The critical Ising model on a torus with a defect line

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Abstract - The critical Ising model in two dimensions with a specific defect line is analyzed to deliver the first exact solution with twisted boundary conditions. We derive exact expressions for the eigenvalues of the transfer matrix and obtain analytically the partition function and the asymptotic expansions of the free energy and inverse correlation lengths for an infinitely long cylinder of circumference \(L_x\). We find that finite-size corrections to scaling are of the form \(a_k/L_x^{k-1}\) for the free energy \(f\) and \(b_k(p)/L_x^{k-1}\) and \(c_\lambda(p)/L_x^{k-1}\) for inverse correlation lengths \(\xi_{p}^{-1}\) and \(\xi_{L_x-p}^{-1}\), respectively, with integer values of \(k\). By exact evaluation we find that the amplitude ratios \(b_k(p)/a_k\) and \(c_\lambda(p)/a_k\) are universal and verify this universal behavior using a perturbative conformal approach.

The study of boundary conditions (BCs) in conformal field theories has attracted much attention in recent decades. This is because of its relevance in string and brane theory on the one hand, and in various problems of statistical mechanics and condensed-matter physics on the other. A classification of conformal boundary conditions on the torus has been given in [1–3]. Computation of the partition function involves identifying the states at the two ends of a cylinder through the trace operation. One may insert a defect line (or seam) into such a system, along a non-contractible circle at the end of the cylinder before closing it into a torus, the effect of which is to twist the boundary conditions. It is important to understand the effects of such a twist, and to do this it is valuable to study model systems, especially those which have exact results [4]. The Ising model is the most prominent example and one of the best studied models of statistical mechanics.

The Ising model with a defect line goes back at least to Bariev [5] and McCoy and Perk [6]. Interpretation in terms of conformal invariance was started by Turban [7] and Henkel and Patkós [8]. It turns out that for a system of \(n\) equidistant infinitely long defect lines, the conformal spectrum, hence the modular invariant partition function, is given in terms of SO(2n)-Kac-Moody-Virasoro algebras, of central charge \(c = n\), see [9] for the explicit formulas, including the partition function. Reference [1] takes up the same question in terms of an orbifold construction. See also ref. [10], Chapt. 15. Modified boundary conditions and defect lines also arise in distinct physical situations. For instance, different types of boundary critical phenomena can be identified in uniaxial antiferromagnets on distinct surfaces [11] and the local magnetisation exponent in the Ising model with a defect line has been measured as the corner exponent of a 3D Ising model [12].

Our goal is not to discuss defect lines in general, but rather to specialise to a very peculiar type, which is the only one which can be treated in terms of unitary conformal characters.

Since Onsager obtained the exact solution of the two-dimensional Ising model with cylindrical BCs in 1944 [13], there have been continuous attempts to treat different two-dimensional topologies [14–19].

At the critical point \(T_c\) the asymptotic finite-size scaling behavior of the critical free energy \(f_N\) and the inverse correlation lengths \(\xi^{-1}\) of an infinitely long 2D cylinder of finite circumference \(N\) has the form [20]

\[
\lim_{N \to \infty} [N^2(f_N - f_\infty)] = A, \quad \lim_{N \to \infty} N\xi^{-1} = D_n, \tag{1}
\]

where \(f_\infty\) is the bulk free energy and \(A\) and \(D_n\) are the universal constants which may depend on the BCs. The values of \(A\) and \(D_n\) are known to be related to the conformal anomaly \(c\), the conformal weights of the ground state
\( \Delta_0, \Delta_0, \) and the scaling dimension of the \( n \)-th scaling field \( x_n = \Delta_n + \Delta_0 \) of the theory [20],

\[
A = 2\pi \zeta \left( \frac{c}{12} + \Delta_0 + \Delta_0 \right), \quad D_n = 2\pi \zeta x_n, \quad (2)
\]

where \( \zeta \) is the anisotropy parameter. The principle of unitarity of the underlying field theory restricts, through the Kac formula, the possible values of \( c, \Delta_0 \) and \( \Delta_0 \). For the 2D Ising model, we have \( c = 1/2 \) and the only possible values are \( \Delta_0, \Delta_0 = 0, 1/16, 1/2 \). Therefore, there are six different boundary universal classes with \((\Delta_0, \Delta_0) = (0, 0)\) for periodic BCs; \((\Delta_0, \Delta_0) = \left( \frac{1}{16}, \frac{1}{16} \right)\) for antiperiodic BCs; and BCs with \((\Delta_0, \Delta_0) = \left( \frac{1}{2}, \frac{1}{2} \right); \left( \frac{1}{16}, 0 \right); \left( \frac{1}{2}, 0 \right)\) and \( \left( \frac{1}{16}, \frac{1}{2} \right) \). Past efforts have focused mainly on periodic and antiperiodic BCs. In this paper we consider \((\Delta_0, \Delta_0) = (0, 0)\), which for the Ising quantum chain is called duality twisted BCs and has been considered in [21]. In what follows we will give the first exact solution of the 2D Ising model with twisted BCs at the critical point. New universal amplitude ratios for finite-size corrections of the two-dimensional Ising model with periodic, antiperiodic, free, fixed and mixed boundary conditions have been recently presented [22]. In this letter we present universal amplitude ratios for the duality twisted BCs universality class.

The Ising model on a lattice with periodic BCs and with a specific defect line (seam) was formulated in [19,23]. For the Ising model the seams are labelled by the Kac labels \((r, s)\). There are six possible partition functions \( Z_{(r,s)}(q) \) for the Ising model with seams labelled by \((r, s) = (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\) and for the three of them, namely, for the seams \((r, s) = (1, 1), (1, 2)\) and \((1, 3)\), the partition functions \( Z_{(r,s)}(q) \) are obtained numerically to very high precision in [23]. In particular for \( Z_{(1,2)}(q) \) they obtain

\[
Z_{(1,2)}(q) = \left[ \chi_0 + \chi_\frac{1}{2} \right] \chi_\frac{1}{2} \left[ \chi_0 + \chi_\frac{1}{2} \right]^*, \quad (3)
\]

where \( \chi_0 \equiv \chi_0(q), \chi_\frac{1}{2} \equiv \chi_{\frac{1}{2}}(q) \) and \( \chi_\frac{1}{2} \equiv \chi_{\frac{1}{2}}(q) \) are the chiral characters and \( q \) is the modular parameter. The \((r, s) = (1, 1)\) and \((1, 3)\) seams reproduce the well-known partition function of the Ising model with periodic and antiperiodic boundary conditions respectively. In what follows we will show that the BC given by the seam \((r, s) = (1, 2)\) corresponds to duality twisted BC.

Let us consider a square lattice rotated by 45 degrees (see fig. 1), in which each row has \( L \) faces and each column has \( M \) faces. We insert one defect line (seam) along the final column. The lattice thus consists of \( 2L - 1 \) regular (zigzagging) columnar edges and one defect line and \( 2M \) regular (zigzagging) row edges. Periodic BCs are imposed in both directions and in fig. 1 this is represented as identifying the light (dark) nodes on the first row (column) with the respective light (dark) nodes on the last row (column). The physical dimensions of the lattice, \( L_x \) and \( L_y \), are given by

\[
L_x = \sqrt{2} \left( L - \frac{1}{2} \right); \quad L_y = \sqrt{2} M. \quad (4)
\]

\[\text{Fig. 1: The lattice is constructed with} \ 2L \text{ regular faces in each row and} \ 2M \text{ regular faces in each column. The lattice has two sublattices which can be even or odd. The heights are even on the even sublattice and odd on the odd sublattice. On the even sublattice the heights are fixed to the value 2. On the odd sublattice we identify the state} \ a = 1 \text{ with the usual Ising state and} \ a = 3 \text{ with the usual Ising state. The odd sublattice can be considered as the square lattice rotated by 45 degrees, in which each row has} \ L \text{ faces and each column has} \ M \text{ faces. The faces on the odd sublattice are two times larger than regular faces. The line defect is inserted along the rightmost columnar edge (dashed line) before closing the lattice into a torus with periodic BCs. Periodic BCs are imposed in both directions.}\]

The finite-size partition function for the Ising model \( Z_{L_x,L_y} \) can be written as \( Z_{L_x,L_y} = \sum_s \exp \left( J \sum_{ij} s_i s_j + K \sum_{ij} s_i s_j \right) \), where the first sum within the parenthesis is over NW-SE edges, and the second sum over NE-SW edges and spin variable \( s_i \) can take the two values \( \pm 1 \). Since we restrict ourselves to the critical Ising model, we have \( \sinh(2J) \sinh(2K) = 1 \). This condition can be conveniently parameterized by introducing a so-called spectral parameter \( u \), so that \( \sinh(2J) = \cot(2u) \), \( \sinh(2K) = \tan(2u) \), with \( 0 < u < \pi / 4 \). The anisotropy parameter is \( \zeta \) related to the spectral parameter \( u \) through \( \zeta = \sin 4u \). Now partition function can be rewritten in the following form:

\[
Z_{L_x,L_y}(u) = Tr \left[ T(u) \right]^M = \sum_n e^{-L_x E_n(u)}, \quad (5)
\]

where the sum is over all eigenvalues of a transfer matrix \( T(u) \), written as \( e^{-E_n(u)} \) and \( E_n(u) = \frac{2\pi i}{L_x} E_n(u) \).

Conformal invariance predicts that the leading finite-size corrections to the energies \( E_n \) take the form [3]

\[
E_n(u) = L_x f_\infty + 2\pi i \frac{\Delta_n + \bar{k}_n - \frac{c}{24}}{L_x} e^{-iu} + \left( \Delta_n + \bar{k}_n - \frac{c}{24} \right) e^{iu} + o \left( \frac{1}{L_x} \right), \quad (6)
\]

where \( \Delta_n \) and \( \bar{k}_n \) are the conformal weights, \( k_n, \bar{k}_n \in N \) are label descendent levels and \( g \) is Coxeter number.
The Boltzmann weights of the model are prescribed to the faces of a regular square lattice [23]
\[ W \left( \begin{array}{cc} d & c \\ a & b \end{array} \right\vert u \right) = s_1(-u)\delta_{ac} + s_0(u)\sqrt{\psi_a \psi_c / \psi_b} \delta_{bd}. \] (7)

Here \( a, b, c, d \) are the spin states, \( u(0 < u < \lambda) \) is the spectral parameter, \( s_k(u) = \sin((u+k\lambda)/\sin\lambda) \), \( \lambda = \frac{\pi}{y} \) is the crossing parameter and \( \psi_a \) are the entries of Perron-Frobenius eigenvector of the adjacency matrix \( G \). The Ising model is related to the Dynkin diagram \( A_3 \) (see fig. 2) whose Coxeter number \( g = 4 \).

The \( A_3 \) model is one of the Andrews-Baxter-Forrester models [24] \( A_f \) with \( L = 3 \). In the \( A_3 \) model, the spins \( a, b, c, d, \ldots \) assigned to the sites of the lattice take heights from the set \( 1, 2, 3 \) (from \( A_3 \) Dynkin diagram) and satisfy the adjacency condition that heights on adjacent sites must differ by \( \pm 1 \). In the \( A_3 \) model, the square lattice has two sublattices which can be even or odd. The heights are even on the even sublattice and odd on the odd sublattice. On the even sublattice the heights are fixed to the value \( 2 \).

On the odd sublattice we identify the state \( u \) as the usual + Ising state and \( \theta \) as the state. From the form of (9) one can easily obtain the largest eigenvalue \( \Lambda(u) \) of the double-row transfer matrices \( U' \) of the Ising model with duality twisted BCs we can obtain from the functional equation on \( U(u)V(u + \pi/4) \) that
\[ \Lambda = e^{2iu} \prod_{k=1}^{L-1} \left( e^{2iu} \sin \varphi_k + \mu_k e^{-2iu} \cos \varphi_k \right)^2, \] (9)

where \( \tilde{u} = u - \frac{\pi}{8} \), \( \varphi_k = \frac{\pi(2k-1)}{2(2L-1)} \) and \( \mu_k \) can be chosen arbitrarily so we have \( 2^{L-1} \) eigenvalues. The remaining eigenvalues can be found by taking the complex conjugate of (9). From eq. (9) one can easily obtain the expressions for the absolute value of the eigenvalue \( |\Lambda| \) and the argument \( \theta \). Let us now consider the largest eigenvalue \( \Lambda_0 \), which corresponds to the case when all \( \mu_k = 1 \). The derivation of the asymptotic expansion of \( \log \Lambda_0 = \log |\Lambda_0| + i\theta_0 \) can be divided into two parts. First, with the help of the Euler-Maclaurin summation formula we can derive the asymptotic expansion of the logarithm of the absolute value of \( \Lambda_0 \),
\[ \frac{\sqrt{2}}{\pi} \log |\Lambda_0| = -L_x f_{\infty} \sum_{k=0}^{\infty} B_{2k+2} f(2k+1) (0) \left( \frac{\pi}{L_x} \right)^{2k+1} , \]

where \( B_{2k+2} \) are the Bernoulli numbers, \( f(x) = \frac{1}{2} \log [1 + \sin(4u) \sin x] \) and \( f_{\infty} = -\frac{1}{2} \int_0^{\pi} f(x) dx \). Next, with the help of the Boole summation formula [25], the asymptotic expansion of the argument \( \theta_0 \) can be written in the form
\[ \theta_0 = -\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} E_{2n+1} (0) g(2n+1) (0) \left( \frac{\pi}{L_x} \right)^{2n+1} , \]
where $g(x) = \frac{1}{2} \arctan \left( \frac{\cos(4u) \cos x}{\sin(4u) + \sin x} \right)$ and $E_{2n+1}(0)$ are the Euler polynomials. Thus, the leading finite-size corrections to the ground-state energies $E_0 = \frac{x^3}{2} E_0 = -\frac{x^3}{2} \log \Lambda_0$ take the form
\[
E_0 = L_x f_\infty + \frac{2\pi i}{L_x} \left( \frac{1}{24} e^{-4iu} + \frac{1}{48} e^{4iu} \right).
\]
(10)

On the other hand, $E_0$ is given by eq. (6) with $n = 0$. Comparing eq. (10) with eq. (6) for the Ising model with $c = 1/2$ and $g = 4$ we see that $\Delta = \frac{1}{192}$ and $\Delta = 0$. Thus, we have shown that duality twisted BCs with $(\Delta, \bar{\Delta}) = (\frac{1}{192}, 0)$ correspond to the BC with seam $(r, s) = (1, 2)$.

Let us now consider the other eigenvalues given by eq. (9), which correspond to some combination of the $\mu_k$. Denote by $\Lambda(p)$ the eigenvalue with $\mu_p = -1$ and the remaining $\mu_k$ having the value $\mu_k = +1$. The leading finite-size corrections to the excited state energies $E_p = \frac{x^3}{2} E_p = -\frac{x^3}{2} \log \Lambda(p)$ and $E_{L-p} = \frac{x^3}{2} E_{L-p} = -\frac{x^3}{2} \log \Lambda(L-p)$ can be written in the form
\[
E_p = L_x f_\infty + \frac{2\pi i}{L_x} \left( \frac{1}{24} e^{-4iu} + \frac{25}{48} - p \right) e^{4iu},
\]
(11)
\[
E_{L-p} = L_x f_\infty + \frac{2\pi i}{L_x} \left( \frac{1}{24} + p \right) e^{-4iu} + \frac{1}{48} e^{4iu},
\]
(12)
where $L_x$ is given by eq. (4). Comparing eqs. (11), (12) with eq. (6) for the Ising model with $c = 1/2$ and $g = 4$ we can see that for the excited state $E_p$ we have $\Delta = 1/16$, $k_n = 0$ and $\Delta = 1/2$, $k_n = p - 1$ and for the excited state $E_{L-p}$ we have $\Delta = 1/16$, $k_n = p$ and $\Delta = 0$, $k_n = 0$.

Now we have all the necessary information to start the calculations of partition function for the Ising model with duality twisted BCs $Z_{L_x L_y}$. For large $L_x$ and $L_y$ (always keeping the ratio $L_y/L_x$ constant) we have
\[
Z_{L_x L_y} = \sum_{\Lambda} \Lambda^M \approx e^{-L_x f_\infty} Z_{(1,2)}(q),
\]
where $Z_{(1,2)}(q)$ is the universal conformal partition function. Let us now find the general form of the eigenvalues with significant input in the partition function. From (9) we can see that these are eigenvalues for which almost all $\mu_k = 1$ and some $\mu_k$ are allowed to take the value $-1$ only if $k \ll L$ or $L - k \ll L$. Any “significant” eigenvalue will be specified by two sets of indexes $K = \{k_1, k_2, \ldots, k_m\}$ and $\bar{K} = \{\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_m\}$, where $k_i \equiv L - k_i$ with $k_1 < k_2 < k_3 < \cdots < k_m \ll L$ and $\bar{k}_1 < \bar{k}_2 < \bar{k}_3 < \cdots < \bar{k}_m \ll L$ so that $\mu_{k_2} = -1$, $\mu_{k_1} = -1$ and the other $\mu_i$’s are $+1$. Thus, from above it is easy to get
\[
Z_{(1,2)}(q) = \sum_{N} q^{-\frac{k_1}{2} + \frac{\mu_1}{2} + \sum_{i=1}^{m} k_i q^{-\frac{k_{i+1}}{2} + \sum_{i=1}^{m} (k_i - \frac{1}{2})},
\]
(13)
where $q = e^{-\frac{\pi i}{L_x} e^{-4iu}}$ is a modular parameter. For further calculations it is convenient to introduce the occupation numbers $\varepsilon_k = \frac{1-\mu_k}{2}$ and $\xi_k = \frac{1+\mu_k}{2}$. In terms of these quantities the universal conformal partition function (13) can be rewritten as
\[
Z_{(1,2)}(q) = \sum_{\{k\}, \{\xi\}} q^{-\frac{k_1}{2} + \frac{\mu_1}{2} + \sum_{i=1}^{m} k_i q^{-\frac{k_{i+1}}{2} + \sum_{i=1}^{m} (k_i - \frac{1}{2})} \prod_{k=1}^{\infty} \left( 1 + q^k \right) \prod_{k=1}^{\infty} \left( 1 + q^{k - \frac{1}{2}} \right).
\]

Taking into account the expressions for the chiral characters $\chi_0, \chi_1, \chi_{\pm}$ and adding the conjugate part of the eigenvalue set, we can get the universal conformal partition function given by eq. (3). Thus, we have obtained analytically the universal conformal partition function for the Ising model with duality twisted BCs $Z_{(1,2)}(q)$ which confirms the numerical result of [23].

Now we will present the new set of the universal amplitude ratios. Let us denote the free energy per spin $f$, the inverse correlation lengths $\xi_p$ and $\xi_{L-p}$ of our critical Ising model as $L_x f = -\frac{x^3}{2} \log(|\Lambda_0|)$, $\xi_p^{-1} = \frac{x^3}{2} \log(\frac{\Lambda_0}{\Lambda_p})$ and $\xi_{L-p}^{-1} = \frac{x^3}{2} \log(\frac{\Lambda_0}{\Lambda_{L-p}})$. We find that subdominant finite-size corrections to scaling should be to the form $a_k/L_x^{2k-1}$ for the free energy $f$ and $b_k(L_x)/L_x^{2k-1}$ and $c_k/L_x^{2k-1}$ for inverse correlation lengths $\xi_p^{-1}$ and $\xi_{L-p}^{-1}$, respectively, where the coefficients $a_k, b_k, c_k$ can be written in the following form:
\[
a_k = \frac{\pi^{2k-1} B_{2k}}{2^{2k-1} (2k)!} f^{(2k-1)}(0),
\]
\[
b_k(p) = \frac{\pi^{2k-1} (2p - 1)^{2k-1}}{2^{2k-1} (2k - 1)!} f^{(2k-1)}(0),
\]
\[
c_k(p) = \frac{2^{k+1} p^{2k-1} (2k - 1)!}{(2k - 1)!} f^{(2k-1)}(0).
\]

The coefficients $a_1, b_1(p)$ and $c_1(p)$ are universal and related to the conformal anomaly number ($c$), the conformal weights $(\Delta, \bar{\Delta})$, and the scaling dimensions of the $p$-th scaling fields of the theory. The coefficients $a_k, b_k(p)$ and $c_k(p)$ for $k \geq 2$ are non-universal, but ratios of these coefficients $r_p(k) = b_k(p)/a_k$ and $r_{L-p}(k) = c_k(p)/a_k$ are universal and given by
\[
r_p(k) = \frac{4k(2p - 1)^{2k-1}}{B_{2k}};
\]
\[
r_{L-p}(k) = \frac{4k(2p)^{2k-1}}{B_{2k}}.
\]

Thus, we have obtained a new set of the universal amplitude ratios $r_p(k)$ and $r_{L-p}(k)$. The case for $k = 1$ is trivial, since $r_p(1), r_{L-p}(1)$ are the ratios of universal coefficients $a_1, b_1(p)$ and $c_1(p)$. The case $k \geq 2$ is non-trivial. Below we will show that the ratios $r_p(2)$ and $r_{L-p}(2)$ are given by
\[
r_p(2) = -240(2p - 1)^3,
\]
\[
r_{L-p}(2) = -1920 p^3
\]
can be obtained from conformal field theory. The finite-size corrections to eq. (6) can in principle be computed in
perturbative conformal field theory. In general, any critical lattice Hamiltonian will contain correction terms to the fixed-point Hamiltonian $H = \langle H/2 \rangle_{\zeta} + \sum_k g_k \int_{L_k/2}^{L_k} d\phi_k(v) dv$, where $g_k$ is a non-universal constant and $\phi_k(v)$ is a perturbative conformal field with scaling dimension $x_k$. To the first order in the perturbation, the energy gaps $(\mathcal{E}_n - \mathcal{E}_0)$ and the ground-state energy $(\mathcal{E}_0)$ can be written as

$$\mathcal{E}_n - \mathcal{E}_0 = \frac{2\pi}{L_x} \zeta x_n + 2\pi \sum_k g_k (C_{\text{unk}} - C_{\text{0k}}) \left( \frac{2\pi}{L_x} \right)^{x_k - 1},$$

$$\mathcal{E}_0 = \mathcal{E}_{0,c} + 2\pi \sum_k g_k C_{\text{0k}} \left( \frac{2\pi}{L_x} \right)^{x_k - 1},$$

where $C_{\text{unk}}$ are universal structure constants. In the case of the cylinder the spectrum of the Hamiltonian are built by the irreducible representation $\Delta, \bar{\Delta}$ of two commuting Virasoro algebras $L_n$ and $\bar{L}_n$. The leading finite-size corrections ($1/L_x^2$) can be described by the Hamiltonian with a single perturbative conformal field $\phi_1(v) = L_x^2 \bar{\phi}_1(v) + \bar{L}_x^2(v)$ with scaling dimension $x_1 = 4$ [10]. Thus, the ratios $r_n(2)$ are indeed universal and given by

$$r_n(2) = \frac{C_{n1n} - C_{010}}{C_{010}}. \quad (16)$$

The universal structure constants $C_{n1n}$ can be obtained from the matrix elements $\langle n|\phi_1(0)|n\rangle = (2\pi/L_x)^2 C_{n1n}$ [26], which, for non-degenerate states, have already been determined by Reinicke [27]:

$$C_{n1n} = (\Delta + r) \left( \frac{2 + c}{12} + \frac{r(2\Delta + r)(5\Delta + 1)}{(\Delta + 1)(2\Delta + 1)} \right) + \left( \frac{c}{24} \right)^2 + \frac{11c}{1440} + \frac{r}{36} \left[ r^2(5c - 8) - (5c + 28) \right] \delta_{\Delta,0} + (\Delta \to \bar{\Delta}, r \to \bar{r}), \quad (17)$$

where $\Delta$ and $\bar{\Delta}$ are the highest conformal weights, $r, \bar{r} \in \mathbb{R}$ are label descendant levels. A state will be labelled by $|\eta\rangle = (\Delta, r; \bar{\Delta}, \bar{r})$. Let us consider the case $c = 1/2$. For the two-dimensional Ising model with duality twisted BCs the ground state $|0\rangle$, the excited state $|p\rangle$ and the excited state $|L - p\rangle$ are given by

$$|0\rangle = |\Delta = \frac{1}{16}, r = 0; \bar{\Delta} = 0, \bar{r} = 0\rangle, \quad (18)$$

$$|p\rangle = |\Delta = \frac{1}{16}, r = 0; \bar{\Delta} = \frac{1}{2}, \bar{r} = p - 1\rangle, \quad (19)$$

$$|L - p\rangle = |\Delta = \frac{1}{16}, r = p; \bar{\Delta} = 0, \bar{r} = 0\rangle. \quad (20)$$

For non-degenerate states the universal structure constants $C_{n1n}$ ($n = 0, p$ and $L - p$) can be obtained from eq. (17) and for ratios $r_p(2)$ and $r_{L-p}(2)$ one can obtain

$$r_p(2) = -240(2p - 1)^3, \quad (21)$$

$$r_{L-p}(2) = \frac{1920}{17} (16p^2 + 3p - 2). \quad (22)$$

Thus, we can see that eq. (21) coincides with eq. (14) for all values of $p$, while eq. (22) coincides with eq. (15) only for $p = 1$ and 2. The excited states $|L - p\rangle$ (for $p \geq 3$) are degenerated and one cannot apply the Reinicke formula (17). For degenerated states the calculations of the universal structure constants $C_{n1n}$ is not straightforward, but for the case of the Ising model it can be done. For the Ising model we have calculated the universal structure constants $C_{n1n}$ for degenerated states and find that the results are in complete agreement with eq. (15) for all values of $p$.

In this letter we considered the Ising model with duality twisted BCs. We have shown that BCs with $(\Delta, \bar{\Delta}) = \left( \frac{1}{16}, 0 \right)$ correspond to the sean $(r, s) = (1, 2)$. We derive exact expressions for all eigenvalues of the transfer matrix for the critical Ising model with the duality twisted BCs. We reproduce by an exact calculation the exact formula for the universal conformal partition function $Z_{(1,2)}(q)$, for which there is also an anterior numerical confirmation [23]. We find that the ratio of the subdominant finite-size corrections to scaling in the asymptotic expansion of the free energy $f$ and the inverse correlation lengths $\xi^{-1}$ and $\xi_{L-p}^{-1}$ are universal and give their exact value. We verify this universal behavior using a perturbative conformal approach.

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