ON THE BASIS POLYNOMIALS IN THE THEORY OF
PERMUTATIONS WITH PRESCRIBED UP-DOWN
STRUCTURE

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ABSTRACT. We study the polynomials which enumerate the permutations \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \) of the elements 1, 2, \ldots, \( n \) with the condition \( \pi_1 < \pi_2 < \ldots < \pi_{n-m} \) (or \( \pi_1 > \pi_2 > \ldots > \pi_{n-m} \)) and prescribed up-down points \( n-m, n-m+1, \ldots, n-1 \) in view of an important role of these polynomials in theory of enumeration the permutations with prescribed up-down structure similar to the role of the binomial coefficients in the enumeration of the subsets of a finite set satisfying some restrictions.

1. Introduction

D. Andre [2] first considered (1881) the problem of the enumerating the alternating permutations \( \pi = (\pi_1, \ldots, \pi_n) \) of the numbers 1, 2, \ldots, \( n \) for which ups and downs are alternating:

\[ \pi_1 < \pi_2 > \pi_3 < \ldots \]

This problem has a highly aesthetic solution: the exponential generating function of such permutations is the sum of tangent and secant. But only after a century (1968) I.Niven [11] considered a general problem of the enumerating the permutations with given up-down structure. For permutation \( \pi = (\pi_1, \ldots, \pi_n) \), the sequence \( (q_1, q_2, \ldots, q_{n-1}) \), where

\[ q_j = \text{sign}(\pi_{j+1} - \pi_j) = \begin{cases} 1, & \text{if } \pi_{j+1} > \pi_j \\ -1, & \text{if } \pi_{j+1} < \pi_j \end{cases} \]

is called a Niven’s signature. For example, \( a = (2, 1, 5, 4, 3) \) has the signature \((-1, 1, -1, -1, -1)\).

Denote \([q_1, q_2, \ldots, q_{n-1}]\) the number of permutations having the Niven’s signature \((q_1, q_2, \ldots, q_{n-1})\). In view of symmetry we have

\[ [q_1, q_2, \ldots, q_{n-1}] = [-q_{n-1}, -q_{n-2}, \ldots, -q_1]. \]

Niven obtained the following basic result.
Theorem 1. [11] Let in the signature \((q_1, q_2, \ldots, q_{n-1})\) the indices of those \(q_i\) which are +1 be \(k_1 < k_2 < \ldots < k_m\) (if such \(q_i\) do not exist then assume \(m = 0\)). Put in addition \(k_0 = 0, \ k_{m+1} = n\). Then

\[
[q_1, q_2, \ldots, q_{n-1}] = \text{det}N,
\]

where \(N = \{n_{ij}\}\) is the square matrix of order \(m + 1\) in which

\[
n_{ij} = \binom{k_i}{k_{j-1}}, \quad i, j = 1, 2, \ldots, m + 1.
\]

After this celebrated Niven’s result and until now there has been a series of articles by many authors. We mention only ten papers in chronological order: N.G.Bruijn, 1970 [5], H.O.Foulkes, 1976 [7], L.Carlitz, 1978 [6], G.Viennot, 1979 [17], C.L.Mallows and L.A.Shepp, 1985 [9], V.Arnold, 1990 [3], V.S.Shevelev, 1996 [14], G.Szpiro, 2001 [16], B.Shapiro, M.Shapiro and A.Vainshtein, 2005 [12], F.C.S.Brown, T.M.A. Fink and K.Willbrand, 2007 [4].

According to the de Bruijn-Viennot algorithm [5], [17] the calculating of \([q_1, q_2, \ldots, q_{n-1}]\) could be realized using the following numerical triangle. At the top of the triangle put 1. Write 0 to the right (left) if \(q_{n-1}\) is 1(−1). For example, if \(q_{n-1} = 1\) then the first two elements of the triangle are: \(1 \ 0\).

Summing these elements we write the sum to the left: \(1 \ 1 \ 0\). Now the following 0 we write to the right of the last element (to the left of the first element) if \(q_{n-2}\) is 1(−1). For example, if \(q_{n-2} = 1\) then we have \(1 \ 1 \ 0 \ 0\).

Now the third row is obtained by summing each element in the third row from the right to the left, first element being 0, with the elements left and above it in the second row:

\[
1 \\
1 \ 0 \\
1 \ 0 \ 0
\]

In case of \(q_{n-2} = -1\) we have \(1 \ 0 \ 0\) and the third row is obtained by summing each element in the third row from the left to the right, first element being 0, with the elements right and above it in the second row:
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The process continues until the n-th row which corresponds to $q_1$. Now the sum of elements of the n-th row is equal to $[q_1, q_2, \ldots, q_{n-1}]$.

Example 1. For signature $(-1, 1, 1, -1, 1)$ we have the triangle

\[
\begin{array}{cccc}
1 & 0 \\
0 & 1 & 1 \\
2 & 2 & 0 \\
5 & 3 & 1 & 0 & 0 \\
0 & 5 & 8 & 9 & 9 & 9 \\
\end{array}
\]

Therefore, $[-1, 1, 1, -1, 1] = 0 + 5 + 8 + 9 + 9 + 9 = 40$. In order to obtain a weight generalization let us consider a matrix function which we call "alternant" (cf.\[13\]). If a permutation $\pi$ has the signature $(q_1, q_2, \ldots, q_{n-1})$ then we write $\pi \in (q_1, q_2, \ldots, q_{n-1})$. Furthermore, if $\pi_i = j$ then to the two-dimensional point $(i, j)$ assign the "weight" $a_{ij}$.

Let $A = (a_{ij})$ be an $n \times n$ matrix. Denote

\[alt_{(q_1, \ldots, q_{n-1})}A = \sum_{\pi \in (q_1, \ldots, q_{n-1})} \prod_{i=1}^{n} a_{i\pi_i}.
\]

Let $A_{1j}$, $j = 1, \ldots, n$, be $(n-1) \times (n-1)$ matrix which is obtained from $A$ by the deletion of the first row and the $j$-th column. Denote $A_{1j}^{(+1)}(A_{1j}^{(-1)})$ the matrix which is obtained from $A_{1j}$ by replacing the $j-1$ first (the $n-j$ last) elements of its first row by 0’s. Then from the Viennot’s algorithm we deduce the following expansion of the alternant by the first row of the matrix.

**Theorem 2.** (cf.\[13\])

\[alt_{(q_1, \ldots, q_{n-1})}A = \sum_{j=1}^{n} a_{1j}alt_{(q_2, \ldots, q_{n-1})}A_{1j}^{(q_1)}.
\]

Note that if $A = J_n - n \times n$ matrix composed of 1’s only, then
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\[ alt_{(q_1, \ldots, q_{n-1})} A = [q_1, \ldots, q_{n-1}] \]

In case of arbitrary \((0,1)\)matrix \(A\), Theorem 2 gives enumeration the permutations having signature \((q_1, \ldots, q_{n-1})\) with restriction on positions. For example, if \(I_n\) is \((n \times n)\) identity matrix, then \(alt_{(q_1, \ldots, q_{n-1})}(J - I)\) gives the number of such permutations without fixed points.

Note that, by (5)\(\ref{5}\)

\[ alt_{(1)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22}, \quad alt_{(-1)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{12}a_{21}. \]

Example 2.

\[
alt_{(1,-1,1)}J_4 = alt_{(1,-1,1)} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = alt_{(-1,1)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \\
+ alt_{(-1,1)} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + alt_{(-1,1)} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + alt_{(-1,1)} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \\
= \left( alt_{(1)} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + alt_{(1)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + alt_{(1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) + \\
+ \left( alt_{(1)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + alt_{(1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) + alt_{(1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0 + 1 + 1 + 1 + 1 + 1 = 5, \\
while \\
alt_{(1,-1,1)}(J_4 - I) = alt_{(1,-1,1)} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = alt_{(-1,1)} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \\
alt_{(-1,1)} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
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\[ +\text{alt}_{(-1,1)} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \text{alt}_{(-1,1)} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \]

\[ = \left( \text{alt}_{(1)} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 \end{pmatrix} + \text{alt}_{(1)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 \end{pmatrix} \right) + \text{alt}_{(1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2. \]

Note that alternant is also useful for enumeration the permutations with some additional conditions. For example, if it is necessary to enumerate the permutations \( \pi \) with signature \((q_1, q_2, \ldots, q_{n-1}) \) for which \( \pi_1 = l, \pi_n = m \) then we should calculate \( \text{alt} J^{(l,m)}_n \) where \( J^{(l,m)}_n \) is obtained from \( J_n \) by replacing all 1's of the first and the last rows by 0’s except the l-th 1 and the m-th 1 correspondingly. For example, there are only 2 such permutations in the case of the signature \((-1, 1, 1, -1, 1)\), \( n = 6, l = 2, m = 6 \).

Let us now introduce an index of the Niven’s signature in the following way: the integer \( k = k_n \) is called the index of the signature \((q_1, q_2, \ldots, q_{n-1})\) if \((n - 1)\)-digit binary representation of \( k \) is

\[ k = \sum_{i=1}^{n-1} q_i 2^{n-i-1}, \tag{9} \]

where

\[ q_i = \begin{cases} 1, & \text{if } q_i = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{10} \]

(cf [13], [14], where as an index was taken \( k + 1 \)). Denote \( S^{(k)}_n \) the set of permutations of elements 1, 2, \ldots, \( n \) having the index \( k \), and put

\[ \binom{n}{k} = |S^{(k)}_n|. \tag{11} \]

Let \( k \in [2^{t-1}, 2^t) \) and the \((n - 1)\)-digit binary expansion of \( k \) \((9)\) has a form:

\[ k = \underbrace{0 \ldots 0}_{n-t-1} \underbrace{1 \ldots 0}_{s_2-s_1-1} \underbrace{1 \ldots 0}_{s_3-s_2-1} \underbrace{1 \ldots 1}_{s_4-s_3-1} \underbrace{0 \ldots 0}_{s_{m-1}-s_m-1} \underbrace{1 \ldots 0}_{t-s_m} \tag{12} \]

where
1 = s_1 < s_2 < \ldots < s_m

are places of 1’s after \( n - t - 1 \) 0’s before the first 1.

From Theorem 1 follows an important formula.

**Theorem 3.** For index \( k \in [2^{t-1}, 2^t) \) we have

\[
\binom{n}{k} = \left| \begin{array}{cccccccc}
1 & 1 & 0 & 0 & \ldots & 0 \\
1 & (n - t + s_2 - 1) & (s_2 - 1) & 1 & 0 & \ldots & 0 \\
1 & (n - t + s_3 - 1) & (s_3 - 1) & (n - t + s_3 - 1) & 1 & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & (n - t + s_m - 1) & (s_m - 1) & (n - t + s_m - 1) & (s_m - s_2) & (n - s_m + 1) & 1 & \ldots & 1 \\
1 & \binom{n}{t} & (n + 1 - s_2) & (t + 1 - s_2) & (t + 1 - s_3) & \ldots & (n + 1 - s_m) & \ldots & (n + 1 - s_m)
\end{array} \right|
\]

In [14] using a techniques of permanents the following explicit formula was proved.

**Theorem 4.** [14]. For \( k \in [2^{t-1}, 2^t) \) we have

\[
\binom{n}{k} = (-1)^m + \sum_{p=1}^{m} (-1)^{m-p} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq m} \left( \frac{n}{t + 1 - s_1} \right) \prod_{r=2}^{p} \left( \frac{n - t - s_i - 1}{s_{i_r} - s_{i_{r-1}}} \right)
\]

Most likely, (15) is the first nondeterminant formula in a closed form for the number of permutations with prescribed up-down structure (cf. [16], [4]).

The Thue-Morse sequence [10], [8] is defined by

\[
\tau_n = (-1)^{\sigma(n)}
\]

where \( \sigma(n) \) denotes the number of 1’s in the binary representation of \( n \).

Thus, from (15) immediately follows an interesting arithmetical property of \( \binom{n}{k} \).
Theorem 5.\cite{13}. If all different from 1 divisors of $n$ are larger than $\lfloor \log_2 k \rfloor$ then

\begin{equation}
\begin{bmatrix} n \\ k \end{bmatrix} \equiv \tau_k \pmod{n} \tag{17}
\end{equation}

Remark 1. It is evident that the validity of (17) does not depend on the fact whether $k$ is a constant or a function of $n$.

Remark 2. From Theorem 5 and (16) it follows that if $n$ has only sufficiently large different from 1 divisors then the number of permutations of $n$ elements with arbitrary prescribed up-down structure is $\pm 1 \pmod{n}$. Quite recently a special case of this result for a prime $n$ was reproduced in \cite{4}.

2. Basis polynomials

As it follows (15) if $k$ does not depend on $n$ then $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in $n$ of the degree

\begin{equation}
t = \lfloor \log_2(2k) \rfloor \tag{18}
\end{equation}

Indeed, the degree of the polynomial in the interior sum of (15) equals to

\[ t + 1 - s_{i_p} + (s_{i_2} - s_{i_1}) + (s_{i_3} - s_{i_2}) + \ldots + (s_{i_p} - s_{i_{p-1}}) = t + 1 - s_{i_1} \leq t \]

and the equality is attained in those summands of the sum in which $i_1 = 1$.

Let us draw an analogy with the binomial coefficients $\binom{n}{k}$.

1a. $\binom{n}{k}$ is the number of subsets of the cardinality $k$ of a set of $n$ elements.

1b. $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of permutations of $n$ elements having the up-down index $k$.

2a. Each subset of a set of $n$ elements is contained in the number of $\binom{n}{k}$ subsets for some value of $k$.

2b. Each permutation of $n$ elements is contained in the number of $\begin{bmatrix} n \\ k \end{bmatrix}$ permutations for some value of the up-down index $k$.\[13\]
3a. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

3b. $\sum_{k=0}^{2^{n-1}-1} \binom{n}{k} = n!$.

4a. $\binom{n}{n-k} = \binom{n}{k}$.

4b. In view of (2), $\binom{n}{k} = \binom{2^{n-1}-1}{2^{n-1}-1-k}$.

5a. $\binom{n}{0} = \binom{n}{n} = 1$.

5b. $\binom{n}{0} = \binom{2^{n-1}-1}{2^{n-1}-1} = 1$.

The latter equality corresponds to the identity permutation.

6. The central binomial coefficients and the "central" numbers $\binom{n}{k}$ are equal one to another. Indeed, below (section 3) we prove that

$$\binom{2n}{2^{n-1}-1} = \binom{2n-1}{n-1},$$

$$\binom{2n+1}{2^n-1} = \binom{2n}{n}.$$  

In view of this analogy, we call $\binom{n}{k}$, $k = 1, 2, \ldots$, the basis polynomial in theory of permutations with prescribed up-down structure.

Note that $\binom{n}{k}$, just as $\binom{n}{k}$, generally is not a polynomial if $k$ is a function of $n$.

**Example 3.** In case of alternating permutations $\pi_1 < \pi_2 > \pi_3 < \ldots$ we have the sequence of indices $\{k_{n-1}\}$ such that

$$k_1 = 1, \ k_2 = 2, \ k_3 = 5, \ k_4 = 10, \ k_5 = 21, \ldots$$

Here

$$k_n - k_{n-2} = 2^{n-1}, \ n \geq 3,$$

whence

$$(19) \quad k_n = \frac{2^{n+2} - 3 + (-1)^{n+1}}{6}, \ n = 1, 2, \ldots$$
Thus, from the classical Andre’s result we obtain

\[
\sum_{n=0}^{\infty} \left\{ \frac{n + 1}{2^{n+2} - 3 + (-1)^{n+1}} \right\} \frac{x^n}{n!} = \tan x + \sec x,
\]

where we put \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \).

From (20) we have some values of \( \begin{pmatrix} n \\ m \end{pmatrix} \):

\[
\begin{align*}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & = 1, \\
\begin{pmatrix} 2 \\ 1 \end{pmatrix} & = 1, \\
\begin{pmatrix} 3 \\ 2 \end{pmatrix} & = 2, \\
\begin{pmatrix} 4 \\ 5 \end{pmatrix} & = 5, \\
\begin{pmatrix} 5 \\ 10 \end{pmatrix} & = 16, \\
\begin{pmatrix} 6 \\ 21 \end{pmatrix} & = 61, \ldots
\end{align*}
\]

It is well-known that these values are explicitly expressed with help of the absolute values of the Bernoulli and Euler numbers.

3. Another general formula for basis polynomials

Here we use formulas (12)-(14) for obtaining more simple general explicit formula. It is important for the subsequent development of our theory to present the results in the form of linear combination of the binomial coefficients \( \binom{n}{i} \). First consider several special cases.

a) \( m = 1, k = 2^{t-1} \). Then

\[
\begin{pmatrix} n \\ 2^{t-1} \end{pmatrix} = \binom{1}{1} \binom{n}{t} = \binom{n}{t} - 1
\]

b) \( m = 2, k = 2^{t-1} + 2^{u-1} \). Here \( s_1 = 1, s_2 = t - u + 1 \) and we have

\[
\begin{pmatrix} n \\ 2^{t-1} + 2^{u-1} \end{pmatrix} = \binom{1}{1} \begin{vmatrix} 1 & 1 & 0 \\ 1 & \binom{n-u}{t-u} & 1 \\ 1 & \binom{n}{t} & \binom{n}{u} \end{vmatrix} = \binom{n-u}{t-u} \binom{n}{u} - \binom{n}{t} - \binom{n}{u} + 1.
\]

Notice that

\[
\binom{n-u}{t-u} \binom{n}{u} = \binom{t}{u} \binom{n}{t}.
\]
Therefore,

\begin{equation}
\left\{ \begin{array}{c}
2^{t-1} + 2^{u-1} \\
\end{array} \right\} = \left( \begin{array}{c}
t \\
u \\
\end{array} \right) - 1 \left( \begin{array}{c}
t \\
u \\
\end{array} \right) + 1.
\end{equation}

\begin{equation}
c) \quad m = 3, \quad k = 2^{t-1} + 2^{u-1} + 2^{v-1}. \text{ Here } s_1 = 1, \quad s_2 = t-u+1, \quad s_3 = t-v+1
\end{equation}

and we have

\[
\left\{ \begin{array}{c}
2^{t-1} + 2^{u-1} + 2^{v-1} \\
\end{array} \right\} = \left| \begin{array}{ccc}
1 & 1 & 0 \\
1 & (n-u) & 0 \\
1 & (n-v) & (n-v) \\
1 & (n) & (n) \\
\end{array} \right| = \\
\left| \begin{array}{ccc}
1 & 1 & 0 \\
1 & (n-v) & 1 \\
1 & t-v & (n-v) \\
1 & (n) & (n) \\
\end{array} \right| = \\
\left( \begin{array}{c}
n-u \\
t-u \\
(n-v) \\
(n_u) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) - \\
\left( \begin{array}{c}
n-u \\
t-u \\
(n-v) \\
(n_u) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) - \\
\left( \begin{array}{c}
n-u \\
t-u \\
(n-v) \\
(n_u) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) - \\
\left( \begin{array}{c}
n-u \\
t-u \\
(n-v) \\
(n_u) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) - 1.
\end{equation}

Notice that

\begin{equation}
\left( \begin{array}{c}
n-u \\
t-u \\
(n-v) \\
(n_u) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) = \left( \begin{array}{c}
t \\
u \\
(n) \\
(n) \\
\end{array} \right),
\end{equation}

\begin{equation}
\left( \begin{array}{c}
n-u \\
t-u \\
(n-v) \\
(n_u) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) = \left( \begin{array}{c}
t \\
u \\
(n) \\
(n) \\
\end{array} \right),
\end{equation}

\begin{equation}
\left( \begin{array}{c}
n-v \\
t-v \\
(n) \\
(n) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) = \left( \begin{array}{c}
t \\
u \\
(n) \\
(n) \\
\end{array} \right),
\end{equation}

\begin{equation}
\left( \begin{array}{c}
n-v \\
t-v \\
(n) \\
(n) \\
\end{array} \right) \left( \begin{array}{c}
n-v \\
u-v \\
v-1 \\
u-1 \\
\end{array} \right) = \left( \begin{array}{c}
t \\
u \\
(n) \\
(n) \\
\end{array} \right).
\end{equation}

Therefore,
\[ \begin{align*}
\left\{ \binom{n}{2^t-1 + 2^{u-1} + 2^{v-1}} \right\} &= \left( \binom{t}{u} \binom{u}{v} - \binom{t}{u} \binom{t}{v} + 1 \right) \binom{n}{t} - \\
&\quad \quad \quad - \left( \binom{u}{v} - 1 \right) \binom{n}{u} + \binom{n}{v} - 1.
\end{align*} \] (28)

At last, for arbitrary \( m \), by investigation of the structure of diagonals of the general determinant of kind a), b), c) and using the above-like combinatorial identities we obtain the following theorem.

**Theorem 6.** If

\[ k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}, \quad t_1 > t_2 > \ldots > t_m \] (29)

then

\[ \left\{ \binom{n}{k} \right\} = (-1)^m \left( 1 - \sum_{i=1}^{m} \binom{n}{t_i} + \sum_{1 \leq i < j \leq m} \binom{n}{t_i} \binom{t_i}{t_j} - \\
\quad \quad \quad - \sum_{1 \leq i < j \leq l \leq m} \binom{n}{t_i} \binom{t_i}{t_j} \binom{t_j}{t_l} + \ldots + (-1)^m \binom{n}{t_1} \prod_{j=1}^{m-1} \binom{t_i}{t_{i+1}} \right). \] (30)

It is clear that in (30) \( \left\{ \binom{n}{k} \right\} \) is presented as a linear combination of \( \binom{n}{i} \), \( 0 \leq i \leq t_1 \). More exactly, as (30) shows, \( \left\{ \binom{n}{k} \right\} \) is a linear combination of elements of the last row of the determinant (14), that does not follow from (14) directly.

Indeed, in case of (29) we have

\[ s_1 = 1, \quad s_i = t_1 - t_i + 1, \quad i = 2, 3, \ldots, m. \]

In (14) \( t_1 = t \) and, consequently,

\[ t + 1 - s_i = t_i, \quad i = 1, 2, \ldots, m. \]

Thus, \( \left\{ \binom{n}{k} \right\} \) is an alternating sum of some elementary symmetric polynomials of binomial coefficients. It is also a polynomial in \( n \) of degree \( t_1 = \lfloor \log_2(2k) \rfloor \).
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Note that another form of Theorem 6 is

**Theorem 7.** In conditions (29) we have

\[
\left\{ \binom{n}{k} \right\} = (-1)^m + \sum_{p=1}^{m} c_p \binom{n}{t_p},
\]

where

\[
c_p = (-1)^m \left( -1 + \sum_{j=p+1}^{m} \binom{t_p}{t_j} - \sum_{p+1 \leq l \leq m} \binom{t_p}{t_j} \binom{t_j}{t_l} + \ldots + \right)
\]

\[
+ (-1)^{m-p-1} \prod_{j=p+1}^{m} \binom{t_j-1}{t_j}.
\]

In particular,

\[
c_m = (-1)^{m+1}
\]

\[
c_{m-1} = (-1)^m \left( -1 + \binom{t_{m-1}}{t_m} \right)
\]

\[
c_{m-2} = (-1)^m \left( -1 + \binom{t_{m-2}}{t_{m-1}} + \binom{t_{m-2}}{t_m} - \binom{t_{m-2}}{t_{m-1}} \binom{t_{m-1}}{t_m} \right)
\]

etc.

**Example 4.**

Let \( k = 2^m - 1 \). Then according to (29) we have

\[
t_m = 1, \quad t_{m-1} = 2, \ldots, \quad t_1 = m
\]

and by (33) we find

\[
c_m = (-1)^{m+1}
\]

\[
c_{m-1} = (-1)^m (-1 + 2) = (-1)^m
\]

\[
c_{m-2} = (-1)^m (-1 + 3 + 3 - 3 \cdot 2) = (-1)^{m+1}
\]

and by induction

\[
c_m = -c_{m-1} = c_{m-2} = \ldots = (-1)^{m-1} c_1 = (-1)^{m-1},
\]

i.e.

\[
c_p = (-1)^{p-1}.
\]

Thus, by (31) we have
The latter identity is proved easily by induction over $m$.

In particular, putting in (34) $n = 2m$ and $n = 2m + 1$ we have

$$\left\{ \begin{array}{l}
2m \\
2^m - 1
\end{array} \right\} = \binom{2m - 1}{m}, \quad \left\{ \begin{array}{l}
2m + 1 \\
2^m - 1
\end{array} \right\} = \binom{2m}{m}.$$  

This proves the analogy for the "central" number $\left\{ n \right\}$ and the central binomial coefficients.

Comparing with (14) we obtain an identity:

$$\left| \begin{array}{cccccccc}
1 & 1 & 0 & 0 & \ldots & 0 \\
1 & \binom{n - m + 1}{m - 1} & 1 & 0 & \ldots & 0 \\
1 & \binom{n - m + 2}{2} & \binom{n - m + 2}{1} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \binom{n}{m} & \binom{n - 1}{m - 1} & \binom{n - 1}{m - 2} & \ldots & 1 \\
\end{array} \right| = \binom{n - 1}{m}.$$  

Formulas (31)-(33) allow rather effectively to calculate basis polynomials $\left\{ n \right\}$. Nevertheless, there exists a recursion relation which is more effective for calculations.

### 4. Recursion relation for basis polynomials.

Consider now $\left\{ n \right\}$ from a more formal point of view as a polynomial (30) or (31). If $k$ has larger digits than $n - 1$ then $\left\{ n \right\}$ loses its combinatorial sense and could take even negative values. Nevertheless, the formal values of $\left\{ n \right\}$ are useful since the coefficients $c_p$ (32) could be represented as some values of the basis polynomials.
Theorem 8. In conditions (29) in (31) we have

$$c_p = \left\{ \frac{t_p}{k - 2t_p - 1} \right\}$$

Proof. By (29)

$$k_1 = k - 2t_p - 1 = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_{p-1} - 1} + 2^{t_{p+1} - 1} + \ldots + 2^{t_m - 1}.$$  

Using (30) for $k_1$ and substituting $n = t_p$ we obtain

$$\left\{ \frac{t_p}{k - 2t_p - 1} \right\} = (-1)^{m-1} \left( 1 - \sum_{i=p+1}^{m} \left( \frac{t_p}{t_i} \right) + \sum_{p+1 \leq i < j \leq m} \left( \frac{t_p}{t_i} \frac{t_j}{t_j} \right) - \ldots \right)$$

and comparing with (32) gives the theorem.

From Theorems 7, 8 we obtain a very simple recursion relation.

Theorem 9. If $k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}$ then

$$\begin{equation} \left\{ \frac{n}{k} \right\} = (-1)^m + \sum_{p=1}^{m} \left\{ \frac{t_p}{k - 2t_p - 1} \right\} \binom{n}{t_p} \end{equation}$$

Example 5. Knowing $\left\{ \frac{n}{j} \right\}, j \leq 20$, to find $\left\{ \frac{n}{21} \right\}$. We have

$$21 = 2^{5-1} + 2^{3-1} + 2^{1-1}, \quad t_1 = 5, \quad t_2 = 3, \quad t_3 = 1.$$  

By (37) we obtain

$$\begin{equation} \left\{ \frac{n}{21} \right\} = -1 + \left\{ \frac{5}{5} \right\} \binom{n}{5} + \left\{ \frac{3}{17} \right\} \binom{n}{3} + \left\{ \frac{1}{20} \right\} \binom{n}{1}. \end{equation}$$

Using formulas (see Appendix)

$$\begin{align*}
\left\{ \frac{n}{5} \right\} &= 2 \binom{n}{3} - \binom{n}{1} + 1 \\
\left\{ \frac{n}{17} \right\} &= 4 \binom{n}{5} - \binom{n}{1} + 1 \\
\left\{ \frac{n}{20} \right\} &= 9 \binom{n}{5} - \binom{n}{3} + 1,
\end{align*}$$
we conclude that
\[
\begin{align*}
\binom{n}{5} &= 2 \cdot 10 - 5 + 1 = 16 \\
\binom{3}{17} &= -2 \\
\binom{1}{20} &= 1
\end{align*}
\]
and by (38) we find
\[
\binom{n}{21} = 16 \binom{n}{5} - 2 \binom{n}{3} + \binom{n}{1} - 1.
\]

5. Another determinant formula for basis polynomials

In conditions (29) the determinant (14) has the form (cf.3,a,b,c)):

\[
(39) \quad \binom{n}{k} = \begin{vmatrix}
1 & 1 & 0 & 0 & \ldots & 0 \\
1 & \frac{n-t_2}{t_1-t_2} & 1 & 0 & \ldots & 0 \\
1 & \frac{n-t_3}{t_1-t_3} & \frac{n-t_3}{t_2-t_3} & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \frac{n-t_m}{t_1-t_m} & \frac{n-t_m}{t_2-t_m} & \frac{n-t_m}{t_3-t_m} & \ldots & 1 \\
1 & \frac{n}{t_1} & \frac{n}{t_2} & \frac{n}{t_3} & \ldots & \frac{n}{t_m}
\end{vmatrix}
\]

Note that this determinant possesses an astonishing property. If to replace the lower triangular submatrix with the main diagonal of 1’s by the upper one such that the elements \(\binom{n-t_j}{t_i-t_j} (i < j)\) are mapped to elements \(\binom{t_i}{t_j}\) which are symmetric respectively the diagonal of 1’s (and which do not depend on \(n(!)\)) then the determinant does not change its value. If in addition to interchange the places of the first and last rows then we obtain the following result.
Theorem 10. \( k = 2^{h-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}, \ t_1 > t_2 > \ldots > t_m, \) then

\[
\begin{vmatrix}
1 & (n) & (n) & (n) & \ldots & (n) \\
1 & 1 & (t_1) & (t_1) & \ldots & (t_m) \\
1 & 0 & 1 & (t_2) & \ldots & (t_m) \\
1 & 0 & 0 & 1 & \ldots & (t_m) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 1
\end{vmatrix}
\]

(40) \[ \binom{n}{k} = (-1)^m \]

Proof. One can prove this formula using the analysis of the structure of the diagonals and the comparison with (30).

In the case of alternating permutations when \( k = k_n \) is defined by (19) we obtain an \((m+1) \times (m+1)\) determinant representation of numbers (21).

Thus, for \( n = 2m \) we have an identity for the Euler numbers \( E_{2m}, \ m \geq 1 \) (cf. [1], Table 23.2, [15], A000364). We drop \((-1)^m\) in order to take account of the sign of \( E_{2m} \).

\[
E_{2m} =
\begin{vmatrix}
1 & (2m) & (2m) & (2m) & \ldots & (2m) \\
1 & 1 & (2m-1) & (2m-1) & \ldots & (2m-1) \\
1 & 0 & 1 & (2m-3) & \ldots & (2m-3) \\
1 & 0 & 0 & 1 & \ldots & (2m-5) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 1
\end{vmatrix}
\]

(41)

Analogously, putting \( n = 2m - 1, \ m \geq 2 \) for the Bernoulli numbers \( B_{2m} \) (cf. [1], Table 23.2) we have the following determinant of an \( m \times m \) matrix:
\[(42)\]

\[
\frac{B_{2m}(2^{2m-1})2^{2m}}{2m} = \begin{vmatrix}
1 & (2m - 1) & (2m - 1) & (2m - 1) & \cdots & (2m - 1) \\
1 & 1 & (2m - 2) & (2m - 4) & \cdots & (2m - 2) \\
1 & 0 & 1 & (2m - 2) & \cdots & (2m - 2) \\
1 & 0 & 0 & 1 & \cdots & (2m - 2) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 1
\end{vmatrix}
\]

The numbers on the left hand side are the tangent numbers \([15], A000182)\).

**Example 6.** For \(m = 1\) we have

\[
E_2 = \begin{vmatrix}
1 & 2 \\
1 & 1
\end{vmatrix} = -1,
\]

for \(m = 2\) we have

\[
\frac{B_4}{4}16 \cdot 15 = \begin{vmatrix}
1 & 3 \\
1 & 2
\end{vmatrix} = -2,
\]

which corresponds to \(B_4 = \frac{-1}{30}\).

For \(m = 2\) we have also

\[
E_4 = \begin{vmatrix}
1 & 4 & 4 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{vmatrix} = 5,
\]

for \(m = 3\) we have

\[
\frac{B_6}{6}64 \cdot 63 = \begin{vmatrix}
1 & 5 & 5 \\
1 & 4 & 2 \\
1 & 0 & 1
\end{vmatrix} = 16,
\]

which corresponds to \(B_6 = \frac{1}{32}\).
6. An identity for partial sums of the basis polynomials

**Theorem 11.** For $1 \leq r \leq n - 1$ we have

\[ \sum_{k=0}^{2^r-1} \binom{n}{k} = n(n-1) \ldots (n-r+1) \]  

**Proof.** Sum (43) enumerates the permutations with the $n-r-1$ fixed down points: 1, 2, \ldots, $n-r-1$. Let us form an arbitrary permutation $\pi$ of such kind. We start with position $n-r+1$. We can choose value of $\pi_{n-r+1}$ by $n$ ways, $\pi_{n-r+2}$ by $n-1$ ways, \ldots, $\pi_{n}$ by $n-(r-1)$ ways. After that $\pi_1 > \pi_2 > \ldots > \pi_{n-r}$ are defined uniquely. Thus, we obtain (43)\footnote{\textsuperscript{43}}

**Example 7.** For $r = 3$ we have (see Appendix)

\[ \sum_{k=0}^{7} \binom{n}{k} = 1 + \binom{n}{1} - 1 + \binom{n}{2} - 1 + \binom{n}{2} - \binom{n}{1} + 1 + \binom{n}{3} - 1 + \binom{2}{3} \binom{n}{1} - 1 \binom{n}{2} + 1 + (2 \binom{n}{3} - \binom{n}{2} + 1) + \binom{n}{3} - \binom{n}{2} + \binom{n}{1} - 1 = 6 \binom{n}{3} = n(n-1)(n-2). \]

7. On the positive integer zeros of the basis polynomials

**Theorem 12.** If $k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}$, $t_1 > t_2 > \ldots > t_m \geq 1$ then the integers $t_1, t_2, \ldots, t_m$ are roots of the basis polynomial $\binom{n}{k}$.

**Proof.** Substituting in (40) $n = t_j$, $j = 1, 2, \ldots, m$, we obtain a determinant with two the same rows. \footnote{\textsuperscript{40}}

Furthermore, the following results are obtained directly from (30):

\[ \binom{1}{k} = (-1)^m \left( 1 - \binom{1}{t_m} \right) = \begin{cases} (-1)^m, & \text{if } t_m \geq 2, \\ 0, & \text{if } t_m = 1 \end{cases}; \]

\[ \binom{2}{k} = (-1)^m \left( 1 - \binom{2}{t_{m-1}} - \binom{2}{t_m} + \binom{2}{t_{m-1}} \binom{2}{t_m} \right) = \begin{cases} (-1)^m, & \text{if } t_m \geq 3 \\ 0, & \text{if } t_m = 2 \\ 0, & \text{if } t_m = 1, \ t_{m-1} = 2, \\ (-1)^{m-1}, & \text{if } t_m = 1, \ t_{m-1} \geq 3 \end{cases}; \]
\[
\{3\over k\} = (-1)^m \left(1 - {3\choose t_{m-2}} - {3\choose t_{m-1}} - {3\choose t_m} + {3\choose t_{m-2}} {3\choose t_{m-1}} + {3\choose t_{m-2}} {3\choose t_m} + {3\choose t_{m-1}} {3\choose t_m}\right) = \\
\begin{cases}
(-1)^{m}, \text{ if } t_m \geq 4 \\
0, \text{ if } t_m = 3 \\
0, \text{ if } t_m = 2, \ t_{m-1} = 3 \\
2(-1)^{m-1}, \text{ if } t_m = 2, \ t_{m-1} > 3 \\
0, \text{ if } t_m = 1, \ t_{m-1} = 2, \ t_{m-2} = 3 \\
4(-1)^{m}, \text{ if } t_m = 1, \ t_{m-1} = 3, \ t_{m-2} > 3 \\
0, \text{ if } t_m = 1, \ t_{m-1} = 2, \ t_{m-2} = 3 \\
2(-1)^{m-1}, \text{ if } t_m = 1, \ t_{m-1} > 3
\end{cases};
\]

etc.

Thus, we note that
\[
\{1\over k\} = 0, \text{ if } k \equiv 1( \mod 2) \\
\{2\over k\} = 0, \text{ if } k \equiv 2, 3( \mod 4) \\
\{3\over k\} = 0, \text{ if } k \equiv 4, 5, 6, 7( \mod 8)
\]

etc.

In general we obtain the following result.

Theorem 13. If

\[
k \equiv 2^{h-1} + j \mod 2^h, \ h \in \mathbb{N},
\]

where \( j \in [0, 2^{h-1}) \) then \( \{h\over k\} = 0.\)

Proof. By (44) we have

\[
k = l2^h + 2^{h-1} + j, \ l \geq 0, \ 0 \leq j \leq 2^{h-1} - 1.
\]

Therefore, if \( k = 2^{t_{l-1}} + 2^{t_{l-2}} + \ldots + 2^{t_m} \) then there exists \( p \in [1, m] \) such that \( t_p = h \) and the result follows from Theorem 12. \( \blacksquare \)
Remark 3. The conversion of Theorem 13 will be obtained at the end of this article.

As a corollary from Theorem 13 it follows a more attractive statement.

Theorem 14. Let \( k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1} \) and \( 1 \leq i \leq \log_2(2k) \). If \( i \neq t_p \), \( p = 1, 2, \ldots, m \), then \( \{ \frac{i}{k - 2^{i-1}} \} = 0. \)

Proof. Let \( t_l < i < t_{l-1} \). Then

\[
\begin{align*}
k - 2^{i-1} &= 2^{t_1-1} + 2^{t_2-1} + \ldots + (2^{t_{i-1}-1} - 2^{i-1}) + 2^{t_{p-1}} + \ldots + 2^{t_m-1} = \\
&= 2^{t_1-1} + 2^{t_2-1} + \ldots + (2^{t_{i-2}} + 2^{t_{i-3}} + \ldots + 2^{i-1}) + 2^{t_{l-1}} + \ldots + 2^{t_m-1} \\
&\equiv 2^{i-1} + 2^{t_{l-1}} + \ldots + 2^{t_m-1} \mod 2^i
\end{align*}
\]

and the theorem directly follows from Theorem 13. ■

8. Another algorithm of evaluation of basis polynomials

Theorem 15. If \( k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1} \), \( t_1 > t_2 > \ldots > t_m \geq 1 \) then

\[
\binom{n}{k} = a_1 \binom{n}{t_m} + a_2 \binom{n}{t_{m-1}} + \ldots + a_m \binom{n}{t_1} + (-1)^m,
\]

where integers \( a_i \), \( i = 1, 2, \ldots, m \), are defined by the system of the linear equations

\[
\begin{align*}
&\left\{ \begin{array}{l}
a_1 + (-1)^m = 0 \\
an_1 + a_2 + (-1)^m = 0 \\
a_{m-1} + a_2 + a_3 + (-1)^m = 0 \\
\vdots \\
a_1 + a_2 + \ldots + a_m + (-1)^m = 0
\end{array} \right. \\
\end{align*}
\]

Proof. From \( (40) \) follows a representation \( (45) \). Substituting in \( (45) \), \( n = t_m, t_{m-1}, \ldots, t_1 \) and using Theorem 13 we obtain system \( (46) \). ■
Example 8. Let us find \( \binom{n}{26} \). We have
\[
26 = 2^5 - 1 + 2^4 - 1 + 2^2 - 1.
\]
Thus, \( t_1 = 5, \ t_2 = 4, \ t_3 = 2, \ m = 3 \). By (46)
\[
\begin{align*}
  a_1 - 1 &= 0 \\
  6a_1 + a_2 - 1 &= 0 \\
  10a_1 + 5a_2 + a_3 - 1 &= 0
\end{align*}
\]
whence \( a_1 = 1, \ a_2 = -5, \ a_3 = 16 \). Consequently, by (45)
\[
\binom{n}{26} = \binom{n}{2} - 5 \binom{n}{4} + 16 \binom{n}{5} - 1.
\]

9. Another recursion relation for basis polynomials

Theorem 16. Let \( k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}, \ t_1 > t_2 > \ldots > t_m \geq 1 \).
Then for \( l > t_1 \) we have
\[
\binom{n}{k + 2^{l - 1}} = \binom{l}{k} \binom{n}{l} - \binom{n}{k}.
\]

Proof. By the latter theorem,
\[
\binom{n}{k + 2^{l - 1}} = b_1 \binom{n}{t_m} + b_2 \binom{n}{t_{m-1}} + \ldots + b_m \binom{n}{t_1} + b_{m+1} \binom{n}{l} + (-1)^{m+1},
\]
where \( b_i, \ i = 1, 2, \ldots, m + 1 \), are defined by the following system:
\[
\begin{align*}
  b_1 + (-1)^{m+1} &= 0 \\
  \binom{t_{m-1}}{t_m} b_1 + b_2 + (-1)^{m+1} &= 0 \\
  \ldots & \ldots \\
  \binom{t_1}{t_m} b_1 + \binom{t_1}{t_{m-1}} b_2 + \ldots + \binom{t_1}{t_2} b_{m-1} + b_m + (-1)^{m+1} &= 0 \\
  \binom{l}{t_m} b_1 + \binom{l}{t_{m-1}} b_2 + \ldots + \binom{l}{t_1} b_m + b_{m+1} + (-1)^{m+1} &= 0
\end{align*}
\]
By comparing the first \( m \) equations of (49) with (46) we conclude that
(50) \[ b_i = -a_i, \; i = 1, 2, \ldots, m \]

and by the \((m + 1)th\) equation of (49) and by (45) we find that

(51) \[ b_{m+1} = -\binom{l}{k}. \]

Now from (45), (48), (50) and (51) we obtain

\[
\binom{n}{k} + \binom{n}{k + 2^{-1}} = \binom{l}{k} \binom{n}{l}
\]

and (47) follows. \(\blacksquare\)

Example 9. Starting with \(\binom{n}{0} = 1\) and putting \(k = 0, \; l = 1\) we obtain

\[
\binom{n}{1} = \binom{1}{0} \binom{n}{1} - \binom{n}{0} = \binom{n}{1} - 1.
\]

Furthermore, we consecutively find:

\[
\begin{align*}
\text{putting } k = 0, \; l = 2, & \quad \binom{n}{2} = \binom{2}{0} \binom{n}{2} - \binom{n}{0} = \binom{n}{2} - 1, \\
\text{putting } k = 1, \; l = 2, & \quad \binom{n}{3} = \binom{2}{1} \binom{n}{2} - \binom{n}{1} = \binom{n}{2} - \binom{n}{1} + 1, \\
\text{putting } k = 0, \; l = 3, & \quad \binom{n}{4} = \binom{3}{0} \binom{n}{3} - \binom{n}{0} = \binom{n}{3} - 1, \\
\text{putting } k = 1, \; l = 3, & \quad \binom{n}{5} = \binom{3}{1} \binom{n}{3} - \binom{n}{1} = 2 \binom{n}{3} - \binom{n}{1} + 1
\end{align*}
\]

e tc.

10. Characteristic conditions for a basis polynomial

Let \(P(n)\) be a polynomial. It is evident that we have

\[ P(n) = C \binom{n}{k} \]

with a constant \(C\) if and only if \(P(r) = 0, \; r = 0, 1, \ldots, k, \) and \(k = \text{deg}P(n).\)

Concerning \(\binom{n}{k}\) we have the following result. Put
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\[ P^1(n) = P(n) - P(n-1) \]

and let \( P^r(n) \) be the \( r \)-th difference of \( P(n) \).

**Theorem 17.** For a polynomial \( P(n) \) there exists a nonnegative integer \( k \) and a constant \( C \neq 0 \) such that

\[
P(n) = C \binom{n}{k}
\]

if and only if the following conditions satisfy:

\[
P(0) \neq 0, \ P^r(r)P(r) = 0, \ r = 1, 2, \ldots, l,
\]

where \( l = \text{deg} P(n) \).

**Proof.** From a simple argument it follows that

\[
P(n) = P(0) + \sum_{r=1}^{l} P^r(r) \binom{n}{r}.
\]

If all \( P^r(r) = 0, \ r = 1, \ldots, l \), then we put \( k = 0, \ C = P(0) \). If \( P^r(r) \neq 0 \) for \( r = t_1 > t_2 > \ldots > t_m \geq 1 \) then by (54)

\[
P(n) = P(0) + \sum_{i=1}^{m} b_{m+1-i} \binom{n}{t_i},
\]

where

\[
b_i = P^{t_i}(t_i), \ i = 1, 2, \ldots, m.
\]

putting

\[
\frac{b_i}{P(0)}(-1)^m = a_i, \ i = 1, 2, \ldots, m
\]

we have

\[
\frac{(-1)^m}{P(0)} P(n) = (-1)^m + \sum_{i=1}^{m} a_{m+1-i} \binom{n}{t_i}
\]

and according to (53) \( P(t_i) = 0, \ i = 1, 2, \ldots, m \). Thus, by Theorem 15, the polynomial (55) is \( \binom{n}{k} \) with
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\[ k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}. \]

The converse statement is evident as well according to Theorem 12. ■

**Example 10.** Consider \( P(n) = n^3 - 3n^2 + 2n - 6 \). We have

\[
P^1(n) = 3n^2 - 9n + 6, \quad P^2(n) = 6n - 12, \quad P^3(n) = 6,
\]

thus,

\[
P^1(1)P(1) = P^2(2)P(2) = P^3(3)P(3) = 0
\]

and by Theorem 17, \( P(n) = C \left\{ \begin{array}{c} n \\ k \end{array} \right\} \). Since only \( P^3(3) \neq 0 \) then \( m = 1 \) and \( t_1 = 3 \). Therefore, \( k = 4 \) and \( C = -P(0) = 6 \). Thus, \( P(n) = 6 \left\{ \begin{array}{c} n \\ 4 \end{array} \right\} \).

11. **ON GENERATING FUNCTION OF THE BASIS POLYNOMIALS**

Let for any \( n \in \mathbb{N} \)

\[
F(n, x) = \sum_{k=0}^{\infty} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k, \quad |x| < 1.
\]

Put

\[
\tau(x) = \sum_{k=0}^{\infty} \tau_k x^k, \quad |x| < 1.
\]

where \( \tau \) is the Thue-Morse sequence \( (16) \).

**Theorem 18.** For every \( n \in \mathbb{N} \) the quotient \( \frac{F(n, x)}{\tau(x)} \) is a rational function.

**Proof.** It follows from Theorems \( [9] \) and \( [14] \) that

\[
\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \tau_k + \sum_{1 \leq i \leq \log_2(2k)} \left( \begin{array}{c} n \\ i \end{array} \right) \left\{ \begin{array}{c} i \\ k - 2^{i-1} \end{array} \right\}, \quad k \geq 1.
\]

Note that for \( k = 0 \) we have

\[
\left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = 1 = \tau_0.
\]

Therefore, by (56)–(58) we obtain
\[ F(n, x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \tau(x) + \sum_{k \geq 1} \sum_{1 \leq i \leq \log_2(2k)} \binom{n}{k-2^i-1} x^k = \]

\[ = \tau(x) + \sum_{i=1}^{n} \binom{n}{i} \sum_{k=2^i-1}^{\infty} x^k \binom{i}{k-2^i-1} = \]

\[ = \tau(x) + \sum_{i=1}^{n} \binom{n}{i} \sum_{r=0}^{i} \binom{i}{r} x^{r+2^i-1} = \]

(59) \[ = \tau(x) + \sum_{i=1}^{n} \binom{n}{i} x^{2^i-1} F(i, x), \quad |x| < 1. \]

(59) gives us a recursion formula for \( F(n, x) \):

(60) \[ \left( 1 - x^{2^{n-1}} \right) F(n, x) = \tau(x) + \sum_{i=1}^{n-1} \binom{n}{i} x^{2^i-1} F(i, x), \quad |x| < 1. \]

Put

(61) \[ F(n, x) = \tau(x) \frac{P_n(x)}{(1-x)(1-x^2)\ldots(1-x^{2^{n-2}})} \quad (|x| < 1) \]

Then we obtain a recursion formula for \( P_n(x) \):

\[ P_n(x) = \frac{1}{1-x^{2^{n-1}}} \left( (1-x)(1-x^2)\ldots(1-x^{2^{n-1}}) + \right. \]

(62) \[ \left. + \sum_{i=1}^{n-1} \binom{n}{i} (1-x^i)(1-x^{i+1})\ldots(1-x^{2^{n-1}}) x^{2^i-1} P_i(x) \right) \]

Here it is not expedient to cancel \( 1-x^{2^{n-1}} \) without additional conventions. In particular, from (62) we obtain

\[ P_1(x) = \frac{1}{1-x}(1-x) = 1 \]

\[ P_2(x) = \frac{1}{1-x^2}((1-x)(1-x^2) + 2(1-x^2)x) = 1 + x \]

\[ P_3(x) = 1 + 2x + 2x^2 + x^3, \]

\[ P_4(x) = 1 + 3x + 5x^2 + 3x^3 + 3x^4 + 5x^5 + 3x^6 + x^7, \]
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\[ P_5(x) = 1 + 4x + 9x^2 + 6x^3 + 9x^4 + 16x^5 + 11x^6 + 4x^7 + 4x^8 + 11x^9 + 16x^{10} + 9x^{11} + 6x^{12} + 9x^{13} + 4x^{14} + x^{15}, \]

\[ P_6(x) = 1 + 5x + 14x^2 + 10x^3 + 19x^4 + 35x^5 + 26x^6 + 10x^7 + 14x^8 + 40x^9 + \]

\[ + 61x^{10} + 35x^{11} + 26x^{12} + 40x^{13} + 19x^{14} + 5x^{15} + 5x^{16} + 19x^{17} + 40x^{18} + \]

\[ + 26x^{19} + 35x^{20} + 61x^{21} + 40x^{22} + 14x^{23} + 10x^{24} + 26x^{25} + 35x^{26} + 19x^{27} + \]

\[ + 5x^{28} + 14x^{29} + 5x^{30} + x^{31}, \]

etc.

By simple induction we see that \( P_n(x) \) is a polynomial in \( x \) of degree \( 2^{n-1} - 1 \). Thus, the theorem follows from \( \text{(61)} \).

But \( \text{(61)} \) gives us more. Since

\[ (1 - x)(1 - x^2) \ldots (1 - x^{2^n-2}) = \sum_{k=0}^{2^{n-1}-1} \tau_k x^k = \tau(x) + o(x^{2^{n-1}-1}). \]

Therefore, from \( \text{(61)} \) it follows that

\[ P_n(x) = F(n, x)(1 + o(x^{2^{n-1}-1})) \]

and since \( P_n(x) \) is a polynomial of degree \( 2^{n-1} - 1 \) then by \( \text{(56)} \) we conclude that the following statement is true.

**Theorem 19.** Polynomial \( P_n(x) \) which is defined recursively by \( \text{(62)} \) is equal to

\[ P_n(x) = \sum_{k=0}^{2^{n-1}-1} \binom{n}{k} x^k, \]

and for every \( n \in \mathbb{N} \) we have an identity
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\[
\sum_{k=0}^{2^{n-1}-1} \binom{n}{k} x^k \quad \sum_{k=0}^{\infty} \binom{n}{k} x^k = \frac{1}{\prod_{i=n-1}^{\infty} (1-x^{2^i})}.
\]

12. Third recursion relation for basis polynomials

For any \( k \in \mathbb{N} \) let us consider the set \( A_k \) of those positive integers \( i \leq \log_2(2k) \) for which

\[
\left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k + 1}{2^i} - 1 \right\rceil.
\]

The common values of expressions (65) denote by \( \lambda(k; i) \).

Theorem 20. \( \{n_k\} = \tau_k + \sum_{i \in A_k} \binom{n}{i} \left\{ k - 2^{i-1} - \lambda(k; i)2^i \right\} \tau_k \lambda(k; i). \)

Proof. Taking into account (63) and comparing coeff. \( k \leq 2^{n-1} - 1 \) in both sides of (62) we obtain

\[
\{n_k\} = \tau_k + \sum_{i=A_k} \binom{n}{i} \sum_{l=0}^{n-1} \tau_l \left\{ k - 2^{i-1} - 2^i l \right\},
\]

where the summing is over those values of \( l \geq 0 \) for which

\[2^{i-1} + 2^i l \leq k, \quad k - 2^{i-1} - 2^i l \leq \deg P_i(x) = 2^{i-1} - 1.\]

Consequently,

\[l \in \left[ \frac{k + 1}{2^i} - 1, \frac{k}{2^i} - \frac{1}{2} \right],\]

Nevertheless, the length of this segment equals to \( \frac{1}{2} - \frac{1}{2^i} \). This means that it contains only value of \( l \geq 0 \) only in the case \( \left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k + 1}{2^i} - 1 \right\rceil \geq 0.\]

Example 11. Let \( k = 2^m \). Then \( i \leq \log_2(2k) = m + 1 \). If \( i \leq m \) then \( \left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = 2^{m-i} - 1 \) while \( \left\lceil \frac{k + 1}{2^i} - 1 \right\rceil = 2^{m-i} \). It is left to consider the case \( i = m + 1 \) for which

\[\left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k + 1}{2^i} - 1 \right\rceil = 0.\]
Thus, by (66) we have

\[
\left\{ \frac{n}{2^m} \right\} = -1 + \left( \frac{n}{m+1} \right) \left\{ m+1 \right\} = \left( \frac{n}{m+1} \right) - 1.
\]

13. SEQUENCES WITH A FIXED VALUE OF THE UPPER INDEX OF BASIS POLYNOMIALS

We have already seen (Theorem 13) that

\[(67) \quad \left\{ \frac{a}{n} \right\} = 0, \text{ if } n \equiv 2^{a-1} + c \pmod{2^a}, \ c \in [0, 2^{a-1}).\]

Now we shall prove the following result.

**Theorem 21.** Let \(a, b \in \mathbb{N}, \ r \geq a, \ c \in [0, 2^{a-1}).\) Then for \(n = 2^rb + c\) we have

\[(68) \quad \left\{ \frac{a}{n} \right\} = \tau_b \left\{ \frac{a}{c} \right\}.
\]

**Proof.** Consider \(k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}, \ t_1 > t_2 > \ldots > t_m \geq 1.\) Comparing Theorems 9 and 20, we conclude that

\[(69) \quad A_k = \{t_1, t_2, \ldots, t_m\},
\]

and for \(i \in A_k\)

\[(70) \quad \left\{ \frac{k - 2^{i-1} - \lambda(k; i)2^i}{k - 2^{i-1}} \right\} = \tau_{\lambda(k; i)} \left\{ \frac{i}{k - 2^{i-1}} \right\}.
\]

In particular, for \(i = t_j\) we have

\[
\lambda(k; t_j) = \left\lfloor \frac{k}{2^{t_j}} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{k + 1}{2^{t_j}} - 1 \right\rfloor.
\]

Thus,

\[
\lambda(k; t_j) = \left\lfloor \frac{2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_j-1} + \ldots + 2^{t_m-1}}{2^{t_j}} - \frac{1}{2} \right\rfloor = 2^{t_1-t_j-1} + 2^{t_2-t_j-1} + \ldots + 2^{t_{j-1}-t_j-1}
\]

(71)
and consequently,

\[ \tau_{\lambda(k;t_j)} = (-1)^{j-1}. \]

Therefore, by (70) for \( i = t_j \) and (71) we find

\[ (-1)^{j-1} \left\{ k - 2 t_1 - 1 - 2 t_2 - 1 - \ldots - 2 t_{j-1} - 1 - 2 t_j - 1 \right\} = \left\{ k - 2 t_j - 1 \right\} \]

or, taking into account that \( k = 2 t_1 - 1 + \ldots + 2 t_m - 1, \)

\[ (-1)^{j-1} \left\{ 2 t_{j+1} - 1 + \ldots + 2 t_m \right\} = \left\{ 2 t_1 - 1 + \ldots + 2 t_j - 1 + 2 t_{j+1} - 1 + \ldots + 2 t_m - 1 \right\} \]

Putting here

\[ t_j = a, \quad 2 t_{j+1} - 1 + \ldots + 2 t_m - 1 = c \in [0, 2 a - 1), \]

\[ t_j - 1 = r \geq a, \quad 2 t_1 - 1 + \ldots + 2 t_j - 1 = 2 r b (b \in \mathbb{N}), \]

\[ n = 2 r b + c, \]

we write (74) in the form of (68). \( \blacksquare \)

To Theorem 21 it is worth to add that by (15)

\[ {\binom{0}{n}} = \tau_n. \]

Since in Theorem 21 \( n - c = 2 r b \) then \( \tau_b = \tau_{n-c} \). Therefore, Theorem 21 one can write in the following form.

**Theorem 22.**

\[ \begin{cases} a \\ n \end{cases} = \tau_{n-i} \begin{cases} a \\ i \end{cases}, \quad n \equiv i \pmod{2^a}, \quad i = 0, 1, 2, \ldots, 2^a - 1. \]
In particular, taking into account (67) and (75), we obtain the following sequences:

\[
\left\{ \frac{0}{n} \right\} : 1, -1, -1, 1, -1, 1, -1, \ldots
\]

\[
\left\{ \frac{1}{n} \right\} : 1, 0, -1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots
\]

\[
\left\{ \frac{2}{n} \right\} : 1, 1, 0, 0, -1, 0, 0, -1, -1, 0, 0, 1, 1, \ldots
\]

\[
\left\{ \frac{3}{n} \right\} : 1, 2, 2, 1, 0, 0, 0, -1, -2, -2, -1, 0, 0, 0, \ldots
\]

\[
\left\{ \frac{4}{n} \right\} : 1, 3, 5, 3, 3, 5, 3, 1, 0, \ldots, 0, -1, -3, -5, -3, -3, \ldots
\]

\[
\left\{ \frac{5}{n} \right\} : 1, 4, 9, 6, 9, 16, 11, 4, 11, 16, 9, 6, 9, 4, 1, 0, \ldots, 0, -1, \ldots
\]

\[
\left\{ \frac{6}{n} \right\} : 1, 5, 14, 10, 19, 35, 26, 10, 14, 40, 61, 35, 26, 40, 19, 5, 5, 19, 40, 26, 35, 61, 40, 14, 10, 26, 35, 19, 10, 14, 5, 1, 0, \ldots, 0, -1, -5, -14, -10, -19, -35, \ldots
\]

etc.

Note that, the bold-faced numbers are coefficient of the corresponding polynomials \( P_i(x) \), \( i = 1, 2, \ldots \) which are defined by recursion (62).

If consider the concatenation sequence of the all bold-faced numbers (cf. [15], A060351):

\[
\left\{ \frac{1}{0} \right\}, \left\{ \frac{2}{0} \right\}, \left\{ \frac{2}{1} \right\}, \left\{ \frac{3}{0} \right\}, \left\{ \frac{3}{1} \right\}, \left\{ \frac{3}{3} \right\}, \left\{ \frac{3}{2} \right\}, \left\{ \frac{4}{3} \right\}, \left\{ \frac{4}{1} \right\}, \left\{ \frac{4}{2} \right\}, \ldots
\]

(76)

then it is easy to see that it is the sequence

\[
\left\{ \frac{\lfloor \log_2 k \rfloor + 1}{k - 2\lfloor \log_2 k \rfloor} \right\}_{k=1}^{\infty}.
\]

(77)

This sequence is closely connected with asymptotics of \( \left\{ \frac{n}{k} \right\} \).
Theorem 23.\[ \binom{n}{k} \sim \left( \frac{\log_2 k + 1}{k - 2^{\lfloor \log_2 k \rfloor}} \right) \left( \frac{n}{\lfloor \log_2 k \rfloor + 1} \right)^n \sim \frac{1}{\lfloor \log_2 k + 1 \rfloor!} \left( \frac{\log_2 k + 1}{k - 2^{\lfloor \log_2 k \rfloor}} \right)^{n \lfloor \log_2 k \rfloor + 1} (n \to \infty) \]

\( (78) \)

**Proof.** The theorem follows directly from (58).\[ \square \]

Thus, according to (78), the first coefficients of \( \binom{n}{k}, k \geq 1 \), as linear combinations of binomial coefficients (see Appendix) form sequence (76).

In addition, note that from Theorem 22 and (62) follows the conversion of Theorem 13 and, thus, we have the following statement.

**Theorem 24.** If \( k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}, t_1 > t_2 > \ldots > t_m \geq 1 \), then \( t_i, i = 1, 2, \ldots, m \), are only positive integer roots of the polynomials \( \binom{n}{k} \).

**Remark 4.** Some observations on the basis polynomials have been done by the author as early as 1993 [13] with the calculation of some first polynomials. But only in the current paper we give a foundation of a more perfect theory of these polynomials.

14. **Some open problems**

1. We conjecture that all real roots of the basis polynomials are rational.
2. We conjecture that a polynomial \( \binom{n}{k}, k \geq 1 \), has only real roots if and only if the number of 0’s in the binary expansion of \( k \) less that 2. In view of Theorem 12 this condition is sufficient (since there is no a place for two conjugate complex roots). Therefore, it is left to prove its necessity.
   We verified this conjecture up to \( k = 32 \). We have polynomials \( \binom{n}{k} \) with only real roots for

   \[ k = 1, 2, 3, 5, 6, 7, 11, 13, 14, 15, 23, 27, 29, 30, 31, \ldots \]

   (cf. sequence A 089633 [15]).
3. It is interesting to investigate the sequence \( \{k_m\} \) for which the polynomials \( \binom{n}{k_m} \) have a root \( n = -1 \). The first values of \( k_m \) are: 2, 5, 8, 11, 23, \ldots
4. Let $D_n^{(a)}$ be the number of alternating permutations without fixed points (i.e. $\pi(i) \neq i, \ i = 1, 2, \ldots, n$). We conjecture that
\[
\lim_{n \to \infty} \frac{D_n^{(a)}}{a_n} = e^{-1},
\]
where $a_n$ is the sequence \[21\].

5. Let $S^{(a)}(n, l)$ be the number of alternating permutations having $l$ cycles (the absolute value of the "alternating" Stirling numbers of the first kind). We conjecture that for a fixed $l$
\[
\lim_{n \to \infty} \frac{nS^{(a)}(n, l)}{a_n(ln(n))^{l-1}} = \frac{1}{(l-1)!},
\]
where $a_n$ is the sequence \[21\]. The latter means that for each $l$ the events "a permutation is alternative" and "a permutation has $l$ cycles" are asymptotically independent.

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Appendix. List of the first 32 basis polynomials

\[
\begin{align*}
\{n\,0\} &= 1 \\
\{n\,1\} &= \binom{n}{1} - 1 \\
\{n\,2\} &= \binom{n}{2} - 1 \\
\{n\,3\} &= \binom{n}{2} - \binom{n}{1} + 1 \\
\{n\,4\} &= \binom{n}{3} - 1 \\
\{n\,5\} &= 2\left(\binom{n}{3} - \binom{n}{1+1}\right) \\
\{n\,6\} &= 2\left(\binom{n}{3} - \binom{n}{2}\right) + 1 \\
\{n\,7\} &= \binom{n}{3} - \binom{n}{2} + \binom{n}{1} - 1 \\
\{n\,8\} &= \binom{n}{4} - 1 \\
\{n\,9\} &= 3\left(\binom{n}{4} - \binom{n}{1}\right) + 1 \\
\{n\,10\} &= 5\left(\binom{n}{4} - \binom{n}{2}\right) + 1 \\
\{n\,11\} &= 3\left(\binom{n}{4} - \binom{n}{2}\right) + \binom{n}{1} - 1 \\
\{n\,12\} &= 3\left(\binom{n}{4} - \binom{n}{3}\right) + 1 \\
\{n\,13\} &= 5\left(\binom{n}{4} - 2\binom{n}{3}\right) + \binom{n}{1} - 1 \\
\{n\,14\} &= 3\left(\binom{n}{4} - 2\binom{n}{3}\right) + \binom{n}{2} - 1 \\
\{n\,15\} &= \binom{n}{4} - \binom{n}{3} + \binom{n}{2} - \binom{n}{1} + 1 \\
\{n\,16\} &= \binom{n}{5} - 1 \\
\{n\,17\} &= 4\left(\binom{n}{5} - \binom{n}{1}\right) + 1 \\
\end{align*}
\]
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\[
\begin{align*}
\{ \binom{n}{18} \} &= 9 \left( \binom{n}{5} \right) - \binom{n}{2} + 1 \\
\{ \binom{n}{19} \} &= 6 \left( \binom{n}{5} \right) - \binom{n}{2} + \binom{n}{1} - 1 \\
\{ \binom{n}{20} \} &= 9 \left( \binom{n}{5} \right) - \binom{n}{3} + 1 \\
\{ \binom{n}{21} \} &= 16 \left( \binom{n}{5} \right) - 2 \binom{n}{3} + \binom{n}{1} - 1 \\
\{ \binom{n}{22} \} &= 11 \left( \binom{n}{5} \right) - 2 \binom{n}{3} + \binom{n}{2} - 1 \\
\{ \binom{n}{23} \} &= 4 \left( \binom{n}{5} \right) - \binom{n}{3} + \binom{n}{2} - \binom{n}{1} + 1 \\
\{ \binom{n}{24} \} &= 4 \left( \binom{n}{5} \right) - \binom{n}{4} + 1 \\
\{ \binom{n}{25} \} &= 11 \left( \binom{n}{5} \right) - 3 \binom{n}{4} + \binom{n}{1} - 1 \\
\{ \binom{n}{26} \} &= 16 \left( \binom{n}{5} \right) - 5 \binom{n}{4} + \binom{n}{2} - 1 \\
\{ \binom{n}{27} \} &= 9 \left( \binom{n}{5} \right) - 3 \binom{n}{4} + \binom{n}{2} - \binom{n}{1} + 1 \\
\{ \binom{n}{28} \} &= 6 \left( \binom{n}{5} \right) - 3 \binom{n}{4} + \binom{n}{3} - 1 \\
\{ \binom{n}{29} \} &= 9 \left( \binom{n}{5} \right) - 5 \binom{n}{4} + 2 \binom{n}{3} - \binom{n}{1} + 1 \\
\{ \binom{n}{30} \} &= 4 \left( \binom{n}{5} \right) - 3 \binom{n}{4} + 2 \binom{n}{3} - \binom{n}{2} + 1 \\
\{ \binom{n}{31} \} &= \left( \binom{n}{5} \right) - \binom{n}{3} + \binom{n}{2} - \binom{n}{1} - 1
\end{align*}
\]

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