REAL COMPONENTS OF MODULAR CURVES

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ABSTRACT. We study the real components of modular curves. Our main result is an abstract group-theoretic description of the real components of a modular curve defined by a congruence subgroup of level $N$ in terms of the corresponding subgroup of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. We apply this result to several families of modular curves (such as $X_0(N)$, $X_1(N)$, etc.) to obtain formulas for the number of real components. Somewhat surprisingly, the multiplicative order of 2 modulo $N$ has a strong influence in many cases: for instance, if $N$ is an odd prime then the real locus of $X_1(N)$ is connected if and only if $-1$ and 2 generate $(\mathbb{Z}/N\mathbb{Z})^\times$.

CONTENTS

1. Introduction 1
2. Real Fuchsian groups 4
3. Components for real Fuchsian groups 6
4. The graph associated to a real subgroup of $\text{SL}_2(R)$ 11
5. Components for real congruence groups 20
6. Examples 22
References 30
Appendix A. Tables 31

1. INTRODUCTION

This article is a study of the real components of modular curves. Our main result is an abstract group-theoretic description of the real components of a modular curve defined by a congruence subgroup of level $N$ in terms of the corresponding subgroup of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. We apply this result to several families of modular curves to obtain formulas for the number of real components.

1.1. Description of main result. We now describe our main result in some detail. We begin in §2 and §3 with a general study of real components of upper half-plane quotients. To this end, let $\Gamma$ be a Fuchsian group and let $X_\Gamma = \mathfrak{h}^*/\Gamma$ be the associated quotient, including cusps. We assume throughout that $X_\Gamma$ is compact. So far, $X_\Gamma$ does not have any real structure. To obtain one, we assume that we have an anti-holomorphic involution $c$ of the upper half-plane ("complex conjugation") which preserves $\Gamma$. Thus $c$ descends to $X_\Gamma$, and we define $X_\Gamma(\mathbb{R})$ to be its fixed locus. By the "real components" of $X_\Gamma$, we mean the connected components of the space $X_\Gamma(\mathbb{R})$. Any real component is diffeomorphic to the circle.

Any point on $X_\Gamma(\mathbb{R})$ lifts to a point $z$ in $\mathfrak{h}^*$ satisfying $cz = \gamma z$ for some $\gamma \in \Gamma$; in fact, this $\gamma$ is "admissible," meaning it satisfies $\gamma^2 = 1$. For such $\gamma$, let $C_\gamma$ denote the locus in $\mathfrak{h}^*$ defined by the equation $cz = \gamma z$. If $\Gamma$ acts freely on $\mathfrak{h}$ with compact quotient, it is not difficult to see that mapping $\gamma$ to the image in $X_\Gamma$ of $C_\gamma$ defines a bijection between admissible twisted conjugacy classes in $\Gamma$ and real components of $X_\Gamma$. (We say that $\gamma$ and $\gamma'$ are "twisted conjugate" if $\gamma = \delta^\gamma \gamma' \delta^{-1}$ for some $\delta \in \Gamma$.) For general $\Gamma$, this result is no longer true: cusps and elliptic points interfere. To remedy
this, we define a graph $\Xi_\Gamma$ whose vertices are the real cusps and real elliptic points of even order on $X_\Gamma$ and whose edges are the admissible twisted conjugacy classes of $\Gamma$. This graph is allowed to have edges which connect to no vertices, which we picture as circles. We then show that $\Xi_\Gamma$ is homeomorphic to $X_\Gamma(R)$.

In §3 we set aside Fuchsian groups and put ourselves in the following abstract situation: we have a ring $R$ of finite characteristic (satisfying an additional hypothesis at 2), an involution $C \in \GL_2(R)$ of determinant $-1$ ("complex conjugation") and a subgroup $G$ of $\SL_2(R)$ stable under conjugation by $C$ and containing $-1$. To this data, we associate a graph $\Xi_G$. The vertices of this graph are of two types: elliptic and parabolic. The definition of $\Xi_G$ is somewhat involved in the presence of elliptic vertices, so for the purposes of the present discussion let us ignore them. The parabolic vertices of $\Xi_G$ are the basis vectors in $R^2$ which are eigenvectors of $Cg$ for some admissible $g \in G$, modulo the action of $G$. (Admissibility in this setting is defined just as before.) Two parabolic vertices $x$ and $y$ are connected if $\langle x, y \rangle$ is $\pm 1$ or $\pm 2$ and there exists $g \in G$ such that

$$Cg + 1 = \left(\frac{2}{\langle x, y \rangle}\right) \langle -, y \rangle x.$$ 

The quantity in parentheses is always taken to be $\pm 1$ or $\pm 2$. Given $x$, $y$ and $g$ as above, one has $Cgx = x$ and $Cgy = -y$, though the above identity is stronger than these two in general. We give an explanation for the pairing condition in §4. The main result of §3 states that the graph $\Xi_G$ is a union of cycles, i.e., every vertex has valence two and there are no infinitely long paths.

In §3 we turn to the study of real components for congruence subgroups of $\SL_2(Z)$. Thus let $\Gamma$ be a subgroup of $\SL_2(Z)$ containing $\Gamma(N)$ for some $N$, and let $c$ be a complex conjugation on $\Gamma$ preserving $\Gamma$. We must assume that $c$ also preserves $\Gamma(N)$; without this assumption, the behavior of the real locus is very different. Associated to $\Gamma$ we have a matrix $C$ in $\GL_2(R)$, which, in this setting, we show belongs to $\GL_2(Z)$. Let $G$ be the image of $\Gamma$ in $\SL_2(Z/NZ)$. We can then build two graphs: $\Xi_\Gamma$, constructed in §3 and $\Xi_G$, constructed in §4. Our main theorem is that these two graphs are naturally isomorphic. This gives a description of $X_\Gamma(R)$ purely in terms of $G$ and $C$.

1.2. Sample of specific results. In §6 we apply the theory we have developed to several families of modular curves to obtain formulas for the number of real components. We state two of those results here.

**Proposition 1.2.1.** Let $N$ be a positive integer. If $N$ is a power of 2 then $X_0(N)$ has one real component. Otherwise, let $n$ be the number of distinct odd primes factors of $N$ and let $\epsilon$ be 1 if $N$ is divisible by 8 and 0 otherwise. Then $X_0(N)$ has $2^{n+\epsilon-1}$ real components.

This result was suggested by Frank Calegari based on computations of William Stein [St]. To state our second formula, let us introduce some notation. Let $\phi(N)$ denote the cardinality of $(\mathbb{Z}/N\mathbb{Z})^\times$, as usual. For an odd integer $N$, let $\psi(N)$ denote the order of the quotient of the group $(\mathbb{Z}/N\mathbb{Z})^\times$ by the subgroup generated by $-1$ and 2. We then have:

**Proposition 1.2.2.** Let $N$ be a positive integer, and write $N = 2^r N'$ with $N'$ odd. Then the number of real components of $X_1(N)$ is given by:

$$\begin{cases}
\psi(N') & \text{if } r \leq 1 \\
\frac{1}{4} \phi(N) & \text{if } r \geq 2 \text{ and } N \neq 4 \\
1 & \text{if } N = 4.
\end{cases}$$

Given the current state of Artin’s primitive root conjecture, it seems that one knows that $-1$ and 2 generate $(\mathbb{Z}/N\mathbb{Z})^\times$ for infinitely many prime numbers $N$ only under the assumption of the generalized Riemann hypothesis. Thus one can show that the real locus of $X_1(N)$ is connected for infinitely many prime $N$ only under this assumption as well.
1.3. Real component groups of modular Jacobians. Real components of a curve are closely related to real components of its Jacobian: if $X/R$ is a smooth projective curve with a real point, then the set of real components of its Jacobian is an $F_2$-vector space whose dimension is one less than the number of real components of $X$. See [GH] for a proof. Thus our theorems on the real components of modular curves can be translated to theorems about the structure of the real component group of modular Jacobians. (In fact, the computations of [ST] are for Jacobians.)

1.4. The role of the number 2. As we have seen, the number 2 plays a special role in the study of real components of congruence groups. For instance, in the abstract setting of two parabolic vertices $x$ and $y$ to be connected we require $(x,y)$ to be $\pm 1$ or $\pm 2$. To explain the source of this condition, let $\Gamma$ be a subgroup of $SL_2(\mathbb{Z})$ without elliptic points and stable under the “standard” complex conjugation $c_0$ given by $x+iy \mapsto x-iy$. Then an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\Gamma$ is admissible if and only if $a = d$. Assuming $\gamma$ is admissible (and $c \neq 0$), the curve $C_\gamma$ contains the two cusps $-a\pm 1$. We thus have an edge in $\Xi_\Gamma$ between these two cusps, and so there is a corresponding edge in $\Xi_G$. Recall that parabolic vertices of $\Xi_G$ are represented by elements of $(\mathbb{Z}/N\mathbb{Z})^2$. A cusp $p/q \in \mathbb{Q}P^1$ of $\Gamma$ corresponds to the parabolic vertex represented by $(p,q)$ in $(\mathbb{Z}/N\mathbb{Z})^2$, assuming the fraction $p/q$ is in lowest terms. We therefore see that the edge of $\Xi_G$ corresponding to $\gamma$ connects the two parabolic vertices

$$\left(\frac{-a + 1}{d_1}, \frac{c}{d_1}\right), \quad \left(\frac{-a - 1}{d_2}, \frac{c}{d_2}\right),$$

where $d_1 = \gcd(-a + 1, c)$ and $d_2 = \gcd(-a - 1, c)$. The inner product of these two vectors is

$$\frac{2c}{d_1d_2},$$

which is equal to $\pm 1$ or $\pm 2$. This fact from elementary number theory is therefore responsible for the role of 2.

1.5. Some further results and questions. We now list some further results we establish:

- Our proof of the structure theorem for $\Xi_G$ can easily be adapted into an efficient algorithm to compute real components of modular curves corresponding to real congruence groups.
- If $\Gamma$ is a real congruence subgroup of odd level without real elliptic points of even order, then this algorithm simplifies into a formula: the number of real components of $X_\Gamma$ is half the number of orbits of multiplication by 2 on the set of real cusps. (Here multiplication by 2 takes a cusp $p/q$, in lowest terms, to $p'/q'$, where $p'$ and $q'$ are coprime integers equivalent to $2p$ and $2q$ modulo the level.)
- We establish a product formula for the graph $\Xi$. Namely, if $G_i$ is a subgroup of $SL_2(R_i)$ stable under $C_i$, for $i = 1, 2$, and $G = G_1 \times G_2$ is the resulting subgroup of $SL_2(R_1 \times R_2)$, then $\Xi_G$ can be recovered as a sort of product of $\Xi_{G_1}$ and $\Xi_{G_2}$. In certain circumstances, this allows one to reduce computation of $\Xi_G$ to the case of prime power level, which is often easier; see the example of $X_0(N)$ in [4] for a good illustration of this.
- In [4] we classify admissible elements of a real Fuchsian group into several types according to their twisted centralizers. In [4] we give examples showing that every type can occur, even for congruence subgroup of $SL_2(\mathbb{Z})$.

Here are some questions we have:
• We construct the graph $\Xi_G$ in great generality: $G$ can be any “real” subgroup of $\text{SL}_2(R)$, for any ring $R$. However, these graphs are only directly related to modular curves when $R = \mathbb{Z}/N\mathbb{Z}$. Do these graphs have any meaning for other rings?

• There are certain real modular curves that do not fit into our theory, such as those corresponding to non-congruence subgroups, or the twisted form of $X_0(N)$ discussed in §6. Can our theory be generalized to accommodate these cases?

• In [Sh2], Shimura showed that the canonical model of a Shimura curve which is not a modular curve does not have any real points. However, twisted forms of such Shimura curves can have real points. Can our theory be extended to cover these curves?

• Assuming 2 is a primitive root modulo infinitely many primes, there is no bound to the number of cusps that can occur on real components of $X_1(N)$. However, no real component of any modular curve we have examined contains more than 18 elliptic points of even order. How many elliptic points of even order can occur on a real component?

1.6. Notation and terminology. We freely use terminology and basic results concerning group actions on the upper half plane. See the first chapter of [Sh] for background. For a ring $R$, we write $\text{PSL}_2(R)$ to mean $\text{SL}_2(R)/\{\pm 1\}$, which is typically not the set of $R$-points of the variety $\text{PSL}_2$. We use the term “graph” in a very general (even vague) sense; unless otherwise specified, we allow loops (edges from a vertex to itself), multiple edges between two vertices and even edges which do not contain any vertices (which we picture as circles). We write $\phi(N)$ for the order of $(\mathbb{Z}/N\mathbb{Z})^\times$ and $\psi(N)$ for the order of the quotient of this group by the subgroup generated by $-1$ and 2.

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2. Real Fuchsian groups

In this section, we introduce the notion of a real Fuchsian group, which is a pair consisting of a discrete subgroup of $\text{PSL}_2(\mathbb{R})$ and a “complex conjugation” of the upper-half plane under which the group stable.

2.1. Complex conjugations. Let $\mathfrak{h}$ denote the upper half-plane. A complex conjugation on $\mathfrak{h}$ is an anti-holomorphic involution $c : \mathfrak{h} \to \mathfrak{h}$. For example, the map $c_0 : \mathfrak{h} \to \mathfrak{h}$ defined by

$$c_0(x + iy) = -x + iy$$

is a complex conjugation, which we call the standard complex conjugation. It is distinguished in two ways. First, under the isomorphism of $\mathfrak{h}$ with the unit disc given by the Cayley transformation, $c_0$ corresponds to the usual complex conjugation on the disc. Second, a modular form $f$ on $\mathfrak{h}$ has real Fourier coefficients if and only if $f(c_0z) = f(z)$. This follows easily from the identity $q(z) = q(c_0z)$, where $q(z) = e^{2\pi iz}$. Despite the appearance of $c_0$ as preferred, it will be convenient for us to have the flexibility of allowing arbitrary complex conjugations.

Let $c$ be a complex conjugation on $\mathfrak{h}$. If $\gamma$ is an element of $\text{PSL}_2(\mathbb{R})$ then $c\gamma c$ is a holomorphic automorphism of $\mathfrak{h}$, and therefore given by an element $\gamma^c$ of $\text{PSL}_2(\mathbb{R})$. The map $\gamma \mapsto \gamma^c$ is a group automorphism of $\text{PSL}_2(\mathbb{R})$ of order 2. We call an element $\gamma$ of $\text{PSL}_2(\mathbb{R})$ admissible (with respect to $c$) if $\gamma^c = \gamma^{-1}$. The map $c\gamma$ of $\mathfrak{h}$ is a complex conjugation if and only if $\gamma$ is admissible with respect to $c$. If $c'$ is a second complex conjugation, then $c' = c\gamma$ for some admissible $\gamma$.

Given an element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\text{PGL}_2(\mathbb{R})$ of negative determinant and an element $z$ of $\mathfrak{h}$, we put

$$gz = \frac{az + b}{cz + d}.$$
where \( \overline{z} \) denotes the usual complex conjugate of the element \( z \in \mathbb{CP}^1 \). The above formula extends the action of \( \text{PSL}_2(\mathbb{R}) \) on \( h \) to an action of all of \( \text{PGL}_2(\mathbb{R}) \). Every anti-holomorphic automorphism of \( h \) is given by an element of \( \text{PGL}_2(\mathbb{R}) \) of negative determinant.

It follows from the above discussion that complex conjugations on \( h \) correspond to elements of \( \text{PGL}_2(\mathbb{R}) \) of negative determinant which square to the identity. Given a complex conjugation \( c \), there is a unique matrix \( C \in \text{GL}_2(\mathbb{R}) \) (up to signs) of trace 0 and determinant \(-1\) which induces \( c \) on \( h \); we call either of \( \pm C \) the matrix associated to \( c \). For example, the matrix associated to \( c_0 \) is given by

\[
C_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Note that \( \gamma \) is admissible with respect to \( c \) if and only if the matrix \( C\gamma \) has trace 0. If \( C \) and \( C' \) are two elements of \( \text{GL}_2(\mathbb{R}) \) of trace 0 and determinant \(-1\) then there exists \( \gamma \in \text{PSL}_2(\mathbb{R}) \) such that \( \gamma C \gamma^{-1} = \pm C' \). It follows that if \( c \) and \( c' \) are two complex conjugations then there exists \( \gamma \in \text{PSL}_2(\mathbb{R}) \) such that \( c' = \gamma c \gamma^{-1} \). In particular, every complex conjugation is conjugate to \( c_0 \), and therefore has a fixed point (in fact, an entire line) in \( h \).

### 2.2. Real Fuchsian groups

A **real Fuchsian group** is a pair \((\Gamma, c)\) consisting of a discrete subgroup \( \Gamma \) of \( \text{PSL}_2(\mathbb{R}) \) and a complex conjugation \( c \) on \( h \) such that \( \Gamma \) is stable under the involution \( \gamma \mapsto \gamma^c \) of \( \text{PSL}_2(\mathbb{R}) \). Let \((\Gamma, c)\) be a real Fuchsian group. Let \( h^* \) be the space obtained by adding the cusps of \( \Gamma \) to \( h \) and put \( X_\Gamma = h^*/\Gamma \). We assume throughout that \( X_\Gamma \) is compact. We typically regard \( X_\Gamma \) as just a topological space, though sometimes we will use the additional structure it has (e.g., smooth manifold). Since \( \Gamma \) is stable under \( c \) the set of cusps of \( \Gamma \), and thus \( h^* \), is stable under \( c \). We then find that \( c \) descends to an automorphism of \( X_\Gamma \), which we denote by \( c \) and still call complex conjugation.

A **real point** of \( X_\Gamma \) is a point fixed by \( c \). We write \( X_\Gamma(\mathbb{R}) \) for the set of real points of \( X_\Gamma \). A simple argument shows that \( X_\Gamma(\mathbb{R}) \) is diffeomorphic to a non-empty disjoint union of circles; it is non-empty since \( c \) fixes a point in \( h \). By a **real component** of \( X_\Gamma \), we mean one of these circles.

A real Fuchsian group \((\Gamma, c)\) gives a complex orbifold \( X_\Gamma \) with a finite set of distinguished points (the cusps) together with an anti-holomorphic involution \( c \). There is an obvious notion of isomorphism for such pairs \((X_\Gamma, c)\), and it is natural to ask how this is reflected in terms of \((\Gamma, c)\). Clearly, if \( \sigma \) is an admissible element of \( \Gamma \) then \((\Gamma, c\sigma)\) gives rise to the same quotient \((X_\Gamma, c)\). Also, it is clear that if \( \gamma \) is an element of \( \text{PSL}_2(\mathbb{R}) \) then \((\gamma^1, \gamma^{-1})\) gives rise to an isomorphic quotient. In fact, this is all that can happen. Precisely, we define two real Fuchsian groups \((\Gamma_1, c_1)\) and \((\Gamma_2, c_2)\) to be **equivalent** if \((\Gamma_1, c_1) = (\Gamma_2\gamma^{-1}, \gamma c_2\gamma^{-1})\) for some \( \gamma \in \text{PSL}_2(\mathbb{R}) \) and some \( \sigma \in \Gamma_2 \) admissible with respect to \( c_2 \). One can then show that \((X_{\Gamma_1}, c_1)\) and \((X_{\Gamma_2}, c_2)\) are isomorphic if and only if \((\Gamma_1, c_1)\) and \((\Gamma_2, c_2)\) are equivalent.

As a final remark, suppose that \( \Gamma \) is a discrete subgroup of \( \text{PSL}_2(\mathbb{R}) \) such that \( X_\Gamma \) is compact and \( \overline{c} \) is an anti-holomorphic involution of \( X_\Gamma \) (respecting the cusps and orbifold structure). It is natural to ask if \((X_\Gamma, \overline{c})\) comes from a real Fuchsian group. A necessary condition is that \( \overline{c} \) has a fixed point on \( X_\Gamma \). In fact, this is sufficient: if \( \overline{c} \) has a fixed point then there is a complex conjugation \( c \) on \( h \) lifting \( \overline{c} \) such that \( \Gamma \) is stable under the involution \( \gamma \mapsto \gamma^c \) of \( \text{PSL}_2(\mathbb{R}) \). Then \((\Gamma, c)\) is a real Fuchsian group giving rise to \((X_\Gamma, \overline{c})\).

### 2.3. Twisted forms

We say that two real Fuchsian groups \((\Gamma_1, c_1)\) and \((\Gamma_2, c_2)\) are **twisted forms** of each other if \( \Gamma_1 \) and \( \Gamma_2 \) are conjugate. This is equivalent to asking for an isomorphism \( X_{\Gamma_1} \rightarrow X_{\Gamma_2} \) respecting all the structure, except for complex conjugation.

Suppose that \((\Gamma, c)\) is a real Fuchsian group. We would like to understand its twisted forms, up to equivalence. A simple argument shows that every twisted form is equivalent to one of the form \((\Gamma, c\sigma)\), where \( \sigma \) is an admissible element of \( \text{Norm}(\Gamma) \), the normalizer of \( \Gamma \) in \( \text{PSL}_2(\mathbb{R}) \). Furthermore, two twisted forms \((\Gamma, c\sigma_1)\) and \((\Gamma, c\sigma_2)\) are equivalent if and only if \( \sigma_1 \) and \( \sigma_2 \) are “twisted conjugate” in \( \text{Norm}(\Gamma) \Gamma \), i.e., there exists \( \gamma \in \text{Norm}(\Gamma) \) such that \( \gamma^c \sigma_1 \gamma^{-1} \) and \( \sigma_2 \) map to the same element in \( \text{Norm}(\Gamma) \Gamma \).
The reader familiar with algebraic geometry may expect that twisted forms of \((X_\Gamma, c)\) are classified by \(H^1(G, \text{Aut}(\Gamma))\), where \(G\) is the two element group generated by \(c\). This is not quite true, since our twisted forms all have real points. In fact, the set of equivalence classes of twisted forms (in our sense) is a subset of \(H^1(G, \text{Aut}(\Gamma))\). The set \(H^1(G, \text{Aut}(\Gamma))\) is identified with the set of twisted conjugation classes of elements \(g \in N\Gamma/\Gamma\) satisfying \(g^c = g^{-1}\). The image of the set of twisted forms, in our sense, consists of those elements \(g\) which lift to an element \(\gamma \in N\Gamma\) which still satisfies \(\gamma^c = \gamma^{-1}\).

3. Components for real Fuchsian groups

In this section we define a graph \(\Xi_\Gamma\) associated to a real Fuchsian group \(\Gamma\) and show that it is homeomorphic to \(X_\Gamma(\mathbb{R})\).

3.1. The graph \(\Xi_\Gamma\) and the main theorem. We fix for the entirety of §3 a real Fuchsian group \((\Gamma, c)\). We let \(X_\Gamma = \mathfrak{h}^*/\Gamma\) be the corresponding quotient, which we assume, as always, to be compact. We let \(\pi : \mathfrak{h}^* \to X_\Gamma\) be the quotient map. In this section, we define the graph \(\Xi_\Gamma\) and state the main theorem of §3.

Before defining \(\Xi_\Gamma\), we must introduce some terminology. We say that a point of \(X_\Gamma\) is special if it is a real cusp or a real elliptic point of even order. For an element \(\gamma \in \Gamma\) we let \(C_\gamma\) denote the locus in \(\mathfrak{h}^*\) consisting of points \(z\) which satisfy \(\gamma z = cz\). We say that two elements \(\gamma\) and \(\gamma'\) of \(\Gamma\) are twisted conjugate if there exists \(\delta \in \Gamma\) such that \(\gamma = \delta^{-1}\gamma'\delta\). Note that \(\delta C_\gamma = C_{\delta^{-1}\gamma\delta^{-1}}\), so if \(\gamma\) and \(\gamma'\) are twisted conjugate then \(\pi(C_\gamma) = \pi(C_{\gamma'})\). Furthermore, note that if \(\gamma\) is admissible then so is any twisted conjugate of \(\gamma\).

We now define the graph \(\Xi_\Gamma\). The vertex set of \(\Xi_\Gamma\) is the set of special points of \(X_\Gamma\). The edge set of \(\Xi_\Gamma\) is the set of admissible twisted conjugacy classes. A special point \(z\) belongs to the edge corresponding to \(\gamma\) if \(z\) belongs to \(\pi(C_\gamma)\). Actually, this definition must be slightly amended, as follows: the edge corresponding to \(\gamma\) forms a loop at the vertex \(z\) if (and only if) \(\pi(C_\gamma)\) contains an open neighborhood of \(z\) in \(X_\Gamma(\mathbb{R})\). We will give a clearer definition of the graph in §3.5. Note that \(\Xi_\Gamma\) can have loops, multiple edges between vertices and edges without any vertices.

We can now state the main theorem of §3.

Theorem 3.1.1. The graph \(\Xi_\Gamma\) is naturally homeomorphic to \(X_\Gamma(\mathbb{R})\).

In fact, this homeomorphism is the identity map on the vertex set of \(\Xi_\Gamma\), and maps the interior of the edge corresponding to \(\gamma\) homeomorphically to the interior of the set \(\pi(C_\gamma)\). This theorem will take most of §3 to prove.

Remark 3.1.2. It is possible to give a definition of \(\Xi_\Gamma\) which is more group-theoretic than the one we give above, making less reference to the space \(X_\Gamma(\mathbb{R})\). For instance, the vertex set can be described in terms of certain subgroups of \(\Gamma\) (the stabilizers of special points). However, we have not found this alternate definition to be so clear or useful. For congruence subgroups, however, we will come to a very useful, but very different, group-theoretic characterization of \(\Xi_\Gamma\).

3.2. The curves \(C_\gamma\). We now prove a few simple results about the loci \(C_\gamma\) introduced above. We first define some variants. Let \(\gamma \in \Gamma\). We define \(\overline{C}_\gamma\) to be the locus in \(\overline{\mathfrak{h}}\) (the union of \(\mathfrak{h}\) and \(\mathbb{RP}^1\)) defined by the equation \(\gamma z = cz\). The locus \(C_\gamma\) is then \(\overline{C}_\gamma \cap \mathfrak{h}^*\). We also put \(C_\gamma^- = \overline{\mathfrak{h}} \cap \mathfrak{h}\). Clearly, any pre-image in \(\mathfrak{h}^*\) of a point in \(X_\Gamma(\mathbb{R})\) lies on one of the curves \(C_\gamma\).

The most basic fact about the \(C_\gamma\) is the following:

Proposition 3.2.1. If \(\gamma\) is admissible then \(\overline{C}_\gamma\) is a semi-circle meeting \(\overline{\mathfrak{h}}\) at two distinct points. If \(\gamma\) is not admissible then \(C_\gamma\) is empty.
Proof. Let $\gamma \in \Gamma$ be given and write

$$C\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Suppose $z \in \mathfrak{h}^*$ satisfies $cz = \gamma z$. Then $(C\gamma)^2 = \gamma^c \gamma$ stabilizes $z$, and so $|\text{tr}(\gamma^c \gamma)| \leq 2$. A short computation shows that $\gamma^c \gamma$ has trace $2 + (a + d)^2$, and so $a = -d$. Thus if $C\gamma$ is non-empty then $C\gamma$ has trace 0, and so $\gamma$ is admissible.

Conversely, suppose that $\gamma$ is admissible (i.e., $a = -d$). It follows from our previous discussion that $c\gamma$ is a complex conjugation, and thus conjugate to $c_0$, and so fixes a semi-circle. However, we prefer to be a bit more explicit. A short computation shows that $z = x + iy$ belongs to $C\gamma$ if and only if

$$cx^2 + cy^2 - 2ax + b = 0.$$ 

When $c = 0$, this simply becomes $x = \frac{b}{2a}$, which is indeed a semi-circle intersecting $\mathbb{R}P^1$ at two distinct points (namely, 0 and $\infty$). When $c \neq 0$, we can write this equation as

$$(x - \frac{a}{c})^2 + y^2 = \frac{1}{cz},$$ 

which is the equation of a circle of radius $\frac{1}{c}$ centered at the real point $\frac{a}{c}$. It thus meets $\mathbb{R}P^1$ at two distinct points, namely $\frac{a \pm 1}{c}$. $\square$

We note that if $\gamma$ is admissible then $\overline{C\gamma}$ is homeomorphic to a closed interval, $C\gamma_z$ to an open interval and $C\gamma$ to an interval, either open, half-open or closed depending on if the endpoints of $\overline{C\gamma}$ are cusps. (In fact, we will see that $C\gamma$ is never half-open.)

The following simple result will be often used, so we state it explicitly:

**Proposition 3.2.2.** Let $\gamma$ be admissible. Then any element of $\Gamma$ stabilizing every point of $C\gamma$ is the identity.

*Proof.* The set of elliptic points of $\Gamma$ is discrete in $\mathfrak{h}$, and so $C\gamma$ contains non-elliptic points of $\mathfrak{h}$. Such points, by definition, have trivial stabilizer in $\Gamma$. $\square$

**Proposition 3.2.3.** If $\gamma$ and $\delta$ are admissible and $C\gamma = C\delta$ then $\gamma = \delta$.

*Proof.* The transformation $\delta^{-1} \gamma$ stabilizes every element of the semi-circle $C\gamma$, and is thus the identity. $\square$

3.3. The space $C\gamma / Z\gamma$. For $\sigma$ and $\gamma$ in $\Gamma$, we call $\sigma^c \gamma \sigma^{-1}$ the *twisted conjugate* of $\gamma$ by $\sigma$. Any twisted conjugate of an admissible element is again admissible, and we have $\sigma C\gamma = C\sigma^c \gamma \sigma^{-1}$. We let $Z\gamma$ denote the *twisted centralizer* of $\gamma$, i.e., the set of elements $\sigma \in \Gamma$ such that $\sigma^c \gamma \sigma^{-1} = \gamma$; it is a group. Fix for the rest of this section an admissible element $\gamma$ of $\Gamma$. By Proposition 3.2.3, $Z\gamma$ consists of exactly those elements $\sigma$ such that $\sigma C\gamma = C\gamma$. By Proposition 3.2.2, no non-trivial element of $Z\gamma$ can fix every point of $C\gamma$, and so the map $Z\gamma \to \text{Aut}(C\gamma)$ is injective.

**Lemma 3.3.1.** The group $Z\gamma$ is either trivial, cyclic of order two, infinite cyclic or infinite dihedral (i.e., the semi-direct product $\mathbb{Z} / 2\mathbb{Z} \ltimes \mathbb{Z}$).

*Proof.* These are the only groups which act admit a proper discontinuous actions on the open interval. $\square$

**Lemma 3.3.2.** The even order elliptic points on $C\gamma$ correspond bijectively to the order two elements of $Z\gamma$, with $z$ corresponding to $\sigma$ if $\sigma$ stabilizes $z$.

*Proof.* Let $z$ be an even order elliptic point on $C\gamma$ and let $\sigma \in \Gamma_z$ be the unique element of order two. One readily verifies that $\tau \mapsto \gamma^{-1} \tau^c \gamma$ defines an automorphism $\Gamma_z \to \Gamma_z$. Since $\sigma$ is the unique element of $\Gamma_z$ of order two, it is mapped to itself under this automorphism. We thus have $\sigma \in Z\gamma$.
Conversely, let $\sigma \in Z_\gamma$ have order two. Let $x$ and $y$ be the two points of $\overline{C}_\gamma$ on $\mathbb{RP}^1$. Since $\sigma$ has finite order, it cannot stabilize elements of $\mathbb{RP}^1$, and so it must switch $x$ and $y$. It follows that $\sigma$ induces an orientation reversing automorphism of $\overline{C}_\gamma$. It therefore has a unique fixed point $z \in C_\gamma^0$, which, by definition, is elliptic of even order.

\[ \square \]

**Lemma 3.3.3.** If $Z_\gamma$ is infinite then $C_\gamma$ contains no cusps and is homeomorphic to an open interval. Any infinite order element of $Z_\gamma$ is hyperbolic and induces an orientation preserving map of $C_\gamma$.

**Proof.** Let $\delta$ be an element of $Z_\gamma$ of infinite order. If $\delta$ induced an orientation reversing map of $C_\gamma^0$ then it would have a fixed point on $C_\gamma^0$, and thus be elliptic, and thus have finite order. Since this is not the case, $\delta$ must be orientation preserving. Let $x$ and $y$ be the two points of $\overline{C}_\gamma$ on $\mathbb{RP}^1$. As $\delta$ stabilizes each of $x$ and $y$, it is a hyperbolic transformation. This shows that $x$ and $y$ cannot be cusps, since the stabilizer of a cusp consists solely of parabolic elements.

By a fundamental domain for the action of $Z_\gamma$ on $C_\gamma$ we mean an open subset $\mathcal{F} \subset C_\gamma$ with the following three properties: (1) $\mathcal{F}$ is homeomorphic to an open interval; (2) no two elements of $\mathcal{F}$ are equivalent under $Z_\gamma$; and (3) every element of $C_\gamma^0$ is equivalent under $Z_\gamma$ to an element of the closure of $\mathcal{F}$. One can easily see that a fundamental domain exists by considering each of the four possibilities for $Z_\gamma$ in turn.

**Lemma 3.3.4.** Let $\mathcal{F} \subset C_\gamma$ be a fundamental domain for the action of $Z_\gamma$. Then $\pi$ is injective on $\mathcal{F}$.

**Proof.** We first claim that $\pi$ is a local homeomorphism on $\mathcal{F}$. To see this, let $x$ be a point in $\mathcal{F}$. We can then pick an open neighborhood of $x$ homeomorphic to $\mathbb{R}$ and an open neighborhood of $\pi(x)$ homeomorphic to $\mathbb{R}$ such that $\pi$ corresponds to the map $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^n$, where $n$ is the order of $\Gamma_x$. It follows from Lemma 3.3.2 or its proof, that the points to the left and right of an even order elliptic point on $C_\gamma$ are equivalent under $Z_\gamma$. Thus $\mathcal{F}$ contains no elliptic points of even order. Since odd powers are local homeomorphisms, the claim follows.

Suppose now that $\pi$ is not injective on $\mathcal{F}$. We thus have a local homeomorphism from $\mathcal{F}$, an open interval, to a connected component of $X_\Gamma(\mathbb{R})$, a circle, which is not injective. It follows that there are disjoint open intervals $U$ and $V$ in $\mathcal{F}$ such that $\pi(U) = \pi(V)$. Pick $z \in U$ and $z' \in V$ non-elliptic such that $\pi(z) = \pi(z')$. Let $\tau$ be an element of $\Gamma$ such that $\tau z' = z$. Then $z$ belongs to both $C_\gamma$ and $\tau C_\gamma = C_\gamma$, and so $\delta = \gamma^{-1} \tau \gamma$ stabilizes $z$. Since $\tau$ is non-elliptic, $\delta$ must be the identity element, and so $\tau$ belongs to $Z_\gamma$. This contradicts $\mathcal{F}$ being a fundamental domain for $Z_\gamma$. We conclude that $\pi$ is injective on $\mathcal{F}$.

**Lemma 3.3.5.** Let $f : (0, 1) \to \mathfrak{h}$ be a continuous map such that as $t$ approaches 1, $f(t)$ converges to an element $x$ in $\mathbb{RP}^1$ (for the topology on $\mathbb{CP}^1$) and $\pi(f(t))$ converges in $X_\Gamma$. Then $x$ is a cusp of $\Gamma$.

**Proof.** Pick $z \in \mathfrak{h}^*$ such that $\pi(f(t))$ converges to $\pi(z)$ as $t$ approaches 1. If $z$ is a cusp, let $U$ be the union of $\{ z \}$ with an open disc tangent to $\mathbb{RP}^1$ at $z$ containing no elliptic points and such that no two of its points are equivalent under $\Gamma$, and put $\Gamma' = 1$. Note that $U$ is open for the topology on $\mathfrak{h}^*$. If $z$ is elliptic, let $U$ be an open neighborhood of $z$ in $\mathfrak{h}^*$ stable under $\Gamma' = \Gamma_z$ such that $\pi$ is injective on $U/\Gamma'$ and which contains no elliptic points other than $z$ and no cusps. If $z$ is neither elliptic nor cuspidal, let $U$ be an open neighborhood of $z$ containing no elliptic or cuspidal points such that no two of its points are equivalent under $\Gamma$, and put $\Gamma' = 1$.

Now, $\pi(U)$ is an open neighborhood of $\pi(z)$, so for $t$ sufficiently close to 1 we have $\pi(f(t)) \in \pi(U)$. It follows that for such $t$ we can find $\gamma(t) \in \Gamma/\Gamma'$, necessarily unique, such that $\gamma(t)f(t)$ belongs to $U$. It is clear that $t \mapsto \gamma(t)$ is locally constant, and therefore constant. Let $\gamma \in \Gamma$ be such that $\gamma(t) = \gamma$ for all $t$ close to 1. Since $\gamma f(t)$ belongs to $U$ for all $t$ close to 1, it follows that $z$ must be a cusp. Furthermore, $\gamma f(t)$ must converge to $z$ as $t$ approaches 1, since the closure of $U$ in $\mathbb{CP}^1$ only
intersects $\mathbb{RP}^1$ at $z$. It follows that $f(t)$ converges to $\gamma^{-1}z$ at $t$ approaches 1, and so $x = \gamma^{-1}z$ is a cusp.

**Lemma 3.3.6.** If $Z_\gamma$ is finite then $C_\gamma$ contains two cusps and is homeomorphic to a closed interval.

**Proof.** Assume first that $Z_\gamma$ is trivial. Then $C^0_\gamma$ is a fundamental domain for the action of $Z_\gamma$ on $C^0_\gamma$, and so $\pi$ is injective on $C^0_\gamma$. Let $f : (0, 1) \to C^0_\gamma$ be a homeomorphism. Since $\pi \circ f$ is a continuous injection from an interval to a circle, the limit of $\pi(f(t))$ as $t$ tends to 1 exists. Of course, as $t$ approaches 1, $f(t)$ converges to an element $x$ of $\mathbb{RP}^1$. Lemma 3.3.5 shows that $x$ is a cusp. Since $C_\gamma$ is closed in $\mathfrak{h}^*$, it contains $x$. Looking at the behavior near $t = 0$, we find another cusp on $C_\gamma$.

The case where $Z_\gamma$ has order 2 is similar. Let $\sigma$ be the non-trivial element of $Z_\gamma$. By Lemma 3.3.2, $\sigma$ fixes a unique element $z$ on $C^0_\gamma$. Let $C_1$ and $C_2$ be the two connected components of $C^0_\gamma \setminus \{z\}$. Both $C_1$ and $C_2$ are fundamental domains for the action of $Z_\gamma$ on $C^0_\gamma$. Choose a homeomorphism $f : (0, 1) \to C_1$ such that $f(t)$ converges to $z$ as $t$ approaches 0. Arguing as in the previous paragraph, the limit of $f(t)$ as $t$ approaches 1 is then a cusp on $C_\gamma$. Looking at $C_2$, we find a second cusp on $C_\gamma$.

Recall that a point of $X_\Gamma$ is special if it belongs to $X_\Gamma(\mathbb{R})$ and is a cusp or an elliptic point of even order; we extend this terminology to points of $\mathfrak{h}^*$ as well.

**Lemma 3.3.7.** Let $\mathcal{F}$ be a fundamental domain for the action of $Z_\gamma$ on $C^0_\gamma$. Then $\mathcal{F}$ contains no special points. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in $C_\gamma$. Then $\overline{\mathcal{F}}$ is a closed interval and exactly one of the following is true: the two boundary points of $\overline{\mathcal{F}}$ are equivalent under $Z_\gamma$; or, each boundary point of $\overline{\mathcal{F}}$ is special.

**Proof.** We proceed in cases. First say that $Z_\gamma$ is trivial. Then $C_\gamma$ is a closed interval whose endpoints are cusps (by Lemma 3.3.6), while $C^0_\gamma$ contains no cusps (obvious) or elliptic points (by Lemma 3.3.2). Since $\mathcal{F} = C^0_\gamma$ and $\overline{\mathcal{F}} = C_\gamma$, the proposition follows in this case.

Next, say that $Z_\gamma$ is cyclic of order two. Then $C_\gamma$ is a closed interval whose endpoints are cusps (by Lemma 3.3.6). The curve $C^0_\gamma$ contains a unique even order elliptic point $z$ (by Lemma 3.3.2). The space $C^0_\gamma \setminus \{z\}$ contains two connected components, and $\mathcal{F}$ is equal to one of them. Clearly, $\mathcal{F}$ contains no cusp or even order elliptic point. The closure $\overline{\mathcal{F}}$ is the closed interval between $z$ and one of the two cusps on $C_\gamma$. The two boundary points are obviously inequivalent under $Z_\gamma$.

Now say that $Z_\gamma$ is infinite cyclic. We can then find a generator $\delta$ of $Z_\gamma$ and a point $z \in C^0_\gamma$ such that $\mathcal{F}$ is the open interval between $z$ and $\delta z$. It follows that $\overline{\mathcal{F}}$ is the closed interval between $z$ and $\delta z$, and so its two endpoints belong to the same orbit of $Z_\gamma$. The interval $\mathcal{F}$ contains no elliptic points (by Lemma 3.3.2) and no cusps (obvious). For the same reasons, the boundary points of $Z_\gamma$ are not elliptic points or cusps.

Finally, say that $Z_\gamma$ is infinite dihedral. The order two elements of $Z_\gamma$ fall into two conjugacy classes. It follows from Lemma 3.3.2 that there are two $Z_\gamma$-orbits of even order elliptic elements on $C^0_\gamma$. The set $\mathcal{F}$ is the open interval between two consecutive even order elliptic elements $x$ and $y$; thus $\mathcal{F}$ contains no even order elliptic points (and of course it contains no cusps). The closure $\overline{\mathcal{F}}$ is the closed interval between $x$ and $y$. Necessarily, $x$ and $y$ belong to different orbits under $Z_\gamma$.

The quotient $C_\gamma / Z_\gamma$ is compact, and therefore either a closed interval or a circle. A point on the boundary of $C_\gamma / Z_\gamma$ is special, while the interior of $C_\gamma / Z_\gamma$ contains no special points. The map $\pi$ is injective on the interior of $C_\gamma / Z_\gamma$.

**Proposition 3.3.8.**

**Proof.** This follows easily from the preceding lemma and Lemma 3.3.4.

**3.4. Classification of admissible elements.** It is useful to classify the admissible elements of $\Gamma$ into four types, according to their twisted centralizers. To this end, let $\gamma$ be an admissible element.
We say that $\gamma$ is Type 1 if $Z_{\gamma}$ is trivial. In this case, $C_{\gamma}/Z_{\gamma}$ is a closed interval whose boundary points are cusps. The image $\pi(C_{\gamma})$ is a closed interval if the two cusps on $C_{\gamma}$ are inequivalent under $\Gamma$, and a circle otherwise. (We say that $\gamma$ is Type 1a in the first case and Type 1b in the second.)

We say that $\gamma$ is Type 2 if $Z_{\gamma}$ is cyclic of order 2. In this case, $C_{\gamma}/Z_{\gamma}$ is a closed interval, with one endpoint a cusp and the other an even order elliptic point. The image $\pi(C_{\gamma})$ is a closed interval.

We say that $\gamma$ is Type 3 if $Z_{\gamma}$ is infinite cyclic. In this case, $C_{\gamma}/Z_{\gamma} = \pi(C_{\gamma})$ is a circle containing no special points.

We say that $\gamma$ is Type 4 if $Z_{\gamma}$ is infinite dihedral. In this case, $C_{\gamma}/Z_{\gamma}$ is a closed interval whose two endpoints are even order elliptic elements. The image $\pi(C_{\gamma})$ is a closed interval if these two elliptic elements are inequivalent under $\Gamma$, and a circle otherwise. (We say that $\gamma$ is Type 4a in the first case and Type 4b in the second.)

If $\gamma$ and $\gamma'$ are twisted conjugate then they are of the same type. We extend the terminology of types to the sets $C_{\gamma}$ or $\pi(C_{\gamma})$, e.g., we say $\pi(C_{\gamma})$ is Type 1 if $\gamma$ is. In §6 we will give examples which show that all the above behaviors actually occur.

3.5. Proof of Theorem 3.1.1. We now prove Theorem 3.1.1. We first give a clearer definition of the vertex-edge relationships in $\Xi_{\Gamma}$. Let $\gamma$ be an admissible element of $\Gamma$. As we have seen, $C_{\gamma}/Z_{\gamma}$ is either a circle or a closed interval. In the first case, the edge of $\Xi_{\Gamma}$ corresponding to $\gamma$ contains no vertices. In the second case, the two boundary points of $C_{\gamma}/Z_{\gamma}$ are special points, and their images in $X_{\Gamma}$ are the two special points on the edge corresponding to $\gamma$ in $\Xi_{\Gamma}$. It is perfectly possible that these two special points of $X_{\Gamma}$ coincide, and in this case the edge corresponding to $\gamma$ forms a loop.

Given the above description of $\Xi_{\Gamma}$, Theorem 3.1.1 is an immediate consequence of the following result.

**Proposition 3.5.1.** Mapping $\gamma$ to the image in $X_{\Gamma}$ of the interior of $C_{\gamma}/Z_{\gamma}$ induces a bijection

$$\{\text{admissible twisted conjugacy classes of } \Gamma\} \rightarrow \pi_0(X_{\Gamma}(R) \setminus Z),$$

where $Z \subset X_{\Gamma}(R)$ is the set of special points.

**Proof.** Let $\gamma$ be an admissible element of $\Gamma$. Then $C_{\gamma}/Z_{\gamma}$ is either a circle or a closed interval whose endpoints are special points. The interior of $C_{\gamma}/Z_{\gamma}$ contains no special points, so it maps into $X_{\gamma}(R) \setminus Z$. If $C_{\gamma}/Z_{\gamma}$ is a circle then $\pi(C_{\gamma})$ is necessarily a component of $X_{\Gamma}(R) \setminus Z$. If $C_{\gamma}/Z_{\gamma}$ is a closed interval, then its endpoints map into $Z$, and so the image of its interior is a connected component of $X_{\Gamma}(R) \setminus Z$. Thus, in all cases, the image of the interior of $C_{\gamma}/Z_{\gamma}$ is mapped to a full connected component of $X_{\Gamma}(R) \setminus Z$.

If $\gamma' = \sigma^c\gamma\sigma^{-1}$ is a twisted conjugate of $\gamma$, then $C_{\gamma'} = \sigma C_{\gamma}$ and so the interiors of $C_{\gamma}/Z_{\gamma}$ and $C_{\gamma'}/Z_{\gamma'}$ have the same image in $X_{\Gamma}(R) \setminus Z$. This shows that mapping $\gamma$ to the image of the interior of $C_{\gamma}/Z_{\gamma}$ induces a well-defined map

$$\Phi : \{\text{admissible twisted conjugacy classes of } \Gamma\} \rightarrow \pi_0(X_{\Gamma}(R) \setminus Z).$$

We now show that $\Phi$ is a bijection.

We first show that $\Phi$ is surjective. Let $x$ be an element of $X_{\Gamma}(R) \setminus Z$. Let $z$ be a lift of $x$ to $h^*$. Then $z$ belongs to $C_{\gamma}$ for some $\gamma$. Since $z$ is not special, it maps to the interior of $C_{\gamma}/Z_{\gamma}$. Thus $x$ belongs to the image of the interior of $C_{\gamma}/Z_{\gamma}$, and so $\Phi$ is surjective.

We now show that $\Phi$ is injective. Let $\gamma$ and $\delta$ be admissible elements of $\Gamma$ and suppose that the interiors of $C_{\gamma}/Z_{\gamma}$ and $C_{\delta}/Z_{\delta}$ map to the same component of $X_{\Gamma}(R) \setminus Z$. Let $x$ be an element of this component which is not an odd order elliptic element (and thus not elliptic or cuspidal). Lift $x$ to an element $z$ of $C_{\gamma}$ and $z'$ of $C_{\delta}$. Pick $\tau$ in $\Gamma$ so that $z = \tau z'$. Then $z$ belongs to both $C_{\gamma}$ and $\tau C_{\delta} = C_{\tau^{-1}\delta\tau^{-1}}$, and so $\gamma^{-1} \sigma \delta \tau^{-1}$ stabilizes $z$. Since $z$ is neither elliptic nor cuspidal it has...
trivial stabilizer, and so \( \gamma^{-1} \tau^c \delta \tau^{-1} = 1 \). This shows that \( \gamma \) is a twisted conjugate of \( \delta \), and so \( \Phi \) is injective. This completes the proof. \( \square \)

We note the following corollary of the theorem.

**Corollary 3.5.2.** The number of special points is equal to the number of admissible twisted conjugacy classes of type 1, 2 and 4.

### 3.6. Local behavior at a special point.

Let \( x \) be a special point in \( \mathfrak{b}^* \). Let \( S_x \) denote the set of curves \( C_\gamma \) which contain \( x \). Given such a curve \( C_\gamma \), a small neighborhood of \( x \) in \( C_\gamma \) maps under \( \pi \) to one side of \( \pi(x) \). We can thus define an equivalence relation \( \sim \) on \( S_x \) by declaring \( C_\gamma \) and \( C_\delta \) equivalent if small neighborhoods of \( x \) in them map to the same side of \( \pi(x) \). There are clearly two equivalence classes. The group \( \Gamma_x \) acts on \( S_x \), the element \( \sigma \in \Gamma_x \) taking \( C_\gamma \) to \( \sigma C_\gamma \). We have the following result:

**Proposition 3.6.1.** The equivalence classes for \( \sim \) on \( S_x \) are exactly the orbits of \( \Gamma_x \).

*Proof.* It is clear that any two elements in an orbit of \( \Gamma_x \) belong to the same equivalence class. We now prove the other direction. Thus let \( C_\gamma \) and \( C_\delta \) be two elements of \( S_x \) which are equivalent under \( \sim \). We can then find non-elliptic points \( z \in C_\gamma^0 \) and \( z' \in C_\delta^0 \) which are in small neighborhoods of \( x \) and equivalent under \( \Gamma \). Pick \( \tau \) so that \( \tau z' = z \). Then \( \tau C_\delta = C_\gamma \), and so \( \tau \) induces a homeomorphism \( C_\delta/Z_\delta \to C_\gamma/Z_\gamma \). Each of these spaces is a closed interval, with one endpoint being represented by \( x \). Since \( z \) and \( z' \) are both close to \( x \), and \( \tau \) maps \( z' \) to \( z \), it follows that \( \tau \) takes \( x \in C_\delta/Z_\delta \) to \( x \in C_\gamma/Z_\gamma \). We thus find that \( \tau(x) \) belongs to \( Z_\gamma x \). Therefore, by replacing \( \tau \) with an element of \( Z_\gamma \), we find \( \tau C_\delta = C_\gamma \) and \( \tau x = x \). This shows that \( C_\gamma \) and \( C_\delta \) are equivalent under \( \Gamma_x \). \( \square \)

**Lemma 3.6.2.** Let \( z \) be a point on \( C_\gamma \) and let \( \sigma \) be any element of \( \Gamma_x \). Then \( \gamma^{-1} \sigma^c \gamma = \sigma^{-1} \).

*Proof.* Clearly, \( z \) belongs to \( C_\gamma \sigma \), and so \( \gamma \sigma \) is admissible. We therefore have

\[
\gamma^{-1} \sigma^c = (\gamma \sigma)^c = (\gamma \sigma)^{-1} = \sigma^{-1} \gamma^{-1},
\]

which proves the lemma. \( \square \)

**Proposition 3.6.3.** Let \( C_\gamma \) contain \( x \) and let \( \sigma \) belong to \( \Gamma_x \). Then \( C_\gamma \sim C_\gamma \sigma \) if and only if \( \sigma \) belongs to \( 2 \Gamma_x \).

*Proof.* For \( \tau \in \Gamma_x \), we have \( \tau C_\gamma = C_{\tau C_\gamma \tau^{-1}} = C_{\gamma \tau^{-2}} \). Thus \( C_{\gamma \sigma} = \tau C_\gamma \) if and only if \( \sigma = \tau^{-2} \). It follows that \( C_{\gamma \sigma} \sim C_\gamma \) if and only if \( \sigma \) is the square of an element of \( \Gamma_x \). \( \square \)

**Corollary 3.6.4.** Let \( C_\gamma \) contain \( x \), and let \( \sigma \) be a generator of \( \Gamma_x \). Then \( C_\gamma \) and \( C_\gamma \sigma \) are inequivalent under \( \sim \).

This corollary is useful when computing the graph \( \Xi \), for if one has found an edge \( \pi(C_\gamma) \) containing the special point \( x \) then the other edge containing \( x \) is given by \( \pi(C_\gamma \sigma) \) where \( \sigma \) generates \( \Gamma_x \). Note that it is possible for \( \pi(C_\gamma) \) and \( \pi(C_\gamma \sigma) \) to coincide; when this happens, \( \pi(C_\gamma) \) forms a loop at \( x \).

### 4. The graph associated to a real subgroup of \( \text{SL}_2(R) \)

In this section, we associate a graph to a “real” subgroup of \( \text{SL}_2(R) \), where \( R \) is a finite characteristic ring, with some additional hypotheses at 2. The main result we prove about this graph is that it is a union of cycles. We also prove a result describing how the construction behaves under direct product and inverse image.
4.1. The graph \( Ξ_G \). Let \( R \) be a ring, let \( U \) be a free rank two \( R \)-module with a non-degenerate symplectic pairing \( ⟨,⟩ \) and let \( C \) be an \( R \)-linear involution of \( U \) of determinant \(-1\). We write \( g \mapsto g^c \) for the involution of \( \text{SL}(U) \) induced by conjugation by \( C \). Let \( G \) be a subgroup of \( \text{SL}(U) \) containing \(-1\) and stable under \( c \) (this is what we mean by a real subgroup). We aim to define a graph \( Ξ_G \) associated to this data. This graph will depend on \( C \), despite its absence from the notation.

We say that an element of \( U \) is a \textit{basis vector} if the \( R \)-submodule it generates is a summand. We call an element \( g \) of \( G \) \textit{admissible} if \( g^c = g^{-1} \). Note that \( g \) is admissible if and only if \((Cg)^2 = 1\). Let \( \tilde{V}_p \) denote the subset of \( U/\{±1\} \) consisting of elements which are represented by some basis vector \( x \) satisfying \( Cgx = x \) for some admissible \( g \in G \). We call elements of \( \tilde{V}_p \) (and often the elements of \( U \) representing them) \textit{parabolic vertices}. We represent parabolic vertices graphically with a solid dot.

Let \( T \) denote the set of triples \([x, y; z]\) of basis vectors of \( U \) satisfying the following two conditions:

(a) We have \( ⟨x, z⟩ = ⟨z, y⟩ = 1 \).
(b) We have \( x + y = wz \) for some \( w \in \{1, 2\} \).

We refer to \( w \) as the \textit{weight} of the triple \([x, y; z]\). Note that \( w = ⟨x, y⟩ \). We define the \textit{complementary weight} \( w' \) to be 2 or 1 depending on if \( w \) is 1 or 2. Note that if \( w = 1 \) or if \( R \) has no non-zero 2-torsion then \( z \) is uniquely determined from \( x \) and \( y \).

Given \([x, y; z]\) in \( T \) put \( ρ([x, y; z]) = [z, z−w'x; y] \). One readily verifies that \( ρ \) maps \( T \) to itself. The map \( ρ \) interchanges the weight and complementary weight, i.e., \( wp = w' \) and \( wρp = w \). A short computation shows that \( ρ \) has order 8; in fact, \( ρ^4([x, y; z]) = [−x, −y; −z] \). The natural action of \( G \) on \( T \) commutes with \( ρ \).

For an element \([x, y; z]\) of \( T \), consider the following condition:

(c) There exists an element \( g \) of \( G \) such that
\[
Cg − 1 = w'⟨−, x⟩y, \quad Cy + 1 = w'⟨−, y⟩x.
\]

Of course, the element \( g \) is uniquely determined by \( x \) and \( y \). In fact, (c) holds if and only if the endomorphism of \( U \) defined by \( u \mapsto Cu + w'(u, x)Cy \) belongs to \( G \). If condition (c) is satisfied then the element \( g \) is necessarily admissible and we have \( Cgx = x \) and \( Cy = −y \), showing that \( x \) and \( y \) are parabolic vertices. One readily verifies that \([x, y; z]\) satisfies (c) if and only if \( ρ^2([x, y; z]) \) does.

A \textit{geodesic} is an element of \( T/(ρ^2) \) satisfying condition (c). Thus, a geodesic is represented by a triple \([x, y; z]\) satisfying conditions (a)–(c), and the triples \([x, y; z]\) and \( ρ^2([x, y; z]) \) represent the same geodesic. The weight of a geodesic is well-defined. We denote geodesics graphically by either a single or double line, depending on if the weight is one or two.

The map \( ρ \) descends to an involution of \( T/(ρ^2) \). However, it need not take geodesics to geodesics. We say that two geodesics \textit{intersect} if they form an orbit of \( ρ \). By definition, intersecting geodesics have complementary weights. We have the following observation:

**Lemma 4.1.1.** Let \([x, y; z]\) be a geodesic. Then \( ρ([x, y; z]) \) is a geodesic if and only if \( G \) contains the map \( σ \) defined by \( σ(x) = −y \) and \( σ(z) = z−w'y \). (Note: \( σ(y) = x \).)

**Proof.** Let \( g \) and \( h \) be defined by
\[
Cg − 1 = w'⟨−, x⟩y, \quad Ch − 1 = w⟨−, z⟩(z−w'x).
\]
A short computation shows that \( CgCh = g^{-1}h = σ \). Since \( g \) belongs to \( G \), we find that \( h \) belongs to \( G \) if and only if \( σ \) does. \( \square \)

An \textit{elliptic vertex} is an unordered pair of intersecting geodesics. We think of the elliptic vertex as the intersection of the two geodesics, and use corresponding terminology (e.g., we say that the geodesics contain the elliptic vertex). By definition, a geodesic contains at most one elliptic point.
We write $\tilde{V}_e$ for the set of elliptic points. We represented elliptic vertices graphically with a hollow dot.

We now define a graph $\tilde{\Xi}$. The vertex set of $\tilde{\Xi}$ is the disjoint union of $\tilde{V}_p$ and $\tilde{V}_e$. The edges of $\tilde{\Xi}$ come from geodesics, as follows. If $[x, y; z]$ is a geodesic containing no elliptic points then it contributes an edge between $x$ and $y$ in $\tilde{\Xi}$. If $[x, y; z]$ is a geodesic containing an elliptic point $p$, then it contributes an edge between $x$ and $p$, as well as one between $p$ and $y$. The edges of $\tilde{\Xi}$ are undirected. We assign each edge weight 1 or 2 according to the weight of the geodesic giving rise to it. Note that it is possible that there is more than one edge between two vertices of $\tilde{\Xi}$. However, $\tilde{\Xi}$ contains no loops.

One easily verifies that $G$ acts on $\tilde{\Xi}$. With this in mind, we can make our main definition:

**Definition 4.1.2.** The graph $\Xi = \Xi_G$ is the quotient of $\tilde{\Xi}$ by $G$.

For the sake of clarity, let us elaborate on the definition slightly. The vertex set of $\Xi$ is the quotient of the vertex set of $\tilde{\Xi}$ by $G$. The action of $G$ on $\tilde{\Xi}$ takes parabolic vertices to parabolic vertices and elliptic vertices to elliptic vertices, so there is a notion of “elliptic” and “parabolic” for vertices $\Xi$. The edge set of $\Xi$ is the quotient of the edge set of $\tilde{\Xi}$ by the action of $G$. The action of $G$ on the edges of $\tilde{\Xi}$ respects weight, and so the edges of $\Xi$ have a weight. We note that each geodesic of $\tilde{\Xi}$ maps to a single edge of $\Xi$.

4.2. **Invariance under inverse image.** Let $R \to R_0$ be a surjection of rings. We assume that 3 and 4 are non-zero in $R_0$, and thus in $R$. Let $U$ be a free $R$-module of rank 2 with complex conjugation $C$, let $U_0 = U \otimes_R R_0$ and let $C_0$ be the induced complex conjugation on $U_0$. Let $G_0$ be a subgroup of $\text{SL}(U_0)$ stable under $C_0$ and let $G$ be its inverse image in $\text{SL}(U)$. Clearly, $G$ is stable under $C$. One therefore has graphs $\Xi = \Xi_G$ and $\Xi_0 = \Xi_{G_0}$. The purpose of this section is to prove the following theorem:

**Theorem 4.2.1.** Assume that the map $\text{SL}_2(R) \to \text{SL}_2(R_0)$ is surjective. Then $\Xi$ and $\Xi_0$ are isomorphic.

The hypothesis implies that every element of $G_0$ lifts to one in $G$. We note that the hypothesis is automatic if the kernel of $R \to R_0$ is nilpotent, since $\text{SL}_2$ is a smooth group scheme. To prove the theorem we proceed in a series of lemmas. For $x \in U$ we write $\overline{x}$ for its image in $U_0$.

**Lemma 4.2.2.** Any basis vector of $U_0$ can be lifted to a basis vector of $U$.

*Proof.* Let $\overline{\pi}$ be a basis vector of $U_0$. Pick $\overline{v} \in U_0$ with $\langle \overline{u}, \overline{v} \rangle = 1$ and pick $x$ and $y$ in $U$ with $\langle x, y \rangle = 1$. Then there exists a unique element $\overline{\pi} \in \text{SL}(U_0)$ such that $\overline{\pi} = \overline{u}x$ and $\overline{v} = \overline{u}y$. Since $\text{SL}(U) \to \text{SL}(U_0)$ is surjective, we can lift $\overline{\pi}$ to an element $g$ of $\text{SL}(U)$. We can then take $v = gx$. □

**Lemma 4.2.3.** Let $\overline{\pi}$ and $\overline{\tau}$ be elements of $U_0$ satisfying $\overline{\langle \pi, \tau \rangle} = 1$ and let $u$ be a lift of $\overline{\pi}$ to a basis vector of $U$. Then there exists a lift $v$ of $\overline{\tau}$ with $\langle u, v \rangle = 1$.

*Proof.* Let $v_0$ be any lift of $v$ and let $z \in U$ be such that $\langle u, z \rangle = 1$. We have $\langle u, v_0 \rangle = 1 + \alpha$, where $\alpha$ belongs to the kernel of $R \to R_0$. We can take $v = v_0 - \alpha z$. □

**Lemma 4.2.4.** Every parabolic vertex of $\tilde{\Xi}_0$ lifts to one of $\tilde{\Xi}$.

*Proof.* Let $\overline{\pi}$ be a parabolic vertex of $\tilde{\Xi}_0$ and let $x$ be a lift of $\overline{\pi}$ to a basis vector of $U$ (possible by Lemma 4.2.2). Let $\overline{g} \in G_0$ be such that $C_0 \overline{g} \overline{\pi} = \overline{\pi}$. Pick $\overline{\pi} \in U_0$ with $\overline{\langle \pi, \tau \rangle} = 1$, so that $C_0 \overline{g} \overline{\pi} = \overline{\pi} + C \alpha \overline{\tau}$ for some $\overline{\tau} \in R_0$. Let $z$ be a lift of $\overline{\tau}$ satisfying $\langle x, z \rangle = 1$ (possible by Lemma 4.2.3) and let $a$ be a lift of $\overline{\pi}$. Let $g$ be the endomorphism of $U$ defined by $gx = Cx$ and $gz = C(z + ax)$. Then $g$ has determinant 1 and reduces to $\overline{g}$, and thus belongs to $G$. Thus $x$ is parabolic, completing the proof. □
Lemma 4.2.5. Two parabolic vertices in $U$ are equivalent if and only if their images in $U_0$ are.

Proof. Let $x$ and $y$ be parabolic vertices in $U_0$ and let $\bar{x} = \bar{h}\bar{y}$ for some $\bar{h} \in G_0$. Let $h \in G$ be a lift of $\bar{h}$. Replacing $y$ by $hy$, we may assume that $x$ and $y$ have the same image in $U_0$. Let $\{x, z\}$ be a basis of $U$ and write $y = ax + bz$. Note that $a$ and $b$ map to 1 and 0 in $R_0$. Since $y$ is a basis vector, $a$ and $b$ generate the unit ideal of $R$, and so we have an expression $pa - qb = 1$ for some $p, q \in R$. Let $g$ be the endomorphism of $U$ given by the matrix

$$\begin{pmatrix} a & (1-a)q \\ b & 1 + p(1-a) \end{pmatrix}$$

in the basis $\{x, z\}$. Then $g$ has determinant 1 and induces the identity map of $U_0$, and thus belongs to $G$. Since $gx = y$, we see that $x$ and $y$ are equivalent. \qed

Lemma 4.2.6. Every geodesic in $\Xi_0$ lifts to one in $\Xi$.

Proof. Let $[x, y; z]$ be a geodesic in $\Xi_0$ of weight $w$. Let $x$ and $z$ be basis vectors of $U$ lifting $\bar{x}$ and $\bar{z}$ and satisfying $(x, z) = 1$ (possible by Lemmas 4.2.2 and 4.2.3). Put $y = wz - x$. The endomorphism $C(1 + w'(-, x)y)$ of $U$ has determinant 1 and lifts the endomorphism $C_0(1 + w'(-, \bar{x})\bar{y})$ of $U_0$. By hypothesis, the latter belongs to $G_0$, and so we find that the former belongs to $G$. This shows that $[x, y; z]$ is a geodesic of $\Xi$, which completes the proof. \qed

Lemma 4.2.7. Two geodesics in $\Xi$ are equivalent if and only if their images in $\Xi_0$ are.

Proof. Let $[x_1, y_1; z_1]$ and $[x_2, y_2; z_2]$ be two geodesics of $\Xi$ whose images in $\Xi_0$ are equivalent. After possibly apply some power of $\rho^2$, we have an element $\overline{\sigma} \in G_0$ such that $[\overline{x_1}, \overline{y_1}; \overline{z_1}] = \overline{[x_2, y_2; z_2]}$ holds in $\Xi_0$. Letting $g$ be a lift of $\overline{\sigma}$ to $G$ and replacing $[x_2, y_2; z_2]$ with $g[x_2, y_2; z_2]$, we can assume that $x_1$ and $x_2$ reduce to the same element of $U$, as do $z_1$ and $z_2$. Now, let $h$ be the endomorphism of $U$ defined by $h(x_1) = x_2$ and $h(z_1) = z_2$. Then $h$ induces the identity map of $U_0$, and thus belongs to $G$. Since $y_1 = w_1 z_1 - x_1$ and $y_2 = w_2 z_2 - x_2$ and $w_1 = w_2$, we find that $h(y_1) = y_2$. Thus $h[x_1, y_1; z_1] = [x_2, y_2; z_2]$, which shows that the two geodesics are equivalent. \qed

Lemma 4.2.8. A geodesic in $\Xi$ intersects another geodesic if and only if its image in $\Xi_0$ does.

Proof. Let $[x, y; z]$ be a geodesic of $\Xi$ of weight $w$. By Lemma 4.4.1, $[x, y; z]$ intersects another geodesic if and only if the map $\sigma$ of $U$ defined by $\sigma(x) = -y$ and $\sigma(z) = z - w'y$ belongs to $G$. Similarly, $[\overline{x}, \overline{y}; \overline{z}]$ intersects another geodesic if and only if the map $\overline{\sigma}$ of $U_0$ defined by $\overline{\sigma}(x) = -\overline{y}$ and $\overline{\sigma}(\overline{y}) = \overline{z} - w'\overline{y}$ belongs to $G_0$. As $\sigma$ reduces to $\overline{\sigma}$, it follows that $\sigma$ belongs to $G$ if and only if $\overline{\sigma}$ belongs to $G_0$. This proves the lemma. \qed

We can now prove the theorem:

Proof of Theorem 4.2.7. By Lemmas 4.2.4 and 4.2.5, the equivalence classes of parabolic vertices of $\Xi$ and $\Xi_0$ are in bijection. By Lemmas 4.2.6 and 4.2.7, the equivalence classes of geodesics in $\Xi$ and $\Xi_0$ are in bijection. From this and Lemma 4.2.8, the equivalences classes of elliptic points in $\Xi$ and $\Xi_0$ are in bijection. Since these bijections are obviously compatible with how edges are constructed, we find that $\Xi$ and $\Xi_0$ are isomorphic. \qed

4.3. The main theorem. A cycle is a connected graph on finitely many vertices in which each vertex has valence 2. We consider a single vertex with a self-edge (i.e., a loop) to be a cycle of length 1. Our main result about $\Xi$ is the following theorem. For our ultimate applications, this theorem is in fact not logically necessary; nonetheless, we feel it is worth including.

Theorem 4.3.1. Suppose $R = \mathbb{Z}/2^r\mathbb{Z} \times R_1$ where $r$ is a non-negative integer and $R_1$ is a ring of odd characteristic. Then the graph $\Xi$ is a union of cycles.
By Theorem 4.2.1, it suffices to prove the theorem when \( r \geq 1 \). We thus make this assumption, to streamline some arguments. In the following section, we give a shorter proof when \( r = 0 \) that yields a stronger result. We let \( t_i \) be the element \((i^2, 0)\) of \( R \), and put \( t = t_{r-1} \). Thus \( t \) is the unique non-zero 2-torsion element of \( R \). We let \( p : R \to \mathbb{Z}/2^r\mathbb{Z} \) and \( \pi : \Xi \to \Xi \) be the projection maps. We proceed with several lemmas.

**Lemma 4.3.2.** Every parabolic vertex belongs to a geodesic.

**Proof.** Let \( x \in U \) be a basis vector and let \( g \in G \) be an admissible element such that \( Cgx = x \). Let \( u \in U \) be such that \( \langle x, u \rangle = 1 \) and write \( Cgu = -u + ax \) for some \( a \in R \). First suppose that \( a \) belongs to \( 2R \), and write \( a = 2b \). Put \( y = u - bx \) and \( z = x + y \). Then \([x, y; z]\) is a weight one geodesic. Now suppose that \( a \) belongs to \( 1 + 2R \), and write \( a = 2b + 1 \). Put \( y = 2u - ax \) and \( z = u - bx \). Then \([x, y; z]\) is a weight two geodesic. \( \square \)

**Lemma 4.3.3.** Every parabolic vertex of \( \Xi \) has valence at least two.

**Proof.** Let \( x \) be a parabolic vertex of \( \Xi \), and let \([x, y; z]\) be a geodesic to which \( x \) belongs (which exists by Lemma 4.3.2). If \( x \) is equivalent to \( y \) and \([x, y; z]\) does not contain an elliptic vertex, then \( \pi(x) \) is contained in a loop and thus has valence at least two. We may thus assume that either \( x \) and \( y \) are inequivalent, or that they are equivalent by an element \( \sigma \) satisfying \( \sigma(x) = -y \) and \( \sigma(y) = x \). We must produce a geodesic inequivalent to \([x, y; z]\) which contains \( x \).

**Case 1:** \([x, y; z]\) has weight one. Observe that \([x, y + tx; z + tx]\) is a geodesic. We can therefore pick \( i \geq 0 \) minimal so that \([x, y + t_i x; z + t_i x]\) is a geodesic. Now, if \([x, y + t_i x; z + t_i x]\) is inequivalent to \([x, y; z]\) then we are done. Thus assume the two are equivalent, and let \( h \in G \) be such that \( h[x, y; z] = [x, y + t_i x; z + t_i x] \). Now, either \( hx = ±x \) and \( hy = ±(y + t_i x) \) or else \( hx = ±(y + t_i x) \). In the latter case, \( \sigma \) is available, and replacing \( h \) with \( h\sigma \) moves us to the first case. We may thus assume we are in the first case. Replacing \( h \) with \(-h\), if necessary, we can assume that \( hx = x \) and \( hy = y + t_i x \). Let \( g \in G \) be such that \( Cgx = x \) and \( Cg(y + t_i x) = -(y + t_i x) \). We have \( Cghx = x \) and \( Cghy = -y - t_i x \). If \( i > 0 \) then we have \( Cgh(y + t_{i-1} x) = -(y + t_{i-1} x) \), which shows that \([x, y + t_{i-1} x; z + t_{i-1} x]\) is a geodesic, contradicting the minimality of \( i \). Thus we have \( i = 0 \). Put \( y' = 2y + t_0 x \) and \( z' = y + ax \), where \( 2a = 1 + t_0 \). Then \([x, y'; z']\) forms a geodesic of weight two, and is thus inequivalent to \([x, y; z]\).

**Case 2:** \([x, y; z]\) has weight two. Observe that \([x, y + z ; tx]\) is a geodesic. We can therefore pick \( i \geq 1 \) minimal so that \([x, y + t_i x; z + t_i x]\) is a geodesic (the base case corresponding to \( i = r \); note that \( t_r = 0 \)). If \([x, y + t_i x; z + t_i x]\) is inequivalent to \([x, y; z]\) we are done. Thus assume the two are equivalent and let \( h \in G \) be such that \( h[x, y; z] = [x, y + t_i x; z + t_i x] \). As in the previous case, after possibly replacing \( h \) with \( ±h \) or \( ±h\sigma \), we can assume that \( hx = x \) and \( hy = y + t_i x \). Of course, \( hz = z + t_{i-1} x \). Let \( g \in G \) be such that \( Cg - 1 = (-x)(y + t_i x) \). Suppose \( i \geq 2 \). Then \( Cgh - 1 = (-x)(y + t_{i-1} x) \), which shows that \([x, y + t_{i-1} x; z + t_{i-2} x]\) is a geodesic, contradicting the minimality of \( i \). Now suppose that \( i = 1 \). Let \( a \in R \) be such that \( 2a = 1 - t_0 \), and put \( y' = z - ax \). Then \([x, y'; z + y']\) is a weight one geodesic (note that \( Cgh - 1 = 2(-x)y' \)), and thus inequivalent to \([x, y; z]\). \( \square \)

**Lemma 4.3.4.** Let \([x, y_1; z_1]\), \([x, y_2; z_2]\) and \([x, y_3; z_3]\) be three geodesics in \( \Xi \). Then there exists \( g \in G \) such that \( gx = x \) and \( gy_i = y_j \) and \( gz_i = z_j \) for some \( i \neq j \).

**Proof.** Let \( w_i = \langle x, y_i \rangle \) and let \( k \) be the number of indices \( i \in \{1, 2, 3\} \) for which \( w_i = 2 \). We permute the \( y_i \) and \( z_i \) so that the weight one geodesics appear first. We proceed in four cases, depending on the value of \( k \). We let \( g_i \in G \) be such that \( Cg_ix - 1 = w_i(-x)y_i \).

**Case 0:** \( k = 0 \). With respect to the basis \( \{x, y_1\} \) of \( U \) we have

\[
 x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad y_2 = \begin{pmatrix} a \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} b \\ 1 \end{pmatrix}
\]
for some \(a\) and \(b\) in \(R\), and
\[
g_1^{-1}g_2 = \left( \begin{array}{cc} 1 & -2a \\ 0 & 1 \end{array} \right), \quad g_1^{-1}g_3 = \left( \begin{array}{cc} 1 & -2b \\ 0 & 1 \end{array} \right).
\]
Note that \(z_i = x + y_i\) for each \(i\). We must show that one of the matrices
\[
\begin{array}{ccc}
1 & a \\ 0 & 1 \\
1 & b \\ 0 & 1 \\
1 & a - b \\ 0 & 1 
\end{array}
\]
belongs to \(G\). If \(p(a) = 0\) then there is an integer \(n\) such that \(na = 2^k a\); the matrix \((g_1^{-1}g_2)^{-n}\) is then the first one in (1). A similar argument works if \(p(b) = 0\). We may thus assume that both \(p(a)\) and \(p(b)\) are non-zero. We now proceed in two cases depending on how \(p(a)\) and \(p(b)\) compare.

First, suppose that \(p(a)\) and \(p(b)\) generate the same ideal of \(\mathbb{Z}/2^r\mathbb{Z}\). We can find \(n, m \in \mathbb{Z}\) such that \(2na + 2mb = a - b\). To do this, first solve in \(R/2^r R = \mathbb{Z}/2^r \mathbb{Z}\) using the fact that \(p(a) - p(b)\) belongs to the ideal generated by \(2p(a)\), then solve in \(R_3\) using the fact that 2 is invertible, and finally use the Chinese remainder theorem. The matrix \((g_1^{-1}g_2)^{-n}(g_1^{-1}g_3)^{-m}\) is then the third in (1).

Now suppose that \(p(a)\) and \(p(b)\) generate different ideals of \(\mathbb{Z}/2^r\mathbb{Z}\) — say \(p(b)\) belongs to the one generated by \(2p(a)\). We can then find \(n, m \in \mathbb{Z}\) such that \(2na + 2mb = -b\), again using the Chinese remainder theorem. The matrix \((g_1^{-1}g_2)^{-n}(g_1^{-1}g_3)^{-m}\) is then the second in (1).

**Case 1:** \(k = 1\). With respect to the basis \(\{x, y_1\}\) of \(U\) we have
\[
x = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad y_1 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad y_2 = \left( \begin{array}{c} a \\ 1 \end{array} \right), \quad y_3 = \left( \begin{array}{c} b \\ 2 \end{array} \right)
\]
for some \(a\) and \(b\) in \(R\), and
\[
g_1^{-1}g_2 = \left( \begin{array}{cc} 1 & -2a \\ 0 & 1 \end{array} \right), \quad g_1^{-1}g_3 = \left( \begin{array}{cc} 1 & -b \\ 0 & 1 \end{array} \right).
\]
Note that \(z_i = x + y_i\) for \(i \neq 3\). We must show that the matrix
\[
\left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right)
\]
belongs to \(G\). Since \(y_3\) is a basis vector, we find that \(p(b)\) is a unit of \(\mathbb{Z}/2^r\mathbb{Z}\); in particular, \(p(a)\) is a multiple of \(p(b)\). We can therefore find \(n, m \in \mathbb{Z}\) such that \(2na + mb = a\), and so \((g_1^{-1}g_2)^{-n}(g_1^{-1}g_3)^{-m}\) is the required matrix.

**Case 2:** \(k = 2\). With respect to the basis \(\{x, y_1\}\) of \(U\) we have
\[
y_2 = \left( \begin{array}{c} a \\ 2 \end{array} \right), \quad z_2 = \left( \begin{array}{c} a' \\ 1 \end{array} \right), \quad y_3 = \left( \begin{array}{c} b \\ 2 \end{array} \right), \quad z_3 = \left( \begin{array}{c} b' \\ 1 \end{array} \right)
\]
for some \(a, a', b, b'\) in \(R\) with \(2a' = a + 1\) and \(2b' = b + 1\), and
\[
g_1^{-1}g_2 = \left( \begin{array}{cc} 1 & -a \\ 0 & 1 \end{array} \right), \quad g_1^{-1}g_3 = \left( \begin{array}{cc} 1 & -b \\ 0 & 1 \end{array} \right).
\]
We must show that
\[
\left( \begin{array}{cc} 1 & a' - b' \\ 0 & 1 \end{array} \right)
\]
belongs to \(G\). Since \(y_2\) is a basis vector, \(p(a)\) is a unit of \(\mathbb{Z}/2^r\mathbb{Z}\), and so \(p(a' - b')\) is a multiple of \(p(a)\). We can therefore find \(n, m \in \mathbb{Z}\) with \(na - mb = a' - b'\). The matrix \((g_1^{-1}g_2)^{-n}(g_1^{-1}g_3)^m\) then works.

**Case 3:** \(k = 3\). With respect to the basis \(\{x, z_1\}\) of \(U\) we have
\[
x = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad y_i = \left( \begin{array}{c} a_i \\ 2 \end{array} \right), \quad z_i = \left( \begin{array}{c} a_i' \\ 1 \end{array} \right)
\]
for \(a_i\) and \(a'_i\) in \(R\) satisfying \(2a'_i = a_i + 1\) (and, of course, \(a'_1 = 0\)), and
\[
Cg_i = \begin{pmatrix} 1 & -a_i \\ -1 & 1 \end{pmatrix}, \quad g_i^{-1}g_j = \begin{pmatrix} 1 & a_i - a_j \\ 1 & 1 \end{pmatrix}.
\]

It suffices to show that \(G\) contains some matrix of the form
\[
\begin{pmatrix} 1 & a'_i - a'_j \\ 1 & 1 \end{pmatrix},
\]
with \(i \neq j\). Now, if \(c_1, c_2\) and \(c_3\) are three elements of \(\mathbb{Z}/2^c\mathbb{Z}\) which sum to 0 then \(c_i\) is a multiple of 2\(c_j\) for some \(i \neq j\). It follows that, after possibly relabeling the indices, \(p(a'_1) - p(a'_3)\) is a multiple of 2(\(p(a'_1) - p(a'_2)\)) = \(p(a_1) - p(a_2)\). We can therefore find \(n, m \in \mathbb{Z}\) such that
\[n(a_1 - a_2) + m(a_1 - a_3) = a'_1 - a'_3.\]
The matrix \((g_i^{-1}g_j)^n(g_i^{-1}g_j)^m\) then works. \(\square\)

**Lemma 4.3.5.** If \([x, y; z]\) and \([x, y_1; z_1]\) are geodesics of \(\tilde{\Xi}\), and \(x\) and \(y\) are equivalent under \(G\), then either \([x, y; z]\) contains an elliptic point or \([x, y; z]\) and \([x, y_1; z_1]\) are equivalent.

**Proof.** Let \(h \in G\) be such that \(hy = x\). Then \([x, y; z], h[x, y; z] = [x, hx; hz]\) and \([x, y_1; z_1]\) are three geodesics containing \(x\). By the Lemma 4.3.4, we can find \(h' \in G\) fixing \(x\) and carrying one of these three geodesics to another. Now, if \(h'[x, y; z] = [x, y_1; z_1]\) or \(h'h[x, y; z] = [x, y_1; z_1]\) then \([x, y; z]\) and \([x, y_1; z_1]\) are equivalent and we are done. Thus assume that \(h'h[x, y; z] = [x, y; z]\). We must show that \([x, y; z]\) contains an elliptic point.

Put \(\sigma = h'h\), so that \(\sigma(y) = x\). The identity \(\sigma[x, y; z] = [x, y; z]\) only holds in \(\mathcal{T}/(p^2)\). In \(\mathcal{T}\) itself, we have \(\sigma[x, y; z] = [\sigma(x), x; \sigma(z)]\), and so \(p^2\sigma[x, y; z] = [x, x - w\sigma(z); \sigma(z - w'x)]\), where
\[w = (x, y).\]
Clearly, this must equal \([x, y; z]\) in \(\mathcal{T}\) itself. We thus have \(x - w\sigma(z) = y\), which yields \(\sigma(x) = y\). We also have \(\sigma(z - w'x) = z\), which yields \(\sigma(z) = z - w'y\). We conclude from Lemma 4.1.1 that \([x, y; z]\) contains an elliptic point. \(\square\)

We now complete the proof of the theorem.

**Proof of Theorem 4.3.1.** Let \(x\) be a parabolic vertex of \(\tilde{\Xi}\). By Lemma 4.3.5, if \(\pi(x)\) belongs to a loop (i.e., there is a geodesic \([x, y; z]\) not containing an elliptic point but with \(\pi(x) = \pi(y)\)) then any two geodesics containing \(x\) are equivalent. Thus \(\pi(x)\) has valence two. Suppose then that \(\pi(x)\) does not belong to a loop. By Lemmas 4.3.3 and 4.3.4, there are exactly two equivalence classes of geodesics containing \(x\). Each contributes one to the valence of \(\pi(x)\), and so \(\pi(x)\) has valence two. Thus all parabolic vertices of \(\Xi\) have valence two. By definition, each elliptic vertex of \(\tilde{\Xi}\) has valence four, being contained in two edges of each weight. The edges of equal weight are equivalent (by Lemma 4.1.1), and so each elliptic vertex of \(\Xi\) has valence two. We have thus shown that all vertices of \(\Xi\) have valence two.

It remains to show that the components of \(\Xi\) are finite. Let \([x, y; z]\) be a geodesic of \(\tilde{\Xi}\) and let \(U'\) be the \(\mathbb{Z}\)-submodule of \(U\) generated by \(x, y\) and \(z\). Let \(e = \pi([x, y; z])\) be the edge of \(\Xi\) corresponding to \([x, y; z]\). We claim that any edge of \(\Xi\) neighboring \(e\) is represented by a geodesic whose components belong to \(U'\). Thus let \(e'\) be an edge neighboring \(e\). First suppose that \(e\) and \(e'\) meet at \(\pi(x)\). Examining the proof of Lemma 4.3.3, we see that we can find a geodesic of \(\tilde{\Xi}\) which is inequivalent to \([x, y; z]\) but whose components belong to \(U'\). This geodesic must map under \(\pi\) to \(e'\), which proves the claim in this case. If \(e\) and \(e'\) meet at \(\pi(y)\) the argument is similar. Now suppose that \(e\) and \(e'\) meet at an elliptic vertex. Then \(e'\) is the image of the geodesic \(\rho[x, y; z]\), whose components do indeed belong to \(U'\). This completes the proof of the claim. It now follows from induction that any parabolic vertex of \(\Xi\) in the same component as \(x\) is represented by an element of \(U'\). Since \(U'\) is a finite set, it follows that there are only finitely many parabolic vertices in the component containing \(x\), and therefore that this component is finite (since elliptic vertices only connect to parabolic vertices). This completes the proof. \(\square\)
4.4. **Theorem 4.3.1 in odd characteristic.** We now establish the following strengthening of Theorem 4.3.1 in odd characteristic:

**Proposition 4.4.1.** Suppose that $R$ has odd characteristic. Then every vertex of $\Xi$ belongs to exactly one edge of each weight (and to no loops).

Note that in any geodesic $[x, y; z]$, the element $z$ is uniquely determined from $x$ and $y$. Therefore, we drop the third component of geodesics from notation in this section. If $x$ and $y$ are basis vectors then $[x, y]$ forms a geodesic if and only if there exists $g \in G$ with $C_g x = x$ and $C_g y = y$. The following lemma, combined with Theorem 4.3.1, establishes the proposition.

**Lemma 4.4.2.** Every parabolic vertex of $\Xi$ belongs to at least one edge of each weight.

**Proof.** Let $x$ be a parabolic vertex of $\Xi$ and pick $g \in G$ admissible so that $C_g x = x$. Let $u \in U$ be such that $(x, u) = 1$, and write $C_g u = -u + ax$ with $a \in R$. Put $y = u - \frac{1}{2}ax$. Then $C_g y = -y$, and so $[x, y]$ forms a geodesic of $\Xi$, of weight 1. As $C_g(2y) = -(2y)$, we find that $[x, 2y]$ forms a geodesic of weight 2. This completes the proof. □

As the proof of Theorem 4.3.1 is quite involved, we offer the following lemma which, together with the above lemma, directly establishes the proposition — except for the statement about loops. (One can give a similar direct argument to deal with loops.)

**Lemma 4.4.3.** Every parabolic vertex of $\Xi$ belongs to at most one edge of each weight.

**Proof.** It suffices to show that if $[x, y]$ and $[x, y']$ are geodesics of $\Xi$ of equal weight then there exists $h \in G$ such that $hx = x$ and $hy = y'$. Let $g \in G$ be such that $C_g x = x$, $C_g y = y$ and similarly define $g'$. Since $\{x, y\}$ forms a basis of $U$, we can write $y' = y + ax$ for some $a \in R$. We have $C_g C_{g'} = g^{-1} g'$, and so $g^{-1} g' x = x$ and $g^{-1} g' y' = y' - 2ax$. Let $n$ be an integer which is equal to $-\frac{1}{2}$ in $R$. Then $(g^{-1} g')^n x = x$ while $(g^{-1} g')^n y' = y$. This completes the proof. □

4.5. **Modular graphs.** It will be convenient in what follows to formalize some notions. A **modular graph** is an undirected graph which is a disjoint union of cycles, and such that each vertex is assigned one of two types (elliptic or parabolic) and each edge is assigned a weight in $\{1, 2\}$. We furthermore require that every edge contain a parabolic vertex. We allow multiple edges between vertices. We say that a modular graph is **regular** if every vertex belongs to exactly one edge of each weight. Of course, $\Xi_G$ is a modular graph by Theorem 4.3.1, and regular when $R$ has odd characteristic by Proposition 4.4.1.

For a modular graph $\Xi$, write $V_p(\Xi)$ for the set of parabolic vertices, $V_e(\Xi)$ for the set of elliptic vertices and $V(\Xi)$ for $V_p(\Xi) \cup V_e(\Xi)$. Write $E(\Xi) \subset V(\Xi) \times V(\Xi)$ for the set of edges, and let $s$ and $t$ (source and target) be the two maps $E(\Xi) \rightarrow V(\Xi)$. We encode the fact that $\Xi$ is undirected in an involution $\tau : E(\Xi) \rightarrow E(\Xi)$ satisfying $t(\tau(e)) = s(e)$. We picture $e$ and $\tau(e)$ as the same edge.

Let $\Xi_1$ and $\Xi_2$ be two modular graphs. We define a new graph $\Xi$ as follows. We define $V_p(\Xi)$ to be $V_p(\Xi_1) \times V_p(\Xi_2)$ and we define $V_e(\Xi)$ to be $V_e(\Xi_1) \times V_e(\Xi_2)$. The set $E(\Xi)$ is the subset of $E(\Xi_1) \times E(\Xi_2)$ consisting of those pairs $(e_1, e_2)$ of equal weight which satisfy one of the following conditions:

- The vertices $s(e_1)$ and $s(e_2)$ are of the same type, as are $t(e_1)$ and $t(e_2)$.
- The vertices $s(e_1)$ and $s(e_2)$ are parabolic, while one of $t(e_1)$ and $t(e_2)$ is parabolic and the other is elliptic.

Let $(e_1, e_2)$ be an element of $E(\Xi)$. In the first case above, we define $\tau(e_1, e_2)$ to be $(\tau(e_1), \tau(e_2))$, we define $s(e_1, e_2)$ to be $(s(e_1), s(e_2))$ and we define $t(e_1, e_2)$ to be $(t(e_1), t(e_2))$. Suppose now we are in the second case, with $t(e_1)$ is parabolic and $t(e_2)$ elliptic. We define $\tau(e_1, e_2)$ to be $(\tau(e_1), e_2)$, we define $s(e_1, e_2)$ to be $(s(e_1), s(e_2))$ and we define define $t(e_1, e_2)$ to be $(t(e_1), s(e_2))$. Of course, we define the weight of $(e_1, e_2)$ to be the weight of $e_1$, which agrees with the weight of $e_2$. We
call $\Xi$ the product of $\Xi_1$ and $\Xi_2$ and denote it by $\Xi_1 \ast \Xi_2$. The product is commutative (up to isomorphism) and distributes over disjoint union.

**Proposition 4.5.1.** Let $\Xi_1$ and $\Xi_2$ be modular graphs, one of which is regular. Then $\Xi_1 \ast \Xi_2$ is a modular graph.

**Proof.** Assume that $\Xi_2$ is regular. Let $x_1$ be a vertex of $\Xi_1$ and $x_2$ a vertex of $\Xi_2$ of the same type as $x_1$. Let $e_1$ be an edge of $\Xi_1$ with $s(e_1) = x_1$. Then there exists a unique edge $e_2$ of $\Xi_2$ with $s(e_2) = x_2$ and having the same weight as $e_2$. Furthermore, if $(e_1, e_2)$ is an edge of $\Xi$ with source $(x_1, x_2)$, and if $s(e_1) = x_1$, then $(e_1, e_2)$ is an edge of $\Xi$ with source $(x_1, x_2)$. Furthermore, if $(e_1, e_2)$ is an edge of $\Xi$ with source $(x_1, x_2)$ then $s(e_1) = x_1$. This shows that there is a bijection between edges of $\Xi$ sourced at $(x_1, x_2)$ and edges of $\Xi_1$ sourced at $x_1$. Since every vertex of $\Xi_1$ has valence two, it follows that every vertex of $\Xi$ has valence two as well. (Note that $\Xi$ is undirected.) Furthermore, it is clear that if $(x_1, x_2)$ and $(y_1, y_2)$ belong to the same component of $\Xi$ then $x_1$ and $y_1$ belong to the same component of $\Xi_1$ and $x_2$ and $y_2$ belong to the same component of $\Xi_2$. Since the components of $\Xi_1$ and $\Xi_2$ are finite, this shows that the components of $\Xi$ are finite as well. This completes the proof. □

The modular graph

\[ \Xi \]

is the unique one with one parabolic vertex, one elliptic vertex, one weight one edge and one weight two edge. It is the identity for the product $\ast$. In fact, $\ast$ is best thought of as a sort of fiber product over the above graph.

### 4.6. Behavior of $\Xi$ under direct products.

Let $R$, $U$, etc., be as in the first paragraph of \[4.3.1\] and let $R'$, $U'$, etc., be defined similarly. Suppose that $R'$ has odd characteristic. Let $\Xi = \Xi_G$ and $\Xi' = \Xi_{G'}$, so that $\Xi$ and $\Xi'$ are modular graphs, with $\Xi'$ regular. Let $\Xi'' = \Xi_{G \times G'}$ be the graph associated to $G \times G'$, a subgroup of $\text{SL}(U \times U')$. We then have the following result:

**Theorem 4.6.1.** The graph $\Xi''$ is isomorphic to the product $\Xi \ast \Xi'$. 

**Proof.** The graph $\Xi''$ is a modular graph by Theorem \[4.3.1\], while $\Xi \ast \Xi'$ is a modular graph by Proposition \[4.5.1\]. Furthermore, it is clear that there is a natural bijection between $V_p(\Xi'')$ and $V_p(\Xi \ast \Xi')$. Thus, to demonstrate the proposition it suffices to show that the following two statements: (1) if two parabolic vertices of $\Xi''$ are connected then the corresponding parabolic vertices of $\Xi \ast \Xi'$ are as well; (2) if two parabolic vertices of $\Xi''$ are connected to a common elliptic vertex then the corresponding vertices of $\Xi \ast \Xi'$ connect to a common elliptic vertex as well.

We now prove statement (1). Thus suppose that $x$ and $y$ are parabolic vertices of $\Xi$ and $x'$ and $y'$ are parabolic vertices of $\Xi'$ such that $(x, x')$ and $(y, y')$ are connected in $\Xi''$. It follows that, after replacing these vertices with equivalent ones, we have a geodesic $[(x, x'), (y, y'); (z, z')]$ in $\Xi''$ which does not contain an elliptic point. It is clear then that $[x, y; z]$ and $[x', y'; z']$ are geodesics in $\Xi$ and $\Xi'$ of equal weights. If both contained elliptic points, then a short argument shows that $[(x, x'), (y, y'); (z, z')]$ would contain an elliptic point; we conclude that one of the two geodesics does not contain an elliptic point. Suppose that $[x, y; z]$ does not contain an elliptic point, so that there is an edge between $x$ and $y$ in $\Xi$. If $[x', y'; z']$ also does not contain an elliptic point, then there is an edge between $x'$ and $y'$ in $\Xi'$, and thus an edge between $(x, x')$ and $(y, y')$ in $\Xi \ast \Xi'$. If $[x', y']$ contains an elliptic point, then we get an edge in $\Xi \ast \Xi'$ between $(x, x')$ and $(y, y')$. Thus, in all cases, $(x, x')$ and $(y, y')$ are connected in $\Xi \ast \Xi'$.

We now prove statement (2). Thus suppose that $[(x_1, x'_1), (y_1, y'_1); (z_1, z'_1)]$ and $[(x_2, x'_2),(y_2, y'_2); (z_2, z'_2)]$ are intersecting geodesics in $\Xi''$. Then $[x_1, y_1; z_1]$ and $[x_2, y_2; z_2]$ are intersecting geodesics in $\Xi$, and $[x'_1, y'_1; z'_1]$ and $[x'_2, y'_2; z'_2]$ are intersecting geodesics in $\Xi'$. It follows that $(x_1, x'_1)$ and $(x_2, x'_2)$ connected to a common elliptic point in $\Xi \ast \Xi'$, as was to be shown. This completes the proof. □
Remark 4.6.2. In many cases, one has a group of the form \(±(G \times G')\) where \(G\) and \(G'\) do not contain \(-1\). The above theorem does not apply to describe the graph associated to this group in terms of \(G\) and \(G'\). However, there is an analogous, though more complicated, result. In fact, to a group \(G\) which does not contain \(-1\) one can associate a graph that is a sort of cover of \(Ξ_{±G}\), and there is a product theorem for such graphs.

5. Components for real congruence groups

The purpose of this section is to describe the real components of \(X_Γ\) when \(Γ\) is a congruence subgroup of \(\text{PSL}_2(\mathbb{Z})\) with an appropriate real structure, in terms of the group theory of the corresponding subgroup of \(\text{SL}_2(\mathbb{Z}/N\mathbb{Z})\).

5.1. Real congruence groups. A real congruence group is a real Fuchsian group \((Γ, c)\) where \(Γ\) is a subgroup of \(\text{PSL}_2(\mathbb{Z})\) for which there exists an integer \(N \geq 1\) such that \(Γ\) contains \(Γ(N)\) and \(Γ(N)\) is stable by \(c\). (Recall that \(Γ(N)\) is the subgroup of \(\text{PSL}_2(\mathbb{Z})\) consisting of matrices which are congruent to the identity modulo \(N\).)

Proposition 5.1.1. Let \((Γ, c)\) be a real congruence group. Then \(c = c_0σ\) for some element \(σ \in \text{PSL}_2(\mathbb{Z})\).

Proof. Of course, \(c = c_0σ\) for some \(σ \in \text{PSL}_2(\mathbb{R})\). Since \(Γ(N)\) is closed under both \(γ \mapsto γ^c\) and \(γ \mapsto γ^c_0\), it follows that \(σ\) normalizes \(Γ(N)\). As \(\text{PSL}_2(\mathbb{Z})\) is the full normalizer of \(Γ(N)\) in \(\text{PSL}_2(\mathbb{R})\), we find that \(σ\) belongs to \(\text{PSL}_2(\mathbb{Z})\), as was to be shown. \(\square\)

Corollary 5.1.2. The matrix \(C\) associated to \(c\) (see 2.7) belongs to \(\text{GL}_2(\mathbb{Z})\).

Proof. Write \(c = c_0σ\) with \(σ \in \text{SL}_2(\mathbb{Z})\), per the proposition. As we showed, the matrix \(C\) is given by \(C_0σ\). Since both \(C_0\) and \(σ\) belong to \(\text{GL}_2(\mathbb{Z})\), the result follows. \(\square\)

Proposition 5.1.3. Let \((Γ, c)\) be a real congruence group and let \(γ \in Γ\) be admissible. Then the curve \(C_γ\) contains two cusps, and is therefore homeomorphic to a closed interval.

Proof. Let \(γ \in Γ\) be admissible with respect to \(c\) and write

\[ C_γ = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \]

with \(a, b\) and \(c\) integers. By Proposition 3.2.1 (or, rather, its proof), \(C_γ\) intersects \(\mathbb{RP}^1\) at the two points \(\frac{a+1}{c}\), both of which belong to \(\mathbb{QP}^1\) and are thus cusps of \(Γ\). \(\square\)

Corollary 5.1.4. Every component of \(X_Γ(\mathbb{R})\) contains a cusp.

Corollary 5.1.5. Every admissible element of \(Γ\) is Type 1 or Type 2.

Combining the above with Corollary 3.5.2, we obtain the following:

Corollary 5.1.6. The number of admissible twisted conjugacy classes in \(Γ\) is equal to the number of special points on \(X_Γ\).

Corollary 5.1.4 is extremely useful, because the set of cusps is easy to understand, and so to determine \(Ξ_Γ\) we just need to determine how the cusps (and elliptic points) are connected. We remark that the above results do not hold if we drop the assumption that \(Γ(N)\) is stable by \(c\): there exist real Fuchsian groups \((Γ, c)\) with \(Γ\) a congruence subgroup of \(\text{PSL}_2(\mathbb{Z})\) which have Type 3 elements (see 6.3).
5.2. The main theorem. Let \((\Gamma, c)\) be a real congruence group. Let \(\Xi\) denote the graph associated to \(\Gamma\) in \(\mathbb{R}\mathbb{C}\). Let \(\Xi'\) denote the graph associated to \(\Gamma\) in \(\mathbb{R}\mathbb{C}\) (taking \(R = \mathbb{Z}, U = \mathbb{Z}^2\), \(\langle , \rangle\) the standard symplectic pairing on \(U, C\) the matrix associated to \(c\) and \(G = \Gamma\)). The following is the main result of [5].

**Theorem 5.2.1.** Let \((\Gamma, c)\) be a real congruence group. Then \(\Xi' = \Xi\).

Before proceeding, let us note the following corollary. Let \(G_0\) be the image of \(\Gamma\) in \(\text{SL}_2(\mathbb{Z}/N\mathbb{Z})\), let \(U_0 = U/NU\) and let \(\Xi_0\) be the graph associated to this data in \(\mathbb{R}\mathbb{C}\). By Theorem 4.2.1, we have \(\Xi = \Xi_0\). (Note: the surjectivity of the map \(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})\) is proved in [Sh, Lem. 1.38].) We therefore obtain the following corollary, which computes the real locus of the curve \(X_\Gamma\) in terms of the subgroup of \(\text{SL}_2(\mathbb{Z}/N\mathbb{Z})\) to which \(\Gamma\) corresponds.

**Corollary 5.2.2.** We have \(\Xi' = \Xi_0\).

We now begin on the proof of the theorem. Let \(U'\) denote the set of basis vectors in \(U\). There is a bijection

\[ i : \mathbb{Q}^1 \to U'/\{\pm 1\}, \]

defined by mapping a point \([p : q]\) in \(\mathbb{Q}^1\) with \(p\) and \(q\) coprime to the vector \((p, q)\) in \(U\). This map is \(\Gamma\)-equivariant, and transforms the action of \(c\) on the left to that of \(C\) on the right. It follows that the cusps of \(X_\Gamma\) (i.e., the orbits of \(\Gamma\) on \(\mathbb{Q}^1\)) are identified via this map with \(U'/\Gamma\). Furthermore, a cusp belongs to \(X_\Gamma(R)\) if and only if the corresponding element of \(U'/\Gamma\) is invariant under \(C\). This shows that the parabolic vertices of \(\Xi'\) and \(\Xi\) are in natural bijection.

We now prove two lemmas, from which the theorem easily follows.

**Lemma 5.2.3.** Let \(\gamma \in \Gamma\) be admissible and let \(\overline{x}\) and \(\overline{y}\) be the two cusps on \(C_\gamma\). Then there exists a unique geodesic \([x, y; z]\) of \(\Xi\) such that the images of \(x\) and \(y\) in \(U'/\{\pm 1\}\) coincide with \(i(\overline{x})\) and \(i(\overline{y})\). This geodesic is characterized by the identity \(C_\gamma - 1 = w'(-x, y)\).

**Proof.** Write

\[ C_\gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \]

Suppose \(c \neq 0\). Then, after possibly switching \(\overline{x}\) and \(\overline{y}\),

\[ \overline{x} = \frac{a - 1}{c}, \quad \overline{y} = \frac{a + 1}{c}. \]

(See the proof of Proposition 5.1.3.) Let \(d_1 = \text{gcd}(a - 1, c)\) and \(d_2 = \text{gcd}(a + 1, c)\). Then, by definition,

\[ i(\overline{x}) = \pm \left( \frac{a - 1}{d_1}, \frac{c}{d_1} \right), \quad i(\overline{y}) = \pm \left( \frac{a + 1}{d_2}, \frac{c}{d_2} \right), \]

and so

\[ \langle i(\overline{x}), i(\overline{y}) \rangle = \pm \frac{2c}{d_1d_2}. \]

An elementary argument shows that \(d_1d_2\) is equal to either \(\pm c\) or \(\pm 2c\). We can thus pick lifts \(x\) and \(y\) of \(i(\overline{x})\) and \(i(\overline{y})\) to \(U\) so that \(w = \langle x, y \rangle\) is 1 or 2. From the identities \(\gamma\overline{x} = c\overline{x}\) and \(\gamma\overline{y} = c\overline{y}\), we conclude that \(C_\gamma x = \pm x\) and \(C_\gamma y = \mp y\). Replacing \(\gamma\) with \(-\gamma\) if necessary, we can assume that \(C_\gamma x = x\) and \(C_\gamma y = -y\), and so \(C_\gamma - 1 = w'(-x, y)\). Let \(z\) be the unique element of \(U\) satisfying \(x + y = wz\). Then \([x, y; z]\) is a geodesic of \(\Xi\). If \([x', y'; z']\) is another geodesic such that \(x'\) lifts \(i(\overline{x})\) and \(y'\) lifts \(i(\overline{y})\), then necessarily \(x' = \pm x\) and \(y' = \pm y\), and so \([x', y'; z'] = [x, y; z]\). If \(b \neq 0\) the proof is similar. When \(b = c = 0\) one can argue directly. This completes the proof.

**Lemma 5.2.4.** Let \(\gamma_1 \neq \gamma_2\) be admissible. Then the curves \(C_{\gamma_1}\) and \(C_{\gamma_2}\) intersect at an even order elliptic point if and only if the geodesics of \(\Xi\) corresponding to \(\gamma_1\) and \(\gamma_2\) intersect (in the sense of [4.7]).
Proof. Let \( \overrightarrow{e}_i \) and \( \overrightarrow{f}_i \) be the cusps on \( C_{\gamma_i} \) and let \([x_i, y_i; z_i]\) be the corresponding geodesic of \( \Xi \). If \([x_1, y_1; z_1]\) and \([x_2, y_2; z_2]\) intersect, then by Lemma 4.1.1 there exists \( \sigma \in \Gamma \) such that \( \sigma(\overrightarrow{e}_i) = \overrightarrow{f}_i \) and \( \sigma(\overrightarrow{f}_i) = \overrightarrow{e}_i \) and \( \sigma^2 = -1 \). Thus \( \sigma \) induces an orientation-reversing homeomorphism of \( C_{\gamma_i} \), and so \( C_{\gamma_i} \) contains a fixed point of \( \sigma \). As \( \sigma \) has only one fixed point in the upper half-plane, an elliptic point of even order, this point belongs to \( C_{\gamma_1} \cap C_{\gamma_2} \).

Now suppose that \( C_{\gamma_1} \) and \( C_{\gamma_2} \) intersect at an elliptic point, and let \( \sigma \in \Gamma \) generate the stabilizer of this elliptic point. Then \( \sigma(\overrightarrow{e}_i) = \overrightarrow{f}_i \) and \( \sigma(\overrightarrow{f}_i) = \overrightarrow{e}_i \) and \( \gamma_2 = \gamma_1 \sigma \). Replacing \( \sigma \) with \( -\sigma \) if necessary, we have \( \sigma(x_i) = -y_i \) and \( \sigma(y_i) = x_i \). It follows that \( \sigma(z_i) = z_i - w_i y_i \). Thus by Lemma 4.1.1 \([x_3, y_3; z_3]\) is a geodesic. We claim that it coincides with \([x_2, y_2; z_2]\). To see this, let \( \gamma' \) be defined by \( C\gamma' = 1 = w'_3\langle -,x_3,y_3 \rangle \). Then, by the first paragraph, \( C\gamma' \) meets \( C_{\gamma_1} \) at an even order elliptic point. Thus \( \gamma' = \gamma_1 \sigma \), which shows that \( \gamma' = \gamma_2 \). Since \([x_2, y_2; z_2]\) and \([x_3, y_3; z_3]\) are both associated to \( \gamma_2 \), they must coincide.

We now complete the proof of the theorem.

Proof of Theorem 5.2.1 We have shown that the parabolic vertices of \( \Xi \) are in bijection with the real cusps of \( \Gamma \) and that the geodesics of \( \Xi \) are in bijection with the curves \( C_{\gamma} \) with \( \gamma \) admissible. Furthermore, two geodesics intersect if and only if the corresponding curves intersect at an even order elliptic point. It now follows simply from the constructions of \( \Xi' \) and \( \Xi \) that they are isomorphic. \( \square \)

5.3. A result on real elliptic points. The following result rules out even order real elliptic points in many examples.

Proposition 5.3.1. Let \((\Gamma, c)\) be a real congruence group with \( c = c_0 \) and such that \( \Gamma \) is contained in \( \Gamma_0(N) \) for some \( N > 2 \). Then \( \Gamma \) has no real elliptic point of even order.

Proof. Let \( z \in \mathfrak{h} \) be a real even order elliptic point for \( \Gamma \). Let \( \sigma \) be a non-trivial order two element of \( \Gamma \) stabilizing \( z \). Since every even order elliptic point for \( \text{PSL}_2(\mathbb{Z}) \) belongs to the orbit of \( i \), we can write \( z = \tau(i) \), for some \( \tau \in \text{PSL}_2(\mathbb{Z}) \). We then have \( \sigma = \tau \sigma_0^{-1} \tau^{-1} \), where

\[
\sigma_0 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Since \( z \) is real, it belongs to \( C_{\gamma} \) for some \( \gamma \in \Gamma \). It follows that \( i \) is contained on the curve \( \tau^{-1} C_{\gamma} = C_{(\tau\gamma)^{-1} \tau^{-1}} \). However, one can verify that if \( \delta \) is an element of \( \text{PSL}_2(\mathbb{Z}) \) such that \( C_{\delta} \) contains \( i \), then \( \delta \) is either \( 1 \) or \( \sigma_0 \). We thus find that \( \gamma \) is either \( \tau \tau^{-1} \) or \( \tau \sigma_0 \tau^{-1} \); clearly, \( \gamma \sigma \) is the other.

We have thus shown that if \( \Gamma \) has a real even order elliptic element, then there exists \( \tau \in \text{PSL}_2(\mathbb{Z}) \) such that \( \tau \sigma_0^{-1} \tau^{-1} \) and \( \tau \sigma_0 \tau^{-1} \) both belong to \( \Gamma \). Writing \( \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), we find that the bottom left entry of \( \tau \sigma_0^{-1} \tau^{-1} \) is \( -(c^2 + d^2) \), while the bottom left entry of \( \tau \sigma_0 \tau^{-1} \) is \( -2cd \). Thus both \( c^2 + d^2 \) and \( 2cd \) are divisible by \( N \). On the other hand, \( c \) and \( d \) are coprime. This cannot happen unless \( N \) is either 1 or 2. \( \square \)

6. Examples

We now give some examples. Throughout this section, \( N \) denotes a positive integer. We write \( N = 2^t N' \) where \( N' \) is odd. When \( N \) is even, we let \( t = 2^{r-1} N' \) be the unique non-zero 2-torsion element of \( \mathbb{Z}/N\mathbb{Z} \). We mostly work with congruence subgroups \( \Gamma \) of level \( N \). We always denote by \( G \) the corresponding subgroup of \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \).
6.1. The full modular group. We begin with the simplest example, namely \( \Gamma = \text{SL}_2(\mathbb{Z}) \) and \( c = c_0 \). We could apply our theory to this example, but we find it clearer to reason directly. The curve \( X_\Gamma \) has genus 0 and a real point and is therefore isomorphic to \( \mathbb{P}^1 \) over \( \mathbb{R} \). It follows that \( X_\Gamma(\mathbb{R}) \) is a circle. Put
\[
\sigma = \left( \begin{array}{cc}
1 & \gamma \\
-1 & 1
\end{array} \right).
\]
Then \( C_1 \) is the positive imaginary axis, while \( C_\sigma \) is the unit circle in the upper half-plane. The two curves intersect at the elliptic point \( i \). From the standard description of the fundamental domain of \( \Gamma \), we see that the loci \( C_1 \) and \( C_\sigma \) are inequivalent, and that \( \pi(C_1) \) and \( \pi(C_\sigma) \) intersect only at \( \pi(i) \) and \( \pi(\infty) \). It follows that the images of \( \pi(C_1) \) and \( \pi(C_\sigma) \) cover the entire real locus. The picture is thus:

In this picture, we have indicated the weights of the two edges, which can be computed by taking the pairing of the two cusps on each curve. (For \( C_1 \) the two cusps are 0 and \( \infty \), which pair to 1, while for \( C_\sigma \) they are \( \pm 1 \), which pair to 2.)

It is instructive to visualize walking around this circle while at the same time moving in the upper half-plane. Start at \( \infty \) in the upper half-plane, which is the parabolic vertex in the above graph. As we move down the imaginary axis towards \( i \), we approach the elliptic vertex along the weight one edge. Of course, we reach \( i \) when we reach the elliptic vertex. If we continue to move down the imaginary axis after reaching \( i \) then we start to move backwards along the weight one edge, reaching the parabolic vertex when we hit 0. To move along the weight two edge, we must turn at \( i \) and move along the unit circle (say clockwise). If we follow this to the real axis we reach 1, which is the parabolic vertex. However, this picture is not completely satisfying since we have returned to a different cusp. Instead, one can travel clockwise from \( i \) to \( e^{2\pi i/6} \), the elliptic point of order 3, and then turn and move up the line with real part \( 1/2 \). This leaves \( C_\sigma \) but moves onto an equivalent curve. Traveling all the way up, we return to \( \infty \).

6.2. The curve \( X^+(N) \). Let \( \Gamma = \Gamma^+(N) \) denote the real congruence group (\( \Gamma(N), c_0 \)) and let \( X = X^+(N) \) denote the corresponding quotient. We have \( G = \{ \pm 1 \} \). The space \( X \) parameterizes pairs \((E,i)\) where \( E \) is an elliptic curve and \( i \) is an isomorphism of \( E[N] \) with \((\mathbb{Z}/N\mathbb{Z}) \oplus \mu_N \) under which the Weil pairing corresponds to the standard \( \mu_N \)-valued pairing on the target. In what follows, we analyze the structure of the graph \( \Xi = \Xi_\Gamma \). Note that for \( N \neq 1 \) the group \( \Gamma \) has no elliptic elements. We assume \( N > 1 \) in what follows.

Suppose that \( r = 0 \), i.e., \( N \) is odd, and \( N \neq 1 \). The real cusps are then represented by the vectors \((a,0)\) and \((0,a)\), with \( a \) coprime to \( N \). There are thus \( \phi(N) \) real cusps. The local structure of \( \Xi \) at the vertex \((a,0)\) is given as follows:

\[
\begin{array}{cccc}
(0,4b) & (0,2b) & (0,b) & (0,b/2) \\
(a/2,0) & (a,0) & (2a,0) & (4a,0)
\end{array}
\]

where \( ab = 1 \) in \( \mathbb{Z}/N\mathbb{Z} \). We thus see that every cycle of \( \Xi \) contains a vertex of the form \((*,0)\) and that \((a,0)\) and \((a',0)\) belong to the same cycle if and only if \( a' \) is of the form \( \pm 2^n a \). It follows that there are \( \psi(N) \) cycles.

Now suppose that \( r = 1 \), i.e., \( 2 \) divides \( N \) exactly. If \( N = 2 \) then there are two real cusps in a single cycle. Thus assume \( N \neq 2 \), i.e., \( N' \neq 1 \). The real cusps are then represented by the vectors \((a,0)\) and \((0,a)\), with \( a \) coprime to \( N \), and the vectors \((a,t)\) and \((t,a)\), with \( a \) coprime to \( N' \). There
are $3\phi(N)$ real cusps. Every cycle of $\Xi$ has six cusps, and is of the form

\[
\begin{align*}
&\text{(ea, t)} \quad \text{(eb, t)} \\
&\text{(a, t)} \quad \text{(a, 0)} \quad \text{(b, 0)} \\
&\text{(t, a)} \quad \text{(t, b)}
\end{align*}
\]

with $a$ coprime to $N$, $ab = 1$ and $e = 1 - t$. Note that $e^2 = e$ and $t^2 = t$. There are $\frac{1}{2}\phi(N)$ cycles.

Finally suppose that $r > 1$, i.e., $N$ is divisible by 4. The real cusps are then represented by the vectors $(a, 0), (0, a), (a, t)$ and $(t, a)$, with $a$ coprime to $N$. There are thus $2\phi(N)$ real cusps. Every cycle of $\Xi$ has exactly four cusps and is of the form

\[
\begin{align*}
&\text{(t, b)} \quad \text{(a, t)} \\
&\text{(a, 0)} \quad \text{(0, b)}
\end{align*}
\]

where $ab = 1$. The number of cycles is $\frac{1}{2}\phi(N)$.

The results are summarized in the following proposition:

**Proposition 6.2.1.** Let $N > 1$ be an integer. Then $X^+(N)$ contains no elliptic points. The number of real cusps on $X^+(N)$ is given by

\[
\phi(N) \times \begin{cases} 
1 & \text{if } N \text{ is odd} \\
3 & \text{if } 2 \mid N \text{ but } N \neq 2 \\
2 & \text{if } 4 \mid N \text{ or } N = 2.
\end{cases}
\]

The number of real components of $X^+(N)$ is given by

\[
\begin{cases} 
\psi(N) & \text{if } N \text{ is odd} \\
\frac{1}{2}\phi(N) & \text{if } N \neq 2 \text{ is even} \\
1 & \text{if } N = 2.
\end{cases}
\]

In all cases, each component has the same structure.

6.3. **The curve** $X^-(N)$. Let $c_1$ denote the complex conjugation of $h$ given by $z \mapsto 1/z$. Let $\Gamma = \Gamma^-(N)$ denote the real congruence group $(\Gamma(N), c_1)$ and let $X = X^-(N)$ denote the corresponding quotient. It is not difficult to see that every real form of $\Gamma(N)$ is equivalent to either $\Gamma^+(N)$ or $\Gamma^-(N)$, and that these two are equivalent to each other if and only if $N$ is odd. The space $X^-(N)$ parameterizes pairs $(E, i)$ where $E$ is an elliptic curve and $i$ is an isomorphism of $E[N]$ with the extension of $\mathbb{Z}/N\mathbb{Z}$ by $\mu_N$ corresponding under Kummer theory to $-1$ (in $\mathbb{R}^\times/(\mathbb{R}^\times)^N$ or even $\mathbb{Q}^\times/(\mathbb{Q}^\times)^N$) under which the Weil pairing corresponds to the standard pairing. (The same description applies to $X^+(N)$, except one uses the extension corresponding to $1 \in \mathbb{R}^\times/(\mathbb{R}^\times)^N$.) Of course, $\Gamma^-(N)$ has no elliptic points for $N \neq 1$. We assume $N > 1$ in what follows.

When $r = 0$ the real locus of $X^-(N)$ is isomorphic to that of $X^+(N)$. Thus assume $r > 0$. The real cusps are represented by vectors of the form $(a, a)$ and $(a, -a)$ where $a$ is prime to $N$. When $N = 2$ there is thus a single real cusp, and it has a self-edge of weight 2. Now assume $N \neq 2$. There are thus $\phi(N)$ real cusps. Every cycle has two vertices and is of the form

\[
\begin{align*}
&\text{(a, a)} \\
&\text{(b, -b)}
\end{align*}
\]

where $ab = 1$. There are thus $\frac{1}{2}\phi(N)$ cycles.

The results are summarized in the following proposition:
Proposition 6.3.1. Let \( N > 1 \) be an integer. Then \( X^{-}(N) \) contains no elliptic points. The number of real cusps on \( X^{-}(N) \) is given by \( \phi(N) \). The number of real components of \( X^{-}(N) \) is given by

\[
\begin{cases} 
\psi(N) & \text{if } N \text{ is odd} \\
\frac{1}{2}\phi(N) & \text{if } N \neq 2 \text{ is even} \\
1 & \text{if } N = 2.
\end{cases}
\]

In all cases, each real component has the same structure.

Note that \( X^{+}(N) \) and \( X^{-}(N) \) have the same number of real components, even though their real components have different structure.

6.4. The curve \( X_1(N) \). Let \( \Gamma = \Gamma_1(N) \) denote the real congruence group \((\Gamma_1(N), c_0)\) and let \( X = X_1(N) \) be the corresponding quotient. The group \( G \) is the subgroup of \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) consisting of matrices of the form

\[
\begin{pmatrix} 
\pm 1 & \ast \\
\pm 1 & 
\end{pmatrix}.
\]

The space \( X \) parameterizes pairs \((E, P)\) where \( E \) is an elliptic curve and \( P \) is a point on \( E \) of exact order \( N \). The group \( \Gamma \) has no even order elliptic elements for \( N > 2 \).

Suppose first that \( r = 0 \). The real cusps of \( \Gamma \) are represented by vectors of the form \((a, 0)\) and \((0, a)\), with \( a \) prime to \( N \). This representation is unique, except for the action of \( \pm 1 \). It follows that there are \( \phi(N) \) real cusps. Clearly, \((a, 0)\) is connected in \( \Xi_G \) to both \((0, b)\) and \((0, 2b)\), where \( b = a^{-1} \). The description of \( \Xi \) is thus exactly the same as it was for \( \Gamma(N) \) (either form).

Now suppose that \( r = 1 \). When \( N = 2 \) there are two real cusps and one real elliptic point of even order in a single component. The picture is:

Now suppose \( N \neq 2 \), so that there are no elliptic vertices. The real cusps are represented by vectors of the form \((a, 0)\), \((0, a)\), \((1, 2a)\) and \((a, t)\) with \( a \) prime to \( N \). Replacing \( a \) by \(-a\) in each of the four forms yields an equivalent parabolic vertex, and this operation captures all equivalences. There are thus \( \frac{1}{2}\phi(N) \) cusps of each kind, for a total of \( 2\phi(N) \) real cusps. Let \( \epsilon = 2 + t \), so \( \epsilon = 1 \) modulo 2 and \( \epsilon = 2 \) modulo \( N' \). The local picture at \((a, 0)\) is then

Here \( ab = 1 \). We thus see that \((a, 0)\) is connected by a chain of length 4, involving each type of cusp exactly once, to \((\epsilon a, 0)\). It follows that the number of cycles is the cardinality of \((\mathbb{Z}/N\mathbb{Z})^\times /\langle -1, \epsilon \rangle\), which coincides with \( \psi(N') \).

Finally, suppose that \( r \geq 2 \). When \( N = 4 \), there are three real cusps and one cycle, as follows:

Now suppose \( N \neq 4 \). The real cusps of \( \Gamma \) are represented by vectors of the form \((a, 0)\), \((0, a)\), \((1, 2a)\) and \((a, t)\) with \( a \) prime to \( N \). There are \( \frac{1}{2}\phi(N) \) cusps of each of the first and second forms and
\[ \frac{1}{2} \phi(N) \] cusps of each of the third and forth forms, for a total of \( \frac{3}{2} \phi(N) \) real cusps. Every cycle of \( \Xi \) contains exactly six cusps, and is of the form

\[ \begin{array}{c}
(a,t) & (0,b) \\
(0,b+t) & (a,0) \\
(a+t,0) & (1,2b)
\end{array} \]

where \( ab = 1 \). It follows that there are \( \frac{1}{2} \phi(N) \) cycles.

The above results are summarized in the following proposition:

**Proposition 6.4.1.** Let \( N \geq 1 \) be an integer, let \( \Gamma = \Gamma_1(N) \) and let \( c = c_0 \). Write \( N = 2^r N' \) with \( N' \) odd. Then \( X_\Gamma \) contains no even order elliptic points if \( N > 2 \), and exactly one even order real elliptic point if \( N \) is 1 or 2. The number of real cusps on \( X_\Gamma \) is given by

\[
\phi(N) \times \begin{cases} 
1 & \text{if } r = 0 \\
2 & \text{if } r = 1 \\
\frac{3}{2} & \text{if } r \geq 2.
\end{cases}
\]

The number of real components of \( X_\Gamma \) is given by

\[
\psi(N') \quad \text{if } r = 0 \text{ or } r = 1 \\
\frac{1}{4} \phi(N) \quad \text{if } r \geq 2 \text{ and } N \neq 4 \\
1 \quad \text{if } N = 4.
\]

In all cases, each component has the same structure.

6.5. **The curve** \( X_0(N) \). Let \( \Gamma = \Gamma_0(N) \) denote the real congruence group \((\Gamma_0(N), c_0)\) and let \( X = X_0(N) \) be the corresponding quotient. The group \( G \) is the subgroup of \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) consisting of upper triangular matrices. The space \( X \) parameterizes elliptic curves together with a cyclic subgroup of order \( N \). The set of cusps of \( \Gamma \) is identified with \( \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \). The action of \( C \) on \( \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \) is multiplication by \(-1\). The space \( X \) can have elliptic points, but as shown in Proposition 5.3.1 for \( N > 2 \) none of them are real.

Write \( G_N \) for the group \( G \) discussed above and \( \Xi_N \) for the corresponding graph. If \( N = p_1^{e_1} \cdots p_n^{e_n} \) then \( G_N = G_{p_1^{e_1}} \times \cdots \times G_{p_n^{e_n}} \), and each \( G_{p_i^{e_i}} \) contains \(-1\). Thus \( \Xi_N = \Xi_{p_1^{e_1}} \ast \cdots \ast \Xi_{p_n^{e_n}} \), where \( \ast \) is the product discussed in §4.5. The graphs \( \Xi_{p^e} \) are as follows:

Here \( p \) is an odd prime, \( e \geq 1 \) and \( r > 2 \) and \( x = [1 : 2] \) and \( y = [1 : 2^{r-1}] \). Let \( \Xi_{\text{odd}} \) denote the leftmost graph, so that any \( \Xi_{p^e} \) with \( p \) odd is isomorphic to \( \Xi_{\text{odd}} \). Now, \( \Xi_{\text{odd}} \ast \Xi_{\text{odd}} \) is isomorphic to \( \Xi_{\text{odd}} \ast \Xi_{\text{odd}} \). It follows that if \( N \) is an odd integer with \( n \) distinct prime factors, then \( \Xi_N \) is isomorphic to a disjoint union of \( 2^{n-1} \) copies of \( \Xi_{\text{odd}} \). In particular, \( \Xi_N \) has \( 2^n \) vertices (all parabolic) and \( 2^n-1 \) cycles. Now, if \( N = 2^r N' \) with \( r > 0 \) and \( N' \) odd with \( n \) distinct prime factors
then $\Xi_N$ is isomorphic to a disjoint union of $2^{n-1}$ copies of $\Xi_{2r} \ast \Xi_{\text{odd}}$. These products are as follows:

$$
\Xi_{\text{odd}} \ast \Xi_2 \\
(\infty,\infty) \\
(\infty,0) \\
(0,\infty) \\
(0,0)
$$

$$
\Xi_{\text{odd}} \ast \Xi_4 \\
(\infty,\infty) \\
(\infty,0) \\
(x,0) \\
(x,\infty) \\
(0,0) \\
(0,\infty)
$$

$$
\Xi_{\text{odd}} \ast \Xi_{2r} \\
(\infty,\infty) \\
(\infty,0) \\
(x,0) \\
(x,\infty) \\
(0,0) \\
(0,\infty)
$$

In particular, $\Xi_N$ has $2^{n+1}$, $3 \cdot 2^n$ or $2^{n+2}$ vertices, depending on if $r = 1$, $r = 2$ or $r \geq 3$ and $2^{n-1}$ or $2^n$ cycles, depending on if $r = 1, 2$ or $r \geq 3$. These results are summarized in the following proposition:

**Proposition 6.5.1.** Let $N \geq 1$ be an integer, write $N = 2^r N'$, with $N'$ odd, and let $n$ be the number of distinct prime factors of $N'$. The space $X_0(N)$ has a real elliptic point only if $N$ is 1 or 2, in which case it has exactly 1. The number of real cusps on $X_0(N)$ is given by

$$
2^n \times \begin{cases} 
1 & \text{if } r = 0 \\
2 & \text{if } r = 1 \\
3 & \text{if } r = 2 \\
4 & \text{if } r \geq 3
\end{cases}
$$

The number of real components of $X_0(N)$ is 1 if $N$ is a power of 2 and is given by

$$
2^{n-1} \times \begin{cases} 
1 & \text{if } r \leq 2 \\
2 & \text{if } r \geq 3
\end{cases}
$$

otherwise. In all cases, each component has the same structure.

6.6. **The curve** $X_{\text{split}}(N)$. Let $G_N$ be the normalizer of the diagonal subgroup of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. If $N$ is a prime power then $G_N$ consists of matrices of the form

$$
\begin{pmatrix} a & \alpha \\ \alpha^{-1} & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} -\alpha^{-1} & a \\ a & -\alpha \end{pmatrix},
$$

with $a$ a unit of $\mathbb{Z}/p^\ell \mathbb{Z}$. Let $\Gamma_{\text{split}}(N)$ denote the inverse image of $G_N$ in $\text{SL}_2(\mathbb{Z})$, as well as the real congruence group $(\Gamma_{\text{split}}(N), c_0)$, and let $X_{\text{split}}(N)$ denote the corresponding quotient. Let $\Xi_N$ denote the graph corresponding to $G_N$. If $N = \prod p_i^{e_i}$, then $G = \prod G_{p_i^{e_i}}$, and so the graph $\Xi_N$ decomposes into a product as well. Thus it suffices to analyze $\Xi_{p^\ell}$ for prime $p$.

First suppose that $p$ is an odd prime. Let $v = (0,1)$, and for a unit $a$ of $\mathbb{Z}/p^\ell \mathbb{Z}$ let $u_a = (1,a)$. Then every parabolic vertex of $\Xi$ is represented by either $v$ or $u_a$. The basis vectors $u_a$ and $u_{-a}$ represent the same parabolic vertex, and there are no other equivalences. We thus find that there are $1 + \frac{1}{2} \phi(p^\ell)$ parabolic vertices. The graphs for $p^\ell = 3$ and $p^\ell = 5$ are as follows:

Suppose now that $p^\ell > 5$. Let $a$ be a unit of $\mathbb{Z}/p^\ell \mathbb{Z}$ and let $b$ be its inverse. If $a^2 \neq \pm1, \pm\frac{1}{4}$ then the three vertices $u_a$, $u_b$ and $u_{b/4}$ are inequivalent, and the local picture at $u_a$ is as follows:
The local picture near \( u_1 \) is as follows:

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
u_1/4 & u_1 & v & u_1/2 & u_2
\end{array}
\]

The above vertices are distinct unless \( p^e \) if 7 or 9, in which case the two end vertices are identified. Now suppose that \( a^2 = -1 \). Such an element exists if and only if \( p \) is 1 modulo 4. The local picture is then:

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
u_{a/4} & u_a & u_{a/2} & u_{2a}
\end{array}
\]

The above vertices are distinct (note \( p \) cannot be 7 or 9 since \( p \) is 1 modulo 4). The total number of real elliptic points is 3 or 1 according to whether \( p \) is 1 or 3 modulo 4. In contrast to previous cases, not all cycles of \( \Xi \) have the same structure: there are either one or two special cycles (those containing the elliptic points). If \( p \) is 3 modulo 4, there is one cycle with elliptic points. If \( p \) is 1 modulo 4 then all three elliptic points belong to the same cycle if \( \sqrt{-1} \) is a power of 2 (or, equivalently, if the multiplicative order of 2 is divisible by 4), and otherwise belong to two distinct cycles. Let \( \sim \) be the equivalence relation on \((\mathbb{Z}/p^e\mathbb{Z})^\times \) given by

\[
x \sim y \text{ if } x = \pm 4^n y^\pm 1.
\]

Let \( \epsilon \) be 1 if the multiplicative order of 2 is divisible by 4. Then the number of cycles of \( \Xi \) is the cardinality of \((\mathbb{Z}/p^e\mathbb{Z})^\times / \sim \) minus \( \epsilon \). (This formula is valid even when \( p^e \) is 3 or 5.)

Now suppose that \( p = 2 \), and let \( 2^r \) be the prime power in question. The graph for \( r = 1 \) is as follows:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
(1,0) & (1,1)
\end{array}
\]

Now suppose that \( r > 1 \). Let \( v = (1,0) \) and \( v' = (1,t) \). For a unit \( a \) of \( \mathbb{Z}/2^r\mathbb{Z} \), let \( u_a = (1,a) \). Then every parabolic vertex is represented by \( v, v' \) or some \( u_a \). The basis vectors \( u_a \) and \( u_{-a} \) are equivalent. There are no other equivalences. We thus find that there are \( 2 + \frac{1}{2}\phi(2^r) \) parabolic vertices. Let \( a \) be unit of \( \mathbb{Z}/2^r\mathbb{Z} \) and let \( b \) be its inverse If \( a \) is not \( \pm 1 \) or \( \pm 1 + t \) then \( u_a \) and \( u_b \) are inequivalent and connected by two weight two edges:

\[
\begin{array}{c}
u_a & u_b
\end{array}
\]

If \( r > 1 \) then the vertex \( u_{1+t} \) belongs to a loop of weight 2. (When \( r = 1 \), the vertex \( u_{1+t} \) is equivalent to \( u_1 \).) The remaining vertices form a 5-cycle:

\[
\begin{array}{c}
u_1 & v' & v
\end{array}
\]

We thus find that there are 2 elliptic vertices. If \( r = 2 \) there is a single cycle, the above 5-cycle. If \( r > 2 \) then there are \( 1 + \frac{1}{2}\phi(2^r) \) cycles. As was the case for odd \( p \), not all cycles have the same structure, but there are at most two special cycles.

To determine \( \Xi_N \) we must now take the product of the \( \Xi_{p^e} \). This is somewhat involved, and we have not worked out the answer theoretically. We have, however, written a program that computes these graphs; it is available at \[Sn\]. We mention two computational findings here. First, up to \( N = 4000 \), no real component of \( X_{\text{split}}(N) \) has more than 18 elliptic points of even order; the maximum 18 is attained several times, first at \( N = 255 \) where there is a cycle with 30 parabolic vertices and 18 elliptic vertices. Does this bound hold for all \( N \)?

Before mentioning the second computational finding, first note that for any real congruence group of odd level any component has an even number of vertices. We have seen several real components of even level with three vertices, and \( X_{\text{split}}(2^r) \) has a component with five vertices when \( r > 1 \). It is thus natural to wonder if one can have real components containing other odd numbers of vertices. Indeed, this is the case. For instance, \( X_{\text{split}}(10) \) has a single real component, with nine vertices,
while $X_{\text{split}}(26)$ has a single real component, with 17 vertices. For $N \leq 4000$, the largest component with an odd number of vertices occurs on $X_{\text{split}}(3994)$ and has 2001 vertices. It seems likely that with larger values of $N$ one will find arbitrarily large components with an odd number of vertices.

6.7. A Type 1b class. So far, we have only seen essentially one example of a loop: it occurred in the graph of $\Gamma_{\text{split}}(2^r)$ for $r > 2$, and had weight two. We now give a simpler example, which has weight one. Let $g$ be the matrix
\[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\]
in $\text{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Then $g$ has order 3. Let $\Gamma$ be the subgroup of $\text{SL}_2(\mathbb{Z})$ consisting of matrices which reduce modulo 2 to $g$ or $g^2$, and let $c = c_0$. Then $(\Gamma, c)$ is a real congruence group. The only element of $\Gamma$ of finite even order is $-1$. Indeed, if $\gamma$ is an element of even order then its reduction modulo 2 is the identity, and it therefore belongs to $\Gamma(2)$; thus $\gamma = \pm 1$, as claimed. Thus $X_{\Gamma}$ has no elliptic points. The group generated by $g$ acts transitively on the non-zero vectors in $(\mathbb{Z}/2\mathbb{Z})^2$, and so $X_{\Gamma}$ has a single cusp, which is real. It follows that $\Xi_{\Gamma}$ has a single parabolic vertex connected to itself. We thus see that the twisted conjugacy class of the identity element has Type 1b. This class always has weight one.

6.8. A twisted form of $X_0(N)$. Let $\Gamma = \Gamma_0(N)$. The matrix
\[
\tau = \begin{pmatrix} \sqrt{N} & -\sqrt{N}^{-1} \\ 1 & 1 \end{pmatrix}
\]
belongs to $\text{SL}_2(\mathbb{R})$, is admissible with respect to $c_0$ and normalizes $\Gamma$. It follows that $c = c_0\tau$ is a complex conjugation under which $\Gamma$ is stable. Thus $(\Gamma, c)$ is a real Fuchsian group, and a twisted form of $X_0(N)$; however, it is not a real congruence group. Explicitly, $c$ is given by $z \mapsto z = (N\tau)^{-1}$. A matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
is admissible with respect to $c$ if and only if $c = -Nb$.

**Proposition 6.8.1.** Let $\Gamma$ be as above, and let $\gamma \in \Gamma$ be admissible for $c$. Then $C_\gamma$ contains a cusp if and only if $N$ is a square.

**Proof.** Let
\[
\gamma = \begin{pmatrix} a & b \\ -Nb & d \end{pmatrix}.
\]
Then the limit points of $C_\gamma$ on $\mathbb{RP}^1$ are
\[
-\frac{b}{a} \pm \frac{1}{av\sqrt{N}}.
\]
These points are rational if and only if $N$ is a square. □

**Proposition 6.8.2.** If $N$ is divisible by 4 or a prime congruent to 3 mod 4 but is not a square then every admissible element of $\Gamma$ is of Type 3.

**Proof.** The conditions guarantee that $\Gamma$ lacks even order elliptic points. □

**Proposition 6.8.3.** If $N = 5$ then $\Gamma$ has two admissible twisted conjugacy classes, both of Type 4a.

**Proof.** The curve $X_{\Gamma}$ has exactly two elliptic points of order two, represented by $\frac{1}{5}(\pm 2 + i)$. Both of these points have modulus $1/5$. The curve $C_\gamma$, with $g = 1$, is exactly $|z|^2 = 1/5$. Thus both elliptic points are real. As $X_{\Gamma}$ has genus 0 and a real point, it has one real component. We have seen that there are no real cusps. This completes the proof. □
Proposition 6.8.4. If $N = 2$ then $\Gamma$ has a single admissible twisted conjugacy class of Type 4b.

Proof. The curve $X_\Gamma$ has a single elliptic point of order two. Since the elliptic point is unique, it must be real. As $X_\Gamma$ has genus 0 and has a real point, it is isomorphic (over $\mathbb{R}$) to $\mathbb{P}^1$; thus $X_\Gamma$ has a single real component. This completes the proof. \hfill \Box

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Appendix A. Tables

In this appendix we give tables of real components for certain modular curves. In the tables, $g$ denotes the genus of the curve in question, $\pi_0$ the number of real components, $p$ the number of real cusps and $e$ the number of real elliptic points of even order. In all but one case, there are no elliptic points when the level is greater than two, and we omit $e$.

| N  | g | $\pi_0$ | p | N  | g | $\pi_0$ | p | N  | g | $\pi_0$ | p |
|----|---|--------|---|----|---|--------|---|----|---|--------|---|
| 1  | 0 | 1      | 1 | 2  | 0 | 1      | 2 | 3  | 0 | 1      | 2 |
| 2  | 0 | 1      | 2 | 3  | 0 | 1      | 2 | 4  | 0 | 1      | 3 |
| 3  | 0 | 1      | 2 | 4  | 0 | 1      | 3 | 5  | 0 | 1      | 2 |
| 4  | 0 | 1      | 3 | 6  | 0 | 1      | 4 | 7  | 0 | 1      | 2 |
| 5  | 0 | 1      | 2 | 8  | 0 | 1      | 4 | 9  | 0 | 1      | 2 |
| 6  | 0 | 1      | 4 | 10 | 0 | 1      | 4 | 11 | 1 | 1      | 2 |
| 7  | 0 | 1      | 2 | 12 | 0 | 1      | 6 | 13 | 0 | 1      | 2 |
| 8  | 0 | 1      | 4 | 14 | 1 | 1      | 4 | 15 | 1 | 2      | 4 |
| 9  | 0 | 1      | 2 | 16 | 0 | 1      | 4 | 17 | 1 | 1      | 2 |
| 10 | 0 | 1      | 4 | 18 | 0 | 1      | 4 | 19 | 1 | 1      | 2 |
| 11 | 1 | 1      | 2 | 20 | 1 | 1      | 6 | 21 | 1 | 2      | 4 |
| 12 | 0 | 1      | 6 | 22 | 2 | 1      | 4 | 23 | 2 | 1      | 2 |
| 13 | 0 | 1      | 2 | 24 | 1 | 2      | 8 | 25 | 0 | 1      | 2 |
| 14 | 1 | 1      | 4 | 26 | 2 | 1      | 4 | 27 | 1 | 1      | 2 |
| 15 | 1 | 2      | 4 | 28 | 2 | 1      | 6 | 29 | 2 | 1      | 2 |
| 16 | 0 | 1      | 4 | 30 | 3 | 2      | 8 |    |    |        |    |

The curve $X_0(N)$. See [6.25] for details.
The curves $X^\pm(N)$. The two curves have the same genus and number of real components. The column $p_\pm$ indicates the number of real cusps on $X^\pm(N)$. See §6.2 and §6.3 for details.

| $N$ | $g$ | $\pi_0$ | $p_+$ | $p_-$ | $N$ | $g$ | $\pi_0$ | $p_+$ | $p_-$ | $N$ | $g$ | $\pi_0$ | $p_+$ | $p_-$ |
|-----|-----|--------|------|------|-----|-----|--------|------|------|-----|-----|--------|------|------|
| 1   | 0   | 1      | 1    | 1    | 2   | 0   | 1      | 2    | 1    | 3   | 0   | 1      | 2    | 2    |
| 2   | 0   | 1      | 2    | 1    | 3   | 0   | 1      | 2    | 2    | 4   | 0   | 1      | 4    | 2    |
| 5   | 0   | 1      | 4    | 4    | 6   | 1   | 1      | 6    | 2    | 7   | 3   | 1      | 6    | 6    |
| 8   | 5   | 2      | 8    | 4    | 9   | 10  | 1      | 6    | 6    | 10  | 13  | 2      | 12   | 4    |
| 12  | 25  | 2      | 8    | 4    | 13  | 50  | 1      | 12   | 12   | 14  | 49  | 3      | 18   | 6    |
| 15  | 73  | 1      | 8    | 8    |     |     |        |      |      |     |     |        |      |      |

The curve $X_1(N)$. See §6.4 for details.
The curve $X_{\text{split}}(N)$. See §6.6 for details.

The genus of $X_{\text{split}}(N)$: According to [Sh, Prop. 1.40], the genus of $X_{\text{split}}(N)$ is given by

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2},$$

where $\mu$ is the index of $\Gamma_{\text{split}}(N)$ in $SL_2(\mathbb{Z})$ and $\nu_2$, $\nu_3$ and $\nu_\infty$ are the number of equivalence classes of even order elliptic points, odd order elliptic points and cusps. We do not know a reference in the literature containing formulas for these quantities, so we give some here. The index is given by

$$\mu = N^2 \prod_{p|N} \left(1 - \frac{1 + p^{-1}}{2}\right).$$

The functions $\nu_2$, $\nu_3$ and $\nu_\infty$ are multiplicative functions of $N$. On prime powers, we have

$$\nu_\infty(p^e) = \begin{cases} p^{e-1} \left(\frac{p+1}{2}\right) & \text{if } p^e \neq 2, \\ p^{e-1} \left(\frac{p-1}{2}\right) & \text{if } p^e = 2. \end{cases}$$

and

$$\nu_2(p^e) = \begin{cases} p^{e-1} \left(\frac{p-1}{2}\right) + 1 & \text{if } p = 1 \quad (\text{mod } 4) \\ p^{e-1} \left(\frac{p+1}{2}\right) & \text{if } p = 3 \quad (\text{mod } 4) \\ p^{e-1} & \text{if } p = 2. \end{cases}$$

The value of $\nu_3(p^e)$ is 1 if $p$ is 1 modulo 3 and 0 otherwise (assuming $e > 0$).