On the product of cocycles in a polyhedral complex

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Abstract. We construct an algorithm for multiplying cochains in a polyhedral complex. It depends on the choice of a linear functional on the ambient space. The cocycles form a subring in the ring of cochains, the coboundaries form an ideal in the ring of cocycles, and the quotient ring is the cohomology ring. The multiplication algorithm depends on the geometry of the cells of the complex. For simplicial complexes (the simplest geometry of cells), it reduces to the well-known Čech algorithm. Our algorithm is of geometric origin. For example, it applies in the calculation of mixed volumes of polyhedra and the construction of stable intersections of tropical varieties. In geometry it is customary to multiply cocycles with values in the exterior algebra of the ambient space. Therefore we assume that the ring of values is supercommutative.

Keywords: product of cocycles, polyhedral complex, polyhedron, tropical variety.

§ 1. Introduction

1.1. Description of results. Let $X$ be a finite set of closed convex polyhedra of dimension $\leq k$ in a real vector space $V$. These polyhedra are referred to as cells. We say that $X$ is a $k$-dimensional polyhedral complex ($\mathbb{P}$-complex) if the following conditions hold.

1) Every face of every cell is a cell.

2) Any non-empty intersection of two cells is a common face.

Let $r_p$ be a function on the set of oriented $p$-dimensional cells with values in a ring $S$ such that $r_p(\Delta)$ changes sign under reversal of the orientation of $\Delta$. Such a function is referred to as a $p$-dimensional cochain (or, depending on the context, a $p$-dimensional chain) of the $\mathbb{P}$-complex $X$ with values in $S$. We write $C^p(X,S)$ and $C_p(X,S)$ for the $S$-modules of $p$-cochains and $p$-chains respectively. The boundary maps $d: C^p(X,S) \to C^{p+1}(X,S)$ and $\partial: C_p(X,S) \to C_{p-1}(X,S)$ are defined in the usual way. In what follows, speaking of cochains and cocycles (but not chains and cycles!) of a $\mathbb{P}$-complex $X$, we always assume that all cells of $X$ are compact.

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Our main result is the construction of an algorithm (depending on the choice of a \(v \in V^*\)) for multiplying cochains of a \(\mathbb{P}\)-complex \(X\), where \(V^*\) is the space of linear functionals on \(V\). For any choice of \(v\), the cocycles form a subring and the coboundaries form an ideal in the ring of cocycles. The quotient ring of cocycles over coboundaries is independent of the choice of \(v\) (Theorem 2.2) and coincides with the cohomology ring of the \(\mathbb{P}\)-complex (Theorem 2.3).

The product in cohomology was discovered independently by Kolmogorov and Alexander and published at the Moscow topology conference in 1935. However, both of their derivations of the formula for the product of cocycles in a simplicial complex contained mistakes. The formulae were erroneous. The first correct definition of the product in the cohomology of a simplicial complex was proposed by Čech [1] (apparently during the same conference). When the \(\mathbb{P}\)-complex \(X\) is simplicial, the algorithm suggested here coincides with Čech’s algorithm. The equivalence of the Kolmogorov–Alexander geometric definition and Čech’s definition was proved by Whitney in 1938; see [2]. The details of this story are given in [3].

Varieties related to the geometry of convex polyhedra are sometimes studied in topology (see [4], for example). Such varieties can be endowed with polyhedral decompositions in accordance with their origin. Our algorithm enables us to multiply cohomology classes of such varieties without passing to simplicial refinements, that is, without violating the original geometry of the problem.

The algorithm is of geometric origin. Such algorithms are used in convex geometry, for example, to calculate mixed volumes of polyhedra or construct stable intersection of tropical varieties. In these applications one multiplies cocycles with values in the exterior algebra of the space. Therefore we assume the ring of values \(S\) to be supercommutative. Recall that a supercommutative ring (a super-ring) is a \(\mathbb{Z}_2\)-graded ring whose homogeneous elements satisfy the identity \(xy = (-1)^{|x||y|}yx\), where \(|x|\) is the parity (that is, the \(\mathbb{Z}_2\)-degree) of \(x\). (Commutative rings are superrings without odd elements.) When using the notation \(|r|\) in what follows, we always assume that the parities of all values of the cochain \(r\) coincide and are equal to \(|r|\).

In the geometric applications mentioned above, one sometimes deals with homotopically trivial \(\mathbb{P}\)-complexes, and it turns out to be important that the product of certain special cycles is independent of the choice of the parameter. Therefore taking products of cocycles requires more care than taking products of cohomology classes. Similar care regarding products of cocycles is sometimes needed in topology [5] (see also Assertion 1.3 in the second part of §1).

We distinguish two subsets \(D' \supset D\) in \(V^*\). Each of them is a finite union of subspaces of codimension 1. The product of cocycles does not change when the parameter \(v\) varies within a connected component of \(V^* \setminus D\) (see Theorem 2.1). For values of the parameter in different components, the product of cocycles may differ by a coboundary (see Theorem 2.2). The product of cochains also remains the same when \(v\) varies within a connected component of \(V^* \setminus D'\). If \(k = 1\), then \(D = \emptyset\). If \(k = 2, 3\), then \(D\) consists of the subspaces orthogonal to the edges of the \(\mathbb{P}\)-complex \(X\). In the second part of the introduction we describe some situations
of geometric origin when the product of cocycles is independent of the choice of the parameter \( v \).

Let \( r_p \) (resp. \( r_q \)) be a \( p \)-cochain (resp. \( q \)-cochain) of a \( \mathbb{P} \)-complex \( X \). We first assume that \( X \) is a simplicial complex with enumerated vertices (this labeling of vertices is a parameter). Čech’s rule for the product of cochains is given by

\[
(r_p \circ r_q)([u_0, \ldots, u_{p+q}]) = r_p([u_0, \ldots, u_p])r_q([u_p, u_{p+1}, \ldots, u_{p+q}]),
\]

where \([u_0, \ldots, u_m]\) stands for the \( m \)-dimensional simplex with increasing vertices \( u_0, \ldots, u_m \).

For an arbitrary \( \mathbb{P} \)-complex \( X \) and any \( v \in V^* \), the value of the product of cochains \( r_p \circ v r_q \) on a \((p + q)\)-dimensional cell \( \gamma \) is defined by the formula

\[
(r_p \circ v r_q)(\gamma) = \sum_{(\delta, \lambda) \in \mathcal{P}(p, q, \gamma, v)} r_p(\delta) r_q(\lambda),
\]

where \( \mathcal{P}(p, q, \gamma, v) \) is a certain subset (depending on the choice of \( v \in V^* \)) of the set of pairs of oriented faces \( (\delta, \lambda) \) of dimensions \( p, q \) in the polyhedron \( \gamma \). Our algorithm for multiplying cochains is an algorithm for choosing the subset \( \mathcal{P}(p, q, \gamma, v) \). It is described in §2. We now give examples to illustrate the choice of \( \mathcal{P}(p, q, \gamma, v) \) as well as the use of (2) in geometry when \( p = q = 1 \).

**Example 1.1.** Suppose that \( r \) and \( s \) are 1-cochains of a \( \mathbb{P} \)-complex \( X \), \( v \in V^* \), \( \gamma \in X \) and \( \dim \gamma = 2 \). Draw the polygon \( \gamma \) on its tangent plane \( \mathbb{P}_\gamma \) (the figure on the left). Then its cotangent plane \( \mathbb{P}_{\gamma}^* \) (the figure on the right) contains a point \( v_\gamma \)

![Figure 1](image.png)

which is equal to the restriction of the functional \( v \in V^* \) to \( \mathbb{P}_\gamma \). The exterior normals to the sides of the polygon are depicted twice on the plane \( \mathbb{P}_{\gamma}^* \) and begin at the points 0 and \( v_\gamma \). The intersections of the normals in the first set with those in the second determine the pairs of oriented sides belonging to \( \mathcal{P}(p, q, \gamma, v) \) (the set \( \mathcal{P}(p, q, \gamma, v) \) depends on the choice of \( v \in V^* \)). In Fig. 1, these pairs are \( (\delta, \beta) \) and \( (\lambda, \alpha) \), and (2) takes the form \( (r \circ v s)(\gamma) = r(\delta)s(\beta) + r(\lambda)s(\alpha) \).

**Example 1.2.** Denote the exterior algebra of a space \( V \) by \( \bigwedge^* V \) and define a cocycle \( r_1 \in C^1(X, \bigwedge^* V) \) by putting \( r_1(\alpha) = \alpha \), where \( \alpha \in X \) is any oriented edge of
the $\mathbb{P}$-complex. Then the 2-cocycle $r_1 \sim_v r_1$ is independent of the choice of $v$ (see Assertion 1.1 in § 1.2). It follows that the area of $\gamma$ is equal to the sum of the areas of the triangles (depending on the choice of $v$) formed by the pairs of vectors in $\mathcal{P}(p, q, \gamma, v)$. In the figure, these are the pairs of vectors $(\delta, \beta)$ and $(\lambda, \alpha)$. A multidimensional version (stated in the second part of the introduction) of this assertion is equivalent to the tropical analogue of the Bernstein–Kushnirenko theorem [6].

Suppose that the vertices of $X$ can be arranged in order of increasing values of the functional $v$. If $\gamma$ is a simplex, then the set $\mathcal{P}(p, q, \gamma, v)$ consists of a single element and (2) takes the form (1). Multiplication algorithms of the form (2) are used in the geometry of polyhedra, tropical geometry, the theory of toric varieties and complex analysis (see [7]–[13]).

In §2 we state the main results (Theorems 2.1–2.3) and necessary definitions. The proofs of these theorems are reduced to the corresponding assertions on the intersection of cycles in a $\mathbb{P}$-complex (see §3). These assertions are of independent interest and have their own applications (see [10]–[13]). The relation between the product of cocycles and the intersection of cycles is established by means of the construction of local duality. For some $\mathbb{P}$-complexes there is a geometric construction of the dual $\mathbb{P}$-complex. The cochain complex of a $\mathbb{P}$-complex is isomorphic to the chain complex of the dual $\mathbb{P}$-complex. Every $\mathbb{P}$-complex consists of $\mathbb{P}$-subcomplexes for which the dual $\mathbb{P}$-complexes exist. The cochain complex of an arbitrary $\mathbb{P}$-complex is accordingly glued from the chain complexes of locally dual $\mathbb{P}$-complexes.

In the proofs of Theorems 2.1 and 2.2 we use the simplest version of duality: the $\mathbb{P}$-complex formed by the faces of a convex polyhedron is dual to the fan of dual cones of this polyhedron. The cochain complex of this convex polyhedron is accordingly isomorphic to the chain complex of the dual fan of cones. To prove Theorem 2.3, we use a less well-known duality construction related to the notion of a regular partition of a polyhedron [8] (see §6.1). Let $\phi$ be a convex piecewise-linear function on a convex polygon $\gamma$. The domains of linearity of $\phi$ form a partition of $\gamma$. Such partitions $(\gamma, \phi)$ of polyhedra are called regular partitions. For example, all simplicial partitions of a polyhedron are regular. The $\mathbb{P}$-complexes formed by the faces of the polyhedra occurring in the partition are dual to those formed by the domains of linearity of convex piecewise-linear functions on the conjugate space $V^*$. The duality is given by the Legendre transform of convex piecewise-linear functions. For example, the Legendre transform of the support function of a polyhedron provides its tautological regular partition $(\gamma, 0)$. The assertion locally dual to Theorem 2.3 describes a connection between the homology of a $\mathbb{P}$-complex and the homology of its Bergman fan. The construction of the Bergman fan of a $\mathbb{P}$-complex (see §5) is analogous to that of the Bergman fan of an algebraic variety [14] or a tropical variety [12].

This paper has two direct sources. The first is the construction of the intersection of tropical varieties or the intersection of cycles with values in the exterior algebra [10]–[12]. The second is an unpublished text of Khovanskii, where tropical varieties are regarded as cocycles of regular partitions of convex polyhedra. Thus we encounter a dual interpretation of the intersection of tropical varieties. This
interpretation transforms intersections of varieties into products of cocycles. I am grateful to A. Khovanskii for useful discussions and an explanation of the results in the text mentioned above.

1.2. Products of cocycles in the geometry of polyhedra. In this part of the introduction we mainly present previously known geometric results. All of these results deal with products of cocycles of \( \mathbb{P} \)-complexes. Assertions directly related to Theorems 2.1–2.3 are given with appropriate comments. This part of the introduction is not used in what follows.

Let \( U \) be a bounded domain in an oriented \( q \)-dimensional affine subspace of \( V \) and let \( \beta_U \in \bigwedge^q V \) be a polyvector such that \( \int_U \omega = \omega(\beta_U) \) for all exterior forms \( \omega \in \bigwedge^q V^* \). The polyvector \( \beta_U \) changes sign when the orientation of \( U \) is reversed. We regard \( \beta_U \) as the \( q \)-dimensional volume of \( U \).

Let \( X \) be a \( \mathbb{P} \)-complex in \( V \) and let \( S = \bigwedge^* V \) be the exterior algebra of \( V \). We write \( V_\delta \) for the subspace generated by the differences of points in a cell \( \delta \in X \). Then \( \beta_\delta \in \bigwedge^q V_\delta \). We define a \( q \)-dimensional cochain \( \text{vol}_X^p(\delta) = q!\beta_\delta \) with values in \( \bigwedge^q V \supset \bigwedge^q V_\delta \). It follows from Pascal’s equations for the \((q+1)\)-dimensional cells of \( X \) that \( \text{vol}_X^q \) is a cocycle with values in the superring \( S \). (Pascal’s equation for a convex polyhedron \( \Delta \) says that \( \sum \lambda v_\lambda = 0 \), where \( \lambda \) is any top-dimensional face of \( \Delta \) and \( v_\lambda \) is its exterior normal vector of length equal to the area of \( \lambda \)).

Assertion 1.1. *For every \( v \) we have \( \text{vol}_p^X \prec v \text{vol}_q^X = \text{vol}_{p+q}^X \).*

The formula (2) reduces Assertion 1.1 to the case of a \( \mathbb{P} \)-complex \( X \) consisting of the faces of a convex polyhedron. In this case, the assertion is equivalent to the corresponding assertion on the intersection of cycles in the \( \mathbb{P} \)-complex consisting of the cones dual to the faces of the polyhedron. This last assertion was proved in [9] and [10]. Theorem 2.3 (see Remark 2.3) reduces Assertion 1.1 to the case when the polyhedron is a simplex.

Put \( R^*(X,S) = R^0(X,S) \oplus \cdots \oplus R^n(X,S) \), where \( R^q(X,S) \) consists of those \( q \)-cocycles whose values on the \( q \)-dimensional cells \( \delta \) belong to \( \bigwedge^q V_\delta \). For example, \( \text{vol}_p^X \in R^p(X,S) \). Then, defining a product \( \prec_v \) by the formula (2), we endow \( R^*(X,S) \) with the structure of a commutative ring (the commutative property follows from Proposition 2.1, 1); see §2. The following assertion is a corollary of Theorem 2.2 (see Remark 2.2).

Assertion 1.2. *The ring structure of \( R^*(X,S) \) is independent of the choice of \( v \).*

When the \( \mathbb{P} \)-complex \( X \) is formed by the faces of a convex \( n \)-dimensional polyhedron, Assertion 1.1 gives an algorithm for calculating the volume and mixed volumes of polyhedra in terms of their edges [7]. It follows from Assertion 1.1 that \( \text{vol}_n^X = (\text{vol}_1^X)^n \). Hence (2) gives rise to an algorithm for calculating the volume of a polyhedron in terms of its edges. The algorithm proceeds as follows.

1) Distinguish certain \( n \)-tuples of edges of the polyhedron (these tuples of edges depend on the choice of \( v \in V^* \)).
2) Sum the volumes of the corresponding parallelepipeds.
The analogous algorithm for calculating mixed volumes runs as follows. Let $\gamma$ be the Minkowski sum of polyhedra $\gamma_1, \ldots, \gamma_n$. We define 1-cocycles $vol_{X,k}^1$ by declaring that they take the following values on the edges of $\gamma$. Every edge $\lambda$ can be written uniquely as a Minkowski sum $\lambda_1 + \cdots + \lambda_n$, where $\lambda_k$ is an edge or a vertex of $\gamma_k$. We put $vol_{X,k}^1(\lambda) = 0$ when $\lambda_k$ is a vertex and $vol_{X,k}^1(\lambda) = \lambda_k$ when $\lambda_k$ is an edge. Then the $n$-vector $vol_{X,1}^1 \cdot \cdots \cdot vol_{X,n}^1$ is the product of $n!$ and the mixed volume of the polyhedra $\gamma_1, \ldots, \gamma_n$.

**Assertion 1.3.** Let $\gamma$ be a simple integral polyhedron and $M$ the corresponding toric variety (see [15]). Then the ring $R^*(X, S)$ coincides with the cohomology ring of $M$ (see [9]).

Suppose that $X$ is a $\mathbb{P}$-complex in $V$ and $a_1, \ldots, a_p$ are $V$-valued 0-cochains. Assume that $da_i \in R^1(X, S)$. For example, if $a(\lambda) = \lambda$, then $da = vol_X^1 \in R^1(X, S)$. We regard the product $a_1 \cdots a_p$ as a 0-cochain $\prod(a_1, \ldots, a_p)$ with values in the symmetric algebra of $V$.

**Assertion 1.4.** The product of cocycles $da_1 \cdot \cdots \cdot da_p$ with values in $\Lambda^* V$ is completely determined by the values of the 0-cochain $\prod(a_1, \ldots, a_p)$.

It follows from Assertion 1.2 that the product $da_1 \cdot \cdots \cdot da_p$ is well defined. If $\gamma \in X$ and $\dim \gamma = p$, then (2) yields that $(da_1 \cdot \cdots \cdot da_p)(\gamma)$ depends only on the values of the cochains $a_i$ at the vertices of $\gamma$. This reduces Assertion 1.4 to the case when the $\mathbb{P}$-complex $X$ consists of the faces of a convex polyhedron. In this case, Assertion 1.4 yields the following result.

**Assertion 1.5.** The mixed volume of polyhedra is completely determined by the product of their support functions [11].

**Proof.** Indeed, take $\gamma = \gamma_1 + \cdots + \gamma_n$ and suppose that the $\mathbb{P}$-complex $X$ consists of the faces of $\gamma$. Every vertex $\lambda \in \gamma$ can be written uniquely as a sum $\lambda_1 + \cdots + \lambda_n$ of vertices of the polyhedra-summands. Put $a_i(\lambda) = \lambda_i$. Then the product $\prod(a_1, \ldots, a_p)$, regarded as a polynomial on $V^*$, coincides on the dual cone of $\lambda$ with the product of the support functions of $\gamma_1, \ldots, \gamma_n$. Since $da_i = vol_{X,i}^1$, the desired result now follows from the algorithm (described above) for the calculation of mixed volumes. □

Assertion 1.5 does not hold for arbitrary convex bodies; see [13].

When $X$ is a $\mathbb{P}$-complex in $\mathbb{C}^n$, the product of cocycles with values in $\Lambda^* \mathbb{C}^n$ also gives rise to geometric corollaries. For example, let $\mathbb{C}^n_\mathbb{R}$ be the realified complex vector space $\mathbb{C}^n$ and let $\rho: \Lambda^* \mathbb{C}^n_\mathbb{R} \rightarrow \Lambda^* \mathbb{C}^n$ be the extension of the identification $\mathbb{C}^n_\mathbb{R} \rightarrow \mathbb{C}^n$ to a ring homomorphism. We put $vol_{X,C}^p(\gamma) = \rho(vol_X^p(\gamma))$. Then $vol_{X,C}^p$ is a cocycle with values in the ring $\Lambda^* \mathbb{C}^n$. Assertions 1.1, 1.4 also hold for the cocycles $vol_{X,C}^p$. Among their corollaries are some properties (analogous to those of mixed volumes stated above) of mixed pseudo-volumes of polyhedra in the complex space [16]–[18]. Moreover, the product of cocycles with values in $\Lambda^* \mathbb{C}^n$ is a convenient tool for describing the action of the complex Monge–Ampère operator on piecewise-linear functions in $\mathbb{C}^n$ [12], [13].
§ 2. Definitions and main theorems

In this section we state the main results (Theorems 2.1–2.3) on the product of cochains of a $\mathcal{P}$-complex. Theorems 2.1 and 2.2 will be proved in § 4 when we deduce their chain analogues in § 3. The local analogue of Theorem 2.3 is an assertion (given in § 5) on the relation between the homology of a $\mathcal{P}$-complex and that of its Bergman fan. Theorem 2.3 will be proved in § 6.

**Definition 2.1.** The intersection of $\mathcal{P}$-complexes $\mathcal{X}_1, \ldots, \mathcal{X}_k$ is the $\mathcal{P}$-complex $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_k$ consisting of the cells $\delta_1 \cap \cdots \cap \delta_k$ with $\delta_i \in \mathcal{X}_i$.

**Definition 2.2.** We say that $\mathcal{P}$-complexes $\mathcal{X}_1, \ldots, \mathcal{X}_k$ in $V$ are transversal if, for any cells $\delta_i \in \mathcal{X}_i$ with non-empty intersection $\delta_1 \cap \cdots \cap \delta_k$, we have

$$\text{codim} \cap_i V_{\delta_i} = \sum_i \text{codim} V_{\delta_i}.$$ 

The rule for choosing the set of summands in (2) uses the construction of the fan of cones dual to a polyhedron. (A fan of cones is a $\mathcal{P}$-complex whose cells are convex cones with vertex 0.) In what follows we always write $V_\gamma$ for the subspace of $V$ spanned by the differences of points of the cell $\gamma$ in a $\mathcal{P}$-complex in $V$.

Let $\gamma \subset V$ be a convex polyhedron. By definition, the *dual cone* $\delta^*$ of a face $\delta \subset \gamma$ consists of all points $u$ in the dual space $V^*$ such that $\max_{v \in \delta} u(v)$ is attained for every $v \in \delta$. The cone $\delta^*$ lies in the orthogonal complement of $V_\delta$ and $\dim \delta^* = \dim V - \dim \delta$.

Let $X$ be a $\mathcal{P}$-complex in $V$. We write $X_\gamma$ for the $\mathcal{P}$-complex in $V_\gamma$ formed by the faces of a polyhedron $\gamma \in X$. Suppose that $\dim \gamma = (p + q)$. Write $(X_\gamma^*)^q$ (resp. $(X_\gamma^*)^p$) for the $q$-dimensional (resp. $p$-dimensional) skeleton of the fan of cones $X_\gamma^*$ dual to $\gamma$ in the space $(V_\gamma)^*$.

**Definition 2.3** (convenient functionals). A point $v \in V^*$ is said to be $(p, q)$-convenient if the $\mathcal{P}$-complexes $\pi_\gamma(v) + (X_\gamma^*)^q$ and $(X_\gamma^*)^p$ are transversal for every $\gamma \in X$, where $\pi_\gamma : V^* \to (V_\gamma)^*$ is the projection conjugate to the embedding $V_\gamma \to V$. Put $\mathcal{U}(X) = \bigcap_{p,q} \mathcal{U}_{p,q}(X)$, where $\mathcal{U}_{p,q}(X)$ is the set of $(p, q)$-convenient points. The points in $\mathcal{U}(X)$ are said to be convenient.

**The rule for choosing the set $\mathcal{P}(p, q, \gamma, v)$ in the definition of the $v$-product of cochains** (formula (2) in § 1) is as follows. Suppose that $v \in \mathcal{U}_{p,q}(X)$. Then the $\mathcal{P}$-complex $\left( \pi_\gamma(v) + (X_\gamma^*)^q \right) \cap X_\gamma^*$ is finite and consists of points of the form $(\pi_\gamma(v) + \delta^*) \cap \lambda^*$, where the pairs $(\delta, \lambda)$ vary in a certain set (depending on the choice of $v$) of pairs of faces of dimensions $(p, q)$ in $\gamma$. We choose orientations of the cells $\delta$ in such a way that the orientations of $V_\gamma$ and $V_\delta \oplus V_\lambda$ are compatible with the isomorphism $V_\gamma = V_\delta \oplus V_\lambda$. Denote the resulting set of ordered pairs of oriented cells by $\mathcal{P}(p, q, \gamma, v)$. The summands in (2) are thus constructed. *If the point $v$ is convenient, then the $v$-product $r \ast_v s$ of any cochains $r$, $s$ is defined.*

**Lemma 2.1.** If $\gamma$ is a simplex, then the set $\mathcal{P}(p, q, \gamma, v)$ consists of a single element $(\delta, \lambda)$, $\dim \delta \cap \lambda = 0$ and $v(x) \leq v(y)$ for all $x \in \delta$, $y \in \lambda$. 
Corollary 2.1. If the values of $v$ at the vertices of a simplicial complex $X$ are distinct and the vertices are ordered by increasing $v$, then the products of cochains defined in (1) and (2) coincide.

Corollary 2.2. If points $u,v$ lie in the same connected component of the set $U_{p,q}(X)$ of $(p,q)$-convenient points, then $r_p \sim_u r_q = r_p \sim_v r_q$ for all $r_p \in C^p(X,S)$, $r_q \in C^q(X,S)$.

Proposition 2.1. If $v \in U_{p,q}(X)$ and $r_p,r_q$ are cochains of dimensions $p,q$, then
1) $r_p \sim_v r_q = (-1)^{pq+1}r_p \sim_v r_q$;
2) $d(r_p \sim_v r_q) = dr_p \sim_v r_q + (-1)^p r_p \sim_v dr_q$.

Corollary 2.3. Let $r_p$, $r_q$ be closed cochains. Then the following assertions hold.
1) The cochain $r_p \sim_v r_q$ is closed.
2) If $r_p = dr_{p-1}$, then $r_p \sim_v r_q = d(r_{p-1} \sim_v r_q)$.

Definition 2.4 (special triples of cells).\footnote{This is close to the notion of ‘1-regular pair of simplices’ in [5].} We write $\Lambda_{p,q}$ for the set of triples of cells $(\delta,\lambda,\mu)$ of dimensions $p,q,p+q$ such that $\delta \cup \lambda \subset \mu$ and $\dim V_\delta \cap V_\lambda \geq 1$. If $(\delta,\lambda,\mu) \in \Lambda_{p,q}$, $\dim V_\delta \cap V_\lambda = 1$ and $\delta \cup \lambda \subset \gamma$, where $\gamma$ is a $(p+q-1)$-dimensional face of $\mu$, then we call $(\delta,\lambda,\gamma)$ a special triple of cells.

Example 2.1. Suppose that $\gamma$ is not a maximal cell of the $\mathbb{P}$-complex $X$. Then the following triples of the form $(\delta,\lambda,\gamma)$ are special:
- $\dim \gamma = 1$: $(\gamma,\gamma,\gamma)$;
- $\dim \gamma = 2$: $(\delta,\gamma,\gamma)$, where $\delta$ is a side of the polygon $\gamma$;
- $\dim \gamma = 3$: $(\delta,\gamma,\gamma)$, where $\delta$ is an edge of $\gamma$, and $(\delta,\lambda,\gamma)$, where $\delta$ and $\lambda$ are non-parallel 2-dimensional faces of $\gamma$.

Definition 2.5 (the discriminant of a $\mathbb{P}$-complex). Write $V_{\delta,\lambda}^*$ for the orthogonal complement of the subspace $V_\delta \cap V_\lambda$ in $V^*$. The union of the hyperplanes $V_{\delta,\lambda}^* \subset V^*$ over all special triples of cells $(\delta,\lambda,\gamma)$ is denoted by $D(X)$ and is called the discriminant of the $\mathbb{P}$-complex $X$.

Theorem 2.1. Let $r_p$, $r_q$ be closed cochains. If convenient points $u$, $v$ belong to the same connected component of $V^* \setminus D(X)$, then $r_p \sim_u r_q = r_p \sim_v r_q$.

Definition 2.6. Suppose that $(\delta,\lambda,\mu) \in \Lambda_{p,q}$. If $\dim V_\delta \cap V_\lambda = 1$, then the hyperplane $V_{\delta,\lambda}^* \subset V^*$ is said to be inconvenient. An inconvenient plane is said to be discriminant if it belongs to the discriminant. If $(u + \delta^*) \cap \lambda^* \neq \emptyset$, where $\delta^*$, $\lambda^*$ are the cones dual to the faces of $\mu$ in the space $V^*$, and if $\dim V_\delta \cap V_\lambda > 1$, then the point $u \in V^*$ is said to be dangerous.

Corollary 2.4. The discriminant $D(X)$ lies in the union of the inconvenient hyperplanes. The codimension of the set of dangerous points is greater than 1. The set $D'(X)$ of non-dangerous inconvenient points has codimension 1 and is contained in the union of the inconvenient hyperplanes. The codimension of the set $D'(X) \setminus D(X)$ is also equal to 1.
When \( \phi \in V^* \) we define the following notion of compatible orientations of the cells \( \delta, \lambda, \gamma \) simultaneously for all special triples \( (\delta, \lambda, \gamma) \) with \( \phi \notin V^*_{\delta, \lambda} \) (recall that \( V^*_{\delta, \lambda} = (V_\delta \cap V_\lambda) \perp \)). The functional \( \phi \) determines an orientation of every 1-dimensional subspace \( V_\delta \cap V_\lambda \). Compatible orientations are chosen in such a way that the isomorphism \( V_\gamma / (V_\delta \cap V_\lambda) = V_\delta / (V_\delta \cap V_\lambda) \oplus V_\lambda / (V_\delta \cap V_\lambda) \) is an isomorphism of oriented spaces.

**Definition 2.7** (special cochains). Suppose that \( r_p \in C^p(X, S) \), \( r_q \in C^q(X, S) \), \( (\delta, \lambda, \gamma) \) is a special triple of cells, \( \dim \delta = p \), \( \dim \lambda = q \) and \( \phi \notin V^*_{\delta, \lambda} \). We write \( \varphi_{r_p, r_q}^{\delta, \lambda, \gamma} : \phi \) for the \( (p + q - 1) \)-cochain which is equal to \( r_p(\delta) r_q(\lambda) \) on \( \gamma \) (we use here the compatible orientations defined above) and is equal to 0 on all other cells of \( X \).

**Condition 2.1.** Let \( H \) be a discriminant hyperplane. Suppose that the following conditions hold for points \( u, v, \kappa \in V \).

1) \( [u, v] \setminus \kappa \subset U_{p, q} \), where \( \kappa \) is an interior point of the closed interval \([u, v] \).
2) The point \( \kappa \) is non-dangerous and belongs to \( H \).
3) If \( \kappa \) belongs to an inconvenient hyperplane \( G \), then \( G = H \).

If \( H = V^*_{\delta, \lambda} \), then it follows from Condition 2.1, 3) that \( (\kappa + \delta^*) \cap \lambda^* \subset H \), where \( \delta^*, \lambda^* \) are the cones dual to the faces \( \delta, \lambda \) of \( \gamma \). Moreover, if \( (\kappa + \delta^*) \cap \lambda^* \neq \emptyset \), then \( H = V^*_{\delta, \lambda} \). It follows from Condition 2.1, 2) that the intersection \( (\kappa + \delta^*) \cap \lambda^* \) is either empty or transversal in \( H \). We write \( S_{p, q}^H(\kappa) \) for the set of all special triples \( (\delta, \lambda, \gamma) \) such that \( (\kappa + \delta^*) \cap \lambda^* \neq \emptyset \) and \( \dim \delta = p \), \( \dim \lambda = q \).

**Theorem 2.2.** Suppose that \( r_p, r_q \) are cocycles of dimensions \( (p, q) \), \( C, D \) are connected components of \( V^* \setminus D(X) \) whose boundaries intersect each other along an \((n - 1)\)-dimensional cone in \( H \), and take \( u \in C, v \in D \). Then

\[
 r_p \sim_v r_q - r_p \sim_u r_q = \sum_{(\delta, \lambda, \gamma) \in S_{p, q}^H(\kappa)} d\varphi_{r_p, r_q}^{\delta, \lambda, \gamma ; v - u}.
\]

**Corollary 2.5.** If convenient points \( u, v \) belong to connected components of \( V^* \setminus D(X) \) whose boundaries intersect each other along an \((n - 1)\)-dimensional cone, then \( r_p \sim_u r_q - r_p \sim_v r_q \) is a coboundary for all cocycles \( r_p, r_q \).

**Corollary 2.6.** For every convenient \( v \) the operation of \( \sim_v \)-multiplication of cocycles determines a \( v \)-independent structure of multiplication in the cohomology of the \( \mathbb{P} \)-complex \( X \).

**Remark 2.1.** It follows from Theorem 2.1 that the cochain on the right-hand side of (3) (in contrast to the set \( S_{p, q}^H(\kappa) \)) is independent of the choice of \( \kappa \).

**Remark 2.2.** Suppose that \( S = \bigwedge^* V \) and \( r_p \in \mathcal{R}^p(X, S) \), \( r_q \in \mathcal{R}^q(X, S) \) (see § 1). Then \( r_p(\delta) r_q(\lambda) = 0 \) for every special triple \( (\delta, \lambda, \gamma) \). Hence \( \varphi_{r_p, r_q}^{\delta, \lambda, \gamma ; \phi} = 0 \). Therefore Assertion 1.2 (see § 1) follows from Theorems 2.1 and 2.2.

**Definition 2.8.** When a \( \mathbb{P} \)-complex \( X \) is a refinement of a \( \mathbb{P} \)-complex \( Y \), we define a map \( \text{res}: C^*(X, S) \rightarrow C^*(Y, S) \) acting on any cochain \( r_p \in C^p(X, S) \) by the formula \( \text{res}(r_p)(\gamma) = \sum_{X \ni \delta \subset \gamma} r_p(\delta) \), where \( \gamma \in Y \) is an arbitrary \( p \)-dimensional cell.
Corollary 2.7. The map res is a homomorphism of complexes.

Proposition 2.2. Suppose that the \( \mathbb{P} \)-complex \( X \) is a simplicial refinement of \( Y \). Then the homomorphism of complexes \( \text{res}: C^*(X, S) \rightarrow C^*(Y, S) \) induces an isomorphism \( H^*(X, S) \rightarrow H^*(Y, S) \) of cohomology \( S \)-modules.

A proof of Proposition 2.2 is given in \( \S \) 6.3.

Theorem 2.3. Suppose that the \( \mathbb{P} \)-complex \( X \) is a simplicial refinement of \( Y \), \( v \in U(Y) \cap U(X) \) and \( r_p, r_q \) are cocycles of \( X \). Then \( \text{res}(r_p) \bowtie v \cdot \text{res}(r_q) - \text{res}(r_p \bowtie v \cdot r_q) \) is a coboundary of \( Y \).

It follows from Theorem 2.3 and Proposition 2.2 that the cohomology \( \bowtie v \)-product coincides with the Kolmogorov–Alexander product for every convenient \( v \).

Remark 2.3. Theorem 2.3 still holds if we replace the condition of simpliciality of \( X \) by the following weaker condition: the \( \mathbb{P} \)-complex \( X \) is a regular refinement of \( Y \) (see \( \S \) 6). Moreover, the proof of Theorem 2.3 in \( \S \) 6.2 shows that if \( S = \bigwedge^* V \) and \( r_p, r_q \in \mathcal{R}^*(X, S) \) (see Assertion 1.2), then \( \text{res}(r_p) \bowtie \text{res}(r_q) = \text{res}(r_p \bowtie r_q) \). The cocycles \( \text{vol}_p^Y \) were defined in \( \S \) 1 and, by construction, \( \text{res}(\text{vol}_p^X) = \text{vol}_p^Y \). Therefore Assertion 1.1 in \( \S \) 1 reduces to the case when the \( \mathbb{P} \)-complex \( X \) consists of the faces of a simplex.

Remark 2.4. In \( \S \) 6.2 we give a formula for the coboundary \( \text{res}(r_p) \bowtie v \cdot \text{res}(r_q) - \text{res}(r_p \bowtie v \cdot r_q) \) which is analogous to the formula (3) in Theorem 2.2.

§ 3. Intersections of chains and cycles

Our derivation of the results in \( \S \) 2 is based on the properties of intersections of chains of \( \mathbb{P} \)-complexes. In \( \S \) 3.1 we define the intersection of chains of transversal \( \mathbb{P} \)-complexes. In \( \S \) 3.2 we construct the intersection index of cycles of complementary dimensions in an arbitrary pair of \( \mathbb{P} \)-complexes. Its properties are used in \( \S \) 3.4 to deduce the chain analogues of Theorems 2.1 and 2.2. In this section we use the terminology of \( \S \) 2 (discriminant, convenient point and so on) to denote new notions. These notions are compatible with their counterparts in \( \S \) 2 via the duality of \( \mathbb{P} \)-complexes in \( \S \) 6. Intersections of chains of a \( \mathbb{P} \)-complex are also considered in \( \S \) 5.

3.1. Intersections of chains of transversal \( \mathbb{P} \)-complexes. Let \( \mathcal{X}_1, \ldots, \mathcal{X}_k \) be transversal \( \mathbb{P} \)-complexes (Definition 2.2) in an oriented space \( V \), \( s_i \) a \( p_i \)-dimensional chain of the \( \mathbb{P} \)-complex \( \mathcal{X}_i \), and \( \delta_i \in \mathcal{X}_i \) an oriented \( p_i \)-dimensional cell.

We put \( l = p_1 + \cdots + p_k - (k - 1)n \) and define the intersection of the chains \( s_i \) with values in \( C^l(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_k) \) (see Definitions 2.2, 2.1 in \( \S \) 2) as
\[
(s_1 \cap \cdots \cap s_k)(\delta_1 \cap \cdots \cap \delta_k) = s_1(\delta_1) \cdots s_k(\delta_k),
\]
where the orientations of the cells \( \delta_i \) and the cell \( \delta_1 \cap \cdots \cap \delta_k \) are compatible in the usual sense, that is, in terms of the isomorphism \( V/V_{\delta_1} \cap \cdots \cap \delta_k = \bigoplus_i V/V_{\delta_i} \).

Corollary 3.1. The intersection of chains of transversal \( \mathbb{P} \)-complexes is associative: \( (s_1 \cap s_2) \cap s_3 = s_1 \cap (s_2 \cap s_3) \).
Proposition 3.1. Let $\partial$ stand for the boundary operator of a chain complex. Then $\partial(s_1 \cap \cdots \cap s_k) = \sum_i (-1)^{p_1 + \cdots + p_i - 1} s_1 \cap \cdots \cap \partial s_i \cap \cdots \cap s_k$.

Proof. Let $b_{i,j}$ be one of the $(p_i - 1)$-dimensional faces of the $p_i$-dimensional polyhedron $\delta_i$. It follows from the transversality condition that the boundary of the cell $\delta_1 \cap \cdots \cap \delta_k$ consists of polyhedra of the form $\delta_1 \cap \cdots \cap b_{i,j} \cap \cdots \cap \delta_k$. The intersection of any two such polyhedra is either empty or a common proper face. The desired result follows. $\square$

Given transversal $\mathbb{P}$-complexes $\mathcal{X}_1$, $\mathcal{X}_2$, we consider the chain complex $C_*(\mathcal{X}_1, \mathcal{X}_2; \partial)$ with components

$$C_m(\mathcal{X}_1, \mathcal{X}_2; \partial) = \bigoplus_{p+q=m} C_p(\mathcal{X}_1) \otimes C_q(\mathcal{X}_2)$$

and differential

$$\partial(s_p \otimes s_q) = \partial s_p \otimes s_q + (-1)^p s_p \otimes \partial s_q.$$ 

Consider the map

$$I_{\mathcal{X}_1, \mathcal{X}_2} : C_*(\mathcal{X}_1, \mathcal{X}_2; \partial) \to C_*(\mathcal{X}_1 \cap \mathcal{X}_2; \partial)$$

which acts on $C_p(\mathcal{X}_1) \otimes C_q(\mathcal{X}_2)$ by the formula $s_p \otimes s_q \mapsto s_p \cap s_q$.

Corollary 3.2. The map $I_{\mathcal{X}_1, \mathcal{X}_2}$ is a homomorphism of complexes.

Corollary 3.3. Suppose that the chains $s_i$ are closed. Then the following assertions hold.

1) The chain $s_1 \cap \cdots \cap s_k$ is closed.

2) If $s_1 = \partial s$, then $s_1 \cap \cdots \cap s_k = \partial(s \cap s_2 \cap \cdots \cap s_k)$.

3.2. The intersection index of shifted cycles. Suppose that $\dim V = n = p+q$ and $\mathcal{X}$, $\mathcal{Z}$ are $\mathbb{P}$-complexes in $V$. Given any $v \in V$ and $r_p \in C_p(\mathcal{X}, S)$, we define a chain $(v+r_p)$ of the shifted $\mathbb{P}$-complex $v+\mathcal{X}$ by the formula $(v+r_p)(v+\delta) = r_p(\delta)$. For every convenient shift $v$ (see Definition 3.1 below) and any chains $r_p \in C_p(\mathcal{X}, S)$, $r_q \in C_q(\mathcal{Z}, S)$ we shall construct the intersection index $\text{ind}_v(r_p, r_q)$ of the chains $v+r_p$ and $r_q$. Theorems 3.1, 3.2 describe the dependence of the intersection index of cycles on the shift $v$. Their proofs are based on a geometric construction which will be described in some detail because it is used in the proofs of the other theorems. In §5.2 we transfer these theorems to the case $p+q \geq n$ with $\mathcal{X} = \mathcal{Z}$. In this case, the values of the intersection index are cycles of the Bergman fan $\mathcal{X}_\infty$.

Definition 3.1. Let $\mathcal{X}^p$ and $\mathcal{Z}^q$ be the $p$-dimensional and $q$-dimensional skeletons of the $\mathbb{P}$-complexes $\mathcal{X}$ and $\mathcal{Z}$ respectively. A point $v \in V$ is said to be convenient if the $\mathbb{P}$-complexes $v+\mathcal{X}^p$, $\mathcal{Z}^q$ are transversal. The set of convenient points is denoted by $\mathcal{U}_{p,q}(\mathcal{X}, \mathcal{Z})$.

Definition 3.2. For $v \in \mathcal{U}_{p,q}(\mathcal{X}, \mathcal{Z})$ we define $\text{ind}_v(r_p, r_q)$ as the sum of the values of the 0-chain $(v+r_p) \cap r_q$ of the $\mathbb{P}$-complex $(v+\mathcal{X}^p) \cap \mathcal{Z}^q$ over all 0-dimensional cells.

Corollary 3.4. If points $u, v$ belong to the same connected component of $\mathcal{U}_{p,q}(\mathcal{X}, \mathcal{Z})$, then $\text{ind}_u(r_p, r_q) = \text{ind}_v(r_p, r_q)$. 

Corollary 3.5. If $\mathbb{P}$-complexes $\mathcal{X}$, $\mathcal{Z}$ are fans of cones, then $\text{ind}_{t\nu}(s_p,s_q) = \text{ind}_\nu(s_p,s_q)$ for every $t > 0$.

The index $\text{ind}_\nu(r_p,r_q)$ depends only on the skeletons $\mathcal{X}^p$, $\mathcal{Z}^q$. Therefore we assume in all definitions and proofs below (but not in the statements of the assertions) that $\dim \mathcal{X} = p$, $\dim \mathcal{Z} = q$.

Definition 3.3. Given any pair of cells $(\delta, \lambda)$, we write $V_{\delta,\lambda}$ for the subspace of $V$ generated by the subspaces $V_\delta$, $V_\lambda$. If $(v+\delta) \cap \lambda \neq \emptyset$ and $\text{codim} V_{\delta,\lambda} > 1$, then the point $v$ is said to be dangerous.

The set of dangerous points is contained in a finite union of affine subspaces of codimension greater than 1.

Definition 3.4. If $\dim \delta + \dim \lambda = n$, $(v+\delta) \cap \lambda \neq \emptyset$ and $\text{codim} V_{\delta,\lambda} = 1$, then the affine hyperplane $H_{\delta,\lambda} = v + V_{\delta,\lambda}$ is said to be inconvenient.

Definition 3.5. A pair of cells $(\delta \in \mathcal{X}, \lambda \in \mathcal{Z})$ of dimensions $(p,q)$ is called a Bergman pair if the cell $(v+\delta) \cap \lambda$ is a 1-dimensional ray for some $v$.

Definition 3.6. Let $(\delta, \lambda)$ be a Bergman pair and fix an orientation of the space $V_{\delta,\lambda}$. Then we put $d^{r_p,r_q}(\delta, \lambda) = r_p(\delta)r_q(\lambda)$ for all $r_p \in C^p(\mathcal{X})$, $r_q \in C^q(\mathcal{Z})$.

Here the orientations of cells are chosen in such a way that we have an isomorphism of oriented spaces $V_{\delta,\lambda}/\mathcal{N} = V_\delta/\mathcal{N} \oplus V_\lambda/\mathcal{N}$, where $\mathcal{N}$ is the 1-dimensional subspace of $V$ parallel to the ray $(v+\delta) \cap \lambda$ with the corresponding orientation.

Definition 3.7. If $(\delta, \lambda)$ is a Bergman pair, then the hyperplane $H_{\delta,\lambda}$ is said to be discriminant. The discriminant $D_{p,q}(\mathcal{X}, \mathcal{Z})$ is the union of all discriminant hyperplanes.

Corollary 3.6. All inconvenient non-dangerous points lie in the union of the inconvenient hyperplanes. Every inconvenient hyperplane contains an open subset consisting of inconvenient points. The codimension of the set of inconvenient points not belonging to $D_{p,q}(\mathcal{X}, \mathcal{Z})$ is equal to 1.

Condition 3.1. Suppose that the points $u,v,\kappa \in V$ satisfy the following conditions.

1) $([u,v] \setminus \kappa) \subset \mathcal{U}_{p,q}$, where $\kappa$ is an interior point of the closed interval $[u,v]$.

2) The point $\kappa$ is non-dangerous and lies in the hyperplane $H$.

3) If $\kappa$ belongs to an inconvenient hyperplane, then this hyperplane coincides with $H$.

Theorem 3.1. Suppose that $r_p, r_q$ are cycles of the $\mathbb{P}$-complexes $\mathcal{X}$, $\mathcal{Z}$ and the hyperplane $H$ is non-discriminant. Then $\text{ind}_u(r_p,r_q) = \text{ind}_v(r_p,r_q)$.

Corollary 3.7. The intersection index $\text{ind}_u(r_p,r_q)$ of shifted cycles is constant for all $u \in \mathcal{C} \cap \mathcal{U}_{p,q}(\mathcal{X}, \mathcal{Z})$, where $\mathcal{C}$ is a connected component of $V \setminus D_{p,q}(\mathcal{X}, \mathcal{Z})$.

Suppose that $u,v$ lie in distinct connected components of $V \setminus D_{p,q}(\mathcal{X}, \mathcal{Z})$. Then the hyperplane $H$ in Condition 3.1, 3) is discriminant. We endow the affine hyperplane $H$ with a co-orientation determined by the direction of the interval $[u,v]$. Using the orientation $V$, we fix the orientation of $H$ corresponding to this co-orientation.
Theorem 3.2. Let \( r_p \) and \( r_q \) be cycles of the \( \mathbb{P} \)-complexes \( \mathfrak{X} \) and \( \mathfrak{Z} \). Suppose that the points \( u, v, \kappa \) are chosen in accordance with the conditions listed above. Then

\[
\text{ind}_v(r_p, r_q) - \text{ind}_u(r_p, r_q) = \sum_{(\lambda, \delta) \in S^H_\kappa} d^{r_p, r_q}(\delta, \lambda),
\]

where \( S^H_\kappa \) is the set of Bergman pairs \((\delta, \lambda)\) such that \( H_{\delta, \lambda} = H \) and \((\kappa + \delta) \cap \lambda \neq \emptyset\).

Comparing Theorems 3.1 and 3.2, we obtain the following assertion.

Corollary 3.8. The right-hand side of (4) depends only on the connected components \( \mathcal{C}, \mathcal{D} \) of \( V \setminus D_{p,q}(\mathfrak{X}, \mathfrak{Z}) \) containing the points \( u, v \) respectively.

Corollary 3.9. Let \( K \) be a curve connecting convenient points \( u, v \). Suppose that

i) almost all points of \( K \) (except for finitely many) are convenient;

ii) \( K \) does not pass through pairwise intersections of inconvenient hyperplanes;

iii) the intersections of \( K \) with discriminant hyperplanes are transversal.

Then

\[
\text{ind}_v(r_p, r_q) - \text{ind}_u(r_p, r_q) = \sum_{H \subset D_{p,q}, H \cap K \neq \emptyset} D^{r_p, r_q}_H,
\]

where \( D^{r_p, r_q}_H \) is the right-hand side of (4) (it is assumed that the orientation of the discriminant hyperplane \( H \) is compatible with that of the intersection \( H \cap K \)).

Proof of Theorems 3.1, 3.2. It follows from Conditions 3.1, 1)–3) on the choice of the points \( u, v, \kappa \) and the hyperplane \( H \) that the \( \mathbb{P} \)-complex \((\kappa + \mathfrak{X}) \cap \mathfrak{Z}\) consists of cells of dimensions 0, 1. Moreover, if \( \dim((\kappa + \delta) \cap \lambda) = 1 \), then \((\kappa + \delta) \cap \lambda \) is contained in an affine hyperplane parallel to \( H \) (this follows from Condition 3.1, 3)). Therefore all the cells of the \( \mathbb{P} \)-complex \((\kappa + \mathfrak{X}) \cap \mathfrak{Z}\) are contained in a finite union of affine hyperplanes parallel to \( H \). Let \( G \) be one of them.

Let \( Y_1 \) be the union of all the 1-dimensional cells of the \( \mathbb{P} \)-complex \((\kappa + \mathfrak{X}) \cap \mathfrak{Z}\) that lie in \( G \) and are not affine lines. (In what follows we perturb the \( \mathbb{P} \)-complex \((\kappa + \mathfrak{X}) \cap \mathfrak{Z}\) by replacing \( \kappa \) by a close point \( x \in [u, v] \). Under such perturbations, cells-lines disappear without a trace.) Thus \( Y_1 \) is a polygonal arc with finitely many edges (it consists of closed intervals and rays). We put \( Y = Y_1 \cup Y_0 \), where \( Y_0 \) is the set of zero-dimensional cells of \((\kappa + \mathfrak{X}) \cap \mathfrak{Z}\) belonging to \( G \setminus Y_1 \).

We write \( x_-, x_+ \in U_{p,q}(\mathfrak{X}, \mathfrak{Z}) \) for points that are close to \( \kappa \) and belong to the intervals \([u, \kappa], [\kappa, v]\) respectively.

Suppose that \( y \in Y_0 \). Then one of the following cases occurs:

i) \( y = (\kappa + \delta) \cap \lambda \), where \( \dim \delta = p, \dim \lambda = q - 1 \);

ii) \( y = (\kappa + \delta) \cap \lambda \), where \( \dim \delta = p - 1, \dim \lambda = q \);

iii) \( y = (\kappa + \text{Int} \delta) \cap \text{Int} \lambda \), where \( \dim \delta = p, \dim \lambda = q \).

We first consider case i). It follows from Condition 3.1, 3) that the set of \( q \)-dimensional cells of \( \mathfrak{Z} \) containing \( \lambda \) splits into two disjoint subsets: the subset of cells \( \{\lambda_i\} \) that intersect \( x_- + \delta \), and the subset of cells \( \{\lambda^j\} \) that intersect \( x_+ + \delta \). Therefore, when \( x \) travels from \( x_- \) to \( x_+ \), some cells of the form \((x + \mathfrak{X}) \cap \mathfrak{Z}\) lying near \( y \) change in the following way: the set of points \((x_- + \delta) \cap \lambda_i\) is replaced by the set \((x_+ + \delta) \cap \lambda^j\). The corresponding variation of the value of \( \text{ind}_x(r_p, r_q) \)
can be written as \( r_p(\delta) \sum r_q(\lambda^j) - r_p(\delta) \sum r_q(\lambda_i) \). Since the chain \( r_q \) is closed, this variation is equal to zero.

In case ii) we similarly use the closedness of \( r_p \) and obtain that the variation of \( \text{ind}_x(r_p, r_q) \) in a neighbourhood of \( y \) is also equal to zero.

In case iii), the point \( y \) is shifted without changing its multiplicity.

Thus \( \text{ind}_x(r_p, r_q) \) is not changed in a neighbourhood of \( y \in Y_0 \).

Let \( y = (\kappa + \delta_q) \cap \lambda_y \) be a vertex of the polygonal arc \( Y_1 \). Then \( \dim \delta_y + \dim \lambda_y = n - 1 \). To be definite, assume that \( \dim \delta_y = p \), \( \dim \lambda_y = q - 1 \).

Let \( [y, y_1], \ldots, [y, y_k] \) be all the edges of \( Y_1 \) containing the vertex \( y \) (when \( [y, y_1] \) is a ray, we put \( y_1 = \infty \)). Then it follows from the condition \( ([u, v] \setminus \kappa) \subset U_{p,q}(X, 3) \) that the set of \( q \)-dimensional cells of \( X \) containing the cell \( \lambda_y \) splits into three disjoint parts: the cells \( \{\lambda_i\} \) that intersect \( x_+ + \delta_y \), the cells \( \{\lambda^j\} \) that intersect \( x_+ + \delta_y \), and the cells \( \gamma^y_1, \ldots, \gamma^y_k \) that contain the edges \( [y, y_1], \ldots, [y, y_k] \) respectively.

Therefore, when \( x \) travels from \( x_- \) to \( x_+ \), the part of the set of cells of the form \( (x + X) \cap 3 \) lying near \( y \) changes in the following way: the subset \( \{(x + \delta_y) \cap \lambda_i\} \) is replaced by the set \( \{(x + \delta) \cap \lambda^j\} \). The corresponding variation of \( \text{ind}_x(r_p, r_q) \) is equal to \( r_p(\delta_y) \left( \sum r_q(\lambda^j) - \sum r_q(\lambda_i) \right) \). Since the chain \( r_q \) is closed, this variation is equal to \( r_p(\delta_y) r_q(\gamma^y_1) + \cdots + r_p(\delta_y) r_q(\gamma^y_k) \).

When \( \dim \delta_y = p - 1 \) and \( \dim \lambda_y = q \), we similarly use the closedness of \( r_p \) and obtain the variation \( r_p(\gamma^y_1) r_q(\lambda^y) + \cdots + r_p(\gamma^y_k) r_q(\lambda^y) \), where \( \gamma^y_1, \ldots, \gamma^y_k \) are the \( p \)-dimensional cells containing \( \delta \) and the edges \( [y, y_1], \ldots, [y, y_k] \) (respectively) of \( Y_1 \).

Let \( z = y_i \) be a vertex adjacent to \( y \) and let \( [y, z] \) be the edge that connects them. Then \( \delta_z = \delta_y \). Therefore the formulae obtained for the variation of \( \text{ind}_x(r_p, r_q) \) near the points \( y \) and \( z \) contain a common summand \( r_p(\delta_y) r_q(\gamma^y_i) \) occurring with opposite signs in these formulae. Hence every compact edge of the polygonal arc \( Y_1 \) contributes zero to the variation of \( \text{ind}_x(r_p, r_q) \). When all the edges of \( Y_1 \) are compact, we thus have \( \text{ind}_{x_+}(r_p, r_q) = \text{ind}_{x_-}(r_p, r_q) \). This proves Theorem 3.1.

When the hyperplane \( H \) is discriminant, some edges of \( Y_1 \) are rays. These rays are intersections of cells of the form \( (\kappa + \delta) \cap \lambda \), where \( (\delta, \lambda) \in S^H_\kappa \). Hence we see (Definition 3.6) that \( \text{ind}_{x_+}(r_p, r_q) - \text{ind}_{x_-}(r_p, r_q) \) is equal to the right-hand side of (4). This proves Theorem 3.2.

### 3.3. Localizations and factorizations

The constructions of localization and factorization reduce assertions on intersections of cycles of a \( \mathbb{P} \)-complex (in §3.6) to the case when the \( \mathbb{P} \)-complex is a fan of cones.

Suppose that \( X \) is a \( k \)-dimensional \( \mathbb{P} \)-complex in an oriented \( n \)-dimensional space \( V \), \( \delta \in X \) and \( x \in \text{Int}(\delta) \). A non-empty intersection of any cell \( \lambda \in X \) containing \( \delta \) with a small neighbourhood of \( x \) is an open part of a set of the form \( x + K_\lambda \), where \( K_\lambda \) is a cone that does not depend on the choice of \( x \) and contains the subspace \( V_\delta \). The cones \( K_\lambda \) form a fan of cones. We denote this fan by \( X_\delta \) and call it the \( \delta \)-localization of the \( \mathbb{P} \)-complex \( X \). The subspace \( V_\delta \) is a minimal cone in \( X_\delta \). We define a map \( X_\delta \to X \) by the formula \( K_\lambda \mapsto \lambda \) and call it the \( \delta \)-localization map of the \( \mathbb{P} \)-complex. The \( \delta \)-localization map of chains \( C_+(X) \to C_+(X_\delta) \) is the map \( s \mapsto s_\delta(K_\lambda) = s(\lambda) \).
Every cone \( K_\lambda \in \mathfrak{X}_\delta \) is the pre-image of a unique cone \( K_\lambda^f \) in \( V/V_\delta \) under the projection map \( \pi_\delta : V \to V/V_\delta \). Let the cell \( \delta \) be oriented. Choose the corresponding orientation of the space \( V/V_\delta \). Endow the cell \( \lambda \) and the cone \( K_\lambda \) with compatible orientations by means of the chosen orientation of \( \delta \). The cones of the form \( K_\lambda^f \) form a fan of cones \( \mathfrak{X}_\delta^f \) in the oriented space \( V/V_\delta \). The cone \( K_\lambda^f \) is called the \( \delta \)-factorization of the cell \( \lambda \), and the fan \( \mathfrak{X}_\delta^f \) is called the \( \delta \)-factorization of the \( \mathbb{P} \)-complex \( \mathfrak{X} \). We define the \( \delta \)-factorization map of chains \( C_{i+m}(\mathfrak{X}) \to C_i(\mathfrak{X}_\delta^f) \) (here \( m = \dim \delta \)) by the formula \( s \mapsto s_\delta^f \), where \( s_\delta^f(K_\lambda^f) = s(\lambda) \).

**Lemma 3.1.** 1) The map \( s \mapsto s_\delta^f \) is a homomorphism of chain complexes \((C_*(\mathfrak{X}), \partial) \to (C_*(\mathfrak{X}_\delta), \partial)\).

2) Suppose that \( m = \dim \delta, k = \dim \mathfrak{X} \). Then the map \( s_\delta \mapsto s_\delta^f \) is an isomorphism of complexes

\[
(0 \to C_k(\mathfrak{X}_\delta) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_m(\mathfrak{X}_\delta) \to 0) \to (C_*(\mathfrak{X}_\delta^f), \partial).
\]

**Corollary 3.10.** The map \( s \mapsto s_\delta^f \) determines a homomorphism of complexes

\[
(0 \to C_k(\mathfrak{X}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_m(\mathfrak{X}) \to 0) \to (C_*(\mathfrak{X}_\delta^f), \partial).
\]

**Corollary 3.11.** Let \( s \) (resp. \( t \)) be a \((p+1)\)-chain (resp. \( p \)-chain) of \( \mathfrak{X} \). Then \( \partial s = t \) if and only if \( (\partial s_\delta^f)(0) = t_\delta^f(0) \) for every \( p \)-dimensional cell \( \delta \in \mathfrak{X} \).

### 3.4. Main assertions on the intersection of cycles of a \( \mathbb{P} \)-complex

Here we fix a \( \mathbb{P} \)-complex \( \mathfrak{X} \) and numbers \( p, q \) such that \( p+q = n+l, l \geq 0 \). In §3.4 we use the notation of §3.2, redefining all terms in the list ‘convenient point, dangerous point, inconvenient hyperplane, discriminant hyperplane, discriminant, Bergman pair of cells’ by the following algorithm.

1) Consider an oriented \( l \)-dimensional cell \( \gamma \in \mathfrak{X} \).

2) Apply the corresponding definitions to the pair of \( \mathbb{P} \)-complexes consisting of the \((p-l)\)-dimensional and \((q-l)\)-dimensional skeletons \((\mathfrak{X}_\delta^f)^{p-l}, (\mathfrak{X}_\gamma^f)^{q-l}\) of the \( \gamma \)-factorization of \( \mathfrak{X} \) (these \( \mathbb{P} \)-complexes lie in the space \( V/V_\gamma \)).

3) Let \( \pi_\gamma : V \to V/V_\gamma \) be the projection map. We say that
   i) a point \( v \) is convenient if \( \forall \gamma \) the point \( \pi_\gamma(v) \) is convenient;
   ii) a point \( v \) is dangerous if \( \exists \gamma \) such that the point \( \pi_\gamma(v) \) is dangerous;
   iii) a hyperplane \( H \) is inconvenient if \( \exists \gamma \) such that the hyperplane \( \pi_\gamma H \) is inconvenient;
   iv) a pair of cells \((\delta, \lambda)\) of the \( \mathbb{P} \)-complex \( \mathfrak{X} \) of dimensions \((p, q)\) is a Bergman pair if \( \dim(\delta \cap \lambda) = l + 1 \) and there is an \( l \)-dimensional cell \( \gamma \) belonging to \( \delta \cap \lambda \) (that is, \( \delta \cap \lambda \) is not an affine subspace);
   v) a hyperplane \( H \) is discriminant if \( \exists \gamma \) such that the hyperplane \( \pi_\gamma H \) is discriminant.

By this definition, every inconvenient hyperplane is a vector subspace (not an affine subspace as in §3.2). In particular, every discriminant hyperplane coincides with the subspace \( V_{\delta, \lambda} \) for some Bergman pair \( \delta, \lambda \). The discriminant \( D_{p,q}(\mathfrak{X}) \) is therefore a union of subspaces of codimension 1.
Definition 3.8 \((v\text{-intersection of chains})\). Suppose that \(s_p \in C_p(\mathbf{x})\), \(s_q \in C_q(\mathbf{x})\) and \(v \in U_{p,q}\). Put \((s_p \cap_v s_q)(\gamma) = \text{ind}_{\pi_\gamma(v)}((s_p)_{\gamma}^l, (s_q)_{\gamma}^l)\), where \(s \to s_{\gamma}^l\) is the \(\gamma\)-factorization map of chains as defined in §3.3.

Corollary 3.12. If \(v\) is sufficiently small, then for every \(l\)-dimensional cell \(\gamma\) we have \((s_p \cap_v s_q)(\gamma) = \sum s_p(\delta) s_q(\lambda)\), where the sum is taken over all pairs of cells \(\delta, \lambda \in \mathbf{x}\) such that \(\dim \delta = p\), \(\dim \lambda = q\), \(\delta \cap \lambda = \gamma\) and \((v + \delta) \cap \lambda \neq \emptyset\).

Corollary 3.13. Suppose that \(l = 0\). If the shift \(v\) is sufficiently small, then \(\text{ind}_v(s_p, s_q)\) is equal to the sum of values of the 0-chain \(s_p \cap_v s_q\) at all 0-dimensional cells of the \(\mathbb{P}\)-complex \(\mathbf{x}\). Moreover, suppose that \(\mathbf{x}\) is a fan of cones. Then \(\text{ind}_v(s_p, s_q) = (s_p \cap_v s_q)(0)\) for every convenient \(v\).

Corollary 3.14. The localization and factorization homomorphisms of chain complexes preserve the \(v\)-intersection of chains.

Corollary 3.15. Suppose that the points \(u\) and \(v\) are connected by a curve lying in the set \(U_{p,q}(\mathbf{x})\) of convenient points. Then \(s_p \cap_u s_q = s_p \cap_v s_q\).

Theorems 3.3 and 3.4 below on the dependence of the \(v\)-intersection of cycles on a parameter are corollaries of Theorems 3.1, 3.2.

Condition 3.2. Suppose (as in Theorems 3.1, 3.2) that the following conditions hold for the points \(u, v, \kappa \in V\).

1) \([u, v] \setminus \kappa \subset U_{p,q}(\mathbf{x})\), where \(\kappa\) is an interior point of the closed interval \([u, v]\.\)

2) The point \(\kappa\) is non-dangerous and lies in the hyperplane \(H\).

3) If \(\kappa\) belongs to an inconvenient hyperplane, then this hyperplane coincides with \(H\).

Theorem 3.3. If the chains \(s_p, s_q\) are closed and the hyperplane \(H\) is non-discriminant, then \(s_p \cap_u s_q = s_p \cap_v s_q\).

Proof. The assertion \((s_p \cap_u s_q)(\gamma) = (s_p \cap_v s_q)(\gamma)\) can be reduced to the case when the \(\mathbb{P}\)-complex is a fan of cones and \(p + q\) is equal to the dimension of the fan. Indeed, replace the \(\mathbb{P}\)-complex \(\mathbf{x}\) by its \(\gamma\)-factorization \(\mathbf{x}_{\gamma}^l\), and the hyperplane \(H\) by its image in \(V/V_{\gamma}\). Then, by Corollary 3.14, it remains to prove that \(((s_p)_{\gamma}^l \cap_{\pi_\gamma(u)} (s_q)_{\gamma}^l)(0) = ((s_p)_{\gamma}^l \cap_{\pi_\gamma(v)} (s_q)_{\gamma}^l)(0)\), where \((s_p)_{\gamma}^l, (s_q)_{\gamma}^l\) are the factorizations of \(s_p, s_q\) (see §3.3). The last equality is a direct corollary of Theorem 3.1 for the pair \(\mathbf{x}^p, \mathbf{x}^q\) of \(\mathbb{P}\)-complexes. □

For \(\phi \in V\) we now construct compatible orientations of the cells \(\delta, \lambda, \delta \cap \lambda\) simultaneously for all Bergman pairs \((\delta, \lambda)\) with \(\phi \notin V_{\delta, \lambda}\). The vector \(\phi\) determines a co-orientation (and hence an orientation) of the space \(V_{\delta, \lambda}\). We choose compatible orientations in such a way that the isomorphism

\[V_{\delta, \lambda} / (V_{\delta} \cap V_{\lambda}) = V_{\delta} / (V_{\delta} \cap V_{\lambda}) \oplus V_{\lambda} / (V_{\delta} \cap V_{\lambda})\]

is an isomorphism of oriented spaces.

Definition 3.9 (Bergman chains). Suppose that \((\delta, \lambda)\) is a Bergman pair of points, \(\mu = \delta \cap \lambda\), \(s_p \in C_p(\mathbf{x})\), \(s_q \in C_q(\mathbf{x})\) and \(\phi \notin V_{\delta, \lambda}\). Write \(\zeta_{s_p, s_q}^{\lambda, \lambda; \phi}\) for the \((l + 1)\)-chain
which is equal to \( s_p(\delta)s_q(\lambda) \) at the cell \( \mu \) (we use here the compatible orientations described above) and 0 at the other cells of \( X \).

Suppose that the points \( u, v \) belong to distinct connected components \( C, D \) of \( V \setminus D_{p,q}(X) \). Then the hyperplane \( H \) (in Condition 3.2, 3)) is discriminant. We fix the orientation of \( H \) compatible with the co-orientation determined by the given direction of the interval \([u, v]\).

**Theorem 3.4.** Let the chains \( s_p, s_q \) be closed. Then

\[
s_p \cap_v s_q - s_p \cap_u s_q = \sum_{(\delta, \lambda) \in S_{H,\kappa,p,q}} \partial \zeta_{s_{sp},s_{sq}}^{\delta,\lambda;v-u},
\]

where \( S_{H,\kappa,p,q} \) is the set of all Bergman pairs of cells \((\delta, \lambda)\) such that \( V_{\delta,\lambda} = H \) and \((\kappa + \delta) \cap \lambda \neq \emptyset\).

**Proof.** As in the proof of Theorem 3.3, the assertion

\[
(s_p \cap_v s_q)(\gamma) - (s_p \cap_u s_q)(\gamma) = \sum_{(\delta, \lambda) \in S_{H,\kappa,p,q}} \partial \zeta_{s_{sp},s_{sq}}^{\delta,\lambda;v-u}(\gamma)
\]

can be reduced to the case when the \( \mathbb{P} \)-complex is a fan of cones and \( p + q \) is equal to the dimension of the fan. In this case, the desired assertion is a direct corollary of Theorem 3.2 for the pair \( X^p, X^q \) of \( \mathbb{P} \)-complexes. \( \square \)

**Corollary 3.16.** The right-hand side of (5) depends only on the choice of the connected components \( C, D \) containing \( u, v \).

**Theorem 3.5.** Suppose that \( v \in U_{p,q}(X) \). Then

\[
\partial(s_p \cap_v s_q) = \partial s_p \cap_v s_q + (-1)^p s_p \cap_v \partial s_q.
\]

**Proof.** Using Corollary 3.11 from §3.3, we can reduce the desired equality to the case when the \( \mathbb{P} \)-complex \( X \) is a fan of cones and \( p + q = n + 1 \).

Consider the 1-chain \( t = (v + s_p) \cap s_q \) of the \( \mathbb{P} \)-complex \((v + X) \cap X\). The support of \( t \) is a polygonal arc \( \Theta \) with finitely many edges. It is formed by 1-dimensional cells of the \( \mathbb{P} \)-complex \((v + X) \cap X\). Assume that the minimal cone of the fan \( X \) is zero (otherwise we easily see that both sides of the desired equality are equal to zero). Then the polygonal arc \( \Theta \) has the following properties.

**Property 3.1.** The edges are either intervals or rays.

**Property 3.2.** The ray-edges are shifts of edges (1-dimensional cones) of \( X \).

**Property 3.3.** Fix an orientation of each edge in such a way that the ray-edges are directed ‘towards infinity’. Then every edge \( \theta \) of the polygonal line \( \Theta \) is endowed with the weight \( t(\theta) \in S \).

**Property 3.4.** If \( l \) is an edge of \( X \), then \((s_p \cap_v s_q)(l) = \sum t(\theta)\), where the sum is taken over all the ray-edges of \( \Theta \) that are parallel to \( l \).
The left-hand side of the desired equality

$$\partial(s_p \cap_w s_q)(0) = (\partial s_p \cap_w s_q + (-1)^ps_p \cap_w \partial s_q)(0)$$

is by definition equal to the sum of the values of the chain $s_p \cap_w s_q$ on the ray-edges of $\mathfrak{X}$ oriented towards infinity. Therefore (by Property 3.4) the left-hand side is equal to the sum of the weights of the ray-edges of $\Theta$.

We endow every vertex of $\Theta$ with a weight equal to the sum of the weights of the outgoing edges minus the sum of the weights of the incoming edges. It follows from Corollary 3.2 (in §3.1) that the 0-cochain $\partial s_p \cap s_q + (-1)^ps_p \cap \partial s_q$ of the $\mathbb{P}$-complex $(v + \mathfrak{X}) \cap \mathfrak{X}$ is the set of all vertices with these weights. By definition, this means that the right-hand side of (6) coincides with the sum of the weights of all the vertices of $\Theta$. It remains to show that the sum of the weights of the vertices is equal to the sum of the weights of the ray-edges.

The weight of each interval-edge occurs with opposite signs in the weights of its endpoints. Therefore the weights of the interval-edges cancel each other when we sum the weights of all the vertices, and only the sum of the weights of the ray-edges remains. □

§4. Proofs of Theorems 2.1, 2.2

Suppose that $X$ is a $\mathbb{P}$-complex in an $n$-dimensional space $V$, $\gamma \in X$ and $\dim \gamma = p + q$, and let $\gamma^\Psi$ be the $\mathbb{P}$-complex in $V_\gamma$ formed by all faces of $\gamma$ (the superscript $\Psi$ is used for compatibility with the notation in §6). Choose an orientation of the cell $\gamma$ and the corresponding orientation of the space $V_\gamma$. Let $\gamma^\Psi$ be the fan of dual cones of the polyhedron $\gamma$. This fan lies in the space $V_{\gamma}^\Psi$ dual to $V_\gamma$. We denote the dual cone of a face $\delta \subset \gamma$ by $\delta^* \in \gamma^\Psi$. There is a one-to-one map of $\mathbb{P}$-complexes $\gamma^\Psi \to \gamma^\Psi$ given by the formula $\delta \mapsto \delta^*$.

Moreover, by construction, $\lambda \subset \delta \iff \lambda^* \supset \delta^*$. We introduce compatible orientations of dual cells as follows. The space $(V_{\gamma}^\Psi)^\delta$ is dual to $V_{\gamma}/V_\delta$. Hence these spaces have compatible orientations. We now define compatible orientations of $V_{\gamma}/V_\delta$ and $V_\delta$ using the orientation of $V_\gamma$. Then the orientations of the spaces $V_\delta$ and $(V_{\gamma}^\Psi)^\delta$ or, equivalently, of the cells $\delta$ and $\delta^*$, are compatible. For every cochain $r \in C^k(\gamma^\Psi, S)$ we define a chain $\nu_{\gamma, \gamma}(r) \in C_{p+q-k}(\gamma^\Psi, S)$ by the formula $(\nu_{\gamma, \gamma}(r)) (\delta^*) = r(\delta)$.

Lemma 4.1. The map $\nu_{\gamma, \gamma} : C^\ast(\gamma^\Psi) \to C^\ast(\gamma^\Psi)$ is an isomorphism of complexes.

It follows from Lemma 4.1 that Proposition 2.1 in §2 is a corollary of Theorem 3.5 in §3.4.

We write $o_\gamma : C^\ast(X, S) \to C^\ast(\gamma^\Psi, S)$ for the restriction map of cochains. The map $o_\gamma$ is a homomorphism of chain complexes. For $s \in C^\ast(X, S)$ we define a map $R_{\gamma}^\Psi : C^\ast(X, S) \to C^\ast(\gamma^\Psi, S)$ by the formula $R_{\gamma}^\Psi(s) = \nu_{\gamma, \gamma}(o_\gamma(s))$. Notice that $R_{\gamma}^\Psi(s) : C^k(X, S) \to C_{p+q-k}(\gamma^\Psi, S)$. Let $\pi_\gamma : V^\ast \to V_{\gamma}^\Psi$ be the projection map. We shall use the following facts, which follow from the definitions given in §§2 and 3.4.

Property 4.1. If $v \in U_{p,q}(X)$, then $\pi_\gamma(v) \in U_{p,q}(\gamma^\Psi)$.

Property 4.2. If the point $\kappa \in V^\ast$ is non-dangerous for the $\mathbb{P}$-complex $X$, then the point $\pi_\gamma(\kappa)$ is non-dangerous for the $\mathbb{P}$-complex $\gamma^\Psi$. 
Property 4.3. If a hyperplane $H \subset V_\gamma^*$ is inconvenient for the $\mathbb{P}$-complex $\gamma^\Psi$, then the hyperplane $\pi_\gamma^{-1}H$ is inconvenient for the $\mathbb{P}$-complex $X$.

Property 4.4. If a hyperplane $H \subset V_\gamma^*$ is discriminant for the $\mathbb{P}$-complex $\gamma^\Psi$, then the hyperplane $\pi_\gamma^{-1}H$ is discriminant for the $\mathbb{P}$-complex $X$.

Property 4.5. $(r_p \cdot_v r_q)(\gamma) = (R_\gamma^\Psi(r_p) \cap_{\pi_\gamma(v)} R_\gamma^\Psi(r_q))(0)$.

Proof of Theorem 2.1. By Properties 4.1–4.4, the points $\pi_\gamma(u), \pi_\gamma(v), \pi_\gamma(\kappa)$ and the cycles $R_\gamma^\Psi(r_p), R_\gamma^\Psi(r_q)$ satisfy the hypotheses of Theorem 3.3. It follows from Property 4.5 that

$$(r_p \cdot_v r_q - r_p \cdot_u r_q)(\gamma) = (R_\gamma^\Psi(r_p) \cap_{\pi_\gamma(v)} R_\gamma^\Psi(r_q) - R_\gamma^\Psi(r_p) \cap_{\pi_\gamma(u)} R_\gamma^\Psi(r_q))(0).$$

Now, using Theorem 3.3 for the fan of cones $\gamma^\Psi$, we obtain the desired assertion. □

We now proceed to prove Theorem 2.2. Let $\{u, v, \kappa, H, S^H_{p,q}(\kappa)\}$ be any data satisfying the hypotheses of the theorem. We consider the corresponding data $\{\pi_\gamma(u), \pi_\gamma(v), \pi_\gamma(\kappa), \pi_\gamma(H), S_\gamma^{\pi_\gamma(H)}(\pi_\gamma(\kappa))\}$ for the $\mathbb{P}$-complex $\gamma^\Psi$.

Lemma 4.2. Suppose that $\delta, \lambda, \mu \in \gamma^\Psi$. Then $(\delta, \lambda, \mu) \in S_\gamma^{\pi_\gamma(H)}(\pi_\gamma(\kappa))$ if and only if $(\delta, \lambda, \mu) \in S^H_{p,q}(\kappa)$.

Corollary 4.1. Let $\Theta$ and $\Theta_\gamma$ be the cochains on the right-hand side of (3) in Theorem 2.2 for the $\mathbb{P}$-complexes $X$ and $\gamma^\Psi$ respectively. Then $\Theta(\gamma) = \Theta_\gamma(\gamma)$.

The values of the left-hand sides of (3) at the cell $\gamma$ coincide for the $\mathbb{P}$-complexes $X$ and $\gamma^\Psi$ by the definition of the $v$-product of cochains. This reduces Theorem 2.2 to the case of $\mathbb{P}$-complexes of the form $\gamma^\Psi$. In this case, we easily see that the local duality map $\nu_{\gamma^\Psi, \gamma} : C^*(\gamma^\Psi) \to C_*(\gamma^\Psi)$ reduces Theorem 2.2 to Theorem 3.4.

§ 5. Bergman fans and a local analogue of Theorem 2.3

In § 5.1 we define the Bergman fan $\mathcal{X}_\infty$ of a $\mathbb{P}$-complex $\mathcal{X}$ and construct a homomorphism of complexes $\beta : C_*(\mathcal{X}, S) \to C_*(\mathcal{X}_\infty, S)$ which is locally dual to the homomorphism $\text{res}$ in Theorem 2.3. In § 5.2 we prove an assertion relating the homology of a $\mathbb{P}$-complex to that of its Bergman fan. This assertion is a local analogue of Theorem 2.3.

5.1. The Bergman fan of a $\mathbb{P}$-complex. Let $\delta$ be a convex polyhedron in an oriented $n$-dimensional space $V$. We write $\delta_\infty$ for the convex polyhedral cone formed by the limit points of the family of polyhedra $t\delta$ as $t \to +0$. The cone $\delta_\infty$ lies in the subspace $V_\delta$. When $\delta$ is bounded, we have $\delta_\infty = 0$. Otherwise $\dim \delta_\infty > 0$.

Definition 5.1. A polyhedron $\delta$ is said to be conelike if $\dim \delta_\infty = \dim \delta$.

Lemma 5.1. Every face of $\delta_\infty$ is a cone of the form $\lambda_\infty$, where $\lambda$ is a conelike face of $\delta$.

Proof. Among the faces $\mu \subset \delta$ with $\mu_\infty = \lambda_\infty$ we choose a face of minimal dimension. This is the desired face $\lambda$. □
Lemma 5.2. If \( \dim \delta_\infty = \dim \delta - 1 \), then there are precisely two faces \( \kappa, \lambda \) of \( \delta \) such that \( \kappa_\infty = \lambda_\infty = \delta_\infty \).

Proof. The image of \( \delta \) under the projection map \( V \to V/V_{\delta_\infty} \) is the closed interval \([u, v]\). Then the faces \( \kappa, \lambda \) are pre-images of the points \( u, v \). \( \square \)

Given any \( k \)-dimensional \( \mathbb{P} \)-complex \( \mathcal{X} \), we put \( S_\infty = \bigcup_{\delta \in \mathcal{X}} \delta_\infty \). For every \( \varphi \in S_\infty \) we define \( L(\varphi) \) as the set of cells \( \delta \in \mathcal{X} \) such that \( \varphi \in \delta_\infty \). Two points \( \varphi, \psi \in S_\infty \) are said to be equivalent if \( L(\varphi) = L(\psi) \). The equivalence classes are convex cones. They form a fan of cones \( \mathcal{X}_\infty \) with \( \dim \mathcal{X}_\infty \leq k \).

Definition 5.2. The fan \( \mathcal{X}_\infty \) is called the Bergman fan of the \( \mathbb{P} \)-complex \( \mathcal{X} \).

Corollary 5.1. If the \( \mathbb{P} \)-complex \( \mathcal{X} \) is a fan of cones, then \( \mathcal{X}_\infty = \mathcal{X} \).

Definition 5.3. We define a map \( \beta: C_*(\mathcal{X}, S) \to C_*(\mathcal{X}_\infty, S) \) in the following way. Suppose that \( r_p \in C_p(\mathcal{X}, S) \), \( \gamma \in \mathcal{X}_\infty \), \( \dim \gamma = p \) and \( \varphi \in \text{Int}(\gamma) \). Then we put \( (\beta r_p)(\gamma) = \sum_{\delta \in L(\varphi), \dim \delta = p} r_p(\delta) \).

By construction, the value of \( (\beta r_p)(\gamma) \) is independent of the choice of \( \varphi \in \text{Int}(\gamma) \).

The use of Bergman fans in the proof of Theorem 2.3 (see §6) is based on the following fact. When we pass to the dual \( \mathbb{P} \)-complex (Definition 6.2 in §6.1), the map \( \text{res} \) (Definition 2.8 in §2) is transformed into the map \( \beta \). Theorems 5.2, 5.3 below comprise a local version of Theorem 2.3.

Theorem 5.1. The map \( \beta \) is a homomorphism of complexes.

Proof. Let \( \mu \) be an oriented \( m \)-dimensional cone in \( \mathcal{X}_\infty \). Then \( \mu \subset \lambda_\infty \), where \( \lambda \) is an \( m \)-dimensional conelike cell of the \( \mathbb{P} \)-complex \( \mathcal{X} \). We choose the orientation of \( \lambda \) that is compatible with the orientation of \( \mu \). Perhaps \( \lambda \) is a face of some cell \( \delta \) of dimension \( m + 1 \). Then we choose the orientation of \( \delta \) that is compatible with the orientation of \( \lambda \). Let \( \Delta_\mu \) be the set of all oriented cells \( \delta \in \mathcal{X} \) (the cell \( \lambda \) is not fixed). We write \( \Delta^c_\mu \) for the subset of conelike cells in \( \Delta_\mu \). Then, by construction, we have \( (\partial \beta r_{m+1})(\mu) = \sum_{\delta \in \Delta^c_\mu} r_{m+1}(\delta) \) for all \( r_{m+1} \in C_{m+1}(\mathcal{X}) \).

If \( \delta \in \Delta_\mu \setminus \Delta^c_\mu \), that is, \( \delta \) is not conelike, then Lemma 5.2 shows that the cell \( \delta \) has precisely two \( m \)-dimensional faces \( \kappa, \lambda \) such that \( \mu \subset \kappa_\infty \) and \( \mu \subset \lambda_\infty \). It follows that \( \delta \) occurs in \( \Delta_\mu \) twice with opposite orientations. Thus we obtain that

\[
(\beta \partial r_{m+1})(\mu) = \sum_{\delta \in \Delta_\mu} r_{m+1}(\delta) = \sum_{\delta \in \Delta^c_\mu} r_{m+1}(\delta) = (\partial \beta r_{m+1})(\mu). \square
\]

5.2. The Bergman fan and the intersection of cycles. Suppose that \( p + q = n + l \), where \( l \geq 0 \). For any cycles \( r_p, r_q \in C_*(\mathcal{X}, S) \) and \( v \in V \), we shall define the \( v \)-intersection \( (v + r_p) \cap_V r_q \) with values in \( C_*(\mathcal{X}_\infty, S) \) and prove analogues of Theorems 3.1, 3.2. To do this, we repeat their derivation in a more general situation. Therefore we slightly modify the list of definitions given in §3.2.

Definition 5.4. Suppose that \( \delta \in \mathcal{X}^p \) and \( \lambda \in \mathcal{X}^q \), where \( \mathcal{X}^m \) is the \( m \)-dimensional skeleton of \( \mathcal{X} \).

1) A point \( v \in V \) is said to be convenient if the \( \mathbb{P} \)-complexes \( v + \mathcal{X}^p, \mathcal{X}^q \) are transversal. The set of convenient points is denoted by \( \mathcal{U}_{p,q}(\mathcal{X}) \).
2) If \((v + \delta) \cap \lambda \neq \emptyset\) and \(\text{codim} \, V_{\delta,\lambda} > 1\), then the point \(v\) is said to be \textit{dangerous}.

3) If \((v + \delta) \cap \lambda \neq \emptyset\) and \(\text{codim} \, V_{\delta,\lambda} = 1\), then the affine hyperplane \(H_{\delta,\lambda} = v + V_{\delta,\lambda}\) is said to be \textit{inconvenient}.

4) Suppose that \(\dim \delta = p, \dim \lambda = q\) and \((v + \delta) \cap \lambda\) is an \((l + 1)\)-dimensional conelike cell for some \(v\). Then \((\delta, \lambda)\) is called a \textit{Bergman} pair of cells, and the inconvenient hyperplane \(H_{\delta,\lambda}\) is said to be \textit{discriminant}. The union of all discriminant hyperplanes is called the \textit{discriminant} and is denoted by \(D_{p,q}(\mathcal{X})\).

**Definition 5.5.** For \(v \in U_{p,q}(\mathcal{X})\) we define a chain \(r_p \cap_u r_q \in C_l(\mathcal{X}_\infty, S)\) in the following way. Suppose that \(\Delta \in \mathcal{X}_\infty\), \(\dim \Delta = l\). Then \((r_p \cap_u r_q)(\Delta)\) is equal to the sum of the values of the \(l\)-chain \((v + r_p) \cap r_q\) of the \(\mathbb{P}\)-complex \((v + \mathcal{X}^p) \cap \mathcal{X}^q\) over all \(l\)-dimensional cells \(\delta \in (v + \mathcal{X}^p) \cap \mathcal{X}^q\) such that \(\delta \in V_{\delta,\lambda} \supseteq \Delta\).

**Corollary 5.2.** If two points \(u, v\) belong to the same connected component of \(U_{p,q}(\mathcal{X})\), then \(r_p \cap_u r_q = r_p \cap_v r_q\).

**Corollary 5.3.** If \(\mathcal{X}\) is a fan of cones, then \(r_p \cap_u r_q = r_p \cap_v r_q\).

Comparing Definitions 5.3, 5.5 and Corollary 3.12, we obtain the following lemma.

**Lemma 5.3.** If \(v\) is sufficiently close to zero, then \(r_p \cap_u r_q = \beta(r_p \cap_v r_q)\).

For small \(u\), the following assertion is a corollary of Lemma 5.3.

**Lemma 5.4.** If the chains \(r_p, r_q\) are closed, then \(r_p \cap_u r_q\) is a cycle of the \(\mathbb{P}\)-complex \(\mathcal{X}_\infty\).

**Proof.** Suppose that \(\Lambda \in \mathcal{X}_\infty\) and \(\dim \Lambda = l - 1\). Write \(M(\Lambda, u)\) for the set of pairs of cells \(\mu, \gamma \in \mathcal{X}\) such that \(\dim (u + \mu) \cap \gamma = l\) and there is an \((l - 1)\)-dimensional face \(v \subset (u + \mu) \cap \gamma\) with \(v \in V_{\delta,\lambda}\). Let \(M_c(\Lambda, u)\) be the subset of \(M(\Lambda, u)\) consisting of all pairs \((\mu, \gamma)\) for which the cell \((u + \mu) \cap \gamma\) is conelike. Then, using Definition 5.5, we obtain

\[
(\partial(r_p \cap u r_q))(\Lambda) = \sum_{(\mu, \gamma) \in M_c(\Lambda, u)} ((u + r_p) \cap r_q)((u + \mu) \cap \gamma),
\]

where \((u + r_p) \cap r_q\) (resp. \((u + \mu) \cap \gamma\)) is regarded as a chain (resp. a cell) of the \(\mathbb{P}\)-complex \((u + \mathcal{X}^p) \cap \mathcal{X}^q\).

It follows from Lemma 5.2 that the sum over \(M_c(\Lambda, u)\) on the right-hand side of (7) can be replaced by the same sum over the larger set \(M(\Lambda, u)\). Then the closedness of the chain \(r_p \cap_u r_q\) follows from Corollary 3.3. □

**Definition 5.6** (Bergman chains). Suppose that \(v \in U_{p,q}(\mathcal{X})\), \((\delta, \lambda)\) is a Bergman pair, \(\phi \notin V_{\delta,\lambda}\) and \((v + \delta) \cap \lambda\) is an \((l + 1)\)-dimensional conelike cell. Put \(\Delta_{\delta,\lambda} = (v + \delta) \cap \lambda\) (the cone \(\Delta_{\delta,\lambda}\) is independent of the choice of \(v\)). We define an \((l + 1)\)-chain \(\Xi_{r_p,r_q}^\delta,\lambda;\phi\) of the fan \(\mathcal{X}_\infty\) by putting \(\Xi_{r_p,r_q}^\delta,\lambda;\phi(\Lambda) = 0\) when \(\Lambda \notin \Delta_{\delta,\lambda}\) and \(\Xi_{r_p,r_q}^\delta,\lambda;\phi(\Lambda) = r_p(\delta)r_q(\lambda)\) when \(\Lambda \subseteq \Delta_{\delta,\lambda}\). In the last equality we assume that the orientations of \(\delta, \lambda\) and \(\Lambda\) are compatible in the following sense. The vector \(\phi\) determines a co-orientation (and hence an orientation) of \(V_{\delta,\lambda}\). The orientations are compatible if the isomorphism \(V_{\delta,\lambda} = V_{\delta} / V_{\Delta_{\delta,\lambda}} \oplus V_{\lambda} / V_{\Delta_{\delta,\lambda}} \oplus V_{\Delta_{\delta,\lambda}}\) is an isomorphism of oriented spaces.
Suppose that the points $u, v, \kappa \in V$ satisfy the following conditions.

**Condition 5.1.** $([u, v] \setminus \kappa) \subset \mathcal{U}_{p,q}(\mathfrak{X})$, where $\kappa$ is an interior point of the closed interval $[u, v]$.

**Condition 5.2.** The point $\kappa$ is non-dangerous and lies in the hyperplane $H$.

**Condition 5.3.** If $\kappa$ belongs to an inconvenient affine hyperplane, then this hyperplane coincides with $H$.

**Theorem 5.2.** If the hyperplane $H$ is non-discriminant, then $r_p \cap_u^\beta r_q = r_p \cap_u^\beta r_q$.

**Theorem 5.3.** If the hyperplane $H$ is discriminant, then

$$r_p \cap_v^\beta r_q - r_p \cap_u^\beta r_q = \sum_{(\lambda, \delta) \in S^H_{\kappa}} \partial \Xi^{\delta,\lambda;v-y-u}_{r_p,r_q},$$

where $S^H_{\kappa}$ is the set of Bergman pairs $(\delta, \lambda)$ such that $H_{\delta,\lambda} = H$ and $(\kappa + \delta) \cap \lambda \neq \emptyset$.

To give a proof of Theorem 2.3 in §6.2, it is convenient to combine Theorems 5.2 and 5.3 in the following way.

**Corollary 5.4.** Let $K$ be a smooth curve connecting two convenient points $u, v$. Assume that $K$ does not pass through pairwise intersections of discriminant hyperplanes and all intersections of $K$ with the discriminant $D_{p,q}(\mathfrak{X})$ are transversal. Given any $\kappa \in K \cap D_{p,q}(\mathfrak{X})$, we write $H_\kappa$ for the discriminant hyperplane containing $\kappa$, and $\bar{\kappa}$ for the velocity vector of the curve $K$ at the point $\kappa$. Then

$$r_p \cap_v^\beta r_q - r_p \cap_u^\beta r_q = \sum_{\kappa \in K \cap D_{p,q}(\mathfrak{X})} \sum_{(\lambda, \delta) \in S^H_{\kappa}} \partial \Xi^{\delta,\lambda;v-y}_{r_p,r_q},$$

**Proof of Theorems 5.2, 5.3.** Let $\Delta$ be an $l$-dimensional cone of the fan $\mathfrak{X}_\infty$. We write $M(x, \Delta)$ for the set of pairs of cells $(\delta \in X^p, \lambda \in X^q)$ such that $((x + \delta) \cap \lambda)_\infty \supseteq \Delta$, $\dim((x + \delta) \cap \lambda) = l$ and the cell $(x + \delta) \cap \lambda$ is not an affine subspace. If $x \in \mathcal{U}_{p,q}(\mathfrak{X})$ and $(\delta, \lambda) \in M(x, \Delta)$, then $\dim \delta = p$ and $\dim \lambda = q$. If $(\delta, \lambda) \in M(\kappa, \Delta)$, then $\dim \delta + \dim \lambda$ is equal to $p + q$ or $p + q - 1$. Let $x_-, x_+ \in \mathcal{U}_{p,q}(\mathfrak{X})$ be points close to $\kappa$ and lying on the intervals $[u, \kappa], [\kappa, v]$ respectively. The difference between the sets $M(x_-, \Delta)$ and $M(x_+, \Delta)$ can be described in the following way.

Put

$$M^l_{-1} = \{ (\delta, \lambda) \in M(\kappa, \Delta) : \dim \delta + \dim \lambda = p + q - 1 \},$$

$$M^p_{-1,q-1} = \{ (\delta, \lambda) \in M^l_{-1} : \dim \delta = p, \dim \lambda = q - 1 \}$$

and

$$M^q_{-1,q-1} = M^l_{-1} \setminus M^p_{-1,q-1}.$$

**Lemma 5.5.** Suppose that $(\delta, \lambda) \in M^p_{-1,q-1}$. Then the set of $q$-dimensional cells containing $\lambda$ splits into the following disjoint parts:

i) the cells $\mu_{-\delta,\lambda}$ such that $(\delta, \mu_{-\delta,\lambda}) \in M(x_-, \Delta)$;

ii) the cells $\mu_{+\delta,\lambda}$ such that $(\delta, \mu_{+\delta,\lambda}) \in M(x_+, \Delta)$;

iii) the cells $\mu_{\delta,\lambda}$ such that $\dim((\kappa + \delta) \cap \mu_{\delta,\lambda}) = l + 1$ and $\mu_{\delta,\lambda} \subset \kappa + V_{\delta,\lambda}$. 

Suppose that \( \delta, \lambda \in M^{n-1,q}_\Delta \), we obtain a similar splitting \( \{\mu_{-\lambda}^{\delta,\lambda}\} \cup \{\mu_{+\lambda}^{\delta,\lambda}\} \cup \{\mu^{\delta,\lambda}\} \) of the set of \( p \)-dimensional cells containing \( \delta \).

By construction, cells of the form \( \mu_{-\lambda}^{\delta,\lambda} \) disappear and cells of the form \( \mu_{+\lambda}^{\delta,\lambda} \) arise when we pass from \( M(x_-,\Delta) \) to \( M(x_+\Delta) \). It is geometrically clear that no other changes in \( M(x,\Delta) \) can occur when we pass from \( x_- \) to \( x_+ \).

If \( (\delta, \lambda) \in M^{p-1,q}_\Delta \), then the closedness of the chain \( r_q \) implies that

\[
\sum_{\mu_{+\lambda}^{\delta,\lambda}} r_p(\delta) r_q(\mu_{+\lambda}^{\delta,\lambda}) - \sum_{\mu_{-\lambda}^{\delta,\lambda}} r_p(\delta) r_q(\mu_{-\lambda}^{\delta,\lambda}) = \sum_{\mu^{\delta,\lambda}} r_p(\delta) r_q(\mu^{\delta,\lambda}).
\]

But if \( (\delta, \lambda) \in M^{p-1,q}_\Delta \), then the closedness of \( r_p \) similarly implies that

\[
\sum_{\mu_{+\lambda}^{\delta,\lambda}} r_p(\mu_{+\lambda}^{\delta,\lambda}) r_q(\lambda) - \sum_{\mu_{-\lambda}^{\delta,\lambda}} r_p(\mu_{-\lambda}^{\delta,\lambda}) r_q(\lambda) = \sum_{\mu^{\delta,\lambda}} r_p(\mu^{\delta,\lambda}) r_q(\lambda).
\]

In (10), (11) we assume that the orientations of the cells \( \mu^{\delta,\lambda} \) are compatible with the orientation of their common face \( \lambda \) in (10) or \( \delta \) in (11).

Thus, if the pair \( (\delta, \lambda) \in M^{l-1}_\Delta \) gives rise to no cells of the form \( \mu^{\delta,\lambda} \), then we 'locally’ have \( r_p \cap_{x_-} r_q = r_p \cap_{x_+} r_q \).

Suppose that \( (\delta, \lambda) \in M^{l-1}_\Delta \). When \( (\delta, \lambda) \in M^{p,q-1}_\Delta \), we put \( z_{\delta,\lambda} = (\kappa + \delta) \cap \mu^{\delta,\lambda} \). When \( (\delta, \lambda) \in M^{p,q-1}_\Delta \), we put \( z_{\delta,\lambda} = (\kappa + \mu^{\delta,\lambda}) \cap \lambda \). The \((l+1)\)-dimensional cell \( z_{\delta,\lambda} \) of the \( \mathbb{P} \) complex \( (\kappa + \mathbb{P}^n) \cap \mathbb{P}^q \) will be referred to as an interval (by analogy with the proofs of Theorems 3.1, 3.2). We endow the interval \( z_{\delta,\lambda} \) with a weight \( m_{\delta,\lambda} = r_p(\mu^{\delta,\lambda}) r_q(\lambda) \) when \( (\delta, \lambda) \in M^{p,q-1}_\Delta \) or \( (\delta, \lambda) \in M^{p,q-1}_\Delta \) respectively.

The cell \( (\kappa + \delta) \cap \lambda \) is referred to as an end of the interval \( z_{\delta,\lambda} \). The interval \( z_{\delta,\lambda} \) is said to be bounded if the cell \( z_{\delta,\lambda} \) is not conelike. We claim that every bounded interval has exactly two ends.

Indeed, to be definite, assume that \( (\delta, \lambda) \in M^{p,q-1}_\Delta \). By Lemma 5.2, the cell \( (\kappa + \delta) \cap \mu^{\delta,\lambda} \) contains precisely two \( l \)-dimensional faces \( \xi, \zeta \) such that \( \xi_\infty \supseteq \Delta \) and \( \zeta_\infty \supseteq \Delta \). One of these faces (say, \( \xi \), to be definite) coincides with the cell \( (\kappa + \delta) \cap \lambda \).

The face \( \zeta \) is also of the form \( (\kappa + \delta_1) \cap \lambda_1 \), where \( (\delta_1, \lambda_1) \in M^{l-1}_\Delta \). Indeed, if \( \zeta = (\kappa + \delta) \cap \lambda_1 \), where \( \lambda_1 \) is a face of \( \mu^{\delta,\lambda} \), then \( \delta_1 = \delta \) and \( (\delta_1, \lambda_1) \in M^{p,q-1}_\Delta \). In this case, \( \mu^{\delta,\lambda} \) is a cell of the form \( \mu^{\delta_1,\lambda_1} \). But if \( \zeta = (\kappa + \delta_1) \cap \mu^{\delta,\lambda} \), where \( \delta_1 \) is a face of \( \delta \), then \( \lambda_1 = \mu^{\delta,\lambda} \) and \( (\delta_1, \lambda_1) \in M^{p,q-1}_\Delta \). In this case, \( \delta \) is a cell of the form \( \mu^{\delta_1,\lambda_1} \). We thus see that the cell \( (\delta_1, \lambda_1) \) is also an end of the interval \( z_{\delta,\lambda} \) or, in other words, \( z_{\delta,\lambda} = z_{\delta_1,\lambda_1} \).

By construction, the weights (defined above) of bounded intervals satisfy \( m_{\delta,\lambda} = -m_{\delta_1,\lambda_1} \). Summing the formulae (10), (11) over all pairs \( (\delta, \lambda) \in M^{l-1}_\Delta \), we obtain

\[ \text{Product of cocycles in a polyhedral complex} 351 \]
that \((r_p \cap^\beta r_q - r_p \cap^\alpha r_q)(\Delta)\) is equal to the sum of the weights \(m_{\delta,\lambda}\) over all unbounded intervals \(z_{\delta,\lambda}\).

When the hyperplane \(H\) is non-discriminant, all intervals \(z_{\delta,\lambda}\) are bounded. Thus we complete the proof of Theorem 5.2.

When the hyperplane \(H\) is discriminant, the right-hand side of (8) in Theorem 5.3 is equal to the sum of the weights of the unbounded intervals. This proves Theorem 5.3. □

§ 6. Regular partitions and proof of Theorem 2.3

In § 6.1 we describe a local duality construction related to regular partitions of \(\mathbb{P}\)-complexes. In § 6.2 we give a proof of Theorem 2.3. For completeness, we give a proof of Proposition 2.2 (see § 2) in § 6.3.

6.1. Regular partitions. Let \(\Phi\) be a polyhedral partition of a \(\mathbb{P}\)-complex \(Y\). We write \(Y^\Phi\) for the \(\mathbb{P}\)-complex consisting of the faces of \(\Phi\). The \(\mathbb{P}\)-complex \(Y^\Phi\) is called a refinement of \(Y\). Given any \(\gamma \in Y\), we write \(\gamma^\Phi\) for the \(\mathbb{P}\)-complex consisting of the cells \(\lambda \in Y^\Phi\) contained in \(\gamma\). If the partition \(\Phi\) is regular (see Definition 6.1 below), then there is a \(\mathbb{P}\)-complex \(\gamma^\Phi\) dual to \(\gamma^\Phi\). The cochain complex \(C^*(\gamma^\Phi)\) is isomorphic to the chain complex \(C_*(\gamma^\Phi)\). Thus the complex \(C^*(Y^\Phi)\) is ‘glued’ from chain complexes of the form \(C_*(\gamma^\Phi)\). This construction is used in the proof of Theorem 2.3.

Let \(\phi\) be a convex piecewise-linear function on a convex polyhedron \(\gamma \subset V\). The domains of linearity of \(\phi\) form a partition \(\Phi\) of \(\gamma\). Such partitions of polyhedra are said to be regular [8]. The \(\mathbb{P}\)-complex formed by the faces of \(\Phi\) is denoted by \(\gamma^\Phi\). The partition of a polyhedron consisting of all of its faces is said to be tautological. When the function \(\phi\) is linear, the regular partition \(\Phi\) is tautological.

Remark 6.1. The Legendre transform of a convex piecewise-linear function \(f\) on the space \(V^*\) (dual to \(V\)) is a convex piecewise-linear function on some convex polyhedron in \(V\) (depending on \(f\)) and is equal to infinity on the exterior of this polyhedron. Conversely, every convex piecewise-linear function on any convex polyhedron (and hence every regular partition of any polyhedron) arises in this way. For example, the function \(\phi\) with \(\phi(x) = 0\) for \(x \in \gamma\) and \(\phi(x) = \infty\) for \(x \notin \gamma\), is the Legendre transform of the support function of \(\gamma\).

Lemma 6.1. Every partition of a polyhedron into simplices is regular.

Indeed, every strictly convex function on a polyhedron can be restricted to the vertices of the partition and extended by linearity to every simplex of the partition.

Definition 6.1. Suppose that we are given a convex piecewise-linear function \(\phi_\gamma\) on every cell \(\gamma\) of a \(\mathbb{P}\)-complex \(Y\). If, for every pair of cells \(\delta, \lambda \in Y\), the domains of linearity of the functions \(\phi_\delta\) and \(\phi_\lambda\) on the cell \(\delta \cap \lambda\) coincide, then the resulting partition \(\Phi\) of \(Y\) is said to be regular and the \(\mathbb{P}\)-complex \(Y^\Phi\) is called a regular refinement of \(Y\).

Corollary 6.1. Zero functions \(\psi_\gamma\) determine a regular partition \(\Psi\). In this case, \(Y^\Psi = Y\).
The partition $\Psi$ and the refinement $Y^\Psi$ of a $\mathbb{P}$-complex $Y$ are said to be tautological.

**Corollary 6.2.** Simplicial refinements of a $\mathbb{P}$-complex are regular.

Let $\gamma$ be a convex polyhedron in an oriented $n$-dimensional space $V$, $\Phi$ the regular partition of $\gamma$ determined by a function $\phi$, and $\Gamma = \{ v \oplus \phi(v) : v \in \gamma \}$ the graph of $\phi$. We denote the convex hull of this graph in the space $V \oplus \mathbb{R}$ by $\text{conv}(\Gamma)$. Let $\mathcal{X}$ be the dual fan of cones of the polyhedron $\text{conv}(\Gamma)$ in the space $V^* \oplus \mathbb{R}$ dual to $V \oplus \mathbb{R}$. The intersections of the cones in $\mathcal{X}$ with the affine subspace $V^* \oplus (-1)$ of the space $V^* \oplus \mathbb{R}$ form a $\mathbb{P}$-complex $\mathcal{X}^*$ in $V^* \oplus (-1)$. Since $\phi$ is convex, it follows that for $\delta \in \gamma^\Phi$ the set $\{(v, \phi(v)) : v \in \delta\}$ is a face of $\text{conv}(\Gamma)$. The intersection of $V^* \oplus (-1)$ and the cone dual to this face forms a cell $\delta^* \in \mathcal{X}^*$. We have $\dim \delta + \dim \delta^* = n$. The cells $\delta \in \gamma^\Phi$ and $\delta^* \in \mathcal{X}^*$ are said to be dual.

**Definition 6.2.** We embed the $\mathbb{P}$-complex $\mathcal{X}^*$ in the space $V^*$ using the identification $v^* \oplus (-1) \sim v^*$. The resulting $\mathbb{P}$-complex in $V^*$ is said to be dual to the $\mathbb{P}$-complex $\gamma^\Phi$ and is denoted by $\gamma^\Phi$.

**Corollary 6.3.** Let $\Psi$ (resp. $\Phi$) be the tautological (resp. an arbitrary regular) partition of $\gamma$. Then $(\gamma^\Phi)_\infty = \gamma^\Phi$.

The map $\delta \mapsto \delta^*$ is a one-to-one map of $\mathbb{P}$-complexes $\gamma^\Phi \to \gamma^\Phi$. Moreover, by construction, $\lambda \subset \delta \iff \lambda^* \supset \delta^*$. We endow dual cells with compatible orientations in the following way. The space $(V^*)_\delta^*$ is dual to $V/V_\delta$. Therefore the orientations of these spaces are compatible. Then we endow $V/V_\delta$ and $V_\delta$ with compatible orientations using the orientation of $V$. Then the orientations of the spaces $V_\delta$ and $(V^*)_\delta^*$ (or, equivalently, of the cells $\delta$ and $\delta^*$) are also compatible.

All assertions in the rest of § 6.1 are simple corollaries of definitions.

**Lemma 6.2.** For every cochain $r \in C^p(\gamma^\Phi, S)$ define a chain $\nu_{\Phi, r} \in C_{n-p}(\gamma^\Phi, S)$ by the formula $(\nu_{\Phi, r})(\delta^*) = r(\delta)$. Then the map $\nu_{\Phi, r} : C^*(\gamma^\Phi, S) \to C_*(\gamma^\Phi, S)$ is an isomorphism of complexes.

**Remark 6.2.** When $\phi$ is the Legendre transform of a convex piecewise-linear function $f$ on $V^*$ (see Remark 6.1), the $\mathbb{P}$-complex $\gamma^\Phi$ consists of domains of linearity of $f$.

**Corollary 6.4.** For all cells $\delta \in \gamma^\Phi$ and $\delta^* \in \gamma^\Phi$, the subspace $(V^*)_\delta^* \subset V^*$ is an orthogonal complement of the subspace $V_\delta \subset V$.

**Corollary 6.5.** The maximal and minimal dimensions of cells of the $\mathbb{P}$-complex $\gamma^\Phi$ are equal to $n$ and $n - \dim \gamma$ respectively. The cell of minimal dimension in $\gamma^\Phi$ is unique and is a vector subspace.

**Lemma 6.3.** Suppose that $\delta \in \gamma^\Phi$. Then the $\delta^*$-localization of the $\mathbb{P}$-complex $\gamma^\Phi$ coincides with the fan of dual cones of the polyhedron $\delta$. In particular, if $\psi$ is the zero function on $\gamma$, then the $\mathbb{P}$-complex $\gamma^\Phi$ coincides with the fan of dual cones of $\gamma$.

**Corollary 6.6.** For every $v \in \mathcal{U}(\gamma^\Phi)$, the isomorphism $\nu_{\Phi, r} : C^*(\gamma^\Phi, S) \to C_*(\gamma^\Phi, S)$ sends the operation of $v$-product of cochains in the $\mathbb{P}$-complex $\gamma^\Phi$ to the operation of $v$-intersection of chains in the $\mathbb{P}$-complex $\gamma^\Phi$. 
Let $\Phi$ be a regular partition of the $\mathbb{P}$-complex $Y$ determined by functions $\phi_\gamma: \gamma \to \mathbb{R}$, and let $Y^\Phi$ be the corresponding regular refinement of $Y$. Consider the restriction map of cochains $o_\gamma: C^*(Y^\Phi, S) \to C^*(\gamma^\Phi, S)$.

**Definition 6.3.** For every $r \in C^*(Y^\Phi, S)$ we put $R^\Phi_r(r) = \nu_{\Phi, \gamma} o_\gamma(r)$. The chain $R^\Phi_r(r) \in C_*(\gamma^\Phi, S)$ is called the $\gamma$-coordinate of the cochain $r$. We write $R^\Phi: C^*(Y^\Phi, S) \to \prod_{\gamma \in Y} C_*(\gamma^\Phi, S)$ for the map sending each cochain to the set of its $\gamma$-coordinates. Put $C_*(Y, S; \Phi) = R^\Phi(C^*(Y^\Phi, S))$.

**Corollary 6.7.** Define the boundary operator $\partial: C_*(Y, S; \Phi) \to C_*(Y, S; \Phi)$ by the formula $\partial: \{s_\gamma\} \mapsto \{\partial s_\gamma\}$. Then the map $R^\Phi: C^*(Y^\Phi, S) \to C_*(Y, S; \Phi)$ is an isomorphism of complexes.

**Corollary 6.8.** For every $u \in U(Y^\Phi)$, the isomorphism $R^\Phi$ sends the $u$-product of cochains to the set of $u$-intersections of their coordinates.

Given any $\mu \subset \gamma \in Y$, we write $o_{\gamma, \mu}: C^*(\gamma^\Phi, S) \to C^*(\mu^\Phi, S)$ for the isomorphism of restriction of cochains. Put $c_{\gamma, \mu} = \nu_{\Phi, \gamma} o_{\gamma, \mu} \nu_{\Phi, \gamma}^{-1}$. Then the map $C_{\gamma, \mu}: C_*(\gamma^\Phi, S) \to C_*(\mu^\Phi, S)$ is a homomorphism of chain complexes.

**Lemma 6.4.** A set of chains $\{s_\gamma \in C_*(\gamma^\Phi, S): \gamma \in Y\}$ belongs to $C_*(Y, S; \Phi)$ if and only if $c_{\delta, \mu}s_\delta = c_{\lambda, \mu}s_\lambda$ for all cells $\mu, \delta, \lambda \in Y$.

Let $\Psi$ (resp. $\Phi$) be the tautological (resp. an arbitrary regular) partition of $\gamma$ and let $\beta_\gamma: C_*(\gamma^\Phi, S) \to C_*(\gamma^\Psi, S)$ be the homomorphism of complexes in §5.1 (see Definition 5.3 and Corollary 6.3).

**Lemma 6.5.** Let

$$
u_{\Psi, \gamma}: C^*(\gamma^\Psi, S) \to C_*(\gamma^\Psi, S),$$

$$\nu_{\Phi, \gamma}: C^*(\gamma^\Phi, S) \to C_*(\gamma^\Phi, S)$$

be the isomorphisms of complexes in Lemma 6.2 and let $\text{res}_\gamma: C^*(\gamma^\Phi, S) \to C^*(\gamma^\Psi, S)$ be the homomorphism in the hypothesis of Theorem 2.3. Then the diagram

$$
\begin{array}{ccc}
C^*(\gamma^\Phi, S) & \xrightarrow{\nu_{\Phi, \gamma}} & C_*(\gamma^\Phi, S) \\
\text{res}_\gamma \downarrow & & \downarrow \beta_\gamma \\
C^*(\gamma^\Psi, S) & \xrightarrow{\nu_{\Psi, \gamma}} & C_*(\gamma^\Psi, S)
\end{array}
$$

is commutative.

**Corollary 6.9.** The homomorphisms $\beta_\gamma$ together determine a homomorphism of complexes $\beta: C_*(Y, S; \Phi) \to C_*(Y, S; \Psi)$. The diagram

$$
\begin{array}{ccc}
C^*(Y^\Phi, S) & \xrightarrow{R^\Phi} & C_*(Y, S; \Phi) \\
\text{res} \downarrow & & \downarrow \beta \\
C^*(Y, S) & \xrightarrow{R^\Phi} & C_*(Y, S; \Psi)
\end{array}
$$

is commutative.
6.2. Proof of Theorem 2.3. The proof has the following structure. Let \( \Phi \) be an arbitrary regular partition of the \( \mathbb{P} \)-complex \( Y, Y^\Phi \) the corresponding regular refinement of \( Y \), and \( r_p, r_q \) cocycles of the \( \mathbb{P} \)-complex \( Y^\Phi \). (Every simplicial refinement is regular by Lemma 6.2. The assumption of the simpliciality of \( Y^\Phi \) is not used in the proof.) We first show that for every \( \gamma \in Y \), Theorem 2.3 for the regular refinement \( \gamma^\Phi \) of the \( \mathbb{P} \)-complex \( \gamma^\Psi \) follows from the results in \( \S 5 \). We recall (see Corollary 6.3) that \( (\gamma^\Phi)_\infty = \gamma^\Psi \), where \( \Psi \) is the tautological partition of \( Y \).

Let \( R^\Phi_\gamma(r_p), R^\Phi_\gamma(r_q) \) be the \( \gamma \)-coordinates of the cocycles \( r_p, r_q \) of the \( \mathbb{P} \)-complex \( Y^\Phi \). Consider the homomorphism of complexes \( \beta_\gamma: C_*(\gamma^\Phi, S) \rightarrow C_*(\gamma^\Psi, S) \) in \( \S 5.1 \) (Definition 5.3). It will be proved below that if \( u \in U_{p,q}(Y) \cap U_{p,q}(Y^\Phi) \) is chosen sufficiently general and sufficiently close to 0, then for every \( \gamma \in Y \) we have

\[
(\beta_\gamma R^\Phi_\gamma(r_p)) \cap_u (\beta_\gamma R^\Phi_\gamma(r_q)) - \beta_\gamma (R^\Phi_\gamma(r_p) \cap_u R^\Phi_\gamma(r_q)) = \partial \Theta_\gamma,
\]

where \( \Theta_\gamma \in C_{n-p-q+1}(\gamma^\Psi, S) \) (we recall that the cycles \( R^\Phi_\gamma(r_p), R^\Phi_\gamma(r_q) \) are the \( \gamma \)-coordinates of the cocycles \( r_p, r_q \) and their dimensions are equal to \( n - p, n - q \)). By Lemma 6.5 and Corollary 6.9, the equalities (14) are actually a dual form of Theorem 2.3 for the \( \mathbb{P} \)-complex \( \gamma^\Psi \) and its regular partition \( \gamma^\Phi \). In accordance with the assertions in \( \S 6.1 \), to prove the theorem, it suffices to globalize (14), that is, to establish the following assertions.

1) \( \beta_\gamma R^\Phi_\gamma(r_p) \cap_u \beta_\gamma R^\Phi_\gamma(r_q) = R^\Phi_\gamma(\text{res}(r_p) \cup_u \text{res}(r_q)) \).
2) \( \beta_\gamma (R^\Phi_\gamma(r_p) \cap_u R^\Phi_\gamma(r_q)) = R^\Phi_\gamma \text{res}(r_p \cup_u r_q) \).
3) There is a cochain \( \Theta \in C^{p+q-1}(Y, S) \) such that \( \Theta_\gamma = R^\Phi_\gamma \Theta \).

The first two assertions follow directly from Lemma 6.5 and Corollaries 6.9, 6.8. The third assertion will be proved when we define the chain \( \Theta_\gamma \) (see Corollary 6.11).

To obtain a formula of type (14), we apply Corollary 5.4. Choose a point \( u \in V^* \), \( t > 0 \), \( v = tu \) and a smooth curve \( K \subset V^* \) connecting \( u \) and \( v \) in such a way that \( u \) is sufficiently close to 0 and \( K \) satisfies the following conditions for every \( \gamma \in Y \).

i) \( u \in U(\gamma^\Psi) \cap U(\gamma^\Phi) \).
ii) The hypotheses of Corollary 5.4 with \( \mathfrak{X} = \gamma^\Phi \) hold for \( K \).

Condition i) implies the following assertion.

**Lemma 6.6.** Suppose that \( \gamma \in Y, \delta, \lambda \in \gamma^\Phi \) and the polyhedron \( (v + \delta) \cap \lambda \) is conelike. If \( t \) is sufficiently large, then the cells \( \delta, \lambda \) are also conelike.

If \( t \) is sufficiently large, then \( v \in U(\gamma^\Psi) \cap U(\gamma^\Phi) \). Therefore, using Lemma 6.6, we obtain the following assertion.

**Corollary 6.10.** Let \( s_p, s_q \) be chains of the \( \mathbb{P} \)-complex \( \gamma^\Phi \). If \( t \) is sufficiently large, then \( s_p \cap_{\gamma^\Phi} s_q = \beta_{\gamma}(s_p) \cup_u \beta_{\gamma}(s_q) \).

Suppose that \( \gamma \in Y \) and \( t \) is sufficiently large. Put \( \{\kappa\}_\gamma = K \cap D_{p,q}(\gamma^\Phi) \) and let \( H_\kappa \) be the discriminant hyperplane of the \( \mathbb{P} \)-complex \( \gamma^\Phi \) such that \( H_\kappa \) contains the point \( \kappa \in \{\kappa\}_\gamma \). By Corollary 5.4 for the \( \mathbb{P} \)-complex \( \gamma^\Phi \), we obtain that

\[
R^\Phi_\gamma(r_p) \cap_v \beta_\gamma R^\Phi_\gamma(r_q) - R^\Phi_\gamma(r_p) \cap_u \beta_\gamma R^\Phi_\gamma(r_q) = \partial \sum_{\kappa \in \{\kappa\}_\gamma} \sum_{(\delta, \lambda) \in S^H_\kappa} \Xi_{R^\Phi_\gamma(r_p), R^\Phi_\gamma(r_q)},
\]

where

\[
\Xi_{R^\Phi_\gamma(r_p), R^\Phi_\gamma(r_q)} = \sum_{\kappa \in \{\kappa\}_\gamma} \sum_{(\delta, \lambda) \in S^H_\kappa} \Xi_{R^\Phi_\gamma(r_p), R^\Phi_\gamma(r_q)}.
\]
Rewrite the first summand on the left-hand side of (15) using Corollary 6.10:
\[ R^\phi_\gamma(r_p) \cap_{\ast} R^\phi_\gamma(r_q) = \beta_\gamma(R^\phi_\gamma(r_p)) \cap_{\ast} \beta_\gamma(R^\phi_\gamma(r_q)). \]

Rewrite the second summand on the left-hand side of (15) using Lemma 5.3:
\[ R^\phi_\gamma(r_p) \cap_{\ast} R^\phi_\gamma(r_q) = \beta_\gamma(R^\phi_\gamma(r_p)) \cap R^\phi_\gamma(r_q). \]

Thus (15) becomes a formula of type (14), where
\[
\Theta_\gamma = \sum_{\kappa \in \{\kappa\}_\gamma} \sum_{(\delta, \lambda) \in S^H_\kappa} \Xi_{\delta,\lambda,\pi} R^\phi_\gamma(r_p), R^\phi_\gamma(r_q). \tag{16}
\]

To complete the proof of Theorem 2.3, it remains to construct a cochain \( \Theta \in C^{p+q-1}(Y, S) \) with \( \gamma \)-coordinates \( \Theta_\gamma \in C_{n-p-q+1}(\gamma \Psi, S) \).

Suppose that \( p+q \leq \ell \) and let \( \delta \) (resp. \( \lambda \)) be a \( p \)-dimensional (resp. \( q \)-dimensional) cell of the \( \mathbb{P} \)-complex \( Y^\Phi \) satisfying \( \dim V_\delta \cap V_\lambda = 1 \). Assume that there is a \((p+q-1)\)-dimensional cell \( \mu \in Y \) with \( \mu \supset \delta \cup \lambda \), that is, \( \delta, \lambda, \mu \in \Phi^\phi \) (there can be only one such cell). If \( V^* \ni \phi \notin V_\delta \cap V_\lambda \), then the functional \( \phi \) determines an orientation of the 1-dimensional subspace \( V_\delta \cap V_\lambda \). Choose compatible orientations of the cells \( \delta, \lambda, \mu \) in such a way that the isomorphism
\[
V_\mu/(V_\delta \cap V_\lambda) = V_\delta/(V_\delta \cap V_\lambda) \oplus V_\lambda/(V_\delta \cap V_\lambda)
\]
is an isomorphism of oriented spaces. Given any \( r_p \in C^p(Y^\Phi, S) \), \( r_q \in C^q(Y^\Phi, S) \), we define a cochain \( \Theta^\delta,\lambda,\phi_{r_p,r_q} \in C^{p+q-1}(Y, S) \) by putting \( \Theta^\delta,\lambda,\phi_{r_p,r_q}(\mu) = r_p(\delta)r_q(\lambda) \) and \( \Theta^\delta,\lambda,\phi_{r_p,r_q}(\gamma) = 0 \) when \( \gamma \neq \mu \).

Remark 6.3. The definition of \( \Theta^\delta,\lambda,\phi_{r_p,r_q} \) is close to that of the special cochain \( \vartheta^\delta,\lambda,\gamma,\phi_{r_p,r_q} \) in §2 (Definition 2.7). They differ in the following respect. In the definition of \( \Theta^\delta,\lambda,\phi_{r_p,r_q} \), the cells \( \delta, \lambda \) and the cell \( \mu \) belong to distinct \( \mathbb{P} \)-complexes.

Lemma 6.7. Suppose that \( \phi \notin V_\delta \cap V_\lambda \). Then \( R^\psi_\gamma(\Theta^\delta,\lambda,\phi_{r_p,r_q}) = \Xi_{\delta,\lambda,\phi} R^\phi_\gamma(r_p), R^\phi_\gamma(r_q) \). In other words, the \( \gamma \)-coordinate of the cochain \( \Theta^\delta,\lambda,\phi_{r_p,r_q} \in C^{p+q-1}(Y, S) \) is equal to the chain \( \Xi_{\delta,\lambda,\phi} R^\phi_\gamma(r_p), R^\phi_\gamma(r_q) \in C_{n-p-q+1}(\gamma \Psi, S) \) for every \( \gamma \in Y \).

Proof. If \( \delta \cup \lambda \not\subseteq \gamma \), then the chains \( \Xi_{\delta,\lambda,\phi} R^\phi_\gamma(r_p), R^\phi_\gamma(r_q) \) are equal to zero by the construction of \( R^\phi_\gamma(\Theta^\delta,\lambda,\phi_{r_p,r_q}) \). Suppose that \( \delta \cup \lambda \subseteq \mu \subseteq \gamma \), where \( \dim \mu = p+q-1 \). Let \( \delta^*, \lambda^* \) be the cells of \( \gamma^\Phi \) dual to \( \delta, \lambda \) and let \( \mu^* \) be the cone of \( \gamma^\Psi \) dual to the face \( \mu \). Then, by definition, the \( \gamma \)-coordinate of \( \Theta^\delta,\lambda,\phi_{r_p,r_q} \) is the chain of the fan of cones \( \gamma^\Psi \) which is equal to \( r_p(\delta)r_q(\lambda) \) at the cone \( \mu^* \) and vanishes at the other cones of this fan. On the other hand, each of the cells \( \delta^*, \lambda^* \) contains a shift of \( \mu^* \). To complete the proof, use the definition of the chain \( \Xi_{\delta,\lambda,\phi} R^\phi_\gamma(r_p), R^\phi_\gamma(r_q) \) (see Definition 5.6). \( \square \)

We define the cochain \( \Theta \in C^{p+q-1}(Y, S) \) in the following way (the sets \( \{\kappa\}_\gamma \) and \( S^H_\kappa \) are defined in §6.2):
\[
\Theta = \sum_{\gamma \in Y} \sum_{\kappa \in \{\kappa\}_\gamma} \sum_{(\delta, \lambda) \in S^H_\kappa} \Theta^\delta,\lambda,\phi_{r_p,r_q}. \]
Then the assertion needed to complete the proof of the theorem follows from Lemma 6.7.

**Corollary 6.11.** For every $\gamma \in Y$, the chain $\Theta_\gamma$ in (16) is the $\gamma$-coordinate of the cochain $\Theta$.

### 6.3. Proof of Proposition 2.2

Suppose that the $\mathbb{P}$-complex $X$ is a regular refinement of the $\mathbb{P}$-complex $Y$ (the simpliciality of $X$ is not used in the proof). Recall that for every $s_p \in C^p(X, S)$, the value of the cochain $\text{res}(s_p) \in C^p(Y, S)$ at a $p$-dimensional cell $\lambda \in Y$ is defined by the formula $\text{res}(s_p)(\lambda) = \sum_{\delta \subset X, \delta \subset \lambda} s_p(\delta)$.

**Proposition 2.2** asserts that the homomorphism of complexes $\text{res}: C^* (X, S) \to C^* (Y, S)$ induces an isomorphism of cohomology $S$-modules $H^*(X, S) \to H^*(Y, S)$.

Consider the exact sequence of complexes

$$0 \to Z(Y, X) \to C^*(X, S) \xrightarrow{\text{res}} C^*(Y, S) \to 0.$$ 

Applying the corresponding cohomology exact sequence, we obtain that Proposition 2.2 is equivalent to the following assertion.

**Lemma 6.8.** All cohomology modules of the complex $Z(Y, X)$ are equal to zero.

**Proof.** Take any $r_p \in (Z(Y, X))^p$ with $dr_p = 0$. Assume that for every cell $\Delta \in Y$ of dimension $\geq p$ there is a cochain $s^\Delta_{p-1} \in (Z(Y, X))^{p-1}$ such that $(r_p - ds^\Delta_{p-1})(\delta) = 0$ for $\delta \subset \Delta$. Then, successively using the transformation $r_p \mapsto r_p - ds^\Delta_{p-1}$ for all cells $\Delta$, we obtain that $r_p = ds$ for $s \in (Z(Y, X))^{p-1}$. The existence of the chain $s^\Delta_{p-1}$ described above follows from the next assertion. $\square$

**Lemma 6.9.** If the $\mathbb{P}$-complex $Y$ consists of all faces of a convex polyhedron, then Lemma 6.8 holds.

**Proof.** Clearly, Proposition 2.2 holds in this case. Hence Lemma 6.8, which is equivalent to this proposition, also holds. $\square$

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