THE ROME APPROACH TO CHIRALITY

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Abstract

Some general considerations on the problem of non perturbative definition of Chiral Gauge Theories are presented and exemplified within the particular proposal known as the Rome Approach.

1 Introduction

Gauge invariance starts as a classical concept: Vector and Chiral symmetries are on the same ground. In Quantum Field Theory, on the contrary, there is a deep difference between them, due to the lack of a chiral invariant regularization.

This fact is not merely a mathematical fancy, but is subject to direct experimental observation, e.g. in the $\pi_0 \to \gamma\gamma$ decay and similar. Also, the global structure of the Standard Model is deeply affected by the non existence of a gauge invariant regularization. In fact already in QCD, although a Vector Theory, the Chiral Classification of Local Observables (Current Algebra) is a complicated problem. Non perturbative computations
in Chiral Gauge Theories could clarify fundamental issues, as the possibility of dynamical Higgs mechanism, Baryon non conservation, the question of Naturalness.

How can we quantize chiral gauge theories?

Several approaches have been explored:

1) Non gauge invariant quantization (Rome approach)\cite{1, 2} based on the Bogolubov method

2) Gauge invariant quantization (J. Smit and P. Swift, S. Aoki, \ldots)

3) Other degrees of freedom.

Mirror Fermions (I. Montvay\cite{3})

4) \ldots and other dimensions

(D. Kaplan, R. Narayanan and H. Neuberger, S. Randjbar-Daemi and J. Strathdee\cite{4})

5) Fine-Grained Fermions

(G. Schierholz\cite{5}, G. t Hooft\cite{6}, P. Hernandez and R. Sundrum\cite{7})

\section{Quantization of Chiral Gauge Theories}

In order to quantize a theory we have to go through several steps:

\begin{itemize}
    \item Definition of the Target Theory
    \item Regularization
    \item Renormalization
\end{itemize}

\subsection{Target Theory}

We have, first of all, to decide what is the theory we are aiming at, i.e. the so called Target Theory. The formal (continuum) theory we want to reproduce is:

\begin{equation}
L = L_G + L_{gf}
\end{equation}

\begin{align*}
L_G &= \bar{\psi}_L \hat{D} \psi_L + \frac{1}{4} F^a_{\mu\nu} F^{a}_{\mu\nu} + \bar{\psi}_R \hat{\partial} \psi_R \\
L_{gf} &= \frac{1}{2\alpha_0} (\partial_\mu A^0_\mu)(\partial_\nu A^0_\nu) + \bar{c} \partial_\mu D_\mu c
\end{align*}

\footnote{Within this class falls also the formulation of the Zaragoza group\cite{3}.}
where $D_{\mu}$ denotes the covariant derivative

$$ D_{\mu} = \partial_{\mu} + ig_{0}A_{\mu}^{a}T^{a} \quad (2) $$

In Eq.(2), the $T^{a}$’s are the appropriate generators of the gauge group $G$, $g_{0}$ and $\alpha_{0}$ denote the bare coupling and gauge fixing parameter, respectively. The rest of the notation is self-explanatory.

A few comments are in order here:

- Presence of Gauge Fixing

As we will see later, it is rather difficult to dispose of it. Our general attitude is that it makes no harm. We are aware of a general argument by Neuberger[13] which shows that the sum over Gribov copies on a finite lattice is such that the expectation value of any gauge invariant quantity assumes the form of an indeterminate expression $0/0$. Neuberger argument applies, however, in situations in which the lattice regularization is exactly gauge invariant. In the present case gauge invariance is recovered only in continuum limit and a crucial ingredient of the argument, i.e. compactness, is lost. Of course this point deserves further investigation.

- No Higgs degrees of freedom are present in Eq.(1), but they could be easily added.

- The particular gauge group $SU(2)$ has been considered in order to avoid Local Anomalies without the need to introduce other fermions. Of course such a theory is probably affected by the Witten Global Anomaly[14], but this, of course, does not show up in perturbative checks of the method.

- The presence of fictitious, non interacting degrees of freedom, $\psi_{R}$, is useful to limit the form of the counterterms. They complicate the issue of Dynamical Fermion Non-Conservation and will be disposed off later.

The most important informations encoded in the Target Theory, are represented by its symmetries. In the present case they are:

a) BRST[15]

If we write the gauge fixing in the linearized form:

$$ L_{gf} = \frac{\alpha_{0}}{2}(\lambda^{a}\lambda^{a}) + i\lambda^{a}(\partial_{\mu}A_{\mu}^{a}) + \bar{c}\partial_{\mu}D_{\mu}c \quad (3) $$

it can be shown that $L_{G}$ and $L_{gf}$ are separately invariant under the BRST transformations, defined on the basic fields as

$$ \delta_{\text{BRST}}\psi_{L} = i\epsilon g_{0}c^{a}T^{a}\psi_{L} $$
\[
\delta \bar{\psi}_L \equiv \epsilon \delta_{\text{BRST}} \bar{\psi}_L = i \epsilon g_{0} \bar{\psi}_L T^a c^a \\
\delta \psi_R = \delta \bar{\psi}_R = 0 \\
\delta A^a_\mu \equiv \epsilon \delta_{\text{BRST}} A^a_\mu = \epsilon (D_\mu c)^a \\
\delta \lambda^a = 0 \\
\delta c^a \equiv \epsilon \delta_{\text{BRST}} c^a = -\frac{1}{2} \epsilon g_{0} f_{abc} c^b c^c \\
\delta \bar{c}^a \equiv \epsilon \delta_{\text{BRST}} \bar{c}^a = \epsilon \lambda^a
\]

where \(\epsilon\) is a grassmannian parameter. In fact \(L_{gf}\) is automatically BRST invariant as a consequence of nilpotency:

\[
\delta^2_{\text{BRST}} = 0
\]  

Other (global) Symmetries.

b) Vector-like:

\[
\psi_L \rightarrow V \psi_L \\
\psi_R \rightarrow V \psi_R \\
A_\mu \rightarrow V A_\mu V^+ \\
V \in G
\]

c) Shift Symmetry, that is the symmetry under the shift of the antighost field:

\[
\bar{c}(x) \rightarrow \bar{c}(x) + \text{const.}
\]

d) Global rotation of the right handed fields:

\[
\psi_R \rightarrow V \psi_R \\
\psi_L \rightarrow \psi_L \\
A_\mu \rightarrow A_\mu
\]

As usual, the invariance of the lagrangian implies an infinite set of identities on the Green’s Functions:

\[
\langle \Phi_1(x_1) \ldots \Phi_n(x_n) \rangle \equiv \int d\mu e^{S_{cl} \Phi_1(x_1) \ldots \Phi_n(x_n)}
\]

In particular BRST invariance implies:

\[
\langle \delta_{\text{BRST}} (\Phi_1(x_1) \ldots \Phi_n(x_n)) \rangle = 0
\]
2.2 Regularization

Once the Target Theory has been defined, in order to set up a consistent quantization scheme, a regulator must be introduced. All the following considerations are not tied to a particular regularization. They are quite general features of any known regularization scheme. However lattice discretization is very peculiar since it also allows the rather unique opportunity to perform systematic nonperturbative numerical explorations. This is why, in the following, we will exemplify the Rome approach in a Lattice Discretization setup.

Therefore, we first of all regularize the theory discretizing it in presence of gauge fixing:

\[
L_0 = \left( \frac{1}{2a} \right) \sum_\mu [\bar{\psi}_L(x + \mu) U_\mu(x) \gamma_\mu \psi_L(x) \\
+ \bar{\psi}_R(x + \mu) \gamma_\mu \psi_R(x) + h.c.] \\
+ \left( \frac{1}{2a^4 g_0^2} \right) \sum_{\mu,\nu} Tr(P_{\mu,\nu} - 1)
\]

where \( P_{\mu,\nu} \) denotes the plaquette formed with the link variables \( U_\mu \).

The general difficulties inherent to the quantization of a Chiral Gauge Theory, manifest themselves, in this case, in the form of the so called Doubling Problem.

In fact the naive discretization of the Dirac action in Eq.(11) leads to a (inverse) Fermion Propagator of the form:

\[
S^{-1}(p) = \frac{1}{a} \sum_{\mu=1}^4 \gamma_\mu \sin (ap_\mu)
\]

The problem with Eq.(12) is that it implies an unwanted proliferation of Fermion species usually referred to as the Doubling Problem. A general solution has been proposed by Wilson\[16\] and it consists in adding to the fermion action the so-called Wilson term:

\[
L_W = \left( \frac{-r}{2a} \right) \sum_\mu \{ [\bar{\psi}_L(x + \mu) \psi_R(x) \\
+ \bar{\psi}_L(x) \psi_R(x + \mu) - 2\bar{\psi}_L(x) \psi_R(x)] + h.c. \}
\]

which reads, in the continuum, as:

\[
L_W \approx ar \bar{\psi}_L \Box \psi_R + h.c.
\]
The presence of the Wilson term modifies Eq. (12) as:

\[ S^{-1}(p) = \frac{1}{a} \sum_{\mu=1}^{4} \gamma_\mu \sin (ap_\mu) + \frac{r}{a} \sum_{\mu=1}^{4} \sin^2 \left( \frac{ap_\mu}{2} \right) \]  

(15)

In this way the doubling problem is avoided.

However the Wilson term leaves us with an unwanted chiral symmetry breaking. This is a very general fact as expressed by the:

**Nielsen-Ninomiya Theorem**: Any local, chiral symmetric, bilinear action implies Spectrum Doubling.

The whole problem of quantizing Chiral Gauge Theories is precisely to understand the effect of such regularization-induced chirality breaking.

### 2.3 Regularization-Induced Symmetry Breaking

In the language of renormalization theory, \( L_W \) is a so-called “irrelevant” term: its presence can be compensated by finite or power divergent counterterms.

Since this a central point (and a very inconvenient one) let us discuss in more detail the origin and the manifestation of this phenomenon.

We start with a theory with a given symmetry group \( G \), broken at the level of the cutoff, say \( a \).

As an example we may think of a \( \lambda \phi^4 \) theory (symmetric under \( \phi \rightarrow -\phi \)) with an additional \( O_5 \approx \phi^5 \) term in the lagrangian.

Of course, in order to be really a correction of order \( a \) to start with, \( O_5 \) should be a finite (i.e. renormalized) operator in the continuum limit \( (a \rightarrow 0) \) in order to avoid an immediate back-reaction giving rise to counter-terms \( \phi \) and \( \phi^3 \).

The theory is defined by the action:

\[ S(\phi) = S_{sym}(\phi) + a \int d^4x O_5(x) \]  

(16)

We can now expand any Green’s function in powers of \( a \int dx O_5 \) and consider, as a particular example, the three-point Green’s function (expected to vanish in the symmetric theory):

\[ \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \phi(x_1)\phi(x_2)\phi(x_3)(a \int dx O_5)^n \rangle \]  

(17)
First order correction:

\[ \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_{(1)} = a \int dy \langle \phi(x_1)\phi(x_2)\phi(x_3)O_5(y) \rangle \] (18)

This is fine (i.e. \( a \to 0 \)) since we assumed that the single insertion of \( O_5 \) is finite.

The next interesting contribution, in this particular example, is the third order correction:

\[
\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_{(3)} \approx a^3 \int dy_1dy_2dy_3 \langle \phi(x_1)\phi(x_2)\phi(x_3)O_5(y_1)O_5(y_2)O_5(y_3) \rangle
\] (19)

The contribution coming from Eq.(19) is hardly of order \( a^3 \). In fact the integrals in the r.h.s. of Eq.(19) get contributions from the region where the \( y \)'s are close together, which can be estimated through the Operator Product Expansion as:

\[
\int dy_1dy_2dy_3O_5(y_1)O_5(y_2)O_5(y_3) \approx \frac{1}{a^4} \int dy_1O_3(y_1)
\] (20)

where \( O_3 \) is, in the present case a renormalized version of \( \phi^3 \).

This integration region gives therefore rise to a linearly divergent contribution (as expected from power-counting) of the form:

\[
\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_{(3)} \approx \frac{1}{a} \int dy_1 \langle \phi(x_1)\phi(x_2)\phi(x_3)O_3(y_1) \rangle
\] (21)

Depending on the particular regularization employed, the appearance of these contributions can be shifted to higher orders in Perturbation Theory, but can hardly be eliminated, unless some exact selection rule is operating at the level of the regularized theory.

In this situation the only way to get a sensible continuum limit is to add to the action all possible symmetry breaking (and non-Lorentz invariant, in the case of the Lattice Discretization) counterterms with dimension \( D \leq 4 \).

This discussion is directly relevant to the case in which Gauge Symmetry is violated by the regulator, at least in the case in which the theory is defined within some specific gauge.
2.4 The Rome Approach to Chirality

In the case of a Target Theory as in Eq.(1) we have plenty of possible counterterms. In fact, defining the vector field $A_\mu$ as e.g.:

$$a g_0 A_\mu \equiv \frac{U_\mu - U_\mu^+}{2i}$$

we have:

- Counterterms with $D = 2$:
  $$- \frac{\delta \mu^2}{2} A_\mu^a(x) A_\mu^a(x)$$

No ghost mass counterterms arise because of the shift symmetry.

- Counterterms with $D = 3$:
  $$\delta M \left[ \bar{\psi}_L(x) \psi_R(x) + h.c. \right]$$

- Examples of counterterms with $D = 4$:
  Fermion vertices counterterms:
  $$- i \delta g_R \bar{\psi}_R T^a A_\mu^a \gamma^\mu \psi_R$$
  $$- i \delta g_L \bar{\psi}_L T^a A_\mu^a \gamma^\mu \psi_L$$

Non minimal terms in $A_\mu$, $\bar{c}^a$ and $c^c$:

$$\frac{\left( \partial \mu A_\mu^a \right)^2}{f^{abc} \partial \mu A_\mu^a A_\mu^b A_\mu^c}$$

$$\sum \partial \mu A_\mu^a \partial \mu A_\mu^a$$

$$\delta g_{gh} f_{abc} \bar{c}^a \partial \mu \left( A_\mu^b c^c \right)$$

The presence of the counterterm Eq.(27) is very important because it signals an irreversible breaking of geometry. We will come back later on this point.

The strategy is to fix the values of the counterterms as follows.

First of all compute (e.g. non perturbatively):

$$\langle \Phi_1(x_1) \ldots \Phi_n(x_n) \rangle = \int DU_\mu D\bar{\psi} D\psi D\bar{c} Dc e^{S_0 + S_W - \frac{1}{4g_0^2} \int d^4x (\partial \mu A_\mu^a)^2 + S_{\text{ghost}} + S_{\text{c.t.}}} \Phi_1(x_1) \ldots \Phi_n(x_n)$$

(28)
then tune the values of the counterterms imposing the BRST Identities Eq. (10). This is at best possible up to order $a$ (and impossible if there are unmatched anomalies).

By this procedure we define a bare chiral theory with parameters $g_0$ and $\alpha_0$ implicitly defined by the BRST transformations.

It is now possible to carry out the usual non-perturbative renormalization, by fixing the bare parameters to reproduce given finiteness conditions (on physical quantities and/or Green functions).

The procedure just outlined is completely non perturbative. However perturbation theory may be practically helpful. In fact the theory so defined should be asymptotically free and we may distinguish two different kinds of counterterms:

- **Dimensionless counterterms:**
  \[ \delta Z = f(g_0, \alpha_0) \] (29)

  The value of these counterterms can be reliably estimated from perturbation theory.

- **Dimensionful counterterms:**
  \[ \delta M = \frac{1}{a} f(g_0, \alpha_0) \] (30)

  These counterterms are essentially non perturbative. In fact exponentially small contributions to $f$ can be rescued in the continuum limit by the factor $\frac{1}{a}$:

  \[ \frac{1}{a} f(g_0, \alpha_0) \approx e^{-\frac{1}{g_0}} \approx \Lambda \] (31)

  where $\Lambda$ is the usual scale defined through dimensional transmutation.

  The scheme just described has been checked in perturbation theory at 1-loop and reproduces the results of continuum perturbation theory.

2.5 Are Ghosts (and Gauge-Fixing) Unavoidable?

G.C.Rossi, R.Sarno and R.Sisto computed the ghost counterterm defined in Eq.(27) at two loops (in dimensional regularization) and found that $\delta g_{gh} \neq 0$. Thus (at least if we trust Perturbation Theory) we cannot invert the Faddeev-Popov procedure and remove the gauge fixing. This is the signal that the gauge geometry is irreversibly lost.
2.6 Possible Obstructions

Several points may go wrong during the implementation of the program just described. In particular the procedure will not work in presence of:

- Non cancelled perturbative anomalies.
- Non perturbative anomalies\[^{[14]}\].
- The symmetry defined by Eq.(8) must be realized à la Wigner. This is not a trivial requirement as the example of QCD clearly shows. In fact in QCD Chiral symmetry can be recovered by an appropriate tuning of the quark masses, but the phase in which it is recovered is a completely dynamical issue.

3 Gauge Averaging

An interesting proposal to deal with a gauge non-invariant regulator is the so-called method of Gauge Averaging (D.Foerster, H.Nielsen and M.Ninomiya\[^{[13]}\], J.Smit and P.Swift\[^{[4]}\], S.Aoki\[^{[5]}\],.....).

The basic idea is to make the Wilson term, or any other gauge non-invariant term, invariant through the introduction of an additional degree of freedom in the form of the angular part of a scalar Higgs-like field $\Omega(x)$ with $\Omega(x) \in G$ as:

$$a\bar{\psi}_R \partial^2 (\Omega^+ \psi_L)$$  \hspace{1cm} (32)

This theory is now exactly invariant under the gauge transformations $g \in G_1$:

$$\Omega \rightarrow g \Omega \equiv \Omega^g$$
$$\psi_L \rightarrow g \psi_L \equiv \psi_L^g$$  \hspace{1cm} (33)
$$\psi_R \rightarrow \psi_R$$
$$U_{\mu} \rightarrow g^+(x + \mu) U_{\mu} g(x) \equiv U_{\mu}^g$$

In this way any action can be made invariant under $G_1$:

$$\int DUD\bar{\psi} D\Omega e^{S_{NI}(U^g, \psi^g, \bar{\psi}^g)}$$  \hspace{1cm} (34)

However the group $G_1$ is not the physical gauge group because the Target Theory does not contain any scalar field and $\Omega(x)$ cannot be identified with a physical Higgs field: switching off the gauge interaction we should get back a free fermion theory. Moreover
the gauge average seems to produce theories with too many relevant parameters. We must remember, at this point, that the correct theory should, instead, be invariant under the physical gauge group:

\[ \Omega \rightarrow \Omega \]
\[ \psi_L \rightarrow g \psi_L \equiv \psi_L^g \]
\[ \psi_R \rightarrow \psi_R \]
\[ U_\mu \rightarrow g^+(x + \mu) U_\mu g(x) \equiv U_\mu^g \]

or:

\[ \Omega \rightarrow g\Omega \equiv \Omega^g \]
\[ \psi_L \rightarrow \psi_L \]
\[ \psi_R \rightarrow \psi_R \]
\[ U_\mu \rightarrow U_\mu \]

If this is the case, then \( \Omega(x) \) can be transformed into the identity and decoupled completely.

A possible strategy\(^2\) to decouple \( \Omega(x) \) is to add to the action (and adjust) all the relevant \( G_1 \)-invariant counterterms.

Among these we have, for example:

\[ \delta S \approx \frac{\kappa}{2} \int d^4x (D(A)_\mu \Omega(x))^2 \]

which provides both a mass term for the gauge field \( A_\mu \) and a kinetic term for \( \Omega(x) \).

It is possible to show\(^2\) that the decoupling of \( \Omega(x) \) can be achieved by, first of all, gauge fixing the theory:

\[
\langle O \rangle = \frac{1}{Z} \int D\Omega \int DUD\psi D\bar{\psi} D\bar{c} Dce^{S_{NI}(U,\psi,\bar{\psi},\bar{c})} + S_g(\bar{c},c,U) O \left( U, \psi, \bar{\psi} \right)
\]

and then tuning the parameters in such a way that the integrand becomes \( \Omega \)-independent. This procedure turns out to be exactly equivalent to impose the BRST identities. The \( \Omega \)-integration can be dropped and we are back to the Rome approach.

Suppose, instead, we try to integrate the theory without any gauge-fixing. Then we could try to argue as follows.
We start by decomposing the action as:

\[ S = S_{GI} + a \int d^4y W(y) \]  

where \( S_{GI} \) is the gauge-invariant part (the theory with the doublers in the physical spectrum) and \( W(x) \) is the "irrelevant" dimension 5 Wilson term. If we denote by \( O_{GI} \) any (multi-) local gauge-invariant operator, we have, for its expectation value (at least formally):

\[ \langle O_{GI} \rangle \equiv \int DUD\bar{\psi}e^{S_{GI} + a \int d^4y W(y)} \langle O_{GI} \rangle = \sum_{n=0}^{\infty} \frac{a^n}{n!} \int D\Omega \langle O_{GI}(\int d^4y W^\Omega(y))^n \rangle \]  

where we have introduced an (harmless) integration over the fictitious variable \( \Omega(x) \). In fact this operation is well defined within any Lattice discretization. The \( \Omega \) integration is compact and obeys the rules coming from group theory. We have, for instance:

\[ \int D\Omega \Omega_{ij}(x)\Omega_{kl}^\dagger(y) = \delta_{xy}\delta_{il}\delta_{jk} \]  

where all the \( \delta \)'s are Kronecker \( \delta \)'s, since we are on a lattice.

The correction linear in \( W \), \( \langle O_{GI} \rangle_{(1)} \), in Eq.(39) vanishes trivially in virtue of the gauge average. On the contrary, for the second order correction, \( \langle O_{GI} \rangle_{(2)} \), we have:

\[ \langle O_{GI} \rangle_{(2)} \approx a^2 \int dy_1dy_2 \int D\Omega \langle O_{GI} W^{\Omega}(y_1) W^{\Omega}(y_2) \rangle \approx a^{10} \sum_{y_1,y_2} \int D\Omega \langle O_{GI} W^{\Omega}(y_1) W^{\Omega}(y_2) \rangle \]  

In the last equality we resorted to the explicit lattice notation for the integral in order to keep track correctly of the powers of \( a \).

From Eq.(39) we know that the \( \Omega \)-integration makes the two \( W \) insertions stick together, giving rise to a (non-)renormalized gauge invariant operator of dimension 10, \( 0_{10} \):

\[ \langle O_{GI} \rangle_{(2)} \approx a^{10} \sum_{y_1} \langle O_{GI} O_{10}(y_1) \rangle \approx a^6 \int dy_1 \langle O_{GI} O_{10}(y_1) \rangle \]  

\[ \approx a^6 \int dy_1 \langle O_{GI} O_{10}(y_1) \rangle \]
where we reintroduced the continuum notation.

The gauge-invariant operator $O_{10}$, defined by Eq.(43), mixes (with power divergent coefficients) with gauge-invariant operators of lower dimension:

$$O_{10} \approx \sum_k \frac{1}{a^{10-k}} O_k$$

(44)

The factor $a^6$ in Eq.(43) selects from the mixing, defined in Eq.(44), all the gauge-invariant operators of dimension 4, $O_4$, or less, with appropriate coefficients.

We thus get, for instance:

$$\langle O_{GI} \rangle_{(2)} \approx \int dy_1 \langle O_{GI} O_4(y_1) \rangle$$

(45)

This procedure can be carried out order by order in the $W$ insertion and the conclusion is that the effect of the gauge average in the computation of gauge-invariant observables can be reabsorbed by a redefinition of the parameters already present in the gauge-invariant part of the action Eq.(39), $S_{GI}$. This argument seems to suggest that, after the gauge average, the expectation value of any gauge-invariant observable, $O_{GI}$ will suffer again from the doublers contribution.

Is this argument safe? This is not completely clear. In fact, although this argument is non-perturbative, the order by order expansion in $W$ may be questioned. Certainly this argument could fail in presence of spontaneous symmetry breaking. In fact in such situations an explicit breaking of the symmetry is needed to select a particular vacuum (the one aligned along the direction of the breaking term) and the formal expansion in powers of the symmetry breaking term could easily cause troubles connected with the failure of the cluster property. In the gauge case this should not cause any problem because we know, from Elitzur’s theorem[21] that the local symmetry does not suffer spontaneous symmetry breaking.

It could, however, be argued that the expansion in the Wilson term may be non-analytic: after all the Wilson term modifies in a dramatic way the physical spectrum of the theory. Although this possibility cannot be disproved in general, it is instructive to examine what happens in a very simple, completely solvable case.

Consider a free fermion theory, in presence of the Wilson term, written, for notational simplicity, in the continuum language:

$$L = \int dx \bar{\psi}(x) \not{D} \psi(x) + ra \int dx \bar{\psi}_L \not{D} \psi_R + h.c.$$  

(46)
Suppose we want to compute, in such a theory, the correlator $\Pi(q^2)$ of the vector current $j_\mu(x) \equiv \bar{\psi}(x) \gamma_\mu \psi(x)$:

$$\Pi(q^2)(q_\mu q_\nu - q^2 \delta_{\mu\nu}) \equiv \int \frac{d^4x}{(2\pi)^4} \langle j_\mu(x)j_\nu(0) \rangle e^{iqx} \quad (47)$$

As well known $\Pi(q^2)$ has a logarithmic divergence proportional to the number of particles running in the loop. Therefore the coefficient will be different in the theory with $r \neq 0$ and the one with $r = 0$ because of the presence of the doublers. If, when $r \neq 0$, we put:

$$\Pi_{r \neq 0}(q^2) \approx \beta \log(a^2q^2) + \text{finite terms} \quad (48)$$

then, in the case $r = 0$ we have:

$$\Pi_{r=0}(q^2) \approx 2^4\beta \log(a^2q^2) + \text{finite terms} \quad (49)$$

precisely because in this case the doublers will contribute. In Eqs. (48), (49), the coefficient $\beta$ is independent of $r$, while the finite terms in Eq. (48) show a logarithmic singularity as $r \to 0$.²

Let us see how this behaviour can be obtained by expanding the theory with $r \neq 0$ in powers of $r$. We have:

$$\Pi_{r \neq 0}(q^2) = \Pi_{r=0}(q^2) + \sum_{n=1}^{\infty} \frac{r^n}{n!} \Pi_{(n)}(q^2) \quad (50)$$

where $\Pi_{(n)}(q^2)$ denotes the contribution to $\Pi(q^2)$ coming from the insertion of $n$ Wilson terms. These insertions are infrared divergent, but the contribution to the infrared divergence comes only by the doublers. In fact the Wilson term is of order $q^2$ for $q^2 \approx 0$, but is of order 1 for $q^2$ around the momenta of any of the doublers. As a consequence, for small $a$, we have, (for $n > 0$):

$$\frac{1}{n!}\Pi_{(n)}(q^2) \approx \frac{(-1)^{n+1}}{n} \frac{2^4 - 1}{\beta} \frac{1}{(a^2q^2)^n} \quad (51)$$

We get, therefore, from Eq. (50):

$$\Pi_{r \neq 0}(q^2) \approx \Pi_{r=0}(q^2) + (2^4 - 1)\beta \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{r^n}{(a^2q^2)^n} \approx$$

$$\approx 2^4\beta \log(a^2q^2) + (2^4 - 1)\beta \log(1 + \frac{r}{(a^2q^2)}) \approx$$

$$\approx \beta \log(a^2q^2)$$

²The argument is completely analogous to the one used in the computation of the effective potential.
Eq. (52) shows that the cancellation of the doubler contribution does not necessarily require a non-analytic behavior in $r$.

4 Fermion non conservation

In the approach just outlined, only Fermion Number conserving Green’s functions can be defined at the Lattice level.

This does not necessarily imply Fermion conservation.

In fact through Cluster Factorization we can define Green’s functions related to Fermion violating processes.

We may compute, for example:

$$\langle O_{\Delta F=2}(x)O_{\Delta F=-2}(y) \rangle$$

and consider the limit:

$$\lim_{x-y \to \infty} \langle O_{\Delta F=2}(x)O_{\Delta F=-2}(y) \rangle = \langle O_{\Delta F=2}(x) \rangle \langle O_{\Delta F=-2}(y) \rangle$$

(54)

It is, however, more consistent and interesting to formulate the theory from the start without redundant degrees of freedom, corresponding to the Target Theory:

$$L = L_g + L_{gf}$$

$$L_g = \bar{\chi}_{\alpha} \not{D}^{\beta} \chi_{\beta} + \frac{1}{4} F_{\mu \nu}^{a} F_{\mu \nu}^{a}$$

$$L_{gf} = \frac{1}{2\alpha_0} \left( \partial_{\mu} A_{\mu}^{a} \right) \left( \partial_{\mu} A_{\mu}^{a} \right) + \bar{c} \partial_{\mu} D_{\mu} c$$

(55)

The Fermionic euclidean functional integration is now performed over the independent Grassmann variables $\bar{\chi}_{\dot{\alpha}}$ and $\chi_{\alpha}$.

In the discretization process we have again the doubling phenomenon, and it can be avoided through a Majorana-Wilson term of the form:

$$L_W = a \left( \chi_{\alpha} \partial_{\mu} \partial_{\mu} \chi_{\alpha} + \bar{\chi}_{\dot{\alpha}} \partial_{\mu} \partial_{\mu} \bar{\chi}_{\dot{\alpha}} \right)$$

(56)

The Majorana-Wilson term is still irrelevant and induces finite or power divergent non-Lorentz invariant and Fermion number violating counterterms with $D \leq 4$, to be fixed again by BRST identities.
This scheme has been checked in 1-loop perturbation theory by G. Travaglini\textsuperscript{25} and it works fine.

Within such a formulation it is now possible to write down fermion number violating Green’s functions that are order $a$ in perturbation theory, but can be enhanced and promoted to finite objects in presence of non-perturbative configurations as, for example, instantons.

## 5 Concluding Remarks

- Wilson-Yukawa theories were not discussed in this talk, but they can be (and, in fact, have been) treated along the same lines\textsuperscript{3}.
  - Fine Tuning and Naturalness.

  The problems outlined in this talk are not necessarily merely technical. In fact if fine tuning would turn out to be a really necessary ingredient for the definition of a Chiral Gauge Theory, this may cast serious doubts on the Naturalness concept in a purely field theoretical framework, with possible far reaching implications on the nature of the more fundamental theory of which quantum field theory could be considered a low energy approximation.

  - Gribov problems?

    Within the Rome approach to Chirality a rather fundamental technical ingredient is represented by the gauge fixing. The presence of the ambiguities due to the Gribov phenomenon still represents a serious challenge within a non-perturbative framework. Certainly much more (difficult) work has to be done in order to clarify this important issue.

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