Pathwise Blowup of space-time fractional SPDEs

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Abstract

The finite time blowup in the almost sure sense of a class of space-time fractional stochastic partial differential equations is discussed. Both the cases of white noise and colored noise are considered. The sufficient and necessary condition between the blowup and Osgood condition is obtained when the spatial domain is bounded. And the sufficient condition for the blowup is obtained when the spatial domain is the whole space. The results in this paper could be regarded as extensions to some results in Foondun and Nualart, 2021.

Keywords: stochastic partial differential equations, pathwise blowup, space-time white noise, space colored noise, Osgood condition.

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1 Introduction

In this paper, we continue our research on the blowup of solutions to stochastic partial differential equations (SPDEs). In our previous work [5], the finite time blowup in $L^2$ sense of solutions to the white or colored noise driven SPDEs with Bernstein functions of the Laplacian were investigated. Inspired by Foondun and Nualart [9], we are going to study the finite time blowup of space-time fractional SPDEs in the almost sure sense in this paper. Our results in this work could be regarded as extension to some of the theorems in [9]. It should also be mentioned that some blowup results about a class of space-time fractional SPDEs were obtained in Desalegn et al. [6], where the fractional operation on the time variable was only considered on the $u_t(x)$ term on the left side of the equations and only the white noise was discussed.

In this paper, we investigate the more general space-time fractional SPDEs and discuss the case of the white noise as well as the colored noise.

We start by considering the space-time fractional SPDEs in a ball $B := B(0, 1)$

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + I_{1-\beta}^t[b(u_t(x)) + \sigma \mathcal{F}(t, x)], \quad t > 0 \quad \text{and} \quad x \in B;$$

$$u_t(x) = 0, \quad x \in \mathbb{R}^d \setminus B, \quad t > 0;$$

$$u_t(x)|_{t=0} = u_0(x).$$

(1.1)

Here $\sigma$ is a positive constant, $-(-\Delta)^{\alpha/2}$ denotes the generator of $\alpha$-stable Lévy process killed upon exiting the ball $B$, $\partial_t^\beta$ is the Caputo derivative, and $I_{1-\beta}^t$ is the fractional integral operator (see Section 2 for the
precise definition). The noise $\dot{F}$, when not space-time white noise is taken to be spatially colored which is white in time and has a spatial correlation given by the Riesz kernel. That is,

$$E(\dot{F}(t,x)\dot{F}(s,y)) = \delta_0(t-s)f(x-y),$$

where $f(x) = |x|^{-\eta}, 0 < \eta < d$. Suppose that $b$ satisfies the Osgood condition: for some $a > 0$

$$\int_a^\infty \frac{ds}{b(s)} < \infty. \quad (1.2)$$

When $\eta = \min(2, \beta^{-1})\alpha$, the mild solution of equation (1.1) is given in the sense of Walsh \[15\] as follows (see \[12\] for a motivation to study equations of this type and \[7\] for the proof of the existence of solutions):

$$u_t(x) = \int_B G_B(t,x,y)u_0(y)\,dy + \int_B \int_0^t b(u_s(y))G_B(t-s,x,y)\,ds\,dy$$

$$+ \sigma \int_B \int_0^t G_B(t-s,x,y)F(ds\,dy), \quad (1.3)$$

where $G_B(t,x,y)$ is the fundamental solution of equation (1.1) when $b = 0$ and $\sigma = 0$.

The eigenfunctions $\{\phi_n : n \in \mathbb{N}\}$ of fractional Laplacian $\frac{1}{\alpha}(-\Delta)^{\alpha/2}$ in $B$ form an orthonormal basis for $L^2(B)$. Let $E_\beta(z) = \sum_{n=0}^\infty \frac{1}{\Gamma(1+n\beta)} z^n$ denote the Mittag-Leffler function for $\beta \in (0,1)$. We have an eigenfunction expansion of the kernel

$$G_B(t,x,y) = \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta)\phi_n(x)\phi_n(y). \quad (1.4)$$

See, for example, Chen et al. \[3\] and Meerschaert et al. \[11\].

We will make the following assumption throughout the paper.

**Assumption 1.1.** The function $b : \mathbb{R} \to \mathbb{R}_+$ is nonnegative, locally Lipschitz and nondecreasing on $(0,\infty)$ and the initial condition $u_0(x)$ is nonnegative and continuous.

Now, we are ready to present our first theorem and its proof will be given in Section 3.

**Theorem 1.2.** Suppose that Assumption 1.1 holds. If $\beta > 1/2$, and $b$ is convex and satisfies the Osgood condition (1.2), then the solution to (1.1) blows up in finite time with positive probability. Conversely, if the solution blows up in finite time with positive probability, then $b$ satisfies the Osgood condition (1.2).

**Remark 1.3.** This result could be regarded as an extension of Theorem 1.5 in \[9\]. Theorem 1.2 still holds when the noise is replaced by space-time white one. The proof has similar steps as in the proof of Theorem 1.2, see Bonder and Groisman \[1\].

Now, we turn to the case of the whole space. Consider the equation with the space-time white noise:

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + I_t^{1-\beta}[b(u_t(x)) + \sigma \dot{W}(t,x)], \quad \nu > 0, t > 0, x \in \mathbb{R}^d;$$

$$u_t(x)|_{t=0} = u_0(x), \quad (1.5)$$

where the initial datum $u_0$ is a nonnegative, continuous, and bounded function, $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian with $\alpha \in (0,2], \dot{W}(t,x)$ is a space-time white noise with $x \in \mathbb{R}^d$, and $\nu$ is a positive constant.

When $d = \min(2, \beta^{-1})\alpha$, the mild solution of equation (1.5) is given in the sense of Walsh \[15\] as follows (see \[12\] )

$$u_t(x) = \int_{\mathbb{R}^d} G(t,x,y)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^d} G(t-s,x,y)b(u_s(y))\,dy\,ds + \int_0^t \int_{\mathbb{R}^d} G(t-s,x,y)\,W(dy\,ds),$$

where $G(t,x,y)$ is the heat kernel of (1.5) when $b = 0$ and $\sigma = 0$.

Now, we are ready to present our second theorem and its proof is given in Section 4.
Theorem 1.4. Suppose that Assumption 1.1 holds. If $b$ satisfies the Osgood condition, then almost surely, there is no global solution to equation to (1.5).

Remark 1.5. This theorem could be regarded as an extension of Theorem 1.4 in [9]. One of the interesting questions is that if the blowup of the solution in finite time in the almost sure sense can indicate that $b$ satisfies the Osgood condition (1.2). Unluckily, we have not found a way to answer it. Meanwhile, we notice that this is also an open question even when the equation is the classical stochastic heat equation, as stated in [9].

We notice that there are very some interesting discussions, at the end of Section 1 in [9], about the non-existence of the solutions to some SPDEs for any $t > 0$, that is the non-existence of the local solution. We would like to make the same claim as Foondun and Nualart did that such non-existence results of the local solutions are not our interests in this paper and our main scope is the non-existence of the global solution. We refer the readers to [9] for more discussions.

The organization of the paper is as follows. Some preliminary results that are needed for the proofs of two theorems are given Section 2. Proofs of Theorem 1.2 and Theorem 1.4 are provided in Sections 3 and 4, respectively.

2 Preliminaries

We give some preliminaries that will be essential to the proofs of the main results. Let $\beta \in (0, 1)$, $\partial_t ^\beta$ is the Caputo fractional derivative which first appeared in [2] and is defined by

$$\partial_t ^\beta u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_r u_r(x) \frac{dr}{(t-r)^\beta}.$$  

For $\gamma > 0$, the fractional order integral is defined by

$$I_t ^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau.$$  

We have some useful estimates for the Mittag-Leffler function next.

Lemma 2.1. We have

$$\inf_{s > 0} e^{\lambda_1 s} E_\beta(-\lambda_1 s^\beta) \in (0, 1].$$

Proof. Recall that (cf. [14, Theorem 4])

$$E_\beta(-x) \geq \frac{1}{1 + \Gamma(1-\beta)x}, \quad x > 0.$$  

We have

$$E_\beta(-\lambda_1 s^\beta) \geq \frac{1}{1 + \Gamma(1-\beta)\lambda_1 s^\beta} \geq C_{\beta, \lambda_1} e^{-\lambda_1 s}, \quad s > 0,$$

where

$$C_{\beta, \lambda_1} := \inf_{s > 0} \frac{e^{\lambda_1 s}}{1 + \Gamma(1-\beta)\lambda_1 s^\beta} \in (0, 1].$$

This implies the desired assertion. \qed

Denote by $p^\alpha(t, x, y)$ the transition density of a symmetric stable process $X_t$ in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$. Denote by $E_t$ the inverse $\beta$-stable subordinator. Then the time-changed Brownian motion $X_{E_t}$ has a transition density

$$G(t, x, y) = G(t, x - y) = E p^\alpha(E_t, x, y) = \int p^\alpha(s, x, y) \mathbb{P}(E_t \in ds).$$

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Lemma 2.2. Let $d \geq 1$. We have

$$\inf_{x \in B(0,1), r \in (0,1)} \int_{B(0,1)} G(r, x, y) \, dy > 0$$

Proof. If $x \in B(0,1)$ and $r \in (0,1)$, it is easy to verify that

$$B \left( (1 - r^{\beta/\alpha}/2)x, r^{\beta/\alpha}/2 \right) \subset B(0,1) \cap B(x, r^{\beta/\alpha}).$$

By [8, Lemma 2.1 (a)] there is some $c_0 > 0$ such that

$$G(t, x, y) \geq c_0 t^{-\beta d/\alpha}$$

whenever $|x - y| \leq t^{\beta/\alpha}$.

Then for any $x \in B(0,1)$ and $r \in (0,1)$,

$$\int_{B(0,1)} G(r, x, y) \, dy \geq \int_{B(0,1) \cap B(x, r^{\beta/\alpha})} c_0 r^{-\beta d/\alpha} \, dy$$

$$= c_0 r^{-\beta d/\alpha} \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \left( \frac{r^{\beta/\alpha}}{2} \right)^d$$

$$= \frac{c_0 \pi^{d/2}}{2^d \Gamma(d/2 + 1)}.$$

Lemma 2.3 (Lemma 1 in [12]). For $d < 2 \alpha$,

$$\int_{\mathbb{R}^d} [G(t, x)]^2 \, dx = C^* t^{-\beta d/\alpha},$$

where $C^* > 0$ is a constant depending only on $d, \alpha, \beta$.

In the following, we consider

$$g_\beta(t, x) := \int_0^t \int_{\mathbb{R}^d} G(t - r, x - y) W(dr, dy).$$

This is the random part of the mild solution to (1.5) when $\sigma \equiv 1$.

Lemma 2.4. Suppose $d < \min(2, \beta^{-1}) \alpha$. For each fixed $x \in \mathbb{R}^d$, almost surely,

$$\limsup_{t \to \infty} \frac{g_\beta(t, x)}{\sqrt{K t^{1 - \beta d/\alpha} \log \log t}} = 1.$$

Proof. Since $\{g_\beta(t, x), \ t \geq 0\}$ is a Gaussian process for each fixed $x$, we need to calculate the variance of $g_\beta(t, x)$. By the Plancherel theorem and using Lemma 2.3,

$$\mathbb{E}[g_\beta(t, x) g_\beta(t, x)] = \int_{\mathbb{R}^d} \int_0^t [G(t - r, x - y)]^2 \, dy \, dr = \frac{C^*}{1 - \beta d/\alpha} t^{1 - \beta d/\alpha}.$$ 

Hence by equation (5) in Lai [10] we get the result using the fact that the variance of $g_\beta(t, x)$ is given by Lemma 2.3

$$\limsup_{t \to \infty} \frac{g_\beta(t, x)}{\sqrt{K t^{1 - \beta d/\alpha} \log \log t}} = 1,$$

where $K := \frac{2C^*}{1 - \beta d/\alpha}$. \qed
Proposition 2.5 (Proposition 2 [12]). Suppose $d < \min(2, \beta^{-1})\alpha$, and $k \geq 2$. Then there exists $c > 1$ such that the following moment estimates for time increments and spatial increments hold.

(i). For $s \leq t$,
\[ c^{-1}|t-s|^{\left(1-\frac{d\beta}{\alpha}\right)\frac{k}{2}} \leq \mathbb{E} \left[ |g_\beta(t, x) - g_\beta(s, x)|^k \right] \leq c|t-s|^{\left(1-\frac{d\beta}{\alpha}\right)\frac{k}{2}}. \]

(ii). For $x, y \in \mathbb{R}^d$,
\[ c^{-1}|x-y|^k \leq \mathbb{E} \left[ |g_\beta(t, x) - g_\beta(t, y)|^k \right] \leq c|x-y|^\min\left\{ \left(\frac{\alpha-d\beta}{4}\right)^{-\frac{k}{2}} \right\}. \]

As a consequence of Proposition 2.5 and classical Garsia’s lemma we have the following estimate.

Proposition 2.6. For all $k \geq 2$, there exists $A_k > 0$ such that for any $n \geq 1$,
\[ \mathbb{E} \left[ \sup_{s, t \in [n, n+2], x, y \in B(0,1)} |g_\beta(t, x) - g_\beta(s, y)|^k \right] \leq A_k 2^{k(1-\beta d/\alpha)/2}. \]

Let $\Psi(t) := \sqrt{Kt^{1-\beta d/\alpha} \log \log t}$. Now using this lemma, we get the next proposition.

Proposition 2.7. Almost surely,
\[ \sup_{s, t \in [n, n+2], x, y \in B(0,1)} \frac{|g_\beta(t, x) - g_\beta(s, y)|}{\Psi(n)} \to 0, \quad \text{as} \quad n \to \infty. \]

Proof. Proposition 2.6 implies that for $k \geq 2\alpha/(\alpha - \beta d)$
\[ \mathbb{E} \left[ \sum_{n=1}^{\infty} \sup_{s, t \in [n, n+2], x, y \in B(0,1)} \frac{|g_\beta(t, x) - g_\beta(s, y)|^k}{\Psi(n)^k} \right] \leq \sum_{n=1}^{\infty} A_k 2^{k(1-\beta d/\alpha)/2} < \infty. \]

This gives the desired result.

Using Proposition 2.7 we get the following result.

Proposition 2.8. Almost surely, there exists a sequence $t_n \to \infty$ such that
\[ \inf_{h \in [0,1], x \in B(0,1)} g_\beta(t_n + h, x) \to \infty \quad \text{as} \quad n \to \infty. \]

Proof. Fix $x_0 \in B(0,1)$. Choose $\omega$ such that Lemma 2.4 and Proposition 2.7 hold. Then we get
\[ \inf_{h \in [0,1], x \in B(0,1)} g_\beta(t + h, x) = g_\beta(t, x_0) + \inf_{h \in [0,1], x \in B(0,1)} \left[ g_\beta(t + h, x) - g_\beta(t, x_0) \right] \geq g_\beta(t, x_0) + \inf_{h \in [0,1], x \in B(0,1)} - \left[ g_\beta(t + h, x) - g_\beta(t, x_0) \right] \]
\[ = \frac{g_\beta(t, x_0)}{\Psi(t)} \Psi(t) + \sup_{h \in [0,1], x \in B(0,1)} \left[ \frac{|g_\beta(t + h, x) - g_\beta(t, x_0)|}{\Psi(t)} \right] \Psi([t]) \]
\[ = \min\{\Psi(t), \Psi([t])\} \left[ \frac{g_\beta(t, x_0)}{\Psi(t)} - \sup_{h \in [0,1], x \in B(0,1)} \left[ \frac{|g_\beta(t + h, x) - g_\beta(t, x_0)|}{\Psi(t)} \right] \right]. \]

Now using Lemma 2.4 and Proposition 2.7, we can choose a suitable sequence $t_n$ to finish the proof. □
3 Proof of Theorem 1.2

Proof of Theorem 1.2. (1) To prove the first assertion, we shall borrow an idea used in the proof of Theorem 1.3 in Foondun and Nualart [9].

a) Set

\[ Y_t := \int_B u(t, x)\phi_1(x) \, dx. \]

From equation (1.4), we can easily get

\[ \int_B G_B(t, x, y)\phi_1(x) \, dx = E_{\beta}(-\mu_1 t^\beta)\phi_1(y). \]

It is a well-know fact that \( \phi_1(x) > 0 \) for \( x \in B \).

Then after using the stochastic Fubini theorem and the decomposition above for the heat kernel \( G_B(t - s, x, y) \), from the mild solution (1.3), we get

\[ Y_t = E_{\beta}(-\lambda_1 t^\beta)Y_0 + \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) b(Y_s - E_{\beta}(-\lambda_1 s^\beta) Y_0) \, ds + \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) \int_B \phi_1(y) F(dy) \, ds. \]

Since \( b \) is convex, using the Jensen inequality, we can get

\[ \int_B b(u_s(y))\phi_1(y) \, dy \geq b(Y_s) \geq b(Y_s - E_{\beta}(-\lambda_1 s^\beta)Y_0), \]

where the second inequality follows from the assumptions that \( b \) is nondecreasing and \( Y_0 \geq 0 \). We also have

\[ \int_0^t \int_B \phi_1(y) F(dy) \, ds = \sqrt{\kappa} B_t, \]

where \( B_t \) is a Brownian motion and

\[ \kappa := \int_{B \times B} \phi_1(y)\phi_1(z)|y - z|^{-\eta} \, dy \, dz, \]

where \( \eta \) is the Riesz kernel exponent.

Therefore we obtain

\[ Y_t \geq E_{\beta}(-\lambda_1 t^\beta)Y_0 + \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) b(Y_s - E_{\beta}(-\lambda_1 s^\beta) Y_0) \, ds + \sqrt{\kappa} \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) \, dB_s. \]

By Lemma 2.1, there exists \( c = c(\beta, \lambda_1) > 0 \) such that

\[ Y_t \geq E_{\beta}(-\lambda_1 t^\beta)Y_0 + c \int_0^t e^{-\lambda_1(t-s)} b(Y_s - E_{\beta}(-\lambda_1 s^\beta) Y_0) \, ds + \sqrt{\kappa} \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) \, dB_s. \]

b) To use the comparison principle, consider

\[ Z_t = E_{\beta}(-\lambda_1 t^\beta) Z_0 + c \int_0^t e^{-\lambda_1(t-s)} b(Z_s - E_{\beta}(-\lambda_1 s^\beta) Z_0) \, ds + \sqrt{\kappa} \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) \, dB_s \]

with \( Z_0 = Y_0 \). We have

\[ e^{\lambda_1 t} Z_t = e^{\lambda_1 t} E_{\beta}(-\lambda_1 t^\beta) Z_0 + c \int_0^t e^{\lambda_1 s} b(Z_s - E_{\beta}(-\lambda_1 s^\beta) Z_0) \, ds + \sqrt{\kappa} e^{\lambda_1 t} \int_0^t E_{\beta}(-\lambda_1(t - s)^\beta) \, dB_s. \]
This implies that
\[
\begin{align*}
    dZ_t &= d\{e^{-\lambda t}(e^{\lambda t}Z_t)\} = -\lambda_1 Z_t \, dt + e^{-\lambda t} \, d\{e^{\lambda t}Z_t\} \\
    &= -\lambda_1 Z_t \, dt + c b(Z_t - E_\beta(-\lambda_1 t^\beta)Z_0) \, dt + \sqrt{\kappa} \, dB_t \\
    &\quad + \left(\lambda_1 E_\beta(-\lambda_1 t^\beta)Z_0 + \frac{dE_\beta(-\lambda_1 t^\beta)}{dt} Z_0 \right) \, dt \\
    &\quad + \sqrt{\kappa} \left( \int_0^t \left[ \lambda_1 E_\beta(-\lambda_1 (t-s)^\beta) + \frac{dE_\beta(-\lambda_1 (t-s)^\beta)}{dt} \right] \, dB_s \right) \, dt.
\end{align*}
\]

Setting \( U_t := Z_t - E_\beta(-\lambda_1 t^\beta)Z_0 \), we get
\[
dU_t = -\lambda_1 U_t \, dt + c b(U_t) \, dt + \sqrt{\kappa} \, dB_t + \sqrt{\kappa \xi(t)} \, dt, \tag{3.1}
\]
where
\[
\xi(t) := \int_0^t \left[ \lambda_1 E_\beta(-\lambda_1 (t-s)^\beta) + \frac{dE_\beta(-\lambda_1 (t-s)^\beta)}{dt} \right] \, dB_s.
\]

\( c \) Consider
\[
dV_t = -\lambda_1 V_t \, dt + c b(V_t) \, dt + \sqrt{\kappa} \, dB_t
\]
with \( V_0 = U_0 \). By the Feller test, \( V_t \) explodes in finite time a.s. This means that there exists (deterministic) \( T < \infty \) such that
\[
\mathbb{P} \left( \text{there exists } S = S(\omega) \leq T \text{ such that } \lim_{t\uparrow S} V_t = \infty \right) > 0. \tag{3.2}
\]

\( d \) Let
\[
h(t) := \lambda_1 E_\beta(-\lambda_1 t^\beta) + \frac{dE_\beta(-\lambda_1 t^\beta)}{dt} = \lambda_1 E_\beta(-\lambda_1 t^\beta) - \beta \lambda_1 t^\beta - 1 E_\beta(-\lambda_1 t^\beta), \quad t > 0,
\]
and \( h(0) := 0 \). Since \( \beta > 1/2 \), \( h \in L^2_{\text{loc}}(\mathbb{R}) \). Note that
\[
\xi(t) = \int_0^t h(t-s) \, dB_s.
\]

By Lemma 4.1 in Appendix,
\[
\tilde{B}_t := B_t + \int_0^t \xi(s) \, ds, \quad 0 \leq t \leq T,
\]
is a Brownian motion under the weighted probability measure \( R_T \mathbb{P} \), where
\[
R_T := \exp \left[ - \int_0^T \xi(s) \, dB_s - \frac{1}{2} \int_0^T |\xi(s)|^2 \, ds \right].
\]

Rewrite (3.1) as
\[
dU_t = -\lambda_1 U_t \, dt + c b(U_t) \, dt + \sqrt{\kappa} \, d\tilde{B}_t.
\]

Then we know that the distribution of \( (U_t)_{0 \leq t \leq T} \) under \( R_T \mathbb{P} \) coincides with that of \( (V_t)_{0 \leq t \leq T} \) under \( \mathbb{P} \). By (3.2), we conclude that
\[
(R_T \mathbb{P}) \left( \text{there exists } S = S(\omega) \leq T \text{ such that } \lim_{t\uparrow S} U_t = \infty \right) > 0.
\]

This implies
\[
\mathbb{P} \left( \text{there exists } S = S(\omega) \leq T \text{ such that } \lim_{t\uparrow S} U_t = \infty \right) > 0. \tag{3.3}
\]
By the comparison principle, \( Y_t \geq Z_t = U_t + \beta_1 (-\lambda_1 t^\beta) Z_0 \). Consequently, (3.3) holds with \( U_t \) replaced by \( Y_t \), and the proof is now finished.

(2) Next, we follow the line of the proof of Theorem 1.3 in Foondun and Nualart [9] (with crucial changes) to prove the second assertion. Let

\[
T := \sup \left\{ t \geq 0 : \sup_{x \in B(0,1)} |u_t(x)| < \infty \right\}.
\]

Since the solution blows up in finite time with positive probability, we can find a set \( A \) satisfying \( P(A) > 0 \) such that for any \( \omega \in A \), we have \( T(\omega) < \infty \). In the following we write \( T = T(\omega) < \infty \) (drop the variable \( \omega \)). Note that \( T \) is the blowup time.

Recall that (\( \sigma = 1 \))

\[
u_t(x) = \int_{B(0,1)} G_B(t, x, y) u_0(y) \, dy + \int_0^t \int_{B(0,1)} b(u_s(y)) G_B(t - s, x, y) \, dy \, ds
\]

\[
+ \int_0^t \int_{B(0,1)} G_B(t - s, x, y) F(dy \, ds)
\]

\[
=: \sum_{i=1}^3 I_i(t, x).
\]

Since the initial value \( u_0 \) is bounded, we find that

\[
|I_1(t, x)| \leq \int_{B(0,1)} G_B(t, x, y)|u_0(y)| \, dy \leq \|u_0\|_\infty \int_{B(0,1)} G_B(t, x, y) \, dy = \|u_0\|_\infty, \quad \forall t \in [0, T], \forall x \in B(0,1).
\]

Set \( Y_t := \sup_{x \in B(0,1)} u_t(x) \). Since \( b \) is nondecreasing, one has

\[
I_2(t, x) \leq \int_0^t \int_{B(0,1)} b(Y_s) G_B(t - s, x, y) \, dy \, ds = \int_0^t b(Y_s) \, ds, \quad \forall t \in [0, T).
\]

Noting that \( I_3 \) is continuous almost surely, there is some \( M > 0 \) such that

\[
\sup_{t \in [0, T], x \in B(0,1)} |I_3(t, x)| \leq M.
\]

Substituting these estimates into the first formula, we get

\[
u_t(x) \leq \|u_0\|_\infty + \int_0^t b(Y_s) \, ds + M, \quad \forall t \in [0, T], \forall x \in B(0,1).
\]

Taking supremum over \( x \in B(0,1) \),

\[
Y_t \leq \tilde{M} + \int_0^t b(Y_s) \, ds, \quad \forall t \in [0, T),
\]

where \( \tilde{M} := \|u_0\|_\infty + M \). Consider

\[
Z_t = \tilde{M} + \int_0^t b(Z_s) \, ds
\]

with \( Z_0 = Y_0 \). By the comparison principle, \( Y_t \leq Z_t \). Since \( Y_t \) blows up at time \( t = T \), so does \( Z_t \). This immediately implies that \( b \) satisfies the so-called Osgood condition by the classical ODE theory (cf. [13]). \( \square \)
4 Proof of theorem 1.4

Proof of Theorem 1.4. Recall the mild formulation

\[ u_t(x) = \int_{\mathbb{R}^d} G(t, x, y)u_0(y) \, dy + \int_0^t \int_{\mathbb{R}^d} G(t - s, x, y)b(u_s(y)) \, dy \, ds + \int_0^t \int_{\mathbb{R}^d} G(t - s, x, y)W(dy \, ds). \]

Let \( \{t_n\} \) be a sequence such that \( t_n \to \infty \) as \( n \to \infty \) and Proposition 2.8 holds. Since \( b \) and \( u_0 \) are nonnegative,

\[ u_{t+t_n}(x) = \int_{\mathbb{R}^d} G(t + t_n, x, y)u_0(y) \, dy + \int_0^{t+t_n} \int_{\mathbb{R}^d} G(t - s, x, y)b(u_s(y)) \, dy \, ds + \int_0^{t+t_n} \int_{\mathbb{R}^d} G(t - s, x, y)W(dy \, ds) \]

\[ \geq \int_{\mathbb{R}^d} G(t + t_n, x, y)u_0(y) \, dy + \int_0^t \int_{\mathbb{R}^d} G(t - s, x, y)b(u_{s+t_n}(y)) \, dy \, ds + \int_0^t \int_{\mathbb{R}^d} G(t - s, x, y)W(dy \, ds) \]

\[ =: \sum_{i=1}^3 I_i(t, x, n). \]

Let

\[ Y_{t,n} := \inf_{y \in B(0,1)} u_{t+t_n}(y). \]

By Lemma 2.2 for any \( x \in B(0,1) \),

\[ I_2(t, x, n) \geq \int_0^t \int_{B(0,1)} G(t - s, x, y)b(u_{s+t_n}(y)) \, dy \, ds \]

\[ \geq \int_0^t b(Y_{s,n}) \int_{B(0,1)} G(t - s, x, y) \, dy \, ds \]

\[ \geq c \int_0^t b(Y_{s,n}) \, ds. \]

Note that for any \( x \in B(0,1) \) and \( t \in (0,1) \),

\[ I_3(t, x, n) = g_\beta(t_n + t, x) \geq \inf_{h \in [0,1],x \in B(0,1)} g_\beta(t_n + h, x). \]

Then we conclude that for any \( x \in B(0,1), t \in (0,1) \) and \( n \in \mathbb{N} \)

\[ u_{t+t_n}(x) \geq \inf_{h \in [0,1],x \in B(0,1)} g_\beta(t_n + h, x) + c \int_0^t b(Y_{s,n}) \, ds \]

Taking infimum over \( x \in B(0,1) \), we know that for any \( t \in (0,1) \) and \( n \in \mathbb{N} \),

\[ Y_{t,n} \geq \inf_{h \in [0,1],x \in B(0,1)} g_\beta(t_n + h, x) + c \int_0^t b(Y_{s,n}) \, ds \]

Consider for \( t \in (0,1) \) and \( n \in \mathbb{N} \)

\[ Z_{t,n} = \inf_{h \in [0,1],x \in B(0,1)} \left[ \int_{\mathbb{R}^d} G(t + t_n, x, y)u_0(y) \, dy + g_\beta(t_n + h, x) \right] + c \int_0^t b(Z_{s,n}) \, ds \quad (4.1) \]
By the comparison principle, \( Y_{t,n} \geq Z_{t,n} \).

Suppose that the solution does not blow up in finite time. Then for any \( n \in \mathbb{N} \), \( Y_{t,n} \) does not blow up. This means that the blow up time of \( Z_{t,n} \) has to be greater than 1. By the classical ODE theory (cf. \( [13] \) and (4.1),

\[
\int_{\inf t \in [0,1], x \in B(0,1)} g_{\beta}(t_n+h,x) \frac{ds}{b(s)} > 1.
\]

Letting \( n \to \infty \), it follows from the Osgood condition that the left hand side tends to zero. This leads to a contradiction and thus finishes the proof.

**Appendix**

Let \( B_t \) be a standard Brownian motion on \( \mathbb{R} \). Let \( h \in L^2_{loc}(\mathbb{R}_+) \), and set for \( t \geq 0 \)

\[
\xi(t) := \int_0^t h(t-s) dB_s, \quad R_t := \exp \left[ - \int_0^t \xi(s) dB_s - \frac{1}{2} \int_0^t |\xi(s)|^2 ds \right].
\]

**Lemma 4.1.** For any \( T > 0 \), \( \{R_t\}_{0 \leq t \leq T} \) is a uniformly integrable martingale.

**Proof.** For \( n \in \mathbb{N} \), let

\[
\tau_n := \inf \left\{ t \geq 0 : \int_0^t |\xi(s)|^2 ds \geq n \right\}.
\]

By the Girsanov theorem, for any \( t \in (0,T) \),

\[
\tilde{B}_s := B_s + \int_0^s \xi(r) dr, \quad 0 \leq s \leq t,
\]

is a Brownian motion under the weighted probability measure \( R_{\tau_n} \mathbb{P} \). Using the elementary inequality that \( |x-y|^2 \leq 2x^2 + 2y^2 \) for \( x, y \in \mathbb{R} \), we obtain that for \( s \in [0,t] \)

\[
\mathbb{E}_{R_{\tau_n}} \left[ |\xi(s \wedge \tau_n)|^2 \right] = \mathbb{E}_{R_{\tau_n}} \left[ \int_0^{s \wedge \tau_n} h(s \wedge \tau_n - r) d\tilde{B}_r - \int_0^{s \wedge \tau_n} h(s \wedge \tau_n - r) \xi(r) dr \right]^2 \\
\leq 2 \mathbb{E}_{R_{\tau_n}} \left[ \left( \int_0^{s \wedge \tau_n} h(s \wedge \tau_n - r) d\tilde{B}_r \right)^2 \right] + 2 \mathbb{E}_{R_{\tau_n}} \left[ \left( \int_0^{s \wedge \tau_n} h(s \wedge \tau_n - r) \xi(r) dr \right)^2 \right] \\
= 2I_1 + 2I_2.
\]

By the Itô isometry, one has

\[
I_1 = \mathbb{E}_{R_{\tau_n}} \left[ \int_0^{s \wedge \tau_n} |h(s \wedge \tau_n - r)|^2 dr \right] = \mathbb{E}_{R_{\tau_n}} \left[ \int_0^{s \wedge \tau_n} |h(r)|^2 dr \right] \\
\leq \int_0^t |h(r)|^2 dr \leq \int_0^t |h(r)|^2 dr =: A_t.
\]

It follows from the Hölder inequality that

\[
I_2 \leq \mathbb{E}_{R_{\tau_n}} \left[ \int_0^{s \wedge \tau_n} |h(s \wedge \tau_n - r)|^2 dr \int_0^{s \wedge \tau_n} |\xi(r)|^2 dr \right] \\
= \mathbb{E}_{R_{\tau_n}} \left[ \int_0^{s \wedge \tau_n} |h(r)|^2 dr \cdot \int_0^{s \wedge \tau_n} |\xi(r)|^2 dr \right] \\
\leq \int_0^t |h(r)|^2 dr \cdot \mathbb{E}_{R_{\tau_n}} \left[ \int_0^{s \wedge \tau_n} |\xi(s \wedge \tau_n)|^2 dr \right] \\
\leq A_t \cdot \int_0^t \mathbb{E}_{R_{\tau_n}} \left[ |\xi(s \wedge \tau_n)|^2 \right] dr.
\]

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Then we get
\[ E_{R_t \wedge \tau_n P} \left[ |\xi(s \wedge \tau_n)|^2 \right] \leq 2 A_t + 2 A_t \cdot \int_0^s E_{R_t \wedge \tau_n P} \left[ |\xi(r \wedge \tau_n)|^2 \right] \, dr, \quad 0 \leq s \leq t, \]
which, together with the Gronwall inequality, implies that
\[ E_{R_t \wedge \tau_n P} \left[ |\xi(s \wedge \tau_n)|^2 \right] \leq 2 A_t e^{2 A_t s}. \]
This yields that for \( t \in (0, T] \)
\[ E \left[ R_t \log R_t \wedge \tau_n \right] = E_{R_t \wedge \tau_n P} \left[ - \int_0^{t \wedge \tau_n} \xi(s) \, d\tilde{B}_s + \frac{1}{2} \int_0^{t \wedge \tau_n} |\xi(s)|^2 \, ds \right] 
= E_{R_t \wedge \tau_n P} \left[ \frac{1}{2} \int_0^{t \wedge \tau_n} |\xi(s)|^2 \, ds \right] 
= \frac{1}{2} \int_0^t E_{R_t \wedge \tau_n P} \left| \xi(s \wedge \tau_n) \right|^2 \, ds 
\leq A_t \int_0^t e^{2 A_t s} \, ds = \frac{1}{2} (e^{2 A_t t} - 1). \]
Since \( \lim_{n \to \infty} \tau_n = \infty \), it holds from the Fatou lemma that
\[ E \left[ R_t \log R_t \right] \leq \frac{1}{2} (e^{2 A_t t} - 1). \]
Thus,
\[ \sup_{t \in [0, T]} E \left[ R_t \log R_t \right] \leq \frac{1}{2} (e^{2 A_T T} - 1) < \infty, \]
and this completes the proof.

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