Dynamical energy transfer in ac driven quantum systems

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We analyze the time-dependent energy and heat flows in a resonant level coupled to a fermionic continuum. The level is periodically forced with an external power source that supplies energy into the system. Based on the tunneling Hamiltonian approach and scattering theory, we discuss the different contributions to the total energy flux. We then derive the appropriate expression for the dynamical dissipation, in accordance with the fundamental principles of thermodynamics. Remarkably, we find that the dissipated heat can be expressed as a Joule law with a universal resistance that is constant at all times.

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Quite generally, energy flows through a physical system coupled to a power source. In the last decades, typical system sizes have been reduced to the nanoscale and, as a consequence, energy transfer is to be treated quantum-mechanically [1]. Fundamental aspects of light-powered biological energy transport [2] and, ultimate refrigeration protocols [3] have been recently uncovered using quantum mechanical principles. However, most discussions are limited to stationary or time-averaged properties [4–8].

Time-dependent quantum transport reveals the dynamical scales that dominate charge transfer across phase-coherent conductors [10, 11]. A prominent example is the experimentally realized quantum capacitor, which exhibits a pure ac response [12, 13]. Applied time-periodic potentials also become a crucial tool to generate directed transport of charge and spin in spatially asymmetric ratchet-like systems [14, 15] and to control matter tunneling in Bose-Einstein condensates [16]. Furthermore, the study of ac-driven quantum systems sheds light on the role of fluctuating forces in nanoelectromechanical resonators [17, 18]. Several aspects related to time-dependent energy transport in electronic systems have been also investigated. Heat production in nanoscale engines is discussed in Refs. [19, 20], while molecular heat pumping against thermal gradients is proposed in Ref. [21]. Furthermore, the concept of local temperature in ac pumps has been generalized in Ref. [22] whereas universal thermal resistance has been predicted for low-temperature dynamical transport in Ref. [23].

Here, we aim at the time-resolved energy production and redistribution in ac-driven quantum coherent electron systems. We show that the coupling between the different parts of the system not only provides a necessary mechanism for particle exchange, like in the case of charge transport, but also contributes to the energy transport. This contribution is of ac nature. Though the time-average of this energy vanishes, it allows for a temporary energy storage. Therefore, the coupling region can be referred to as an energy reactance, which only affects peak power developed in the dynamics. Our goal is also to discuss which portion of the time-resolved energy can be identified as heat, in accordance to the fundamental laws of the thermodynamics.

To be more precise let us consider a simple but generic model, the resonant level model sketched in Fig. 1. It describes a localized fermion (the impurity) coupled to a fermionic band of continuous density of states (the reservoir). This model has been widely used across disciplines to study asymmetric atomic spectra [24], dissipative quantum mechanics [25] and resonant-tunneling semiconductor heterostructures [26], to name a few. Transitions from the quantum level to the reservoir yield a finite lifetime to the localized fermion which can be represented with a Lorentzian density of states. We consider the case in which the level is attached to a harmonically driven

FIG. 1: Energy diagram of the system under consideration. A single electronic level (the impurity with charge e) is coupled to a Fermi sea (the reservoir with chemical potential µ). Energy is supplied into the system by a power source (amplitude Vac and frequency Ω) attached to the quantum level. Thus, energy rates are created not only at the impurity (Wp) but also at the reservoir (WC) and in the contact region (WT).

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power source as in Fig. 1. Then, the Hamiltonian reads,

$$\mathcal{H} = \mathcal{H}_C + \mathcal{H}_T + \mathcal{H}_D(t), \quad (1)$$

where $\mathcal{H}_C = \sum_k \varepsilon_k c_k^\dagger c_k$ is the continuum of electron states with wavevector $k$ and band energy $\varepsilon_k$, $\mathcal{H}_T = \sum_k (w_k d_k^\dagger + h.c.)$ describes the tunneling hybridization between propagating electrons and the localized fermion with coupling amplitude $w_k$, and $\mathcal{H}_D(t) = \varepsilon_d(t)d^\dagger d$ represents the impurity Hamiltonian with a time-dependent energy level $\varepsilon_d(t) = \varepsilon_0 + V_{ac} \cos(\Omega t)$, $\varepsilon_0$ being the energy of the bare level. This model can be implemented, e.g., using an electronic terminal coupled to a quantum dot acting as an artificial impurity [12, 13] which, in turn, is interacting with a nearby capacitive gate with harmonic driving potential $V_{ac} \cos(\Omega t)$, where $V_{ac}$ and $\Omega$ are the ac amplitude and frequency, respectively. Our model is also relevant for fermionic gases of cold atoms [27] in periodically driven optical lattices [28]. For definiteness, we take a single reservoir in the spinless case but the model can be straightforwardly generalized to account for multiple leads and spinful electrons.

The Hamiltonian given by Eq. (1) conserves the number of particles but not the total energy. We can write,

$$\frac{d\langle \mathcal{H} \rangle}{dt} = W_C(t) + W_T(t) + W_D(t) + P(t), \quad (2)$$

where the energy fluxes (energy per unit time) are $W_C(t) = i\langle [\mathcal{H}, \mathcal{H}_C]/\hbar \rangle$, $W_T(t) = i\langle [\mathcal{H}, \mathcal{H}_T]/\hbar \rangle$ and $W_D(t) = i\langle [\mathcal{H}, \mathcal{H}_D]/\hbar \rangle$, and fulfill $W_C(t) + W_T(t) + W_D(t) = 0$. The term $P(t) = \langle \partial \mathcal{H}_D/\partial t \rangle$ is the power developed by the ac forces. Importantly, energy transport contains an additional term as compared to charge transport. In the latter case, the current conservation condition reads $I_C(t) + I_D(t) = 0$ where the electronic currents (charge per unit time) in the reservoir and the quantum level are given, respectively, by $I_C(t) = ie\langle [\mathcal{H}, \sum_k c_k^\dagger c_k]/\hbar \rangle$ and $I_D(t) = ie\langle [\mathcal{H}, d^\dagger d]/\hbar \rangle$. There is no particle flux associated to the coupling Hamiltonian $\mathcal{H}_T$ (although the currents must, of course, be calculated in the presence of $\mathcal{H}_T$). In stark contrast, the energy flux in the reservoir, $W_C(t)$, cannot be solely inferred from that in the impurity, $W_D(t)$, but necessitates knowledge on how energy is absorbed or desorbed in the contact region, $W_T(t)$. This crucial fact introduces some ambiguity in the definition of the concept of heat current, as shown below.

The different energy fluxes entering Eq. (2) can be computed in terms of the retarded $\mathcal{G}'(t, t') = -i\theta(t - t')\{d(t), d^\dagger(t')\}$ and lesser $\mathcal{G}^\less(t, t') = i[d^\dagger(t')d(t)]$ Green functions. We find that the energy flux entering the reservoir at time $t$ reads [29]

$$W_C = -2\text{Re} \int_0^\infty \frac{d\varepsilon}{\hbar} \Gamma(\varepsilon) \left[ i\mathcal{G}'(t, \varepsilon)\varepsilon + \mathcal{G}^\less(t, \varepsilon) \Theta(\varepsilon) \right], \quad (3)$$

where $\mathcal{G}'(t, t') = \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t-t')} \mathcal{G}(t, \varepsilon)$ and $\Theta(\varepsilon) = \int \frac{d\varepsilon'}{2\pi} e^{\varepsilon'(t-t')}$. In Eq. (3), $f(x) = 1/[1 + e^{(x-\mu)/k_BT}]$ is the Fermi-Dirac distribution with background temperature $T$, the chemical potential $\mu$, and $\Gamma(\varepsilon) = 2\pi \sum_k |w_k|^2 \delta(\varepsilon - \varepsilon_k)$ is the resonance width due to coupling to the continuous set of states. For definiteness, we consider a model for the continuum with a flat density of states, corresponding to a constant $\Gamma$. We emphasize that Eq. (3) is completely general and valid to all orders in $\Omega$ and $V_{ac}$. Moreover, it would be valid even in the presence of Coulomb interactions acting on the spatially localized region.

Following the same procedure, we find for the impurity energy flux the expression

$$W_D(t) = -\varepsilon_d(t)IC(t)/e, \quad (4)$$

where $IC(t) = -2e\text{Re} \int d\varepsilon/\hbar \Psi^\dagger (t, \varepsilon)[i\mathcal{G}'(t, \varepsilon)f(\varepsilon) + \mathcal{G}^\less(t, \varepsilon)]$ is the charge current measured in the reservoir. Equation (3) has a rather simple interpretation. Let $n_d(t)$ be the expected value of the particle number at the localized site. Then, its total energy rate of change is $d[n_d(t)n_d(t)]/dt$, which consists of two terms, namely, the ac source power $P(t) = n_d(t)d\varepsilon_d(t)/dt$ and the energy flux $W_D = \varepsilon_d(t)dn_d(t)/dt = -\varepsilon_d(t)IC(t)/e$, since $I_D(t) \equiv edn_d(t)/dt = -IC(t)$.

Finally, we determine the energy flux associated with the region that mixes continuous and localized states, $W_T = -W_C - W_D$. It reads,

$$W_T(t) = 2\text{Re} \int_0^\infty \frac{d\varepsilon}{\hbar} \partial_\varepsilon \mathcal{G}'(t, \varepsilon) \Gamma(\varepsilon), \quad (5)$$

with $\mathcal{G}'(t, \varepsilon) = \sum_m e^{-i\varepsilon t} \mathcal{G}(t, \varepsilon)$. It is easy to verify that Eq. (5) is a purely ac contribution and vanishes in the limit $\Omega \to 0$. Thus, for applied static fields or for time-averaged ac transport, this special contribution to the system’s energy flow is zero. The quantity $W_T$ will be nonzero only for systems exhibiting a dynamical response. In a quantum-dot setup, the tunnel barrier coupling the dot and the contact lead would periodically store and release energy in response to a nearby ac field, thereby the term energy reactance.

To gain further insight into the physical significance of $W_T$, we now resort to the scattering-matrix formalism applied to quantum transport. Equivalence between Green-function and scattering matrix approaches has been proven in Ref. [30] for averaged time-dependent quantities. But because $W_T$ precisely vanishes in the stationary limit, we now analyze the full time-dependent energy flux by considering the energy current density operator $\rho_E = \Psi^\dagger \mathcal{H} \Psi$, where $\mathcal{H} = -\hbar^2 \nabla^2/2m + U(t, \vec{r})$ is the first-quantized version of Eq. (1) and $U$ is the full electronic potential which includes externally applied time-dependent fields. Then, $\rho_E$ satisfies the continuity equation [31]

$$\partial_t \rho_E + \nabla \cdot W_E = S_E, \quad (6)$$
where \( W_E = (\hbar/4m^2)|\Psi^*\mathcal{H}\nabla^2\Psi - \nabla\Psi^*\mathcal{H}\Psi + \text{h.c.}| \) is the symmetrized energy flux and \( S_E = \Psi^*\partial_t U\Psi \) is the source term accounting for the explicit time dependence of \( U \).

As is customary (see, e.g., Ref. [34]), we introduce the field operator \( \hat{\Psi} \sim \int dz e^{-i\epsilon t/\hbar} e^{ikz\hat{a}^d(\epsilon) + e^{-i\epsilon t/\hbar}\hat{b}(\epsilon)} \) at the cross section \( x \)-position through which the flux is measured. Then, the energy flux is expressed as

\[
W_E(t) = \sum_{n,q} e^{-i\epsilon n t} \int dz \frac{\varepsilon_q + \varepsilon_{n+q}}{2\hbar} G^F(\varepsilon_q, \varepsilon) G^F(\varepsilon_{n+q}, \varepsilon) \\
\times \left[ f(\varepsilon_q) - f(\varepsilon) \right],
\]

where the Floquet scattering matrix relates the lead outgoing flux operators \( \hat{b} \) to the incoming ones \( \hat{a} \) via \( \hat{b}(\epsilon) = \sum_n S^F(\epsilon, \varepsilon_n) \hat{a}(\varepsilon_n) \) and \( \varepsilon_n = \epsilon + n\hbar\Omega \).

Remarkably, if we now insert the generalized Fisher-Lee relation \( S^F(\varepsilon_m, \varepsilon_n) = \delta_{m,n} - i\Gamma G(m-n, \varepsilon_n) \) into Eq. (7) we find \( W_E(t) = W_C(t) + \frac{1}{2}W_T(t) \).

This relation states that in the presence of time-dependent fields the energy fluxes entering the reservoir predicted by scattering theory and the Green function tunneling Hamiltonian approach surprisingly differ by a term \( \frac{1}{2}W_T \). Note that this departure occurs for dynamical energy transport only. In the case of time-dependent particle currents or time-averaged energy fluxes the correspondence between the two theoretical frameworks is exact, i.e., \( W_E = W_C \) with the notation \( \langle \ldots \rangle = \int_0^T \ldots dt/\tau \), being \( \tau = 2\pi/\Omega \).

What is the origin of the discrepancy in Eq. (8)? Let us turn back to Fig. 1 and examine the role of the contact region. While the scattering approach considers propagating electrons with potential energy described by the single function \( U \), the resonant level model considers partitions of the energy contributions as in Eq. (1), similarly to Bardeen’s picture of tunneling [34]. Clearly, the mixing Hamiltonian \( \mathcal{H}_T \) contains creation and destruction operators associated to degrees of freedom of electrons within the continuum as well as within the localized state. When separating the full setup into a reservoir and the driven localized part, it is then natural to split \( \mathcal{H}_T \) symmetrically, contributing equally to these two pieces. The point we would like to make here is that Eq. (8) shows that one should carefully examine how heat fluxes are measured in a given setup before attempting a detailed comparison with theory.

A concomitant question is which portion of the energy flux can be identified as heat. In stationary systems, where the heat transport is accompanied by the particle transport, the heat flux between the localized system and the reservoir is defined from the change in the energy stored in the reservoir subtracting the convective term originated by the particle flow [35]. Such definition was also adopted for the dc component of the heat flux in time-dependent driven systems [19], obtaining the same description within the frameworks of the Green function and scattering matrix formalisms. However, there is an ambiguity in defining heat in the time domain. Specifically, Eq. (7) suggests that the appropriate definition is

\[
\dot{Q}(t) = (W_E(t) - \mu I_C(t))/e = W_C(t) + \frac{1}{2}W_T(t) - \mu I_C(t)/e,
\]

while Eq. (2) implies the heat flow definition \( \dot{Q} = W_C(t) - \mu I_C(t)/e \).

We resort to the basic principles of thermodynamics in order to argue that Eq. (7) is the most meaningful definition of heat flux in the time-domain. Since the reservoir is a macroscopic system, a suitable interpretation of the different portions of its internal energy under slow variations of the driven localized part, would lead to the definition of heat. We proceed along the lines of a textbook analysis [29], identifying as the reservoir the terms of the Hamiltonian \( \mathcal{H} \) containing operators \( c_k^\dagger, c_k \) and as the driven system those depending on \( d^\dagger, d \). The tunneling part \( \mathcal{H}_T \) contains both, hence, it is natural to consider the symmetric splitting \( \mathcal{H} = \mathcal{H}_C + \frac{1}{2}\mathcal{H}_T \) describing the reservoir and \( \mathcal{H}_S = \mathcal{H}_D + \frac{1}{2}\mathcal{H}_T \) defining the driven system. We then evaluate the rate of change of the internal energy \( \langle \mathcal{H} \rangle = \langle \mathcal{H}_C \rangle - \frac{1}{2} \sum_{k} \langle \varepsilon_k - \varepsilon_d(t) \rangle n_k \), with \( n_k = \langle c_k^\dagger c_k \rangle \), which leads us to interpret the quantity \( \mathcal{D}\langle \mathcal{H}_T \rangle = -\sum_{k} \langle \varepsilon_k - \varepsilon_d(t) \rangle \delta n_k \) as the chemical work due to particle flow through the contact. Hence, in accordance to the first principle of thermodynamics, an appropriate definition for the heat exchange in the reservoir induced by slow variations of the driven system is \( \delta\langle \mathcal{H}_T \rangle = \delta\langle \mathcal{H}_C \rangle + \mathcal{D}\langle \mathcal{H}_T \rangle /2 - \mu \delta N_C \), with \( N_C = \sum_{k} n_k \), as suggested by Eq. (9). In what follows we also show that this expression is also in agreement with the second law of thermodynamics, while this is not the case of the alternative definition \( \dot{Q} \).

We focus on the slow driving regime and consider, for simplicity, zero temperature \( (T = 0) \). Then, an exact analysis can be performed by means of an expansion in powers of \( \Omega \) for the Green functions (or equivalently of the scattering matrix) [37]:

\[
\mathcal{G}'(t, \varepsilon) = \mathcal{G}'(t, \varepsilon) + \frac{i\hbar}{2} \partial_\varepsilon \mathcal{G}'(t, \varepsilon) + \ldots
\]

\[
\mathcal{G}'(t, \varepsilon) = [\varepsilon - \varepsilon_d(t) + i\Gamma/2]^{-1}
\]

is the frozen Green function describing the regime in which the electron instantaneously adjusts its potential to the ac field. Considering the expansion of \( \mathcal{G} \) up to \( O(\Omega) \) yield heat fluxes exact up to \( O(\Omega^2) \) [29]. We find \( \dot{Q}(t) = \dot{Q}^{(1)}(t) + \dot{Q}^{(2)}(t) \), where
the first and second order terms in $\Omega$ are, respectively,

$$
\dot{Q}^{(1)}(t) = \int \frac{d\varepsilon}{\hbar} \left[ \mu - \varepsilon \frac{\partial f}{\partial \varepsilon} \rho^f(t, \varepsilon) \right] \frac{d\varepsilon_d}{dt},
$$

(11)

$$
\dot{Q}^{(2)}(t) = -\frac{1}{2} \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \left[ (\mu - \varepsilon) \frac{d}{dt} \left[ \rho^f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \right] \right] + \left[ \rho^f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \right]^2.
$$

(12)

Here $\rho^f(t, \varepsilon) = -2\text{Im}[G_f(t, \varepsilon)] = |G_f(t, \varepsilon)|^2 \Gamma = -i\partial_x S_f S_f$ is the local density of states and $S_f(t, \varepsilon)$ the frozen scattering matrix, i.e., the stationary scattering matrix with time-dependent parameters.

Both the first-order term $\dot{Q}^{(1)}(t)$ and the first term of $\dot{Q}^{(2)}(t)$ vanish at $T = 0$ since $-\partial_x f = \delta(\varepsilon - \mu)$. The component $\dot{Q}^{(2)}(t)$, which is second order in $\Omega$, represents the leading-order to the dissipated power in the reservoir. At $T = 0$, Eq. (12) reduces to $\dot{Q}^{(2)}(t) = [\rho^f(t, \mu) \frac{d\varepsilon_d}{dt}]^2 / 2$.

Evaluating the charge current up to the first order in $\Omega$, we find, $I_C^{(1)}(t) = (\varepsilon / \hbar) \rho^f(t, \mu) \frac{d\varepsilon_d}{dt}$, which implies

$$
\dot{Q}^{(2)}(t) = R_q |I_C^{(1)}(t)|^2,
$$

(13)

with $R_q = \hbar / 2e^2$ the relaxation resistance quantum. Since $R_q$ is a manifestly positive quantity at all times, the heat flux given by Eq. (4) represents the heat dissipated into the cold reservoir when the system is coupled to the ac driving force. Therefore, Eq. (4) agrees with the second law of thermodynamics.

We reinforce our conclusion by comparing with the heat rate of change given by $\dot{Q}$. Thus, we evaluate $W_T$ up to second order in $\Omega$:

$$
W_T^{(1)}(t) = 2 \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \left[ \rho^f(t, \varepsilon) (\varepsilon - \varepsilon_d(t)) \frac{d\varepsilon_d}{dt} \right],
$$

$$
W_T^{(2)}(t) = -\int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \frac{d}{dt} \left[ \rho^f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \right] (\varepsilon - \varepsilon_d(t)) \frac{d\varepsilon_d}{dt}.
$$

(14)

Within the weak driving regime, $\dot{Q}^{(t)}(t) = \dot{Q}(t) - [W_T^{(1)}(t) + W_T^{(2)}(t)] / 2$, which at $T = 0$ contains contributions $\propto \Omega$ and $\propto \Omega^2$. Defining the resistance $\tilde{R}(t)$ from the relation $\dot{Q}^{(t)}(t) = |I_C^{(1)}(t)|^2 \tilde{R}(t)$, we find that $\tilde{R}(t)$ is non-universal and depends on time. In fact, it is not even positive definite and then $\dot{Q}^{(t)}(t)$ cannot be interpreted as a dissipated heat. We illustrate in Fig. 4 the behavior of the two expressions of the heat flux for different amplitudes of the driving potential $V_{ac}$ for a reservoir at $T = 0$ and small driving frequencies. The inset shows that, as a function of time, $\dot{Q}^{(t)}(t)$ is always positive whereas $\dot{Q}^{(t)}(t)$ may attain negative values. The main panel shows $\dot{Q}^{(t)}(t)$ and $\dot{Q}^{(t)}(t)$ as a function of $I_C(t)^2$ within the slow driving regime. In the first case, we observe a linear function with the universal slope $R_q$. In contrast, in the second case we observe a non-universal behavior, including negative values of $\tilde{R}(t)$. The two definitions of heat, however, lead to the same result when averaged in time, $\overline{\dot{Q}}^{(t)} = \overline{\dot{Q}^{(t)}}$ and, therefore, only a pure dynamical measurement would be able to distinguish both.

In conclusion, we have discussed the dynamical heat generation in a resonant level system due to coupling to an external time-dependent potential and highlighted the important role played by the energy associated with the coupling region. The latter is unique to dynamical energy transport. By recourse to an adiabatic expansion valid for the slow-driving regime, we have found that an appropriate expression of the dynamical heat flux that agrees with the fundamental principles of thermodynamics requires to take into account the work associated to particles flowing through the tunneling region. Importantly, the time-dependent flux is instantaneously given by a Joule law with universal resistance. Our results are relevant for recent developments in the energetics of atomic systems and nanostructures.

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Dynamical energy transfer in ac driven quantum systems: Supplementary Information

This supporting document describes further details of the derivation of the time-dependent energy fluxes, a comparison with the scattering matrix approach and the heat flow for the slow driving case.

**Time-dependent energy fluxes**

We consider a simple model of a harmonically driven resonant level, which is coupled to a single reservoir consisting of a fermionic band of continuous state with a flat density, described by the Hamiltonian of Eq. (1)

In order to define the energy fluxes entering each part of the system, we analyze the evolution in time of the total energy:

\[
\frac{d}{dt} \langle H \rangle = \frac{d}{dt} \langle H_C \rangle + \frac{d}{dt} \langle H_T \rangle + \frac{d}{dt} \langle H_D \rangle,
\]

(S1)

The energy current flowing into the reservoir drives its energy variation, and as a consequence we define the energy flux entering the reservoir \( W_C \) as

\[
W_C \equiv \frac{d}{dt} \langle H_C \rangle = i \hbar \langle [H, H_C] \rangle.
\]

(S2)

The same occurs for the energy variation stored in the coupling region. Thus, the energy flow reads

\[
W_T \equiv \frac{d}{dt} \langle H_T \rangle = i \hbar \langle [H, H_T] \rangle.
\]

(S3)

On the other hand, the rate of the energy in the driven system involves the energy flux entering it and the power developed by the time dependent voltage applied to the resonant level,

\[
\frac{d}{dt} \langle H_D \rangle = i \hbar \langle [H, H_D] \rangle + \langle \frac{\partial H_D}{\partial t} \rangle.
\]

(S4)

From this expression, we identify the term involving the commutator of operators as the energy flux entering the level \( W_D \equiv i \hbar \langle [H, H_D] \rangle \), and the power applied by the external field \( P(t) = \langle \frac{\partial H_D}{\partial t} \rangle \).

Therefore, we can write

\[
\frac{d}{dt} \langle H \rangle = W_C(t) + W_T(t) + W_D(t) + P(t).
\]

(2)

The energy of the full system is not conserved due to the presence of a power source,

\[
\frac{d}{dt} \langle H \rangle = \langle \frac{\partial H_D}{\partial t} \rangle = \langle \frac{\partial H_D}{\partial t} \rangle = P(t),
\]

(S5)

and then combining Eqs. (2) and (S5), the energy fluxes fulfill the condition \( W_C(t) + W_T(t) + W_D(t) = 0 \).

Following definition (S2), we start computing the energy flux entering the reservoir as a function of time,

\[
W_C = \frac{i}{\hbar} \langle [H, H_C] \rangle = \frac{i}{\hbar} \langle [H_T, H_C] \rangle
= -\frac{i}{\hbar} \sum_k \varepsilon_k [w_k \langle c_k^\dagger(t)d(t) \rangle - \langle d^\dagger(t)c_k(t) \rangle].
\]

(S6)

Taking quantum-mechanical averages and using the definition \( G_k^< (t', t) = i \langle c_k^\dagger(t')d(t) \rangle \), the variation in time of the energy stored in the reservoir can be written as follows:

\[
W_C = -\frac{2}{\hbar} \sum_k \varepsilon_k \text{Re} \{w_k G_k^< (t, t)\}.
\]

(S7)

From Dyson equation and Langreth rules (see Refs. [1, 2]), the above expression can be expressed as follows:

\[
W_C = -\frac{2}{\hbar} \sum_k \left| w_k \right|^2 \varepsilon_k \int dt_1 [G^r(t, t_1)G_k^< (t_1, t) + G_k^< (t, t_1)G_k^>(t_1, t)]
\]

(S8)
being
\[ g^r(t_1, t) = -i\theta(t - t_1)\{d(t), d^\dagger(t_1)\}, \]
\[ g^< (t_1, t) = i\{d^\dagger(t_1)d(t)\}, \] (S9)
and
\[ g^r_\Sigma(t_1, t) = \int \frac{d\varepsilon}{2\pi} f(\varepsilon)\gamma_k(\varepsilon)e^{-i\varepsilon(t_1 - t)/\hbar}, \]
\[ g^<_\Sigma(t_1, t) = \int \frac{d\varepsilon}{2\pi} \int \frac{d\varepsilon'}{2\pi} \frac{\gamma_k(\varepsilon')}{\varepsilon - \varepsilon' - i0^+}e^{-i\varepsilon(t_1 - t)/\hbar}, \]
\[ \gamma_k(\varepsilon) = 2\pi\delta(\varepsilon - \varepsilon_k). \] (S10)
Substituting in Eq. (S8):
\[ W_C = -\frac{2}{\hbar}\text{Re}\{\int dt_1 \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t_1 - t)/\hbar} \left[iG^r(t, t_1)\Gamma f(\varepsilon)\varepsilon + \int \frac{d\varepsilon'}{2\pi} G^<(t, t_1)\Gamma \frac{\varepsilon'}{\varepsilon - \varepsilon' - i0^+}\right]\}, \] (S11)
where \( \Gamma = \sum_k |w_k|^2\gamma_k(\varepsilon) \) is the hybridization width due to coupling to the reservoir, and \( f(\varepsilon) = 1/[1 + e^{(\varepsilon - \mu)/k_B T}] \) the Fermi-Dirac distribution. We also assume a reservoir with a wide-band density of states, such that \( \Gamma \) is approximately independent of \( \varepsilon \).

Introducing the Fourier representation for the Green function
\[ G(t, t_1) = \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t_1 - t)/\hbar} G(t, \varepsilon), \] (S12)
we obtain Eq. (3) of the main text
\[ W_C = -2\text{Re}\{\int \frac{d\varepsilon}{\hbar} \Gamma \left[iG^r(t, \varepsilon)f(\varepsilon)\varepsilon + G^<(t, \varepsilon)\Theta(\varepsilon)\right]\}, \] (3)
where
\[ \Theta(\varepsilon) = \int \frac{d\varepsilon'}{2\pi} \frac{\varepsilon'}{\varepsilon - \varepsilon' - i0^+}. \] (S13)

We now consider the Keldysh equation for the lesser Green function:
\[ g^< (t, \varepsilon) = \int dt_1 \int dt_2 \int dt_3 G^r(t, t_2)\Sigma^<(t_2 - t_3)G^a(t_3, t_1)e^{i\varepsilon(t_1 - t_1)/\hbar}, \] (S14)
with \( \Sigma^<(\varepsilon) = if(\varepsilon)\Gamma, \) and taking into account the fact that the forcing is periodic in time we introduce the following Floquet-Fourier representation for the Green function \( \tilde{G} \):
\[ \tilde{G}^r(t, \varepsilon) = \sum_n e^{-i\varepsilon n\Omega} \tilde{G}(n, \varepsilon), \] (S15)
\( \Omega \) being the fundamental driving frequency. Substituting Eqs. (S14) and (S15) in Eq. (3) we obtain:
\[ W_C(t) = -2\text{Re}\{\sum_n e^{-i\varepsilon n\Omega} \int \frac{d\varepsilon}{\hbar} \Gamma \left[i\tilde{G}^r(t, \varepsilon)f(\varepsilon)\varepsilon \right. \]
\[ +i\Gamma \sum_n \tilde{G}^r(t + n, \varepsilon - n\Omega)\tilde{G}^r(n, \varepsilon - n\Omega)f(\varepsilon - n\Omega)\Theta(\varepsilon)\} \}. \] (S16)
Thus, after working with the above expression we finally find
\[ W_C(t) = -\sum_n e^{-i\varepsilon n\Omega} \int \frac{d\varepsilon}{\hbar} \left[i\tilde{G}^{r\star}(-l, \varepsilon)\Gamma[(\varepsilon - lh\Omega)f(\varepsilon - lh\Omega) - \varepsilon f(\varepsilon)] - \right. \]
\[ \left. \sum_n [(\varepsilon + lh\Omega/2)f(\varepsilon - lh\Omega) - \varepsilon f(\varepsilon)]\tilde{G}^r(t + n, \varepsilon - n\Omega)\Gamma^2\tilde{G}^{r\star}(n, \varepsilon - n\Omega)\right]. \] (S17)
Following a similar procedure, we can compute the other fluxes entering Eq. (2). The energy current flowing through the contact between the reservoir and the impurity reads

\[
W_T(t) = \sum_k \left[ w_k \frac{d}{dt} \langle d_k^\dagger c_k \rangle + \hbar c \right] = 2 \sum_k \text{Im} \left\{ w_k \frac{dG_k^<(t,t)}{dt} \right\} \\
= \int \frac{d\varepsilon}{\hbar} \Omega f(\varepsilon) \sum_l \left[ 2\text{Im}\{e^{-i\Omega \varepsilon} G^r(l,\varepsilon)\} + \sum_n \text{Re}\{e^{-i\Omega \varepsilon} G^r(l+n,\varepsilon)G^{<\ast}(n,\varepsilon)\} \right]. \tag{S18}
\]

It is easy to prove that the last term \( \sum_{l,n} \text{Re}\{e^{-i\Omega \varepsilon} G^r(l+n,\varepsilon)G^{<\ast}(n,\varepsilon)\} \) vanishes,

\[
\sum_{l>0} \sum_n \left( l \text{Re}\{e^{-i\Omega \varepsilon} G^r(l+n,\varepsilon)G^{<\ast}(n,\varepsilon)\} - l \text{Re}\{e^{i\Omega \varepsilon} G^r(-l+n,\varepsilon)G^{<\ast}(n,\varepsilon)\} \right) = 0.
\]

As a consequence, the variation of the energy stored in the contact is

\[
W_T(t) = \int \frac{d\varepsilon}{\hbar} \Omega f(\varepsilon) \sum_l 2\text{Im}\{e^{-i\Omega \varepsilon} G^r(l,\varepsilon)\}. \tag{S19}
\]

Combining this expression with Eq. (S15), we find

\[
W_T(t) = 2\text{Re}\left( \int \frac{d\varepsilon}{\hbar} \partial_t G^r(t,\varepsilon) \Gamma f(\varepsilon) \right). \tag{5}
\]

Finally, for the energy flux entering the impurity we find

\[
W_D = \frac{i}{\hbar} \langle [\mathcal{H}, \mathcal{H}_D] \rangle = \varepsilon_d(t) \frac{d}{dt} \frac{dn_d(t)}{dt}, \tag{S20}
\]

where \( n_d(t) = \langle d^\dagger(t)d(t) \rangle \) is the number of particles present in the resonant level, which is related to the charge current measured in the reservoir \( I_C \) due to the conservation of the charge

\[
e^{-\frac{dn_d(t)}{dt}} = -I_C(t). \tag{S21}
\]

Thus,

\[
W_D = -\varepsilon_d(t) I_C(t)/e. \tag{4}
\]

We apply the same procedure as before to obtain an expression for the charge current flowing through the contact between the reservoir and the system, defined as

\[
I_C(t) = \frac{ie}{\hbar} \langle [\mathcal{H}, \sum_k c_k^\dagger c_k] \rangle = -\frac{2e}{\hbar} \text{Re}\{w_k G_k^<(t,t)\} = \frac{e}{\hbar} \text{Im}\{ \int d\varepsilon \Gamma \left[ 2G^r(t,\varepsilon) f(\varepsilon) + G^<(t,\varepsilon) \right] \}. \tag{S22}
\]

Using Eq. (S15), this quantity can be written as

\[
I_C(t) = -\frac{e}{\hbar} \sum_l e^{-i\Omega \varepsilon} \int d\varepsilon \{iG^{<\ast}(-l,\varepsilon)\Gamma[f(\varepsilon - l\hbar\Omega) - f(\varepsilon)] - \sum_n [f(\varepsilon - n\hbar\Omega) - f(\varepsilon)]G^r(l+n,\varepsilon - n\hbar\Omega)\Gamma^2G^{<\ast}(n,\varepsilon - n\hbar\Omega) \}. \tag{S23}
\]
Comparison with the scattering matrix approach

Previous works \cite{3, 5} have proven the existence of a simple relation between the scattering matrix elements and the Fourier coefficients for the Green function. For periodically driven systems, the Fisher-Lee formula for stationary systems has been generalized as

\[ S^F(\epsilon_m, \epsilon_n) = \delta_{m,n} - i \Gamma G^*(\epsilon_{m-n}, \epsilon_n), \tag{S24} \]

where \( S^F(\epsilon, \epsilon) \) are the Floquet scattering matrix elements, and \( \epsilon_n = \epsilon + n \hbar \Omega \). The above relation leads to expressions for the time-averaged charge and energy currents that are equivalent in both formalisms. The aim of the present section is to show in detail the derivation of Eq. (8) of the main text.

To start, according to Ref. \cite{4} the time-dependent energy flow within the scattering matrix formalism reads:

\[ W_E(t) = \sum_{n,q} e^{-i\hbar \Omega t} \int d\epsilon \frac{\epsilon_n + \epsilon_{n+q}}{2\hbar} S^{F^*}(\epsilon, \epsilon) S^F(\epsilon_{n+q}, \epsilon)[f(\epsilon) - f(\epsilon_q)], \tag{7} \]

and taking into account the relation \( \int \epsilon \Gamma \left( \epsilon - \frac{l}{2} \hbar \Omega \right) \text{Im} \left\{ \mathcal{G}^*(l + n, \epsilon - n \hbar \Omega) \right\} \text{Im} \left\{ \mathcal{G}^*(l, \epsilon) \right\} \] \( \text{Im} \left\{ \mathcal{G}^*(l, \epsilon) \right\} \]

Comparing with the expression \( \text{Im} \left\{ \mathcal{G}^*(l, \epsilon) \right\} \) it can be seen that

\[ W_E(t) - W_C(t) = \sum_{l} e^{-i\hbar \Omega t} \int d\epsilon \left\{ i \mathcal{G}^*(l, \epsilon) \Gamma \left( \epsilon - \frac{l}{2} \hbar \Omega \right) \right\} [f(\epsilon) - f(\epsilon_l)] - \sum_{n} \frac{l \hbar \Omega}{2} f(\epsilon)[\mathcal{G}^*(l + n, \epsilon - n \hbar \Omega) \Gamma \mathcal{G}^*(l, \epsilon)] \]

\[ = - \sum_{n} e^{-i\hbar \Omega t} \frac{l \hbar \Omega}{2} f(\epsilon) \Gamma \left\{ \mathcal{G}^*(l, \epsilon) + i \mathcal{G}^*(l, \epsilon) \right\} \]

\[ = \int \frac{d\epsilon}{\hbar} f(\epsilon) \hbar \Omega \sum_{l} f(\epsilon) \Gamma \left\{ \mathcal{G}^*(l, \epsilon) \right\}. \tag{S25} \]

Comparing with the expression \( \text{Im} \left\{ \mathcal{G}^*(l, \epsilon) \right\} \) it can be seen that

\[ W_E(t) - W_C(t) = \frac{1}{2} W_T(t). \tag{8} \]

This is a surprising result because in addition to presenting a discrepancy between the energy fluxes predicted by scattering theory and the tunneling Hamiltonian model it also states that the difference is \( \frac{1}{2} W_T(t) \) related to the energy flowing through the contact. This result does not contradict the exact correspondence within the stationary limit, i.e., \( \frac{W_E}{W_C} = \frac{W_C}{W_T} \), because \( W_T \) vanishes when it is averaged over time.

Heat flow for slow driving

For slow driving, an exact solution of the Dyson equation up to \( O(\Omega) \) can be obtained by expanding the Green functions in powers of \( \Omega \) \( \cite{6} \)

\[ \mathcal{G}^r(t, \epsilon) = \mathcal{G}^r_f(t, \epsilon) + \frac{i \hbar}{2} \partial_t \partial_\epsilon \mathcal{G}^r_f(t, \epsilon), \tag{10} \]

where \( \mathcal{G}^r_f = [\epsilon - \epsilon_d(t) + i\Gamma / 2]^{-1} \) is the frozen Green function, and its derivatives are

\[ \partial_\epsilon \mathcal{G}^r_f(t, \epsilon) = -\mathcal{G}^r_f(t, \epsilon)^2 \]

\[ \partial_t \mathcal{G}^r_f(t, \epsilon) = -\partial_\epsilon \mathcal{G}^r_f(t, \epsilon) \frac{\epsilon_d(t)}{dt}. \tag{S27} \]
In this section we present the expressions for both definitions of the heat flow

\[ \dot{Q}(t) = W_E(t) - \mu I_C(t)/e \]  \hspace{1cm} (S28)

\[ \dot{Q}(t) = W_C(t) - \mu I_C(t)/e = \dot{Q}(t) - \frac{1}{2} W_T(t) \]  \hspace{1cm} (11)

within that approximation.

Thus, we just need to compute the energy flux entering the reservoir \( W_E \) and the charge current \( I_C \) at low frequencies. We can express those quantities as a sum of linear and quadratic terms in the driving frequency \( \Omega \), respectively

\[ W_E(t) = W_E^{(1)}(t) + W_E^{(2)}(t) \]
\[ I_C(t) = I_C^{(1)}(t) + I_C^{(2)}(t). \]  \hspace{1cm} (S29)

The zero order term for those quantities vanishes and is not present in the above expansion.

Next, we expand \( f(\varepsilon + n\hbar\Omega) \sim f(\varepsilon) + (\partial f/\partial \varepsilon)n\hbar\Omega + (\partial^2 f/\partial \varepsilon^2)(n\hbar\Omega)^2/2 \) and

\[ G(n, \varepsilon) \sim G^{(0)}(n, \varepsilon) + \hbar\Omega G^{(1)}(n, \varepsilon) \]  \hspace{1cm} (S30)

with

\[ G^{(0)}(n, \varepsilon) = \int_0^\tau \frac{dt}{\tau} G_f(t, \varepsilon) e^{in\Omega t} \]
\[ \Omega G^{(1)}(n, \varepsilon) = \int_0^\tau \frac{dt}{\tau} i \partial_t G_f(t, \varepsilon) e^{in\Omega t}, \]  \hspace{1cm} (S31)

where \( \tau = 2\pi/\Omega \). Substituting these expansions into Eqs. (S23) and (S25), and performing an inverse Fourier transformation, we can express the energy and charge currents in terms of the frozen Green function:

\[ W_E^{(1)}(t) = -\int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \rho_f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \]
\[ W_E^{(2)}(t) = \frac{1}{2} \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \{ \varepsilon \frac{d}{dt} \left[ (\rho_f(t, \varepsilon))^2 \frac{d\varepsilon_d}{dt} \right] - \left( \rho_f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \right)^2 \}, \]  \hspace{1cm} (S32)

and

\[ I_C^{(1)}(t) = -\frac{e}{\hbar} \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \rho_f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \]
\[ I_C^{(2)}(t) = \frac{e}{2\hbar} \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \frac{d}{dt} \left[ (\rho_f(t, \varepsilon))^2 \frac{d\varepsilon_d}{dt} \right]. \]  \hspace{1cm} (S33)

where we have defined the frozen density of state \( \rho_f(t, \varepsilon) = -2\text{Im} \{G_f(t, \varepsilon)\} = \Gamma |G_f(t, \varepsilon)|^2 \).

Now, we can expand the heat flow \( \dot{Q}(t) \) in the same fashion and obtain from Eq. (S28)

\[ \dot{Q}^{(1)}(t) = \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} (\mu - \varepsilon) \rho_f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \]  \hspace{1cm} (12)

\[ \dot{Q}^{(2)}(t) = -\frac{1}{2} \frac{1}{2} \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \left( (\mu - \varepsilon) \frac{d}{dt} \left[ (\rho_f(t, \varepsilon))^2 \frac{d\varepsilon_d}{dt} \right] + \left( \rho_f(t, \varepsilon) \frac{d\varepsilon_d}{dt} \right)^2 \right). \]  \hspace{1cm} (2)

At temperature \( T = 0 \), since \( \partial_c f \sim \delta(\varepsilon - \mu) \), both the first order term \( \dot{Q}^{(1)}(t) \) and the first term of \( \dot{Q}^{(2)}(t) \) vanish. In contrast, the last term of the second order component of the heat is equal to the power developed by the external voltage \( P(t) \), since

\[ P(t) = \langle \frac{\partial H_D}{\partial t} \rangle = \frac{d\varepsilon_d}{dt} \langle n_d(t) \rangle, \]  \hspace{1cm} (S34)

which in the low frequency approximation reads

\[ P(t) = \frac{d\varepsilon_d}{dt} \frac{1}{2} \int \frac{d\varepsilon}{\hbar} \frac{\partial f}{\partial \varepsilon} \left( \rho_f(t, \varepsilon)^2 \frac{d\varepsilon_d}{dt} \right). \]  \hspace{1cm} (S35)
To finalize, in order to compute the alternative definition of heat $\dot{Q}(t)$ within this approximation, we have to evaluate $W_T$ up to second order in $\Omega$. Following the same procedure from Eq. (S19) we find:

$$W_T^{(1)}(t) = -2 \int \frac{d\varepsilon}{h} f(\varepsilon) \text{Re}\{\frac{\partial G^r_f(t, \varepsilon)}{\partial \varepsilon}\} \frac{d\varepsilon_d}{dt}$$
$$= 2 \int \frac{d\varepsilon}{h} \frac{\partial f(\varepsilon)}{\partial \varepsilon} \left[ \rho^f(t, \varepsilon)(\varepsilon - \varepsilon_d(t)) \frac{d\varepsilon_d}{dt} \right]$$

$$W_T^{(2)}(t) = 2 \int \frac{d\varepsilon}{h} f(\varepsilon) \frac{d}{dt} \text{Im}\{G^r_f(t, \varepsilon) \frac{\partial G^r_f(t, \varepsilon)}{\partial t}\}$$
$$= - \int \frac{d\varepsilon}{h} \frac{\partial f(\varepsilon)}{\partial \varepsilon} \frac{d}{dt} \left[ (\rho^f(t, \varepsilon))^2 (\varepsilon - \varepsilon_d(t)) \frac{d\varepsilon_d}{dt} \right],$$

(S36)

where we have used the relation between real and imaginary parts of the Green function

$$\text{Re}\{G^r_f(t, \varepsilon)\} = -2\text{Im}\{G^r_f(t, \varepsilon)\}(\varepsilon - \varepsilon_d(t)) = \rho^f(t, \varepsilon)(\varepsilon - \varepsilon_d(t)).$$

(S37)

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