We discuss transfer-function realization for multivariable holomorphic functions mapping the unit polydisk or the right polyhalfplane into the operator analogue of either the unit disk or the right halfplane (Schur/Herglotz functions over either the unit polydisk or the right polyhalfplane) which satisfy the appropriate stronger contractive/positive real part condition for the values of these functions on commutative tuples of strict contractions/strictly accretive operators (Schur–Agler/Herglotz–Agler functions over either the unit polydisk or the right polyhalfplane). As originally shown by Agler, the first case (polydisk to disk) can be solved via unitary extensions of a partially defined isometry constructed in a canonical way from a kernel decomposition for the function (the lurking-isometry method). We show how a geometric reformulation of the lurking-isometry method (embedding of a given isotropic subspace of a Kreın space into a Lagrangian subspace—the lurking-isotropic-subspace method) can be used to handle the second two cases (polydisk to halfplane and polyhalfplane to disk), as well as the last case (polyhalfplane to halfplane) if an additional growth condition at $\infty$ is imposed. For the general fourth case, we show how a linear-fractional-transformation change of variable can be used to arrive at the appropriate symmetrized nonhomogeneous Bessmertny˘ı long-resolvent realization. We also indicate how this last result recovers the classical integral representation formula for scalar-valued holomorphic functions mapping the right halfplane into itself.

1. Introduction

For $\mathcal{U}$, $\mathcal{Y}$ coefficient separable Hilbert spaces, we define the operator-valued Schur class $S(\mathcal{U}, \mathcal{Y})$ (over the unit disk $\mathbb{D}$) to consist of all holomorphic functions $S$ on the unit disk $\mathbb{D}$ with values in the closed unit ball of the space $L(\mathcal{U}, \mathcal{Y})$ of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$, i.e., subject to $\|S(\zeta)\| \leq 1$ for all $\zeta \in \mathbb{D}$. The following result linking the theories of holomorphic functions, linear operators, and input/state/output linear systems is now well known (see e.g. [10] for a full discussion where multivariable extensions are also treated).

Theorem 1.1. Given a function $S: \mathbb{D} \to L(\mathcal{U}, \mathcal{Y})$, the following are equivalent.

1. $S \in S(\mathcal{U}, \mathcal{Y})$. 

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(2) The de Branges–Rovnyak kernel

\[ K_S(\omega, \zeta) = \frac{I - S(\omega)^*S(\zeta)}{1 - \overline{\zeta} \omega} \]

is a positive kernel on \( \mathbb{D} \), i.e., there is an auxiliary Hilbert space \( X \) and a holomorphic function \( H: \mathbb{D} \to \mathcal{L}(U, X) \) which gives rise to a Kolmogorov decomposition for \( K_S \):

\[ K_S(\omega, \zeta) = H(\omega)^*H(\zeta). \]

(3) \( S \) has a unitary transfer-function realization, i.e., there is an auxiliary Hilbert state space \( X \) and a unitary colligation matrix

\[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} X \\ H \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix} \]

so that

\[ S(\zeta) = D + \zeta C(I - \zeta A)^{-1}B \quad \text{for} \quad \zeta \in \mathbb{D}. \]

\( (3') \) Condition (3) above holds where the colligation matrix \( U \) is taken to be any of (i) coisometric, (ii) isometric, or (iii) contractive.

It is natural to seek extensions of the Schur class to the multivariable setting where the disk \( \mathbb{D} \) is replaced by the polydisk

\[ \mathbb{D}^d = \{ \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d : |\zeta_k| < 1 \text{ for } k = 1, \ldots, d \}. \]

We therefore define the \( d \)-variable Schur class \( S_d(U, Y) \) to consist of holomorphic functions \( S: \mathbb{D}^d \to \mathcal{L}(U, Y) \) subject to \( \|S(\zeta)\| \leq 1 \) for all \( \zeta \in \mathbb{D}^d \). It was the profound observation of Agler [1] that, unless \( d \leq 2 \), a characterization of \( S_d(U, Y) \) of the same form as Theorem 1.1 is not possible. Instead, we define what is now called the Schur–Agler class, denoted as \( SA_d(U, Y) \), to consist of holomorphic functions \( S: \mathbb{D}^d \to \mathcal{L}(U, Y) \) such that \( \|S(T_1, \ldots, T_d)\| \leq 1 \) whenever \( T = (T_1, \ldots, T_d) \) is a commutative \( d \)-tuple of strict contraction operators on a fixed separable infinite-dimensional Hilbert space \( K \). Here the functional calculus defining \( S(T_1, \ldots, T_d) \) can be given by

\[ S(T_1, \ldots, T_d) = \sum_{n \in \mathbb{Z}_+^d} S_n \otimes T^n \]

(convergence in the strong operator topology) where \( S(\zeta) = \sum_{n \in \mathbb{Z}_+^d} S_n \zeta^n \) is the multivariable Taylor expansion for \( S \) centered at the origin \( 0 \in \mathbb{D}^d \) and where we use standard multivariable notation:

\[ \zeta^n = \zeta_1^{n_1} \cdots \zeta_d^{n_d}, \quad T^n = T_1^{n_1} \cdots T_d^{n_d} \text{ if } n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d. \]

The following result due to Agler [1] (see also [2, 13]) has had a profound impact on the subject over the years.

**Theorem 1.2.** Given a function \( S: \mathbb{D}^d \to \mathcal{L}(U, Y) \), the following are equivalent.

1. \( S \in SA_d(U, Y) \).
2. \( S \) has an Agler decomposition in the sense that there exists \( d \) \( \mathcal{L}(Y) \)-valued positive kernels \( K_1, \ldots, K_d \) on \( \mathbb{D}^d \) such that

\[ I - S(\omega)^*S(\zeta) = \sum_{k=1}^d (1 - \overline{\omega_k}\zeta_k)K_k(\omega, \zeta). \quad (1.1) \]
(3) $S$ has a unitary Givone–Roesser $d$-dimensional transfer-function realization, i.e., there is an auxiliary Hilbert state space $X$ with a $d$-fold orthogonal direct-sum decomposition $X = X_1 \oplus \cdots \oplus X_d$ together with a unitary colligation matrix
\[ \mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} X' \\ Y \end{bmatrix} \to \begin{bmatrix} X' \\ Y \end{bmatrix} \]
so that
\[ S(\zeta) = D + C(I - P(\zeta)A)^{-1}P(\zeta)B \text{ for } \zeta \in \mathbb{D}^d. \] (1.2)
where we have set
\[ P(\zeta) = \zeta_1 P_1 + \cdots + \zeta_d P_d \]
where $P_k$ is the orthogonal projection of $X$ onto $X_k$ for each $k = 1, \ldots, d$.

(3') Condition (3) above holds where the colligation matrix $\mathbf{U}$ is taken to be any of (i) coisometric, (ii) isometric, or (iii) contractive.

The goal of this paper is to study parallel results for an assortment of linear-fractional transformed versions of the Schur–Agler class. Specifically, we seek analogous characterizations of the following classes of holomorphic functions:

(1) The Herglotz–Agler class over the unit polydisk $\mathbb{D}^d$, denoted $\mathcal{HA}(\mathbb{D}^d, \mathcal{L}(\mathcal{U}))$, consisting of all holomorphic $\mathcal{L}(\mathcal{U})$-valued functions $\mathbf{f}$ on $\mathbb{D}^d$ such that
\[ \frac{1}{2}i(\mathbf{f}(\mathbf{A}) - \mathbf{f}(\mathbf{A})^*) \geq cI \text{ for some constant } c > 0, k = 1, \ldots, d. \]

(2) The Schur–Agler class over the right polyhalfplane $\Pi^d$, denoted $\mathcal{SA}(\Pi^d, \mathcal{L}(\mathcal{U}))$, consisting of all holomorphic $\mathcal{L}(\mathcal{U})$-valued functions $\mathbf{f}$ on $\Pi^d$ such that
\[ \mathbf{f}(\mathbf{A}) + \mathbf{f}(\mathbf{A})^* \geq 0 \text{ for all } \mathbf{A} = (A_1, \ldots, A_d) \text{ of operators on } K (i.e., such that } A_k + A_k^* \geq cI \text{ for some constant } c > 0, k = 1, \ldots, d. \]

(3) The Herglotz–Agler class over $\Pi^d$, denoted $\mathcal{HA}(\Pi^d, \mathcal{L}(\mathcal{U}))$, consisting of all holomorphic functions $\mathbf{f}$ on $\Pi^d$ such that
\[ \mathbf{f}(\mathbf{A}) + \mathbf{f}(\mathbf{A})^* \geq 0 \text{ for all } \mathbf{A} = (A_1, \ldots, A_d) \text{ of operators on } K. \]

(4) The Nevanlinna–Agler class over the upper polyhalfplane
\[ (i\Pi)^d = \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \frac{z_k - \overline{z}_k}{2i} > 0 \text{ for } k = 1, \ldots, d \}, \]
denoted by $\mathcal{NA}((i\Pi)^d, \mathcal{L}(\mathcal{U}))$, consisting of all holomorphic $\mathcal{L}(\mathcal{U})$-valued functions $\tilde{f}$ on $(i\Pi)^d$ such that
\[ \frac{1}{2i} (\tilde{f}(\tilde{A}) - \tilde{f}(\tilde{A})^*) \geq 0 \text{ whenever } \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_d) \text{ is a commutative } d\text{-tuple of operators on } K, \text{ each with strictly positive-definite imaginary part (i.e., such that } \frac{1}{2i}(A_k - A_k^*) \geq cI \text{ for some constant } c > 0, k = 1, \ldots, d. \]

To be consistent with the more detailed notation used for these variants of the Schur–Agler class, we will also use the notation $\mathcal{SA}(\mathbb{D}^d, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$ for the Schur–Agler class $\mathcal{SA}(\Pi^d, \mathcal{L}(\mathcal{Y}))$ over the polydisk $\mathbb{D}^d$ discussed above.
We note that our convention is to use the term Herglotz for functions with values having positive real part, and Nevanlinna for functions with values having positive imaginary part; we recognize that these conventions are by no means universal (see e.g. [6]).

For the single-variable case such realization results have been explored in a systematic way in [35] and [12]. For the multivariable setting, apart from the now classical Schur–Agler class over the polydisk $\mathcal{SA}_d(U, Y)$, the only results along these lines which we are aware of are those in the recent paper of Agler–McCarthy–Young [3] and of Agler–Tully-Doyle–Young [4, 5].

The approach in [35] (in the single-variable setting) is to use a linear-fractional-transformation (LFT) change of variables (on the domain and/or the range side) to reduce the desired result to the corresponding result for the Schur class over the unit disk. This is also the main tool in [3, 4, 5]: use an LFT Cayley-transform change of variables to reduce results for the Nevanlinna–Agler class to the corresponding known results for the Schur–Agler class. However the procedure is rather intricate due to the added subtleties involved in handling points at infinity in the multivariable case.

In contrast, the approach in [12] is to apply a projective version of the lurking isometry argument (roughly, a lurking-isotropic-subspace argument in a Krein-space setting) to arrive at the desired realization result via a direct but unified Krein-space geometric argument. One of the main contributions of the present work is to extend this approach to the multivariable setting. The main difficulty is to guarantee that a naturally defined isotropic subspace is actually a graph space with respect to a system of coordinates not coming from a fundamental decomposition of the ambient Krein space. We show how this difficulty can be overcome for the case of the $D^d$-Herglotz–Agler class and $H^2$-Schur–Agler class. For the $D^d$-Herglotz–Agler class, we are able to overcome the difficulty only in a special case (associated with the imposition of a growth condition at infinity), thereby recovering parallel results from [3]. For the most general $D^d$-Herglotz–Agler function $f$, we follow the LFT change-of-variable approach of [4] combined with the more general realization formalism (Schur complement of an operator pencil) suggested by the work of Bessmertnyi (see [18, 19, 20, 21, 28]) to arrive at a realization formula for the most general $D^d$-Herglotz–Agler function. We note that the original Bessmertnyi class involved additional symmetries leading to strong rigidity results. It was conjectured in [9] that an appropriate weakening of the metric conditions for the Bessmertnyi operator pencil should lead to a representation for the most general $D^d$-Herglotz–Agler function. Here we show that this conjecture is correct once one identifies the appropriate modification: one must allow the nonhomogeneous skew-adjoint term in the nonhomogeneous Bessmertnyi operator pencil to be unbounded (more precisely, a certain flip II-impedance-conservative system node in the sense of [35]).

There has been a lot of work on transfer-function realization for the single-variable Schur and Herglotz classes over the right half plane. The most influential for our point of view toward multivariable generalizations is the work of Arov–Nudelman [7] and of Staffans and collaborators (see [34, 35, 12, 29, 37] as well as the treatise [36] and the references there). There is also a complementary approach to such realization theory (upper halfplane rather than right halfplane version) with emphasis on the theory of selfadjoint extensions of densely defined symmetric
operators on a Hilbert space (see [14, 15, 16] as well as the recent book [6] and the references there).

The paper is organized as follows. Section 2 highlights the main ideas from Krein-space geometry and from infinite-dimensional systems theory, in particular, the idea of a system node, which will be used in the later sections. Section 3 presents our results for the Herglotz–Agler class over the polydisk while Section 4 does the same for the Schur–Agler class over the right polyhalfplane. Section 5 presents our results for the restricted Herglotz–Agler class over the right polyhalfplane where a growth condition at infinity is imposed on the functions to be realized. With this added restriction, the lurking-isotropic-subspace method from [12] adapts well to lead to a classical type realization (but with an in general unbounded II-impedance-conservative system node) for the $\Pi^d$-Herglotz–Agler function. Section 6 identifies the nonhomogeneous unbounded Bessmertnyıı operator pencils which then lead to a realization for the most general Herglotz–Agler function over $\Pi^d$. We also mention that the results parallel to the results of this paper for the four classes under discussion ($\mathbb{D}^d$-Schur–Agler, $\mathbb{D}^d$-Herglotz–Agler, $\Pi^d$-Schur–Agler and $\Pi^d$-Herglotz–Agler) for the rational matrix-valued (Cayley) inner case, where the emphasis is on obtaining realizations with finite-dimensional state space, are obtained in our companion paper [11].

We shall have occasion to need a Cayley transform (with both scalar and operator argument) acting between the right halfplane and the unit disk. Following the conventions in [28] and [11], we shall make use of the following version:

$$\zeta \in \mathbb{D} \mapsto w = \frac{1 + \zeta}{1 - \zeta} \in \Pi, \quad \text{with inverse given by}$$

$$w \in \Pi \mapsto \zeta = \frac{w - 1}{w + 1} \in \mathbb{D}. \quad (1.3)$$

For $\zeta = (\zeta_1, \ldots, \zeta_d)$ a point in the unit polydisk $\mathbb{D}^d$, we continue to use the notation

$$\frac{1 + \zeta}{1 - \zeta} := \left( \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d} \right) \quad (1.4)$$

for the corresponding point in the right polyhalfplane $\Pi^d$. Similarly, given a point $w = (w_1, \ldots, w_d)$ in the right polyhalfplane $\Pi^d$, we use the notation

$$\frac{w - 1}{w + 1} := \left( \frac{w_1 - 1}{w_1 + 1}, \ldots, \frac{w_d - 1}{w_d + 1} \right) \quad (1.5)$$

for the associated point in the polydisk.

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2. Preliminaries

2.1. Decompositions of the identity. Given a Hilbert space $\mathcal{X}$, we shall say that a collection of $d$ operators $(Y_1, \ldots, Y_d)$ on $\mathcal{X}$ forms a $d$-fold positive decomposition of the identity $I_{\mathcal{X}}$ if each $Y_k$ is a selfadjoint contraction ($0 \leq Y_k \leq I_{\mathcal{X}}$ for $1 \leq k \leq d$) which together sum up to the identity ($\sum_{k=1}^d Y_k = I_{\mathcal{X}}$). In case $(Y_1, \ldots, Y_d) = (P_1, \ldots, P_d)$ consists of orthogonal projection operators (necessarily with pairwise orthogonal ranges) we shall say that $(P_1, \ldots, P_d)$ forms a $d$-fold spectral decomposition of $I_{\mathcal{X}}$. Note that $d$-fold spectral decompositions $(P_1, \ldots, P_d)$
arise in the realization formula for the Schur–Agler class in Theorem 1.2. We shall see that the more general positive decompositions are needed for the realization formulas for functions in the $\Pi^d$-Schur–Agler class and in the $\Pi^d$-Herglotz–Agler class, as already discovered in [3, 4, 5].

From the definitions we see that any spectral decomposition $(P_1, \ldots, P_d)$ is also a positive decomposition. There is also a result in the converse direction: if $(Y_1, \ldots, Y_d)$ is a positive decomposition of $I_X$, then there exist a Hilbert space $\tilde{X}$, a spectral decomposition $(P_1, \ldots, P_d)$ of $I_{\tilde{X}}$, and an isometric embedding $\iota: X \to \tilde{X}$ such that

$$Y_k = \iota^* P_k \iota \quad \text{for } k = 1, \ldots, d.$$ 

This can be seen as a consequence of the Naimark dilation theorem (apply [30, Theorem 4.6] with the measurable space $X$ taken to be the finite set \{k ∈ \mathbb{N}: 1 ≤ k ≤ d\}). To prove the result for this simple case of the Naimark dilation theorem, simply define an isometric embedding of $X$ into $\bigoplus_{i=1}^d X$ by

$$\iota = \begin{bmatrix} Q_1 \\ \vdots \\ Q_d \end{bmatrix}$$

where $Q_k$ provides a factorization $Y_k = Q_k^* Q_k$ and take $P_k$ equal to the projection onto the $k$-th block in the direct-sum space $\bigoplus_{i=1}^d X$.

2.2. Basics on the geometry of Kreı́n spaces. In this Section we review some basics about the geometry of Kreı́n spaces and Kreı́n-space operator theory which we shall need in the sequel. Other resources on this topic is a similar survey section in the paper [12] as well as the more complete treatises [22, 8].

A Kreı́n space by definition is a linear space $K$ endowed with an indefinite inner product $[\cdot, \cdot]$ which is complete in the following sense: there are two subspaces $K_+$ and $K_-$ of $K$ such that the restriction of $[\cdot, \cdot]$ to $K_+ \times K_+$ makes $K_+$ a Hilbert space while the restriction of $-[\cdot, \cdot]$ to $K_- \times K_-$ makes $K_-$ a Hilbert space, and $K = K_+[+]K_-$ is a $[\cdot, \cdot]$-orthogonal direct sum decomposition of $K$. In this case the decomposition $K = K_+[+]K_-$ is said to form a fundamental decomposition for $K$. Fundamental decompositions are never unique except in the trivial case where one of $K_+$ or $K_-$ is equal to the zero space.

Unlike the case of Hilbert spaces where closed subspaces all look the same, there is a rich geometry for subspaces of a Kreı́n space. A subspace $M$ of a Kreı́n space $K$ is said to be positive, isotropic, or negative depending on whether $[u, u] ≥ 0$ for all $u ∈ M$, $[u, u] = 0$ for all $u ∈ M$ (in which case it follows that $[u, v] = 0$ for all $u, v ∈ M$ as a consequence of the Cauchy-Schwarz inequality), or $[u, u] ≤ 0$ for all $u ∈ M$. Given any subspace $M$, we define the Kreı́n-space orthogonal complement $M^\perp$ to consist of all $v ∈ K$ such that $[u, v] = 0$ for all $u ∈ K$. Note that the statement that $M$ is isotropic is just the statement that $M ⊂ M^\perp$. If it happens that $M = M^\perp$, we say that $M$ is a Lagrangian subspace of $K$.

Examples of such subspaces arise from placing appropriate Kreı́n-space inner products on the direct sum $H_1 \oplus H_2$ of two Hilbert spaces and looking at graphs of operators of an appropriate class.
Example 2.1. Suppose that $\mathcal{H}'$ and $\mathcal{H}$ are two Hilbert spaces and we take $\mathcal{K}$ to be the external direct sum $\mathcal{H'} \oplus \mathcal{H}$ with inner product
\[
\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H'} \oplus \mathcal{H}} = \left\langle \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & -I_{\mathcal{H}} \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H'} \oplus \mathcal{H}}
\]
where $\langle \cdot, \cdot \rangle_{\mathcal{H'} \oplus \mathcal{H}}$ is the standard Hilbert-space inner product on the direct-sum Hilbert space $\mathcal{H'} \oplus \mathcal{H}$. In this case it is easy to find a fundamental decomposition: take $\mathcal{K}_+ = \left[ \begin{matrix} \mathcal{H} \\ \{0\} \end{matrix} \right]$ and $\mathcal{K}_- = \left[ \begin{matrix} \{0\} \\ \mathcal{H}' \end{matrix} \right]$. Now let $T$ be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}'$ and let $\mathcal{M}$ be the graph of $T$:
\[
\mathcal{M} = \mathcal{G}_T = \left\{ \begin{bmatrix} Tx \\ x \end{bmatrix} : x \in \mathcal{H} \right\} \subset \mathcal{K}.
\]
Then a good exercise is to work out the following facts:
- $\mathcal{G}_T$ is negative if and only if $\|T\| \leq 1$.
- $\mathcal{G}_T$ is isotropic if and only if $T$ is isometric ($T^*T = I_{\mathcal{H}}$).
- $\mathcal{G}_T$ is Lagrangian if and only if $T$ is unitary: $T^*T = I_{\mathcal{H}}$ and $TT^* = I_{\mathcal{H}'}$.

Example 2.2. Let $\mathcal{H}$ be a Hilbert space and set $\mathcal{K}$ equal to the direct-sum space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ with indefinite inner product given by
\[
\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H} \oplus \mathcal{H}} = \left\langle \begin{bmatrix} 0 & I_{\mathcal{H}} \\ I_{\mathcal{H}} & 0 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H} \oplus \mathcal{H}}.
\]
In this case a choice of fundamental decomposition is not so obvious; one such choice is
\[
\mathcal{K}_+ = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} : x \in \mathcal{H} \right\}, \quad \mathcal{K}_- = \left\{ \begin{bmatrix} -y \\ y \end{bmatrix} : y \in \mathcal{H} \right\}.
\]
Then the exercise parallel to that suggested in Example 2.1 is to work out the following: given a closed operator $T \in L(\mathcal{H})$ with dense domain $D(T) \subset \mathcal{H}$, let
\[
\mathcal{G}_T = \left\{ \begin{bmatrix} Tx \\ x \end{bmatrix} : x \in D(T) \right\} \subset \mathcal{K}
\]
be its graph space. Then:
- $\mathcal{G}_T$ is negative if and only if $T$ is dissipative: $\langle Tx, x \rangle + \langle x, Tx \rangle \leq 0$ for all $x \in D(T)$.
- $\mathcal{G}_T$ is maximal negative, i.e., $\mathcal{G}_T$ is negative and is not contained in any properly larger negative subspace, if and only if $T$ is maximal dissipative, i.e., $T$ is dissipative and has no proper dissipative extension. An equivalent condition is $T$ is dissipative and the operator $I + T$ is onto (or equivalently $wI + T$ is onto for all $w$ in the right halfplane $\Pi^+$) (see [31]).
- $\mathcal{G}_T$ is isotropic if and only if $T$ is skew-symmetric or $T \subset -T^*$, i.e., $\langle Tx, x \rangle_{\mathcal{H}} + \langle x, Tx \rangle_{\mathcal{H}} = 0$ for all $x \in D(T)$.
- $\mathcal{G}_T$ is Lagrangian if and only if $T$ is skew-adjoint or $T = -T^*$ (i.e., $T \subset -T^*$ and $T$ and $T^*$ have the same domain: $y, z \in \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{H}} = \langle x, z \rangle_{\mathcal{H}}$ implies that $y \in D(T)$ and $z = -Ty$. A closely related result is proved in Corollary 2.7 below.

We shall have use for the following connection between Examples 2.1 and 2.2. Consider Example 2.1 for the case where $\mathcal{H}' = \mathcal{H}$ and call this Krein space $\mathcal{K}_1$. Let
\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -I_H \\ -I_H & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \]
consists of bounded operators, imposition of the initial condition \( x(0) = 0 \) and application of the Laplace transform

\[
\hat{x}(w) = \int_0^\infty e^{-wt}x(t)\,dt, \quad \hat{u}(w) = \int_0^\infty e^{-wt}u(t)\,dt, \quad \hat{y}(w) = \int_0^\infty e^{-wt}y(t)\,dt
\]
leads to the input-output relation in the frequency domain

\[
\hat{y}(w) = T_\Sigma(w)\hat{u}(w)
\]

where

\[
T_\Sigma(w) = D + C(wI - A)^{-1}B
\]
is the transfer function of the linear system \( \Sigma \). The converse question of when an \( L(U, Y) \)-valued function \( f \) on the right halfplane \( \Pi \) can be realized as \( f = T_\Sigma \) for some system \( \Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) has generated much interest over the years. The obvious necessary condition is that \( w \mapsto f(w) \) be analytic on some right halfplane but this is not sufficient if we limit our attention to systems \( \Sigma \) consisting only of bounded operators \( A, B, C, D \). While it is clear that the right generalization of the state operator \( A \) is that it should be the generator of a \( C_0 \)-semigroup, exactly how to handle the remaining operators \( B, C, D \) so as to get a meaningful theory containing compelling examples of interest was not so clear, but some progress was made already in the 1970s (see [26, 27]). It is now understood that a useful notion of generalized system matrix \( \Sigma \) is that associated with so-called well-posed systems.

Roughly, a well-posed linear system is an \( i/s/o \) linear system for which the integral form of the system operators \( A, B, C, D \) satisfies natural compatibility conditions and the integral form of the system matrix

\[
\begin{bmatrix} 0 & 0 \\ \mathcal{D} & \mathcal{D} \end{bmatrix} : \begin{bmatrix} x(0) \\ u|_{[0,t]} \end{bmatrix} \mapsto \begin{bmatrix} x(t) \\ y|_{[0,t]} \end{bmatrix}
\]

makes sense as a bounded operator from \( \mathcal{X} \oplus L^2([0, t]) \) to \( \mathcal{X} \oplus L^2([0, t]) \) for each \( t > 0 \) (see [36] for complete details). The “right” infinitesimal object (the analogue of the system matrix \( \Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) appearing in (2.2)) is the notion of system node defined as follows; this notion is well laid out in the work of Staffans [36, 35] where it is acknowledged that much of the idea was already anticipated in the earlier work of Salamon [32] and Smuljan [33].

We first make some preliminary observations. A system node \( \Sigma \) is still an operator from \( \mathcal{X} \oplus U \) to \( \mathcal{X} \oplus Y \) but now allowed to be unbounded with some domain \( \mathcal{D}(\Sigma) \subset \mathcal{X} \oplus U \). We may then split \( \Sigma \) in the form

\[
\Sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix}
\]

where \( \Sigma_1 : \mathcal{D}(\Sigma) \to \mathcal{X} \) and \( \Sigma_2 : \mathcal{D}(\Sigma) \to Y \). However we allow the possibility that \( \mathcal{D}(\Sigma) \) does not split \( \mathcal{D}(\Sigma) = \begin{bmatrix} \mathcal{D}(\Sigma)_1 \\ \mathcal{D}(\Sigma)_2 \end{bmatrix} \) as the direct sum of a linear manifold \( \mathcal{D}(\Sigma)_1 \) in \( \mathcal{X} \) with a linear manifold \( \mathcal{D}(\Sigma)_2 \) in \( U \). To keep the parallel with the classical case, we therefore write

\[
\Sigma = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}
\]
with the notation $A\&B$ (and similarly $C\&D$) suggesting that the common domain of $A\&B$ and $C\&D$ in $X \oplus U$ may not have a splitting. However $A\&B$ will have a splitting $[A;X - B]$ where $A;X$ and $B$ are operators mapping the spaces $X$ and $U$ into a larger “rigged” space $X_{-1}$ which is part of a so-called Gelfand triple defined as follows.

We assume that $A$ is an in general unbounded closed operator with dense domain $\mathcal{D}(A) := X_1$ in $X$ and with nonempty resolvent set. Then $X_1$ is a Hilbert space in its own right with respect to the $X_1$-norm given by

$$\|x\|_1 := \|(\alpha I - A)x\|_X$$

where $\alpha$ is any fixed number in the resolvent set of $A$. While the norm depends on the choice of $\alpha$, any other choice $\alpha'$ of $\alpha$ leads to the same space but with an equivalent norm, as long as $\alpha$ and $\alpha'$ are in the same connected component of the resolvent set of $A$. If $A$ is the generator of a $C_0$-semigroup (the most interesting case for us), one can take the connected component of the resolvent set to be a right half plane. Note that $(\alpha I - A)|_{X_1}$ can be viewed as an isometry from $X_1$ onto $X$.

We next construct another Hilbert space $X_{-1}$ as the completion of $X$ in the norm

$$\|x\|_{-1} := \|(\alpha I - A)^{-1}x\|_X.$$ 

If $\{x_n\}_{n \in \mathbb{Z}_+}$ is a sequence in $X_1$ converging to $x \in X$ in $X$-norm, then the sequence $\{(\alpha I - A)x_n\}_{n \in \mathbb{Z}_+}$ is Cauchy in $X_{-1}$-norm and hence converges to an element $x^{-1} \in X_{-1}$. One can check that this element $x^{-1} \in X_{-1}$ is independent of the choice of sequence $\{x_n\} \subset X_1$ converging in $X$-norm to $x$; we denote this element $y \in X_{-1}$ by $y = (\alpha I - A)x$ and then define an extension $A|_{X} : X \to X_{-1}$ by

$$A|_{X} : x \mapsto \alpha x - (\alpha I - A)x \in X_{-1} \text{ if } x \in X.$$ 

We then have that the extended operator $(\alpha I - A)|_{X}$ is an isometry from $X$ onto $X_{-1}$ and we have the nested inclusions $X_1 \subset X \subset X_{-1}$ with continuous and dense injections. We will on occasion simplify the notation $A|_{X}$ to simply $A$ when the meaning is clear; thus for $x \in X$ and $x$ not necessarily in $X_1 = \mathcal{D}(A)$, the element $Ax$ is still defined but as an element of $X_{-1}$.

It is also useful to note the role of these spaces in duality pairings. First, we note that the constructions in the previous paragraph can be carried out using the operator $A^*$ in place of $A$. When this is done we get spaces $X_1^* = (\pi I - A^*)^{-1}X$ (with the norm $\|x\|_{1,*} := \|(\pi I - A^*)x\|_X$ and $X_{-1}^*$ equal to the completion of $X$ in the $X_{-1}$-norm $\|x\|_{-1,*} := \|(\pi I - A^*)^{-1}x\|_X$ with the properties that $(\pi I - A^*)^{-1}$ is an isometry from $X$ onto $X_1^*$ and $(\pi I - A^*)$ extends to an isometry from $X$ onto $X_1^*$ with the nesting $X_1^* \subset X \subset X_1^*$. Given any $x \in X$, we can view $x$ as a linear functional on $X_1^*$ using the $X$-pairing:

$$\ell_x(x_1^*) := \langle x_1^*, x \rangle_X \text{ for } x_1^* \in X_1^*. \quad (2.4)$$
If we write \( x_1^* = (\pi I - A^*)^{-1} y \) with \( y \in \mathcal{X} \), then

\[
|\ell_x(x_1^*)| = |(\langle (\pi I - A^*)^{-1} y, x_1^* \rangle)_{\mathcal{X}}|
\]

\[
= |\langle y, (\alpha I - A)^{-1} x_1^* \rangle_{\mathcal{X}}|
\]

\[
\leq \|y\|_{\mathcal{X}} \| (\alpha I - A)^{-1} x_1^* \|_{\mathcal{X}}
\]

\[
= \|x_1^*\|_{1,\star} \|x_1^*\|_{-1}
\]

with equality if we take \( x_1^* = (\pi I - A^*)^{-1} (\alpha I - A)^{-1} x \) (or \( y = (\alpha I - A)^{-1} x \)). We conclude that the linear-functional norm of \( \ell_x \) is equal to the \( \mathcal{X}_{-1} \)-norm of \( \mathcal{X} \):

\[
\|\ell_x\|_{(\mathcal{X}_1)^*} = \|x_1^*\|_{-1}
\]

and \( \mathcal{X} \) can be identified with a subspace of \( (\mathcal{X}_1)^* \). It is not difficult to see that this subspace is dense and hence, after taking completions, we have that \( \mathcal{X}_{-1} \) is naturally isomorphic to the dual space of \( \mathcal{X}_1 \) via the \( \mathcal{X} \)-pairing \( (2.4) \). For our application to infinite-dimensional linear systems, in practice the unbounded closed operator \( A \) in this construction will also be taken to be the generator of a \( C_0 \)-semigroup, so as to make sense of a differential equation of the form \( \frac{dx}{dt}(t) = Ax(t) \).

We are now ready to introduce the notion of *system node*.

**Definition 2.4.** By a *system node* \( \Sigma \) on the collection of three Hilbert spaces \((\mathcal{U}, \mathcal{X}, \mathcal{Y})\), we mean a linear operator

\[
\Sigma := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \mathcal{D}(\Sigma) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}
\]

such that:

1. \( \Sigma \) is a closed operator with domain \( \mathcal{D}(\Sigma) \) dense in \( \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \).
2. If we define an operator \( A \) with domain \( \mathcal{D}(A) = \{x \in \mathcal{X}: [\begin{smallmatrix} x \\ 0 \end{smallmatrix}] \in \mathcal{D}(\Sigma)\} \) by

\[
Ax = \Sigma [\begin{smallmatrix} x \\ 0 \end{smallmatrix}]
\]

for \( x \in \mathcal{D}(A) \),

then \( A \) is the generator of a \( C_0 \)-semigroup on \( \mathcal{X} \).
3. Let \( A|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_{-1} \) be the extension of the operator \( A : \mathcal{X}_1 = \mathcal{D}(A) \to \mathcal{X} \) (as in \( (2.5) \)) as described in the previous paragraph. Then there is a bounded linear operator \( B : \mathcal{U} \to \mathcal{X}_{-1} \) so that we recover the operator \( A \& B \) as the restriction of the operator \( [A|_{\mathcal{X}} \quad B] : [\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}] \to \mathcal{X}_{-1} \) to \( \mathcal{D}(\Sigma) \):

\[
\Sigma = [A|_{\mathcal{X}} \quad B] |_{\mathcal{D}(\Sigma)}.
\]

4. \( C \& D \) is a bounded operator from \( \mathcal{D}(\Sigma) \) to \( \mathcal{Y} \)

\[
C \& D \in L(\mathcal{D}(\Sigma), \mathcal{Y})
\]

where \( \mathcal{D}(\Sigma) \) carries the graph norm.
5. The domain \( \mathcal{D}(\Sigma) \) of \( \Sigma \) is characterized as

\[
\mathcal{D}(\Sigma) = \left\{ [\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} : A|_{\mathcal{X}} x + Bu \in \mathcal{X} \right\}.
\]

A consequence of this definition of system node is the following fact:

*Given \( u \in \mathcal{U} \), there exists \( x_u \in \mathcal{D}(\Sigma) \) so that \( [\begin{smallmatrix} x_u \\ u \end{smallmatrix}] \in \mathcal{D}(\Sigma) \).*
Indeed, it suffices to take $x_u = ((\alpha I - A)|_X)^{-1}Bu$. To see this, we use the criterion in part (5) of Definition 2.4 to check that this $[x_u]$ is in $\mathcal{D}(\Sigma)$:

$$\begin{align*}
[A|_X \quad B] \left[ x_u \right] &= A|_X \left( (\alpha I - A)|_X \right)^{-1} Bu + Bu \\
&= (A - \alpha I)|_X \left( (\alpha I - A)|_X \right)^{-1} Bu + \alpha \cdot \left( (\alpha I - A)|_X \right)^{-1} Bu + Bu \\
&= \alpha \cdot \left( (\alpha I - A)|_X \right)^{-1} Bu \in \mathcal{X}.
\end{align*}$$

In the definition of system node, we took pains to write the top component in the form $A\&B$ to indicate that its domain in $[\mathcal{X}]$ does not split; yet in the end we found a splitting by extending to a larger space $[\mathcal{X}_{\alpha}]$ and writing $A\&B$ as the restriction of an extended operator $[A|_X \quad B]: [\mathcal{X}_{\alpha}] \to \mathcal{Y}$ whose domain does split. Similarly, there is at least a partial splitting for the operator $C\&D: \mathcal{D}(\Sigma) \to \mathcal{Y}$. Indeed, we have seen that $X_1 = \mathcal{D}(A) = \{x \in \mathcal{X} : [x] \in \mathcal{D}(\Sigma)\}$ is a dense subset of $\mathcal{X}$. We may therefore define an operator $C: X_1 \to \mathcal{Y}$ by

$$Cx = C\&D \left[ x \right] \text{ for } x \in X_1. \quad (2.8)$$

Since $C\&D$ is a bounded operator from $\mathcal{D}(\Sigma)$ (graph norm) to $\mathcal{Y}$, it follows that $C$ so defined is a bounded operator from $X_1$ (graph norm induced by the operator $A$) to $\mathcal{Y}$. In practice we assign no independent meaning to the $D$ in $C\&D$ except under some additional hypotheses (e.g., for the case of a regular system—see the paper of Weiss [39] or [36, Section 5.6]). To this point we have at least versions of all the usual constituents for a linear input/state/output linear system:

- $A: X_1 \to \mathcal{X}$ main operator or state dynamics,
- $B: \mathcal{U} \to X_{-1}$ input or control operator,
- $C: X_1 \to \mathcal{Y}$ output or observation operator,
- $C\&D: \mathcal{D}(\Sigma) \to \mathcal{Y}$ combined observation/feedthrough operator, \quad (2.9)

and we lack in general an independent well-defined feedthrough operator $D$.

The formula for $x_u$ in (2.7) can use any point $w$ in the connected component (e.g., an appropriate right half plane) of the resolvent set of $A$ containing $\alpha$. For clarity, let us write $x_u(w) = (wI - A)|_X^{-1}Bu$ to indicate the dependence of $x_u$ on the point $w$ in the right half plane. From the fact that $[x_u(w)] \in \mathcal{D}(\Sigma) = \mathcal{D}(C\&D)$ it follows that the expression

$$T_\Sigma(w): u \mapsto C\&D \left[ x_u(w) \right] = C\&D \left[ (wI - A)|_X^{-1} Bu \right]$$

is well defined and defines the transfer function of the system node. Let $\alpha$ be any fixed point in the resolvent set of $A$. Then we can recover the value of the transfer function at any point $w$ in the same connected component of the resolvent set of $A$ from its value at the fixed point $\alpha$ according to the recipe

$$\begin{align*}
T_\Sigma(w)u &= (T_\Sigma(w) - T_\Sigma(\alpha))u + T_\Sigma(\alpha)u \\
&= C(x_u(w) - x_u(\alpha)) + T_\Sigma(\alpha)u \\
&= (\alpha - w)C(wI - A)^{-1}(\alpha I - A)|_X^{-1}Bu + T_\Sigma(\alpha)u \quad (2.10)
\end{align*}$$

Conversely, start with any semigroup generator $A$ on $\mathcal{X}$ with domain $\mathcal{D}(A) = X_1$ with induced Gelfand rigging $X_1 \subset \mathcal{X} \subset X_{-1}$, any input operator $B: \mathcal{U} \to X_{-1}$,
and an output operator \( C : X_1 \to Y \) along with a value \( T_\Sigma(\alpha) \in \mathcal{L}(U, Y) \) for the transfer function at the point \( \alpha \). Define

\[
\mathcal{D}(\Sigma) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : A x + B u \in X \right\}
\]

with

\[
A \& B = \begin{bmatrix} A & B \end{bmatrix} \mid \mathcal{D}(\Sigma), \quad C \& D \begin{bmatrix} x \\ u \end{bmatrix} = C(x - x_u(\alpha)) + T_\Sigma(\alpha) u.
\]

Then \( \Sigma \) so defined is a system node with value of its transfer function at \( \alpha \) equal to the prescribed value \( T_\Sigma(\alpha) \).

Note here that \( x - x_u(\alpha) \) is in \( X_1 \) since \( \begin{bmatrix} x \\ u \end{bmatrix} \) and \( \begin{bmatrix} x_u(\alpha) \\ u \end{bmatrix} \) are in \( \mathcal{D}(\Sigma) \) and hence so also is \( \begin{bmatrix} x - x_u(\alpha) \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix} - \begin{bmatrix} x_u(\alpha) \\ u \end{bmatrix} \) (2.12) resulting in \( x - x_u(\alpha) \in X_1 \). Moreover, we recover the transfer function \( T_\Sigma \) at a general point from \( A, B, C \) and \( T_\Sigma(\alpha) \) via the formula (2.10). (see [34, Lemma 2.2] for more complete details).

Given a system node \( \Sigma \), it is possible to make sense of the associated system of differential equations

\[
\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \Sigma \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad x(0) = x_0
\]

as long as \( u \in C^2([0, \infty); U) \) and \( \begin{bmatrix} x(0) \\ u(0) \end{bmatrix} \in \mathcal{D}(\Sigma) \) (see [29, Proposition 2.6]). Application of the Laplace transform

\[
x(t) \mapsto \hat{x}(w) := \int_0^\infty e^{-wt} x(t) \, dt
\]

to the system equations (2.13) leads us to the input-output property of the transfer function (2.10):

\[
\hat{y}(w) = C(wI - A)^{-1} x(0) + T_\Sigma(w) \hat{u}(w)
\]

for \( w \) with sufficiently large real part.

We mention that any well-posed linear system (2.3) is the integral form of the dynamical system associated with a system node (see e.g. [36]); however there are system nodes for which the associated dynamical system (2.13) is not well-posed (i.e., one or more of the block operators \( \mathcal{B}_0, \mathcal{C}_0, \mathcal{D}_0 \) appearing in (2.3) fail to exist as bounded operators between the appropriate spaces), despite the fact that the infinitesimal form of the system equations (2.13) does make sense.

The following examples of system nodes will be useful in the sequel. In this discussion we make use of Kre˘ın-space geometry notions discussion in Section 2.2.

**Example 2.5.** Suppose that \( \Sigma = \begin{bmatrix} A \& B \\ C \& D \end{bmatrix} : \begin{bmatrix} X \\ \mathcal{U} \end{bmatrix} \supset \mathcal{D}(\Sigma) \to \begin{bmatrix} X \\ Y \end{bmatrix} \) is a closed operator such that its graph space

\[
\mathcal{G}(\Sigma) := \begin{bmatrix} A \& B \\ C \& D \end{bmatrix} \mid \mathcal{D}(\Sigma) \subset \begin{bmatrix} X \\ \mathcal{U} \end{bmatrix} \]
is a Lagrangian subspace of $\mathcal{K} := \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{X} \oplus \mathcal{U}$, where $\mathcal{K}$ is given a Krein space structure using the signature operator

$$\mathcal{J} = \begin{bmatrix} 0 & 0 & I_X & 0 \\ 0 & I_Y & 0 & 0 \\ I_X & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_U \end{bmatrix}.$$  

Then $\Sigma$ is a system node (see [12, Proposition 4.9]). In fact, Schur-class functions over the right halfplane $\Pi$ are characterized as those functions $s$ having a realization $s(w) = T_{\Sigma}(w)$ as in (2.10) with a system node $\Sigma$ of this form (see [12, Theorem 4.10]). These systems are also characterized by the energy-balance property that the block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in (2.3) is assumed only to be contractive, we refer to the recent work of Malinen, Staffans, and Weiss ([29], [37], [38]). In particular, the result of [37] is that a linear operator $S : [\mathcal{X} \ U] \rightarrow [\mathcal{X} \ U]$ is a $\Pi$-scattering passive system node if and only if it is closed with its graph equal to a maximal $J'$-negative subspace of $\mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{X} \oplus \mathcal{U}$. Much of this work also has a focus of fitting physical examples into this framework; a recent accomplishment was to fit Maxwell’s equations into this framework (see [40]).

Example 2.6. Suppose that $\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : [\mathcal{X} \ U] \supset \mathcal{D}(\Sigma) \rightarrow [\mathcal{X} \ U]$ is a closed operator with output space $\mathcal{Y}$ taken to be the same as the input space $\mathcal{U}$ such that

1. its graph $G(\Sigma)$ is a Lagrangian subspace of $\mathcal{K}$, but where now $\mathcal{K}$ is given the Krein-space inner product induced by the signature operator $\mathcal{J}'$ given by

$$\mathcal{J}' = \begin{bmatrix} 0 & 0 & I_X & 0 \\ 0 & 0 & 0 & -I_U \\ I_X & 0 & 0 & 0 \\ 0 & -I_U & 0 & 0 \end{bmatrix},$$

and

2. for each $u \in \mathcal{U}$ there is an $x_u \in \mathcal{X}$ so that $\begin{bmatrix} x_u \\ u \end{bmatrix}$ is in $\mathcal{D}(\Sigma)$.

Then $\Sigma$ is a system node (see [12, Proposition 4.11]). In fact, Herglotz functions over the right halfplane which also satisfy the growth condition at infinity

$$\lim_{t \rightarrow +\infty} t^{-1} f(t)u = 0$$

are characterized as those functions $f$ having a realization $f(w) = T_{\Sigma}(w)$ as in (2.10) with a system node $\Sigma$ of this form (see [12, Theorem 4.12]). The trajectories $(u(t), x(t), y(t))$ satisfy the alternative energy-conservation law

$$\|x(t)\|^2_{\mathcal{X}} - \|x_0\|^2_{\mathcal{X}} = 2 \int_0^t \text{Re} \langle y(t), u(t) \rangle_{\mathcal{U}} dt.$$
and \( \Sigma \) is called a \textit{\( \Pi \)-impedance-conservative system node} (see [35]). As the transfer function for a \( \Pi \)-impedance-conservative system node need not be bounded in the right halfplane, it follows that \( \Pi \)-impedance-conservative system nodes need not be well-posed in general (see [34]).

In connection with Example 2.6 we shall have use for the following additional fact.

**Corollary 2.7.** Suppose that \( Y: \mathcal{D}(Y) \subset [X \mid U] \to [X \mid U] \) is a closed densely defined operator such that

1. \( Y \) is skew-adjoint: \( Y = -Y^* \), and
2. for each \( u \in U \) there is an \( x_u \in X \) such that \( \begin{bmatrix} x_u \\ u \end{bmatrix} \in \mathcal{D}(Y) \).

Set \( J'_0 = \begin{bmatrix} I_X & 0 \\ 0 & -I_U \end{bmatrix} \) and set \( \Sigma := -Y J'_0 \). Then \( \Sigma \) is a \( \Pi \)-impedance-conservative system node as in Example 2.6.

Conversely, if \( \Sigma \) is a \( \Pi \)-impedance-conservative system node as in Example 2.6, then \( Y := -\Sigma J'_0: \mathcal{D}(Y) \subset [X \mid U] \to [X \mid U] \) is a closed operator with the dense domain

\[
\mathcal{D}(Y) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in [X \mid U] : \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(\Sigma) \right\}
\]

satisfying conditions (1) and (2).

**Proof.** Given any closed, densely defined operator \( \Sigma: \mathcal{D}(\Sigma) \subset [X \mid U] \to [X \mid U] \), the following translation of the definition of adjoint operator to graph spaces is well known:

\[
\left( \begin{bmatrix} \Sigma \\ I \end{bmatrix} \mathcal{D}(\Sigma) \right)^\perp = \begin{bmatrix} I \\ -\Sigma^* \end{bmatrix} \mathcal{D}(\Sigma^*)
\]

where here the orthogonal complement is with respect to the standard Hilbert space inner product. More generally, compute the \( J' \)-orthogonal complement (where \( J' = \begin{bmatrix} 0 & J'_0 \\ J'_0 & 0 \end{bmatrix} \)) by the definition (2.14) of \( J' \) as follows:

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \left( \begin{bmatrix} \Sigma \\ I \end{bmatrix} \mathcal{D}(\Sigma) \right)^\perp_{J'} \iff J' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} J'_0 y_2 \\ J'_0 y_1 \end{bmatrix} \in \left( \begin{bmatrix} \Sigma \\ I \end{bmatrix} \mathcal{D}(\Sigma) \right)^\perp = \begin{bmatrix} I \\ -\Sigma^* \end{bmatrix} \mathcal{D}(\Sigma^*)
\]

\[
\iff \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \begin{bmatrix} -J'_0 \Sigma^* J'_0 \\ I \end{bmatrix} \mathcal{D}(\Sigma^* J'_0).
\]

Thus condition (1) in Example 2.6 for \( \Sigma \) to be a \( \Pi \)-impedance-conservative system node translates to

\[
\begin{bmatrix} \Sigma \\ I \end{bmatrix} \mathcal{D}(\Sigma) = \begin{bmatrix} -J'_0 \Sigma^* J'_0 \\ I \end{bmatrix} \mathcal{D}(\Sigma^* J'_0)
\]

or simply

\[
\Sigma = -J'_0 \Sigma^* J'_0.
\]

An equivalent condition is: the operator \( Y := \Sigma J'_0 \) (or equivalently \( -Y = -\Sigma J'_0 \)) is skew-adjoint: \( Y = -Y^* \).
Conversely, if $Y$ is any skew-adjoint operator with dense domain in $[X, U]$, then
\[ \Sigma = -YJ' \] satisfies condition (1) in Example 2.6. Since \[ \begin{bmatrix} x \\ u \end{bmatrix} \in D(\Sigma) \] if and only if \[ \begin{bmatrix} x \\ -u \end{bmatrix} \in D(Y), \] condition (2) in Example 2.6 is equivalent to condition (2) in the statement of the corollary. \qed

The next result gives a model for $\Pi$-impedance-conservative system nodes as described in Corollary 2.7.

**Proposition 2.8.** Let $(T, V_0, R)$ be a triple of operators such that:

1. $T$ is a densely defined skew-adjoint operator on the Hilbert space $X$,
2. $V_0 \in \mathcal{L}(U, X)$ is a bounded linear operator from the input-output space $U$ into $X$,
3. $R$ is a bounded skew-adjoint operator on $U$.

Define an operator $\Sigma: D(\Sigma) \subset [X, U] \rightarrow [X, U]$ as follows. Set
\[ D(\Sigma) = \{ \begin{bmatrix} x \\ u \end{bmatrix} \in [X, U] : x - V_0u \in D(T) \} \] (2.15)
and then define
\[ \Sigma \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} T(x - V_0u) + V_0u \\ V_0^*x + V_0^*T(x - V_0u) + Ru \end{bmatrix} \in [X, U] \text{ for } \begin{bmatrix} x \\ u \end{bmatrix} \in D(\Sigma). \] (2.16)

Equivalently, $\Sigma$ can be defined as the system node constructed from the data
\[ A = T \in \mathcal{L}(X_1, X), \quad B = (I - T)V_0 \in \mathcal{L}(U, X_{-1}), \quad C = V_0^*(I - T)^* \in \mathcal{L}(X_1, U), \quad \alpha = 1 \text{ and } T\Sigma(1) = V_0^*V_0 + iR \in \mathcal{L}(U) \] (2.17)
according to the recipe (2.11). Then $\Sigma$ is a $\Pi$-impedance-conservative system node.

Conversely, any $\Pi$-impedance-conservative system node arises in this way from a triple of operators $T, V_0, R$ satisfying conditions (1), (2), and (3) above.

**Remark 2.9.** We note that, in case $T$ is bounded, the operator $\Sigma$ given by (2.15) and (2.16) is simply
\[ \Sigma = \begin{bmatrix} I & 0 \\ 0 & V_0^* \end{bmatrix} \begin{bmatrix} T & I - T \\ (I - T)^* & -T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_0 \end{bmatrix} = \begin{bmatrix} T & I - T \\ I + T & -T \end{bmatrix} \] (2.18)
One way to make sense of the first term in this formula for the general case where $T$ is allowed to be unbounded is as follows. We may view $\tilde{\Sigma} = \begin{bmatrix} T & I - T \\ (I - T)^* & -T \end{bmatrix}$ as an operator from $[X, X]$ to $[X_{-1}, X_{-1}]$, where $X_{-1}$ is the rigged level-(-1) space associated with the skew-adjoint operator $T$ as explained in discussion at the beginning of this section. It is natural to introduce a domain
\[ D = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \tilde{\Sigma}(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, T(x_1 - x_2) \in X \right\}. \] (2.19)
and define an operator \( \tilde{\Sigma}_0 : \mathcal{D} \subset \mathcal{X} \to \mathcal{X} \) by \( \tilde{\Sigma}_0 = \tilde{\Sigma} |_\mathcal{D} \). Then we may define the operator \( \Sigma \) via the formula (2.18) with domain given by

\[
\mathcal{D}(\Sigma) = \left\{ \left[ \begin{array}{c} x \\ u \end{array} \right] : \left[ \begin{array}{cc} I & 0 \\ 0 & V_0 \end{array} \right] \left[ \begin{array}{c} x \\ u \end{array} \right] \in \mathcal{D}(\tilde{\Sigma}_0) \right\}.
\]

**Proof of Proposition 2.8.** We first show that \( \Sigma \) defined as in (2.15), (2.16) satisfies conditions (1) and (2) in Corollary 2.7. As for condition (2), note that if we set \( x_u = V_0 u \) for each \( u \in \mathcal{U} \), then \( x_u - V_0 u = 0 \in \mathcal{D}(T) \), so condition (2) is satisfied. It remains to verify condition (1).

Toward this end, by the representation (2.18) for \( \Sigma \) as explained in Remark 2.9, it suffices to show that the operator

\[
\Sigma' := \left[ \begin{array}{cc} -I & 0 \\ 0 & I \end{array} \right] \left[ \begin{array}{ccc} T & & \end{array} \right] \left[ \begin{array}{cc} I - T \\ I + T & -T + R \end{array} \right] = \left[ \begin{array}{cc} -T & T - I \\ T + I & -T + R \end{array} \right]
\]

with domain \( \mathcal{D} \) as in (2.19) is skew-adjoint. Note that \( \left[ \begin{array}{c} y_1 \\ y_2 \end{array} \right] \in \mathcal{D} \left( \left[ \begin{array}{cc} -T & T - I \\ T + I & -T + R \end{array} \right] \right) \) means that the sesquilinear form

\[
\left\langle \left[ \begin{array}{ccc} T & & \\ & I - T & \\ & & I + T \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right], \left[ \begin{array}{c} y_1 \\ y_2 \end{array} \right] \right\rangle
= \langle T(x_1 - x_2), -y_1 + y_2 \rangle - \langle x_2, y_1 \rangle + \langle x_1, y_2 \rangle + \langle Rx_2, y_2 \rangle
\]

defined for \( \left[ \begin{array}{c} x_2 \\ y_2 \end{array} \right] \in \mathcal{D} \left( \left[ \begin{array}{cc} T & T - I \\ T + I & -T + R \end{array} \right] \right) \) is continuous in the argument \( \left[ \begin{array}{c} x_2 \\ y_2 \end{array} \right] \) in the \( \mathcal{X} \oplus \mathcal{X} \) norm. As the second, third, and fourth terms are automatically bounded in the \( \left[ \begin{array}{c} x_2 \\ y_2 \end{array} \right] \)-argument and the element \( x_1 - x_2 \) is an arbitrary element of \( \mathcal{D}(T) \) (e.g., take \( x_2 = 0 \) and \( x_1 \in \mathcal{D}(T) \)), it follows that necessarily \( y_2 - y_1 \in \mathcal{D}(T^*) = \mathcal{D}(T) \) (i.e., \( \left[ \begin{array}{c} y_1 \\ y_2 \end{array} \right] \in \mathcal{D} \)) and the calculation above continues as

\[
\left\langle \left[ \begin{array}{ccc} T & & \\ & I - T & \\ & & I + T \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right], \left[ \begin{array}{c} y_1 \\ y_2 \end{array} \right] \right\rangle
= \langle x_1 - x_2, T^* (y_2 - y_1) \rangle - \langle x_2, y_1 \rangle + \langle x_1, y_2 \rangle + \langle Rx_2, y_2 \rangle
= \left\langle \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right], \left[ \begin{array}{c} -T y_2 + T y_1 \end{array} \right] \right\rangle = \left\langle \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right], \left[ \begin{array}{c} -T y_2 + T y_1 \end{array} \right] \right\rangle
\]

where we use \( R \) is bounded with \( R = -R^* \) in the last two steps. The skew-adjointness of the operator \( \Sigma' \) now follows as wanted. As a consequence of Corollary 2.7 it follows in particular that \( \Sigma \) is a system node. The equivalence of the original definition (2.16) of \( \Sigma \) with the alternative formulation based on the data set (2.17) is a simple consequence of the identities (2.11) along with computation of the value of the transfer function at the point \( 1 \in \Pi: T_\Sigma(1) = V_0^* V_0 + i R \); this in turn is a routine verification which we leave to the reader.

For the converse statement, let \( \Sigma' = \left[ \begin{array}{cc} A' & B' \\ C' & D' \end{array} \right] \in \Pi \) be any \( \Pi \)-impedance-conservative system node. As \( \left[ \begin{array}{cc} A' & B' \\ C' & D' \end{array} \right] \) is skew-adjoint, from the duality theory of system nodes (see e.g. [36]) one can see that necessarily \( A' = -A'^* \) and \( C' = B'^* \); here \( B' \in \mathcal{L}(\mathcal{U}, \mathcal{X}'_1) \), \( C' \in \mathcal{L}(\mathcal{X}'_1, \mathcal{U}) \) and computation of the adjoint \( B'^* \) is with respect to the duality pairing between \( \mathcal{X}'_1 \) and \( \mathcal{X}'_1 \) via the \( \mathcal{X}' \) pairing (2.4) (note that \( \mathcal{X}' = \mathcal{X}'_1 \) since \( A' = -A'^* \)). Set \( T = A' \). As \( T \) is skew-adjoint, both \( 1 \) and \( -1 \) are in the resolvent set of \( T \) and we may define a bounded operator \( V_0 \in \mathcal{L}(\mathcal{U}, \mathcal{X}') \)
by \( V_0 = (I - T)^{-1}B' \). We then have \( B' = (I - T)V_0 \) and \( C' = V_0^*(I + T) \). We now have all the ingredients to define another II-impedance-conservative system node
\[
\Sigma_0 = \begin{bmatrix} I & 0 \\ 0 & V_0^* \end{bmatrix} \begin{bmatrix} T & I - T \\ I + T & -T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_0 \end{bmatrix}
\]
as in (2.15), (2.16) (with \( R \) taken equal to zero). When we write \( \Sigma_0 \) in the form
\[
\Sigma_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix},
\]
we see that the construction gives that \( \mathcal{D}(\Sigma') = \mathcal{D}(\Sigma_0) \), i.e., \( \mathcal{D}(A_0 \& B_0) = \mathcal{D}(A' \& B') \), with \( A_0 \& B_0 = A' \& B' \), so
\[
A_0 := A_0 \& B_0|_{\mathcal{D}(\Sigma_0)^c} = A' \& B'|_{\mathcal{D}(\Sigma_0)^c} : A', \quad B_0 = B' \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1}),
\]
\[
C_0 := C_0 \& D_0|_{\mathcal{D}(\Sigma_0)^c} = C' \& D'|_{\mathcal{D}(\Sigma_0)^c} : C'.
\]

As observed in (2.12), for any fixed choice of \( \alpha \in \Pi \) (e.g., \( \alpha = 1 \)) an element \( \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(\Sigma') \) can be decomposed as
\[
\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x - (\alpha I - A)^{-1}Bu \\ 0 \end{bmatrix} + \begin{bmatrix} (\alpha I - A)^{-1}Bu \\ u \end{bmatrix}
\]
where each summand is again in \( \mathcal{D}(\Sigma') = \mathcal{D}(\Sigma_0) \). Then making use of the formulas (2.11) we compute
\[
\Sigma \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ C(x - (\alpha I - A)^{-1}Bu) + \begin{bmatrix} 0 \\ T_{\Sigma_0}(\alpha)u \end{bmatrix} \end{bmatrix}
\]
and similarly
\[
\Sigma_0 \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ C(x - (\alpha I - A)^{-1}Bu) + \begin{bmatrix} 0 \\ T_{\Sigma_0}(\alpha)u \end{bmatrix} \end{bmatrix}
\]
from which we read off
\[
\Sigma \begin{bmatrix} x \\ u \end{bmatrix} = \Sigma_0 \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ (T_{\Sigma_0}(\alpha) - T_{\Sigma_0}(\alpha))u \end{bmatrix}.
\]
Thus \( \Sigma \Sigma_0 + \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \) with \( R := T_{\Sigma}(\alpha) - T_{\Sigma_0}(\alpha) \) equal to a bounded operator on \( \mathcal{U} \). As both \( \Sigma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \) and \( \Sigma_0 \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \) are skew-adjoint, it is necessarily the case that \( R = -R^* \) as well.

We need a generalization of the formula for the transfer function (2.10) to the setting where the resolvent \((wI - A)^{-1}\) is replaced by the structured resolvent \((Y(w) - A)^{-1}\). For the statement of this result it is convenient to assume that \( A \) is maximal dissipative. Recall from the discussion in Example 2.2 above that a densely defined closed Hilbert-space operator \( A \) is maximal dissipative if it is dissipative \((\text{Re} \langle Ax, x \rangle \leq 0 \text{ for all } x \in \mathcal{D}(A)\) and \( I + A \) is onto; it then follows that \( wI + A \) is onto for all \( w \in \Pi \) and that \( A \) is the generator of a contractive semigroup (see [31]).

**Proposition 2.10.** Let \( A \) be a maximal dissipative operator on the Hilbert space \( \mathcal{X} \) (and hence \( A \) is the generator of a contractive \( C_0 \)-semigroup with resolvent set containing the right halfplane \( \Pi \)), and suppose that \((Y_1, \ldots, Y_d)\) is a d-fold positive decomposition of \( I_X \). For \( w = (w_1, \ldots, w_d) \in \mathbb{C}^d \), we set \( Y(w) = w_1Y_1 + \cdots + w_dY_d \). Then the following observations hold true:
(1) For $w \in \Pi^d$, $Y(w) - A$ is invertible with
\[
\|(Y(w) - A)^{-1}\| \leq \frac{1}{\min_j \Re w_j}.
\] (2.20)

(2) For $w \in \Pi^d$, the operator $(Y(w) - A)^{-1}$ is a bicontinuous bijection from $\mathcal{X}$ onto $\mathcal{X}_1$.

(3) For $w \in \Pi^d$, the operator $Y(w) - A : \mathcal{X}_1 \to \mathcal{X}$ has an extension
\[
(Y(w) - A)|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_{-1}
\]
which is a bicontinuous bijection from $\mathcal{X}$ onto $\mathcal{X}_{-1}$.

(4) For any system node $\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ containing $A$ as its state operator/semi-group generator, it holds that
\[
\left(\left( (Y(w) - A)|_{\mathcal{X}} \right)^{-1} Bu \right) \in \mathcal{D}(\Sigma).
\]

Proof. (1) For $w = (w_1, \ldots, w_d) \in \Pi^d$ and $x \in \mathcal{D}(A)$, we have
\[
\|(Y(w) - A)x\| \cdot \|x\| \geq \|((Y(w) - A)x, x)\|
\]
\[
\geq \Re((Y(w) - A)x, x)
\]
\[
\geq \Re(Y(w)x, x)
\]
\[
= \sum_{k=1}^{d} \langle \Re w_k Y_k x, x \rangle
\]
\[
\geq \min_j(\Re w_j) \sum_{k=1}^{d} \langle Y_k x, x \rangle
\]
\[
= \min_j(\Re w_j)\|x\|^2.
\] (2.21)

Thus $Y(w) - A$ is bounded below and has a left inverse. A similar argument with $Y(w) - A$ replaced by $(Y(w) - A)^*$ shows that $Y(w) - A$ also has a right inverse. The computation (2.21) gives the estimate (2.20).

(2) Next note that for $w \in \Pi^d$ and $x \in \mathcal{X}$,
\[
A(Y(w) - A)^{-1} x = -x + Y(w)(Y(w) - A)^{-1} x \in \mathcal{X}
\]
from which we conclude that $(Y(w) - A)^{-1}$ maps $\mathcal{X}$ into $\mathcal{X}_1$. Conversely if $x \in \mathcal{X}_1$, then $y = (Y(w) - A)x \in \mathcal{X}$ and we recover $x$ as $x = (Y(w) - A)^{-1} y$. We conclude that $(Y(w) - A)^{-1}$ maps $\mathcal{X}$ bijectively to $\mathcal{X}_1$. The fact that $(Y(w) - A)^{-1}$ is bicontinuous is then a consequence of the open mapping theorem.

(3) We use the dual version of a result from part (2): for $w \in \Pi^d$,
\[
Y(w)^* - A^* : \mathcal{X}_1^* \to \mathcal{X}
\]
is a bicontinuous bijection from $\mathcal{X}_1^*$ (with the $(1, \ast)$-norm) onto $\mathcal{X}$. If we then take adjoints with respect to the $\mathcal{X}$-pairing, we get an operator
\[
(Y(w)^* - A^*)^* : \mathcal{X} \to \mathcal{X}_{-1}
\]
which must also be a bicontinuous bijection, but now from $\mathcal{X}$ to $\mathcal{X}_{-1}$. Since the duality is with respect to the $\mathcal{X}$-pairing (2.4), it is clear that this map provides an extension of the operator $Y(w) - A : \mathcal{X}_1 \to \mathcal{X}$:
\[
(Y(w) - A)|_{\mathcal{X}} := (Y(w)^* - A^*)^* : \mathcal{X} \to \mathcal{X}_{-1}.
\]
(4) To show that \( \left[ (Y(w) - A)_{\chi}^{-1}Bu \right] \) is in \( \mathcal{D}(\Sigma) \), by the characterization of \( \mathcal{D}(\Sigma) \) in Definition 2.4 we need only show that

\[
[A_{\chi} B] \left[ (Y(w) - A)_{\chi}^{-1}Bu \right] \in \mathcal{X}.
\]

(2.22)

As \( Bu \in \mathcal{X}_{-1} \), \( (Y(w) - A)^{-1} \) maps \( \mathcal{X}_{-1} \) to \( \mathcal{X} \) by part (3) and \( Y(w) \) is a bounded operator on \( \mathcal{X} \), we conclude that

\[
Y(w)(Y(w) - A)_{\chi}^{-1}Bu \in \mathcal{X}.
\]

We now rewrite the expression in (2.22) as

\[
[A_{\chi} B] \left[ (Y(w) - A)_{\chi}^{-1}Bu \right]
= \left[ (A - Y(w))_{\chi} + Y(w) \right] (Y(w) - A)_{\chi}^{-1}Bu + Bu
= Y(w)(Y(w) - A)_{\chi}^{-1}Bu
\]

to conclude that (2.22) holds as desired. 

□

Remark 2.11. We note that as a consequence of property (4) in Proposition 2.10 it is possible to define the transfer function associated with the structured resolvent \( (Y(w) - A)^{-1} \) via

\[
T_{\Sigma}(Y_k)(w) = \left[ C \& D \right] \left[ (Y(w) - A)_{\chi}^{-1}Bu \right].
\]

It is tempting to view this as the transfer function for a multidimensional linear system

\[
\sum_{k=1}^{d} Y_k \frac{\partial \chi}{\partial t_k}(t) = Ax(t) + Bu(t) \quad \text{and} \quad y(t) = Cx(t) + Du(t).
\]

(2.23)

where \( t = (t_1, \ldots, t_d) \) (a continuous-time version of a multidimensional linear system of Fornasini–Marchesini type—see [25]). However, in general it is not clear how to extend the operators \( Y_1, \ldots, Y_d \in \mathcal{L}(\mathcal{X}) \) to operators \( Y_k_{\chi} \) in a sensible way. Special cases where this is possible are: (1) the case where \( B \) and \( C \) are bounded, i.e., \( B \in \mathcal{L}(U, \mathcal{X}) \) and \( C \in \mathcal{L}(\mathcal{X}, Y) \): in this situation the system (2.23) makes sense with state space taken simply to be \( \mathcal{X} \), and (2) in case the operators \( Y_k \) commute with \( A \): in this case one can extend \( Y_k \) to \( \mathcal{X}_{-1} \) via the formula

\[
Y_k_{\chi} = (\alpha I - A)Y_k(\alpha I - A)^{-1} : \mathcal{X}_{-1} \to \mathcal{X}_{-1}.
\]

We note that continuous-time counterparts of Fornasini–Marchesini models of a somewhat different form have been considered in the literature (see [23] and the references there).

3. The Herglotz–Agler class over the polydisk

In this section we present our results for the Herglotz–Agler class over the polydisk \( \mathbb{D}^d \).

Theorem 3.1. Given a function \( F : \mathbb{D}^d \to \mathcal{L}(U) \), the following are equivalent.

(1) \( F \) is in the Herglotz–Agler class \( \mathcal{HA}(\mathbb{D}^d, \mathcal{L}(U)) \).
(2) $F$ has a $\mathbb{D}^d$-Herlotz–Agler decomposition, i.e., there exist $\mathcal{L}(\mathcal{U})$-valued positive kernels $K_1, \ldots, K_d$ on $\mathbb{D}^d$ such that

$$F(\omega)^* + F(\zeta) = \sum_{k=1}^{d} (1 - \overline{\zeta}_k \zeta_k) K_k(\omega, \zeta).$$  \hspace{1cm} (3.1)

(3) There exist a Hilbert space $\mathcal{X}$, a $d$-fold spectral decomposition $(P_1, \ldots, P_d)$ of $I_{\mathcal{X}}$ with associated operator pencil $P(\zeta) = \zeta_1 P_1 + \cdots + \zeta_d P_d$ (see Section 2.1), and a bounded colligation matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$$

with block matrix entries $A, B, C, D$ satisfying the relations

$$A^* A = AA^* = I_{\mathcal{X}}, \quad B = AC^*, \quad D + D^* = CC^* \quad (= B^*B)$$ \hspace{1cm} (3.2)

such that

$$F(\zeta) = D + C(I - P(\zeta)A)^{-1}P(\zeta)B.$$ \hspace{1cm} (3.3)

Proof. (1) $\Rightarrow$ (2): Use that $F \in \mathcal{H}(\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{U})))$ if and only if $S(\zeta) := [F(\zeta) - I][F(\zeta) + I]^{-1}$ is in the Schur–Agler class $\mathcal{S}(\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{U})))$. Then $S$ has a Schur–Agler decomposition

$$I - S(\omega)^* S(\zeta) = \sum_{k=1}^{d} (1 - \overline{\zeta}_k \zeta_k) \tilde{K}_k(\omega, \zeta)$$

for $\mathcal{L}(\mathcal{U})$-valued positive kernels $\tilde{K}_1, \ldots, \tilde{K}_d$ on $\mathbb{D}^d$. A routine computation gives

$$I - S(\omega)^* S(\zeta) = 2[F(\omega)^* + I]^{-1} (F(\omega)^* + F(\zeta)) [F(\zeta) + I]^{-1}.$$

This leads us to the $\mathbb{D}^d$-Herlotz–Agler decomposition (3.1) with

$$K_k(\omega, \zeta) = \frac{1}{2} [F(\omega)^* + I] \tilde{K}_k(\omega, \zeta) [F(\zeta) + I].$$

Notice that Agler in [1] proved the implications (1)$\Rightarrow$(2) in Theorems 1.2 and 3.1 simultaneously, while we show here that they are, in fact, equivalent.

(2) $\Rightarrow$ (3): Since each kernel $K_k$ is positive, each $K_k$ has a Kolmogorov decomposition, i.e., there is a function $H_k : \mathbb{D}^d \rightarrow \mathcal{L}(\mathcal{X}_k, \tilde{\mathcal{X}}_k)$ (for some auxiliary Hilbert space $\tilde{\mathcal{X}}_k$) so that we have the factorization $K_k(\omega, \zeta) = H_k(\omega)^* H_k(\zeta)$. We let $H(\zeta) = \text{CoL}_{\zeta=1,\ldots,d}[H_k(\zeta)]$ be the associated block column matrix function defining a function on $\mathbb{D}^d$ with values in $\mathcal{L}(\mathcal{U}, \tilde{\mathcal{X}})$, where we set $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_1 \oplus \cdots \oplus \tilde{\mathcal{X}}_d$ (written as columns). We set $P(\zeta) = \left[ \begin{array}{c} \zeta_1 I_{\mathcal{X}_1} \\
\vdots \\
\zeta_d I_{\mathcal{X}_d} \end{array} \right]$. We now may rewrite the Agler decomposition (3.1) in the form

$$F(\omega)^* + F(\zeta) = H(\omega)^* (I_{\tilde{\mathcal{X}}} - P(\omega)^* P(\zeta)) H(\zeta).$$ \hspace{1cm} (3.4)

We consider the subspace

$$\tilde{\mathcal{G}} = \text{span} \left\{ \begin{bmatrix} H(\zeta) \\ F(\zeta) \\ P(\zeta) H(\zeta) \\ I_U \end{bmatrix} : u \in \mathcal{U}, \zeta \in \mathbb{D}^d \right\} \subset \begin{bmatrix} \tilde{\mathcal{X}} \\ \mathcal{U} \\
\tilde{\mathcal{X}} \\ \mathcal{U} \end{bmatrix}$$
where the ambient space $\tilde{X} \oplus U \oplus \tilde{X} \oplus U$ is given the Krein-space inner product induced by the signature matrix $\tilde{J}$ given by

$$\tilde{J} = \begin{bmatrix} -I_{\tilde{X}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_U \\ 0 & 0 & I_{\tilde{X}} & 0 \\ 0 & I_U & 0 & 0 \end{bmatrix}.$$ 

Then one can check

$$\left\langle \begin{bmatrix} -I_{\tilde{X}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_U \\ 0 & 0 & I_{\tilde{X}} & 0 \\ 0 & I_U & 0 & 0 \end{bmatrix} \begin{bmatrix} H(\zeta) \\ F(\zeta) \\ P(\zeta)H(\zeta) \\ I_U \end{bmatrix}, \begin{bmatrix} H(\omega) \\ F(\omega) \\ P(\zeta)H(\zeta) \\ I_U \end{bmatrix} u' \right\rangle = \langle -H(\omega)^*H(\zeta) + F(\omega)^* + H(\omega)^*P(\zeta)H(\zeta) + F(\zeta) \rangle u, u' = 0$$

where the last equality follows as a consequence of the Agler decomposition (3.4). We conclude that $\tilde{G}$ is a $\tilde{J}$-isotropic subspace. We then extend $\tilde{G}$ to a $\tilde{J}$-Lagrangian subspace $G$, where the ambient space $\tilde{X} \oplus U \oplus \tilde{X} \oplus U$ extended to a space of the form $X \oplus U \oplus \tilde{X} \oplus U$ with $X \supset \tilde{X}$ and the Krein-space Gramian matrix $J$ of the same block form as $\tilde{J}$ above.

We claim that

$$G \cap \begin{bmatrix} X \\ U \end{bmatrix}_{(0)} = \{0\}.$$ 

Indeed, suppose that $\begin{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \in G$. As $G$ is isotropic, we then must have, for all $u' \in U$ and $\zeta \in \mathbb{D}$,

$$0 = \left\langle J \begin{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} H(\zeta) \\ P(\zeta)H(\zeta) \\ I_U \end{bmatrix} u' \right\rangle = \langle -H(\zeta)^*P_{\tilde{X}}x + u, u' \rangle_U.$$ 

Hence $u = H(\zeta)^*P_{\tilde{X}}x$ for all $\zeta \in \mathbb{D}$; in particular, $u = H(0)^*P_{\tilde{X}}x$. Thus our element of $G$ has the form $\begin{bmatrix} H(0)^*P_{\tilde{X}}x \\ 0 \end{bmatrix}$. We must also have

$$0 = \left\langle J \begin{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} H(0)^*P_{\tilde{X}}x \\ 0 \end{bmatrix} \right\rangle = -\|x\|_{\tilde{X}}^2$$

which enables us to conclude that $x = 0$ and hence also $\begin{bmatrix} H(0)^*P_{\tilde{X}}x \\ 0 \end{bmatrix} = 0$, and the claim follows.

We are now able to conclude that $G$ is a graph space:

$$G = \left\{ \begin{bmatrix} A& B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} : \begin{bmatrix} x \\ u \end{bmatrix} \in D \right\}$$

for some closed linear operator $\begin{bmatrix} A& B \\ C & D \end{bmatrix}$ with domain $D \subset [X]$. Furthermore, by construction the vector $\begin{bmatrix} P(\zeta)H(\zeta)u \\ u \end{bmatrix}$ is in $D$ for all $\zeta \in \mathbb{D}$ and $u \in U$; in particular, by setting $\zeta = 0 \in \mathbb{D}$, we see that $\begin{bmatrix} 0 \\ u \end{bmatrix} \in D$ for all $u \in U$ and hence $D$ splits: $D = \begin{bmatrix} 0 \\ u \end{bmatrix}$ for some linear manifold $D_1 \subset X$. We are now in position to apply Lemma 3.4 from [12] to conclude that in fact $\begin{bmatrix} A& B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is bounded with
domain equal to all of $[x^j_U]$, and moreover the identities (3.2) hold. We have now produced a colligation matrix $U = [A B C D]$ so that

$$G = \begin{bmatrix} A & B \\ C & D \\ I_X & 0 \\ 0 & I_U \end{bmatrix}$$

In particular, it follows that, for any $u \in U$, there is a corresponding $[x']$ in $[x^j_U]$ so that

$$\begin{bmatrix} H(\zeta) \\ F(\zeta) \\ P(\zeta)H(\zeta) \\ I_U \end{bmatrix} u = \begin{bmatrix} A & B \\ C & D \\ I_X & 0 \\ 0 & I_U \end{bmatrix} \begin{bmatrix} x' \\ u \end{bmatrix}.$$  \hspace{1cm} (3.5)

From the bottom two components of (3.5) we read off

$$x' = P(\zeta)H(\zeta)u, \quad u' = u.$$  \hspace{1cm} (3.5a)

Then the top two components of (3.5) give

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P(\zeta)H(\zeta)u \\ u \end{bmatrix} = \begin{bmatrix} H(\zeta)u \\ F(\zeta)u \end{bmatrix} \text{ for all } u \in U$$

which we rewrite as a linear system of operator equations

$$\begin{array}{ll}
AP(\zeta)H(\zeta) + B &= H(\zeta) \\
CP(\zeta)H(\zeta) + D &= F(\zeta).
\end{array}$$  \hspace{1cm} (3.6)

Solving the first equation in (3.6) for $H(\zeta)$ gives

$$H(\zeta) = (I_X - AP(\zeta))^{-1}B.$$  \hspace{1cm} (3.6a)

(We note that the inverse on the right hand side exists since $A$ is unitary and $\|P(\omega)\| < 1$ for $\omega \in D^d$.) Plugging this last expression into the second of equations (3.6) then yields

$$F(\zeta) = D + CP(\zeta)(I - AP(\zeta))^{-1}B$$

and condition (3) in the statement of Theorem 3.1 follows.

$(3) \Rightarrow (2)$: We assume that $F$ has the representation (3.3) where the coefficient matrices $A, B, C, D$ satisfy the relations (3.2). Then we compute

$$F(\zeta)^* + F(\omega) = D^* + B^*(I - P(\zeta)^*A^*)^{-1}P(\zeta)^*C^* + D + CP(\omega)(I - AP(\omega))^{-1}B$$

$$= [D + D^*] + B^*(I - P(\zeta)^*A^*)^{-1}P(\zeta)^*A^*B + B^*AP(\omega)(I - AP(\omega))^{-1}B$$

where we have set $X$ equal to

$$X = (I - P(\zeta)^*A^*) (I - AP(\omega)) + P(\zeta)^*A^*(I - AP(\omega)) + (I - P(\zeta)^*A^*)AP(\omega)$$

and $F(\zeta)^* P(\zeta)^* + F(\omega) - P(\zeta)^*P(\omega) = I - P(\zeta)^*P(\omega)$. We conclude that the Agler decomposition (3.1) holds with

$$K_h(\zeta, \omega) = B^*(I - P(\zeta)^*A^*)^{-1}P_h(I - AP(\omega))^{-1}B,$$
where $P_k = P_{\mathcal{K}_k}$ is the orthogonal projection of $\hat{X}$ onto $\mathcal{K}_k$, and condition (2) in Theorem 3.1 follows.

(2)$\Rightarrow$(1): Given an Agler decomposition (3.1), we may rewrite it in the form (3.4). From the proof of (2)$\Rightarrow$(3)$\Rightarrow$(2) we see that we may assume that $H(\zeta)$ is holomorphic in $\zeta \in \mathbb{D}^d$. If $T = (T_1, \ldots, T_d)$ is a commutative $d$-tuple of strict contractions on $\mathcal{K}$, it is straightforward to verify that the formula (3.4) leads to

$$F(T)^* + F(T) = H(T)^* (I_{X\otimes\mathcal{K}} - P(T)^* P(T)) H(T).$$

From the diagonal form of $P(\zeta) = \sum_{k=1}^d \zeta_k P_k$, we see that $P(T) = \sum_{k=1}^d P_k \otimes T_k$ has $\|P(T)\| < 1$. Hence $F(T)^* + F(T) \geq 0$ and we conclude that $F \in \mathcal{H}\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$.

\[\Box\]

**Remark 3.2.** Given a representation for $F \in \mathcal{H}\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ as in (3.3) and (3.2), let us separate out the selfadjoint and skew-adjoint parts of $F(0) = D$ to rewrite the formula (3.3) as

$$F(\zeta) = \text{Re} F(0) + C(I - P(\zeta)A)^{-1} P(\zeta)B + R$$

where $\text{Re} F(0) = \frac{1}{2}(D + D^*)$ and we set $R = \frac{1}{2}(D - D^*) = -R^*$. From the relations (3.2) we see that

$$F(\zeta) - R = \frac{1}{2} B^* (I + 2A(I - P(\zeta)A)^{-1} P(\zeta)I) B$$

$$= \frac{1}{2} B^* (I + 2AP(\zeta)(I - AP(\zeta))^{-1}) B$$

$$= V^* (I - AP(\zeta))^{-1} (I + AP(\zeta))V,$$

where $V := \frac{1}{\sqrt{2}} B$ is such that $V^* V = \text{Re} F(0)$.

We note that Agler [1] obtained the representation (3.7) for a function in the Herglotz–Agler class $\mathcal{H}\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ starting with the realization $S(\zeta) = D + C(I - P(\zeta)A)^{-1} P(\zeta)B$ (with $U = [A \ B] \in \mathcal{H}\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ unitary as in part (3) of Theorem 1.2) for the associated function

$$S(\zeta) = (F(\zeta) - I)(F(\zeta) + I)^{-1}$$

in the Schur–Agler class $\mathcal{S}\mathcal{A}(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ as follows. The fact that $S$ arises as the Cayley transform of a function in the Herglotz–Agler class implies that $I - S(\zeta) = 2(F(\zeta) + I)^{-1}$ is injective: in particular, $D = S(0)$ has the property that $I - D$ is invertible; moreover, the fact that $U$ is unitary implies that the $U_0 := A + B(I - D)^{-1} C$ is also unitary. This follows from the two identities,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I \\ (I - D)^{-1} C \end{bmatrix} = \begin{bmatrix} U_0 \\ (I - D)^{-1} C \end{bmatrix},$$

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} I \\ (I - D)^{-1} B^* \end{bmatrix} = \begin{bmatrix} U_0^* \\ (I - D)^{-1} B^* \end{bmatrix},$$

which, when combined with that fact that $[A \ B]$ is unitary, imply that both $U_0$ and $U_0^*$ are isometries. In terms of our notation, the result of Agler [1, Proof of theorem 1.8] is that then $F$ has the representation (3.7) with

$$A = U_0, \quad V = \frac{1}{\sqrt{2}} B.$$  

(3.9)
The formula (3.7) can be further adjusted as follows:

\[ F(\zeta) - R = V^*(I - AP(\zeta))^{-1}(I + AP(\zeta))V \]
\[ = V^*(U - P(\zeta))^{-1}(U + P(\zeta))V \]

(3.10)

where we set \( U = A^* \). Here we still have \( V^*V = \text{Re} F(0) \) and \( U = A^* \) is unitary. We shall use this representation for a \( \mathbb{D}^d \)-Herglotz–Agler function in Section 6. For an alternate direct derivation of the realization (3.10) for a \( \mathbb{D}^d \)-Herglotz–Agler function, see [11], where the result is given for the rational matrix-valued case with the additional constraint that \( F \) have zero real part on the unit circle; in this case one can arrange that the state space \( \mathcal{X} \) is finite-dimensional.

4. The Schur–Agler class over the right polyhalfplane

In this section we present our realization results for the Schur–Agler class over the right polyhalfplane \( \Pi^d \).

**Theorem 4.1.** Given a function \( s: \Pi^d \rightarrow \mathcal{L}(U, Y) \), the following are equivalent.

1. \( s \) is in the right polyhalfplane Schur–Agler class \( \mathcal{S}A(\Pi^d, \mathcal{L}(U, Y)) \).
2. \( s \) has a \( \Pi^d \)-Schur–Agler decomposition, i.e., there exist positive kernels \( K_1, \ldots, K_d \) on \( \Pi^d \) such that

\[ I - s(z)^*s(w) = \sum_{k=1}^{d} (\tau_k + w_k)K_k(z, w). \]

(4.1)

for all \( z, w \in \Pi^d \).

3. There exists a state space \( \mathcal{X} \) and a \( d \)-fold positive decomposition of \( I_{\mathcal{X}} \) \( (Y_1, \ldots, Y_d) \) (see Section 2.1) together with a \( \Pi \)-scattering-conservative system node (see Example 2.5)

\[ U = \begin{bmatrix} A&B \\ C&D \end{bmatrix} : \mathcal{D} \subset \begin{bmatrix} \mathcal{X} \\ U \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ Y \end{bmatrix} \]

so that

\[ s(w) = C&D \left[ ((Y(w) - A)_{\mathcal{X}})^{-1}B \right], \]

(4.2)

where we have set \( Y(w) = w_1Y_1 + \cdots + w_dY_d \).

**Proof.** (1)⇒(2): The proof uses the component-wise multivariable Cayley transform from \( \Pi^d \) to \( \mathbb{D}^d \); for this purpose it is convenient to use the condensed notation (1.4) for the point \( \frac{1 + \zeta}{1 - \zeta} = \left( \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d} \right) \) in the polydisk \( \mathbb{D}^d \) associated with the point \( \zeta = (\zeta_1, \ldots, \zeta_d) \in \Pi^d \).

For \( s \in \mathcal{S}A(\Pi^d, \mathcal{L}(U, Y)) \) in the Schur–Agler class over \( \Pi^d \) we associate the function \( S \in \mathcal{S}A(\mathbb{D}^d, \mathcal{L}(U, Y)) \) in the Schur–Agler class over \( \mathbb{D}^d \) via

\[ S(\zeta) = s \left( \frac{1 + \zeta}{1 - \zeta} \right) \]

Then by Theorem 1.2 \( S \) has a \( \mathbb{D}^d \)-Schur–Agler decomposition

\[ I_{\mathcal{U}} - S(\omega)^*S(\zeta) = \sum_{k=1}^{d} (1 - \tau_k \zeta_k)\tilde{K}_k(\omega, \zeta). \]
Using the relation
\[ s(w) = S\left(\frac{w + 1}{w - 1}\right) \]
(where we use the convention (1.5)), we next get that
\[
I_U - s(z)s(w) = \sum_{k=1}^{d} \left(1 - \frac{z_k - 1}{z_k + 1}, \frac{w_k - 1}{w_k + 1}\right) \tilde{K}_k(z, w)
\]
where we set
\[
\tilde{K}_k(z, w) = \tilde{K}_k\left(\frac{z - 1}{z + 1}, \frac{w - 1}{w + 1}\right).
\]
This in turn leads to
\[
I_U - s(z)s(w) = \sum_{k=1}^{d} (z_k + w_k) K_k(z, w)
\]
with \( K_k \) given by
\[
K_k(z, w) = 2\frac{1}{z_k + 1} \tilde{K}_k(z, w) \frac{1}{w_k + 1}
\]
and (2) follows.

(2)⇒(3): We use the Kolmogorov decompositions \( K_k(z, w) = H_k(z)H_k(w) \) (where \( H_k: \Pi^d \to L(U, \tilde{\mathcal{X}}_k) \)) of the positive kernels \( K_k(z, w) \) to rewrite the \( \Pi^d \)-Schur–Agler decomposition (4.1) in the form
\[
I - s(z)s(w) = H(z)(P(z) + P(w))H(w)
\]
where we set \( H(w) = \text{Col} \{ \text{Ran } H(w): w \in \Pi^d \} \subset \sum_{k=1}^{d} \tilde{\mathcal{X}}_k = \tilde{\mathcal{X}} \) and introduce the subspace
\[
\tilde{\mathcal{X}}_0 := \text{span} \left\{ \text{Ran } H(w): w \in \Pi^d \} \subset \sum_{k=1}^{d} \tilde{\mathcal{X}}_k = \tilde{\mathcal{X}} \right\}
\]
and we introduce operators \( \tilde{Y}_1, \ldots, \tilde{Y}_d \) on \( \tilde{\mathcal{X}}_0 \) by
\[
\tilde{Y}_k = P_{\tilde{\mathcal{X}}_k} P_k|_{\tilde{\mathcal{X}}_0}
\]
where \( P_{\tilde{\mathcal{X}}_k} \) is the orthogonal projection of \( \tilde{\mathcal{X}} \) onto its subspace \( \tilde{\mathcal{X}}_0 \) (4.4) and where \( P_k = P_{\tilde{\mathcal{X}}_k} \) is the orthogonal projection of \( \tilde{\mathcal{X}} \) onto \( \tilde{\mathcal{X}}_k \). It is easily verified that \( \tilde{Y}_1, \ldots, \tilde{Y}_d \) form a positive decomposition of the identity on \( \tilde{\mathcal{X}}_0 \).

We next view (4.3) as the statement that the subspace
\[
\tilde{\mathcal{G}} := \text{span} \left\{ \begin{bmatrix} \tilde{Y}(w)H(w) \\ s(w)H(w) \\ I_U \end{bmatrix} u: w \in \Pi^d, u \in U \right\} \subset \begin{bmatrix} \tilde{\mathcal{X}}_0 \\ \tilde{\mathcal{Y}}_0 \\ \tilde{\mathcal{U}}_0 \end{bmatrix} =: \tilde{\mathcal{H}}
\]
is an isotropic subspace of $\tilde{\mathcal{K}}$, where $\tilde{\mathcal{K}}$ is considered as a Krein space with inner product induced by the indefinite Gramian matrix

$$
\tilde{\mathcal{J}} := \begin{bmatrix}
0 & 0 & I_{\tilde{\mathcal{X}}_0} & 0 \\
0 & I_{\tilde{\mathcal{Y}}} & 0 & 0 \\
I_{\tilde{\mathcal{X}}_0} & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{\tilde{\mathcal{U}}}
\end{bmatrix}.
$$

(4.5)

We next check that $\tilde{\mathcal{G}}$ can be expressed as a graph space

$$
\tilde{\mathcal{G}} = \begin{bmatrix}
\tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\
\tilde{\mathcal{C}} & \tilde{\mathcal{D}} \\
I_{\tilde{\mathcal{X}}_0} & 0 \\
0 & I_{\tilde{\mathcal{U}}}
\end{bmatrix}
$$

associated with a closed operator $\tilde{\mathcal{U}} = \begin{bmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{bmatrix}$ with dense domain $\tilde{\mathcal{D}}_0 \subset \tilde{\mathcal{X}}_0 \oplus \tilde{\mathcal{U}}$ mapping into $\tilde{\mathcal{X}}_0 \oplus \tilde{\mathcal{Y}}$.

To this end we first check the necessary condition that

$$
\tilde{\mathcal{G}} \cap \begin{bmatrix}
\tilde{\mathcal{X}}_0 \\
\tilde{\mathcal{Y}} \\
\{0\} \\
\{0\}
\end{bmatrix} = \{0\}
$$

as follows. We suppose that $x' \oplus y' \oplus 0 \oplus 0 \in \tilde{\mathcal{G}}$. As $\tilde{\mathcal{G}}$ is isotropic, it follows that

$$
0 = \langle \tilde{\mathcal{J}} \begin{bmatrix} x' \\ y' \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ 0 \\ 0 \end{bmatrix} \rangle = \|y'\|^2
$$

from which we see that $y' = 0$ and $x' \oplus 0 \oplus 0 \oplus 0 \in \tilde{\mathcal{G}}$. As $\tilde{\mathcal{G}}$ is isotropic, we must then also have

$$
0 = \langle \tilde{\mathcal{J}} \begin{bmatrix} \tilde{\mathcal{Y}}(w)H(w) \\ s(w)H(w) \\ I_{\tilde{\mathcal{U}}}u \\ 0 \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ 0 \\ 0 \end{bmatrix} \rangle = \langle H(w)u, x' \rangle_{\tilde{\mathcal{X}}_0}
$$

for all $u \in \mathcal{U}$ and $w \in \Pi^d$. We now use the condition (4.4) to conclude that necessarily $x' = 0$ as well.

We next observe that (4.3) for $z = w = te$, where $e = (1, \ldots, 1)$, becomes

$$
I - s(te)^*s(te) = 2tH(te)^*H(te),
$$

so for any $u \in \mathcal{U}$ we have

$$
\|H(te)u\|^2 = \frac{1}{2t}(\|u\|^2 - \|s(te)u\|^2) \to 0 \text{ as } t \to \infty.
$$

Therefore

$$
\text{span}_{u,w}\left\{ \begin{bmatrix} H(w) \\ I_{\tilde{\mathcal{U}}} \end{bmatrix}u \right\} \subset \text{span}_{u,w,t}\left\{ \begin{bmatrix} H(w) - H(te) \\ 0 \end{bmatrix}u \right\} \subset \text{span}_{u,w}\left\{ \begin{bmatrix} H(w) \\ 0 \end{bmatrix}u \right\}
$$

$$
= \begin{bmatrix} \tilde{\mathcal{X}}_0 \\ \{0\} \end{bmatrix},
$$

and we now see that $\left\{ \begin{bmatrix} H(w) \\ I_{\tilde{\mathcal{U}}} \end{bmatrix}u \right\}$ has dense span in $\tilde{\mathcal{X}}_0 \oplus \mathcal{U}$. We conclude that $\tilde{\mathcal{G}}$ is indeed a graph space as claimed.
By Proposition 2.5 in [12], we may embed \( \tilde{G} \) into a \( J \)-Lagrangian subspace \( G \) of \( K := X \oplus Y \oplus X \oplus U \) where we may arrange that \( X \) is a Hilbert space containing \( \tilde{X} \) and where we set
\[
J = \begin{bmatrix}
0 & 0 & I_X & 0 \\
0 & I_Y & 0 & 0 \\
I_X & 0 & 0 & 0 \\
0 & 0 & 0 & -I_U
\end{bmatrix}.
\]
(4.6)

Furthermore, it is argued there that one can arrange that this (possibly) enlarged Lagrangian subspace \( G \) is also a graph space:
\[
G \cap \begin{bmatrix}
X \\
Y \\
\{0\} \\
\{0\}
\end{bmatrix} = \{0\}.
\]

Hence there is a closed operator
\[
U = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} : \mathcal{D}(U) \subset [X] \rightarrow [Y] \cup
\]
so that
\[
G = \begin{bmatrix}
U \\
I
\end{bmatrix} \mathcal{D}(U) = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
I_X & 0 \\
0 & I_U
\end{bmatrix} \mathcal{D}(U).
\]
(4.7)

As discussed in Example 2.5, \( U \) is a \( \Pi \)-scattering-conservative system node. It is shown in the proof of Proposition 4.9 from [12] that then the main operator \( A \) of \( U \) (given by (2.5) with \( U \) in place of \( \Sigma \)) is maximal dissipative (as defined in Example 2.2 above). We let \( Y_1, \ldots, Y_d \) be a positive decomposition of the identity on \( X \) which extends \( \tilde{Y}_1, \ldots, \tilde{Y}_d \); e.g., one way to do this is
\[
Y_1|_{X \oplus \tilde{X}_0} = I_{X \oplus \tilde{X}_0}, \quad Y_k|_{X \oplus \tilde{X}_0} = 0 \text{ for } k = 2, \ldots, d,
\]
and extend by linearity. We then set \( Y(w) = w_1Y_1 + \cdots + w_dY_d \).

As \( G \) contains \( \tilde{G} \), we conclude that
\[
\begin{bmatrix}
Y(w)H(w)u \\
s(w)u \\
H(w)u \\
u
\end{bmatrix} \in \begin{bmatrix}
A & B \\
C & D \\
I_X & 0 \\
0 & I_U
\end{bmatrix} \mathcal{D}(U)
\]
for each \( w \in \Pi^d \) and \( u \in U \). Thus, for each such \( w \) and \( u \) there is a \( [x_{w, u} u'] \in \mathcal{D}(U) \) so that
\[
\begin{bmatrix}
Y(w)H(w)u \\
s(w)u \\
H(w)u \\
u
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D \\
I_X & 0 \\
0 & I_U
\end{bmatrix} \begin{bmatrix}
x_{w, u} \\
u_{w, u}'
\end{bmatrix}.
\]
(4.8)

From the bottom two rows of (4.8) we read off
\[
H(w)u = x_{w, u}, \quad u = u_{w, u}'.
\]
Since \( U \) is a II-scattering-conservative system node, a consequence of Proposition 2.10 is that \( Y(w) - A \) is invertible for each \( w \in \Pi^d \) with an extension \( (Y(w) - A)_{|\mathcal{X}} : \mathcal{X} \to \mathcal{X}_{-1} \) having the property that \( ((Y(w) - A)_{|\mathcal{X}})^{-1} : \mathcal{X}_{-1} \to \mathcal{X} \). We may therefore solve the first of equations (4.9) for \( H(w)u \) to get

\[
H(w)u = ((Y(w) - A)_{|\mathcal{X}})^{-1} Bu. \tag{4.10}
\]

and furthermore

\[
\left( (Y(w) - A)_{|\mathcal{X}} \right)^{-1} B u \in \mathcal{D}(C&D) \text{ for each } w \in \Pi^d.
\]

We can now substitute (4.10) into the second of equations (4.9) to arrive at the desired realization formula (4.2) for \( s \), and (3) follows.

(3) \( \Rightarrow \) (2): Assume that \( s \) has a realization as in (4.2). For \( w \in \Pi^d \), set

\[
H(w) = ((Y(w) - A)_{|\mathcal{X}})^{-1} B.
\]

Observe that

\[
A_{|\mathcal{X}} H(w)u + Bu = A \& B \left( (Y(w) - A)_{|\mathcal{X}} \right)^{-1} B u
= A_{|\mathcal{X}} \left( (Y(w) - A)_{|\mathcal{X}} \right)^{-1} Bu + Bu
= -Bu + P(u) \left( (Y(w) - A)_{|\mathcal{X}} \right)^{-1} Bu + Bu
= Y(w) \left( (Y(w) - A)_{|\mathcal{X}} \right)^{-1} Bu
= Y(w) H(w)u.
\]

Combining this with (4.3) gives

\[
U \begin{bmatrix} H(w)u \\ u \end{bmatrix} = \begin{bmatrix} A \& B \\ C&D \end{bmatrix} \begin{bmatrix} H(w)u \\ u \end{bmatrix} = \begin{bmatrix} Y(w)H(w)u \\ s(w)u \end{bmatrix}. \tag{4.11}
\]

By Proposition 4.9 from [12], the fact that \( U \) is a II-scattering-conservative system node tells us that the graph of \( U \) is a Lagrangian subspace of \( \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{X} \oplus \mathcal{U} \) in the Krein-space inner product induced by \( \mathcal{J} \) as in (4.6). In particular it holds that

\[
0 = \left< \mathcal{J} \begin{bmatrix} A \& B \\ C&D \end{bmatrix} \begin{bmatrix} H(w)u \\ u \end{bmatrix}, \begin{bmatrix} A \& B \\ C&D \end{bmatrix} \begin{bmatrix} H(z)u' \\ w' \end{bmatrix} \right>
= \left< \mathcal{J} \begin{bmatrix} Y(w)H(w)u \\ s(w)u \\ H(w)u \\ u \end{bmatrix}, \begin{bmatrix} Y(z)H(z)u' \\ s(z)u' \\ H(z)u' \\ w' \end{bmatrix} \right>
= ([H(z)^*(Y(z)^* + Y(w)))H(w) + (s(z)^*s(w) - I_{\mathcal{U}})] u, u'). \tag{4.12}
\]
By the arbitrariness of \(u, u' \in U\), we conclude that
\[
H(z)^* \left( \sum_{k=1}^{d} (z_k + w_k)Y_k \right) H(w) = I_U - s(z)^*s(w).
\] (4.13)

Then (4.13) leads to the Agler decomposition (4.1) with
\[
K_k(z, w) = H(z)^*Y_kH(w) = \left[ Y_k^{1/2}H(z) \right]^* \left[ Y_k^{1/2}H(w) \right].
\]

(2) \(\Rightarrow\) (1): We write the Agler decomposition (4.1) in the form (4.3). Then, if \(A = (A_1, \ldots, A_d)\) is a commutative \(d\)-tuple of strictly accretive operators, the functional calculus gives
\[
I - s(A)^*s(A) = H(A)^*(P(A)^* + P(A))H(A).
\]

From the diagonal form \(P(w) = w_1P_1 + \cdots + w_dP_d\) of \(P(w)\), we see that \(P(w)\) is positive, and hence \(I - s(A)^*s(A) \geq 0\), or \(\|s(A)\| \leq 1\). We conclude that \(s \in SA(\Pi^d, L(U, Y))\) and (1) follows. \(\square\)

5. The Herglotz–Agler class over the polyhalfplane

In this section we present our realization results for a restricted class of Herglotz–Agler functions over the right polyhalfplane \(\Pi^d\), where a growth condition (5.1) is imposed at infinity.

**Theorem 5.1.** Given a function \(f: \Pi^d \to L(U)\), the following are equivalent:

1. \(f \in H\mathcal{A}(\Pi^d, L(U))\) and also \(f\) satisfies the growth condition at \(+\infty:\)
   \[
   \lim_{t \to +\infty} t^{-1} f(te)u = 0 \quad \text{for each} \quad u \in U.
   \] (5.1)
   where \(e = (1, \ldots, 1) \in \Pi^d\).

2. \(f\) has a \(\Pi^d\)-Herglotz–Agler decomposition, i.e., there exist \(L(U)\)-valued positive kernels \(K_1, \ldots, K_d\) on \(\Pi^d\) such that
   \[
   f(z)^* + f(w) = \sum_{k=1}^{d} (z_k + w_k)K_k(z, w)
   \] (5.2)
   and in addition \(f\) satisfies the growth condition (5.1).

3. There exists a Hilbert state space \(Y\) and a positive decomposition of the identity \((Y_1, \ldots, Y_d)\) on \(Y\) along with an \(\Pi\)-impedance-conservative system node (see Example 2.6)
   \[
   Y = \begin{bmatrix} A&B \\ C&D \end{bmatrix}; \quad D(Y) \subset \begin{bmatrix} X \\ U \end{bmatrix} \to \begin{bmatrix} X \\ U \end{bmatrix}
   \]
   such that
   \[
   f(w) = C \wedge \begin{bmatrix} (Y(w) - A)|X \end{bmatrix}^{-1} B.
   \] (5.3)

**Proof.** We show

(1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (2) \(\Rightarrow\) (1).

(1) \(\Rightarrow\) (2): We note that \(f\) is in the Herglotz–Agler class \(H\mathcal{A}(\Pi^d, L(U))\) if and only if \(s = (f - I)(f + I)^{-1}\) is in the Schur–Agler class \(SA(\Pi^d, L(U))\). Thus by
Theorem 4.1 this s has a $\Pi^d$-Schur–Agler decomposition

$$I - s(z)^*s(w) = \sum_{k=1}^{d}(z_k + w_k)\tilde{K}_k(z, w)$$

for some $\mathcal{L}(\mathcal{U})$-valued positive kernels $\tilde{K}_k$ on $\Pi^d$. A standard computation gives

$$I - s(z)^*s(w) = 2(f(z)^* + I)^{-1}[f(z)^* + f(w)](f(w) + I)^{-1}$$

from which we see that (5.2) holds with

$$K_k(z, w) = \frac{1}{2}(f(z)^* + I)\tilde{K}_k(z, w)(f(w) + I).$$

(2)$\Rightarrow$(3): We use the Kolmogorov decompositions $K_k(z, w) = H_k(z)^*H_k(w)$ of the positive kernels $K_k$ (with $H_k : \Pi^d \to \mathcal{L}(\mathcal{U}, \tilde{X}_k)$ say) to rewrite the Agler decomposition (5.2) in the form

$$f(z)^* + f(w) = H(z)^*(P(z)^* + P(w))H(w)$$

where we have set

$$H(w) = \begin{bmatrix} H_1(w) \\ \vdots \\ H_d(w) \end{bmatrix}, \quad P(w) = \begin{bmatrix} w_1I_{\tilde{X}_1} \\ \vdots \\ w_dI_{\tilde{X}_d} \end{bmatrix}.$$ 

Just as in the proof of Theorem 4.1, we introduce the subspace

$$\tilde{X}_0 := \text{span}\{\text{Ran}H(w): w \in \Pi^d\} \subset \bigoplus_{k=1}^{d} \tilde{X}_k =: \tilde{X}$$

and introduce operators $\tilde{Y}_1, \ldots, \tilde{Y}_d$ on $\tilde{X}_0$ by

$$\tilde{Y}_k = P_{\tilde{X}_0}P_k|\tilde{X}_0.$$ 

We then view (5.2) as the statement that the subspace

$$\tilde{G} = \text{span}\left\{ \begin{bmatrix} \tilde{Y}(w)H(w) \\ f(w) \\ H(w) \end{bmatrix} : u \in \mathcal{U}, w \in \Pi^d \right\} \subset \begin{bmatrix} \tilde{X}_0 \\ \mathcal{U} \\ \tilde{X}_0 \end{bmatrix} =: \tilde{K}$$

is an isotropic subspace of $\tilde{K}$ when $\tilde{K}$ is given the Krein-space inner product induced by the Gramian matrix

$$\tilde{J} = \begin{bmatrix} 0 & 0 & I_{\tilde{X}_0} & 0 \\ 0 & 0 & 0 & -I_{\mathcal{U}} \\ I_{\tilde{X}_0} & 0 & 0 & 0 \\ 0 & -I_{\mathcal{U}} & 0 & 0 \end{bmatrix}. \quad (5.6)$$

We show next that $\tilde{G}$ is a graph space, i.e.,

$$\tilde{G} \cap \begin{bmatrix} \tilde{X}_0 \\ \mathcal{U} \\ \{0\} \end{bmatrix} = \{0\}. \quad (5.7)$$
Toward this end, suppose that \( x \oplus u \oplus 0 \oplus 0 \in \tilde{G} \). Our goal is to show that then necessarily \( x = 0 \) and \( u = 0 \). As \( \tilde{G} \) is \( \tilde{J} \)-isotropic, we necessarily have, for all \( u' \in \mathcal{U} \),

\[
0 = \left\langle \tilde{J} \begin{bmatrix} \tilde{Y}(w)H(w)u' \\ f(w)u' \\ H(w)u' \\ u' \end{bmatrix}, \begin{bmatrix} x \\ u \\ 0 \\ 0 \end{bmatrix} \right\rangle 
= \left\langle \begin{bmatrix} H(w)u' \\ -u' \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle 
= \langle u', H(w)^* x - u \rangle 
\]

from which we conclude that

\[
u = H(w)^* x \text{ for all } w \in \Pi^d. \tag{5.9}\]

We note that, as a consequence of (5.4),

\[
H(te)^* H(te) = \frac{f(te)^* + f(te)}{2t}. 
\]

The growth assumption (5.1) then implies that \( H(te) \to 0 \) strongly as \( t \to +\infty \).

To conclude the proof of (5.7), we now need only specialize (5.9) to the case \( w = te \) and take a weak limit as \( t \to +\infty \) to show that \( u = 0 \). It then follows from (5.8) that \( x \) is orthogonal to \( \text{span}\{H(w)u: w \in \Pi^d, u \in \mathcal{U}\} \). As a consequence of (5.5), this in turn forces \( x = 0 \) and (5.7) follows.

We next embed \( \tilde{G} \) into a \( \tilde{J} \)-Lagrangian subspace \( G \) of \( X \oplus U \oplus X \oplus U \), where \( X \) is a Hilbert space containing \( \tilde{X} \) as a subspace and the indefinite Gramian matrix has the same form as \( \tilde{J} \) in (5.6) above:

\[
J = \begin{bmatrix}
0 & 0 & I_X & 0 \\
0 & 0 & 0 & -I_U \\
I_X & 0 & 0 & 0 \\
0 & -I_U & 0 & 0
\end{bmatrix} \tag{5.10}
\]
in such a way that \( G \) is still a graph subspace. That this is possible follows via a minor adjustment of Proposition 2.5 in [12] as indicated in the proof of Theorem 4.12 there. We also note that condition (2) in Example 2.6 is automatic since the subspace \( \tilde{G} \) satisfies this condition by construction. It then follows that \( G \) has the form (4.7) for a closed operator

\[
U = \begin{bmatrix} A\& B \\
C\& D \end{bmatrix} : \mathcal{D}(U) \subset \begin{bmatrix} X \\ U \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}. 
\]

That \( U \) is a II-impedance-conservative system node follows from the fact that \( G \) is \( J \)-Lagrangian (with \( J \) given by (5.10)). We also extend the positive decomposition of the identity \( \tilde{Y}_1, \ldots, \tilde{Y}_d \) on \( \tilde{X}_0 \) to a positive decomposition of the identity \( Y_1, \ldots, Y_d \) on \( X \) just as in the proof of (2)⇒(3) in Theorem 4.1 above. That we recover \( f(w) \) as the transfer-function for the system node \( U \), i.e., the formula (5.3) holds, now follows exactly as in the proof of Theorem 4.1.

\((3)\Rightarrow(2)\): We follow the proof of (3)⇒(2) in Theorem 4.1. If we define \( H(w) = A\& B \left[ (Y(w)-A_1x)^{-1}B \right] \) (well-defined by Proposition 2.10), we arrive at (4.11) and
(4.12), but with $J$ given by (5.10) rather than by (4.5), leading to the adjusted final conclusion

$$0 = \langle [H(z)^* (Y(z)^* + Y(w))]H(w) - (f(z)^* + f(w))]u, u' \rangle.$$ 

This leads to the Agler decomposition (5.2) with

$$K_k(z, w) = H(z)^* Y_k H(w) = [Y_k^{1/2} H(z)]^*[Y_k^{1/2} H(w)]$$

as in the proof of (3)$\Rightarrow$(2) in Theorem 4.1.

It remains to show that the growth condition (5.1) necessarily holds if $f$ has a realization (5.3) from a $Π$-impedance-conservative system node. To see this, we note that then the single-variable Herglotz function $f(s) ∈ Π$ has an impedance-conservative system-node realization. That the growth condition (5.1) holds now follows from the result for the single-variable case (see Theorem 7.4 in [35]).

(2)$⇒$(1): The proof is parallel to the proofs of (2)$⇒$(1) in Theorems 3.1 and 4.1. Write the Agler decomposition (5.2) in the form (5.4) and observe that the functional calculus gives

$$f(A)^* + f(A) = H(A)^* (P(A)^* + P(A)) H(A).$$

If $A$ is a strictly accretive commutative $d$-tuple, from the diagonal form of $P(w)$ we see that $P(A)^* + P(A) ≥ 0$. We conclude that $f ∈ \mathcal{H}A(Π^d, \mathcal{L}(U))$, and (1) follows.

In [3] a criterion was given for when a $Π^d$-Herglotz–Agler function has a realization involving the structured resolvent $(P(w) - A)^{-1}$ coming from a spectral decomposition $(P_1, \ldots, P_d)$ (so $P(w) = w_1 P_1 + \cdots + w_d P_d$) rather than just a positive decomposition $(Y_1, \ldots, Y_d)$ of the identity). We give our version of a result of this type, with realization in terms of a $Π$-impedance-conservative system node rather than in the form presented in [3].

**Theorem 5.2.** Suppose that $f : Π^d → \mathcal{L}(U)$ is a $Π^d$-Herglotz–Agler function satisfying the growth condition (5.1). Then the following are equivalent:

1. There exists a Hilbert space $\mathcal{X}$ and a $d$-fold spectral decomposition $(P_1, \ldots, P_d)$ of $I_X$ along with a $Π$-impedance-conservative system node

   $$\mathcal{Y} = \left[ \begin{array}{c} A&B \\ C&D \end{array} \right] : \mathcal{D}(U) ⊆ \mathcal{X} → \mathcal{U}$$

   such that

   $$f(w) = C&D \left[ (P(w) - A|_\mathcal{X})^{-1} B \right].$$

2. If $S(ζ) = C(f)(ζ) := \left[ f \left( \frac{1+ζ}{1-ζ} \right) - I \right] \left[ f \left( \frac{1+ζ}{1-ζ} \right) + I \right]^{-1}$ (where by convention

   $$(1.4) \frac{1+ζ}{1-ζ} = \left( \frac{1+ζ_1}{1-ζ_1}, \ldots, \frac{1+ζ_d}{1-ζ_d} \right)$$

   if $ζ = (ζ_1, \ldots, ζ_d) ∈ \mathbb{D}^d)$, then $S$ is in the Schur–Agler class $\mathcal{S}A(\mathbb{D}^d, \mathcal{L}(U))$ and $S$ has a realization as in (1.2) where

   $$\mathcal{U} = \left[ \begin{array}{c} A \\ C \end{array} \right] \in \mathbb{D}$$

   is unitary with the additional property that $1$ is not in the point spectrum of $\mathcal{U}$.

**Proof.** (1)$⇒$(2): We suppose that we are given a $Π$-impedance-conservative system node $\mathcal{Y}$ as in condition (1). We set

$$H(w) = ((P(w) - A|_\mathcal{X})^{-1} B : U \rightarrow \mathcal{X}$$
(well-defined by Proposition 2.10) and verify that
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
H(w)u \\
u
\end{bmatrix} = \begin{bmatrix}
P(w)H(w)u \\
f(w)u
\end{bmatrix} \tag{5.11}
\]
for all \(u \in \mathcal{U}\). Furthermore, working as in the proof of \((3) \Rightarrow (2)\) in Theorem 5.1, we see that \(H(w)\) so defined provides an Agler decomposition for \(f\):
\[
f(z)^* + f(w) = \sum_{k=1}^d (\zeta_k + w_k)H(z)^* P_k H(w).
\]
If we set \(\breve{Y} = \begin{bmatrix}
-(A & B) \\
C & D
\end{bmatrix}\), then the fact that \(Y = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}\) is an impedance-conservative system node means that \(\breve{Y}\) is skew-adjoint (see Corollary 2.7): \(\breve{Y}^* = -\breve{Y}\). Moreover we can rewrite the identity (5.11) in the form
\[
\breve{Y} \begin{bmatrix}
H(w)u \\
u
\end{bmatrix} = \begin{bmatrix}
-P(w)H(w)u \\
f(w)u
\end{bmatrix} \tag{5.12}
\]
As \(\breve{Y}\) is skew-adjoint, easily verified properties of the Cayley transform imply that \(U := (\breve{Y} - I)(\breve{Y} + I)^{-1}\) is unitary and the point 1 is not in the point spectrum of \(U\) (this is another version of Proposition 2.3). It remains only to check that \(U := \begin{bmatrix} A & B \end{bmatrix}\) provides a \(d^d\)-scattering-conservative realization of \(S := C(f)\).

We rewrite (5.12) in terms of \(U\) as
\[
(I + U)(I - U)^{-1} \begin{bmatrix}
H(w)u \\
u
\end{bmatrix} = \begin{bmatrix}
-P(w)H(w)u \\
f(w)u
\end{bmatrix}
\]
or equivalently
\[
(I + U) \begin{bmatrix}
H(w)u \\
u
\end{bmatrix} = (I - U) \begin{bmatrix}
-P(w)H(w)u \\
f(w)u
\end{bmatrix}
\]
We reorganize this using linearity to get
\[
U \begin{bmatrix}
(P(w) - I)H(w)u \\
-(f(w) + I)u
\end{bmatrix} = \begin{bmatrix}
(P(w) + I)H(w)u \\
-(f(w) - I)u
\end{bmatrix} \tag{5.13}
\]
We introduce \(L(U, \mathcal{X}_k)\)-valued functions \(\tilde{H}_k\) on the polydisk \(\mathbb{D}^d\) according to the relation (where we again use the convention (1.5))
\[
H_k(w) := P_k H(w) = \frac{1}{w_k + 1} \tilde{H}_k \left( \frac{w - 1}{w + 1} \right) (f(w) + I).
\]
Then we note that
\[
(P(w) - I)H(w) = \sum_{k=1}^d (w_k - 1) P_k H(w)
\]
\[
= \sum_{k=1}^d \frac{w_k - 1}{w_k + 1} \tilde{H}_k(\zeta)(f(w) + I)
\]
\[
= \sum_{k=1}^d \zeta_k \tilde{H}_k(\zeta)(f(w) + I)
\]
\[
= P(\zeta) \tilde{H}(\zeta)(f(w) + I)
\]
where we set
\[ \zeta = \frac{w - 1}{w + 1} \] (as in (1.5)) for \( w \in \Pi^d \)
and
\[ \tilde{H}(\zeta) = \sum_{k=1}^{d} \tilde{H}_k(\zeta). \]

Similarly one can verify that
\[
(P(w) + I)H(w) = \sum_{k=1}^{d} \frac{w_k + 1}{w_k + 1} \tilde{H}_k(\zeta)(f(w) + I) = \tilde{H}(\zeta)(f(w) + I),
\]
and we have arrived at the pair of identities
\[
(P(w) - I)H(w) = \tilde{H}(\zeta)(I + f(w))\]
\[
(P(w) + I)H(w) = \tilde{H}(\zeta)(I + f(w)).
\]

Fix a vector \( v \in U \). Define \( u_w = (I + f(w))^{-1}v \) so \( v = (I + f(w))u_w \). Then (5.13) with \( u = u_w \) can be rewritten in the form
\[
U \begin{bmatrix} P(\zeta)\tilde{H}(\zeta)v \\ \tilde{H}(\zeta)v \\ -S(\zeta)v \end{bmatrix} = \begin{bmatrix} \tilde{H}(\zeta)v \\ -S(\zeta)v \end{bmatrix}.
\] (5.14)

Writing out \( U = [A \ B \ C \ D] \), one can now solve (5.14) in the standard way to arrive at
\[
S(\zeta) = D + CP(\zeta)(I - AP(\zeta))^{-1}B,
\]
i.e., the unitary colligation matrix \( U \) with the additional property that \( U \) does not have 1 as an eigenvalue provides a \( \mathbb{D}^d \)-scattering-conservative realization for \( S = C(f) \).

(2)\(\Rightarrow\)(1): We suppose that \( S = C(f) \) has a \( \mathbb{D}^d \)-scattering-conservative realization (1.2) where the associated unitary colligation matrix \( U = [A \ B \ C \ D] \) does not have 1 as an eigenvalue. Then we know that \( U \) also has the defining property
\[
U \begin{bmatrix} P(\zeta)\tilde{H}(\zeta) \\ \tilde{H}(\zeta) \\ S(\zeta) \end{bmatrix} u = \begin{bmatrix} \tilde{H}(\zeta) \\ S(\zeta) \end{bmatrix} u
\] (5.15)
for all \( u \in U \) and \( \zeta \in \mathbb{D}^d \) where \( \tilde{H}(\zeta) = \begin{bmatrix} \tilde{H}_1(\zeta) \\ \vdots \\ \tilde{H}_d(\zeta) \end{bmatrix} \) provides a \( \mathbb{D}^d \)-Schur–Agler decomposition (1.1). Since 1 is not an eigenvalue of \( U \) by assumption we may form the Cayley transform
\[
\tilde{Y} := (I + U)(I - U)^{-1} \text{ with } \mathcal{D}(\tilde{Y}) = \text{Ran}(I - U).
\]

By Proposition 2.3, \( \tilde{Y} \) is skew-adjoint:
\[
\tilde{Y} = -\tilde{Y}^*.
\] (5.16)

By construction we have
\[
\tilde{Y} : (I - U) \begin{bmatrix} x \\ u \end{bmatrix} \mapsto (I + U) \begin{bmatrix} x \\ u \end{bmatrix}.
\] (5.17)
From (5.15) we note that
\[
(I - U) \begin{bmatrix} P(\zeta) \bar{H}(\zeta) \end{bmatrix} u = \begin{bmatrix} (P(\zeta) - I) \bar{H}(\zeta) u \\ (I - S(\zeta)) u \end{bmatrix},
\]
\[
(I + U) \begin{bmatrix} P(\zeta) \bar{H}(\zeta) \end{bmatrix} u = \begin{bmatrix} (P(\zeta) + I) \bar{H}(\zeta) \\ (I + S(\zeta)) u \end{bmatrix}.
\]
Notice that \((I - S(\zeta)) = 2(f(w) + I)^{-1}\) is invertible. Hence (5.17) leads to
\[
\tilde{Y} \begin{bmatrix} (P(\zeta) - I) \bar{H}(\zeta)(I - S(\zeta))^{-1} \\ (I - S(\zeta))^{-1} u \end{bmatrix} = \begin{bmatrix} (P(\zeta) + I) \bar{H}(\zeta)(I - S(\zeta))^{-1} u \\ (I + S(\zeta)) u \end{bmatrix}. \tag{5.18}
\]
Let us set
\[
H_k(w) = \frac{1}{w_k + 1} \bar{H}_k \left( \frac{w - 1}{w + 1} \right) (I + f(w))
\]
where we use the convention (1.5) as usual, and note that
\[
I + f(w) = 2(I - S(\zeta))^{-1}.
\]
Then we compute
\[
(P(\zeta) - I) \bar{H}(\zeta)(I - S(\zeta))^{-1} = \sum_{k=1}^{d} \left( \frac{w_k - 1}{w_k + 1} - 1 \right) \bar{H}_k(\zeta)(I - S(\zeta))^{-1}
\]
\[
= - \sum_{k=1}^{d} \frac{2}{w_k + 1} \bar{H}_k(\zeta)(I - S(\zeta))^{-1}
\]
\[
= - \sum_{k=1}^{d} H_k(w) = -H(w)
\]
and similarly
\[
(P(\zeta) + I) \bar{H}(\zeta)(I - S(\zeta))^{-1} = \sum_{k=1}^{d} \left( \frac{w_k - 1}{w_k + 1} + 1 \right) \bar{H}_k(\zeta)(I - S(\zeta))^{-1}
\]
\[
= \sum_{k=1}^{d} \frac{2w_k}{w_k + 1} \bar{H}_k(\zeta)(I - S(\zeta))^{-1}
\]
\[
= \sum_{k=1}^{d} w_k H_k(w) = P(w)H(w).
\]
From these identities we see that the identity (5.18) is equivalent to
\[
\tilde{Y} \begin{bmatrix} -H(w) \\ I \end{bmatrix} u = \begin{bmatrix} P(w)H(w) \\ f(w) \end{bmatrix} u. \tag{5.19}
\]
In particular, for \(u \in \mathcal{U}\) then there is an \(x_{u} = -H(w) u \in \mathcal{X}\) so that \(\frac{x_{u}}{w} \in \mathcal{D}(\tilde{Y})\).

As we have already observed that \(\tilde{Y}\) is skew-adjoint (see (5.16)), we now have all the hypotheses needed in order to apply Corollary 2.7 to conclude that \(\tilde{Y} = \tilde{Y}^\dagger = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}\) is a II-impedance-conservative system node. Moreover the relation (5.19) leads to the relation
\[
Y \begin{bmatrix} H(w) \\ I \end{bmatrix} u = \begin{bmatrix} P(w)H(w) \\ f(w) \end{bmatrix} u.
\]
We now recover \( f(w) \) as the transfer-function for the \( \Pi \)-impedance-conservative system node \( Y \) exactly as in the proof of (2)\( \Rightarrow \) (3) in Theorem 5.1. \( \square \)

**Remark 5.3.** The implication (2)\( \Rightarrow \) (1) in Theorem 5.2 is due essentially to Agler–McCarthy–Young [3] (without explicit reference to \( \Pi \)-impedance-conservative system nodes), where the result is worked out for the scalar-valued case in the Nevanlinna–Agler (rather than Herglotz–Agler) setting.

### 6. Bessmertny˘ı long resolvent representations for Herglotz–Agler functions

Bessmertny˘ı long-resolvent representations were introduced by Bessmertny˘ı in connection with the study of general rational matrix functions of several variables, with a special symmetrized form of such a representation handling functions \( f \) in the \( \Pi^d \)-Herglotz–Agler class having an extension to \( \Omega^d := \bigcup_{\lambda \in \mathbb{T}} \lambda \Pi^d \subset \mathbb{C}^d \) satisfying additional symmetry conditions (see [17, 18, 19, 20, 21, 28] and our companion paper [11] for more detail). In [9], a relaxation of the symmetrized Bessmertny˘ı long-resolvent representation was proposed which handles more general Herglotz–Agler-class functions (i.e., the homogeneity property is discarded).

Given a \( 2 \times 2 \)-block operator pencil
\[
\begin{bmatrix}
V_{11}(w) & V_{12}(w) \\
V_{21}(w) & V_{22}(w)
\end{bmatrix} =: V(w) = V_0 + w_1 V_1 + \cdots + w_d V_d \in \mathcal{L}(\mathcal{U} \oplus \mathcal{X})
\]
we define the transfer function \( f_V(w) \) associated with the operator pencil \( V \) by
\[
f_V(w) := V_{11}(w) - V_{12}(w) V_{22}(w)^{-1} V_{21}(w).
\]
wherever the formula makes sense. Let us say that a representation \( f(w) = f_V(w) \) of the form (6.2) for a given \( f \) is a \( \mathcal{B} \)-realization if the operator pencil \( V(w) \) satisfies
(1) the \( \mathcal{B} \)-symmetry condition \( V(w) = -V(-w)^* \), namely
\[
V_0 + V_0^* = 0, \quad V_k = V_k^* \quad \text{for} \quad k = 1, \ldots, d,
\]
and (2) the \( \mathcal{B} \)-positivity condition
\[
V_k = V_k^* \geq 0 \quad \text{for} \quad k = 1, \ldots, d \quad \text{with} \quad \sum_{k=1}^d V_{22,k} \text{ strictly positive definite.}
\]
It was shown in [9] that any function \( f \) of the form (6.2) with pencil \( V(w) \) satisfying conditions (6.3) and (6.4) is in the Herglotz–Agler class \( \mathcal{H}_A(\Pi^d, \mathcal{L}(\mathcal{U})) \).

We note that the realization (5.3) in Theorem 5.1 can formally be considered as a \( \mathcal{B} \)-realization with operator pencil \( V(w) = V_0 + w_1 V_1 + \cdots + w_d V_d \) given by
\[
V_0 = \begin{bmatrix} D & C \\ -B & A \end{bmatrix}, \quad V_k = \begin{bmatrix} 0 & 0 \\ 0 & Y_k \end{bmatrix} = V_k^* \geq 0 \quad \text{for} \quad k = 1, \ldots, d,
\]
meeting all the constraints (6.3)–(6.4) for a \( \mathcal{B} \)-realization of \( f \) with the exception that \( V_0 \) is unbounded. We now make precise how a general Herglotz–Agler-class function (i.e., with the growth condition at infinity (5.1) removed) can be completely characterized in terms of possession of a nonhomogeneous Bessmertny˘ı-type representation of the form (6.2) subject to (6.3) and (6.4) but with possibly unbounded skew-adjoint operator \( V_0 \).
We shall consider unbounded pencils of the form (6.1) but with the constant term
\[ V_0 = \begin{bmatrix} V_{0,11} & V_{0,12} \\ V_{0,21} & V_{0,22} \end{bmatrix} \]
a possibly unbounded operator on \([U X]\) satisfying what we shall call the Herglotz–Agler system node properties:

(HA1) \( V_0 \) is skew-adjoint on \([U X]\).

(HA2) For each \( u \in U \) there is an \( x_u \in X \) so that \([u x_u]\) ∈ \( \mathcal{D}(V_0) \).

The gist of conditions (HA1) and (HA2) is that the reorganized colligation matrix
\[ \begin{bmatrix} A&B \\ C&D \end{bmatrix} = \begin{bmatrix} 0 & -I_X \\ I_U & 0 \end{bmatrix} V_0 \begin{bmatrix} 0 & I_U \\ I_X & 0 \end{bmatrix} \]
is an impedance-conservative system node as discussed in Example 2.6. To reduce the number of subscripts and to suggest the connection with system nodes in the work of Staffans et al. [29, 35, 36, 12], we shall use the notation encoded in the following formal definition.

**Definition 6.1.** We shall say that the pencil (6.1) is a Herglotz–Agler operator pencil if the following conditions hold:

1. \( V_0 \) has the form
\[ V_0 = \begin{bmatrix} D&C \\ -(B&A) \end{bmatrix} := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A&B \\ C&D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \]

where \([A&B] \begin{bmatrix} C&D \end{bmatrix}\) is an impedance-conservative system node as in Example 2.6.

2. The homogeneous part of the pencil, \( V_H(w) := \sum_{k=1}^d w_k V_k \) has each
\[ V_k = \begin{bmatrix} V_{k,11} & V_{k,12} \\ V_{k,21} & V_{k,22} \end{bmatrix} \]
a bounded positive semidefinite operator on \( U ⊕ X \) with the sum having the form
\[ V_H(e) = \sum_{k=1}^d \begin{bmatrix} V_{k,11} & V_{k,12} \\ V_{k,21} & V_{k,22} \end{bmatrix} = \begin{bmatrix} V_{H,11}(e) & 0 \\ 0 & I \end{bmatrix} \]
where we use the notation \( e = (1, \ldots, 1) \) as in (5.1).

Let us suppose that
\[ V(w) = V_0 + V_H(w) = \begin{bmatrix} D&C \\ -(B&A) \end{bmatrix} + \begin{bmatrix} V_{H,11}(w) & V_{H,12}(w) \\ V_{H,21}(w) & V_{H,22}(w) \end{bmatrix} \]
is a Herglotz–Agler operator pencil. Thus in particular
\[ \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} V_0 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \]
has all the properties delineated in Definition 2.4, with the additional property that \( V_0 = -V_0^* \) (see Example 2.6). In particular we may write
\[ B&A = \begin{bmatrix} B & A|x \end{bmatrix} |_{\mathcal{D}(V_0)}. \]

Here \( A|x : X \to X_1 \) is the extension of the skew-adjoint operator \( A \) defined by
\[ \mathcal{D}(A) = \left\{ x : \begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathcal{D}(V_0) \right\} \text{ with } Ax = -V_0 \begin{bmatrix} 0 \\ x \end{bmatrix} \]
and with $B: \mathcal{U} \to \mathcal{X}_{-1}$ constructed via

$$Bu = B\&A \begin{bmatrix} u \\ x_u \end{bmatrix} - A_{x|x} x$$

where $x_u$ is any choice of vector in $\mathcal{X}$ such that $\begin{bmatrix} x_u \\ u \end{bmatrix} \in \mathcal{D}(V_0)$; it can be shown that $B$ is well defined, i.e., the formula for $Bu$ is independent of the choice of $x_u \in \mathcal{X}$. Furthermore, the domain of $V_0$ is the same as the domain of $B\&A$ and has the precise characterization

$$\mathcal{D}(V_0) = \mathcal{D}(B\&A) = \left\{ \begin{bmatrix} u \\ x \end{bmatrix} \in \mathcal{U} \times \mathcal{X} : Bu + A_{x|x} x \in \mathcal{X} \right\}.$$  

The following proposition gives some additional key properties for Herglotz–Agler operator pencils.

**Proposition 6.2.** Suppose that

$$V(w) = V_0 + V_H(w) := \begin{bmatrix} D&C \\ -(B\&A) \end{bmatrix} + \begin{bmatrix} V_{H,11}(w) & V_{H,12}(w) \\ V_{H,21}(w) & V_{H,22}(w) \end{bmatrix}$$

is a Herglotz–Agler operator pencil. Then, for a given $w \in \Pi^4$, the formal Bessmertnyi transfer function (6.2),

$$f_V(w) = V_{11}(w) - V_{12}(w) V_{22}(w)^{-1} V_{21}(w)$$

$$= (D + V_{H,11}(w)) - (C + V_{H,12}(w))(V_{H,22}(w) - A)^{-1}(-B + V_{H,21}(w)),$$

can be interpreted as a bounded operator on $\mathcal{U}$ equal to the sum of five well-defined bounded terms

$$f_V(w) = f_{V,1}(w) + f_{V,2}(w) + f_{V,3}(w) + f_{V,4}(w) + f_{V,5}(w) \quad (6.6)$$

where

$$f_{V,1}(w) = D&C \left[ (V_{H,22}(w) - A)^{-1} B \right],$$

$$f_{V,2}(w) = V_{H,12}(w) \left[ (V_{H,22}(w) - A)^{-1} V_{H,21}(w) \right),$$

$$f_{V,3}(w) = -C(V_{H,22}(w) - A)^{-1} V_{H,21}(w),$$

$$f_{V,4}(w) = -V_{H,12}(w)(V_{H,22}(w) - A)^{-1} V_{H,21}(w),$$

$$f_{V,5}(w) = V_{H,11}(w). \quad (6.7)$$

Directly in terms of the Bessmertnyi pencil $V(w)$, we have, for each $w \in \Pi^4$,

$$\begin{bmatrix} I \\ -V_{22}(w)^{-1} V_{21}(w) \end{bmatrix} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(V_0)), \quad (6.8)$$

$$V_{11}(w) \& V_{12}(w) \in \mathcal{L}(\mathcal{D}(V_0), \mathcal{U}), \quad (6.9)$$

and we recover $f_V(w)$ as the composition of bounded operators

$$f_V(w) = V_{11}(w) \& V_{12}(w) \begin{bmatrix} I \\ -V_{22}(w)^{-1} V_{21}(w) \end{bmatrix} \in \mathcal{L}(\mathcal{U}). \quad (6.10)$$

**Proof.** Analysis of $f_{V,1}$: It follows from part (4) of Proposition 2.10 that the operator $\begin{bmatrix} ((V_{H,22}(w)^{-1} - A)^{-1} B \end{bmatrix}$ maps $\mathcal{D}(V_0)$ into $\mathcal{D}(V_0)$. Furthermore, one can check that $\begin{bmatrix} ((V_{H,22}(w)^{-1} - A)^{-1} B \end{bmatrix}$ is bounded as an operator from $\mathcal{U}$ to $\mathcal{D}(V_0)$ (with $\mathcal{D}(V_0)$ equipped with the graph norm of $V_{H,22}(w)$). As $D&C$ has domain equal to
$D(V_0)$, we see that $D&C$ maps $D(V_0)$ into $U$. Moreover, part (4) of Definition 2.4 assures us that $D&C$ is bounded as an operator from $D(V_0)$ into $U$. We conclude that $f_{V,1}(w) = D&C \cdot \left[ ((V_{H,22}(w) - A|_X)^{-1}B \right] \in \mathcal{L}(U).

Analysis of $f_{V,2}$: By part (2) of Proposition 2.10, $((V_{H,22}(w) - A|_X)^{-1}B$ maps $U$ boundedly into $X$. As $V_{H,12}(w)$ is bounded as an operator from $X$ into $U$, it follows that $f_{V,2}(w) = V_{H,12}(w) \cdot (V_{H,22}(w) - A|_X)^{-1}B$ is bounded as an operator on $U$.

Analysis of $f_{V,3}$: It is a consequence of part (2) of Proposition 2.10 that the operator $(V_{H,22}(w) - A)^{-1}$ is bounded from $X$ into $X_1$. From the definition (2.8) of $C$, we see that $C$ maps $X_1$ boundedly into $U$. Since also $V_{H,21}(w)$ is bounded as an operator from $U$ to $X$, it follows that $f_{V,3}(w) = -C \cdot (V_{H,22}(w) - A)^{-1} \cdot V_{H,21}(w)$ defines a bounded operator on $U$.

Analysis of $f_{V,4}$: Note that $V_{H,21}(w) \in \mathcal{L}(U,X)$, by part (3) of Proposition 2.10 $(V_{H,22}(w) - A)^{-1}$ is bounded as an operator from $X$ into $X_1$ and hence as an operator from $X$ into itself, and $V_{H,12}(w)$ is bounded as an operator from $X$ into $U$. It follows that $f_{V,4}(w)$, as a composition of bounded operators, is bounded as an operator on $U$.

Analysis of $f_{V,5}$: This is the easiest term: $V_{H,11}(w)$ is a bounded operator on $U$ from the definition of Herlotz-Agler pencil (Definition 6.1).

Verification of formula (6.10): We first write out $V(w)$ in terms of constant term and homogeneous part:

$$V(w) = V_0 + V_{H,k}(w) = \begin{bmatrix} D&C \\ -(B&A) \end{bmatrix} + \begin{bmatrix} V_{H,11}(w) \\ V_{H,21}(w) \end{bmatrix}.$$

Thus

$$V_{11}(w) + V_{12}(w) = D&C + \begin{bmatrix} V_{H,11}(w) \\ V_{H,12}(w) \end{bmatrix}.$$ 

The first term maps $D(U_0)$ boundedly into $U$ while the second term is bounded from the larger space $[U]_X$ into $U$. It follows that the sum indeed is bounded from $D(U_0)$ into $U$, verifying property (6.9). Similarly,

$$\begin{bmatrix} I \\ -V_{22}(w)^{-1}V_{21}(w) \end{bmatrix} = \begin{bmatrix} I \\ (V_{H,22}(w) - A)^{-1}(B - V_{H,21}(w) \end{bmatrix} = \begin{bmatrix} I \\ (V_{H,22}(w) - A)^{-1}B \end{bmatrix} - \begin{bmatrix} 0 \\ (V_{H,22}(w) - A)^{-1}V_{H,21}(w) \end{bmatrix}.$$

The first term maps $U$ boundedly into $D(U_0)$ as a consequence of part (4) of Proposition 2.10 while the second term maps $U$ boundedly into $\left[ \frac{U}{X} \right] \subset D(U_0)$ (and hence also boundedly into $D(U_0)$) by part (2) of Proposition 2.10; this verifies property (6.8). Thus the composition in the formula (6.10) defines a bounded operator on $U$. Working out the various pieces in detail, we see that the result agrees with the formula for $f_{V}(w)$ in (6.6). $\square$

The following is the main result of this section.

Theorem 6.3. Given a function $f : \Pi^d \rightarrow \mathcal{L}(U)$, the following are equivalent:

1. $f$ is in the Herlotz–Agler class $\mathcal{HA}(\Pi^d, \mathcal{L}(U))$. 


(2) \( f \) has a \( \Pi^d \)-Herglotz–Agler decomposition, i.e., there exist \( \mathcal{L}(U) \)-valued positive kernels \( K_1, \ldots, K_d \) on \( \Pi^d \) such that
\[
f(z)^* + f(w) = \sum_{k=1}^{d} (\xi_k + w_k)K_k(z, w).
\]

(3) There exists a Hilbert space \( \mathcal{X} \) and a Herglotz–Agler pencil
\[
V(w) = V_0 + VH(w) = [D&\mathcal{C} \\
\mathcal{C} & -(B&A)] + \begin{bmatrix} VH_{11}(w) & VH_{12}(w) \\
VH_{21}(w) & VH_{22}(w) \end{bmatrix}
\]
such that \( f(w) = f_V(w) \) (with \( f_V(w) \) as in (6.6)–(6.7) or (6.10)).

Proof. (1) \( \Rightarrow \) (2): This is already done in the proof of Theorem 5.1.

(1) or (2) \( \Rightarrow \) (3): We start with the representation for a Herglotz–Agler function \( F \) over the polydisk \( \mathbb{D}^d \):
\[
F(\zeta) = R + V^*(U - P(\zeta))^{-1}(U + P(\zeta))V
\]
where \( U \) is unitary, \( P(\zeta) = \zeta_1 P_1 + \cdots + \zeta_d P_d \) is a spectral decomposition of the identity on the state space \( \mathcal{X} \), \( R = \frac{F(0) - F(1)}{2} \) and \( V^*V = \frac{F(0) + F(1)}{2} \) as in formula (3.10). We use the tuple version of the mutually inverse Cayley changes of variable (1.3):
\[
\zeta \in \mathbb{D}^d \mapsto \frac{1 + \zeta}{1 - \zeta} \in \mathbb{D}^d, \quad w \in \mathbb{D}^d \mapsto \frac{w - 1}{w + 1} \in \mathbb{D}^d,
\]
with the conventions (1.4) and (1.5) in force. If we set \( F(\zeta) = f \left( \frac{1 + \zeta}{1 - \zeta} \right) \), then \( F \) is in the Herglotz–Agler class over the polydisk \( \mathbb{D}^d \) and hence we can represent \( F(\zeta) \) as in (6.11). Moreover, we recover \( f(w) \) from \( F(\zeta) \) via \( f(w) = F \left( \frac{w - 1}{w + 1} \right) \). This leads to the formula
\[
f(w) = R + V^*M(w)V
\]
where we set
\[
M(w) = \left( U - P \left( \frac{w - 1}{w + 1} \right) \right)^{-1} \left( U + P \left( \frac{w - 1}{w + 1} \right) \right).
\]
We compute further
\[
M(w) = (U - (P(w) - I)(P(w) + I)^{-1})^{-1} (U + (P(w) - I)(P(w) + I)^{-1})
\]
\[
= (P(w)U + U - P(w) + I)^{-1} (P(w)U + U + P(w) - I)
\]
\[
= (P(w)(U - I) + (U + I))^{-1} (P(w)(U + I) + (U - I))
\]
Let us split out the eigenspace of \( U \) for eigenvalue \( z = 1 \) (if any) by writing \( U \) in the form
\[
U = \begin{bmatrix} I & 0 \\ 0 & U_0 \end{bmatrix}
\]
with respect to the decomposition \( \mathcal{X} = \mathcal{X}^{(1)} \oplus \mathcal{X}^{(0)} \) (\( \mathcal{X}^{(1)} \) equal to the 1-eigenspace for \( U \) and \( \mathcal{X}^{(0)} \) equal to the orthogonal complement of \( \mathcal{X}^{(1)} \) in \( \mathcal{X} \)). Then \( U_0 \) is unitary but does not have 1 as an eigenvalue. We have
\[
U - I = \begin{bmatrix} 0 & 0 \\ 0 & U_0 - 1 \end{bmatrix}, \quad U + I = \begin{bmatrix} 2I & 0 \\ 0 & U_0 + I \end{bmatrix}.
\]
and hence
\[ M(w) = \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & U_0 - I \end{bmatrix} + \begin{bmatrix} 2I & 0 \\ 0 & U_0 + I \end{bmatrix} \right)^{-1} \]
\[ \cdot \left( P(w) \begin{bmatrix} 2I & 0 \\ 0 & U_0 - I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & U_0 + I \end{bmatrix} \right) \]

Let us set
\[ T = (I + U_0)(I - U_0)^{-1}. \]
Then \( T \) is a possibly unbounded skew-adjoint operator on \( X_0 \) (by Proposition 2.3) and we have the following relations between \( T \) and \( U_0 \):
\[ U_0 = (T - I)(T + I)^{-1} = I - 2(T + I)^{-1} \]
\[ \Rightarrow U_0 - I = -2(T + I)^{-1}, \quad (U_0 - I)^{-1} = -\frac{1}{2}(T + I). \]

We may then continue the computation (6.13) to get
\[ M(w) = \left[ \begin{bmatrix} 2I \\ 0 \end{bmatrix} (U_0 - I)^{-1} \right] \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & (U_0 + I)(U_0 - I)^{-1} \end{bmatrix} \right)^{-1} \]
\[ \cdot \left( P(w) \begin{bmatrix} 2I & 0 \\ 0 & U_0 - I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & U_0 + I \end{bmatrix} \right) \]
\[ = \left[ \begin{bmatrix} 0 & 0 \\ 0 & -(T + I) \end{bmatrix} \right] \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} \right)^{-1} \]
\[ \cdot \left( P(w) \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) \left( I \begin{bmatrix} 0 & 0 \\ 0 & -(T + I)^{-1} \end{bmatrix} \right) \]
\[ = \left[ \begin{bmatrix} 0 & 0 \\ 0 & -(T + I)^\ast \end{bmatrix} \right] N(w) \left[ \begin{bmatrix} I & 0 \\ 0 & -(T + I)^\ast \end{bmatrix} \right] \]
(6.14)

where, due to the identity \(-(T + I)^\ast = T - I\) arising from \( T = -T^\ast \), \( N(w) \) is given by
\[ N(w) = \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} \right)^{-1} \]
\[ \cdot \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} \right) \left( I \begin{bmatrix} 0 & (I - T^2)^{-1} \end{bmatrix} \right) \]
(6.15)

If we write out the block matrix decomposition
\[ P(w) = \begin{bmatrix} P_{11}(w) & P_{10}(w) \\ P_{01}(w) & P_{00}(w) \end{bmatrix} = \sum_{k=1}^{d} w_k \begin{bmatrix} P_{k,11} & P_{k,10} \\ P_{k,01} & P_{k,00} \end{bmatrix} \]
of \( P(w) \) with respect to the decomposition \( X = \begin{bmatrix} X^{(1)} \\ X^{(0)} \end{bmatrix} \) of \( X \), we can write out more explicitly
\[ \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} \right)^{-1} = \left[ I - P_{10}(w)(P_{00}(w) - T)^{-1} \right]^{-1} \]
\[ = \left[ I - P_{10}(w)(P_{00}(w) - T)^{-1} \right]. \]
\[
P(w) \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P_{11}(w) & -P_{10}(w)T \\ P_{01}(w) & I - P_{00}(w)T \end{bmatrix}
\]

and from (6.15) we see that \(N(w)\) is given by

\[
N(w) = \begin{bmatrix} I & -P_{10}(w)(P_{00}(w) - T)^{-1} \\ 0 & (P_{00}(w) - T)^{-1} \end{bmatrix} \begin{bmatrix} P_{11}(w) & -P_{10}(w)T(I - T^2)^{-1} \\ P_{01}(w) & (I - P_{00}(w)T)(I - T^2)^{-1} \end{bmatrix}.
\]

(6.16)

At this stage it is convenient to introduce the Gelfand triple (or rigging) of \(X^{(0)}\) associated with the (possibly unbounded) skew-adjoint operator \(T\):

\[
\mathcal{X}^{(0)}_1 := \text{Dom} T = \text{Ran}(I - T)^{-1},
\]

\[
\mathcal{X}^{(0)}_{-1} := \text{completion of } \mathcal{X}^{(0)} \text{ in } \mathcal{X}^{(0)}_1 \text{-norm: } \|x\|_{-1} = \|(I - T)^{-1}x\|_{X^{(0)}} \text{ for } x \in \mathcal{X}^{(0)}.
\]

Then we see that \((I - T)^{-1}\) is well defined as an element of \(\mathcal{L}(\mathcal{X}^{(0)}_1, \mathcal{X}^{(0)}_0)\) and of \(\mathcal{L}(\mathcal{X}^{(0)}_{-1}, \mathcal{X}^{(0)}_0)\). A careful inspection of the formula (6.16) for \(N(w)\) shows that

\[
N(w): \begin{bmatrix} \mathcal{X}^{(1)}_1 \\ \mathcal{X}^{(0)}_0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^{(1)}_1 \\ \mathcal{X}^{(0)}_0 \end{bmatrix}
\]

from which it follows that the formula (6.14) gives sense for \(M(w)\) as an element of \(\mathcal{L}(\mathcal{X}^{(1)}_0 \oplus \mathcal{X}^{(0)}_1)\). However the formula (6.16) (and (6.15)) lacks symmetry. To fix this we introduce the operator \(J\) by

\[
J := -T(I - T^2)^{-1},
\]

(6.18)

Thus \(J = -J^*\) and we can consider \(J\) as an element of \(\mathcal{L}(\mathcal{X}^{(0)}_0, \mathcal{X}^{(1)}_0)\) as well as \(\mathcal{L}(\mathcal{X}^{(0)}_1, \mathcal{X}^{(0)}_0)\). While the operator \(I - P_{00}(w)T\) makes sense as an element of \(\mathcal{L}(\mathcal{X}^{(0)}_1, \mathcal{X}^{(0)}_0)\) (as well as \(\mathcal{L}(\mathcal{X}^{(0)}_0, \mathcal{X}^{(0)}_1)\)), the individual terms in the additive decomposition

\[
I - P_{00}(w)T = (I - T^2) + (T^2 - P_{22}(w))T
\]

(6.19)

make sense only as elements in \(\mathcal{L}(\mathcal{X}^{(0)}_1, \mathcal{X}^{(0)}_0)\). Nevertheless, we proceed to get a more symmetric formula for \(N(w)\) as follows. Note first that the decomposition (6.19) leads to

\[
(I - P_{00}(w)T)(I - T^2)^{-1} = I - (P_{00}(w) - T)T(I - T^2)^{-1}
\]

\[
= I + (P_{00}(w) - T)J: \mathcal{X}^{(0)} \rightarrow \mathcal{X}^{(0)}.
\]

From (6.16) and the definition (6.18) of \(J\), we then have

\[
N(w) = \begin{bmatrix} I & -P_{10}(w)(P_{00}(w) - T)^{-1} \\ 0 & (P_{00}(w) - T)^{-1} \end{bmatrix} \begin{bmatrix} P_{11}(w) & -P_{10}(w)J \\ P_{01}(w) & I + (P_{00}(w) - T)J \end{bmatrix} \begin{bmatrix} P_{11}(w) - P_{10}(w)(P_{00}(w) - T)^{-1}P_{01}(w) & -P_{10}(w)(P_{00}(w) - T)^{-1} \\ (P_{00}(w) - T)^{-1}P_{01}(w) & (P_{00}(w) - T)^{-1} + J \end{bmatrix}
\]

(6.20)

which a priori makes sense only as an operator from \(\begin{bmatrix} \mathcal{X}^{(1)}_1 \\ \mathcal{X}^{(0)}_0 \end{bmatrix}\) to \(\begin{bmatrix} \mathcal{X}^{(1)}_1 \\ \mathcal{X}^{(0)}_0 \end{bmatrix}\) rather than to \(\begin{bmatrix} \mathcal{X}^{(1)}_1 \\ \mathcal{X}^{(0)}_0 \end{bmatrix}\) as in (6.17), due to the decoupling of an \(\infty - \infty\) cancellation occurring in the application of the decomposition (6.19). This in turn leads to difficulties.
in understanding $M(w)$ as a bounded operator on $\mathcal{X}^{(1)} \oplus \mathcal{X}^{(0)}$ from the formula (6.14).

Continuation of the analysis with an extra assumption: Assuming for the moment that $T$ is bounded (as is the case for the situation where the state space $\mathcal{X}$ is finite-dimensional as in the setting discussed in [11]), this difficulty does not occur and we may continue as follows. From (6.20) we see that

$$N(w) = \begin{bmatrix} P_{11}(w) & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} P_{10}(w) \\ -I \end{bmatrix} (P_{00}(w) - T)^{-1} \begin{bmatrix} P_{01}(w) \\ I \end{bmatrix}.$$  \hfill (6.21)

If we block-decompose the operator $V: \mathcal{U} \to \mathcal{X} = \begin{bmatrix} \mathcal{X}^{(1)} \\ \mathcal{X}^{(0)} \end{bmatrix}$ as $V = \begin{bmatrix} V_1 \\ V_0 \end{bmatrix}$ and then combine (6.12) with (6.14) and (6.21) while noting the simplification

$$(I + T)J(I + T)^* = (I + T)[-(I + T)^{-1}T(I - T)^{-1}](I + T)^* \quad \text{(by (6.18))}$$

we arrive at

$$f(w) = R + V_1^* P_{11}(w)V_1 - V_0^* TV_0$$

$$- [V_1^* P_{10}(w) + V_0^*(I + T)[(P_{00}(w) - T)^{-1}[P_{01}(w)V_1 - (I + T)^*V_0].$$  \hfill (6.22)

We have arrived at a Bessmertny˘ı long-resolvent representation for $f$

$$f(w) = f_V(w)$$

where the operator pencil $V(w) = \begin{bmatrix} V_{11}(w) & V_{12}(w) \\ V_{21}(w) & V_{22}(w) \end{bmatrix}$ is given by

$$V_{11}(w) = R + V_0^*(I + T)J(I + T)^*V_0 + V_1^* P_{11}(w)V_1,$$

$$V_{12}(w) = V_0^*(I + T) + V_1^* P_{10}(w),$$

$$V_{21}(w) = -(I + T)^*V_0 + P_{01}(w)V_1,$$

$$V_{22}(w) = -T + P_{00}(w).$$

Thus the associated linear pencil

$$V(w) = \begin{bmatrix} V_{11}(w) & V_{12}(w) \\ V_{21}(w) & V_{22}(w) \end{bmatrix} = V_0 + w_1 V_1 + \cdots + w_d V_d$$  \hfill (6.23)

has coefficients

$$V_0 = \begin{bmatrix} R - V_0^*TV_0 & V_0^*(I + T) \\ -(I + T)^*V_0 & -T \end{bmatrix}$$

$$= \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} V_0^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -T & I + T \\ -T & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ V_0 & I \end{bmatrix},$$  \hfill (6.24)

$$V_k = \begin{bmatrix} V_1^* P_{k,11} V_1 \\ V_1^* P_{k,10} V_0 \\ P_{k,01} V_1 \\ P_{k,00} \end{bmatrix} \text{ for } k = 1, \ldots, d.$$  \hfill (6.25)

Moreover, it is easily checked that $V_0$ is skew-adjoint and that $P_k \geq 0$ for each $k$ with $\sum_{k=1}^d P_{k,00} = I_{\mathcal{X}^{(0)}}$ (since $\sum_{k=1}^d P_k = I_{\mathcal{X}}$), and hence $V(w)$ is a Herglotz–Agler pencil and Theorem 6.3 is completely proved in case $T = (I + U_0)(I - U_0)^{-1}$ is bounded on $\mathcal{X}^{(0)}$.

Back to the general case: For the general case (where $T$ is allowed to be unbounded), the formula (6.25) for $V_k$ still makes good sense and the $V_k$’s meet
property (2) in Definition 6.1. The next step is to make sense of the formula (6.24) for \( \mathbf{V}_0 \).

The first term in the formula (6.24) for \( \mathbf{V}_0 \) can always be added in later so we focus on the second term \( \mathbf{V}_0' \):

\[
\mathbf{V}_0' = \begin{bmatrix}
\mathbf{V}_0^* & 0 \\
0 & \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
-T & I + T \\
-(I + T)^* & -T
\end{bmatrix}
\begin{bmatrix}
\mathbf{V}_0 & 0 \\
0 & \mathbf{I}
\end{bmatrix}.
\]

We view \( \mathbf{V}_0' \) as a possibly unbounded operator with dense domain in \( \mathcal{D}(\mathbf{V}_0') \) given by

\[
\mathcal{D}(\mathbf{V}_0') = \left\{ \left. \begin{bmatrix}
u \\
x
\end{bmatrix} \in \mathcal{H} : x - \mathbf{V}_0u \in \mathcal{D}(T) \right\}.
\]

Then the flip of \( \mathbf{V}_0' \), namely

\[
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix} \mathbf{V}_0' \begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & \mathbf{V}_0^*
\end{bmatrix}
\begin{bmatrix}
T & I - T \\
I + T & -T
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & \mathbf{V}_0
\end{bmatrix}
\]

has exactly the form of the model II-impedance-conservative system node given in Proposition 2.8. We can now conclude that \( \mathbf{V}(w) \) given by (6.23), (6.24), (6.25) is indeed a Herglotz–Agler pencil. It remains only to verify that \( f(w) = f_{\mathbf{V}}(w) \).

From the formula (6.12) for \( f \) combined with the formula (6.14) for \( M(w) \) and the formula (6.20) for \( \mathcal{N}(w) \), we know that \( f(w) \) has the representation

\[
f(w) = R + \mathbf{V}_1^* P_{11}(w) \mathbf{V}_1 - \mathbf{V}_1^* P_{00}(w)(P_{00}(w) - T)^{-1} P_{01}(w) \mathbf{V}_1
\]

\[
- \mathbf{V}_0^*(I + T)(P_{00}(w) - T)^{-1} P_{01}(w) \mathbf{V}_1
\]

\[
+ \mathbf{V}_1^* P_{10}(w)(T(I + T)^{-1} + (P_{00}(w) - T)^{-1}(I - P_{00}(w)T)(I + T)^{-1}) \mathbf{V}_0
\]

\[
+ \mathbf{V}_0^*(I + T)(P_{00}(w) - T)^{-1}(I - P_{00}(w)T)(I + T)^{-1} \mathbf{V}_0.
\]

(6.26)

On the other hand the Bessmertnyi transfer function associated with the pencil \( \mathbf{V} \)

(6.23) can be written as

\[
f_{\mathbf{V}}(w) = [\mathbf{V}_{11}(w) & \mathbf{V}_{12}(w)]
\]

\[
- \mathbf{V}_{22}(w)^{-1} \mathbf{V}_{21}(w)
\]

\[
= R + \mathbf{V}_0^* [-T + (I + T)(P_{00}(w) - T)^{-1}(I - T)] \mathbf{V}_0
\]

\[
+ \mathbf{V}_1^* P_{11}(w) \mathbf{V}_1 - \mathbf{V}_1^* P_{10}(w)(P_{00}(w) - T)^{-1} P_{01}(w) \mathbf{V}_1
\]

\[
- \mathbf{V}_0^*(I + T)(P_{00}(w) - T)^{-1} P_{01}(w) \mathbf{V}_1
\]

\[
+ \mathbf{V}_0^* P_{00}(w)(P_{00}(w) - T)^{-1}(I - T) \mathbf{V}_0.
\]

(6.27)

Note that care must be taken in writing the first term of the expression after the \( R \)

term: the individual expressions \(-\mathbf{V}_0^* T \mathbf{V}_0\) and \( \mathbf{V}_0^*(I + T)(P_{00}(w) - T)^{-1}(I - T) \mathbf{V}_0\)

make no sense since the operators \(-T\) and \((I + T)(P_{00}(w) - T)^{-1}(I - T)\) map \( S(P_{00}) \) into \( S(P_{00}) \) and \( \mathbf{V}_0^* \in \mathcal{L}(\mathcal{S}(P_{00}), \mathcal{U}) \) has no extension to \( S(P_{00}) \); as we shall see in detail below, the combination \(-T + (I + T)(P_{00}(w) - T)^{-1}(I - T)\) fortuitously maps \( S(P_{00}) \) back into itself so that the combined term \( \mathbf{V}_0^*(T + (I + T)(P_{00}(w) - T)^{-1}(I - T)) \mathbf{V}_0\)

makes good sense as a bounded operator on \( \mathcal{U} \) for each \( w \in \mathbb{C}^2 \); roughly speaking, this is where we couple back together the \( \infty - \infty \) cancellation introduced earlier to make our formulas once again make sense. Note also that the formula (6.27) agrees with the formula (6.22) once one takes care to rearrange the terms in (6.22) so that the result makes sense as a well-defined bounded operator on \( \mathcal{U} \) defining the operator \( f(w) \).
By the analysis done above with the extra assumption imposed, we see that the
two expressions (6.26) and (6.27) agree in the special case where the skew-adjoint
operator \( T \) is bounded. Once the operator \(-T + (I + T)(P_{00}(w) - T)^{-1}P_{01}(w)(I - T)\)
is exhibited more explicitly as a bounded operator on \( X^{(0)} \) (even in the case where
\( T \) itself is unbounded), it is possible to verify the equality of the two expressions
(6.26) and (6.27) by approximating the unbounded case by the bounded case and
then taking limits. As this is really about algebra, however, perhaps more satisfying
is to verify the equality between (6.26) and (6.27) directly by brute-force algebra.

Toward this goal, we note that each term in (6.26) can be paired with an identical
term in (6.27) once we establish the validity of the two identities:

\[
\begin{align*}
(I + T)(P_{00}(w) - T)^{-1}(I - P_{00}(w)T)(I + T)^{-1} \\
= -T + (I + T)(P_{00}(w) - T)^{-1}(I - T) \\
P_{10}(w) [T(I + T)^{-1} + (P_{00}(w) - T)^{-1}(I - P_{00}(w)T)(I + T)^{-1}] \\
= P_{10}(w)(P_{00}(w) - T)^{-1}(I - T).
\end{align*}
\] (6.28)

In particular, (6.28) demonstrates how the expression
\(-T + (I + T)(P_{00}(w) - T)^{-1}(I - T)\) actually defines a bounded operator on \( X \).
These two identities can be verified directly by brute-force algebra; we leave the details to the reader (or as an
exercise for MATHEMATICA). This concludes the proof of (1) or (2) \( \Rightarrow \) (3) in Theorem
6.3.

(3) \( \Rightarrow \) (2): We assume that \( f(w) = f_X(w) \) for a Herglotz–Agler pencil

\[
V(w) = \begin{bmatrix} V_{11}(w) & V_{12}(w) \\ V_{21}(w) & V_{22}(w) \end{bmatrix} = \begin{bmatrix} D&C \\ -B&A \end{bmatrix} + \begin{bmatrix} V_{H,11}(w) & V_{H,12}(w) \\ V_{H,21}(w) & V_{H,22}(w) \end{bmatrix}.
\]

Thus, as explained in Proposition 6.2, if we set \( x_w = -V_{22}(w)^{-1}V_{21}(w) \) for \( w \in \Pi^d \),
then for each \( u \in \mathcal{U} \) we have

\[
\begin{bmatrix} u \\ x_wu \end{bmatrix} \in \mathcal{D} \left( \begin{bmatrix} V_{11}(w) & V_{12}(w) \\ V_{21}(w) & V_{22}(w) \end{bmatrix} \right)
\]

and

\[
f(w)u = (V_{11}(w) & V_{12}(w)) \begin{bmatrix} u \\ x_wu \end{bmatrix} \tag{6.29}
\]

We may also compute

\[
(V_{21}(w) & V_{22}(w)) \begin{bmatrix} u \\ -V_{22}(w)^{-1}V_{21}(w)u \end{bmatrix} = [V_{21}(w) & V_{22}(w)] \begin{bmatrix} u \\ -V_{22}(w)^{-1}V_{21}(w)u \end{bmatrix} \quad \text{(as a vector in } X^{(0)}_{-1})
\]

\[
= V_{21}(w)u - V_{21}(w)u = 0.
\]

Thus (6.29) can be expanded to the identity

\[
\begin{bmatrix} V_{11}(w) & V_{12}(w) \\ V_{21}(w) & V_{22}(w) \end{bmatrix} \begin{bmatrix} u \\ x_wu \end{bmatrix} = \begin{bmatrix} f(w) \\ 0 \end{bmatrix} u. \tag{6.30}
\]
In addition to \( u \in U \) and \( w \in \Pi^d \), choose another pair \( u' \in U \) and \( z \in \Pi^d \) and consider the sesquilinear form
\[
Q(z, w)[u, u'] := \left \langle \begin{bmatrix} V_{11}(w) & V_{12}(w) \\ V_{21}(w) & V_{22}(w) \end{bmatrix} [u] , [u'] \right \rangle + \left \langle \begin{bmatrix} u \\ x_wu \end{bmatrix} , \begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix} [u'] \right \rangle.
\]
As a consequence of (6.30) we see that
\[
Q(z, w)[u, u'] = \left \langle \begin{bmatrix} f(w)u \\ u' \end{bmatrix} , \begin{bmatrix} u \\ x_wu \end{bmatrix} \right \rangle + \left \langle \begin{bmatrix} u \\ x_wu \end{bmatrix} , \begin{bmatrix} f(z)u' \\ 0 \end{bmatrix} \right \rangle = ((f(z) + f(w)) u, u'). \tag{6.31}
\]
On the other hand, from the decomposition of \( V(w) \) as \( V(w) = V_0 + V_H(w) \) with \( V_0 = -V_0^* \), we have
\[
Q(z, w)[u, u'] = \left \langle \begin{bmatrix} u \\ x_wu \end{bmatrix} , \begin{bmatrix} u' \\ x_zu' \end{bmatrix} \right \rangle + \left \langle \begin{bmatrix} u \\ x_wu \end{bmatrix} , V(z) [u'] \right \rangle = \left \langle (V_H(z) + V_H(w)) \begin{bmatrix} I \\ -V_{22}(w)^{-1}V_{21}(w) \end{bmatrix} u, \begin{bmatrix} I \\ -V_{22}(z)^{-1}V_{21}(z) \end{bmatrix} u' \right \rangle \]
\[
= \sum_{j=1}^d (\bar{z}_j + w_j) (H(z)^* V_j H(w) u, u'). \tag{6.32}
\]
where we set \( H(w) = \begin{bmatrix} I \\ -V_{22}(w)^{-1}V_{21}(w) \end{bmatrix} \in \mathcal{L}(U, U \oplus X) \). Combining (6.31) and (6.32) gives us
\[
f(z)^* + f(w) = \sum_{j=1}^d (\bar{z}_j + w_j) H(z)^* V_j H(w)
\]
where \( V_j \geq 0 \) on \( U \oplus X \) by assumption, and (2) follows.

We next illustrate Theorem 6.3 by looking at some special cases.

**Special case 1:** \( V_1 = 0 \). We note that the case \( V_1 = 0 \) in the proof of Theorem 6.3 is exactly the case where the representation (6.27) for \( f \) collapses to
\[
f(w) = R + V_0^*[-T + (I + T)(P_{00}(w) - T)^{-1}(I - T)] V_0. \tag{6.33}
\]
where \( P_{00}(w) = w_1 P_{1,00} + \cdots + w_d P_{d,00} \) is a positive decomposition of \( I_{V_0^*} \), i.e., \( f \) has a representation exactly as in part (3) of Theorem 5.1. In general, from the property (6.5) for a Herglotz–Agler pencil, we have
\[
V_H(\epsilon e) = t V_H(e) = t \begin{bmatrix} V_{H,11}(e) & 0 \\ 0 & I \end{bmatrix}.
\]
Thus, in the general representation (6.27) for \( f \), \( f(\epsilon e) \) takes on the simplified form
\[
f(\epsilon e) = R + V_0^*[-T + (I + T)(tI - T)^{-1}(I - T)] V_0 + t V_1^* V_1.
\]
We have already seen that
\[
\lim_{t \to +\infty} \frac{1}{t} \left [ R + V_0^*(-T + (I + T)(tI - T)^{-1}(I - T)) V_0 \right ] = 0.
\]
We conclude that in general
\[
\lim_{t \to +\infty} \frac{1}{t} f(\epsilon e) = V_1^* V_1.
\]
Thus the growth condition at $\infty$ (5.1) is equivalent to the condition that $V_1 = 0$. In this way we arrive at Theorem 5.1 as a corollary of Theorem 6.3.

**Special case 2:** $X^{(1)} = \{0\}$. This corresponds to the case where $P_k = P_{k,00}$ and $P_{00}(w) = w_1 P_{1,00} + \cdots + w_d P_{d,00}$ is a spectral decomposition (not just a positive decomposition) of $I_{X^{(0)}}$. Then the representation (6.27) collapses again to (6.33), but this time with the stronger property that $w_1 P_{1,00} + \cdots + P_{d,00} w_d$ is a spectral decomposition of $I_{X^{(0)}}$, i.e., exactly the conclusion of part (1) of Theorem 5.2. On the other hand, condition $X^{(1)} = \{0\}$ means that the unitary operator $U$ in the Herglotz representation (3.10) for the function $F \in \mathcal{H} A(\mathbb{D}, \mathcal{U})$ given by

$$F(\zeta) = f \left( \frac{1 + \zeta}{1 - \zeta} \right)$$

does not have 1 as an eigenvalue. On the other hand, condition (1) in Theorem 6.3 is that the colligation matrix $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in the unitary Givone–Roesser representation

$$S(\zeta) = D + C(I - P(\zeta)A)^{-1} P(\zeta)B$$

(6.34) for the $SA(\mathbb{D}, \mathcal{U})$-class function

$$S(\zeta) = \left( f \left( \frac{1 + \zeta}{1 - \zeta} \right) - I \right) \left( f \left( \frac{1 + \zeta}{1 - \zeta} \right) + I \right)^{-1} = (F(\zeta) - I)(F(\zeta) + I)^{-1}$$

(6.35) does not have 1 as an eigenvalue. To see that these conditions match up, we recall from the discussion in Remark 2.11 (note formulas (3.7), (3.9) and (3.10)) that we have the following connection between the unitary operator $U$ in the Herglotz representation (3.10) for $F(\zeta)$ and the unitary colligation matrix $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ generating the Givone–Roesser representation (6.34) for $S(\zeta)$ given by (6.35):

$$U^* = U_0 := A - B(I - D)^{-1}C.$$

As was observed in Remark 3.2, $I - D$ is injective since $I - D = I - S(0) = 2(F(0) + I)^{-1}$. By the theory of Schur complements, given that $I - D$ is injective, then $I - U$ is injective if and only if its Schur complement $U_0 = I - B(I - D)^{-1}C$ is injective. As $U_0$ is unitary, this in turn is equivalent to $U = (U_0)^*$ not having 1 as an eigenvalue. In this way we recover Theorem 5.2 as a corollary of Theorem 6.3.

**Special case 3:** $X^{(0)} = \{0\}$. In this case, $M(w) = P(w)$, and the representation (6.27) collapses to

$$f(w) = V(w) = R + V^* P(w)V.$$

**Special case 4:** $R = 0$, $U = U^*$, $V_0 = 0$. In this case, $T = T^* = -T^*$, hence $T = 0$, and the linear pencil $A(w)$ is homogeneous:

$$V(w) = V_0(w) = \begin{bmatrix} V_1^* & 0 \\ 0 & I \end{bmatrix} P(w) \begin{bmatrix} V_1 & 0 \\ 0 & I \end{bmatrix}.$$

Moreover, $V(e) = \begin{bmatrix} V_1^* V_1 & 0 \\ 0 & I \end{bmatrix}$, as in (6.5). The representation

$$f(w) = V_1^* (P_{11}(w) - P_{10}(w) P_{00}(w)^{-1} P_{01}(w)) V_1$$

(6.36)

$$= V_{H,11}(w) - V_{H,12}(w) V_{H,22}(w)^{-1} V_{H,21}$$

(6.37)
is then Bessmertnyǐ’s long-resolvent representation in the infinite-dimensional setting, as in [28], i.e., $f$ belongs to the Bessmertnyǐ class $\mathcal{B}_d(\mathcal{U})$.

**Remark 6.4.** Notice that for a function $f \in \mathcal{B}_d(\mathcal{U})$ one can always find a homogeneous Bessmertnyǐ pencil $V_H(w)$ satisfying the condition (6.5). First of all, $V_{H,22}(e)$ must be invertible. Indeed, if $\ker V_{H,22}(e) \neq \{0\}$, then the maximum principle and positivity of the coefficients $V_k$, $k = 1, \ldots, d$, force $V_H(w)$ to have the form

$$
V_H(w) = \begin{bmatrix}
V_{H,11}(w) & \tilde{V}_{H,12}(w) & 0 \\
V_{H,21}(w) & \tilde{V}_{H,22}(w) & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
U \\
\operatorname{Ran} V_{H,22}(e) \\
\ker V_{H,22}(e)
\end{bmatrix} \rightarrow \begin{bmatrix}
U \\
\operatorname{Ran} V_{H,22}(e) \\
\ker V_{H,22}(e)
\end{bmatrix},
$$

in contradiction with condition (6.5) for a Herglotz-Agler pencil. Therefore, one can replace the pencil $V_H(w)$ by the pencil

$$
\tilde{V}_H(w) = \begin{bmatrix}
I & 0 & 0 \\
0 & V_{H,22}(e)^{-1} & 0 \\
0 & 0 & V_{H,22}(e)^{-1}
\end{bmatrix} V_H(w) \begin{bmatrix}
I & 0 & 0 \\
0 & V_{H,22}(e)^{-1} & 0 \\
0 & 0 & V_{H,22}(e)^{-1}
\end{bmatrix},
$$

which satisfies the condition (6.5) and provides another Bessmertnyǐ transfer-function realization of $f$:

$$
f(w) = \tilde{V}_{H,11}(w) - \tilde{V}_{H,12}(w) \tilde{V}_{H,22}(w)^{-1} \tilde{V}_{H,21}.
$$

In fact, the special case described in Remark 6.4 covers the whole class $\mathcal{B}_d(\mathcal{U})$. In the following theorem, we collect the characterizations of this class from [28], together with the two additional characterizations: via its image in the class $\mathcal{H}_A(\mathcal{U})$ under the Cayley transform over the variables, and as this special case of Theorem 6.3.

**Theorem 6.5.** Given a function $f : \Pi^d \rightarrow \mathcal{L}(\mathcal{U})$, the following are equivalent:

1. $f$ is in the Herglotz–Agler class $\mathcal{H}_A(\Pi^d, \mathcal{L}(\mathcal{U}))$ and can be extended to a holomorphic $\mathcal{L}(\mathcal{U})$-valued function on $\Omega_d := \bigcup_{\lambda \in \mathbb{T}} (\lambda \Pi)^d$ which satisfies the following conditions:
   a. Homogeneity: $f(\lambda w_1, \ldots, \lambda w_d) = \lambda f(w_1, \ldots, w_d)$ for every $\lambda \in \mathbb{C} \setminus \{0\}$ and $w = (w_1, \ldots, w_d) \in \Omega_d$.
   b. Real Symmetry: $f(\overline{w}_1, \ldots, \overline{w}_d) = f(w_1, \ldots, w_d)^*$ for every $w \in \Omega_d$.
2. $f$ has a $\Pi^d$-Bessmertnyǐ decomposition, i.e., there exist $\mathcal{L}(\mathcal{U})$-valued positive kernels $K_1, \ldots, K_d$ on $\Pi^d$ such that the identity

$$
f(w) = \sum_{k=1}^d w_k K_k(z, w), \quad z, w \in \Pi^d,
$$

holds, or equivalently, the following two identities hold:

$$
f(z)^* + f(w) = \sum_{k=1}^d (\overline{z}_k \pm w_k) K_k(z, w), \quad z, w \in \Pi^d.
$$

3. $f$ belongs to the Bessmertnyǐ class $\mathcal{B}_d(\mathcal{U})$, i.e., there exist a Hilbert space $\mathcal{X}$ and a Bessmertnyǐ pencil

$$
V(w) = V_H(w) = \begin{bmatrix}
V_{H,11}(w) & V_{H,12}(w) \\
V_{H,21}(w) & V_{H,22}(w)
\end{bmatrix} = \sum_{k=1}^d w_k V_k,
$$
where $V_k \in \mathcal{L}(\mathcal{D})$ are positive semidefinite operators, so that (6.5) holds and $f(w) = f_V(w)$ (with $f_V(w)$ as in (6.37)).

(4) The double Cayley transform of $f$,

$$S(\zeta) = C(f) := \left[f\left(\frac{1+\zeta}{1-\zeta}\right) - I\right] \left[f\left(\frac{1+\zeta}{1-\zeta}\right) + I\right]^{-1},$$

belongs to the Schur–Agler class $\mathcal{S}(\mathbb{D}, \mathcal{L}(\mathcal{D}))$ and has a unitary Givone–Roesser realization (1.2) with a unitary and self-adjoint colligation matrix $U = U^* = U^{-1}$.

(5) The Cayley transform of $f$ over the variables,

$$F(\zeta) = f\left(\frac{1+\zeta}{1-\zeta}\right),$$

belongs to the Herglotz–Agler class $\mathcal{H}(\mathbb{D}, \mathcal{L}(\mathcal{D}))$ and has a representation (6.11) with $R = 0, U = U^{-1} = U^*$, and Ran $V$ contained in the 1-eigenspace of $U$.

(6) There exist a Hilbert space $\mathcal{X}$, its subspaces $\mathcal{X}^{(0)}$, $\mathcal{X}^{(1)}$ with $\mathcal{X} = \mathcal{X}^{(1)} \oplus \mathcal{X}^{(0)}$, a decomposition of $I_{\mathcal{X}}, P(w) = w_1 P_1 + \cdots + w_d P_d = \begin{bmatrix} P_{11}(w) & P_{10}(w) \\ P_{01}(w) & P_{00}(w) \end{bmatrix}$ (with respect to the two-fold decomposition of $\mathcal{X}$), and $V_i \in \mathcal{L}(U, \mathcal{X}^{(i)})$, such that $f$ has the form (6.36).

Proof. The equivalence of statements (1), (2), (3), and (4) has been proved in [28].

(5)⇒(6) has been shown above, in the first paragraph of Special case 4. This is an application of the construction in the proof of Theorem 6.3 to this special case.

(6)⇒(3) is obvious.

(4)⇒(5). The function $F$ is related to $S = C(f)$ as in (3.8). Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a unitary colligation matrix providing the transfer-function realization for $S$ as in (4), i.e., $U^* = U$, together with the spectral decomposition of $I_{\mathcal{X}}, P(\zeta) = \zeta_1 P_1 + \cdots + \zeta_d P_d$. As it was shown in Remark 3.2, $F$ has a representation (3.10) with skew-adjoint $R$, unitary $U = U_0 = A^* - C^*(I - D^*)^{-1} B^*$ and $V = \frac{1}{\sqrt{\lambda}} B$.

Now we also have that $R = R^* = -R^*$, which means that $R = 0$, and that $U = U^* = A - B(I - D)^{-1} B^*$. It follows that the space $\mathcal{X}$ has an orthogonal decomposition $\mathcal{X} = \mathcal{X}^{(1)} \oplus \mathcal{X}^{(0)}$, where $\mathcal{X}^{(1)}$ is the 1-eigenspace of $U$ and $\mathcal{X}^{(0)}$ is the (-1)-eigenspace of $U$. With respect to this decomposition, let us write

$$U = U_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{10} \\ A_{01} & A_{00} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}.$$ 

Since $A_{00}$ is a self-adjoint and contractive, the identity

$$-I = A_{00} + B_0 (I - D)^{-1} B_0^*$$

is possible only if $B_0 = 0$. Thus $\text{im} V \subseteq \mathcal{X}^{(1)}$, which completes the proof. \hfill \Box

**Special case 5: the single-variable case.** When we specialize Theorem 6.3 to the single-variable case, some simplifications occur. In particular the Herlgotz pencil (6.23) involves only two operators, namely the flip of a Π-impedance-conservative system node which can be assumed to be the canonical form (6.24)

$$U_0 = \begin{bmatrix} R - V_0^* T V & V_0^* (I + T) \\ -(I + T)^* V_0 & -T \end{bmatrix},$$
along with a single positive operator necessarily of the diagonal form
\[ U_1 = \begin{bmatrix} V_1^*V_1 & 0 \\ 0 & I \end{bmatrix}. \]

Then the pencil \( U(w) \) has the form
\[ U(w) = \begin{bmatrix} R - V_0^*TV_0 & V_0^*(I + T) \\ -(I + T)^*V_0 & -T \end{bmatrix} + w \begin{bmatrix} V_1^*V_1 & 0 \\ 0 & I \end{bmatrix} \]
and the Bessmertnyí transfer-function realization for \( f \in H(\Pi, L(U)) \) becomes
\[ f(w) = R + wV_1^*V_1 + V_0^*(-T + (I + T)(wI - T)^{-1}(I + T)^*)V_0. \quad (6.38) \]

For simplicity, let us now assume that \( U = C \) (so \( f \) is scalar-valued). Then \( R \in C \) is just a purely imaginary number and \( V_0^*V_0 \) is an operator on \( C \) and so can be identified with a nonnegative real number \( \alpha \) (the image of the operator \( V_0^*V_0 \) acting on \( 1 \in C \)). From the representation (6.38), we see that there is no harm in cutting the state space \( X(0) \) (on which \( T \) is acting) down to the smallest subspace reducing for \( T \) which contains the range of the rank-1 operator \( V_0 \), i.e., we may assume that \( V_0 \cdot 1 \) is a cyclic vector for \( T \). Then the spectral theorem tells us that there is a measure \( \nu \) on the imaginary line \( iR \) so that \( T \) is unitarily equivalent to \( M_{-\zeta} : f(\zeta) \mapsto -\zeta f(\zeta) \)
acting on \( L^2(\nu) \). Without loss of generality we take the cyclic vector \( V_0^*1 \) to be the function \( \frac{1}{1-\zeta} \). As this function must be in \( L^2(\nu) \), we conclude that \( \frac{1}{1-\zeta} \in L^2(\nu) \), i.e., that \( \frac{1}{1+|\zeta|^2} d\nu(\zeta) \) is a finite measure. When this is done then we see that the adjoint operator \( V_0^* : L^2(\nu) \mapsto C \) is given by
\[ V_0^* : f(\zeta) \mapsto \int_{iR} \frac{1}{1-\zeta} f(\zeta) d\nu(\zeta). \]

Then the Bessmertnyí realization (6.38) for \( f(w) \) collapses to the integral representation formula
\[ f(w) = \alpha w + R + \int_{iR} \left[ \frac{\zeta}{1 + |\zeta|^2} + \frac{1}{\zeta + w} \right] d\nu(\zeta). \]

This agrees with the classical Nevanlinna integral representation for holomorphic functions taking the right halfplane into itself. Actually the formula is usually stated for holomorphic functions taking the upper halfplane into itself (see [24, Theorem 1 page 20]); however the correspondence \( \tilde{f}(\omega) \mapsto f(w) := -i\tilde{f}(iw) \) between \( \tilde{f} \) in the Nevanlinna class and \( f \) in the Herglotz class enables one to easily convert one integral representation to the other. We also point out that our proof (starting with the Herglotz representation (3.10) for the Herglotz function \( F(\zeta) \) on the disk and then separating out the 1-eigenspace of the unitary operator \( U \) in that representation) is just an operator-theoretic analogue of the proof of the integral representation formula in [24], where one starts with the integral Herglotz representation
\[ F(\zeta) = R + \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} d\mu(t) \]
and then separates out any point mass of \( \mu \) at the point 1 on the circle.
Remark 6.6. In recent work [4], Agler–Tully-Doyle–Young obtain a realization formula for the most general scalar-valued Nevanlinna–Agler function on the upper polyhalfplane. It is a straightforward matter to adjust the formulas to the right polyhalfplane setting which we have here and to extend the results to the operator-valued case. The result amounts to combining our formulas (6.12), (6.14) and (6.15), i.e.,
\[ f(w) = R + V^* \begin{bmatrix} I & 0 \\ 0 & -(I + T) \end{bmatrix} \left( P(w) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} \right)^{-1} \cdot \left( -P(w) \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) \left( I - (I - T)^{-1} \right) V. \] (6.39)

However the associated Bessmertny˘ pencil
\[ V(w) = \begin{bmatrix} R \\ \left( -P(w) \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) \left( I - (I - T)^{-1} \right) V \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -T \end{bmatrix} \]

lacks the symmetry properties of what we are calling a Herglotz–Agler operator pencil (see Definition 6.1). Hence there is no easy analogue of the proof of (3) ⇒ (2) in Theorem 6.3 and it is not all transparent from the presentation (6.39) why the resulting function (6.39) has a $\Pi^d$-Herglotz–Agler decomposition (5.2); indeed, it takes several pages of calculations in [4] (see Propositions 3.4 and 3.5 there) to arrive at this result.

There are other results in [4] and [5] using realization theory to characterize various types of boundary behavior of the function $f$ at infinity; we do not go into this topic here.

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DEPARTMENT OF MATHEMATICS, VIRGINIA TECH, BLACKSBURG, VA, 24061
E-mail address: joball@math.vt.edu

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, 3141 CHESTNUT ST., PHILADELPHIA, PA, 19104
E-mail address: dmitryk@math.drexel.edu