The classical and quantum dynamics of the inhomogeneous Dicke model and its Ehrenfest time

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We show that in the few-excitation regime the classical and quantum time-evolution of the inhomogeneous Dicke model for \( N \) two-level systems coupled to a single boson mode agree for \( N \gg 1 \). In the presence of a single excitation only, the leading term in an \( 1/N \)-expansion of the classical equations of motion reproduces the result of the Schrödinger equation. For a small number of excitations, the numerical solutions of the classical and quantum problems become equal for \( N \) sufficiently large. By solving the Schrödinger equation exactly for two excitations and a particular inhomogeneity we obtain \( 1/N \)-corrections which lead to a significant difference between the classical and quantum solutions at a new time scale which we identify as an Ehrenfest time, given by \( \tau_E = \sqrt{N/\langle g^2 \rangle} \), where \( \sqrt{\langle g^2 \rangle} \) is an effective coupling strength between the two-level systems and the boson.

I. INTRODUCTION

The recent experimental advances on cold atoms in optical cavities, Bose-Einstein condensation of exciton polaritons, and observation of vacuum Rabi oscillations in semiconductor microcavities renewed interest in light-matter interaction in the quantum coherent regime. These studies were motivated by an observation made by Dicke long ago who realized that radiation from \( N \) identical two-level systems (spins \( 1/2 \)) cannot be treated as a sum of \( N \) independent radiative processes but rather as a collective quantum phenomenon that involves all \( N \) spins and a photon mode even on the level of perturbation theory. Also, several schemes based on light-matter interaction to couple spatially separated spins that had been originally proposed as an element of a quantum computing device were recently improved by a suggestion to use qubits constructed out of many spins to enhance coupling with the optical mode due to the superradiant effect.

For instance, considerable attention was paid imanently to the \( \sqrt{N} \)-enhancement of the light-matter coupling. In typical set-ups the spins are spatially separated, therefore the excitation energies of different spins may be different as they are affected by local forces that typically vary across the sample. The coupling strength to the light mode also varies as different spins are inhomogeneous but with constant Zeeman energies. If the Zeeman energies \( \epsilon_j \) are also inhomogeneous but spread narrower than the threshold given by \( \Omega \) this single frequency acquires a small Lamb-like shift, whereas for a spread exceeding \( \Omega \) the boson decays completely in time.

In this paper we show that the solution to the classical Hamilton equations of motion matches the solution of the time-dependent Schrödinger equation when the number of spins is large, i.e. \( N \gg 1 \), while the number of excitations \( p \) is still small, i.e. \( p \ll N \). For a single excitation \( (p=1) \) the leading order in an \( 1/N \)-expansion of the classical equations agrees with the quantum one. For a few excitations such correspondence does not hold, but for \( p=2,3 \) the numerical solutions of both classical equations of motion and Schrödinger equation agree for \( N \gg 1 \). It is plausible to assume that in leading \( 1/N \)-order the same correspondence holds for \( p \gg 3 \). The numerical treatment of the Schrödinger equation with a large number of spins is possible since the Fock space scales only as a power of \( N \) \( (N^2, N^3, \ldots) \) in the few-excitation subspaces.

As the classical equations of motion for \( p > 1 \) can also be mapped on the Schrödinger equation in the single excitation subspace in leading \( 1/N \)-order the already available quantum result can be used to analyze the classical equations of motion for few excitations \( (p \ll N) \). For \( p \) excitations with \( p > 1 \) we obtain the dynamics by simply rescaling the solution derived in by \( \sqrt{N} \). This extends the single-excitation quantum solution to the case of few excitations when \( N \gg 1 \).

To assess the validity of the classical approximation for \( p > 1 \) excitations we solve the Schrödinger equation exactly in the two-excitation subspace with inhomogeneity in the coupling constants only and compare it with the classical solution. When \( N \) is small both solutions are completely different. For large \( N \) we perform
an 1/N-expansion of the quantum solution and recover the classical result in leading order. Subleading 1/N-corrections cause deviations between quantum and classical dynamics which become significant at a large time scale $\tau_E = \sqrt{N/(g^2)}$ for $p \ll N$. We refer to this time scale as an Ehrenfest time, defined here as the time where the quantum dynamics starts to differ from the classical dynamics.

Also, having found a quantum solution for $p = 2$ we study it separately and in particular compare it with the $p = 1$ quantum dynamics. We find that inhomogeneity of the coupling constants results in a different spectrum when $N$ is finite: in the subspace with $p = 1$ there is only one harmonic mode with a single frequency in the time-dependent occupation number of the boson, and for $p = 2$ there are $N$ discrete harmonic modes that form a continuum spectrum in the limit of large $N$. Such a mechanism can lead to destructive interference, thus to decay, of the excitations caused solely by the inhomogeneity of the coupling constants when $p > 1$. But, as pointed out already, for $p = 2$ we find that the leading 1/N-term recovers the single frequency dynamics in accordance with the classical solution. The decay due to inhomogeneous coupling constants thus manifests itself only in the first subleading 1/N-correction. We find that this contribution is an oscillatory mode with frequency $\Omega/2$ and a slowly decaying envelope. The decay behavior is essentially non-exponential with a long power-law tail and the decay time is $\tau_D \sim \sqrt{N/(g^2)}$, where $\sqrt{g^2}$ is a characteristic coupling. This decay occurs on the same time scale as the Ehrenfest time $\tau_E$ defined above. Thus, it can be described correctly only by the Schrödinger equation (and not by the classical one).

In our theoretical analysis we assume the following ideal experiment. The spin bath is prepared in the ground state, e.g. dynamically or by the thermal cooling. The non-equilibrium dynamics of the boson is then initialized by a short radiation pulse from an external source which populates the boson mode with a few excitations like in $^{18, 20}$. The dissipation of the boson mode, e.g. leakage of the photons through the mirrors that define an optical cavity can be used to detect the dynamics, similarly to the measurements performed on semiconductor quantum well microcavities $^{23, 21}$, for the limiting case where the cavity leakage time exceeds the internal time scale.

The rest of the paper is organized as follows. In Section II we discuss general properties of the inhomogeneous Dicke model. In Section III we quote the already known solution to the Schrödinger equation in the single excitation subspace. In Section IV we construct the classical analog of the inhomogeneous Dicke model. Section V contains the exact solution of the Schrödinger equation in the two-excitation subspace for the inhomogeneous couplings only. In Section VI we compare the numerical solution of the classical and the quantum equations of motions for two and three excitations in the limit of many spins. Section VII contains a discussion of applicability of the classical approximation. In the Appendix we give some details on the calculation of the 1/N-correction.

II. INHOMOGENEOUS DICKE MODEL

The Hamiltonian for the Dicke model that describes the interaction between a set of $N$ spins $1/2$ with excitation energies $\epsilon_j$ and a single bosonic mode of frequency $\omega$ is given by

$$H = \omega b^\dagger b + \sum_{j=1}^{N} \epsilon_j S_j^z + \sum_{j=1}^{N} g_j (S_j^+ b + S_j^- b^\dagger),$$

(1)

where $S_j^\pm = S_j^x \pm iS_j^y$, $S_j^z$ are spin 1/2 operators, $b (b^\dagger)$ are the standard Bose annihilation (creation) operators. The coupling constants $g_j$ are typically given as dipole matrix elements and thus are, in general, complex numbers. Since their phases can be eliminated by a unitary transformation, we treat $g_j$ as real and positive numbers.

In the present paper we assume that the boson mode is tuned in resonance with the spins $\langle \epsilon_j \rangle = \omega$, where $\langle \ldots \rangle = \sum_j \ldots /N$. If the boson mode is strongly detuned, $|\langle \epsilon_j \rangle - \omega| \gg \sqrt{(g^2)}$, the interaction between them is weak and the model Eq. (1) can be analyzed perturbatively $^{21}$. Also note that the inhomogeneities of $g_j$ and/or $\epsilon_j$ forbids to represent the Hamiltonian Eq. (1) in terms of the total angular momentum operators $J_\alpha = \sum_j S_j^\alpha$, $\alpha = x,y,z$. .

The total number of spin-boson excitations, $L = n + \sum_j S_j^z$, is conserved by the model Eq. (1), where $n = b^\dagger b$ is the bosonic occupation number. The eigenvalue $c$ of $L$ labels the subspace of the Hamiltonian with a given total number of excitations.

We restrict ourselves to a small number of excitations, $p \ll N$. In the following we assume that the spins can be prepared in the ground state with each spin in its low Zeeman state. The bosonic mode is assumed to be occupied by $p$ bosons initially, the time evolution is restricted to the subspace with $c = -N/2 + p$. Then the leakage of the boson mode to the outside world can be used to monitor the time dynamics of the system by detecting the leaked mode at given subsequent instances in time.

III. SINGLE EXCITATION

The time dynamics of Eq. (1) for a single excitation was analyzed in detail in Ref. $^{17}$. Here we only quote the explicit form of the corresponding Schrödinger equation and the main results derived from it.

The time evolution is restricted the the subspace with $c = -N/2 + 1$ and is described by the general state

$$|\Psi (t) \rangle = \alpha (t) |\uparrow, 1 \rangle + \sum_{j=1}^{N} \beta_j (t) |\uparrow_j, 0 \rangle,$$

(2)
where \( \alpha(t) \) and \( \beta_j(t) \) are normalized amplitudes, 
\[ |\alpha(t)|^2 + \sum_j |\beta_j(t)|^2 = 1, \]
of finding either a state with one boson and no spin excitations present or a state with no boson and the \( j \)-th spin excited (flipped). As initial condition we will assume throughout (with one exception discussed at the end) that initially only bosonic excitations are present while each spin is in its individual ground state, i.e. \( \alpha(t=0) = 1 \). The state \( |\Psi(t)\rangle \) from Eq. (2) describes then the time evolution of an initial product state \( |\beta, 1\rangle \) into an entangled state formed by a coherent superposition of \( N-1 \) states, where each \( |\beta_j, 0\rangle \) contains an excited spin and no boson. This entangled state can be viewed as a (para-) magnon state in the uniform limit. In other words, the initial bosonic excitation gets coherently spread out over the entire system in course of time.

Inserting \( |\Psi(t)\rangle \) from Eq. (2) into the time-dependent Schrödinger equation we get
\[ -i\dot{\alpha}(t) = \sum_j g_j \beta_j(t), \]
\[ -i\dot{\beta}_k(t) = (\epsilon_j - \omega) \beta_k(t) + g_k \alpha(t). \]
This set of coupled equations can be solved explicitly via Laplace transformation. We use the same approach to solve the Schrödinger equation in the two excitation subspace in Section V of this paper.

If the number of spins is large, \( N \gg 1 \), the sum over \( j \) in the exact solution of Eq. (3) can be substituted by an integral. In this continuum limit the discrete set of \( \epsilon_j \) and \( g_j \) become continuous variables characterized by distribution functions \( Q(g) \) and \( P(\epsilon) \). Any distribution function of \( g \) results only in a renormalized coupling constant \( \sqrt{\langle g^2 \rangle} \) and the dynamics of the boson is not affected in any other way.

Different distribution functions of \( \epsilon \) result in qualitatively different regimes of the dynamics. Let us choose \( P(\epsilon) \) as a rectangular pulse shape of width \( \Delta \) centered around \( \omega \),
\[ P(\epsilon) = \theta(-\epsilon + \omega + \Delta/2) \theta(\epsilon - \omega + \Delta/2), \]
where \( \theta(x) \) is the Heaviside step-function. It was shown that when the inhomogeneity is below a certain threshold, \( \Delta/\Omega \ll 1 \), where \( \Omega = \sqrt{\langle g^2 \rangle} \) is the collective Rabi frequency, the boson excitation, \( \langle n \rangle = |\alpha(t)|^2 \), does not decay, i.e.
\[ \langle n(t) \rangle = \cos^2(\Omega t). \]

The corrections to this result are small and are on the order of \( \Delta/\Omega \). In the opposite limit, \( \Delta/\Omega \gg 1 \), the spins act as the thermal bath at zero temperature. The bosonic excitation decays completely and exponentially,
\[ \langle n(t) \rangle = \exp(-t/t_2), \]
with the decay time \( t_2 = 2\Delta/\pi\Omega^2 \). In the intermediate regime, \( \Delta \simeq \Omega \), the decay is partial and the decay law is a combination of exponential and inverse-power laws.

\section*{IV. CLASSICAL ANALOGY}

Here we construct a classical version of the inhomogeneous Dicke model. Using Dirac’s analogy, we change the boson operator \( b \) in the model Eq. (1) to a classical complex variable \( a = a_x + ia_y \), and the spin operators \( S_j \) to a set of \( N \) vectors \( C_j = (C^x_j, C^y_j, C^z_j) \) of length \( |C_j| = 1/2 \). These classical degrees of freedom obey the Poisson bracket relations which are obtained from the bosonic and spin commutation relations via the ansatz
\[ [\cdot, \cdot] \rightarrow -i[\cdot, \cdot]: [C_\alpha, C_\beta]_{cl} = -\epsilon_\alpha \beta_\gamma C_\gamma, \]
and \( [a, a^*]_{cl} = i \).

The Hamilton equation of motion for the \( j \)-th spin, \( \dot{C}_j = [H, C_j]_{cl} \), is a Bloch equation
\[ \dot{C}_j = B_j \times C_j, \]
where the in-plane component of the effective magnetic field is the complex bosonic field, and the perpendicular component is the single spin excitation energy, \( B_j = (2g_j a_x, 2g_j a_y, \epsilon_j - \omega) \). The Hamilton equation of motion for \( a \) is a feedback to the bosonic field from the in-plane component of all spins,
\[ \dot{a} = -i \sum_j g_j C_j^z, \]
where \( C_j^z = C_j^x - iC_j^y \). Generally, these differential equations can be solved numerically with the initial conditions \( C_j(0) = (0, 0, -1/2) \) and \( a(0) = \sqrt{\rho} \) to obtain the time-dependent solution \( C_j(t) \) and \( a(t) \) explicitly. The time-dependent value of the bosonic field is \( \langle n(t) \rangle = |a(t)|^2 \).

For a small number of excitations, \( p \ll N \), Eq. (7) simplifies. The quantity \( L = |a|^2 + \sum_j C_j^z = -N/2 + p \) is conserved during the evolution governed by Eqs. (7, 8).

Thus, at any instance of time \( \sum_j C_j^z \approx -N/2 \), i.e. if the dynamics starts with only a few bosonic excitations, the spins cannot ‘flip’ during the evolution. Using the approximation \( C_j^z(t) \approx -1/2 \), the equation for \( C_j^z(t) \) drops out from Eq. (7) and the remaining two equations are
\[ \dot{C}_j^z = -i(\epsilon_j - \omega) C_j^z - ig_j a. \]

When \( p = 1 \), the Hamilton Eqs. (9, 10) with the initial conditions above coincide formally with the Schrödinger Eq. (8). By direct comparison we can establish the correspondence between the quantum mechanical amplitudes and the classical variables: the classical field \( \alpha \) is the amplitude \( \epsilon \) and the in-plane component of the spin vector \( C_j^x \) is the amplitude \( \beta_j \). Note that the classical spins are not averages of the spin operators, \( \langle S_j \rangle \equiv 0 \), but instead they are connected with the quantum mechanical amplitudes. The solution of the dynamical Eqs. (8, 9) is the same as the solution of Eq. (7). With the number of excitations at the initial time is \( p > 1 \), but still \( p \ll N \), the mapping of the classical equations on Eq. (9), approximation (10), still holds but...
the initial condition for Eq. (3) is different: \( \alpha (0) = \sqrt{p} \).
This difference results then only in a renormalization of \( \langle n(t) \rangle \) obtained from Eq. (3) by \( p \).

When the number of excitations is large \( p \geq N \), the \( z \)-components of the classical spins deviate significantly from their initial values during the evolution and the approximation \( 4 \) is not valid. In this regime the classical Eqs. \( 7 \) \( 8 \) have to be solved numerically.

V. TWO-EXCITATION REGIME AND INHOMOGENEOUS CouPLING Constants

In this section we consider the dynamics of two excitations for a system with inhomogeneous coupling constants \( g_{ij} \) but constant Zeeman energies \( \epsilon_j = \omega \).

The time evolution of the two excitations is restricted to the subspace with \( c = -N/2 + 2 \) and is described by the general state

\[
|\Psi (t)\rangle = \alpha (t) |2, \psi\rangle + \sum_j \beta_j |1, \psi, \uparrow\rangle + \sum_{i>j} \gamma_{ij} |0, \psi, \uparrow, \uparrow\rangle,
\]

where \( \alpha (t) \), \( \beta_j (t) \), and \( \gamma_{ij} (t) \) are the normalized amplitudes, \( |\alpha (t)|^2 + \sum_j |\beta_j (t)|^2 + \sum_{i>j} |\gamma_{ij}|^2 = 1 \), of the state with two bosonic excitations, a state with one bosonic excitation and the \( j \)th spin excited, and a state with no bosonic excitation and the \( i \)th and \( j \)th spins excited (with \( i \neq j \)). The amplitude \( \gamma_{ij} \) is defined such that \( \gamma_{ij} = 0 \) if \( j \leq i \).

The conservation law can be used to simplify the Hamiltonian Eq. \( \Pi \). We subtract \( \omega L \) from Eq. \( \Pi \), which only changes an irrelevant overall phase of \( |\Psi (t)\rangle \), to eliminate the first two terms. Note that the second term will not be zero away from the resonance \( \omega \neq \epsilon \). Inserting \( |\Psi (t)\rangle \) into the time-dependent Schrödinger equation we then obtain,

\[
-i\dot{\alpha} (t) = \sqrt{2} \sum_j g_{ij} \beta_j (t),
\]

\[
-i\dot{\beta}_k (t) = \sqrt{2} g_k \alpha (t) + \sum_{j<k} g_{jk} \beta_j (t) + \sum_{j>k} g_j \gamma_{jk} (t),
\]

\[
-i\dot{\gamma}_{kl} (t) = (g_k \beta_l (t) + g_l \beta_k (t)) (1 - \delta_{kl}).
\]

The initial condition, \( \alpha (0) = 1 \) and \( \beta_j (0) = \gamma_{ij} (0) = 0 \), which we further assume corresponds to the doubly occupied boson mode at the initial time. The physical observable of interest is the time-dependent value of the boson occupation number \( \langle n(t) \rangle \), which can be expressed in terms of the amplitudes \( \alpha (t) \) and \( \beta_j (t) \) as

\[
\langle n(t) \rangle = 2 |\alpha (t)|^2 + \sum_j |\beta_j (t)|^2,
\]

where \( \langle \ldots \rangle = \langle \Psi (t)| \ldots |\Psi (t) \rangle \) is the time-dependent expectation value.

A. General solution

We use the Laplace transform, \( A(s) = \int_0^\infty dt A(t) e^{st} \), to solve the set of equations, Eq. \( \Pi \). In the Laplace domain Eq. \( \Pi \) is a set of linear algebraic equations,

\[
-i (s\alpha - 1) = \sqrt{2} \sum_j g_{ij} \beta_j
\]

\[
-is\beta_k = 2 g_k \alpha + \sum_{j<k} g_{jk} \gamma_{jk} + \sum_{j>k} g_j \gamma_{jk}
\]

\[
-is\gamma_{kl} = (g_k \beta_l + g_l \beta_k) (1 - \delta_{kl})
\]

that can be explicitly solved. The substitution of \( \alpha (s) \) and \( \gamma_{kl} (s) \) as functions of \( \beta_k (s) \), that are obtained from the first and the last lines, into the middle line gives the following set of equations for \( \beta_k \) only,

\[
\left( -s^2 + 2 g_k^2 - \sum_j g_j^2 \right) \beta_k / g_k = \sum_j g_j \beta_j - i \sqrt{2}.
\]

Each \( \beta_k (s) \) is easily found from the above equation since \( \sum_j g_j \beta_j (s) \) on the right hand side is the same in each equation for all \( \beta_k (s) \). Then the sum is found self-consistently and we obtain the solution for the amplitude as

\[
\beta_k (s) = \frac{-i \sqrt{2} g_k}{-s^2 + 2 g_k^2 - N \langle g^2 \rangle} \frac{1}{1 - 3 \left( \frac{g_k^2}{g_k^2 + 2 g_{ij}^2 - N \langle g^2 \rangle} \right)},
\]

where the average is the sum over all spins \( \langle \ldots \rangle = (\sum_j \ldots) / N \). The other two amplitudes are found from the first and the third lines of Eq. \( \Pi \) by substitution of the above solution for \( \beta_k (s) \),

\[
\alpha (s) = 1 - \frac{1}{s} \left( \frac{g_k^2}{g_k^2 + 2 g_{ij}^2 - N \langle g^2 \rangle} \right),
\]

\[
\gamma_{kl} (s) = \frac{i}{s} (g_k \beta_l + g_l \beta_k) (1 - \delta_{kl}).
\]

The main focus of our interest will be on Eqs. \( \Pi \) \( 10 \) as the observable quantity \( \langle n(t) \rangle \) depends only on \( \alpha (t) \) and \( \beta_k (t) \). These time-dependent amplitudes can be obtained from Eqs. \( \Pi \) \( 10 \) by the inverse Laplace transform. The analytic structure of Eqs. \( \Pi \) \( 10 \) is governed, in general, by a set of poles given by the roots of denominators which depend on a particular set of \( g_{ij} \). For instance, if the number of spins \( N \) is small there are \( 2N \) conjugated complex roots. The inverse Laplace transforms of \( \alpha (s) \) and \( \beta_k (s) \) will be a sum of \( N \) discrete harmonic modes in contrast to the single excitation dynamics where in such a setup there is just a single pair of roots independent of the particular set of \( g_{ij} \), see Section III, and there is only a single harmonic mode in the dynamics of the boson occupation number, see Eq. \( \Pi \). Such a result marks a qualitative difference in the dynamics of the single- and two-excitation subspaces.
B. Time-evolution in the continuum limit of many spins

In this section we study the limit of many spins, i.e. $N \gg 1$. The sum over $j$ in Eqs. (15-16) can be substituted by an integral over a distribution function of $g$, $\sum_j \rightarrow \int_0^\infty dg Q(g)$ … In the continuum limit some poles can merge together, forming branch cuts, and some poles can separate themselves from the others. The inverse Laplace transform of the branch cuts will become a decay function in the time domain and the separate poles will contribute a set of harmonic modes.

The analytic structure of $\alpha(s)$ and $\beta_k(s)$ explicitly depends on the particular form of $Q(g)$.

C. Uniform distribution function

To be specific we consider a set of coupling constants which are uniformly distributed from a minimum value $g = g_0 - \xi$ to a maximum value $g = g_0$.

$$Q_1(g) = \theta(-g + g_0) \theta(g - g_0 + \xi) / \xi. \quad (18)$$

The coupling constants cannot be negative, so $\xi$ can vary from $\xi = 0$ (e. g. all couplings are the same and are equal to $g_0$) to $\xi = g_0$ (e. g. the couplings are evenly distributed from 0 to $g_0$), see Fig. 1b. A useful property of this distribution function $Q_1$ is that a small and a large inhomogeneity can be analyzed on the same footing. Another distribution function will be considered in the next subsection.

Turning the sum in Eqs. (15-16) into an integral and using $Q_1(g)$, we obtain

$$\left\langle \frac{g^2 N}{s^2 + 2g_j^2 - N \langle g^2 \rangle} \right\rangle = -\frac{N}{2} \left( \frac{\sqrt{-s^2 - N \langle g^2 \rangle}}{\sqrt{2} \xi} \arctan \left( \frac{\sqrt{2} \xi \sqrt{-s^2 - N \langle g^2 \rangle}}{s^2 + N \langle g^2 \rangle - 2g_0 (g_0 - \xi)} \right) + 1 \right), \quad (19)$$

product in Eq. (16). Note that $s_0$ was obtained using a $1/N$-expansion and is independent of $\xi$ in leading order. In the first subleading $1/N$-order $s_0$ depends on $\xi$,

$$s_0 = 2 \sqrt{N \langle g^2 \rangle - \frac{3}{10} g_0^2}, \quad (20)$$

when $\xi = g_0$, and

$$s_0 = 2 \sqrt{N \langle g^2 \rangle - \frac{1}{2} g_0^2}, \quad (21)$$

when $\xi = 0$. The inverse Laplace transform of the functions with poles is a sum over the corresponding residues, $\alpha_p(t) = \sum_{s=0,\pm is_0} \text{Res}_s \alpha(s) e^{st}$, and it gives

$$\alpha_p(t) = \frac{1}{2} + \frac{1}{2} \cos s_0 t + \frac{1}{N} \delta \alpha(\xi). \quad (22)$$

Here, the leading term is independent of $\xi$ unlike the first $1/N$-correction, $\delta \alpha(0) = -\frac{27}{20} (1 - \cos (s_0 t))$ and $\delta \alpha(0) = -\left( \frac{3}{4} - \frac{1}{2} \cos (s_0 t) \right)$. We refer to Appendix A for the calculation.

The expression in Eq. (19) has four branch points. Two are given by the square root, $s = \pm is_1$ where $s_1 = \sqrt{N \langle g^2 \rangle}$. The remaining two are given by arctan. Solving the equation

$$\frac{\sqrt{2} \xi \sqrt{-s^2 - N \langle g^2 \rangle}}{s^2 + N \langle g^2 \rangle - 2g_0 (g_0 - \xi)} = \pm i \quad (23)$$

Figure 1: Distribution functions of $g$ that are used to evaluate the sums in Eqs. (15-16). (a) The uniform distribution function $Q_1(g)$ has a maximum coupling strength $g_0$ and a width $\xi$ which can vary from 0 (i.e. homogeneous coupling constants) to $g_0$ (i.e. maximally inhomogeneous coupling constants). (b) The sawtooth distribution function $Q_2(g)$ describes a non-uniform spread of the coupling constants $g_j$ from 0 to $g_0$.\[\begin{align}
\langle g^2 \rangle &= g_0^2 - \xi g_0 + \xi^2 / 3.
\end{align}\]
where the dimensionless integral $I(t)$ describes the time-
The leading 1/N-correction, \( \delta n(t) = 4\alpha(t)\delta\alpha(t) + 2 \sum_j \beta_j(t)\delta\beta_j(t) \), is qualitatively different for \( \xi = 0 \) and \( \xi > 0 \). When the coupling constants are homogeneous, \( \xi = 0 \),
\[
\delta n(t) = - \left( 2 (1 + \cos(s_0 t)) + \frac{1}{8} (1 - \cos(2s_0 t)) \right)
\]
is an oscillatory function where the second harmonic with the doubled frequency \( 2s_0 \) appears in addition to the main frequency \( s_0 \) of the leading term. For the case of maximally inhomogeneous coupling constants, \( \xi = g_0 \), the function
\[
\delta n(t) = 1.8 \sin^2(s_0 t) - 2 \sin(s_0 t) I(t)
\]
contains a decaying contribution, where \( I(t) \) is the decay function from Eq. (28). Up to the time scale \( \tau_g \) the term \( \sin(s_0 t) I(t) \sim \sin(s_0 t) \cos(s_1 t) \) is a harmonic mode with the third frequency \( s_0 + s_1 \) in addition to \( s_0 \) and \( 2s_0 \). This mode can only be observed if the short-time regime with \( t < \tau_g \) is accessible to measurement.

D. Sawtooth distribution function \( Q_2(g) \)

Here we study another distribution of the coupling constants. Assuming that there are more spins at the nodes of the cavity mode, so that the stronger coupling constants are more favorable, we consider a sawtooth-like distribution function with its maximum at the largest coupling strength \( g_0 \), \( Q_2(g) = 2g/g_0 \theta (g - g_0) \theta (g) \), see Fig. (1).
Replacing the sums in Eqs. \([15\ 16]\) by integrals,

\[
\frac{g_0^2 N}{s^2 + 2g_0^2 - N \langle g^2 \rangle} = \frac{N}{2} \left( 1 - \frac{s^2 + N \langle g^2 \rangle}{2g_0^2} \ln \left( \frac{s^2 + N \langle g^2 \rangle - 2g_0^2}{s^2 + N \langle g^2 \rangle - 2g_0^2} \right) \right),
\]

where \(\langle g^2 \rangle = g_0^2/2\).

The analytic structure of Eqs. \([15\ 16]\) with the sum from the above equation is the same as with the sum from Eq. \([19]\) obtained using \(Q_1 (g)\). There are three poles \((\alpha (s)\) has three poles and \(\beta_k (s)\) has only two as in the previous subsection) and two branch cuts, see Fig. \(2\).

Thus, the inverse Laplace transforms of \(\alpha (s)\) and \(\beta_k (s)\) also have two contributions, i.e. \(\alpha (t) = \alpha_p (t) + \alpha_n (t)\) and \(\beta_k (t) = \beta_p^k (t) + \beta_n^k (t)\).

Similarly to the previous subsection, there is a pole at \(s = \pm i s_0\), where \(s_0 = 2\sqrt{N \langle g^2 \rangle}\) agrees in leading \(1/N\)-order with what was obtained in the previous subsection.

The four branch points, which are due to the logarithm in Eq. \((34)\), are found from

\[
s^2 + N \langle g^2 \rangle \ln \left( s^2 + N \langle g^2 \rangle - 2g_0^2 \right) = 0, \infty,
\]

as \(s = \pm is_{1,2}\) where \(s_1 = \sqrt{N \langle g^2 \rangle}\) and \(s_2 = \sqrt{N \langle g^2 \rangle - 2g_0^2}\). These also agree with what we have already found in the previous subsection when the coupling constants were maximally inhomogeneous, i.e. \(\xi = g_0\).

The sums over the residues give the main contribution to the inverse Laplace transforms. The leading \(1/N\)-terms in \(\alpha (t)\) and \(\beta_k (t)\) agree with the leading terms in Eq. \((22)\) and Eq. \((25)\), where \(\langle g^2 \rangle\) has to be calculated using the sawtooth distribution function \(Q_2 (g)\) instead of \(Q_1 (g)\). The contributions from the branch cuts also appear only in the first subleading \(1/N\)-order. The main features of the time decay are similar to that of Eq. \((28)\). Indeed, the decay time \(\tau_g\) and the frequency of the fast oscillating term in Eq. \((28)\) for times \(t < \tau_g\) result from the same branch points, \(\pm is_{1,2}\), as in the previous subsection.

As the amplitudes \(\alpha (t)\) and \(\beta_k (t)\) are similar to the ones in the previous subsection, the boson number \(\langle n(t)\rangle\) is also given by Eq. \((31)\). The leading \(1/N\)-term depends on \(Q_2 (g)\) only through the effective coupling constant \(\sqrt{\langle g^2 \rangle}\), and the leading \(1/N\)-correction contains a decay term.

VI. SUBSPACE OF TWO AND THREE EXCITATIONS

In this section we compare numerically the solution of the Schrödinger equation with one of the classical equations of motion Eqs. \((7\ 8)\) for \(p = 2, 3\) excitations. We start from writing down the Schrödinger equation in these two subspaces explicitly.

The time evolution of two excitations is restricted to the subspace with \(c = -N/2 + 2\) and is described by the general state Eq. \((10)\). The Schrödinger equation for an arbitrary set of \(\epsilon_j\) and \(g_j\) in this subspace is similar to Eq. \((11)\),

\[
-i \dot{\alpha} = \sqrt{2} \sum_j g_j b_j \tag{36}
\]

\[
-i \dot{\beta}_k = (\epsilon_k - \omega) \beta_k + \sqrt{2} g_k \alpha + \sum_{j<k} g_j \gamma_{kj} + \sum_{j>k} g_j \gamma_{jk} \tag{37}
\]

\[
-i \dot{\gamma}_{kl} = [(\epsilon_k - \omega) + (\epsilon_l - \omega)] \gamma_{kl} + (g_k \beta_l + g_l \beta_k) (1 - \delta_{kl})\tag{38}
\]

with the initial condition \(\alpha (0) = 1, \beta_k (0) = 0, \gamma_{kl} (0) = 0\).

The subspace of three excitations is labeled by \(c = -N/2 + 3\) and is described by the general state,

\[
|\Psi (t)\rangle = \alpha (t) |3, \downarrow\rangle + \sum_j b_j |2, \downarrow \uparrow_j\rangle + \sum_{i>j} \gamma_{ij} |1, \uparrow \uparrow_i \uparrow_j\rangle + \sum_{i>j>r} \eta_{rij} |0, \uparrow \uparrow_i \uparrow_j \uparrow_r\rangle, \tag{39}
\]

where \(\alpha (t), \beta_j (t), \gamma_{ij} (t),\) and \(\eta_{rij} (t)\) are the normalized amplitudes, \(\alpha (t)\), \(\sum_j |\beta_j (t)|^2 + \sum_{i>j} |\gamma_{ij}|^2 + \sum_{i>j>r} |\eta_{rij}|^2 = 1\), of the state with three bosonic excitations, a state with two bosonic excitations and the \(j^\text{th}\) spin excited, a state with one bosonic excitation and the \(i^\text{th}\) and \(j^\text{th}\) spins excited, and a state with no bosonic excitations and the \(i^\text{th}\), \(j^\text{th}\) and \(k^\text{th}\) spins excited. The amplitude \(\gamma_{ij}\) is defined such that \(\gamma_{ij} = 0\) if \(j \geq i\), and \(\eta_{rij}\) is defined such that \(\eta_{rij} = 0\) if the inequality \(i > j > r\) is
not satisfied. The Schrödinger equation in this subspace is

\[
- i \dot{\alpha} = \sqrt{3} \sum_j g_j \beta_j,
\]

\[
- i \dot{\beta}_k = (\epsilon_k - \omega) \beta_k + \sqrt{3} g_k \alpha + \sqrt{2} \sum_{j<k} g_j \gamma_{kj} + \sqrt{2} \sum_{j>k} g_j \gamma_{jk},
\]

\[
- i \dot{\gamma}_{kl} = \left[ (\epsilon_k - \omega) + (\epsilon_l - \omega) \right] \gamma_{kl} + \sqrt{2} (g_k \beta_l + g_l \beta_k) \left( 1 - \delta_{kl} \right) + \sum_{j>k>l} g_j \eta_{jkl} + \sum_{k>j>l} g_j \eta_{jkl} + \sum_{k>j>l} g_j \eta_{jkl},
\]

\[
- i \dot{\eta}_{klm} = \left[ (\epsilon_k - \omega) + (\epsilon_l - \omega) + (\epsilon_m - \omega) \right] \gamma_{kl} + g_k \gamma_{lm} + g_l \gamma_{km} + g_m \gamma_{lm}
\]

with the initial conditions \( \alpha(0) = 1, \beta_k(0) = 0, \) and \( \gamma_{kl}(0) = 0. \) The physical observable of interest is again the time-dependent boson number which can be expressed in terms of the amplitudes \( \alpha(t), \beta_j(t), \) and \( \gamma_{ij}(t) \) as

\[
\langle n(t) \rangle = 3 |\alpha(t)|^2 + 2 \sum_j |\beta_j(t)|^2 + \sum_j |\gamma_{ij}(t)|^2. \tag{39}
\]

Unlike before for \( p = 1, \) the classical equations of motion Eqs. (7, 8) and the Schrödinger Eqs. (38) are not equivalent in the subspaces of \( p = 2, 3. \) Solving these equations numerically, we compare the time-dependent boson number, given by Eqs. (12, 13), for \( p = 2, 3 \) with the square modulus of the classical field \( \alpha(t) \) obtained from Eqs. (7, 8). In the large-\( N \) limit the solutions of both classical and quantum equations have exactly the same form.

In Fig. 4 we give a detailed comparison of the different types of inhomogeneities characterized by the distribution functions \( P(\epsilon) \) and \( Q_1(g) \) for a large number of spins, i.e. \( N = 200. \) The solutions of Eq. (38) and Eqs. (7, 8) with \( p = 2 \) are compared in Fig. 4 (a)-c), and the solutions of Eq. (38) and Eqs. (7, 8) with \( p = 3 \) are compared in Fig. 4 (d)-f). The regime with inhomogeneous coupling constants only, characterized by \( Q_1(g) \) from Eq. (13) with \( \xi = g_0, \) and \( P(\epsilon) \) from Eq. (4) with \( \Delta = 0, \) is shown in Fig. 4 (b) and e). The regime of strongly inhomogeneous Zeeman energies, \( \Delta/\Omega = 10, \) is shown in Fig. 4 (c) and f). Finally, an intermediate regime with \( \Delta/\Omega = 2.2 \) is shown in Fig. 4 (a) and (d).

As the classical and quantum solutions coincide in all regimes, the classical equations can be used to find the time-dynamics of the boson occupation number. This is quite remarkable since the classical equations are significantly simpler to solve than the Schrödinger equation, both analytically and numerically. When the number of excitation is small, \( p \ll N, \) the classical equations can be mapped to the Schrödinger equation in the one-excitation subspace \( p = 1 \) in leading \( 1/N \)-order, see Eqs. (3). The Schrödinger equation for this case was already solved. A larger number \( p \) of excitations changes only the initial condition of Eq. (3) from \( \alpha(0) = 1 \) to \( \alpha(0) = \sqrt{n}. \) This difference can be accounted for by a simple rescaling by \( p \) of the bosonic occupation number \( \langle n(t) \rangle \) that was obtained in the single-excitation subspace for all regimes. The explicit results were given in Sec. III, see Eqs. (4, 6), but we do not show them in Fig. 4.

In conclusion, the analysis in Ref. 17 is also applicable to the case with more than one excitation, provided \( p \ll N \) and \( N \gg 1. \)

**VII. APPLICABILITY OF THE CLASSICAL APPROXIMATION**

The classical approximation in the few-excitation sector is exact when the number of spins \( N \) is infinite. For finite but still large \( N's, \) i.e. \( N \gg 1, \) the time-evolution of the classical system deviates from the quantum system by a small amount. The goal of this section is to analyze these finite-size deviations.

**A. Ehrenfest time**

One way to quantify the difference between the classical and the quantum solution is to identify the Ehrenfest time \( \tau_E \) at which they deviate significantly from each other. To this end, we compare the boson number \( \langle n(t) \rangle, \) obtained from the classical equations (in the few-excitation approximation), Eqs. (3, 4), with the exact quantum solution in the two-excitation subspace obtained in Sec. V. The solution to the classical equations Eq. (5) in this regime is \( n = 2 \cos^2(\Omega t), \) where \( \Omega = \sqrt{N} (g^2). \) From Eq. (21) the solution to the Schrödinger equation is \( \langle n \rangle = 2 \cos^2(\xi_0 t/2) \) in leading \( 1/N \)-order. Both solutions are single harmonic modes with frequencies that also match in leading \( 1/N \)-order, where \( \xi_0 = 2 \sqrt{N} (g^2). \) In first subleading \( 1/N \)-order, the correction to \( \xi_0 \) depends explicitly on \( Q_1(g), \) see Eqs. (20, 21). For \( \xi = 0, \) the expansion of Eq. (21) gives \( \xi_0 = 2 \sqrt{N} (g^2) - \sqrt{4 (g^2)^3}/N, \) and for \( \xi = g_0 \) expanding
Figure 4: Numerical solutions of the Schrödinger Eqs. (36, 38) and classical equations of motion Eqs. (7, 8) for a large number of spins, \( N = 200 \), and \( p = 2, 3 \) number of excitations. The inhomogeneities are characterized by the distributions \( P(\epsilon) \) and \( Q_1(\xi) \). In plots a) and d): \( \Delta/\Omega = 2.2 \) and \( \xi = 0 \); in plots b) and e): \( \Delta = 0 \) and \( \xi = g_0 \); in plots c) and f): \( \Delta/\Omega = 10 \) and \( \xi = 0 \).
Eq. (20) we get \( s_0 = 2\sqrt{N (g^2)} - 3\sqrt{3 (g^2)/10N} \). Thus, at the time scale

\[
\tau_E = \sqrt{\frac{N}{(g^2)}}
\] (40)

the phase difference between the two harmonic modes, with frequencies \( \Omega \) and \( s_0/2 \), is comparable to \( 2\pi \). Hence, the difference between classical and quantum solutions is significant for any value of \( \xi \) at this time scale \( \tau_E \), to which we refer as Ehrenfest time. The numerical prefactor depends on the particular type of inhomogeneity. The explicit calculations in Sec. V for typical distribution functions show that these numerical prefactors are of order one.

There is also a \( 1/N \)-correction to the amplitude of \( \langle n(t) \rangle \) coming from the Schrödinger equation, see Eq. (33). This correction contains a decaying contribution (proportional to \( I(t) \) given in Eq. (28)) which, again, marks a qualitative difference between the quantum and classical time-dynamics in the two-excitation subspace. In particular, \( I(t) \) decays at the characteristic time scale \( \tau_g \) given in Eq. (29) which is equal to the Ehrenfest time \( \tau_E \) introduced above. Thus, we see that this difference in the amplitude (although it is only a \( 1/N \)-correction) is another manifestation of the quantum nature of the system where the time-dynamics for times exceeding the Ehrenfest time can be described correctly only by the Schrödinger equation (and not by the classical one).

So far we have been using the approximate classical Eqs. (5 9) for few excitations, thereby neglecting the deviations of the \( z \)-component of the classical spins from \( C_j = -1/2 \). To estimate the quality of this approximation, we use the result obtained in Ref. 29 for the homogeneous classical system. The solution of the unapproximated Eqs. (7 8) with \( g_j \equiv g_0 \) and \( \epsilon_j \equiv \omega \) is an elliptic function of time \( \tau \). When the number of excitations is small, \( p \ll N \), this elliptic function can be expanded into a harmonic series with a leading term that reproduces the solution of the approximate Eqs. (8 9). The frequency of the leading harmonic term matches the frequency \( \Omega = \sqrt{N g_0} \) from Eq. (5) in leading \( 1/N \)-order, but it also contains corrections on the order of \( g_0/\sqrt{N} \) like Eq. (31). Such corrections, however, are irrelevant as they become only sizable at the Ehrenfest time \( \tau_E \) the time beyond which the classical solution fails and the true time-dynamics must be described anyway by the Schrödinger equation.

### B. Initial spin excitations

Up to now we have focussed on a particular initial condition with excitations being initially present only in the boson mode. In contrast, a different initial condition was considered in Ref. 16, whereby the dynamics starts from an initial state with no boson present but, say, with the \( i^{th} \) spin excited. Considering homogeneous systems, it was found that during the time-evolution this \( i^{th} \) spin remains excited if the total number of spins is large, regardless of how strong the spin-boson coupling is. The corresponding expectation value is \( \langle 0, \uparrow \downarrow | S_i^z (t) | 0, \uparrow \downarrow \rangle = 1/2 - (1 - \cos (\Omega t)) / N \). This result was associated with the effect of “radiation trapping”, and it can also be obtained with the classical approximation. The corresponding initial condition \( C_{i \neq i} (0) = \langle 0, \uparrow \uparrow | S_i^z (t) | 0, \uparrow \uparrow \rangle = 0 \), \( C_i (0) = \langle 0, \uparrow \uparrow | S_i^z (t) | 0, \uparrow \uparrow \rangle = 0 \), and \( a (0) = 0 \) is a fixed point of the classical Eqs. (7 8). Indeed, the effective magnetic field for each spin \( B_j = \langle 0, \uparrow \uparrow | [H, \hat{S}_j^z] | 0, \uparrow \uparrow \rangle \) only has an \( x \)-component, and therefore the vector-product of two parallel vectors vanishes, i.e. \( B_j \times C_j = 0 \). The dynamics of the classical field \( a \) is frozen as \( C_j^x = 0 \). Quantum corrections to this result show up only in the first \( 1/N \)-correction. Thus, the classical approximation is also valid for a different initial condition in the few-excitation regime.

### VIII. CONCLUSIONS

In this paper we have shown that the solution to the classical Hamilton equations of motion of the inhomogeneous Dicke model coincide with the solution of the time-dependent Schrödinger equation when the number of spins \( N \) is large and the number of excitations \( p \) is small, \( p \ll N \). For a single excitation the leading \( 1/N \)-order of the classical solution coincides with the quantum solution. For a few excitations such correspondence does not hold but for \( p = 2, 3 \) excitations the numerical solutions of both classical equations of motion and Schrödinger equation coincide when the number of spins is large. It is plausible to conjecture that the same correspondence holds for \( p > 3 \) in leading \( 1/N \)-order.

To assess the validity of the classical approximation for \( p > 1 \) excitations, we have solved the Schrödinger equation exactly in the two-excitation subspace with inhomogeneous coupling constants only and compared the result with the classical solutions. For large \( N \), we performed an \( 1/N \)-expansion of the solution to the Schrödinger equation and recovered the classical solution in leading order. Subleading \( 1/N \)-corrections cause small deviations of the classical from the quantum solution, that, at a large time scale, make the difference between the two significant. This defines the Ehrenfest time that we identify in the limit of \( p \ll N \) as \( \tau_E = \sqrt{N / (g^2)} \).

Analyzing the solution to the Schrödinger equation for \( p = 2 \), we compared it with the solutions to the Schrödinger equation for \( p = 1 \). We have found that the boson occupation number in the two-excitation subspace exhibits a multi-frequency dynamics due to the inhomogeneous couplings only, which, unlike in the single-excitation subspace, can lead to a decay in the limit of large \( N \). But the leading term of an \( 1/N \)-expansion recovers the single-frequency dynamics. The decay due to the inhomogeneity shows up only in the first subleading \( 1/N \)-correction. We find that this contribution is an
oscillatory mode with frequency $\frac{4}{1} \Omega$ and a slow decay envelope. The decay is essentially non-exponential with a long power-law tail, and the decay time is $\tau_\alpha \sim \sqrt{N/(g^2)}$, where $\sqrt{g^2}$ is a characteristic coupling, the numerical prefactor is of order one for the special case of uniformly distributed coupling constants.

The decay due to an inhomogeneous coupling to a spin bath, which is unavoidable as the spins are located at different positions of the cavity mode (with different amplitudes of the electromagnetic field), is similar to the decay of an electron spin coupled to a bath of nuclear spins through the hyperfine interaction. In the dynamics of a cavity mode this mechanism can be neglected when only a few excitations are present in the system (for instance, in the few-photon spectroscopy experiments in Ref. [3]), but may lead to a significant decay in a system with many excitations present initially (such as, for instance, in a bath of nuclear spins coupled to a cavity).

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Appendix A: $1/N$ corrections to $\alpha_p(t)$ and $\beta_k(t)$

In this appendix we calculate the first $1/N$ correction to the pole contributions to the inverse Laplace transform of $\alpha(t)$ and $\beta_k(t)$ from Sec. V.

If $\xi = 0$, i.e. for homogeneous coupling constants, the solution to Eq. (13) simplifies. Substituting $g_j = g_0$ into Eqs. (13), we get

$$\beta_k(s) = \frac{i \sqrt{2g_0}}{s^2 + (4N - 2) g_0^2},$$

(A1)

and

$$\alpha(s) = \frac{1}{s^2 + (4N - 2) g_0^2}.$$  

(A2)

These expressions have only poles, but no branch points: Two poles for $\beta_k(s)$, $s = \pm i \sqrt{2g_0 \sqrt{N - 1 \frac{1}{2}}}$, and three for $\alpha(s)$, $s = 0, \pm i \sqrt{2g_0 \sqrt{N - 1 \frac{2}{3}}}$, see Fig. 2. The inverse Laplace transform is given by residues only, $\alpha(t) = \alpha_p(t)$ and $\beta_k(t) = \beta_k^p(t)$,

$$\beta_k^p(t) = \frac{i \sin \left(2g_0 \sqrt{N - 1 \frac{1}{2}} t \right)}{\sqrt{2N - 1}}$$

(A3)

and

$$\alpha_p(t) = \frac{N - 1}{2N - 1} + \frac{N}{2N - 1} \cos \left(2g_0 \sqrt{N - 1 \frac{1}{2}} t \right).$$  

(A4)

Expanding the above expression in $1/N$, we obtain the corrections $\delta \alpha(0)$ and $\delta \beta_k(0)$ in Eqs. (22, 25).

To calculate the $1/N$-correction when $\xi = g_0$, i.e. for maximum inhomogeneity, we expand Eq. (19) up to the second order in $1/N$ at the poles, $s = \pm is_0$, $s_0 = \sqrt{4N (g^2)}$, and also account for the second order corrections that come from the positions of the poles, $s_0 = \sqrt{4N (g^2)} - \frac{g^2}{2g_0}$.

Performing this procedure we write the residues of $\alpha(s)$ at $s = \pm is_0$ as

$$\text{Res}_{s = \pm is_0} \alpha(s) e^{st} = \frac{1}{\pm is_0} \frac{N_0 + \delta N}{D_0 + \delta D} e^{\pm is_0 t},$$

(A5)

where

$$D_0 = -\frac{2(\pm is_0) N g_0^2}{(-s_0^2 + N (g^2))^2},$$

(A6)

$$N_0 = 1 + \frac{N g_0^2}{3(-s_0^2 + N (g^2))},$$

(A7)

are the denominator and numerator obtained using just $s_0 = \sqrt{4N (g^2)}$. Further,

$$\delta D = -\frac{2(\pm is_0) N g_0^2}{(-s_0^2 + N (g^2))^2} \frac{12 g_0^2}{5(-s_0^2 + N (g^2))},$$

(A8)

$$\delta N = \frac{2N g_0^4}{5(-s_0^2 + N (g^2))^2}$$

(A9)

are the first $1/N$-corrections. The average $\langle g^2 \rangle = g_0^2/3$ is evaluated using $Q_1(g)$ with $\xi = g_0$.

Performing the summation over the two poles $\pm is_0$ we get

$$\sum_{\pm is_0} \text{Res}_{s = \pm is_0} \alpha(s) e^{st} = \frac{2}{i s_0} \frac{N_0 + \delta N}{D_0 + \delta D} \cos (s_0 t),$$

(A10)

and expand it in the small parameter as

$$\sum_{\pm is_0} \text{Res}_{s = \pm is_0} \alpha(s) e^{st} = \frac{2}{i s_0} \frac{N_0 + \delta N}{D_0 + \delta D} \cos (s_0 t),$$

(A11)

where the first two terms in the product still have to be expanded in the small correction to $s_0 = \sqrt{4N \langle g^2 \rangle}$, and the last two have to be calculated using only the leading term $s_0 = \sqrt{4N \langle g^2 \rangle}$.

The first two terms, which have to be calculated with $s_0 = \sqrt{4N \langle g^2 \rangle} - \frac{g^2}{2g_0}$, are

...
\[
\frac{2}{i s_0} \frac{N_0}{D_0} = \frac{2}{i \sqrt{4N \langle g^2 \rangle}} \left( 1 + \frac{3g_0^2}{20N \langle g^2 \rangle} \right) \left( \frac{2}{3} - \frac{2}{5N} \right) \frac{-g_0^2}{i2\sqrt{4 \langle g^2 \rangle}} \left( 1 - \frac{3}{5N} \right) \left( 1 + \frac{9}{20N} \right)
\]
\[\approx \frac{1}{2} \left( 1 + \frac{9}{10N} \right) \left( 1 - \frac{6}{5N} \right) \approx \frac{1}{2} \left( 1 - \frac{3}{10N} \right) \quad (A12)\]

and the last two, which have to be calculated with \( s_0 = \sqrt{4N \langle g^2 \rangle} \), are
\[
\left( 1 + \frac{\delta N}{N_0} \right) \left( 1 - \frac{\delta D}{D_0} \right) = \left( 1 + \frac{3}{5N} \right) \left( 1 + \frac{12}{5N} \right) \approx 1 + \frac{3}{N} \quad (A13)\]

Finally, the contribution from the poles \( \pm is_0 \) is
\[
\sum_{\pm is_0} \text{Res}_{s=\pm is_0} \alpha e^{st} = \frac{1}{2} \cos (s_0t) + \frac{27}{20N} \cos (s_0t) \quad (A14)\]

The residue of \( \alpha (s) \) at \( s = 0 \) is also expanded in the small corrections,
\[
\text{Res}_{s=0} \alpha (s) = \frac{N_0 + \delta N}{D_0} \cdot \frac{N_0}{D_0} \left( 1 + \frac{\delta N}{N_0} \right) \left( 1 - \frac{\delta D}{D_0} \right) \quad (A15)\]

where
\[
\frac{\delta N}{N_0} = \frac{9}{5N}, \quad \frac{\delta D}{D_0} = \frac{36}{5N} \quad (A16)\]

And the contribution from the pole \( s = 0 \) is
\[
\text{Res}_{s=0} \alpha (s) = \frac{1}{2} \cdot \frac{27}{20N} \quad (A17)\]

The sum of Eqs. (A14) (A17) gives the correction \( \delta \alpha (g_0) \) from Eq. (22).

Then, we calculate the corrections to \( \beta_k (t) = \text{Res}_{s=\pm is_0} \beta_k (s) e^{st} \). Expanding the residues at \( s = \pm is_0 \), where \( s_0 = \sqrt{4N \langle g^2 \rangle} - \frac{2}{3}g_0^2 \), we get
\[
\text{Res}_{s=\pm is_0} \beta_k (s) e^{st} = \frac{e^{\pm is_0t}}{s_0 - N \langle g^2 \rangle + 2g_k^2 D_0 + \delta D} \quad (A18)\]

where \( D_0 \) and \( \delta D \) have already been calculated, see Eqs. (A6) (A8). Similarly to the calculation of \( \alpha (s) \) we again expand and get
\[
\text{Res}_{s=\pm is_0} \beta_k (s) e^{st} = -i\sqrt{2} g_k \frac{e^{\pm is_0t}}{N_0} \left( 1 - \frac{2 (g_k / g_0)^2}{N} + \frac{6}{5N} \right) \frac{Ng_0^2}{4 \sqrt{N \langle g^2 \rangle}} \left( 1 - \frac{3}{5N} \right) \left( 1 + \frac{12}{5N} \right) e^{\pm is_0t} \quad (A19)\]

The sum over these residues,
\[
\beta_k^0 (g_0) = \frac{ig_k \sqrt{2}}{2 \sqrt{N \langle g^2 \rangle}} \left( 1 - \frac{2 (g_k / g_0)^2}{N} + \frac{6}{5N} \right) \left( 1 + \frac{9}{5N} \right) \sin (s_0t) \quad (A20)\]

is the correction from Eq. (20) for \( \xi = g_0 \).

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