COMMUTATORS OF FLOWS AND FIELDS

MARKUS MAUHART

PETER W. MICHIR

Institut für Mathematik, Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.

Abstract. The well known formula 
\[ [X, Y] = \frac{1}{2} \partial^2 \partial t^2 |_0 (Fl^t_Y \circ Fl^X_t \circ Fl^Y_t \circ Fl^X_t) \]
for vector fields \( X, Y \) is generalized to arbitrary bracket expressions and arbitrary curves
of local diffeomorphisms.

Let \( M \) be a smooth manifold. It is well known that for vector fields \( X, Y \in \mathfrak{X}(M) \) we have
\[ 0 = \frac{\partial}{\partial t} |_0 (Fl^t_Y \circ Fl^X_t \circ Fl^Y_t \circ Fl^X_t), \]
\[ [X, Y] = \frac{1}{2} \partial^2 \partial t^2 |_0 (Fl^t_Y \circ Fl^X_t \circ Fl^Y_t \circ Fl^X_t). \]

We give the following generalization:

1. **Theorem.** Let \( M \) be a manifold, let \( \varphi^i : \mathbb{R} \times M \supset U_{\varphi^i} \rightarrow M \) be smooth mappings for \( i = 1, \ldots, k \) where each \( U_{\varphi^i} \) is an open neighborhood of \( \{0\} \times M \) in \( \mathbb{R} \times M \), such that each \( \varphi^i_t \) is a diffeomorphism on its domain, \( \varphi^0 = 1d_M \), and \( \frac{\partial}{\partial t} |_0 \varphi^i_t = X_i \in \mathfrak{X}(M) \). We put \( [\varphi^i_t, \varphi^j_t] := (\varphi^j_t)^{-1} \circ (\varphi^i_t)^{-1} \circ \varphi^i_t \circ \varphi^j_t \). Then for each formal bracket expression \( B \) of length \( k \) we have
\[ 0 = \frac{\partial^\ell}{\partial t^\ell} |_0 B(\varphi^1_t, \ldots, \varphi^k_t) \quad \text{for } 1 \leq \ell < k, \]
\[ B(X_1, \ldots, X_k) = \frac{1}{k!} \frac{\partial^k}{\partial t^k} |_0 B(\varphi^1_t, \ldots, \varphi^k_t) \in \mathfrak{X}(M) \]
in the sense explained in 3 below.

In fact this theorem is a special case of the more general theorem 10 below. The somewhat unusual choice of the commutator of flows is explained by the fact that the bracket on the Lie algebra of the diffeomorphism group is the negative of the usual Lie bracket of vector fields.

1991 Mathematics Subject Classification. 58F25.
Key words and phrases. Commutators, flows, vector fields.

1Supported by Project P 7724 PHY of ‘Fonds zur Förderung der wissenschaftlichen Forschung’

Typeset by \texttt{AMS-\LaTeX}\n
1
2. Lemma. Let \( c : \mathbb{R} \to M \) be a smooth curve. If \( c(0) = x \in M \), \( c'(0) = 0, \ldots, c^{(k-1)}(0) = 0 \), then \( c^{(k)}(0) \) is a well defined tangent vector in \( T_xM \) which is given by the derivation \( f \mapsto (f \circ c)^{(k)}(0) \) at \( x \).

Proof. We have

\[
((f \circ g) \circ c)^{(k)}(0) = (f \circ (g \circ c))^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} (f \circ c)^{(j)}(0)(g \circ c)^{(k-j)}(0)
\]

since all other summands vanish: \( (f \circ c)^{(j)}(0) = 0 \) for \( 1 \leq j < k \). \( \square \)

3. Curves of local diffeomorphisms. Let \( \varphi : \mathbb{R} \times M \supset U_0 \to M \) be a smooth mapping where \( U_\varphi \) is an open neighborhood of \( \{0\} \times M \) in \( \mathbb{R} \times M \), such that each \( \varphi_t \) is a diffeomorphism on its domain and \( \varphi_0 = Id_M \). We say that \( \varphi_t \) is a curve of local diffeomorphisms through \( Id_M \).

From lemma 2 we see that if \( \frac{\partial^j}{\partial t^j} |_{t=0} \varphi_t = 0 \) for all \( 1 \leq j < k \), then \( X := \frac{1}{k!} \frac{\partial^k}{\partial t^k} |_{t=0} \varphi_t \) is a well defined vector field on \( M \). We say that \( X \) is the first non-vanishing derivative at 0 of the curve \( \varphi_t \) of local diffeomorphisms. We may paraphrase this as \( (\frac{\partial}{\partial t} |_{t=0}) f = \sum_{j=0}^{k} \frac{\partial^j}{\partial t^j} |_{t=0} \varphi_t f \).

4. Natural vector bundles. See [KMS, 6.14]. Let \( \mathcal{M}_{f \!m} \) denote the category of all smooth \( m \)-dimensional manifolds and local diffeomorphisms between them. A vector bundle functor or natural vector bundle is a functor \( F \) which associates a vector bundle \((F(M), p_M, M)\) to each manifold \( M \) and a vector bundle homomorphism

\[
\begin{array}{ccc}
F(M) & \xrightarrow{F(f)} & F(N) \\
p_M & \downarrow & \downarrow p_N \\
M & \xrightarrow{f} & N
\end{array}
\]

to each \( f : M \to N \) in \( \mathcal{M}_{f \!m} \), which covers \( f \) and is fiber wise a linear isomorphism.

If \( f \) is the embedding of an open subset of \( N \) then this diagram turns out to be a pullback diagram. We also point out that \( f \mapsto F(f) \) maps smoothly parameterized families to smoothly parameterized families, see [KMS, 14.8]. Assuming this property all vector bundle functors were classified by [T]; They correspond to linear representations of higher jet groups, they are associated vector bundles to higher order frame bundles, see also [KMS, 14.8].

Examples of vector bundle functors are tangent and cotangent bundles, tensor bundles, and also the trivial bundle \( M \times \mathbb{R} \) which will give us theorem 1.

5. Pullback of sections. Let \( F \) be a vector bundle functor on \( \mathcal{M}_{f \!m} \) as described in 4. Let \( M \) be an \( m \)-manifold and let \( \varphi_t \) be a curve of local diffeomorphisms...
through $Id_M$ on $M$. Then the flow $\varphi_t$, for fixed $t$, is a diffeomorphism defined on an open subset $U_{\varphi_t}$ of $M$. The mapping

$$
\begin{align*}
F(M) & \xleftarrow{p_M} F(U_{\varphi_t}) \xrightarrow{F(\varphi_t)} F(M) \\
M & \xleftarrow{\varphi_t} U_{\varphi_t} \xrightarrow{\varphi_t} M
\end{align*}
$$

is then a vector bundle isomorphism.

We consider a section $s \in C^\infty(F(M))$ of the vector bundle $(F(M), p_M, M)$ and we define for $t \in \mathbb{R}$

$$
\varphi_t^* s := F(\varphi_t^{-1}) \circ s \circ \varphi_t.
$$

This is a local section of the bundle $F(M)$. For each $x \in M$ the value $(\varphi_t^* s)(x) \in F(M)_x := p_M^{-1}(x)$ is defined, if $t$ is small enough. So in the vector space $F(M)_x$ the expression $\frac{d}{dt}|_0 (\varphi_t^* s)(x)$ makes sense and therefore the section $\frac{d}{dt}|_0 (\varphi_t^* s)$ is globally defined and is an element of $C^\infty(F(M))$. If $\varphi_t = Fl^X$ is the flow of a vector field $X$ on $M$ this section

$$
L_X s := \frac{d}{dt}|_0 (Fl^X)^* s
$$

is called the Lie derivative of $s$ along $X$. It satisfies $L_X L_Y - L_Y L_X = L_{[X,Y]}$, see [KMS, 6.20].

6. Lemma. Let $\varphi_t$ be a smooth curve of local diffeomorphisms through $Id_M$ with first non-vanishing derivative $kX = \partial^k_{X}|_0 \varphi_t$. Then for any vector bundle functor $F$ and for any section $s \in C^\infty(F(M))$ we have the first non-vanishing derivative

$$
k! L_X s = \partial^k_{X}|_0 \varphi_t^* s.
$$

Proof. This is again a local question, so let $x \in M$. We choose a complete Riemannian metric on $M$ and we denote by $U_k$ the open ball with radius $r_k > 0$ and center $x$ for this metric, and let $\overline{U}_k$ be its closure. Since $\varphi_0 = Id_M$ we may choose a chart $(U, u : U \rightarrow \mathbb{R}^m)$ of $M$ with $x \in U$ and $u(U) = \mathbb{R}^m$, radii $r_0 > r_1 > r_2 > r_3 > r_4 > 0$ and $\varepsilon > 0$ such that the following hold: $\varphi$ is defined and smooth on $((-2\varepsilon, 2\varepsilon) \times U_0)$, $\varphi([-\varepsilon, \varepsilon] \times \overline{U_1}) \subset U$, $\varphi((-\varepsilon, \varepsilon) \times \overline{U_2}) \supset \overline{U_3}$, and $\varphi((-\varepsilon, \varepsilon) \times \overline{U_4}) \subset \overline{U_3}$. Let $E$ be the set of all $f \in C^\infty(U_1, U)$ such that $f|_{\overline{U_2}}$ is a diffeomorphism onto its image, $f(U_2) \supset \overline{U_3}$, and $f(\overline{U_k}) \subset U_3$. Then via the linear isomorphism $u_* : C^\infty(U_1, U) \rightarrow C^\infty(U_1, \mathbb{R}^m)$ which we suppress from now on, the set $E$ is an open subset of the Fréchet space $C^\infty(U_1, \mathbb{R}^m)$ for the compact $C^\infty$-topology, since the closures $\overline{U_k}$ are compact for each $r_k > 0$ by completeness of the metric.

By cartesian closedness [FK, 4.4.13] or [KMb, 1.8] the curve $\hat{\varphi} : (-\varepsilon, \varepsilon) \rightarrow C^\infty(U_1, \mathbb{R}^m)$ is smooth and takes values in the open subset $E$. 

Claim. Let $L(C^\infty(F(M)), C^\infty(F(U_4)))$ denote the space of all bounded linear mappings between the convenient vector spaces indicated which are equipped with the compact $C^\infty$-topology, and let $P : C^\infty(U_1, \mathbb{R}^m) \supset \mathcal{E} \to L(C^\infty(F(M)), C^\infty(F(U_4)))$ be the mapping given by $P(f)(s) = f^*s = F(f^{-1} \circ f)$. Then $P$ is smooth.

First we check that $P$ takes values in the space of bounded (i.e. smooth) linear mappings. We have to check that $P(f)$ maps smooth curves in $C^\infty(F(M))$ to smooth curves in $C^\infty(F(U_4))$. A curve $c : \mathbb{R} \to C^\infty(F(M))$ is smooth if and only if the composition $4.4.7$, see also $[\text{KM}b, 1.7.2]$, the assignment $\varphi \mapsto \varphi \circ c_c$ respects smoothly parameterized families.

By the smooth uniform boundedness principle $[\text{FK}, \text{remark on page 89, also 4.4.7}$, see also $[\text{KM}b, 1.7.2]$, the assignment $t \mapsto P(f_t)$ is smooth as a mapping

$$(0, \epsilon) \to L(C^\infty(F(M)), C^\infty(F(U_4)))$$

if and only if the composition

$$(0, \epsilon) \to L(C^\infty(F(M)), C^\infty(F(U_4))) \xrightarrow{ev} C^\infty(F(U_4))$$

is smooth for each $s \in C^\infty(F(M))$. We have already checked this condition, so the claim follows.

Now the smooth curve $\varphi$ takes values in $\mathcal{E}$, so we may compute for $1 \leq \ell \leq k$ as follows:

$$\partial^\ell_0 \varphi^*_t s = \partial^\ell_0 (ev_s \circ P \circ \varphi)(t) = d(ev_s \circ P)(\varphi_0)(\partial^\ell_0 \varphi_t) + 0$$

since each other term contains a derivative at 0 of $\varphi_t$ of order less than $\ell$ which is 0, and thus we get $\partial^\ell_0 \varphi^*_t s = 0$ for $\ell < k$ and

$$\partial^k_0 \varphi^*_t s = k! d(ev_s \circ P)(Id_{U_2})(\partial^k_0 \varphi_t) = k! d(ev_s \circ P)(Id_{U_2})(k! X) = k! d(ev_s \circ P)(Id_{U_2})(\partial_t 0 \Phi^X_t) = k! \partial_t 0 (ev_s \circ P \circ \Phi^X_t)(t) = k! \partial_t 0 (\Phi^X_t)^s s = k! \mathcal{L}_X s. \square$$
7. Lemma. Let $M$ be a smooth manifold and let $F$ be a vector bundle functor on $\mathcal{MF}_m$. Let $\varphi_t, \psi_t$ be curves of local diffeomorphisms through $\text{Id}_M$ and let $s \in C^\infty(F(M))$ be a section of the vector bundle $F(M) \to M$. Then we have

$$\partial^k_t|_0(\varphi_t \circ \psi_t)^* s = \sum_{j=0}^k \binom{k}{j} (\partial^j_t|_0(\varphi_t)^* \circ \partial^{k-j}_t|_0(\psi_t)^*) s.$$ 

Also the multinomial version of this formula holds:

$$\partial^k_t|_0(\varphi_t \circ \cdots \circ \varphi_t)^* s = \sum_{j_1 + \cdots + j_k = k} \frac{k!}{j_1! \cdots j_k!} (\partial^{j_1}_t|_0(\varphi_t)^*) \cdots (\partial^{j_k}_t|_0(\varphi_t)^*) s.$$ 

Proof. We only prove the binomial version. The question is local on $M$, so let $U$ be an open neighborhood of some point $x$ in $M$ such that $\varphi$ is defined and smooth on $(-\varepsilon, \varepsilon) \times U$. From the claim in the proof of lemma 6 we know that $t \mapsto \varphi_t^*$ is a smooth curve in the convenient vector space $L(C^\infty(F(M)), C^\infty(F(U)))$ of all bounded linear mappings.

Now let $V \subset M$ be an open neighborhood of $x$ such that $\psi$ is defined on $(-\varepsilon, \varepsilon) \times V$ and $\psi((-\varepsilon, \varepsilon) \times V) \subseteq U$. By the arguments just given the mapping $t \mapsto \psi_t^*$ is a smooth mapping $(-\varepsilon, \varepsilon) \to L(C^\infty(F(U)), C^\infty(F(V)))$ also. Composition

$$L(C^\infty(F(M)), C^\infty(F(U))) \times L(C^\infty(F(U)), C^\infty(F(V))) \to L(C^\infty(F(M)), C^\infty(F(V)))$$

is smooth and bilinear, see [FK, 4.4.16] and we may just apply the Leibniz formula for higher derivatives of bilinear expressions of functions. We evaluate first at $s \in C^\infty(F(M))$ and then at $x \in M$ to obtain the formula. 

8. Lemma. Let $\varphi_t$ be a curve of local diffeomorphisms through $\text{Id}_M$ with first non-vanishing derivative $k!X = \partial^k|_0 \varphi_t$. Then the inverse curve of local diffeomorphisms $\varphi_t^{-1}$ has first non-vanishing derivative $-k!X = \partial^k|_0 \varphi_t^{-1}$.

Proof. For we have $\varphi_t^{-1} \circ \varphi_t = \text{Id}$, so by lemma 7 we get for $1 \leq j \leq k$

$$0 = \partial^j_t|_0(\varphi_t^{-1} \circ \varphi_t)f = \sum_{i=0}^j \binom{j}{i} (\partial^i_t|_0 \varphi_t^*) (\partial^{j-i} (\varphi_t^{-1})^*) f =$$

$$\partial^j_t|_0 \varphi_t^* (\varphi_t^{-1})^* f + \varphi_t^{-1}|_0 (\varphi_t^{-1})^* f,$$

i.e. $\partial^j_t|_0 \varphi_t^* f = -\partial^j_t|_0 (\varphi_t^{-1})^* f$ as required.
9. Lemma. Let \( M \) be a manifold, let \( F \) be a vector bundle functor, let \( s \) be a smooth section of \( F(M) \), let \( \varphi \) be a curve of local diffeomorphisms through \( Id_M \) with first non-vanishing derivative \( m!X = \partial_t^m|0|\varphi_1 \), and let \( \psi \) be a curve of local diffeomorphisms through \( Id_M \) with first non-vanishing derivative \( n!Y = \partial_t^n|0|\psi_1 \).

Then the curve of local sections \([\varphi_1, \psi_1]^s = (\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t \circ \varphi_t)^s\) has first non-vanishing derivative
\[
(m + n)!L_{[X,Y]}s = \partial_t^{m+n}|0|\varphi_1, \psi_1|^s.
\]

Proof. From lemmas 6 and 8 we have the following first non-vanishing derivatives
\[
(1)

\begin{align*}
m!L_Xs &= \partial_t^m|0|\varphi_1^s, \\
n!L_Ys &= \partial_t^n|0|\psi_1^s, \\
m!L_{-X}s &= \partial_t^m|0|(\varphi_t^{-1})^s, \\
n!L_{-Y}s &= \partial_t^n|0|(\psi_t^{-1})^s.
\end{align*}
\]

By the multinomial version of lemma 7 we have
\[
A_Ns = \partial_t^N|0|(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t \circ \varphi_t)^s
= \sum_{i+j+k+l=N} \frac{N!}{i!j!k!l!} (\partial_i^{|0|\varphi_t^s})(\partial_j^{|0|\psi_t^s})(\partial_k^{(\varphi_t^{-1})^s})(\partial_l^{(\psi_t^{-1})^s})s.
\]

Let us suppose that \( 1 \leq n \leq m \), the case \( m \leq n \) is similar. If \( N < n \) all summands are 0. If \( N = n \) we have by lemma 8
\[
A_Ns = (\partial_t^n|0|\varphi_1^s) + (\partial_t^n|0|\psi_1^s) + (\partial_t^n|0|(\varphi_t^{-1})^s) + (\partial_t^n|0|(\psi_t^{-1})^s) = 0.
\]

If \( n < N \leq m \) we have, using again lemma 8
\[
A_Ns = \sum_{j+k+\ell=N} \frac{N!}{j!k!\ell!} (\partial_j^{|0|\varphi_t^s})(\partial_k^{(\varphi_t^{-1})^s})(\partial_\ell^{(\psi_t^s)})s + \delta_N^m ((\partial_t^m|0|\varphi_t^s) + (\partial_t^m|0|(\psi_t^{-1})^s))s
\]
\[
= (\partial_t^N|0|(\psi_t^{-1} \circ \psi_t)^s) + 0 = 0.
\]

Now we come to the difficult case \( m, n < N \leq m + n \).
\[
A_Ns = \partial_t^N|0|(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t \circ \varphi_t)^s + \left(\begin{array}{c} n \\ m \end{array}\right)(\partial_t^m|0|\varphi_t^s)(\partial_t^{N-m}|0|(\varphi_t^{-1} \circ \psi_t^{-1} \circ \psi_t)^s)
\]
\[
+ (\partial_t^N|0|\varphi_t^s) = 0 + (\partial_t^N|0|\varphi_t^s)
\]

by lemma 7, since all other terms vanish, see (4) below. By lemma 7 again we get:
\[
\partial_t^N|0|(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t)^s = \sum_{j+k+\ell=N} \frac{N!}{j!k!\ell!} (\partial_j^{|0|\varphi_t^s})(\partial_k^{(\varphi_t^{-1})^s})(\partial_\ell^{(\psi_t^{-1})^s})s
\]
\[
(3)

\begin{align*}
&= \sum_{j+k+\ell=N} \left(\begin{array}{c} N \\ j \end{array}\right)(\partial_j^{|0|\varphi_t^s})(\partial_k^{(\varphi_t^{-1})^s})s + \left(\begin{array}{c} N \\ m \end{array}\right)(\partial_t^m|0|\varphi_t^s)(\partial_t^{N-m}|0|(\varphi_t^{-1})^s)
\end{align*}
\]

\[
= \delta_{m+n}(m + n)|0|(L_XL_Y - L_YL_X)s + \partial_t^N|0|(\varphi_t^{-1})^s
\]

\[
= \delta_{m+n}(m + n)|L_{[X,Y]}s + \partial_t^N|0|(\varphi_t^{-1})^s
\]
From the second expression in (3) one can also read off that

\[
\partial_t^{N-m}|_0(\psi^{-1}_t \circ \varphi_t^{-1} \circ \psi_t)^* s = \partial_t^{N-m}|_0(\varphi_t^{-1})^* s.
\]

If we put (3) and (4) into (2) we get, using lemmas 7 and 8 again, the final result which proves lemma 9:

\[
A_N s = \delta_{m+n} N! \langle X,Y \rangle_s + \partial_t^N|_0(\varphi_t^{-1})^* s
+ \binom{N}{m} (\partial_t^m|_0(\varphi_t^{-1})^*) s + (\partial_t^N|_0(\varphi_t^{-1})^*) s
= \delta_{m+n} N! \langle X,Y \rangle_s + \partial_t^N|_0(\varphi_t^{-1})^* s
= \delta_{m+n} N! \langle X,Y \rangle_s + 0. \quad \square
\]

10. Theorem. Let \( M \) be a manifold, let \( \varphi^i \) be smooth curves of local diffeomorphisms through \( \text{Id}_M \) for \( i = 1, \ldots, j \) with non-vanishing first derivative \( \partial^k_t|_0(\varphi_t^i) = k! X_i \in \mathfrak{X}(M) \). Let \( F \) be a vector bundle functor and let \( s \in C^\infty(F(M)) \) be a section. Then for each formal bracket expression \( B \) of length \( j \) we have

\[
0 = \partial^\ell_t|_0(\varphi_t^1, \ldots, \varphi_t^j)^* s \quad \text{for } 1 \leq \ell < k,
\]

\[
\mathcal{L}_B(X_1, \ldots, X_k) s = \frac{1}{k!} \partial^k_t|_0(\varphi_t^1, \ldots, \varphi_t^j)^* s \in C^\infty(F(M)),
\]

where \( k = k_1 + \cdots + k_j \).

Proof. Apply lemma 9 recursively. \( \square \)

11. Proposition. Let \( \varphi \) be a curve of local diffeomorphisms through \( \text{Id}_M \) with first non-vanishing derivative \( k! X = \partial^k_t|_0(\varphi_t) \). Then the curve of local vector fields \( (\partial_t^k \varphi_t) \circ \varphi_t^{-1} \) has as first non-vanishing derivative

\[
k! X = \partial^{k-1}_t|_0((\partial_t^k \varphi_t) \circ \varphi_t^{-1})�
\]

Proof. Using lemma 7 for \( f \in C^\infty(M, \mathbb{R}) \) we have for \( 1 \leq \ell < k \):

\[
\partial^\ell_t|_0((\partial_t^k \varphi_t) \circ \varphi_t^{-1}) f = \partial^\ell_t|_0(\varphi_t^{-1})^* \partial_t^k f
= \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} (\partial^j_t|_0(\varphi_t^{-1})^*) (\partial^{\ell-j}_t|_0(\varphi_t^{-1})^*) f
= (\varphi_0^{-1})^* (\partial^\ell_t|_0(\varphi_t^{-1})^*) f + 0
= \delta^\ell_t k! \mathcal{L}_X f. \quad \square
12. Corollary. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. For $g, h \in G$ we consider the group commutator $[g, h] = g h g^{-1} h^{-1}$. Then for any bracket expression $B$ of length $k$ and $X_i \in \mathfrak{g}$ we have

$$k! B(X_1, \ldots, X_k) = \partial^k_0 |_0 B(\exp t X_1, \ldots, \exp t X_k)$$

$$= \partial^k_{t-1} |_0 \left( T \lambda \exp B(\exp t X_1, \ldots, \exp t X_k)^{-1} \partial_t B(\exp t X_1, \ldots, \exp t X_k) \right),$$

where $\lambda_g$ denotes left translation by $g$.

The first equation is a generalization of the well known ‘Trotter product formula’, i.e. the case of $B = [\cdot, \cdot]$.

Proof. The flow of the left invariant vector field $L_X$ corresponding to $X \in \mathfrak{g}$ is the right translation $\rho \exp t X$ by $\exp t X$, so we just apply theorem 1 to get

$$k! B(L_{X_1}, \ldots, L_{X_k}) = \partial^k_0 |_0 B(\rho \exp t X_1, \ldots, \rho \exp t X_k)$$

$$= \partial^k_{t-1} |_0 \rho(B(\exp t X_1, \ldots, \exp t X_k)),$$

(1)

where in the first line the commutator of flows is applied, and in the second line the group commutator with reversed order. Evaluating both sides at $e \in G$ gives the first formula. From (1) and proposition 11 we get

$$k! B(X_1, \ldots, X_k) =$$

$$= \partial^k_{t-1} |_0 \left( \partial_t + \rho \left( B(\exp t X_1, \ldots, \exp t X_k) \right) \circ \rho \left( B(\exp t X_1, \ldots, \exp t X_k) \right)^{-1} \right).$$

We evaluate this at $e \in G$ and get

$$k! B(X_1, \ldots, X_k) =$$

$$= \partial^k_{t-1} |_0 \left( \partial_t B(\exp t X_1, \ldots, \exp t X_k) \right) \circ B(\exp t X_1, \ldots, \exp t X_k)^{-1} \right)$$

$$= \partial^k_{t-1} |_0 \left( T \lambda \exp B(\exp t X_1, \ldots, \exp t X_k)^{-1} \partial_t B(\exp t X_1, \ldots, \exp t X_k) \right).$$

□

References

[FK] Frölicher, Alfred; Kriegl, Andreas, *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.

[KMS] Kollár, Ivan; Michor, Peter W.; Slovak, J., *Natural operators in differential geometry*, Springer-Verlag, 1993.

[KMa] Kriegl, Andreas; Michor, Peter W., *A convenient setting for real analytic mappings*, Acta Mathematica 165 (1990), 105–159.

[KMb] Kriegl, A.; Michor, P. W., *Aspects of the theory of infinite dimensional manifolds*, Differential Geometry and Applications 1(1) (1991).

[KMc] A. Kriegl, Michor, P. W., *The Convenient Setting of Global Analysis*, AMS, 1997.

[KN] Kriegl, Andreas; Nel, Louis D., *A convenient setting for holomorphy*, Cahiers Top. Géo. Diff. 26 (1985), 273–309.
[M] Mauhart, Markus, *Iterierte Lie Ableitungen und Integrabilität*, Diplomarbeit, Universität Wien, 1990.

[T] Terng, Chu Lian, *Natural vector bundles and natural differential operators*, American J. of Math. **100** (1978), 775–828.

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: MICHOR@AVIRAP.BITNET