Global Regularity Criterion for the 3D Incompressible Navier–Stokes Equations Involving the Velocity Partial Derivative

Tian Li Li and Wen Wang

1Department of Basic Education, Anhui Vocational and Technical College, Hefei 230011, China
2School of Mathematics and Statistics, Hefei Normal University, Hefei 230601, China

Correspondence should be addressed to Tian Li Li; litianli87423@163.com and Wen Wang; wwen2014@mail.ustc.edu.cn

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In this paper, we study the regularity of the weak solutions for the incompressible 3D Navier–Stokes equations with the partial derivative of the velocity. By the embedded technology, we prove that the weak solution $u$ is regular on $(0, T]$ if
\[ z^3 u \in L^p(0, T; L^q(R^3)) \] with
\[ \left( \frac{2}{p} + \frac{3}{q} \right) \leq \left( \frac{70}{37} \right) + \left( \frac{15}{37} \cdot q \right), \quad \left( \frac{15}{4} \right) \leq q \leq \infty, \] or
\[ \left( \frac{2}{p} + \frac{3}{q} \right) \leq \left( \frac{34}{19} \right) + \left( \frac{9}{19} \cdot q \right), \quad \left( \frac{9}{4} \right) \leq q \leq \infty. \]

1. Introduction and the Main Result

This paper focuses on the following three-dimensional incompressible Navier–Stokes (N-S) equations:
\[
\begin{aligned}
&u_t + u \cdot \nabla u + \nabla p = \Delta u, \\
&\nabla \cdot u = 0,
\end{aligned}
\]
with
\[
\begin{aligned}
u(x, 0) = u_0(x),
\end{aligned}
\]
where $u$ and $p$ denote the velocity field and the pressure, respectively, and $u_0(x)$ is the initial fluid which satisfied $\nabla \cdot u_0 = 0$.

The existence of weak solutions of N-S equations was proved by Leray [1] and Hopf [2]. However, the existence of 3D global regular solutions is still an open question. Prodi [3] and Serrin [4] first considered the regularity of solutions. They, respectively, proved that the weak solution of 3D N-S equations is regular when the exponents $p$ and $q$ satisfy
\[ u \in L^p(0, T; L^q(R^3)) = L^2_t L^3_x \] with $2/p + 3/q = 1, \quad 3 \leq q \leq \infty. \] (2)

In 1995, Veiga [5] generalized the result to
\[ \nabla u \in L^p_t L^q_x, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 3 \leq q \leq \infty. \] (3)

When $\nabla u, \nabla u_3, u_3$, and the like satisfy a certain integrable condition, the weak solution is regular, and a large number of results are obtained (for details, refer to [6–16]). And Penel and Pokorný [13], Kukavica and Ziane [14], Cao [15], and Zhang [16], respectively, proved that the weak solution is regular on $(0, T]$ when the weak solution satisfies the following conditions:
\[
\begin{aligned}
&\partial_t u \in L^p_t L^q_x, \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty, \\
&\partial_t u \in L^p_t L^q_x, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 9 \leq q \leq 3, \\
&\partial_t u \in L^p_t L^q_x, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{27}{16} \leq q \leq \frac{5}{2}, \\
&\partial_t u \in L^p_t L^q_x, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{1.56 \sqrt{37}}{4} - 3 \leq q \leq 3.
\end{aligned}
\]

Recently, Zhang, Yuan, and Zhou in [17] proved if
\[ \partial_t u \in L^p_t L^q_x, \quad \frac{2}{p} + \frac{3}{q} = \frac{8}{5} + \frac{3}{5q} \leq 1.75, \quad 4 \leq q \leq \infty, \] or
We shall give the proof of our main result in the third part. In order to facilitate reading, we will give the necessary preparatory knowledge in the following section.

2. Preliminaries

Throughout this text, $C$ stands for a generic positive constant which may differ in value from one line to another. We use $\|\cdot\|_{L^p}$ to denote the norm of the Lebesgue space $L^p (1 \leq p \leq \infty)$ as follows:

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}^3} |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty} = \esssup_{x \in \mathbb{R}^3} |f(x)|, \quad p = \infty.$$  

Definition 1 (see [19]). Assume $u_0 \in L^2 (\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0, T > 0$. The measurable function $u$ defined on $(0, T] \times \mathbb{R}^3$ is called the weak solution of equation (1) if

(1) $u \in L^\infty (0, T; L^2 (\mathbb{R}^3)) \cap L^2 (0, T; H^1 (\mathbb{R}^3))$

(2) $\nabla \cdot u = 0$ and $\forall \varphi \in C_0^\infty ((0, T) \times \mathbb{R}^3)$ satisfy

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla \varphi \, dx \, dt = 0,$$

and equation (1) holds in the sense of distributions.

For $\forall \varphi \in C_0^\infty ((0, T) \times \mathbb{R}^3)$ and $\nabla \cdot \varphi = 0$ satisfy

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t \varphi + (u \cdot \nabla) \varphi) \cdot u \, dx \, dt + \int_{\mathbb{R}^3} u_0 \cdot \varphi(x, 0) \, dx$$

$$= \int_0^T \int_{\mathbb{R}^3} \nabla u : \nabla \varphi \, dx \, dt,$$

where $A: B = \sum_{i,j=1}^3 a_{ij} b_{ij}$, $A = (a_{ij}), B = (b_{ij})$.

(3) The strong energy inequality, that is,

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq \|u(0)\|_{L^2}^2, \quad \forall 0 < t < T.$$  

Lemma 1 (see [13]).

$$\|p\|_{L^r} \leq C \|u\|_{L^r}, \quad 1 < r < \infty.$$  

Lemma 2 (Sobolev embedding inequality).

$$\|u\|_{L^q} \leq C \|\partial_t u\|_{L^2}^{(1/3)} \|\partial_x u\|_{L^2}^{(1/3)} \|\partial_y u\|_{L^2}^{(1/3)}, \quad 1 < r \leq \infty.$$  

3. Proof of Main Results

In this part, we give the proof of main results. In order to prove Theorems 1 and 2, we thank the results in [13]. In [13], Penel and Pokorný showed that if

\[ u_3 \in L^{\left(2\varepsilon/(3-\varepsilon)\right)}(0, T; L^1(R^3)), \quad 3 < s \leq \infty, \]

\[ \partial_3 u_1, \partial_3 u_4 \in L^{\left(2q/(3q-3)\right)}(0, T; L^1(R^3)), \quad \frac{3}{2} < q \leq \infty, \quad (18) \]

then the weak solution of the N-S equations is regular on \((0, T]\).

For arbitrary \((15/4) \leq q \leq \infty\) and \((9/4) \leq q \leq \infty\), we have

\[ L^{(37q/(35q-48))}(0, T; L^q(R^3)) \subset L^{(2q/(3q-3))}(0, T; L^q(R^3)), \]

\[ L^{(19q/(17q-24))}(0, T; L^q(R^3)) \subset L^{(2q/(3q-3))}(0, T; L^q(R^3)), \]

respectively.

\[ (19) \]

\[ \int_{R^3} u \cdot \nabla u_3 |u_3| |u_3| dx = \frac{1}{3} \int_{R^3} u \cdot \nabla |u_3|^3 dx \]

\[ = \frac{1}{3} \int_{R^3} \nabla \cdot u |u_3|^3 dx = 0, \]

\[ -\int_{R^3} \Delta u_3 |u_3| |u_3| dx = \int_{R^3} (\nabla u_3) \cdot \nabla |u_3| |u_3| dx + \int_{R^3} (\nabla u_3) \cdot \nabla |u_3| |u_3| dx \]

\[ = \frac{1}{2} \int_{R^3} (\nabla |u_3|^2) \cdot \nabla |u_3| |u_3| dx + \int_{R^3} |\nabla u_3|^2 |u_3| |u_3| dx \]

\[ \geq \frac{4}{9} \int_{R^3} (|\nabla |u_3|^2|) |u_3| |u_3| dx \]

Proof of Theorem 1. We estimate the right side of (21). By using the H"older inequality, the Young inequality, (14), (16), and (17), we get that

\[ |I| \leq C \left\| \partial_3 u_1 \right\|_{L^q} \left\| u_3 \right\|_{L^q}^2 \\
\leq C \left\| u_1 \right\|_{L^{3q/(3q-3)}} \left\| \partial_3 u_1 \right\|_{L^1} \left\| u_3 \right\|_{L^q}^2 \\
\leq C \left\| u_1 \right\|_{L^{(3q/(3q-3))}} \left\| \partial_3 u_1 \right\|_{L^1} \left\| u_3 \right\|_{L^q}^2 \\
\leq C \left\| \partial_1 u_4 \right\|_{L^{(5q/(5q-5))}} \left\| \partial_3 u_4 \right\|_{L^1} \left\| u_3 \right\|_{L^q}^2 \\
\leq C \left\| u_1 \right\|_{L^{(2q/(2q-2))}} \left\| \partial_3 u_1 \right\|_{L^1} \left\| u_3 \right\|_{L^q}^2 \\
\leq C \left\| u_1 \right\|_{L^{(37q/(35q-48))}} \left\| \partial_3 u_1 \right\|_{L^1} \left\| u_3 \right\|_{L^q}^2 \\
\leq C \left( \left\| u_1 \right\|_{L^q} + \left\| \partial_3 u_1 \right\|_{L^q} \right) \left\| u_3 \right\|_{L^q}^2. \]

Substituting (23) into (21), we have

\[ \frac{d}{dt} \left\| u_3 \right\|_{L^q}^2 \leq C \left( \left\| \nabla u_1 \right\|_{L^q} + \left\| \partial_3 u_1 \right\|_{L^q} \right) \left\| u_3 \right\|_{L^q}^2. \]

Dividing both sides by \( \left\| u_3 \right\|_{L^q}^2 \), and integrating with respect to \( t \) imply that

\[ \int_0^T \frac{d}{dr} \left\| u_3 \right\|_{L^q}^2 \leq C \left( \int_0^T \left\| \nabla u_1 \right\|_{L^q} + \left\| \partial_3 u_1 \right\|_{L^q} \right) \left\| u_3 \right\|_{L^q}^2 \]

\[ = \left\| \sqrt{\frac{d}{dr}} \left\| u_3 \right\|_{L^q} \right\|_{L^q} \leq C \left( \left\| u_3 \right\|_{L^q} + \left\| \partial_3 u_1 \right\|_{L^q} \right) \left\| u_3 \right\|_{L^q}^2. \]

We deduce from (9) and (14) that

\[ \left\| u_3 \right\|_{L^{(37q/(35q-48))}} \leq C. \]

It is available from type (23) that

\[ \frac{4}{9} \left\| \nabla u_1 \right\|_{L^q}^2 \leq C \left( \left\| u_1 \right\|_{L^q} + \left\| \partial_3 u_1 \right\|_{L^q} \right) \left\| u_3 \right\|_{L^q}^2. \]
So, by the embedding inequality, we get
\[
\|u_3\|_{L^p(0,T; L^q(R^d))} \leq \|u_3\|_{L^p(0, T; L^q(R^d))}^{(2/3)} = \|u_3\|_{L^p((0,T; L^q(R^d))}^{(2/3)} \leq \|u_3\|_{L^p((0,T; L^q(R^d))}^{(2/3)} = \|u_3\|_{L^p((0,T; L^q(R^d))}^{(2/3)} \leq C.
\]
(28)

So, Theorem 1 is proved. □

**Proof of Theorem 2.** We have another estimation for the right side of (21). By using the Hölder inequality, the Young inequality, (14), (15), and (17), we have
\[
[I] \leq \int_{R^d} |p||\partial_3 u_3||u_3|\,dx
\[
\leq C \int_{R^d} \|p\|_{L^{1/(q-3)}(R^d)} \|\partial_3 u_3\|_{L^q(\partial \Omega)} \|u_3\|_{L^3(\partial \Omega)} \,dx
\[
\leq C \int_{R^d} \|u_3\|_{L^{1/(q-3)}(R^d)} \|\partial_3 u_3\|_{L^q(\partial \Omega)} \,dx
\[
\leq C \int_{R^d} \|u_3\|_{L^{1/(q-3)}(R^d)} \|\partial_3 u_3\|_{L^q(\partial \Omega)} \,dx
\[
\leq C \int_{R^d} \|u_3\|_{L^{1/(q-3)}(R^d)} \|\partial_3 u_3\|_{L^q(\partial \Omega)} \,dx
\[
\leq C \int_{R^d} \|u_3\|_{L^{1/(q-3)}(R^d)} \|\partial_3 u_3\|_{L^q(\partial \Omega)} \,dx
\[
\leq C \left( \|u_3\|^2_{L^2} + \|\partial_3 u_3\|_{L^q(\partial \Omega)} \right) \|u_3\|_{L^3}.
\]
(29)

Inserting (29) into (21), one has
\[
\frac{d}{dt}\|u_3\|_{L^3} \leq C \left( \|u_3\|^2_{L^2} + \|\partial_3 u_3\|_{L^q(\partial \Omega)} \right) \|u_3\|_{L^3}.
\]
(30)

Dividing both sides by \(\|u_3\|_{L^3}\) and integrating with respect to \(t\) imply that
\[
\int_0^T \|u_3\|_{L^3} \,dt \leq C \int_0^T \left( \|u_3\|^2_{L^2} + \|\partial_3 u_3\|_{L^q(\partial \Omega)} \right) \,dt.
\]
(31)

By (10) and (14), we have
\[
\|u_3\|_{L^\infty(0,T; L^3(R^d))} \leq C.
\]
(32)

The same can be proved:
\[
u_3 \in L^3(0, T; L^3(R^d)).
\]
(33)

So, Theorem 2 is proved. □

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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