Rate of convergence to self-similarity for the fragmentation equation in $L^1$ spaces

María J. Cáceres∗ José A. Cañizo†
Stéphane Mischler‡

December 20, 2010

Abstract
In a recent result by the authors [1] it was proved that solutions of the self-similar fragmentation equation converge to equilibrium exponentially fast. This was done by showing a spectral gap in weighted $L^2$ spaces of the operator defining the time evolution. In the present work we prove that there is also a spectral gap in weighted $L^1$ spaces, thus extending exponential convergence to a larger set of initial conditions. The main tool is an extension result in [4].

1 Introduction
In a recent paper [1] we have studied the speed of convergence to equilibrium for solutions of equations involving the fragmentation operator and first-order differential terms. In this paper we will focus on the case of self-similar fragmentation given by

\begin{align}
\partial_t g_t(x) &= -x \partial_x g_t(x) - 2g_t(x) + \mathcal{L}g_t(x) \\
g_0(x) &= g_{in}(x) \quad (x > 0).
\end{align}

Here the unknown is a function $g_t(x)$ depending on time $t \geq 0$ and on size $x > 0$, which represents a density of units (usually particles, cells or polymers) of size $x$ at time $t$, and $g_{in}$ is an initial condition. The fragmentation operator $\mathcal{L}$ acts on a function $g = g(x)$ as

$$\mathcal{L}g(x) := \mathcal{L}_+g(x) - B(x)g(x),$$

∗Departamento de Matemática Aplicada, Universidad de Granada, E18071 Granada, Spain. Email: caceres@ugr.es
†Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain. Email: canizo@mat.uab.es
‡IUF and CEREMADE, Univ. Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris CEDEX 16, France. Email: mischler@ceremade.dauphine.fr
where the positive part $\mathcal{L}_+$ is given by
\[
\mathcal{L}_+ g(x) := \int_x^\infty b(y, x) g(y) \, dy.
\] (1.3)

The coefficient $b(y, x)$, defined for $y > x > 0$, is the fragmentation coefficient, and $B(x)$ is the total fragmentation rate of particles of size $x > 0$. It is obtained from $b$ through
\[
B(x) := \int_0^x \frac{y}{x} b(x, y) \, dy \quad (x > 0).
\] (1.4)

We refer to [1, 5, 7, 2, 6, 8] for a motivation of (1.1) in several applications and a general survey of the mathematical literature related to it.

We call $T$ the operator on the right hand side of (1.1a), this is,
\[
Tg(x) := -x \partial_x g(x) - 2g(x) + \mathcal{L}g(x) \quad (x > 0),
\] (1.5)
acting on a (sufficiently regular) function $g$ defined on $(0, +\infty)$.

Assumptions on the fragmentation coefficient In order to use the results in [1] we will make the following hypotheses on the fragmentation coefficient $b$:

**Hypothesis 1.1.** For all $x > 0$, $b(x, \cdot)$ is a nonnegative measure on the interval $[0, x]$. Also, for all $\psi \in C_0([0, +\infty))$, the function $x \mapsto \int_{[0,x]} b(x, y) \psi(y) \, dy$ is measurable.

**Hypothesis 1.2.** There exists $\kappa > 1$ such that
\[
\int_0^x b(x, y) \, dy = \kappa B(x) \quad (x > 0).
\] (1.6)
Hypothesis 1.3. There exists $0 < B_m < B_M$ satisfying

$$2B_m x^{\gamma-1} \leq b(x, y) \leq 2B_M x^{\gamma-1} \quad (0 < y < x)$$  \hspace{1cm} (1.7)

for some $0 < \gamma < 2$.

This implies the following useful bound, as remarked in [1, Corollary 6.4]:

**Lemma 1.4.** Consider a fragmentation coefficient $b$ satisfying Hypotheses 1.1–1.3. There exists a strictly decreasing function $k \mapsto p_k$ for $k \geq 0$ with $\lim_{k \to +\infty} p_k = 0$, $p_k > 1$ for $k \in [0, 1)$, $p_1 = 1$, $0 < p_k < 1$ for $k > 1$, and such that

$$\int_0^x y^k b(x, y) \, dy \leq p_k x^k B(x) \quad (x > 0, \, k > 0).$$  \hspace{1cm} (1.9)

**Main results** The main result of the present work is a spectral gap of $T$ on weighted $L^1$ spaces.

**Theorem 1.5.** Assume hypotheses 1.1–1.3. For any $1/2 < m < 1$ there exists $1 < M < 2$ such that the operator (1.5) has a spectral gap in the space $X := L^1(x^m + x^M)$. More precisely, there exists $\alpha > 0$ and a constant $C \geq 1$ such that, for all $g$ in $X$ with $\int x g = 1$

$$\|g_t - G\|_X \leq C e^{-\alpha t} \|g_{in} - G\|_X \quad (t \geq 0).$$

2 Preliminaries

In this section we gather some known results from previous works.

2.1 Previous results on the spectral gap of $T$

A result like Theorem 1.5 was proved in [1], but in the $L^2$ space with weight $xG^{-1}$. This is summarized in the following theorem:

**Theorem 2.6** (1). Assume Hypotheses 1.1–1.3 and consider $G$ the self-similar profile with $\int x G = 1$. The operator $T$ given by (1.5) has a spectral gap in the space $H = L^2(xG^{-1})$.

More precisely, there exists $\beta > 0$ such that for any $g_{in} \in H$ with $\int x g = 1$ the solution $g \in C([0, \infty); L^1(x \, dx))$ to equation (1.1) satisfies

$$\|g_t - G\|_H \leq e^{-\beta t} \|g_{in} - G\|_H \quad (t \geq 0).$$
2.2 Bounds for the self-similar profile

We recall the following result from [1, Theorem 3.1]:

**Theorem 2.7.** Assume Hypotheses 1.1–1.3 on the fragmentation coefficient \( b \), and call \( \Lambda(x) := \int_0^x B(s) \, ds \). Let \( G \) be the self-similar profile with \( \int x \, G = 1 \).

For any \( \delta > 0 \) and any \( a \in (0, B_m/B_M) \), \( a' \in (1, +\infty) \) there exist constants \( C' = C'(a', \delta), \, C = C(a) > 0 \) such that

\[
C' \, e^{-a' \Lambda(x)} \leq G(x) \leq C \, e^{-a \Lambda(x)} \quad \text{for} \quad x > 0.
\] (2.10)

**Remark 2.8.** In the case \( b(x, y) = 2 \, x^{\gamma - 1} \) (so \( B(x) = x^{\gamma} \)), the profile \( G \) has the explicit expression \( G(x) = e^{-x^{\gamma}} \) for \( \gamma > 0 \). This motivates the choice of \( e^{-a \Lambda(x)} \) as functions for comparison. For a general \( b(x, y) \) no explicit form is available.

**Proof.** Everything but the lower bound of \( G \) for small \( x \) is proved in [1, Section 3]. For the lower bound, we calculate as follows:

\[
\partial_x \left( x^2 \, e^{\Lambda(x)} \, G(x) \right) = x \, e^{\Lambda(x)} \int_x^\infty b(y, x) \, G(y) \, dy \quad (x > 0),
\] (2.11)

which implies that \( x^2 \, e^{\Lambda(x)} \, G(x) \) is a nondecreasing function. Hence, it must have a limit as \( x \to 0 \), and this limit must be 0 since we know \( x \, G(x) \) is integrable.

Then, integrating (2.11), and for \( 0 < z < 1 \),

\[
z^2 \, e^{\Lambda(z)} \, G(z) = \int_0^z x \, e^{\Lambda(x)} \int_x^\infty b(y, x) \, G(y) \, dy \, dx
\]

\[
= \int_0^\infty G(y) \int_0^{\min\{z, y\}} b(y, x) \, x \, e^{\Lambda(x)} \, dx \, dy
\]

\[
\geq 2B_m \int_0^\infty y^{\gamma - 1} \, G(y) \int_0^{\min\{z, y\}} x \, dx \, dy
\]

\[
= B_m \int_0^\infty y^{\gamma - 1} \, G(y) (\min\{z, y\})^2 \, dy
\]

\[
\geq B_m z^2 \int_z^\infty y^{\gamma - 1} \, G(y) \, dy
\]

\[
\geq B_m z^2 \int_1^\infty y^{\gamma - 1} \, G(y) \, dy = Cz^2 \quad (0 < z < 1).
\] (2.12)

Notice that the number \( \int_1^\infty y^{\gamma - 1} \, G(y) \, dy \) is strictly positive, as the profile \( G \) is strictly positive everywhere (see [2, 3, 1]). This proves the lower bound on \( G(x) \) for \( 0 < x < 1 \), and completes the proof.

2.3 A general spectral gap extension result

Our proof is based on the following result from [4], which was already used in [1] for an extension to an \( L^2 \) space with a polynomial weight:
Theorem 2.9. Consider a Hilbert space $H$ and a Banach space $X$ (both over the field $\mathbb{C}$ of complex numbers) such that $H \subset X$ and $H$ is dense in $X$. Consider two unbounded closed operators with dense domain $T$ on $H$, $\Lambda$ on $X$ such that $\Lambda|_H = T$. On $H$ assume that

1. There is $G \in H$ such that $TG = 0$ with $\|G\|_H = 1$;
2. Defining $\psi(f) := \langle f, G \rangle_H$, the space $H_0 := \{f \in H; \psi(f) = 0\}$ is invariant under the action of $T$.
3. $T - a$ is dissipative on $H_0$ for some $a < 0$, in the sense that $\forall g \in D(T) \cap H_0 \quad ((T - a)g, g)_H \leq 0$,
4. $T$ generates a semigroup $e^{tT}$ on $H$;
5. there exists a continuous linear form $\Psi : X \rightarrow \mathbb{R}$ such that $\Psi|_H = \psi$; and $\Lambda$ decomposes as $\Lambda = A + B$ with
6. $A$ is a bounded operator from $X$ to $H$;
7. $B$ is a closed unbounded operator on $X$ (with same domain as $D(\Lambda)$ the domain of $\Lambda$) and the semigroup $e^{tB}$ it generates satisfies, for some constant $C \geq 1$, that

$$\forall t \geq 0, \forall g \in X \text{ with } \Psi(g) = 0, \quad \|e^{tB}g\|_X \leq C\|g\|_X e^{at}.$$  

Then, for any $a' \in (a, 0)$ there exists $C_{a'} \geq 1$ such that

$$\forall t \geq 0, \forall g \in X, \quad \|e^{tA}g - \Psi(g)G\|_X \leq C_{a'}\|g - \Psi(g)G\|_X e^{a't}.$$  

3 Proof of the main theorem

The proof consists is an application of Theorem 2.9. For this, we consider the Hilbert space $H := L^2(x^m + x^M)$, where $G$ is the unique self-similar profile with $\int G = 1$, and define $\psi(g) := \int xg$. Due to our previous results [1] we know that $T$ and $\psi$ satisfy points 1–4 of Theorem 2.9.

As the larger space we take $X = L^1(x^m + x^M)$, with $1/2 < m < 1 < M$, to be precised later. Observe that, due to the bounds on $G$ from Theorem 2.7,

$$\|g\|_X \leq \int_0^\infty (x^m + x^M)|g(x)| \, dx$$

$$\leq \left( \int_0^\infty g(x)^2 \frac{x}{G(x)} \, dx \right)^{1/2} \left( \int_0^\infty (x^{m-\frac{1}{2}} + x^{M-\frac{1}{2}})^2 G(x) \, dx \right)^{1/2} = C\|g\|_H,$$
and hence \( H \subseteq X \). Similarly,
\[
\int_0^\infty x |g(x)| \, dx \leq \int_0^\infty (x^m + x^M) |g(x)| \, dx,
\]
which allows us to define \( \Psi : X \to \mathbb{R}, \Psi(g) := \int x g \), and proves that \( \Psi \) is continuous on \( X \). Obviously \( \Psi|_H = \psi \), so point 5 of Theorem 2.9 is also satisfied.

Consider \( \Lambda \) the unbounded operator on \( X \) given by the same expression (1.5) (with domain a suitable dense subspace of \( X \) which makes \( \Lambda \) a closed operator). To prove the remaining points 6 and 7 we use the following splitting of \( \Lambda \), taking real numbers \( 0 < \delta < R \) to be chosen later:
\[
A g(x) := \mathcal{L}^{+,s} g(x) := \int_x^\infty b_{R, \delta}(y, x) g(y) \, dy
\]
\[
\Lambda = A + B,
\]
\[
B g := \Lambda g - A g,
\]
where we denote \( b_{R, \delta}(x, y) := b(x, y) 1_{x \geq \delta} 1_{y \leq R} \). We define
\[
\mathcal{L}^{+,r} g := \mathcal{L}^+_g - \mathcal{L}^{+,s} g
\]
\[
= \int_x^\infty b(y, x) \left( 1 - 1_{1_{y \geq \delta} 1_{y \leq R} \leq 0} \right) g(y) \, dy
\]
\[
= \int_x^\infty b(y, x) 1_{y \leq \delta} g(y) \, dy + \int_x^\infty b(y, x) 1_{y \geq \delta} 1_{y \leq R} g(y) \, dy
\]
\[
=: \mathcal{L}^{+,r}_1 g + \mathcal{L}^{+,r}_2 g
\]
so we may write \( B \) as
\[
Bg = -2g - x \partial_x g - Bg + \mathcal{L}^{+,r}_1 g + \mathcal{L}^{+,r}_2 g.
\]

First, let us prove that \( A \) is bounded from \( X \) to \( H \). We compute
\[
\|Ag\|_H^2 = \int_0^\infty x (\mathcal{L}^{+,s} g)^2 G(x)^{-1} \, dx
\]
\[
\leq (2BM)^2 \left( \sup_{[0,R]} x G(x)^{-1} \right) \int_0^R \left( \int_{\max(x, \delta)}^\infty y^{-1} g(y) \, dy \right)^2 \, dx
\]
\[
\leq C_R \left( \int_0^\infty y^{-1} g(y) \, dy \right)^2 \leq C_R \delta \|g\|_X^2,
\]
which shows \( A : X \to H \) is a bounded operator. Notice that we have used here the lower bound \( G(x) \geq Cx \) for \( x \) small, proved in Theorem 2.7.

Then, let us prove that one can choose \( 0 < \delta < R \) appropriately so that \( B \) satisfies point 7 of Theorem 2.9 for some \( a < 0 \). It is enough to prove that, for \( g \) a real function in the domain of \( \Lambda \) (the same as the domain of \( B \)),
\[
\int_0^\infty \text{sign}(g(x)) Bg(x) (x^m + x^M) \, dx \leq a \|g\|_X,
\]
since then one can obtain \( \text{(2.13)} \) with \( C = 1 \) by considering the time derivative of the \( L^1 \) norm of \( e^{tB}g \). If we have this for any real \( g \), it is easy to show it also holds for a complex \( g \) and some constant \( C \geq 1 \). So, we take \( g \) real and in the domain of \( \Lambda \), and calculate as follows for any \( k > 0 \), using \( \text{(3.16)} \):

\[
\int_0^\infty \text{sign}(g(x)) Bg(x) x^k \, dx \leq (k-1) \int_0^\infty x^k |g| \, dx - \int_0^\infty B(x)x^k |g| \, dx + \int_0^\infty |L_1^{+} g| x^k \, dx + \int_0^\infty |L_2^{+} g| x^k \, dx, \tag{3.18}
\]

where the first term is obtained from the terms \(-2g - \partial_x g\) through an integration by parts. We give separately some bounds on the last two terms in \( \text{(3.18)} \). On one hand, we have

\[
\int_0^\infty |L_1^{+} g| x^k \, dx \leq \int_0^\infty x^k \int_0^\infty b(y, x) \mathbf{1}_{y \leq \delta} |g(y)| \, dy \, dx \\
\leq \int_0^\delta |g(y)| \left( \int_0^y x^k b(y, x) \, dx \right) \, dy \\
\leq 2BM \int_0^\delta |g(y)| B(y) y^k \, dy \\
\leq p_k B_m \delta^\gamma \int_0^\delta |g(y)| y^k \, dy, \tag{3.19}
\]

where we have used \( \text{(1.9)} \). On the other hand, and again due to \( \text{(1.9)} \),

\[
\int_0^\infty |L_2^{+} g| x^k \, dx \leq \int_0^\infty x^k \int_0^\infty b(y, x) \mathbf{1}_{x \geq R} \mathbf{1}_{y \geq \delta} |g(y)| \, dy \, dx \\
\leq \int_0^\infty x^k \int_{x=R}^\infty b(y, x) \mathbf{1}_{x \geq R} \mathbf{1}_{y \geq \delta} |g(y)| \, dy \, dx \\
\leq \int_R^\infty |g(y)| \left( \int_R^y x^k b(y, x) \, dx \right) \, dy \\
\leq p_k \int_R^\infty |g(y)| y^k B(y) \, dy. \tag{3.20}
\]
Hence, from (3.18) and the bounds (3.19)–(3.20) we obtain
\[
\int_0^\infty Bg(x) \text{sign}(g(x))(x^m + x^M) \, dx 
\leq (m - 1) \int_0^\infty x^m |g| \, dx + (M - 1) \int_0^\infty x^M |g| \, dx 
- \int_0^\infty B(x)(x^m + x^M) |g| \, dx 
+ p_m B_m \delta \gamma \int_0^\delta x^m |g(x)| \, dx + p_m \int_R^\infty x^m B(x) |g(x)| \, dx 
+ p_M B_m \delta \gamma \int_0^\delta x^M |g(x)| \, dx + p_M \int_R^\infty x^M B(x) |g(x)| \, dx. \quad (3.21)
\]
We have to choose \(1/2 < m < 1 < M < 2\) so that this is bounded by \(-C\|g\|_X\) for some positive constant \(C\). First, fix any \(m\) with \(1/2 < m < 1\), and take \(0 < \delta < 1\) small enough such that
\[
p_m B_m \delta \gamma < \frac{1 - m}{4}, \quad B_m \delta \gamma < \frac{1 - m}{4}.
\]
(Which can be done due to \(\gamma > 0\).) Then, as \(p_M < 1\) and \(x^M < x^m\) for \(x < \delta < 1\),
\[
\int_0^\infty Bg(x) \text{sign}(g(x))(x^m + x^M) \, dx 
\leq -\frac{1 - m}{2} \int_0^\infty x^m |g| \, dx + (M - 1) \int_0^\infty x^M |g| \, dx 
- \int_0^\infty B(x)(x^m + x^M) |g| \, dx 
+ p_m \int_R^\infty x^m B(x) |g(x)| \, dx + p_M \int_R^\infty x^M B(x) |g(x)| \, dx. \quad (3.22)
\]
Now, take \(R_0 > 0\) such that \(B(x) > 2 > M\) for \(x \geq R_0\). Then, choose \(1 < M < 2\) such that \((M - 1)x^M < \frac{1 - m}{4} x^m\) for \(0 < x < R_0\). Then whatever \(R\) is we have from (3.21):
\[
\int_0^\infty Bg(x) \text{sign}(g(x))(x^m + x^M) \, dx 
\leq -\frac{1 - m}{4} \int_0^{R_0} x^m |g| \, dx - \int_{R_0}^R x^M |g| \, dx 
- \int_R^\infty (B(x) - M + 1)x^M |g| \, dx 
+ p_m \int_R^\infty x^m B(x) |g(x)| \, dx + p_M \int_R^\infty x^M B(x) |g(x)| \, dx. \quad (3.23)
\]
Finally, choose $R > 1$ such that
\[-(B(x)(1 - p_M) - M + 1)x^M + pmx^m \leq -x^M \quad \text{for } x > R.\]

With this, and continuing from (3.23),
\[
\int_0^\infty Bg(x) \text{sign}(g(x))(x^m + x^M) \, dx \\
\leq \frac{1 - m}{4} \int_0^{R_0} x^m |g| \, dx - \int_0^\infty x^M |g| \, dx \\
\leq -C \|g\|_X, \quad (3.24)
\]

for some number $C = C(m, M, R_0) > 0$. This shows point that $\mathcal{B}$ is dissipative with constant $-C$, and hence point 7 of Theorem 2.9 holds with $\alpha = -C$. A direct application of Theorem 2.9 then proves our result, Theorem 1.5, with $\alpha := \min\{\beta, C\}$ (where $\beta$ is the one appearing in Theorem 2.6).

Acknowledgments. The first two authors acknowledge support from the project MTM2008-06349-C03-03 DGI-MCI (Spain) and the Spanish-French project FR2009-0019. The second author is also supported by the project 2009-SGR-345 from AGAUR-Generalitat de Catalunya. The third author acknowledges support from the project ANR-MADCOF.

References

[1] M. J. Cáreres, J. A. Cañizo, and S. Mischler. Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations. Journal de Mathématiques Pures et Appliquées, to appear, 2011 (preprint arXiv:1010.546).

[2] M. Doumic-Jauffret and P. Gabriel. Eigenelements of a general aggregation-fragmentation model. Mathematical Models and Methods in the Applied Sciences, 20(5):757–783, 2010.

[3] M. Escobedo, S. Mischler, and M. Rodríguez Ricard. On self-similarity and stationary problem for fragmentation and coagulation models. Ann. Inst. H. Poincaré Anal. Non Linéaire, 22(1):99–125, 2005.

[4] M. P. Gualdani, S. Mischler, and C. Mouhot. Factorization for non-symmetric operators and exponential H-theorem. Preprint, Jun 2010.

[5] J. A. J. Metz and O. Diekmann. The Dynamics of Physiologically Structured Populations, volume 68 of Lecture notes in Biomathematics. Springer, 1st edition, August 1986.

[6] P. Michel. Existence of a solution to the cell division eigenproblem. Mathematical Models and Methods in Applied Sciences, 16(1 supp):1125–1153, July 2006.
[7] B. Perthame. *Transport equations in biology.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.

[8] B. Perthame and L. Ryzhik. Exponential decay for the fragmentation or cell-division equation. *Journal of Differential Equations, 210*(1):155–177, March 2005.