On algebras of three-dimensional quaternionic harmonic fields

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Abstract
A quaternionic field is a pair $p = \{\alpha, u\}$ of function $\alpha$ and vector field $u$ given on a 3d Riemannian manifold $\Omega$ with the boundary. The field is said to be harmonic if $\nabla \alpha = \text{rot} u$ in $\Omega$. The linear space of harmonic fields is not an algebra w.r.t. quaternion multiplication. However, it may contain the commutative algebras, what is the subject of the paper. Possible application of these algebras to the impedance tomography problem is touched on.

Key words: quaternion harmonic fields, commutative Banach algebras, reconstruction of manifolds.
MSC: 30F15, 35Qxx, 46Jxx.

In memory of Gennadii Markovich Henkin

0 Introduction

Motivation. Let $(\Omega, g)$ be a smooth compact Riemannian manifold with the boundary $\Gamma$, $\Delta_g$ the Beltrami-Laplace operator, $u = u^f(x)$ a solution of the problem

$$
\begin{align*}
\Delta_g u &= 0 &\text{in } \Omega \\
u &= f &\text{on } \Gamma,
\end{align*}
$$

1 everywhere in the paper ‘smooth’ means $C^\infty$-smooth
\[ \Lambda : f \mapsto \partial_\nu u \big|_\Gamma \] the Dirichlet-to-Neumann operator, \( \nu \) the outward normal on \( \Gamma \). An \textit{impedance tomography problem} (ITP) is to recover \((\Omega, g)\) via the given \( \Lambda \).

For the case \( \dim \Omega = 2 \), an algebraic approach to ITP is proposed in [1]. Its key device is the commutative Banach algebra of analytic functions \( \mathcal{A}(\Omega) := \{ w = \varphi + \psi i \mid \varphi, \psi \in C(\Omega), d \psi = \star d \varphi \text{ in } \Omega \setminus \Gamma \} \). (0.1)

The Gelfand spectrum \( \widehat{\mathcal{A}(\Omega)} := \Omega^C \) (the set of homomorphisms \( \mathcal{A}(\Omega) \to \mathbb{C} \)) of this algebra is homeomorphic to the manifold: \( \Omega^C \cong \Omega \). The algebra of boundary values \( \mathcal{A}(\Gamma) = \{ w \big|_\Gamma \mid w \in \mathcal{A}(\Omega) \} \) is isometrically isomorphic to \( \mathcal{A}(\Omega) \). Therefore, the spectra of these algebras are canonically homeomorphic: \( \widehat{\mathcal{A}(\Gamma)} \cong \Omega^C \). In the mean time, \( \mathcal{A}(\Gamma) \) is determined by the DN-operator \( \Lambda \). The latter enables one to recover \( \Omega \) up to homeomorphism by the scheme \( \Lambda \Rightarrow \mathcal{A}(\Gamma) \Rightarrow \widehat{\mathcal{A}(\Gamma)} \cong \Omega^C \cong \Omega \) (see [1, 3] for more detail).

For \( \dim \Omega \geq 3 \), the known results on ITP [5, 8, 9] concern to a certain specific class of admissible metrics \( g \). In the mean time, the attempt to extend the algebraic approach encounters the following obstacle. The relevant multidimensional analog of \( \mathcal{A}(\Omega) \) is the space of differential forms satisfying the Cauchy-Riemann condition \( d \psi = \star d \varphi \) [2, 4]. Unfortunately, this space is not an algebra. However, at least in the 3d-case, it may possess of certain algebraic properties, which are the subject of our paper. We hope for utility of these properties for the future progress in ITP.

\textbf{Contents.} We deal with the case \( \dim \Omega = 3 \). A \textit{quaternion field} on \( \Omega \) is a pair \( p = \{ \alpha, u \} \) of (real valued) function \( \alpha \) and vector field (section of \( T \Omega \)) \( u \). The space \( \mathcal{C}(\Omega) = \{ p \mid \alpha \in C(\Omega), u \in \tilde{C}(\Omega) \} \) with the sup-norm is a noncommutative Banach algebra w.r.t. the multiplication \( pq = \{ \alpha \beta - u \cdot v, \alpha v + \beta u + u \wedge v \} \), where \( q = \{ \beta, v \} \), \( \cdot \) and \( \wedge \) are the point-wise inner and vector products in the tangent spaces \( T\Omega_m \). This space contains the (sub)space of \textit{harmonic fields} \( \mathcal{Q}(\Omega) = \{ p \in \mathcal{C}(\Omega) \mid \nabla \alpha = \text{rot } u \} \), which is not a (sub)algebra: generically \( p, q \in \mathcal{Q}(\Omega) \) does not imply \( pq \in \mathcal{Q}(\Omega) \).

However, we show that, under certain conditions on the metric \( g \), the space \( \mathcal{Q}(\Omega) \) may contain the commutative algebras \( \mathcal{A}_e(\Omega) \) associated with the geodesic fields \( e \) and similar to the above mentioned 2d algebras \( \mathcal{A}(\Omega) \). These algebras determine a \textit{quaternionic spectrum} \( \Omega^H \), which is a candidate for the role of relevant 3d analog of the 2d Gelfand spectrum \( \Omega^C \).
From the viewpoint of Algebra, the case $\Omega \in \mathbb{R}^3$ is richer in content: the space $\mathcal{Q}(\Omega)$ contains a subspace $\tilde{\mathcal{Q}}(\Omega)$, which is an AH-module [7], whereas the spectrum $\Omega^\mathbb{H}$ is well defined and homeomorphic to $\Omega$.

It is the possible homeomorphism $\Omega^\mathbb{H} \cong \Omega$, which enables us to hope for application of the quaternionic spectrum to the 3d ITP. We mean the reconstruction of $\Omega$ by the scheme: $\Lambda \Rightarrow$ an isometric copy $\tilde{\mathcal{Q}}(\Omega)$ of the space $\mathcal{Q}(\Omega) \Rightarrow$ its spectrum $\tilde{\Omega}^\mathbb{H} \cong \Omega^\mathbb{H} \cong \Omega$.

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1 Harmonic quaternion fields

Quaternions. • By $\mathbb{H} = \{q = a + Pi + Qj + Rk \mid a, P, Q, R \in \mathbb{R}\}$ we denote the quaternion algebra endowed with the standard linear operations and multiplication determined by the table $i^2 = j^2 = k^2 = -1; \ ij = k, \ jk = i, \ ki = j$ (see, e.g, [6]). Also, one defines the involution $q \mapsto \overline{q} = a - Pi - Qj - Rk$ and modulus $|q| = (\overline{q}q)^{\frac{1}{2}} = (a^2 + P^2 + Q^2 + R^2)^{\frac{1}{2}}$. We denote $\Re q = a, \ \Im q = Pi + Qj + Rk$, and call elements of the subspace $\mathbb{I} = \{q \in \mathbb{H} \mid \Re q = 0\}$ the imaginary quaternions.

• Let $E$ be a real oriented 3-dimensional Euclidean space, $\cdot$ and $\wedge$ the inner and vector products in $E$. A pair $q = \{\alpha, u\}$ with $\alpha \in \mathbb{R}$ and $u \in E$ is said to be a geometric quaternion. We denote $\Re q = \alpha, \ \Im q = u$.

The 4d linear space $\mathcal{C}$ of such pairs endowed with the component-wise summation, multiplication

$$qp := \{\alpha\beta - u \cdot v, \alpha v + \beta u + u \wedge v\}$$

(here $p = \{\beta, v\}$), involution $q \mapsto \overline{q} = \{\alpha, -u\}$, and modulus $|q| = (\overline{q}q)^{\frac{1}{2}} = (\alpha^2 + |u|^2_E)^{\frac{1}{2}}$ is an algebra. Elements of the subspace $\mathcal{I} = \{q \in \mathbb{H} \mid \Re q = 0\}$ are named by imaginary (geometric) quaternions.

Choosing an orthonormal basis $e_1, e_2, e_3 \in E$ and representing $u = Ae_1 + Be_2 + Ce_3$, one determines the isometric isomorphism between the algebras
\( \mathbb{C} \) and \( \mathbb{H} \) by \( \{ \alpha, Ae_1 + Be_2 + Ce_3 \} \leftrightarrow \alpha + Ai + Bj + Ck \), the isomorphism mapping \( \mathbb{I} \) onto \( \mathbb{I} \). By this, we identify \( \mathbb{C} \) and \( \mathbb{H} \).

- As well as \( \mathbb{H} \), algebra \( \mathbb{C} \) is noncommutative. However, it contains commutative subalgebras of the form
  \[
  \mathcal{A}_e = \{ p = \{ \varphi, \psi e \} \mid \varphi, \psi \in \mathbb{R}; \ e \in E, \ |e| = 1 \}; \\
  \mathcal{A}_0 = \{ q = \{ \alpha, 0 \} \mid \alpha \in \mathbb{R} \},
  \]  
  which are isometrically isomorphic to \( \mathbb{C} \) (by \( \mathbb{C} \ni p \leftrightarrow \varphi + \psi i \in \mathbb{C} \)) and \( \mathbb{R} \) respectively. As is easy to see, any commutative subalgebra in \( \mathbb{C} \) is of the form (1.2).

Vector analysis. Let \( \Omega \) be a smooth oriented Riemannian manifold, \( \dim \Omega = 3 \), \( g \) the metric tensor, \( \mu \) the Riemannian volume 3-form, \( \star \) the Hodge operator, \( \nabla_u \) the covariant derivative. On such a manifold, the intrinsic operations of vector analysis are well defined on smooth functions and vector fields (sections of the tangent bundle \( T\Omega \)). Following [12], we recall their definitions.

- For a field \( u \), one defines the conjugate 1-form \( u^\sharp \) by \( u^\sharp(v) = g(u, v), \ \forall v \). For a 1-form \( f \), the conjugate field \( f^\sharp \) is defined by \( g(f^\sharp, u) = f(u), \ \forall u \).
- The scalar product \( \cdot : \{ \text{fields} \} \times \{ \text{fields} \} \rightarrow \{ \text{functions} \} \) is defined pointwise by \( u \cdot v = g(u, v) \). The vector product \( \wedge : \{ \text{fields} \} \times \{ \text{fields} \} \rightarrow \{ \text{fields} \} \) is defined pointwise by \( g(u \wedge v, w) = \mu(u, v, w), \ \forall w \).
- The gradient \( \nabla : \{ \text{functions} \} \rightarrow \{ \text{fields} \} \) and divergence \( \text{div} : \{ \text{fields} \} \rightarrow \{ \text{functions} \} \) are defined by \( \nabla \alpha = (d\alpha)^2 \) and \( \text{div} u = \star d u^\sharp \) respectively, where \( d \) is the exterior derivative.
- The rotor maps \{ fields \} to \{ fields \} by \( \text{rot} u = (\star d u^\sharp)^2 \). Recall the basic identities: \( \text{div} \text{rot} = 0 \) and \( \text{rot} \nabla = 0 \). The equalities
  \[
  \nabla \alpha = \text{rot} u \quad \text{and} \quad d\alpha = \star d u^\sharp
  \]
  are equivalent. By analogy with the Cauchy-Riemann conditions in (0.1), they are called the CR-conditions.

- The Laplacian \( \Delta : \{ \text{functions} \} \rightarrow \{ \text{functions} \} \) is \( \Delta = \text{div} \nabla \).

For smooth functions \( \alpha, \beta \) and fields \( u, v \), the following relations hold:

\[
\nabla \alpha \beta = \beta \nabla \alpha + \alpha \nabla \beta; \quad \nabla u \cdot v = \nabla_u u + \nabla_u v + v \wedge \text{rot} u + u \wedge \text{rot} v; \\
\text{rot} \alpha v = \nabla \alpha \wedge v + \alpha \text{rot} v; \quad \text{rot} (u \wedge v) = \nabla_u u - \nabla_u v - (\text{div} u)v + (\text{div} v)u; \\
\text{div} u \wedge v = v \cdot \text{rot} u - u \cdot \text{rot} v; \quad \text{div} \alpha v = \nabla \alpha \cdot v + \alpha \text{div} v
\]  

(1.3)
In what follows we deal with a compact $\Omega$ with the boundary $\Gamma$. By $C(\Omega)$ and $\vec{C}(\Omega)$ we denote the Banach spaces of continuous functions and vector fields endowed with the standard sup-norms.

**Quatérrion fields.** A quaternion field is a pair $q = \{\alpha, u\}$, where $\alpha = \Re q$ and $u = \Im q$ are a function and vector field given in $\Omega$. The space of pairs $\mathcal{C}(\Omega) = \{q \mid \alpha \in C(\Omega), u \in \vec{C}(\Omega)\}$ with the point-wise summation and multiplication (1.1), and the norm
\[
\|q\| = \sup_{x \in \Omega} |q(x)| = \sup_{x \in \Omega} \left( |\alpha(x)|^2 + |u(x)|^2 \right)^{1/2}
\]
is a noncommutative Banach algebra; in particular, $\|qp\| \leq \|q\|\|p\|$ does hold. The set of imaginary fields $\mathcal{I}(\Omega) = \{q \in \mathcal{C}(\Omega) \mid \Re q = 0\}$ is a subspace but not a subalgebra in $\mathcal{C}(\Omega)$.

Elements of the subspace
\[
\mathcal{Q}(\Omega) = \{q \in \mathcal{C}(\Omega) \mid \nabla \alpha = \text{rot} \ u \ \text{in} \ \Omega \setminus \Gamma\}
\]
are said to be harmonic fields. Also, we introduce the subspace
\[
\hat{\mathcal{Q}}(\Omega) = \{q \in \mathcal{C}(\Omega) \mid \nabla \alpha = \text{rot} \ u, \ \text{div} \ u = 0 \ \text{in} \ \Omega \setminus \Gamma\} \subset \mathcal{Q}(\Omega)
\]
and call its elements pure harmonic fields.

## 2 Axial algebras

Neither $\mathcal{Q}(\Omega)$ nor $\hat{\mathcal{Q}}(\Omega)$ are the (sub)algebras in $\mathcal{C}(\Omega)$ since, generically, multiplication (1.1) does not preserve harmonicity. However, as will be shown, under some conditions on the manifold $\Omega$, these subspaces may contain commutative algebras. These algebras are of the main our interest.

**Formulas.** Let $p = \{\alpha, u\}, q = \{\beta, v\}$ be smooth quaternion fields in $\Omega$. A field $\varepsilon(p) = \nabla \alpha - \text{rot} \ u$ is said do be a harmonic residual of $p$. By this definition, one has $\varepsilon|_{\mathcal{Q}(\Omega)} = 0$. Using (1.3), one derives the following equalities:
\[
\varepsilon(pq) =
\beta \varepsilon(p) + \alpha \varepsilon(q) + v \wedge \varepsilon(p) + u \wedge \varepsilon(q) + (\text{div} u)v - (\text{div} v)u - 2\nabla_v u , \quad (2.1)
\]
\[
\text{div} \ \Im(pq) = \text{div} (\alpha v + \beta u + u \wedge v) =
\alpha \text{div} v + \beta \text{div} u + u \cdot \varepsilon(q) + v \cdot \varepsilon(p) + 2v \cdot \text{rot} \ u . \quad (2.2)
\]
If \( p, q \in \mathcal{Q}(\Omega) \) then \( \varepsilon(p) = \varepsilon(q) = 0 \), and these equalities imply
\[
\varepsilon(pq) = (\text{div } u)v - (\text{div } v)u - 2\nabla_v u, \tag{2.3}
\]
\[
(\alpha v + \beta u + u \wedge v) = \alpha \text{div } v + \beta \text{div } u + 2v \cdot \text{rot } u. \tag{2.4}
\]

For \( p, q \in \dot{\mathcal{Q}}(\Omega) \), we get
\[
\varepsilon(pq) = -2\nabla_v u, \quad \text{div } (\alpha v + \beta u + u \wedge v) = 2v \cdot \text{rot } u. \tag{2.5}
\]

**Algebras** \( \mathcal{A}_e(\Omega) \). Assume that \( \mathcal{A} \subset \mathcal{Q}(\Omega) \) is an algebra and \( p = \{\varphi, h\} \in \mathcal{A}, \ h \neq 0 \). Such a field \( p \) has to possess the following properties.

- The relation \( \nabla \varphi = \text{rot } h \) leads to \( \text{div } \nabla \varphi = \Delta \varphi = 0 \), so that \( \varphi \) is harmonic in \( \Omega \setminus \Gamma \) and, hence, \( \varphi \neq 0 \) almost everywhere.
- Since \( p^2 \in \mathcal{A} \), one has \( \varepsilon(p^2) = 0 \), and (2.3) for \( q = p \) implies \( \nabla_h h = 0 \). Writing \( h = \psi e \) with a smooth \( \psi \) and \( |e| = 1 \), we have
\[
0 = \nabla_h h = \psi \left[ (\nabla_e \psi) e + \psi \nabla_e e \right],
\]
whereas \( e \cdot \nabla_e e = 0 \) holds. This implies
\[
\nabla_e \psi = 0, \quad \nabla_e e = 0 \tag{2.6}
\]
and means that the lines of the vector field \( h \) are the geodesics and \( |h| = |
\psi| = \text{const} \) along each line.
- Since \( p^2 = \{\varphi^2 - \psi^2, 2\varphi \psi e\} \in \mathcal{Q}(\Omega) \), the same arguments, which have led to the first equality in (2.6), imply \( \nabla_e [2\varphi \psi] = 0 \) and lead to \( \nabla_e \varphi = 0 \). Thus, we have
\[
\nabla_e \psi = \nabla_e \varphi = 0. \tag{2.7}
\]
- The scalar component of the quaternion field \( p^2 \in \mathcal{Q}(\Omega) \) must be harmonic. Hence, we have
\[
0 = \Delta(\varphi^2 - \psi^2) = 2 \left[ \varphi \Delta \varphi + |\nabla \varphi|^2 - \psi \Delta \psi - |\nabla \psi|^2 \right] =
= 2 \left[ |\nabla \varphi|^2 - \psi \Delta \psi - |\nabla \psi|^2 \right], \quad \text{i.e.,} \quad |\nabla \varphi|^2 = \psi \Delta \psi + |\nabla \psi|^2.
\]
By the latter equality, with regard to \( \varphi \neq \text{const} \), we conclude that \( \psi \neq 0 \) almost everywhere. Indeed, assuming the opposite, there is a set \( A \subset \Omega \)
such that \( \text{mes} A > 0 \) and \( \psi|_A = 0 \). Let \( A^* \subset A \) be the set of density points of \( A \), so that \( \text{mes} A = \text{mes} A^* > 0 \). If \( a \in A \) and \( \nabla \psi(a) \neq 0 \) then the set \( A \) is a smooth surface near \( a \) and, hence, \( a \) is not a density point of \( A \). Hence, \( \nabla \psi|_{A^*} = 0 \), i.e., \( \psi \) and \( \nabla \psi \) vanish on \( A^* \) simultaneously. Therefore, \( |\nabla \varphi|^2 = \psi \Delta \psi + |\nabla \psi|^2 = 0 \) on \( A^* \). In the mean time, \( \nabla \varphi|_{A^*} = 0 \) for a harmonic \( \varphi \) yields \( \varphi = \text{const} \) by general results of elliptic theory (see, e.g., [10]). Thus, we arrive at a contradiction. So, \( \psi \neq 0 \) almost everywhere.

• By the use of the third equality in (2.1), we have

\[
\nabla \varphi = \text{rot} e = \nabla \psi \land e + \psi \text{rot} e.
\] (2.8)

Multiplying by \( e \), with regard to \( e \perp [\nabla \psi \land e] \) and (2.7), we arrive at the relation \( e \cdot \text{rot} e = 0 \), which is a vectorial form of the Frobenius integrability condition. Hence, we conclude that

\[
e = \nabla \tau \quad \text{locally in } \Omega.
\] (2.9)

Therefore, \textit{locally}, the level surfaces \( S^c = \{ x \in \Omega \mid \tau(x) = c \}, -\delta < c < \delta \) are geodesically parallel, whereas \( \tau(x) = \pm \text{dist}(x, S_0) \).

• By (2.9), relation (2.8) takes the form \( \nabla \varphi = \nabla \psi \land \nabla \tau \), which is equivalent to \( \nabla \psi = \nabla \tau \land \nabla \varphi \). Applying div to the latter, with regard to fourth equality in (1.3) we get \( \Delta \psi = 0 \). So, we have

\[
\Delta \varphi = \Delta \psi = 0 \quad \text{and} \quad \nabla \psi = \nabla \tau \land \nabla \varphi \quad \text{locally in } \Omega.
\] (2.10)

Thus, any \( p \in \mathcal{A} \subset \mathcal{Q}(\Omega) \) is represented in the form

\[
p = \{ \varphi, \psi \nabla \tau \} \quad \text{locally in } \Omega
\] (2.11)

with \( |\nabla \tau| = 1 \) and \( \varphi, \psi \) satisfying (2.10).

As is easy to verify, the converse is also true in the following sense. If \( \mathcal{Q}(\Omega) \) contains a field \( p = \{ \varphi, h \} \) of the form (2.11), then there is a commutative algebra \( \mathcal{A}_c(\Omega) \subset \mathcal{Q}(\Omega) \), which consists of the elements \( q = \{ \lambda, \mu \nabla \tau \} \) and

\[
p q = qp = \{ \varphi \lambda - \psi \mu, (\varphi \mu + \psi \lambda) \nabla \tau \}
\]

holds.

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2Recall that \( x \in A \) is a \textit{density point} if \( \lim_{r \to 0} \frac{\text{mes} A \cap B_r[x]}{\text{mes} B_r[x]} = 1 \), where \( B_r[x] \) is a ball of radius \( r \) and center \( x \). If \( A \) is measurable then almost every \( x \in A \) is a density point: see, e.g., [13].
Algebra $\mathcal{A}_e(\Omega)$ is evidently related with $\mathcal{A}_e(1.2)$, what motivates similarity in notation. Moreover, it is closely related with the analytic function algebras (0.1). Indeed, let $p$ be of the form (2.11). Define a complex-valued function $w = \varphi + \psi i$; let $w^c = w|_{S^c} = \varphi^c + \psi^c i$ be its traces on the level sets of $\tau$. Then, by virtue of the second equality in (2.10), $\varphi^c$ and $\psi^c$ turn out to be conjugated by Cauchy-Riemann, in the same sense as in (0.1). Therefore, $w^c$ belongs to $\mathcal{A}(S^c)$, whereas $\mathcal{A}_e(\Omega)$ is stratified as $\mathcal{A}_e(\Omega) = \bigcup_c \mathcal{A}(S^c)$.

• Assume in addition that $p \in \mathcal{A}_e(\Omega) \cap \dot{Q}(\Omega)$, so that $\text{div} \Im p = \text{div} \psi \nabla \tau = 0$. In such a case, one has

$$0 = \text{div} \psi \nabla \tau \overset{\text{(1.3)}}{=} \nabla \psi \cdot \nabla \tau + \psi \Delta \tau \overset{\text{(2.10)}}{=} \psi \Delta \tau$$

and, hence, $\Delta \tau = 0$ holds. Thus, if $p$ is a pure harmonic field then the corresponding distant function $\tau$ is harmonic. The converse is also true.

We say $\mathcal{A}_e(\Omega)$, as well as the corresponding functions satisfying (2.10) and associated with geodesic vector fields $e$, to be axial algebras and axial harmonic functions ($e$ is an axis).

• Elements of axial algebras obey the maximum module principle: for their elements $p = \{\varphi, h\}$, the relation

$$\max_\Omega |p| = \max_\Gamma |p|$$

(2.12)

is valid. Indeed, by $p \in \mathcal{A}_e(\Omega)$ one has $\Delta \varphi = 0$, $\text{rot} h = 0$ and $\nabla_h h = 0$, which implies

$$\Delta |p|^2 = \text{div} \nabla (\varphi^2 + h \cdot h) \overset{\text{(1.3)}}{=} 2 [\varphi \Delta \varphi + |\nabla \varphi|^2 + \text{div} (\nabla_h h + h \wedge \text{rot} h)] =$$

$$= 2 [||\nabla \varphi||^2 + \text{div} \nabla_h h + |\text{rot} h|^2 - h \wedge \text{rot} \text{rot} h] = 2 [||\nabla \varphi||^2 + |\text{rot} h|^2] > 0.$$  

Hence, $|p|^2$ is a subharmonic function and, as such, attains its maximum at the boundary. Therefore, the same is valid for $|p|$.

Admissible metrics. The question arises of which $\Omega$ the axial algebras do exists. The exhausting answer is not known, and the following is some considerations on this point.

• Here we provide an example of an algebra $\mathcal{A}_e(\Omega)$. Our construction is of local character.
Take a smooth surface $S \subset \Omega$. Let $\sigma^1, \sigma^2$ be the local coordinates on $S$ and $\tau := \pm \text{dist} (\cdot, S)$. Near $S$ the semi-geodesic coordinates $\sigma^1, \sigma^2, \tau$ are regular. Assume that the metric $g$ is of the form

$$ds^2 = d\tau^2 + \rho(\tau) g_{ik}(\sigma^1, \sigma^2) \sigma^i \sigma^k$$

(2.13)

with a smooth positive conformal factor $\rho$. Choose two functions $\varphi^0, \psi^0$ on $S$ related via the CR-conditions $d\psi^0 = * d\varphi^0$ (w.r.t. the induced metric $g|_S$). Extend them to a neighborhood in $\Omega$ by $\varphi(\sigma^1, \sigma^2, \tau) = \varphi^0(\sigma^1, \sigma^2), \psi(\sigma^1, \sigma^2, \tau) = \psi^0(\sigma^1, \sigma^2)$. As is easy to recognize, $\varphi$ and $\psi$ satisfy (2.10), whereas the field $p = \{\varphi, \psi \nabla \tau\}$ is harmonic and inscribed in the corresponding algebra $\mathcal{A}_e(\Omega)$.

It is not improbable that this example exhausts all possible cases. If so, in order for algebras $\mathcal{A}_e(\Omega)$ to exist, the metric in $\Omega$ must be of the structure (2.13) along the proper directions $e$'s. In particular, such algebras do exist in the spaces of constant curvature.

- As to a pure harmonic $p$, the condition $\Delta \tau = 0$ turns out to be very restrictive. For instance, it cannot be realized in the 3d sphere $S^3$ even locally.

3 \textbf{H}-spectrum

Axial algebras in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be an (open) bounded domain with the smooth boundary $\Gamma$. In this case, there is a rich reserve of algebras $\mathcal{A}_e(\Omega)$.

- Fix an $O \in \mathbb{R}^3$ and the polar coordinate system $\phi, \theta, r$ with the pole $O$. Recall that, in the polar coordinates, the $\mathbb{R}^3$-metric takes the form (2.13).

For a point $x$, by $\vec{r}(x)$ we denote its radius-vector applied at $x$, so that $|\vec{r}| = r$. Thus, we have a spherically-symmetric geodesic field $e = r^{-1} \vec{r}$ with $\text{div} e(x) = 2r^{-1}$.

Denote $S^2_O = \{x \in \mathbb{R}^3 \mid r(x) = \text{dist} (x, O) = c\}$. For a point $x = x(\phi, \theta, r) \neq O$, denote by $\pi(x) \in S^1_O$ its geodesic projection on the unit sphere, i.e., the point with the coordinates $(\phi, \theta, 1)$.

Let $\Omega$ and $O$ be such that the chosen polar system is regular in a neighborhood of $\Omega$. In this case, the domain is regularly stratified: $\Omega = \cup_c [\Omega \cap S^2_O]$.

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3 S.V.Ivanov, private communication
4 S.V.Ivanov, private communication
Let $\varphi^1, \psi^1$ be two functions on $S^1_\Omega$ (of variables $\phi, \theta$) continuous in $\pi(\Omega)$ and provided $d\psi^1 = \star d\varphi^1$ in $\pi(\Omega)$. In $\Omega$, define the functions $\varphi = \varphi^1(\pi(x)), \psi = \psi^1(\pi(x))$. Then, just by construction, the quaternion field $p = \{\varphi, \psi e\}$ turns out to be harmonic and belongs to $\mathcal{Q}(\Omega)$. The fields of this form constitute the axial algebra $\mathcal{A}_e(\Omega)$. In the mean time, the constructed $p$’s are not pure harmonic, i.e., $\mathcal{A}_e(\Omega) \not\subset \mathcal{Q}(\Omega)$ since $\text{div} e = \nabla \psi \cdot e + \psi \text{div} e = \psi 2r^{-1} \neq 0$.

For a fixed $\Omega$, varying properly the position of the pole $O$, we “prospect” the domain by the radial fields $e$ and get a rich family of the algebras $\mathcal{A}_e(\Omega) \subset \mathcal{Q}(\Omega)$.  

• Such a family becomes even richer if one changes spheres by planes. Fix an $\omega, |\omega| = 1$, and denote $\Pi^c_\omega = \{x \in \mathbb{R}^3 | x \cdot \omega = c\}$. Then the domain is stratified as $\bar{\Omega} = \bigcup_c [\bar{\Omega} \cap \Pi^c_\omega]$, whereas $\pi(\Omega) = \{x - (x \cdot \omega) \omega | x \in \Omega\}$ is its projection onto $\Pi^0_\omega$.

Choose two functions $\varphi^0, \psi^0$ on $\Pi^0_\omega$ continuous in $\pi(\Omega)$ and such that $d\psi^0 = \star d\varphi^0$ in $\pi(\Omega)$. In $\Omega$, define the functions $\varphi = \varphi^0(\pi(x)), \psi = \psi^0(\pi(x))$. The quaternion field $p = \{\varphi, \psi e\}$ is harmonic and belongs to $\mathcal{Q}(\Omega)$. Such fields constitute the axial algebra $\mathcal{A}_\omega(\Omega)$ (here $\omega$ is understood as a constant vector field $e \equiv \omega$ in $\mathbb{R}^3$). Moreover, by virtue of $\text{div} \omega = 0$, its elements turn out to be pure harmonic, so that $\mathcal{A}_\omega(\Omega) \subset \mathcal{Q}(\Omega)$ holds.

Thus, there is the family of pure harmonic axial algebras indexed by the unit vectors $\omega$.

• By calculations quite analogous to the ones which have led to (2.12), one can show that all elements of the space $\mathcal{Q}(\Omega)$ obey the maximum module principle.

**AH-structure on $\mathcal{Q}(\Omega)$**. A specific feature of the case $\Omega \subset \mathbb{R}^3$ is that the quaternion fields are canonically identified with the $\mathbb{H}$-fields (H-valued functions) by the correspondence $\{\alpha, u\} \equiv \alpha + (u \cdot e_1)j + (u \cdot e_2)j + (u \cdot e_3)k$, where $e_1, e_2, e_3$ is a fixed orthonormal basis in $\mathbb{R}^3$. As a consequence, the space $\mathcal{Q}(\Omega)$ is provided with some additional algebraic structure. We describe it, keeping the notions and terminology of the paper [7].

• Begin with a portion of abstract definitions.

Let a real linear space $\mathcal{U}$ be a (left) $\mathbb{H}$-module with the action $u \mapsto au$ for $u \in \mathcal{U}, a \in \mathbb{H}$.

By $\mathcal{U}^\times$ we denote the $\mathbb{H}$-dual space, i.e., the space of linear maps $f : \mathcal{U} \to \mathbb{H}$, which satisfy $f(ap) = af(p), a \in \mathbb{H}$. Let us call such $f$’s $\mathbb{H}$-functionals. Note that $\mathcal{U}^\times$ is a left $\mathbb{H}$-module with the action $(bf)(p) = f(p)b, b \in \mathbb{H}$. 

\[ \text{Ah-structure on } \mathcal{Q}(\Omega). \]
If $U$ is a normed space then $U^\times$ is also endowed with the norm $\|f\|_\times = \sup_{\|u\|=1} |f(u)|$.

Let $U' \subset U$ be a subspace. Note that we do not require $U'$ to be invariant w.r.t. $\mathbb{H}$-action. Define $U^\dagger = \{ f \in U^\times \mid f(u) \in \mathbb{I} \text{ for all } u \in U' \}$.

We say the pair $\{U, U'\}$ to be an $AH$-module (augmented $\mathbb{H}$-module) if $f(u) = 0$ for all $f \in U^\dagger$ implies $u = 0$. This means that the subspace $U^\dagger$ possesses a totality property: it distinguishes elements of $U$.

Now, let us show that in $\mathbb{R}^3$ the pure harmonic quaternion fields constitute an $AH$-module.

Define the action of $\mathbb{H}$ on $\hat{Q}(\Omega)$. Fix an $a \in \mathbb{H}$ and denote by $\tilde{a}(\cdot) = a$ the constant quaternion field in $\Omega$. Now, for $p = \{\varphi, h\} \in \hat{Q}(\Omega)$ we put $ap = \tilde{a}(\cdot)p(\cdot)$ (point-wise). By (2.5), one has

$$\varepsilon(ap) = -2\nabla_h \Im \tilde{a} = 0, \quad \text{div} \Im (ap) = 2h \cdot \text{rot} \Im \tilde{a} = 0,$$

just because $\tilde{a}$ is constant. Hence, $ap \in \hat{Q}(\Omega)$, so that the $\mathbb{H}$-action is well defined. Thus, $\hat{Q}(\Omega)$ is an $\mathbb{H}$-module.

Let $[\hat{Q}(\Omega)]^\times$ be the $\mathbb{H}$-dual space.

Take $[\hat{Q}(\Omega)]' = \{ p \in \hat{Q}(\Omega) \mid \Re p = 0 \} = \hat{Q}(\Omega) \cap \mathcal{I}(\Omega)$. This subspace consists of the fields $p = \{0, h\}$ such that $\text{div} h = 0$ and $\text{rot} h = 0$. Recall that $[\hat{Q}(\Omega)]^\dagger = \{ f \in [\hat{Q}(\Omega)]^\times \mid f(u) \in \mathbb{I} \text{ for all } u \in [\hat{Q}(\Omega)]' \}$. This subspace contains the quaternion Dirac measures $\theta_m$, which are associated with points $m \in \bar{\Omega}$ and act by $\theta_m(p) = p(m) \in \mathbb{H}$. These measures distinguish elements of $\hat{Q}(\Omega)$: if $\theta_m(p) = 0$ for all $m$ then $p = 0$. Hence, moreover, the wider set $[\hat{Q}(\Omega)]^\dagger$ does distinguish elements, i.e., possesses the totality property.

Summarizing, we conclude that $\{\hat{Q}(\Omega), [\hat{Q}(\Omega)]'\}$ is an $AH$-module.

Denote $\Theta(\Omega) = \{ \theta_m \mid m \in \bar{\Omega} \} \subset [\hat{Q}(\Omega)]^\times$. For any $\theta_m$ and $p \in \hat{Q}(\Omega)$, one has

$$|\theta_m(p)| = |p(m)| \leq \sup_{\Omega} |p(\cdot)| = \|p\|_{\hat{Q}(\Omega)}$$

that implies $\|\theta_m\|_\times \leq 1$. In the mean time, the sup is attained on $p = \{1, 0\}$. Hence, $\|\theta_m\|_\times = 1$, i.e., $\Theta(\Omega)$ is embedded to the unit sphere of the dual space $[\hat{Q}(\Omega)]^\times$. This sphere is compact w.r.t. the $*$-topology determined by the point-wise convergence of $\mathbb{H}$-functionals on elements of $\hat{Q}(\Omega)$. As one can show, in this topology $\Theta(\Omega)$ is a compact set. Moreover, the bijection $\Theta(\Omega) \ni \theta_m \leftrightarrow m \in \bar{\Omega}$ is a homeomorphism of topological spaces.
Choose a unit vector $\omega$; let $y, z \in \mathcal{A}_\omega(\Omega)$. Recall that $\mathcal{A}_\omega(\Omega)$ is a commutative algebra. For any $\theta_m$ one has
\[
\theta_m(yz) = (yz)(m) = y(m)z(m) = \theta_m(y)\theta_m(z),
\]
i.e., the $\mathbb{H}$-functional $\theta_m$ is multiplicative on $\mathcal{A}_\omega(\Omega)$.

- The above mentioned properties of the Dirac measures characterize the set $\Theta(\Omega)$. Namely, let us define $\Omega^\mathbb{H} \subset [\mathcal{Q}(\Omega)]^\times$ as a set of $\mathbb{H}$-functionals of the norm 1, which act multiplicatively on each algebra $\mathcal{A}_\omega(\Omega)$. Then, modifying properly the arguments of [7] (section 3.4), one can show that $\Omega^\mathbb{H} = \Theta(\Omega)$.

We say $\Omega^\mathbb{H}$ to be an $\mathbb{H}$-spectrum of the domain $\Omega \subset \mathbb{R}^3$. As it follows from the aforesaid, the $\mathbb{H}$-spectrum is homeomorphic to the domain.

- By Gelfand, the null-subspace $\text{Ker} \left[ \theta_m|_{\mathcal{A}_\omega(\Omega)} \right]$ corresponds to the maximal ideal in $\mathcal{A}_\omega(\Omega)$, which consists of (axial) analytic functions $w = \varphi + \psi i$ vanishing on the straight line, which passes through $m \in \bar{\Omega}$ in parallel to $\omega$. This line intersects the projection $\pi(\bar{\Omega})$ of the domain $\bar{\Omega}$ onto the orthogonal plane $\Pi_0^\omega$ at the point $\pi(m)$. As a result, also by Gelfand, each algebra $\mathcal{A}_\omega(\Omega)$ determines this projection up to homeomorphism. Moreover, it determines $\pi(\bar{\Omega})$ (as a 2d Riemannian manifold) up to conformal equivalence [1].

Comments and conjectures. • In the generic case of 3d $\Omega$, the vector parts $u$ of the fields $p = \{\varphi, u\} \in \mathcal{Q}(\Omega)$ take values in tangent spaces $T\Omega_m$ but not in $\mathbb{R}^3$, and no canonical way is seen to identify fields $p$ with $\mathbb{H}$-values functions (as in the case of $\Omega \subset \mathbb{R}^3$). Surely, one can turn $\mathbb{R} \times T\Omega_m$ into $\mathbb{H}$ by choosing a local frame but the analog of the constant fields $\tilde{a}$, which determine the action of $\mathbb{H}$ on $\mathcal{Q}(\Omega)$, does not appear. As a consequence, the AH-structure disappears, what reduces the options of the algebraic approach to ITP.

However, one can define the $\mathbb{H}$-spectrum as follows. Beginning with the space $\mathcal{Q}(\Omega)$, we introduce the dual space $[\mathcal{Q}(\Omega)]^\times$ of $\mathbb{R}$-linear operators from $\mathcal{Q}(\Omega)$ to $\mathbb{H}$. Then, we define $\Omega^\mathbb{H} \subset [\mathcal{Q}(\Omega)]^\times$ as the unit norm elements, which act multiplicatively on the axial algebras $\mathcal{A}_e(\Omega) \subset \mathcal{Q}(\Omega)$ (if the latter do exist; otherwise, we put $\Omega^\mathbb{H} = \emptyset$). However, the question arises whether such a definition is rich in content. In particular, can one hope for the homeomorphism $\Omega^\mathbb{H} \cong \Omega$?

Perhaps, a relevant general definition of $\Omega^\mathbb{H}$ may be extracted from the D.Quillen paper [11], which interprets D.Joyce’s constructions in algebraic geometry terms. Unfortunately, the author of the given paper is not quite educated in Algebra to understand what is written there.
• There is a case, in which $Q(\Omega)$ contains at least one algebra $A_e(\Omega)$, whereas the $H$-spectrum is well defined and available for reconstruction of $\Omega$. Let the manifold be cylindric, i.e., $\Omega = M \times [0, 1]$, whereas the metric $g$ is of the form (2.13) with $\rho = \text{const}$. Then there is the algebra $A_e(\Omega)$ with the axial field $e$, which has the lines $\{m\} \times [0, 1]$, $m \in M$. In such a situation, the "cross-section" $M$ can be recovered as the Gelfand spectrum $M^C$ of algebra $A_e(\Omega)$ (even if $g_M$ is not a simple metric). If $A_e(\Omega)$ is a unique commutative algebra in $Q(\Omega)$ then $\Omega^H \simeq M^C$ holds.

In this connection, note that just a combination of the results of [1, 2] and the given paper leads to the following assertion on the uniqueness in ITP.

**Proposition 1.** Let a 3d manifold $(\Omega, g^\dagger)$ be such that

1. $\Omega = M \times I$, where $M$ is a simply connected compact 2d manifold with boundary, $I = [0, 1]$;
2. the metric $g^\dagger$ is of the form (2.13) with $\rho \equiv 1$, $0 \leq \tau \leq 1$ ($\sigma^1, \sigma^2$ are the local coordinates on $M$).

Let $g$ be a metric on $\Omega$, $\Lambda_g$ and $\Lambda_g^\dagger$, the DN-operators, corresponding to the metrics $g$ and $g^\dagger$. In such a case, if $\Lambda_g = \Lambda_g^\dagger$, then $g = \Phi \ast g^\dagger$, where $\Phi : \Omega \rightarrow \Omega$ is a diffeomorphism, which acts identically on $\partial \Omega$.

**Proof (sketch).**

□ Since $M$ is simply connected, $\Omega$ is also simply connected (as a 3d manifold). Therefore, $\Lambda_g$ determines the operator $\vec{\Lambda}_g$ (a "magneto-static" DN-map: see [2]).

Given the pair $\Lambda_g, \vec{\Lambda}_g$ one can determine the traces of the fields $p \in \dot{Q}(\Omega)$ at $\partial \Omega$. Checking whether the trace of $p^2$ belongs to $Q(\Omega)|_{\partial \Omega}$, one can select the "algebraic" $p$’s and determine the traces of all the algebras $A_e(\Omega)$.

The traces of elements of the algebra $A_w(\Omega)$ corresponding to the field of geodesics $\{\{m\} \times I \mid m \in M\}$ are of specific form. As such, they can be selected and provide the trace algebra $A_w(\Omega)|_{\partial \Omega}$.

Owing to the maximum module principle, the latter algebra determines the "invisible" algebra $A_w(\Omega)$ up to isometric isomorphism. Hence, we can find the Gelfand spectrum of $A_w(\Omega)|_{\partial \Omega}$ and, thus, determine $(M, \lambda g_M)$, where $\lambda = \lambda(m)$ is an (unknown) conformal factor [1].

The symbol of $\Lambda_g$ determines $g|_{\partial \Omega}$ (G.Uhlmann et al). In particular, it determines $g|_{M \times \{0\}}$ and, hence, fixes the factor $\lambda$. Thus, the metric $g^\dagger$ is recovered. ■
So, at least in the dimension 3, the assumption on the injectivity of the ray transform accepted in [5] is unnecessary. Note that if \( \Lambda \) is given in addition to \( \Lambda \), then the assumption on \( M \) to be simply connected can be also cancelled.

A curious point is that, at the current stage of understanding the ITP, the class of admissible metrics is the same for the geometrical optic approach and our algebraic approach. Perhaps, there are some intimate relations between GO-solutions and algebras \( \mathcal{A}_e(\Omega) \).

- It is not improbable that there are the cases, when a finite number of the axial algebras do exist and is available for reconstruction. Also, perhaps, the reconstruction by [7], where the author deals with the even-dimensional hypercomplex manifolds, may be adapted for a class of the impedance tomography problems. We plan to touch upon this subject in future papers.

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