Coherent Neutrino Interactions in a Dense Medium

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Abstract

Motivated by the effect of matter on neutrino oscillations (the MSW effect) we study in more detail the propagation of neutrinos in a dense medium. The dispersion relation for massive neutrinos in a medium is known to have a minimum at nonzero momentum $p \sim G_F \rho / \sqrt{2}$. We study in detail the origin and consequences of this dispersion relation for both Dirac and Majorana neutrinos both in a toy model with only neutral currents and a single neutrino flavour and in a realistic “Standard Model” with two neutrino flavours. We find that for a range of neutrino momenta near the minimum of the dispersion relation, Dirac neutrinos are trapped by their coherent interactions with the medium. This effect does not lead to the trapping of Majorana neutrinos.

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I. INTRODUCTION

Motivated by the effect of matter on neutrino oscillations (the MSW effect \[1\]), there have been several works in recent years aimed at understanding in a more complete way the propagation of one or more flavours of massive neutrinos in matter. One of the first papers along these lines was the paper of Mannheim in 1987 \[2\] whose main purpose was to derive the MSW effect from a Field Theoretic starting point. Mannheim used second quantization techniques to derive the wave functions and dispersion relations of two flavours of both Dirac and Majorana neutrinos propagating in a medium in which there was a finite density of electrons. Mannheim then analyzed his result in the ultrarelativistic regime and recovered the standard MSW results.

Both the work of Mannheim and the work of Nieves \[3\] and of Nötzold and Raffelt \[4\] showed that the entire MSW effect could be reliably analyzed with a modified Dirac (or Majorana) equation by adding to the frequency of the electron neutrino a term proportional to the density ($\sqrt{2}G_F\rho$ in the standard MSW scenario). This term is analogous to a chemical potential term for the electron neutrino though it couples only to the chiral left handed part of a possibly massive neutrino.

In 1991 Panteleone \[5\] used such a modified Dirac equation to study the behaviour of neutrinos in supernova cores. He first analyzed the case of a single neutrino flavour without restricting the momenta to be relativistic. At this point he noticed a very unusual characteristic of the neutrino dispersion relation. The dispersion relation had a minimum at a nonzero neutrino momentum. Thus for a range of neutrino momentum the neutrino’s phase velocity and group velocity are in opposite directions! Panteleone later analyzed the case of two and three flavours of neutrinos. He noticed that the neutral currents did not decouple in general although his detailed analysis was carried out only in the high energy regime in which they do decouple.

In this paper we provide a synthesis of many of the above results. We are particularly interested in effects at low neutrino momentum and in effects due to the minimum of the
dispersion relation at nonzero momentum. We begin in Section II by examining a simple model with only a single neutrino flavour in which the neutrino propagates in a background of electrons. We take the neutrino-electron interaction to be mediated only by neutral current interactions. This model allows for a careful analysis of the Field Theory aspects of neutrino propagation in a medium. We pay special attention to the minimum of the dispersion relation which occurs at non-zero momentum and analyze its effects on neutrino interactions. We will find that the minimum energy for Dirac neutrinos in the medium will generically be less than the rest mass of the neutrino in the vacuum so that very low energy neutrinos are effectively trapped by the medium. A similar effect has been studied from a different point of view in the work of Loeb 1[6], although his effect has an entirely different origin. Our analysis of the trapping of neutrinos will bring up many interesting questions and puzzles which will need to be resolved in order to have a complete understanding of the problem. We will also examine the case of Majorana neutrinos and find that in general Majorana neutrinos are not trapped.

In Sec. III we extend our analysis to a more realistic model which has two neutrino flavours and in which there are both neutral and charged current interactions. In this case the dispersion relations are governed by quartic equations for Dirac neutrinos and quadratic equations for Majorana neutrinos. We will again be most interested in examining the form of the dispersion relations in the medium and their effects on neutrino propagation. Again we will find that the Dirac case leads quite generically to neutrino trapping while the Majorana case can have no trapping. We will also make a few remarks regarding the oscillations of neutrinos in a medium, noting that even for a single neutrino there could in principle be oscillations in the probability to detect the neutrino, due to the differing phase velocities of

1Loeb studies the problem from the point of view of the neutrino’s “index of refraction” in the medium. A radially varying density leads to a force on the neutrino which allows it to have bound orbits in the medium.
the helicity eigenstates.

We conclude in Sec. IV with a brief discussion of our results and some concluding remarks.

II. A SIMPLE MODEL WITH ONE NEUTRINO FLAVOUR

Many of the interesting effects which we will discuss, namely those due to the minimum of the dispersion relation which occurs at non-zero momentum, occur already in a simple model with only a single neutrino flavour. It is useful, then, to first consider a simplified model in which a Dirac neutrino propagates in an electron “gas” to which it couples only via the neutral current interaction. The case of a Majorana neutrino is somewhat more subtle and will be considered subsequently. Our model may be described by the following Lagrangian

\[
\mathcal{L} = \bar{\psi}_\nu \left( i \not{D}^+ - m_\nu \right) \psi_\nu + \bar{\psi}_e \left( i \not{D}^- - m_e \right) \psi_e - \frac{1}{4} F^2 + \frac{m^2 Z^2}{2} - \mu_e \bar{\psi}_e \psi_e, \quad (2.1)
\]

where

\[
D^\pm_\mu = \partial_\mu \pm i \frac{g}{2 \sqrt{2}} Z_\mu (1 - \gamma^5), \quad (2.2)
\]

\[
F_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu. \quad (2.3)
\]

The chemical potential term in the Lagrangian is included in order to give a non-zero value to the electron density; that is

\[
\rho_e = \langle \psi_e^\dagger \psi_e \rangle \neq 0. \quad (2.4)
\]

In order to study the propagation of a neutrino in this medium, we compute the neutrino self-energy. To one loop there are three diagrams, shown in Fig. 2\textsuperscript{2}. All effects due to the

\textsuperscript{2}The electron loop in this diagram is, of course, equal to the electron density to all orders. The neutrino loop and the \( Z_0 \) loop lead to a term proportional to the neutrino density which, as we have pointed out, is nonzero. This leads to a small correction which can easily be included but which we shall ignore.
non-zero electron density come from the electron loop in Fig. (a) which is easily calculated and yields

\[ \Sigma = -\frac{G}{\sqrt{2}} \rho_e \gamma^0 (1 - \gamma^5), \]  

(2.5)
in which we have defined \( G = g^2 / (4\sqrt{2}m_Z^2) \) in analogy with the usual Fermi coupling constant, \( G_F \). From the self-energy one may obtain the neutrino propagator in the usual way by summing a geometric series. For constant \( \rho_e \) the resulting expression is given in momentum space by

\[ G_\nu(p) = \frac{1}{\not{p} - m_\nu - \Sigma}. \]  

(2.6)

If \( \rho_e \) depends explicitly on \( x \) the propagator may still be formally written in position space as

\[ G_\nu(x) = \frac{1}{i\not{\partial} - m - \Sigma(x)}. \]  

(2.7)
The effective action is then given, to this order, by

\[ S_{\text{eff}} = \int d^4x \bar{\psi}_\nu \left[ G_\nu^{-1}(x) \right] \psi_\nu \]  

(2.8)

\[ = \int d^4x \bar{\psi}_\nu \left[ i\not{\partial} - m + \frac{G}{\sqrt{2}} \rho_e \gamma^0 (1 - \gamma^5) \right] \psi_\nu. \]  

(2.9)

Variation of the effective action leads finally to an effective “Dirac equation,” given by

\[ \left[ i\not{\partial} - m + \alpha \gamma^0 (1 - \gamma^5) \right] \psi_\nu = 0, \]  

(2.10)
in which we have defined \( \alpha = G \rho_e / \sqrt{2} \). For constant electron density, the presence of the “chiral potential” in this expression leads to a shift in the frequency by \( \alpha \), but only for the left-handed (chiral) piece. This shift in the frequency is precisely the “index of refraction” familiar from the MSW effect. Once we have derived the dispersion relations, it will be clear that the shift in energy for the neutrino is opposite that for the anti-neutrino. If the neutrino is “attracted” by the medium, then the anti-neutrino is “repelled” by it.

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3This shift comes from a term in the effective Hamiltonian proportional to \( \psi_\nu^\dagger \psi_\nu \) which equals the number density of neutrinos minus the number density of anti-neutrinos.
We have noted above that the chemical potential $\mu_e$ in the Lagrangian (2.1) gives rise to a non-zero electron density, $\rho_e \equiv \langle \bar{\psi}_e \psi_e \rangle$. A similar calculation of $\rho_\nu \equiv \langle \bar{\psi}_\nu \psi_\nu \rangle$ for the effective Lagrangian defined by Eq. (2.9) shows that, at least naively, there appears also to be a non-zero density of neutrinos in this medium (provided, of course, that the effective chemical potential $\alpha$ is larger than $m$). It is clear that this arises due to the potential term proportional to $\gamma^0$ in the effective Lagrangian, since this term looks exactly like a chemical potential for the chiral left-handed neutrinos. How one handles this apparent density of neutrinos can drastically affect the MSW effect. Suppose, for example, that we were to insist that in the sun $\rho_\nu \equiv \langle \bar{\psi}_\nu \psi_\nu \rangle = 0$. In order to implement this, we may choose to introduce a “counter” chemical potential into the original Lagrangian which would exactly cancel the neutrino density generated by the interactions with the electrons in the medium. It turns out, however, that this would change the MSW result quite dramatically. In fact, if our theory had a “vector” instead of a “chiral” potential, we would kill the entire MSW effect by doing this. As we shall discuss in more detail below, the correct thing to do is to accept the fact that in equilibrium there is a non-zero density of neutrinos of very low momentum due to their attractive interaction with the medium [7].

Further insight into the physics of our model can be gained by examining the equations of motion following from the Lagrangian in Eq. (2.1). Varying the Lagrangian with respect to the $Z^\mu$ field leads to

$$\partial_\nu F^\nu \mu + m_Z^2 Z^\mu = -\frac{g}{2\sqrt{2}} J^\mu,$$

(2.11)

where

$$J^\mu = \bar{\psi}_e \gamma^\mu (1 - \gamma^5) \psi_e - \bar{\psi}_\nu \gamma^\mu (1 - \gamma^5) \psi_\nu.$$

(2.12)

If $\rho_e = \langle \bar{\psi}_e \psi_e \rangle$ is constant, then (2.11) leads to

$$\langle Z^0 \rangle = -\frac{g \rho_e}{2\sqrt{2} m_Z^2},$$

(2.13)

that is, the $Z^0$ field has gained a vacuum expectation value. The equation of motion for the neutrino field is then given by
\[
\left[ i \partial - m - \frac{g}{2\sqrt{2}} (Z^0) \gamma^0 (1 - \gamma^5) \right] \psi_\nu = 0, \tag{2.14}
\]

which is equivalent to the effective Dirac equation of (2.10) once the value for the \(Z^0\) expectation value, Eq. (2.13), is inserted. From this point of view, then, the left-handed neutrino sees a mean (coherent) “scalar potential,” \(\langle Z^0 \rangle\). From the point of view of the field theoretic calculation above, it is clear why the \(Z^0\) field has developed a vacuum expectation value. The one-loop diagram corresponding to \(\langle Z^0 \rangle\) is simply the electron loop, corresponding to \(\rho_e\), with a \(Z\) propagator attached. This coherent \(Z^0\) field is analogous to the electric field which surrounds a static charge distribution and is due to the net weak charge of the medium.

**A. Solution to the Dirac Equation**

In order to study the propagation of neutrinos over macroscopic distances, it suffices to study the effective Dirac equation given in Eq. (2.10). The propagator defined in Eq. (2.7) contains this information, but it also encodes the off-shell behaviour of the neutrino. For a medium with constant density, it is straightforward to solve the Dirac equation in momentum space by employing the chiral representation, so that

\[
\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}, \tag{2.15}
\]

in which the upper and lower components correspond to the left and right chiral projections, respectively. In this representation, the Dirac equation becomes

\[
\begin{pmatrix}
-m & \omega - \vec{\sigma} \cdot \vec{p} \\
\omega + 2\alpha + \vec{\sigma} \cdot \vec{p} & -m
\end{pmatrix}
\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0. \tag{2.16}
\]

where \(\alpha = G \rho_c / \sqrt{2}\). Without loss of generality we may choose \(\vec{p} = p \hat{z}\), so that \(\chi_{L,R}\) (and hence also \(\psi\)) may be chosen to be eigenstates of \(\sigma_3\), the spin projection in the \(z\) direction. That is,

\[
\sigma_3 \chi_{L,R} = s \chi_{L,R}, \tag{2.17}
\]
where $s=\pm 1$. Solving for the energy yields four solutions

$$\omega = -\alpha \pm \sqrt{(p + \alpha s)^2 + m^2}. \quad (2.18)$$

These dispersion relations are plotted in Fig. 2 both for $m=0$ (dashed curves) and $m\neq 0$ (solid curves.) Several key features of these plots should be noted. First of all, the “negative energy” states are, in this case, those which are unbounded from below as the momentum is increased. In the second quantized theory the correct energy of such a state is just the negative of its energy eigenvalue. We also note that when $m=0$ there are “level crossings.” These are avoided for $m\neq 0$ by level repulsion due to the mixing of the levels.

The most noteworthy feature of these dispersion relations (discussed previously by Pantaleone) is the fact that the minima of the dispersion relations occur at non-zero values of the momentum, $p=\pm \alpha$, instead of at the origin. One interesting consequence of this fact is that the neutrino can have a vanishing group velocity at non-zero momenta. Furthermore, for $|p|<|\alpha|$, the neutrino’s group velocity, $d\omega/dp$, is in a direction opposite to its momentum!

This will play an important role in understanding the reflection of neutrinos at the boundary of the medium. Another interesting feature of these curves is that the minimum energy $\omega_{\text{min}}=-\alpha + m$ is less than the neutrino mass. Thus it is possible to produce a neutrino in the medium which has $\omega<m$. Such a neutrino will not have enough energy to survive in the vacuum and will thus be trapped by the medium. We shall examine these peculiar features of the neutrino dispersion relations in detail below.

Finally, we note that for high momentum, the solution corresponding to negative helicity has energy

$$\omega \approx p + \frac{m^2}{2p} - 2\alpha, \quad (2.19)$$

which is just the usual MSW result. By way of contrast, the positive helicity solution approaches its vacuum value of $\omega \approx p + m^2/2p$. This illustrates the spin-dependence of the interaction. For high momentum, the left-handed (chiral) states are nearly equivalent to the negative helicity eigenstates, so the potential (which is left-handed) affects only the negative, and not the positive, helicity eigenstates.
B. Neutrino Trapping

Let us now consider a low energy neutrino which is produced inside the medium and which then tries to escape to the vacuum. We take the medium to have $\alpha > 0$. The dispersion relations for this neutrino both inside and outside the medium are illustrated in Fig. 3. In this figure the neutrino energy as a function of its momentum is plotted for zero density (outside the medium) (Fig. 3(a)), low density (Fig. 3(b)), intermediate density (Fig. 3(c)) and high density (Fig. 3(d)). The negative energy solutions to the Dirac Equation are also plotted, for reasons which will become apparent momentarily.

Let us begin by looking at the case of low electron density, $0 < \alpha < m$. This region is characterized by the fact that the minimum of the neutrino dispersion relation lies above the vacuum energy ($E=0$) but below the neutrino mass ($E=m$). In this case a neutrino which is produced inside the medium with momentum $p$ such that $E(p) > m$ will escape from the medium (although there will also be some amplitude for reflection). If, on the other hand, its momentum $p$ is close to $\alpha$ so that $E(p) < m$ (as is shown, for example, by the point A in Fig. 3(b)), then this neutrino is trapped in the medium since there is no Energy level corresponding to its energy if it escapes into the vacuum (Fig. 3(a)). The way this total reflection is realized in practice is fascinating. Suppose the neutrino is incident normal to the interface of the medium and the vacuum. We might then be concerned that the neutrino has to flip its spin in order to find an energy level with negative momentum even in the medium (see Fig. 2). It is straightforward to show, however, that the neutrino’s spin must be conserved in such a reaction. The resolution of this apparent problem is that the reflected neutrino indeed has a positive momentum as shown by the point B in Fig. 3(b). The reason this corresponds to a reflected wave is that the neutrino’s group velocity is negative in this region so that the neutrino travels back into the medium. It is straightforward to extend these arguments to the case of non-normal incidence [8].

\[^{4}\text{Note that for } \alpha < 0 \text{ the medium would trap anti-neutrinos instead of neutrinos.}\]
The case of intermediate electron density, $m<\alpha<2m$, is shown in Fig. 3(c) and is characterized by the fact that the minimum neutrino energy in the medium is less than zero but greater than the largest negative energy state outside the medium. In this case the lowest energy of the system occurs when all the neutrino states below $E=0$ are filled so that the mean neutrino density $\rho_\nu$ is nonzero. This is consistent with the field theoretic calculation of this density which was discussed previously in this paper. In this case the neutrinos with $m>E>0$ would be trapped as in the previous case but no neutrinos with $E<0$ could be produced in the medium since all corresponding levels are full. One could imagine a *nonequilibrium* (higher energy) situation in which these levels are not full. In this case neutrinos with energy $E<0$ (due to their attractive interaction with the medium) could be produced (by some external reaction) and they would also be trapped. Whether such a situation arises in any practical case depends on the dynamics of the formation of this region.

Finally we turn to the case of high electron density for which $\alpha>2m$. This situation is shown in Fig. 3(d) and is characterized by the fact that the minimum of the neutrino dispersion relation inside the medium has a lower energy than the maximum energy of the negative energy Dirac states in the vacuum. In this case the potential difference between the vacuum and the medium is strong enough to induce pair production of neutrino–antineutrino pairs at the interface. If the neutrino levels below $E=-m$ are not filled initially then this pair production would eventually lead to the filling of these levels. If a neutrino with energy below $-m$ is produced by another mechanism during this time, it would be trapped by the medium.

In all the cases discussed above it is simple to write down the criterion which must be

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In this case the reflection amplitude would also have a contribution from pair production which would allow a spin flip at the boundary. This contribution will be the dominant contribution in the case of massless neutrinos.
met in order that a neutrino be trapped. The dispersion relation for a negative spin (relative to \( \hat{z} \)) neutrino is given by

\[
\omega = -\alpha + \sqrt{(p - \alpha)^2 + m^2},
\]

so that the condition for trapping is

\[
-\alpha + \sqrt{(p - \alpha)^2 + m^2} < m
\]

or

\[
p < p_{\text{trap}} \equiv \alpha + \sqrt{\alpha^2 + 2\alpha m}.
\]

Thus, neutrinos produced in this medium with momentum \( p < p_{\text{trap}} \) will not have enough energy to survive in the vacuum and will be trapped.

Before proceeding it is useful to get some idea of the overall magnitude of the effect of neutrino trapping. Setting \( G \approx G_F \) and \( m \approx 10^{-3}\text{eV} \) (which is a mass relevant for the MSW-resolution of the solar neutrino problem), we find that \( p_{\text{trap}} \approx 10^{-8}\text{eV} \) in the sun (for which \( \alpha \approx 10^{-12}\text{eV} \)) and \( p_{\text{trap}} \approx 100\text{eV} \) in a supernova (for which \( \alpha \approx 100\text{eV} \)). The phenomenon of trapping is quite remarkable when we consider that the mean free path (which increases with decreasing momentum) of a neutrino with \( p \approx 10^{-8}\text{eV} \) in the sun is on the order of \( 10^{20} \) solar radii. Such a neutrino would thus have no chance of being “incoherently” trapped in the sun (say by back-scattering from nuclei), but would still be trapped by the coherent process which we are discussing. We note furthermore that in the case of incoherent trapping the scattering cross section is typically dependent on the mass of the target particle, whereas for coherent trapping this is not the case. Loeb has discussed a similar effect (albeit of different origin) for neutron stars and has estimated that in general this effect will add an extra 30 kg to the mass of the star.

\[\square\]

C. The Majorana Case

In the case of Majorana neutrinos, there is only a single (left-handed) field, \( \chi_L \). The effective Lagrangian is given by
\[ L_{\text{eff}} = \overline{\psi}_M \left[ \frac{1}{2}(i\partial - m) + \alpha \gamma^0(1 - \gamma^5) \right] \psi_M, \]  

where \[ \psi_M = \begin{pmatrix} \chi_L \\ -i\sigma^2\chi_L^* \end{pmatrix} \]  
in the chiral representation. In general one needs to be somewhat careful when dealing with Majorana fermions. For example, even at the classical level, the fields need to be taken as Grassman-valued, or else the mass term disappears. The dispersion relations in this case can be obtained by solving the equations of motion, as was first done by Mannheim [2]. The reader is referred to Mannheim's paper for details of the calculation (see also Ref. [10].) The resulting expression for the negative helicity neutrino is given by  

\[ \omega = \pm \sqrt{(|\vec{p}| - 2\alpha)^2 + m^2}. \]  

In this case the energy has a minimum value, \( \omega = m \), which occurs at \( |\vec{p}| = 2\alpha \). In fact, the dispersion relation in matter is identical to that in vacuum except for a lateral shift to the right. This implies in particular that, in contradistinction to the Dirac case, a neutrino cannot have an energy less than \( m \) and there is thus no trapping of Majorana neutrinos in the medium.

### III. DISPERSION RELATIONS FOR TWO NEUTRINO FLAVOURS

We turn now to consider a more realistic scenario in which there are two neutrino flavours and in which there are both neutral current and charged current couplings to the medium. We have in mind, of course, the Standard Electroweak Model with massive neutrinos. We shall first derive the quartic equation governing the dispersion relations in the Dirac case and examine the solutions in some representative cases. We shall see again that in this case there is neutrino trapping. We then examine the Majorana case, in which the dispersion relations are quadratic and are thus readily analyzed but in which both trapping and pair
production at the surface of the sun are absent. We shall then describe briefly an alternate model which has been studied in the literature which has no chiral coupling but still yields a minimum in the dispersion relation at non-zero momentum. Finally, we comment on neutrino oscillations in these models.

A. Dirac Case

We begin with the Dirac Equation in the mass basis for a pair of massive Dirac neutrinos with both neutral and charged current coupling to a medium:

\[
\{ p - M + (\beta - \alpha Q) \gamma^0 (1 - \gamma_5) \} \psi = 0
\]

(3.1)

where \( M \) is the diagonal \( 2 \times 2 \) mass matrix, \( \beta \propto \rho G_F \) is the contribution of the neutral current which couples only to the left handed neutrinos and \( \alpha \propto \rho_e G_F \) represents the charged current contribution which couples only to \( \nu_e \). This coupling is assured by the mixing matrix

\[
Q = \begin{pmatrix}
\cos^2(\theta) & \sin(\theta) \cos(\theta) \\
\sin(\theta) \cos(\theta) & \sin^2(\theta)
\end{pmatrix}.
\]

(3.2)

\( \alpha \) and \( \beta \) may be calculated by computing the one loop contributions to the neutrino self-energy in the background medium. The Feynman diagrams corresponding to these processes are shown in Fig. 4 and yield

\[
\alpha = \frac{G_F}{\sqrt{2}} \rho_e
\]

(3.3)

and

\[
\beta = -\frac{G_F}{\sqrt{2}} \sum_f (T^f_3 - 2Q^f) \sin^2 \theta_W \rho_f
\]

(3.4)

in which the sum in (3.4) runs over all fermions in the medium, \( T^f_3 \) is the third component of the fermion’s weak isospin and \( Q^f \) is its charge. If there are appreciable densities of anti-particles in the medium, then \( \rho_f \) needs to be replaced by \( \rho_f - \rho_f^\dagger \) in these expressions.

In the Chiral representation we write
\[ \psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \]  

(3.5)

and the Dirac Equation becomes

\[ (\omega - \vec{\sigma} \cdot \vec{p}) \chi_R = M\chi_L, \quad \{(\omega + \vec{\sigma} \cdot \vec{p}) + 2(\beta - \alpha Q)\}\chi_L = M\chi_R. \]  

(3.6)

For simplicity we may assume the momentum to be in the \( \hat{z} \) direction in which case the solutions to the Dirac Equation will be eigenstates of \( \sigma_3 \):

\[ \chi_L = \begin{pmatrix} L_+ \\ 0 \end{pmatrix}, \quad \chi_R = \begin{pmatrix} R_+ \\ 0 \end{pmatrix}; \quad \text{or} \quad \chi_L = \begin{pmatrix} 0 \\ L_- \end{pmatrix}, \quad \chi_R = \begin{pmatrix} 0 \\ R_- \end{pmatrix} \]  

(3.7)

leading to the equations:

\[ \left\{ \omega^2 - p^2 + 2(\omega \mp p)(\beta - \alpha Q) - M^2 \right\} L_\pm = 0, \]  

(3.8)

\[ R_\pm = \frac{1}{(\omega \mp p)} ML_\pm. \]  

(3.9)

To find the energy eigenvalues we rewrite Eq. (3.8) as:

\[ \left\{ \omega^2 - p^2 - \mu^2 + 2(\omega \mp p)\beta - 2\alpha(\omega \mp p)N_\mp \right\} L_\pm = 0 \]  

(3.10)

where \( \mu^2 = \langle m^2 \rangle = (m_1^2 + m_2^2)/2 \) is the mean squared mass and

\[ N_\mp = \begin{pmatrix} \cos^2(\theta) - \xi_\mp & \sin(\theta) \cos(\theta) \\ \sin(\theta) \cos(\theta) & \sin^2(\theta) + \xi_\mp \end{pmatrix} \]  

(3.11)

with

\[ \xi_\mp = \frac{\Delta^2}{4\alpha(\omega \mp p)} \]  

(3.12)

and \( \Delta^2 = m_2^2 - m_1^2 \). It thus remains only to find the eigenvalues of \( N_\mp \).

The eigenvalues of \( N_\mp \) are:

\[ \lambda_1^\mp = \frac{1}{2} \left( 1 + \sqrt{1 - 4 \xi_\mp (\cos(2\theta) - \xi_\mp)} \right) \]  

(3.13)

\[ \lambda_2^\mp = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \xi_\mp (\cos(2\theta) - \xi_\mp)} \right). \]  

(3.14)
Plugging these back into the Dirac Equation leads to the following quartic equation for the energy eigenvalues:

$$\left[ \omega^2 - p^2 - \mu^2 + (2\beta - \alpha)(\omega - sp) \right]^2 = \alpha^2(\omega - sp)^2 - \alpha\Delta^2 \cos(2\theta)(\omega - sp) + \frac{1}{4}\Delta^4, \quad (3.15)$$

in which \( s=\pm 1 \) is the eigenvalue of \( \sigma_3 \), the spin projection in the \(+z\) direction. In special cases this expression reduces to those found in the papers of Mannheim (in which the neutral current contribution has been left out) and Pantaleone (in which one of the masses has been set to zero.)

Equation (3.15) is the main quartic equation whose eight solutions (for \( s=\pm 1 \)) lead to the eight dispersion relations (four positive energy and four negative energy) in the medium. It may not at first be obvious that all eight of the solutions \( \omega \) of (3.15) corresponding to a fixed value of the momentum are real. This is, however, the case, which may be seen as follows. (The proof is equally straightforward for any number of flavours, so we will do it immediately in the general case.) In the general case the Dirac Equation in the mass basis becomes

$$\left\{ \vec{p} - M + (\beta - \alpha U^\dagger N_e U) \gamma^0 (1 - \gamma_5) \right\} \psi = 0 \quad (3.16)$$

in which \( U \) is the mixing matrix in flavour space and \( N_e=\text{diag}(1,0,\ldots,0) \) is also a matrix in flavour space. Pre-multiplying this expression by \( \gamma^0 \) leads to the eigenvalue equation

$$\mathcal{N} \psi = \omega \psi, \quad (3.17)$$

where

$$\mathcal{N} = \gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 M - (\beta - \alpha U^\dagger N_e U) (1 - \gamma_5). \quad (3.18)$$

Since \( \mathcal{N} \) is hermitian for real \( \vec{p} \), the eigenvalues of Eq. (3.17) are guaranteed to be real. This completes the proof.

It is best to analyze the expression governing the dispersion relations, Eq. (3.15), by first considering some special cases. The very simplest case is when \( m_1=m_2=0 \), which yields
\[ \omega = p + (\beta - \alpha)(s - 1) \]  
(3.19)

\[ \omega = -p - (\beta - \alpha)(s + 1) \]  
(3.20)

for the electron neutrinos and

\[ \omega = p + \beta(s - 1) \]  
(3.21)

\[ \omega = -p - \beta(s + 1) \]  
(3.22)

for the muon neutrinos. These expressions are easy to understand. Four of the dispersion relations are unchanged from their values in the vacuum, since positive helicity neutrinos are also right-handed (chiral) and are thus unaffected by the left-handed Standard Model interactions. The remaining four dispersion relations are displaced vertically from their vacuum values by amounts proportional to their couplings to the medium. Note that only the dispersion relation corresponding to \( \nu_e \) is affected by the charged current contribution, \( \alpha \).

Another simple case occurs when the coupling \( \theta \) is set to zero. In this case the dispersion relations for \( \nu_e \) and \( \nu_\mu \) decouple, as one would expect, and we find

\[ \omega = -(\beta - \alpha) \pm \sqrt{(p + s(\beta - \alpha))^2 + m_1^2} \]  
(3.23)

and

\[ \omega = -\beta \pm \sqrt{(p + s\beta)^2 + m_2^2} \]  
(3.24)

for the electron and muon neutrinos, respectively. These expressions are in exact agreement with what we found in the single-neutrino case in Sec. II A. Once again the dispersion relations corresponding to the massless case undergo “level repulsion” when a finite mass is added.

Since the equations governing the dispersion relations in the two-flavour Dirac case are quartic it is difficult, in general, to obtain analytic expressions for these dispersion relations. Of course quartic equations are analytically solvable and we know that in our case all the
solutions are real. In general, however, no practical insight can be gained by examining the analytic expressions of these solutions. One approach which is helpful in understanding the dispersion relations if the coupling $\theta$ is not too large is to use a graphical approach. One begins by looking at the solutions when $\theta=0$ in which case the two neutrino flavours decouple and we can use the solutions derived in the previous section. Thus, for example, in Fig. 3(a) the dotted curves represent the solutions for $\theta=0$. Note that the dispersion relations for the two flavours of neutrinos cross at some points. When $\theta$ is “turned on” we expect that these levels will repel and will lead to a curve similar to the solid curve in that figure. The solid curve is, in fact, the solution to the quartic equation when $\theta=0.2$. This graphical method is reasonably accurate when $\theta$ is small but can be used as a guide even for larger values of $\theta$.

One interesting feature in the two-neutrino case is that the heavier neutrino, which would have been trapped in certain cases if the neutrinos were decoupled, can now “leak out” due to its coupling to the lighter mass eigenstate. That is, only states with energy less than the mass of the lightest mass eigenstate are strictly “trapped” now.

It is also possible to derive approximate solutions of the quartic equations if the neutrinos are relativistic. In that case approximate solutions are given by

$$\omega \simeq p - (2\beta - \alpha) + \frac{\mu^2}{2p} \pm \frac{1}{4p} \left[ (4\alpha p - \Delta^2 \cos(2\theta))^2 + \Delta^4 \sin^2(2\theta) \right]^{1/2},$$  \hspace{1cm} (3.25)

$$\omega \simeq -p - (2\beta - \alpha) - \frac{\mu^2}{2p} \mp \frac{1}{4p} \left[ (4\alpha p + \Delta^2 \cos(2\theta))^2 + \Delta^4 \sin^2(2\theta) \right]^{1/2},$$  \hspace{1cm} (3.26)

$$\omega \simeq \pm \left( p + \frac{m_{1,2}^2}{2p} \right),$$  \hspace{1cm} (3.27)

where the corrections to the above expressions go like $\alpha\mu^2/p^2$, $\beta\mu^2/p^2$ and $\mu^4/p^3$. Of these expressions, (3.25) gives the energy of the negative-helicity particle eigenstates and (3.26) gives the energy of the positive-helicity anti-particle eigenstates. These are in agreement with the usual result and show that the neutral current contribution “factorizes” in the relativistic limit; that is, the difference between the two negative-helicity particle energies is independent of $\beta$. Furthermore, it is clear that, if $\alpha>0$ (which occurs if the background contains more electrons than positrons) then a resonance can occur when $4\alpha p = \Delta^2 \cos(2\theta)$. This is the
well-known MSW resonance. The remaining dispersion relations, given in Eq. (3.27), are unchanged from their vacuum values (since the potential is left-handed) and are related to the positive-helicity neutrinos and negative-helicity anti-neutrinos.

**B. Majorana Case**

The case of Majorana neutrinos is interesting for two reasons. First of all, it is the favoured realistic scenario in models which have massive neutrinos, for example in models which employ the “see-saw” mechanism. Secondly, it turns out that the equations governing the dispersion relations are *quadratic* rather than *quartic*, which means that in principle they should be easier to analyze.

The calculation proceeds in a manner similar to that followed in Sec. II C, the only complication being the additional mixing in flavour space. We omit the details and simply present the result. The negative-helicity dispersion relations in this case are determined by the equation

\[
\left(\omega^2 - p^2 - \Delta^2_+(p)\right)\left(\omega^2 - p^2 - \Delta^2_-(p)\right) = 0,
\]

(3.28)

where

\[
\Delta^2_{\pm}(p) = \frac{1}{2} \left(m_1^2 + m_2^2 + 4p(\alpha - 2\beta) + 4\alpha(\alpha - 2\beta) + 8\beta^2\right)
\]

\[\pm \frac{1}{2} \left\{ \left[(m_2^2 - m_1^2)\cos(2\theta) - 4\alpha p - 4\alpha(\alpha - 2\beta)\right]^2
\]

\[+ (m_2 - m_1)^2 \left[ (m_1 + m_2)^2 + 4\alpha^2 \right] \sin^2(2\theta) \right\}^{1/2}.
\]

(3.29)

Thus the four solutions are

\[
\omega = \pm \sqrt{p^2 + \Delta^2_+(p)},
\]

(3.30)

\[
\omega = \pm \sqrt{p^2 + \Delta^2_-(p)}.
\]

(3.31)

These again reduce to Mannheim’s result if we set $\beta=0$ [2]. It is interesting to note that these solutions are not functions only of $m_1^2 + m_2^2$ and $m_2^2 - m_1^2$, as is the case in the relativistic regime.
Fig. 5(b) shows a plot of the dispersion relations for Majorana neutrinos in a medium. The dotted and solid curves correspond to the cases with no coupling and with $\theta = 0.2$, respectively. Again the curves with non-zero coupling are similar to those with no coupling, except for the “level repulsion” which occurs in the former case. The parameters in this plot are identical to those in the analogous plot for Dirac neutrinos shown in Fig. 5(a). Clearly the dispersion relations are quite different in the two cases.

In all cases examined the minimum of the dispersion relations is always greater than or equal to the minimum mass and so again there appears to be no trapping in the Majorana case.

For relativistic neutrinos the exact expressions for the energies may be simplified somewhat to give

$$\omega \simeq p - (2\beta - \alpha) + \frac{\mu^2}{2p} \pm \frac{1}{4p} \left[(4\alpha p - \Delta^2 \cos(2\theta))^2 + \Delta^4 \sin^2(2\theta)\right]^{1/2}, \quad (3.32)$$

$$\omega \simeq -p + (2\beta - \alpha) - \frac{\mu^2}{2p} \pm \frac{1}{4p} \left[(4\alpha p - \Delta^2 \cos(2\theta))^2 + \Delta^4 \sin^2(2\theta)\right]^{1/2}, \quad (3.33)$$

the first of which is in agreement with the analogous expression, Eq. (3.25), for negative helicity neutrinos in the Dirac case.

C. The Vector Model

In the models considered so far, the fact that the dispersion relations have minima at non-zero values of the momentum is due to the chiral nature of the potential in the effective Dirac equation. That is, since the potential depends on the spin, the curves are displaced to the right or left depending on whether the neutrino’s spin is parallel or anti-parallel to its momentum. As we have seen, this phenomenon occurs for a single neutrino flavour and persists when another flavour is added. It is amusing to note that it is possible to obtain a minimum at non-zero momentum even with a purely vector interaction, although this effect requires the presence of at least two neutrino fields. Such a model was studied several years ago by Chang and Zia [12]. The effective Lagrangian is in this case given by
\[
\left\{ \psi - M - \alpha Q \gamma^0 \right\} \psi = 0, 
\]

(3.34)

where the matrix \( Q \) is as defined in Eq. (3.12). The above equation is similar to Eq. (3.1) except for the absence of the \((1 - \gamma^5)\) factor which was present in that case. We have also set \( \beta = 0 \) for simplicity. The equations governing the dispersion relations may be derived in a manner similar to the Dirac case above to yield

\[
\left[ \left( \omega - \alpha \cos^2 \theta \right)^2 - p^2 - m_1^2 \right] \left[ \left( \omega - \alpha \sin^2 \theta \right)^2 - p^2 - m_2^2 \right] \\
= 2\alpha^2 \sin^2 \theta \cos^2 \theta \left( \omega^2 + p^2 - \alpha \omega + m_1 m_2 \right) + \alpha^4 \sin^4 \theta \cos^4 \theta 
\]

(3.35)

which is independent of the spin and is symmetric under \( p \rightarrow -p \). A-priori it might then seem impossible to generate a minimum at non-zero \( p \). Indeed, for a single neutrino flavour this is the case. For two flavours, however, something very interesting can happen. Suppose we first set \( \theta \) to zero and imagine increasing \( \alpha \) by so much that the negative energy \( \nu_e \) solution overlaps with the positive energy \( \nu_\mu \) solution. When a non-zero coupling is included, these levels repel each other and minima develop near the former crossing points, symmetrically placed about the origin. This feature is illustrated in Fig. 6. Thus in this case as well it is possible to have minima in the dispersion relations at non-zero values of the momentum. Note that this case is still somewhat different from the chiral cases which we have studied above, since the first and second derivatives at the origin are zero and negative, respectively, corresponding to a negative effective mass at the origin. This is not the case in chiral theories.

\textbf{D. Neutrino Oscillations}

We have so far mostly restricted our attention to an investigation of the forms of the dispersion relations themselves and have not considered in detail the effects that these would have on the oscillations of neutrinos. We first note that in the relativistic regime, the standard MSW results are recovered; that is, (i) the neutral current contribution factorizes
and (ii) the negative-helicity states obtain the appropriate dispersion relations in matter while the positive-helicity states revert to their vacuum dispersion relations.

For non-relativistic neutrinos, however, the situation is in some sense far more interesting. One novel effect which arises in the Dirac case purely as a result of the chiral nature of the potential is that in principle one could observe neutrino oscillations with only a single neutrino flavour. This could happen since, for non-relativistic neutrinos, the left-handed interactions responsible for producing neutrinos would produce both negative- and positive-helicity neutrinos. Since these propagate with different phase velocities in the medium, they would in general get out of phase with each other, producing oscillations in the probability to detect left-handed neutrinos. For relativistic neutrinos this effect disappears since the amplitude to produce and detect a positive-helicity neutrino becomes negligible. Note, however, that the difference in phase velocities remains and would lead to oscillations if only positive-helicity neutrinos could be produced and detected.

The generalization of this effect to the two-neutrino case gives the result that in general there could be oscillations between four different states. This would lead to an oscillation probability which is a superposition of four different oscillation curves.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have examined the coherent interactions of a neutrino with a background medium by examining the solutions of the effective Dirac equation for the neutrino. A close analysis revealed that the dispersion relations corresponding to such a Dirac equation have a non-trivial form, even in the simple case in which there is only a single neutrino flavour. In particular, we have examined the interesting effects which arise due to the minimum of the dispersion relation which occurs for non-zero momentum. We have shown that, quite generally for Dirac neutrinos, the minimum value of the energy is less than the neutrino’s mass, which implies that for any such background there will be trapping of very low energy neutrinos. In cases in which the strength of the potential exceeds twice the rest mass of
the neutrino in vacuum, neutrino–antineutrino pairs are produced by the electron density gradient at the boundary and this affects the reflection amplitude of a low energy neutrino in the medium. Our analysis of the case of a single Majorana neutrino flavour showed that Majorana neutrinos are not trapped by the medium.

We have also presented a study of the dispersion relations for Dirac and Majorana neutrinos for the Standard Model with two neutrino flavours. In this case we found that the trapping phenomenon persists in the Dirac case but is absent in the Majorana case. The neutral current contribution to the oscillation probabilities “factorizes” in the relativistic regime, but not in the non-relativistic case. We saw, in fact, that in principle neutrino oscillations could occur with only a single flavour of neutrino, due to the different phase velocities of the helicity eigenstates.

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REFERENCES

[1] L. Wolfenstein, Phys. Rev. D 17 (1978) 2369; Phys. Rev. D 20 (1979) 2634;
    S.P. Mikheyev and A.Yu. Smirnov, Yad. Fiz. 42 (1985) 1441 [Sov. J. Nucl. Phys. 42 (1985) 913]; Il Nuovo Cimento C 9 (1986) 17.

[2] P.D. Mannheim, Phys. Rev. D 37 (1988) 1935.

[3] J.F. Nieves, Phys. Rev. D 40 (1989) 866.

[4] D. Nötzold and G. Raffelt, Nucl. Phys. B 307 (1988) 924.

[5] J. Pantaleone, Phys. Lett. B 268 (1991) 227; Phys. Rev. D 46 (1992) 510.

[6] A. Loeb, Phys. Rev. Lett. 64 (1990) 115.

[7] Note that while the trapping effect which we discuss may be difficult to investigate directly via experiment, there could be less direct effects due to the trapped neutrinos which are nevertheless of interest. There has, for example, been a recent discussion concerning the possible effect of trapped (Dirac) neutrinos on long-range interactions in stars: A.Yu. Smirnov and F. Vissani, `hep-ph/9604443`.
    A.Yu. Smirnov, `hep-ph/9611465`.
    As. Abada, M.B. Gavela and O. Pène, Phys. Lett. B 387 (1996) 315;
    E. Fischbach, Ann. Phys. 247 (1996) 213.

[8] K.A. Kiers, “A study of neutrino propagation and oscillations both in vacuum and in dense media,” Ph.D. thesis at the University of British Columbia, 1996.

[9] An interesting discussion on the interplay between particle reflection at an interface and pair production may be found in: A. Hansen and F. Ravndal, Phys. Scripta 23 (1981) 1036.

[10] P.D. Mannheim, Int. J. Theor. Phys. 23 (1984) 643.

[11] P.B. Pal and T.N. Pham, Phys. Rev. D 40 (1989) 259.
[12] L.N. Chang and R.K.P. Zia, Phys. Rev. D 38 (1988) 1669.
FIG. 1. One-loop diagrams contributing to the neutrino self-energy in the model of Sec. II.

FIG. 2. Dispersion relations for the model considered in Sec. II A, with $\alpha=3$ and $m=0,1$, in arbitrary units. The dashed and solid curves correspond, respectively, to the $m=0$ and $m\neq0$ cases. Note how the curves in the massive case result from the level repulsion at the level crossings of the massless case.
FIG. 3. Dispersion relations for a neutrino (a) outside and (b)-(d) inside the medium in the simple model of Sec. II A, with \( m = 1 \) and \( \alpha = 0, 0.8, 1.3 \) and 2.5 for (a), (b), (c) and (d), respectively.
FIG. 4. Density-dependent one-loop diagrams contributing to the neutrino self-energy in a realistic model with both neutral-current and charged-current couplings. In (a) the “f” stands for the contributions due to all fermions in the medium which have neutral current couplings. In (b) we have assumed that the medium contains electrons and positrons, but no other charged leptons.
FIG. 5. Dispersion relations for two neutrinos in (a) the Dirac case (negative helicity) and (b) the Majorana case. In both cases we have set $\alpha=1.0$, $\beta=2.5$, $m_1=0.5$ and $m_2=2.0$, in arbitrary units. The dotted and solid curves correspond to $\theta=0$ and 0.2, respectively.
FIG. 6. Dispersion relations in the two-neutrino vector model, with $\alpha=3.0$, $m_1=0.5$ and $m_2=1.0$, in arbitrary units. The dotted and solid curves correspond to $\theta=0$ and 0.2, respectively.