How much entropy is produced in strongly coupled Quark-Gluon Plasma (sQGP) by dissipative effects?

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We argue that estimates of dissipative effects based on the first-order hydrodynamics with shear viscosity are potentially misleading because higher order terms in the gradient expansion of the dissipative part of the stress tensor tend to reduce them. Using recently obtained sound dispersion relation in thermal $N=4$ supersymmetric plasma, we calculate the resummed effect of these high order terms for Bjorken expansion appropriate to RHIC/LHC collisions. A reduction of entropy production is found to be substantial, up to an order of magnitude.

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Hydrodynamical description of matter created in high energy collisions have been proposed by Landau [1] more than 50 years ago, motivated by large coupling at small distance, as followed from the beta functions of QED and scalar theories known at the time. Hadronic matter is of course described by QCD, in which the coupling runs in the opposite way. And yet, recent RHIC experiments have shown spectacular collective flows, well described by relativistic hydrodynamics. More specifically, one observed three types of flow: (i) outward expansion in transverse plane, or radial flow, (ii) azimuthal asymmetry or “elliptic flow” [2, 3], as well as recently proposed (iii) “conical flow” from quenched jets [4]. These observation lead to conclusion that QGP at RHIC is a near-perfect liquid, in a strongly coupled regime [5]. The issue we discuss below is at what “initial time” one is able to start hydrodynamical description of heavy ion collisions, without phenomenological/theoretical contradictions.

Phenomenologically, it was argued in [2, 3] that elliptic flow is especially sensitive to $\tau_0$. Indeed, ballistic motion of partons may quickly erase the initial spatial anisotropy on which this effect is based. In practice, hydrodynamics at RHIC is usually used starting from time $\tau_0 \sim 1/2 fm$, otherwise the observed ellipticity is not reproduced.

Can one actually use hydrodynamics reliably at such short time? How large is $\tau_0$ compared to a relevant “microscopic scales” of sQGP? How much dissipation occurs in the system at this time? As a measure of that, we will calculate below the ratio of the amount of entropy produced in the system at this time? As a measure of that, we will calculate below the ratio of the amount of entropy produced in the system at this time. As a measure of that, we will calculate below the ratio of the amount of entropy produced in the system at this time?

To set up the problem, let us start with a very crude dimensional estimate. If we think that the QCD effective coupling is large $\alpha_s \sim 1$ and the only reasonable microscopic length is given by temperature $T_0$, then the relevant micro-to-macro ratio of scales is simply $T_0/\tau_0$. With $T_0 \sim 400 MeV$ at RHIC, one finds this ratio to be close to one. We are then lead to a pessimistic conclusion: at such time application of any macroscopic theory, thermo or hydro-dynamics, seems to be impossible, since order one corrections are expected.

Let us then do the first approximation, including the explicit viscosity term to the first order. Zeroth order (in mean free path) stress tensor used in the ideal hydrodynamics has the form

$$T^{(0)}_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}$$  (1)

while dissipative corrections are induced by gradients of the velocity field. The well known first order corrections are due to shear ($\eta$) and bulk ($\xi$) viscosities

$$\delta T^{(1)}_{\mu\nu} = \eta (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \Delta_{\mu\nu} \nabla_\rho u_\rho) + \xi (\Delta_{\mu\nu} \nabla_\rho u_\rho)$$  (2)

In this equation the following projection operator onto the matter rest frame was used:

$$\nabla_\mu \equiv \Delta_{\mu\nu} \partial_\nu, \quad \Delta_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu$$  (3)

The energy-momentum conservation $\partial_\mu T_{\mu\nu}$ at this order corresponds to Navier-Stokes equation.

Because colliding nuclei are Lorentz-compressed, the largest gradients at early time are longitudinal, along the beam direction. The expansion at this time can be approximated by well known Bjorken rapidity-independent setup [6], in which hydrodynamical equations depend on only one coordinate – proper time $\tau = \sqrt{t^2 - x^2}$.

$$\frac{1}{\epsilon + p} \frac{dc}{d\tau} = \frac{1}{s} \frac{ds}{d\tau} = -\frac{1}{\tau} \left( 1 - \frac{4/3}{\epsilon + p} \right)$$  (4)

where we have introduced the entropy density $s = (\epsilon + p)/T$. Note that for traceless $T_{\mu\nu}$ (conformally invariant plasma), the bulk viscosity $\xi = 0$.

For reasons which will become clear soon, let us compare this eqn to another problem, in which large longitudinal gradients appear as well, namely sound wave in the medium. The dispersion relation (the pole position) for a sound wave with frequency $\omega$ and wave vector $q$ is, at small $q$

$$\omega = c_s q - \frac{i}{2} q^2 \Gamma_s, \quad \Gamma_s = \frac{4}{3} \frac{\eta}{\epsilon + p}$$  (5)

Notice that the right hand side of [4] contains precisely the same combination of viscosity and thermodynamical
parameters as appears in the sound attenuation problem: the length $\Gamma_s$, which measures directly the magnitude of the dissipative corrections. At proper times $\tau \sim \Gamma_s$ one has to abandon the hydrodynamics altogether, as the dissipative corrections cannot be ignored.

For the entropy production the first correction to the ideal case is $(1 - \Gamma_s/\tau)$. Since the correction to one is negative, it reduces the rate of the entropy decrease with time. Equivalently statement is that the total positive sign shows that some amount of entropy is generated by the dissipative term. Danielewicz and Gyulassy \cite{7} have analyzed eq. (4) in great details considering various values of $\eta$. Their results indicate that the entropy production can be substantial.

Our present study is motivated by the following argument. If the hydrodynamical description is forced to begin at early time $\tau_0$ which is not large compared to the intrinsic micro scale $1/T$, then limiting dissipative effects to the first gradient only ($\delta T^{(1)}_{\mu\nu}$) is parametrically not justified and higher order terms have to be accounted for. Ideally those effects need to be resummed. As a first step, however, we may attempt to guess their sign and estimate the magnitude.

Formally one can think of the dissipative part of the stress tensor $\delta T_{\mu\nu}$ as expended in a series containing all derivatives of the velocity field $u$, $\delta T^{(1)}_{\mu\nu}$ being the first term in the expansion. In general 3+1 dimensional case there are many structures, each entering with a new and independent viscosity coefficient. We call them “higher order viscosities” and the expansion is somewhat similar to twist expansion. For 1+1 Bjorken problem, the appearance of the extra terms modifies eq. (4), which can be written as a series in inverse proper time

$$\frac{\partial T(\tau s)}{s(\tau T)} = 4 \frac{\eta}{s} \left[ \frac{1}{3} \left( \frac{1}{\tau T} \right)^2 + \sum_{n=2}^{\infty} \frac{c_n}{(T/T)^{2n}} \right]$$  \hspace{1cm} (6)

We have put $T$ here simply for dimensional reasons: clearly $T/\tau$ is a micro-to-macro scale ratio which determines convergence of these series and the total amount of produced entropy. Similarly, the sound wave dispersion relation becomes nonlinear as we go beyond the lowest order:

$$\omega = \Re[\omega(q)] + i \Im[\omega(q)];$$  \hspace{1cm} (7)

$$\frac{\Re[\omega]}{2\pi T} = c_s \frac{q}{2\pi T} + \sum_{n=1}^{\infty} r_n \left( \frac{q}{2\pi T} \right)^{2n+1};$$

$$\frac{\Im[\omega]}{2\pi T} = -\frac{4\pi \eta}{s} \left[ \frac{1}{3} \left( \frac{q}{2\pi T} \right)^2 + \sum_{n=2}^{\infty} \eta_n \left( \frac{q}{2\pi T} \right)^{2n} \right]$$

Based on T-parity arguments we keep only odd (even) powers of $q$ for the real (imaginary) parts of $\omega$. The coefficients $c_n$, $r_n$ and $\eta_n$ are related since they originate from the very same gradient expansion of $T^{(1)}_{\mu\nu}$. Although both the entropy production series above and sound absorption should converge to sign-definite answer, the coefficients of the series may well be of alternating sign (as we will see shortly).

Clearly, keeping these next order terms can be useful only provided there is some microscopic theory which would make it possible to determine the values of the high order viscosities. For strongly coupled QCD plasma this information is at the moment beyond current theoretical reach, and we have to rely on models. A particularly useful and widely studied model of QCD plasma is $\mathcal{N} = 4$ supersymmetric plasma, which is also conformal (CFT). The AdS/CFT correspondence \cite{8} (see \cite{9} for review) relates the strongly coupled gauge theory description to weakly coupled gravity problem in the background of AdS$_5$ black hole metric. Remarkably, certain information on higher order viscosities in the CFT plasma can be read of from the literature and we exploit this possibility below.

The viscosity-to-entropy ratio ($\eta/s = 1/4\pi$) deduced from AdS \cite{10} turns out to be quite a reasonable approximation to the values appropriate for the RHIC data description. Thus one may hope that the information on the higher viscosities gained from the very same model can be well trusted as a model for QCD. Admittedly having no convincing argument in favor, we simply assume that the viscosity expansion of the QCD plasma displays very similar behavior, both qualitative and quantitative, as its CFT sister.

Our estimates are based on the analysis of the quasi-normal modes in the AdS black hole background due to Kovtun and Starinets \cite{11}. The dispersion relation for the sound mode, calculated in ref. \cite{11}, is shown in Fig.1 The real and imaginary parts of $\omega$ correspond to the expressions given in (7). At $q \to 0$ they agree with the leading order hydrodynamical dispersion relation (4).

The first important observation is that the next order coefficient $\eta_2$ is negative, reducing the effect of the first one when gradients are large. The second is that $|\Im[\omega]|$ has maximum at $q/2\pi T \sim 1$, and at large $q$ the imaginary part starts to decrease. This means that the expansion (4) has a radius of convergence $q/2\pi T \sim 1$.

In order to estimate the effect of higher viscosities on the entropy production in the Bjorken setup we first identify $\tau$ in (4) with $2\pi/q$ in (4). Second we identify the coefficients $c_n$ with $\eta_n$. Both sound attenuation and entropy production in question are one dimensional problems associated with the same longitudinal gradients and presumably the same physics. In practice we use the curve for the imaginary part of $\omega$ (Fig. 1) as an input for the right hand side of (4).

The numerical results are shown in Figs. 2 and 3 in which we compare our estimates with the “conventional” shear viscosity results from (4). To be fully consistent with the model we set $\eta/s = 1/4\pi$. We also set the initial temperature $T_0 = 300$ MeV while the standard equation of state $s = 4 k_{SB} T^3$. For the coefficient $k_{SB}$ we use the “QCD” value

$$k_{SB} = \frac{\pi^2}{90} \left( 2(N_c - 1)^2 + \frac{7}{2} N_f n_f \right); \hspace{1cm} n_f = 3; \hspace{1cm} N_c = 3$$

Fig. 2 presents the results for entropy production as a
function of proper time for two initial times \( \tau_0 = 0.2 \text{ fm} \) and \( \tau_0 = 0.5 \text{ fm} \). The dashed lines correspond to the first order result \( \mathcal{E}_1 \) while the solid curves include the higher order viscosity corrections. Noticeably there is a dramatic effect toward reduction of the entropy production as we start the hydro evolution at earlier times (the effect is almost invisible on the temperature profile). This is the central message of the present paper.

Fig. 3 illustrates the relative amount of entropy produced during the hydro phase as a function of initial proper time. If the fist order hydrodynamics is launched at very early times, the hydro phase produces too large amount of entropy, up to 250%. (Such a large discrepancy is not seen in the RHIC data.) In sharp contrast, the results from the resummed viscous hydrodynamics is very stable, and does not produce more than some 25% of initial entropy, even if pushed to start from extremely early times. The right figure displays the absence of any pathological explosion at small \( \tau_0 \).

It is worth commenting that we carried the analysis using the minimal value for the ratio \( \eta/s = 1/4\pi \). We expect that if this ratio is taken larger, the discrepancy between the first order dissipative hydro and all orders will be even stronger.

Before concluding this paper we note that a practical implementation of relativistic viscous hydrodynamics had followed Israel-Stewart second order formalism (for recent publications see \[12\]) in which one introduces additional parameter \( \tau_0 \), the relaxation time for the system. Then the dissipative part of the stress tensor is found as a solution of an evolution equation, with the relaxation time being its parameter. For the Bjorken setup, the dissipative tensor thus obtained has all powers in \( 1/\tau \) and might resemble the expansion in \( T \) and \( \ll 1 \). The use of AdS/CFT may shed light on the interrelation between the two approaches: the first step in this direction has been made recently \[13\], resulting in numerically very small relaxation time.

Finally, why can it be that macroscopic approaches like hydrodynamics can be rather accurate at such a short time scale? Trying to answer this central question one should keep in mind that \( 1/T \) is not the shortest microscopic scale. The inter-parton distance is much smaller, \( \sim 1/(T \ast N_{dof}^{1/3}) \) where the number of effective degrees of freedom \( N_{dof} \sim 40 \) in QCD while \( N_{dof} \sim N_c^2 \rightarrow \infty \) in the AdS/CFT approach.

In summary, we have argued that the higher order dissipative terms strongly reduce the effect of the usual viscosity. Therefore an “effective” viscosity-to-entropy ratio found from comparison Navier-Stokes results to experiment, can even be below the (proposed) lower bound of \( 1/4\pi \). We conclude that it is not impossible to use a hydrodynamic description of RHIC collision starting from very early times. In particular, our study suggests that the final entropy observed and its “primordial” value obtained right after collision should indeed match, with an
accuracy of 10-20 percent.

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