Correlation Minor Norm as a Detector and Quantifier of Entanglement

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In this paper we develop an approach for detecting entanglement, which is based on measuring quantum correlations and constructing a correlation matrix. The correlation matrix is then used for defining a family of parameters, named Correlation Minor Norms, which allow one to detect entanglement. This approach generalizes the computable cross-norm or realignment (CCNR) criterion, and moreover requires measuring a state-independent set of operators. Furthermore, we illustrate a scheme which yields for each Correlation Minor Norm a separable state that maximizes it. The proposed entanglement detection scheme is believed to be advantageous in comparison to other methods because correlations have a simple, intuitive meaning and in addition they can be directly measured in experiment. Moreover, it is demonstrated to be stronger than the CCNR criterion. We also illustrate the relation between the Correlation Minor Norm and entanglement entropy for pure states. Finally, we discuss the relation between the Correlation Minor Norm and quantum discord, and briefly discuss possible generalizations for multipartite scenarios.

I. INTRODUCTION

The last three decades have seen significant advancement in development of promising quantum technologies, both from theoretical and practical aspects. These technologies often utilize quantum entanglement in order to gain advantage compared to classical technologies. Thus, the practical ability to detect entanglement and quantify its strength in a given system, is essential for the advancement of quantum technologies. Entanglement detection in many-body quantum systems is also of major interest [1, 2].

This has led researchers to seek simple ways to detect entanglement, preferably, ones which may be used in practice. For example, the Peres-Horodecki criterion [3] is a necessary condition for a state to be separable; however, it is sufficient only in the 2 × 2 and 2 × 3 dimensional cases [4, 5].

Another important concept is an entanglement witness, which is a measurable quantum property (i.e. a bounded Hermitian operator), such that its expectation value is always non-negative for separable states [4]. For any entangled state, there is at least one entanglement witness which would achieve a negative expectation value in this state; however, to use an entanglement witness in order to detect entanglement, one must measure a specific operator tailored to the state. Moreover, an approach to quantify entanglement using entanglement witnesses can be found in [6].

In [7–12], a construction of a quantum correlation matrix was demonstrated, and it was shown that this matrix may be utilized to detect entanglement. In [13, 14], a quantum correlation matrix has allowed the authors to derive generalized uncertainty relations, as well as a novel approach for finding bounds on nonlocal correlations. This matrix is the correlation matrix of a vector of quantum observables; thus, it may have complex entries. In [15] it was demonstrated that such a matrix allows one to construct new Bell parameters and find their Tsirelson bounds. Another approach for Bell parameters based on covariance can be found in [16].

Indeed, quantum correlations are subtly related to entanglement, e.g. pure product states are always uncorrelated. This is not true for mixed states; Separable mixed states may admit quantum correlations between remote parties [17]. These correlations are due to noncommutativity of quantum operators; thus, they allude to a different quantum property aside of entanglement, known as quantum discord [18–22].

In [23, 24], an approach for detecting entanglement using symmetric polynomials of the state’s Schmidt coefficients has been studied. It was shown to be a generalization of the well-known CCNR criterion (computable cross-norm or realignment; first defined in [25, 26]), according to which the sum of all Schmidt coefficients is no greater than 1 for any separable state. The symmetric polynomial approach equips each one of these polynomials with some upper bound; if the polynomial exceeds its bound then it follows that the state is entangled. Thus, the sum of all Schmidt coefficients with the upper bound 1 is a special case of this approach.

In this paper, we construct for a given quantum state its quantum correlation matrix, and examine the norms of its compound matrices. Since the compound matrix in our case is constructed from minors of a certain correlation matrix, we call the proposed entanglement detectors “Correlation Minor Norms”. Since these norms are invariant under orthogonal transformations of the observables, they can be regarded as a family of physical scalars which can be readily derived from bipartite correlations. Next, for each Correlation Minor Norm (CMN) we find an upper bound, such that if the CMN exceeds this bound it implies that the state is entangled. Our proposed method is shown to generalize the symmetric polynomial approach. We also provide results and conjectures regarding the states that saturate the bounds. Finally, we explore how the CMN relates to entanglement entropy and quantum discord, and discuss possible generalizations for multipartite scenarios.

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II. CONSTRUCTION OF THE CORRELATION MATRIX

Let two remote parties, Alice and Bob, share a quantum system in $\mathcal{H}_A \otimes \mathcal{H}_B$, the tensor product of Hilbert spaces. Denote $d_A \triangleq \dim \mathcal{H}_A$, $d_B \triangleq \dim \mathcal{H}_B$, and let $A \triangleq \{ A_i \}_{i=1}^{d_A^2}$ be an orthonormal basis of the (real) vector space of $d_A \times d_A$ Hermitian operators, w.r.t. the Hilbert-Schmidt inner product. Similarly, $B \triangleq \{ B_j \}_{j=1}^{d_B^2}$ is an orthonormal basis of the $d_B \times d_B$ Hermitian matrices. Note that such a basis always exists, since the real vector space of $d \times d$ Hermitian matrices is simply the real Lie algebra $\mathfrak{u}(n)$, which is known to have dimension $n^2$. Here we regard $u(A)$, $u(B)$ simply as inner product spaces, ignoring their Lie algebraic properties. Consequently, we require the normalization $\text{tr}(A_i A_j) = \delta_{ij}$ (and similarly for Bob) - without the factor of 2, which is normally taken to make the structure constants more convenient. For example, for $d = 3$ one could take $A_0 = \frac{1}{\sqrt{3}}I$ and $A_i = \frac{1}{\sqrt{2}}\gamma_i$ for all $i = 1, \ldots, 8$, where $\gamma_i$ are the (“standard-normalization”) Gell-Mann matrices.

The (cross-)correlation matrix of $A, B$, denoted by $C$, is defined by:

$$C_{ij} \triangleq \langle A_i \otimes B_j \rangle = \text{tr} (\rho A_i \otimes B_j)$$

(1)

where $\rho$ is the density matrix shared by Alice and Bob. As we shall see in the next section, the information contained in $C$ regarding the strength of nonlocal correlations is encoded entirely in its singular values. An equivalent characterization is provided by a noteworthy relation between the singular value decomposition (SVD) of $C$ and the operator-Schmidt decomposition of the underlying state, which we describe hereinafter. The operator-Schmidt decomposition of any state $\rho$ is defined as its unique decomposition of the form:

$$\rho = \sum_{k=1}^{d^2} \lambda_k G_k \otimes H_k$$

(2)

where each $\lambda_k \geq 0$ is a real scalar, and the sets $\{ G_k \}$ and $\{ H_k \}$ are orthonormal sets of $d_A \times d_A, d_B \times d_B$ Hermitian matrices respectively. It can be shown that the singular values of $C$ are precisely the Schmidt coefficients $\lambda_k$; moreover, the sets $\{ G_k \}$ and $\{ H_k \}$ are related to the sets $\{ A_i \}$ and $\{ B_j \}$ through the orthogonal matrices $U, V$ of the SVD, respectively. Extended definitions and proof may be found in Appendix A.1.

III. CORRELATION MINOR NORM

The goal of this work is to produce physical scalars from $C$ that would allow for entanglement detection (and possibly quantification). In the context of this paper, a scalar is considered to be physical if it is invariant under any orthogonal transformation of the set of measurements. For instance, if Alice’s observables correspond to measurements of spin in a given orthogonal set of directions, then an orthogonal transformation describes a rotation of Alice’s entire lab; similarly, Bob’s lab may be rotated independently of Alice’s. Mathematically, transformations of this type are described by:

$$C \rightarrow U_A C U_B^T, \quad U_A \in O(d_A), U_B \in O(d_B)$$

(3)

where $O(d_A)$ and $O(d_B)$ are the groups of real orthogonal $d_A \times d_A$ and $d_B \times d_B$ matrices respectively. Introducing into the SVD of $C$, written as $C = V_A \Sigma V_B^T$, yields:

$$V_A \Sigma V_B^T \rightarrow U_A V_A \Sigma V_B^T U_B^T.$$  

(4)

Since $U_A$ and $V_A$ are elements of $O(d_A)$ (and similarly for the matrices with the subscript $B$), we may observe that a general orthogonal transformation of $C = V_A \Sigma V_B^T$ reduces to the substitution of $V_A$ and $V_B$ by any other elements of their respective orthogonal groups. Thus, it is clear that any physical scalar derived from $C$ can only depend on its singular values, i.e. the operator-Schmidt coefficients.

The simplest candidates for scalars produced by a matrix are its trace, determinant, and any type of matrix norm. However, $\text{tr}(C)$ is not a physical scalar in the sense described above; and a broad class of matrix norms are given as special cases of the scalars constructed in this section. Thus, for now we wish to consider $\det C$. The determinant of a quantum cross-correlation matrix can help detect entanglement, and may also serve as a measure of entanglement for two-qubit pure states (see Appendix A.3) and two-mode Gaussian states [27, 29].

However, in more general scenarios, there are states in which the mutual information between Alice and Bob stems from specific subspaces of their respective vector spaces (in pure states, the dimension of these subspaces is given by the Schmidt rank). To accommodate these cases, one should go over all possible subspaces of some given dimension and consider the determinant of the matrix comprised of correlations between their basis elements. Then, one could construct a measure as some function of all those determinants. One way of doing so is treat them as entries of a matrix and take its norm.

In light of the observations above, we define the Correlation Minor Norm with parameters $h$ and $p = 2$:

$$M_{h,2} \triangleq \sqrt{\sum_{B \in \{ \binom{d_B^2}{h} \}} \sum_{S \in \{ \binom{d_A^2}{h} \}} |\det C_{R,S}|^2}$$

(5)

where $\binom{d}{h}$ denotes the set of $b$-combinations of $[d]$ (this notation is common in the Cauchy-Binet formula), $C_{R,S}$ is the matrix whose rows are the rows of $C$ at indices from $R$ and whose columns are the columns of $C$ at indices from $S$, and $1 \leq h \leq \min \{d_A^2, d_B^2\}$.

Note that $M_{h,2}$ is the Frobenius norm of a matrix $N$ of size $\binom{d_A^2}{h} \times \binom{d_B^2}{h}$, defined by:

$$N_{ij} \triangleq \det C_{R,S}$$

(6)
where we have numbered the sets’ elements:
\[
\begin{align*}
\left[\frac{d_A^2}{h}\right] & \triangleq \left\{ R_1, \ldots, R\left(\frac{d_A^2}{h}\right) \right\} \\
\left[\frac{d_B^2}{h}\right] & \triangleq \left\{ S_1, \ldots, S\left(\frac{d_B^2}{h}\right) \right\}.
\end{align*}
\]

Such a matrix \( N \) is known as the \( h \)-th compound matrix of \( C \), and is denoted by \( C_h (C) \). Now, recall the Schatten \( p \)-norm of any matrix \( M \) is defined by \( \|M\|_p := \|\sigma (M)\|_p \), i.e., the vector \( p \)-norm of the vector composed of the singular values of \( M \). Schatten \( p \)-norms lead to a generalization of the definition \(5\): for \( p \in [1, \infty) \), define the Correlation Minor Norm with parameters \( h \) and \( p \) as:
\[
M_{h,p} = \|C_h (C)\|_p, \tag{7}
\]
i.e., it is the Schatten \( p \)-norm of the \( h \)-th compound matrix of the correlation matrix \( C \). Substituting the known relation between the singular values of any matrix and its compound matrix (see Appendix \( B \)), one obtains the following formula for computing the Correlation Minor Norm:
\[
M_{h,p} = \left( \sum_{R \in \left[\frac{d^2}{h}\right]} \prod_{k \in R} [\sigma_k (C)]^p \right)^{1/p}. \tag{8}
\]

where \( d = \min \{d_A, d_B\} \), and \( \sigma_k (C) \) denotes the \( k \)-th singular value of \( C \). This implies that \( M_{h,p} \) is indeed a physical scalar. Note that the Schatten \( p \)-norm of \( C \) itself is obtained as a special case, for \( h = 1 \).

IV. ENTANGLEMENT DETECTION USING THE CORRELATION MINOR NORM

For general mixed states, there are a few known links between Schmidt coefficients and entanglement detection; the best-known is probably the CCNR criterion: If \( \sum_{k=1}^{d^2} \lambda_k > 1 \), then \( \rho \) is entangled \(30\). The Correlation Minor Norm allows for an equivalent formulation: if \( M_{h=1,p=1} > 1 \), then \( \rho \) is entangled.

The CCNR criterion has an additional immediate consequence regarding the CMN: since \( M_{h,p} \) is a monotonically increasing function of the operator-Schmidt coefficients \( \lambda_k \), there is an upper bound for the value it may obtain without violating the inequality \( \sum_{k=1}^{d^2} \lambda_k \leq 1 \). Thus, for all \( h \) and \( p \), there exists some positive number \( B = B (d_A, d_B, h, p) \) with the property: if \( \rho \) is separable, then \( M_{h,p} \leq B (d_A, d_B, h, p) \) with the property: if \( \rho \) is separable, then \( M_{h,p} \leq B (d_A, d_B, h, p) \). This implies the Correlation Minor Norm can be used to detect entanglement by the following procedure: given a state \( \rho \), the corresponding correlation matrix \( C \) is obtained - either by computation or by direct measurement; then, the SVD of \( C \) is used to find the singular values, and these are substituted in \(8\) to compute the desired CMN, \( M_{h,p} \); and finally, \( M_{h,p} \) is compared with \( B (d_A, d_B, h, p) \). If \( M_{h,p} \leq B (d_A, d_B, h, p) \), we cannot deduce anything. However, if \( M_{h,p} > B (d_A, d_B, h, p) \), we infer the state \( \rho \) is entangled. The remainder of this section deals with results regarding the upper bounds \( B (d_A, d_B, h, p) \).

In \(23\), Lupo et al. generalize the CCNR criterion in the following way: they construct all symmetric polynomials of the Schmidt coefficients \( \lambda_k \) of \( \rho \), and find bounds on these assuming \( \rho \) is separable. The \( h \)-th symmetric polynomial is identical to \( M_{h=1,p=1} \). A more recent work \(24\) which cites \(23\), makes the following important claim: assuming \( d_A = d_B \), they find a tight bound on the \( h \)-th symmetric polynomial (for separable states), and prove that as an entanglement detector it is no stronger than the CCNR criterion. Since the conjectures presented in this section imply this is true for the CMN with \( p = \infty \) as well, we conjecture that for \( d_A = d_B \) and any value of \( p \), the CMN is no stronger than the CCNR criterion as an entanglement detector.

However, in the case where \( d_A \neq d_B \), it seems the CMN may detect entanglement in cases where CCNR does not. Let us define following \(31-33\), a state in Filter Normal Form (FNF) as a state \( \rho \) for which any traceless Alice-observable \( A \) and any traceless Bob-observable \( B \) have vanishing expectation values; i.e., \( (A \otimes 1)\rho = (1 \otimes B)\rho = 0 \). Then, we have the following result:

**Theorem 1.** Assume \( D := \max \{d_A, d_B\} \leq d^2 \) and \( h > 1 \). Then, for any separable state in FNF Normal Form:
\[
M_{h,p=1} \leq S_h (\alpha, \beta, \ldots, \beta) \tag{9}
\]

where \( \alpha := 1/\sqrt{Dd} \), \( \beta := \sqrt{\frac{D-1}{D(d^2-1)}}, \beta = 1/\sqrt{D(d^2-1)}, \) and \( S_h \) is the \( h \)-th symmetric polynomial in \( d^2 \) variables.

Proof may be found in Appendix \( D2 \). Moreover, we conjecture the following theorem still holds with the assumption of the state being in FNF removed. If proven, this conjecture would have explained the upper bounds presented in \(23\) for \( d_A \neq d_B \), which had been found numerically.

Before presenting the next result, let us introduce quantum designs \(34\). A quantum design in dimension \( b \) with \( v \) elements is simply a set of \( v \) orthogonal projections \( \{P_k\}_{k=1}^v \) on \( \mathbb{C}^b \). A quantum design is regular with \( r = 1 \) if all projections are pure (i.e. one-dimensional); it is coherent if the sum \( \sum_k P_k \) is proportional to the identity operator; and it has degree \( 1 \) if there exists \( \mu \in \mathbb{R} \) such that \( \forall k \neq l, \text{tr} (P_k P_l) = \mu \). If a quantum design has all three qualities, then \( \mu = \frac{1}{d^2-1} \).

A regular, coherent, degree-1 quantum design with \( r = 1 \) having \( v \) elements, is simply a set of \( v \) “equally spaced” pure states in the same space. For example, such a quantum design in dimension \( d \) containing \( d^2 \) elements is known as a symmetric, informationally complete, positive operator-valued measure (SIC-POVM) \(35\).

The following theorem tells us how to construct a separable state saturating \(9\) using quantum designs:

**Theorem 2.** Let \( \{P_{A_k}^A\}_{k=1}^{d^2}, \{P_{B_k}^B\}_{k=1}^{d^2} \) be sets of pure projections comprising regular, coherent, degree-1 quantum designs with \( r = 1 \), in dimensions \( d_A, d_B \) respectively and having \( d^2 \)
elements each. Define a state:
\[ \rho = \frac{1}{d^2} \sum_{k=1}^{d^2} P_k^A \otimes P_k^B \] (10)

Then, the operator-Schmidt coefficients of \( \rho \) are \( \alpha \) with multiplicity one and \( \beta \) with multiplicity \( d^2 - 1 \).

The proof appears in Appendix [D3]. Note the last two theorems have the following special case: for \( h = d^2 \), they imply that the above state maximizes the product of all Schmidt coefficients; i.e., it maximizes \( \mathcal{M}_{h=d^2,p} \) for all \( p \).

Furthermore, we have similar claims for \( p = \infty \).

**Theorem 3.** Let \( \rho \) be a separable state in FNF, and \( h \geq \sqrt{Dd} \). Then:
\[ \mathcal{M}_{h,p=\infty} \leq \frac{1}{\sqrt{Dd}} \left[ \frac{D - 1}{D} \frac{d - 1}{d} \right]^{k_{\alpha,\beta}}. \] (11)

Proof may be found in Appendix [D4]. As in Theorem 1, we conjecture this theorem still holds without the assumption that \( \rho \) is in FNF. Evidence for why we believe this conjecture to be true may be found in Appendix [D5]. The following theorem yields a way of saturating the bound (11):

**Theorem 4.** Let \( \{ P_k^A \}_{k=1}^h \), \( \{ P_k^B \}_{k=1}^h \) be sets of pure projections comprising regular, coherent, degree-1 quantum designs with \( r = 1 \), in dimensions \( d_A, d_B \) respectively and having \( h \) elements each. Define a state:
\[ \rho = \frac{1}{h} \sum_{k=1}^{h} P_k^A \otimes P_k^B \] (12)

Then, the operator-Schmidt coefficients of \( \rho \) are \( \alpha \) with multiplicity one and \( \beta' = \sqrt{\frac{D-1}{D} \frac{d-1}{d}} \) with multiplicity \( h - 1 \).

The proof appears in Appendix [D6]. Note the coherence of \( \{ P_k^{A/B} \} \) ensures the state (12) is in FNF. Moreover, the constants \( \mu_{A/B} := \frac{h - d_A d_B}{d_A d_B (h - 1)} \) enter the operator-Schmidt coefficients (and thus the upper bound (11)) elegantly: \( \beta' = \sqrt{\frac{1 - \mu_A}{h} \frac{1 - \mu_B}{h}} \).

We hypothesize that upper bounds over \( \mathcal{M}_{h,p} \) for any value of \( p \) may be characterized using quantum designs. If this hypothesis is proven, then separable states built using such quantum designs are, in a way, on the “edges” of the convex separable set. However, one should note that quantum designs in a given dimension with a given number of elements do not always exist; the above theorems hold only in the cases where they do exist.

![Figure 1](image_url)

Figure 1. The dashed line illustrates the von-Neumann entanglement entropy \( S(\rho_A) \), while the solid lines plot linear functions of correlation minor norms, for a family of two-qutrit states with pure-state-Schmidt coefficients \( s_1 = s_2 = \frac{\cos \alpha}{\sqrt{2}} \) and \( s_3 = \sin \alpha \) (where \( 0 \leq \alpha \leq \pi/2 \)). The blue line corresponds to \( 2 \mathcal{M}_{h=2,p=1/3} \), and the red line is \( \mathcal{M}_{h=1,p=1} = 1 \). These functions where chosen as simple modifications of the CMNs, such that they would agree with \( S(\rho_A) \) on the “boundary”, i.e. \( \alpha = 0, \pi/2 \).

V. FURTHER RESULTS AND OPEN QUESTIONS

A. Relation to entanglement entropy for pure states

Let \( |\psi\rangle \) be a pure state, and let \( s_1, \ldots, s_d \) denote its “pure-state-Schmidt coefficients” (i.e., the ones arising when writing the Schmidt decomposition for pure states of \( |\psi\rangle \)). Then, its operator-Schmidt coefficients are \( s_k s_l \), i.e. all the pairwise products of pure-state-Schmidt coefficients (if \( k \neq l \), \( s_k s_l \) appears as an operator-Schmidt coefficient with multiplicity \( 2 \); for proof please refer to Appendix [A2]).

For pure states, the Correlation Minor Norm is linked to the state’s Schmidt rank by the following observation: for all \( r \in [d] \), \( \mathcal{M}_{r,\rho} \neq 0 \) iff the state’s pure-state-Schmidt rank is at least \( r \). Thus, the Correlation Minor Norm may be used to find the Schmidt rank in pure states. Fig. [1] illustrates a comparison between \( \mathcal{M}_{h=(d-1)/2,p=2} \) and entanglement entropy. Their similarity supports the notion that \( \mathcal{M}_{h,p} \) and entanglement entropy. However, not all Correlation Minor Norms are useful for this purpose; in fact, the relation between operator-Schmidt coefficients and pure-state-Schmidt coefficients implies:
\[ \mathcal{M}_{h=1,p=2}^2 = \sum_{k,l} \mu_k^2 = \sum_{k,l} (s_k s_l)^2 = \left( \sum_k s_k^2 \right)^2 = 1. \] (13)

Thus, \( \mathcal{M}_{h=1,p=2} = 1 \) for any pure state, be it separable or entangled.

B. Improving on the CCNR criterion

In this section, we shall present an entangled state which may be detected by the CMN, but cannot be detected by
the CCNR criterion. First, let \( \rho_0 \) be the state for
d\_A = 3, d\_B = 2; and let \( \rho_1 := |\psi\rangle \langle \psi| \), where \( \psi := (|11\rangle + |20\rangle) / \sqrt{2} \). The state is constructed as follows:

\[
\rho_p = p\rho_1 + (1 - p) \rho_0. \tag{14}
\]

For \( p = 0.295 \) the state is entangled (easily verifiable by the PPT criterion). However, it is not detected by the CCNR criterion: \( M_{h=1,p=1} = 0.9981 < 1 \); and it is detected by the CMN: \( M_{h=2,p=1} = 0.3509 \), exceeding the bound \( \frac{2+3\sqrt{2}}{18} \approx 0.3468 \).

C. Relation to quantum discord?

Figure 2 plots the CMNs with \( p = 1 \) and the quantum discord for the two-qubit Werner states. Note that the Werner state with \( c = 1/3 \) has the highest discord among all two-qubit separable states [36]; moreover, it is precisely the state maximizing the CMNs with \( p = 1 \). This raises the question of how general this observation is; i.e., in which cases does the construction illustrated in Theorem 2 also yield a state of maximal discord amongst separable states? Furthermore, can the CMN (one or more of them), in these cases, be said to measure the strength of quantum correlations?

D. Multipartite scenarios?

In [37], the authors consider detection of genuine multipartite entanglement and non-full-separability using correlation tensors. Specifically, they consider tensors comprising all multipartite correlations between orthonormal bases to the traceless observables; and they find upper bounds on norms of matricizations of these tensors, such that exceeding these bounds implies the state is genuine multipartite entangled, or non-full-separably.

This paper may hint as to how our work may be generalized to the multipartite case: one could consider the full correlation tensor (i.e. correlations between bases to the entire space of observables, not just the traceless ones); then, the CMN with parameters \( h, p \) may be defined as the Schatten-\( p \)-norm of the \( h \)th compound matrix of a certain matricization of this tensor. The bounds shown in [37] could then be utilized to find two upper bounds on each of the CMNs - one for non-genuinely-entangled states, and another for fully-separable states. The question of which matricization should be used remains to be determined. Moreover, further work is required to find the states saturating these bounds.

VI. CONCLUSIONS

The tasks of entanglement detection and quantification are important for basic quantum science, as well as various quantum technologies. The current work was motivated by the following question: since bipartite entanglement can be characterized by correlations between all of the parties' observables, can it also be quantified via some norm of these correlations? As demonstrated by our results, the answer is likely to be affirmative.

We have defined the Correlation Minor Norm and explored its characteristics. This has allowed us to propose an approach for detecting entanglement both in pure and mixed states.

Furthermore, it was shown that for pure states, the Correlation Minor Norm allows one to determine the Schmidt rank, and perhaps also quantify the strength of quantum correlations. Given the dimensions of the two parties’ respective systems, one may choose a single set of operators which can be used for detecting entanglement in any state, be it pure or mixed.

Some directions for future research may include finding a quantitative connection between the correlation minor norm and quantum discord. Moreover, since the correlation matrix contains exactly the same information as the density matrix, perhaps one could develop dynamical equations for the Correlation Minor Norm.

Another possible generalization is considering multipartite systems, where the correlation matrix is replaced by a correlation tensor; applying appropriate generalizations of determinants and norms for tensors could yield generalizations of the CMN. These may answer some interesting questions, such as, whether the same quantum designs we considered here could still be used to saturate the separable-state bounds in multipartite systems.

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Appendix A: Entanglement detection using the quantum correlation matrix

1. The Operator-Schmidt Decomposition

Given any state \( \rho \) (either separable or entangled), one may write down the following unique decomposition:

\[
\rho = \sum_{k=1}^{d^2} \lambda_k G_k \otimes H_k \quad (A1)
\]

where \( d := \min \{d_A, d_B\} \), each \( \lambda_k \geq 0 \) is a real scalar, and the sets \( \{G_k\} \) and \( \{H_k\} \) form orthonormal bases of the \( d_A \times d_B \) Hermitian matrices. Note this is not necessarily a “separable decomposition”, since \( G_k, H_k \) are not compelled to be positive semi-definite.

Let us assume that \( \lambda_k \) are in non-increasing order. We shall demonstrate that the SVD of the cross-correlation matrix \( C \) is equivalent to the Operator-Schmidt Decomposition.

**Theorem.** Given a state \( \rho \), let \( C \) be the second moment matrix of the orthonormal sets \( \{A_i\}, \{B_j\} \), defined by \( C_{ij} = \langle A_i | B_j \rangle \rho \). Let \( C \) have the SVD \( C = U \Sigma V^T \) with singular values \( \sigma_1 \geq \ldots \geq \sigma_d \). Then, the unique decomposition \( (A1) \) of \( \rho \) satisfies the following:

1. \( \lambda_k = \sigma_k \)
2. \( G_k = \sum_{i=1}^{d^2} U_{ik} A_i \)
3. \( H_k = \sum_{j=1}^{d^2} V_{jk} B_j \).

Proof outline: since \( \{A_i \otimes B_j\}_{i,j} \) comprise a basis to the set of \( d^2 \otimes d^2 \) Hermitian matrices, the matrix \( C \) suffices in order to fully characterize \( \rho \). Thus, since the Operator-Schmidt decomposition is unique, all is left to do is verify that \( \rho \) with the above Operator-Schmidt decomposition reproduces the same correlations \( C_{ij} \), which is straightforward. All of this is well-known.

2. Operator-Schmidt decomposition for pure states

Let \( |\psi\rangle \) be a pure state given in its pure-state-Schmidt decomposition:

\[
|\psi\rangle = \sum_{k=1}^{d} s_k |\phi_k\rangle \otimes |\xi_k\rangle \quad (A2)
\]

The appropriate density matrix:

\[
\rho = |\psi\rangle \langle \psi| = \sum_{k,l=1}^{d} s_k s_l |\phi_k\rangle \langle \phi_l| \otimes |\xi_k\rangle \langle \xi_l| \quad (A3)
\]

Let us fix \( k, l \) s.t. \( k < l \). \( k \) and \( l \) appear in two terms of the sum: \( s_k s_l \{ |\phi_k\rangle \langle \phi_l| \otimes |\xi_k\rangle \langle \xi_l| + |\phi_l\rangle \langle \phi_k| \otimes |\xi_l\rangle \langle \xi_k| \} \). We wish to write down the parenthesized expression in the form \( G_{kl} \otimes H_{kl} + G_{lk} \otimes H_{lk} \), where \( G_{kl}, H_{kl}, G_{lk}, H_{lk} \) are all trace-normalized Hermitian operators, and \( \text{tr} \{G_{kl} G_{lk}\} = \text{tr} \{H_{kl} H_{lk}\} = 0 \). Indeed, this is achieved by setting:

\[
G_{kl} := \frac{|\phi_k\rangle \langle \phi_l| + |\phi_l\rangle \langle \phi_k|}{\sqrt{2}}; \\
H_{kl} := \frac{|\xi_k\rangle \langle \xi_l| + |\xi_l\rangle \langle \xi_k|}{\sqrt{2}}; \\
G_{lk} := i \{ |\phi_k\rangle \langle \phi_l| - |\phi_l\rangle \langle \phi_k| \};
\]

\[
H_{lk} := - i \{ |\xi_k\rangle \langle \xi_l| - |\xi_l\rangle \langle \xi_k| \} \quad (A4)
\]

By supplementing the notations \( G_{kk}, H_{kk} \) are both orthonormal sets of operators, \( \{A_i\}, \{B_j\} \) are the operator-Schmidt decomposition of \( \rho \); thus, \( \{s_k s_l\} \) are its operator-Schmidt coefficients.

3. \( \det C \) for two-qubit pure states

Let \( |\psi\rangle \) be a two-qubit pure state, given in its pure-state-Schmidt decomposition:

\[
|\psi\rangle = s_1 |\phi_1\rangle \otimes |\xi_1\rangle + s_2 |\phi_2\rangle \otimes |\xi_2\rangle \quad (A6)
\]

From the previous subsection, its operator-Schmidt coefficients are \( s_1^2, s_1 s_2, s_2 s_1, s_2^2 \). Moreover, from Appendix \( A1 \), these are also the singular values of its correlation matrix. Thus:

\[
\det C = \prod_k \sigma_k (C) = s_1^4 s_2^4 = (s_1^2 s_2^2)^2 = \left[ s_1^2 (1 - s_1^2) \right]^2 \quad (A7)
\]

where the final transition follows from the normalization condition \( s_1^2 + s_2^2 = 1 \). An interesting observation is that \( \sqrt{\det C} \) is proportional to the interferometric distinguishability measure studied in \([38, 42]\); moreover, \([39]\) illustrates the striking resemblance between this measure and entanglement entropy. Thus, for two-qubit pure states, \( \det C \) indeed quantifies entanglement. As an aside, we note that the distinguishability measure is generalized for a certain family of Gaussian states in \([43]\).

Appendix B: CMN and SVD

In order to compute the CMN, one should seek a relation between the singular values of given matrix, and the singular values of its compound matrices. Such a relation is known \([44]\):
Lemma. Let $E$ be a $n \times n$ matrix. The singular values of $C_h (E)$, are the $\binom{n}{h}$ possible products $\sigma_i \cdots \sigma_i$.

Which implies:

$$\| C_h (E) \|_p = \left( \sum_{R \in \binom{[n]}{h}} \prod_{k \in R} |\sigma_k (E)|^p \right)^{1/p}$$  \hspace{1cm} (B1)

where:

$$\binom{[n]}{h} \triangleq \{ R \in 2^{[n]} : |R| = h \}$$ \hspace{1cm} (B2)

i.e., $\binom{[n]}{h}$ denotes the set of subsets of $[n]$ having cardinality $h$. Thus we obtain the following formula for the correlation minor norm, using only the singular values of the second moment matrix:

$$\mathcal{M}_{h,p} = \left( \sum_{R \in \binom{[d^2]}{k}} \prod_{k \in R} |\sigma_k (C)|^p \right)^{1/p}.$$ \hspace{1cm} (B3)

Note that the CMN yields another formulation for the CCNR criterion:

$$\forall \rho \in S, \quad \mathcal{M}_{h=1, p=1} \leq 1,$$ \hspace{1cm} (B4)

where $S$ denotes the set of separable states. The CMN also allows for a new formulation of the CM criterion [10]: For any separable state in FNF, $\mathcal{M}_{h=1, p=1} \leq \frac{1 + \sqrt{(D-1)(d-1)}}{\sqrt{Dd}}$.

Note the RHS is strictly smaller than 1 iff $D \neq d$.

### Appendix C: The operator-Schmidt decomposition of a separable state

Assume $D = d_A \geq d_B = d$. We wish to find the Schmidt coefficients of the following density matrix:

$$\rho = \sum_{k=1}^{n} p_k O_k \otimes Q_k.$$ \hspace{1cm} (C1)

1. Aside: $n = d^2$

First, let us prove we can always assume that $n = d^2$ (however, $O_k, Q_k$ are not necessarily pure): Suppose $n > d^2$. It suffices to show we can always transform [C1] to a similar state with $n - 1$. Since the $Q_k$ all belong to the space of $d \times d$ Hermitian matrices, they must be linearly dependent; i.e., thus, one of them (w.l.g. it is $Q_n$) may be written as a linear combination of the others:

$$\exists c_1, \ldots, c_{n-1} : Q_n = \sum_{k=1}^{n-1} c_k Q_k.$$ \hspace{1cm} (C2)

where $\text{tr} (Q_n) = 1$ implies $\sum c_k = 1$. Plugging this into (C1) yields:

$$\rho = \sum_{k=1}^{n-1} p_k O_k \otimes Q_k + p_n O_n \otimes Q_n = \sum_{k=1}^{n-1} (p_k O_k + p_n c_k O_n) \otimes Q_k = \sum_{k=1}^{n-1} \underbrace{(p_k + p_n c_k)}_{\bar{\rho}_k} O_k \otimes Q_k,$$

To conclude the proof, one should verify $\sum_{k=1}^{n-1} \bar{\rho}_k = 1$ and $\text{tr} (\bar{O}_k) = 1$. This is straightforward so we don’t show it here.

2. Realignment and correlation in Bloch vector representation

Let us write the realigned density matrix:

$$\rho_R = \sum_{k=1}^{n} p_k \text{vec} O_k \text{vec} Q_k.$$ \hspace{1cm} (C4)

Now, we shall write down $\rho_R \rho_R$ as a “superoperator” $\hat{\mathcal{P}}$ - i.e., its operates on $d \times d$ Hermitian operators:

$$\hat{\mathcal{P}} = \sum_{k=1}^{n} p_k O_k \otimes Q_k$$ \hspace{1cm} (C5)

here the tensor product sign $\otimes$ has a meaning closer to its original one, rather than its regular abuse in quantum information theory; that is, it “wants” to act on a $d \times d$ Hermitian operator with the Hilbert-Schmidt inner product as follows:

$$(A \otimes B) C = \langle B, C \rangle A = \text{tr} (B^\dagger C) A$$ \hspace{1cm} (C6)

where $B, C$ are both all $d \times d$ (Hermitian) operators. For the sake of simplicity, we switch to the Bloch representation of the operators, satisfying the following properties:

1. Each operator $Q_k$ is written as $Q_k = \frac{1}{\sqrt{d}} q_k^\mu \hat{\sigma}_\mu$, where

   $\mu = 0, 1, \ldots, d^2 - 1$, $\hat{\sigma}_0 = 1_d / \sqrt{d}$, and the other $\hat{\sigma}_i$ are (traceless) $d \times d$ Hermitian operators s.t. all $\hat{\sigma}_\mu$ are an orthogonal set (w.r.t. the Hilbert-Schmidt inner product), satisfying:

   $$\text{tr} (\hat{\sigma}_\nu \hat{\sigma}_\mu) = \delta_{\nu\mu}$$

   and the $q_k^\mu$ are real numbers, given by:

   $$q_k^\mu = \sqrt{d} \cdot \text{tr} (\hat{\sigma}_\mu Q_k).$$ \hspace{1cm} (C7)

   Note that $q_k^0 = \text{tr} (Q_k) = 1$.

2. This notation allows one to compute the Hilbert-
Schmidt inner product of two operators $U = \frac{1}{\sqrt{d}} u^\sigma \hat{\sigma}_\nu$ and $V = \frac{1}{\sqrt{d}} v^\mu \hat{\sigma}_\mu$ as follows:

$$
(U, V)_{HS} = \mathrm{tr} (UV) = \frac{1}{d} \mathrm{tr} (u^\sigma \hat{\sigma}_\nu v^\mu \hat{\sigma}_\mu) = \frac{1}{d} \mathrm{tr} (\hat{\sigma}_\nu u^\sigma v^\mu \hat{\sigma}_\mu) = \frac{1}{d} \mathrm{tr} (u^\sigma v^\mu) = \frac{1}{d} d \mu v_\mu.
$$

And similarly for the operators $O_k$ (where $d$ is replaced with $D := \max \{d_A, d_B\} = d_A$):

$$
O_k = \frac{1}{\sqrt{D}} O_k \hat{\xi}_\gamma, \quad \mathrm{tr} (O_k \hat{\xi}_\gamma) = \delta_{\gamma \eta}, \quad O_k^\gamma = \sqrt{D} \mathrm{tr} (O_k \hat{\xi}_\eta).
$$

Once Hermitian operators are represented by column vectors (using the bases $\{\xi_\gamma\}$, $\{\hat{\sigma}_\mu\}$), the superoperator $\hat{\mathcal{P}}$ may once again be written as a $D^2 \times d^2$ matrix:

$$
\mathcal{C}^{\gamma \mu} = \frac{1}{\sqrt{D} d} \sum_{k=1}^n p_k O_k^\gamma g_k^\mu = \frac{1}{\sqrt{D} d} \mathcal{O} \mathcal{P} \mathcal{Q}^T, \quad (C9)
$$

where $O_k^\gamma := O_k^\gamma$, $Q_k^\gamma := q_k^\gamma$ are the matrices with columns comprised of the Bloch vectors of $O_i$, $Q_k$ respectively; and $\mathcal{P} := \mathrm{diag} [p_1, \ldots, p_n]$. Later on, it shall be useful to consider the matrix $\mathcal{R} := \mathcal{C}^T \mathcal{C}$, since its eigenvalues are the squared singular values of $\mathcal{C}$:

$$
\mathcal{R} = \frac{1}{D d} (\mathcal{O} \mathcal{P} \mathcal{Q}^T)^T \mathcal{O} \mathcal{P} \mathcal{Q}^T = \frac{1}{D d} \mathcal{O} \mathcal{Q} \mathcal{P} \mathcal{Q}^T \mathcal{O}^T. \quad (C10)
$$

### 3. Separability

Up until this point we still haven’t used the separability of $\rho$; it manifests in the fact that the operators $O_k$, $Q_1$ all represent states, implying:

$$
\forall k, \quad \begin{aligned}
\mathrm{tr} (O_k) &= 1 \\
\mathrm{tr} (O_k^2) &\leq 1 \\
\mathrm{tr} (Q_k^2) &\leq 1
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
O_k^0 &= q_k^0 = 1 \\
O_k^\mu &= q_k^\mu \leq D,
\end{aligned}
\quad (C11)

where only the Greek indices are summed upon. I.e., the first row of $\mathcal{O}$, $\mathcal{Q}$ is all ones; and the main diagonals of $\frac{1}{2} \mathcal{O}^T \mathcal{O}$, $\frac{1}{2} \mathcal{Q}^T \mathcal{Q}$ are bounded by one. Let us denote $\mathcal{O}_+$, $\mathcal{Q}_+$ as the matrices obtained by removing the all-ones first rows from $\mathcal{O}$, $\mathcal{Q}$ respectively. It would be useful to unify the two conditions, by writing down the summation explicitly:

$$
\forall k, \quad 1 \geq \mathrm{tr} (O_k^2) = \frac{1}{D} \sum_{\mu=0}^{D^2-1} O_k^\mu O_k^\mu = \frac{1}{D} \left( 1 + \sum_{\mu=1}^{D^2-1} O_k^\mu O_k^\mu \right) = \frac{1}{D} \left( 1 + [\mathcal{O}_+^T \mathcal{O}_+]_{kk} \right), \quad (C12)
$$

implying:

$$
\forall k, \quad [\mathcal{O}_+^T \mathcal{O}_+]_{kk} \leq D - 1 \quad (C13)
$$

and similarly:

$$
\forall k, \quad [\mathcal{Q}_+^T \mathcal{Q}_+]_{kk} \leq d - 1. \quad (C14)
$$

Finally, we note that:

$$
r^\gamma = \mathcal{C}_{\gamma 0} = \frac{1}{\sqrt{D d}} \sum_k p_k O_k^\gamma \quad (C15)
$$

implying:

$$
r = \frac{1}{\sqrt{D d}} \mathcal{O}_+ \mathcal{P}. \quad (C16)
$$

This is unsurprising, since $[1, r^T]$ is the Bloch vector of $\rho_A = \sum_k p_k O_k$. Similarly:

$$
s = \frac{1}{\sqrt{D d}} \mathcal{Q}_+ \mathcal{P}. \quad (C17)
$$

### 4. FNF

Let us assume that Alice and Bob choose their orthonormal observables such that $A_1 = \frac{1}{\sqrt{d_A}} A_{d_A}$ and $B_1 = \frac{1}{\sqrt{d_B}} B_{d_B}$, i.e. the trivial measurements. Note this implies that all the other observables $A_i, B_j$ are traceless. Given this assumption, we are motivated to introduce the following notation (similar to [12]):

$$
\mathcal{C} = \begin{bmatrix}
\frac{1}{\sqrt{D d}} & s \\
\mathcal{R} & T
\end{bmatrix}, \quad (C18)
$$

i.e.: $r_i := \langle A_i \otimes 1 \rangle / \sqrt{\nu_B}$, $s_j := \langle 1 \otimes B_j \rangle / \sqrt{\nu_A}$, and $T$ is the correlation matrix of only traceless observables. A state $\rho$ is said to be in FNF if $r = 0$ and $s = 0$. Any state may be transformed to FNF (using SLOCC), such that the original state is separable iff the transformed state is separable.

We note:

$$
T = \frac{1}{\sqrt{d d}} \mathcal{O}_+ \mathcal{Q}_+ \mathcal{Q}_+^T. \quad (C19)
$$

A recent paper [11] has used similar ideas to construct a necessary and sufficient separability criterion; in fact, they state that a correlation matrix $T$ describes a separable state in FNF iff it admits a decomposition of the form [C12], where $P = \mathrm{diag} p$ is diagonal, real, non-negative and has unit trace; and $\mathcal{O}_+ \mathcal{P} = 0$, $\mathcal{Q}_+ \mathcal{P} = 0$. The latter conditions are related to the FNF: from [C1], we observe that $\rho$ is in FNF iff $\sum_k p_k O_k$, $\sum_k p_k Q_k$ are both proportional to the identity; considering this statement in Bloch vector terms readily implies these conditions.
Appendix D: Proving the upper bounds

1. Preliminaries

Observe the following:

\[
\sum_{j=1}^{d^2-1} \sigma_j (T) = \frac{1}{\sqrt{Dd}} \sum_{k=1}^{d^2-1} \sigma_k \left( O_+ P Q_+^T \right) = \frac{1}{\sqrt{Dd}} \left\| O_+ \sqrt{P} \sqrt{P} Q_+^T \right\|_1 \leq \frac{1}{\sqrt{Dd}} \left\| O_+ \sqrt{P} \right\|_2 \left\| Q_+ \sqrt{P} \right\|_2
\]

where the final transition follows from Hölder's inequality. Let us find bounds on the 2-norms:

\[
\left\| O_+ \sqrt{P} \right\|_2^2 = \text{tr} \left( \sqrt{P} O_+^T O_+ \sqrt{P} \right) = \text{tr} \left( PO_+^T O_+ P \right) = \sum_{k=1}^{d^2} \rho_k [O_+^T O_+]_{kk} \leq D - 1
\]

and similarly,

\[
\left\| Q_+ \sqrt{P} \right\|_2^2 \leq d - 1.
\]

Substitution of the latter two in (D1) yields

\[
\sum_{k=1}^{d^2-1} \sigma_k \leq \sqrt{\frac{D - 1}{D} \frac{d - 1}{d}}.
\]

Note this inequality is equivalent to the separability criterion defined by de Vicente in [12] (the dV criterion), de Vicente defines a matrix $T$ similar to our $T$; in fact, $T = \frac{Dd}{(d - 1)}$. The dV criterion states that for any separable state,

\[
\|T\|_F \leq \sqrt{\frac{Dd (D - 1) (d - 1)}{4}},
\]

where $\|\cdot\|_F$ denotes the Ky-Fan norm, i.e. the Schatten 1-norm (also known as the trace norm or nuclear norm). (D5) implies:

\[
\sum_{j=1}^{d^2-1} \sigma_j (T) = \|T\|_F = \frac{2}{Dd} \|T\|_F \leq \sqrt{\frac{D - 1}{D} \frac{d - 1}{d}}.
\]

2. Proof of the upper bound of $M_{h,p=1}$

In this subsection, we wish to use the results of the previous section to find bounds on $M_{h,p=1} = \| C_h (C) \|$ for separable states. Clearly, the tight upper bound for $h = 1$ is $M_{h=1,p=1} \leq 1$ (CCNR). Here we add three other assumptions: one regarding the domain of $h - h > 1$; another is $D \leq d^2$; and finally, we assume the state is in FNF.

Thus, $\sigma_0 = 1/\sqrt{Dd}$, and we obtain:

\[
M_{h,p=1} = S_h \left( 1/\sqrt{Dd}, \sigma_1, \ldots, \sigma_{d^2-1} \right) = \frac{1}{\sqrt{Dd}} S_{h-1} (\sigma_1, \ldots, \sigma_{d^2-1}) + S_h (\sigma_1, \ldots, \sigma_{d^2-1})
\]

Let us denote $s := \sum_{k=1}^{d^2-1} \sigma_k$ and $\beta := \frac{1}{\sqrt{Dd}} \sqrt{\frac{D - 1}{d}}$. Clearly $s \leq \beta (d^2 - 1)$. Moreover, the vectors $\vec{e} := (\sigma_1, \ldots, \sigma_{d^2-1})$ and $\vec{e} := \frac{1}{\sqrt{d}} (1, \ldots, 1)$ both sum up to $s$; thus, $\vec{e} \succeq e^\otimes$ (denotes majorization). Since the symmetric polynomials $S_h$ are Schur concave, we obtain:

\[
S_h (\sigma_1, \ldots, \sigma_{d^2-1}) \leq S_h \left( \frac{s}{d^2 - 1}, \ldots, \frac{s}{d^2 - 1} \right).
\]

Next, we use the fact that $S_h$ is monotonically increasing in each of its variables, alongside the inequality $\frac{s}{d^2 - 1} \leq \beta$, to obtain:

\[
S_h (\sigma_1, \ldots, \sigma_{d^2-1}) \leq S_h (\beta, \ldots, \beta).
\]

Substitution in (D7) yields:

\[
M_{h,p=1} \leq \alpha S_{h-1} (\beta, \ldots, \beta) + S_h (\beta, \ldots, \beta) = S_h (\alpha, \beta, \ldots, \beta)
\]

where $\beta$ is always repeated $d^2 - 1$ times.

3. Saturating the upper bound of $M_{h,p=1}$

In our special construction of $\rho$ from Theorem 2, $n = d^2$ and $\forall k, \rho_k = 1/d^2$. The following additional assumptions follow from $\{O_k\}, \{Q_l\}$ being regular, coherent, degree-1 quantum designs with $r = 1$ and $d^2$ elements:

\[
\frac{1}{D} \left[ O^T O \right]_{kl} = \langle O_k, O_l \rangle = \begin{cases} 1; & k = l \\ \mu_A; & k \neq l \end{cases},
\]

\[
\frac{1}{d} \left[ Q^T Q \right]_{kl} = \langle Q_k, Q_l \rangle = \begin{cases} 1; & k = l \\ \mu_B; & k \neq l \end{cases}
\]

where $\mu_A/B = \frac{d^2 - d_{A/B}}{d_{A/B} (d^2 - 1)}$. Coherence has the following additional implications:

\[
\sum_{k=1}^{d^2} O_k = \frac{d}{D} 1_D, \quad \sum_{k=1}^{d^2} Q_k = d 1_d
\]

\[
\Rightarrow \sum_{k=1}^{d^2} O_k = \sum_{k=1}^{d^2} Q_k = \begin{cases} d^2; & \mu = 0 \\ 0; & \mu \neq 0 \end{cases}
\]

(D12)
Similarly, for all $O$ and for all $P$ where we have plugged $D$ into (D11). Diagonalization of this matrix is rather straightforward; it is not difficult to obtain that it has two distinct eigenvalues:

$$\lambda_0 = \text{with multiplicity } 1, \text{ where the eigenspace is spanned by } 1 := (1, \ldots, 1)^T; \quad \text{and }$$

$$\lambda_1 = \frac{d^2 (D - 1)}{d^2 - 1} \text{ with multiplicity } d^2 - 1 \text{ and eigenspace }\Lambda := \text{span } \{1\}^\perp.$$  

This demonstrates that $O^T \omega_+^T$ behaves as a scalar matrix, when its domain is restricted to $\Lambda \subset \mathbb{R}^{d^2}$. Thus, our next step would be showing that $Q^T_+ : \mathbb{R}^{d^2-1} \rightarrow \mathbb{R}^{d^2}$ performs exactly this restriction.

In other words, we wish to prove that $\text{Im } Q^T_+ = \Lambda$. First note $Q^T_+$ has an empty kernel, since otherwise there exists a nonzero vector orthogonal to all vectors in $\{q_1^+\}$; this would have implied that $\{q_k^+\}$ do not span the entire $d^2 - 1$-dimensional space of traceless Hermitian $d \times d$ operators, contradicting them comprising a SIC-POVM. Thus $\ker Q^T_+ = 0$, and from the rank-nullity theorem $Q^T_+$ must have rank $d^2 - 1$. Hence, demonstrating that $\text{Im } Q^T_+ \subset \Lambda$ would complete the proof. Let $v \in \mathbb{R}^{d^2-1}$; indeed, direct computation yields:

$$1 \cdot Q^T_+ v = Q_+ 1 \cdot v = 0,$$

where we have used (D13). Thus, for all $v \in \mathbb{R}^{d^2-1}$ we have:

$$O_+^T O_+ Q^T_+ v = \frac{d^2 (D - 1)}{d^2 - 1} Q^T_+ v,$$

implying:

$$O_+^T O_+ Q^T_+ = \frac{d^2 (D - 1)}{d^2 - 1} Q^T_+.$$

To complete the proof, we note that since $\{Q_k\}$ comprises a SIC-POVM, the columns of $Q_+$ form a (real) equiangular tight frame in dimension $d^2 - 1$ with $d^2$ elements [45]; thus, its frame operator $Q_+^T Q_+$ is scalar; more specifically, it satisfies:

$$Q_+ Q^T_+ = A 1_{d^2 - 1},$$

where $A$ is readily found by taking the trace of both sides:

$$(d^2 - 1) A = \text{tr } (Q_+ Q^T_+) = \text{tr } (Q^T_+ Q^T_+) = d^2 (d - 1),$$

where we have used (D11) again. Multiplying (D21) by $Q_+$ from the left yields:

$$Q_+ O_+^T O_+ Q^T_+ = \frac{d^2 (D - 1)}{d^2 - 1} Q_+ Q^T_+ = \frac{d^2 (D - 1)}{d^2 - 1} \frac{d^2 (d - 1)}{d^2 - 1} 1_{d^2 - 1},$$

which, when plugged into (D17), concludes the proof.

4. Proof of the upper bound of $M_{h, p = \infty}$

In this subsection we prove the bound on $M_{h, p = \infty} = ||C_h||_\infty$ for separable states in FNF. Clearly, the tight upper bound for $h = 1$ is $M_{h=1, p=\infty} \leq 1$. 

Thus, \( \sigma_0 = 1/\sqrt{Dd} \), and we obtain:

\[
\mathcal{M}_{h,p=\infty} = \| C_h (\mathcal{C}) \|_\infty = \frac{1}{\sqrt{Dd}} \| C_{h-1} (T) \|_\infty = \\
= \frac{1}{\sqrt{Dd}} \left\| C_{h-1} \left( \frac{1}{\sqrt{Dd}} O P Q^T \right) \right\|_\infty = \\
= (Dd)^{-h/2} \prod_{k=1}^{h-1} \sigma_k (O P Q^T) \leq \\
\leq (Dd)^{-h/2} (h-1)^{-(h-1)} \left[ \sum_{k=1}^{h-1} \sigma_k (O P Q^T) \right]^{h-1} .
\]

(D25)

Let us find a bound on the sum:

\[
\sum_{k=1}^{h-1} \sigma_k (O P Q^T) \leq \sum_{k=1}^{d^2} \sigma_k (O P Q^T) \leq \\
\leq \sqrt{(D-1)(d-1)}
\]

where we have used (D4). To conclude, substituting in (D25) obtains the bound:

\[
\mathcal{M}_{h,p=\infty} \leq \frac{1}{\sqrt{Dd}} \left[ \frac{D-1}{D(h-1)(d(h-1))} \right]^{\frac{h-1}{2}} .
\]

(D27)

5. Evidence to support Theorem\textsuperscript{B} without assuming FNF

\( \mathcal{M}_{h,p=\infty} \) is a monotonically non-decreasing differentiable function of the singular values \( \hat{\sigma} = (\sigma_0, \ldots, \sigma_{d^2-1}) \). The constraints on the domain of \( \hat{\sigma} \) are rather complicated and we do not know them all. However, we know some of them:

\[
\sigma_0 \geq 1/\sqrt{Dd}
\]

(D28)

\[
\forall j \in \{1, \ldots, d^2 - 1\}, \quad \sigma_j \geq 0
\]

(D29)

\[
\sum_{j=0}^{d^2} \sigma_j \leq 1
\]

(D30)

\[
\sum_{j=1}^{d^2-1} \sigma_j \leq \frac{D-1}{D} \frac{d-1}{d}
\]

(D31)

Furthermore, we know from numerical simulations that (D30) and (D31) cannot be saturated simultaneously (for \( Dd \leq h^2 \)); in fact, if it seems that the latter is saturated, then the state must be in FNF (thus saturating (D28) instead). Assume that this statement holds in general, and that no other constraints on \( \hat{\sigma} \) are relevant for global maxima analysis of \( \mathcal{M}_{h,p=\infty} \) - i.e., no other constraints need be saturated to obtain its global maxima; then, the theorem holds.

Since \( \mathcal{M}_{h,p=\infty} \) is monotonically increasing, one of the constraints (D30), (D31) must be saturated in a global maximum; otherwise, any one of the \( \sigma_k \) could be increased, thus increasing the value of \( \mathcal{M}_{h,p=\infty} \) without leaving the domain. According to our assumption, if (D31) is saturated the state is in FNF, which is the case we already treated. Thus, assume (D30) is saturated. If more than \( h \) singular values are nonzero, the point cannot be a global maximum, since we can increase the largest singular value while decreasing the smallest nonzero singular value, thus leaving (D30) saturated while increasing \( \mathcal{M}_{h,p=\infty} \). Thus, we may treat \( \mathcal{M}_{h,p=\infty} \) as a function depending only on the \( h \) largest singular values:

\[
f (\vec{\sigma}) = \prod_{k=0}^{h-1} \sigma_k,
\]

(D32)

And we are currently considering a global maximum \( \vec{\sigma}' = (\sigma_0, \ldots, \sigma_{h-1}) \) s.t. \( \sum_{k=0}^{h-1} \sigma_k = 1 \). Clearly, non of the \( \sigma_k \) can be zero - otherwise \( \vec{\sigma}' \) is a minimum rather than a maximum. Thus, of all the above constraints, \( \vec{\sigma}' \) saturates only (D30). Thus, it should be a local maximum of the following function constructed using a Lagrange multiplier:

\[
g (\vec{\sigma}, \lambda) := \prod_{k=0}^{h-1} \sigma_k - \lambda \left( \sum_{k=0}^{h-1} \sigma_k - 1 \right).
\]

(D33)

Thus, the partial derivatives with respect to \( \sigma_k \) should vanish:

\[
0 = \frac{\partial g}{\partial \sigma_l} = \prod_{k \neq l} \sigma_k - \lambda
\]

(D34)

implying that for all \( l \), \( \prod_{k \neq l} \sigma_k = \lambda \); but that could only happen if \( \sigma_0 = \ldots = \sigma_{h-1} = 1/h \). Substituting \( \sigma_0 = 1/h \) in (D28) would have implied \( h \leq \sqrt{Dd} \). If this is an equality, we are again in FNF; otherwise, it contradicts one of our initial assumptions. Thus, the only possible global maximum is the one obtained in FNF, for which Theorem\textsuperscript{B} holds.

6. Saturating the upper bound of \( \mathcal{M}_{h,p=\infty} \)

In our special construction of \( \rho \) from Theorem\textsuperscript{A} \( n = h \) and \( \forall k, p_k = 1/h \). Since \( \{O_k\} \), \( \{Q_k\} \) are regular, coherent, degree-1 quantum designs with \( r = 1 \) and \( h \) elements, (D11) still holds; the only difference is that in this case, \( \mu_{A/B} = \frac{h-d_{A/B}}{d_{A/B}(h-1)} \). As before, coherence has an additional implication:

\[
\sum_{k=1}^{h} O_k = \frac{h}{D} I_D, \quad \sum_{k=1}^{h} Q_k = \frac{h}{d} I_d
\]

(D35)

in matrix notation:

\[
\mathcal{O}I = \mathcal{Q}I = \begin{bmatrix} h & \vdots & \vdots \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \end{bmatrix}, \quad \mathcal{O} = \mathcal{Q} = 0,
\]

(D36)
where \( 1 \) is the vector whose \( h \) entries all equal 1. Substituting these implications allows one to obtain:

\[
\mathcal{R}_{\mu
u} = \frac{1}{h^2d} \sum_{k=1}^{h} q_k^\mu q_k^\nu + \frac{\mu\lambda}{h^2d} \sum_{k\neq l} q_k^\mu q_l^\nu. \tag{D37}
\]

Moreover, we have:

\[
\mathcal{R}_{00} = \frac{1}{h^2d} \sum_{k,l=1}^{h} (O_k, O_l) q_k^0 q_l^0 = \frac{1}{h^2d} \sum_{k=1}^{h} O_k \sum_{l=1}^{h} O_l = 1 = \frac{1}{\mathcal{D}^2d} (1_D, 1_D) = \frac{1}{d_A d_B}. \tag{D38}
\]

and for all \( \nu \neq 0 \):

\[
\mathcal{R}_{0\nu} = \frac{1}{h^2d} \sum_{k,l=1}^{h} (O_k, O_l) q_k^\nu q_l^0 = \frac{1}{h^2d} \sum_{l=1}^{h} \left( \sum_{k=1}^{h} (O_k, O_l) q_l^\nu \right) q_0^0 = 0. \tag{D39}
\]

Similarly, for all \( \mu \neq 0, \mathcal{R}_{\mu 0} = 0 \). Thus, \( \lambda_0 = \mathcal{R}_{00} = \frac{1}{d_A d_B} \) is an eigenvalue. To conclude the proof, we need to show that the submatrix of \( \mathcal{R} \) without the first row and column - again, \( \mathcal{T}^T \mathcal{T} \) - is composed of two diagonal blocks, one being a nontrivial scalar matrix and the other is the zero matrix.

We commence in a manner similar to what we have done in subsection [D3], writing down \( \mathcal{T}^T \mathcal{T} \):

\[
\mathcal{T}^T \mathcal{T} = \frac{1}{Ddh^2} \mathcal{Q}_+ \mathcal{O}^T_+ \mathcal{O}_+ \mathcal{Q}_+^T,
\]

and computing \( \mathcal{Q}_+^T \mathcal{O}_+ \):

\[
[\mathcal{O}_+^T \mathcal{Q}_+]_{kl} = [\mathcal{Q}_+^T \mathcal{O}]_{kl} - 1 = \begin{cases} D - 1, & k = l \\ \frac{-D - 1}{h - 1}, & k \neq l \end{cases} \tag{D41}
\]

This \( h \times h \) matrix has the eigenvalues:

1. \( \lambda_0 \) = with multiplicity 1, where the eigenspace is spanned by \( 1 := (1, \ldots, 1)^T \); and

2. \( \lambda_1 = \frac{h(D-1)}{h-1} \) with multiplicity \( h - 1 \) and eigenspace \( \Lambda := (\text{span} \{1\})^\perp \).

Next, we consider \( \mathcal{Q}_+^T : \mathbb{R}^{d^2-1} \rightarrow \mathbb{R}^h \). This time it does not have an empty kernel. However, it turns out we need not show that \( \text{Im} \mathcal{Q}_+^T = \Lambda \). It suffices to show \( \text{Im} \mathcal{Q}_+^T \subset \Lambda \). Indeed, this follows simply as before:

\[
\forall \mathbf{v} \in \mathbb{R}^{d^2-1}, \quad \mathbf{1} \cdot \mathcal{Q}_+^T \mathbf{v} = \mathcal{Q}_+ \mathbf{1} \cdot \mathbf{v} = \mathbf{0}, \tag{D42}
\]

where the last transition is just [D36]. Thus we obtain:

\[
\mathcal{O}_+^T \mathcal{Q}_+ = \frac{h(D-1)}{h-1} \mathcal{Q}_+^T. \tag{D43}
\]

To conclude the proof, we must analyze \( \mathcal{Q}_+ \mathcal{Q}_+^T \). Since for \( h < d^2 \) the projections \( \{Q_k\} \) do not comprise a SIC-POVM, \( \mathcal{Q}_+ \mathcal{Q}_+^T \) is no longer a frame operator of a tight frame, and thus not necessarily scalar. However, we may use the fact that \( \mathcal{Q}_+ \mathcal{Q}_+^T \) and \( \mathcal{Q}_+^T \mathcal{Q}_+ \) have the same non-zero eigenvalues (i.e., squares of the singular values of \( \mathcal{Q}_+ \)). Therefore, our next step would be computing \( \mathcal{Q}_+^T \mathcal{Q}_+ ^T \):

\[
[\mathcal{Q}_+^T \mathcal{Q}_+]_{kl} = [\mathcal{Q}_+^T \mathcal{Q}]_{kl} - 1 = \begin{cases} d - 1; & k = l \\ -\frac{d - 1}{h - 1}; & k \neq l \end{cases} \tag{D44}
\]

As before, it is straightforward to note this \( h \times h \) matrix has two eigenvalues: \( \lambda_0 = 0 \) with multiplicity 1, and \( \lambda_1 = \frac{h(d-1)}{h-1} \) with multiplicity \( h - 1 \). Thus, \( \mathcal{Q}_+ \mathcal{Q}_+^T \) has the eigenvalues \( \lambda_1 \) with multiplicity \( h - 1 \), and \( 0 \) with multiplicity \( d^2 - h \).

To conclude, we observed the following:

1. \( \mathcal{T}^T \mathcal{T} = \frac{1}{Ddh^2} \mathcal{Q}_+ \mathcal{Q}_+^T \),

2. \( \mathcal{Q}_+ \mathcal{Q}_+^T \) has precisely \( h - 1 \) nonzero eigenvalues, which all equal \( \lambda_1 = \frac{h(d-1)}{h-1} \).

Thus, \( \mathcal{T}^T \mathcal{T} \) also has \( h - 1 \) nonzero eigenvalues, which all equal \( \lambda' = \frac{D - 1}{D(h-1)} \frac{d - 1}{d(h-1)} \). Consequently, the \( h \) largest singular values of \( C \) are \( \sigma_0 = 1/\sqrt{Dh} \) with multiplicity 1, and \( \sqrt{D} \) with multiplicity \( h - 1 \); and the CMN is their product, that is:

\[
\mathcal{M}_{h, p=\infty} = \frac{1}{\sqrt{Dh}} \left[ \frac{D - 1}{D(h-1)} \frac{d - 1}{d(h-1)} \right]^{\frac{h-1}{h}} \tag{D45}
\]

which concludes the proof.
