UNIVERSAL TORSORS AND VALUES OF QUADRATIC POLYNOMIALS REPRESENTED BY NORMS

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Abstract. Let $K/k$ be an extension of number fields, and let $P(t)$ be a quadratic polynomial over $k$. Let $X$ be the affine variety defined by $P(t) = N_{K/k}(z)$. We study the Hasse principle and weak approximation for $X$ in two cases. For $[K : k] = 4$ and $P(t)$ irreducible over $k$ and split in $K$, we prove the Hasse principle and weak approximation. For $k = \mathbb{Q}$ with arbitrary $K$, we show that the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one.

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1. Introduction

Let $K/k$ be an extension of number fields of degree $n$. When can values of a polynomial $P(t)$ over $k$ be represented by norms of elements of $K$? To answer this question, we study solutions $(t, z) \in k \times K$ of the equation

$$P(t) = N_{K/k}(z).$$

This is closely related to the problem of studying the Hasse principle and weak approximation (see the end of this introduction for a review of this terminology) on a smooth proper model $X^c$ of the affine hypersurface $X \subset \mathbb{A}^1_k \times \mathbb{A}^n_k$ with coordinates $(t, z) = (t, z_1, \ldots, z_n)$ defined by (1), via a choice of a basis $\omega_1, \ldots, \omega_n$ of $K$ over $k$, with $N_{K/k}(z) = N_{K/k}(z_1 \omega_1 + \cdots + z_n \omega_n)$.

Colliot-Thélène conjectured that the Brauer–Manin obstruction to weak approximation is the only one on $X^c$ (see [CT03]). This conjecture is known in the case where $P(t)$ is constant, thanks to work of Sansuc [San81]; if additionally $K/k$ is cyclic, it is known that the Hasse principle (proved by Hasse himself [Has30, p. 150]) and weak approximation hold. Other known cases of Colliot-Thélène’s conjecture, in some cases leading to a proof of the Hasse principle and weak approximation, include the class of Châtelet surfaces ($[K : k] = 2$ and $\deg(P(t)) \leq 4$) [CTSanSD87a, CTSanSD87b], a class of singular cubic hypersurfaces ($[K : k] = 3$ and $\deg(P(t)) \leq 3$) [CTSanSD87a, CTSanSD87b].

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Theorem 2. We also generalize Theorem 2 to a large class of multivariate combination of their analytic work with descent theory gives our more general from \cite[Theorem 2]{BHB11} leads to their restriction to $K$ number theory, inspired by work of Fouvry and Iwaniec \cite{FI97}. While one Theorem 2] proves weak approximation using sieve methods from analytic \cite{BHB11, (1.5)} is the restriction of a universal torsor $T$ over $X$ to $U$, or a product of $T_U$ with a quasi-split torus. For the variety $Y$, \cite[Theorem 2]{BHB11} proves weak approximation using sieve methods from analytic number theory, inspired by work of Fouvry and Iwaniec \cite{FI97}. While one step in Browning’s and Heath-Brown’s deduction of \cite[Theorem 1]{BHB11} from \cite[Theorem 2]{BHB11} leads to their restriction to $[K : Q] = 4$, the combination of their analytic work with descent theory gives our more general Theorem 2. We also generalize Theorem 2 to a large class of multivariate polynomials $P(t_1, \ldots, t_\ell) \in Q[t_1, \ldots, t_\ell]$.

Terminology. For an algebraic variety $Z$ defined over a number field $k$, one says that the Hasse principle holds if $\prod_{v \in \Omega_k} Z(k_v) \neq \emptyset$ (where $\Omega_k$ is the set of places of $k$ and $k_v$ is the completion of $k$ at $v$) implies $Z(k) \neq \emptyset$. One says that weak approximation holds if $Z(k)$ is dense in $\prod_{v \in \Omega_k} Z(k_v)$ with the product topology, via the diagonal embedding.

If $Z$ is smooth and proper, one says that the Brauer–Manin obstruction to the Hasse principle is the only one if $(\prod_{v \in \Omega_k} Z(k_v))^{\text{Br}(Z)} \neq \emptyset$ implies that $Z(k) \neq \emptyset$, and that the Brauer–Manin obstruction to weak approximation is the only one if $Z(k)$ is dense in $(\prod_{v \in \Omega_k} Z(k_v))^{\text{Br}(Z)}$. Here $(\prod_{v \in \Omega_k} Z(k_v))^{\text{Br}(Z)}$ is the set of all $(z_v) \in \prod_{v \in \Omega_k} Z(k_v)$ satisfying $\sum_{v \in \Omega_k} \text{inv}_v(A(z_v)) = 0$ for each $A$ in the Brauer group $\text{Br}(Z) = H^2_k(Z, \mathbb{G}_m)$ of $Z$, where the map $\text{inv}_v : \text{Br}(k_v) \to Q/Z$ is the invariant map from local class field theory.

Let $X$ be the variety defined by $f$ and let $U \subset X$ be the open subvariety given by $P(t) \neq 0$. We will prove that the variety $Y$ defined by $\{f(t) = 0\}$ is the restriction of a universal torsor $T$ over $X$ to $U$, or a product of $T_U$ with a quasi-split torus. For the variety $Y$, \cite[Theorem 2]{BHB11} proves weak approximation using sieve methods from analytic number theory, inspired by work of Fouvry and Iwaniec \cite{FI97}. While one step in Browning’s and Heath-Brown’s deduction of \cite[Theorem 1]{BHB11} from \cite[Theorem 2]{BHB11} leads to their restriction to $[K : Q] = 4$, the combination of their analytic work with descent theory gives our more general Theorem 2. We also generalize Theorem 2 to a large class of multivariate polynomials $P(t_1, \ldots, t_\ell) \in Q[t_1, \ldots, t_\ell]$.

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\cite{CTS89} and the case where $K/k$ is arbitrary and $P(t)$ is split over $k$ with at most two distinct roots \cite{HB92, CTS03, SJ11}. Finally, if one admits Schinzel’s hypothesis, then the conjecture is known for $K/k$ cyclic and $P(t)$ arbitrary \cite{SD98}. See \cite[Introduction]{CT93} and \cite[Section 1]{BHB11} for a more detailed discussion of these results and the difficulties of this problem.

In a recent preprint, Browning and Heath-Brown proved the conjecture for $k = Q$, $[K : Q] = 4$ and $\deg(P(t)) = 2$ with $P(t)$ irreducible over $k$ and split over $K$. Their main result \cite[Theorem 1]{BHB11} answers a question of Colliot-Thélène (see \cite[Section 2]{CT03}) in the case $k = Q$. We give a very short and elementary proof of this result for an arbitrary number field $k$. It is independent of the work of Browning and Heath-Brown and uses the fibration method in a simple and classical case.

**Theorem 1.** Let $P(t)$ be a quadratic polynomial that is irreducible over a number field $k$ and split in $K$ with $[K : k] = 4$. Then the Hasse principle and weak approximation hold for the variety $X \subset A^2_k$ defined by $f$.

If the ground field is $Q$, we can prove a much more general result based on the analytic work of Browning and Heath-Brown in \cite[Theorem 2]{BHB11} and the descent method of Colliot-Thélène and Sansuc:

**Theorem 2.** Let $k = Q$ and $K$ be any number field. Let $P(t) \in Q[t]$ be an arbitrary quadratic polynomial. Then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only obstruction on any smooth proper model of $X \subset A^{n+1}_Q$ defined by $f$.
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2. Quadratic polynomials represented by a quartic norm

In this section, we give a very short proof of Theorem 1 that is independent of the work of Browning and Heath-Brown [BHB11] and generalizes it from \( \mathbb{Q} \) to an arbitrary number field \( k \). Let \( K/k \) be an extension of degree 4. Let \( P(t) \in k[t] \) be an irreducible quadratic polynomial that is split in \( K \). Using a change of variables if necessary, we can assume that \( P(t) = c(t^2 - a) \), with \( c \in k^* \), where \( a \in k^* \) is not a square and \( \sqrt{a} \in K \). Write \( L = k(\sqrt{a}) \subset K \).

Proof of Theorem 1. Let \( U = \{(t, z) : P(t) \neq 0\} \subset X \). Let \( S \subset A^2_k \) be the conic defined by the affine equation \( c = N_{L/k}(w) \) and let \( p : U \to S \) be the morphism defined by

\[
(t, z) \mapsto (t - \sqrt{a})^{-1} N_{K/L}(z).
\]

The morphism \( p \) is smooth.

Let \( S^c \) be a smooth compactification of \( S \). Then there exists a smooth compactification \( X^c \) of \( X \) such that \( p \) extends to \( X^c \to S^c \). The conic \( S \) satisfies weak approximation. We can assume that \( X^c \) has points everywhere locally; otherwise there is nothing to prove. This implies that \( S^c(k) \neq \emptyset \) and even \( S(k) \neq \emptyset \). The fiber of \( p \) over a rational point \( w \in S(k) \) is defined by the equation \( t - \sqrt{a} = w^{-1} N_{K/L}(x) \). This \( k \)-variety is isomorphic to a smooth quadric of dimension 3: writing \( x = (x_1 + x_2 \sqrt{a}) + (x_3 + x_4 \sqrt{a}) \beta \) for some \( \beta \in L \) such that \( K = L(\beta) \), Lemma 1 below shows that

\[
w^{-1} N_{K/L}(x) = f_0(x_1, \ldots, x_4) + f_1(x_1, \ldots, x_4) \sqrt{a}
\]

for some quadratic forms \( f_0 \) and \( f_1 \) (over \( k \)) of rank 4, which depend on \( w \). Hence the fiber is isomorphic to the affine \( k \)-hypersurface given by an equation \( f_1(x_1, \ldots, x_4) = -1 \), which is indeed a smooth quadric of dimension 3. Therefore, Theorem 1 holds by [CTSanSD87a, Proposition 3.9]. \( \square \)

Now we show that the quadratic forms in the proof of Theorem 1 have rank 4.

Lemma 1. Let \( K/k \) be a quartic extension of fields of characteristic 0 which contains a quadratic subextension \( L = k(\sqrt{a}) \). Take \( \beta \in K \) with \( K = L(\beta) \). Let \( \rho \in L^* \). Let \( y = (y_1 + y_2 \sqrt{a}) + (y_3 + y_4 \sqrt{a}) \beta \) be a \( K \)-variable. If

\[
\rho N_{K/L}(y) = f_0(y_1, \ldots, y_4) + f_1(y_1, \ldots, y_4) \sqrt{a},
\]

then the quadratic forms \( f_0 \) and \( f_1 \) have rank 4.

Proof. Write \( \beta = \sqrt{u + v \sqrt{a}} \) for some \( u, v \in k \). Elementary computations give

\[
N_{K/L}(y) = g_0(y_1, \ldots, y_4) + g_1(y_1, \ldots, y_4) \sqrt{a}
\]
with
\[ g_0(y_1, \ldots, y_4) = y_1^2 + ay_2^2 - u(y_3^2 + ay_4^2 - 2ay_3y_4), \]
\[ g_1(y_1, \ldots, y_4) = 2y_1y_2 - 2uy_3y_4 - v(y_3^2 + ay_4^2). \]

Multiplying by \( \rho = \rho_0 + \rho_1\sqrt{a} \neq 0 \) (where \( \rho_0, \rho_1 \in K \)), we see that \( f_0 \) and \( f_1 \) are of the form \( \lambda g_0 + \mu g_1 \) for some \( (\lambda, \mu) \in k^2 \) with \( (\lambda, \mu) \neq (0, 0) \). Then

\[ \lambda g_0(y_1, \ldots, y_4) + \mu g_1(y_1, \ldots, y_4) = q_0(y_1, y_2) + q_1(y_2, y_4) \]

with
\[ q_0(y_1, y_2) = \lambda y_1^2 + 2\mu y_1y_2 + a\lambda y_2^2, \]
\[ q_1(y_3, y_4) = -(\lambda u + \mu v)y_3^2 - 2(\lambda av + \mu u)y_3y_4 - a(\lambda u + \mu v)y_4^2. \]

Clearly \( q_0 \) and \( q_1 \) have rank 2 since \( a \not\in k^{\times 2} \) implies that

\[ \text{disc}(q_0) = \lambda^2a - \mu^2 \neq 0, \quad \text{disc}(q_1) = -(\lambda^2a - \mu^2)(v^2a - u^2) \neq 0. \]

The result follows. \( \square \)

**Remark 1.** The analog of Theorem 1 holds for global fields of positive characteristic different from 2 as well. Indeed, it is not hard to see that our arguments and the proof of [CTSanSD87a, Proposition 3.9] remain valid for such fields.

### 3. Universal torsors

The basic strategy is based on the following result, which reduces the problem of the Hasse principle and weak approximation on a variety to the same questions on its universal torsors, where we have no Brauer–Manin obstructions. This kind of result has been proved essentially by Colliot-Thélène and Sansuc in their seminal paper [CTSan87]. However, they developed their theory under the simplifying assumption that the varieties involved are proper. Skorobogatov developed a variant under less stringent assumptions in [Sko99]. Descent on open varieties also features in [CTSko00] and [CT03]. We will use the following variant:

**Proposition 1.** Let \( Z \) be a smooth, geometrically rational variety over a number field \( k \) with algebraic closure \( \overline{k} \). Let \( \overline{Z} = Z \times_k \overline{k} \). Assume furthermore that \( \overline{k}(Z)^{\times} = \overline{k}^\times \), that \( \text{Pic}(\overline{Z}) \) is free of finite rank, that \( Z \) has universal torsors and that there is an open subvariety \( U \subset Z \) such that the restriction \( \mathcal{T}|_U \) satisfies weak approximation for any universal \( Z \)-torsor \( \mathcal{T} \). Then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only one for any smooth proper model \( Z^c \) of \( Z \).

The condition \( \overline{k}(Z)^{\times} = \overline{k}^\times \) means that the only invertible regular functions on \( Z \) are the constant ones.

Let us explain how to obtain this result from the existing results in the literature. Let \( (x_v) \in \prod_{v} Z^c(k_v))^{Br(Z^c)} \). For any finite set \( S \) of places of \( k \), we must find an \( x \in Z^c(k) \) that is arbitrarily close to \( x_v \) for all \( v \in S \).

Because of our assumptions on \( Z \), \( Br_1(Z)/Br_0(Z) \cong H^1(k, \text{Pic}(\overline{Z})) \) is finite, where \( Br_1(Z) \) is the kernel of the natural map \( Br(Z) \to Br(\overline{Z}) \) and \( Br_0(Z) \) is the image of \( Br(k) \to Br(Z) \). We note that the variety \( Z^c \) is smooth, proper and geometrically rational, so that \( Br(Z^c) = 0 \). Therefore,
we can apply [CTSk00] Proposition 1.1] to conclude that $Z(\mathbb{A}_k)^{Br_1}(Z)$ is dense in $\prod_v Z^r(k_v)^{Br(Z^r)}$, and we can choose $(y_v) \in Z(\mathbb{A}_k)^{Br_1}(Z)$ such that $y_v$ is as close as we wish to $x_v$ for all $v \in S$.

By assumption, we can find a universal torsor $f: T \to Z$ and an adelic point $(t_v) \in T(\mathbb{A}_k) \subset \prod_v T(k_v)$ such that $f((t_v)) = (y_v)$ using descent theory [Sko99 Theorem 3]. Since $T$ is smooth over $X$, the implicit function theorem implies that there exists $(u_v) \in \prod_v T_U(k_v)$ such that $u_v$ is arbitrarily close to $t_v$ for all places $v \in S$.

As weak approximation holds on $T_U$ by assumption, we find $u \in T_U(k)$ such that $u$ is as close as we wish to $u_v$ for all $v \in S$. Since $f$ is continuous, the point $x = f(u)$ has the required properties.

The main result of this section is concerned with the existence of universal torsors [CTSan87] (2.0.4) over $X$ as in [1] and their local description.

Let us recall some more definitions. If $k$ is a field and if $A$ is an étale $k$-algebra, then the $k$-variety $R_{A/k}(\mathbb{G}_m,A)$ is defined via its functor of points: take $R_{A/k}(\mathbb{G}_m,A)(B) = (A \otimes_k B)^\times$ functorially for every $k$-algebra $B$. The norm map $N_{A/k}$ is defined as in [Bou98] §12.2. We denote the absolute Galois group of $k$ by $\Gamma_k$.

**Proposition 2.** Let $K/k$ be an extension of fields of degree $n$. Let $P(t)$ be an irreducible separable polynomial of degree $r$ over $k$.

The variety $X \subset \mathbb{A}_k^{n+1}$ defined by (2) is smooth and geometrically integral, with $\text{Pic}(\mathbb{X})$ free of finite rank and $\mathbb{X}[X]^\times = \mathbb{K}^\times$. Let $U$ be the open subset of $X$ defined by $P(t) \neq 0$. Then $\text{Pic}(\bar{U}) = 0$.

Let $c \in \mathbb{K}^\times$ be the leading coefficient of $P(t)$, let $L$ be the field $k[t]/(P(t))$ and let $\eta$ be the class of $t$ in $L$. Let $A = L \otimes_k K$. For any universal torsor $\mathcal{T}$ over $X$, there exists a solution $(\rho, \xi) \in L^\times \times K^\times$ of the equation $cN_{L/k}(\rho) = N_{K/k}(\xi)$ such that $\mathcal{T}_U$ (its restriction to $U$) is isomorphic to the subvariety of $\mathbb{A}_k^1 \times R_{A/k}(\mathbb{G}_m,A)$ (with coordinates $(t, z)$) given by the equation

$$t - \eta = \rho N_{A/L}(z).$$

Using only the basic definitions, it is easy to see that one can specialize equation (2) as follows in the two “extreme” cases:

(a) If $P(t)$ splits completely in $K$, then $\mathcal{T}_U$ is isomorphic to the subvariety of $\mathbb{A}_k^1 \times (R_{K/k}(\mathbb{G}_m,K))^r$ (with coordinates $(t, x_1, \ldots, x_r)$) given by the equation

$$t - \eta = \rho \prod_{i=1}^r \sigma_i^{-1}(N_{K/k}(x_i))$$

where $\sigma_1, \ldots, \sigma_r$ is a set of representatives of $\Gamma_k/\Gamma_L$.

(b) If $P(t)$ remains irreducible in $K$, then $\mathcal{T}_U$ is isomorphic to the subvariety of $\mathbb{A}_k^1 \times R_{F/k}(\mathbb{G}_m,F)$ (with coordinates $(t, x)$) given by the equation

$$t - \eta = \rho N_{F/L}(x)$$

where $F = L \cdot K$. 

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The proof of Proposition 2 will occupy most of the remainder of this section. The $\kappa$-variety $\overline{X}$ can be described by an equation of the form

$$c \prod_{i=1}^{r} (t - \eta_i) = u_1 \cdots u_n$$

(5)

where $\eta_1, \ldots, \eta_r$ are the embeddings of $\eta$ in $\kappa$. We note that $X$ is smooth because $P(t)$ is separable. Consider the morphism $p : X \to \mathbb{A}_k^1$ given by $(t, x) \mapsto t$. Over $\kappa$, it has precisely $r$ reducible fibers $X_i$, for $i = 1, \ldots, r$, over $t = \eta_i$. Each of these has $n$ irreducible components $D_{i,j} = \{t = \eta_i, u_j = 0\}$ for $j = 1, \ldots, n$. Let $U_0$ be the open subset of $\mathbb{A}_k^1$ where $P(t) \neq 0$ and let $U = p^{-1}(U_0) \subset X$. We have

$$\mathcal{U} = U \times_k \kappa \cong (\mathbb{A}_k^1 \setminus \{\eta_1, \ldots, \eta_r\}) \times \mathbb{G}^{n-1}_{m, \kappa},$$

so that $\text{Pic}(\mathcal{U}) = 0$.

We have $\overline{X}[\mathbb{K}] = \overline{\mathbb{K}}$. Indeed, the generic fiber of $\overline{X}$ is $\mathbb{A}_k^1 \setminus \{\eta_1, \ldots, \eta_r\}$, so that $\text{Pic}(\mathcal{U}) = 0$. Therefore, any $f \in \overline{X}[\mathbb{K}]$ has the form $f = g(t)u_1^{m_1} \cdots u_n^{m_n}$ with $g \in k(t)$ and $m_1, \ldots, m_n \in \mathbb{Z}$. If $g(t)$ has a root or pole in some $t_0 \notin \{\eta_1, \ldots, \eta_r\}$, then $f$ or $f^{-1}$ is not regular in a point on $p^{-1}(t_0)$. Otherwise, we have

$$g(t) = c' \prod_{i=1}^{r} (t - \eta_i)^{e_i}$$

for some $c' \in \overline{\mathbb{K}}$ and $e_1, \ldots, e_r \in \mathbb{Z}$. Then

$$\text{div}(f) = \sum_{i=1}^{r} \sum_{j=1}^{n} (e_i + m_j)D_{i,j},$$

so $f \in \overline{X}$ if and only if $e_1 = \cdots = e_r = -m_1 = \cdots = -m_n$. By (5), this is equivalent to saying that $f$ is a constant in $\overline{\mathbb{K}}$.

By descent theory [CTSan87, Corollary 2.3.4], universal torsors over $X$ exist if and only if the exact sequence of $\Gamma_k$-modules

$$1 \to \overline{\mathbb{K}} \to \overline{\mathbb{K}}[U] \to \overline{\mathbb{K}}[U]/\overline{\mathbb{K}} \to 1$$

(6)

is split.

It is easy to see that the abelian group $\overline{\mathbb{K}}[U]/\overline{\mathbb{K}}$ is free of rank $r + n - 1$, generated by the classes of the functions $t - \eta_1, \ldots, t - \eta_r, u_1, \ldots, u_n$ with an obvious $\Gamma_k$-action and the relation

$$\sum_{i=1}^{r} [t - \eta_i] - \sum_{j=1}^{n} [u_j] = 0$$

(7)

because of the equation defining $X$.

The exact sequence (6) is split if and only if the classes can be lifted to $\overline{\mathbb{K}}[U]$ in a $\Gamma_k$-equivariant way, via a map

$$\phi : \overline{\mathbb{K}}[U]/\overline{\mathbb{K}} \to \overline{\mathbb{K}}[U], \quad [t - \eta] \mapsto \rho^{-1}(t - \eta), \quad [u_1] \mapsto \xi^{-1}u_1$$

(8)

where $\rho \in L^\times$ and $\xi \in K^\times$. Because of the unique relation (7), the pair $(\rho, \xi) \in L^\times \times K^\times$ defines such a splitting if and only if

$$cN_{L/k}(\rho) = N_{K/k}(\xi).$$

(9)
We now want to apply [CTSan87, Theorem 2.3.1, Corollary 2.3.4] for the local description of universal torsors over $X$. We will describe a morphism of tori $d : M \to T$ such that its dual map of characters fits into the following commutative diagram of $\Gamma_k$-equivariant homomorphisms.

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \overline{\mathcal{O}} & \longrightarrow \text{Pic}(X) & \longrightarrow & 0 \\
& \searrow^{i} & \downarrow^{d} & \downarrow^{\overline{d}} & \downarrow^{j} & \longrightarrow & & \\
1 & \longrightarrow & \mathcal{O}(U)^{\times} / \overline{\mathcal{O}}^{\times} & \longrightarrow & \text{Div}_X(U_X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & 0
\end{array}
$$

Here, the second row is exact because Pic($\overline{U}$) = 0 and $\overline{k}[X]^{\times} = \overline{k}^{\times}$.

The $\Gamma_k$-module $\mathcal{O}(U)^{\times} / \overline{\mathcal{O}}^{\times}$ is isomorphic to the module of characters of the algebraic $k$-torus $T \subset R_{L/k}(\mathbb{G}_{m,L}) \times R_{K/k}(\mathbb{G}_{m,K})$ with coordinates $(z_1, z_2)$ given by

$$
N_{L/k}(z_1) = N_{K/k}(z_2).
$$

Indeed, the character group $\mathcal{O}$ is the quotient of $\mathbb{Z}[\Gamma_k/\Gamma_L] \oplus \mathbb{Z}[\Gamma_k/\Gamma_K]$ with the diagonal $\Gamma_k$-action by the relation

$$
\sum_{\sigma \Gamma_L \in \Gamma_k/\Gamma_L} \sigma \Gamma_L = \sum_{\gamma \Gamma_K \in \Gamma_k/\Gamma_K} \gamma \Gamma_K.
$$

The isomorphism $i : \mathcal{O} \to \mathcal{O}(U)^{\times} / \overline{\mathcal{O}}^{\times}$ is given by

$$
i(\sigma \Gamma_L) = [t - \sigma(\eta)], \quad i(\gamma \Gamma_K) = [\gamma(u_1)].$$

The abelian group $\text{Div}_X(U_X)$ is free of rank $rn$, generated by $D_{i,j}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$. There is a bijection $\Gamma_k/\Gamma_L \times \Gamma_k/\Gamma_K \to \{ D_{i,j} \}$ defined by $(\sigma \Gamma_L, \gamma \Gamma_K) \mapsto \{ t = \sigma(\eta), \gamma(u_1) = 0 \}$ that is compatible with the action of $\Gamma_k$, acting diagonally on the left hand side. Recalling $A = L \otimes_k K$, this shows that $\text{Div}_X(U_X)$ is isomorphic to the module of characters of the $k$-torus $M = R_{A/k}(\mathbb{G}_{m,A})$. Let $j : \widehat{M} \to \text{Div}_X(U_X)$ be this isomorphism.

Consider the homomorphism $\text{div} : \mathcal{O}(U)^{\times} / \overline{\mathcal{O}}^{\times} \to \text{Div}_X(U_X)$ that maps a function to its divisor. We have

$$
\text{div}(t - \eta) = \sum_{j=1}^{n} D_{1,j}, \quad \text{div}(u_1) = \sum_{i=1}^{r} D_{i,1}.
$$

Now $\text{div}$ induces a homomorphism on the character modules $\widehat{d} : \mathcal{O} \to \widehat{M}$. The dual of this homomorphism is then given by the morphism of $k$-tori

$$
d : M \to T, \quad \mathbf{z} \mapsto (N_{A/L}(z), N_{A/K}(z)).
$$

Let $S$ be the Néron–Severi torus dual to the $\Gamma_k$-module $\text{Pic}(\overline{X})$, so that we have an exact sequence of tori

$$
1 \to S \to M \to T \to 1.
$$

This makes $M$ into a $T$-torsor under $S$.

We now describe the map $U \to T$ induced by the splitting $\phi$ as in (8) by a choice of $(\rho, \xi) \in L^{\times} \times K^{\times}$ satisfying (9). The induced map is given by

$$
U \to T, \quad (t, x) \mapsto (\rho^{-1}(t - \eta), \xi^{-1}x),
$$

where $\rho^{-1}(t - \eta)$ is the norm of $t$ with respect to the extension $L/k$. This completes the description of the universal torsor over $X$. 

and it is easy to see that the image is in $T$ using the equation of $X$ and the condition $[1]$. Therefore, the image of $U$ in $T$ is isomorphic to the subvariety of $\mathbb{A}_k^1 \times T$ with coordinates $(t, z_1, z_2)$ defined by
\[ t - \eta = \rho z_1. \]

By [CTSan87] Theorem 2.3.1, Corollary 2.3.4, any universal torsor $\mathcal{T}_U$ over $U$ is the pullback of a torsor $M$ from $T$ to $U$. Our computations show that it is isomorphic to the subvariety of $\mathbb{A}_k^1 \times R_{G/A}(\mathbb{G}_m,A)$ with coordinates $(t, z)$ defined by $[3]$. This completes the proof of Proposition 2.

**Remark 2.** One can determine equations for universal torsors $T$ over the smooth locus $X_{sm}$ of the variety $X$ defined by $[1]$ even if $P(t)$ is not irreducible over $k$; note that $X$ is not smooth if $P(t)$ is not separable. Then Pic($X_{sm}$) is a finitely generated (but not necessarily free) abelian group. So $T$ will be a torsor over $X_{sm}$ under the group of multiplicative type that is dual to Pic($X_{sm}$).

The result is as follows: Assume that
\[ P(t) = cP_1(t)^{e_1} \cdots P_d(t)^{e_d} \]
for $c \in k^\times$, some irreducible monic polynomials $P_i(t) \in k[t]$ and positive integers $e_i$. Write $L_i = k[t]/(P_i(t))$ and let $\eta_i$ be the class of $t$ in $L_i$. For $i = 1, \ldots, d$, consider the étale $L_i$-algebra $A_i = L_i \otimes_k K$. Let $U \subset X_{sm}$ be the open subvariety given by $P(t) \neq 0$. For any universal torsor $T$ over $X_{sm}$, there exists a solution $(\rho_1, \ldots, \rho_d, \xi) \in L_1^\times \times \cdots \times L_d^\times \times K^\times$ of the equation
\[ cN_{L_1/k}(\rho_1)^{e_1} \cdots N_{L_d/k}(\rho_d)^{e_d} = N_{K/k}(\xi) \]
such that $\mathcal{T}_U$ is isomorphic to the subvariety of $\mathbb{A}_k^1 \times \prod_{i=1}^d R_{A_i/k}(\mathbb{G}_m,A_i)$ with coordinates $(t, z_1, \ldots, z_d)$ given by the system of equations
\[ t - \eta_i = \rho_i N_{A_i/L_i}(z_i) \quad \text{for} \quad 1 \leq i \leq d. \]

The proof is a straightforward generalization of the proof of Proposition 2.

Note that [HBSko02] Theorem 2.2 is a special case of this result.

In case that $k$ is a number field, the following result links the existence of universal torsors as in Proposition 2 to the absence of Brauer–Manin obstructions on $X$.

This can also be deduced from general results ([Sk01 Proposition 6.1.4] and [CTSko00 Proposition 1.1]). However, our more elementary proof easily generalizes to the setting of Remark 2 where these general results do not apply.

**Lemma 2.** Let $P(t)$ be an irreducible polynomial over a number field $k$. Let $K/k$ be an extension of finite degree $n$. Let $X \subset \mathbb{A}_k^{n+1}$ be the variety defined by $[1]$. A universal torsor over $X$ exists if there is no Brauer–Manin obstruction to the Hasse principle on a smooth proper model of $X$.

**Proof.** Consider the variety $E \subset R_{L/k}(\mathbb{G}_m,L) \times R_{K/k}(\mathbb{G}_m,K)$ defined by the equation $cN_{L/k}(z_1) = N_{K/k}(z_2)$, corresponding to the equality $[1]$. We have a natural map $U \to E$ defined by $(t, x) \mapsto (t - \eta, x)$. It is clear that $E$ is a principal homogeneous space of the torus $T \subset R_{L/k}(\mathbb{G}_m,L) \times R_{K/k}(\mathbb{G}_m,K)$ defined by $N_{L/k}(z_1) = N_{K/k}(z_2)$.
By Hironaka’s theorem, there exist a smooth compactification $E^c$ of $E$, a smooth compactification $U^c$ of $U$ and a morphism $U^c \to E^c$ extending the map $U \to E$ above. Our assumption implies
\[
\left( \prod_v U^c(k_v) \right)^{Br(U^c)} \neq \emptyset \quad \text{and hence} \quad \left( \prod_v E^c(k_v) \right)^{Br(E^c)} \neq \emptyset.
\]
Since the Brauer–Manin obstruction to weak approximation is the only one for compactifications of homogeneous spaces of linear algebraic groups [San81, Theorem 8.12], we have $E(k) \neq \emptyset$, i.e., there exists $(\rho, \xi) \in L^\times \times K^\times$ satisfying (9). Now the existence of a universal torsor over $X$ follows from [CTS87, Corollary 2.3.4].

4. Quadratic polynomials represented by a norm over $\mathbb{Q}$

Let $k = \mathbb{Q}$. As before, we can assume without loss of generality that $P(t) = c(t^2 - a)$ with $c \in \mathbb{Q}^\times$ and $a \in \mathbb{Q}$, but now we do not assume that $P(t)$ is split in $K$. Using the deep work of Browning and Heath-Brown and our description of universal torsors, we can prove the following result:

Proposition 3. If the quadratic polynomial $P(t)$ is irreducible over $\mathbb{Q}$, then the restriction $\mathcal{T}_U$ of each universal torsor $\mathcal{T}$ over $X$ as in Proposition 2 satisfies weak approximation.

Proof. Assume that $P(t)$ is split in $K$. Consider $\mathcal{T}_U \subset \mathbb{A}_k^1 \times (R_{K/k}(\mathbb{G}_{m,K}))^2$ defined by equation (3) in the case $r = 2$. For any $\sigma \in \Gamma_k$, we have $\sigma(L) = L$, and for any $x \in L$, we have $\sigma(x) = \sigma^{-1}(x)$. Therefore, (3) can be rewritten as
\[
t - \sqrt{a} = \rho N_{K/L}(x_1) \cdot \sigma(N_{K/L}(x_2)),
\]
where $\sigma \in \Gamma_k$ with $\sigma(\sqrt{a}) = -\sqrt{a}$.

The variety determined by this equation is isomorphic to the subvariety $Y$ of $\mathbb{A}_k^1 \times (R_{K/k}(\mathbb{G}_{m,K}))^2$ defined by the equation
\[
N_{K/k}(w)(t - \sqrt{a}) = \rho N_{K/L}(y),
\]
via the substitution
\[
w = x_2^{-1}, \quad y = x_1 x_2^{-1}
\]
with inverse
\[
x_1 = w^{-1} y, \quad x_2 = w^{-1}
\]
using $N_{K/k}(x_2) = N_{L/k}(N_{K/L}(x_2)) = N_{K/L}(x_2) \cdot \sigma(N_{K/L}(x_2))$. This is exactly [BHB11, equation (1.5)]. Weak approximation then holds on $Y$ because of [BHB11, Theorem 2].

Assume now that $P(t)$ remains irreducible over $K$ and write $F = K \cdot L$, where $L = k(\sqrt{a})$. Choose some $\sigma \in \Gamma_K$ such that $\sigma \notin \Gamma_F = \Gamma_L \cap \Gamma_K$, so $\sigma \notin \Gamma_L$. Therefore, $\sigma$ is a representative of the non-trivial class both in $\Gamma_K/\Gamma_F$ and in $\Gamma_L/\Gamma_F$.

Let $\gamma_1, \ldots, \gamma_n$ be a set of coset representatives of $\Gamma_L/\Gamma_F$. We claim that a set of representatives of $\Gamma_k/\Gamma_F$ is given by $\gamma_1, \ldots, \gamma_n, \gamma_1 \sigma, \ldots, \gamma_n \sigma$. Indeed, if $\gamma_1 \sigma \Gamma_F = \gamma_2 \sigma \Gamma_F$, then we have $\sigma^{-1} \gamma_j^{-1} \gamma_i \sigma \in \Gamma_F = \Gamma_L \cap \Gamma_K$. Since $L/k$ is Galois, this gives $\gamma_j^{-1} \gamma_i \in \sigma \Gamma_L \sigma^{-1} = \Gamma_{\sigma(L)} = \Gamma_L$, so $\gamma_j^{-1} \gamma_i \in \sigma \Gamma_K \sigma^{-1} = \Gamma_K$ since $\sigma \in \Gamma_K$. Hence $\gamma_j^{-1} \gamma_i \in \Gamma_L \cap \Gamma_K = \Gamma_F$, so $\gamma_i \Gamma_F = \gamma_j \Gamma_F$, which implies
Furthermore, if $\gamma_i \sigma \Gamma_F = \gamma_j \Gamma_F$, then $\gamma_j^{-1} \gamma_i \sigma \in \Gamma_F \subset \Gamma_L$, which contradicts the fact that $\gamma_i, \gamma_j \in \Gamma_L$, but $\sigma \notin \Gamma_L$. Finally, $\gamma_i \Gamma_F = \gamma_j \Gamma_F$ only for $i = j$ by construction. This proves the claim.

Therefore, $N_{F/k}(w) = N_{F/L}(w)N_{F/L}(\sigma(w))$. We note that $\sigma$ induces an automorphism over $k$ of the variety $R_{F/k}(\mathbb{G}_{m,F})$: this is clear from the functor-of-points description of $R_{F/k}(\mathbb{G}_{m,F})$.

Using this observation, we see that the variety $Y' \subset \mathbb{A}_k^1 \times (R_{F/k}(\mathbb{G}_{m,F}))^2$ with coordinates $(t, w, y)$ defined by

$$N_{F/k}(w)(t - \sqrt{a}) = \rho N_{F/L}(y)$$

(i.e. equation \text{(13)} with $K$ replaced by $F$) is isomorphic to the product $T_U \times R_{F/k}(\mathbb{G}_{m,F})$ with coordinates $(t, x, y)$ subject to \text{(4)}. The isomorphism is defined by the map

$$(t, w, y) \mapsto (t, (w \sigma(w))^{-1} y, w),$$

the inverse substitution being given by

$$(t, x, y) \mapsto (t, y, xy \sigma(y)).$$

Since $Y'$ satisfies weak approximation by \cite[Theorem 2]{BHB11} and since $R_{F/k}(\mathbb{G}_{m,F})$ is rational and therefore has non-trivial $k_v$-points for any place $v$, this implies that $T_U$ satisfies weak approximation.

\textbf{Proof of Theorem 2.} If $P(t)$ is split over $\mathbb{Q}$ with two distinct roots, then Theorem 2 is a special case of \cite[Theorem 1.1]{HBSko02}. If it is split over $\mathbb{Q}$ with one double root, $U \subset X$ as in Proposition 2 is a principal homogeneous space of a torus, and Theorem 2 holds by \cite{San81}.

Next, assume that $P(t)$ is irreducible over $\mathbb{Q}$. Assume that there is no Brauer–Manin obstruction to the Hasse principle on a smooth and proper model of $X$. Then Lemma 2 shows that universal torsors $T_U$ over $X$ exist. By Proposition 3, $T_U$ satisfies weak approximation. Proposition 2 shows that $\mathbb{F}[X]^\times = \mathbb{F}^\times$ and that Pic($X$) is free of finite rank. Then an application of Proposition 1 gives the result.

\textbf{Corollary 1.} If the quadratic polynomial $P(t) \in \mathbb{Q}[t]$ is not split in the Galois closure of $K/\mathbb{Q}$, then the Hasse principle and weak approximation hold on any smooth proper model of $X \subset \mathbb{A}_{\mathbb{Q}}^{n+1}$ defined by \text{(7)}.

\textbf{Proof.} By \cite[Theorem 2.2]{Wei12}, the smooth proper model $X^c$ satisfies $\text{Br}(X^c) = \text{Br}_0(X^c)$, so the result follows immediately from Theorem 2.

Finally, we generalize Theorem 2 to equations involving a multivariate polynomial $P(t_1, \ldots, t_{\ell})$, using techniques developed by Harari in \cite{Har97}:

\textbf{Corollary 2.} Let $P_0, P_1, P_2$ be polynomials in $\ell - 1$ variables $t_2, \ldots, t_{\ell}$ over $\mathbb{Q}$ of arbitrary degree satisfying

$$\text{gcd}(P_0(t_2, \ldots, t_{\ell}), P_1(t_2, \ldots, t_{\ell}), P_2(t_2, \ldots, t_{\ell})) = 1.$$

Let $K$ be an arbitrary number field of degree $n = [K : \mathbb{Q}]$. Then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only obstruction on any smooth proper model of $X \subset \mathbb{A}_{\mathbb{Q}}^{n+\ell}$ defined by the equation

$$t_1^2 \cdot P_2(t_2, \ldots, t_{\ell}) + t_1 \cdot P_1(t_2, \ldots, t_{\ell}) + P_0(t_2, \ldots, t_{\ell}) = N_K/\mathbb{Q}(x).$$
Proof. Consider the projection $\pi : X \to \mathbb{A}_{\mathbb{Q}}^{\ell-1}$ defined by $(t, x) \mapsto (t_2, \ldots, t_\ell)$ and consider the closed subset

$$F = \{ P_0(t_2, \ldots, t_\ell) = P_1(t_2, \ldots, t_\ell) = P_2(t_2, \ldots, t_\ell) = 0 \}$$

of $\mathbb{A}_{\mathbb{Q}}^{\ell-1}$, which is of codimension at least 2 by assumption.

The fibers of $\pi$ over $\mathbb{A}_{\mathbb{Q}}^{\ell-1}\setminus F$ are geometrically integral. The fiber over each rational point in this set is defined by $P(t_1)$ of degree at most 2. By Theorem 2 for quadratic $P(t_1)$, by rationality for linear $P(t_1)$ and by [San81] for constant $P(t_1)$, this has the property that the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only obstruction on any smooth proper model.

The generic fiber of $\pi$ is a rational variety. Therefore, the result follows by an application of [Har97, Théorème 3.2.1].

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