A NOTE ON STRONGLY REAL BEAUVILLE $p$-GROUPS

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Abstract. We give an infinite family of non-abelian strongly real Beauville $p$-groups for any odd prime $p$ by considering the lower central quotients of the free product of two cyclic groups of order $p$. This is the first known infinite family of non-abelian strongly real Beauville $p$-groups.

1. Introduction

A Beauville surface of unmixed type is a compact complex surface isomorphic to $(C_1 \times C_2)/G$, where $C_1$ and $C_2$ are algebraic curves of genus at least 2 and $G$ is a finite group acting freely on $C_1 \times C_2$ and faithfully on the factors $C_i$ such that $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$. Then the group $G$ is said to be a Beauville group.

The condition for a finite group $G$ to be a Beauville group can be formulated in purely group-theoretical terms.

Definition 1.1. For a couple of elements $x, y \in G$, we define

$$\Sigma(x, y) = \bigcup_{g \in G} \left( \langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right),$$

that is, the union of all subgroups of $G$ which are conjugate to $\langle x \rangle$, to $\langle y \rangle$ or to $\langle xy \rangle$. Then $G$ is a Beauville group if and only if the following conditions hold:

(i) $G$ is a 2-generator group.
(ii) There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of $G$ such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a Beauville structure for $G$.

Definition 1.2. Let $G$ be a Beauville group. We say that $G$ is strongly real if there exists a Beauville structure $\{\{x_1, y_1\}, \{x_2, y_2\}\}$ such that there exist an automorphism $\theta \in \text{Aut}(G)$ and elements $g_i \in G$ for $i = 1, 2$ such that

$$g_i \theta(x_i) g_i^{-1} = x_i^{-1} \quad \text{and} \quad g_i \theta(y_i) g_i^{-1} = y_i^{-1}$$

for $i = 1, 2$. Then the Beauville structure is called strongly real Beauville structure.

In practice, it is convenient to take $g_1 = g_2 = 1$.

In 2000, Catanese [3] proved that a finite abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, where $n > 1$ and $\text{gcd}(n, 6) =$

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1. Since for any abelian group the function $x \mapsto -x$ is an automorphism, the following result is immediate.

**Lemma 1.3.** Every abelian Beauville group is a strongly real Beauville group.

Thus, there are infinitely many abelian strongly real Beauville $p$-groups for $p \geq 5$.

Recall that the only known infinite family of Beauville 2-groups was constructed in [1]. However, one of the main results in [1] shows that these Beauville 2-groups are not strongly real. On the other hand, in [5], Fairbairn has recently given the following examples of strongly real Beauville 2-groups. The groups

$$G = \langle x, y \mid x^8 = y^8 = [x^2, y^2] = (x^i y^j)^4 = 1 \text{ for } i, j = 1, 2, 3 \rangle,$$

and

$$G = \langle x, y \mid (x^i y^j)^4 = 1 \text{ for } i, j = 0, 1, 2, 3 \rangle$$

are strongly real Beauville groups of order $2^{13}$ and $2^{14}$, respectively.

If $p \geq 3$ there is no known example of a non-abelian strongly real Beauville $p$-group. Thus, up to now the only examples of strongly real Beauville $p$-groups are the abelian ones and the two groups given above.

In this paper, we give infinitely many non-abelian strongly real Beauville $p$-groups for any odd prime $p$. To this purpose, we work with the lower central quotients of the free product of two cyclic groups of order $p$. The main result of this paper is as follows.

**Theorem A.** Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order $p$ for an odd prime $p$, and let $i = k(p - 1) + 1$ for $k \geq 1$. Then the quotient $F/\gamma_{i+1}(F)$ is a strongly real Beauville group.

Note that in [7], it was recently shown that all $p$-central quotients of the free product $F = \langle x, y \mid x^p, y^p \rangle$ are Beauville groups. Observe that since $F/F'$ has exponent $p$, the lower central series and $p$-central series of $F$ coincide.

2. Proof of the main theorem

In this section, we give the proof of Theorem A. Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order $p$. We begin by stating a lemma regarding the existence of an automorphism of $F$ which sends the generators to their inverses. The proof is left to the reader.

**Lemma 2.1.** Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order $p$. Then the map

$$\theta : F \rightarrow F$$

$$x \mapsto x^{-1},$$

$$y \mapsto y^{-1},$$

is an automorphism of $F$.  

Before we proceed, we will introduce some results regarding the Nottingham group which will help us to determine some properties of $F$.

The *Nottingham group* $\mathcal{N}$ over the field $\mathbb{F}_p$, for odd $p$, is the (topological) group of normalized automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series. For any positive integer $k$, the automorphisms $f \in \mathcal{N}$ such that $f(t) = t + \sum_{i \geq k+1} a_i t^i$ form an open normal subgroup $\mathcal{N}_k$ of $\mathcal{N}$ of index $p^{k-1}$.

Observe that $|\mathcal{N}_k : \mathcal{N}_{k+1}| = p$ for all $k \geq 1$. We have the commutator formula

$$[\mathcal{N}_k, \mathcal{N}_\ell] = \begin{cases} \mathcal{N}_{k+\ell}, & \text{if } k \not\equiv \ell \pmod{p}, \\ \mathcal{N}_{k+\ell+1}, & \text{if } k \equiv \ell \pmod{p} \end{cases}$$

(see [2], Theorem 2). Thus the lower central series of $\mathcal{N}$ is given by

$$(1) \quad \gamma_i(\mathcal{N}) = \mathcal{N}_{r(i)}, \quad \text{where } r(i) = i + 1 + \left\lfloor \frac{i-2}{p-1} \right\rfloor.$$

As a consequence, $|\gamma_i(\mathcal{N}) : \gamma_{i+1}(\mathcal{N})| \leq p^2$, and we have ‘diamonds’ of order $p^2$ if and only if $i$ is of the form $i = k(p-1) + 1$ for some $k \geq 0$. In other words, the diamonds in the lower central series of $\mathcal{N}$ correspond to quotients $\mathcal{N}_{kp+1}/\mathcal{N}_{kp+3}$.

Recall that by Remark 3 in [2], $\mathcal{N}$ is topologically generated by the elements $a \in \mathcal{N}_1 \setminus \mathcal{N}_2$ and $b \in \mathcal{N}_2 \setminus \mathcal{N}_3$ given by $a(t) = t(1-t)^{-1}$ and $b(t) = t(1-2t)^{-1/2}$, which are both of order $p$.

In the following lemma, we need a result of Klopsch [8] formula (3.4) regarding the centralizers of elements of order $p$ of $\mathcal{N}$ in some quotients $\mathcal{N}/\mathcal{N}_k$. More specifically, if $f \in \mathcal{N}_k \setminus \mathcal{N}_{k+1}$ is of order $p$, then for every $\ell = k+1+pn$ with $n \in \mathbb{N}$, we have

$$(2) \quad C_{\mathcal{N}/\mathcal{N}_\ell}(f\mathcal{N}_\ell) = C_{\mathcal{N}(f)\mathcal{N}_{\ell-k}/\mathcal{N}_\ell}.$$

**Lemma 2.2.** Put $G = \mathcal{N}/\mathcal{N}_{kp+3}$ and $N_i = \mathcal{N}_i/\mathcal{N}_{kp+3}$ for $1 \leq i \leq kp+3$. If $\alpha$ is the image of $a$ in $G$, then the set $\{[\alpha, g] \mid g \in G\}$ does not cover $N_{kp+1}$.

**Proof.** To prove the lemma, we will show that $\{[\alpha, g] \mid g \in G\} \cap N_{kp+2} = 1$. Assume that $[\alpha, g] \in N_{kp+2}$ for some $g \in G$. Since $a \in \mathcal{N}_1 \setminus \mathcal{N}_2$ is of order $p$, it follows from (2) that

$$C_{\mathcal{N}/N_{kp+2}}(a\mathcal{N}_{kp+2}) = C_{\mathcal{N}(a)\mathcal{N}_{kp+1}/\mathcal{N}_{kp+2}}.$$

Thus we can write $g = ch$, with $[\alpha, c] = 1$ and $h \in N_{kp+1}$. Then $[\alpha, g] = [\alpha, h] \in [G, N_{kp+1}] = 1$, since $N_{kp+1}$ is central in $G$. \hfill $\square$

**Lemma 2.3.** Put $H = F/\gamma_{i+1}(F)$, where $i = k(p-1) + 1$ for $k \geq 1$ and $H_i = \gamma_i(F)/\gamma_{i+1}(F)$. If $u$ and $v$ are the images of $x$ and $y$ in $H$, respectively, then the sets $\{[u, h] \mid h \in H\}$ and $\{[v, h] \mid h \in H\}$ do not cover $H_i$.

**Proof.** Let $G = \mathcal{N}/\mathcal{N}_{kp+3}$, and let us call $\alpha$ and $\beta$ the images of $a$ and $b$ in $G$, respectively. Since $\alpha$ and $\beta$ are of order $p$ and $\gamma_{i+1}(G) = 1$, the map

$$\psi: H \longrightarrow G$$

$$u \longmapsto \alpha$$

$$v \longmapsto \beta,$$

is well-defined and an epimorphism.
By Lemma 2.2, the set of commutators of $\alpha$ does not cover the subgroup $\gamma_i(G) = N_{kp+1}$. It then follows that the set $\{[u, h] \mid h \in H\}$ does not cover $H_i$. Since the roles of $u$ and $v$ are symmetric, we also conclude that the set $\{[v, h] \mid h \in H\}$ does not cover $H_i$, as desired.

\[ \square \]

To prove the main result, we need the following three lemmas.

**Lemma 2.4.** Let $G = \langle a, b \rangle$ be a 2-generator $p$-group and $o(a) = p$, for some prime $p$. Then

\[ \left( \bigcup_{g \in G} \langle a \rangle^g \right) \cap \left( \bigcup_{g \in G} \langle b \rangle^g \right) = 1. \]

**Proof.** We assume that $x = (a^i)^g = (b^j)^h$ for some $i, j \in \mathbb{Z}$ and $g, h \in G$, and prove that $x = 1$. In the quotient $G/G\Phi(G) = \Phi(G)/Phi(G) = (\Phi(G) \cup \bar{b})$, we have $\Phi(G) \cup \bar{b} = \bar{g}$ implying that $x \in \Phi(G)$. On the other hand, $x \in \langle a \rangle^g$, where $a^g$ is of order $p$ and $a^g \notin \Phi(G)$. It then follows that $x = 1$.

\[ \square \]

**Lemma 2.5.** [6] Let $G$ be a finite $p$-group and let $x \in G \setminus \Phi(G)$ be an element of order $p$. If $t \in \Phi(G) \setminus \{[x, y] \mid y \in G\}$ then

\[ \left( \bigcup_{g \in G} \langle x \rangle^g \right) \cap \left( \bigcup_{g \in G} \langle xt \rangle^g \right) = 1. \]

**Lemma 2.6.** [2] Let $G_1 \rightarrow G_2$ be a group homomorphism, let $x_1, y_1 \in G_1$ and $x_2 = \psi(x_1), y_2 = \psi(y_1)$. If $o(x_1) = o(x_2)$ then the condition $\langle x_2 \rangle \cap \langle y_2 \rangle = 1$ implies that $\langle x_1 \rangle \cap \langle y_1 \rangle = 1$ for $g, h \in G_1$.

Let $H = F/\gamma_i+1(F)$ and let $u$ and $v$ be the images of $x$ and $y$ in $H$, respectively. In order to prove the main theorem, we need to know the order of $uv$. We first recall a result of Easterfield [4] regarding the exponent of $\Omega_j(G)$. More precisely, if $G$ is a $p$-group, then for every $j, k \geq 1$, the condition $\gamma_k(p-1)+1(G) = 1$ implies that

\[ \exp \Omega_j(G) \leq p^{j+k-1}. \]

If we set $k = \left\lfloor \frac{1}{p-1} \right\rfloor$, we have $\gamma_k(p-1)+1(H) \leq \gamma_i+1(H) = 1$. Then by [4], we get $\exp H \leq p^k$, and hence $o(\theta^i+1) \leq p^k$. Indeed, we will show that $o(\theta^{p-1}+\ldots+\theta+1) = p^k$. To this purpose, we also need to introduce a result regarding $p$-groups of maximal class with some specific properties.

Let $G = \langle s \rangle \rtimes A$ where $s$ is of order $p$ and $A \cong \mathbb{Z}_p^{p-1}$. The action of $s$ on $A$ is via $\theta$, where $\theta$ is defined by the companion matrix of the $p$th cyclotomic polynomial $x^{p-1} + \ldots + x + 1$. Then $G$ is the only infinite pro-$p$ group of maximal class. Since $s^p = 1$ and $\theta^{p-1}+\ldots+\theta+1$ annihilates $A$, this implies that for every $a \in A$,

\[ (sa)^p = s^pa^{s^{p-1}+\ldots+s+1} = 1. \]

Thus all elements in $G \setminus A$ are of order $p$.

Let $P$ be a finite quotient of $G$ of order $p^{i+1}$ for $i \geq 2$. Let us call $P_1$ the abelian maximal subgroup of $P$ and $P_j = [P_1, P_{j-1}, P] = \gamma_j(P)$ for
Let $j \geq 2$. Then one can easily check that $\exp P_j = p^{\lceil \frac{j+1}{p-1} \rceil}$ and every element in $P_j \setminus P_{j+1}$ is of order $p^{\lceil \frac{j+1}{p-1} \rceil}$.

Let $s \in P \setminus P_1$ and $s_1 \in P_1 \setminus P'$. Since all elements in $P \setminus P_1$ are of order $p$ and $\gamma_i(P) = 1$, the map

$$
\psi : H \to P
$$

$$
\psi(u) = s^{-1},
$$

$$
\psi(v) = ss_1,
$$

is well-defined and an epimorphism. Then we have $o(uv) \geq o(s_1) = p^k$, and this, together with $\exp H = p^k$, implies that $o(uv) = p^k$.

We are now ready to give the proof of main theorem.

**Theorem 2.7.** Let $p \geq 3$, and let $i = k(p - 1) + 1$ for $k \geq 1$. Then the quotient $F/\gamma_i(F)$ is a strongly real Beauville group.

**Proof.** Let $H$ and $H_i$ be as defined in Lemma 2.3. Let $u$ and $v$ be the images of $x$ and $y$ in $H$, respectively. By Lemma 2.3 there exist $w, z \in H_i$ such that $w \not\in \{[u, h] \mid h \in H\}$ and $z \not\in \{[v, h] \mid h \in H\}$. Observe that $w$ and $z$ are central elements of order $p$ in $H$. We claim that $\{u, v\}$ and $\{(uw)^{-1}, vz\}$ form a Beauville structure in $H$. Let $X = \{u, v, uv\}$ and $Y = \{(uw)^{-1}, vz, u^{-1}vw^{-1}z\}$.

Assume first that $x \in X$ is of order $p$, and let $y \in Y$. If $\langle x\Phi(H) \rangle \neq \langle y\Phi(H) \rangle$ in $H/\Phi(H)$, then by Lemma 2.4, $\langle x \rangle^g \cap \langle y \rangle^h = 1$ for every $g, h \in H$. Otherwise, we are in one of the following two cases: $x = u$ and $y = (uv)^{-1}$, or $x = v$ and $y = vz$. Then the condition $\langle x \rangle^g \cap \langle y \rangle^h = 1$ follows by Lemma 2.5.

We now assume that $x = uv$. Again applying Lemma 2.4 we get $\langle x \rangle^g \cap \langle y \rangle^h = 1$ where $y = (uv)^{-1}$ or $y = vz$, which is of order $p$. Thus we are only left with the case when $x = uv$ and $y = u^{-1}vw^{-1}z$. Recall that the map $\psi : H \to P$ is an epimorphism such that $\psi(u) = s^{-1}$ and $\psi(v) = ss_1$. Then $\psi(u^{-1}vw^{-1}z)$ is an element outside $P_1$, which is of order $p$. Thus $\langle \psi(u^{-1}vw^{-1}z) \rangle \cap \langle s_1 \rangle = 1$ for all $g, h \in H$. Since $o(uv) = o(s_1)$, the condition $\langle x \rangle^g \cap \langle y \rangle^h = 1$ for all $g, h \in H$ follows by Lemma 2.6. This completes the proof that $G$ is a Beauville group.

We next show that the Beauville structure $\{(u, v), (uv)^{-1}, vz\}$ is strongly real. By Lemma 2.4 we know that the map $\theta$ is an automorphism of $F$. Since $\theta(\gamma_n(F)) = \gamma_n(\theta(F)) = \gamma_n(F)$ for all $n \geq 1$, the map $\theta$ induces an automorphism $\overline{\theta} : H \to H$ such that $\overline{\theta}(u) = u^{-1}$ and $\overline{\theta}(v) = v^{-1}$. Now we only need to check if $\overline{\theta}((uv)^{-1}) = uv$ and $\overline{\theta}(vz) = (vz)^{-1}$. Note that

$$
\overline{\theta}((uv)^{-1}) = \overline{\theta}(uv)u = u\overline{\theta}(u),
$$

and

$$
\overline{\theta}(vz) = v^{-1}\overline{\theta}(z) = \overline{\theta}(z)v^{-1}
$$

where the last equalities follow from the fact that both $w$ and $z$ are central in $H$. Thus it suffices to see that $\overline{\theta}(w^{-1}) = w$ and $\overline{\theta}(z) = z^{-1}$.

Note that $H_i$ is generated by the commutators of length $i$ in $u$ and $v$. Since $i$ is odd and $H_i \leq Z(G)$, it follows that

$$
\overline{\theta}([x_{j_1}, x_{j_2}, \ldots, x_{j_l}]) = [x_{j_1}^{-1}, x_{j_2}^{-1}, \ldots, x_{j_l}^{-1}] = [x_{j_1}, x_{j_2}, \ldots, x_{j_l}]^{-1},
$$

for all $j_1, j_2, \ldots, j_l \leq i$.
where each $x_j$ is either $u$ or $v$. Hence the automorphism $\theta$ sends the generators of $H_i$ to their inverses. Since $H_i$ is abelian, this implies that for every $t \in H_i$ we have $\theta(t) = t^{-1}$. \hfill \Box

\textbf{References}

[1] N. Barker, N. Boston, N. Peyerimhoff, and A. Vdovina, An infinite family of 2-groups with mixed Beauville structures, \textit{Int. Math. Res. Notices} \textbf{11} (2015), 3598–3618.

[2] R. Camina, The Nottingham group, in New Horizons in Pro-p Groups, editors M. du Sautoy, D. Segal, A. Shalev, \textit{Progress in Mathematics}, Volume 184, Birkhäuser, 2000, pp. 205–221.

[3] F. Catanese, Fibred surfaces, varieties isogenous to a product and related moduli spaces, \textit{Amer. J. Math.} \textbf{122} (2000), 1–44.

[4] T.E. Easterfield, The orders of products and commutators in prime power groups, \textit{Proc. Cambridge Phil. Soc.} \textbf{36} (1940), 14–26.

[5] B. Fairbairn, More on strongly real Beauville groups, in Symmetries in Graphs, Maps, and Polytopes, editors J. Siran, R. Jajcay, \textit{Springer Proceedings in Mathematics & Statistics}, Volume 159, Springer, 2016, pp. 129–146.

[6] G.A. Fernández-Alcober and Ş. Gül, Beauville structures in finite $p$-groups, preprint, available at \texttt{arXiv:1507.02942v2 [math.GR]}.

[7] Ş. Gül, Beauville structures in $p$-central quotients, to appear in \textit{J. Group Theory}, \texttt{arXiv:1604.06031 [math.GR]}.

[8] B. Klopsch, Automorphisms of the Nottingham group, \textit{J. Algebra} \textbf{223} (2000), 37–56.

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