, The signless Laplacian spectral radius of graphs with no intersecting triangles, Linear Algebra Appl. 618 (2021) 12–21.