On an Intermediate Bivariant Theory for $C^*$-algebras, I

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Abstract

We construct a new bivariant theory, that we call $KE$-theory, which is intermediate between the $KK$-theory of G. G. Kasparov, and the $E$-theory of A. Connes and N. Higson. For each pair of separable graded $C^*$-algebras $A$ and $B$, acted upon by a locally compact $\sigma$-compact group $G$, we define an abelian group $KE_G(A, B)$. We show that there is an associative product $KE_G(A, D) \otimes KE_G(D, B) \to KE_G(A, B)$. Various functoriality properties of the $KE$-theory groups and of the product are presented. The new theory has a simpler product than $KK$-theory and there are natural transformations $KK_G \to KE_G$ and $KE_G \to E_G$. The complete description of these maps will form the substance of a second paper.

Key words: $C^*$-algebras, $KK$-theory, Kasparov product, $K$-theory.

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0 Introduction

In the 1960’s, the algebraic topology of manifolds produced one of the profound theorems of XXth century mathematics: the index theorem of Atiyah and Singer ([AtSi63], [Pal]). The conceptual proof given in [AtSi1] is based on a cohomology theory invented by M. Atiyah and F. Hirzebruch — namely $K$-theory. Using hints coming from various generalizations of the index theorem, Atiyah [Atiy69] also proposed a way of defining cycles of the dual theory — namely $K$-homology. The only thing left open by Atiyah was the definition of the equivalence relation that would make these cycles into a group. This issue was resolved by G. G. Kasparov [Kas75]. He succeeded in creating (see also [Kas81]) a bivariant theory — named $KK$-theory — which associates to any two $C^*$-algebras $A$ and $B$ a group $KK(A,B)$. His theory generalizes both $K$-theory for compact manifolds (obtained when $A$ is $\mathbb{C}$, and $B$ is the continuous functions on the manifold) and $K$-homology (obtained when $A$ is the continuous functions on the manifold, and $B$ is $\mathbb{C}$). For a very well written account of the origins of $KK$-theory see the introductory sections of [Hig87a] and [Hig90a].

Besides a wealth of functorial properties, the key feature of $KK$-theory is the existence for any separable $C^*$-algebras $A$, $B$, and $D$ of an associative product map $KK(A,D) \otimes KK(D,B) \to KK(A,B)$. Following an approach indicated by J. Cuntz ([Cu82], [Cu84]), N. Higson [Hig87a] gave the following description of $KK$-theory: it is the universal category with homotopy invariance, stability, and split-exactness. This category has separable $C^*$-algebras as objects, elements of $KK$-groups as morphisms, and the Kasparov product as the composition of morphisms.

In a subsequent paper [Hig90b] Higson described the universal category with homotopy invariance, stability, and exactness. The resulting new theory — named $E$-theory — has become important in $C^*$-algebra theory after A. Connes and N. Higson [CoHig90] described it concretely in terms of asymptotic morphisms. (An asymptotic morphism between two $C^*$-algebras is a family of maps between the two, indexed by $[1, \infty)$, which satisfies the conditions
of a $*$-homomorphism in the limit at $\infty$.) The description of $KK$-theory and $E$-theory using category theory implies, in a rather abstract and algebraic way, the existence of a map $KK(A, B) \to E(A, B)$, for any two $C^*$-algebras $A$ and $B$. This map is an isomorphism when $A$ is nuclear $\text{Sk}88$. Similar descriptions of the universality property for the equivariant theories are also known: for the equivariant $KK$-theory $\text{Kas}88$ under the action of a group see $\text{Thms}98$, for the equivariant $E$-theory under the action of a group see $\text{GHT}$, and for both theories under the action of a groupoid see $\text{Pop}$.

Equivariant $KK$-theory and $E$-theory have become essential tools in $C^*$-algebra theory because of their use in solving topological/geometrical problems, notably cases of the Novikov conjecture $\text{Kas}88$, $\text{Rsn84}$, and the Baum-Connes conjecture $\text{BC}82$, $\text{BCH}$.

In this paper a new theory is constructed, that we call $KE$-theory, which is intermediate between $KK$-theory and $E$-theory. It applies to $C^*$-algebras that are separable, graded, and admit an action of a locally compact σ-compact Hausdorff group. For such a group $G$, and for any two such $G$-$C^*$-algebras $A$ and $B$, the resulting abelian group is denoted by $KE_G(A, B)$. The $KE$-theory recovers in a rather direct way the ordinary $K$-theory of ungraded $C^*$-algebras. The $KE$-theory groups satisfy some of the good functorial properties of the other two bivariant theories, and there exists an associative product $KE_G(A, D) \otimes KE_G(D, B) \to KE_G(A, B)$. We have also proved the existence of two natural transformations, $\Theta: KK_G(A, B) \to KE_G(A, B)$ and $\Xi: KE_G(A, B) \to E_G(A, B)$, which preserve the product structures. Their composition $\Xi \circ \Theta$ provides an explicit construction of the map $KK \to E$, abstractly known to exist because of the universality properties of the two theories (as we mentioned above). The complete analysis of the maps $\Theta$ and $\Xi$ will form the substance of a subsequent paper. The idea of constructing a theory intermediate between $KK$-theory and $E$-theory was suggested by V. Lafforgue (private communication to N. Higson).

Intermediate theories between $KK$-theory and $E$-theory appear also in the work of J. Cuntz $\text{Cu97}$, $\text{Cu98}$. Our construction is different in initial motivation, concrete realization, and final goal: we wanted to produce a solid framework for another proof to the Baum-Connes conjecture for a-T-menable groups $\text{HgKas97}$, $\text{HgKas01}$. Details for this application will be given elsewhere.

The paper is structured as follows. In Section 1 we briefly review the essential definitions, theorems and constructions related to $KK$-theory. We also use it to set up notation. Section 2 constructs the new $KE$-theory. In subsection 2.1 we introduce and study its ‘cycles’, which we call asymptotic Kasparov modules. They are appropriate families of pairs, indexed by $[1, \infty)$. Each pair consists of a Hilbert module and an operator on it, that are put together in a field satisfying conditions that resemble those appearing in $KK$-theory. An example of such cycle, motivated by the $K$-homology class of the Dirac operator on a spin manifold, consists of a $C^*$-algebra $A$, a Hilbert space $H$ (constant family), a $*$-homomorphism $\varphi: A \to B(H)$, and a family $\{F_t\}_{t \in [1, \infty)}$ of bounded linear operators on $H$ satisfying:

(aKm1) $F_t = F_t^*$, for all $t$;
(aKm2) $\| [F_t, \varphi(a)] \| \to \infty$, for all $a \in A$;
(aKm3) $\varphi(a)(F_t^2 - 1)\varphi(a)^* \geq 0$, modulo compact operators and operators which converge in norm to zero.

Such a family $\{(H, F_t)\}_{t \in [1, \infty)}$ is an asymptotic Kasparov $(A, \mathbb{C})$-module. Axiom (aKm2) encodes the pseudo-locality of first order elliptic differential operators, and axiom (aKm3) is
supposed to encode the Fredholm property of elliptic operators on smooth manifolds. The definition can be also adapted to include a group action, and in subsection 2.2 we define, for a locally compact group $G$ and two graded separable $G$-$C^*$-algebras $A$ and $B$, the group $K_E(G, A, B)$ of homotopy equivalence classes of asymptotic Kasparov $G$-$(A, B)$-modules. Various functoriality properties of these groups are proved in the remaining part of the section.

In Section 3 the product in $K_E$-theory is constructed using the notions of ‘two-dimensional’ connection and quasi-central approximate unit. Let $G$ be a locally compact group, and $A_1, A_2, B_1, B_2, D$ be $G$-$C^*$-algebras. As in $KK$-theory, in its most general form, the product is a map

$$KE_G(A_1, B_1 \otimes D) \otimes KE_G(D \otimes A_2, B_2) \rightarrow KE_G(A_1 \otimes A_2, B_1 \otimes B_2), \quad (x, y) \mapsto x_{\sharp_D} y.$$ 

Insight about the product in the new theory can be obtained by looking at the particular case when $B_1 = B_2 = D = \mathbb{C}$, which corresponds to the external product in $K$-homology. Consider two asymptotic Kasparov modules as described above: $\{(H_1, F_{1,t})\}_t \in KE(A_1, \mathbb{C})$, and $\{(H_2, F_{2,t})\}_t \in KE(A_2, \mathbb{C})$. Their product is

$$\{(H_1 \otimes H_2, F_{1,t} \otimes 1 + 1 \otimes F_{2,t})\}_t \in KE(A_1 \otimes A_2, \mathbb{C}).$$

The reader familiar with $KK$-theory will notice that no Kasparov Technical Theorem was used in our construction. The general case is more involved, but we hope that it is still simpler than in $KK$-theory. Our method is summarized in Overview 3.7. In subsection 3.4 we analyze the algebra behind the product. We show that the product is associative and its various compatibilities with the functoriality of $KE$-groups are worked out. The stability of $KE$-theory is an easy consequence of the corresponding property of $KK$-theory. Subsection 3.3 plays the role of an appendix to this section. It contains the proof of Theorem 3.9 used to construct the product.

In the short Section 4 we present the axioms of asymptotic Kasparov modules from the perspective of two concrete examples. In subsection 4.1 we show that the $KE$-theory groups recover the ordinary $K$-theory for trivially graded $C^*$-algebras, and in subsection 4.2 we compute $KE_{\Gamma}(\mathbb{C}, \mathbb{C})$, for a discrete group $\Gamma$.

In Section 5 we define the two natural transformations $\Theta : KK_G \rightarrow KE_G$ and $\Xi : KE_G \rightarrow E_G$, whose composition gives an explicit characterization of the map from $KK$-theory into $E$-theory. We also briefly discuss what we were able to prove related to the exactness property: $KE$-theory has Puppe exact sequences and is split-exact. Full details will appear elsewhere.

All definitions and results (theorems, propositions, corollaries, lemmas, important remarks and examples) are numbered in each section in order of appearance. This convention dictates that a reference of the type $m.n$ sends to the result $n$ from section $m$. All equations and diagrams are numbered after the section. The end of a proof is marked by $\blacksquare$.

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1 Preliminaries: review and notation

The focus of this section is to briefly review the essential definitions, theorems and constructions related to KK-theory. We also use it to set up notation. Among the covered topics we mention: tensor products of $C^*$-algebras, Hilbert modules and tensor products of Hilbert modules, group actions, approximate units, and Kasparov’s Technical Theorem. A short overview of $KK$-theory, including its product, is given in subsection 1.4.

The standing assumption for the entire paper is: we shall work in the category $C^*-$alg, whose objects are the separable and $(\mathbb{Z}_2\mathcal{g})$-graded $C^*$-algebras [Blck 14.1], and whose morphisms are $\ast$-homomorphisms that preserve the grading.

1.1 $C^*$-algebras, Hilbert modules and tensor products

Given a separable graded $C^*$-algebra $A$, the commutator of two elements $a, b \in A$ is: $[a, b] = ab - (-1)^{\partial a \partial b}ba$. The $C^*$-algebra of complex numbers, $\mathbb{C}$, is trivially graded. As a general rule, given a locally compact space $X$, the $C^*$-algebra $C_0(X)$ of complex valued continuous functions on $X$ vanishing at infinity, will be trivially graded.

All the tensor products are graded [Blck 14.4]. The minimal $C^*$-algebra tensor product is denoted by $\otimes$, the maximal one by $\otimes_{max}$. For two $C^*$-algebras $A_1$ and $A_2$, there is a transposition isomorphism $A_1 \otimes A_2 \simeq A_2 \otimes A_1$, given on elementary tensors by $a_1 \otimes a_2 \mapsto (-1)^{\partial a_1 \partial a_2}a_2 \otimes a_1$. We recall also two of the identities that hold true with graded tensor products: $(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\partial a_2 \partial b_1}a_1b_1 \otimes a_2b_2$, and $(a_2 \otimes a_2)^* = (-1)^{\partial a_1 \partial a_2}a_1^* \otimes a_2^*$, for all $a_1, b_1 \in A_1$, and $a_2, b_2 \in A_2$.

Let $L = [1, \infty)$, and $LL = [1, \infty) \times [1, \infty)$. For any $C^*$-algebra $B$ and any locally compact space $X$, the $C^*$-algebra $B(X)$ of $B$-valued continuous functions on $X$ vanishing at infinity is $B(X) = C_0(X, B) = C_0(X) \otimes B$. We further simplify and write: $BL = C_0(L, B)$, $BLL = C_0(\left[0, 1\right], B)$.

Given a Hilbert $B$-module $E$ [Lan Ch.1], the $\ast$-algebra of adjointable operators on $E$ (see [Kas80], [Kas80], [Lan]) is denoted by $B(E)$. The closed ideal of ‘compact’ operators on $E$ is denoted by $\mathcal{K}(E)$. It is generated by the rank-one operators $\theta_{\xi, \eta}(\xi) = \xi(\eta, \cdot)$, for $\xi, \eta, \zeta \in E$.

Let $E_1$ and $E_2$ be graded Hilbert modules over $B_1$ and $B_2$, respectively. The completion $E_1 \otimes E_2$ of the algebraic tensor product $E_1 \otimes E_2$ with respect to the $B_1 \otimes B_2$-valued semi-inner product $\langle \xi_1, \eta_1, \xi_2, \eta_2 \rangle = (-1)^{\partial a_1(\partial b_1 + \partial b_2)}(\xi_1, \xi_2) \otimes (\eta_1, \eta_2)$ is a Hilbert $B_1 \otimes B_2$-module, called the external tensor product of $E_1$ and $E_2$. If $\varphi : B_1 \to B_2$ is a $\ast$-homomorphism, we can also construct the internal tensor product $E_1 \otimes_{\varphi} E_2$ of $E_1$ and $E_2$. (The notation $E_1 \otimes_{\varphi} E_2$ will also be used.) It is the Hilbert $B_2$-module obtained as completion of the algebraic tensor product $E_1 \otimes_{\varphi} E_2$ with respect to the $B_2$-valued semi-inner product $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi(\xi_1, \xi_2)(\eta_2) \rangle$. In both cases the grading is $\partial(\xi \otimes \eta) = \partial \xi + \partial \eta$. For details we refer the reader to [Kas80], [Lan, Ch.4], or [Blck, Secs.13,14].

Given two Hilbert modules $E_1$ and $E_2$, there is an embedding $B(E_1) \otimes B(E_2) \to B(E_1 \otimes E_2)$, given by $(F_1 \otimes F_2)(\xi \otimes \eta) = (-1)^{\partial a_1 \partial b_2}F_1(\xi) \otimes F_2(\eta)$. Its restriction to compact operators gives an isomorphism $\mathcal{K}(E_1) \otimes \mathcal{K}(E_2) \simeq \mathcal{K}(E_1 \otimes E_2)$. In the case of internal tensor product of Hilbert modules, we only have a natural graded $\ast$-homomorphism $B(E_1) \to B(E_1 \otimes_{\varphi} E_2)$, $F \mapsto F \otimes_{B_1} 1$, $(F \otimes_{B_1} 1)(\xi \otimes \eta) = F(\xi) \otimes \eta$. 

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Given a Hilbert \( B \)-module \( \mathcal{E} \) and a space \( X, \mathcal{E}(X) \) is the Hilbert \( B(X) \)-module \( C_0(X) \otimes \mathcal{E} \) (external tensor product of Hilbert modules). We shall use the notation: \( \mathcal{E}L = C_0(L) \otimes \mathcal{E} = \{ \mathcal{E} \}_t = \) constant family indexed by \([1, \infty)\), \( \mathcal{E}LL = C_0(LL) \otimes \mathcal{E} = \) constant family indexed by \([1, \infty) \times [1, \infty)\).

The multiplier algebra \( M(A) \) of a \( C^* \)-algebra \( A \) is the largest \( C^* \)-algebra in which \( A \) embeds as an essential ideal ( [Blck, Sec.12], [Lan, Ch.2]). We recall the following two facts about multipliers: \( M(\mathcal{K}(\mathcal{E})) \simeq \mathcal{B}(\mathcal{E}) \), for any Hilbert \( B \)-module \( \mathcal{E} \) ([Blck, 13.4.1], [Lan, 2.4]), and \( M(C_0([1, \infty), \mathcal{K}(\mathcal{E}))) \simeq C^b([1, \infty), \mathcal{B}_{str}(\mathcal{E})) \), where \( \mathcal{B}_{str}(\mathcal{E}) \) denotes the strict topology ([APT, 3.4]).

### 1.2 Group actions

As reference for this section see [Kas88, Sec.1]. Besides being separable and graded, the \( C^* \)-algebras that we consider have an additional structure: the action of a group by automorphisms. A standing assumption for the entire paper is the following: *all groups are supposed to be locally compact, \( \sigma \)-compact and Hausdorff.* Given such a group \( G \) and a \( C^* \)-algebra \( A \), an action of \( G \) on \( A \) is a group homomorphism \( G \to \text{Aut}(A) \), where \( \text{Aut}(A) \) is the group of automorphisms of \( A \), with no topology on it. An element \( a \in A \) is called \( G \)-continuous if the map \( G \to A, g \mapsto g(a) \) is continuous. We denote by \( G\text{-}C^*\text{-alg} \) the category with objects the separable graded \( C^* \)-algebras equipped with \( G \)-action compatible with the grading and having all the elements \( G \)-continuous, and with morphisms the equivariant \(*\)-homomorphisms. The objects of \( G\text{-}C^*\text{-alg} \) are called \( G\text{-}C^*\text{-algebras}. \) The action of any group \( G \) on \( \mathbb{C} \) is trivial.

**Definition 1.1.** Given a group \( G \), a \( G\text{-}C^*\text{-alg} \) \( B \), and a Hilbert \( B \)-module \( \mathcal{E} \), an action of \( G \) on \( \mathcal{E} \), or a \( G \)-action, is an action of \( G \) by grading preserving linear automorphisms such that:

(i) \( G \times \mathcal{E} \to \mathcal{E} \), \( (g, \xi) \mapsto g(\xi) \), is continuous in the norm topology of \( \mathcal{E} \); (ii) \( g(\xi b) = g(\xi)g(b) \); and (iii) \( (g(\xi), g(\eta)) = (g(\xi), g(\eta)) \), for all \( \xi, \eta \in \mathcal{E}, b \in B, g \in G \). We call such a Hilbert module \( \mathcal{E} \) a \( G\text{-}B \)-module.

Given an action of \( G \) on \( \mathcal{E} \), there is an induced action of \( G \) on \( \mathcal{B}(\mathcal{E}) \) as follows: \( g(T)(\xi) = g(T(g^{-1}\xi)) \), for all \( g \in G, T \in \mathcal{B}(\mathcal{E}), \) and \( \xi \in \mathcal{E} \). In this way, for any \( G\text{-}C^*\text{-alg} \) \( B \), there is a canonical induced action on \( M(B) \). Let \( \mathcal{E}_1 \) be a Hilbert \( D \)-module, with a \( G \)-action, and \( \mathcal{E}_2 \) be a \( G\text{-}(D,B) \)-module. The action of \( G \) on the internal tensor product \( \mathcal{E}_1 \otimes_D \mathcal{E}_2 \) is given by \( g(\xi \otimes_D \eta) = g(\xi) \otimes_D g(\eta) \), for all \( \xi \in \mathcal{E}_1, \eta \in \mathcal{E}_2 \). This implies, for \( T \in \mathcal{B}(\mathcal{E}_1) \), that \( g(T \otimes_D 1) = g(T) \otimes_D 1 \).

The standard Hilbert \( G \)-space is \( \mathcal{H}_G = L^2(G) \oplus L^2(G) \oplus \ldots \), with infinitely many summands, graded alternately even and odd, and equipped with the left regular representation of \( G \). Let \( \mathcal{K} = \mathcal{K}(\mathcal{H}_G) \) be the compact operators on \( \mathcal{H}_G \). For any \( G\text{-}C^*\text{-alg} \) \( B \), the standard Hilbert \( G\text{-}B \)-module is \( \mathcal{H}_B = l^2 \otimes L^2(G) \otimes (B \otimes B^{op}) \).

### 1.3 Quasi-central approximate units, and Kasparov’s Technical Theorem

Recall that a \( C^* \)-algebra is called \( \sigma \)-unital if it has a countable approximate unit.
Definition 1.2. Let $G$ be a group. Consider an inclusion $I \subset B \subset A$, where $A$ is a $G$-$C^*$-algebra, $B$ is a $\sigma$-unital $G$-$C^*$-subalgebra of $A$, and $I$ is a $\sigma$-unital $G$-ideal of $A$. A quasi-invariant quasi-central approximate unit for $I$ in $B$ (abbreviated q.i.q.c.a.u.) is a continuous family $\{u_t\}_{t \in [1,\infty)}$ of positive, increasing, even elements of $I$ satisfying:

(a.u.) $\| xu_t - x \| \overset{t \to \infty}{\longrightarrow} 0$, for all $x \in I$;
(b.u.) $\| yu_t - uy \| \overset{t \to \infty}{\longrightarrow} 0$, for all $y \in B$; and
(c.i.) $\| g(u_t) - u_t \| \overset{t \to \infty}{\longrightarrow} 0$, uniformly on compact subsets of $G$.

Proposition 1.3. Let $G$ be a group. A quasi-invariant quasi-central approximate unit exists for any closed $G$-invariant ideal $I$ of a $G$-$C^*$-algebra $A$.

For a proof see [Kas88, Lemma 1.4], or [GHT, 5.3]. Without a group action, the existence of quasi-central approximate units is proved in [Pdr, 3.12.14], or [Arv, Thm.1]. Such approximate units exist for any $I \subset A$, but in this paper we need a countable approximate unit $\{u_n\}_n$ (which by interpolation gives the family $\{u_1\}$), and this justifies the presence of the separable subalgebra $B$. It is usually clear from the context what $B$ is (the biggest subalgebra that one needs in each particular application!), and we shall usually omit mention of it.

The following result in pure $C^*$-algebra theory is due to Kasparov. His initial proof [Kas81, Sec.3] was complicated. N. Higson [Hg87b] gave an elegant proof in the non-equivariant case based on the notion of quasi-central approximate unit. The statement that follows is Theorem 1.5 of [Kas88]. We shall often abbreviate the result as KTT.

Kasparov’s Technical Theorem. Let $J$ be a $\sigma$-unital $G$-$C^*$-algebra. Assume that $E_1$ and $E_2$ are subalgebras of $M(J)$, $E_1$ with $G$-action and having all the elements $G$-continuous, such that: (i) $E_1$, $E_2$ are $\sigma$-unital, (ii) $E_1, E_2 \subset J$. Assume also that $\Delta$ is a subset of $M(J)$, separable in the norm topology, consisting of $G$-continuous elements, and satisfying:

(iii) $[\Delta, E_1] \subset E_1$. Further assume that $\phi : G \rightarrow M(J)$ is a bounded function, such that:

(iv) $E_1 \phi(G) \subset J$, $\phi(G) E_1 \subset J$, and (v) $g \mapsto a \phi(g)$, $g \mapsto \phi(g)a$ are norm continuous on $G$, for any $a \in E_1 + J$. Then there exist $G$-continuous positive even elements $M, N \in M(J)$ with the properties: (1) $M + N = 1$; (2) $M E_1 \subset J$, $N E_2 \subset J$; (3) $[M, \Delta] \subset J$; (4) $g(M - M) \in J$, for all $g \in G$; (5) $N \phi(G) \subset J$, $\phi(G) N \subset J$; and (6) $g \mapsto N \phi(g)$, $g \mapsto \phi(g) N$ are norm continuous on $G$.

1.4 $KK$-theory

Taking into account the fact that some constructions in $KE$-theory are motivated by $KK$-theory constructions, and in order to have the paper self-contained, we present in this subsection a quick review of the theory of Gennadi Kasparov. The $KK$-theory groups were introduced and studied in [Kas73, Kas81], the equivariant ones under the action of a group in [Kas74, Kas88], under the action of a Hopf algebra in [BaaSk], and under the action of a groupoid in [EG99]. We follow the equivariant presentation of Kasparov [Kas88].

Definition 1.4. Consider a group $G$, and two graded separable $G$-$C^*$-algebras $A$ and $B$. A Kasparov $G$-$(A, B)$-module is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a Hilbert $B$-module, admitting a $G$-action and an action of $A$ via a $*$-homomorphism $\varphi : A \rightarrow \mathcal{B}(\mathcal{E})$, and $F \in \mathcal{B}(\mathcal{E})$ is an odd
$G$-continuous operator such that for every $a \in A$ and $g \in G$

$$(F - F^*)\varphi(a), [F, \varphi(a)], (F^2 - 1)\varphi(a), \text{ and } (g(F) - F)\varphi(a) \text{ all belong to } \mathcal{K}(\mathcal{E}). \quad (1)$$

The set of all Kasparov $G$-$(A, B)$-modules will be denoted by $\text{kk}G(A, B)$. A Kasparov $G$-$(A, B)$-module $(\mathcal{E}, F)$ is said to be degenerate if for all $a \in A$ and $g \in G$: $(F - F^*)\varphi(a) = 0$, $[F, \varphi(a)] = 0$, $(F^2 - 1)\varphi(a) = 0$, and $(g(F) - F)\varphi(a) = 0$. Whenever there is no risk of confusion, we shall write $a$ instead of $\varphi(a)$.

**Definition 1.5.** An element $(\mathcal{E}, F)$ of $\text{kk}G(A, B[0, 1])$ gives by ‘evaluation at $s$’ a family

$$(\mathcal{E}_s, F_s) \in \text{kk}G(A, B) \mid s \in [0, 1]$$

with $\mathcal{E}_s = \mathcal{E} \otimes_{ev_s} B$, $F_s = F \otimes_{ev_s} 1$. Such an element $(\mathcal{E}, F)$ and the family that it generates are called a homotopy between $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$.

**Definition 1.6.** The set $KKG(A, B)$ is defined as the quotient of $\text{kk}G(A, B)$ by the equivalence relation generated by homotopy. Given an element $x = (\mathcal{E}, F) \in \text{kk}G(A, B)$, its class in $KKG(A, B)$ will be denoted by $x$. The addition of two Kasparov $G$-$(A, B)$-modules is given by the obvious notion of direct sum.

Under the above defined addition $KKG(A, B)$ becomes an abelian group. The following elements play an important role in the theory: $1 = 1_C \in KKG(\mathbb{C}, \mathbb{C})$, the class of the Kasparov module $(\mathbb{C}, 0)$, and $1_A \in KKG(A, A)$, the class of the Kasparov module $(A, 0)$. Given $A$, $B$, and $D$, there is a map $\sigma_D : KKG(A, B) \to KKG(A \otimes D, B \otimes D)$, $(\mathcal{E}, F) \mapsto (\mathcal{E} \otimes D, F \otimes 1)$.

**Definition 1.7.** ([CoSk Def.A.1], [Sk84 Def.8]) Assume that the following elements are given: a Hilbert $D$-module $\mathcal{E}_1$, a Hilbert $(D, B)$-module $\mathcal{E}_2$, and $F_2 \in \mathcal{B}(\mathcal{E}_2)$. Let $\mathcal{E} = \mathcal{E}_1 \otimes_{D} \mathcal{E}_2$. An operator $F \in \mathcal{B}(\mathcal{E})$ is called an $F_2$-connection for $\mathcal{E}_1$ if it has the same degree as $F_2$ and if it satisfies for every $\xi \in \mathcal{E}_1$:

$$(T_\xi F_2 - (-1)\partial_\xi \partial F_2 \mathcal{E} T_\xi) \in \mathcal{K}(\mathcal{E}_2, :\mathcal{E}:) \quad \text{and} \quad (F_2 T_\xi^* - (-1)\partial_\xi \partial F_2 T_\xi^* \mathcal{E}) \in \mathcal{K}(\mathcal{E}, \mathcal{E}_2). \quad (2)$$

Here $T_\xi \in \mathcal{B}(\mathcal{E}_2, :\mathcal{E}:)$ is defined by $T_\xi(\eta) = \xi \otimes \eta$, for $\eta \in \mathcal{E}_2$.

The properties of connections are listed in [Sk84 Prop.9]. Here is the one that interests us most (the equivariant part is contained in [Kas88 Lemma 2.6]):

**Lemma 1.8.** Consider the notation of the previous definition. If $F_2$ satisfies, for all $d \in D$, $[F_2, d] \in \mathcal{K}(\mathcal{E}_2)$, then an $F_2$-connection $F$ exists for any countably generated $\mathcal{E}_1$. If $d F_2$ and $F_2 d$ are $G$-continuous for any $d \in D$, then $F(K \otimes_D 1)$ and $(K \otimes_D 1) F$ are $G$-continuous, for any $K \in \mathcal{K}(\mathcal{E}_1)$.

**Definition 1.9.** ([CoSk Thm.A.3], [Sk84 Def.10]) Let $A$, $B$, $D$ be $G$-$C^*$-algebras, $x = (\mathcal{E}_1, F_1) \in \text{kk}G(A, D)$, $y = (\mathcal{E}_2, F_2) \in \text{kk}G(D, B)$, $\mathcal{E} = \mathcal{E}_1 \otimes_{D} \mathcal{E}_2$. Denote by $F_1 \uparrow_D F_2$ the set of operators $F \in \mathcal{B}(\mathcal{E})$ satisfying:

1. $(\mathcal{E}, F) \in \text{kk}G(A, B)$;
2. $F$ is an $F_2$-connection for $\mathcal{E}_1$; and
3. $a[F_1 \otimes_D 1, F] a^* \geq 0$, modulo $\mathcal{K}(\mathcal{E})$, for all $a \in A$. 

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For any $F \in F_1 \sharp_D F_2$, the pair $z = (\mathcal{E}, F)$ will be called the product of $x$ and $y$. We shall use the notation $z = x \sharp_D y$. (The same notation $\sharp$ will also be used to designate the product in the new $KE$-theory. We hope that it will be clear from the context to what theory a certain product belongs to.)

**Theorem 1.10.** Let $G$ be a group, and $A$, $B$, and $D$ be separable graded $G$-$C^*$-algebras. The product $\sharp_D$ exists, is unique up to homotopy, and defines a bilinear pairing:

$$KK_G(A, D) \otimes KK_G(D, B) \xrightarrow{\sharp_D} KK_G(A, B), \ (x, y) \mapsto x \sharp_D y.$$  

**Proof.** ([Kas88, Thm.2.11], [Sk84, Thm.12]) As in the definition above, let $x = (\mathcal{E}, F_1) \in kk_G(A, D)$, $y = (\mathcal{E}_2, F_2) \in kk_G(D, B)$, $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$. Let $\mathcal{F}$ be an $F_2$-connection for $\mathcal{E}_1$. Apply KTT for: $J = \mathcal{K}(\mathcal{E})$; $E_1 = \mathcal{K}(\mathcal{E}_1) \otimes_D 1 + \mathcal{K}(\mathcal{E})$; $E_2 = \mathcal{K}(\mathcal{E}_2)$, $\phi(g) = g(\mathcal{F}) - \mathcal{F}$. With the elements $M$ and $N$ so obtained, define:

$$F = M \sharp (F_1 \otimes_D 1) + N \sharp \mathcal{F}.$$  

(4)

Then $F$ satisfies the conditions of Definition 1.9, and consequently the class $(\mathcal{E}, F)$ represents the product $x \sharp_D y$ of $x$ and $y$. □

**Example 1.11 (External product in $KK$-theory).** Let $A_1$, $A_2$, $B_1$, $B_2$ be $G$-$C^*$-algebras. The external product is the map:

$$KK_G(A_1, B_1) \otimes KK_G(A_2, B_2) \xrightarrow{\sharp_c} KK_G(A_1 \otimes A_2, B_1 \otimes B_2).$$  

(5)

Let $x = (\mathcal{E}_1, F_1) \in kk_G(A_1, B_1)$, $y = (\mathcal{E}_2, F_2) \in kk_G(A_2, B_2)$, $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ (external product of Hilbert modules). We shall still apply Kasparov’s Technical Theorem, as in the proof of the theorem above, but things are simpler because as $F_2$-connection we can choose $\mathcal{F} = 1 \otimes F_2$. Let: $J = \mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{E}_2)$; $E_1 = \mathcal{K}(\mathcal{E}_1) \otimes A_2 + J$; $E_2 = A_1 \otimes \mathcal{K}(\mathcal{E}_2)$; $\Delta = \mathcal{E}_1 \otimes 1 \otimes F_2$, and $A_1 \otimes A_2$; and $\varphi \equiv 0$. With the elements given by KTT, define:

$$F = M \sharp (F_1 \otimes 1) + N \sharp (1 \otimes F_2).$$  

(6)

Then $F$ satisfies the conditions of Definition 1.9, and consequently the class $(\mathcal{E}, F)$ represents the product $x \sharp_c y$ of $x$ and $y$.

We make the remark that even in the external product case one cannot in general obtain the product without the ‘partition of unity’ provided by KTT. See nevertheless Section 10.7 of [HigRos] for the example of elliptic differential operators where the ‘ideal’ formula $F = F_1 \otimes 1 + 1 \otimes F_2$ holds true. The search for a theory in which such a simple product always exists and which is well suited to deal with elliptic operators provided leads towards the new $KE$-theory. See Section 3 and especially subsection 3.1.
2 \( KE \)-theory: definitions and functorial properties

In this section we introduce the new bivariant theory. Its cycles are appropriate families of pairs, indexed by \([1, \infty)\). Each pair consists of a Hilbert module and an operator on it, and they are put together in a field satisfying conditions that resemble those appearing in \( KK \)-theory. Various functoriality properties of the theory are discussed.

2.1 Asymptotic Kasparov modules

**Definition 2.1.** Consider a group \( G \), and two \( G \)-\( C^* \)-algebras \( A \) and \( B \). A continuous field of \( G \)-\((A,B)\)-modules is a countably generated \( G \)-\((A,BL)\)-module, that is a Hilbert \( BL \)-module \( \mathcal{E} \), admitting a \( G \)-action and a left action of \( A \) through an equivariant \(*\)-homomorphism \( \varphi : A \to BL(\mathcal{E}) \). (We recall the notation: \( L = [1, \infty) \), \( BL = C_0(L) \otimes B = C_0(L,B) \).) We omit \( G \) in the non-equivariant case.

A continuous field \( \mathcal{E} \) of \( G \)-\((A,B)\)-modules may be thought of as a family \( \{\mathcal{E}_t\}_{t \in [1,\infty)} \) of Hilbert \( B \)-modules, each acted on the left by \( A \) and \( G \), satisfying certain continuity conditions for the left and right actions. Indeed, for any \( t \in [1,\infty) \), let \( ev_t : BL \to B, ev_t(f \otimes b) = f(t)b \), be the evaluation \(*\)-homomorphism at \( t \). We obtain the Hilbert \( G \)-\((A,B)\)-module \( \mathcal{E}_t = \mathcal{E} \otimes_{ev_t} B \), with inner product \( \langle \xi \otimes b, \xi' \otimes b' \rangle_t = b^* ev_t(\langle \xi, \xi' \rangle) b' \). The \( A \)-action an each \( \mathcal{E}_t \) is \( \varphi_t : A \to BL(\mathcal{E}_t) \), \( \varphi_t(a) = \varphi(a) \otimes_{ev_t} 1 \). Whenever there is no risk of confusion, we shall write \( a \) instead of \( \varphi(a) \), and \( a_t \) instead of \( \varphi_t(a) \). It is also the case that an operator \( F \in BL(\mathcal{E}) \) gives a family \( \{F_t\}_{t \in [1,\infty)} = \{F \otimes_{ev_t} 1\}_{t \in [1,\infty)} \). When \( \mathcal{E} = \mathcal{E}_*L \), for a fixed Hilbert \( B \)-module \( \mathcal{E}_* \), the function \( L \to BL(\mathcal{E}_*) \), \( t \mapsto F_t \), is ‘bounded and \(*\)-strong continuous’ [Hg90a, 3.16], i.e. the family \( \{F_t\}_t \) is norm bounded, and for each \( \xi \in \mathcal{E} \) the functions \( t \mapsto F_t(\xi) \) and \( t \mapsto F_t^*(\xi) \) are norm continuous. Indeed, we have: \( BL(\mathcal{E}_*) = BL(C_0(L) \otimes \mathcal{E}_*) = M(C_0(L, \mathcal{K}(\mathcal{E}_*))) = C_b(L, \mathcal{B}_{str}(\mathcal{E}_*)) \), and strict continuity is \(*\)-strong continuity. On \( BL(\mathcal{E}_*) \) \(*\)-strong continuity is weaker than norm continuity.

For the remaining part of this subsection we assume no group action. Given any Hilbert \( BL \)-module \( \mathcal{E} \), besides the adjointable operators \( BL(\mathcal{E}) \) on \( \mathcal{E} \) and the compact operators \( \mathcal{K}(\mathcal{E}) \), two other ideals will play an important role in our presentation:

**Definition 2.2.** The closed ideal of locally compact-valued families of operators is

\[
\mathcal{C}(\mathcal{E}) = \{ F \in BL(\mathcal{E}) \mid F \in \mathcal{K}(\mathcal{E}), \text{ for all } F \in C_0(L) \}. 
\]

(Here \( C_0(L) \) is viewed as a sub-\( C^* \)-algebra of \( BL(\mathcal{E}) \) as follows: let \( \{b_n\}_n \) be an approximate unit for \( B \), then for \( \xi \in \mathcal{E} \), let \( f(\xi) = \lim_{n \to \infty} \xi(f \otimes b_n) \).) The closed ideal of vanishing families of operators is

\[
\mathcal{J}(\mathcal{E}) = \{ F \in BL(\mathcal{E}) \mid \lim_{t \to \infty} ||F_t|| = 0 \}.
\]

**Lemma 2.3.** \( \mathcal{K}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E}) \).

*Proof.* The inclusion \( \mathcal{K}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E}) \) is clear. Let \( F \in \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E}) \). From the fact that \( F \in \mathcal{J}(\mathcal{E}) \) it follows that for every positive integer \( n \) there exists \( t_n \) such that \( ||F_t|| < 2^{-n} \), for all \( t \geq t_n \). Consider a partition of unity for \( L \), \( \{\chi_0, \chi_1, \cdots, \chi_n, \cdots\} \), subordinated to the cover.
\{(1, t_1 + 2^{-1}) \cup \{(t_n, t_{n+1} + 2^{-n-1})|n = 1, 2, \ldots\}. Then \( F = F \cdot 1 = \sum_{n=0}^{\infty} F \cdot \chi_n \in \mathcal{K}(\mathcal{E}) \), due to the fact that each term \( F \cdot \chi_n \) of the sum is compact \((F \in \mathcal{C}(\mathcal{E}))\), and of norm less than \(2^{-n}\) (for \(n \geq 1\)).

**Lemma 2.4.** If \( \mathcal{E} = \mathcal{E}_t \mathcal{L} \) is a constant family of Hilbert \( B \)-modules, then any \( F \in \mathcal{C}(\mathcal{E}) \) generates a norm-continuous family of operators \( \{F_t\}_{t} \) in \( \mathcal{K}(\mathcal{E}_t) \).

*Proof.* We first notice that the elements of \( \mathcal{K}(\mathcal{E}) \) generate norm-continuous families of operators. This is because any \( \xi \in \mathcal{E}_t \mathcal{L} \) is a norm-continuous section vanishing at infinity in the constant field of Hilbert modules \( \{\mathcal{E}_t\}_t \). Consequently the generators \( \theta_{\xi, \eta}, \xi, \eta \in \mathcal{E}_t \mathcal{L} \), of \( \mathcal{K}(\mathcal{E}) \) are norm-continuous. Now, given \( F \in \mathcal{C}(\mathcal{E}) \), the continuity of the family \( \{F_t\}_t \) that it generates is a local property. For any \( t_0 \), choose \( f \in C_c(L) \), \( f \equiv 1 \) in a neighborhood of \( t_0 \). The definition of \( \mathcal{C}(\mathcal{E}) \) says that \( Ff \in \mathcal{K}(\mathcal{E}) \), and consequently \( Ff \) is a norm-continuous family. This gives the norm-continuity of \( \{F_t\}_t \) at \( t_0 \).

**Remark 2.5.** \( \mathcal{C}(\mathcal{E}) \) does not coincide with \( \{F \in \mathcal{B}(\mathcal{E}) \mid F_t \in \mathcal{K}(\mathcal{E}_t), \text{ for all } t \} \). Indeed, it is not difficult to construct a \(*\)-strongly continuous family \( \{P_t\}_{t \in [1, \infty)} \) of rank-one projections on an infinite dimensional Hilbert space which is not norm continuous.

We summarize the relations between these various ideals in the following diagram:

\[
\begin{array}{ccc}
\mathcal{B}(\mathcal{E}) & \overset{\mathcal{C}(\mathcal{E}) \overset{\mathcal{J}(\mathcal{E})}{\searrow} \mathcal{K}(\mathcal{E})}{\mathcal{J}(\mathcal{E})}
\end{array}
\]

**Definition 2.6.** Let \( A \) and \( B \) be graded separable \( C^*\)-algebras (with no group action). An *asymptotic Kasparov \((A, B)\)-module* is a pair \((\mathcal{E}, F)\), where \( \mathcal{E} \) is a continuous field of \((A, B)\)-modules, and \( F \in \mathcal{B}(\mathcal{E}) \) is odd and satisfies for any \( a \in A \):

\begin{align*}
(aKm1) & \quad (F - F^*) \varphi(a) \in \mathcal{J}(\mathcal{E}); \\
(aKm2) & \quad [F, \varphi(a)] \in \mathcal{J}(\mathcal{E}); \text{ and} \\
(aKm3) & \quad \varphi(a)(F^2 - 1) \varphi(a)^* \geq 0, \text{ modulo } \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}).
\end{align*}

The set of all asymptotic Kasparov \((A, B)\)-modules will be denoted by \( ke(A, B) \).

**Remark 2.7.** Compare these axioms with the ones that a Kasparov module \((\mathcal{E}, F)\) must satisfy (Definition \(\square\)). It is worth noticing that the third axiom of a Kasparov module, \((F^2 - 1)\varphi(a) \in \mathcal{K}(\mathcal{E})\), can be replaced (at least when \( \|F\| \leq 1 \)) by \( \varphi(a)(F^2 - 1)\varphi(a)^* \geq 0 \), modulo \( \mathcal{K}(\mathcal{E}) \), which looks more like our \((aKm3)\).

**Remark 2.8.** We introduce the following notation: given two operators \( T, T' \in \mathcal{B}(\mathcal{E}) \), then \( T \sim T' \) if \((T - T') \in \mathcal{J}(\mathcal{E})\). With this convention \((aKm1)\) reads \((F - F^*)\varphi(a) \sim 0 \), and \((aKm2)\) reads \([F, \varphi(a)] \sim 0 \), for all \( a \in A \).
Remark 2.9. In terms of families we can rephrase the conditions of Definition 2.4 as follows: \( \{ \mathcal{E}_t \}_{t \in [1, \infty]} \) is a family of Hilbert \((A, B)\)-modules, \( \{ F_t \}_{t \in [1, \infty]} \) is a bounded \( \ast \)-strong continuous family of odd operators, meaning that for each continuous section \( \xi = \{ \xi_t \}_t \) the maps \( t \mapsto F_t(\xi_t) \) and \( t \mapsto F^*_t(\xi_t) \) are continuous sections of the field \( \{ \mathcal{E}_t \}_{t \in [1, \infty]} \), and for each \( a \in A \)

(aKm1') \( \| (F_t - F^*_t) a_t \| \xrightarrow{t \to \infty} 0 \);

(aKm2') \( \| [F_t, a_t] \| \xrightarrow{t \to \infty} 0 \); and

(aKm3') \( a_t (F^2_t - 1) a^*_t \geq K^a_t \), for a family \( \{ K^a_t \}_t \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}) \) depending on \( a \).

(Here \( a_t \) denotes \( \varphi_t(a) = \varphi(a) \otimes_{_{c_*}} 1 \).)

Example 2.10. Given a \( \ast \)-homomorphism \( \psi : A \to B \), we form the asymptotic Kasparov \((A, B)\)-module \((\mathcal{E}, F)\), where \( \mathcal{E} = BL \) and \( F = 0 \). The representation of \( A \) is \( \varphi = 1 \otimes \psi \). Axioms (aKm1) and (aKm2) are trivially satisfied, and (aKm3) follows from the fact that \( \varphi(a) \varphi(a)^* \in \mathcal{C}(\mathcal{E}) \) (the family \( \{ K^a_t \}_t \), is constant, \( K^a_t = -\psi(a) \varphi(a)^* \in B \simeq \mathcal{K}(B) \), for all \( t \in [1, \infty) \) and all \( a \in A \). More generally, given a \( \ast \)-homomorphism \( \psi : A \to \mathcal{K}(\mathcal{H}) \otimes B \), with \( \mathcal{H} \) a countable generated Hilbert space, we form the asymptotic Kasparov \((A, B)\)-module \((\mathcal{H}_B, 0)\), with constant action of \( A \) on ‘fibers’ as above. In this situation \( K^a_t = -\psi(a) \varphi(a)^* \in \mathcal{K}(\mathcal{H}) \otimes B \simeq \mathcal{K}(\mathcal{H}_B) \). This simple but fundamental example implies the following principle: if a Kasparov module \((\mathcal{E}, F)\) with \( F = 0 \) exists, then an asymptotic Kasparov module can be constructed from it, namely \((\mathcal{E}L, 0)\).

Example 2.11 (The \( K \)-homology class of the Dirac operator). Let \( M^{2n} \) be an even-dimensional, complete, spin\(^c\)-manifold, with spinor bundle \( S = S_M \), and Dirac operator \( D = D_M \). \( D \) is essentially self-adjoint, and whenever functional calculus is used \( D \) actually denotes the closure \( \overline{D} = D^* \). The fundamental asymptotic Kasparov \((C_0(M), \mathbb{C})\)-module is constructed as follows: \( \mathcal{E} = \{ L^2(M, S) \}_{t \in [1, \infty)} \), constant family; the action of \( C_0(M) \) is the same on each ‘fiber’, by multiplication operators \( \varphi_t(f) = M_f \); and \( F = \{ \chi(\frac{1}{t}D) \}_{t \in [1, \infty)} \), where \( \chi \) is a normalizing function \( i.e. \chi : \mathbb{R} \to [-1, 1] \) is odd, smooth, and \( \lim_{t \to \pm \infty} \chi(x) = \pm 1 \); for example one could take \( \chi(x) = x/(1 + x^2)^{1/2} \). Let us show that this is an asymptotic Kasparov module. (For a thorough exposition of elliptic operators on manifolds see [HgRo, Chaps.10,11]. This reference also explains the terminology that we use in this example.)

- \( F \in \mathcal{B}(\mathcal{E}) \). Indeed, this is implied by the norm continuity of \( t \mapsto \chi(\frac{1}{t}D) \).

- \( F \) satisfies (aKm1). As noted above, when we write \( D \) we actually mean \( \overline{D} = D^* \), which is self-adjoint, and the functional calculus gives \( F = F^* \).

- \( F \) satisfies (aKm2). Let \( f \in C_c(\infty)(M) \). Then \( [\frac{1}{t}D, f] = \frac{1}{t} (DF - fD) = \frac{1}{t} \nabla f \xrightarrow{t \to \infty} 0 \), in norm (\( \nabla f \) represents Clifford multiplication by the vector field \( \nabla f \)). This gives ([Hg91]):

\[
[\left(\frac{1}{t}D \pm i\right)^{-1}, f] = \left(\frac{1}{t}D \pm i\right)^{-1} f - f \left(\frac{1}{t}D \pm i\right)^{-1} = \left(\frac{1}{t}D \pm i\right)^{-1} \left\{ f \left(\frac{1}{t}D \pm i\right) - \left(\frac{1}{t}D \pm i\right) f \right\} \left(\frac{1}{t}D \pm i\right)^{-1} = \left(\frac{1}{t}D \pm i\right)^{-1} \left(\frac{1}{t} \nabla f \right) \left(\frac{1}{t}D \pm i\right)^{-1} \xrightarrow{t \to \infty} 0. \tag{10}
\]

It follows that we obtain norm convergence \( [\phi(\frac{1}{t}D), f] \xrightarrow{t \to \infty} 0 \), for all \( \phi \in C_0(\mathbb{R}) \), \( f \in C_0(M) \). The significance is that the asymptotic Kasparov module that we construct will not depend on the normalizing function, any two such having difference in \( C_0(\mathbb{R}) \). Moreover it suffices
now to prove (aKm2) for one particular normalizing function \( \chi_0 \). We choose it such that its distributional Fourier transform \( \hat{\chi}_0 \) is compactly supported, and \( s \mapsto s \hat{\chi}_0(s) \) is smooth (or in \( L^1(\mathbb{R}) \)). (Such functions exist: see \[HgRo\], 10.9.3.) We know, basically from Stone's theorem, that:

\[
\langle \chi_0(D)u, v \rangle = \int_{\mathbb{R}} \langle e^{isD}u, v \rangle \hat{\chi}_0(s) \, ds, \quad \text{for all } u, v \in C_c^\infty(M, \mathbb{S}).
\]

(11)

Consider for the moment a function \( f \in C^\infty(M) \) which takes values in \( S^1 \subset \mathbb{C} \) (i.e. \( M_f \) is an unitary operator), and such that \( \nabla f \) is also a bounded operator. We have:

\[
[\chi_0(\frac{1}{T}D), f] = \chi_0(\frac{1}{T}D) f - f \chi_0(\frac{1}{T}D) = f \left( f^{-1} \chi_0(\frac{1}{T}D) f - \chi_0(\frac{1}{T}D) \right)
\]

(12)

Putting together (11) and (12), we obtain:

\[
\langle [\chi_0(\frac{1}{T}D), f]u, v \rangle = \int_{\mathbb{R}} \langle (e^{ist^{-1}Df} - e^{ist^{-1}D})u, \overline{f}v \rangle \hat{\chi}_0(s) \, ds.
\]

(13)

By our first computation of this paragraph, \( f^{-1}Df - D = f^{-1}[D, f] = f^{-1}\nabla f \) is a bounded operator. In accordance with \[HgRo\], Lemma 10.3.6], applied to \( T_1 = \frac{1}{T}f^{-1}Df \) and \( T_2 = \frac{1}{T}D \), we have:

\[
\|e^{ist_1} - e^{ist_2}\| \leq |s| \|T_1 - T_2\|, \quad \text{for all } s \in \mathbb{R}.
\]

(14)

Because of (14), the inner product in the integral of (13) equals \( |s| \times \text{smooth function} \). The required norm asymptotic commutation now follows:

\[
\| [\chi_0(\frac{1}{T}D), f] \| \leq \frac{1}{T} \|\nabla f\| \int_{\mathbb{R}} |s| \hat{\chi}_0(s) \, ds.
\]

The computation made in the last part of the argument above is \[HgRo\], Prop. 10.3.7.

Finally, any arbitrary non-zero \( f \in C_c^\infty(M) \) can be written as a linear combination of functions on \( M \) which are \( S^1 \)-valued. Indeed, \( f = \text{Re}(f) + i \text{Im}(f) \), and for a real valued \( f \neq 0 \) one writes:

\[
f = \left( \|f\|/2 \right) \left( \left( f/\|f\| + i\sqrt{1 - f^2/\|f\|^2} \right) + \left( f/\|f\| - i\sqrt{1 - f^2/\|f\|^2} \right) \right).
\]

We are through (due to the density of \( C_c^\infty(M) \) in \( C_0(M) \)).

- \( F \) satisfies (aKm3). The standard theory of elliptic first order differential operators shows that \( f \left( \chi^2(\frac{1}{T}D) - 1 \right) \) is compact for \( f \in C_0(M) \). It follows that \( f \left( F^2 - 1 \right) \overline{f} = 0 \), modulo \( \mathcal{C}(\mathcal{E}) \). (The norm continuity of \( t \mapsto F_t \) was used again here.)

**Remark 2.12.** Given an asymptotic Kasparov \( (A, B) \)-module \( (\mathcal{E}, F) \) then \( (\mathcal{E}, (F + F^*)/2) \) is another such object. Indeed, the only axiom which is not clear is (aKm3). It reduces to showing that \( (F + F^*)^2/4 \geq (F^2 + (F^*)^2)/2 \), which in turn is equivalent to the obvious \( (F - F^*) (F - F^*)^* \geq 0 \).
2.2 The $KE$-theory groups

In this subsection we define the new bivariant theory and we study some of its functorial properties. A group (locally compact, $\sigma$-compact, Hausdorff) is assumed to act continuously on all the objects under study. We start with an extension of our previous Definition 2.10 to the equivariant context.

Definition 2.13. Consider a group $G$, and two graded separable $G$-$C^*$-algebras $A$ and $B$. An asymptotic Kasparov $G$-$(A,B)$-module is a pair $(E,F)$, where $E$ is a continuous field of $G$-$(A,B)$-modules (see Definition 2.1), and $F \in \mathcal{B}(E)$ is an odd $G$-continuous operator that satisfies $(\text{aKm1})$, $(\text{aKm2})$, $(\text{aKm3})$ of Definition 2.10 and the extra condition:

$(\text{aKm4})$ \[ (g(F) - F) \varphi(a) \in \mathcal{J}(E), \text{ for all } g \in G, a \in A. \]

In terms of families this last condition reads:

$(\text{aKm4}')$ \[ \| (g(F_t) - F_t) a_t \| \xrightarrow{t \to \infty} 0, \text{ for all } g \in G, a \in A, \text{ and with } a_t = \varphi_t(a). \]

The set of all asymptotic Kasparov $G$-$(A,B)$-modules is denoted by $ke_G(A,B)$.

Example 2.14. Consider an equivariantly split exact sequence of $G$-$C^*$-algebras

\[ 0 \to B \overset{j} \to D \overset{s} \to A \to 0, \]

meaning that all the $*$-homomorphisms are equivariant and that $p \circ s = id_A$. Let $\omega : D \to M(B) = \mathcal{B}(B)$ be the canonical extension of the inclusion $B \to M(B)$ (the construction of the extension given in the proof of [Lance, Prop.2.1] is equivariant). Let $\{u_t\}_t$ be a quasi-invariant quasi-central approximate unit for $B \subset \omega(D) \subset M(B)$ (recall Definition 1.2). We associate to the above extension the asymptotic Kasparov $G$-$(D,B)$-module $\{(B \oplus B^{op}, \left\{0_{1-ut} \ 1-ut\right\})\}_t$, where the action of $D$ is constant on fibers $\varphi_t : D \to \mathcal{B}(B \oplus B^{op})$, $\varphi_t(d) = \left(\omega(d) \ 0 \ 0 \ (\omega \circ op)(d)\right)$. Its class in $KE_G(D,B)$ is the splitting morphism of the exact sequence (see [Black, 17.1.2b] and [Cegala, Sec.5]).

Example 2.15 (The Bott element). Let $V$ be a separable Euclidean space, and denote by $\mathcal{A}(V)$ the non-commutative $C^*$-algebra used by Higson-Kasparov-Trout in their proof of Bott periodicity ([Higson, Def.3.3], [Higson-Kasparov, Def.4.1]). One considers $C_0(\mathbb{R})$ graded by even and odd functions. For a finite dimensional affine subspace $V_a$ of $V$, denote by $V_a^\mathbb{R}$ its linear support, and by $\mathcal{A}(V_a) = C_0(\mathbb{R}) \otimes C_0(V_a, \text{Cliff}(V_a^\mathbb{R}))$. The $C^*$-algebra $\mathcal{A}(V)$ is defined as the direct limit over all finite dimensional affine spaces $V_a \subset V$ of $\mathcal{A}(V_a)$: $\mathcal{A}(V) = \varinjlim \mathcal{A}(V_a)$. Then let $\beta : C_0(\mathbb{R}) \to \mathcal{A}(V)$ be the $*$-homomorphism given by the inclusion $(0) \subset V$, and use it to construct a family of $*$-homomorphisms $\{\beta_t\}_{t \in [1,\infty)} : C_0(\mathbb{R}) \to \mathcal{A}(V)$, $\beta_t(f) = \beta(f_t)$, where $f_t(x) = f(t^{-1}x)$. For each $t$ extend $\beta_t$ to a $*$-homomorphism $\overline{\beta_t} : C_0(\mathbb{R}) \to M(\mathcal{A}(V))$. Consider $\lambda(x) = x/(1 + x^2)^{1/2}$ and define $F_t = \overline{\beta_t}(\lambda) \in M(\mathcal{A}(V))$. Further assume that a group $G$ acts isometrically and by affine transformations on $V$. We associate the asymptotic Kasparov $G$-$(C,\mathcal{A}(V))$-module $\{\mathcal{A}(V), F_t\}_t$, where the action of $C$ is constant on fibers $\varphi_t : C \to \mathcal{B}(\mathcal{A}(V))$, $\varphi_t(1) = 1$. We notice that, for each $t$, $F_t$ is odd and self-adjoint (because $\lambda$ has these properties), and that $\{F_t\}_t$ is actually a norm continuous family of operators. This shows that (aKm1) is satisfied, (aKm2) is trivial, and (aKm4) follows from the asymptotic equivariance of $\{\beta_t\}_t$ ([Higson-Kasparov, Def.4.3]). Finally, to see that (aKm3) holds true, note that $F_t^2 - 1 = -\beta_t(1/(1 + x^2)) \in \mathcal{A}(V) = \mathcal{K}(\mathcal{A}(V))$. Consequently $F_t^2 - 1 = 0$, modulo $\mathcal{E}(\mathcal{A}(V)L)$. 

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**Definition 2.16.** An element \( (\mathcal{E}, F) \) of \( ke_G(A, B[0, 1]) \) gives, by ‘evaluation at \( s \)’, a family \( \{ (\mathcal{E}_s, F_s) \in ke_G(A, B) \mid s \in [0, 1] \} \), with \( \mathcal{E}_s = \mathcal{E} \otimes_{\mathcal{E}_s} BL, F_s = F \otimes_{\mathcal{E}_s} 1 \). Such an element \( (\mathcal{E}, F) \) and the family that it generates are called a **homotopy** between \( (\mathcal{E}_0, F_0) \) and \( (\mathcal{E}_1, F_1) \). An **operator homotopy** is a homotopy \( \{ (\mathcal{E}_s, F_s) \mid s \in [0, 1] \} \), with \( s \mapsto F_s \) being norm continuous. Note that \( \mathcal{E} \), and the action of \( A \) on it, are constant throughout an operator homotopy.

**Example 2.17.** Each \( (\mathcal{E}_0, F_0) = \{(\mathcal{E}_{0,t}, F_{0,t})\}_{t} \in ke_G(A, B) \) is homotopic to any of its ‘translates’ \( \{(\mathcal{E}_{0,t+N}, F_{0,t+N})\}_{t} \). It can also be ‘stretched’ by a homotopy to \( (\mathcal{E}_1, F_1) = \{(\mathcal{E}_{0,h(t)}, F_{0,h(t)})\}_{t} \), for any increasing bijective function \( h : [1, \infty) \to [1, \infty) \).

**Definition 2.18.** An asymptotic Kasparov \( G-(A, B) \)-module \( (\mathcal{E}, F) \) is said to be **degenerate** if for all \( a \in A \) and \( g \in G \): \( F = F^* \), \( [F, \varphi(a)] = 0 \), \( (g(F) - F)\varphi(a) = 0 \), and \( \varphi(a) (F^2 - 1) \varphi(a)^* \geq 0 \), modulo \( \mathcal{J}(\mathcal{E}) \).

**Remark 2.19.** We want to comment on the definition of degenerate elements. The first three conditions are identical with the ones for a degenerate Kasparov module (Definition 1.4), but in (aKm3) we require positivity modulo \( \mathcal{J}(\mathcal{E}) \). In this way, for example, the generator of \( KE(\mathbb{C}, \mathbb{C}) \) will be described by \( \mathcal{C}(\mathcal{H}L)/\mathcal{J}(\mathcal{H}L) \), which corresponds to the Fredholm index as invariant. This result is required by the dimension axiom that any homology theory has to satisfy.

**Lemma 2.20.** If \( (\mathcal{E}, F) \) is degenerate, then it is homotopic to the 0-module \((0,0)\).

**Proof.** The pair \((C_0([0,1]) \otimes \mathcal{E}, 1 \otimes F)\), with \( A \) acting as \( 1 \otimes \varphi \), is a degenerate asymptotic Kasparov \((A,BI)\)-module, which gives a homotopy between \((\mathcal{E}, F)\) and \((0,0)\). ■

**Definition 2.21.** Given \((\mathcal{E}, F)\) and \((\mathcal{E}, F')\) in \( ke_G(A, B) \), we say that \( F' \) is a ‘small perturbation’ of \( F \) if \( (F - F') \varphi(a) \in \mathcal{J}(\mathcal{E}) \), for all \( a \in A \).

**Lemma 2.22.** Consider \((\mathcal{E}, F)\) in \( ke_G(A, B) \), and \( F' \) a ‘small perturbation’ of \( F \). Then \((\mathcal{E}, F)\) and \((\mathcal{E}, F')\) are operatorially homotopic.

**Proof.** Indeed, the straight line segment between \( F \) and \( F' \) is an operator homotopy: \( F = \{ sF + (1 - s)F' \}_{s \in [0,1]} \). We note that it is the same proof as in KK-theory for ‘compact perturbations’ ([Bick}, Def.17.2.4]. ■

**Corollary 2.23.** Any \((\mathcal{E}, F)\) in \( ke_G(A, B) \) is homotopic to \((\mathcal{E}, (F + F^*)/2)\).

**Proof.** \((F + F^*)/2\) is a ‘small perturbation’ of \( F \). ■

From the corollary above it follows that (aKm1) can be strengthened: in Definitions 2.6 and 2.13 we could consider only self-adjoint operators \( F \). Other changes are possible too.

A less trivial example of homotopy is provided by the next result (compare with [Sk84, Lemma 11]). Despite the simplicity of its proof, it will be very useful when we shall analyze in depth the product in KE-theory.

**Lemma 2.24.** Let \( \mathcal{E} \) be a continuous field of G-(A, B)-modules. Consider two asymptotic Kasparov \((A, B)\)-modules \((\mathcal{E}, F)\), \((\mathcal{E}, F')\) in \( ke_G(A, B) \), such that \( \varphi(a) [F, F'] \varphi(a)^* \geq 0 \), modulo \( \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}) \), for all \( a \in A \). Then \((\mathcal{E}, F)\) and \((\mathcal{E}, F')\) are (operatorially) homotopic.
Proof. Put $F_s = \cos(s \pi/2)F + \sin(s \pi/2)F'$, for $s \in [0, 1]$. Then the family $\{(\mathcal{E}, F_s)\}_s$ realizes the required homotopy.  

Definition 2.25. The set $KE_G(A, B)$ is defined as the quotient of $ke_G(A, B)$ by the equivalence relation generated by homotopy. (We shall omit $G$ in the non-equivariant case.) Given $x = (\mathcal{E}, F) \in ke_G(A, B)$, its class in $KE_G(A, B)$ will be denoted by $[x]$. The addition of two asymptotic Kasparov $G$-(\mathcal{A}, \mathcal{B})$-modules $(\mathcal{E}, F_1)$ and $(\mathcal{E}, F_2)$ is defined by $(\mathcal{E}_1, F_1) + (\mathcal{E}_2, F_2) = (\mathcal{E}_1 \oplus \mathcal{E}_2, F_1 \oplus F_2) \in ke_G(A, B)$.

Theorem 2.26. With the notation of the previous definition, $KE_G(A, B)$ is an abelian group.

Proof. The argument is similar to the one for $KK$-theory – see [Sk84, Prop.4]. The inverse of $(\mathcal{E}, F)$ is $(E^op, -U F U^*)$, where $E^op$ is $E$ with the opposite grading, $U : E \to E^op$ is the identity, and $A$ acts on $E^op$ as $a(U \xi) = U((-1)^{g_a} a \xi)$.

Definition 2.27. For any group $G$, $1 = 1_c \in KE_G(C, C)$ is the class of the identity $*$-homomorphism $\psi = id : C \to C$, i.e. the class of $(C_0(L), 0)$, with trivial action on $C_0(L)$. Note that 1 has nothing to do with the abelian group structure. More generally, given a $G$-$C^*$-algebra $A$, the element $1_A \in KE_G(A, A)$ is the class of the identity $*$-homomorphism $\psi = id : A \to A$ (as in Example 2.10), i.e. the class of $(AL, 0)$. Given an equivariant $*$-homomorphism $\psi : A \to B$ or more generally $\psi : A \to \mathcal{K} \otimes B$, its class in $KE_G(A, B)$ is denoted by $[\psi]$.

2.3 Functoriality properties

We discuss next some of the functoriality properties of the $KE$-groups. They are similar to the ones that the $KK$-theory groups satisfy.

(a) Given a $*$-homomorphism $\psi : A_1 \to A$, we obtain a map:
\[ \psi^* : ke_G(A, B) \to ke_G(A_1, B), (\mathcal{E}, F) \mapsto (\psi^* \mathcal{E}, F). \]
Here $\psi^* \mathcal{E}$ denotes the same Hilbert module $\mathcal{E}$, but with left action by $A_1$ given by the composition $\varphi \circ \psi : A_1 \to \mathcal{B}(\mathcal{E})$. We observe that $\psi^*$ respects direct sums, and homotopy of asymptotic Kasparov modules. Consequently we get a well-defined map, denoted by the same symbol, at the level of groups: $\psi^* : KE_G(A, B) \to KE_G(A_1, B)$. It is clear that for $*$-homomorphisms $A_2 \xrightarrow{\omega} A_1 \xrightarrow{\psi} A$ we have $(\psi \circ \omega)^* = \omega^* \circ \psi^*$.

(b) Let $\psi : B \to B_1$ be a $*$-homomorphism. Using $1 \otimes \psi : BL \to B_1 L$, we obtain a map:
\[ \psi_* : ke_G(A, B) \to ke_G(A, B_1), (\mathcal{E}, F) \mapsto (\mathcal{E} \otimes_{1 \otimes \psi} B_1 L, F \otimes_{1 \otimes \psi} 1). \]
This map also respects direct sums, and homotopy of asymptotic Kasparov modules, and so gives a well-defined map: $\psi_* : KE_G(A, B) \to KE_G(A, B_1)$.

(c) For any $G$-$C^*$-algebra $D$ there is a map:
\[ \sigma_D : ke_G(A, B) \to ke_G(A \otimes D, B \otimes D), (\mathcal{E}, F) \mapsto (\mathcal{E} \otimes D, F \otimes 1). \] (15)
It passes to quotients and gives a map $\sigma_D : KE_G(A, B) \to KE_G(A \otimes D, B \otimes D)$. Indeed, we verify first that the axioms for asymptotic Kasparov modules are satisfied.

- $F \otimes 1$ satisfies (aKm1). $(F \otimes 1 - (F \otimes 1)^*) (a \otimes d) = (F - F^*) a \otimes d \in \mathcal{J}(\mathcal{E}) \otimes D \subseteq \mathcal{J}(\mathcal{E} \otimes D)$.
- $F \otimes 1$ satisfies (aKm2). $(F \otimes 1) (a \otimes d) - (-1)^{\partial a + \partial d} (a \otimes d) (F \otimes 1) = [F, a] \otimes d \in \mathcal{J}(\mathcal{E}) \otimes D \subseteq \mathcal{J}(\mathcal{E} \otimes D)$.
- $F \otimes 1$ satisfies (aKm3). $(a \otimes d) (F^2 \otimes 1) (a^* \otimes d^*) = aF^2 a^* \otimes dd^* \geq aa^* \otimes dd^*$, modulo $\mathcal{C}(\mathcal{E}) \otimes D + \mathcal{J}(\mathcal{E} \otimes D) \subseteq \mathcal{C}(\mathcal{E} \otimes D) + \mathcal{J}(\mathcal{E} \otimes D)$. The last inclusion follows from the isomorphism $\mathcal{K}(\mathcal{F} \otimes D) \simeq \mathcal{K}(\mathcal{F}) \otimes D$, where $\mathcal{F}$ is any Hilbert module.
- $F \otimes 1$ satisfies (aKm4). $(g(F \otimes 1) - F \otimes 1) (a \otimes d) = (g(F) - F) a \otimes d \in \mathcal{J}(\mathcal{E} \otimes D)$.

Finally, $\sigma_D$ sends homotopic Kasparov modules to homotopic asymptotic Kasparov modules, and this shows that $\sigma_D$ is well defined at the level of groups.

**Proposition 2.28 (Homotopy invariance).** The bifunctor $KE_G(A, B)$ is homotopy invariant in both variables:

(a) let $\psi_0, \psi_1 : A_1 \to A$ be homotopic $*$-homomorphisms; then, for any $B$, $\psi_0^* = \psi_1^* : KE_G(A_1, B) \to KE_G(A, B)$;

(b) let $\psi_0, \psi_1 : B \to B_1$ be homotopic $*$-homomorphisms; then, for any $A$, $\psi_0^* = \psi_1^* : KE_G(A, B) \to KE_G(A, B_1)$.

**Proof.** Once again we may follow the same proof as in $KK$-theory.

(a) Let $\psi : A_1 \to A[0, 1]$ be a homotopy between $\psi_0$ and $\psi_1$. If $(\mathcal{E}, F) \in ke_G(A, B)$, then $\psi^* (\sigma_{G[0, 1]} ((\mathcal{E}, F))) \in ke_G(A_1, B[0, 1])$ gives a homotopy between $\psi_0^* ((\mathcal{E}, F))$ and $\psi_1^* ((\mathcal{E}, F))$.

(b) Let $\psi : B \to B_1[0, 1]$ be a homotopy between $\psi_0$ and $\psi_1$. Because $ev_0$ and $ev_1$ are essential $*$-homomorphisms, it follows that $\psi_{i,*} = ev_{i,*} \circ \psi_*$, for $i = 0, 1$. Consequently, given $(\mathcal{E}, F) \in ke_G(A, B)$, $\psi_* ((\mathcal{E}, F))$ gives a homotopy between $\psi_{0,*} ((\mathcal{E}, F))$ and $\psi_{1,*} ((\mathcal{E}, F))$.

2.4 Some technical results

We conclude this section with a technical result (namely Lemma 2.29), three definitions, and a ‘diagonalization’ process, that will be used in the definition of the product in Section 3. Recall that any self-adjoint element $x$ of a $C^*$-algebra can be written as a difference of two positive elements $x = x_+ - x_-$, with $x_+ x_- = x_- x_+ = 0$. The element $x_-$ is called the negative part of $x$.

**Lemma 2.29.** Let $A$ and $B$ be separable $G$-$C^*$-algebras. Given $(\mathcal{E}, F) \in ke_G(A, B)$, there exists a self-adjoint element $u \in \mathcal{C}(\mathcal{E})^{(0)}$ satisfying:

(i) $[u, F] \in \mathcal{J}(\mathcal{E})$;

(ii) $[u, a] \in \mathcal{J}(\mathcal{E})$, for all $a \in A$;

(iii) $(1 - u^2) (a (F^2 - 1) a^*) \in \mathcal{J}(\mathcal{E})$, for all $a \in A$; and

(iv) $(g(u) - u) \in \mathcal{J}(\mathcal{E})$, for all $g \in G$. 

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Proof. Consider a dense subset \( \{a_n\}_{n=1}^{\infty} \) in \( A \), and an appropriate (see below) cover of \([1, \infty)\) by closed intervals \( \{I_n\}_{n=0}^{\infty} \), of the form \( I_n = [t_n, t_{n+2}] \), with \( t_0 = 1 \), and \( \{t_n\}_{n} \) being a strictly increasing sequence with \( \lim_{n \to \infty} t_n = \infty \). Choose a partition of unity \( \{\mu_n\}_{n=0}^{\infty} \) in \( C_0([1, \infty)) \) subordinated to this cover. For each positive integer \( n \), let \( r_n : BL \to B(I_n) \) be the restriction \(*\)-homomorphism, and use it to define the restriction of \( \mathcal{E} \) and \( F \) to \( I_n \): \( \mathcal{E}|_{I_n} = (r_n)_* (\mathcal{E}), \ F|_{I_n} = (r_n)_* (F) \).

Let \( u_{0,0} \) be an arbitrary even self-adjoint element of \( \mathcal{K}(\mathcal{E}|_{I_0}) \). For each \( n \geq 1 \), apply Proposition 1.3 to construct a quasi-invariant approximate unit \( \{u_{n,k}\}_{k=1}^{\infty} \) for \( \mathcal{K}(\mathcal{E}|_{I_n}) \), which is quasi-central for \( F|_{I_n}, A|_{I_n} \), and \( \{ (a (F^2 - 1)a^*)_- |_{I_n} \mid a \in A \} \). There exists an index \( k_n \) such that \( \| [u_{n,k_n}, F] \| < 1/n, \| [u_{n,k_n}, a_m] \| < 1/n, \| (1 - u_{n,k_n}^2) (a_m(F^2 - 1)a_m^*)_- \| < 1/n \), for \( m = 1, 2, \ldots, n \). (For the third inequality, recall that (aKm3) implies \( (a_m(F^2 - 1)a_m^*)_- \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}) \), with the \( \mathcal{C}(\mathcal{E}) \) part restricting to an element of \( \mathcal{K}(\mathcal{E}|_{I_n}) \), and the \( \mathcal{J}(\mathcal{E}) \) part having norm \( < 1/2n \) by our initial choice of the partition \( \{I_n\}_{n} \).) Define: \( u = \sum_{n=0}^{\infty} \mu_n u_{n,k_n} \in \mathcal{C}(\mathcal{E}) \). We observe that (i) is satisfied, and that (ii) and (iii) hold true for all the elements of the dense subset \( \{a_n\}_{n} \) of \( A \). A density argument finishes the proof. To have (iv) satisfied, one uses quasi-invariance, and a similar argument after choosing a dense subset \( \{g_n\}_{n=1}^{\infty} \) of \( G \). ■

Remark. The diagram 3 shows that the operators that appear in (i) and (ii) of the lemma above actually belong to \( \mathcal{K}(\mathcal{E}) \). 

Definition 2.30. A section of \([1, \infty) \times [1, \infty)\) is any increasing continuous function \( h : [1, \infty) \to [1, \infty) \), with \( h(1) = 1 \), \( \lim_{t \to \infty} h(t) = \infty \), differentiable on \([1, \infty) \), except maybe for a countable set of points where it has finite one-sided derivatives. (Note that the differentiability assumption is just a convenience.)

Lemma 2.31. Given a countable family \( \{h_n\}_{n} \) of sections of \([1, \infty) \times [1, \infty)\), one can find a suitable strictly increasing sequence of numbers \( \{1 = x_0, x_1, x_2, \ldots, x_n, \ldots\} \), with \( \lim_{n \to \infty} x_n = \infty \), and a section \( h \) satisfying the following condition: for each \( n, h \geq h_i \), for \( i = 1, 2, \ldots, n, \) over the closed interval \([x_{n-1}, x_n]\).

Proof. The definition of a section implies the existence, for each \( h_i \), of a sequence \( \{1 = x_0^i, x_1^i, \ldots, x_n^i, \ldots\} \), such that \( h_i \) is differentiable on \((x_{n-1}^i, x_n^i)\), for each positive integer \( n \), and has finite one-sided derivatives at the end points. Let \( h(1) = 1 \), and for each integer \( n \geq 1 \) define:

\[
x_n = \max_{1 \leq i \leq n} \{x_n^i\},
\]

\[
m_n = \max_{1 \leq i \leq n} \sup_{t \in [x_{n-1}, x_n]} \{h'_i(t)\}, \text{ (one-sided derivatives included),}
\]

\[
H_n = \max \{0, h_{n+1}(x_n) - (h(x_{n-1}) + m_n (x_n - x_{n-1}))\}.
\]

Define \( h \) on \((x_{n-1}, x_n)\) by:

\[
h(t) = h(x_{n-1}) + \left( m_n + \frac{H_n}{x_n - x_{n-1}} \right) (t - x_{n-1}).
\]
Definition 2.32. Consider a Hilbert BLL-module $\mathcal{E}$. Given a section $h$ of $[1, \infty) \times [1, \infty)$ as in Definition 2.30, consider the restriction $*$-homomorphism: $\text{Res}_h : BLL \to BL$, $f \mapsto f|_{\text{graph}(h)}$. (The parameter $t \in L$ in $BL$ is such that $(t, h(t)) \in \text{graph}(h) \subset [1, \infty) \times [1, \infty)$.) The restriction of $\mathcal{E}$ to the graph of $h$ is the Hilbert $BL$-module $\mathcal{E}_h := (\text{Res}_h)_*(\mathcal{E}) = \mathcal{E} \otimes_{\text{Res}_h} BL$. Consider now any operator $F \in \mathcal{B}(\mathcal{E})$. The restriction of $F$ to the graph of $h$ is the operator $F_h := (\text{Res}_h)_*(F) = F \otimes_{\text{Res}_h} 1 \in \mathcal{B}(\mathcal{E}_h)$.

Definition 2.33. Given a Hilbert BLL-module $\mathcal{E}$, let

$$J(\mathcal{E}) = \{ F \in \mathcal{B}(\mathcal{E}) | \lim_{t_1, t_2 \to \infty} \| F(t_1, t_2) \| = 0 \}.$$ 

Here $(t_1, t_2)$ designates a point in $LL = [1, \infty) \times [1, \infty)$, and the limit is taken when both $t_1$ and $t_2$ approach infinity. Note that if $F \in J(\mathcal{E})$ then $F_h \in J(\mathcal{E}_h)$ for any section $h$ of $[1, \infty) \times [1, \infty)$.

3 $KE$-theory: construction of the product

In this section the product is defined and various properties, including its associativity, are proved.

3.1 A motivational example

Let $G$ be a locally compact $\sigma$-compact Hausdorff group, and $A_1$, $A_2$, $B_1$, $B_2$, $D$ be separable $G$-$C^*$-algebras. The aim is to construct a certain bilinear map

$$KE_G(A_1, B_1 \otimes D) \otimes KE_G(D \otimes A_2, B_2) \to KE_G(A_1 \otimes A_2, B_1 \otimes B_2).$$  \hfill (16)

This will be the product in $KE$-theory (compare with the product in $KK$-theory and in $E$-theory), and its construction is based on the particular case when $B_1 = A_2 = \mathbb{C}$. The intuition, based on examples coming from $K$-homology and $K$-theory, is that the product should have the form:

$$((\mathcal{E}_1, F_1), (\mathcal{E}_2, F_2)) \mapsto (\mathcal{E}_1 \boxtimes \mathcal{E}_2, F_1 \boxtimes 1 + 1 \boxtimes F_2),$$  \hfill (17)

where $\boxtimes$ is a certain ‘tensor product.’ Kasparov \cite{Kas75}, \cite{Kas81} succeeded to overcome the serious technical difficulties that arise in making sense of (17). We start our approach by providing a construction of the product (16) in the case when $D = \mathbb{C}$, known as external product. By doing so, we shall present a case when the formula (17) is actually correct. We shall also see the axioms (aKm1) - (aKm4) at work, and understand some of the difficulties involved in the general construction.

Example 3.1 (External product). Consider elements $(\mathcal{E}_1, F_1) \in ke_G(A_1, B_1)$ and $(\mathcal{E}_2, F_2) \in ke_G(A_2, B_2)$. Construct the $(A_1 \otimes A_2, B_1 L \otimes B_2 L)$-module $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ (external tensor product of Hilbert modules), and $F = F_1 \otimes 1 + 1 \otimes F_2 \in \mathcal{B}(\mathcal{E})$. The claim is that the restriction $(\text{Res}_h)_*(\mathcal{E}, F)$ to the graph of any section $h$ satisfies (aKm1) — (aKm4). Indeed, due to the inclusions $J(\mathcal{E}_1) \otimes \mathcal{B}(\mathcal{E}_2) \subset J(\mathcal{E})$ and $\mathcal{B}(\mathcal{E}_1) \otimes J(\mathcal{E}_2) \subset J(\mathcal{E})$, it is easy to see that
\[(F - F^*)a, [F, a], (g(F) - F)a \in \mathcal{J}(\mathcal{E})\), for all \(a = a_1 \otimes a_2 \in A\) (recall Definition 2.33 for the meaning of \(\mathcal{J}(\mathcal{E})\)). We also have:

\[
(a_1 \otimes a_2)(F^2 - 1)(a_1 \otimes a_2)^*
= (a_1 \otimes a_2)(F^2 \otimes 1 + 1 \otimes F^2 - 1)(a_1 \otimes a_2)^*
\]

\[
= \begin{cases}
  a_1(F_1^2 - 1)a_1^* \otimes a_2a_2^* + a_1a_1^* \otimes a_2F_2^2a_2^* \ge 0, & \text{modulo } J_1 = (\mathcal{C}(\mathcal{E}_1) + \mathcal{J}(\mathcal{E}_1)) \otimes \mathcal{B}(\mathcal{E}_2), \\
  a_1F_2^2a_1^* \otimes a_2a_2^* + a_1a_1^* \otimes a_2(F_2^2 - 1)a_2^* \ge 0, & \text{modulo } J_2 = \mathcal{B}(\mathcal{E}_1) \otimes (\mathcal{C}(\mathcal{E}_2) + \mathcal{J}(\mathcal{E}_2)).
\end{cases}
\]

Apply Lemma 3.2, with \(J_1, J_2\) ideals in \(\mathcal{B}(\mathcal{E}_1) \otimes \mathcal{B}(\mathcal{E}_2)\), to see that \((a_1 \otimes a_2)(F^2 - 1)(a_1 \otimes a_2)^* \ge 0\), modulo \(J_1J_2 \subseteq \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})\). There is only one thing left: in order to obtain a right \((B_1 \otimes B_2)\)-module (and not an \((B_1 \otimes B_2)LL\)-module as \(\mathcal{E}\) is) we restrict \(\mathcal{E}\) and \(F\) to the graph of \(h(t) = t\). It is clear that \(F_b\) satisfies \((aKm1)-(aKm4)\). The class of \((Res)_*(\mathcal{E}, F)\) in \(KE_G(A_1 \otimes A_2, B_1 \otimes B_2)\) is called the external product of \((\mathcal{E}_1, F_1)\) and \((\mathcal{E}_2, F_2)\).

\text{Compare with Example [1.11].}

\text{Conclusion.} The external product of two asymptotic Kasparov G-modules \(\{(\mathcal{E}_1, t, F_1, t)\}_t\) and \(\{(\mathcal{E}_2, t, F_2, t)\}_t\) will be the asymptotic Kasparov G-module \(\{(\mathcal{E}_1 \otimes \mathcal{E}_2, t, F_1, t \otimes 1 + 1 \otimes F_2, t)\}_t\).

In the above example we used:

\textbf{Lemma 3.2.} Let \(J_1\) and \(J_2\) be closed ideals of the \(C^*\)-algebra \(A\). If \(a \ge 0 \mod J_1\), and \(a \ge 0 \mod J_2\), then \(a \ge 0 \mod J_1J_2 = J_1 \cap J_2\).

\text{Proof.} Given a \(C^*\)-subalgebra \(B\) and a closed ideal \(I\) of \(A\), then \((B + I)\) is a \(C^*\)-subalgebra of \(A\), and the map \((B + I)/I \to B/(B \cap I)\) is a *-isomorphism (see [Pders, 1.5.8]). Assume that \(a \notin J_1\), otherwise interchange the roles of \(J_1\) and \(J_2\) in the argument below (\(a \in J_1 \cap J_2\) being trivial). Consider \(B\) to be the \(C^*\)-subalgebra generated by \(J_2\) and \(a\), and \(I = J_1\). We obtain the *-isomorphism: \((B + J_1)/J_1 \to B/(B \cap J_1) = B/(J_1 \cap J_2), b + J_1 \mapsto b + J_1J_2\). It sends the positive element \(a + J_1\) to a positive element, namely \(a + J_1J_2\). \(\blacksquare\)

## 3.2 Two-dimensional connections

As in Kasparov’s \(KK\)-theory, the general product will involve tensor products of Hilbert modules. Given a Hilbert \(DL\)-module \(\mathcal{E}_1\) and a Hilbert \(BL\)-module \(\mathcal{E}_2\), their tensor product (internal or external) will be a continuous field of modules over \([1, \infty) \times [1, \infty)\) (to be precise, it will be a module over the algebra \(BL\) or \((D \otimes B)LL\)). We shall call such modules over \([1, \infty) \times [1, \infty)\), and corresponding families of operators, ‘two-dimensional.’ The ones indexed by \([1, \infty)\) are ‘one-dimensional.’ Our construction of the product will be based on an appropriate notion of connection, which is going to be a ‘two-dimensional’ operator. The original definition of connection, on which ours is modelled, appears in [CoSk, Def.A.1] and [Sk84, Def.8] (see Definition 1.7).

\textbf{Definition 3.3.} Assume that the following elements are given: a Hilbert \(DL\)-module \(\mathcal{E}_1\), a Hilbert \((D, BL)\)-module \(\mathcal{E}_2\), and \(F_2 \in \mathcal{B}(\mathcal{E}_2)\). Consider the Hilbert \(BLL\)-module \(\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2L\), with \(\mathcal{E}_2L = C_0(L) \otimes \mathcal{E}_2\). An operator \(F \in \mathcal{B}(\mathcal{E})\) is called an \(F_2\)-connection for \(\mathcal{E}_1\) if it has the same degree as \(F_2\) and if it satisfies, for every \(compactly\ supported\ \xi\ in\ \mathcal{E}_1\),

\[
(T_\xi \ (1 \otimes F_2) - (-1)^{\partial \xi} \partial F_2 F T_\xi) \in \mathcal{J}(\mathcal{E}_2L, \mathcal{E}),
\]
and 
\[
(1 \otimes F_2) T^*_\xi - (-1)^{\partial \xi \cdot F_2} T^*_\xi \in \mathcal{J}(\mathcal{E}, \mathcal{E}_2L).
\]

Here \(T_\xi \in \mathcal{B}(\mathcal{E}_2L, \mathcal{E})\) is defined by \(T_\xi (g \otimes \eta) = \xi \otimes_{DL} (g \otimes \eta)\), for \(g \in C_0(L)\), and \(\eta \in \mathcal{E}_2\). Moreover \(\mathcal{J}(\mathcal{E}_2L, \mathcal{E}) = \{T \in \mathcal{B}(\mathcal{E}_2L, \mathcal{E}) \mid \lim_{t_1, t_2 \to \infty} \|T(t_1, t_2)\| = 0\}\), and \(\mathcal{J}(\mathcal{E}, \mathcal{E}_2L)\) is defined similarly.

**Remark.** The above two conditions which a connection must satisfy are better remembered through the gradedly commutative modulo \(\mathcal{J}\) diagrams

\[
\begin{array}{ccc}
\mathcal{E}_2L & \xrightarrow{1 \otimes F_2} & \mathcal{E}_2L \\
\downarrow T_\xi & & \downarrow T_\xi \\
\mathcal{E} & \xrightarrow{E} & \mathcal{E}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\mathcal{E}_2L & \xrightarrow{1 \otimes F_2} & \mathcal{E}_2L \\
\uparrow T_\xi & & \uparrow T_\xi \\
\mathcal{E} & \xleftarrow{E} & \mathcal{E}
\end{array}
\]

**Proposition 3.4.** Consider the notation of the previous definition, with \(\varphi_2 : D \to \mathcal{B}(\mathcal{E}_2)\) denoting the left action of \(D\) on \(\mathcal{E}_2\). If \(F_2\) satisfies, for all \(d \in D\), \(\varphi_2(d) \in \mathcal{J}(\mathcal{E}_2)\), then an \(F_2\)-connection exists for any countably generated \(\mathcal{E}_1\).

**Proof.** According with the Stabilization Theorem [Kas80, Thm.2], there exists an element \(V \in \mathcal{B}(\mathcal{E}_1, \mathcal{H}(DL)^\sim)\) of degree 0 such that \(V^* V = 1\). (This follows from the isomorphism \(\mathcal{E}_1 \oplus \mathcal{H}(DL)^\sim \cong \mathcal{H}(DL)^\sim\) Assume first that the unit of \((DL)^\sim\) acts as identity on \(\mathcal{E}_2L\). There is then an obvious isomorphism \(W : \mathcal{H}(DL)^\sim \otimes (DL)^\sim \mathcal{E}_2L \to \mathcal{H} \otimes \mathcal{E}_2L\), given on elementary tensors by \(W((v \otimes f) \otimes (DL)^\sim \eta) = v \otimes f \eta\), for \(v \in \mathcal{H}, f \in (DL)^\sim, \eta \in \mathcal{E}_2L\). (In \(\mathcal{H} \otimes \mathcal{E}_2L\) the tensor product is an external one.) We obtain an \(F_2\)-connection \(E\) by imposing the commutativity of the diagram below:

\[
\begin{array}{ccc}
\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2L & \xrightarrow{F} & \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2L \\
V \otimes_{(DL)^\sim}^{-1} & & V^* \otimes_{(DL)^\sim}^{-1} \\
\mathcal{H}(DL)^\sim \otimes (DL)^\sim \mathcal{E}_2L & \xrightarrow{W} & \mathcal{H}(DL)^\sim \otimes (DL)^\sim \mathcal{E}_2L \\
W & & W^{-1} \\
\mathcal{H} \otimes \mathcal{E}_2L & \xleftarrow{1 \otimes (1 \otimes F_2)} & \mathcal{H} \otimes \mathcal{E}_2L
\end{array}
\]

i.e.

\[
\begin{align*}
E &= (V^* \otimes 1) W^{-1} (1 \otimes (1 \otimes F_2)) W (V \otimes 1) \quad \text{(19)}
\end{align*}
\]

We shall verify only one of the conditions for an \(F_2\)-connection (the other one being similar). Let \(\xi\) be a compactly supported homogeneous section of \(\mathcal{E}_1\), and \(V(\xi) = \sum_{i=1}^\infty e_i \otimes f_i\), where \(\{e_i\}_{i=1}^\infty\) is an orthonormal basis in \(\mathcal{H}\), and \(\sum_{i=1}^\infty f_i^* f_i < \infty\) in \(DL\). We have of course \(\partial \xi = \).
\( \partial e_i + \partial f_i \), and supp\((f_i) \subseteq \text{supp}(\xi) \). A direct computation gives for any \( \eta \in \mathcal{E}_2 L \):

\[
W (V \otimes 1) \left( T_\xi (1 \otimes F_2) - (-1)^\partial_x \partial F_2 \sum\frac{\partial \xi}{\partial F_2} \right) (\eta)
\]

\[
= W( V(\xi) \otimes (DL) - (1 \otimes F_2)(\eta)) - (-1)^{\partial_x \partial F_2} (1 \otimes (1 \otimes F_2)) W( V(\xi) \otimes (DL) - \eta)
\]

\[
= \sum_{i=1}^\infty e_i \otimes f_i (1 \otimes F_2)(\eta) - (-1)^{\partial x \partial F_2} \sum_{i=1}^\infty e_i \otimes (1 \otimes F_2)(f_i \eta)
\]

\[
= \sum_{i=1}^\infty e_i \otimes [f_i, 1 \otimes F_2](\eta).
\]

Consequently, it remains to show the convergence of the last infinite sum and that it belongs to \( \mathcal{J}(\mathcal{E}_2 L, \mathcal{E}) \). This is accomplished by proving the convergence in operator norm of the partial sums \( S_I = \sum_{i=1}^I e_i \otimes [f_i, 1 \otimes F_2] \), using the expression given after the second equal sign in the above computation. The desired result follows because the partial sums belong to \( \mathcal{J}(\mathcal{E}_2 L, \mathcal{H} \otimes \mathcal{E}_2 L) \). (The last observation uses the hypothesis on \( F_2 \) and on \( \xi \).)

Fix \( \varepsilon > 0 \). We have:

\[
\| (S_{I+k} - S_I)(\eta) \| = \| \sum_{i=I+1}^{I+k} e_i \otimes f_i (1 \otimes F_2)(\eta) - (-1)^{\partial x \partial F_2} \sum_{i=I+1}^{I+k} e_i \otimes (1 \otimes F_2)f_i(\eta) \|
\]

\[
\leq \| \sum_{i=I+1}^{I+k} e_i \otimes f_i (1 \otimes F_2)(\eta) \| + \| \sum_{i=I+1}^{I+k} e_i \otimes (1 \otimes F_2)f_i(\eta) \|.
\]

Now:

\[
\alpha^2 = \| \langle (1 \otimes F_2)(\eta), \sum_{i=I+1}^{I+k} f_i^* f_i \rangle \| \leq \sum_{i=I+1}^{I+k} \| f_i^* f_i \| \cdot \| F_2 \|^2 \cdot \| \eta \|^2.
\]

Choose \( I \) such that \( \| \sum_{i \in \Omega} f_i^* f_i \| \leq \varepsilon^2 / (4\| F_2 \|^2) \), for every finite set \( \Omega \) which does not intersect \( \{1, 2, \ldots, I \} \). Next:

\[
\beta^2 = \| \langle \eta, \sum_{i=I+1}^{I+k} f_i^* (1 \otimes F_2)^* f_i (\eta) \rangle \|
\]

\[
\leq \| F_2^* F_2 \| \cdot \| \sum_{i=I+1}^{I+k} f_i^* f_i \| \cdot \| \eta \|^2 = \| F_2 \|^2 \cdot \| \sum_{i=I+1}^{I+k} f_i^* f_i \| \cdot \| \eta \|^2.
\]

For the chosen \( I \), we obtain: \( \alpha + \beta \leq (\varepsilon / 2 + \varepsilon / 2) \| \eta \| \). Consequently, \( \| S_{I+k} - S_I \| \leq \varepsilon \), for all positive integers \( k \). This proves the norm convergence of the double sum, and the proposition in the case when the unit of \((DL)\) acts as identity on \( \mathcal{E}_2 L \).

In the general case, the equation \((19)\) needs replaced by:

\[
\mathcal{F} = (V^* \otimes 1) W^{-1} \left( (1 \otimes (1 \otimes F_2)) (DL) \right) W (V \otimes 1),
\]

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where \( W : \mathcal{H}_{DL} \otimes_{DL} \mathcal{E}_2 L \to \mathcal{H} \otimes (DL \cdot \mathcal{E}_2 L) \), and \((1 \otimes (1 \otimes F_2)|_{DL}) \in \mathcal{B}(\mathcal{H} \otimes (DL \cdot \mathcal{E}_2 L))\). We recall the definition of the restriction operator \((1 \otimes F_2)|_{DL}\) of \(1 \otimes F_2\) to the closed (but not necessarily complemented) subspace \(DL \cdot \mathcal{E}_2 L\):

\[
(1 \otimes F_2)|_{DL} = \sum_{n=1}^{\infty} (1 \otimes \varphi_2)(\delta_n^{1/2}) (1 \otimes F_2) (1 \otimes \varphi_2)(\delta_n^{1/2}),
\]

where \(\{u_n\}_{n=1}^{\infty}\) is an approximate unit for \(DL\), \(\delta_n = u_n - u_{n-1}\), \(n = 1, 2, \ldots\), and \(u_0 = 0\). The computations are now longer, but there is no new idea involved in the proof.

The next result gathers some useful properties of connections (compare with [Sk84, Prop.9]). The same notation as in Definition 3.3 is used.

**Proposition 3.5.** (i) Let \( F \) be an \( F_2 \)-connection for \( \mathcal{E}_1 \), and \( F' \) be an \( F_2' \)-connection for \( \mathcal{E}_1 \). Then \((F + F')\) is an \((F_2 + F_2')\)-connection for \( \mathcal{E}_1 \), and \((F \cdot F')\) is an \((F_2 F_2')\)-connection for \( \mathcal{E}_1 \).

(ii) The linear space of 0-connections for \( \mathcal{E}_1 \) is

\[
\{ F \in \mathcal{B} (\mathcal{E}) \mid (K \otimes_{DL} 1) F, F(K \otimes_{DL} 1) \in \mathcal{J}(\mathcal{E}), \text{ for all } K \in \mathcal{K} (\mathcal{E}_1) \}.
\]

**Proof.** Both (i) and (ii) follow immediately from the definition of connection.

**Lemma 3.6.** Consider the notation of Definition 3.3 and assume that a separable set \( K \subset \mathcal{C}(\mathcal{E}_1) \) is given. Then there exists a section \( h_{00} \) of \([1, \infty) \times [1, \infty)\) such that for any other section \( h \geq h_{00} \) the following holds:

\[
(\text{Res}_h)_* ( [k \otimes_{DL} 1, F] ) \in \mathcal{J} ( (\text{Res}_h)_* ( \mathcal{E} ) ), \text{ for all } k \in K.
\]

**Proof.** Choose a dense subset \( \{ k_n \}_{n=1}^{\infty} \) of \( K \). Assume that one is able to find for each \( k_n \) a section \( h_n \) such that \((\text{Res}_h)_* ( [k_n \otimes_{DL} 1, F] ) \in \mathcal{J} ( (\text{Res}_h)_* ( \mathcal{E} ) )\), for any \( h \geq h_n \). Apply the diagonalization process described in Lemma 2.3 to obtain a section \( h_{00} \) which makes the conclusion true for all \( k_n \)’s. A density argument shows that the result holds for all \( k \in K \).

Consequently it is enough to construct a section that works for a single element \( k \in K \). As in the proof of (27) in the Technical Theorem (Subsection 1.3), one uses a partition of unity for \( L \), an approximation of \( k \otimes_{DL} 1 \) by finite sums \( \sum_i T_{\xi_i} T_{\eta_i}^* \), with \( \xi_i, \eta_i \in \mathcal{E}_1 \), and the properties of connections that \( F \) satisfies.

### 3.3 Construction of the product

We are now ready to give the construction of the product (16) in the case when \( B_1 = A_2 = \mathbb{C} \). Before stating the main theorem we present an overview of the proof.

**Overview 3.7.** Consider two asymptotic Kasparov modules \((\mathcal{E}_1, F_1) \in k\mathcal{E}_G (A, D)\) and \((\mathcal{E}_2, F_2) \in k\mathcal{E}_G (D, B)\). Their product, which is an element in \( k\mathcal{E}_G (A, B)\), is obtained by performing the following sequence of steps.

**Step 1.** Find a self-adjoint \( u \in \mathcal{C}(\mathcal{E}_1)^{(0)} \) such that:

1. \([u, F_1] \in \mathcal{J}(\mathcal{E}_1)\),

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2. \([u, a] \in \mathcal{J}(\mathcal{E}_1)\), for all \(a \in A\),

3. \((1 - u^2)(a (F_1^2 - 1) a^*) \in \mathcal{J}(\mathcal{E}_1)\), for all \(a \in A\),

4. \((g(u) - u) \in \mathcal{J}(\mathcal{E}_1)\), for all \(g \in G\).

**Step 2.** Define \(\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L\). Find \(F = F^*\) an \(F_2\)-connection for \(\mathcal{E}_1\), and define \(F = F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) F\). (The self-adjointness of \(F\) is just a convenience.)

**Step 3.** Choose a section \(h_{00}\) of \([1, \infty) \times [1, \infty)\) such that the restrictions of the following operators to the graph of any other section \(h \geq h_{00}\) are in \(\mathcal{J}((\text{Res} h)_*, (E))\):

5. \([u \otimes_{DL} 1, F]\),

6. \([u F_1 \otimes_{DL} 1, F]\),

7. \([u a \otimes_{DL} 1, F]\), for all \(a \in A\).

**Step 4.** Find \(h_0 \geq h_{00}\) such that the restriction to the graph of any \(h \geq h_0\) of:

8. \((u \otimes_{DL} 1) (F^2 - 1) (u \otimes_{DL} 1)\) is positive modulo \(\mathbb{C}((\text{Res} h)_*, (E)) + \mathcal{J}((\text{Res} h)_*, (E))\),

9. \((u \otimes_{DL} 1) (g(F) - F)\) is in \(\mathcal{J}((\text{Res} h)_*, (E))\), for all \(g \in G\).

Once a triple \((u, F, h_0)\) satisfying (1)–(9) is constructed, the conclusion is that the restriction of \((\mathcal{E}, F)\) to the graph of any \(h \geq h_0\) gives an asymptotic Kasparov \(G\)-(\(A, B\))-module \((\mathcal{E}_h, F_h)\), that we call a product of \((\mathcal{E}_1, F_1)\) by \((\mathcal{E}_2, F_2)\):

\[
\mathcal{E}_h = (\text{Res} h)_* (\mathcal{E}) = (\text{Res} h)_* (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L),
\]

\[
F_h = (\text{Res} h)_* (F) = (\text{Res} h)_* (F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) F) = F_1 \otimes_{D,h} 1 + 1 \otimes_{D,h} F_2.
\]  

(20)

The notation \(F_1 \otimes_{D,h} 1 = (\text{Res} h)_* (F_1 \otimes_{DL} 1)\), and \(1 \otimes_{D,h} F_2 = (\text{Res} h)_* ((u \otimes_{DL} 1) F)\) is suggested by the form of the product in the external product case. Note that in terms of families \(\mathcal{E}(t, h)\) reads:

\[
(\mathcal{E}_h, F_h) = \left\{ (\mathcal{E}_{1,t} \otimes_D \mathcal{E}_{2,h(t)}, F_{1,t} \otimes_D 1 + (u_t \otimes_D 1) F_{(t,h(t))}) \right\}_{t \in [1, \infty)}.
\]  

(21)

**Remark 3.8.** We do not have an axiomatic definition of the product as in [Sk84, Def.10], [CoSk, Thm.A.3] (see Definition 1.9), so the situation is more like in \(E\)-theory.

The following theorem guarantees that Steps 1–4 of Overview 3.7 can be performed. Its proof will be given in Subsection 3.3.

**Theorem 3.9 (Technical Theorem).** Let \(G\) be a locally compact \(\sigma\)-compact Hausdorff group, and let \(A, B,\) and \(D\) be separable graded \(G\)-\(C^*\)-algebras. Consider two asymptotic Kasparov modules \((\mathcal{E}_1, F_1) \in \text{ker}_G(A, D)\) and \((\mathcal{E}_2, F_2) \in \text{ker}_G(D, B)\). There exists a triple \((u, F, h_0)\), with \(u\) a self-adjoint element of \(\mathcal{C}(0)(\mathcal{E}_1)\), \(F\) an \(F_2\)-connection for \(\mathcal{E}_1\), and \(h_0\) a section of \([1, \infty) \times [1, \infty)\), as in Overview 3.7, such that for any other section \(h \geq h_0\)

\[
(\mathcal{E}_h, F_h) = (\text{Res} h)_* (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) F)
\]

is an asymptotic Kasparov \(G\)-(\(A, B\))-module.
We can now give the definition of the product map in KE-theory in the form of:

**Theorem 3.10.** With the notation of the above theorem, the map \( ((\mathcal{E}_1, F_1), (\mathcal{E}_2, F_2)) \mapsto (\mathcal{E}_{h_0}, F_{h_0}) \) passes to quotients and defines the product map:

\[
KE_G(A, D) \otimes KE_G(D, B) \xrightarrow{s_D} KE_G(A, B), \quad (x, y) \mapsto x \ast_D y.
\]  

(22)

**Proof.** The notation is that of Overview 3.1. (I) Independence of \( h \). For any two \( h_1, h_2 \geq h_0 \) we have a homotopy between \((\mathcal{E}_{h_1}, F_{h_1}) \) and \((\mathcal{E}_{h_2}, F_{h_2}) \) given by the explicit formula:

\[
\{ (\text{Res}_{s(1-s)h_2})_*(\mathcal{E}), (\text{Res}_{s(1-s)h_2})_*(F) \} \}_{s \in [0,1]'.
\]

(II) Independence of the triple \((u, \mathcal{E}, h_0)\). (a) As above, one can construct a homotopy between two asymptotic Kasparov modules corresponding to different \( h_0 \)'s satisfying Step 4. This proves the independence of \( h_0 \). (b) In order to show independence of \( \mathcal{E} \), consider two \( F_2 \)-connections \( \mathcal{E} \) and \( \mathcal{E}' \) and the same \( u \). Now \( (\mathcal{E} - \mathcal{E}') \) is a 0-connection, and Proposition 3.3(ii) implies that there exists a section \( h \) such that \((\text{Res}_h)_*( (u \otimes_{DL} 1)_\mathcal{E} - (u \otimes_{DL} 1)_{\mathcal{E}'} ) \in \mathcal{G}(\mathcal{E}_h) \). Further modify \( h \) such that both \( F_h \) and \( F'_h \) give elements in \( k\mathcal{E}_G(A, B) \). Lemma 2.22 applies and gives a homotopy between \( F_h \) and \( F'_h \). (c) To show independence of \( u \), choose two different such elements \( u \) and \( u' \), both satisfying the requirements of Step 1, same \( \mathcal{E} \), and an \( h \) that works for both choices. We obtain a homotopy by the formula:

\[
\{ (\text{Res}_h)_* (F_1 \otimes_{DL} 1 + (s(u \otimes_{DL} 1) + (1-s)(u' \otimes_{DL} 1))^E) \} \}_{s \in [0,1]'.
\]

Combining (a), (b), and (c) above we get that the homotopy class of the element \((\mathcal{E}_h, F_h)\) constructed in Theorem 3.9 does not depend on the triple \((u, \mathcal{E}, h_0)\).

(III) Passage to quotients. Our goal is to show that the homotopy class of the product does not depend on the representatives in the class of \((\mathcal{E}_1, F_1)\) and \((\mathcal{E}_2, F_2)\), respectively. Consider \((\mathcal{E}_1, F_1) \in k\mathcal{E}_G(A, D[0,1])\) a homotopy between \((\mathcal{E}_{1,0}, F_{1,0}) \) and \((\mathcal{E}_{1,1}, F_{1,1}) \). A product \((\mathcal{E}, F)\) of \((\mathcal{E}_1, F_1) \) by \( \sigma_{C[0,1]}((\mathcal{E}_2, F_2)) \) represents a homotopy between the product of \((\mathcal{E}_{1,0}, F_{1,0}) \) by \((\mathcal{E}_2, F_2) \) and a product of \((\mathcal{E}_{1,1}, F_{1,1}) \) by \((\mathcal{E}_2, F_2) \). Consider now \((\mathcal{E}_2, F_2) \in k\mathcal{E}_G(D, B[0,1]) \). A product \((\mathcal{E}, F)\) of \((\mathcal{E}_1, F_1) \) by \((\mathcal{E}_2, F_2) \) represents a homotopy between the product of \((\mathcal{E}_{1,0}, F_{1,0}) \) by \((\mathcal{E}_{2,0}, F_{2,0}) \) and a product of \((\mathcal{E}_1, F_1) \) by \((\mathcal{E}_{2,1}, F_{2,1}) \). We obtain that the map from the statement does pass to a well-defined map at the level of KE-theory groups.

Using Theorem 3.10 and the map \( \sigma \), we are now in position to construct the general product \((\mathcal{E}_1, F_1) \) mentioned at the very beginning of this section (compare with the definition in KK-theory [Kas88, Def.2.12]).

**Definition 3.11.** Let \( G \) be a group, and let \( A_1, A_2, B_1, B_2, D \) be G-C*-algebras. The general product in KE-theory is the map

\[
KE_G(A_1, B_1 \otimes D) \otimes KE_G(D \otimes A_2, B_2) \rightarrow KE_G(A_1 \otimes A_2, B_1 \otimes B_2),
\]

(19)

defined by:

\[
x \ast_D y = \sigma_{A_2}(x) \ast_{B_1 \otimes D \otimes A_2} \sigma_{B_1}(y).
\]

(23)

The external product corresponds to \( D = \mathbb{C} \).
This subsection is concluded by showing that, in the case of external product, the asymptotic Kasparov module constructed in Example 3.1 is homotopic with the one given by the general product of Definition 3.1. This will show that Example 3.1 really represents the construction of a product, and not merely of some other asymptotic Kasparov module. Let \( x \in KEG(A_1, B_1) \) be represented by \((E_1, F_1)\), and \( y \in KEG(A_2, B_2) \) be represented by \((E_2, F_2)\). According with Definition 3.1, \( x \hat{\otimes} y = \sigma_{A_2}(x) \otimes_{B_1 \otimes A_2} \sigma_{B_1}(y) \). Now, \( \sigma_{A_2}(x) \) is represented by \((E_1 \otimes A_2, F_1 \otimes 1)\), and \( \sigma_{B_1}(y) \) is represented by \((B_1 \otimes E_2, 1 \otimes F_2)\). (Bear in mind the details related to the graded tensor product of Hilbert modules [Blck, 14.4.4].) To obtain a module that represents the product we follow the steps given in Overview 3.7. The element \( u \) of Step 1 can be chosen of the form \( \{ \tilde{u}_i \otimes \alpha_{h(t)} \}_{t} \), with \( \{ \tilde{u}_i \}_{t} \) a q.i.q.c.a.u. for \( \mathcal{K}(E_1) \), \{\( \alpha_t \}_{t} \) an a.u. for \( A_2 \), and \( h \) an arbitrary section. In Step 2 we identify \( \mathcal{E} \) with \( \mathcal{E}_1 \otimes A_2 \mathcal{E}_2 \), which is a Hilbert \((B_1L \otimes B_2L)\)-module, acted on the left by \( A_1 \otimes A_2 \). As two-dimensional connection we can take the constant field \( \{ 1 \otimes F_2, t \}_{(t, t_2) \in L} \). With the choices and identifications made so far, any section \( b_{00} \) will do in Step 3. In Step 4 choose a section \( h \) that makes the restriction to its graph an asymptotic Kasparov module:

\[
(\mathcal{E}_h, F_h) = \{ (\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,h(t)}, F_{1,t} \otimes 1 + \tilde{u}_t \otimes \alpha_{h(t)} F_{2,h(t)}) \}_{t} \in ke_G(A_1 \otimes A_2, B_1 \otimes B_2).
\]

Lemma 7.24 applies and gives a homotopy between \((\mathcal{E}_h, F_h)\) and

\[
(\mathcal{E}^\prime_h, F^\prime_h) = \{ (\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,h(t)}, F_{1,t} \otimes 1 + 1 \otimes F_{2,h(t)}) \}_{t} \in ke_G(A_1 \otimes A_2, B_1 \otimes B_2).
\]

Finally we notice that \( \{ (\mathcal{E}_{2,h(t)}, F_{2,h(t)}) \}_{t} \) is just another representative of \( y \), obtained by ‘stretching’ (Example 2.17) the initial representative \((\mathcal{E}_2, F_2) = \{ (\mathcal{E}_{2,t}, F_{2,t}) \}_{t} \). Consequently, using two homotopies, we succeeded to show that the product \( \hat{\otimes} \) of Definition 3.1 is what we called external tensor product in Example 3.1.

### 3.4 Properties of the product

We study in this subsection some of the properties of the product in \( KE \)-theory. They are very similar to those that the Kasparov product satisfies in \( KK \)-theory. For our first result compare with [Kas88, Thm.2.14].

**Theorem 3.12.** The product \( \hat{\otimes} \) satisfies the following functoriality properties:

(i) it is bilinear;

(ii) it is contravariant in \( A \), i.e. \( f^*(x) \hat{\otimes} y = f^*(x \hat{\otimes} y) \), for any \(*\)-homorphism \( f : A_1 \rightarrow A \), \( x \in KEG(A, D) \), and \( y \in KEG(D, B) \);

(iii) it is covariant in \( B \), i.e. \( g_*(x \hat{\otimes} y) = x \hat{\otimes} g_*(y) \), for any \(*\)-homorphism \( g : B \rightarrow B_1 \), \( x \in KEG(A, D) \), and \( y \in KEG(D, B) \);

(iv) it is functorial in \( D \), i.e. \( f_*(x) \hat{\otimes} y = x \hat{\otimes} f_*(y) \), for any \(*\)-homorphism \( f : D_1 \rightarrow D_2 \), \( x \in KEG(A, D_1) \), and \( y \in KEG(D_2, B) \);

(v) \( \sigma_{D_1}(x \hat{\otimes} y) = \sigma_{D_1}(x) \hat{\otimes}_{D \otimes D_1} \sigma_{D_1}(y) \), for \( x \in KEG(A, D) \) and \( y \in KEG(D, B) \).
Proof. (i) Let \( x = \{(E_1, F_1)\} \in KE_G(A, D) \), \( y_1 = \{(E_2, F_2)\} \), \( y_2 = \{(E'_2, F'_2)\} \in KE_G(D, B) \). Then: \( x \overset{D}{\otimes} y_1 = \{(\text{Res}_{h_1})_*((E_1 \otimes_{DL} E_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) F))\} \), \( x \overset{D}{\otimes} y_2 = \{(\text{Res}_{h_2})_*((E_1 \otimes_{DL} E'_2, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) F'))\} \). Let \( h = \sup\{h_1, h_2\} \). Using \( E_1 \otimes_{DL} (E_2 \oplus E'_2) \) \( L \simeq \) \( (E_1 \otimes_{DL} E_2 L) \oplus (E_1 \otimes_{DL} E'_2 L) \), the definition of connection shows that \( (\mathcal{F} \oplus \mathcal{F}') \) is an \( (F_2 + F'_2) \)-connection for \( E_1 \). It is clear that:

\[
x \overset{D}{\otimes} (y_1 + y_2) = \{(\text{Res}_h)_*((E_1 \otimes_{DL} (E_2 \oplus E'_2) L, F_1 \otimes_{DL} (1 \oplus 1) + (u \otimes_{DL} (1 \oplus 1)) (\mathcal{F} \oplus \mathcal{F}'))\} \]

The linearity in the first variable is simpler.

(ii,iii,iv) A proof using the definition of the product can be given as for (i) above, but these properties are also a direct consequence of the associativity of the product (Theorem 3.14) and of the following remark: \( f^*(x) = \{(f^*_y)_A x\} \) and \( \sigma_{D_1}(x \overset{D}{\otimes} y) \) is represented by the restriction of \( (E_1 \otimes_{DL} E_2 L) \otimes_{D_1} (F_1 \otimes_{DL} 1) \otimes 1 + (u \otimes_{DL} 1) (\mathcal{F} \otimes 1) \) to the graph of a section \( h \). Let \( E_1' = E_1 \otimes_{D_1} D_1, E_2' = E_2 \otimes_{D_1} D_1 \), \( D' = D \otimes D_1 \). The product \( \sigma_{D_1}(x \overset{D}{\otimes} y) \overset{D}{\otimes} \sigma_{D_1}(y) \) is represented by the restriction of \( (E_1' \otimes_{D'} E_2 L, F_1 \otimes 1 \otimes_{D'} 1 + (u \otimes_{D'} 1) (\mathcal{F} \otimes 1) \). Under the identification \( E_1' \otimes_{D'} E_2 L \simeq (E_1 \otimes_{DL} E_2 L) \otimes_{D_1} D_1 \), we can take \( \mathcal{F}' = \mathcal{F} \otimes 1 \). Given any quasi-invariant approximate unit \( \tilde{d} = \{d_t\} \) for \( D_1 \), we can choose \( \tilde{u} = u \otimes \tilde{d} \in C(0)(E_1 \otimes_{D_1} D_1) \). Finally, after considering a common section for both products, Lemma 2.24 applies and gives a homotopy between the two representatives.

Remark. In the proof of the next theorem the language of elementary calculus will be used again in order to ‘visualize’ the construction of a double product in \( KE \)-theory. A 3D-cartesian coordinate system is assumed, with \( LLL \) viewed as ‘octant’ in this system. The quotations marks required by such imprecise, but suggestive we hope, terminology will be dropped.

Definition 3.13. A 3D-section is a function \( h : L \rightarrow LL, t \mapsto (h_2(t), h_3(t)) \), with \( h_2 \) and \( h_3 \) ordinary sections.

Theorem 3.14 (Associativity of the product). Let \( A, B, D, \) and \( E \) be \( G-C^*-\)algebras. Then, for any \( x_1 \in KE_G(A, D) \), \( x_2 \in KE_G(D, E) \), and \( x_3 \in KE_G(E, B) \),

\[
(x_1 \overset{D}{\otimes} x_2) \overset{E}{\otimes} x_3 = x_1 \overset{D}{\otimes} (x_2 \overset{E}{\otimes} x_3).
\]

Proof. Assume that \( x_1, x_2, x_3 \) are represented by \( (E_1, F_1) \in ke_G(A, D) \), \( (E_2, F_2) \in ke_G(D, E) \), \( (E_3, F_3) \in ke_G(E, B) \), respectively. We shall use the notation: \( E_{12} = E_1 \otimes_{DL} E_2 L, E_{23} = E_2 \otimes_{EL} E_3 L, E = E_1 \otimes_{DL} E_2 L \otimes_{ELL} E_3 LL \), \( x_{123} = (x_1 \overset{D}{\otimes} x_2) \overset{E}{\otimes} x_3 \), \( x_{123} = x_1 \overset{D}{\otimes} (x_2 \overset{E}{\otimes} x_3) \). An inner product \( (\xi \otimes_{DL} \eta \otimes_{EL} \zeta) \in E \) is abbreviated as \( (\xi \otimes_{D} \eta \otimes_{E} \zeta) \), and similarly for operators on \( E \). In \( LLL \), the first copy of \( L \) and the first coordinate \( t_1 \) correspond to \( E_1 \), the second copy of \( L \) and the second coordinate \( t_2 \) correspond to \( E_2 \), and the third copy of \( L \) and the third coordinate \( t_3 \) correspond to \( E_3 \).

We first describe the product \( x_{123} \). As explained in the previous subsection, \( x_1 \overset{D}{\otimes} x_2 \) is constructed from a triple \( (u_1, F_{12}, h_{12}) \), with \( u_1 \in C(E_1) \), \( F_{12} \) an \( F_2 \) connection for \( E_1 \), and \( h_{12} \)
a section in the \((t_1, t_2)\)-plane. It is represented by \((E_{12, h_{12}}, F_{12, h_{12}}) = (\text{Res}\, h_{12})_\ast ((E_{12}, F_{12}))\), where

\[
F_{12} = F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)F_{12}.
\]

The product \(x_{12, 3}\) is constructed from a triple \((u_{12, h_{12}}, F_{12, 3}, h_3)\), with \(u_{12, h_{12}} \in \mathcal{C}(E_{12, h_{12}}),\) \(F_{12, 3}\) an \(F_3\)-connection for \(E_{12, h_{12}}\), and \(h_3\) a section in the ‘surface’ \(\Sigma_1 = \{ (t_1, t_2, t_3) \in LLL \mid t_2 = h_{12}(t_1) \}\). It is represented by the restriction to the graph of \(h_3\) of \((E_{12, h_{12}} \otimes_{EL} E_3L, F_{12, h_{12}} \otimes_{EL} 1 + (u_{12, h_{12}} \otimes_{EL} 1)F_{12, 3})\). There is a simpler way of describing a representative. Define the 3D-section \(h(t) = (h_{12}(t), h_3(t))\). Consider the three-dimensional objects \(E\) and

\[
F = F_1 \otimes_D 1 \otimes E 1 + (u_1 \otimes_D 1 \otimes E 1)(F_{12} \otimes_E 1) + (u_2 \otimes_E 1)F_E,
\]

with \(u_1, F_{12}\) as before, \(u_{12} \in \mathcal{C}(E_{12})\), and \(F\) a three-dimensional \(F_3\)-connection for \(E_{12}\). (Such a three-dimensional connection is a straightforward generalization of our definition for two-dimensional connection. See \((25)\) for one of the defining, commutative up to \(J\), diagrams.) The product is represented by the restriction of \((E, F)\) to the graph of \(h\).

Similarly, \(x_{2, h_{23}}x_3\) is constructed from a triple \((u_2, F_{23, h_{23}})\), with \(u_2 \in \mathcal{C}(E_2), F_{23}\) an \(F_3\)-connection for \(E_2\), and \(h_{23}\) a section in the \((t_2, t_3)\)-plane. It is represented by \((E_{23, h_{23}}, F_{23, h_{23}}) = (\text{Res}\, h_{23})_\ast ((E_2, F_{23}))\), where

\[
F_{23} = F_2 \otimes_{EL} 1 + (u_2 \otimes_{EL} 1)F_{23}.
\]

The product \(x_{1, 23}\) is constructed from a triple \((u_1, F_{1, 23}, h'_{23})\), with the same \(u_1\) as before, \(F_{1, 23}\) an \(F_{23, h_{23}}\)-connection for \(E_1\), and \(h'_{23}\) a section in the ‘surface’ \(\Sigma_2 = \{ (t_1, t_2, t_3) \in LLL \mid t_3 = h_{23}(t_2) \}\). Let \(h'\) be the 3D-section whose graph is given by the graph of \(h'_{23}\). We can describe a representative for \(x_{1, 23}\) by the restriction to the graph of \(h'\) of

\[
F' = F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)F_{1, 23},
\]

with \(F_{1, 23}\) an \(F_{23}\)-connection for \(E_1\). The properties of connections given in Proposition \(3.3\) imply that we can take \(F_{1, 23} = F_{12} \otimes_E 1 + U_2F'\), where \(U_2\) is an \((u_2 \otimes_{EL} 1)\)-connection for \(E_1\), and \(F'\) is an \(F_{23}\)-connection for \(E_1\). The best way to see this choice for \(F_{1, 23}\) is through the diagram below, which represents the first of the two diagrams \((28)\) for the connections under discussion (the other one being constructed in a similar way):

\[
\begin{array}{ccc}
(E_3L) & \xrightarrow{1 \otimes (1 \otimes F_{1})} & (E_3L) \\
| & \downarrow & | \\
\quad f_1 \otimes T_\eta & \ \\
| & \downarrow & | \\
(E_2 \otimes_{EL} E_3L) & \xrightarrow{1 \otimes F_{23}} & (E_2 \otimes_{EL} E_3L) \\
| & \downarrow & | \\
T_\zeta & \ \\
| & \downarrow & | \\
\quad \mathcal{E} & \xrightarrow{F'} & \quad \mathcal{E} \\
| & \downarrow & | \\
\quad \mathcal{E} & \xrightarrow{U_2} & \quad \mathcal{E}
\end{array}
\]

(In the diagram: \(f_1 \in C_0(L)\), \(\eta \in \mathcal{E}_2, \xi \in \mathcal{E}_1\). We also have made the identification: \(E_1 \otimes_{DL} (E_2L \otimes_{ELL} E_3LL) \simeq \mathcal{E} \simeq (E_1 \otimes_{DL} E_2L) \otimes_{ELL} E_3LL\).) The bottom squares of \((24)\) show that
\[ \frac{U_2 F'}{F_3} \text{ is indeed a } (u_2 \otimes_{E_1} 1)F_{23}\text{-connection for } \mathcal{E}_1. \] The left squares of (24) are nothing but an \( F_3\text{-connection for } \mathcal{E}_{12}: \]

\[
\begin{array}{ccc}
(\mathcal{E}_3)_{LL} & \xrightarrow{(1 \otimes 1) \otimes F_3} & (\mathcal{E}_3)_{LL} \\
T_{\xi \otimes DL(t_1 \otimes \eta)} & \downarrow & T_{\xi \otimes DL(t_1 \otimes \eta)} \\
\mathcal{E} & \xrightarrow{\mathcal{E}' = \mathcal{E}} & \mathcal{E}
\end{array}
\tag{25}
\]

The outcome of all the above is the following: \( x_{12,3} \) and \( x_{1,23} \) can be represented by the restriction of three dimensional pairs \( (\mathcal{E}, F) \) and \( (\mathcal{E}, F') \), where

\[
F = F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(F_{12} \otimes_E 1) + (u_2 \otimes_E 1)F, \]

\[
F' = F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(F_{12} \otimes_E 1) + (u_1 \otimes_D 1 \otimes_E 1)U_2 F.
\tag{26}
\]

to the graphs of appropriate sections \( h \) and \( h' \), respectively. We complete the proof by showing that \( h \) and \( h' \) can be chosen the same, and that \( F \) and \( F' \) are homotopic.

The proof of Technical Theorem given in Subsection 3.5 (see also the remark that follows that proof) shows that, while the section \( h_0 \) that appears in the triple \( (u, F, h_0) \) used to define the product of two \( KE\text{-modules} \) is an important element, the ‘right decay conditions’ actually hold true on a two dimensional object, namely over \( \cup_{n=0}^{\infty}[T_{1,n}, T_{1,n+1}] \times [T_{2,n}, \infty) \), or over \{ \( (t_1, t_2) \in LL \mid t_2 \geq h_0(t_1) \} \). (Notation as in the proof of Technical Theorem.) This implies that in the computation of a product the section is important only through the fact that it captures the behavior when both \( t_1 \to \infty \) and \( t_2 \to \infty \). This observation is summarized as:

**Lemma 3.15.** The products \((x_1 \overset{t_1}{\otimes}_D x_2) \overset{t_2}{\otimes}_E x_3\) and \( x_1 \overset{t_2}{\otimes}_D (x_2 \overset{t_3}{\otimes}_E x_3)\) can be computed by restricting the operators of (24) to a common 3D-section \( h \).

We need one more result:

**Lemma 3.16.** Define: \( J_0(\mathcal{E}) = \{ F \in \mathcal{B}(\mathcal{E}) \mid \lim_{t_1, t_2, t_3 \to \infty} \| F(t_1, t_2, t_3) \| = 0 \} \). (Here \( t_1, t_2, t_3 \to \infty \) means \( t_i \to \infty \) for \( i = 1, 2, 3 \).) Then \( [u_1 \otimes_D 1 \otimes_E 1, U_2] \in J_0(\mathcal{E}) \), and \( u_{12} \) can be chosen such that \([u_{12} \otimes_E 1, (u_1 \otimes_D 1 \otimes_E 1)U_2] \in J_0(\mathcal{E})\).

**Proof.** Modulo an element in \( J(\mathcal{E}_1) \otimes_D 1 \otimes_E 1 \subset J_0(\mathcal{E}) \), \((u_1 \otimes_D 1 \otimes_E 1)\) can be approximated on compact intervals in \( t_1\)-variable by finite sums \( \sum_i (T_{\xi_i, T_{\eta_i}^* \otimes E_1}) \), with \( \xi_i, \eta_i \in \mathcal{E}_1 \), compactly supported. (See the proof of Technical Theorem in Subsection 3.5.) This implies:

\[
[u_1 \otimes_D 1 \otimes_E 1, U_2] \\
\sim \sum_i ((T_{\xi_i} T_{\eta_i}^* \otimes E_1)U_2 - U_2(T_{\xi_i} T_{\eta_i}^* \otimes E_1)) \quad \text{modulo } J_0(\mathcal{E})
\]

\[
\sim (-1)^{\partial_{\eta_i}} \sum_i (T_{\xi_i} (1 \otimes (u_2 \otimes_{E_1} 1))T_{\eta_i}^* - T_{\xi_i} (1 \otimes (u_2 \otimes_{E_1} 1))T_{\eta_i}^*) \quad \text{modulo } J_0(\mathcal{E})
\]

\[
= 0.
\]

This proves the first inclusion. For the second one, use the same approximation for \((u_1 \otimes_D 1 \otimes_{E_1} 1)\) as above to see that, modulo \( J_0(\mathcal{E}) \), \([u_1 \otimes_D 1 \otimes_{E_1} 1, U_2] \) is an element of \( \mathcal{B}(\mathcal{E}_{12}) \otimes_{E_1} 1 \). The claimed asymptotic-commutativity follows by actually imposing it as an extra requirement for \( u_{12} \) (besides the conditions that appear in Step 1, Overview 3.7).
This last lemma implies that $a[F, F'] a^* \geq 0$, modulo $\mathcal{J}(\mathcal{E}_h)$, for any section $h$, and consequently Lemma 2.24 gives the required homotopy. We have showed that $x_{12,3} = x_{1,23}$ in $KE_G(A, B)$, and this completes the proof of Theorem 3.14.

\[\square\]

\textit{Remark.} There is another way to see the homotopy between the operators from (26). It uses the following result, whose proof is left to the reader:

\textit{Lemma 3.17.} $(u_1 \otimes_D 1 \otimes_E 1)U_2$ satisfies the (properly modified) conditions of Step 1, Overview 3.4, that $(u_{12} \otimes_E 1)$ satisfies.

Consequently, the straight line homotopy \{ $(1-s)(u_{12} \otimes_E 1) + s(u_1 \otimes_D 1 \otimes_E 1)U_2$ \}_{s \in [0,1]} can be used to give a homotopy between $F$ and $F'$.

Recall from Definition 2.27 that $1 = 1_C \in KE_G(\mathbb{C}, \mathbb{C})$ is the class of the identity homomorphism $\psi = \text{id} : \mathbb{C} \to \mathbb{C}$. For the next result compare with [Kas81, Thm.4.5], [Sk84, Prop.17].

\textbf{Proposition 3.18.} Let $A$ and $B$ be separable $G$-$C^*$-algebras, then

\[1_C \sharp_C x = x \sharp_C 1_C = x, \text{ for any } x \in KE_G(A,B).\]

\textit{Proof.} One equality is easy. We have: $x \sharp_C 1_C \overset{\text{def}}{=} x \sharp_B \sigma_B(1_C) = x \sharp_B 1_B$. Let $x$ be represented by $(\mathcal{E}, F)$ and $1_B$ be represented by $(BL, 0)$. As $0$-connection for $\mathcal{E}$ we can take the $0$ operator, and we can restrict to $h(t) = t$ in the construction of the product to obtain:

\begin{align*}
(\text{Res}_{h*})(\mathcal{E} \otimes_{BL} BLL) &\sim \mathcal{E}, \text{ via } (\text{Res}_{h*})(\xi \otimes_{BL} (f \otimes g \otimes b)) \mapsto (\xi \cdot (fg \otimes b)), \\
(\text{Res}_{h*})(F \otimes_{BL} 1) &\sim F \text{ (under the previous isomorphism)}.
\end{align*}

Consequently $x \sharp_B 1_B = x$.

For the other equality, we start with: $1_C \sharp_C x \overset{\text{def}}{=} \sigma_A(1_C) \sharp_A x = 1_A \sharp_A x$. Let $1_A$ be represented by $(AL, 0)$. Consider a quasi-invariant approximate unit $\{u_n\}_{n=1}^{\infty}$ for $A$, and construct an element $u = \{u_t\}_{t \in [1,\infty)} \in C^0(\mathcal{E})$ by interpolating the $u_n$'s: $u_t = (1 - \{t\})u_{[t]} + \{t\}u_{[t]+1}$, with $[t]$ denoting the greatest integer smaller that $t$, and $\{t\} = t - [t]$. We shall exhibit a homotopy between $(\mathcal{E}, F)$ and a representative of the product $1_A \sharp_A x$ constructed using $u$.

If $A$ is unital, consider the projection $\varphi(1) = P \in B(\mathcal{E})$. With the identification $AL \otimes_{AL} \mathcal{E}L \simeq (P\mathcal{E})L$, and after choosing $u \equiv 1$ and $h(t) = t$ in the definition of the product, we obtain as representative of $1_A \sharp_A x$ the asymptotic Kasparov module $(P\mathcal{E}, PF)$. There is an operator homotopy between $(\mathcal{E}, F)$ and $\mathcal{E}, PF = (P\mathcal{E}, PF) \oplus ((1-P)\mathcal{E}, 0)$, with the second summand being degenerate. This proves that $(\mathcal{E}, F)$ represents the product.

Assume now that A is not unital. Let $A^\sim$ be the unitization of $A$, with $1$ acting as identity on $\mathcal{E}$. Following [Sk84, Prop.17], let $A[0,1]$ be the $G$-$(A, A^\sim[0,1])$-module:

\[\hat{A}[0,1] = \{ f : [0,1] \to A^\sim \mid f(1) \in A \} \subseteq A^\sim[0,1].\]

Notice that $A$ acts as multiplication by constant functions. Let $\tilde{\mathcal{E}} = \hat{A}[0,1]L \otimes_{A^\sim[0,1]}(\mathcal{E}[0,1])L$, and let $\tilde{F}$ be an $(1 \otimes F)$-connection for $\hat{A}[0,1]L$. Consider $\tilde{u} = \{(1-s)1 + su\}_{s \in [0,1]} \in \tilde{F}$.
\( \mathcal{C}^{(0)}(\overline{A[0,1]}L) \). Finally, let \( \tilde{h} \) be any section of \( LL \) that makes \( (\tilde{\mathcal{E}}_k, \tilde{F}_k) = (\text{Res}_{\tilde{h}})_* \left( (\mathcal{E}, (u \otimes 1)\tilde{F}) \right) \) an asymptotic Kasparov \( G-(A,B[0,1]) \)-module. Then \( (\tilde{\mathcal{E}}_{k,0}, \tilde{F}_{k,0}) \) is homotopic (via a ‘stretching’) with \( (\mathcal{E}, F) \), and \( (\tilde{\mathcal{E}}_{k,1}, \tilde{F}_{k,1}) \) represents \( 1_A \sharp_A x \).

**Remark.** Theorem 3.14 and Proposition 3.18 imply that, for any \( G-C^* \)-algebra \( A \), \( KE_G(A,A) \) is a ring with unit.

The following notion is important in further studying the properties of \( KE \)-theory and in applications.

**Definition 3.19.** Let \( D_1 \) and \( D_2 \) be \( G-C^* \)-algebras. An element \( \alpha \in KE_G(D_1, D_2) \) is called \( KE \)-equivalence (or invertible) if there exists an element \( \beta \in KE_G(D_2, D_1) \) such that \( \alpha \sharp_{D_2} \beta = 1_{D_1} \) and \( \beta \sharp_{D_1} \alpha = 1_{D_2} \). If such an element \( \alpha \) exists then \( D_1 \) and \( D_2 \) are called \( KE \)-equivalent. (See [Kas88, 2.17], [Blck, 19.1].)

We use \( KE \)-equivalence to state a result that bears considerable theoretical significance:

**Theorem 3.20 (Stability in \( KE \)-theory).** For any \( G-C^* \)-algebra \( A \) and \( A \otimes \mathcal{K}(\mathcal{H}_G) \) are \( KE \)-equivalent.

The proof follows from the corresponding result in \( KK \)-theory, as explained in Corollary 3.4. Another proof can be given by rephrasing [Kas88, 2.18] in terms of \( KE \)-theory groups.

**Corollary 3.21.** For any separable \( G-C^* \)-algebras \( A \) and \( B \), we have

\[
KE_G(A, B) \simeq KE_G(A, B \otimes \mathcal{K}(\mathcal{H}_G)) \simeq KE_G(A \otimes \mathcal{K}(\mathcal{H}_G), B) \simeq KE_G(A \otimes \mathcal{K}(\mathcal{H}_G), B \otimes \mathcal{K}(\mathcal{H}_G)).
\]

We end this section by defining in \( KE \)-theory (as it is the case in \( KK \)-theory and \( E \)-theory) the higher order groups. We recall that \( \mathcal{C}_{+n} \) is the Clifford algebra of \( \mathbb{R}^n \), i.e., the universal algebra with odd generators \( \{e_1,...,e_n\} \) satisfying \( e_ie_j + e_je_i = +2\delta_{ij} \), for \( 1 \leq i,j \leq n \), \( e_i^2 = +e_i \), and \( \|e_i\| = 1 \). (The grading is the standard one, and the notation coincides with the one from [Kas75]. The adjoint and the norm refer to the fact that \( \mathcal{C}_{+n} \) can be given the structure of a \( C^* \)-algebra.)

**Definition 3.22.** \( KE^*_G(A,B) = KE_G(A,B \otimes \mathcal{C}_{+n}) \), for \( n = 1,2,... \).

### 3.5 The proof of the technical theorem

In this subsection the following is proved:

**Technical Theorem (Theorem 3.9).** Let \( G \) be a locally compact \( \sigma \)-compact Hausdorff group, and let \( A \), \( B \), and \( D \) be separable graded \( G-C^* \)-algebras. Consider two asymptotic Kasparov modules \((\mathcal{E}_1,F_1) \in ke_G(A,D)\) and \((\mathcal{E}_2,F_2) \in ke_G(D,B)\). There exists a triple \((u,\underline{\mathcal{E}}_2,h_0)\), with \( u \) a self-adjoint element of \( \mathcal{C}^{(0)}(\overline{\mathcal{E}}_1) \), \( \underline{\mathcal{E}}_2 \) an \( F_2 \)-connection for \( \mathcal{E}_1 \), and \( h_0 \) a section of \([1,\infty) \times [1,\infty)\), as in Overview 3.4, such that for any other section \( h \geq h_0 \)

\[
(\mathcal{E}_h,F_h) = (\text{Res}_{\tilde{h}})_*(\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) \underline{\mathcal{E}}_2)
\]

is an asymptotic Kasparov \( G-(A,B) \)-module.
Proof. We shall justify Steps 1-4 of the Overview [3.7]. Step 1, in which $u$ is constructed, is nothing but Lemma 2.29 applied to $(E_1, F_1)$. The existence of the connection $E = E^*$ in Step 2 follows from Proposition 3.4 (and after choosing $F_2 = F_2^*$). As it will become clear from the proof, the self-adjointness of $E$ is just a convenience. It enables us to reduce some of the computations to the unified requirements of Step 3. So far we succeeded to create the pair of ‘two-dimensional’ objects $(E, F) = (E_1 \otimes_{DL} E_2, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1) F)$. For Step 3, we obtain $h_{00}$ by applying Lemma 3.3 for the set $K = \{u, uF_1, ua_1, ua_2, \ldots, ua_n, \ldots\}$, where $\{a_n\}_{n=1}^\infty$ is a dense subset of $A$. The essential Step 4 is concerned with finding an appropriate section $h_0$ such that $(E_{h_0}, F_{h_0}) = (\text{Res}_{h_0})_* ((E, F))$ will be the asymptotic Kasparov $G$-$(A, B)$-module which represents the product. For this to happen, the axioms (aKm1)–(aKm4) must be satisfied. The tensor products that appear below are all inner (over $DL$), but the $C^*$-algebra will be omitted in order to simplify the writing.

- The simple computation: $(F - F^*)(a \otimes 1) = (F_1 \otimes 1 + (u \otimes 1) F - F_1^* \otimes 1 - F^* (u \otimes 1))(a \otimes 1) = (F_1 - F_1^*)a \otimes 1 + [u \otimes 1, F](a \otimes 1)$, shows that (aKm1) for $F_h$ is satisfied for any $h \geq h_{00}$, due to (aKm1) for $F_1$, and (5) of Step 3.

- Next, given $a \in A$, we have $[F_1 a \otimes 1] = F_1 a \otimes 1 + (u \otimes 1) F (a \otimes 1) - (-1)^{\partial a} a F_1 \otimes 1 - (-1)^{\partial a} (au \otimes 1) F = [F_1, a] \otimes 1 - (-1)^{\partial a} [ua \otimes 1, F] + (-1)^{\partial a} ([u, a] \otimes 1) F + [u \otimes 1, F](a \otimes 1)$. Consequently (aKm2) for $F_h$ is also satisfied for any $h \geq h_{00}$, because of (aKm2) for $F_1$, and (2), (5), and (7).

- For (aKm3), it is noted that:

$$a \left( F^2 - 1 \right) a^* = (a \otimes 1) \left( F_1^2 \otimes 1 + (u \otimes 1) F (F_1 \otimes 1) + (F_1 \otimes 1)(u \otimes 1) F - (u \otimes 1) F [F_1, u \otimes 1] \right) (a^* \otimes 1)$$

$$\sim (au) F_1^2 (au)^* \otimes 1 + (1 - u^2) (a (F_1^2 - 1) a^*) \otimes 1 - (a \otimes 1)(u \otimes 1) F [F, u \otimes 1] (a^* \otimes 1)$$

$$+ (a \otimes 1) \left( ([F_1, u] \otimes 1) F + [u F_1 \otimes 1, F] + [u \otimes 1, F] (F_1 \otimes 1) \right) (a^* \otimes 1)$$

$$+ (a \otimes 1) (u \otimes 1) (F^2 - 1)(u \otimes 1) (a \otimes 1)^* \text{, modulo } \mathcal{J}(E_1) \otimes_{DL} 1.$$

(For the second equality $\sim$ above, we used (1) and (2) of Step 1, and the self-adjointness of $u$.) The restriction of the first six terms to any $h \geq h_{00}$ will give a positive element modulo $\mathcal{J}(E_h)$, because of (3), (5) and (6). So we shall have (aKm3) satisfied provided that

$$(u \otimes 1) (F^2 - 1)(u \otimes 1) \text{ restricts to a positive element modulo } \mathcal{C}(E_h) + \mathcal{J}(E_h). \quad (27)$$

Showing (27) is a critical point in the construction. Let $\{I_n\}_{n=0}^\infty$ be a cover of $[1, \infty)$ by closed intervals of the form $I_n = [t_n, t_{n+2}]$, with $t_0 = 1$, and $\{t_n\}_n$ being a strictly increasing sequence with $\lim_{n \to \infty} t_n = \infty$. Let $T_{1,n} = t_n$, for $n \geq 0$, and $T_{2,0} = 1$. If $\{\mu_n\}_{n=0}^\infty$ is a partition of unity subordinated to this cover, then $u \otimes 1 = \sum_{n=0}^\infty (\mu_n u \otimes 1)$. For each $n \geq 1$, we can approximate $(\mu_n u \otimes 1)$ by a self-adjoint finite rank operator

$$K_n = \sum_{i=1}^{N_n} T_{\xi_i} T_{\eta_i}^* = \sum_{i=1}^{N_n} T_{\eta_i}^* T_{\xi_i}^*, \text{ with } \xi_i, \eta_i \in E_1|_{I_n}, \text{ for } i = 1, 2, \ldots, N_n, \quad (28)$$

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and such that \( \| (\mu_n u \otimes 1) - K_n \| < 1/(24n(\| F_2 \|^2 + 1)) \). Note that:

\[
K_n (F_2^2 - 1) K_n^* = \sum_{i,j=1}^{N_n} T_{\xi_i} T_{\eta_j}^* (F_2^2 - 1) T_{\eta_j} T_{\xi_i}^*
\sim \sum_{i,j=1}^{N_n} T_{\xi_i} T_{\eta_j}^* (1 \otimes F_2^2 - 1) T_{\xi_j}, \quad \text{modulo } \mathcal{J}(\mathcal{E})
\]

(29)

\[
= \sum_{i,j=1}^{N_n} T_{\xi_i} \langle \eta_i, \eta_j \rangle \left( (1 \otimes F_2^2 - 1) T_{\xi_j}^* \right), \quad \text{modulo } \mathcal{J}(\mathcal{E}).
\]

(30)

There exists \( \tau_{n,1} \) such that \( \| (F_2^2 T_{\eta_j} - T_{\eta_j} (1 \otimes F_2^2))_{(t_1,t_2)} \| < 1/(12nN_n^3) \), for all \( \eta_j \), all \( t_1 \in I_n \), and all \( t_2 > \tau_{n,1} \). This implies that the error of the commutation that was used for the second line of equation (29) is smaller than \( 1/(12n) \), in norm and when restricted to the graph of any section \( h \) whose values on \( I_n \) are bigger than \( \tau_{n,1} \). Using the characterization of positive operators on Hilbert modules [Lan, 4.1] that generalizes the familiar one from Hilbert space theory, we see that the matrix \( P = (\langle \eta_i, \eta_j \rangle) \in M_{N_n}(DL) \) is positive. Consequently \( P = QQ^* \), with \( Q = (d_{ij}) \), and we get:

\[
\sum_{i,j=1}^{N_n} T_{\xi_i} \langle \eta_i, \eta_j \rangle \left( (1 \otimes F_2^2 - 1) T_{\xi_j}^* \right) = \sum_{i,j=1}^{N_n} T_{\xi_i} \left( \sum_{k=1}^{N_n} d_{ik} d_{jk}^* \right) \left( (1 \otimes F_2^2 - 1) T_{\xi_j}^* \right)
\]

(30)

\[
\sim \sum_{k=1}^{N_n} \left( \sum_{i,j=1}^{N_n} T_{\xi_i} d_{ik} \right) \left( (1 \otimes F_2^2 - 1) \left( \sum_{j=1}^{N_n} T_{\xi_j} d_{jk} \right)^* \right), \quad \text{modulo } \mathcal{J}(\mathcal{E}).
\]

There exists \( \tau_{n,2} \) such that \( \| (d_{jk}, 1 \otimes F_2^2)_{(t_1,t_2)} \| < 1/(12nN_n^3) \), for all \( d_{jk} \), all \( t_1 \in I_n \), and all \( t_2 > \tau_{n,2} \). This implies that the error due to asymptotic commutativity \((aKm2)\) for \( F_2 \), used to obtain the second line of equation (30) is smaller than \( 1/(12n) \), in norm and when restricted to the graph of any section \( h \) whose values on \( I_n \) are bigger than \( \tau_{n,2} \). Let \( \{\delta_m\}_m \) be an approximate unit in \( D \). Because of \((aKm3)\) for \( F_2 \),

\[
\sum_{k=1}^{N_n} \left( \sum_{i=1}^{N_n} T_{\xi_i} d_{ik} \right) (1 \otimes \delta_m (F_2^2 - 1) \delta_m) \left( \sum_{j=1}^{N_n} T_{\xi_j} d_{jk} \right)^*
\]

(31)

is positive modulo \( \mathcal{C}(\mathcal{E}|_{I_n}) + \mathcal{J}(\mathcal{E}|_{I_n}) \). Choose \( m_0 \) such that the entire sum from (31) approximates the one from the second line of (30) by \( 1/(12n) \).

Let \( T_{2,n} = \max\{\tau_{n,1}, \tau_{n,2}, T_{2,(n-1)} + 1\} \). (To be precise, there is also an \( \tau_{n,3} \) coming from \((aKm4)\) to be taken into account, but we ignore it for the moment.) Once the sequence \( \{T_{2,n}\}_n \) has been constructed, we define \( h_0 \) on \([T_{1,n}, T_{1,(n+1)}]\) as the linear function satisfying \( h_0(T_{1,n}) = T_{2,n} \) and \( h_0(T_{1,(n+1)}) = T_{2,(n+1)} \). The estimates above show that the restriction to the graph of \( h_0|_{I_n} \) of \((\mu_n u \otimes 1) (F_2^2 - 1) (\mu_n u \otimes 1)^* \) is positive modulo \( \mathcal{C}(\mathcal{E}_{h_0}) \), with an error which is smaller than \( 1/(3n) \), in norm. At most three such terms are non-zero over \( I_n \), this proves (27) for any \( h \geq h_0 \), and consequently \( F_h \) satisfies \((aKm3)\).
• Finally, for any \( g \in G \), we have:

\[
(g(F) - F)(a \otimes 1) = (g(F_1 \otimes 1) + g(u \otimes 1) g(F) - (F_1 \otimes 1) - (u \otimes 1) F)(a \otimes 1)
\]

\[
= (g(F_1) - F_1) a \otimes 1 + ((g(u) - u) \otimes 1) g(F)(a \otimes 1)
\]

\[
+ (u \otimes 1)(g(F) - F)(a \otimes 1).
\]

Due to (aKm4) for \( F_1 \) and (4) of Step 1, the first two terms put no extra constraints on \( h_0 \). For the third one, \( u \otimes 1 \) can be approximated, as in the proof of (aKm3), on each interval \( I_n \), by a finite sum \( \sum \xi T_{\xi} T_{\eta}^* \). A simple computation shows that \( g T_{\eta}^* = T_{g(\eta)}^* \). Consequently:

\[
T_{\xi} T_{\eta}^* (g(F) - F) = T_{\xi} g(g^{-1}(T_{\eta}^*) F) - T_{\xi} T_{\eta}^* F
\]

\[
\sim (-1)^{\partial n} T_{\xi} g(F_2 T_{g^{-1}(\eta)}^*) - (-1)^{\partial n} T_{\xi} F_2 T_{\eta}^*, \ \text{modulo } \mathcal{J}(\mathcal{E})
\]

\[
= (-1)^{\partial n} T_{\xi} (g(F_2) - F_2) T_{\eta}^*.
\]

Further modification (increase) of \( h_0 \), using (aKm4) for \( F_2 \), will make the above errors go to zero when restricted to the graph of \( h_0 \). (This is the place where the \( \tau_{n,3} \) mentioned when we defined \( T_{2,n} \) makes its appearance.) This shows that (aKm4) holds for \( F_h \), for any \( h \geq (\text{new } h_0) \), and the proof of the Technical Theorem is complete.

**Remark.** The only important fact that \( h_0 \) encodes in the construction of the product is a certain behavior that occurs when \( t_1 \to \infty \) and \( t_2 \to \infty \), with \( h_0 \) correlating \( t_1 \) and \( t_2 \). We have noticed that certain decay properties hold true on entire ‘stripes’ \([T_{1,n}, T_{1,n+1}] \times [T_{2,n}, \infty)\), and not only on the graph of \( h_0 \). This observation is used in the proof of the associativity of the product (see Lemma 3.13), where it allows us to focus on the analysis of the operators that appear in the construction rather than on the sections.

## 4 KE-theory: some examples

We further investigate, by means of examples, the significance the axioms (aKm1)–(aKm4) that lie at the foundation of KE-theory. A consequence of our discussion is that non-equivariant KE-theory groups recover the ordinary K-theory for trivially graded \( C^* \)-algebras.

### 4.1 A non-equivariant example: K-theory

In this subsection the \( C^* \)-algebras are trivially graded (ungraded), and there is no group action. We consider that there is some merit in the proof of the next result:

**Proposition 4.1.** Let \( B \) be an ungraded separable \( C^* \)-algebra, then

\[
KE^*_*(\mathbb{C}, B) \simeq KK^*_*(\mathbb{C}, B) \simeq K_*(B), \ \text{for } * = 0, 1.
\]

**Proof.** The second group isomorphism, \( KK^*_*(\mathbb{C}, B) = K_*(B) \), is well-known (see for example [Blick, 17.5.6, 17.5.7]). Consequently the main point behind this proposition is the following: we shall show that the axioms of Kasparov modules (Definition 3.4 (i)) can be successively modified, in the case when \( A = \mathbb{C} \), to give the axioms (aKm1–3) of asymptotic Kasparov
For the inverse map, let $\psi$ and direct sums):

Consequently, in the homotopy class of the initial Kasparov module we find a representative $\tilde{\gamma}$ modules. To define the inverse map $\gamma$ in the homotopy class of the initial Kasparov module $\psi$, we need to construct the group homomorphism $\alpha$, such that $\alpha(1) = \text{id}$, $F = F^*$, and $(F^2 - 1/2) \geq 0$, modulo $\mathfrak{K}(\mathcal{E})$.

(See Remark 2.7.) To construct the group homomorphism $\alpha: KK(\mathbb{C}, B) \rightarrow \tilde{KK}(\mathbb{C}, B)$ we recall some of the standard simplifications of the axioms that a Kasparov module has to satisfy [Blick, 17.4]. Let $(\mathcal{E}, F) \in kk(\mathbb{C}, B)$ be an arbitrary Kasparov module. By replacing $F$ with $F' = (F + F^*)/2$ we find a homotopic module $(\mathcal{E}, F')$ with the operator self-adjoint.

Next, consider the projection $\varphi(1) = P \in \mathcal{B}(\mathcal{E})$. The pair $(\mathcal{E}, F')$ is operator homotopic to $(\mathcal{E}, PFP') = (P\mathcal{E}, PFP') + ((1 - P)\mathcal{E}, 0)$, with the second summand being degenerate. Consequently, in the homotopy class of the initial Kasparov module we find a representative $(\tilde{\mathcal{E}}, \tilde{F}) = (P\mathcal{E}, PFP')$, with $1 \in \mathbb{C}$ acting as identity, $\tilde{F}$ self-adjoint, and $\tilde{F}^2 = 1 \geq 1/2$, modulo $\mathfrak{K}(\tilde{\mathcal{E}})$. This defines the group homomorphism $\alpha$ (all the changes above preserve homotopies and direct sums):

$$\alpha : KK(\mathbb{C}, B) \rightarrow \tilde{KK}(\mathbb{C}, B), (\mathcal{E}, F) \mapsto (\tilde{\mathcal{E}}, \tilde{F}).$$

For the inverse map, let $\psi : \mathbb{R} \rightarrow \mathbb{R}$, be $\psi(x) = -1$, for $x \leq -1/\sqrt{2}$, $\psi(x) = \sqrt{2}x$, for $x \in (-1/\sqrt{2}, 1/\sqrt{2})$, and $\psi(x) = 1$, for $x \geq 1/\sqrt{2}$. Define

$$\alpha' : (\tilde{\mathcal{E}}, \tilde{F}) \mapsto (\tilde{\mathcal{E}}, \psi(\tilde{F})).$$

The only non-trivial checking is $\psi(\tilde{F})^2 - 1 = 2\tilde{F}^2 - 1 \geq 0$, modulo $\mathfrak{K}(\tilde{\mathcal{E}})$. We observe that $[\psi(\tilde{F}), \tilde{F}] \geq 0$, and consequently both compositions $\alpha' \circ \alpha$ and $\alpha \circ \alpha'$ give results homotopic with the initial module. It follows that $\alpha$ is an isomorphism, with $\alpha^{-1} = \alpha'$.

Define $\tilde{KE}(\mathbb{C}, B)$ to be the abelian group (under direct sum) of homotopy classes of asymptotic Kasparov $(\mathbb{C}, B)$-modules $(\tilde{\mathcal{E}}, \tilde{F})$ satisfying the extra conditions:

$$\varphi(1) = \text{id}, \tilde{F} = \tilde{F}^\circ, \text{ and } (\tilde{F}^2 - 1/2) \geq 0, \text{ modulo } \mathfrak{C}(\tilde{\mathcal{E}}).$$

The map $\gamma : \tilde{KE}(\mathbb{C}, B) \rightarrow KE(\mathbb{C}, B)$ is the forgetting map at the level of asymptotic Kasparov modules. To define the inverse $\gamma'$, let $(\tilde{\mathcal{E}}, \tilde{F})$ be an arbitrary asymptotic Kasparov module. We can make the action of $\mathbb{C}$ unital as in $KK$-theory: there is a homotopy followed by a ‘small perturbation’ connecting $(\tilde{\mathcal{E}}, \tilde{F})$ with $(\tilde{\mathcal{E}}', \tilde{F}'') = (P\tilde{\mathcal{E}}, PFP)$, where $P = \varphi(1)$. As we
have already observed in Corollary 2.23, there is a homotopy from this last pair to another one \((\hat{E}', \hat{F}')\), with \(\hat{F}'\) self-adjoint. Finally, (aKm3) implies that \((\hat{F}_t')^2 - 1 \geq U_t + V_t\), with
\(U = \{U_t\}_t \in \mathcal{C}(\hat{E})\) and \(V = \{V_t\}_t \in \mathcal{B}(\hat{E})\). Let \(T\) be such that \(\|V_t\| < 1/2\), for all \(t > T\). It follows that \((\hat{F}_t')^2 - 1/2 \geq U_t\), for \(t > T\). We define \(\gamma'\) via a ‘translation’ (see Example 2.17):
\[
\gamma' : \{ (\hat{E}_t, \hat{F}_t) \}_t \mapsto \{ (\hat{E}_{t+T}, \hat{F}_{t+T}) \}_t.
\]
All the operations used to define \(\gamma'\) preserve homotopies and direct sums, and consequently both \(\gamma' \circ \gamma\) and \(\gamma \circ \gamma'\) are identity, and \(\gamma^{-1} = \gamma'\).

Finally, define
\[
\beta : \hat{K}\hat{K}(\mathbb{C}, B) \to \hat{K}\hat{E}(\mathbb{C}, B), \ (\hat{E}, \hat{F}) \mapsto \{ (\hat{E}_t, \hat{F}_t) \}_t \text{ (constant family)}, \quad (36)
\]
and
\[
\beta' : \hat{K}\hat{E}(\mathbb{C}, B) \to \hat{K}\hat{K}(\mathbb{C}, B), \ (\hat{E}, \hat{F}) = \{ (\hat{E}_t, \hat{F}_t) \}_t \mapsto \{ (\hat{E}_1, \hat{F}_1) \} \text{ (the ‘fiber’ at } t = 1).\]
The composition \(\beta' \circ \beta = \text{id}\) is obvious. Let now \((\hat{E}, \hat{F}) = \{ (\hat{E}_t, \hat{F}_t) \}_t\) be an element of \(\hat{K}\hat{E}(\mathbb{C}, B)\). There exists a homotopy \((\mathcal{E}, \mathcal{F})\) between \((\hat{E}, \hat{F})\) and \((\beta \circ \beta'((\hat{E}, \hat{F})) = \{ (\hat{E}_1, \hat{F}_1) \}_t\) given by explicit formulas:
\[
\mathcal{E}_{t,s} = \hat{E}_{s+(1-s)t}, \quad \mathcal{F}_{t,s} = \hat{F}_{s+(1-s)t}, \quad \text{for } s \in [0, 1], t \in [1, \infty).
\]
This proves that \(\beta\) is also an isomorphism, with \(\beta^{-1} = \beta'\).

The claimed isomorphism is \(\gamma \circ \beta \circ \alpha : \hat{K}\hat{K}(\mathbb{C}, B) \to \hat{K}\hat{E}(\mathbb{C}, B)\). Finally we get:
\[
\hat{K}\hat{K}(\mathbb{C}, B) \overset{\text{def}}{=} KK(\mathbb{C}, B) \overset{\text{as above}}{=} KE(\mathbb{C}, B) \overset{\text{def}}{=} KE^{1}(\mathbb{C}, B). \quad \blacksquare
\]

4.2 An equivariant example: \(KE(\Gamma, \mathbb{C})\), for \(\Gamma\) discrete

The next result is similar with Remark 2, after [Kas88, 2.15], namely the dual of Green-Julg theorem in \(KK\)-theory:

**Proposition 4.2.** Let \(\Gamma\) be a discrete group and \(A\) a separable \(\Gamma\)-\(C^*\)-algebra, then \(KE(\Gamma, \mathbb{C}) = KE(C^*(\Gamma, A), \mathbb{C})\).

**Proof.** The presentation of crossed products contained in [Dvds, Ch.8] should be enough to follow the argument below. We start by choosing \((\mathcal{E}, F) \in ke_\Gamma(A, \mathbb{C})\). Using the Stability Theorem [3.20] and the Stabilization Theorem [Kas80, Thm.2], we can assume that \(\mathcal{E} = HL\), for a fixed Hilbert space \(H\) (see [78]). The field of Hilbert spaces \(\mathcal{E}\) is endowed with a unitary action \(U : \Gamma \to \mathcal{B}(\mathcal{E})\), and an equivariant *-representation \(\varphi : A \to \mathcal{B}(\mathcal{E})\). In terms of families \((\mathcal{E}, F)\) gives bounded and *-strong continuous families \(\{ U_t : \Gamma \to \mathcal{U}(H) \}_t\), and \(\{ \varphi_t : A \to \mathcal{B}(H) \}_t\).

We denote \(U_t(g)\) by \(g_t \in \mathcal{U}(H)\), for each \(t \in [1, \infty)\). The equivariance of each \(\varphi_t\) implies that we actually have a covariant representation of the dynamical system \((A, \Gamma)\). Consequently we can construct actions \(\tilde{\varphi}_t : C_c(\Gamma, A) \to \mathcal{B}(H)\) in the usual way:
\[
\tilde{\varphi}_t(f) = \sum_{g \in \Gamma} \varphi_t(a_g) g_t, \text{ for } f = \sum_{g \in \Gamma} a_g \delta_g \in C_c(\Gamma, A).
\]
Note that \( \{ \tilde{\varphi}_t(f) \}_{t} \) is bounded and \( * \)-strong continuous for each \( f \in C_c(\Gamma, A) \). By the norm density of \( C_c(\Gamma, A) \) in \( C^*(\Gamma, A) \) we obtain representations \( \tilde{\varphi} : C^*(\Gamma, A) \to \mathcal{B}(\mathcal{E}) \). We claim that with this representation \( \tilde{\varphi} : C^*(\Gamma, A) \to \mathcal{B}(\mathcal{E}) \) the asymptotic Kasparov module \((\mathcal{E}, F)\) gives an element \((\tilde{E}, \tilde{F}) = (\mathcal{E}, \tilde{\varphi}, F)\) in \( ke(C^*(\Gamma, A), \mathbb{C}) \). It is enough to check the axioms for \( f = a_g \delta_g \in C_c(\Gamma, A) \).

- \( \tilde{F} \) satisfies (aKm1). \((\tilde{F} - \tilde{F}^*)\tilde{\varphi}(f) = (F - F^*)\varphi(a_g)g \sim 0\), by (aKm1) for \( F \).
- \( \tilde{F} \) satisfies (aKm2). Indeed: \( [\tilde{F}, \tilde{\varphi}(f)] = F\varphi(a_g)g - (1)^{\partial a} \varphi(a_g)gF = [F, \varphi(a_g)]g + \varphi(a_g)(F - g(F))g \sim 0 \), by (aKm2) and (aKm4) for \( F \).
- \( \tilde{F} \) satisfies (aKm3).

\[
\tilde{\varphi}(f)(\tilde{F}^2 - 1)\tilde{\varphi}(f)^* = \varphi(a_g)g(F^2 - 1)g^{-1}\varphi(a_g)
\sim \varphi(a_g)(F^2 - 1)\varphi(a_g) \quad \text{by (aKm4) for } F
\geq 0, \text{ modulo } \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}), \quad \text{by (aKm3) for } F.
\]

The computation above shows also that a homotopy in \( ke_\Gamma(A, C([0, 1])) \) is sent to a homotopy in \( ke(C^*(\Gamma, A), C([0, 1])) \). We obtain in this way a group homomorphism \( KE_\Gamma(A, \mathbb{C}) \to KE(C^*(\Gamma, A), \mathbb{C}), [(\mathcal{E}, F)] \mapsto [(\tilde{E}, \tilde{F})] \). Now for the inverse group homomorphism, consider and asymptotic Kasparov module \((\tilde{E}, \tilde{F}) \in ke(C^*(\Gamma, A), \mathbb{C})\), where \( \tilde{E} = HL \), for a fixed Hilbert space \( H \), and \( \tilde{\varphi} : C^*(\Gamma, A) \to \mathcal{B}(\mathcal{E}) \). Let \( \{ a_n \} \) be an approximate unit for \( A \). We obtain by a standard construction bounded and \( * \)-strong continuous families of representations \((t \in [1, \infty)):\)

\[
\varphi_t : A \to \mathcal{B}(H), \varphi_t(a) = \tilde{\varphi}_t(a \delta_e), \text{ with } e \text{ the unit of } \Gamma,
\]
and

\[
U_t : \Gamma \to \mathcal{U}(H), U_t(g) = \lim_{n \to \infty} \tilde{\varphi}_t(a_n \delta_g), \text{ for } g \in \Gamma.
\]

Denote by \((\mathcal{E}, F)\) the pair \((\tilde{E}, \tilde{F})\) with the actions of \( \Gamma \) and \( A \) obtained in this way. The claim is that \( (\mathcal{E}, F) \) belongs to \( ke_\Gamma(A, \mathbb{C}) \), and the only non-trivial axiom to be checked is (aKm4).

We have:

\[
(g(F) - F)\varphi(a) = \lim_{n \to \infty} \left( \tilde{\varphi}(a_n \delta_g) \tilde{F} \tilde{\varphi}(a_n \delta_{g^{-1}}) - \tilde{F} \right) \tilde{\varphi}(a \delta_e)
= \lim_{n \to \infty} \left( \tilde{\varphi}(a_n \delta_g) [\tilde{F}, \tilde{\varphi}(a_n \delta_{g^{-1}})]
+ (-1)^{\partial a} \left( \tilde{\varphi}(a_n \delta_g) \tilde{\varphi}(a \delta_e) - \tilde{\varphi}(a \delta_e) \tilde{\varphi}(a_n \delta_g) \right) \right)
\sim 0, \text{ by (aKm2) for } \tilde{F}.
\]

Again homotopies in \( ke(C^*(\Gamma, A), C([0, 1])) \) are sent to homotopies in \( ke_\Gamma(A, C([0, 1])) \), and it is clear that we obtain in this way the inverse group homomorphism. ■

**Note.** Using Bott periodicity we also have: \( KE_\Gamma^4(A, \mathbb{C}) \cong KE_\Gamma(A \otimes C_{+1}, \mathbb{C}) \cong KE(C^*(\Gamma, A \otimes C_{+1}), \mathbb{C}) \cong KE(C^*(\Gamma, A) \otimes C_{+1}, \mathbb{C}) \cong KE_\Gamma^4(C^*(\Gamma, A), \mathbb{C}) \).
5 Final remarks: the relation with $KK$-theory and $E$-theory

Assume that a group $G$ (locally compact, $\sigma$-compact, Hausdorff) is given. In this final section we construct two functors: $\Theta : KK_G \rightarrow KE_G$, and $\Xi : KE_G \rightarrow E_G$. The three categories have all the same objects: the separable and graded $G$-$C^*$-algebras. The morphisms of $KK_G$ ([Hg87a, Hg90b, Thms98]) are the $KK$-theory groups, with composition given by the Kasparov product (see Theorem 4.10). The morphisms of $KE_G$ are the $KE$-theory groups, with composition given by the product defined in Section 3. The morphisms of $E_G$ ([GHT, HgKas97]) are the $E$-theory groups, with the corresponding composition product. Both functors are the identity on objects.

One consequence of the existence of these two functors is the construction of an explicit natural transformation, namely the composition $\Xi \circ \Theta$, between $KK$-theory and $E$-theory. This transformation preserves the product structures of the two theories. This connecting functor is roughly:

$$KK_G(A, B) \xrightarrow{\Theta} KE_G(A, B) \xrightarrow{\Xi} E_G(A, B)$$

$$(\mathcal{E}, F) \quad \mapsto \quad \{ (\mathcal{E}, (1 - u_t)F(1 - u_t)) \}_{t} \quad \mapsto \quad \{ f \otimes a \hat{\otimes} f((1 - u_t)F(1 - u_t))a \}_{t} \quad \text{(37)}$$

Here $(\mathcal{E}, F)$ is a Kasparov module, $\{u_t\}_t$ is a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E})$, and $\{\hat{\varphi}_t\}_t : C_0((-1, 1)) \otimes A \rightarrow B \otimes \mathcal{K}$ is an asymptotic family. The suggestive but somewhat imprecise (see subsection 5.2) formula of the composition in (37), namely $\Xi \circ \Theta : KK_G(A, B) \rightarrow E_G(A, B)$, $(\mathcal{E}, F) \mapsto \{ f \otimes a \hat{\otimes} f((1 - u_t)F(1 - u_t))a \}_{t}$, appears also in [Pop00, 4.5.1], in the context of groupoid actions.

5.1 The map $KK_G \rightarrow KE_G$

Let $G$ be a group, $A$ and $B$ be $G$-$C^*$-algebras. Consider $(\mathcal{E}, F) \in kk_G(A, B)$. This means that $\mathcal{E}$ is a graded Hilbert $G$-$B$-module, acted on by $A$, and $F \in \mathcal{B}(\mathcal{E})$ is an odd operator such that $(F - F^*)a, [F, a], (F^2 - 1)a, (gF - F)a$ belong to $\mathcal{K}(\mathcal{E})$, for all $a \in A, g \in G$. Denote by $C^*(\mathcal{K}(\mathcal{E}), A, F)$ the smallest $C^*$-subalgebra of $\mathcal{B}(\mathcal{E})$ that contains $\mathcal{K}(\mathcal{E}), \varphi(A)$, and $F$, and let $u = \{u_t\}_{t \in [1, \infty)}$ be a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}), A, F) \subset \mathcal{B}(\mathcal{E})$. It will be convenient, at least for notational purposes, to regard $u$ as an element of $\mathcal{C}(\mathcal{E}L)$. We make the notation: $\hat{\mathcal{E}} = \mathcal{E}L$ (constant family of modules), and $\hat{F} = \{(1 - u_t)F(1 - u_t)\}_t = (1 - u)F(1 - u)$.

Claim 5.1. $\{ (\mathcal{E}, (1 - u_t)F(1 - u_t)) \}_{t} = (\hat{\mathcal{E}}, \hat{F})$ is an asymptotic Kasparov $G$-$A, B$-module.

With this result at our disposal, the connection between the $KK$-theory and $KE$-theory groups is given by the following two results:

Theorem 5.2. Let $G$ be a group, $A$ and $B$ be $G$-$C^*$-algebras. Consider $(\mathcal{E}, F) \in kk_G(A, B)$, and let $u = \{u_t\}_{t \in [1, \infty)}$ be a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E}) \subset \mathcal{B}(\mathcal{E})$. The map

$$\Theta : kk_G(A, B) \rightarrow ke_G(A, B), \quad (\mathcal{E}, F) \mapsto \{ (\mathcal{E}, (1 - u_t)F(1 - u_t)) \}_{t \in [1, \infty)}; \quad \text{(38)}$$
passes to quotients and gives a group homomorphism $\Theta : KK_G(A,B) \rightarrow KE_G(A,B)$.

**Theorem 5.3.** $\Theta : KK_G \rightarrow KE_G$ is a functor.

One consequence is worth noticing:

**Corollary 5.4.** A $KK$-equivalence is sent by $\Theta$ into a $KE$-equivalence. In particular we obtain that $A$ and $A \otimes \mathcal{K}(H_G)$ are $KE$-theory equivalent, for any $G$-$C^*$-algebra $A$, and that the $KE$-theory groups satisfy Bott periodicity.

### 5.2 The map $KE_G \rightarrow E_G$

The $E$-theory groups were introduced and studied in [CoHg89, CoHg90], the equivariant ones under the action of a group in [GHT], and under the action of a groupoid in [Pop]. We use here the approach taken in [HgKas97, Sec.2]. Let $S$ be the $C^*$-algebra $C_0(\mathbb{R})$ graded by even and odd functions.

**Definition 5.5.** ([HgKas97, Def.2.2]) We denote by $E_G(A,B)$ the set of all homotopy equivalence classes of asymptotic families from $SA \otimes \mathcal{K}(H_G) = S \otimes A \otimes \mathcal{K}(H_G)$ to $B \otimes \mathcal{K}(H_G)$: $E_G(A,B) = \big\{ SA \otimes \mathcal{K}(H_G), B \otimes \mathcal{K}(H_G) \big\}$.

Our construction of the connecting map between $KE$-theory and $E$-theory is performed via a description of the $E$-theory groups which involves $C_0((-1,1))$ instead of $S$. Such a modification seems more appropriate when working with bounded operators. As with $S = C_0(\mathbb{R})$, the $C^*$-algebra $C_0((-1,1))$ will be graded by even and odd functions.

Let $G$ be a group, $A$ and $B$ be $G$-$C^*$-algebras. We consider first a particular case of asymptotic Kasparov $(A,B)$-modules: $(\mathcal{E}, F) = \{(\mathcal{E}_t, F_t)\}_{t \in \mathbb{R}} \in k e_G(A,B)$, where $\mathcal{E}_t$ is a fixed Hilbert $G$-$B$-module acted upon by $A$ through the $*$-homomorphism $\varphi : A \rightarrow \mathcal{B}(\mathcal{E}_t)$ (or through a family of $*$-homomorphisms $\varphi_t : A \rightarrow \mathcal{B}(\mathcal{E}_t)$, but the argument remains unchanged). This means that $F_t = F^*_t \in \mathcal{B}(\mathcal{E}_t)$ is an odd self-adjoint operator, for every $t$, such that $[F_t, a]$, $(g(F_t) - F_t)a$ converge in norm to 0 as $t \rightarrow \infty$, for all $a \in A$, $g \in G$, and that $a(F_t^2 - 1)a^* \geq 0$, modulo compacts, with an error that converges in norm to 0 as $t \rightarrow \infty$.

**Claim 5.6.** The family of maps

$$\phi_F = \{ \phi_{F,t} \}_{t \in [1, \infty)} : C_0((-1,1)) \otimes A \rightarrow \mathcal{K}(\mathcal{E}_t), \ f \otimes a \mapsto \phi_{F,t}(f) a,$$

for $f \in C_0((-1,1))$, $a \in A$, is an asymptotic family, in the sense of $E$-theory [GHT, Def.1.3].

The asymptotic family constructed above indicates that a ‘$C_0((-1,1))$-picture’ of $E$-theory is in order. The next lemma is the first step towards such a characterization.

**Lemma 5.7.** Let $A$ and $D$ be $G$-$C^*$-algebras, and consider an equivariant asymptotic family $\phi_F = \{ \phi_{F,t} \}_{t : C_0((-1,1)) \otimes A \rightarrow D}$. Then there exists a unique, up to homotopy, equivariant asymptotic family $\psi_F = \{ \psi_{F,t} \}_{t : SA \rightarrow D}$ such that the diagram

\[
\begin{array}{ccc}
C_0((-1,1)) \otimes A & \xrightarrow{\phi_F} & D \\
\text{inclusion} \downarrow & & \parallel \\
SA & \xrightarrow{\psi_F} & D
\end{array}
\]

produces a $K_0$-equivalence.
commutes up to homotopy.

To discuss the general case we mention the following possible simplification in the definition of asymptotic Kasparov modules:

**Proposition 5.8.** Given two $G$-$C^*$-algebras $A'$ and $B'$, let $A = A' \otimes \mathcal{K}(\mathcal{H}_G)$ and $B = B' \otimes \mathcal{K}(\mathcal{H}_G)$. Then, in the definition of $KE_G(A, B)$ it is enough to consider modules of the form $(\mathcal{H}_BL, F)$.

The proposition implies that the previous construction of the asymptotic morphism associated to an asymptotic Kasparov module with constant ‘fibers’ can be carried over the general case. Consider an arbitrary Kasparov module $(\mathcal{E}, F) \in ke_G(A, B)$. We can construct an asymptotic morphism $\phi : C_0((-1, 1)) \otimes A \to \mathcal{C}(\mathcal{E})/\mathcal{K}(\mathcal{E})$. This in turn gives an asymptotic morphism:

$$\phi \otimes 1 : C_0((-1, 1)) \otimes A \otimes \mathcal{K}(L^2(G)) \to \mathcal{C}(\mathcal{E} \otimes L^2(G))/\mathcal{K}(\mathcal{E} \otimes L^2(G)).$$

(40)

By ignoring the action of $G$, apply the Stabilization Theorem ([Kas80, Thm.2], with $G=\{e\}$) to get a non-equivariant isometry $V : \mathcal{E} \to \mathcal{H}_BL$. Apply next the Fell’s trick to construct an equivariant $BL$-linear isometry $W : \mathcal{E} \otimes L^2(G) \to \mathcal{H}_BL \otimes L^2(G)$. Use it, and the fact that now we have a constant field $\mathcal{H}_BL$ of modules, to transform the asymptotic morphism $\phi \otimes 1$ of (40) into an asymptotic family:

$$\phi_F : C_0((-1, 1)) \otimes A \otimes \mathcal{K}(L^2(G)) \longrightarrow \mathcal{K}(\mathcal{H}_B) \otimes \mathcal{K}(L^2(G)).$$

(41)

After tensoring with $\mathcal{K}$, we can use Lemma [7] to obtain an asymptotic morphism $\psi_F : SA \otimes \mathcal{K} \longrightarrow B \otimes \mathcal{K}$. The connection between $KE$-theory and $E$-theory is given by:

**Theorem 5.9.** For any group $G$, and any two $G$-$C^*$-algebras $A$ and $B$, the map $\Xi : (\mathcal{E}, F) \mapsto \psi_F$, from asymptotic Kasparov $G$-(A, B)-modules to asymptotic families from $SA \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$, passes to quotients and gives a natural group homomorphism

$$\Xi : KE_G(A, B) \to E_G(A, B), \ \Xi((\mathcal{E}, F)) = [\psi_F].$$

(42)

We next have:

**Theorem 5.10.** $\Xi : KE_G \longrightarrow E_G$ is a functor.

### 5.3 Puppe sequences and long exact sequences in $KE$-theory

The last subsection of the paper discusses some partial results that we have obtained related to the characterization of the excision in $KE$-theory. Further study is necessary, but at least we can mention two facts about the non-equivariant groups: the existence of Puppe sequences [Rsn82] and the split-exactness [CuSk].

**Theorem 5.11.** Let $A$, $B$, $D$ be graded separable $C^*$-algebras and $\varphi : A \to B$ a $*$-morphism, then we have exact sequences

$$KE(D, A(0, 1)) \xrightarrow{S_{\varphi}} KE(D, B(0, 1)) \xrightarrow{p} KE(D, C_{\varphi}) \xrightarrow{\varphi} KE(D, A) \xrightarrow{e} KE(D, B).$$

and

$$KE(A(0, 1), D) \xleftarrow{S_{\varphi}} KE(B(0, 1), D) \xleftarrow{\varphi} KE(C_{\varphi}, D) \xleftarrow{\varphi} KE(A, D) \xleftarrow{e} KE(B, D).$$
Proof. In the statement $C\varphi$ is the mapping cone of $\varphi$, $C\varphi = \{(a, f) \mid a \in A, f \in B[0, 1), \varphi(a) = f(0)\}$, $p : C\varphi \to A$ is the projection onto the first factor, and $i : B(0, 1) \to C\varphi$ is the inclusion $i(f) = (0, f)$. The justification of the theorem is a minor modification of [CuSk, Thm.1.1]. ■

**Theorem 5.12.** Consider a short exact sequence of separable graded $C^*$-algebras

\[
0 \longrightarrow J \overset{j}\longrightarrow A \overset{q}\longrightarrow B \longrightarrow 0,
\]

such that $q$ admits a completely positive (grading preserving and norm decreasing) cross-section. Then six-term exact sequences exist in $KE$-theory:

\[
\begin{array}{c}
KE(D, J) \xrightarrow{j^*} KE(D, A) \xrightarrow{q^*} KE(D, B) \\
\delta \uparrow \downarrow \delta
\end{array}
\]

and

\[
\begin{array}{c}
KE(J, D) \xleftarrow{j^*} KE(A, D) \xleftarrow{q^*} KE(B, D) \\
\delta \downarrow \delta \uparrow \delta
\end{array}
\]

Proof. Use Theorem 2.11 and Corollary 3.4 to get a $KE$-equivalence from the $KK$-equivalence [CuSk, Thm.2.1] given by $e : J \to C_q$, $e(x) = (j(x), 0)$. ■

Full details and proofs for the results contained in this section will form the substance of a second paper.

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