A QUIVER CONSTRUCTION OF SYMMETRIC CRYSTALS

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Abstract. In the papers [EK1], [EK2] and [EK3] with Masaki Kashiwara, the author introduced the notion of symmetric crystals and presented the Lascoux-Leclerc-Thibon-Ariki type conjectures for the affine Hecke algebras of type $B$. Namely, we conjectured that certain composition multiplicities and branching rules for the affine Hecke algebras of type $B$ are described by using the lower global basis of symmetric crystals of $V_{\theta}(\lambda)$. In the present paper, we prove the existence of crystal bases and global bases of $V_{\theta}(0)$ for any symmetric quantized Kac-Moody algebra by using a geometry of quivers (with a Dynkin diagram involution). This is analogous to George Lusztig’s geometric construction of $U_{v^{-1}}$ and its lower global basis.

1. Introduction

1.1. Let $K_n^{\text{AHA}}$ be the Grothendieck group of the affine Hecke algebra $H_n(q)$ of type $A_n$ and set $K^{\text{AHA}} = \bigoplus_{n \geq 0} K_n^{\text{AHA}}$. Generalizing the LLT conjecture [LLT] for the Hecke algebra of type $A$, S. Ariki [Ari] proved that $K^{\text{AHA}} \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to $U_{v^{-1}}(g)$ as $U_{v^{-1}}(g)$-modules. Here $g = \widehat{sl}_{\ell-1}$ or $\mathfrak{gl}_{\infty}$ according that the parameter $q$ of the affine Hecke algebras of type $A$ is a primitive $\ell$-th root of unity or not a root of unity. This isomorphism sends the irreducible modules of the affine Hecke algebras to the specialization of the upper global basis of $U_{v^{-1}}(g)$ at $v = 1$. His proof is based on two results in the geometric representation theory. One is the equivariant $K$-theoretic description of the irreducible and standard modules of the affine Hecke algebras by Chriss-Ginzburg and Kazhdan-Lusztig, and the other is G. Lusztig’s geometric construction [Lus1] of the lower global basis of $U_{v^{-1}}(g)$. Lusztig’s theory is summarized as follows.

Let $g$ be a symmetric Kac-Moody algebra and $I$ an index set of simple roots of $g$. For a fixed set of arrows $\Omega$, we consider $(I, \Omega)$ as a (finite) oriented graph. We call $(I, \Omega)$ a quiver. For an $I$-graded vector space $V$, we define the moduli space of representations of quiver $(I, \Omega)$ by

$$E_{V, \Omega} = \bigoplus_{i, j, \Omega} \text{Hom}(V_i, V_j).$$

The algebraic group $G_V = \prod_{i \in I} GL(V_i)$ acts on $E_{V, \Omega}$. Lusztig introduced a certain full subcategory $\mathcal{D}_{V, \Omega}$ of $\mathcal{D}(E_{V, \Omega})$ where $\mathcal{D}(E_{V, \Omega})$ is the bounded derived category of constructible complexes of sheaves on $E_{V, \Omega}$ (for the definition, see section 3). Let $K(\mathcal{D}_{V, \Omega})$ be the Grothendieck group of $\mathcal{D}_{V, \Omega}$. He constructed the induction operators $f_i$ and the restriction operators $e'_i$ on the Grothendieck group $K_{\Omega} := \bigoplus_{V} K(\mathcal{D}_{V, \Omega})$, where $V$ runs over the isomorphism classes of $I$-graded vector spaces. He proved the following theorem.

Theorem 1.1 (Lusztig).

(i) The operators $e'_i$ and $f_i$ define the action of the reduced $v$-analogue $B_v(g)$ of $g$ on $K_{\Omega} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v)$, and $K_{\Omega} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v)$ is isomorphic to $U_{v^{-1}}(g)$ as a $B_v(g)$-module.

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Similarly to Lusztig’s arguments, we consider a certain full subcategory condition on $E$ basis of $V$ criterion of crystals in Theorem 2.14. We use this in our proof of existence of the crystal perverse sheaves and Fourier-Sato-Deligne transforms. Especially, we recall an important crystal bases and the global bases, the notion of symmetric crystals and known results of

In section 2, we recall some results on the quantum enveloping algebras, the theory of the

This paper is organized as follows.

In section 3, we give a quick review on Lusztig’s construction of $U_v^{-}(g)$ and $F_1.2$ using a criterion of crystals (Theorem 2.14) and certain estimates for the actions of $\theta$ and its Grothendieck group $I$ analogous result of Theorem 1.1.

(ii) The simple perverse sheaves in $\bigoplus V \mathcal{D}_{V, \Omega}$ give the lower global basis of $U_v^{-}(g)$.

1.2. Recently in [EK1] and [EK2] with M. Kashiwara, the author presented an analogue of the LLTA conjecture for the affine Hecke algebra of type $B$. In [EK2], we considered $U_v(g)$ and its Dynkin diagram involution $\theta$ and constructed an analogue $B_\theta(g)$ of the reduced $v$-analogue $B_v(g)$ (for the definition, see Definition 2.3 below). We gave a $B_\theta(g)$-module $V_\theta(\lambda)$ for a dominant integral weight $\lambda$ such that $\theta(\lambda) = \lambda$, which is an analogue of the $B_v(g)$-module $U_v^{-}(g)$ (for the definition, see Definition 2.10 below). We defined the notion of symmetric crystals and conjectured the existence of the global basis. In the case $g = g_{\infty}$, $I = Z_{odd}$, $\theta(i) = -i$ and $\lambda = 0$, we constructed the PBW type basis and the lower (and upper) global basis parametrized by the $\theta$-restricted multi-segments. We conjectured that irreducible modules of the affine Hecke algebras of type $B$ are described by the global basis associated to the symmetric crystals.

1.3. In this paper, we construct the lower global basis for the symmetric crystals by using a geometry of quivers (with a Dynkin diagram involution). Hence for any symmetric quantized Kac-Moody algebra $U_v(g)$, we establish the existence of a crystal basis and a global basis for $V_\theta(0)$.

We introduce the notion of $\theta$-quivers. This is a quiver $(I, \Omega)$ with an involution $\theta : I \rightarrow I$ (and $\theta : \Omega \rightarrow \Omega$) satisfying some conditions (see Definition 4.1). This notion is partially motivated by Syu Kato’s construction [K] of the irreducible representations of the affine Hecke algebras of type $B$.

We also introduce the $\theta$-symmetric $I$-graded vector spaces. This is an $I$-graded vector space $V = (V_i)_{i \in I}$ endowed with a non-degenerate symmetric bilinear form such that $V_i$ and $V_j$ are orthogonal if $j \neq \theta(i)$. For a $\theta$-quiver $(I, \Omega)$ and a $\theta$-symmetric $I$-graded vector space $V$, we define the moduli space $\mathcal{E}_{V, \Omega}$ of representations of $(I, \Omega)$ adding a skew-symmetric condition on $E_{V, \Omega}$ with respect to the involution $\theta$.

Similarly to Lusztig’s arguments, we consider a certain full subcategory $\mathcal{D}_{V, \Omega}^\theta$ of $\mathcal{D}(\mathcal{E}_{V, \Omega})$ and its Grothendieck group $\mathcal{K}_{V, \Omega}^\theta$. We define the induction operators $F_i$ and the restriction operators $E_i$ on $\mathcal{K}_{\Omega}^\theta := \bigoplus V \mathcal{K}_{V, \Omega}^\theta$ where $V$ runs over the isomorphism classes of the $\theta$-symmetric $I$-graded vector spaces. We prove the following main theorem which is an analogous result of Theorem 5.1.

**Theorem 1.2** (Theorem 5.12). $\mathcal{K}_{\Omega}^\theta \otimes_{Z_{[v, v^{-1}]}} Q(v) \cong V_\theta(0)$ as $B_\theta(g)$-modules. The simple perverse sheaves in $\mathcal{K}_{\Omega}^\theta$ give a lower global basis of $V_\theta(0)$.

Though Lusztig proved Theorem 1.2 using some inner product on $K_{\Omega}$, we prove Theorem 1.2 using a criterion of crystals (Theorem 2.14) and certain estimates for the actions of $E_i$ and $F_i$ on simple perverse sheaves (Theorem 5.3).

This paper is organized as follows.

In section 2, we recall some results on the quantum enveloping algebras, the theory of the crystal bases and the global bases, the notion of symmetric crystals and known results of perverse sheaves and Fourier-Sato-Deligne transforms. Especially, we recall an important criterion of crystals in Theorem 2.14. We use this in our proof of existence of the crystal basis of $V_\theta(0)$.

In section 3, we give a quick review on Lusztig’s construction of $U_v^{-}(g)$ and its lower global basis.

In section 4, we introduce the notion of $\theta$-quivers and $\theta$-symmetric $I$-graded vector spaces. We define the category $\mathcal{D}_{V, \Omega}^\theta$ and the induction operators $F_i$ and the restriction operators
We calculate actions of $E_i$ and $F_i$ on $\mathcal{O}_{\mathcal{V}, \Omega}$. We also prove that $E_i$ and $F_i$ commute with the Fourier-Sato-Deligne transforms.

In section 5, we introduce the Grothendieck group $^\theta K_\Omega$ and show three key results. First, we calculate the commutation relations of $E_i$ and $F_i$. Second, we give certain estimates of coefficients with respect to the action of $E_i$ and $F_i$ on simple perverse sheaves. These estimates satisfy the condition in Theorem 2.14. Third, we prove the invariance of simple perverse sheaves with respect to the Verdier duality functor. Combining these results we prove the main theorem.

**Remark 1.3.** We give two remarks on a difference from the “folding” procedure and an overlap with perverse sheaves arising from graded Lie algebras by Lusztig.

(i) Our construction is completely different from Lusztig’s construction, “Quiver with automorphisms”, in his book [Lus3, Chapter 12-14].

He considered actions $a : I \to I$ and $a : H \to H$ induced from a finite cyclic group $C$ generated by $a$. Put an orientation $\Omega$ such that out($a(h)) = a(out(h)$ and in($a(h)) = a(in(h)$). He said this orientation “compatible”. Let $V^a$ be the category of $I$-graded vector spaces $V$ such that $\dim V_i = \dim V_{a(i)}$ for any $i \in I$. For $V \in V^a$, $a$ induces a natural automorphism on $E_V, \Omega$ and a functor $a^* : \mathcal{D}(E_V, \Omega) \to \mathcal{D}(E_V, \Omega)$. He introduced “C-equivariant” simple perverse sheaves $(B, \phi)$, where $B$ is a perverse sheaf on $E_V, \Omega$ and $\phi : a^*B \cong B$. Then he proved that the set $\bigcup_{V \in V^a} B_V, \Omega$ of $C$-equivariant perverse sheaves gives a lower global basis of $U^-\theta(\mathfrak{g})$. Here $\mathfrak{g}$ has a non-symmetric Cartan matrix which is obtained by the “folding” procedure with respect to the $C$-action on $I$.

But in our construction, a $\theta$-orientation is not a compatible orientation. Moreover the most essential difference is that his construction has no skew-symmetric condition in our sense. Hence the set of simple perverse sheaves $^\theta \mathcal{D}(E_V, \Omega)$ and the space $^\theta K_\Omega \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v) \cong V_\theta(0)$ are different from $B_V, \Omega$ and $U^-\theta(\mathfrak{g})$, respectively. The explicit crystal structure of $V_\theta(0)$ is unknown except for the case $\mathfrak{g} = \mathfrak{gl}_\infty$, $I = \mathbb{Z}_{odd}$ and $\theta(i) = -i$ in $\mathbb{Z}K_2$.

(ii) In some special case, the lower global basis constructed in this paper is obtained by Lusztig ([Lus4] and [Lus5]). Let us consider the case $G = SO(2n, \mathbb{C})$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $T$ a fixed maximal torus of $G$. Set $\varepsilon_{2i-1}$ ($1 \leq i \leq n$) the fundamental characters of $T$. Assume $q \in \mathbb{C}^*$ is not a root of unity. We choose a semisimple element $s \in T$ such that $\varepsilon_{2i-1}(s) \in q^{\mathbb{Z}_{odd}}$ for any $i$ and put $d_{2i-1} = \{j | \varepsilon_{2j-1}(s) = q^{2i-1}\}$. Then the centralizer $G(s)$ of $s$ acts on

$$\mathfrak{g}_2 := \{X \in \mathfrak{g} \mid sXs^{-1} = q^2X\}$$

which has finitely many $G(s)$-orbits. Lusztig considered the category $\mathcal{D}(\mathfrak{g}_2)$ of semisimple $G(s)$-equivariant complex on $\mathfrak{g}_2$ and constructed the canonical basis $B(\mathfrak{g}_2)$ of $K(\mathfrak{g}_2)$ which is the Grothendieck group of $\mathcal{D}(\mathfrak{g}_2)$.

On the other hand, let us consider the $\theta$-symmetric vector space $V$ such that $\text{wt}(V) = \sum_{i=1}^{n} d_{2i-1}(\alpha_{2i-1} + \alpha_{-2i+1})$ and the following $\theta$-quiver of type $A_{2n}$ and the $\theta$-orientation $\Omega$:

![Diagram](https://via.placeholder.com/150)

$$-2n + 1 \quad -5 \quad -3 \quad -1 \quad 1 \quad 3 \quad 5 \quad 2n - 1$$
In this case, we have $G(s) = \prod_{i=1}^{n} GL(d_{2i-1}) = \mathcal{g}\mathcal{V} \text{ and } \mathcal{g}_2 \cong \mathcal{g}\mathcal{V},\mathcal{O}$. Thus the set $\mathcal{g}\mathcal{V},\mathcal{O}$ of simple perverse sheaves coincide with $\mathcal{B}(\mathcal{g}_2)$.

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2. Preliminaries

2.1. Quantum enveloping algebras.

2.1.1. Quantum enveloping algebras and reduced $v$-analogue. We shall recall the quantized universal enveloping algebra $U_v(\mathcal{g})$. In this paper, we treat only the symmetric Cartan matrix case. Let $I$ be an index set (for simple roots), and $Q$ the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i \in I}$. Let $(\cdot, \cdot) : Q \times Q \to \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i) = 2$ and $(\alpha_i, \alpha_j) \in \mathbb{Z}_{\leq 0}$ for $i \neq j$. Let $v$ be an indeterminate and set $K := \mathbb{Q}(v)$. We define its subrings $A_0$, $A_\infty$ and $A$ as follows.

\[ A_0 = \{ f \in K \mid f \text{ is regular at } v = 0 \}, \]
\[ A_\infty = \{ f \in K \mid f \text{ is regular at } v = \infty \}, \]
\[ A = \mathbb{Q}[v, v^{-1}]. \]

**Definition 2.1.** The quantized universal enveloping algebra $U_v(\mathcal{g})$ is the $K$-algebra generated by elements $e_i, f_i$ and invertible elements $t_i$ ($i \in I$) with the following defining relations.

1. The $t_i$'s commute with each other.
2. $t_i e_i t_i^{-1} = v^{(\alpha_i, \alpha_i)} e_i$ and $t_i f_i t_i^{-1} = v^{-(\alpha_i, \alpha_i)} f_i$ for any $i, j \in I$.
3. $[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{v - v^{-1}}$ for $i, j \in I$.
4. (v-Serre relation) For $i \neq j$,

\[ \sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \sum_{k=0}^{b} (-1)^{k} f_i^{(k)} f_j f_i^{(b-k)} = 0. \]

Here $b = 1 - (\alpha_i, \alpha_j)$ and 
\[ e_i^{(k)} = e_i^k/[k]_v!, \quad f_i^{(k)} = f_i^k/[k]_v!, \quad [k]_v = (v^k - v^{-k})/(v - v^{-1}), \quad [k]_v! = [1]_v \cdots [k]_v. \]

Let us denote by $U_v^-(\mathcal{g})$ the subalgebra of $U_v(\mathcal{g})$ generated by the $f_i$’s. Let $e_i'$ and $e_i''$ be the operators on $U_v^-(\mathcal{g})$ defined by

\[ [e_i, a] = \frac{(e_i'' a) t_i - t_i^{-1} e_i' a}{v - v^{-1}} \quad (a \in U_v^-(\mathcal{g})). \]

These operators satisfy the following formulas similar to derivations:

\[ e_i'(ab) = (e_i'a)b + (\text{Ad}(t_i)a)e_i'b. \]

The algebra $U_v^-(\mathcal{g})$ has a unique symmetric bilinear form $(\cdot, \cdot)$ such that $(1, 1) = 1$ and

\[ (e_i'a, b) = (a, f_i'b) \quad \text{for any } a, b \in U_v^-(\mathcal{g}). \]
It is non-degenerate. The left multiplication operator \( f_j \) and \( e'_i \) satisfy the commutation relations

\[
e'_i f_j = v^{-(a_i, a_j)} f_j e'_i + \delta_{ij}, \quad e'_i f_j = f_j e'_i + \delta_{ij} \operatorname{Ad}(t_i),
\]

and the \( e'_i \)'s satisfy the \( v \)-Serre relations.

**Definition 2.2.** The reduced \( v \)-analogue \( B_v(\mathfrak{g}) \) of \( \mathfrak{g} \) is the \( \mathbb{Q}(v) \)-algebra generated by \( e'_i \) and \( f_i \).

2.1.2. **Review on crystal bases and global bases of \( U_v^- \).** Since \( e'_i \) and \( f_i \) satisfy the \( v \)-boson relation, any element \( a \in U_v^- (\mathfrak{g}) \) can be uniquely written as

\[
a = \sum_{n \geq 0} f^n_i a_n \quad \text{with} \quad e'_i a_n = 0.
\]

Here \(
\left( f^n_i \right) = \frac{f^n_i}{[n]_i!}
\).

**Definition 2.3.** We define the modified root operators \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( U_v^- (\mathfrak{g}) \) by

\[
\tilde{e}_i a = \sum_{n \geq 1} f^{n-1}_i a_n, \quad \tilde{f}_i a = \sum_{n \geq 0} f^{n+1}_i a_n.
\]

**Theorem 2.4 ([K91]).** We define

\[
L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_v^- (\mathfrak{g}),
\]

\[
B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod vL(\infty) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \right\} \subset L(\infty)/vL(\infty).
\]

Then we have

1. \( \tilde{e}_i L(\infty) \subset L(\infty) \) and \( \tilde{f}_i L(\infty) \subset L(\infty) \),
2. \( B(\infty) \) is a basis of \( L(\infty)/vL(\infty) \),
3. \( \tilde{f}_i B(\infty) \subset B(\infty) \) and \( \tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\} \).

We call \( (L(\infty), B(\infty)) \) the crystal basis of \( U_v^- (\mathfrak{g}) \).

**Definition 2.5.** We define \( \varepsilon_i (b) := \max \{ m \in \mathbb{Z}_{\geq 0} \mid \varepsilon_i^m b \neq 0 \} \) for \( i \in I \) and \( b \in B(\infty) \).

Let \( - \) be the automorphism of \( \mathbb{K} \) sending \( v \) to \( v^{-1} \). Then \( \mathbb{A}_0 \) coincides with \( \mathbb{A}_\infty \).

Let \( V \) be a vector space over \( \mathbb{K} \), \( L_0 \) an \( \mathbb{A} \)-submodule of \( V \), \( L_\infty \) an \( \mathbb{A}_\infty \)-submodule, and \( V_\mathbb{A} \) an \( \mathbb{A} \)-submodule. Set \( E := L_0 \cap L_\infty \cap V_\mathbb{A} \).

**Definition 2.6 ([K91]).** We say that \( (L_0, L_\infty, V_\mathbb{A}) \) is balanced if each of \( L_0 \), \( L_\infty \) and \( V_\mathbb{A} \) generates \( V \) as a \( \mathbb{K} \)-vector space, and if one of the following equivalent conditions is satisfied.

1. \( E \to L_0/vL_0 \) is an isomorphism,
2. \( E \to L_\infty/vL_\infty \) is an isomorphism,
3. \( (L_0 \cap V_\mathbb{A}) \oplus (v^{-1} L_\infty \cap V_\mathbb{A}) \to V_\mathbb{A} \) is an isomorphism.
4. \( \mathbb{A}_0 \otimes_{\mathbb{Q}} E \to L_0, \mathbb{A}_\infty \otimes_{\mathbb{Q}} E \to L_\infty, \mathbb{A} \otimes_{\mathbb{Q}} E \to V_\mathbb{A} \) and \( \mathbb{K} \otimes_{\mathbb{Q}} E \to V \) are isomorphisms.

Let \( - \) be the ring automorphism of \( U_v (\mathfrak{g}) \) sending \( v, t_i, e_i, f_i \) to \( v^{-1}, t_i^{-1}, e_i, f_i \).

Let \( U_v (\mathfrak{g})_\mathbb{A} \) be the \( \mathbb{A} \)-subalgebra of \( U_v (\mathfrak{g}) \) generated by \( e_i^{(n)}, f_i^{(n)} \) and \( t_i \). Similarly we define \( U_v^- (\mathfrak{g})_\mathbb{A} \).

**Theorem 2.7.** \( (L(\infty), L(\infty)^-, U_v^- (\mathfrak{g})_\mathbb{A}) \) is balanced.
Let \( G^{low} : L(\infty)/vL(\infty) \sim \to E := L(\infty) \cap L(\infty)^- \cap U^-(g)_A \)
be the inverse of \( E \sim \to L(\infty)/vL(\infty) \). Then \( \{G^{low}(b) \mid b \in B(\infty)\} \)
forms a basis of \( U^-(g) \).
We call it a (lower) global basis. It is first introduced by G. Lusztig ([Lus1]) under
the name of “canonical basis” for the A, D, E cases.

Definition 2.8. Let
\[ \{G^{up}(b) \mid b \in B(\infty)\} \]
be the dual basis of \( \{G^{low}(b) \mid b \in B(\infty)\} \) with respect to the inner product \((\cdot, \cdot)\). We call it
the upper global basis of \( U^-(g) \).

2.2. Symmetric Crystals. Let \( \theta \) be an automorphism of \( I \) such that \( \theta^2 = \text{id} \) and
\((\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j) \).
Hence it extends to an automorphism of the root lattice \( Q \) by
\( \theta(\alpha_i) = \alpha_{\theta(i)} \), and induces an automorphism of \( U_v(g) \).

Definition 2.9. Let \( B_\theta(g) \) be the \( K \)-algebra generated by \( E_i, F_i \), and invertible elements
\( T_i(i \in I) \) satisfying the following defining relations:

(i) the \( T_i \)'s commute with each other,
(ii) \( T_{\theta(i)} = T_i \) for any \( i \),
(iii) \( T_iE_jT_i^{-1} = v^{(\alpha_i, \alpha_j)}E_j \) and \( T_iF_jT_i^{-1} = v^{(\alpha_i, \alpha_j)}F_j \) for \( i, j \in I \),
(iv) \( E_iF_j = v^{(\alpha_i, \alpha_j)}F_jE_i + (\delta_{i,j} + \delta_{\theta(i), j})T_i \) for \( i, j \in I \),
(v) the \( E_i \)'s and the \( F_i \)'s satisfy the \( v \)-Serre relations.

We set \( F_i^{(n)} = F_i^n/n! \).

Proposition 2.10 ([EK2, Proposition 2.11]). Let
\[ \lambda \in P_+ := \{\lambda \in \text{Hom}(Q, \mathbb{Q}) \mid \lambda(\alpha_i) \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\} \]
be a dominant integral weight such that \( \theta(\lambda) = \lambda \).

(i) There exists a \( B_\theta(g) \)-module \( V_\theta(\lambda) \) generated by a non-zero vector \( \phi_\lambda \) such that
(a) \( E_i\phi_\lambda = 0 \) for any \( i \in I \),
(b) \( T_i\phi_\lambda = v^{(\alpha_i, \lambda)}\phi_\lambda \) for any \( i \in I \),
(c) \( \{u \in V_\theta(\lambda) \mid E_iu = 0 \text{ for any } i \in I\} = K\phi_\lambda \).
Moreover such a \( V_\theta(\lambda) \) is irreducible and unique up to an isomorphism.

(ii) There exists a unique non-degenerate symmetric bilinear form \((\cdot, \cdot)\) on \( V_\theta(\lambda) \) such that
\( (\phi_\lambda, \phi_\lambda) = 1 \) and \( (E_iu, v) = (u, F_iv) \) for any \( i \in I \) and \( u, v \in V_\theta(\lambda) \).

(iii) There exists an endomorphism - of \( V_\theta(\lambda) \) such that \( \bar{\phi}_\lambda = \phi_\lambda \) and \( av = \bar{a}v, F_i\bar{v} = F_i\bar{v} \)
for any \( a \in K \) and \( v \in V_\theta(\lambda) \).

Hereafter we assume further that
\[ \text{there is no } i \in I \text{ such that } \theta(i) = i. \]

In [EK2], we conjectured that \( V_\theta(\lambda) \) has a crystal basis. This means the following. Since
\( E_i \) and \( F_i \) satisfy the \( v \)-boson relation \( E_iF_i = v^{-(\alpha_i, \alpha_i)}F_iE_i + 1 \), we define the modified root operators:
\[ \bar{E}_i(u) = \sum_{n \geq 1} F_i^{(n)}u_n \text{ and } \bar{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)}u_n, \]
when writing \( u = \sum_{n \geq 0} F_i^{(n)}u_n \) with \( E_iu_n = 0 \). Let \( L_\theta(\lambda) \) be the \( A_0 \)-submodule of \( V_\theta(\lambda) \)
genrated by \( \bar{F}_{i_1} \cdots \bar{F}_{i_\ell}\phi_\lambda \) \( (\ell \geq 0 \text{ and } i_1, \ldots, i_\ell \in I) \), and let \( B_\theta(\lambda) \) be the subset
\[ \{\bar{F}_{i_1} \cdots \bar{F}_{i_\ell}\phi_\lambda \mid \ell \geq 0, i_1, \ldots, i_\ell \in I\} \]
Conjecture 2.11. Let \( \lambda \) be a dominant integral weight such that \( \theta(\lambda) = \lambda \).

1. \( \tilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda) \) and \( \tilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda) \),
2. \( B_\theta(\lambda) \) is a basis of \( L_\theta(\lambda) / vL_\theta(\lambda) \),
3. \( \tilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda) \), and \( \tilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \sqcup \{0\} \),
4. \( \tilde{F}_i \tilde{E}_i(b) = b \) for any \( b \in B_\theta(\lambda) \) such that \( \tilde{E}_i b \neq 0 \), and \( \tilde{E}_i \tilde{F}_i(b) = b \) for any \( b \in B_\theta(\lambda) \).

Moreover we conjectured that \( V_\theta(\lambda) \) has a global crystal basis. Namely we have

Conjecture 2.12. \( (L_\theta(\lambda), \tilde{L}_\theta(\lambda), V_\theta(\lambda)^\text{low}) \) is balanced. Here \( V_\theta(\lambda)^\text{low} := U_v^-(\mathfrak{g}) A \phi_\lambda \).

Example 2.13. Suppose \( \mathfrak{g} = \mathfrak{gl}_\infty \), the Dynkin diagram involution \( \theta \) of \( I \) defined by \( \theta(i) = -i \) for \( i \in I = \mathbb{Z}_{\text{odd}} \).

\[ \begin{array}{ccccccc}
\bullet & & \bullet & & \bullet & & \bullet \\
& & \theta & & \theta & & \\
\bullet & & \bullet & & \bullet & & \bullet \\
& & \theta & & \theta & & \\
\bullet & & \bullet & & \bullet & & \bullet \\
\end{array} \]

And assume \( \lambda = 0 \). In this case, we can prove

\[ V_\theta(0) \cong U_v^- / \sum_{i \in I} U_v^-(f_i - f_{\theta(i)}). \]

Moreover we can construct a PBW type basis, a crystal basis and an upper and lower global basis on \( V_\theta(0) \) parametrized by "the \( \theta \)-restricted multisegments". For more details, see [EK2].

2.3. Criterion for crystals. Let \( K[e, f] \) be the ring generated by \( e \) and \( f \) with the defining relation \( ef = v^{-2}fe + 1 \). We call this algebra the \( v \)-boson algebra. Let \( P \) be a free \( \mathbb{Z} \)-module, and let \( \alpha \) be a non-zero element of \( P \). Let \( M \) be a \( K[e, f] \)-module. Assume that \( M \) has a weight decomposition \( M = \oplus \xi \in \mathcal{P} \mathfrak{M}_\xi \) and \( eM_\lambda \subset M_{\lambda + \alpha} \) and \( fM_\lambda \subset M_{\lambda - \alpha} \). Assume the following finiteness conditions:

for any \( \lambda \in \mathcal{P} \), \( \dim M_\lambda < \infty \) and \( M_{\lambda + n\alpha} = 0 \) for \( n \gg 0 \).

Hence for \( u \in M \), we can write \( u = \sum_{n \geq 0} f^{(n)} u_n \) with \( eu_n = 0 \). We define endmorphisms \( \tilde{\varepsilon} \) and \( \tilde{\eta} \) of \( M \) by

\[ \tilde{\varepsilon} u = \sum_{n \geq 1} f^{(n-1)} u_n, \quad \tilde{\eta} u = \sum_{n \geq 0} f^{(n+1)} u_n. \]

Let \( B \) be a crystal with weight decomposition by \( P \) in the following sense. We have \( \text{wt}: B \to P \), \( \tilde{f}: B \to B \), \( \tilde{e}: B \to B \sqcup \{0\} \) and \( \varepsilon: B \to \mathbb{Z}_{\geq 0} \) satisfying the following properties, where \( B_\lambda = \text{wt}^{-1}(\lambda) \):

(i) \( \tilde{f} B_\lambda \subset B_{\lambda - \alpha} \) and \( \tilde{e} B_\lambda \subset B_{\lambda + \alpha} \sqcup \{0\} \) for any \( \lambda \in \mathcal{P} \),
(ii) \( \tilde{f} \varepsilon b = b \) if \( \varepsilon b \neq 0 \), and \( \tilde{e} \circ \tilde{f} = \text{id}_B \),
(iii) for any \( \lambda \in \mathcal{P}, B_\lambda \) is a finite set and \( B_{\lambda + n\alpha} = \phi \) for \( n \gg 0 \),
(iv) \( \varepsilon(b) = \max\{n \geq 0 \mid \tilde{\varepsilon}^n b \neq 0\} \) for any \( b \in B \).

Set \( \text{ord}(a) = \sup\{n \in \mathbb{Z} \mid a \in v^n A_0\} \) for \( a \in K \). We understand \( \text{ord}(0) = \infty \). Let \( \{G(b)\}_{b \in B} \) be a system of generators of \( M \) with \( G(b) \in M_{\text{wt}(b)} \). Assume that we have expressions:

\[ eG(b) = \sum_{b' \in B} E_{b, b'} G(b), \quad fG(b) = \sum_{b' \in B} F_{b, b'} G(b). \]
Now consider the following conditions for these data, where \( \ell = \varepsilon(b) \) and \( \ell' = \varepsilon(b') \):

1. \( \text{ord}(F_{b,b'}) \geq 1 - \ell' \),
2. \( \text{ord}(E_{b,b'}) \geq -\ell' \),
3. \( F_{b,b} \in v^{-\ell}(1 + vA_0) \),
4. \( E_{b,b} \in v^{1-\ell}(1 + vA_0) \),
5. \( \text{ord}(F_{b,b'}) > 1 - \ell' \) if \( \ell < \ell' \) and \( b' \neq \bar{f}b \),
6. \( \text{ord}(E_{b,b'}) > -\ell' \) if \( \ell < \ell' + 1 \) and \( b' \neq \bar{c}b \).

**Theorem 2.14** ([EK2 Theorem 4.1, Corollary 4.4]). Assume the conditions \((2.1) - (2.6)\).

Let \( L \) be the \( A_0 \)-submodule \( \sum_{b \in B} A_0 G(b) \) of \( M \). Then we have \( \bar{c}L \subset L \) and \( \bar{f}L \subset L \). Moreover we have

\[
\bar{c}G(b) \equiv G(\bar{c}b) \mod vL, \quad \bar{f}G(b) \equiv G(\bar{f}b) \mod vL
\]

for any \( b \in B \). Here we understand \( G(0) = 0 \).

In [EK2], this theorem is proved under more general assumptions.

### 2.4. Perverse Sheaves.

#### 2.4.1. Perverse Sheaves.

In this paper, we consider algebraic varieties over \( \mathbb{C} \). Let \( \mathcal{D}(X) \) be the bounded derived category of constructible complexes of sheaves on an algebraic variety \( X \). We denote by \( \mathcal{D}^{\leq 0}(X) \) (resp. \( \mathcal{D}^{\geq 0}(X) \)) the full subcategory of \( \mathcal{D}(X) \) consisting of objects \( L \) satisfying \( H^k(L) = 0 \) for \( k > 0 \) (resp. \( k < 0 \)). Put \( \mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n] \) and \( \mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n] \).

For a morphism \( f : X \to Y \) of algebraic varieties \( X \) and \( Y \), let \( f^* \) be the inverse image, \( f_! \) the direct image with proper support and \( D : \mathcal{D}(X) \to \mathcal{D}(X) \) the Verdier duality functor.

**Lemma 2.15.**

(i) Suppose that \( f : X \to Y \) is smooth with the fiber dimension \( d \). Then \( D(f^*L) \cong f^*D(L)[2d] \) for \( L \in \mathcal{D}(Y) \).

(ii) Suppose that \( f : X \to Y \) is proper. Then \( D(f_!L) \cong f_!D(L) \) for \( L \in \mathcal{D}(X) \).

Let \( (p\mathcal{D}^{\leq 0}(X), p\mathcal{D}^{\geq 0}(X)) \) be the perverse t-structure and \( \text{Perv}(X) := p\mathcal{D}^{\leq 0}(X) \cap p\mathcal{D}^{\geq 0}(X) \).

**Lemma 2.16.** Suppose \( L \in p\mathcal{D}^{\leq 0}(X) \) and \( K \in p\mathcal{D}^{\geq 0}(X) \), then \( H^j(\mathbb{R}\mathcal{H}om(L, K)) = 0 \) for \( j < 0 \), namely \( \mathbb{R}\mathcal{H}om(L, K) \in \mathcal{D}^{\geq 0}(X) \).

Let \( pH^k(\_ ) \) be the \( k \)-th perverse cohomology sheaf. We say that an object \( L \) in \( \mathcal{D}(X) \) is semisimple if \( L \) is isomorphic to the direct sum \( \bigoplus_k pH^k(L)[-k] \) and if each \( pH^k(L) \) is a semisimple perverse sheaf. Assume that we are given an action of a connected algebraic group \( G \) on \( X \). A semisimple object \( L \) in \( \mathcal{D}(X) \) is said to be \( G \)-equivariant if each \( pH^i(L) \) is a \( G \)-equivariant perverse sheaf.

**Lemma 2.17.**

(i) Suppose that \( f : X \to Y \) is smooth with connected fibers of dimension \( d \). Then we have a fully faithful functor \( \text{Perv}(Y) \to \text{Perv}(X) \) given by \( K \mapsto f^*K[d] \). Moreover if \( K \) is simple, then \( f^*K[d] \) is simple.
(ii) Let $G$ be a connected algebraic group of dimension $d$ and $\text{Perv}_G(X)$ the category of $G$-equivariant perverse sheaves. Suppose that $f : X \to Y$ is a principal $G$-bundle. The functors

$$\text{Perv}(Y) \to \text{Perv}_G(X) : K \mapsto f^* K[d]$$

and

$$\text{Perv}_G(X) \to \text{Perv}(Y) : L \mapsto (\mathcal{H}^{-d} f_* L)$$

define an equivalence of categories, quasi-inverse to each other. Moreover if $K$ is a semisimple object of $\mathcal{D}(Y)$, then $f^* K$ is a $G$-equivariant semisimple object in $\mathcal{D}(X)$. Conversely, if $L$ is a $G$-equivariant semisimple object of $\mathcal{D}(X)$, then there is a unique semisimple object $K \in \mathcal{D}(Y)$ such that $L \cong f^* K$.

We denote by $1_X$ the constant sheaf on $X$.

**Lemma 2.18** ([BBD], [lus3]).

1. Let $f : X \to Y$ be a projective morphism with $X$ smooth. Then $f_! 1_X \in \mathcal{D}(Y)$ is semisimple.
2. Let $f : X \to Y$ be a morphism. Assume that there exists a partition $X = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_m$ such that $X \leq j = X_0 \cup X_1 \cup \cdots \cup X_j$ is closed for $j = 0, 1, \ldots, m$. Assume that, for each $j$, the restriction $f_j : X_j \to Y$ of $f$ decomposes as $X_j \xrightarrow{f_j'} Z_j \xrightarrow{f_j''} Y$ such that $Z_j$ is smooth, $f_j''$ is an affine bundle and $f_j'$ is projective. Then $f_! 1_X \in \mathcal{D}(Y)$ is semisimple. Moreover, we have $f_! 1_X \cong \bigoplus_j (f_j)_! 1_{X_j}$.

2.4.2. Simple objects. Let $Y$ be an irreducible variety and $U$ a Zariski open subset of $Y$. Set $Z := Y \setminus U$ and $i : Z \hookrightarrow Y$.

**Proposition 2.19.** For $F \in \text{Perv}(U)$, there exists a unique perverse sheaf $\pi F$ on $Y$ satisfying

1. $\pi F|_U \cong F$,
2. $i^*(\pi F) \in \mathcal{P}^{\leq -1}(Z)$,
3. $i'_!(\pi F) \in \mathcal{P}^{\geq 1}(Z)$.

We call $\pi F$ the minimal extension of $F$. We have the following properties of the minimal extension:

1. $\pi F$ has neither non-trivial subobject nor non-trivial quotient object whose support is contained in $Z$.
2. If $F$ is simple, then $\pi F$ is simple.
3. For the Verdier duality functors $D_Y$ and $D_U$, we have $D_Y(\pi F) \cong \pi (D_U(F))$.

Let $X$ be a variety, $Y$ an irreducible locally closed smooth subvariety of $X$. For a simple local system $L$ on $Y$, the minimal extension $\pi L[\dim Y]$ is called the intersection cohomology complex of $Y$. We can regard $\pi L[\dim Y]$ as a simple perverse on $X$ whose support is the closure $\overline{Y}$ of $Y$. Conversely, any simple object in $\text{Perv}(X)$ is obtained in this way.

**Theorem 2.20** ([BBD]). For a simple perverse sheaf $F$ on $X$, there exist an irreducible closed subvariety $\overline{Y}$ and an simple local system $L$ on $Y$ such that $F \cong \pi L[\dim Y]$. Moreover, for simple perverse sheaves $F_1$ and $F_2$, we have $\text{Ext}^0(F_1, F_2) = \text{Hom}_{\text{Perv}(X)}(F_1, F_2) = \mathbb{C}$ or 0 according that $F_1$ and $F_2$ are isomorphic or not.
2.4.3. Fourier-Sato-Deligne transforms. Let $E \to S$ be a vector bundle and $E^* \to S$ the dual vector bundle. Hence $\mathbb{C}^* \times$ acts on $E$ and $E^*$. We say that $L \in \mathcal{D}(E)$ is monodromic if $H^\dagger(L)$ is locally constant on every $\mathbb{C}^*$-orbit of $E$. Let $\mathcal{D}_{\text{mono}}(E)$ be the full subcategory of $\mathcal{D}(E)$ consisting of monodromic objects. Then we can define the Fourier transform
\[ \Phi_{E/S} : \mathcal{D}_{\text{mono}}(E) \to \mathcal{D}_{\text{mono}}(E^*). \]

We will use the following properties of $\Phi$.

**Proposition 2.21** (e.g. [KS], [Lau]).
(1) For $K \in \mathcal{D}_{\text{mono}}(E)$, we have $\Phi_{E^*/S} \circ \Phi_{E/S}(K) \cong a^*K$, where $a : E \to E$ is the multiplication by $-1$ on each fiber of $E$.
(2) For a perverse sheaf $K \in \mathcal{D}_{\text{mono}}(E)$, $\Phi_{E/S}(K)$ is a perverse sheaf in $\mathcal{D}_{\text{mono}}(E^*)$.
(3) Let $E_1$ and $E_2$ be two vector bundles over $S$ with rank $r_1$ and $r_2$. Let $f : E_1 \to E_2$ be a morphism of vector bundles and $f^* : E_2^* \to E_1^*$ the transpose of $f$. Then we have
\[ \Phi_{E_2/S} \circ f^* \cong (f^*)^* \circ \Phi_{E_1/S}|_{r_2 - r_1}, \quad (f^*)_! \circ \Phi_{E_2/S} \cong \Phi_{E_1/S} \circ f^*[r_1 - r_2]. \]
(4) Suppose that $E_1 \to S_1$ and $E \to S$ are two vector bundles. If the following two diagrams
\[ \begin{array}{ccc}
E_1 & \xrightarrow{f_E} & E \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{\rho} & S
\end{array} \quad \begin{array}{ccc}
E_1^* & \xrightarrow{f_{E^*}} & E^* \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{\rho} & S
\end{array} \]
are Cartesian, then we have
\[ \Phi_{E/S} \circ (f_E)_! \cong (f_{E^*})^* \circ \Phi_{E_1/S_1}, \quad \Phi_{E_1/S_1} \circ (f_E)^* \cong (f_{E^*})^* \circ \Phi_{E/S}. \]
(5) The Fourier transforms commute with the Verdier duality functors.

2.5. Quivers. Let $I$ and $\alpha_i$’s be as in 2.1

**Definition 2.22.** A quiver $(I, H)$ associated with the symmetric Cartan matrix is a following data:

(i) a set $H$,
(ii) two maps out, in: $H \to I$ such that out($h$) $\neq$ in($h$) for any $h \in H$,
(iii) an involution $h \mapsto \overline{h}$ on $H$ satisfying out($\overline{h}$) $=$ in($h$) and in($\overline{h}$) $=$ out $h$,
(iv) $\{ h \in H | \text{out}(h) = i, \text{in}(h) = j \} = -(\alpha_i, \alpha_j)$ for $i \neq j$.

An orientation of a quiver $(I, H)$ is a subset $\Omega$ of $H$ such that $\Omega \cap \overline{\Omega} = \phi$ and $\Omega \cup \overline{\Omega} = H$. For a fixed orientation $\Omega$, we call a vertex $i \in I$ a sink if out($h$) $\neq i$ for any $h \in \Omega$.

**Definition 2.23.** Let $\mathcal{V}$ be the category of $I$-graded vector spaces $V = (V_i)$, with morphisms being linear maps respecting the grading. Put $\text{wt}(V) = \sum_{i \in I} \text{dim } V_i \alpha_i$.

Let $S_i$ be an $I$-graded vector space such that $\text{wt}(S_i) = \alpha_i$.

**Definition 2.24.** For $V \in \mathcal{V}$ and a subset $\Omega$ of $H$, we define
\[ E_{V, \Omega} : = \bigoplus_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}). \]

The algebraic group $G_V = \prod_{i \in I} GL(V_i)$ acts on $E_{V, \Omega}$ by $(g, x) \mapsto gx$ where $(gx)_h = g_{\text{in}(h)}x_hg_{\text{out}(h)}^{-1}$.

The group $(\mathbb{C}^*)^\Omega$ also acts on $E_{V, \Omega}$ by $x_h \mapsto c_hx_h$ ($h \in \Omega, c_h \in \mathbb{C}^*$).

For $x \in E_{V, \Omega}$, an $I$-graded subspace $W \subset V$ is $x$-stable if $x_h(W_{\text{out}(h)}) \subset W_{\text{in}(h)}$ for any $h \in \Omega$. 

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Note that \( E_{S_i, \Omega} \cong \{ \text{pt} \} \).

3. A Review on Lusztig’s Geometric Construction

We give a quick review on Lusztig’s theory in [Lus1] and [Lus2] (cf. [Lus3]). For a sequence \( i = (i_1, \ldots, i_m) \in I^m \) and a sequence \( a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m \), a flag of type \((i, a)\) is by definition a finite decreasing sequence \( F = (V = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m = \{0\}) \) of \( I \)-graded subspaces of \( V \) such that the \( I \)-graded vector space \( \mathcal{F}_{-1}/\mathcal{F}_i \) vanishes in degrees \( \neq i \) and has dimension \( a_i \) in degree \( i \). We denote by \( T_{i, a}^t \) the set of pairs \((x, F)\) such that \( x \in \mathcal{E}_V \) and \( F \) is an \( x \)-stable flag of type \((i, a)\). The group \( G_V \) acts on \( T_{i, a}^t \). The first projection \( \pi_{i, a} : T_{i, a}^t \rightarrow \mathcal{E}_V \) is a \( G_V \)-equivariant projective morphism.

By Lemma 2.18 \( L_{i, a}^t : = (\pi_{i, a})_!(\tilde{T}_{i, a}^t) \in \mathcal{D}(\mathcal{E}_V) \) is a semisimple complex. We define \( \mathcal{D}_V \) as the set of the isomorphism classes of simple perverse sheaves \( L \in \mathcal{D}(\mathcal{E}_V) \) satisfying the following property: \( L \) appears as a direct summand of \( L_{i, a}^t[d] \) for some \( d \) and \((i, a)\). We denote by \( \mathcal{D}_V \) the full subcategory of \( \mathcal{D}(\mathcal{E}_V) \) consisting of all objects which are isomorphic to finite direct sums of complexes of the form \( L[d] \) for various \( L \in \mathcal{D}_V \) and various integers \( d \). Any complex in \( \mathcal{D}_V \) is \( G_V \times (\mathbb{C}^\times)^{\Omega} \)-equivariant.

Let \( T, W, V \) be \( I \)-graded vector spaces such that \( \text{wt}(V) = \text{wt}(W) + \text{wt}(T) \). We consider the following diagram

\[
\begin{array}{ccc}
E_{T, \Omega} \times E_{W, \Omega} & \overset{p_1}{\longrightarrow} & E'_{\Omega} \\
\downarrow & & \downarrow \\
E''_{\Omega} & \overset{p_2}{\longrightarrow} & E_{W, \Omega} \\
\downarrow & & \downarrow \\
E_{V, \Omega} & \overset{p_3}{\longrightarrow} & E_{V, \Omega}.
\end{array}
\]

Here \( E''_{\Omega} \) is the variety of \((x, W)\) where \( x \in \mathcal{E}_V \) and \( W \) is an \( x \)-stable \( I \)-graded subspace of \( V \) such that \( \text{wt}(W) = \text{wt}(W) \). The variety \( E_{\Omega} \) consists of \((x, W, \varphi^W, \varphi^T)\) where \((x, W) \in E''_{\Omega}, \varphi^W : W \cong V, \varphi^T : T \cong V/W\). The morphisms \( p_1, p_2 \) and \( p_3 \) are given by \( p_1(x, W, \varphi^W, \varphi^T) = (x|_T, x|_W) \), \( p_2(x, W, \varphi^W, \varphi^T) = (x, W) \) and \( p_3(x, W) = x \). Then \( p_1 \) is smooth with connected fibers, \( p_2 \) is a principal \( G_T \times G_W \)-bundle, and \( p_3 \) is projective. For a \( G_T \)-equivariant semisimple complex \( K_T \) and a \( G_W \)-equivariant semisimple complex \( K_W \), there exists a unique semisimple complex \( K'' \) satisfying \( p_1^*(K_T \boxtimes K_W) = p_2^* K'' \). We define \( K_T \cdot K_W : = (p_3^*)_!(K'') \in \mathcal{D}(\mathcal{E}_V) \).

For an \( I \)-graded subspace \( U \) of \( V \) such that \( V/U \cong T \), we also consider the following diagram

\[
\begin{array}{ccc}
E_{T, \Omega} \times E_{U, \Omega} & \overset{p}{\longrightarrow} & E(U, V) \\
\downarrow & & \downarrow \\
E_{V, \Omega} & \overset{e_{i}}{\longrightarrow} & E_{V, \Omega}.
\end{array}
\]

Here \( E(U, V) \) is the variety of \( x \in \mathcal{E}_V \) such that \( U \) is \( x \)-stable. For \( K \in \mathcal{D}(\mathcal{E}_V) \), we define \( \text{Res}_{T, U}(K) : = p_{i}^*(K) \).

We define \( K_{V, \Omega} \) as the Grothendieck group of \( \mathcal{D}_V \). It is the additive group generated by the isomorphism classes \((L)\) of objects \( L \in \mathcal{D}_V \) with the relation \((L) = (L') + (L'')\) when \( L \cong L' \oplus L'' \). The group \( K_{V, \Omega} \) has a \( \mathbb{Z}[v, v^{-1}] \)-module structure by \( v(L) = (L[1]) \) and \( v^{-1}(L) = (L[-1]) \) for \( L \in \mathcal{D}_V \). Hence, \( K_{V, \Omega} \) is a free \( \mathbb{Z}[v, v^{-1}] \)-module with a basis \( \{(L)|L \in \mathcal{D}_V\} \). We define \( K_{\Omega} \) as \( \bigoplus \mathcal{V} K_{V, \Omega} \) where \( \mathcal{V} \) runs over the isomorphism classes of \( I \)-graded vector spaces. Recall that \( S_i \) is an \( I \)-graded vector space such that \( \text{wt}(S_i) = \alpha_i \).

Then we can define the induction \( f_i : K_{W, \Omega} \rightarrow K_{V, \Omega} \) and the restriction \( e_i' : K_{V, \Omega} \rightarrow K_{W, \Omega} \) by

\[
f_i(K) : = v^\dim W_i + \sum_{j \in \Omega} \dim W_j (1_{S_i} \ast K), \quad e_i'(K) : = v^{-\dim W_i + \sum_{j \in \Omega} \dim W_j} \text{Res}_{S_i, V}(K).
\]

Then Lusztig’s main theorem is stated as follows.

**Theorem 3.1** (Lusztig).
(i) The operators $e_i$ and $f_i$ define the action of the reduced $\nu$-analogue $B_\nu(g)$ of $g$ on $K_\Omega \otimes_{\mathbb{Z}[\nu,\nu^{-1}]} \mathbb{Q}(\nu)$. The $B_\nu(g)$-module $K_\Omega \otimes_{\mathbb{Z}[\nu,\nu^{-1}]} \mathbb{Q}(\nu)$ is isomorphic to $U_\nu^-(g)$. The involution induced by the Verdier duality functor coincides with the bar involution on $U_\nu^-(g)$.

(ii) The simple perverse sheaves in $\mathbb{X} \mathbb{P} \mathbb{V}_\Omega$ give a lower global basis of $U_\nu^-(g)$.

4. Quivers with an involution $\theta$

Definition 4.1. A $\theta$-quiver is a data:

1. a quiver $(I, H)$,
2. involutions $\theta: I \to I$ and $\theta: H \to H$,

satisfying

(a) $\text{out}(\theta(h)) = \theta(\text{in}(h))$ and $\text{in}(\theta(h)) = \theta(\text{out}(h))$,
(b) If $\theta(\text{out}(h)) = \text{in}(h)$, then $\theta(h) = h$,
(c) $\theta(h) = \theta(h)$,
(d) There is no $i \in I$ such that $\theta(i) = i$

A $\theta$-orientation is an orientation of $(I, H)$ such that $\Omega$ is stable by $\theta$.

From the assumption (d), any vertex $i$ is a sink with respect to some $\theta$-orientation $\Omega$.

Example 4.2. We give two $\theta$-orientations for the case of Example 2.13. The vertex $1$ is a sink in the right example.

Example 4.3. Our definition of a $\theta$-quiver contains the case of type $A_1^{(1)}$. The following three figures are three $\theta$-orientations in this case.

Definition 4.4. A $\theta$-symmetric $I$-graded vector space $V$ is an $I$-graded vector space endowed with a non-degenerate symmetric bilinear form $(\cdot, \cdot): V \times V \to \mathbb{C}$ such that $V_i$ and $V_j$ are orthogonal if $j \neq \theta(i)$. For an $I$-graded subspace $W$ of $V$, we set

$W^\perp := \{ v \in V \mid (v, w) = 0 \text{ for any } w \in W \}$.

Hence $(W^\perp)_{\theta(i)} \cong (V_i/W_i)^*$. 

Note that if $W \supset W^\perp$, then $W/W^\perp$ has a structure of $\theta$-symmetric $I$-graded vector space. Note that two $\theta$-symmetric $I$-graded vector spaces with the same dimension are isomorphic.

Definition 4.5. Let $(I, H)$ be a $\theta$-quiver. For a $\theta$-symmetric $I$-graded vector space $V$ and a $\theta$-stable subset $\Omega$ of $H$, we define

$^\theta E_{V, \Omega} := \{ x \in E_{V, \Omega} \mid x_{\theta(h)} = -^t x_h \in \text{Hom}(V_{\theta(\text{in}(h))}, V_{\theta(\text{out}(h))}) \text{ for any } h \in \Omega \}$.

The algebraic group $^\theta G_V := \{ g \in G_V \mid g_i^{-1} = g_{\theta(i)} \text{ for any } i \}$ naturally acts on $^\theta E_{V, \Omega}$. Set $(\mathbb{C}^\times)^{\Omega, \theta} := \{ (c_h)_{h \in \Omega} \mid c_h \in \mathbb{C}^\times \text{ and } c_{\theta(h)} = c_h \}$. The group $(\mathbb{C}^\times)^{\Omega, \theta}$ also acts on $^\theta E_{V, \Omega}$ by $x_h \mapsto c_h x_h \text{ (} h \in \Omega \text{)}$. These two actions commute with each other.
Definition 4.6. For a $\theta$-symmetric $I$-graded vector space $V$, a sequence $i = (i_1, \ldots, i_{2m}) \in I^{2m}$ such that $\theta(i_\ell) = i_{2m-\ell+1}$ and a sequence $a = (a_1, \ldots, a_{2m}) \in \mathbb{Z}_{\geq 0}^m$ such that $a_{2m-\ell+1} = a_\ell$, we say that a flag of $I$-graded subspace of $V$

$$F = (V = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} \supset \cdots \supset F^{2m} = \{0\})$$

is of type $(i, a)$ if

(i) $\dim(F^{\ell-1}/F^{\ell})_i \bigg\{ \begin{array}{ll}
    a_\ell & (i = i_\ell) \\
    0 & (i \neq i_\ell)
\end{array}$,

(ii) $F^{2m-\ell} = (F^{\ell})^\perp$.

Then we have $\text{wt} V = \sum_{1 \leq \ell \leq 2m} a_\ell \alpha_i$. We denote by $\mathcal{F}_{i,a}$ the set of flags of type $(i, a)$.

For $x \in \theta E_{V,\Omega}$, a flag $F$ of type $(i, a)$ is $x$-stable if $F^\ell (\ell = 1, \ldots, 2m)$ are $x$-stable. We define

$$\theta \mathcal{F}_{i,a,\Omega} : = \{ (x, F) \in \theta E_{V,\Omega} \times \theta \mathcal{F}_{i,a} \mid F \text{ is } x\text{-stable} \}.$$ 

The group $\theta G_V$ naturally acts on $\theta \mathcal{F}_{i,a}$ and $\theta \mathcal{F}_{i,a,\Omega}$.

Note that $x : V \to V \cong V^*$ in $\theta E_{V,\Omega}$ may be regarded as a skew-symmetric form on $V$, and the condition that $F$ is $x$-stable is equivalent to the one $x(F^\ell, F^{2m-\ell}) = 0$ for any $\ell$.

The following lemma is obvious.

Lemma 4.7. The variety $\theta \mathcal{F}_{i,a,\Omega}$ is smooth and irreducible. The first projection $\theta \pi_{i,a} : \theta \mathcal{F}_{i,a,\Omega} \to \theta E_{V,\Omega}$ is $\theta G_V \times (\mathbb{C}^*)^{\Omega, \theta}$-equivariant and projective.

4.2. Perverse sheaves on $\theta E_{V,\Omega}$. Let $\Omega$ be a $\theta$-orientation. By Lemma 4.7 and Lemma 2.18

$$\theta L_{i,a,\Omega} : = (\theta \pi_{i,a})(1_s \mathcal{F}_{i,a,\Omega})$$

is a semisimple complex in $\mathcal{D}(\theta E_{V,\Omega})$.

Definition 4.8. We define $\theta \mathcal{P}_{V,\Omega}$ as the set of the isomorphism classes of simple perverse sheaves $L$ in $\mathcal{D}(\theta E_{V,\Omega})$ satisfying the property: $L$ appears in $\theta L_{i,a,\Omega}[d]$ as a direct summand for some integer $d$ and $(i, a)$. We denote by $\theta \mathcal{P}_{V,\Omega}$ the full subcategory of $\mathcal{D}(\theta E_{V,\Omega})$ consisting of objects which are isomorphic to finite direct sums of $L[d]$ with $L \in \theta \mathcal{P}_{V,\Omega}$ and $d \in \mathbb{Z}$.

Note that any object in $\theta \mathcal{P}_{V,\Omega}$ is $\theta G_V \times (\mathbb{C}^*)^{\Omega, \theta}$-equivariant.

4.3. Multiplications and Restrictions. Fix $\theta$-symmetric and $I$-graded vector spaces $V$ and $W$, and an $I$-graded vector space $T$ such that $\text{wt}(V) = \text{wt}(W) + \text{wt}(T) + \theta(\text{wt}(T))$. We consider the following diagram

$$\text{E}_T,\Omega \times \theta \text{E}_W,\Omega \xrightarrow{p_1} \theta \text{E}'_\Omega \xrightarrow{p_2} \theta \text{E}''_\Omega \xrightarrow{p_3} \theta \text{E}_{V,\Omega}.$$ 

Here $\theta \text{E}''_\Omega$ is the variety of $(x, V)$ where $x \in \theta \text{E}_{V,\Omega}$ and $V$ is an $x$-stable $I$-graded subspace of $V$ such that $V \supset V^\perp$ and $\text{wt}(V/V) = \text{wt}(T)$, and we denote by $\theta \text{E}'_\Omega$ the variety of $(x, V, \varphi_W, \varphi_T)$ where $(x, V) \in \theta \text{E}''_\Omega$, $\varphi_W : W \sim V/V^\perp$ is an isomorphism of $\theta$-symmetric $I$-graded vector spaces and $\varphi_T : T \sim V/V$ is an isomorphism of $I$-graded vector spaces.

We define $p_1, p_2$ and $p_3$ by $p_1(x, V, \varphi_W, \varphi_T) = (x_T, x_W)$, $p_2(x, V, \varphi_W, \varphi_T) = (x, V)$ and $p_3(x, V) = x$. Here the morphism $x_W, x_T$ are defined by

$$x^W_h = \varphi^W_{\text{in}(h)} \circ (x|_{V/V^\perp}) \circ \varphi^W_{\text{out}(h)}, \quad x^T_h = \varphi^T_{\text{in}(h)} \circ (x|_{V/V}) \circ \varphi^T_{\text{out}(h)}.$$
Then $p_1$ is smooth with connected fibers, $p_2$ is a principal $\mathbf{G}_T \times {}^\theta \mathbf{G}_W$-bundle and $p_3$ is projective.

For a $\mathbf{G}_T$-equivariant semisimple object $K_T \in \mathcal{D}_{T,\Omega}$ and a $\mathbf{G}_W$-equivariant semisimple object $K_W \in \mathcal{D}_{W,\Omega}$, there exists a unique semisimple object $K'' \in \mathcal{D}(^\theta \mathbf{E}_W)$ satisfying $p_1^*(K_T \boxtimes K_W) = p_2^*(K'').$

**Definition 4.9.** We define $K_T \ast K_W : = (p_3)_!(K'') \in \mathcal{D}(^\theta \mathbf{E}_{V,\Omega}).$

Next, we fix an $I$-graded vector space $U$ such that

$$V \supset U \supset U \in \{0\}.$$  

We also fix an isomorphism $W \cong U/\perp$ as $\theta$-symmetric $I$-graded vector spaces and an isomorphism $T \cong V/U$ as $I$-graded vector spaces. We consider the following diagram

$$E_{T,\Omega} \times {}^\theta E_{W,\Omega} \overset{p}{\longrightarrow} {}^\theta E(W, V)_{\Omega} \overset{\iota}{\longrightarrow} {}^\theta E_{V,\Omega},$$

where

$${}^\theta E(W, V)_{\Omega} = \{x \in {}^\theta E_{V,\Omega} \mid U \text{ is } x\text{-stable}\}$$

and $p(x) = (x^T, x^W)$, $\iota(x) = x$.

**Definition 4.10.** For $K \in \mathcal{D}(^\theta \mathbf{E}_{V,\Omega})$, we define $\text{Res}_{T, W}(K) : = p_{1*}(K).$

**Proposition 4.11.** Let $V$ and $W$ be $\theta$-symmetric $I$-graded vector spaces such that $\text{wt } V = \text{wt } W + \alpha_i + \alpha_{\theta(i)}$. For $a \in \mathbb{Z}_{\geq 0}$, let $S^a_i$ be an $I$-graded vector space such that $\text{wt } (S^a_i) = a\alpha_i$.

(i) Suppose $^\theta L_{i, a; \Omega} \in \mathcal{D}(^\theta \mathbf{E}_{W,\Omega}).$ We have

$$1.S^a_i \ast L_{i, a; \Omega} = L_{i, i, \theta(i), (a, a, a), \Omega},$$

for $a \in \mathbb{Z}_{\geq 0}$.

(ii) Suppose $^\theta L_{i, a; \Omega} \in \mathcal{D}(^\theta \mathbf{E}_{V,\Omega})$ and $a_\ell > 0$ for all $\ell$ such that $i_\ell = i$. For $1 \leq k \leq 2m$ such that $i_k = i$, we define $a^{(k)} = (a_1^{(k)}, \ldots, a_{2m}^{(k)})$ by $a_\ell^{(k)} = a_\ell - \delta_{\ell, k} - \delta_{\ell, 2m-k+1}$ and we set

$$M_k(i, a^{(k)}) = \sum_{i_\ell = i, \ell < k} a_\ell^{(k)} + \sum_{k < \ell, h \in \text{out}(h) = i, \text{in}(h) = i_\ell} a_\ell^{(k)}. $$

Then we have

$$\text{Res}_{S_i, W}(^\theta L_{i, a; \Omega}) = \bigoplus_{i_k = i} [^\theta L_{i, a^{(k)}; \Omega}] [-2M_k(i, a^{(k)})].$$

**Proof.** (1) We consider the following diagram:

$$\begin{array}{ccc}
\theta \widetilde{F}_{i, a; \Omega} & \overset{p_1}{\longrightarrow} & \theta \widetilde{E}_{\Omega} \\
\theta \pi_{i, a} & \underset{\square \rho}{\longleftarrow} & \theta \pi_{i, a} \\
\theta E_{W, \Omega} & \overset{p_2}{\longrightarrow} & \theta E'_{\Omega} \\
\theta \pi_{i, a} & \overset{p_3}{\longrightarrow} & \theta E''_{\Omega} \\
\end{array}$$

where

$$\theta \widetilde{E}_\Omega : = \{(x, F, \varphi^W) \mid (x, F) \in \theta \widetilde{F}_{i, \theta(i), (a, a, a), \Omega}, \varphi^W : W \cong \mathbb{F}^1 / \mathbb{F}^{2m+1}\}.$$

Here, $\rho'' : \theta \widetilde{F}_{i, \theta(i), (a, a, a)} \rightarrow \theta E''_{\Omega}$ is given by $(x, F) \mapsto (x, F^1)$. Then $\rho''$ is projective and $p_3 \circ \rho'' = \theta \pi_{i, \theta(i), (a, a, a)}$. Hence $\rho''(1_{\theta \widetilde{F}_{i, \theta(i), (a, a, a)}; \Omega})$ is semisimple and $^\theta L_{i, i, \theta(i), (a, a, a); \Omega} = (\pi_{i, \theta(i), (a, a, a)}; (1_{\theta \widetilde{F}_{i, \theta(i), (a, a, a)})}) = (p_3)_!(\rho''(1_{\theta \widetilde{F}_{i, \theta(i), (a, a, a)}; \Omega}))$. On the other hand, we have

$$p_2^*(\rho''(1_{\theta \widetilde{F}_{i, \theta(i), (a, a, a)}; \Omega})) = \rho_1'(p_2)^*1_{\theta \widetilde{F}_{i, \theta(i), (a, a, a)}; \Omega} = \rho_1'(p_3)^*1_{\theta \pi_{i, a; \Omega}} = p_1^*(\rho_1)^*1_{\theta \pi_{i, a; \Omega}} = p_1^*(\theta \pi_{i, a; \Omega}) = p_1^*(\theta L_{i, a; \Omega}).$$
Hence we have $1_{S_i} \ast \theta L_{i,a,\Omega} = (p_3)_! \rho_! \circ (1_{S_i})_{(i,\theta(i)),(a,a,a)} = \theta L_{(i,\theta(i)),(a,a,a)}$.

(2) Set $\theta F(W,V) = \{(x,F) \in \theta F_{i,a,\Omega} \mid U \text{ is } x\text{-stable}\}$ and $\theta F_{i,a,\Omega} = \{F \in \theta F_{i,a,\Omega} \mid F^k \subset \Omega, F^{k-1} \not\subset U\}$. We define

$$\theta F_{i,a,\Omega}(W,V) = \{(x,F) \in \theta F(W,V) \mid F \in \theta F_{i,a,\Omega}\}.$$ 

Then the locally closed smooth subvarieties $\theta F_{i,a,\Omega}(W,V)$ ($1 \leq k \leq 2m, i_k = i$) give a partition $\theta F(W,V)$.

For a flag $F$ of $V$, we define the flag $F|_{U/U^\perp}$ by

$$F|_{U/U^\perp} = (U/U^\perp = (F^0 \cap U)/(F^0 \cap U^\perp) \supset \cdots \supset (F^{2m} \cap U)/(F^{2m} \cap U^\perp) = \{0\}).$$

Note that for $(x,F) \in \theta F_{i,a,\Omega}(W,V)$,

$$\dim(F^j \cap U_j) = \dim(F^j - \delta(j = i, \ell < k),$$

$$\dim(F^j \cap (U^\perp)_j) = \delta(2m - \ell \geq k, j = \theta(i)).$$

We have

$$\dim((F|_{U/U^\perp})^{\ell-1}/(F|_{U/U^\perp})^\ell)_j = \dim(F^{\ell-1}/F^\ell)_j - \delta(j = i, \ell = k) - \delta(j = \theta(i), 2m - \ell = k - 1).$$

Hence the flag $F|_{U/U^\perp}$ is a flag of type $(i,a^{(k)})$. Therefore $(x,F) \mapsto (x|_{U/U^\perp}, F|_{U/U^\perp})$ defines $f_{a^{(k)}} : \theta F_{i,a^{(k)},\Omega} \to \theta F_{i,a,\Omega}$. We obtain the following diagram:

$$\begin{array}{cccc}
\theta F_{i,a^{(k)},\Omega} & \xrightarrow{f_{a^{(k)}}} & \theta F_{i,a,\Omega} & \xrightarrow{\theta F(W,V)} & \theta F_{i,a,\Omega} \\
\theta E_{W,\Omega} & \xrightarrow{p} & \theta E(W,V)_{\Omega} & \xrightarrow{\theta F(W,V)} & \theta E_{V,\Omega}
\end{array}$$

Claim. The morphism $f_{a^{(k)}}$ is an affine bundle of rank $M_k(i,a^{(k)})$.

Proof. Fix $(x,W,F_W) \in \theta F_{i,a^{(k)},\Omega}$. Note that $(U^\perp)_j = \{0\}$ and $U_j \cong W_j$ for $j \neq \theta(i)$. If $F \in \theta F_{i,a^{(k)}}$ satisfies $F|_{U/U^\perp} = F_W$, we have

$$F^\ell_i = F^\ell_{W,i} (\ell \geq k), \quad F^\ell_i = F^\ell_{W,i} + F^{k-1}_i (\ell < k), \quad F^\ell_{\theta(i)} = (F^{2m-k+1})^\perp$$

and $F^\ell_j = F^\ell_{W,j} (j \neq i, \theta(i))$. A subspace $F^{k-1}_i$ is parametrized by a one-dimensional subspace $F^{k-1}_{W,i} \subset V_i/F^{k-1}_{W,i}$ such that $F^{k-1}_{W,i} \not\subset U_i/F^{k-1}_{W,i}$. Hence the fibers of $\theta F_{i,a^{(k)}} \to \theta F_{i,a^{(k)},\Omega} : F \mapsto F|_{U/U^\perp}$ at $F_W$ is isomorphic to $A^{\dim(V_i/F^{k-1}_{W,i})-1}$. Note that

$$\dim(V_i/F^{k-1}_{W,i}) - 1 = \sum_{\ell < k, i_\ell = i} a_{\ell} = \sum_{\ell < k, i_\ell = i} a^{(k)}_{\ell}.$$ 

Fix a flag $F \in \theta F_{i,a,\Omega}$ such that $F|_{U/U^\perp} = F_W$. Note that $V_i \supset U_i \cong W_i$, $V_{\theta(i)} = U_{\theta(i)}$ and $V_j = U_j \cong W_j$ for $j \neq i, \theta(i)$. Assume that $x \in \theta E_{W,\Omega}$ satisfies the condition that $F$ is $x$-stable and $x|_{U/U^\perp} = x_W$.

First, suppose that $h \in \Omega$ satisfies $\text{out}(h) \neq i$ and $\text{in}(h) \neq \theta(i)$. Then $x_h$ coincides with the composition $V_{\text{out}(h)} \to U_{\text{out}(h)}/(U^\perp)_{\text{out}(h)} \cong W_{\text{out}(h)} \xrightarrow{x_W,h} W_{\text{in}(h)} \cong U_{\text{in}(h)} \subseteq V_{\text{in}(h)}$. Hence, for such an $h \in \Omega$, $x_h$ is uniquely determined by $x_W$ and $x$ stabilizes the flag $F$. 


Second, suppose that $h \in \Omega$ satisfies $\text{out}(h) = i$. Take $v \in F_i^{k-1}$ such that $v \notin U_i$. If $\text{in}(h) \neq \theta(i)$, $x_h$ is parametrised by $x_h(v) \in F_{\text{in}(h)}^{k-1}$. Note that

$$\dim F_{\text{in}(h)}^{k-1} = \sum_{\ell \geq k, i_\ell = \text{in}(h)} a_{\ell} = \sum_{\ell > k, i_\ell = \text{in}(h)} a_{\ell}^{(k)},$$

because $\text{in}(h) \neq i, \theta(i)$, $i_k = i$ and $\ell \neq k, 2m - k + 1$. If $\text{out}(h) = i$ and $\text{in}(h) = \theta(i)$, we can regard $x_h$ as a skew-symmetric form on $V_i$. Since $\text{F}_i^\ell = F_{W,i}^\ell + \delta(\ell < k)\text{C}_v$, the skew-symmetric condition on $x$ is equal to the condition $x(v, F_i^{2m-k+1} + \text{C}_v) = 0$. Then $x_h$ is parametrized by $(V_i/(F_i^{2m-k+1} + \text{C}_v))^*$. Since $v \notin F_i^{2m-k+1}$ if only if $2m - k + 1 \geq k$, we have

$$\dim (V_i/(F_i^{2m-k+1} + \text{C}_v))^* = \dim (V/F_i^{2m-k+1})^* - \delta(2m - k + 1 \geq k)$$

$$= \left( \dim F_{\theta(i)}^{k-1} \right) - \delta(2m - k + 1 \geq k) = \left( \sum_{\ell \geq k, i_\ell = \theta(i)} a_{\ell} \right) - \delta(2m - k + 1 \geq k).$$

Since $i_k = i \neq \theta(i)$, $i_{2m-k+1} = \theta(i)$, we have $a_{\ell} = a_{\ell}^{(k)} + \delta(\ell = 2m - k + 1)$ if $i_{\ell} = \theta(i)$. Thus we obtain

$$\dim (V_i/(F_i^{2m-k+1} + \text{C}_v)) = \sum_{\ell > k, i_\ell = \theta(i)} a_{\ell}^{(k)}.$$ 

Set

$$\Omega_0 := \{ h \in \Omega \mid \text{out}(h) = i, \text{in}(h) = \theta(i) \}, \quad \Omega_1 := \{ h \in \Omega \mid \text{out}(h) = i, \text{in}(h) \neq \theta(i) \}.$$ 

The morphism $\tilde{F}_i(W, V) \to \{ F \in \theta F_i^{(k)} | F|_{U/U^\perp} = F_W \}$ is an affine bundle and its fiber dimension is equal to

$$\sum_{h \in \Omega_1} \dim(F_{\text{in}(h)}^{k-1}) + \sum_{h \in \Omega_0} \dim\{ V_i/(F_i^{2m-k+1} + \text{C}_v) \}$$

$$= \sum_{h \in \Omega_1, \ell > k, i_\ell \neq \theta(i)} a_{\ell}^{(k)} + \sum_{h \in \Omega_0, \ell > k, i_\ell = \theta(i)} a_{\ell}^{(k)} = \sum_{h \in \Omega_0, \ell > k} a_{\ell}^{(k)}.$$ 

Thus the rank of $f_{a(k)}$ is equal to

$$\dim(V_i/F_i^{k-1}) - 1 + \sum_{h \in \Omega_0, \ell > k} a_{\ell}^{(k)} = \sum_{i_{\ell} = i, \ell < k} a_{\ell}^{(k)} + \sum_{h \in \Omega_0, h_{i\ell} < h} a_{\ell}^{(k)} = M_k(i, a^{(k)}).$$

By this claim, we have $(f_{a(k)}):1_{\tilde{F}_i(W, V)} = 1_{\tilde{F}_i^{(k)}(W, V)}[-2M_k(i, a^{(k)})]$. By Lemma 2.18(2), we obtain

$$\text{Res}_{i, W}(\theta L_{i,a,\Omega}) = (\theta_{i,a,\Omega})_i 1_{\tilde{F}_i^{(k)}(W, V)} = \bigoplus_k (\theta_{i,a^{(k)},\Omega})_i (f_{a^{(k)} }):1_{\tilde{F}_i(W, V)}$$

$$= \bigoplus_k \theta_{i,a^{(k)},\Omega}[-2M_k(i, a^{(k)})].$$

**Lemma 4.12.** Let $T^1$ and $T^2$ be $I$-graded vector spaces. Let $W$ and $V$ be $\theta$-symmetric $I$-graded vector spaces such that $\text{wt} V = \text{wt} T^1 + \theta(\text{wt} T^1) + \text{wt} T^2 + \theta(\text{wt} T^2) + \text{wt} W$. For $G^\theta$-equivariant semisimple objects $L_j \in \mathcal{D}(E_i, T^j, \Omega)$ ($j = 1, 2$) and a $G^\theta_W$-equivariant semisimple object $L \in \mathcal{D}(E_W, \Omega)$, we have $(L_1 * L_2) * L \cong L_1 * (L_2 * L)$. 

\[\square\]
Proof. Let $T^{12}$ be an $I$-graded vector space such that $\text{wt} \, T^{12} = \text{wt} \, T^1 + \text{wt} \, T^2$. Let $W^2$ be a $\theta$-symmetric $I$-graded vector space such that $\text{wt} \, W^2 = \text{wt} \, T^2 + \theta(\text{wt} \, T^2) + \text{wt} \, W$. We denote by $\mathcal{F}$ the variety of pairs $(x, F)$ where $x \in \theta E_{V, \Omega}$ and $F = (V \supset F^1 \supset F^2 \supset F^3 \supset F^4 \supset \{0\})$ is an $x$-stable flag such that $F^3 = (F^2)^\perp$, $F^4 = (F^1)^\perp$, $F^1/F^4 \cong W^2$ and $F^2/F^3 \cong W$ as $\theta$-symmetric $I$-graded vector spaces. Let $\mathcal{F}$ be the variety of pairs $(x, F, \varphi_{W}, \varphi_{W^2}, \varphi_1, \varphi_2, \varphi_{T^2})$ where $(x, F) \in \mathcal{F}$ and $\varphi_{W^2} : F^1/F^4 \cong W^2$, $\varphi_{W} : F^2/F^3 \cong W$ as $\theta$-symmetric $I$-graded vector spaces, and $\varphi_1 : V/F^1 \cong T^1$, $\varphi_2 : V/F^2 \cong T^{12}$ and $\varphi_{T^2} : F^1/F^2 \cong T^2$ as $I$-graded vector spaces.

We consider the following diagram:

\[
\begin{array}{ccc}
E_{T^1, \Omega} \times E_{T^2, \Omega} \times \theta E_{W, \Omega} & \xrightarrow{u_1} & \mathcal{F} \\
& \xrightarrow{u_2} & \mathcal{F} \\
& \xrightarrow{u_3} & \theta E_{V, \Omega}.
\end{array}
\]

Here $u_1(x, F, \varphi_{W}, \varphi_{W^2}, \varphi_1, \varphi_2, \varphi_{T^2}) = (x^1, x^2, x_W)$, where $x_W, x^1$ and $x^2$ are the restrictions of $x$ to $W$, $T^1$ and $T^2$ through the isomorphism $\varphi_W, \varphi_1$ and $\varphi_2$ respectively, and $u_2$ and $u_3$ are natural projections. Note that $u_1$ is smooth with connected fibers, $u_2$ is a principal $G_{T^1} \times G_{T^2} \times G_W$-bundle and $u_3$ is projective. Then, for $L \in \theta \mathcal{F}_{W, \Omega}$, there exists a unique semisimple object $L'' \in \mathcal{D} \mathcal{F}$ such that $u_1^1(L_1 \boxtimes L_2 \boxtimes L) = u_2^*L''$, we define $K$ by $(u_3)L''$.

We shall prove $K \cong L_1 * (L_2 * L)$ and $K \cong (L_1 * L_2) * L$.

First, $L_2 * L$ is defined by the following diagram:

\[
\begin{array}{ccc}
E_{T^2, \Omega} \times \theta E_{W, \Omega} & \xrightarrow{q_1} & E_2' \\
& \xrightarrow{q_2} & E_2'' \\
& \xrightarrow{q_3} & \theta E_{W^2, \Omega}.
\end{array}
\]

Here $E''_2$ is the variety of $(y, V)$ where $y \in \theta E_{W^2, \Omega}$ and $V$ is an $y$-stable $I$-graded vector subspace of $W^2$ such that $V \supset V^\perp$ and $\text{wt}(W^2/V) = \text{wt}(T^2)$, and $E'_2$ is the variety of $(y, V, \psi_{W}, \psi_{T^2})$ where $(y, V) \in E''_2$ and $\psi_W : V/V^\perp \cong W$ and $\psi_{T^2} : W^2/V \cong T^2$. For $L''_2 \in \mathcal{D} \mathcal{E}_2''$ such that $q_2^1(L_2 \boxtimes L) = q_2^2L''_2$, we have $(q_3)L''_2 = L_2 * L$. We consider the diagram

\[
\begin{array}{ccc}
E_{T^1, \Omega} \times E_{T^2, \Omega} \times \theta E_{W, \Omega} & \xrightarrow{\tilde{q}_1} & E_{T^1, \Omega} \times E'_2 \\
& \xrightarrow{\tilde{q}_2} & E_{T^1, \Omega} \times E''_2 \\
& \xrightarrow{\tilde{q}_3} & \theta E_{W^2, \Omega},
\end{array}
\]

and denote by $L''_1' \equiv L_1 \boxtimes L''_2 \in \mathcal{D}(E_{T^1, \Omega} \times E''_2)$. Then $\tilde{q}_3^1(L_1 \boxtimes L_2 \boxtimes L) = \tilde{q}_1^2L''_1'$ and $(\tilde{q}_3)L''_1' = L_1 \boxtimes (L_2 * L)$.

Second, $L_1 * (L_2 * L)$ is defined by the following diagram:

\[
\begin{array}{ccc}
E_{T^1, \Omega} \times \theta E_{W^2, \Omega} & \xrightarrow{p_1} & E' \\
& \xrightarrow{p_2} & E'' \\
& \xrightarrow{p_3} & \theta E_{V, \Omega}.
\end{array}
\]

Here $E''$ is the variety of $(y, V)$ where $y \in \theta E_{V, \Omega}$ and $V$ is an $y$-stable $I$-graded vector subspace of $V$ such that $V \supset V^\perp$ and $\text{wt}(V/V) = \text{wt}(T^1)$, and $E'$ is the variety of $(y, V, \psi_{W^2}, \psi_{T^1})$ where $(y, V) \in E''$ and $\psi_{W^2} : V/V^\perp \cong W^2$ and $\psi_{T^1} : V/V \cong T^1$. For $K'' \in \mathcal{D}(E'')$ such that $p_3^1(L_1 \boxtimes (L_2 * L)) = p_3^2K''$, we have $L_1 * (L_2 * L) = (p_3)K''$.

Set $E'_1 = E_{T^1, \Omega} \times E'_2$, $E''_1 = E_{T^1, \Omega} \times E''_2$, $E_{12} = E_{T^1, \Omega} \times E_{T^2, \Omega} \times \theta E_{W, \Omega}$ and $E_2 = E_{T^1, \Omega} \times \theta E_{W^2, \Omega}$.
We consider the following diagram:

where $E = \mathcal{F} \times_{E''} E'$. Here $s_3(x, F) = (x, F^1)$, $t_2(x, F, \varphi \mathbf{W}, \varphi \mathbf{W}_2, \varphi_1, \varphi_2) = (x, F, \varphi \mathbf{W}_2, \varphi_1)$, $r_2(x, F, \varphi \mathbf{W}_2, \varphi_1) = (x, F)$ and $t_3(x, F, \varphi \mathbf{W}_2, \varphi_1) = (x, F^1, \varphi \mathbf{W}_2, \varphi_1)$. We define $r_1$ and $v_1$ by

$$r_1(x, F, \varphi \mathbf{W}_2, \varphi_1) = (x^1, x \mathbf{W}_2, \varphi \mathbf{W}_2(F_2/F_4)),$$

$$v_1(x, F, \varphi \mathbf{W}, \varphi \mathbf{W}_2, \varphi_1, \varphi_2, \varphi \mathbf{t}_2) = (x^1, x \mathbf{W}_2, \varphi \mathbf{W}_2(F_2/F_4), \psi \mathbf{W}, \psi_2),$$

where $x \mathbf{W}_2, \psi \mathbf{W}$ and $\psi_2$ are natural morphism induced by using $\varphi \mathbf{W}, \varphi \mathbf{W}_2$ and $\varphi_2$. We have $t_3^*r_1^*L_1^f = v_1^*t_2^*L_1 = u_1^*(L_1 \otimes L_2 \otimes L) = u_3^*L'' = t_3^*r_2^*L''$. Since $t_2$ is a $G \mathbf{T}_2 \times \mathbf{G} \mathbf{W}_1$-principal bundle, we obtain $r_1^*L'' = r_2^*L''$. Therefore $p_2^*(s_3)|L'' = (t_3|r_2^*L'' = (t_3|r_2^*L'' = p_1|q_3)^*L'' = p_1^*(L_1 \otimes (L_2 * L))$. Thus $(p_3)|(r_3)|L'' = L_1 * (L_2 * L)$. We have $K = (u_3)|L'' = L_1 * (L_2 * L)$. Similarly, we obtain $K \cong (L_1 * L_2) * L$. Thus the claim follows.

4.4. Restriction functor $E_i$, Induction functors $F_i$ and $F_i^{(a)}$. We consider the following diagram

$$E_{T, \Omega} \times \theta \mathbf{E}_{\mathbf{W}, \Omega} \xrightarrow{p_1} \theta \mathbf{E}_{\Omega} \xrightarrow{p_2} \theta \mathbf{E}_{\Omega'} \xrightarrow{p_3} \theta \mathbf{E}_{\mathbf{V}, \Omega}.$$

**Lemma 4.13.** Suppose $T = S_i$. Let $d_{p_1}$ and $d_{p_2}$ be the dimension of the fibers of $p_1$ and $p_2$, respectively. The we have

$$d_{p_1} - d_{p_2} = \dim \theta \mathbf{E}_{\Omega'} - \dim \theta \mathbf{E}_{\mathbf{W}, \Omega} = \dim \mathbf{W}_i + \sum_{h \in \Omega: \text{out}(h) = i} \dim \mathbf{W}_{\text{in}(h)}.$$

**Proof.** For a vector space $V$, we denote by $\text{Alt}(V)$ the set of all skew-symmetric linear maps $V \to V^*$. Let $P(V)$ denote the projective space of hyperplanes of $V$. Set $\Omega_0 = \{h \in \Omega \mid \theta(h) = h\}$, $\Omega_1 = \Omega \setminus \Omega_0$. We have

$$\dim \theta \mathbf{E}_{\mathbf{W}, \Omega} = \frac{1}{2} \sum_{h \in \Omega_1} \dim \mathbf{W}_{\text{out}(h)} \dim \mathbf{W}_{\text{in}(h)} + \sum_{h \in \Omega_0} \dim \text{Alt}(\mathbf{W}_{\text{out}(h)}).$$

We set

$$\begin{align*}
\Omega_{10} & = \{h \in \Omega_1 \mid \text{out}(h) \neq i, \text{in}(h) \neq i\}, \\
\Omega_{11} & = \{h \in \Omega_1 \mid \text{out}(h) = i\}, \\
\Omega_{12} & = \{h \in \Omega_1 \mid \text{in}(h) = i\}, \\
\Omega_{00} & = \{h \in \Omega_0 \mid \text{out}(h), \text{in}(h) = (i, \theta(i)) \text{ or } (\theta(i), i)\}, \\
\Omega_{01} & = \Omega_0 \setminus \Omega_{00}.
\end{align*}$$

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Then \( \Omega_1 = \Omega_{10} \sqcup \Omega_{11} \sqcup \Omega_{12} \) and \( \Omega_0 = \Omega_{00} \sqcup \Omega_{01} \). Note that \( \theta \) gives bijections \( \Omega_{10} \to \Omega_{10} \) and \( \Omega_{11} \to \Omega_{12} \). Therefore we have

\[
\dim^\theta \mathbf{E}^\prime_{\Omega} = \dim \mathbf{P}(V_i) + \frac{1}{2} \sum_{h \in \Omega_{10}} \dim W_{\text{out}(h)} \dim W_{\text{in}(h)} + \\
+ \sum_{h \in \Omega_{11}} \dim V_i \dim W_{\text{in}(h)} + \sum_{h \in \Omega_{01}} \dim \text{Alt}(W_{\text{out}(h)}) + \\
+ \sum_{h \in \Omega_{\text{out}(h)} = i, \text{in}(h) = \theta(i)} \dim \text{Alt}(V_i) + \sum_{h \in \Omega_{\text{out}(h)} = \theta(i), \text{in}(h) = i} \dim \text{Alt}(W_i).
\]

Since \( \dim V_i = \dim W_i + 1 \) and \( \dim \text{Alt}(V_i) = \dim \text{Alt}(W_i) = \dim W_i \), we conclude

\[
\dim^\theta \mathbf{E}^\prime_{\Omega} - \dim^\theta \mathbf{E}_{\Omega, V} = \dim W_i + \sum_{h \in \Omega_{11}} \dim W_{\text{in}(h)} + \sum_{h \in \Omega_{\text{out}(h)} = i, \text{in}(h) = \theta(i)} (\dim \text{Alt}(V_i) - \dim \text{Alt}(W_i)) + \\
\sum_{h \in \Omega_{\text{out}(h)} = i, \text{in}(h) = \theta(i)} \dim W_i = \dim W_i + \sum_{h \in \Omega_{\text{out}(h)} = i} \dim W_{\text{in}(h)}.
\]

\( \square \)

**Definition 4.14.**

(i) For \( T = S_i \) and a \( \theta \mathbf{G}_V \)-equivariant semisimple object \( K \) in \( \mathcal{D}_{\mathbf{E}_\Omega} \), we define the operator \( F_i \) by

\[
F_i(K) = (1_{S_i} \ast K)[d_{F_i}]
\]

where

\[
d_{F_i} = d_{p_1} - d_{p_2} = \dim W_i + \sum_{h \in \Omega_{\text{out}(h)} = i} \dim W_{\text{in}(h)}.
\]

(ii) For \( T = S_i \), we define the functor \( E_i : \mathcal{D}(\theta \mathbf{E}_{V, \Omega}) \to \mathcal{D}(\theta \mathbf{E}_{\Omega}) \) by

\[
E_i(K) = \text{Ress}_{\mathbf{E} \to \mathbf{E}}(K)[d_{E_i}]
\]

where

\[
d_{E_i} = d_{F_i} - 2 \dim W_i = - \dim W_i + \sum_{h \in \Omega_{\text{out}(h)} = i} \dim W_{\text{in}(h)}.
\]

By Proposition 4.11, \( E_i \) and \( F_i \) induce the restriction functor \( \mathcal{D}_{\mathbf{E}_{V, \Omega}} \to \mathcal{D}_{\mathbf{E}_{\Omega}} \), induction functor \( \mathcal{D}_{\mathbf{E}_{\Omega}} \to \mathcal{D}_{\mathbf{E}_{V, \Omega}} \), respectively.

**Definition 4.15.** For \( a \in \mathbb{Z}_{>0} \), let \( W \) and \( V \) be \( \theta \)-symmetric 1-graded vector spaces such that \( \text{wt}(V) = \text{wt}(W) + a(\alpha_i + \alpha_{\theta(i)}) \). For a \( \theta \mathbf{G}_W \)-equivariant semisimple object \( L \in \mathcal{D}_{\mathbf{E}_{\Omega}} \), we define \( F_i^{(a)}(L) = 1_{S_i} \ast L[d_a] \) where

\[
d_a = a \left( \dim W_i + \sum_{h \in \Omega_{\text{out}(h)} = i} \dim W_{\text{in}(h)} \right) + \frac{a(a - 1)}{2} \# \{ h \in \Omega_{\text{out}(h) = i, \text{in}(h) = \theta(i)} \}.
\]

We call \( F_i^{(a)} \) the \( a \)-th divided power of \( F_i \).

By Proposition 4.11 (1), we have the following lemma.
Lemma 4.16. The object $^\theta L_{1,a;\Omega}$ is isomorphic to $F_i^{(a_1)} F_i^{(a_2)} \cdots F_i^{(a_m)} 1_{pt}$ up to shift.

Lemma 4.17. The operator $F_i^{(a)}$ gives a functor $^\theta \mathcal{W}_\Omega \to ^\theta \mathcal{V}_\Omega$ and satisfy $F_i F_i^{(a)} = F_i^{(a)} F_i = [a+1]_* F_i^{(a+1)}$.

Proof. By Proposition 4.11(1), $F_i^{(a)}$ gives a functor $^\theta \mathcal{W}_\Omega \to ^\theta \mathcal{V}_\Omega$. We have

$$F_i F_i^{(a)} (L) = F_i (1_{S_i^a} * L)[d_a] = 1_{S_i^a} * (1_{S_i^a} * L) v^{d_a + d}$$

where

$$d = \dim W_i + a + \sum_{h \in \Omega : \text{out}(h) = i, \text{in}(h) \neq \theta(i)} \dim W_{\text{in}(h)} + \sum_{h \in \Omega : \text{out}(h) = i, \text{in}(h) = \theta(i)} (\dim W_{\theta(i)} + a)$$

$$= \dim W_i + a + \sum_{h \in \Omega : \text{out}(h) = i} \dim W_{\text{in}(h)} + a \sharp \{h \in \Omega \mid \text{out}(h) = i, \text{in}(h) = \theta(i)\}.$$

Note that $1_{S_i^a} * 1_{S_i^a} = (1 + v^{-2} + \cdots + v^{-2a}) 1_{S_i^{a+1}} = [a+1]_* v^{-a} 1_{S_i^{a+1}}$ in $E_{S_i^{a+1},\Omega}$. By Lemma 4.12 we have

$$F_i F_i^{(a)} (L) = [a+1]_* v^{-a} v^{d_a + d} 1_{S_i^{a+1}} * L$$

$$= [a+1]_* v^{-a} v^{d_a + d + d_{a+1}} F_i^{(a+1)} (L).$$

Since

$$d_{a+1} + a = (a+1) \left( \dim W_i + \sum_{h \in \Omega : \text{out}(h) = i} \dim W_{\text{in}(h)} \right) + a$$

$$+ \left( \frac{a(a-1)}{2} + a \right) \sharp \{h \in \Omega \mid \text{out}(h) = i, \text{in}(h) = \theta(i)\}$$

we conclude $F_i F_i^{(a)} = [a+1]_* F_i^{(a+1)}$. \hfill \Box

4.5. Commutativity with Fourier transforms. For two $\theta$-orientations $\Omega$ and $\Omega'$, we have $\Omega \setminus \Omega' = \Omega' \setminus \Omega$. Then we can regard $^\theta \mathcal{E}_{V,\Omega} \to ^\theta \mathcal{E}_{V,\Omega'\Omega'}$ and $^\theta \mathcal{E}_{V,\Omega'} \to ^\theta \mathcal{E}_{V,\Omega\Omega'}$ as vector bundles and they are the dual bundles to each other by the form $\sum_{h \in \Omega \cap \Omega'} \text{tr}(x_h x_h^{-1})$ on $^\theta \mathcal{E}_{V,\Omega} \times ^\theta \mathcal{E}_{V,\Omega'}$. We say that $L \in \mathcal{D}(^\theta \mathcal{E}_{V,\Omega})$ is $(C^\times)^{\Omega,\theta}$-monodromic if $H^j(L)$ is locally constant on every $(C^\times)^{\Omega,\theta}$-orbit on $^\theta \mathcal{E}_{V,\Omega}$. Let $\mathcal{D}_{(C^\times)^{\Omega,\theta}}(^\theta \mathcal{E}_{V,\Omega})$ be the full subcategory of $\mathcal{D}(^\theta \mathcal{E}_{V,\Omega})$ consisting of $(C^\times)^{\Omega,\theta}$-monodromic objects. Hence we have the Fourier transform $\Phi^\Omega_{\Omega'} : \mathcal{D}_{(C^\times)^{\Omega,\theta}}(^\theta \mathcal{E}_{V,\Omega}) \to \mathcal{D}_{(C^\times)^{\Omega',\theta}}(^\theta \mathcal{E}_{V,\Omega'})$. The following lemma is obvious.

Lemma 4.18. For three $\theta$-orientations $\Omega, \Omega'$ and $\Omega''$, we have

$$\Phi^\Omega_{\Omega''} \circ \Phi^\Omega_{\Omega'} \cong a^* \circ \Phi^\Omega_{\Omega''} : \mathcal{D}_{(C^\times)^{\Omega,\theta}}(^\theta \mathcal{E}_{V,\Omega}) \to \mathcal{D}_{(C^\times)^{\Omega'',\theta}}(^\theta \mathcal{E}_{V,\Omega''})$$

where $a : ^\theta \mathcal{E}_{V,\Omega''} \to ^\theta \mathcal{E}_{V,\Omega'}$ is defined by $x_h \mapsto -x_h$ or $x_h$ according that $h \in \Omega'' \cap \overline{\Omega'} \cap \Omega$ or not. In particular, $\mathcal{D}_{(C^\times)^{\Omega,\theta}}(^\theta \mathcal{E}_{V,\Omega})$ does not depend on $\Omega$.

Since any object in $^\theta \mathcal{W}_{V,\Omega}$ is $^\theta \mathcal{G}_{V} \times (C^\times)^{\Omega,\theta}$-equivariant, it is a monodromic object. By the commutativity between $E_i, F_i$ and $(C^\times)^{\Omega,\theta}$-action, the functors $E_i$ and $F_i$ preserve the category $(C^\times)^{\Omega,\theta}$-monodromic objects.
Theorem 4.19. Let $V$ and $W$ be $\theta$-symmetric $I$-graded vector spaces such that $\text{wt } V = \text{wt } W + \alpha_i + \alpha_{\theta(i)}$, and $\Omega$ and $\Omega'$ be two $\theta$-symmetric orientations.

(1) Let $F_i^\Omega$ and $F_i^{\Omega'}$ be the induction functors with respect to $\Omega$ and $\Omega'$, respectively. For a $^\theta G_W$-equivariant semisimple object $L \in \^\theta D_W$, we have $\Phi_{V, \Omega'}^{\Omega'} \circ F_i^\Omega(L) \cong F_i^{\Omega'} \circ \Phi_{W, \Omega}^{\Omega'}(L)$.

(2) Let $E_i^\Omega$ and $E_i^{\Omega'}$ be the restriction functors with respect to $\Omega$ and $\Omega'$, respectively. For a $^\theta G_V$-equivariant semisimple object $K \in \^\theta D_W$, we have $\Phi_{W, \Omega}^{\Omega'} \circ E_i^{\Omega'}(K) \cong E_i^\Omega \circ \Phi_{V, \Omega}^{\Omega'}(K)$.

(3) The Fourier transform $\Phi_{V, \Omega}^{\Omega'}$ gives an isomorphism between $^\theta D_{V, \Omega}$ and $^\theta D_{V, \Omega'}$ and an equivalence between $^\theta D_{V, \Omega}$ and $^\theta D_{V, \Omega'}$.

Proof. (1) Let us define the fibre products $E_1, E_2, E_3, E_1', E_2'$ and $E_3'$ by

$$
E_1' = \theta E_{W, \Omega} \times_{\theta E_{W, \Omega \cap \Omega'}} \theta E_{\Omega, \Omega'}, \quad E_1' = \theta E_{W, \Omega} \times_{\theta E_{W, \Omega \cap \Omega'}} \theta E_{\Omega, \Omega'},
$$
$$
E_2' = \theta E_{\Omega, \Omega'} \times_{\theta E_{V, \Omega}} \theta E_{V, \Omega}, \quad E_2' = \theta E_{\Omega, \Omega'} \times_{\theta E_{V, \Omega}} \theta E_{V, \Omega},
$$
$$
E_3' = \theta E_{\Omega, \Omega'} \times_{\theta E_{V, \Omega}} \theta E_{V, \Omega}, \quad E_3' = \theta E_{\Omega, \Omega'} \times_{\theta E_{V, \Omega}} \theta E_{V, \Omega}.
$$

Note that $E_1'$ and $E_2'$ are the dual vector bundle of $E_1$ and $E_2$ over $\theta E_{\Omega \cap \Omega'}$, respectively, and $E_3'$ is the dual vector bundle of $E_3$ over $\theta E_{\Omega \cap \Omega'}$. We denote by $\Phi_{E_j} : \mathcal{D}(C^{\times})_{\alpha_j, \beta_j, \gamma_j}(E_j) \to \mathcal{D}(C^{\times})_{\alpha_j, \beta_j, \gamma_j}(E_j')$ ($j = 1, 2, 3$) and $\Phi' : \mathcal{D}(C^{\times})_{\alpha_j, \beta_j, \gamma_j}(E_j') \to \mathcal{D}(C^{\times})_{\alpha_j, \beta_j, \gamma_j}(E_j'^*)$ the Fourier transforms. For simplicity, we denote by $\Phi_{V, W}, \Phi_{V, W}'$ instead of $\Phi_{V, \Omega}^{\Omega'}, \Phi_{W, \Omega}^{\Omega'}$, respectively.

We denote by $u_1$ and $u_1'$ the projections $E_1 \to \theta E_{W, \Omega}$ and $E_1' \to \theta E_{W, \Omega'}$, respectively. Let $\bar{\pi}_1, \bar{\pi}_1', \iota_2$ and $\iota_2'$ be the canonical maps $\theta E_{\Omega} \to E_1, \theta E_{\Omega'} \to E_1', \theta E_{\Omega} \to E_2$ and $\theta E_{\Omega'} \to E_2'$, respectively. Then we obtain the following Cartesian diagram of the vector bundles on $\theta E_{\Omega \cap \Omega'}$:

\[
\begin{array}{ccc}
\theta E_{\Omega'} & \xrightarrow{\iota_2'} & E_2' \\
\bar{\pi}_1' \downarrow & & \downarrow \iota_2 \\
E_1' & \xrightarrow{\bar{\pi}_1} & (E_{\Omega}^*)'
\end{array}
\]

Moreover let $u_3$ and $u_3'$ be the projections $E_3 \to \theta E_{V, \Omega}$ and $E_3' \to \theta E_{V, \Omega'}$, respectively, $\bar{\pi}_2, \bar{\pi}_2', \iota_3$ and $\iota_3'$ the canonical maps $\theta E_{\Omega} \to E_3$ and $\theta E_{\Omega'} \to E_3'$, respectively. We obtain the...
By Proposition 2.21, we have

\[ \Phi_{\mathbf{V}}(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_3}(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_2^* L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \]

and

\[ \Phi_\Phi(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_3}(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_2^* L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \]

Thus we obtain \( \Phi_\Phi(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \).

Let \( L \) be \( \mathbf{G}_W \)-equivariant semisimple complex on \( \mathbf{E}_{\mathbf{W}, \Omega} \), \( L'' \) a unique semisimple complex such that \( p_2^* L'' = p_1^* L \), and \( K'' \) a unique semisimple complex such that \( p_1^* \Phi_\Phi(L) = p_2^* K'' \).

By Proposition 2.21, we have

\[ \Phi_{E_3}(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_2^* L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \]

where

\[ d_1 = \text{rank}(E_1) - \text{rank}(\mathbf{E}_{\mathbf{W}, \Omega}), \quad d_2 = \text{rank}(E_2) - \text{rank}(\mathbf{E}_{\mathbf{W}, \Omega}) \]

Hence \( \Phi_{E_3}(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_2^* L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \).

We have

\[ \Phi_\Phi(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_3}(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_2^* L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \]

where

\[ d = d_1 + d_2 + \sum_{i=1}^{\Omega'} \text{dim} W_i - \sum_{i=1}^{\Omega''} \text{dim} W_i \]

Now we suppose \( \Omega' \neq \Omega'' \) and put out \( (h, \theta(h)) \) and \( \text{out}(h) = k, \text{in}(h) = \ell \). When \( k = i \), we have \( \sum_{i=1}^{\Omega'} \text{dim} W_i = \sum_{i=1}^{\Omega''} \text{dim} W_i = \text{dim} W_i \). If \( \ell \neq \theta(i) \), we have \( d_2 = 0 \) and \( d_1 = \text{dim} W_i \). If \( \ell = \theta(i) \), we have \( d_2 = 0 \) and \( d_1 = \text{dim} W_i \). Thus we obtain \( \Phi_\Phi(\langle (1) \rangle_{\Omega'}; L'') = \Phi_{E_2}(\langle (1) \rangle_{\Omega'}; p_1^* L) \).

When \( k = \theta(i) \), we can prove the claim by the same way.
(2) We may suppose $\Omega \setminus \Omega' = \{ h, \theta(h) \}$ and put $\out(h) = k, \in(h) = \ell$.

We consider the following diagram:

$$
\begin{array}{c}
\theta E_{W,\Omega} \rightarrow \theta E(W, V)_{\Omega} \rightarrow \theta E_{V,\Omega} \\
\theta E_{W,\Omega'} \rightarrow \theta E(W, V)_{\Omega' \Omega'} \rightarrow \theta E_{V,\Omega' \Omega'} \\
\theta E_{W,\Omega} \rightarrow \theta E(W, V)_{\Omega} \rightarrow \theta E_{V,\Omega'} \\
\theta E_{W,\Omega'} \rightarrow \theta E(W, V)_{\Omega' \Omega'} \rightarrow \theta E_{V,\Omega' \Omega'} \\
\end{array}
$$

If $k, \ell \neq i, \theta(i)$, the above four diagrams are cartesian. Then the commutativity is clear.

When $k = i$, we consider the two fiber products by the following:

$$
\begin{array}{c}
\theta E_{W,\Omega} \rightarrow \theta E(W, V)_{\Omega} \rightarrow \theta E_{V,\Omega} \\
\theta E_{W,\Omega} \rightarrow \theta E(W, V)_{\Omega} \rightarrow \theta E_{V,\Omega'} \\
\theta E_{W,\Omega'} \rightarrow \theta E(W, V)_{\Omega'} \rightarrow \theta E_{V,\Omega' \Omega'} \\
\theta E_{W,\Omega} \rightarrow \theta E(W, V)_{\Omega} \rightarrow \theta E_{V,\Omega'} \\
\end{array}
$$

where

$$E : = \theta E_{W,\Omega} \times_{\theta E_{W,\Omega' \Omega'}} \theta E(W, V)_{\Omega' \Omega'}, \quad E' : = \theta E_{V,\Omega'} \times_{\theta E_{V,\Omega' \Omega'}} \theta E(W, V)_{\Omega' \Omega'}.$$  

We can regard $E$ and $E'$ as the dual vector bundle of $\theta E(W, V)_{\Omega'}$ and $\theta E(W, V)_{\Omega}$ on $\theta E(W, V)_{\Omega' \Omega'}$ respectively. We can regard $r_2$ as the transpose of $q_2$. We denote by $\Phi$ and $\Phi'$ the Fourier transforms

$$
\Phi : \mathcal{D}_{(C^\infty)^{\alpha, \theta_{-\text{mono}}}}(E) \rightarrow \mathcal{D}_{(C^\infty)^{\alpha, \theta_{-\text{mono}}}}(\theta E(W, V)_{\Omega'}), \\
\Phi' : \mathcal{D}_{(C^\infty)^{\alpha, \theta_{-\text{mono}}}}(\theta E(W, V)_{\Omega}) \rightarrow \mathcal{D}_{(C^\infty)^{\alpha, \theta_{-\text{mono}}}}(E').
$$

Then, for $K \in \theta E_{V,\Omega}$ we have

$$
\Phi_W(p_i \tau^* K) = p'_i \Phi((q_2 \tau^* K) = p'_i r'_2 \tau \Phi'(\tau^* K)[d] = p'_i r'_2 \tau \Phi_V(K)[d] = p'_i \tau \Phi_V(K)[d],
$$

where $d = \rank(E) - \rank(\theta E(W, V)_{\Omega})$. If $\ell \neq \theta(i)$, we have $\rank(E) = \dim W_i \dim W_{\ell}$ and $\rank(\theta E(W, V)_{\Omega}) = \dim V_i \dim V_{\ell}$. Since $V_i = W_{\ell}$, we have $d = \dim W_{\ell}$. If $\ell = \theta(i)$, we have $\rank(E) = \dim \Alt(W_i, W_{\theta(i)})$ and $\rank(\theta E(W, V)_{\Omega}) = \dim \Alt(V_i, V_{\theta(i)})$. Then $d = \dim W_{\theta(i)}$. Since $\Omega \setminus \Omega' = \{ i \rightarrow \ell, \theta(\ell) \rightarrow \theta(i) \}$, we have $d_{E_{i'}} - \dim W_{\ell} = d_{E_{i'}}$. Thus $\Phi_W \circ E_{i'}(K) = E_{i'} \circ \Phi_V$. 

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When \( k = \theta(i) \), we obtain the following diagram:

\[
\begin{array}{ccc}
\theta E_{W,\Omega} & \xrightarrow{p} & \theta E(W, V)_{\Omega} \\
\downarrow & & \downarrow \\
\theta E_{W,\Omega^\prime} & \xrightarrow{\iota_2} & \theta E_{V,\Omega^\prime}
\end{array}
\]

\[
\begin{array}{ccc}
\theta E_{W,\Omega} & \xrightarrow{\iota_2} & \theta E_{W,\Omega^\prime} \\
\downarrow & & \downarrow \\
\theta E_{W,\Omega^\prime} & \xrightarrow{p'} & \theta E(W, V)_{\Omega^\prime}
\end{array}
\]

Here

\[ F' = \theta E_{V,\Omega} \times_{\theta E_{v,\Omega,v'}} \theta E(W, V)_{\Omega^\prime}, \]

\[ F'' = \theta E_{W,\Omega} \times_{\theta E_{W,\Omega^\prime}} \theta E(W, V)_{\Omega^\prime}. \]

We regard \( p' \) as the transpose of \( \iota_2 \). Hence we can prove the claim by the similar way.

(3) The claim follows from Proposition 2.21(2) and the commutativity of \( F_\iota \) and \( \Phi_{V,\Omega}^{\Omega'\prime} \).

Similarly, we can prove the commutativity of \( F_\iota^{(\alpha)} \) and the Fourier transforms. We omit the proof.

**Proposition 4.20.** Let \( W \) and \( V \) be \( \theta \)-symmetric \( I \)-graded vector spaces such that \( \text{wt}(V) = \text{wt}(W) + a(\alpha_i + \alpha_{\theta(i)}) \). Let \( F_\iota^{(\alpha)} \) and \( F_\iota^{(\alpha)\prime} \) be the \( a \)-th divided powers with respect to \( \theta \)-orientations \( \Omega \) and \( \Omega' \), respectively. For a \( \theta G_W \)-equivariant semisimple object \( L \in \mathcal{O}_W, \) we have \( \Phi_{V,\Omega}^{\Omega'\prime} \circ F_\iota^{(\alpha)}(L) \cong F_\iota^{(\alpha)\prime} \circ \Phi_{W,\Omega}^{\Omega'\prime}(L) \).

5. A Geometric Construction of Symmetric Crystals

5.1. Grothendieck group. For a \( \theta \)-orientation \( \Omega \) and a \( \theta \)-symmetric and \( I \)-graded vector space \( V \), we define \( \theta K_{V,\Omega} \) as the Grothendieck group of \( \mathcal{O}_{V,\Omega} \). Namely \( \theta K_{V,\Omega} \) is generated by \( L \in \mathcal{O}_{V,\Omega} \) with the relation \( (L) = (L') + (L'') \) when \( L \cong L' \oplus L'' \). This is a \( \mathbb{Z}[v, v^{-1}] \)-module by \( v(L) = (L[1]) \) and \( v^{-1}(L) = (L[-1]) \) for \( L \in \mathcal{O}_{V,\Omega} \). Hence, \( \theta K_{V,\Omega} \) is a free \( \mathbb{Z}[v, v^{-1}] \)-module with a basis \( \{ (L) \mid L \in \mathcal{O}_{V,\Omega} \} \) for another \( \theta \)-symmetric and \( I \)-graded vector space \( V' \) such that \( \text{wt} V = \text{wt} V' \), we have \( \theta K_{V,\Omega} \cong \theta K_{V',\Omega} \). We define

\[ \theta K_{\Omega} := \bigoplus \theta K_{V,\Omega} \]

where \( V \) runs over the isomorphism classes of \( \theta \)-symmetric \( I \)-graded vector spaces. For two \( \theta \)-orientations \( \Omega \) and \( \Omega' \), the Fourier transform induces an equivalence \( \mathcal{O}_{V,\Omega} \rightarrow \mathcal{O}_{V,\Omega'} \) and the isomorphism \( \theta K_{V,\Omega} \cong \theta K_{V,\Omega'} \). Therefore \( \theta K_{\Omega} \cong \theta K_{\Omega'} \).

We set \( \theta K = \theta K_{\Omega}, \theta \mathcal{P}_V = \theta \mathcal{P}_{V,\Omega} \). By Lemma 4.18, they are well-defined.

5.2. Actions of \( E_i \) and \( F_i \). The functors \( E_i \) and \( F_i^{(\alpha)} \) induce the action on \( \theta K_{\Omega} \). Since \( E_i \) and \( F_i \) commute with the Fourier transforms, they also act on \( \theta K \). The submodule \( \theta K' = \sum_{(i, a)} \mathbb{Z}[v, v^{-1}] (\theta L_{i, a, \Omega}) \subset \theta K \) is stable by \( E_i \) and \( F_i \) by Proposition 4.11. We define

\[ T_{ia}(\theta K_{V,\Omega}) = v^{-(a, \text{wt} V)} \text{id}_{\theta K_{V,\Omega}}. \]
where

\[ E \text{ and } W \]

Proof. We take \( \theta \)-symmetric \( I \)-graded vector spaces \( W, V, U \) and \( X \) such that \( \dim(W) = \dim(W) + \alpha_j + \alpha_{\theta(j)} \), \( \dim(U) = \dim(W) + \alpha_i + \alpha_{\theta(i)} \) and \( \dim(X) = \dim(W) + \alpha_j + \alpha_{\theta(j)} + \alpha_i + \alpha_{\theta(i)} \).

We consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{D}(\theta E_{W,\Omega}) & \overset{F_j}{\longrightarrow} & \mathcal{D}(\theta E_{V,\Omega}) \\
E_i \downarrow & & \downarrow E_i \\
\mathcal{D}(\theta E_{U,\Omega}) & \overset{F_j}{\longrightarrow} & \mathcal{D}(\theta E_{X,\Omega})
\end{array}
\]

First, we have

\[ E_i F_j \theta L_{i,a;\Omega} = \delta_{ij} \theta L_{(i,i,\theta(i)),(0,a,0);\Omega}[c_a] + \delta_{\theta(i),j} \theta L_{(\theta(i),i,i),(0,a,0);\Omega}[c_{a_0}] + \bigoplus_{a'} \theta L_{(j,i,\theta(j)),(1,a',1);\Omega}[c_{a'}], \]

where

\[ c_a = \dim W_i + \sum_{i \rightarrow \eta} \dim W_\eta - \dim X_i + \sum_{i \rightarrow \xi} \dim X_\xi - 2M_1((i, i, \theta(i)), (0, a, 0)), \]

\[ c_{a_0} = \dim W_{\theta(i)} + \sum_{\theta(i) \rightarrow \eta} \dim W_\eta - \dim X_i + \sum_{i \rightarrow \xi} \dim X_\xi - 2M_{2m+1}((\theta(i), i, i), (0, a, 0)), \]

\[ c_{a'} = \dim W_j + \sum_{j \rightarrow \eta} \dim W_\eta - \dim X_i + \sum_{i \rightarrow \xi} \dim X_\xi - 2M_{k+1}((j, i, \theta(j)), (1, a', 1)). \]

Here \( a' \) runs over the sequences \( a^{(k)} (1 \leq k \leq m, i_k = i, \theta(i)) \).

If \( i = j \), we have \( c_a = 0 \) by \( W = X \) and

\[ M_1((i, i, \theta(i)), (0, a, 0)) = \sum_{i \rightarrow i_\xi} a_\xi = \sum_{i \rightarrow \eta} \dim W_\eta. \]

If \( \theta(i) = j \), we have

\[ c_{a_0} = \sum_{\eta \rightarrow i} \dim W_\eta + \sum_{i \rightarrow \eta} \dim W_\eta - 2 \dim W_i = -(\alpha_i, \dim(V)) \]

by \( W = X \),

\[ M_{2m+1}((\theta(i), i, i), (0, a, 0)) = \sum_{i_\xi = i} a_\xi = \dim W_i \]

and \( \sum_{\theta(i) \rightarrow \eta} \dim W_\eta = \sum_{\eta \rightarrow i} \dim W_\eta. \)

On the other hand, we have

\[ F_j E_i \theta L_{i,a;\Omega} = \bigoplus_{a'} \theta L_{(j,i,\theta(j)),(1,a',1);\Omega}[d_{a'}], \]

where

\[ d_{a'} = - \dim U_i + \sum_{i \rightarrow \xi} \dim U_\xi + \dim U_j + \sum_{j \rightarrow \eta} \dim U_\eta - 2M_k(i, a'). \]
and \( a' \) runs over the sequences \( a^{(k)} \) \((1 \leq k \leq m, i_k = i, \theta(i))\).

We have

\[
M_{k+1}((j, i, \theta(j)), (1, a', 1)) - M_k(i, a') = \begin{cases} 
1 + \#\{i \xrightarrow{\Omega} \theta(i)\} & (j = i) \\
0 & (j = \theta(i)) \\
\#\{i \xrightarrow{\Omega} \theta(j)\} & (j \neq i, \theta(i))
\end{cases}
\]

and

\[
\left(\dim W_j + \sum_{j \rightarrow \eta} \dim W_j - \dim X_i + \sum_{i \rightarrow \xi} \dim X_\xi\right) \\
- \left(\dim U_i + \sum_{i \rightarrow \xi} \dim U_\xi + \dim U_j + \sum_{j \rightarrow \eta} \dim U_\eta\right) \\
= \begin{cases} 
2\#\{i \xrightarrow{\Omega} \theta(i)\} & (j = i) \\
\#\{i \xrightarrow{\Omega} \theta(i)\} + \#\{\theta(i) \xrightarrow{\Omega} i\} & (j = \theta(i)) \\
\#\{i \xrightarrow{\Omega} j\} + \#\{j \xrightarrow{\Omega} i\} + 2\#\{i \xrightarrow{\Omega} \theta(j)\} & (j \neq i, \theta(i))
\end{cases}
\]

Thus \( c_{a'} - d_{a'} = -(\alpha_i, \alpha_j) \). We conclude

\[
E_i F_j (\theta L_{1,a;\Omega}) - v^{-(\alpha_i, \alpha_j)} F_j E_i (\theta L_{1,a;\Omega}) = \delta_{ij} (\theta L_{1,a;\Omega}) + \delta_{\theta(i),j} T_i (\theta L_{1,a;\Omega}).
\]

The relations \( T_i E_j T_i^{-1} = v^{(\alpha_i + \alpha_{\theta(i)}, \alpha_j)} E_j \) and \( T_i F_j T_i^{-1} = v^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j \) are obvious. \( \square \)

5.3. **Key estimates of coefficients.** Let \( \Omega \) be a \( \theta \)-orientation and suppose that a vertex \( i \) is a sink. For a \( \theta \)-symmetric \( I \)-graded vector space \( V \) and \( r \in \mathbb{Z}_{\geq 0} \), we define

\[
^{\theta}E_{V,\Omega,r} := \left\{ x \in ^{\theta}E_{V,\Omega} \left| \dim \text{Coker} \left( \bigoplus_{h \in \Omega; \text{in}(h) = i} V_{\text{out}(h)} \to V_i \right) = r \right. \right\}.
\]

Then we have \( ^{\theta}E_{V,\Omega} = \bigsqcup_{r \geq 0} ^{\theta}E_{V,\Omega,r} \) and \( ^{\theta}E_{V,\Omega,\geq r} := \bigsqcup_{r \geq r} ^{\theta}E_{V,\Omega,r} \) is a closed subset of \( ^{\theta}E_{V,\Omega} \).

**Definition 5.2.** For \( L \in ^{\theta}\mathcal{P}_V \) and \( i \in I \), choose a \( \theta \)-orientation \( \Omega \) such that \( i \) is a sink with respect to \( \Omega \), and regard \( L \) as an element of \( ^{\theta}\mathcal{P}_{V,\Omega} \). We define \( \varepsilon_i(L) \) as the largest integer \( r \) satisfying \( \text{Supp}(L) \subset ^{\theta}E_{V,\Omega,\geq r} \). This does not depend on the choice of \( \Omega \).

Note that \( 0 \leq \varepsilon_i(L) \leq \dim V_i \).

We shall prove the following key estimates with respect to \( F_i(L) \) and \( E_i(L) \).

**Theorem 5.3.** Assume that \( \theta \)-symmetric and \( I \)-graded vector spaces \( V \) and \( W \) satisfy \( \text{wt} \ V = \text{wt} \ W + \alpha_i + \alpha_{\theta(i)} \). Fix a \( \theta \)-orientation \( \Omega \) such that the vertex \( i \) is a sink.

(1) For \( L \in ^{\theta}\mathcal{P}_{W,\Omega} \), there exists a unique simple perverse sheaf \( L_0 \in ^{\theta}\mathcal{P}_{V,\Omega} \) such that \( \varepsilon_i(L_0) = \varepsilon_i(L) + 1 \) and

\[
F_i(L) = [\varepsilon_i(L) + 1]v(L_0) + \sum_{L' \in ^{\theta}\mathcal{P}_{V,\Omega}: \varepsilon_i(L') > \varepsilon_i(L) + 1} a_{L'}(L')
\]

for \( a_{L'} \in v^{2-\varepsilon_i(L')}[v] \).

We define the map \( \tilde{F}_i: ^{\theta}\mathcal{P}_W \cong ^{\theta}\mathcal{P}_{W,\Omega} \to ^{\theta}\mathcal{P}_{V,\Omega} \cong ^{\theta}\mathcal{P}_V \) by \( \tilde{F}_i(L) = L_0 \). It does not depend on the choice of \( \Omega \).
(2) Let $K \in \mathcal{P}_{V, \Omega}$. If $\varepsilon_i(K) > 0$, there exists a unique simple perverse sheaf $K_0 \in \mathcal{P}_{W, \Omega}$ such that $\varepsilon_i(K_0) = \varepsilon_i(K) - 1$ and

$$
E_i(K) = v^{1-\varepsilon_i(K)}(K_0) + \sum_{K' \in \mathcal{P}_{W, \Omega} : \varepsilon_i(K') > \varepsilon_i(K) - 1} b_{K'}(K')
$$

for $b_{K'} \in v^{-\varepsilon_i(K')+1}\mathbb{Z}[v]$. Here we regard $K_0 = 0$ if $\varepsilon_i(K) = 0$.

We define the map $\tilde{E}_i : \mathcal{P}_V \rightarrow \mathcal{P}_{W, \Omega} \uplus \{0\} \simeq \mathcal{P}_W \uplus \{0\}$ by $\tilde{E}_i(K) = K_0$ if $\varepsilon_i(K) > 0$ and $\tilde{E}_i(K) = 0$ if $\varepsilon_i(K) = 0$. It does not depend on the choice of $\Omega$.

**Proof.** (1) We consider the diagram

$$
\theta E_{\Omega, \Omega} \xrightarrow{p_1} \theta E_{\Omega} \xrightarrow{p_2} \theta E_{\Omega, \Omega} \xrightarrow{p_3} \theta E_{V, \Omega, \Omega}.
$$

Since $i$ is a sink, we have $p_1^{-1}(\theta E_{V, \Omega, \Omega}) = p_2^{-1}(\theta E_{V, \Omega, \Omega})$ for any integer $r$. Especially, for $L \in \mathcal{P}_{W, \Omega}$, $p_3p_2(p_1^{-1}(\text{Supp } L)) \subset \theta E_{V, \Omega, \Omega, \geq \varepsilon_i(L) + 1}$. For $r$, set $\theta E_{\Omega, r} = p_3^{-1}(\theta E_{V, \Omega, \Omega})$. Then $p_2^{-1}(\theta E_{\Omega, r}) = p_1^{-1}(\theta E_{V, \Omega, \Omega - 1})$. We set $\theta E_{\Omega, r, \leq i} = \bigcup_{i < r} \theta E_{\Omega, r}$. Then $\theta E_{\Omega, r, \leq i}$ is an open subset of $\theta E_{\Omega}$. If $p_3(x, V) = x \in \theta E_{V, \Omega, r}$, $V$ is a codimensional subspace of $V_i$ which contains the $(\dim V - r)$-dimensional subspace $\sum_{i(h) = h} \text{Im } x_h$ of $V_i$. Therefore $\theta E_{\Omega, r} \rightarrow \theta E_{V, \Omega, \Omega, r}$ is a $\mathbb{P}^{r-1}$-bundle. For $L \in \mathcal{P}_{W, \Omega}$, there is a unique simple perverse sheaf $L'' \in \mathcal{P}(\theta E_{\Omega})$ such that $p_1^*L(d_{p_1} - d_{p_2}) = p_3^*L''$ and $(p_3)_*L'' = F_i(L)$. For $x \in \theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L) + 1}$, the action of the stabilizer $G_{V, x} \subset \text{G}_V$ of $x$ on $p_3^{-1}(x)$ is transitive. Since $L''$ is $\theta G_V$-equivariant, $L''$ is a constant sheaf on any fibers of $p_3$ over $\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L) + 1}$.

We restrict $L''$ to the open subset $\theta E_{\Omega, \Omega, \leq \varepsilon_i(L) + 1}$. There exists a unique simple perverse sheaf $J_0$ on $\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L) + 1}$ such that $L''|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L) + 1}} = p_3^*J_0[\varepsilon_i(L)]$. Hence $(p_3)_*L''|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L) + 1}} = (p_3)_*p_3^*J_0[\varepsilon_i(L)] = [\varepsilon_i(L) + 1]_vJ_0$. Let $L_0$ be the minimal extension of $J_0$. Then $L_0$ is a simple perverse sheaf on $\theta E_{V, \Omega}$. Since $F_iL$ is semisimple, we have

$$
F_i(L) = [\varepsilon_i(L) + 1]_v(L_0) + \sum a_{L'}(L'),
$$

where $L' \in \mathcal{P}_{V, \Omega}$ satisfies $\text{Supp}(L') \subset \theta E_{V, \Omega, \Omega, \geq \varepsilon_i(L) + 1}$, or $\varepsilon_i(L') > \varepsilon_i(L)$.

To prove $a_{L'} \in v^{2-\varepsilon_i(L')}\mathbb{Z}[v]$, we restrict $R \mathcal{H}om(p_3)_*L''$, $L'$ to the open subset $\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}$. Write $F_iL = \bigoplus J \otimes M_J$, where $M_J \in \mathcal{D}(pt)$ is the multiplicity space of $J$ in the expansion of $F_iL$. Then

$$
R \mathcal{H}om((p_3)_*L'', L')|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}} = \bigoplus J R \mathcal{H}om(J, L')|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}} \otimes M_J^*
$$

$$
\supset R \mathcal{H}om(L', L')|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}} \otimes M_L^*.
$$

On the other hand, since $p_3$ is a $\mathbb{P}^{\varepsilon_i(L') - 1}$-bundle on $\theta E_{V, \Omega, \Omega, L'}$ and $\text{Supp}(L') \cap \theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')} \subset \theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}$, we have

$$
R \mathcal{H}om((p_3)_*L'', L')|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}} = (p_3)_* R \mathcal{H}om(L'', p_3^*L')|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}} = (p_3)_* R \mathcal{H}om(L'', p_3^*L'[\varepsilon_i(L') - 1])|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}}[\varepsilon_i(L') - 1].
$$

Since $p_3(L'[\varepsilon_i(L') - 1])|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}}$ is a perverse sheaf, we have

$$
R \mathcal{H}om(L'', p_3^*L'[\varepsilon_i(L') - 1])|_{\theta E_{V, \Omega, \Omega, \leq \varepsilon_i(L')}} \in \mathcal{D}^0
$$

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Moreover since $\text{Supp}(L'') \supseteq \text{Supp}(p_3^*L')$, we have

$$H^0(\mathbf{R} \mathcal{H}om(L'', p_3^*L'[\varepsilon_i(L') - 1])|_{p_3^{-1}(\emptyset_{E, \Omega, \leq \varepsilon_i(L')})} = 0.$$ 

Therefore $\mathbf{R} \mathcal{H}om(L'', p_3^*L'[\varepsilon_i(L') - 1])|_{p_3^{-1}(\emptyset_{E, \Omega, \leq \varepsilon_i(L')})} \in \mathcal{D}^{>0}(pt)$ and its direct image of $p_3$ is contained in $\mathcal{D}^{>0}$. Thus we obtain $\mathbf{R} \mathcal{H}om((p_3)_*L'', L')|_{p_3^{-1}(\emptyset_{E, \Omega, \leq \varepsilon_i(L')})} \in \mathcal{D}^{>1-\varepsilon_i(L')}$. Since $H^0(\mathbf{R} \mathcal{H}om(L', L')) \neq 0$, we conclude $M_L^* \in \mathcal{D}^{>1-\varepsilon_i(L')}(pt)$. Hence $a_{L'} \in v^{2-\varepsilon_i(L')}\mathbb{Z}[v]$.

(2) Recall the following diagram:

$$E_{T, \Omega} \times \theta E_{W, \Omega} \xrightarrow{p} \theta E(W, V) \xrightarrow{t} \theta E_{V, \Omega}.$$ 

Since $i$ is a sink, for a fixed $x_W \in \theta E_{W, \Omega}$, $x \in \theta E_{V, \Omega}$ is uniquely determined by the condition that $U$ is $x$-stable and $x$ induces $x_W$ on $U/U \perp \cong W$. Therefore we have $\theta E_{W, \Omega} \cong \theta E(W, V)_{\Omega}$. We have a section $s$ of $p_1: \theta E_{V, \Omega} \to \theta E_{W, \Omega}$ by $x_W \mapsto (x, U, \varphi_W)$ where $\varphi_W: U/U \perp \cong W$ is a given isomorphism of $\emptyset$-symmetric $I$-graded vector spaces. We consider the following diagram:

$$\begin{array}{ccc}
\theta E_{V, \Omega} & \xrightarrow{q} \theta E_{W, \Omega} \\
\downarrow p_1 & & \downarrow t \\
\theta E_{W, \Omega} & \xrightarrow{s} \theta E(W, V)_{\Omega}
\end{array}$$

For $K \in \theta \mathcal{P}_{V, \Omega}$, we have $E_iK = s^*q^*K[-\dim W_i]$. Assume that $\varepsilon_i(K) > 0$. Since $\text{Supp}(K) \subset \theta E_{V, \Omega, \leq \varepsilon_i(K)}$, $K|_{\emptyset_{E, V, \leq \varepsilon_i(K)}}$ is a simple perverse sheaf. Since $q$ is smooth on $\theta E_{V, \Omega, \leq \varepsilon_i(K)}$, the restriction $q^*K[d_3]|_{q^{-1}(\emptyset_{E, V, \leq \varepsilon_i(K)})}$ is a $\emptyset_{G, V}$-equivariant perverse sheaf, where $d_3$ is the fiber dimension of $q$ on $\theta E_{V, \Omega, \leq \varepsilon_i(K)}$. Note that $p_1$ is an affine bundle on $\theta E_{W, \Omega, \leq \varepsilon_i(K)}$. If $x \in \theta E(W, V)_{\Omega}$ induces $x_W \in \theta E_{W, \Omega, \leq \varepsilon_i(K)}$, the stabilizer $\emptyset_{G, V}$ acts transitively on the fiber of $p_1$ at $x_W$. Thus $q^*K[d_3]|_{q^{-1}(\emptyset_{E, V, \leq \varepsilon_i(K)})}$ is constant on any fibers of $p_1$. Hence $s^*q^*K[d_3 - d_{p_3}]|_{\emptyset_{E, W, \Omega, \leq \varepsilon_i(K)}}$ is a simple perverse sheaf. Here

$$d_{p_3} - d_q = d_{p_1} - d_{p_2} - (\varepsilon_i(K) - 1) = \dim W_i + 1 - \varepsilon_i(K).$$

Therefore we obtain

$$E_i(K) = v^{1-\varepsilon_i(K)}(K_0) + \sum_{K' \in \emptyset \mathcal{P}_{W, \Omega} : \varepsilon_i(K') > \varepsilon_i(K) - 1} b_{K'}(K'),$$

where $K_0$ is the minimal extension of $s^*q^*K[d_3 - d_{p_3}]|_{\emptyset_{E, W, \Omega, \leq \varepsilon_i(K)}}$.

We shall prove $b_{K'} \in v^{1-\varepsilon_i(K')}\mathbb{Z}[v]$.

Since $q^*K[-\dim W_i]$ and $p_3^*E_iK$ are constant along the fibers of $p_1$, and $s^*q^*K[-\dim W_i] = s^*p_3^*E_iK$, we obtain $q^*K[-\dim W_i] = p_3^*E_iK$. We have $q^*K[-\dim W_i] = \oplus K''p_i^*K'' \otimes M_{K''}$, where $M_{K''}$ is the multiplicity space of $K''$ in $E_iK$. Since there is a unique semisimple object $L_{K''} \in \mathcal{D}(\emptyset_{E, \Omega})$ such that $p_3^*K'' = p_3^*L_{K''}$, we have $p_3^*p_3^*K[-\dim W_i] = \oplus K''p_3^*L_{K''} \otimes M_{K''}$.

We obtain $p_3^*K[-\dim W_i] = \oplus K''L_{K''} \otimes M_{K''}$. 

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Therefore we have
\[
\oplus_{K'} \mathcal{R} \text{Hom}(L_{K''} \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}} \rho_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}, L_{K'} \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}) \otimes M_{K''} \\
= \mathcal{R} \text{Hom}(p_3^* K[\dim W] \lceil p_3^{-1}(\theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}), L_{K'} \lceil p_3^{-1}(\theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}})) \\
= \mathcal{R} \text{Hom}(K[\dim W] \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}} \rho_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}, (p_3)_* L_{K'} \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}) \\
= \mathcal{R} \text{Hom}(K[\dim W] \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}, F_i(K') \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}} [-\dim W]) \\
= \mathcal{R} \text{Hom}(K \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}, F_i(K') \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}.
\]

By the claim of (1), \( F_i K' \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}} \in \mathcal{P}^{\geq -\varepsilon_i(K')}(\theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}) \).
Since \( \text{Supp}(K) \subseteq \text{Supp}(F_i K') \), we have \( \mathcal{R} \text{Hom}(K \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}, F_i(K') \lceil \theta_{E_{V,\Omega, \leq \varepsilon_i(K')+1}}) \in \mathcal{G}^{\geq 1 - \varepsilon_i(K')} \).
Therefore \( \mathcal{R} \text{Hom}(L_{K'}, L_{K'}) \otimes M_{K'} \in \mathcal{G}^{\geq 1 - \varepsilon_i(K')} \), which implies \( M_{K'} \in \mathcal{G}^{\geq 1 - \varepsilon_i(K')} \).
Hence \( b_{K'} \in v^{1 - \varepsilon_i(K')} \mathbb{Z} \) is proved.
In the case \( \varepsilon_i(K) = 0 \), we can prove similarly \( b_{K'} \in v^{1 - \varepsilon_i(K')} \mathbb{Z} \).

**Lemma 5.4.** Suppose \( \text{wt} V \neq 0 \). For any \( L \in \mathcal{P}_{E_{V,\Omega}} \), there exists \( i \in I \) such that \( \varepsilon_i(L) > 0 \).

**Proof.** If \( V \neq \{0\} \), there exists an integer \( d \), \( i = (i_1, \ldots, i_{2m}) \) and \( a \) such that \( L[d] \) appears in a direct summand of \( \theta_{E_{I,a,\Omega}} \). We may assume \( a_1 > 0 \). Then, taking \( \Omega \) such that \( i_1 \) is a sink, we have \( \text{Supp}(L) \subseteq \text{Supp}(\theta_{E_{I,a,\Omega}}) \subseteq \theta_{E_{V,\Omega} \geq 1} \). By the definition of \( \varepsilon_i \), we have \( \varepsilon_i(L) \neq 0 \). \( \square \)

**Lemma 5.5.** For \( L \in \mathcal{P}_V \), we have \( \tilde{E}_i \tilde{F}_i(L) = (L) \), and if \( \tilde{E}_i(L) \neq 0 \), we have \( \tilde{F}_i \tilde{E}_i(L) = L \).

**Proof.** We assume that \( i \) is a sink.

Following the recall diagram:

\[
\begin{array}{ccc}
\theta_{E_{V,\Omega}} & \overset{\text{q} = p_3^* p_2}{\longrightarrow} & \theta_{E_{V,\Omega}} \\
\downarrow s & & \downarrow \iota \\
\theta_{E_{W,\Omega}} & \overset{\sim}{\longrightarrow} & \theta_{E(W, V)}
\end{array}
\]

For \( L \in \mathcal{P}_{E_{W,\Omega}} \), take simple perverse sheaf \( L'' \in \mathcal{G}(\theta_{E_{W,\Omega}'}) \) such that \( p_3^* L[\dim W] = p_3^* L'' \) and \( (p_3)_* L'' = F_i L \), then \( (p_3)_* L'' \cong [\varepsilon_i(L) + 1]|_V F_i L \) on \( \theta_{E_{V,\Omega} \leq \varepsilon_i(L)+1} \). On the other hand, since \( L'' \cong p_3^* F_i L |_{\varepsilon_i(L)} \) on \( p_2^{-1}(\theta_{E_{V,\Omega} \leq \varepsilon_i(L)+1}) \), we have \( q^* F_i L \cong p_3^* L''[-\varepsilon_i(L)] \)
\( p_1^* [\dim W - \varepsilon_i(L)] \) on \( p_1^{-1}(\theta_{E_{W,\Omega} \leq \varepsilon_i(L)}) \) on \( \theta_{E_{V,\Omega} \leq \varepsilon_i(L)+1} \). Then we have \( s^* q^* F_i L = L[\dim W - \varepsilon_i(L)] \) on \( \theta_{E_{W,\Omega} \leq \varepsilon_i(L)+1} \). Note that \( E_i F_i L = s^* q^* F_i L[-\dim W] \). We obtain \( E_i (F_i L) = L[-\varepsilon_i(L)] \) on \( \theta_{E_{W,\Omega} \leq \varepsilon_i(L)} \). Hence \( \tilde{E}_i \tilde{F}_i(L) = (L) \).
Conversely, take \( K \in \mathcal{P}_{E_{V,\Omega}} \) such that \( \varepsilon_i(K) > 0 \). By the similar argument in the proof of Theorem [2], we have \( p_3^* E_i K = q^* K[-\dim W] \).
Hence we obtain \( p_1^* E_i K[\dim W] = q^* K[\varepsilon_i(K) - 1] \) on \( p_1^{-1}(\theta_{E_{E_{W,\Omega} \leq \varepsilon_i(K)+1}}) \).
Since \( p_3^* K[\varepsilon_i(K) - 1] \) is a simple perverse sheaf on \( p_3^{-1}(\theta_{E_{E_{V,\Omega} \leq \varepsilon_i(K)}) \), we have \( F_i E_i K = (p_3)_* p_3^* K[\varepsilon_i(K) - 1] \).
Then we have \( \tilde{F}_i \tilde{E}_i(K) = (K) \). \( \square \)

5.4. **Verdier duality functor.** The Verdier duality functor \( D: \mathcal{G}(\theta_{E_{V,\Omega}}) \rightarrow \mathcal{G}(\theta_{E_{V,\Omega}}) \) satisfies \( D(L[d]) = D(L)[-d] \) for \( L \in \mathcal{G}(\theta_{E_{V,\Omega}}), d \in \mathbb{Z} \). Then \( D \) induces the involution \( v^{\pm 1} \rightarrow v^{\mp 1} \).

**Proposition 5.6.**
(i) \( D(\theta L_{i, a; \Omega}) = \eta L_{i, a; \Omega}[2 \dim \theta F_{i, a; \Omega}] \).
(ii) For any \( L \in \mathfrak{P}_{V, \Omega} \), we have \( D(F_i L) = F_i D(L) \).
(iii) For any \( L \in \mathfrak{P}_{V, \Omega} \), we have \( D(L) \cong L \).

Proof. (i) and (ii) follow from the general property of the Verdier duality functor (see Lemma 2.15).

To prove (iii), we use the induction on \( wt V \).

When \( wt V = 0 \), the claim is clear by \( \mathfrak{P}_{V, \Omega} = \{ 1_{pt} \} \) and \( D(1_{pt}) = 1_{pt} \).

Suppose \( wt V \neq 0 \). By Lemma 5.14, there exists \( i \) such that \( \varepsilon_i(L) > 0 \). We shall prove \( D(L) = L \) by the descending induction on \( \varepsilon_i(L) \). By Theorem 5.3 and Lemma 5.5, we have

\[
F_i(\tilde{E}_i L) = [\varepsilon_i(L)]_{\varepsilon} + \sum_{L' \in \mathfrak{P}_{V, \Omega} : \varepsilon(L') > \varepsilon_i(L)} a_{L'}(L').
\]

By the induction hypothesis on \( wt V \), \( D(\tilde{E}_i L) = \tilde{E}_i L \). Hence the lefthand side is \( D \)-invariant by (ii). We restrict \( F_i(\tilde{E}_i L) \) on the open subset \( \theta E_{V, \Omega, \varepsilon_i(L)} \). Then it is isomorphic to \( [\varepsilon_i(L)]_{\varepsilon} \theta F_{V, \Omega, \varepsilon_i(L)} \) and \( D \)-invariant. Since \( L \) is the minimal extension of \( L |_{\theta E_{V, \Omega, \varepsilon_i(L)}} \), \( L \) is \( D \)-invariant.

Remark 5.7. By the result of (iii), we have \( a_{L'}(v) = a_{L'}(v^{-1}) \) in Theorem 5.3 (1).

Lemma 5.8. For \( L \in \mathfrak{P}_{V, \Omega} \), we have

\[
F_i^{(a)}(L) = \left[ \varepsilon_i(L) + a \right] v (\tilde{F}_i a L) + \sum_{L' : \varepsilon_i(L') > \varepsilon_i(L) + a} c_{L'}(L').
\]

with \( c_{L'} \in \mathbb{Z}[v, v^{-1}] \).

Proof. We shall prove the claim by the induction on \( a \). If \( a = 1 \), the claim follows from Theorem 5.3. If \( a > 1 \), by the induction hypothesis and Theorem 5.3, we have

\[
F_i F_i^{(a)}(L) = \left[ \varepsilon_i(L) + a \right] v F_i (\tilde{F}_i a L) + \sum_{L' : \varepsilon_i(L') > \varepsilon_i(L) + a} c_{L'} F_i (L')
\]

\[
= [a + 1] v \left[ \varepsilon_i(L) + a + 1 \right] v (\tilde{F}_i a + 1 L) + \sum_{L'' : \varepsilon_i(L'') > \varepsilon_i(L) + a + 1} d_{L''}(L'').
\]

where \( d_{L''} \in \mathbb{Q}(v) \). Hence

\[
F_i^{(a+1)}(L) = \left[ \varepsilon_i(L) + a + 1 \right] v (\tilde{F}_i a + 1 L) + \sum_{L'' : \varepsilon_i(L'') > \varepsilon_i(L) + a + 1} d_{L''}(L'').
\]

On the other hand, since \( F_i^{(a+1)}(L) = 1_{S_{a+1}} \cdot L[d_{a+1}] \) is semisimple, we conclude \( d_{L''} \in \mathbb{Z}[v, v^{-1}] \).

Proposition 5.9. We have \( \theta K = \sum \mathbb{Z}[v, v^{-1}] F_i^{(a_1)} \cdots F_i^{(a_k)} 1_{\{ pt \}} \).

Proof. For \( L \in \mathfrak{P}_{V, \Omega} \) such that \( wt V \neq 0 \), there exists \( i \) such that \( \varepsilon_i(L) > 0 \). We shall prove that \( (L) \) is contained in \( \sum \mathbb{Z}[v, v^{-1}] F_i^{(a_1)} \cdots F_i^{(a_k)} 1_{\{ pt \}} \) by the induction on \( wt V \) and the descending induction on \( \varepsilon_i(L) \). We have

\[
F_i^{(\varepsilon_i(L))}(\tilde{E}_i^{\varepsilon_i(L)} L) = (L) + \sum_{L' \in \mathfrak{P}_{V, \Omega} : \varepsilon_i(L') > \varepsilon_i(L)} c_{L'}(L').
\]
by Lemma 5.8 and Lemma 5.3. By the induction hypothesis, we have \( c_{L'}(L') \) and \( \tilde{E}_i^{\varepsilon_i(L')L} \) are contained in \( \sum Z[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{(pt)} \). Thus \( (L) \in \sum Z[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{(pt)} \).

5.5. Main Theorem. Let us recall
\[
\theta K' : = \sum_{(i,a)} Z[v,v^{-1}] (\theta L_{i,a} ) = \sum Z[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{(pt)} \subset \theta K.
\]

Theorem 5.10.
(i) \( \theta K = \theta K' \).
(ii) For \( L \in \theta \mathcal{P} \), we define \( wt(L) = -wt V \). Then \( (wt, \tilde{E}_i, \tilde{F}_i, \varepsilon_i) \) gives a crystal structure on \( \theta \mathcal{P} : = \sqcup_{V} \theta \mathcal{P}_V \) in the sense of section 2.3. Here \( V \) runs over all isomorphism classes of \( \theta \)-symmetric 1-graded vector spaces.
(iii) Let \( \mathcal{L} \) be the \( A_0 \)-submodule \( \sum_{(L) \in \theta \mathcal{P}} A_0(L) \) of \( \theta K \). Then \( \{(L) \mod v \mathcal{L} | L \in \theta \mathcal{P} \} \) gives a crystal basis of \( \theta K \). Especially, the actions of modified root operators \( \tilde{E}_i \) and \( \tilde{F}_i \) on \( \mathcal{L}/v \mathcal{L} \) are compatible with the actions of \( \tilde{E}_i \) and \( \tilde{F}_i \) on \( \theta \mathcal{P} \) introduced in Theorem 5.3.

Proof. (i) is nothing but Proposition 5.9.
(ii) By the definition of \( \varepsilon_i(L), \tilde{F}_i \) and \( \tilde{E}_i \), and Lemma 5.5, we conclude that \( (wt, \tilde{E}_i, \tilde{F}_i, \varepsilon_i) \) gives a crystal structure on \( \theta \mathcal{P} : = \sqcup_{V} \theta \mathcal{P}_V \) in the sense of section 2.3(i)-(iv). By the estimates in Theorem 5.3 the actions of \( E_i \) and \( F_i \) in \( (L) \in \theta \mathcal{P} \) satisfy the conditions 2.12 in section 2.3. Thus we obtain the claim.
(iii) follows from Theorem 2.14.

Lemma 5.11. We have \( \{ v \in \theta K \mid E_i v = 0 \quad \text{for any} \quad i \in I \} = Z[v,v^{-1}] 1_{(pt)} \).

Proof. Suppose that \( E_i (\sum a_L(L)) = 0 \) for any \( L \). Then \( a_L \in v^c Z[v] \) for some \( c \). Put \( \bar{a}_L = v^{-c} a_L \in Z[v] \). By the definition of the modified root operators and Theorem 5.10(iii), we have \( \tilde{E}_i (\sum \bar{a}_L(L)) = 0 \). Specializing \( v \) to 0, we have \( \bar{a}_L(0) = 0 \) if \( E_i \neq 0 \). But for any \( L \) such that \( wt(L) \neq 0 \), we exist \( i \in I \) such that \( \varepsilon_i(L) > 0 \). Hence we obtain \( \bar{a}_L \in v Z[v] \) and hence \( a_L \in v^{c+1} Z[v] \). By the induction on \( c \), we have \( a_L \in v^c Z[v] \) for any \( c \). Thus we conclude \( a_L = 0 \) for \( wt(L) \neq 0 \).

Theorem 5.12.
(i) \( \theta K \otimes_{Z[v,v^{-1}]} Q(v) \cong V_{\theta}(0) \) as a \( B_{\theta}(g) \)-module. The involution induced by the Verdier duality functor coincides with the bar involution on \( V_{\theta}(0) \).
(ii) \( \{(L) \mid L \in \theta \mathcal{P} \} \) gives the lower global basis on \( V_{\theta}(0) \).

Proof. (i) By Proposition 5.11 to check the defining relations of \( B_{\theta}(g) \), we only need to prove the \( v \)-Serre relations. Put
\[
S_e = \sum_{k=0}^{b} (-1)^k E_i^{(k)} E_j E_i^{-k}, \quad S_f = \sum_{k=0}^{b} (-1)^k F_i^{(k)} F_j F_i^{-k}
\]
and note that \( F_k S_e = S_e F_k \) and \( E_k S_f = S_f E_k \) for any \( k \in I \).

Since \( \theta K \) is generated by \( F_{k_i}^{(n)} \)'s from \( \phi : = 1_{(pt)} \) and \( S_e \phi = 0 \), we have \( S_e v = 0 \) for any \( v \in \theta K \). We show \( S_f L = 0 \) for any \( L \in \theta \mathcal{P}_V, \Omega \) by the induction on \( wt \mathcal{V} \). If \( wt(S_f(L)) \neq 0 \), we have \( E_L S_f(L) = S_f E_L(L) = 0 \) for any \( k \in I \) by applying the induction hypothesis to \( E_L(L) \). Since \( wt(S_f(L)) \neq 0 \), we have \( S_f(L) = 0 \) by Lemma 5.11. Hence \( \theta K \) is a \( B_{\theta}(g) \)-module. Note that \( \tilde{T}_i 1_{(pt)} = 1_{(pt)} \) for any \( i \in I \). We conclude \( \theta K \cong V_{\theta}(0) \) by Lemma 5.11 and the characterization of \( V_{\theta}(0) \) in Proposition 2.10.
We already know that $\mathcal{L} = \sum_{L \in \theta \mathscr{P}} \mathbf{A}_0(L)$ is a crystal lattice and $\{(L) \mod v\mathcal{L}\}$ is a basis of $\mathcal{L}/v\mathcal{L}$. Note that $\sum_{L \in \theta \mathscr{P}} \mathbb{Z}[v, v^{-1}](L)$ is stable under the actions of $E_i$’s and $F_i^{(a)}$’s by Lemma 5.8 and $L$ is $D$-invariant, namely bar-invariant. Moreover $\{(L) \mid L \in \theta \mathscr{P}\}$ is a basis of the $\mathbf{A}_0$-module $\mathcal{L}$ and also a basis of the $\mathbb{Z}[v, v^{-1}]$-module $\theta K$. Hence we conclude that $\{(L) \mid L \in \theta \mathscr{P}\}$ gives the lower global basis on $V_\theta(0)$.

\textbf{Corollary 5.13.} For any Kac-Moody algebra $\mathfrak{g}$ with a symmetric Cartan matrix, the $B_\theta(\mathfrak{g})$-module $V_\theta(0)$ has a crystal basis and a lower global basis, namely Conjecture 2.11 and Conjecture 2.12 is true if $\lambda = 0$.

\textbf{Example 5.14.} Let us consider the case $\mathfrak{g} = \mathfrak{sl}_3$, $I = \{\pm 1\}$ and $\theta(i) = -i$. Fix a $\theta$-symmetric orientation $-1 \xrightarrow{\Omega} 1$. For a $\theta$-symmetric $I$-graded vector space $V$ such that $\text{wt}(V) = n(a_{i-1} + a_1)$, $\theta E_{V, \Omega}$ is the set of skew symmetric matrix $x$ of size $n$. Its $\theta G_V$-orbits are parametrized by the rank $2r$ ($0 \leq r \leq \lfloor \frac{n}{2} \rfloor$) of $x$. We denote $\mathcal{O}_n^\theta$ by the orbit consisting of $n \times n$ skew symmetric matrices $x$ of rank $2r$. Note that any simple local system on each $\theta G_V$-orbit is trivial. Let us denote $IC_r^n$ by the simple perverse sheaves corresponding to the orbit $\mathcal{O}_r^n$. Note that $\varepsilon_IC_r^n = n - 2r$.

Let $W$ be a $\theta$-symmetric $I$-graded vector space such that $\text{wt}(W) = (n - 1)(a_{i-1} + a_1)$. We consider the diagram:

\[ \begin{array}{cccc}
\theta E_{W, \Omega} & \xrightarrow{p_1} & \theta E_{\Omega}^{\prime} & \xrightarrow{p_2} \theta E_{\Omega}^{\prime\prime} & \xrightarrow{p_3} \theta E_{V, \Omega}.
\end{array} \]

Note that the fibers of $p_3$ on $\mathcal{O}_r^n$ is isomorphic to $\mathbb{P}^{n-1-2r}$. Then

\[ F_1(\text{IC}_r^n) = [n - 2r]_v(\text{IC}_r^n) + \sum_{k=0}^{r-1} a_{k,n}(\text{IC}_k^n) \]

where $a_{k,n} \in v^{2-n+2k}$. We obtain the crystal graph:

\[ \begin{array}{c}
\text{IC}_0 \xrightarrow{1} \text{IC}^1 \xrightarrow{1} \text{IC}_0 \\
\text{IC}_0 \xrightarrow{-1} \text{IC}^2 \xrightarrow{1} \text{IC}_0 \\
\text{IC}_0 \xrightarrow{-1} \text{IC}^3 \xrightarrow{1} \text{IC}_0 \\
\text{IC}_0 \xrightarrow{-1} \text{IC}^4 \xrightarrow{1} \text{IC}_0 \\
\text{IC}_0 \xrightarrow{-1} \text{IC}^5 \xrightarrow{1} \text{IC}_0 \\
\end{array} \]

Therefore we recover the crystal graph parametrized by ”$\theta$-restricted multi-segments” in [EK2] Example 4.7 (1).

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