LARGE DEVIATION PRINCIPLE FOR EMPIRICAL MEASURES OF MULTITYPE RANDOM NETWORKS

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Abstract. In this article we study the stochastic block model also known as the multi-type random networks (MRNs). For the stochastic block model or the MRNs we define the empirical group measure, empirical cooperative measure and the empirical locality measure. We derive large deviation principles for the empirical measures in the weak topology. These results will form the basis of understanding asymptotics of the evolutionary and co-evolutionary processes on the stochastic block model.

1. Introduction

Networks are complex objects and may be sometimes difficult to study. The role of network structure in information Science, Social Science and behavioural science is well understood. In behavioural Science (such as Sociology, Economics) and applied statistics the study of social bonds among actors is a classical field, known as social network analysis. See Wasserman and Faust (1994). Modelling the evolution of networks is central to our understanding of large communication systems, and more general, modern economics and social systems. See Hellman and Staudigl (2014). The research on social network and economic networks is interdisciplinary and the number of proposed models is huge.

Often the sites in a network can be classified to belong to certain groups. A most common feature of social networks is the phenomenon of homophily. Thus, sites of similar attributes are more likely to be linked with each other. Fienberg, Meyer and Wasserman (1985) introduced stochastic blocking models, where sites are categorized to come from a certain subgroup. Every subgroup may have its own law of network formation. More recently, this type of networks has also been used in the field of Economics, see example (Golub, 2012), where it has been called a multitype random networks.

In this paper we use large deviation technique to obtain asymptotic results for multi-type random networks. See, Doku-Amponsah and Moerters (2010), Doku-Amponsah (2015) for similar results for coloured random graphs. To be specific, we find in this article joint large deviation principles for empirical group measure and the empirical co-operative measure, and the empirical co-operative measure and the empirical locality measure of the multitype random networks in the weak topology. We obtain these results using (Doku-Amponsah and Moerters 2010, Theorem 2.1 and Theorem 2.3).

Our main motivation for studying this model is two-fold. In one hand we try to improve the presentations of the LDPs for the inhomogeneous random graphs by other researcher and on the other hand we working towards understanding the asymptotics of the evolution and co-evolution processes on random networks. Thus, LDPs develop for empirical measures of the multitype random networks.

Mathematics Subject Classification : 94A15, 94A24, 60F10, 05C80
Keywords: Multitype random network, large deviation principles, empirical measures, relative entropy.
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networks will form the basis for understanding evolution as the building block of the construction of co-evolution model of networks and play.

1.1 Empirical Measures of the Multitype Random Networks

By $\Omega$ we denote a finite alphabet and denote by $\mathcal{N}(\Omega)$ the space of counting measure on $\Omega$ equipped with the discrete topology. By $\mathcal{M}(\Omega)$ we denote the space of probability measures on $\Omega$ equipped with the weak topology and $\mathcal{M}_*(\Omega)$ the space of finite measures on $\Omega$ equipped with the weak topology.

Let $\kappa_n, \ell_n : \Omega \times \Omega \to [0, \infty)$ be continuous functions and $\eta : \Omega \to (0, 1)$ a probability measure. The Multitype Random Network is defined as follows:

(i) Partition the set of sites, $[n] = \{1, 2, 3, ..., n\}$, into finitely many types $\Omega = \{a_1, a_2, a_3, ..., a_m\}$ independently according to the type law $\eta$.

(ii) For a couple of sites $(i, j)$ take the intensities of link formation and link destruction to be $w_{ij}^{(n)}(a, b) = (1 - A_{ij})\kappa_n(a, b)$, $\nu_{ij}^{(n)}(a, b) = A_{ij}\ell_n(a, b)$, whenever site $i$ is a member of group $a$, and site $j$ is a member of group $b$.

We shall consider $Z := \{(Z(i), i \in \Omega), E\}$ under the joint law of the type and graph, and interpret $Z(i)$ as the type of site $i$ and $Z$ as multi-type random graph. We shall refer to the multitype random graph $Z$ as symmetric if both functions $\kappa_n$ and $\ell_n$ are symmetric otherwise we call it asymmetric multitype random graph.

This model is a special case of the inhomogeneous random graph, see example Hellman and Staudigl [13], where edge specific probabilities between members of group $a$ and group $b$ are given by

$$p_n(a, b) = \frac{\kappa_n(a, b)}{\kappa_n(a, b) + \ell_n(a, b)}.$$ 

The multi-type random graph, unlike the inhomogeneous random graph, is completely specified by the grouping of the sites, and the group-specific link-success probabilities.

With each multitype random graph $Z$ we associate a probability distribution, the empirical type measure $L^1 \in \mathcal{M}(\Omega)$, by

$$L^1(a) := \frac{1}{n} \sum_{j=1}^{n} \delta_{Z(j)}(a), \quad \text{for } a \in \Omega,$$

and a finite measure, the empirical cooperative measure $L^2 \in \mathcal{M}_*(\Omega \times \Omega)$, by

$$L^2(a, b) = \begin{cases} \frac{1}{n} \sum_{(i,j) \in E} [\delta(Z(i), Z(j)) + \delta(Z(j), Z(i))] \left( a, b \right) & \text{if } Z \text{ is symmetric} \\ \frac{2}{n} \sum_{(i,j) \in E} \delta(Z(i), Z(j)) \left( a, b \right) & \text{if } Z \text{ is asymmetric.} \end{cases}$$

Notice, $\|L^2\| = 2|E|/n$ in both the symmetric and asymmetric cases of the multitype random graph. We define a probability distribution, the empirical neighbourhood measure $M^1 \in \mathcal{M}(\Omega \times \mathcal{N}(\Omega))$, by

$$M^1(a, l) := \frac{1}{n} \sum_{j=1}^{n} \delta_{Z(j), L(j)}(a, l), \quad \text{for } (a, l) \in \Omega \times \mathcal{N}(\Omega),$$

where $L(i) = (l^i(b), b \in \Omega)$ and $l^i(b)$ is the number of sites of type $b$ linked to site $i$. 
2. Main Results

We assume throughout the paper that the intensities of link formation and link destruction satisfies
\[ n\kappa(a,b)/\ell_n(a,b) \to \kappa(a,b)/\ell(a,b), \] while \( \kappa, \ell : \Omega \times \Omega \to [0, \infty) \). We write \( \frac{\pi}{T} \rho \otimes \rho(a,b) = \frac{k(a,b)}{\ell(a,b)} \rho(a)\rho(b) \) for \( a, b \in \Omega \) and
\[ \mathcal{D}_{\kappa/\ell}(\pi \| \rho) := H(\pi \| \frac{\pi}{T} \rho \otimes \rho) + \| \frac{\pi}{T} \rho \otimes \rho - \pi \| \).
It is not hard to see that \( \mathcal{D}_{1}(\pi \| \omega) \geq 0 \) and equality holds if and only if \( \pi = \frac{\omega}{\ell} \omega_1 \otimes \omega_1 \).

**Theorem 2.1.** Suppose \( G \) is a multitype random network with grouping law \( \eta : \Omega \to [0,1] \). Assume, that the intensities of link formation and link destruction are
\[ w_{ij}(a,b) = (1 - A_{ij})\kappa_n(a,b), \quad \nu_{ij}(a,b) = A_{ij}\ell_n(a,b), \]
where \( \kappa : \Omega \times \Omega \to [0, \infty) \) and \( \ell : \Omega \times \Omega \to (0, \infty) \), whenever \( i \) is a member of group \( a \) and site \( j \) is a member of group \( b \), for \( a, b \in \Omega \). Then, the pair \((L^2, M)\) an LDP obeys in the space \( \mathcal{M}(\Omega \times \Omega) \times \mathcal{M}(\Omega \times \mathcal{N}(\Omega)) \) with speed \( n \) and with good rate function
\[ J^1(\pi, \omega) = \begin{cases} H(\omega \| q^1) + H(\omega_1 \| \eta) + \frac{1}{2}\mathcal{D}_{2}(\pi \| \omega_1) & \text{if } (\pi, \omega) \text{ consistent and } \omega_1 = \pi_2, \\ \infty & \text{otherwise.} \end{cases} \]
where
\[ q^1(a, \rho) := \omega_1(a) \prod_{b \in \Omega} \frac{e^{-\left[\pi(a,b)/\mu_1(a)\right]}}{\rho(b)!} \left[\frac{\pi(a,b)/\omega_1(a)}{\rho(b)}\right]^\rho(b) \]

**Corollary 2.2.** Suppose \( \deg_Z \) is the degree measure of \( Z \), an Erdos-Renyi graph model. Assume, that the intensities of link formation and link destruction are
\[ w_{ij} = (1 - A_{ij})\kappa, \quad \nu_{ij} = A_{ij}\ell. \]

- Then, \( \deg_Z \) satisfies an LDP in the space \( \mathcal{M}(N \cup \{0\}) \) with speed \( n \) and good rate function
\[ \lambda(d) = \begin{cases} H(d \| q(d)) + \frac{1}{2} \langle d \rangle \log \left( \frac{\ell(d)}{\kappa} \right) - \frac{1}{2} \langle d \rangle + \frac{\kappa}{d^2} & \text{if } \langle d \rangle < \infty, \\ \infty & \text{if } \langle d \rangle = \infty. \end{cases} \quad (2.1) \]
where \( q(c) = \frac{e^{-c/k}}{k!} \), \( k = 0, 1, 2, \ldots \) and \( \langle d \rangle = \sum_{k=0}^{\infty} kd(k) \).

- Then, the proportion of isolated vertices, \( \deg_Z(0) \) satisfies an LDP in \([0,1]\) with good rate function
\[ \lambda_z = z \log z + \log \left( \frac{z}{t} \right) (1-z/2) - (1-z) \left[ \log \left( \frac{z}{t} \right) - \frac{(a-\frac{1}{e})(1-z)^2}{2} \right], \]
where \( t = t(z) \) is the unique positive solution of \( 1 - e^{-t} = \frac{z}{t} (1-z) \).

**Remark 1** Note, typically, as \( n \to \infty \), the proportion of isolated sites in an Erdos-Renyi graph converges to \( e^{-n\kappa/\ell} \) in probability, i.e.
\[ \lim_{n \to \infty} \mathbb{P}\left\{ \left| \deg_X(0) - e^{-n\kappa/\ell} \right| \geq \varepsilon \right\} = 0. \]

**Remark 2** Note, typically, as \( n \to \infty \), the proportion of isolated sites in an Erdos-Renyi graph converges to \( e^{-n\kappa/\ell} \) in probability, i.e.
\[ \lim_{n \to \infty} \mathbb{P}\left\{ \left| \deg_X(0) - e^{-n\kappa/\ell} \right| \geq \varepsilon \right\} = 0. \]
Theorem 2.3. Suppose \( Z \) is a multitype random network with grouping law \( \eta : \Omega \to [0, 1] \). Assume, that the intensities of link formation and link destruction are

\[ w_{ij}(a, b) = (1 - A_{ij})\kappa(a, b), \quad \ell_{i,j}(a, b) = A_{ij}\ell(a, b), \]

where \( \kappa : \Omega \times \Omega \to [0, \infty) \) and \( \ell : \Omega \times \Omega \to (0, \infty) \), whenever site \( i \) is a member of group \( a \) and site \( j \) is a member of group \( b \) for \( a, b \in \Omega \). Then, \( L^2 \) obeys the following large deviation principle:

- for any \( F \) closed subset of \( \mathcal{M}(\Omega \times \Omega) \), we have

\[
\limsup_{n \to \infty} \frac{1}{n} n \log P_\omega \left\{ L^2 \in F \right\} \leq - \inf_{\pi \in F} I_1(\omega, \pi)
\]

Remark 3 Note, typically, as \( n \to \infty \), the proportion of isolated sites in an Erdos-Renyi graph converges to \( e^{-n\kappa/\ell} \) in probability, i.e.

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \left| \text{deg}_X(0) - e^{-n\kappa/\ell} \right| \geq \varepsilon \right\} = 0.
\]

Remark 4 We observe that as \( n \to \infty \), given \( \langle d \rangle = \kappa/\ell \) the degree distribution converges \( q_{\kappa/\ell} \) in probability, i.e.

\[
\lim_{n \to \infty} \sup_{k \in \mathbb{N} \setminus \{0\}} \mathbb{P} \left\{ \left| \text{deg}_Z(k) - \frac{e^{-\kappa/\ell}(\kappa/\ell)^k}{k!} \right| \geq \varepsilon \right\} = 0.
\]

Theorem 2.4. Suppose \( Z \) is a multitype random network with grouping law \( \eta : \Omega \to [0, 1] \). Assume, that the intensities of link formation and link destruction are

\[ w_{ij}(a, b) = (1 - A_{ij})\kappa(a, b), \quad \ell_{i,j}(a, b) = A_{ij}\ell(a, b), \]

where \( \kappa : \Omega \times \Omega \to [0, \infty) \) and \( \ell : \Omega \times \Omega \to (0, \infty) \), whenever site \( i \) is a member of group \( a \) and site \( j \) is a member of group \( b \) for \( a, b \in \Omega \). Then, the pair \( (L^1, L^2) \) obeys an LDP in the space \( \mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \) with speed \( n \) and with good rate function

\[ I(\rho, \pi) = H(\rho \parallel \eta) + \frac{1}{2} \delta_{\kappa/\ell}(\pi \parallel \rho) \]

Remark 5 We observe that as \( n \to \infty \), given \( L^1 = \rho \), the number edges per site \( L^2 \) converges \( \frac{\kappa}{\ell} \rho \otimes \rho \) in probability, i.e.

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \left| L^2 - \frac{\kappa}{\ell} \rho \otimes \rho \right| \geq \varepsilon \left| L^1 = \rho \right\} = 0.
\]

We write \( I_1(\rho, \pi) = \frac{1}{2} \delta_{\kappa/\ell}(\pi \parallel \rho) \) and state the following Large deviation principle for \( L^2 \).

Theorem 2.5. Suppose \( Z \) is a multitype random network with grouping law \( \eta : \Omega \to [0, 1] \). Assume, that the intensities of link formation and link destruction are

\[ w_{ij}(a, b) = (1 - A_{ij})\kappa(a, b), \quad \ell_{i,j}(a, b) = A_{ij}\ell(a, b), \]

where \( \kappa : \Omega \times \Omega \to [0, \infty) \) and \( \ell : \Omega \times \Omega \to (0, \infty) \), whenever site \( i \) is a member of group \( a \) and site \( j \) is a member of group \( b \) for \( a, b \in \Omega \). Then, \( L^2 \) obeys the following large deviation principle:

- for any \( F \) closed subset of \( \mathcal{M}(\Omega \times \Omega) \), we have

\[
\limsup_{n \to \infty} \frac{1}{n} n \log P_\omega \left\{ L^2 \in F \right\} \leq - \inf_{\pi \in F} I_1(\omega, \pi)
\]
for any $\Gamma$ open subset of $\mathcal{M}(\Omega \times \Omega)$, we have
\[
\liminf_{n \to \infty} \frac{1}{n} \log P_\omega \left\{ L^2 \in F \right\} \geq - \inf_{\pi \in F} I_1(\omega, \pi)
\]

### 3. Proof of Theorem 2.5

The technique use in this section is routed in spectral potential theory and it is a summary of many large deviation theories. To begin, for any multitype random network $z$ conditioned to have empirical type measure $L^1 = \omega$ we define the spectral potential
\[
\rho_{\kappa/\ell}(g, \omega) = -\langle (1 - e^g), \kappa \ell \omega \otimes \omega \rangle
\]
We observe from [5] that $\rho_{\kappa/\ell}$ possesses all the remarkable properties mention by [1] and the following Lemma holds for $L^2$.

**Lemma 3.1.** Suppose $Z$ is a multitype random network with grouping law $\eta : \Omega \to [0, 1]$. Assume, that the intensities of link formation and link destruction are
\[
w_{ij}(a, b) = (1 - A_{ij})\kappa(a, b), \quad \ell_{i,j}(a, b) = A_{ij}\ell(a, b),
\]
where $\kappa : \Omega \times \Omega \to [0, \infty)$ and $\ell : \Omega \times \Omega \to (0, \infty]$, whenever site $i$ is a member of group $a$ and site $j$ is a member of group $b$ for $a, b \in \Omega$. Then, $L^2$ obeys the following local large deviation principle:

- for any $\pi \in \mathcal{M}(\Omega \times \Omega)$ and a number $\epsilon > 0$, there is a weak neighbourhood $B_\pi$ such that
  \[
P_\omega \left\{ L^2 \in B_\pi \right\} \leq e^{-nI_1(\omega, \pi) - n\epsilon + o(n)}
  \]
- for any $\pi \in \mathcal{M}(\Omega \times \Omega)$, any number $\epsilon > 0$ and a fine neighbourhood $B_\pi$, we have the asymptotic estimate
  \[
P_\omega \left\{ L^2 \in B_\pi \right\} \geq e^{-nI_1(\omega, \pi) + n\epsilon - o(n)}
  \]

**Lemma 3.2** below, which provides a summaries of the properties of the Kullback action was first presented in the paper [5]. To state it, we denote by $C$ the space of continuous functions $g : \Gamma \times \Gamma \to \mathbb{R}$.

**Lemma 3.2** ([5]). The following holds for the Kullback action or divergence function $S_{\kappa/\ell}(\pi \parallel \omega)$.

1. $S_{\kappa/\ell}(\pi \parallel \omega) = \frac{1}{2} \sup_{g \in C} \left\{ \langle g, \pi \rangle - \rho_{\kappa/\ell}(g, \omega) \right\}$.
2. The function $S_{\kappa/\ell}(\pi \parallel \omega)$ is lower semi-continuous on the space $\mathcal{P}_*(\Gamma \times \Gamma)$.
3. For any real $c$, the set $\left\{ \nu \in \mathcal{P}_*(\Gamma \times \Omega) : S_{\kappa/\ell}(\pi \parallel \omega) \leq c \right\}$ is weakly compact.

Please we refer to [1] for similar result and proof for the empirical measures on measurable spaces. Note that Lemma 3.2 (i) above implies the so-called variational principle. See, example [10].

### 3.1 Proof of Lemma 3.1

We write $p_n(a, b) := \frac{\kappa_n(a, b)}{\kappa_n(a, b) + \ell_n(a, b)}$ and $\tilde{p}_n(a, b) := \frac{\tilde{\kappa}_n(a, b)}{\tilde{\kappa}_n(a, b) + \tilde{\ell}_n(a, b)}$. Recall the definition of $\tilde{\kappa}_n$ from [4] and note from Lemma 3.2 that, for any $\epsilon > 0$ there exists a function $g \in \mathcal{M}(\Omega \times \Omega)$ such that
\[
S_{\kappa/\ell}(\pi \parallel \omega) - \frac{\epsilon}{2} < \langle g, \pi \rangle - \rho_{\kappa/\ell}(g, \omega).
\]
We define the probability distribution $\tilde{P}_n$ by
Now we define a neighbourhood of the functional $\pi$ as follows:

$$B_{\pi} = \left\{ \varpi \in \mathcal{M}(\Omega \times \Omega) : \langle g, \varpi \rangle > \langle g, \pi \rangle - \frac{\varepsilon}{2} \right\}.$$ 

Therefore, under the condition $L^1 \in B_{\pi}$ we have that

$$\frac{dP_{\omega}(z)}{dP_{\omega}(Z)} < e^{\frac{1}{2}(\rho_{n/\ell}(g, \omega) - (g, \pi) + \frac{\varepsilon}{2})} e^{-n\delta_{n/\ell}(\pi \parallel \omega) + n\varepsilon}.$$ 

Hence, we have

$$P_{\omega}\left\{ z \in \mathcal{G}([n], \Omega) | L^2_z \in B_{\pi} \right\} \le \int_{\mathcal{G}([n], \Omega)} \mathbb{1}_{\{L^2_z \in B_{\pi}\}} dP_{\omega}(z) \le \int_{\mathcal{G}([n], \Omega)} \mathbb{1}_{\{L^2_y \in B_{\pi}\}} e^{-n\delta_{n/\ell}(\pi \parallel \omega) - n\varepsilon} dP_{\omega}(z) \le e^{-n\delta_{n/\ell}(\pi \parallel \omega)/2 - n\varepsilon}.$$ 

Note that $\delta_{n/\ell}(\pi \parallel \omega) = \infty$ implies Lemma 3.1(ii) and so it suffice to prove that for a probability measure of the form $\pi = e^{\sigma} T_{\omega} \otimes \omega$, where the Kullback action $\delta_{n/\ell}(\pi \parallel \omega) = \langle g, \pi \rangle + \langle (1 - e^{\sigma}) \frac{\omega}{\ell \omega} \otimes \omega \rangle$ is finite. Fix any number $\varepsilon > 0$ and any neighbourhood $B_{\pi} \subset \mathcal{L}(\Omega \times \Omega)$. We define the sequence of sets

$$\tilde{\mathcal{G}}([n], \Omega) := \left\{ y \in \mathcal{G}([n], \Omega) : L^2_y \in B_{\pi}, \left| \langle g, L^2_y \rangle - \langle g, \pi \rangle \right| \le \frac{\varepsilon}{2} \right\}.$$ 

Observe that, for all $z \in \mathcal{G}([n], \Omega)$ we have

$$\frac{dP_{\omega}(z)}{dP_{\omega}(Z)} = e^{-n\left(\frac{1}{2} L^2, \tilde{g}\right) - n\left(\frac{1}{2} L^1 \otimes L^1, \tilde{h}_n\right) + \frac{1}{2} L^1 \otimes \tilde{h}_n} > e^{-n\left(\frac{1}{2} \pi, \log \frac{\pi}{\ell \omega \otimes \omega}\right) - n\frac{1}{2} \left(\frac{\omega}{\ell \omega} \otimes \omega, (1 - \frac{\pi}{\ell \omega \otimes \omega})\right)}.$$ 

This gives

$$P_{\omega}\left(\tilde{\mathcal{G}}([n], \Omega) \right) = \int_{\tilde{\mathcal{G}}([n], \Omega)} dP_{\omega}(z) \ge \int_{\tilde{\mathcal{G}}([n], \Omega)} e^{-n\frac{1}{2} \pi, \log \frac{\pi}{\ell \omega \otimes \omega}} - n\frac{1}{2} \left(\frac{\omega}{\ell \omega} \otimes \omega, (1 - \frac{\pi}{\ell \omega \otimes \omega})\right) + \frac{\varepsilon}{2} dP_{\omega}(z) \ge e^{-n\delta_{n/\ell}(\pi \parallel \omega)/2 + n\varepsilon} P_{\omega}\left(\tilde{\mathcal{G}}([n], \Omega) \right).$$
Using the law of large numbers we have \( \lim_{n \to \infty} \tilde{P}_\omega(\tilde{G}(\lfloor n \rfloor, \Omega)) = 1 \) which completes the proof.

The proof of Theorem 2.5 below, follows from Lemma 3.1 above using similar arguments as in [11, p. 544].

### 3.2 Proof of Theorem 2.5

**Proof.** Note that the empirical link measure is a finite measure and so belongs to some ball in \( B^* (\Omega \times \Omega) \).
Hence, without loss of generality we may assume that the set \( \Omega \) in Theorem 2.5(ii) is relatively compact.
See Lemma 3.1 (iii). Choose any \( \varepsilon > 0 \).
Then for every functional \( \pi \in \mathcal{M}(\Omega \times \Omega) \) we can find a weak neighbourhood such that the estimate of Lemma 3.1(i) holds. We choose from all these neighbourhood a finite cover of \( F \) and sum up over the estimate in Theorem 3.1(i) to obtain

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n \{ L^2_z \in F \} \leq - \inf_{\pi \in F} I_1(\omega, \pi) + \varepsilon.
\]

As \( \varepsilon \) was arbitrarily chosen and the lower bound in Lemma 3.1(ii) implies the lower bound in Theorem 2.5(ii) we have the desired results which ends the proof of the Theorem.

\[\square\]

### 4. Proof of Theorem 2.2, Proof of Theorem 2.1 and Corollary

#### 4.1 Proof of Theorem 2.2

Note that \( dP_n(\omega, \pi) := dP_\omega(\pi) dP(\omega) \) and the sequence of \( P_n \) probability measures are exponentially tight. See [7].
Moreover the function \( I_1, I \) are lower semicontinuous on the space of measures \( \mathcal{M}(\Omega \times \Omega) \).
Therefore, by the Sanov Theorem, see [8], Theorem 2.5 and the Theorem for mixing, see [2, Theorem] we \((L^1, L^2)\) satisfies an LDP with convex rate function \( I \) which completes the proof.

#### 4.2 Proof of Theorem 2.1

Similarly, we note that the law of the pair could be written as
\( dP^m(\pi, \omega) = dP_{\mu, \pi}(\omega) dP_n(\mu, \pi) \) and also the probability measures \( P(\Omega) \) are exponentially tight.
See [7].
Furthermore, the function \( J^1(\pi, \omega) \) and \( J(\mu, \pi) \) are lower semi-continuous on the space of measures on \( \mathcal{M}(\Omega \times \Omega) \times \mathcal{M}(\Omega \times N'(\Omega)) \).
Therefore, by Theorem 2.2 and the Theorem for mixing large deviation principles, see [2, Theorem], we have that \((L^2, M)\) obeys an LDP with convex good rate function \( J^1 \).
This ends the proof of the Theorem.

#### 4.3 Corollary 2.2

Corollary 2.2(i). The proof of Corollary 2.2(i) follows from Theorem 2.1 if we take \( p_n \to \kappa/\ell \) and note from the Doku-Amponsah [7, P.p. ] that \( \lambda(d) = \{ J^1(\pi, \mu) : \langle d \rangle \leq \kappa/\ell, \mu = d \} \) which produces the rate function in Corollary 2.2(i).

Corollary 2.2(ii). Similarly, the proof of the (ii) part follows from the Corollary 2.2(i) by applying the contraction principle to the linear mapping \( U(d) = d(0) \) and solve the variational problem \( \xi(y) = \inf \{ \lambda(d) : d(0) = y \} \) as in Doku-Amponsah [6, pp. ] to obtain the desired form of the rate function in Corollary 2.2(ii).
This completes the proof of the Corollary.

### Conflict of Interest

The author declares that he has no conflict of interest.
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