EMBEDDED DESINGULARIZATION FOR ARITHMETIC SURFACES – TOWARD A PARALLEL IMPLEMENTATION

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Abstract. We present an approach for algorithmic embedded desingularization of arithmetic surfaces bearing in mind implementability. Our algorithm is based on work by Cossart-Jannsen-Saito, though we present a variant using a refinement of the order instead of the Hilbert-Samuel function as a measure for the complexity of the singularity. We particularly focus on aspects arising when working in mixed characteristic. The essential building block is the computation of the locus of maximal order in the arithmetic setting. Furthermore, we exploit the algorithm’s natural parallel structure rephrasing it in terms of Petri nets for use in the parallelization environment GPI-Space with Singular as computational back-end.

1. Introduction

Resolution of singularities in all its facets has captured the attention of many algebraic geometers since the end of the 19th century. Hironaka’s monumental work [40] solved the long standing problem of desingularization in characteristic zero in arbitrary dimension in 1964, whereas the problem is still wide open in positive and mixed characteristic in higher dimension. With the aim to device a viable approach in positive characteristic, Hironaka’s characteristic zero approach has been analyzed and rephrased over the decades. Starting with the first constructive formulations, (see in particular [52, 3]) and even more algorithmic simplified approaches (as for example [12, 28, 53]) to the point that nowadays there are accessible, concise introductions to the subject available (like [23, 46]). Around the turn of the century the algorithmization of resolution of singularities had reached a level which allowed first prototype implementations [8, 34], of which the latter was sufficiently efficient to also permit various applications [32, 33, 30].

In positive characteristic, there exist only results in small dimensions and in very special cases [5, 6, 7, 47, 51]. While the surface case has been treated in many different ways [1, 2, 14, 16, 22, 25, 42, 45, 48], it was not until the end of the last decade that the dimension 3 case (non-embedded) was solved in full generality [17, 18, 19, 20, 21]. In mixed characteristic, in particular the approaches of Lipman [48] and Cossart-Jannsen-Saito [16] exemplify two different ways to deal with the task: using a combination of normalization and blow-up steps in the first case and relying solely on blowing up at suitably chosen centers in the second. In Lipman’s approach the normalization in mixed characteristic proved to be a bottleneck in practice [31]; the Cossart-Jannsen-Saito (CJS) algorithm, on the other hand, provides the structural advantage of only using a single kind of birational morphism.

The motivation behind our study of the CJS-algorithm is threefold: first, we need a sufficiently powerful tool for desingularization in this case to enter a more structured, experiment-driven approach to the comparison of desingularization and valuation theoretic approach started in [35]; second, key parts of the algorithm to determine the center can also be used for an alternative approach to normalization; last, but not least, the quest for a solution to the desingularization problem in higher dimension makes it tempting to experimentally search for cases where a straightforward generalization of an existing algorithm for surfaces to higher dimensions reveals new interesting phenomena. Section 6 of this article provides some more details on the last two motivations.

The goal of this article is to introduce a variant of the approach of Cossart, Jannsen and Saito to embedded desingularization in dimension 2, which is sufficiently explicit to allow implementation. The underlying ideas of the CJS-algorithm can be traced back to Hironaka’s Bowdoin lectures [42] for hypersurfaces, for which Cossart later closed a gap in [15]. To further outline this approach in slightly more detail, we first need to fix the
setting and notation for embedded resolution in mixed characteristic: Let $X$ be a reduced excellent Noetherian scheme of dimension two, embedded in some excellent regular scheme $Z$. Note that we do not fix a base field, explicitly allowing the case of a scheme over a Dedekind ring. We may assume without loss of generality that $Z$ is connected (and hence that $Z$ is irreducible) since $Z$ is regular and we may safely treat each component separately. Throughout the article, we fix

$$N := \dim(Z).$$

In contrast to CJS, we avoid the use of the Hilbert-Samuel function $H_X(x)$ (loc. cit. Definition 1.28) as measure for the complexity of the singularity of $X$ at $x$ and pursue the idea of using the order of $I(X)$ at $x$ instead. More precisely, we introduce a refined order $\nu_{\text{ref}}(x)$ (Definition 2.4), for which the computation of the maximal locus is not as expensive as for the Hilbert-Samuel function. In Example 4.12 we show that this leads to a different resolution process. The idea of replacing the Hilbert-Samuel function by the order is by no means new, as it can e.g., be found in [12]. More recently, also the multiplicity has been used as a replacement for the Hilbert-Samuel function in [13], but we will not follow that train of thought here.

The article is structured to first cover the theoretical background, before formulating the algorithms first in a sequential way and then as Petri-nets. More precisely, section 2 discusses our variant of the algorithm of Cossart-Jannsen-Saito without assuming that the reader is already deeply familiar with desingularization algorithms. This section therefore also recalls important results and constructions from [16] and [22], and explains the complete algorithm illustrating important facts by examples, wherever necessary. In particular, it introduces the main invariant, the refined order of an ideal.

As the main measure for the complexity of a singularity is the order of the corresponding ideal in our setting, the computation of the locus of maximal order is the very heart of the algorithm. As this is not a completely trivial task in mixed characteristic, section 3 is devoted to treating this in detail. This section does not assume that the reader is familiar with the notion of Hasse derivatives, which are indispensable for the computation, and hence also discusses them briefly before rendering the construction of the locus of maximal refined order completely algorithmic.

Based on the theoretical discussions in sections 2 and 3, we are then prepared to reformulate our version of the CJS-approach in terms of sequential algorithms in section 4. In here, we apply particular care to the identification of bad primes or more precisely potentially bad primes, which we refer to as interesting primes (i.e., prime numbers above which there possibly lies a component of the maximal locus). At this point, it is also important to note that we distinguish between horizontal and vertical components in the maximal locus. The former are components that can be seen at every prime, while each vertical component is associated to a single particular prime $p$ and is not seen above any prime $q \neq p$. In contrast to [16], we give precedence to the horizontal components and consider the vertical ones only when no more horizontal components are left.

Given the sequential algorithms and the naturally parallel structure due to the use of coverings by charts at different points in the algorithms, we rephrase the algorithms in terms of Petri-nets in section 5 to make them accessible to the parallel workflow management system GPI-Space. As we also do not assume familiarity of the reader with the formalism of Petri nets, we give a very brief overview at the beginning of the section, before stating the nets which represent the main parts of the algorithms.

The last section is then devoted to an outlook which details the possible applications already mentioned as motivations for this article.

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1 In fact, if one is only interested in understanding the local situation, it is always possible to reduce all consideration to the embedded setting, see [16] Remark 4.22.

2 The idea to try to resolve first all horizontal components and then to handle the vertical components has been pointed out by H. Hironaka during a private conversation at a conference in 2006 at the ICTP in Trieste, Italy.
Notational Remark: throughout the article, \(\mathbb{N} = \{0,1,2,\ldots\}\) and we often use multi-index notation, i.e., for \((\underline{x}) = (x_1,\ldots,x_m)\) and \(\underline{a} = (a_1,\ldots,a_m) \in \mathbb{N}^m\), we abbreviate \(\underline{x}^{\underline{a}} = x_1^{a_1} \cdots x_m^{a_m}\).

2. A VARIANT OF THE ALGORITHM OF COSSART, JANNSSEN AND SAITO

We present a variant of the algorithm by Cossart, Jannsen, Saito \([16]\). In contrast to their approach, we do not consider the Hilbert-Samuel function \(H_X : X \to \mathbb{N}^0\) as measure for the complexity of the singularities, but a refinement \(\nu_{\text{ref}} : X \to \mathbb{N}\) of the order (Definition 2.4). The benefit is that with our modification the implementation of the algorithm runs in reasonable time, whereas the computation of the Hilbert-Samuel locus is far more expensive. (cf. \([50]\)). Furthermore, we make a distinction between the vertical components (i.e., those lying completely in one fiber above some prime \(p \in \mathbb{Z}\)) in the maximal locus of \(\nu_{\text{ref}}\) and the horizontal ones (which appear in every fiber).

Let us point out that in this section we do not make use of the additional structure imposed on \(Z\). Hence the constructed resolution procedure can be applied for any excellent irreducible regular scheme \(Z\).

From the viewpoint of resolution of singularities, a map \(\mu : X \to S\) into a totally\(^3\) ordered set \((S, \leq)\) has to fulfill the following properties in order to be a decent measure for the complexity of a singularity and its improvement during the resolution process:

(A) \(\mu\) distinguishes singular and regular points.

(B) \(\mu\) is (Zariski) upper semi-continuous: the sets \(X_\mu(\geq s) := \{x \in X \mid \mu(x) \geq s\}\) are closed, for every \(s \in S\).

(C) \(\mu\) is infinitesimally upper semi-continuous: \(\mu\) does not increase if we blow up a regular center \(D\) contained in the locus, where \(\mu\) is maximal.

(D) \(\mu\) can only decrease finitely many times until all points are regular.

Therefore the order itself is not an appropriate measure since (A) fails to hold, in general, see Example 2.7. In our setting, we shall use \(S = \mathbb{N}^2\) equipped with the lexicographical order. Hence, for us, (C) implies (D).

Recall the following criterion for upper semi-continuity:

Lemma 2.1 ([16] Lemma 1.34(a)). A map \(\mu : X \to S\), with \((S, \leq)\) a totally ordered set, is upper semi-continuous if and only if the following holds:

1. If \(x, y \in X\) with \(x \in \overline{\{y\}}\), then \(\mu(x) \geq \mu(y)\).
2. For all \(y \in X\) there is a dense open subset \(U \subset \overline{\{y\}}\) such that \(\mu(x) = \mu(y)\) for all \(x \in U\).

Since we assume that \(\mu\) cannot increase under suitable blowing ups and that \(\mu\) can only decrease finitely many times until the strict transform of \(X\) is regular, it is sufficient to construct a finite sequence of blowing ups such that the maximal value achieved by \(\mu\) decreases strictly. This implies then embedded resolution of singularities for \(X \subset Z\). More precisely, we need to answer the following question: Given \(X \subset Z\), can we find a finite sequence of blowing ups,

\[
Z =: Z_0 \leftarrow B\ell_{D_0}(Z_0) =: Z_1 \leftarrow \cdots \leftarrow B\ell_{D_{a-1}}(Z_{a-1}) =: Z_a
\]

\[
X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_a,
\]

in centers \(D_i\) contained in the locus, where \(\mu\) is maximal on \(X_i\), \(0 \leq i \leq a - 1\), \((X_i\) denotes the strict transform of \(X\) in \(Z_i\)) such that eventually

\[
\max\{\mu(x_a) \mid x_a \in X_a\} < \max\{\mu(x) \mid x \in X\}?
\]

As a first step we introduce our refinement \(\nu_{\text{ref}}\) of the order and then show that the above properties hold for \(\nu_{\text{ref}}\) in order to justify its use.

\(^3\)In fact, it is sufficient if the set is partially ordered if we require that \(\mu\) can only achieve finitely many maximal values on \(X\), cf. \([16]\) Lemma 1.34(c) and the preceding paragraph. For example, this appears if the Hilbert-Samuel function is used.
Definition 2.2. Let $x \in X$ (not necessarily closed). We denote by $(R = \mathcal{O}_{Z,x}, m, k = R/m)$ the local ring of $Z$ at $x$ (which is excellent and regular) and by $J \subset R$ the ideal which defines $X$ locally at $x$.

1. The order of $X$ at $x$ is defined as the order of $J$ at $m$,
   \[ \operatorname{ord}_x(X) := \operatorname{ord}_m(J) := \sup\{ t \in \mathbb{N} \mid J \subset m^t \} \]
   For an element $f \in J$, we also abbreviate $d(f) := \operatorname{ord}_m(f)$.

2. The maximal order of $X$ is defined as
   \[ \max-\operatorname{ord}(X) := \sup\{ \operatorname{ord}_x(X) \mid x \in X \} \]
   and the maximal order locus of $X$ is the locus of points of order $\max-\operatorname{ord}(X)$,
   \[ \operatorname{Max-\operatorname{ord}}(X) := \{ x \in X \mid \operatorname{ord}_x(X) = \max-\operatorname{ord}(X) \}. \]

3. The initial form of $f$ (with respect to $m$) is defined as
   \[ \operatorname{in}_m(f) := f \mod m^{d(f)+1} \in \operatorname{gr}_m(R), \]
   where $\operatorname{gr}_m(R) = \bigoplus_{t \geq 0} m^t/m^{t+1}$ denotes the associated graded ring of $R$ at $m$.

4. The initial ideal of $J$ at $m$ is defined as the ideal $\operatorname{In}_m(J)$ in $\operatorname{gr}_m(R)$ generated by the initial forms of the elements in $J$,
   \[ \operatorname{In}_m(J) := \langle \operatorname{in}_m(f) \mid f \in J \rangle. \]

Recall that the order is an upper semi-continuous function, see [40] Chapter III §3 Corollary 1 p. 220. Furthermore, the graded $\operatorname{gr}_m(R)$ is isomorphic to a polynomial ring in $n$ variables over the residue field $k$ (even in mixed characteristic), where
\[ n := n_x := \dim(R). \]
In the graded ring one of the variables may correspond to a prime number, e.g. $R = \mathbb{Z}[x]/(73,x)$. Moreover, $\operatorname{In}_m(J)$ is a homogeneous ideal in $\operatorname{gr}_m(R)$. For $\mathfrak{N} := \operatorname{In}_m(m) \subset \operatorname{gr}_m(R)$, we have
\[ \min\{ \operatorname{ord}_m(f) \mid f \in J \} = \operatorname{ord}_m(J) = \operatorname{ord}_{\mathfrak{N}}(\operatorname{In}_m(J)). \]

Observation 2.3. Let the situation be as in the previous definition. Set $I_1 := \operatorname{In}_m(J)$. If $\operatorname{ord}_{\mathfrak{N}}(I_1) = 1$, we can consider its image $I_1'$ in the degree 1 slice of $\operatorname{gr}_m(R)$, which yields a subspace of the finite dimensional $k$-vector space $\operatorname{gr}_m(R)_{1} = m/m^2$. We can thus find a basis, say $F_1, \ldots, F_a$ of $I_1'$, for some $a := a_x \in \mathbb{N}$. Either $I_1 = \langle F_1, \ldots, F_a \rangle$ or $\operatorname{ord}_{\mathfrak{N}}(H) > 1$ for all elements $H$ in the ideal $I_{a+1}$ generated by the set $I_1 \setminus \langle F_1, \ldots, F_a \rangle$ in $\operatorname{gr}_m(R)$. We define
\[ d_x := \begin{cases} 1, & \text{if } I_1 = \langle F_1, \ldots, F_a \rangle \\ \operatorname{ord}_{\mathfrak{N}}(I_{a+1}) > 1, & \text{otherwise}. \end{cases} \]

For each $F_i \in \operatorname{In}_m(J)$, we can choose a lift $f_i \in J$ with $\operatorname{in}_m(f_i) = F_i, 1 \leq i \leq a$. We have $\operatorname{ord}_m(f_i) = \operatorname{ord}_{\mathfrak{N}}(F_i) = 1$ and $(f_1, \ldots, f_a)$ forms a regular sequence, i.e., $f_i \notin (f_1, \ldots, f_{i-1})$, for $1 \leq i \leq a$. Hence $\mathcal{Y} := V(f_1, \ldots, f_a)$ defines a $(n-a)$-dimensional regular subvariety of $Z := \operatorname{Spec}(R)$ which contains $X = V(J)$. This implies
\[ a_x \leq n_x - \dim X \leq N = \dim(Z). \]

The following are equivalent:

1. the first inequality is strict, $a_x < n_x - \dim X$ (i.e., $\dim X < \dim \mathcal{Y}$).
2. $d_x > 1$, and
3. $x$ is a singular point of $X$.

At a closed point corresponding to a maximal ideal $m_x$ the value of $n_x - a_x$ coincides with the embedding dimension of $X$ at $x$,
\[ n_x - a_x = \operatorname{embdim}(X). \]
However, this idea needs to be refined slightly to also allow local considerations at non-maximal prime ideals. This is taken into account in the following definition. The reasons for the change will become apparent in Example 2.5 directly after it.

Definition 2.4. Let $X \subset Z$ be as above.
Remark 2.6. (1) The reader familiar with the notion of Hironaka’s \( X \)
\( \text{Spec}(k[t, v, y, z]) \) be the hypersurface defined by the polynomial \( f = t^2 - y^2z \). Its singular locus is the curve \( C := V(t, y) \), whose generic point we denote by \( \eta \). By \( x \) we denote the closed point \( V(t, y, z) \).

We have \( a_x = a_\eta = 0 \) and \( d_x = d_\eta = 2 \) coincides with the order of \( f \) at the respective point. Therefore \( \nu_\text{ref}(x) = \nu_\text{ref}(\eta) = (3, 2) \) since \( N = \dim(Z) = 3 \), and \( \text{Max-}\nu(X) = V(t, y) \).

On the other hand, if we used \( \nu_{hyp}(x) = (a_x, a_\eta, d_x, d_\eta) \) instead of \( \nu_\text{ref} \) we would see the following: \( n_x = \dim(O_{Z,x}) = 3 \) and \( n_\eta = \dim(O_{Z,\eta}) = 2 \), since \( z \) becomes an element of the residue field at \( \eta \) and we see that \( \text{trdeg}((O_{Z,\eta}/m_{Z,\eta}) : (O_{Z,x}/m_{Z,x})) = 1 \). Hence we have \( n_x - a_x = 3 > 2 = n_\eta - a_\eta \), i.e., \( \nu_{hyp}(x) = (3, 2) > (2, 2) = \nu_{hyp}(\eta) \). In fact, this computation holds for any closed point \( x' \) contained in \( C \) instead of \( x \). Since \( k \) is infinite, there are infinitely many closed points on \( C \) and the maximal locus of \( \nu_{hyp} \) is not closed.

Another interesting aspect about this example is the following: if we blow up the closed point \( x \) then the strict transform of \( f \) at the point with parameters \( (t', v', z') = (t, y, z) \) is given by \( f' = t^2 - y^2z' \) and we have thus encountered a loop.

Remark 2.6. (1) The reader familiar with the notion of Hironaka’s \( \nu^* \)-invariant ([40] Chapter III Definition 1, p. 205, see also [16] Definition 1.17) will recognize a connection. Observation 2.3 is precisely the beginning of a possible way to construct a so called standard basis for \( J \) and thus to determine \( \nu^*(J, R) \), see [16] the paragraph before Remark 1.5. Set \( I := \text{In}_m(J) \), then

\[
\nu^*(J, R) = (\nu^1(I), \nu^2(I), \ldots, \nu^a(I), \nu^{a+1}(I), \nu^{a+2}(I), \ldots) = (1, 1, \ldots, 1, d_x, d_x^* + 2, \ldots) \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}},
\]

(2) If \( X = V(f) \) is an affine hypersurface then \( \nu_\text{ref} \) determines the order of \( X \) and vice versa. On the other hand, the behavior of the Hilbert-Samuel function is also detected by the order in this case ([16] Theorem 2.10). Hence the maximal loci considered by the original [16]-algorithm and by our variant coincide for hypersurfaces. Nonetheless, the algorithms may differ in their way of resolving the maximal locus since this is in general not a hypersurface.

(3) Since we assume \( \dim(X) = 2 \) and \( X \) reduced, \( \text{Max-}\nu(X) \) has at most dimension one. In particular, \( \text{Max-}\nu(X) \) itself has at most isolated singularities.

Example 2.7. Let \( R = \mathbb{Z}[t, v, w, y, z_\pm] \), where \( \mathfrak{m} = \langle p, t, v, w, y, z \rangle \), for \( p \in \mathbb{Z} \) prime. We consider the variety \( X := V(J) \subset \text{Spec}(R) \), where

\[
J := \langle p, t, v^2 - y^3, z^5 - y^2w^5 \rangle \subset R.
\]
Then, \( \ord_m(J) = 1 \), although the point \( x \) in \( X \) corresponding to \( m \) is singular. Hence, the order cannot distinguish regular and singular points.

On the other hand, since \( \ord_m(x^2 - y^2) = 2 \) is maximal and \( \ord_m(z^5 - y^2w^5) = 5 \geq 2 \), we have \( \nu_{\ref}(x) = (4, 2) \). In fact, \( \max-\nu(X) = (4, 2) \) and \( \max-\nu(X) = V(p, t, v, y, z) \).

Note: if we put \( P := (p, t, v, y, z) \), then \( \ord_1(z^5 - y^2w^5) = 2 < 5 = \ord_1(z^5 - y^2w^5) \), a difference which is undetected by \( \nu_{\ref} \).

**Proposition 2.8.** The map \( \nu_{\ref} = \nu_{\ref,X,Z} : X \to \mathbb{N}^2 \) distinguishes regular from singular points (A), is Zariski upper semi-continuous (B), is infinitesimally upper semi-continuous (C), and can only decrease finitely many times until it detects regularity (D).

**Proof.** Since \( \nu_{\ref} \) takes values in \( \mathbb{N}^2 \), (D) is implied by (C). Further, we have seen in Observation 2.3 that \( d_{\ref} \) distinguishes singular and regular points. Hence (A) holds.

For (C), we may assume that \( X \) is singular. Let \( x \in \max-\nu(X) \). As before, \( J \) denotes the ideal in \( R := \mathcal{O}_{Z, x} \) defining \( X \) locally at \( x \). Let \( f_1, \ldots, f_a, f_{\ell} \in J \) be generators for \( J \) such that \( \ord_m(f_i) = 1 \) and \( f_i \notin (f_1, \ldots, f_{i-1}) \), for \( 1 \leq i \leq a \), and \( \ord_1(f_j) > 1 \), for \( j > a \) (cf. Observation 2.3). In particular, \( V(f_1, \ldots, f_a) \) is regular.

Let \( \pi : Z' := \beta \mathcal{D}(Z) \to Z \) be a weakly permissible blowing up with center \( D \subset \max-\nu(X) \) containing \( x \). Denote by \( X' \) the strict transform of \( X \) in \( Z' \). We have to show \( \nu_{\ref}(x') \leq \nu_{\ref}(x) \), for every \( x' \in \pi^{-1}(x) \). (Note that \( \dim(Z') = \dim(Z) \)).

Above the point \( x \), the strict transform of \( X \) is given by the strict transforms of the elements \( h \in J \). Since \( X \) is singular, we can choose \( h \in J \) such that \( \ord_m(h) = d_{\ref} \), where \( \overline{\mathfrak{m}} \) and \( \overline{\mathfrak{h}} \) denote the images of the respective objects in \( R/(f_1, \ldots, f_a) \). Without loss of generality, we may assume that \( h \) is of order \( d_{\ref} \) at \( x \) (since we may suppose that \( h \) is in normal form with respect to \( (f_1, \ldots, f_a) \) in the sense of standard bases and Mora’s algorithm). Note that \( D \) is a weakly permissible center for the hypersurface \( V(h) \subset Z \) at least in a Zariski-open neighborhood of \( x \) (i.e., it is regular and contained in \( \max-\nu(V(h)) \) — in fact the latter coincides with the maximal order locus of \( V(h) \)). It is a well-known fact (see [16] Theorem 2.10, using loc. cit. Definitions 1.26, 1.17(2)) that the order of a hypersurface does not increase under such blowing ups. This implies \( \nu_{\ref}(x') = \nu_{\ref}(x) \) and hence (C).

It remains to prove (B), which is the upper semi-continuity of \( \nu_{\ref} \). For this, we will make use of Lemma 2.1: Let \( x, y \in X \subset Z \) with \( x \in \{y\} \). We have to show that the inequality \( \nu_{\ref}(x) \geq \nu_{\ref}(y) \) holds. Let \( U_0 = \Spec(S) \) be an affine open set containing \( x \) and \( y \). Let \( f_1, \ldots, f_a \in S \) be such that each \( f_i \) is of order one at \( x \) and they define \( a_y = a \) (cf. Observation 2.3). By the upper semi-continuity of the order, the order of each \( f_i \) does not increase by passing from \( x \) to \( y \). Since \( y \in X \), the order of \( f_i \) has to be at least one at \( y \). Hence, we have \( a_y \geq a_x = a \) and if the inequality is strict then \( \nu_{\ref}(y) < \nu_{\ref}(x) \).

Suppose \( a_y = a_x \). We stay in the same notation \( (J \subset S) \) the ideal defining \( X \cap U_0 \) in \( U_0 = \Spec(S) \) containing \( x \) and \( y \). Let \( h \in J \) be an element of order \( d_{\ref} \) at \( x \) (cf. proof of (C)). By choosing \( U_0 \) sufficiently small, we may assume that \( V(f_1, \ldots, f_a) \subset U_0 \) is regular. Let us denote by \( \overline{\mathfrak{m}} \) the image of \( h \) in \( S/(f_1, \ldots, f_a) \). Note that \( x, y \in V(f_1, \ldots, f_a) \). We have that the order of \( \overline{\mathfrak{m}} \) at \( x \) is \( d_{\ref} \). Again, by the upper semi-continuity of the order, we obtain that the order of \( \overline{\mathfrak{m}} \) at \( y \) is at most \( d_{\ref} \). Therefore, \( d_y \leq d_x \), i.e., \( \nu_{\ref}(y) \leq \nu_{\ref}(x) \). In other words, Lemma 2.1(1) holds.

For Lemma 2.1(2), let \( y \in X \). We need to show that there exists a dense open subset \( U \subset \{y\} \) such that \( \nu_{\ref}(x) = \nu_{\ref}(y) \) for all \( x \in U \). Let \( U_0 = \Spec(S) \) be an affine neighborhood of \( y \) in \( Z \) and denote by \( J \subset S \) the ideal defining \( X \). Using Observation 2.3 at \( y \) and analogous notation to before, we choose generators \( (f_1, \ldots, f_a, f_{a+1}, \ldots, f_r) \) for \( J \) in \( S \) such that \( V(f_1, \ldots, f_a) \) is regular and \( N - a \) is the first entry of \( \nu_{\ref}(y) \). (Note: We can choose \( U_0 \) sufficiently small for this). From the construction of \( d_y \), it follows that \( d_y \) coincides with the order of \( J \cdot S/(f_1, \ldots, f_a) \) at \( y \). The latter is upper semi-continuous and by Lemma 2.1 there exists a dense open subset \( V \subset V(f_1, \ldots, f_a) \) such that the order at every point in \( V \) is equal to \( d_y \). This yields then a dense open set \( U \subset \{y\} \) such that \( \nu_{\ref}(x) = \nu_{\ref}(y) \)

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4It may, of course happen that the maximal refined order of \( h \) exceeds \( d_{\ref} \) somewhere else, but this can only happen on a Zariski-closed subset not meeting \( x \). Hence it is possible to avoid this set by taking a suitable Zariski-open neighbourhood of \( x \).
for all $x \in U$. Hence Lemma 2.1(2) holds. As a conclusion we obtain that $\nu_{\text{ref}}$ is upper semi-continuous. 

The resolution process provides us with more data than just the strict transform $X'$ of $X$ in $Z'$. Namely, there are also the exceptional divisors $E'$ of the blowing ups. Additionally to the above, one imposes for an embedded resolution of singularities that the strict transform of $X$ has at most simple normal crossings with $E'$ and that all centers in (2.1) meet the exceptional divisors transversally.

Furthermore, in order to attain a canonical resolution of singularities one has to take the exceptional divisors into account. For example, the singular locus of $X := V(x^2 - y^2z^2)$ (which coincides with Max-$\nu(X)$) consists of the two lines $L_1 := V(x, y)$ and $L_2 := V(x, z)$. A priori there is no way to distinguish them and we need to blow up their intersection $V(x, y, z)$. At first glance the situation did not change locally at the point with coordinates $(x', y', z') = (0, 0, 0)$. But, in fact, $V(x', y')$ is contained in the exceptional divisor, whereas $V(x', y')$ is the strict transform of $L_1$. Thus we can distinguish the lines and pick one of them as the next center. (For example, the [16]-algorithm will choose $L_1$ since it is "older" than the other component, see also Example 6.1).

It is a general philosophy in resolution of singularities to encode the information on the exceptional divisors in some way. Following [16], we do this using a so called boundary $B$ ([16] Definition 3.3) and adapt the notion of $B$-permissible centers (loc. cit. Definition 3.5) for our situation.

**Definition 2.9.** Let $X \subset Z$ be as before.

1. A boundary on $Z$ is a collection $B = \{B_1, \ldots, B_d\}$ of regular divisors on $Z$ such that the associated divisor $\text{div}(B) = B_1 \cup \cdots \cup B_d$ is a simple normal crossings divisor. For $x \in Z$, we let $B(x) := \{B_i \mid x \in B_i\}$ be the boundary components passing through $x$.

2. A closed subscheme $D \subset X$ is called $B$-weakly permissible if it is weakly permissible and has at most simple normal crossings with the boundary $B$ (i.e., $D$ is regular, contained in Max-$\nu(X)$, and intersects $\text{div}(B)$ transversally).

We also say that the corresponding blowing up $\pi : Z' := \text{Bl}_D(Z) \rightarrow Z$ with center $D$ is a $B$-weakly permissible blowing up. The transform $B'$ of the boundary $B$ under $\pi$ is defined to be the union of the strict transforms of the components of $B$ and the exceptional divisor $E'_x$ of $\pi$ (i.e., $B' = \{B'_1, \ldots, B'_d, E'_x\}$ with the obvious notations).

We distinguish the boundary components in two sets (cf. [16] Lemmas 3.7 and 3.14): Suppose we are at the beginning of the resolution process and we consider some point $x$. Then we define all boundary components passing through $x$ to be old, $O(x) := B(x)$.

Let $\pi : Z' \rightarrow Z$ be a blowing up of $Z$ in a regular center $D \subset X$ containing $x$. Let $x' \in X'$ be a point on the strict transform of $X$ which lies above $x$, $\pi(x') = x$.

- If $\nu_{\text{ref}}(x') < \nu_{\text{ref}}(x)$ then the singularity improved at $x'$ and we treat the situation as if we are at the beginning. Hence $O(x') := B'(x')$.

- Suppose $\nu_{\text{ref}}(x') = \nu_{\text{ref}}(x)$. Then we define the old boundary components $O(x')$ to be the subset of $B'(x')$ consisting of the strict transforms of the elements in $O(x)$ which pass thorough $x'$. All other components are called new, $N(x') := B'(x') \setminus O(x')$.

The reason for this distinction is that we have some control on the new boundary components and how they behave with respect to $X$. On the other hand, we loose this control once the singularity strictly improved, i.e., once $\nu_{\text{ref}}$ decreases strictly.

**Definition 2.10.** Let $X \subset Z$ be as before and let $B$ be a boundary on $Z$. We define the log-refined order by

$$\nu_{\text{ref}}^O = \nu_{\text{ref}, X, Z}^O : \begin{array}{c} X \\ x \end{array} \rightarrow (\mathbb{N}^3, \leq_{\text{lex}})$$

$$\nu_{\text{ref}}(x), |O(x)|).$$

For $\tilde{A} \in \mathbb{N}^3$, we set

$$\nu_{\geq \tilde{A}}^O(X) := \{x \in X \mid \nu_{\text{ref}}^O(x) \geq \tilde{A}\},$$
\[
\max-\nu^O(X) := (\alpha, \delta, \sigma) := \max\{\nu^O_{\text{ref}}(x) \mid x \in X\},
\]

\[
\text{Max-}\nu^O(X) := \begin{cases} 
\nu^O_{\text{max-}\nu^O(X)}(X), & \text{if } \delta > 1 \text{ or } \sigma > 0, \\
X, & \text{if } \delta = 1 \text{ and } \sigma = 0.
\end{cases}
\]

Let us explain how to obtain \(\nu^O_{\geq(A,b)}(X)\) from \(\nu^O_{\geq A}(X)\), for \(A \in \mathbb{N}^2\), \(b \in \mathbb{N}\):

**Construction 2.11.** Let the situation be as in the previous definition and, in particular, let \(B = \{B_1, \ldots, B_d\}\). Let \(\tilde{A} = (A, b) \in \mathbb{N}^3\) with \(A \in \mathbb{N}^2\), \(b \in \mathbb{N}\). If \(b > d\) or \(b > N = \dim(Z)\), then \(\nu^O_{\geq A}(X) = \emptyset\) is closed.

Hence, let us assume that we are in none of these two cases. The upper semi-continuity of \(\nu_{\text{ref}}\) (Proposition 2.8), implies that \(\nu^O_{\geq A}(X) = \{x \in X \mid \nu_{\text{ref}}(x) \geq A\}\) is closed. We define the closed set

\[
Y_0 := \bigcup_{(i_1, \ldots, i_b)} (\nu^O_{\geq A}(X) \cap B_{i_1} \cap \ldots \cap B_{i_b}).
\]

Suppose that there is an irreducible component \(W\) of \(Y_0\) such that we do not have \(\nu^O_{\text{ref}}(w) \geq \tilde{A}\), for every \(w \in W\). Then we replace \(W\) in \(Y_0\) by

\[
W_0 := \bigcup_{j : W \cap B_j} (W \cap B_j).
\]

If we have that \(W \subset B_j\), for every \(j \in \{1, \ldots, d\}\), then \(W_0 = \emptyset\). Furthermore, the dimension of each irreducible component of \(W_0\) is strictly smaller than \(\dim(W)\) if \(W_0 \neq \emptyset\). Hence, after finitely many iterations this procedure has to end. Let us denote by \(Y_\ast\) be the resulting closed set. By construction, we have that \(Y_\ast = \nu^O_{\geq A}(X)\).

In particular, we obtain that \(\nu^O_{\text{ref}}\) is an upper semi-continuous function.

By combining the latter with Proposition 2.8, we get

**Proposition 2.12.** The map \(\nu^O_{\text{ref}} : X \to \mathbb{N}^3\) distinguishes regular from singular points (A), is Zariski upper semi-continuous (B), is infinitesimally upper semi-continuous (C), and can only decrease finitely many times until it detects regularity (D).

**Proof.** By Proposition 2.8, (A) holds. The previous construction implies property (B) and, since \(\nu^O_{\text{ref}}\) takes values in \(\mathbb{N}^3\), (D) follows. It remains to show (C).

A \(B\)-weakly permissible blowing up is in particular weakly permissible and thus \(\nu^O_{\text{ref}}\) does not increase at points \(x'\) lying above \(x\). Furthermore, if \(\nu^O_{\text{ref}}(x') = \nu^O_{\text{ref}}(x)\) then the definition of \(O(x')\) implies \(|O(x')| \leq |O(x)|\). Therefore we obtain (C).

\(\square\)

**Definition 2.13.** Let \(X \subset Z\) be as above. We distinguish the irreducible components of \(\text{Max-}\nu^O(X)\) as follows: If \(Z\) contains a field, then all irreducible components are defined to be vertical.

Suppose \(Z\) does not contain a field. Let \(Y \subset \text{Max-}\nu^O(X)\) be an irreducible component. We say \(Y\) is a horizontal component of \(\text{Max-}\nu^O(X)\) if \(Y \times_{\mathbb{Z}} \mathbb{Q} \neq \emptyset\). Otherwise, \(Y\) is called a vertical component. We denote by \(\text{Max-}\nu^O_{\text{hor}}(X)\) the set of horizontal components of \(\text{Max-}\nu^O(X)\).

**Example 2.14.** Consider \(Z = \text{Spec}(\mathbb{Z}[x, y])\) with empty boundary. Let

\[
X = V(x^2 - y^45^7(y - 2)^4).
\]

We leave it as an exercise to the reader to show that

\[
\text{Max-}\nu^O(X) = \text{Max-}\nu(X) = \text{Max-}\text{ord}(X) = V(x, y) \cup V(x, 5) \cup V(x, y - 2).
\]

By the previous definition, \(V(x, y)\) is a horizontal component and \(V(x, 5)\) is a vertical component lying above the prime 5. Furthermore, the third component, \(V(x, y - 2)\), is also a horizontal component of \(\text{Max-}\nu^O(X)\) since it is non-empty after passing to \(\text{Spec}(\mathbb{Q}[x, y])\). Note: \(V(x, y + p - 2, p) \subset V(x, y - 2)\) since \(y - 2 = (y + p - 2) - p\), for every prime \(p \in \mathbb{Z}\).
Definition 2.15. Let \( \pi : Z' := B \ell D(Z) \to Z \) be a blowing up with some irreducible regular center \( D \). Let \( Y' \subset Z' \) be an irreducible subscheme contained in the exceptional divisor \( E' := \pi^{-1}(D) \). \( Y' \subset E' \). Then \( Y' \) dominates \( D \) if its image under \( \pi \) is dense in \( D \).

If the center \( D \) is not necessarily irreducible, we say \( Y' \) dominates \( D \) if there exists an irreducible component \( D_0 \) of \( D \) which is dominated by \( Y' \).

For example, let \( Z = \mathbb{A}^3_S \) with coordinates \((x, y, z)\), \( S \) any base, \( D = D_0 = V(x, y) \). We consider the chart with coordinates \((x', y', z) := (x, \frac{y}{z}, z)\). In this local situation, the exceptional divisor \( E' \) is \( V(x') \), and the component \( V(x^2, y) \subset E' \) dominates the center, whereas \( V(x', z) \subset E' \) does not, the closure of its image in \( Z \) is \( V(x, y, z) \).

We are now prepared to formulate our variant of the [16]-algorithm. We deliberately formulate this construction in a way close to the original one [16] Remark 5.29.

Construction 2.16. Let \( X \) be a reduced excellent Noetherian scheme of dimension two, embedded in a regular scheme \( Z, X \subset Z \). Let \( \mathcal{B} \) be a boundary on \( Z \).

**Horizontal case:** Suppose \( \text{Max}-\nu^O_{\text{hor}}(X) \neq \emptyset \), we set

\[ Y_0 := \text{Max}-\nu^O_{\text{hor}}(X), \]

Since \( X \) is reduced, \( \dim(Y_0) < \dim(X) \) unless \( X \) is regular in which case the construction of a desingularization is complete. By the induction on dimension we find a finite sequence of \( \mathcal{B} \)-weakly permissible blowing ups,

\[
\begin{align*}
Z &= Z_0 \leftarrow \cdots \leftarrow Z_{m_0} := Z' \\
\mathcal{B} &= \mathcal{B}_0 \leftarrow \cdots \leftarrow \mathcal{B}_{m_0} := \mathcal{B}',
\end{align*}
\]

such that the strict transform \( Y_0' \) of \( Y_0 \) in \( Z' \) is a \( \mathcal{B}' \)-weakly permissible center for \( X_{m_0} \). The next center chosen according to the construction is \( D_{m_0} := Y_0' \),

\[ Z' = Z_{m_0} \leftarrow B \ell D_{m_0}(Z_{m_0}) := Z_{m_0+1} \]

\[ \mathcal{B}' = \mathcal{B}_{m_0} \leftarrow \mathcal{B}_{m_0+1}. \]

Let \( X_{m_0+1} \) be the strict transform of \( X_0 := X \) under the previous sequence of blowing ups. If \( \text{max}-\nu^O(X_{m_0+1}) < \text{max}-\nu^O(X_0) \) then the singularity improved strictly and the construction restarts with the new maximal value for \( \nu^O_{\text{ref}} \) and the corresponding data. Further, if \( \text{max}-\nu^O(X_{m_0+1}) = \text{max}-\nu^O(X_0) \) and \( \text{Max}-\nu^O_{\text{hor}}(X_{m_0+1}) = \emptyset \), we go to the vertical case below.

Suppose \( \text{max}-\nu^O(X_{m_0+1}) = \text{max}-\nu^O(X_0) \) and \( \text{Max}-\nu^O_{\text{hor}}(X_{m_0+1}) \neq \emptyset \). In order to determine the next center we need to study the locus \( Y_{m_0+1} := \text{Max}-\nu^O_{\text{hor}}(X_{m_0+1}) \). We have a decomposition

\[ Y_{m_0+1} = Y_{m_0+1}^{(0)} \cup Y_{m_0+1}^{(1)} \cup \cdots \cup Y_{m_0+1}^{(m_0+1)}, \]

where:

- For \( 1 \leq i \leq m_0 \), \( Y_{m_0+1}^{(i)} \) denotes the union of irreducible components in \( Y_{m_0+1} \) that first occurred after the \( i \)-th blowing up of the algorithm ("label \( i \) components")
- \( Y_{m_0+1}^{(i)} \) are those irreducible components of \( Y_{m_0+1} \) dominating the center \( D_{m_0} \) of the last blowing up. These inherit the label of \( D_{m_0} \).
- \( Y_{m_0+1}^{(m_0+1)} \) are those irreducible components of \( Y_{m_0+1} \) lying above the center \( D_{m_0} \) but not dominating. Hence we assign the label \( m_0 + 1 \) to them.

We also say, \( Y_{m_0+1}^{(j)} \) are the irreducible components of \( Y_{m_0+1} \) with labels \( j \), \( 0 \leq j \leq m_0 + 1 \).

Let \( k := \min\{j \in \{0, \ldots, m_0 + 1\} \mid Y_{m_0+1}^{(j)} \neq \emptyset \} \). In the next step, the algorithm repeats the previous, with \( Y_{m_0+1}^{(k)} \) now in the place of \( Y_0 \).

**Vertical case:** Suppose \( \text{Max}-\nu^O_{\text{hor}}(X) = \emptyset \). We set

\[ Y_0 := \text{Max}-\nu^O(X), \]

and proceed analogous to the horizontal case using \( \text{Max}-\nu^O \) instead of \( \text{Max}-\nu^O_{\text{hor}} \). Note that the vertical case is stable until \( \text{max}-\nu^O \) decreases strictly (see the remark below).
Remark 2.17. As we explained before, an irreducible components \( Y' \) in \( \maxv{O}(X') \) inherits the label of a component \( D \) in \( \maxv{O}(X) \) if \( Y' \) dominates \( D \). In order to obtain this situation, we must blow up an entire irreducible component of \( \maxv{O}(X) \).

Set \( Y_0 := \maxv{O}(X) \subset \mathbb{Z} \). During the blowing ups in (2.4), where we prepare \( Y_0 \) to become \( B \)-weakly permissible, the centers are strictly contained in some irreducible components. Therefore, even if a new irreducible component \( Y' \) in \( \maxv{O}(X') \) dominates the center, it never dominates a whole irreducible component of \( Y_0 = \maxv{O}(X) \). Thus we assign to \( Y' \) a new label.

An irreducible component \( Y' \subset \maxv{O}(X') \) lying above a vertical component of \( D \subset \maxv{O}(X) \) is always vertical. Hence, once \( \maxv{O}_{\ref{end })(X) \text{ is empty, it remains so until } \maxv{O}(X') < \maxv{O}(X) \) decreases. In other words, in our variant of the [16]-algorithm, we first handle the horizontal components and whenever no horizontal component exists, we consider the vertical ones.

3. The locus of refined maximal order

Let \( X \subset \mathbb{Z} \) be as before. In this section we discuss a method to compute the maximal order locus \( \maxord(X) \) of \( X \) from a theoretical viewpoint. We deduce how to construct the maximal refined order locus \( \maxr{O}(X) \), from which we obtain its log-variant \( \maxv{O}(X) \) by applying Construction 2.11. Within this, we may restrict ourselves to a finite affine covering \( \bigcup_{i \in I} U_i \) of \( Z \). The respective loci are then obtained by gluing together the relevant affine pieces. Note, however, that the maximal value of ord, \( \nu_{\ref{end })(X \cap U_i \text{ is not necessarily a global maximum for } X \}. \)

We pass to the local situation in a fixed \( U_i \): Let \( Z = \Spec(A[x]) = \mathbb{A}^m_\mathbb{Z} \), where \( A \) is a field or a principal ideal Dedekind domain, and \( \langle x \rangle := \langle x_1, \ldots, x_m \rangle \). Let \( X = V(J) \subset \mathbb{Z} \) be the affine variety defined by a non-zero ideal \( J \subset \mathbb{Z} \). We introduce new variables \( (X) = (T_1, \ldots, T_m) \) and consider

\[
F(X + T) = F(X_1 + T_1, \ldots, X_m + T_m) = \sum_{\mathbf{a} \in \mathbb{Z}^m_{\geq 0}} F_{\mathbf{a}}(X) T^\mathbf{a}.
\]

The Hasse-Schmidt derivative of \( F \) by \( X^\mathbf{a} \) is defined by the coefficient of \( T^\mathbf{a} \) in the previous expansion,

\[
\frac{\partial}{\partial X^\mathbf{a}} F(X) = F_{\mathbf{a}}(X).
\]

For any \( \mathbf{a} = (a_1, \ldots, a_m), \mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}^m_{\geq 0} \) and \( \lambda \in \mathbb{Q} \), we have

\[
\frac{\partial}{\partial X^\mathbf{a}} \lambda X^\mathbf{b} = \lambda \binom{\mathbf{b}}{\mathbf{a}} X^{\mathbf{b} - \mathbf{a}},
\]

where \( \binom{\mathbf{b}}{\mathbf{a}} = \binom{b_1}{a_1} \cdots \binom{b_m}{a_m} \) and \( \binom{b}{a} = 0 \) if \( b < a \). In particular, \( \partial X^\mathbf{a} = 1 \).

Let \( Y \) be one of the variables and \( a, b \in \mathbb{Z}_{\geq 0} \). Since \( \binom{b}{a} \in \mathbb{Z} \), we can relate the Hasse-Schmidt derivatives to usual derivatives via the following symbolic computations:

\[
\frac{\partial}{\partial Y^a} Y^b = \binom{b}{a} Y^{b-a} = \frac{b!}{a!(b-a)!} Y^{b-a} = \frac{1}{a!} \left( \frac{\partial}{\partial Y} \right)^a Y^b.
\]

More generally, we have, for \( \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0} \), that

\[
\frac{\partial}{\partial X^\mathbf{a}} = \frac{1}{a_1! \cdots a_m!} \left( \frac{\partial}{\partial X_1} \right)^{a_1} \cdots \left( \frac{\partial}{\partial X_m} \right)^{a_m}.
\]

In fact, the above construction is valid if we replace the field \( k \) by any domain \( R \). For \( B = A[x] \), the usual derivatives by the variables \( x_i \) and the Hasse-Schmidt derivatives generate the \( B \)-module \( \text{Diff}_A(B) \) of \( A \)-linear differential operators.
Using Hasse-Schmidt derivatives, we can recall a first connection between the order and differential operators:

**Lemma 3.1.** Let $R$ be a domain and $S = R[\overline{z}]$, for $(\overline{z}) = (x_1, \ldots, x_m)$. Set $I_0 = (\overline{z}) \subset R$. We have that, for every $f \in R \setminus \{0\}$,

$$\text{ord}_{I_0}(f) = \min \left\{ i \in \mathbb{N} | \exists a \in \mathbb{N}^m : |a| = i \land \frac{\partial}{\partial x^a} f \notin I_0 \right\}.$$

**Proof.** Recall that $\text{ord}_{I_0}(f) = \sup \{ i \in \mathbb{N} | f \in I_0^i \}$. Set $d := \text{ord}_{I_0}(f) < \infty$. Since $B$ is a polynomial ring over $A$, we have that $f = \sum \lambda_a \overline{z}^a$, for $a \in \mathbb{N}^m$ and some $\lambda_a \in R$ of which only finitely many are non-zero. By definition of the order at $I_0 = (\overline{z})$,

$$d = \min \{ i \in \mathbb{N} | \exists a \in \mathbb{N}^m : |a| = i \land \lambda_a \neq 0 \}.$$

Let $a_0 \in \mathbb{N}^m$ be such that $|a_0| = d$ and $\lambda_{a_0} \neq 0$. We have that

$$\frac{\partial}{\partial x^a} f = \lambda_{a_0} + \sum_{b \neq 0} \lambda_{a_0 + b} \left( \frac{a_0 + b}{a_0} \right) \overline{z}^b \notin I_0.$$

By the minimality of $d$, the assertion follows. \qed

At a closed point, e.g., given by the maximal ideal $(p, x_1, \ldots, x_m)$, for some prime element $p \in A$, the connection with differential operators is best seen in the associated graded ring, where $p$ corresponds to some variable, say $P$.

**Example 3.2.** Consider the hypersurface $X := V(f) \subset \mathbb{A}^2 =: Z$ given by the polynomial

$$f = 12 - uv^2 \in \mathbb{Z}[u, v].$$

Since $12 = 3 \cdot (1 + 3)$ the order of $f$ at the maximal ideal $\mathfrak{m} := (3, u, v)$ is one. Therefore $X$ is not singular at the point corresponding to $\mathfrak{m}$, although $X$ is singular in the fiber modulo 3. The initial form of $f$ with respect to $\mathfrak{m}$ is

$$\text{in}_{\mathfrak{m}}(f) = P \in k_m[P, U, V],$$

where $P := 3 \mod \mathfrak{m}^2$, $U := u \mod \mathfrak{m}^2$, and $V := v \mod \mathfrak{m}^2$, and $k_m = \mathbb{Z}/3$. As we can see the derivative of $\text{in}_{\mathfrak{m}}(f)$ by $P$ is non-zero.

On the other hand, $12 = 2^2 \cdot (1 + 2)$ and at the point corresponding to the maximal ideal $\mathfrak{n} := (2, u, v)$, we have $\text{ord}_{\mathfrak{n}}(f) = 2$. The initial form is

$$F := \text{in}_{\mathfrak{n}}(f) = Q^2 \in k_n[Q, U, V],$$

where $Q = 2 \mod \mathfrak{n}^2$ ($U, V$ analogous to above) and $k_n = \mathbb{Z}/2$. The derivative by $Q$ is $2Q = 0$ in the graded ring. (We also introduced the letter $Q$ for 2 mod $\mathfrak{n}^2$ in order to avoid confusion between the powers of the element 2 mod $\mathfrak{n}^2$ and a factor 2 = 0 in $k_n$). Of course, we have $\frac{\partial}{\partial Q}(\frac{\partial}{\partial Q} F) = 0 \notin \mathfrak{n}$, but we have that $\frac{\partial}{\partial Q} Q^2 = 1 \notin \mathfrak{n}$. This shows the connection between the order of $f$ at $\mathfrak{n}$ and differential operators.

Nonetheless, we need also so called *derivatives by constants*, i.e., with respect to elements in the field itself, in order to compute the locus of maximal order. This is related to the notion of p-bases (see [39], 0TV (21.1.9), or [27] section 2.2).

**Definition 3.3.** Let $S$ be a ring containing a field of characteristic $p$ and $e \in \mathbb{N}$, $e > 0$. A family $(s_i)_{i \in I}$ in $S$ is called $p^e$-independent (resp. a $p^e$-basis) of $S$ if the family of monomials $\overline{x}^a$ with $a \in \mathbb{N}^I$, $0 \leq a_i < p^e$ (i \in I), is a free family (resp. a basis) for the $S^{p^e}$-module $S$.

Note that the family $(s_i)_{i \in I}$ is not necessarily finite. If the Frobenius $F : S \to S$, $b \mapsto b^p$, is injective (e.g., if $S$ is a domain) and $(s_i)_{i \in I}$ is a $p$-basis, then the monomials $\overline{x}^a$ with $0 \leq a_i < p^e$ form a basis for $B$ considered as $B^{p^e}$-module ([27] Remark (2.2.3)).

If $S = k[X_1, \ldots, X_m]$ is a polynomial ring over a field $k$ and if $(\lambda_i)_{i \in I}$ is a $p^e$-basis for $k$, then the family $(\lambda_i, X_j)_{i \in I, j \in \{1, \ldots, m\}}$ is a $p^e$-basis for $S$ ([27] Remark (2.2.7)).

Two crucial results are
Proposition 3.4 ([27] Proposition (2.2.5)). Let $S$ be a ring containing a field of characteristic $p$. Let $(s_i)_{i \in I}$ be a $p$-basis of $S$ and suppose that the Frobenius $F : S \to S$ is injective. For every multiindex $a \in \mathbb{N}^{(I)}$, there exists $\mathcal{D}_a \in \text{Diff}_F^m(S)$ with

$$\mathcal{D}_a(x^b) = \left( \begin{array}{c} b \\ a \end{array} \right) x^{b-a},$$

for all $b \in \mathbb{N}^{(I)}$. Furthermore, every $\mathcal{D} \in \text{Diff}_F^m(S)$ is a (possibly infinite) sum

$$\mathcal{D} = \sum_{|a| \leq m} c_a \mathcal{D}_a$$

with unique coefficients $c_a \in S$.

Lemma 3.5 ([36] Lemma 1.2.3, p. III-4 (cf. [27] Lemma (2.3.1))). Let $R$ be a ring, $S$ be an $R$-algebra, and $\mathcal{D} \in \text{Diff}_F^m(S)$. For an ideal $I \subseteq S$ and $\ell \geq i$, we have

$$\mathcal{D}(I^\ell) \subseteq I^{\ell-i}.$$ 

Example 3.6. Let $S = k[X, Y]$, where $k$ is a non-perfect field of characteristic two. Consider $F = X^2 - \lambda Y^2$ with $\lambda \in k \setminus k^2$. The element $\lambda$ can be extended to a 2-basis for $k$. We fix a 2-basis $(\lambda_i)_{i \in I}$ of $k$ containing $\lambda$. Then we can speak about the derivative $\frac{\partial}{\partial \lambda}$.

The maximal order of $F$ is 2 and we have

$$\frac{\partial}{\partial \lambda}(X^2 - \lambda Y^2) = Y^2.$$ 

Since all other derivatives are zero, this implies that the locus of maximal order is the closed point $V(X, Y)$.

We now come to the computation of the maximal order locus.

Construction 3.7. Let $J \subset B = A[\mathfrak{a}]$ be a non-zero ideal, where $A$ is a field or a principal ideal Dedekind domain. Set $X := V(J)$. Let

$$d := \max\{\text{ord}_s(X) \mid x \in X\} \geq 1.$$ 

For $d = 1$, the locus of maximal order is $X$. Hence let us suppose $d > 1$.

If $B$ contains a field $k$, $k \subseteq B$, then, using Proposition 3.4 and Lemma 3.5, we get that the maximal order locus for $X = V(J)$ is given by

$$\Delta^{d-1}(J) := V(\mathcal{D}f \mid f \in J, \mathcal{D} \in \text{Diff}_F^{d-1}(B)).$$

Suppose $B$ does not contain a field (i.e., $A$ is of mixed characteristic). Let $F := \text{Frac}(A)$ be the field of fractions of $A$ and set $BF := B \otimes_A F = F[\mathfrak{a}]$. If the maximal order of $V(J \cdot BF)$ is equal to $d$, then $\Delta^{d-1}(J \cdot BF)$ provides the horizontal irreducible components of the locus of maximal order of $X = V(J)$.

For the vertical components, let $p \in A$ be a prime element. We pass from $B = A[\mathfrak{a}]$ to $B_p := B \otimes_A \text{gr}_p(A) = (\text{gr}_p(A))[\mathfrak{a}]$. An element $\sum f_a \mathfrak{a}^a \in B$ is mapped to $\sum F_a \mathfrak{a}^a \in B_p$, where $F_a = F_a(P) = \text{in}_p(f_a) \in \text{gr}_p(A) \cong k_p[P]$ with $P := \text{in}_p(p) = p \mod p^2$ and $k_p := A/p$ the residue field. Note that $B' \cong k_p[P, \mathfrak{a}]$ is a polynomial ring over the field $k_p$. If the maximal order of $V(J \cdot B_p)$ is equal to $d$, then $\Delta^{d-1}(J \cdot B_p)$ provides the vertical irreducible components of the locus of maximal order of $X = V(J)$ with $I \cap A = (p)_A$.

Applying this procedure for every prime $p \in A$, we eventually obtain $\text{Max-ord}(X)$.

Remark 3.8. (1) Since the max order locus has only finitely many irreducible components, there are only finitely many primes that need to be considered. An essential point in implementing the resolution algorithm is to detect these "interesting primes". In the next section we address this issue for $A = \mathbb{Z}$.

(2) The case $A = \mathbb{Z}$ also explains why we need to pass to $\text{gr}_p(A)$. Let $p \in \mathbb{Z}$ be any prime number, e.g., $p = 2$. There cannot exist a derivation by 2 since every derivation is $\mathbb{Z}$-linear. Assume there exists $\mathcal{D} \in \text{Diff}_F(B)$ being the derivation by 2. Then $\mathcal{D}$ certainly has to fulfill $\mathcal{D}(2) = 1$. But the $\mathbb{Z}$-linearity implies

$$1 = \mathcal{D}(2) = 2\mathcal{D}(1) = 2 \cdot 0 = 0.$$

Contradiction.
Using Construction 3.7, we can describe how to determine the locus of maximal refined order \( \text{Max-}\nu(X) \) of \( X \). As before, we reduce to the affine case and then glue the components.

**Construction 3.9.** Let \( J \subset B = \mathbb{A}[f] \) be a non-zero ideal, where \( A \) is a field or a principal ideal Dedekind domain. Set \( X = V(J) \). Suppose
\[
\text{max-}\nu(X) = (\alpha, \delta) \in \mathbb{N}^2.
\]

We provide an inductive construction of \( \text{Max-}\nu(X) \) depending on
\[
a := a(Z) := N - \alpha = \dim(Z) - \alpha \geq 0 \quad \text{(recall Definition 2.4)}.
\]
If \( a = 0 \), then \( \text{Max-}\nu(X) \) coincides with the locus of maximal order \( \text{Max-ord}(X) \) and we may apply Construction 3.7.

Suppose \( a(Z) \geq 1 \). Then, \( J \) is of order one at every point of \( X \). In this case, it is possible to descend in the dimension of the ambient space locally at every point. Hence, for every point \( q \) in \( \text{Spec}(B) \), there exists a differential operator \( \partial = \partial(q) \) such that \( (\partial J) \cdot B_{t_q} = B_{t_q} \), where \( \partial J = \langle \partial h \mid h \in J \rangle \) and \( B_{t_q} \) denotes the localization of \( B \) at the ideal of \( q \).

Let \( (f_1, \ldots, f_r) \) be a set of generators for \( J \). The latter implies that we find finitely many \((i(1), j(1)), \ldots, (i(\tau), j(\tau))\) such that
\[
X \subset \bigcup_{\ell=1}^{\tau} D\left(\frac{\partial f_{i(\ell)}}{\partial x_{j(\ell)}}\right).
\]
This provides a (refined) open covering of \( X \) and in each \( U_\ell := D\left(\frac{\partial f_{i(\ell)}}{\partial x_{j(\ell)}}\right) \), we have that
\[
X_\ell := X \cap U_\ell \subset V(f_{j(\ell)}) \cap U_\ell =: Z_\ell.
\]
Since \( \dim(Z_\ell) = \dim(Z) - 1 \), we have \( a(Z_\ell) = a(Z) - 1 \) (since \( \nu_{\text{ref}} \) and hence \( \text{max-}\nu(X) \) does not depend on the embedding). By induction, we know how to determine \( \text{Max-}\nu(X_\ell) \) and \( \text{max-}\nu(X_\ell) \). If \( \text{max-}\nu(X_\ell) = \text{max-}\nu(X) \), then \( \text{Max-}\nu(X_\ell) \) contributes to \( \text{Max-}\nu(X) \). Otherwise, we can discard the chart \( U_\ell \). By gluing together the relevant affine pieces, we obtain then \( \text{Max-}\nu(X) \).

Finally, if \( \text{max-}\nu^{O}(X) = (\alpha, \delta, \sigma) \in \mathbb{N}^3 \), we can apply Construction 2.11 to obtain from \( \text{Max-}\nu(X) \) the locus of maximal log-refined order \( \text{Max-}\nu^{O}(X) \).

**Observation 3.10.** For practical reason, we like to point out some facts about computing the maximal locus after blowing up: Let \( X \subset Z \) be as before. Suppose \( \text{Max}(X) \subset X \) is a locus that we may compute from \( X \) (e.g., \( \text{Max}(X) = \text{Max-}\nu^{O}(X) \), or \( \text{Max}(X) \) being the (maximal) Hilbert-Samuel locus of \( X \); cf. [16], Definition 1.35, p.27).

Let \( \pi : Z' = \text{Bl}_D(Z) \to Z \) be the blowing up in some regular center \( D \subset \text{Max}(X) \) and denote by \( X' \) the strict transform of \( X \) under \( \pi \). Set \( E := \pi^{-1}(D) \), the exceptional divisor of the blowing up. The locus \( \text{Max}(X') \) consists of two types of irreducible components:

1. The strict transform \( Y' \) of \( Y := \text{Max}(X) \) is contained in \( \text{Max}(X') \).
2. We may have created new components. But we know that each of them has to be contained in the exceptional divisor \( E \).

Since we computed \( Y \) before, we can simply determine its strict transform under \( \pi \) in order to obtain the components of \( \text{Max}(X') \) of type (1). Therefore it is only left to determine the new components, i.e., those of type (2). The fact that these are contained in \( E \) then helps to simplify the local computations.

4. **Algorithmic formulation and examples**

In the previous sections the theoretical side of the desingularization algorithm by Cossart-Jannsen-Saito has been outlined to give us a solid basis for implementation aspects. We have seen that the main loop is a sequence of blowing ups of the arithmetic surface at suitably chosen centers.

From the computational point of view, it is standard to represent the given schemes by means of a covering with affine charts, i.e., as ideals in a polynomial ring. In each of these
charts, we then encounter a situation \( I_Z \subset I_X \subset I_Y \subset R[\mathbf{X}] \) at the beginning and with the boundary \( B \) on \( Z \) represented by \( I_Z \subset I_B \subset R[\mathbf{X}] \) in later steps of the construction.

In general, the underlying base ring \( R \) can be any Dedekind domain or field, as long as the arithmetic operations are practically computable; for the purpose of this section, however, we shall concentrate on the case of \( R = \mathbb{Z} \). As the choice of centers is controlled by the locus of maximal value of \( \nu_{\text{ref}}^O \), intersections of charts and patching are not an issue in this process. Moreover, blowing up at a given regular center is a well-understood standard technique, see e.g. [26] or [29]. We can hence focus completely on the practical aspects of the choice of centers in this section.

**Remark 4.1.** As we have seen before, the key ingredient for the choice of the center is the computation of \( \text{Max-}O^X(X) \) which was discussed from the theoretical point of view in the last section. More precisely, the center is constructed as follows:

**Step 1:** If the maximal order of \( X \) is one, the variety can locally be embedded into a smooth hypersurface and it thus suffices to find an open covering such that the use of a single such hypersurface on an open set is possible. The equation of this hypersurface can then be added to the generators of the ideal of the ambient space \( Z \). Iterating this process, we descend in ambient space as long as we have not reached to minimal appearing \( a_x \) at any point \( x \in X \). At these points the first entry of the invariant \( N - a_x \) attains a maximal value. (Cf. Constructions 3.7 and 3.9).

**Step 2:** Again computing the locus of maximal order of \( X \) with the new ambient space \( Z \) resulting from step 1, we reach the locus of maximal refined order, i.e., the points where \( \nu_{\text{ref}}(x) = (N - a_x, d_x) \) has maximal value.

**Step 3:** By taking into account the exceptional divisors, we obtain the locus of maximal log-refined order (cf. Construction 2.11).

**Step 4:** We label the irreducible components of \( \text{Max-}O^X(X) \) using the history of the resolution process and detect the locus of components which have smallest label (cf. Construction 2.16 and the remark after it). Since \( X \) is reduced and of dimension two, the latter components have at most dimension one. Hence, we either blow up the entire smallest label locus, or we have to prepare it to become weakly permissible by blowing up closed points.

Step 1 and Step 2 are standard techniques that already appear over fields of characteristic zero (cf. [12]). Moreover, they have been realized in implementations (see [8, 31]) and it requires at most minor modifications to adapt the algorithms to our situation. No special features of positive or mixed characteristic appear in these steps. The same is true for the labeling process in Step 4 which is of combinatorial nature and only requires some diligent book keeping.

Therefore, we completely focus on the algorithm to determine the locus of maximal order in the arithmetic setting, see Algorithm 3. The approach presented in the previous section is purely theoretical as it requires local computations at each point. The analogous problem already occurs in the by far simpler case of desingularization over a field of characteristic zero and we mimic the approach taken there. To this end, we first state the algorithm in characteristic zero before developing a similar approach in the arithmetic case, see Algorithm 1: MaxOrd (char \( K = 0 \)) (p.15).

Note that we only require that \( Z \) is equidimensional and not necessarily irreducible. Since \( Z \) is regular, all irreducible components are disjoint. From a theoretical viewpoint, it suffices to solve the resolution problem on each component separately and hence one may reduce to the case \( Z \) irreducible. In praxis, we may have to deal with \( Z \) having several irreducible components if we pass to a smaller dimensional ambient space as in Construction 3.9.

**Remark 4.2** ([11] Remark 3.3). As the minor \( \det(M) \) is only invertible on \( D(\det(M)) \) (from step 12 of Algorithm 1 onwards), the differentiation in step 16 is more subtle than it seems at first glance. We start by determining a square matrix \( A \) satisfying

\[
A \cdot M = q \cdot E_{\text{codim}(Z)},
\]

where \( q = \det(M) \). On \( D(q) \), \( \frac{1}{q} \cdot A \) is precisely the inverse matrix of \( M \). As system of parameters \( y_1, \ldots, y_s \), we use the one induced by \( \{ x_i \mid i \text{ not a column in } M \} \). Let
Algorithm 1 MaxOrd (char $K = 0$)

Input: $I_Z \subseteq \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[x]$ such that $Z := V(I_Z)$ is equidimensional and regular, $I_X \subseteq \mathbb{C}[x]$ such that $I_Z \subset I_X$, i.e., $\emptyset \neq X := V(I_X) \subset Z$

Output: $(m, I)$, where

\[ m = \text{max-ord}(X) \]

$I$ ideal describing Max-ord($X$)

1. $I_{\text{temp}} = I_X$, $I_{\text{max}} = \{1\}$, $\text{maxord} = 0$
2. if $I_Z = \emptyset$ then
3. while $I_{\text{temp}} \neq \emptyset$ do
4. $I_{\text{max}} = I_{\text{temp}}$
5. $I_{\text{temp}} = I_{\text{temp}} + \langle \frac{\partial f_i}{\partial y_j} \mid 1 \leq j \leq n \text{ and } 1 \leq i \leq r \rangle$, where $f_1, \ldots, f_r$ generate $I_{\text{temp}}$
6. $\text{maxord} = \text{maxord} + 1$
7. return (maxord, $I_{\text{max}}$)
8. $L = \{ \text{codim}(Z) \text{ square submatrices of Jacobian matrix of } I_Z \}$
9. choose a subset $L_1 \subset L$ such that $X \subset \bigcup_{M \in L_1} D(\det(M))$
10. $\text{maxord} = 1$, $\text{thisord} = 0$, $I_{\text{max}} = \emptyset$
11. for $M \in L_1$ do
12. denote by $y_1, \ldots, y_s$ the system of parameters on $Z \cap D(\det(M))$ induced by variables not corresponding to columns of $M$
13. $I_{\text{temp}} = I_X$
14. while $I_{\text{temp}} + I_Z \neq \emptyset$ do
15. $I_{\text{old}} = I_{\text{temp}}$
16. $I_{\text{temp}} = I_{\text{temp}} + \langle \frac{\partial f_i}{\partial y_j} \mid 1 \leq j \leq s \text{ and } 1 \leq i \leq r \rangle$, where $f_1, \ldots, f_r \in \mathbb{C}[x]$ correspond to generators of $I_{\text{temp}} \cdot \mathbb{C}[x]/I_Z$
17. $I_{\text{temp}} = \text{sat}(I_{\text{temp}}, \det(M))$
18. $\text{thisord} = \text{thisord} + 1$
19. if $\text{thisord} \geq \text{maxord}$ then
20. if $\text{thisord} = \text{maxord}$ then
21. $I_{\text{max}} = I_{\text{max}} \cap I_{\text{old}}$
22. else
23. $\text{maxord} = \text{thisord}$, $I_{\text{max}} = I_{\text{old}}$
24. $\text{thisord} = 0$
25. return (maxord, $I_{\text{max}}$)

$q_1, \ldots, q_s \in \mathbb{C}[x]$ be a set of generators for $I_Z$. Further, we choose a set of generators $f_1, \ldots, f_r \in \mathbb{C}[x]/I_Z$ for the ideal $I_X \cdot \mathbb{C}[x]/I_Z$ and choose representatives $f_1, \ldots, f_r \in \mathbb{C}[x]$ for these. For simplicity of presentation, we assume in the next formula that $M$ involves precisely the last columns of the Jacobian matrix so that the indices of $y_i$ and the corresponding $x_i$ coincide. Then the chain rule provides the following derivatives:

\[ q \frac{\partial f_i}{\partial y_j} = q \cdot \frac{\partial f_i}{\partial x_j} - \sum_{k \text{ column of } M} \frac{\partial q}{\partial x_j} A_{ik} \frac{\partial f_i}{\partial x_k} \mod I_Z. \]

To discard the extra factor $q$ or more geometrically all components inside $V(q)$, we then need to saturate the resulting ideal with $(q)$.

In the arithmetic setting, however, we also need to consider derivatives with respect to a prime, as we have seen in the previous section. More precisely, we first need to determine the primes above which we might find components of Max-ord($X$) (resp. Max-$\nu^O(X)$). The following example shows that even in rather trivial cases, it is not feasible to read off the relevant primes from a given set of generators:

Example 4.3. Consider the ideals $I_Z = \langle g \rangle \subset \langle y, f \rangle = I_X \subset \mathbb{Z}[x, y, z]$ where $f = 3x - y + 7z$ and $g = x - 4y + 6z$. If only the primes appearing in the generators are considered, it may seem that at most the primes 2, 3 and 7 are relevant. However, passing to a different
second generator of $I_X$ it becomes apparent that the locus of order at least 2 actually has a contribution in the fibre above 11, as $f - 3g = (3x - y + 7z) - 3 \cdot (x - 4y + 6z) = 11 \cdot (y - z)$, which is certainly of order 2 locally at points like for example $V(x, y, z, 11)$. It is easy to observe, but not relevant for the example that 2 is also the highest appearing order.

To systematically determine the primes relevant for the locus of maximal order, we now try to mimic the previous algorithm which had been formulated in the case of a field of characteristic zero. Over $\mathbb{Z}$, we cannot expect to obtain the correct locus from this construction -- not even using Hasse-Schmidt derivatives -- as we are still missing the ‘derivative’ with respect to the prime. On the other hand, the algorithm for computing Max-ord($X$) is blind to vertical components of it (cf. Definition 2.13) as long as there are still horizontal components of this locus. The latter behaviour is not a flaw, but in tune with the fact that we have given horizontal components precedence over vertical ones in the description of the resolution algorithms in the previous section.

In an ad-hoc notation, we shall call the primes arising from the algorithm interesting as the bad primes will eventually appear among those in the course of the resolution, but not all of them arising primes need to be bad. In Algorithm 2: InterestingPrimes (p.16), we present an algorithm that detects the set of of interesting primes.

**Algorithm 2 InterestingPrimes**

**Input:** $I_Z \subset \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}[x]$ such that $Z := V(I_Z)$ is equidimensional and regular, $I_X \subset \mathbb{Z}[x]$ such that $I_Z \subset I_X$, i.e., $\emptyset \neq X := V(I_X) \subset Z$

**Output:** $S = \{p_1, \ldots, p_s\}$ interesting primes for $I_X \supset I_Z$

1: resultlist = $\emptyset$

2: if $(I_Z \cap \mathbb{Z})! = \langle 0 \rangle$ then

3: resultlist = primefactors(generator of principal ideal $I_Z \cap \mathbb{Z}$) (as set)

4: return resultlist

5: $I_{\text{temp}} = I_X, I_{\text{int}} = \langle 0 \rangle$

6: if $I_Z = \langle 0 \rangle$ then

7: while $I_{\text{int}} = \langle 0 \rangle$ do

8: $I_{\text{temp}} = I_{\text{temp}} + (\frac{\partial f_i}{\partial x_j} | 1 \leq j \leq n \text{ and } 1 \leq i \leq r)$, where $f_1, \ldots, f_r$ generate $I_{\text{temp}}$

9: $I_{\text{int}} = I_{\text{temp}} \cap \mathbb{Z}$

10: resultlist = primefactors(generator of principal ideal $I_{\text{int}}$) (as set)

11: return resultlist

12: $L = \{\text{codim}(Z) \text{ square submatrices of Jacobian matrix of } I_Z\}$

13: choose a subset $L_1 \subset L$ such that $X \subset \bigcup_{M \in L_1} D(\text{det}(M))$

14: for $M \in L_1$ do

15: denote by $y_1, \ldots, y_s$ the system of parameters on $Z \cap D(\text{det}(M))$ induced by variables not corresponding to columns of $M$

16: $I_{\text{temp}} = I_X, I_{\text{int}} = \langle 0 \rangle$

17: while $I_{\text{int}} = \langle 0 \rangle$ do

18: $I_{\text{temp}} = I_{\text{temp}} + (\frac{\partial f_i}{\partial y_j} | 1 \leq j \leq s \text{ and } 1 \leq i \leq r)$, where $f_1, \ldots, f_r \in \mathbb{Z}[x]$ correspond to generators of $I_{\text{temp}} \cdot \mathbb{Z}[x]/I_Z$

19: $I_{\text{int}} = (I_{\text{temp}} + I_Z) \cap \mathbb{Z}$

20: resultlist = resultlist $\cup$ primefactors(generator of principal ideal $I_{\text{int}}$) (as sets)

21: return resultlist

The reader observes that in Algorithm 1 the criterion for the end of the while loop is $I_{\text{temp}} + I_Z = \langle 1 \rangle$, while here it is $I_{\text{int}} \neq \langle 0 \rangle$. Recall that $I_{\text{int}} = (I_{\text{temp}} + I_Z) \cap \mathbb{Z}$. In the situation of Algorithm 1, i.e., over a field of characteristic zero, $I_{\text{int}} \neq \langle 0 \rangle$ is equivalent to $I_{\text{temp}} + I_Z = \langle 1 \rangle$.

Following our variant of the Cossart-Jannsen-Saito algorithm (Construction 2.16), all primes are good if there are horizontal components in maximal log-refined order locus. We adapt this for Max-ord($X$), i.e., all primes are good if the latter has horizontal components.
we continue the process: Algorithm, but without changing the output. Suppose the while loop in line 3 (resp. 13) of irreducible component, say \( W \). Since there are no horizontal components in Max-ord(\( X \)), we have that for every irreducible component, say \( W \subseteq \mathbb{Z}[x] \). Assume that the maximal order locus of \( X \) has only vertical components, i.e., Max-ord(\( X \)) hor = \( \emptyset \). Then the set of interesting primes \( S = \{p_1, \ldots, p_n\} \) resulting from Algorithm 2 contains all bad primes. 

**Proof.** Since there are no horizontal components in Max-ord(\( X \)), we have that for every irreducible component, say \( W \subseteq \mathbb{Z}[x] \), there exists a prime \( p = p(W) \in \mathbb{Z} \) such that \( W \subseteq V(p) \). In other words, \( p \) can be chosen to be part of a set of generators for the ideal of \( W \). In particular, \( p \) is a bad prime.

Our goal is to explain how Algorithm 2 collects \( p \) in \( S \). For this, we slightly extend the algorithm, but without changing the output. Suppose the while loop in line 3 (resp. 13) of Algorithm 2 stops. Then a generator of the principal ideal \( I_{\text{int}} \subseteq \mathbb{Z} \) provides \( S \). Nonetheless, we continue the process:

\[
\begin{align*}
I_{\text{temp}}^n &= I_{\text{temp}}^{old} + \left( \frac{\partial f_i}{\partial x_j} \right) | 1 \leq j \leq n \text{ and } 1 \leq i \leq r \\
I_{\text{int}}^n &= (I_{\text{temp}}^{new} + I_Z) \cap \mathbb{Z} \\
S_{\text{new}} &= S_{\text{old}} \cup \text{primefactors}(\text{generator of principal ideal } I_{\text{int}}^{new}) \text{ (as sets)}
\end{align*}
\]

with \( f_1, \ldots, f_r \in \mathbb{Z}[x] \) corresponding to generators of \( I_{\text{temp}}^{old} \cdot \mathbb{Z}[x]/I_Z \). If we have that \( I_{\text{temp}}^{new} \not\subseteq I_{\text{temp}}^{old} \), we repeat. After finitely many steps, this procedure stabilizes, that means, we will have \( I_{\text{temp}}^{new} \subseteq I_{\text{temp}}^{old} \). We first claim that

\[
S_{\text{final}} = S.
\]

This follows since \( I_{\text{temp}}^{old} \subseteq I_{\text{temp}}^{new} \) and \( (a, b) = (\gcd(a, b)) \subseteq \mathbb{Z} \), for \( a, b \in \mathbb{Z} \). (Take \( a \) to be a \( \text{generator for } I_{\text{int}}^{old} \) and \( b \) to be one for \( I_{\text{int}}^{new} \)).

Set \( d := \text{max-ord}(X) \). Let \( (g_1, \ldots, g_t, f_1, \ldots, f_r) \) be a set of generators for \( I_X \) such that \( I_Z = (g_1, \ldots, g_t) \) and that the images of \( (f_1, \ldots, f_r) \) in \( \mathbb{Z}[x]/I_Z \) generate \( I_X \cdot \mathbb{Z}[x]/I_Z \). Since \( d = \text{max-ord}(X) \) and \( W \subseteq \text{Max-ord}(X) \), there exists an element \( f_* \in \{f_1, \ldots, f_r\} \) such that

\[
f_* \in I_{W}^n \setminus I_{W}^{d+1}
\]

By assumption, \( f_* \) is in normal form\(^5\) with respect to \( (g_1, \ldots, g_t) \). Since \( p \in I_W \) and Max-ord(\( X \)) hor = \( \emptyset \), we may choose \( f_* \) such that a monomial of the form

\[
p^b x^a \quad \text{with } |a| < d \text{ and } b + |a| \geq d,
\]

for \( a = (a_1, \ldots, a_n) \), appears with non-zero coefficient. We choose \( f_* \) and \( a \) such that \( |a| \) is minimal. At least, after applying \( a \)-times the derivatives by \( x_i \) to \( f_* \), we obtain an element of the form

\[
A p^b + H \in I_{\text{temp}}^{new} \quad \text{for some } H \in (x) \text{ and } A \in \mathbb{Z}_+.
\]

Recall that we modified the process and continue until we have \( I_{\text{temp}}^{old} \subseteq I_{\text{temp}}^{new} \) and that this did not change the resulting set \( S \). This justifies that we can apply all these derivatives.

If \( H = 0 \), we are done. Suppose this is not the case. By continuing the process, we can eliminate step by step the respective elements in \( H \) of largest degree. Eventually, we obtain \( H = 0 \) and hence \( p \in S \).

Note that it is important \( f_* \) is in normal form with respect to \( I_Z \):

**Example 4.5.** Let \( I_Z = (u - x^m) \subseteq I_X = (f) + I_Z \subseteq \mathbb{Z}[u, x, y, z] \), for some \( m \in \mathbb{Z}_{\geq 2} \) and \( f = \tau^3 u - y^3 + xz^3 \).

The Jacobian matrix of \( I_Z \) is \( (1, 0, 0, 0) \). Hence, the only possible submatrix is \( M = 1 \) which corresponds to the variable \( u \). The system of parameters on \( Z \cap D(\det(M)) = \mathbb{Z} \) induced by the variables not corresponding to the columns of \( M \) is \( (x, y, z) \).

---

\(^5\)Again in the sense of standard bases and Mora’s algorithm.
The algorithm puts $I_{\text{temp}} = \langle f \rangle$, and adds derivatives with respect to $(x, y, z)$ to the generators until $I_{\text{int}} = (I_{\text{temp}} + I_Z) \cap \mathbb{Z}$ is non-zero. That means, after the first step, we add
\[
\frac{\partial f}{\partial x} = z^3, \quad \frac{\partial f}{\partial y} = 5y^4, \quad \frac{\partial f}{\partial z} = 3xz^2. \quad \text{and } I_{\text{temp}}^{(1)} = (7^9 u - y^5, z^3, 5y^4, 3xz^2).
\]
Since $\gamma_{\text{int}}^{(1)} = (0)$, we repeat and obtain
\[
I_{\text{temp}}^{(2)} = (7^9 u - y^5, z^3, 3z^2, 5y^4, 4 \cdot 5y^3, 2 \cdot 3xz) \quad \text{and } I_{\text{int}}^{(2)} = (0).
\]
Next, we get
\[
I_{\text{temp}}^{(3)} = (7^9 u - y^5, z^3, 3z^2, 2 \cdot 3z, 5y^4, 4 \cdot 5y^3, 3 \cdot 4 \cdot 5y^2, 2 \cdot 3x) \quad \text{and } I_{\text{int}}^{(3)} = (0).
\]
Eventually, $I_{\text{int}}^{(4)} = (2 \cdot 3)$ and the detected set of primes is $\{2, 3\}$.

On the other hand, we can use instead of $f$ the following element as part of the generators:
\[
f_* := f - 7^9(u - x^m) = 7^9x^m - y^5 + xz^3.
\]
From this, we see that, if $m \geq 4$, the maximal refined order is 4 and the locus is $V(u, x, y, z)$, i.e., it has a horizontal component and we do not have to worry for the interesting primes.

If $m \in \{2, 3\}$, then the maximal refined order is still 4 and it is attained at the closed point $V(7, u, x, y, z)$, but not at any other point of $V(u, x, y, z)$. In particular, 7 is a bad prime that was not detected before. The reason for this is that $f$ is not in normal form with respect to $u - x^m$, while $f_*$ is. For example, for $m = 2$, taking two times the derivative of $f_* = 7^9x^2 - y^5 + xz^3$ by $x$, we obtain the factor $2 \cdot 7^9 \in I_{\text{temp}}$.

Furthermore, the condition that there is no horizontal component is crucial. This can already be seen in the previous example, but let us give another one which is simpler:

**Example 4.6.** Consider $I_X = \langle x^2 - 5^3 y^3 \rangle \subset \mathbb{Z}[x, y]$ and $I_Z = (0)$. It is not hard to see that Max-ord$(X) = V(x, y) \cup V(x, 5)$. After the first pass through the loop starting at line 3, we get
\[
I_{\text{temp}} = \langle x^2 - 5^3 y^3, 2x, 3 \cdot 5^3 y^2 \rangle \quad \text{and } I_{\text{int}} = (0).
\]
We stay in the loop and obtain after a second run:
\[
I_{\text{temp}} = \langle x^2 - 5^3 y^3, 2x, 2 \cdot 3 \cdot 5^3 y^2, 2 \cdot 3 \cdot 5^3 y, (x^2, 2, y^2) \rangle \quad \text{and } I_{\text{int}} = (2) \neq (0).
\]
Hence, 2 is the only interesting prime. In particular, we do not detect 5. This is not a contradiction since $V(x, y)$ is a horizontal component.

We now pick up the toy example of before to illustrate the algorithm to determine the interesting primes in the case $I_Z \neq (0)$:

**Example 4.7.** Let $I_Z = \langle g \rangle \subset \langle g, f \rangle = I_X \subset \mathbb{Z}[x, y, z]$ where $f = 3x - y + 7z$ and $g = x - 4y + 6z$. At this point, we observe that we can use the generator $g$ of $I_Z$ to eliminate the variable $x$ as it has the coefficient 1, which also means that the corresponding 1-minor $q$ of the Jacobian matrix has the value 1. This implies that we can use $(y, z)$ to obtain a global system inducing a regular system of parameters $(y_1, y_2)$ at each point of $Z$. Using $q = 1$, the derivatives of $f_1 := f$ are:
\[
\frac{\partial f_1}{\partial y_1} = \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \frac{\partial f}{\partial x} = -1 - (-4) \cdot 3 = 11
\]
and
\[
\frac{\partial f_1}{\partial y_2} = \frac{\partial f}{\partial z} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial x} = 7 - 6 \cdot 3 = -11.
\]
This shows that indeed the only interesting prime in this example is 11.

To see, how the algorithm proceeds, let us consider a more complex example:

**Example 4.8.** Let $I_Z := (0) \subset I_X := (3^2 x^2 - 5^2 y^2) \subset \mathbb{Z}[x, y]$. Here it is easy to check that the maximal order 2 is attained precisely at $V(5, x) \cup V(3, y) \cup V(x, y)$, where the first two components are vertical and the last one is horizontal. After the first pass through the while loop starting at line 3, we have
\[
I_{\text{temp}} = \langle 3^2 x^2 - 5^2 y^2, 2 \cdot 3^2 x, 2 \cdot 5^2 y, (x^2 + z^2, 2) \rangle \quad \text{and } I_{\text{int}} = (0).
\]
After the second pass we obtain
\[
I_{\text{temp}} = \langle 3^2 x^2 - 5^2 y^2, 2 \cdot 3^2 x, 2 \cdot 5^2 y, (x^2 + z^2, 2) \rangle \quad \text{and } I_{\text{int}} = (2) \neq (0).
\]
Thus, we leave the loop and return only the interesting prime 2 which is not bad. This is due to the presence of the horizontal component \(V(x, y)\). (An algorithm to properly determine this component will be discussed in Algorithm 3)

After blowing up with center \(V(x, y)\), we consider the \(X\)-chart, i.e., the chart with coordinates \((x', y') = (x, \frac{1}{y})\). Here, again \(I_X = \langle 0 \rangle\), but \(I_X = \langle 3^2 - 5^2y'^2 \rangle\). Running the algorithm as before, we obtain

\[
I_{\text{temp}} = \langle 3^2 - 5^2y'^2, 2 \cdot 5^2y' \rangle,
\]

which already contains an integer \(z = 2 \cdot 3^2\) after the first pass; i.e.,

\[
I_{\text{int}} = \langle 2 \cdot 3^2 \rangle \neq \langle 0 \rangle
\]

and the loop stops. Hence, we have collected the interesting primes 2 and 3, where of course 2 is still not bad, but 3 really leads to a vertical component. In the other chart, we then see the bad prime 5 in complete analogy to the discussed chart.

The above phenomena is one of the reasons why we vary the original Cossart-Jannsen-Saito algorithm and give a preference to horizontal components (cf. Construction 2.16).

The studious reader may study other special cases of the following generalization of the previous example: \(I_Z := \langle 0 \rangle \subset I_X := \langle p^a x^m - q^b y^b \rangle \subset \mathbb{Z}[x, y]\), for \(a \geq m, b \geq m\), and distinct primes \(p, q \in \mathbb{Z}\), i.e., \((p, q) = 1\). We observe that the locus of order at least \(m\) is \(V(x, y) \cup V(p, y) \cup V(q, x)\), but \(m\) is not necessarily the maximal order of \(X\).

**Remark 4.9.** In principle we can use Hasse-Schmidt derivatives (cf. the idea of the computation in Remark 4.10 and the corresponding Algorithm 4 below) instead of the usual derivatives in the preceding algorithm. The ratio of primes contributing to \(\text{Max-ord}(X)\) among the interesting primes would then be higher, because we would not pick up as many primes originating directly from exponents. However, we would like to postpone the rather technical discussion of the computation of Hasse-Schmidt derivatives with respect to a system of parameters in the presence of a non-trivial \(\mathbb{Z}\) as long as possible and first focus on the central task of determining the locus of maximal order.

Given a list of interesting primes and the ideals \(I_X\) and \(I_Z\), we are now ready to determine \(\text{Max-ord}(X)\), see Algorithm 3: MaxOrdArith (p.20). As this requires working locally at an interesting prime \(p\) and differentiating with respect to it, we need to rewrite each coefficient \(c\) in the form \(\frac{c}{p^\ell} \cdot \alpha\), where \(\ell\) is chosen maximal under the condition that \(p^\ell\) divides \(c\) and \(\alpha\) is a new variable. Note that it is not necessary to compute a full \(p\)-adic expansion, as locally in the stalk above \(p\) we see a unit \(\frac{c}{p^\ell}\) due to the choice of \(\ell\).

In line 6 of Algorithm 3, we consider systems of generators for \(I_X\) and \(I_Z\). This choice does not affect the result since we eventually replace \(P\) again by \(p\) in line 11 (resp. 30) and since we are only interested in the maximal order locus locally at \(p\) at this step.

In Algorithm 3, there is still one black box to be described further: HasseDeriv. In contrast to derivatives in the characteristic zero case, we cannot compute them iteratively. Instead, the Hasse-Schmidt derivatives of a given polynomial \(f(\underline{x})\) are extracted from \(f(\underline{x}+\underline{t})\) as the coefficients of the respective \(\underline{t}^m\), as was already mentioned above (see (3.1)). In view of the need to compute derivatives with respect to a regular system of parameters for a non-trivial \(\mathbb{Z}\), we are forced to reconsider the arguments of Remark 4.2:

**Remark 4.10.** In the setting analogous to Remark 4.2, we again consider a minor \(q = \det(M)\) of the Jacobian matrix of \(Z\) and restrict the considerations to \(D(q)\), where \(q\) is invertible. As before, we can compute the usual derivatives w.r.t. \(y\). To pass to Hasse derivatives, however, we have to recall the symbolic identity, which we encountered on page 10:

\[
\frac{\partial}{\partial Y^a} Y^b = \frac{1}{a!} \left( \frac{\partial}{\partial Y} \right)^a Y^b.
\]

This stresses that first order Hasse derivatives coincide with usual derivatives and it allows us to transform the higher usual derivatives into Hasse derivatives at each step of the computation by applying the appropriate correction factor.

In the algorithm to compute the list of Hasse derivatives, we thus carry along the information from which derivative a given polynomial originated and reuse this when passing...
Algorithm 3 MaxOrdArith

Input: $I_Z \subseteq \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}[x]$ such that $Z := V(I_Z)$ is equidimensional and regular, $I_X \subseteq \mathbb{Z}[x]$ such that $I_Z \subset I_X$, i.e., $\emptyset \neq X := V(I_X) \subset Z$

Output: $(m, L)$, where

$m = \text{max-ord}(X)$
$L$ list, where $L[i] = (p_i, I_i)$ such that
$\text{Max-ord}(X) = \bigcup_i V(I_i)$ and $I_i$ has been detected locally at $p_i$

1. $\text{MaxOrd0} = \text{MaxOrd}(I_Z \otimes \mathbb{Q}, I_X \otimes \mathbb{Q})$
2. $\text{maxord} = \text{MaxOrd0}[1]$, $I_{\text{max}} = \text{MaxOrd0}[2] \cap \mathbb{Z}[x]$
3. RetList[1] = $(0, I_{\text{max}})$
4. PrimeList = InterestingPrimes($I_Z, I_X$)
5. for $p \in \text{PrimeList}$ do
6. replace each coefficient $c \in \mathbb{Z}$ appearing in the generators of $I_X$ and $I_Z$ by $\frac{c}{p^\ell}$ with
   $\ell$ maximal and $P$ a new variable, denote by $J_X$ and $J_Z$ the resulting ideals in $\mathbb{Z}[\frac{x}{P}]$
7. if $I_Z == (0)$ then
8. $\text{DiffList} = \text{HasseDeriv}(J_Z, J_X, (\frac{x}{P}, 0))$/i-th entry = up to i-th derivatives of $J_X$
9. $m = \text{size}(\text{DiffList})$
10. for $i \in \{1, \ldots, m\}$ do
11. $\text{DiffList}[i] = \text{ideal} (\text{substitute}(\text{DiffList}[i], P, p))$
12. while $\text{DiffList}[m] == (1)$ do
13. $m = m - 1$
14. if $m >= \text{maxord}$ then
15. $I_{\text{max}} = \text{DiffList}[m]$
16. if $m > \text{maxord}$ then
17. RetList = $\emptyset$, $\text{maxord} = m$
18. RetList[1] = $(p, I_{\text{max}})$
19. else
20. RetList[\text{size}(\text{RetList}) + 1] = $(p, I_{\text{max}})$
21. else
22. $L = \{\text{codim}(Z) \text{ square submatrices of Jacobian matrix of } J_Z\}$
23. choose a subset $L_1 \subset L$ such that $V(J_X) \subset \bigcup_{M \in L_1} D(\text{det}(M))$
24. $\text{locord} = 1$
25. for $M \in L_1$ do
26. fix (y, P) system of parameters on $V(J_Z) \cap D(\text{det}(M))$ as in line 12 Algorithm 1
27. $\text{DiffList} = \text{HasseDeriv}(J_Z, J_X, (\frac{x}{P}, P), M)$
28. $m = \text{size}(\text{DiffList})$
29. for $i \in \{1, \ldots, m\}$ do
30. $\text{DiffList}[i] = \text{ideal} (\text{substitute}(\text{DiffList}[i], P, p))$
31. while $\text{DiffList}[m] == (1)$ do
32. $m = m - 1$
33. if $m > \text{locord}$ then
34. $I_{\text{max}} = \text{DiffList}[m]$
35. $\text{locord} = m$
36. else if $m == \text{locord}$ then
37. $I_{\text{max}} = I_{\text{max}} \cap \text{DiffList}[m]$
38. if $\text{locord} >= \text{maxord}$ then
39. if $\text{locord} > \text{maxord}$ then
40. RetList = $\emptyset$, $\text{maxord} = \text{locord}$
41. RetList[1] = $(p, I_{\text{max}})$
42. else
43. RetList[\text{size}(\text{RetList}) + 1] = $(p, I_{\text{max}})$
44. return $(\text{maxord}, \text{RetList})$

...to the next derivative. What we are not allowed to apply in this approach is a change of
generators of intermediate ideals. Therefore, we need to carry along both the list of already computed derivatives and the corresponding saturated ideal.

In Algorithm 4: HasseDeriv (p.21), the previous considerations have been reformulated closer to an implementable algorithm.

**Algorithm 4 HasseDeriv**

**Input:** $I_Z = \langle g_1, \ldots, g_r \rangle \subset \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}[x]$ such that $Z := V(I_Z)$ is equidimensional and regular, $I_X = I_Z + \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{Z}[x]$, i.e., $\emptyset \neq X := V(I_X) \subset Z$ - system of parameters on $Z \cap D(\det(M))$

$\text{M codim}(Z)$ square submatrix of Jacobian matrix of $I_Z$

**Output:** list RetList such that $\text{RetList}[i] = I_Z + \langle \partial f / \partial x | f \in I_X, |a| \leq i \rangle$

1: if $I_Z == \langle 0 \rangle$ then
2: for $j \in \{1, \ldots, r\}$ do
3: $F_j(y, 0) = f_j(x_1 + t_1, \ldots, x_n + t_n)$
4: $i = 1$, tempid = $g_1, \ldots, g_r, f_1, \ldots, f_r$
5: while $(i == 1) \text{ OR } (\text{tempid} != \text{RetList}[-1])$ do
6: RetList[i] = tempid
7: for $a \in \{b \in \mathbb{Z}_{\geq 0}^n | |b| == i \}$ do
8: tempid = tempid, $\{ \text{coefficients of } t^a \text{ in } F_1, \ldots, F_r \}$
9: $i = i + 1$
10: return RetList
11: $I_{\text{temp}} = I_X$, Null = $\langle 0, \ldots, 0 \rangle$ (Null has #y entries)
12: for $1 \leq i \leq r$ do
13: $L[i] = \{f_i, \text{Null} \}$
14: old = 0, cur = $r$
15: while $I_{\text{temp}} \cap Z == \langle 0 \rangle$ do
16: for old < $i$ < cur do
17: $(f_{\text{temp}}, \text{note}) = L[i]$
18: for $y_j \in y$ do
19: note[j] = note[j] + 1
20: $f_{\text{temp}} = \frac{1}{\text{note}[j]} \frac{\partial f_{\text{temp}}}{\partial y_j}$

(precise formulation of $\frac{\partial f_{\text{temp}}}{\partial y_j}$ in Remark 4.2 - these are usual derivatives)
21: $L[\text{size}(L)+1] = (f_{\text{temp}}, \text{note})$
22: $I_{\text{temp}} = I_{\text{temp}} + \langle f_{\text{temp}} \rangle$
23: $I_{\text{temp}} = \text{sat}(I_{\text{temp}}, \det(M))$
24: RetList[size(RetList)+1] = $I_{\text{temp}}$
25: old = cur, cur = size(L)
26: return RetList

Even if we start with a trivial ambient space, i.e., $I_Z = \langle 0 \rangle$, it may become non-trivial after blowing up:

**Example 4.11.** Consider the hypersurface $X$ over $Z$ given by

$$f = 3^2 5^2 + 5xy + x^3 y^3 \in \mathbb{Z}[x, y].$$

Let us point out that the ambient space is $Z = \text{Spec}(\mathbb{Z}[x, y])$. We assume that the boundary is empty. The maximal order achieved for $f$ is 2 and

$$\text{Max-}\nu^0(f) := \text{Max-}\nu^0(X) = V(3, x, y) \cup V(5, x) \cup V(5, y),$$

where the first component is disjoint from the other two. All get label 0. One observes that Max-\nu(f) is not weakly permissible and its singular locus is $V(5, x, y)$. Therefore we blow up with center $V(5, x, y)$. Consider the $Y$-chart, where we set $q := \frac{2}{y}$ and $x' := \frac{x}{y}$ and $y' = y$. 

It is important to keep in mind the transform of the ambient space, which is non-trivial in this chart: \( Z' = \text{Spec}(\mathbb{Z}[x', y', q]/(5 - qy')) \). The exceptional component is given by \( V(y') \), hence the boundary is \( B = \{ V(y') \} \). The strict transform of \( f \) is
\[
f' = 3y^2 + q(x'y' + x^3y').
\]
Note that \( \text{Max-}\nu^O(f') \) consists of the strict transform of \( \text{Max-}\nu^O(f) \) in \( Z' \) and components contained in the exceptional divisor. Hence we obtain
\[
\text{Max-}\nu^O(f') = V(q, x') \cup V(q, y'),
\]
The first component is the strict transform of \( V(5, x) \) and has label 0. The second is contained in the exceptional component and thus gets the new label 1.

Note: In the computations, the component \( V(3, x', y') \) of the maximal order locus has been handled, but we do not see it in this chart as its intersection with the non-trivial ambient space is empty here: \((3, x', y', 5 - qy') = (1) \) as \( \gcd(3, 5) = 1 \).

Since \( V(q, x') \) is weakly permissible the algorithm chooses this component as the center for the next blowing up. We leave the other charts and the remaining resolution process to the reader.

Let us point out again that our variant is different from the original [16]-resolution. Since we are using \( \nu_{\text{red}} \) instead of the Hilbert-Samuel function, we cannot control the maximal value of the Hilbert-Samuel function along our variant of the resolution process:

**Example 4.12.** Let \( X \) be the affine variety over \( \mathbb{Z} \) defined by the ideal
\[
J = \langle x^2 - y^{17}, p^5 - y^2z^6 \rangle \subset \mathbb{Z}[x, y, z],
\]
where \( p \in \mathbb{Z} \) is a prime number. The ambient space is \( Z = \mathbb{A}^3_\mathbb{Z} \) and we assume that the boundary is empty \( B = \emptyset \). We have \( \text{max-}\nu(X) = (4, 2) \) and
\[
\text{Max-}\nu^O(X) = \text{Max-}\nu(X) = V(p, x, y).
\]

In contrast to this, the maximal Hilbert-Samuel locus is \( V(p, x, y, z) \).

The algorithm proposes as first center \( V(p, x, y) \). After blowing up, we consider in the \( Y \)-chart, where we use the notation \( p' := \frac{q}{y}, x' := \frac{p}{y}, y' = y, \) and \( z' := z \). The strict transform of the ambient space is \( Z' = \text{Spec}(\mathbb{Z}[p', x', y', z']/(p - p'y')) \) and \( X' \) is given by the ideal
\[
J' = \langle x'^2 - y'^{15}, p^5y^3 - z'^6 \rangle.
\]
Hence, \( \text{max-}\nu(X') = \text{max-}\nu(X) = (4, 2) \). Note that the maximal order of the second polynomial increased from 5 to 6 after the blowing up. This implies that the maximal value achieved by the Hilbert-Samuel function increased\(^6\), which is not surprising since we have blown up a center that is not contained in the maximal Hilbert-Samuel locus.

We have \( \text{Max-}\nu^O(J') = V(x', y', z') \) which is a weakly permissible and hence the next center. We leave the other charts and the remaining resolution process to the reader.

5. Exploiting the Naturally Parallel Structure

In this section, we discuss the parallelization of the algorithm for finding the center of the chosen resolution strategy. The ubiquitous use of coverings by charts already suggests a high potential for parallelization; instead of starting with an arbitrarily chosen covering by a subset of the minors of the Jacobian matrix of \( I_Z \), we can also start with all possible choices, run them in parallel and terminate as soon as all of \( X \) has been covered. As we aim to use GPI-Space \cite{GPI-Space} as the workflow-management system for managing and scheduling our parallel approach, we rephrase our algorithms in the language of Petri nets, which is slightly unusual for algebraic geometry, but has already proved useful in \cite{Frühbis-Krüger2014}.

\(^6\)The precise value is \( (1, 4, 9, 16, 25, 36, 48, \ldots) \), whereas it was \( (1, 4, 9, 16, 25, 35, 45, \ldots) \) before blowing up.
5.1. **Petri nets and parallel implementation.** Petri nets are models of distributed systems. In its basic form, a Petri net is a bipartite, directed, finite graph. Its vertices are *places*, which are denoted by circles in graphical representation, and *transitions*, which are denoted by rectangles. The edges of the graph are called *arcs*. The places from which there is an arc to a certain transition are called the *input places* of that transition. There can also be *tokens* on the places, depicted by small solid circles. A token can be thought of carrying a piece of information (formally: it is of a certain *color*) that is being processed by the Petri net. A configuration of tokens is called a *marking* of the net. *Colored Petri nets* are in fact an extension to the original concept where the tokens on one place are indistinguishable. There are actually numerous extensions of Petri nets; some of them provide important features for describing computations which cannot be phrased with the language of basic Petri nets. We describe the properties of the Petri nets which are used in GPI-Space. The latter is a parallel environment that handles the workflow management of our implementation.

A transition of a Petri net is called *enabled* if there is at least one token at each of its input places. It can *switch* (or *fire*). When it switches, the transition consumes one token from every input place and places new tokens on every output place. Note that the Petri net itself makes no statement when an enabled transition will switch (if it does at all) and which tokens it will consume if there are several to choose from. In this sense, Petri nets are executed in a non-deterministic way (the actual choices are up to the underlying implementation).

Let us now consider an extremely simplified example to get used to the terminology of Petri nets:

**Example 5.1.** Consider a composition of functions $X \xrightarrow{f} Y \xrightarrow{g} Z$ between some sets $X, Y, Z$. This translates into the following Petri net (only showing places and transitions):

![Petri net diagram](image)

So far, we are hiding the tokens. If there is a token in $p_X$, it can be consumed by the transition $f$ which then provides at least one token in $p_Y$, see below for further explanations. A snapshot of the Petri net might look as follows:

![Petri net snapshot](image)

That means currently, there are three tokens in $p_X$ waiting to be consumed by $f$ and one in $p_Z$. Transition $g$ cannot fire, as there are no token in its input place $p_Y$, whereas transition $f$ is enabled as a consequence of the tokens in $p_X$. Upon firing, transition $f$ consumes one token from place $p_X$, representing an element $x \in X$, and produces one token on place $p_Y$, representing a token $f(x) \in Y$.

Example 5.1 also shows further properties of Petri nets: One important point is that all dependencies are local, at least in theory. Every transition only needs to know about its input and output places, for both the data (color of tokens) and control dependency (presence of tokens). There is no need for any kind of global synchronization or global clock signal. This also means that in a real setting (i.e., in GPI-Space through which the computer executes the Petri net), there is usually no global state during execution that can be described with a marking. In the model, each switching is an atomic, instantaneous process, i.e., it is a step that can not be further divided into smaller parts and it is processed immediately without time consumption. In a real system, the execution of a transition will take a finite amount of time and there will be several transitions executing at a given point in time, in general. However, it is always possible to give an equivalent switching sequence without overlaps.
This leads to the second important point: Petri nets automatically come with parallelism. There is both data parallelism, that is, there can be several instances of one transition running at the same time (on different CPU cores or even on different machines), but also task parallelism, that is, instances of different transitions can be running simultaneously.

Keeping this background in mind, we can now state a Petri net corresponding to a simplified version of the embedded resolution process:

**Example 5.2.** Roughly speaking a resolution of singularities via blowing ups in regular centers has the following structure: Given $X \subset Z$ as in the previous sections, first determine the center $C$ of the first blowing up. Then, if $X$ is not already resolved (i.e., if $C \subseteq X$), we blow up at the center $C$. This process is then repeated for the transform of $X \subset Z$ under the blowing up. Eventually, we reach the point that all singularities are resolved – at least in those cases, where desingularization by a finite number of blowing ups is not an open problem.

As we mentioned before, the resolution data is collected in local affine charts. In other words, given the initial data $X \subset Z$, we first have to split it into affine charts, each of which corresponds to a token (in place a below). Furthermore, after blowing up (i.e., at place c below), we also have to split the data into finitely many affine charts (i.e., we produce several tokens from one). Those of the affine charts (i.e., tokens) that are resolved are detected at place b and glued with the already handled resolved charts. The resulting token holding the glued object is put into place d, ready for gluing the next chart to it. If the token on place d already describes the entire resolution of $X$, the transition Heureka fires and initiates the clean-up and termination of the Petri net. The output is an Embedded Resolution of Singularities $ERS(X, Z, B)$, where $B$ is some boundary on $Z$.

These explanations provide the simplified Petri net, where we use the obvious notation. We mark the beginning (resp. end) by an upper (resp. lower) half-disc.

Note that we have omitted several details in the above Petri Net, e.g. the use of place d can only work properly, if a token representing the empty set is placed there at the beginning. To discuss all technicalities of the use of Petri nets, such as the use of conditions and parameter dependent behaviour, is far beyond the scope of the current article which deliberately puts its focus on the mathematical side. For an introduction to parallelization using Petri nets for Algebraic Geometers we refer to section 3 and 4 of [9]. We only state simplified Petri nets in the following sense: only the essential steps in the respective algorithm are presented, some of the transitions may themselves represent Petri nets, like FindCenter above, whereas others, like BlowUp, represent code which is executed in the back-end SINGULAR [26] and seen as atomic from the perspective of GPI-Space. Moreover, we take the liberty to add annotations for the data (i.e., coloring of tokens) at those places, where it contributes to a better understanding, and for the arcs, where the transition is subject to a condition or where more than one output token is created.
5.2. Petri nets within the desingularization process. In Example 5.2, we have already seen the inherent parallelity through the use of charts, but we have not yet discussed the very heart of the algorithm: the choice of center. The following Petri net describes the algorithm FindCenter, which is itself of sequential nature and directly corresponds to the steps outlined in section 4 as can easily be observed from its Petri net. The input- and output-tokens of this net are precisely of the structure of the respective tokens at the input and output place for FindCenter in the Petri net in Example 5.2.

Let us point out that, in the Petri net, \( Y \) denotes the locus of those irreducible components in \( \text{Max-}^O(X) \) that are of lowest label.

The transitions IntersecExc and Labeling are implemented in the back-end Singular and hence seen as atomic in this context. The purpose of CenterY is the choice of a center to resolve \( Y \), which is of dimension at most \( \dim(X) - 1 \). In a general setting, this is a recursive call to the resolution. But in the special case of \( \dim(X) = 2 \), this only handles at most a finite number of singular points of \( Y \) and we consider it as simple call to Singular. Note that we will get \( C = Y \) at some step of the algorithm, i.e., that the entire locus \( Y \) is the next center.

This leaves two important building blocks to discuss: InterestPrimes and MaxRefOrd, both of which exhibit a naturally parallel structure. We first discuss the Petri net of InterestPrimes which corresponds to the algorithm InterestingPrimes of the previous section:

\[
(X, Z, B) = (X_{af}, Z_{af}, B_{af})
\]

\[
\langle m \rangle = I_Z \cap \mathbb{Z}
\]

\[
\langle \ldots, M, m \rangle
\]

\[
\ldots, I_{\text{temp}}, I_{\text{int}}
\]

\[
\ldots, \{p_i, \ldots, p_l\}
\]

Note that for consistency of coloring at place a, the matrix \( M \) is initialized to the zero matrix in the very first transition of the net. Split is the covering by affine charts arising...
from the minors of the Jacobian matrix of $I_Z$, contains a call to SINGULAR and fills in a correct value for the matrix $M$ for each of the generated tokens. The iteration of purely formally forming the derivatives is represented by FormalDer, which relies on functionality of the back-end SINGULAR again. Having found an integer $m \neq 0$, we need to find the prime factors, combine the knowledge from all charts and return the result.

The next transition in the Petri net of FindCenter is the computation of the locus of maximal refined order MaxRefOrd, which is represented by the next Petri net. In reality, the output place of InterestPrimes and the input place of MaxRefOrd are identical; for presentation as separate Petri nets, we therefore start MaxRefOrd by a transition which actually only places its input token into its output place $a$.

As the first entry of the desingularization invariant represents the number of descents upon finding maximal order one, the main part of this Petri net is the loop which computes the maximal order locus and then passes to the descent in ambient dimension, if the maximal order was 1. If the maximal order exceeded 1, the transition Glue&Collect fires instead, compares the maximal order to the current maximal value, identifies the current maximal order and places the locus of this order (as computed up to this point) into the place $c$. Note that the zero ideal has order $\infty$, which also triggers Glue&Collect, but the transition itself handles this special case appropriately, as it first compares the first entry of the invariant.

\[
(X, Z, B, S) = (X_{af}, Z_{af}, B_{af}, S_{af})
\]

\[
\downarrow
\]

\[
\text{MaxOrdArithm}
\]

\[
\text{Descent}
\]

\[
\text{Glue & Collect}
\]

\[
\text{Heureka}
\]

\[
(X_{af}, Z_{af})
\]

\[
\downarrow
\]

\[
\text{many}
\]

\[
\max-\nu(X_{af})
\]

\[
\text{Glue & Collect}
\]

\[
\text{Heureka}
\]

\[
\text{max-\nu}(X_{af})
\]

\[
\max-\nu(X_{af}) = 1
\]

\[
\max-\nu(X_{af}) > 1
\]

\[
X \text{ not covered}
\]

\[
X \text{ covered}
\]

\[
\text{Drop}
\]

In this Petri net, we are still hiding another key step, the calculation of the locus of maximal order in the arithmetic case, i.e., MaxOrdArith. This is the last of the Petri nets we are presenting here and starts by calling MaxOrd0, already available functionality to compute the locus of maximal order in characteristic zero.

In the absence of interesting primes, this already terminates the algorithm. Otherwise, the stalk at each interesting prime needs to be considered separately giving rise to at least $\#S$ tokens at place $b$ for each of which the prime $p$ has already been replaced by a symbol $P$. Note that at each of the stalks a covering by minors of the Jacobian matrix may become necessary. The computation of Hasse derivatives in HasseDer is a task for the back-end SINGULAR, as are the computations after resubstitution of $P$ by $p$.

In the transition Glue&Collect, the computed maximal order of the token from place $d$ is compared to the maximal order assigned to the token from place $e$ (technically, we again need to make sure that place $e$ holds an initialization token corresponding to the empty set at the beginning, but this is not shown here). According to the result of the comparison, the one of lower value is dropped or the union of the computed loci is formed in case of equality. This comprises a gluing step for charts by maximal minors and the collecting of the stalks at primes. At the same time, the prime $p$ is discarded from $S$ allowing us to leave the loop as soon as $S$ is empty. The transition Drop then just drops unnecessary parts of the coloring to reach conformity with the coloring of tokens at place $a$. 
Finally, let us discuss a technicality which we neglected so far. We heavily work with coverings of the given space. In some extensions of Petri nets, transitions can consume or produce a variable number of tokens, as is required to produce any kind of covering by charts. In GPI-Space, however, this is not directly supported, but it is possible to use auxiliary constructions like the following one:

Here, the computation in transition Cover gives a (finite) set of results for which the number of elements is not known before. The transition produces one single token on the intermediate place $p_L$, containing a list of elements. The Extract transition takes a (non-empty) list from $p_L$, extracts a single element and places it on $p_{out}$, while a token with the list of remaining elements is put on $p_L$ again. A token with an empty list will then be discarded by transition Remove. Altogether, this net produces the desired number of tokens on $p_{out}$, all resulting from one token on $p_{in}$.

While this approach is inherently sequential, it is possible to create a parallelizable version of it which successively splits larger lists into smaller lists.

6. Outlook

6.1. Normalization in the arithmetic case. Up to this point, all considerations in the article have followed the general structure of Hironaka style desingularization, i.e., using suitably chosen smooth centers for resolving by blowing up. There is another classical
approach to resolution of singularities for surfaces (even in mixed characteristic): Lipman introduced a desingularization process in [48], which starts by a normalization of the surface followed by a sequence of suitable blow ups.

Over a field, normalization can be achieved algorithmically following the ideas of Grauert (see [44]), for which state-of-the-art algorithms and implementations are available (see for instance [38], [10]). Crucial to all of these is the use of the ideal of the singular locus as an approximation of the non-normal locus. In the arithmetic case, the same theoretical ideas still hold, but on the practical side the approximation of the non-normal locus causes a bottleneck: if it is done by means of the vanishing of the minors of the Jacobian matrix, it is by far too coarse, as it does not only detect the singular points, but also fibre singularities. Recall the first lines of Example 3.2 for an illustration of the difference between singular point and singularity of the fibre.

On the other hand, the discussion of the locus Max-$\nu(X)$ also points to another potential way to approximate the non-normal locus: Instead of the locus Max-$\nu(X)$ corresponding to the value max-$\nu(X) = (\alpha, \delta)$, we can consider the locus $V_{\geq (\alpha, 2)}$ which obviously contributes to the singular locus. Unfortunately, there is no reason why it should be the whole singular locus. Away from it there may be points with a smaller first entry (i.e., locally allowing a smooth ambient space $Z$ of smaller dimension), but a second entry again exceeding 2, which implies that $X$ is not smooth at these points. As there can only be finitely many descends in ambient dimension, it seems promising to determine the singular locus by first determining $Y_1 = V_{\geq (\alpha, 2)}(X)$, then pass to an open cover of $X \setminus Y_1$ by principal open sets, determine $Y_2 = V_{\geq (\alpha-1, 3)}(X)$ on each of these open sets and iterate the process. After finitely many descends of ambient space, this yields a finer approximation of the non-normal locus and might hence help to counteract the current bottleneck in the normalization in the arithmetic setting.

6.2. Schemes of higher dimension. As it is already mentioned in [16] Remark 5.29, the formulation of the algorithm (see Construction 2.16) allows an arbitrary dimensional scheme $X$ instead of a two-dimensional one. But termination of the algorithm is not known in higher dimensions (even in the situation over a field of characteristic zero), neither for the original version nor for our variant.

While the newly created components in the locus of maximal singularity (i.e., those with label 1 or larger) are well-behaved in dimension two, there is less control in higher dimensions, see [22] Example 2.5. Furthermore, one has to be more careful when dealing with the labels of irreducible components, see [22] Example 2.6. Since the mentioned examples are hypersurfaces they apply for both variants of the algorithm.

In the papers studying the algorithm ([42], [15], [16], [22]) Hironaka’s characteristic polyhedron [41] plays a crucial role as a measure for the improvement of the singularity during the resolution process. Another difficulty in higher dimension is to obtain a similar control on the characteristic polyhedron or to give an alternative invariant measuring the improvement.

Therefore an implementation of the algorithm could be extremely useful for investigations in this direction. In particular, it will be easier to test numerous examples on termination of the algorithm, to search for patterns in order to develop new invariants, and to explore the behavior of potential invariants.

Finally, one may think that choosing always the oldest irreducible components seems quite arbitrary. In fact, as the following example shows, we cannot modify the algorithm by considering always the newest irreducible component.

Example 6.1. Let $k$ be a field of characteristic zero and $\mathcal{B} = \emptyset$. Let $X = V(f) \subset \mathbb{A}^5_k$ be the affine variety given by the polynomial

$$f = t^5 + x^4y^2z^4 + w^{10}z^6.$$

We have max-$\nu(X) = (5, 5)$ and Max-$\nu^{\mathcal{O}}(X) = V(t, x, z) \cup V(t, y, z) \cup V(t, x, y, w)$. All components get label 0 and the algorithm blows up with center the origin $D_0 := V(t, x, y, z, w)$.

In the affine chart with coordinates $(t', x', y', z', w') = (\frac{t}{z}, \frac{x}{z}, \frac{y}{z}, z, w)$ the strict transform $X'$ of $X$ is given by (for simplicity, we abuse notation and call the coordinates again $(t, x, y, z, w)$)

$$f' = t^5 + x^4y^2z^5 + w^{10}z^{11}.$$
We have max-ν(\(X\)) = (5, 5) and Max-ν\(^O\)(\(X\)) = \(V(t, z) \cup V(t, x, y, w)\). Since \(V(t, z)\) is contained in the exceptional divisor and the center is not an entire irreducible component of Max-ν\(^O\)(\(X\)), the component \(V(t, z)\) gets the label 1.

Suppose we modify Construction 2.16 such that we always consider the irreducible components with highest label. Then the center of the next blowing up is \(D_1 := V(t, z)\).

In the affine chart with coordinates \((t', x', y', z', w') = \left(\frac{t}{z}, x, y, z, w\right)\), we obtain that the strict transform \(X''\) is given by
\[
f' = t^5 + x^4y^2 + w^{10}z^6.
\]

Further, max-ν(\(X''\)) = (5, 5), and Max-ν\(^O\)(\(X''\)) = \(V(t, x, y, z) \cup V(t, x, y, w)\), where \(V(t, x, y, z)\) has label 2 and \(V(t, x, y, w)\) has label 0. The modified algorithm chooses \(D_2 := V(t, x, y, z)\) as the next center.

In the affine chart with coordinates \((t', x', y', z', w') = \left(\frac{t}{z}, \frac{x}{y}, \frac{y}{z}, z, w\right)\) the strict transform \(X'(3)\) is given by
\[
f^{(3)} = t^5 + x^4y^2 + w^{10}z^6.
\]

Again, max-ν(\(X'(3)\)) = (5, 5), and we have Max-ν\(^O\)(\(X'(3)\)) = \(V(t, x, y, z) \cup V(t, x, y, w)\), where the component \(V(t, x, y, z)\) gets label 3 and \(V(t, x, y, w)\) has label 0. Hence \(D_3 := V(t, x, y, z)\) is the center of the next blowing up.

In the affine chart with coordinates \((t', x', y', z', w') = \left(\frac{t}{z}, \frac{x}{x'}, \frac{y}{y'}, z, w\right)\) the strict transform \(X'(4)\) is given by
\[
f^{(4)} = t^5 + x^4y^2 + w^{10}z^6.
\]

But \(f^{(4)} = f''\). Thus we have created a loop and the modified algorithm never ends.

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