ESTIMATES OF LEBESGUE CONSTANTS
VIA FOURIER TRANSFORMS. MANY DIMENSIONS.

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ABSTRACT. This is an attempt of a comprehensive survey of the results in which estimates of the norms of linear means of multiple Fourier series, the Lebesgue constants, are obtained by means of estimating the Fourier transform of a function generating such a method. Only few proofs are given in order to illustrate a general idea of techniques applied. Among the results are well known elsewhere as well as less known or published in an unacceptable journals and several new unpublished results.
Introduction.

0.1. The purpose of this work is to give a survey of results, and partly of proofs, dealing with estimates of $L^1$-norms of linear means of multi-dimensional Fourier series, via estimates of Fourier transforms of functions generating these means. Such norms are called, in a general sense, the Lebesgue constants. One might think that Lebesgue constants in the multiple case are, in essence, always estimated by means of Fourier transforms. Though the answer would be “no, not always”, such estimates are indeed the central part of the theory, while other methods are at present only in the periphery of the area. As for the area in general, one is reminded the words which were said on a similar field in the marvellous book of K. M. Davis and Y.-C. Chang [DC]: “a topic now in disrepute due to its difficulty”. Nevertheless, several enthusiasts continue to spent their time seeking to find a pearl in this sea. From time to time some interesting results appear giving hope for new growth of popularity of this part of Fourier Analysis.

My teacher Professor R. M. Trigub seeing me returning again and again to these problems once proposed that I should try to write a survey on Lebesgue constants. I am very grateful to him for “infecting” me with this tempting idea, and for repeatedly encouraging me to continue with the work. Surely, there are several related surveys already in the literature (see [Zh, AIN, AAP, Go, Dy3]), and several important books on different aspects of Fourier Analysis; those of Stein (see [SW, S3]) should be mentioned first. But many interesting sides of our somewhat special topic were not touched at all in these comprehensive works. Nevertheless, I was unsure if there was need for an additional survey, until once Professor E. M. Stein asked me whether it is possible to outline results on Lebesgue constants in which the Fourier transform methods are involved. This question gave me a clear starting point for which I would like to express my gratitude to Professor E. M. Stein.

The outline of this survey is as follows. After preliminary notations in §1 we give a proof of one V. A. Yudin’s result. This proof illustrates a general approach to such estimates. We also give several results for the Bochner-Riesz means which were prototypes to Yudin’s result and to some other general results that are also given in this survey. In §2 we give very general results mostly due to E. Belinskii. In §3 we investigate various generalizations of the Bochner-Riesz means. The next §4 is also devoted to generalizations of the Bochner-Riesz means. What is preserved is the spherical nature of summation. In §5 a collection of “polyhedral” results is given. It is shown that there are sufficiently interesting problems in this case. In §6 results are considered in which partial sums or more general linear means are taken with respect to “hyperbolic crosses”. In §7 we give as an Appendix other results which are not proved by means of the Fourier transform methods. We tried to mention as many as possible results on the topic; some of them are very recent.

My friends and colleagues Professors E. Belinskii, A. Podkorytov and M. Skopina were the readers of earlier variants of this work. Many improvements are due to their precise remarks and helpful discussions. I would like to acknowledge their efforts and sincere interest.

Finally I wish to mention that only one person deserves “acknowledgements” for possible mistakes, misprints and all poor passages. “It’s Me, O Lord”, as R. Kent entitled his book.
0.2. Let \( f \) be an integrable function on \( T^n \), where \( T = (-\pi, \pi] \), 2\(\pi\)-periodic in each variable. Consider a Fourier series of this function

\[
\sum_k \hat{f}(k) e^{ikx}
\]

where \( x = (x_1, ..., x_n) \) is a point in the real \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \), the lattice of points in \( \mathbb{R}^n \) with integer coordinates, \( kx = k_1x_1 + ... + k_nx_n \) is the scalar product, and

\[
\hat{f}(k) = (2\pi)^{-n} \int_{T^n} f(x) e^{-ikx} dx
\]

is the \( k \)-th Fourier coefficient of function \( f \).

Kolmogorov’s famous result [K] says that the series (0.1) may be divergent at every point of \( T^n \). Thus, it is quite natural to consider a sequence of linear operators

\[
L_N^\lambda : f(x) \mapsto L_N^\lambda(f; x) = \sum_k \lambda\left(\frac{k}{N}\right) \hat{f}(k) e^{ikx},
\]

where \( \lambda \) is a bounded measurable function (of course, it should be defined at the points of type \( \frac{k}{N} \)), and to study its properties in order to derive some information about the function \( f \), or a certain space of such functions. The basic information one receives from the behavior of the norms of these operators. When the operators map \( C(T^n) \) into \( C(T^n) \), or \( L^1(T^n) \) into \( L^1(T^n) \), and \( \lambda \equiv 1 \) on some set and vanishes outside (or, in the other words, \( \lambda \) is an indicator function of this set) the norms are traditionally called the Lebesgue constants. This term is frequently saved for the general situation.

0.3. It is well-known [SW,Ch.VII,Th.3.4] that the operator \( L_N^\lambda \) is bounded if and only if the series

\[
\sum_k \lambda\left(\frac{k}{N}\right) e^{ikx}
\]

is a Fourier series of some measure \( \mu_N \), and \( \| L_N^\lambda \| = \| \mu_N \| \). And if this series is the Fourier series of an integrable function, the following relation takes place:

\[
\| L_N^\lambda \| = (2\pi)^{-n} \int_{T^n} \left| \sum_k \lambda\left(\frac{k}{N}\right) e^{ikx} \right| dx.
\]

This occurs, for instance, when \( \lambda \) is boundedly supported.

If to take formally an integral instead of the sum on the right-hand side of (0.4), and to fulfil some simple computations (not less formal for a moment), considering \( k \) as a continuous parameter, one can expect something like \((2\pi)^{-n} \int_{T^n} \left| \hat{\lambda}(x) \right| dx\) on place of the right-hand side of (0.4). Here

\[
\hat{\lambda}(x) = \int \lambda(u)e^{-iux} du
\]
is the Fourier transform of $\lambda$. This process which looks so natural and so attractive (and so short!) can be very subtle and sometimes very cumbersome in reality. We will see below that sometimes it does not valid in some sense!

0.4. Very often such transformations turn on very special problems of Number Theory, or Differential Geometry (see, e.g., the work [S2] as a representative example in a series of Stein’s works devoted to the role of the Gaussian curvature in Fourier Analysis. Besides this, in different parts of his recent book [S3] the place of this notion among other important things may be seen).

It is written in [DC] that “the best trick will be to transfer problems” (of convergence of Fourier series) ”from Fourier series to Fourier integrals. This is good because it is easier to compute an integral explicitly than to sum a series in a closed form. On the other hand, this is bad because the integrals defining Fourier transforms do not converge absolutely”.

We try to consider some problems in which ”this is good” (or at least we think so). Even in the cases when ”this is bad” (in the mentioned sense), we obtain certain information studying the order of ”badness”.

1. Spherical partial sums, and some generalizations.

1.1. Let us begin with a very exemplary case. Consider the spherical partial sums $S_N$ of the Fourier series of a function $f$:

$$S_N(f; x) = \sum_{|k| \leq N} \hat{f}(k) e^{ikx},$$

where $|k|^2 = k_1^2 + \cdots + k_n^2$. It is well-known that the norms of operators

$$S_N : f(x) \mapsto S_N(f; x)$$

taking $C(T^n)$ into $C(T^n)$ (or $L^1(T^n)$ into $L^1(T^n)$ that is the same) equal to

$$\|S_N\| = (2\pi)^{-n} \int_{T^n} \left| \sum_{|k| \leq N} e^{ikx} \right| dx.$$

The following ordinal estimate holds.

**Theorem 1.1.** There exist positive constants $C_1$, and $C_2$, where $C_1 < C_2$, depending only on $n$, such that

$$C_1 N^{n-1} \leq \|S_N\| \leq C_2 N^{n-1}.$$

**Proof.** We give an outline of the proof which illustrates a method, rather general and having the passage to the Fourier transform described in the Introduction as the basic construction.

The estimate from below was obtained firstly by V.A.Ilyin [I] (for expansions corresponding to the Laplace operator), and after that two-sided estimates were obtained in [Ba2] and [IA]. All those proofs were rather complicate. We will follow a very simple proof of the upper estimate proposed by Yudin [Y1] (in more general situations; the earlier proof of the upper estimate in (1.2) due to H.Shapiro [Sh].
essentially is almost the same). Then, using one trick proposed in [I], we will adjust this proof for the estimate from below as well.

If $I_k$ is the cube with the edge of length 1 and the center at the point $k$, and $B_N = \bigcup_{k:|k| \leq N} I_k$, then

$$\int_{B_N} e^{ixu} du = \sum_{k:|k| \leq N} \int_{I_k} e^{ixu} du$$

$$= \sum_{k:|k| \leq N} e^{ikx} \prod_{j=1}^{n} \frac{2\sin \frac{x_j}{2}}{x_j} = \prod_{j=1}^{n} \frac{2\sin \frac{x_j}{2}}{x_j} \sum_{k:|k| \leq N} e^{ikx}.$$ 

Thus, we obtain

$$\|S_N\| \leq (2\pi)^{-n} \left(\frac{\pi}{2}\right)^n \left|\int_{T^n} \int_{B_N} e^{ixu} du \right| dx$$

$$\leq 4^{-n} \int_{T^n} \left|\int_{|u| \leq N} e^{ixu} du \right| dx + 4^{-n} \int_{T^n} \left|\int_{D_N} e^{ixu} du \right| dx,$$

(1.3)

where $D_N$ is a symmetric difference of the sets $B_N$ and $|u| \leq N$. Taking into account that $\text{mes} D_N \leq CN^{n-1}$ and applying the Cauchy-Schwarz inequality to the last summand on the right-hand side of (1.3), we obtain by virtue of Parseval’s equality that

$$\|S_N\| \leq 4^{-n} \int_{T^n} \left|\int_{|u| \leq N} e^{ixu} du \right| dx + O(N^{n-1})$$

$$= 4^{-n} \int_{NT^n} \left|\int_{\mathbb{R}^n} \chi_1(u)e^{ixu} du \right| dx + O(N^{n-1})$$

(1.4)

$$= 4^{-n} \int_{NT^n} |\hat{\chi}_1(x)| dx + O(N^{n-1}),$$

where $\chi_1$ is the indicator function of the unit ball $|u| \leq 1$. Now standard computations of $\hat{\chi}_1$ via Bessel functions and consequent integration (see, e.g., [SW], Ch.7) complete the upper estimate in (1.2).

In order to obtain the lower estimate, let us introduce a small parameter $\varepsilon$, $0 < \varepsilon < 1$, as it was done in [I]. We obtain, instead of (1.1),

$$\|S_N\| \geq (2\pi)^{-n} \int_{T^n} \left|\sum_{k:|k| \leq N} e^{ikx} \right| dx.$$
Now we can repeat the operations similar to (1.3), (1.4), but with estimates from below and signs "−" instead of "+", on appropriate places. This will give us

\[ \|S_N\| \geq (2\pi)^{-n} \int_{\varepsilon N T^n} |\hat{\chi}_1(x)| \, dx - C_3 \varepsilon^\frac{n}{2} N^{\frac{n-1}{2}}. \]

The afore-mentioned computations via Bessel functions will yield here

\[ \|S_N\| \geq \left( C_4 \varepsilon^{\frac{n}{2}} - C_3 \varepsilon^{\frac{n}{2}} \right) N^{\frac{n-1}{2}}. \]

It remains only to choose \( \varepsilon \) such that \( C_1 = C_4 \varepsilon^{\frac{n}{2}} - C_3 \varepsilon^{\frac{n}{2}} > 0 \). The proof is complete. \( \Box \)

In the less known paper by A. Podkorytov [P0] similar technique was elaborated independently. This allowed to obtain the following interesting result. To indicate that partial sums correspond to certain set \( B \) we will denote them by

\[ S_B(f; x) = \sum_{k \in B} \hat{f}(k)e^{ikx}. \]

**Theorem 1.2.** Assume that the set \( B \subset \mathbb{R}^n \) satisfies the following conditions:

1) \( B \subset [-N_1, N_1; -N_2, N_2; \ldots; -N_n, N_n] \), where \( N_1 \geq N_2 \geq \ldots \geq N_n \geq 0 \).

2) For all \( j = 1, \ldots, n \) and all \( x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \) the set

\[ B_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = \{x_j : (x_1, \ldots, x_j, \ldots, x_n) \in B \} \]

is either empty or is an interval.

Then

\[ \|S_B\| = O\left( \sqrt{N_2 \cdots N_n} \left( 1 + \ln \frac{N_1}{N_2} \right) \right). \]

1.2 This way, or certain of its modifications, is used in many results dealing with the estimates of Lebesgue constants of partial sums as well as the linear means of Fourier series. We have already mentioned that such was Yudin’s estimate from above [Y1], for more general sets \( B \) generating the corresponding partial sums, namely those which are balanced (with each point \( x \) the whole set \( \delta x, |\delta| \leq 1 \), belongs to \( B \)), and having the finite upper Minkowski measure:

\[ \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{mes}\{x : \rho(x, \partial B) < \varepsilon\} < \infty, \]

where \( \rho(x, y) \) is the distance between two points \( x, y \in \mathbb{R}^n \), and \( \rho(x, \partial B) = \inf_{y \in \partial B} \rho(x, y) \).

The same method was applied for estimate from below in [L2], where conditions are less restrictive (above all, they are local) than in the earlier paper [CaS] and the later papers [Br1, Br2].
Theorem 1.3. ([L2, L3]) Let the boundary of the region $B$ contain a simple (non-intersecting) piece of a surface of smoothness $\lceil \frac{n+2}{2} \rceil$ in which there is at least one point with non-vanishing principal curvatures. Then there exists a positive constant $C$ depending only on $B$ such that

$$\int_{\mathbb{T}^n} \left| \sum_{k \in NB \cap \mathbb{Z}^n} e^{ikx} \right| dx \geq CN^{\frac{n-1}{2}}$$

for large $N$.

We will see below a generalization of this theorem (see Theorem 3.2). And now let us give one related two-dimensional result.

Theorem 1.4. ([Gu]) Assume that a convex set $B$ is included into $\mathbb{T}^2$. Then for sufficiently large $N$ the inequality

$$\|S_{NB}\| \geq CN^{\frac{1}{2}} \left( \int_{-\pi}^{\pi} \sqrt{\rho(\varphi)} \, d\varphi \right)^2$$

holds, where $\rho(\varphi)$ is the curvature radius of $\partial B$ at the point where $\max\{x_1 \cos \varphi + x_2 \sin \varphi : x \in B\}$ is attained.

If $\lim \inf_{N \to \infty} \frac{\|S_{NB}\|}{N^{\frac{1}{2}}} = 0$ then the boundary of the set $B$ is degenerate, i.e. for almost all directions $\varphi$ its curvature radius is equal to zero.

What is of special interest in this result, as well as in [P7], is the fact that no assumptions on smoothness of $\partial B$ are involved. Of course, convexity itself gives some minimal smoothness. Then, only two-dimensional statements are obtained in [P7] and [Gu]. This gives rise to the question: what are minimal smoothness assumptions for such estimates in case of arbitrary dimension.

2. Some general estimates of Lebesgue constants.

2.1. Certainly, we were speaking about an outline, and in more general and more complicate situations each step can be cumbersome and entailed with bigger technical difficulties. For example, even for the case of Fejer means of partial sums generated by convex sets, a proof of the boundedness of the norms of corresponding operators in [P1] is rather complicate, and all the time refers to non-trivial estimates of Fourier transforms.

2.2. Let us give now one Belinskii’s result in which this method is realized on a very high level of generality. Belinskii was apparently the first began a systematic study of connections between summability and integrability of the Fourier transform of a function generating a method of summability, in the multi-dimensional case.

Theorem 2.1. ([Be2]) Let $\lambda$ be a bounded measurable function with a compact support in $\mathbb{R}^d$. Then there exists a positive constant $C$ depending only on $\lambda$ such that

$$\int_{\mathbb{T}^d} \left| \sum_{k \in NB \cap \mathbb{Z}^d} e^{ikx} \right| dx \geq CN^{\frac{d-1}{2}},$$

for large $N$. 

We will see below a generalization of this theorem (see Theorem 3.2). And now let us give one related two-dimensional result.
support. Then for the norms of a sequence of linear operators (0.2) we have

\[ \|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)} \leq (2\pi)^{-n} \int_{T^n} \prod_{j=1}^{n} \frac{x_j}{2N \sin \frac{x_j}{2N}} |\hat{\lambda}(x)| \, dx \]

(2.1)

\[ + \sum_{j=1}^{m-1} \left( \frac{\pi}{2} \right)^{(j+1)n} \int_{T^n} \left| \frac{x}{N} \right|^j \, dx \]

\[ + \frac{\pi^{nm+n/2}}{2^{nm-n/2}} \int_{\frac{1}{2\pi}T^n} \cdots \int_{\frac{1}{2\pi}T^n} \left( \sum_{k} \left| \Delta^{m}_{N}(\lambda; \frac{u_1}{N}, ..., \frac{u_m}{N}) \right|^2 \right)^{\frac{1}{2}} \, du_1 \cdots du_m, \]

\[ C_p \|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)} \geq \left\{ (2\pi)^{-n} \int_{\varepsilon T^n} \prod_{j=1}^{n} \frac{x_j}{2N \sin \frac{x_j}{2N}} |\hat{\lambda}(x)|^p \, dx \right\}^{\frac{1}{p}} \]

(2.2)

\[ - \sum_{j=1}^{m-1} \left( \frac{\pi}{2} \right)^{(j+1)n} \left\{ \int_{\varepsilon T^n} |\hat{\lambda}(x)|^p \left| \frac{x}{N} \right|^j \, dx \right\}^{\frac{1}{p}} \]

\[ - \frac{\pi^{nm+n/2}}{2^{nm-n/2}} \varepsilon^{n \frac{2-p}{p-1}} \int_{\frac{1}{2\pi}T^n} \cdots \int_{\frac{1}{2\pi}T^n} \left( \sum_{k} \left| \Delta^{m}_{N}(\lambda; \frac{u_1}{N}, ..., \frac{u_m}{N}) \right|^2 \right)^{\frac{1}{2}} \, du_1 \cdots du_m. \]

Here, \( \varepsilon, 0 < \varepsilon \leq 1 \), is an arbitrary real number, \( m \) is integer, and \( 1 \leq p \leq 2 \). The \( m \)-th difference \( \Delta^m z(\lambda; h_1, ..., h_m) \) is defined recursively by the formulas

\[ \Delta^1 z(\lambda; h_1) = \lambda(z + h_1) - \lambda(z); \]

\[ \Delta^m z(\lambda; h_1, ..., h_m) = \Delta^{m-1}_{z+h_m}(\lambda; h_1, ..., h_{m-1}) - \Delta^{m-1}_{z}(\lambda; h_1, ..., h_{m-1}), \]

with \( h_j, z \in \mathbb{R}^n \). When \( p > 2 \), in view of duality (see, e.g., [SW,Ch.1,Th.3.20]), the estimate (2.2) still valids with \( p' = \frac{p}{p-1} \) instead of \( p \). We give the proof of the main case: the estimate from below for \( p = 1 \).

**Proof.** We follow the argument from [Be2]. We have by definition

\[ \|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)} = \sup_{\|f\|_{L^p(T^n)} \leq 1} \left\{ (2\pi)^{-2n} \int_{T^n} \left| f(x-u) \sum_{k} \lambda \left( \frac{k}{N} \right) e^{ikx} \right|^p \, dx \right\}^{\frac{1}{p}}. \]

For \( p = 1 \) we have

\[ \|L_N^\lambda\| = \sup_{\|f\|_{L^p(T^n)} \leq 1} \left\{ (2\pi)^{-2n} \int_{T^n} \left| f(x-u) \sum_{k} \lambda \left( \frac{k}{N} \right) e^{ikx} \right| \, dx \right\}. \]

(0.4)
For \( p > 1 \), assume without loss of generality that \( \lambda = 0 \) when \( |x_j| > 1 \), \( j = 1, 2, \ldots, n \), and set

\[
f(x_1, \ldots, x_n) = \frac{N^{n-\frac{1}{p}}}{C_p} \prod_{j=1}^{n} D_N(x_j)
\]

where \( D_N(x_j) \) is the Dirichlet kernel. A constant \( C_p \) is chosen to provide \( \|f\|_{L^p(T^n)} \leq 1 \). We obtain

\[
C_p\|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)} \geq N^{n-\frac{1}{p}} \left\{ (2\pi)^{-n} \int_{T^n} \left| \sum_k \lambda \left( \frac{k}{N} \right) e^{ikx} \right|^p \right\}^{\frac{1}{p}}.
\]

Using the obvious equality

\[
\int_{k_j - \frac{1}{2}}^{k_j + \frac{1}{2}} e^{ix_j \cdot v_j} dv_j = e^{ik_j x_j} \frac{2 \sin \frac{x_j}{2}}{x_j}, \quad j = 1, 2, \ldots, n,
\]

replace the sum by the Fourier transform of \( \lambda \) as in [Y1] (see also [Zg], Ch.V, Th.2.29):

\[
C_p\|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)}
\]

\[
= N^{n-\frac{1}{p}} \left\{ (2\pi)^{-n} \int_{T^n} \left| \prod_{j=1}^{n} \frac{x_j}{2 \sin \frac{x_j}{2}} \sum_k \right| \frac{x_j}{2 \sin \frac{x_j}{2}} \int_{k + \frac{1}{2} \pi T^n} \lambda \left( \frac{k}{N} \right) e^{ixv} dv \right|^p dx \right\}^{\frac{1}{p}}.
\]

Obviously that for \( 0 < \varepsilon < 1 \)

\[
C_p\|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)}
\]

\[
\geq N^{n-\frac{1}{p}} \left\{ (2\pi)^{-n} \int_{\varepsilon T^n} \left| \prod_{j=1}^{n} \frac{x_j}{2 \sin \frac{x_j}{2}} \sum_k \right| \frac{x_j}{2 \sin \frac{x_j}{2}} \int_{k + \frac{1}{2} \pi T^n} \lambda \left( \frac{k}{N} \right) e^{ixv} dv \right|^p dx \right\}^{\frac{1}{p}}.
\]

The simple inequality \( \frac{2}{\pi} t \leq \sin t \), whis \( 0 \leq t \leq \frac{\pi}{2} \), and Minkowski’s inequality yield
\[ C_p \|L_N^\lambda\|_{L^p(T^n) \to L^p(T^n)} \]

\[ \geq N^n \frac{1-p}{p} \left\{ (2\pi)^{-n} \int_{\epsilon T^n} \left| \prod_{j=1}^{n} \frac{x_j}{2\sin \frac{x_j}{2}} \sum_{k} \int_{k+\frac{1}{2\pi} T^n} \lambda \left( \frac{v}{N} \right) e^{ixv} \, dv \right|^p \, dx \right\}^{\frac{1}{p}} \]

\[ - N^n \frac{1-p}{p} \left\{ (2\pi)^{-n} \left( \frac{\pi}{2} \right)^n \int_{\epsilon T^n} \left[ \sum_k \int_{k+\frac{1}{2\pi} T^n} \left[ \lambda \left( \frac{k}{N} \right) - \lambda \left( \frac{v}{N} \right) \right] e^{ixv} \, dv \right] \right\}^{\frac{1}{p}} \]

Summing and change of variables reduces the first term on the right-hand side to the form claimed. Let us estimate the second term. Change of variables \( v_j \to v_j + k_j \), for \( j = 1, \ldots, n \), and generalized Minkowski’s inequality yield

\[ N^n \frac{1+n}{p} \left( \frac{\pi}{2} \right)^n \left\{ (2\pi)^{-n} \int_{\epsilon T^n} \left[ \sum_k \int_{k+\frac{1}{2\pi} T^n} \left[ \lambda \left( \frac{k}{N} \right) - \lambda \left( \frac{v}{N} \right) \right] e^{ixv} \, dv \right] \right\}^{\frac{1}{p}} \]

\[ \leq N^n \frac{1+n}{p} (2\pi)^{-n} \left( \frac{\pi}{2} \right)^n \int_{\frac{1}{2\pi} T^n} \left\{ \int_{\epsilon T^n} \left[ \sum_k \left[ \lambda \left( \frac{k}{N} \right) - \lambda \left( \frac{k+v}{N} \right) \right] e^{i x k} \right]^p \, dx \right\}^{\frac{1}{p}} \, dv \]

\[ = N^n \frac{1+n}{p} (2\pi)^{-n} \left( \frac{\pi}{2} \right)^n \int_{\frac{1}{2\pi} T^n} \left\{ \int_{\epsilon T^n} \left[ \sum_k \Delta_{x} \left( \lambda; \frac{u}{N} \right) e^{i x k} \right]^p \, dx \right\}^{\frac{1}{p}} \, du_1. \]

Note that the inner integral on the right-hand side is of the same form as in the beginning of the proof, so the same argument is applicable to it. In what follows we need only estimates from above. Taking into account that

\[ \int_{\mathbb{R}^n} \left[ \lambda \left( \frac{u_2}{N} \right) - \lambda \left( u_2 + \frac{u_1}{N} \right) \right] e^{-i u_2 x} \, du_2 \]

\[ = \int_{\mathbb{R}^n} \lambda(u_2) [e^{-i u_2 x} - e^{-i (u_2 - \frac{u_1}{N}) x}] \, du_2 \]

\[ = [1 - e^{i \frac{u_1}{N} x}] \int_{\mathbb{R}^n} \lambda(u_2) e^{-i u_2 x} \, du_2 = [1 - e^{i \frac{u_1}{N} x}] \hat{\lambda}(x), \]
we obtain

\[ N^{n - \frac{2}{p}} (2\pi)^{-\frac{n}{2}} \left( \frac{\pi}{2} \right)^n \int_{\mathbb{T}^n} \left\{ \int_{\mathbb{T}^n} \left| \sum_k \Delta_{\frac{\lambda}{N}}^1 (\lambda; \frac{u_1}{N}) e^{ikx} \right|^p dx \right\}^{\frac{1}{p}} du_1 \]

\[ \leq N^n \left( \frac{\pi}{2} \right)^{2n} \int_{\mathbb{T}^n} \left\{ \int_{\mathbb{T}^n} \left| [1 - e^{i\frac{\alpha}{N} x}] \lambda(x) \right|^p dx \right\}^{\frac{1}{p}} du_1 \]

\[ + N^n \left( \frac{\pi}{2} \right)^{2n} \int_{\mathbb{T}^n} \left\{ \int_{\mathbb{T}^n} \left| \sum_k \Delta_{\frac{\lambda}{N}}^2 (\lambda; \frac{u_1}{N}, \frac{u_2}{N}) e^{ikx} \right|^p dx \right\}^{\frac{1}{p}} du_1 du_2. \]

Since \( \sin t \leq t \) for \( t > 0 \), we get using the Cauchy-Schwarz inequality

\[ |1 - e^{i\frac{u_1}{N} x}| \leq \frac{|u_1|}{N} |x|. \]

Hence

\[ \int_{\mathbb{T}^n} \left\{ \int_{\mathbb{T}^n} \left| [1 - e^{i\frac{\alpha}{N} x}] \lambda(x) \right|^p dx \right\}^{\frac{1}{p}} du_1 \leq \int_{\mathbb{T}^n} \left\{ \int_{\mathbb{T}^n} \left| \frac{x}{N} \right|^p |\lambda(x)| dx \right\}^{\frac{1}{p}}. \]

Repeating the same computations \( m - 2 \) times, we obtain (2.2) with the remainder term

\[ N^{n - \frac{2}{p}} (2\pi)^{-\frac{n}{2}} \left( \frac{\pi}{2} \right)^{nm} \int_{\mathbb{T}^n} \left\{ \int_{\mathbb{T}^n} \left| \sum_k \Delta_{\frac{\lambda}{N}}^m (\lambda; \frac{u_1}{N}, \ldots, \frac{u_m}{N}) e^{ikx} \right|^p dx \right\}^{\frac{1}{p}} du_1...du_m. \]

In order to complete the proof apply Hölder’s inequality with the power \( \frac{2}{p} \) to the inner integral, and then Parseval’s equality. The opposite inequality \( (p = 1) \) can be obtained similarly. \( \square \)

2.3. To show the strength of this theorem, we can say that not only Theorem 1.1 follows from it as a technical corollary, but the following more general result as well.

Let \( \lambda(x) = R_\alpha = (1 - |x|^2)^{\alpha}. \) Corresponding linear means generated by this function are called the Bochner-Riesz means. An unbreakable interest to these means was initiated by Bochner’s famous work [Bc].

**Theorem 2.2.** ([I, Ba2, IA]) There exist positive constants \( C_1 \) and \( C_2 \), where \( C_1 < C_2 \), depending only on \( n \) and \( \alpha \), such that for \( 0 \leq \alpha < \frac{n-1}{2} \)

\[ C_1 N^{n - \frac{2}{p} - \alpha} \leq \left\| F_{\alpha}^{R_\alpha} \right\|_{L^1(T^n) \to L^1(T^n)} \leq C_2 N^{n - \frac{2}{p} - \alpha}. \]

Moreover the following estimate due to Babenko [Ba2, [Ba3] follows from Theorem 2.1 in the same manner.
Corollary 2.1. The following estimate holds:

\[ \left\| L_{N}^{R_{\alpha}} \right\|_{L^{p}(T^{n}) \to L^{p}(T^{n})} \geq C \left( \frac{N^{p(\alpha_{p} - \alpha)} - 1}{p(\alpha_{p} - \alpha)} \right)^{\frac{1}{p}}, \]

where \( 1 \leq p \leq \frac{2n}{n+1} \), and \( 0 \leq \alpha < \alpha_{p} = \frac{n}{p} - \frac{n+1}{2} \). For \( \alpha = \alpha_{p} \) this estimate should be understood as a limit as \( \alpha \to \alpha_{p} \), and gives a logarithmic order of growth.

Proof. The Fourier transform of \( R_{\alpha} \) is very well known (see e.g., [SW], Ch.4):

\[ \hat{R}_{\alpha}(x) = 2^{-\frac{n}{2} + \alpha} \pi^{-\frac{n}{2}} \Gamma(\alpha + 1) |x|^{-\frac{n}{2} - \alpha} J_{\frac{n}{2} + \alpha}(|x|) \]

where \( J_{\nu} \) is the Bessel function of the first kind and order \( \nu \). Let us estimate in (2.1) summands with the Fourier transform. In each one extend the domain of integration to the ball of the radius \( \sqrt{n\pi N} \) and pass to spherical coordinates. We get

\[ \left\| L_{N}^{R_{\alpha}} \right\| \leq C \int_{0}^{\sqrt{n\pi N}} \left| \frac{J_{\frac{n}{2} + \alpha}(r)}{r^{\frac{n}{2} + \alpha}} \right| r^{n-1} dr + R, \]

where \( R \) denotes the last summand in (2.1) or (2.2). We will estimate it separately. Let us use the following asymptotic formulas for Bessel functions (see e.g., [BE], 7.12(8),7.13.1(3)):

\[ J_{\nu}(t) = \left( \frac{t}{2} \right)^{\nu} \frac{1}{\Gamma(\nu + 1)} + O(|t|^{\nu+2}), \quad \text{as} \quad t \to 0. \]

\[ J_{\nu}(t) = \left( \frac{2}{\pi t} \right)^{\frac{1}{2}} \cos(t - \frac{\nu\pi}{4} - \frac{\pi}{4}) + O(t^{-\frac{3}{2}}), \quad \text{as} \quad t \to \infty. \]

These yield

\[ \int_{0}^{\sqrt{n\pi N}} \left| \frac{J_{\frac{n}{2} + \alpha}(r)}{r^{\frac{n}{2} + \alpha}} \right| r^{n-1} dr \leq C \frac{N^{\frac{n-1}{2} - \alpha} - 1}{\frac{n-1}{2} - \alpha} + O(N^{\frac{n-2}{2}}). \]

Let us estimate now the remainder \( R \).

\[ \sup_{u_{1}, \ldots, u_{m} \in \frac{1}{N} T^{n}} \left\{ \sum_{k} \left| \frac{N}{2} \right| \left( R_{\alpha} u_{1} N, \ldots, u_{m} N \right) \right\}^{\frac{1}{2}} \]

\[ = \sup_{u_{1}, \ldots, u_{m} \in \frac{1}{N} T^{n}} \left\{ \sum_{|k| < N-m-2} + \sum_{N-m-2 \leq |k| \leq N} \right\}^{\frac{1}{2}}. \]

Estimate each summand in the second sum by maximal value of the function at these points. Since \( R_{\alpha} \) is monotone increasing near the origin we have

\[ \sum_{|k| < N-m-2} \leq C N^{-2\alpha} \sum_{|k| < N-m-2} 1 \leq C N^{n-1-2\alpha}. \]
(the latter value follows from the well known estimates of the number of points of $\mathbb{Z}^n$

in the $n$-dimensional ball of radius $N$. The mean-value theorem for the directional
derivative yields that the first sum is

$$N^{-2m} \sum_{|k|<N-m-2} \left| \frac{\partial^m R_\alpha}{\partial u_1...\partial u_m} \left| \frac{k}{N} + \theta_1 \frac{u_1}{N} + ... + \theta_m \frac{u_m}{N} \right| \right|^2.$$ 

Choose $m$ such that $m \geq \alpha + 1$. If $\alpha$ is integer than the derivative is bounded and

$$\sum_{|k|<N-m-2} \leq C N^{n-2-2\alpha}.$$ 

If $\alpha$ is fractional than we have by estimating the derivative by its maximal value

on the interval:

$$\sum_{|k|<N-m-2} \leq C N^{-2m} \sum_{|k|<N-m-2} \left( 1 - \frac{|k| + m + 2}{N^2} \right)^{2(\alpha-m)}.$$ 

In view of monotonicity of the function it is possible to integrate in place of sum-

ming. This gives

$$\sum_{|k|<N-m-2} \leq C N^{n-1-2\alpha},$$

and finally

$$(2.8) \quad R \leq C N^{\frac{n-1}{2} - \alpha}.$$ 

Use now (2.5), (2.6), and (2.8). The formula (2.2) yields

$$||L^*_N||_{L^p(T^n) \to L^p(T^n)} \geq C_1 \left[ \frac{N^p(\alpha_p - \alpha_p - 1)}{p(\alpha_p - \alpha)} \right]^{\frac{1}{p}} e^{\alpha_p - \alpha} - C_2 N^{\alpha_p - \alpha} e^{\alpha_p + \frac{1}{2}}.$$ 

Choose $\varepsilon$ so that

$$C_1 e^{\alpha_p - \alpha} - C_2 e^{\alpha_p + \frac{1}{2}} \geq C > 0,$$

and this completes the proof. \[1\]

Remark 2.1. Observe that (2.7) and (2.8) give the right-hand side of (2.3).

The original proofs and the way of derivation them from Theorem 2.1 are un-

comparable as for hardness. For instance, properties of Riemann’s Zeta Function

is the main tool in [Ba2]. It is more convenient for us to give Corollary 2.2 in the

next section.

2.4. Let us give some other results of Belinskii [Be1]. These results are based

on the Poisson summation formula and technique of estimating trigonometric sums

and integrals via the Fourier transform.

Consider a certain function $\lambda(x)$ bounded on $\mathbb{R}^n$ and continuous at the points

of $\mathbb{Z}^n$, and construct the formal trigonometric series for $f \in C(T^n)$

$$(2.9) \quad \sum_{k \in \mathbb{Z}^n} \lambda(k) \hat{f}(k) e^{ikx}.$$ 

Denote

$$U_N(f; x) = (2\pi)^{-n} \int_{T^n} f(x + u) \sum_{m} \lambda \left( \frac{m}{N} \right) e^{-imu} du.$$ 

The following two propositions are very useful in many applications. They are the
corollaries to Theorem 1 in [Be1]. We do not formulate the theorem itself because
just these propositions have proved to concentrate its main possibilities.
Proposition 2.1. Suppose $U_N(f; x)$ is defined as above. If $\lambda \in C(\mathbb{R}^n)$ and $\hat{\lambda} \in L^1(\mathbb{R}^n)$, then
$$
\|U_N\| = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{\lambda}(u)| \, du + \theta (2\pi)^{-n} \int_{\mathbb{R}^n \setminus N_\mathbb{T}^n} |\hat{\lambda}(u)| \, du,
$$
where $-2 \leq \theta \leq 0$, and, in general, the constant $\theta$ depends on $N$.

Proposition 2.2. If $\lambda$ is boundedly supported and continuous, then
$$
\sup_N \|U_N\| = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{\lambda}(u)| \, du.
$$

All the norms here are the uniform norms. Note, that similar upper estimate, in the case when both $\lambda$ and $\hat{\lambda}$ are integrable, may be found in [SW, Ch.VII, Sect.2]:
$$
\int_{\mathbb{T}^n} \left| \sum_k \lambda \left( \frac{k}{N} \right) e^{ikx} \right| \, dx \leq \int_{\mathbb{R}^n} |\hat{\lambda}(x)| \, dx.
$$

2.5. In many questions dealing with summability some assumptions on $\lambda$, connected with bounded variation, are rather natural. We can notice several such works in the one-dimensional case, say, [H, Te, Be0, T2]. Let us give one result of Trigub, which is quite general and seems to be very useful in many cases when one needs to pass from Fourier series to Fourier integrals. We need some well-known notions. The Vitali variation is defined as follows (see e.g., [AC]). Let $\lambda$ be a complex-valued function and
$$
\Delta_u \lambda(x) = \left( \prod_{j=1}^n \Delta_u \lambda_j \right) \lambda(x),
$$
$$
\Delta_u \lambda(x) = \lambda(x) - \lambda(x_1, \ldots, x_{j-1}, x_j + u, x_{j+1}, \ldots, x_n),
$$
be a “mixed” difference with respect to parallelepiped $[x, x + u]$. Let us take an arbitrary number of non-overlapping parallelepipeds, and form a mixed difference with respect to each of them. Then the Vitali variation is
$$
V(\lambda) = \sup \sum |\Delta_u \lambda(x)|
$$
where the least upper bound is taken over all the sets of such parallelepipeds. For smooth functions $\lambda$
$$
V(\lambda) = \int_{\mathbb{R}^n} \left| \frac{\partial^n \lambda(x)}{\partial x_1 \cdots \partial x_n} \right| \, dx.
$$
The Tonelli variation is something another [To]. Roughly saying, a function is of bounded Tonelli variation if it has a bounded variation in each variable, and these variations are integrable as functions of the rest variables. For smooth function $\lambda$ it is equal to
$$
\int \sum_{j=1}^n \left| \frac{\partial \lambda(x)}{\partial x_j} \right| \, dx.
$$
Let us write $\lambda \in V_0$ if its Vitali variation is bounded and \( \lim_{|x| \to \infty} \lambda(x) = 0 \). In this case the function is of bounded variation with respect to any smaller number of variables.
Theorem 2.3. ([T2, T3]) The following relations hold:

a) For each \( \lambda \in V_0 \), and for every \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \), \( \varepsilon_j > 0 \), \( j = 1, \ldots, n \),

\[
\sup_{0 < |u_j| \leq \frac{\pi}{\varepsilon_j}} \left| \int_{\mathbb{R}^n} \lambda(x) e^{-iux} \, dx - \prod_{j=1}^n \varepsilon_j \sum_k \lambda(\varepsilon_{1k}, \ldots, \varepsilon_{nk}) e^{-i(\varepsilon_{1k} u_1 + \cdots + \varepsilon_{nk} u_n)} \right| \leq CV(\lambda) \sum_{j=1}^n \varepsilon_j \prod_{q \neq j} \frac{1}{|u_q|}.
\]

(2.12)

b) If, moreover, \( \lambda \) has also a bounded Tonelli variation, dominated by \( V(\lambda) \), then in (2.12) it is possible to put \( \frac{1}{1 + |u_j|} \) instead of \( \frac{1}{|u_q|} \).

c) If \( \lambda \) satisfies a) and b) then for \( N = (N_1, \ldots, N_n) \) and \( \frac{k}{N} = (\frac{k_1}{N_1}, \ldots, \frac{k_n}{N_n}) \)

\[
\|L^N_\lambda\| = (2\pi)^{-n} \int_{|x_j| \leq \pi N_j} |\hat{\lambda}(x)| \, dx + \theta V(\lambda) \sum_{j=1}^n \prod_{q \neq j} \ln(N_q + 1), \quad |\theta| \leq C.
\]

(2.13)

In this theorem the constants \( C \) depend only on \( n \), and integrals and sums should be treated in the Cauchy sense. We will return to functions of bounded variation while considering the radial case.

3. Generalizations of the Bochner-Riesz means.

Let us consider another generalization of the linear means described earlier. We mean the estimates (2.3) for more general means of Bochner-Riesz type. They are strongly connected with a support of the function generating these means.

3.1. The strongest estimates from above were obtained by Colzani and Soardi [CoS]. Their method is the direct generalization of that used by Yudin [Y1] and lies in the framework of our topic.

Suppose \( S \subset \mathbb{R}^n \) is an open bounded set whose boundary \( \partial S \) has finite upper Minkowski measure. Let us consider complex-valued bounded functions \( \lambda \) on \( \mathbb{R}^n \) satisfying the following assumptions:

(3.1) \( \lambda(x) = 0 \) if \( x \) does not belong to \( S \);

there exist an integer \( m \geq 0 \) and real numbers \( \alpha > -\frac{1}{2} \) and \( \beta > -\frac{3}{2} \) such that

(3.2) \( \lambda \in C^{m+1}(S) \);

(3.3) \( |D^\xi \lambda(x)| \leq C \rho(x, \partial S)^\alpha \) if \( \xi_1 + \cdots + \xi_n = m \) and \( x \in S \);

(3.4) \( |D^\xi \lambda(x)| \leq C \rho(x, \partial S)^\beta \) if \( \xi_1 + \cdots + \xi_n = m + 1 \) and \( x \in S \).

If (3.1)–(3.4) are satisfied with \( m \geq 1 \), \( \lambda \) must also satisfy

(3.5) \( \lambda \in C^{m-1}(\mathbb{R}^n) \).

Since \( \lambda \) is supposed bounded, we may assume \( \alpha \geq 0 \) whenever \( m = 0 \). Let

(3.6) \( \gamma = \min \left( 1, \alpha + \frac{1}{2}, \beta + \frac{3}{2} \right) \).

If \( \beta = -\frac{1}{2} \) and \( \alpha \geq \frac{1}{2} \), let

(3.7) \( \lambda \in C^{m+2}(S) \);

(3.8) \( |D^\xi \lambda(x)| \leq C \rho(x, \partial S)^{-\frac{3}{2}} \) if \( \xi_1 + \cdots + \xi_n = m + 2 \) and \( x \in S \).
Theorem 3.1. ([CoS]) Let \( S \) be as above and \( \lambda \) satisfies (3.1)-(3.5), and, in addition, (3.7), (3.8) when \( \beta = -\frac{1}{2} \) and \( \alpha \geq \frac{1}{2} \). Let \( p_c = \frac{2n}{n+2(m+\gamma)} \). Then for all \( N > 2 \):

a) If \( m + \gamma \leq \frac{n}{2} \)

\[
\| L^N \lambda \|_p \leq C_p N^{\frac{n}{2} - (m+\gamma)} \quad \text{if} \quad 1 \leq p < p_c,
\]
\[
\| L^N \lambda \|_p \leq C_p N^{\frac{n-1}{p}} \ln^{\frac{1}{p}} N \quad \text{if} \quad p = p_c,
\]
\[
\| L^N \lambda \|_p \leq C_p N^{\frac{p-1}{p}} \quad \text{if} \quad p_c < p \leq 2.
\]

b) If \( m + \gamma > \frac{n}{2} \)

\[\| L^N \lambda \|_1 \leq C.\]

M. Vignati [V] generalized these results to the case of nonisotropic metrics in \( \mathbb{R}^n \). V. A. Yudin [Y3] showed that these estimates cannot be asymptotically improved for \( N \to \infty \) in the class of sets considered.

3.2. Special examination of general conditions for lower estimates was begun in [Y2], where the lower bound \( \ln^n N \) for the order of growth of Lebesgue constants of "all reasonable" partial sums is established, namely, for those generated by convex sets which may contain a certain ball inside. Such an investigation was continued, as it was mentioned above, in [CaS], and then in [L2, L3] (see Theorem 1.3). The recent result from [LRZ] generalizes the left-hand inequality in (2.3) in the spirit of Theorem 1.3.

Let \( S = \text{supp} \lambda \) be the support of a function \( \lambda(x) \), where \( S \) is not necessarily a compactum. In what following we will be interested in functions \( \lambda(x) = \lambda_{r,\alpha}(x) \), which are \( r \)-smooth inside \( S \), and may be represented in a certain neighborhood of \( \partial S \) as follows:

\[
\lambda_{r,\alpha}(x) = f(x)(\rho(x))^\alpha,
\]

where \( f \in C^r(\mathbb{R}^n) \) and does not vanish on \( \partial S \), while \( \rho(x) = 0 \) if \( x \notin S \), and \( \rho(x) = \rho(x, \partial S) \) if \( x \in S \). Notice, that \( \rho(x) \) is a smooth function in a neighborhood of \( \partial S \) when \( x \in S \) (see e.g., [Gi, Appendix B]). It should be mentioned that in [CoS] the following obvious consequence of (3.1)-(3.5) is proved:

Lemma 3.1. Suppose \( S \subset \mathbb{R}^n \) is a bounded open set such that \( S \) has finite upper Minkowski measure and \( \lambda \) is a bounded complex-valued function on \( \mathbb{R}^n \) satisfying (3.1)-(3.5). Then there exists a constant \( C \) such that

\[|\lambda(x)| \leq A \rho(x, \partial S)^{\alpha + m} \quad \text{for all} \quad x \in S.\]

The following theorem shows that the generalizations cited of the norms of the Bochner-Riesz means for upper estimates and for lower estimates, respectively, are not far one from another.

Theorem 3.2. ([LRZ]) Suppose that there exist an open set \( U \) and a hypersurface \( V \) of smoothness \( r > \max(1, \frac{n-1}{2} + \alpha) \), where \( 0 \leq \alpha < \frac{n-1}{2} \), with non-vanishing principal curvatures, such that \( \partial S \cap U = V \). Suppose, further, that in \( U \cap S \) we have \( \lambda(x) = \lambda_{r,\alpha}(x) \). Then there exists a positive constant \( C_{S,\lambda} \) depending only on \( S \) and \( \lambda \) such that

\[\| L^N \lambda \| \geq C_{S,\lambda} N^{\frac{n-1}{2} - \alpha}\]

for large \( N \).

3.3. We want to outline three knot points on which the proof of Theorem 7 is based. The first one was proved in a discussion of the author and Belinskii.
Lemma 3.2. ([L2, LRZ]) Let $K$ be a set in $\mathbb{R}^n$ and $\psi$ be a bounded measurable function with support in $K$. Then for every point $x_0 \in \mathbb{R}^n$, for every ball $B_\delta(x_0)$ of radius $\delta$ centred at $x_0$, and for every function $\varphi$ supported in $B_\delta(x_0)$ and having the Fourier transform integrable over all $\mathbb{R}^n$, there exists a constant $C$, depending only on $\varphi$, such that

$$
\|L^\psi_K\| \geq C\|L^\varphi_{K \cap B_\delta(x_0)}\|.
$$

Proof. We have

$$
(3.10) \quad \|L^\psi_K\| = \sup_{\|f\| \leq 1} \|L^\psi_K(f; \cdot)\| \geq \sup_{\|T_{B_\delta(x_0)}\| \leq 1} \|L^\psi_K(T_{B_\delta(x_0)}; \cdot)\|,
$$

where $T_{B_\delta(x_0)}$ denotes all the trigonometric polynomials with spectrum in $B_\delta(x_0)$. According to [Be1, Corollary 2] the following inequality holds for every $f \in C(\mathbb{T}^n)$:

$$
\left\| L^\phi_{B_\delta(x_0)}(f; \cdot) \right\| \leq (2\pi)^{-n} \|\hat{\varphi}\|_{L^1(\mathbb{R}^n)} \|f\|.
$$

Since the image of $L^\phi_{B_\delta(x_0)}$ is only a part of all polynomials $T_{B_\delta(x_0)}$, it follows from (3.10) that with some constants

$$
\|L^\psi_K\| \geq \sup_{\|L^\phi_{B_\delta(x_0)}(f; \cdot)\| \leq 1} \left\| L^\psi_K \left( L^\phi_{B_\delta(x_0)}(f; \cdot) \right) \right\| \geq C \sup_{\|f\| \leq 1} \left\| L^\psi_{K \cap B_\delta(x_0)}(f; \cdot) \right\| = C \left\| L^\psi_{K \cap B_\delta(x_0)} \right\|.
$$

The lemma is proved. \(\Box\)

This lemma is of certain interest by itself, but mainly as a tool for some estimates from below. It is more or less clear that after its application we can pass to estimates (from below) of the Lebesgue constants of a method of summability, generated by a function with the support possessing in global the properties which are local for the given set in Theorem 7. A similar way to make “global from local” may be found in [Se].

The next step of the proof is the application of Theorem 2.1, more precisely the lower estimate for $p = 1$. After that we need appropriate asymptotic estimates of the Fourier transform of the functions considered. Let us formulate in full such a result strongly based on estimates of singularities of the Radon transform, due to Ramm and Zaslavsky (see [RZ1, RZ2]).

Theorem 3.3. ([LRZ], see also [RZ] and [RK]) Let $S$ be the compact support of a function $\lambda(x) = \lambda_{r, \alpha}(x)$ with $\alpha \geq 0$ and $r > \max(1, \frac{n-1}{2} + \alpha)$. Let $S$ be convex, with the $r$-smooth boundary $\partial S$, and suppose the principal curvatures of $\partial S$ never vanish. Let $\theta \in \mathbb{R}^n$ be a vector on the unit sphere, $x^+(\theta)$ and $x^-(\theta)$ be the (uniquely defined) points of $\partial S$ at which the function $\theta_1 x_1 + \ldots + \theta_n x_n$ attains maximum and minimum on $\partial S$, respectively. Then for $t \to +\infty$

$$
\tilde{\lambda}(t\theta) = t^{-\alpha - \frac{n+1}{2}} \left( \Xi^+ (\theta) e^{itx^+(\theta) \theta} + \Xi^- (\theta) e^{itx^-(\theta) \theta} + o(1) \right),
$$

$$
\Xi^\pm (\theta) = (2\pi)^{\frac{n-1}{2}} \Gamma(\alpha + 1) e^{\pm i\frac{\alpha + 1}{2}} f(x^\pm(\theta)) \left( 1 + o(1) \right),
$$

where $f$ is the spherical mean of a bounded measurable function $f$. If $S$ is strictly convex and $\lambda_{r, \alpha}(x) = \lambda_{r, \alpha}$ is independent of $x$, then

$$
\Xi^\pm (\theta) = (2\pi)^{\frac{n-1}{2}} \Gamma(\alpha + 1) e^{\pm i\frac{\alpha + 1}{2}} f(x^\pm(\theta)) + o(1).
$$

If $S$ is a smooth surface, then

$$
\Xi^\pm (\theta) = (2\pi)^{\frac{n-1}{2}} \Gamma(\alpha + 1) e^{\pm i\frac{\alpha + 1}{2}} f(x^\pm(\theta)) + o(1).
$$

If $S$ is a polyhedron and $\lambda_{r, \alpha}(x) = \lambda_{r, \alpha}$ is independent of $x$, then

$$
\Xi^\pm (\theta) = (2\pi)^{\frac{n-1}{2}} \Gamma(\alpha + 1) e^{\pm i\frac{\alpha + 1}{2}} f(x^\pm(\theta)) + o(1).
$$

If $S$ is a polyhedron and $\lambda_{r, \alpha}(x) = \lambda_{r, \alpha}$ is independent of $x$, then

$$
\Xi^\pm (\theta) = (2\pi)^{\frac{n-1}{2}} \Gamma(\alpha + 1) e^{\pm i\frac{\alpha + 1}{2}} f(x^\pm(\theta)) + o(1).
$$

Theorem 3.3. ([LRZ], see also [RZ] and [RK]) Let $S$ be the compact support of a function $\lambda(x) = \lambda_{r, \alpha}(x)$ with $\alpha \geq 0$ and $r > \max(1, \frac{n-1}{2} + \alpha)$. Let $S$ be convex, with the $r$-smooth boundary $\partial S$, and suppose the principal curvatures of $\partial S$ never vanish. Let $\theta \in \mathbb{R}^n$ be a vector on the unit sphere, $x^+(\theta)$ and $x^-(\theta)$ be the (uniquely defined) points of $\partial S$ at which the function $\theta_1 x_1 + \ldots + \theta_n x_n$ attains maximum and minimum on $\partial S$, respectively. Then for $t \to +\infty$
where the remainder term is small uniformly in $\theta$, and $x^{\pm}(\theta)$ is the Gaussian curvature of $\partial S$ at the points $x^{\pm}(\theta)$, respectively.

This result continues and develops the well-known asymptotic estimate for the characteristic function of a convex set [GGV]. There is an "almost all" gap between Theorem 8 and the result in [P6] in the two-dimensional case. We must mention that many authors use one result of Herz [Hz] for estimates of Fourier transforms. But smoothness assumptions in this work are essentially more restrictive than those in [GGV] (and of course in Theorem 3.3) since the author was interested in sharp estimate for the remainder term. This explains, for example, the smoothness conditions in [CaS] or [Br1, Br2]. Of course we omit a number of technical details and estimates, not easy by no means.

3.4. Let us pay our attention to the following circumstance. One can see that in many results cited the value $\frac{n-1}{2}$ behaved like a star actor on the stage. It is not an occasional event, and this number is called "critical order" for the Bochner-Riesz means. Let us compare Theorem 2.2 with the following well-known result of Stein:

**Theorem 3.4. ([S1])** The following asymptotic formula holds:

\[
\|L_{\frac{R_{n-1}}{N}}\|_{L_1(T^n)\to L_1(T^n)} = \omega_n \ln N + O(1).
\]

This asymptotics was obtained as a corollary to some general estimates of the difference between the corresponding kernel

\[
\sum_{|k|\leq N} R_{n-1} \left( \frac{k}{N} \right) e^{ikx}
\]

and its integral analog. The constant $\omega_n$ was not indicated explicitly. Here the Lebesgue constants of the Bochner-Riesz means lose their power rate of growth, and behave as the Lebesgue constants of one-dimensional partial sums. This likeness is not casual. Before formulating one recent generalization of Theorem 3.4 we want to mention that (3.11) is a relatively simple corollary to Theorem 2.1.

**Corollary 2.2.** The following asymptotic formula holds:

\[
\|L_{\frac{R_{n-1}}{N}}\|_{L_1(T^n)\to L_1(T^n)} = \frac{4\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \ln N + O(1).
\]

**Proof.** ([Be2]) By Theorem 2.1 and (2.5), (2.6), and (2.8) we have

\[
\|L_{\frac{R_{n-1}}{N}}\| = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{1\leq|x|\leq N} \left| \prod_{j=1}^{n} x_j \frac{\cos(|x| - \frac{\pi n}{2})}{2N \sin \frac{x_j}{2N} |x|^n} \right| dx + O(1).
\]

The following relation

\[
\frac{x_j}{2N \sin \frac{x_j}{2N}} - 1 = O\left(\frac{x_j^2}{N^2}\right)
\]
and estimates from the proof of Corollary 2.1 imply

\[ \|L_N^{R_{n-1}}\| = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{1\leq|x|\leq N} \left| \frac{\cos(|x| - \frac{\pi n}{2})}{|x|^n} \right| dx + O(1). \]

We obtain after passage to spherical coordinates

\[ \|L_N^{R_{n-1}}\| = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{1\leq|x|\leq N} \left| \frac{\cos(|x| - \frac{\pi n}{2})}{|x|^n} \right| dx + O(1) \]

\[ = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \frac{2\pi^2}{\Gamma\left(\frac{n}{2}\right)} \int_1^N \left| \frac{\cos(t - \frac{\pi n}{2})}{t} \right| dt + O(1). \]

It is well-known that the last integral is

\[ \frac{2}{\pi} \ln N + O(1) \]

(see e.g., [Z], Vol.1, Ch.2), and this completes the proof. □

Furthermore, again Theorem 2.1 and certain technique connected with Theorem 3.3 allow us to obtain the mentioned generalization as follows.

**Theorem 3.5.** ([L5]) Let \( S \) be the compact support of a function \( \lambda = \lambda_n, n-1 \), with the \( n \)-smooth boundary \( \partial S \). Assume that \( S \) is convex and the principal curvatures of \( \partial S \) never vanish. Then there exists a positive constant \( C_{S,\lambda} \) depending only on \( S \) and \( \lambda \) such that

\[ (3.12) \quad \|L_N^\lambda\|_{L_1(T^n)\to L_1(T^n)} = C_{S,\lambda} \ln N + o(\ln N) \]

for large \( N \).

**Remark 3.1.** The following formula is given in [L5] to calculate \( C_{S,\lambda} \):

\[ C_{S,\lambda} = (2\pi)^{-\frac{n+3}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_0^{2\pi} \int_0^{2\pi} |(-1)^n \phi^+(\theta) e^{it} + \phi^-(\theta)| dt \]

where \( \phi^\pm(\theta) = f(x^\pm(\theta))(x^\pm(\theta))^{\frac{1}{2}} \) (cf. Theorem 3.3). Simple calculations yield for the Lebesgue constants of the usual Bochner-Riesz means the same constant as in Corollary 2.2.

**Remark 3.2.** It is obvious that if, in the conditions of Theorem 10, to take \( \lambda = \lambda_{r,\alpha} \), with \( r > n \) and \( \alpha > \frac{n-1}{2} \), we will obtain \( \|L_\lambda\| = O(1) \) (cf. b) in Theorem 5).
4. "Radial" results.

A lot of attention to the Bochner-Riesz means and certain of their generalizations has been given. But we have not speak yet about one more peculiarity of the Bochner-Riesz means. We mean the fact that they are generated by the function $R_\alpha$ which is radial, that is depending only on $|x|$. Such functions play a special role in Fourier Analysis, and there are many ways to exploit the radiality.

4.1. Now we want to consider one special class of radial functions, close, in many respects, to the Bochner-Riesz means of critical order.

Let $\lambda(x) = \lambda_0(|x|)$ be a radial function satisfying the following conditions:

(4.1) $\lambda_0 \in C[0, \infty)$, $\lambda_0 \in C^{\left[\frac{n-2}{2}\right]}(0, \infty)$,

(4.2) $t^r \lambda_0^{(r)}(t) \to 0$ as $t \to 0$, $r = 1, 2, \ldots, \left[\frac{n-2}{2}\right]$ ($n > 3$),

(4.3) $\Lambda(t) = t^{\frac{n-2}{2}} \lambda_0^{\left(\frac{n-2}{2}\right)}(t)$, $t^r \lambda_0^{(r)}(t) \to 0$ as $t \to \infty$, $r = 0, 1, 2, \ldots, \left[\frac{n-2}{2}\right]$,

(4.4) $\Lambda$ has a bounded variation $V_\Lambda$ on $[0, \infty)$.

Here the fractional derivative is understood in the Weil sense (see e.g., [BE]), namely, for a function $g$ defined on $[0, \infty)$

$$W_\alpha(g; t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty g(u)(u-t)^{\alpha-1} du$$

is the Weil integral of fractional order $\alpha$, and for $0 < \alpha < 1$

$$g^{(\alpha)}(t) = \frac{d}{dt} W_{1-\alpha}(g; t)$$

is the fractional derivative of order $\alpha$. For $\gamma = r + \alpha$, $r = 1, 2, \ldots, 0 < \alpha < 1$,

$$g^{(\gamma)}(t) = \frac{d^r}{dt^r} g^{(\alpha)}(t)$$

is the fractional derivative of order $\gamma$.

**Theorem 4.1.** ([BL1, BL2]) Let $\lambda(x) = \lambda_0(|x|)$ be a radial function satisfying (4.1)–(4.4). Then

(4.5) $\|L_N\|_{L^1(T^n) \to L^1(T^n)} = (2\pi)^{-n} \int_{|x| \leq \pi N} |\hat{\lambda}(x)| \, dx + O(V_\Lambda + |\lambda(0)|)$.

**Proof.** It suffices to prove that the series

(0.3) $\sum \lambda\left(\frac{k}{N}\right) e^{ikx}$
is the Fourier series of an integrable function in order \((0.4)\) to be true. Consider

\[
\left| ||L_N^\lambda|| - (2\pi)^{-n} \int_{|x| \leq \pi N} |\widehat{\lambda}(x)| \, dx - (2\pi)^{-n} |\lambda(0)| \right| 
= R_N \leq (2\pi)^{-n} \int_{T^n} \left| \sum_{k \neq 0} \lambda \left( \frac{k}{N} \right) e^{ikx} - \Phi_N(x) \right| \, dx,
\]

where

\[
\Phi_N(x) = \begin{cases} 
N^n \widehat{\lambda}(Nx), & |x| \leq \pi, \\
0, & x \in T^n \setminus \{ x : |x| \leq \pi \}.
\end{cases}
\]

Let us calculate the \(k\)-th Fourier coefficient of the periodic in each variable continuation of this function. No confusion will result saving the same notation. In [BL1, BL2] the following results were obtained: if a function \(\lambda\) satisfies conditions \((4.1)-(4.4)\) its Fourier transform can be calculated by the following relation:

\[
(4.6) \quad \widehat{\lambda}(x) = \frac{(2\pi)^{\frac{n}{2}} (-1)^{\left| \frac{n}{2} \right|}}{\Gamma\left( \frac{n-1}{2} \right)} |x|^{1-\frac{\nu}{2}} \int_0^\infty \Lambda(t) t^{\frac{\nu}{2}} Q(|x| t) \, dt,
\]

where \(Q(r) = \int_0^1 (1 - s)^{\frac{\nu - 3}{2}} s^{\frac{\nu}{2}} J_{\frac{\nu}{2} - 1} (rs) \, ds\), and the inverse formula holds:

\[
(4.7) \quad \lambda(x) = \lim_{A \to \infty} (2\pi)^{-n} \int_{|u| \leq A} \widehat{\lambda}(u) e^{ixu} \, du.
\]

For generalizations of this result see [L6], [L7]. For \(|k| > 0\) \((4.6)\) and \((4.7)\) yield

\[
\widehat{\Phi}_N(k) = (2\pi)^{-n} \int_{|u| \leq \pi} N^n \widehat{\lambda}(Nu) e^{-iku} \, du
= (2\pi)^{-n} \int_{|u| \leq \pi N} \widehat{\lambda}(u) e^{-i\frac{k}{N} u} \, du
= \lambda\left( -\frac{k}{N} \right) - (2\pi)^{-n} \int_{|u| > \pi N} \widehat{\lambda}(u) e^{-i\frac{k}{N} u} \, du.
\]

For \(k = 0\), the passage to spherical coordinates and \((4.6)\) yield

\[
\widehat{\Phi}_N(0) = (2\pi)^{-n} \int_{|u| \leq \pi N} \widehat{\lambda}(u) \, du = \frac{2^{n-1} (-1)^{\left| \frac{n}{2} \right|}}{\Gamma\left( \frac{n}{2} \right) \Gamma\left( \frac{n-1}{2} \right)} \int_0^{\pi^2} \int_0^\infty \Lambda(t) t^{\frac{\nu}{2}} Q(rt) \, dt.
\]

Use the well-known formula (see e.g., [BE], 7.2.8(50),(51))

\[
(4.8) \quad \frac{d}{dt} t^{\pm \nu} J_\nu(t) = \pm t^{\pm \nu} J_\nu(t) \pm 1(t).
\]
and denote $q(r) = \frac{1}{0} (1 - s)^{\frac{n-3}{2}} s^{\frac{n-1}{2}} J_{\frac{n-1}{2}}(rs) ds$. This implies

$$\hat{\Phi}_N(0) = \frac{2^{n-1}(-1)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}) \Gamma(\frac{n-1}{2})} (\pi N)^{\frac{n}{2}} \int_0^{\pi N} r^{\frac{n}{2}} dr \int_0^\infty \Lambda(t) t^{\frac{n}{2} - 1} \left[ q(\pi Nt) - \alpha_2(\pi Nt)^{-\frac{n}{2}} \right] dt.$$  

For $q$ the following asymptotic relation was obtained in [BL2] (see also [L6, L7]):

$$(4.9) \quad q(r) = \alpha_1 r^{-\frac{n+1}{2}} J_{n-\frac{1}{2}}(r) + \alpha_2 r^{-\frac{n}{2}} + O(r^{-\frac{n+2}{2}})$$

as $r \to \infty$, where $\alpha_1$ and $\alpha_2$ are some constants. Integrating by parts one obtains

$$\hat{\Phi}_N(0) = \frac{2^{n-1}(-1)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}) \Gamma(\frac{n-1}{2})} (\pi N)^{\frac{n}{2}} \left\{ \Lambda(t) \int_0^t r^{\frac{n}{2} - 1} \left[ q(\pi Nr) - \alpha_2(\pi Nr)^{-\frac{n}{2}} \right] dr \right\} \bigg|_0^\infty$$

$$- \int_0^\infty \left\{ \int_0^t r^{\frac{n}{2} - 1} \left[ q(\pi Nr) - \alpha_2(\pi Nr)^{-\frac{n}{2}} \right] dr \right\} d\Lambda(t).$$

In order to get $|\hat{\Phi}_N(0)| \leq CV_\Lambda$ it suffices, taking into account (4.4), to prove the boundedness of the value

$$\sup_{N, t} (\pi N)^{\frac{n}{2}} \left| \int_0^t r^{\frac{n}{2} - 1} \left[ q(\pi Nr) - \alpha_2(\pi Nr)^{-\frac{n}{2}} \right] dr \right|$$

$$= \sup_{N, t; \pi Nt > 1} \left| \int_1^{\pi Nt} r^{\frac{n}{2} - 1} \left[ q(\pi Nr) - \alpha_2(\pi Nr)^{-\frac{n}{2}} \right] dr \right| + O(1).$$

It follows from (4.9) that the right-hand side is equal to

$$\sup_{N, t; \pi Nt > 1} \left| \alpha_1 \int_1^{\pi Nt} r^{-\frac{n}{2}} J_{n-\frac{1}{2}}(r) dr + O(\int_1^{\pi Nt} r^{-2} dr) \right| + O(1)$$

$$= \sup_{N, t; \pi Nt > 1} \left| \alpha_1 \int_1^{\pi Nt} r^{-\frac{n}{2}} J_{n-\frac{1}{2}}(r) dr \right| + O(1).$$

The asymptotic formula (2.6) makes the claimed estimate obvious. Thus

$$R_N \leq (2\pi)^{-n} \left| \sum_{k \in \mathbb{Z} 

\text{such that } 2\pi k \neq \pi Nt} \int_0^{2\pi} \int_0^{2\pi} \hat{\lambda}(u) e^{-i\frac{k}{N} u} du \right| e^{ikx} dx + CV_\Lambda.$$
Apply the Cauchy-Schwarz inequality to the outer integral and the Parseval equality. We get

\[ R_N \leq C \left\{ \sum_{k \neq 0} \left| k \right|^{2-n} \left| \int_{\pi N}^{\infty} J_{\frac{n}{2}-1} \left( \frac{|k|}{N} r \right) r \, dr \right| \right\}^{\frac{1}{2}} + CV_A. \]

The Cauchy-Poisson formula (see e.g., [Bc1], Th. 56) and (4.6) yield

\[ R_N \leq CN^{\frac{n}{2}-1} \left\{ \sum_{k \neq 0} \left| k \right|^{2-n} \left| \int_{\pi N}^{\infty} J_{\frac{n}{2}-1} \left( \frac{|k|}{N} r \right) r \, dr \right| \right\}^{\frac{1}{2}} + CV_A. \]

Integration by parts in \( t \) implies:

\[ R_N \leq CN^{\frac{n}{2}-1} \left\{ \sum_{k \neq 0} \left| k \right|^{2-n} \left| \int_{\pi N}^{\infty} J_{\frac{n}{2}-1} \left( \frac{|k|}{N} r \right) \Lambda(t) \frac{d}{dt} q(rt) \, dt \right| \right\}^{\frac{1}{2}} + CV_A. \]

After applying generalized Minkowski’s inequality and (4.4) we get, as above, that it suffices to prove the boundedness of the value

\[ \sup_{N,t} N^{\frac{n}{2}-1} \left\{ \sum_{k \neq 0} \left| k \right|^{2-n} \left| \int_{\pi N}^{\infty} J_{\frac{n}{2}-1} \left( \frac{|k|}{N} r \right) q(rt) \, dr \right| \right\}^{\frac{1}{2}} \]

\[ = \sup_{N,t} N^{\frac{n}{2}-1} \left\{ \sum_{k \neq 0} \left| k \right|^{2-n} \left| t^{\frac{n}{2}} \int_{\pi N}^{\infty} J_{\frac{n}{2}-1} \left( \frac{|k|}{N} r \right) \frac{r}{2} J_{\frac{n}{2}} \left( \frac{|k|}{N} r \right) \, dr \right| \right\}^{\frac{1}{2}}. \]

Integrate by parts using (4.8) and obtain

\[ \sup_{N,t} N^{\frac{n}{2}} \left\{ \sum_{k \neq 0} \left| k \right|^{-n} \left| t^{\frac{n}{2}} \int_{\pi N}^{\infty} J_{\frac{n}{2}} \left( \frac{|k|}{N} r \right) q(rt) \, dr \right|^{\frac{1}{2}} \right\} + t^{\frac{n}{2}+1} \int_{\pi N}^{\infty} J_{\frac{n}{2}} \left( \frac{|k|}{N} r \right) \frac{1}{2} \left| (1-s) \frac{n-3}{2} s^{\frac{n}{2}} J_{\frac{n}{2}+1}(rts) \, ds \right|^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \]

Relations (4.8) and (4.9) and convergence of the series \( \sum_{k \neq 0} \left| k \right|^{-n-1} \) imply the boundedness of the integrated terms. Further, integration by parts and (4.10) yield

\[ \int_{0}^{1} (1-s) \frac{n-3}{2} s^{\frac{n}{2}} J_{\frac{n}{2}+1}(rts) \, ds = \alpha_3 r^{-\frac{n}{2}} \sin \left( r - \frac{\pi n}{2} \right) + O(r^{-\frac{n+2}{2}}). \]
Estimates with the remainder term are trivial. Let us estimate
\[
\sup_{N,t} N^{-\frac{n+1}{2}} \left\{ \sum_{k \neq 0} |k|^{-n} \left| \int \frac{1}{r} \int_0^\infty r^{-\frac{n+1}{2}} j_n \left( \frac{|k|}{N} r \right) \sin(r t - \frac{\pi n}{2}) \, dr \right| \right\}^{\frac{1}{2}}.
\]

Again integrate by parts and get
\[
\sup_{N,t} N^{-\frac{n+1}{2}} \left\{ \sum_{k \neq 0} |k|^{-n} \left| \int \frac{1}{r} \int_0^\infty r^{-\frac{n+1}{2}} j_n \left( \frac{|k|}{N} r \right) \cos(r t - \frac{\pi n}{2}) \, dr \right| \right\}^{\frac{1}{2}}.
\]

The integrated terms are simply estimated. Apply now (2.6) to the last integral.

Estimates for the remainder term are obvious. Using also simple trigonometric
identities we have to estimate
\[
(4.10) \quad \sup_{N,t} N^{-\frac{n+1}{2}} \left\{ \sum_{k \neq 0} |k|^{1-n} \left| \int \frac{1}{r} \int_0^\infty r^{-\frac{n+1}{2}} \sin \left( \frac{|k|}{N} t - \frac{\pi n}{t} \right) \, dr \right| \right\}^{\frac{1}{2}}.
\]

Notice that estimates for similar values with \( \sin \left( \frac{|k|}{N} t + \frac{\pi n}{t} \right) \) or \( \cos \left( \frac{|k|}{N} t - \frac{\pi n}{t} \right) \) on place of \( \sin \left( \frac{|k|}{N} t - \frac{\pi n}{t} \right) \) are the same. Assume that \( Nt \) is big enough. Split the sum in (4.10) into three ones: 1 \( \leq |k| < Nt - 1 \), \( Nt - 1 \leq |k| \leq Nt + 1 \), and \( Nt + 1 < |k| < \infty \).

Integration by parts implies for the integral in (4.10) the estimate
\[
N^{-\frac{n+1}{2}} \left| \frac{|k|}{N} - t \right|^{-1} = N^{-\frac{n+1}{2}} \left| \frac{k - Nt}{N} \right|^{-1}.
\]

Therefore the boundedness of the following sums:
\[
\sum_{1 \leq |k| < Nt - 1} |k|^{1-n} \left( Nt - |k| \right)^{-2} \quad \text{and} \quad \sum_{Nt + 1 < |k| < \infty} |k|^{1-n} \left( |k| - Nt \right)^{-2}
\]

has to be established when estimating with respect to the first and third domains.

This is easy to see passing to integrals instead of sums. For the second one we obtain
\[
\sup_{N,t} N^{-\frac{n+1}{2}} \left\{ \sum_{Nt-1 \leq |k| \leq Nt+1} |k|^{1-n} \left| \int_0^\infty \int_0^\infty r^{-\frac{n+1}{2}} \sin \left( \frac{|k|}{N} t - \frac{\pi n}{t} \right) \, dr \right| \right\}^{\frac{1}{2}}
\]
\[
\leq N^{-\frac{n+1}{2}} \left\{ \sum_{Nt-1 \leq |k| \leq Nt+1} |k|^{1-n} \left| \int_0^\infty r^{-\frac{n+1}{2}} \, dr \right| \right\}^{\frac{1}{2}}
\]
\[
\leq C \sum_{Nt-1 \leq |k| \leq Nt+1} |k|^{1-n} \leq C.
\]
When $Nt$ is small similar estimates valid after splitting the sum into two ones: $1 \leq |k| \leq 3$ and $3 < |k| < \infty$. The proof is complete. □

**Remark 4.1.** One can see that (0.3) is the Fourier series of a function not only from $L^1(T^n)$ but from $L^2(T^n)$ as well.

**Remark 4.2.** Observe that besides other applications, say, to approximation on the class of functions with bounded polyharmonic operator, Theorem 4.1 allows to obtain (3.11) as a simple corollary once more. Indeed, conditions (4.1)–(4.4) are verified easily. Then the estimates are similar to those in the proof of Corollary 2.2, and of course with the same constant.

So we have several different approaches with (3.11) as intersection. This fact cannot be senseless?!

4.2. One of the features of radial functions is that they combine, in a certain sense, some properties of the multi-dimensional case and some properties of the one-dimensional case. In our situation it may be expressed by the following relation.

**Theorem 4.2.** ([BL1]) Let $\lambda$ be a function satisfying (4.1)–(4.4) and, moreover,

$$
\int_0^1 \frac{|\Lambda(t)|}{t} \, dt < \infty.
$$

Then we have for $n \geq 2$

$$
\int_{1 \leq |x| \leq N} \hat{\lambda}(x) \, dx = \frac{2^{n+1} \pi^n}{\Gamma\left(\frac{n}{2}\right)} \int_1^N \left| \int_0^\infty \Lambda(t) \sin(st - \frac{\pi n}{2}) \, dt \right| ds
$$

$$
+ O(V_\Lambda + ||\Lambda||_{C[0,\infty)} + \int_0^1 \frac{|\Lambda(t)|}{t} \, dt).
$$

**Remark 4.3.** The condition (4.12) is sharp and cannot be removed.

Hence, we can apply many one-dimensional results dealing with the behavior of Fourier transforms to such functions $\lambda$. Let us give such an example.

**Proposition 4.1.** ([BL1]) If $\lambda$ satisfy (4.1)–(4.4) and (4.12), and $\Lambda$ has at least one point of discontinuity, then

$$
\|L^\lambda_N\|_{L^1(T^n) \to L^1(T^n)} = M(\Lambda) \ln N + o(\ln N),
$$

where $M(\Lambda)$ is an average of some almost periodic function built in accordance with $\Lambda$.

This result as well as Theorem 4.1 are generalizations of one-dimensional results in [Be0].

Some other conditions allowing a similar connection between the radial case and the one-dimensional one were obtained by Podkorytov. Let us formulate them.
Theorem 4.3. ([P4]) Let \( \lambda_0 \in C[0, \infty) \) and \( \lambda_0(t) = 0 \) for \( t \geq 1 \). Then the following integrals converge simultaneously:

\[
\int_{\mathbb{R}^n} |\hat{\lambda}(x)| \, dx
\]

and

\[
\int_0^\infty \left| \int_0^1 s^{\frac{n-1}{2}} \lambda_0(t) \cos(2\pi st - \frac{\pi(n-1)}{4}) \, dt \right| \, ds.
\]

If to compare the latter two theorems, one can see that conditions (4.1)–(4.4) and (4.12) is the price payed for \( \lambda \) should not be necessarily boundedly supported and having the integrable Fourier transform.

4.3. Let us describe one more ”radial” result due to Trigub. The estimate (2.11) plays an important role in its proof. Consider a function \( \lambda_0(t) \) in \([0, \pi]\) and expand it in the cosine series:

\[
\lambda_0(t) \sim \sum_{j=0}^{\infty} a_j \cos jx.
\]

Theorem 4.4. ([T2]) Let \( \lambda_0 \in C(\frac{n-1}{2})[0, \pi] \), and \( \lambda_0^{(r)}(\pi) = 0 \), \( 0 \leq r \leq \lfloor \frac{n-1}{2} \rfloor \). Then

\[
(4.14) \quad \sup_N \int_{\mathbb{T}^n} \left| \sum_{|k| \leq N} \lambda_0 \left( \frac{|k|\pi}{N} \right) e^{ikx} \right| \, dx \leq C \sum_{j=0}^{\infty} j^{\frac{n-1}{2}} |a_j| \ln(j+1).
\]

It is supposed that the series on the right-hand side converges and \( C \) depends only on \( n \). It suffices to supplement (4.14) with the condition \( \lambda_0(0) = 1 \) for the summability on the whole class of periodic continuous functions.

It may be shown that for \( a_j, j \geq 1 \), with alternating signs, the opposite inequality holds provided \( n = 1 \pmod{4} \). This generalizes the corresponding one-dimensional result [T1]. It is possible to consider coefficients \( b_j \) of the sine expansion of \( \lambda_0 \), instead of \( a_j \), as well.

5. ”Polyhedral” results.

5.1. Let us quote another extract from the book [DC]: ”Concentric polygons are an obvious thing to try, but this turns out to be no more interesting than repeating several one-dimensional results. It doesn’t give any new mathematics, and it avoids having to think deeply about Fefferman’s result.\(^1\) To avoid thinking about a subject is almost always a mistake; at best you are in for some big surprises later on”.

This passage is not absolute truth even if one speaks about polygons (or, maybe more exactly, polyhedra) with the sides parallel to coordinate planes. In this case one frequently obtains nothing more than the product of one-dimensional estimates. But even there the above-mentioned quotation contradicts the fact that there exists another Fefferman’s bright result [F1], which gives an example of continuous

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\(^1\)The famous solution of the multiplier problem for the ball in [F2] is meant.
function with rectangularly divergent partial sums. And if to consider more general objects being within the frame of "polyhedral" case, one can find many non-trivial problems. We will touch some of them connected with our topic.

We must say that, in general, this case has a "logarithmic" nature. For the case of arbitrary parallelogram it was shown by Belinskii [Be2]; that the same method proves the more general case was also mentioned here. This was realized later in [P3, Bb]. More precisely, there exist two positive constants $C_1, C_2$, with $C_1 < C_2$, such that for each polyhedron $E$

\begin{equation}
C_1 \ln^N N \leq \int_{T^2} | \sum_{k \in NE} e^{ikx} | dx \leq C_2 \ln^N N.
\end{equation}

Thus, we see an essential difference between this case and the "ball" case (1.2). In the latter case the Lebesgue constants have a power growth. We intend to pay our attention to two important problems around polyhedra.

5.2. One of them deals with a quite natural question: whether some sets of spectrum of partial sums exist for which the norms of the corresponding operators (1.1) have an intermediate (between (5.1) and (1.2)) rate of growth with respect to $N$-dilations of these sets. The positive answer to this question was given by Podkorytov (similar results were given in [YY2]).

Let $C_1$ and $C_2$ denote, as above, positive constants such that $C_1 < C_2$.

**Theorem 5.1.** ([P4])

1. For any $p > 2$ there exists a compact, convex set $E$ for which

\begin{equation}
C_1 \ln^p N \leq \int_{T^2} | \sum_{k \in NE} e^{ikx} | dx \leq C_2 \ln^p N, \quad N \geq 2.
\end{equation}

2. For any $p \in (0, \frac{1}{2})$ and $\alpha > 1$ there exists a compact, convex set $E$ for which

\begin{equation}
C_1 N^p \ln^{-\alpha p} N \leq \int_{T^2} | \sum_{k \in NE} e^{ikx} | dx \leq C_2 N^p \ln^{2-2p} N, \quad N \geq 2.
\end{equation}

It is well worth noticing that these sets look like polyhedra with an infinite number of sides (of course, becoming smaller and smaller), and the sketch of the proof, outlined when proving Theorem 1, was realized there with taking into account very subtle technical peculiarities.

5.3. The second question is rather natural too and asks whether it is possible to write a certain asymptotic relation instead of (5.1). Some partial cases were investigated by Daugavet [D], Kuznecova [Ku], Skopina [Sk0, Sk2]. For example Kuznetsova generalized Daugavet’s result as follows. What tells both these results from many others is that not dilations of certain fixed domain are taken. Namely, the following result is true.

**Theorem 5.2.** ([Ku1, Ku2]) Let $B_{N_1, N_2} = \{(k_1, k_2) : |k_1| \frac{N_1}{N_2} + |k_2| \frac{N_2}{N_1} \leq 1\}$. The asymptotic equality

\[ ||S_{B_{N_1, N_2}} || = 32 \pi^{-4} \ln N_1 \ln N_2 - 16 \pi^{-4} \ln^2 N_2 + O(\ln N_2) \]
holds uniformly with respect to all natural \( N_1, N_2 \), and \( l = \frac{N_2}{N_1} \).

The case \( l = 1 \) is the mentioned result of Daugavet.

An unexpected result was obtained again by Podkorytov [P6]. He has shown that there are two main cases. The first one, to which the afore-mentioned asymptotic results may be referred, deals with the polygons (we are speaking about two-dimensional results) with integer, or rational vertices. Of course, those times any constant, the same for all vertices, are appropriate. In this case one can show that the estimates change insignificantly if, instead of sums, to consider the corresponding integrals, that is, the Fourier transform of the indicator function of the corresponding set. This circumstance allows to obtain the logarithmic asymptotics, namely, \( \int_{T^2} |\hat{\chi}_{NE}(x)|\,dx \) is equivalent to \( C \ln^2 N \).

In the second case, that is, for any dilation at least one vertex has some irrational coordinates, the situation changes qualitatively: the upper limit and the lower limit, as \( N \to \infty \), are different, and a limit of \( \int_{T^2} \sum_{k \in NE} e^{ikx} \,dx \), as \( N \to \infty \), does not exist. In other words, in this case the behavior of the Fourier transform of the indicator function of \( NE \) does not express the nature of the behavior of the corresponding partial sums. In [P6] the quantitative estimate of this phenomenon is given at once. Namely, for the triangles

\[
E = E_\alpha = \{(u, v) : 0 \leq u \leq 1, \quad 0 \leq v \leq \alpha u\}
\]

the following theorem holds.

**Theorem 5.3.**

1. \( \int_{T^2} \sum_{k \in NE_\alpha} e^{ikx} \,dx = \int_{T^2} |\hat{\chi}_{NE_\alpha}(x)|\,dx + \int_0^{2\pi} \sum_{j=0}^N \{\alpha j\} e^{ijt} \,dt + O(\ln N \ln \ln N) \)

   where \( \{\ldots\} \) is the symbol of the fractional part.

2. There exists irrational \( \alpha \) such that

\[
\lim_{N \to \infty} \frac{1}{\ln^2 N} \int_0^{2\pi} \sum_{0 \leq j \leq N} \{\alpha j\} e^{ijt} \,dt > 0.
\]

The main defect of this theorem is that it is true only for \( \alpha \) from very scarce set and nothing know about other \( \alpha \). In a recent paper of F. Nazarov and A. Podkorytov [NP] this uncertainty is partly removed. Namely, the following is true. Denote by \( I_N(\alpha) \) the integral from (5.4).

**Statement 5.1.** Let \( \alpha \) be irrational. Then

1. \( 0 < C_1 \leq \lim_{N \to \infty} \frac{I_N(\alpha)}{\ln^2 N} \leq C_2. \)

2. \( \lim_{N \to \infty} \frac{I_N(\alpha)}{\ln^2 N} = 0 \) if and only if \( \alpha \) is a Liouville number, that is if and only if for each \( M > 0 \) there exist fractions \( \frac{p}{q} \) (\( q \geq 2 \)) such that \( |\alpha - \frac{p}{q}| \leq \frac{1}{q^M} \).

3. If \( |\alpha - \frac{p}{q}| \leq \frac{1}{q^M} \) for some \( M > 2 \) and as many fractions \( \frac{p}{q} \) as infinite (\( q \geq 2 \)), then the fraction \( I_N(\alpha) \) has no limit as \( N \to \infty \).
4. The integral $I_N(\alpha)$ is concentrated on a set of small measure, namely for all $N \geq 2$ and $\alpha$ irrational there exists a set $E = E(N, \alpha) \subset \mathbb{T}$ such that $\operatorname{mes}(E) \leq e^{-\sqrt{\ln N}}$ while
\[
\int_{\mathbb{T}\setminus E} \left| \sum_{0 \leq j \leq N} \{\alpha j\} e^{ijt} \right| dt \leq C \ln^2 N.
\]

5. There exist numbers $0 < \omega \leq \Omega < \infty$ such that for almost all $\alpha$
\[
\omega = \lim_{N \to \infty} \frac{I_N(\alpha)}{\ln^2 N} \quad \text{and} \quad \Omega = \lim_{N \to \infty} \frac{I_N(\alpha)}{\ln^2 N}.
\]

5.4. Observe that Podkorytov in [P2] and Skopina in [Sk1, Sk2] gave some asymptotic estimates for more general linear means in the cases which we may treat as "polyhedral" as well. Let
\[
\rho(x) = \rho_E(x) = \inf\{\alpha > 0 : \frac{x}{\alpha} \in E\}
\]
be the Minkowski functional of a set $E$ and
\[
L^\lambda_N(f; x) = L^\lambda_N(f; x) = \sum_{k \in NE} \lambda\left(\frac{\rho(k)}{N}\right) \hat{f}(k)e^{ikx}.
\]

**Theorem 5.4.** ([P2]) Let $E$ be a polyhedron starlike with respect to the origin, which is an interior point of it, and $\lambda \in C[0, \infty)$ be supported on $[0, 1]$.

1. If the extension of at least one of the faces of the polyhedron $E$ passes through the origin, then
\[
\sup_{N} \|L^\lambda_N\| = \infty
\]
and consequently there exists an $f \in C(\mathbb{T}^n)$ such that
\[
\lim_{N \to \infty} |L^\lambda_N(f; 0)| = \infty.
\]

2. If the extension of all faces do not pass through the origin, then the convergence of the integral
\[
F_n(\lambda) = \int_{\mathbb{R}} |d\hat{\lambda}(r)| \frac{\ln^{n-1}(2 + |r|)}{1 + |r|} dr,
\]
where
\[
d\hat{\lambda}(r) = \int_{0}^{1} e^{-irt} d\lambda(t),
\]
is sufficient for the norms $\|L^\lambda_N\|$ to be bounded, and consequently $L^\lambda_N(f; \cdot)$ converge uniformly to $f$ as $N \to \infty$ for all $f \in C(\mathbb{T}^n)$.

Some results for "polyhedral" functions $\lambda$ are obtained in [Sk2, Sk3] in the form like in Theorem 4.1. In particular, the following relation holds.
**Theorem 5.5.** ([Sk3]) Let $E$ be an $n$-dimensional polyhedron with rational vertices, starlike with respect to the origin, and the origin does not lie at extension of any face of the polyhedron. Let $\lambda(x) = \lambda_E$. Then

$$\|L_N^\lambda\|_{L_1(T^n) \to L_1(T^n)} = (2\pi)^{-n} \int_{N^n} |\hat{\lambda}(x)| \, dx + O(\lambda_0 + |\lambda_0(0)|) \ln^{n-1} N.$$ 

On the base of this theorem, it is possible to find the main term of

$$\|L_N^\lambda\|_{L_1(T^n) \to L_1(T^n)}$$

in the form suitable for calculations. It is shown in [Sk3], that:

*If $E$ is a convex symmetric $2l$-polygon, and $\lambda_0 \in C[0,1] \cap C^1[0,1)$ is such that $\lambda_0(t) \geq 0$, $\lambda_0(1) = 0$, $\lambda_0'(t)$ and $(t-1)\lambda_0'(t)$ is monotone decreasing, then

$$\|L_N^\lambda\|_{L_1(T^2) \to L_1(T^2)} = \frac{16l}{\pi^4} \int_1^N \frac{\lambda_0(1 - \frac{1}{x}) \ln x}{x} \, dx + O\left(\int_1^N \frac{\lambda_0(1 - \frac{1}{x})}{x} \, dx + \lambda_0(0)\right).$$

This allows us to obtain the logarithmic asymptotics providing some estimates of remainder values. It is found that the constant in the main term depends on geometric properties of the polyhedron. It is shown in [Sk1], that the Lebesgue constants grow as $\left(\frac{2}{\pi}\right)^{2n} \ln^n N$ for parallelepipeds, and as $\frac{2(n+1)^2}{\pi^{n+1}} \ln^n N$ for simplexes. More precisely, let for $N = 0, 1, 2, \ldots$, and $0 \leq p \leq N$ the means $L_N^\lambda$ are defined as follows:

$$\lambda(x) = \begin{cases} 1, & \text{for } x \in (N-p)E, \\ \frac{N+1-p(x)}{p+1}, & \text{for } x \in NE \setminus (N-p)E, \\ 0, & \text{for } x \not\in NE, \end{cases}$$

where $E$ is the same as in Theorem 5.5. She proved that the norms of such operators are

$$\|L_N^\lambda_E\| = (2\pi)^{-n} \int_{T^n} |\hat{\lambda}(x)| \, dx + \Sigma,$$

where

$$|\Sigma| \leq C_{P,n} \frac{1}{p+1} (\ln(N+2))^{n-1}.$$ 

6. "Hyperbolic" results.

Since the appearance of Babenko’s paper [Ba1] interest has continued in various questions of Approximation Theory and Fourier Analysis in $\mathbb{R}^n$ connected with studying linear means with harmonics in “hyperbolic crosses”

$$\Gamma(N, \gamma) = \{k \in \mathbb{Z}^n : h(N, k, \gamma) = \prod_{j=1}^n (|k_j|/N)^{\gamma_j} \leq 1, \quad \gamma_j > 0, j = 1, \ldots, n\}.$$
We are interested in the hyperbolic means of Bochner-Riesz type of order $\alpha \geq 0$

$$L_\Gamma^{\alpha}(N,\gamma) : f(x) \mapsto \sum_{k \in \Gamma(N,\gamma)} (1 - h(N,k,\gamma))^\alpha \hat{f}(k)e^{ik\cdot x}.$$ 

Hyperbolic Bochner-Riesz means (for the two-dimensional Fourier integrals with $\gamma_1 = \gamma_2 = 2$) appeared firstly in the paper of El-Kohen [EK] in connection with the study of its $L^p$-norms. By the way, his result was not sharp, and shortly after was strengthened by Carbery [C].

Hyperbolic partial sums $L_\Gamma(N,\gamma) = L_0^{0}(N,\gamma)$ were investigated separately earlier. 

The exact degree of growth for them $\|L_\Gamma^{\alpha}(N,\gamma)\| \asymp N^{n-1/2}$ (cf. Theorem 1.1) was established in the two-dimensional case independently by Belinskii [Be2] and by A.A. and V.A. Yudins [YY1], and afterwards was generalized to the case of arbitrary dimension by Liflyand [L1]. For $\alpha > 0$ the following relations hold.

**Theorem 6.1.** ([L4])

1) For $\alpha < \frac{n-1}{2}$ we have $\|L_\Gamma^{\alpha}(N,\gamma)\| \asymp N^{n-1/2-\alpha}$.

2) $\|L_\Gamma^{\frac{n-1}{2}}(N,\gamma)\| = \omega_{n,\gamma} \ln N + O(\ln^{n-1} N)$.

3) For $\alpha > \frac{n-1}{2}$ we have $\|L_\Gamma^{\alpha}(N,\gamma)\| = \omega_{n,\gamma,\alpha} \ln^{n-1} N + O(\ln^{n-2} N)$.

Here and below $\omega$ with subscripts denotes, generally saying, different constants depending only on the indicated indices.

Observe that the critical order $\frac{n-1}{2}$ is the same as in the spherical case. But if for the values lower than the critical one the orders of growth of the Lebesgue constants coincide, the difference between (3.11) and 2) in Theorem 6.1 is obvious as well as for orders greater than $\frac{n-1}{2}$: in this case the Lebesgue constants of the usual Bochner-Riesz spherical means are bounded. The lower estimate in the case 1) follows also from Theorem 3.2. In order to establish Theorem 6.1, especially 2) and 3), we need the following

**Theorem 6.2.** ([L4]) For the norms of operators

$$\bar{L}_\Gamma^{\alpha}(N,\gamma) : f(x) \mapsto \sum_{|k_j| \leq N, j=1,...,n} (1 - h(N,k,\gamma))^\alpha \hat{f}(k)e^{ik\cdot x}$$

the following asymptotic equality is true

$$\|\bar{L}_\Gamma^{\alpha}(N,\gamma)\| = \omega_{n,\gamma,\alpha} \ln^{n-1} N + O(\ln^{n-2} N).$$

This is a strengthening of Kivinukk’s result [Ki], who was the first found the effect of influence of the smoothness at the corner points on the order drop of a logarithmic growth, as compared with the Lebesgue constants of cubic partial sums, and established double-sided ordinal inequalities.

It should be said that these theorems are proved by step by step passage from sums to corresponding integrals. This leads to the Fourier transform of a function generating the method of summability under consideration.
7. Appendix

In this section we collected results on Lebesgue constants which are not (explicitly) connected with the Fourier transform methods. Let us start with one result due to Trigub [T4].

Let \( \{ e_j \}_{j=1}^n \) be the standard basis in \( \mathbb{R}^n \), \( M_0 = (1, \ldots, n) \), and \( q = \sum q_j e_j \) where the \( q_j \) are natural numbers \( (j \in M_0) \); analogously \( h = \sum h_j e_j \) where the \( h_j \) are also natural numbers. Set

\[
\Delta_h \lambda_k = \lambda_k - \lambda_{k+h_j e_j}
\]

(the difference operator with stepsize \( h_j \) in the direction \( e_j \)) and

\[
\Delta^q_h \lambda_k = \left( \prod_{j \in M_0} \Delta^q_{h_j} \right) \lambda_k
\]

("mixed" difference in the direction of all axes).

**Theorem 7.1.** For every \( p \in [1, 2) \) and \( q \), there exists a constant \( C \), depending only on \( p, q \) and \( m \), such that

\[
\int_{\mathbb{T}^n} \left| \sum_{-N_j \leq k_j \leq N_j} \lambda_k e^{ikx} \right|^p dx \leq C \prod_{j} (N_j+1)^{\frac{p-2}{2}} \sum_{0 \leq s_j \leq [\log_2(N_j+1)]} \frac{2^{\frac{2-p}{2} s_j} \sum_{k} |\Delta^q_h \lambda_k|^2}{2^s_j}
\]

where \( \lambda_k \) is taken to equal 0 for \( k_j \neq [-N_j, N_j] \) in the sum \( \sum_k \), while \( h = h(s, N) \) is defined by the following conditions

\[
\frac{N_j+1}{3 \cdot 2^s_j} \leq h_j \leq \frac{5(N_j+1)}{6 \cdot 2^s_j}, \quad \frac{N_j+1}{3 \cdot 2^s_j} \leq h_j \leq \frac{N_j+1}{2^s_j}
\]

according as \( s_j < [\log_2(N_j+1)] \) or \( s_j = [\log_2(N_j+1)] \).

In several corollaries sufficient conditions are given to ensure that the Lebesgue constants have a given rate of growth. This is done in terms of smoothness of a function generating the sequence \( \{ \lambda_k \} \), namely \( \lambda_k = \lambda_N, k = \lambda_N(k_1, \ldots, k_n) \).

Let us go on to results in which the operator of taking of partial sums is unbounded. The first one is due to Belinskii [Be5] (see also [MP]).

Let \( l_1, l_2, \ldots, l_k \), with \( k < n \), be a family of linear independent vectors in \( \mathbb{R}^n \). Consider the sets

\( P_j = \{ m \in \mathbb{Z}^n : |l_j m| \leq 1 \} \)

and put

\[
P = \bigcap_{j=1}^k P_j
\]

and

\[
P_0 = \bigcap_{j=1}^k \{ x \in \mathbb{R}^n : l_j x = 0 \}.
\]
Theorem 7.2. 1. If in $P_0$ there exists a sublattice of $\mathbb{Z}^n$ of dimension $n - k$, then
\[ ||S_{NP}|| \asymp \ln^k N \quad \text{as} \quad N \to \infty. \]

2. If there not exist anyone such sublattice in $P_0$ then the operator $S_{NP}$ is unbounded for each $N > 0$.

Remark. If in the first case of Theorem 7.2 for a set considered an asymptotics is proved (see Section 5 or also [Ku2]) then one gets the asymptotics in Theorem 7.2 as well.

The next theorem, due to Belinskii and Liflyand [BL3], is of the same nature but deals with hyperbolic partial sums (see also Section 6). Let $L_j(x) = l_{j1}x_1 + \ldots + l_{jn}x_n$, $j = 1, 2, \ldots, n$, be linear forms with nonsingular coefficient matrix $\Lambda = \{l_{jk}\}$, $1 \leq j, k \leq n$, $\det \Lambda \neq 0$, and
\[ H = \{x \in \mathbb{R}^n : \prod_{j=1}^n |L_j(x)| \leq 1\}. \]

We call the matrix $\Lambda$ rational if each row of it consists of integers, up to a common factor. In the contrary case, the matrix is said to be irrational.

Theorem 7.3. 1. If the matrix $\Lambda$ is rational, then
\[ ||S_{NH}|| \asymp N^{-\frac{1}{n-1}}. \]

2. If $\Lambda$ is irrational, then there exists an integer $N_0$ such that the operator $S_{NH}$ is unbounded for all $N > N_0$.

In both theorems the second parts are proved by using the following result from [Ru1] (see also [Ru2], Th.3.1.3):
If the operator of taking partial sums with respect to some dilation of a given set is bounded than this set may be represented as a finite union of cosets of discrete subgroups of the lattice $\mathbb{Z}^n$.

To get a contradiction with this statement some theorems from Geometric Number Theory are used (see [Ca]).

For $n = 2$ Theorem 7.3 was earlier obtained by Belinskii (see [Be3, Be4]). The proof was essentially two-dimensional and relied on some other results in Number Theory, in particular an exact value $N_0$ was indicated. Nevertheless even in this case the second part of Theorem 7.3 cannot be established for more general sets $H = \{ \prod_{j=1}^n |L_j|^{\gamma_j} \leq 1 \}$, since there are no corresponding results in Number Theory.

The following result obtained by A. Yudin and V. Yudin [YY1] is closely connected with the result of Podkorytov given above in Theorem 1.2. The latter is even easier for proving the estimate from above for the Lebesgue constants of hyperbolic partial sums (see Section 6) though it was not applied to this and the author knows the proof from Podkorytov’s private communication. In contrary, in [YY1] what follows was invented especially for "hyperbolic" estimates. Thus, let $U \subset \mathbb{Z}^n$ be a bounded set and $t \in \mathbb{Z}^n$. We set
\[ U_t = \{k \in \mathbb{Z}^n : k - t \in U\} \]
and
\[ \omega(t, U) = 2|U| - |U \cap U_t| - |U \cap U_{-t}|, \]
where $|U|$ denotes the number of points in $U$. 
Theorem 7.4. Let numbers \( L_1 \leq L_2 \) be such that
\[
\omega(he_j, U) \leq L_1 h \quad \text{and} \quad \omega(he_r, U) \leq L_2 h
\]
for some natural numbers \( r, j \in M_0 \), where \( r \neq j \), and every natural number \( h \). Then
\[
\|S_U\| \leq \frac{1}{2} \left( \frac{L_1}{L_2} \right)^{\frac{1}{2}} \log_2 \frac{L_2}{L_1} + \frac{3}{2} - \sqrt{2} L_1^{\frac{1}{2}}.
\]

The Lebesgue constants of step-wise hyperbolic crosses have been considered in many papers, together with various applications of such estimates. These problems were discussed by Temlyakov [Tm], Galeev [Ga1, Ga2], and Belinskii. For example it was shown by Belinskii in [Be6] that if \( H_N \) is defined as
\[
H_N = \bigcup \{ m \in \mathbb{Z}^n : 2^{s_j} \leq |m_j| < 2^{s_j+1} \}
\]
for \( s \in \mathbb{Z}^n \cap [0, \infty)^n \) such that \( 0 \leq s_1 + \ldots + s_n \leq N \), and \( N = 1, 2, \ldots \), then
\[
\|S_{H_N}\| \asymp N^{n+\frac{n-1}{2}}
\]
as \( N \to \infty \).

We now need to introduce some new notation to be able to formulate further results due to M. Dyachenko. Let \( A_2 \) be the class of bounded sets \( U \subset \mathbb{Z}^n \) such that if \( m \in U \), then
\[
\mathbb{Z}^n \cap \prod_{j=1}^n [\min(m_j, 0), \max(m_j, 0)] \subset U,
\]
and let us define \( A_1 \) by
\[
A_1 = \{ U \cap (0, \infty)^n, \quad \text{where} \quad U \in A_2 \}.
\]
We also define \( M_1 \) as the class of \( n \)-dimensional sequences
\[
a = \{a_m\} = \{a_{m_1,\ldots,m_n}\}_{m_1,\ldots,m_n=1}^\infty
\]
such that \( 1 \leq k_j \leq m_j \) implies that \( a_k \geq a_m \geq 0 \). Set
\[
\prod(x) = \prod_{j=1}^n (|x_j| + 1),
\]
and it is then possible to give the following assertions:

Theorem 7.5. Given \( U \in A_1 \) or \( U \in A_2 \) and a number \( p \in [1, 2n/(n+1)] \), then
\[
\|S_U\|_{L^p} \leq C_{p,n} \max_{m \in U} \left( \prod(m) \right)^{n-1/2n}.
\]
We note that Theorem 7.5 yields the upper bound of Theorem 6.1 in the case 1) for \( \alpha = 0 \) and \( \gamma_1 = \ldots = \gamma_n = 0 \).
Corollary 7.1. The following inequalities are satisfied under the hypotheses of Theorem 7.5:

\[ \| S_U \|_{L^p} \leq C_{p,m} |U|^{\frac{p-1}{2n}} \]

and

\[ \| S_U \|_{L^p} \leq C_{p,m} \left( \sum_{m \in U} \left( \prod_{m \in U} (|m|) \right)^{\frac{2n}{p+1}} \right)^{\frac{n+1}{2n}}. \]

The first inequality of Corollary 7.1 was proved in [Dy2] for \( p = 1 \) and \( U \in A_1 \) with the constant \( C_{1,n} = 50n^3 \). In [Dy1] such an estimate was obtained with an additional logarithmic factor. Some other estimates for \( p > 1 \) as well as some open problems can be found in the survey [Dy3], Sect.3.

Mention also results of Ustina on the two-dimensional Hausdorff method (see [U]).

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