FLOW SOLUTIONS OF TRANSPORT EQUATIONS

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Abstract. Under general assumptions on the velocity field, it is possible to construct a flow that is forward untangled. Once such a flow has been selected, the associated transport problem is well-posed.

1. Introduction

Transport processes are ubiquitous in the natural and engineering sciences. There are two complementary ways of describing such phenomena: Lagrangian and Eulerian. The Lagrangian approach tracks the motion of transported quantities (mass, charge). Its mathematical formulation involves ordinary differential equations and flow maps. The Eulerian approach is based on the density of the transported quantity, which is a function of time and space. Its evolution is described by a first-order pde called the continuity equation. The two approaches are closely related.

In this paper, we consider transport along a velocity field \( b \) in a spatial domain \( \Omega \subset \mathbb{R}^d \). The continuity equation for the density \( \rho \geq 0 \) then takes the form

\[
\partial_t \rho + \text{div}(\rho b) = 0 \quad \text{in} \quad (0,T) \times \Omega. \tag{1.1}
\]

We assume that initial data \( \rho(0,\cdot) = \bar{\rho} \) is given with \( \bar{\rho} \) a nonnegative Borel measure. Equation (1.1) must be interpreted in the weak (distributional) sense; see below. For simplicity, we will not consider boundary conditions allowing for in/outflow through the boundary \( \partial \Omega \) or time-dependent spatial domains. While these generalizations are very relevant for many applications, here we just insist that the transported quantity, described through its density \( \rho \), does not exit the domain \( \Omega \).

On the Lagrangian side, we are instead concerned with solutions to

\[
\dot{\gamma}(t) = b(t,\gamma(t)), \quad \gamma(0) = x \tag{1.2}
\]

for \( t \in [0,T] \) and \( x \in \Omega \). If (1.2) can be solved for all \( x \), then we define the flow

\[
X : [0,T] \times \Omega \rightarrow \mathbb{R}^d, \quad X(t,x) := \gamma_x(t) \quad \text{where} \quad \gamma_x \text{ is the solution of (1.2).} \tag{1.3}
\]

The solution of (1.1) can then be recovered though the push-forward

\[
\rho(t,\cdot) := X(t,\cdot) \# \bar{\rho} \quad \text{for all} \quad t \in [0,T]. \tag{1.4}
\]

This representation may not be unique if the flow is not.

The existence of solutions to (1.2) locally in time can be established whenever \( b \) is continuous in space, by Cauchy-Peano theorem. In order for \( \gamma(t) \) to remain inside of \( \Omega \), it is necessary that at the boundary \( \partial \Omega \) the velocity field \( b \) points back

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into \( \Omega \). Moreover, a mild assumption on the growth of \( b(t, x) \) as \( |x| \to \infty \) ensures that solutions of (1.2) exist for all times \( t \); see Section 3 for details. If the map \( x \mapsto b(t, x) \) is Lipschitz continuous, then the solutions of (1.2) are unique.

On the other hand, if the velocity field \( b \) is not Lipschitz in space, then existence and uniqueness for (1.1)/(1.2) are much more subtle. Some regularity of \( b \) is needed to establish that solutions do exist at all. One possible strategy is to approximate the continuity equation (1.1) by a parabolic regularization, as is done in [6]. In order to establish convergence to a solution of (1.1) (thereby proving existence) one must show that the commutator \( \text{div}((P_s \varrho)(b) - P_s(\text{div}(\varrho b))) \) vanishes strongly as \( s \to 0 \), where \( P_s \) denotes a suitable regularizing semigroup. A sufficient condition for this is Sobolev regularity of \( b \); see DiPerna-Lions [31]. One possible strategy for showing uniqueness of solutions is to establish the renormalization property, which means that if \( \varrho \) satisfies the continuity equation (1.1), then \( \beta(\varrho) \) satisfies
\[
\partial_t \beta(\varrho) + \text{div}(\beta(\varrho)b) + (\beta(\varrho)\varrho - \beta(\varrho)) \text{div}b = 0 \quad \text{in} \quad (0, T) \times \Omega,
\]
for smooth functions \( \beta \). This is a consequence of the chain rule if all quantities are smooth. In general, it requires a more sophisticated argument; see [6]. Typically we will not distinguish between a measure that is absolutely continuous with respect to Lebesgue measure and its Radon-Nikodým derivative, to simplify notation.

Quite often, constructing approximate solutions \( b_\varepsilon \) of (1.1) also involves approximating the velocity field, e.g., by convolution \( b_\varepsilon := b * \varphi_\varepsilon \) with some mollifier \( \varphi_\varepsilon \). Since typically \( b_\varepsilon \) only converges weak* in the sense of measures it raises the issue of passing to the limit in the approximate “momentum” \( \varrho_\varepsilon b_\varepsilon \). There are two scenarios: If \( b \) is sufficiently smooth and \( b_\varepsilon \to b \) strongly in the sup-norm, for example, then the velocity can become part of the test function. Alternatively, one can consider the convergence \( \varrho_\varepsilon b_\varepsilon \to m \) weak* in the sense of measures first. One then has to argue that \( m \) can be disintegrated in the form \( \varrho b' \), which is often done by establishing that \( m \) is absolutely continuous with respect to \( \varrho \) and using Radon-Nikodým theorem. In this case, it can happen that \( b' = b \) only outside of a set of measure zero.

The well-posedness result [31] was extended by Ambrosio [3] to the case with \( b \) a function of bounded variation. A crucial assumption was that the spatial divergence of \( b \) be essentially bounded, which prevents the density \( \varrho \) from vanishing (formation of vacuum) and from concentrating (formation of singular measures). In particular, if \( \varrho \) is absolutely continuous with respect to \( d \)-dimensional Lebesgue measure \( \mathcal{L}^d \), then so is \( \varrho(t, \cdot) \) for all times \( t \). Alternatively, instead of imposing an \( \mathcal{L}^\infty \)-bound on \( \text{div}b \) one can simply assume that \( b \) is such that
\[
C^{-1} \leq \varrho(t, x) \leq C \quad \text{for all} \quad (t, x) \in (0, T) \times \Omega,
\]
with constant \( C > 0 \). Assumption (1.5) is strictly weaker than bounding \( \text{div}b \); see Remark 3.15 in [14]. We then say that the velocity \( b \) is nearly incompressible.

As a tool for establishing the existence of flows for (1.2) Ambrosio introduced what is nowadays known as the superposition principle: there exists a Borel probability measure \( \eta \) on the space \( \mathcal{C}([0, T]; \mathbb{R}^d) \) of continuous curves such that
\[
\varrho(t, \cdot) = e_t \# \eta \quad \text{for all} \quad t \in [0, T]
\]
where \( e_t(\gamma) := \gamma(t) \) is the evaluation map. Moreover, the measure \( \eta \) is concentrated on the set of curves that are a.e. solutions of (1.2); see Definition 3.1. In particular, the superposition principle implies the existence of solutions of (1.2) for \( \varrho \)-a.e. \( x \in \Omega \). It follows from an abstract decomposition result for currents (cf. [43]) because the
vector measure \( \varrho(1, b) \) satisfying (1.1) can be interpreted as a normal current. It is important to realize that \( \varrho \) and \( b \) must be compatible in the sense that the velocity field is tangential to the set where the density is concentrated; see [15]. These results have been generalized to metric measure spaces, making contact to the theory of Dirichlet forms; see [6] and the references therein.

The current state of the theory can be summarized as follows: If there is existence of solutions of (1.1) in a suitable class of functions, then existence of solutions of (1.2) for a.e. \( x \in \Omega \) can be derived from the superposition principle. If additionally we have uniqueness for (1.1), then one can prove the existence of a unique regular Lagrangian flow (which is a flow that preserves absolute continuity of the density). A sufficient condition for this is Sobolev- or BV-regularity of the velocity field \( b \) plus near incompressibility. We refer the reader to [27, 4, 6, 10]. The latter reference also contains a proof of Bressan’s conjecture, which is a stability result for regular Lagrangian flows that will be discussed in detail in Section 4.1.

The formal adjoint to the continuity equation is the transport equation

\[
\partial_t u + b \cdot \nabla u = 0 \quad \text{in } (0, T) \times \Omega, \tag{1.6}
\]

with suitable initial/final data. In our opinion, the most natural way to approach (1.6) is to reinterpret the equation in the form

\[
\partial_t (\varrho u) + \text{div}(\varrho ub) = 0 \quad \text{in } (0, T) \times \Omega \tag{1.7}
\]

where \( \varrho \) solves the continuity equation (1.1). By a solution of (1.6) we will therefore mean a function \( u \) that is integrable with respect to the space-time measure

\[
\sigma(dx, dt) := \varrho(t, dx) dt
\]

and satisfies (1.7) in the distributional sense. This is the setting for the compressible Euler equations of gas dynamics, for example, where the Eulerian velocity \( \mathbf{v}(t, \cdot) \in L^2(\Omega, \varrho(t, \cdot)) \) for all \( t \),

the square-integrability expressing the fact that the kinetic energy is finite. If the velocity field \( b \) is nearly incompressible in the sense of (1.5), then integrability of \( u \) can be assumed with respect to the \((d + 1)\)-dimensional Lebesgue measure.

On the other hand, if there exists a flow \( \mathbf{X} \) as in (1.3) and if the map \( x \mapsto \mathbf{X}(t, x) \) is invertible, then the solution \( u \) of (1.6) is trivial: just consider

\[
u(t, z) := \bar{u}(t, \mathbf{X}(t, \cdot)^{-1}(z)) \quad \text{for } t \in [0, T], \, z \in \mathbf{X}(t, \Omega)
\]

where \( \bar{u} := u(0, \cdot) \) is some initial data. Indeed the transport equation (1.6) simply expresses the fact that \( u \) must be constant along the integral curves of \( \mathbf{b} \).

More generally, if we are to solve

\[
\partial_t u + b \cdot \nabla u + cu = f \quad \text{in } (0, T) \times \Omega, \tag{1.8}
\]

for suitable functions \( c \) and \( f \), then the function

\[
U(t, x) := u(t, \mathbf{X}(t, x))
\]
satisfies the new equation \( \partial_t U + CU = F \) where

\[
C(t, x) := c(t, \mathbf{X}(t, x)) \quad \text{and} \quad F(t, x) := f(t, \mathbf{X}(t, x)) \tag{1.9}
\]
for all \( t \in [0, T] \) and \( x \in \Omega \). Recall that \( \partial_t X(t, x) = b(t, X(t, x)) \) since the flow \( X \) is built from solutions of (1.2). This simplifies the problem considerably: instead of solving a partial differential equation, we only consider a family of ODEs.

The problem of solving (1.8) therefore decomposes into two parts:

1. Geometry: Compute the flow \( X \) and the density \( \varrho \).
2. Transport: Solve the simplified equation \( \partial_t U + CU = F \).

Notice that the two steps are largely independent: The first step only depends on the velocity field \( b \), not on the data for (1.8), for instance. On the other hand, in the second step the geometry of the transport is completely obscured; it enters only through the redefined functions (1.9), which depend on the flow \( X \). The goal of this paper is to study how this point of view can be exploited to establish well-posedness for the transport equation (1.8). The paper has two parts:

- In Section 3 we investigate conditions that ensure the existence of a flow map \( X \). Different from the regularity assumptions outlined above, which imply the existence of a unique regular Lagrangian flow, we will consider more general (less regular) velocity fields and then select a suitable family of integral curves (see [24, 23]), from which we can build the flow.
- In Section 4 we introduce the notion of flow solutions of (1.8) and then use a generalized Lax-Milgram theorem to establish existence and uniqueness. We discuss possible numerical approximations using the (discontinuous) Petrov-Galerkin framework, along the lines of [20].

2. Notation

Let us introduce notation and general results that are used throughout the paper.

2.1. Set-Valued Maps. Let \( \Omega \) and \( E \) be two nonempty metric spaces.

**Definition 2.1.** We denote by \( \mathcal{P}(E) \) the collection of subsets of \( E \).

- A map \( S: \Omega \to \mathcal{P}(E) \) will be called a set-valued map.
- The sets \( S(x) \subset E \), with \( x \in \Omega \), are called the values of \( S \), and
  \[
  S(K) := \bigcup_{x \in K} S(x) \quad \text{for all } K \subset \Omega
  \]
  is the image of \( K \) under \( S \).
- We call domain and graph of \( S \) the sets
  \[
  \text{dom}(S) := \{ x \in \Omega: S(x) \neq \emptyset \},
  \text{graph}(S) := \{ (x, \gamma) \in \Omega \times E: x \in \Omega, \gamma \in S(x) \}.
  \]
- We call weak inverse and strong inverse of \( S \) the maps
  \[
  S^{-}(A) := \{ x \in \Omega: S(x) \cap A \neq \emptyset \},
  S^{+}(A) := \{ x \in \Omega: S(x) \subset A \}
  \quad \text{for all } A \subset E.
  \]

Note that \( S^{-}(\emptyset) = S^{+}(\emptyset) = \emptyset \) and \( S^{+}(A) \subset S^{-}(A) \) for all \( A \subset E \).

**Definition 2.2.** Let \( S: \Omega \to \mathcal{P}(E) \) be a set-valued map with \( \text{dom}(S) \neq \emptyset \). Then

- \( S \) is upper semicontinuous (abbreviated u.s.c.) at \( x \in \Omega \) if for any open set \( V \subset E \) with \( S(x) \subset V \), there exists a neighborhood \( U \subset \Omega \) of \( x \) with
  \[
  U \subset S^{+}(V) \iff S(U) \subset V.
  \]
We say that $S$ is u.s.c. if $S$ is upper semicontinuous at every $x \in \Omega$.

- $S$ is compact-valued if $S(x)$ is compact in $E$ for all $x \in \Omega$.

**Lemma 2.3.** Consider a set-valued map $S: \Omega \rightarrow \mathcal{P}(E)$ such that $\text{dom}(S) = \Omega$. The following statements are equivalent:

1. $S$ is u.s.c.
2. $S^+(V)$ is open in $\Omega$ whenever $V \subset E$ is open.
3. $S^-(W)$ is closed in $\Omega$ whenever $W \subset E$ is closed.

If $S$ is compact-valued and u.s.c., and $K \subset \Omega$ is compact, then $S(K)$ is compact.

**Proof.** We proceed in three steps (see also [38]).

**Step 1.** We first observe that

$$
\Omega \setminus S^\pm(A) = S^\mp(E \setminus A) \quad \text{for all } A \subset E.
$$

Indeed for all $x \in \Omega$ and $A \subset E$ we have

$$
x \in \Omega \setminus S^-(A) \iff x \notin S^-(A) \\
\iff S(x) \cap A = \emptyset \\
\iff S(x) \subset E \setminus A \iff x \in S^+(E \setminus A).
$$

The statement with $S^-$ and $S^+$ interchanged can be proved analogously.

**Step 2.** Suppose that $S$ is upper semicontinuous and $V \subset E$ open. By definition of strong inverse, for every $x \in S^+(V)$ we have $S(x) \subset V$. By definition of u.s.c., there exists a neighborhood $U \subset \Omega$ of $x$ with $U \subset S^+(V)$. Hence $(1) \implies (2)$.

To prove that $(2) \implies (3)$, consider a closed subset $W \subset E$. Then

$$
E \setminus W \text{ is open} \implies S^+(E \setminus W) \text{ is open} \\
\implies \Omega \setminus S^+(E \setminus W) \text{ is closed in } \Omega.
$$

Because of Step 1, it follows that

$$
S^-(W) = S^-(E \setminus (E \setminus W)) = \Omega \setminus S^+(E \setminus W) \text{ is closed in } \Omega.
$$

Finally, we prove that $(3) \implies (1)$. For given $x \in \Omega$ consider any open subset $V \subset E$ with $S(x) \subset V$. Then $E \setminus V$ is closed, and it follows that

$$
\Omega \setminus S^+(V) = S^-(E \setminus V) \text{ is closed in } \Omega,
$$

thus $S^+(V)$ is open; see again Step 1. Moreover, we have that

$$
S(x) \cap (E \setminus V) = \emptyset \implies x \notin S^-(E \setminus V) \implies x \in S^+(V),
$$

by definition of weak inverse. Since $S^+(V)$ is open there is a neighborhood $U \subset \Omega$ of $x$ with $U \subset S^+(V)$. As $x \in \Omega$ was arbitrary, we conclude that $S$ is u.s.c.

**Step 3.** Assume that $S$ is u.s.c. and compact-valued and that $K \subset \Omega$ is compact. Let $\{V\alpha: \alpha \in \Lambda\}$ be an open covering of $S(K)$. For any given $x \in \Omega$, the set

$$
S(x) \text{ is compact and } S(x) \subset \bigcup_{\alpha \in \Lambda} V\alpha.
$$

Therefore there exists a subcovering $\{V\alpha_k: k = 1, \ldots, n(\omega)\}$ such that

$$
S(x) \subset \bigcup_{k=1}^{n(\omega)} V\alpha_k =: W_x \quad \text{and} \quad W_x \subset E \text{ is open}.
$$
Since $S$ is u.s.c., we have that $S^+(W_x)$ is open; see Step 2. Then \( \{ S^+(W_x) : x \in K \} \) is an open covering of the compact set $K$ and there exist $x_1, \ldots, x_m \in K$ with
\[
K \subset \bigcup_{i=1}^{m} S^+(W_{x_i}).
\] (2.1)

By definition of strong inverse, we have that $x \in S^+(A)$ implies $S(x) \subset A$, thus
\[
S(S^+(A)) = \bigcup_{x \in S^+(A)} S(x) \subset A \text{ for all } A \subset E.
\]

Taking the image of either side of (2.1) under $S$ (see Definition 2.1), we obtain
\[
S(K) \subset S \left( \bigcup_{i=1}^{m} S^+(W_{x_i}) \right) = \bigcup_{i=1}^{m} S(S^+(W_{x_i})) \subset \bigcup_{i=1}^{m} W_{x_i}.
\]

Since every $W_x$ is a finite union of sets $V_\alpha$ we have indeed found a finite subcovering of $S(K)$ taken from the open covering \( \{ V_\alpha : \alpha \in \Lambda \} \). This proves the result. \( \square \)

2.2. Measurability. Here we discuss measurability of set-valued maps.

**Definition 2.4.** Suppose that $(A, \mathcal{A})$ is a measurable space and that $E$ is a topological space. We say that a set-valued map $S : A \rightarrow \mathcal{P}(E)$ is
- **weakly measurable** if $S^-(V) \in \mathcal{A}$ for each open subset $V \subset E$;
- **measurable** if $S^-(W) \in \mathcal{A}$ for each closed subset $W \subset E$.

A measurable selector from $S$ is a measurable function $f : A \rightarrow E$ such that $f(x) \in S(x)$ for all $x \in A$.

**Lemma 2.5.** Suppose that $(A, \mathcal{A})$ is a measurable space and $E$ is a metrizable space. For a set-valued map $S : A \rightarrow \mathcal{P}(E)$ we have the following:

1. If $S$ is measurable, then $S$ is weakly measurable.
2. If $S$ is compact-valued and weakly measurable, then $S$ is measurable.

**Proof.** We refer the reader to Lemma 18.2 in [2]. \( \square \)

**Lemma 2.6.** Suppose that $(A, \mathcal{A})$ is a measurable space and $E$ a separable metrizable space. Consider a sequence of weakly measurable set-valued maps $S_n : A \rightarrow \mathcal{P}(E)$ with closed values such that, for each $x \in A$, there exits a $k \in \mathbb{N}$ with $S_k(x)$ compact.

Then the intersection set-valued map $I : A \rightarrow \mathcal{P}(E)$, defined as
\[
I(x) := \bigcap_{n=1}^{\infty} S_n(x) \text{ for all } x \in A
\]
is measurable (hence weakly measurable).

**Proof.** We refer the reader to Lemma 18.4 in [2]. \( \square \)

**Definition 2.7 (Polish Space).** A topological space $S$ is called **completely metrizable** if there exists a distance $d$ that is compatible with the topology s.t. $(S, d)$ is complete. A topological space is called **Polish** if it is separable and completely metrizable.

Equivalently, a topological space is Polish if it has a countable dense subset and is homeomorphic to a complete metric space. Notice that Polish-ness only requires the existence of at least one complete distance compatible with the given topology. There may be other distances that are not complete. The unit interval $(0, 1)$ in $\mathbb{R}$,
for instance, which is open in the usual topology (therefore not complete), is Polish because it is homeomorphic to $\mathbb{R}$ whose usual metric is complete.

**Theorem 2.8** (Measurable Maximum Theorem). Let $(A, \mathcal{A})$ be a measurable space and $E$ a separable metrizable space. Let $\Gamma : A \to \mathcal{P}(E)$ be a weakly measurable set-valued map with nonempty compact values, and suppose that $f : A \times E \to \mathbb{R}$ is a Carathéodory function. Define the value function $m : A \to \mathbb{R}$ by

$$m(x) := \max_{\gamma \in S(x)} f(x, \gamma) \quad \text{for all } x \in A,$$

and the set-valued map $\Gamma_* : A \to \mathcal{P}(E)$ of maximizers by

$$\Gamma_*(x) := \{ \gamma \in \Gamma(x) : f(x, \gamma) = m(x) \} \quad \text{for all } x \in A.$$

Then

1. the value function $m$ is measurable;
2. the argmax function $\Gamma_*$ has nonempty and compact values;
3. the argmax function $\Gamma_*$ is measurable and admits a measurable selector.

**Proof.** We refer the reader to Theorem 18.19 in [2].

2.3. **Uniqueness.** We discuss methods to separate objects in metric spaces.

**Definition 2.9.** We say that a Banach space $E$ has the Radon-Nikodým property (abbreviated RNP) if the fundamental theorem of calculus holds for $E$-valued maps: If $f : [a, b] \to E$ is absolutely continuous, then there exists a Bochner integrable function $g : [a, b] \to E$ with the property that

$$f(t) = f(a) + \int_a^t g(s) \, ds \quad \text{for all } t \in [a, b].$$

Then $f$ is differentiable for a.e. $t \in [a, b]$ with derivative $f' = g$.

Recall that a function $f : [a, b] \to E$ is called absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum \|f(b_i) - f(a_i)\| < \varepsilon$ for every finite collection $\{(a_i, b_i)\}$ of disjoint intervals in $[a, b]$ with $\sum(b_i - a_i) < \delta$. We say that $f$ is Lipschitz continuous if there exists $M$ such that $\|f(t) - f(s)\| \leq M|t - s|$ for all $s, t \in [a, b]$. Clearly, any Lipschitz continuous function is absolutely continuous.

The following result is concerned with uniqueness of the inverse Laplace transform for Banach space-valued functions. It is known as Lerch’s theorem.

**Lemma 2.10.** Suppose that $E$ is a Banach space with the Radon-Nikodým property. Let $\text{Lip}_0(\mathbb{R}_+; E)$ be the space of Lipschitz continuous functions $F : \mathbb{R}_+ \to E$ such that $F(0) = 0$. Here $\mathbb{R}_+ := [0, \infty)$. If a function $F \in \text{Lip}_0(\mathbb{R}_+; E)$ satisfies

$$\int_0^\infty \exp(-\mu_n t) \, dF(t) = 0 \quad \text{for all } n \in \mathbb{N} \tag{2.2}$$

(in the sense of the Riemann-Stieltjes integral), for a sequence of distinct complex numbers $\mu_n$ such that $\text{Re} \, \mu_n \geq \mu > 0$ for some $\mu > 0$ and

$$\sum_{n=1}^N \left( 1 - \frac{|\mu_n - 1|}{|\mu_n + 1|} \right) \to \infty \quad \text{as } N \to \infty, \tag{2.3}$$

then $F(t) = 0$ for all $t \in \mathbb{R}_+$. 
Remark 2.11. If $E$ has the Radon-Nikodym property and $F \in \text{Lip}_0(\mathbb{R}_+; E)$, then
\[
\int_0^\infty g(t) \, dF(t) = \int_0^\infty g(t) F'(t) \, dt
\]
for all $g \in L^1(\mathbb{R}_+)$ continuous. In this case, the Laplace-Stieltjes transform (2.2) reduces to the Laplace transform. Otherwise, it is a proper generalization. Separable dual spaces and reflexive spaces have the Radon-Nikodym property; see [17].

Proof of Lemma 2.10. There exists an isometric isomorphism between $\text{Lip}_0(\mathbb{R}_+; E)$ and the space of bounded linear maps from $L^1(\mathbb{R}_+)$ to $E$ (Riesz-Stieltjes representation): To any $F \in \text{Lip}_0(\mathbb{R}_+; E)$ associate $T_F : L^1(\mathbb{R}_+) \to E$ such that
\[
T_F g := \int_0^\infty g(t) \, dF(t)
\]
and $T_F 1_{[0,t]} := F(t)$ for all $t \in \mathbb{R}_+$. By density, the map $T_F$ is uniquely determined by these assumptions. On the other hand, the family of functions $S := \{ \exp(-\lambda_n t) : n \in \mathbb{N} \}$, with complex numbers $\lambda_n$ as above, is total in $L^1(\mathbb{R}_+)$, which means precisely that the only bounded linear map on $L^1(\mathbb{R}_+)$ that vanishes on $S$ is the zero functional. We refer the reader to Corollary 1.3 in [9] for additional information. □

We now turn to separating classes of functions on topological spaces.

Definition 2.12. Let $(X, \tau)$ be a topological space and $M$ a collection of $\mathbb{R}$-valued Borel measurable functions on $X$. We say that $M$ separates points if for any $x, y \in X$ with $x \neq y$ there exists a $g \in M$ with $g(x) \neq g(y)$. We say that $M$ strongly separates points if for any $x \in X$ and any neighborhood $O_x$ of $x$ there exists a finite collection $\{g_1, \ldots, g_k\} \subset M$ such that $\inf_{y \not\in O_x} \max_{1 \leq i \leq k} |g_i(x) - g_i(y)| > 0$.

Lemma 2.13. Let $(X, \tau)$ be a topological space with countable basis and suppose that a subset $M \subset C(X; \mathbb{R})$ strongly separates points. Then there exists a countable collection $\{g_i\}_{i \in \mathbb{N}} \subset M$ that also strongly separates points. Moreover, this collection can be taken closed under either multiplication or addition if $M$ is.

Proof. We refer the reader to Lemma 2 in [12]. □

Theorem 2.14. Let $(X, \tau)$ be a topological space and $M$ a countable collection of $\mathbb{R}$-valued Borel measurable functions of $X$ that is closed under multiplication and strongly separates points. If $\mu$ is any Borel probability measure on $\mathbb{S}$, then
\[
\left( \int_S g \, d\mu = 0 \quad \text{for all } g \in M \right) \implies \mu = 0.
\]

Proof. We refer the reader to Theorem 11(c) in [12]. □

Remark 2.15. We can apply Lemma 2.13 to a Polish space $(\mathbb{S}, d)$, with $M$ the family of Lipschitz continuous functions with bounded support. Notice that the topology of a Polish space (which is a separable metric space) does have a countable basis. Moreover, $M$ strongly separates points on $\mathbb{S}$ since the set
\[
\left\{ (1 - kd(\cdot, y))_+ : y \in \mathbb{S}, k \in \mathbb{N} \right\}
\]
belongs to $\mathcal{M}$ and strongly separates points. Then there exists a countable collection $\{g_i\}_{i \in \mathbb{N}}$ of elements of $\mathcal{M}$ that strongly separates points. It follows that
\[
\left( \int_S g_i \, d\mu = 0 \quad \text{for all } i \in \mathbb{N} \right) \implies \mu = 0,
\]
where $\mu$ is any Borel probability measure on $S$; see Theorem 2.14. Our choice of $\mathcal{M}$ is motivated by the theory of continuity equations on metric spaces, where the satisfaction of the transport ODE is defined by testing against a class of Lipschitz continuous functions. We refer the reader to [6] for further information.

3. Measurable Semi-Processes

Starting with the seminal work by DiPerna-Lions [31], a lot of effort has been devoted to identifying successively weaker conditions that ensure the existence and, in particular, the uniqueness of (regular) flows for (1.2). These conditions typically come in the form of regularity assumptions on the velocity field $b$.

Here we will explore a different approach. We make only minimal assumptions on the regularity of the velocity field $b$ that ensure existence of solutions to (1.2). Since then uniqueness is typically lost, in order to be able to anyway define a flow map $X$, we select among all solutions of (1.2) a suitable family of integral curves, one for each starting point $x \in \Omega$. While the resulting flow $X$ may not be unique, it will still have the crucial semi-group property. Using the push-forward formula (1.4) we obtain a solution of the continuity equation (1.1). Again we do not expect uniqueness of solutions to (1.1). Instead our approach amounts to selecting a suitable one (namely one that has associated to it a flow map with good properties). This presents an alternative to the regularity-based approach to transport problems.

Notice that the idea of selecting a suitable solution among many possible ones is at the heart of the theory of hyperbolic conservation laws where recent breakthroughs by De Lellis, Szekelyhidi, and others have conclusively demonstrated that uniqueness cannot be expected for some fundamental physical models, such as the Euler/Navier-Stokes equations of gas dynamics; see [29, 28, 22]. This has reinvigorated the quest for suitable entropy conditions that among all weak solutions of the equations would pick the one with physical relevance. The selection principle here is different insofar as the procedure is motivated purely by mathematical considerations.

3.1. Differential Inclusions. Let $\Omega \subset \mathbb{R}^d$ be closed. In the following, open/closed subsets of and neighborhoods in $\Omega$ will always be understood with respect to the relative topology. Let us first clarify our solution concept for (1.2).

**Definition 3.1.** An a.c. solution (also called Carathéodory solution) of (1.2) is an absolutely continuous map $\gamma : [0, T] \to \Omega$ with the property that
\[
\gamma(t) = x + \int_0^t b(s, \gamma(s)) \, ds \quad \text{for all } t \in [0, T].
\]

By Cauchy-Peano theorem, if the velocity field $b$ is continuous in space, then a.c. solutions of (1.2) exist for $T$ sufficiently small. But generally there is no uniqueness. A standard counterexample is the following initial value problem
\[
\dot{\gamma} = 2\gamma^{1/2}, \quad \gamma(0) = 0,
\]
which has infinitely many solutions of the form
\[ \gamma(t) = \begin{cases} 0 & \text{if } t < c \\ (t - c)^2 & \text{if } t \geq c \end{cases} \quad \text{for all } t \in [0, T], \]
for any \( c \geq 0 \). If \( b \) is discontinuous in space, then even existence for (1.2) may no longer be given, as the following example demonstrates: If the velocity field
\[ b(x) := \begin{cases} +1 & \text{for } x \leq 0 \\ -1 & \text{for } x > 0 \end{cases} \]
then there exists no solution of (1.2) for initial data \( x = 0 \).

In order to have a robust existence theory at our disposal, instead of the differential equation (1.2) we will consider differential inclusions of the form
\[ \dot{\gamma}(t) \in F(t, \gamma(t)) \quad \text{for } s \leq t \leq T, \quad \gamma(s) = x \]
with \((s, x) \in [0, T] \times \Omega\) and \( F: [s, T] \times \Omega \rightarrow \mathcal{P}(\mathbb{R}^d) \); see Definition 2.1.

**Definition 3.2.** An \emph{a.c. solution} (also called Carathéodory solution) of (3.1) is an absolutely continuous map \( \gamma: [s, T] \rightarrow \Omega \) with the property that
\[ \dot{\gamma}(t) \in F(t, \gamma(t)) \quad \text{for a.e. } t \in [s, T]. \]

In order for a.c. solutions of (3.1) to remain inside of \( \Omega \) it is necessary that the velocity field at the boundary does not point into the complement of \( \Omega \).

**Definition 3.3.** We define the \emph{tangent cone} to \( \Omega \) at the point \( x \) as
\[ T_x \Omega := \left\{ v \in \mathbb{R}^d : \liminf_{\lambda \to 0^+} \lambda^{-1} d(x + \lambda v, \Omega) = 0 \right\} \quad \text{for all } x \in \Omega. \]
Here \( d(x, y) := \|x - y\| \) for \( x, y \in \mathbb{R}^d \) is the induced distance.

We then require that \( b(t, x) \in T_x \Omega \) for all \( x \in \partial \Omega \). Note that \( T_x \Omega = \mathbb{R}^d \) if \( x \in \hat{\Omega} \).

We can now state the main existence result (recall Definitions 2.2 and 2.4).

**Theorem 3.4.** Let \( I := [s, T] \). Suppose that a set-valued map \( F: I \times \Omega \rightarrow \mathcal{P}(\mathbb{R}^d) \) is given with nonempty closed convex values and with the following properties:

1. For all \( t \in I \), the map \( x \mapsto F(t, x) \) is upper semicontinuous.
2. For all \( x \in \Omega \), the map \( t \mapsto F(t, x) \) is measurable.
3. For all \((t, x) \in [s, T] \times \Omega\), we have \( F(t, x) \cap T_x \Omega \neq \emptyset \).
4. There exists a function \( c \in L^1(I) \) such that
\[ \|F(t, x)\| \leq c(t)(1 + |x|) \quad \text{for all } (t, x) \in I \times \Omega, \quad \text{where} \quad \|F(t, x)\| := \sup \{|y| : y \in F(t, x)\}. \]

Then there exists an a.c. solution of (3.1) for every \( x \in \Omega \).

**Proof.** We refer the reader to Theorem 5.2 in [30]. \( \square \)

**Remark 3.5.** By Gronwall’s lemma and assumption (3.2), we have
\[ |\gamma(t)| \leq |x| + \int_s^t c(r) \, dr \exp \left( \int_s^t c(r) \, dr \right) \]
for all \( s \leq t \leq T \). In particular, the solutions of (3.1) remain bounded.

The following topological properties of the solution set of (3.1) will be crucial.
Theorem 3.6. For any \( x \in \Omega \) let \( \Gamma(s, x) \) be the set of a.c. solutions of (3.1).

(1) For all \( x \) the set \( \Gamma(s, x) \) is nonempty and compact.

(2) The map \( x \mapsto \Gamma(s, x) \) is upper semicontinuous.

Proof. We refer the reader to Theorem 7.1 in [30]. \( \square \)

Remark 3.7. We remark that (3.3) is equivalent to the more common definition

\[ F(t, x) := \bigcap_{\delta > 0} \bigcap_{\mathcal{L}^d(N) = 0} \overline{\text{conv}} b(t, B_\delta(x) \setminus N) \text{ for } (t, x) \in [0, T] \times \Omega, \quad (3.5) \]

which was introduced by Filippov; see [33, 34]. \( F(t, x) \) is the smallest closed convex set, any neighborhood of which contains the values of \( b(t, z) \) for almost all \( z \in \Omega \). We will call Filippov solution of the differential equation (1.2) any a.c. solution of the differential inclusion (3.1) with \( F \) defined by (3.3).

Proposition 3.8. Let \( I = [s, T] \) for some \( T > 0 \) and \( \Omega \subset \mathbb{R}^d \) closed. Suppose that a velocity field \( b \in L_\infty^{\text{loc}}(I \times \Omega; \mathbb{R}^d) \) is given with the following properties:

(1) There exists a function \( c \in L^1(I) \) such that

\[ |b(t, x)| \leq c(t)(1 + |x|) \text{ for all } (t, x) \in I \times \Omega. \quad (3.6) \]

(2) For all \( (t, x) \in [s, T] \times \partial \Omega \), we have \( F(t, x) \cap \partial \Omega \neq \emptyset \), where the set-valued function \( (t, x) \mapsto F(t, x) \) is defined in (3.3); see also Definition 3.3.

Then the map \( F \) satisfies the conditions of Theorem 3.4.

Remark 3.9. For a.e. \( t \in I \) the function \( x \mapsto b(t, x) \) is measurable. By Lusin’s theorem, it is therefore continuous outside a set of measure zero. At each point of continuity, it holds that \( F(t, x) = \{b(t, x)\} \), which is therefore true for a.e. \( x \in \Omega \).
Remark 3.10. The Filippov construction (3.5) changes the velocity field, which may be undesired in some applications. On the other hand, the density $\rho$ and the velocity field $b$ are often constructed simultaneously from an approximation, e.g., in the construction of solutions to the compressible Euler equations.

Remark 3.11. Filippov solutions may not be unique. Sufficient for uniqueness is that $b$ satisfies a one-sided Lipschitz condition; see [35] for details. Then the solution of the continuity equation is unique as well [39, 16]. There is a stability result for the corresponding flow [11], which will be discussed in Section 4.1.

Remark 3.12. From a measure theory point of view it would be natural to define the essential supporting function as the approximate upper limit

$$h(t, x, \xi) := \text{ap lim sup}_{y \to x} \left( \xi \cdot b(t, y) \right)$$

for $(t, x) \in I \times \Omega$ and $\xi \in S^{d-1}$; see Definition A.4. This can be rewritten as

$$h(t, x, \xi) = \lim_{\delta \to 0} \left( \inf_{C \in \mathcal{C}} \sup_{y \in B_\delta(x) \cap C} \left( \xi \cdot b(t, y) \right) \right)$$

(3.7)

where $\mathcal{C}$ is the class of measurable subsets of $\Omega$ with density 1 at $x$; see Proposition A.5. This class contains in particular sets of the form $\Omega \setminus N$ with $\mathcal{L}^d(N) = 0$, which implies that (3.7) is not bigger than the $h(t, x, \xi)$ defined in terms of (3.4). We do not know, however, whether the substitution of (3.7) in (3.3) would result in an upper semicontinuous set-valued function, as needed for Theorem 3.4.

Proof of Proposition 3.8. The result is well-known; see [35, 7, 30, 37], for instance. In these references, the arguments are often only sketched, therefore we provide a detailed proof for the reader’s convenience; see also [36]. Note first that $h_\delta(t, x, \xi)$ is well-defined for every $x \in \Omega$ and a.e. $t \in I$. Then (3.6) implies the bound

$$|h_\delta(t, x, \xi)| \leq c(t)(1 + |x| + \delta)$$

for all $(t, x) \in I \times \Omega$, for all $\xi \in S^{d-1}$ and $\delta > 0$. It is straightforward to check that the map $\xi \mapsto h_\delta(t, x, \xi)$ is one-homogeneous and convex, and the same is true for its pointwise limit $h(t, x, \xi)$. It follows that for all $x \in \Omega$ and a.e. $t \in I$ the set $F(t, x)$ is nonempty and convex, satisfying (3.2). Since $F(t, x)$ is also closed, it is in fact compact. Recall that in this case, weakly measurable and measurable are equivalent; see Lemma 2.5.

For the remaining proof, we proceed in two steps.

Step 1. The function $x \mapsto b(t, x)$ with $x \in \Omega$ is measurable for a.e. $t \in I$. We claim that for any such $t$, the map $x \mapsto h_\delta(t, x, \xi)$ is upper semicontinuous, for every $\xi \in S^{d-1}$ and $\delta > 0$. We must prove the following inequality:

$$\limsup_{y \to x} h_\delta(t, y, \xi) \leq h_\delta(t, x, \xi).$$

(3.8)

We consider any sequence of $y_k \in \Omega$ such that $|y_k - x| \leq 1/k$ and

$$\lim_{k \to \infty} h_\delta(t, y_k, \xi) = \limsup_{y \to x} h_\delta(t, y, \xi).$$

Then $B_\delta(y_k) \subset B_{\delta + 1/k}(x)$, which implies that

$$\text{ess sup}_{z \in B_\delta(y_k) \cap \Omega} (\xi \cdot b(t, z)) \leq \text{ess sup}_{z \in B_{\delta + 1/k}(x) \cap \Omega} (\xi \cdot b(t, z)).$$

(3.9)
Since the balls $B_{\delta+1/k}(x)$ form a nested and decreasing sequence of sets, the right-hand side of (3.9) converges to $h_\delta(t,x,\xi)$ as $k \to \infty$. This proves (3.8). We now use that the pointwise inf of a family of upper semicontinuous functions is upper semicontinuous. This follows from the fact that for u.s.c. functions the preimages of open intervals of the form $(-\infty, \alpha)$ with $\alpha \in \mathbb{R}$ are open, which implies that the preimages of such intervals under pointwise infima of u.s.c. functions are unions of open sets, thus open again. Notice that in the definition of $h(t,x,\xi)$ we can replace $\lim_{\delta \to 0}$ by $\inf_{\delta > 0}$ because $h_\delta(t,x,\xi)$ is nonincreasing as $\delta$ gets smaller. We conclude that $x \mapsto h(t,x,\xi)$ is upper semicontinuous, for all $\xi \in S^{d-1}$ and a.e. $t \in I$.

In order to prove that the set-valued map $x \mapsto F(t,x)$ is upper semicontinuous, for a.e. $t \in I$, we first prove that the function has closed graph. Consider again a sequence of $y_k \in \Omega$ with $y_k \to x$ as $k \to \infty$ and suppose that $v_k \in F(t,y_k)$ for all $k$ with $v_k \to v$. By definition of $F(t,y_k)$ we have $\xi \cdot v_k \leq h(t,y_k,\xi)$ for all $\xi \in S^{d-1}$. By upper semicontinuity of the essential support function, it follows that

$$\xi \cdot v = \lim_{k \to \infty} \xi \cdot y_k \leq \limsup_{k \to \infty} h(t,y_k,\xi) \leq h(t,x,\xi).$$

Since this holds for every $\xi \in S^{d-1}$, we conclude that $v \in F(t,x)$.

It remains to prove that the closed graph property implies upper semicontinuity; see Corollary 1.1 in [7]. For simplicity, we define the set-valued map $S(x) := F(t,x)$ for all $x \in \Omega$ and $t$ as above. Recall that $S$ has compact values. Let us fix $x_0 \in \Omega$ for the following. For any open subset $V \subset E$ with $S(x_0) \subset V$, we must establish the existence of a neighborhood $U \subset \Omega$ of $x_0$ such that $S(U) \subset V$. It will be sufficient to consider $S$ restricted to a bounded neighborhood $U_0 \subset \Omega$ of $x_0$, for which $S(U_0)$ is contained in a compact subset $K \subset E$. This is possible because of (3.2).

We consider the complement $W := K \setminus V$, which is again a compact subset of $E$. By assumption, $S(x_0) \subset V$. Therefore, for any $y \in W$ we have $y \not\in S(x_0)$, and thus $(x_0,y) \not\in \text{graph}(S)$. Since $\text{graph}(S)$ is closed, there exist neighborhoods $N(y)$ of $y$ and $N_y(x_0)$ of $x_0$, (relatively) open in $E$ and $\Omega$, respectively, such that

$$\text{graph}(S) \cap (N_y(x_0) \times N(y)) = \emptyset \quad \text{for all } y \in W.$$ 

In particular, it follows that $S(x) \cap N(y) = \emptyset$ for every $x \in N_y(x_0)$. Since $W$ is compact, it can be covered by finitely many $N(y_i)$ where $y_i \in W$ and $i = 1, \ldots, n$. We now define $U := U_0 \cap \bigcap_{i=1}^n N_{y_i}(x_0)$, which is (relatively) open in $\Omega$. For $x \in U$ we have $S(x) \subset K = V \cup W$ because $x \in U_0$. On the other hand, it holds

$$S(x) \cap N(y_i) = \emptyset \quad \text{for all } i = 1, \ldots, n,$$

because $x \in N_{y_i}(x_0)$. Since the $N(y_i)$ cover $W$, we conclude that $S(x) \cap W = \emptyset$ for all $x \in U$, thus $S(U) \subset V$. As $x_0 \in \Omega$ was arbitrary, $S$ is upper semicontinuous.

**Step 2.** We now prove measurability of the map $t \mapsto F(t,x)$ for any fixed $x \in \Omega$. Since there is no upper semicontinuity in time, a new argument different from the one in Step 1 is needed. We first prove that the essential supporting function

$$t \mapsto h(t,x,\xi) \quad \text{for } t \in I$$

is measurable, for every fixed $\xi \in S^{d-1}$. To simplify notation, for given $\delta > 0$ we write

$$A := B_\delta(x), \quad \mu := \frac{\mathcal{L}^d(\Omega)}{|A \cap \Omega|}, \quad \text{and } f_\delta(x) := \xi \cdot b(t,x).$$
Notice that \( \mu(A) = 1 \), by construction. For any \( r > 0 \) the function
\[
t \mapsto \int_A \exp \left( rf_t(y) \right) \mu(dy) =: \phi(t, r)
\]
is measurable for any \( r > 0 \); see Corollary 3.3.3 in [13]. Recall that the composition of a measurable function with a continuous one is measurable. Then also the function
\[
t \mapsto \frac{1}{r} \log \left( \phi(t, r) \right)
\]
is measurable for any \( r > 0 \). (3.10)

We now claim that for a.e. \( t \in I \) we have
\[
\lim_{r \to \infty} \frac{1}{r} \log \left( \phi(t, r) \right) = \operatorname{ess \ sup}_{y \in A} f_t(y) =: \alpha_t.
\] (3.11) Here the essential sup is taken with respect to \( \mu \). Assume first that \( \alpha_t > 0 \). Then we estimate \( \phi(t, r) \leq \exp(r \alpha_t) \), from which we get (3.11) with inequality \( \leq \).

To prove the converse, we need to estimate \( \phi(t, r) \) from below. We have
\[
\phi(t, r) \geq \exp(r \beta) \mu(A \cap \{ f_t \geq \beta \}) + \int_{A \cap \{ f_t < \beta \}} \exp \left( rf_t(y) \right) \mu(dy) =: \phi_\beta(t, r)
\]
for \( 0 < \beta < \alpha_t \) and \( r > 0 \). The first term in \( \phi_\beta(t, r) \) grows unboundedly as \( r \to \infty \) since \( \mu(A \cap \{ f_t \geq \beta \}) > 0 \) for all \( \beta > 0 \) strictly less than the ess sup of \( f_t \). Here we have used the assumption \( \alpha_t > 0 \). The second term in \( \phi_\beta(t, r) \) is nonnegative.

For a.e. \( y \in A \) and all \(-1 < h < 1\) it holds
\[
\exp \left( (r + h)f_t(y) \right) - \exp \left( rf_t(y) \right) = f_t(y) \exp \left( (r + \theta h)f_t(y) \right)
\]
for some \( \theta \in [0, 1] \), by the mean value theorem. It follows that
\[
\left| \exp \left( (r + h)f_t(y) \right) - \exp \left( rf_t(y) \right) \right| \leq \| f_t \|_{L^\infty(A)} \exp \left( (r + 1) \| f_t \|_{L^\infty(A)} \right),
\]
which is finite for a.e. \( t \). By dominated convergence, we obtain for \( h \to 0 \) that
\[
\partial_r \phi_\beta(t, r) = \beta \exp(r \beta) \mu(A \cap \{ f_t \geq \beta \}) + \int_{A \cap \{ f_t < \beta \}} f_t(y) \exp \left( rf_t(y) \right) \mu(dy).
\]

Using de l’Hôpital rule and canceling \( \exp(r \beta) \), we now obtain the identity
\[
\lim_{r \to \infty} \frac{1}{r} \log \left( \phi_\beta(t, r) \right) = \lim_{r \to \infty} \frac{\partial_r \phi_\beta(t, r)}{\phi_\beta(t, r)}
\]
(3.12)
\[
= \lim_{r \to \infty} \frac{\beta \mu(A \cap \{ f_t \geq \beta \}) + \int_{A \cap \{ f_t < \beta \}} f_t(y) \exp \left( r(f_t(y) - \beta) \right) \mu(dy)}{\mu(A \cap \{ f_t \geq \beta \}) + \int_{A \cap \{ f_t < \beta \}} \exp \left( r(f_t(y) - \beta) \right) \mu(dy)}.
\]
Note that \( |f_t(y) \exp(r(f_t(y) - \beta))| \leq \| f_t \|_{L^\infty(A)} \) for every \( y \in A \cap \{ f_t < \beta \} \), with a similar estimate for the integrand in the denominator of (3.12). The two integrals in (3.12) converge to zero as \( r \to \infty \), by dominated convergence. Thus
\[
\lim_{r \to \infty} \frac{1}{r} \log \left( \phi(t, r) \right) \geq \beta \quad \text{for all } 0 < \beta < \alpha_t.
\]
This proves the identity (3.11) in the case \( \alpha_t > 0 \).

If \( \alpha_t \leq 0 \), then we define the function \( g_t := f_t - (\alpha_t - 1) \) and rewrite
\[
\phi(t, r) = \exp \left( r(\alpha_t - 1) \right) \int_A \exp \left( rg_t(y) \right) \mu(dy) \quad \text{for all } r > 0.
\]
Recall that $\mu(A) = 1$. We have $\text{ess sup}_{x \in A} g(t,y) = 1$, which is positive. Then

$$
\lim_{r \to \infty} \frac{1}{r} \log \phi(t,r) = (\alpha_t - 1) + \lim_{r \to \infty} \frac{1}{r} \log \left( \int_A \exp (rg_t(y)) \mu(dy) \right)
$$

$$
= (\alpha_t - 1) + 1 = \alpha_t,
$$

where we have used identity (3.11) with $g_t$ in place of $f_t$.

Combining (3.10) and (3.11), we conclude that the map

$$
t \mapsto \text{ess sup}_{y \in B_t(x) \cap \Omega} (\xi \cdot b(t,y)) = h_\delta(t,x,\xi) \quad \text{for } t \in I \text{ is measurable (3.13)}
$$

since the pointwise limits of sequences of measurable functions are measurable. The essential supporting function $t \mapsto h(t,x,\xi)$, which is obtained from (3.13) by sending $\delta \to 0$, is measurable as well, for all $x \in \Omega$ and $\xi \in S^{d-1}$ fixed.

For $x \in \Omega$ fixed we define $F_x(t) := F(t,x)$ for all $t \in I$. We must show that

$$
F_x^-(B) = \{ t \in I : F_x(t) \cap B \neq \emptyset \} \text{ is measurable for all } B \subset \mathbb{R}^d \text{ open.}
$$

We follow Theorem 8.3.1 in [8] and consider first the case of an open ball $B = B_r(y)$ centered at $y \in \mathbb{R}^d$ with radius $r > 0$. Then $F_x(t) \cap B_r(y)$ is nonempty if and only if the Euclidean distance $d(y,F_x(t))$ is strictly less than $r$. Thus

$$
F_x^-(B) = \{ t \in I : d(y,F_x(t)) < r \}. \quad (3.14)
$$

Since the set $F(t)$ is closed and convex, the distance $d(y,F_x(t))$ can be expressed in terms of $h(t,x,\xi)$ as follows (see Theorem 8.2.14 in [8]):

$$
d(y,F_x(t)) = d(0,F_x(t) - y)
$$

$$
= - \inf_{\xi \in S^{d-1}} \sup_{a \in F_x(t)} (\xi \cdot (a - y)) = \sup_{\xi \in S^{d-1}} (\xi \cdot y - h(t,x,\xi)). \quad (3.15)
$$

It suffices to consider in (3.15) the sup over a countable set $S$ that is dense in $S^{d-1}$. Indeed it is known that a convex function defined on an open convex set $U$ in a normed vector space is Lipschitz continuous on any compact subset of $U$; see [40]. If the sup in (3.15) is attained at $\xi \in S^{d-1}$, then there exists a sequence of $\xi_k \in S$ with $\xi_k \to \xi$ and $h(t,x,\xi_k) \to h(t,x,\xi)$ as $k \to \infty$, by density of $S$.

Since the function $t \mapsto h(t,x,\xi)$ is measurable for every $x \in \Omega$ and $\xi \in S$ fixed, it follows that the distance (3.15), as the pointwise sup of countably many measurable functions, is measurable in $t$ for every $y \in \mathbb{R}^d$. This implies that (3.14) is measurable for any $B_r(y)$. For a general open set $B \subset \mathbb{R}^d$ we have the identity

$$
F_x^-(B) = \bigcup_{n \in \mathbb{N}} F_x^-(B_{r_n}(y_n)), \quad (3.16)
$$

with $(B_{r_n}(y_n))_n$ a countable family of open balls whose union equals $B$. Thus (3.16) is measurable for all $B \subset \mathbb{R}^d$ open, which proves that $F_x$ is measurable.

3.2. Selection of Flow Map. Here is the main result of Section 3

**Theorem 3.13.** Let $I := [0,T]$ for some $T > 0$ and $\Omega \subset \mathbb{R}^d$ closed. Suppose that a velocity $b \in L^\infty(I \times \Omega; \mathbb{R}^d)$ is given with the properties listed in Proposition 3.8. Then there exists a map $X : I \times I \times \Omega \to \Omega$ such that

1. For all $0 \leq s \leq t \leq T$, the map $x \mapsto X(t,s,x)$ is Borel measurable.
2. For all $(s,x) \in I \times \Omega$, the map $t \mapsto X(t,s,x)$ is a Filippov solution of

$$
\dot{\gamma}(t) = b(t,\gamma(t)) \quad \text{for } s \leq t \leq T, \quad \gamma(s) = x. \quad (3.17)
$$
(3) The flow has the semigroup property: for all \( x \in \Omega \) it holds
\[
X(t, r, x) = X(t, s, X(s, r, x)) \quad \text{for } 0 \leq r \leq s \leq t \leq T. \tag{3.18}
\]

**Remark 3.14.** Theorem 3.13 does not state that the map \( x \mapsto X(t, s, x) \) is injective for all \( 0 \leq s \leq t \leq T \), nor that flows lines are unique for all \( x \in \Omega \). The semigroup property (3.18) does imply, however, that whenever two characteristics starting a distinct locations in \( \Omega \) meet at some later time, then the integral curves will coincide from that time on. Indeed suppose that \( x_1, x_2 \in \Omega \) and \( r_1, r_2, s \in I \) are such that
\[
X(s, r_1, x_1) = X(s, r_2, x_2) =: x_m,
\]
with \( s \geq r_1, r_2 \). Applying (3.18) twice, we observe that
\[
X(t, r_1, x_1) = X(t, s, x_m) = X(t, r_2, x_2) \quad \text{for all } s \leq t \leq T.
\]

Using language from [10], we say the flow lines are *forward untangled*.

**Remark 3.15.** We emphasize again that while the theory outlined in the Introduction provides sufficient conditions in terms of regularity of \( b \) so that flow lines are unique, here we make only minimal assumptions on \( b \), which cannot rule out non-uniqueness. But then we select integral curves so that we are still able to define a flow.

**Proof of Theorem 3.13.** The proof follows the approach developed in [24, 23]. It is based on a repeated application of the Measurable Maximum Theorem. Therefore we start by particularizing Theorem 2.8 to the situation at hand: Recall that \( \Omega \subset \mathbb{R}^d \) is closed. We consider the measurable space
\[
(A, A) := (\Omega, \mathcal{B}(\Omega)) \quad \text{with } \mathcal{B}(\Omega) \text{ the Borel } \sigma\text{-algebra.}
\]

As mentioned before, open/closed sets and neighborhoods in \( \Omega \) are understood with respect to the relative topology. We then consider the Banach spaces
\[
E := \mathcal{C}([s, T]; \mathbb{R}^d) \quad \text{endowed with the sup-norm,}
\]
with \( s \in I \). Notice that \( \mathcal{C}([s, T]; \Omega) \) is a separable metrizable space, as required by Theorem 2.8. We define the set-valued map \( x \mapsto \Gamma(s, x) \) with \( x \in \Omega \) as
\[
\Gamma(s, x) := \{ \gamma \in \mathcal{C}([s, T]; \mathbb{R}^d) : \gamma \text{ is a Filippov solution of (3.17)} \}.
\]
Then the conditions of Theorem 2.8 are fulfilled, for any \( s \in I \) fixed:

(A1) **For all** \( x \in \Omega \) **the set** \( \Gamma(s, x) \) **is nonempty and compact.**

This follows from Theorem 3.6 and Proposition 3.8 which checks that the assumptions of Theorem 3.4 are satisfied. Compactness (with respect to the sup-norm) follows from Arzelà-Ascoli theorem and the fact that solutions of (3.17) are Lipschitz continuous since the velocity \( b \) is locally bounded.

(A2) **The map** \( x \mapsto \Gamma(s, x) \) **for** \( x \in \Omega \) **is weakly (Borel) measurable.**

By Theorem 3.6 the map \( x \mapsto \Gamma(s, x) \) is upper semicontinuous, hence
\[
\Gamma(s, \cdot)^{-1}(W) \text{ is closed in } \Omega \text{ whenever } W \subset \mathcal{C}([s, T]; \mathbb{R}^d) \text{ is closed;}
\]
recall the equivalent definitions of upper semicontinuity in Lemma 2.3. This implies that the map is measurable with respect to the Borel algebra \( \mathcal{B}(\Omega) \), which is generated by the closed subsets of \( \Omega \); see Definition 2.4. On the other hand, measurability implies weak measurability; see Lemma 2.5.
We will apply Theorem 2.8 to functionals of the form
\[ f_{\lambda,\varphi}(\gamma) := \int_s^T \exp(-\lambda t)\varphi(\gamma(t)) \, dt \quad \text{for all } \gamma \in \mathcal{C}([s, T]; \mathbb{R}^d) \] (3.19)
with \( \lambda > 0 \) and \( \varphi \in \mathcal{C}_b(\Omega) \), for any \( s \in [0, T] \). The functional \( f_{\lambda,\varphi} \) is well-defined. It is linear and continuous with respect to the topology induced by the sup-norm. Thus \( f_{\lambda,\varphi} \) is, in fact, a Carathéodory function \( \Omega \times E \rightarrow \mathbb{R} \) (which does not explicitly depend on the position \( x \in \Omega \)). Applying Theorem 2.8 we obtain that the set-valued map \( \Gamma_{\lambda,\varphi} : I \times \Omega \rightarrow \mathcal{P}(E) \) of maximizers of \( f_{\lambda,\varphi} \), defined by
\[ \Gamma_{\lambda,\varphi}(s, x) := \text{argmax} \left\{ f_{\lambda,\varphi}(\gamma) : \gamma \in \Gamma(s, x) \right\} \quad \text{for } (s, x) \in I \times \Omega, \] (3.20)
again has the following properties, for any \( s \in I \) fixed:

1. For all \( x \in \Omega \) the set \( \Gamma_{\lambda,\varphi}(s, x) \) is nonempty and compact.

2. The map \( x \mapsto \Gamma_{\lambda,\varphi}(x) \) for \( x \in \Omega \) is weakly measurable.

In particular, we can now repeat the procedure above, substituting the maximizers-valued function \( \Gamma_{\lambda,\varphi} \) for \( \Gamma \), with a different choice of \( \lambda' > 0 \) and \( \varphi' \in \mathcal{C}_b(\Omega) \). Notice that since \( \Gamma(s, x) \) is compact and \( f_{\lambda,\varphi} \) continuous, the sup of \( f_{\lambda,\varphi} \) is attained, which explains why the set of maximizers \( \Gamma_{\lambda,\varphi}(s, x) \) is nonempty for all \((s, x)\). It is compact because it is a closed (by continuity of \( f_{\lambda,\varphi} \)) subset of the compact set \( \Gamma(s, x) \).

The approach above does not directly use that the family of curves consists of solutions of the differential equation (3.17). We know highlight two crucial properties of these curves and show that they are preserved under the maximization:

(A3a) For any \( 0 \leq r \leq s \leq T \) and any \( \gamma \in \Gamma(r, x) \) with \( x \in \Omega \), we have that
\[ \gamma|_{[r, s]} \in \Gamma\left(s, \gamma(s)\right). \]

Indeed, any flow line starting at time \( r \) at the position \( x \) is described by a curve in \( \mathcal{C}([r, T]; \Omega) \) that is a Filippov solution of the differential equation on that time interval. That is, the function is absolutely continuous, thus differentiable a.e., and satisfies a differential inclusion for almost every time. This is preserved when restricted to a smaller time interval \([r, T] \), and that part of the curve then belongs to the funnel of flow lines that start at time \( s \) at position \( \gamma(s) \). This follows immediately from the definition.

(A3b) For any \( 0 \leq r \leq s \leq T \), any \( \gamma \in \Gamma(r, x) \) with \( x \in \Omega \), and any \( \eta \in \Gamma(s, \gamma(s)) \), we have that the spliced curve \( \gamma \bowtie \eta \) belongs to \( \Gamma(r, x) \), where
\[ \gamma \bowtie \eta(t) := \begin{cases} \gamma(t) & \text{for } r \leq t \leq s, \\ \eta(t) & \text{for } s \leq t \leq T. \end{cases} \] (3.21)

The curves \( \gamma \) and \( \eta \) are absolutely continuous on \([r, T]\) and \([s, T]\), respectively. Restricting \( \gamma \) to \([r, s]\) and joining it with \( \eta \) at time \( s \), we preserve the absolute continuity because \( \gamma(s) = \eta(s) \), by construction. Therefore \( \gamma \bowtie \eta \) is again differentiable a.e. and satisfies the required differential inclusion for almost all times in \([r, T]\), which proves the claim. Note that if there was a unique solution starting at time \( s \) at location \( \gamma(s) \), then \( \eta \) would have to coincide with \( \gamma \) on the time interval \([s, T]\), thus \( \gamma \bowtie \eta = \gamma \). The splicing of curves is possible because our solution concept requires only that the differential inclusion/equation be satisfied almost everywhere in time.
We claim that (A3a)/(A3b) hold with $\Gamma$ replaced by $\Gamma_{\lambda,\varphi}$; recall (3.20) and (3.19). Consider a curve $\gamma \in \Gamma_{\lambda,\varphi}(r, x)$ with $x \in \Omega$. We must show that $\gamma_{[s,T]}$ minimizes the functional $f_{\lambda,\varphi}$ over the set $\Gamma(s, \gamma(s))$, for any $0 \leq r \leq s \leq T$. Note that this is not obvious since the maximization problems for different $s$ are defined over different sets, so the corresponding maximizers need not be related. We know, however, that for every $\eta \in \Gamma(s, \gamma(s))$, the spliced curve $\gamma \bowtie \eta$ belongs to $\Gamma(r, x)$ because of (A3b). Since $\gamma$ maximizes $f_{\lambda,\varphi}$ over the set $\Gamma(r, x)$, we have

$$\int_r^T \exp(-\lambda t)\varphi(\gamma(t))\,dt = f_{\lambda,\varphi}(\gamma) \geq f_{\lambda,\varphi}(\gamma \bowtie \eta) \quad (3.22)$$

By definition (3.21), the second integral can be decomposed as

$$\int_r^T \exp(-\lambda t)\varphi(\gamma \bowtie \eta(t))\,dt = \int_r^s \exp(-\lambda t)\varphi(\gamma(t))\,dt + \int_s^T \exp(-\lambda t)\varphi(\eta(t))\,dt.$$  

Decomposing the first integral in (3.22) and simplifying, we find that

$$f_{\lambda,\varphi}(\gamma_{[s,T]}) = \int_s^T \exp(-\lambda t)\varphi(\gamma(t))\,dt \geq \int_s^T \exp(-\lambda t)\varphi(\eta(t))\,dt$$

$$= f_{\lambda,\varphi}(\eta) \quad \text{for all } \eta \in \Gamma(s, \gamma(s)),$$

which proves that indeed $\gamma_{[s,T]} \in \Gamma_{\lambda,\varphi}(s, \gamma(s))$. This is statement (A3a).

Consider now $\gamma \in \Gamma_{\lambda,\varphi}(r, x)$ and $\eta \in \Gamma_{\lambda,\varphi}(s, \gamma(s))$ for some $s \in I$. We must show that the spliced curve $\gamma \bowtie \eta$ maximizes $f_{\lambda,\varphi}$ over $\Gamma(r, x)$. Since we have just proved that $\gamma_{[s,T]} \in \Gamma_{\lambda,\varphi}(s, \gamma(x))$ and by choice of $\eta$, we have the following identity:

$$\int_r^T \exp(-\lambda t)\varphi(\gamma(t))\,dt = \max_{\zeta \in \Gamma(s, \gamma(x))} f_{\lambda,\varphi}(\zeta) = \int_s^T \exp(-\lambda t)\varphi(\eta(t))\,dt.$$  

Decomposing the integrals again, we obtain

$$f_{\lambda,\varphi}(\gamma \bowtie \eta) = \int_r^s \exp(-\lambda t)\varphi(\gamma(t))\,dt + \int_s^T \exp(-\lambda t)\varphi(\eta(t))\,dt$$

$$= \int_r^s \exp(-\lambda t)\varphi(\gamma(t))\,dt + \int_s^T \exp(-\lambda t)\varphi(\gamma(t))\,dt = f_{\lambda,\varphi}(\gamma).$$

Since $\gamma$ maximizes $f_{\lambda,\varphi}$ over $\Gamma(r, x)$, so does $\gamma \bowtie \eta$. This is statement (A3b).

To finish the proof, we proceed in three steps.

**Step 1.** Let $\{g_i\}_{i \in \mathbb{N}}$ be a countable collection of Lipschitz continuous functions of bounded support that are closed under multiplication and strongly separates points on the separable metric space $\Omega \subset \mathbb{R}^d$, endowed with the Euclidean distance; see Remark 2.15. Consider also a sequence $(\mu_n)_{n \in \mathbb{N}}$ of distinct complex numbers $\mu_n$ satisfying condition (2.3). We now fix some enumeration

$$(\lambda_k, \varphi_k) := (\mu_n, g_{i_k})$$

of all pairs $(\mu_n, g_{i_k})$. We define recursively the sets $\Gamma_0(s, x) := \Gamma(s, x)$ and

$$\Gamma_k(s, x) := \text{argmax} \left\{ f_{\lambda_k,\varphi_k}(\gamma) : \gamma \in \Gamma_{k-1}(s, x) \right\} \quad \text{with } (s, x) \in I \times \Omega,$$

for all $k \in \mathbb{N}$. Finally, we define the intersections $\Gamma_\infty(s, x) := \bigcap_{k \in \mathbb{N}} \Gamma_k(s, x)$. 

As explained above, all $\Gamma_k$ have the properties (A1)-(A3b). Since $\mathcal{C}([s,T];\mathbb{R}^d)$ is a Banach space (complete) with respect to uniform convergence, and since the sets $\Gamma_k(s,x)$ are decreasing, nonempty, and compact, the intersection $\Gamma(s,x)$ is again nonempty (and compact), for every $(s,x) \in I \times \Omega$. Because of Lemma 2.6, we conclude that for all $s \in I$, the map $x \mapsto \Gamma(s,x)$ is (Borel) measurable.

**Step 2.** We claim that for each $(s,x) \in I \times \Omega$, the set $\Gamma(s,x)$ contains precisely one curve, i.e., one curve $\gamma \in \mathcal{C}([s,T];\mathbb{R}^d)$ that is a Filippov solution of (3.17). Indeed fix any $(s,x)$ and consider $\gamma, \eta \in \Gamma(s,x)$. By construction, we have

$$f_{\mu_n, g_i}(\gamma) = f_{\mu_n, g_i}(\eta) \quad \text{for all } i, n \in \mathbb{N}.$$  

Defining $F_{\gamma,i}(t) := \int_0^t g_i(\gamma(r)) \, dr$ for $t \geq 0$ (extending $g_i(\gamma)$ by zero outside $[s,T]$), we obtain a function in $\text{Lip}_0(\mathbb{R}^d;\mathbb{R})$. Notice that $\gamma$ remains bounded (see Remark 3.5) and $g_i$ has bounded support. We define $F_{\eta,i}$ in a similar way. Then

$$\int_0^\infty \exp(-\mu_n t) \, d(F_{\gamma,i} - F_{\eta,i})(t) = \int_s^T \exp(-\mu_n t) \left( g_i(\gamma(t)) - g_i(\eta(t)) \right) \, dt = 0$$

for all $n \in \mathbb{N}$, for any $i \in \mathbb{N}$; see Remark 2.11. Applying Lemma 2.10, we get

$$F_{\gamma,i}(t) = F_{\eta,i}(t) \quad \text{for all } t \geq 0,$$

which implies that $g_i(\gamma(t)) = g_i(\eta(t))$ for a.e. $t \in [s,T]$, by differentiation. But since both $\gamma$ and $\eta$ are continuous on $[s,T]$, and $g_i$ is continuous as well, we get

$$g_i(\gamma(t)) = g_i(\eta(t)) \quad \text{for all } t \in [s,T] \text{ and all } i \in \mathbb{N}.$$

The collection $\{g_i\}_{i \in \mathbb{N}}$ strongly separates points, by assumption. Hence $\gamma(t) = \eta(t)$ for all $t \in [s,T]$. This proves that $\Gamma(s,x)$ contains only one curve.

**Step 3.** For all $(s,x) \in \Omega$, we now define the flow

$$X(t,s,x) := \gamma(t) \quad \text{for all } t \in [s,T],$$

where $\gamma$ is the unique curve in $\Gamma(s,x)$. Then $t \mapsto X(t,s,x)$ is a Filippov solution of (3.17), by construction. Moreover, since the map $x \mapsto X(t,s,x)$ is the composition of the (Borel) measurable set-valued function $x \mapsto \Gamma(s,x)$ with the evaluation

$$e_t : \mathcal{C}([s,T];\mathbb{R}^d) \longrightarrow \Omega$$

defined by $e_t(\gamma) := \gamma(t)$ for all $t \in [s,T]$, which is continuous, it is Borel measurable. It only remains to establish the semigroup property (3.18). Notice that if $\gamma \in \Gamma_k(r,x)$ for some $x \in \Omega$, then $\gamma \in \Gamma_k(r,x)$ for all $k \in \mathbb{N}$, by definition of $\Gamma_k(r,x)$. Then

$$\gamma|_{[s,T]} \in \Gamma_k(s,\gamma(s)) \quad \text{for all } 0 \leq r \leq s \leq T \text{ and } k \in \mathbb{N}$$

because of property (A3a), which implies that $\gamma|_{[s,T]} \in \Gamma(s,\gamma(s))$. But $\Gamma(s,\gamma(s))$ contains exactly one curve, which we have used to define the map $t \mapsto X(t,s,\gamma(s))$. Since $X(t,r,x) = \gamma(t)$ for all $0 \leq r \leq t \leq T$ (in particular, for $t = s$), we get

$$X(t,t,x) = \gamma(t) = X(t,s,\gamma(s)) = X(t,s,X(s,r,x))$$

for all $0 \leq r \leq s \leq t \leq T$ and $x \in \Omega$. This completes the proof. \hfill \Box

**Corollary 3.16.** Let $I := [0,T]$ for some $T > 0$ and $\Omega \subset \mathbb{R}^d$ closed. Suppose that a velocity field $b \in L^\infty_{\text{loc}}(I \times \Omega;\mathbb{R}^d)$ as in Theorem 3.13 is given, with associated flow map $X$ as defined there. For any finite Borel measure $\mu \geq 0$ on $\Omega$, we define

$$\varrho(t,\cdot) := X(t,0,\cdot)^\# \mu \quad \text{for all } t \in I,$$  

(3.23)
where \( \# \) denotes the push-forward operator. Then \((\varrho, b)\) solves the continuity equation \( \{1.1\} \) in distributional sense, with initial data \( \varrho(0, \cdot) = \bar{\varrho} \).

**Proof.** This follows from standard arguments; see [27], for instance. Note that \( \{3.23\} \) is well-defined since the map \( x \mapsto X(t, 0, x) \) is Borel measurable, for all \( t \in I \). \( \square \)

**Remark 3.17.** After possibly modifying it on a set of measure zero, we may assume that the map \( t \mapsto \varrho(t, \cdot) \) for \( t \in I \) is continuous with respect to weak* convergence of measures; see Lemma 3.7 in [27]. If the measure \( \varrho(t, \cdot) \) is absolutely continuous with respect to \( L^d \), then we use the same symbol also for its Radon-Nikodým density.

### 4. Transport Equations

In this section, we first consider a very simple transport equation with constant velocity field and investigate the well-posedness of its variational formulation. Then we will introduce the notion of flow solution to the transport equation \( \{1.8\} \) and establish their existence and uniqueness. Finally, we will discuss stability properties and possible numerical approximations using the Petrov-Galerkin framework.

We start with the simple transport equation

\[
\partial_t U + CU = F \quad \text{in } Q := (0, T) \times D, \\
U(0, \cdot) = \bar{U} \quad \text{in } D, \tag{4.1}
\]

with \( T > 0 \) and \( D \subset \mathbb{R}^d \) a bounded Lipschitz domain, for

\[
C \in \mathcal{L}^\infty(Q), \quad F \in \mathcal{L}^2(Q), \quad \bar{U} \in \mathcal{L}^2(D). \tag{4.2}
\]

Formally \( \{4.1\} \) can be interpreted as a family of ODEs and solved explicitly by

\[
U(t, x) = \left( \bar{U}(x) + \int_0^t F(s, x) \exp \left( I(s, x) \right) ds \right) \exp \left( - I(t, x) \right), \tag{4.3}
\]

for \( t \in [0, T] \), with \( x \in D \) fixed. We are going to rephrase the problem in variational form, which is more suitable for the transport equation \( \{1.8\} \).

**Remark 4.1.** We may assume that \( C(t, x) \geq 0 \) for a.e. \((t, x)\). Indeed if \( U \) is a solution of \( \{4.1\} \), then the scaled function \( \tilde{U}(t, x) := \exp(-\lambda t)U(t, x) \) satisfies

\[
\partial_t \tilde{U} + (C + \lambda)\tilde{U} = F \quad \text{in } Q, \\
\tilde{U}(0, \cdot) = \bar{U} \quad \text{in } D, 
\]

where \( \tilde{F}(t, x) := \exp(-\lambda t)F(t, x) \) for a.e. \((t, x)\). Then \( C + \lambda \geq 0 \) if \( \lambda \geq \|C\|_{\mathcal{L}^\infty(Q)} \). In particular, we can consider the case that \( C \) vanishes.

For any \( U, V \in \mathcal{C}(\bar{Q}) \cap \mathcal{C}^1(Q) \) we define

\[
\mathcal{B}_U U := \partial_t U + CU \quad \text{and} \quad \mathcal{B}_V^* V := -\partial_t V + CV, \tag{4.4}
\]

with \( \mathcal{B}_V^* \) the formal adjoint of \( \mathcal{B}_V \). Defining the spaces

\[
\mathcal{C}^1_\pm(Q) := \{ U \in \mathcal{C}(\bar{Q}) \cap \mathcal{C}^1(Q) : U(s_\pm, \cdot) = 0 \}
\]

with \( s_- := 0, s_+ := T \), and using Green’s Theorem, we have that

\[
\langle \mathcal{B}_U U, V \rangle_{\mathcal{L}^2(Q)} = \langle U, \mathcal{B}_V^* V \rangle_{\mathcal{L}^2(Q)} \quad \text{for all } U \in \mathcal{C}^1_-(Q) \text{ and } V \in \mathcal{C}^1_+(Q),
\]
where \( \langle \cdot , \cdot \rangle_{\mathcal{L}^2(Q)} \) is the \( \mathcal{L}^2(Q) \)-inner product. One can show that the graph norm
\[
\|V\| : = \|\mathcal{B}^*_0 V\|_{\mathcal{L}^2(Q)} \tag{4.5}
\]
is indeed a norm on \( \mathcal{C}^1_+(Q) \) since
\[
\|V\|_{\mathcal{L}^2(Q)} \leq 2T\|\mathcal{B}^*_0 V\|_{\mathcal{L}^2(Q)} \quad \text{for all } V \in \mathcal{C}^1_+(Q);
\]
see Proposition 2.2(i) in [21]. (Note that the flow associated to the constant velocity field in (4.1) is trivially \( L^2 \)-filling, as defined there.) Let \( \mathcal{V} \) be the closure of \( \mathcal{C}^1_+(Q) \) with respect to (4.5), which is a Hilbert space with inner product/norm
\[
\langle V, W \rangle_{\mathcal{V}} := \langle \mathcal{B}^* V, \mathcal{B}^* W \rangle_{\mathcal{L}^2(Q)} \quad \text{and} \quad \|V\|_{\mathcal{V}} := \|\mathcal{B}^* V\|_{\mathcal{L}^2(Q)}
\]
for all \( V, W \in \mathcal{V} \). Here \( \mathcal{B}^*: \mathcal{V} \to \mathcal{L}^2(Q) \) denotes the continuous extension of \( \mathcal{B}^*_0 \) from \( \mathcal{C}^1_+(Q) \) to \( \mathcal{V} \). We define \( \mathcal{B}: \mathcal{L}^2(Q) \to \mathcal{V}' \) by duality as \( \mathcal{B} := (\mathcal{B}^*)^* \).

We can now present the variational formulation of (4.1), which will be the basis of the adaptive Petrov-Galerkin approximation in the Section 4.2. Let
\[
b(U, V) := \langle U, \mathcal{B}^* V \rangle_{\mathcal{L}^2(Q)} \quad \text{for all } U \in \mathcal{L}^2(Q) \text{ and } V \in \mathcal{V}. \tag{4.6}
\]
Moreover, define the linear functional
\[
\ell(V) := \langle F, V \rangle_{\mathcal{L}^2(Q)} + \int_D \tilde{U}(x)V_0(x) \, dx \quad \text{for all } V \in \mathcal{V}, \tag{4.7}
\]
where \( V_0 \) is the strong \( \mathcal{L}^2(D) \)-trace of the test function \( V \) on the boundary \( \{0\} \times D \).

The existence of traces was proved in Proposition 2.4 in [26], for instance.

**Theorem 4.2.** For functions \( C \geq 0, F \) and initial data \( \tilde{U} \) as in (4.2), let the bilinear form \( b \) and the functional \( \ell \) be defined by (4.6) and (4.7), respectively. Then
\[
\sup_{U \in \mathcal{L}^2(Q)} \sup_{V \in \mathcal{V}} \frac{b(U, V)}{\|U\|_{\mathcal{L}^2(Q)}\|V\|_{\mathcal{V}}} = \inf_{U \in \mathcal{L}^2(Q)} \sup_{V \in \mathcal{V}} \frac{b(U, V)}{\|U\|_{\mathcal{L}^2(Q)}\|V\|_{\mathcal{V}}} = 1, \tag{4.8}
\]
so that the continuity and inf-sup-conditions for the form \( b \) are satisfied with optimal constant 1. This is equivalent to the following stability estimates:
\[
\|\mathcal{B}\|_{\mathcal{L}(\mathcal{L}^2(Q), \mathcal{V}')} = \|\mathcal{B}^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{L}^2(Q))} = 1,
\]
\[
\|\mathcal{B}^*\|_{\mathcal{L}(\mathcal{V}, \mathcal{L}^2(Q))} = \|(\mathcal{B}^*)^{-1}\|_{\mathcal{L}(\mathcal{L}^2(Q), \mathcal{V}')} = 1.
\]
Here \( \mathcal{L} \) denotes spaces of bounded linear maps, and \( \mathcal{V}' \) is the dual of \( \mathcal{V} \). In particular, there exists a unique \( U \in \mathcal{L}^2(Q) \) that solves equation (4.1) in the sense that
\[
b(U, V) = \ell(V) \quad \text{for all } V \in \mathcal{V}, \tag{4.9}
\]
and the stability estimate \( \|U\|_{\mathcal{L}^2(Q)} \leq \|\ell\|_{\mathcal{V}'} \) holds.

Moreover, for any \( W \in \mathcal{L}^2(Q) \) we have that
\[
\|U - W\|_{\mathcal{L}^2(Q)} = \|R_W\|_{\mathcal{V}'}, \tag{4.10}
\]
with residual \( R_W \in \mathcal{V}' \) defined for all \( V \in \mathcal{V} \) as
\[
R_W(V) := b(U - W, V) = \ell(V) - b(W, V).
\]
Therefore the approximation error can be controlled in terms of the residual.

**Proof.** This is [21] Theorem 2.5, which is based on [26]. Then (4.10) follows from
\[
\|U - W\|_{\mathcal{L}^2(Q)} \left\{ \begin{array}{ll} \leq & \sup_{V \in \mathcal{V}} \frac{b(U - W, V)}{\|V\|_{\mathcal{V}}} = \sup_{V \in \mathcal{V}} \frac{R_W(V)}{\|V\|_{\mathcal{V}}} = \|R_W\|_{\mathcal{V}'}, \end{array} \right.
\]
which is a consequence of the continuity and the inf-sup-condition (4.8). \( \Box \)
Remark 4.3. Given assumptions (4.12) on the data, the map $U$ in (4.3) is well-defined for $t \in [0, T]$ and a.e. $x \in D$. In particular, $U$ belongs to $L^2(Q)$, it is differentiable with respect to $t$, and $\partial_t U = -CU + F$ belongs to $L^2(Q)$ as well. Using integration by parts, one finds that $U$ satisfies the variational formulation (4.9). Since variational solutions are unique, the $U$ in (4.3) coincides with the unique solution to (4.9).

As explained in the Introduction, we want to solve transport equations of the form (1.8) by decoupling geometry and transport.

Definition 4.4. Let $I := [0, T]$ for $T > 0$, $D \subset \mathbb{R}^d$ a bounded Lipschitz domain, and $Q := (0, T) \times D$. Consider $b \in L^\infty(Q; \mathbb{R}^d)$ and $X : Q \rightarrow D$ such that

1. For all $t \in I$, the map $x \mapsto X(t, x)$ is Borel measurable.
2. For all $x \in D$, the map $t \mapsto X(t, x)$ is a Filippov solution of
   \[
   \dot{\gamma}(t) = b(t, \gamma(t)) \quad \text{for } t \in I, \quad \gamma(0) = x.
   \]
3. The flow lines are forward untangled: for $s \in I$ and $x_1, x_2 \in \bar{D}$
   \[
   X(s, x_1) = X(s, x_2) \implies X(t, x_1) = X(t, x_2) \quad \text{for all } s \leq t \leq T.
   \]

For all $t \in I$, we define a density $\varrho(t, \cdot)$ as in (3.23) with initial measure $\bar{\varrho} = L^d, D$, and the space-time measure $\sigma(dx, dt) := \varrho(t, dx) dt$ on $Q$. Consider now

\[
\begin{align*}
    c & \in L^\infty(\bar{Q}, \sigma), \quad f \in L^2(\bar{Q}, \sigma), \quad \bar{u} \in L^2(D). 
\end{align*}
\]

We say that $u \in L^2(\bar{Q}, \sigma)$ is a flow-solution of
\[
\begin{align*}
    \partial_t u + b \cdot \nabla u + cu &= f \quad \text{in } \bar{Q}, \\
    u(0, \cdot) &= \bar{u} \quad \text{in } D,
\end{align*}
\]
if the following is true: for all $\varphi \in C(\bar{Q})$ it holds
\[
\iint_{\bar{Q}} u(t, z) \varphi(t, z) \varrho(t, dz) dt = \iint_{\bar{Q}} U(t, x) \varphi(t, X(t, x)) dx dt,
\]
where $U \in L^2(Q)$ is a variational solution of (4.9) (satisfying (4.9)) with
\[
\begin{align*}
    C(t, x) &:= c(t, X(t, x)), \\
    F(t, x) &:= f(t, X(t, x)), \\
    \bar{U}(x) &:= \bar{u}(x)
\end{align*}
\]
for $(t, x) \in (0, T) \times D$. We assume that $c \geq 0 \sigma$-a.e.; cf. Remark 4.1.

As was shown in Theorem 3.13, a flow $X$ as in Definition 4.4 does exist, but may be non-unique. Once the flow has been selected we have well-posedness:

Theorem 4.5. Under the assumptions listed in Definition 4.4 there exists a unique flow-solution of the initial value problem (4.12) of the transport equation.

Proof. Our assumptions (4.11) imply that Theorem 4.2 can be applied. Indeed recall that the push-forward formula (3.23) implies the identity
\[
\iint_{\bar{Q}} g(t, X(t, x)) dx dt = \iint_{\bar{Q}} g(t, z) \varrho(t, dz) dt
\]
for all $g \in L^\infty(I \times \Omega)$ with bounded support. From this, we get
\[
\iint_{\bar{Q}} |F(t, x)|^2 dx dt = \iint_{\bar{Q}} |f(t, z)|^2 \varrho(t, dz) dt
\]
(using a suitable approximation argument), which is finite. In a similar way, one can check that the other functions in (4.14) satisfy (4.2), as required. \qed
Notice that the function

\[ V(t,x) := \zeta(t,X(t,x)) \quad \text{for } (t,x) \in Q \]  

(4.16)

belongs to the test space \( \mathcal{Y} \), for any \( \zeta \in \mathcal{C}(\bar{Q}) \cap \mathcal{C}^1(Q) \). Indeed since the function \( V \) is bounded and \( Q \) has finite Lebesgue measure, and since

\[ \partial_t V(t,x) = \partial_t \zeta(t,X(t,x)) + b(t,X(t,x)) \cdot \nabla \zeta(t,X(t,x)), \]  

(4.17)

by definition of the flow \( X \), the claim follows from the boundedness of \( b \). We can use this test function in the variational formulation (4.9) and obtain

\[
\int_Q \int_U(t,x) \left\{ - \partial_t \zeta(t,X(t,x)) - b(t,X(t,x)) \cdot \nabla \zeta(t,X(t,x)) \right. \\
+ C(t,x) \zeta(t,X(t,x)) \left. \right\} \, dx \, dt
\]

\[ = \int_Q F(t,x) \zeta(t,X(t,x)) \, dx \, dt + \int_D \bar{U}(x) \zeta(x) \, dx. \]  

(4.18)

Because of (4.14) and (4.15) (and another approximation argument), this gives

\[
\int_Q u(t,z) \left\{ - \partial_t \zeta(t,z) - b(t,z) \cdot \nabla \zeta(t,z) + c(t,z) \zeta(t,z) \right\} \varrho(t,dz) \, dt
\]

\[ = \int_Q f(t,z) \zeta(t,z) \varrho(t,dz) \, dt + \int_D \bar{\varrho}(z) \zeta(z) \, \bar{\varrho}(dz), \]

which is the weak formulation of the initial value problem

\[ \partial_t (\varrho u) + \nabla \cdot (\varrho u b) + c \varrho u = \varrho f \quad \text{in } \bar{Q}, \]

\[ (\varrho u)(0,\cdot) = \varrho \bar{u} \quad \text{in } D. \]

Since \( \varrho \) is a weak solution of the continuity equation

\[ \partial_t \varrho + \nabla \cdot (\varrho b) = 0 \quad \text{in } \bar{Q}, \]

\[ \varrho(0,\cdot) = \bar{\varrho} \quad \text{in } D, \]

we formally get (4.12) using the product rule. Therefore the notion of flow-solution is indeed a suitable solution concept for the transport equation (4.12).

Remark 4.6. If there exist \( s \in I \) and \( x_1, x_2 \in \bar{D} \) such that \( X(s,x_1) = z = X(s,x_2) \), then the solution \( u \) at \( (s,z) \) is found as the superposition of the values of \( u \) along all incoming flow lines, because of (4.13). This is similar to the sticky particle dynamics for pressureless gas dynamics; see [18]. In particular, if all flow lines starting from a set with positive Lebesgue measure merge at \( (s,z) \), then the density \( \varrho \) becomes singular, forming a Dirac measure at \( z \), for instance. Since the flow lines are assumed to be forward untangled, however, we have that \( C(t,x_1) = C(t,x_2) \) and \( F(t,x_1) = F(t,x_2) \) for all \( s \leq t \leq T \); recall (4.14). The ODEs (4.11) at the points \( x_1 \) and \( x_2 \), respectively, are therefore identical after the merging; see Figure 1.

We have the following compactness result for flow solutions of (4.12).

**Theorem 4.7.** Let \( I := [0,T] \) for \( T > 0 \), \( D \subset \mathbb{R}^d \) a bounded Lipschitz domain, and \( Q := (0,T) \times D \). Consider a sequence of velocity fields \( b_n \in L^\infty(\bar{Q};\mathbb{R}^d) \) with corresponding flows \( X_n : \bar{Q} \to \bar{D} \) whose flow lines are forward untangled. For all
Suppose that $c_n \geq 0$ a.e.; cf. Remark 4.4.

We now make the following assumptions:

1. There exist $b \in L^\infty(\bar{Q};\mathbb{R}^d)$ and an associated flow $X: \bar{Q} \rightarrow \bar{D}$ with

   \[ b_n \rightarrow b, \quad X_n \rightarrow X \] pointwise a.e. in $\bar{Q}.$

   The flow lines of $X$ are forward untangled.

2. There exists $\bar{u} \in L^2(D)$ such that

   \[ \bar{u}_n \rightarrow \bar{u} \quad \text{weakly in } L^2(D). \]

3. There exist $c \in L^\infty(\bar{Q},\sigma)$ and $f \in L^2(\bar{Q},\sigma)$ such that

   \[ C_n \rightarrow C, \quad F_n \rightarrow F \] pointwise a.e. in $Q,$ respectively, and with density $\rho$ and space-time measure $\sigma$ as in Definition 4.4.

There exists a subsequence (not relabeled) and $U \in L^2(Q)$ with $U_n \rightarrow U$ weakly in $L^2(Q)$ such that $u \in L^2(\bar{Q},\sigma)$ defined by (4.13) is a flow-solution of (4.12).

Proof. The weak convergences (4.21) and (4.23) imply uniform boundedness of the linear forms $\ell_n$ in $\mathcal{Y}'.$ Because of the uniform stability estimate $\|U_n\|_{L^2(\bar{Q})} \leq \|\ell_n\|_{\mathcal{Y}}$, (see Theorem 4.2), it follows that the sequence of $U_n$ is uniformly bounded, thus...
weakly precompact. We can therefore extract a subsequence (not relabeled) such that \( U_n \rightharpoonup U \) for some \( U \in L^2(Q) \). Notice that for any \( \varphi \in C(\bar{Q}) \), the map

\[
(t,x) \mapsto \varphi(t,X_n(t,x)) \quad \text{for} \quad (t,x) \in \bar{Q}
\]

is uniformly bounded in \( L^\infty(\bar{Q}) \) and converges pointwise a.e. This implies that

\[
\sigma_n \rightharpoonup \sigma \quad \text{weak}^* \text{ in the sense of measures},
\]

for some \( \sigma \in \mathcal{M}(\bar{Q}) \) that can be disintegrated in the form \( \sigma(dz,dt) =: \varrho(t,dz) \, dt \). Similarly, we have the convergence

\[
\sigma_n u_n \rightharpoonup m \quad \text{weak}^* \text{ in the sense of measures},
\]

for some finite measure \( m \in \mathcal{M}(\bar{Q}) \).

In particular, the functions \( U \in L^2(Q) \) and \( u \in L^2(\bar{Q},\sigma) \) are related by (4.13).

It remains to check that \( U \) is a solution of the variational problem (4.9). This follows from passing to the limit in the integrals defining the bilinear forms \( b_n \) and the right-hand sides \( \ell_n \). For instance, for any \( V \in \mathcal{Y} \) the maps

\[
(t,x) \mapsto C_n(t,x) V(t,x) \quad \text{for} \quad (t,x) \in Q
\]

converge **strongly** in \( L^2(Q) \) because the first factor remains bounded and converges pointwise a.e., by assumption (4.22). It follows that

\[
\int_Q U_n(t,x) C_n(t,x) V(t,x) \, dx \, dt \rightharpoonup \int_Q U(t,x) C(t,x) V(t,x) \, dx \, dt
\]

because \( U_n \rightharpoonup U \) weakly in \( L^2(Q) \). In a similar way, we find that

\[
\int_Q U_n(t,x) \partial_t V(t,x) \, dx \, dt \rightharpoonup \int_Q U(t,x) \partial_t V(t,x) \, dx \, dt
\]

\[
\int_Q F_n(t,x) V(t,x) \, dx \, dt \rightharpoonup \int_Q F(t,x) V(t,x) \, dx \, dt
\]

\[
\int_D \bar{U}_n(x) V_0(x) \, dx \rightharpoonup \int_D \bar{U}(x) V_0(x) \, dx,
\]

where \( V_0 \) is the strong \( L^2(D) \)-trace of \( V \in \mathcal{Y} \) on the boundary \( \{0\} \times D \). Thus

\[
b(U,V) = \lim_{n \to \infty} b_n(U_n,V) = \lim_{n \to \infty} \ell_n(V) = \ell(V) \quad \text{(4.24)}
\]

for all \( V \in \mathcal{Y} \). This concludes the proof of the theorem.

\[
\square
\]

**Remark 4.8.** A sufficient condition for (4.22) and (4.23) is that \( c_n \) and \( f_n \)

- are Borel measurable,
- uniformly bounded in \( L^\infty(\bar{Q}) \),
- converge pointwise a.e. to \( c \) and \( f \), respectively.
In this case, for any $V \in \mathcal{Y}$ the maps

$$(t, x) \mapsto C_n(t, x)V(t, x) = c_n(t, X_n(t, x))V(t, x) \quad \text{for } (t, x) \in Q$$

converge strongly in $\mathcal{L}^2(Q)$ because the first factor remains bounded and converges pointwise a.e., by assumptions (4.20) and (4.22). The same applies to

$$(t, x) \mapsto F_n(t, x) = f_n(t, X_n(t, x)) \quad \text{for } (t, x) \in Q.$$ 

We can then pass to the limit in the integrals

$$\int_Q U_n(t, x)c_n(t, X(t, x))V(t, x) \, dx \, dt \longrightarrow \int_Q U(t, x)c(t, X(t, x))V(t, x) \, dx \, dt,$$

$$\int_Q f_n(t, X(t, x))V(t, x) \, dx \, dt \longrightarrow \int_Q f(t, X(t, x))V(t, x) \, dx \, dt.$$

We can then argue as above to obtain (4.24).

If there exists a constant $C > 0$ such that the densities $\varrho_n$ satisfy the inequalities (4.25) below uniformly in $n$, then a sufficient condition for (4.23) is for $f_n$ to converge strongly in $\mathcal{L}^2(Q)$. In this case, integration with respect to the space-time measure $\sigma_n$ is equivalent to integration against the $(d + 1)$-dimensional Lebesgue measure on $Q$. Indeed, let us define the truncation operator $T_R : \mathbb{R} \rightarrow \mathbb{R}$ as

$$T_R(s) := \begin{cases} s & \text{if } |s| \leq N \\ Ns/|s| & \text{otherwise} \end{cases} \quad \text{for all } s \in \mathbb{R}.$$ 

We can then estimate

$$\int_Q \left| (1 - T_R)\left( F_n(t, x) \right) \right|^2 \, dx \, dt = \int_Q \left| (1 - T_R)\left( f_n(t, X_n(t, x)) \right) \right|^2 \, dx \, dt$$

$$= \int_Q \left| (1 - T_R)\left( f_n(t, z) \right) \right|^2 \varrho_n(t, dz) \, dt \leq C \int_Q \left| (1 - T_R)\left( f_n(t, z) \right) \right|^2 \, dz \, dt,$$

which is arbitrarily small uniformly in $n$ if $R > 0$ is large enough. For the function $T_R(F_n)$ we can argue as above. This gives $F_n \rightarrow F$ strongly in $\mathcal{L}^2(Q)$.

**Remark 4.9.** As in the proof of Theorem 4.7, we find that the maps

$$V(t, x) := \zeta(t, X_n(t, x)) \quad \text{for } (t, x) \in Q$$

converge strongly in $\mathcal{Y}$ to the function $V$ defined in (4.16), for any $\zeta \in \mathcal{C}(\overline{Q}) \cap \mathcal{C}^1(\overline{Q})$. Indeed note that the derivative $\partial_tV_n$ is of the form (4.17). We can then combine the uniform boundedness of $(\partial_t\zeta, \nabla \zeta)$ and the velocity fields $b_n$, with the pointwise convergence (4.20) to obtain strong convergence in $\mathcal{L}^2(Q)$ for both $V$ and its partial derivative $\partial_t V$. This enables us to pass to the limit in the analogue of (4.18).

**4.1. Stability of Flow Map.** We are interested in the situation where the sequence of flows $X_n$ in Theorem 4.7 is generated by an approximation $b_n$ of the velocity field, e.g., in a numerical method. A recent result by Bianchini and Bonicatto [11] provides strong convergence in $\mathcal{L}^1(Q)$ if the sequence of (approximate) velocity fields is uniformly of bounded variation and nearly incompressible.

**Definition 4.10.** We say that a velocity $b \in \mathcal{L}^\infty(Q; \mathbb{R}^d)$ is **nearly incompressible** if there exists a function $\varrho \in \mathcal{L}^\infty(Q)$ and a constant $C > 0$ such that

$$C^{-1} \leq \varrho(t, x) \leq C \quad \text{for a.e. } (t, x) \in Q,$$

(4.25)
where $\varrho$ is a distributional solution of the continuity equation (1.1). We will assume in the following that the initial data is given by the constant function: $\varrho(0,x) = 1$ for all $x \in \Omega$, in which case we will refer to $\varrho$ as a density associated to $b$. We will also say that $\varrho$ is nearly incompressible if (4.25) holds.

**Remark 4.11.** A sufficient condition for near incompressibility is that $\text{div} \, b$ is bounded. Indeed the continuity equation can (formally) be rewritten as

$$
\partial_t \varrho + b \cdot \nabla \varrho = -\varrho \text{div} \, b \quad \text{in } Q,
$$

so that the change of log $\varrho$ along the integral curves of $b$ is given by $-\text{div} \, b$. In general, near incompressibility is strictly weaker since it allows for certain discontinuities in the velocity field, as is relevant for hyperbolic conservation laws, for instance.

**Remark 4.12.** Near incompressibility encodes important information about the flow: On the one hand, the lower bound in (4.25) (absence of vacuum) implies that there are no sets of positive Lebesgue measure that are not reached by the flow. This is a weak form of surjectivity. On the other hand, the upper bound in (4.25) (absence of concentrations) guarantees that no sets of positive Lebesgue measure are mapped to small sets such as a single point. This is a weak form of injectivity.

We can now state the main stability result for approximate flow maps.

**Theorem 4.13.** Let $I := [0,T]$ for $T > 0$, $D \subset \mathbb{R}^d$ a bounded Lipschitz domain, and $Q := (0,T) \times D$. Consider a velocity field $b \in L^1((0,T); BV(D; \mathbb{R}^d))$ bounded and nearly incompressible. Then there exists a unique regular Lagrangian flow (RLF) for $b$, i.e., a flow $X: \bar{Q} \rightarrow \bar{D}$ associated to $b$ such that for a.e. $t \in (0,T)$

$$
A \subset \bar{D} \text{ Borel with } |A| = 0 \quad \Rightarrow \quad |\{x \in \bar{D}: X(t,x) \in A\}| = 0.
$$

Here $| \cdot |$ denotes the $d$-dimensional Lebesgue measure.

Moreover, consider a sequence of velocities $b_n$ with the following properties:

- For all $n \in \mathbb{N}$, we have $b_n \in L^1((0,T); BV(D; \mathbb{R}^d))$.
- The quantity $\|b_n\|_{L^\infty(Q)}$ is bounded uniformly in $n$ and $b_n \rightharpoonup b$ strongly in $L^1(Q)$.
- There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$
C^{-1} \leq \varrho_n(t,x) \leq C \quad \text{for a.e. } (t,x) \in Q,
$$

where $\varrho_n$ is defined by (3.23), with $X_n$ the RLF generated by $b_n$.

Then $X_n \rightharpoonup X$ strongly in $L^1(Q)$.

The statement of Theorem 4.13 was first formulated in [19] and is nowadays known as Bressan’s conjecture. It was proved very recently by Bianchini and Bonicatto [10]. The proof establishes the existence of a set of untangled flow lines that partition the space-time domain and induce the disintegration of the continuity equation into one-dimensional transport problems along integral curves. One can then show that the flow has the renormalization property, from which Bressan’s conjecture can be deduced; see Theorem 3.22 in [27] for details. It would be very interesting to make the stability/convergence of the approximate flows $X_n$ more quantitative, along the lines of [41], for example; see also [25] for additional information.
Remark 4.14. Another stability result for approximate flows was derived in [11] for Filippov solutions for monotone velocity fields, which satisfy the inequality

\[ (b(t, y) - b(t, x), y - x) \leq \alpha(t)|y - x|^2 \]  

(4.26)

for a.e. \( t \in I \) and \( x, y \in \mathbb{R}^d \), where \( \alpha \in \mathcal{L}^1(I) \). Rescaling time, one can reduce the problem to the case of zero right-hand side in (4.26); cf. Remark 4.1. The condition (4.26) is sufficient to ensure uniqueness of Filippov solutions; see [34]. In the context of transport equations it has been studied in [39] and [16]. Note that (4.26) corresponds to a compressive situation where \( \text{div} \, b \) may become a negative measure.

4.2. Petrov-Galerkin Approximation. To approximate the variational problem (4.9), an optimally stable Petrov-Galerkin method can be used, which we describe now. We follow the presentation in [21]. Let \( \mathcal{V} \subset \mathcal{L}^2(Q) \) and \( \mathcal{V} \subset \mathcal{W} \) be suitable trial and test spaces (e.g., finite dimensional finite element spaces). Suppose that \( \mathcal{W}, \mathcal{V} \) are Hilbert (sub)spaces. Then we aim to find \( U \in \mathcal{W} \) such that

\[ b(U, V) = f(V) \quad \text{for all } V \in \mathcal{V}. \]  

(4.27)

Arguing as for Theorem 4.2, one can show that (4.27) has a unique solution if

\[ \gamma \geq \sup_{U \in \mathcal{W}} \sup_{V \in \mathcal{V}} \frac{b(U, V)}{\| U \|_{\mathcal{L}^2(Q)} \| V \|_{\mathcal{W}}} \geq \inf_{U \in \mathcal{W}} \inf_{V \in \mathcal{V}} \frac{b(U, V)}{\| U \|_{\mathcal{L}^2(Q)} \| V \|_{\mathcal{W}}} \geq \beta \]  

(4.28)

for constants \( \beta > 0 \) and \( \gamma < \infty \). Since \( \mathcal{W} \subset \mathcal{L}^2(Q) \) and \( \mathcal{V} \subset \mathcal{W} \), by assumption, the upper bound in (4.28) holds with \( \gamma = 1 \) because of (4.8).

Let \( U \in \mathcal{L}^2(Q) \) and \( U_\mathcal{W} \in \mathcal{W} \) be unique solutions of (4.9) and (4.27), respectively. Since \( \mathcal{V} \subset \mathcal{W} \) we obtain the Galerkin orthogonality

\[ b(U - U_\mathcal{W}, V) = 0 \quad \text{for all } V \in \mathcal{V}. \]  

(4.29)

The inf-sup-condition (4.28) now implies that, for any \( W \in \mathcal{W} \),

\[ \beta \| U_\mathcal{W} - W \|_{\mathcal{L}^2(Q)} \leq \sup_{V \in \mathcal{V}} \frac{b(U_\mathcal{W} - W, V)}{\| V \|_{\mathcal{W}}} = \sup_{V \in \mathcal{V}} \frac{b(U - W, V)}{\| V \|_{\mathcal{W}}} \leq \| U - W \|_{\mathcal{L}^2(Q)}. \]

We have used (4.29) and the continuity of \( b \) with \( \gamma = 1 \). Then

\[ \| U - U_\mathcal{W} \|_{\mathcal{L}^2(Q)} \leq \| U - W \|_{\mathcal{L}^2(Q)} + \| U_\mathcal{W} - W \|_{\mathcal{L}^2(Q)} \leq (1 + \beta^{-1}) \| U - W \|_{\mathcal{L}^2(Q)}, \]

and so the error can be estimated by the best approximation of \( U \) in \( \mathcal{W} \):

\[ \| U - U_\mathcal{W} \|_{\mathcal{L}^2(Q)} \leq (1 + \beta^{-1}) \inf_{W \in \mathcal{W}} \| U - W \|_{\mathcal{L}^2(Q)}. \]

Note the dependence on the inf-sup-constant \( \beta > 0 \) from (4.28). As the trial space \( \mathcal{W} \) approaches \( \mathcal{L}^2(Q) \), the approximation \( U_\mathcal{W} \) converges to the exact solution \( U \).

In order to compute \( \beta \), for any \( U \in \mathcal{W} \) we want to solve

\[ \sup_{V \in \mathcal{V}} \frac{b(U, V)}{\| U \|_{\mathcal{L}^2(Q)} \| V \|_{\mathcal{W}}} = \beta. \]  

(4.30)

We rewrite (4.30) as a constrained maximization problem

\[ \sup_{V \in \mathcal{V}} \frac{b(U, V)}{\| U \|_{\mathcal{L}^2(Q)} \| V \|_{\mathcal{W}}} = \sup \{ b(U, V) : V \in \mathcal{V}, \| V \|_{\mathcal{W}} = 1 \} / \| U \|_{\mathcal{L}^2(Q)}. \]
which can be solved using Lagrange’s method: We consider the Lagrangian
\[ L_U(V, \lambda) := b(U, V) + \lambda \left( 1 - \|V\|^2_Y \right) \] (4.31)
with \( V \in \mathcal{Y} \) and \( \lambda \in \mathbb{R} \). We denote by \((V_0, \lambda_0)\) the critical points of (4.31), which are characterized by the fact that the first variations of (4.31) vanish. Thus
\[ b(U, W) - 2\lambda_0 b(V_0, W)_{\mathcal{Y}} = 0 \quad \text{for all } W \in \mathcal{Y}, \] (4.32)
and the normalization condition \( \|V_0\|^2_Y = 1 \) holds. Using \( W = V_0 \) in (4.32), we find that \( 2\lambda_0 = b(U, V_0) \). Defining \( V_U := 2\lambda_0 V_0 \), we have the identity
\[ b(U, W) = \langle V_U, W \rangle_{\mathcal{Y}} \quad \text{for all } W \in \mathcal{Y}, \] (4.33)
which shows that \( V_U \) is actually the unique element in \( \mathcal{Y} \) representing the bounded linear map \( V \mapsto b(U, V) \) on the Hilbert space \( \mathcal{Y} \), in the sense of the Riesz representation theorem. Note that, by definition and (4.33), we have the identities
\[ \langle U, \mathcal{B}^* V_U \rangle_{\mathcal{Z}^2(Q)} = b(U, V_U) = \|V_U\|^2_Y = \langle \mathcal{B}^* V_U, \mathcal{B}^* V_U \rangle_{\mathcal{Z}^2(Q)}. \] (4.34)

It is convenient to make the choice that
\[ \mathcal{U} := \mathcal{B}^* (\mathcal{Y}), \] (4.35)
in which case (4.34) implies that \( U = \mathcal{B}^* V_U \), or \( V_U = (\mathcal{B}^*)^{-1} U \) because the linear map \( \mathcal{B}^* \) is bounded invertible; see Theorem 4.2 above. We call the map \( \tau : \mathcal{U} \to \mathcal{Y} \), defined by \( \tau(U) := V_U \) for all \( U \in \mathcal{U} \), the trial-to-test map. Then
\[ \sup_{V \in \mathcal{Y}} \frac{b(U, V)}{\|U\|_{\mathcal{Z}^2(Q)} \|V\|_{\mathcal{Y}}} = \frac{b(U, V_U)}{\|U\|_{\mathcal{Z}^2(Q)} \|V_U\|_{\mathcal{Y}}} = \frac{\langle U, (\mathcal{B}^*)^{-1} U \rangle_{\mathcal{Z}^2(Q)}}{\|U\|_{\mathcal{Z}^2(Q)} \|\mathcal{B}^*(\mathcal{B}^*)^{-1} U\|_{\mathcal{Z}^2(Q)}} = 1 \]
for all \( U \in \mathcal{U} \). Taking the inf / sup in \( U \), we obtain the continuity estimate and the inf-sup-condition with constant 1. Therefore the method is optimally stable.

In order to realize an optimally conditioned and thus optimally stable Petrov-Galerkin method that is also computationally feasible, it was proposed in [21] to first choose a conformal finite-dimensional test space \( \mathcal{Y}_h \subset \mathcal{U} \), which depends on some discretization parameter \( h > 0 \), and then to define the trial space through (4.35). Possible choices for \( \mathcal{Y}_h \) include spaces of sufficiently smooth functions such as splines or broken spaces of finite elements on suitable triangulations of the domain. It is then proved that the inf-sup-property holds uniformly in \( h \). The authors also consider the dependence of the variational problem on a compact set of parameters and prove compactness of the solution set (analogous to our Theorem 4.7).

A more sophisticated framework was developed in [20] utilizing a discontinuous Petrov-Galerkin approximation. In this case, the unknown is a pair of \( U \in \mathcal{Z}^2(Q) \) as above and an additional function that captures the jumps across the boundaries of the triangulation cells. This is motivated by the fact that for generic \( \mathcal{Z}^2(Q) \)-functions the trace onto hyperplanes may not be defined. The authors prove near optimal stability of this approximation with discretized data.

**Remark 4.15.** The discussion above suggests a numerical method for solving the initial value problem (4.12) that consists of two steps:

1. In physical space-time approximate the velocity field \( b \) using finite elements on a suitable triangulation. Compute the corresponding flow \( X \).
2. Solve the variational formulation of the transport equation \( \partial_t U + CU = F \), with functions \( C \) and \( F \) defined by (4.14).
The triangulation of the first step induces a decomposition of \( Q := (0, T) \times D \) for the second step, as cells are mapped by the flow \( X \). It is possible to iterate the two steps adaptively, taking into account the error estimates in [20], for instance. The implementation of this approach will be considered in a future publication.

**Appendix A. Geometric Measure Theory**

Let \((X, d)\) be a metric space.

**Definition A.1.** Suppose that \( \mu \) is a Borel regular measure on \( X \) without atoms, such that every bounded set has finite \( \mu \)-measure. If \( C \subset X \) satisfies

\[
\lim_{r \to 0} \frac{\mu(B_r(x) \cap C)}{\mu(B_r(x))} = \alpha
\]

with \( x \in X \), then we say that \( C \) has density \( \alpha \) at \( x \), where \( \alpha \in [0, 1] \).

**Lemma A.2.** Consider \( \mu \)-measurable subsets \( C_n \subset X \), \( n \in \mathbb{N} \), that have density 1 at \( x \in X \). Then every finite intersection of sets \( C_n \) has density 1 at \( x \).

**Proof.** The set \( C := C_1 \cap C_2 \) is \( \mu \)-measurable. For any \( x \in X \) and \( r > 0 \) we have

\[
\mu(B_r(x) \cap (C_1 \cap C_2)) = \mu(B_r(x) \cap C_1) + \mu(B_r(x) \cap C_2) - \mu(B_r(x) \cap (C_1 \cup C_2)).
\]

The last term on the right-hand side can be estimated from below by \( -\mu(B_r(x)) \). We now divide by \( \mu(B_r(x)) \) and let \( r \to 0 \) to obtain

\[
\frac{\mu(B_r(x) \cap (C_1 \cap C_2))}{\mu(B_r(x))} \geq \frac{\mu(B_r(x) \cap C_1)}{\mu(B_r(x))} + \frac{\mu(B_r(x) \cap C_2)}{\mu(B_r(x))} - 1 \to 1
\]

if \( C_1, C_2 \) have density 1 as \( x \). An induction argument then shows that the intersection of any finite number of sets with density 1 at \( x \) still has density 1 at \( x \). \( \square \)

**Definition A.3.** Suppose that \( f \) is a function mapping some subset \( E \subset X \) into a Hausdorff space \( Y \). We call \( y \in Y \) the approximate limit of \( f \) at \( x \) if, for every neighborhood \( W \) of \( y \) in \( Y \), the set \( X \setminus f^{-1}(W) \) has density 0 at \( x \). The approximate limit, if it exists, is unique and denoted by \( \text{ap lim}_{z \to x} f(z) := y \). We say that \( f \) is approximately continuous at \( x \) if \( x \in E \) and \( \text{ap lim}_{z \to x} f(z) = f(x) \).

**Definition A.4.** Suppose that \( f \) is a function mapping some subset \( E \subset X \) into \( \mathbb{R} := \mathbb{R} \cup \{\pm \infty\} \). We define the approximate upper limit of \( f \) at \( x \) as

\[
\text{ap lim sup}_{z \to x} f(z) := \inf_U U_E
\]

where \( U_E \) is the set of all numbers \( t \) such that

\[
E_t := \{z \in E : f(z) > t\}
\]

has density 0 at \( x \).

We refer the reader to [32] for further information. In particular, it is proved there (see Theorem 2.9.13) that if \( Y \) is a separable metric space, then \( f \) is \( \mu \)-measurable if and only if \( f \) is approximately continuous at \( \mu \) almost every point of \( X \).

**Proposition A.5.** Let \((X, d)\) and \( \mu \) be as in Definition A.1. Suppose that \( E \subset X \) and \( f : E \to \mathbb{R} \) are \( \mu \)-measurable and that \( E \) has density 1 at \( x \in X \). Then

\[
\text{ap lim sup}_{z \to x} f(z) = \inf_{C \in C} \left( \limsup_{z \to x \text{ in } C} f(z) \right) = \lim_{\delta \to 0} \left( \inf_{C \in C} \sup_{z \in B_\delta(x) \cap C} f(z) \right)
\]

(A.2)
where \( C \) denotes the collection of all measurable subsets of \( X \) that have density 1 at \( x \). A subset \( C \subset X \) has density 1 at \( x \) if (A.1) holds with \( \alpha = 1 \).

Proof. The first equality in (A.2) was established in [42] for a special case. We give a proof for the second identity. The argument is similar to the one in [42].

By definition of \( \limsup \), we must prove that

\[
\inf_{C \in C} \left( \lim_{\delta \to 0} \sup_{z \in B_\delta(x) \cap C} f(z) \right) = \lim_{\delta \to 0} \left( \inf_{C \in C} \sup_{z \in B_\delta(x) \cap C} f(z) \right) =: \lambda. \tag{A.3}
\]

We have (A.3) with inequality \( \geq \) because for each \( \delta > 0 \) we can optimize \( C \in C \). In particular, if \( \lambda = +\infty \), then also the left-hand side of (A.3) equals \( +\infty \).

To prove the converse inequality, we observe first that the map

\[
\delta \mapsto \inf_{C \in C} \sup_{z \in B_\delta(x) \cap C} f(z)
\]

is nonincreasing as \( \delta > 0 \) gets smaller. \( \tag{A.4} \)

Indeed let \( \varepsilon > 0 \) and \( \delta_1 < \delta_2 \) be given. Pick \( C_2 \in C \) such that

\[
\sup_{z \in B_{\delta_2}(x) \cap C_2} f(z) \leq \inf_{C \in C} \sup_{z \in B_{\delta_2}(x) \cap C_2} f(z) + \varepsilon. \tag{A.5}
\]

Since \( B_{\delta_1}(x) \subset B_{\delta_2}(x) \) we can estimate

\[
\inf_{C \in C} \sup_{z \in B_{\delta_1}(x) \cap C} f(z) \leq \sup_{z \in B_{\delta_1}(x) \cap C_2} f(z) \leq \sup_{z \in B_{\delta_2}(x) \cap C_2} f(z),
\]

which can be combined with (A.5) to prove (A.4).

We will now construct a set \( C \in C \) with the property that

\[
\lim_{\delta \to 0} \sup_{z \in B_\delta(x) \cap C} f(z) \leq \lim_{\delta \to 0} \left( \inf_{C \in C} \sup_{z \in B_\delta(x) \cap C} f(z) \right) = \lambda, \tag{A.6}
\]

with \( \lambda < +\infty \). For any \( n \in \mathbb{N} \) pick \( C_n \in C \) such that

\[
\sup_{z \in B_{1/n}(x) \cap C_n} f(z) \leq \inf_{C \in C} \sup_{z \in B_{1/n}(x) \cap C} f(z) + 1/n. \tag{A.7}
\]

Then define the sets \( E_n := \bigcap_{k=1}^n C_k \), which are measurable and have density 1 at \( x \) because of Lemma A.2. There exist \( \varepsilon_n > 0 \) with

\[
\mu(B_\delta(x) \cap E_n) > (1 - 1/n)\mu(B_\delta(x)) \quad \text{for all } 0 < \delta < \varepsilon_n.
\]

Choose a decreasing sequence of numbers \( \delta_n > 0 \) such that

\[
\delta_n < \min\{\varepsilon_n, 1/n\} \quad \text{and} \quad \mu(B_{\delta_{n+1}}(x)) \leq \mu(B_{\delta_n}(x))/n \tag{A.8}
\]

for every \( n \in \mathbb{N} \). This is possible since \( \mu(B_\delta(x)) \to 0 \) as \( \delta \to 0 \) because \( \mu \) does not have atoms and \( \lim_{\delta \to 0} \mu(B_\delta(x)) = \mu(\bigcap_{\delta > 0} B_\delta(x)) \). We now define the set

\[
C_* := \bigcup_{n=1}^\infty (B_{\delta_n}(x) \setminus B_{\delta_{n+1}}(x)) \cap E_n.
\]
We prove first that $C_*$ has density 1 at $x$. Indeed for any $\delta \in (0, \delta_1]$ there exists a unique index $n$ such that $\delta_{n+1} < \delta \leq \delta_n$. Then we can estimate
\[
\mu(B_\delta(x) \cap C_*) \geq \mu\left((B_\delta(x) \setminus B_{\delta_n+1}(x)) \cap E_n\right) + \mu\left((B_{\delta_n+1}(x) \setminus B_{\delta_{n+2}}(x)) \cap E_{n+1}\right)
\]
\[
\geq \left(1 - \frac{1}{n}\right)\mu(B_\delta(x)) - \mu(B_{\delta_{n+1}}(x)) + \left(1 - \frac{1}{n+1}\right)\mu(B_{\delta_{n+1}}) - \mu(B_{\delta_{n+2}}(x))
\]
\[
\geq \left(1 - \frac{1}{n}\right)\mu(B_\delta(x)) - \frac{2}{n+1}\mu(B_{\delta_{n+1}}(x)),
\]
where we have used (A.8). Since $\delta_{n+1} < \delta$ and $2/(n+1) < 2/n$ we obtain
\[
\mu(B_\delta(x) \cap C_*) \geq \left(1 - \frac{3}{n}\right)\mu(B_\delta(x))
\]
for all $\delta$ as above. Letting $n \to \infty$, we find that $C_*$ has density 1 at $x$.

Because of (A.7), we have that
\[
f(z) \leq \inf_{C \in \mathcal{C}} \sup_{z \in B_{1/n}(x) \cap C} f(z) + 1/n \quad \text{for all } z \in B_{\delta_n}(x) \cap C_*,
\]
which is contained in $B_{1/n}(x) \cap C_n$. Sending $n \to \infty$, we obtain inequality (A.6). In particular, if $\lambda = -\infty$, then also the left-hand side of (A.3) equals $-\infty$. □

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