Parameter Estimation of Sigmoid Superpositions: Dynamical System Approach

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Abstract

Superposition of sigmoid function over a finite time interval is shown to be equivalent to the linear combination of the solutions of a linearly parameterized system of logistic differential equations. Due to the linearity with respect to the parameters of the system, it is possible to design an effective procedure for parameter adjustment. Stability properties of this procedure are analyzed.

Keywords: Neural Networks, Control Theory, Adaptive Control, Learning Algorithms

1 Introduction

Static base functions are used in a variety of universal function-approximation schemes. Their general form runs as follows: Let a given continuous function $g(t)$ be defined over a compact time interval $[0, T]$. There will be a function $y(t)$, represented as

$$y(t) = \sum_{i=1}^{n} c_i f(a_i t + b_i),$$

in which $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function and for any given $\varepsilon > 0$, there are values of $n, a_i, b_i,$ and $c_i$, such that for all $t \in [0, T],$

$$|g(t) - y(t)| \leq \varepsilon.$$ 

Among the functions $f(\cdot)$ for which approximation of $g(t)$ can be proven, the Gaussian and the sigmoid are the most well-known ones. Approximation by sigmoid is often favored for, amongst
others, its very good rate of convergence with respect to the number \( n \) of additive terms in equation (1) [4]. Recent results [11] have shown that
\[
\int_0^t (g(\tau) - y(\tau))^2 d\tau = O\left(\frac{1}{n^2}\right).
\]
Another advantage is that convergence is also possible in Sobolev space, implying the existence of an optimal approximator for derivatives of function \( g(t) \) [18]; [17].

In spite of significant progress in the fields of nonlinear optimization and neural networks (a comprehensive review of a neural learning algorithms is given in [15]) an estimation of the unknown values of parameters \( a_i, b_i, c_i \) in (1) is still a difficult problem. Simple local optimization strategies, involving gradient descent, fail to converge because of nonconvexity of the function with respect to the parameters; global search algorithms [19]; [14] are prohibitively expensive computationally [39], and second-order search algorithms rely on assumptions relating to the error surface that are not always met, for instance uniqueness of the extremum [41].

In order to address the parameter adjustment problem, simplifying assumptions have been made [7]. This approach, for instance, requires that the values of each additive term \( f(a_i t + b_i) \) in (1) over \([0, T]\) be known. Under this assumption convergence to a global minimum could be proven. The method was shown to have a very fast speed of convergence. However, the requirement that the value of each term be known imposes severe restrictions on the applicability of this method. Following a different strategy, in recent years several new methods have been proposed which are capable of avoiding local minima by modifying the learning criterion (see, for instance [22]). Yet, these methods cannot guarantee that the estimates of the unknown values of the parameters \( a_i, b_i, c_i \) converge to their true values (up to permutations). In our view the underlying problem with these conventional methods is that, whereas they use error minimization for approximating a solution, they lack an explicit model of error dynamics. We will propose a novel approach to estimate the values of the parameters in (1) utilizing elements of classical control theory.

In this approach the values of function \( g(t) \) are interpreted as reference signals, the outputs of a dynamical system called reference system. The reference signal is used in the explicit definition of an error function as, for instance, the difference with a tracking signal. This signal, in turn, is considered the output \( y(\theta, t) : \Omega_\theta \times R \to R, \theta \in \Omega_\theta, \Omega_\theta \times R \to R, \theta \in \Omega_\theta \) of a dynamical system called tracking system with parameter vector \( \theta = (a^T : b^T : c^T) \ a, b, c \in R^n \) to be determined. Thus the problem of function approximation is transformed into one of finding a suitable parameterization for a given tracking system.

A similar strategy was used in [38], [1] for different purposes. In these studies the resulting equations remained nonlinear in their parameters. The presently proposed transformation, however, will enable us to represent the problem in terms of a nonlinear system that is linear in its
parameters. The linearity allows us to apply conventional methods of adaptive control theory for stabilizing the error dynamics and thus facilitate finding the optimal solution. For this purpose, the learning problem is formulated as one of adaptive tracking (or equivalently, synchronization between reference and tracking system). To this problem we can apply the method of Lyapunov functions, extending parameter space \( \Omega_\theta \) to \( \{ \alpha, \beta, C, x(t) | \alpha, \beta, C \in \mathbb{R}^n, x(t) : R \rightarrow \mathbb{R}^n \} \), and use a simple rule for parameter adjustment in the enhanced system dynamics. This provides us with a method potentially more powerful than, for instance, gradient descent, which operates entirely in the original parameter space by relying on the contraction theorem.

It should be mentioned, however, that the problem of parameter value identification has not completely been solved even for our case of linearly parameterized, nonlinear systems. The solutions available in the literature are formulated either for linear systems [20]; [24]; [32] or for some special classes of nonlinear plants, assuming full state measurement [10] or the possibility to transform the system into an output injection form [26]; [27]. We do not wish to impose any such restrictions. Instead we exploit the possibility to extend both the reference and tracking signals to be repeated periodically starting from the same initial conditions. By doing so we significantly simplify the problem of searching for the optimal values of unknown parameters.

A strategy similar to the one proposed is often used in iterative learning control [2]; [3]; [29]; [33] mostly for determining a feed-forward control term which is defined as a function of time. The time-variability of the solution severely reduces the significance of these methods for our problem. Nevertheless, there are several approaches that can be applied to search for unknown parameters within an iterative learning control framework [31]; [16]; [36]. These approaches, however, according to our knowledge, are either designed for linear dynamical systems or when dealing with nonlinear systems cannot guarantee to stop at the non-local solution. This motivates us not only to show the possibility to transform the entire problem of static nonlinear optimization into dynamics one but also to provide an algorithm to estimate the unknown parameters of the resulting linearly parameterized system of nonlinear differential equations.

The first step in our approach will be the selection of a “base function” for the reference and tracking systems, suitable for representing a broad class of functions. We have chosen the logistic differential equation [37]. We will start off by providing an existence proof for approximation in this system. The next step will be the specification of an algorithm for parameter adjustment that effectively finds the optimal solution in an interesting domain of functions. We consider this problem for systems with unperturbed conditions as well as with time-varying parameters. The former constitutes a method for representing scalar functions in one variable, for instance time; the latter provides a method for representing functions with multiple inputs. Finally, the viability
of the approach is demonstrated in examples comparing it to gradient descent.

The paper is organized as follows. In Section 2 we formulate the problem and introduce the class of systems to be analyzed. In Section 3 we investigate the dynamic abilities of the system and prove the approximation properties of the system. In Section 4 we introduce the schemes to adjust the unknown parameters of the system. In Section 5 we discuss multi-dimensional approximation problems and show the possibility to utilize the same technique for approximation of a system of nonlinear differential equations with arbitrary smooth right-hand sides. Section 6 contains simulation results for illustrative examples. Section 7 concludes the paper.

2 Problem Formulation

Although the sigmoidal function approximation scheme has several attractive features, the most important obstacle on the way to its implementation remains the absence of an algorithm that guarantees convergence to an optimal solution. We suggest a strategy to turn the problem of searching for the parameter values of the static nonlinear parameterized map \( f(\alpha, \beta, c, t), \alpha, \beta, c \in R^n \) into one of searching for linear parameter values of a system of nonlinear differential equations:

\[
\dot{x} = \sum_{i=1}^{n} \xi_{1,i}(x)\alpha_i + \sum_{i=1}^{n} \xi_{2,i}(x)\beta_i, \quad y(x) = Cx,
\]

where \( x \in R^n, \alpha = (\alpha_1, \ldots, \alpha_n)^T, \beta = (\beta_1, \ldots, \beta_n)^T \in R^n, \xi_{1,i} : R^n \to R^n, \xi_{2,i} : R^n \to R^n \) are continuous functions, \( C \in R^{n1} \). Therefore, the first problem to be addressed is the existence of such a transformation. The proposed solution uses differential logistic equations to realize system \( f \). This means we will approach function \( g(t) \) with a weighted sum \( y(x(t)) \), for which we then have to deal with the issue of identifying the parameter values of (2). To this purpose, in control-theoretic terms, system \( f \) is considered the reference system, whereas the tracking system will have the following description:

\[
\dot{\hat{x}} = \sum_{i=1}^{n} \xi_{1,i}(\hat{x})\hat{\alpha}_i + \sum_{i=1}^{n} \xi_{2,i}(\hat{x})\hat{\beta}_i + \eta(y(x), y(\hat{x}), t), \quad y(\hat{x}) = \hat{C}\hat{x},
\]

where \( \hat{x} \in R^n, \hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n)^T, \hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_n)^T \in R^n, \hat{C} \in R^n \). Note the similarity in structure between tracking and reference system, except for an error function \( \eta : R^3 \to R^n \), added to the tracking system. In what follows symbols \( x(t), \hat{x}(t) \) denote the solutions of differential equations \( f, \hat{f} \) with parameters \( \alpha, \beta \) (\( \hat{\alpha} \) and \( \hat{\beta} \)) and starting from initial conditions \( x_0 \). Sometimes in order to stress this dependence explicitly we will write \( x(\alpha, \beta, x_0, t) \) or \( \hat{x}(\hat{\alpha}, \hat{\beta}, x_0, t) \).

\(^1\)We would like to note that dimensions of the vectors \( \alpha \) and \( \beta \) are not necessarily equal to \( n \). Although we do not discuss any other parameterization, a variety of alternative descriptions with different parameterizations is possible.
As both the reference and tracking systems are described in the same manner, it is natural to consider the combined system, which couples reference to tracking system via output \( y(x(t)) \) through the error function \( \eta(y(x), y(\hat{x})) \):

\[
\dot{x} = \sum_{i=1}^{n} \xi_{1,i}(x) \alpha_i + \sum_{i=1}^{n} \xi_{2,i}(x) \beta_i, \quad y(x) = Cx,
\]

\[
\dot{\hat{x}} = \sum_{i=1}^{n} \xi_{1,i}(\hat{x}) \hat{\alpha}_i + \sum_{i=1}^{n} \xi_{2,i}(\hat{x}) \hat{\beta}_i + \eta(y(x), y(\hat{x}), t), \quad y(\hat{x}) = \hat{C}\hat{x}. \tag{4}
\]

It is possible then to estimate the unknown parameters \( \alpha, \beta, C \) of the reference system. We start out by assuming that the only uncertainties are in the vectors \( \alpha \) and \( \beta \), while vector \( C \) is supposed to be known. We will propose an algorithm for parameter adjustment that is capable of finding the solution. Our learning algorithm will belong to the following class:

\[
\dot{\hat{\alpha}} = A(y(x), y(\hat{x}), \hat{x});
\]

\[
\dot{\hat{\beta}} = B(y(x), y(\hat{x}), \hat{x}), \tag{5}
\]

where operators \( A(\cdot) \) and \( B(\cdot) \) are to be determined on the basis of the speed-gradient algorithm \cite{12}. If this strategy works, an extension would be to consider cases where the reference system does not represent function \( g(t) \) completely (i.e. systems with unmodeled dynamics).

Thus, the questions to be addressed are: is it possible (at least in theory) to transform a problem of nonlinear static optimization into a problem of searching for linearly parameterized nonlinear differential equations? If so, then how to estimate the parameters of this nonlinear dynamical system in order to obtain qualitative approximation? The next sections will provide us with the answers.

### 3 Approximation with Logistic Differential Equations

Let the following system be given:

\[
\dot{x}_1 = \alpha_1 x_1(1 - \beta_1 x_1);
\]

\[
\dot{x}_2 = \alpha_2 x_2(1 - \beta_2 x_2);
\]

\[
\vdots = \vdots
\]

\[
\dot{x}_n = \alpha_n x_n(1 - \beta_n x_n);
\]

\[
y(x) = C^T x = \sum_{i} c_i x_i, \quad x_i(0) = \Delta_i, \tag{6}
\]

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) is a state vector, \( \alpha_i \in \mathbb{R} \), are parameters of system \((6)\), \( y \) is an output function, \( C = (c_1, \ldots, c_n)^T \in \mathbb{R}^n \) is a vector of parameters associated with output \( y \), \( x_i(0) \in \mathbb{R} \) are initial conditions.
We begin our investigation by asking the question: what dynamics can the autonomous system (6) produce as a function of \( t \)? The answer to this question is formulated in the following theorem:

**Theorem 1** Let continuously differentiable function \( g(t) : \mathbb{R} \rightarrow \mathbb{R} \) be given. Then for any \( \varepsilon > 0 \), \( 0 < T < \infty \) and \( t \in [0, T] \) there are such numbers \( n, \alpha_i, \beta_i, c_i \) and initial conditions \( x_i(0) = \Delta_i \) that the following inequality holds:

\[
|y(x(t)) - g(t)| \leq \varepsilon.
\]

Theorem 1 proof is quite straightforward and is based on the known fact that solution of the logistic differential equation of the first order can be given by a sigmoidal function [23]. Nevertheless, in order to make the paper self-contained we present the proof in the Appendix. Proofs of the subsequent theorems and lemmas are given in the Appendix as well.

**Remark 1** It follows from Theorem 1 proof that it is possible to transform the problem of non-linear function approximation by static sigmoidal functions into a problem of choosing initial conditions and parameters \( \alpha_i \) and \( c_i \) of dynamical system (6), where parameters \( \alpha_i \) enter (6) linearly. One can observe, in addition, that under an appropriate linear transformation \( x_i \rightarrow x_i/c_i \) \((c_i \neq 0)\) we can get rid of uncertainties in \( C \) (see Remark 1 after Lemma 2 in Appendix 1) and replace system (6) by

\[
\begin{align*}
\dot{x}_i &= \alpha_i x_i + \beta_i x_i^2; \\
y(x) &= \sum_i x_i, \quad x_i(0) = \Delta_i/c_i,
\end{align*}
\]

where \( \alpha_i \) and \( \beta_i \) are to be determined. We formulate this

**Corollary 1** Let system (7) and continuous differentiable function \( g(t) : \mathbb{R} \rightarrow \mathbb{R} \) be given. Then for any \( \varepsilon > 0 \), \( 0 < T < \infty \) and \( t \in [0, T] \) there are such numbers \( n, \alpha_i, \beta_i \) and initial conditions \( x_i(0) \) that the following inequality holds:

\[
|y(x(t)) - g(t)| \leq \varepsilon.
\]

This result will allow us to turn the problem of determining the nonlinear parameters of a static function into a problem of determining the linear parameters \( \alpha_i, \beta_i \) of system (7). The restrictions are that the values \( x_i(0) \) will have to be known.

**Remark 2** Theorem 1 proves that there is a one-to-one transformation of a function approximation problem in terms of static sigmoidal functions to one in terms of differential logistic equations. The latter, therefore, shares all the advantages of the former, including the very good convergence rate [11] and its application in Sobolev space [17].
Theorem 1 merely states the existence of parameters $\alpha_i$ and $c_i$ of system (6) (or $\alpha_i$ and $\beta_i$ of system (7)) that ensure arbitrarily small errors between the system output and the reference function $g(t)$. It does not answer the question how to derive the parameters. However, the linearity of the system in its parameters simplifies our task. We will show in Section 5 that in the multidimensional case the resulting system will be linearly parameterized as well. In the next section we will turn to the issue of how to find the values of the parameters $\alpha_i$ that yield minimum errors.

4 Parameter Adjustment Algorithm

The question is whether it is possible to estimate the unknown parameter values $\alpha_i$, $\beta_i$ for which $g(t) - y(\hat{x}(t)) = 0$ for $t \in [0, T]$, utilizing the linear parameterization of system (7). For designing the estimation algorithm the following strategy was used: first, it is assumed that the only uncertainties are in the linear parameters $\alpha_i$, $\beta_i$, initial conditions $x(0)$ are assumed to be known. We formulate this in Assumption 1. First, our main algorithm is presented. Second, after this algorithm is given we extend it to the cases where the reference system does not represent the function $g(t)$ completely, i.e., with unmodeled dynamics. It will be possible to invoke Theorem 1 and show that any function that merely is approached by reference system dynamics can still effectively be modelled by the tracking system, albeit within a margin of tolerance.

In order to proceed with the analysis we would like to introduce the following assumption:

**Assumption 1** Let continuous function $g(t)$, number of equations $n$ and initial conditions $x_i(0)$ be given. There exist such parameter values $\alpha_i$ and $\beta_i$ that for any $t \in [0, T]$ the following equality holds for system (7) solutions:

$$g(t) - \sum_{i=1}^{n} c_i x_i(t) = 0.$$ 

Assumption 1 states that the reference signal $g(t)$ can be represented by the output of system (7):

$$g(t) = \sum_{i=1}^{n} c_i x_i(\alpha_i, \beta_i, x_i(0), t).$$

The coefficients $c_i$ can be equal to the unity.

In order to make the presentation more clear and compact, we would like to introduce a notational assumption regarding the tracking and reference systems. Let us redefine the system
equations, denoting the right-hand side of (7) by $C$ where $K$ is that we need the tracking system "to copy" the reference dynamics along a manifold $y$ and tracking system (7) can be written in the following form:

$$\alpha = \sum_{i=1}^{n} \xi_{1,i}(x)\alpha_i + \sum_{i=1}^{n} \xi_{2,i}(x)\beta_i,$$

$$\dot{\alpha} = \sum_{i=1}^{n} \xi_{1,i}(x)\dot{\alpha}_i + \sum_{i=1}^{n} \xi_{2,i}(x)\dot{\beta}_i + \eta(y(x),y(\hat{x}),t), \ y(\hat{x}) = \hat{C}\hat{x},$$

where $C = \hat{C} = (1, \ldots, 1)^T$. Hence, to complete the definitions of reference and tracking systems one needs to determine $\eta(y(x),y(\hat{x}),t)$. One possible way to do this is to define the function $\eta(y(x),y(\hat{x}),t)$ as follows:

$$\eta(y(x),y(\hat{x}),t) = K(t)(y(\hat{x}) - y(x)),$$

where $K(t) = (k_1(t), \ldots, k_n(t))^T$ and $k_i(t)$ are to be specified later. The reason for such a structure is that we need the tracking system "to copy" the reference dynamics along a manifold $y(x) - y(\hat{x}) = 0$. Thus, an aggregated system which contains both the reference system for signal $g(t)$ and tracking system (7) can be written in the following form:

$$\dot{x} = \sum_{i=1}^{n} \xi_{1,i}(x)\alpha_i + \sum_{i=1}^{n} \xi_{2,i}(x)\beta_i, \ y(x) = Cx,$$

$$\dot{\hat{x}} = \sum_{i=1}^{n} \xi_{1,i}(\hat{x})\dot{\alpha}_i + \sum_{i=1}^{n} \xi_{2,i}(\hat{x})\dot{\beta}_i + K(t)(y(\hat{x}) - y(x)), \ y(\hat{x}) = \hat{C}\hat{x}, \quad (8)$$

As has been mentioned in the beginning of the section, we would like to obtain such estimates of the parameters $\alpha_i, \beta_i$, that $g(t) - y(\hat{x}(t)) = 0$ over time-interval $[0,T]$. It was proposed in Section 2 to utilize conventional speed-gradient like techniques to design the learning or adaptation rule. For these methods, the parameters are supposed to be adjusting on-line, that is in the same time-scale as the reference and tracking systems evolve. In general, it may take much more time than $T$ (the length of the interval $[0,T]$) for the estimates $\hat{\alpha}_i, \hat{\beta}_i$ to converge to $\alpha_i, \beta_i$. However, the function $g(t)$ may not be defined for $t > T$, and even if it is well defined over $[T,\infty)$ then equivalence $y(x(\alpha,\beta,t,x_0)) \equiv y(\hat{x}(\hat{\alpha},\hat{\beta},t,x_0))$ for $t > T$ does not imply that $g(t) = y(\hat{x}(\hat{\alpha},\hat{\beta},t,x_0))$ for any $t \in [0,T]$. 


In addition, we note that logistic equations (6) can be very unstable and may have finite escape time depending on the vectors $\alpha$ and $\beta$. For the reference system this is not important as we assumed that every solution $x_i$ of (6) can be described by a sigmoid function and therefore is bounded. For the tracking system, however, stability becomes very crucial. It is very well possible that during $\dot{\alpha}$ and $\dot{\beta}$ adjustment and due to the term $K(t)(y(x) - y(\hat{x}))$ in (8) the state $\dot{x}$ of the reference system can reach infinity in finite time thus making the whole system unstable.

Taking these considerations into account, it is necessary to redesign the reference and tracking systems in such a way that: 1) $y(x(\alpha, \beta, t, x_0)) \rightarrow y(\dot{x}(\dot{\alpha}, \ddot{x}, t, x_0))$ as $t \rightarrow \infty$ implies that $|g(t) - y(\dot{x}(\dot{\alpha}, \ddot{x}, t, x_0))| < \varepsilon$ for any $\varepsilon > 0$ and arbitrary $t \in [0, T]$; and 2) the state $\dot{x}$ of the tracking system remains bounded for any $t > 0$.

Our proposed solution to problem 1) is to let the reference signal $g(t)$ be repeated periodically (see Fig. 1, where the initial signal $g(t)$ is extended periodically along axis $t$). Periodicity can be achieved by introducing special terms ($\lambda$ and $\sigma$ below) into the systems right-hand sides that will periodically force the states to move to $x_0$ (with period $T_1 = T + \Delta T_2$, where $\Delta T_2$ is amount of time needed to reach $x_0$). In order to solve problem 2) we have to make sure that state $\dot{x}$ of the tracking system is bounded for any $t > 0$. This can be achieved if we force the states of both systems to move to $x_0$ as soon as $||\dot{x}||$ exceeds certain bound $D$. Roughly speaking, one can add time-varying negative feedback to both reference and tracking systems, thus making the point $x_0$ globally asymptotically stable for both systems and, in addition, allowing the output $y(x(\alpha, \beta, t, x_0))$ of the reference system to coincide periodically with the segments of trajectory $g(t)$ defined over $[0, T]$.

In order to satisfy these requirements we introduce the next

**Assumption 2** There is a positive constant $l_0 > 0$ and function $\lambda : R^2 \rightarrow R$

$$
\lambda(t, D) = \begin{cases} 
0, & t \in [(j - 1)T_1, jT_1 - \Delta T_2) \text{ and } ||\dot{x}(t)|| < D \\
1, & t \in [jT_1 - \Delta T_2, jT_1) \text{ or } ||\dot{x}(t)|| \geq D 
\end{cases}, \quad j \in \{1, 2, \ldots, \infty\},
$$

such that the reference signal is given by the following system:

$$
\dot{x} = \left(\sum_{i=1}^{n} \xi_{1,i}(x)\alpha_{i} + \sum_{i=1}^{n} \xi_{2,i}(x)\beta_{i}\right)(1 - \lambda(t, D)) - \lambda(t, D)\sigma(x - x(0))
$$

$$
y(x(t)) = \bar{g}(t),
$$

where $\sigma(\cdot)$ is a signum function:

$$
\sigma(\cdot) = (\sigma_1(\cdot), \ldots, \sigma_n(\cdot))^T : \sigma_i(x - x(0)) = \begin{cases} 
1, & x_i - x_i(0) > 0; \\
0, & x_i - x_i(0) = 0; \\
-1, & x_i - x_i(0) < 0,
\end{cases}
$$

$l_0 \geq D/\Delta T_2$, $\bar{g}(t_1)$, $t_1 \in [0, \infty)$ is an extension of $g(t)$, $t \in [0, T]$ and $T_1 = T + \Delta T_2$. 


Assumption 2 requires an inclusion of several extra parameters and functions into the generating system right-hand side. Additional restrictions are to be introduced just to make sure that for each $t = jT_1$, the following holds:

$$x_i(t) = x_i(jT_1) = \hat{x}_i(t) = \hat{x}_i(jT_1) = x_i(0), \ j = \{1, 2, \ldots, \infty\}, \ i \in \{1, \ldots, n\}.$$

Taking into account Assumption 2 and the fact that the tracking system is designed to copy the structure of the reference system, we can write the combined reference and tracking systems as follows:

\[
\begin{align*}
\dot{x} &= \left( \sum_{i=1}^{n} \xi_{1,i}(x)\alpha_i + \sum_{i=1}^{n} \xi_{2,i}(x)\beta_i \right) (1 - \lambda(t, D)) - \lambda(t, D)I_0\sigma(x - x(0)) \\
\dot{\hat{x}} &= \left( \sum_{i=1}^{n} \xi_{1,i}(\hat{x})\hat{\alpha}_i + \sum_{i=1}^{n} \xi_{2,i}(\hat{x})\hat{\beta}_i + K(t)(y(\hat{x}) - y(x)) \right) (1 - \lambda(t, D)) - \lambda(t, D)I_0\sigma(\hat{x} - x(0)) \\
\hat{y}(t) &= y(\hat{x}(t)) = \hat{C}^T\hat{x}(t); \\
y(t) &= y(x(t)) = C^Tx(t).
\end{align*}
\]

(9)

Before we introduce an adjustment rule for the tracking system let us formulate the following lemma:

**Lemma 1** Let system (9) be given and $\hat{C}^T \neq 0$. Consider

\[
|\hat{C}^T \sum_{i=1}^{n} (\alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x))) (1 - \lambda(t, D))) + \epsilon \sum_{i=1}^{n} k_i\hat{c}_i
\]

Then for any given constant $\delta > 0$ there exist $k_i = k_i^* \in R$ such that

\[
|\hat{C}^T \sum_{i=1}^{n} (\alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x))) (1 - \lambda(t, D))) + \epsilon \sum_{i=1}^{n} k_i^*\hat{c}_i < 0
\]

(10)

for any $\epsilon > \delta$.

According to Lemma 1 for any positive $\delta > 0$ the existence of the coefficients $k_i^*$ satisfying inequality (10) is guaranteed. This property is very important for the subsequent analysis. In fact, it states that the error function $e = \hat{y}(t) - y(t)$ is attracted to the domain $|e| \leq \delta$ at $\hat{\alpha} = \alpha$, $\hat{\beta} = \beta$, $\lambda(t, D) = 0$ and $k_i(t) = k_i^*$ as

\[
\dot{e} = \left( \hat{C}^T \sum_{i=1}^{n} (\alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x))) + \epsilon \sum_{i=1}^{n} k_i^*\hat{c}_i \right) (1 - \lambda(t, D))
\]

and

\[
\frac{d}{dt}(0.5e^2) = ee < 0, \ \forall|e| > \delta.
\]

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Let us introduce the adjustment rules for parameters $\hat{\alpha}_i$, $\hat{\beta}_i$:
\begin{align}
\dot{\hat{\alpha}}_i &= -\gamma e(t)S_\delta(e)\hat{C}^T\xi_{1,i}(\hat{x})(1 - \lambda(t, D)), \\
\dot{\hat{\beta}}_i &= -\gamma e(t)S_\delta(e)\hat{C}^T\xi_{2,i}(\hat{x})(1 - \lambda(t, D)), \\
S_\delta(e) &= \begin{cases} 
1, & |e| > \delta \\
0, & |e| \leq \delta .
\end{cases}
\end{align}

(11)

where $e(t) = \dot{y}(t) - y^*(t)$ is the tracking error, $\gamma > 0$ is a positive constant.

The stability properties of system (9) with algorithm (11) are formulated in:

**Theorem 2** Let Assumptions 1, 2 hold, vector $\hat{C} \neq 0$, and function $K(t) = (k_1(t), \ldots, k_n(t))^T$ in (9) be given by the following system of differential equations
\begin{align}
\dot{\hat{\alpha}}_i &= -\gamma S_\delta(e)\hat{C}^T\xi_{1,i}(\hat{x})(1 - \lambda(t, D)).
\end{align}

(12)

Then for any positive $\gamma > 0$ all trajectories of system (9) are bounded, and there exists $t_1 > 0$ such that for any $t > t_1$ the following inequality holds:
\begin{align}
|y(x) - y(\hat{x})| < \delta + \delta_1, \delta_1 > 0.
\end{align}

**Remark 3** Theorem 2 guarantees that function $e(t)\lambda(t, D)$ in system (9) converges to the domain $|e(t)\lambda(t, D)| < \delta$, where constant $\delta$ is defined in learning algorithm (11). Formally, $|e(t)\lambda(t, D)| < \delta$ does not automatically imply that estimates $\hat{\alpha}, \hat{\beta}$ converge to the point $\hat{\alpha} = \alpha, \hat{\beta} = \beta$ in the parameter space. Nevertheless, according to formula (10) (see Appendix, proof of Theorem 2), one can derive the following estimate of how close we are to the solution
\begin{align}
||\hat{\alpha}(t_0) - \alpha||^2_{\gamma^{-1}} + ||\hat{\beta}(t_0) - \beta||^2_{\gamma^{-1}} - ||\bar{\alpha}(t) - \alpha||^2_{\gamma^{-1}} - ||\bar{\beta}(t) - \beta||^2_{\gamma^{-1}} \\
\geq ||K(t) - k^*||^2_{\gamma^{-1}} - ||K(t_0) - k^*||^2_{\gamma^{-1}} + 2\int_{t_0}^{t} S_\delta(e)|\epsilon(\tau)\lambda(\tau, D)\sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1 d\tau.
\end{align}

(13)

Equation (13) may be taken to reflect the quality of estimation of the unknown parameters $\alpha$ and $\beta$. In particular, if we choose $K(t_0) = 0$, then
\begin{align}
||\hat{\alpha}(t_0) - \alpha||^2_{\gamma^{-1}} + ||\hat{\beta}(t_0) - \beta||^2_{\gamma^{-1}} - ||\bar{\alpha}(t) - \alpha||^2_{\gamma^{-1}} - ||\bar{\beta}(t) - \beta||^2_{\gamma^{-1}} \\
\geq ||K(t) - k^*||^2_{\gamma^{-1}} - ||k^*||^2_{\gamma^{-1}} + 2\int_{t_0}^{t} S_\delta(e)|\epsilon(\tau)\lambda(\tau, D)\sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1 d\tau.
\end{align}

Therefore, the smaller the norm $||K(t)||$, the greater is the chance that the difference
\begin{align}
||\hat{\alpha}(t_0) - \alpha||^2_{\gamma^{-1}} + ||\hat{\beta}(t_0) - \beta||^2_{\gamma^{-1}} - ||\bar{\alpha}(t) - \alpha||^2_{\gamma^{-1}} - ||\bar{\beta}(t) - \beta||^2_{\gamma^{-1}}.
\end{align}

(14)

is nonnegative. On the other hand, given the values of $\delta, \delta_1, D, \hat{C} \hat{C}$ and bounds for $\alpha, \beta$, one can explicitly estimate vector $k^*$, satisfying inequality (10). Hence in this case formula (13) gives
explicit bounds for the deviations of the estimates \( \hat{\alpha}, \hat{\beta} \) with respect to \( \alpha \) and \( \beta \). Furthermore, for known \( k^* \) it is possible to get rid of time-varying coefficients \( k_i(t) \) in (9), replacing them by \( k_i^* \). In this case difference (14) is positive if \( |e(t)\lambda(t, D)| \) exceeds \( \delta \) at some time \( t_1 \).

In general in order to ensure the positiveness of difference (14) for a given parameterization of the reference system, it is necessary to consider more carefully the dynamics of the following deviation \( \rho = \hat{x} - x \) at \( \hat{\alpha} = \alpha \) and \( \hat{\beta} = \beta \) over the time intervals where \( \lambda(t, D) = 0 \):

\[
\dot{\rho} = \left( \sum_{i=1}^{n} \alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x)) \right) + K(t)\hat{C}^T(\hat{x} - x)
\]

Functions \( \xi_{1,i} \) and \( \xi_{2,i} \) are differentiable with respect to their arguments. Therefore there exist such \( \Xi_{1,i}(\hat{x}, x) \) and \( \Xi_{2,i}(\hat{x}, x) \) that the following equalities hold:

\[
\Xi_{1,i}(\hat{x}, x)(\hat{x} - x) = \xi_{1,i}(\hat{x}) - \xi_{1,i}(x)
\]

\[
\Xi_{2,i}(\hat{x}, x)(\hat{x} - x) = \xi_{2,i}(\hat{x}) - \xi_{2,i}(x)
\]

Then derivative \( \dot{\rho} \) can be written in the following form

\[
\dot{\rho} = \left( \sum_{i=1}^{n} \alpha_i\Xi_{1,i}(\hat{x}, x) + \beta_i\Xi_{2,i}(\hat{x}, x) + K(t)\hat{C}^T \right) \rho
\]

It can be derived from Theorem 2 proof that the existence of a positive function \( V(y(\hat{x}), y(x)) \) with time derivative \( \dot{V} \) at \( \hat{\alpha} = \alpha \), \( \hat{\beta} = \beta \) satisfying,

\[
\dot{V}(y(\hat{x}), y(x)) = -W(y(\hat{x}) - y(x))
\]

where \( W(\cdot) \) is a positive definite function, guarantees monotonic increase of the difference (14). Therefore, if one can find vector \( K(t) \) such that it asymptotically stabilizes system (15) for the given domain of parameters \( \alpha, \beta \), and furthermore, inequality (16) holds, then the positiveness of difference (14) is guaranteed. The problem of determining \( K(t) \) however is not very easy to solve, especially for nonlinear systems. Even for linear ones, a similar problem known in the literature as the Brockett problem\(^2\) \[6\] has positive solutions at present for systems of second and third order \[21, 30\]. Nevertheless, despite the obvious difficulties, we believe that the question of searching for the suitable \( K(t) \) ensuring inequality (16) for system (15) could be an achievable goal for future studies.

\(^2\)Let the following triplet of matrixes be given \( A, B, C \in \mathbb{R}^{n \times n} \). Under what conditions does a time-variant matrix \( K(t) \) exist such that system

\[
\dot{x} = Ax + BK(t)Cx, \quad x \in \mathbb{R}^n
\]

is asymptotically stable?
It is desirable to note that Theorem 2 requires the validity of Assumption 1. Assumption 1 allowed us to model the function $g(t)$ by a reference system of the same structure as the tracking one. This feature has been exploited in the proof of the theorem and played an important role in order to guarantee convergence of errors to a neighborhood of the origin. This assumption may be too restrictive as it requires strict equivalence between reference and tracking signals for $\hat{\alpha} = \alpha, \hat{\beta} = \beta$. We are now ready to abandon this assumption by invoking Theorem 1 again.

If Assumption 1 does not hold this leads to nonzero error $\varepsilon(t)$ between the output $y(x) = C^T x(t)$ of the reference system (26) and signal $g(t)$ to be tracked:

$$\varepsilon(t) = \sum_{i=1}^{n} c_i x_i(t) - g(t).$$

Let us assume that $g(t)$ is continuously differentiable, then $\varepsilon(t)$ is differentiable as well. We denote its first derivative by $d\varepsilon(t)$:

$$\frac{d}{dt}(y(x(t)) - g(t)) = \sum_{i=1}^{n} c_i \dot{x}_i - \dot{g}(t) = d\varepsilon(t).$$

Due to the compactness of the interval $[0, T]$ we can conclude that derivative $d\varepsilon(t)$ is bounded:

$$|d\varepsilon(t)| < s.$$

Let us derive the error $e(t) = y(\hat{x}) - g(t) = y(\hat{x}) + \varepsilon(t) - y(x)$ dynamics taking into account that $C = \hat{C}$ and, in addition, that function $\varepsilon(t)$ can be considered as an unmeasured disturbance subtracted from the output $y(x(t))$ generated by the reference system (26):

$$\dot{e} = \hat{C}^T \left( \sum_{i=1}^{n} \hat{\alpha}_i \xi_{1,i}(\hat{x}) - \alpha_i \xi_{1,i}(x) + \hat{\beta}_i \xi_{2,i}(\hat{x}) - \beta_i \xi_{2,i}(x) \right) (1 - \lambda(t, D)) - d\varepsilon(t) + \hat{C}^T (K(t)(y(\hat{x}) - y(x) + \varepsilon(t))(1 - \lambda(t, D)) + l_0(\sigma(x - x_0) - \sigma(\hat{x} - x_0))\lambda(t, D))$$

The only difference between error dynamics according to Assumption 1 and the expression given in (18) is in the term $d\varepsilon(t) + \hat{C}^T K(t)\varepsilon(t)$ which represents the unmodeled dynamics of $g(t)$.

There are several ways to deal with such an uncertainty. One of them is to include a dead-zone into the parameter adjustment scheme and chose $K(t) = \text{const}$. The algorithms with a dead-zone will have the same form as (11):

$$\dot{\hat{\alpha}}_i = -\gamma e(t)S_\delta(e)\hat{C}^T \xi_{1,i}(\hat{x})(1 - \lambda(t, D)),$$

$$\dot{\hat{\beta}}_i = -\gamma e(t)S_\delta(e)\hat{C}^T \xi_{2,i}(\hat{x})(1 - \lambda(t, D)),$$

$$S_\delta(e) = \begin{cases} 1, & |e| > \delta \\ 0, & |e| \leq \delta \end{cases}.$$
except that the width $\delta$ of the dead-zone is to depend on the bounds for $d\varepsilon(t)$ and $C^TK\varepsilon(t)$. Theoretical analysis of the stability of the whole system with learning rule (19) can be done in the same manner as with (11).

It is clear that the tolerance of the resulting learning process will depend on the dead-zone width $\delta$, which is exactly the upper bound of $d\varepsilon(t) + C^TK\varepsilon(t)$. Therefore, in general, applicability of the proposed learning rules strongly depends on a smoothness of $\varepsilon(t)$ (in the sense of the maximum absolute value of its first derivative). We may deal with this issue by referring to the properties of this approximation scheme in Sobolev space [17]: [18]. It can be shown that for any arbitrary small $\delta_2 > 0$ there exists a network that can approximate a given reference function $g(t)$ such that both derivative $d\varepsilon(t)$ and $\varepsilon(t)$ satisfy the following estimation: $|d\varepsilon(t) + C^TK\varepsilon(t)| < \delta_2$. Hence, learning algorithm (19) will still be applicable even in the presence of nonzero differentiable error $\varepsilon(t)$ between the reference signal and outputs of the tracking system at $\hat{\alpha} = \alpha$, $\hat{\beta} = \beta$. What value of $\delta$ is admissible will depend on the dimension of the system.

## 5 Discussion

Here we discuss multi-dimensional extensions with an eye for possible neural network applications of our approach. Theorem 1 states that any continuous function of $t$ can be approximated over time interval $[0, T]$ by a linear combination of the solutions of system (7). It is desirable to note that we can choose function $g(t)$ in such a way that the following equality holds:

$$g(t) = \tilde{g}(\xi(t)), \quad (20)$$

where $g \in C^1$, $\xi(t)$ is a smooth function of $t$. Let us suppose that system (7) realizes function $\tilde{g}(\xi)$. This means that

$$\tilde{g}(\xi) = \sum_{i=1}^{n} c_i x_i(\xi),$$

where $\dot{x}_i = \alpha_i x_i(1 - \beta_i x_i)$. Then we consider function $\tilde{g}(\xi)$ as a function of time $t$ which satisfies equation (20). Therefore due to formula (20) we can write:

$$\tilde{g}(\xi(t)) = \sum_{i=1}^{n} c_i x_i(\xi(t)).$$

Moreover

$$\dot{\tilde{g}}(t) = \frac{d}{dt} \tilde{g}(\xi(t)) = \frac{\partial}{\partial \xi} \tilde{g}(\xi) \frac{\partial}{\partial t} \xi(t) = \sum_{i=1}^{n} c_i x_i(1 - \beta_i x_i) \dot{\xi}.$$  

Hence under the following assumptions: $\dot{g}(t) = \dot{\tilde{g}}(t)$ at $t = 0$ and $g(0) = \tilde{g}(\xi(0))$ we can see that linear combination $\sum_{i=1}^{n} c_i x_i(t)$ of the solutions of system

$$\dot{x}_i = \alpha_i x_i(1 - \beta_i x_i) \dot{\xi}(t)$$
realizes function \(g(t)\) and vice-versa. This simple observation suggests how to extend the result to the multi-dimensional case. It is possible to consider a reference function \(g(\xi_1, \ldots, \xi_m)\) with \(m\) inputs as a function of time \(t\): \(g(\xi_1(t), \ldots, \xi_m(t))\). Then a system which realizes function \(g(\xi_1(t), \ldots, \xi_m(t))\) can be represented in the following form:

\[
\dot{x}_i = \left(\sum_{j=1}^{m} \alpha_{i,j} \dot{\xi}_j(t)\right) x_i(1 - \beta_{i,j} x_i);
\]

\[
y(\hat{x}(t)) = \sum_{i=1}^{n} c_i x_i(t). \tag{21}
\]

If we return to the approximation problem we may observe on account of Theorem 1 that system (21) is able to approximate a given function \(g(\xi_1, \ldots, \xi_m)\) over a given compact domain in such a way that for a particular trajectory \((\xi_1(t), \ldots, \xi_m(t))\) and any given constant \(\varepsilon > 0\) there exist parameters \(\alpha_{i,j}, \beta_{i,j}, c_i\), initial conditions and number \(n\) satisfying the following:

\[
|g(\xi_1(t), \ldots, \xi_m(t)) - y(\hat{x}(t))| \leq \varepsilon.
\]

Curve \(\xi(t)\) should be designed in such a way that good approximation along the curve \(\xi(t)\) implies good approximation along the whole surface. Intuitively, this depends on the degree to which the curve ”covers” the space. In other words, the more complex curve \((\xi_1(t), \ldots, \xi_m(t))\) is, the better the approximation that can be achieved over the given compact interval.

An important consequence of this description is that a system of coupled logistic differential equations (21) may realize an approximation of a nonlinear time-invariant system of the following type:

\[
\dot{y} = \chi(y), \tag{22}
\]

where \(\chi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n\) is an arbitrary smooth function. Let us explain this. Denote:

\[
\mathcal{F}(x, b, c, t) = \sum_{i=1}^{n} c_i f(a_i t + b_i).
\]

Consider system (21) for \(m = 1\) and replace \(\dot{\xi}(t)\) by \(\xi(t)\):

\[
\dot{x}_i = \alpha_i \xi(t) x_i(1 - \beta_i x_i);
\]

\[
y(x(t)) = \sum_{i=1}^{n} c_i x_i(t) = \mathcal{F}(\alpha, \beta, x_0, C, \int_{0}^{t} \xi(\tau) d\tau). \tag{23}
\]

One may substitute function \(y(t)\) in (23) instead of \(\xi(t)\). This leads immediately to the following equations:

\[
\dot{\hat{x}}_i = \alpha_i y(t) x_i(1 - \beta_i x_i);
\]

\[
y(t) = \mathcal{F}(\alpha, \beta, x_0, C, \int_{0}^{t} y(\tau) d\tau). \tag{24}
\]
Denoting \( z(t) = \int_0^t y(\tau) d\tau \) and taking into account that \( y = \sum_{i=1}^n c_i x_i \) we can rewrite system (24) in the following manner:

\[
\dot{x}_i = \alpha_i \left( \sum_{j=1}^n c_j x_j \right) x_i (1 - \beta_i x_i);
\]

\[
\dot{z} = \sum_{i=1}^n c_i x_i,
\]

(25)

where the new output function \( z(t) \) satisfies the following differential equation:

\[
\dot{z} = \mathcal{F}(\alpha, \beta, x_0, C, z).
\]

\( \mathcal{F}(\alpha, \beta, x_0, C, z) \) may realize function \( \chi(z) \) with given tolerance subject to the choice of the parameters \( \alpha, \beta, x_0, C \) and the number of equations in (25). In the same fashion one can derive the results for \( m > 1 \) and obtain the corresponding systems for differential equations:

\[
\dot{z}_i = \mathcal{F}_i(\alpha, \beta, x_0, C, z_1, z_2, \ldots, z_i, \ldots, z_n),
\]

thus approximating (22).

There are two important observations to be made regarding system (25). First, one may notice that system (25) is a specific instance of the Cohen-Grossberg model [8]. Therefore, it is possible to claim that Cohen-Grossberg models of several differential equations, each of which has relatively simple description (for instance, coupled logistic differential equations), in principle, are capable of approximating every nonlinear dynamical system with smooth right-hand sides (subject to appropriate choice of the number of differential equations, initial conditions and parameters). Furthermore, the learning algorithms, introduced in the paper can be applied to these models as well, and their stability may be proven in the same fashion. Second, it is desirable to notice that this approach allows us to introduce an alternative learning technique to that of backpropagation through time [40], albeit for continuous-time systems. A detailed discussion of these topics is beyond the scope of the present paper.

The algorithms introduced in the paper guarantee that under certain circumstances the estimates \( \hat{\alpha}, \hat{\beta} \) approach to a domain around \( \alpha, \beta \). Still, they cannot guarantee that \( \hat{\alpha} \to \alpha \) and \( \hat{\beta} \to \beta \). An interesting problem, therefore, is whether it is possible to design a tracking system that guarantees convergence of \( \hat{\alpha}, \hat{\beta} \) to \( \alpha \) and \( \beta \) respectively. This problem in our opinion is closely related to the problem of adaptive observer design [28] for the reference system in (9):

\[
\dot{x} = \left( \sum_{i=1}^n \xi_{1,i}(x) \alpha_i + \sum_{i=1}^n \xi_{2,i}(x) \beta_i \right) (1 - \lambda(t, D)) - \lambda(t, D) \sigma(x - x(0))
\]

\[
y(x(t)) = C^T x.
\]

(26)
A prerequisite for applying the corresponding method is that these systems are transformed into the *canonical observable form* \[5\]. For nonlinear systems that are linear in parameters necessary and sufficient conditions for this have been given \[25\]. These conditions do not hold, however, for the parameterizations of type \(26\). Therefore, the question remains open, whether is it possible to find such linearly parameterized nonlinear system and corresponding output function \(y(x)\), such that 1) its parameters can be transformed by one-to-one mapping into those of sigmoid superposition, and 2) the parameterization of this system obeys assumptions introduced in work \[25\] (see Theorem 3.1). If one finds such a suitable parameterization, then the problem of finding the “true” parameters (subject to permutations) can be solved effectively.

6 Examples

In this section we illustrate the theoretical results with examples. First we consider application of Theorem \[2\] to the search for unknown parameter values of a single sigmoid function and then show the effectiveness of our method in comparison with the conventional schemes for two-dimensional optimization problem. In addition we illustrate our method with the results of computer simulations performed for a system consisting of 10 sigmoidal functions.

6.1 Example 1

Let us illustrate the possibility to search for the parameters \(\alpha_i\) and \(c_i\) simultaneously. As has been suggested in Section 3, instead of the parameters \(\alpha_i\) and \(c_i\) we will deal with \(\alpha_i/c_i\) and \(\beta_i = \alpha_i/c_i\). Reference function \(g(t)\) has been chosen to satisfy:

\[
g(t, \alpha, c) = \frac{c}{1 + e^{-\alpha t + 2.944}},
\]

where \(c = 2, \alpha = 2/3\). We design the reference and tracking systems as follows:

\[
\begin{align*}
\dot{x} &= (\alpha x - \beta x^2)(1 - \lambda(t)) - \lambda(t)(l_0\sigma(x - x(0))) \\
\dot{x} &= (\alpha\dot{x} - \beta\dot{x}^2)(1 - \lambda(t)) - \lambda(t)(l_0\sigma(\dot{x} - x(0))) - K(t)e,
\end{align*}
\]

(27)

where \(\alpha = 2/3, \beta = 1/3, l_0 = 1, x(0) = 0.1, K(t) = 0.2, e = \dot{x} - x\). Function \(\lambda(t)\) was chosen to be a periodic function with period \(T = 10\) sec, pulse width is 1 sec and unit amplitude (one may easily check that this parameter setting ensures exact matching between function \(g(t)\) and \(x(t)\) over time interval \([0, 9]\)).

Adaptation rules to adjust the parameters \(\hat{\alpha}\) and \(\hat{\beta}\) may be written as follows:

\[
\begin{align*}
\dot{\hat{\alpha}} &= -0.2e(t)\dot{x}(t)(1 - \lambda(t)); \\
\dot{\hat{\beta}} &= 0.2e(t)\dot{x}^2(t)(1 - \lambda(t)).
\end{align*}
\]

(28)
In order to make the example more illustrative we would like to compare the performance of algorithm (28) with a conventional pattern-by-pattern gradient scheme:

\[
\begin{align*}
\dot{\hat{\alpha}} &= -0.2e(t) \frac{\partial g(t, \hat{\alpha}, \hat{c})}{\partial \hat{\alpha}} \\
\dot{\hat{c}} &= -0.2e(t) \frac{\partial g(t, \hat{\alpha}, \hat{c})}{\partial \hat{c}}
\end{align*}
\]  

(29)

and batch rule:

\[
\begin{align*}
\dot{\hat{\alpha}} &= -0.2 \frac{\partial J(\hat{\alpha}, \hat{c})}{\partial \hat{\alpha}} \\
\dot{\hat{c}} &= -0.2 \frac{\partial J(\hat{\alpha}, \hat{c})}{\partial \hat{c}},
\end{align*}
\]

(30)

where

\[J(\hat{\alpha}, \hat{c}) = \int_0^\infty (g(\tau, \hat{\alpha}, \hat{c}) - g(\tau, \alpha^*, c^*))^2 d\tau\]

Results of such a comparison are shown if Figures 2-5. In Figure 2 there are two trajectories of the parameters \(\hat{\alpha}(t)\) and \(\hat{c}(t)\) in two-dimensional space. The first curve is obtained from the trajectories of \(\hat{\alpha}(t) = \hat{\alpha}(t), \hat{c}(t) = \hat{\alpha}(t)/\hat{\beta}(t)\) and results from algorithm (28) with initial conditions \(\hat{\alpha}(0) = -3, \hat{\beta}(0) = 1\). Curve 2 is a solution of (29) starting from initial conditions \(\hat{\alpha}(0) = -3, \hat{c}(0) = -3\). It can be seen that algorithm (28) reaches the global minimum. Conventional gradient descent fails to do so. It appears unstable and goes through a neighborhood of the global minimum along a valley. This process is shown in Fig. 2. In addition, algorithm (28) is much faster than (29) (see Fig. 3 for details).

Figure 4 reflects another interesting feature of algorithm (28). Whereas the conventional gradient algorithm starting from \(\hat{\alpha}(0) = 3, \hat{c}(0) = -3\) goes towards the goal along the isolines (Curve 2), algorithm (28) does not stick to isolines. Instead, it goes through infinity in the coordinates \(\hat{\alpha}, \hat{c}\). This is not because of any singularities with respect to the coordinates \(\hat{\alpha}, \hat{\beta}\) but is due simply to the transformation \(\hat{c} = \hat{\alpha}/\hat{\beta}\), when \(\hat{\beta}\) goes through zero.

Figure 5 contains the trajectories of the solutions obtained with algorithm (30). Curve 1 shows the trajectory corresponding to initial conditions \(\hat{\alpha}(0) = -3, \hat{c}(0) = -3\), Curve 2 is related to initial conditions \(\hat{\alpha}(0) = 3, \hat{c}(0) = -3\). It is easy to see that this algorithm gets stuck in local minima.

The performance of algorithm (28) is not surprising because it uses information about the system properties in a more intelligent way than gradient descent methods do. In addition some coordinate transformation has been used and the process of searching for the minimum is organized in a different coordinate system. All the results relating to stability, however, remain true for the functions which may be represented by a superposition of sigmoid function only.
6.2 Example 2

In addition to the simple example of the previous section which merely illustrates the design procedure for the parameters adjustment rules proposed in the paper, we would like to present more supporting results of computer simulation of our algorithms for a larger number of functions in superposition. We consider the sum of 10 sigmoid functions

\[ g(t, \alpha, C) = \sum_{i=1}^{10} \frac{c_i}{1 + e^{-\alpha_i t + b_i}}, \]

where parameters \( b_i \) and \( c_i \) are assumed to be known and \( t \in [0, T] \). According to the results presented, this sum is equivalent to the solutions of the corresponding system of logistic equations (6) with known \( \beta_i, c_i \) and initial conditions. The only uncertainties are in parameters \( \alpha_i \). First, we extend the reference signal \( g(t) \) to be periodically repeated over \([0, \infty)\):

\[ \tilde{g}(t) = \begin{cases} 
  g(t), & t \leq T \\
  0, & T < t < T + \Delta T_2, \\
  g(t - T - \Delta T_2), & t > T + \Delta T_2 
\end{cases} \]

Then we design the tracking system

\[ \dot{x}_i = \hat{\alpha}_i \dot{x}_i (1 - \dot{x}_i)(1 - \lambda(t, D)) + k_i(t) e(1 - \lambda(t, D)) - \lambda(t, D) l_0 \sigma(\dot{x}_i - x_i(0)) \]  

(31)

and adaptation algorithm

\[ \dot{\alpha}_i = -\gamma S_4(e) \dot{x}_i (1 - \dot{x}_i)(1 - \lambda(t, D)) \]

\[ \dot{k}_i(t) = -\gamma S_6(e) e^2 c_i (1 - \lambda(t, D)) \]  

(32)

where \( D = 10 \) (taking into account that \(|x_i| \leq 1\) we have to choose \( D > 1 \)), \( \lambda(t, D) \) is a \( T + \Delta T_2 \) periodic function with the pulse width \( \Delta T_2, \delta = 0.0001, \gamma = 0.001, T = 2, \Delta T_2 = 1, l_0 = 10 \). Initial conditions \( x_i(0) \) and parameters \( c_i \) were randomly chosen and their exact values are given below:

\[
\begin{align*}
x_1(0) &= 0.1 & c_1 &= 3 \\
x_2(0) &= 0.2 & c_2 &= 5 \\
x_3(0) &= 0.3 & c_3 &= -3 \\
x_4(0) &= 0.2 & c_4 &= 0.5 \\
x_5(0) &= 0.5 & c_5 &= -1 \\
x_6(0) &= 0.1 & c_6 &= 2 \\
x_7(0) &= 0.7 & c_7 &= -0.7 \\
x_8(0) &= 0.2 & c_8 &= 5.5 \\
x_9(0) &= 0.6 & c_9 &= -3 \\
x_{10}(0) &= 0.4 & c_{10} &= 2 
\end{align*}
\]

One could choose the functions \( k_i(t) \) to be equal to some constants over \([0, \infty)\). This however would require knowledge of the exact value for a width of the dead-zone (parameter \( \delta \)) in the adjustment algorithm for this particular set of \( k_i(t) \).
We simulated tracking system (31) with algorithm (32) for 400 trials, choosing the initial conditions for the estimates \( \hat{\alpha}(0) \) randomly in the hypercube \([0, 12]^10\) for every trial, initial conditions for \( k_i(t) \) were set to zero. Each trial consisted of 10000 periods (epoch) and each epoch lasted for \( T + \Delta T_2 = 3 \text{ seconds} \). In order to check the sensitivity of the approach to the numerical integration we used a simple Euler’s method of the first order with integration step \( \delta t = 0.0001 \) seconds to approximate the solutions of \( \dot{x}_i(t) \), \( \dot{\alpha}_i(t) \) and \( k_i(t) \). In order to judge effectiveness of our algorithm we introduced the following criteria:

\[
d(t) = \sqrt{\sum_{i=1}^{10} (\hat{\alpha}_i(t) - \alpha_i)^2}
\]

\[
R(t) = \frac{(T+\Delta T_2)/\Delta t}{\sum_{i=0}^{(T+\Delta T_2)/\Delta t} e(t - T - \Delta T_2 + i\Delta t)^2 \Delta t} \left( \frac{T + \Delta T_2}{T + \Delta T_2} \right)
\]

The histograms of distributions of distances \( d(t) \) and performance indices \( R(t) \) computed in the end of each trial are shown in Fig. 6 and 7, respectively (we made sure that \( d(0) - d((T+\Delta T_2)10000) > 0 \) for every trial). It can be clearly seen from the figures that after application of the algorithm (32) the distributions of the distances \( d((T+\Delta T_2)10000) \) and \( R((T+\Delta T_2)10000) \) are significantly shifted to the left towards zero.

7 Conclusion

In this work the problem of estimating the parameters for a function represented by sigmoid superposition has been analyzed. The key to our proposal is the transformation of this static nonlinearity into a linear combination of solutions of a system of differential equations. These equations are linear in parameters but nonlinear with respect to the state variables. We considered the dynamics of an unperturbed system of differential logistic equations. It was found that a linear combination of the system solutions may realize any continuous function over interval \([0, T]\) with given tolerance \( \varepsilon > 0 \). This tolerance can be made arbitrary small as a function of the number of equations, with corresponding parameters and initial conditions. In addition, we showed that a system of logistic equations with time-varying parameters can realize a function with multiple inputs. The results enabled us to consider a system with coupled equations via output function \( y(\bar{x}) \) as a generator of almost any dynamical system as long as it is smooth in its state and output variables.

The linearity of the resulting system with respect to its unknown parameters allowed us to apply conventional methods and ideas of adaptive control in order to estimate their values for a given reference function. Extension of both the reference and tracking signals to be repeatable
(periodic) over \([0, \infty)\) interval played a crucial role in our analysis. This feature makes it possible to use known matching conditions (or certainty equivalence) to design the adaptation algorithms. Stability analysis has been performed for the learning schemes introduced.

The current algorithm is able to produce the estimates that approach the true values of unknown system parameters within a bounded domain. However, convergence to these true values cannot be guaranteed. It should be mentioned, however, that the problem of finding a flawless algorithm is all but solved by our proposal. The most difficult hurdles to knock down were shown to be the boundedness of solutions and the problem of determining the maximum amplitude of unmodeled dynamics (when the reference signal is not exactly a superposition of sigmoid function). Though we offered possible solution to these issues in the present paper, more effective ones may still exist. Finding these may be a topic for future research.

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Theorem 1 proof. We prove the theorem in 3 steps. First, we transform the original system \((6)\) into a system with its right-hand side depending on one set of parameters \((\alpha = (\alpha_1, \ldots, \alpha_n)^T\) only instead of the two sets \(\alpha\) and \(\beta\)). Second, for each \(x_i, i \in \{1, \ldots, n\}\) we show that the solution \(x_i(t)\) belongs to the interval \([0, 1]\) for any \(x_i(0) \in (0, 1)\); \(x(t)\) is a monotonic and sigmoidal function with parameters depending on \(\alpha\) and initial conditions. Therefore, to conclude the proof it is sufficient to apply a widely-known result\(^3\) from approximation theory \([9]; [13]\).

Let us start with

\begin{equation}
\begin{align*}
\dot{x}_1 &= \alpha_1 x_1 (1 - x_1); \\
\dot{x}_2 &= \alpha_2 x_2 (1 - x_2); \\
\cdots &= \cdots \\
\dot{x}_n &= \alpha_n x_n (1 - x_n); \\
\end{align*}
\end{equation}

\(^3\)Let \(f\) be any continuous sigmoidal function. Then finite sums of the form: \(\sum_{i=1}^{n} c_i f(a_i x + b_i), a_i \in \mathbb{R}^n, x \in \mathbb{R}^n, b_i \in \mathbb{R}\) are dense in \(C(I_n)\).
\[ y(x) = C^T x = \sum_i \frac{C_i}{\beta_i} \hat{x}_i, \quad \hat{x}_i(0) = \beta_i \Delta_i, \quad (33) \]

**Lemma 2** proof. The proof is a routine procedure. Let us calculate \( \dot{\hat{x}}_i = \beta_i \hat{x}_i \):

\[ \dot{\hat{x}}_i = \beta_i \hat{x}_i = \alpha_i \beta_i x_i (1 - \beta_i x_i) = \alpha_i \hat{x}_i (1 - \hat{x}_i). \]

The rest of the lemma proof is quite obvious and we skipped it. The lemma is proven.

**Remark 4** It is desirable to note that the linear transformation \( \hat{x}_i = \beta_i x_i \) is one-to-one, and for any system \( \text{(33)} \) we can derive its transformed version in the form of system \( \text{(7)} \) by the inverse transformation \( x_i = 1/\beta_i \hat{x}_i \). Therefore in the rest of the proof we will deal with system \( \text{(33)} \). In addition, it is always possible to make a transformation such that the resulting \( \alpha_i \) will be positive. Furthermore, given system \( \text{(33)} \), one can choose such linear transformation \( \hat{x}_i = 1/C_i x_i \) that the transformed system obeys

\[ \begin{align*}
\hat{x}_1 &= \alpha_1 \hat{x}_1 (1 - \beta_1 C_1 \hat{x}_1); \\
\hat{x}_2 &= \alpha_2 \hat{x}_2 (1 - \beta_2 C_2 \hat{x}_2); \\
\vdots &= \ldots \\
\hat{x}_n &= \alpha_n \hat{x}_n (1 - \beta_n C_n \hat{x}_n); \\
y(x) &= C^T x = \sum_i \frac{C_i}{C_i} \hat{x}_i = \sum_i \hat{x}_i, \quad \hat{x}_i(0) = \Delta_i/C_i, 
\end{align*} \]

thus eliminating the parametric uncertainties in output function \( y(\hat{x}) \) and replacing them by the parametric uncertainties of linearly parameterized system \( \text{(34)} \) with known output function \( y(\hat{x}) \).

Let us consider the properties of each \( i \)-th equation of system \( \text{(33)} \). We formulate the next lemma:

**Lemma 3** Let the following differential equation be given:

\[ \dot{x} = kx(1 - x), \quad k \neq 0, \quad (35) \]

and \( x(t) \) is a solution of system \( \text{(35)} \) for initial condition \( x(0) = x_0, x_0 \in (0, 1) \). Then the next statements hold for equation \( \text{(35)} \):

1) \( x(t) \) is a monotonic function with respect to \( t > 0 \);
2) \( x(t) \to 1 \) at \( t \to \infty \) for \( k > 0 \) and \( x_0 \in (0, 1) \); \( x(t) \to 0 \) at \( t \to \infty \) for \( k < 0 \) and \( x_0 \in (0, 1) \)
3) \( x(t) \) is unique for any \( t > 0 \) and initial condition \( x_0 \in (0, 1) \).
Lemma proof. Statement 1) of the lemma proof is obvious and therefore has been skipped here (see, for example [37]). Let us prove statement 2) of the lemma. We consider the following function:

\[ V(x) = 0.5(x - 1)^2. \] (36)

It is clear that function \( V(x) \) is well-defined and positive definite for any \( x > 0 \). Moreover, \( V(x) \to \infty \) at \( x \to \infty \) and \( V(x) = 0 \) at \( x = 1 \). These facts allow us to consider function \( V \) as Lyapunov’s candidate for system (35). Let us calculate \( \dot{V} \):

\[ \dot{V} = (x - 1)\dot{x} = -kx(1 - x)^2 \leq 0. \]

We observe that \( V > 0 \) and \( \dot{V} = -kx(1 - x)^2 < 0 \) for \( x > 0, x \neq 1 \). For any \( x \in (0, 1) \), \( V(x(0)) - V(x(t)) > 0 \) and therefore \( x(t) > x(0) \). Hence the next inequality holds:

\[ \dot{V} = (x - 1)\dot{x} \leq -kx(0)(1 - x)^2. \]

This can be written as follows:

\[ \dot{V} \leq -kx(0)2V(x). \]

Hence \( V \to 0 \) asymptotically, and \( x(t) \to 1 \) at \( t \to \infty \) for any \( x \in (0, 1) \). To prove the second part of statement 2, where \( k < 0 \), it is sufficient to consider the following Lyapunov’s candidate \( V(x) = 0.5x^2 \). Its derivative satisfies the following equation: \( \dot{V}(x) = kx^2(1 - x) \) and is obviously negative definite over \( x \in [0, 1) \).

Uniqueness of \( x(t) \) follows directly from the continuity of equation (35) right part [34]. Lemma is proven.

Regarding lemma we observe that system [38] solutions for \( \alpha_i > 0 \) are completely defined by the choice of initial conditions \( \hat{x}_i(0) \). This means that if \( \hat{x}_i(t + \tau) \) and \( \check{x}_i(t) \) are solutions of system (38) and \( \hat{x}_i(t + \tau) = \check{x}_i(t) \) for any \( t \geq 0 \), then

\[ \hat{x}_i(t + \tau) = \check{x}_i(t) \Leftrightarrow \hat{x}_i(\tau) = \check{x}_i(0). \]

In other words, for each solution \( \hat{x}_i(t) \) time-shift is equivalent to choice of initial conditions. Moreover, it is easy to see that for any \( \tau \in (-\infty, \infty) \) and \( \hat{x}_i(0) \in (0, 1) \) there is an initial condition \( \hat{x}_i(0) \) such that \( \check{x}_i(t + \tau) = \check{x}_i(t) \).

All we have to prove now is that \( \hat{x}_i(t) \) is a sigmoidal function. Let us consider \( \hat{x}_i \). As it follows from system equations, \( \hat{x}_i(t) \) time-derivative is:

\[ \frac{\partial\hat{x}_i(t)}{\partial t} = \alpha_i\hat{x}_i(t)(1 - \hat{x}_i(t)). \]
then
\[
\dot{x}_i(t) = \int \alpha_i \dot{x}_i(t)(1 - \dot{x}_i(t))dt = f(\alpha_i t + b_i) + D,
\]
where
\[
f(\alpha_i t + b_i) = \frac{1}{1 + e^{-(\alpha_i t + b_i)}}, \quad D = 0.
\]
As initial conditions of system (33) completely define time-shifts of the solutions \(\dot{x}_i(t)\), coefficients \(b_i\) in (37) depend on initial conditions \(\dot{x}_i(0)\) only.

We just proved that \(i\)-th solution of system (33) can be written in the following manner:
\[
\dot{x}_i(t) = f(\alpha_i t + b_i),
\]
where \(b_i \in (-\infty, \infty)\), \(b_i = f^{-1}(\dot{x}_i(0))\) depends on \(\dot{x}_i(0) \in (0, 1)\) explicitly and \(f(\cdot)\) is the sigmoid function. Let us consider output \(y(x)\) of system (33):
\[
y(x) = \sum_{i=1}^{n} \left( \frac{C_i}{\beta_i} f(\alpha_i t + b_i) \right).
\]
We denote \(\dot{c}_i = C_i/\beta_i\), so \(y(x)\) can be written in the form:
\[
\sum_{i=1}^{n} (\dot{c}_i f(\alpha_i t + b_i)).
\]
Therefore, due to [9], for any \(\varepsilon > 0\) and \(g(t) \in C_{[0,T]}^{1}\) there are such \(n, \dot{c}_i\) and \(b_i\) that the following inequality holds:
\[
|\sum_{i=1}^{n} (\dot{c}_i f(\alpha_i t + b_i)) - g(t)| \leq \varepsilon
\]
for \(t \in [0, T]\). To conclude the proof, it is sufficient to notice that parameters \(\alpha_i, \beta_i\) and initial conditions \(\Delta_i\) can be restored from \(b_i\) and \(\dot{c}_i\). \textit{The theorem is proven.}

\textit{Lemma 1 proof.} The lemma proof is trivial. Trajectories \(x(t)\) and \(\dot{x}(t)\) of (9) are bounded, then sum
\[
|\dot{C}^T \sum_{i=1}^{n} (\alpha_i (\xi_{1,i}(\dot{x}) - \xi_{1,i}(x)) + \beta_i (\xi_{2,i}(\dot{x}) - \xi_{2,i}(x))) (1 - \lambda(t, D))| < D_2,
\]
where \(D_2 > 0\). Therefore the coefficients \(k_i^*\) (if exist) should satisfy the following inequality
\[
\frac{D_2}{\varepsilon} < \frac{D_2}{\delta} < -\sum_{i=1}^{n} k_i^* \dot{c}_i
\]
for \(\varepsilon > \delta > 0\). Vector \(\dot{C} \neq 0\), hence there exists at least one \(\dot{c}_i \neq 0\). Therefore there exists at least one vector \(k^* = (k_1^*, \ldots, k_n^*)^T\) such that
\[
\dot{C}^T k^* < -\frac{D_2}{\delta}
\]
Therefore inequality (10) is satisfied for every $\epsilon > \delta > 0$. The lemma is proven.

**Theorem 2 proof.** According to the theorem assumptions vector $\hat{C} \neq 0$. Therefore, from Lemma it follows that there exist coefficients $k^*_i$ such that

$$|\hat{C}^T \sum_{i=1}^{n} (\alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x))) \left(1 - \lambda(t, D)\right)| + \epsilon \sum_{i=1}^{n} k^*_i \hat{c}_i < 0$$

for any $\epsilon > \delta - \delta_1$, where $\delta > \delta_1 > 0$. Define the following set of time intervals:

$$\Delta_{t,0} = \{\Delta(2i, 2i + 1) = [t_{2i}, t_{2i+1}]|\lambda(t, D) = 0 \forall t \in [t_{2i}, t_{2i+1}], i \in \mathcal{N}, t_0 < t_1 \ldots < t_j < t_{j+1} < t_{j+2} < \ldots\}.$$

$$\Delta_{t,1} = \{\Omega(2i + 1, 2i + 2) = (t_{2i+1}, t_{2i+2})|\lambda(t, D) = 1 \forall t \in (t_{2i+1}, t_{2i+2}), t_1 < t_2 \ldots < t_j < t_{j+1} < t_{j+2} < \ldots\}.$$

Consider the following positive-definite function

$$V(e, \hat{\alpha}, \hat{\beta}, K) = \int_0^\epsilon S_\delta(\nu) \nu d\nu + 0.5\|\hat{\alpha} - \alpha\|_{\gamma - 1}^2 + 0.5\|\hat{\beta} - \beta\|_{\gamma - 1}^2 + 0.5\|K(t) - k^*_i\|_{\gamma - 1}^2,$$

where $k^*_i$ satisfy inequality (38) for every $\epsilon > \delta - \delta_1$. Its time-derivative over the set $\Delta_{t,0}$ can be expressed as follows

$$\frac{d}{dt}V(e, \hat{\alpha}, \hat{\beta}, K) = S_\delta(e) \left(\epsilon \hat{c} - \sum_{i=1}^{n} \left((\hat{\alpha}_i - \alpha_i)e\hat{C}^T \xi_{1,i}(\hat{x}) - (\hat{\beta}_i - \beta_i)e\hat{C}^T \xi_{2,i}(\hat{x}) - (k_i(t) - k^*_i)e^2 \hat{c}_i\right)\right)$$

It is clear that $\dot{V} = 0$ for any $|e| < \delta$ as $S_\delta(e) \equiv 0$ for all $|e| < \delta$. Let $|e| \geq \delta$, then

$$\dot{V} = S_\delta(e) \left(\epsilon \hat{c} - \sum_{i=1}^{n} \left((\hat{\alpha}_i - \alpha_i)e\hat{C}^T \xi_{1,i}(\hat{x}) - (\hat{\beta}_i - \beta_i)e\hat{C}^T \xi_{2,i}(\hat{x}) - (k_i(t) - k^*_i)e^2 \hat{c}_i\right)\right) -$$

$$S_\delta(e) \left(\sum_{i=1}^{n} (\hat{\alpha}_i - \alpha_i)e\hat{C}^T \xi_{1,i}(\hat{x}) - (\hat{\beta}_i - \beta_i)e\hat{C}^T \xi_{2,i}(\hat{x}) - (k_i(t) - k^*_i)e^2 \hat{c}_i\right)\right) =$$

$$S_\delta(e) \left(\sum_{i=1}^{n} \hat{C}^T \alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \hat{C}^T \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x)) + k^*_i \hat{c}_i e\right)$$

$$\leq S_\delta(e) |e| \left(\sum_{i=1}^{n} |\hat{C}^T \alpha_i(\xi_{1,i}(\hat{x}) - \xi_{1,i}(x)) + \hat{C}^T \beta_i(\xi_{2,i}(\hat{x}) - \xi_{2,i}(x))| + \sum_{i=1}^{n} k^*_i \hat{c}_i \delta_1 \right) \leq S_\delta(e) |e| \sum_{i=1}^{n} k^*_i \hat{c}_i \delta_1 \leq 0.$$  (39)

(In order to get the last inequality note that sum $\sum_{i=1}^{n} k^*_i \hat{c}_i$ must be negative.) Taking into account that $\dot{V}$ is not positive over $[t_{2i}, t_{2i+1}]$ and that $e(t_i) = 0$ (because the states of both reference and
tracking systems are forced to move to $x(0)$ over $\Delta_{t,1}$, one can write

$$V(e(t_{2i}), \dot{\alpha}(t_{2i}), \dot{\beta}(t_{2i}), \dot{K}(t_{2i})) - V(e(t_{2i+1}), \dot{\alpha}(t_{2i+1}), \dot{\beta}(t_{2i+1}), \dot{K}(t_{2i+1}))$$

$$= 0.5\|\dot{\alpha}(t_{2i}) - \alpha\|_{\gamma^{-1}}^2 + 0.5\|\dot{\beta}(t_{2i}) - \beta\|_{\gamma^{-1}}^2 + 0.5\|K(t_{2i}) - k^*\|_{\gamma^{-1}}^2 -$$

$$0.5\|\dot{\alpha}(t_{2i+1}) - \alpha\|_{\gamma^{-1}}^2 - 0.5\|\dot{\beta}(t_{2i+1}) - \beta\|_{\gamma^{-1}}^2 - 0.5\|K(t_{2i+1}) - k^*\|_{\gamma^{-1}}^2$$

$$> \int_{t_{2i}}^{t_{2i+1}} S_\delta(e)|e(\tau)| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau + \int_{0}^{e(t_{2i+1})} S_\delta(\nu)\nu d\nu.$$

Consider the following series:

$$W(n) = 0.5 \sum_{i=0}^{n} \left(\|\dot{\alpha}(t_{2i}) - \alpha\|_{\gamma^{-1}}^2 + \|\dot{\beta}(t_{2i}) - \beta\|_{\gamma^{-1}}^2 + \|K(t_{2i}) - k^*\|_{\gamma^{-1}}^2 -\right.$$

$$\|\dot{\alpha}(t_{2i+1}) - \alpha\|_{\gamma^{-1}}^2 - \|\dot{\beta}(t_{2i+1}) - \beta\|_{\gamma^{-1}}^2 - \|K(t_{2i+1}) - k^*\|_{\gamma^{-1}}^2\right).$$

One can notice that

$$\|\dot{\alpha}(t_{2i+1}) - \alpha\|_{\gamma^{-1}}^2 + \|\dot{\beta}(t_{2i+1}) - \beta\|_{\gamma^{-1}}^2 + \|K(t_{2i+1}) - k^*\|_{\gamma^{-1}}^2 -$$

$$\|\dot{\alpha}(t_{2i+2}) - \alpha\|_{\gamma^{-1}}^2 + \|\dot{\beta}(t_{2i+2}) - \beta\|_{\gamma^{-1}}^2 + \|K(t_{2i+2}) - k^*\|_{\gamma^{-1}}^2$$

as vectors $\dot{\alpha}, \dot{\beta}$ and $K$ remain constant over intervals $\Delta_{t,1}$. Therefore

$$W(n) = 0.5 \left(\|\dot{\alpha}(t_0) - \alpha\|_{\gamma^{-1}}^2 + \|\dot{\beta}(t_0) - \beta\|_{\gamma^{-1}}^2 + \|K(t_0) - k^*\|_{\gamma^{-1}}^2 -\right.$$

$$\|\dot{\alpha}(t_{2n+1}) - \alpha\|_{\gamma^{-1}}^2 - \|\dot{\beta}(t_{2n+1}) - \beta\|_{\gamma^{-1}}^2 - \|K(t_{2n+1}) - k^*\|_{\gamma^{-1}}^2\right)$$

$$> \sum_{i=0}^{n} \int_{t_{2i}}^{t_{2i+1}} S_\delta(e)|e(\tau)| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau + \sum_{i=0}^{n} \int_{0}^{e(t_{2i+1})} S_\delta(\nu)\nu d\nu > 0. \quad (40)$$

Given that $x(t), \dot{x}(t)$ are bounded we can conclude that $\dot{\alpha}, \dot{\beta}$ and $\dot{K}(t)$ are bounded and hence $\dot{\alpha}, \dot{\beta}, K(t)$ are bounded. Furthermore, the following inequality holds

$$0.5 \left(\|\dot{\alpha}(t_0) - \alpha\|_{\gamma^{-1}}^2 + \|\dot{\beta}(t_0) - \beta\|_{\gamma^{-1}}^2 + \|K(t_0) - k^*\|_{\gamma^{-1}}^2 -\right.$$

$$\|\dot{\alpha}(t_{2n+1}) - \alpha\|_{\gamma^{-1}}^2 - \|\dot{\beta}(t_{2n+1}) - \beta\|_{\gamma^{-1}}^2 - \|K(t_{2n+1}) - k^*\|_{\gamma^{-1}}^2\right) > \sum_{i=0}^{n} \int_{t_{2i}}^{t_{2i+1}} S_\delta(e)|e(\tau)| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau$$

$$= \int_{0}^{t_{2n+1}} S_\delta(e)\lambda(\tau, D)|e(\tau)| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau > 0.$$

Hence

$$0 < \int_{0}^{\infty} S_\delta(e)\lambda(\tau, D)|e(\tau)| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau < \infty.$$

Let us consider the following time-intervals $\Delta_i = [\tau_{2i}, \tau_{2i+1}] : |e|\lambda(t, D) \geq \delta \forall t \in \Delta_i, i \in \{0, 1, \ldots, \infty\}$. As $|e(t)| > \delta$ it is clear that

$$\infty > \int_{0}^{\infty} S_\delta(e)\lambda(\tau, D)|e(\tau)| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau > \int_{0}^{\infty} S_\delta(e)\lambda(\tau, D)|\delta| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1|d\tau$$

$$= \sum_{i=0}^{\infty} \Delta_i |\delta| \sum_{j=1}^{n} k_j^* \hat{c}_j|\delta_1| > 0.$$
Then series
\[ \sum_{i=0}^{\infty} \Delta_i |\delta \sum_{j=1}^{n} k_j^* \hat{c}_j | \delta_i \]
converges and, therefore, \( \Delta_i \to 0 \) as \( i \to \infty \). In order to finish the proof of the theorem, it is sufficient to consider the error function \( e(t) \) over intervals \( \Delta_i \). Derivative \( \dot{e} \) is bounded (say \(|\dot{e}| < D_3\)) as vectors \( x, \hat{x}, \hat{\alpha}, \hat{\beta}, K(t) \) are bounded. Therefore for any \( t \in \Delta_i \):

\[
|e(t)\lambda(t, D)| = |e(\tau_{2i}) + \int_{\tau_{2i}}^{\tau_{2i+1}} \dot{e}(\tau) d\tau| \leq |e(\tau_{2i})| + |\int_{\tau_{2i}}^{\tau_{2i+1}} \dot{e}(\tau) d\tau|
\]

\[
\leq |e(\tau_{2i})| + |\Delta_i| D_3 = \delta + \Delta_i D_3.
\]

Then
\[
\lim_{t \to \infty} \sup |e(t)\lambda(t, D)| = \delta.
\]

Hence for any arbitrary small \( \delta_1 > 0 \) there exists such \( t_1 \) that
\[
|e(t)\lambda(t, D)| < \delta + \delta_1
\]
for any \( t > t_1 \). The theorem is proven.
Figure 1: Periodical extension of the reference signal $g(t)$ defined over $[0, T]$
Figure 2: Trajectories $\hat{\alpha}(t), \hat{c}(t)$ in system (27) with algorithm (28) (Curve 1) and algorithm (29) (Curve 2) starting from point $(-3, -3)$. Global minimum is marked by circle.
Figure 3: Trajectories $\dot{\alpha}(t), \dot{c}(t)$ in system (27) with algorithm (28) (Curve 1) and algorithm (29) (Curve 2) starting from point $(-3, -3)$. The trajectories have been shown for time interval $[0, 900]$ sec.
Figure 4: Trajectories $\hat{a}(t), \hat{c}(t)$ in system (27) with algorithm (28) (Curve 1) and algorithm (29) (Curve 2) starting from point $(3, -3)$. Algorithm (28) ensures that the estimates reach a neighborhood of the global minimum in very short time and then to approach it with oscillations in the parameter space (blob-like part of the trajectory).
Figure 5: Trajectories $\hat{\alpha}(t), \hat{c}(t)$ in system (27) with batch gradient algorithm (30) starting from point $(-3, -3)$ (Curve 1) and $(3, -3)$ (Curve 2). None reaches the global minimum (marked by circle)
Figure 6: Histograms of the distributions of the distances $d((T + \Delta T_2)10000)$ (plot a) and $d(0)$ (plot b) for 400 trials with random initial conditions for the estimates $\hat{\alpha}_i(0)$. 
Figure 7: Histograms of the distributions of the performance indices $R((T + \Delta T_2)10000)$ (plot a) and $R(0)$ (plot b) for 400 trials with random initial conditions for the estimates $\hat{\alpha}_i(0)$. 