Dualization in Lattices Given by Ordered Sets of Irreducibles

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Abstract

Dualization of a monotone Boolean function on a finite lattice can be represented by transforming the set of its minimal 1 to the set of its maximal 0 values. In this paper we consider finite lattices given by ordered sets of their meet and join irreducibles (i.e., as a concept lattice of a formal context). We show that in this case dualization is equivalent to the enumeration of so-called minimal hypotheses. In contrast to usual dualization setting, where a lattice is given by the ordered set of its elements, dualization in this case is shown to be impossible in output polynomial time unless P = NP. However, if the lattice is distributive, dualization is shown to be possible in subexponential time.

1 Introduction

A monotone Boolean function on a finite lattice can be given by the set of minimal 1 values or by the set of its maximal 0 values. Dualization is the transformation of the set of minimal 1 values of a Boolean function to the set of its maximal 0 values or vice versa. Since dualization is equivalent to many important problems in computer and data sciences \cite{4, 17}, the paper \cite{8} on quasi-polynomial dualization algorithm for Boolean lattices was an important breakthrough. It paved the way to generalizations to various classes of structures where subexponential dualization in output-polynomial time is possible, among them dualization on lattices given by ordered sets of their elements or by products of simple lattices, like chains \cite{4, 5}.

A well-known fact is that every lattice is determined up to isomorphism by the ordered set of its meet (infimum) and join (supremum) irreducible elements \cite{9}. These elements cannot be represented as meets (joins) of other elements that are larger (smaller) then them. On diagram of finite lattices these elements have one upper (lower) neighbor. In this paper we consider finite lattices given by ordered sets of their meet and join irreducibles, known as concept lattices \cite{9, 1, 12}. We show that dualization for representation of this type is impossible in output polynomial time unless P = NP. However, in an important particular case where the lattice is distributive, we propose a subexponential algorithm.

Dualization in the considered case is not only of theoretical interest. Actually, this study was motivated by a practical problem of enumerating minimal hypotheses, which is a problem of learning specific type of classifiers from positive and negative examples.
Hypotheses or JSM-hypotheses were proposed by V.K.Finn \[6, 7\] and formalized in terms of Formal Concept Analysis in \[14, 11, 15\]. The set of minimal hypotheses is classification equivalent to the set of all hypotheses, thus making a condensed representation of the latter. The set of all hypotheses can be generated with polynomial delay \[15\], however, the problem of generating minimal hypotheses with polynomial delay remained an open one for long time. In this paper we show that dualization on lattices given by the ordered set of its irreducible elements is equivalent to enumeration of minimal hypotheses, thus complexity results concerning minimal hypotheses and dualization can be mutually translated to one another.

In what follows we shall use the notation of Formal Concept Analysis \[9\], which provides a convenient language and necessary results for lattices given by ordered sets of irreducible elements.

The rest of the paper is organized as follows: In the second section we give most important definitions and relate dualization problem to the problem of enumerating minimal hypotheses. In the third section we prove the main intractability result on impossibility of enumerating minimal hypotheses and dualization in output polynomial time unless P = NP. In the fourth section we conclude by discussing the implication of the results for the problem of dualizing monotone Boolean functions. In the fifth section we consider a particular case of distributive lattices and show that subexponential dualization algorithm is possible in this case.

\section{Basic definitions}

A partial ordered set \((L, \leq)\) is called a lattice if any pair of it elements has an infimum (meet) and a supremum (join). Equivalently, a lattice is an algebra \((L, \land, \lor)\) with the following properties of \(\land\) and \(\lor\):

\begin{align*}
    &L1 \quad x \lor x = x, \quad x \land x = x \quad \text{(idempotence)} \\
    &L2 \quad x \lor y = y \lor x, \quad x \land y = y \land x \quad \text{(commutativity)} \\
    &L3 \quad x \lor (y \lor z) = (x \lor y) \lor z, \quad x \land (y \land z) = (x \land y) \land z \quad \text{(associativity)} \\
    &L4 \quad x = x \land (x \lor y) = x \lor (x \land y) \quad \text{(absorption)}
\end{align*}

A lattice is called complete if every subset of it has meet and join. A lattice is distributive if for any \(x, y, z \in L\)

\[x \land (y \lor z) = (x \land y) \lor (x \land z)\]

The following elements of a lattice are very important in our work. An element \(x \in L\) is called infimum-irreducible (or meet-irreducible) if \(x \neq \bigwedge_{y \geq x} y\), i.e., \(x\) is not represented by the intersection of any elements above it. Dually, an element \(x \in L\) is called supremum-irreducible (or join-irreducible) if \(x \neq \bigvee_{y \leq x} y\), i.e., \(x\) is not represented by the union of any elements below it. Meet- (join-) irreducible elements have only one upper (lower) neighbor in the lattice diagram.

For Formal Concept Analysis we use the standard definitions and facts from \[9\]. Let \(G\) and \(M\) be sets, called the set of objects and attributes, respectively. Let \(I\) be a relation
has the attribute \( m \). The triple \( \mathcal{K} = (G, M, I) \) is called a (formal) context and is naturally represented by a cross-table, where rows stay for objects, columns stay for attributes and crosses stay for pairs \((g, m)\) \( \in I \). If \( A \subseteq G, B \subseteq M \) are arbitrary subsets, then the Galois connection is given by the following derivation operators:

\[
A' = \{ m \in M \mid gIm \forall g \in A \} \\
B' = \{ g \in G \mid gIm \forall m \in B \}
\]

The pair \((A, B)\), where \( A \subseteq G, B \subseteq M, A' = B \), and \( B' = A \) is called a (formal) concept (of the context \( \mathcal{K} \)) with extent \( A \) and intent \( B \) (in this case we have also \( A'' = A \) and \( B'' = B \)). Formal concepts are ordered by the following relation

\[
(A_1, B_1) \leq (A_2, B_2) \text{ iff } A_1 \subseteq A_2(B_2 \subseteq B_1),
\]

this partial order being a complete lattice on the set of all concepts. This lattice is called a concept lattice \( \mathcal{L}(G, M, I) \) of the context \((G, M, I)\).

The set of join-irreducible elements of a concept lattice \( \mathcal{L}(G, M, I) \) is contained in the set of object concepts, which have the form \((g'', g')\), \( g \in G \). Dually, the set of meet-irreducible elements of a concept lattice is contained in the set of attribute concepts, which have the form \((m', m'')\), \( m \in M \). An object \( g \) is called reducible if \( g' = M \) or \( \exists X \subseteq G \setminus \{g\} : g' = \bigcap_{j \in X} j' \), i.e., the respective row of the context cross-table is either full or is an intersection of some other rows. If \( g \) is not reducible, then \((g'', g')\) is a join-irreducible element of \( \mathcal{L}(G, M, I) \). Dually, an attribute \( m \) is called reducible if \( m' = G \) or \( \exists Y \subseteq M \setminus \{m\} : m' = \bigcap_{j \in Y} j' \), i.e. the respective column of the context cross-table is either full or is an intersection of some other columns. If \( m \) is not reducible, then \((m', m'')\) is a meet-irreducible element of \( \mathcal{L}(G, M, I) \).

The Basic Theorem of FCA \cite{FCA} says in particular that every finite lattice \((L, \lor, \land)\) can be represented as a concept lattice \( \mathcal{L}(J(L), M(L), \leq) \), where \( J(L) \) is the set of all join-irreducible elements of \( L, M(L) \) is the set of meet-irreducible elements of \( L \), and \( \leq \) is the natural partial order of \((L, \lor, \land)\).

A set of attributes \( B \) is implied by a set of attributes \( A \), or implication \( A \rightarrow B \) holds, if all objects from \( G \) that have all attributes from the set \( A \) also have all attributes from the set \( B \), i.e. \( A' \subseteq B' \). Implications obey Armstrong rules

\[
\frac{X \rightarrow Y}{X \lor Z \rightarrow Y}, \quad \frac{X \rightarrow Y, Y \cup Z \rightarrow W}{X \cup Z \rightarrow W},
\]

and a minimal subset of implications from which all other implications can be deduced by means of Armstrong rules is called an implication base. In \cite{DG} a characterization of cardinality-minimum implication base (Duquenne-Guigues base) was given.

Now we present a learning model from \cite{6,7} in terms of FCA \cite{14,11,15}. This model complies with the common paradigm of learning from positive and negative examples (see, e.g. \cite{11}, \cite{15}): given a positive and negative examples of a “target attribute”, construct a generalization of the positive examples that would not cover any negative example.
Let \( t \) be target attribute, different from attributes from the set \( M \), which correspond to structural attributes of objects. For example, in pharmacological applications the structural attributes can correspond to particular subgraphs of molecular graphs of chemical compounds.

Input data for learning can be represented by sets of positive, negative, and undetermined examples. Positive examples (or \((+)-\) examples) are objects that are known to have the target attribute \( t \) and negative examples (or \((-)-\) examples) are objects that are known not have this attribute.

**Definition 2.1.** Consider positive context \( \mathbb{K}_+ = (G_+, M, \mathcal{I}_+) \) and negative context \( \mathbb{K}_- = (G_-, M, \mathcal{I}_-) \). The context \( \mathbb{K}_\pm = (G_+ \cup G_-, M \cup \{w\}, \mathcal{I}_+ \cup \mathcal{I}_- \cup G_+ \times \{w\}) \) is called a training context. The derivation operator in this context is denoted by superscript \((\cdot)^\pm\).

**Definition 2.2.** A subset \( H \subseteq M \) is called a positive (or \((+)-\))-hypothesis of training context \( \mathbb{K}_\pm \) if \( H \) is intent of \( \mathbb{K}_+ \) and \( H \) is not a subset of any intent of \( \mathbb{K}_- \). For \( k \in \mathbb{N} \cup \{0\} \) a subset \( H \subseteq M \) is called a \( k \)-weak positive (or \( k(+)\))-hypothesis of training context \( \mathbb{K}_\pm \) if \( H \) is intent of \( \mathbb{K}_+ \) and \( |H^+ \cap G^-| \leq k \).

Obviously, a positive hypothesis is a 0-weak hypothesis. In the same way negative (or \((-)-\)) hypotheses are defined.

Besides classified objects (positive and negative examples), one usually has objects for which the value of the target attribute is unknown. These examples are usually called undetermined examples, they can be given by a context \( \mathbb{K}_\tau := (G_\tau, M, \mathcal{I}_\tau) \), where the corresponding derivation operator is denoted by \((\cdot)^\tau\).

Hypotheses can be used to classify the undetermined examples: If the intent \( g^\tau := \{m \in M \mid (g, m) \in I_\tau\} \) of an object \( g \in G_\tau \) contains a positive, but no negative hypothesis, then \( g^\tau \) is classified positively. Negative classifications are defined similarly. If \( g^\tau \) contains hypotheses of both kinds, or if \( g^\tau \) contains no hypothesis at all, then the classification is contradictory or undetermined, respectively. In this case one can apply probabilistic techniques.

In [11], [15] it was argued that one can restrict to minimal (w.r.t. inclusion \( \subseteq \)) hypotheses, positive as well as negative, since an object intent \( g^\tau \) obviously contains a positive hypothesis if and only if it contains a minimal positive hypothesis.

**Definition 2.3.** For \( k \in \mathbb{N} \cup \{0\} \) if the set of \( k(+)\)-hypotheses is not empty, then \( H \) is a minimal \( k(+)\)-hypothesis iff \( H \) is a \((k)\)-hypothesis and \( F \) is not a \((k+)\)-hypothesis for any \( F \subseteq H \). In case the set of \( k(+)\)-hypotheses is empty, we put the set of minimal \( k(+)\)-hypothesis consisting of the only set \( M \).

The latter condition is needed technically for dualization: without it not every monotone Boolean function would be dualizable.

**Example.** Consider the following training context, where \( m_0 \) is the target attribute, the set of attributes is \( M = \{m_1, \ldots, m_6\} \), the set of negative examples is \( G = \{g_1, g_2, g_3\} \), the set of positive examples is \( G_+ = \{g_4, \ldots, g_9\} \) and the incidence relation is given by the following cross table:
Here, we have $2^3 = 8$ minimal hypotheses: \{m_1, m_2, m_3\}, \{m_1, m_2, m_6\}, \{m_1, m_5, m_3\}, \{m_1, m_5, m_6\}, \{m_4, m_2, m_3\}, \{m_4, m_6, m_3\}, \{m_4, m_5, m_3\}, \{m_4, m_5, m_6\}.

In what follows we will also need the following definition from FCA, which is important in constructing “hard cases” for FCA-related complexity problems.

**Definition 2.4.** Let $G = \{g_1, \ldots, g_n\}$ and $M = \{m_1, \ldots, m_n\}$ be sets with same cardinality. Then the context $\mathbb{K} = (G, M, \mathcal{I}_\neq)$ is called contranominal scale, where $\mathcal{I}_\neq = G \times M - \{(g_1, m_1), \ldots, (g_n, m_n)\}$.

The contranominal scale has the following property, which we will use later: for any $H \subseteq M$ one has $H'' = H$ and $H' = \{g_i \mid m_i \notin H, 1 \leq i \leq n\}$.

### 3 Enumeration of minimal hypotheses

Here we discuss algorithmic complexity of enumerating all minimal hypotheses. Note that there is an obvious algorithm for enumerating all hypotheses (not necessarily minimal) with polynomial delay [15]. This algorithm is an adaptation of an algorithm for computing the set of all concepts, where the branching condition is changed to include the additional condition $|H^+ \cap G^-| \leq k$.

**Problem:** Minimal hypotheses enumeration (MHE)

**INPUT:** Positive and negative contexts $\mathbb{K}_+ = (G_+, M, \mathcal{I}_+), \mathbb{K}_- = (G_-, M, \mathcal{I}_-)$

**OUTPUT:** All minimal hypotheses of $\mathbb{K}_\pm$

Unfortunately, this problem cannot be solved in output polynomial time unless $P = NP$. In order to prove this result we study complexity of the following decision problem.

**Problem:** Additional minimal hypothesis (AMH)

**INPUT:** Positive and negative contexts $\mathbb{K}_+ = (G_+, M, \mathcal{I}_+), \mathbb{K}_- = (G_-, M, \mathcal{I}_-)$ and a set of minimal hypotheses $\mathcal{H} = \{H_1, \ldots, H_k\}$.

**QUESTION:** Is there an additional minimal hypothesis $H$ of $\mathbb{K}_\pm$ i.e. minimal hypothesis $H$ that is $H \notin \mathcal{H}$.

**Lemma 3.1.** AMH is in P iff MHE can be solved in output polynomial time.
Algorithm 1 FindNewMinH(\(K_+, K_-, H\))

Require: \(\text{DecideAMH}(K_+, K_-, H) = \text{true}\)
1: for \(g \in G_+\) do
2: \(G^g_+ \leftarrow \{g^+ \cap h^+ \mid h \in G_+\}\)
3: \(I^g_+ \leftarrow \{(g, m) \mid m \in g, g \in G^g_+\}\)
4: \(G^g_- \leftarrow \{g^+ \cap h^- \mid h \in G_-\}\)
5: \(I^g_- \leftarrow \{(g, m) \mid m \in g, g \in G^g_-\}\)
6: \(K^g_+ \leftarrow K(G^g_+, M \cap g^+, I^g_+\)
7: \(K^g_- \leftarrow K(G^g_2, M \cap g^-, I^g_-\)
8: \(\mathcal{H}^g \leftarrow \{h \mid h \subseteq g^+, h \in \mathcal{H}\}\)
9: if DecideAMH\((K^g_+, K^g_-, \mathcal{H}^g)\) then
10: \(\text{return} \ \text{FindNewMinH}(K^g_+, K^g_-, \mathcal{H}^g)\)
11: end if
12: end for
13: return \(M\)

Proof. (\(\Leftarrow\)) Assume there is an output polynomial algorithm \(A\) that generates all minimal hypotheses in time \(pol(|G_+|, |M|, |I_+|, |G_-|, |I_-|, N)\), where \(N\) is the number of minimal hypotheses. Use this algorithm to construct \(A'\) that makes first \(p(|G_+|, |M|, |I_+|, |G_-|, |I_-|, k + 1)\) steps of \(A\). Clearly, if there is more than \(k\) minimal hypotheses, then \(A'\) generates \(k + 1\) minimal hypotheses, hence we can solve AMH in polynomial time. The lines 2 and 8

\((\Rightarrow)\) Now suppose there is a function \(\text{DecideAMH}(K_+, K_-, H)\) that solves AMH problem instance in time \(O(t)\). We can use \(\text{Algorithm}[7]\) to find additional minimal hypothesis if there is one. Clearly line 2 to line 8 can be computed in time \(O((|G_+| + |G_-|)|M|)\). Also note that the total number of recursive calls can not be greater than \(|M|\). Thus, time complexity of the \(\text{Algorithm}[7]\) is \(O((|G_+| + |G_-|)|M|^2t)\).

Now we prove the correctness. First note that since hypotheses are closed in \(K_+\) the additional minimal hypothesis must be a subset of some \(g^+, g \in G_+\), or it could be \(M\). By definition the context \(K^g_+\) defines exactly all closed sets of \(K\) that are subsets of \(g^+\). It remains to note that at the last recursive call of \(\text{Algorithm}[7]\) no AMH\((K^g_+, K^g_-, \mathcal{H}^g)\) holds for any \(g \in G_+\). Thus, the only possible additional minimal hypothesis that can be returned is \(M\).

Now we prove \(NP\)-completeness of AMH through the reduction of the most known \(NP\)-complete problem – satisfiability of CNF – to AMH.

Problem: CNF satisfiability (SAT)

INPUT: A Boolean CNF formula \(f(x_1, \ldots, x_n) = C_1 \land \ldots \land C_k\)

QUESTION: Is \(f\) satisfiable?

Consider an arbitrary CNF instance \(C_1, \ldots, C_k\) with variables \(x_1, \ldots, x_n\), where \(C_i = (l_i \lor \ldots \lor l_{ir})\), \(1 \leq i \leq k\) and \(l_{ir} \in \{x_1, \ldots, x_n\} \cup \{\neg x_1, \ldots, \neg x_n\}\) \((1 \leq i \leq k, 1 \leq j \leq r_i)\) are some variables or their negations called literals. From this instance we construct a positive context \(K_+ = (G_+, M, I_+)\) and a negative context \(K_- = (G_-, M, I_-)\). Define

\[
M = \{C_1, \ldots, C_k\} \cup \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}
\]
\[ G_+ = \{ g_{x_1}, g_{-x_1}, \ldots, g_{x_n}, g_{-x_n} \} \cup \{ g_{C_1}, \ldots, g_{C_k} \} \]

\[ G_- = \{ g_1, \ldots, g_n \} \]

The incidence relation of the positive context is defined by \( \mathcal{I}_+ = \mathcal{I}_C \cup \mathcal{I}_\neq \cup \mathcal{I}_- \), where

\[ \mathcal{I}_C = \{ (g_{x_i}, C_j) \mid x_i \notin C_j, 1 \leq i \leq n, 1 \leq j \leq k \} \cup \{ (g_{-x_i}, C_j) \mid \neg x_i \notin C_j, 1 \leq i \leq n, 1 \leq j \leq k \} \]

\[ \mathcal{I}_\neq = \{ (g_{x_1}, g_{-x_1}, \ldots, g_{x_n}, g_{-x_n}) \times \{ x_1, \neg x_1, \ldots, x_n, \neg x_n \} \}
\]

\[ - \{ (g_{x_1}, x_1), (g_{-x_1}, \neg x_1), \ldots, (g_{x_n}, x_n), (g_{-x_n}, \neg x_n) \} \]

\[ \mathcal{I}_- = \{ (g_{C_1}, C_1), \ldots, (g_{C_k}, C_k) \} \]

that is for \( i \)-th clause \( C_i^+ \cap \{ g_{x_1}, g_{-x_1}, \ldots, g_{x_n}, g_{-x_n} \} \) is the set of literals not included in \( C_i \), \( \mathcal{I}_\neq \) is the relation of contranominal scale.

The incidence relation of the negative context is given by \( \mathcal{I}_- = \mathcal{I}_C \) where

\[ \mathcal{I}_C = G_- \times \{ x_1, \neg x_1, \ldots, x_n, \neg x_n \} \]

\[ - \{ (g_1, x_1), (g_1, \neg x_1), \ldots, (g_n, x_n), (g_n, \neg x_n) \} \]

| \( K_+ \) | \( C_1 \) | \( C_2 \) | \( \cdots \) | \( C_k \) | \( x_1 \) | \( \neg x_1 \) | \( \cdots \) | \( x_n \) | \( \neg x_n \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( g_{x_1} \) | \( g_{-x_1} \) | \( \vdots \) | \( g_{x_n} \) | \( g_{-x_n} \) | \( \mathcal{I}_C \) | \( \mathcal{I}_\neq \) | \( \mathcal{I}_- \) | \( \mathcal{I}_C \) |
| \( g_{C_1} \) | \( \vdots \) | \( g_{C_k} \) |
| \( K_+ \) | \( g_1 \) | \( \vdots \) | \( g_n \) | \( \mathcal{I}_C \) |

As the set of minimal hypotheses we take \( \mathcal{H} = \{ \{ C_1 \}, \{ C_2 \}, \ldots, \{ C_k \} \} \). It is easy to see that \( K_\pm \) with \( \mathcal{H} \) is a correct instance of AMH.

If a hypothesis (not necessary minimal) is not included in \( \mathcal{H} \) we will call it additional.

**Proposition 3.2.** If \( H \) is an additional minimal hypothesis of \( K_\pm \) then
\[ H \subseteq \{ x_1, \neg x_1, \ldots, x_n, \neg x_n \} \]

**Proof.** Suppose \( H \not\subseteq \{ x_1, \neg x_1, \ldots, x_n, \neg x_n \} \), then since \( H \) is not empty there is some \( C_i \in H \), \( 1 \leq i \leq k \). But \( H \) is a minimal hypothesis and thus it does not contain any hypothesis. Hence \( H = C_i \) and this contradicts the fact that \( H \) is an additional minimal hypothesis. \( \square \)
For any $H \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ that satisfies $\{x_i, \neg x_i\} \not\subseteq H$ for any $1 \leq i \leq n$ we define the truth assignment $\varphi_H$ in a natural way:

$$\varphi_H(x_i) = \begin{cases} \text{true}, & \text{if } x_i \in H; \\ \text{false}, & \text{if } x_i \notin H; \end{cases}$$

In the case $\{x_i, \neg x_i\} \subseteq H$ for some $1 \leq i \leq n$, $\varphi_H$ is not defined.

Symmetrically, for a truth assignment $\varphi$ define the set $H_\varphi = \{x_i \mid \varphi(x_i) = \text{true}\} \cup \{\neg x_i \mid \varphi(x_i) = \text{false}\}$.

Below, for $H \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ we will denote the complement of $H$ in $\{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ by $\overline{H}$.

**Proposition 3.3.** If a subset $H \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ is not contained in the intent of any negative example (i.e. $\forall g \in G^-, H \not\subseteq g^−$), then $\varphi_{\overline{H}}$ is defined. Conversely, for a truth assignment $\varphi$ the set $\overline{H}_\varphi$ is not contained in the intent of any negative concept.

The proof is straightforward.

The following theorem proves NP-hardness of AMH.

**Theorem 3.4.** AMH has a solution if and only if SAT has a solution.

**Proof.** ($\Rightarrow$) Let $H$ be an additional minimal hypothesis of $\mathbb{K}_\pm$. First note that by Proposition 3.2 and Proposition 3.3 the truth assignment $\varphi_{\overline{H}}$ is correctly defined. Since $H$ is a nonempty concept intent of $\mathbb{K}_\pm$, Proposition 3.2 together with the fact that $I_\varphi$ is the relation of contranominal scale implies $H^+ = \{g_{x_i} \mid x_i \in \overline{H}\} \cup \{g_{\neg x_i} \mid \neg x_i \in \overline{H}\}$. Now $H^{++} \cap \{C_1, C_2, \ldots, C_k\} = \emptyset$, hence for any $C_i$ ($1 \leq i \leq k$) there is some $g_l \in H^+$ such that $g_l \notin C_i^+$. According to the definition of $I_C$ the latter means that literal $l$ belongs to clause $C_i$. Thus $f(\varphi_{\overline{H}}) = \text{true}$.

($\Leftarrow$) Let $\varphi$ be a truth assignment and $f(\varphi) = \text{true}$. Define $H = \overline{H}_\varphi$. Note that $H^+ = \{g_{x_i} \mid x_i \in H\} \cup \{g_{\neg x_i} \mid \neg x_i \in H\}$, because $I_\varphi$ is the relation of contranominal scale and $H \cap g_{C_i} = \emptyset$, $1 \leq i \leq k$. Suppose that $C_i \in H^{++}$ for some $1 \leq i \leq k$. This is equivalent to $H^+ \subseteq C_i^+$. Hence, by definition of $I_C$, there is no literal $l \in H_\varphi$ such that $l \in C_i$. Therefore, the clause $C_i$ does not hold and this contradicts that $\varphi$ satisfies CNF $f$. Thus $H^{++} = H$ and $H$ is a hypothesis. Since $H$ does not contain any $\{C_i\}$, it must contain additional minimal hypothesis. \hfill $\square$

**Corollary 3.5.** MHE cannot be solved in output polynomial time, unless $P = NP$.

## 4 Dualizing monotone Boolean functions on lattices

Let $\mathfrak{B}$ be a complete lattice and $f$ be a monotone Boolean function on it. Without loss of generality we can assume that $\mathfrak{B}$ is a concept lattice $\mathfrak{B}(G, M, I)$ of the corresponding formal context $\mathfrak{K}(G, M, I)$. Then $A \subseteq B \Rightarrow f((A, A')) \leq f((B, B'))$. It is known that any monotone Boolean function on a lattice is uniquely given by its minimal 1 values, i.e. by the set $\{(A, A') \mid (A, A') \in \mathfrak{B}, f((A, A')) = 1, f((B, B')) = 0 \forall B \subset A\}$. We can represent the
set of minimal 1 values of a monotone Boolean function as the set of minimal hypotheses of the learning context defined by $K_+$ and $K_-$, where $K_+ = K$ and object intents of $K_-$ are precisely maximal 0 values of $f$. Symmetrically, a learning context $K_{±}$ specifies a monotone Boolean function $f$ on concept lattice of $K_+$ such that maximal 0 values of $f$ are (inclusion) maximal object intents of $K_-$. Consider the following

**Problem:** Minimal true values enumeration (MTE)

**INPUT:** A formal context $K$ and a set of maximal 0 values of monotone Boolean function $f$ on the concept lattice of $K$.

**OUTPUT:** Set of minimal 1 values of $f$.

From Corollary 3.5 it follows that MTE cannot be solved in output polynomial time unless $P = NP$. Note that in the case of Boolean lattice this problem is polynomially equivalent to Monotone Boolean Dualism and minimal hypotheses in this case can be enumerated in output quasi-polynomial time $O(N^{\log N})$, where $N$ is $|input\ size| + |output\ size|$ (see [8]).

In database theory a closure of a set of attributes $A$ is defined by means of iterated applications of functional dependencies with premises contained in $A$. Same type of closure, by means of implications instead of functional dependencies, is known in FCA. More precisely, applying $\text{imp}(A) = A \cup \{B \mid D \rightarrow B, D \subseteq A\}$ iteratively to $A$ by putting at each next step $A ::= \text{imp}(A)$ until saturation, one obtains implicational closure of $A$, which is equal to $A''$ [9]. So, the set of all implications of a context defines the closure operator $(\cdot)''$, closed subsets of attributes, which together with the respective closed subsets of objects (extents) give the concept lattice. Hence, instead of defining a lattice by the ordered set of its irreducible elements, one can define it in terms of the set of all valid implications of the respective formal context, or, equivalently, by its implication base. This consideration poses another setting of the dualization problem, where the lattice – instead of the set of positive examples $G_+$ – is given by its implications or implication base, and one has to dualize the monotone function given by the set of examples $G_-$. When the lattice is Boolean, its implication base is empty [9], so one has to dualize the set of examples $G_-$, which can be considered as a monotone DNF, where disjunction goes over objects – elements of $G_-$ – which themselves can be taken as conjunctions of the respective attributes. When the lattice is distributive, its minimum implication base has one-element premises [9] (hence, the number of implications in the base is not larger than $|M|$), so it can easily be computed from the context in polynomial time, and vice versa. Therefore, the dualization on lattices given by implication bases for distributive lattices is polynomially equivalent to the dualization on lattices given by contexts (ordered sets of irreducible elements), which we study in the next section. The study of dualization problems for lattices given by implication bases is motivated by simple linear-time reciprocal translations of implications to functional dependencies [10] and propositional Horn theories [1].

In [13] it has been proven that the following problem is NP-hard:

**Problem:** Incremental maximal model (IME)

**INPUT:** Horn theory $\Phi$ and a set of its maximal models $S$.

**QUESTION:** Is there another maximal model of $\Phi$ not contained in $S$?

In terms of FCA a Horn theory corresponds to a set of implications $\mathcal{J}$ and maximal models correspond to inclusion maximal closed sets of $\mathcal{J}$, or object intents, that are not $M$.  

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In the dualization setting maximal closed sets are dual to the singleton set \{M\}. Thus we get the following corollary

**Problem:** Maximal 0 value enumeration (MZE)

**INPUT:** A lattice \(L(J)\) given by an implication base \(J\) and a set of minimal 1 values of monotone Boolean function \(f\) on the lattice \(L(J)\).

**OUTPUT:** Set of maximal 0 values of \(f\).

**Corollary 4.1.** A solution of MZE is impossible in output polynomial time unless \(P = NP\).

5 Dualization on distributive lattices

We assume that a distributive lattice is represented as a lattice \(L(P)\) of uppersets (order filters) of a poset \(P\), and poset \(P\) is given by a matrix \(n \times n\). It is well known that any distributive lattice has such a representation \([1, 12, 9]\). Note that one can use formal context representation of the distributive lattice as well, since the size of the corresponding formal context \((P, P, \leq)\) is polynomial in \(n\), and our dualization algorithm is subexponential.

We treat the elements of \(L = L(P)\) as subsets of \(P\) (since they are uppersets of \(P\)), so for two uppersets \(A, B \in L(P)\) \(A \leq B\) means that \(A \subseteq B\). For an element \(p \in P\), the smallest (by set inclusion) upperset that contains \(p\) is denoted by \(\uparrow p\), and the smallest downset that contains \(p\) is denoted by \(\downarrow p\).

For two antichains \(A \subseteq L\), and \(B \subseteq L\) we say \((A, B)\) has property (\(\ast\)) if

\[ a \not\leq b \text{ for any } a \in A, b \in B. \]

Let \(A\) and \(B\) be antichains of a distributive lattice \(L(P)\). We say that \(A\) and \(B\) are dual iff \(A, B\) satisfy property (\(\ast\)) and for any \(X \in L\) either \(A \subseteq X\) for some \(A \in A\) or \(X \subseteq B\) for some \(B \in B\). Equivalently, there is a monotone Boolean function \(f\) on \(L\) such that \(A\) is the set of minimal 1 values of \(f\) and \(B\) is the set of maximal 0 values of \(f\). Further on we will call a triple of the form \(((A, B), P)\) dualization problem input. Note that in the degenerate cases where \(A = \emptyset\) or \(B = \emptyset\) the duality can easily be tested in polynomial time. If \(A\) is empty, then \(B\) is dual to \(A\) iff \(B = \{P\}\). If \(B\) is empty, then \(A\) is dual to \(B\) iff \(A = \{p\} | \uparrow p = \{p\}, p \in P\). Let us call the algorithm that tests duality in these two degenerate cases \(EasyTest((A, B), P)\).

We will also use the notion of frequency of an element \(p \in P\). Let \(C\) be some set of subsets of \(P\) (i.e. \(C \subseteq 2^P\)), then the frequency of \(p\) in \(C\) is the fraction of elements of \(C\) that contain \(p\):

**Definition 5.1.** \(freq_C(p) = |\{C \in C \mid p \in C\}|/|C|\).

Also for convenience we define the quantities \(N = |A| + |B|\), and \(m = \max_{p \in P} (|\uparrow p| + |\downarrow p|)\).

5.1 Algorithm

Here we describe a subexponential algorithm for testing duality on a distributive lattice. The structure of the algorithm is close to one in \([8]\). The algorithm decomposes the initial
problem instance into smaller instances and solves them recursively. In order to keep the total number of recursive calls subexponential at each decomposition step, the algorithm tries to select an element of $\mathcal{P}$ such that either it is frequent or it has a large fraction of successors of predecessors.

Algorithm 2 TestDuality((A, B), $\mathcal{P}$)

Require: $A, B \subseteq \mathcal{L}(\mathcal{P})$
1: if $A = \emptyset$ or $B = \emptyset$ then
2: return EasyTest((A, B), $\mathcal{P}$)
3: end if
4: $n \leftarrow |\mathcal{P}|$
5: $m \leftarrow \max_{p \in \mathcal{P}} (|\uparrow p| + |\downarrow p|)$
6: if $m > n^{1/3}$ then
7: $p \leftarrow \arg\max_{p \in \mathcal{P}} (|\uparrow p| + |\downarrow p|)$
8: else
9: $p \leftarrow \arg\max_{p \in \mathcal{P}} (\max(freq_A(p), freq_B(p)))$
10: end if
11: return TestDuality((A$^p_1$, B$^p_1$), $\mathcal{P} \setminus \uparrow p$) $\land$ TestDuality((A$^p_2$, B$^p_2$), $\mathcal{P} \setminus \downarrow p$)

To describe decomposition performed by our algorithm we define the following four sets:

$A^p_1 = \{ \uparrow p \mid A \in A \}$, $B^p_1 = \{ \downarrow p \mid B \in B \}$;

$A^p_2 = \{ A \mid p \not\in A, A \in A \}$, $B^p_2 = \{ B \mid \downarrow p \mid B \in B \}$.

Note that $B^p_0 = \{ \uparrow p \mid \uparrow p \subseteq B, B \in B \}$, and $A^p_2 = \{ A \mid \downarrow p \cap A = \emptyset, A \in A \}$. The following lemma proves the correctness of Algorithm 2.

Lemma 5.1. For any $p \in \mathcal{P}$, $A$ and $B$ are dual iff the following two conditions hold:

$A^p_1$ and $B^p_1$ are dual on $\mathcal{L}(\mathcal{P} \setminus \uparrow p)$,
$A^p_2$ and $B^p_2$ are dual on $\mathcal{L}(\mathcal{P} \setminus \downarrow p)$.

Proof. ($\Rightarrow$) Let us fix arbitrary $X \in \mathcal{L}$. Consider two possible cases: $p \in X$ and $p \not\in X$. If $p \in X$ then since $A^p_1$ and $B^p_1$ are dual either $A \subseteq X \setminus \uparrow p$ for some $A \in A^p_1$, or $X \setminus \uparrow p \subseteq B_1$ for some $B_1 \in B^p_1$. Clearly $X \setminus \uparrow p \subseteq B_1$ implies $X \subseteq B_1 \cup \uparrow p \in B$. On the other hand $A \in A^p_1$ implies that there is $A \in A$ such that $A \uparrow p$, and hence $A \subseteq X$ (since $\uparrow p \subseteq X$).

If $p \not\in X$ then since $A^p_2$ and $B^p_2$ are dual either $A \subseteq X \setminus \downarrow p$ for some $A \in A^p_2$, or $X \setminus \downarrow p \subseteq B_2$ for some $B_2 \in B^p_2$. By definition $B_2 \in B^p_2$ implies that there is $B \in B$ such that $B_2 = B \setminus \downarrow p$. Note that $A_2 \in A$, and $X = X \setminus \downarrow p \subseteq B_2 \subseteq B$.

($\Rightarrow$) Let us prove that $A^p_1$ and $B^p_1$ are dual. Consider arbitrary $X \in \mathcal{L}(\mathcal{P} \setminus \uparrow p)$. Because $A$ and $B$ are dual on $\mathcal{L}(\mathcal{P})$ either $A \subseteq X \cup \uparrow p$ for some $A \in A$, or $X \cup \uparrow p \subseteq B$ for some $B \in B$. If $A \subseteq X \cup \uparrow p$ then $A \setminus \uparrow p \subseteq X$ (since $\uparrow p \cap X = \emptyset$). If $X \cup \uparrow p \subseteq B$ then $X \subseteq B \setminus \uparrow p$, and by definition $B \setminus \uparrow p \in B^p_1$. It is easy to check that $(A^p_1, B^p_1)$ has property $(\ast)$.

Now we prove that $A^p_2$ and $B^p_2$ are dual. Consider arbitrary $X \in \mathcal{L}(\mathcal{P} \setminus \downarrow p)$. Note that $X \in \mathcal{L}(\mathcal{P})$. Because $A$ and $B$ are dual on $\mathcal{L}(\mathcal{P})$ either $A \subseteq X$ for some $A \in A$, or $X \subseteq B$
for some $B \in \mathcal{B}$. If $A \subseteq X$ then $p \notin A$, and $A \in \mathcal{A}^p$. If $X \subseteq B$ then $X \subseteq B \setminus \downarrow p$ (since $\downarrow p \cap X = \emptyset$). It is easy to check that $(\mathcal{A}_2^p, \mathcal{B}_2^p)$ has property $(\ast)$. \hfill \Box

The following lemma helps to establish the lower bound on the frequency of the most frequent element of $\mathcal{P}$.

**Lemma 5.2.** If $\mathcal{A}$ and $\mathcal{B}$ are dual then

$$\sum_{A \in \mathcal{A}} \frac{(3/4)^{|A|/m^2}}{|A|} + \sum_{B \in \mathcal{B}} e^{-(n-|B|)/m} \geq 1$$

**Proof.** To prove this bound we use the 'method of expectations' similar to one in [8], but with a more tricky probability distribution. Suppose we fixed some probability distribution of $X \in \mathcal{L}$. Let us denote the expected number of $A \in \mathcal{A}, A \subseteq X$ by $E_A$, and the expected number of $B \in \mathcal{B}, X \subseteq B$ by $E_B$. Antichains $\mathcal{A}$ and $\mathcal{B}$ are dual iff for any $X \in \mathcal{L}$ either $A \subseteq X$, for some $A \in \mathcal{A}$, or $X \subseteq B$, for some $B \in \mathcal{B}$. Thus if $\mathcal{A}$ and $\mathcal{B}$ are dual, then $E_A + E_B \geq 1$. By linearity of expectations $E_A = \sum_{A \in \mathcal{A}} E_A$, where $E_A$ is probability that $A \subseteq X$. Similarly, $E_B = \sum_{B \in \mathcal{B}} E_B$, where $E_B$ is the probability that $X \subseteq B$. Unlike to the Boolean lattice case, no analytical expression for $E_A$ and $E_B$ is known (even the existence of a polynomial approximation algorithm is an open question [3]), but we can find upper bounds for $E_A, A \in \mathcal{A}$ and $E_B, B \in \mathcal{B}$.

In order to generate random (but not uniform) element $X \in \mathcal{L}$ we select each $p \in \mathcal{P}$ with probability $1/m$. Suppose elements $p_1, p_2, \ldots, p_r$ have been selected, then the resulting upperset $X \in \mathcal{L}$ is defined as $X = \uparrow p_1 \cup \uparrow p_2 \cup \ldots \cup \uparrow p_r$.

For a given upperset $A \in \mathcal{A}$ let us bound probability that $A \subseteq X$. For each $p \in \mathcal{P}$ we associate an event $E_p$ that $p \in X$. Note that $P_r(E_p) \geq (1 - 1/m)^m \geq 1/4$ (since $m \geq 2$). Consider any maximal set $\{a_1, a_2, \ldots, a_k\} \subseteq A$ such that events $E_{a_1}, E_{a_2}, \ldots, E_{a_k}$ are mutually independent. It is easy to see that $k \geq |A|/m^2$. Since event $A \subseteq X$ happens if $E_{a_1} \wedge E_{a_2} \wedge \ldots \wedge E_{a_k}$ we have

$$P_r(A \subseteq X) \leq \prod_{1 \leq i \leq k} (1 - P_r(E_{a_i})) \leq (1 - 1/4)^{|A|/m^2}.$$

To bound $E_B$, note that for any $B \in \mathcal{B}$, the probability $P_r(X \subseteq B) = P_r(X \cap (\mathcal{P} \setminus B) = \emptyset)$. This probability is exactly $(1 - 1/m)^{|\mathcal{P} \setminus B|} = (1 - 1/m)^{n-|B|} \leq e^{-(n-|B|)/m}$ \hfill \Box

**Corollary 5.3.** If $\mathcal{A}$ and $\mathcal{B}$ are dual, then at least one of the following statements is true

- $\exists p \in \mathcal{P} : freq_A(p) \geq \frac{1}{m \log N}$
- $\exists p \in \mathcal{P} : freq_B(p) \geq \frac{1}{m^2 \log N}$

**Proof.** Let $k_A = \min_{A \in \mathcal{A}} |A|/m^2$, $k_B = \min_{B \in \mathcal{B}} (n-|B|)/m$, and $k = \min(k_A, k_B)$. By Lemma 5.2 $\sum_{A \in \mathcal{A}} (3/4)^{|A|/m^2} + \sum_{B \in \mathcal{B}} (3/4)^{(n-|B|)/m} \geq 1$. Hence $(3/4)^k N \geq 1$ which yields $k \leq \log_{4/3} N$. Since $(\mathcal{A}, \mathcal{B})$ has property $(\ast)$, for any $A \in \mathcal{A}, B \in \mathcal{B}$ the intersection $A \cap \overline{B}$ is nonempty. If $|A| = km^2$, then there is some $a \in A$ such that $freq_A(a) \geq 1/(km^2) \geq 1/(m^2 \log N)$. Similarly, if $|B| = km$, then there is some $b \notin B$ such that $freq_B(b) \geq 1/(km) \geq 1/(m \log N)$. \hfill \Box

**Theorem 5.4** (Time complexity of the dualization algorithm). Algorithm $\mathcal{P}$ decides duality in time $2^{O(n^{0.67}\log^3(|A|+|\mathcal{B}|))}$. 

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Proof. In order to bound the number of recursive calls occurred during an execution of Algorithm 2 we consider the following problem volume quantity: \( \text{vol}(A, B, P) = |A| \cdot |B| \cdot n \). Dualization problem \((A, B, P)\) branches into two subproblems \((A_p^1, B_p^0, P \uparrow p)\) and \((A_p^0, B_p^1, P \downarrow p)\). Let us denote the volumes of these problems by \( \text{vol}_1 \), \( \text{vol}_2 \), and \( \text{vol}_3 \), respectively. By Corollary 5.3 either \( \text{vol}_1 \leq (1 - \frac{m}{2n^1}) \text{vol} \) or \( \text{vol}_2 \leq (1 - \frac{m}{2n^2}) \text{vol} \). Moreover, in case of line 7 of the Algorithm 2, \( m = |\uparrow p| + |\downarrow p| \geq n^{1/3} \), which implies either \( \text{vol}_1 \leq (n - \frac{n}{2})/n \cdot \text{vol} \leq (1 - \frac{1}{2n^{2/3}}) \text{vol} \), or \( \text{vol}_2 \leq (1 - \frac{1}{2n^{2/3}}) \text{vol} \). Thus we have the following bound on the number of recursive calls: \( A(\text{vol}) \leq A((1 - \frac{1}{2n^{2/3}}) \text{vol}) + A(\text{vol} - 1) + 1 \). In [8] it has been proven that solution \( A(v) \) of the recurrence \( A(v) \leq 1 + A((1 - \varepsilon) v) + A(v - 1), A(1) = 1 \) can be bounded by \( A(v) \leq (3 + 2 \varepsilon) \log v / \varepsilon \). Substituting \( \varepsilon = \frac{1}{2n^{2/3} \log N} \) yields \( A(v) \leq (3 + 2N^{2/3} n^{1/3}) O((\log N + \log n) n^{2/3} \log N) \leq 2 O((\log N + \log n) n^{2/3} \log N) \leq 2 O(n^{0.67} \log^3 N) \).

6 Conclusion

In this paper we have studied the dualization problem on a lattice given by ordered sets of its irreducible elements (as a concept lattice). For this representation, the dualization problem has complexity different from that in case of explicit lattice representation by an ordered set of all its elements. We have shown that the dualization problem for a lattice given by the ordered set of its irreducible elements (concept lattice) is equivalent to the enumeration of minimal hypotheses, which is not possible in output polynomial time unless P=NP. For the case of distributive lattices the enumeration was shown to be possible in subexponential time. It is still open whether monotone Boolean dualization on a distributive lattice can be solved in quasi-polynomial time, or this problem cannot be solved in output polynomial time unless P = NP. The complexity of dualization for other important classes of lattices, such as modular, also remains an open question for the case where the lattice is given by the ordered set of its irreducible elements.

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