Stability estimates for the complex Monge-Ampère and Hessian equations

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Abstract
A new proof for stability estimates for the complex Monge-Ampère and Hessian equations is given, which does not require pluripotential theory. A major advantage is that the resulting stability estimates are then uniform under general degenerations of the background metric in the case of the Monge-Ampère equation, and under degenerations to a big class in the case of Hessian equations.

1 Introduction

Stability estimates for a non-linear partial differential equation are estimates for how much the solution can vary, given the size of the variation of the right hand side. Clearly, they are of great theoretical as well as practical importance. Such estimates had been obtained by Kolodziej [11] for the complex Monge-Ampère equation, by Dinew and Kolodziej [3] for complex Hessian equations, and by Dinew and Zhang [5] for Monge-Ampère equations when the background metric is not necessarily Kähler, but just big. In all cases, the proofs made extensive use of pluripotential theory and the background metric was fixed. It remained an open question whether these estimates can be established without pluripotential theory, and whether they can be made uniform under degenerations of the background metric, a situation which arises frequently in geometric applications.
In [8], the authors developed a method for obtaining sharp $L^\infty$ estimates for the complex Monge-Ampère equation without pluripotential theory. As explained in greater detail there, the method of [8] built on works of Wang, Wang, Zhou [13] and of Chen and Cheng [2], particularly on the last two authors’ idea of considering an associated complex Monge-Ampère equation. It achieved the stated goal of giving an alternate PDE proof of $L^\infty$ estimates for the complex Monge-Ampère equation, but it also went considerably beyond in, on one hand, applying to more general fully non-linear equations, and on the other hand, allowing the background metrics to degenerate. It can also give sharp gradient estimates [9], improving on the estimates in e.g. [1, 6, 7, 10, 12].

The main goal of the present paper is to obtain stability estimates for the complex Monge-Ampère and Hessian equations which are uniform under degenerations. We use the method of [8]. We recover in the process the stability estimates of [3, 5, 11], this time without pluripotential theory. Our estimates are also uniform under general degenerations of the background metric in the case of the Monge-Ampère equation, and under degenerations to a big class in the case of Hessian equations. Thus we answer in the positive both questions asked above.

2 Statement of the main results

Let $(X, \omega)$ be a compact Kähler manifold, $\chi$ a closed and non-negative $(1, 1)$-form, and set

$$\omega_t = \chi + t \omega, \quad t \in (0, 1].$$

(2.1)

Let $f, h \in C^\infty$ are smooth functions normalized by

$$\int_X e^f \omega^n = \int_X e^h \omega^n = \int_X \omega^n = 1,$$

and consider the following complex Hessian equations

$$\left( \omega_t + i \bar{\partial} \partial u_t \right)^k \wedge \omega^{n-k} = c_t e^f \omega^n, \quad \left( \omega_t + i \bar{\partial} \partial v_t \right)^k \wedge \omega^{n-k} = c_t e^h \omega^n$$

(2.2)

with the constants $c_t$ given by $c_t = \int_X \omega_t^k \wedge \omega^{n-k}$. We normalize $u_t$ and $v_t$ so that

$$\max_X (u_t - v_t) = \max_X (v_t - u_t).$$

We will consider three cases:

I : $k = n$, and $t \in (0, 1]$,  
II : $1 \leq k < n$, and $\chi = 0$, $t = 1$,  
III : $1 \leq k < n$, and $t \in (0, 1]$, $\chi$ is big i.e. $\int_X \chi^n > 0$,

which correspond respectively to the Monge-Ampère equations with degenerations, the $k$-th Hessian equation with a fixed background metric, and the $k$-th Hessian equation with degenerations. Different cases correspond to different choices of test functions, and constants. So we will treat the cases separately, when necessary.

For each case, we will make the following assumptions and choice of constants,

I : $\|e^h\|_{L^1(\log L)^{p_1}(\omega^n)}, \|e^f\|_{L^1(\log L)^{p_1}(\omega^n)} \leq K$, $p_1 > n$

II : $\|e^h\|_{L^{p_2}(\omega^n)}, \|e^f\|_{L^{p_2}(\omega^n)} \leq K$, $p_2 > \frac{n}{k}$, and $q_2 = \frac{p_2-1}{p_2-n/k}$
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III : \( \| e^h \|_{L^p(\omega^n)} , \| e^f \|_{L^p(\omega^n)} \leq K , \quad p_3 > \frac{n^2}{K} \), and \( q_3 = \frac{p_3-1}{p_3-n/k} \) for a fixed constant \( K > 0 \).

The equations (2.2) admit smooth solutions by [4, 14]. By [8] (also [3, 11]), under the above assumptions, the oscillations oscut and osctv of the solutions are uniformly bounded independently of \( t \) in all three cases I, II, and III. Let \( \beta_0 > 1 \) denote such an upper bound depending only on \( n, k, \omega, \chi, p_a \) and \( 10K \).

To start with, we will define a positive function \( \gamma_a(r) \) with \( \gamma_a(r) \to 0 \) as \( r \to 0 \). Each case has different choice of such a function. We define \( \gamma_a \) case by case:

I : \( \gamma_1(r) = r^{\frac{1}{q_1} - n + 1} \), where \( \delta_1 = \frac{p_1-n}{p_1n} < \frac{1}{n} \),

II : \( \gamma_2(r) = r^{(n+1)q_2-n} \),

III : \( \gamma_3(r) = r^{(n+1)q_3-n} \).

Theorem 1 Let the assumptions and notations be as above. Then we have in all three cases listed in (2.3)

\[
\sup_X |u_t - v_t| \leq C \| e^f - e^h \|_{L^1(\omega^n)}^{1/(n+3+\sigma_a)},
\]

where in each case, \( \sigma_a > 0 \) is the power of \( r \) in \( \gamma_a(r) \), i.e. \( \gamma_a(r) = r^{\sigma_a} \), and \( C \) is a constant depending only on \( n, k, \omega, \chi, K > 0 \) and \( p_a \).

We observe that this theorem improves on all results known so far. More specifically in case I, we get uniform stability for a degenerating family, and it does not even matter whether \( \chi \) is big or not. Kolodziej [11] proved this case for a fixed Kähler metric, and Dinew and Zhang [5] proved it for a fixed big class. In case II, we slightly sharpen the known stability result in [3], where the RHS in the inequality is \( \| e^f - e^h \|_{L^q} \) for some \( q' > 1 \), while we are able to prove the inequality for \( q' = 1 \). In case III, we obtain a uniform stability theorem for Hessian equations when the class remains big. This is completely new, and relies in particular on the uniform \( L^\infty \) estimate in [8] for solutions with degenerating big classes.

We note that the exponent \( 1/(n+3+\sigma_a) \) is not sharp in general. The sharp exponent can be obtained by replacing Lemma 1 below by a result from [5]. We leave the details to the interested readers.

3 Proof of theorem 1

For the proof, it is convenient to restate the theorem as follows. Under the above assumptions, if in each case we have

\[
\| e^f - e^h \|_{L^1(\omega^n)} \leq \gamma_a(r)r^{n+3},
\]

then there is a small \( r_0 > 0 \) such that for all \( 0 < r \leq r_0 \)

\[
\sup_X |u_t - v_t| \leq Cr,
\]

for all \( t \) listed in each case in (2.3) and \( C \) depends only on \( n, k, \omega, \chi, K > 0 \) and the corresponding \( p_a \) in each case. This is the version which we shall prove.

We begin with a lemma due to Kolodziej [11]. The proof is almost identical to that in [11], but since we would like to avoid the use of pluripotential theory, some additional smoothing is needed in the proof, and we provide a full proof of this lemma.
Choose a small \( \tilde{r}_0 \in (0, \frac{1}{10}) \) such that \( \gamma_a(\tilde{r}_0)\tilde{r}_0^n \leq \frac{1}{3} \). We then fix an \( 0 < r < \tilde{r}_0 \). We remark that all relevant constants are independent of \( r \). Later on we will choose an even smaller \( r_0 > 0 \).

By switching the roles of \( u_t \) and \( v_t \) if necessary, we may assume

\[
\int_{\{v_t \leq u_t\}} (e^{f_t} + e^{h_t}) \omega^n \leq 1.
\]

Denote \( E_j := \{v_t \leq u_t - j \beta_0 r\} \). The next lemma states that over the set \( E_2 \), the integral of \( e^h \) is small.

**Lemma 1** In each case \( a = I, II, III \), we have

\[
\int_{E_2} e^h \omega^n \leq C_0 \gamma_a(r) r^n,
\]

for some constant \( C_0 = 1 + \frac{2}{(\frac{1}{2})^{1/8} - 1} \).

**Proof** We calculate

\[
\int_{E_0} e^h \omega^n = \frac{1}{2} \int_{E_0} (e^f + e^h) + (e^h - e^f) \omega^n \leq \frac{1}{2} \left( 1 + \frac{1}{5} \right) = \frac{3}{5}.
\]

Take a sequence of positive smooth functions \( \tau_j \) that converge pointwise to \( \chi_{E_0} \) such that \( \tau_j \equiv 1 \) on \( E_0 \), and such \( \tau_j \) can be chosen for example as 1 in the tubular neighborhood \( T_{1/j}(E_0) \), as \( 1/j \) on \( X \setminus T_{2/j}(E_0) \), and as between \( 1/j \) and 1 in the annulus \( T_{2/j}(E_0) \setminus T_{1/j}(E_0) \). Hence we assume \( 0 < \tau_j \leq 1 \). Consider a sequence of smooth positive functions

\[
e^{h_j} = \frac{3}{2} \tau_j e^h + c_j (1 - \tau_j) e^h,
\]

where \( c_j > 0 \) are chosen so that \( \int_X e^{h_j} \omega^n = 1 \). It is not hard to see from (3.2) that for \( j >> 1 \), \( \frac{1}{20} \leq c_j \leq 3 \). Hence when \( j >> 1 \)

- In case \( I \), we have \( \|e^{h_j}\|_{L^1(\log L)^{p_1}} \leq 5 K \),
- In case \( II \), we have \( \|e^{h_j}\|_{L^{p_2}} \leq 5 K \),
- In case \( III \), we have \( \|e^{h_j}\|_{L^{p_3}} \leq 5 K \).

We solve the following Hessian equations which are known to admit unique smooth solutions \([4, 14]\),

\[
(\omega_t + i \partial \bar{\partial} \rho_j)^k \wedge \omega^{n-k} = c_t e^{h_j} \omega^n, \quad \text{sup} \ \rho_j = 0 \text{ and } \omega_t + i \partial \bar{\partial} \rho_j \in \Gamma_k,
\]

where \( \Gamma_k \) is the usual open convex cone in \( k \)-th Hessian equations. By the choice of \( \beta_0 \), we have \(-\beta_0 \leq \rho_j \leq 0 \) (see [8]). The following Newton inequality holds pointwise for any \( 1 \leq l \leq k \)

\[
\omega_{t, u_t}^l \wedge \omega_{t, \rho_j}^{k-l} \wedge \omega^{n-k} \geq \left( \frac{\omega_{t, u_t}^k \wedge \omega^{n-k}}{\omega^n} \right)^{l/k} \left( \frac{\omega_{t, \rho_j}^k \wedge \omega^{n-k}}{\omega^n} \right)^{(k-l)/k} \omega^n.
\]

Then on the set \( E_0 \setminus G \) where \( G = \{e^f \leq (1 - r^2)e^h\} \), we have

\[
\omega_{t, u_t}^l \wedge \omega_{t, \rho_j}^{k-l} \wedge \omega^{n-k} \geq c_t \left( 1 - r^2 \right)^{l/k} \left( \frac{3}{2} \right)^{(k-l)/k} e^h \omega^n.
\]
It follows that on $E_0 \setminus G$

$$\omega_{t,r\rho_j+(1-r)u_t}^k \wedge \omega^{n-k} = \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} r^{k-l}(1-r)^l \omega_{t,u_t}^j \wedge \omega_{t,r\rho_j}^{k-l} \wedge \omega^{n-k}$$

$$\geq c_t \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} r^{k-l}(1-r)^l(1-r^2)^{l/k} \left(\frac{3}{2}\right)^{(k-l)/k} e^h \omega^n$$

$$= c_t (1-r)(1-r^2)^{1/k} + r \left(\frac{3}{2}\right)^{1/k} e^h \omega^n \geq c_t (1 + b_0 r) e^h \omega^n,$$

(3.3)

where $b_0 = \frac{1}{2} \left(\left(\frac{3}{2}\right)^{1/k} - 1\right) > 0$, since $r$ is chosen to be small.

Note that by (3.1)

$$r^2 \int_G e^h \omega^n \leq \int_G (e^h - e^f) \omega^n \leq \gamma_a(r) r^{n+3},$$

which implies

$$\int_G e^h \omega^n \leq \gamma_a(r) r^{n+1}.$$  \hspace{1cm} (3.4)

Adding the same constant to $u_t$ and $v_t$, we may assume without loss of generality $-\beta_0 \leq u_t \leq 0$. The following inclusion relation holds from the definition

$$E_2 \subset E := \{ v_t \leq r \rho_j + (1-r)u_t - \beta_0 r \} \subset E_0.$$  

All functions involved are smooth so by the comparison principle and (3.3),

$$c_t (1 + b_0 r) \int_{E \setminus G} e^h \omega^n \leq \int_E \omega_{t,r\rho_j+(1-r)u_t}^k \wedge \omega^{n-k}$$

$$\leq \int_E \omega_{v_t}^k \wedge \omega^{n-k} = c_t \int_{E \setminus G} e^h \omega^n + c_t \int_G e^h \omega^n.$$

Combined with (3.4) this implies

$$\int_{E \setminus G} e^h \omega^n \leq \frac{1}{b_0} \gamma_a(r) r^n.$$

It follows that

$$\int_{E_2} e^h \omega^n \leq \int_{E \setminus G} e^h \omega^n + \int_G e^h \omega^n \leq (1 + \frac{1}{b_0}) \gamma_a(r) r^n.$$  

The Lemma is proved. \hfill \Box

We now come to the proof of Theorem 1 proper. We normalize $u_t$ as in the proof of Lemma 1. For $s \geq 0$, we set $\Omega_s = \{ v_t \leq (1-r)u_t - 3 \beta_0 r - s \}$. Note that $\Omega_s \subset E_2$ for any $s \geq 0$.

We follow the same strategy as in [8]. We choose a sequence of smooth positive functions $\eta_j : \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\eta_j(x) = x + \frac{1}{j}, \quad \text{when } x \geq 0,$$  \hspace{1cm} (3.5)
and 
\[ \eta_j(x) = \frac{1}{2^j}, \quad \text{when } x \leq -\frac{1}{j}, \]
and \( \eta_j(x) \) lies between 1/2 and 1/j for \( x \in [-1/j, 0] \). Clearly \( \eta_j \to \eta_\infty(x) = x \cdot \chi_{\mathbb{R}_+}(x) \) pointwise as \( j \to \infty \).

We solve the complex Monge-Ampère equations

\[
(\omega_t + i \partial \bar{\partial} \psi_j)^n = c_t^n/j \eta_j(-v_t + (1 - r)u_t - 3\beta_0r - s) A_{s,j} e^{\frac{n}{\Lambda} h} \omega^n, \quad \limsup \psi_j = 0.
\]

As \( j \to \infty \) we have by the dominated convergence theorem

\[
A_{s,j} = \frac{c_t^n}{V_t} \int_X (\eta_j(-v_t + (1 - r)u_t - 3\beta_0r - s)) e^{\frac{n}{\Lambda} h} \omega^n \to A_s,
\]

where the constant \( A_s \) is defined by

\[
A_s := \frac{c_t^n}{V_t} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0r - s) e^{\frac{n}{\Lambda} h} \omega^n.
\]

Consider \( \Phi = -\varepsilon (-\psi_j + \Lambda)^{\frac{n}{\pi + 1}} + (-v_t + (1 - r)u_t - 3\beta_0r - s) \) where

\[
\varepsilon = \left( \frac{k(n+1)}{n^2} \right)^{\frac{n}{\pi + 1}} c(n,k) \left( \frac{1}{\pi + 1} A_{s,j} \right)^{\frac{1}{\pi + 1}}, \quad \text{where } c(n,k) \text{ is the one in (3.6).}
\]

\[
\Lambda = \left( \frac{n}{n+1} \right)^{n+1} c(n,k) \frac{A_{s,j}}{r^{n+1}}
\]

Suppose \( \sup \Phi = \Phi(x_0) \) for some point \( x_0 \) in \( X \). If \( x_0 \notin \Omega_s^c \), then by definition \( \Phi(x_0) < 0 \). Otherwise \( x_0 \in \Omega_s^c \). We calculate as in [8]. First note that for \( G^{ij} = \frac{\partial}{\partial (\omega_t,v_t)_j} \log \sigma_k(\omega_t,v_t) \)

\[
\det G^{ij} \geq c(n,k)c_i^\frac{n}{\Xi} e^{-\frac{\varepsilon}{\Lambda}}
\]

for some computable constant \( c(n,k) > 0 \). It follows that at \( x_0 \)

\[
0 \geq G^{ij}(\Phi)_{i,j}(x_0)
\]

\[
\geq \frac{n^2\varepsilon}{n+\delta} (-\psi_j + \Lambda)^{-\frac{1}{\pi + 1}} \left( \det G \cdot \det \omega_t, \psi_j \right)^{1/n} - k + \left( r - \frac{ne}{n+1} \Lambda^{-\frac{1}{\pi + 1}} \right) G^{ij}(\omega_t)_{i,j}
\]

\[
\geq \frac{n^2\varepsilon}{n+1} (-\psi_j + \Lambda)^{-\frac{1}{\pi + 1}} c(n,k)^{1/n} \left( \frac{-v_t + (1 - r)u_t - 3\beta_0r - s}{A_{s,j}} \right)^{1/n} - k.
\]

By the choice of \( \varepsilon \) and \( \Lambda \), we deduce that \( \Phi(x_0) \leq 0 \). Thus \( \Phi \leq 0 \) on \( X \) and this implies

\[
\int_{\Omega_s} \exp \left\{ c_0 \left( \frac{-v_t + (1 - r)u_t - 3\beta_0r - s}{A_{s,j}^{1/(n+1)}} \right)^{\frac{n+1}{n}} \right\} \omega^n \leq \int_{\Omega_s} \exp \left( -\alpha_0 \psi_j + \alpha_0 \Lambda \right) \omega^n
\]

\[
\leq C \exp \left\{ \frac{A_{s,j}}{r^{n+1}} \right\}
\]

(3.7)
for some small $c_0 = c_0(n, k, \omega, \chi) > 0$, $C = C(n, k, \omega, \chi) > 0$, and $\alpha_0$ a fixed number satisfying $0 < \alpha_0 < \alpha(X, \omega)$, where $\alpha(X, \omega)$ is the $\alpha$-invariant of $(X, \omega)$. Letting $j \to \infty$ gives

$$
\int_{\Omega_s} \exp \left\{ c_0 \left( \frac{(-v_t + (1 - r)u_t - 3\beta_0 r - s)}{A_s^{1/(n+1)}} \right)^{\frac{n+1}{n}} \right\} \omega^n \leq C \exp \left( \frac{A_s}{r^{n+1}} \right). \quad (3.8)
$$

To estimate $\frac{A_s}{r^{n+1}}$ in (3.8), we need to consider separately the three cases I, II and III.

- **Case I.** In this case $k = n$ and $c_t = V_t$. By Lemma 1 we deduce that

$$
\frac{A_s}{r^{n+1}} = \frac{c_t^{n/k}}{V_t} \cdot \frac{1}{r^{n+1}} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0 r - s)e^{\frac{2}{h}} \omega^n
$$

$$
\leq \frac{1}{r^{n+1}} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0 r)e^{\frac{2}{h}} \omega^n
$$

$$
\leq \frac{C(n, \beta_0)}{r^{n+1}} \int_{E_2} e^{\frac{2}{h}} \omega^n \leq C_0C(n, \beta_0)\gamma_1(r)r^{-1}
$$

$$
\leq C(n, \beta_0) \text{ by the choice of } \gamma_1(r).
$$

- **Case II.** In this case, under our normalization, $V_1 = c_1 = 1$. As in Case I, we have

$$
\frac{A_s}{r^{n+1}} = \frac{1}{r^{n+1}} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0 r - s)e^{\frac{2}{h}} \omega^n
$$

$$
\leq \frac{1}{r^{n+1}} \int_{E_2} e^{\frac{2}{h}} \omega^n = \frac{C(n, \beta_0)}{r^{n+1}} \int_{E_2} e^{(\frac{2}{h} - 1)h} e^{\frac{2}{h}} \omega^n
$$

$$
\leq \frac{C(n, \beta_0)}{r^{n+1}} \left( \int_{E_2} e^{\frac{q_2}{2} - (\frac{2}{h} - 1)h} e^{\frac{2}{h}} \omega^n \right)^{1/q_2} \left( \int_{E_2} e^{h} \right)^{1/q_2}
$$

$$
\leq \frac{C(n, \beta_0)}{r^{n+1}} \left( \int_{E_2} e^{p_2h} \omega^n \right)^{1/q_2} \left( \int_{E_2} e^{h} \right)^{1/q_2}
$$

$$
\leq C(n, k, \beta_0, K)\gamma_2(r)^{\frac{1}{q_2}}r^{-\frac{n}{q_2} - n - 1} = C(n, k, \beta_0, K),
$$

where in the third line we have used the fact that $\Omega_s \subset E_2$, and the integrand is nonnegative and bounded above by a constant $C(n, \beta_0)$, where we recall $\beta_0 > 1$ is a fixed upper bound of $\|u_t\|_{L^\infty}$ and $\|v_t\|_{L^\infty}$; in the fourth line $q_2^*$ satisfies $\frac{1}{q_2} + \frac{1}{q_2^*} = 1$; and in the last equation we use the choice of the function $\gamma_2(r)$.

- **Case III.** We note that since $[\chi]$ is big, $V_t \geq \int_{\chi} \chi^n > 0$, hence $\frac{c_t^{n/k}}{V_t} \leq C_{\omega, \chi}$ for a uniform $C_{\omega, \chi} = C_{\omega, \chi}(n, k) > 0$ which we will fix throughout the proof below. Then we have

$$
\frac{A_s}{r^{n+1}} = \frac{c_t^{n/k}}{V_t} \cdot \frac{1}{r^{n+1}} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0 r - s)e^{\frac{2}{h}} \omega^n
$$

$$
\leq \frac{C(n, \omega, \chi, k, \beta_0)}{r^{n+1}} \left( \int_{E_2} e^{p_3 h} \omega^n \right)^{1/q_3} \left( \int_{E_2} e^{h} \right)^{1/q_3}
$$

$$
\leq C(n, k, \omega, \chi, K)\gamma_3(r)^{\frac{1}{q_3}}r^{-\frac{n}{q_3} - n - 1} = C(n, k, \omega, \chi, K),
$$

\(\square\)
where \( \frac{1}{q_3} + \frac{1}{q_1^2} = 1 \) and in the last identity we use the choice of the function \( \gamma_3(r) \).

So for all cases I, II and III, we get from (3.8) that

\[
\int_{\Omega_s} \exp \left\{ c_0 \left( -v_t + (1 - r)u_t - 3\beta_0r - s \right) A_s^{1/(n+1)} \right\} \omega^n \leq C, \tag{3.9}
\]

for some constant \( C > 0 \) depending on \( n, k, \chi, \omega, K \) and the exponents \( p_1, p_2, p_3 \) in each case, respectively. In particular this \( C \) is independent of the choice of \( r \in (0, \tilde{r}_0) \).

We choose \( p > n \) as \( p = p_1 \) in case I, and arbitrary and large \( p > n \) in cases II and III.

Define \( \eta : \mathbb{R}_+ \to \mathbb{R}_+ \) by \( \eta(x) = (\log (1 + x))^p \). Note that \( \eta \) is a strictly increasing function with \( \eta(0) = 0 \), and let \( \eta^{-1} \) be its inverse function. If we let

\[
\Psi := \frac{c_0}{2} \left( -v_t + (1 - r)u_t - 3\beta_0r - s \right) A_s^{1/(n+1)}, \tag{3.10}
\]

then we have for any \( z \in \Omega_s \), by the generalized Young’s inequality with respect to \( \eta \),

\[
\Psi(z)^p e^{\frac{\eta h(z)}{n}} \leq \int_0^{\Psi(z)^p} \eta(x) dx + \int_0^{\eta^{-1}(y)} \eta^{-1}(y) dy 
\leq e^{\frac{\eta h(z)}{n}} (1 + |h(z)|)^p + C(p) e^{2\eta h(z)}.
\]

We integrate both sides in the inequality above over \( z \in \Omega_s \), and get by (3.9) that

\[
\int_{\Omega_s} \Psi(z)^p e^{\frac{\eta h(z)}{n}} \omega^n \leq \| e^h \|_{L^{n/k}(\log L)^p} + C,
\]

where the constant \( C > 0 \) depends only on \( n, k, \omega, \chi, p, K \). In view of the definition of \( \Psi \), this implies

\[
\int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0r - s)^{\frac{(n+1)p}{n}} e^{\frac{\eta h}{n}} \omega^n \leq C A_s^{\frac{p}{n}} \left( \| e^h \|_{L^{n/k}(\log L)^p} + 1 \right). \tag{3.11}
\]

It follows from the Hölder inequality that

\[
A_s = \frac{c_t^{n/k}}{V_t} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0r - s) e^{\frac{\eta h}{n}} \omega^n 
\leq \left( \frac{c_t^{n/k}}{V_t} \int_{\Omega_s} (-v_t + (1 - r)u_t - 3\beta_0r - s)^{\frac{(n+1)p}{n}} e^{\frac{\eta h}{n}} \omega^n \right)^{\frac{n}{n+q}} \left( \frac{c_t^{n/k}}{V_t} \int_{\Omega_s} e^{\frac{\eta h}{n}} \omega^n \right)^{\frac{1}{q}} 
\leq C A_s^{\frac{p}{n+q}} \left( \| e^h \|_{L^{n/k}(\log L)^p} + 1 \right)^{\frac{n}{n+q}} \left( \frac{c_t^{n/k}}{V_t} \int_{\Omega_s} e^{\frac{\eta h}{n}} \omega^n \right)^{\frac{1}{q}},
\]

where \( q > 1 \) satisfies \( \frac{n}{p(n+1)} + \frac{1}{q} = 1 \), i.e. \( q = \frac{p(n+1)}{p(n+1)-n} \). The inequality above yields

\[
A_s \leq C \left( \| e^h \|_{L^{n/k}(\log L)^p} + 1 \right)^{1/p} \left( \frac{c_t^{n/k}}{V_t} \int_{\Omega_s} e^{\frac{\eta h}{n}} \omega^n \right)^{\frac{1+n}{qn}} = B_0 \left( \frac{c_t^{n/k}}{V_t} \int_{\Omega_s} e^{\frac{\eta h}{n}} \omega^n \right)^{1+\delta_0}. \tag{3.12}
\]

Observe that the exponent of the integral on the right hand of (3.12) satisfies

\[
\frac{1+n}{qn} = \frac{pn+p-n}{pn} = 1+\delta_0 > 1, \quad \text{for} \quad \delta_0 := \frac{p-n}{pn} > 0.
\]
We remark that $\delta_0$ can be chosen to be close to $1/n$ in cases II and III by picking $p$ large enough. Furthermore, we note that

$$B_0 := C \left( \|e^b\|_{L^{n/(\log L)^p}} + 1 \right)^{1/p} \quad (3.13)$$

is a constant depending only on $n$, $k$, $\omega$, $\chi$, $K$, and the exponents $p_1$, $p_2$, $p_3$ in each case, respectively, and in particular, it is independent of $r$ with $r \in (0, \bar{r}_0)$.

If we define

$$\phi(s) = \frac{c_{1 \ell}^{n/k}}{V_\ell} \int_{\Omega_s} e^{\frac{n\ell}{\chi} \omega^n},$$

then (3.12) shows that if $\Omega_{s+s'} \neq \emptyset$ then

$$s' \phi(s + s') \leq B_0 \phi(s)^{1+\delta_0}, \quad \text{for all } s' \geq 0 \text{ and } s \geq 0. \quad (3.14)$$

We now choose $r_0 < \bar{r}_0$ small in each case as follows.

**Case I.** We choose $r_0 > 0$ small so that for $r \in (0, r_0)$

$$B_0 \phi(0)^{\delta_0} \leq B_0 \left( \int_{E_2} e^{\frac{n\ell}{\chi} \omega^n} \right)^{\delta_0} \leq B_0 C_0^{\delta_0} (\gamma_1(r) r^n)^{\delta_0} \leq B_0 C_0^{\delta_0} (\gamma_1(r_0) r_0^n)^{\delta_0} \leq \frac{1}{2} \quad (3.15)$$

and $\phi(0) \leq C_0 \gamma_1(r) r^n < \bar{C} r^{1/\delta_0}$ by Lemma 1 for some uniform $\bar{C}$.

**Case II.** We choose $r_0 > 0$ small so that for all $r \in (0, r_0)$

$$B_0 \phi(0)^{\delta_0} \leq B_0 \left( \int_{E_2} e^{\frac{n\ell}{\chi} \omega^n} \right)^{\delta_0} \leq B_0 \left( \int_{E_2} e^{p_2 \frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_2^*} \left( \int_{E_2} e^{\frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_2} \leq B_0 C_0^{\delta_0/q_2} \left( \int_{E_2} e^{p_2 \frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_2^*} (\gamma_2(r) r^n)^{\delta_0/q_2} \leq B_0 C_0^{\delta_0/q_2} \left( \int_{E_2} e^{p_2 \frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_2^*} (\gamma_2(r_0) r_0^n)^{\delta_0/q_2} \leq \frac{1}{2},$$

where $\frac{1}{q_2} + \frac{1}{q_2^*} = 1$ and we also have

$$\phi(0) \leq C_0^{1/q_2^*} \left( \int_{E_2} e^{p_2 \frac{n\ell}{\chi} \omega^n} \right)^{1/q_2^*} (\gamma_2(r) r^n)^{1/q_2} < \bar{C} r^{1/\delta_0}$$

for some uniform $\bar{C} > 0$ by the definition of $\gamma_2(r)$.

**Case III.** We choose $r_0 > 0$ small so that for $r \in (0, r_0)$

$$B_0 \phi(0)^{\delta_0} \leq B_0 C_{\omega,\chi}^{\delta_0} \left( \int_{E_2} e^{\frac{n\ell}{\chi} \omega^n} \right)^{\delta_0} \leq B_0 C_{\omega,\chi}^{\delta_0} \left( \int_{E_2} e^{p_3 \frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_3^*} \left( \int_{E_2} e^{\frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_3} \leq B_0 C_{\omega,\chi}^{\delta_0/q_3} C_0^{\delta_0/q_3} \left( \int_{E_2} e^{p_3 \frac{n\ell}{\chi} \omega^n} \right)^{\delta_0/q_3} (\gamma_3(r) r^n)^{\delta_0/q_3}$$
\[
\leq B_0 C_{\omega, \chi} C_0^{\delta_0/3} \left( \int_{E_2} e^{p_3 h} \omega^n \right)^{\delta_0/3} \left( \gamma_3 (r_0) r_0^n \right)^{\delta_0/3} \leq \frac{1}{2},
\]

where \( \frac{1}{q_3} + \frac{1}{q_3} = 1 \) and we also have

\[
\phi(0) \leq C_{\omega, \chi} C_0^{1/3} \left( \int_{E_2} e^{p_3 h} \omega^n \right)^{1/3} \left( \gamma_3 (r) r^n \right)^{1/3} < \hat{C} r^{1/\delta_0}
\]

by the choice of \( \gamma_3 (r) \).

It is clear that in all cases, \( r_0 \) and \( \hat{C} \) depend only on the given data, namely, \( n, k, \omega, \chi, K \) and \( p_a \), and we have \( B_0 \phi(0) \delta_0 \leq \frac{1}{2} \) and \( \phi(0) \leq \hat{C} r^{1/\delta_0} \).

Define a sequence of increasing real numbers \( (s_j) \) inductively such that \( s_0 = 0 \) and

\[
s_{j+1} = \sup \left\{ s > s_j \mid \phi(s) > \frac{1}{2} \phi(s_j) \right\}.
\]

Then we can show that (see [8]) \( \phi(s_j) \leq 2^{-j} \phi(s_0) \) and

\[
s_{j+1} - s_j \leq 2 B_0 2^{-j} \phi(0)^{\delta_0}.
\]

Thus the limit \( S_\infty = \lim_{j \to \infty} s_j \) satisfies

\[
S_\infty \leq \frac{2 B_0}{1 - 2^{-\delta_0}} \phi(0)^{\delta_0} \leq \frac{2 B_0 \delta_0}{1 - 2^{-\delta_0}} r = \hat{C} r.
\]

Hence the set \( \Omega_{\hat{C} r} = \emptyset \), and we conclude that

\[
v_t \geq (1 - r) u_t - 3 \beta_0 r - \hat{C} r, \quad \text{or equivalently} \quad v_t - u_t \geq -C r,
\]

for some uniform constant \( C > 0 \) depending only on the given data. By the normalization \( \max(u_t - v_t) = \max(v_t - u_t) \), it is clear that \( v_t - u_t \leq C r \). The proof of Theorem 1 is complete.

**Data sharing** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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