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ABSTRACT
The analysis of the oscillatory mixed convection flow of electrically conducting fluid along a nonconducting horizontal circular cylinder in the presence of variable density is performed. The density of electrically conducting fluid is assumed to be an exponential function of temperature. The governing boundary layer equations are transformed by adopting primitive variable transformation, which is integrated numerically by employing the finite difference method. The influence of the various physical parameters, density/temperature parameter \( m \), mixed-convection parameter \( \lambda \), magnetic force parameter \( \xi \), magnetic Prandtl number \( \gamma \), and Prandtl number \( \text{Pr} \), is interpreted graphically and numerically. The impact of these pertinent parameters on velocity, temperature, and magnetic field profiles at positions \( \alpha = \pi/6, \pi/3 \), and \( \pi \) on the surface of a nonconducting cylinder is examined and then used to compute oscillatory skin friction, heat transfer, and current density. From these results, it is concluded that an increase in the density/temperature parameter \( m \) means an increase in the velocity of the fluid particles due to an increase in the buoyancy forces. Due to this reason, a good response in steadiness and amplitude of oscillation is noted at an angle \( \pi/3 \) for heat transfer and current density and velocity field at \( \pi/6 \). Furthermore, it is noticed that the decrease in the density parameter \( m \) leads to a sharp increase in the velocity of fluid at the position \( \alpha = \pi/6 \) for a lower Prandtl number \( \text{Pr} = 0.1 \).

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NOMENCLATURE

| Symbol | Definition |
|--------|------------|
| \( u, v \) | velocity along the xy-direction (m s\(^{-1}\)) |
| \( H_x, H_y \) | magnetic field along the xy-direction (T) |
| \( \mu \) | dynamic viscosity (kg m\(^{-1}\) s\(^{-1}\)) |
| \( \nu \) | kinematic viscosity (m\(^2\) s\(^{-1}\)) |
| \( \rho \) | density (kg m\(^{-3}\)) |
| \( g \) | gravitational acceleration (m s\(^{-2}\)) |
| \( \beta \) | thermal expansion coefficient (K\(^{-1}\)) |
| \( \nu_m \) | magnetic permeability (H m\(^{-1}\)) |
| \( \alpha \) | thermal diffusivity (m\(^2\) s\(^{-1}\)) |
| \( T \) | temperature (K) |
| \( C_p \) | specific heat (J kg\(^{-1}\) K\(^{-1}\)) |
| \( T_\infty \) | ambient fluid temperature |
| \( R_e \) | Reynolds number |
| \( G_r \) | Grashof number |

Greek letters

| Symbol | Definition |
|--------|------------|
| \( \tau \) | shear stress (Pa) |
| \( \xi \) | magnetic force parameter |
| \( \lambda \) | mixed convection parameter |
| \( \theta \) | dimensionless temperature |
| \( \gamma \) | magnetic Prandtl number |
| \( \text{Pr} \) | Prandtl number |
| \( \sigma \) | electrical conductivity (s m\(^{-1}\)) |

I. INTRODUCTION

The interaction of temperature dependent density has many important practical applications in engineering fields such as dams, large-scale plumbing projects, ships, submarines, and oil floats. Glauert presented a note on the electrically conducting boundary conditions.
layer in a uniform flow on a magnetic plate analytically. He showed that the strength of the magnetic field develops boundary layer separation. Chawla et al. presented a classical solution of the hydromagnetic flow by using the Karman-Pohlhausen method and series solution. Grieben et al. investigated the effect of the magnetohydrodynamic (MHD) pressure gradient on an electrically conducting boundary layer flow. He observed the behavior of the tangential component of the magnetic field at the wall. Drazin and Moore mathematically talked about the stable and incompressible flow of fluid with temperature-density over an obstacle. They depicted that there is no critical internal Froude number for the existence of a steady flow along the dipole and the vertical wall. Mohanty et al. has discussed the steady boundary layer flow along a semi-infinite magnetic plate in the presence of harmonic oscillations of amplitude in the magnitude of a magnetic field. Lin et al. numerically explored specific calculations for moist air flowing on thin liquid film in the tube and detected that velocity gradually becomes distorted at a certain axial location for buoyancy effects at a higher temperature. Anderson et al. has numerically solved the non-Newtonian, shear-thickening MHD flow problem for different values of the power-law index and magnetic parameter. He concluded that magnetic field tends to make boundary layers thinner with increasing wall friction.

In general, most research projects of the convective flow along the vertical surface were restricted, where the temperature-difference is very small between the fluid and surface. In this case, density can be treated as a variable in the buoyancy term of the momentum equation by using Boussinesq approximations. Liu and Dane developed one and two dimensional models to simulate the unstable flow, and solute-transport equations were solved numerically. Chassaing et al. investigated the temperature density effects in turbulent fluid, fluid mechanics, and its applications. They gave some quality information about variable density effects in turbulence. For a large temperature difference, Azzam numerically examined the radiation effects on the MHD flow of an optically thick and electrically conducting fluid past a moving semi-infinite vertical surface. He graphically examined the velocity shapes, temperature profiles, and heat-flux for different parameters. Abo-Eldahab and Azzam numerically concluded the magnetohydrodynamic flow of an optically thick gray fluid at an inclined surface for higher concentration and temperature difference. The fluid density is an exponential function of temperature and concentration. Cortell presented a numerical study of the flow of the magnetic power-law fluid over a stretching-surface taking into account the uniform transverse magnetic-field. Azziz numerically considered the mixed convection flow over a continuous moving-flat surface for high temperature differences with radiation. He showed that the fluid-velocity increases when buoyancy forces increase and decreases as temperature decreases. Laaroussi et al. constructed a numerical simulation of a vertical parallel-surface with evaporation of thin liquid films on wetted walls. They observed upward velocities at the wall regions due to stronger solute buoyancy-forces near the damped surfaces.

Salem et al. numerically illustrated the effects of variable-density on a hydromagnetic mixed convection flow of a non-Newtonian fluid at a vertical surface for higher temperature. They found that the velocity decreases for higher values of the magnetic-field parameter as the density-parameter increases. Siddiqa et al. numerically analyzed the radiative heat-flux having a temperature-dependent density on a permeable surface with the effects of thermal radiation. Khan et al. numerically examined the heat generation/absorption effects on the three dimensional Oldroyd-B nanofluid flow towards a bidirectional stretching surface. They observed that the concentration profile decreases with the increase in Brownian motion, while an opposite behavior is noted for the higher thermophoresis parameter \( N_t \). Morel et al. discussed the validity of systematic groundwater flow simulations of Boussinesq equations with a variable-density. Liu et al. explored the numerical outcomes of the variable-density, saturated variable flow, and transportation model through the unsaturated aquifer system and porous-media. They found that the hydraulic governing parameters disturb the distribution of the water substance in the unsaturated zone. Khan and Khan explored the analytical solution of the three dimensional radiative flow of the Burgers’ fluid with the impact of thermophoresis by employing the homotopy method. A problem on the steady 3D flow of the Burgers’ fluid in the presence of homogeneous-heterogeneous reactions has been formulated numerically. Ashraf et al. reported a numerical explanation for the 2-dimensional flow on a vertical magnetized-surface, when viscosity and thermal conductivity are functions of temperature. They examined the outcomes of thermal-conductivity and the viscosity parameter on the flow structure and heat mass transfer characteristics.

Khan et al. analytically scrutinized the effect of the heterogeneous-homogeneous process on the 3D Burgers’ fluid for non-Fourier’s heat and non-Fick’s mass-flux models. Siddiqa et al. investigated the effect of temperature density on the natural-convection flow on a circular disk. They observed that the buoyancy forces enhance the velocity shapes as volumetric and density parameters increase with temperature. Khan et al. investigated the characteristics of classical Fourier’s and Fick’s models in Sisko fluid numerically. It was noted that the concentration boundary layer thickness decreased for a higher value of the Schmidt number. A rotating flow of an Oldroyd-B fluid has been numerically examined for improved Cattaneo-Christov heat-mass models by using the homotopy technique. Almagro et al. graphically performed the direct numerical simulations of variable-density, low-speed, turbulent, and mixing layers. The magnetohydrodynamic boundary layer flow of the generalized Burgers’ fluid with heterogeneous-homogeneous reactions has been studied. They depicted that the temperature and concentration profiles show opposite effects for a larger value of the Hartmann number. Khan et al. analyzed the problem of melting heat and mass transfer characteristics of the generalized Burgers’ fluid with the impact of nonlinear radiative heat flux. Sultan et al. reported the novel characteristics of the magnetomixed convection flow of the non-Newtonian nanofluid in the presence of infinite shear rate viscosity and entropy generation. The idea of a curved surface subjected to unsteady the Sisko nanofluid has been introduced to develop a mathematical model featuring thermophoresis and Brownian motion. They concluded that the velocity profile of the Sisko magnetonanofluid enhanced for augmented values of the curvature parameter. Sacia et al. studied the unsteady mixed convection flow in a square cavity with constant heat-flux applied to a part of the bottom wall.

Khan et al. numerically considered a convective heat transfer analysis for the non-Newtonian nanofluid subjected to
Brownian and thermophoretic diffusions. A melting heat transport phenomenon for the Falkner-Skan flow of nanofluid has been analyzed by Khan et al. They concluded that velocity profiles boost for higher velocity and melting rate parameters. Alzahrani and Alshomrani developed a numerical method to analyze the mixed-convection flow of the Oldroyd-B fluid with radiative heat transfer. Khan and Ali has studied a mathematical model of entropy generation for cross fluid in the presence of quartic autocatalysis. They found that heat-mass mechanisms were affected by rates of entropy generation. From the proceeding literature survey, it is observed that the oscillating mixed convection flow along a nonconducting horizontal circular cylinder. The form of dimensionless boundary layer equations for velocity, temperature, and magnetic field applied normal to the horizontal cylindrical surface, θ is the dimensionless temperature variable, and y is the magnetic Prandtl number, which are, respectively, mentioned below:

\[ \xi = \frac{\mu H_0^2}{\rho U_\infty x}, \quad P_r = \frac{v}{\alpha}, \quad \gamma = \frac{v}{\nu_m}, \quad \alpha = \frac{\kappa}{\rho C_p}, \]

\[ \lambda = \frac{G_\alpha}{Re_i^2}, \quad Re_i = \frac{U_m L}{\nu}, \quad G_\alpha = \frac{g \beta A TL^3}{\nu^2}. \]

Here, we consider the stream velocity \( U(\tau) = 1 + \varepsilon e^{i\omega t} \), with \( |\varepsilon| \ll 1 \), where \( \varepsilon \) is the small amplitude oscillating component, and \( \omega \) is the frequency factor. The velocity, magnetic, and temperature components \( u, v, h_x, h_y \) and \( \theta \) can be written as the sum of steady and unsteady components as shown below:

\[ \bar{u} = u_0 + \varepsilon u_1 e^{i\omega t}, \quad \bar{v} = v_1 + \varepsilon v_1 e^{i\omega t}, \quad \bar{h}_x = h_{x0} + \varepsilon h_{x1} e^{i\omega t}, \quad \bar{h}_y = h_{y0} + \varepsilon h_{y1} e^{i\omega t}. \]

Expanding Eq. (A) into a Taylor series and neglecting higher-order terms, one gets

\[ \rho = \rho_\infty (1 - \beta (T_\infty - T_\infty)). \]

The form of dimensionless boundary layer equations for velocity, temperature, and magnetic field are

\[ \frac{\partial \bar{u}}{\partial \tau} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{\bar{m}}{\rho_\infty} \frac{\partial^2 \bar{u}}{\partial y^2} + \bar{\xi} \bar{e}^{\bar{m}} \left( \frac{i \hbar_x}{\partial y} + \bar{h}_y \frac{\partial}{\partial y} \right), \]

\[ \frac{\partial \bar{h}}{\partial \tau} + \bar{u} \frac{\partial \bar{h}}{\partial x} + \bar{v} \frac{\partial \bar{h}}{\partial y} = \frac{\bar{m}}{\rho_\infty} \frac{\partial^2 \bar{h}}{\partial y^2}, \]

\[ \frac{\partial \bar{T}}{\partial \tau} + \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} = \frac{\bar{m}}{\rho_\infty} \frac{\partial^2 \bar{T}}{\partial y^2}. \]

II. THE BASIC PHYSICAL EQUATIONS

Consider a two-dimensional structure of the boundary layer flow mechanism over a horizontal nonconducting circular cylinder. As shown in Fig. 1, x-distance runs along the surface, y-distance is normal to the surface, and \( u, v \) are the corresponding velocity components. Here, \( H_\tau \) is the magnetic field component along the x-direction, \( H_y \) is the component of magnetic field along the y-direction, \( T \) is the temperature field, and \( U(x, \tau) \) is the external flow velocity. Furthermore, the magnetic field acts in the direction normal to the surface of the nonconducting horizontal cylinder. The fluid density is assumed to be reduced exponentially with temperature as follows:

\[ \rho = \rho_\infty e^{-\beta (T - T_\infty)}, \]

where

\[ \beta = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_{\rho}. \]
Equations (15)–(19) with boundary conditions in (20) are oscillating boundary layer equations along a nonconducting circular cylinder. We separate unsteady Eqs. (15)–(20) into real and imaginary forms by considering oscillating Stokes conditions,

\[ u_t = u_1 + iu_2, \quad v_t = v_1 + iv_2, \quad \theta_t = \theta_1 + i\theta_2, \]

\[ \bar{h}_1 = h_{1x} + ih_{1y}, \quad \bar{h}_2 = h_{2x} + ih_{2y}. \]

Using Eq. (21) in Eqs. (15)–(20), we obtained the following system of real and imaginary equations:

**Real Part:**

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \]

\[ -\omega u_2 + \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + v_1 \frac{\partial u_1}{\partial y} - h_{1y} \frac{\partial u_1}{\partial y} = \frac{\epsilon^{m_0}}{y} \frac{\partial^2 h_1}{\partial y^2}, \]

\[ -h_{1y} \frac{\partial u_1}{\partial y} - h_{2y} \frac{\partial u_2}{\partial y} = \frac{\epsilon^{m_0}}{y} \frac{\partial^2 h_1}{\partial y^2}, \]

\[ \frac{\partial h_{1x}}{\partial x} + \frac{\partial h_{1y}}{\partial y} = 0, \]

\[ -\omega \theta_2 + u_1 \frac{\partial \theta_1}{\partial x} + u_2 \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \theta_1}{\partial y} = \frac{\epsilon^{m_0}}{P} \frac{\partial^2 \theta_1}{\partial y^2}, \]

with appropriate boundary conditions

\[ u_1 = v_1 = 0, \quad h_{1y} = h_{1x} = 0, \quad \theta_1 = 1 \quad \text{at} \quad y = 0, \]

\[ u_i \rightarrow 1, \quad \theta_i \rightarrow 0, \quad h_{1y} \rightarrow 1 \quad \text{as} \quad y \rightarrow \infty. \]

**Imaginary Part:**

\[ \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \]

\[ \omega u_1 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + v_1 \frac{\partial u_2}{\partial y} + v_2 \frac{\partial u_2}{\partial y} = \frac{\epsilon^{m_0}}{P} \frac{\partial^2 \theta_1}{\partial y^2}, \]

\[ \frac{\partial h_{2x}}{\partial x} + \frac{\partial h_{2y}}{\partial y} = 0, \]

\[ \omega \theta_1 + u_1 \frac{\partial \theta_2}{\partial x} + u_2 \frac{\partial \theta_2}{\partial y} + v_1 \frac{\partial \theta_2}{\partial y} + v_2 \frac{\partial \theta_2}{\partial y} = \frac{\epsilon^{m_0}}{P} \frac{\partial^2 \theta_1}{\partial y^2}, \]

with appropriate boundary conditions

\[ u_1 = v_1 = 0, \quad h_{2y} = h_{1x} = 0, \quad \theta_1 = 1 \quad \text{at} \quad y = 0, \]

\[ u_i \rightarrow 1, \quad \theta_i \rightarrow 0, \quad h_{1y} \rightarrow 1 \quad \text{as} \quad y \rightarrow \infty. \]
III. SOLUTION METHODOLOGY

The governing dimensionless steady and unsteady equations are solved by applying a very efficient finite-difference method. By using primitive-variable formulation, we can find the numerical results of the given coupled model. The details of the numerical scheme and the solution procedure are given.

A. Primitive variable formulation

To adopt this formulation, we use the formulation given in (34) for both dependent and independent variables of steady equations in a convenient form,

\[ u_1(x,y) = U_1(X,Y), \quad v_1(x,y) = x \frac{\partial Y}{\partial X} V_1(X,Y), \]

\[ h_{12}(x,y) = x \frac{\partial Y}{\partial X} \varphi_{12}(X,Y), \quad h_{22}(x,y) = \varphi_{22}(X,Y), \]

\[ \theta_1(x,y) = \theta_1(X,Y), \quad Y = x \frac{\partial Y}{\partial X}, \quad X = x. \]

Steady Part:

By using (34) in (9)–(14), the transformed primitive equations are

\[ X \frac{\partial U_1}{\partial X} - \frac{Y}{2} \frac{\partial U_1}{\partial Y} + \frac{\partial V_1}{\partial Y} = 0, \]

\[ X U_1 \frac{\partial U_1}{\partial X} + \left[ V_1 - \frac{Y}{2} U_1 \right] \frac{\partial U_1}{\partial Y} + \left[ Y \frac{\partial \varphi_{12}}{\partial Y} + \left( \frac{\varphi_{12}}{2 \varphi_{22}} \right) \frac{\partial \varphi_{22}}{\partial Y} \right] \frac{\partial \varphi_{12}}{\partial Y} = \lambda \left[ \frac{e^{m \alpha} - 1}{1 - e^{-m}} + \theta_1 \sin \alpha \right], \]

\[ X \frac{\partial \varphi_{22}}{\partial X} - \frac{Y}{2} \frac{\partial \varphi_{22}}{\partial Y} + \frac{\partial \varphi_{12}}{\partial Y} = 0, \]

\[ X U_1 \frac{\partial \varphi_{12}}{\partial X} + \left[ V_1 - \frac{Y}{2} U_1 \right] \frac{\partial \varphi_{12}}{\partial Y} + X \varphi_{22} \frac{\partial U_1}{\partial Y} \left( \frac{\varphi_{12}}{2 \varphi_{22}} \right) \frac{\partial U_1}{\partial Y} = \frac{e^{m \alpha}}{Y} \frac{\partial^2 \varphi_{12}}{\partial Y^2}, \]

\[ \frac{X U_1 \partial \theta_1}{\partial X} + \left[ V_1 - \frac{Y}{2} U_1 \right] \frac{\partial \theta_1}{\partial Y} + \frac{e^{m \alpha}}{P_r} \frac{\partial^2 \theta_1}{\partial Y^2} = \frac{e^{m \alpha}}{Y} \frac{\partial^2 \varphi_{12}}{\partial Y^2}. \]

with boundary conditions as

\[ U_i = V_i = 0, \quad \varphi_{12} = \varphi_{22} = 0, \quad \theta_1 = 1 \quad \text{at} \quad Y = 0, \]

\[ U_1 \rightarrow 1, \quad \theta_1 \rightarrow 0, \quad \varphi_{12} \rightarrow 1 \quad \text{as} \quad Y \rightarrow \infty. \] (40)

Real Part:

By using the given formulation in (41) for the real part to reduce to a convenient form,

\[ u_1(x,y) = U_1(X,Y), \quad v_1(x,y) = x \frac{\partial Y}{\partial X} V_1(X,Y), \]

\[ h_{12}(x,y) = x \frac{\partial Y}{\partial X} \varphi_{12}(X,Y), \quad h_{22}(x,y) = \varphi_{22}(X,Y), \]

\[ \theta_1(x,y) = \theta_1(X,Y), \quad Y = x \frac{\partial Y}{\partial X}, \quad X = x. \] (41)

By using (41) in (22)–(27), the convenient primitive equations are

\[ X \frac{\partial U_1}{\partial X} - \frac{Y}{2} \frac{\partial U_1}{\partial Y} + \frac{\partial V_1}{\partial Y} = 0, \]

\[ X \left[ U_1 \frac{\partial U_1}{\partial X} + U_1 \frac{\partial U_1}{\partial X} + \left( \frac{Y}{2} - U_1 \right) \frac{\partial U_1}{\partial Y} + \left( \frac{Y}{2} - U_1 \right) \frac{\partial U_1}{\partial Y} - \omega X U_1 \right] \frac{\partial^2 U_1}{\partial Y^2} + \frac{e^{m \alpha}}{Y} \frac{\partial^2 \varphi_{12}}{\partial Y^2} + \left( \frac{\varphi_{12}}{2 \varphi_{22}} \right) \frac{\partial \varphi_{22}}{\partial Y} \frac{\partial \varphi_{12}}{\partial Y} \left( \frac{\varphi_{12}}{2 \varphi_{22}} \right) \frac{\partial \varphi_{22}}{\partial Y} = \lambda \left[ \frac{e^{m \alpha} - 1}{1 - e^{-m}} + \theta_1 \sin \alpha \right], \]

\[ X \frac{\partial \varphi_{22}}{\partial X} - \frac{Y}{2} \frac{\partial \varphi_{22}}{\partial Y} + \frac{\partial \varphi_{12}}{\partial Y} = 0, \]

\[ X \left[ U_1 \frac{\partial U_1}{\partial X} + U_1 \frac{\partial U_1}{\partial X} + \left( \frac{Y}{2} - U_1 \right) \frac{\partial U_1}{\partial Y} + \left( \frac{Y}{2} - U_1 \right) \frac{\partial U_1}{\partial Y} - \omega X U_1 \right] \frac{\partial^2 U_1}{\partial Y^2} = \lambda \left[ \frac{e^{m \alpha} - 1}{1 - e^{-m}} + \theta_1 \sin \alpha \right]. \]

with boundary conditions as

\[ U_1 = V_1 = 0, \quad \varphi_{12} = \varphi_{22} = 0, \quad \theta_1 = 1 \quad \text{at} \quad Y = 0, \]

\[ U_1 \rightarrow 1, \quad \theta_1 \rightarrow 0, \quad \varphi_{12} \rightarrow 1 \quad \text{as} \quad Y \rightarrow \infty. \] (47)

Imaginary Part:

By using the given formulation in (48) for the imaginary part to reduce to a convenient form,

\[ u_2(x,y) = U_2(X,Y), \quad v_2(x,y) = x \frac{\partial Y}{\partial X} V_2(X,Y), \]

\[ h_{12}(x,y) = x \frac{\partial Y}{\partial X} \varphi_{12}(X,Y), \quad h_{22}(x,y) = \varphi_{22}(X,Y), \]

\[ \theta_2(x,y) = \theta_2(X,Y), \quad Y = x \frac{\partial Y}{\partial X}, \quad X = x. \] (48)
By using (48) in (28)–(33), the reduced system of primitive equations is

\[
X \frac{\partial U_2}{\partial X} + Y \frac{\partial U_2}{\partial Y} + \frac{\partial V_2}{\partial Y} = 0, \tag{49}
\]

\[
X \left[ U_1 \frac{\partial U_2}{\partial X} + U_2 \frac{\partial U_2}{\partial X} + \left( V_1 - \frac{Y}{2} U_1 \right) \frac{\partial U_2}{\partial Y} + \frac{V_2}{2} \right] \frac{\partial U_2}{\partial Y} + \omega X U_1 = e^{m0} \frac{\partial^2 U_2}{\partial Y^2} + e^{m0} \xi X \left( \frac{\partial^2 \varphi_{s2}}{\partial X^2} + \frac{\partial^2 \varphi_{s3}}{\partial X^2} \right) \left( \frac{\partial \varphi_{s3} - \frac{Y}{2} \varphi_{s2}}{\partial Y} \right) + \lambda \left( \frac{e^{m0} \varphi_{s2}}{1 - e^{-m0}} + \theta_1 \sin \alpha \right), \tag{50}
\]

\[
X \frac{\partial \varphi_{s1}}{\partial X} - \frac{Y}{2} \frac{\partial \varphi_{s1}}{\partial Y} + \frac{\partial \varphi_{s1}}{\partial Y} = 0, \tag{51}
\]

\[
X \left[ U_1 \frac{\partial \varphi_1}{\partial X} + U_2 \frac{\partial \varphi_1}{\partial X} + \left( V_1 - \frac{Y}{2} U_1 \right) \frac{\partial \varphi_1}{\partial Y} + \frac{V_2}{2} \right] \frac{\partial \varphi_1}{\partial Y} + \omega X \varphi_1 = X \left( \frac{\partial^2 \varphi_{s1}}{\partial X^2} + \frac{\partial^2 \varphi_{s1}}{\partial X^2} \right) \left( \frac{\partial \varphi_{s1} - \frac{Y}{2} \varphi_{s2}}{\partial Y} \right) + \frac{\partial \varphi_{s2} - \frac{Y}{2} \varphi_{s2}}{\partial Y} \frac{\partial U_2}{\partial Y} = e^{m0} \frac{\partial^2 \varphi_1}{\partial Y^2}, \tag{52}
\]

\[
X \left[ U_1 \frac{\partial \theta_1}{\partial X} + U_2 \frac{\partial \theta_1}{\partial X} + \left( V_1 - \frac{Y}{2} U_1 \right) \frac{\partial \theta_1}{\partial Y} + \frac{V_2}{2} \right] \frac{\partial \theta_1}{\partial Y} + \omega X \theta_1 = \frac{e^{m0} \varphi_{s2}}{P_r} \frac{\partial^2 \theta_1}{\partial Y^2}, \tag{53}
\]

with boundary conditions as

\[
U_2 = V_2 = 0, \quad \theta_2 = \varphi_{s2} = 0, \quad \theta_2 = 1 \quad \text{at} \quad Y = 0, \tag{54}
\]

\[
U_2 \to 0, \quad \theta_2 \to 0, \quad \varphi_{s2} \to 0 \quad \text{as} \quad Y \to \infty.
\]

IV. COMPUTATIONAL TECHNIQUE

The numerical results of the transformed equation given in (35)–(54) are obtained with the help of a finite-difference method. We apply central-difference along the y-axis and backward-difference along the x-axis to these equations. We obtain a system of algebraic equations with unknown variables \( U, V, \theta, \) and \( \varphi \). These variables have coefficients in the form of a tridiagonal matrix. To find the numerical solutions of these unknown variables, we use the Gaussian elimination technique. The obtained results are used in the following equations to find oscillatory skin friction, heat transfer, and magnetic flux from different locations of a nonconducting horizontal circular cylinder.

\[
\tau_w = \left( \frac{\partial U}{\partial Y} \right)_{y=0} + \epsilon[A_1] \cos(\omega t + \alpha_1), \tag{55}
\]

\[
q_w = \left( \frac{\partial \theta}{\partial Y} \right)_{y=0} + \epsilon[A_1] \cos(\omega t + \alpha_1), \tag{56}
\]

\[
j_w = \left( \frac{\partial \varphi}{\partial Y} \right)_{y=0} + \epsilon[A_1] \cos(\omega t + \alpha_m). \tag{57}
\]

where

\[
A_1 = \left( u_1^2 + u_2^2 \right)^{\frac{1}{2}}, \quad A_1 = \left( \theta_1^2 + \theta_2^2 \right)^{\frac{1}{2}}, \quad A_m = \left( \varphi_{s1}^2 + \varphi_{s2}^2 \right)^{\frac{1}{2}}.
\]

\[
\alpha_1 = \tan^{-1} \left( \frac{u_1}{u_1} \right), \quad \alpha_1 = \tan^{-1} \left( \frac{\theta_1}{\theta_1} \right), \quad \alpha_m = \tan^{-1} \left( \frac{\varphi_{s2}}{\varphi_{s1}} \right),
\]

where, \( A_1, A_1, \) and \( A_m \) are the amplitudes of skin friction, heat transfer, and current density, respectively, while \( \alpha_1, \alpha_1, \) and \( \alpha_m \) are the phase angles of skin friction, heat transfer, and current density, respectively.

V. RESULTS AND DISCUSSION

The present analysis based on an oscillatory mixed convection flow of electrically conducting fluid along a nonconducting horizontal circular cylinder in the presence of variable density is performed. The density of the fluid is assumed to decrease exponentially with temperature. In order to analyze the model results, the system of dimensionless boundary layer differential equations are reduced to a convenient form by using primitive variable transformation with the finite difference technique for various values of physical parameters, such as the density/temperature parameter \( m \), magnetic force parameter \( \xi \), mixed convection parameter \( \lambda \), magnetic Prandtl number \( \gamma \), and Prandtl number \( \rho \). The impact of these pertinent parameters on velocity, temperature rate, and magnetic field is computed and then used to examine the oscillatory behavior of skin friction, heat transfer, and current density at each position \( \alpha = \pi/\rho, \pi/3, \) and \( \pi \) of the nonconducting horizontal circular cylinder. The dimensionless density equation (A) in a convenient form can be written as

\[
\rho = \rho_0 e^{-m}. \tag{58}
\]

Since \( \rho \) varies from 0, at the edge of the boundary layer, to 1 along the normal to the nonconducting horizontal circular cylinder, from this equation, it is clear that the mixed convection flow is studied, where \( m \) cannot be identically zero.

The influence of the density/temperature parameter \( m \) on velocity, temperature, and magnetic field profiles are presented in Figs. 2(a)–2(c). Figure 2(a) shows that dimensionless velocity increases as the density/temperature parameter \( m \) increases at the position \( \alpha = \pi/6 \) for the Prandtl number \( Pr = 7.0 \). Figures 2(b) and 2(c) predicted that the temperature distribution and magnetic profile is decreasing at \( \alpha = \pi/3 \) for a higher value of the density/temperature parameter \( m = 0.5 \), while other parameters are kept constant. An increase in the density/temperature parameter \( m \) means an increase in the velocity of the fluid particles due to an increase in the buoyancy forces. Fluid attains two forces by increasing \( m \) in which the first force increases the velocity of the fluid by increasing the buoyancy forces, and second force decreases the velocity of the fluid due to decrease in temperature distribution. The numerical values of the velocity, temperature distribution, and magnetic field profiles plotted for different values of the Prandtl number \( Pr = 0.1, 0.71 \) and 7.0 for a higher value of \( m = 1.5 \) are shown in Figs. 3(a)–3(c). It has to be noticed that decrease in the density parameter \( m \) leads to a sharp increase in the velocity of the fluid at the position \( \alpha = \pi/6 \) for a lower Prandtl number \( Pr = 0.1 \), as shown in Fig. 3(a). It is also observed that an increase in the Prandtl number \( Pr \) leads
FIG. 2. Geometric interpretation of the (a) velocity profile, (b) temperature distribution, and (c) magnetic field profile at $\alpha = \pi/6, \pi/3, \pi$ for three values of the density/temperature parameter $m = 0.1, 0.3, 0.5$ while $\gamma = 0.01$, $\xi = 1.0$, $Pr = 7.0$, and $\lambda = 1.5$.

FIG. 3. Geometric interpretation of the (a) velocity profile, (b) temperature distribution, and (c) magnetic field profile at $\alpha = \pi/6, \pi/3, \pi$ for three values of the Prandtl number $Pr = 0.1, 0.71$ and 7.0 while $\gamma = 0.01$, $\xi = 1.0$, $\lambda = 1.5$, and $m = 0.1$.

to a decrease in the velocity and temperature profile, as shown in Figs. 3(a) and 3(b), and at each position $\alpha = \pi/6, \pi/3, \pi$ of the nonconducting horizontal circular cylinder and maximum behavior in the magnetic field, as shown in Fig. 3(c). Physically, it is possible because thermal conductivity decreases as $Pr$ increases, which has a lower magnitude of frictional forces between the viscous layers. Figures 4(a)–4(c) plot against different values of the mixed-convection parameter $\lambda$ at three positions $\alpha = \pi/6, \pi/3, \pi$ of the nonconducting horizontal circular cylinder, while other parameters are fixed. It illustrates maximum behavior in the velocity field for a

FIG. 4. Geometric interpretation of the (a) velocity profile, (b) temperature distribution, and (c) magnetic field profile at $\alpha = \pi/6, \pi/3, \pi$ for three values of the mixed-convection number $\lambda = 0.1, 0.4$ and 0.9 while $\gamma = 0.01$, $\xi = 1.5$, $Pr = 7.0$, and $m = 0.1$. 
higher value of \( \lambda = 0.9 \) at the position \( \alpha = \pi/6 \), as shown in Fig. 4(a), but a lower temperature and magnetic rate, as shown in Figs. 4(b) and 4(c). In addition, it is noted that a sharp increase in the velocity profile is due to the higher magnetic force parameter \( \xi \). At a higher value of the magnetic force parameter \( \xi \), as shown in Figs. 5(a)–5(c), it can be concluded that the velocity increases sharply near the cylinder surface for a maximum value of \( m \), attaining a maximum asymptotic value as \( \xi \) increases at the position \( \alpha = \pi/3 \). It can be seen that the temperature distribution and magnetic profile shows uniform behavior at each position with increase in \( \xi \), attaining an asymptotic value. In addition, it is found that the momentum and thermal boundary layer thickness increase with \( \xi \). It happens because the magnetic force parameter is the ratio of magnetic energy to kinetic energy, so with the increase in the magnetic force parameter \( \xi \), the magnetic energy is increased, while the kinetic energy is reduced. The distinct behavior in velocity and magnetic field is observed for lower values of the magnetic Prandtl number \( \gamma \) at three positions \( \alpha = \pi/6, \pi/3, \) and \( \pi \) of the nonconducting horizontal circular cylinder, as shown in Figs. 6(a) and 6(c), but uniform behavior in the temperature rate, as shown in Fig. 6(b). From these figures, it is examined that the velocity profile attains maximum behavior for \( \gamma = 0.03 \) and decreases to an asymptotic value at the position \( \alpha = \pi/6 \). The reason is that by increasing the magnetic Prandtl number, the magnetic diffusion is decreased, which is responsible for the abovementioned mechanism.

Figures 7(a)–7(c) give the magnitude of oscillations of skin friction, heat transfer, and current density for two values of the density/temperature parameter \( m \) at three positions \( \alpha = \pi/6, \pi/3, \) and \( \pi \) of the nonconducting horizontal circular cylinder. It can be concluded that the amplitude of oscillation of skin friction is minimum, but a good response in heat transfer and current density is obtained. This happens because an increase in density variation with temperature leads to an increase in the buoyancy force. It is due to the reason that the buoyancy force acts like the driving force accelerating the fluid flow for a higher temperature difference. The effect of the Prandtl number \( Pr \) is analyzed to obtained magnitude of oscillations at different positions, as shown in Figs. 8(a)–8(c). A good response in the amplitude of oscillation in heat transfer and current density is predicted at the position \( \alpha = \pi/3 \) for a lower Prandtl number, in addition to a good agreement in skin-friction for a higher value of the density/temperature parameter. This phenomenon is expected because an increase in \( Pr \) tends to increase density variation with temperature which enhances the buoyancy force. The effect of the mixed convection parameter \( \lambda \) is illustrated in Figs. 9(a)–9(c),
and maximum oscillating behavior of skin friction, heat transfer, and current density at $\alpha = \pi/3$ is noted. An increase in $\lambda$ tends to increase the amplitude of oscillation of heat transfer and current density. Figures 10(a)–10(c) presented that the geometric interpretation of oscillations in skin-friction is minimum and uniform as the magnetic force parameter $\xi$ increases, whereas heat transfer and current density attain good amplitude as $\xi$ increases. This result was expected because an increase in $\xi$ means an increase in the Lorentz-force which opposes the flow and velocity of the fluid decrease. This Lorentz force increases the friction between fluid layers which leads
VI. CONCLUDING REMARKS

We have studied the oscillatory mixed convection flow of electrically conducting fluid along a nonconducting horizontal circular cylinder in the presence of variable density. The density of the fluid is considered as exponentially varying with temperature. Due to this mechanism, the solution of the transformed model is accurate and valid for high temperature difference. The boundary layer equations are transformed to a convenient form for numerical results by adopting primitive variable transformation with a finite difference technique. The numerical results are presented graphically for different values of physical parameters and found to be in good agreement. The influence of various pertinent parameters on velocity, temperature, and magnetic field velocity are observed and then used to analyze the oscillatory behavior of skin friction, heat transfer, and current density at different positions of the nonconducting horizontal circular cylinder. It is concluded that the dimensionless velocity increases as the density/temperature parameter $m$ increases at the position $\alpha = \pi/6$ for the Prandtl number $Pr = 7.0$. An increase in the density/temperature parameter $m$ means an increase in the velocity of the fluid particles due to an increase in the buoyancy forces. It illustrates the maximum behavior in the velocity field for a higher value of $\lambda = 0.9$ at the position $\alpha = \pi/6$, but the temperature and magnetic rate is lower. It is concluded that the velocity increases sharply near the cylinder surface for a maximum value of $m$, attaining a maximum asymptotic value as $\xi$ increases at the position $\alpha = \pi/3$. The distinct behavior in velocity and magnetic field is observed for lower values of the magnetic Prandtl number $\gamma$ at three positions $\alpha = \pi/6, \pi/3$, and $\pi$ of the nonconducting horizontal circular cylinder. A good response in the amplitude of oscillation in heat transfer and current density is predicted at the position $\alpha = \pi/3$ for a lower Prandtl number, which is also in good agreement with
skin-friction for a higher value of the density/temperature parameter. Fluctuation in skin-friction increases for a higher value of the density/temperature parameter $m = 5.0$ and uniform skin friction for both values of $\gamma$ are noted. This phenomenon is expected because an increase in $m$ tends to enhance the amplitude due to buoyancy force. In addition, it has to be mentioned that the buoyancy force acts like a driving force accelerating the fluid flow for a higher temperature difference.

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