Topics in String Tachyon Dynamics

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Abstract

We review some aspects of string tachyon dynamics with special emphasis on effective actions and K-theory interpretation.

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1 Introduction

In the last few years a new understanding of the dynamical role of tachyons in string theory has started to emerge ([1] to [17]). For the simplest open bosonic string there exist already a lot of evidence on tachyon condensation [1, 6, 7, 8, 11, 12]. The tachyon vacuum expectation value characterizing this condensate exactly cancel the open string one loop contribution to the cosmological constant, what we now understand as the $D_{25}$ filling brane tension. The vacuum defined by this condensate is naturally identified with the closed string vacua. Precise computations of the tachyon potential supporting this picture has been carried out both in open string field theory [7, 8, 9, 10, 12, 13, 14, 15] and in background independent open string field theory [18, 19, 20, 21, 22, 23]. At this level of understanding two main problems remain open. First of all we have the problem of the closed tachyon that survives as an unstability of the closed string vacua defined by the open string tachyon condensate. Secondly we lack a precise understanding of the dynamical mechanism by which the $U(1)$ gauge open degrees of freedom are decoupled from the closed string spectrum.

Concerning the problem of the closed string tachyon, the $\sigma$-model beta functions [24, 25] indicate that closed tachyon condensation creates a contribution to the cosmological constant of the same type generated by working with non critical dimensions. The well known result about the $c = 1$ barrier in the context of linear dilaton backgrounds [26] could indicate a sort of unstability that reduces drastically the space time dimensions until reaching the safe $D = 2$.

With respect to the problem of the fate of $U(1)$ gauge degrees of freedom after open tachyon condensation - a short of confinement of open degrees of freedom into closed spectrum - there are two formal hints. One is the suggestion of a trivial nilpotent BRST charge of type $ac_0$, for $c$ the ghost field, around the background defined by the tachyon condensate [27]. The other hint comes from observing that the open string effective Born-
Infeld lagrangian, is multiplied by a factor $e^{-T}$ with $T = \infty$ defining the open tachyon condensate \cite{28, 29, 30, 23}.

In the context of more healthy superstrings without tachyons, the phenomena of tachyon condensation shed some new light on the solitonic interpretation of the $D$-branes. We have two main examples corresponding to pairs $D_p - D_p$ brane-antibrane which will support an open tachyon on the world volume spectrum and the case of configurations of unstable non-BPS $D$-branes. In both cases tachyon condensation will allow us to interpret stable BPS $D$-branes as topologically stable extended objects, solitons, of the auxiliary gauge theory defined on the world volume of the original configuration of unstable $D$-branes.

The mechanism of decay into a closed string vacua by tachyon condensation can be used to define a new algebraic structure to characterize $D$-brane stability and $D$-brane charges, namely K-theory (\cite{31} to \cite{35}). The main ingredient in order to go to K-theory is the use of stability equivalence with respect to creation-annihilation of branes. In type IIB $D_p$-branes of space codimension $2k$ are related to $K(B^{2k}, S^{2k-1})$ and for type IIA $D_p$-branes of space codimension $2k+1$ are related to $K^{-1}(B^{2k+1}, S^{2k})$. The characterization of $K(X, Y)$ in terms of triplets \cite{36} $(E, F, \alpha)$ with $E, F$ vector bundles on $X$ and $\alpha$ an isomorphism $\alpha : E |_Y \rightarrow F |_Y$ makes specially clear the mathematical meaning of the open tachyon field as defining the isomorphism $\alpha$. A similar construction in terms of pairs $(E, \alpha)$ with $\alpha$ an automorphism of $E$ can be carried out for the definition of the higher $K^{-1}$-group \cite{32}.

Finally we would like to point out to some striking similarities between the topological characterization of stable $D_p$-branes in type IIA string and gauge fixing singularities for unitary gauges \cite{37} of the type of 'tHooft’s abelian projection \cite{40}. Can we learn something of dynamical relevance from this analogy?. After the discovery of asymptotic freedom, the Holy Grial of high energy physics is the solution of the confinement problem. The abelian projection gauge was originally suggested in \cite{40} as a first step towards a quantitative approach to confinement i.e. to the computation of the magnetic monopole condensate. The analogy between stable $D_p$-branes ($p \leq 6$) in type IIA and the magnetic monopoles
associated with the abelian projection gauge singularities seems to indicate, as the stringy analog of confinement, the decay of the gauge vacua associated with a configuration of unstable $D_9$-filling branes into a closed string vacua populated of stable $D_p$-branes. Another interesting lesson we learn from the analogy is that as it is the case with magnetic monopoles in the abelian projection, that should be considered as physical degrees of freedom independently of what is the phase, confinement, Higgs or Coulomb of the underlying gauge theory, the same should be true concerning $D_p$-branes in type IIA, independently of the concrete form of the open tachyon potential. What is relevant to characterize the “confinement” closed string phase is the “dualization” of the original open gauge string degrees of freedom into RR closed string fields whose sources are stable $D_p$-branes. Finally and from a different point of view another hint suggested by this analogy is the potential relevance of the higher K-group $K^{-1}$ to describe gauge fixing singularities in ordinary gauge theories. Maybe the answer to the natural question why the higher K-group $K^{-1}$ is pointing out to some hidden “M-theoretical” meaning of the gauge $\theta$-parameter.

The present review is not intended to be complete in any sense. Simply covers the material presented by one of us (C.G.) during the 4th SIGRAV School on Contemporary Relativity and Gravitational Physics and 2001 School on Algebraic Geometry and Physics. Como May 2001.

2 Why tachyons?

In quantizing string theory in flat Minkowski space-time there are two constants that should be fixed by consistency, namely the normal ordering constant appearing in the

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2Some parts of these lectures were also presented in: II Workshop on Non-commutative Geometry, String Theory, and Particle Physics. Rabat May 2001 and in the Workshop New Interfaces between Geometry and Physics. Miraflores June 2001.
mass formula:

\[ M^2 = \frac{4}{\alpha'}(N - a) \]  

(1)

and the dimension D of the space-time. These two constants determines the Virasoro anomaly

\[ [L_m, L_n] = (m - n)L_{m+n} + A(D, a, m)\delta_{m+n} \]  

(2)

with

\[ A(D, a, m) = \frac{D}{12}(m^3 - m) + \frac{1}{6}(m - 13m^3) + 2am \]  

(3)

and

\[ L_m = L^{(\text{matter})}_m + L^{(\text{ghosts})}_m - a\delta_m \]  

(4)

Impossing \( A(D, a, m) = 0 \) implies the standard constraints on the bosonic string, namely \( D = 26 \) and \( a = 1 \).

The first consequence of the non vanishing normal ordering constant \( a \) is that the \((mass)^2\) of the ground state \( (N = 0) \) is negative i.e. it is a tachyon. In spite of that there is a good consequence of this normal ordering value, namely, the existence at the first level of a massless vector boson in the open case and the massless graviton in closed case.

A priori the only consistency requirement we should impose is absence of negative norm ghost states in the physical Hilbert space. This allow us to relax the condition on D and \( a \) to \( D \leq 26 \) and \( a \leq 1 \).

Although in these conditions the open string theory is perfectly healthy at tree level we will find unitarity problems for higher order corrections, more precisely singularity cuts for one loop non planar diagrams. In the closed string case the problems at one loop will show up as lack of modular invariance. Thus we will reduce ourselves to critical dimension \( D = 26 \) and \( a = 1 \).

One important place where the normal ordering constant appears in string theory is in the definition of the BRST operator:

\[ Q = \sum_m (L_m c_{-m} - \frac{1}{2} \sum_n (m - n)c_{-m} c_{-n} b_{m+n}) \]  

(5)
with \( b, c \) the usual ghost system for the bosonic string. The charge \( Q \) can be written in a more compact way as:

\[
Q = \sum_m (L_m^{\text{matter}} + \frac{1}{2} L_m^{\text{ghosts}}) c_m - a \delta_m c_m 
\] (6)

Notice that the contribution of the normal ordering constant to \( Q \) is simply \( ac_0 \). This quantity defines by itself a BRST charge -since it is trivially nilpotent \( c_0^2 = 0 \)- with trivial cohomology\(^3\).

In standard quantum field theory a tachyon is not such an unfamiliar object. A good example is for instance the Higgs field if we perturb around the wrong vacua \( \langle \phi \rangle = 0 \). In this sense the presence of a tachyon usually means that we are perturbing around an unstable vacua. In a physically sensible situation we expect the system roll down to some stable vacua where automatically the tachyon will disappears. In the bosonic string it is not clear at all if this is the case since we still lack a powerful tool to study the string theory off shell. The only real procedure to address this issue is of course string field theory.

In superstring theories with space-time supersymmetry i.e. type I, type II or heterotic, the tachyons are projected out by imposing GSO. However even in these cases open string tachyons can appear if we consider non-BPS Dirichlet D-branes. In those cases the open tachyon is associated with unstabilities of these non-BPS D-branes.

### 3 Tachyons in AdS: The \( c = 1 \) barrier

A simple way to see the unstabilities induced by tachyonic fields with negative \( (mass)^2 \), is to compute their contribution to the energy in flat Minkowski space-time. Generically the energy is defined by

\[
E = \int d^{n-1}x dr \sqrt{g} g^{\mu \nu} \partial_{\mu} \phi^* \partial_{\nu} \phi + m^2 \phi^* \phi \] (7)

\(^3\)This is the BRST operator recently suggested in \([27]\) to describe the cohomology around the open tachyon condensate.
where $n$ is the space-time dimension. The condition of finite energy requires an exponential falloff $\phi \sim e^{-\lambda r}$ at infinity with $\lambda > 0$. The energy of a field fluctuation with this falloff at infinity goes like $E \sim (\lambda^2 + m^2)$. Thus if $m^2 < 0$ this energy can become negative for small enough $\lambda$, which means unstability. This is not necessarily the case if we consider curved space time.

For $AdS_n$ the metric can be written as:

$$ds^2 = e^{2ky}dx_{n-1}^2 + dy^2$$  \hspace{1cm} (8)

with the curvature radius being:

$$R = \frac{1}{k}$$  \hspace{1cm} (9)

For simplicity let us consider fluctuations of the field depending only on the $y$ coordinate. The condition of finite energy requires now an exponential falloff $\phi \sim e^{-\lambda y}$ for $y \to \infty$ with

$$\lambda > \frac{k(n-1)}{2}$$  \hspace{1cm} (10)

As before the contribution to the energy will go as $E \sim (\lambda^2 + m^2)$ and therefore we get positive energy for tachyon fields with $m^2 = -a$ if

$$a \leq \frac{(n-1)^2}{4R^2}$$  \hspace{1cm} (11)

This bound on the tachyon mass in $AdS_n$ is known as Breitenlohner- Freedmann bound [11].

In the case of string theory the contribution to the energy of closed string tachyons goes like:

$$E = \int d^{d-1}x dr \sqrt{g} e^{-2\Phi} [g^{\mu\nu} \partial_\mu T \partial_\nu T + m^2 T^2]$$  \hspace{1cm} (12)

with $m^2 = -\frac{4}{\alpha'}$. The field $\Phi$ in (12) is the dilaton field. We will be interested in working in flat Minkowski space-time of dimension $n$. The dilaton $\sigma$-model beta function equation

$$\frac{n-26}{6\alpha'} + (\nabla \Phi)^2 - \frac{1}{2}(\nabla^2 \Phi) = 0$$  \hspace{1cm} (13)
implies a lineal dilaton behavior of type:

\[ \Phi = y \sqrt{\frac{n - 26}{6\alpha'}} \]  

(14)

for some arbitrary coordinate \( y \). Let us now consider tachyon fluctuations on this background depending only on coordinate \( y \). Using the same argument that with \( AdS_n \) we get the bound on the tachyon mass \( m^2 = -a \):

\[ a \leq \frac{n - 26}{6\alpha'} \]  

(15)

Thus in order to saturate this bound for the closed string tachyon \( a = \frac{4}{\alpha'} \) we need \( n = 2 \). This is the famous \( c = 1 \) barrier, namely only for space time dimension equal two or smaller the closed string tachyon is not inducing any unstability.

Notice that from the point of view of the tachyon mass bound, linear dilaton for dimension \( n \) behaves as \( AdS_n \) with curvature radius given by:

\[ R^2 = \frac{3(n - 1)^2\alpha'}{2(n - 26)} \]  

(16)

4 Tachyon \( \sigma \)-model beta functions

The partition function for the bosonic string in a closed tachyon background is given by:

\[ Z(T) = \int Dxe^{\frac{-1}{2\pi\alpha'} \int d^2\sigma \sqrt{h(x)\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} + T(x)}} \]  

(17)

The first thing we notice is that the tachyon term \( \int \sqrt{h}T(x) \) is clearly non invariant with respect to Weyl rescalings of the world-sheet metric. The strategy we will follow would be to fix \( h_{\alpha\beta} = e^{2\phi} \eta_{\alpha\beta} \) in (17) and to impose invariance with respect to changes of \( \phi \) for the quantum corrected \( \sigma \)-model. We will use a background field \( x^\mu_0 \) with \( x^\mu = x^\mu_0 + \xi^\mu \) and such that \( \partial_\mu T(x_0) = 0 \). In these conditions we get at one loop in the \( \sigma \)-model:

\( \text{For solutions to the bosonic beta function interpolating between AdS and linear dilaton see references [38, 39]} \)
where by $<\xi^\mu \xi^\nu>$ we indicate the one loop of quantum fluctuations (see figure 1) and where $\Lambda$ is the ultraviolet cutoff for the one loop integration.

Next we need to relate the Weyl factor $\phi$ with the cutoff $\Lambda$. A dilatation of the world-sheet metric induces a change $\Lambda \to \lambda \Lambda$ and $e^\phi \to \lambda e^\phi$, thus we can identify $e^\phi$ with $\Lambda$. Doing this we get from (18):

$$\Lambda^2[T(x_0) + \frac{\alpha'}{2} \partial_\mu \partial_\nu T(x_0) \eta^{\mu\nu} \log \Lambda]$$

(20)

Expanding (20) in powers of $\log \Lambda$ we get, at first order in $\log \Lambda$, that independence of Weyl rescalings requires

$$\beta_T \equiv 2T(x_0) + \frac{\alpha'}{2} \partial_\mu \partial_\nu T(x_0) \eta^{\mu\nu} = 0$$

(21)

which is the definition of the closed string tachyon beta function.

Repeating exactly the same steps for the open string tachyon we get instead of (21):

$$\beta_T^o \equiv T(x_0) + \alpha' \partial_\mu \partial_\nu T(x_0) \eta^{\mu\nu} = 0$$

(22)

If we interpret (21) and (22) as equations of motion they correspond to tachyonic space time fields of $(mass)^2$ respectively $-\frac{4}{\alpha'}$ and $-\frac{1}{\alpha'}$. 

Figure 1: One loop contribution to the tachyon beta function.
What we learn from this simple exercise is that the tachyonic nature of background $T$ introduced in (17) is tied to the simple fact that $\int_\Sigma \sqrt{h} T$ is not Weyl invariant. Notice that although the usual dilaton term $\int_\Sigma \sqrt{h} \Phi R^{(2)}$ is not Weyl invariant it depends on $\phi$ only through $(\partial \phi)^2$ terms.

5 Open strings and cosmological constant: The Fischler-Susskind mechanism

5.1 Fischler-Susskind mechanism: Closed string case

Let us start considering one loop divergences in the critical $D = 26$ closed bosonic string. For simplicity we will reduce ourselves to amplitudes with $M$ external tachyons. Divergences for this amplitude will arise in the limit where all the $M$ external tachyon insertions coalesce (see figure 2).

Figure 2: Relevant topology to describe the limit where the insertion points coalesce.
The amplitude is given by:

\[ A(1, 2, ..., M) = \int \frac{d^2\tau}{(Im\tau)^2} C(\tau) F(\tau) \]  

(23)

where

\[ C(\tau) = \left( \frac{Im\tau}{2} \right)^{-12} e^{4\pi Im\tau} |f(e^{2i\pi\tau})|^{-48} \]  

(24)

and

\[ F(\tau) = \kappa^M Im\tau \int \prod_{r<s} d^2\nu_r \prod_{r<s} (\chi_{rs})^{\frac{k_r k_s}{2}} \]  

(25)

Expression (23) is invariant under \( SL(2, \mathbb{Z}) \) modular transformations

\[ \tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1 \]  

(26)

Integration in (23) is reduced to the fundamental domain \( F \). Using the conformal Killing vector on the torus we have fixed the position \( \nu_M \) of one external tachyon. It is convenient to define the new variables:

\[ \varepsilon\eta_r \equiv \nu_r - \nu_M \quad r = 1, ..., M - 2 \]  

(27)

\[ \eta_{M-1} \equiv \nu_{M-1} - \nu_M = \varepsilon e^{i\phi} \]  

(28)

with \( \varepsilon \) and \( \phi \) real variables. The Jacobian of the transformation is:

\[ \prod_{r=1}^{M-1} d^2\nu_r = i\varepsilon^{2M-3} d\varepsilon d\phi \prod_{r<s} d^2\eta_r \]  

(29)

In the limit where \( \nu_{rs} = \nu_r - \nu_s \sim 0 \) the Green function \( \chi_{rs} \) in (23) behaves like:

\[ \chi_{rs} \sim 2\pi |\nu_{rs}| \]  

(30)

Expanding in this limit the integrand in (23) in powers of \( \varepsilon \) the leading divergence is:

\[ \kappa^M \int_0^1 \frac{d\varepsilon}{\varepsilon^3} d\phi \prod_{r<s} d^2\eta_r \prod_{1\leq r \leq s \leq M-1} |\eta_r - \eta_s|^\frac{k_r k_s}{2} \int \frac{d^2\tau}{(Im\tau)} C(\tau) \]  

(31)
where we have used the on shell condition for the closed tachyon

$$\sum_{1 \leq r \leq s \leq M - 1} k_r k_s = -4M$$  

(32)

The amplitude (31) correspond to the propagation of a closed tachyon along the neck. The next subleading term in the expansion goes like $\frac{1}{\varepsilon}$ and correspond to the propagation along the neck of a massless dilaton. Thus divergent contribution to the amplitude can be written like:

$$\int_0^1 \frac{d\varepsilon}{\varepsilon} A_0(k = 0, 1...M) \kappa J$$  

(33)

where $A_0$ is the genus zero amplitude for M external tachyons and one dilaton at zero momentum and where $\kappa J$ is proportional to the genus one dilaton tadpole (see figure 2):

$$\kappa J = \kappa \int_F \frac{d^2 \tau}{(Im \tau)^2} C(\tau)$$  

(34)

The original idea of Fischler-Susskind mechanism [42] consist in absorbing the genus one divergence (33) into a renormalization of the world sheet $\sigma$-model lagrangian, namely:

$$\eta_{\mu\nu} \partial x^\mu \partial x^\nu \rightarrow \eta_{\mu\nu}[1 + \kappa^2 J \int_0^1 \frac{d\varepsilon}{\varepsilon} \partial x^\mu \partial x^\nu]$$  

(35)

The factor $\kappa^2$ in (34) appears because we want to use this counterterm on the sphere to cancel a genus one divergence. Recall that generic genus one amplitudes goes like $\kappa^M$ while genus zero amplitudes goes like $\kappa^{M-2}$. 

Figure 3: Dilaton tadpole graph.
Obviously the renormalized lagrangian defined in (35) explicitly breaks conformal invariance. Introducing a cutoff $\Lambda$ in the $\varepsilon$-integration the corresponding $\sigma$-model beta function is:

$$\beta^{(1)}_{\mu\nu} = \kappa^2 J_{\eta \mu\nu} \sim \frac{\delta L_R}{\delta \log \Lambda} \quad (36)$$

for $L_R$ the renormalized lagrangian defined in (35). In principle we can generalize (36) to curved space time just replacing $\eta_{\mu\nu}$ by $G_{\mu\nu}$. Once we do that we can compensate the $\sigma$-model beta function coming from $\sigma$-model one loop effects:

$$\langle \log \Lambda \rangle R_{\mu\nu} \partial x^\mu \partial x^\nu \quad (37)$$

with the genus one contribution, by imposing:

$$R_{\mu\nu} = \kappa^2 J G_{\mu\nu} \quad (38)$$

In summary the main message of Fischler-Susskind mechanism is that $\sigma$-model divergences can be compensated by string loop divergences. We have shown that this is the case case at least at genus one. Including the dilaton field and using the well known relation

$$\kappa = e^\Phi \quad (39)$$

we will get instead of (38):

$$R_{\mu\nu} - 2 \nabla_\mu \nabla_\nu \Phi = e^{2\Phi} J G_{\mu\nu} \quad (40)$$

### 5.2 Open string contribution to the cosmological constant: The filling brane

This time we will consider the open string one loop amplitude for $M$ external on shell open tachyons (see figure 4)

In the planar case this amplitude in given by:

$$A(1, 2, ..., M) = g^M \int_0^1 \prod_{r=1}^{M-1} \theta(\nu_{r+1} - \nu_r) d\nu_r \int_0^1 dq q^{-2} [f(q^2)]^{-24} \prod_{r<s} [\Psi_{rs}]^{k_r k_s} \quad (41)$$
The divergences of this amplitude appear in the $q \to 0$ limit corresponding to shrinking to zero size the hole of the annulus. The structure of divergences can be read from the annulus vacuum to vacuum amplitude:

$$Z^{(1)}_0 = \int_0^1 \frac{dq}{q} q^{-2} [f(q^2)]^{-24} = \int_0^1 \frac{dq}{q^3} [1 + (26 - 2)q^2 + ...]$$

(42)

Extending Fischler-Susskind to (42) is equivalent to reproduce the coefficient of the divergences in terms of expectation values of certain operators on the disc [13]. The divergence $26 \int_0^1 \frac{dq}{q}$ is easily reproduced by:

$$\int_0^1 \frac{dq}{q} e^\Phi < \eta_{\mu\nu} \partial x^\mu \partial x^\nu >_{disc}$$

(43)

where we have included the dilaton factor required for matching the one loop and disc amplitudes. The divergence $\int_0^1 \frac{dq}{q^3}$ correspond to:

$$\int_0^1 \frac{dq}{q^3} e^\Phi < 1_d >_{disc}$$

(44)

The logaritmic divergence $-2 \int_0^1 \frac{dq}{q}$ comes from the contribution of ghosts to the annulus partition function. The correct way to reproduce this divergence is in terms of the ghost dilaton vertex operator $D^{(\text{ghost})}(k = 0)$ as
\[
\int_0^1 \frac{dq}{q} e^\Phi <D^{(\text{ghost})}(k = 0) >_{\text{disc}}
\] (45)

In fact the representation (45) of the divergence \(-2 \int_0^1 \frac{dq}{q}\) is a direct consequence of the dilaton theorem [44]:

\[
< \int D_{\text{ghost}}(z, \bar{z}) \Phi(p_1) \ldots \Phi(p_n) >_\Sigma \sim 2g - 2 + n < \Phi(p_1) \ldots \Phi(p_n) >_\Sigma
\] (46)

with

\[
D_{\text{ghost}}(z, \bar{z}) = \frac{1}{2} c \partial^2 c - c \partial^2 \bar{c}
\] (47)

Let us concentrate on (43), the Fischler-Susskind counterterm needed to cancel this divergence induces a contribution to the \(\beta_{\mu\nu}\) \(\sigma\)-model beta function proportional to

\[
e^\Phi \frac{\alpha'}{ \alpha'} \eta_{\mu\nu}
\] (48)

In order to reproduce this term we need to add to the closed string effective lagrangian the open string cosmological constant term:

\[
\frac{1}{\kappa^2} \int d^{26}x e^{-\Phi} \sqrt{g} + \ldots
\] (49)

The reader can easily recognize in (49) the first term in the expansion of the \(D - 25\) filling brane Born-Infeld lagrangian:

\[
S_{BI} = T_{25} e^{-\Phi} \int d^{26}x \sqrt{g + b + F} = T_{25} e^{-\Phi} \int d^{26}x \sqrt{g + \ldots}
\] (50)

with \(T_{25}\) the filling brane tension given by \(\sim \frac{1}{\alpha' \kappa^2}\).

Thus we learn that the \(D - 25\) filling brane tension simply represents the open string contribution to the cosmological constant.

Before finishing this section let us just summarize in the following table the different string contributions to the cosmological constant:
| $D \neq D_{Cr}$ | $\Lambda_{Cr} \sim e^{-2\Phi \frac{D-D_{Cr}}{6\alpha'}}$ |
|-----------------|--------------------------------------------------|
| Closed string divergences | $\Lambda_c \sim J$ |
| Open string divergences | $\Lambda_o \sim e^{-\Phi \frac{1}{\alpha'}}$ |

Tachyon condensation is strongly connected with these string contributions to the cosmological constant. Generically closed tachyon condensation could change the value of $\Lambda_{Cr}$ and open tachyon condensation, according to Sen's conjecture, can cancel $\Lambda_o$.

6 The effective action

6.1 A warming up exercise

Let us start with the following open string action

$$S(a) = \int_\Sigma d^2 \sigma \sqrt{h} h^{\alpha\beta} \partial_{\alpha} x^\mu \partial_{\beta} x^\nu \eta_{\mu\nu} + \int_{\partial \Sigma} d\theta a$$  \hspace{1cm} (51)

with $a$ some constant and $\Sigma$ the disc.

We will fix a world sheet metric $h_{\alpha\beta} = e^{2\Phi} \eta_{\alpha\beta}$, thus the open string tachyon term in (51) is $\int_{\partial \Sigma} d\theta ae^\phi$.

The partition function $Z(a)$ is simply defined by:

$$Z(a) = \int Dxe^{-S(a)}$$  \hspace{1cm} (52)

If as usual we identify $e^\phi$ as the ultraviolet cutoff we get the beta function for $a$:

$$\beta_a = -a$$  \hspace{1cm} (53)

The effective action will be defined, in this trivial case, by:

$$\frac{\partial I(a)}{\partial a} = G_{TT} \beta_a$$  \hspace{1cm} (54)
with \( \beta_a \) given in (53) and \( G_{TT} \) the Zamolodchikov metric defined by the open string amplitude on the disc of two open tachyon vertex operators at zero momentum:

\[
G_{TT}(a) = \langle 1_d, 1_d \rangle_{\text{disc}}(a) \tag{61}
\]

with the expectation value in (61) computed for the action (51). In our case and assuming decoupling of ghosts it is obvious that \( G_{TT} \) is equal to \( e^{-a} Z(0) \). Thus using (54) we get for the effective action \( I(a) \) the following relation:

\[
\frac{\partial I(a)}{\partial a} = G_{TT} \beta_a = -e^{-a} a Z(0) \tag{62}
\]

which can be trivially integrated to:

\[
I(a) = (1 + a) Z(a) = (1 + a) e^{-a} Z(0) \tag{63}
\]

For a formal derivation of (54) see [45]. Very briefly the proof is as follows. Let us define a family of two dimensional field theories

\[
L = L_0 + \lambda_i u^i(\xi) \tag{55}
\]

parametrized by \( \lambda_i \). The generating functional \( Z(\lambda_1 \ldots \lambda_n) \) can be expanded in powers of \( \lambda \). At order \( N \) we have

\[
Z^N = \int d^2 \xi_1 \ldots d^2 \xi_N < u_{n_1}(\xi_1) \ldots u_{n_N}(\xi_N) > \lambda_1 \ldots \lambda_n \tag{56}
\]

Using the OPE we get the logaritmic contribution:

\[
Z^n = \int d^2 \xi_1 \ldots d^2 \xi_N d^2 \xi f_{n_1 n_2 m} \frac{1}{|\xi|^2} \lambda_1 \ldots \lambda_n < u_m(\xi) u_{n_3}(\xi_3) \ldots u_{n_N}(\xi_N) > \tag{57}
\]

from (57) we read the beta function \( \beta_m \):

\[
\beta_m = \frac{d \lambda_m}{d \log \Lambda} = f_{mn_1 n_2} \lambda_{n_1} \lambda_{n_2} \tag{58}
\]

for \( \lambda_m^R = \lambda_m^B + f_{mn_1 n_2} \lambda_{n_1} \lambda_{n_2} \log \Lambda \) with \( \Lambda \) the ultraviolet cutoff in the integration (57). Defining now the effective action:

\[
\Gamma(\lambda) = \sum \lambda^i_1 \ldots \lambda^i_N < u_{i_1} \ldots u_{i_N} > \tag{59}
\]

we get

\[
\frac{\partial \Gamma(\lambda)}{\partial \lambda_m} = \sum \lambda^i_1 \ldots \lambda^i_N C^{i_1 \ldots i_N}_{m} < u_m u_e > = \sum \beta_e G_{me} \tag{60}
\]

where we use a generalized OPE and expression (58) for the beta functions. In this section we will use (60) to define the effective action.
This is a extremely interesting result since it defines a non trivial potential for the tachyon constant $a$ (figure 5), namely

$$V(a) = (1 + a)e^{-a} \quad (64)$$

This potential have two extremal points at $a = \infty$ and $a = 0$.

![Figure 5: Open string Tachyon potential](image)

The interpretation of the two extremal points in $V(a)$ is by no means obvious. The extremal point $a = 0$ is the standard open string vacua with vanishing expectation value of the open tachyon. It is a maximum reflecting the existence of open tachyons in the string spectrum. The extremal point $a = \infty$ is a bit more mysterious since apparently it describes a stable vacua (up to tunneling processes to $a = -\infty$) of the open string in flat Minkowski space time and without open tachyons. What is the physical meaning of this vacua?

### 6.2 The effective action

Next we will consider, following ref [18, 19, 20, 21, 22, 23], a slightly more complicated action:

$$S(a, u_i) = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2 \sigma \sqrt{h} h^{\alpha\beta} \partial_{\alpha} x^\mu \partial_{\beta} x^\nu \eta_{\mu\nu} + \int_{\partial\Sigma} d\theta \sqrt{h} T(x) \quad (65)$$

with

$$T(x) = a + \sum u_i x_i^2 \quad (66)$$
Identifying, as usual, the ultraviolet cutoff with the world sheet Weyl factor we get at one loop in the \( \sigma \)-model

\[
\Lambda[a + u_i \alpha' \log \Lambda]
\]

from which we derive the beta function \( \beta_a \):

\[
\beta_a = -a - \sum_i \alpha' u_i
\]

At this point we are interpreting \( x_i \) in (66) as representing quantum fluctuations i.e. \( \alpha' u_i = \frac{\partial^2 T}{\partial x_0 \partial x_0} \) and \( T(x_0) = a \) for some background \( x_0 \). Thus we should replace in (63) \( u_i \) by \( u_i \alpha' \).

In addition to \( \beta_a \) we have at tree level

\[
\Lambda[\alpha' u_i x_i^2]
\]

which implies a beta function

\[
\beta_{u_i} = -u_i
\]

Using these tools we can define the effective action by:

\[
dI = \frac{\partial I}{\partial a} da + \frac{\partial I}{\partial u_i} du_i
\]

with

\[
\frac{\partial I}{\partial a} = G_{aa} \beta_a + G_{au_i} \beta_{u_i}
\]

\[
\frac{\partial I}{\partial u_i} = G_{u_i u_j} \beta_{u_j} + G_{u_i a} \beta_a
\]

where the “metric” factors are defined by:

\[
G_{aa} = \int_0^{2\pi} d\theta < 1_d, 1_d >_{\text{disc}}
\]

\[
G_{au_i} = \int_0^{2\pi} d\theta < 1_d, x_i^2 >_{\text{disc}}
\]

\[
G_{u_i u_j} = \int_0^{2\pi} d\theta < x_i^2, x_j^2 >_{\text{disc}}
\]
In terms of the partition function $Z(a, u_i)$

$$Z(a, u_i) = \int dx e^{-S(a, u_i)}$$  \hspace{1cm} (77)

we get from (71)-(72)

$$dI = d(\sum \alpha' u_i Z - \sum u_j \frac{\partial Z}{\partial u_j} + (1 + a) Z)$$  \hspace{1cm} (78)

where we have used:

$$G_{au_i} = \frac{\partial Z}{\partial u_i}$$  \hspace{1cm} (79)

$$G_{u_i u_j} = \frac{\partial^2 Z}{\partial u_i \partial u_j}$$  \hspace{1cm} (80)

Integrating (78) we get:

$$I = (\sum \alpha' u_i - \sum u_j \frac{\partial}{\partial u_j} + (1 + a))Z(a, u_i)$$  \hspace{1cm} (81)

as the definition of the effective action. In this formal derivation we have assumed complete decoupling of ghosts. Notice that the contribution $1 + a + \sum \alpha' u_i$ comes directly from the beta function $\beta_a$ defined in (68) while the contribution $\sum u_j \frac{\partial}{\partial u_j}$ comes from the $\beta_{u_i}$ defined in (70). We can rewrite (81) in a more compact way as:

$$I = (1 + \beta_a \frac{\partial}{\partial a} + \sum \beta_{u_i} \frac{\partial}{\partial u_i})Z(a, u_i)$$  \hspace{1cm} (82)

where we have used $Z(a, u_i) = e^{a\tilde{Z}(u_i)}$.

The next step is to compute $Z(a, u_i)$. In order to do that we need the Green function on the disc satisfying the boundary conditions

$$n_\alpha \partial^a x^i + u_i x^i = 0$$  \hspace{1cm} (83)

on $\partial \Sigma$ with $n_\alpha$ a normal vector to the boundary. This Green function is given by:

$$G(z, w) = -\log |z - w|^2 - \log |1 - z\bar{w}|^2 + \frac{2}{u} - 2u \sum \frac{1}{k(k + u)}((z\bar{w})^k + (\bar{z}w)^k)$$  \hspace{1cm} (84)
Integrating
\[ \frac{\partial Z}{\partial u_i} = \int_0^{2\pi} d\theta < x_i^2 >_{disc} \]  \hspace{1cm} (85)

and using
\[ < x_i^2 > = \lim_{\epsilon \to 0} G'_{R}(\theta, \theta + \epsilon) \]  \hspace{1cm} (86)

for the renormalized Green function
\[ G'_{R}(\theta, \theta) = \frac{2}{u} - 4u \sum_k \frac{1}{k(k + u)} \]  \hspace{1cm} (87)

we get:
\[ Z(a, u_i) = e^{-a} \prod_i \sqrt{\alpha' u_i} \Gamma(\alpha' u_i) e^{\gamma \alpha' u_i} \]  \hspace{1cm} (88)

for \( \gamma \) the Euler constant. For small \( u_i \) we can approximate (88) by:
\[ Z(a, u_i) \sim e^{-a} \prod_i \frac{1}{\sqrt{\alpha' u_i}} \quad u_i \to 0 \]  \hspace{1cm} (89)

In this limit we get from (82):
\[ I(a, u_i) \sim (1 + a)e^{-a} \prod_i \frac{1}{\sqrt{\alpha' u_i}} + \alpha'(\sum u_i)e^{-a} \prod_i \frac{1}{\sqrt{\alpha' u_i}} + \ldots \]  \hspace{1cm} (90)

We can now compare the first term with:
\[ T_{25} \int d^{26}x(1 + T)e^{-T} \]  \hspace{1cm} (91)

for \( T = a + \sum u_i x_i^2 \), obtaining the well known result on the filling brane tension
\[ T_{25} = \frac{1}{(2\pi \alpha')^{13}} \]  \hspace{1cm} (92)

Next terms in (90) correspond to the kinetic term for the open tachyon
\[ T_{25} \int d^{26}x e^{-T} \partial T \partial T \]  \hspace{1cm} (93)

In order to define a potential we can change variables
\[ T \to \Phi = 2e^{-\frac{T}{2}} \]  \hspace{1cm} (94)
In these new variables the tachyon lagrangian becomes:

\[ S = T_{25} \int d^{26}x [\alpha' \partial \Phi \partial \Phi + V(\Phi)] \]  

(95)

with (see figure 6)

\[ V(\Phi) = \frac{\Phi^2}{4} (1 - \log \frac{\Phi^2}{4}) \]  

(96)

The extremal corresponding to \( T = \infty \) is \( \Phi = 0 \). The effective mass of the tachyon around this extremal is:

\[ m^2 = \frac{\partial^2 V(\Phi)}{\partial \Phi \partial \Phi} |_{\Phi=0} = \infty \]  

(97)

The extremal \( T = 0 \) i.e. \( \Phi = 2 \) is a maximum reproducing the standard open tachyon mass

\[ m^2 = \frac{\partial^2 V(\Phi)}{\partial \Phi \partial \Phi} |_{\Phi=2} = -\frac{1}{\alpha'} \]  

(98)

![Figure 6: Open string Tachyon potential V(\Phi)](image)

As we see for equation (97) open tachyon condensation at \( T = \infty \) induces an infinite mass for the open tachyon. Using the string mass formula (1) we can interpret this as an effective normal ordering constant \( a = -\infty \). If we do that the dominant contribution to the BRST charge (5) is just the comohologicaly trivial BRST charge \( Q = c_0 \).

This heuristic argument indicates in agreement with Sen’s conjecture that no open string degrees of freedom survive once the tachyon condenses to \( T = \infty \). In summary we can
interpret the vacuum defined by the \( T = \infty \) condensate as the closed string vacua. The closed string tachyon can be interpreted as associated with the quantum unstability due to tunneling processes from \( \Phi = 0 \) to \( \Phi = \infty \).

### 6.3 Non-critical dimension and tachyon condensation

The space time lagrangian for the open tachyon is given by:

\[
S = T_{25} \int d^{26}xe^{-T}[\alpha'\partial T\partial T + (1 + T)]
\]  

(99)

The corresponding equation of motion is

\[
2\alpha'\partial^\mu\partial_\mu T - \alpha'\partial^\mu T\partial_\mu T + T = 0
\]  

(100)

A soliton solution for the equation (100) is given by:

\[
T(x) = a + \sum u_i x_i^2
\]  

(101)

with \( u_i = \frac{1}{4\alpha'} \) or \( u_i = 0 \) and \( a = -n \) for \( n \) the number of non vanishing \( u_i \)'s. In terms of the field \( \Phi \) defined in (94) the profile of the solution looks like the one depicted in figure 7.

![Figure 7: Soliton shape](image)

This can be interpreted in a first approximation as \( D - (25 - n) \) soliton brane.
In principle we can try to play the same game but including the effect of a non trivial
dilaton. The simplest example will be of course to work with non critical dimension $n$ and
a linear dilaton background

$$\Phi = qy$$  \hspace{1cm} (102)

with $q = \sqrt{\frac{n-26}{6 \alpha'}}$. Inspired by the Liouville picture of non critical strings we take the linear
dilaton depending only on one coordinate $y$. The lagrangian including the effect of the
dilaton would be most like:

$$S = T_{25} \int d^{26}x e^{-\Phi} e^{-T[\alpha' \partial T \partial T + (1 + T)]}$$  \hspace{1cm} (103)

The equation of motion becomes:

$$2\alpha' \partial^\mu \partial_\mu T - \alpha' \partial^\mu \partial_\mu T - \alpha' \partial^\mu \Phi \partial_\mu T + T = 0$$  \hspace{1cm} (104)

As solution we can try

$$T(x) = a + \sum u_i x_i^2 \hspace{0.5cm} a = -m \hspace{0.5cm} u_i = \frac{1}{4\alpha'} \hspace{0.5cm} i = 1...m$$  \hspace{1cm} (105)

with $u_y = 0$. This soliton defines a $D - (n - m - 1)$-brane that extends on the “Liouville”-
direction. Notice that we have not soliton solutions for $u_y \neq 0$ which seem to imply that
tachyon condensation is not taking place in the Liouville direction. This lead us to suggest
the following conjecture: In non critical open strings open tachyon condensation can not
take place in the Liouville direction.

A trivial corolary of the previous conjecture is that in space-time dimension equal two
tachyon condensation does not take place, which is consistent with the fact that tachyons
in $D = 2$ with the linear dilaton turn on are not real tachyons.

7 D-branes, tachyon condensation and K-theory
7.1 Extended objects and topological stability

Let us start considering a gauge theory with a Higgs field $\Phi$:

$$L = L_0(A^\mu, \Phi) + V(\Phi)$$  \hspace{1cm} (106)

for some Higgs potential $V(\Phi)$. Necessary conditions for the existence of topologically stable extended objects of space codimension $p$ is the non triviality of the homotopy group

$$\Pi_{p-1}(V)$$  \hspace{1cm} (107)

for $V$ the manifold of classical vacua of lagrangian (106).

In fact for an extended object of codimension $p$ the condition of finite density of energy implies that at the infinity region in the transversal directions -whose topology is $S^{p-1}$- the field configuration must belong to the vacuum manifold $V$. Hence we associate with each configuration of finite density of energy a map

$$\Psi : S^{p-1} \to V$$  \hspace{1cm} (108)

whose topological classification is defined by the homotopy group (107).

The simplest example of vacuum manifold corresponding to spontaneous breaking of symmetry $G \to H$ is the homogeneous space

$$V = G/H$$  \hspace{1cm} (109)

So 'tHooft-Polyakov monopole for instance is defined for $G = SU(2)$ and $H = U(1)$ by the topological condition $\Pi_2(G/H) = Z$ which coincide with its magnetic charge.

7.2 A gauge theory analog for D-branes in type II strings

We know that in type II strings we have extended objects which are RR charged and stable, namely the D-branes. For type IIA we have $D_p$-branes with $p$ even and for type
IIB $D_p$-branes with $p$ odd. Since we are working in critical 10 dimensional space-time the space codimension of those $D_p$-branes is odd $2k + 1$ for type IIA and even $2k$ for type IIB.

We will consider now the following formal problem. Try to get two gauge Higgs lagrangians $L^{IIA(IIB)}(A_\mu, \Phi)$ such that it can be stabilized a one to one map between type II D-branes and topological stable extended objects for those lagrangians in the sense defined in previous section. We will denote this formal gauge theory the gauge theory analog of the type II strings.

Of course the hint for answering this question is Sen’s tachyon condensation conjecture for type II strings. We will present first this construction in the case of type IIB strings.

### 7.2.1 Sen’s conjecture for type IIB

In type IIB strings we have well defined $D_9$ filling branes. Since they are charged under the RR sector we can define the corresponding $D_9$-antibranes. As it is well known the low energy physics on the world volume of a set of $N$ $D_9$-branes is a $U(N)$ gauge theory without open tachyons. In fact the open tachyon is projected out by the standard GSO projection

$$ (-1)^F = +1 $$

for $F$ the world sheet fermion number operator. The situation changed when we consider $N$ $D_9$-branes and $N$ $D_9$-antibranes. In this case the theory on the world volume is $U(N) \times U(N)$ and not $U(2N)$ due to the fact that the GSO projection on open string states with end points at a $D_9$-brane and a $D_9$-antibrane is the opposite, namely

$$ (-1)^F = -1 $$

This projection eliminates from the spectrum the massless gauge vector bosons that will enhance the $U(N) \times U(N)$ gauge symmetry to $U(2N)$. In addition the projection (111) is not killing the tachyon in the $(9, \bar{9})$ and $(\bar{9}, 9)$ open string sectors. Thus the gauge theory associated with the configuration of $N$ $D_9$-branes and $N$ $D_9$-antibranes is a
$U(N) \times U(N)$ gauge theory with a Higgs field, the open tachyon, in the bifundamental $(N, \bar{N})$ representation.

This gauge theory will be a natural starting point for defining the gauge analog model in the case of type IIB strings.

Of course in order to get a rigorous criterion for the topological stability of extended objects in this gauge theory we need to know the potential for the open tachyon. This potential is something that at this point we don’t know how to calculate in a rigorous way. However we can assume that a tachyon condensation is generated with a vacuum expectation value

$$< T > = T_0$$

(112)

with $T_0$ diagonal and with equal eigenvalues. If this condensate takes place then the vacuum manifold is simply

$$V = \frac{U(N) \times U(N)}{U_D(N)} \sim U(N)$$

(113)

Thus the condition for topological stability for extended objects of space codimension $2k$ will be:

$$\Pi_{2k-1}(U(N)) \neq 0$$

(114)

which by Bott periodicity theorem:

$$\Pi_j(U(k)) = \begin{cases} Z & \text{j odd } j < 2k \\ 0 & \text{j even } j < 2k \end{cases}$$

(115)

we know is the case for big enough $N$.

The simplest example will be to take $k = 1$ corresponding to the extended object of the type of a $D_7$-brane. The condition of finite energy density defines a map from $S^1$ into $U(N)$. For just one pair of $D_9 - D_9$ configuration we get $\Pi_1(U(1)) = Z$, with this winding number representing the “magnetic charge” of the $D_7$-brane that looks topologically like a vortex line (see figure 8).
If we go to the following brane namely the $D_5$-brane we have $k = 2$ and we need a non vanishing homotopy group $\Pi_3(U(N))$. The minimum $N$ for which this is possible according to (115) is $N = 2$ i.e. two pairs of $D_9 - \bar{D}_9$-branes. We can understand what is happening in two steps. First of all we get a configuration of two $D_7$-branes and from this the $D_5$-brane.

For $k = 3$ we need non vanishing homotopy group $\Pi_5(U(N))$. The natural $N$ we should choose is dictated by the step construction namely $N = 4$. In general for codimension $2k$ we will consider a gauge group $U(2^{k-1})$.

### 7.3 K-theory version of Sen’s conjecture

The configuration of $D_q - \bar{D}_q$-branes naturally defines a couple of $U(N)$ vector bundles $(E, F)$ on space time. The main idea of Sen’s tachyon condensation is that a configuration characterized by the couple of vector bundles $(E, E)$ with a topologically trivial tachyon field configuration decays into the closed string vacua for type IIB string theory. This is exactly the same type of phenomena we have discussed in section 7 for the bosonic string. The phenomena naturally leads to consider, as far as we are concerned with $D$-brane charges, instead of the couple of bundles $(E, F)$ the equivalence class defined by (31)

\[(E, F) \sim (E \oplus G, F \oplus G)\]  

(116)
for $\oplus$ the direct sum of bundles. This is precisely the definition of the K-group of vector bundles on the space-time $X$, $K(X)$.

Let us here recall that the space $A = V_{cc}(X)$ of vector bundles on $X$ is a semigroup with respect to the operation of direct sum. The way to associate with $A$ a group $K(A)$ is as the quotient space in $A \times A$ defined by the equivalence relation

$$(m, n) \sim (m', n')$$

if $\exists p$ such that $m + n' + p = n + m' + p$ which is precisely what we are doing in the definition (116).

A different but equivalent way to define $K(A)$ for $A = V_{cc}(X)$ is as the set of equivalence classes in $V_{cc}(X)$ defined by the equivalence relation:

$$E \sim F \text{ if } \exists G : E \oplus G = F \oplus G$$

where the “=” means isomorphism.

It is convenient for our purposes to work with the reduced K-group $\tilde{K}(X)$ which is defined by

$$Ker[K(X) \to K(p)]$$

for $p$ a point in $X$. Notice that $K(p)$ is just the group of integer numbers $\mathbb{Z}$. This is the group naturally associated with the semigroup $V_{cc}(p) = N$ where $N$ here parametrizes the different dimensions of the vector bundles in $V_{cc}(p)$.

In order to characterize $D_p$-branes in type IIB in terms of K-theory we will need to consider the group $K(X, Y)$. We will consider $X$ a compact space and $Y$ also compact and contained in $X$.

In order to define $K(X, Y)$ we will use triplets $(E, F, \alpha)$ where $E$ and $F$ are vector bundles on $X$ and where $\alpha$ is an isomorphism:

$$\alpha : E \mid_Y \to F \mid_Y$$
of the vector bundles $E$ and $F$ reduced to the subspace $Y$.

The definition of $K(X, Y)$ requires to define elementary triplets. An elementary triplet is given by $(E, F, \alpha)$ with $E = F$ and $\alpha$ homotopic to the identity in the space of automorphisms of $E|_Y$. Once we have defined elementary triplets the equivalence relation defining $K(X, Y)$ is:

$$\sigma = (E, F, \alpha) \quad \sigma' = (E', F', \alpha')$$

$$\sigma \sim \sigma' \text{ iff } \exists \text{ elementary triplets } \tau \text{ and } \tau' \text{ such that }$$

$$\sigma + \tau = \sigma' + \tau'$$

where

$$\sigma + \tau = (E \oplus G, F \oplus G, \alpha \oplus 1_d)$$

Once we have defined $K(X, Y)$ we can try to put in this language the topological characterization of a $D_p$-brane of space codimension $2k$. Namely we will take as $Y$ the “boundary” region in transversal directions i.e. $S^{2k-1}$. As space $X$ we will take the ball $B^{2k}$. The tachyon field transforming in the bifundamental representation will define on $S^{2k-1}$ an isomorphism between the two vector bundles $E, F$ defined by the starting configurations of $D_9-D_{\bar{9}}$-branes. Finally the homotopy class of this map will define the charge of the $D_q$-brane of space codimension $2k$. The K-group we define in this way is

$$K(B^{2k}, S^{2k-1})$$

Now we can use the well known relation:

$$K(B^{2k}, S^{2k-1}) = \tilde{K}(B^{2k}/S^{2k-1})$$

where $X/Y$ is defined by contracting $Y$ to a point.

It is easy to see that

$$B^{2k}/S^{2k-1} \sim S^{2k}$$
Thus we can associate with type IIB $D_p$-branes of space codimension $2k$ elements in $\tilde{K}(S^{2k})$.

In order to define the tachyon field in this case we will specify the isomorphism $\alpha$. For codimension $2k$ let us consider the $2^k \times 2^k$ gamma matrices $\Gamma_i$ ($i = 0 \ldots 2k$). Let $v$ a vector in $C^{2k}$. The isomorphism $\alpha$ is defined by

$$\alpha(x, v) = (x, x_i \Gamma^i(v))$$

for $x \in S^{2k-1}$. The tachyon field is defined by

$$T(x) \big|_{x \in S^{2k-1}} = x_i \Gamma^i$$

7.4 Type IIA strings

Next we will define a gauge analog for type IIA strings. The gauge-Higgs lagrangian will be defined in terms of a configuration of $D_9$-branes. For type IIA $D_9$-branes are not BPS and therefore they are unstable. The manifestation of this unstability is the existence of an open tachyon field $T$ transforming in the adjoint representation. The gauge group for a configuration of $N$ $D_9$-branes is $U(N)$. Notice that in the case of the type IIA we can not use $D_9$-antibranes since for type IIA $D_9$-branes are not RR-charged.

We can now follow the same steps that in the type IIB case, namely to look for a tachyon potential and to compute the even homotopy groups of the corresponding vacuum manifold. Instead of doing that we will approach the problem from a different point of view, interpreting the D-branes of type IIA as topological defects associated with the gauge fixing topology. In order to describe this approach we need first to review some know facts about gauge fixing topology for non abelian gauge theories.

7.4.1 ’tHooft’s abelian projection

An important issue in the quantization of non abelian gauge theories is to fix the gauge. By an unitary gauge we means a procedure to parametrize the space of gauge “orbits” i.e.
the space of physical configurations

\[ R/G \]  \hspace{1cm} (129)

for \( R \) the total space of field configurations, in terms of physical degrees of freedom where by that we mean those that contribute to the unitary S-matrix. This in particular means a ghost free gauge fixing.

In reference \[40\] 'tHooft suggested a way to fix the non abelian gauge invariance in a unitary way. This type of gauge fixing known as “abelian projection” reduce the physical degrees of freedom to a set of \( U(1) \) photons and electrically charged vector bosons.

In addition to these particles there is an extra set of dynamical degrees of freedom we need to include in order to have a complete description of the non abelian gauge theory. These extra degrees of freedom are magnetic monopoles that appear as a consequence of the topology of the gauge fixing.

More precisely let \( X \) be a field transforming in the adjoint representation

\[ X \rightarrow gXg^{-1} \]  \hspace{1cm} (130)

The field \( X \) can be a functional \( X(A) \) of the gauge field \( A \) or some extra field in the theory. The way to fix the gauge is to impose \( X \) to be diagonal

\[
X = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_N
\end{pmatrix}
\]  \hspace{1cm} (131)

The residual gauge invariance for a \( U(N) \) gauge theory is \( U(1)^N \) i.e. gauge transformations of the type

\[
g = \begin{pmatrix}
e^{i\alpha_1} \\
\vdots \\
e^{i\alpha_N}
\end{pmatrix}
\]  \hspace{1cm} (132)

The degrees of freedom of this gauge are:
• $N$-U(1) photons
• $\frac{1}{2} N(N-1)$ charged vector bosons
• $N$ scalars fields $\lambda_i$

Gauge fixing singularities will appear whenever two eigenvalues coincide

$$\lambda_i = \lambda_{i+1}$$ \hspace{1cm} (133)

Notice that we can fix the gauge imposing $X$ to be diagonal and that $\lambda_i > \lambda_{i+1} > \lambda_{i+2} > \ldots$. What is the physical meaning of these gauge fixing singularities?

First of all it is easy to see that generically these gauge fixing singularities have codimension 3 in space. In particular this means that if we are working in four dimensional space time they behave as pointlike particles.

Secondly if we consider the field $X$ in a close neighborhood of the singular point before gauge fixing

$$X = \begin{pmatrix}
D_1 & 0 & 0 \\
0 & \lambda + \epsilon_3 & \epsilon_1 - i\epsilon_2 \\
0 & \epsilon_1 + i\epsilon_2 & \lambda - \epsilon_3 \\
0 & 0 & D_2
\end{pmatrix}$$ \hspace{1cm} (134)

we can write the small two by two matrix in (134) as:

$$X = \lambda 1_d + \epsilon_i \sigma_i$$ \hspace{1cm} (135)

for $\sigma_i$ the Pauli matrices. The field $\epsilon(x)$ is equal to zero at the singular point and in a close neighborhood can be written as:

$$\epsilon(x) = \sum_{i=1}^{3} x_i \sigma_i$$ \hspace{1cm} (136)

We can easily relate this field to a magnetic monopole. In fact let us consider $S^2$ in $R^3$ and let us define the field on $S^2$:

$$X(x) \mid_{x \in S^2} = \sum_{i=1}^{3} x_i \sigma_i$$ \hspace{1cm} (137)
Clearly \( X^2(x) \mid_{x \in S^2} = 1_d \), thus we can define the projector:

\[
\Pi_\pm = \frac{1}{2}(1 \pm X(x))
\] (138)

The trivial bundle \( S^2 \times C^2 \) decomposes into:

\[
S^2 \times C^2 = E_+ \oplus E_-
\] (139)

where the line bundles \( E_\pm \) are defined by the action of the projection \( \Pi_\pm \) on \( C^2 \). The associated principal bundle to \( E_+, E_- \) define the magnetic monopoles.

In summary the gauge fixing singularities of gauge (131) corresponding to two equal eigenvalues should be interpreted as pointlike magnetically charged particles. It is important to stress that the existence of these magnetic monopoles is completely independent of being in a Higgs or confinement phase.

7.4.2 The \( D_6 \)-brane

Here we will repeat the discussion in 7.4.1 but for the \( U(N) \) gauge theory defined by a configuration of \( D_9 \) unstable filling branes. We will use the open tachyon field transforming in the adjoint representation to fix the gauge. By imposing \( T \) to be diagonal we reduce the theory to pure abelian degrees of freedom in addition to magnetically charged objects of space codimension 3 that very likely can be identified with \( D_6 \)-branes.

Using expression (137) and replacing the \( X \) field by the open tachyon we find that in the close neighborhood of a codimension 3 singular region the tachyon field is represented by:

\[
T(x) \mid_{x \in S^2} = \sum_{i=1}^{3} x_i \sigma_i
\] (140)

which is precisely the representation of the tachyon field around a \( D_6 \)-brane suggested in reference [32].
7.4.3 K-theory description

The data we can naturally associate with a configuration of $D_9$-branes in type IIA is a couple $(E, T)$ with $E$ a vector bundle and $T$ the open tachyon field. We will translate these data into a more mathematical language using the higher K-group $K^{-1}(X)$ [31, 32].

In order to define $K^{-1}(X)$ we will start with couple $(E, \alpha)$ with $E$ a vector bundle on $X$ and $\alpha$ an automorphism of $E$. As we did in the definition of $K(X, Y)$ we define elementary pairs $(E, \alpha)$ if $\alpha$ is homotopic to the identity within automorphisms of $E$. Using elementary pairs $(E, \alpha) = \tau$ we define the equivalence relation

$$\sigma \sim \sigma'$$

iff $\exists \tau, \tau'$ elementary such that

$$\sigma \oplus \tau = \sigma' \oplus \tau'$$

We can now define $K^{-1}(X, Y)$ as pairs $(E, \alpha) \in K^{-1}(X)$ such that $\alpha|_Y = 1_d$.

As before we will use the tachyon field $T$ to define the automorphism $\alpha$. In codimension 3 we got in the previous section that

$$T(x)|_{x \in S^2} = \sum_{i=1}^{3} x_i \sigma_i$$

Clearly $T^2(x)|_{x \in S^2} = 1_d$ and therefore if we define

$$\alpha = e^{iT}$$

and we identify $Y = S^2$ we get the condition

$$\alpha|_Y = 1_d$$

used in the definition of $K^{-1}(X, Y)$. Thus we associate the $D_p$-branes of codimension $2k+1$ with elements in

$$K^{-1}(B^{2k+1}, S^{2k})$$
Using again the relation
\[ K^{-1}(B^{2k+1}, S^{2k}) = \tilde{K}^{-1}(B^{2k+1}/S^{2k}) \] (147)
and
\[ \tilde{K}^{-1}(X) = \tilde{K}(SX) \] (148)
for SX the reduced suspension of X (in particular SS^n = S^{n+1}) we conclude that D_p-branes in type IIA are associated with
\[ \tilde{K}^{-1}(S^{2k+1}) = \tilde{K}(S^{2k+2}) \] (149)

The reader can wonder at this point in what sense to work with K-theory is relevant for this analysis. The simplest answer comes from remembering the group structure of \( K^{-1}(X) \).

The group structure of in \( K^{-1}(X) \) is associated with the definition of inverse. Namely the inverse of \((E, \alpha)\) is \((E, \alpha^{-1})\). The reason is that
\[ (E, \alpha) \oplus (E, \alpha^{-1}) = (E \oplus E, \alpha \oplus \alpha^{-1}) \] (150)
where
\[ \alpha \oplus \alpha^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \] (151)

Now there is a homotopy transforming matrix \([151]\) into the identity
\[
\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} (t) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix}
\] (152)
such that
\[
\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} (t = 1) = \begin{pmatrix} \alpha \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = 1_d
\] (153)

What this homotopy means is again Sen’s tachyon condensation conjecture. In fact if we associate a \( D_6 - D_6 \) brane configuration with a matrix \( T \) with two pairs of eigenvalues...
(λ_i = λ_{i+1}) and (λ_j = λ_{j+1}) equal. This configuration is - because of homotopy (152) - topologically equivalent to the vacuum. We see once more how Sen’s condensation is at the core of the K-theory promotion of V_{ec}(X) from a semigroup into a group.

8 Some final comments on gauge theories

The data associated with a gauge theory and a unitary gauge fixing of the type used in the abelian projection can be summarized in the same type of couples used to define the higher K-group \( K^{-1}(X) \), namely a \( U(N) \) vector bundle \( E \) and an automorphism \( \alpha \). In this sense gauge fixing topology is translated into the homotopy class of \( \alpha \) within the automorphisms of \( E \). In standard four dimensional gauge theories the gauge fixing topology is described in terms of magnetic monopoles and antimonopoles. In principle we have different types of magnetic monopoles charged with respect to the different \( U(1) \)'s in the Cartan subalgebra. The group theory meaning of \( K^{-1}(X) \) is reproduced, at the gauge theory level, by the homotopy (152) that is telling us that monopole-antimonopole pairs, although charged with respect to different \( U(1) \)'s in the Cartan subalgebra, annihilate into the vacuum, very much in the same way as, by Sen’s tachyon condensation, a pair of brane-antibrane decay into the vacuum.

In what we have denoted the gauge theory analog of type II strings, namely a gauge-Higgs lagrangian with topologically stable extended objects in one to one correspondence with type II stable \( D_p \)-branes, it is apparently absent one important dynamical aspect of \( D \)-filling brane configurations. In fact in the case of unstable filling branes the decay into the vacuum comes together with the process of cancelation of the filling brane tension and thus with the “confinement” of “electric” open string degrees of freedom. The resulting state is a closed string vacua with stable \( D_p \)-branes that are sources of RR fields which are part of the the closed string spectrum. The dynamics we lack in the gauge theory analog...
is on one side the equivalent of the confinement of open string degrees of freedom and on
the other side the RR closed string interpretation of the dual field created by the \( D_p \)-brane
topological defects. Very likely the gauge theory interpretation of this two phenomena can
shed some light on the quark confinement problem.

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\[6\] i.e. the concept of a filling brane tension which is intimately related to the open-closed relation in
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