Euler–Lagrange equations for variational problems involving the Riesz–Hilfer fractional derivative

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ABSTRACT
In this paper, we obtain the Euler-Lagrange equations for different kind of variational problems with the Lagrangian function containing the Riesz-Hilfer fractional derivative. Since the Riesz-Hilfer fractional derivative is a generalization for the Riesz-Riemann-Liouville and the Riesz-Caputo derivative, then our results generalize many recent works in which the Lagrangian function involving the Riesz-Riemann-Liouville or the Riesz-Caputo derivative. We also study the problem in the presence of delay derivatives and establish a version for Noether theorem in the Riesz-Hilfer sense. In order to achieve our aims we derive some formulas to integration by parts for the Riesz-Hilfer fractional derivative. In the last section, examples are given to clarify the possibility of applicability of our results. In order to clarify the significant conclusions of the paper, we refer to our techniques enable to study many different variational problems containing the Riesz-Hilfer derivative.

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1. Introduction
There are many applications for differential equations and inclusions of fractional order in various fields [1–4], and many results are obtained on this subject, one can see, e.g. [5–14] and the references therein. Furthermore, the field of calculus of variation have importance in engineering, optimal control theory, and pure and applied mathematics, see for example, [15–18]. The calculus of variations with fractional derivative is initiated with the work of Riewe [19, 20]. It deals with variational problems where the Lagrangian function containing fractional derivative or fractional integrals. In order to mention some recent works on fractional variational problems, Agrawal [21] derived a necessary conditions for functionals, containing multiple of left and right Riemann–Liouville fractional derivative (RLFD), to have an extremum. Agrawal [22]. considered a Lagrangian containing the Riesz–Caputo fractional derivative (RCFD). Almeida [23] considered different variational problems involving Caputo fractional derivatives (CFD). Almeida [24] obtained the necessary conditions for a pair function-time to be an optimal solution, when the Lagrangian function involving (RCFD) and the interval of integration is contained in the interval of fractional derivative. Odzijewicz et al. [25], obtained the Euler–Lagrange equations for functionals containing Caputo and combined Caputo fractional derivatives. Sayevand et al. [26], considered delay fractional variation problems with isoperimetric and holomorphic constraints and involving (CFD). Tavares et al. [27] studied two variational problems involving (RCFD) and a state time delay. Almiada et al. [28] considered functional containing distributed—order fractional derivatives. For more results on fractional variational calculus, we refer to [29–40].

On the other hand, Hilfer [41] introduced the Hilfer fractional derivative (HFD), which includes (RLFD) and (CFD). Agrawal et al. [42] developed the fractional Euler–Lagrange equations for functionals containing (HFD) with three parameter fractional.

For new and important developments for searching exact and numerical solutions for nonlinear partial differential equations by used a kind of mathematical methods, we refer to [43–57].

In this paper, and in order to generalize many results cited above, we give the notations for the Riesz–Hilfer fractional derivative (RHFD) which includes Riesz–Riemann–Liouville fractional derivative (RRLFD) and the Riesz–Caputo fractional derivatives (RCFD), and hence we obtain the Euler–Lagrange equations for different kind of fractional variational problems, with a Lagrangian containing (RHFD). In Section 2, we derive some formulas to integration by parts for (RHFD). In Section 3, we consider a simple fractional variational problem, then we take the case when the interval of integration of the functional is contained in the interval.

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of fractional derivative. In Section 4, we obtain the Euler–Lagrange equations for isoperimetric problems, in which the function eligible for the extremization of a given definite integral is required to conform with certain restrictions. In Section 5, we study the problem in the presence of delay derivatives. In Section 6, we obtain the conditions that assure a pair of function-time to be an optimal solution. In Section 7, we establish a version for Noether theorem in the Riesz–Hilfer sense is given. Finally, we give examples to clarify the possibility of applicability of our results.

To the best of our knowledge, up to now, no work has reported on fractional variational problems, with a Lagrangian function involving (RHFD). Moreover, since (RHFD) is generalization for (RRLFD) and (RCFD), then our work generalizes many works mentioned above and allow to consider other variational problems, where the Lagrangian function involving (RHFD).

2. Preliminaries and notations

For any natural number m let

\[ AC^m[a, b] := \{ f : f \text{ has continuous derivatives up to order } m \text{ on } [a, b] \} \]

In the sequel we use the following notations:

(1) (FVP) is the fractional variational problem
(2) (RLFI) is the Riemann–Liouville fractional integral.
(3) (LRLFI) is the left-sided Riemann–Liouville fractional integral.
(4) (RRLFI) is the right-sided Riemann–Liouville fractional integral.
(5) (LRLFD) is the left-sided Riemann–Liouville fractional derivative.
(6) (RRLFD) is the right-sided Riemann–Liouville fractional derivative.
(7) (RFI) is the Riesz fractional integral.
(8) (LHFD) is the left-sided Hilfer fractional derivative.

We recall some concepts on fractional calculus [1, 4].

**Definition 2.1:** The (LRLFI) of order \( \mu > 0 \) for a Lebesgue integrable function \( h : [a, b] \rightarrow \mathbb{R} \) is given by:

\[
\alpha^\mu_{a} h(t) := \frac{1}{\Gamma(\mu)} \int_{a}^{t} (t - s)^{\mu - 1} h(s) \, ds, \quad t \in [a, b],
\]

where \( \Gamma \) is the Euler gamma function.

**Definition 2.2:** The (RRLFI) of order \( \mu > 0 \) for a Lebesgue integrable function \( h : [a, b] \rightarrow \mathbb{R} \) is defined as:

\[
\tau^\mu_{b} h(t) := \frac{1}{\Gamma(\mu)} \int_{t}^{b} (s - t)^{\mu - 1} h(s) \, ds, \quad t \in [a, b].
\]

It is known that \( \alpha^\mu_{a} \left( \alpha_{a}^{\mu} h(t) \right) = \alpha_{a}^{\mu + \beta} h(t) \), and \( \tau_{b}^{\mu} \left( \tau_{b}^{\mu} h(t) \right) = \tau_{b}^{\mu + \beta} h(t) \), \( \beta, \mu > 0 \). If \( \mu = 0 \), we set \( \alpha_{a}^{0} h(t) = h(t) \), \( t \in [a, b] \).

**Lemma 2.1 ([1]):** Let \( \mu > 0 \) and \( h \in L^p[a, b] \), \( \eta \in L^r[a, b] \) such that \( p \geq 1, s \geq 1 \) and \( 1/p + 1/r + 1/s \leq 1 + 1/ \mu \) \( (p > 1, s > 1, \text{in the case when } 1/p + 1/s < 1 + 1/ \mu) \). Then

\[
\int_{a}^{b} h(t) (\alpha_{a}^{\mu} \eta)(t) \, dt = \int_{a}^{b} \eta(t) (\tau_{b}^{\mu} h(t)) \, dt.
\]

**Definition 2.3:** The (RFI) of order \( \mu > 0 \) for a function \( h \in L^1[a, b] \), is defined as:

\[
\tau_{b}^{\mu} h(t) := \frac{1}{2} \left( \alpha_{a}^{\mu} h(t) + \tau_{b}^{\mu} h(t) \right), \quad t \in [a, b].
\]

Notice that

\[
\alpha_{a}^{\mu} h(t)
\]

\[
= \frac{1}{\Gamma(\mu)} \left[ \int_{a}^{t} (t - s)^{\mu - 1} h(s) \, ds + \int_{t}^{b} (s - t)^{\mu - 1} h(s) \, ds \right],
\]

\[
= \frac{1}{\Gamma(\mu)} \left[ \int_{a}^{t} |t - s|^{\mu - 1} h(s) \, ds + \int_{t}^{b} |t - s|^{\mu - 1} h(s) \, ds \right],
\]

\[
= \frac{1}{\Gamma(\mu)} \int_{a}^{b} |t - s|^{\mu - 1} h(s) \, ds, \quad t \in [a, b].
\]

In what follows \( \mu \) denotes to a positive real number and \( m \) is the smallest natural number such that \( \mu \leq m \).

**Definition 2.4:** Let \( h \in L[a, b] \) such that \( \alpha_{a}^{m-\mu} h \in AC^m[a, b] \). The (LRLFD) of order \( \mu \) for \( h \) at \( t \in [a, b] \) is given by

\[
a \mathcal{D}^{\mu}_{a} h(t) = \begin{cases} \frac{d^{m}}{dt^{m}} \left( \alpha_{a}^{m-\mu} h(t) \right), & \text{if } m - 1 < \mu < m, \\ \frac{d^{m}}{dt^{m}} h(t), & \text{if } \mu = m. \end{cases}
\]

**Definition 2.5:** Let \( h \in L^1((a, b], E) \) such that \( \tau_{b}^{m-\mu} h \in AC^m((a, b], E) \). The right-sided Riemann–Liouville fractional derivative of order \( \mu \) for \( h \) at \( t \in [a, b] \) is defined by

\[
\mathcal{D}^{\mu}_{b} h(t) = \begin{cases} (-1)^{m} \frac{d^{m}}{dt^{m}} \left( \tau_{b}^{m-\mu} h(t) \right), & \text{if } m - 1 < \mu < m, \\ (-1)^{m} \frac{d^{m}}{dt^{m}} h(t), & \text{if } \mu = m. \end{cases}
\]
Definition 2.6: Let \( h \in L^1[a,b] \) such that \( aD_{t}^{\mu-m}h, \) \( aD_{t}^{\mu-m}h \in AC^m[a,b] \). The (RRLFD) of order \( \mu \) for \( h \) is given by

\[
\frac{\partial D_{b}^{\mu} h(t)}{\partial t} = \frac{1}{2} [aD_{t}^{\mu}h(t) + (-1)^{m} C_{t}D_{b}^{\mu} h(t)], \quad t \in [a,b].
\]

Definition 2.7: The left-sided Caputo fractional derivative of order \( \mu \) for \( h \in AC^m[a,b] \) is given at \( t \in [a,b] \) by

\[
C_{a}D_{t}^{\mu} h(t) := \begin{cases} \int_{a}^{t} \frac{(t-s)^{\mu-m} h(s)}{\Gamma(\mu-m+1)} ds, & \text{if } m-1 < \mu < m, \\ \frac{d^{m}h}{dt^{m}}(t), & \text{if } \mu = m. \end{cases}
\]

Definition 2.8: The right-sided Caputo fractional derivative of order \( \mu \) for \( h \in AC^m[a,b] \) is given at \( t \in [a,b] \) by

\[
C_{b}D_{t}^{\mu} h(t) := \begin{cases} \int_{a}^{t} \frac{(t-s)^{\mu-m} h(s)}{\Gamma(\mu-m+1)} ds, & \text{if } m-1 < \mu < m, \\ \frac{d^{m}h}{dt^{m}}(t), & \text{if } \mu = m. \end{cases}
\]

Definition 2.9: Let \( h \in AC^m[a,b] \). The (RCFD) of order \( \mu > 0 \) for \( h \in AC^m[a,b] \) is given by

\[
\frac{\partial D_{b}^{\mu} h(t)}{\partial t} = \frac{1}{2} [aD_{t}^{\mu}h(t) + (-1)^{m} C_{t}D_{b}^{\mu} h(t)], \quad t \in [a,b].
\]
Lemma 2.1, it yields

\[ \int_a^b h(t) D_t^\beta \eta(t) \, dt = \int_a^b h(t) D_t^\beta(D_t^\beta \eta(t)) \, dt \]

\[ = \int_a^b (D_t^\beta \eta(t)) D_t^\beta h(t) \, dt \]

\[ = \int_a^b \eta(t) D_t^\beta D_t^\beta h(t) \, dt \]

\[ = \int_a^b \eta(t) D_t^\beta h(t) \, dt \] \hspace{1cm} (17)

Similarly, one can obtain

\[ \int_a^b h(t) t D_t^\beta \eta(t) \, dt \]

\[ = \int_a^b h(t) t D_t^\beta(D_t^\beta \eta(t)) \, dt \]

\[ = \int_a^b (D_t^\beta \eta(t)) D_t^\beta h(t) \, dt \]

\[ = \int_a^b \eta(t) D_t^\beta D_t^\beta h(t) \, dt \]

\[ = \int_a^b \eta(t) D_t^\beta h(t) \, dt \] \hspace{1cm} (18)

From the definition of \( R^H D_t^\beta \eta(t) \), (17) and (18) it yields (16).

**Corollary 2.1:** (1) If we put \( \beta = 0 \) in (16), we obtain relation (20) in [22] and the integration rule by parts formula in [24]. In fact

\[ \int_a^b h(t) R^H D_t^\beta \eta(t) \, dt \]

\[ = \int_a^b h(t) R^H D_t^\beta \eta(t) \, dt \]

\[ = - \int_a^b h(t) R^H D_t^\beta \eta(t) \, dt \]

\[ + \frac{1}{2} \eta(t) D_t^\beta h(t) |_a^b + \frac{1}{2} \eta(t) D_t^\beta h(t) |_a^b \]

\[ = - \int_a^b h(t) R^H D_t^\beta \eta(t) \, dt + \frac{1}{2} \eta(t) D_t^\beta h(t) |_a^b. \] \hspace{1cm} (19)

(2) If we put \( \beta = 1 \) in (16) we obtain relation (21) in [22]. In fact

\[ \int_a^b h(t) R^H D_t^1 \eta(t) \, dt \]

\[ = \int_a^b h(t) R^H D_t^1 \eta(t) \, dt \]

\[ = - \int_a^b h(t) R^H D_t^1 \eta(t) \, dt \]

\[ + \frac{1}{2} h(t) D_t^1 \eta(t) |_a^b \]

\[ = - \int_a^b h(t) R^H D_t^1 \eta(t) \, dt + h(t) D_t^1 \eta(t) |_a^b. \] \hspace{1cm} (20)

We need to the following lemma in the third section.

**Lemma 2.3:** If \( \mu \in (0, 1] \), \( \beta \in (0, 1] \) and \( h, \eta : [a, b] \to \mathbb{R} \) are continuously differentiable and \( r \in (a, b) \), then

\[ \int_a^r h(t) R^H D_t^\beta \eta(t) \, dt \]

\[ = - \int_a^r \eta(t) R^H D_t^\beta h(t) \, dt \]

\[ + \frac{1}{2} \left[ \int_a^r \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right] \]

\[ + \frac{1}{2} \left[ \int_a^r \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right] \]

\[ + \frac{1}{2} \left[ \int_a^r \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right] \]

\[ - \frac{1}{2} \left[ \int_a^r \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right] \]

\[ = - \int_a^r h(t) R^H D_t^\beta \eta(t) \, dt + \frac{1}{2} \left[ \int_a^r \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right]. \] \hspace{1cm} (21)

and

\[ \int_r^b h(t) R^H D_t^\beta \eta(t) \, dt \]

\[ = - \int_r^b h(t) R^H D_t^\beta \eta(t) \, dt \]

\[ + \frac{1}{2} \left[ \int_r^b \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right] \]

\[ + \frac{1}{2} \left[ \int_r^b \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right] \]

\[ - \frac{1}{2} \left[ \int_r^b \eta(t) (a D_t^{\beta - 1} - a D_t^{\beta - 1}) h(t) \, dt \right]. \]
+ \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt \\
+ \frac{1}{2} \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) - t D_0^{\mu - \beta} h(t) \, dt \\
+ \frac{1}{2} \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
+ a_t^b h(t) a t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
+ a_t^b h(t) a t D_t^{\mu - \beta} h(t) \, dt \\
- t D_0^{\mu - \beta} h(t) \, dt.

(22)

Proof: Let \( q = (1 - \mu)(1 - \beta) \). According to (17) we have

\[
\int_a^b h(t) a D_t^{\mu - \beta} \eta(t) \, dt \\
= \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt \\
+ \frac{1}{2} \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
+ a_t^b h(t) a t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
- t D_0^{\mu - \beta} h(t) \, dt.
\]

and this proves (21). Similarly we prove the validity of (22). In view of (17) and (18) we get

\[
\int_a^b h(t) a D_t^{\mu - \beta} \eta(t) \, dt \\
= \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt - \frac{1}{2} \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt \\
+ \frac{1}{2} \int_a^b \eta(t) t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
+ a_t^b h(t) a t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
- t D_0^{\mu - \beta} h(t) \, dt.
\]

(26)

Also, relation (18) leads to

\[
\int_a^b h(t) D_t^{\mu - \beta} \eta(t) \, dt \\
= \int_a^b h(t) D_t^{\mu - \beta} \eta(t) \, dt - \int_a^b h(t) D_t^{\mu - \beta} \eta(t) \, dt \\
+ \frac{1}{2} \int_a^b h(t) D_t^{\mu - \beta} \eta(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
+ a_t^b h(t) a t D_t^{\mu - \beta} h(t) \, dt \\
- r_t^b h(t) (a t D_t^{\mu - \beta} - r D_t^{\mu - \beta} h(t)) \, dt \\
- t D_0^{\mu - \beta} h(t) \, dt.
\]

(24)

It follows from Equations (23) and (24) that

\[
\int_a^b h(t) D_0^{\mu - \beta} \eta(t) \, dt \\
= \frac{1}{2} \left[ \int_a^b h(t) a D_t^{\mu - \beta} \eta(t) \, dt - \int_a^b h(t) t D_t^{\mu - \beta} \eta(t) \, dt \right] \\
= - \int_a^b h(t) D_0^{\mu - \beta} \eta(t) \, dt.
\]

(27)

It follows from (26) and (27) that

\[
\int_a^b h(t) D_0^{\mu - \beta} \eta(t) \, dt \\
= - \int_a^b h(t) D_0^{\mu - \beta} \eta(t) \, dt.
\]
\[ + \frac{1}{2} \left( \int_a^b \eta(t) (a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} - r D_{b}^{\mu,1-\beta} h(t)) \, dt \right) + t \partial^{(1-\mu)}_b \eta(t) \| h(t) \|_0^b \]
\[ + \int_a^b \eta(t) (a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} - r D_{b}^{\mu,1-\beta} h(t)) \, dt \]
\[ + t \partial^{\mu}_b (t) - \partial^{(1-\mu)}_b \eta(t) \big|_0^b. \]

So, (22) is true. \[\blacksquare\]

**Remark 2.3:** If we add Equations (21) and (22), then we obtain (16). This means that the obtained results in Lemmas 2.2 and 2.3 are compatible.

We need to the following basic Lemma [18].

**Lemma 2.4:** If \( G : [a, b] \rightarrow \mathbb{R} \) is a continuous function and \( G(t) \eta(t) \big|_0^b \) for every choice of the continuously differentiable function \( \eta \) for which \( \eta(a) = \eta(b) = 0 \), we conclude that \( G(t) = 0 \), for any \( t \in [a, b] \).

### 3. Euler–Lagrange equation for a simple fractional variational problem involving the Riesz–Hilfer derivative

Let \( \mu \in (0, 1] \) and \( \beta \in [0, 1) \).

**Theorem 3.1:** Assume that the first and second partial derivatives of a Lagrangian function \( L : [a, b] \times \mathbb{R}^{3n} \rightarrow \mathbb{R} \) with respect to all its arguments are continuous. Consider a functional of the form

\[ J[q_1, \ldots, q_n] = \int_a^b \left[ L(t, q_1(t), \ldots, q_n(t), \dot{q}_1(t), \ldots, \dot{q}_n(t), a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_1(t), \ldots, a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_n(t)) \right] \, dt, \]

defined on the set of functions \( q_1, \ldots, q_n \) which are continuously differentiable and such that \( a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_i(t) \) is continuous in \([a, b]\) and that

\[ q_i(a) = q_i^a \quad \text{and} \quad q_i(b) = q_i^b, \quad i = 1, 2, \ldots, n. \]

Then a necessary condition for the functional (29) attains an extremum at \( q_i \), \( i = 1, 2, \ldots, n \), is that \( q_i \) satisfy the following Euler–Lagrange equations:

\[ \frac{\partial L}{\partial q_i}(t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(t) - a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta}(t) = 0, \quad \forall t \in [a, b]. \]

If \( \beta = 1 \), then the functions \( q_i \) should be satisfy the transversality condition:

\[ \left. \frac{\partial L}{\partial a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_i}(t) \right|_{t=a} = \left. \frac{\partial L}{\partial a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_i}(t) \right|_{t=b} = 0. \]

**Proof:** It is known that the necessary condition for \( q_i \), \( i = 1, 2, \ldots, n \) to be extremum, is given by

\[ \frac{\partial J(q_i + \epsilon \eta_i)}{\partial \epsilon} \bigg|_{\epsilon=0} = 0, \quad \forall i = 1, 2, \ldots, n. \]

where \( \eta_i \) are arbitrary continuously differentiable functions for which

\[ \eta_i(a) = \eta_i(b) = 0. \]

That is

\[ 0 = \int_a^b \sum_{i=1}^n \left[ \frac{\partial L}{\partial q_i} \eta_i(t) + \frac{\partial L}{\partial \dot{q}_i} \eta_i(t) + \frac{\partial L}{a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_i} (a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} \eta_i(t)) \right] \, dt \]

Put \( q = (1 - \mu)(1 - \beta) \) and using Lemma 2.2, it follows that

\[ \int_a^b \frac{\partial L}{a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_i} (a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} \eta_i(t)) \, dt \]

\[ = - \int_a^b \eta_i(t) a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} z_i(t) \, dt \]

\[ + \frac{1}{2} t \partial^{\mu}_b \eta_i(t) - \partial^{(1-\mu)}_b \eta_i(t), \quad i = 1, 2, \ldots, n. \]

defined on the set of functions \( z_1, \ldots, z_n \) which are continuously differentiable and such that \( a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} z_i(t) \) is continuous in \([a, b]\) and that

\[ z_i(a) = z_i^a \quad \text{and} \quad z_i(b) = z_i^b, \quad i = 1, 2, \ldots, n. \]

Then a necessary condition for the functional (29) attains an extremum at \( z_i \), \( i = 1, 2, \ldots, n \), is that \( z_i \) satisfy the following Euler–Lagrange equations:

\[ \frac{\partial L}{\partial q_i}(t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(t) + a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta}(t) = 0, \quad \forall t \in [a, b]. \]

If \( \beta = 1 \), then Equation (32) leads to

\[ t \partial^{\mu}_b \eta_i(t) - \partial^{(1-\mu)}_b \eta_i(t) + \frac{1}{2} t \partial^{(1-\mu)}_b \eta_i(t) = 0, \]

\[ \int_a^b a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} z_i(t) \, dt \]

Then, by (37)–(39), Equation (36) becomes

\[ \int_a^b \frac{\partial L}{a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} q_i} (a \partial^{\mu,1-\beta}_b D_{b}^{\mu,1-\beta} \eta_i(t)) \, dt \]

Moreover, Equations (34) leads to
\[ \int_a^b \frac{\partial L}{\partial q_i'} \eta_i(t) \, dt = \int_a^b \frac{d}{dt} \frac{\partial L}{\partial q_i} \eta_i(t) \, dt - \int_a^b \frac{d}{dt} \frac{\partial L}{\partial q_i} \eta_i(t) \, dt \]
\[ = - \int_a^b \frac{d}{dt} \frac{\partial L}{\partial q_i} \eta_i(t) \, dt. \] (41)

It yields from (35),(40) and (41) that
\[ 0 = \sum_{i=1}^n \int_a^b \left[ \frac{\partial L}{\partial q_i'}(t) - \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right)(t) - \frac{\partial \mathcal{R} H_{b}^{\mu \beta}}{\partial q_i} \eta_i(t) \right] \eta_i(t) \, dt, \]
\[ \forall i = 1, 2, \ldots, n, \] (42)
and this should be true for all admissible functions \( \eta_i \).

Since the functions \( \eta_i \) are independent, then we can choose for any fixed \( i \), \( \eta_k(t) = 0, \forall t \in [a, b], k \neq i \). Hence Lemma 2.4, gives us
\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) = \frac{\partial \mathcal{R} H_{b}^{\mu \beta}}{\partial q_i} q_i, \]
\[ \forall i = 1, 2, \ldots, n, \] (43)
and the proof is completed. \( \blacksquare \)

Remark 3.1: If \( \beta = 0 \) in the previous theorem and \( \beta = 1 \) then we obtain Theorem 1 in [22].

Now we extend Theorem 3.1 when the interval of integration is contained in the interval of fractional derivative.

**Theorem 3.2:** Assume that the Lagrangian function \( L: [a, b] \times \mathbb{R} \to \mathbb{R} \) be as in Theorem 3.1. Consider the functional
\[ J(q_1, \ldots, q_n) = \int_A^B L(t, q_1(t), \ldots, q_n(t), \frac{\partial \mathcal{R} H_{b}^{\mu \beta}}{\partial q_i} q_i(t)), \]
\[ \ldots, \frac{\partial \mathcal{R} H_{b}^{\mu \beta}}{\partial q_i} q_i(t) \, dt, \]
\[ \text{defined on the set of functions } q_1, \ldots, q_n \text{ which are continuously differentiable and have continuous Riesz–Hilfer derivatives of order } \mu \text{ and of type } \beta \text{ in } [a, b], \text{ where } [A, B] \subseteq [a, b]. \]

Then necessary conditions for the functions \( q_i, i = 1, 2, \ldots, n, \) which satisfy (30) to be an extremum of the functional given by (31) are that \( q_i, i = 1, 2, \ldots, n, \) satisfy the following Euler–Lagrange equations:
\[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) = \frac{\partial \mathcal{R} H_{b}^{\mu \beta}}{\partial q_i} q_i, \]
\[ \forall t \in [a, b], \] (45)
\[ \mathcal{D}_{b_{\mu \beta}} q_i(t) - \mathcal{D}_{a_{\mu \beta}} q_i(t) = 0, \]
\[ \forall t \in [a, A], \] (46)

Utilizing the rule of integration by parts (21) and (22) and taking into account the assumptions \( \eta_i(A) = \eta_i(B) = 0, i = 1, 2, \ldots, n, \) we get
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) \eta_i(t) \, dt = \sum_{i=1}^n \int_A^B \left( \frac{\partial L}{\partial q_i} \right) \eta_i(t) \, dt \]
\[ \text{for any } t \in [A, B]. \]
If $\beta = 1$, then equation (48) implies that

$$
\begin{align*}
&= \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{b}^{(1-\mu)} - \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} - \eta(t)_{A}^{(1-\mu)} + 2 B \int_{0}^{1} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{b}^{(1-\mu)} + 2 \int_{0}^{1} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{A}^{(1-\mu)} = 0. 
\end{align*}
$$

Using equations (53)–(55), equation (52) becomes

$$
\frac{dJ}{de} = \sum_{i=1}^{n} \left[ \int_{0}^{1} A \eta(t)_{i}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) - \int_{0}^{1} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{i}^{(1-\mu)} \right] dt 
+ \int_{0}^{1} B \eta(t)_{i}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) dt
$$

$$
= \sum_{i=1}^{n} \left[ \int_{0}^{1} A \eta(t)_{i}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) - \int_{0}^{1} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{i}^{(1-\mu)} \right] dt 
+ \int_{0}^{1} B \eta(t)_{i}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) dt
$$

If $\beta \in (0, 1)$, then $(1-\beta)(1-\mu) \neq 0 \neq \beta(1-\mu)$, and consequently, the continuity of $\partial L/(a \frac{\partial H D_b}{\partial \mu})$ and $\eta(t)$ implies to

$$
0 = t_{b}^{(1-\beta)(1-\mu)} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{b}^{(1-\mu)} - t_{A}^{(1-\beta)(1-\mu)} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} - \eta(t)_{A}^{(1-\mu)} + 2 B t_{b}^{(1-\beta)(1-\mu)} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{b}^{(1-\mu)} + 2 \int_{0}^{1} B \eta(t)_{A}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) dt 
$$

$$
= t_{b}^{(1-\beta)(1-\mu)} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{b}^{(1-\mu)} - t_{A}^{(1-\beta)(1-\mu)} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} - \eta(t)_{A}^{(1-\mu)} + 2 B t_{b}^{(1-\beta)(1-\mu)} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{b}^{(1-\mu)} + 2 \int_{0}^{1} B \eta(t)_{A}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) dt 
$$

$$
= \sum_{i=1}^{n} \left[ \int_{0}^{1} A \eta(t)_{i}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) - \int_{0}^{1} \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \eta(t)_{i}^{(1-\mu)} \right] dt 
+ \int_{0}^{1} B \eta(t)_{i}^{(1-\beta)} \left( \frac{\partial L}{\partial a \frac{\partial H D_b}{\partial \mu}} \right) dt
$$
appropriate choices of derivatives of a Lagrangian function \( L \) and a function \( z \).

Theorem 4.1: In this section we consider a problem in which the eligible integral is required to satisfy certain restrictions that are added to the usual conditions.

\[
\begin{align*}
- \frac{1}{2} \int_{a}^{b} \eta_i(t) \frac{d}{dt} \left( \frac{\partial L}{\partial z_i} \right) dt - \frac{1}{2} \int_{a}^{b} \eta_i(t) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_i} \right) dt & \quad \text{for } \beta = 0, \\
- \frac{1}{2} \int_{a}^{b} \eta_i(t) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_i} \right) dt & \quad \text{where } L = \int_{a}^{b} L(t, z(t), \dot{z}(t)) dt.
\end{align*}
\]

and

\[
\int_{a}^{b} \eta_i(t) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_i} \right) dt = l.
\]

If \( L \) is continuous on \([a, b]\), then, necessary conditions for \( J(q_1, q_2, \ldots, q_n) \) to have an extremum at \( q_i^* \), \( i = 1, 2, \ldots, n \), which satisfies the boundary conditions (30), such that

\[
\int_{a}^{b} \eta_i(t) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_i} \right) dt = l
\]

are that \( q_i^* \) satisfy the following Euler–Lagrange equations

\[
\frac{\partial L}{\partial z_i} - \frac{\partial}{\partial \dot{z}_i} \left( \frac{\partial L}{\partial \dot{z}_i} \right) + \lambda \frac{\partial z}{\partial q_i} = 0,
\]

\( \forall i = 1, 2, \ldots, n. \)

If \( \beta = 1 \), then \( q_i^* \) should satisfy the transversality conditions

\[
\eta_i(t_{1}) = \frac{\partial L}{\partial \dot{z}_i} \bigg|_{t_{1}} = 0,
\]

and

\[
\lambda \bigg|_{t_{2}} = 0,
\]

where \( \lambda \) is the Lagrange’s multiplier whose value can be determined by the conditions on \( L \) and \( z \).

Proof: To derive the necessary conditions let

\[
q_i(t) = q_i^*(t) + \epsilon_i \eta_i(t) + \epsilon_i \zeta_i(t), \quad i = 1, 2, \ldots, n.
\]

where \( \eta_i \) and \( \zeta_i \) are arbitrary continuously differentiable functions for which

\[
\eta_i(a) = \eta_i(b) = \zeta_i(a) = \zeta_i(b) = 0, \quad i = 1, 2, \ldots, n.
\]

Inserting (64) in (59) and (60), respectively, we get

\[
J[\epsilon_1, \epsilon_2] = \int_{a}^{b} L(t, q_1(t) + \epsilon_1 \eta_1(t) + \epsilon_2 \zeta_1(t), \ldots, q_n(t) + \epsilon_1 \eta_n(t) + \epsilon_2 \zeta_n(t)) dt
\]

and

\[
J[\epsilon_1, \epsilon_2] = \int_{a}^{b} g(t, q_1(t) + \epsilon_1 \eta_1(t) + \epsilon_2 \zeta_1(t), \ldots, q_n(t) + \epsilon_1 \eta_n(t) + \epsilon_2 \zeta_n(t) dt = l.
\]

Clearly, the parameters \( \epsilon_1 \) and \( \epsilon_2 \) are not independent because \( \int_{a}^{b} L(t, q_1(t), \ldots, q_n(t), \dot{q}_1(t), \ldots, \dot{q}_n(t)) dt = \int_{a}^{b} g(t, q_1(t), \ldots, q_n(t)) dt = l \). Since \( q_i^* \), \( i = 1, 2, \ldots, n \) are assumed to be the actual extremizing functions, we
have \( J(\epsilon_1, \epsilon_2) \) is extremum with respect to \( \epsilon_1 \) and \( \epsilon_2 \) which satisfy (65), when \( \epsilon_1 = \epsilon_2 = 0 \). According to the method of Lagrange multipliers we introduce

\[
J'(\epsilon_1, \epsilon_2) = J(\epsilon_1, \epsilon_2) + \lambda J(\epsilon_1, \epsilon_2),
\]

where \( \lambda \) is the Lagrange's multiplier. Then, according to the method of Lagrange multipliers we must have

\[
\frac{\partial J'}{\partial \epsilon_1} = \frac{\partial J'}{\partial \epsilon_2} = 0 \text{ when } \epsilon_1 = \epsilon_2 = 0.
\]

It follows, by applying the rule of integration by parts (16),

\[
0 = \frac{\partial J'}{\partial \epsilon_1} = \sum_{i=1}^{n} \int_{a}^{b} \left\{ \frac{\partial L}{\partial q_{i}} \eta_{i} + \frac{\partial L}{\partial b D_{b}^{\mu_{i}}} \left( \beta b D_{b}^{\mu_{i}} q_{i} \right) \right\} dt
+ \lambda \int_{a}^{b} \sum_{i=1}^{n} \frac{\partial g}{\partial q_{i}} \beta \left( \sum_{j=1}^{n} \beta_{j} \right) dt,
\]

and

\[
0 = \frac{\partial J'}{\partial \epsilon_2} = \sum_{i=1}^{n} \int_{a}^{b} \left\{ \frac{\partial L}{\partial q_{i}} \beta \left( \sum_{j=1}^{n} \beta_{j} \right) + \frac{\partial g}{\partial q_{i}} \right\} dt.
\]

Consequently from (57)–(59), we get

\[
0 = \frac{\partial J'}{\partial \epsilon_1} = \sum_{i=1}^{n} \int_{a}^{b} \left\{ \frac{\partial L}{\partial q_{i}} \eta_{i} + \frac{\partial L}{\partial b D_{b}^{\mu_{i}}} \left( \beta b D_{b}^{\mu_{i}} q_{i} \right) \right\} dt
+ \lambda \int_{a}^{b} \sum_{i=1}^{n} \frac{\partial g}{\partial q_{i}} \eta_{i} dt,
\]

and

\[
0 = \frac{\partial J'}{\partial \epsilon_2} = \sum_{i=1}^{n} \int_{a}^{b} \left\{ \frac{\partial L}{\partial q_{i}} \beta \left( \sum_{j=1}^{n} \beta_{j} \right) + \frac{\partial g}{\partial q_{i}} \right\} dt.
\]

Since the functions \( \eta_{i} \) and \( \varsigma_{i} \) are independent, the proof is finished.

5. Fractional variational problem with delay

We study the case when there is a delay on the system. Let \( \tau \in (0, b - a) \) and consider the functional

\[
T[q_{1}, \ldots, q_{n}]
= \int_{a}^{b} L(t, q_{1}(t), \ldots, q_{n}(t), \dot{q}_{1}, \ldots, \dot{q}_{n}, \ddot{q}_{1}, \ldots, \ddot{q}_{n}, q_{1}(t - \tau), \ldots, q_{n}(t - \tau), q_{1}(t - \tau), \ldots, q_{n}(t - \tau)) dt,
\]

where \( q_{i} : [a, b] \rightarrow \mathbb{R}, i = 1, 2, \ldots, n \) are continuously differentiable and \( b D_{b}^{\mu_{i}} q_{i}(t) \) is continuous in \([a, b]\) and \( L : [a, b] \times \mathbb{R}^{3n} \rightarrow \mathbb{R} \) is a Lagrangian function.

Theorem 5.1: Assume that the first and second partial derivatives of a Lagrangian function \( L : [a, b] \times \mathbb{R}^{3n} \rightarrow \mathbb{R} \) with respect to all of its arguments are continuous and \( \tilde{\eta}_{i} : [a - \tau, 0] \rightarrow \mathbb{R}, i = 1, 2, \ldots, n \) are continuous functions. Then a necessary condition for the functional (75) subject to boundary conditions

\[
q_{i}(t) = \tilde{q}_{i}(t), t \in [a - \tau, 0] \quad \text{and} \quad q_{i}(b) = q_{i}^{0}, i = 1, 2, \ldots, n
\]

achieves an extremum at \( q_{i}, i = 1, 2, \ldots, n \), is that \( q_{i} \) sat-
isfy following Euler–Lagrange equations

\[
\frac{\partial L}{\partial q_i} (t) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} (t) \right) - \frac{\partial H}{\partial \dot{q}_i} (t) q_i (t) = 0,
\]

for \( t \in [a, b - r] \),

\[
\frac{\partial L}{\partial q_i} (t) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} (t) \right) - \frac{\partial H}{\partial \dot{q}_i} (t) q_i (t) = 0,
\]

for \( t \in [b - r, b] \),

If \( \beta = 1 \), then \( q_i \) should be verify the transversality condition

\[
\left. \frac{\partial L}{\partial \beta \dot{q}_i^\beta} q_i \right|_{t=b} = \left. \frac{\partial L}{\partial \beta \dot{q}_i^\beta} q_i \right|_{t=a} = 0.
\]

**Proof:** We follow the approach discussed in the proof of Theorem 3.1, the necessary condition for \( q_i \), \( i = 1, 2, \ldots, n \) to be extremum, is given by

\[
\left. \frac{\partial J(q_i + \epsilon \eta_i)}{\partial \epsilon} \right|_{\epsilon=0} = 0, \quad \forall i = 1, 2, \ldots, n.
\]

where \( \eta_i \) are arbitrary continuously differentiable functions for which

\[
\eta_i(t) = 0 \text{ if } a - r \leq t \leq a - r \quad \text{and} \quad \eta_i(b) = 0,
\]

\( i = 1, 2, \ldots, n \).

Then

\[
\sum_{i=1}^{n} \int_{a}^{b} \left[ \frac{\partial L}{\partial q_i} (t) \eta_i(t) + \frac{\partial L}{\partial \dot{q}_i} (t) \eta_i'(t) + \frac{\partial L}{\partial \beta \dot{q}_i^\beta} q_i (t) \right] dt = 0.
\]

In the fourth and fifth term making the change of variables for \( t-r \) and taking into account that \( \eta_i(t) = 0 \), for \( t \in [a - r, a] \), we obtain

\[
\int_{a}^{b} \left[ \frac{\partial L}{\partial q_i} (t-r) \eta_i(t-r) + \frac{\partial L}{\partial \dot{q}_i} (t-r) \eta_i'(t-r) \right] dt = 0.
\]

It follows from (82) and (83) that

\[
\sum_{i=1}^{n} \int_{a}^{b} \left[ \frac{\partial L}{\partial q_i} (t) \eta_i(t) + \frac{\partial L}{\partial \dot{q}_i} (t) \eta_i'(t) + \frac{\partial L}{\partial \beta \dot{q}_i^\beta} q_i (t) \right] dt = 0.
\]

Using the usual rule integrating by parts and Equations (21), (22), equation (84) becomes

\[
\sum_{i=1}^{n} \int_{a}^{b} \left[ \frac{\partial L}{\partial q_i} (t) \eta_i(t) + \frac{\partial L}{\partial \dot{q}_i} (t) \eta_i'(t) + \frac{\partial L}{\partial \beta \dot{q}_i^\beta} q_i (t) \right] dt = 0.
\]

This equation reduces to

\[
0 = \sum_{i=1}^{n} \int_{a}^{b} \left[ \frac{\partial L}{\partial q_i} (t) \eta_i(t) + \frac{\partial L}{\partial \dot{q}_i} (t) \eta_i'(t) + \frac{\partial L}{\partial \beta \dot{q}_i^\beta} q_i (t) \right] dt.
\]
Theorem 6.1: Suppose that the first and second partial derivatives of a Lagrangian function $L : [a, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ with respect to all its arguments are continuous. Consider a functional of the form

$$J[q, T] = \int_a^T L(t, q(t), a \partial_t \frac{\partial L}{\partial q} q(t)) \, dt,$$

defined on the set of pairs $(q, T)$, where $T \in [a, b)$ and $q$ is continuously differentiable, $\partial_\alpha \frac{\partial L}{\partial q} q(t)$ $(t) = 1, 2, \ldots, n$ is continuous in $[a, \infty)$ and satisfies the boundary condition $q(a) = q^*$. Then necessary conditions for the functional (89) achieves an extremum at a pair $(q^*, S)$ are

$$L(T, q(S) + a \partial_\alpha \frac{\partial L}{\partial q} q^*(S)) = 0,$$

$$\frac{\partial L}{\partial q^*} = a \partial_\alpha \frac{\partial \partial_\alpha \frac{\partial L}{\partial q} q^*}{\partial q^*} = 0, \quad t \in [a, S].$$

If $\beta = 0$, then the following transversality condition should be hold

$$a \partial_\alpha \frac{\partial L}{\partial q^*} |_{t=0} = s \frac{\partial L}{\partial q} |_{t=S} = 0.$$

If $\beta = 1$, then the following transversality condition should be hold

$$\frac{\partial L}{\partial q^*} |_{t=0} = \frac{\partial L}{\partial q} |_{t=S} = 0.$$

Proof: Let $\epsilon > 0$ and define a family of curves $q(t) = q^*(t) + \epsilon v(t)$, where $v$ is an arbitrary continuously differentiable functions for which $v(\alpha) = 0$. Let $\Delta T$ be a positive real number. Then the function

$$J[q^* + \epsilon v, S + \epsilon \Delta T] = \int_a^{S+\Delta T} L(t, q^*(t) + \epsilon v(t), a \partial_t \frac{\partial L}{\partial q} q^*(t) + \epsilon a \partial_\alpha \frac{\partial L}{\partial q} q^*(t)) \, dt,$$

depends on $\epsilon$ only. Since $J$ admits an extremum at $(q^*, S)$ then the necessary condition for which $J[\epsilon]$ achieves a minimum, is

$$\frac{dJ}{d\epsilon}(0) = 0.$$

Applying Liebniz integral rule we get

$$\frac{dJ[\epsilon]}{d\epsilon} = \Delta T \left[ L(S + \epsilon \Delta T, q^* + \epsilon \Delta T) + \epsilon v(S + \epsilon \Delta T), a \partial_\alpha \frac{\partial L}{\partial q} q^*(S + \epsilon \Delta T) + \epsilon a \partial_\alpha \frac{\partial L}{\partial q} q^*(S + \epsilon \Delta T) \right]$$

$$+ \int_a^{S+\epsilon \Delta T} \frac{\partial L}{\partial q} v(t) + \frac{\partial L}{\partial q} a \partial_\alpha \frac{\partial L}{\partial q} a \partial_\alpha \frac{\partial L}{\partial q} q^*(t) \, dt.$$

6. Optimal time problem

In this section, we find the necessary conditions for a variational problem to have a extremum on an optimal time.

Theorem 6.1: Suppose that the first and second partial derivatives of a Lagrangian function $L : [a, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ with respect to all its arguments are continuos. Consider
From (97) and (96) one obtains
\[0 = \Delta T L(S, q(S)) + \int_a^S \left[ \frac{\partial L}{\partial q} v(t) + \frac{\partial L}{\partial D_b^\mu q} (\nu^{Rb}_D v(t)) \right] \, dt. \tag{98}\]

Since setting \(\varepsilon\) equal to zero is equivalent to replacing \(q_i\) and \(\nu^{Rb}_D q_i\) by \(a q_i\) and \(D_b^\mu q_i\), the last equation becomes
\[0 = \Delta T L(S, q^*(S), D_b^\mu q(S)) + \int_a^S \left[ \frac{\partial L}{\partial q} v(t) + \frac{\partial L}{\partial D_b^\mu q} (\nu^{Rb}_D v(t)) \right] \, dt. \tag{99}\]

Now, to simplify the notations we put \(z(t) = 0\). Then, applying relation (20) to and get
\[\int_a^S z(t) D_b^\mu v(t) \, dt = 0. \tag{100}\]

It follows from (99) and (100) that
\[\Delta T L(S, q(S)) + \int_a^S v(t) \left[ \frac{\partial L}{\partial q} - \frac{\partial L}{\partial D_b^\mu q} z(t) \right] \, dt + \frac{1}{2} \int_a^S v(t) (a D_t^\mu - D_b^\mu z(t)) \, dt - \int_a^S v(t) (a b D_t^\mu - b D_b^\mu z(t)) \, dt + t s \left[ (\nu^{Rb}_D v(t)) (a D_t^\mu - D_b^\mu v(t)) \right] = 0. \tag{101}\]

Remark that if \(\beta \in (0, 1)\), then by the continuity of \(v\) and \(z = 0\), the continuity of \(v\) and (93), one obtains
\[t s \left[ (\nu^{Rb}_D v(t)) (a D_t^\mu - D_b^\mu v(t)) \right] = 0. \tag{102}\]

If \(\beta = 0\), then from the assumption \(v(0) = 0\) and (93), we get
\[t s \left[ (\nu^{Rb}_D v(t)) (a D_t^\mu - D_b^\mu v(t)) \right] = 0. \tag{103}\]

If \(\beta = 1\), then it follows from (94) that
\[t s \left[ (\nu^{Rb}_D v(t)) (a D_t^\mu - D_b^\mu v(t)) \right] = 0. \tag{104}\]

Using equations (102)–(108), equation (101) becomes
\[\Delta T L(S, q(S), D_b^\mu q(S)) + \int_a^S v(t) \left[ \frac{\partial L}{\partial q} - \frac{\partial L}{\partial D_b^\mu q} z(t) \right] \, dt = 0. \tag{109}\]
Theorem 7.1: If the functional (110) satisfies equation (111), where \( \bar{q}(t) \) is given by (112), then for any \( t \in [a, b] \)
\[
\frac{\partial L}{\partial q} (t, q(t)) + \frac{\partial L}{\partial \dot{q}} (t, q(t)) = 0,
\]
and this is equivalent to
\[
\frac{d^2}{dt^2} q^*(t) + \frac{R H D_{0}^{\mu,1}}{2} (t^2 - t) = 0, \quad \text{for any } t \in [0, 1].
\]
Now according to property (2.1), p. 71, in [1] we get
\[
\frac{R H D_{0}^{\mu}}{2} (t^2 - t) = \frac{1}{2} \left[ 0 D_{0}^{\mu} (t^2 - t) - t D_{0}^{\mu} (t^2 - t) \right]
\]
\[
= \frac{1}{2} \left[ 0 D_{0}^{\mu} (t^2 - t) - t D_{0}^{\mu} ((1 - t)^2 - (1 - t)) \right]
\]
By applying the boundary condition, we get

\[ q(s) = \frac{t^{4-\mu}}{\Gamma(5-\mu)} - \frac{t^{3-\mu}}{2\Gamma(4-\mu)} - \frac{(1-t)^{4-\mu}}{\Gamma(5-\mu)} + \frac{(1-t)^{3-\mu}}{2\Gamma(4-\mu)}, \]

for \( s \in \left[ \frac{3}{4}, 1 \right] \), (131)

and

\[ p\left(\frac{7}{4}\right) = 0, \quad p(2) = \frac{3}{\Gamma(5-\mu)} - \frac{3}{2\Gamma(6-\mu)} - \frac{1}{\Gamma(4-\mu)} - \frac{4}{3}, \]

(132)

By applying Theorem 5.1 the function \( p \) must satisfy the following conditions:

\[ 2p''(s) + 2s = \frac{RH \, D_2^{\mu,1}}{1} (6s^2 - s^3 - 11s + 6), \]

(133)

for \( s \in \left[ \frac{1}{4}, \frac{7}{4} \right] \) and

\[ p''(s) + s = \frac{RH \, D_2^{\mu,1}}{1} (6s^2 - s^3 - 11s + 6), \]

(134)

for \( s \in \left[ \frac{7}{4}, 2 \right] \). Observe that, according to property (2.1) in [1] it yields

\[ \frac{RH \, D_2^{\mu,1}}{1} (6s^2 - s^3 - 11s + 6) = \frac{8}{2} D_2^{\mu}(6s^2 - s^3 - 11s + 6) - \frac{3}{2} D_2^{\mu}(6s^2 - s^3 - 11s + 6) \]

\[ = \frac{1}{2} \left[ D_2^{\mu}(3-1)^2 - (s-1)^2 - 2(s-1) \right] \]

\[ + \frac{1}{2} \left[ D_2^{\mu}(2-s)^3 - (2-s) \right] \]

\[ = \frac{3(s-1)^2}{\Gamma(3-\mu)} - \frac{3(s-1)^3}{2\Gamma(4-\mu)} - \frac{(s-1)^{1-\mu}}{\Gamma(2-\mu)} + \frac{3(2-s)^3}{2\Gamma(4-\mu)} - \frac{(2-s)^{1-\mu}}{2\Gamma(2-\mu)}. \]

(135)

Inserting the expression (135) into (133) and (134) and taking into account the boundary conditions (131) and (122), we get after some manipulations

\[ p(s) = \frac{3(s-1)^{4-\mu}}{2\Gamma(5-\mu)} - \frac{3(s-1)^{5-\mu}}{4\Gamma(6-\mu)} - \frac{(s-1)^{3-\mu}}{2\Gamma(4-\mu)} - \frac{3(2-s)^{5-\mu}}{4\Gamma(6-\mu)} - \frac{(2-s)^{3-\mu}}{2\Gamma(4-\mu)} - \frac{(s-1)^{1-\mu}}{\Gamma(2-\mu)} + \frac{(2-s)^{1-\mu}}{\Gamma(2-\mu)} + \frac{1}{4} \left[ \frac{3}{\Gamma(4-\mu)} \frac{4}{4} \right]^{3-\mu} - \frac{1}{3} \left( \frac{1}{4} \right)^{3-\mu} \]

\[ + \frac{2(s-1)^3}{9} \left( \frac{7}{4} \right)^{3-\mu}, \quad s \in \left[ \frac{7}{4}, 2 \right]. \]
The graph of the solution function given by (128) for different values of $\mu$. (b) The graph of the solution function given by (131), (136) and (137) for different values of $\mu$.

\[ + \frac{2(s-1)}{9} \left( \frac{7}{4} \right)^3, \quad s \in \left[ \frac{1}{4}, 1 \right], \quad (136) \]

and

\[ p(s) = \frac{1}{2} \left[ \frac{3\Gamma(3)}{\Gamma(5-\mu)} (s-1)^{4-\mu} - \frac{\Gamma(4)}{\Gamma(6-\mu)} (s-1)^{5-\mu} \right] - \frac{2\Gamma(2)}{\Gamma(4-\mu)} (s-1)^{3-\mu} + \frac{\Gamma(4)}{\Gamma(6-\mu)} (2-s)^{5-\mu} + \frac{\Gamma(2)}{\Gamma(4-\mu)} (2-s)^{3-\mu} - \frac{s^3}{6} + 2(s-2) \left[ \frac{\Gamma(3)}{\Gamma(5-\mu)} \left( \frac{3}{4} \right)^{4-\mu} - \frac{\Gamma(4)}{\Gamma(6-\mu)} \left( \frac{3}{4} \right)^{3-\mu} \right] \times \left( 3^{-\mu} - \frac{2\Gamma(2)}{\Gamma(4-\mu)} \left( \frac{3}{4} \right)^{3-\mu} + \frac{\Gamma(4)}{\Gamma(6-\mu)} \left( \frac{1}{4} \right)^{5-\mu} \right) \right] + \frac{\Gamma(2)}{\Gamma(4-\mu)} \left( \frac{1}{4} \right)^{5-\mu} - \frac{\Gamma(4)}{\Gamma(6-\mu)} \left( \frac{1}{4} \right)^{3-\mu} - (s-2) \frac{4 \left( \frac{7}{4} \right)^3}{3}, \quad s \in \left[ \frac{7}{4}, 2 \right], \quad (137) \]

The graph of the function $p$ is clarified in Figure 1(b) for different values of $\mu$.

**Remark 8.1:** According to Remark 2.2, if $\beta = 0$, then $\frac{R^H_{\mu}}{D^H_{\alpha}} q(t) = \int_a^b D^\mu_{\alpha} q(t) ; t \in [a, b]$, and hence, when $\mu = 1$, we get $\frac{R^H_{\mu}}{D^H_{\alpha}} q(t) = dq/dt ; t \in [a, b]$. So, the figures (a) and (b) emphasize that when $\mu$ approach to the value one, the solution function curve approaches to the solution function curve if the Riesz–Hilfer derivative is replaced by the first derivative.

**Example 8.3:** Fractional Lagrangian for RLC.

In this example we consider a simple loop current that is described by a fractional Lagrangian. We assume that this single loop circuit involves a capacitor $C$, a resistor $R$ and an inductor $\chi$. The fractional Lagrangian for this loop take the form

\[ L = \frac{\chi}{2} \left[ \frac{dq}{dt} \right]^2 - \frac{1}{2c_1} q^2(t) + \frac{i\beta R}{2} D^\mu_{\alpha} q(t), \quad (138) \]

where, $q$ is the charge. According to Theorem 3.1, Euler–Lagrange equation corresponding to (138) is

\[ \frac{q}{c} + \chi \frac{d^2 q}{dt^2} + R D^\mu_{\alpha} \left( \frac{i\beta R}{2} \right) (t) = 0. \quad (139) \]

**Example 8.4:** Let $r \in (0, b-a), h : [a, b] \rightarrow \mathbb{R}$ be such that the function $z(t) = \frac{R^H_{\mu}}{D^H_{\alpha}} h(t)$ is continuously differentiable. Consider the functional

\[ J[q] = \int_a^b \left( \frac{1}{2} q^2(t) + h(t) \frac{R^H_{\mu}}{D^H_{\alpha}} q(t) + \frac{1}{2} (q'(t-r))^2 \right) dt, \quad (140) \]

with the boundary conditions

\[ q(t) = \psi(t), \quad t \in [a-r, a] \quad \text{and} \quad q(b) = q^b. \quad (141) \]

According to Theorem 5.1, the functional $J$ has an extremum at a function $q$ if $q$ satisfies:

\[ q(t) - \frac{R^H_{\mu}}{D^H_{\alpha}} h(t) - q''(t-r) = 0, \quad \text{for} \ t \in [a, b-r], \quad (142) \]

and

\[ q(t) - \frac{R^H_{\mu}}{D^H_{\alpha}} h(t) = 0, \quad \text{for} \ t \in [b-r, b]. \quad (143) \]

**Remark 8.2:** Like in many papers, see for example [21], the numerical methods are more suitable to find the solutions of FVP.
9. Results and discussion

As mentioned earlier, variational problems and fractional calculus have many applications in different branches in engineering and mathematics, moreover, the Riesz–Hilfer fractional derivative (RHFD) is a generalization for the Riesz–Riemann–Liouville and the Riesz–Caputo derivative. The results that we obtained are to find Euler–Lagrange equations for various of fractional variational problems with the Lagrangian function containing (RHFD), and hence our results generalize many recent papers in the literature, for example, [22–25, 36]. On other hand, our technique allows to generalize some works, such as the obtained results in [26] to the case when the functional involving (RHFD). As we mentioned in Corollary 2.1, relations (20) and (21) in [22] are particular cases of Lemma 2.1, and if \( \beta = 0 \) and \( i = 1 \) in Theorem 3.1, then we obtain Theorem 3.1, in [22]. Moreover, if we put \( \beta = 0 \) in both Theorems 3.2 and 6.1, we obtain Theorems 4.1 and 7.1, respectively, in [24].

10. Conclusion

Euler–Lagrange equations for different kind of fractional variational problems with the Lagrangian function containing the Riesz–Hilfer fractional derivative are obtained. Since the Riesz–Hilfer fractional derivative is a generalization for the Riesz–Riemann–Liouville and the Riesz–Caputo derivative, then our results generalize many recent works in which the Lagrangian function involving the Riesz–Riemann–Liouville or the Riesz–Caputo derivative. Fractional variational problem in the presence of delay derivatives is considered. Moreover, a version for Noether theorem in the Riesz–Hilfer sense is established. Necessary conditions for a pair function-time to be an optimal solution to the problem are investigated. Examples are given to illustrate the applicability of the obtained results. Furthermore, our obtained results generalize some existing results such as Theorem 1 in [22] and Theorems 3 and 6 in [24]. Also, the technique used in the present paper enable to extend the results in [26, 28, 31] when the treated problems in these works involving Riesz–Hilfer fractional derivative. Moreover, this work, may be, encourages to study partial differential equations containing Riesz–Hilfer fractional derivative.

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