Combination Properties of Weakly Contracting Systems

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Consider an autonomous system with state $x \in \mathbb{R}^n$ and dynamics

$$\dot{x} = f(x).$$

We are interested in the limiting behaviour of such a system as $t \to \infty$, and how this behaviour is preserved under interconnection. We assume that the system evolves on a compact and simply-connected strictly forward-invariant set $\mathcal{X}$. I.e., any solution starting on the boundary of $\mathcal{X}$ at $t = 0$ is in the interior of $\mathcal{X}$ for $t > 0$. Define $J(x) = \frac{\partial f}{\partial x}$ as the system’s Jacobian, and $J_s(x) = \frac{1}{2}(J(x) + J(x)')$ as its symmetric part.

For a symmetric $n \times n$ matrix $H$, we define $\lambda_1(H) \geq \lambda_2(H) \ldots \lambda_n(H)$ as the eigenvalues of $H$ in non-increasing order. We also define a function giving the sum of the $k$ largest eigenvalues:

$$S_k(H) := \sum_{j=1}^{k} \lambda_j(H)$$

We are interested in systems satisfying the following property:

$$S_k(J_s(x)) < 0 \quad \forall x$$

Such a property allows one to bound the Hausdorff dimension of an attractor of the system, and to our knowledge was first investigated by Douady and Oesterlé, and subsequently studied by Smith and Leonov and colleagues (see, e.g., [3], [4]). Extensions include non-autonomous systems and the addition of a Lyapunov-like function to reduce conservatism.

In particular, if $k = 2$, i.e. $\lambda_1(J_s) + \lambda_2(J_s) < 0$, then all bounded trajectories converge to an equilibrium (not necessarily unique), i.e. the dimension of any attractor is zero. Following Leonov, we refer to this property as weak contraction and a system having this property is a weakly contracting system.

By contrast, a contracting system (in the identity metric) has the property that $S_1(J_s(x)) < 0$, i.e. the largest eigenvalue $\lambda_1(J_s(x))$ is negative.

Note that if it is known that the system has a unique equilibrium and trajectories are bounded, then the condition $\lambda_1 + \lambda_2 < 0$ guarantees convergence to that equilibrium, using a weaker condition than the full contraction condition $\lambda_1 < 0$. 
Remark 1. For computational methods, it is important to note that \( S_k(H) \) has a representation, due to Ky Fan, as

\[
S_k(H) = \max_{V \in \mathcal{V}_k} \text{Tr}(V H V')
\]

where \( \mathcal{V}_k \) is the set of all \( k \times n \) matrices with orthonormal rows, i.e. satisfying \( V V' \) is the identity matrix of size \( k \). From this construction, it is clear that \( S_k \) is convex, since it is the maximum of an infinite family of functions linear in \( M \). In fact, \( S_k(H) \) can be represented as a linear matrix inequality \([6, p. 238]\).

Convergence to an equilibrium is obviously unchanged by coordinate transformation, so we will also call a system weakly contracting if there exists a nonsingular matrix \( T \) such that \( S_2(T J(x) T^{-1} + (T J(x) T^{-1})') < 0 \) for all \( x \). When we need to be specific, we say the system is \textit{weakly contracting under} \( T \).

0.1 Example

A simple damped pendulum

\[
\ddot{y} + b \dot{y} + \sin y = 0
\]

naturally has a cylindrical phase space with two equilibria, however if the angle \( y \) is considered as a real number then there are equilibria at \( k\pi, k \in \mathbb{Z} \). Defining a state \( x = [y, \dot{y}]' \) we have the Jacobian

\[
J = \begin{bmatrix}
0 & 1 \\
-\cos(x_1) & -b
\end{bmatrix}
\]

and symmetric part:

\[
J_s = \frac{1}{2} \begin{bmatrix}
0 & \frac{1}{2}(1 - \cos(x_1)) \\
\frac{1}{2}(1 - \cos(x_1)) & -b
\end{bmatrix}
\]

For a two-dimensional system, the condition is trivially given by the trace of \( J_s \):

\[
\lambda_1 + \lambda_2 = -b < 0.
\]

Hence every initial condition converges to an equilibrium. Notice that an initial condition that starts on one of the unstable equilibria stays there, and any other converges to a stable equilibrium.

For planar systems, weak contraction is related to the Bendixson criterion for non-existence of limit cycles, but of course weak contraction extends to higher-order systems.

1 Combination Properties

In the spirit of \([3]\), we remark that it is straightforward to show that certain combinations of contracting and weakly contracting systems preserve the weak contraction property, and therefore the have the property that all solutions converge to an equilibrium. We assume that the combination does not result in the system leaving the set on which the weak contraction condition holds.
1.1 Parallel Interconnection

**Theorem 1.** Suppose two systems \( f_a \) and \( f_b \) are weakly contracting under the same \( T \), then so is the system

\[
\dot{x} = \alpha f_a(x) + \beta f_b(x)
\]

where \( \alpha, \beta \) are non-negative constants, and \( \alpha + \beta > 0 \).

**Proof.** The symmetric part of the Jacobian is

\[
J_{ab}^s = \alpha J_a^s + \beta J_b^s
\]

Take \( \gamma = \alpha + \beta \) then clearly \( J_{ab}^s < 0 \iff J_{ab}^s / \gamma < 0 \). But the latter is a convex combination of \( J_a^s \) and \( J_b^s \) both of which are negative by assumption, and \( S_2 \) is a convex function, \( J_{ab}^s < 0 \).

By induction, this can be extended to non-negative combinations of any number of weakly contracting systems.

1.2 Skew-Symmetric Feedback Interconnection

Consider a skew-symmetric interconnection of a weakly contracting system \( f_a \) and a contracting system \( f_b \), with the Jacobian

\[
J_{ab}^s(x) = \begin{bmatrix}
J_a^s(x) & G(x) \\
-G(x)' & J_b^s(x)
\end{bmatrix}
\]

for arbitrary \( G(x) \). We define \( \lambda_i^a = \lambda_i(J_a^s) \) and likewise for \( \lambda_i^b \).

**Theorem 2.** If \( \lambda_1^a + \lambda_1^b < 0 \) then the feedback interconnection is weakly contracting.

**Proof.** The symmetric part of the interconnection’s Jacobian is

\[
J_{ab}^s(x) = \begin{bmatrix}
J_a^s(x) & 0 \\
0 & J_b^s(x)
\end{bmatrix}
\]

and so each eigenvalue \( \lambda_i^a, \lambda_j^b, i = 1, 2, ..., j = 1, 2, ... \) of \( J_a^s(x), J_b^s(x) \) is an eigenvalue of \( J_{ab}^s \).

Either \( \lambda_1^b \leq \lambda_2^a \) or \( \lambda_1^b > \lambda_2^a \). In the first case the combined system has the same \( \lambda_1 + \lambda_2 \) as system \( a \). In the second case, \( \lambda_1^b \) becomes the new \( \lambda_2^{ab} \) and so \( \lambda_1^a + \lambda_1^b < 0 \) implies weak contraction.

By induction, this can be extended to non-negative combinations of any number of systems. This can easily be extended to more complex feedback interconnections using state transformations, following [7].

It is clear from the proof that a less restrictive test (although possibly less convenient) would be that the sum of the two largest of \( \lambda_1^a, \lambda_2^a, \lambda_1^b, \lambda_2^b \) should be less than zero.
1.3 Hierarchical Interconnections

Consider a hierarchical connection of systems with the Jacobian

\[ J^{ab}(x) = \begin{bmatrix} J^a(x) & G(x) \\ 0 & J^b(x) \end{bmatrix} \]

for arbitrary \( G(x) \), with either system \( a \) or \( b \) weakly contracting, and the other contracting.

**Theorem 3.** If \( \lambda_a^1 + \lambda_b^1 < 0 \) then the hierarchical interconnection is weakly contracting.

**Proof.** Let \( n \) and \( m \) be the state dimensions of Systems \( a \) and \( b \), respectively. Consider the family of transformations

\[ T = \begin{bmatrix} I_n & 0 \\ 0 & \epsilon I_m \end{bmatrix} \]

where \( \epsilon > 0 \) is a parameter, and \( I_r \) is the \( r \)-dimensional identity matrix. The symmetric part of the interconnection’s Jacobian is

\[ J^{ab}_s(x) = \begin{bmatrix} J^a_s(x) & \frac{\epsilon}{2} G(x) \\ \frac{\epsilon}{2} G(x)^t & J^b_s(x) \end{bmatrix} \]

and clearly for each \( x \) there is a sufficiently small \( \epsilon \) such that the eigenvalues of \( J^{ab}_s \) can be brought arbitrarily close to the eigenvalues of the matrix

\[ \begin{bmatrix} J^a_s(x) & 0 \\ 0 & J^b_s(x) \end{bmatrix} \]

and since \( \mathcal{X} \) is compact, one can find an \( \epsilon \) that works uniformly for \( x \in \mathcal{X} \). The remainder of the proof follows exactly that of Theorem 2. \( \square \)

The remarks following the skew-symmetric interconnection also apply here.

2 Remarks

Just as contraction can be understood in terms of differential line elements and Riemannian distances, weak contraction can be understood in terms of shrinking of differential planar areas. A proof of the non-existence of limit cycles follows from this reasoning: assume a closed curve \( \gamma \) is preserved under the flow of the system, then consider the two-dimensional submanifold of smallest area having \( \gamma \) as a boundary (like a film of soapy water on a loop for blowing bubbles). Since this area must shrink under the flow of the system, this contradicts the preservation of \( \gamma \). The extension to nonlinear metrics and a Riemannian area integral is natural.

The non-existence of more complex (chaotic) behaviour follows from the fact that the weak contraction conditions are open – i.e. the strict inequality means
they are preserved under small perturbations of \( f \) – and the Pugh closing lemma which states, roughly speaking, that any chaotic attractor can be perturbed by a small amount to result in a closed cycle.

It is interesting to note that transverse contracting systems satisfy \( \lambda_1 + \lambda_2 < 0 \) for some choice of metric. Transverse contracting systems have the property that all solutions converge to the same limit set, which is either a unique equilibrium or a unique limit cycle.

This may seem to contradict the above results, but the key point is that convergence to a limit cycle can only occur if the domain of contraction is not a simply connected set. In fact, if a system is transverse contracting on a simply connected set, then all solutions converge to a unique equilibrium.

In biology and robotics (e.g. locomotion), natural behaviours of dynamic systems include convergence to an equilibrium (unique or not) and oscillation, so weak contraction and transverse contraction seem to be useful frameworks for studying interconnection of biological systems. Synchronisation is a special case of weak contraction in which \( \lambda_1 = 0 \) and the rate of convergence to a synchronised state is given by \( \lambda_2 \).

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