On nonlinear oscillations of time-periodic Hamiltonian systems in the presence of double fourth-order resonances

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Abstract. The motion of a time-periodic Hamiltonian system with two degrees of freedom in a neighborhood of a linearly stable equilibrium is investigated. A double fourth-order resonance (of fundamental and combination types) is assumed to be realized in the system, or the parameters of the problem are close to the resonance values. The problem of the existence and number of resonant periodic motions in a small neighborhood of the equilibrium is investigated; the conditions for their stability in the linear approximation are analyzed. The results are applied to the problem of the motion of a dynamically symmetric satellite (a rigid body) near its stationary rotation (cylindrical precession) in the central Newtonian gravitational field in an elliptical orbit of arbitrary eccentricity.

1. Introduction
The study of nonlinear oscillations of non-autonomous, time-periodic Hamiltonian systems in the vicinity of the equilibrium position in the presence of multiple resonances (of different orders) is a new and urgent problem of the theory of nonlinear oscillations. The cases of multiple resonances are often found in problems of classical and celestial mechanics. However, until recently, they were excluded from consideration due to their complexity and the lack of developed research algorithms. Each case of multiple resonances requires separate consideration, it is a multiparametric problem that depends on the coefficients of the perturbed motion Hamiltonian normalized in the vicinity of the resonant point and the values of the resonant detunings. This paper studies the case of double fourth-order resonances (of fundamental and combination types) in the Hamiltonian systems under study. The question of the existence, number, and stability (in the linear approximation) of periodic motions of the system that are analytical in a small parameter is solved. For the general case, an algorithm for investigation is proposed, which is then applied to one of the points of the considered multiple resonances in the problem of the motion of a dynamically symmetric satellite (a rigid body) with mass geometry of a plate near its stationary rotation (cylindrical precession) in the central Newtonian gravitational field in an elliptical orbit of arbitrary eccentricity.

2. Statement of the problem
We consider the motion of a non-autonomous, $2\pi$-periodic in time Hamiltonian system with two degrees of freedom. Let the phase space origin $q_j = p_j = 0$ ($j = 1, 2$) be the equilibrium
position of the system. It is stable in the linear approximation, and the characteristic exponents
$\pm i \lambda_j \ (j = 1, 2)$ of the linearized system of equations of perturbed motion are purely imaginary.

Let there be no resonances in the system up to the third order inclusive, but there is a double
fourth-order resonance (of fundamental and combination types), for which the values $4 \lambda_1$ and
$\lambda_1 + 3 \lambda_2$ (or $\lambda_1 - 3 \lambda_2$) are integers or close to integers. A list of possible variants of the pairs
$\lambda_1, \lambda_2$ for which these combinations are integers (the cases of exact resonance) is presented in
[1], where the procedure for normalizing the Hamiltonian $H(q_j, p_j, t)$ of perturbed motion in terms
up to the fourth degrees inclusive with respect to perturbations is described in detail. The resulting terms of the fifth and higher degrees with respect to perturbations are periodic in time with a period $24 \pi$.

Let us consider the cases close to the resonances under study. We pass to a small neighborhood
of the phase space origin, of the order of $\varepsilon \ (< \varepsilon \ll 1)$, assuming that $q_j = \varepsilon q'_j, \ p_j = \varepsilon p'_j$
($j = 1, 2$). We introduce the resonant detunings $\mu_j \ (j = 1, 2)$ by the formulas $\lambda_j = \lambda_{j0} - \varepsilon^2 \mu_j$,
where $\lambda_{j0}$ are the values of $\lambda_j$ in the case of the exact resonance. Let $\tau = \varepsilon^2 t$ be the
new independent variable. In the symplectic polar coordinates $R_j, \Phi_j$ ($q'_j = \sqrt{2R_j} \sin \Phi_j,$
$p'_j = \sqrt{2R_j} \cos \Phi_j, \ j = 1, 2$) the normalized Hamiltonian is written as

$$
H_\pm = -\mu_1 R_1 - \mu_2 R_2 + c_{20} R_1^2 + c_{11} R_1 R_2 + c_{02} R_2^2
+ a R_1^2 \cos 4\Phi_1 + b R_1^{1/2} R_2^{3/2} \cos (\Phi_1 \pm 3\Phi_2) + O(\varepsilon).
$$

(1)

Here $O(\varepsilon)$ is a set of terms that are periodic in $\tau$, with a period $24 \pi \varepsilon^2$. The resonance
coefficients $a$ and $b$ are assumed to be positive, which can always be achieved by shifting the angular coordinates $\Phi_1$ and $\Phi_2$. The signs + and − correspond to the cases of strong and weak
combination resonance respectively. The conditions for stability and instability in the presence
of one strong resonance of fundamental or combination type are written out in [2]. In the presence of one weak combination resonance, the trivial equilibrium of the system is formally
stable.

In this paper, for the systems with Hamiltonian functions (1), the problem of the existence
and number of resonant periodic motions that are analytic with respect to a small parameter is
investigated. The conditions for their stability in the linear approximation are analyzed. The results obtained are used in the problem of the motion of a dynamically symmetric satellite (a
rigid body) with mass geometry of a plate near its stationary rotation (cylindrical precession)
in the central Newtonian gravitational field in an elliptical orbit of arbitrary eccentricity.

### 3. Study of resonant periodic motions

At the first stage, we consider the approximate autonomous systems whose Hamiltonian
functions are obtained from (1) after dropping the last term. Equating to zero the partial
derivatives of the approximate Hamiltonian functions with respect to $R_j$ and $\Phi_j \ (j = 1, 2)$, we
compose a system of equations for finding nontrivial equilibria of the approximate systems.

These systems of equations have particular solutions for which $R_j = 0$, and the equilibrium
values of $R_1$ and $\Phi_1$ correspond to the equilibrium positions of a Hamiltonian system with one
degree of freedom at a fourth-order resonance. Such systems were previously studied in detail
[3]. This case will not be considered in this paper.

Other equilibrium positions satisfy the conditions

$$
\sin 4\Phi_1 = 0, \ \sin (\Phi_1 \pm 3\Phi_2) = 0
$$

(2)

and the system of equations

$$
-2\mu_1 u_1 + 4(c_{20} + a\delta_1)u_1^3 + 2c_{11} u_1 u_2^2 + b\delta_2 u_2^3 = 0, \ -\mu_2 + c_{11} u_2^2 + 2c_{02} u_2^2 + \frac{3}{2} b\delta_2 u_1 u_2 = 0
$$

(3)
where \( u_j = \sqrt{R_j}, (u_j > 0, \ j = 1, 2) \) and
\[
\delta_1 = \text{sign} \cos 4\Phi_1, \ \delta_2 = \text{sign} \cos(\Phi_1 \pm 3\Phi_2) \ (\delta_j = 1 \ or \ -1).
\]

To reduce the number of parameters, we introduce new notation
\[
\nu_j = \frac{\mu_j}{b\delta_2} \ (j = 1, 2), \ C_{20} = \frac{c_{20} + a\delta_1}{b\delta_2}, \ C_{11} = \frac{c_{11}}{b\delta_2}, \ C_{02} = \frac{c_{02}}{b\delta_2}
\]
and rewrite system (3) in the form
\[
-2\nu_1 u_1 + 4C_{20}u_1^3 + 2C_{11}u_1u_2 + u_2^3 = 0, \ -\nu_2 + C_{11}u_1^2 + \frac{3}{2}u_1u_2 + 2C_{02}u_2^2 = 0.
\tag{4}
\]
The parameters \( \nu_1, \nu_2, C_{20}, C_{11}, C_{02} \) can take values of any sign.

3.1. Cases \( \nu_1 = \nu_2 = 0 \)

In the case \( \nu_1 = \nu_2 = 0 \) of the exact resonance, equations (4) are homogeneous (of the third and second degrees) with respect to \( u_1 \) and \( u_2 \). Assuming \( u_2 = xu_1 \ (x > 0) \) and making a reduction, we get a system of equations
\[
x^3 + 2C_{11}x^2 + 4C_{20} = 0, \ 2C_{02}x^2 + \frac{3}{2}x + C_{11} = 0.
\tag{5}
\]
Analysis of the structure and coefficients of the cubic equation of the system shows that for \( C_{20} < 0 \) and any \( C_{11} \) this equation has one positive real root, and for
\[
C_{20} > 0, \ C_{11} < 0, \ 0 < C_{20} < \frac{8}{27} |C_{11}|^3,
\]
two such roots. In other cases, the required roots are absent.

The quadratic equation in (5) has one or two positive real roots under the conditions \( C_{11}C_{02} < 0 \) or
\[
C_{11} < 0, \ C_{11} < 0, \ C_{11}C_{02} < \frac{9}{32}
\]
respectively. In other cases, there are no required solutions.

Common roots of two equations (5) exist if their resultant in \( x \) equal to
\[
r = 256C_{02}^3C_{20}^2 - (128C_{11}^2C_{02}^2 - 144C_{11}C_{02} + 27)C_{20}^2 + 4C_{11}^3(4C_{11}C_{02} - 1),
\tag{6}
\]
vanishes. The equation \( r = 0 \) defines the relation between the coefficients \( C_{ij} \) for which solutions of system (5) exist in the studied case of exact resonance.

Consider equation (6) as quadratic with respect to \( C_{20} \), with discriminant \( D = -(32C_{11}C_{02} - 9)^3 \). For \( C_{11}C_{02} > 9/32 \) this equation has no real solutions (as the quadratic equation in (5)), and for
\[
C_{11}C_{02} < 9/32
\tag{7}
\]
it has two real solutions \( C_{20} = C_{20}^{(1,2)} \ (C_{20}^{(1)} < C_{20}^{(2)}) \).

Under the condition (7), the condition of the positivity of the common root \( x = x_+ \) of two equations (5) has been checked. It was found that in the case \( C_{11}C_{02} < 0 \), the common positive root of these equations corresponds to the smaller root \( C_{20}^{(1)} \), and in the case of \( 0 < C_{11}C_{02} < 9/32 \) each root \( C_{20}^{(1)} \) and \( C_{20}^{(2)} \) has one common positive root of equations (5). For the found positive \( x_+ \), we get the family of solutions \( (u_1, x_+u_1) \) of system (4) (for \( \nu_1 = \nu_2 = 0 \)); here \( u_1 \) is an arbitrary positive number.
3.2. Cases \( \nu_1 \neq 0, \nu_2 = 0 \) and \( \nu_1 = 0, \nu_2 \neq 0 \)
If \( \nu_1 \neq 0, \nu_2 = 0 \), we choose the real positive roots \( x = x_\ast \) of the quadratic equation in (5). By substituting \( u_2 = x_\ast u_1 \) into the first equation (4), we get the equation

\[
u_1 \left( x_\ast^3 + 2C_{11}x_\ast^2 + 4C_{20} \right) > 0.
\] (9)

This inequality allows you to choose the sign of the value \( \nu_1 \) for each found \( x_\ast \) and obtain the corresponding solution \((u_1, x_\ast u_1)\) of system (4) when \( \nu_2 = 0 \). There may be two, one, or none of these solutions.

Note that the sign of the denominator in (8) (and the sign of \( \nu_1 \)) changes for parameter values that satisfy the condition \( r = 0 \), where \( r \) is set in (6).

In the case \( \nu_1 = 0, \nu_2 \neq 0 \), similarly, we choose the real positive roots \( x = x_\ast \) of the cubic equation from (5), according to the results of Section 3.1. Then substituting \( u_2 = x_\ast u_1 \) into the second equation (4), we get

\[
u_2(2C_{02}x_\ast^2 + \frac{3}{2}x_\ast + C_{11}) > 0.
\]

provided \( \nu_2(2C_{02}x_\ast^2 + \frac{3}{2}x_\ast + C_{11}) \). This inequality determines the sign of \( \nu_2 \) for each root \( x_\ast \) obtained. Next, we compose the corresponding pairs of solutions \((u_1, x_\ast u_2)\) of system (4) with \( \nu_1 = 0 \), their number is two, one, or zero.

The sign of the square trinomial \( 2C_{02}x_\ast^2 + \frac{3}{2}x_\ast + C_{11} \) (and the parameter \( \nu_2 \)) changes if \( r = 0 \).

3.3. General case \( \nu_1 \neq 0, \nu_2 \neq 0 \)
To study the general case \( \nu_1 \neq 0, \nu_2 \neq 0 \), we multiply the first equation in (4) by \( \nu_2 \), the second one by \( -2\nu_1 u_1 \) and add the results term by term. In this case, the terms linear in \( u_j \) disappear, and the resulting corollary equation is a homogeneous equation of the third degree with respect to \( u_1 \) and \( u_2 \). Making, as before, the substitution \( u_2 = x_\ast u_1 \) and carrying out the reduction, we arrive at a cubic equation for \( x \) as follows:

\[
f(x) = \nu_2 x^3 - 2(2\nu_1 C_{02} - C_{11} \nu_2)x^2 - 3\nu_1 x + 2(2\nu_2 2C_{20} - \nu_1 C_{11}) = 0.
\] (10)

The sign of the discriminant of equation (10) coincides with the sign of the homogeneous form of the fourth degree in \( \nu_1 \) and \( \nu_2 \) of the form

\[
\Delta = a_0\nu_1^4 + a_1\nu_1^3\nu_2 + a_2\nu_1^2\nu_2^2 + a_3\nu_1\nu_2^3 + a_4\nu_2^4,
\] (11)

\[
a_0 = -4C_{02}^2(32C_{11})C_{02} - 9, \quad a_1 = 192C_{11}^2 C_{02}^2 + 256C_{02}^3 C_{20} - 144C_{11} C_{02} + 27, \\
a_2 = -12(8C_{11}^3 C_{02} + 32C_{11} C_{20} C_{02} - 3C_{11}^2 - 18C_{20} C_{02}), \quad a_3 = 16(C_{11}^2 + 12C_{02} C_{20}), \\
a_4 = -4C_{20}(27C_{20} + 8C_{11}^3).
\]

For \( \Delta > 0 \) or \( \Delta < 0 \), equation (10) has three or one real root, respectively.

The fourth-degree polynomial

\[
a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4,
\]

obtained by substitution in (11) \( s = \nu_1/\nu_2 \), has 4, 2 or 0 real roots \( s = s_\ast \). To these real roots the straight lines \( \nu_1 = s_\ast \nu_2 \) correspond, dividing the plane of the parameters \( \nu_1, \nu_2 \) into regions with alternating signs of the value \( \Delta \); the number of these regions is 8, 4 or 1.
Choose a point \((\nu_1, \nu_2)\) in one of the regions and calculate the corresponding real roots of equation (10). Note that at the symmetric point \((-\nu_1, -\nu_2)\), this equation has the same roots, since when replacing \(\nu_j \rightarrow -\nu_j\) \((j = 1, 2)\), it does not change. Among the found roots, we select only positive ones.

Let \(x = x_*\) be one of these roots. Let’s make a substitution \(u_2 = x_*u_1\), for example, in the first equation of system (4). We obtain relation (8) for \(u_1\); it should be considered together with the condition (9), which determines the sign of \(\nu_1\). This means that of the two symmetric points \((\pm \nu_1, \pm \nu_2)\) in the parameter plane \(\nu_1, \nu_2\), only one corresponds to the positive pair \((u_1, x_*u_1)\) of solutions to system (4).

Thus, in the regions \(\Delta < 0\), the number of solutions of system (4) can be equal to 1 or 0, and in the regions \(\Delta > 0\) it takes the values 3, 2, 1 or 0.

Note that system (4) contains four systems corresponding to different variants of the pairs of \((\delta_1, \delta_2)\). The findings apply to each of these variants.

A more detailed analysis of the number of equilibrium points is difficult due to the large number of parameters. It is better to carry out such an analysis for a specific Hamiltonian system, when the coefficients of the normal form of the Hamiltonian of perturbed motion are specific numbers, and the equilibrium points are functions of the resonant detunings \(\nu_1, \nu_2\) only.

3.4. On the stability study of equilibrium positions

To study the stability of the equilibrium positions of the approximate systems at hand, we set the perturbations of the variables in (1)

\[
\Phi_j = \Phi_{j0} + y_j, \quad R_j = R_{j0} + Y_j \quad (j = 1, 2)
\]

relative to their equilibrium values given by systems (2), (3).

The quadratic in \(y_j, Y_j\) part of the Hamiltonian of perturbed motion is represented as

\[
H_2 = \gamma_1 y_1^2 + \gamma_2 y_1 y_2 + \gamma_3 y_2^2 + \gamma_4 Y_1^2 + \gamma_5 Y_1 Y_2 + \gamma_6 Y_2^2.
\]

The coefficients \(\gamma_k\) in the notation introduced at the beginning of Section 3 have the form

\[
\gamma_1 = -\frac{1}{2}u_1^4(\delta_2 x^3 b + 16a\delta_1), \quad \gamma_2 = -3 buoy_1^4x^3\delta_2, \quad \gamma_3 = -\frac{9}{2}b\delta_2^4x^6, \quad \gamma_4 = \frac{1}{8}b\delta_2(8C_{20} - x^3), \quad \gamma_5 = \frac{1}{4}b\delta_2(3x + 4C_{11}), \quad \gamma_6 = \frac{1}{8x}\delta_2(8C_{02}x + 3)
\]

Here \(x\) is the root of equation (10).

The characteristic equation of the corresponding linearized equations of motion is biquadratic:

\[
\lambda^4 + \alpha \lambda^2 + \beta = 0, \quad (12)
\]

\[
\alpha = \frac{u_1^4}{4}[b^2x^6 - 18b^2x^4 - 8b(C_{20}b + 3C_{11}b + 9C_{02}b - 2\delta_2\delta_1a)x^3 - 27b^2x^2 - 128ab\delta_1\delta_2C_{20}],
\]

\[
\beta = 36ab^0\delta_1\delta_2u_1^8x^2[-2C_{02}x^4 - 3x^3 - 6C_{11}x^2 + 4(4C_{02}C_{20} - C_{11})x + 6C_{20}]
\]

When the conditions

\[
\alpha > 0, \quad \beta > 0, \quad d = \alpha^2 - 4\beta > 0 \quad (13)
\]

are met, the roots of equation (12) are purely imaginary, and the equilibrium under study is stable in the linear approximation. If the sign of at least one inequality in (13) is reversed, then
this equilibrium is unstable, and not only in the linearized system, but also in the complete nonlinear system.

A joint consideration of the condition \( \beta > 0 \) and equation (10) shows that the sign of \( \beta \) changes only for the values \( \nu_1 \) and \( \nu_2 \) satisfying the relation \( \Delta = 0 \) (\( \Delta \) is defined in (11)), i.e., when passing through the straight lines \( \nu_1 = s_1 \nu_2 \), by which the plane of the parameters \( \nu_1, \nu_2 \) is divided into regions with a different number of equilibrium points (see Section 3.3). Within each such region, the sign of \( \beta \) is preserved.

In the regions where \( \beta < 0 \), the studied equilibria are unstable. In the regions with \( \beta > 0 \), the study should be continued. The nature of stability of the equilibria changes for the values \( \nu_1 \) and \( \nu_2 \) that satisfy the conditions \( d = 0 \) and \( \alpha > 0 \), where \( x \) is the root of equation (10).

The real positive roots of the polynomial \( d \) (of the twelfth degree) are calculated as a function of \( x \). Each such root is substituted into equation (10), the result is the equation of a straight line passing through the origin in the parameter plane \( \nu_1, \nu_2 \), which represents the desired set of points for the condition \( d = 0 \). Similarly, we obtain straight lines on which \( \alpha = 0 \), each of them lies inside the domain with \( d < 0 \) (and \( \beta > 0 \)). Further, regions of linear stability (where \( \alpha > 0 \), \( d > 0 \)) and regions of instability are determined.

3.5. On resonant periodic motions of the complete systems

Let us return to the complete non-autonomous system with Hamiltonian functions (1). Since the roots of the characteristic equation (12) are of the order of unity, and the perturbations (in the terms \( O(\varepsilon) \) of the Hamiltonian functions) occur with a frequency of the order of \( \varepsilon^{-2} \), the non-resonant case of the Poincaré theory of periodic motions takes place [4]. Therefore, in the considered \( \varepsilon \)-neighborhood of the phase space origin, the described equilibrium positions of the approximate systems generate analytic in \( \varepsilon \), periodic in \( \tau \) (with a period \( 24\pi\varepsilon^{-2} \)) motions of the complete systems with Hamiltonian functions (1). In the initial variables, they correspond to \( 24\pi \)-periodic in \( t \) motions, analytic in the small parameter.

Due to the continuity in \( \varepsilon \) of the characteristic exponents of the linearized equations of perturbed motion, linearly stable and unstable equilibrium positions of the approximate systems generate periodic motions of the complete systems, which are also linearly stable and unstable, respectively.

4. On resonant periodic motions of a satellite-plate

As an application, we consider the problem of the motion of a dynamically symmetric satellite (rigid body) with mass geometry of a plate in the vicinity of its stationary rotation (cylindrical precession) in the central Newtonian gravitational field in an elliptical orbit of arbitrary eccentricity \( e \). Let \( \beta = r_0/\omega_0 \), where \( r_0 \) is the projection of the absolute angular velocity of the satellite onto the dynamic symmetry axis (\( r_0 = \text{const} \)), and \( \omega_0 \) is the mean motion of the center of masses.

Earlier in [5] a detailed, linear and nonlinear, stability analysis of this stationary rotation was carried out. In particular, in the parameter plane of \( e, \beta \) the curves of fourth-order resonances were obtained in the linearly stability regions, on which the stability and instability criteria were verified, except for the points of multiple resonances. In [1], for the part of the multiple fourth-order resonances points found, the formal stability of the cylindrical precession of the satellite-plate was shown.

In this paper, for the described satellite problem, we investigate a number of points of multiple fourth-order resonances of the type under consideration. Here are the results of the study for one of these points.

For the values \( e = 0.607388193, \beta = 0.801227648 \), a multiple fourth-order resonance is realized in the system, for which \( 3\lambda_1 + \lambda_2 = 7 \), and \( 4\lambda_2 = 3 \). In the vicinity of the resonance
point, the Hamiltonian of the perturbed motion can be reduced to the form

\[
H = -\mu_1 R_1 - \mu_2 R_2 + 29.29863892R_1^2 + 32.31780430R_1R_2 + 1.566609140R_2^2 + 21.54284368R_1^{3/2}R_2^{1/2}\cos(3\Phi_1 + \Phi_2) + 0.5412543461R_2^2\cos 4\Phi_2 + O(\varepsilon).
\]

Analysis shows that in the parameter plane of \(\mu_1\), \(\mu_2\) for \(\mu_1 \leq 0\), the corresponding approximate system has no equilibrium points.

The results of the study in the region \(\mu_1 > 0\) are presented in Fig.1 and Fig.2 for the cases \(\delta_1 = 1\) and \(\delta_1 = -1\), respectively. The region under consideration is divided into subregions, the boundaries of which are rays defined by the equations \(\nu_1 = \sigma\nu_2\). Solid lines are rays separating regions with different numbers of equilibrium points, dashed lines are rays, passing through which changes the character of stability of equilibrium points (at \(d = 0\)).

The coefficients \(\sigma\) for rays shown in Fig.1 are equal (when moving along the figure from top to bottom)

\[
7.967227898, \ 7.177269868, \ 3.151237683, \ 2.248887479
\]

In Fig.2, the corresponding coefficients are equal to

\[
16.03278171, \ 16.01504914, \ 2.479236236, \ 0.6316575824.
\]

For the values \(\delta_1 = \delta_2 = 1\) in regions \(2', 2, 2''\), and \(3\) in Fig.1 there is one unstable equilibrium point of the approximate system. For the pair \(\delta_1 = -\delta_2 = 1\), there is also one unstable equilibrium point in region 1 in Fig.1. In regions \(2', 2, 2''\) there are two such points. In region 2 both of them are unstable, but in regions \(2'\) and \(2''\), one of them is unstable and the other linearly stable.

In the case \(\delta_1 = \delta_2 = -1\), in region 1 in Fig.2 there is one linearly stable equilibrium point, in regions 2 and \(2'\) there are two points, unstable and linearly stable. If \(\delta_1 = -\delta_2 = -1\), then the system has one equilibrium point in regions \(2', 2, 3,\) and \(3'\). In regions 2 and 3 it is unstable, and in regions \(2'\) and \(3'\) it is linearly stable.

Thus, the approximate system under study has an even (from 0 to 6) number of equilibrium positions, depending on the values of \(\mu_1\) and \(\mu_2\). So, for parameter values near the resonant point in the parameter plane, there are 0, 2, 4 or 6 \(2\pi\)-periodic in \(\nu\) motions of the satellite-plate in the vicinity of its cylindrical precession generated by these equilibria.
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