An Improved Trickle down Theorem for Partite Complexes

Dorna Abdolazimi
University of Washington, Seattle, WA, USA

Shayan Oveis Gharan
University of Washington, Seattle, WA, USA

Abstract

We prove a strengthening of the trickle down theorem for partite complexes. Given a \((d+1)\)-partite \(d\)-dimensional simplicial complex, we show that if “on average” the links of faces of co-dimension 2 are \(\frac{1-d}{k+1}\) (one-sided) spectral expanders, then the link of any face of co-dimension \(k\) is an \(O(\frac{1-d}{k+1})\) (one-sided) spectral expander, for all \(3 \leq k \leq d+1\). For an application, using our theorem as a black-box, we show that links of faces of co-dimension \(k\) in recent constructions of bounded degree high dimensional expanders have spectral expansion at most \(O(1/k)\) fraction of the spectral expansion of the links of the worst faces of co-dimension 2.

2012 ACM Subject Classification Theory of computation → Expander graphs and randomness extractors; Theory of computation → Error-correcting codes; Theory of computation → Random walks and Markov chains

Keywords and phrases Simplicial complexes, High dimensional expanders, Trickle down theorem, Bounded degree high dimensional expanders, Locally testable codes, Random walks

Digital Object Identifier 10.4230/LIPIcs.CCC.2023.10

Funding Dorna Abdolazimi: Research supported by NSF grant CCF-1907845 and Air Force Office of Scientific Research grant FA9550-20-1-0212. Shayan Oveis Gharan: Research supported by Air Force Office of Scientific Research grant FA9550-20-1-0212, Simons Investigator grant.

Acknowledgements The discussion that initiated this work took place at the DIMACS Workshop on Entropy and Maximization. We would like to thank the DIMACS center and the workshop organizers for making this happen. In particular, we would like to thank Tali Kaufman for raising the question of an improved trickle down theorem for sparse simplicial complexes in that workshop. We also would like to thank Ryan O’Donnell and Kevin Pratt for helpful discussions on high dimensional expanders based on Chevalley groups.

1 Introduction

A simplicial complex \(X\) on a finite ground set \([n] = \{0, \ldots, n\}\) is a downwards closed collection of subsets of \([n]\), i.e. if \(\tau \in X\) and \(\sigma \subset \tau\), then \(\sigma \in X\). The elements of \(X\) are called faces, and the maximal faces are called facets. We say that a face \(\tau\) is of dimension \(k\) if \(|\tau| = k+1\) and write \(\dim(\tau) = k\). A simplicial complex \(X\) is a pure \(d\)-dimensional complex if every facet has dimension \(d\). In this paper, all simplicial complexes are assumed to be pure. Given a \(d\)-dimensional complex \(X\), for any \(0 \leq i \leq d\), define \(X(i) = \{\tau \in X : \dim(\tau) = i\}\). Moreover, the co-dimension of a face \(\tau \in X\) is defined as \(\text{codim}(\tau) = d - \dim(\tau)\). For a face \(\tau \in X\), define the link of \(\tau\) as the simplicial complex \(X_\tau = \{\sigma \setminus \tau : \sigma \in X, \sigma \supset \tau\}\). Note that \(X_\tau\) is a \((\text{codim}(\tau) - 1)\)-dimensional complex.

A \((d+1)\)-partite complex is a \(d\)-dimensional complex such that \(X(0)\) can be (uniquely) partitioned into sets \(T_0 \cup \cdots \cup T_d\) such that for every facet \(\tau \in X(d)\), we have \(|\tau \cap T_i| = 1\) for all \(i \in [d]\). The type of any face \(\tau \in X\) is defined as type(\(\tau\)) = \(\{i \in [d] : |\tau \cap T_i| = 1\}\).
We equip $X$ with a probability distribution $\pi$ supported on all facets of $X$ and we denote this pair by $(X, \pi)$. For a face $\tau \in X$, $\pi$ induces a conditional distribution $\pi_\tau$ on facets of $X_\tau$ where for each facet $\sigma \in X_\tau$, 

$$\pi_\tau(\sigma) = \frac{\pi(\sigma \cup \tau)}{\sum_{\text{facet } \sigma' \in X_\tau} \pi(\sigma' \cup \tau)}.$$ 

For each face $\tau$ of co-dimension at least 2 the 1-skeleton of $(X_\tau, \pi_\tau)$ is a weighted graph with vertices $X_\tau(0)$, edges $X_\tau(1)$, and edge weights given by $\mathbb{P}_{\sigma \sim \pi_\tau}[\{x, y\} \subseteq \sigma]$ for each edge $\{x, y\}$. Note that when $\tau$ is of co-dimension 2, the complex $(X_\tau, \pi_\tau)$ is itself a weighted graph. We say that a complex $X$ is totally connected if the 1-skeleton of the link of any face $\tau$ of co-dimension at least 2 is connected.

**Definition 1** (Local Spectral High Dimensional Expander). We say that the link of a face $\tau$ of co-dimension at least 2 of a $d$-dimensional (weighted) complex $(X, \pi)$ is a $\gamma$-spectral expander if the second largest eigenvalue of the simple random walk on the 1-skeleton of $(X_\tau, \pi_\tau)$ is at most $\gamma$. We say that $(X, \pi)$ is a $(\gamma_2, \gamma_3, \ldots, \gamma_{d+1})$-local spectral expander if the link of any face $\tau$ of co-dimension at least 2 is a $\gamma_{\text{codim}(\tau)}$-spectral expander. When the complex $(X, \pi)$ is clear in the context, for an integer $2 \le k \le d + 1$, we write $\gamma_k$ to denote the largest $2$nd eigenvalue of the simple random walk on the 1-skeleton of all links of faces of co-dimension $k$ of the complex.

Over the last few years, the study of local spectral high dimensional expanders (HDX) has revolutionized several areas of Math and theoretical computer science, namely in analysis of Markov chains [4, 3, 6, 1], coding theory [9], and elsewhere [2, 11, 10]. One can generally divide the family of HDXes studied in recent works into two groups: (i) Dense Complexes. Here we have a HDX with exponentially large number of facets, i.e., $|X(0)|^{\Omega(d)}$. One typically encounters these objects in studying Markov Chain Monte Carlo technique where we use a Markov Chain to sample from an exponentially large probability distribution. Perhaps the simplest such family is the complex of all independent sets of a matroid. (ii) Sparse/Ramanujan Complexes. Here we have a HDX where every vertex (of $X(0)$) only appear in constant number of facets, independent of $|X(0)|$. See, [15, 13, 17] for explicit constructions. These objects have been useful in constructing double samplers [11], agreement testers [8, 7], or locally testable codes [9].

One of the main aspects of local spectral expanders is their “local to global phenomenon”, often referred to as the Garland’s method or the trickle down theorem [18].

**Theorem 2** (Trickle Down Theorem). Given a totally connected complex $(X, \pi)$, if $\gamma_2 \le \frac{1 - \delta}{d^2}$ for some $0 < \delta \le 1$, then $\gamma_k \le \frac{1 - \delta}{d^2(1 - 2\delta)} \le \frac{1 - \delta}{2d}$ for all $2 \le k \le d$.

The trickle down theorem has found numerous applications in proving bounds on local spectral expansion of simplicial complexes. To invoke the theorem one needs to inspect all faces of co-dimension 2 to find the worst $2$nd eigenvalue. If we get lucky and this number is below $1/d$, then, the trickle down theorem kicks in and inductively bounds the spectral expansion of all links of the complex.

There are, however, two pitfalls for the theorem: i) The required bound on $\gamma_2$ is too small and often not satisfiable. In particular, for many dense complexes in counting and sampling applications that satisfy $\gamma_k = O(1/k)$ for $k \ge \Omega(d)$ (see e.g., [3, 6]), the links of faces of co-dimension 2 are only $\Theta(1)$-spectral expanders. ii) Even if $\gamma_2 \ll 1/d$, the trickle down theorem only implies $\gamma_k \simeq \gamma_2$, i.e., $\gamma_k$ does not increase too much as $k$ increases. This is in contrast with the fact that, for many dense complexes, one can observe a steep decrease in spectral expansion as the co-dimension increases, i.e., $\gamma_k \lesssim \gamma_2/k$. 

...
Such a decrease has not been known for any sparse complex. This led some experts to conjecture that, perhaps, dense and sparse complexes exhibit a different pattern of local spectral expansion; in particular, unlike dense HDX, local spectral expansion does not decay for sparse complexes.

In this paper, we prove a generalization of the trickle down theorem for partite complexes that shows that even if $\gamma_2 = \Theta(1)$, but “on average” the links of faces of co-dimension 2 are $< 1/d$-spectral expanders, then we have $\gamma_k \leq O(1/k)$ for all $3 \leq k \leq d + 1$. Surprising to us, our average condition is satisfied by some recent construction of (sparse) bounded degree high dimensional expanders [13, 17]. In particular, as we explain below, one can use our theorem to prove a significantly better local spectral expansion for the Kaufman-Opennheim construction in a black-box manner.

### 1.1 Main Contribution

We start by stating two special cases of our theorem. We need the following definition.

> **Definition 3.** Given a $(d+1)$-partite complex $(X, \pi)$ with parts $[d]$, for every $i \in [d]$, define

$$\Delta(X, \pi)(i) = |\{j \in [d] \setminus i : \exists \tau \text{ of type } [d] \setminus \{i, j\} \text{ s.t. } \lambda_2(P_{\tau}) > 0\}|,$$

i.e. $\Delta(X, \pi)(i)$ is the number of parts $j \neq i$ for which there exists a face of type $[d] \setminus \{i, j\}$ whose link is not a 0-spectral expander. Moreover, define $\Delta(X, \pi) = \max_{i \in [d]} \Delta(X, \pi)(i)$. We drop the subscripts $(X, \pi)$ when the complex is clear in the context.

> **Theorem 4.** Let $(X, \pi)$ be a $(d+1)$-partite (weighted) totally connected complex. For some $0 < \delta < 1$, assume that

$$\gamma_2 \leq \frac{\delta^2}{10(1 + \ln \Delta)} \quad \text{and} \quad \gamma_2 \leq \frac{1 - \delta}{\Delta + \ln \Delta}.$$

Then, the link of any face $\tau$ of co-dimension $k$ of $X$ has spectral expansion

$$\left\{ \begin{array}{ll}
\frac{c(1-\delta)}{k^8} & \text{if } k \geq \Delta, \\
\frac{c(1-\delta)}{k^{8+\ln k}} & \text{if } k < \Delta,
\end{array} \right.$$

for some constant $c \leq 2$ that depends on $\delta$.

Note that, for $\Delta = d$, this theorem retrieves Theorem 2 up to a lower order term in the condition on $\gamma_2$ and a constant in the bounds on local spectral expansions.

When $\Delta \ll d$, this theorem is a significant improvement over Theorem 2. Roughly speaking, this theorem says that, if the complex has many faces of co-dimension 2 whose links are 0-expanders, one needs to satisfy a much weaker condition on $\gamma_2$ to get $O(1/k)$-spectral expansion for faces of co-dimension $k$. In other words, the faces of co-dimension 2 that have perfect spectral expansion can compensate for faces of co-dimension 2 that have bad spectral expansion.

Next, we state the second special case of our theorem. For every integers $1 \leq n$, let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th harmonic number. Moreover, for any $1 \leq i \leq n$ define $H_n(i) = \sum_{j=i}^{n} \frac{1}{j}$ and let $H_n(0) = H_n(1)$.
10:4 An Improved Trickle down Theorem for Partite Complexes

▶ Theorem 5. Let \((X, \pi)\) be a \((d+1)\)-partite (weighted) totally connected complex. For any distinct \(i, j \in [d]\), let \(\epsilon_{(i,j)} = \max_{r:\text{type}(r)=|\{}i\}, j\}} \lambda_2(P_r)\) be the 2nd largest eigenvalue of the simple random walk matrices on \((X_\tau, \pi_\tau)\) for all \(\tau\) of type \([d] \setminus \{i, j\}\). For some \(0 < \delta < 1\), assume that for every \(i \in [d]\),

\[
\epsilon_{(i,j)} \cdot H_d \leq \frac{\delta^2}{10}, \forall j \neq i \quad \text{and}
\sum_{\ell=1}^{d} \epsilon_{(i,j)} \cdot \frac{H_d(\ell-1)}{d} \leq 1 - \delta,
\]

where \(j_0, \ldots, j_d\) is an ordering of \([d] \setminus i\) such that \(\epsilon_{(i,j_0)} \leq \cdots \leq \epsilon_{(i,j_d)}\). Then, \(X\) is \((\frac{c(1-\delta)}{\delta}, \ldots, \frac{(1-\delta)}{d\delta})\)-local spectral expander for some constant \(c \leq 2\) that depends on \(\delta\).

We remark that for every \(i \in [d]\), \(1 \leq \sum_{\ell=1}^{d} \frac{H_d(\ell-1)}{d} \leq 1 + \ln d\). So, roughly speaking, the last condition can be seen as \(\mathbb{E}_{\ell}[\epsilon_{(i,j)}] \leq 1 - \delta\) for every \(i \in [d]\), where the expectation is weighted according to \(\frac{H_d(\ell)}{d}\). This is an improvement over the stronger condition in Theorem 2.

Now, we state the main theorem.

▶ Theorem 6 (Main). Let \((X, \pi)\) be a \((d+1)\)-partite (weighted) totally connected complex. For any distinct \(i, j \in [d]\), let \(\epsilon_{(i,j)} = \max_{r:\text{type}(r)=|\{}i\}, j\}} \lambda_2(P_r)\) be the 2nd largest eigenvalue of the simple random walk matrices on \((X_\tau, \pi_\tau)\) for all \(\tau\) of type \([d] \setminus \{i, j\}\). For some \(0 < \delta < 1\), assume that for every \(i \in [d]\),

\[
\epsilon_{(i,j)} \cdot H_{\Delta - 1} \leq \frac{\delta^2}{10}, \forall j \neq i \quad \text{and}
\sum_{\ell=1}^{\Delta(i)} \epsilon_{(i,j)} \cdot \frac{H_{\Delta(i)-1}(\ell-1)}{1} \leq 1 - \delta,
\]

where \(j_0, \ldots, j_d\) is an ordering of \([d] \setminus i\) such that \(\epsilon_{(i,j_0)} \leq \cdots \leq \epsilon_{(i,j_d)}\). Then, (the link of the emptyset of) \(X\) is a \((\frac{c(1-\delta)}{\delta}, \ldots, \frac{(1-\delta)}{d\delta})\)-expander for \(c = \frac{2(1+\frac{\delta^2}{1+\delta})}{1+\delta^2}\).

▶ Remark 7. If, for some \(\delta > 0\), the conditions of the above theorem hold for a complex \((X, \pi)\), then the conditions also hold for the same \(\delta\) for all links \((X_\tau, \pi_\tau)\) (of faces of co-dimension at least 2). Therefore, this theorem implies that \(X\) is \((\frac{c(1-\delta)}{\delta}, \ldots, \frac{(1-\delta)}{d\delta})\)-local spectral expander for \(c = \frac{2(1+\frac{\delta^2}{1+\delta})}{1+\delta^2}\). One can prove tighter bounds if they apply this theorem to any link \((X_\tau, \pi_\tau)\) individually and possibly use better bounds on \(\Delta(\tau)\).

Proof of Theorem 4. Fix a face \(\tau\) of co-dimension \(k\). For brevity we abuse notation and write \(\Delta\) denote \(\Delta(\tau, \pi)\). If \(k \geq \Delta\) the statement follows from the above remark. In particular, for any \(i, j \in [d]\)

\[
\epsilon_{(i,j)} \cdot H_{\Delta-1} \leq \gamma_2 \cdot H_{\Delta-1} \leq \gamma_2 \cdot (1 + \ln \Delta) \leq \frac{\delta^2}{10},
\]

\[
\sum_{\ell=1}^{\Delta(i)} \epsilon_{(i,j)} \cdot H_{\Delta(i)-1}(\ell-1) \leq \gamma_2(\Delta + \ln \Delta) \leq 1 - \delta,
\]

So, we can apply Theorem 6.
Otherwise, to bound the spectral expansion of \((X_\tau, \pi_\tau)\), let \(\delta_k = 1 - (1 - \delta) \frac{k + \ln k}{\Delta + \ln \Delta} \geq \delta\). For \(i, j \in [d]\)
\[
\epsilon_{(i,j)} \cdot H_{\Delta, r-1} \leq \gamma_2 \cdot H_{k-1} \leq \frac{\delta^2 \cdot H_{k-1}}{10(1 + \ln \Delta)} \leq \frac{\delta^2}{10},
\]
\[
\sum_{\ell=1}^{\Delta(i)} \epsilon_{(i,j)} \cdot H_{\Delta, (i)-1} \leq \gamma_2(k + \ln k) \leq \frac{(1 - \delta)(k + \ln k)}{\Delta + \ln \Delta} = 1 - \delta_k.
\]
Therefore, applying Theorem 6 to \((X_\tau, \pi_\tau)\), we obtain that \((X_\tau, \pi_\tau)\) is a \(c(1 - \delta_k)\)-expander. \(\blacksquare\)

**Applications to Graph Coloring**

Consider a graph \(G = ([n], E)\) with degree function \(\Delta : [n] \to \mathbb{Z}_{\geq 0}\) and maximum degree \(\Delta\), paired with a collection of color lists \(\{L(i)\}_{i \in [n]}\) satisfying \(L(i) \geq \Delta(i) + (1 + \eta)\Delta\) for all \(i \in [n]\) and for some \(0 < \eta \leq 0.9\) such that \(\frac{1 + \ln \Delta}{3} \leq \frac{\eta^2}{40}\).

We define the \((n+1)\)-partite coloring complex \(X(G, L)\) specified by the following facets: \(\{i, \sigma(i)\}_{i \in [n]}\) is a facet if and only if \(\sigma\) is a proper \(L\)-coloring of \(G\), i.e., \(\sigma(i) \in L(i)\) for each \(i \in [n]\) and \(\sigma(i) \neq \sigma(j)\) if \(\{i, j\} \in E\). It is not hard to see that if \(\{i, j\} \notin E\), then \(\epsilon_{(i,j)} = 0\). Moreover, if \(\{i, j\} \in E\), then \(\epsilon_{(i,j)} \leq \frac{1}{(1 + \eta)\Delta} + \frac{1}{(1 + \eta)^2\Delta^2}\) (see Theorem 4.4 in [1]).

Once can verify that if we apply the above theorem to the coloring complex \(X(G, L)\) with \(\delta = \frac{\eta}{2}\), we get that \(X(G, L)\) is a \((\frac{d}{\sqrt{p}}, \frac{\eta}{2}, \ldots, \frac{\eta}{2\sqrt{p}})\)-local spectral expander, and thus the Glauber dynamics for sampling a random proper coloring mixes in polynomial time. This retrieves (up to constants) a theorem proved in [1].

**Applications to Sparse High Dimensional Expanders**

Kaufman and Oppenheim [13] obtained a simple construction of sparse \((d + 1)\)-partite complexes with \(|X(0)| \geq p^s\) for any integer \(s > d\) and prime power \(p\) such that every \(x \in X(0)\) is in at most \(p^{O(d)}\) many facets (hence the degree is independent of \(s\)). They argued that for any non-consecutive pair of parts \(i, j \in [d]\), i.e., \(j \neq i + 1\) and \(i \neq j + 1\) (mod \(d + 1\)), we have \(\epsilon_{(i,j)} = 0\) but \(\epsilon_{(i,i+1)} \leq \frac{1}{\sqrt{p}}\) for any \(i \in [d]\) (\(i + 1\) is taken modulo \(d + 1\)). Consequently, \(\Delta(i) = 2\) for any \(i \in [d]\). Then, using Theorem 2, they show that the complex is a \((\frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p^2 - d - 2}}, \ldots, \frac{1}{\sqrt{p^d - d})})\)-local spectral expander for \(p > (d - 2)^2\). Simply plugging in these values into the above theorem, for \(\delta = 1 - \frac{\eta\sqrt{p}}{\sqrt{p^d - d})}\) and \(p \geq 193\) (independent of \(d\)) the assumptions of the theorem are satisfied. The resulting complex is \((\frac{2\sqrt{p}}{\sqrt{p^2 - d + 1}}, \ldots, \frac{2\sqrt{p}}{\sqrt{p^d - d})})\)-local spectral expander for \(c \approx 1.15\). In other words, not only does the Kaufman–Oppenheim construction give a HDX for constant values of \(p\) independent of \(d\), but also its local spectral expansion improves inverse linearly with the co-dimension. O’Donnell and Pratt [17] constructed \((d+1)\)-partite (sparse) high-dimensional expanders, with unbounded dimension \(d\), via root systems of simple Lie Algebras, namely families \(A_d\) for \(d \geq 1\), \(B_d\) for \(d \geq 2\), \(C_d\) for \(d \geq 3\) and \(D_d\) for \(d \geq 4\). For explicit descriptions of these root systems, see e.g. [5, Sec. 3.6]. O’Donnell and Pratt [17] showed that, similar to the Kaufman–Oppenheim construction, the resulting \(d\)-dimensional complex \(X\) satisfies \(|X(0)| \geq p^{\Theta(m)}\) whereas every vertex is only in \(p^{\Theta(d)}\) many facets and for any \(i, j \in [d]\), \(\epsilon_{i,j} \leq \frac{\sqrt{2}}{\sqrt{p}}\). Then, using Theorem 2 they concluded that the complex is a \((\frac{1}{\sqrt{p^2 - d + 1}}, \ldots, \frac{1}{\sqrt{p^d - d})})\)-local spectral expander. Upon further inspection of the explicit set of roots, one can verify that \(\Delta \leq 2\) for complexes based on \(A_d, B_d, C_d, D_d\) root systems and \(\Delta \leq 3\) for the \(D_d\) root system. Plugging in these values in the above theorem and setting \(\delta = 1 - 2\sqrt{2/p}\) for \(A_d, B_d, C_d\)
An Improved Trickle down Theorem for Partite Complexes

complexes and \( \delta = 1 - 3.5\sqrt{2/p} \) for the \( D_d \) complex, if \( p \geq 376 \) for \( A_d, B_d, C_d \) complexes and \( p \geq 729 \) for the \( D_d \) complex, we get that these complexes are \( (\frac{c}{\sqrt{p}}, \ldots, \frac{c'}{\sqrt{p}}) \)-local spectral expander for some constant \( c' > 1 \).

The well known Ramanujan complexes, also known as LSV complexes, are generalizations of Ramanujan graphs that were introduced by Lubotsky, Samuels, and Vishne in [14] and explicitly constructed in [16]. Any \( d \)-dimensional LSV complex \( X \) that is \( q \)-thick for some fixed prime power \( q \) and \( d \geq 2 \) has a bounded degree (the number of facets that contain each \( x \in X(0) \) only deponents on \( q \) and \( d \), and is constant in the size of the ground set \( n \) which can be arbitrarily large) (e.g. see [12]). Moreover, the link of every proper face of type \( S \) is a spherical building complex in which \( \Delta(i) = |\{ j \neq i : \epsilon_{i,j} > 0 \}| \) is at most 2 for every \( i \in [d] \setminus S \). Furthermore, the worst expansion among links co-dimension 2 is \( \frac{c}{\sqrt{q}} \), for some constant \( c \) independent of \( q, d, n \). So, there is a constant \( q_0 \) such that if \( q \geq q_0 \), Theorem 6 implies that the link of any (proper) face of \( X \) of co-dimension \( k \) is a \( \frac{c}{\sqrt{k+1}} \)-spectral expander for some constant \( c' > 0 \) independent of \( q, d, n \). This improves over the bound \( \frac{C(d)}{\sqrt{q}} \) proved in [12], where \( C(d) \geq 2^d(d+1)! \).

1.2 Proof Overview

At a high-level, our proof builds on the matrix trickle down framework introduced in the work of the authors with Liu [1]. The Oppenheim’s trickle down theorem follows from an inductive argument that derives a bound on the second eigenvalue of the simple walk on 1-skeleton of each link \((X_\tau, \pi_\tau)\) using the largest second eigenvalue of the simple walk on the 1-skeleton of links \((X_{\tau'}, \pi_{\tau'})\) for all faces \( \tau' \supset \tau \) of size \( |\tau| + 1 \). The reason that one has to take the largest 2nd eigenvalue as opposed to the average in each inductive step is that the eigenspaces of these simple walks are very different. The matrix trickle down framework overcomes this issue by substituting the scalar bounds on the second eigenvalues with matrices that upper bound the transition probability matrices of the simple walks on the 1-skeletons of links. However, as opposed to Oppenheim’s trickle down theorem, the matrix trickle down framework cannot be applied in a black-box manner to bound the spectral expansion of the 1-skeletons of all links only by bounding the spectral expansion of the 1-skeletons of links of faces of co-dimension 2. The main result of this paper can be seen as applying the matrix trickle down framework with a carefully chosen set of upper-bound matrices to prove an improved trickle down theorem for partite complexes that can be applied in the same black-box fashion, just known an “average” second eigenvalue.

Our technical contribution in this paper are twofold: First, we observe that for any two disjoint sets of parts \( S, T \subseteq [d] \), if the links of all faces of co-dimension 2 whose types intersect with both \( S, T \) are 0-spectral expanders, then for any \( \sigma \in X \) of type \( S \) and \( \tau \) of type \( T \) we get

\[
P_{\eta \sim \pi}[\sigma \subset \eta | \tau \subset \eta] = P_{\eta \sim \pi}[\sigma \subset \eta] \quad \text{and} \quad P_{\eta \sim \pi}[\tau \subset \eta | \sigma \subset \eta] = P_{\eta \sim \pi}[\tau \subset \eta],
\]

namely, the conditional distributions on these types are independent (see Lemma 18 for details). This observation significantly simplifies invoking the Matrix trickle down framework. Armed with this tool, we invoke the matrix trickle down theorem using a carefully chosen family of (diagonal) matrices as the matrix bounds. These matrices are recursively defined based on an “average” of the spectral expansions of the links of all faces of co-dimension 2, See the proof of Theorem 6 for the construction of these matrices.
2 Preliminaries

For any integer \( n \geq 0 \), let \([n] = \{0, \ldots, n\}\). When it is clear from context, we write \( x \) to denote a singleton \( \{x\}\). Given a set \( S \), we write \( v \in S^R \) and \( M \in R^{S \times S} \) to respectively denote a vector and a matrix indexed by \( S \). Given a probability distributions \( \mu \) over a set \( S \), we may view \( \mu \) as a vector in \( R^S \). For a \( n \times n \) matrix \( M \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \), define \( \rho(A) = \max_{1 \leq i \leq n} |\lambda_i| \).

Graphs

Given a graph \( G = (V, E) \), for any \( v \in V \), let \( \Delta_G(v) \) be the degree of \( v \) in \( G \), and let \( \Delta_G \) be the maximum degree of \( G \). Moreover, given a subset \( S \subseteq V \), \( G[S] \) denotes the induced subgraph of \( G \) on the set of vertices \( S \). For any \( S \subseteq V \), define \( G_S = G[V \setminus S] \). For simplicity of notation, when \( G \) is clear from context, we denote \( \Delta_G(v) \) by \( \Delta(v) \) for any \( v \in V \), and for any \( S \subseteq V \), we denote \( \Delta_{G_S}(v) \) by \( \Delta_S(v) \) for any \( v \in V \setminus S \). Similarly, we denote the maximum degree of \( G \) and \( G_S \) by \( \Delta \) and \( \Delta_S \) respectively. Moreover, when \( G \) is clear from context, we write \( u \sim v \) if \( u, v \) are adjacent vertices in \( G \) and \( u \sim_S v \) if \( u, v \in V \setminus S \) and \( u \sim v \).

We say that a graph \( G = (V, E) \) paired with a weight function \( w : E \to R_{\geq 0} \) is \( \epsilon \)-expander if \( \lambda_2(P) \leq \epsilon \), where \( P \in R^{V \times V} \) is the transition probability matrix of the simple random walk on \( (G, w) \) defined as \( P(x, y) = \frac{w(x, y)}{\sum_z w(x, z)} \) for any \( x, y \in V \). For such a graph we write \( d_w(x) = \sum_{y \sim x} w(x, y) \) to denote the weighted degree of a vertex \( x \) and \( \text{vol}(S) = \sum_{x \in S} d_w(x) \) to denote the volume of a set \( S \subseteq V \).

2.1 Linear Algebra

\( \blacktriangleright \) Lemma 8 (Cheeger’s Inequality). For any graph \( G = (V, E) \) with weights \( w : E \to R_{\geq 0} \) and any \( S \subseteq V \),
\[
\frac{w(E(S, S^c))}{\min\{\text{vol}(S), \text{vol}(S^c)\}} \leq \sqrt{2(1 - \lambda_2)}
\]
where \( \lambda_2 \) is the second largest eigenvalue of the simple random walk on \( G \).

\( \blacktriangleright \) Lemma 9 (Expander Mixing Lemma). Given a (weighted) graph \( G = (V, E, w) \), for any set \( S \subseteq V \),
\[
\left| w(E(S)) - \frac{\text{vol}(S)^2}{\text{vol}(V)} \right| \leq \lambda_2 \text{vol}(S),
\]
where \( \lambda_2 \) is the second largest eigenvalue of the simple random walk on \( G \).

2.2 Simplicial Complexes

We say that a simplicial complex \( X \) is gallery connected if for any face \( \tau \) of co-dimension at least 2 and any pair of facets \( \sigma, \sigma' \) of \( X_{\tau} \) there is a sequence of facets of \( X_{\tau}, \sigma = \sigma_0, \sigma_1, \ldots, \sigma_{\ell} = \sigma' \), such that for all \( 0 \leq i < \ell \), \( |\sigma_i \Delta \sigma_{i+1}| = 2 \). It is shown in [18, Prop 3.6] that if \( X \) is totally connected, then it is gallery connected.

\( \blacktriangleright \) Lemma 10. Consider a totally connected \((d + 1)\)-partite complex \( X \) with parts indexed by \([d]\). For any \( S \subseteq [d] \), The induced subgraph of the 1-skeleton of \( X \) on vertices of type \( S \) is connected.
10:8 An Improved Trickle down Theorem for Partite Complexes

Proof. Take \( x, y \) of type \( i, j \in S \) and facets \( \eta, \eta' \) such that \( x \in \eta, y \in \eta' \). Total connectivity implies that there is a sequence \( \eta = \eta_1, \ldots, \eta_t = \eta' \) such that \( \eta_i \cap \eta_{i+1} \neq \emptyset \) for all \( 1 \leq i \leq t - 1 \). Let \( \sigma_1 \subseteq \eta_1, \ldots, \sigma_t \subseteq \eta_t \) be faces of type \( \{i, j\} \). Then \( \sigma_1, \ldots, \sigma_t \) gives a path between \( x, y \).

Given a (weighted) complex \( (X, \pi) \), for integer \(-1 \leq i \leq \dim(X) - 1\), \( \pi \) induces a distribution \( \pi_i \) on \( X(i) \),

\[
\pi_i(\sigma) = \frac{1}{(\dim(X)+1)} \Pr_{\tau \sim \pi}[\sigma \subset \tau] \quad \forall \sigma \in X(i).
\]

Let \( P_{(X,\pi),\tau} \in \mathbb{R}^{X(0) \times X(0)} \) denote the transition probability matrix of the simple random walk on the 1-skeleton of \( (X, \pi, \tau) \) padded with zeros outside the \( X_\tau(0) \times X_\tau(0) \) block, i.e.,

\[
P_{(X,\pi),\tau}(x,y) = \sum_{\{x,y\} \subseteq \tau} P_{\pi}(x,y) = \begin{cases} \Pr_{\pi \sim \tau}\[x,y\] & \text{for } x,y \in X_\tau(0), \\ 0 & \text{otherwise.} \end{cases}
\]

When the weighted complex \( (X, \pi) \) is clear from context, we write \( P_\tau \) to denote \( P_{(X,\pi),\tau} \). For any \( \tau \) of co-dimension at least 2, we define the diagonal matrix \( \Pi_{(X,\pi),\tau} \in \mathbb{R}^{X(0) \times X(0)} \) as follows:

\[
\Pi_{(X,\pi),\tau}(x,x) = \pi_{\tau,0}(x) \quad \text{for } x \in X_\tau(0), \quad \Pi_{(X,\pi),\tau}(x,x) = 0 \quad \text{otherwise.}
\]

When \( (X, \pi) \) is clear from context, we write \( \Pi_\tau \) to denote \( \Pi_{(X,\pi),\tau} \). Note that \( \Pi_\tau P_\tau \) is a symmetric matrix.

Given a \((d+1)\)-partite complex, we say that an \( x \in X(0) \) is of type \( i \) and write type\( (x) = i \) if \( x \in T_i \). Similarly, the type of a face \( \tau \in X \) is defined as type\( (\tau) = \{i \in [d] : |\tau \cap T_i| = 1\} \). The following facts hold for weighted partite complexes.

\begin{itemize}
\item \textbf{Observation 11.} Consider a weighted \((d+1)\) partite complex \((X, \pi)\) and a face \( \tau \) of co-dimension \( k \geq 1 \). We have \( k \pi_{\tau,0}(x) = \Pr_{\pi \sim \tau}[x \in \sigma] \) for all \( x \in X_\tau(0) \).
\item \textbf{Observation 12.} Consider a weighted \((d+1)\) partite complex \((X, \pi)\) with parts indexed by \([d]\) and a face \( \tau \) of co-dimension \( k \geq 1 \). For any \( i \in [d] \),

\[
\sum_{x : \text{type}(x) = i} \Pr_{\pi \sim \tau}[x \in \sigma] = 1.
\]
\end{itemize}

The following definition is useful for proving the main theorem.

\begin{itemize}
\item \textbf{Definition 13.} For any \((d+1)\)-partite complex \((X, \pi)\) with parts indexed by \([d]\), define a graph \( G_{(X,\pi)} \) on the set of vertices \([d]\), where any distinct \( i, j \in [d] \) are adjacent in \( G_{(X,\pi)} \) if there exists \( \tau \) of type \([d] \setminus \{i,j\}\) such that the second eigenvalue of \((X_\tau, \pi_\tau)\) is positive.
\item \textbf{Remark 14.} For any \((d+1)\)-partite complex \((X, \pi)\) with parts indexed by \([d]\), for every \( i \in [d] \), \( \Delta(i) \) (see Definition 3) is the degree of \( i \) in graph \( G_{(X,\pi)} \) and \( \Delta \) is the maximum degree of \( G_{(X,\pi)} \).
\end{itemize}

Note that if \( \text{codim}(\tau) = k \), the link \( X_\tau \) is a \( k \)-partite complex with parts indexed by \([d] \setminus S\). One can verify that given a face \( \tau \) of type \( S \), the set of edges of \( G_{(X_\tau,\pi_\tau)} \) is a subset of the edges of \( G_{(X,\pi)} \), i.e., the induced subgraph of \( G_{(X,\pi)} \) on \([d] \setminus S\). When \((X, \pi)\) is clear from context, we write \( G \) for \( G_{(X,\pi)} \) and \( G_S \) for \( G_{(X,\pi)} \).

\section*{Product of Weighted Complexes}

Given weighted complexes \((Y_1, \mu_1), \ldots, (Y_t, \mu_t)\) defined on disjoint ground sets and of dimensions \(d_1, \ldots, d_t\) respectively, and a weighted complexes \((X, \pi)\) of dimension \(d\), we write \( (X, \pi) = (Y_1, \mu_1) \times \cdots \times (Y_t, \mu_t) \) if \( X(d) = \{ \cup_{i \in [t]} \tau_i : \tau_1 \in Y_1(d_1), \ldots, \tau_t \in Y_t(d_t) \} \) and \( \pi(\cup_{i \in [t]} \tau_i) = \prod_{i \in [t]} \mu_i(\tau_i) \) for all \( \tau_1 \in Y_1(d_1), \ldots, \tau_t \in Y_t(d_t) \). We denote the generating polynomial of \((X, \pi)\) by \( g_{(X,\pi)} \), i.e.,

\[
g_{(X,\pi)} = \sum_{\tau \subseteq X(d)} \pi(\tau) \prod_{x \in \tau} x \tau.
\]

One can verify that \( g_{(X,\pi)} = g_{(X_1,\mu_1)} \times \cdots \times g_{(X_t,\mu_t)} \) and only if \( g_{(X,\pi)} = g_{(X_1,\mu_1)} \times \cdots \times g_{(X_t,\mu_t)} \). Note that this is true because we assume that for any weighted simplicial complex, the given distribution on facets is non-zero on all facets.
Matrix Trickle Down Theorem

We use the following theorem which is the main technical theorem in [1].

**Theorem 15** ([1, Thm III.5]). Let \((X, \pi)\) be a totally connected weighted complex. Suppose \(\{M_\tau \in \mathbb{R}^{X(0) \times X(0)}\}_{\tau \in X(\leq d - 2)}\) is a family of symmetric matrices satisfying the following:

1. **Base Case:** For every \(\tau\) of co-dimension 2, we have the spectral inequality
   \[
   \Pi_\tau P_\tau - 2\pi_{\tau,0}\pi_{\tau,0}^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.
   \]

2. **Recursive Condition:** For every \(\tau\) of co-dimension at least \(k \geq 3\), at least one of the following holds: \(M_\tau\) satisfies
   \[
   M_\tau \preceq \frac{k - 1}{3k - 1}\Pi_\tau \quad \text{and} \quad \mathbb{E}_{x \sim \pi_\tau} M_{\tau \cup \{x\}} \preceq M_\tau - \frac{k - 1}{k - 2} M_\tau \Pi_\tau^{-1} M_\tau.
   \]
   Or, \((X_\tau, \pi_{\tau,k-1})\) is a product of weighted simplicial complexes \((Y_1, \mu_1), \ldots, (Y_t, \mu_t)\) and for every \(\eta \in X_\tau(k - 1)\),
   \[
   M_\tau = \bigoplus_{1 \leq i \leq t, d_i \geq 1} \frac{d_i (d_i + 1)}{k(k - 1)} M_{\tau \cup \eta_i - 1},
   \]
   where \(\eta_i = \eta \setminus Y_i(0)\).

Then for every \(\tau \in X(\leq d - 2)\), we have the bound \(\lambda_2(\Pi_\tau P_\tau) \leq \rho(\Pi_\tau^{-1} M_\tau)\).

### 3 Simplifying Matrix Trickle Down’s Conditions to Scalar Inequalities

In this section, given a \((d + 1)\)-partite complex \((X, \pi)\), we apply the matrix trickle down theorem to derive a set of conditions on a family of vectors \(\{f_S \in \mathbb{R}^{|S|}\}_{S \subset [d], |S| < d}\) that will guarantee that \(\lambda_2(P_\tau) \leq \frac{\max_{i \in [d]} f_S(i)}{k - 1}\) for all \(k \geq 2\) and co-dimension \(k\) and type \(S\). We prove the following theorem.

**Theorem 16.** Consider a totally connected \((d + 1)\)-partite complex \((X, \pi)\) with parts indexed by \([d]\) and graph \(G = G(X, \pi)\). Suppose we are given a family of vectors \(\{f_S \in \mathbb{R}^{|S|}\}_{S \subset [d], |S| < d}\) such that for all \(S \subset [d]\) of size \((d + 1) - k\), the support of \(f_S\) is a subset of \([d]\ \backslash \ S\), and the following holds:

- If \(G_S\) is disconnected, then \(f_S = \sum_{1 \leq i \leq \ell} f_{[d] \setminus I_i}\), where \(I_1 \cup \cdots \cup I_\ell\) are the vertices of the connected components of \(G_S\). Note that if all connected components are of size 1, then \(f_S = 0\).

- Otherwise if \(G_S\) is connected, we have \(\max_{i \in [d]} f_S(i) \leq \frac{(k - 1)^2}{3k - 1}\) and
  (i) **Base Case:** If \(k = 2\), then for every face \(\tau\) of type \(S\), \(\lambda_2(P_\tau) \leq \max_{i \in [d] \setminus S} f_S(i)\).
  (ii) **Recursive Condition:** If \(k \geq 3\), then
   \[
   \sum_{j \in [d] \setminus (S \cup i)} f_{S \cup j}(i) \leq (k - 2)f_S(i) - f_S^2(i),
   \]
   for all \(i \in [d] \setminus S\).

Then, for all \(k \geq 2\) and \(\tau\) of co-dimension \(k\) and type \(S\), \(\lambda_2(P_\tau) \leq \frac{\max_{i \in [d]} f_S(i)}{k - 1}\).

The main sets of conditions in the above theorem are the inequalities in Item i and Item ii. To get some intuition about these conditions, it is helpful to compare the above with the standard trickle down theorem (Theorem 2). There, one shows that if \(\lambda_2(P_{\tau \cup \{x\}}) \leq \lambda\) for all \(x \in X_\tau(0)\), then \(\lambda_2(P_\tau) \leq \alpha\), where satisfies

\[
\lambda \leq \alpha - \alpha^2(1 - \lambda).
\]
Then, Theorem 2 follows by recursively applying this inequalities.

In the above theorem, instead of a single upper bound on \( \lambda_2(P_\tau) \) for faces \( \tau \) of co-dimension 2, one bounds the expansion of the links of all faces of co-dimension 2 of each type separately, allowing higher degrees of freedom. For any face \( \tau \) of type \( S \) and co-dimension \( k = |S| \), the function \( \frac{f_S(i)}{i-1} \) will serve as the diagonal entries of a matrix upper-bound \( P_\tau \).

Then, the inequality \( \frac{\sum_{i=1}^{d} \sum_{j=0}^{d} f_{S\cup j}(i)}{d} \leq f_S(i) - \frac{f_S^2(i)}{k-2} \) is the natural analogue of (4) which requires \( f_S \) to be at least “the average” of \( f_{S\cup j} \) for all \( j \in [d] \setminus S \). Then, (5) implies that for any face \( \mu \) of type \( S \) and its associated graph \( G \), we get

Then, Theorem 2 follows by recursively applying this inequalities.

Proof. Note that \((X, \pi)\) is a weighted bipartite graph with parts \( S, T \). Let \( G_{S,T} = G((y, z) = A(y, z) \) for \( y \in S, z \in T \) and 0 on other vertices. Moreover, let \( A_{S,T} = A - A_{S,T} \). Then, for any vector \( v \in \mathbb{R}^{X(0) \times X(0)} \), we get \( A = A_{S,T} + \lambda v \). Then, if \( Av = \lambda v \), then \( Av' = -\lambda v' \), for \( v' = (v_0 + v_T) \). So if \( \mu \) is an eigenvalue of \( A \), then \( -\mu \) is also an eigenvalue of \( A \). Thus, if \((X, \pi)\) is 0-expander, the rank of \( A \) is 2. This implies that there are vectors \( w_S \in \mathbb{R}^S \) and \( w_T \in \mathbb{R}^T \) such that \( \pi((y, z)) = A(y, z) = A(z, y) = w_S(y)w_T(z) \) for \( y \in S, z \in T \). Without loss of generality, assume \( \|w_S\|_1 = \|w_T\|_1 = 1 \). Then, for any \( y \in S \) and \( z \in T \), we have \( \frac{\sum_{i \in S} \pi_i(y, z)) = s(y) = w_S(y) \). Similarly, \( \pi_i(z) = w_T(z) \). Thus, \( \pi((y, z)) = \pi_y(z) \pi_z(y) \). This finishes the proof.

Lemma 17. Consider a 2-partite complex \((X, \pi)\) with parts \( S, T \). If \((X, \pi)\) is 0-expander, then \((X, \pi) = (X_x, \pi_z) \times (X_y, \pi_y) \) for any \( y \in S \) and \( z \in T \).

Proof. We prove the statement by induction on \( d \). For \( d = 1 \), the statement simply follows from Lemma 17. Now, assume that \( d > 1 \). If \( |I| = 1 \) for all \( 1 \leq i \leq \ell \), then \( \ell \geq 3 \). In this case, let \( S = I_1 \cup I_2 \). Otherwise, WLOG assume that \( |I_1| \geq 2 \) and let \( S = I_1 \). First, we show that \( g_{(X, \pi)} \) can be written as \( g(x, \pi) = h \cdot h' \), where \( h \) is a polynomial in \( \{z_y : \text{type}(y) \in I \setminus S\} \) and \( h' \) is a polynomial in terms of variables in \( \{z_y : \text{type}(y) \in I \setminus S\} \). By induction hypothesis, for any \( i \in S, x \in T_i \), and any face \( \sigma \in X \) of type \( S \) such that \( x \in \sigma \)

\[
\frac{\partial x \cdot g_{(X, \pi)}}{f_x} = f_x \cdot g_x
\]

where \( f_x \) is a polynomial in terms of variables in \( \{z_y : \text{type}(y) \in I \setminus S\} \) and \( g_x \) is a polynomial in terms of variables in \( \{z_y : \text{type}(y) \in I \setminus S\} \). Now, take arbitrary \( i, j \in S \) such that \( i \neq j \). Then, (5) implies that for any face \( (x, y) \) of type \( \{i, j\} \)

\[
\frac{\partial x \cdot \partial y \cdot g_{(X, \pi)}}{f_x} = (\partial_x f_x) g_x = (\partial_x f_x g_y)
\]

It thus follows that \( g_x \) is a multiple of \( g_y \). One can see this simply by substituting 1 for all variables in \( \{z_y : \text{type}(y) \in S \setminus \{i, j\}\} \). Moreover, since \( g_x \) and \( g_y \) are generating polynomials of distributions, i.e., the coefficients sum up to 1, we get \( g_x = g_y \). Therefore, we get that
for any distinct $x, y$ such that type$(x), \text{type}(y) \in S$ and $\{x, y\}$ is a face, $g^x = g^y$. Applying Lemma 10, we get $g^x = g^y$ for all $x, y \in \cup_{i \in S} T_i$. Thus, there exist a polynomial $h$ in variables $\{z_y : \text{type}(y) \in S \setminus i\}$ such that we can rewrite (5) for any $x$ with type$(x) \in S$ as
\[
\partial_{z_x} g(X, \pi) = f^x \cdot h,
\]
where $f^x$ is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in S \setminus i\}$. Finally, since $X$ is a partite complex,
\[
|S| g(X, \pi) = \sum_{i \in S} \sum_{x \in T_i} z_x \partial_{z_x} g(X, \pi) = h \cdot \sum_{i \in S} \sum_{x \in T_i} z_x f^x = h \cdot h',
\]
where $h' = \sum_{i \in S} \sum_{x \in T_i} z_x f^x$ is a polynomial in $\{z_y : \text{type}(y) \in S\}$. It remains to show that for any face $\sigma$ of type $S$, we have $h = g(X, \pi_\sigma)$, and for any $\tau$ of type $[d] \setminus S$, we have $h' = g(X, \pi_\tau)$. Fix arbitrary faces $\sigma$ of type $S$ and $\tau$ of type $[d] \setminus S$. Noting that $g(X, \pi)$ is a multiple of $h \cdot h'$, and that $h'$ is a polynomial in variables associated to elements whose types are in $S$ and $h$ is in variables associated to elements whose types are in $[d] \setminus S$, we conclude that $h'$ has a monomial that is a multiple of $\prod_{x \in \tau} z_x$ and $h$ has a monomial that is a multiple of $\prod_{x \in \sigma} z_x$.

First, take $(\prod_{x \in \sigma} \partial_{z_x})$ from both sides of (6). We get that $g(X, \pi_\sigma)$ is a positive multiple of $h$. Similarly, taking $(\prod_{x \in \tau} \partial_{z_x})$ from both sides of (6), we get that $g(X, \pi_\tau)$ is a positive multiple of $h'$. Thus, noting that the coefficients of generating polynomials sum up to 1, we get $h = g(X, \pi_\sigma)$ and $h' = g(X, \pi_\tau)$ as desired. Repeating the same argument inductively on the complex $(X, \pi_\sigma)$ proves the claim.

Now we are ready to prove Theorem 16.

**Proof of Theorem 16.** We apply Theorem 15. For every $S \subset [d]$ such that $|S| < d$, define a diagonal matrix $D_S \in \mathbb{R}^{X(0) \times X(0)}$ as $D_S(x, x) = f_S(\text{type}(x))$ for all $x \in X(0)$. We prove that the conditions of Theorem 15 hold for $M_T = \frac{\Pi_T D_T}{k-1}$ for an arbitrary face $\tau \in X$ of co-dimension at least $k \geq 2$ and type $S$. If $G_S$ is connected, $\max_{i \in [d]} f_S(i) \leq \frac{(k-1)^2}{3k^2}$ holds by assumption. If $G_S$ is disconnected, $\max_{i \in [d]} f_S(i) \leq \frac{(k-1)^2}{3k^2}$ follows from the assumptions that $f_S = \sum_{\ell \leq \ell : |I_{\ell}| \geq 2} f_{[\ell]}$, where $I_1 \cup \cdots \cup I_{\ell}$ are the vertices of connected components of $G_S$.

That is because the supports of vectors $f_{[\ell]}$, are disjoint by assumption and $\frac{(k-1)^2}{3k^2}$ is an increasing function for $k \geq 2$. So, we get $D_T \leq \frac{(k-1)^2}{3k^2} I$, and thus, $M_T \leq \frac{k-1}{3k^3} \Pi_T$. To prove the rest of the conditions hold, first assume that $k = 2$. If $G_S$ is two disconnected vertices, we get $f_S = 0$, and therefore, $D_S = 0$. Thus, we get $\Pi_T M_T - \pi_{\sigma,0} \pi_{\eta,0} \leq 0 = \Pi_T D_S = M_T$, as desired. If $G_S$ is connected, the base case assumption (Item i) implies that $\lambda_2(P_T) \leq D_S(x, x)$ for all $x \in X_T$. Therefore, $\Pi_T M_T - \pi_{\sigma,0} \pi_{\eta,0} \leq \Pi_T D_S = M_T$. Now, assume $k \geq 3$.

First assume that $G_S$ is disconnected and $G[I_1], \ldots, G[I_{\ell}]$ are its connected components for some partition $I_1 \cup \cdots \cup I_{\ell}$ of $[d] \setminus S$. Fix any $\sigma \in X_T(k-1)$. By Lemma 18, $(X_T, \pi_\sigma) = (X_{\cup_{j \neq \sigma} I_j}, \pi_{\cup_{j \neq \sigma} I_j}) \times (X_{\sigma I_j}, \pi_{\sigma I_j})$ where for every $1 \leq j \leq \ell$, $\sigma_j$ is a subset of $\sigma$ that has type $[d] \setminus (S \cup I_j)$. Therefore, we get $Pr_{\eta \sim \pi_{\cup_{j \neq \sigma} I_j}}[x \in \eta] = Pr_{\eta \sim \pi_{\sigma}}[x \in \eta]$ for all $1 \leq j \leq \ell$ and $x \in X_{\cup_{j \neq \sigma} I_j}$.

Combining this with Observation 11, we get $k_j \cdot \pi_{\cup_{j \neq \sigma} I_j, 0}(x) = k \cdot \pi_{\sigma,0}(x)$, where $k_j = |I_j|$ for all $1 \leq j \leq \ell$. Thus we can write
\[
\sum_{1 \leq j \leq |I_j|} \frac{(k_j - 1)k_j}{(k-1)k} M_{\cup_{j \neq \sigma} I_j, \text{def of } M_{\cup_{j \neq \sigma} I_j}} \sum_{1 \leq j \leq |I_j|} \frac{(k_j - 1)k_j}{(k-1)k} \Pi_{\cup_{j \neq \sigma} I_j} D_{[\ell]}|I_j| = \frac{\Pi_T}{k-1} \sum_{1 \leq j \leq |I_j|} D_{[\ell]}|I_j| = \frac{\Pi_T D_S}{k-1} = M_T.
\]
where in the second to last equality, we used the fact that $\sum_{1 \leq j \leq \ell(i)} f_{i,j} = f_i$, and thus $\sum_{1 \leq j \leq \ell(i)} f_{i,j} = D_S$ by definition of $D_S$. Now, assume that $G_S$ is connected. It is enough to show that $E_{x \sim \tau, M\tau \cup Z \subseteq M \tau - M \tau \Pi^{-1} M \tau}$. This is equivalent to showing that for any $x \in X_\tau(0)$

$$E_{y \sim \tau, 0} \left[ (\Pi_{1}^{\tau} \Pi_{\tau \cup y} D_{S, type}(y)) (x, x) \right] \leq \frac{D_S(x, x)}{k - 1} - \frac{D_S^2(x, x)}{(k - 2)(k - 1)} \quad (7)$$

One can check that for any $x \in X_\tau(0)$ of type $i$

$$E_{y \sim \tau, 0} \left[ (\Pi_{1}^{\tau} \Pi_{\tau \cup y} D_{S, type}(y)) (x, x) \right] = \sum_{y \in X_{\tau \cup x}(0)} \Pr_{y \sim \tau \cup x, y \in S\tau \cup x(0)} [y \in \sigma] D_{S, type}(y) (x, x)$$

$$= \sum_{j \in \{\} \setminus S} \frac{f_{\tau \cup j} (i)}{(k - 1)(k - 2)} \sum_{y \in X_{\tau \cup x(0)} : y \sim \tau \cup x, type(y) = j} \Pr_{y \sim \tau \cup x, y \in S\tau \cup x(0)} [y \in \sigma]$$

where in the last equality, we used Observation 12. Thus, substituting $D_S(x, x) = f_S(type(x))$ in the RHS of (7), it is enough to show that for any $i \in [d] \setminus S$

$$\sum_{j \in \{\} \setminus S} f_{\tau \cup j} (i) \leq f_S(i) \frac{f_S^2(i)}{(k - 1)(k - 2)} \geq \frac{f_S^2(i)}{(k - 1)(k - 2)}$$

which holds by assumption Item ii.

4 Proof of Main Theorem

We are ready to prove Theorem 6.

**Proof of Theorem 6.** We find a family of vectors $\{f_S \in \mathbb{R}^{[d]} \}_{S \subseteq [d], |S| < d}$ that satisfy the conditions of theorem Theorem 16. Let $G = G_{(X, x)}$. Based on the conditions of Theorem 16, vectors $\{f_S \in \mathbb{R}^{[d]} \}_{S \subseteq [d], |S| < d}$ can be defined as functions of $\{\epsilon_{i,j} \}_{i, j \in [d], i \neq j}$. Recall that edges of $G$ capture pairs $\{i, j\}$ for which $\epsilon_{i,j} > 0$. Assign every edge $\{i, j\}$ of $G$ with weight $\epsilon_{i,j}$. We restrict our attention to functions that are very local with respect to $G$, i.e. for every $S$ and $i \in [d] \setminus S$, we assume $f_S(i)$ only depends on $\Delta_S(i)$ and the weights of edges adjacent to $i$ in $G_S$ if $\Delta(i) > 1$. It turns out that if $\Delta(i) = 1$, we would need to also take into account the degree of the unique neighbor of $i$. More formally, consider the following family of vectors $\{f_S \in \mathbb{R}^{[d]} \}_{S \subseteq [d], |S| < d}$: for any $S \subseteq [d]$ such that $|S| < d$, let $f_S$ be of the following form: for any $i \in S$, let $f_S(i) = 0$, and for any $i \in [d] \setminus S$ define

$$f_S(i) = \begin{cases} 0 & \text{if } \Delta_S(i) = 0, \\ \epsilon_{i,j} \cdot g_{i,j}(\Delta_S(j)) & \text{if } \Delta_S(i) = 1 \land i \sim_S j, \\ \sum_{j \sim_S i} \epsilon_{i,j} \cdot h_i(\Delta_S(i)) & \text{if } \Delta_S(i) \geq 2, \end{cases}$$

where for every $i \in [d]$ and $j \sim i$, functions $g_{i,j}, h_i : \{1, \ldots, \Delta \} \to \mathbb{R}_{\geq 0}$ are defined later in a way that guarantees that $\{f_S \}_{S \subseteq [d], |S| < d}$ satisfies the assumptions of Theorem 16 (see (10), (12)).
First, consider the case that $G_S$ is disconnected. Note that for any $S, S' \subset [d]$ such that $|S|, |S'| < d$, if $\{j \in [d] : j \sim i\} = \{j \in [d] : j \sim S'\}$ for some $i \notin S, S'$, then $f_S(i) = f_{S'}(i)$.

Let $I_1, \ldots, I_\ell$ be the vertices of connected components of $G_S$. Since the neighborhood of each vertex in any connected component of $G_S$ is the same as its neighborhood in $G_S$, we get $f_S = \sum_{1 \leq |I_i| \geq 2} f_{\{I_i\}}$.

Now, assume $G_S$ is connected. Take an arbitrary $k \geq 2$ and $S \subset [d]$ of size $(d + 1) - k$. First we verify the set of conditions given in Item i and Item ii. First, assume that $k = 2$. Let $[d] \backslash S = \{i, j\}$. By definition of $\epsilon_{(i,j)}$, for any $\tau$ of type $S$, $\lambda_2(P_{ \tau}) \leq \epsilon_{(i,j)}$. Thus, if we define $g_{\ell, t}(1) = 1$ for all distinct $\ell, t \in [d]$, then we get $\lambda_2(P_{ \tau}) \leq \epsilon_{(i,j)}g_{j, i}(1) = f_S(i) = f_S(j)$, as desired. Now, assume that $k \geq 3$. Fix an arbitrary $i \in [d] \backslash S$. Our goal is to define $g_{j, i}, h_i : \{1, \ldots, \Delta\} \rightarrow \mathbb{R}_{\geq 0}$ for all $j \sim i$ such that $g_{j, i}(1) = 1$ for all $j \sim i$ and the following inequality is satisfied:

$$\sum_{j \in [d] \backslash \{S, i\}} f_{S \cup j}(i) \leq (k - 2)f_S(i) - f^2_S(i).$$

(8)

To keep the notation concise, relabel the elements such that $i$ is relabeled to 0 and $\epsilon_{\{0,1\}} \geq \cdots \geq \epsilon_{\{0,d\}}$. Moreover, define $\epsilon_j = \epsilon_{\{0,j\}}$ for any $j \in [d] \backslash 0$.

**Case 1:** $\Delta_S(0) = 1$, and $j \sim 0$. Since $G_S$ is connected and $(d + 1) - |S| \geq 3$, we have $\Delta_S(j) \geq 2$. Define $t = \Delta_S(j)$. We have

$$\sum_{\ell \in [d] \backslash \{S, i\}} f_{S \cup \ell}(0) = f_{S \cup i}(0) + \sum_{\ell \in [d] \backslash \{S, i\} : \ell \sim \tau \not\sim \ell} f_{S \cup \ell}(0) + \sum_{\ell \in [d] \backslash \{S, i\} : \ell \not\sim \ell} f_{S \cup \ell}(0)$$

$$= 0 + (t - 1) \cdot \epsilon_j \cdot g_{0, j}(t - 1) + (k - t - 1) \cdot \epsilon_j \cdot g_{0, j}(t).$$

On the other hand, $(k - 2)f_S(0) - f_S(0)^2 = (k - 2) \cdot \epsilon_j \cdot g_{0, j}(t) - \epsilon_j^2 \cdot g_{0, j}^2(t)$. So it is enough to satisfy

$$(t - 1) \cdot \epsilon_j \cdot (g_{0, j}(t) - g_{0, j}(t - 1)) \geq \epsilon_j^2 \cdot g_{0, j}^2(t).$$

(9)

Now, define $g_{0, j} : \{1, \ldots, \Delta\} \rightarrow \mathbb{R}_{\geq 0}$ as follows: recall that we defined $g_{0, j}(1) = 1$. For any $2 \leq \ell \leq \Delta$, let

$$g_{0, j}(\ell) = 1 + 1.3 \cdot \epsilon_j \cdot H_{\ell - 1}.$$  

(10)

Using assumption (1), $\epsilon_j H_{\Delta - 1} \leq \epsilon_j^2 \leq \frac{1}{10}$. Thus

$$\epsilon_j^2 \cdot g_{0, j}(\ell) \leq \epsilon_j^2 \cdot g_{0, j}(1 + 1.3\epsilon_j (1 + H_{\Delta - 1}))^2 < 1.3\epsilon_j^2.$$

Substituting $g_{0, j}(\ell)$ according to (10) and using the above bound, one can verify that (9) holds.

**Case 2:** $\Delta_S(0) \geq 2$. For simplicity of notation, define $t = \Delta_S(0)$ and $\alpha = \sum_{j : j \sim 0} \epsilon_j$. Define $h_0(1) = \max_{j : j \sim 0} g_{0, j}(\Delta)$.

$$\sum_{j \in [d] \backslash \{S, i\}} f_{S \cup j}(0) = \sum_{j \in [d] \backslash \{S, i\} : j \sim 0} f_{S \cup j}(0) + \sum_{j \in [d] \backslash \{S, i\} : j \not\sim 0} f_{S \cup j}(0)$$

$$\leq \left( \sum_{j \in [d] \backslash \{S, i\} : j \sim 0} (\alpha - \epsilon_{\{0,j\}}) \right) \cdot h_0(t - 1) + (k - t - 1) \cdot \alpha \cdot h_0(t)$$

$$= (t - 1) \cdot \alpha \cdot h_0(t - 1) + (k - t - 1) \cdot \alpha \cdot h_0(t).$$
An Improved Trickle down Theorem for Partite Complexes

We need to prove (11) for a carefully chosen \( c \). Let \( \beta \) be such that \( h_0(t) = \frac{h_0(1)}{\beta} \). We get \( h_0(t - 1) = \frac{h_0(1)}{\beta + c(\sum_{j=1}^{t} \epsilon_j)} \), and thus,

\[
(t - 1)(h_0(t) - h_0(t - 1)) = \frac{h_0(1) \cdot c \sum_{j=1}^{t} \epsilon_j}{\beta \cdot (\beta + c(\sum_{j=1}^{t} \epsilon_j))}
\]

Note that \( \alpha \cdot h_0^2(t) = \frac{\alpha h_0^2(t)}{\beta} \). Thus, to satisfy (11), it is enough to show that

\[
\beta \cdot c \cdot \left( \sum_{j=1}^{t} \epsilon_j \right) \geq \alpha \cdot h_0(1) \cdot \left( \beta + c(\sum_{j=1}^{t} \epsilon_j) \right).
\]

Note that

\[
h_0(1) \leq \max_{j \sim 0} g_{0,j}(\Delta) = 1 + 1.3 \epsilon_1 H_{\Delta - 1} \leq 1 + 1.3 \delta^2 / 10.
\]

Moreover, \( \sum_{j=1}^{t} \epsilon_j \geq \sum_{j \sim 0} \epsilon_j = \alpha \). Thus, letting \( c = 1 + c' \delta \) for some \( c' > 0 \) that we choose later, it is enough to show that

\[
\beta \cdot (c' - 0.13) \delta \geq (1 + 0.13 \delta) \frac{(1 + c' \delta) \sum_{j=1}^{t} \epsilon_j}{t - 1}.
\]

Using \( \sum_{j=1}^{t} \epsilon_j \leq 2 \epsilon_1 \leq \frac{\delta^2}{10} \) by (1), it is enough to show that

\[
\beta \cdot (c' - 0.13) \delta \geq (1 + 0.13 \delta)(1 + c' \delta) \delta / 5.
\]

On the other hand,

\[
\beta \geq 1 - (1 + c' \delta) \left( \sum_{j=1}^{\Delta(0)} \epsilon_j H_{\Delta(0)-1}(j - 1) \right) \geq 1 - (1 + c' \delta)(1 - \delta) = \delta(1 - c' + c' \delta),
\]

Equivalently, it suffices to satisfy

\[
(t - 1)(h_0(t) - h_0(t - 1)) \geq \alpha \cdot h_0^2(t).
\]
Thus, to satisfy (14), it is enough to show that $(1 - c' + c'\delta)(c' - 0.13\delta) \geq (1 + 1.13\delta)(1 + c'\delta)\frac{1}{5}$. Letting $c' = \frac{1}{2}$, this inequality holds for every $0 < \delta < 1$. This establishes Equation (8). So we verified conditions Item i and Item ii are satisfied.

To show that all conditions of Theorem 16 are satisfied, it remains to show that $\max_{i \in [d]} f_S(i) \leq \frac{(k-1)^2}{100k^2}$. Note that $\sum_{j,\ell \in \tau} \epsilon_{i,j} \leq \Delta_S \cdot \epsilon_1 \leq \Delta_S \cdot \frac{\delta^2}{10}$ for all $i \in [d] \setminus S$.

Thus, we get $\max_{i \in [d]} f_S(i) \leq \Delta_S \cdot \frac{\delta^2}{10} \max_{i \in [d] \setminus S} h_i(\Delta_S(i))$. Moreover, using (13) and (15) with $c' = \frac{1}{2}$ (we can write this inequality for every $i$), we get

$$h_i(\Delta_S(i)) \leq h_i(\Delta(i)) \leq \frac{1 + \frac{\delta^2}{10}}{\delta(\frac{1}{2} + \frac{\delta}{2})},$$

Thus, we can write

$$\max_{i \in [d]} f_S(i) \leq \Delta_S \cdot \frac{\delta^2}{10} \frac{1 + \frac{\delta^2}{10}}{\delta(\frac{1}{2} + \frac{\delta}{2})} \leq \frac{\Delta_S}{5} \leq \frac{k - 1}{5} \leq \frac{(k-1)^2}{3k} \cdot \delta,$$

as desired. So we proved that $\{f_S\}_{S \subseteq [d] : |S| < d}$ satisfies the conditions of Theorem 16. Now, we are ready to bound $\lambda_2(P_{\tau})$ for any face $\tau$ of co-dimension $k \geq 2$ and type $S$. First, we show that for every $i \in [d] \setminus S$, $\sum_{j : j \sim_{\sim} i} \epsilon_{i,j} \leq 1 - \delta$. Note that

$$\sum_{\ell = 1}^{\Delta(i)} H_{\Delta(i) - 1}(\ell - 1) = \sum_{\ell = 2}^{\Delta(i)} \frac{\ell}{\ell - 1} = 2 + \sum_{\ell = 3}^{\Delta(i)} \frac{\ell}{\ell - 1} \geq \Delta(i).$$

Thus, we can write

$$\sum_{j \sim_{\sim} i} \epsilon_{i,j} \leq \left(\sum_{\ell = 1}^{\Delta(i)} \frac{H_{\Delta(i) - 1}(\ell - 1)}{\Delta(i)}\right) \left(\sum_{j \sim_{\sim} i} \epsilon_{i,j}\right) \leq \sum_{\ell = 1}^{\Delta(i)} \frac{H_{\Delta(i) - 1}(\ell - 1) \cdot \epsilon_{i,j}}{\Delta(i)} \leq 1 - \delta.$$

where we assumed that $i_1, \ldots, i_d$ is an ordering of $[d] \setminus S$ such that $\epsilon_{i_1} \leq \cdots \leq \epsilon_{i_d}$. Using this inequality and (16), we get

$$\lambda_2(P_{\tau}) \leq \frac{\max_{i \in [d] \setminus S} f_S(i)}{k - 1} \leq \frac{\max_{i \in [d] \setminus S} \left(\sum_{j \sim_{\sim} i} \epsilon_{i,j}\right) \cdot h_i(\Delta_S(i))}{k - 1} \leq \frac{(1 - \delta) \cdot \max_{i \in [d] \setminus S} h_i(\Delta(i))}{k - 1} \leq \frac{(1 - \delta) \cdot \frac{2(1 + \frac{\delta^2}{10})}{\Delta_S(i)}}{k - 1},$$

as desired.

---

**References**

1. Dorna Abdolazimi, Kuikui Liu, and Shayan Oveis Gharan. A matrix trickle-down theorem on simplicial complexes and applications to sampling colorings. In *FOCS*, pages 161–172. IEEE, 2021.

2. Vedat Levi Alev, Fernando Granha Jeronimo, and Madhur Tulsiani. Approximating constraint satisfaction problems on high-dimensional expanders. In David Zuckerman, editor, *FOCS*, pages 180–201. IEEE Computer Society, 2019.

3. Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. Spectral independence in high-dimensional expanders and applications to the hardcore model. *SIAM Journal on Computing*, FOCS20:1–37, 2020. doi:10.1137/20M1367696.
An Improved Trickle down Theorem for Partite Complexes

---

4. Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid. In Moses Charikar and Edith Cohen, editors, *STOC*, pages 1–12. ACM, 2019.

5. Roger Carter. Simple groups of Lie type. John Wiley & Sons, 1989.

6. Zongchen Chen, Kuikui Liu, and Eric Vigoda. Optimal mixing of glauber dynamics: entropy factorization via high-dimensional expansion. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC*, pages 1537–1550. ACM, 2021.

7. Yotam Dikstein and Irit Dinur. Agreement testing theorems on layered set systems. In David Zuckerman, editor, *FOCS*, pages 1495–1524. IEEE Computer Society, 2019. doi: 10.1109/FOCS.2019.00088.

8. I. Dinur and T. Kaufman. High dimensional expanders imply agreement expanders. In *FOCS*, pages 974–985, 2017.

9. Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. Locally testable codes with constant rate, distance, and locality. In Stefano Leonardi and Anupam Gupta, editors, *STOC*, pages 357–374. ACM, 2022.

10. Irit Dinur, Yuval Filmus, Prahladh Harsha, and Madhur Tulsiani. Explicit sos lower bounds from high-dimensional expanders. In James R. Lee, editor, 12th *Innovations in Theoretical Computer Science Conference, ITCS 2021, January 6-8, 2021, Virtual Conference*, volume 185 of *LIPIcs*, pages 38:1–38:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021.

11. Irit Dinur, Prahladh Harsha, Tali Kaufman, Inbal Livni Navon, and Amnon Ta-Shma. List-decoding with double samplers. *SIAM J. Comput.*, 50(2):301–349, 2021.

12. Shai Evra and Tali Kaufman. Bounded degree cosystolic expanders of every dimension. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pages 36–48, 2016.

13. Tali Kaufman and Izhar Oppenheim. Construction of new local spectral high dimensional expanders. In Ilias Diakonikolas, David Kempe, and Monika Henzinger, editors, *STOC*, pages 773–786. ACM, 2018. doi: 10.1145/3188745.3188782.

14. Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of ramanujan complexes of type $A_d$. *European Journal of Combinatorics*, 26(6):965–993, 2005.

15. Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of ramanujan complexes of type $A_d$. *European Journal of Combinatorics*, 26(6):965–993, 2005. Combinatorics and Representation Theory. doi:10.1016/j.ejc.2004.06.007.

16. Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Ramanujan complexes of type $A_d$. *Israel Journal of Mathematics*, 149:267–299, 2005.

17. Ryan O’Donnell and Kevin Pratt. High-Dimensional Expanders from Chevalley Groups. In Shachar Lovett, editor, 37th *Computational Complexity Conference (CCC 2022)*, volume 234 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 18:1–18:26, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

18. Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders part i: Descent of spectral gaps. *Discrete and Computational Geometry*, 59(2):293–330, 2018.