Autonomous three-dimensional Newtonian systems which admit Lie and Noether point symmetries

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Abstract
We determine the autonomous three-dimensional Newtonian systems which admit Lie point symmetries and the three-dimensional autonomous Newtonian Hamiltonian systems which admit Noether point symmetries. We apply the results in order to determine the two-dimensional Hamiltonian dynamical systems which move in a space of constant non-vanishing curvature and are integrable via Noether point symmetries. The derivation of the results is geometric and can be extended naturally to higher dimensions.

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1. Introduction

The Lie and Noether point symmetries of the equations of motion of a dynamical system provide a systematic method for the determination of invariants and first integrals (see [1] for a review). In a recent work [2], we have determined the autonomous two-dimensional Newtonian systems which admit Lie and Noether point symmetries. In this work we extend this study to the autonomous three-dimensional (3D) Newtonian systems, that is, we consider the equations of motion

\[ \ddot{x}^\mu = F^\mu (x^\nu) , \mu = 1, 2, 3, \]  

and compute the form of the functions \( F^\mu (x^\nu) \) for which (1) admits Lie point symmetries (in addition to the trivial one \( \partial_t \)).

Subsequently, we assume the system to be Hamiltonian with Lagrangian

\[ L(x^\mu, \dot{x}^\nu) = \frac{1}{2} \delta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V(x^\mu) , \]

where \( \delta_{\mu\nu} \) is the Euclidian 3D metric and \( V(x^\mu) \) is the potential function, and determine the potential functions \( V(x^\mu) \) for which the Lagrangian admits at least one Lie or Noether point symmetries.
symmetry (in addition to the trivial $\partial_t$). Because the derivation is based solely on geometric arguments, the results can be generalized in a straightforward manner in $E^n$.

Using the fact that a space of constant curvature of dimension $n - 1$ can be embedded in a flat space of dimension $n$, we apply the results in $E^3$ in order to determine the dynamical systems which move in a two-dimensional space of a constant non-vanishing curvature and are Liouville integrable via Noether point symmetries.

The structure of the paper is as follows. In section 2 we give the basic definitions concerning the collineations in a Riemannian space. In section 3 we present two theorems which relate the Lie and the Noether point symmetry algebras of the equations of motion of a dynamical system moving in an $n$-dimensional Riemannian space with the projective and the homothetic algebra of the space, respectively. In section 4 we determine the autonomous Newtonian systems which admit Lie point symmetries. In section 5 we determine the subset of the systems which admit Noether point symmetries. In section 6 we apply the results to determine the Newtonian Hamiltonian dynamical systems which move in a two-dimensional space of constant non-vanishing curvature and admit Noether point symmetries. Finally, in section 7 we draw our conclusions.

2. Collineations of Riemannian spaces

A collineation in a Riemannian space is a vector field $X$ which satisfies an equation of the form

$$\mathcal{L}_X A = B$$

(3)

where $\mathcal{L}_X$ denotes Lie derivative [3], $A$ is a geometric object (not necessarily a tensor) defined in terms of the metric and its derivatives (e.g. connection coefficients, Ricci tensor, curvature tensor, etc) and $B$ is an arbitrary tensor with the same tensor indices as $A$. The collineations in a Riemannian space have been classified by Katzin et al [4]. In the following we use only certain collineations.

A conformal Killing vector (CKV) is defined by the relation

$$\mathcal{L}_X g_{ij} = 2\psi(x^k)g_{ij}.$$  

(4)

If $\psi = 0$, $X$ is called a Killing vector (KV), if $\psi$ is a non-vanishing constant, $X$ is a homothetic vector (HV) and if $\psi_{,ij} = 0$, $X$ is called a special conformal Killing vector (SCKV). A CKV is called proper if it is not a KV, HV or a SCKV.

A projective collineation (PC) is defined by the equation

$$\mathcal{L}_X \Gamma^i_{jk} = 2\phi_{,i}^j\delta_{ij}.$$ 

(5)

If $\phi = 0$, then PC is called an affine collineation (AC) and if $\phi_{,ij} = 0$ a special projective collineation (SPC). A proper PC is a PC which is not an AC, HV, KV or SPC. The PCs form a Lie algebra whose ACs, HVs and KVs are subalgebras. It has been shown that if a metric admits a SCKV, then it also admits a SPC, a gradient HV and a gradient KV [5].

In the following we shall need the symmetry algebra of spaces of constant curvature. In [6] it has been shown that the PCs of a space of constant non-vanishing curvature consist of proper PCs and KVs only, and if the space is flat, then the algebra of the PCs consists of KVs/HV/ACs and SPCs. Note that the algebra of KVs is common in both cases.
3. Lie and Noether point symmetries versus collineations

We review briefly the basic definitions concerning the Lie and Noether point symmetries of systems of second-order ordinary differential equations (ODEs):

\[ \dddot{x}_i = \omega_i(t, x^j, \dot{x}_j). \]  

(6)

A vector field \( X = \xi(t, x^j) \partial_t + \eta^i(t, x^j) \partial_i \) in the augmented space \( \{t, x^j\} \) is the generator of a Lie point symmetry of the system of ODEs (6) if the following condition is satisfied [7]:

\[ X^{[2]}(\dddot{x}_i - \omega_i(t, x^j, \dot{x}_j)) = 0, \]  

(7)

where \( X^{[2]} \) is the second prolongation of \( X \) defined by the formula

\[ X^{[2]} = \xi \partial_t + \eta^i \partial_i + (\dot{\eta}^i - \dot{x}_i \dot{\xi}) \partial_{\dot{x}_i} + (\dddot{\eta}^i - \dddot{x}_i \dddot{\xi}) \partial_{\dddot{x}_i}. \]  

(8)

Condition (7) is equivalent to condition [8]:

\[ [X^{[1]}, A] = \lambda(x^a) A, \]  

(9)

where \( X^{[1]} \) is the first prolongation of \( X \) and \( A \) is the Hamiltonian vector field

\[ A = \partial_t + \dot{x} \partial_x + \omega_i(t, x^j, \dot{x}_j) \partial_{\dot{x}_j}. \]  

(10)

If the system of ODEs results from a first-order Lagrangian \( L = L(t, x^j, \dot{x}_j) \), then a Lie symmetry \( X \) is a Noether symmetry of the Lagrangian, if the additional condition is satisfied,

\[ X^{[1]} L + L \frac{d\xi}{dt} \partial_{\dot{x}_i} = \frac{dG}{dt}, \]  

(11)

where \( G = G(t, x^i) \) is the Noether gauge function. To each Noether symmetry there corresponds a first integral (a Noether integral) [8] of the system of equations (6) which is given by the formula

\[ I = \xi E - \frac{\partial L}{\partial \dot{x}_i} \eta^i + G, \]  

(12)

where \( E \) is the Hamiltonian.

Using the standard Lie method the Lie point symmetry condition (7) for the second-order system

\[ \dddot{x}_i + \Gamma^i_{jk} \dot{x}_j \dot{x}_k + F_i(x^j) = 0 \]  

(13)

is computed in the following geometric form [1]:

\[ L_\eta F_i + 2 \xi_{,ij} F^j + \eta^i_{,ij} = 0 \]  

(14)

\[ (\xi_{,ik} \delta^k_j + 2 \xi_{,ij} \delta^k_j) F_k + 2 \eta^i_{,ij} - \xi_{,ij} \delta^k_j = 0 \]  

(15)

\[ L_\eta \Gamma^i_{jk} = 2 \xi_{,ij} \delta^k_j \]  

(16)

\[ \xi_{(,ij)\delta^k_j} = 0. \]  

(17)

Equation (17) means that \( \xi_{,ij} \) is a gradient KV of \( g_{ij} \). Equation (16) means that \( \eta^i \) is a PC of the metric with projective function \( \xi_{,ij} \). The remaining two equations are the constraint conditions, which relate the components \( \xi^i, \eta^i \) of the Lie point symmetry vector with the vector \( F^i \). Equation (14) gives

\[ (L_\eta \delta^j_i) F_j + g^{ij} L_\eta F_j + 2 \xi_{,ij} \delta^j_i F_j + \eta^j_{,ij} = 0. \]  

(18)

The use of an algebraic computing program (e.g. Lie) does not reveal directly the Lie symmetry conditions in this geometric form. The ‘solution’ of these conditions is given in [2]. For the convenience of the reader, we repeat this solution in a concise form.
This equation restricts \( \eta^i \) further because it relates it directly to the metric symmetries. Finally, equation (15) gives

\[
- \delta^i_j \xi, \tau + (\xi, \delta^i_j + 2 \delta^j_i \xi, \delta^k_j) F^k + 2 \eta^i_{,j} + 2 \Gamma^i_{jk} \eta^k_{,j} = 0. \tag{19}
\]

We conclude that the Lie symmetry equations are equations (18) and (19), where \( \xi(t, x) \) is a gradient KV of the metric \( g_{ij} \) and \( \eta^i(t, x) \) is a SPC of the metric \( g_{ij} \) with a projective function \( \xi, \tau \). The above equation leads to the following theorem which relates the Lie point symmetries of an autonomous dynamical system ‘moving’ in a Riemannian space with the collineations of the space\(^2\).

**Theorem 1.** The Lie point symmetries of the equations of motion of an autonomous dynamical system moving in a Riemannian space with metric \( g_{ij} \) (of any signature) under the action of the force \( F^i(x) \) (13) are given in terms of the generators \( Y^i \) of the special projective Lie algebra of the metric \( g_{ij} \).

If the force \( F^i \) is derivable from a potential \( V(x^i) \), so that the equations of motion follow from the standard Lagrangian

\[
L(x^i, \dot{x}^i) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^i) \tag{20}
\]

with Hamiltonian

\[
E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V(x^i), \tag{21}
\]

then the Noether point conditions (11) for the Lagrangian (20) are

\[
V, k \eta^k + V \xi, \tau = -f, \tau \tag{22}
\]

\[
\eta^i_{,j} g_{ij} - \xi, j V = f, j \tag{23}
\]

\[
L_\tau g_{ij} = 2 \frac{1}{2} \xi, j g_{ij} \tag{24}
\]

\[
\xi, k = 0. \tag{25}
\]

Equation (25) implies \( \xi = \xi(t) \) and reduces the system as follows:

\[
L_\tau g_{ij} = 2 \frac{1}{2} \xi, j g_{ij} \tag{26}
\]

\[
V, k \eta^k + V \xi, \tau = -f, \tau \tag{27}
\]

\[
\eta^i_{,i} = f, i. \tag{28}
\]

Equation (26) implies that \( \eta^i \) is a CKV of the metric provided \( \xi, \tau \neq 0 \). Because \( g_{ij} \) is independent of \( t \) and \( \xi = \xi(t) \), the \( \eta^i \) must be a HV of the metric. This means that \( \eta^i(t, x) = T(t) Y^i(x) \), where \( Y^i \) is a HV. If \( \xi, \tau = 0 \), then \( \eta^i \) is a KV of the metric. Equations (27) and (28) are the constraint conditions, which the Noether symmetry and the potential must satisfy for the former to be admitted. These lead to the following theorem\(^3\).

**Theorem 2.** The Noether point symmetries of the Lagrangian (20) are generated from the homothetic algebra of the metric \( g_{ij} \).

More specifically, concerning the Noether symmetries, we have the following [2].

\(^2\) This theorem contains various cases which can be found in the detailed version of the theorem given in [2]. It is important for the comprehension of this paper that the reader will consult the detailed version of the theorem.

\(^3\) The detailed version of this theorem is given in [2].
All autonomous systems admit the Noether symmetry $\frac{\partial}{\partial t}$ whose Noether integral is the Hamiltonian $E$. For the rest of the Noether symmetries we consider the following cases.

**Case 1.** Noether point symmetries generated by the homothetic algebra.

The Noether symmetry vector and the Noether function $G(t,x^i)$ are

$$X = 2\psi_Y t \partial_t + Y^i \partial_i, \quad G(t,x^i) = pt, \quad (29)$$

where $\psi_Y$ is the homothetic factor of $Y^i (\psi_Y = 0$ for a KV and 1 for the HV) and $p$ is a constant, provided the potential satisfies the condition

$$L_Y V + 2\psi_Y V + p = 0. \quad (30)$$

**Case 2.** Noether point symmetries generated by the gradient homothetic Lie algebra, i.e. both KV and the HV are gradient.

In this case the Noether symmetry vector and the Noether function are

$$X = 2\psi_Y \int T(t) dt \partial_t + T(t) H^i \partial_i, \quad G(t,x^i) = T, H^i(x^i) + p \int T dt, \quad (31)$$

where $H^i$ is a gradient HV or a gradient KV, the function $T(t)$ is computed from the relation $T^\mu = m T$, where $m$ is a constant and the potential satisfies the condition

$$L_H V + 2\psi_Y V + m H + p = 0. \quad (32)$$

Concerning the Noether integrals we have the following result (not including the Hamiltonian).

**Corollary 3.** The Noether integrals (12) of case 1 and case 2 are, respectively,

$$I_{C_1} = 2\psi_Y t E - g_{ij} Y^i \dot{x}^j + pt, \quad (33)$$

$$I_{C_2} = 2\psi_Y \int T(t) dt E - g_{ij} H^i \dot{x}^j + T, H + p \int T dt, \quad (34)$$

where $E$ is the Hamiltonian (21).

We remark that theorems 1 and 2 do not apply to generalized symmetries [9, 10].

**4. Lie point symmetries of 3D autonomous Newtonian systems**

In this section, we determine the forces $F^\mu = F^\mu(x^i)$ for which the equations of motion (11) admit Lie point symmetries (in addition to the trivial $\partial_t$). To do that, we need the special projective algebra of the Euclidian 3D metric

$$ds^2 = dx^2 + dy^2 + dz^2. \quad (35)$$

This algebra consists of 15 vectors\(^4\) as follows: six KVs $\partial_\mu, x_\nu \partial_\mu$, nine ACs $x_\mu \partial_\mu$, $x_\nu \partial_\mu$ and three SPCs $x_\mu^2 \partial_\mu + x_\nu x_\sigma \partial_\sigma$, where\(^5\) $\mu \neq \nu \neq \sigma$,

$$r(\alpha) = \text{arctan} \left( \frac{x_\nu}{x_\mu} \right) \quad \text{and} \quad R, \theta, \phi \text{ are spherical coordinates}.$$

In the computation of Lie symmetries we consider only the linearly independent vectors of the special projective group\(^6\).

\(^4\) These vectors are not all linearly independent, i.e. the HV and the rotations are linear combinations of the ACs.

\(^5\) If $x_\mu = x$, then $\{x_\nu = y, x_\sigma = z\}$ or $\{x_\nu = z, x_\sigma = y\}$.

\(^6\) We do not consider their linear combinations because the resulting Lie symmetries are too many; on the other hand, they can be computed in the standard way.
In tables 1 and 2 we list the Lie point symmetries and the functional dependence of the

4.1. Lie point symmetries for non-conservative forces

Immediately, we recognize that this dynamical system is the well-known and important
generalized Kepler Ermakov system (see [11]). A different representation of

For this representation from table 2 line 4, we have
\[ F' = -\frac{m}{4}(x_\mu, x_\nu, x_\sigma) + \left( \frac{1}{x_{\mu\nu}} f \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right), \frac{1}{x_{\mu\nu}} g \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right), \frac{1}{x_{\mu\nu}} h \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \right). \] (38)

which leads again to the autonomous Kepler–Ermakov system.

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**Table 1.** Case A1. Lie point symmetries generated by the affine algebra.

| Lie symmetry | \( F_\mu(x_\nu, x_\sigma, x_\rho) \) | \( F_\nu(x_\mu, x_\rho, x_\sigma) \) | \( F_\rho(x_\mu, x_\nu, x_\sigma) \) |
|--------------|-----------------------------------|-----------------------------------|-----------------------------------|
| \( t \partial_t + \partial_x \) | \( e^{-\partial_x} f(x_\nu, x_\rho) \) | \( e^{-\partial_x} g(x_\nu, x_\sigma) \) | \( e^{-\partial_x} h(x_\nu, x_\rho) \) |
| \( t \partial_t + \partial_x \partial_x \) | \( e^{-\partial_x} f(r_{\mu\nu}, x_\rho) \) | \( e^{-\partial_x} g(r_{\mu\nu}, x_\sigma) \) | \( e^{-\partial_x} h(r_{\mu\nu}, x_\rho) \) |
| \( t \partial_t + R \partial_R \) | \( x_{\mu\nu}^{-1} f \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) | \( x_{\mu\nu}^{-1} g \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) | \( x_{\mu\nu}^{-1} h \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) |
| \( t \partial_t + x_\mu \partial_\mu \) | \( x_{\mu\nu}^{-1} f(x_\nu, x_\rho) \) | \( x_{\mu\nu}^{-1} g(x_\nu, x_\sigma) \) | \( x_{\mu\nu}^{-1} h(x_\nu, x_\rho) \) |

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**Table 2.** Case A2. Lie point symmetries generated by the gradient homothetic algebra.

| Lie symmetry | \( F_\mu(x_\nu, x_\sigma, x_\rho) \) | \( F_\nu(x_\mu, x_\rho, x_\sigma) \) | \( F_\rho(x_\mu, x_\nu, x_\sigma) \) |
|--------------|-----------------------------------|-----------------------------------|-----------------------------------|
| \( t \partial_t \) | \( f(x_\nu, x_\rho) \) | \( g(x_\nu, x_\sigma) \) | \( h(x_\nu, x_\rho) \) |
| \( t^2 \partial_t + t R \partial_R \) | \( \frac{1}{2} f \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) | \( \frac{1}{2} g \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) | \( \frac{1}{2} h \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) |
| \( e^{x_\mu R} \partial_\mu \) | \( -m x_\mu + f(x_\nu, x_\rho) \) | \( g(x_\nu, x_\sigma) \) | \( h(x_\nu, x_\rho) \) |
| \( \frac{1}{\sqrt{m}} e^{x_\mu R} \partial_\mu \pm e^{x_\mu R} R \partial_R \) | \( -\frac{m}{2} x_\mu + \frac{1}{2} f \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) | \( -\frac{m}{2} x_\nu + \frac{1}{2} g \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) | \( -\frac{m}{2} x_\nu + \frac{1}{2} h \left( \frac{x_\mu}{x_\nu}, \frac{x_\nu}{x_\mu} \right) \) |
Table 3. Case A1. Lie point symmetries generated by the affine algebra (conservative force).

| Lie \(\forall (x, y, z)\) | \(d = 0\) | \(d = 2\) | \(d \neq 0, 2\) |
|--------------------------|---------|---------|---------|
| \(\frac{4}{3} t \delta_\mu + \partial_\mu\) | \(c_1 x_\mu + f(x_\mu, x_\nu)\) | \(e^{-2x_\mu} f(x_\mu, x_\nu)\) | \(e^{-d x_\mu} f(x_\mu, x_\nu)\) |
| \(\frac{4}{3} t \delta_\mu + \partial_\mu\) | \(c_1 x_\mu + f(x_\mu, x_\nu)\) | \(c^{-2h_{\mu \nu}} f(r_{\mu \nu}, x_\sigma)\) | \(c^{-2h_{\mu \nu}} f(r_{\mu \nu}, x_\sigma)\) |
| \(\frac{4}{3} t \delta_\mu + R \delta_\mu\) | \(x^2 f\left(\frac{x_{\mu}}{x_{\nu}}, \frac{x_{\nu}}{x_{\mu}}\right)\) | \(c_1 \ln(x_\mu) + f\left(\frac{x_{\mu}}{x_{\nu}}, \frac{x_{\nu}}{x_{\mu}}\right)\) | \(\chi^2 f\left(\frac{x_{\mu}}{x_{\nu}}, \frac{x_{\nu}}{x_{\mu}}\right)\) |
| \(\frac{4}{3} t \delta_\mu + x_\mu \partial_\mu\) | \(c_1 x_\mu + f(x_\mu, x_\nu)\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) |
| \(\frac{4}{3} t \delta_\mu + x_\mu \partial_\mu\) | \(c_1 x_\mu + c_2 (x_\mu + z_\nu)^2 + f(x_\mu)\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) |

Table 4. Case A2. Lie point symmetries generated by the gradient homothetic algebra (conservative force).

| Lie | \(V(x, y, z)\) | Lie | \(V(x, y, z)\) |
|-----|----------------|-----|----------------|
| \(t \partial_\mu\) | \(c_1 x_\mu + f(x_\mu, x_\nu)\) | \(e^{\nu_\mu, \nu_\mu} \partial_\mu\) | \(-\frac{\nu_\mu}{2} x_\mu + c_1 x_\mu + f(x_\mu, x_\nu)\) |
| \(t^2 \partial_\mu + t R \delta_\mu\) | \(\frac{1}{2} f\left(\frac{x_{\mu}}{x_{\nu}}, \frac{x_{\nu}}{x_{\mu}}\right)\) | \(\frac{1}{2} e^{\nu_\mu, \nu_\mu} \partial_\mu + e^{\nu_\mu, \nu_\mu} R \delta_\mu\) | \(-\frac{\nu_\mu}{2} (x_\mu + x_\nu) + f\left(\frac{x_{\mu}}{x_{\nu}}, \frac{x_{\nu}}{x_{\mu}}\right)\) |

4.2. Lie point symmetries for conservative forces

In this section we assume that the force is given by the potential \(V = V(x^\mu)\) and repeat the calculations. Again we ignore the linear combinations of Lie symmetries for each case. We state the results in tables 3 and 4.

Case B1/B2. In this case the potential is of the form \(V(x, y, z) = \frac{\omega^2}{2} (x^2 + y^2 + z^2) + p(x + y + z)\), where \(\omega\) and \(p\) are constants.

From tables 3 and 4 we infer that the isotropic oscillator admits 24 Lie point symmetries generating the \(SL(5, R)\), as many as the free particle [12].

5. 3D autonomous Newtonian systems which admit Noether point symmetries

In this section using theorem 2 we determine all autonomous Newtonian Hamiltonian systems with Lagrangian

\[
L = \frac{1}{2} (x^2 + y^2 + z^2) - V(x, y, z),
\]

which admit a non-trivial Noether point symmetry. This problem has been considered previously in [13, 14]; however, as we shall show, the results in these works are not complete. We note that the Lie symmetries of a conservative system are not necessarily Noether symmetries. The inverse is of course true.

Before we continue we note that the homothetic algebra of the Euclidian 3D space \(E^3\) has dimension 7 and consists of three gradient KVs \(\delta_\mu\), with a gradient function \(x_\mu\), three non-gradient KVs \(x_\mu \partial_\mu - x_\mu \partial_\nu\) generating the rotational algebra \(so(3)\) and a gradient \(HV\) \(H' = R \delta_\mu\) with a gradient function \(H = \frac{1}{2} R^2\), where \(R^2 = x^\mu x_\mu\). According to theorem 2, we have to consider the following cases.

5.1. Case 1. Noether symmetries generated from the homothetic algebra

The Noether point symmetries generated from the homothetic algebra, i.e. the non-gradient \(so(3)\) elements included, are shown in table 5.

The corresponding Noether integrals are computed easily from relation (33) of corollary 3. In tables A1 and A2 we give a complete list of the potentials resulting from the linear combinations of the elements of the homothetic algebra.
5.2. Case 2. Noether point symmetries generated from the gradient homothetic algebra

The Noether symmetries generated from the gradient homothetic algebra are listed in Table 6.

As before, the Noether integrals corresponding to these Noether point symmetries are computed from relation (34) of Corollary 3. In Table A3, we give the potential functions which result from the linear combinations of the elements of the gradient homothetic algebra. From the table we infer that the isotropic linear forced oscillator admits 12 Noether point symmetries, as many as the free particle.

As has been remarked above, the determination of the Noether point symmetries admitted by an autonomous Newtonian–Hamiltonian system has been considered previously in [14]. Our results extend the results of [14] and coincide with them if we set the constant \( p = 0 \).

For example, in page 12 case 1 and page 15 case 6 of [14], the terms \(-p x_{\mu} + f(x^{\mu}, x^{\nu})\) and \(p \arctan(l(\theta, \phi))\) are missing, respectively. Furthermore, the potential given in pages/lines 12/1, 13/2 and 13/3 of [14] admits Noether symmetries only when \( \lambda = 0 \) and \( b_{1,2}(t) = \text{const} \). This is due to the fact that the vectors given in [14] are KVs, and in order to have \( b_{j} \neq 0 \) they must be given by case 2 of Theorem 2 above, that is, the KVs must be gradient. However, the KVs used are linear combinations of translations and rotations which are non-gradient.

We remark that from the above results we are also able to give, without any further calculations, the Lie and the Noether point symmetries of a dynamical system ‘moving’ in a 3D flat space whose metric has a Lorenzian signature simply by taking one of the coordinates to be complex, for example by setting \( x^{1} = i x^{1} \).

6. Motion on the two-dimensional sphere

The first application of the results of Section 5 is the determination of Lie and Noether point symmetries admitted by the equations of motion of a Newtonian particle moving in a two-dimensional space of constant non-vanishing curvature.

Before we continue, it is useful to recall some facts concerning spaces of constant curvature. Consider a \((n + 1)\)-dimensional flat space with a fundamental form

\[
ds^{2} = \sum_{\alpha} c_{\alpha} (dx^{\alpha})^{2} \quad a = 1, 2, \ldots, n + 1,
\]
where $c_a$ are real constants. The hypersurfaces defined by
\[ \sum a c_a (d^{a})^2 = e R_0^2, \]
where $R_0$ is an arbitrary constant and $e = \pm 1$ are called fundamental hyperquadrics of the space. When all coefficients $c_a$ are positive, the space is Euclidian and $e = +1$. In this case there is one family of hyperquadrics which is the hyperspheres. In all other cases (excluding the case when all $c_a$s are negative) there are two families of hyperquadrics corresponding to the values $e = +1$ and $e = -1$. It has been shown that in all cases the hyperquadrics are spaces of constant curvature (see [15, p 202]).

One way to work is to consider in the above results $R = \text{constant}$. However, in order to demonstrate the application of theorem 2 in practice, we choose to work in the standard way. We use spherical coordinates which are natural in the case of spaces of constant curvature.

We consider an autonomous dynamical system moving in the two-dimensional sphere (Euclidian $(\varepsilon = 1)$ or Hyperbolic $(\varepsilon = -1)$) with Lagrangian [16]:
\[ L(\phi, \theta, \dot{\phi}, \dot{\theta}) = \frac{1}{2} (\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2) - V(\theta, \phi), \]  
(40)
where
\[ \sin \phi = \begin{cases} \sin \phi & \varepsilon = 1 \\ \sinh \phi & \varepsilon = -1 \end{cases}, \quad \cos \phi = \begin{cases} \cos \phi & \varepsilon = 1 \\ \cosh \phi & \varepsilon = -1 \end{cases}. \]

The equations of motion are
\[ \ddot{\phi} - \sin \phi \cos \phi \dot{\theta}^2 + V,\phi = 0 \]  
(41)
\[ \ddot{\theta} + 2 \frac{\cos \phi}{\sin \phi} \dot{\theta} \dot{\phi} + \frac{1}{\sin^2 \phi} V,\phi = 0. \]  
(42)

We note that the Lagrangian (40) is of the form (20) with the metric $g_{\mu \nu}$ to be the metric of a space of constant curvature. Therefore, theorem 2 applies and we use it to find the potentials $V(\theta, \phi)$ for which additional Noether point symmetries, hence, Noether integrals are admitted.

The homothetic algebra of a metric of spaces of constant curvature consists only of non-gradient KVs (hence $\psi = 0$) as follows:

(a) $\varepsilon = 1$ (Euclidian case)
\[ CK_1^1 = \sin \theta \partial_{\theta} + \cos \theta \cot \phi \partial_{\phi}, \quad CK_2^1 = \cos \theta \partial_{\theta} - \sin \theta \cot \phi \partial_{\phi}, \quad CK_3^1 = \partial_{\theta} \]  
(43)

(b) $\varepsilon = -1$ (Hyperbolic case)
\[ CK_1^h = \sin \theta \partial_{\theta} + \cos \theta \coth \phi \partial_{\phi}, \quad CK_2^h = \cos \theta \partial_{\theta} - \sin \theta \coth \phi \partial_{\phi}, \quad CK_3^h = \partial_{\theta}. \]  
(44)

Because we have only non-gradient KVs, according to theorem 2 only case 1 survives. Therefore, the Noether vectors and the Noether function are
\[ X = CK_{\varepsilon \phi} \partial_{\phi}, \quad f = pt, \]  
(45)
provided the potential satisfies the condition
\[ L_{CK} V + p = 0. \]  
(46)
The first integrals given by (33) are
\[ \phi_{II} = -g_{ij} CK_{\varepsilon \phi}^{j} + pt \]  
(47)
and are time dependent if $p \neq 0$. 

9
We consider two cases, the case 6.1. Noether symmetries

Noether symmetries are produced by the non-gradient KVs with Lie algebra

where

point symmetries will be (including Kepler potential and

Table 7. Noether symmetries/integrals and potentials for the Lagrangian (40).

| Noether symmetry       | \( V(\theta, \phi) \)                  | Noether integral |
|------------------------|--------------------------------------|------------------|
| \( CK_{,a} \)          | \( F(\cos \theta \sin \phi) \)       | \( I_{ CK_{,a} } \) |
| \( CK_{,b} \)          | \( F(\sin \theta \sin \phi) \)       | \( I_{ CK_{,b} } \) |
| \( CK_{,c} \)          | \( F(\phi) \)                        | \( I_{ CK_{,c} } \) |
| \( aCK_{,a} + bCK_{,b} \) | \( F\left(\frac{1+i\sin^2 \theta}{\sin \phi \tan \theta}\right)\) | \( aI_{ CK_{,a} } + bI_{ CK_{,b} } \) |
| \( aCK_{,a} + bCK_{,b} \) | \( F\left(a \cos \theta \sin \phi - b \sin \cos \phi\right)\) | \( aI_{ CK_{,a} } + bI_{ CK_{,b} } \) |
| \( aCK_{,a} + bCK_{,b} \) | \( F\left(a \sin \theta \sin \phi - b \cos \phi\right)\) | \( aI_{ CK_{,a} } + bI_{ CK_{,b} } \) |
| \( aCK_{,a} + bCK_{,b} + cCK_{,c} \) | \( F\left((a \cos \theta - b \sin \phi) \sin \phi - c \cos \phi\right)\) | \( aI_{ CK_{,a} } + bI_{ CK_{,b} } + cI_{ CK_{,c} } \) |

6.1. Noether symmetries

We consider two cases, the case \( V(\theta, \phi) = \text{constant} \) which concerns the geodesics of the space, and the case \( V(\theta, \phi) \neq \text{constant} \).

For the case of geodesics, it has been shown [17] that the Noether point symmetries are the three elements of \( so(3) \) with corresponding Noether integrals:

\[
I_{ CK_{,a} } = \dot{\phi} \sin \theta + \dot{\theta} \cos \theta \sin \phi \cos \phi \quad (48)
\]
\[
I_{ CK_{,b} } = \dot{\phi} \cos \theta - \dot{\theta} \sin \theta \sin \phi \cos \phi \quad (49)
\]
\[
I_{ CK_{,c} } = \dot{\theta} \sin^2 \phi. \quad (50)
\]

These integrals are in involution with the Hamiltonian hence the system is Liouville integrable.

In the case \( V(\theta, \phi) \neq \text{constant} \), we find the results of table 7.

The first integrals which correspond to each potential of table 7 are in involution with the Hamiltonian and independent. Hence the corresponding systems are integrable. From table 7 we infer the following result.

**Proposition 4.** A dynamical system with Lagrangian (40) has one, two or four Noether point symmetries hence Noether integrals.

**Proof.** For the case of the free particle, we have the maximum number of four Noether symmetries (the rotation group \( so(3) \) plus the \( \partial_0 \)). In the case the potential is not constant the Noether symmetries are produced by the non-gradient KVs with Lie algebra

\[ [X_1, X_2] = C^{\alpha}_{\beta} X_{\beta}, \]

where \( C_{12} = C_{31} = C_{23} = 1 \) for \( \varepsilon = 1 \) and \( C_{21} = C_{13} = C_{32} = 1 \) for \( \varepsilon = -1 \). Because the Noether point symmetries form a Lie algebra and the Lie algebra of the KVs is semi-simple, the system will admit either none, one or three Noether symmetries generated from the KVs. The case of three is when \( V(\theta, \phi) = V_0 \), that is, the case of geodesics; therefore the Noether point symmetries will be (including \( \partial_0 \)) either one, two or four.

We note that the two important potentials of celestial mechanics, that is, \( V_1 = -\frac{\cos \phi}{\sin \phi} \), \( V_2 = \frac{1}{2} \sin^2 \phi \), which according to Bertrand’s theorem [16, 18, 19] produce closed orbits on the sphere, are included in table 7. Hence, the dynamical systems they define are Liouville integrable via Noether point symmetries \( CK_{,a} \). The potential \( V_1 \) corresponds to the Newtonian Kepler potential and \( V_2 \) is the analogue of the harmonic oscillator. We also note that our results
Tables A1, A2 and A3 give the three-dimensional potentials which admit Noether point symmetries resulting from linear combinations of the elements of the homothetic group. In table A2,

\[ \lambda(\phi, \theta) = \left( (a^2 + b^2) \cos \phi - bc \tan \theta \sin \phi + cM_1 \right) \times \left\{ M_2 \left[ -b^2M_2^2 - 2b\tan \theta \sin \phi M_1 - a^2 \sin^2 \phi \tan^2 \theta \right] \right\}^{-\frac{1}{2}} \]

and \( M_1 = \frac{1}{\cos \theta} \sqrt{\sin^2 \phi \left( 2\cos^2 \theta - 1 \right)} \), \( M_2 = \sqrt{a^2 + b^2 + c^2} \).

7 We thank one of the referees for bringing this reference to our attention.

contain those of [16] if we consider the correspondence\(^7\) \( S_k(r) \rightarrow \sin \phi, C_k(r) \rightarrow \cos \phi, \theta \rightarrow \phi, v_r \rightarrow \dot{\phi}, v_\theta \rightarrow \dot{\theta} \).

We emphasize that the potentials listed in table 7 concern dynamical systems with Lagrangian (40) which are integrable via Noether point symmetries. It is possible that there exist integrable Newtonian dynamical systems for potentials not included in these tables, for example systems which admit only dynamical symmetries [9, 10] with integrals quadratic in momenta [20]. However, these systems are not integrable via Noether point symmetries.

7. Conclusion

We have determined the three-dimensional (3D) Newtonian dynamical systems which admit Lie point symmetries and the 3D Hamiltonian Newtonian dynamical systems which admit Noether point symmetries. These results complete previous results [13, 14] concerning the Noether point symmetries of the 3D Newtonian dynamical systems and extend previous work on the two-dimensional case [2, 21]. We note that, due to the geometric derivation and the tabular presentation, the results can be extended easily to higher dimensional flat spaces; however, at the cost of convenience because the linear combinations of the symmetry vectors increase dramatically. In a subsequent work, we shall apply the results obtained here to study the integrability of the 3D Hamiltonian Kepler–Ermakov system [11] and generalize it in a Riemannian space.

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Appendix

Tables A1, A2 and A3 give the three-dimensional potentials which admit Noether point symmetries resulting from linear combinations of the elements of the homothetic group.

Table A1. Linear combinations of two vector fields for case 1.

| Noether symmetry | \( V(x, y, z) \) |
|------------------|------------------|
| \( a\partial_x + b\partial_y \) | \(-\frac{a}{2}x + f(x'^e - \frac{b}{2}x^e, x^e)\) |
| \( a\partial_x + b(x, \partial_y - x_y, \partial_e) \) | \(-\frac{a}{2} \arctan(\frac{\partial_x}{\partial_y}) + f(\frac{1}{2}f(x, y) + \frac{b}{2}x^e, x^e)\) |
| \( a(x_y, \partial_y - x_y, \partial_e) \) | \( \frac{a}{2} \arctan(\frac{\partial_x}{\partial_y}) \) |
| \( +b(x, \partial_y - x_y, \partial_e) \) | \(+ \frac{1}{2} f(x, \frac{1}{2}x, \phi^2(1 - \frac{1}{2} + \frac{2}{a} \theta^2 + x^2)\) |
| \( 2\partial_x \partial_y + a\partial_x + bR\partial_y \) | \(-\frac{a}{2} \ln f(x, y, \frac{1}{2} \ln f(x, y) + \frac{b}{a} \theta^2)\) |
| \( 2\partial_x \partial_y + a\partial_x + bR\partial_y \) | \(-\frac{1}{2} \ln f(\theta, y, \frac{1}{2} \ln f(x, y) + \frac{b}{a} \theta^2)\) |
Table A2. Linear combination of three vector fields for case 1.

| Noether symmetry | $V(x, y, z)$                      |
|------------------|----------------------------------|
| $a \partial_x + b \partial_y + c \partial_z$ | $-\frac{x}{2} y + f(x^2 - \frac{1}{2} x^4, x^2 - \frac{1}{2} x^4)$ |
| $a \partial_x + b \partial_y + c(x \partial_y - x_y \partial_x)$ | $\frac{1}{2} \arctan \left( \frac{\sqrt{x}}{\sqrt{a + b x^2}} \right)$ |
| $a \partial_x + b \partial_y + c \partial_y + c(x \partial_y - x_y \partial_x)$ | $+ f(x^2 + x^2 (1 - \frac{1}{x^2})) + (\frac{2}{x} + \frac{1}{x^2} x_y) x_y + \frac{1}{x^2} x_y$ |

so(3) linear combination

2ct $\partial_t + a \partial_y + b \partial_{(x \partial_y - x_y \partial_x)} + cR \partial_R$

$\frac{1}{1 - \frac{1}{x^2}} f(\theta_{(x \partial_y - x_y \partial_x)}) - \frac{1}{2} \ln r_{(x \partial_y - x_y \partial_x)} + \frac{1}{2} \ln |x^2 + \frac{1}{x^2} x_y|$

2$\partial t \partial_t + (a \partial_y + b \partial_y + c \partial_y + b \partial_y + c \partial_y + cR \partial_R)$

$\frac{1}{1 - \frac{1}{x^2}} f(\theta(\theta_{(x \partial_y - x_y \partial_x)}) - \frac{1}{2} \ln r(\theta_{(x \partial_y - x_y \partial_x)})$

Table A3. Linear combination of vector fields for case 2.

| Noether symmetry | $V(x, y, z)$                      |
|------------------|----------------------------------|
| $2 \partial t \partial t + (a \partial_y + b \partial_y + c \partial_y + cR \partial_R)$ | $- \frac{2}{3} R^2 + f(x^2 - \frac{1}{2} x^4, x^2 - \frac{1}{2} x^4)$ |
| $2 \partial t \partial t + (a \partial_y + b \partial_y + c \partial_y + cR \partial_R)$ | $- \frac{2}{3} (x^2 + x^2 (1 - \frac{1}{x^2})) + (\frac{2}{x} + \frac{1}{x^2} x_y) x_y + \frac{1}{x^2} x_y$ |

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