Motivated by an intention to apply partial wave analysis to systems containing photons, we construct eigenvectors of the distance of separation between two photons. A choice is made that makes the case of two photons most closely resemble the case of two massive particles. The treatment is relativistic. An Hermitian separation observable can then be defined, unlike a position operator for a single photon, which cannot be defined. We test these results on a model system of two photons meant to describe the initial or final state of a scattering experiment. We find that the separation probability density gives a meaningful picture of localization.

I. INTRODUCTION

It has long been known that it is not possible to construct point-localized state vectors for individual photons that satisfy the criteria of Newton and Wigner [1]. We consider that these criteria capture the essence of what it means to have a point-localized state vector in quantum mechanics, so we have no intent, in this paper, of challenging those criteria. Some authors (Kiessling and Tahvildar-Zadeh [2], Bialynicki-Birula and Bialynicka-Birula [3], Hawton [4], Pryce [5]) claim to have produced a position operator for the photon, but these are defined to act on proposed wavefunctions rather than on the basis \{ |k, \lambda \rangle \} of momentum-helicity eigenvectors. In this paper, the observable we introduce has matrix elements between direct products of photon momentum-helicity eigenvectors.

The reason for this negative result is not the masslessness of the photon, as point-localized state vectors for a hypothetical massless, spin-0 particle could be constructed. Instead, the reason is the limited helicity spectrum, \( \lambda = \pm 1 \), of the photon (Wightman [6]). Because of the way eigenvectors of helicity rotate, it is not possible to form an irreducible representation of rotations about the localization point without having an extra, \( \lambda = 0 \), state.

Yet it is clear from physical results that photons can be partially localized in wavepacket states. From a theoretical perspective, we would like to perform partial wave analysis on, for example, two-particle states including one or two photons. To find the phase shifts requires finding the asymptotic form, as the separation grows without bound, of amplitudes involving basis vectors with particle separation as an eigenvalue.

Thus we are led to find eigenvectors of photon separation rather than individual positions. We will do this using eigenvectors of total four-momentum (\( P \)), total centre of mass (CM) frame angular momentum (quantum numbers \( J \) and \( M \)) and two CM-frame helicities (\( \lambda_{1}^{CM} \) and \( \lambda_{2}^{CM} \)). In these basis vectors, the rotation properties of the helicity-carrying photons are “hidden”. The state vectors rotate only according to the \( J^{2} = J(J+1) \) representations in the CM frame.

A reasonable choice, introduced as a postulate, is made to quantify these concepts. Then the consequences of this choice are calculated for a model system. The result is a useful measure of separation distance for two photons, indistinguishable, for the case considered, from the result for two massive particles.

Throughout this paper, we use Heaviside-Lorentz units, in which \( \hbar = c = \epsilon_{0} = \mu_{0} = 1 \).

II. THE SEPARATION OF TWO MASSIVE PARTICLES

The following construction can only be done for two massive particles, and provides a comparison with the one- or two-photon cases. We want to define separation eigenvectors in terms of eigenvectors of total four-momentum, \( P \rightarrow P^{\mu} \) for \( \mu = 0, 1, 2, 3 \) and total CM-frame angular momentum (quantum numbers \( J, M \)), since we can construct similar eigenvectors for the one- and two-photon cases, allowing that comparison.

We consider two different individual particle translations applied to a free two-particle eigenvector of the individual
momenta and the rest-frame spin z-components (eigenvalues $\mu_1$ and $\mu_2$ for spins $s_1$ and $s_2$, respectively):

$$U(T_1(+r/2))U(T_2(-r/2))\left|\frac{1}{2}P + p, \mu_1, \frac{1}{2}P - p, \mu_2; f\right\rangle = \left|\frac{1}{2}P + p, \mu_1, \frac{1}{2}P - p, \mu_2\right\rangle e^{-ip\cdot r}.$$  \hfill (1)

(We are using a “noncovariant” orthonormalization for the basis vectors (Fong and Rowe [7]). See eq. (6) below.)

This transformation changes the separation of the two particles by $r$ but leaves the average position unchanged. So we define simultaneous eigenvectors of separation in the CM frame (where the argument of the spherical harmonic is here a unit vector, not an operator). We note that

$$r^\text{CM} = i \frac{\partial}{\partial p} = i \frac{\partial}{\partial p_1} - i \frac{\partial}{\partial p_2}$$

and $\hat{P}$ commute and both commute with the spin operators.

We form eigenvectors of the magnitude of the separation distance and of orbital angular momentum in the CM frame

$$\left|r^\text{CM}, 0, \mu_1, \mu_2; f\right\rangle = \int \frac{d^4p}{(2\pi)^2} \left|+p, \mu_1, -p, \mu_2; f\right\rangle e^{-ip\cdot r^\text{CM}},$$

noting that

$$\hat{r}^\text{CM} = i \frac{\partial}{\partial p} = i \frac{\partial}{\partial p_1} - i \frac{\partial}{\partial p_2}$$

and $\hat{P}$ commute and both commute with the spin operators.

We form eigenvectors of the magnitude of the separation distance and of orbital angular momentum in the CM frame,

$$\left|r^\text{CM}, l, m, 0, \mu_1, \mu_2; f\right\rangle = \int \frac{d^2r^\text{CM}}{(2\pi)^2} \left|r^\text{CM}, 0, \mu_1, \mu_2; f\right\rangle Y_{lm}(\hat{r}^\text{CM}),$$

where the argument of the spherical harmonic is here a unit vector, not an operator. We note that $\hat{L} = \hat{r} \times \hat{p}$, defined in the CM frame, also commutes with $\hat{P}$ and the spin operators.

Note that these basis vectors will have calculable but very complicated Lorentz transformation properties, as with any position eigenvectors (Newton and Wigner [1]). For our purposes it will suffice to ignore the total three-momentum label in what follows.

Then we combine the spins using Clebsch-Gordan coefficients (Messiah [8]), producing a set of representations with quantum numbers $S$ in the range $|s_1 - s_2| \leq S \leq s_1 + s_2$. Finally we combine this added spin angular momentum with the orbital angular momentum to find

$$\left|r^\text{CM}, J, M, 0, l, S; f\right\rangle = \int \frac{d^2r^\text{CM}}{(2\pi)^2} \left|r^\text{CM}, 0, \mu_1, \mu_2; f\right\rangle Y_{lm}(\hat{r}^\text{CM}) \langle s_1 s_2 \mu_1 \mu_2 | S M_S \rangle \langle l S m M_S | J M \rangle \text{ for } J \geq |s_1 - s_2|,$$

with summations implied over $\mu_1, \mu_2, m$ and $M_S$.

If we choose the orthonormalization

$$\langle +p_a, \mu_1a, -p_a, \mu_2a; f| p_b, \mu_1b, -p_b, \mu_2b; f \rangle = \delta_{\mu_1a\mu_1b} \delta_{\mu_2a\mu_2b} \delta^3(p_a - p_b),$$

then we find

$$\langle r^\text{CM}_a, J_a, M_a, l_a, S_a; f| r^\text{CM}_b, J_b, M_b, l_b, S_b; f \rangle = \delta_{J_aJ_b} \delta_{M_aM_b} \delta_{l_al_b} \delta_{S_aS_b} \delta(r^\text{CM}_a - r^\text{CM}_b),$$

confirming $r^\text{CM}$ as a Hermitian observable. Note that if we were dealing with identical bosons or fermions, symmetrization or antisymmetrization, respectively, could be applied.

A similar construction could be used to form eigenvectors of momentum magnitude and orbital angular momentum in the $P = 0$ CM frame, starting with

$$\left|p^\text{CM}, l, m, \mu_1, \mu_2; f\right\rangle = \int \frac{d^2p}{(2\pi)^2} \left|p, \mu_1, -p, \mu_2; f\right\rangle p Y_{lm}(\hat{p}).$$

Then we find the relation between eigenvectors of momentum magnitude and eigenvectors of separation magnitude:

$$\langle r^\text{CM}_b, J_b, M_b, l_b, S_b; f| p^\text{CM}_a, J_a, M_a, l_a, S_a; f \rangle = \delta_{J_aJ_b} \delta_{M_aM_b} \delta_{l_al_b} \delta_{S_aS_b} \sqrt{\frac{2}{\pi}} \frac{p^\text{CM}_a}{p^\text{CM}_b} Y_{lm}(p^\text{CM}_a p^\text{CM}_b),$$

\hfill (9)
in terms of the free spherical waves, \( y_i^{(f)}(r, p) = \sqrt{2 \pi} pr j_i(pr) \), familiar from partial wave analysis of the radial, nonrelativistic, Schrödinger equation (Messiah [8]). But nowhere in its derivation here did we specify the nonrelativistic dependence of energy on momentum. They obey the differential equation

\[
\left\{ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right\} y_i^{(f)}(r, p) = p^2 y_i^{(f)}(r, p).
\]

There is no mass in this equation. It is not saying \( p^2/2m \) is the energy for some mass, \( m \). It is merely saying that the square of the individual-particle momenta in the CM frame is \( p^2 \) for all partial waves with \( l \geq 0 \). As an equation in the centre of mass frame, it is consistent with special relativity.

### III. THE SEPARATION OF TWO PHOTONS

For a two-photon state vector, it is not possible to define orbital and spin angular momenta separately, so the previous construction is not possible. Instead, we use the helicity formalism (Jacob and Wick [9], Macfarlane [10]), which produces basis vectors \( |P, J, M, \lambda_1^CM, \lambda_2^CM; f : \gamma \gamma \rangle \). In particular, in the CM frame,

\[
|k_1^{CM}, 0, J, M, \lambda_1^{CM}, \lambda_2^{CM}; f : \gamma \gamma \rangle = \int d^2 k_1^{CM} \left| + k_1^{CM}, \lambda_1^{CM}, -k_1^{CM}, \lambda_2^{CM}; f : \gamma \gamma \rangle e^{-i\lambda_2^{CM}(2\gamma_1^{CM} + \pi)} \sqrt{\frac{2J + 1}{4\pi}} R_0^{(J^*)}_{M_1^{CM} - \lambda_2^{CM}} |k_1^{CM} \rangle,
\]

for \( J \geq |\lambda_1^{CM} - \lambda_2^{CM}| \), where \( \theta_1^{CM} \) and \( \varphi_1^{CM} \) are the spherical polar angles of particle 1 and \( R_1^{(J^*)}_{M_1^{CM} - \lambda_2^{CM}} |k_1^{CM} \rangle \) are matrix elements of the unitary representation of the rotation

\[
R_0(k_1^{CM}) = R_z(+\varphi_1^{CM}) R_y(\theta_1^{CM}) R_z(-\varphi_1^{CM}).
\]

The eigenvalue of total energy is \( 2k^{CM} \), we merely renormalized to eigenvalue \( k^{CM} \). Again we factor out the dependence on total three-momentum in what follows.

Our physical expectation is that the separation probability density has no dependence on spin or helicity, at least for large particle separations. Thus we propose a definition that makes the case of two photons most closely resemble the case of two massive particles (for \( r^{CM} > 0 \)),

\[
(r^{CM}, J_a, M_a, \lambda_1^{CM_a}, \lambda_2^{CM_a}; f : \gamma \gamma | k^{CM}, J_b, M_b, \lambda_1^{CM_b}, \lambda_2^{CM_b}; f : \gamma \gamma \rangle = \delta_{J_a J_b} \delta_{M_a M_b} \delta_{\lambda_1^{CM_a} \lambda_1^{CM_b}} \delta_{\lambda_2^{CM_a} \lambda_2^{CM_b}} \sqrt{\frac{2}{\pi}} k^{CM} r^{CM} j_\ell(k^{CM} r^{CM}),
\]

which defines the separation eigenvectors, in abbreviated form, as

\[
|r^{CM}, Q; f : \gamma \gamma \rangle = \int_0^\infty dk^{CM} |k^{CM}, Q; f : \gamma \gamma \rangle \sqrt{\frac{2}{\pi}} k^{CM} r^{CM} j_\ell(k^{CM} r^{CM}),
\]

guaranteeing orthonormality, since

\[
\int_0^\infty dk \sqrt{\frac{2}{\pi}} kr j_\ell(kr_a) \sqrt{\frac{2}{\pi}} kr j_\ell(kr_b) = \delta(r_a - r_b).
\]

Here \( Q = J, M, \lambda_1^{CM}, \lambda_2^{CM} \) represents the other quantum numbers.

We are left with the question of what to choose for \( \ell \). No orbital angular momentum quantum number appears in these expressions. We argue that a suitable choice is

\[
\ell = \ell(J, \lambda_1^{CM}, \lambda_2^{CM}) = J - |\lambda_1^{CM} - \lambda_2^{CM}|.
\]

It is always integral and its spectrum, from the condition attached to eq. (11), is \( \ell = 0, 1, 2, \ldots \).

The observable with eigenvalues \( r^{CM} \) is then given by

\[
r^{CM} = \int_0^\infty d\rho^{CM} \sum_Q |\rho^{CM}, Q; f : \gamma \gamma \rangle r(\rho^{CM}, Q; f : \gamma \gamma \rangle.
\]

Note that the case of one photon and one massive particle could be easily treated using this method.
From our experience with the massive case, and use of the measure of localization introduced in Hoffmann [11], we expect that the state vector (not yet symmetrized)

\[
|\psi(k_0, R, \lambda_1, \lambda_2)\rangle = \int d^3k_1 \int d^3k_2 \left|\begin{array}{c} \lambda_1, k_1, \lambda_2, k_2 \end{array}\right\rangle e^{-\frac{|k_1-k_0|^2}{4\sigma_k^2}} e^{i k_1 \cdot R/2} e^{-\frac{|k_2+k_0|^2}{4\sigma_k^2}} e^{-i k_2 \cdot R/2}
\]

represents one photon with average momentum \(k_0 = +k_0 \hat{z}\) \((k_0 > 0)\) and a small spread in momentum, chosen so that \(\sigma_k \ll k_0\), partially localized at average position \(-R\hat{z}/2\) with large spatial width, \(\sigma_x\) (with \(\sigma, \sigma_k = \frac{1}{\sqrt{2}}\)), and helicity \(\lambda_1\), and another photon with average momentum \(-k_0\) partially localized around \(+R\hat{z}/2\) with helicity \(\lambda_2\), at time zero. These state vectors are mutually orthogonal for different helicities and are normalized to unity if

\[
\langle k_{1a}, \lambda_{1a}, k_{2a}, \lambda_{2a} | k_{1b}, \lambda_{1b}, k_{2b}, \lambda_{2b} \rangle = \delta_{\lambda_{1a}\lambda_{1b}} \delta_{\lambda_{2a}\lambda_{2b}} \delta^3(k_{1a} - k_{1b}) \delta^3(k_{2a} - k_{2b}).
\]

Using similar methods to those employed in Hoffmann [12], these state vectors can be written on a basis of eigenvectors of total four-momentum and total CM frame angular momentum as

\[
|\psi(k_0, R, \lambda_1, \lambda_2)\rangle = \int_0^\infty dk \int d^3P \sum_{J=|\lambda_1-\lambda_2|}^{\infty} \left|\begin{array}{c} \kappa, P, J, \lambda_1 - \lambda_2, \lambda_1, \lambda_2 \end{array}\right\rangle e^{-(k-k_0)^2/2\sigma_k^2} \left(\frac{\pi \sigma_k^2}{\sigma_k^2}\right)^{\frac{3}{4}} e^{i k \cdot P} e^{i n R} e^{i \sqrt{2 J + 1}} e^{-\frac{3}{4} (J + \frac{1}{2})^2/2} e^{\frac{2J}{4\pi \sigma_k^2}}.
\]

As usual, we can ignore the dependence on \(P\). Note that, in this model, the average total three-momentum vanishes with a small spread, so the frame is effectively a CM frame and we have left off the CM labels from the helicities. We have changed quantum numbers from total energy, \(E\), to \(\kappa = E/2\), which measures the magnitude of momentum of either photon.

The behaviour of the CM-frame \(PJM\) basis vectors under particle exchange, \(E\), is found to be

\[
U(E) |\begin{array}{c} \kappa, 0, J, \lambda_1 - \lambda_2, \lambda_1, \lambda_2 \end{array}\rangle = |\begin{array}{c} \kappa, 0, J, \lambda_1 - \lambda_2, \lambda_2, \lambda_1 \end{array}\rangle (-)^{J+\lambda_1-\lambda_2}.
\]

Then the exchange-symmetric state vector is

\[
|\psi(k_0, R, \lambda_1, \lambda_2), S\rangle = \int_0^\infty dk \sum_{J=|\lambda_1-\lambda_2|}^{\infty} \frac{1}{\sqrt{2}} \left(\begin{array}{c} |\begin{array}{c} \kappa, J, \lambda_1 - \lambda_2, \lambda_1, \lambda_2 \end{array}\rangle + |\begin{array}{c} \kappa, J, \lambda_1 - \lambda_2, \lambda_2, \lambda_1 \end{array}\rangle (-)^{J+\lambda_1-\lambda_2}\end{array}\right) \times e^{-(k-k_0)^2/2\sigma_k^2} \left(\frac{\pi \sigma_k^2}{\sigma_k^2}\right)^{\frac{3}{4}} e^{i k \cdot P} e^{i n R} e^{i \sqrt{2 J + 1}} e^{-\frac{3}{4} (J + \frac{1}{2})^2/2},
\]

very nearly normalized to unity for these largely distinguishable individual-particle wavepackets.

Note that we do not apply symmetrization to the separation eigenvectors, as we want the probabilities of distinguishable measurements. In the overlap between these and the state vectors of eq. (22), only the case \(M = \lambda_1 - \lambda_2\) will contribute. We find

\[
\langle r, J, \lambda_1 - \lambda_2, \lambda_1, \lambda_2 | \psi(p, R, \lambda_1, \lambda_2), S\rangle = A(r, J, \lambda_1, \lambda_2) + \delta_{\lambda_1\lambda_2} A_e(r, J, \lambda_1, \lambda_1),
\]

where

\[
A(r, J, \lambda_1, \lambda_2) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{d\kappa}{\pi \kappa} \frac{1}{J^{|\lambda_1-\lambda_2|}(\kappa) \kappa^2} e^{-(k-k_0)^2/2\sigma_k^2} \left(\frac{\pi \sigma_k^2}{\sigma_k^2}\right)^{\frac{3}{4}} e^{i k \cdot P} e^{i n R} e^{i \sqrt{2 J + 1}} e^{-\frac{3}{4} (J + \frac{1}{2})^2/2}
\]

and

\[
A_e(r, J, \lambda_1, \lambda_1) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{d\kappa}{\pi \kappa} \frac{1}{J(\kappa) \kappa^2} \left(\frac{\pi \sigma_k^2}{\sigma_k^2}\right)^{\frac{3}{4}} e^{i k \cdot P} e^{i n R} e^{-\frac{3}{4} (J + \frac{1}{2})^2/2}.
\]
we chose $R/\sigma_x = 1/\sqrt{\epsilon}$, where $\epsilon = \sigma_k/p$ is the fractional average momentum spread, chosen so small that $\sqrt{\epsilon}$ is also small. A suitable choice is $\epsilon = 0.001$ ($\sqrt{\epsilon} = 0.03$), with sensible results. Then
\[ \frac{R}{\sigma_x} \gg 1 \quad \text{and} \quad pR = \frac{1}{2\epsilon^2} \gg 1. \] (26)
Exploring this regime will simplify our calculations. Exploring the small $R$ regime would require more difficult calculations, and will not be done here.

We will see shortly that the contributions to the amplitudes in eqs. (24,25) come mainly from $\kappa \approx p$ and $r \approx R$. In that region $\kappa r \gg 1$ so we can use the asymptotic approximation (Abramowitz and Stegun [13])
\[ \sqrt{\frac{2}{\pi}} \kappa r J_i(\kappa r) = \sqrt{\frac{2}{\pi}} \sin(\kappa r - \ell \sqrt{\frac{2}{\pi}}). \] (27)
In $\psi = \{\exp(+i\psi) - \exp(-i\psi)\}/2i$, only the $\exp(-i\psi)$ term will contribute significantly, giving a peak at $r = R$. Then we can evaluate the $\kappa$ integrals to find
\[ A(r,J,\lambda_1,\lambda_2) = \frac{1}{\sqrt{2}} e^{(J-|\lambda_1-\lambda_2|-1)\pi/2} \epsilon e^{-i\kappa_0 r} \frac{e^{-(r-R)^2/8\sigma^2}}{(4\pi\sigma^2)^{3/2}} \epsilon \sqrt{2J} + 1 e^{-\epsilon^2(J+\ell^2)/2} \] (28)
and
\[ A_e(r,J,\lambda_1,\lambda_2) = \frac{1}{\sqrt{2}} e^{(J-1)\pi/2} e^{-i\kappa_0 r} \frac{e^{-(r-R)^2/8\sigma^2}}{(4\pi\sigma^2)^{3/2}} (-)^J \epsilon \sqrt{2J} + 1 e^{-\epsilon^2(J+\ell^2)/2}. \] (29)

Then we want the total separation probability density, summed over $J$,
\[ \rho(r,\lambda_1,\lambda_2) = \sum_{J=|\lambda_1-\lambda_2|}^{\infty} |A(r,J,\lambda_1,\lambda_2) + \delta_{\lambda_1,\lambda_2} A_e(r,J,\lambda_1,\lambda_1)|^2 \]
\[ = \frac{e^{-(r-R)^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \sum_{J=|\lambda_1-\lambda_2|}^{\infty} \epsilon^2 (2J+1) e^{-\epsilon^2(J+\ell^2)^2} + \]
\[ \delta_{\lambda_1,\lambda_2} \frac{e^{-(r-R)^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \sum_{J=|\lambda_1-\lambda_2|}^{\infty} (-)^J \epsilon^2 (2J+1) e^{-\epsilon^2(J+\ell^2)^2}. \] (30)
The difference between taking the $J$ sums from $|\lambda_1-\lambda_2|$ and from 0 is at most $O(\epsilon^2)$. The first sum then evaluates to unity while the last sum evaluates to $3.7 \times 10^{-7}$ for $\epsilon = 0.001$. So
\[ \rho(r,\lambda_1,\lambda_2) \rightarrow \frac{e^{-(r-R)^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}, \] (31)
a useful measure of the distribution in separation distance, normalized to unity. We note that a different choice of $\ell$ would still lead to a relative phase of unity between the two terms in eq. (23).

V. CONCLUSIONS

It was our aim to construct eigenvectors of photon separation distance, consistent with quantum mechanics and special relativity. This was done with a choice of free spherical waves in the matrix element between momentum magnitude and separation distance eigenvectors. It was also necessary to choose the “orbital angular momentum” index of those free spherical waves. These can be considered postulates of the theory. They were designed to make the photon case as close as possible to the case of massive particles.

The construction was tested on a model problem meant to describe either the initial or final state of a scattering experiment. We found that this predicted a localized probability density in separation, although in this case the spread of separation was very large.
The consequence is that one can now choose photon wavepackets with an appropriate parameter, $R$, to describe a scattering experiment, knowing that the initial average separation of the wavepackets is meaningfully given by $R$.

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