The Kobayashi-Maskawa Parametrization of Lepton Flavor Mixing and Its Application to Neutrino Oscillations in Matter

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Abstract

We show that the Kobayashi-Maskawa (KM) parametrization of the $3 \times 3$ lepton flavor mixing matrix is a useful language to describe the phenomenology of neutrino oscillations. In particular, it provides us with a convenient way to link the genuine flavor mixing parameters ($\theta_1$, $\theta_2$, $\theta_3$ and $\delta_{KM}$) to their effective counterparts in matter ($\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{\theta}_3$ and $\tilde{\delta}_{KM}$). We rediscover the Toshev-like relation $\sin \tilde{\delta}_{KM} \sin 2\tilde{\theta}_2 = \sin \delta_{KM} \sin 2\theta_2$ in the KM parametrization. We make reasonable analytical approximations to the exact relations between the genuine and matter-corrected flavor mixing parameters in two different experimental scenarios: (a) the neutrino beam energy $E$ is above $O(1)$ GeV and (b) $E$ is below $O(1)$ GeV. As an example, the probability of $\nu_\mu \rightarrow \nu_e$ oscillations and CP-violating effects are calculated for the upcoming NO$\nu$A and Hyper-K experiments.

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1 Introduction

Just like quark flavor mixing, lepton flavor mixing has been observed in a number of neutrino oscillation experiments [1]. The phenomenon of lepton flavor mixing is described by a $3 \times 3$ unitary matrix $V$, the so-called Maki-Nakawaga-Sakata-Pontecorvo (MNSP) matrix [2]. There are totally 9 different parametrizations of $V$ in terms of the rotation angles and phase angles [3]. Among them, the one advocated by the Particle Data Group (PDG) [1] is most popular in accounting for current neutrino oscillation data, and the Fritzsch-Xing (FX) parametrization [3] has a particular merit in describing the running behaviors of neutrino masses and flavor mixing parameters from one energy scale to another by means of the one-loop renormalization-group equations [4]. Is the original Kobayashi-Maskawa (KM) parametrization [5] advantageous to the description of neutrino phenomenology? We shall give an affirmative answer to this question in the present work.

In fact, it has recently been noticed that the KM parametrization of the $3 \times 3$ quark flavor mixing matrix is very useful to link current experimental data to the unitarity triangles—the so-called “Unitarity boomerang” [6]: $\delta_{\text{KM}} \simeq \alpha_{\text{UT}} \simeq 90^\circ$, where $\delta_{\text{KM}}$ is the CP-violating phase in the KM parametrization and $\alpha_{\text{UT}}$ is one of the inner angles of the KM unitarity triangles. A similar relationship was found earlier in the FX parametrization [7]. As the structure of the KM parametrization is partly analogous to that of the FX parametrization and partly analogous to that of the PDG parametrization, we naturally expect that it should also be useful to describe the salient features of neutrino oscillations in vacuum and in matter. In other words, we expect that the KM parametrization can provide us with a simple and convenient link between its parameters and the observable quantities of neutrino oscillations. The main purpose of this paper is just to demonstrate our expectation and offer an alternative description of lepton flavor mixing for both the phenomenology of neutrino oscillations and the building of neutrino mass models. Needless to say, a convenient parametrization is sometimes possible to make the underlying physics more transparent.

The remaining parts of this work are organized as follows:

(1) In section 2 we shall first establish the explicit relations between the flavor mixing parameters in the KM parametrization ($\theta_1$, $\theta_2$, $\theta_3$ and $\delta_{\text{KM}}$) and those in the PDG parametrization ($\theta_{12}$, $\theta_{13}$, $\theta_{23}$ and $\delta$). Because the Majorana CP-violating phases have nothing to do with neutrino oscillations, they will not be taken into account in this work. We find that $\theta_2 = \theta_{23}$ and $\delta_{\text{KM}} = \delta$ hold exactly if the condition $\cos \delta = \sin \theta_{13} \cot \theta_{12} \cot 2\theta_{23}$ (or equivalently $\cos \delta_{\text{KM}} = \tan \theta_3 \cos \theta_1 \cot 2\theta_2$) is satisfied. Because both $\theta_{13}$ and $\theta_3$ are expected to be small, the above condition seems to hint at $\delta \sim \delta_{\text{KM}} \sim \pm 90^\circ$.

(2) Section 3 is devoted to a detailed calculation of the relations between the genuine KM flavor mixing parameters in vacuum ($\theta_1$, $\theta_2$, $\theta_3$ and $\delta_{\text{KM}}$) and their effective counterparts in matter ($\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{\theta}_3$ and $\tilde{\delta}_{\text{KM}}$), given neutrino oscillations in a constant terrestrial matter profile. The so-called Toshev relation $\sin \tilde{\delta} \sin 2\theta_{13} = \sin \delta \sin 2\theta_{23}$ [8] in the PDG parametrization is rediscovered in the KM parametrization: $\sin \tilde{\delta}_{\text{KM}} \sin 2\tilde{\theta}_2 = \sin \delta_{\text{KM}} \sin 2\theta_2$. This interesting result means that the KM parametrization is definitely useful and convenient to describe the phenomenology of neutrino oscillations. To be more explicit, we make analytical approximations for the relations between the genuine and matter-corrected parameters of lepton flavor mixing in two different experimental scenarios: (a) the neutrino beam energy $E$ is above $\mathcal{O}(1)$ GeV; and (b) $E$ is below $\mathcal{O}(1)$ GeV. The accuracy of each approximation is examined by comparing its results with exact numerical calculations. Such analytical results are phenomenologically useful, just like those obtained previously in the PDG parametrization.

(3) For illustration, we consider $\nu_\mu \rightarrow \nu_e$ oscillations in section 4 and calculate the oscillation
probability by means of the KM parametrization. Our expressions are simple and instructive, and they can be used to analyze the upcoming data from T2K \cite{9}, NOνA \cite{10} and Hyper-K \cite{11} experiments.

(4) A brief summary of this work, together with some concluding remarks, is given in section 5.

2 Comparison between the KM and PDG parametrizations

In the framework of 3-generation leptons, the KM and PDG parametrizations of the MNS matrix are given by

$$V_{(KM)} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta_{KM}} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta_{KM}} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta_{KM}} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta_{KM}} \end{pmatrix},$$

$$V_{(PDG)} = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & s_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{pmatrix}, \quad (1)$$

respectively, where \( s_i = \sin \theta_i, \ c_i = \cos \theta_i, \ s_{ij} = \sin \theta_{ij} \) and \( c_{ij} = \cos \theta_{ij} \) (for \( i = 1, 2, 3 \) and \( ij = 12, 23, 13 \)). Comparing the KM parametrization with the PDG parametrization, one can derive the expressions of \( \theta_i \) and \( \delta_{KM} \) in terms of \( \theta_{ij} \) and \( \delta \):

$$\cos \theta_1 = \cos \theta_{12} \cos \theta_{13},$$

$$\tan \theta_2 = \tan \theta_{23} \left| \frac{1 - \sin \theta_{13} \cot \theta_{12} \cot \theta_{23} e^{i\delta}}{1 + \sin \theta_{13} \cot \theta_{12} \tan \theta_{23} e^{i\delta}} \right|,$$

$$\tan \theta_3 = \tan \theta_{13} \csc \theta_{12},$$

$$\sin \delta_{KM} = \sin \delta \frac{1 + \sin^2 \theta_{13} \cot^2 \theta_{12}}{|(1 - \sin \theta_{13} \cot \theta_{12} \cot \theta_{23} e^{i\delta})(1 + \sin \theta_{13} \cot \theta_{12} \tan \theta_{23} e^{i\delta})|}. \quad (2)$$

With the help of Eq. (2) and the current experimental oscillation data \cite{12}, we find the following properties of \( \theta_1 \) and \( \delta_{KM} \):

- \( \theta_1 \approx \theta_{12} \) is a good approximation because of \( \theta_{13} \lesssim 12^\circ \) as constrained by the present experimental data \cite{12}.

- \( \theta_2 \) depends on not only \( \theta_{23} \) but also \( \theta_{13} \) and \( \delta \). \( \theta_2 \approx \theta_{23} \) is a good approximation if \( \delta \) is near \( \pm 90^\circ \), but not good if \( \delta \) is near \( 0^\circ \) or \( \pm 180^\circ \). If one requires \( \theta_2 = \theta_{23} \) to hold exactly and \( \theta_{13} \neq 0 \), a constraint equation in the PDG parametrization must be satisfied:

$$\cos \delta = \sin \theta_{13} \cot \theta_{12} \cot 2\theta_{23}. \quad (3)$$

- When calculating the effective flavor mixing parameters and neutrino oscillation probabilities in matter, one of ten takes \( \theta_{13} \) as an expansion parameter to make analytical approximations because of its small value (e.g., \cite{13}). Since \( \tan \theta_3 = \tan \theta_{13} \csc \theta_{12} \) holds, we can also treat \( \theta_3 \) as a small parameter to do series expansions.

- The experimental data yield \( \theta_{23} \approx 45^\circ \) \cite{12}, leading to \( \sin \delta_{KM} \approx \sin \delta + O(s_{13}^2) \) from Eq. (2). So \( \delta_{KM} \approx \delta \) is a good approximation. Under the condition of Eq. (3), \( \delta_{KM} = \delta \) holds exactly.
We remark that Eq. (3) is a useful constraint that leads exactly to both \( \theta_2 = \theta_{23} \) and \( \delta_{\text{KM}} = \delta \). It can be re-expressed in the KM parametrization:

\[
\cos \delta_{\text{KM}} = \tan \theta_3 \cos \theta_1 \cot \theta_2 .
\]

Moreover, one can see from Eqs. (3) and (4) that \( \theta_2 = \theta_{23} \) is equivalent to \( \delta_{\text{KM}} = \delta \). One special case is the \( \mu-\tau \) flavor symmetry with \( \theta_2 = \theta_{23} = 45^\circ \) and maximal CP violation with \( \delta_{\text{KM}} = \delta = \pm 90^\circ \) \cite{14}. We shall consider this possibility in the next section when discussing matter effects.

We obtain the best-fit values and the 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) ranges of mixing angles \( \theta_1 \), \( \theta_2 \) and \( \theta_3 \) for both the normal hierarchy (NH) and inverted hierarchy (IH) of neutrino masses in Table 1, according to a global analysis of current neutrino data presented in Ref. \cite{12}. Some details of this table should be mentioned. First, we do not list any possible values of the CP phase \( \delta_{\text{KM}} \) because \( \delta_{\text{KM}} \) is not well constrained by current experiments. Ref. \cite{12} gives very loose bounds of \( \delta \), with \( \delta = -110^\circ \)\( (-74^\circ) \) for the NH (IH) at the best fit and \(-227^\circ \) to \(+7^\circ \)\( (-200^\circ \) to \(+43^\circ) \) for the NH (IH) in the 1\( \sigma \) range, without constraints in the 2\( \sigma \) or 3\( \sigma \) ranges. In this case, one may assume the bounds of \( \delta \) as the bounds of \( \delta_{\text{KM}} \) for \( \delta_{\text{KM}} \approx \delta \). Second, because we have little knowledge about \( \delta \), \( \theta_2 \) cannot be well restricted from Eq. (2). Thus we assume \( \delta \) at its best-fit value when we calculate the range of \( \theta_2 \) in Table 1. For a full understanding of the possible values of \( \theta_2 \) and their dependence on \( \delta \), one may see Fig. 1, where we take the NH as an example to show how \( \theta_2 \) changes with \( \delta \).

3 The KM flavor mixing parameters in matter

3.1 General formalism

To understand the phenomenon of neutrino oscillations in the long-baseline experiments, it is necessary to analyze neutrino mixing in matter. In this section, we calculate the effective flavor mixing parameters in a constant terrestrial matter profile \( (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3 \text{ and } \tilde{\delta}_{\text{KM}}) \) and study their relations with the genuine flavor mixing parameters in vacuum \( (\theta_1, \theta_2, \theta_3 \text{ and } \delta_{\text{KM}}) \).

In the flavor basis \( \nu(t) \equiv (\nu_e(t) , \nu_\mu(t) , \nu_\tau(t))^T \), the evolution of neutrinos in matter is described by a Schrödinger-like equation:

\[
i \frac{d}{dt} |\nu(t)\rangle = \tilde{H} |\nu(t)\rangle
\]

with the effective Hamiltonian

\[
\tilde{H} = \frac{\Delta m^2_{31}}{2E} V \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{array} \right) V^\dagger + \left( \begin{array}{ccc} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\]

Here \( V \) is the MNSP matrix, \( \alpha \equiv \Delta m^2_{21}/\Delta m^2_{31} \) is the mass hierarchy parameter with the mass-squared differences \( \Delta m^2_{ij} \equiv m_i^2 - m_j^2 \), and \( A \equiv 2E\alpha/\Delta m^2_{31} \) is a dimensionless variable arising from the matter-induced effective potential \( a \equiv \sqrt{2} G_F N_e \) \cite{15}. \( N_e \) is the number density of electrons in matter, and it can be taken to be half of the number density of nucleons of the Earth. In this case, \( A \) is given by

\[
A \approx 0.085 \left( \frac{2.5 \times 10^{-3} \text{ eV}^2}{\Delta m^2_{31}} \right) \left( \frac{E}{1 \text{ GeV}} \right) \left( \frac{\rho_m}{2.8 \text{ g/cm}^2} \right),
\]
where \( \rho_m \) is the mass density along the path of neutrinos. In most long-baseline neutrino oscillation experiments, \( \rho_m \) is approximately a constant \([9, 10, 11]\). Eq. (6) holds for neutrinos. When considering the evolution of antineutrinos, we have to perform the replacements \( V \rightarrow V^\ast \) and \( a \rightarrow -a \) in the effective Hamiltonian.

In view of Eq. (6), one may define the effective MNSP matrix \( \tilde{V} \) through

\[
\tilde{H} \equiv \frac{\Delta m_{31}^2}{2E} \tilde{V} \left( \begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array} \right) \tilde{V}^\dagger ,
\]

where \( \lambda_i \) are the eigenvalues of the matrix in the square bracket of Eq. (6). The effective matter-corrected mass-squared differences are written as \( \Delta \tilde{m}_{21}^2 = \Delta m_{21}^2 (\lambda_2 - \lambda_1) \) and \( \Delta \tilde{m}_{31}^2 = \Delta m_{31}^2 (\lambda_3 - \lambda_1) \).

To describe the MNSP matrices \( V \) and \( \tilde{V} \) in the KM parametrization, we need three real rotation matrices and one diagonal phase matrix \( O_1, O_2, O_3 \) and \( U_\delta \):

\[
O_1 = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix}, \quad O_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix}, \quad U_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -e^{i\delta_{KM}} \end{pmatrix},
\]

in which \( O_1 \) is a rotation matrix in the (1,2) plane and \( O_2 \) and \( O_3 \) are rotation matrices in the (2,3) plane. Thus \( V \) and \( \tilde{V} \) can be parametrized as

\[
V = O_2 U_\delta O_1 O_3, \quad \tilde{V} = \tilde{O}_2 \tilde{U}_\delta \tilde{O}_1 \tilde{O}_3,
\]

with the flavor mixing parameters in vacuum (\( \theta_1, \theta_2, \theta_3 \) and \( \delta_{KM} \)) and in matter (\( \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3 \) and \( \tilde{\delta}_{KM} \)), respectively.

In the representation of the rotation matrices and phase matrix, Eq. (6) can be rewritten as

\[
\tilde{H} = \frac{\Delta m_{31}^2}{2E} O_2 U_\delta M U_\delta^T O_2^T
\]

with

\[
M = O_1 O_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} O_3^T O_1^T + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Being a real symmetric matrix, \( M \) can be diagonalized by an orthogonal matrix \( \tilde{V} \equiv \tilde{O}_2 \tilde{O}_1 \tilde{O}_3 \) with three rotation angles \( \tilde{\theta}_1, \tilde{\theta}_2, \) and \( \tilde{\theta}_3 \). Thus, \( \tilde{H} \) is diagonalized by \( \tilde{V}' \equiv O_2 U_\delta \tilde{O}_2 \tilde{O}_1 \tilde{O}_3 \). After a phase transformation, one can derive \( \tilde{V} = U' \tilde{V}' = U'O_2 U_\delta \tilde{O}_2 \tilde{O}_1 \tilde{O}_3 \) with an additional unphysical phase matrix \( U' \). Comparing it with Eq. (9), we find the following relations:

\[
\tilde{\theta}_1 = \theta_1, \quad \tilde{\theta}_3 = \theta_3, \quad \tilde{s}_2^2 = c_2^2 s_2^2 + s_2^2 c_2^2 + 2 c_2 s_2 c_2 \tilde{s}_2 \cos \tilde{\delta}_{KM},
\]

and

\[
\sin \tilde{\delta}_{KM} \sin 2\tilde{\theta}_2 = \sin \delta_{KM} \sin 2\theta_2,
\]

where \( \tilde{s}_2 \equiv \sin \tilde{\theta}_2 \), and \( \tilde{c}_2 \equiv \cos \tilde{\theta}_2 \).
Eq. (13) gives a simple relationship between the genuine and matter-corrected flavor mixing parameters, which is similar to the Toshev relation in the PDG parametrization

\[ \sin \tilde{\delta} \sin 2\tilde{\theta}_{23} = \sin \delta \sin 2\theta_{23}, \tag{14} \]

in which the tildes always stand for the parameters in matter. We point out that such a similar relationship exists in not only the PDG parametrization, but also the KM parametrization and another parametrization denoted as P7 in Ref. [3].† Generally, we call all of them the Toshev relations.

We end this part with a direct application of this relation. Given the \( \mu-\tau \) symmetry with \( \theta_2 = 45^\circ \) and maximal CP violation with \( \delta_{KM} = \pm 90^\circ \) in vacuum, both sides of the Toshev relation equal \( \pm 1 \), then \( |\sin \tilde{\delta}_{KM}| = |\sin 2\tilde{\theta}_2| = 1 \), leading to \( \tilde{\theta}_2 = 45^\circ \) and \( \tilde{\delta}_{KM} = \pm 90^\circ \). Thus, we have proved that the \( \mu-\tau \) symmetry and maximal CP violation keep unchanged when matter effects are taken into account. Together with the PDG and P7 parametrizations, we rewrite the overall invariance of the \( \mu-\tau \) symmetry and maximal CP violation in three parametrizations as follows:

\[
\theta_{23} = 45^\circ, \delta = \pm 90^\circ \iff \tilde{\theta}_{23} = 45^\circ, \tilde{\delta} = \pm 90^\circ \\
\iff \theta_2 = 45^\circ, \delta_{KM} = \pm 90^\circ \iff \tilde{\theta}_2 = 45^\circ, \tilde{\delta}_{KM} = \pm 90^\circ \\
\iff \theta'_{23} = 45^\circ, \delta' = \pm 90^\circ \iff \tilde{\theta}'_{23} = 45^\circ, \tilde{\delta}' = \pm 90^\circ. \tag{15} 
\]

Ref. [16] provides another proof to this claim in the PDG parametrization, where a special basis of the neutrino fields is taken.

### 3.2 Analytical approximations

Our approximate formulation is based on two premises:

- From Table 1, one can derive \( s_3^2 \approx 0.04 \) (0.05) for the NH (IH). So \( s_3^2 \sim |\alpha| \approx 0.03 \) provides a reliable basis for our analytical approximation.

- Given small \( \alpha \), two scenarios should be considered separately: (a) \( E \) is above \( O(1) \) GeV; and (b) \( E \) is below \( O(1) \) GeV.

The reason for distinguishing between scenarios (a) and (b) is simple. In scenario (b), where \( |\alpha| \gtrsim |A| \) holds, we have to regard \( A \) as a small parameter; but in scenario (a) with \( |\alpha| \ll |A| \), we do not have to do so.

†The P7 parametrization of the MNSP matrix is given by

\[
V = \begin{pmatrix}
    c_{12}c_{13} & s_{12} & -c_{13}s_{13} \\
    -s_{12}c_{13} - c_{12}s_{13}e^{-i\delta'} & c_{12}c_{23} + s_{13}s_{23}e^{-i\delta'} & s_{13}s_{23} + c_{13}s_{23}e^{-i\delta'} \\
    s_{12}c_{13} + s_{13}s_{23}e^{-i\delta'} & -c_{12}s_{23} & -s_{12}s_{13}s_{23} + c_{13}c_{23}e^{-i\delta'}
\end{pmatrix},
\]

where \( s_{ij} = \sin \theta_{ij} \) and \( c_{ij} = \cos \theta_{ij} \). In this parametrization, the Toshev relation is expressed as \( \sin \tilde{\delta} \sin 2\tilde{\theta}_{23} = \sin \delta \sin 2\theta_{23} \). The reason for the Toshev relation being satisfied is that the first rotation matrix on the right-hand side of the MNSP matrix is a (2,3) rotation matrix in all of these three parametrizations, and it commutes with the effective potential.
Scenario (a)

Our first step is to diagonalize $M$ in Eq. (11). The symmetric matrix $M$ can be decomposed into two terms in series of $\alpha$:

$$M = M^{(0)} + \alpha M^{(1)}.$$  

(16)

The first term $M^{(0)}$ is a singular matrix and can be strictly diagonalized. The second term $\alpha M^{(1)}$ is a perturbation and can be diagonalized in series of $\alpha$ order by order. In general, the eigenvalues and eigenvectors of $M$ are expressed as

$$\lambda_i = \lambda_i^{(0)} + \alpha \lambda_i^{(1)} + \cdots,$$

$$v_i = v_i^{(0)} + \alpha v_i^{(1)} + \cdots,$$

respectively. And the orthogonal matrix $\hat{V}$ is written as $\hat{V} = (v_1, v_2, v_3)$. Here one must pay attention to the detail of how to derive a proper order of $\lambda_i$ (or $v_i$). In order to derive the proper order, we have to consider the cases of $A < 1$ and $A > 1$ separately. For $A < 1$, we formulate the series expansion in $\alpha$ and derive the eigenvalues and eigenvectors to the first order. The eigenvalues are given by

$$\lambda_1 = \alpha \frac{c_2^2}{1 - s_1^2 s_3^2},$$

$$\lambda_2 = \frac{1}{2} \left[ 1 + A - C + \alpha s_1^2 c_3^2 \frac{C + (1 - A + 2 A s_1^2 s_3^2)}{C(1 - s_1^2 s_3^2)} \right],$$

$$\lambda_3 = \frac{1}{2} \left[ 1 + A + C + \alpha s_1^2 c_3^2 \frac{C - (1 - A + 2 A s_1^2 s_3^2)}{C(1 - s_1^2 s_3^2)} \right]$$

(18)

with

$$C = \sqrt{(1 - A)^2 + 4 A s_1^2 s_3^2}.$$  

(19)

The eigenvectors are too complicated to be listed here.

Up to $O(s_3^2)$, Eq. (15) can be simplified to

$$\lambda_1 = \alpha c_1^2,$$

$$\lambda_2 = A - \frac{A s_1^2 s_3^2}{1 - A} + \alpha s_1^2,$$

$$\lambda_3 = 1 + \frac{A s_1^2 s_3^2}{1 - A}.$$  

(20)

The orthogonal matrix $\hat{V}$ turns out to be

$$\hat{V} = \begin{pmatrix}
\alpha c_1 s_1 & -1 + \frac{s_1^2 s_3^2}{2(1 - A)^2} & -s_1 s_3 \\
1 - \frac{1}{2} c_2^2 s_3^2 & \alpha c_1 s_1 & \frac{c_1 s_1 s_3^2}{1 - A} \\
-c_1 s_3 & \frac{s_1 s_3}{1 - A} & 1 - \frac{s_1^2 s_3^2}{2(1 - A)^2} - \frac{1}{2} c_1^2 s_3^2
\end{pmatrix}.$$  

(21)
whose three rotation angles are given by \( \cot \hat{\theta}_1 = \alpha c_1 s_1 / A \), \( \tan \hat{\theta}_2 = -c_1 s_3 \) and \( \tan \hat{\theta}_3 = s_1 s_3 / (1 - A) \). Substituting the angles into Eqs. \( 12 \) and \( 13 \), we obtain the approximate expressions of the matter-corrected flavor mixing parameters:

\[
\begin{align*}
\cot \check{\theta}_1 &= \frac{\alpha \sin 2\theta_1}{2A}, \\
\sin \check{\theta}_2 &= \sin \theta_2 \sqrt{1 - 2\epsilon \cos \delta_{\text{KM}} \cot \theta_2 + \epsilon^2 \cot^2 \theta_2}, \\
\tan \check{\theta}_3 &= \frac{\sin \theta_1 \sin \theta_3}{1 - A}, \\
\sin \check{\delta}_{\text{KM}} &= \frac{(1 + \epsilon^2) \sin \delta_{\text{KM}}}{\sqrt{(1 + 2\epsilon \cos \delta_{\text{KM}} \tan \theta_2 + \epsilon^2 \tan^2 \theta_2) (1 - 2\epsilon \cos \delta_{\text{KM}} \cot \theta_2 + \epsilon^2 \cot^2 \theta_2)}},
\end{align*}
\]

in which \( \epsilon = c_1 s_3 \). One can see that \( \cot \check{\theta}_1 \) and \( \tan \check{\theta}_3 \) are suppressed by \( \alpha \) or \( s_3 \), leading to \( \check{\theta}_1 \sim 90^\circ \) and \( \check{\theta}_3 \sim 0 \). The most interesting result comes from \( \check{\theta}_2 \) and \( \check{\delta}_{\text{KM}} \). First, \( \check{\theta}_2 \approx \theta_2 \) and \( \check{\delta}_{\text{KM}} \approx \delta_{\text{KM}} \) are good approximations for \( \epsilon \) is suppressed by \( s_3 \). In other words, the matter effects on \( \theta_2 \) and \( \delta_{\text{KM}} \) are small. Second, such small matter effects are independent of \( A \) in the approximation made above. This is a peculiar feature of the KM parametrization of \( \check{V} \).

Note that Eq. \( 22 \) only holds for neutrinos with \( A < 1 \). For neutrinos with \( A > 1 \), the replacement of \( \tan \check{\theta}_3 \Rightarrow \cot \check{\theta}_3 \) in Eq. \( 22 \) should be made.

For antineutrinos with \( -A < 1 \), the approximate expressions are given by

\[
\begin{align*}
\tan \check{\theta}_1 &= \frac{\sin \theta_1 \sin \theta_3}{1 + A}, \\
\sin \check{\theta}_2 &= \cos \theta_2 \sqrt{1 + 2\epsilon \cos \delta_{\text{KM}} \tan \theta_2 + \epsilon^2 \tan^2 \theta_2}, \\
\cot \check{\theta}_3 &= \frac{\alpha (1 + A) \cos \theta_1}{A \sin \theta_3}, \\
\sin \check{\delta}_{\text{KM}} &= \frac{(1 + \epsilon^2) \sin \delta_{\text{KM}}}{\sqrt{(1 + 2\epsilon \cos \delta_{\text{KM}} \tan \theta_2 + \epsilon^2 \tan^2 \theta_2) (1 - 2\epsilon \cos \delta_{\text{KM}} \cot \theta_2 + \epsilon^2 \cot^2 \theta_2)}},
\end{align*}
\]

in which \( \epsilon = c_1 s_3 [1 - \alpha (1 + A) / (As_3^2)] \). Eq. \( 23 \) is different from Eq. \( 22 \) because the order of \( \lambda_1 \) and \( \lambda_2 \) has been exchanged. Given small \( \theta_3 \), \( \check{\theta}_1 \sim 0 \), \( \check{\theta}_3 \sim 90^\circ \) and \( \check{\theta}_2 \sim 90^\circ - \theta_2 \) roughly hold. As \( \epsilon \) is \( A \)-dependent, \( \check{\theta}_2 \) and \( \check{\delta}_{\text{KM}} \) obviously rely on \( A \).

For antineutrinos with \( -A > 1 \), the replacement of \( \tan \check{\theta}_1 \Rightarrow \cot \check{\theta}_1 \) in Eq. \( 23 \) should be made.

The magnitude of the intrinsic CP violation in neutrino oscillations depends only upon the Jarlskog invariant \[17\]. It is calculated via \( \mathcal{J} = \text{Im} \left( V_{e1} V_{\mu2}^* V_{e2}^* V_{\mu1}^* \right) \). In the KM parametrization, one has \( \mathcal{J} = (1/8) \sin \theta_1 \sin 2\theta_1 \sin 2\theta_2 \sin 2\theta_3 \sin \delta_{\text{KM}} \) in vacuum and \( \check{\mathcal{J}} = (1/8) \sin \check{\theta}_1 \sin 2\check{\theta}_1 \sin 2\check{\theta}_2 \sin 2\check{\theta}_3 \sin \check{\delta}_{\text{KM}} \) in matter. They satisfy the Naumov identity \( \check{\mathcal{J}} \Delta \bar{m}_{21}^2 \Delta \bar{m}_{31}^2 \Delta \bar{m}_{32}^2 = \mathcal{J} \Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2 \) \[18\] \[19\]. One can derive \( \check{\mathcal{J}} \) from Eq. \( 13 \) as

\[
\check{\mathcal{J}} = \frac{\alpha}{AC (1 - \sin^2 \theta_1 \sin^2 \theta_3)} \mathcal{J}
\]

for neutrinos. Suppressed by \( \alpha \), the Jarlskog invariant in matter is much smaller than that in vacuum. \( \check{\mathcal{J}} \) obtains its relative maximum \( \check{\mathcal{J}}_m^a \approx \alpha \mathcal{J} / (2s_1 s_3) = (1/8) \alpha \sin 2\theta_1 \sin 2\theta_2 \cos \theta_3 \sin \delta_{\text{KM}} \) at \( A \approx 1 \),
where \( C \) is near its minimal value. The series expansion in \( s_3 \) gives the leading-order result of \( \tilde{J} \)

\[
\tilde{J} = \frac{\alpha}{A(1-A)} J.
\]  (25)

It can also be obtained from Eq. \( (22) \). Eq. \( (25) \) does not hold for \( A \sim 1 \). To derive the correct result of \( \tilde{J} \) for antineutrinos, one has to make the replacements \( A \rightarrow -A \) and \( J \rightarrow -J \).

**Scenario (b)**

Considering about \( |A| \lesssim |\alpha| \), we rewrite Eq. \( (16) \) as

\[
M = M^{(0)} + AM^{(1)} + \alpha M^{(1)},
\]  (26)

where \( M^{(0)} + AM^{(1)} \equiv M^{(0)} \). \( M^{(0)} \) has two degenerate eigenvalues 0 and one eigenvalue 1. To the first order of \( \alpha \), the eigenvalues are expressed as

\[
\lambda_1 = \frac{1}{2} \left[ A(1 - s_1^2 s_3^2) + \alpha - D \right],
\]

\[
\lambda_2 = \frac{1}{2} \left[ A(1 - s_1^2 s_3^2) + \alpha + D \right],
\]

\[
\lambda_3 = 1 + As_1 c_3 s_3,
\]  (27)

in which \( D = \sqrt{\left[ A(1 - s_1^2 s_3^2) + \alpha \right]^2 - 4\alpha A c_1^2 s_1} \). \( M \) is diagonalized by

\[
\hat{V} = O_1 O_3 \begin{pmatrix} \cos \vartheta & -\sin \vartheta & 0 \\ \sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & h_{13} \\ 0 & 1 & h_{23} \\ -h_{13} & -h_{23} & 1 \end{pmatrix}
\]  (28)

with one possibly large rotation angle \( \vartheta \) at the zeroth order in \( A \) and two small perturbation coefficients \( h_{13} \) and \( h_{23} \) at the first order. A strict calculation yields

\[
\sin \vartheta = \sqrt{\frac{1}{2} + \frac{A \cos 2\theta_1 - \alpha}{2D}},
\]

\[
h_{13} = As_1 s_3 (c_1 \sin \vartheta + s_1 c_3 \cos \vartheta),
\]

\[
h_{23} = As_1 s_3 (c_1 \cos \vartheta - s_1 c_3 \sin \vartheta).
\]  (29)

Finally, by using Eqs. \( (12) \) and \( (13) \) and taking account of \( s_3^2 \sim \alpha \), we obtain the flavor mixing parameters in matter:

\[
\sin 2\tilde{\theta}_1 = \frac{\alpha}{D} \sin 2\theta_1,
\]

\[
\sin \tilde{\theta}_2 = \sin \theta_2 \sqrt{\frac{1 - 2\kappa \cos \delta_{\text{KM}} \cot \theta_2 + \kappa^2 \cot^2 \theta_2}{1 + \kappa^2}},
\]

\[
\tan \tilde{\theta}_3 = \frac{\sin \theta_1 \sin \theta_3}{\sin(\theta_1 + \vartheta)},
\]

\[
\sin \tilde{\delta}_{\text{KM}} = \frac{(1 + \kappa^2) \sin \delta_{\text{KM}}}{\sqrt{(1 + 2\kappa \cos \delta_{\text{KM}} \tan \theta_2 + \kappa^2 \tan^2 \theta_2) (1 - 2\kappa \cos \delta_{\text{KM}} \cot \theta_2 + \kappa^2 \cot^2 \theta_2)}},
\]  (30)

where \( \kappa = s_3 \sin \vartheta \csc(\theta_1 + \vartheta) \). If one sets \( A \rightarrow 0 \), then \( D \rightarrow \alpha \), \( \vartheta \rightarrow 0 \) and all the parameters in matter return to those in vacuum. For a nonzero \( A \), \( \tilde{\theta}_1 \) may have a remarkable deviation from \( \theta_1 \). For
example, if $A = \alpha \cos 2\theta_1$, then $D \approx \alpha \sin 2\theta_1$ and $\sin 2\tilde{\theta}_1 \approx 1$, leading to $\tilde{\theta}_1 \approx 45^\circ$. But if $A = \alpha$, then $D \approx 2\alpha s_1$ and $\sin 2\tilde{\theta}_1 \approx \cos \theta_1$, leading to $\tilde{\theta}_1 \approx 45^\circ + \theta_1/2$. $\tilde{\theta}_3$ remains small though it receives some corrections. $\tilde{\theta}_3$ and $\tilde{\delta}_{\text{KM}}$ have small $A$-dependent corrections suppressed by $s_3$.

Eq. (30) holds for neutrinos with the NH. One can replace $\vartheta$ with $90^\circ - \vartheta$, $-\vartheta$ and $\vartheta - 90^\circ$ but keep Eq. (30) unchanged to derive the correct results for neutrinos with the IH, antineutrinos with the NH and antineutrinos with the IH, respectively.

The Jarlskog invariant reads

$$\tilde{J} = \frac{\alpha}{D} J$$

for the NH. Since $D \sim A \sim \alpha$, $\tilde{J}$ is of the same order as $J$. Therefore, the CP violation for $E < \mathcal{O}(1)$ GeV is not suppressed significantly by matter effects. $\tilde{J}$ gets its relative maximum $\tilde{J}_{m}^\alpha \approx J \csc 2\theta_1 = (1/8) \sin \theta_1 \sin 2\theta_2 \sin 2\tilde{\theta}_3 \sin \delta_{\text{KM}}$ at $A \approx \alpha \cos 2\theta_1$, which is even greater than $J$. One can obtain the results in the IH case by changing $\alpha$ to $-\alpha$.

In Table 2 we show how to obtain the expressions of the matter-induced flavor mixing parameters in all the possible cases we have discussed in both scenarios.

### 3.3 Numerical Analysis

In Fig. 2 we make a comparison of the analytical and numerical results for the matter-corrected flavor mixing parameters. The analytical expressions in different intervals of $A$ are given in Table 2. We have utilized Eq. (7) with $\rho_m = 2.8 \text{ g/cm}^3$ and replaced $A$ with the neutrino energy $E$. The values of the other input parameters are taken to be $\Delta m_{21}^2 = 7.59 \times 10^{-5}$ eV$^2$, $\Delta m_{31}^2 = 2.5 ( -2.4 ) \times 10^{-3}$ eV$^2$, $\theta_1 = 34.5^\circ (34.6^\circ)$, $\theta_2 = 49.3^\circ (43.2^\circ)$, $\theta_3 = 11.6^\circ (12.9^\circ)$ and $\delta_{\text{KM}} = -110^\circ (-74^\circ)$ for the NH (IH), according to the best-fit values in Ref. [12] and Table 1. Some comments and discussions are in order.

- In most cases, the analytical results in scenario (a) fit well with their numerical results for $E > \mathcal{O}(1)$ GeV and the analytical results in scenario (b) fit well with their numerical results for $E < \mathcal{O}(1)$ GeV. So $|\alpha| \ll |A|$ corresponds to $E > \mathcal{O}(1)$ GeV, and $|\alpha| \gg |A|$ corresponds to $E < \mathcal{O}(1)$ GeV.

- The expression of $\tilde{\theta}_3$ for neutrinos with the NH and that of $\tilde{\theta}_1$ for antineutrinos with the IH are not good approximations in the interval $9 \text{ GeV} < E < 15 \text{ GeV}$. Both $\tilde{\theta}_3$ and $\tilde{\theta}_1$ increase rapidly from a small angle to near $90^\circ$ when $E$ is running from $9 \text{ GeV}$ to $15 \text{ GeV}$. $\tilde{\theta}_3$ reaches $45^\circ$ at $E \approx 12 \text{ GeV}$ (or $A \approx 1$) for neutrinos with the NH, while $\tilde{\theta}_1$ reaches $45^\circ$ at the same energy for antineutrinos with the IH.

- The analytical approximations of $\tilde{\theta}_2$ and $\tilde{\delta}_{\text{KM}}$ are in good agreement with their numerical results. For neutrinos with $E > \mathcal{O}(1)$ GeV, the numerical results confirm that $\tilde{\theta}_2$ and $\tilde{\delta}_{\text{KM}}$ have small deviations from the vacuum parameters $\theta_2$ and $\delta_{\text{KM}}$ and are nearly independent of the neutrino beam energy $E$.

In Fig. 3 we compare the analytical results of the Jarlskog invariant $\tilde{J}$ in Eqs. (24), (25) and (31) with the numerical results. The input values are taken the same as in Fig. 2. The analytical approximations fit well with the numerical results, except for Eq. (25) in the interval $9 \text{ GeV} < E < 15 \text{ GeV}$. The numerical results show that the relative maximum of the Jarlskog invariant $\tilde{J}_{m}^\alpha \approx -0.003$ takes place at $E \approx 12 \text{ GeV}$ (or $A \approx 1$) for neutrinos with the NH and antineutrinos with the IH, and
\[ \mathcal{J}_m^b \approx -0.03 \text{ at } E \approx 0.1 \text{ GeV (or } A \approx \alpha \cos 2\theta_1) \text{ for neutrinos with both the NH and IH. They verify our analytical calculations of } \mathcal{J}_m^a \text{ and } \mathcal{J}_m^b \text{ in the above discussions.} \]

4 The probability of \( \nu_\mu \to \nu_e \) oscillations

The probability of \( \nu_\mu \to \nu_e \) oscillations in matter has been calculated in the PDG parametrization \[13, 20, 21, 22\] but not yet in the KM parametrization. In this section, we calculate it in the KM parametrization. Both scenarios (a) and (b) will be considered.

In vacuum, the oscillation probability of \( \nu_\mu \to \nu_e \) is given by

\[
P(\nu_\mu \to \nu_e) = -4 \sum_{i>j}^3 \text{Re} (V_{\mu i} V_{\nu_j}^*) \sin \Delta_{ij} - 8J \prod_{i>j}^3 \sin \Delta_{ij}, \quad (32)
\]

in which \( \Delta_{ij} = \Delta m_{ij}^2 L/(4E) \). In the KM parametrization, one has

\[
P(\nu_\mu \to \nu_e) = \left( \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 2\theta_3 + J \sin^2 \theta_3 \cos \delta_{\text{KM}} + 2 \sin^2 \theta_1 \cos^2 \theta_2 \sin^4 \theta_3 \right) \sin^2 \Delta_{32}
\]

\[
+ J \cos (\Delta_{31} + \delta_{\text{KM}}) \sin \Delta_{32} \sin \Delta_{21} + \sin^2 \theta_1 \cos \theta_2 \cos^2 \theta_3 \sin^2 \Delta_{21}
\]

\[
+ 2 \sin^2 \theta_2 \cos^2 \theta_1 \sin^2 \theta_3 \cos \Delta_{31} \sin \Delta_{32} \sin \Delta_{21}, \quad (33)
\]

where \( J \equiv 8J / \sin \delta_{\text{KM}} = \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_3 \). Replacing \( \delta_{\text{KM}} \) with \(-\delta_{\text{KM}}\), one may obtain the expression of \( P(\bar{\nu}_\mu \to \bar{\nu}_e) \). Their difference is a measure of the intrinsic CP violation:

\[
\Delta P(\nu_\mu \to \nu_e) \equiv P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e) = -2J \sin \delta_{\text{KM}} \sin \Delta_{31} \sin \Delta_{32} \sin \Delta_{21}. \quad (34)
\]

By replacing the parameters in vacuum with those in matter, one can achieve a similar expression of the oscillation probability in matter \( \tilde{P}(\nu_\mu \to \nu_e) \) from Eq. (33). In scenario (a), with the help of Eq. (22), \( \tilde{P}(\nu_\mu \to \nu_e) \) will be re-expressed in terms of the genuine mixing parameters and \( A \). It is not difficult to derive the following approximate formula from Eq. (33) by replacing \( \Delta_{32} \) and \( \Delta_{21} \) with \( \sin(1 - A) \Delta_{31} \) and \( \alpha \sin A \Delta_{31} \), respectively:

\[
\tilde{P}(\nu_\mu \to \nu_e) = \left( \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 2\theta_3 + J \sin^2 \theta_3 \cos \delta_{\text{KM}} \right) \frac{\sin^2(1 - A) \Delta_{31}}{(1 - A)^2}
\]

\[
+ \alpha J \cos (\Delta_{31} + \delta_{\text{KM}}) \frac{\sin A \Delta_{31} \sin(1 - A) \Delta_{31}}{A}
\]

\[
+ \alpha^2 \sin^2 \theta_2 \cos^2 \theta_1 \cos^2 \theta_3 \frac{\sin^2 A \Delta_{31}}{A^2}. \quad (35)
\]

Here \( \Delta_{31} = \tilde{\Delta}_{31}, (1 - A) \Delta_{31} = \tilde{\Delta}_{32} \) and \( A \Delta_{31} = \tilde{\Delta}_{21} \) hold to the leading order in \( \alpha \) and \( s_3 \), and Eq. (35) holds to the second order in \( \alpha \) and the third order in \( s_3 \). The second term in the first line is a term of \( s_3^3 \). We keep it in view of the sizable value of \( \theta_3 \). The CP violation is included in the second line, which is suppressed by \( \alpha \). The term of \( \alpha^2 \) in the third line is reserved for comparing with some former works in the PDG parametrization, in which the term of \( \alpha^2 \) is often kept (e.g., [13]). But given the current experimental data, this term is meaningless unless we keep the terms of \( s_{13}^4 \) or \( \alpha s_{13}^2 s_{13}^2 [20] \). The divergences for \( A \to 0 \) and 1 are absent because of the suppressions by sine functions in \( \sin A \Delta_{31} \) and
\[\sin(\hat{\Delta} - 1)\Delta.\] One can see from Eq. (33) that the expression of the \(\nu_\mu \to \nu_e\) oscillation probability in the KM parametrization is similar to that in the PDG parametrization. By replacing \(\delta_{\text{KM}}\) with \(-\delta_{\text{KM}}\) and \(A\) with \(-A\) in Eq. (35), one can obtain the expression of \(\hat{P}(\bar{\nu}_\mu \to \bar{\nu}_e)\) and then their difference \(\Delta \hat{P}(\nu_\mu \to \nu_e) \equiv \hat{P}(\nu_\mu \to \nu_e) - \hat{P}(\bar{\nu}_\mu \to \bar{\nu}_e)\). \(\Delta \hat{P}(\nu_\mu \to \nu_e)\) consists of both the intrinsic CP violation and matter-induced contribution and is sometimes dominated by the latter.

In scenario (b), one can also make the similar replacements as in scenario (a) and obtain the approximate expression of \(\hat{P}(\nu_\mu \to \nu_e)\) in terms of the vacuum parameters and \(A\). But because of the complexity of \(\theta\), it is difficult for us to simplify the expression to a form as a simple function of \(A\). So we do not write out the expression here. We use another method proposed in Ref. [22] to obtain an alternative simple formula. It reads

\[
\hat{P}(\nu_\mu \to \nu_e) = P(\nu_\mu \to \nu_e) + 2A \sin \Delta_{31}(\sin \Delta_{31} - \Delta_{31} \cos \Delta_{32})
\times (\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 2\theta_3 + J \sin^2 \theta_3 \cos \delta_{\text{KM}}) .
\]

(36)

With \(\delta_{\text{KM}} \Rightarrow -\delta_{\text{KM}}\) and \(A \Rightarrow -A\), one may arrive at \(\hat{P}(\bar{\nu}_\mu \to \bar{\nu}_e)\). Their difference is given by

\[
\Delta \hat{P}(\nu_\mu \to \nu_e) = \Delta P(\nu_\mu \to \nu_e) + 4A \sin \Delta_{31}(\sin \Delta_{31} - \Delta_{31} \cos \Delta_{32})
\times (\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 2\theta_3 + J \sin^2 \theta_3 \cos \delta_{\text{KM}}) .
\]

(37)

In Eq. (37), the intrinsic CP violation and the matter-induced contribution are separated to two different parts. The former is dominant if \(\delta_{\text{KM}}\) is not too small.

Finally, we compare the analytical results of \(\hat{P}(\nu_\mu \to \nu_e)\) and \(\Delta \hat{P}(\nu_\mu \to \nu_e)\) with the numerical results to show the validity of our approximations. We take the future long-baseline neutrino oscillation experiments NO\(\nu\)A as an example in scenario (a) and Hyper-K as an example in scenario (b). The former will use the neutrino beams with \(E \sim 2\) GeV, which is compatible with scenario (a). And the latter will use the neutrino beams with \(E \sim 0.6\) GeV, which is compatible with scenario (b). Different baseline lengths \(L\) and matter densities \(\rho_m\) have been input in different experiments. \(L = 810\) km and \(\rho_m = 2.8\) g/cm\(^3\) are taken in Fig. 4 for the NO\(\nu\)A experiment [10], and \(L = 295\) km and \(\rho_m = 2.6\) g/cm\(^3\) are taken in Fig. 5 for the Hyper-K experiment [11]. Other input parameters are taken the same as in Fig. 2. We see that the numerical results confirm the precision of the analytical approximations of \(\hat{P}(\nu_\mu \to \nu_e)\) and \(\Delta \hat{P}(\nu_\mu \to \nu_e)\) in both scenarios.

5 Conclusions

In this paper, we have used the KM parametrization to study the lepton flavor mixing and neutrino oscillations in matter. We re-discover the Toshev-like relation in the KM parametrization and prove that the \(\mu-\tau\) symmetry with \(\theta_2 = 45^\circ\) and maximal CP violation with \(\delta_{\text{KM}} = \pm 90^\circ\) keep unchanged when matter effects are taken into account. We have presented the approximate expressions of the matter-corrected flavor mixing parameters. Different methods have been chosen in two scenarios for the neutrino energy above \(\mathcal{O}(1)\) GeV and below \(\mathcal{O}(1)\) GeV. Finally, we have calculated the probability of \(\nu_\mu \to \nu_e\) oscillations as an application in both scenarios. Below we compare the main features of the KM parametrization with those of the PDG parametrization:

- The genuine flavor mixing parameters in the KM parametrization \(\theta_1, \theta_2, \text{ and } \delta_{\text{KM}}\) are approximately equal to the corresponding parameters in the PDG parametrization \(\theta_{12}, \theta_{23}, \text{ and } \delta\),

12
respectively. Although \( \theta_3 \) is not close to \( \theta_{13} \), it is small enough to do the series expansion in both \( \theta_3 \) and \( \Delta m^2_{21}/\Delta m^2_{31} \) by treating \( \sin^2 \theta_3 \sim \Delta m^2_{21}/\Delta m^2_{31} \).

- For neutrinos with \( E \) above \( \mathcal{O}(1) \) GeV, the corrections to the rotation angle \( \bar{\theta}_2 \) and the CP phase \( \delta_{\text{KM}} \) induced by matter effects are small and nearly independent of the matter density and the neutrino energy. This is a salient feature of the KM parametrization.

- The analytical expressions of the oscillation probabilities in Eqs. (33) and (35) are similar to the corresponding expressions in the PDG parametrization.

It is well known that a good parametrization brings much convenience to the description of physical quantities. In this paper, we have explored the KM parametrization to study the neutrino phenomenology.

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Table 1: The mixing angles in the KM parametrization translated from the results obtained in the PDG parametrization [12]. The upper (lower) row corresponds to the normal hierarchy (inverted hierarchy) of neutrino masses. The best-fit value $\delta = -110^\circ (-74^\circ)$ have been taken for simplicity.

| Parameter | $\theta_1$ | $\theta_2$ (at its best fit) | $\theta_3$ |
|-----------|------------|-------------------------------|------------|
| Best fit  | $34.5^\circ$ | $49.3^\circ$ | $11.6^\circ$ |
| (34.6°)   | (43.2°)    | (12.9°)                      |            |
| $1\sigma$ range | $33.4^\circ$ - $35.8^\circ$ | $44.8^\circ$ - $53.0^\circ$ | $9.4^\circ$ - $14.0^\circ$ |
|           | (33.5° - 36.0°) | (39.8° - 45.8°) | (10.4° - 15.3°) |
| $2\sigma$ range | $32.1^\circ$ - $37.4^\circ$ | $41.9^\circ$ - $55.0^\circ$ | $6.8^\circ$ - $16.0^\circ$ |
|           | (32.2° - 37.5°) | (38.4° - 47.1°) | (7.6° - 16.8°) |
| $3\sigma$ range | $31.4^\circ$ - $38.2^\circ$ | $39.7^\circ$ - $56.8^\circ$ | $3.5^\circ$ - $17.6^\circ$ |
|           | (31.4° - 38.4°) | (37.8° - 48.2°) | (3.5° - 18.6°) |

Table 2: The replacements for obtaining the proper expressions of the effective matter-corrected flavor mixing parameters in different cases.

| NH       | IH       |
|----------|----------|
| $\nu$ $A \lesssim \alpha$ | Eq. [30]  | $|A| \lesssim |\alpha|$ | Eq. [30], $\vartheta \Rightarrow 90^\circ - \vartheta$ |
| $\alpha \ll A < 1$ | Eq. [22] | $|\alpha| \ll |A|$ | Eq. [22] |
| $1 < A$ | Eq. [22], $\theta_3 \Rightarrow 90^\circ - \theta_3$ |
| $\bar{\nu}$ $A \lesssim \alpha$ | Eq. [30], $\vartheta \Rightarrow -\vartheta$ | $|A| \lesssim |\alpha|$ | Eq. [30], $\vartheta \Rightarrow -\vartheta - 90^\circ$ |
| $\alpha \ll A$ | Eq. [23] | $|\alpha| \ll |A|$ | $< 1$ | Eq. [23] |
| $1 < |A|$ | Eq. [23], $\theta_3 \Rightarrow 90^\circ - \theta_3$ |

Figure 1: The dependence of $\theta_2$ on $\delta$ according to Eq. [2], where the best-fit values and the $1\sigma$, $2\sigma$ and $3\sigma$ ranges of $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$ [12] have been input.
Figure 2: The analytical (dashed line) and numerical (solid line) results of the matter-corrected flavor mixing parameters. The genuine flavor mixing parameters are taken as the best-fit values in Ref. [12] and Table 1: $\Delta m_{21}^2 = 7.59 \times 10^{-5}$ eV$^2$, $\Delta m_{31}^2 = 2.5 (-2.4) \times 10^{-3}$ eV$^2$, $\theta_1 = 34.5^\circ (34.6^\circ)$, $\theta_3 = 11.6^\circ (12.9^\circ)$, $\theta_2 = 49.3^\circ (43.2^\circ)$ and $\delta_{\text{KM}} = -110^\circ (-74^\circ)$ for the NH (IH). The mass density of matter is assumed to be $\rho_m = 2.8$ g/cm$^3$. 
Figure 3: The analytical results of Eqs. (25) and (31) (dashed line), Eq. (24) (circle curve) and numerical results (solid line) of the Jarlskog invariant. All the input parameters are taken the same as in Fig. 2.
Figure 4: The analytical (dashed line) and numerical (solid line) results of the oscillation probability $\tilde{P}(\nu_\mu \rightarrow \nu_e)$ and the difference $\Delta \tilde{P}(\nu_\mu \rightarrow \nu_e)$ with a length of baseline $L = 810$ km and matter density $\rho_m = 2.8$ g/cm$^3$ in the NO$\nu$A experiment [10]. The genuine flavor mixing parameters ($\theta_1$, $\theta_2$, $\theta_3$ and $\delta_{\text{KM}}$) and mass-squared differences ($\Delta m^2_{21}$ and $\Delta m^2_{31}$) are taken the same as in Fig. 2.

Figure 5: The analytical (dashed line) and numerical (solid line) results of the oscillation probability $\tilde{P}(\nu_\mu \rightarrow \nu_e)$ and the difference $\Delta \tilde{P}(\nu_\mu \rightarrow \nu_e)$ in the Hyper-K experiment [11]. Compared with Fig. 4, a length of baseline $L = 295$ km and matter density $\rho_m = 2.6$ g/cm$^3$ are taken. The genuine flavor mixing parameters and mass-squared differences are taken the same as in Fig. 2.