On a relation between the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ and a Schur–Weyl representation of the infinite symmetric group

N. V. Tsilevich* A. M. Vershik*

March 7, 2014

Abstract

We prove that there is a natural grading-preserving isomorphism of $\mathfrak{sl}_2$-modules between the basic module of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ (with the homogeneous grading) and a Schur–Weyl module of the infinite symmetric group $\mathfrak{S}_\infty$ with a grading defined through the combinatorial notion of the major index of a Young tableau, and study the properties of this isomorphism. The results reveal new and deep interrelations between the representation theory of $\widehat{\mathfrak{sl}_2}$ and the Virasoro algebra on the one hand, and the representation theory of $\mathfrak{S}_\infty$ and the related combinatorics on the other hand.

1 Introduction

In this paper we reveal some new and deep interrelations between two well developed branches of representation theory: the representation theory of the infinite symmetric group and that of the affine Lie and Virasoro algebras. Our starting point was the analogy observed in [10] between the decomposition [3] of a so-called Schur–Weyl representation of the infinite symmetric group $\mathfrak{S}_\infty$...
into irreducibles,
\[ \mathcal{X} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k, \]
and the decomposition (17) of the basic representation of the affine Lie algebra \( \widehat{\mathfrak{sl}_2} \) into irreducible representations of the Virasoro algebra Vir,
\[ \mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2); \]
in these formulas \( \Pi_k \) is an irreducible representation of \( \mathfrak{S}_N \), \( L(1, k^2) \) is an irreducible representation of Vir, and \( M_{2k+1} \) is the \((2k+1)\)-dimensional irreducible representation of \( \mathfrak{sl}_2 \); in both cases, the operator algebras generated by the actions of \( \mathfrak{S}_N \) or Vir and \( \mathfrak{sl}_2 \) are mutual commutants. This analogy suggested that there should be a natural action of Vir in the \( \mathfrak{S}_N \)-module \( \mathcal{X} \), or, equivalently, a natural action of \( \mathfrak{S}_N \) in the \( \widehat{\mathfrak{sl}_2} \)-module \( \mathcal{H}_0 \). The aim of this paper is to describe and study the underlying natural isomorphism of \( \mathfrak{sl}_2 \)-modules.

For this, we use the result of B. Feigin and E. Feigin [2] that the level 1 irreducible highest weight representations of \( \widehat{\mathfrak{sl}_2} \) can be realized as certain inductive limits of tensor powers \( (\mathbb{C}^2)^{\otimes N} \) of the two-dimensional irreducible representation of \( \mathfrak{sl}_2 \). The construction of [2] is based on the notion of the fusion product of representations, whose main ingredient is, in turn, a special grading in the space \( (\mathbb{C}^2)^{\otimes N} \). A key observation underlying the results of this paper is that the fusion product under consideration can be realized in an \( \mathfrak{S}_N \)-module so that this special grading essentially coincides with a well-known combinatorial characteristic of Young tableaux called the major index (see Sec. 4 and Theorem 1). Thus our results provide, in particular, a kind of combinatorial description of the fusion product and show that the combinatorial notion of the major index of a Young tableau has new and rich representation-theoretic meaning. For instance, Corollary 3 in Sec. 7 shows that the so-called stable major indices of infinite Young tableaux are the eigenvalues of the Virasoro \( L_0 \) operator, the Gelfand–Tsetlin basis of the Schur–Weyl module being its eigenbasis.

The paper is organized as follows. In Secs. 2 and 3 we briefly reproduce the necessary background on the notion of Schur–Weyl duality and the fusion product of representations, respectively. Section 4 contains our finite-dimensional Theorem 1 with combinatorial interpretation of the fusion
product grading via the major index of Young tableaux. In Sec. 5 we prove its infinite-dimensional version, our main Theorem 2, which states that the grading-preserving isomorphism of $\mathfrak{sl}_2$-modules constructed in Theorem 1 extends, through the corresponding inductive limits, to a grading-preserving isomorphism of $\mathfrak{sl}_2$-modules between the basic $\mathfrak{sl}_2$-module $L_{0,1}$ and the Schur–Weyl module $\mathcal{X}$. The remaining part of the paper is devoted to studying the key isomorphism in more detail. With this aim, in Sec. 6 we describe the Fock space realizations of the involved representations of $\mathfrak{sl}_2$ and Vir, and then, in Sec. 7, prove some properties of our isomorphism (Theorem 3).

For definiteness, in what follows we consider only the even case $N = 2n$. The odd case can be treated in exactly the same way; instead of the basic representation $L_{0,1}$, it leads to the other level 1 highest weight representation $L_{1,1}$ of $\hat{\mathfrak{sl}}_2$.

Acknowledgments. The authors are grateful to Igor Frenkel for many inspiring discussions, and also to Boris Feigin and Evgeny Feigin for introducing into the notion of fusion product.

2 Infinite-dimensional Schur–Weyl duality

In [10], the notion of infinite-dimensional Schur–Weyl duality was introduced. Namely, starting from the classical Schur–Weyl duality

$$((\mathbb{C}^2)^\otimes N) = \sum_{k=0}^{n} M_{2k+1} \otimes \pi_k,$$

where $\pi_k$ is the irreducible representation of the symmetric group $\mathfrak{S}_N$ corresponding to the two-row Young diagram $\lambda^{(k)} = (n + k, n - k)$ and $M_{2k+1}$ is the $(2k + 1)$-dimensional irreducible representation of the special linear group $SL(2, \mathbb{C})$, we consider so-called Schur–Weyl embeddings $((\mathbb{C}^2)^\otimes N) \hookrightarrow ((\mathbb{C}^2)^\otimes (N+2))$ that preserve this Schur–Weyl structure, i.e., respect both the actions of $SL(2, \mathbb{C})$ and $\mathfrak{S}_N$, and the inductive limits of chains

$$((\mathbb{C}^2)^\otimes 0) \hookrightarrow ((\mathbb{C}^2)^\otimes 2) \hookrightarrow ((\mathbb{C}^2)^\otimes 4) \hookrightarrow \ldots$$

Such an inductive limit has the form

$$\mathcal{X} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k,$$
where $\Pi_k$ is an irreducible representation of the infinite symmetric group $\mathfrak{S}_N$ (an inductive limit of the sequence of irreducible representations $\pi_k$ of $\mathfrak{S}_N$); the operator algebras generated by the actions of $\mathfrak{S}_N$ and $SL(2, \mathbb{C})$ are mutual commutants.

3 Fusion product

The notion of the fusion product of finite-dimensional representations of $\mathfrak{sl}_2$ was introduced in [3]. Given an $\mathfrak{sl}_2$-representation $\rho$ and $z \in \mathbb{C}$, let $\rho(z)$ be the evaluation representation of the polynomial current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, defined as $(x \otimes t^i)v = z^i \cdot xv$ for $x \in \mathfrak{sl}_2$, $v \in \rho$. Now, given a collection $\rho_1, \ldots, \rho_N$ of irreducible representations of $\mathfrak{sl}_2$ with lowest weight vectors $v_1, \ldots, v_N$, and a collection $z_1, \ldots, z_N$ of pairwise distinct complex numbers, we consider the tensor product of the corresponding evaluation representations: $V_N = \rho_1(z_1) \otimes \ldots \otimes \rho_n(z_N)$. The crucial step is introducing a special grading in $V_N$ by setting

$$
V_N^{(m)} = \mathcal{U}(m)(e \otimes \mathbb{C}[t])(v_1 \otimes \ldots \otimes v_n) \subset V_N,
$$

where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the raising operator in $\mathfrak{sl}_2$ and $\mathcal{U}(m)$ is spanned by homogeneous elements of degree $m$ in $t$. In other words, $V_N^{(m)}$ is spanned by the monomials of the form

$$
e_{i_1} \ldots e_{i_k}, \quad i_1 + \ldots + i_k = m,
$$

where $e_j = e \otimes t^j$. Then we consider the corresponding filtration on $V_N$:

$$
V_N^{(\leq m)} = \bigoplus_{k \leq m} V_N^{(k)}.
$$

The fusion product of $\rho_1, \ldots, \rho_N$ is the graded representation with respect to the above filtration:

$$
V_N^* = \text{gr } V_N = V_N^{(\leq 0)} \oplus V_N^{(\leq 1)}/V_N^{(\leq 0)} \oplus V_N^{(\leq 2)}/V_N^{(\leq 1)} \oplus \ldots.
$$

The space $V_N^*[k] = V_N^{(\leq k)}/V_N^{(\leq k-1)}$ is the subspace of elements of degree $k$, and elements of the form $x \otimes t^l \in \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ send $V_N^*[k]$ to $V_N^*[k + l]$. The degree of an element with respect to this grading will be denoted by $\deg$. 

\[4\]
It is proved in [3] that $V_N^*$ is an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n)$-module that does not depend on $z_1, \ldots, z_N$ provided that they are pairwise distinct. Moreover, $V_N^*$ is isomorphic to $\rho_1 \otimes \ldots \otimes \rho_N$ as an $\mathfrak{sl}_2$-module.

We apply this construction to the case where $\rho_1 = \ldots = \rho_N = M_2$ with $M_2 = \mathbb{C}^2$ being the two-dimensional irreducible representation of $\mathfrak{sl}_2$ with the lowest weight vector $v_0$. In this case,

$$V_N^* \simeq (\mathbb{C}^2)^{\otimes N} \quad \text{as an $\mathfrak{sl}_2$-module.}$$

We equip $V_N^*$ with the inner product such that the corresponding representation of $\mathfrak{sl}_2$ is unitary.

Consider the decomposition of $V_N^*$ into irreducible $\mathfrak{sl}_2$-modules:

$$V_N^* = \bigoplus_{k=0}^n M_{2k+1} \otimes \mathcal{M}_k.$$

By the classical Schur–Weyl duality (11), we know that the multiplicity space $\mathcal{M}_k$ is the space of the irreducible representation $\pi_k$ of $\mathfrak{S}_N$. On the other hand, it inherits the grading from $V_N^*$:

$$\mathcal{M}_k = \bigoplus_{i \geq 0} \mathcal{M}_k[i], \quad (5)$$

where $\mathcal{M}_k[i] = \mathcal{M}_k \cap V[i]$. Consider the corresponding $q$-character

$$\text{ch}_q \mathcal{M}_k = \sum_{i \geq 0} q^i \dim \mathcal{M}_k[i].$$

It was proved in [5] that

$$\text{ch}_q \mathcal{M}_k = q^{N(N-1)/2} \cdot K_{\lambda_k,1^N}(1/q), \quad (6)$$

where $K_{\lambda,\mu}$ is the Kostka–Foulkes polynomial (see [7 Sec. III.6]).

4 Major index and the tableaux realization of the fusion product

Let $T_N$ be the set of all standard Young tableaux of length $N$ with at most two rows.
As was proved in [6],

\[ K_{\lambda,1^N}(q) = \sum_{\tau \in [\lambda]} q^{c(\tau)}, \tag{7} \]

where \([\lambda]\) is the set of standard Young tableaux of shape \(\lambda\) and \(c(\tau)\) is the so-called charge of a tableau \(\tau \in T_N\), defined as the sum of \(i \leq N - 1\) such that in \(\tau\) the element \(i + 1\) lies to the right of \(i\) (see [7]).

It is more convenient for our purposes to use another statistic on Young tableaux, namely, the major index, defined as follows (see [9, Sec. 7.19]):

\[ \text{maj}(\tau) = \sum_{i \in \text{des}(\tau)} i, \]

where, for \(\tau \in T_N\),

\[ \text{des}(\tau) = \{ i \leq N - 1 : \text{the element } i + 1 \text{ in } \tau \text{ lies lower than } i \} \]

is the descent set of \(\tau\). Obviously, for \(\tau \in T_N\) we have \(\text{maj}(\tau) = \frac{N(N-1)}{2} - c(\tau)\).

Then it follows from (6) and (7) that

\[ \dim M_k[i] = \#\{ \tau \in [(n+k,n-k)] : \text{maj}(\tau) = i \}. \tag{8} \]

Denote by \(\mathcal{X}_N\) the space \((\mathbb{C}^2)^{\otimes N} = \sum_{k=0}^n M_{2k+1} \otimes \pi_k\) (see [1]) in which the irreducible representation \(\pi_k\) of \(\mathfrak{S}_N\) is realized in the space spanned by the standard Young tableaux of shape \((n+k,n-k)\) equipped with the standard inner product under which the representation is unitary. Note that this is an \(\mathfrak{sl}_2\)-module endowed additionally with the grading \(\text{maj}\).

**Theorem 1.** There is a grading-preserving unitary isomorphism of the fusion product \(V_N^*\) (with the grading \(\tilde{\deg}\)) and the space \(\mathcal{X}_N\) (with the grading \(\text{maj}\)) as \(\mathfrak{sl}_2\)-modules such that the multiplicity space \(M_k\) is spanned by the standard Young tableaux \(\tau\) of shape \((n+k,n-k)\) (and hence \(M_k[i]\) is spanned by \(\tau\) with \(\text{maj}(\tau) = i\)).

**Proof.** Follows from the fact that the fusion product \(V_N^*\) is isomorphic to \((\mathbb{C}^2)^{\otimes N}\) as an \(\mathfrak{sl}_2\)-module and equation (8). \(\square\)

**Remark 1.** Observe that the isomorphism from Theorem 1 is not unique.
Remark 2. The isomorphism from Theorem 1 determines an action of the symmetric group $\mathfrak{S}_N$ on the space $V_N^*$. It does not coincide with the original action of $\mathfrak{S}_N$ on $\mathbb{C}^{\otimes N}$.

Given $\tau \in T_N$, let $k(\tau)$ be half the difference of the lengths of the first and the second row of $\tau$. Then, in view of the Schur–Weyl duality, we can write

$$V_N^* = \bigoplus_{\tau \in T_N} M_{2k(\tau)+1}(\tau),$$

where $M_{2k(\tau)+1}(\tau)$ is the $(2k(\tau)+1)$-dimensional $\mathfrak{sl}_2$-module parametrized by $\tau$ as an element of the multiplicity space.

5 Embeddings and the limit

It is proved in [2] that there is an embedding

$$j_N : V_N^* \to V_{N+2}^*$$

equivariant with respect to the action of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-n})$, and the corresponding inductive limit

$$\mathcal{V} = \lim(V_N, j_N)$$

is isomorphic to the basic representation $L_{0,1}$ of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. This embedding satisfies

$$\tilde{\deg}(j_N x) = \tilde{\deg}(x) - (N + 1). \quad (9)$$

Now consider the following natural embedding $i_N : T_N \to T_{N+2}$: given a standard Young tableau $\tau$ of length $N$, its image $i_N(\tau)$ is the standard Young tableau of length $N + 2$ obtained from $\tau$ by adding the element $N + 1$ to the first row and the element $N + 2$ to the second row.

Note that $i_N$ is, obviously, a Schur–Weyl embedding in the sense of [10] (see Sec. 2). Let $\mathcal{X}$ be the corresponding inductive limit (3). Then $\Pi_k$ is the discrete representation of the infinite symmetric group $\mathfrak{S}_{\infty}$ associated with the tableau

$$\tau_k = \begin{array}{ccccccc}
1 & 2 & \ldots & 2k & 2k + 1 & 2k + 3 & \ldots \\
2k + 2 & 2k + 4 & \ldots 
\end{array}, \quad (10)$$
which can be realized in the space (which, by abuse of notation, will also be denoted by \( \Pi_k \)) spanned by the infinite two-row Young tableaux tail-equivalent to \( \tau_k \) (we denote the set of such tableaux by \( T_k \)). In what follows, the tableaux \( \tau_k \) will be called principal.

Obviously,
\[
\text{maj}(i_N(\tau)) = \text{maj}(\tau) + (N + 1). \tag{11}
\]
Given \( N = 2n \) and \( \tau \in T_N \), denote \( r_N(\tau) = n^2 - \text{maj}(\tau) \). Then \( r_{N+2}(i_N(\tau)) = r_N(\tau) \), so that we have a well-defined grading on the space \( \Pi = \bigoplus_{k=0}^{\infty} \Pi_k \):
\[
r(\tau) = \lim_{n \to \infty} r_{2n}([\tau]_{2n}) = \lim_{n \to \infty} (n^2 - \text{maj}([\tau]_{2n})), \tag{12}
\]
where \([\tau]_l\) is the initial part of length \( l \) of the infinite tableau \( \tau \). We will call \( r(\tau) \) the stable major index of \( \tau \). Obviously, \( r(\tau_k) = k^2 \).

Our main theorem is the following.

**Theorem 2.** The grading-preserving unitary isomorphism of \( \mathfrak{sl}_2 \)-modules described in Theorem 1 extends to a grading-preserving unitary isomorphism of \( \mathfrak{sl}_2 \)-modules between the spaces \( V \) and \( X \):
\[
V \simeq X = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k. \tag{13}
\]

Thus in the Schur–Weyl module \( X \), which is an \( \mathfrak{sl}_2 \)-module and an \( \mathfrak{S}_N \)-module, there is also a structure of the basic \( \widehat{\mathfrak{sl}}_2 \)-module \( L_{0,1} \). The corresponding grading is given by the stable major index \([12]\), that is, for \( w = x \otimes \tau \in M_{2k+1} \otimes \Pi_k \), we have \( \deg w = r(\tau) \).

**Remark.** As mentioned in the introduction, we consider in detail only the even case just for simplicity of notation. Considering instead of \([2]\) the chain \((\mathbb{C}^2)^{\otimes 1} \to (\mathbb{C}^2)^{\otimes 3} \to (\mathbb{C}^2)^{\otimes 5} \to \ldots\) and reproducing exactly the same arguments, we will obtain a grading-preserving isomorphism of the corresponding Schur–Weyl representation with the other level 1 highest weight representation \( L_{1,1} \) of \( \widehat{\mathfrak{sl}}_2 \).

**Proof.** Since we are now considering \( \mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}] \) instead of \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \), we should slightly modify the previous constructions to take the minus sign into account. Namely, instead of \([5]\) we now have \( \mathcal{M}_k = \bigoplus_{i \geq 0} \mathcal{M}_k[-i] \), and the isomorphism of Theorem 1 identifies \( \mathcal{M}_k[-i] \) with the space spanned by
the tableaux $\tau$ of shape $(n + k, n - k)$ such that $\text{maj}(\tau) = i$. Denote this isomorphism between $V_N^*$ and $X_N$ by $\rho_N$. Observe that the only conditions we impose on $\rho_N$ are as follows: (a) $\rho_N$ is a unitary isomorphism of $\mathfrak{sl}_2$-modules and (b) $\rho_N \circ \text{deg} = -\text{maj}.$

Now, to prove Theorem 2, we need to show that we can choose a sequence of isomorphisms $\rho_N$ such that the diagram

$$
\begin{array}{ccc}
V_N^* & \xrightarrow{\rho_N} & X_N \\
\downarrow j_N & & \downarrow i_N \\
V_{N+2}^* & \xrightarrow{\rho_{N+2}} & X_{N+2}
\end{array}
$$

is commutative for all $N$. We use induction on $N$. The base being obvious, assume that we have already constructed $\rho_N$, and let us construct $\rho_{N+2}$.

We have $V_{N+2}^* = j_N(V_N^*) \oplus (j_N(V_N^*))^\perp$. On the first subspace, we set $\rho_{N+2}(x) := i_N(\rho_N(j_N^{-1}(x)))$. On the second one, we define it in an arbitrary way to satisfy the desired conditions (a) and (b). The fact that this definition is correct and provides us with a desired isomorphism between $V_{N+2}^*$ and $X_{N+2}$ follows from (9) and (11).

**Corollary 1.** The embedding $j_N : V_N^* \to V_{N+2}^*$ is equivariant with respect to the action of the symmetric group $\mathfrak{S}_N$ (see Remark 2 after Theorem 1). Thus the limit space $\mathcal{V}$, isomorphic to $L_{0,1}$, has the structure of a representation of the infinite symmetric group $\mathfrak{S}_\infty$.

Let $\omega_{-2k}$ be the lowest vector in $M_{2k+1}$. Then a natural basis of $\mathcal{V}$ is $\{e^m_0 \omega_{-2k} \otimes \tau : m = 0, 1, \ldots, 2k, \tau \in \mathcal{T}_k\}$. Denoting $\mathcal{V}_k = M_{2k+1} \otimes \Pi_k$ and $\mathcal{V}_k[0] = \{v \in \mathcal{V}_k : h_0 v = 0\}$, we have $\mathcal{V}_k[0] = e_0^k \omega_{-2k} \otimes \Pi_k$, so that we may identify $\mathcal{V}_k[0]$ with $\Pi_k$ by the correspondence

$$
c(t) \cdot e_0^k \omega_{-2k} \otimes t \leftrightarrow t, \quad t \in \Pi_k,
$$

where $c(t)$ is a normalizing constant. Thus we have

$$
\mathcal{V}[0] := \{v \in \mathcal{V} : h_0 v = 0\} \longleftrightarrow \Pi = \bigoplus_{k=0}^{\infty} \Pi_k, \quad (14)
$$

where $\Pi$ is the space spanned by all infinite two-row Young tableaux with “correct” tail behavior, i.e., tail-equivalent to $\tau_k$ (see (10)) for some $k$.

Our aim in the remaining part of the paper is to study the isomorphism from Theorem 2 in more detail. For this, we first describe the Fock space realization of the basic $\mathfrak{sl}_2$-module and the fusion product.
6 The Fock space

6.1 The Fock space and the level 1 highest weight representations of \( \widehat{sl}_2 \)

Let \( \mathcal{F} \) be the fermionic Fock space constructed as the infinite wedge space over the linear space with basis \( \{ u_k \}_{k \in \mathbb{Z}} \cup \{ v_k \}_{k \in \mathbb{Z}} \). That is, \( \mathcal{F} \) is spanned by the semi-infinite forms

\[
u_{i_1} \wedge \ldots \wedge u_{i_k} \wedge v_{j_1} \wedge \ldots \wedge v_{j_l} \wedge u_{N} \wedge v_{N} \wedge u_{N-1} \wedge v_{N-1} \wedge \ldots,
\]

\( N \in \mathbb{Z}, \ i_1 > \ldots > i_k > N, \ j_1 > \ldots > j_l > N, \)

and is equipped with the inner product in which such monomials are orthonormal. Let \( \phi_k \) be the exterior multiplication by \( u_k \) and \( \psi_k \) be the exterior multiplication by \( v_k \), and denote by \( \phi_k^* \), \( \psi_k^* \) the corresponding adjoint operators. Then this family of operators satisfies the canonical anticommutation relations (CAR):

\[
\phi_k \phi_k^* + \phi_k^* \phi_k = 1, \quad \psi_k \psi_k^* + \psi_k^* \psi_k = 1,
\]

all the other anticommutators being zero.

Consider the generating functions

\[
\phi(z) = \sum_{i \in \mathbb{Z}} \phi_i z^{-(i+1)}, \quad \psi(z) = \sum_{i \in \mathbb{Z}} \psi_i z^{-(i+1)}, \quad \phi^*(z) = \sum_{i \in \mathbb{Z}} \phi_i^* z^i, \quad \psi^*(z) = \sum_{i \in \mathbb{Z}} \psi_i^* z^i.
\]

Let \( a_\phi^0 \) and \( a_\psi^0 \) be the systems of bosons constructed from the fermions \( \{ \phi_k \} \) and \( \{ \psi_k \} \), respectively:

\[
a_\phi^0 = \sum_{n=1}^{\infty} \phi_n \phi_n^* - \sum_{n=0}^{\infty} \phi_n^* \phi_{-n}, \quad a_\phi^0 = \sum_{k \in \mathbb{Z}} \phi_k \phi_k^*, \quad n \neq 0,
\]

and similarly for \( a_\psi \). They satisfy the canonical commutation relations (CCR)

\[
[a_\phi^0, a_m^\phi] = n \delta_{n, -m}, \quad [a_n^\psi, a_m^\psi] = n \delta_{n, -m}, \quad (15)
\]
i.e., form a representation of the Heisenberg algebra \( \mathfrak{A} \). Denote

\[
\phi(z) = \sum_{n \in \mathbb{Z}} a_\phi^0 z^{-(n+1)}, \quad \psi(z) = \sum_{n \in \mathbb{Z}} a_n^\psi z^{-(n+1)}.
\]
Let $V$ be the operator in $\mathcal{F}$ that shifts the indices by 1:

$$V(w_{i_1} \wedge w_{i_2} \wedge \ldots) = V_0(w_{i_1}) \wedge V_0(w_{i_2}) \wedge \ldots, \quad V_0(u_i) = u_{i+1}, \quad V_0(v_i) = v_{i-1}.$$ 

The vacuum vector in $\mathcal{F}$ is $\Omega = u_{-1} \wedge v_{-1} \wedge u_{-2} \wedge v_{-2} \wedge \ldots$. We also consider the family of vectors

$$\Omega_0 = \Omega, \quad \Omega_{2n} = V^{-n}\Omega_0, \quad n \in \mathbb{Z}.$$ 

In the space $\mathcal{F}$ we have a canonical representation of the affine Lie algebra $\widehat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, which is given by the following formulas. Given $x \in sl_2$, denote $X(z) = \sum_{n \in \mathbb{Z}} x_n z^{-(n+1)}$. Then

$$E(z) = \psi(z)\phi^*(z), \quad F(z) = \phi(z)\psi^*(z),$$

$$h_n = a^\psi_{-n} - a^\phi_{-n}, \quad d = \frac{h_0^2}{2} + \sum_{n=1}^{\infty} h_{-n}h_n, \quad c = 1.$$ 

We have

$$\mathcal{F} = \mathcal{H}_0 \otimes \mathcal{K}_0 + \mathcal{H}_1 \otimes \mathcal{K}_1,$$

where $\mathcal{H}_0 \simeq L_{0,1}$ and $\mathcal{H}_1 \simeq L_{1,1}$ are the irreducible level 1 highest weight representations of $\widehat{sl}_2$ and $\mathcal{K}_0$ and $\mathcal{K}_1$ are the multiplicity spaces. Observe also that

$$e_{-(N+1)}\Omega_{-N} = \Omega_{-(N+2)}.$$ 

Note that the operators $a_n = \frac{1}{\sqrt{2}}h_n$ satisfy the CCR (15), i.e., form a system of free bosons, or generate the Heisenberg algebra $\mathfrak{h}_h$. The vectors $\{\Omega_{2n}\}_{n \in \mathbb{Z}}$ introduced above are exactly singular vectors for this Heisenberg algebra: $h_k \Omega_m = 0$ for $m < 0$, $h_0 \Omega_m = m \Omega_m$. The representation of $\mathfrak{h}_h$ in $\mathcal{H}_0$ breaks into a direct sum of irreducible representations:

$$\mathcal{H}_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_0[2k], \quad (16)$$

where $\mathcal{H}_0[2k]$ is the charge $2k$ subspace, i.e., the eigenspace of $h_0$ with eigenvalue $2k$:

$$\mathcal{H}_0[2k] = \{v \in \mathcal{H}_0 : h_0v = 2kv\} = \mathbb{C}[h_0, h_1, \ldots] \Omega_{2k}.$$
6.2 The representation of the Virasoro algebra associated with the basic representation of $\hat{\mathfrak{sl}_2}$

Given a representation of the affine Lie algebra $\hat{\mathfrak{sl}_2}$, we can use the Sugawara construction to obtain the corresponding representation of the Virasoro algebra $\text{Vir}$. It can also be described in the following way. As noted above, the operators $a_n = \frac{1}{\sqrt{2}} h_n$ form a system of free bosons. Given such a system, a representation of $\text{Vir}$ can be constructed as follows ([4, Ex. 9.17]):

$$L_n = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n}, \quad n \neq 0; \quad L_0 = \sum_{j=1}^{\infty} a_{-j} a_j.$$  

Thus we obtain a representation of $\text{Vir}$ in $\mathcal{F}$ and, in particular, in $\mathcal{H}_0$. In this representation, the algebras generated by the operators of $\text{Vir}$ and $\mathfrak{sl}_2 \subset \hat{\mathfrak{sl}_2}$ are mutual commutants, and we have the decomposition

$$\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2),$$

(17)

where $M_{2k+1}$ is the $(2k+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ and $L(1, k^2)$ is the irreducible representation of $\text{Vir}$ with central charge 1 and conformal dimension $k^2$.

The charge $k$ subspace $\mathcal{H}_0[k]$ contains a series of singular vectors $\xi_{k,m}$ of $\text{Vir}$ with energy $(k + m)^2$:

$$L_n \xi_{k,m} = 0 \text{ for } n = 1, 2, \ldots, \quad L_0 \xi_{k,m} = (k + m)^2.$$  

Let us use the so-called homogeneous vertex operator construction of the basic representation of $\hat{\mathfrak{sl}_2}$ (see [4, Sec. 14.8]). In this realization,

$$E(z) = \Gamma_-(z) \Gamma_+(z) z^{-h_0} V^{-1}, \quad F(z) = \Gamma_+(z) \Gamma_-(z) z^{h_0} V,$$  

(18)

where

$$\Gamma_{\pm}(z) = \exp \left( \mp \sum_{j=1}^{\infty} \frac{z^j}{j} h_{\pm j} \right)$$

and the operators $\Gamma_{\pm}(z)$ satisfy the commutation relation

$$\Gamma_+(z) \Gamma_-(w) = \Gamma_-(w) \Gamma_+(z) \left( 1 - \frac{z}{w} \right)^2.$$  

(19)
Using the boson–fermion correspondence (see [4, Ch. 14]), we can identify \( \mathcal{H}_0 \) with the space \( \Lambda \otimes \mathbb{C}[q, q^{-1}] \), where \( \Lambda \) is the algebra of symmetric functions (see [7]). In particular, consider the charge 0 subspace \( \mathcal{H}[0] = \mathcal{H}_0[0] \), which is identified with \( \Lambda \). We can use the following representation of the Heisenberg algebra generated by \( \{h_n\}_{n \in \mathbb{Z}} \):

\[
h_n \leftrightarrow 2n \frac{\partial}{\partial p_n}, \quad h_{-n} = p_n, \quad n > 0, \tag{20}
\]

where \( p_j \) are Newton’s power sums. Then the corresponding Virasoro operators are

\[
L_n = \sum_{r=n+1}^{\infty} p_{n-r} \cdot r \frac{\partial}{\partial p_r} + \sum_{r=1}^{n-1} r(n-r) \cdot \frac{\partial}{\partial p_r} \frac{\partial}{\partial p_{n-r}}, \\
L_{-n} = \sum_{r=1}^{\infty} p_{n+r} \cdot r \frac{\partial}{\partial p_r} + \frac{1}{4} \sum_{r=1}^{n-1} p_r p_{n-r}, \quad n > 0. \tag{21}
\]

Note that the representation (20) of the Heisenberg algebra, and hence the representation (21) of the Virasoro algebra, are not unitary with respect to the standard inner product in \( \Lambda \). To make it unitary, we should consider the inner product in \( \Lambda \) defined by

\[
\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \cdot z_\lambda \cdot 2^{l(\lambda)}, \tag{22}
\]

where \( p_\lambda \) are the power sum symmetric functions, \( z_\lambda = \prod_i i^{m_i} m_i! \) for a Young diagram \( \lambda \) with \( m_i \) parts of length \( i \), and \( l(\lambda) \) is the length (number of nonzero rows) of \( \lambda \).

Denote the singular vectors of Vir in \( \mathcal{H}[0] \) by \( \xi_m := \xi_{0,m} \). According to a result by Segal [8], in the symmetric function realization (20),

\[
\xi_n \leftrightarrow c \cdot s_{(n^n)}, \tag{23}
\]

where \( s_{(n^n)} \) is the Schur function indexed by the \( n \times n \) square Young diagram and \( c \) is a numerical coefficient.

### 6.3 Fusion product and the Fock space

It is shown in [2] that

\[
V^*_{2n} \simeq \mathbb{C}[e_0, \ldots, e_{-(2n-1)}] \Omega_{-2n} \subset \mathcal{F}
\]
as an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-2n})$-module, the embedding $j_{2n}$ under this isomorphism coincides with the natural inclusion

$$\mathbb{C}[e_0, \ldots, e_{-(2n-1)}] \Omega_{-2n} \subset \mathbb{C}[e_0, \ldots, e_{-(2n+1)}] \Omega_{-2(n+1)},$$

and the limit space $V$ coincides with $\mathcal{H}_0$.

Using results of [2], one can easily prove the following lemma.

**Lemma 1.** A basis in $F_{2n} = \mathbb{C}[e_0, \ldots, e_{-(2n-1)}] \Omega_{-2n}$ is

$$\{e_0^{i_0} e_1^{i_1} \ldots e_{-(2n-1)}^{i_{2n-1}} : 0 \leq k \leq 2n - (i_0 + \ldots + i_{2n-1})\} \Omega_{-2n}.$$

Observe that under this “fusion–Fock” correspondence, the charge 0 subspace $\mathcal{H}[0]$ is identified with $\mathcal{V}[0]$. It follows from Lemma 1 that a basis of $F_{2n}[0] = F_{2n} \cap \mathcal{H}[0]$ is

$$\{\prod e_0^{i_0} e_1^{i_1} \ldots e_{-(2n)}^{i_{2n}} : i_0 + i_1 + \ldots + i_n = n\} \Omega_{-2n}. \quad (24)$$

### 7 The key isomorphism in more detail

Comparing (13) and (17), we obtain the following result.

**Corollary 2.** The space $\Pi_k$ of the discrete representation of the infinite symmetric group corresponding to the tableau $\tau_k$ has a natural structure of the Virasoro module $L(1,k^2)$.

Our aim is to study this Virasoro representation in $\Pi_k$ (or, which is equivalent, the corresponding representation of the infinite symmetric group in the Fock space). In particular, from the known theory of the basic module $L_{0,1}$, we immediately obtain the following result.

**Corollary 3.** In the above realization of the Virasoro module $L(1,k^2)$, the Gelfand–Tsetlin basis in $\Pi_k$ (which consists of the infinite two-row Young tableaux tail-equivalent to $\tau_k$) is the eigenbasis of $L_0$, and the eigenvalues are given by the stable major index $r$:

$$L_0 \tau = r(\tau) \tau.$$

Note that, in view of (14) and the remark after Lemma 1, the charge 0 subspace $\mathcal{H}[0]$ is identified with the space $\Pi$ spanned by all infinite two-row Young tableaux with “correct” tail behavior. Thus we obtain the following corollary.
Corollary 4. The space $\Pi$, which is the countable sum of discrete representations of the infinite symmetric group $\mathfrak{S}_\mathbb{N}$, has a structure of an irreducible representation of the Heisenberg algebra $\mathfrak{A}$.

On the other hand, as mentioned above, $\mathcal{H}[0]$ can be identified with the algebra of symmetric functions $\Lambda$ via (20). Denote by $\Phi$ the obtained isomorphism between $\Pi$ and $\Lambda$, which thus associates with every tableau $\tau \in \Pi$ a symmetric function $\Phi(\tau) \in \Lambda$ such that $r(\tau) = \deg \Phi(\tau)$.

Denote by $T^{(N)}$ the (finite) set of two-row tableaux that coincide with some $\tau_n$, $n = 0, 1, \ldots$, from the $N$th level. Let $\Pi^{(N)}$ be the subspace in $\Pi$ spanned by all $\tau \in T^{(N)}$. It follows from all the above identifications that $\Pi^{(2k)} \leftrightarrow F_{2k}[0]$.

Theorem 3. Under the isomorphism $\Phi$,

1) the principal tableaux (10) correspond to the Schur functions with square Young diagrams:
$$\Phi(\tau_k) = \text{const} \cdot s_{(k)};$$

2) the subspace $\Pi^{(2k)}$ correspond to the subspace $\Lambda_{k \times k}$ of $\Lambda$ spanned by the Schur functions indexed by Young diagrams lying in the $k \times k$ square; the correspondence between the Schur function basis in $\Lambda_{k \times k}$ and the basis (24) in $\Pi^{(2k)} \simeq F_{2k}[0]$ is given by formula (29) below.

Proof. We follow Wasserman’s [11] proof of Segal’s result (23). Let $0 \leq i_1, \ldots, i_k \leq k$. Then, obviously,
$$e_{-i_1} \ldots e_{-i_k} \Omega_{-2k} = \left[ \prod_{j=1}^k z_j^{i_j - 1} \right] E(z_k) \ldots E(z_1) \Omega_{-2k},$$
where by $[\text{monomial}] F(z_1, \ldots, z_m)$ we denote the coefficient of this monomial in $F(z_1, \ldots, z_m)$. Now, using the representation (18), the commutation relation (19), and the obvious facts that $V^{-k} \Omega_{-2k} = \Omega_0$ and $\Gamma_+(z) \Omega_0 = \Omega_0$, we obtain
$$E(z_k) \ldots E(z_1) \Omega_{-2k} = \prod_{j=1}^k z_j^{2(k-j)} \prod_{1 \leq j < i \leq k} \left( 1 - \frac{z_i}{z_j} \right)^2 \Gamma_-(z_k) \ldots \Gamma_-(z_1) \Omega_0.$$

Observe that, in view of (20) and the well-known fact from the theory of symmetric functions, $\Gamma_-(z)$ is exactly the generating function of the complete
symmetric functions. Hence, expanding the product $\Gamma(z_k)\ldots\Gamma(z_1)\Omega_0$ by the Cauchy identity ([7, I.4.3]) and making simple transformations, we obtain

$$E(z_k)\ldots E(z_1)\Omega_{-2k} = (-1)^{k(k-1)/2} \prod_{j=1}^{k} z_j^{k-1} a_\delta(z) a_\delta(z^{-1}) \sum_{\lambda : l(\lambda) \leq k} s_\lambda(z^{-1}) s_\lambda,$$

where

$$a_\delta(z) = \prod_{1 \leq i < j \leq k} (z_i - z_j) = \det[z_i^{k-j}]_{1 \leq i,j \leq k}$$

is the Vandermonde determinant, $a_\delta(z^{-1})$ is the similar determinant for the variables $z^{-1} = (z_1^{-1}, \ldots, z_k^{-1})$, $l(\lambda)$ is the length of the diagram $\lambda$ (the number of nonzero rows), $s_\lambda(z^{-1})$ is the Schur function calculated at the variables $z^{-1}$, and $s_\lambda$ is the Schur function as an element of $\Lambda$ identified with $H[0]$.

Thus we have

$$e_{-i_1} \ldots e_{-i_k} \Omega_{-2k} = (-1)^{k(k-1)/2} \cdot [1] \left( \prod_{j=1}^{k} z_j^{k-i_j} a_\delta(z) a_\delta(z^{-1}) \sum_{\lambda} s_\lambda(z^{-1}) s_\lambda \right).$$

First consider the case where $i_1 = \ldots = i_k = m$. Then, by the definition of the Schur functions [7 I.3.1],

$$\prod z_j^{k-i_j} a_\delta(z) = \det[z_i^{2k-m-j}]_{1 \leq i,j \leq k} = a_\delta(z) s_{((k-m)k)}(z),$$

where $((k-m)k)$ is the rectangular Young diagram with $k$ rows of length $k - m$, and the standard orthogonality relations imply that

$$e_{-m} \Omega_{-2k} = (-1)^{k(k-1)/2} k! \cdot s_{((k-m)k)}.$$ (25)

Since $\xi_k = e_0^k \Omega_{-2k}$, for $m = 0$ this is Segal’s result [23], which we have now extended to the case of rectangular diagrams. It is easy to see that the singular vector of Vir in $\mathcal{V}_k[0]$ is just $e_0^k \omega_{-2k} \otimes \tau_k$, so that the first claim of the theorem follows.

We now turn to the case of $i_1, \ldots, i_k$ that are not necessarily equal. For convenience, set $\tilde{e}_{p} := e_{-(k-p)}$, $0 \leq p \leq k$. Given $0 \leq \alpha_1, \ldots, \alpha_k \leq k$, we have

$$\tilde{e}_{\alpha_1} \ldots \tilde{e}_{\alpha_k} \Omega_{-2k} = [1] \left( \prod_{j=1}^{k} z_j^{\alpha_j} a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_\lambda \right),$$ (26)
where \(a_{\lambda+\delta}(x) = \det[x_i^{\lambda_j+k-j}]_{1 \leq i, j \leq k} = s_\lambda(x)a_\delta(x)\). Consider a Young diagram \(\mu = (\mu_1, \ldots, \mu_k) = (0^r_0, 1^{r_1}, 2^{r_2}, \ldots)\). Let us sum (26) over all different permutations \(\alpha = (\alpha_1, \ldots, \alpha_k)\) of the sequence \((\mu_1, \ldots, \mu_k)\). Note that the operators \(e_j\) commute with each other, so that the left-hand side does not depend on the order of the factors. In the right-hand side, \(\sum_{\alpha} \prod z_{j_\alpha}^{\alpha_j} = m_\mu(z)\), a monomial symmetric function. Thus we have

\[
\frac{k!}{\prod_{j=0}^{k} r_j!} \tilde{e}_{\mu_1} \cdots \tilde{e}_{\mu_k} = [1] \left( m_\mu(z)a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1})s_\lambda \right).
\]  

(27)

Let \(\nu\) be a Young diagram with at most \(k\) rows and at most \(k\) columns, i.e., \(\nu \subset (k^k)\). We have

\[
s_\nu(z) = \sum_{\mu} K_{\nu\mu} m_\mu(z),
\]  

(28)

where \(K_{\nu\mu}\) are Kostka numbers. It is well known that \(K_{\nu\mu} = 0\) unless \(\mu \leq \nu\), where \(\leq\) is the standard ordering on partitions: \(\mu \leq \nu \iff \mu_1 + \ldots + \mu_i \leq \nu_1 + \ldots + \nu_i\) for every \(i \geq 1\). In particular, \(\mu_1 \leq \nu_1 \leq k\). Besides, since we consider only \(k\) nonzero variables \(z_1, \ldots, z_k\), it also follows that \(m_\mu(z) = 0\) unless \(l(\mu) \leq k\). Thus the sum in (28) can be taken only over diagrams \(\mu \subset (k^k)\), for which equation (27) holds. Multiplying this equation by \(K_{\nu\mu}\) and summing over \(\mu\) yields

\[
\sum_{\mu=(0^r_0, 1^{r_1}, 2^{r_2}, \ldots) \subset (k^k)} \frac{k!}{\prod_{j=0}^{k} r_j!} K_{\nu\mu} \tilde{e}_{\mu_1} \cdots \tilde{e}_{\mu_k} = [1] \left( s_\nu(z)a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1})s_\lambda \right).
\]  

By the orthogonality relations, the right-hand side is equal to \(k!s_\nu\). Thus we obtain the following formula:

\[
s_\nu = \sum_{\mu=(0^r_0, 1^{r_1}, 2^{r_2}, \ldots) \subset (k^k)} \frac{K_{\nu\mu}}{\prod_{j=0}^{k} r_j!} e_{-(k-\mu_1)} \cdots e_{-(k-\mu_k)} \Omega_{-2k}.
\]  

(29)

Observe that for rectangular diagrams this formula reduces to (25). Indeed, for \(\nu = ((k-m)^k)\), all diagrams \(\mu\) with \(\mu < \nu\) have \(l(\mu) > k\), hence the only nonzero term in the right-hand side of (29) corresponds to \(\mu = \nu\), with \(K_{\nu\nu} = 1\) and \(r_j = k!\delta_{j,k-m}\).

It follows from (29) that \(\Lambda_{k \times k} \subset \Pi^{2k}\). On the other hand, the generating functions for the tableaux from \(T^{(2k)}\) and for the Young diagrams lying in
the $k \times k$ square coincide:

$$\sum_{\tau \in T^{(2k)}} q^{r(\tau)} = \sum_{\lambda \subset (k^k)} q^{\lambda} = \left[ \begin{array}{c} \frac{2k}{k} \\ q \end{array} \right],$$

where $|\lambda|$ is the number of cells in a Young diagram $\lambda$ and $\left[ \begin{array}{c} \frac{2k}{k} \\ q \end{array} \right]$ is the $q$-binomial coefficient (the equation for Young diagrams can be found in [1, Theorem 3.1]; for tableaux, it can be deduced from the known results on the major index given, e.g., in [3]). This implies, in particular, that $\dim \Lambda_{k \times k} = \dim \Pi^{2k}$ and completes the proof.

References

[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976.

[2] B. Feigin and E. Feigin, Integrable $\widehat{\mathfrak{sl}_2}$-modules as infinite tensor products, in: *Fundamental Mathematics Today*, Independent Univ. of Moscow, Moscow (2003), pp. 304–334.

[3] B. Feigin and S. Loktev, On generalized Kostka polynomials and quantum Verlinde rule, in: *Differential Topology, Infinite-Dimensional Lie algebras, and Applications*, Amer. Math. Soc. Transl. Ser. 2, 194 (1999), pp. 61–79.

[4] V. G. Kac, *Infinite-Dimensional Lie Algebras*, 3rd edition, Cambridge Univ. Press, Cambridge, 1990.

[5] R. Kedem, Fusion products, cohomology of $GL_N$ flag manifolds, and Kostka polynomials, *Int. Math. Res. Not.* 2004, No. 25, 1273-1298 (2004).

[6] A. Lascoux and M. P. Schützenberger, Sur une conjecture de H. O. Foulkes, *C. R. Acad. Sci. Paris* 286A, 323–324 (1978).

[7] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, New York, 1995.

[8] G. B. Segal, Unitary representations of some infinite-dimensional groups, *Comm. Math. Phys.* 80 (1981), 301–342.
[9] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Univ. Press, Cambridge, 2001.

[10] N. Tsilevich and A. Vershik, Infinite-dimensional Schur–Weyl duality and the Coxeter–Laplace operator, PDMI preprint 16/2012 (2012), to appear in *Comm. Math. Phys.*

[11] A. Wasserman, Direct proof of the Feigin–Fuchs character formula for unitary representations of the Virasoro algebra, [arXiv:1012.6003v](http://arxiv.org/abs/1012.6003v).