Considerations on some neglected but important issues concerning the Internal Linear Combination method in Astronomy

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ABSTRACT

Although the internal linear combination method (ILC) is a technique widely used for the separation of the Cosmic Microwave Background signal from the Galactic foregrounds, its characteristics are not yet well defined. This can lead to misleading conclusions about the actual potentialities and limits of such approach in real applications. Here we discuss briefly some facts about ILC that to our knowledge are not fully worked out in literature and yet have deep effects in the interpretation of the results.

Key words. Methods: data analysis – Methods: statistical – Cosmology: cosmic microwave background

1. Introduction

A widely used approach for the separation of the cosmic microwave background (CMB) from the diffuse Galactic background is the internal linear combination method (ILC). For instance, this method was adopted in the reduction of the data from the Wilkinson Microwave Anisotropy Probe (WMAP) satellite for CMB observations (Bennett et al. 2003). Its success is due to the fact that, among the separation techniques, ILC calls for the smallest number of a priori assumptions. If the data are in the form of maps, taken at different frequencies and containing \( N_o \) pixels each, the model on which ILC is based is

\[
S^{(i)}(p) = S_c(p) + S_f^{(i)}(p) + N^{(i)}(p).
\]

Here, \( S^{(i)}(p) \) provides the value of the \( p \)th pixel for a map obtained at channel \( i \), whereas \( S_c(p) \), \( S_f^{(i)}(p) \) and \( N^{(i)}(p) \) are the contributions due to the CMB, the diffuse Galactic foreground and the experimental noise, respectively. Although not necessary, often it is assumed that all of these contributions are representable by means of stationary random fields. Moreover, without loss of generality, for ease of notation the random fields are supposed as realization of zero-mean spatial processes. The basic idea behind model (1) is that, contrary to the components that form the Galactic background, ILC exploits this fact averaging \( N_o \) images \( \{ S^{(i)}(p) \}_{i=1}^{N_o} \) and giving a specific weight \( w_i \) to each of them so as to minimize the impact of the foreground and noise (Bennett et al., 2003). This means to look for a solution of type

\[
\hat{S}_c(p) = \sum_{i=1}^{N_o} w_i S^{(i)}(p).
\]

If the constraint \( \sum_{i=1}^{N_o} w_i = 1 \) is imposed, Eq. (2) becomes

\[
\hat{S}_c(p) = S_c(p) + \sum_{i=1}^{N_o} w_i [S_f^{(i)}(p) + N^{(i)}(p)].
\]

Now, from this equation it is clear that, for a given pixel \( p \), the only variable terms are in the sum. Hence, under the assumption of independence of \( S_c(p) \) from \( S_f^{(i)}(p) \) and \( N^{(i)}(p) \), the weights \( \{ w_i \} \) have to minimize the variance of \( \hat{S}_c(p) \), i.e.

\[
\{ w_i \} = \arg \min_{\{ w_i \}} \mathrm{VAR}[\hat{S}_c(p)] + \mathrm{VAR} \left[ \sum_{i=1}^{N_o} w_i (S_f^{(i)}(p) + N^{(i)}(p)) \right],
\]

where \( \mathrm{VAR}[s(p)] \) is the expected variance of \( s(p) \).

If \( S^{(i)} \) denotes a row vector such as \( S^{(i)} = [S^{(i)}(1), S^{(i)}(2), \ldots, S^{(i)}(N_p)] \) and the \( N_o \times N_p \) matrix \( S \) is defined as

\[
S = \begin{pmatrix}
S^{(1)} \\
S^{(2)} \\
\vdots \\
S^{(N_o)}
\end{pmatrix},
\]

then Eq. (1) becomes

\[
S = \hat{S}_c + S_f + N.
\]
In this case, the weights are given by (Eriksen et al. 2004)

\[ w = \frac{C^{-1}_S}{I^T C^{-1}_S I}, \tag{7} \]

where \( C_S \) is the \( N_o \times N_o \) cross-covariance matrix of the random processes that generate \( S \), i.e.

\[ C_S = E[SS^T], \tag{8} \]

and \( 1 = (1, 1, \ldots, 1)^T \) is a column vector of all ones. Here, \( E[.] \) denotes the expectation operator. Hence, the ILC estimator takes the form

\[ \hat{C}_S = w^T S, \tag{9} \]

\[ = \alpha 1^T C^{-1}_S 1, \tag{10} \]

with \( 1^T w = 1 \) and the scalar quantity \( \alpha \) given by

\[ \alpha = [1^T C^{-1}_S 1]^{-1}. \tag{11} \]

In practical applications, matrix \( C_S \) is unknown and has to be estimated from the data. Typically, this is done by means of the estimator

\[ \hat{C}_S = \frac{1}{N_p} SS^T. \tag{12} \]

In this case, the ILC estimator is given by Eqs. (9)-(11) with \( C_S \) and \( w \) replaced, respectively, by \( \hat{C}_S \) and

\[ \hat{w} = \frac{\hat{C}^{-1}_S 1}{1^T \hat{C}^{-1}_S 1}. \tag{13} \]

2. Some unfocussed points about ILC

In spite of its popularity, various questions concerning ILC appear not yet well fixed. A first issue is linked to the fact that the estimation of the power-spectrum of CMB is a problem deeply different from the separation of this component from the Galactic foreground. Indeed, the estimation of the second-order properties of a stochastic signal, though contaminated by noise, is an easier task than its recovery. Here, the point is that, if one is interested in the spatial distribution of the CMB emission, then in Eq. (1) the term \( S_{ij}^{(p)}(p) + N_{ij}^{(p)}(p) \) cannot be considered as a single noise component. Often such assumption is made (e.g., see Eriksen et al. 2004; Hinshaw et al. 2007; Delabrouille et al. 2003; Dick et al. 2009) since in this way the problem is reduced to the separation of two components only and no a priori information on the “global” noise is required. Actually, this procedure can lead to wrong conclusions. For example, since all of the component in the mixtures \( S \) are supposed to be the realization of a zero-mean random processes, from Eq. (9) one could derive that

\[ E[\hat{C}_S | S] = \hat{C}_S + w^T E[S] + w^T E[N] = \hat{C}_S, \tag{14} \]

i.e. the ILC estimator is unbiased. This is not correct: the claim that \( \hat{C}_S \) is unbiased requires to prove that

\[ E[\hat{C}_S | S, S] = \hat{C}_S + w^T S + w^T E[N] = \hat{C}_S. \tag{15} \]

The reason is that \( S_i \) is a fixed realization of a random process. There is no way to obtain another one. Even if observed many times (under the same experimental conditions) the foreground components (for instance the Galaxy) will always appear the same. Only the noise component \( N \) will change. Indeed, apart from very specific situations, in general ILC has to be expected to provide biased estimates of CMB (Vio & Andreani 2008). Here, we stress that the biased results provided by ILC is a well established fact only relatively to the estimation of the power-spectrum of CMB (e.g. Saha et al. 2008).

A second issue, often neglected in literature, is the belief that, in order to work, ILC does not require any hypothesis about \( S_i \). Again, this is not correct. As proved by Vio & Andreani (2008), even in the case of noise-free data, one may hope to obtain an unbiased estimate of the CMB only if \( S_i \) is given by a linear mixture of the contribution of \( N_c \) physical processes \( \{S_j\}_{j=1}^{N_c} \), i.e.

\[ S_i = \sum_{j=1}^{N_c} a_{ij} S_j, \tag{16} \]

with \( a_{ij} \) constant coefficients and \( N_c < N_o \). This means that for the \( j \)th physical process a template \( \hat{S}_j \) is assumed to exists which is independent of the specific channel “ \( i \)”. Moreover, the number of these channels has to be greater than that of the physical processes. Inserting Eq. (16) into Eq. (6) one obtains

\[ S = A \mathbf{S} + \mathbf{N}, \tag{17} \]

with

\[ A = \left( \begin{array}{ccc} \mathbf{S}_1 & \mathbf{S}_2 & \cdots & \mathbf{S}_{N_c} \\ 1 & a_{11} & a_{12} & \cdots & a_{1N_c} \\ 1 & a_{21} & a_{22} & \cdots & a_{2N_c} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & a_{N_{c}1} & a_{N_{c}2} & \cdots & a_{N_{c}N_{c}} \end{array} \right), \tag{18} \]

and

\[ \mathbf{S} = \left( \begin{array}{c} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_{N_c} \end{array} \right). \tag{19} \]

This is the only model that ILC is able to handle. This fact should suggest extreme caution when dealing with questions as the nonGaussianity of CMB. Indeed, the results produced by ILC are extremely sensitive to the presence of components that are not modelizable by means of Eqs. (17)-(19). By instance, without renouncing the cosmic origin of CMB, this could explain the spatial associations claimed by Verschuur (2007) between the interstellar neutral hydrogen (HI) emission morphology and small-scale structure observed by the Wilkinson Microwave Anisotropy Probe (WMAP).

A third issue is that, using the weights \( w \), it is implicitly assumed that \( C_S \) is a well conditioned matrix. However, there are various practical situations where this condition is not satisfied. By instance, this can be expected for high signal-to-noise ratio observations at high Galactic latitude, i.e. when noise \( \mathbf{N} \) is negligible and CMB is by far

\[ C_S \] has to be substituted with \( C_S - C_N \).
the dominant component with $a_{ij} \ll 1$, $i = 1, 2, \ldots, N_o$, $j = 1, 2, \ldots, N_c$. Indeed, since
\[ C_S = AC \otimes A^T, \]
when $A$ is badly conditioned it has to be expected that the same holds for $C_S$. As a consequence, an imprecise calibration of the $N_o$ maps, in such a way that the true value of the entries in the first column of $A$ are different from “1”, will be amplified with catastrophic consequences. This is what lead Dick et al. (2009) to the conclusion that, in the case of high signal-to-noise ratio observations, ILC is extremely sensitive to calibration errors. If true, this should be a quite troublesome situation since it limits the usefulness of the observations in a high signal-to-noise ratio regime. Actually, this problem can be easily avoided if in Eq. (7) $C_S^{-1}$ is substituted with the corresponding Moore-Penrose inverse $C_S^\dagger$. These arguments hold also when the weights $\hat{w}$ are used and $\hat{C}_S$ substitutes $C_S$ in Eq. (13). This is clearly visible in Figs. 1 and 2 that show the results provided by ILC in the case of a simulated high Galactic latitude observation at three different frequencies, say 30, 44, and 70 GHz. The region analyzed is a square patch (400 $\times$ 400 pixels) with side of about 24°, centered at $l = 90^\circ$, $b = 45^\circ$ (Galactic coordinates). Three components have been considered, i.e. CMB, synchrotron and dust. More details can be found in Vio et al. (2003). The effects of the different point spread function for the various frequencies have been neglected. The top panels of Fig. 1 shows the templates relative to the physical processes that have been used to form the three linear mixtures shown in the bottom panel of the same figure. No noise has been added. Perfect frequency scaling has been assumed, i.e. model (17)-(19) holds, but a calibration error of 10% has been imposed in all of the channels. The assumption of perfect frequency scaling constitutes only an approximation. However, since CMB by far constitutes the dominant component, this is of secondary importance. In fact, the condition number of the resulting $\hat{C}_S$ is $1.6 \times 10^6$, i.e. this matrix is ill-conditioned. The results obtainable by ILC using the original weights (13) and their version computed using $\hat{C}_S^\dagger$ are shown in Fig. 2. It is evident that, if not properly addressed, the ill-conditioning of $\hat{C}_S$ has catastrophic consequences on the quality of the separation.

3. Conclusions

From the analysis presented above, it is evident that the results provided by ILC have to be interpreted with extreme caution. This techniques suffers many drawbacks that, if not properly taken into account, can lead to misleading if not wrong conclusions. In particular, ILC should be used only in situations where matrix $\hat{C}_S$ is (close to be) singular, i.e. only when one can be certain that model (17)-(19) holds with $N_o > N_c + 1$. In the contrary case, the separation operated through ILC has to be expected inaccurate.

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Fig. 1. Original templates (top panels) and the corresponding three mixtures (bottom panels) used in the numerical experiments described in the text.
Fig. 2. ILC solutions obtained using, respectively, the Moore-Penrose inverse of $\tilde{C}_S$ (top-left panel) and the original weights (13) (bottom-left panel). For reference, the target CMB component is also presented (top-right panel).