Automorphic pairs of distributions on $\mathbb{R}$, and Maass forms of real weights

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Abstract

We give a correspondence between automorphic pairs of distributions on $\mathbb{R}$ and Dirichlet series satisfying functional equations and some additional analytic conditions. Moreover, we show that the notion of automorphic pairs of distributions on $\mathbb{R}$ can be regarded as a generalization of automorphic distributions on smooth principal series representations of the universal covering group of $SL(2, \mathbb{R})$. As an application, we prove Weil type converse theorems for automorphic distributions and Maass forms of real weights.

1 Introduction

Our goal is to give a detailed exposition of the theory of automorphic pairs of distributions on $\mathbb{R}$. Automorphic pairs of distributions on $Sym(n, \mathbb{R})$ were first introduced by Suzuki [Su] (in the name of “distribution with automorphy”), and extended by Tamura [Ta] to more general prehomogeneous vector spaces. They proved that the Dirichlet series associated with automorphic pairs of distributions have the meromorphic continuations to the whole complex plane, and satisfy the functional equations. Their method is based on the idea in the theory of prehomogeneous vector spaces (namely, the combination of the summation formula and the local functional equation), and it is powerful for the study of the analytic properties of Dirichlet series. For an exploration of the possibility of automorphic pairs of distributions, we study the simplest case precisely in this paper. As an application, we give Weil type converse theorems for automorphic distributions and Maass forms of real weights, which are generalizations of the results of the previous paper [MSSU].

Let us recall Suzuki’s result in [Su] for the simplest case $Sym(1, \mathbb{R}) = \mathbb{R}$ with slight modification. Let $L_i = u_{i1} + u_{i2} \mathbb{Z}$ with $u_{i1} \in \mathbb{R}$ and $u_{i2} \in \mathbb{R}^*$ for $i = 1, 2$. We call such $L_i$ a shifted lattice in $\mathbb{R}$ in this paper. Let $\mu$ and $\nu$ be

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complex numbers. An automorphic pair of distributions on \( \mathbb{R} \) for \((L_1, L_2)\) with the automorphic factor \( J_{\mu, \nu}(x) = e^{-s \sqrt{T} |x|/|l|} |x|^{-2\nu - 1} \) \((x \in \mathbb{R}^\times)\) is a pair \((T_{\alpha_1}, T_{\alpha_2})\) of distributions on \( \mathbb{R} \) defined by Fourier expansions

\[
T_{\alpha_i}(f) = \sum_{l \in L_i} \alpha_i(l) \mathcal{F}(f)(l), \quad (f \in C_0^\infty(\mathbb{R}), \ i = 1, 2) \tag{1.1}
\]
such that the coefficient \( \alpha_i \) is a polynomial growth function on \( L_i \) and the following equality holds:

\[
T_{\alpha_1}(f) = T_{\alpha_2}(f_{\mu, \nu, \infty}) \quad (f \in C_0^\infty(\mathbb{R}) \text{ with support in } \mathbb{R}^\times). \tag{1.2}
\]

Here \( C_0^\infty(\mathbb{R}) \) is the space of smooth functions on \( \mathbb{R} \) with compact support, \( \mathcal{F} \) denotes the Fourier transformation, and \( f_{\mu, \nu, \infty} \) is a function in \( C_0^\infty(\mathbb{R}) \) characterized by

\[
f_{\mu, \nu, \infty}(x) = J_{\mu, \nu}(x)f(-1/x) \quad (x \in \mathbb{R}^\times).
\]

Let \( \mathcal{A}(L_1, L_2; J_{\mu, \nu}) \) be the space of automorphic pairs of distributions on \( \mathbb{R} \) for \((L_1, L_2)\) with the automorphic factor \( J_{\mu, \nu} \). Suzuki proved that, if \((T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu, \nu})\), then Dirichlet series

\[
\xi_+^{(\alpha_1)}(s) = \sum_{0 < l \in L_1} \frac{\alpha_1(l)}{l^s}, \quad \xi_-^{(\alpha_1)}(s) = \sum_{0 < l \in L_1} \frac{\alpha_1(l)}{|l|^s} \quad (\text{Re}(s) \gg 0, \ i = 1, 2)
\]

have the meromorphic continuations to the whole \( s \)-plane and satisfy the functional equation

\[
E(s) \left( \begin{array}{c} \Xi_+^{(\alpha_1)}(s) \\ \Xi_-^{(\alpha_1)}(s) \end{array} \right) = \Sigma_{\mu} E(-s - 2\nu + 1) \left( \begin{array}{c} \Xi_+^{(\alpha_2)}; -s - 2\nu + 1 \\ \Xi_-^{(\alpha_2)}; -s - 2\nu + 1 \end{array} \right), \tag{1.3}
\]

where \( \Xi_{\pm}^{(\alpha_1)}(s) = (2\pi)^{-s} \Gamma(s) \xi_{\pm}^{(\alpha_1)}(s) \) and

\[
E(s) = \left( \begin{array}{cc} e^{\pm \sqrt{-1}Ts/2} & e^{-\pm \sqrt{-1}Ts/2} \\ e^{-\pm \sqrt{-1}Ts/2} & e^{\pm \sqrt{-1}Ts/2} \end{array} \right), \quad \Sigma_{\mu} = \left( \begin{array}{cc} 0 & e^{\pm \sqrt{-1}T\mu/2} \\ e^{-\pm \sqrt{-1}T\mu/2} & 0 \end{array} \right).
\]

In this paper, we have three purposes related to automorphic pairs of distributions on \( \mathbb{R} \). The first purpose is to make Suzuki’s result more precise. In Theorem 2.2, we give a correspondence between automorphic pairs of distributions on \( \mathbb{R} \) and Dirichlet series with nice properties. More precisely, for polynomial growth functions \( \alpha_1 \) and \( \alpha_2 \) respectively on \( L_1 \) and \( L_2 \), a pair \((T_{\alpha_1}, T_{\alpha_2})\) defined by (1.1) is in \( \mathcal{A}(L_1, L_2; J_{\mu, \nu}) \) if and only if the Dirichlet series \( \xi_{\pm}^{(\alpha_1)}(s) \) and \( \xi_{\pm}^{(\alpha_2)}(s) \) satisfy the following conditions (1) and (2):

1. The functions \( \Xi_{\mp}^{(\alpha_1)}(s) \) have the meromorphic continuations to the whole \( s \)-plane, and the functional equation (1.3) holds. Moreover,

\[
E(s) \left( \begin{array}{c} \Xi_+^{(\alpha_1)}(s) \\ \Xi_-^{(\alpha_1)}(s) \end{array} \right) + \left( \frac{\alpha_1(0)}{s} - 1 \right) \left( \begin{array}{c} \alpha_1(0) \\ \alpha_2(0) \end{array} \right) (s + 2\nu - 1) \Sigma_{\mu} = \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

is entire. Here, for \( i = 1, 2 \), we understand \( \alpha_i(0) = 0 \) if \( 0 \notin L_i \).
Our proof is based on Suzuki’s idea in [Su], that is, the combination of the summation formula (1.2) and the local functional equation. By a careful argument using certain test functions, we derive from (1.2) not only the functional equation (1.3) but also the entireness of (1.4) and the estimate in (2). Conversely, we derive (1.2) assuming (1) and (2) by using the Mellin inversion formula, similar to the proof of the converse theorem for holomorphic modular forms.

The second purpose of this paper is to clarify the relation between automorphic pairs of distributions on $\mathbb{R}$ and automorphic distributions on smooth principal series representations of the universal covering group $\widetilde{G}$ of $G = SL(2, \mathbb{R})$. Automorphic distributions are distributional realizations of automorphic forms, and those theory is developed by Miller and Schmid ([MS1], [MS2], [MS3]). Let us explain briefly our results for the second purpose, which are given in §2.5, §2.6, §2.7 and §2.8.

Let $(\rho, I_{\mu,\nu}^\infty)$ be a smooth principal series representation of $\widetilde{G}$, and we call a continuous functional on $I_{\mu,\nu}^\infty$ a distribution on $I_{\mu,\nu}^\infty$ in this paper. For the shifted lattice $L_i = u_{1i} + u_{2i} \mathbb{Z}$ ($i = 1, 2$), we define the dual lattice $L_i^\vee$ of $L_i$ by $L_i^\vee = u_{2i}^{-1} \mathbb{Z}$, and define a character $\omega_{L_i} : L_i^\vee \to \mathbb{C}^\times$ by $\omega_{L_i}(t) = e^{2\pi \sqrt{-1} t_{i,1}}$ ($t \in L_i^\vee$). A distribution $\lambda$ on $I_{\mu,\nu}^\infty$ is said to be automorphic for $(L_1, L_2)$ if $\lambda$ satisfies

$$\lambda(\rho(\tilde{u}(t_1))F) = \omega_{L_1}(t_1)\lambda(F) \quad \lambda(\rho(\tilde{w}u(t_2)w^{-1})F) = \omega_{L_2}(t_2)\lambda(F)$$

for $t_1 \in L_1^\vee$, $t_2 \in L_2^\vee$ and $F \in I_{\mu,\nu}^\infty$. Here $\tilde{u}(x)$ and $\tilde{w}$ are the lifts of $u(x) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ and $w = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ to $\widetilde{G}$, respectively (cf. §2.4). For a distribution $\lambda$ on $I_{\mu,\nu}^\infty$, we define maps $T^\lambda$ and $T^{\lambda\infty}$ from $C_0^\infty(\mathbb{R})$ to $\mathbb{C}$ by

$$T^\lambda(f) = \lambda(\iota_{\mu,\nu}(f)), \quad T^{\lambda\infty}(f) = e^{\pi \sqrt{-1} t_{i,1}}\lambda(\rho(\tilde{w})\iota_{\mu,\nu}(f)) \quad (f \in C_0^\infty(\mathbb{R})).$$

Here $\iota_{\mu,\nu}(f)$ is the function in $I_{\mu,\nu}^\infty$ characterized by $\iota_{\mu,\nu}(f)(\tilde{w}u(-x)) = f(x)$ ($x \in \mathbb{R}$). Then $\lambda \mapsto (T^\lambda, T^{\lambda\infty})$ defines an injective map from the space of automorphic distributions on $I_{\mu,\nu}^\infty$ for $(L_1, L_2)$ to $\mathcal{A}(L_1, L_2; J_{\mu,\nu})$. Moreover, for $(T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu,\nu})$, there is an automorphic distribution $\lambda$ on $I_{\mu,\nu}^\infty$ for $(L_1, L_2)$ such that $(T^\lambda, T^{\lambda\infty}) = (T_{\alpha_1}, T_{\alpha_2})$ if and only if the associated Dirichlet series $\xi_{\pm}(\alpha_1; s)$ and $\xi_{\pm}(\alpha_2; s)$ satisfy the following condition (3):

(3) For $i = 1, 2$, the functions $(s - 1)(s + 2\nu - 1)\xi_{\pm}(\alpha_1; s)$ are entire if $0 \in L_{3-i}$, and the functions $\xi_{\pm}(\alpha_i; s)$ are entire if $0 \not\in L_{3-i}$.

As an remarkable fact, “Mittag–Leffler” theorem of Knopp [Kn, Theorem 2] gives Dirichlet series which satisfy the conditions (1), (2) and do not satisfy the condition (3). Hence, we know that there are automorphic pairs of distributions on $\mathbb{R}$ not coming from automorphic distributions on $I_{\mu,\nu}^\infty$. This fact is indicated by Professor Fumihiro Sato.
The third purpose of this paper is to give Weil type converse theorems for automorphic distributions and Maass forms of real weights. As an analogue of the converse theorem of Weil [We] for holomorphic modular forms of level $N$, some mathematicians studied converse theorems for Maass forms of level $N$. Diamantis–Goldfeld [DG] gave a converse theorem characterizing linear combinations of metaplectic Eisenstein series, by considering the twists of Dirichlet series by Dirichlet characters including imprimitive characters. Neururer–Oliver [NO] gave a converse theorem characterizing Maass forms of weight 0, and succeeded to avoid the twists by imprimitive characters. In the previous paper [MSSU], following the approach of Diamantis-Goldfeld, we gave converse theorems characterizing automorphic distributions and Maass forms of integral and half-integral weights. In this paper, we generalize the result of [MSSU] to real weights.

Let us explain our converse theorems, more precisely. Until the end of the introduction, we assume $\mu \in \mathbb{R}$. Let $N$ be a positive integer, and let $\tilde{\Gamma}_0(N) = \varpi^{-1}((\Gamma_0(N)))$ with the covering map $\varpi: \tilde{G} \to G$. Let $v$ be a multiplier system on $\Gamma_0(N)$ of weight $\mu$, and we denote by $\tilde{\chi}_v$ the character of $\tilde{\Gamma}_0(N)$ corresponding to $v$ (cf. §2.11). A distribution $\lambda$ on $I_{\mu,\nu}^\infty$ is said to be automorphic for $\tilde{\Gamma}_0(N)$ with character $\tilde{\chi}_v$ if $\lambda$ satisfies

$$\lambda(\rho(\tilde{\gamma})F) = \tilde{\chi}_v(\tilde{\gamma})\lambda(F) \quad (\tilde{\gamma} \in \tilde{\Gamma}_0(N), \ F \in I_{\mu,\nu}^\infty).$$

We define shifted lattices $L$ and $\hat{L}$ by $L = u + \mathbb{Z}$ and $\hat{L} = N^{-1}(\hat{u} + \mathbb{Z})$ with $0 \leq u, \hat{u} < 1$ determined by $v(u(1)) = e^{2\pi \sqrt{-1}u}$ and $v(u(N)w^{-1}) = e^{2\pi \sqrt{-1}\hat{u}}$. Then an automorphic distribution on $I_{\mu,\nu}^\infty$ for $\tilde{\Gamma}_0(N)$ with $\tilde{\chi}_v$ is automorphic for $(L, \hat{L})$. Our converse theorem (Theorem 2.18 (ii)) gives a condition for automorphic distributions on $I_{\mu,\nu}^\infty$ for $(L, \hat{L})$ to be automorphic for $\tilde{\Gamma}_0(N)$ with $\tilde{\chi}_v$ in terms of twists of the associated Dirichlet series by Dirichlet characters. Our strategy of the proof is slightly different from that in the previous paper [MSSU]. In [MSSU], we investigate the relation between automorphic distributions and Dirichlet series directly, generalizing the local functional equation to the line model of smooth principal series representations. In this paper, we investigate the relation through the intermediary of automorphic pairs of distributions on $\mathbb{R}$, and do not use the generalized local functional equation. Since the Poisson transform of an automorphic distribution is a Maass form, we also obtain a converse theorem for Maass forms of level $N$.

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Notation

We denote by \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{C} \) the ring of rational integers, the rational number field, the real number field, and the complex number field, respectively. For a subset \( S \) of \( \mathbb{R} \) and \( t \in \mathbb{R} \), we denote by \( S \geq t \), \( S > t \), \( S \leq t \), and \( S < t \) the subsets of \( S \) consisting of all numbers \( r \) satisfying \( r \geq t \), \( r > t \), \( r \leq t \), and \( r < t \), respectively. For \( x \in \mathbb{R}^\times \), we set \( \text{sgn}(x) = x/|x| \).

The real part, the imaginary part, the complex conjugate, and the absolute value of a complex number \( z \) are denoted by \( \text{Re}(z) \), \( \text{Im}(z) \), \( \bar{z} \), and \( |z| \), respectively. For \( z \in \mathbb{C}^\times \), we denote by \( \text{arg} z \) the principal argument of \( z \), namely, the argument of \( z \) satisfying \( -\pi < \text{arg} z \leq \pi \). Moreover, we set \( z^s = |z|^s e^{\sqrt{-1}s \text{arg} z} \) for \( z \in \mathbb{C}^\times \) and \( s \in \mathbb{C} \).

For an open subset \( S \) of \( \mathbb{R} \), we denote by \( C(S) \) and \( C^\infty(S) \) the spaces of continuous functions and smooth functions on \( S \), respectively. We denote by \( C_0(S) \) the subspace of \( C(S) \) consisting of all functions \( f \) with compact support, and let \( C_0^\infty(S) = C^\infty(S) \cap C_0(S) \). Moreover, \( C^n(S) \) denotes the subspace of \( C(S) \) consisting of all \( n \) times continuously-differentiable functions on \( S \) for \( n \in \mathbb{Z}_{\geq 0} \). We denote by \( S(\mathbb{R}) \) the space of rapidly decreasing functions on \( \mathbb{R} \).

We denote by \( L^1(\mathbb{R}) \) the space of integrable functions on \( \mathbb{R} \) with respect to the usual Lebesgue measure. For \( f \in C(S) \), we denote by \( \text{supp}(f) \) the support of \( f \). For \( f \in C^n(S) \), we denote by \( f^{(n)} \) the \( n \)-th derivative of \( f \). Here the 0-th derivative \( f^{(0)} \) is just the original function \( f \). We denote simply by \( f' \) and \( f'' \) the first derivative \( f^{(1)} \) and the second derivative \( f^{(2)} \) of \( f \), respectively. In this paper, we regard \( f \in C(\mathbb{R}^\times) \) as an element of \( C(\mathbb{R}) \) by setting \( f(0) = \lim_{x \to 0} f(x) \) if the limit \( \lim_{x \to 0} f(x) \) exists.

For a set \( S \), we denote by \( S^2 \) and \( M_2(S) \) the sets of 2 \( \times \) 1- and 2 \( \times \) 2-matrices whose entries in \( S \), respectively. The unit matrix of size 2 is denoted by \( 1_2 \).

2 Main results

For the sake of readability, we summarize the main theorems and important propositions in this section. Most of the proofs are given in later sections.

2.1 Periodic distributions on \( \mathbb{R} \)

A \( \mathbb{C} \)-linear map \( T: C_0^\infty(\mathbb{R}) \to \mathbb{C} \) is called a distribution on \( \mathbb{R} \) if and only if, for any \( u > 0 \), there are \( m \in \mathbb{Z}_{\geq 0} \) and \( c > 0 \) such that

\[
|T(f)| \leq c \sum_{i=0}^{m} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|
\]

\((f \in C_0^\infty(\mathbb{R}) \text{ with } \text{supp}(f) \subset \{x \in \mathbb{R} \mid |x| \leq u\}).

We denote by \( D'(\mathbb{R}) \) the space of distributions on \( \mathbb{R} \).
For $f \in L^1(\mathbb{R})$, we define the Fourier transform $\mathcal{F}(f)$ of $f$ by

$$\mathcal{F}(f)(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi i xy} \, dx \quad (y \in \mathbb{R}).$$

Let $L$ be a subset of $\mathbb{R}$ of the form $L = u_1 + u_2 \mathbb{Z}$ with $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}^\times$. We call such $L$ a shifted lattice in $\mathbb{R}$. We define the dual lattice $L^\vee$ of $L$ by

$$L^\vee = \{ t \in \mathbb{R} \mid (l_1 - l_2) t \in \mathbb{Z} \quad (l_1, l_2 \in L) \} = u_2^{-1} \mathbb{Z},$$

and define a character $\omega_L: L^\vee \to \mathbb{C}^\times$ by $\omega_L(t) = e^{2\pi i l t} \quad (t \in L^\vee)$ with $l \in L$. Here the definition of $\omega_L$ does not depend on the choice of $l$. We note that $L$ is a lattice in $\mathbb{R}$ if and only if $0 \in L$, and the condition $0 \in L$ implies that $\omega_L$ is the trivial character. We define a subspace $\mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$ of $\mathcal{D}'(\mathbb{R})$ by

$$\mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L) = \{ T \in \mathcal{D}'(\mathbb{R}) \mid T(s_t(f)) = \omega_L(-t)T(f) \quad (t \in L^\vee, \ f \in C^\infty_0(\mathbb{R})) \},$$

where $s_t(f)(x) = f(x + t) \ (x \in \mathbb{R})$. Here we note that $\mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$ is the space of periodic distributions on $\mathbb{R}$ if $0 \in L$.

For $r \in \mathbb{R}$, let $\mathfrak{M}_r(L)$ be the space of functions $\alpha: L \to \mathbb{C}$ satisfying $\alpha(l) = O(|l|^r) \quad (|l| \to \infty)$. Let $\mathfrak{M}(L) = \bigcup_{r \in \mathbb{R}} \mathfrak{M}_r(L)$. For $\alpha \in \mathfrak{M}(L)$, it is convenient to set $\alpha(0) = 0$ if $0 \notin L$. For $\alpha \in \mathfrak{M}(L)$, we define $T_\alpha \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$ by

$$T_\alpha(f) = \sum_{l \in L} \alpha(l)\mathcal{F}(f)(l) \quad (f \in C^\infty_0(\mathbb{R})).$$

**Proposition 2.1.** Let $L$ be a shifted lattice in $\mathbb{R}$. Then the $\mathbb{C}$-linear map $\mathfrak{M}(L) \ni \alpha \mapsto T_\alpha \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$ is bijective.

**Proof.** Let $L = u_1 + u_2 \mathbb{Z}$ with $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}^\times$. We define a bijectiv map

$$\mathfrak{M}(L) \ni \alpha \mapsto \alpha_{(u_1,u_2)} \in \mathfrak{M}(\mathbb{Z}) \quad (2.1)$$

by $\alpha_{(u_1,u_2)}(n) = \alpha(u_1 + u_2 n)/u_2 \quad (n \in \mathbb{Z})$. Since $\mathcal{D}'(\mathbb{Z}; \omega_Z)$ is the space of periodic distributions, it is known that

$$\mathfrak{M}(\mathbb{Z}) \ni \alpha \mapsto T_\alpha \in \mathcal{D}'(\mathbb{Z}; \omega_Z) \quad (2.2)$$

is bijective (see, for example, Friedlander and Joshi [Fr, §8.5]). We define a bijective map

$$\mathcal{D}'(\mathbb{Z}; \omega_Z) \ni T \mapsto T \circ i_{(u_1,u_2)} \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L) \quad (2.3)$$

by $i_{(u_1,u_2)}(f)(x) = e^{2\pi i u_1 x/u_2} f(x/u_2) \quad (x \in \mathbb{R}, \ f \in C^\infty_0(\mathbb{R}))$.

Since $\mathfrak{M}(L) \ni \alpha \mapsto T_\alpha \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$ is the composite of the bijective maps $(2.1)$, $(2.2)$ and $(2.3)$, we obtain the assertion. \hfill \Box

For $T \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$, there is a unique $\alpha \in \mathfrak{M}(L)$ such that

$$T(f) = \sum_{l \in L} \alpha(l)\mathcal{F}(f)(l) \quad (f \in C^\infty_0(\mathbb{R})) \quad (2.4)$$

by Proposition 2.1. We call the expression $(2.4)$ the Fourier expansion of $T$. 6
2.2 Automorphic pairs of distributions on $\mathbb{R}$

Let $\mu, \nu \in \mathbb{C}$. We define a function $J_{\mu, \nu}$ on $\mathbb{R}^\times$ by

$$J_{\mu, \nu}(x) = e^{-\text{sgn}(x)\pi \sqrt{-\mu/2|x|} - 2\nu - 1} \quad (x \in \mathbb{R}^\times)$$

with $\text{sgn}(x) = x/|x|$. We define a map $C(\mathbb{R}^\times) \ni f \mapsto f_{\mu, \nu, \infty} \in C(\mathbb{R}^\times)$ by

$$f_{\mu, \nu, \infty}(x) = J_{\mu, \nu}(x)f(-1/x) \quad (x \in \mathbb{R}^\times).$$

Here we note that the following equalities hold:

$$(f_{\mu, \nu, \infty})_{\mu, \nu, \infty} = f, \quad f_{\mu+2, \nu, \infty} = -f_{\mu, \nu, \infty} \quad (f \in C(\mathbb{R}^\times)). \quad (2.5)$$

In this paper, if $\lim_{x \to 0} f(x)$ exists, we regard $f \in C(\mathbb{R}^\times)$ as an element of $C(\mathbb{R})$ by setting $f(0) = \lim_{x \to 0} f(x)$. In particular, we regard $f \in C_0^\infty(\mathbb{R}^\times)$ as an element of $C_0^\infty(\mathbb{R})$ by setting $f(0) = 0$. Let $A(J_{\mu, \nu})$ be the space of pairs $(T_1, T_2)$ of distributions on $\mathbb{R}$ such that

$$T_1(f) = T_2(f_{\mu, \nu, \infty}) \quad (f \in C_0^\infty(\mathbb{R}^\times)). \quad (2.6)$$

For two shifted lattices $L_1$ and $L_2$ in $\mathbb{R}$, we define a space $A(L_1, L_2; J_{\mu, \nu})$ by

$$A(L_1, L_2; J_{\mu, \nu}) = A(J_{\mu, \nu}) \cap \left( \mathcal{D}'(L_1^\times \setminus \mathbb{R}; \omega_{L_1}) \times \mathcal{D}'(L_2^\times \setminus \mathbb{R}; \omega_{L_2}) \right)$$

$$= \{(T_{\alpha_1}, T_{\alpha_2}) \in A(J_{\mu, \nu}) \mid \alpha_1 \in \mathfrak{M}(L_1), \; \alpha_2 \in \mathfrak{M}(L_2)\}.$$

Here the second expression follows from Proposition 2.1. We call an element of $A(L_1, L_2; J_{\mu, \nu})$ an automorphic pair of distributions on $\mathbb{R}$ for $(L_1, L_2)$ with the automorphic factor $J_{\mu, \nu}$.

Because of (2.5), we note that $(T_2, T_1) \in A(J_{\mu, \nu})$ and $(T_1, -T_2) \in A(J_{\mu+2, \nu})$ hold for $(T_1, T_2) \in A(J_{\mu, \nu})$. Moreover, we have $(T_2, T_1) \in A(L_1, L_2; J_{\mu, \nu})$ and $(T_1, -T_2) \in A(L_1, L_2; J_{\mu+2, \nu})$ for $(T_1, T_2) \in A(L_1, L_2; J_{\mu, \nu})$.

2.3 Dirichlet series associated with automorphic pairs

Let $r \in \mathbb{R}$. Let $L$ be a shifted lattice in $\mathbb{R}$. For $\alpha \in \mathfrak{M}_r(L)$, we define Dirichlet series $\xi_+(\alpha; s)$ and $\xi_-(\alpha; s)$ by

$$\xi_+(\alpha; s) = \sum_{0 < l \in L} \frac{\alpha(l)}{|ls|^s}, \quad \xi_-(\alpha; s) = \sum_{0 < l \in L} \frac{\alpha(l)}{|l||s|^s}. \quad (2.7)$$

It is easy to see that these series converge absolutely and are holomorphic functions on $\text{Re}(s) > r + 1$. We set

$$\Xi_+(\alpha; s) = (2\pi)^{-s} \Gamma(s) \xi_+(\alpha; s), \quad \Xi_-(\alpha; s) = (2\pi)^{-s} \Gamma(s) \xi_-(\alpha; s). \quad (2.8)$$

For $s \in \mathbb{C}$ and $\mu \in \mathbb{C}$, we define matrices $E(s)$ and $\Sigma_\mu$ by

$$E(s) = \begin{pmatrix} e^{\pi\sqrt{-Ts}/2} & e^{-\pi\sqrt{-Ts}/2} \\ e^{-\pi\sqrt{-Ts}/2} & e^{\pi\sqrt{-Ts}/2} \end{pmatrix}, \quad (2.9)$$
\[ \Sigma_\mu = \begin{pmatrix} 0 & e^{-\pi\sqrt{-1}\mu/2} \\ e^{-\pi\sqrt{-1}\mu/2} & 0 \end{pmatrix}. \] (2.10)

Let \( L_1 \) and \( L_2 \) be two shifted lattices in \( \mathbb{R} \). Let \( \alpha_1 \in \mathfrak{M}(L_1) \), \( \alpha_2 \in \mathfrak{M}(L_2) \) and \( \mu, \nu \in \mathbb{C} \). We consider the following conditions [D2-1] and [D2-2] on \( \Xi \):

**[D2-1]** The functions \( \Xi \) have the meromorphic continuations to the whole \( s \)-plane, and the \( \mathbb{C}^2 \)-valued function

\[
E(s) \left( \begin{array}{c} \Xi_+(\alpha_1; s) \\ \Xi_-(-\alpha_1; s) \end{array} \right) + \left( \frac{\alpha_1(0)}{s} - 12 - \frac{\alpha_2(0)}{s + 2\nu - 1} \Sigma_\mu \right) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 0
\]

is entire. Here, for \( j \in \{1, 2\} \), we understand \( \alpha_j(0) = 0 \) if \( 0 \notin L_j \). Moreover, \( \Xi_\pm(\alpha_1; s) \) and \( \Xi_\pm(\alpha_2; s) \) satisfy the functional equation

\[
E(s) \left( \begin{array}{c} \Xi_+(\alpha_1; s) \\ \Xi_-(-\alpha_1; s) \end{array} \right) = \Sigma_\mu E(-s - 2\nu + 1) \left( \begin{array}{c} \Xi_+(\alpha_2; -s - 2\nu + 1) \\ \Xi_-(-\alpha_2; -s - 2\nu + 1) \end{array} \right).
\]

When \( \Xi_\pm(\alpha_1; s) \) have the meromorphic continuations to the whole \( s \)-plane, we consider the following conditions [D2-1] and [D2-2] on \( \Xi_\pm(\alpha_1; s) \):

**[D2-1]** For any \( \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \sigma_1 < \sigma_2 \), the functions \( \Xi_\pm(\alpha_1; s) \) satisfy

\[
\Xi_\pm(\alpha_1; \sigma_1) = O \left( e^{-\pi|\text{Im}(s)|/2 + \sqrt{|\text{Im}(s)|}} \right) \quad (|s| \to \infty)
\]

uniformly on \( \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \).

**[D2-2]** For any \( \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \sigma_1 < \sigma_2 \), there is \( c_0 \in \mathbb{R}_{>0} \) such that

\[
\Xi_\pm(\alpha_1; s) = O(e^{c_0|s|}) \quad (|s| \to \infty)
\]

uniformly on \( \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \).

Here we note that the condition [D2-1] is stronger than [D2-2].

**Theorem 2.2.** Let \( L_1 \) and \( L_2 \) be two shifted lattices in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \).

(i) Let \( (T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu, \nu}) \) with \( \alpha_1 \in \mathfrak{M}(L_1) \) and \( \alpha_2 \in \mathfrak{M}(L_2) \). Then \( \Xi_\pm(\alpha_1; s) \) and \( \Xi_\pm(\alpha_2; s) \) satisfy the conditions [D1] and [D2].

(ii) Let \( \alpha_1 \in \mathfrak{M}(L_1) \) and \( \alpha_2 \in \mathfrak{M}(L_2) \) such that \( \Xi_\pm(\alpha_1; s) \) and \( \Xi_\pm(\alpha_2; s) \) satisfy the conditions [D1] and [D2]. Then \( (T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu, \nu}) \).

A proof of Theorem 2.2 is given in §5. The statement (i) is proved in §5.2 and §5.3. The statement (ii) is proved in §5.4.

**2.4. The universal covering group of \( SL(2, \mathbb{R}) \)**

Let \( G = SL(2, \mathbb{R}) \), and we fix an Iwasawa decomposition \( G = UAK \), where \( U = \{u(x) \mid x \in \mathbb{R}\} \), \( A = \{a(y) \mid y \in \mathbb{R}_{>0}\} \) and \( K = \{k(\theta) \mid \theta \in \mathbb{R}\} \) with

\[
u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}, \quad k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]
Let $\mathfrak{H}$ be the upper half plane $\{ z = x + \sqrt{-1}y \mid x \in \mathbb{R}, \; y \in \mathbb{R}_{>0}\}$. We set 
\[ gz = \frac{az + b}{cz + d}, \quad J(g, z) = cz + d \quad (g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \; z \in \mathfrak{H}). \]

Since $K = \{ g \in G \mid g\sqrt{-1} = \sqrt{-1} \}$ and $u(x)a(y)\sqrt{-1} = z$ $(z = x + \sqrt{-1}y \in \mathfrak{H})$, we have $G/K \simeq \mathcal{H}$ via the map induced from $G \ni g \mapsto g\sqrt{-1} \in \mathfrak{H}$. It is easy to see that the following cocycle condition holds:
\[ J(g_1g_2, z) = J(g_1, g_2z)J(g_2, z) \quad (g_1, g_2 \in G, \; z \in \mathfrak{H}). \]

The universal covering group of $G$ is constructed in [Br, §2.2]. For convenience, we introduce a slightly modified construction here. Let 
\[ \tilde{G} = \{ (g, \theta) \in G \times \mathbb{R} \mid J(g, \sqrt{-1})e^{\sqrt{-1}\theta} = |J(g, \sqrt{-1})| \}, \]
and regard $\tilde{G}$ as a group by the operation
\[ (g_1, \theta_1)(g_2, \theta_2) = (g_1g_2, \theta_1 + \theta_2 - \text{arg} \frac{J(g_1, g_2\sqrt{-1})}{J(g_1, \sqrt{-1})}) \quad (2.12) \]
for $(g_1, \theta_1), (g_2, \theta_2) \in \tilde{G}$. Then $\tilde{G}$ is the universal covering group of $G$ with the covering map $\tilde{\pi} : \tilde{G} \ni (g, \theta) \mapsto g \in G$. We fix a section $G \ni g \mapsto \gamma g \in \tilde{G}$ by $\gamma g = (g, -\text{arg} J(g, \sqrt{-1}))$.

We define subgroups $\tilde{U}, \tilde{A}$ and $\tilde{K}$ of $\tilde{G}$ by
\[ \tilde{U} = \{ \tilde{u}(x) \mid x \in \mathbb{R} \}, \quad \tilde{A} = \{ \tilde{a}(y) \mid y \in \mathbb{R}_{>0} \}, \quad \tilde{K} = \{ \tilde{k}(\theta) \mid \theta \in \mathbb{R} \} \]
with $\tilde{u}(x) = (u(x), 0), \tilde{a}(y) = (a(y), 0)$ and $\tilde{k}(\theta) = (k(\theta), \theta)$. Then we have an Iwasawa decomposition $\tilde{G} = \tilde{U}\tilde{A}\tilde{K}$. More precisely, for $\tilde{g} = (g, \theta) \in \tilde{G}$, we have
\[ \tilde{g} = \tilde{u}(\text{Re}(g\sqrt{-1}))\tilde{a}(\text{Im}(g\sqrt{-1}))\tilde{k}(\theta). \quad (2.13) \]

The center of $\tilde{G}$ is given by $\tilde{M} = \{ \tilde{k}(m\pi) \mid m \in \mathbb{Z} \}$. Let $\tilde{P} = \tilde{U}\tilde{A}\tilde{M}$ and
\[ \hat{w} = \tilde{k}(-\pi/2) = (w, -\pi/2), \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]
Then we have a Bruhat decomposition $\tilde{G} = \tilde{P}\hat{w}\tilde{U} \sqcup \tilde{P}$. More precisely, for $\tilde{g} = (g, \theta) \in \tilde{G}$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have
\[ \tilde{g} = \begin{cases} \tilde{u}(a/c)\tilde{a}(1/c^2)\tilde{k}(\theta + \text{arg}(J(g, \sqrt{-1})/c))\hat{w}\tilde{a}(d/c) & \text{if } c \neq 0, \\ \tilde{u}(b/d)\tilde{a}(a/d)\tilde{k}(\theta) & \text{if } c = 0. \end{cases} \quad (2.14) \]
Here we note that $\theta + \text{arg}(J(g, \sqrt{-1})/c) \in \pi\mathbb{Z}$ if $c \neq 0$, and $\theta \in \pi\mathbb{Z}$ if $c = 0$. For $x \in \mathbb{R}$, we set
\[ \tilde{u}(x) = \hat{w}\tilde{u}(-x)\hat{w}^{-1} = (\tilde{u}(x), -\text{arg}(1 + \sqrt{-1}x)), \quad \tilde{u}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}. \]

We denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $G = SL(2, \mathbb{R})$ and by $\cdot, \cdot$ the bracket product on $\mathfrak{g}$. Let $\{H, E_+, E_-\}$ be the standard basis of $\mathfrak{g}$ defined by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. $$

We denote by $\mathfrak{g}_\mathbb{C}$ the complexification $\mathfrak{sl}(2, \mathbb{C})$ of $\mathfrak{g}$, and by $U(\mathfrak{g}_\mathbb{C})$ the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. The Lie algebra $\mathfrak{g}$ can be regarded as the Lie algebra of $\tilde{G}$. We denote by $\exp$ the exponential map from $\mathfrak{g}$ to $\tilde{G}$.

### 2.5 Distributions on principal series representations

We denote by $C(\tilde{G})$ and $C^\infty(\tilde{G})$ the spaces of continuous functions and smooth functions on $\tilde{G}$, respectively. We define the action $\rho$ of $\tilde{G}$ on $C(\tilde{G})$ by

$$(\rho(\tilde{g})F)(\tilde{h}) = F(\tilde{h}\tilde{g}) \quad (\tilde{g}, \tilde{h} \in \tilde{G}, \ F \in C(\tilde{G})).$$

We define a seminorm $|\cdot|_K$ on $C(\tilde{G})$ by

$$|F|_K = \sup_{\tilde{g} \in \tilde{K}} |F(\tilde{g})| = \sup_{-\pi \leq \tilde{\theta} < \pi} |F(\tilde{g}(\tilde{\theta}))| \quad (F \in C(\tilde{G})).$$

We define the action $\rho$ of $\mathfrak{g}$ on $C^\infty(\tilde{G})$ by

$$(\rho(X)F)(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} F(\tilde{g} \exp(tX)) \quad (X \in \mathfrak{g}, \ \tilde{g} \in \tilde{G}, \ F \in C^\infty(\tilde{G})), $$

and extend this action to $U(\mathfrak{g}_\mathbb{C})$ in the usual way. For $X \in U(\mathfrak{g}_\mathbb{C})$, we define a seminorm $Q_X$ on $C^\infty(\tilde{G})$ by

$$Q_X(F) = |\rho(X)F|_K \quad (F \in C^\infty(\tilde{G})).$$

Let $\mu, \nu \in \mathbb{C}$. Let $I_{\mu, \nu}$ be the subspace of $C(\tilde{G})$ consisting of all functions $F$ such that

$$F(\tilde{u}(x)\tilde{a}(y)\tilde{k}(m\pi)y) = e^{\pi \nu \sqrt{-1} \mu} e^{\nu+1/2} F(\tilde{g}) \quad (x \in \mathbb{R}, \ y \in \mathbb{R}_{>0}, \ m \in \mathbb{Z}, \ \tilde{g} \in \tilde{G}).$$

By the decomposition (2.13), for $\tilde{g} = (g, \theta) \in \tilde{G}$ and $F \in I_{\mu, \nu}$, we have

$$F(\tilde{g}) = e^{\sqrt{-1} \mu + \arg J(g, \sqrt{-1}) + \arg J(g, \sqrt{-1})} \Im(\sqrt{-1} \nu + 1/2) F(\tilde{k}(\arg J(g, \sqrt{-1}))).$$

Since $\tilde{k}(\arg J(g, \sqrt{-1})) \in \tilde{K}$, we know that $F = 0$ if and only if $|F|_K = 0$ for $F \in I_{\mu, \nu}$. Hence $|\cdot|_K$ is a norm on $I_{\mu, \nu}$. We give $I_{\mu, \nu}$ the topology induced by the norm $|\cdot|_K$. Then it is easy to see that $I_{\mu, \nu}$ is complete.

We set $I_{\mu, \nu}^\infty = I_{\mu, \nu} \cap C^\infty(\tilde{G})$, and give $I_{\mu, \nu}^\infty$ the topology induced by the seminorms $Q_X$ ($X \in U(\mathfrak{g}_\mathbb{C})$). Then we know that $I_{\mu, \nu}^\infty$ is a Fréchet space by the standard argument (quite similar to the proof of [Wa, Lemma 1.6.4 (1)]). We call $(\rho, I_{\mu, \nu}^\infty)$ a smooth principal series representation of $\tilde{G}$.

We define an involution $I_{\mu, \nu}^\infty \ni F \mapsto F_\infty \in I_{\mu, \nu}^\infty$ by $F_\infty = e^{\pi \sqrt{-1} \mu/2} \rho(\tilde{w}) F$. Here we note that, for $F \in I_{\mu, \nu}^\infty$, the substitution $\mu \to \mu + 2$ causes the substitution $F_\infty \to -F_\infty$ although $I_{\mu+2, \nu}^\infty = I_{\mu, \nu}^\infty$ as a $\tilde{G}$-module.
According to the decomposition (2.14), we define an injective $C$-linear map $\iota_{\mu,\nu} : C^\infty_0(\mathbb{R}) \to I_{\mu,\nu}^\infty$ by
\[
\iota_{\mu,\nu}(f)(\tilde{g}) = \begin{cases} 
\mathrm{e}^{\sqrt{-1}\mu(\theta + \arg(J(g,\sqrt{-1})/c))}|c|^{-2\nu-1}f(-d/c) & \text{if } c \neq 0, \\
0 & \text{if } c = 0 
\end{cases} \tag{2.16}
\]
for $f \in C^\infty_0(\mathbb{R})$ and $\tilde{g} = (g,\theta) \in \tilde{G}$ with $g = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in G$. By the definition of $I_{\mu,\nu}^\infty$ and the density of $\bar{F}w\bar{U}$ in $\tilde{G}$, for $f \in C^\infty_0(\mathbb{R})$, we know that $\iota_{\mu,\nu}(f)$ is the function in $I_{\mu,\nu}^\infty$ characterized by the equality
\[
\iota_{\mu,\nu}(f)(\tilde{w}\tilde{u}(-x)) = f(x) \quad (x \in \mathbb{R}). \tag{2.17}
\]
That is, for $F \in I_{\mu,\nu}^\infty$ and $f \in C^\infty_0(\mathbb{R})$, we have
\[
F = \iota_{\mu,\nu}(f) \quad \text{if and only if} \quad F(\tilde{w}\tilde{u}(-x)) = f(x) \quad (x \in \mathbb{R}). \tag{2.18}
\]
For $f \in C^\infty_0(\mathbb{R})$ and $x \in \mathbb{R}$, we have
\[
(\iota_{\mu,\nu}(f))_\infty(\tilde{w}\tilde{u}(-x)) = \mathrm{e}^{\pi\sqrt{-1}\mu/2}\iota_{\mu,\nu}(f)\bar{\Pi}(x)\bar{K}(-\pi) = f_{\mu,\nu,\infty}(x). \tag{2.19}
\]
Hence, by the characterization (2.18), we have
\[
\iota_{\mu,\nu}(f_{\mu,\nu,\infty}) = (\iota_{\mu,\nu}(f))_\infty \quad (f \in C^\infty_0(\mathbb{R}^\times)). \tag{2.20}
\]

**Lemma 2.3.** Let $\mu,\nu \in \mathbb{C}$, $t \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$.

(i) For $f \in C^\infty_0(\mathbb{R})$, the following equalities hold:
\[
\rho(\tilde{u}(t))\iota_{\mu,\nu}(f) = \iota_{\mu,\nu}(s_{-t}(f)), \quad \rho(\bar{a}(y))\iota_{\mu,\nu}(f) = y^{-\nu-1/2}\iota_{\mu,\nu}(f|_{y^{-1}}),
\]
where $f_{y^{-1}}(x) = f(y^{-1}x)$ $(x \in \mathbb{R})$.

(ii) For $F \in I_{\mu,\nu}^\infty$, the following equalities hold:
\[
(\rho(\bar{\Pi}(t))F)_\infty = \rho(\tilde{u}(-t))F_\infty, \quad (\rho(\tilde{u}(t))F)_\infty = \rho(\overline{\Pi}(-t))F_\infty,
\]
\[
(\rho(\bar{a}(y))F)_\infty = \rho(\bar{a}(1/y))F_\infty.
\]

**Proof.** The statement (i) follows from the characterization (2.18) and
\[
(\rho(\tilde{u}(t))\iota_{\mu,\nu}(f)(\tilde{w}\tilde{u}(-x)) = \iota_{\mu,\nu}(f)(\tilde{w}\tilde{u}(-x+t)) = f(x-t) = s_{-t}(f)(x),
\]
\[
(\rho(\bar{a}(y))\iota_{\mu,\nu}(f)(\tilde{w}\tilde{u}(-x)) = \iota_{\mu,\nu}(f)(\tilde{w}\tilde{u}(-x)\bar{a}(y)) = \iota_{\mu,\nu}(f)(\bar{a}(1/y))\tilde{w}\tilde{u}(-x/y) = y^{-\nu-1/2}f(x/y) = y^{-\nu-1/2}f_{y^{-1}}(x)
\]
for $f \in C^\infty_0(\mathbb{R})$ and $x \in \mathbb{R}$. The statement (ii) follows from the equalities $\tilde{w}\tilde{\Pi}(t) = \tilde{u}(-t)\tilde{w}$, $\tilde{w}\tilde{u}(t) = \overline{\Pi}(-t)\tilde{w}$ and $\bar{a}(y) = \bar{a}(1/y)\bar{w}$. \qed
In this paper, we call a continuous $\mathbb{C}$-linear map $\lambda: I_{\mu,\nu}^\infty \to \mathbb{C}$ a distribution on $I_{\mu,\nu}^\infty$, and denote by $I_{\mu,\nu}^{-\infty}$ the space of distributions on $I_{\mu,\nu}^{-\infty}$. Since the topology of $I_{\mu,\nu}^\infty$ is induced by the seminorms $Q_X (X \in U(g_c))$, we note that a $\mathbb{C}$-linear map $\lambda: I_{\mu,\nu}^\infty \to \mathbb{C}$ is continuous if and only if there exist $c > 0$, $m \in \mathbb{Z}_{>0}$ and $X_1, X_2, \cdots, X_m \in U(g_c)$ such that

$$|\lambda(F)| \leq c \sum_{i=1}^{m} Q_{X_i}(F) \quad (F \in I_{\mu,\nu}^\infty). \quad (2.21)$$

For $\lambda \in I_{\mu,\nu}^{-\infty}$, we define $\mathbb{C}$-linear maps $T^{\lambda}: C_0^\infty(\mathbb{R}) \to \mathbb{C}$ and $\lambda_{\infty}: I_{\mu,\nu}^\infty \to \mathbb{C}$ by

$$T^{\lambda}(f) = \lambda(t_{\mu,\nu}(f)) \quad (f \in C_0^\infty(\mathbb{R})), \quad \lambda_{\infty}(F) = \lambda(F_{\infty}) \quad (F \in I_{\mu,\nu}^\infty).$$

For $F \in I_{\mu,\nu}^\infty$, we call a pair $(f_1, f_2)$ a partition of $F$ if and only if $f_1$ and $f_2$ are functions in $C_0^\infty(\mathbb{R})$ such that $t_{\mu,\nu}(f_1) + t_{\mu,\nu}(f_2)_{\infty} = F$. It is easy to see that any function $F$ in $I_{\mu,\nu}^\infty$ has a partition (cf. Lemma 6.2), although a partition is not unique. For $(T_1, T_2) \in A(J_{\mu,\nu})$ and $F \in I_{\mu,\nu}^\infty$, we set

$$\Lambda(T_1, T_2)(F) = T_1(f_1) + T_2(f_2), \quad (2.22)$$

where $(f_1, f_2)$ is a partition of $F$. The following proposition is proved in §6.1.

**Proposition 2.4.** Let $\mu, \nu \in \mathbb{C}$.

(i) $T^{\lambda} \in \mathcal{D}'(\mathbb{R})$, $\lambda_{\infty} \in I_{\mu,\nu}^{-\infty}$ and $\lambda_{\infty} = \lambda$ for $\lambda \in I_{\mu,\nu}^{-\infty}$.

(ii) The definition (2.22) does not depend on the choice of a partition $(f_1, f_2)$ of $F$. Moreover, $\Lambda(T_1, T_2) \in I_{\mu,\nu}^{-\infty}$ for $(T_1, T_2) \in A(J_{\mu,\nu})$.

(iii) The $\mathbb{C}$-linear map $I_{\mu,\nu}^{-\infty} \ni \lambda \mapsto (T^{\lambda}, \lambda_{\infty}) \in A(J_{\mu,\nu})$ is bijective, and its inverse map is given by $(T^{\lambda}, \lambda_{\infty}) \in A(J_{\mu,\nu}) \mapsto (T_1, T_2)$.

Let $L$ be a shifted lattice in $\mathbb{R}$. We define subspaces $(I_{\mu,\nu}^{-\infty})_L$ and $(I_{\mu,\nu}^{-\infty})_{L_{\text{quasi}}}$ of $I_{\mu,\nu}^{-\infty}$ by

$$(I_{\mu,\nu}^{-\infty})_L = \{ \lambda \in I_{\mu,\nu}^{-\infty} \mid \lambda(\rho(\tilde{u}(t))F) = \omega_L(t)\lambda(F) \quad (t \in L^\vee, F \in I_{\mu,\nu}^\infty) \},$$

$$(I_{\mu,\nu}^{-\infty})_{L_{\text{quasi}}} = \{ \lambda \in I_{\mu,\nu}^{-\infty} \mid T^{\lambda} \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L) \}. $$

By Lemma 2.3 (i) and Proposition 2.4 (i), we have $T^{\lambda} \in \mathcal{D}'(L^\vee \setminus \mathbb{R}; \omega_L)$ for $\lambda \in (I_{\mu,\nu}^{-\infty})_L$. Hence, $(I_{\mu,\nu}^{-\infty})_L$ is a subspace of $(I_{\mu,\nu}^{-\infty})_{L_{\text{quasi}}}$.

### 2.6 The Jacquet integrals and the Fourier expansions

In this subsection, we introduce the Jacquet integral, and the Fourier expansions in terms of them. Let $\mu \in \mathbb{C}$, $y \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{R})$. It is easy to see that $f_{\mu,\nu,\infty}$ is integrable and $\mathcal{F}(f_{\mu,\nu,\infty})(y)$ is a holomorphic function of $\nu$ on $\text{Re}(\nu) > 0$. For $m \in \mathbb{Z}_{\geq 0}$, we denote by $\delta^{(m)}$ the $m$-th derivative of the Dirac delta distribution, that is, $\delta^{(m)}(f) = (-1)^m f^{(m)}(0) \quad (f \in C_0^\infty(\mathbb{R}))$. 

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Lemma 2.5. Let $\mu \in \mathbb{C}$, $y \in \mathbb{R}^\times$ and $f \in C_0^\infty(\mathbb{R})$.

(i) As a function of $\nu$, $\mathcal{F}(f_{\mu,\nu,\infty})(y)$ has the holomorphic continuation to the whole $\nu$-plane.

(ii) As a function of $\nu$, $\mathcal{F}(f_{\mu,\nu,\infty})(0)$ has the meromorphic continuation to the whole $\nu$-plane. Moreover,

$$\mathcal{F}(f_{\mu,\nu,\infty})(0) - \sum_{i=0}^{n} (\sqrt{-1})^{i} \frac{2 \cos(\frac{\pi(i+\mu)}{2})}{\nu!} \delta(i)(f)$$

is holomorphic on $\text{Re}(\nu) > -(n+1)/2$ for any $n \in \mathbb{Z}_{\geq 0}$.

A proof of Lemma 2.5 is given in §6.2. For $\mu, \nu \in \mathbb{C}$, $y \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{R})$, we define the twisted Fourier transform $\mathcal{F}_{\mu,\nu,\infty}(f)(y)$ of $f$ by

$$\mathcal{F}_{\mu,\nu,\infty}(f)(y) = \begin{cases} 
\mathcal{F}(f_{\mu,\nu,\infty})(y) & \text{if } y \neq 0 \text{ or } -2\nu \notin \mathbb{Z}_{\geq 0}, \\
\lim_{s \to -\nu/2} \left( \mathcal{F}(f_{\mu,s,\infty})(0) - (\sqrt{-1})^n \frac{2 \cos(\frac{\pi(n+\mu)}{2})}{\nu!(2s+n)} \delta(n)(f) \right) & \text{if } y = 0 \text{ and } -2\nu = n \text{ with some } n \in \mathbb{Z}_{\geq 0}
\end{cases}$$

with the meromorphic continuation in Lemma 2.5. The following lemma is proved in §6.2.

Lemma 2.6. Let $\mu, \nu \in \mathbb{C}$ and $y \in \mathbb{R}$. A map $C_0^\infty(\mathbb{R}) \ni f \mapsto \mathcal{F}_{\mu,\nu,\infty}(f)(y) \in \mathbb{C}$ is a distribution on $\mathbb{R}$. Moreover, we have

$$\mathcal{F}_{\mu,\nu,\infty}(f)(y) = \mathcal{F}(f_{\mu,\nu,\infty})(y) \quad (f \in C_0^\infty(\mathbb{R}^\times)). \quad (2.23)$$

Let $\mu, \nu \in \mathbb{C}$ and $l \in \mathbb{R}$. When $\text{Re}(\nu) > 0$, we define the Jacquet integral $\mathcal{J}_l \in I_{\mu,\nu}^\infty$ by

$$\mathcal{J}_l(F) = \int_{-\infty}^{\infty} F(\tilde{w}\bar{u}(-x)) e^{2\pi \sqrt{-1}lx} \, dx \quad (F \in I_{\mu,\nu}^\infty).$$

By (2.17) and (2.19), we have

$$\mathcal{J}_l(F) = \mathcal{F}(f_1)(l) + \mathcal{F}_{\mu,\nu,\infty}(f_2)(l) \quad (F \in I_{\mu,\nu}^\infty), \quad (2.24)$$

where $(f_1, f_2)$ is a partition of $F$. By Lemma 2.6, we note that a pair of $f \mapsto \mathcal{F}(f)(l)$ and $f \mapsto \mathcal{F}_{\mu,\nu,\infty}(f)(l)$ is in $\mathcal{A}(J_{\mu,\nu})$. Hence, we can extend the definition of $\mathcal{J}_l \in I_{\mu,\nu}^\infty$ to general $\nu \in \mathbb{C}$ by the expression (2.24) and Proposition 2.4. Here this extension of the Jacquet integral $\mathcal{J}_l$ is essentially same as the extension of the Fourier transformation in [MSSU, §2.1 and §2.3] except for the case of $l = 0$ and $-2\nu \in \mathbb{Z}_{\geq 0}$.

For $m \in \mathbb{Z}_{\geq 0}$, we define $\delta_{\infty}^{(m)} \in I_{\mu,\nu}^\infty$ by

$$\delta_{\infty}^{(m)}(F) = (\rho(E_+)^m F_{\infty})(\tilde{w}) \quad (F \in I_{\mu,\nu}^\infty). \quad (2.25)$$
We denote by \( \mathfrak{M}(\mathbb{Z}_{\geq 0}) \) the space of functions \( \beta: \mathbb{Z}_{\geq 0} \to \mathbb{C} \) such that \( \beta(m) = 0 \) for all but finitely many \( m \in \mathbb{Z}_{\geq 0} \). For \( \alpha \in \mathfrak{M}(L) \) and \( \beta \in \mathfrak{M}(\mathbb{Z}_{\geq 0}) \), we define a map \( \lambda_{\alpha,\beta}: I_{\mu,\nu}^\infty \to \mathbb{C} \) by

\[
\lambda_{\alpha,\beta}(F) = \sum_{l \in L} \alpha(l)J_l(F) + \sum_{m=0}^{\infty} \beta(m)\delta^{(m)}_\infty(F) \quad (F \in I_{\mu,\nu}^\infty).
\]

**Proposition 2.7.** Let \( L \) be a shifted lattice in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \). Then, for \( \alpha \in \mathfrak{M}(L) \) and \( \beta \in \mathfrak{M}(\mathbb{Z}_{\geq 0}) \), a map \( \lambda_{\alpha,\beta} \) is a distribution in \( (I_{\mu,\nu}^{-\infty})_{L}^{\text{quasi}} \) such that \( T^{\lambda_{\alpha,\beta}} = T_\alpha \). Moreover, the \( \mathbb{C} \)-linear map

\[
\mathfrak{M}(L) \times \mathfrak{M}(\mathbb{Z}_{\geq 0}) \ni (\alpha, \beta) \mapsto \lambda_{\alpha,\beta} \in (I_{\mu,\nu}^{-\infty})_{L}^{\text{quasi}} \tag{2.26}
\]

is bijective.

A proof of Proposition 2.7 is given in §6.4. By Proposition 2.7, for a distribution \( \lambda \) in \( (I_{\mu,\nu}^{-\infty})_{L}^{\text{quasi}} \), there is a unique \((\alpha, \beta) \in \mathfrak{M}(L) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})\) such that

\[
\lambda(F) = \sum_{l \in L} \alpha(l)J_l(F) + \sum_{m=0}^{\infty} \beta(m)\delta^{(m)}_\infty(F) \quad (F \in I_{\mu,\nu}^\infty). \tag{2.27}
\]

We call the expression (2.27) the Fourier expansion of \( \lambda \).

For a subset \( S \) of \( \mathbb{Z}_{\geq 0} \), we denote by \( \mathfrak{M}(S) \) a subspace of \( \mathfrak{M}(\mathbb{Z}_{\geq 0}) \) consisting of all functions \( \beta \) such that \( \beta(m) = 0 \) unless \( m \in S \). Then, for \( \alpha \in \mathfrak{M}(L) \) and \( \beta \in \mathfrak{M}(S) \), the Fourier expansion of \( \lambda_{\alpha,\beta} \) is given as follows:

\[
\lambda_{\alpha,\beta}(F) = \sum_{l \in L} \alpha(l)J_l(F) + \sum_{m \in S} \beta(m)\delta^{(m)}_\infty(F) \quad (F \in I_{\mu,\nu}^\infty).
\]

Let \( \mathfrak{M}(L)_{\mu,\nu}^0 \) be the subset of \( \mathfrak{M}(L) \) consisting of all functions \( \alpha \) satisfying

\[
(\alpha(0) = 0 \text{ if } 0 \in L, -2\nu \in \mathbb{Z}_{\geq 0} \text{ and } \mu - 2\nu - 1 \not\in 2\mathbb{Z} ).
\]

We define a subset \( S_\nu(L) \) of \( \mathbb{Z}_{\geq 0} \) by

\[
S_\nu(L) = \begin{cases} 
\emptyset & \text{if } 0 \not\in L, \\
\{0,-2\nu\} \cap \mathbb{Z}_{\geq 0} & \text{if } 0 \in L.
\end{cases}
\]

The following proposition is proved in §6.4.

**Proposition 2.8.** Let \( L \) be a shifted lattice in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \). Let \( \alpha \in \mathfrak{M}(L) \) and \( \beta \in \mathfrak{M}(\mathbb{Z}_{\geq 0}) \). Then we have \( \lambda_{\alpha,\beta} \in (I_{\mu,\nu}^{-\infty})_L \) if and only if \( \alpha \in \mathfrak{M}(L)_{\mu,\nu}^0 \) and \( \beta \in \mathfrak{M}(S_\nu(L)) \).

### 2.7 Automorphic distributions for shifted lattices

Let \( L_1 \) and \( L_2 \) be two shifted lattices in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \). We define subspaces \( (I_{\mu,\nu}^{-\infty})_{L_1,L_2} \) and \( (I_{\mu,\nu}^{-\infty})_{L_1,L_2}^{\text{quasi}} \) of \( I_{\mu,\nu}^{-\infty} \) by

\[
(I_{\mu,\nu}^{-\infty})_{L_1,L_2} = \{ \lambda \mid \lambda \in (I_{\mu,\nu}^{-\infty})_{L_1}, \lambda_\infty \in (I_{\mu,\nu}^{-\infty})_{L_2} \}
\]

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For the natural map $\lambda$ such that $S_\lambda$ is bijective. Moreover, we have $\lambda \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$ for $\lambda \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$,

$$
(I_{\mu,\nu}^{-\infty})_{L_1,L_2} \ni \lambda \mapsto (T^\lambda, T^{\lambda\infty}) \in \mathcal{A}(L_1, L_2; J_{\mu,\nu})
$$

is bijective. Moreover, we have $\lambda_\infty \in (I_{\mu,\nu}^{-\infty})_{L_2,L_1}$ for $\lambda \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$, and $\lambda_\infty \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$ for $\lambda \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$.

For a shifted lattice $L$ in $\mathbb{R}$, we take $\lambda_{\alpha,\beta}$ ($\alpha \in \mathfrak{M}(L)$, $\beta \in \mathfrak{M}([0,\infty))$, $\mathfrak{M}(L)^0_{\mu,\nu}$ and $S_{\nu}(L)$ as in §2.6.

**Theorem 2.9.** Let $L_1$ and $L_2$ be two shifted lattices in $\mathbb{R}$. Let $\mu, \nu \in \mathbb{C}$.

(i) Let $\lambda \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$. There are unique $(\alpha_i, \beta_i) \in \mathfrak{M}(L_i) \times \mathfrak{M}([0,\infty))$ ($i = 1, 2$) such that $\lambda = \lambda_{\alpha_i,\beta_i}$ and $\lambda_{\alpha_i,\beta_i} = \lambda_{\alpha_2,\beta_2}$. Moreover, if $\lambda \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$, then, for $i = 1, 2$, we have $(\alpha_i, \beta_i) \in \mathfrak{M}(L_i)^0_{\mu,\nu} \times \mathfrak{M}(S_{\nu}(L_i))$ and

$$(\beta_i(0) = 0 \text{ if } -2\nu \in \mathbb{Z}_{>0}, \mu - 2\nu - 1 \in 2\mathbb{Z} \text{ and } (0 \not\in L_{3-i} \text{ or } -2\nu > 1).$$

(ii) Let $(\alpha_i, \beta_i) \in \mathfrak{M}(L_i) \times \mathfrak{M}([0,\infty))$ ($i = 1, 2$) such that $(\lambda_{\alpha_i,\beta_i})_{\infty} = \lambda_{\alpha_2,\beta_2}$. Then we have $\lambda_{\alpha_i,\beta_i} \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$. Moreover, if $(\alpha_i, \beta_i) \in \mathfrak{M}(L_i)^0_{\mu,\nu} \times \mathfrak{M}(S_{\nu}(L_i))$ ($i = 1, 2$), then we have $\lambda_{\alpha_i,\beta_i} \in (I_{\mu,\nu}^{-\infty})_{L_1,L_2}$.

Theorem 2.9 (ii) follows immediately from Propositions 2.7 and 2.8. A proof of Theorem 2.9 (i) is given in §6.5.

Let $(\alpha_i, \beta_i) \in \mathfrak{M}(L_i) \times \mathfrak{M}([0,\infty))$ ($i = 1, 2$). When $\xi_{\pm}(\alpha_i; s)$ (or their completion $\Xi_{\pm}(\alpha_i; s)$) have the meromorphic continuations to the whole $s$-plane for $i = 1, 2$, we consider the following conditions [D3] and [D4] on $\xi_{\pm}(\alpha_i; s)$, $\xi_{\pm}(\alpha_2; s)$, $\beta_2$ and $\beta_2$:

[D3] For any $m \in \mathbb{Z}_{\geq 0}$ and $i = 1, 2$, the functions $\xi_{\pm}(\alpha_i; s)$ satisfy

$$
\text{Res}_{s=m+1} (\xi_{\pm}(\alpha_i; s) + (-1)^m \xi_{-}(\alpha_i; s)) = 2(2\pi \sqrt{-1})^m \beta_{3-i}(m).
$$

[D4] For $i = 1, 2$, the functions $(s-1)(s+2\nu-1)\xi_{\pm}(\alpha_i; s)$ are entire if $0 \in L_{3-i}$, and the functions $\xi_{\pm}(\alpha_i; s)$ are entire if $0 \not\in L_{3-i}$.

Here we note that $\beta_2$ and $\beta_2$ are uniquely determined by $\alpha_2$ and $\alpha_1$, respectively, if [D3] holds. If [D3] and [D4] hold, we have $\beta_i(m) = 0$ for $i = 1, 2$ and $m \in \mathbb{Z}_{\geq 0}$ unless $0 \in L_{3-i}$, and $m \in \{0, -2\nu\}$. Let [D1], [D2-1] and [D2-2] be the conditions in §2.3.
Theorem 2.10. Let $L_1$ and $L_2$ be two shifted lattices in $\mathbb{R}$. Let $\mu, \nu \in \mathbb{C}$. Let $(\alpha_1, \beta_1) \in \mathfrak{M}(L_1) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$ and $(\alpha_2, \beta_2) \in \mathfrak{M}(L_2) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$.

(i) Assume $(\lambda_{\alpha_1, \beta_1})_{\infty} = \lambda_{\alpha_2, \beta_2}$. Then the conditions [D1], [D2-1] and [D3] hold. Moreover, if $\lambda_{\alpha_1, \beta_1} \in (I_{\mu, \nu})_{L_1, L_2}$, then the condition [D4] holds.

(ii) Assume that [D1], [D2-2] and [D3] hold. Then $\lambda_{\alpha_1, \beta_1} \in (I_{\mu, \nu})_{\text{quasi}}^{\text{quasi}}_{L_1, L_2}$ and $(\lambda_{\alpha_1, \beta_1})_{\infty} = \lambda_{\alpha_2, \beta_2}$. If [D4] also holds, then $\lambda_{\alpha_1, \beta_1} \in (I_{\mu, \nu})_{L_1, L_2}$.

In order to prove Theorem 2.10, we use the following proposition, which is proved in §6.6.

Proposition 2.11. Let $L_1$ and $L_2$ be two shifted lattices in $\mathbb{R}$. Let $\mu, \nu \in \mathbb{C}$. Let $(\alpha_1, \beta_1) \in \mathfrak{M}(L_1) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$ (i = 1, 2) such that $(\lambda_{\alpha_1, \beta_1})_{\infty} = \lambda_{\alpha_2, \beta_2}$. Let $\xi_{\pm}(\alpha_1; s)$ be the Dirichlet series, which is meromorphically continued to the whole $s$-plane by Theorem 2.2 (i). If $-2\nu \notin \mathbb{Z}_{\geq 0}$, then the functions

$$
\xi_{\pm}(\alpha_1; s) - \sum_{m=0}^{\infty} \frac{(\pm 2\pi \sqrt{-1})^m \beta_2(m)}{s - m - 1} - \frac{2\Gamma(2\nu) \cos\left(\frac{\pi(2\nu+\mu)}{2}\right) \alpha_2(0)}{(2\pi)^{2\nu}(s + 2\nu - 1)}
$$

are entire. If $-2\nu = n$ with some $n \in \mathbb{Z}_{\geq 0}$, then the functions

$$
\xi_{\pm}(\alpha_1; s) - \sum_{m=0}^{\infty} \frac{(\pm 2\pi \sqrt{-1})^m \beta_2(m)}{s - m - 1} + \frac{(2\pi)^n \cos\left(\frac{\pi(n+\mu)}{2}\right) \alpha_2(0)}{n!} + \frac{(2\pi)^n \cos\left(\frac{\pi(n+\mu)}{2}\right) \alpha_2(0)}{n!(s - n - 1)^2}
$$

are entire. Here we understand $\alpha_2(0) = 0$ if $0 \notin L_2$, and double signs are in the same order.

Proof of Theorem 2.10. First, we will prove the statement (i). Assume that $(\lambda_{\alpha_1, \beta_1})_{\infty} = \lambda_{\alpha_2, \beta_2}$ holds. Then we have $(T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu, \nu})$ by Propositions 2.4 and 2.7. Hence, [D1] and [D2-1] hold by Theorem 2.2 (i). Since $(\lambda_{\alpha_2, \beta_2})_{\infty} = \lambda_{\alpha_1, \beta_1}$ also holds, we know that [D3] holds by Proposition 2.11. If $\lambda_{\alpha_1, \beta_1} \in (I_{\mu, \nu})_{L_1, L_2}$, then [D4] holds by Theorem 2.9 (i) and Proposition 2.11.

Next, we will prove the statement (ii). Assume that [D1], [D2-2] and [D3] hold. Then we have $(T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu, \nu})$ by [D1], [D2-2] and Theorem 2.2 (ii). Let $\lambda = \Lambda(T_{\alpha_1}, T_{\alpha_2}) \in (I_{\mu, \nu})_{\text{quasi}}^{\text{quasi}}_{L_1, L_2}$. By Proposition 2.7, there are $\beta_1', \beta_2' \in \mathfrak{M}(\mathbb{Z}_{\geq 0})$ such that $\lambda = \lambda_{\alpha_1, \beta_1'}$ and $(\lambda_{\alpha_1, \beta_1'})_{\infty} = \lambda_{\alpha_2, \beta_2'}$. By Proposition 2.11 and [D3], we have $\beta_1 = \beta_1'$ and $\beta_2 = \beta_2'$. Hence, $\lambda_{\alpha_1, \beta_1} \in (I_{\mu, \nu})_{L_1, L_2}$ and $(\lambda_{\alpha_1, \beta_1})_{\infty} = \lambda_{\alpha_2, \beta_2'}$. If [D4] also holds, we have $\lambda_{\alpha_1, \beta_1} \in (I_{\mu, \nu})_{L_1, L_2}$ by Theorem 2.9 (ii) and Proposition 2.11. $\blacksquare$

2.8 Knopp’s result

In this subsection, we show that there exist quasi-automorphic distributions, which are not automorphic distributions, by using the “Mittag-Leffler” theorem of Knopp. The result in this subsection is given by Professor Fumihiro Sato.
Then \( K_1 \) implies \( \alpha \). Theorem 2.12 and \( \alpha \). Hence, \( K_2 \) and \( K_3 \) imply that
we take a Dirichlet series \( \xi \) satisfying
\[ q \]
satisfying the following conditions \[ K_1 \], \[ K_2 \] and \[ K_3 \]:

- \( K_1 \) There is \( r > 0 \) such that \( a_q(m) = O(m^r) \) \( m \to \infty \). In particular, the
Dirichlet series \( \xi_q(s) \) converges absolutely on \( \text{Re}(s) > r + 1 \).
- \( K_2 \) The Dirichlet series \( \xi_q(s) \) has the meromorphic continuation to the whole
\( s \)-plane, and there is a polynomial function \( p(s) \) such that \( p(s)\xi_q(s) \) is an
entire function of finite genus.
- \( K_3 \) The function \( \Xi_q(s) = \pi^{-s} \Gamma(s) \xi_q(s) \) satisfies \( \Xi_q(\kappa - s) = \Xi_q(s) \), and the
function \( \Xi_q(s) - q(s) \) is entire.

Let \( i \in \{0, 1\} \) and \( n \in \mathbb{Z}_{>0} \). Let \( \beta \in \mathcal{M}(\mathbb{Z}_{\geq 0}) \). For \( \kappa = 2n \) and
\[
q(s) = \sum_{m=0}^{\infty} \beta(m) \left( \frac{1}{s - n - i - 2m} + \frac{1}{-s + n - i - 2m} \right),
\]
we take a Dirichlet series \( \xi_q(s) = \sum_{m=1}^{\infty} a_q(m)m^{-s} \) as in Theorem 2.12. Set
\[
\mu = 0, \quad \nu = -n + \frac{1}{2}, \quad L_1 = L_2 = 2^{-1}\mathbb{Z}
\]
and
\[
\alpha_1(l) = \begin{cases} 
    a_q(2l) & \text{if } l > 0, \\
    0 & \text{if } l = 0, \\
    (1)^{\alpha_{n+1}}a_q(-2l) & \text{if } l < 0
\end{cases}
\]
\( \alpha_2 = (1)^n\alpha_1 \).

Then \( K_1 \) implies \( \alpha_1 \in \mathcal{M}(L_1) \) and \( \alpha_2 \in \mathcal{M}(L_2) \). Moreover, we have
\[
\Xi_+(\alpha_1; s) = (1)^{\alpha_{n+1}}\Xi_-(\alpha_1; s) = (1)^n\Xi_+(\alpha_2; s) = (1)^{\alpha_{n+1}}\Xi_-(\alpha_2; s) = \Xi_q(s).
\]

Hence, \( K_2 \) and \( K_3 \) imply that \( \alpha_1, \alpha_2 \) satisfy \( [D1] \) and \( [D2-2] \). By Theorem
2.2 and Proposition 2.4, we have \( (T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{0, \nu}) \) and \( \Lambda(T_{\alpha_1}, T_{\alpha_2}) \in
(I_{0, \nu}^{-\infty})_{L_1, L_2}^{\text{quasi}} \). If we take \( \beta \) so that \( q(s) \) has a pole on \( \mathbb{C} \setminus (\{2n\} \cup \mathbb{Z}_{\leq 1}) \), then
\( \Lambda(T_{\alpha_1}, T_{\alpha_2}) \not\in (I_{0, \nu}^{-\infty})_{L_1, L_2} \) by Theorems 2.9 (i) and 2.10 (i). In particular, we
know that the complement of \( (I_{0, \nu}^{-\infty})_{L_1, L_2} \) in \( (I_{0, \nu}^{\text{quasi}})_{L_1, L_2} \) is infinite dimensional.

2.9 The Poisson transform
Let \( \mu, \nu \in \mathbb{C} \) and \( \kappa \in \mu + 2\mathbb{Z} \). Let \( C^\infty(\mathfrak{f}) \) be the space of smooth functions on
\( \mathfrak{f} \). For \( \phi \in C^\infty(\mathfrak{f}) \) and \( g \in G \), we set
\[
(\phi|_{\kappa}g)(z) = \left( \frac{J(g, z)}{|J(g, z)|} \right)^{-\kappa} \phi(gz) \quad (z \in \mathfrak{f}).
\]
We define the hyperbolic Laplacian $\Omega_\kappa$ of weight $\kappa$ on $\mathfrak{H}$ by
\[
\Omega_\kappa = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \sqrt{-1} \kappa y \frac{\partial}{\partial x} \quad (z = x + \sqrt{-1}y \in \mathfrak{H}).
\] (2.28)

Let $\mathcal{M}_\nu(\mathfrak{H}; \kappa)$ be a subspace of $C^\infty(\mathfrak{H})$ consisting of all functions $\phi$ satisfying
\[
(\Omega_\kappa \phi)(z) = \left( \frac{1}{4} - \nu^2 \right) \phi(z)
\]
and the following condition [M1]:

[M1] There are $c, r > 0$ such that $|\phi(z)| \leq c \left( \frac{|z|^2 + 1}{y} \right)^r \quad (z = x + \sqrt{-1}y \in \mathfrak{H}).$

For a distribution $\lambda$ on $I_{\mu, \nu}^\infty$, we define the Poisson transform $\mathcal{P}_{\nu, \kappa}(\lambda)$ of $\lambda$ by
\[
\mathcal{P}_{\nu, \kappa}(\lambda)(z) = e^{\pi \sqrt{-1} \kappa/2} \lambda(\rho(\tilde{u}(x)\tilde{a}(y)))F_{\nu, \kappa} \quad (z = x + \sqrt{-1}y \in \mathfrak{H}),
\] (2.29)
where $F_{\nu, \kappa}$ is an element of $I_{\mu, \nu}^\infty$ defined by
\[
F_{\nu, \kappa}(\tilde{g}) = \text{Im}(g\sqrt{-1})^{\nu+1/2} e^{\pi \sqrt{-1} \kappa \theta} 
\quad (\tilde{g} = (g, \theta) \in \tilde{G}).
\] (2.30)

**Proposition 2.13.** Let $\mu, \nu \in \mathbb{C}$, and $\kappa \in \mu + 2\mathbb{Z}$.

(i) Let $\lambda \in I_{\mu, \nu}^\infty$. Then we have $\mathcal{P}_{\nu, \kappa}(\lambda) \in \mathcal{M}_\nu(\mathfrak{H}; \kappa)$ and
\[
(\mathcal{P}_{\nu, \kappa}(\lambda)|_g)(z) = e^{\pi \sqrt{-1} \kappa/2} \lambda(\rho(\tilde{g}(x)\tilde{a}(y)))F_{\nu, \kappa}
\] (2.31)
for $g \in G$ and $z = x + \sqrt{-1}y \in \mathfrak{H}$.

(ii) Let $\alpha \in \mathcal{M}(L)_{\mu, \nu}^0$ and $\beta \in \mathcal{M}(S_{\nu}(L))$. For $z = x + \sqrt{-1}y \in \mathfrak{H}$, we have
\[
\mathcal{P}_{\nu, \kappa}(\lambda_{\alpha, \beta})(z) = \sum_{\lambda \in \Lambda} e^{\pi \sqrt{-1} \kappa/2} \lambda(\rho(\tilde{g}(x)\tilde{a}(y))) F_{\nu, \kappa}
\]
\[
\left\{ \begin{array}{ll}
\frac{\pi \nu + 1}{2} l |l|^{\nu - \frac{1}{2}} \alpha(l) W_{\text{sgn}(l)}(4\pi l |y|) e^{2\pi \sqrt{-1}ux} & \text{if } 0 \in L \text{ and } \nu \neq 0, \\
+ cy^{-\nu + \frac{1}{2}} + \begin{cases} 
-2 \cos \left( \frac{\pi \nu}{2} \right) \alpha(0) y^{\frac{1}{2} \log y} & \text{if } 0 \in L \text{ and } \nu = 0, \\
0 & \text{if } 0 \not\in L.
\end{cases} & \text{if } 0 \not\in L.
\end{array} \right.
\] (2.32)

Here $W_{\kappa, \nu}(y)$ is Whittaker’s function ([WW, Chapter 16]), and $c$ is the constant determined by
\[
c = \left\{ \begin{array}{ll}
\frac{2^{1-2\nu} \Gamma(2\nu)}{\Gamma \left( \frac{2\nu + 1 - \kappa}{2} \right) \Gamma \left( \frac{2\nu + 1 + \kappa}{2} \right)} \alpha(0) & \text{if } 0 \in L \text{ and } -2\nu \not\in \mathbb{Z}_{\geq 0}, \\
(1) \frac{2^{n-1} \pi n! (\frac{1}{2})!}{(\frac{1}{2})!} \alpha(0) + (1)^{\frac{n-\kappa}{2}} \beta(n) d(n, \kappa) & \text{if } 0 \in L, -2\nu = n \text{ and } |\kappa| - n + 1 \in 2\mathbb{Z}_{>0} \text{ with some } n \in \mathbb{Z}_{>0}, \\
(1)^{\frac{n-\kappa}{2}} d(n, \kappa) \beta(n) & \text{if } 0 \in L, -2\nu = n \text{ and } \mu + n - 1 \not\in 2\mathbb{Z} \text{ with some } n \in \mathbb{Z}_{>0}, \\
2j(\kappa) \alpha(0) \cos \left( \frac{\pi \nu}{2} \right) + (1)^{\frac{n-\kappa}{2}} \beta(0) & \text{if } 0 \in L, \nu = 0 \text{ and } \mu - 1 \not\in 2\mathbb{Z}, \\
0 & \text{otherwise}.
\end{array} \right.
\]
with

$$d(n, \kappa) = (\sqrt{-1})^n \prod_{j=0}^{n-1} (\kappa + n - 1 - 2j), \quad (2.33)$$

$$j(\kappa) = \sum_{i=0}^{\infty} \left(\frac{1}{2i+1} - \frac{1}{2i+1+\kappa} \right) - \log 2. \quad (2.34)$$

Proofs of the statements (i) and (ii) of Proposition 2.13 are given in §7.2 and §6.5, respectively.

### 2.10 Maass forms of real weights

Let $\Gamma$ be a cofinite subgroup of $SL(2, \mathbb{Z})$ such that $-1/2 \in \Gamma$. Let $\kappa \in \mathbb{R}$. We call $v$ a multiplier system on $\Gamma$ of weight $\kappa$ if and only if $v$ is a map from $\Gamma$ to $\mathbb{C} \times$ satisfying the conditions $v(-1/2) = e^{-\pi \sqrt{-1}}, |v(\gamma)| = 1 \ (\gamma \in \Gamma)$ and

$$v(\gamma_1 \gamma_2) = v(\gamma_1) v(\gamma_2) \frac{J(\gamma_1, \gamma_2; z)}{J(\gamma_1; z)} \quad (\gamma_1, \gamma_2 \in \Gamma, \ z \in \mathcal{A}). \quad (2.35)$$

Here we note that the right hand side does not depend on $z \in \mathcal{A}$.

Let $\mathcal{D} = \{z = x + \sqrt{-1}y \mid -1/2 \leq x \leq 1/2, \ y \geq \sqrt{1-x^2}\}$, which is the closure of the standard fundamental domain of $SL(2, \mathbb{Z}) \backslash \mathcal{A}$. Let $v$ be a multiplier system on $\Gamma$ of weight $\kappa$. Let $\nu \in \mathbb{C}$. Let $M_{\nu}(\Gamma \backslash \mathcal{A}; v, \kappa)$ be a subspace of $C^\infty(\mathcal{A})$ consisting all functions $\phi$ satisfying

$$(\phi | \kappa)(z) = v(\gamma) \phi(z) \quad (\gamma \in \Gamma), \quad (\Omega_{\kappa} \phi)(z) = \left(\frac{1}{4} - \nu^2\right) \phi(z)$$

and the following condition [M2]:

[M2] $\phi$ is of moderate growth at any cusp of $\Gamma$, that is, for any $\gamma \in SL(2, \mathbb{Z})$,

there exist $c, r > 0$ such that $(\phi | \gamma)(z) \leq c \text{Im}(z)^r \quad (z \in \mathcal{D})$.

We call a function in $M_{\nu}(\Gamma \backslash \mathcal{A}; v, \kappa)$ a Maass form for $\Gamma$ of weight $\kappa$ with multiplier system $v$ and eigenvalue $\frac{1}{4} - \nu^2$. The following proposition is proved in §7.1.

**Proposition 2.14.** Retain the notation. Then we have

$$M_{\nu}(\Gamma \backslash \mathcal{A}; v, \kappa) = \left\{ \phi \in M_{\nu}(\mathcal{A}; \kappa) \mid \phi|_{\kappa} \gamma = v(\gamma) \phi \quad (\gamma \in \Gamma) \right\}.$$  

In [Ma, Chapter IV, §2], Maass introduce the Fourier expansion of Maass forms at each cusps. Let $\phi \in M_{\nu}(\Gamma \backslash \mathcal{A}; v, \kappa)$, and we introduce the Fourier expansion of $\phi$ at $\infty$ here. Let $L = R^{-1}(u + \mathbb{Z})$ be a shifted lattice with the real numbers $R, u$ determined by

$$R = \min\{t \in \mathbb{Z}_{>0} \mid u(t) \in \Gamma\}, \quad 0 \leq u < 1, \quad v(u(R)) = e^{2\pi \sqrt{-1}u}. \quad (2.36)$$
Then $\phi$ has the following Fourier expansion at $\infty$

$$
\phi(z) = \sum_{0 \neq l \in L} a(\phi; l) W_{\text{sgn}(l)}(z) e^{2\pi \sqrt{-1} l z} + a(\phi; 0) y^{-\nu + \frac{1}{2}}
+ \begin{cases} 
  b(\phi; 0) y^{\nu + \frac{1}{2}} & \text{if } \nu \neq 0, \\
  b(\phi; 0) y^{\frac{1}{2} \log y} & \text{if } \nu = 0
\end{cases} \quad (z = x + \sqrt{-1} y \in \mathfrak{H})
$$

(2.37)

with $b(\phi; 0), a(\phi; l) \in \mathbb{C}$ ($l \in L$), where $a(\phi; 0) = b(\phi; 0) = 0$ if $0 \notin L$. Here $W_{\kappa, \nu}(y)$ is Whittaker's function ([WW, Chapter 16]).

**Remark 2.15.** Our definition of Maass forms is slightly different from that in Maass [Ma, Chapter IV, §2]. For a smooth function $\phi$ on $\mathfrak{H}$, we note that $\phi \in \mathcal{M}_\nu(\Gamma \backslash \mathfrak{H}; v, \kappa)$ if and only if $f(z, \tau) = \text{Im}(z)^{-\nu - \frac{1}{2}} \phi(z)$ is an automorphic form of the type $(\Gamma, \nu + \frac{1}{2}, \nu + \frac{1}{2}, v)$ in the sense of Maass.

### 2.11 Automorphic distributions for discrete groups

Let $\Gamma$ be a cofinite subgroup of $SL(2, \mathbb{Z})$ such that $-1_2 \in \Gamma$. We define a subgroup $\bar{\Gamma}$ of $\bar{G}$ by $\bar{\Gamma} = \varpi^{-1}(\Gamma) = \{(\gamma, \theta) \in \bar{G} \mid \gamma \in \Gamma\}$. Let $\kappa \in \mathbb{R}$. For a multiplier system $v$ on $\Gamma$ of weight $\kappa$, we set

$$
\hat{\chi}_v(\bar{\gamma}) = v(\gamma) e^{\sqrt{-1} \kappa(\theta + \arg J(\gamma, \sqrt{-1}))} \quad (\bar{\gamma} = (\gamma, \theta) \in \bar{\Gamma}).
$$

(2.38)

Then $\hat{\chi}_v$ is a unitary character of $\bar{\Gamma}$. Moreover, $v \mapsto \hat{\chi}_v$ defines a bijection from the set of multiplier systems on $\Gamma$ of weight $\kappa$ to the set of unitary characters $\hat{\chi}$ of $\bar{\Gamma}$ satisfying $\hat{\chi}(\mu(-1_2)) = e^{-\pi \kappa \sqrt{-1}}$, since its inverse map $\hat{\chi} \mapsto v_{\hat{\chi}}$ is given by $v_{\hat{\chi}}(\gamma) = \hat{\chi}(\gamma) \quad (\gamma \in \Gamma)$.

Let $\mu \in \mathbb{R}$ and $\nu \in \mathbb{C}$. Let $v$ be a multiplier system on $\Gamma$ of weight $\kappa$. We define a subspace $(I_{\mu, \nu})^\mathbb{F} \hat{\chi}_v$ of $I_{\mu, \nu}^\infty$ by

$$
(I_{\mu, \nu})^\mathbb{F} \hat{\chi}_v = \{ \lambda \in I_{\mu, \nu}^\infty \mid \lambda(\rho(\bar{\gamma}) F) = \hat{\chi}_v(\bar{\gamma}) \lambda(F) \quad (\bar{\gamma} \in \bar{\Gamma}, \; F \in I_{\mu, \nu}^\infty) \}.
$$

We call a distribution in $(I_{\mu, \nu})^\mathbb{F} \hat{\chi}_v$ an automorphic distribution on $I_{\mu, \nu}^\infty$ for $\bar{\Gamma}$ with character $\hat{\chi}_v$. For $m \in \mathbb{Z}$, we have

$$
\rho(k(m\pi)) F = e^{\pi \sqrt{-1} \kappa m} F \quad (F \in I_{\mu, \nu}^\infty), \quad \hat{\chi}_v(k(m\pi)) = e^{\pi m \sqrt{-1} \kappa}.
$$

(2.39)

Hence, $(I_{\mu, \nu})^\mathbb{F} \hat{\chi}_v = \{0\}$ unless $\kappa \in \mu + 2\mathbb{Z}$. We assume $\kappa \in \mu + 2\mathbb{Z}$ until the end of this subsection. Then we have the following proposition as an immediate consequence of Propositions 2.13 (i) and 2.14.

**Proposition 2.16.** Retain the notation. Let $\lambda \in (I_{\mu, \nu})^\mathbb{F} \hat{\chi}_v$. Then the Poisson transform $P_{v, \kappa}(\lambda)$ of $\lambda$ is a Maass form in $\mathcal{M}_\nu(\Gamma \backslash \mathfrak{H}; v, \kappa)$.

Let $L = R^{-1}(u + \mathbb{Z})$ be a shifted lattice in $\mathbb{R}$ with the real numbers $R$, $u$ determined by (2.36). Let $\bar{L} = R^{-1}(\bar{u} + \mathbb{Z})$ be a shifted lattice in $\mathbb{R}$ with the real numbers $\bar{R}$, $\bar{u}$ determined by

$$
\bar{R} = \min\{ t \in \mathbb{Z}_{>0} \mid \pi(-t) \in \Gamma \}, \quad 0 \leq \bar{u} < 1, \quad v(\pi(-\bar{R})) = e^{2\pi \sqrt{-1} \bar{u}}.
$$

(2.40)
Then it is obvious that \((I_{\mu,\nu})_{\tilde{\xi}}\) is a subspace of \((I_{\mu,\nu})_{\tilde{\xi}}\). Hence, by Theorem 2.9 (i), for \(\lambda \in (I_{\mu,\nu})_{\tilde{\xi}}\), there are unique \(\alpha \in \mathfrak{M}(L)\), \(\beta \in \mathfrak{M}(S(L))\), \(\hat{\alpha} \in \mathfrak{M}(L)\) and \(\hat{\beta} \in \mathfrak{M}(S(L))\) such that \(\lambda = \lambda_{\alpha,\beta}\) and \(\lambda_{\infty} = \lambda_{\hat{\alpha},\hat{\beta}}\), that is,

\[
\lambda(F) = \lambda_{\alpha,\beta}(F) = \sum_{l \in L} \alpha(l)J_l(F) + \sum_{m \in S(L)} \beta(m)\delta_{\hat{m}}^l(F),
\]

\[
\lambda_{\infty}(F) = \lambda_{\hat{\alpha},\hat{\beta}}(F) = \sum_{l \in \hat{L}} \hat{\alpha}(l)J_l(F) + \sum_{m \in S(L)} \hat{\beta}(m)\delta_{\hat{m}}^l(F)
\]

for \(F \in I_{\mu,\nu}^{\infty}\). We note that (2.32) is the Fourier expansion (2.37) at \(\infty\) for the Maass form \(\mathcal{P}_{\nu,\kappa}(\lambda) = \mathcal{P}_{\nu,\kappa}(\lambda_{\alpha,\beta})\).

### 2.12 Automorphic distributions for \(\tilde{\Gamma}_0(N)\)

Let \(N \in \mathbb{Z}_{>0}\). We define a subgroup \(\tilde{\Gamma}_0(N)\) of \(SL(2, \mathbb{Z})\) by

\[
\tilde{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \bigg| a, b, c, d \in \mathbb{Z}, \ ad - Nbc = 1 \right\},
\]

and set \(\tilde{\mathcal{I}}_0(N) = \varpi^{-1}(\Gamma_0(N)) = \{ (\gamma, \theta) \in \tilde{G} \mid \gamma \in \Gamma_0(N) \}. \) Let \(\kappa \in \mathbb{R}\). Let \(v\) be a multiplier system on \(\Gamma_0(N)\) of weight \(\kappa\), and define the unitary character \(\tilde{\chi}_v\) of \(\tilde{\Gamma}_0(N)\) by (2.38) with \(\Gamma = \Gamma_0(N)\). For \(\Gamma = \Gamma_0(N)\) and \(v\), we take shifted lattices \(L, \tilde{L}\) as in the previous subsection, that is, \(L = u + \mathbb{Z}, \tilde{L} = N^{-1}(\tilde{u} + \mathbb{Z})\) with \(0 \leq u, \tilde{u} < 1\) determined by \(v(u(1)) = e^{2\pi \sqrt{-1}u}\) and \(v(\pi(-N)) = e^{2\pi \sqrt{-1}\tilde{u}}\). Let [D1], [D2-1], [D2-2], [D3] and [D4] be the conditions for \(\alpha_i \in \mathfrak{M}(L_i), \beta_i \in \mathfrak{M}(\mathbb{Z}_{\geq 0})\) \((i = 1, 2)\), which are given in \(\S 2.3\) and \(\S 2.7\).

When \(N = 1\), we note \(v(u(1)) = v(\pi(-1))\) and \(L = \tilde{L}\). As a corollary of Theorem 2.10, we obtain the following.

**Corollary 2.17.** Retain the notation and assume \(N = 1\). Let \(\mu \in \kappa + 2\mathbb{Z}\) and \(\nu \in \mathbb{C}\). Let \((\alpha, \beta) \in \mathfrak{M}(L) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})\). Set \(\alpha_1 = \alpha, \beta_1 = \beta, \alpha_2 = v(w)e^{\pi \sqrt{-1}u/2}\alpha\) and \(\beta_2 = v(w)e^{\pi \sqrt{-1}u/2}\beta\) with \(L_1 = L_2 = L\).

(i) Assume \(\lambda_{\alpha,\beta} = (I_{\mu,\nu})_{\tilde{\xi}}\). Then the conditions [D1], [D2-1], [D3] and [D4] hold for the above \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\).

(ii) Assume that [D1], [D2-2], [D3] and [D4] hold for the above \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\). Then \(\lambda_{\alpha,\beta} = (I_{\mu,\nu})_{\tilde{\xi}}\) and \((\lambda_{\alpha,\beta})_{\infty} = v(w)e^{\pi \sqrt{-1}u/2}\lambda_{\alpha,\beta}\). Moreover, we have \(\mathcal{P}_{\nu,\kappa}(\lambda_{\alpha,\beta}) \in \mathcal{M}_{\nu}(\tilde{\Gamma}_0(1)\setminus \tilde{G}; v, \kappa)\) with the Fourier expansion (2.32).

*Proof.* It is wellknown that \(\Gamma_0(1) = SL(2, \mathbb{Z})\) is generated by \(u(1)\) and \(w\). Hence, by (2.39), for \(\lambda \in (I_{\mu,\nu})_{\tilde{\xi}}\), we know that \(\lambda \in (I_{\mu,\nu})_{\tilde{\xi}}\) if and only if

\[
\lambda(\rho(\tilde{u})F) = v(w)\lambda(F) \quad (F \in I_{\mu,\nu}^{\infty}).
\]

This equality is equivalent to \(\lambda_{\infty} = v(w)e^{\pi \sqrt{-1}u/2}\lambda\). Hence, the assertion follows from Theorem 2.10 together with Propositions 2.16 and 2.13 (ii). \(\square\)
Until the end of this subsection, we assume $N > 1$. Let $d$ be a positive integer coprime to $N$. Let $\psi$ be a Dirichlet character modulo $d$. For $m \in \mathbb{Z}$, we define character sums $\tau_\psi(m)$ and $\hat{\tau}_{\psi,v}(m)$ by

$$
\tau_\psi(m) = \sum_{\substack{c \mod d \quad \text{gcd}(c,d) = 1}} \psi(c)e^{2\pi\sqrt{-1}mc/d},
$$

(2.41)

$$
\hat{\tau}_{\psi,v}(m) = \sum_{\substack{c \mod d \quad \text{gcd}(c,d) = 1}} \overline{\psi(c)}v(\gamma)e^{2\pi\sqrt{-1}(mc+\bar{a}c-ub)/d}
$$

(2.42)

with $\gamma = \left( \begin{array}{cc} a & b \\ Nc & d \end{array} \right) \in \Gamma_0(N)$. Here we note that these definitions do not depend on the choice of a reduced residue system modulo $d$ and $\gamma \in \Gamma_0(N)$.

Let $(\alpha, \beta) \in \mathfrak{M}(L) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$ and $(\hat{\alpha}, \hat{\beta}) \in \mathfrak{M}(\hat{L}) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$. Let $(\alpha_\psi, \beta_\psi)$ be an element of $\mathfrak{M}(L) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$ determined by

$$
\alpha_\psi(l) = \tau_\psi(l-u)\alpha(l) \quad (l \in L),
$$

$$
\beta_\psi = \tau_\psi(0)\beta.
$$

Let $(\hat{\alpha}_{\psi,v}, \hat{\beta}_{\psi,v})$ be an element of $\mathfrak{M}(d^{-2}\hat{L}) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$ determined by

$$
\hat{\alpha}_{\psi,v}(d^{-2}\hat{t}) = \psi(-N)\hat{\tau}_{\psi,v}(Nl-\hat{u}) d^{2\nu-1}\hat{\alpha}(l),
$$

$$
\hat{\beta}_{\psi,v}(m) = \psi(-N)\hat{\tau}_{\psi,v}(0) d^{-2\nu-2m-1}
$$

$$
\times \begin{cases} (\hat{\beta}(0) + 4\cos(D\pi)\hat{\alpha}(0) \log d) & \text{if } 0 \in \hat{L} \text{ and } m = \nu = 0, \\
\hat{\beta}(m) & \text{otherwise} 
\end{cases}
$$

for $l \in \hat{L}$ and $m \in \mathbb{Z}_{\geq 0}$. For $i = 1, 2$, we consider the following conditions [W1-i] and [W2-i]$_{d,\psi}$ for $\alpha, \beta, \hat{\alpha}$ and $\hat{\beta}$:

[W1-i] The conditions [D1], [D2-i], [D3] and [D4] hold for $\alpha_1 = \alpha$, $\beta_1 = \beta$,

$$
\alpha_2 = \hat{\alpha} \text{ and } \beta_2 = \hat{\beta} \text{ with } L_1 = L \text{ and } L_2 = \hat{L}.
$$

[W2-i]$_{d,\psi}$ The conditions [D1], [D2-i], [D3] and [D4] hold for $\alpha_1 = \alpha_\psi$, $\beta_1 = \beta_\psi$,

$$
\alpha_2 = \hat{\alpha}_{\psi,v} \text{ and } \beta_2 = \hat{\beta}_{\psi,v} \text{ with } L_1 = L \text{ and } L_2 = d^{-2}\hat{L}.
$$

Let $\mathbb{P}_N$ be a subset of the set of positive odd prime integers not dividing $N$, such that $\mathbb{P}_N \cap \{am + b \mid m \in \mathbb{Z}\} \neq \emptyset$ for any integers $a, b$ coprime to each other (the existence of such $\mathbb{P}_N$ follows from Dirichlet’s theorem on arithmetic progressions).

**Theorem 2.18.** Let $\mu \in \mathbb{R}$, $\nu \in \mathbb{C}$ and $N \in \mathbb{Z}_{\geq 1}$. Let $\kappa \in \mu + 2\mathbb{Z}$. Let $v$ be a multiplier system on $\Gamma_0(N)$ of weight $\kappa$. Let $L = u + \mathbb{Z}$ and $\hat{L} = N^{-1}(\hat{u} + \mathbb{Z})$ with $0 \leq u, \hat{u} < 1$ determined by $v(u(1)) = e^{2\pi\sqrt{-1}Tu}$ and $v(\overline{\mu}(-N)) = e^{2\pi\sqrt{-1}\bar{\mu}u}$. Let $(\alpha, \beta) \in \mathfrak{M}(L) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$ and $(\hat{\alpha}, \hat{\beta}) \in \mathfrak{M}(\hat{L}) \times \mathfrak{M}(\mathbb{Z}_{\geq 0})$.

(i) Assume $\lambda_{\alpha,\beta} \in (I_{\mu,\nu})\hat{\chi}_{\alpha,\beta}$ and $(\lambda_{\alpha,\beta})_{\infty} = \lambda_{\hat{\alpha},\hat{\beta}}$. Then the condition [W1-i] holds. Moreover, the condition [W2-i]$_{d,\psi}$ holds for any $d \in \mathbb{Z}_{\geq 0}$ coprime
to $N$ and any Dirichlet character $\psi$ modulo $d$.

(ii) Assume that $[W1-2]$ holds. Assume furthermore that $[W2-2]$ holds for any $d \in \mathbb{P}_N$ and any Dirichlet character $\psi$ modulo $d$. Then $\lambda_{\alpha,\beta} \in (I_{\mu,\nu}^{\infty})_{\nu}(N,\tilde{\chi})$ and $(\lambda_{\alpha,\beta})_{\infty} = \lambda_{\alpha,\beta}$. Moreover, we have $\mathcal{P}_{v,\kappa}(\lambda_{\alpha,\beta}) \in \mathcal{M}_{\nu}(\Gamma_0(N) \setminus \mathbb{H}; v, \kappa)$ with the Fourier expansion (2.32).

This theorem is proved in §7.3. Theorem 2.18 (ii) is a Weil type converse theorem for automorphic distributions and Maass forms of real weights, which is a generalization of [MSSU, Theorems 4.2 and 4.3].

3 Preparation

3.1 Some lemmas

Lemma 3.1. Let $S$ be a finite subset of $\mathbb{R}$. For $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $f \in C^1(\mathbb{R} \setminus S)$ and $f' \in L^1(\mathbb{R})$, we have

$$(2\pi \sqrt{-1} y) F(f)(y) = -F(f')(y) \quad (y \in \mathbb{R}).$$

Proof. The assertion follows immediately from integration by parts. \hfill \Box

Lemma 3.2. Let $a, b, c, d \in \mathbb{R}$ such that $\{ x \in \mathbb{R} \mid cx + d > 0 \} \neq \emptyset$. Let $s \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$ and $f \in C^n(\mathbb{R})$. Then we have

$$\frac{d^n}{dx^n} \left( (cx + d)^s f \left( \frac{ax + b}{cx + d} \right) \right)$$

$$= \sum_{i=0}^{n} \binom{n}{i} (ad - bc)^i c^{n-i}(s - n + 1)_{n-i}(cx + d)^{s-n-i} f^{(i)} \left( \frac{ax + b}{cx + d} \right)$$

on $\{ x \in \mathbb{R} \mid cx + d > 0 \}$. Here $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the binomial coefficient, and $(z)_i = \Gamma(z + i)/\Gamma(z) = z(z + 1) \cdots (z + i - 1)$ is the Pochhammer symbol.

Proof. We obtain the assertion by induction with respect to $n$. \hfill \Box

3.2 Test functions

We define a function $\Delta$ on $\mathbb{R}$ by

$$\Delta(t) = \begin{cases} \frac{e^{(4 - t^2)}}{e^{(4 - t^2)} + e^{(4 - 1/t^2)}} & \text{if } t \neq 0, \\
1 & \text{if } t = 0, \end{cases} \quad e(t) = \begin{cases} e^{-1/t} & \text{if } t > 0, \\
0 & \text{if } t \leq 0. \end{cases}$$

Then we note that $\Delta$ is a smooth function on $\mathbb{R}$ with $\text{supp}(\Delta) \subset \{ t \in \mathbb{R} \mid |t| \leq 2 \}$ and has the following properties.

(1) $0 \leq \Delta(t) \leq 1$ and $\Delta(t) = \Delta(-t)$ for $t \in \mathbb{R}$.  

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(2) \( \Delta(t) = 1 \) if and only if \( |t| \leq 1/2 \).

(3) \( \Delta(t) + \Delta(-1/t) = 1 \) for \( t \in \mathbb{R}^\times \).

We define a function \( \varphi \) on \( \mathbb{R} \) by \( \varphi(t) = 0 \) \( (t \leq 0) \) and

\[
\varphi(t) = \frac{1}{2\pi \sqrt{-1}} \int_{\text{Re}(s) = \sigma_0} \exp \left( - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 - \sqrt{-1}sr) \sqrt{|r|}}{(\sqrt{-1}r - s)(1 + r^2)} \, dr \right) t^s \, ds \quad (t > 0).
\]

Here the path of the integration \( \int_{\text{Re}(s) = \sigma_0} \) is the vertical line from \( \sigma_0 - \sqrt{-1} \infty \) to \( \sigma_0 + \sqrt{-1} \infty \) with \( \sigma_0 < 0 \).

By the proof of [Ki, Lemma 4.30], we know that \( \varphi \) is a smooth function on \( \mathbb{R} \) with \( \text{supp}(\varphi) \subseteq \{ t \in \mathbb{R} \mid t \geq 1 \} \) and has the following properties.

(1) For \( n \in \mathbb{Z}_{\geq 0} \), the \( n \)-th derivative \( \varphi^{(n)} \) of \( \varphi \) is bounded.

(2) For \( s \in \mathbb{C} \) such that \( \text{Re}(s) < 0 \), we have

\[
\left| \int_0^\infty \varphi(t) t^{s-1} \, dt \right| > \exp \left( -\sqrt{\text{Im}(s)} - \frac{\sqrt{|\text{Re}(s)|}}{\pi} \int_{-\infty}^\infty \frac{\sqrt{|t|}}{1 + t^2} \, dt \right).
\]

Let \( \varepsilon \in \{ \pm 1 \}, \sigma \in \mathbb{R} \) and \( u > 1 \). We define functions \( \varphi_{\varepsilon, \sigma} \in C^\infty(\mathbb{R}) \) and \( \varphi_{\varepsilon, \sigma, u} \in C^\infty(\mathbb{R}^\times) \) by

\[
\varphi_{\varepsilon, \sigma}(x) = |x|^{-\sigma} \varphi(\varepsilon x), \quad \varphi_{\varepsilon, \sigma, u}(x) = |x|^{-\sigma} \varphi(\varepsilon x) \Delta \left( \frac{\varepsilon x}{2u} \right) \quad (x \in \mathbb{R}^\times).
\]

By the above properties of \( \Delta \) and \( \varphi \), we have

\[
\text{supp}(\varphi_{\varepsilon, \sigma, u}) \subseteq \text{supp}(\varphi_{\varepsilon, \sigma}) \subseteq \{ x \in \mathbb{R} \mid \varepsilon x \geq 1 \},
\]

\[
\text{supp}(\varphi_{\varepsilon, \sigma} - \varphi_{\varepsilon, \sigma, u}) \subseteq \{ x \in \mathbb{R} \mid \varepsilon x \geq u \},
\]

\[
(\varphi_{\varepsilon, \sigma})^{(n)}(x) = O(|x|^{-\sigma}) \quad (|x| \to \infty) \quad (n \in \mathbb{Z}_{\geq 0}).
\]

**Lemma 3.3.** Let \( \sigma \in \mathbb{R} \) and \( n \in \mathbb{Z}_{\geq 0} \). There exists \( \hat{c}_{\sigma, n} > 0 \) such that

\[
| (\varphi_{\varepsilon, \sigma} - \varphi_{\varepsilon, \sigma, u})^{(n)}(x) | \leq \hat{c}_{\sigma, n} |x|^{-\sigma} \quad (x \in \mathbb{R}^\times, \varepsilon \in \{ \pm 1 \}, u > 1).
\]

**Proof.** By direct computation, we have

\[
(\varphi_{\varepsilon, \sigma} - \varphi_{\varepsilon, \sigma, u})^{(n)}(x) = \frac{d^n}{dx^n} \left\{ \varphi_{1, \sigma}(\varepsilon x) - \varphi_{1, \sigma}(\varepsilon x) \Delta \left( \frac{\varepsilon x}{2u} \right) \right\}
\]

\[
= \varepsilon^n (\varphi_{1, \sigma})^{(n)}(\varepsilon x) - \varepsilon^n \sum_{i=1}^{n} \binom{n}{i} (\varphi_{1, \sigma})^{(i)}(\varepsilon x) \Delta^{(n-i)} \left( \frac{\varepsilon x}{2u} \right) (2u)^{-n+i}.
\]

Hence, if we put

\[
\hat{c}_{\sigma, n} = \sup_{x \in \mathbb{R}} |x|^\sigma (\varphi_{1, \sigma})^{(n)}(x) + \sum_{i=1}^{n} \binom{n}{i} \sup_{x \in \mathbb{R}} |x|^\sigma (\varphi_{1, \sigma})^{(i)}(x) \sup_{x \in \mathbb{R}} \Delta^{(n-i)}(x),
\]

the inequality (3.4) holds. \( \square \)
Proposition 3.4. Let $\sigma > 1$ and $n \in \mathbb{Z}_{\geq 0}$. There exists $c_{\sigma,n} > 0$ such that
\[
|y|^n |F(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})(y)| \leq c_{\sigma,n} u^{1-\sigma} \quad (y \in \mathbb{R}, \varepsilon \in \{\pm 1\}, u > 1). \tag{3.5}
\]

Proof. By repeated application of Lemma 3.1, we have
\[
(2\pi \sqrt{-1})^n F(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})(y) = (-1)^n F((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})^{(n)})(y).
\]
Hence we have
\[
|y|^n |F(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})(y)| = \frac{1}{(2\pi)^n} \left| F((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})^{(n)})(y) \right|
\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \left| (\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})^{(n)}(x) \right| dx \leq \frac{\tilde{c}_{\sigma,n} u^{1-\sigma}}{(2\pi)^n (\sigma - 1)}.
\]
Here the last inequality follows from (3.2), Lemma 3.3 and
\[
\int_{-\infty}^{\infty} |x|^{-\sigma} dx = \int_{-\infty}^{-u} |x|^{-\sigma} dx + \int_{-\infty}^{u} |x|^{-\sigma} dx = \frac{u^{1-\sigma}}{\sigma - 1}.
\]
Therefore, we obtain (3.5) with $c_{\sigma,n} = \frac{\tilde{c}_{\sigma,n}}{(2\pi)^n (\sigma - 1)}$. \hfill \Box

Proposition 3.5. Let $n \in \mathbb{Z}_{\geq 0}$, $\mu, \nu \in \mathbb{C}$ and $\sigma > 2 \text{Re}(\nu) + 2n$. There exists $c_{\mu,\nu,\sigma,n} > 0$ such that
\[
|y|^n |F((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})(y)| \leq c_{\mu,\nu,\sigma,n} u^{-\sigma + 2 \text{Re}(\nu) + 2n} \quad (y \in \mathbb{R}, \varepsilon \in \{\pm 1\}, u > 1). \tag{3.6}
\]

Proof. By (3.2), we have
\[
\text{supp}((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty}) \subset \{x \in \mathbb{R} \mid 0 \leq -\varepsilon x \leq 1/u\}. \tag{3.7}
\]
For $-\varepsilon x > 0$, we have
\[
(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty}(x) = e^{\varepsilon x \sqrt{1-\mu}/2} (-\varepsilon x)^{-2\nu-1} (\varphi_{1,\sigma} - \varphi_{1,\sigma,u}) \left( \frac{1}{-\varepsilon x} \right).
\]
By Lemma 3.2 with $a = d = 0$, $b = 1$, $c = -\varepsilon$, $s = -2\nu - 1$ and $f = \varphi_{1,\sigma} - \varphi_{1,\sigma,u}$, we have
\[
((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})^{(n)}(x) = e^{\varepsilon x \sqrt{1-\mu}/2} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (-2\nu - n)_{n-i} \times (-\varepsilon x)^{-2\nu-1-n-i} (\varphi_{1,\sigma} - \varphi_{1,\sigma,u})^{(i)} \left( \frac{1}{-\varepsilon x} \right)
\]
for $-\varepsilon x > 0$. By Lemma 3.3, for $0 < -\varepsilon x < 1$, we have
\[
\left| ((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})^{(n)}(x) \right| \leq \tilde{c}_{\mu,\nu,\sigma,n} |x|^\sigma |2\text{Re}(\nu) - 1 - 2n| \tag{3.8}
\]
with
\[ \tilde{c}_{\mu,\nu,n} = e^{\pi \text{Im}(\mu)/2} \sum_{i=0}^{n} \binom{n}{i} \left| (-2\nu - n)_{n-i} \right| \tilde{c}_{\sigma,i}. \]

By repeated application of Lemma 3.1, we have
\[ (2\pi \sqrt{-1})^n \mathcal{F}(\mathcal{F}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})(y) = (-1)^n \mathcal{F}(\mathcal{F}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})^{(n)}(y). \]

Hence we have
\[ |y|^n \left| \mathcal{F}(\mathcal{F}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})(y) \right| = \frac{1}{(2\pi)^n} \left| \mathcal{F}(\mathcal{F}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})^{(n)}(y) \right| \]
\[ \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \left| ((\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty})^{(n)}(x) \right| \, dx \leq \frac{\tilde{c}_{\mu,\nu,n} u^{-\sigma+2\text{Re}(\nu)+2n}}{(2\pi)^n (\sigma - 2\text{Re}(\nu) - 2n)}. \]

Here the last inequality follows from (3.7), (3.8) and
\[ \int_{0}^{1/u} |x|^{\sigma-2\text{Re}(\nu)-1-2n} \, dx = \int_{-1/u}^{0} |x|^{\sigma-2\text{Re}(\nu)-1-2n} \, dx = \frac{u^{-\sigma+2\text{Re}(\nu)+2n}}{\sigma - 2\text{Re}(\nu) - 2n}. \]

Therefore, we obtain (3.6) with \( c_{\mu,\nu,n} = \frac{\tilde{c}_{\mu,\nu,n}}{(2\pi)^n (\sigma - 2\text{Re}(\nu) - 2n)}. \)

\[ \text{Corollary 3.6. Let } \mu, \nu \in \mathbb{C}, \varepsilon \in \{\pm 1\} \text{ and } \sigma > 2\text{Re}(\nu). \text{ For } n \in \mathbb{Z} \text{ such that } \sigma - 2\text{Re}(\nu) > 2n \geq 0, \text{ we have} \]
\[ \mathcal{F}((\varphi_{\varepsilon,\sigma})_{\mu,\nu,\infty})(y) = O(|y|^{-n}) \quad (|y| \to \infty). \]

\[ \text{Proof. Since } \varphi_{\varepsilon,\sigma,u} \in C_0^\infty(\mathbb{R}^\times), \text{ we have } \mathcal{F}((\varphi_{\varepsilon,\sigma,u})_{\mu,\nu,\infty}) \in \mathcal{S}(\mathbb{R}) \text{ for } u > 1. \]

Hence, the assertion follows immediately from Proposition 3.5. \qed

\section{The local zeta functions}

\subsection{The definition of the local zeta functions}

Let \( \varepsilon \in \{\pm 1\} \) and \( f \in C(\mathbb{R}^\times). \) For \( s \in \mathbb{C} \), we set
\[ \Phi_{\varepsilon}(f; s) = \int_{0}^{\infty} f(\varepsilon t)t^{s-1} \, dt = \Phi_{\varepsilon,+}(f; s) + \Phi_{\varepsilon,-}(f; s) \]
with
\[ \Phi_{\varepsilon,+}(f; s) = \int_{1}^{\infty} f(\varepsilon t)t^{s-1} \, dt, \quad \Phi_{\varepsilon,-}(f; s) = \int_{0}^{1} f(\varepsilon t)t^{s-1} \, dt. \]

The integral \( \Phi_{\varepsilon,+}(f; s) \) converges absolutely and defines a holomorphic function on \( \text{Re}(s) < -r \) if there is \( r \in \mathbb{R} \) such that \( f(x) = O(|x|^r) \) \( (|x| \to \infty) \). The integral \( \Phi_{\varepsilon,-}(f; s) \) converges absolutely and defines a holomorphic function on \( \text{Re}(s) > -r \) if there is \( r \in \mathbb{R} \) such that \( f(x) = O(|x|^r) \) \( (|x| \to 0) \). We call \( \Phi_{\varepsilon}(f; s) \) the \textit{local zeta function}.
4.2 The meromorphic continuation of $\Phi_{\pm 1}(f; s)$

Let $\varepsilon \in \{\pm 1\}$ and $f \in \mathcal{S}(\mathbb{R})$. Then $\Phi_{\varepsilon}(f; s)$ converges absolutely and defines a holomorphic function on $\text{Re}(s) > 0$. In this subsection, we give the meromorphic continuation of $\Phi_{\varepsilon}(f; s)$. For $s \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$, we set

$$
\Phi_{\varepsilon, m}^m(f; s) = \int_0^1 \left( f(\varepsilon t) - \sum_{n=0}^{m-1} \frac{\delta^{(n)}(f)}{n!} (-\varepsilon t)^n \right) t^{s-1} dt
$$

with $\delta^{(n)}(f) = (-1)^n f^{(n)}(0)$. Here we note that $\Phi_{\varepsilon, m}^m(0, s) = \Phi_{\varepsilon, -}(f; s)$. Maclaurin’s theorem asserts that, for $m \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, there exists $0 < \theta < 1$ such that

$$
\int_0^x f(t) dt = x f(x) - \sum_{n=0}^{m-1} \frac{(-\varepsilon)^n \delta^{(n)}(f)}{n!} x^n.
$$

(4.1)

Hence, we know that the integral $\Phi_{\varepsilon, m}^m(f; s)$ converges absolutely and defines a holomorphic function on $\text{Re}(s) > -m$.

Using the equality $\frac{1}{s+n} = \int_0^1 t^{s+n-1} dt$, we have

$$
\Phi_{\varepsilon}(f; s) = \Phi_{\varepsilon, +}(f; s) + \Phi_{\varepsilon, -}(f; s) + \sum_{n=0}^{m-1} \left( \frac{(-\varepsilon)^n \delta^{(n)}(f)}{n!} \right) \Phi_{\varepsilon, -}(f; s)
$$

(4.2)

for $m \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$ such that $\text{Re}(s) > 0$. The expression (4.2) gives the meromorphic continuation of $\Phi_{\varepsilon}(f; s)$ to $\text{Re}(s) > -m$. Since a non-negative integer $m$ can be chosen arbitrarily, we have the meromorphic continuation of $\Phi_{\varepsilon}(f; s)$ to the whole $s$-plane. The local zeta function $\Phi_{\varepsilon}(f; s)$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and has poles of order at most 1 at $s = -n$ ($n \in \mathbb{Z}_{\geq 0}$).

4.3 The local zeta distributions

In this subsection, we explain that the local zeta functions can be regarded as distributions. Let $\varepsilon \in \{\pm 1\}$. Let $u > 1$. Then, for $s \in \mathbb{C}$ and $f \in C_0^\infty(\mathbb{R})$ such that $\text{supp}(f) \subset \{x \in \mathbb{R} \mid |x| \leq u\}$, we have

$$
|\Phi_{\varepsilon, +}(f; s)| = \left| \int_1^u f(\varepsilon t) t^{s-1} dt \right| \leq \left( \int_1^u t^{\text{Re}(s)-1} dt \right) \sup_{x \in \mathbb{R}} |f(x)|.
$$

Hence $C_0^\infty(\mathbb{R}) \ni f \mapsto \Phi_{\varepsilon, +}(f; s) \in \mathbb{C}$ is a distribution on $\mathbb{R}$ for $s \in \mathbb{C}$.

Let $m \in \mathbb{Z}_{\geq 0}$. By Maclaurin’s theorem (4.1), we have

$$
|\Phi_{\varepsilon, -}^m(f; s)| = \left| \int_0^1 \left( f(\varepsilon t) - \sum_{n=0}^{m-1} \frac{\delta^{(n)}(f)}{n!} (-\varepsilon t)^n \right) t^{s-1} dt \right|
$$

\[
\leq \frac{1}{m!} \left( \int_0^1 t^{\text{Re}(s)+m-1} dt \right) \sup_{x \in \mathbb{R}} |f^{(m)}(x)| = \frac{1}{m!(\text{Re}(s)+m)} \sup_{x \in \mathbb{R}} |f^{(m)}(x)|
\]
for \( f \in C^\infty_0(\mathbb{R}) \) and \( s \in \mathbb{C} \) such that \( \text{Re}(s) > -m \). Hence \( C^\infty_0(\mathbb{R}) \ni f \mapsto \Phi^\epsilon_{t,-}(f; s) \in \mathbb{C} \) is a distribution on \( \mathbb{R} \) for \( s \in \mathbb{C} \) such that \( \text{Re}(s) > -m \).

Therefore, because of the expression (4.2), we know that \( C^\infty_0(\mathbb{R}) \ni f \mapsto \Phi_{\epsilon}(f; s) \in \mathbb{C} \) is a distribution on \( \mathbb{R} \) for \( s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \).

### 4.4 The local functional equation

In this subsection, we introduce some equations for the local zeta functions. The equation in the following proposition is called the local functional equation.

**Proposition 4.1** ([Ki, Proposition 4.21]). For \( f \in S(\mathbb{R}) \), we have

\[
(\Phi_1(F(f); s), \Phi_{-1}(F(f); s)) = (2\pi)^{-s} \Gamma(s)(\Phi_1(f; -s + 1), \Phi_{-1}(f; -s + 1))E(s).
\]

Here \( E(s) \) is the matrix defined by (2.9).

**Lemma 4.2.** Let \( \epsilon \in \{\pm 1\} \) and \( f \in C_0(\mathbb{R}^\times). \) Then \( \Phi_{\epsilon}(f; s) \) is entire and is bounded on a vertical strip \( \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \) for \( \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \sigma_1 < \sigma_2 \). Moreover, we have the equality

\[
(\Phi_1(f_{\mu,\nu}; s), \Phi_{-1}(f_{\mu,\nu}; s)) = (\Phi_1(f; -s + 2\nu + 1), \Phi_{-1}(f; -s + 2\nu + 1))\Sigma_\mu
\]

for \( \mu, \nu \in \mathbb{C} \). Here \( \Sigma_\mu \) is the matrix defined by (2.10).

**Proof.** The entireness of \( \Phi_{\epsilon}(f; s) \) is obvious. For \( \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \sigma_1 < \sigma_2 \), the function \( \Phi_{\epsilon}(f; s) \) is bounded on a vertical strip \( \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \), since

\[
|\Phi_{\epsilon}(f; s)| = \left| \int_{0}^\infty f(\epsilon t) t^{s-1} dt \right| \leq \int_{1}^\infty |f(\epsilon t)| t^{\sigma_2-1} dt + \int_{0}^{1} |f(\epsilon t)| t^{\sigma_1-1} dt.
\]

By direct computation, we have

\[
\Phi_{\epsilon}(f_{\mu,\nu}; s) = \int_{0}^\infty f_{\mu,\nu}(\epsilon t) t^{s-1} dt = e^{-\epsilon \pi \sqrt{-\mu}/2} \int_{0}^\infty f(-\epsilon t^{-1}) t^{-\nu-2} dt
\]

\[
= e^{-\epsilon \pi \sqrt{-\mu}/2} \Phi_{-\epsilon}(f; -s + 2\nu + 1).
\]

Here the third equality follows from the substitution \( t \to t^{-1} \). This implies that the equality in the statement holds.

### 4.5 The local zeta function for \( \varphi_{\epsilon,\sigma} \)

Let \( \epsilon \in \{\pm 1\} \) and \( \sigma > 1 \). Let \( \varphi_{\epsilon,\sigma} \in C^\infty(\mathbb{R}) \) and \( \varphi_{\epsilon,\sigma,u} \in C^\infty_0(\mathbb{R}^\times) \) \((u > 1)\) be the functions in §3.2. We consider here the local zeta function for \( \varphi_{\epsilon,\sigma} \). By (3.1) and (3.3), the integral \( \Phi_1(\varphi_{1,\sigma}; s) = \Phi_{-1}(\varphi_{-1,\sigma}; s) \) converges absolutely and defines a holomorphic function on \( \text{Re}(s) < \sigma \). By (3.1), we have \( \Phi_{-1}(\varphi_{1,\sigma}; s) = \Phi_1(\varphi_{-1,\sigma}; s) = 0 \) \((s \in \mathbb{C})\). By the property (2) of \( \varphi \) in §3.2, for \( s \in \mathbb{C} \) such that \( \text{Re}(s) < \sigma \), we have

\[
|\Phi_1(\varphi_{1,\sigma}; s)| > \exp \left(-\sqrt{\text{Im}(s)} - \frac{\sqrt{\sigma - \text{Re}(s)}}{\pi} \int_{-\infty}^{\infty} \sqrt{|t|} \frac{dt}{1 + t^2} \right). \quad (4.3)
\]
Hence, for $\varepsilon_1 \in \{\pm 1\}$, the integral $\Phi_{\varepsilon_1}(\mathcal{F}(\varphi_{\varepsilon,\sigma}); s)$ converges absolutely and defines a holomorphic function on $\text{Re}(s) > 0$.

**Lemma 4.3.** Retain the notation. For $s \in \mathbb{C}$ such that $\text{Re}(s) > 0$, we have

$$
\Phi_{\varepsilon_1}(\mathcal{F}(\varphi_{\varepsilon,\sigma}); s), \Phi_{-1}(\mathcal{F}(\varphi_{\varepsilon,\sigma}); s)) = (2\pi)^{-s}\Gamma(s)(\Phi_{\varepsilon_1}(\varphi_{\varepsilon,\sigma}; -s + 1), \Phi_{-1}(\varphi_{\varepsilon,\sigma}; -s + 1))E(s).
$$

**Proof.** Let $\varepsilon_1, \varepsilon \in \{\pm 1\}$ and $u > 1$. Let $s \in \mathbb{C}$ such that $\text{Re}(s) > 0$. By (3.1), we have $\Phi_{-\varepsilon}(\varphi_{\varepsilon,\sigma,u}; s) = 0$. Hence, applying Proposition 4.1 to $f = \varphi_{\varepsilon,\sigma,u}$, we have

$$
\Phi_{\varepsilon_1}(\mathcal{F}(\varphi_{\varepsilon,\sigma,u}); s) = (2\pi)^{-s}\Gamma(s)e^{\varepsilon_1 \pi \sqrt{-1} s/2}\Phi_{\varepsilon}(\varphi_{\varepsilon,\sigma,u}; -s + 1).
$$

Since $|x|^\sigma (|\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u}|(x)) \leq |\varphi(x)|$ for $x \in \mathbb{R}$, we have

$$
|x|\sigma (|\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u}|(x)) \leq c_\varphi \quad (x \in \mathbb{R})
$$

with $c_\varphi = \sup_{x \in \mathbb{R}} |\varphi(x)|$. We take $n \in \mathbb{Z}_{\geq 0}$ so that $\text{Re}(s) < n$. Using (3.2), (4.5), (4.6) and Proposition 3.4, we have

$$
|\Phi_{\varepsilon_1}(\mathcal{F}(\varphi_{\varepsilon,\sigma}); s) - (2\pi)^{-s}\Gamma(s)e^{\varepsilon_1 \pi \sqrt{-1} s/2}\Phi_{\varepsilon}(\varphi_{\varepsilon,\sigma}; -s + 1)|
$$

$$
= \left|\Phi_{\varepsilon_1}(\mathcal{F}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u}); s) - (2\pi)^{-s}\Gamma(s)e^{\varepsilon_1 \pi \sqrt{-1} s/2}\Phi_{\varepsilon}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u}; -s + 1)\right|
$$

$$
\leq \int_0^\infty |\mathcal{F}(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})(\varepsilon_1 t)| t^{\text{Re}(s) - 1} dt
$$

$$
+ (2\pi)^{-\text{Re}(s)}e^{-\varepsilon_1 \pi \text{Im}(s)/2}\left|\Gamma(s)\int_0^\infty |(\varphi_{\varepsilon,\sigma} - \varphi_{\varepsilon,\sigma,u})(\varepsilon t)| t^{-\text{Re}(s)} dt\right|
$$

$$
\leq \left(\int_0^1 c_{\varphi,u}u^{1-\sigma \text{Re}(s)-1} dt + \int_1^{\infty} c_{\varphi,n}u^{1-\sigma \text{Re}(s)-n-1} dt\right)
$$

$$
+ (2\pi)^{-\text{Re}(s)}e^{-\varepsilon_1 \pi \text{Im}(s)/2}\left|\Gamma(s)\int_0^\infty c_\varphi t^{-\text{Re}(s)-\sigma} dt\right|
$$

$$
\leq \left(\frac{c_{\varphi,0}}{\text{Re}(s)} + \frac{c_{\varphi,n}}{n - \text{Re}(s)}\right) u^{1-\sigma} + \frac{c_\varphi(2\pi)^{-\text{Re}(s)}e^{-\varepsilon_1 \pi \text{Im}(s)/2}\left|\Gamma(s)\right| u^{-\text{Re}(s)-\sigma+1}}{\text{Re}(s) + \sigma - 1}.
$$

Since this inequality holds for any $u > 1$, we have

$$
\Phi_{\varepsilon_1}(\mathcal{F}(\varphi_{\varepsilon,\sigma}); s) = (2\pi)^{-s}\Gamma(s)e^{\varepsilon_1 \pi \sqrt{-1} s/2}\Phi_{\varepsilon}(\varphi_{\varepsilon,\sigma}; -s + 1).
$$

This implies that the equality in the statement holds.

5 Dirichlet series

In this section, we give a proof of Theorem 2.2.
5.1 Definition of zeta integrals

Let $L$ be a shifted lattice in $\mathbb{R}$. Let $r \in \mathbb{R}$, $\alpha \in \mathfrak{M}_r(L)$ and $\sigma > r + 1$. Let $f$ be a function in $C(\mathbb{R}^\times)$ satisfying the following condition:

$$x \mapsto |x|^{\sigma} f(x) \text{ is a bounded function on } \mathbb{R}^\times.$$  \hspace{1cm} (5.1)

It is convenient to introduce a series

$$\vartheta(\alpha, f; t) = \sum_{0 \neq l \in L} \alpha(l) f(tl) \quad (t > 0).$$

Lemma 5.1. We take $L$, $r$, $\alpha$ and $\sigma$ as above. For $f \in C(\mathbb{R}^\times)$ satisfying (5.1), the series $\vartheta(\alpha, f; t)$ converges absolutely and defines a continuous function on $\mathbb{R}_{>0}$. Moreover, there is a constant $C_{\alpha, \sigma} > 0$ such that

$$\sum_{0 \neq l \in L} |\alpha(l) f(tl)| \leq C_{\alpha, \sigma} \sup_{x \in \mathbb{R}^\times} (|x|^{\sigma} |f(x)|) t^{-\sigma}$$  \hspace{1cm} (5.2)

$t > 0$, $f \in C(\mathbb{R}^\times)$ satisfying (5.1).

Proof. Since $\alpha \in \mathfrak{M}_r(L)$, there is $C_{\alpha} > 0$ such that $|\alpha(l)| \leq C_{\alpha} |l|^r$ $(0 \neq l \in L)$. Hence, for $t > 0$ and $0 \neq l \in L$, we have

$$|\alpha(l) f(tl)| \leq C_{\alpha} |l|^r \sup_{x \in \mathbb{R}^\times} (|x|^{\sigma} |f(x)|) |t|^\sigma \leq C_{\alpha} |l|^{-(\sigma - r)} \sup_{x \in \mathbb{R}^\times} (|x|^{\sigma} |f(x)|) t^{-\sigma}.$$ 

Since $\sum_{0 \neq l \in L} |l|^{-(\sigma - r)}$ converges, we obtain the assertion.  \hspace{1cm} $\square$

For $s \in \mathbb{C}$, we define the zeta integral $Z(\alpha, f; s)$ by

$$Z(\alpha, f; s) = \int_0^\infty \vartheta(\alpha, f; t) t^{s-1} dt = Z_+(\alpha, f; s) + Z_-(\alpha, f; s)$$

with

$$Z_+(\alpha, f; s) = \int_1^\infty \vartheta(\alpha, f; t) t^{s-1} dt, \quad Z_-(\alpha, f; s) = \int_0^1 \vartheta(\alpha, f; t) t^{s-1} dt.$$ 

Lemma 5.2. We take $L$, $r$, $\alpha$ and $\sigma$ as above. Let $f$ be a function in $C(\mathbb{R}^\times)$ satisfying (5.1). We take $C_{\alpha, \sigma} > 0$ as in Lemma 5.1.

(i) The integral $Z_+(\alpha, f; s)$ converges absolutely and defines a holomorphic function on $\text{Re}(s) < \sigma$. For $s \in \mathbb{C}$ such that $\text{Re}(s) < \sigma$, we have

$$|Z_+(\alpha, f; s)| \leq \frac{C_{\alpha, \sigma}}{\sigma - \text{Re}(s)} \sup_{x \in \mathbb{R}^\times} (|x|^{\sigma} |f(x)|).$$

(ii) The integral $Z_-(\alpha, f; s)$ converges absolutely and defines a holomorphic function on $\text{Re}(s) > \sigma$. For $s \in \mathbb{C}$ such that $\text{Re}(s) > \sigma$, we have

$$|Z_-(\alpha, f; s)| \leq \frac{C_{\alpha, \sigma}}{\text{Re}(s) - \sigma} \sup_{x \in \mathbb{R}^\times} (|x|^{\sigma} |f(x)|).$$
Lemma 5.5. We take $L, r$ and $\alpha$ as above. Let $\sigma_2 > \sigma_1 > r + 1$. Let $f$ be a function in $C(\mathbb{R}^\infty)$ such that $x \mapsto |x|^r f(x)$ is a bounded function on $\mathbb{R}^\infty$ for $i \in \{1, 2\}$. Then the zeta integral $Z(\alpha, f; s)$ converges absolutely and defines a holomorphic function on $\sigma_1 < \text{Re}(s) < \sigma_2$. Moreover, for $s \in \mathbb{C}$ such that $\sigma_1 < \text{Re}(s) < \sigma_2$, we have

$$Z(\alpha, f; s) = (\Phi_1(f; s), \Phi_{-1}(f; s)) \left( \frac{\xi_+(\alpha; s)}{\xi_-(\alpha; s)} \right).$$

Proof. The former part of the statement follows immediately from Lemma 5.2. The equality (5.3) is obtained as follows:

$$Z(\alpha, f; s) = \int_0^\infty \vartheta(\alpha, f; t)t^s \frac{dt}{t} = \int_0^\infty \sum_{0 \neq l \in L} \alpha(l)(t) t^s \frac{dt}{t}$$

$$= \sum_{0 < l \in L} \alpha(l) \int_0^\infty f(t) t^s \frac{dt}{t} + \sum_{0 > l \in L} \alpha(l) \int_0^\infty f(t) t^s \frac{dt}{t}$$

$$= \xi_+(\alpha; s)\Phi_1(f; s) + \xi_- (\alpha; s) \Phi_{-1}(f; s) = (\Phi_1(f; s), \Phi_{-1}(f; s)) \left( \frac{\xi_+(\alpha; s)}{\xi_-(\alpha; s)} \right).$$

Here the fourth equality follows from the substitution $t \rightarrow t|l|^{-1}$. □

5.2 Functional equations

In this subsection, we give a proof of the following proposition, which is the former part of Theorem 2.2 (i).

Proposition 5.4. Let $L_1$ and $L_2$ be two shifted lattices in $\mathbb{R}$. Let $\mu, \nu \in \mathbb{C}$. Let $(T_{\alpha_1}, T_{\alpha_2}) \in A(L_1, L_2; J_{\mu, \nu})$ with $\alpha_1 \in \mathbb{M}(L_1)$ and $\alpha_2 \in \mathbb{M}(L_2)$. Then $\Xi_\pm(\alpha_1; s)$ and $\Xi_\pm(\alpha_2; s)$ satisfy the condition [D1] in §2.3.

In order to prove this proposition, we prepare some lemmas. For $t > 0$ and a function $f$ on $\mathbb{R}$ or $\mathbb{R}^\infty$, we set $f_{[t]}(x) = f(tx)$.

Lemma 5.5. Let $\mu, \nu \in \mathbb{C}$, $y \in \mathbb{R}$ and $t > 0$.

(i) $\mathcal{F}(f)(ty) = t^{-1} \mathcal{F}(f_{[t^{-1}]})(y)$ for $f \in L^1(\mathbb{R})$.

(ii) $(f_{[t^{-1}]})_{\mu, \nu, \infty} = t^{2\nu+1}(f_{\mu, \nu, \infty})_{[t]}$ for $f \in C(\mathbb{R}^\infty)$.

Proof. For $f \in L^1(\mathbb{R})$, we have

$$\mathcal{F}(f)(ty) = \int_{-\infty}^{\infty} f(x)e^{2\pi i tx} dx = t^{-1} \mathcal{F}(f_{[t^{-1}]})(y).$$

Here the second equality follows from the substitution $x \rightarrow t^{-1}x$. For $x \in \mathbb{R}^\infty$ and $f \in C(\mathbb{R}^\infty)$, we have

$$t^{2\nu+1}(f_{\mu, \nu, \infty})_{[t]}(x) = e^{-\text{sgn}(x)\pi \sqrt{-t} \mu / 2} |x|^{-2\nu - 1} f(-1/(tx)) = (f_{[t^{-1}]})_{\mu, \nu, \infty}(x).$$

Here the first equality follows from $\text{sgn}(tx) = \text{sgn}(x)$. □
Lemma 5.6. We use the notation in Proposition 5.4. Then we have
\[
\sum_{l \in L_1} \alpha_1(l)F(f)(tl) = t^{2\nu-1} \sum_{l \in L_2} \alpha_2(l)F(f_{\mu,\nu,\infty})(t^{-1}l) \quad (t > 0, \ f \in C_0^\infty(\mathbb{R}^\times)).
\]

Proof. By Lemma 5.5 and the relation (2.6), we have
\[
\sum_{l \in L_1} \alpha_1(l)F(f)(tl) = t^{-1} \sum_{l \in L_1} \alpha_1(l)F(f_{l^{-1}})(l) = t^{-1}T_{\alpha_1}(f_{l^{-1}}) = t^{-1}T_{\alpha_2}(f_{(\mu,\nu,\infty)}) = t^{2\nu}T_{\alpha_2}(f_{(\mu,\nu,\infty)}) = t^{2\nu} \sum_{l \in L_2} \alpha_2(l)F((f_{\mu,\nu,\infty})(t))(l) = t^{2\nu-1} \sum_{l \in L_2} \alpha_2(l)F(f_{\mu,\nu,\infty})(t^{-1}l)
\]
for \( t > 0 \) and \( f \in C_0^\infty(\mathbb{R}^\times) \). Hence, we obtain the assertion. \( \square \)

Lemma 5.7. We use the notation in Proposition 5.4. Let \( f \in C_0^\infty(\mathbb{R}^\times) \). Take \( r \in \mathbb{R} \) so that \( \alpha_1 \in \mathfrak{N}_r(L_1) \). Then, for \( s \in \mathbb{C} \) such that \( \Re(s) > \max\{r+1,0\} \), we have
\[
Z(\alpha_1, F(f); s) = Z_+(\alpha_1, F(f); s) + Z_+(\alpha_2, F(f_{\mu,\nu,\infty}); -s - 2\nu + 1) - \frac{\alpha_1(0)F(f)(0)}{s} + \frac{\alpha_2(0)F(f_{\mu,\nu,\infty})(0)}{s + 2\nu - 1}.
\]

Proof. By Lemma 5.6, we have
\[
\vartheta(\alpha_1, F(f); t) = t^{2\nu-1}\vartheta(\alpha_2, F(f_{\mu,\nu,\infty}); t^{-1}) - \alpha_1(0)F(f)(0) + t^{2\nu-1}\alpha_2(0)F(f_{\mu,\nu,\infty})(0).
\] (5.4)

If \( \Re(s) \) is sufficiently large, we have
\[
Z_-(\alpha_1, F(f); s) = \int_0^1 \vartheta(\alpha_1, F(f); t)t^s dt/t
\]
\[
= \int_0^1 \vartheta(\alpha_2, F(f_{\mu,\nu,\infty}); t^{-1})t^{s+2\nu-1} dt/t
\]
\[
- \int_0^s \alpha_1(0)F(f)(0)t^s dt/t + \int_0^1 \alpha_2(0)F(f_{\mu,\nu,\infty})(0)t^{s+2\nu-1} dt/t
\]
\[
= Z_+(\alpha_2, F(f_{\mu,\nu,\infty}); -s - 2\nu + 1) - \frac{\alpha_1(0)F(f)(0)}{s} + \frac{\alpha_2(0)F(f_{\mu,\nu,\infty})(0)}{s + 2\nu - 1}.
\]

Here the second equality follows from (5.4), and the third equality follows from the substitution \( t \to t^{-1} \) in the first term. The assertion follows from this equality and Lemma 5.2. \( \square \)

Proof of Proposition 5.4. Let \( f \in C_0^\infty(\mathbb{R}^\times) \). By Lemma 5.7, we have
\[
Z(\alpha_1, F(f); s) = Z_+(\alpha_1, F(f); s) + Z_+(\alpha_2, F(f_{\mu,\nu,\infty}); -s - 2\nu + 1) - \frac{\alpha_1(0)F(f)(0)}{s} + \frac{\alpha_2(0)F(f_{\mu,\nu,\infty})(0)}{s + 2\nu - 1}
\] (5.5)

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for \( s \in \mathbb{C} \) such that \( \text{Re}(s) \) is sufficiently large. Since the first 2 terms on the right-hand side of (5.5) are entire functions by Lemma 5.2 (i), the expression (5.5) gives the meromorphic continuation of \( Z(\alpha_1, \mathcal{F}(f); s) \) to the whole \( s \)-plane. By Lemma 5.3 and Proposition 4.1, we have

\[
Z(\alpha_1, \mathcal{F}(f); s) = (\Phi_1(f; -s + 1), \Phi_{-1}(f; -s + 1)) E(s) \left( \begin{array}{c}
\Xi_+ (\alpha_1; s) \\
\Xi_- (\alpha_1; s)
\end{array} \right) \tag{5.6}
\]

for \( s \in \mathbb{C} \) such that \( \text{Re}(s) \) is sufficiently large. Let \( s_0 \in \mathbb{C} \). For \( \varepsilon \in \{\pm 1\} \), we define the function \( \Delta_{s_0, \varepsilon} \) in \( C_0^\infty (\mathbb{R}^\times) \) by

\[
\Delta_{s_0, \varepsilon}(x) = |x|^{s_0} \Delta(\varepsilon x - 3) \quad (x \in \mathbb{R}^\times),
\]

where \( \Delta \) is the function defined in §3.2. Then \( \Phi_1(\Delta_{s_0, 1}; -s_0 + 1) > 0 \) and

\[
\left( \begin{array}{cc}
\Phi_1(\Delta_{s_0, 1}; s) & \Phi_{-1}(\Delta_{s_0, 1}; s) \\
\Phi_1(\Delta_{s_0, -1}; s) & \Phi_{-1}(\Delta_{s_0, -1}; s)
\end{array} \right) = \Phi_1(\Delta_{s_0, 1}; s) |_{2} \tag{5.7}
\]

for \( s \in \mathbb{C} \). Hence, we have

\[
\Phi_1(\Delta_{s_0, 1}; -s + 1) E(s) \left( \begin{array}{c}
\Xi_+ (\alpha_1; s) \\
\Xi_- (\alpha_1; s)
\end{array} \right) = \left( \begin{array}{c}
Z(\alpha_1, \mathcal{F}(\Delta_{s_0, 1}); s) \\
Z(\alpha_1, \mathcal{F}(\Delta_{s_0, -1}); s)
\end{array} \right),
\]

and \( \Phi_1(\Delta_{s_0, 1}; -s + 1) \neq 0 \) on a sufficiently small neighborhood of \( s_0 \). Since \( s_0 \) can be chosen arbitrarily, we obtain the meromorphic continuations of \( \Xi_+ (\alpha_1; s) \) and \( \Xi_- (\alpha_1; s) \) to the whole \( s \)-plane. Moreover, since

\[
\mathcal{F}(\Delta_{0,1})(0) = \mathcal{F}(\Delta_{0,-1})(0) = \Phi_1(\Delta_{0,1}; 1),
\]

\[
\mathcal{F}((\Delta_{2\nu + 1, \varepsilon})_{\mu, \nu, \infty})(0) = e^{\pi \sqrt{1 - \mu^2} \nu} \Phi_1(\Delta_{-2\nu + 1, \varepsilon}; 2\nu) \quad (\varepsilon \in \{\pm 1\}),
\]

we know that (2.11) is entire.

Since \((T_{\alpha_1}, T_{\alpha_2}) \in \mathcal{A}(L_1, L_2; J_{\mu, \nu})\) if and only if \((T_{\alpha_2}, T_{\alpha_1}) \in \mathcal{A}(L_2, L_1; J_{\mu, \nu})\), by Lemma 5.7, we note that \( Z(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); -s - 2\nu + 1) \) is equal to the right-hand side of (5.5). Hence we obtain the equality

\[
Z(\alpha_1, \mathcal{F}(f); s) = Z(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); -s - 2\nu + 1). \tag{5.8}
\]

By Lemmas 4.2, 5.3 and Proposition 4.1, we have

\[
Z(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); s)
= (\Phi_1(f; s + 2\nu), \Phi_{-1}(f; s + 2\nu)) \Sigma_\mu E(s) \left( \begin{array}{c}
\Xi_+ (\alpha_2; s) \\
\Xi_- (\alpha_2; s)
\end{array} \right). \tag{5.9}
\]

By (5.6), (5.8) and (5.9) with \( f = \Delta_{s_0, \varepsilon} \) (\( s_0 \in \mathbb{C}, \varepsilon \in \{\pm 1\} \)) and (5.7), we obtain the functional equation in [D1]. \( \square \)
5.3 The estimate on vertical strips

In this subsection, we complete a proof of Theorem 2.2 (i). We use the test functions $\varphi_{\varepsilon, \sigma} \in C^\infty(\mathbb{R})$ and $\varphi_{\varepsilon, \sigma, u} \in C^\infty_0(\mathbb{R}^\times)$ in §3.2.

**Lemma 5.8.** We use the notation in Proposition 5.4. Let $r$ be a real number such that $\alpha_1 \in \mathcal{M}_r(L_1)$ and $\alpha_2 \in \mathcal{M}_r(L_2)$. Let $\varepsilon \in \{\pm 1\}$, and take $\sigma > 1$ and $n \in \mathbb{Z}$ so that $\sigma - 2 \text{Re}(\nu) > 2n \geq 2 \max\{r + 1, 0\}$. For $s \in \mathbb{C}$ such that $\text{Re}(s) > \max\{n, -n - 2\text{Re}(\nu) + 1\}$, we have

$$Z(\alpha_1, F(\varphi_{\varepsilon, \sigma}); s) = Z(\alpha_1, F(\varphi_{\varepsilon, \sigma}); s) + Z(\alpha_2, F((\varphi_{\varepsilon, \sigma})_{\mu, \nu, \infty}); -s - 2\nu + 1)$$

$$- \frac{\alpha_1(0)F(\varphi_{\varepsilon, \sigma}(0))}{s} + \frac{\alpha_2(0)F((\varphi_{\varepsilon, \sigma})_{\mu, \nu, \infty}(0))}{s + 2\nu - 1}.$$  

**Proof.** By (4.4), Corollary 3.6 and Lemma 5.2, we know that the integrals $Z_+(\alpha_1, F(\varphi_{\varepsilon, \sigma}); s)$, $Z_-(\alpha_1, F(\varphi_{\varepsilon, \sigma}); s)$ and $Z_+(\alpha_2, F((\varphi_{\varepsilon, \sigma})_{\mu, \nu, \infty}); s)$ converge absolutely and define holomorphic functions on the whole $s$-plane, $\text{Re}(s) > n$ and $\text{Re}(s) < n$, respectively. Let $u > 1$. Because of the proof of Lemma 5.7, for $s \in \mathbb{C}$ such that $\text{Re}(s) > \max\{r + 1, 0\}$, we have

$$Z_-(\alpha_1, F(\varphi_{\varepsilon, \sigma, u}); s) = Z_+(\alpha_2, F((\varphi_{\varepsilon, \sigma, u})_{\mu, \nu, \infty}); -s - 2\nu + 1)$$

$$- \frac{\alpha_1(0)F(\varphi_{\varepsilon, \sigma}(0))}{s} + \frac{\alpha_2(0)F((\varphi_{\varepsilon, \sigma, u})_{\mu, \nu, \infty}(0))}{s + 2\nu - 1}.$$  

(5.10)

Let $s \in \mathbb{C}$ such that $\text{Re}(s) > \max\{n, -n - 2\text{Re}(\nu) + 1\}$. Then we have

$$\left| Z_-(\alpha_1, F(\varphi_{\varepsilon, \sigma}); s) - Z_+(\alpha_2, F((\varphi_{\varepsilon, \sigma})_{\mu, \nu, \infty}); -s - 2\nu + 1) \right|$$

$$+ \left| \frac{\alpha_1(0)F(\varphi_{\varepsilon, \sigma}(0))}{s} - \frac{\alpha_2(0)F((\varphi_{\varepsilon, \sigma})_{\mu, \nu, \infty}(0))}{s + 2\nu - 1} \right|$$

$$\leq \frac{C_{\alpha_1, n}}{\text{Re}(s) - n} \sup_{x \in \mathbb{R}^\times} |x|^n |F(\varphi_{\varepsilon, \sigma, u})(x)|$$

$$+ \frac{C_{\alpha_2, n}}{n - \text{Re}(-s - 2\nu + 1)} \sup_{x \in \mathbb{R}^\times} |x|^n |F((\varphi_{\varepsilon, \sigma} - \varphi_{\varepsilon, \sigma, u})_{\mu, \nu, \infty})(x)|$$

$$+ \frac{|a_1(0)|}{|s|} |F(\varphi_{\varepsilon, \sigma} - \varphi_{\varepsilon, \sigma, u})(0)| + \frac{|a_2(0)|}{|s + 2\nu - 1|} |F((\varphi_{\varepsilon, \sigma} - \varphi_{\varepsilon, \sigma, u})_{\mu, \nu, \infty}(0)|$$

$$\leq \frac{C_{\alpha_1, n} c_{\sigma, n} u^{1 - \sigma}}{\text{Re}(s) - n} + \frac{C_{\alpha_2, n} c_{\mu, \nu, \sigma, n} u^{-(\sigma - 2\text{Re}(\nu) - 2n)}}{n - \text{Re}(-s - 2\nu + 1)}$$

$$+ \frac{|a_1(0)|}{|s|} c_{\sigma, 0} u^{1 - \sigma} + \frac{|a_2(0)|}{|s + 2\nu - 1|} c_{\mu, \nu, \sigma, 0} u^{-(\sigma - 2\text{Re}(\nu))}.$$
Here the first equality follows from (5.10), the first inequality follows from Lemma 5.2, and the second inequality follows from Propositions 3.4 and 3.5. Since this inequality holds for any $u > 1$, we have
\[
Z_-(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s)) = Z_+(\alpha_2, \mathcal{F}((\varphi_{\epsilon, \sigma})_{\mu, \nu, \infty}; -s - 2\nu + 1)) - \frac{\alpha_1(0)\mathcal{F}(\varphi_{\epsilon, \sigma}; 0)}{s} + \frac{\alpha_2(0)\mathcal{F}((\varphi_{\epsilon, \sigma})_{\mu, \nu, \infty}; 0)}{s + 2\nu - 1}.
\]
By this equality, we obtain the assertion. \hfill \Box

**Proof of Theorem 2.2 (i).** Because of Proposition 5.4, it suffices to prove that the meromorphically continued function $\Xi_+(\alpha_1; s)$ satisfies the condition [D2-1] in §2.3. Let $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 < \sigma_2$. We take $r \in \mathbb{R}$ so that $\alpha_1 \in \mathcal{M}_r(L_1)$ and $\alpha_2 \in \mathcal{M}_r(L_2)$. Let $\varepsilon \in \{\pm 1\}$.

By Lemma 5.3 and Lemma 4.3, for $s \in \mathbb{C}$ such that $\text{Re}(s) > n$, we have
\[
Z(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s)) = (\Phi_1(\mathcal{F}(\varphi_{\epsilon, \sigma}; s)), \Phi_{-1}(\mathcal{F}(\varphi_{\epsilon, \sigma}; s))) \left( \begin{array}{c} \xi_+(\alpha_1; s) \\ \xi_-(\alpha_1; s) \end{array} \right)
= (\Phi_1(\varphi_{\epsilon, \sigma}; -s + 1), \Phi_{-1}(\varphi_{\epsilon, \sigma}; -s + 1)) E(s) \left( \begin{array}{c} \Xi_+(\alpha_1; s) \\ \Xi_-(\alpha_1; s) \end{array} \right).
\]
Since $\Phi_1(\varphi_{\epsilon, \sigma}; s) = \Phi_{-1}(\varphi_{-1, \sigma}; s)$ and $\Phi_{-1}(\varphi_{1, \sigma}; s) = \Phi_1(\varphi_{1, \sigma}; s) = 0$, we have
\[
\left( \begin{array}{c} \Xi_+(\alpha_1; s) \\ \Xi_-(\alpha_1; s) \end{array} \right) = \frac{1}{\Phi_1(\varphi_{1, \sigma}; -s + 1)} E(s)^{-1} \left( \begin{array}{c} Z(\alpha_1, \mathcal{F}(\varphi_{1, \sigma}; s)) \\ Z(\alpha_1, \mathcal{F}(\varphi_{-1, \sigma}; s)) \end{array} \right). \tag{5.11}
\]
By Lemma 5.8, we have
\[
Z(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s)) = Z_+(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s)) + Z_+(\alpha_2, \mathcal{F}((\varphi_{\epsilon, \sigma})_{\mu, \nu, \infty}; -s - 2\nu + 1)) - \frac{\alpha_1(0)\mathcal{F}(\varphi_{\epsilon, \sigma}; 0)}{s} + \frac{\alpha_2(0)\mathcal{F}((\varphi_{\epsilon, \sigma})_{\mu, \nu, \infty}; 0)}{s + 2\nu - 1}
\]
for $s \in \mathbb{C}$ such that $\text{Re}(s) > \max\{n, -n - 2\text{Re}(\nu) + 1\}$. This expression gives the meromorphic continuation of $Z(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s))$ to $\text{Re}(s) > -n - 2\text{Re}(\nu) + 1$, and the inequality
\[
|Z(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s))| \leq \int_1^\infty |\vartheta(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; t))| t^\sigma_2 \frac{dt}{t} + \int_1^\infty |\vartheta(\alpha_2, \mathcal{F}((\varphi_{\epsilon, \sigma})_{\mu, \nu, \infty}; t))| t^{-\sigma_1 - 2\text{Re}(\nu) + 1} \frac{dt}{t} + \left| \frac{\alpha_1(0)\mathcal{F}(\varphi_{\epsilon, \sigma}; 0)}{s} \right| + \left| \frac{\alpha_2(0)\mathcal{F}((\varphi_{\epsilon, \sigma})_{\mu, \nu, \infty}; 0)}{s + 2\nu - 1} \right|
\]
for $s \in \mathbb{C}$ such that $\sigma_1 \leq \text{Re}(s) \leq \sigma_2$. Hence, we have
\[
Z(\alpha_1, \mathcal{F}(\varphi_{\epsilon, \sigma}; s)) = O(1) \quad (|s| \to \infty) \quad \text{uniformly on } \sigma_1 \leq \text{Re}(s) \leq \sigma_2.
\]
By (4.3), (5.11) and this estimate, we obtain the assertion. \hfill \Box

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5.4 The converse theorem

In this subsection, we give a proof of Theorem 2.2 (ii).

**Lemma 5.9.** We use the notation in Theorem 2.2 (ii). Then, for any $f \in C_c^\infty(\mathbb{R}^\times)$, the zeta integrals $Z(\alpha_1, F(f); s)$ and $Z(\alpha_2, F(f_{\mu,\nu,\infty}); s)$ satisfy the following conditions $[Z1]$ and $[Z2]$: 

$[Z1]$ The zeta integral $Z(\alpha_1, F(f); s)$ has the meromorphic continuation to the whole $s$-plane, and the function 

$$Z(\alpha_1, F(f); s) + \frac{\alpha_1(0)F(f)(0)}{s} - \frac{\alpha_2(0)F(f_{\mu,\nu,\infty})(0)}{s + 2\nu - 1}$$

is entire. Moreover, $Z(\alpha_1, F(f); s)$ and $Z(\alpha_2, F(f_{\mu,\nu,\infty}); s)$ satisfy the functional equation 

$$Z(\alpha_1, F(f); s) = Z(\alpha_2, F(f_{\mu,\nu,\infty}); -s - 2\nu + 1).$$

$[Z2]$ For any $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 < \sigma_2$, there is some $c_0 \in \mathbb{R}_{>0}$ such that 

$$Z(\alpha_1, F(f); s) = O(e^{c_0|s|}) \quad (|s| \to \infty) \quad \text{uniformly on } \sigma_1 \leq \text{Re}(s) \leq \sigma_2.$$ 

**Proof.** By Lemmas 4.2, 5.3 and Proposition 4.1, we have 

$$Z(\alpha_1, F(f); s) = (\Phi_1(f; -s + 1), \Phi_{-1}(f; -s + 1))E(s)\left(\frac{\Xi_1(\alpha_1; s)}{\Xi_2(\alpha_1; s)}\right),$$

$$Z(\alpha_2, F(f_{\mu,\nu,\infty}); s) = (\Phi_1(f; s + 2\nu), \Phi_{-1}(f; s + 2\nu))\Sigma_{\mu}E(s)\left(\frac{\Xi_1(\alpha_2; s)}{\Xi_2(\alpha_2; s)}\right)$$

for $s \in \mathbb{C}$ with a sufficiently large real part. Using these equalities, Lemma 4.2 and $F(f)(0) = \Phi_1(f; 1) + \Phi_{-1}(f; 1)$, we obtain $[Z1]$ and $[Z2]$ from the conditions $[D1]$ and $[D2-2]$ in §2.3. 

---

We recall the Mellin inversion formula. For a continuous function $\vartheta$ on $\mathbb{R}_{>0}$, we define the Mellin transform $\mathcal{M}(\vartheta)$ by 

$$\mathcal{M}(\vartheta)(s) = \int_0^\infty \vartheta(t)t^{s-1}dt \quad (s \in \mathbb{C}).$$

**Lemma 5.10 (The Mellin inversion formula).** Let $\vartheta$ be a continuous function on $\mathbb{R}_{>0}$. Assume that there exists $\sigma \in \mathbb{R}$ such that 

$$\int_0^\infty |\vartheta(t)|t^{\sigma-1}dt < \infty, \quad \int_{\text{Re}(s) = \sigma}|\mathcal{M}(\vartheta)(s)|ds < \infty. \quad (5.14)$$

Here the path of the integration $\int_{\text{Re}(s) = \sigma}$ is the vertical line from $\sigma - \sqrt{-1}\infty$ to $\sigma + \sqrt{-1}\infty$. Then we have $\vartheta(t) = \frac{1}{2\pi\sqrt{-1}}\int_{\text{Re}(s) = \sigma}\mathcal{M}(\vartheta)(s)t^{-s}ds \quad (t > 0).$
Proof. We derive this lemma from the Fourier inversion formula (See, for example, [Bu, §1.5]).

Lemma 5.11. Let $L$ be a shifted lattice in $\mathbb{R}$. Let $r \in \mathbb{R}$, $\alpha \in \mathfrak{M}_{r}(L)$, $f \in \mathcal{S}(\mathbb{R})$ and $\sigma > \max\{0, r + 1\}$. Then

$$Z(\alpha, f; s) = O(|s|^{-\alpha}) \quad (|s| \to \infty) \quad \text{on } \Re(s) = \sigma \quad (5.15)$$

for any $n \in \mathbb{Z}_{\geq 0}$. Moreover, we have

$$\vartheta(\alpha, f; t) = \frac{1}{2\pi \sqrt{-1}} \int_{\Re(s) = \sigma} Z(\alpha, f; s) t^{-s} \, ds \quad (t > 0). \quad (5.16)$$

Proof. We prove (5.15) by induction with respect to $n$. Let $\sigma > \max\{0, r + 1\}$. When $n = 0$, the estimate (5.15) holds since

$$|Z(\alpha, f; s)| \leq \int_{0}^{\infty} |\vartheta(\alpha, f; t)| t^{\sigma} \frac{dt}{t}$$

for $s \in \mathbb{C}$ such that $\Re(s) = \sigma$. If $\Re(s) = \sigma$, we have

$$Z(\alpha, f; s) = \int_{0}^{\infty} \sum_{\lambda \not\in L} \alpha(l) f(\lambda l) t^{\lambda - 1} \, dt = \int_{0}^{\infty} \sum_{\lambda \not\in L} \alpha(l) f(\lambda l) \left( \frac{d}{dt} t^{\lambda} \right) \, dt$$

$$= -\int_{0}^{\infty} \sum_{\lambda \not\in L} \alpha(l) f(\lambda l) \frac{t^{\lambda}}{s} \, dt = -\frac{1}{s} Z(\alpha', f'; s + 1) \quad (5.17)$$

with $\alpha' \in \mathfrak{M}_{r+1}(L)$ defined by $\alpha'(l) = \lambda l(l \in L)$. Here the third equality follows from integration by parts. From the expression (5.17), we know that (5.15) for $n = n_0 + 1$ holds if (5.15) for $n = n_0$ holds. This complete the proof of (5.15) for all $n \in \mathbb{Z}_{\geq 0}$.

Because of Lemma 5.1 and the estimate (5.15), we note that $\vartheta(t) = \vartheta(\alpha, f; t)$ satisfies the condition (5.14). Since $\mathcal{M}(\vartheta)(s) = Z(\alpha, f; s)$, we obtain (5.16) by the Mellin inversion formula (Lemma 5.10).

Lemma 5.12 (The Phragmen–Lindelöf theorem). Let $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 < \sigma_2$. Let $\Theta(s)$ be a holomorphic function on a domain containing a vertical strip $\sigma_1 \leq \Re(s) \leq \sigma_2$. Assume that there are $c_0 \in \mathbb{R}_{> 0}$ and $c_1 \in \mathbb{R}$ such that

$$\Theta(s) = O(e^{c_0|s|}) \quad (|s| \to \infty) \quad \text{uniformly on } \sigma_1 \leq \Re(s) \leq \sigma_2, \quad \Theta(s) = O(|s|^{c_1}) \quad (|s| \to \infty) \quad \text{on } \Re(s) = \sigma_1 \text{ and } \Re(s) = \sigma_2.$$

Then $\Theta(s) = O(|s|^{c_2}) \quad (|s| \to \infty) \quad \text{uniformly on } \sigma_1 \leq \Re(s) \leq \sigma_2$.

Proof. See, for example, Miyake [Mi, Lemma 4.3.4].

Proof of Theorem 2.2 (ii). Let $f \in C^\infty_0(\mathbb{R}^\times)$. By Lemma 5.9, we know that $\alpha_1, \alpha_2$ and $f$ satisfy the conditions [Z1] and [Z2]. Let $\sigma$ be a sufficiently large real number. By Lemma 5.11, we have

$$Z(\alpha_1, \mathcal{F}(f); s) = O(|s|^{-\alpha}) \quad (|s| \to \infty) \quad \text{on } \Re(s) = \sigma,$$
for any \( n \in \mathbb{Z}_{\geq 0} \). By the functional equation (5.13) in [Z1], we note that this estimate also holds on \( \text{Re}(s) = -\sigma - 2\text{Re}(\nu) + 1 \). By these estimates, [Z2] and the entireness of (5.12) in [Z1], we know that

\[
\Theta(s) = s(s + 2\nu - 1)Z(\alpha_1, \mathcal{F}(f); s)
\]

satisfies the assumption in the Phragmen–Lindelöf theorem (Lemma 5.12) with \( \sigma_1 = -\sigma - 2\text{Re}(\nu) + 1 \), \( \sigma_2 = \sigma \) and \( c_1 = -n \) for any \( n \in \mathbb{Z}_{\geq 0} \). This implies that

\[
Z(\alpha_1, \mathcal{F}(f); s) = O(|s|^{-n}) \quad (|s| \to \infty)
\]

uniformly on \( -\sigma - 2\text{Re}(\nu) + 1 \leq \text{Re}(s) \leq \sigma \), for any \( n \in \mathbb{Z}_{\geq 0} \). Hence, by Lemma 5.11 and the shift of the contour, we have

\[
\vartheta(\alpha_1, \mathcal{F}(f); t) = \frac{1}{2\pi\sqrt{-1}} \int_{\text{Re}(s) = \sigma} Z(\alpha_1, \mathcal{F}(f); s)t^{-s} \, ds
\]

Moreover, we have

\[
\begin{align*}
&\frac{1}{2\pi\sqrt{-1}} \int_{\text{Re}(s) = -\sigma - 2\text{Re}(\nu) + 1} Z(\alpha_1, \mathcal{F}(f); s)t^{-s} \, ds \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\text{Re}(s) = -\sigma - 2\text{Re}(\nu) + 1} Z(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); -s - 2\nu + 1)t^{-s} \, ds \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\text{Re}(s) = \sigma} Z(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); s)t^{s+2\nu-1} \, ds \\
&= t^{2\nu-1} \vartheta(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); t^{-1}).
\end{align*}
\]

(5.19)

Here the first equality follows from the functional equation (5.13) in [Z1], the second equality follows from the substitution \( s \to -s - 2\nu + 1 \), and the third equality follows from Lemma 5.11. Combining the equalities (5.18) and (5.19), we have

\[
\vartheta(\alpha_1, \mathcal{F}(f); t) + \alpha_1(0)\mathcal{F}(f)(0) = t^{2\nu-1}\{\vartheta(\alpha_2, \mathcal{F}(f_{\mu, \nu, \infty}); t^{-1}) + \alpha_2(0)\mathcal{F}(f_{\mu, \nu, \infty})(0)\}.
\]

This equality for \( t = 1 \) is \( T_{\alpha_1}(f) = T_{\alpha_2}(f_{\mu, \nu, \infty}) \), and complete the proof. \( \square \)

6 Distributions on \( I_{\mu, \nu}^{\infty} \)

6.1 The correspondence between \( \mathcal{A}(J_{\mu, \nu}) \) and \( I_{\mu, \nu}^{-\infty} \).

In this subsection, we give a proof of Proposition 2.4. For \( j \in \mathbb{Z}_{\geq 0} \), we denote by \( \mathcal{U}_j(\mathfrak{g}_C) \) the subspace of \( \mathcal{U}(\mathfrak{g}_C) \) spanned by the products of \( j \) or less elements of
Let \( \exp: \mathfrak{g} \to G \) be the exponential map, that is,

\[
\exp(X) = \sum_{i=0}^{\infty} \frac{1}{i!}X^i \quad (X \in \mathfrak{g}).
\]

**Lemma 6.1.** Let \( \mathfrak{g} \in \mathfrak{g} \). Let

(i) \( \tilde{\exp}(X) = (s\exp(\frac{1}{m}X))^{m} \),

(ii) \( \tilde{\exp}(X)\tilde{g}^{-1} = \exp(\text{Ad}(g)X) \),

(iii) \( \tilde{\exp}(tH) = \tilde{a}(e^{tH}), \tilde{\exp}(tE_{+}) = \tilde{u}(t) \),

(iv) \( \tilde{\exp}(t(E_{+} - E_{-})) = \tilde{\kappa}(t) \).

**Proof.** Let \( \mathfrak{g} \in \mathfrak{g} \) and \( \tilde{g} = (g, \theta) \in \tilde{G} \). Since \( \mathbb{R} \ni t \mapsto \tilde{g}\tilde{\exp}(tX)\tilde{g}^{-1} \in \tilde{G} \) is a continuous group homomorphism, we note that \( \tilde{g}\tilde{\exp}(tX)\tilde{g}^{-1} \) is contained in a neighborhood \( \tilde{G} \) of \((1,0)\) for \( t \in \mathbb{R} \) such that \( |t| \) is sufficiently small. Since

\[
\varpi(\tilde{g}\tilde{\exp}(tX)\tilde{g}^{-1}) = g\exp(tX)g^{-1} = \exp(t\text{Ad}(g)X),
\]

we have \( \tilde{g}\tilde{\exp}(tX)\tilde{g}^{-1} = \ast\exp(t\text{Ad}(g)X) \) for \( t \in \mathbb{R} \) such that \( |t| \) is sufficiently small. Hence, for a sufficiently large positive integer \( m \), we have

\[
\tilde{g}\tilde{\exp}(X)\tilde{g}^{-1} = (\tilde{g}\tilde{\exp}(\frac{1}{m}X)\tilde{g}^{-1})^{m} = (\ast\exp(\frac{1}{m}\text{Ad}(g)X))^{m}.
\]

The statement (i) follows from this equality in the case of \( \tilde{g} = (1,0) \). The statement (ii) follows from this equality and the statement (i). The statement (iii) and (iv) follow from the statement (i). \( \square \)

**Lemma 6.2.** Let \( \mu, \nu \in \mathbb{C} \). Let \( F \in \mathcal{I}_{\mu, \nu}^{\infty} \), and set

\[
f_{1}(x) = F(\tilde{w}(\tilde{u}_{-}x))\Delta(x), \quad f_{2}(x) = F_{\infty}(\tilde{w}(\tilde{u}_{-}x))\Delta(x) \quad (x \in \mathbb{R}), \quad (6.1)
\]

where \( \Delta \) is the function defined in §3.2. Then \( (f_{1}, f_{2}) \) is a partition of \( F \), and satisfies \( \text{supp}(f_{i}) \subset \{ t \in \mathbb{R} \mid |t| \leq 2 \} \) \((i = 1, 2)\).

**Proof.** Let \( \tilde{g} = (g, \theta) \in \tilde{G} \) with \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G \). By definition, we have

\[
\iota_{\mu, \nu}(f_{1})(\tilde{g}) = \begin{cases} F(\tilde{g})\Delta(-d/c) & \text{if } c \neq 0, \\
0 & \text{if } c = 0, \end{cases}
\]

\[
\iota_{\mu, \nu}(f_{2})(\tilde{g})_{\infty} = \begin{cases} F(\tilde{g})\Delta(c/d) & \text{if } d \neq 0, \\
0 & \text{if } d = 0. \end{cases}
\]

By \( \Delta(0) = 1, \Delta(x) + \Delta(-1/x) = 1 \) \((x \in \mathbb{R}^{\times})\), \( \text{supp}(\Delta) \subset \{ t \in \mathbb{R} \mid |t| \leq 2 \} \) and these equalities, we know that \( (f_{1}, f_{2}) \) is a partition of \( F \). \( \square \)
Lemma 6.3. Let \( \mu, \nu \in \mathbb{C} \). Let \((T_1, T_2) \in \mathcal{A}(f_\mu, \nu)\). Let \((f_1, f_2)\) and \((f_3, f_4)\) be two partitions of \( F \in I^\infty_{\mu, \nu} \). Then \( T_1(f_1) + T_2(f_2) = T_1(f_3) + T_2(f_4) \).

Proof. By (2.17), (2.19) and the definition of partitions, we have

\[
F(\tilde{w}\tilde{u}(x)) = f_1(x) + (f_2)_{\mu, \nu, \infty}(x) = f_3(x) + (f_4)_{\mu, \nu, \infty}(x) \quad (x \in \mathbb{R}).
\]

Hence, \( f_1 - f_3 = -(f_2 - f_4)_{\mu, \nu, \infty} \). This equality implies \( f_2 - f_4 \in C_0^\infty(\mathbb{R}^+) \) since \( f_1, f_2, f_3, f_4 \in C_0^\infty(\mathbb{R}) \). Therefore, by (2.6), we have

\[
(T_1(f_1) + T_2(f_2)) - (T_1(f_3) + T_2(f_4)) = T_1(f_1 - f_3) + T_2(f_2 - f_4) = T_1(f_1 - f_3 + (f_2 - f_4)_{\mu, \nu, \infty}) = T_1(0) = 0.
\]

Hence, we obtain the assertion. \( \square \)

Lemma 6.4. Let \( \mu, \nu \in \mathbb{C} \) and \( u > 0 \). For \( F \in I^\infty_{\mu, \nu} \), the inequalities

\[
\sup_{-u \leq x \leq u} |F(\tilde{w}\tilde{u}(x))| \leq (1 + u^2)^{\Re(\nu)+1/2}|F|_K, \quad (6.2)
\]

\[
|F|_K \leq e^{\pi\Im(\mu)/2}|F|_K \quad (6.3)
\]

hold. For \( f \in C_0^\infty(\mathbb{R}) \) with \( \text{supp}(f) \subset \{ x \in \mathbb{R} \mid |x| \leq u \} \), we have

\[
|\iota_{\mu, \nu}(f)|_K \leq e^{\pi\Im(\mu)/2}(1 + u^2)^{\Re(\nu)+1/2} \sup_{x \in \mathbb{R}} |f(x)|. \quad (6.4)
\]

Proof. Let \( F \in I^\infty_{\mu, \nu} \). Since

\[
\tilde{w}\tilde{u}(x) = \tilde{u}\left(\frac{x}{1 + x^2}\right) \tilde{u}\left(\frac{1}{1 + x^2}\right) \tilde{k}(\arg(\sqrt{-1} - x)),
\]

we have \( F(\tilde{w}\tilde{u}(x)) = (1 + x^2)^{-\nu-1/2} F(\tilde{k}(\arg(\sqrt{-1} - x))) \) for \( x \in \mathbb{R} \). By this equality, we obtain (6.2). The inequality (6.3) follows from

\[
F_{\infty}(\tilde{k}(\theta)) = e^{\pi\sqrt{-1}\mu/2} F(\tilde{k}(\theta - \pi/2)) = e^{-\pi\sqrt{-1}\mu/2} F(\tilde{k}(\theta + \pi/2)) \quad (\theta \in \mathbb{R}).
\]

The inequality (6.4) follows from

\[
\iota_{\mu, \nu}(f)(\tilde{k}(\theta)) = \begin{cases} 
   e^{\pi\sqrt{-1}\mu}|\sin \theta|^{-2\nu-1} f(1/\tan \theta) & \text{if } 0 < \theta < \pi, \\
   |\sin \theta|^{-2\nu-1} f(1/\tan \theta) & \text{if } -\pi < \theta < 0, \\
   0 & \text{if } \theta = \pi \mathbb{Z} 
\end{cases}
\]

\[
\frac{1}{1 + u^2} \leq \sin^2 \theta \leq 1 \quad \text{if } \left| \frac{1}{\tan \theta} \right| \leq u,
\]

for \( \theta \in \mathbb{R} \) and \( f \in C_0^\infty(\mathbb{R}) \). \( \square \)
Lemma 6.5. Let $\mu, \nu \in \mathbb{C}$.

(i) $(\rho(E_+)F)(\tilde{w}u(-x)) = -\frac{d}{dx} F(\tilde{w}u(-x))$ $(x \in \mathbb{R})$ for $F \in I^\infty_{\mu,\nu}$.

(ii) Let $f \in C_0^\infty(\mathbb{R})$. Then $\rho(E_+)^t_{\mu,\nu}(f) = -t_{\mu,\nu}(f')$ and

$$
\rho(H)^t_{\mu,\nu}(f) = t_{\mu,\nu}(f_H) \quad \text{with} \quad f_H(x) = -2x f'(x) - (2\nu + 1)f(x),
$$

$$
\rho(E_-)^t_{\mu,\nu}(f) = t_{\mu,\nu}(f_{E_-}) \quad \text{with} \quad f_{E_-}(x) = x^2 f'(x) + (2\nu + 1)xf(x).
$$

**Proof.** The statement (i) follows immediately from Lemma 6.1 (iii). Let $x \in \mathbb{R}$.

By the statement (i) and (2.17), we have $(\rho(E_+)^t_{\mu,\nu}(f))(\tilde{w}u(-x)) = -f'(x)$.

Hence, we have $\rho(E_+)^t_{\mu,\nu}(f) = -t_{\mu,\nu}(f')$ by the characterization (2.18).

By Lemma 6.1 (iii) and (2.14), we have

$$
\tilde{w}u(-x) \exp(tE_-) = \tilde{w}u(-x)\tilde{u}(t) = \tilde{u}\left(\frac{-t}{1-tx}\right) \tilde{a}\left(\frac{1}{(1-tx)^2}\right) \tilde{w}u\left(\frac{-x}{1-tx}\right)
$$

for $t \in \mathbb{R}$ such that $|t|$ is sufficiently small. Using this equality, we have

$$
\left.\frac{d}{dt}\right|_{t=0} (\rho(E_-)^t_{\mu,\nu}(f))(\tilde{w}u(-x)) = \left.\frac{d}{dt}\right|_{t=0} \left((1-tx)^{-2\nu-1}f\left(\frac{x}{1-tx}\right)\right)
$$

$$
= x^2 f'(x) + (2\nu + 1)xf(x) = f_{E_-}(x).
$$

Hence, we have $\rho(E_-)^t_{\mu,\nu}(f) = t_{\mu,\nu}(f_{E_-})$ by the characterization (2.18). We obtain $\rho(H)^t_{\mu,\nu}(f) = t_{\mu,\nu}(f_H)$ by the formulas for $\rho(E_+)$ and $\rho(E_-)$ with $\rho(H) = \rho(E_+) \circ \rho(E_-) - \rho(E_-) \circ \rho(E_+)$. 

**Lemma 6.6.** Let $\mu, \nu \in \mathbb{C}$. Then $Q_X(F_\infty) \leq e^{\pi|\text{Im}(\mu)|/2} Q_{\text{Ad}(\nu^{-1})X}(F)$ for $F \in I^\infty_{\mu,\nu}$ and $X \in U(\mathfrak{g}_\mathbb{C})$.

**Proof.** The assertion follows immediately from (6.3) and Lemma 6.1 (ii).

**Lemma 6.7.** Let $\mu, \nu \in \mathbb{C}$. For $i \in \mathbb{Z}_{\geq 0}$, there is a constant $c_{\Delta,\nu,i}$ such that

$$
\sup_{x \in \mathbb{R}} \left| \frac{d^i}{dx^i} (F(\tilde{w}u(-x))\Delta(x)) \right| \leq c_{\Delta,\nu,i} \sum_{j=0}^i Q_{(E_+)^j}(F) \quad (F \in I^\infty_{\mu,\nu}).
$$

**Proof.** Let $F \in I^\infty_{\mu,\nu}$ and $i \in \mathbb{Z}_{\geq 0}$. By Lemma 6.5 (i), we have

$$
\frac{d^i}{dx^i} (F(\tilde{w}u(-x))\Delta(x)) = \sum_{j=0}^i \binom{i}{j} (-1)^j (\rho(E_+)^j F)(\tilde{w}u(-x))\Delta^{(i-j)}(x)
$$

for $x \in \mathbb{R}$. By (6.2) and $\text{supp}(\Delta) \subset \{x \in \mathbb{R} \mid |x| \leq 2\}$, we have

$$
\sup_{x \in \mathbb{R}} \left| \frac{d^i}{dx^i} (F(\tilde{w}u(-x))\Delta(x)) \right| \leq \sum_{j=0}^i \binom{i}{j} \left( \sup_{x \in \mathbb{R}} \Delta^{(i-j)}(x) \right)
$$

$$
\times 5^{||\text{Re}(\nu)||+1/2} Q_{(E_+)^j}(F).
$$

Hence, we obtain the assertion.
Lemma 6.8. Let $\mu, \nu \in \mathbb{C}$ and $j \in \mathbb{Z}_{\geq 0}$. For $X \in \mathcal{U}_j(\mathfrak{g}_C)$ and $u > 0$, there is a constant $c_{X,u} > 0$ such that

$$Q_X(t_{\mu,\nu}(f)) \leq c_{X,u} \sum_{i=0}^{j} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|$$

$$(f \in C_{0}^{\infty}(\mathbb{R}) \text{ with } \text{supp}(f) \subset \{x \in \mathbb{R} \mid |x| \leq u\}).$$

Proof. Let $X \in \mathcal{U}_j(\mathfrak{g}_C)$ and $u > 0$. Since $\{H, E_+, E_-\}$ is a basis of $\mathfrak{g}_C$, it follows from Lemma 6.5 (ii) that there are polynomial functions $p_{X,i} (0 \leq i \leq j)$ on $\mathbb{R}$ such that

$$\rho(X) t_{\mu,\nu}(f) = \sum_{i=0}^{j} t_{\mu,\nu}(p_{X,i}) f^{(i)}$$

$(f \in C_{0}^{\infty}(\mathbb{R})$.)

Hence, for $f \in C_{0}^{\infty}(\mathbb{R})$ with $\text{supp}(f) \subset \{x \in \mathbb{R} \mid |x| \leq u\}$, we have

$$Q_X(t_{\mu,\nu}(f)) \leq e^{\pi |\text{Im}(\mu)| \left(1 + u^2\right) |\text{Re}(\nu)|^{1/2}} \sum_{i=0}^{j} \left( \sup_{-u \leq x \leq u} |p_{X,i}(x)| \right) \sup_{x \in \mathbb{R}} |f^{(i)}(x)|$$

by (6.4). The assertion follows from this inequality. \qed

Proof of Proposition 2.4. First, we will prove the statement (i). Let $\lambda \in I^{-\infty}_{\mu,\nu}$ and $u > 0$. Then there exist $c > 0$, $m \in \mathbb{Z}_{\geq 0}$ and $X_1, X_2, \cdots, X_m \in \mathcal{U}(\mathfrak{g}_C)$ satisfying (2.21). Take $j \in \mathbb{Z}_{\geq 0}$ so that $X_1, X_2, \cdots, X_m \in \mathcal{U}_j(\mathfrak{g}_C)$. By Lemma 6.8, there are constants $c_{X_i,u} > 0 (1 \leq i \leq m)$ such that

$$|T^{\lambda}(f)| = |\lambda(t_{\mu,\nu}(f))| \leq e^{c \sum_{i=1}^{m} Q_{X_i}(t_{\mu,\nu}(f))} \leq \left( c \sum_{i=1}^{m} c_{X_i,u} \right) \sum_{i=0}^{j} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|$$

$(f \in C_{0}^{\infty}(\mathbb{R}) \text{ with } \text{supp}(f) \subset \{x \in \mathbb{R} \mid |x| \leq u\})$.

This implies $T^{\lambda} \in \mathcal{D}'(\mathbb{R})$. By Lemma 6.6, we have

$$|\lambda_{\infty}(F)| = |\lambda(F_{\infty})| \leq e^{c \sum_{i=1}^{m} Q_{X_i}(F_{\infty})} \leq c e^{\pi |\text{Im}(\mu)| |\text{Re}(\nu)|^{1/2}} \sum_{i=1}^{m} Q_{\text{Ad}(w^{-1})X_i}(F)$$

for $F \in I_{\mu,\nu}^{\infty}$. This implies $\lambda_{\infty} \in I^{-\infty}_{\mu,\nu}$. The equality $(\lambda_{\infty})_{\infty} = \lambda$ follows from $(F_{\infty})_{\infty} = F$ ($F \in I_{\mu,\nu}^{\infty}$).

Next, we will prove the statement (ii). By Lemma 6.3, we know that the definition (2.22) does not depend on the choice of a partition. Let $(T_1, T_2) \in \mathcal{A}(J_{\mu,\nu})$. Then there are $m \in \mathbb{Z}_{\geq 0}$ and $c > 0$ such that

$$|T_1(f)| \leq c \sum_{i=0}^{m} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|, \quad |T_2(f)| \leq c \sum_{i=0}^{m} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|$$

$(f \in C_{0}^{\infty}(\mathbb{R}) \text{ with } \text{supp}(f) \subset \{x \in \mathbb{R} \mid |x| \leq 2\})$. 

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By Lemmas 6.6 and 6.7, there are constants $c_{\Delta,\nu,i} > 0$ ($1 \leq i \leq j$) such that

$$|\Lambda(T_1, T_2)(F)| = |T_1(f_1) + T_2(f_2)| \leq c \sum_{i=0}^{m} \left( \sup_{x \in \mathbb{R}} \left| f_1^{(i)}(x) \right| + \sup_{x \in \mathbb{R}} \left| f_2^{(i)}(x) \right| \right)^i$$

$$\leq c \sum_{i=0}^{m} c_{\Delta,\nu,i} \sum_{j=0}^{i} \left( Q(E_j) \right) + e^{\pi|\text{Im}(\mu)|/2} Q(-E_{-j}) (F)$$

for $F \in L^\infty_{\mu,\nu}$ with the partition $(f_1, f_2)$ of $F$ defined by (6.1). This implies $\Lambda(T_1, T_2) \in L^\infty_{\mu,\nu}$.

Finally, we will prove the statement (iii). By definition, we have

$$\lambda = T^\lambda, T^\lambda = T_1, \quad (\Lambda(T_1, T_2))_\infty = \Lambda(T_2, T_1)$$

for $\lambda \in L^{-\infty}_{\mu,\nu}$ and $(T_1, T_2) \in A(J_{\mu,\nu})$. The statement (iii) follows from the statement (i), (ii) and these equalities. \qed

### 6.2 The twisted Fourier transforms

In this subsection, we give proofs of Lemmas 2.5, 2.6, and prepare some lemmas for the twisted Fourier transformation.

**Proof of Lemma 2.5.** Let $\mu \in \mathbb{C}$, $y \in \mathbb{R}^+$ and $f \in C_0^\infty(\mathbb{R})$. By direct computation, for $\nu \in \mathbb{C}$, we have $(f_{\mu,\nu,\infty})'' = (D_\nu f)_{\mu,\nu+1,\infty}$, where $D_\nu$ is the differential operator defined by

$$(D_\nu f)(x) = x^2 f''(x) + (2\nu + 2) \{2xf'(x) + (2\nu + 1)f(x)\}. \tag{6.5}$$

By Lemma 3.1, we have $F(f_{\mu,\nu,\infty})(y) = \frac{1}{(2\pi \sqrt{-1})^2} F((D_\nu f)_{\mu,\nu+1,\infty})(y)$ for $\nu \in \mathbb{C}$ such that $\text{Re}(\nu) > 0$. By repeated application of this equality, we have

$$F(f_{\mu,\nu,\infty})(y) = \frac{1}{(2\pi \sqrt{-1})^{2m}} F((D_{\nu+m-1}D_{\nu+m-2}\cdots D_\nu f)_{\mu,\nu+m,\infty})(y) \tag{6.6}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$ such that $\text{Re}(\nu) > 0$. This expression gives the holomorphic continuation of $F(f_{\mu,\nu,\infty})(y)$ to $\text{Re}(\nu) > -m$. Since a non-negative integer $m$ can be chosen arbitrarily, we obtain the statement (i).

By direct computation, for $\nu \in \mathbb{C}$ such that $\text{Re}(\nu) > 0$, we have

$$F(f_{\mu,\nu,\infty})(0) = e^{\pi\sqrt{-1}\nu/2} \Phi_1(f; 2\nu) + e^{-\pi\sqrt{-1}\nu/2} \Phi_{-1}(f; 2\nu). \tag{6.7}$$

Hence, by the results in §4.2, we obtain the statement (ii). \qed

The twisted Fourier transform $F_{\mu,\nu,\infty}(f)$ of $f \in C_0^\infty(\mathbb{R})$ is defined in §2.6 using the meromorphic continuation in Lemma 2.5.
Lemma 6.9. Let $\mu, \nu \in \mathbb{C}$ and $y \in \mathbb{R}^\times$. For $u > 0$ and $m \in \mathbb{Z}_{\geq 0}$ such that $\text{Re}(\nu) > -m$, we have

$$|\mathcal{F}_{\mu, \nu, \infty}(f)(y)| \leq e^{\pi|\text{Im}(\mu)|/2}u^{2\text{Re}(\nu)+2m}(\text{Re}(\nu) + m)(2\pi|y|)^{2m}\sup_{x \in \mathbb{R}}|(D_{\nu+m-1}D_{\nu+m-2} \cdots D_{\nu}f)(x)|$$

$$(f \in C^\infty_0(\mathbb{R}) \text{ with } \text{supp}(f) \subset \{x \in \mathbb{R} | |x| \leq u\})$$

where $D_{\nu}$ is the differential operator defined by (6.5).

Proof. Let $\mu, \nu \in \mathbb{C}$, $y \in \mathbb{R}^\times$, $u > 0$ and $f \in C^\infty_0(\mathbb{R})$ with $\text{supp}(f) \subset \{x \in \mathbb{R} | |x| \leq u\}$. For $\nu \in \mathbb{C}$ such that $\text{Re}(\nu) > 0$, we have

$$|\mathcal{F}(f_{\mu, \nu, \infty})(y)| = \sum_{\varepsilon \in \{\pm 1\}} e^{\pi\sqrt{-1}\varepsilon y/2} \int_{1/u}^{\infty} t^{-2\nu-1}f(\varepsilon/t)e^{-2\pi\sqrt{-1}\varepsilon yt} dt$$

$$\leq 2e^{\pi|\text{Im}(\mu)|/2} \left( \int_{1/u}^{\infty} t^{-2\text{Re}(\nu)-1} dt \right) \sup_{x \in \mathbb{R}}|f(x)| = \frac{e^{\pi|\text{Im}(\mu)|/2}u^{2\text{Re}(\nu)}}{\text{Re}(\nu)} \sup_{x \in \mathbb{R}}|f(x)|.$$

The assertion follows immediately from this inequality and (6.6). \qed

Proof of Lemma 2.6. Let $\mu, \nu \in \mathbb{C}$. By (6.7) and the results in §4.3, we know that $C^\infty_0(\mathbb{R}) \ni f \mapsto \mathcal{F}_{\mu, \nu, \infty}(f)(0) \in \mathbb{C}$ is a distribution on $\mathbb{R}$. By Lemma 6.9, we know that $C^\infty_0(\mathbb{R}) \ni f \mapsto \mathcal{F}_{\mu, \nu, \infty}(f)(y) \in \mathbb{C}$ is a distribution on $\mathbb{R}$ for $y \in \mathbb{R}^\times$. The equality (2.23) follows from the definition and $\delta^{(n)}(f) = 0$ ($f \in C^\infty_0(\mathbb{R}^\times)$). \qed

For $t > 0$ and a function $f$ on $\mathbb{R}$ or $\mathbb{R}^\times$, we define a function $f_{[t]}$ as in §5.2, that is, $f_{[t]}(x) = f(tx)$.

Lemma 6.10. Let $\mu, \nu \in \mathbb{C}$, $y \in \mathbb{R}$, $t > 0$ and $f \in C^\infty_0(\mathbb{R})$.

(i) $\delta^{(n)}(f_{[t]}) = t^n\delta^{(n)}(f)$ for $n \in \mathbb{Z}_{\geq 0}$.

(ii) $\mathcal{F}_{\mu, \nu, \infty}(f_{[t^{-1}]})(y) = t^{2\nu}\mathcal{F}_{\mu, \nu, \infty}(f)(t^{-1}y)$ if either $2\nu \notin \mathbb{Z}_{\geq 0}$ or $y \neq 0$ holds.

(iii) For $n \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{F}_{\mu, -n/2, \infty}(f_{[t^{-1}]})(0) = t^{-n}\mathcal{F}_{\mu, -n/2, \infty}(f)(0)$$

$$+ (\sqrt{-1})^n \frac{2\cos(\pi(n+1)/2)}{n!}\delta^{(n)}(f)t^{-n}\log t.$$

Proof. We obtain the statement (i) by direct computation. By Lemma 5.5, if $\text{Re}(\nu) > 0$, we have

$$\mathcal{F}_{\mu, \nu, \infty}(f_{[t^{-1}]}) (y) = \mathcal{F}((f_{[t^{-1}]}_{\mu, \nu, \infty}))(y) = t^{2\nu+1}\mathcal{F}((f_{\mu, \nu, \infty})_{[t]})(y)$$

$$= t^{2\nu}\mathcal{F}(f_{\mu, \nu, \infty})(t^{-1}y) = t^{2\nu}\mathcal{F}_{\mu, \nu, \infty}(f)(t^{-1}y).$$

Hence, by the uniqueness of the analytic continuation as a function of $\nu$, we obtain the statement (ii). For $n \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{F}_{\mu, -n/2, \infty}(f_{[t^{-1}]})(0) - t^{-n}\mathcal{F}_{\mu, -n/2, \infty}(f)(0)$$

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For Lemma 6.11. Let

\[ \frac{\pi(n+\mu)}{2} \delta(n)(f_{t^{-1}}) \]

\[ - t^{2s} \left( \frac{\pi(n+\mu)}{2} \frac{2 \cos \left( \frac{\pi(n+\mu)}{2} \right)}{n!} \delta(n)(f) \right) \]

\[ = \left( \frac{-1}{2} \right)^n \frac{2 \cos \left( \frac{\pi(n+\mu)}{2} \right)}{n!} \delta(n)(f) \lim_{s \to -n/2} \frac{t^{2s} - t^{-n}}{2s + n}. \]

Here the second equality follows from the statements (i) and (ii). By l'Hôpital's rule, we have

\[ \lim_{s \to -n/2} \frac{t^{2s} - t^{-n}}{2s + n} = t^{-n} \log t, \]

and obtain the statement (iii). \( \square \)

### 6.3 The Jacquet integrals and the delta distribution

In this subsection, we introduce some properties of the distributions \( \mathcal{J}_t \) and \( \delta^\infty(m) \) on \( I_{\mu,\nu}^\infty \), which are defined by (2.24) and (2.25), respectively.

**Lemma 6.11.** Let \( \mu, \nu \in \mathbb{C} \) and \( m \in \mathbb{Z}_{\geq 0} \). Then \( \delta^\infty(m) = \Lambda(0, \delta^\infty(m)) \), that is,

\[ \delta^\infty(m)(F) = \delta(m)(f_2) \quad (F \in I_{\mu,\nu}^\infty \text{ with a partition } (f_1, f_2)). \] (6.8)

Here \( 0, \delta(m) \) is regarded as an element of \( A(J_{\mu,\nu}) \).

**Proof.** By (2.16), for \( f \in C_0^\infty(\mathbb{R}) \), there is a neighborhood of \( (1_2, 0) \) in \( \tilde{G} \) on which \( \imath_{\mu,\nu}(f) \) is equal to 0. Hence, by Lemmas 6.1 and 6.5, we have

\[ \delta^\infty(m)(F) = (\rho(E_+)^m \imath_{\mu,\nu}(f_1))(\tilde{w}) + (\rho(E_-)^m \imath_{\mu,\nu}(f_2))(\tilde{w}) \]

\[ = e^{-\pi \sum \mu_i/2} (\rho(-E_-)^m \imath_{\mu,\nu}(f_1))(1_2, 0) + (-1)^m \imath_{\mu,\nu}(f_2)(m)(\tilde{w}) \]

\[ = 0 + (-1)^m f_2(m)(0) = \delta(m)(f_2) \]

for \( F \in I_{\mu,\nu}^\infty \) with a partition \( (f_1, f_2) \). \( \square \)

**Proposition 6.12.** Let \( \mu, \nu \in \mathbb{C}, t \in \mathbb{R} \) and \( F \in I_{\mu,\nu}^\infty \).

(i) For \( n \in \mathbb{Z}_{\geq 0} \), we have

\[ \delta^\infty(n)(\rho(\tilde{u}(t))F) = \sum_{i=0}^{n} \binom{n}{i} (-2\nu - n)_{n-i} (-t)^{n-i} \delta^\infty(i)(F), \]

where \( (z)_i = \Gamma(z+i)/\Gamma(z) = z(z+1) \cdots (z+i-1) \) is the Pochhammer symbol.

(ii) If either \( l \neq 0 \) or \( 2\nu \notin \mathbb{Z}_{\geq 0} \) holds, we have \( J_t(\rho(\tilde{u}(t))F) = e^{2\pi \sqrt{-1}t} J_t(F) \).

(iii) If \( -2\nu = n \) with some \( n \in \mathbb{Z}_{\geq 0} \), we have

\[ J_0(\rho(\tilde{u}(t))F) = J_0(F) + (\sqrt{-1})^n \sum_{i=0}^{n-1} \frac{2 \cos \left( \frac{\pi(n+i)}{2} \right)}{i!(n-i)} (-t)^{n-i} \delta^\infty(i)(F). \]

In particular, if \( \nu = 0 \), we have \( J_0(\rho(\tilde{u}(t))F) = J_0(F) \).
Proof. Since \( \tilde{w}u(-x)\tilde{n}(-t) = \tilde{u}\left(\frac{t}{1 + tx}\right) \tilde{\gamma}\left(\frac{1}{(1 + tx)^2}\right) \tilde{w}\tilde{u}\left(\frac{-x}{1 + tx}\right) \) for \( x \in \mathbb{R} \) such that \( |x| \) is sufficiently small, by Lemma 6.5 (i), we have
\[
\delta_n(\rho(\tilde{u}(t))F) = (\rho(E_+)^n(\rho(\tilde{u}(t))F)_\infty)(\tilde{w}) = (\rho(E_+)^n\rho(\tilde{n}(-t))F)\infty)(\tilde{w})
\]
\[
= (-1)^n \frac{d^n}{dx^n} \bigg|_{x=0} F_\infty(\tilde{w}u(-x)\tilde{n}(-t))
\]
\[
= (-1)^n \frac{d^n}{dx^n} \bigg|_{x=0} (1 + tx)^{-2\nu - 1} F_\infty(\tilde{w}\tilde{u}\left(\frac{-x}{1 + tx}\right))
\]
for \( n \in \mathbb{Z}_{\geq 0} \). Hence, by Lemma 3.2 with \( a = d = 1, b = 0, c = t, s = -2\nu - 1 \) and \( f(x) = F_\infty(\tilde{w}u(-x)) \), we obtain the statement (i).

Let \((f_1, f_2)\) be a partition of \( F \). We set \( F^{[s]} = \iota_{\mu,s}(f_1) + (\iota_{\mu,s}(f_2))_\infty \in F_{\mu,s} \) \((s \in \mathbb{C})\). We take a partition \((f_1^{[s]}, f_2^{[s]})\) of \( \rho(\tilde{u}(t))F^{[s]} \) as in Lemma 6.2, that is,
\[
f_1^{[s]}(x) = (\rho(\tilde{u}(t))F^{[s]})(\tilde{w}u(-x))\Delta(x), \quad f_2^{[s]}(x) = (\rho(\tilde{u}(t))F^{[s]})_\infty(\tilde{w}u(-x))\Delta(x)
\]
for \( x \in \mathbb{R} \). We consider the Jacquet integrals
\[
\mathcal{J}_l(F^{[s]}) = \mathcal{F}(f_1)(l) + F_{\mu,s,\infty}(f_2)(l), \quad \mathcal{J}_l(\rho(\tilde{u}(t))F^{[s]}) = \mathcal{F}(f_1^{[s]})(l) + F_{\mu,s,\infty}(f_2^{[s]})(l).
\]
(6.9) (6.10)
It is easy to see that the right hand sides of (6.9) and (6.10) are both holomorphic on \( \mathbb{C} \) if \( l \neq 0 \), and on \( \mathbb{C} \setminus \frac{1}{2} \mathbb{Z}_{\leq 0} \) if \( l = 0 \), as functions of \( s \). If \( \text{Re}(s) > 0 \), we have
\[
\mathcal{J}_l(\rho(\tilde{u}(t))F^{[s]}) = \int_{-\infty}^{\infty} F^{[s]}(\tilde{w}u(-x + t))e^{2\pi \sqrt{-1}lx}dx = e^{2\pi \sqrt{-1}lt} \mathcal{J}_l(F^{[s]}),
\]
where the second equality follows from the substitution \( x \to x + t \). By the uniqueness of the analytic continuation as a function of \( s \), we have
\[
\mathcal{J}_l(\rho(\tilde{u}(t))F^{[s]}) = e^{2\pi \sqrt{-1}lt} \mathcal{J}_l(F^{[s]})
\]
if either \( l \neq 0 \) or \( 2s \notin \mathbb{Z}_{\leq 0} \) holds. Since \( F^{[v]} = F \), we obtain the statement (ii).

Assume \(-2\nu = n\) with some \( n \in \mathbb{Z}_{\geq 0} \). By (2.23) and (6.8), we have
\[
\mathcal{J}_0(\rho(\tilde{u}(t))F) = \lim_{s \to -n/2} \left( \mathcal{J}_0(\rho(\tilde{u}(t))F^{[s]}) - (\sqrt{-1})^n \frac{2\cos(\frac{\pi(n+\mu)}{2})}{n!(2s + n)} \delta_{\infty}^{(n)}(\rho(\tilde{u}(t))F^{[s]}) \right).
\]
\[
\mathcal{J}_0(F) = \lim_{s \to -n/2} \left( \mathcal{J}_0(F^{[s]}) - (\sqrt{-1})^n \frac{2\cos(\frac{\pi(n+\mu)}{2})}{n!(2s + n)} \delta_{\infty}^{(n)}(F^{[s]}) \right).
\]
The statement (iii) follows from the statement (i), (ii) and these expressions. □
Proposition 6.13. Let $\mu, \nu \in \mathbb{C}$, $y \in \mathbb{R}_{>0}$, $l \in \mathbb{R}$ and $F \in I_{\mu, \nu}^\infty$.

(i) $\delta^{(m)}_\infty(\rho(\tilde{a}(y))F) = y^{\nu}y^{\nu+1/2}\delta^{(m)}_\infty(F)$ for $m \in \mathbb{Z}_{\geq 0}$.

(ii) $\mathcal{J}_l(\rho(\tilde{a}(y))F) = y^{-\nu+1/2}\mathcal{J}_l y(F)$ if either $-2\nu \not\in \mathbb{Z}_{\geq 0}$ or $l \neq 0$ holds.

(iii) If $-2\nu = n$ with some $n \in \mathbb{Z}_{\geq 0}$, then we have

$$\mathcal{J}_0(\rho(\tilde{a}(y))F) = y^{\frac{n+1}{2}}\mathcal{J}_0(F) - (\sqrt{-1})^n \frac{2\cos\left(\frac{\pi(n+\nu)}{2}\right)}{n!}\delta^{(n)}_\infty(F)y^{\frac{n+1}{2}}\log y.$$

Proof. Let $(f_1, f_2)$ be a partition of $F \in I_{\mu, \nu}^\infty$. By Lemma 2.3, We have

$$\rho(\tilde{a}(y))F = \rho(\tilde{a}(y))(t_{\mu, \nu}(f_1) + t_{\mu, \nu}(f_2))_\infty$$

$$= y^{-\nu-1/2}t_{\mu, \nu}((f_1)_y-1) + y^{\nu+1/2}(t_{\mu, \nu}((f_2)_y)_\infty).$$

This implies that $(y^{-\nu-1/2}(f_1)_{[y]}^y, y^{\nu+1/2}(f_2)_{[y]})$ is a partition of $\rho(\tilde{a}(y))F$. Hence, by (6.8) and (2.24), we have

$$\delta^{(m)}_\infty(\rho(\tilde{a}(y))F) = y^{-\nu+1/2}\delta^{(m)}((f_2)_{[y]}),$$

$$\mathcal{J}_l(\rho(\tilde{a}(y))F) = y^{-\nu-1/2}\mathcal{J}_l((f_1)_{[y]}^y-1) + y^{\nu+1/2}\mathcal{J}_l t_{\mu, \nu}((f_2)_{[y]})_\infty.$$ 

Applying Lemmas 5.5, 6.10, and comparing with (6.8) and (2.24), we obtain the assertion. \qed

6.4 The Fourier expansions

In this subsection, we give proofs of Propositions 2.7 and 2.8.

Proof of Proposition 2.7. Let $L$ be a shifted lattice in $\mathbb{R}$. Let $\mu, \nu \in \mathbb{C}$.

First, let $\alpha \in \mathcal{M}(L)$, $\beta \in \mathcal{M}(\mathbb{Z}_{\geq 0})$, and we will prove that $\lambda_{\alpha, \beta} \in \mathcal{M}(L)$ and $T^{\lambda_{\alpha, \beta}} = T_{\alpha}$. We define a $\mathbb{C}$-linear map $(T_{\alpha})_{\mu, \nu, \infty}: C_0^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$(T_{\alpha})_{\mu, \nu, \infty}(f) = \sum_{l \in L} \alpha(l)\mathcal{F}_{\mu, \nu, \infty}(f)(l) \quad (f \in C_0^\infty(\mathbb{R})). \quad (6.11)$$

By Lemmas 2.6 and 6.9, the right-hand side of (6.11) converges absolutely, and defines a distribution on $\mathbb{R}$, By Lemma 2.6, we have

$$(T_{\alpha})_{\mu, \nu, \infty}(f) = \sum_{l \in L} \alpha(l)\mathcal{F}(f_{\mu, \nu, \infty})(l) = T_{\alpha}(f_{\mu, \nu, \infty}) \quad (f \in C_0^\infty(\mathbb{R})). \quad (6.12)$$

This equality implies $(T_{\alpha}, (T_{\alpha})_{\mu, \nu, \infty}) \in \mathcal{A}(J_{\mu, \nu})$. Moreover, by (2.24), we have

$$\Lambda(T_{\alpha}, (T_{\alpha})_{\mu, \nu, \infty})(F) = \sum_{l \in L} \alpha(l)\mathcal{J}_l(F) \quad (F \in I_{\mu, \nu}^\infty).$$

By this equality and Lemma 6.11, we have

$$\lambda_{\alpha, \beta} = \Lambda(T_{\alpha}, (T_{\alpha})_{\mu, \nu, \infty} + \sum_{m=0}^{\infty} \beta(m)\delta^{(m)}). \quad (6.13)$$

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This implies \( \lambda_{\alpha,\beta} \in (I_{\mu,\nu})_{\mu,\nu}^{\text{quasi}} \) and \( T^{\lambda_{\alpha,\beta}} = T_\alpha \) by Proposition 2.4.

Next, we will prove that (2.26) is bijective. Let \( \lambda \in (I_{\mu,\nu})_{\mu,\nu}^{\text{quasi}} \). By Proposition 2.1, there is a unique \( \alpha \in \mathfrak{M}(L) \) such that \( T^{\lambda} = T_\alpha \). Hence, because of (6.13) and Proposition 2.4, it suffices to show that there is a unique \( \beta \in \mathfrak{M}(\mathbb{Z}_{\geq 0}) \) satisfying \( T^{\lambda_{\infty}} = (T_\alpha)_{\mu,\nu,\infty} + \sum_{m=0}^{\infty} \beta(m)\delta^{(m)} \). By Proposition 2.4 and (6.12), we have

\[ T^{\lambda_{\infty}}(f) = T^{\lambda}(f_{\mu,\nu,\infty}) = T_\alpha(f_{\mu,\nu,\infty}) = (T_\alpha)_{\mu,\nu,\infty}(f) \quad (f \in C^\infty_c(\mathbb{R}^\times)). \]

Hence, \( T^{\lambda_{\infty}} - (T_\alpha)_{\mu,\nu,\infty} \) is a distribution on \( \mathbb{R} \) whose support is contained in \( \{0\} \), and there is a unique \( \beta \in \mathfrak{M}(\mathbb{Z}_{\geq 0}) \) satisfying \( T^{\lambda_{\infty}} = (T_\alpha)_{\mu,\nu,\infty} + \sum_{m=0}^{\infty} \beta(m)\delta^{(m)} \) by [Fr, Theorem 3.2.1].

Let \( \mu, \nu \in \mathbb{C} \). For \( m \in \mathbb{Z}_{\geq 0} \), we set \( F_m = (\iota_{\mu,\nu}(\Delta_m))_{\infty} \in I_{\mu,\nu}^{\infty} \) with

\[ \Delta_m(x) = x^m \Delta(x) \quad (x \in \mathbb{R}), \quad (6.14) \]

where \( \Delta \) is the function defined in §3.2. Proposition 2.8 follows immediately from Proposition 2.7 and the following lemma.

**Lemma 6.14.** Let \( L \) be a shifted lattice in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \), \( t \in L' \) and \( F \in I_{\mu,\nu}^{\infty} \). Let \( \alpha \in \mathfrak{M}(L) \) and \( \beta \in \mathfrak{M}(\mathbb{Z}_{\geq 0}) \).

(i) Assume \( -2\nu \notin \mathbb{Z}_{\geq 0} \). Then

\[
\lambda_{\alpha,\beta}(\rho(\bar{u}(t))F) = \omega_L(t)\lambda_{\alpha,\beta}(F) + (1 - \omega_L(t)) \sum_{m=0}^{\infty} \beta(m)\delta^{(m)}(F) + \sum_{m=1}^{\infty} \beta(m) \left( \sum_{i=0}^{m-1} \binom{m}{i} (-2\nu - m)_m (-t)^{m-i} \delta^{(i)}(F) \right).
\]

In particular,

\[
\lambda_{\alpha,\beta}(\rho(\bar{u}(t))F_0) = \omega_L(t)\lambda_{\alpha,\beta}(F_0) + (1 - \omega_L(t)) \beta(0) + \sum_{m=1}^{\infty} \beta(m)(-2\nu - m)_m (-t)^m.
\]

(ii) Assume \( -2\nu = n \) with some \( n \in \mathbb{Z}_{\geq 0} \). Then

\[
\lambda_{\alpha,\beta}(\rho(\bar{u}(t))F) = \omega_L(t)\lambda_{\alpha,\beta}(F) + (1 - \omega_L(t)) \sum_{m=0}^{\infty} \beta(m)\delta^{(m)}(F) + \sum_{m=1}^{n-1} \beta(m) \left( \sum_{i=0}^{m-1} \binom{m}{i} (-1)^m \delta^{(i)}(F) \right) + \sum_{m=1}^{n-1} \beta(m) \left( \sum_{i=0}^{m-1} \binom{m}{i} \frac{n-i-1}{n-m-1} \delta^{(i)}(F) \right).
\]

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Here we understand $\alpha(0) = 0$ if $0 \not\in L$. In particular,

$$
\lambda_{\alpha,\beta}(\rho(u(t))F_0) = \omega_L(t)\lambda_{\alpha,\beta}(F_0) + (1 - \omega_L(t))\beta(0) + \alpha(0) \frac{2\cos\left(\frac{\pi(n+\mu)}{2}\right)}{n(\sqrt{-1})^n} t^n + \sum_{m=1}^{n-1} \frac{\beta(m)(n-1)!}{(n-m-1)!} (-1)^m t^m - n,
$$

$$
\lambda_{\alpha,\beta}(\rho(u(t))F_n) = \omega_L(t)\lambda_{\alpha,\beta}(F_n) + (-1)^n(1 - \omega_L(t))\beta(n) n! + \sum_{m=n+1}^{\infty} (-1)^m \beta(m)! t^{m-n}.
$$

Proof. This lemma follows from Proposition 6.12 and the equalities

$$(n-m)_{m-i} = \begin{cases} (n-i-1)! & \text{if } i \leq m < n, \\ (n-m-1)! & \text{if } n \leq i \leq m, \\ (-1)^{m-i}(m-n)! & \text{if } i < n \leq m, \\ 0 & \text{if } i < n \leq m \end{cases}$$

and $\delta^{(n)}_{\infty}(F_m) = \begin{cases} (-1)^m m! & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases}$ for $m, n, i \in \mathbb{Z}_{\geq 0}$. \hfill \Box

### 6.5 The explicit calculation of the Jacquet integrals

In this subsection, we give proofs of Theorem 2.9 (i) and Proposition 2.13 (ii). Proposition 2.13 (ii) is an immediate consequence of Propositions 6.12, 6.13 and the following proposition.

**Proposition 6.15.** Let $\mu, \nu \in \mathbb{C}$ and $\kappa \in \mu + 2\mathbb{Z}$. Let $F_{\nu,\kappa}$ be an element of $I^\infty_{\mu,\nu}$ defined by (2.30).

(i) Let $l \in \mathbb{R}$. Then

$$
e^{-\frac{l l}{2}} \delta^{(0)}_{\infty}(F_{\nu,\kappa}) = (-1)^{\frac{\mu - \nu}{2}},
$$

$$
e^{-\frac{l l}{2}} \mathcal{J}_l(F_{\nu,\kappa}) = \begin{cases} \frac{2^{1-2\nu} \Gamma(2\nu)}{\Gamma\left(\frac{2\nu+1+\kappa}{2}\right) \Gamma\left(\frac{2\nu+1+\kappa}{2}\right)} & \text{if } l = 0 \text{ and } -2\nu \not\in \mathbb{Z}_{\geq 0}, \\ \frac{-\pi^{\nu+\frac{1}{2}} l^{\nu-\frac{1}{2}}}{\Gamma\left(\frac{2\nu+1+\text{sgn}(l)\kappa}{2}\right)} W_{\text{sgn}(l)}^{\nu}(4\pi |l|) & \text{if } l \neq 0. \\ \end{cases}
$$

(ii) Assume $-2\nu = n$ with some $n \in \mathbb{Z}_{\geq 0}$. Then

$$
e^{-\frac{l l}{2}} \delta^{(n)}_{\infty}(F_{-n/2,\kappa}) = (-1)^{\frac{n-\nu}{2}} d(n, \kappa),
$$

$$
e^{-\frac{l l}{2}} \mathcal{J}_0(F_{-n/2,\kappa})$$

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where \( d(n, \kappa) \) and \( j(\kappa) \) are defined by (2.33) and (2.34), respectively.

**Proof.** The equality \( e^{\pi \sqrt{-1}\kappa/2} \delta_{\infty}^{(0)}(F_{\nu, \kappa}) = (-1)^{\frac{n-\nu}{2}} \) follows from the definition. We take a partition \( (f_{\nu, \kappa}, 1, f_{\nu, \kappa}, 2) \) of \( F_{\nu, \kappa} \) as in Lemma 6.2, that is,

\[
f_{\nu, \kappa}(x) = F_{\nu, \kappa}(\tilde{w}u(-x))\Delta(x), \quad f_{\nu, \kappa, 2}(x) = (F_{\nu, \kappa})_{\infty}(\tilde{w}u(-x))\Delta(x),
\]

where \( \Delta \) is the function defined in §3.2. Then we have

\[
\mathcal{J}_l(F_{\nu, \kappa}) = \mathcal{F}(f_{\nu, \kappa, 1})(l) + \mathcal{F}_{\nu, \kappa}(f_{\nu, \kappa, 2})(l).
\]

By this expression, we know that \( \mathcal{J}_l(F_{\nu, \kappa}) \) is a holomorphic function of \( \nu \) on \( \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}_{\leq 0} \) if \( l = 0 \), and on \( \mathbb{C} \) if \( l \neq 0 \). Because of the uniqueness of the analytic continuation, in order to prove the equality for \( e^{\pi \sqrt{-1}\kappa/2} \mathcal{J}_l(F_{\nu, \kappa}) \) in the statement (i), it suffices to show the case of \( \text{Re}(\nu) > 0 \). When \( \text{Re}(\nu) > 0 \), we have

\[
e^{\pi \sqrt{-1}\kappa/2} \mathcal{J}_l(F_{\nu, \kappa}) = e^{\pi \sqrt{-1}\kappa/2} \int_{-\infty}^{\infty} F_{\nu, \kappa}(\tilde{w}u(-x))e^{2\pi \sqrt{-1}lx}dx
\]

\[
e^{\pi \sqrt{-1}\kappa/2} \int_{-\infty}^{\infty} (1 + x^2)^{-\nu/2}e^{-\sqrt{-1}\kappa\arg(-x+\sqrt{-1})}e^{2\pi \sqrt{-1}lx}dx
\]

\[
= \int_{-\infty}^{\infty} (1 - \sqrt{-1}x)^{-\nu/2} (1 + \sqrt{-1}x)^{-\nu/2} e^{2\pi \sqrt{-1}lx}dx.
\]

The last integral is calculated by Maass [Ma, Chapter IV, §3], and we obtain the equality for \( e^{\pi \sqrt{-1}\kappa/2} \mathcal{J}_l(F_{\nu, \kappa}) \) in the statement (i).

Assume \(-2\nu = n\) with some \( n \in \mathbb{Z}_{\geq 0} \). By direct computation, we have

\[
e^{\pi \sqrt{-1}n/2} \delta_{\infty}^{(0)}(F_{-n/2, \kappa}) = e^{\pi \sqrt{-1}n/2}(-1)^{\frac{n-\nu}{2}}(\rho(E_+)^n F_{-n/2, \kappa})(\tilde{w})
\]

\[
e^{\pi \sqrt{-1}n/2}(-1)^{n+\frac{n-\nu}{2}} \left. \frac{d^n}{dx^n} F_{-n/2, \kappa}(\tilde{w}u(-x)) \right|_{x=0}
\]

\[
= (-1)^{n+\frac{n-\nu}{2}} \left. \frac{d^n}{dx^n} \right|_{x=0} \left( 1 - \sqrt{-1}x \right)^{\frac{n-\nu-1}{2}} (1 + \sqrt{-1}x)^{-\frac{n-\nu+1}{2}}
\]

\[
= (-1)^{\frac{n-\nu}{2}} d(n, \kappa).
\]

The equality for \( e^{\pi \sqrt{-1}n/2} \mathcal{J}_0(F_{-n/2, \kappa}) \) in the case of \( \mu + n - 1 \in 2\mathbb{Z} \) follows from the statement (i). We have

\[
e^{\pi \sqrt{-1}n/2} \mathcal{J}_0(F_{0, \kappa}) = e^{\pi \sqrt{-1}n/2} \lim_{s \to 0} \left( \mathcal{J}_0(F_{s, \kappa}) - \frac{\cos(\frac{\pi u}{2})}{s} \delta_{\infty}^{(0)}(F_{s, \kappa}) \right)
\]

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Lemma 6.16. Proposition 2.8. Hence, our proof is completed by the following lemma. 

\[ \lambda \in I \]

\[ \lambda \in \mu, \nu \]

and (6.15). Hence, we obtain the statement (ii). 

The third equality follows from l'Hospital's rule, and the fourth equality follows from the expansion 

\[ \Gamma(s) = \sum_{i=0}^{\infty} \left( \frac{1}{s+i} - \frac{1}{i+1} \right) \]

and (6.15). Hence, we obtain the statement (ii). 

\[ \Gamma(s) = \frac{\pi}{\sin(\pi s)} = \frac{\pi}{\cos(\pi(s - \frac{1}{2}))}. \]

(6.15)

Let us prove Theorem 2.9 (i). Let \( L_1 \) and \( L_2 \) be two shifted lattices in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \). Let \( \lambda \in (I_{\mu, \nu})^{\text{quad}} \). By Proposition 2.7, there are unique \((\alpha_i, \beta_i) \in \mathfrak{M}(L_i) \times \mathfrak{N}(\mathbb{Z}_{\geq 0}) \) such that \( \lambda = \lambda_{\alpha_1, \beta_1} \) and \( \lambda_{\alpha_i, \beta_i} = \lambda_{\alpha_2, \beta_2} \). If \( \lambda \in (I_{\mu, \nu})_{L_1, L_2} \), then we have \((\alpha_i, \beta_i) \in \mathfrak{M}(L_i)^0 \times \mathfrak{N}(S_\kappa(L_i)) \) \((i = 1, 2)\) by Proposition 2.8. Hence, our proof is completed by the following lemma.

Lemma 6.16. Let \( L_1 \) be a lattice in \( \mathbb{R} \). Let \( L_2 \) be a shifted lattice in \( \mathbb{R} \). Let \( n \in \mathbb{Z}_{>0} \) and \( \mu \in 1 - n + 2\mathbb{Z} \). Let \((\alpha_1, \beta_1) \in \mathfrak{M}(L_1) \times \mathfrak{N}(S_{-\kappa/2}(L_1)) \). Assume \( \lambda_{\alpha_1, \beta_1} \in (I_{\mu, -\kappa/2})_{L_1, L_2} \) \((\mu > 1)\) and \( \mu > 1 \). Then we have \( \beta_1(0) = 0 \).

Proof. Let \( \kappa \in \mu + 2\mathbb{Z} \) such that \( |\kappa| \leq n - 1 \). Let \( F_{-\kappa/2} \) be the function in \( \Gamma^\infty_{\mu, -\kappa/2} \) defined by (2.30). By Propositions 6.12, 6.13, 6.15 and

\[ \tilde{\mu}(t) = \tilde{u} \left( \frac{t}{t^2 + 1} \right) \tilde{a} \left( \frac{1}{t^2 + 1} \right) \tilde{k}(-\arg(1 + \sqrt{-1}t)), \]

we have 

\[ e^{\pi \sqrt{-\mu/2}} \lambda_{\alpha_1, \beta_1}(\rho(\tilde{\mu}(t))F_{-\kappa/2}) = (1 + t^2)^{\frac{n-1}{2}} e^{-\sqrt{-\kappa} \arg(1 + \sqrt{-1}t)} \beta_1(0) \]

for \( t \in \mathbb{R} \). By this equality and the definition of \((I_{\mu, -\kappa/2})_{L_1, L_2} \), we have

\[ \omega_{L_2}(-t)\beta_1(0) = e^{\pi \sqrt{-\mu/2}} \omega_{L_2}(-t) \lambda_{\alpha_1, \beta_1}(F_{-\kappa/2}) \]

\[ = e^{\pi \sqrt{-\mu/2}} \lambda_{\alpha_1, \beta_1}(\rho(\tilde{\mu}(t))F_{-\kappa/2}) \]

\[ = (1 + t^2)^{\frac{n-1}{2}} e^{-\sqrt{-\kappa} \arg(1 + \sqrt{-1}t)} \beta_1(0) \]

for \( t \in L_2^\perp \). Hence, we have \( \beta_1(0) = 0 \) by the assumption \((n > 1)\) \((n \notin L_2^\perp)\).
6.6 Poles of the Dirichlet series

In this subsection, we give a proof of Proposition 6.11. Proposition 6.11 follows from Euler’s reflection formula (6.15) of the Gamma function, and Proposition 6.17 below.

**Proposition 6.17.** Let \( L_1 \) and \( L_2 \) be two shifted lattices in \( \mathbb{R} \). Let \( \mu, \nu \in \mathbb{C} \). Let \( (\alpha_i, \beta_i) \in \mathfrak{M}(L_i) \times \mathfrak{M}(\mathbb{Z}) \) \((i = 1, 2)\) such that \((\lambda_{\alpha_i, \beta_i})_\infty = \lambda_{\alpha_2, \beta_2}\). We understand that \( \alpha_2(0) = 0 \) if \( 0 \notin L_2 \), and double signs are in the same order.

(i) The functions
\[
\sin(\pi s)\Xi_\pm(\alpha_1; s) - \frac{\cos\left(\frac{\pi(2s+\mu)}{2}\right)\alpha_2(0)}{s + 2\nu - 1}
\]
are entire.

(ii) Let \( m \in \mathbb{Z}_{\geq 0} \), and assume \(-2\nu \neq m\). Then the functions
\[
\Xi_\pm(\alpha_1; s) - \frac{(\pm\sqrt{-1})^m m! \beta_2(m)}{2\pi(s - m - 1)}
\]
are holomorphic at \( s = m + 1 \).

(iii) Let \( m \in \mathbb{Z}_{\geq 0} \), and assume \(-2\nu = m\). Then the functions
\[
\Xi_\pm(\alpha_1; s) + \frac{\pi \sin\left(\frac{\pi(m+\mu)}{2}\right) \alpha_2(0) - (\pm\sqrt{-1})^m m! \beta_2(m)}{2\pi(s - m - 1)} + \frac{\cos\left(\frac{\pi(m+\mu)}{2}\right) \alpha_2(0)}{\pi(s - m - 1)^2}
\]
are holomorphic at \( s = m + 1 \).

In order to prove this proposition, we prepare some lemmas.

**Lemma 6.18.** We use the notation in Proposition 6.17. Take \( r \in \mathbb{R} \) so that \( \alpha_1 \in \mathfrak{M}_r(L_1) \). Let \( f \in C_r^\infty(\mathbb{R}) \). Then, for \( s \in \mathbb{C} \) such that \( \text{Re}(s) > \text{max}\{r+1, 0\} \), we have
\[
Z(\alpha_1, \mathcal{F}(f); s) = Z_+^{r}(\alpha_1, \mathcal{F}(f); s) + Z_+^{\nu}(\alpha_2, \mathcal{F}_{\mu, \nu, \infty}(f); -s - 2\nu + 1)
\]
\[
- \frac{\alpha_1(0) \mathcal{F}(f)(0)}{s} + \frac{\alpha_2(0) \mathcal{F}_{\mu, \nu, \infty}(f)(0)}{s + 2\nu - 1} + \sum_{m=0}^{\infty} \frac{\beta_2(m)\delta^{(m)}(f)}{s - m - 1}
\]
\[
- \left\{ \begin{array}{ll}
(\sqrt{-1})^n \alpha_2(0) \frac{2 \cos\left(\frac{\pi(n+\mu)}{2}\right)}{n!(s - n - 1)^2} \delta^{(n)}(f) & \text{if } -2\nu = n \text{ with some } n \in \mathbb{Z}_{\geq 0}, \\
0 & \text{otherwise}.
\end{array} \right.
\]

**Proof.** By the equality \( \lambda_{\alpha_1, \beta_1}(t_{\mu, \nu}(f_{[t-1]})) = \lambda_{\alpha_2, \beta_2}(t_{\mu, \nu}(f_{[t-1]})) \), we have
\[
\sum_{l \in L_1} \alpha_1(l) \mathcal{F}(f_{[t-1]})(l) = \sum_{l \in L_2} \alpha_2(l) \mathcal{F}_{\mu, \nu, \infty}(f_{[t-1]})(l) + \sum_{m=0}^{\infty} \beta_2(m)\delta^{(m)}(f_{[t-1]})
\]
for $t > 0$. Applying Lemmas 5.5 (i) and 6.10, we have

$$
\sum_{l \in L_1} \alpha_1(l) F(f)(tl) = t^{2\nu-1} \sum_{l \in L_2} \alpha_2(l) F_{\mu,\nu,\infty}(f)(t^{-1}l) + \sum_{m=0}^{\infty} \beta_2(m) \delta(m)(f) t^{-m-1}
$$

$$
\left\{ \begin{array}{ll}
(\sqrt{-1})^n \alpha_2(0) \frac{2 \cos(\frac{\pi(n+\mu)}{2})}{n!} \delta(n)(f) t^{-n-1} \log t & \text{if } -2\nu = n \text{ with some } n \in \mathbb{Z}_{\geq 0}, \\
0 & \text{otherwise}
\end{array} \right.
$$

for $t > 0$. Using this equality instead of Lemma 5.6, we obtain the assertion similar to the proof of Lemma 5.7.

For $m \in \mathbb{Z}_{\geq 0}$, let $\Delta_m$ be the function on $\mathbb{R}$ defined by (6.14).

**Lemma 6.19.** Let $m \in \mathbb{Z}_{\geq 0}$. Then

$$
\Phi_1(\Delta_m; -s + 1) + \frac{1}{s - m - 1} \text{ is entire,} \quad (6.17)
$$

and $\Delta_m$ satisfies the following equalities

$$
\delta^{(n)}(\Delta_m) = \left\{ \begin{array}{ll}
(-1)^m m! & \text{if } m = n, \\
0 & \text{otherwise}
\end{array} \right. \quad (n \in \mathbb{Z}_{\geq 0}), \quad (6.18)
$$

$$
\Phi^{-1}(\Delta_m; s) = (-1)^m \Phi_1(\Delta_m; s), \quad (6.19)
$$

$$
F_{\mu,-m/2,\infty}(\Delta_m)(0) = 2(\sqrt{-1})^m \cos\left(\frac{\pi(\mu-m)}{2}\right)
$$

$$
\times \lim_{s \to m+1} \left( \Phi_1(\Delta_m; -s + 1) + \frac{1}{s - m - 1} \right) \quad (\mu \in \mathbb{C}). \quad (6.20)
$$

**Proof.** The assertion follows from the definition and the results in §4.2.

**Proof of Proposition 6.17.** By Propositions 2.4 and 2.7, we have $(T_{\alpha_1},T_{\alpha_2}) \in \mathcal{A}(L_1,L_2;J_{\mu,\nu})$. Multiplying by

$$
\frac{1}{2\sqrt{-1}} \begin{pmatrix}
\text{e}^{\pi \sqrt{-1}s/2} & \text{-e}^{-\pi \sqrt{-1}s/2} \\
\text{e}^{-\pi \sqrt{-1}s/2} & \text{e}^{\pi \sqrt{-1}s/2}
\end{pmatrix}
$$

from the left and subtracting some entire function, the entire $\mathbb{C}^2$-valued function (2.11) becomes a $\mathbb{C}^2$-valued function whose entries are (6.16). Hence, we obtain the statement (i).

Let $m \in \mathbb{Z}_{\geq 0}$. Multiplying by $(\text{e}^{-\pi \sqrt{-1}(m+1)/2}, \text{e}^{\pi \sqrt{-1}(m+1)/2})$ from the left, the entire function (2.11) becomes

$$
2 \cos\left(\frac{\pi(s-m-1)}{2}\right) \Xi_+^\prime(\alpha_1; s) - (-1)^m \Xi_-(\alpha_1; s)
$$

$$
+ 2 \cos\left(\frac{\pi(m+1)}{2}\right) \frac{\alpha_1(0)}{s} - 2 \sin\left(\frac{\pi(\mu-m)}{2}\right) \frac{\alpha_2(0)}{s + 2\nu - 1}.
$$

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This implies that
\[
\Xi_+(\alpha_1; s) - (-1)^m \Xi_-(\alpha_1; s) - \sin\left(\frac{\pi(\mu-m)}{2}\right) \frac{\alpha_2(0)}{s + 2\nu - 1} \tag{6.21}
\]
is holomorphic at \(s = m + 1\). Hence, in order to prove the statements (ii) and (iii), it suffices to give the principal part of \(\Xi_+(\alpha_1; s) + (-1)^m \Xi_-(\alpha_1; s)\) at \(s = m + 1\). We have
\[
Z(\alpha_1, \mathcal{F}(\Delta_m); s) = (\Phi_1(\Delta_m; -s + 1), \Phi_{-1}(\Delta_m; -s + 1)) E(s) \left( \frac{\Xi_+(\alpha_1; s)}{\Xi_-(\alpha_1; s)} \right)
= -2(\sqrt{-1})^m \sin\left(\frac{\pi(\mu-m)}{2}\right) \Phi_1(\Delta_m; -s + 1)(\Xi_+(\alpha_1; s) + (-1)^m \Xi_-(\alpha_1; s)).
\]
Here the first equality follows from Lemma 5.3 and Proposition 4.1, and the second equality follows from (6.19). On the other hand, we have
\[
Z(\alpha_1, \mathcal{F}(\Delta_m); s) = Z_+(\alpha_1, \mathcal{F}(\Delta_m); s) + Z_+ (\alpha_2, \mathcal{F}_{\mu,\nu,\infty}(\Delta_m); -s - 2\nu + 1)
- \frac{\alpha_1(0)\mathcal{F}(\Delta_m)(0)}{s} + \frac{\alpha_2(0)\mathcal{F}_{\mu,\nu,\infty}(\Delta_m)(0)}{s + 2\nu - 1} + \frac{(-1)^m m! \beta_2(m)}{s - m - 1}
- 2(\sqrt{-1})^m \cos\left(\frac{\pi(\mu-m)}{2}\right) \frac{\alpha_2(0)}{(s - m - 1)^2} \tag{6.22}
\]
if \(-2\nu = m\),
\[
0 \tag{otherwise}
\]
by Lemma 6.18 and (6.18). By these equalities, we find that
\[
2 \sin\left(\frac{\pi(\mu-m)}{2}\right) \Phi_1(\Delta_m; -s + 1)(\Xi_+(\alpha_1; s) + (-1)^m \Xi_-(\alpha_1; s))
+ \frac{\alpha_2(0)\mathcal{F}_{\mu,\nu,\infty}(\Delta_m)(0)}{(\sqrt{-1})^m (s + 2\nu - 1)} + \frac{(-1)^m m! \beta_2(m)}{s - m - 1}
- \left\{ \begin{array}{ll}
2 \cos\left(\frac{\pi(\mu-m)}{2}\right) \frac{\alpha_2(0)}{(s - m - 1)^2} & \text{if } -2\nu = m, \\
0 & \text{otherwise}
\end{array} \right. \tag{6.22}
\]
is holomorphic at \(s = m + 1\). By (6.22) and (6.17), if \(2\nu \neq -m\), we know that
\[
\Xi_+(\alpha_1; s) + (-1)^m \Xi_-(\alpha_1; s) - \frac{(\sqrt{-1})^m m! \beta_2(m)}{\pi(s - m - 1)}
\]
is holomorphic at \(s = m + 1\). This completes the proof of the statement (ii).

Assume \(-2\nu = m\). By (6.17) and (6.20), we find that
\[
\frac{\alpha_2(0)\mathcal{F}_{\mu,-\nu,\infty}(\Delta_m)(0)}{(\sqrt{-1})^m (s - m - 1)} - 2 \cos\left(\frac{\pi(\mu-m)}{2}\right) \frac{\alpha_2(0)}{(s - m - 1)^2}
- \left( \frac{2 \sin\left(\frac{\pi(\mu-m)}{2}\right)}{\pi(s - m - 1)} \right) \frac{2 \cos\left(\frac{\pi(\mu-m)}{2}\right) \alpha_2(0)}{s - m - 1} \Phi_1(\Delta_m; -s + 1) \tag{6.23}
\]
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is holomorphic at \( s = m + 1 \). Subtracting (6.23) from (6.22), we get a function
\[
2 \sin \left( \frac{2(s-m-1)}{2} \right) \Phi_1(\Delta_m; -s+1) \\
\times \left( \Xi_+(\alpha; s) + (-1)^m \Xi_- (\alpha; s) + \frac{2 \cos \left( \frac{\pi(s-m)}{2} \right) \alpha_2(0)}{\pi(s-m-1)^2} \right) + \frac{(\sqrt{-1})^m \beta_2(m)}{s-m-1},
\]
which is holomorphic at \( s = m + 1 \). Hence, by (6.17), we know that
\[
\Xi_+(\alpha; s) + (-1)^m \Xi_- (\alpha; s) + \frac{2 \cos \left( \frac{\pi(s-m)}{2} \right) \alpha_2(0)}{\pi(s-m-1)^2} - \frac{(\sqrt{-1})^m \beta_2(m)}{s-m-1}
\]
is holomorphic at \( s = m + 1 \). This completes the proof of the statement (iii). \( \square \)

7 Automorphic distributions and Maass forms

7.1 Moderate growth functions on \( \tilde{G} \)

In this subsection, we give a proof of Proposition 2.14. Let \( \kappa \in \mathbb{C} \). Let \( C^\infty(\tilde{G}/\tilde{K}; \kappa) \) be the subspace of \( C^\infty(\tilde{G}) \) consisting of all functions \( F \) such that
\[
F(\tilde{g}\tilde{k}(\theta)) = F(\tilde{g}) e^{\sqrt{-1} \kappa \theta} \quad (\tilde{g} \in \tilde{G}, \theta \in \mathbb{R}). \tag{7.1}
\]
We define a \( \mathbb{C} \)-linear map \( C^\infty(\tilde{G}/\tilde{K}; \kappa) \ni F \mapsto \phi_F \in C^\infty(\tilde{H}) \) by
\[
\phi_F(z) = F(\tilde{u}(x)\tilde{a}(y)) \quad (z = x + \sqrt{-1} y \in \tilde{H}). \tag{7.2}
\]
By the Iwasawa decomposition (2.13), we know that this map is bijective, and the inverse map \( C^\infty(\tilde{H}) \ni \phi \mapsto F_\phi \in C^\infty(\tilde{G}/\tilde{K}; \kappa) \) of this map is given by
\[
F_\phi(\tilde{g}) = \phi(g\sqrt{-1}) e^{\sqrt{-1} \kappa \theta} \quad (\tilde{g} = (g, \theta) \in \tilde{G}). \tag{7.3}
\]
Let \( \| \cdot \| \) be the euclidean norm on \( M_2(\mathbb{R}) \simeq \mathbb{R}^4 \), that is,
\[
\|g\| = \sqrt{a^2 + b^2 + c^2 + d^2} \quad \left( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \right).
\]
Then it is well-known and easy to show that
\[
\|g_1 g_2\| \leq \|g_1\| \|g_2\| \quad (g_1, g_2 \in M_2(\mathbb{R})), \tag{7.4}
\]
\[
\|kg\| = \|gk\| = \|g\| \quad (k \in K, g \in M_2(\mathbb{R})). \tag{7.5}
\]
Let [M1] and [M2] be the conditions for \( \phi \in C^\infty(\tilde{H}) \) in §2.9 and §2.10, respectively. We consider the following condition [M3] for \( F \in C^\infty(\tilde{G}) \):

[M3] There are \( c, r > 0 \) such that \( |F(\tilde{g})| \leq c\|g\|^r \quad (g \in \tilde{G}) \).
Let Ωκ be the hyperbolic Laplacian of weight κ on ℳ defined by (2.28). Let C be the Casimir element

\[ C = \frac{1}{4}(H^2 + 2E_+E_1 + 2E_-E_1), \]

which generates the center of \( \mathcal{U}(g) \) as a C-algebra.

**Lemma 7.1.** Retain the notation. Let \( F \in C^\infty(\bar{G}/K; \kappa) \).

(i) We have \( F(\gamma g) = (\phi_F|g)(z) \) for \( g \in G \) and \( x = x + \sqrt{-1}y \in \mathfrak{h} \).

(ii) For \( \nu \in \mathbb{C} \), we have \( \Omega\nu \phi_F = (\frac{1}{4} - \nu^2) \phi_F \) if and only if \( \rho(C)F = (\nu^2 - \frac{1}{4})F \).

(iii) The function \( \phi = \phi_F \) satisfies [M1] if and only if \( F \) satisfies [M3].

(iv) The function \( \phi = \phi_F \) satisfies [M2] if \( F \) satisfies [M3].

**Proof.** Our proof is based on the Iwasawa decomposition \( \bar{G} = \bar{U}\bar{A}\bar{K} \). The statement (i) follows from the equality

\[ \gamma g u(x) \alpha(y) = u(Re(gz))\alpha(Im(gz))k(-arg J(g,z)) \]

for \( g \in G \) and \( x = x + \sqrt{-1}y \in \mathfrak{h} \). The statement (ii) follows from

\[ (\rho(C)F)(u(x)\alpha(y)) = - (\Omega\nu \phi_F)(z) \quad (z = x + \sqrt{-1}y \in \mathfrak{h}) \]

and \( \rho(\bar{k}) \circ \rho(\bar{C}) = \rho(\bar{C}) \circ \rho(\bar{k}) \) \((k \in K)\). The statement (iii) follows from (7.5) and the equality \( \|u(x)a(y)\|^2 = (|z|^2 + 1)/y \) for \( z = x + \sqrt{-1}y \in \mathfrak{h} \). By (7.4), we have

\[ \|\gamma x a(y)\| \leq \|\gamma\| \|u(x)\| \|a(y)\| = \|\gamma\| \sqrt{2 + x^2/\sqrt{y + y^{-1}}} \]

\[ \leq \|\gamma\| \sqrt{2 + \frac{1}{4}} \sqrt{y + \frac{4}{3}y} = \frac{\|\gamma\|\sqrt{2t}}{2} y^{1/2} = \frac{\|\gamma\|\sqrt{2t}}{2} \text{Im}(z)^{1/2} \]

for \( \gamma \in SL(2,\mathbb{Z}) \) and \( z = x + \sqrt{-1}y \in \mathcal{D} \). By this inequality and the statement (i), we obtain the statement (iv). \( \square \)

Let \( \mathcal{M}_\nu(\mathfrak{h}; \kappa) \) be the subspace of \( C^\infty(\mathfrak{h}) \) defined in §2.9. Let \( \mathcal{M}_\nu(\bar{G}/\bar{K}; \kappa) \) be a subspace of \( C^\infty(\bar{G}/\bar{K}; \kappa) \) consisting of all functions \( F \) satisfying [M3] and \( \rho(C)F = (\nu^2 - \frac{1}{4})F \). Then we obtain the following corollary of Lemma 7.1.

**Corollary 7.2.** Let \( \kappa \in \mathbb{C} \). A bijective \( \mathbb{C} \)-linear map from \( \mathcal{M}_\nu(\bar{G}/\bar{K}; \kappa) \) to \( \mathcal{M}_\nu(\mathfrak{h}; \kappa) \) is given by \( F \mapsto \phi_F \), and its inverse map is given by \( \phi \mapsto F_\phi \).

Let us prove Proposition 2.14. We assume \( \kappa \in \mathbb{R} \) until the end of this subsection. Let \( \Gamma \) be a cofinite subgroup of \( SL(2,\mathbb{Z}) \) such that \( -I_2 \in \Gamma \). Let \( v \) be a multiplier system on \( \Gamma \) of weight \( \kappa \). Proposition 2.14 follows immediately from Lemma 7.1 and the following lemma.

**Lemma 7.3.** Retain the notation. Let \( \phi \in C^\infty(\mathfrak{h}) \) satisfying [M2] and the equality \( \phi|\kappa = v(\gamma)\phi \quad (\gamma \in \Gamma) \). Then \( F = F_\phi \) satisfies [M3].
Proof. Let \( \{ \gamma_i \}_{i=1}^d \) be a complete system of representatives of \( \Gamma \setminus SL(2, \mathbb{Z}) \). By the condition [M2], there are \( c_0, r_0 > 0 \) such that

\[
| (\phi|_{\gamma_i})(z) | \leq c_0 y^{r_0} \quad (z = x + \sqrt{-1}y \in \mathcal{D}, \ 1 \leq i \leq d). \tag{7.6}
\]

Let \( \tilde{g} \in \tilde{G} \). Since \( \mathcal{D} \) is the closure of the fundamental domain of \( SL(2, \mathbb{Z}) \setminus \mathfrak{H} \simeq SL(2, \mathbb{Z}) \setminus G/K \), we can take \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL(2, \mathbb{Z}), \ z_0 = x_0 + \sqrt{-1}y_0 \in \mathcal{D} \) and \( \theta \in \mathbb{R} \) so that \( \tilde{g} = \gamma \tilde{u}(x_0) \tilde{a}(y_0) \tilde{k}(\theta) \). By Lemma 7.1 (i), we have

\[
| F_{\phi}(\tilde{g}) | = | F_{\phi}(\gamma \tilde{u}(x_0) \tilde{a}(y_0)) e^{\sqrt{-1}t \theta} | = | (\phi|_{\gamma})(z_0) |. \tag{7.7}
\]

By (7.5) and \( \gamma \tilde{u}(x_0) \tilde{a}(y_0) = \left( \frac{a \sqrt{y_0}}{c}, \frac{b \sqrt{y_0}}{c}, \frac{ax_0 + b}{c}, \frac{dx_0 + c}{c} \right) \), we have

\[
\| g \| = \| \gamma \tilde{u}(x_0) \tilde{a}(y_0) \| \geq \sqrt{(a \sqrt{y_0})^2 + (c \sqrt{y_0})^2} = \sqrt{(a^2 + c^2)y_0} \geq \sqrt{y_0}. \tag{7.8}
\]

We take \( \gamma_0 \in \Gamma \) and \( 1 \leq j \leq d \) so that \( \gamma = \gamma_0 \gamma_j \). Since \( \phi|_{\gamma_0} = v(\gamma_0) \phi \), we have

\[
(\phi|_{\gamma})(z) = e^{\sqrt{-1}t \theta} ((\phi|_{\gamma_0})|_{\gamma_j})(z) = e^{\sqrt{-1}t \theta} v(\gamma_0)(\phi|_{\gamma_j})(z) \quad (z \in \mathfrak{H}) \tag{7.9}
\]

with \( t = \arg J(\gamma_0, \gamma_j z) + \arg J(\gamma_j, z) - \arg J(\gamma, z) \). By (7.6), (7.7), (7.8) and (7.9), we have

\[
| F_{\phi}(\tilde{g}) | = | (\phi|_{\gamma})(z_0) | = | (\phi|_{\gamma_j})(z_0) | \leq c_0 y_0^{r_0} \leq c_0 \| g \|^{2r_0},
\]

and this completes the proof. \( \square \)

### 7.2 Representations of moderate growth

In this subsection, we give a proof of Proposition 2.13 (i).

**Lemma 7.4.** Let \( \mu, \nu \in \mathbb{C} \). For \( \tilde{g} = (g, \theta) \in \tilde{G} \) and \( F \in \mathcal{I}_{\mu, \nu} \), we have

\[
| F(\tilde{g}) | \leq e^{-\mu(\theta + \arg J(\phi, \sqrt{-1}))} \| g \|^{|2\Re(\nu)+1|} | F |_{K}. \]

**Proof.** Let \( \tilde{g} = (g, \theta) \in \tilde{G} \) and set \( x = \Re(g \sqrt{-1}) \) and \( y = \Im(g \sqrt{-1}) \). By (2.13), we have \( \tilde{g} = \tilde{u}(x) \tilde{a}(y) \tilde{k}(\theta) \). Hence, we have

\[
\| g \| = \| u(x) a(y) \| = \sqrt{y + y^{-1} + x^2 y^{-1}} \geq \sqrt{y + y^{-1}}
\]

by (7.5). The assertion follows immediately from this inequality and (2.15). \( \square \)

For \( j \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{U}_j(\mathfrak{g}_C) \) be the space defined in §6.1, that is, the subspace of \( \mathcal{U}(\mathfrak{g}_C) \) spanned by the products of \( j \) or less elements of \( \mathfrak{g}_C \).
Lemma 7.5. For $j \in \mathbb{Z}_{\geq 0}$ and $X \in \mathcal{U}_j(\mathfrak{g}_C)$, there are positive integer $m$, $Y_1, Y_2, \ldots, Y_m \in \mathcal{U}_j(\mathfrak{g}_C)$ and polynomial functions $p_i$ of degree at most $2j$ on $M_2(\mathbb{R}) \cong \mathbb{R}^4$ ($1 \leq i \leq m$) such that

$$
\rho(X) \circ \rho(\tilde{g}) = \sum_{i=1}^{m} p_i(g) \rho(\tilde{g}) \circ \rho(Y_i) \quad (\tilde{g} = (g, \theta) \in \tilde{G}).
$$

Proof. Let $\tilde{g} = (g, \theta) \in G$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 6.1 (ii), we have

$$
\rho(X) \circ \rho(\tilde{g}) = \rho(\tilde{g}) \circ \rho(\text{Ad}(g^{-1})X) \quad (X \in \mathcal{U}(\mathfrak{g}_C)).
$$

By this equality and the formulas

$$
\text{Ad}(g^{-1})H = (ad + bc)H + 2bdE_+ - 2acE_-,
$$

$$
\text{Ad}(g^{-1})E_+ = cdH + d^2E_+ - c^2E_-,
$$

$$
\text{Ad}(g^{-1})E_- = -abH - b^2E_+ + a^2E_-,
$$
on the basis $\{H, E_+, E_-, \}$ of $\mathfrak{g}_C$, we obtain the assertion. \hfill \Box

Proposition 7.6. Let $\mu, \nu \in \mathbb{C}$. For $j \in \mathbb{Z}_{\geq 0}$ and $X \in \mathcal{U}_j(\mathfrak{g}_C)$, there are $c > 0$, $m \in \mathbb{Z}_{>0}$ and $Y_1, Y_2, \ldots, Y_m \in \mathcal{U}_j(\mathfrak{g}_C)$ such that

$$
Q_X(\rho(\tilde{g})F) \leq c\|g\|^{2 \text{Re}(\nu) + 1} + 2j \sum_{i=1}^{m} Q_{Y_i}(F) \quad (g \in G, \ F \in I_{\mu,\nu}^\infty). \quad (7.10)
$$

Proof. By Lemma 7.4 and (7.5), we have

$$
|\rho(\tilde{g})F'(k)| = |F'(k^*g)| \leq e^{3\pi|\text{Im}(\mu)|}\|g\|^{2\text{Re}(\nu) + 1}|F|_K \quad (k \in K, \ g \in \tilde{G}).
$$

This implies that $|\rho(\tilde{g})F|_K \leq e^{3\pi|\text{Im}(\mu)|}\|g\|^{2\text{Re}(\nu) + 1}|F|_K$ ($g \in G$). The assertion follows from this inequality and Lemma 7.5. \hfill \Box

Because of (2.21), we obtain the following as a corollary of Proposition 7.6.

Corollary 7.7. Let $\mu, \nu \in \mathbb{C}$. For $\lambda \in I_{\mu,\nu}^\infty$, there are $j \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$, $X_1, X_2, \ldots, X_m \in \mathcal{U}_j(\mathfrak{g}_C)$ and $c > 0$ such that

$$
\lambda(\rho(\tilde{g})F) \leq c\|g\|^{2 \text{Re}(\nu) + 1} + 2j \sum_{i=1}^{m} Q_{X_i}(F) \quad (g \in G, \ F \in I_{\mu,\nu}^\infty).
$$

Let us prove Proposition 2.13 (i). Let $\mu, \nu \in \mathbb{C}$ and $\kappa \in \mu + 2\mathbb{Z}$. For a distribution $\lambda$ on $I_{\mu,\nu}^\infty$, we define a function $\mathcal{P}_{\nu,\kappa}^G(\lambda)(\tilde{g})$ on $\tilde{G}$ by

$$
\mathcal{P}_{\nu,\kappa}^G(\lambda)(\tilde{g}) = e^{\pi \sqrt{-1}\kappa/2} \lambda(\tilde{g}) F_{\nu,\kappa} \quad (\tilde{g} \in \tilde{G}),
$$

where $F_{\nu,\kappa}$ is an element of $I_{\mu,\nu}^\infty$ defined by (2.30). Since

$$
\mathcal{P}_{\nu,\kappa}(\lambda)(z) = \phi_{\mathcal{P}_{\nu,\kappa}^G(\lambda)}(z) = \mathcal{P}_{\nu,\kappa}^G(\lambda)(\tilde{u}(x)\tilde{h}(y)) \quad (z = x + \sqrt{-1}y \in \mathfrak{h}),
$$

Proposition 2.13 (i) follows immediately from Lemma 7.1 (i), Corollary 7.2 and Proposition 7.8 below.
Proposition 7.8. Let \( \mu, \nu \in \mathbb{C} \), and \( \kappa \in \mu + 2\mathbb{Z} \). Let \( \lambda \) be a distribution on \( I_{\mu,\nu}^\infty \). Then we have \( \mathcal{P}_{\mu,\nu}^G(\lambda) \in \mathcal{M}_{\nu}(\mathbb{G}/\mathbb{K}; \kappa) \).

Proof. By the definition (2.30) of \( I \) expressions of multiplier systems. 

In this subsection, we give a proof of Theorem 2.18. Contrast to the previous subsection, we give a proof of Theorem 2.18. 

7.3 Twists of automorphic distributions

By Corollary 7.7, the function \( F = \mathcal{P}_{\mu,\nu}^G(\lambda) \) satisfies [M3].

7.3 Twists of automorphic distributions

In this subsection, we give a proof of Theorem 2.18. Contrast to the previous subsection, our method here is simplified and does not depend on the explicit expressions of multiplier systems.

Let \( \mu \in \mathbb{R}, \nu \in \mathbb{C} \) and \( N \in \mathbb{Z}_{>0} \). Let \( \kappa \in \mu + 2\mathbb{Z} \). Let \( v \) be a multiplier system on \( \Gamma_0(N) \) of weight \( \kappa \). Let \( L = u + \mathbb{Z} \) and \( \bar{L} = N^{-1}(\bar{u} + \mathbb{Z}) \) with \( 0 \leq u, \bar{u} < 1 \) determined by \( v(u(1)) = e^{2\pi\sqrt{-1}u} \) and \( v(\bar{u}(-N)) = e^{2\pi\sqrt{-1}\bar{u}} \). Let \( P_N \) be a subset of the set of positive odd prime integers not dividing \( N \), such that \( P_N \cap \{am + b \mid m \in \mathbb{Z} \} \neq \emptyset \) for any integers \( a, b \) coprime to each other. Theorem 2.18 follows immediately from Theorems 2.9, 2.10, Propositions 2.13 (ii), 2.16, and Lemmas 7.9, 7.10 below.

Lemma 7.9. Retain the notation.

(i) Let \( \lambda \in (I_{\mu,\nu}^\infty)^{\Gamma_0(N):\tilde{x}_v} \). Then we have

\[
\lambda(\rho(\gamma)F) = \nu(\gamma)\lambda(F) \quad \left( \gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}, \ F \in I_{\mu,\nu}^\infty \right)
\]

for any \( a, b, c, d \in \mathbb{Z} \) such that \( d > 0 \) and \( ad - Nbc = 1 \).

(ii) Let \( \lambda \in (I_{\mu,\nu}^\infty)^{\bar{\Gamma}_0(N):\tilde{x}_v} \), and assume that (7.12) holds for any \( a, b, c, d \in \mathbb{Z} \) such that \( d \in P_N \) and \( ad - Nbc = 1 \). Then we have \( \lambda \in (I_{\mu,\nu}^\infty)^{\tilde{x}_v} \).

Proof. The statement (i) follows from the definition of \( (I_{\mu,\nu}^\infty)^{\Gamma_0(N):\tilde{x}_v} \). Since

\[
\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} u(m) = \begin{pmatrix} a & am + b \\ Nc & Ncm + d \end{pmatrix} \quad (a, b, c, d, m \in \mathbb{Z}),
\]

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we note that, for any element $\tilde{\gamma}$ of $\Gamma_0(N)$, there are some $a, b, c, d, m_1, m_2 \in \mathbb{Z}$ such that $d \in \mathbb{F}_N$, $ad - Nbc = 1$ and

$$\tilde{\gamma} = \left( \begin{array}{cc} a & b \\ Nc & d \end{array} \right) \tilde{u}(m_1) \tilde{k}(m_2 \pi).$$

Hence, the statement (ii) follows from (2.39) and the definition of $(I_{\mu,\nu}^{-\infty})_{L,L}$. \hfill \Box

**Lemma 7.10.** Retain the notation, and let $d \in \mathbb{Z}_{>0}$ coprime to $N$. Let $(\alpha, \beta) \in \mathcal{M}(L)_{\mu,\nu}^0 \times \mathfrak{N}(S_{\nu}(L))$ and $(\hat{\alpha}, \hat{\beta}) \in \mathcal{M}(\hat{L})_{\mu,\nu}^0 \times \mathfrak{N}(S_{\nu}(\hat{L}))$ such that $(\lambda_{\alpha,\beta})_{\infty} = \lambda_{\hat{\alpha},\hat{\beta}}$. For a Dirichlet character $\psi$ modulo $d$, we take $\alpha_{\psi}$, $\beta_{\psi}$, $\hat{\alpha}_{\psi,\nu}$ and $\hat{\beta}_{\psi,\nu}$ as in §2.12. Then $\lambda = \lambda_{\alpha,\beta}$ satisfies (7.12) for any $a, b, c \in \mathbb{Z}$ such that $ad - Nbc = 1$, if and only if, the equality

$$(\lambda_{\alpha_{\psi},\beta_{\psi}})_{\infty} = \lambda_{\hat{\alpha}_{\psi,\nu},\hat{\beta}_{\psi,\nu}}$$

holds for any Dirichlet character $\psi$ modulo $d$.

**Proof.** Let $a, b, c \in \mathbb{Z}$ such that $ad - Nbc = 1$, and we set $\gamma = \left( \begin{array}{cc} a & b \\ Nc & d \end{array} \right)$. Since $s_{\gamma} = \tilde{u}(b/d)\tilde{a}(d^{-2})\tilde{n}(Nc/d)$, by the substitution $\rho(\tilde{a}(d^{-2})\tilde{n}(Nc/d))F \to F$, the equality (7.12) becomes

$$\lambda(\rho(\tilde{u}(b/d))F) = v(\gamma)\lambda(\rho(\tilde{n}(-Nc/d)\tilde{a}(d^{2}))F) \quad (F \in I_{\mu,\nu}^{\infty}).$$

(7.13)

Multiplying the both sides by $e^{-2\pi \sqrt{-1}ub/d}$, and applying the equality $\lambda(F) = \lambda_{\infty}(F_{\infty})$ ($F \in I_{\mu,\nu}^{\infty}$) with Lemma 2.3 (ii) to the right-hand side, the equality (7.13) becomes

$$e^{-2\pi \sqrt{-1}ub/d} \lambda(\rho(\tilde{u}(b/d))F) = v(\gamma)e^{-2\pi \sqrt{-1}ub/d}\lambda_{\infty}(\rho(\tilde{n}(Nc/d)\tilde{a}(d^{-2}))F_{\infty}) \quad (F \in I_{\mu,\nu}^{\infty}).$$

(7.14)

If $\lambda = \lambda_{\alpha,\beta}$, then $\lambda_{\infty} = \lambda_{\hat{\alpha},\hat{\beta}}$ and the equality (7.14) becomes

$$\sum_{l \in L} e^{2\pi \sqrt{-1}(l-\nu)b/d} \alpha(l)J_{l}(F) + \sum_{m \in S_{\nu}(L)} e^{-2\pi \sqrt{-1}ub/d} \beta(m)\delta^{(m)}_{\infty}(F_{\infty})$$

$$= \sum_{l \in L} v(\gamma)e^{2\pi \sqrt{-1}(Nl-ub)/d}d^{2\nu-1}\hat{\alpha}(l)J_{l/d}(F_{\infty})$$

$$+ \begin{cases} 4 \cos \left( \frac{\nu\pi}{2} \right) v(\gamma)e^{-2\pi \sqrt{-1}ub/d}\hat{\alpha}(0)\delta^{(0)}_{\infty}(F_{\infty})d^{-1} \log d & \text{if } 0 \in \hat{L} \text{ and } \nu = 0, \\
0 & \text{otherwise.} \end{cases}$$

$$+ \sum_{m \in S_{\nu}(L)} v(\gamma)e^{-2\pi \sqrt{-1}ub/d}\hat{\alpha}(m)d^{-2\nu-2m-1}\delta^{(m)}_{\infty}(F_{\infty}) \quad (F \in I_{\mu,\nu}^{\infty})$$

(7.15)
by Propositions 6.12 and 6.13. Multiplying by the respective sides of
\[ \psi(b) = \psi(-Nc), \]
and taking the sum over all integers \( b \) in a reduced residue system modulo \( d \) (here \( c \) ranges over a reduced residue system modulo \( d \), since \( ad - Nbc = 1 \)), we have
\[
\sum_{l \in L} \alpha_l(l) J_l(F) + \sum_{m \in S_v(L)} \beta_m(m) \delta_\infty^{(m)}(F) = \sum_{l \in L} \hat{\alpha}_{\psi,v}(l) J_{l/d^2}(F_\infty) + \sum_{m \in S_v(L)} \hat{\beta}_{\psi,v}(m) \delta_\infty^{(m)}(F_\infty) \quad (F \in I_{\mu,\nu}^\infty)
\] (7.16)
for a Dirichlet character \( \psi \) modulo \( d \). Conversely, multiplying by the respective sides of
\[
\frac{1}{\#(\mathbb{Z}/d\mathbb{Z})^\times} \psi(b) = \frac{1}{\#(\mathbb{Z}/d\mathbb{Z})^\times} \psi(-Nc),
\]
and taking the sum over all Dirichlet character \( \psi \) modulo \( d \), we obtain (7.15) from (7.16). Here \( \#(\mathbb{Z}/d\mathbb{Z})^\times \) is the cardinality of \( (\mathbb{Z}/d\mathbb{Z})^\times \), and we use the orthogonality relation
\[
\frac{1}{\#(\mathbb{Z}/d\mathbb{Z})^\times} \sum_{\psi} \psi(m_1) \psi(m_2) = \begin{cases} 1 & \text{if } m_1 \equiv m_2 \mod d, \\ 0 & \text{otherwise} \end{cases} \quad (m_1, m_2 \in \mathbb{Z}),
\]
where the sum \( \sum_{\psi} \) is over all Dirichlet character \( \psi \) modulo \( d \). Therefore, we know that \( \lambda = \lambda_{\alpha,\beta} \) satisfies (7.12) for any \( a, b, c \in \mathbb{Z} \) such that \( ad - Nbc = 1 \), if and only if, (7.16) holds for any Dirichlet character \( \psi \) modulo \( d \). \( \square \)

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