PERIODIC 3-MANIFOLDS AND MODULAR CATEGORIES

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Abstract. A $p$-periodic 3-manifold is a 3-manifold that admits a $\mathbb{Z}_p$-action whose fixed point set is a circle. We give a congruence that relates the quantum invariant of a $p$-periodic 3-manifold associated to any modular category over an integrally closed ground ring and the corresponding quantum invariant of its orbit space.

Introduction

Let $p$ be an odd prime, and let $G$ be the finite cyclic group $\mathbb{Z}_p$. We assume that all 3-manifolds are compact and closed. The quantum invariant of a 3-manifold can be defined using any modular category. In [G], Gilmer was interested in studying the relation between the $SU(2)$-invariants of $p$-periodic 3-manifolds and their quotient manifolds. He obtained a congruence relating these invariants. His result was obtained by using the trace formula of topological quantum field theory (see proposition (3.1)) and studying Gaussian sums. Chbili used the results about the Jones polynomial and the Kauffman multi-bracket of $p$-periodic links to obtain a similar result for rational homology 3-spheres for the $SO(3)$-invariants in [C2]. Also in [C1], he gave similar results for the $SU(3)$ and the MOO-invariants. Moreover in [CL], Chen and Le generalized the above results for rational homology spheres using any complex simple Lie algebra. We give similar results for all 3-manifolds using any modular category over an integrally closed ground ring. Our proof takes place completely in the context of modular categories. We use the surgery descriptions of $p$-periodic 3-manifold and its orbit manifold, obtained in [PS], to prove the result.

In section [1], we give a brief exposition on how to calculate the quantum invariant for any 3-manifold from its surgery description. In section [2], we discuss the $\mathbb{Z}_p$-actions on 3-manifolds and the relation between the link that describes a $p$-periodic 3-manifold and the link that describes its orbit manifold. Some formulas and results regarding the value of colored ribbon graphs under the covariant functor $F$ will be given in section [3]. Finally in section [4], we state and prove the main result.

1. QUANTUM INVARIANTS OF 3-MANIFOLDS

Fix a strict modular category $(\mathcal{V}, \{v_i\}_{i \in I})$ with ground ring $K$ and a rank $\mathcal{D} \in K$.

1.1. Introduction. A result due to Lickorish and Wallace asserts that every closed oriented 3-manifold can be obtained by surgery on $S^3$ along a framed link.

1.2. The $\tau$-invariant of closed 3-manifolds. Let $M$ be a closed oriented 3-manifold obtained by surgery on $S^3$ along a framed link $L$. The $\tau$-invariant of $(M, \Omega)$ associated to $(\mathcal{V}, \mathcal{D})$ where $\Omega$ is a colored ribbon graph in $M$ is given by

$$\tau(M, \Omega) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L) - m - 1}\{L, \Omega\}.$$
Here $\sigma(L)$ is the signature of the linking matrix of the link $L$, and $m$ is the number of components of $L$, and $\Delta = \{U^-\}$ where $U^-$ denotes the diagram for the unknot with a single double point and writhe -1.

We use the notation
\[
\{L, \Omega\} = \sum_{\lambda \in \text{col}(L)} \{L, \Omega\}_\lambda
\]
\[
= \sum_{\lambda \in \text{col}(L)} \prod_{i=1}^{m} \dim(\lambda(L_i)) F(\Gamma(L, \lambda) \cup \Omega),
\]
where $\text{col}(L)$ is the set of all mappings from the set of components of $L$ to $I$ (the set of simple objects), and $\Gamma(L, \lambda)$ is the ribbon graph obtained by coloring the $i$-th component of $L$ by $V_{\lambda(i)}$. Here $F$ is the covariant functor defined in [T, chapter I] which assigns to a $\mathcal{V}$-colored ribbon graph in $\mathbb{R}$ an element of the ground ring. The material of this section is due to Turaev [T].

1.3. The $I$-invariant. We take the definition of the quantum invariant to be as follows
\[
I(M, \Omega) = D\tau(M, \Omega).
\]
Our result is simpler when expressed using this normalization.

**Theorem 1.1.** [Turaev] $\tau(M, \Omega)$ (or $I(M, \Omega)$) is a topological invariant of the pair $(M, \Omega)$.

**Example.** We know that $S^3$ is obtained by doing surgery on the empty link, i.e $I(S^3) = 1$. Also, $S^3$ is obtained by doing surgery along the Hopf link $H$ with framing 0 on both components. Hence, we conclude $\{H\} = D^2$.

**Corollary 1.2.** $\{L, \Omega\}$ is invariant under Kirby sliding.

This corollary is really major part of proof of Theorem 1.1. Finally, the $\tau$-invariant can be recovered in terms of the TQFT-theory $(V, Z)$ which is a functor from the category $C$ whose objects are closed surfaces and 3-manifolds as its morphisms (the surfaces and 3-manifolds have banded links sitting inside of them) to the category of $K$-modules and $K$-linear homomorphisms, where $V$ is the functor on the surfaces and $Z$ is the functor on 3-manifolds. In fact, the assigned value of a closed 3-manifold under $Z$ is a scalar multiplication homomorphism from the base ring to itself and that scalar is the $\tau$-invariant of that manifold. For more details of TQFT see (T, BHMV).

2. Periodic links and periodic 3-manifolds

Let $M$ be a closed oriented 3-manifold that is a result of surgery on $S^3$ along the framed link $L$.

**Definition 2.1.** A framed link $L$ in $S^3$ is said to be $p$-periodic if there exists a $\mathbb{Z}_p$-action on $S^3$, with a fixed point set equal to a circle, that maps $L$ to itself under this action and $L$ is assumed to be disjoint from the circle.

**Definition 2.2.** $M$ is said to be $p$-periodic if there is an orientation preserving $\mathbb{Z}_p$-action with fixed point set equal to a circle, and the action is free outside this circle.

Now we list the following two results from [PS] that will be used in later sections.

**Theorem 2.3.** There is a $\mathbb{Z}_p$-action on $M$ with a fixed point set equal to a circle iff $M$ can be obtained be as a result of surgery on a $p$-periodic link $L$ and $\mathbb{Z}_p$ acts freely on the set of the components of $L$. 
By the positive solution of the Smith conjecture, we can represent any framed $p$-periodic link as a closure of some graph such that the rotation of this graph about the $z$-axis in $\mathbb{R}^3$ (or the circle in $S^3$) by $2\pi/p$ leaves it invariant, i.e. $L = \overline{\Omega}$ (where the bar means the closure of the graph) see figure 1.

![Figure 1. Periodic Link and its Quotient](image_url)

Let $M_* = M/\mathbb{Z}_p$ denote the orbit space, then $M_*$ is obtained by surgery on $S^3$ along the link $L_* = L/\mathbb{Z}_p$.

**Lemma 2.4.** Let $L$ a $p$-periodic link in $S^3$. The following are equivalent

1. $\mathbb{Z}_p$ acts freely on the set of components of $L$;
2. the linking number of each component of the $L_*$ the axis of the action is congruent to zero modulo $p$;
3. the number of components of $L$ is equal to $p$ times the number of components of $L_*$.

3. **SOME RESULTS ABOUT TRACES**

We use two different notions of trace one is the trace of a linear homomorphism (denoted by Trace) in the category of $K$-modules and the other one is the trace of a ribbon graph (denoted by $Tr$) in the category of ribbon graphs defined in [1] Chapter. 1

**Proposition 3.1.**

$$\tau(S^2 \times S^1, \Omega) = \text{Trace}_{V(S^2,0)}(Z(S^2 \times I, \Omega)),$$

where $\Omega$ is a colored ribbon $l \times l$ tangle in $S^2 \times I$.

**Proof.** This is a special case of the Trace Formula for TQFT [BHMV Prop. 1.2] and [1] Ex. 2.8.1.

**Lemma 3.2.**

$$\text{Tr}(\Omega) = \frac{1}{D^2} \sum_{i \in I} \dim(V_i) \text{Trace}_{V(S^2,0+1)}(Z(S^2 \times I, V_i \otimes \Omega)).$$
Proof. Let $H$ stands for the zero-framed Hopf link on both components. We have
\begin{equation}
(3.1) \quad \text{Tr}(F(\Omega)) = F(\Omega) \quad \text{by \cite[Cor. 2.7.2]{1}}
\end{equation}
\begin{align*}
&= \frac{1}{\{H\}} \sum_{\lambda \in \text{col}(H)} \{H\}_{\lambda} F(\Omega), \quad \text{as } \{H\} = \sum_{\lambda \in \text{col}(L)} \{H\} \\
&= \frac{1}{D^2} \sum_{\lambda \in \text{col}(H)} \dim(\lambda) F(\Omega H), \quad \text{as } \{H\} = D^2, \text{ where } H \text{ is unlinked from } \Omega \\
&= \frac{1}{D^2} \sum_{\lambda \in \text{col}(H)} \dim(\lambda) F(\Omega'), \quad \text{using the invariance of } \{L, \Omega\} \\
&\quad \text{under sliding see figure 2} \\
&= \sum_{i \in I} \dim(V_i) D^{-2} \sum_{j \in I} \dim(V_j) F(\Omega') \\
&= \sum_{i \in I} \dim(V_i) \tau(S^2 \times S^1, \Omega_i \otimes \Xi), \quad \text{by formula (1.1)} \\
&= \sum_{i \in I} \dim(V_i) \text{Trace}_{V(S^2 \times S^1, \Omega'_i \otimes \Xi)}, \quad \text{by proposition (3.1)}
\end{align*}

\[ \square \]

**Definition 3.3.** Let $J_p = (p, \dim(V_i)^p - \dim(V_i))$ be the ideal generated by $p$ and $\dim(V_i)^p - \dim(V_i), \forall i \in I$ in $K$.

**Corollary 3.4.** Let $\Omega$ be any colored ribbon graph over any modular category with integrally closed ground ring. Then
\begin{equation}
(3.2) \quad \text{Tr}(\Omega)^p \equiv \text{Tr}(\Omega^p) \pmod{J_p}.
\end{equation}

**Proof.** If we assume
\[ \dim(V_i)^p = \dim(V_i), \]

it follows that
\[ D^{2p} = \sum_{i \in I} \dim(V_i)^{2p} = D^2 \pmod{p}. \]
Hence the result follows from Lemma 3.2 and [CL, Lem. 3.5(i)], which implies that
\[ \text{Trace}(Z^p) \equiv \left[ \text{Trace}(Z) \right]^p \pmod{p}, \]
where Z is an endomorphism of free K-module. □

4. QUANTUM INVARIANTS OF PERIODIC 3-MANIFOLDS

Let M be a 3-manifold that admits a Z_p-action with a fixed point set equal to a circle. Then we are in situation of theorem 2.3, i.e. M is obtained by surgery on S^3 along a framed p-periodic link L (see figure 1). We would like to relate the quantum invariant of M to the quantum invariant of \( M_\ast = M / \mathbb{Z}_p \). Before we do so, we introduce the following.

Definition 4.1. Let L be a p-periodic link, and λ be a coloring of L. If \( \Gamma(L, \lambda) \) is invariant under the rotation of the graph that represents L by \( 2\pi/p \), then λ is called a p-periodic coloring.

Lemma 4.2. Let L be a p-periodic link, such that \( L_\ast = L / \mathbb{Z}_p \). Then
\[ \{L\} \equiv \{L_\ast\}^p \pmod{J_p}. \]

Proof. Let us start with any coloring of L say λ, either λ is p-periodic or not. Let us assume that λ is not p-periodic, i.e \( \Gamma(L, \lambda) \) is not invariant under the rotation by \( 2\pi/p \) about the z-axis. Hence the i-th rotation of \( \Gamma(L, \lambda) \) (the rotation by \( 2\pi/p \)) represents a ribbon graph with the same value under F (since F is an isotopy invariant) and different coloring denoted by \( \lambda_i \). So the term with a non-periodic coloring occurs p times. Hence we reduce the summation on the left-hand side to the periodic colorings. Now the result follows from corollary (3.4) and the fact that the periodic colorings of L are in one-to-one correspondence with the colorings of \( L_\ast \) (by restriction). □

We introduce the notion \( \kappa = \Delta D^{-1} \). Now, we are ready to give a relation between the quantum invariants of M and \( M_\ast \).

Theorem 4.3. Over any modular category with integrally closed ground ring K; we have
\[ I(M) \equiv \kappa^\delta I(M_\ast)^p \pmod{J_p}, \]
for some integer δ.

Proof. We assume that M and \( M_\ast \) are obtained by surgery on S^3 along L and \( L_\ast \) respectively.
\[ I(M) = (\Delta D^{-1})^{\sigma(L)} D^{-pm} \{L\} \]
\[ \equiv (\Delta D^{-1})^{\sigma(L)} D^{-pm} \{L_\ast\}^p \pmod{J_p} \text{ by lemma 1.2} \]
\[ \equiv (\Delta D^{-1})^{\sigma(L)} - \sigma(L_\ast) (\delta D^{-1})^{\sigma(L_\ast)} p(D^{-m})^p \{L_\ast\}^p \pmod{J_p} \]
\[ \equiv \kappa^\delta I(M_\ast)^p \pmod{J_p}. \]
Here \( \delta = \sigma(L) - p\sigma(L_\ast) \). □

Corollary 4.4.
\[ \tau(M) \equiv \kappa^\delta D^{p-1} \tau(M_\ast)^p \pmod{J_p}, \]
where \( \delta \) and \( \kappa \) as defined before.

Before we go to the next corollary, we define the total signature for a knot.
**Definition 4.5.** Suppose $K$ is a knot in a homology sphere $M_*$. Let $\pi: M \rightarrow M_*$ be the $p$-fold cyclic cover branched along $K$. It is known that, we can extend this cover to a cover $W \rightarrow W_*$ of 4-manifolds (where $M = \partial W$ and $M_* = \partial W_*$) branching over the surface $Y$. Let $Y \cdot Y$ denote the self-intersection of $Y$ in $W_*$. In this case, we define the total signature $\sigma_p(K)$

$$
\sigma_p(K) = p\sigma(W_*) - \sigma(W) - \frac{p^2 - 1}{3p} Y \cdot Y
$$

$$
= p\sigma(L_*) - \sigma(L) - \frac{p^2 - 1}{3p} Y \cdot Y.
$$

By a well-known argument, using Novikov additivity and the $G$-signature theory [K], $\sigma_p(K)$ is independent of the choices made.

The following corollary generalizes [G, Th. 3].

**Corollary 4.6.** If $M$ is a $p$-fold branched cyclic cover of a homology sphere along a knot $K$, then

$$
I(M) \equiv \kappa - \sigma_p(K) (mod J_p),
$$

where $\sigma_p(K)$ is the total signature of $K$.

**Proof.** The linking matrix of $L$ describes the intersection form on 4-manifold with boundary $M$ which is a branched cover along a disk with zero self-intersection in a 4-manifold with boundary $M_*$. The corollary now follows by identifying $\delta$ with the total signature of $K$. \qed

**Corollary 4.7.** If $M$ is a $p$-fold branched cyclic cover of $S^3$ along the knot $K$, then

$$
I(M) \equiv \kappa - \sigma_p(K) (mod J_p).
$$

**Remark 4.8.** If $K$ is a knot in $S^3$, $\sigma_p(K)$ can be identified with minus the sum of the Tristram-Levine “$p$-signatures”.

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