The singularity of the anharmonic oscillator

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Abstract. We have shown that a general solution to an anharmonic oscillation equation of a physical pendulum considered as a nonlinearity parameter function has a singularity at zero. This implies that applying different methods of regular decompositions in accordance with this parameter is impossible. We have developed a perturbation theory by using a new parameter of smallness in transformed canonical variables.

1. Introduction.
Anharmonic oscillator is a physical model which is widely used in various fields of physics, e.g. in classical nonlinear optics \cite{1}-\cite{3} anharmonic oscillator is the basis of the phenomenological theory of medium nonlinear susceptibility.

In this paper we have considered anharmonic oscillations of a physical pendulum. This model is quite common and convenient because there is no need in zero dimension of the original equation.

As it is known, anharmonic oscillations of a physical pendulum are described by the equation:
\begin{equation}
\frac{1}{\omega^2} \frac{d^2 \phi}{dt^2} = -\phi + \phi^3 / 6,
\end{equation}
were $\omega$ - is a cyclic frequency, which is determined by the given length of the pendulum. The right part is the expansion of function $-\sin \phi$, in which the first two members are retained.

Now in (1) we pass on to dimensionless time $\tau = \omega t$ and we receive:
\begin{equation}
\dot{\phi} = -\phi + \lambda \phi^3,
\end{equation}
where $\lambda = 1/6$ is an anharmonicity parameter. The point denotes derivation on dimensionless time. Value $\lambda = 1/6$ is the parameter of infinitesimal in regular expansions various methods of \cite{4} - \cite{6}.

It will be shown that the general solution of equation (2) singularity when $\lambda = 0$. This deperturbation of solution means that it is impossible to obtain the solution of the unperturbed oscillator in the form of harmonic oscillations with the passage to the limit $\lambda \to 0$.

Therefore, the general solution of equation (2) cannot be obtained by perturbation theory starting with linear equations. The right side of equation (2) satisfies the Cauchy theorem, so there exists a general solution. As a preliminary in (2) we pass on to a new dependent variable $q = \phi^2 / 2$. So we get:
\begin{equation}
2q \ddot{q} - \dot{q}^2 = -4q^2 + 8\lambda q^3.
\end{equation}

Let us demonstrate the validity of theorem.

Theorem. The general solution of equation (3), considered as a parameter function $\lambda$ has a pole of the first order at zero.
Proof of theorem. Equation (3) is equivalent to the following system of two equations of the first order:

\[
\dot{q} = 2pq,
\]

\[
\dot{p} = -p^2 - 1 + 2\lambda q.
\]

Indeed, by differentiating (4) and substituting (4) and (5) into equation, we find

\[
\dot{q} = -2q + 4\lambda q^2 + 2p^2 q.
\]

Substituting \( p = \dot{q}/2q \) from (4) we get (3). For the proof of theorem from (5) we express:

\[
q = \frac{(p + p^2 + 1)}{2\lambda}.
\]

Now, by substituting (6) into (4), we get an equation of second order for \( p \):

\[
\ddot{p} = 2p(p^2 + 1).
\]

This equation does not contain \( \lambda \), so its general solution \( p = f(t, C_1, C_2) \), after being substituted in (6) proves the theorem. The obvious solution of the last equation is \( p = \frac{\dot{q}}{2q} \), then from (6) it follows that \( q = 1/\lambda \cos^2 \tau \) is a simple illustration of the theorem.

2. The perturbation theory
A consequence of the above theorem is to substitute \( \varphi \to \varphi/\sqrt{\lambda} \) in the original equation (2) which leads (2) to the form:

\[
\ddot{\varphi} = -\varphi + \varphi^3,
\]

which does not contain the parameter of infinitesimal. Considering (8) as the Newton's equation, we find the potential energy:

\[
W = \varphi^2/2 - \varphi^4/4,
\]

which has a maximum value of \( E = 1/4 \) and it may be the parameter of infinitesimal in the new perturbation theory. Now in (9) we pass on to the new dependent variable \( q = \varphi^2/2 \). We obtain the equation:

\[
2q\ddot{q} - \dot{q}^2 = -4q^2 + 8q^3,
\]

which is equivalent to the system of two equations of the first order:

\[
\ddot{q} = 2pq,
\]

\[
\dot{p} = -p^2 + 1 + 2q,
\]

which is Hamilton system with a Hamiltonian:

\[
H = (p^2 + 1)q - q^2,
\]

which allows to apply a well developed theory of canonical transformations with its powerful generating function. It should be noted that we have previously discussed the perturbation theory in the equation of Hamilton-Jacobi [7] in the action-angle variables with the old parameter of infinitesimal \( \lambda \). As a result, we can find corrections to the unperturbed energy quite fast. But these corrections change the coordinates of the action-angle hamper a cumbersome solution of equations with significant coefficients in the Fourier series. The results of this perturbation theory are suitable for quantum anharmonic oscillator with its quasi-classical consideration. We are primarily interested in the law of anharmonic oscillations, which is important, for example, in nonlinear optics for determination of the nonlinear electric susceptibility.

Now in (11), (12), (13) we pass on to the interaction representation, considering \( H_0 = (p^2 + 1)q \) to be a free Hamiltonian. For this case we use the theory of canonical transformations. As it follows from Jacobi-Poincare theorem, if a twice differentiable function \( S(q, p, \tau) \) exists, the one that \( \frac{\partial^2 S}{\partial q \partial \overline{p}} \) is not identical to zero, then transformation \( (q, p) \leftrightarrow (\overline{q}, \overline{p}) \) generated by this function is:

\[
p = \partial S / \partial q; \quad \overline{q} = \partial S / \partial \overline{p}
\]

is a canonical function. And the new Hamiltonian function has the form:
In the theory of canonical transformations for two-dimensional phase space we will consider four types of generating functions that depend on one new and one old variable \([8]\). For further purposes, the most appropriate is the function that depends on the old momentum and new coordinates:

\[
\tau(q, \bar{q}, p, \bar{p}, r) = \frac{\partial F}{\partial q} \bar{p} - \frac{\partial F}{\partial \bar{q}},
\]

and the new Hamiltonian function:

\[
\bar{H}(q, \bar{q}, \bar{p}, \bar{r}) = H(q(q(q, \bar{p}, \bar{q}, r)), p(q(q, \bar{q}, p, r), r) - \frac{\partial F}{\partial r}(p(q(q, \bar{q}, p, r)), \bar{q}, r).
\]

Now we pass on from the dynamic system \(\{q, p, H\}\) to the system \(\{\bar{q}, \bar{p}, \bar{H}\}\) in the interaction representation with the help of the generating function \(F(q, \bar{q}, p, \bar{p}, r)\), the one that the new Hamiltonian function is only determined by the perturbation in the new variables. This is done by using generating functions of the form:

\[
F(q, \bar{q}, p, \bar{p}, r) = \arctan(p + \bar{q} r),
\]

which is a complete integral of the unperturbed equation of Hamilton-Jacobi: \(\partial F / \partial r = (p^2 + 1)F / \partial p\). The clear form of transformations follows from equations (16):

\[
q = q(q(q, \bar{p}, \bar{q}, \bar{r})), \quad p = \tan(p(q(q, \bar{q}, p, r), r), \bar{q}, r),
\]

The form of the new Hamiltonian function follows from equations (17):

\[
\bar{H} = -\bar{q}^2 \cos^4(p - \bar{r}),
\]

New Hamiltonian equations:

\[
\frac{\partial \bar{H}}{\partial \bar{q}} = \frac{\partial \bar{p}}{\partial \bar{p}}, \quad \frac{\partial \bar{p}}{\partial \bar{q}} = \frac{\partial \bar{H}}{\partial \bar{p}},
\]

determine the dynamics of the new canonical variables \(\bar{q}\) and \(\bar{p}\):

\[
\frac{\partial \bar{q}}{\partial \tau} = 4\bar{q}^2 \cos^3(p - \bar{r}) \sin(p - \bar{r}), \quad \frac{\partial \bar{p}}{\partial \tau} = 2\bar{q}^2 \cos^4(p - \bar{r}).
\]

As in the interaction representation the Hamiltonian depends on the time in combination \((p - \bar{r})\), the first integral of the system (22) is easy to find:

\[
\bar{q} - \bar{q}^2 \cos^4(p - \bar{r}) = E.
\]

Indeed, having taken the total time derivative of this expression we receive an identical expression with respect to (20) and (21). From (23) we get:

\[
\bar{q} = \frac{1 - \sqrt{1 - 4E \cos^4(p - \bar{r})}}{2 \cos^3(p - \bar{r})}.
\]

For the final solution we will find \(\bar{p}\) as a function of time. For this let us take a second transformation from \(\{\bar{q}, \bar{p}, \bar{H}\}\) to \(\{Q, P, \hat{H} = 0\}\), which implies that \(Q = E = \text{const}; P = \text{const}\), and the generating function of the transformation \(F(p, Q, r)\) satisfies the equation \(\bar{H}(\partial F / \partial \bar{p}, \bar{p}) = \partial F / \partial r\).

Now expressions

\[
\bar{q} = \partial F / \partial \bar{p}, \quad P = \partial F / \partial Q
\]

in implicit form set the first integrals of the Hamiltonian system (21). The equation for the generating function based on (20) has the form:

\[
\frac{\partial F}{\partial \tau} = \left(\frac{\partial F}{\partial \bar{p}}\right)^2 \cos^4(p - \bar{r}).
\]

Let us factor the full integral \(F(p, Q, r)\) in the series to the powers \(Q = E:\)

\[
F = F_0 + QF_1 + Q^2 F_2 + \ldots
\]
Now let us substitute this expansion into equation (26). By equating expressions with equal powers of \( Q \) on the left and on the right, we get the system of equations:

\[
\frac{\partial F_2}{\partial \tau} = \left( \frac{\partial F_0}{\partial \tau} \right)^2 \cos^4(\bar{p} - \tau), \quad \frac{\partial F_1}{\partial \tau} = -2\left( \frac{\partial F_0}{\partial \tau} \right) \cos^4(\bar{p} - \tau), \quad \frac{\partial F_2}{\partial \tau} = -2\left( \frac{\partial F_0}{\partial \tau} \right) \cos^4(\bar{p} - \tau) + \left( \frac{\partial F_1}{\partial \tau} \right)^2,
\]

etc.

As the original equation (8) was obtained in the first order decomposition of the sinus, then in this system it is sufficient to use the second order. The solution of this system is not difficult: from the first equation we find \( \frac{\partial F_0}{\partial \bar{p}} = 1/\cos^4(\bar{p} - \tau) \), by substituting this into the last two equations and by solving them, we get \( F_1 = \bar{p} - 2\tau, \quad F_2 = \int \cos^4(\bar{p} - \tau)d(\bar{p} - \tau) = -\frac{3}{8}(\bar{p} - \tau) - \frac{1}{4}\sin 2(\bar{p} - \tau) - \frac{1}{32}\sin 4(\bar{p} - \tau) \).

According to the second equation (25) \( P = \frac{\partial F}{\partial Q} \), taking into account (27), we get:

\[
P = P + 2\tau + \frac{3Q}{4}(P + \tau) + \frac{Q}{16}\sin 2(P + \tau) + \frac{Q}{16}\sin 4(P + \tau).
\]

(28)

The problem is solved. Arbitrary constants \( Q \) and \( P \) have a simple physical meaning: \( Q = E \leq 1/4 \); \( P = -\tau_0 \), where \( \tau_0 \) - the initial moment of time. Now we have to go back to the original variable \( \phi \) from the equation (2):

\[
\phi = \sqrt{1 - \frac{1 - 4E \cos^4(\bar{p} - \tau)}{\lambda \cos^2(\bar{p} - \tau)}}.
\]

(29)

Expression (28) should be substituted here, but this would make the final result extremely cumbersome.

3. Conclusions

Thus, the problem of finding the general solution of equation (2) is reduced by using the theory of Hamiltonian systems to the two expressions (28) and (29), the first of which is approximate, and the second is exact. Given the known internal decomposition of the radical in a power series we come to the conclusion that the right-hand side of the formula (29) is a continuous function of infinitesimal time \( \tau \) and of the parameter \( E \) and it has a singularity at zero on the anharmonicity parameter \( \lambda \).

4. References

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