All-order $\alpha'$-expansion of one-loop open-string integrals

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We present a new method to evaluate the $\alpha'$-expansion of genus-one integrals over open-string punctures and unravel the structure of the elliptic multiple zeta values in its coefficients. This is done by obtaining a simple differential equation of Knizhnik–Zamolodchikov–Bernard-type satisfied by generating functions of such integrals, and solving it via Picard iteration. The initial condition involves the generating functions at the cusp $\tau \rightarrow i\infty$ and can be reduced to genus-zero integrals.

INTRODUCTION

Elliptic analogues of polylogarithms [1 2] and multiple zeta values [3] have become a driving force in higher-order computations of scattering amplitudes in quantum field theories and string theories. The study of their differential equations and their connections with modular forms turned into a vibrant research area at the interface of particle phenomenology, string theory and number theory. In the same way as a variety of Feynman integrals has been recently expressed in terms of elliptic polylogarithms and iterated integrals of modular forms [4 5], the low-energy expansion of one-loop open-string amplitudes introduces elliptic multiple zeta values (eMZVs) [6 8].

So far, the appearance of eMZVs in one-loop open-string amplitudes arose from direct integration over the punctures on a genus-one worldsheet of cylinder or Möbius-strip topology. Although there is no conceptual bottleneck in extending the techniques of [6–8] to arbitrary multiplicities and orders in the inverse string tension $\alpha'$, in this letter we will present a new method to evaluate these genus-one integrals which is related to elliptic associators [9] and Tsunogai’s derivations dual to Eisenstein series [10]. The results are given by eMZVs in their minimal form [3 11] and reveal elegant structures in the $\alpha'$-expansions. More details will be given in a longer companion paper [12].

OPEN-STRING INTEGRALS AT GENUS ONE

One-loop string amplitudes are described by correlation functions of vertex operators in a conformal field theory over a genus-one Riemann surface, the torus. The location of the vertex operator associated with the $j$th external string state is parameterized by the coordinates $z_j = u_j + v_j$ with $u_j, v_j \in (0, 1)$, where $\tau$ is the modulus with $\text{Im} \tau > 0$, see figure 1 and we define $z_{ij} \equiv z_i - z_j$.

By suitable involutions of the torus [12], one obtains the surfaces describing the scattering of open-string states, the cylinder and the Möbius strip. The two boundaries of the cylinder will be parameterized by the $A$-cycle $z_i \in (0,1)$ and its displacement $z_i \in \frac{1}{2} + (0,1)$ by half a $B$-cycle, i.e. $u_j \in \{0, \frac{1}{2}\}$ and $dz_j = dv_j$. See figure 2.

The massless $n$-point one-loop amplitudes of the open superstring give rise to integrals of the form ($z_1 = 0$) [9]

$$\int_{C(*)} \left( \prod_{j=2}^{n} dz_j \right) \sum_{i<j} s_{ij} G(z_{ij}, \tau),$$

with differing integration domains $C(*)$ for the cylinder and the Möbius strips. For planar cylinders, we set $* \rightarrow 1, 2, \ldots, n$ and parametrize the domain as

$$C(1, 2, \ldots, n) = \{z_2, \ldots, z_n \in \mathbb{R}, 0 < z_2 < \ldots < z_n < 1\},$$

see figure 2 and [12] for the non-planar analogue with $* \rightarrow \tau + 1, \ldots, n$. Furthermore, in the integrand of [1], $f^{(k)}(z_{ij}, \tau)$ denote the Laurent coefficients of the doubly-periodic Kronecker–Eisenstein series defined by [2 14]

$$\Omega(z, \eta, \tau) = \exp \left( 2\pi i \eta \frac{\text{Im} z}{\text{Im} \tau} \right) \frac{\theta'_{\tau}(0, \tau) \theta_{\tau}(z + \eta, \tau)}{\theta_{\tau}(z, \tau) \theta_{\tau}(\eta, \tau)},$$

$$\Omega(z, \eta, \tau) = \sum_{k=0}^{\infty} \eta^{k-1} f^{(k)}(z, \tau).$$

FIG. 1: We parameterize the torus through the lattice $\mathbb{Z} \times \mathbb{Z}$, with identifications $z \equiv z + 1 \equiv z + \tau$ along the $A$- and $B$-cycle.

FIG. 2: The cylinder parameterization.
The simplest examples of the coefficient functions are $f^{(0)}(z, \tau) = 1$ and $f^{(1)}(z, \tau) = \partial_z \log \theta_1(z, \tau) + 2\pi i \frac{m \tau + \nu}{\Im \tau}$, and higher $f^{(k\geq 2)}(z, \tau)$ do not have any poles in $z$.

Finally, exp \((\sum_{i<j} s_{ij} \mathcal{G}(z_{ij}, \tau))\) in (1) is the Koba–Nielsen factor written in terms of dimensionless Mandelstam invariants $s_{ij} = -2\alpha_k \cdot k_i$ and Green functions $\mathcal{G}(z, \tau)$ subject to the universal differential equation
\[
\partial_v \mathcal{G}(z_{ij}, \tau) = -f^{(1)}(z_{ij}, \tau)
\]
\[
2\pi i \partial_t \mathcal{G}(z_{ij}, \tau) = -f^{(2)}(z_{ij}, \tau) - 2C_2,
\]
where $\partial_v$ is the derivative along the cylinder boundary, and $\zeta_n = \sum_{k=1}^\infty \frac{1}{n^2}$ with $n \geq 2$ denote Riemann zeta values.

### A. Generating functions:

Instead of handling the $\alpha'$-expansion of the individual integrals (1) as in the method of [6,8], we will evaluate the following generating function of integrals (with $\eta_1, \eta_2, \ldots, \eta_n$)
\[
Z_{\eta}^i(1, 2, \ldots, n) = \int \prod_{j=2}^n \mathrm{d}z_j \exp \left( \sum_{i \neq j} s_{ij} \mathcal{G}(z_{ij}, \tau) \right) (6)
\]
\[
\times \Omega(z_{12}, \eta_3, \ldots, \eta_n) \ldots \Omega(z_{n-1, n}, \eta_n, \tau).
\]

The integrands $f^{(k_{ij})}_{ij}$ in (1) relevant to $n$-point open-superstring amplitudes have $k_1 + k_2 + \ldots = n - 4$ and reside at the order of $\eta_3^{-3}$ of [6]. Moreover, $(n \geq 8)$-point integrands additionally involve holomorphic Eisenstein series $G_{z>4}(\tau) = -f^{(0)}(0, \tau)$ multiplying (1) at $k_1 + k_2 + \ldots = n - 4 - \ell$ as seen at the $\eta_3^{-3-\ell}$-order of (6).

Although the cylinder contribution to one-loop open-superstring amplitudes is localized at purely imaginary $\tau$ as drawn in figure 2, we will define and evaluate the integrals (10) for generic $\tau$ in the upper half plane with $\Re \tau \neq 0$. In view of the parental torus, $Z_{\eta}^i(1, 2, \ldots, n)\) and $Z_{\eta}^i(1, 2, \ldots, n, |)$ will be referred to as planar and non-planar $A$-cycle integrals, respectively.

Möbius-strip integrals can be reconstructed by specializing planar $A$-cycle integrals to $\Re \tau = \frac{1}{2}$, and the cancellation of tadpole divergences from one-loop open-superstring amplitudes can be analyzed as in [13].

The $A$-cycle integrand (10) at $n$ points involves $n-1$ factors of the Kronecker–Eisenstein series (4) at different arguments. The second entry $Z_{\eta}^i(A)$ specifies permutations $A = a_1 a_2 \ldots a_n \in S_n$ of these arguments, and $\Omega(\ldots)$ at different $z_{a_j}, \eta_{a_j}$ are related by the Fay identity
\[
\Omega(z_1, \eta_1, \tau) \Omega(z_2, \eta_2, \tau) = \Omega(z_1, \eta_1 + \eta_2, \tau) \Omega(z_2 - z_1, \eta_2, \tau) + \Omega(z_2, \eta_1 + \eta_2, \tau) \Omega(z_1 - z_2, \eta_2, \tau).
\]

Repeated use of (7) and imposing $\eta_j = -\sum_{j=2}^n \eta_j$ only leaves $(n-1)!$ independent permutations of the integrand in (10), and we will use a basis of $Z_{\eta}^i(1, 1, B)$ with permutations $B \in S_{n-1}$ acting on $2, 3, \ldots, n$.

### B. The differential equation:

As will be detailed in [12], the $\tau$-derivatives of (6) can be written as
\[
2\pi i \partial_v Z_{\eta}^i(1, 1, B) = \sum_{C \in S_{n-1}} D_{\eta}^i(B|C) Z_{\eta}^i(1, 1, C),
\]
where the $(n-1)! \times (n-1)!$ matrix $D_{\eta}^i$ is a differential operator w.r.t. $\eta_j$. Its detailed form will be exemplified in the next section and follows from the properties of the Green function, the vanishing of boundary terms $\int \mathrm{d}v_j \partial_{\eta_j} \ldots$ and the mixed heat equation $u|_v (u, v \in \mathbb{R})$
\[
2\pi i \partial_v \Omega(\eta \tau + v, \eta, \tau) = \partial_\eta \partial_\eta \Omega(\eta \tau + v, \eta, \tau).
\]

Most importantly, the form of $D_{\eta}^i(B|C)$ does not depend on the planar or non-planar integration cycle $A$, and its entries are linear in the dimensionless Mandelstam invariants $s_{ij}$ and therefore in $\alpha'$.

Hence, the $\alpha'$-expansion of the genus-one integrals $Z_{\eta}^i$ follows from the solution of (5) via Picard iteration,
\[
Z_{\eta}^i(A|1, 1, B) = \sum_{k=0}^\infty \left( \frac{1}{2\pi i} \right)^k \int_1^\infty \mathrm{d}r_1 \int_1^\infty \mathrm{d}r_2 \ldots \int_1^\infty \mathrm{d}r_{k-1} \int_1^\infty \mathrm{d}r_k
\]
\[
\times \sum_{C \in S_{n-1}} (D_{\eta}^{i*} \ldots D_{\eta}^{i_1}) (B|C) Z_{\eta}^i(A|1, C)
\]
with matrix products $D_{\eta}^{i*} \ldots D_{\eta}^{i_1}$. As an initial value, the degeneration $Z_{\eta}^i(\tau)$ at the cusp $\tau \rightarrow \infty$ will be expressed in terms of disk integrals with two additional punctures from the pinching of the $A$-cycle in figure 1.

As will be detailed in [12], the entire $\tau$-dependence of $D_{\eta}^i$ is carried by Weierstrass functions (with $G_0 = -1$)
\[
\varphi(\eta, \tau) = -\frac{G_0}{\eta^2} + \sum_{k=4}^\infty (k-1) \eta^{k-2} G_k(\tau).
\]

This allows us to decompose
\[
D_{\eta}^i = \sum_{k=0}^\infty (1-k) G_k(\tau) r_{\eta}(\epsilon_k),
\]
where $r_{\eta}(\epsilon_k)$ are $(n-1)! \times (n-1)!$ matrices whose entries are independent of $\tau$, rational functions of $\eta_j$, linear in $s_{ij}$ and may involve second derivatives $\partial_\eta \partial_\eta \eta_j$. Note that $r_{\eta}(\epsilon_2) = 0$ and $r_{\eta}(\epsilon_{2p-1}) = 0$ $\forall p \in \mathbb{N}$ by (11).

### C. The main result:

Based on (12), the open-string integrals (10) can be expressed in terms of iterated Eisenstein integrals
\[
\gamma(k_1, k_2, \ldots, k_r | \tau) = \int_{1}^{\infty} \frac{\mathrm{d}r'}{2\pi i} G_{k_r}(r') \gamma(k_1, \ldots, k_{r-1} | r')
\]
subject to $\gamma(\emptyset | \tau) = 1$ and tangential-base-point regularization [16], e.g. $\gamma(0 | \tau) = \frac{1}{\eta^2\tau}$. As the main result of this work, we can therefore bring the open-string $\alpha'$-expansion into the following elegant form:
\[
Z_{\eta}^i(A|1, 1) = \sum_{r=0}^\infty \sum_{\{k_1, k_2, \ldots, k_r\}} \sum_{\{\epsilon_{k_1}, \epsilon_{k_2}, \ldots, \epsilon_{k_r}\}} \gamma(k_1, k_2, \ldots, k_r | \tau)
\]
\[
\times \prod_{j=1}^r (k_j-1) \sum_{C \in S_{n-1}} r_{\eta}(\epsilon_{k_1}, \epsilon_{k_2}, \ldots, \epsilon_{k_j} k_j) B C Z_{\eta}^i(A|1, C),
\]
where \( r_{p}(\epsilon_{k}, \ldots, \epsilon_{k'}) \equiv r_{p}(\epsilon_{k}) \cdots r_{p}(\epsilon_{k'}) \). Since each order in \( \alpha' \) is expressible in terms of eMZVs, the \( r_{p}(\epsilon_{k}) \) should be matrix representations of Tsung-gai’s derivations \( \epsilon_{k} \) dual to Eisenstein series. In particular, (12) brings the differential equation of \( Z'_{\eta_{2}} \) into the same form as that of the elliptic Knizhnik–Zamolodchikov–Bernard associator, where the derivations \( \epsilon_{k} \) act on its non-commutative arguments.

The decomposition of eMZVs into iterated Eisenstein integrals automatically incorporates all their relations over the rational numbers. Moreover, the derivation of (14) does not rely on any relation among the Mandelstam invariants. The \( n \)-point results of this work are valid for \( \frac{1}{2}n(n-1) \) independent \( s_{ij} \), and one can still impose momentum conservation when applying the \( \alpha' \)-expansion of \( Z'_{\eta_{2}} \) to string amplitudes.

**EXAMPLES FOR DIFFERENTIAL OPERATORS**

In this section, we present \((n<4)\)-point examples of the matrix-valued differential operators \( D'_{\eta_{2}} \) and the four-point case is relegated to the appendix. All-multiplicity expressions as well as detailed derivations of the differential equations can be found in [12].

**A. Two points** allow for a single planar and non-planar \( A \)-cycle integral [9] each,

\[
Z'_{\eta_{2}}(1, 2|1, 2) = \int_{0}^{1} dv_{2} \Omega(v_{12}, \eta_{2}, \tau) e^{s_{12}G(v_{12}, \tau)} \quad (15)
\]

\[
Z'_{\eta_{2}}(\frac{1}{2}, 1, 2) = \int_{0}^{1} dv_{2} \Omega(v_{12}+\frac{2}{3}, \eta_{2}, \tau) e^{s_{12}G(v_{12}+\frac{2}{3}, \tau)}.
\]

Their \( \tau \)-derivatives resulting from [5], [1] and integration by parts w.r.t. \( v_{2} \) take the universal form

\[
2\pi i\partial_{\tau}Z'_{\eta_{2}}(\ast|1, 2) = s_{12}\left(\frac{1}{2}\partial^{2}_{\eta_{2}} - \varphi(\eta_{2}, \tau) - 2\zeta_{2}\right)Z'_{\eta_{2}}(\ast|1, 2),
\]

so one can read off the scalar differential operator in [8] and the resulting representation of the derivations,

\[
D'_{\eta_{2}}(2|2) = s_{12}\left(\frac{1}{2}\partial^{2}_{\eta_{2}} - \varphi(\eta_{2}, \tau) - 2\zeta_{2}\right),
\]

\[
r_{\eta_{2}}(\epsilon_{0}) = s_{12}\left(1 - 2\zeta_{2} - \frac{1}{2}\partial^{2}_{\eta_{2}}\right), \quad r_{\eta_{2}}(\epsilon_{k}\geq 4) = s_{12}\eta_{2}^{k-2}.
\]

Note that various combinations of iterated Eisenstein integrals drop out from the two-point instance of [14] since commutators \([r_{\eta_{2}}(\epsilon_{k_{1}}), r_{\eta_{2}}(\epsilon_{k_{2}})]\) with \( k_{1}, k_{2} \geq 4 \) vanish.

**B. Three points** give rise to \( A \)-cycle integrals

\[
Z'_{\eta_{2}, \eta_{3}}(\ast|1, 2, 3) = \int_{c(\ast)} dz_{2} dz_{3} \Omega(z_{12}, \eta_{2}+\eta_{3}, \tau) \times \Omega(z_{23}, \eta_{3}, \tau) e^{s_{12}G(z_{12}, \tau) + s_{13}G(z_{13}, \tau) + s_{23}G(z_{23}, \tau)} \quad (18)
\]

that mix under \( \tau \)-derivatives \((s_{12}...p) \equiv \sum_{1 \leq i < j} s_{ij}\).

\[
2\pi i\partial_{\tau}Z'_{\eta_{2}, \eta_{3}}(\ast|1, 2, 3) = \left(-2\zeta_{2}s_{123}\right)
\]

\[
+ s_{12}\left[\frac{1}{2}\partial^{2}_{\eta_{2}} - \varphi(\eta_{2}+\eta_{3}, \tau)\right] + s_{13}\left[\frac{1}{2}\partial^{2}_{\eta_{3}} - \varphi(\eta_{3}, \tau)\right] + s_{23}\left[\frac{1}{2}(\partial_{\eta_{2}} - \partial_{\eta_{3}})^{2} - \varphi(\eta_{3}, \tau)\right] + s_{31}\left[\frac{1}{2}(\partial_{\eta_{3}} - \partial_{\eta_{2}})^{2} - \varphi(\eta_{2}, \tau)\right] + s_{13}\left[\varphi(\eta_{2}+\eta_{3}, \tau) - \varphi(\eta_{3}, \tau)\right]Z'_{\eta_{2}, \eta_{3}}(\ast|1, 3, 2),
\]

The resulting matrix entries of the \( 2 \times 2 \) differential operator in [8] read

\[
D'_{\eta_{2}, \eta_{3}}(2, 3|2, 3) = -2\zeta_{2}s_{123} + s_{12}\left[\frac{1}{2}\partial^{2}_{\eta_{2}} - \varphi(\eta_{2}+\eta_{3}, \tau)\right]
\]

\[
+ s_{13}\left[\frac{1}{2}(\partial_{\eta_{2}} - \partial_{\eta_{3}})^{2} - \varphi(\eta_{3}, \tau)\right] + s_{23}\left[\frac{1}{2}\partial^{2}_{\eta_{3}} - \varphi(\eta_{2}, \tau)\right] + s_{31}\left[\frac{1}{2}(\partial_{\eta_{3}} - \partial_{\eta_{2}})^{2} - \varphi(\eta_{2}, \tau)\right],
\]

and the first row is always sufficient to generate the remaining entries via permutations of \( s_{ij} \) and \( \eta_{j} \), e.g.

\[
D'_{\eta_{2}, \eta_{3}}(3, 2|3, 2) = D'_{\eta_{2}, \eta_{3}}(2, 3|2, 3)\left|_{s_{123} \rightarrow s_{123} + s_{12}}\right.
\]

\[
D'_{\eta_{2}, \eta_{3}}(3, 2|3, 2) = D'_{\eta_{2}, \eta_{3}}(2, 3|2, 3)\left|_{s_{123} \rightarrow s_{123} + s_{12}}\right.
\]

One can read off the \( 2 \times 2 \) matrix representations of the derivations \((k \neq 2)\),

\[
r_{\eta_{2}, \eta_{3}}(\epsilon_{k}) = \delta_{k, 0}\left(2\zeta_{2}s_{123} - \frac{1}{2}s_{23}(\partial_{\eta_{2}} - \partial_{\eta_{3}})^{2} - \frac{1}{2}s_{12}\partial^{2}_{\eta_{2}} - \frac{1}{2}s_{13}\partial^{2}_{\eta_{3}}\right)
\]

\[
+ \eta_{2}^{k-2}\left(\frac{1}{2}s_{123}\partial_{\eta_{3}} - \frac{1}{2}s_{123}\partial_{\eta_{2}}\right),
\]

where \([r_{\eta_{2}, \eta_{3}}(\epsilon_{k_{1}}, \epsilon_{k_{2}})]\) no longer vanish individually, and relations in the derivation algebra [10][11][17] hold non-trivially.

**EXAMPLES FOR INITIAL VALUES**

This section is dedicated to the degeneration of \( A \)-cycle integrals [6] at the cusp \( \tau \rightarrow i\infty \) which enters the \( \alpha' \)-expansion [14] as an initial value.

**A. Generalities:** The behaviour of \( A \)-cycle integrals at the cusp is most conveniently studied in the variables

\[
\sigma_{j} = e^{2\pi iz_{j}}, \quad dz_{j} = \frac{d\sigma_{j}}{2\pi i\sigma_{j}}, \quad G_{ij} = 2\pi i\frac{\sigma_{j} - \sigma_{i}}{\sigma_{j} - \sigma_{i}},
\]

where the planar Green function and Kronecker–Eisenstein series degenerate to \((\sigma_{ji} \equiv \sigma_{j} - \sigma_{i})\)

\[
\lim_{\tau \rightarrow \infty} \Omega(v_{ij}, \eta, \tau) = \pi \cot(\pi \eta) + G_{ij}
\]

\[
\lim_{\tau \rightarrow \infty} G(v_{ij}, \tau) = \frac{1}{2} \log(\sigma_{j}) + \frac{1}{2} \log(\sigma_{j}) - \log(\sigma_{ji}).
\]

Their non-planar analogues take an even simpler form,

\[
\lim_{\tau \rightarrow \infty} \Omega(v_{ij} + \frac{2}{\pi}, \eta, \tau) = \frac{\pi}{\sin(\pi \eta)}, \quad \lim_{\tau \rightarrow \infty} G(v_{ij} + \frac{2}{\pi}, \tau) = 0.
\]
Since string-theory applications of [14] involve the coefficients w.r.t. $\eta$, we will need the expansions
\[
\pi \cot(\pi \eta) = \frac{1}{\eta} - 2 \sum_{k=1}^{\infty} \zeta_{2k} \eta^{2k-1}.
\]
(26)
\[
\pi \sin(\pi \eta) = \frac{1}{\eta} + \sum_{k=1}^{\infty} \frac{2^{2k-1} - 1}{2^{2k-2}} \zeta_{2k} \eta^{2k-1}.
\]
(27)

As will be detailed in [12], the $\sigma_j$-integration in $n$-point $Z_{\eta}^{\infty}$ lines up with explicitly known combinations of $N = (n+2)$-point disk integrals [13]
\[
Z_{\eta}^{\text{tree}}(a_1, a_2, \ldots, a_N | 1, 2, \ldots, N) = \int \frac{d\sigma_1 \, d\sigma_2 \ldots \, d\sigma_N}{\text{vol } SL_2(\mathbb{R})} \prod_{i<j}^{N} |\sigma_{ij}|^{-s_{ij}} \prod_{i=N-1}^{N} |\sigma_{iN}|^{-s_{iN}}. \quad (28)
\]

The two extra punctures $n+1 \to +$ and $n+2 \to -$ are associated with Mandelstam invariants
\[
s_{j+} = s_{j-} = -\frac{1}{2} \sum_{1 \leq i < j} n_{ij}, \quad s_{+,+} = \sum_{1 \leq i < j} n_{ij}. \quad (29)
\]

The $\alpha'$-expansion of (27) and therefore $Z_{\eta}^{\infty}$ involves multiple zeta values (MZVs) which can be systematically generated from the all-multiplicity methods of [19, 20].

**B. Two points:** Planar initial values at two points descend from four-point tree-level integrals,
\[
Z_{\eta}^{\infty}(1, 2 | 1, 2) = \pi \cot(\pi \eta) \frac{\pi s_{12}}{2} \sin \left( \frac{\pi s_{12}}{2} \right)
\]
\[
\times \int_{0}^{1} d\sigma_2 \frac{1}{2\pi i \sigma_2} \sigma^{s_{12}/2} (1 - \sigma_2)^{-s_{12}}
\]
\[
= \pi \cot(\pi \eta) \frac{\Gamma(1 - s_{12})}{\Gamma(1 - s_{12})}. \quad (30)
\]

The factor of $2i \sin(\frac{\pi s_{12}}{2})$ and similar trigonometric functions below stem from contour deformations detailed in [12]. The gamma functions with standard $\alpha'$-expansion
\[
\frac{\Gamma(1 - s_{12})}{\Gamma(1 - s_{12})} = \exp \left( \sum_{k=2}^{\infty} \frac{\zeta_k}{k} (1 - 2^{1-k}) s_{12}^k \right)
\]
\[
= 1 + \frac{s_{12}^2}{12} + \frac{1}{4} s_{12}^3 + \frac{19}{160} s_{12}^4 + \mathcal{O}(\alpha'^5)
\]
do not appear in the non-planar counterpart of (29)
\[
Z_{\eta}^{\infty}(1, 2 | 1, 2) = -\frac{\pi}{\sin(\pi \eta)}. \quad (31)
\]

**C. Three points:** Degenerate $A$-cycle integrals at three points introduce five-point disk integrals,
\[
Z_{\eta, \eta, \eta}^{\infty}(1, a_2, a_3 | 1, 2, 3) = \pi^2 \left( \cot(\pi \eta) \cot(\pi \eta) + \frac{s_{13}}{s_{123}} \right) \int_{\text{tree}}(1, a_2, a_3 | 1) + \pi \left( \cot(\pi \eta) + \frac{s_{23}}{s_{12}} \cot(\pi \eta) \right) \int_{\text{tree}}(1, a_2, a_3 | G_{23}), \quad (32)
\]

where
\[
\int_{\text{tree}}(1, a_2, a_3 | 1) = -\frac{1}{2\pi^2} \left[ \sin \left( \frac{\pi}{2} (s_{1a_2} + s_{23}) \right) \sin \left( \frac{\pi}{2} s_{1a_3} \right) \right.
\]
\[
\times \left( Z_{\eta}^{\text{tree}}(+, a_2, a_3, 1, -|+, 2, 3, -, -) \right.
\]
\[
+ Z_{\eta}^{\text{tree}}(+, a_2, a_3, 1, -|+, 3, 2, -, -) \left\} + (2 \leftrightarrow 3) \right)
\]
\[
\int_{\text{tree}}(1, a_2, a_3 | G_{23}) = \frac{1}{2\pi} \left[ \sin \left( \frac{\pi}{2} (s_{1a_2} + s_{23}) \right) \cos \left( \frac{\pi}{2} s_{1a_3} \right) \right.
\]
\[
\times \left( Z_{\eta}^{\text{tree}}(+, a_2, a_3, 1, -|+, 2, 3, -, -) \right.
\]
\[
- Z_{\eta}^{\text{tree}}(+, a_2, a_3, 1, -|+, 3, 2, -, -) \left\} + (2 \leftrightarrow 3) \right).
\]

Their leading low-energy orders read [12]
\[
\int_{\text{tree}}(1, 2, 3 | 1) = -\frac{1}{2} + \left( s_{12}^2 + s_{13}^2 + s_{23}^2 \right) + \mathcal{O}(\alpha'^3) \quad (33)
\]
\[
\int_{\text{tree}}(1, 2, 3 | G_{23}) = \frac{1}{s_{23}} + \left( s_{12}^2 + s_{13}^2 + s_{23}^2 \right)^2 + \mathcal{O}(\alpha'^2)
\]

and exemplify that integrals over $k$ factors of $G_{ij}$ in [23] may have up to $k$ kinematic poles.

Non-planar three-point initial values in turn boil down to four-point disk integrals with $\alpha'$-expansions in [30],
\[
Z_{\eta, \eta, \eta}^{\infty}(1, 2 | 1, 2, 3) = \frac{\pi^2 \cot(\pi \eta)}{\sin(\pi \eta)} \left\{ \frac{\Gamma(1 - s_{12})}{\Gamma(1 - s_{12})} \right\} \quad (34)
\]
\[
Z_{\eta, \eta, \eta}^{\infty}(1, 2 | 1, 3, 2) = \frac{\pi^2}{\sin(\pi \eta) \sin(\pi \eta)} \left\{ \frac{\Gamma(1 - s_{12})}{\Gamma(1 - s_{12})} \right\} \quad (35)
\]

**CONCLUSIONS AND FURTHER DIRECTIONS**

In this letter we presented a method to expand a generating series of genus-one integrals [6] relevant to one-loop open-string amplitudes. At each order in the inverse string tension $\alpha'$, our main result [14] pinpoints the accompanying eMZVs in their minimal and canonical representation via iterated Eisenstein integrals.

Genus-zero integrals relevant to open-string tree amplitudes obey Knizhnik–Zamolodchikov equations with a characteristic linear factor of $\alpha'$ on their right-hand side [19]. This structure is analogous to the $\varepsilon$-form of differential equations among Feynman integrals with dimensional-regularization parameter $\varepsilon$ [2, 21], suggesting a correspondence between $\alpha'$ and $\varepsilon$. By the linearity of the differential operators $D^2_{\eta}$ in $s_{ij} = -2\alpha' k_i \cdot k_j$, the Knizhnik–Zamolodchikov–Bernard-type equation also becomes linear in $\alpha'$. So our results generalize this intriguing correspondence to genus one and provide the string-theory analogue of the $\varepsilon$-form for differential equations of elliptic Feynman integrals [5].

The generating functions $Z_2^\eta$ are expected to comprise any moduli-space integral in massless one-loop amplitudes of open bosonic strings and superstrings upon expansion in $\eta$. Accordingly, they are proposed to generalize the universal disk-integrals [27] that appear in the
double-copy representation of string tree-level amplitudes [18, 22]. Hence, the study of the genus-one integrals \( Z^g_1 \) is an essential step towards universal double-copy structures in one-loop amplitudes of different string theories that generalize those of the superstring [23].

The generating functions \( Z^g_1 \) can be adapted to a closed-string context, encoding the integrals over torus punctures in one-loop amplitudes of type-II, heterotic and closed bosonic string theories. Closed-string analogues of \( Z^g_1 \) will be shown [24] to obey similar differential equations and to shed new light on the properties of modular graph forms [25] including their relation with open-string amplitudes [26].

Moreover, the method of this work to infer moduli-space integrals from differential equations should be applicable at higher loops. In the same way as disk integrals were used as the initial value for our one-loop results, higher-genus integrals in string amplitudes are expected to obey differential equations w.r.t. complex-structure moduli such that their separating and non-separating degenerations set the initial conditions. It would be interesting to explore a differential-equation approach of this type to the higher-genus modular graph functions of [27].

In summary, our new approach to one-loop open-string amplitudes via differential equations connects with state-of-the-art techniques in particle phenomenology and provides explicit matrix representations of profound number-theoretic structures. As will be elaborated in [12], our results manifest important formal properties of string amplitudes such as uniform transcendentality, coaction formulae and the dropout of twisted eMZVs from non-planar open-string amplitudes.

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**APPENDIX: FOUR-POINT EXAMPLES**

This appendix provides further details on the expansion [14] of four-point A-cycle integrals [6].

**A. Differential equation:** The 6 × 6 differential operator \( D^e_\ell = D^e_{\ell_1, \ell_2, \ell_3, \ell_4} \) in [8] is determined by

\[
D^e_\ell(2, 3, 4|2, 3, 4) = \sum_{j=2}^4 \frac{s_{ij}}{2} \partial^2_{\eta_j} + \sum_{2 \leq i < j} \frac{s_{ij}}{2} (\partial_{\eta_i} - \partial_{\eta_j})^2 - s_{12} \wp(\eta_{234}, \tau) - (s_{13} + s_{23}) \wp(\eta_{34}, \tau) - (s_{14} + s_{24} + s_{34}) \wp(\eta_{41}, \tau) - 2 \wp^2(\eta_{1234})
\]

(36)

\[
D^e_\ell(2, 3, 4|2, 4, 3) = (s_{14} + s_{24}) [\wp(\eta_{34}, \tau) - \wp(\eta_{41}, \tau)]
\]

\[
D^e_\ell(2, 3, 4|3, 2, 4) = s_{13} [\wp(\eta_{234}, \tau) - \wp(\eta_{34}, \tau)]
\]

\[
D^e_\ell(2, 3, 4|4, 2, 3) = s_{14} [\wp(\eta_{34}, \tau) - \wp(\eta_{41}, \tau)]
\]

\[
D^e_\ell(2, 3, 4|4, 3, 2) = s_{14} [\wp(\eta_{34}, \tau) - \wp(\eta_{41}, \tau)]
\]

with \( \eta_{i...p} = \eta_i + \eta_j + \ldots + \eta_p \). The corresponding matrix representations of the derivations \((k \neq 2)\)

\[
\begin{align*}
& r_\ell(e_k) = \eta_{234}^{-k} r_\ell(e_{234}) + \sum_{2 \leq i < j} \eta_{i}^{k-2} r_\ell(e_{ij}) + \sum_{i=2}^4 \eta_{i}^{k-2} r_\ell(e_i) \\
& + \delta_{k,0} (2 \wp^2(\eta_{1234}) - \sum_{2 \leq i < j} \frac{s_{ij}}{2} (\partial_{\eta_i} - \partial_{\eta_j})^2 - \sum_{i=2}^4 \frac{s_{i}}{2} \partial^2_{\eta_i}) 16 \times 6
\end{align*}
\]

(37)

can be assembled from \((S_{1234} = s_{14} + s_{24} + s_{34})\)

\[
\begin{pmatrix}
\frac{s_{12}}{s_{12}} & 0 & -s_{13} & -s_{13} & 0 & s_{14} \\
0 & \frac{s_{12}}{s_{12}} & 0 & s_{13} & -s_{14} & -s_{14} \\
-s_{13} & -s_{13} & 0 & \frac{s_{14}}{s_{14}} & 0 & 0 \\
0 & 0 & s_{12} & 0 & s_{14} & 0 \\
-s_{14} & -s_{14} & s_{13} & 0 & \frac{s_{13}}{s_{13}} & 0 \\
-s_{14} & -s_{14} & s_{13} & 0 & s_{14} & 0 \\
\end{pmatrix}
\]

(38)

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
S_{1234} & s_{14} + s_{24} & s_{14} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S_{1234} & s_{14} + s_{34} & s_{14} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(39)

and relabellings.

**B. Initial values:** The four-point integrals in massless one-loop string amplitudes descend from orders of \( Z^g_1 \) with odd homogeneity degree in \( \eta_j \). Since the derivations [37] do not mix odd and even functions of \( \eta_j \), we only spell out the odd part of the planar initial value

\[
Z^\text{grav}_{\ell}(\ast|1, 2, 3, 4)^{\text{odd}}_{\text{planar}} = \pi \cot(\pi \eta_{34}) \text{I}^{\text{free}}_{\ell}(\ast|G_{12}G_{34}) + \pi \cot(\pi \eta_{14}) \text{I}^{\text{free}}_{\ell}(\ast|G_{12}G_{34}) - s_{14} \text{I}^{\text{free}}_{\ell}(\ast|G_{13}G_{2}) + \pi \cot(\pi \eta_{23} + \pi \eta_{34}) \text{I}^{\text{free}}_{\ell}(\ast|G_{12}G_{34}) - s_{14} \text{I}^{\text{free}}_{\ell}(\ast|G_{13}G_{2}) + 6 \wp^2(\pi \cot(\pi \eta_{14} - s_{14} + s_{24} + s_{34}) \text{I}^{\text{free}}_{\ell}(\ast|1) + s \cot(\pi \eta_{234}) \cot(\pi \eta_{34}) \cot(\pi \eta_{4}) \text{I}^{\text{free}}_{\ell}(\ast|1) = 0.
\]
Similar to (34), \( f_{\text{tree}} \) denote combinations of six-point disk integrals \([27]\) which no longer depend on \( \eta_j \), see section 5.5 of \([12]\) for further details.

Non-planar four-point initial values reduce to four- and five-point disk integrals, e.g.

\[
Z^{\infty}_{\eta}(3,4|1,2,3,4) = \frac{\pi^3 \cot(\pi \eta_{234}) \cot(\pi \eta_4)}{\sin(\pi \eta_{234})} \times \frac{\Gamma(1-s_{12}) \Gamma(1-s_{34})}{\Gamma(1-\frac{s_4}{3}) \Gamma(1-\frac{s_3}{4})^2}
\]

(40)

\[
Z^{\infty}_{\eta}(3,4|1,2,3,4) = \frac{\pi}{\sin(\pi \eta_{234})} Z^{\infty}_{\eta_{234},\eta_3}(2,3,4|2,3,4),
\]

see \([32]\) for \( Z^{\infty}_{\eta_{234},\eta_3}(2,3,4|2,3,4) \). By extracting the order of \( \eta_{34} \eta_{34}^{-1} \eta_4 \) from \([14]\), we have checked \([36, 39, 40]\) to reproduce the \( \alpha' \)-expansions of \([6, 7]\) to the order of \( \alpha'^2 \) and \( \alpha'^3 \) in the planar and non-planar sectors, respectively.

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