SUSY QM VIA 2x2 MATRIX SUPERPOTENTIAL

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Abstract

The \( N = 2 \) supersymmetry in quantum mechanics involving two-component eigenfunction is investigated.

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1 Introduction

The algebraic technique of the supersymmetry in quantum mechanics (SUSY QM) formulated by Witten [1], in which the essential idea is based on the Darboux procedure on second-order differential equations, has been extended in order to find the 2x2 matrix superpotential [2, 3, 4, 5].

In this work we show that the superpotential for the SUSY QM with two-component wave functions is a Hermitian matrix, and we consider the application to a planar physical system, a neutron interacting with the magnetic field [6].

2 Supersymmetry for two-component eigenfunction

In this section we consider a non-relativistic Hamiltonian ($H_1$) for a two-component wave function in the following bilinear forms

$$H_1 = A_1^+ A_1^- + E_1^{(0)}$$
$$= -I \frac{d^2}{dx^2} + \left( \frac{d}{dx} W_1(x) \right) + W_1(x) \frac{d}{dx} - W_1^\dagger(x) \frac{d}{dx} + W_1^\dagger W_1(x), \tag{1}$$

$$H_2 = A_1^- A_1^+ + E_1^{(0)}$$
$$= -I \frac{d^2}{dx^2} - \left( \frac{d}{dx} W_1^\dagger(x) \right) - W_1^\dagger(x) \frac{d}{dx} + W_1(x) \frac{d}{dx} + W_1(x) W_1^\dagger, \tag{2}$$

where

$$A_1^- = -I \frac{d}{dx} + W_1(x), \quad A_1^+ = \left( A_1^- \right)^\dagger. \tag{3}$$

So far $W_1(x)$ can be a two by two non-Hermitian matrix, but we will now show that $H_1$ and $H_2$ are exactly the Hamiltonians of the bosonic and fermionic sectors of a SUSY Hamiltonian if and only if the matrix superpotential is a Hermitian one. Indeed (comparing the pair SUSY Hamiltonians $H_\pm$ with the Hamiltonians $H_1$ and $H_2$) we see that only when the hermiticity condition of the $W_1$ is readily satisfied, i.e., $W_1^\dagger = W_1$, we may put $H_1$ in a
bosonic sector Hamiltonian. In this case $H_1 (H_2)$ becomes exactly $H_- (H_+)$ of a SUSY Hamiltonian model, analogous to the Witten model, viz.,

$$H_{SUSY} = -\frac{1}{2} I \frac{d^2}{dx^2} + \frac{1}{2} \{ W^2(x) + W'(x) \sigma_3 \} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix},$$

(4)

where $\sigma_3$ is the Pauli matrix. Only under the hermiticity condition one can to call $W_1 = W(x)$ of a matrix superpotential.

Let $H_\pm$ be the bosonic (-) and fermionic (+) sector Hamiltonians for a two-component eigenstate $\Psi_-$, given by

$$H_\pm = -I \frac{d^2}{dx^2} + V_\pm(x), \quad \Psi_-(x) = \begin{pmatrix} \psi_{-1}(x) \\ \psi_{-2}(x) \end{pmatrix}, \quad E_-^{(0)} = 0,$$

(5)

where $I$ denotes the 2x2 unit matrix and the pair of SUSY potential $V_-(x)$, is a 2x2 matrix potential which may be written in terms of a 2x2 matrix superpotential $W(x)$, viz.,

$$V_\pm(x) = W^2(x) \mp W'(x).$$

(6)

Let us consider the eigenvalue equations for the bosonic and fermionic sector Hamiltonians, viz.,

$$H_\pm \Psi_\pm^{(n)} = E_\pm^{(n)} \Psi_\pm^{(n)}, n = 0, 1, 2, \ldots.$$  

(7)

These systems can exhibit bound and continuous eigenstates under the annihilation conditions

$$A^- \Psi_-^{(0)} = 0, \quad \Psi_-^{(0)}(x) = \begin{pmatrix} \psi_{-1}^{(0)}(x) \\ \psi_{-2}^{(0)}(x) \end{pmatrix}$$

(8)

or

$$A^+ \Psi_+^{(0)} = 0, \quad \Psi_+^{(0)}(x) = \begin{pmatrix} \psi_{+1}^{(0)}(x) \\ \psi_{+2}^{(0)}(x) \end{pmatrix}.$$  

(9)

In this case we see that one cannot put $\Psi_+^{(0)}(x)$ in terms of $\Psi_-^{(0)}(x)$ and vice-versa in a similar manner to the case of one-component eigenfunction system. However, if $\Psi_-^{(0)}(x)$ is normalizable we have
\[
\int_{-\infty}^{+\infty} \left( |\psi_{-1}^{(0)}|^2 + |\psi_{-2}^{(0)}|^2 \right) dx = 1. \tag{10}
\]

Note that in Eq. (5) of ref. [4] the author has taken a particular Hermitian matrix for his superpotential in such a way that the validity of his development is ensured.

Let us now consider the interesting application of the above development for a bidimensional physical system in coordinate space associated to a Neutron with magnetic momentum \( \vec{\mu} = \mu(\sigma_1, \sigma_2, \sigma_3) \) in a static magnetic field [6]. In this case, \( x = \rho > 0 \), the Ricatti equation in matrix form is given by

\[
V_-(\rho) = W'(\rho) + W^2(\rho) = \left( \begin{array}{cc}
\frac{m^2 - \frac{\mu}{\rho^2}}{\rho^2} & \frac{-2F}{\rho} \\
\frac{-2F}{\rho} & \frac{(m+1)^2 - \frac{\mu}{\rho^2}}{\rho^2}
\end{array} \right) - \tilde{E}_1^{(0)}, \tag{11}
\]

which has the following particular solution for the 2x2 matrix superpotential given by

\[
W_m = \left( \begin{array}{cc}
\frac{m+\frac{\mu}{\rho}}{\rho} & \frac{-F}{\rho^2} \\
\frac{-F}{\rho^2} & \frac{m+1}{\rho}
\end{array} \right),
\]

where the energy eigenvalue of the ground state is \( \tilde{E}_1^{(0)} = -\frac{F^2}{2(m+1)^2} \), \( F \propto -\mu I \), \( m = 0, \pm 1, \pm 2, \cdots \), and \( \rho \) is the usual cylindrical coordinate. We are considering the current \( I \) located along the \( z \)-axis, and we have used units with \( \hbar = 1 = mass \). The current \( I \) generate a static magnetic field. Also, note that \( V_-(\rho) \) has zero ground state energy, \( E_-^{(0)} = 0 \), thus SUSY is said to be unbroken.

The algebra of SUSY in quantum mechanics is characterized by one anticommutation and two commutation relations given below

\[
H_{susy} = [Q_-, Q_+], \quad [H_{susy}, Q_\pm]_- = 0 = (Q_-)^2 = (Q_+)^2. \tag{12}
\]

One representation of the \( N = 2 \) SUSY superalgebra is the following

\[
H_{susy} = [Q_-, Q_+]_+ = \begin{pmatrix}
A^+A^- & 0 \\
0 & A^-A^+
\end{pmatrix}_{4x4} = \begin{pmatrix}
H_- & 0 \\
0 & H_+
\end{pmatrix}_{4x4}. \tag{13}
\]

The supercharges \( Q_\pm \) are differential operators of first order and can be given by

\[
Q_- = \begin{pmatrix}
0 & 0 \\
A^- & 0
\end{pmatrix}_{4x4}, \quad Q_+ = \begin{pmatrix}
0 & A^+
\end{pmatrix}_{4x4},
\]

where \( A^\pm \) are 2x2 non-Hermitian matrices given by Eq. (3).
3 Conclusion

In this work we investigate an extension of the supersymmetry in non-relativistic quantum mechanics for two-component wave functions. This leads to 4x4 supercharges and supersymmetric Hamiltonians whose bosonic sectors are privileged with two-component eigenstates.

We have considered the application for a Neutron in interaction with a static magnetic field of a straight current carrying wire [6].

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