On Galois categories & perfectly reduced schemes

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Let $X$ be a scheme.\(^1\) A point $x = \text{Spec} \kappa(x) \to X$ of $X$ has an image Zariski point $x_0 \in X^\text{zar}$ with residue field $\kappa(x_0) \subseteq \kappa(x)$. Let us say that $x$ is a geometric point when $\kappa(x)$ is a separable closure of $\kappa(x_0)$. Geometric points constitute a category [SGA 4_1, Exposé VIII, §7], which we call the Galois category $\text{Gal}(X)$. The morphisms $x \to y$ are specialisations $x \rightsquigarrow y$ – i.e., natural transformations between the corresponding morphisms of topoi $x_* \to y_*$ (or, if you prefer, $x^* \to y^*$). In other words, $\text{Gal}(X)$ is the category of points of the étale topos of $X$. It is a 1-category in which every endomorphism is an automorphism. It comes equipped with a profinite topology; that is, $\text{Gal}(X)$ is a category object in Stone topological spaces.

The Galois category also comes equipped with a conservative functor $\text{Gal}(X) \to X^\text{zar}$, whose target is a poset under specialisation; this functor is continuous for the profinite topologies.\(^2\) Accordingly, $X^\text{zar}$ is the poset of isomorphism classes of objects of $\text{Gal}(X)$.

The profinite category $\text{Gal}(X)$ is determined by the étale topos of $X$, but it also determines it; in fact, if you’re a hyperpolyglot, you can probably deduce this already from Makkai’s Strong Conceptual Completeness Theorem [3]. We took [1] an explicit approach that showed that étale sheaves on $X$ ‘are’ continuous representations $\text{Gal}(X)$, generalising the usual equivalence between the étale cohomology and the Galois cohomology of a field.

If $X^\text{zar} \to P$ is a finite constructible stratification of a scheme, then the Galois $\infty$-category $\text{Gal}(X/P)$ is what you get by localising (in the wholesome $\infty$-categorical sense) the specialisations that occur within any single stratum. The result is a profinite $\infty$-category with a conservative functor to $P$ – what we called a profinite $P$-stratified space [1]. It is the $\infty$-category of points of the $P$-stratified $\infty$-topos. In the extreme case, when $P$ is the trivial poset, $\text{Gal}(X/\ast)$ is the profinite étale homotopy type. Hence $\text{Gal}(X)$ is a complete delocalisation of the étale homotopy type.

When you view $\text{Gal}(X)$ through this lens, you get to interpret it as a profinite stratified space whose underlying space is the profinite étale homotopy type $\text{Gal}(X/\ast)$. Each irreducible closed subscheme $Z \subseteq X$ identifies the closure $[Z]$ of a stratum within $X$. If $Z \subseteq W$ are two irreducible closed subschemes of $X$, then the space of sections of $\text{Gal}(X) \to X^\text{zar}$ over the edge $\eta_Z \to \eta_W$ of the generic points is the deleted tubular neighbourhood\(^3\) of $[Z]$ in $[W]$. This stratified space is a stratified 1-type: the strata and deleted tubular neighbourhoods are all $K(\pi, 1)$’s.

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1We only work with coherent schemes, which out of indolence we just call schemes.

2The topological space $X^\text{zar}$ is a spectral topological space, which is the same thing as a profinite poset.

3This is literally the étale homotopy type of the oriented fibre product $\eta_Z \times_X \eta_W$. 
Example (Fields). If $k$ is a field, then a choice of a separable closure of $k$ identifies an equivalence $\text{Gal} (\text{Spec } k) \cong B G_k$, where $G_k$ is the absolute Galois group of $k$.

Example (Knots and primes). If $A$ is a number ring with fraction field $K$, then $\text{Gal} (\text{Spec } A)$ is a category with (isomorphism classes of) objects the prime ideals of $A$. For each nonzero prime ideal $\mathfrak{p} \in \text{Spec } A$, the automorphisms of $\mathfrak{p}$ can be identified with the absolute Galois group $G_{k(p)}$ of the finite field $k(\mathfrak{p})$. Thus the étale homotopy type of $\text{Spec } A$ is stratified by the various closed strata, each of which is an embedded circle – i.e., a knot $BG_{k(p)}$. The open complement of each $BG_{k(p)}$ is a $BG_p$, where $G_p := \pi_1 (\text{Spec } A \setminus p)$ is the automorphism group of the maximal Galois extension of $K$ that is ramified at most only at $p$ and the infinite primes. Enveloping each knot is a tubular neighbourhood, given by $\text{Gal} (\text{Spec } A_{\text{sh}})$ (sh=strict henselisation), so that the deleted tubular neighbourhood of $BG_{k(p)}$ is a $BG_{K_p}$.

Example (Analytification). If $X$ is a finite type $F$-scheme, where $F$ is $C$, $R$, or any nonarchimedean field, then there is an associated $X^{\text{an}}$ analytic space, which admits a profinite stratification by $X^{\text{an}}$. The category $\text{Gal} (X)$ is the profinite completion of the exit-path $\infty$-category of $X^{\text{an}}$ with this stratification. (We proved this over $C$ [1, Proposition 13.15 & Corollary 13.16], but the same proof will work any time you have access to an Artin Comparison Theorem, which you do in these situations; see [2].)

The perfectly reduced schemes of the title are schemes taken up to universal homeomorphism (Definition 4.2). Grothendieck’s invariance topologique of the étale topos [SGA 4, Exposé VIII, 1.1] ensures that the only kinds of schemes Galois categories can hope to capture in their entirety are the perfectly reduced schemes. This note is the suggestion of a recognition principle that flows in the opposite direction; that is, we aim to read off facts about perfectly reduced schemes from their Galois categories. Our goal is a dictionary between the geometric features of a perfectly reduced scheme (or morphism of such) and the categorical properties of its Galois category (or functor of such); the gnomic section titles are the first few entries in this dictionary.

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1 Open = cosieve & closed = sieve

Let us begin with the obvious.

1.1 Proposition. A monomorphism $U \hookrightarrow X$ of schemes is an open immersion if and only if the induced functor $\text{Gal}(U) \to \text{Gal}(X)$ is equivalent to the inclusion of a cosieve.

Dually, a monomorphism $Z \hookrightarrow X$ of schemes is a closed immersion if and only if $\text{Gal}(Z) \to \text{Gal}(X)$ is equivalent to the inclusion of a sieve.

An interval in an \(\infty\)-category \(C\) is a full subcategory \(D \subseteq C\) such that a morphism \(P \to Q\) of \(D\) factors through an object \(R\) of \(C\) only if \(R\) lies in \(D\).

1.2 Corollary. A monomorphism $W \hookrightarrow X$ of schemes is a locally closed immersion if and only if the induced functor $\text{Gal}(W) \to \text{Gal}(X)$ is equivalent to the inclusion of an interval.

1.3 Corollary. A scheme $X$ is local if and only if $\text{Gal}(X)$ contains a weakly initial object – i.e., an object from which every object receives a morphism. Dually, a scheme $X$ is irreducible if and only if $\text{Gal}(X)$ contains a weakly terminal object – i.e., an object to which every object sends a morphism.

1.4. For any scheme $X$ and any point $x_0 \in X^{\text{zar}}$, the Galois category of the localisation is the fibre product

$$\text{Gal}(X_{(x_0)}) \simeq \text{Gal}(X) \times^{\text{zar}} X_{x_0}^{\text{zar}}.$$

Dually, for any point $y_0 \in X^{\text{zar}}$, the Galois category of the closure $X^{(y_0)}$ of $y_0$ (with the reduced subscheme structure, say) is the fibre product

$$\text{Gal}(X^{(y_0)}) \simeq \text{Gal}(X) \times^{\text{zar}} X_{y_0}^{\text{zar}}.$$

2 Strict localisation = undercategory & strict normalisation = overcategory

2.1 Notation. If $x \to X$ is a point of a scheme $X$, then we write $O^h_{X,x_0}$ for the henselisation of the local ring $O_{X,x_0}$, and we write $O^h_{X,x} \supseteq O^h_{X,x_0}$ for the unique extension of henselian local rings that on residue fields reduces to the field extension $\kappa \supseteq \kappa(x_0)$, where $\kappa$ is the separable closure of $\kappa(x_0)$ in $\kappa(x)$. We will also write

$$X_{(x)} \coloneqq \text{Spec } O^h_{X,x}.$$

We call $X_{(x)}$ the localisation of $X$ at $x$. It is the limit of the factorisations $x \to U \to X$ in which $U \to X$ is étale.

If $x \to X$ is a geometric point, then $O^h_{X,x}$ is the strict henselisation of $O_{X,x_0}$, and $X_{(x)}$ is the strict localisation of $X$ at $x$.

Dually, if $y \to X$ is a point, then we write $X^{(y)}$ for the reduced subscheme structure on the Zariski closure of $y_0$, and we write $X^{(y)}$ for the normalisation of $X^{(y)}$ under $\text{Spec } \kappa$, where $\kappa$ is the separable closure of $\kappa(y_0)$ in $\kappa(y)$. We call $X^{(y)}$ the normalisation of $X$ at $y$.

If $y \to X$ is a geometric point, then we call $X^{(y)}$ the strict normalisation of $X$ at $y$. It is the limit of the factorisations $y \to Z \to X$ in which $Z \to X$ is finite.
2.2. Stefan Schröer [5] has brought us \textit{totally separably closed} schemes, which are integral normal schemes whose function field is separably closed. In other words, a totally separably closed scheme is one of the form $X^{(y)}$ for some geometric point $y \to X$. (In the language of Schröer, $X^{(y)}$ is the total separable closure of the Zariski closure of $y_0$ -- with the reduced subscheme structure -- under $y$.) Schröer has shown that this class of schemes has a number of curious properties:

- If $Z$ is totally separably closed, then for any point $z_0 \in Z^{\text{zar}}$, the local ring $O_{Z, z_0}$ is strictly henselian [5, Proposition 2.6].

- If $Z$ is totally separably closed, then the étale topos and the Zariski topos of $Z$ coincide, so that $\text{Gal}(Z) \cong Z^{\text{zar}}$ [5, Corollary 2.5]. In other words, $\text{Gal}(Z)$ is a profinite poset with a terminal object.

- If $Z$ is totally separably closed and $W$ is irreducible, then any integral morphism $W \to Z$ is radicial [5, Lemma 2.3]. Thus any integral surjection $W \to Z$ is a universal homeomorphism.

- If $Z$ is totally separably closed, then the poset $\text{Gal}(Z) \cong Z^{\text{zar}}$ has all finite nonempty joins [6, Theorem 2.1].

Here now is the basic observation, which follows more or less immediately from the limit descriptions of the strict localisation and the strict normalisation:

\textbf{2.3 Proposition.} Let $X$ be a scheme, and let $x \to X$ and $y \to X$ be two geometric points thereof. The following profinite sets are in (canonical) bijection:

- the set $\text{Map}_{\text{Gal}(X)}(x, y)$ of morphisms $x \to y$ in $\text{Gal}(X)$;
- the set $\text{Mor}_X(y, X_{(x)})$ of lifts of $y$ to the strict localisation $X_{(x)}$;
- the set $\text{Mor}_X(x, X^{(y)})$ of lifts of $y$ to the strict normalisation $X^{(y)}$.

We may thus describe the over- and undercategories of Galois categories:

\textbf{2.4 Corollary.} Let $X$ be a scheme, and let $x \to X$ and $y \to X$ be two geometric points thereof. Then we have

$$\text{Gal}(X)_{x/y} \cong \text{Gal}(X_{(x)}) \quad \text{and} \quad \text{Gal}(X)_{/y} \cong \text{Gal}(X^{(y)}).$$

See also [SGA 4\textsubscript{\textit{ii}}, Exposé VIII, Corollaire 7.6], where the first sentence is proved.

\textbf{2.5 Corollary.} Let $X$ be a scheme. Then $\text{Gal}(X)$ is equivalent to both of the following full subcategories of $X$-schemes:

- the one spanned by the strict localisations of $X$, and
- the one spanned by the strict normalisations of $X$.

Since $\text{Gal}(X^{(y)}) \cong X^{(y)\text{zar}}$, it follows that Galois categories are of a very particular sort:

\textbf{2.6 Corollary.} Let $X$ be a scheme. For any geometric point $y \to X$, the overcategory $\text{Gal}(X)_{/y}$ is a profinite poset with all finite nonempty joins. In particular, every morphism of $\text{Gal}(X)$ is a monomorphism.

\textbf{2.7 Definition.} Let $X$ be a scheme. Then a \textit{witness} is a totally separably closed valuation ring $V$ and a morphism $\gamma : \text{Spec } V \to X$. If $p_0$ is the initial object of $\text{Gal}(V)$ and $p_{\infty}$ is the terminal object of $\text{Gal}(V)$, then we say that $\gamma$ \textit{witnesses} the map $\gamma(p_0) \to \gamma(p_{\infty})$ of $\text{Gal}(X)$.

\textbf{2.8.} Any morphism $x \to y$ of $\text{Gal}(X)$ has a witness: you can always find a local morphism $\text{Spec } V \to (X^{(y)})_{(x)}$ that induces an isomorphism of function fields.
3 Universal homeomorphism = equivalence

Now we arrive at a sensitive question: under which circumstances does a morphism of schemes induce an equivalence of étale topoi or, equivalently, of Galois categories? The well-known theorem here is Grothendieck’s invariance topologique of the étale topos [SGA 4_{II}, Exposé VIII, 1.1], which states that a universal homeomorphism induces an equivalence on étale topoi. Let us reprove this result with the aid of Galois categories; this will also provide us with a partial converse.

3.1 Proposition. Let \( f : X \to Y \) be a morphism of schemes. If \( f \) is radicial, then every fibre of \( \text{Gal}(X) \to \text{Gal}(Y) \) is either empty or a singleton.\(^4\) Conversely, if \( f \) is of finite type, and if every fibre of \( \text{Gal}(X) \to \text{Gal}(Y) \) is either empty or a singleton, then \( f \) is radicial.

Proof. If \( f \) is radicial, then the map \( X^{\text{zar}} \to Y^{\text{zar}} \) is an injection, and for any point \( x_0 \in X^{\text{zar}} \), the map \( B_{\kappa(x_0)} \to B_{\kappa(f(x_0))} \) on fibres is an equivalence since \( \kappa(f(x_0)) \subseteq \kappa(x_0) \) is purely inseparable. So for any geometric point \( y \) with image \( y_0 \), the fibre over \( y \) is a singleton.

Conversely, if \( f \) is of finite type, and if every fibre of \( \text{Gal}(X) \to \text{Gal}(Y) \) is either empty or a singleton, then certainly the map \( X^{\text{zar}} \to Y^{\text{zar}} \) is an injection, whence \( f \) is in particular quasifinite. For any point \( x_0 \in X^{\text{zar}} \), the fibres of the map \( B_{\kappa(x_0)} \to B_{\kappa(f(x_0))} \) are each a singleton, whence it is an equivalence. Now since \( \kappa(f(x_0)) \subseteq \kappa(x_0) \) is a finite extension, it is purely inseparable.

3.2 Example. The finite type hypothesis in the second half of Proposition 3.1 is of course necessary, as any nontrivial extension \( E \subset F \) of separably closed fields induces the identity on trivial Galois categories.

3.3 Corollary. Let \( f : X \to Y \) be a morphism of schemes. If \( f \) is radicial and surjective, then every fibre of \( \text{Gal}(X) \to \text{Gal}(Y) \) is a singleton. Conversely, if \( f \) is of finite type, and if every fibre of \( \text{Gal}(X) \to \text{Gal}(Y) \) is a singleton, then \( f \) is radicial and surjective.

The following is the Valuative Criterion, along with a simple argument [STK, Tag 03K8] that allows one to extend the fraction field of the valuation ring therein.

3.4 Lemma. Let \( f : X \to Y \) be a morphism of schemes. Then the following are equivalent.

- The morphism \( f \) is universally closed.
- For any witness \( y : \text{Spec} V \to Y \) and any diagram

\[
\begin{array}{ccc}
\text{Spec } K & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\text{Spec } V & \longrightarrow & Y \\
\end{array}
\]

in which \( K \) is the fraction field of \( V \), there exists a lift \( \overline{\gamma} : \text{Spec } V \to X \).

3.5 Recollection. A functor \( f : C \to D \) is said to be a right fibration if and only if, for any object \( x \in C \), the induced functor \( C/x \to D/f(x) \) is an equivalence of categories. In this case, one may say that \( C \) is a category fibred in groupoids over \( D \). For any such
right fibration, there is a diagram $F$ of groupoids indexed on $D^\otimes$ such that $C$ is the Grothendieck construction of $F$.

Dually, $f$ is a left fibration if and only if $f^\otimes$ is a right fibration, so that for any object $x \in C$, the induced functor $C_x \to D_{f(x)}$ is an equivalence of categories.

3.6 Proposition. Let $f : X \to Y$ be a morphism of schemes. If $f$ is an integral morphism, then $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration. Conversely, if $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration, then $f$ is universally closed.

Proof. Assume that $f$ is integral. Then for every geometric point $x \to X$, the induced morphism $X^{(x)} \to Y^{(f(x))}$ is also integral, and by [5, Lemma 2.3], it is radicial as well. Hence at the level of Zariski topological spaces, $X^{(x),\text{zar}} \to Y^{(f(x),\text{zar}}$ is an inclusion of a closed subset; since source and target are each irreducible, and the inclusion carries the generic point to the generic point, it is a homeomorphism. (In fact, $X^{(x)} \to Y^{(f(x))}$ is a universal homeomorphism.) Thus

$$\text{Gal}(X)_{/x} \simeq \text{Gal}(X^{(x)}) \simeq X^{(x),\text{zar}} \to Y^{(f(x),\text{zar}} \simeq \text{Gal}(Y^{(f(x))}) \simeq \text{Gal}(Y)_{/f(x)}$$

is an equivalence, whence $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration.

Conversely, assume that $f$ is of finite type and that $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration. We employ Lemma 3.4 to show that $f$ is universally closed; consider a witness $\gamma : \text{Spec } V \to Y$ along with a diagram

$$\begin{array}{ccc}
\text{Spec } K & \xrightarrow{\xi} & X \\
\downarrow & & \downarrow f \\
\text{Spec } V & \xrightarrow{\gamma} & Y
\end{array}$$

in which $K$ is the fraction field of $V$. Let $\psi : y \to f(\xi)$ be the morphism of $\text{Gal}(Y)$ witnessed by $\gamma$, and let $\phi : x \to \xi$ be a lift thereof to $\text{Gal}(X)$. We obtain a square

$$\begin{array}{ccc}
O_{X,Y}^{\text{sh}} & \xrightarrow{\gamma} & V \\
\downarrow & & \downarrow \\
O_{X,X}^{\text{sh}} & \xrightarrow{\xi} & K,
\end{array}$$

and since $O_{X,Y}^{\text{sh}} \to O_{X,X}^{\text{sh}}$ is local, we obtain a lift $\overline{\gamma} : O_{X,X}^{\text{sh}} \to V$, as required.

A universal homeomorphism is a morphism that is radicial, surjective, and universally closed. An equivalence of categories is a right fibration with fibres contractible groupoids. We thus deduce:

3.7 Proposition. Let $f : X \to Y$ be a morphism of schemes. If $f$ is a universal homeomorphism, then $\text{Gal}(X) \to \text{Gal}(Y)$ is an equivalence. Conversely, if $f$ is of finite type, and if $\text{Gal}(X) \to \text{Gal}(Y)$ is an equivalence, then $f$ is a universal homeomorphism (which is necessarily finite).

4 Interlude: perfectly reduced schemes

A reduced scheme receives no nontrivial nilimmersions; a perfectly reduced scheme receives no nontrivial universal homeomorphisms. This is in fact a local condition that can be expressed in very concrete terms:
4.1 Proposition. The following are equivalent for a scheme $X$.

- There exists an affine open covering $\{\text{Spec } A_i\}_{i \in I}$ of $X$ such that for every $i \in I$, the following conditions obtain:
  - for any $f, g \in A_i$, if $f^2 = g^3$, then there is a unique $h \in A_i$ such that $f = h^3$ and $g = h^2$; and
  - for any prime number $p$ and any $f, g \in A_i$, if $f^p = p^e g$, then there is a unique element $h \in A_i$ such that $f = ph$ and $g = h^p$.
- If $X'$ is a reduced scheme and $f : X' \to X$ is a universal homeomorphism, then $f$ is an isomorphism.

4.2 Definition. A scheme that enjoys one and therefore both of the conditions of Proposition 4.1 is said to be perfectly reduced or – in the parlance of [4, Appendix B] and [STK, Tag 0EUL] – absolutely weakly normal.

Let us write $\text{Sch}_{\text{perf}} \subset \text{Sch}_{\text{coh}}$ for the full subcategory of schemes spanned by the perfectly reduced schemes.

4.3. To express this differently, let us define a family of reference universal homeomorphisms. First, let $Y$ denote the cuspidal cubic

$$Y := \text{Spec } \mathbb{Z}[u, v]/(u^2 - v^3).$$

The normalisation $\rho : A^1_Y \to Y$ defined by the equations $u = t^3$ and $v = t^2$ is a universal homeomorphism. Next, for any prime number $p$, set

$$Z_p := \text{Spec } \mathbb{Z}[y, z]/(y^p - p^e z).$$

The normalisation $\tau_p : A^1_{Z_p} \to Z_p$ defined by the equations $y = px$ and $z = x^p$ is a universal homeomorphism. Proposition 4.1 states that a scheme $X$ is perfectly reduced if and only if every point $x \in X$ is contained in a Zariski open neighbourhood $U \subseteq X$ such that the map

$$\text{Mor}(U, A^1_{Z_p}) \to \text{Mor}(U, Y)$$

is a bijection, and for any prime number $p$, the map

$$\text{Mor}(U, A^1_{Z_p}) \to \text{Mor}(U, Z_p)$$

is a bijection.

4.4. Any (quasicompact) open subscheme of a perfectly reduced scheme is perfectly reduced. A reduced $\mathbb{Q}$-scheme is perfectly reduced if and only if it is seminormal. A reduced $\mathbb{F}_p$-scheme is perfectly reduced if and only if the Frobenius morphism is an isomorphism.

4.5 Proposition ([1, Proposition 14.5]). The inclusion $\text{Sch}_{\text{perf}} \hookrightarrow \text{Sch}_{\text{coh}}$ admits a right adjoint $X \mapsto X_{\text{perf}}$ which exhibits $\text{Sch}_{\text{perf}}$ as the colocalisation of $\text{Sch}_{\text{coh}}$ along the class of universal homeomorphisms. In particular, the counit $X_{\text{perf}} \Rightarrow X$ is the initial object in the category of universal homeomorphisms to $X$. We call $X_{\text{perf}}$ the perfection of $X$.

4.6. For reduced $\mathbb{Q}$-schemes, the perfection is the seminormalisation [STK, Tag 0EUT]. For reduced $\mathbb{F}_p$-schemes $X$ the perfection is the limit of $X$ over powers of the Frobenius, as usual.
4.7 Definition. A topological morphism from a scheme \( X \) to a scheme \( Y \) is a morphism \( \phi : X_{\text{perf}} \rightarrow Y \). If \( \phi \) induces an isomorphism \( X_{\text{perf}} \cong Y_{\text{perf}} \), then it is said to be a topological equivalence from \( X \) to \( Y \).

4.8. Let \( X \) and \( Y \) be schemes. Consider the following category \( T(X, Y) \). The objects are diagrams

\[
X \leftarrow X' \rightarrow Y
\]

in which \( X \leftarrow X' \) is a universal homeomorphism. A morphism from \( X \leftarrow X' \rightarrow Y \) to \( X \leftarrow X'' \rightarrow Y \) is a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{X'} & Y \\
\downarrow & & \downarrow \\
X'' & \xleftarrow{X''} & Y
\end{array}
\]

in which the vertical morphism is (of necessity) a universal homeomorphism. The nerve of the category \( T(X, Y) \) is equivalent to the set \( \text{Mor}(X_{\text{perf}}, Y) \cong \text{Mor}(X_{\text{perf}}, Y_{\text{perf}}) \) of topological morphisms from \( X \) to \( Y \).

4.9. The point now is that \( \text{Gal} \), viewed as a functor from \( \text{Sch}_{\text{perf}} \) to categories, is conservative.

4.10 Definition. Let \( P \) be a property of morphisms of schemes that is stable under base change and composition. We will say that a morphism \( f : X \rightarrow Y \) is topologically \( P \) if and only if it is topologically equivalent to a morphism of schemes \( f' : X' \rightarrow Y' \) with property \( P \).

4.11. Let \( P \) be a property of morphisms of schemes that is stable under base change and composition. The class of topologically \( P \) morphisms is the smallest class of morphisms \( P' \) that contains \( P \) and satisfies the following condition: for any commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi} & & \downarrow{\psi} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

in which \( \phi \) and \( \psi \) are universal homeomorphisms, the morphism \( f \) lies in \( P' \) if and only if \( f' \) does.

A morphism \( f : X \rightarrow Y \) of perfectly reduced schemes is topologically \( P \) precisely when it factors as a universal homeomorphism \( X \rightarrow X' \) followed by a morphism \( X' \rightarrow Y \) with property \( P \).

4.12 Example. A morphism \( f : X \rightarrow Y \) of perfectly reduced schemes is topologically radicial, surjective, universally closed, or integral if and only if it is radicial, surjective, universally closed, or integral (respectively).

4.13 Example. A morphism \( f : X \rightarrow Y \) of perfectly reduced schemes is topologically étale if and only if it is étale. Indeed, if \( f' : X' \rightarrow Y \) is étale, then \( X' \) is perfectly reduced [4, B.6(ii)].
5 Finite = right fibration with finite fibres

We’ve already seen that an integral morphism of schemes induces a right fibration of
Galois categories and that a morphism that induces a right fibration of Galois categories
must be universally closed. Let us complete this picture.

Let us begin with an obvious characterisation of quasifinite morphisms. We will say
that a functor has finite fibres if each of its fibres is a finite set.

5.1 Lemma. Let \( f : X \to Y \) be a morphism that is of finite type. Then \( f \) is quasifinite
if and only if \( \text{Gal}(X) \to \text{Gal}(Y) \) has finite fibres.

Since proper quasi finite morphisms are finite, Proposition 3.6 now yields:

5.2 Proposition. Let \( f : X \to Y \) be a morphism that is separated and of finite type.
Then \( f \) is finite if and only if \( \text{Gal}(X) \to \text{Gal}(Y) \) is a right fibration with finite fibres.

6 Étale = left fibration with finite fibres

6.1 Proposition. Let \( f : X \to Y \) be a morphism of schemes. If \( f \) is weakly étale, then
\( \text{Gal}(X) \to \text{Gal}(Y) \) is equivalent to a left fibration. Conversely, if \( X \) and \( Y \) are perfectly
reduced, if \( f \) is of finite presentation, and if \( \text{Gal}(X) \to \text{Gal}(Y) \) is a left fibration with
finite fibres, then \( f \) is étale.

Proof. Assume that \( f \) is weakly étale. Then for any geometric point \( x \to X \), the mor-
phism \( X_{(x)} \to Y_{(f(x))} \) is an isomorphism, whence the functor
\[
\text{Gal}(X)_{x/} \approx \text{Gal}(X_{(x)}) \to \text{Gal}(Y_{(f(x))}) \approx \text{Gal}(Y)_{f(x)/}
\]
is an equivalence, whence \( \text{Gal}(X) \to \text{Gal}(Y) \) is a left fibration.

Conversely, assume that \( X \) and \( Y \) are perfectly reduced, that \( f \) is of finite presen-
tation, and that \( \text{Gal}(X) \to \text{Gal}(Y) \) is a left fibration with finite fibres. So the functor
\( \text{Gal}(X) \to \text{Gal}(Y) \) is classified by a continuous functor \( \text{Gal}(Y) \to \text{Set}^{\text{fin}} \), which in turn
corresponds to a constructible étale sheaf of finite sets on \( Y \), which in particular coin-
dides with the sheaf represented by \( X \). Since the sheaf represented by \( X \) is constructible,
there exists an étale map \( U \to Y \) and an effective epimorphism \( U \to X \) of étale sheaves
on \( Y \). By descent, \( X \to Y \) is étale.

7 Finite étale = Kan fibration with finite fibres

We may as well combine the last two entries in our dictionary.

7.1 Recollection. A Kan fibration is a functor that induces a Kan fibration on nerves.
Equivalently, it is a functor that is both a left and right fibration. Equivalently, it is a
functor \( C \to D \) that is equivalent to the Grothendieck construction applied to a dia-
gram of groupoids indexed on \( D^{op} \) that carries every morphism to an equivalence of
groupoids.

7.2 Proposition. Let \( f : X \to Y \) be a morphism of perfectly reduced schemes that
is separated and of finite presentation. Then \( f \) is finite étale if and only if \( \text{Gal}(X) \to \text{Gal}(Y) \)
is a Kan fibration with finite fibres.

\(^5\)which for our purposes means a finite disjoint union of contractible groupoids
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