A complex interpolation formula for tensor products of vector-valued Banach function spaces

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Abstract

We prove the complex interpolation formula

\[
[X_0(E_0) \hat{\otimes} \varepsilon Y_0(F_0), X_1(E_1) \hat{\otimes} \varepsilon Y_1(F_1)]_\theta = [X_0(E_0), X_1(E_1)]_\theta \hat{\otimes} \varepsilon [Y_0(F_0), Y_1(F_1)]_\theta,
\]

for the injective tensor product of vector-valued Banach function spaces \(X_i(E_i)\) and \(Y_i(F_i)\) satisfying certain geometric assumptions. This result unifies results of Kouba, and moreover, our approach offers an alternate proof of Kouba’s interpolation formula for scalar-valued Banach function spaces.

The following theorem for the complex interpolation of injective tensor products of vector-valued Banach function spaces is proved:

**Theorem.** Let \(X_0(\mu), X_1(\mu), Y_0(\nu), Y_1(\nu)\) be real-valued Banach function spaces, and \([E_0, E_1]\) and \([F_0, F_1]\) interpolation couples of complex Banach spaces with dense intersections. Then for \(0 < \theta < 1\) the equality

\[
[X_0(E_0) \hat{\otimes} \varepsilon Y_0(F_0), X_1(E_1) \hat{\otimes} \varepsilon Y_1(F_1)]_\theta = [X_0(E_0), X_1(E_1)]_\theta \hat{\otimes} \varepsilon [Y_0(F_0), Y_1(F_1)]_\theta
\]

(0.1)

holds algebraically and topologically whenever the Banach lattices \(X_0, X_1, Y_0, Y_1\) are 2-concave and the Banach spaces \(E_i\) and \(F_i\) satisfy one of the following conditions:

1. \(E'_0, E'_1, F'_0\) and \(F'_1\) are type 2 spaces.
2. \(E'_0, E'_1\) are type 2 spaces and \(F_0 = F_1\) is a cotype 2 space.
3. \(E_0 = E_1\) and \(F_0 = F_1\) are cotype 2 spaces.

This is an extension and unification of deep results due to Kouba [Kou91] who proved the preceding interpolation formula if one of the couples \([X_0, X_1]\) and \([E_0, E_1]\), and one of the couples \([Y_0, Y_1]\) and \([F_0, F_1]\) is trivial (i.e. either \(X_0 = X_1 = \mathbb{R}\) or \(E_0 = E_1 = \mathbb{C}\), and either \(Y_0 = Y_1 = \mathbb{R}\) or \(F_0 = F_1 = \mathbb{C}\)). Moreover, following an idea of Pisier [Pi90] and based on variants of the Maurey–Rosenthal Factorization Theorem (see Def99), our approach offers an alternate proof of Kouba’s interpolation formula for complex-valued Banach function spaces.

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spaces: For 2-concave complex-valued Banach function spaces \( X_0(\mu), X_1(\mu), Y_0(\nu), Y_1(\nu) \) and \( 0 < \theta < 1 \)

\[
[X_0 \hat{\otimes}_\varepsilon Y_0, X_1 \hat{\otimes}_\varepsilon Y_1]_\theta = [X_0, X_1]_\theta \hat{\otimes}_\varepsilon [Y_0, Y_1]_\theta. \tag{0.2}
\]

The main ingredients of the proof will be “uniform estimates” of

\[
d_\theta[M_0, M_1] := \|\mathcal{L}(\ell_2, [M_0, M_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_2, M_0), \mathcal{L}(\ell_2, M_1)]_\theta\|, \tag{0.3}
\]

where \([M_0, M_1]\) is an interpolation couple of two \( n \)-dimensional Banach spaces. Such estimates proved to be of independent interest: The facts \( \sup_n d_\theta[\ell^n_1, \ell^n_2] < \infty \) (see \([\Pi90]_n \) and \([\Ko91]_n \), \( 3.5 \)); here it is a consequence of Proposition \( 3 \) and its non-commutative analogue \( \sup_n d_\theta[S^n_1, S^n_2] < \infty \) for finite-dimensional Schatten classes (due to Junge in \([\Ju96]_n \), \( 4.2.6 \)) and based on an extension of Kouba’s formulas for the Haagerup tensor product of operator spaces due to \([\Pi90]_n \) were used in \([\DM98]_n \) in order to study so-called “Bennett–Carl Inequalities” for identity operators between finite-dimensional symmetric Banach sequence spaces as well as their “non-commutative analogues” for identity operators between finite-dimensional unitary ideals.

**Preliminaries**

We shall use standard notation and notions from Banach space theory, as presented e.g. in \([\DJT93]_n, [LT77]_n, [LT79]_n \) and \([\text{TJ89}]_n \); for tensor products of Banach spaces we refer to \([\text{DF93}]_n \). If \( E \) is a Banach space, then \( B_E \) is its (closed) unit ball and \( E' \) its dual, and \( FIN(E) \) stands for the collection of all its finite-dimensional subspaces. As usual \( \mathcal{L}(E, F) \) denotes the Banach space of all (bounded and linear) operators from \( E \) into \( F \) endowed with the operator norm \( \| \cdot \| \). For a Banach space \( E \) of type 2 (resp. cotype 2) we write \( T_2(E) \) (resp. \( C_2(E) \)) for its (Rademacher) type 2 constant (resp. cotype 2 constant), and for \( 1 \leq r \leq \infty \) we denote by \( M^{(r)}(X) \) (resp. \( M_{(r)}(X) \)) the \( r \)-convexity (resp. \( r \)-concavity) constant of an \( r \)-convex (resp. \( r \)-concave) Banach lattice \( X \). Recall that for Banach spaces \( E, F \) the injective norm on \( E \otimes F \) is defined by

\[
\|z\|_{E \hat{\otimes}_\varepsilon F} := \sup \{|(x' \otimes y', z)| \mid x' \in B_{E'}, y' \in B_{F'}\}, \quad z \in E \otimes F,
\]

and with \( E \hat{\otimes}_\varepsilon F \) we denote the completion of \( E \otimes F \) endowed with this norm. We will extensively use the fact that the equality \( E \hat{\otimes}_\varepsilon F = \mathcal{L}(E', F) \) holds isometrically whenever one of the two involved spaces is finite-dimensional.

Let \((\Omega, \Sigma, \mu)\) be a \( \sigma \)-finite and complete measure space, and denote all \((\mu\text{-a.e. equivalence classes of})\) real-valued measurable functions on \( \Omega \) by \( L_0(\mu) \). A Banach space \( X = X(\mu) \) of functions in \( L_0(\mu) \) is said to be a Banach function space if it satisfies the following conditions:

(I) If \( |f| \leq |g| \), with \( f \in L_0(\mu) \) and \( g \in X(\mu) \), then \( f \in X(\mu) \) and \( \|f\|_X \leq \|g\|_X \).

(II) For every \( \Lambda \in \Sigma \) with \( \mu(\Lambda) < \infty \) the characteristic function \( \chi_\Lambda \) of \( \Lambda \) belongs to \( X(\mu) \).

A finite-dimensional real Banach space \( X = (\mathbb{R}^n, \| \cdot \|_X) \) is called an \( n \)-dimensional lattice if
\[ \| \cdot \|_X \text{ is a lattice norm; clearly, } X \text{ then is a Banach function space in the above sense. For Banach function spaces } X_0(\mu), X_1(\mu) \text{ and } 0 < \theta < 1 \text{ the space } X_0^{1-\theta} X_1^\theta \text{ is defined to be the set of functions } f \in L_0(\mu) \text{ for which there exist } g \in X_0 \text{ and } h \in X_1 \text{ such that } |f| = |g|^{1-\theta} |h|^\theta. \]

Together with the norm \[ \|f\|_{X_0^{1-\theta} X_1^\theta} := \{\|g\|_{X_0}^{1-\theta} \cdot \|h\|_{X_1}^\theta \mid |f| = |g|^{1-\theta} \cdot |h|^\theta, g \in X_0, h \in X_1\}, \]

\[ X_0^{1-\theta} X_1^\theta \text{ becomes a Banach function space (with respect to } (\Omega, \Sigma, \mu)) \text{. It can be easily seen (see e.g. } [\text{J}89, \text{p. } 218/219] \text{) that if for } 1 \leq r < \infty \text{ the lattices } X_0 \text{ and } X_1 \text{ are both } r\text{-convex or both } r\text{-concave, then } X_0^{1-\theta} X_1^\theta \text{ also has this property, with} \]

\[ M^{(r)}(X_0^{1-\theta} X_1^\theta) \leq M^{(r)}(X_0)^{1-\theta} \cdot M^{(r)}(X_1)^\theta, \quad (0.4) \]

\[ M^{(r)}(X_0^{1-\theta} X_1^\theta) \leq M^{(r)}(X_0)^{1-\theta} \cdot M^{(r)}(X_1)^\theta, \quad (0.5) \]

respectively.

Let \( X(\mu) \) be a Banach function space and \( E \) a Banach space. A function \( x \) defined on \( \Omega \) with values in \( E \) is said to be strongly measurable if there exists a sequence of strictly simple functions on \( \Omega \) converging to \( x \) almost everywhere; here a function \( y \) on \( \Omega \) with values in \( E \) is called strictly simple if it assumes only finitely many non-zero values, each on a measurable set with finite measure. Then by \( X(E) \) we denote the collection of all strongly measurable functions \( x \) with values in \( E \) for which \( \|x(\cdot)\|_E \in X \). Together with the norm \( \|x\|_{X(E)} := \|\|x(\cdot)\|_E\|_X \), this vector space becomes a Banach space (\( \mathbb{K}\text{-linear whenever } E \text{ is } \mathbb{K}\text{-linear}).

For all information on complex interpolation we refer to \[ [BL78, \text{KPS}82]. \text{Given a (complex) interpolation couple } [E_0, E_1], \text{ we write } E_\Delta := E_0 \cap E_1, \text{ and as usual denote for } 0 < \theta < 1 \text{ the complex interpolation space with respect to } [E_0, E_1] \text{ and } \theta \text{ by } [E_0, E_1]_\theta. \text{ If we speak of a finite-dimensional interpolation couple } [E_0, E_1], \text{ this always means that both spaces have the same finite dimension. Clearly, if } [E_0, E_1] \text{ is an interpolation couple and } X_0(\mu), X_1(\mu) \text{ are Banach function spaces, then } [X_0(E_0), X_1(E_1)] \text{ is an interpolation couple. We will heavily use the following complex interpolation formula due to Calderón [Cal64, 13.6]: For } 0 < \theta < 1 \]

\[ [X_0(E_0), X_1(E_1)]_\theta = (X_0^{1-\theta} X_1^\theta)([E_0, E_1]_\theta) \quad (0.6) \]

holds isometrically whenever \( X_0 \) or \( X_1 \) is \( \sigma\)-order continuous; note that under the assumptions of the theorem all involved Banach function spaces are \( \sigma\)-order continuous (for an argument see Section 4), and clearly this is true for finite-dimensional lattices.

1 The approximation lemma

First we show—similar to \[ [\text{Kou}91], \text{Section 4}—that equalities as stated in the above theorem are of a finite-dimensional nature. In order to make the following more readable, let us introduce the following notation: If \( [E_0, E_1] \) is an interpolation couple, \( E \subset E_\Delta \) a subspace
which is dense in $E_0, E_1$ and $A \subset \text{FIN}(E)$ is cofinal (i.e. for every $G \in \text{FIN}(E)$ there exists $M \in A$ with $G \subset M$), then the triple $([E_0, E_1], E, A)$ is called a cofinal interpolation triple. For $M \in \text{FIN}(E)$ we denote by $M_0$ (resp. $M_1$) the subspace $M$ of $E_0$ (resp. $E_1$) endowed with the induced norm.

The following two lemmas are crucial. The first one is an only slight modification of [Kou91, 4.1]; we omit its proof.

**Lemma 1.** Let $([E_0, E_1], E, A)$ be a cofinal interpolation triple and $0 < \theta < 1$. Then for each $\varepsilon > 0$ and $G \in \text{FIN}(E)$ there exists $M \in A$ such that $G \subset M$ and for all $x \in G$

$$
(1 - \varepsilon) \cdot \|x\|_{[M_0, M_1]_\theta} \leq \|x\|_{[E_0, E_1]_\theta} \leq \|x\|_{[M_0, M_1]_\theta}. \tag{1.1}
$$

If $[M_0, M_1]$ and $[N_0, N_1]$ are finite-dimensional interpolation couples, then we define for $0 < \theta < 1$

$$
d_\theta[M_0, M_1; N_0, N_1] := \|[M_0, M_1]_\theta \otimes_{\varepsilon} [N_0, N_1]_\theta \hookrightarrow [M_0 \otimes_{\varepsilon} N_0, M_1 \otimes_{\varepsilon} N_1]_\theta\|. \tag{1.2}
$$

The second lemma—which for obvious reasons is called “approximation lemma”—reduces the proof of Kouba type formulas (1.1) or (1.2) to uniform estimates of $d_\theta[M_0, M_1; N_0, N_1]$ for cofinally many suitable finite-dimensional subspaces of the underlying infinite-dimensional spaces. Its proof is very close to the proof of [Kou91], but we state it for the convenience of the reader.

**Approximation Lemma 2.** Let $([E_0, E_1], E, A)$ and $([F_0, F_1], F, B)$ be cofinal interpolation triples and $0 < \theta < 1$. If

$$
d_\theta[E_0, E_1; F_0, F_1] := \sup_{M \in A} \sup_{N \in B} d_\theta[M_0, M_1; N_0, N_1] < \infty,
$$

then

$$
[E_0 \hat{\otimes}_{\varepsilon} F_0, E_1 \hat{\otimes}_{\varepsilon} F_1]_\theta = [E_0, E_1]_\theta \hat{\otimes}_{\varepsilon} [F_0, F_1]_\theta.
$$

**Proof.** From the density assumptions we conclude that $E \otimes F$ is dense in $[E_0, E_1]_\theta \hat{\otimes}_{\varepsilon} [F_0, F_1]_\theta$ and in $[E_0 \hat{\otimes}_{\varepsilon} F_0, E_1 \hat{\otimes}_{\varepsilon} F_1]_\theta$, hence it is sufficient to show that for a given $z \in E \otimes F$

$$
\|z\|_{[E_0, E_1]_\theta \hat{\otimes}_{\varepsilon} [F_0, F_1]_\theta} \leq \|z\|_{[E_0 \hat{\otimes}_{\varepsilon} F_0, E_1 \hat{\otimes}_{\varepsilon} F_1]_\theta} \leq d_\theta[E_0, E_1; F_0, F_1] \cdot \|z\|_{[E_0, E_1]_\theta \hat{\otimes}_{\varepsilon} [F_0, F_1]_\theta}. \tag{1.3}
$$

We start with a simple observation to show (1.2). If $[M_0, M_1]$ and $[N_0, N_1]$ are finite-dimensional interpolation couples, then

$$
\|\mathcal{L}(M_0, N_0), \mathcal{L}(M_1, N_1)]_\theta \hookrightarrow \mathcal{L}([M_0, M_1]_\theta, [N_0, N_1]_\theta)\| \leq 1; \tag{1.4}
$$

indeed, consider for $i = 0, 1$ the bilinear mapping

$$
\phi_i : \mathcal{L}(M_i, N_i) \times M_i \to N_i, \quad (T, x) \mapsto Tx,
$$

which clearly has norm 1, hence (1.4) follows from the fact that by bilinear interpolation (see [BL78, 4.4.1]) the interpolated mapping

$$
\phi_\theta : \mathcal{L}(M_0, N_0), \mathcal{L}(M_1, N_1)]_\theta \times [M_0, M_1]_\theta \to [N_0, N_1]_\theta
$$
also has norm \( \leq 1 \). Now (1.2) is a straightforward consequence: Obviously \( C := \{ M \otimes N \mid M \in \mathcal{A}, N \in \mathcal{B} \} \subset \text{FIN}(E \otimes F) \) is cofinal, hence, by Lemma [I] and the fact that the injective norm respects subspaces, there exist \( M \in \mathcal{A} \) and \( N \in \mathcal{B} \) such that \( z \in M \otimes N \) and
\[
\|z\|_{[M \otimes \varepsilon N_0, M_1 \otimes \varepsilon N_1]_\theta} \leq (1 + \varepsilon) \cdot \|z\|_{[E_0 \otimes \varepsilon F_0, E_1 \otimes \varepsilon F_1]_\theta}.
\]

Finally, by the mapping property of the injective norm and (1.4),
\[
\|z\|_{[E_0, E_1]_\theta \otimes \varepsilon [F_0, F_1]_\theta} \leq \|z\|_{[M_0, M_1]_\theta \otimes \varepsilon [N_0, N_1]_\theta} \leq (1 + \varepsilon) \cdot \|z\|_{[E_0 \otimes \varepsilon F_0, E_1 \otimes \varepsilon F_1]_\theta}.
\]

In order to show (1.3) let \( z \in G \otimes H \) for some \( G \in \text{FIN}(E), H \in \text{FIN}(F) \), and choose by Lemma [II] subspaces \( M \in \mathcal{A} \) and \( N \in \mathcal{B} \) such that \( G \subset M, H \subset N \) and
\[
\| (G, \| \cdot \|_{[E_0, E_1]_\theta}) \mapsto [M_0, M_1]_\theta \| \leq \sqrt{1 + \varepsilon},
\]
\[
\| (H, \| \cdot \|_{[F_0, F_1]_\theta}) \mapsto [N_0, N_1]_\theta \| \leq \sqrt{1 + \varepsilon}.
\]
Then, by the mapping property,
\[
\| (G, \| \cdot \|_{[E_0, E_1]_\theta}) \otimes \varepsilon (H, \| \cdot \|_{[F_0, F_1]_\theta}) \| \leq [M_0, M_1]_\theta \otimes \varepsilon [N_0, N_1]_\theta \| \leq 1 + \varepsilon,
\]
hence, since the injective norm respects subspaces,
\[
\|z\|_{[M_0, M_1]_\theta \otimes \varepsilon [N_0, N_1]_\theta} \leq (1 + \varepsilon) \cdot \|z\|_{[E_0, E_1]_\theta \otimes \varepsilon [F_0, F_1]_\theta}.
\]
By the usual interpolation theorem we obtain
\[
\|z\|_{[E_0 \otimes \varepsilon F_0, E_1 \otimes \varepsilon F_1]_\theta} \leq \|z\|_{[M_0 \otimes \varepsilon N_0, M_1 \otimes \varepsilon N_1]_\theta} \leq d_\theta [M_0, M_1; N_0, N_1] \cdot \|z\|_{[M_0, M_1]_\theta \otimes \varepsilon [N_0, N_1]_\theta} \leq (1 + \varepsilon) \cdot d_\theta [E_0, E_1; F_0, F_1] \cdot \|z\|_{[E_0, E_1]_\theta \otimes \varepsilon [F_0, F_1]_\theta}.
\]
\]

\[\Box\]

2 The Hilbert space case

Recall for a finite-dimensional interpolation couple \([E_0, E_1]\) the definition of \( d_\theta[E_0, E_1] \) from (1.3), and note that by the approximation lemma
\[
d_\theta[E_0, E_1] = \sup_n d_\theta[\ell^n_2, \ell^n_2; E_0, E_1].
\]
The main step in the proof of (1.1) is the following estimate:

**Proposition 3.** Let \( X_0, X_1 \) be \( n \)-dimensional lattices and \([E_0, E_1]\) a finite-dimensional interpolation couple. Then for each \( 0 < \theta < 1 \)
\[
d_\theta[X_0(E_0), X_1(E_1)] \leq \sqrt{2} \cdot C_2([E_0, E_1]_\theta) \cdot M_2(X_0)^{1-\theta} \cdot M_2(X_1)^{\theta} \cdot d_\theta[\ell^n_2(E_0), \ell^n_2(E_1)]. \tag{2.1}
\]
Before giving the proof we collect some facts about so-called powers of finite-dimensional lattices. For $0 < r < \infty$ and an $n$-dimensional lattice $X$ with $\mathbf{M}^{(\max(1,r))}(X) = 1$ (recall that $\mathbf{M}^{(1)}(X) = 1$)

$$
\|x\|_r := \|x\|^{1/r} \|x\|_X, \quad x \in \mathbb{R}^n
$$
defines a lattice norm on $\mathbb{R}^n$ (see e.g. [Def99]); the $n$-dimensional lattice $(\mathbb{R}^n, \| \cdot \|_r)$ will be denoted by $X^r$.

**Lemma 4.** Let $X, X_0, X_1$ be $n$-dimensional lattices, $E$ a Banach space, $\lambda \in \mathbb{R}^n$ and $0 < \theta < 1$.

(a) If $\mathbf{M}^{(2)}(X) = 1$, then $\|D_\lambda \otimes \text{id} : \ell^2_0(E) \to X(E)\| \leq \|D_\lambda\| = \|\lambda\|_{((X')^2)^{1/2}}$, where $D_\lambda : \ell^2_0 \to X$ denotes the diagonal operator associated with $\lambda$.

(b) $(X^{1-\theta}_0 X^\theta_1)' = (X^{1-\theta}_0)' (X^\theta_1)'$ isometrically.

(c) For $0 < r < \infty$ let $\mathbf{M}^{(\max(1,r))}(X_0) = \mathbf{M}^{(\max(1,r))}(X_1) = 1$. Then $(X^{1-\theta}_0 X^\theta_1)^r = (X^{1-\theta}_0)^r (X^\theta_1)^r$ isometrically.

**Proof.** (a) For $x \in \ell^2_0(E)$

$$
\|D_\lambda \otimes \text{id} x\|_{X(E)} = \|\lambda_k \cdot \|x_k\|\|_X \leq \|D_\lambda : \ell^2_0 \to X\| \cdot \left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2},
$$

and (note that $\mathbf{M}^{(2)}(X') = \mathbf{M}^{(2)}(X) = 1$)

$$
\|\lambda\|_{((X')^2)^{1/2}} = \|\lambda^2\|_{((X')^2)^{1/2}} = \sup_{\|\mu\|_{(X')^2} \leq 1} \|\lambda^2 \mu\|_{\ell^1_n} = \sup_{\|\mu\|_{X'} \leq 1} \left(\sum_{i=1}^n \lambda_i \mu_i\right) = \sup_{\|\mu\|_{X'} \leq 1} \|\lambda \mu\|_X = \|D_\lambda : \ell^2_0 \to X\|.
$$

(b) By the Calderón formula ([0,6]), the duality theorem [BL78, 4.5.2] and the fact that $Y(\mathbb{C})' = Y(\mathbb{C})$ holds isometrically for every finite-dimensional lattice $Y$, one arrives at the isometric identity

$$
(X^{1-\theta}_0 X^\theta_1)'(\mathbb{C}) = ((X^{1-\theta}_0)' (X^\theta_1)')'(\mathbb{C}),
$$

which clearly implies the above statement.

(c) First note that $\mathbf{M}^{(\max(1,r))}(X^{1-\theta}_0 X^\theta_1) = 1$ by (4.4), hence the power $(X^{1-\theta}_0 X^\theta_1)^r$ is normed. Let $V := (X^{1-\theta}_0 X^\theta_1)^r$ and $W := (X^{1-\theta}_0)' (X^\theta_1)'$. Then, if $|f|^{1/r} = |g|^{1-\theta} \cdot |h|^\theta$, 

$$
\|f\|_W \leq \|g|^{1-\theta} \|X_0^{1-\theta} \cdot |h|^\theta \|X_1^\theta \|X_0^r \leq \left(\|g|^{1-\theta} \cdot |h|^\theta \|X_0^{1-\theta} \cdot |h|^\theta \right)^r,
$$

which clearly implies $\|f\|_W \leq \|f\|_V$. Conversely, let $|f| = |g|^{1-\theta} \cdot |h|^\theta$. Then 

$$
\|f\|_V = \|f|^{1/r} \|X_0^{1-\theta} X_1^\theta \|X_0^{1-\theta} \cdot \|h|^{1/r} \|X_1^\theta \|X_0^{1-\theta} \cdot \|h|^{\theta} = \|g|^{1-\theta} \cdot \|h|^\theta,
$$

hence $\|f\|_V \leq \|f\|_W$. \(\square\)
Another important tool for the proof of (2.1) is a variant of the Maurey–Rosenthal Factorization Theorem (\cite{Mau74}) for vector-valued Banach function spaces given in \cite{Def99}.

**Lemma 5.** Let $X(\mu)$ be a 2-concave Banach function space and $E$ a Banach space of cotype 2. Then each $T \in \mathcal{L}(\ell_2, X(E))$ factorizes as follows:

\[
\begin{array}{c}
\ell_2 \\
\downarrow T \\
X(E) \\
\downarrow R \\
L_2(\mu, E) \\
\end{array}
\]

where $R : \ell_2 \to L_2(\mu, E)$ is an operator and $M_g : L_2(\mu) \to X$ a multiplication operator with respect to $g \in L_0(\mu)$ such that $\|R\| \cdot \|M_g\| \leq \sqrt{2} \cdot C_2(E) \cdot M_{(2)}(X) \cdot \|T\|$.

**Proof.** Let $D_n := \{-1, +1\}^n$, $\mu_n(\{\omega\}) := 1/2^n$ for $\omega \in D_n$ and $\varepsilon_i : D_n \to \{-1, +1\}$ the $i$-th canonical projection. Then for $x_1, \ldots, x_n \in \ell_2$

\[
\left\| \left( \sum_{i=1}^n \|Tx_i(\cdot)\|_E^2 \right)^{1/2} \right\|_X \leq \sqrt{2} \cdot C_2(E) \cdot \left\| \int_{D_n} \left\| \sum_{i=1}^n \varepsilon_i(\omega) \cdot Tx_i(\cdot) \right\|_E \, d\mu_n(\omega) \right\|_X \\
\leq \sqrt{2} \cdot C_2(E) \cdot \int_{D_n} \left\| \sum_{i=1}^n \varepsilon_i(\omega) \cdot Tx_i(\cdot) \right\|_E \, d\mu_n(\omega) \\
= \sqrt{2} \cdot C_2(E) \cdot \int_{D_n} \left\| T \left( \sum_{i=1}^n \varepsilon_i(\omega) \cdot x_i \right) \right\|_X \, d\mu_n(\omega) \\
\leq \sqrt{2} \cdot C_2(E) \cdot \|T\| \cdot \left( \sum_{i=1}^n \|x_i\|_{\ell_2}^2 \right)^{1/2}
\]

(the constant $\sqrt{2}$ comes from the Khinchine–Kahane inequality for the case “$L_2$ versus $L_1$”), hence by \cite[4.4]{Def99} there exists $0 \leq \omega \in L_0(\mu)$ with

\[
\sup_{y \in B_{L_2(\mu)}} \|\omega^{1/2} \cdot y\|_X \leq \sqrt{2} \cdot M_{(2)}(X) \cdot C_2(E) \cdot \|T\| \tag{2.2}
\]

such that for all $x \in \ell_2$

\[
\left( \int_{\Omega} \|Tx(\cdot)\|_E^2 / \omega \, d\mu \right)^{1/2} \leq \|x\|_{\ell_2}. \tag{2.3}
\]

Define the operator $R \in \mathcal{L}(\ell_2, L_2(\mu, E))$ by $Rx := Tx/\omega^{1/2}$ for $x \in \ell_2$ (well-defined by (2.3)) and the multiplication operator $M_g : L_2(\mu) \to X$ with $g := \omega^{1/2}$ (well-defined by (2.2)). Clearly, this produces the desired factorization. 

Now we are prepared for the **Proof of Proposition 3.** Its main idea—the use of factorizations of Maurey–Rosenthal type—is taken from \cite{Pi90}. 

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Without loss of generality we may assume that $M_{(2)}(X_0) = M_{(2)}(X_1) = 1$; indeed, let $Y_0$ and $Y_1$ be the associated renormed lattices such that $M_{(2)}(Y_i) = 1$ and $\|X_i \hookrightarrow Y_i\| \cdot \|Y_i \hookrightarrow X_i\| \leq M_{(2)}(X_i)$ for $i = 0, 1$ (see e.g. [LT79, 1.d.8]). Now consider the factorization

$$\ell_2 \otimes_{\varepsilon} [X_0(E_0), X_1(E_1)]_{\theta} \xrightarrow{id \otimes id} [\ell_2 \otimes_{\varepsilon} X_0(E_0), \ell_2 \otimes_{\varepsilon} X_1(E_1)]_{\theta}$$

and observe that $\|u\| \cdot \|v\| \leq M_{(2)}(X_0)^{1-\theta} \cdot M_{(2)}(X_1)^{\theta}$.

Put $X_\theta := X_0^{1-\theta}X_1^\theta$. Since $[X_0(E_0), X_1(E_1)]_{\theta} = X_\theta([E_0, E_1])_{\theta}$ holds isometrically (see (2.6)) and $M_{(2)}(X_\theta) = 1$ (see (1.3)), by Lemma 5 every operator $T \in L(\ell_2, X_\theta([E_0, E_1])_{\theta})$ factors

$$\ell_2 \xrightarrow{T} X_\theta([E_0, E_1])_{\theta} \xrightarrow{D_\lambda \otimes id} \ell_2^2([E_0, E_1])_{\theta},$$

with $\|R\| \cdot \|D_\lambda\| \leq \sqrt{2} \cdot C_2([E_0, E_1])_{\theta} \cdot \|T : \ell_2 \to [X_0(E_0), X_1(E_1)]_{\theta}\|$. Define $Y_\eta := ((X_\eta')^2)^{1/2}$ for $\eta = 0, 1, \theta$; by Lemma 3 (b),(c) and the Calderón formula (D.3) we have $[Y_0(\mathcal{C}), Y_1(\mathcal{C})]_{\theta} = Y_\theta(\mathcal{C})$. By Lemma 3 (a) the mapping

$$\Phi_\eta : Y_\eta(\mathcal{C}) \to L(\ell_2^2(E_\eta), X_\eta(E_\eta)), \quad \mu \mapsto D_\mu \otimes id$$

has norm $\leq 1$, and consequently the interpolated mapping

$$[\Phi_0, \Phi_1]_{\theta} : [Y_0(\mathcal{C}), Y_1(\mathcal{C})]_{\theta} \to V := [L(\ell_2^2(E_0), X_0(E_0)), L(\ell_2^2(E_1), X_1(E_1))]_{\theta}$$

has norm $\leq 1$. Moreover, by bilinear interpolation ([BL78, 4.4.1]) the mapping

$$U \times V \to W, \quad (u, v) \mapsto v \circ u,$$

where $U := [L(\ell_2, \ell_2^2(E_0)), L(\ell_2, \ell_2^2(E_1))]_{\theta}$ and $W := [L(\ell_2, X_0(E_0)), L(\ell_2, X_1(E_1))]_{\theta}$, also has norm $\leq 1$. Since by definition $\|R\|_U \leq d_\theta[\ell_2^2(E_0), \ell_2^2(E_1)] \cdot \|R\|$, we obtain altogether

$$\|T\|_W = \|(D_\lambda \otimes id) \circ R\|_W \leq \|R\|_U \cdot \|D_\lambda \otimes id\|_V = \|R\|_U \cdot \|[\Phi_0, \Phi_1]_{\theta}(\lambda)\|_V \leq d_\theta[\ell_2^2(E_0), \ell_2^2(E_1)] \cdot \|R\| \cdot \|\lambda\|_{Y_\theta} \leq d_\theta[\ell_2^2(E_0), \ell_2^2(E_1)] \cdot \sqrt{2} \cdot C_2([E_0, E_1])_{\theta} \cdot \|T\|,$$

the desired inequality.
A quick look at (2.3) reveals that in the case $E = E_0 = E_1$ one has

**Corollary 6.** Let $X_0, X_1$ be $n$-dimensional lattices and $E$ a finite-dimensional normed space. Then for $0 < \theta < 1$

$$d_\theta[X_0(E), X_1(E)] \leq \sqrt{2} \cdot C_2(E) \cdot M_{2}(X_0)^{1-\theta} \cdot M_{2}(X_1)^{\theta}. \quad (2.4)$$

For the case that $E_0$ and $E_1$ have different norms, one can use the following upper estimate for $d_\theta[\ell_2^0(E_0), \ell_2^0(E_1)]$ in terms of type 2 constants which is taken from [Kou91, 3.5]: Let $[F_0, F_1]$ be a finite-dimensional interpolation couple. Then

$$d_\theta[F_0, F_1] \leq T_2(F_0')^{1-\theta} \cdot T_2(F_1')^{\theta}. \quad (2.5)$$

Note that the estimate given in (2.5) is slightly different from that in Kouba’s work; we refer the reader to [DM98] for the details.

Using the simple fact that $T_2(\ell_2^0(E_1')) = T_2(E_1')$ (see e.g. [DJK95, 11.12]), (2.3) gives $d_\theta[\ell_2^0(E_0), \ell_2^0(E_1)] \leq T_2(E_0')^{1-\theta} \cdot T_2(E_1')^{\theta}$. Furthermore, by the duality of type and cotype (see e.g. [DJK95, 11.10]) and the interpolative nature of the type 2 constants (see e.g. [J89, (3.8)]) $C_2([E_0, E_1]_\theta) \leq T_2([E_0', E_1']_\theta) \leq T_2(E_0')^{1-\theta} \cdot T_2(E_1')^{\theta}$. Altogether we arrive at

**Corollary 7.** Let $X_0, X_1$ be $n$-dimensional lattices and $[E_0, E_1]$ a finite-dimensional interpolation couple. Then for $0 < \theta < 1$

$$d_\theta[X_0(E_0), X_1(E_1)] \leq \sqrt{2} \cdot M_{2}(X_0)^{1-\theta} \cdot M_{2}(X_1)^{\theta} \cdot (T_2(E_0')^{1-\theta} \cdot T_2(E_1')^{\theta})^2. \quad (2.6)$$

### 3 The finite-dimensional case in general

Our estimates for $d_\theta[X_0(E_0), X_1(E_1); Y_0(F_0), Y_1(F_1)]$ are as follows:

**Proposition 8.** Let $X_0, X_1$ and $Y_0, Y_1$ be $n$-dimensional and $m$-dimensional lattices, respectively, and $[E_0, E_1], [F_0, F_1]$ two arbitrary finite-dimensional interpolation couples. Then for $0 < \theta < 1$

$$d_\theta[X_0(E_0), X_1(E_1); Y_0(F_0), Y_1(F_1)] \leq 16 \cdot [(M_{2}(X_0) \cdot M_{2}(Y_0))^{1-\theta} \cdot (M_{2}(X_1) \cdot M_{2}(Y_1))^{\theta}]^{5/2} \cdot t_\theta[E_0, E_1] \cdot t_\theta[F_0, F_1], \quad (3.1)$$

where, if $G$ represents either $E$ or $F$,

$$t_\theta[G_0, G_1] := \begin{cases} C_2(G)^{5/2} & \text{if } G = G_0 = G_1, \\ T_2(G_0')^{1-\theta} \cdot T_2(G_1')^{\theta}^{7/2} & \text{else.} \end{cases}$$

The proof is based on the following “factorization lemma” which will enable us to use the estimates from the Hilbert space case derived in (2.3) and (2.4) in order to obtain estimates for the general case. As usual we denote by $\Gamma_2$ the Banach operator ideal of all operators $T$ which allow a factorization $T = RS$ through a Hilbert space, together with the norm $\gamma_2(T) := \inf \|R\| \cdot \|S\|$. 

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Lemma 9. Let \([E_0, E_1]\) and \([F_0, F_1]\) be finite-dimensional interpolation couples. Then for \(0 < \theta < 1\)

\[
\|\Gamma_2([E_0, E_1]_\theta', [F_0, F_1]_\theta)\| \leq \|\Gamma_2([E_0, E_1], F_0), \Gamma_2(E_1', F_1)]_\theta\| \leq d_\theta[E_0, E_1] \cdot d_\theta[F_0, F_1].
\]

Proof. Let \(T : [E_0, E_1]_\theta' \to [F_0, F_1]_\theta\) factorize as follows:

\[
\begin{array}{ccc}
[E_0, E_1]_\theta' & \xrightarrow{T} & [F_0, F_1]_\theta \\
R & \downarrow & S \\
\ell_2 & \downarrow & \ell_2
\end{array}
\]

and consider by bilinear interpolation the norm 1 mapping

\[
U \times V \to W, \quad (u, v) \mapsto v \circ u',
\]

where

\[
U := [\mathcal{L}(\ell_2, E_0), \mathcal{L}(\ell_2, E_1)]_\theta, \quad V := [\mathcal{L}(\ell_2, F_0), \mathcal{L}(\ell_2, F_1)]_\theta
\]

and

\[
W := [\Gamma_2(E_0', F_0), \Gamma_2(E_1', F_1)]_\theta.
\]

Then

\[
\|T\|_W = \|SR\|_W \leq \|R'|_U \cdot \|S\|_V \leq d_\theta[E_0, E_1] \cdot d_\theta[F_0, F_1] \cdot \|R'\| \cdot \|S\|
\]

which clearly gives \(\|T\|_W \leq d_\theta[E_0, E_1] \cdot d_\theta[F_0, F_1] \cdot \gamma_2(T)\). \hfill \Box

Another ingredient needed for the proof of Proposition 8 is a simple estimate for the cotype 2 constant of vector-valued Banach function spaces. We omit its straightforward proof (which needs arguments already used in the proof of Lemma 8).

Lemma 10. Let \(X\) be a 2-concave Banach function space and \(E\) a Banach space of cotype 2. Then \(X(E)\) has cotype 2, and \(C_2(X(E)) \leq \sqrt{2} \cdot M_2(X) \cdot C_2(E)\).

With this the proof of Proposition 8 is straightforward:

Proof of Proposition 8. For the moment denote by \(D_\theta\) the norm of the embedding

\[
\Gamma_2([X_0(E_0), X_1(E_1)]_\theta', [Y_0(F_0), Y_1(F_1)]_\theta) \to [\Gamma_2(X_0(E_0'), Y_0(F_0)), \Gamma_2(X_1(E_1'), Y_1(F_1))]_\theta
\]

and \(d_\theta := d_\theta[X_0(E_0), X_1(E_1); Y_0(F_0), Y_1(F_1)]\). Using Pisier’s Factorization Theorem ([Pi86, 4.1] or [DF93, 31.4]), the Calderón formula (1.6), Lemma 8 and the interpolative nature of the 2-concavity constants (see (1.2)) one has

\[
d_\theta \leq (2 \cdot C_2([X_0(E_0), X_1(E_1)]_\theta) \cdot C_2([Y_0(F_0), Y_1(F_1)]_\theta))^{3/2} \cdot D_\theta
\]

\[
= (2 \cdot C_2((X_0^{1-\theta} X_1^{\theta})(E_0, E_1)]_\theta) \cdot C_2((Y_0^{1-\theta} Y_1^{\theta})(F_0, F_1)]_\theta))^{3/2} \cdot D_\theta
\]

\[
\leq 8 \cdot (M_2(X_0^{1-\theta} X_1^{\theta}) \cdot M_2(Y_0^{1-\theta} Y_1^{\theta}) \cdot C_2([E_0, E_1]_\theta) \cdot C_2([F_0, F_1]_\theta))^{3/2} \cdot D_\theta
\]

\[
\leq 8 \cdot ((M_2(X_0) \cdot M_2(Y_0))^{1-\theta} \cdot (M_2(X_1) \cdot M_2(Y_1))^{\theta} \cdot C_2([E_0, E_1]_\theta) \cdot C_2([F_0, F_1]_\theta))^{3/2} \cdot D_\theta.
\]
Now let us start the Proof of the theorem: First observe that if we define
\[ \mathcal{A} := \{ U(M) \mid U \in \text{FIN}_\chi(\mu), M \in \text{FIN}(E_\Delta) \} \]
and
\[ \mathcal{B} := \{ V(N) \mid V \in \text{FIN}_\chi(\nu), N \in \text{FIN}(F_\Delta) \}, \]
then \( ([X_0(E_0), X_1(E_1)], S(\mu, E_\Delta), \mathcal{A}) \) and \( ([Y_0(F_0), Y_1(F_1)], S(\nu, F_\Delta), \mathcal{B}) \) are cofinal interpolation triples whenever \( X_0, X_1 \) and \( Y_0, Y_1 \) have non-trivial concavity. Indeed, these assumptions together with \cite[1.a.5]{LT79} and \cite[1.a.7]{LT79} imply that \( X_0 \) and \( X_1 \) are \( \sigma \)-order continuous, and by \cite[p. 211]{KPS82} it follows that \( S(\mu, E_\Delta) \) is dense in \( X_0(E_0) \) and \( X_1(E_1) \); obviously each \( G \in \text{FIN}(S(\mu, E_\Delta)) \) is contained in some \( U(M) \) with \( U \in \text{FIN}_\chi(\mu) \) and \( M \in \text{FIN}(E_\Delta) \). Moreover, if \( U \) is generated by measurable, pairwise disjoint sets \( A_1, \ldots, A_n \) with finite non-zero measures, then \( \chi_{A_1}, \ldots, \chi_{A_n} \) form a 1-unconditional basis for \( U \), hence \( U \) is a finite-dimensional lattice which is order isometric to \( \mathbb{R}^n \) endowed with a lattice norm under the canonical order.

This now puts us in the position to apply the Approximation Lemma \[ \square \] together with Proposition 3. For \( U \in \text{FIN}_\chi(\mu), V \in \text{FIN}_\chi(\nu), M \in \text{FIN}(E_\Delta) \) and \( N \in \text{FIN}(F_\Delta) \)
\[ d_\theta[U_0(M_0), U_1(M_1); V_0(N_0), V_1(N_1)] \]
\[ \leq 16 \cdot \left[ (M_{2}(U_0) \cdot M_{2}(V_0))^{1-\theta} (M_{2}(U_1) \cdot M_{2}(V_1))^{\theta} \right]^{5/2} \]
\[ \cdot t_\theta[M_0, M_1] \cdot t_\theta[N_0, N_1] \]
\[ \leq 16 \cdot \left[ (M_{2}(X_0) \cdot M_{2}(Y_0))^{1-\theta} (M_{2}(X_1) \cdot M_{2}(Y_1))^{\theta} \right]^{5/2} \]
\[ \cdot t_\theta[E_0, E_1] \cdot t_\theta[F_0, F_1], \]
where the latter inequality follows from the fact that \( M_{2} \) respects sublattices, \( C_{2} \) subspaces and \( T_{2} \) quotients. \[ \square \]

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