Perturbative Approach to Non-renormalizable Theories

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On the perturbatively non-renormalizable and non-perturbatively finite examples (delta-function type potential in non-relativistic quantum mechanics and the mathematical model of the propagator by Redmond and Uretsky in quantum field theory) we illustrate that one can develop a perturbative approach for non-renormalizable theory. The key idea is the introduction of finite number of additional expansion parameters which allows us to eliminate all infinities from the perturbative expressions. The generated perturbative series reproduce the expansions of the exact analytical solutions.

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I. INTRODUCTION

It is well known that divergences itself do not make the main renormalization problem in quantum field theories. One can remove all divergences from any theory performing subtractions, but these subtractions lead to the ambiguous finite parts. One can include these arbitrary terms into the finite number of physical parameters only in renormalizable theories, retaining predictive power of the theory \[2\].

There exists a challenging possibility of non-perturbative finiteness of non-renormalizable theory (We recall the idea about non-perturbative finiteness of quantum gravity (see for example \[3\], \[4\])). If a theory is perturbatively non-renormalizable and non-perturbatively finite in terms of bare or renormalised parameters (in the latter case ambiguities are hidden in the renormalised parameters), then the physical quantities do not contain arbitrary parts and the theory has predictive power. Let us assume that we found out somehow that a perturbatively non-renormalizable theory is non-perturbatively finite, but we do not have exact solutions. The question we want to address is whether we can extract any reliable physical information from perturbative expressions of physical quantities in this theory.

In \[1\] a new perturbative approach to non-renormalizable quantum field theory has been suggested. This method introduces a finite number of additional expansion parameters and assuming non-perturbative finiteness of the theory gives unambiguous series with finite coefficients for all physical quantities. Unfortunately at least at the moment being one can not argue that this series correctly reproduce the features of exact solutions (if they exist).

Due to the absence of exact solutions for physically relevant field-theoretical models the quantum mechanical examples with delta-function type potentials occur to be useful to investigate the above mentioned problem. This kind of potential, having zero-range or contact interaction, seems to be relevant from the point of view of field theory, considering the ultraviolet divergences as a trace of short-distance singularities. Some examples of regularization and renormalization of delta-function potentials in non-relativistic quantum mechanics have been considered in \[1\]–\[3\].

In the present paper we demonstrate that the perturbative approach to non-renormalizable quantum field theories suggested in \[1\] can lead to consistent results. We consider two examples, delta-function (with derivatives) potential in non-relativistic quantum mechanics and a mathematical model of field-theoretical propagator by Redmond and Uretsky \[13\] and show that the resulting series reproduce the expansions of exact analytic expressions.

II. δ-FUNCTION TYPE POTENTIAL IN NON-RELATIVISTIC QUANTUM MECHANICS

We start to elucidate the procedure of the perturbative treatment of non-renormalizable theories on the example of the quantum mechanical problem considered in \[1\]. In this example the amplitude is non-perturbatively finite in terms of two renormalised coupling constants so it contains only two arbitrary parameters which are fixed from two physical quantities. This model is perturbatively non-renormalizable i.e. to remove divergences one has to include an infinite number of additional (counter-)terms into the potential. The standard perturbative renormalization technique leads to the conclusion that the physical quantities depend on an infinite number of arbitrary parameters. The potential has the following form:

\[
<x|V|x'> = [C + C_2 (\nabla^2 + \nabla'^2)] \delta(x - x') \delta(x)
\]

with two (yet) unspecified parameters \(C\) and \(C_2\). One could object that this \(δ\)-type potential is not mathematically well defined. Note that we do not seek much physics in this potential. For our illustrative purposes we could take as our definition of the model the cutoff regularized potential with subsequent removal of cutoff. We would find that our perturbative approach reproduces the results of exact solutions.

The exact formal expression for the scattering amplitude in s-channel for \(E ≥ 0\) is (see \[1\]; we take the particle mass \(\mu = 1\)):

\[
T(E) = \frac{C + C_2^2 I_5 + 2EC_2 (2 - C_2 I_3)}{(C_2 I_3 - 1)^2 - I(E) [C + C_2^2 I_5 + 2EC_2 (2 - C_2 I_3)]}
\]

\(1\)
The integrals

\[ I(E) = 2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E^+ - k^2} = \frac{1}{\pi^2} P \int_0^\infty \frac{dk}{2E - k^2} - \frac{1}{2\pi} (2E)^{\frac{1}{2}}, \]

\[ I_3 = -2 \int \frac{d^3k}{(2\pi)^3}; \quad I_5 = -2 \int \frac{d^3k}{(2\pi)^3}k^2 \]

\((E^+ \equiv E + i\epsilon)\) and \(P\) denotes a principal value prescription) diverge as a linear, third and fifth power of some cut-off regulator. So far, the amplitude requires re-normalization.

In [9] the renormalization is carried out by choosing the scattering length \(a\) and the effective range \(r_e\) as the renormalization parameters and by fixing \(C_0\) and \(C_2\) demanding

\[ \frac{1}{T(E)} = -\frac{1}{2\pi} \left( -\frac{1}{a} + r_e E + O(E^2) - i(2E)^{\frac{1}{2}} \right) \]

Our aim is to analyse the possibility of extracting meaningful physical information from perturbation theory for this example and compare perturbative results with exact ones.

Let us bring up some results for the exact solution. The amplitude \(T(E)\) after simple and lengthy calculations may be expressed as

\[ T(E) = \frac{x(1 + xI_1 + 2Exy)}{1 + xI_1 - 2EI_1yx^2 - xW(E)(1 + xI_1 + 2Ey)}, \quad (2) \]

where

\[ Re(I(E)) = I(0) = I_1 = -2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \]

\(x = 2\pi a, \quad y = r_e/(4\pi)\) and \(W(E) = I(E) - I_1 = iImI(E)\).

Let us introduce the quantity

\[ \alpha^* (\mu^2) = \frac{1}{4gy^2} \left[ T(\mu^2) - T(\mu^2) \mid_{i_1=0} \right] \mid_{W=0} = \frac{\left(\mu^2\right)^2 xI_1}{1 + xI_1 - 2\mu^2I_1yx^2}. \quad (3) \]

In [3] to extract the part which is independent of \(W\) we temporarily considered \(W\) independent from \(\mu^2\) and put it equal to 0.

Now, extracting \(xI_1\) from [3] and substituting into expression [3] we get:

\[ T(E) = \frac{N}{D}, \quad (4) \]

where

\[ N = x + 2Ey^2 - \alpha (\mu^2) \left[ 2y^2 (E - \mu^2) - 4E\mu^2y^2x^3 \right] \quad (5) \]

\[ D = 1 - xW(E) + 2xy (\mu^2 - E) \left[ 2y^2 (E - \mu^2) - 4E\mu^2y^2x^3 \right] \]

\[ - 4y^2x^3 E\mu^2 W(E) \alpha (\mu^2) - 2y^2x^2 W(E) E \quad (6) \]

and

\[ \alpha(E) = \alpha^* (\mu^2) / \mu^2. \quad (7) \]

It is straightforward to check that the substitution of the value of \(\alpha\) from [7] and [3] into [4]-[6] leads to the expression [3] for \(T(E)\).
Below we demonstrate that, although the above discussed model is perturbatively non-renormalizable, we receive the finite perturbative expression for the amplitude by introducing an additional expansion parameter $\alpha$ and this result is in agreement with the exact solution.

One can easily solve the Lippman-Schwinger equation perturbatively for $T(E)$ in $s$-channel and obtain:

$$T(E) = C + 4EC_2 + 2CC_2I_3 + 8ECC_2I + C^2I + C^3I^2 + C^4I^3 + C^5I^4 + C^6I^5 +$$
$$C^2I_5 + 6C^2EI_3 + 4C^2C_2I_3 + 12C^2C_2EI^2 + 6C^3C_2I^2 + 16C^3C_2EI^3 + 16C^2E^2I + \ldots \quad (8)$$

Evidently, the expansion of the exact solution (9) in $C$ and $C_2$ coincides with (8). Designating

$$x = 2\pi a = T|_{E=0}, \quad y = \frac{r_s}{4\pi} = \frac{1}{2\pi^2} \frac{d\text{Re} T(E)}{dE}|_{E=0} + \frac{x}{4\pi^2} \quad (9)$$

we can express $C$ and $C_2$ iteratively from (9) and (10) as power series of $x$ and $y$:

$$C_2 = \frac{1}{2}x^2y - x^3yI_1 + \frac{3}{2}x^4yI_1^2 - 2x^5yI_1^3 - \frac{3}{8}x^4y^2I_3 + \ldots \quad (10)$$

and

$$C = x - x^2I_1 + x^3I_1^2 - x^4I_1^3 - x^5I_1^4 + x^5I_1^5 + 3I_1^3x^4y - x^6I_1^6 - 6x^5yI_1^7I_3 - \frac{1}{4}x^4y^2I_5 + \ldots \quad (11)$$

The substitution of (10) and (11) into (8) leads to the following expression for the amplitude:

$$T(E) = x + x^2W(E) + x^3W(E)^2 + x^6W(E)^3 + x^5W(E)^4 + x^6W(E)^5 + 4EW(E)x^3y +$$
$$2Exy^2 + 6EW(E)^2x^4y + 8EW(E)^3x^5y + 4E^2x^4y^2W(E) + 4E^2x^3y^2I_1 + \ldots \quad (12)$$

Now, in analogy with (12) we define

$$\alpha^* (\mu^2) = \frac{1}{4\pi^2y} [T(\mu^2) - T(\mu^2)|_{I_1=0}] |_{W=0} = \mu^2xI_1 + \ldots \quad (13)$$

Expressing $xI_1$ from (13) and substituting into (12) we get:

$$T(E) = x + x^2W(E) + x^3W(E)^2 + x^6W(E)^3 + x^5W(E)^4 + x^6W(E)^5 + 4EW(E)x^3y +$$
$$2Exy^2 + 6EW(E)^2x^4y + 8EW(E)^3x^5y + 4E^2x^4y^2W(E) + 4E^2x^3y^2\alpha (\mu^2) + \ldots \quad (14)$$

where $\alpha (\mu^2) = \alpha^* (\mu^2) / \mu^2$.

It is straightforward to check that the perturbative series (14) is the expansion of the exact result, given by (12). A tedious calculation shows that the same statement is true in the next order. It should be clear from the above discussion that the generated perturbative series in $x$, $y$ and $\alpha$ reproduces the expansion of the exact solution up to any orders.

To demonstrate that our approach can have practical sense let us make some numerical estimations. For simplicity let us take $\mu = 0$. (15) leads to the following expression:

$$T(E) = \frac{x + 2Exy^2 - 2gy^2\alpha(0)E}{1 - xW(E) - 2xy\alpha(0)E + 2x^2y\alpha(0)W(E) - 2x^2yW(E)E} \quad (15)$$

Let us take the numerical values for $x$, $y$, $\alpha$ and $E$ satisfying

$$xy\alpha(0)E << |ixW(E)| < 1 \quad (16)$$

Under this conditions we can approximate (13) by:

$$T(E) = \frac{x}{1 - xW(E)} \quad (17)$$
Expanding (17) in \( x \) we are led to the convergent series. On the other hand the perturbative expression under condition (16) leads to

\[
T(E) = x + x^2 W(E) + x^3 W(E)^2 + x^4 W(E)^3 + x^5 W(E)^4 + x^6 W(E)^5 + ...
\]  

(18)

This series coincides with the expansion of (17) and first few terms give good approximation to the exact result the next corrections being small. For example, let us take : \( \alpha(0) = 1, x = 1, xW = -0.1i \) and \( xyE = 0.001 \). (15) gives: \( T = 1/(1 + 0.1i) = 0.99(000) - 0.0(9900)i \) and (18) leads to: \( T = (1 - 0.01 + 0.0001 + ...) - i(0.1 - 0.001 + 0.00001 + ...) \) These simple numerical analysis are just to demonstrate that there exists a region of numerical values of the parameters where our perturbative approach is reliable numerically.

III. A MODEL FOR FIELD-THEORETICAL PROPAGATOR

To come closer to the quantum field theory problems let us demonstrate new perturbative approach on the mathematical model of the propagator of non-perturbatively finite and perturbatively non-renormalizable field theory by Redmond and Uretsky [13].

Let us consider the following expression

\[
\frac{1}{p^2 + \mu^2 - i\varepsilon} + \frac{g^2}{M^2} \int_{m_0^2}^{\infty} \frac{dm^2}{(p^2 + m^2 - i\varepsilon) \left[ (1 - g^2 \frac{m^2}{M^2})^2 + g^4 \right]}
\]  

(19)

as a mathematical model of the propagator of some field. Here \( \mu, M \) and \( g \) are some parameters and \( p \) is momenta. For the simplicity we take \( p^2 > 0 \). It is easy to check that the expansion of (14) in terms of \( g^2 \) produces divergences and one has to make an infinite number of subtractions to remove them. Consequently, final finite perturbative expression contains an infinite number of arbitrary parameters, in other words one would have to introduce an infinite number of counter-terms into the Lagrangian. On the other hand it is clear that (13) gives an unique result and it does not contain any ambiguous parameters.

Below we illustrate that the information extracted from non-renormalizable divergent perturbative series reproduces results of exact expression.

We omit the first finite term in (19) and define

\[
G(p^2) = \frac{g^2}{M^2} I(p^2)
\]  

(20)

where

\[
I(p^2) = \int_{m_0^2}^{\infty} \frac{dx}{(p^2 + x) \left[ (1 - g^2 \frac{m^2}{M^2})^2 + g^4 \right]}
\]

Let us introduce a cut-off and calculate the above integral:

\[
\int_{m_0^2}^{\Lambda^2} \frac{dx}{(p^2 + x) \left[ (1 - g^2 \frac{m^2}{M^2})^2 + g^4 \right]}
\]

\[
= \frac{1}{\Delta} \left\{ \frac{2ln p^2 + \Lambda^2}{p^2 + m_0^2} - \frac{g^4 + \left(1 - g^2 \frac{\Lambda^2}{M^2}\right)^2}{g^4 + \left(1 - g^2 \frac{m_0^2}{M^2}\right)^2} \left[ \arctan \frac{1 - g^2 \frac{m_0^2}{M^2}}{g^2} - \arctan \frac{1 - g^2 \frac{\Lambda^2}{M^2}}{g^2} \right] \right\}
\]

\[
= I(p^2, \Lambda)
\]  

(21)

where

\[
\Delta = 1 + 2 \frac{g^2}{M^2} p^2 + g^4 \left(1 + \frac{p^4}{M^4}\right)
\]  

(22)
Taking the limit $\Lambda \to \infty$ in (21) we get

$$
\int_{m_0^2}^{\infty} \frac{dx}{(p^2 + x)^2 + g^4} =
\frac{1}{\Delta} \left\{ 2\ln \frac{M^2}{p^2 + m_0^2} - \ln \frac{g^4}{g^4 + (1 - g^2 \frac{m_0^2}{M^2})^2} + \frac{1 + g^2 \frac{p^2}{M^2}}{g^2} \left[ \arctan \frac{1 - g^2 \frac{m_0^2}{M^2}}{g^2} + \frac{\pi}{2} \right] \right\}
$$

(23)

Note that the regularized expression (21) expanded in terms of $g$ has the following structure:

$$
\sum_{i,j,k=0,1,\ldots} (g^2)^i \left( g^2 \frac{\Lambda^2}{m_0^2} \right)^j \left( g^2 \frac{\Lambda^2}{M^2} \right)^k C_{ijk} (\Lambda^2, M^2, m_0, p^2)
$$

(24)

where $C_{ijk}$ are finite in the $\Lambda \to \infty$ limit. This structure (24) will play an important role in the following development.

In (21) we replace

$$
\ln \frac{p^2 + \Lambda^2}{p^2 + m_0^2} = \ln \frac{\Lambda^2}{m_0^2} + \ln \frac{m_0^2}{p^2 + m_0^2} + \ln \frac{\Lambda^2 + p^2}{\Lambda^2} \to \ln \frac{\Lambda^2}{m_0^2} + \ln \frac{m_0^2}{p^2 + m_0^2}.
$$

Let us define

$$
G (p^2, \Lambda) = \frac{g^2}{M^2} I (p^2, \Lambda),
$$

the “related” quantities (taking the structure (24) into account, where $g^2 \ln \frac{\Lambda^2}{m_0^2}$ is replaced by $g^2 \ln \frac{\Lambda^2}{m_0^2}$ and $g^2 \frac{\Lambda^2}{M^2}$ by $g^2 \frac{\Lambda^2}{M^2}$):

$$
M^2 \Delta (p^2) G^* (p^2, \Lambda) = 2 g^2 \ln \frac{\Lambda^2}{m_0^2} + 2 g^2 \ln \frac{m_0^2}{p^2 + m_0^2} - \ln \frac{g^4 + \left( 1 - g^2 \frac{\Lambda^2}{M^2} \right)^2}{g^4 + \left( 1 - g^2 \frac{m_0^2}{M^2} \right)^2} +
$$

$$
\frac{1 + g^2 \frac{p^2}{M^2}}{g^2} \left[ \arctan \frac{1 - g^2 \frac{m_0^2}{M^2}}{g^2} - \arctan \frac{1 - g^2 \frac{\Lambda^2}{m_0^2}}{g^2} \right]
$$

$$
F (\lambda_1^2, \lambda_2^2, \Lambda^2) = G^* (\lambda_1^2, \Lambda^2) \Delta (\lambda_1^2) - G^* (\lambda_2^2, \Lambda^2) \Delta (\lambda_2^2) =
$$

$$
= \frac{g^2}{M^2} \left\{ -2 \ln \frac{\lambda_1^2 + m_0^2}{\lambda_1^2 + m_0^2} + \frac{\lambda_1^2 - \lambda_2^2}{M^2} \left[ \arctan \frac{1 - g^2 \frac{m_0^2}{\lambda_1^2}}{g^2} - \arctan \frac{1 - g^2 \frac{\Lambda^2}{m_0^2}}{g^2} \right] \right\},
$$

(25)

and

$$
\alpha = \frac{M^2}{g^2 (\lambda_1^2 - \lambda_2^2)} \left( \frac{M^2}{g^2} F (\lambda_1^2, \lambda_2^2, \Lambda^2) + 2 \ln \frac{\lambda_1^2 + m_0^2}{\lambda_2^2 + m_0^2} \right)
$$

(26)

From (25) and (26)

$$
\tan (g^2 \alpha) = \frac{g^2 \frac{g^2}{\frac{\lambda_1^2 + \Lambda^2}{\lambda_1^2}} - g^4 \frac{m_0^2}{M^2}}{g^4 + \left( 1 - g^2 \frac{m_0^2}{M^2} \right) \left( 1 - g^2 \frac{\Lambda^2}{M^2} \right)}.
$$

(27)

Extracting $g^2 \ln \frac{\Lambda^2}{m_0^2}$ and $g^2 \frac{\Lambda^2}{M^2}$ from (27) and

$$
G^* (\lambda_1^2, \Lambda^2) = \frac{g^2}{M^2} I (\lambda_1^2, \Lambda^2)
$$

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and substituting into $G^* (Q^2, \Lambda^2)$ we get:

$$G^* (Q^2, \Lambda^2) =$$

$$= \frac{1}{M^2 \Delta (Q^2)} \left( \Delta (\lambda_1^2) M^2 G^* (\lambda_1^2, \Lambda^2) + 2g^2 \ln \frac{\lambda_2^2}{Q^2 + m_0^2} + \frac{Q^2 - \lambda_1^2}{M^2} \alpha g^4 \right)$$

Taking the limit $\Lambda^2 \to \infty$ and substituting $g_1 = g$, $g_2 = g$ we get:

$$G (Q^2) =$$

$$= \frac{1}{M^2 \Delta (Q^2)} \left( \Delta (\lambda_1^2) M^2 G (\lambda_1^2) + 2g^2 \ln \frac{\lambda_1^2 + m_0^2}{Q^2 + m_0^2} + \frac{Q^2 - \lambda_1^2}{M^2} \alpha g^4 \right). \quad (28)$$

Thus we have expressed the finite quantity $G (Q^2)$ in terms of other finite quantities $g^2$, $G (\lambda_1^2)$ and $\alpha$. It is straightforward to check that by substituting $G (\lambda_1^2)$ and $\alpha$ into (28) we are led to the correct expression (given by (20) and (23)) for $G (Q^2)$.

On the other hand, expansion of (24) in terms of $g^2$ gives the perturbative expression for $G (p^2, \Lambda^2)$:

$$M^2 G (p^2, \Lambda^2) = 2g^2 \ln \frac{\Lambda_0^2}{m_0^2} - 2g^2 \ln \frac{m_0^2 + p^2}{m_0^2} + 2g^4 \Lambda_0^2 - 2g^4 \frac{m_0^2}{M^2} - 4g^4 \frac{p^2}{M^2} \ln \frac{\Lambda^2}{m_0^2} +$$

$$+ 4g^4 \frac{p^2}{M^2} \ln \frac{m_0^2 + p^2}{m_0^2} + 3 \frac{2g^6 \Lambda^4}{M^4} - 3 \frac{2g^6 \Lambda_0^4}{M^4} - 3g^6 \frac{p^2 m_0^2}{M^4} + 3g^6 \frac{p^4 m_0^2}{M^4} + 6g^6 \frac{p^4}{M^4} \ln \frac{\Lambda^2}{m_0^2}$$

$$= 2g^2 \ln \frac{\Lambda_0^2}{m_0^2} - 2g^2 \ln \frac{m_0^2 + p^2}{m_0^2} + 2g^4 \frac{\Lambda_0^2}{M^2} - 2g^4 \frac{m_0^2}{M^2} - 4g^4 \frac{p^2}{M^2} \ln \frac{\Lambda_0^2}{m_0^2} +$$

$$+ 4g^4 \frac{p^2}{M^2} \ln \frac{m_0^2 + p^2}{m_0^2} + 3 \frac{2g^6 \Lambda_0^4}{M^4} - 3 \frac{2g^6 \Lambda^4}{M^4} - 3g^6 \frac{p^2 \Lambda_0^2}{M^4} + 3g^6 \frac{p^4 m_0^2}{M^4} + 6g^6 \frac{p^4}{M^4} \ln \frac{\Lambda_0^2}{m_0^2}$$

where we have replaced $\ln \frac{\Lambda^2}{m_0^2} + \ln \frac{m_0^2}{p^2 + m_0^2}$ by $\frac{\Lambda^2}{m_0^2} + \ln \frac{m_0^2}{p^2 + m_0^2}$ and $\frac{\Lambda_0^2}{m_0^2}$ by $\frac{\Lambda^2}{m_0^2}$ in (28).

Let us introduce “related” quantities (taking the structure (24) into account $g^2 \ln \frac{\Lambda_0^2}{m_0^2}$ is replaced by $g_1^2 \ln \frac{\Lambda_0^2}{m_0^2}$ and $g^2 \frac{\Lambda_0^2}{M^2}$ by $g_2^2 \frac{\Lambda_0^2}{M^2}$ in (28)):

$$M^2 G^* (p^2, \Lambda^2) = 2g_1^2 \ln \frac{\Lambda_0^2}{m_0^2} - 2g^2 \ln \frac{m_0^2 + p^2}{m_0^2} + 2g^2 g_2^2 \frac{\Lambda_0^2}{M^2} - 2g^4 \frac{m_0^2}{M^2} - 4g^4 g_2^2 \frac{p^2}{M^2} \ln \frac{\Lambda_0^2}{m_0^2} +$$

$$+ 4g^4 \frac{p^2}{M^2} \ln \frac{m_0^2 + p^2}{m_0^2} + 3 \frac{2g^6 \Lambda_0^4}{M^4} - 3 \frac{2g^6 \Lambda^4}{M^4} - 3g^6 \frac{p^2 \Lambda_0^2}{M^4} + 3g^6 \frac{p^4 m_0^2}{M^4} + 6g^6 \frac{p^4}{M^4} \ln \frac{\Lambda_0^2}{m_0^2}$$

and

$$M^2 F (\lambda_1^2, \lambda_2^2, \Lambda^2) = M^2 \{ G^* (\lambda_1^2, \Lambda^2) \Delta (\lambda_1^2) - G^* (\lambda_2^2, \Lambda^2) \Delta (\lambda_2^2) \} =$$

$$= g_2^2 \frac{\Lambda^2}{M^2} (\lambda_1^2 - \lambda_2^2) - g^6 \frac{m_0^2}{M^4} (\lambda_1^2 - \lambda_2^2) - g^2 \ln \frac{m_0^2 + \lambda_1^2}{m_0^2 + \lambda_2^2} + \ldots \quad (30)$$

where $\Delta$ is defined by (22).

Expressing $g_1^2 \ln \frac{\Lambda_0^2}{m_0^2}$ from $M^2 G^* (\lambda_1^2, \Lambda^2)$ we get (The structure (24) guarantees that $g_1^2 \ln \frac{\Lambda_0^2}{m_0^2}$ and $g_2^2 \frac{\Lambda_0^2}{M^2}$ can be extracted perturbatively from physical quantities up to arbitrarily large order of expansion parameters):

$$g_1^2 \ln \frac{\Lambda_0^2}{m_0^2} = \frac{1}{2} M^2 G^* (\lambda_1^2, \Lambda^2) + \frac{1}{2} g_2^2 \ln \frac{m_0^2 + \lambda_1^2}{m_0^2} + \frac{\lambda_1^2}{M^2} g_2^2 G^* (\lambda_1^2, \Lambda^2) - g^2 g_2^2 \frac{\Lambda_0^2}{M^2} +$$
\[ + \frac{m_0^2}{M^2} g^4 + \frac{1}{2} \left( 1 + \frac{\lambda_1^2}{M^4} \right) G^* (\lambda_1^2, \Lambda^2) g^4 + \left( 3 \frac{m_0^4}{4 M^4} + \frac{\lambda_1^2 m_0^2}{2 M^4} \right) g^6 - \frac{3}{4} g^2 g^2 \frac{\Lambda^2}{M^4} - \frac{\lambda_1^2}{M^2} g^4 g^2 \frac{\Lambda^2}{2M^2} + \ldots \] (31)

From (30)
\[ g^2 \frac{\Lambda^2}{M^2} = g \frac{m_0^2}{M^2} + \alpha + \ldots \] (32)

where
\[ \alpha = \frac{M^2}{g^2 (\lambda_1^2 - \lambda_2^2)} \left( \frac{M^2}{g^2} F(\lambda_1^2, \lambda_2^2, \Lambda^2) + \ln \frac{\lambda_1^2 + m_0^2}{\lambda_2^2 + m_0^2} \right) \]

Substituting (31) and (32) into perturbative expression of \( G^* (Q^2, \Lambda^2) \), taking \( g_1 = g_2 = g \) in the limit \( \Lambda^2 \to \infty \) we get:
\[ G (Q^2) = G (\lambda_1^2) + \frac{2 g^2}{M^2} \ln \frac{m_0^2 + \lambda_1^2}{m_0^2 + Q^2} + 2 \frac{\lambda_1^2 - Q^2}{M^2} g^2 G (\lambda_1^2) + \]
\[ + \frac{\lambda_1^4 - 3 \lambda_1^2 Q^2 + 4 Q^4}{M^4} G (\lambda_1^2) g^4 + \frac{Q^2 - \lambda_1^2}{M^2} g^4 \alpha + 4 g^4 \frac{Q^2}{M^4} \ln \frac{m_0^2 + Q^2}{m_0^2 + \lambda_1^2} + \]
\[ \frac{2 g^6}{M^2} \ln \frac{m_0^2 + Q^2}{m_0^2 + \lambda_1^2} \left( 1 - 3 \frac{Q^4}{M^4} \right) + \ldots \] (33)

It is easy to check that (33) coincides to the expansion of (28). So the perturbation theory reproduces the results of exact solution.

**IV. CONCLUSIONS**

The perturbative approach to non-renormalizable theories [1] based on introduction of a finite number of additional expansion parameters correctly reproduces the exact results for perturbatively non-renormalizable and non-perturbatively finite quantum mechanical problem. If the model under consideration would be realistic, one could extract the values of the parameters from the observables and compare the predictions of (perturbation) theory with experimental data, while the standard perturbative renormalization technique requires the introduction of infinite number of additional parameters and the theory has no predictive power. The same is true for the mathematical model of field-theoretical propagator by Redmond and Uretsky.

Note that there exists infinite number of choices for additional expansion parameters. This flexibility is the realization of the freedom in choice of renormalization scheme. Surely, the different normalisation schemes are not equivalent from the point of view of numerical convergence. Although in general the problem of numerical convergence of perturbative series (as well, as conventional series, arising in renormalizable theories) remains open, we are optimistic about applications of ideas sketched in [1] to the problems of non-renormalizable quantum field theories and in particular to quantum (Einstein’s) gravity.

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