Basic properties of the natural parametrization for the Schramm-Loewner evolution

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Abstract

The natural parameterization or length for the Schramm-Loewner evolution (SLE$_\kappa$) is the candidate for the scaling limit of the length of discrete curves for $\kappa < 8$. We improve the proof of the existence of the parameterization and use this to establish some new results. In particular, we show that the natural parametrization is independent of domain and it is Hölder continuous with respect to the capacity parametrization. We also give up-to-constants bounds for the two-point Green’s function. Although we do not prove the conjecture that the natural length is given by the appropriate Minkowski content, we do prove that the corresponding expectations converge.

1 Introduction

A number of measures on paths or clusters on two-dimensional lattices arising from critical statistical mechanical models are believed to exhibit some kind of conformal invariance in the scaling limit. Schramm introduced a one-parameter family of such processes, now called the (chordal) Schramm-Loewner evolution with parameter $\kappa$ (SLE$_\kappa$), and showed that these give the only possible limits for conformally invariant processes in simply connected domains satisfying a certain “domain Markov property”. He defined the process as a probability measure on curves from 0 to $\infty$ in $\mathbb{H}$, and then used conformal invariance to define the process in other simply connected domains.

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The definition of the process in $\mathbb{H}$ uses parametrization by half-plane capacity (see Section 2.1 for definitions). Suppose $\gamma : (0, t] \rightarrow \mathbb{H}$ is a (non-crossing) curve parameterized so that $\text{hcap}(\gamma(0, t]) = at$ for some constant $a > 0$. We write $\gamma_t$ for the set of points $\gamma(0, t]$. Let $H_t$ denote the unbounded component of $\mathbb{H} \setminus \gamma_t$ and $g_t : H_t \rightarrow \mathbb{H}$ be the unique conformal transformation with $g_t(z) - z = o(1)$ as $z \rightarrow \infty$. Then the following holds.

- For $z \in \mathbb{H}$, the map $t \mapsto g_t(z)$ is a smooth flow and satisfies the Loewner differential equation
  \[ \partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \]
  where $U_t$ is a continuous function on $\mathbb{R}$.
- If $t, s > 0$, and $\eta(t) = g_t(\gamma(t + r))$,
  \[ as = \text{hcap}(\gamma_{t+s}) - \text{hcap}(\gamma_t) = \text{hcap}(\eta_s). \]

Schramm defined chordal $SLE_\kappa$ to be the solution to the Loewner equation with $a = 2$ and $U_t$ a Brownian motion with variance parameter $\kappa$. An equivalent definition (up to a linear time change) which we use in this paper is to choose $U_t$ to be a standard Brownian motion and $a = 2/\kappa$. It has been shown that a number of discrete random models have $SLE$ as the scaling limit provided that the discrete models are parameterized using (discrete) half-plane capacity. Examples are loop-erased random walk for $\kappa = 2$ [9], Ising interfaces for $\kappa = 3$ [17], harmonic explorer for $\kappa = 4$ [15], percolation interfaces on the triangular lattice for $\kappa = 6$ [16], and uniform spanning trees for $\kappa = 8$ [9].

If $D$ is a simply connected domain with distinct boundary points $z, w$, then chordal $SLE_\kappa$ from $z$ to $w$ in $D$ is defined by taking the conformal image of $SLE_\kappa$ in the upper half plane under a transformation $F : \mathbb{H} \rightarrow D$ with $F(0) = z, F(\infty) = w$. The map $F$ is not unique, but scale invariance of $SLE$ in $\mathbb{H}$ shows that the distribution on paths is independent of the choice. This can be considered as a measure on the curves $F \circ \gamma$ with the induced parametrization or as a measure on curves modulo reparameterization.

While the capacity parametrization is useful for analyzing the curve, it is not the scaling limit of the “natural” parametrization of the discrete models. For example, for loop-erased walks, it is natural to parametrize by the length of the random walk. One can ask whether the curves parameterized by a normalized version of this “natural length” converge to $SLE$ with a different parametrization. The Hausdorff dimension of the $SLE$ paths [11] is $d = 1 + \min\{8/5, 1\}$. It is conjectured, but still unproven, that the “natural length” of an $SLE$ path can be given by an appropriate $d$-dimensional “measure”. If $\kappa \geq 8$, then the paths are plane-filling, and we can choose the measure of $\gamma_t$ to be the area of $\gamma_t$. For the remainder of this paper we consider the case $\kappa < 8$ for which $1 < d < 2$.

A candidate for the natural length of $\gamma(0, t]$ is the $d$-dimensional Minkowski content defined as follows. Let $f(\epsilon)$ be a positive function with $f(\epsilon) \rightarrow 0$ and $\epsilon / f(\epsilon) \rightarrow 0$ as $\epsilon \downarrow 0$. Let
\[ \text{Cont}_d(\gamma_t ; \epsilon, f) = \epsilon^{d-2} \text{Area} \{ z : \text{dist}(z, \gamma_t) \leq \epsilon, \text{dist}(z, \partial D) \geq f(\epsilon) \}, \]

\[ 2 \]
and
\[ \text{Cont}_d(\gamma_t) = \lim_{\epsilon \to 0^+} \text{Cont}_d(\gamma_t; \epsilon, f), \]  \hspace{1cm} (1)

provided that the limit exists and is independent of \( f \). It is not known whether or not this limit exists. For the moment let us assume that it does and, moreover, that the function \( t \mapsto \text{Cont}_d(\gamma_t) \) is continuous and strictly increasing. In this case we can reparameterize \( \gamma \) by
\[ \tilde{\gamma}(t) = \gamma(\sigma_t), \quad \sigma_t = \inf \{ s : \text{Cont}_d(\gamma_s) = t \}. \]

We could now define \( \text{SLE}_\kappa \) to be the measure on curves \( \tilde{\gamma} \) with this “natural” parametrization. Among the properties that this would have are the following.

- Suppose \( \tilde{\gamma}_t \) is an initial segment of an \( \text{SLE}_\kappa \) from 0 to \( \infty \) in \( \mathbb{H} \) and \( \tilde{\gamma}_t \subset D \subset \mathbb{H} \). If we consider \( \tilde{\gamma}_t \) as being an \( \text{SLE}_\kappa \) path in \( D \) instead, the amount of time to traverse \( \tilde{\gamma}_t \) is the same in \( D \) as in \( \mathbb{H} \).

- Suppose \( D \) is a simply connected domain with distinct boundary points \( z, w \); \( F : \mathbb{H} \to D \) is a conformal transformation with \( F(0) = z, F(\infty) = w \); and \( \tilde{\gamma}(t) \) is an \( \text{SLE}_\kappa \) curve in \( \mathbb{H} \) with the natural parametrization. Then
\[ \text{Cont}_d(F \circ \tilde{\gamma}_t) = \int_0^t |F'(\tilde{\gamma}(s))|^d \, ds. \]

In particular, to define \( \text{SLE}_\kappa \) in \( D \) with the natural parametrization, one lets
\[ \tilde{\eta}(t) = F \circ \tilde{\gamma}(\sigma_t), \]
where \( \sigma_t \) is defined by
\[ \int_0^{\sigma_t} |F'(\tilde{\gamma}(s))|^d \, ds = t \]

- Suppose \( D \) is a bounded domain with locally analytic boundary points \( z, w \), and \( \gamma \) is an \( \text{SLE} \) path from \( z \) to \( w \) (using any parametrization). Let \( \gamma^R \) denote the reversed path from \( w \) to \( z \). Then
\[ \text{Cont}_d(\gamma) = \text{Cont}_d(\gamma^R) < \infty. \]

Moreover,
\[ \mathbb{E}[\text{Cont}_d(\gamma)] = c \int_D G_D(\zeta; z, w) \, dA(\zeta), \]  \hspace{1cm} (2)

where \( c \) is a constant (depending only on \( \kappa \)), \( dA \) denotes integration with respect to area, and \( G_D(\zeta; z, w) \) denotes the “Green’s function” for \( \text{SLE}_\kappa \) in \( D \). The function \( G_D \), whose definition is recalled in Section 2.1, satisfies
\[ \lim_{\epsilon \to 0} \epsilon^{d-2} \mathbb{P}\{\text{dist}(z, \gamma_\infty) \leq \epsilon \} = \hat{c} G(z), \]  \hspace{1cm} (3)

where \( \hat{c} = \hat{c}_\kappa \in (0, \infty). \)
It is still open to prove (1). It is known that if we choose
\[ G(z) := G_{\mathbb{H}}(z; 0, \infty) = \text{Im}(z)^{d-2} [\sin \arg(z)]^{4a-1}, \]
then an analogue of (3) holds where distance is replaced with conformal radius. One of the goals of this paper is to establish (3) although our proof does not determine the constant. (See Section 2.1 for precise statements of what is known and what we prove here.) For other simply connected domains, the Green’s function can be computed using the scaling rule
\[ G_D(\zeta; z, w) = |f'(\zeta)|^{2-d} G_{f(D)}(f(\zeta); f(z), f(w)). \]

In [8], a different approach was taken to constructing the natural parametrization, using (2) as the starting point. For ease suppose that \( D \) is a bounded domain and \( z, w \) are distinct boundary points with
\[ \Psi = \int_D G_D(\zeta; z, w) \, dA(\zeta) < \infty. \]
Let \( \gamma \) denote an \( SLE_\kappa \) curve from \( z \) to \( w \) in \( D \). Let us give the curve the capacity parametrization inherited from capacity in \( \mathbb{H} \). Let \( \Theta_t \) denote the “natural length” of \( \gamma_t \) (we expect that this is a multiple of \( \text{Cont}_d[\gamma_t] \) but we do not assume it as such). Then
\[ E[\Theta_\infty | \gamma_t] = \Theta_t + E[\Theta_\infty - \Theta_t | \gamma_t]. \]
Using (2), we see that we would expect
\[ E[\Theta_\infty - \Theta_t | \gamma_t] = \Psi_t := \int_{D_t} G_{D_t}(\zeta; \gamma(t), w) \, dA(\zeta), \]
where \( D_t \) denotes the component of \( D \setminus \gamma_t \) that contains \( w \) in its boundary. For each \( \zeta \) one can see that
\[ M_t(\zeta) := G_{D_t}(\zeta; \gamma(t), w) \]
is a positive local martingale and hence is a supermartingale. Therefore, \( \Psi_t \) is a supermartingale. However, \( N_t := E[\Theta_\infty | \gamma_t] \) should be a martingale. Therefore, we define \( \Theta_t \) to be the unique increasing process such that
\[ \Psi_t + \Theta_t \]
is a martingale. This is a standard Doob-Meyer decomposition.

In order to justify this definition, one needs to prove moment bounds. Indeed, if \( \Psi_t \) were actually a local martingale (which would not be shocking since it is an integral of local martingales), then there would be no nontrivial increasing process that we could add to \( \Psi_t \) to make it a martingale.

• In [8], it was shown that for \( \kappa < 5.0 \cdots \), the process \( \Theta_t \) exists in \( \mathbb{H} \) (the definition has to be modified slightly in \( \mathbb{H} \) because \( \Psi_0 \) as we have defined it above is infinite—this is not very difficult). The necessary second moment bounds were obtained using the reverse Loewner flow. It was shown that for this range of \( \kappa \), there exists \( \alpha_0 = \alpha_0(\kappa) > 0 \) such that the function \( t \mapsto \Theta_t \) is Hölder continuous of order \( \alpha \) for \( \alpha < \alpha_0 \).
In [11], the natural parametrization was shown to exist for all $\kappa < 8$. There the necessary two-point estimates were obtained from estimates on the “two-point Green’s function” [1, 10]. However, the estimates were not strong enough to determine Hölder continuity of the function $\Theta_t$.

This paper continues the study of the natural parametrization which in turn leads to further study of the multi-point Green’s function. We extend the definition of the Green’s function and prove some important bounds; we describe these results in the next section which outlines the paper. The main new results about the natural parametrization are the following.

- We improve the proof in [11] by establishing that for all $\kappa < 8$, the discrete approximations of the natural parametrization converge in $L^1$. Moreover, we show that there exists $\alpha_0 > 0$ such that the $t \mapsto \Theta_t$ is Hölder continuous of order $\alpha < \alpha_0$.

- We prove that the natural parametrization is “independent of domain”. In other words, we prove the first property that we listed that the “natural parametrization” should satisfy.

We do not establish the reversibility of the natural parametrization. It is our hope that the results in this paper will help us in establishing the limit (1).

1.1 Overview of the paper

We start with a review of the Schramm-Loewner evolution (SLE) and the notation we will use in Section 2.1. This includes the definition of the Green’s function $G(z)$ and its relationship to the probability of SLE getting close to a point. Roughly speaking, $G(z)$ is the normalized probability that SLE hits $z$. The next subsection introduces the time-dependent Green’s function $G^t(z)$ which corresponds (again, roughly) to the probability that the SLE path hits $z$ by time $t$ in the capacity parametrization. This function appears implicitly in [8] as $G(z)\phi(t; z)$, but we find it useful to formalize this and to prove some estimates. Section 2.3 studies the two-point Green’s function as introduced in [10] and defines a time-dependent version of it. Two important estimates are stated in this section. Theorem 2.11, which we label as a theorem because we believe the estimate will be useful for others, completes the work in [11] by giving a two-sided up-to-constants estimate for the (time independent) two-point Green’s function. Lemma 2.12 gives a time-dependent version of a two-point estimate from [11]. Both of these estimates are important in our study of the natural parametrization. We delay the proofs of these estimates to Section 4. We do derive some corollaries of these estimates in Section 2.3.

We define the natural parametrization in $\mathbb{H}$ in Section 3.1. Although the definition is the same as that in [8, 11], we phrase the definition in terms of the time-dependent Green’s function. In the next subsection we prove the existence and Hölder continuity of the parametrization. Our proof combines ideas in [8, 11] as well as the Hölder continuity of
the SLE path for $\kappa < 8$. Section 3.3 proves two lemmas that are used in the subsequent subsection to establish the independence of the natural parametrization and the domain. An exact statement of the independence is given in Theorem 3.11.

Section (4) gives proofs of two of the main estimates. These results generalize results from previous papers, and the arguments rely on the work in those papers. Lemma 2.12 extends a result in [11] to time-dependent Green’s functions and uses one fact from that paper. Proposition 4.3, which is an important step in the proof of Theorem 2.11 extends a result in [10]. The main extension is to allow the interior target points to close to the boundary.

The final section gives the proof of (3). This proof uses properties of two-sided radial SLE but is independent of the other results in this paper.

2 Green’s functions for SLE

2.1 Schramm-Loewner evolution (SLE) and notation

In this section we review the Schramm-Loewner evolution and the chordal Green’s function. See [5, 6] for more details.

Suppose that $\gamma : (0, \infty) \to \mathbb{H} = \{x + iy : y > 0\}$ is a curve with $\gamma(0+) \in \mathbb{R}$ and $\gamma(t) \to \infty$ as $t \to \infty$. Let $H_t$ be the unbounded component of $\mathbb{H} \setminus \gamma(0,t)$. Using the Riemann mapping theorem, one can see that there is a unique conformal transformation

$$g_t : H_t \to \mathbb{H}$$

satisfying $g_t(z) - z \to 0$ as $z \to \infty$. It has an expansion at infinity

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}).$$

The coefficient $a(t)$ equals $\text{hc}ap(\gamma(0,t])$ where $\text{hc}ap(A)$ denotes the half-plane capacity from infinity of a bounded set $A$. There are a number of ways of defining $\text{hc}ap$, e.g.,

$$\text{hc}ap(A) = \lim_{y \to \infty} y \mathbb{E}^{iy}[\text{Im}(B_{\tau})],$$

where $B$ is a complex Brownian motion and $\tau = \inf\{t : B_t \in \mathbb{R} \cup A\}$.

We assume that $\gamma$ is a non-crossing curve; by this we mean that for each $t$, $\gamma(t) \in \partial H_t$ and the image $V_t = g_t(\gamma(t))$ is well defined and a continuous function of $t$. If $\gamma$ is simple, then it is non-crossing, but there are non-simple, non-crossing curves. Then $g_t$ satisfies the (chordal) Loewner equation

$$\dot{g}_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z,$$

where $V_t = g_t(\gamma(t))$ is a continuous function.
Conversely, one can start with a continuous real-valued function \( V_t \) and define \( g_t \) by (4). For \( z \in \mathbb{H} \setminus \{0\} \), the function \( t \mapsto g_t(z) \) is well defined up to time \( T_z := \sup\{t : \text{Im}[g_t(z)] > 0\} \). The (chordal) Schramm-Loewner evolution (SLE) (from 0 to \( \infty \) in \( \mathbb{H} \)) is the solution to (4) where \( V_t = -B_t \) is a standard Brownian motion and \( \alpha = 2/\kappa \). There exists a random non-crossing curve \( \gamma \), which is also called SLE, such that \( g_t \) comes from the curve \( \gamma \) as above. Moreover, if \( H_t \) denotes the unbounded component of \( \mathbb{H} \setminus \gamma(0, t) \), then

\[
H_t = \{ z \in \mathbb{H} : T_z > t \}.
\]

We write

\[
f_t(z) = g_t^{-1}(z + V_t).
\]

Throughout this paper we set \( \kappa = 2/a \). There are three phases for the curve [13]: for \( \kappa \leq 4 \), the curve is simple; if \( \kappa \geq 8 \), the curve is plane-filling; and for \( 4 < \kappa < 8 \), the curve has self-intersections but is not space-filling. We will consider \( \kappa < 8 \) in this paper in which case [1] the Hausdorff dimension of \( \gamma_t \) is

\[
d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a}.
\]  

(5)

We will use the scaling property of SLE that we recall in a proposition.

**Proposition 2.1.** Suppose \( U_t \) is a standard Brownian motion and let \( g_t \) be the solution to the Loewner equation (4) with \( V_t = U_t \) producing the SLE_\( \kappa \) curve \( \gamma(t) \). Let \( r > 0 \) and define

\[
\hat{\gamma}(t) = r^{-1} \gamma(r^2 t), \quad \hat{g}_t(z) = r^{-1} g_{r^2 t}(r z), \quad \hat{U}_t = r^{-1} U_{r^2 t}.
\]

Then \( \hat{\gamma}(t) \) has the distribution of SLE_\( \kappa \). Indeed, \( \hat{g}_t(z) \) is the solution to (4) with \( V_t = \hat{U}_t \).

SLE_\( \kappa \) in other simply connected domains is defined by conformal invariance. To be more precise, suppose that \( D \) is a simply connected domain and \( w_1, w_2 \) are distinct points in \( \partial D \). Let \( F : \mathbb{H} \to D \) be a conformal transformation of \( \mathbb{H} \) onto \( D \) with \( F(0) = w_1, F(\infty) = w_2 \). Then the distribution of

\[
\tilde{\gamma}(t) = F \circ \gamma(t),
\]

is that of SLE_\( \kappa \) in \( D \) from \( w_1 \) to \( w_2 \). Although the map \( F \) is not unique, scale invariance of SLE_\( \kappa \) in \( \mathbb{H} \) shows that the distribution is independent of the choice. This measure is often considered as a measure on paths modulo reparameterization, but we can also consider it as a measure on parameterized curves.

If \( \gamma(t) \) is an SLE_\( \kappa \) curve with transformations \( g_t \) and driving function \( U_t \), we write \( \gamma_t = \gamma(0, t), \gamma = \gamma_\infty \), and let \( H_t \) be the unbounded component of \( \mathbb{H} \setminus \gamma_t \). If \( z \in \mathbb{H} \) and \( t < T_z \), we let

\[
Z_t(z) = g_t(z) - U_t, \quad S_t(z) = \sin \arg Z_t(z), \quad T_t(z) = \frac{\text{Im}[g_t(z)]}{|g_t'(z)|}.
\]

(6)

If \( Z_t(z) = X_t(z) + iY_t(z) \), then the Loewner equation can be written as

\[
dX_t = \frac{a X_t(z)}{|Z_t(z)|^2} dt + dB_t, \quad \partial_t Y_t(z) = -\frac{a Y_t(z)}{|Z_t(z)|^2},
\]

\[
dY_t = \frac{a Y_t(z)}{|Z_t(z)|^2} dt - \frac{a Y_t(z)}{|Z_t(z)|^2} dB_t.
\]

(7)
where \( B_t = -U_t \). More generally, if \( D \) is a simply connected domain and \( z \in D \), we let \( \Upsilon_D(z) \) denote \((1/2)\) times the conformal radius of \( D \) with respect to \( z \), that is, if \( F: \mathbb{D} \to D \) is a conformal transformation with \( F(0) = z \), then \(|F'(0)| = 2\Upsilon_D(z)\). Using the Schwarz lemma and the Koebe \((1/4)\)-theorem we see that

\[
\frac{\Upsilon_D(z)}{2} \leq \text{dist}(z, \partial D) \leq 2 \Upsilon_D(z).
\]

It is easy to check that if \( t < T_z \), then \( \Upsilon_t(z) \) as given in (6) is the same as \( \Upsilon_{H_t}(z) \). Also, if \( z \notin \gamma \), then \( \Upsilon(z) := \Upsilon_{T_z}(z) = \Upsilon_D(z) \) where \( D \) denotes the connected component of \( \mathbb{H} \setminus \gamma \) containing \( z \).

Similarly, if \( w_1, w_2 \) are distinct boundary points on a simply connected domain \( D \) and \( z \in D \), we define

\[
S_D(z; w_1, w_2) = \sin[\text{arg } f(z)],
\]

where \( f: D \to \mathbb{H} \) is a conformal transformation with \( f(w_1) = 0, f(w_2) = \infty \). If \( t < T_z \), then \( S_t(z) = S_{H_t}(z; \gamma(t), \infty) \). If \( f: D \to f(D) \) is a conformal transformation,

\[
S_D(z; w_1, w_2) = S_{f(D)}(f(z); f(w_1), f(w_2)).
\]

If \( \partial_1, \partial_2 \) denote the two components of \( \partial D \setminus \{w_1, w_2\} \), then

\[
S_D(z; w_1, w_2) \asymp \min \{ \text{hm}_D(z, \partial_1), \text{hm}_D(z, \partial_2) \}. \tag{7}
\]

Here, and throughout this paper, \( \text{hm} \) will denote harmonic measure; that is, \( \text{hm}_D(z, K) \) is the probability that a Brownian motion starting at \( z \) exits \( D \) at \( K \).

Let

\[
G(z) = |z|^{d-2} \sin^{\frac{d}{2}+\frac{d}{2}-2}(\text{arg } z) = \text{Im}(z)^{d-2} \sin^{4a-1}(\text{arg } z), \tag{8}
\]

denote the \((\text{chordal}) \text{ Green’s function for } SLE_r \) \((\text{in } \mathbb{H} \text{ from } 0 \text{ to } \infty)\). This function first appeared in [13] and the combination \((d, G)\) can be characterized up to multiplicative constant by the scaling rule \( G(rz) = r^{d-2} G(z) \) and the fact that

\[
M_t(z) := |g_t'(z)|^{2-d} G(Z_t(z)) \tag{9}
\]

is a local martingale. More generally, if \( D \) is a simply connected domain with distinct \( w_1, w_2 \in \partial D \), we define

\[
G_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}.
\]

The Green’s function satisfies the conformal covariance rule

\[
G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)).
\]

Note that if \( t < T_z \), then

\[
M_t(z) = G_{H_t}(z; \gamma(t), \infty).
\]
The local martingale $M_t(z)$ is not a martingale because it “blows up” at time $t = T_z$. If we stop it before that time, it is actually a martingale. To be precise, suppose that

$$\tau = \tau_{\epsilon,z} = \inf\{t : Y_t(z) \leq \epsilon\}. \quad (10)$$

Then for every $\epsilon > 0$, $M_{t\wedge \tau}(z)$ is a martingale. Moreover, on the event $\tau = \infty$, we have $M_{\infty}(z) = 0$. Therefore, if $\sigma \leq \tau$ is a stopping time,

$$G(z) = \mathbb{E}[M_\sigma(z)] = \mathbb{E}[|g'_{\sigma}(z)|^{2-d}G(Z(\sigma))] \quad (11)$$

The following is proved in [6] (the proof there is in the upper half plane, but it immediately extends by conformal invariance).

**Proposition 2.2.** Suppose $\kappa < 8$, $z \in D, w_1, w_2 \in \partial D$ and $\gamma$ is a chordal $SLE_\kappa$ path from $w_1$ to $w_2$ in $D$. Let $D_\infty$ denote the component of $D \setminus \gamma$ containing $z$. Then, as $\epsilon \downarrow 0$,

$$\mathbb{P}\{Y_{D_\infty}(z) \leq \epsilon\} \sim c_{\epsilon}^2 \epsilon^{2-d} G_D(z; w_1, w_2), \quad c_{\epsilon} = 2 \left[ \int_0^\pi \sin^{4a} x \, dx \right]^{-1}.$$  

In this paper, we give the analogous result replacing conformal radius with distance to the curve. We do not have an explicit form for the constant $\hat{c}$.

**Theorem 2.3.** Suppose $\kappa < 8$. There exist $0 < \hat{c}, c, u < \infty$ (depending on $\kappa$) such that the following holds. Suppose $D$ is a simply connected domain, $z \in D$, $w_1, w_2 \in \partial D$ and $\gamma$ is a chordal $SLE_\kappa$ path from $w_1$ to $w_2$ in $D$. Then, as $\epsilon \downarrow 0$,

$$\mathbb{P}\{\text{dist}(z, \gamma_\infty) \leq \epsilon\} \sim \hat{c} G_D(z; w_1, w_2) \epsilon^{2-d}, \quad (12)$$

Moreover, if $\epsilon < \text{dist}(z, \partial D)/10$,

$$|\epsilon^{d-2} G_D(z; w_1, w_2)^{-1} \mathbb{P}\{\text{dist}(z, \gamma_\infty) \leq \epsilon\} - \hat{c}| \leq c \left( \frac{\epsilon}{\text{dist}(z, \partial D)} \right)^u.$$  

**Proof.** See Section 5. \[ QED \]

**Corollary 2.4.** Suppose $\kappa < 8$. Let $f(r)$ be a positive function with $f(r) \to 0$ and $r/f(r) \to 0$ as $r \downarrow 0$. Suppose $D$ is a simply connected domain and $w_1, w_2 \in \partial D$. Let $\gamma$ be an $SLE_\kappa$ path from $w_1$ to $w_2$ in $D$ and let

$$Y_\epsilon = \text{Area}\{z \in D : \text{dist}(z, \partial D) \geq f(\epsilon), \text{dist}(z, \gamma_\infty) \leq \epsilon\}.$$  

Then

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{E}[Y_\epsilon] = \hat{c} \int_D G(z; w_1, w_2) \, dA(z),$$

where $\hat{c}$ is the constant in (12).
2.2 Time dependent Green’s function

If \( z \in \mathbb{H} \), then two-sided radial SLE\(_\kappa\) (stopped at \( T_z \), the time at which it reaches \( z \)) is chordal SLE\(_\kappa\) “conditioned to go through \( z \)”. This is conditioning on an event of measure zero, but there are several equivalent ways to make this precise. The term two-sided radial comes from the fact that at the interior point \( z \), there are two paths coming out, \( \gamma[0,T_z] \) and \( \gamma[T_z,\infty) \).

We are only considering the marginal distribution on \( \gamma[0,T_z] \).

The elegant way to define two-sided radial SLE\(_\kappa\) is to say that it is chordal SLE\(_\kappa\) weighted by the local martingale \( M_t(z) \) (see, e.g., [10, Section 2.4]). By the Girsanov theorem, we can also describe this process by giving the drift on the driving function. Indeed, if \( P^*_z \) denotes the probability measure of two-sided radial and \( X_t = X_t(z) = \text{Re}[Z_t(z)] \), then

\[
dX_t = \frac{(1-3a)X_t}{|Z_t|^2} \, dt + dB_t, \quad X_0 = \text{Re}(z),
\]

(13)

where \( B_t \) is a \( P^*_z \)-Brownian motion. (The equation for \( Y_t = Y_t(z) \) is deterministic, and hence the same for as for usual chordal SLE.) The following technical fact is useful.

**Proposition 2.5.** [7] If \( \kappa < 8 \) and \( z \in \mathbb{H} \), then with probability one, two-sided radial SLE\(_\kappa\) is continuous at \( T_z \). In other words, with probability one, \( T_z < \infty \) and

\[
\lim_{t \uparrow T_z} \gamma(t) = \gamma(T_z) = z.
\]

Fix \( z \in \mathbb{H} \) and let \( \tau_\epsilon = \tau_{\epsilon,z} \) be as in [10]. For any \( \delta < \epsilon \), we can define a probability measure \( \nu(\delta,\epsilon) \) on curves \( \gamma(0,\tau_\epsilon) \) by considering SLE\(_\kappa\), conditioning on the event \( \{\tau_\delta < \infty\} \), and then viewing this as a measure on the paths \( \gamma(0,\tau_\epsilon) \). The following proposition which was proved in [10] gives justification to calling two-sided radial SLE, “chordal SLE conditioned to go through \( z \”).

**Proposition 2.6.** As \( \delta \to 0^+ \), the measures \( \nu(\delta,\epsilon) \) converge to two-sided radial SLE\(_\kappa\) stopped at time \( \tau_\epsilon \).

We write \( P^*_z, E^*_z \) for probabilities and expectations with respect to two-sided radial SLE going through \( z \). Let \( \phi(z;t) = P^*_z\{T_z \leq t\} \) denote the distribution function of \( T_z \) under the measure of two-sided radial SLE. The scaling property of SLE implies that \( \phi(rz;r^2t) = \phi(z;t) \). The time-dependent Green’s function \( G^t \) is defined by

\[
G^t(z) = G(z) \phi(z;t).
\]

An equivalent way of defining \( G^t \) is by

\[
E[M_t(z)] = G(z) - G^t(z).
\]

(14)

We have the scaling rule

\[
G^{r^2t}(rz) = G(rz) \phi(rz;r^2t) = r^{d-2} G(z) \phi(z,t) = r^{d-2} G^t(z).
\]
The analogue of (11) is the following: if $\epsilon > 0$ and $\sigma \leq \tau_{e,z}$ is a stopping time, then

$$G^t(z) = \mathbb{E} \left[ |g^\sigma(z)|^{2-d} G^{t-\sigma}(Z_\sigma(z)) \right] = \mathbb{E} \left[ |g^\sigma(z)|^{2-d} G^{t-\sigma}(Z_\sigma(z)); \sigma < t \right],$$  

(15)

where we set $G^s(z) = 0$ if $s \leq 0$.

While the Green’s function $G$ is independent of the parametrization of the SLE path, the time-dependent function $G^t$ is defined in terms of the half-plane capacity. We finish this section by proving some lemmas about $G^t$. It follows from the Loewner equation (see, e.g., [5, Lemma 4.13]), that in time $t$, the diameter of the path $\gamma_t$ is bounded above by

$$2a \left[ t^{1/2} \vee \max\{|U_s| : 0 \leq s \leq t\} \right].$$  

(16)

Hence we also get tail estimates for $\text{diam}[\gamma_t]$. However, for $G^t$, we need tail estimates “conditioned that $T_z < \infty$”, which we prove in the next lemma. The exponents in the estimates are not optimal but they will suffice for our needs.

**Lemma 2.7.** There exists $c < \infty$ such that if $z = x + iy$, then

$$G^t(z) = 0, \quad y^2 \geq 2at,$$

$$G^t(z) \leq c e^{-\frac{y^2}{8t}} G(z).$$

Moreover, there exists $c_1 > 0$ such that if $y^2 \leq 2at$,

$$G^{2t}(z) \geq c_1 e^{-\frac{3y^2}{4t}} G(z),$$

$$G^{100t}(z) \geq c_1 e^{\frac{y^2}{40t}} G^t(z).$$

**Proof.** Without loss of generality we assume that $x \geq 0$. The first inequality follows immediately from the Loewner equation since $\partial_t (|\text{Im} g_t(z)|^2) \geq -2a$. For the remainder of the proof we assume $y^2 \leq 2at$.

Let $\mathbb{P}^* = \mathbb{P}_x^*$ and let $X_t, Z_t, B_t$ be as in (13) with $X_0 = B_0 = x$. Let $T_z = \inf\{t : Z_t = 0\} = \inf\{t : \gamma(t) = z\}$. Then the second and third inequalities are equivalent to

$$\mathbb{P}^*\{T_z \leq t\} \leq c e^{-\frac{y^2}{8t}}.$$  

(17)

$$\mathbb{P}^*\{T_z \leq 2t\} \geq c_1 e^{-\frac{3y^2}{4t}}.$$  

(18)

By scaling, it suffices to prove (17) with $x = 1$ and for $t$ sufficiently small, Let $R = \inf\{t : X_t = 1/4\}, S = \inf\{t : B_t = 1/2\}$. Note that if $t \leq R$, then $B_t \leq X_t + 4rt$ where $r = (3a - 1) \vee 0$. Therefore, if $t \leq 1/(16r),$

$$\mathbb{P}^*\{T_z \leq t\} \leq \mathbb{P}^*\{R \leq t\} \leq \mathbb{P}^*\{S \leq t\}.$$
The reflection principle for Brownian motion implies that
\[ \mathbb{P}^* \{ S \leq t \} \leq \mathbb{P}^* \left\{ \min_{0 \leq s \leq t} (B_s - B_0) \leq -1/2 \right\} \leq 2 \mathbb{P} \{ B_t \geq 1/2 \} \leq c e^{-\frac{1}{8t}}. \]

By scaling it suffices to prove (18) for \( t = 1 \). The estimate is easy to establish if \( x = 0 \); indeed, there exists \( \delta > 0 \) such that if \( z = iy \) with \( y \leq \sqrt{2a} \),
\[ \mathbb{P}^* \{ T_z \leq 3/2 \} \geq \delta. \]
If \( x > 0 \), let \( T = \inf \{ t : X_t = 0 \} \).
The strong Markov property implies that \( \mathbb{P}^* \{ T_z \leq 2 \} \geq \delta \mathbb{P}^* \{ T \leq 1/2 \} \). Hence, it suffices to prove that \( \mathbb{P}^* \{ T \leq 1/2 \mid X_0 = x \} \geq c_1 e^{-5x^2} \), or equivalently, by scaling,
\[ \mathbb{P}^* \{ T \leq t \mid X_0 = 1 \} \leq c_1 e^{-\frac{5}{2t}}. \]
Using (13) and \( a > 1/4 \), we can see that it suffices to establish the last inequality where \( X \) satisfies the Bessel equation
\[ dX_t = \frac{dt}{4X_t} + dB_t, \quad X_0 = 1, \]
and \( T = \inf \{ t : X_t = 0 \} \). For the remainder of this argument we assume this with \( B_0 = 1 \) and write just \( \mathbb{P} \) for the probability measure. Let \( S = \inf \{ t : X_t = 1/2 \} \). Then the strong Markov property and Bessel scaling imply that \( T \) has the same distribution as \( S + (\tilde{T}/4) \) where \( S, \tilde{T} \) are independent and \( \tilde{T} \) has the same distribution as \( T \). Therefore,
\[ \mathbb{P} \{ T \leq t \} \geq \mathbb{P} \{ S \leq t/2 \} \mathbb{P} \{ \tilde{T} \leq t/2 \} = \mathbb{P} \{ S \leq t/2 \} \mathbb{P} \{ T \leq 2t \}. \]
If \( s \leq S \), then \( X_s \leq B_s + \frac{s}{2} \). Therefore, if \( t \leq 2 \),
\[ \mathbb{P} \{ S \leq t/2 \} \geq \mathbb{P} \{ B_{t/2} \leq 0 \} \geq c \sqrt{t} e^{-\frac{1}{4t}}. \]
In particular, there exists \( t_0 > 0 \), such that for \( t \leq t_0 \),
\[ \mathbb{P} \{ T \leq t \} \geq e^{-\frac{1}{8t}} \mathbb{P} \{ T \leq 2t \}. \]
(19)
Since \( \mathbb{P} \{ T \leq t_0/2 \} > 0 \), there exists \( c_1 \) such that for \( t_0/2 \leq t \leq t_0 \),
\[ \mathbb{P} \{ T \leq t \} \geq c_1 e^{-\frac{5}{2t}}. \]
(20)
Using (19) and induction, we see that (20) holds for all \( t \leq t_0 \). This finishes the proof of the third inequality.

The last inequality in the lemma follows from the second and third inequalities as follows.
\[ \frac{G^{100}(z)}{G(z)} \geq c_1 \exp \left\{ -\frac{5x^2}{50t} \right\} = c_1 \exp \left\{ \frac{x^2}{40t} \right\} \exp \left\{ -\frac{x^2}{8t} \right\} \geq c_1 \exp \left\{ \frac{x^2}{40t} \right\} \frac{G^t(z)}{G(z)}. \]
Lemma 2.8. There exists \( \beta > 0, c < \infty \) such that if \( z = 1 + \delta i \) with \( 0 < \delta < 1 \), \( x \geq 4 \), \( 0 \leq t \leq 1 \), and
\[
\sigma' = \sigma'_x = \inf\{ s : |\text{Re}[\gamma(s)]| = x \},
\]
then
\[
\mathbb{P}_z^*\{ \sigma' \leq t \wedge T_z \} \leq ce^{-\beta x^2/t}.
\]

Proof. All constants in this proof are independent of \( x \) and \( \delta \) (but may depend on \( a = 2/\kappa \)).
Let \( \sigma = \inf\{ s : \text{Re}[\gamma(s)] = x \} \). We will prove that
\[
\mathbb{P}_z^*\{ \sigma \leq t \wedge T_z \} \leq ce^{-\beta x^2/t}.
\]
A similar argument, which we omit, establishes the estimate for \( \sigma = \inf\{ s : \text{Re}[\gamma(s)] = -x \} \).

Let
\[
Z_s = X_s + iY_s = g_s(z) - U_s = g_s(z) - g(\gamma(s)),
\]
and \( Z = X + iY = Z_\sigma \). As before, let \( H_s \) be the unbounded component of \( \mathbb{H} \setminus \gamma_s \) and let \( H = H_{\sigma} \). The Loewner equation (4) implies that \( Y \leq -\delta \). Suppose that \( \sigma < T_z \). Let \( \ell \) denote the vertical line segment from \( \gamma(\sigma) \) to \( R \), \( U = H \setminus \ell \), and let \( U' \) be the unbounded component of \( U \). We write \( \{ \sigma < t \wedge T_z \} = E_1 \cup E_2 \) where \( E_1 \ (E_2) \) is the intersection of \( \{ \sigma < t \wedge T_z \} \) with the event that \( z \in U' \) (resp., \( z \notin U' \)).
On the event \( E_1 \), deterministic conformal mapping estimates imply that there exist \( c_1 \) such that
\[
X \leq -c_1 x.
\]
(We leave out the details of this estimate which can be obtained by considering the event that the path of a Brownian motion starting at \( (x/2) + 10i \) and stopped when it leaves \( H \), concatenated with the half-infinite line \( (x/2) + i[10, \infty) \), separates the disk of radius \( \text{dist}(z, \partial H)/2 \) about \( z \) from \( U_{\sigma} \). By considering the image of this event under \( g_{\sigma} \) we can get the estimate.) Therefore,
\[
\mathbb{P}_z^*\{E_1\} \leq \mathbb{P}_z^*\{X_s \leq -c_1 x \text{ for some } s \leq t \wedge T_z \}.
\]
We know from (13) that \( X_t \) satisfies
\[
dX_t = \frac{(1 - 3a) X_t}{|Z_t|^2} dt + dB_t, \quad X_0 = 1,
\]
where \( B_t \) is a \( \mathbb{P}_z^* \)-Brownian motion. By comparison with a Bessel process (we omit the details), we can see that for \( y \geq 2 \),
\[
\mathbb{P}_z^*\{X_s \leq -y \text{ for some } s \leq t \wedge T_z \ | X_0 \geq 1 \} \leq ce^{-\beta' y^2/t},
\]
for appropriate \( c, \beta' \).
We now assume $E_2$ holds. Let $U''$ denote the component of $U$ containing $z$. Except on an event of $\mathbb{P}_z$ measure zero, we know that $\ell \subset \partial U''$. We claim there exists $c_3, \beta_3$ such that for $z' \in U''$ with $\mathfrak{R}(z') \leq 3/2$, we have

$$\text{hm}_{U''}(z', \ell) \leq c_3 e^{-\beta_3 x^2/t}. \quad (21)$$

To see this, note that we can find a crosscut $\ell''$ of the form $(2+y_1i, 2+y_2i)$ with the property that $\ell''$ disconnects $z'$ from $\ell$ in $U''$. Since $\text{hcpt}[\gamma_\sigma] \leq t$, we know (see [4, Theorem 1]) that the area of $U''$ is bounded by $c_4 t$. Also, $\text{dist}(\ell, \ell'') = x - 2 \geq x/2$. Using extremal distance estimates (see, e.g., [5, Lemma 3.74]), we know that the extremal distance between $\ell$ and $\ell''$ in $U'' \setminus \ell''$ is bounded below by $c_5 x^2/t$. The estimate (21) follows from the relationship between harmonic measure and extremal distance.

Let $\Upsilon_t = Y_t/|g'_t(z)|$ denote (one half) times the conformal radius of $z$ in $H_t$. We know that $\Upsilon_0 = \delta$ and $\Upsilon_t \approx 2 \text{dist}(z, \partial H_t)$. Let $\eta_k = \inf \{t : \Upsilon_t \leq 2 - \kappa \delta\}$, and write

$$\mathbb{P}^*_z(E_2) = \sum_{k=1}^{\infty} \mathbb{P}^*_z(E_2 \cap \{\eta_{k-1} \leq \sigma < \eta_k\}).$$

Using the Beurling estimate, (7), and (21), we can see that on the event $E_2 \cap \{\eta_{k-1} \leq \sigma < \eta_k\}$,

$$S_\sigma \leq c 2^{-k/2} \delta e^{-\beta_3 x^2/t}. \quad (22)$$

By the study of two-sided radial $SLE_\kappa$ in the radial parametrization (see [11, Section 2.1.1]) we know that there exist $c_6, \alpha$ such that

$$\mathbb{P}^*_z\{\exists t \leq \eta_1 : S_t \leq \delta r\} \leq cr^\alpha,$$

and for $k \geq 1$,

$$\mathbb{P}^*_z\{\exists \eta_k \leq t \leq \eta_{k+1} : S_t \leq r\} \leq cr^\alpha.$$

Combining this with (22), we get

$$\mathbb{P}^*_z[ E_2 \cap \{\eta_k < \sigma \leq \eta_{k-1}\}] \leq c 2^{-\alpha k/2} e^{-\alpha x^2/t}.$$

By summing over $k$, we see that $\mathbb{P}^*_z(E_2) \leq ce^{-\alpha x^2/t}$. \hfill \Box

**Corollary 2.9.** For every $\epsilon > 0$ there exists $K < \infty$ such that the following holds. Suppose $z = x + iy|x|$ with $x \neq 0, y \leq 1$ and $x^2y^2/a \leq t \leq x^2$. Then

$$\mathbb{P}^*_z\{\text{diam}[\gamma] \geq K x \mid T_z \leq t\} \leq \epsilon.$$

**Proof.** By scaling and symmetry we may assume $x = 1$. By Lemma 2.7, there exist $c_1, \beta_1$ such that

$$\mathbb{P}^*_z\{T_z \leq t\} \geq c_1 e^{-\beta_1/t}.$$
By Lemma 2.8, there exist $c_2, \beta_2$ such that
\[
P^*_z \{ T_z \leq t, \, \operatorname{diam}[\gamma] \geq K \} \leq c_2 e^{-\beta_2 K^2 / t}.
\]
Therefore, if $K \geq 2 \beta_1 / \beta_2$,
\[
P^*_z \{ \operatorname{diam}[\gamma] \geq K \mid T_z \leq t \} \leq (c_2 / c_1) \exp \left\{ \frac{\beta_1 - \beta_2 K^2}{t} \right\} \leq (c_2 / c_1) \exp \left\{ -\frac{\beta_2 K^2}{2t} \right\}
\]
\[\square\]

Let
\[
M^t_s(z) = |g'_s(z)|^{2-d} G^{t-s}(Z_s(z)).
\]
For fixed $t$, $M^t_s(z)$ is a continuous, nonnegative local martingale with $M^t_s(z) = 0$ for $s \geq t$.
Moreover, if $r \leq t$, then
\[
N_s = M^t_s(z) - M^r_s(z) = |g'_s(z)|^{2-d} \left[ G^{t-s}(Z_s(z)) - G^{r-s}(Z_s(z)) \right],
\]
is a martingale for $0 \leq s \leq r$. In analogy to (14) we can see that if $s \leq t$,
\[
\mathbb{E} \left[ M^t_s(z) \right] = G^t(z) - G^s(z).
\]
Let
\[
L(0, t) = \int G^t(z) \, dA(z),
\]
and more generally, if $s < t$,
\[
L(s, t) = \int M^t_s(z) \, dA(z) = \int |g'_s(z)|^{2-d} G^{t-s}(Z_s(z)) \, dA(z).
\]
Here, and throughout this paper, $dA$ refers to integration with respect to 2-dimensional Lebesgue measure (area) and, unless specified otherwise, integrals are over $\mathbb{H}$. Note that
\[
\mathbb{E}[L(s, t)] = \int [G^t(z) - G^s(z)] \, dA(z),
\]
and if $s < t \leq t_1$,
\[
L(s, t) = \mathbb{E} [L(s, t_1) - L(t, t_1) \mid \mathcal{F}_s].
\]
For fixed $t$, $M^t_s(z)$ is a positive supermartingale in $s$. Therefore, we have the following important property.

- For fixed $t$, $L(s, t)$ is a nonnegative supermartingale in $s$ with $L(s, t) = 0$ for $s \geq t$. 

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If \( f_s(w) = g_s^{-1}(w + U_s) \), then the substitution \( z = f_s(w) \) in (24) yields

\[
L(s, s + t) = \int |f'_s(w)|^d G^t(w) dA(w).
\]

If \( r > 0 \), then the scaling rules for SLE imply

\[
L(s, s + r^2 t) = \int |f'_s(w)|^d G^{r^2 t}(w) dA(w)
\]
\[
= r^{d-2} \int |f'_s(w)|^d G^t(w/r) dA(w)
\]
\[
= r^d \int |f'_t(r z)|^d G^t(z) dA(z).
\]

(27)

### 2.3 Two-point Green’s function

The proof of existence of \( \Theta_t \) in [11] uses the two-point Green’s function \( G(z, w) \) introduced in [10]. Our proof of Theorem 3.4 will use a time-dependent version.

The two-point Green’s function \( G(z, w) \) is the normalized probability that \( z \) and \( w \) are both in \( \gamma(0, \infty) \). More precisely, it satisfies

\[
\mathbb{P} \{ \Upsilon_{\infty}(z) < \epsilon, \Upsilon_{\infty}(w) < \delta \} \sim c^2 G(z, w) \epsilon^{2-d} \delta^{2-d}, \quad \epsilon, \delta \downarrow 0.
\]

(28)

It can be written as

\[
G(z, w) = \hat{G}(z, w) + \hat{G}(w, z),
\]

(29)

where \( \hat{G}(z, w) \), roughly speaking, is the probability to visit \( z \) first and then \( w \) later. To be precise, we can write

\[
\hat{G}(z, w) = G(z) \mathbb{E}_z^* [M_{T_z}(w)].
\]

(30)

In [10] it is shown that if one defines \( G(z, w) \) by (29) and (30), then (28) holds. Given (28), we see that there is a corresponding two-point local martingale

\[
M_t(z, w) = |g'_t(z)|^{2-d} |g'_t(w)|^{2-d} G(Z_t(z), Z_t(w)).
\]

Note that \( G \) is symmetric in \( z, w \) but the ordered Green’s function \( \hat{G} \) is not. We will need a bounded time version of the Green’s function, and it is easier to define first the nonsymmetric version.

**Definition** The Green’s function \( \hat{G}^{s,t}(z, w) \) is defined by

\[
\hat{G}^{s,t}(z, w) = G(z) \mathbb{E}_z^* [M_{T_z}^{t-T_z}(w) ; T_z \leq s].
\]

The function \( G^{s,t}(z, w) \) is defined by

\[
G^{s,t}(z, w) = \hat{G}^{s,t}(z, w) + \hat{G}^{t,s}(w, z).
\]
We will be using the symmetric function $G^{t,t}(z, w)$. The rough interpretation of $G^{t,t}(z, w)$ is the normalized probability that the path goes through both $z$ and $w$ and both of the visits occur before time $t$. As for the single point Green’s function, we can give some equivalent formulations that follow immediately from the definition and the scaling rules.

**Proposition 2.10.**

- If $s < t$,
  \[
  \hat{G}^{t,t}(z, w) - \hat{G}^{s,t}(z, w) = \mathbb{E} \left[ |g_s'(z) g_s'(w)|^{2-d} \hat{G}^{t-s, t-s}(Z_s(z), Z_s(w)) \right].
  \]

- If $r > 0$,
  \[
  \hat{G}^{s,t}(z, w) = r^{2(2-d)} \hat{G}^{r^2 s, r^2 t}(rz, rw),
  \]
  \[
  G^{s,t}(z, w) = r^{2(2-d)} G^{r^2 s, r^2 t}(rz, rw).
  \]

The next two results are estimates on the two-point Green’s function that we will use. We delay the proofs until later.

**Theorem 2.11.** There exist $0 < c_1 < c_2 < \infty$ such that if $z, w \in \mathbb{H}$ with $|z| \leq |w|$, then

\[
 c_1 q^{d-2} [S(w) \lor q]^{-\beta} \leq \frac{G(z, w)}{G(z) G(w)} \leq c_2 q^{d-2} [S(w) \lor q]^{-\beta},
\]

where

\[
 q = \frac{|w - z|}{|w|} \leq 2, \quad \beta = (4a - 1) - (2 - d) = \frac{8}{\kappa} - 2 > 0.
\]

**Proof.** See Section 4.2

**Lemma 2.12.** There exists $c < \infty$ such that if $z, w \in \mathbb{H}$ and $s, t > 0$,

\[
 G^{s}(z) G^{t}(w) \leq c G^{cs, ct}(z, w).
\]  

**Proof.** See Section 4.1

We will now prove some consequences of these estimates.

**Lemma 2.13.** There exists $c < \infty$ such that if $z, w \in \mathbb{H}$ with $|z| \leq |w|$,

\[
 G(z, w) \leq c |z|^{d-2} |z - w|^{d-2}.
\]

**Proof.**

\[
 G(z, w) \leq c_2 q^{d-2} [S(w) \lor q]^{-\beta} G(z) G(w) \\
 \leq c q^{d-2} [S(w) \lor q]^{-\beta} |z|^{d-2} |w|^{d-2} S(w)^{\beta} \\
 \leq c |w - z|^{d-2} |z|^{d-2}.
\]

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Lemma 2.14. There exist $c < \infty$, $\beta > 0$ such that if $z = x_z + iy_z$, $w = x_w + iy_w \in \mathbb{H}$, then
\[
G^{t,t}(z, w) \leq c e^{-\beta|w|^2/t} G(z, w).
\] (33)

Proof. Without loss of generality we assume that $|w|^2 \geq |z|^2$. By scaling and symmetry, we may assume $|w| = 2$, and it suffices to prove the estimate for $t$ sufficiently small.

We first suppose that $|z| \geq 1/2$, and let $\sigma = \inf\{t : |\gamma(t)| = 1/4\}$. Then
\[
G^{t,t}(z, w) \leq \mathbb{P}\{\sigma \leq t\} \mathbb{E}[M_\sigma(z, w) | \sigma \leq t].
\]
Using distortion estimates and Theorem 2.11, we see that
\[
M_\sigma(z, w) \asymp G(z, w),
\]
and hence
\[
G^{t,t}(z, w) \leq c \mathbb{P}\{\sigma \leq t\} G(z, w).
\]
Using the Loewner equation, there exists $\delta > 0$ such that if $t \leq t_0$ and $|U_s| \leq \delta$, $0 \leq s \leq t$, then $\sigma > t$. Therefore,
\[
\mathbb{P}\{\sigma \leq t\} \leq \mathbb{P}\left\{\max_{0 \leq s \leq t} |U_s| \geq \delta\right\} \leq c e^{-\beta/t},
\] (34)
for some $c, \beta$. Similarly, for every $\epsilon > 0$, we can find $c_\epsilon, \beta_\epsilon$ such that
\[
G^{t,t}(z, w) \leq c_\epsilon e^{-\beta_\epsilon/t} G(z, w), \quad \epsilon \leq |z|, |w| = 2.
\] (35)

We now suppose $|z| \leq 1/2$ for which Theorem 2.11 implies that $G(z, w) \asymp G(z) G(w)$. Let
\[
\xi = \inf\{t : |\gamma(t)| = 1\},
\]
and note that
\[
G^{t,t}(z, w) \leq \mathbb{E}[M^t_\xi(z, w); \xi \leq t] + G(z) \mathbb{E}^*\{M^t_{T_z}(w); T_z < \xi\].
\]
If $T_z < \xi$, then straightforward estimates show that $|g^t_{T_z}(w)| \asymp 1$, $G_{T_z}(w) \asymp G(w)$, and hence we can use Lemma 2.7 to conclude
\[
\mathbb{E}^*\{M^t_{T_z}(w); T_z < \xi\} \leq c e^{-\beta/t} G(w).
\]
We will now show that
\[
\mathbb{E}[M^t_\xi(z, w); \xi \leq t] \leq c e^{-\beta/t} G(z) G(w).
\]
By (35), it suffices to show this for $t, |z|$ sufficiently small. Let $H = H_\xi$, $g = g_\xi$, $\hat{z} = g(z) - U_\xi, \hat{w} = g(w) - U_\xi, r = \text{dist}(z, \partial H)/y$. We need to show
\[
\mathbb{E}[|g'(z)|^2 - d |g'(w)|^2 - d G^{t,t}(\hat{z}, \hat{w}); \xi \leq t] \leq c e^{-\beta/t} G(z) G(w).
\]
On the event $\{\xi \leq t\}$, we know that $\text{hcap}[\gamma_\xi] \leq t$. In particular, for $t$ sufficiently small, $\text{Im}[\gamma(\xi)] \leq 1/10$. Let $\eta_1, \eta_2$ denote the two open subarcs of the unit half circle with $\gamma(\xi)$
removed. Let \( \eta_2 \) be the smaller arc. Let \( \eta \) be the subarc that disconnects \( z \) from \( \infty \) in \( H \). We consider the two cases \( \eta = \eta_1 \) and \( \eta = \eta_2 \).

Suppose \( \eta = \eta_1 \) so that its length is at least \( \pi/2 \). Since \( |z| \leq 1/2 \), we can use conformal invariance, to see that \( c_1 \leq |\hat{z}|, |\hat{w}| \leq c_2 \). Therefore, \([35]\) implies that
\[
G^{t,t}(\hat{z}, \hat{w}) \leq c e^{-\beta/t} G(\hat{z}, \hat{w}) .
\]
Hence,
\[
\mathbb{E}[|g'(z)|^{2-d} |g'(w)|^{2-d} G^{t,t}(\hat{z}, \hat{w}); \xi \leq t, \eta = \eta_1] \leq c e^{-\beta/t} \mathbb{E}[|g'(z)|^{2-d} |g'(w)|^{2-d} G(\hat{z}, \hat{w})] \leq c e^{-\beta/t} G(z, w).
\]
The second inequality follows from the fact that \( M_t(z, w) \) is a martingale.

Now suppose \( \eta = \eta_2 \) and let \( V \) be the component of \( H \setminus \eta \) containing \( z \). Since \( \text{hcap}[\gamma_\eta] \leq t \), we can deduce \([4]\) that \( \text{Area}(V) \leq ct \). We claim that there exist \( c, \alpha \) such that
\[
S(z) \leq c y r^{1/2} e^{-\alpha/t} . \tag{36}
\]
To see this we use estimates on extremal distance as in the proof of Lemma \(2.8\) to see that if \( \zeta \in V \) with \( |\zeta| \leq 1/2 \), then the probability that a Brownian motion starting at \( \zeta \) exits \( \eta \) is \( O(e^{-\alpha/t}) \) for some \( \alpha \). However, combining the Beurling estimate and the gambler’s ruin estimate, the probability that a Brownian motion starting at \( z \) reaches the circle of radius \( 1/2 \) without leaving \( H \) is \( O(r^{1/2}y) \). This gives \([36]\) and since \( \Upsilon_H(z) \asymp ry \), we get
\[
|g'(z)|^{2-d} G(\hat{z}) = G_H(z; \gamma(\xi), \infty) \leq c (ry)^{d-2} (yr^{1/2} e^{-\alpha/t})^{4a-1} \leq c G(y) e^{-\beta/t} r^{d-2} r^{2a-\frac{d}{2}}.
\]
Hence,
\[
\mathbb{E} \left[ |g'(z) g'(w)|^{2-d} G(\hat{z}, \hat{w}) \mid 2^{-n} \leq r \leq 2^{-n+1}, \eta = \eta_2 \right] \leq c e^{-\beta/t} 2^{-n(d-2)} 2^{-nu} G(z) G(w),
\]
where \( u = 2a - \frac{1}{2} > 0 \). Proposition \([22]\) implies that
\[
\mathbb{P}\{2^{-n} \leq r \leq 2^{-n+1}, \eta = \eta_2\} \leq \mathbb{P}\{2^{-n} \leq r \leq 2^{-n+1}\} \leq c 2^{-n(2-d)},
\]
and hence
\[
\mathbb{E} \left[ |g'(z) g'(w)|^{2-d} G(\hat{z}, \hat{w}); 2^{-n} \leq r \leq 2^{-n+1}, \eta = \eta_2 \right] \leq c e^{-\beta/t} 2^{-nu} G(z) G(w).
\]
The estimate is obtained by summing over \( n \).

We will write \( dA(z, w) = dA(z) dA(w) \) and
\[
\int_{V_1 \times V_2} f(z, w) dA(z, w) = \int_{V_1} \left[ \int_{V_2} f(z, w) dA(w) \right] dA(z),
\]
\[
\int f(z, w) dA(z, w) = \int_{H \times \mathbb{H}} f(z, w) dA(z, w).
\]
Lemma 2.15.

\[
\int G^{1,1}(z, w) \, dA(z, w) < \infty.
\]  

Moreover, there exists \( c < \infty \) such that for every \( \epsilon > 0 \),

\[
\int G^{1,1}(z, w) \mathbf{1}\{|z - w| \leq \epsilon\} \, dA(z, w) \leq c \epsilon^d.
\]

**Proof.** By symmetry it suffices to bound the integrals on the set \( \{ |z| \leq |w| \} \).

If \( z \in \mathbb{H} \), and \( |w| \geq 2|z| \), then (32) and (33) imply that there exist \( \beta > 0 \) such that

\[
\int_{|w| \geq 2|z|} G^{1,1}(z, w) \, dA(w) \leq c \, |z|^{d-2} \int_{|w| \geq 2|z|} e^{-\beta |w|^2} \, |w|^{d-2} \, dA(w)
\]

\[
\leq c \, |z|^{d-2} \int_{2|z|}^{\infty} r^{d-1} e^{-\beta r^2} \, dr
\]

\[
\leq c \, |z|^{d-2} \, e^{-2\beta |z|^2}.
\]

Therefore,

\[
\int G^{1,1}(z, w) \mathbf{1}\{|w| \geq 2|z|\} \, dA(z, w) \leq c \int |z|^{d-2} e^{-2\beta |z|^2} \, dA(z) < \infty.
\]  

Let

\[
V_k = \{ z \in \mathbb{H} : 2^{-k-1} < |z| \leq 2^{-k} \},
\]

and let \( \psi_m(z, w) \) be the indicator that \( |z| \leq |w| \) and \( |z - w| \leq 2^{-m} \). If \( z \in V_0 \) and \( |z| \leq |w| \), Theorem 2.11 implies that

\[
G(z, w) \leq c \, |z - w|^{d-2},
\]

and hence

\[
\int_{V_0 \times \mathbb{H}} G(z, w) \psi_m(z, w) \, dA(z, w) \leq c \int_{|w| \leq 2^{-m}} |w|^{d-2} \, dA(w) \leq c \, 2^{-md}.
\]

More generally, if \( k \) is an integer, we can use the scaling rule for the Green’s function to see that

\[
\int_{V_k \times \mathbb{H}} G(z, w) \psi_m(z, w) \, dA(z, w) = 2^{2k(2-d)} \int_{V_k \times \mathbb{H}} G(2^k z, 2^k w) \psi_{m-k}(2^k z, 2^k w) \, dA(z, w)
\]

\[
= 2^{-2kd} \int_{V_k \times \mathbb{H}} G(2^k z, 2^k w) \psi_{m-k}(2^k z, 2^k w) \, dA(2^k z, 2^k w)
\]

\[
= 2^{-2kd} \int_{V_0 \times \mathbb{H}} G(z, w) \psi_{m-k}(z, w) \, dA(z, w)
\]

\[
\leq c \, 2^{-kd} \, 2^{-md}.
\]
If \( z \in V_k \) and \( w \in V_k \cup V_{k-1} \), then \( |z - w| \leq 2^{-k} + 2^{1-k} < 2^{2-k} \). Setting \( m = k - 2 \) in the last inequality, we see that
\[
\int_{V_k \times (V_k \cup V_{k-1})} G(z, w) \, dA(z, w) \leq c \, 2^{-2kd}.
\]
Combining this with (39), we see that
\[
\int_{|z|,|w| \leq 1} G(z, w) \, dA(z, w) := C_0 < \infty.
\]
The scaling rule gives
\[
\int_{|z|,|w| \leq 2^k} G(z, w) \, dA(z, w) = C_0 \, 2^{2kd}.
\]
Using (33), we get
\[
\int_{|z|,|w| \leq 2^k} G^{1,1}(z, w) \, dA(z, w)
\leq \int_{|z|,|w| \leq 2^{k-1}} G^{1,1}(z, w) \, dA(z, w) + c \, e^{-\beta 2^k} \int_{|z|,|w| \leq 2^k} G(z, w) \, dA(z, w)
\leq \int_{|z|,|w| \leq 2^{k-1}} G^{1,1}(z, w) \, dA(z, w) + c \, e^{-\beta 2^k} C_0 \, 2^{2kd}.
\]
By summing over \( k \in \mathbb{Z} \), we see that
\[
\int G^{1,1}(z, w) \, dA(z, w) < \infty.
\]
Using (33) again, we get
\[
\int_{V_k \times \mathbb{H}} G^{1,1}(z, w) \psi_m(z, w) \, dA(z, w)
\leq c \, e^{-\beta 2^{-2k}} \int_{V_k \times \mathbb{H}} G(z, w) \, dA(z, w)
\leq c \, 2^{-md} e^{-\beta 2^{-2k}} \, 2^{-kd}.
\]
By summing over \( k \in \mathbb{Z} \), we get
\[
\int G^{1,1}(z, w) \psi_m(z, w) \, dA(z, w) \leq c \, 2^{-md}
\]
which gives the second estimate. \( \square \)
3 Natural parametrization

3.1 Natural parametrization in $\mathbb{H}$

Recall that

$$L(s,t) = \int_{M} M_s^t(z) \, dA(z)$$

and if $r < s < t$,

$$\mathbb{E}[L(r,t) - L(s,t) \mid F_r] = \int_{M} M_s^r(z) \, dA(z) = L(r,s) = \int |f'_r(z)|^d G^{s-r}(Z_r(z)) \, dA(z).$$

If we fix $t_0$, then

$$L(t,t_0), \quad 0 \leq t \leq t_0$$

is a positive supermartingale. The *natural parametrization or natural length* $\Theta_t$ is the unique adapted increasing process $\Theta_t$ such that for each $t_0$,

$$L(t,t_0) + \Theta_t, \quad 0 \leq t \leq t_0$$

is a martingale. Although it may appear at first glance that the definition of $\Theta_t$ depends on $t_0$, the fact that $N_t$ in (23) is martingale implies that $\Theta_t$ is independent of $t_0$. In our proof of the existence, it is shown that with probability one for all $0 < s < t < \infty$,

$$\Theta_t - \Theta_s = \lim_{n \to \infty} [\Theta_t^{(n)} - \Theta_s^{(n)}], \quad (40)$$

where

$$\Theta_t^{(n)} = \sum_{j < t 2^n} L \left( \frac{j}{2^n}, \frac{j + 1}{2^n} \right).$$

Our definition here is slightly different but equivalent to the one used in [8, 11]. The equivalence can be seen in that in both cases we derive the formula (40). We note that if $t = k \, 2^{-n}$ for positive integer $k$, then

$$\mathbb{E} \left[ \Theta_t^{(n)} \right] = \mathbb{E}[L(0,t)] = \int G^t(z) \, dA(z) = t^d \int G^1(z) \, dA(z) < \infty. \quad (41)$$

The existence of such a process was established in [8] for $\kappa < 5.0 \cdots$ and for all $\kappa < 8$ in [11]. In this section, we reprove the results with some improvements. We will consider only the case $t_0 = 1$; other values of $t_0$ can be handled using scaling. For the remainder of this section, we let

$$L_t = L(t,1),$$

and recall that $L_t = 0$ if $t \geq 1$. Proof of existence follows from the Doob-Meyer decomposition once we have obtained sufficient bounds on the second moment. The key is the following proposition which is very similar to [11, (16)]. The proof is quick because the hard work is in the estimate (31) on the time-dependent Green's function.
Proposition 3.1. There exists $C < \infty$ such that for every stopping time $T \leq 1$, 
\[ \mathbb{E} \left[ L_T^2 \right] \leq C. \] (42)

Proof. Let $u$ denote the constant $c$ appearing in (31). Let $G^t(z, w) = G^{t,t}(z, w)$. We establish the estimate with 
\[ C = u \int G^u(z, w) \, dA(z, w) \]
which is finite by (37). Let $T$ be a stopping time and let $g = g_T, Z = Z_T$. Using the two-point martingale and (31), we see that for all $z, w \in \mathbb{H}$,
\[
\mathbb{E} \left[ M^1_T(z) M^1_T(w) \right] = \mathbb{E} \left[ \left| g'(z) g'(w) \right| 2^{-d} G^{1-T}(Z(z)) G^{1-T}(Z(w)) \right] \\
\leq u \mathbb{E} \left[ \left| g'(z) g'(w) \right| 2^{-d} G^u(1-T)(Z(z), Z(w)) \right] \\
\leq u G^u(1-T)(z, w) \\
\leq u G^u(z, w).
\]

Therefore,
\[
\mathbb{E} [L_T^2] = \int \mathbb{E} \left[ M^1_T(z) M^1_T(w) \right] \, dA(z, w) \leq u \int G^u(z, w) \, dA(z, w).
\]

With this estimate, we can see that the supermartingale $L_t$ is in the “class DL” and a process $\Theta_t$ exists which makes $L_t + \Theta_t$ a martingale. (See [3, Section 1.4] for relevant facts about the Doob-Meyer decomposition.) Moreover, we can deduce (40) if we weaken our notion of limit to weak-$L^1$ convergence, that is, for every integrable $Y$
\[
\lim_{n \to \infty} \mathbb{E} \left[ (\Theta_T^{(n)} - \Theta_T) Y \right] = 0.
\]

For the restricted range of $\kappa$, an almost sure limit was established as well as an estimate of the Hölder continuity of the paths in [8]. One cannot conclude these stronger results using only the estimate (32). In this section, we will establish the almost sure limit and Hölder continuity for all $\kappa < 8$ by giving a stronger moment bound. Given the existence of $\Theta_t$, we can also write
\[
L(s, t) = \mathbb{E} \left[ \Theta_t - \Theta_s \mid \mathcal{F}_s \right] = \int_{\mathbb{H}} |f'_s(z)|^d G^{t-s}(z) \, dA(z).
\]

Before proving this theorem, we make a comment that will be important when we study the natural parametrization in a smaller domain. Suppose $\{\sigma_s\}$ is a collection of bounded stopping times for the supermartingale $L_t$ such that with probability one $s \mapsto \sigma_s$ is continuous and strictly increasing. Then,
\[
\tilde{L}_s = L_{\sigma_s},
\]
is a time change of a positive supermartingale and hence is a positive supermartingale. A stopping time $T$ for $\tilde{L}_s$ can be considered as a stopping time for $L_t$ and hence we can see
that $\mathbb{E}[\tilde{L}_s^2]$ is uniformly bounded. We can apply the Doob-Meyer theory to $\tilde{L}_s$ and conclude that there is an increasing process $\tilde{\Theta}_s$ such that $\tilde{L}_s + \tilde{\Theta}_s$ is a martingale. Indeed, it is easy to see that $\tilde{\Theta}_s = \Theta_{\sigma_s}$. If $\tilde{F}_s = \mathcal{F}_{\sigma_s}$ and

$$\tilde{\Theta}_s^{(n)} = \sum_{0 \leq j < 2^n} \mathbb{E} \left[ \tilde{L}_{j2^{-n}} - \tilde{L}_{(j+1)2^{-n}} \mid \tilde{F}_{j2^{-n}} \right],$$

then we know that $\tilde{\Theta}_s^{(n)} \to \tilde{\Theta}_s = \Theta_{\sigma_s}$, at least weakly in $L^1$. It is difficult, in general, to give useful expressions for

$$\mathbb{E} \left[ \tilde{L}_{j2^{-n}} - \tilde{L}_{(j+1)2^{-n}} \mid \tilde{F}_{j2^{-n}} \right].$$

However, we will use this with a smooth reparameterization for which we can estimate the conditional expectation.

Our proof of almost sure convergence of $\Theta_t^{(n)} - \Theta_s^{(n)}$ will make use of the Hölder continuity of $SLE_\kappa$ curves with $\kappa \neq 8$ which we now review. We say that a function $f : [0, \infty) \to \mathbb{R}$ is weakly $\alpha$-Hölder if it is continuous at 0 and for every $0 < s_1 < s_2 < \infty$ and $\beta < \alpha$, $f(t), s_1 \leq t \leq s_2$ is Hölder continuous of order $\beta$. The following was proved in [12]; see also [2] where $\alpha^*$ is shown to be optimal.

**Lemma 3.2.** Suppose that $\kappa > 0, \kappa \neq 8$, and let

$$\alpha^*_s = \alpha_s(\kappa) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8 + \kappa}} > 0.$$  

Then with probability one, the $SLE_\kappa$ curve parameterized by half-plane capacity is weakly $\alpha^*_s$-Hölder.

We now state the main theorem in this section.

**Theorem 3.3.** If $\kappa < 8$, then with probability one for all $0 < s < t < \infty$ the limit

$$\Theta_t - \Theta_s = \lim_{n \to \infty} \left[ \Theta_t^{(n)} - \Theta_s^{(n)} \right]$$

exists and $t \mapsto \Theta_t$ is weakly $(\alpha_s d/2)$-Hölder. Moreover, for each $t$, $\Theta_t^{(n)} \to \Theta_t$ in $L^1$.

The bulk of the work in proving this theorem will come in establishing the next result. We will conclude this section by showing how to derive Theorem 3.3 from Theorem 3.4.

**Theorem 3.4.** Suppose $0 < \alpha < \alpha^*_s$ and $K < \infty$. Let $\tau_K = \tau_{K,\alpha}$ denote the stopping time

$$\tau_K = \inf \{ t : \exists s \in [0, t) \text{ with } |\gamma(t) - \gamma(s)| \geq K(t - s)^\alpha \}.$$  

Then if $1 \leq t \leq 2$, the limit

$$\lim_{n \to \infty} \left[ \Theta_t^{(n)} - \Theta_s^{(n)} \right] := Q_K(t),$$

exists in $L^2$ and with probability one. Moreover, with probability one, if $\beta < \alpha d/2$, the function $t \mapsto Q_K(t)$ is Hölder continuous of order $\beta$.  

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Proof. See Section 3.2

Proof of Theorem 3.3 given Theorem 3.4. Suppose \( \alpha < \alpha^* \), \( \beta < \alpha d/2 \), and let \( \tau_K = \tau_{K, \alpha} \) be as in Theorem 3.4. Lemma 3.2 implies that with probability one \( \tau_K > 0 \) for all \( K > 0 \) and

\[
\lim_{K \to \infty} \tau_K = \infty.
\]

Therefore, if \( Q(t) = \lim_{K \to \infty} Q_K(t) \), with probability one, for \( 1 \leq t \leq 2 \),

\[
\lim_{n \to \infty} \left[ \Theta_t^{(n)} - \Theta_1^{(n)} \right] = Q(t).
\]

Moreover, \( t \mapsto Q(t) \) is Hölder continuous of order \( \beta \) on \([1, 2]\).

Using SLE scaling, we can see that Theorem 3.4 implies that with probability one for every positive integer \( m \) and \( 2^{-m} \leq t \), the limit

\[
Q(t; m) = \lim_{n \to \infty} \left[ \Theta_t^{(n)} - \Theta_{2^{-m}}^{(n)} \right]
\]

exists. Moreover, with probability one, for \( t_0 < \infty \), \( t \mapsto Q(t; m) \) is Hölder continuous of order \( \beta \) on \([2^{-m}, t_0]\). Using (11) and Fatou’s lemma, we get \( \mathbb{E}[Q(t; m)] \leq c t^d \). From this we can define by monotonicity

\[
\Theta_t = \lim_{m \to \infty} Q(t; m)
\]

and we can see that \( \mathbb{E}[\Theta_t] \leq c t^d \). Note that if \( 2^{-m} \leq s \leq t \), then

\[
\Theta_t - \Theta_s = Q(t; m) - Q(s; m).
\]

We claim that with probability one, \( \Theta_{0^+} = 0 \). Indeed,

\[
\mathbb{P}\{\Theta_t > \epsilon\} \leq \epsilon^{-1} \mathbb{E}[\Theta_t] \leq c \epsilon^{-1} t^d,
\]

and hence

\[
\sum_{j=1}^{\infty} \mathbb{P}\{\Theta_{2^{-j}} > 2^{-dj/2}\} < \infty.
\]

Therefore, by Borel-Cantelli, we get with probability one \( \Theta_{0^+} = 0 \).

We have shown that \( \Theta_t \) is weakly \( \beta \)-Hölder for every \( \beta < \alpha_d/2 \); hence it is weakly \((\alpha_d/2)\)-Hölder.

\[
\end{proof}
\]

3.2 Proof of Theorem 3.4

We fix \( \alpha < \alpha_* \) and allow all constants to depend on \( \alpha \). As before, let

\[
M_t(z) = |g_t'(z)|^{2-d} G(Z_t(z)),
\]
and if \( 0 \leq s < t \leq \theta \),

\[
\Psi_t = \int M_t(z) \, dA(z),
\]

\[
L(s, t; K) = \mathbb{E} \left[ \Psi_{s \wedge \tau_K} - \Psi_{t \wedge \tau_K} \mid \mathcal{F}_s \right].
\]

The theorem is an example of the Doob-Meyer decomposition. To prove the existence of the decomposition there is an easy step and a hard step: the easy step is to discretize time and give an approximation and the hard step is to take the limit of the approximations. This latter step is not so difficult if one establishes extra moment bounds. In particular, our theorem follows from \([8, \text{Proposition 4.1}]\) once we obtain the following estimate which we derive in this section.

**Proposition 3.5.** For every \( \epsilon > 0 \) and \( K < \infty \), there exists \( c < \infty \) such that if \( \epsilon \leq s < t \leq \frac{1}{\epsilon} \),

\[
\mathbb{E} \left[ L(s, t; K)^2 \right] \leq c (t - s)^{1+\alpha d}.
\]

In \([8]\), a similar estimate was obtained for \( \kappa < 5.0 \cdots \) using the reverse Loewner flow. No such estimate was established in \([11]\) for \( \kappa < 8 \), and this is why the result there was not as strong. The proof we give here uses the usual (forward) Loewner flow, as well as the Hölder continuity of the SLE curve (in the capacity parametrization).

Let

\[
M_t^\theta(z) = |g_t'(z)|^{2-d} G^{\theta-t}(Z_t(z)), \quad \Psi_t^\theta = \int M_t^\theta(z) \, dA(z).
\]

For every \( 0 \leq s \leq t \leq \theta \),

\[
L(s, t; K) = \mathbb{E} \left[ \Psi_{s \wedge \tau_K}^\theta - \Psi_{t \wedge \tau_K}^\theta \mid \mathcal{F}_s \right],
\]

and, in particular,

\[
L(s, t; K) = \mathbb{E} \left[ \Psi_{s \wedge \tau_K}^\theta - \Psi_{t \wedge \tau_K}^\theta \mid \mathcal{F}_s \right].
\]

Note that for fixed \( s \), \( L(s, t; K) \) is increasing in \( t, K \).

**Lemma 3.6.** If \( r > 0 \) and \( 0 \leq s < t \), then the distribution of \( L(s, t; K) \) is the same as that of \( r^{-d} L(r^2 s, r^2 t; r^{2\alpha - 1} K) \). In particular, it has the same distribution as

\[
s^{d/2} L(1, t/s; s^{1-\alpha} K).
\]

**Proof.** We use the scaling rule for \( \text{SLE}_\kappa \). Fix \( r \) and let

\[
\tilde{g}_t(z) = r^{-1} g_{r^2 t}(rz), \quad \tilde{\gamma}(t) = r^{-1} \gamma(r^2 t).
\]

Then \( \tilde{g}_t, \tilde{\gamma} \) have the same distribution as \( g_t, \gamma \). Indeed, \( \tilde{g}_t \) is the solution to the Loewner equation with driving function \( \tilde{U}_t = r^{-1} U_{r^2 t} \). Let

\[
\tilde{Z}_t(z) = \tilde{g}_t(z) - \tilde{U}_t = r^{-1} Z_{r^2 t}(rz).
\]
We define
\[ \tilde{\tau}_K = \inf \{ t : \exists s \leq t : |\tilde{\gamma}(t) - \tilde{\gamma}(s)| = K(t - s)^\alpha \}. \]

Since
\[ \frac{|\tilde{\gamma}(t) - \tilde{\gamma}(s)|}{(t - s)^\alpha} = \frac{|\gamma(r^2 t) - \gamma(r^2 s)|}{r(t - s)^\alpha} = \frac{|\gamma(r^2 t) - \gamma(r^2 s)|}{r^{1 - 2\alpha}(r^2 t - r^2 s)^\alpha}, \]
we get
\[ \tilde{\tau}_K = r^{-2} \tau_{r^2 t}^{-\alpha - 1} K. \]

Let \( \bar{M}_t^\theta(z), \bar{\Psi}_t^\theta \) be defined as above using \( \tilde{g}, \tilde{\gamma} \). Note that
\[ \bar{M}_t^\theta(z) = |\tilde{g}'(z)|^2 G^\theta \tilde{Z}_t(z) \]
\[ = |g'_{r^2 t}(rz)|^2 G^\theta(Z_{r^2 t}(rz) / r) \]
\[ = |g'_{r^2 t}(rz)|^2 r^{-2d} r^{2d} G^{r^2 \theta - r^2 t}(Z_{r^2 t}(rz)) \]
\[ = r^{2d} M_{r^2 t}^{r^2 \theta}(rz), \]
\[ \bar{\Psi}_t^\theta = \int \bar{M}_t^\theta(z) dA(z) \]
\[ = r^{2d} \int M_{r^2 t}^{r^2 \theta}(rz) dA(z) \]
\[ = r^{-d} \int M_{r^2 t}^{r^2 \theta}(rz) dA(rz) = r^{-d} \Psi_{r^2 t}^{r^2 \theta}. \]

Moreover, for each \( K \),
\[ \bar{\Psi}_{t \wedge \tilde{\tau}_K}^{\theta} = r^{-d} \Psi_{t \wedge \tau_{r^2 t}^{-\alpha - 1} K}^{r^2 \theta}. \]

Since \( \tilde{F}_t = \mathcal{F}_{r^2 t} \) we can take conditional expectations and get the result.

\[ \square \]

**Corollary 3.7.** If \( n \) is a positive integer,
\[ \mathbb{E} \left[ L(1, 1 + 2^{-n - 1}; K) \right] \leq 2^{-n} \sum_{j=1}^{2^n} \mathbb{E} \left[ L(1 + (j-1)2^{-n}, 1 + j2^{-n}; K) \right]. \]

**Proof.** Letting \( s = 1 + (j-1)2^{-n} \in [1, 2] \), we have
\[ \mathbb{E} \left[ L(1 + (j-1)2^{-n}, 1 + j2^{-n}; K) \right] = s^d \mathbb{E} \left[ L \left( 1, 1 + \frac{1}{2^n + j - 1}; s^{\frac{1}{2^n} - \alpha} K \right) \right]^2 \]
\[ \geq \mathbb{E} \left[ L(1, 1 + 2^{-(n+1)}; K) \right]. \]

\[ \square \]
Proof of Proposition 3.5. Using Lemma 3.6 we see that it suffices to prove the result when $s = 1$ and $t + s = 2^{-n}$ for some positive integer $n$. By Corollary 3.7 it suffices to prove that for all $K < \infty$,

$$
\sum_{j=1}^{2^n} \mathbb{E} [L(j, n)^2] \leq c_K 2^{-n}\ln d,
$$

where

$$
L(j, n) = L \left(1 + (j - 1)2^{-n}, 1 + j2^{-n}; K\right).
$$

For the remainder, we fix $K$ and allow constants to depend on $K$. We write $G^s(z, w)$ for $G^{s, s}(z, w)$.

Let $\tau = \tau_K$ and

$$
Y_t = \Psi_{(1+t)\wedge \tau}.
$$

The definition of $\tau$ implies that $|\gamma((t + s) \wedge \tau) - \gamma(t \wedge \tau)| \leq Ks^\alpha$. Therefore,

$$
\mathbb{E} \left[ M_{(1+t)\wedge \tau} (z) 1\{|z - \gamma(t)| > Ks^\alpha\} \right] = M_{t\wedge \tau} (z) 1\{|z - \gamma(t)| > Ks^\alpha\}.
$$

In particular, if $t = 1 + (j - 1)2^{-n}$,

$$
\mathbb{E} \left[ Y_{(j-1)2^{-n}} - Y_{j2^{-n}} \mid \mathcal{F}_t \right] \leq \int \left| g_t'(z) \right|^2 dG^{2^{-n}}(Z_t(z)) 1 \left\{ \left| z - \gamma(t) \right| \leq K 2^{-\alpha n} \right\} dA(z).
$$

Hence, if

$$
J_n = \{(z, w) \in \mathbb{R}^d : |z - w| \leq 2K 2^{-\alpha n}\},
$$

then

$$
L(j, n)^2 \leq \int_{J_n} \left| g_t'(z) \right|^2 dG^{2^{-n}}(Z_t(z)) G^{2^{-n}}(Z_t(w)) dA(z, w).
$$

Lemma 2.12 implies that there exists $u < \infty$ such that

$$
L(j, n)^2 \leq u \int_{J_n} \left| g_t'(z) \right|^2 dG^{2^{-n}}(Z_t(z), Z_t(w)) dA(z, w).
$$

Without loss of generality we assume $u = 2^{m_0}$ for some integer $m_0 \geq 3$.

Let

$$
\hat{M}_t(z, w) = |g_t'(z)|^2 |g_t'(w)|^2 G^{u+1-t}(Z_t(z), Z_t(w)),
$$

and note that if $t = j2^{-n}$,

$$
\mathbb{E} \left[ \hat{M}_t(z, w) - \hat{M}_{t+u2^{-n}}(z, w) \right] \geq \mathbb{E} \left[ |g_t'(z)|^2 |g_t'(w)|^2 G^{u2^{-n}}(Z_t(z), Z_t(w)) \right].
$$

(Roughly speaking, the right-hand side corresponds to the normalized probability that the path goes through both $z$ and $w$ between times $t$ and $t + u2^{-n}$.) By summing over $j$, we see that

$$
\sum_{j=1}^{2^n-m_0} \mathbb{E} [L(jm_0, n)^2] \leq c \int_{J_n} G^{u+1}(z, w) dA(z, w).
$$
Similarly, for $k = 0, 1, 2, \ldots, m_0 - 1$, we see that
\[
\sum_{j=1}^{2^{n-m_0}} \mathbb{E}[L(jm_0 + k, n)^2] \leq c \int_{J_n} G^{u+k2^{-n}+1}(z, w) dA(z, w).
\]

By summing over $k$, we see that
\[
\sum_{j=1}^{2^n} \mathbb{E}[L(j, n)^2] \leq c \int_{J_n} G^{2u}(z, w) dA(z, w).
\]

and by using (38) and scaling we see that
\[
\sum_{j=0}^{2^n-1} \mathbb{E}[L(j, n)^2] \leq c 2^{-d\alpha n}.
\]

The next lemma will be used later.

**Lemma 3.8.** There exists $c < \infty$ such that if $s, t > 0$ and $r \leq 1/2$, then
\[
L(s, s + (1 + r)t) - L(s, s + t) \leq c r L(s, s + t).
\]

**Proof.** Let $f(z) = g_s^{-1}(U_s + z)$ and $u = \sqrt{1+r}$. Then,
\[
L(s, s + t) = \int_{\mathbb{H}} |f'(z)|^d G^t(z) dA(z).
\]

\[
L(s, s + (1 + r)t) = \int_{\mathbb{H}} |f'(w)|^d G^{t(1+r)}(w) dA(w)
\]
\[
= u^{d-2} \int_{\mathbb{H}} |f'(w)|^d G^t(w/u) dA(w)
\]
\[
= u^d \int_{\mathbb{H}} |f'(uz)|^d G^t(z) dA(z)
\]

The distortion theorem implies that
\[
|f'(uz)| = |f'(z)| [1 + O(u - 1)] = |f'(z)| [1 + O(r)].
\]

Since $u^d = 1 + O(r)$,
\[
L(s, s + (1 + r)t) = [1 + O(r)] \int_{\mathbb{H}} |f'(uz)|^d G^t(z) dA(z) = L(s, s + t) [1 + O(r)].
\]

\[\square\]
3.3 Two useful lemmas

We have shown that

\[
\Theta_t = \lim_{n \to \infty} \sum_{j \leq t2^n} \int_{H_n} |f_{(j-1)2^{-n}}'(z)|^d G^{2^{-n}}(z) \, dA(z),
\]  

(44)

where the limit is in \( L^1 \). The integral is concentrated on \( z \) near the origin. Proposition 3.9 makes a precise statement of this. Recall Lévy’s theorem on the modulus of continuity of Brownian motion (see, e.g., [3, Theorem 9.5]) which states that if \( U_t \) is a standard Brownian motion and \( t_0 < \infty \), then with probability one

\[
\sup \left\{ \frac{|U_t - U_s|}{\sqrt{|t - s|} \log(t - s)} : 0 \leq s < t \leq t_0 \right\} < \infty.
\]  

(45)

Proposition 3.9. Suppose \( \delta_n \) is a sequence of positive numbers such that

\[
\lim_{n \to \infty} 2^{-n/2} \sqrt{n} \delta_n^{-1} = 0.
\]

Let \( H_n = \{ z \in \mathbb{H} : |z| \leq \delta_n \} \). Then,

\[
\Theta_t = \lim_{n \to \infty} \sum_{j \leq t2^n} \int_{H_n} |f_{(j-1)2^{-n}}'(z)|^d G^{2^{-n}}(z) \, dA(z),
\]

where the limit is in \( L^1 \).

Proof. Let \( \epsilon_n \) be a sequence with

\[
\lim_{n \to \infty} 2^{-n/2} \sqrt{n} \epsilon_n^{-1} = 0,
\]

\[
\lim_{n \to \infty} \epsilon_n \delta_n^{-1} = 0.
\]

Let

\[
\gamma(s)(t) = g_s(\gamma(t + s)) - g_s(\gamma(s)) = g_s(\gamma(t + s)) - U_s,
\]

and let

\[
\lambda_n = \inf \{ t : \exists m \geq n, s \leq t \text{ with } |t - s| \leq 2^{-m} \text{ and } |\gamma(s)(t)| \geq \delta_m \},
\]

\[
\tilde{\lambda}_n = \inf \{ t : \exists m \geq n, s \leq t \text{ with } |t - s| \leq 2^{-m} \text{ and } |U_t - U_s| \geq \epsilon_m \}.
\]

Deterministic estimates using the Loewner equation (see [5, Lemma 4.13]) show that for all \( n \) sufficiently large, \( \tilde{\lambda}_n \leq \lambda_n \). Using (45) can see that with probability one \( \tilde{\lambda}_n \uparrow \infty \) and hence \( \lambda_n \uparrow \infty \).

Note that for fixed \( m \), if \( n \geq m \),

\[
\mathbb{E}[\Theta_{(s+2^{-n})\wedge \lambda_m} - \Theta_{s \wedge \lambda_m} \mid \mathcal{F}_{s \wedge \lambda_m}] \leq \int_{H_n} |f_{s}'(z)|^d G^{2^{-n}}(z) \, dA(z).
\]
Therefore,
\[ \Theta_{t \wedge \lambda_n} \leq \liminf_{n \to \infty} \sum_{j \leq t^{2^n}} \int_{E_n} |f'_{j-1}2^{-n}(z)|^d G^{2^{-n}}(z) dA(z). \]

However $\Theta_{t \wedge \lambda_n}$ converges monotonically to $\Theta_t$ and hence $E[\Theta_t - \Theta_{t \wedge \lambda_n}] \to 0$. Combining with (44), we get the result.

Lemma 3.10. Suppose $\U_t$ is a driving function such that the Loewner equation generates a curve. Then for all $T < \infty$, the functions \{f_s(z) : 0 \leq s \leq T\} are equicontinuous at 0. In other words, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|z| < \delta$ and $t \in [0,T]$, then $|f_t(z) - \gamma(t)| < \epsilon$.

Proof. If $s < t$, define $g_t^s$ by $g_t = g_t^s \circ g_s$ and write $h_t = g_t^{-1}, h_s^t = (g_t^s)^{-1}$. Note that if $s < t$, then $h_s = h_t \circ g_t^s$ and $h_t = h_s \circ g_s^t$.

Since $\U_t$ is uniformly continuous on $[0,T]$, there exists increasing $v(s)$ with $v(0^+) = 0$ such that $|\U_{t+s} - \U_t| \leq v(s), 0 \leq t \leq T - s$. From the Loewner equation (see [3, Lemma 4.13]), one can see that there exists a universal $c < \infty$ such that for all $z$, and $0 \leq t \leq T - s$,

\[ |g_t^{t+s}(z) - z| \leq c[v(s) \vee \sqrt{s}], \quad |h_t^{t+s}(z) - z| \leq c[v(s) \vee \sqrt{s}]. \tag{46} \]

Since the Loewner equation is generated by a curve, we know that for each $t$, $f_t(z)$ can be extended continuously to $\H$ with $f_t(0) = h_t(\U_t) = \gamma(t)$. We claim the following stronger fact: for every $\epsilon > 0$ and $t \in [0,T]$, there exists $\delta = \delta(t, \epsilon) > 0$ such that if $|z| < \delta$ and $|t - s| < \delta$, then $|f_s(z) - \gamma(t)| < \epsilon$. To see this, fix $t$ and find $r$ such that $|f_t(z) - f_t(0)| < \epsilon/2$ for $|z| < r$. Using (46), we can find $\delta < r/4$ so that if $|s - t| < \delta$,

\[ |\U_t - \U_s| < r/4 \]

and

\[ \sup_{w} |g_t^s(w) - w| \leq r/4 \quad \text{if} \quad s < t, \]
\[ \sup_{w} |h_t^s(w) - w| \leq r/4 \quad \text{if} \quad s > t. \]

Then if $t - \delta < s < t$ and $|z| < \delta$, we have

\[ |f_s(z) - f_t(0)| = |h_s(z + \U_s) - f_t(0)| = |h_t(g_t^s(z + \U_s)) - f_t(0)| = |f_t(w) - f_t(0)|, \]

for some $w$ with $|w| < r$. Hence $|f_s(z) - f_t(0)| < \epsilon/2$. A similar argument proves this estimate for $t < s < t + \delta$. Therefore, if $|z| < \delta$ and $|s - t| < \delta$,

\[ |f_s(z) - \gamma(t)| = |f_s(z) - f_t(0)| \leq |f_s(z) - f_s(0)| + |f_s(0) - f_t(0)| < \epsilon. \]

We finish the proof by a standard compactness argument, covering $[0,T]$ by a finite number of intervals $[t - \delta_t, t + \delta_t]$ and then choosing $\delta = \min\{\delta_t\}$. \qed
3.4 Natural parametrization in other domains

Suppose $D$ is a simply connected domain with distinct boundary points $z, w$. Then $SLE_{\kappa}$ from $z$ to $w$ in $D$ is defined (up to reparameterization) by $\eta(t) = F \circ \gamma(t)$, where $\gamma$ is an $SLE_{\kappa}$ in $\mathbb{H}$ from 0 to $\infty$ and $F : \mathbb{H} \to D$ is a conformal transformation with $F(0) = z, F(\infty) = w$. The conformal transformation is unique only up to an initial dilation, but scaling shows that the distribution is independent of the choice. If $\Theta_t$ denotes the natural parametrization for the curve $\gamma$, we define the natural parametrization $\tilde{\Theta}_t$ in $D$ by the scaling rule

$$\tilde{\Theta}_t = \int_0^t |F'(\gamma(s))|^d d\Theta_s.$$  

(47)

A simple argument shows that the joint distribution of $(F \circ \gamma(t), \tilde{\Theta}_t)$, modulo time reparameterization, is independent of the choice of $F$.

This definition can also be given more intrinsically. Suppose for ease that $D$ is a bounded domain with

$$\int_D G_D(\zeta; z, w) \, dA(\zeta) < \infty. \quad (48)$$

Let $\eta(t)$ be an $SLE_{\kappa}$ path from $z$ to $w$ in $D$; the choice of parametrization is not important. Then the natural length in $D$ is the unique increasing process $\tilde{\Theta}_t$ such that

$$\int_D G_{D_t}(\zeta; \eta(t), w) \, dA(\zeta) + \tilde{\Theta}_t,$$

is a martingale. Here $D_t$ represents the component of $D \setminus \eta_t$ whose boundary includes $\eta(t)$ and $w$. In particular, $\eta$ has the natural parametrization if

$$\int_D G_{D_t}(\zeta; \eta(t), w) \, dA(\zeta) + t$$

is a martingale. If (48) does not hold we can define $\tilde{\Theta}$ as we did in the half plane by using some type of cut-off. (In this paper, the cut-off was given by using a time-dependent Green's function; in [8, 11] the cut-off was made by considering the Green’s function in a bounded domain bounded away from the origin.)

If $\gamma[0, t]$ is a curve that lies in both $D_1$ and $D_2$, then it is not immediate from our definition that the natural length of $\gamma_t$ is independent of whether we consider it in $D_1$ or in $D_2$. We will establish this in this section. By conformal invariance, it suffices to prove this for $D_1 = D = \mathbb{H} \setminus V \subset \mathbb{H} = D_2$ with $V$ bounded and $\text{dist}(0, V) > 0$ which we now assume.

Let

$$F : \mathbb{H} \to D$$

be the unique conformal transformation with $F(z) - z = o(1)$ as $z \to \infty$, and let $\Phi = F^{-1}$. If $\gamma$ is an $SLE_{\kappa}$ path from 0 to $\infty$ in $\mathbb{H}$, let

$$\tau_D = \inf\{t : \gamma(t) \in V\}.$$
Suppose $\sigma$ is a stopping time with $\sigma < \tau_D$. There are two probability measures on $\gamma_\sigma$, that of $SLE$ in $D$ and that of $SLE$ in $\mathbb{H}$ (in both cases we are choosing endpoints $0$ and $\infty$). It is known that these two probability measures, considered as measures on the stopped paths $\gamma_\sigma$, are mutually absolutely continuous. The natural length of $\gamma_\sigma$ can be considered using either measure. In the case of $SLE$ in $D$, one defines the length in $\mathbb{H}$ and then uses $F$ and the conformal covariance rule (47) to get the length in $D$. We will show the two definitions are the same. We start by setting up some notation.

Let $\gamma(t)$ be an $SLE_\kappa$ path in $\mathbb{H}$ defined as usual with $\text{hcap}[\gamma_t] = at$. Let $g_t$ and $U_t$ denote the conformal maps and driving function, respectively. We assume $t < \tau_D$, and let

$$D_t = g_t(D), \quad \eta^*(t) = \Phi \circ \gamma(t).$$

Let $H^*_t$ be the unbounded component of $\mathbb{H} \setminus \eta^*_t$, and let $g^*_t$ denote the unique conformal transformation of $H^*_t$ onto $\mathbb{H}$ with $g^*_t(z) - z = o(1)$ as $z \to \infty$. Let

$$\Phi_t = g^*_t \circ \Phi \circ g_t^{-1}, \quad F_t = \Phi_t^{-1}.$$

Then $\Phi_t : D_t \to \mathbb{H}$ is the unique conformal transformation with $\Phi_t(z) - z \to 0$ as $z \to \infty$. Let $\hat{a}(t)$ denote the capacity of the curve viewed in $D$, that is,

$$\hat{a}(t) = \text{hcap} [ \Phi \circ \gamma_t ] = \text{hcap}[\eta^*_t].$$

Then [5, Section 4.6]

$$\hat{a}(t) = a \int_0^t \Phi'_s(U_s)^2 \, ds,$$

and $g^*_t$ satisfies the Loewner equation

$$\partial_t g^*_t(z) = \frac{a \Phi'_t(U_t)^2}{g^*_t(z) - U^*_t},$$

where $U^*_t = g^*_t(\eta^*(t)) = \Phi_t(U_t)$. Let $\delta(t) = \text{dist}(U_t, \mathbb{H} \setminus D_t)$. By Schwarz reflection, $\Phi_t$ can be extended to a conformal transformation on the open disk of radius $\delta(t)$ about $U_t$. The distortion theorem implies that there exists a universal $c < \infty$ such that

$$|\Phi'_t(z)| \leq c \delta(t)^{-1} \Phi'_t(U_t), \quad |z - U_t| \leq \delta(t)/2,$$

$$|\Phi''_t(z)| \leq c \delta(t)^{-2} \Phi'_t(U_t), \quad |z - U_t| \leq \delta(t)/2,$$

$$|\Phi'_t(z) - \Phi'_t(U_t)| \leq c \delta(t)^{-1} \Phi'_t(U_t) |z - U_t|, \quad |z - U_t| \leq \delta(t)/2.$$  

Using the Loewner equation (see [5, Proposition 4.41]), we see that there exist $c$ such that

$$\partial_t \Phi'_t(x) \leq c \delta(t)^{-2} \Phi'_t(x), \quad U_t - \frac{\delta(t)}{2} \leq x \leq U_t + \frac{\delta(t)}{2}.$$  

We will not need the full force of these estimates, but we will use the following consequence. Let $\hat{\delta}_t = \inf \{ \delta(s) : 0 \leq s \leq t \}$, and

$$\Delta(s,t) = \max\{|U_r - U_s| : s \leq r \leq t\}.$$
• For every $\delta > 0$, there exists $\epsilon > 0, c < \infty$ such that if $\delta_t > \delta, 0 \leq s \leq t \leq s + \epsilon$ and $\Delta(s, t) \leq \epsilon$, then

$$|\Phi'_t(U_t) - \Phi'_s(U_s)| \leq c [(t - s) + \Delta(s, t)] \Phi'_s(U_s).$$

Let $\sigma(t) = \inf \{s : \hat{a}(s) = at\}$, and let $\gamma(t) = \gamma(\sigma_t)$ denote the path $\gamma$ reparameterized so that $\operatorname{hcap}[\Phi \circ \gamma_t] = at$. We do this only for $t < \tilde{\tau}_D$, let $\eta(t) = \Phi \circ \tilde{\gamma}(t) = \eta^*(\sigma_t)$, and note that $\eta$ is parameterized so that $\operatorname{hcap}[\eta_t] = at$. Let $\hat{a}_t = g_{\sigma_t}^*, \hat{U}_t = U_{\sigma_t}^*, \hat{\Phi}_t = \Phi_{\sigma_t}, \hat{U}_t = U_{\sigma_t}, \hat{g}_t = g_{\sigma_t}, \hat{f}_t(z) = \hat{g}_t^{-1}(z + \hat{U}_t), \tilde{f}_t(z) = \tilde{g}_t^{-1}(z + \tilde{U}_t), \tilde{F}_t = \tilde{f}_t^{-1} \circ F \circ \tilde{f}_t$. Note that

$$\tilde{f}_t \circ \tilde{F}_t = F \circ \tilde{f}_t.$$ 

Since

$$at = \operatorname{hcap}[\eta_t] = a \int_0^{\sigma_t} \Phi'_s(U_s)^2 ds,$$

we see that

$$\partial_t \sigma_t = K_t \quad \text{where} \quad K_t = \Phi'_t(U_t)^{-2} = \hat{\Phi}'_t(0)^2.$$

Let $\Theta^*_t$ denote the natural parametrization associated to the curve $\eta$ (under the measure of $SLE_\kappa$ from 0 to $\infty$ in $\mathbb{H}$). Recall that

$$\Theta^*_t = \lim_{n \to \infty} \sum_{j < t/2^n} L^* \left( \frac{j - 1}{2^n}, \frac{j}{2^n} \right), \quad (49)$$

where

$$L^*(r, s) = \int_{\mathbb{H}} |\hat{f}_j(z)|^d G^{s-r}(z) dA(z),$$

and the limit is in $L^1$. Using Proposition 3.31, we can also write

$$\Theta^*_t = \lim_{n \to \infty} \sum_{j < t/2^n} J^*(j, n)$$

where

$$J^*(j, n) = \int_{\mathbb{H}_n} |\hat{f}_{j-1,n}(z)|^d G^{2-n}(z) dA(z), \quad \mathbb{H}_n = \{z \in \mathbb{H} : |z| \leq n 2^{-n/2}\}, \quad \hat{f}_{j,n} = \hat{f}_{2^{-n}}.$$
The expression on the right-hand side of (49) is a deterministic function of the curve \( \eta(s) \), \( 0 \leq s \leq t \). Using continuity, it follows that

\[
\hat{\Theta}_t := \int_0^t |F'(\eta(s))|^d \, d\Theta^*_s = \lim_{n \to \infty} \hat{\Theta}_{t,n} \tag{50}
\]

where

\[
\Theta_{t,n} = \sum_{j < \frac{t}{2^n}} J^*(j, n) \left| F' \left( \eta \left( \frac{j - 1}{2^n} \right) \right) \right|^d.
\]

The left-hand side of (50) is the natural parametrization of \( \tilde{\gamma} \) considered as SLE in \( D \). If \( \Theta_t \) is the natural parametrization associated to \( \gamma \), then \( \tilde{\Theta}_t = \Theta_t \sigma_t \) is the length of \( \tilde{\gamma} \) considered as an SLE curve in \( \mathbb{H} \). Having set up the notation, we can now state the main theorem of this section.

**Theorem 3.11.** With probability one, for all \( t < \tau_D \), \( \tilde{\Theta}_t = \hat{\Theta}_t \).

By absolute continuity, the “with probability one” in the statement can be taken either with respect to SLE\( _\kappa \) in \( \mathbb{H} \) or SLE\( _\kappa \) in \( D \). Since both \( \tilde{\Theta}_t \) and \( \hat{\Theta}_t \) are continuous in \( t \) it suffices to show that for each \( t < \tau_D \), \( \tilde{\Theta}_t = \hat{\Theta}_t \) with probability one. To do this we will define a sequence of stopping times \( \tau_m \) with \( \tau_m \uparrow \tau_D \) and prove that for each \( m \) and \( t \)

\[
P \left\{ \tilde{\Theta}_{t \wedge \tau_m} \neq \hat{\Theta}_{t \wedge \tau_m} \right\} = 0. \tag{51}
\]

We let \( \tau_m \) be the first time \( t \) such that one of the following occurs: \( t \geq m \); \( \tilde{\delta}(t) \leq 1/m \); \( \tilde{\Phi}'(U_t) \geq m \); \( \tilde{\Phi}'(U_t) \leq 1/m \); \( |F'(\eta(t))| \geq m \); \( |F'(\eta(t))| \leq 1/m \). It is easy to check from our distortion estimates above that \( \tau_m \uparrow \tau \). For the remainder, we fix \( m \) and write \( \tau = \tau_m \). All constants, implicit or explicit, may depend on \( m \). Also, we only need consider \( t \leq m \), since for \( t > m \), \( t \wedge \tau = m \wedge \tau \).

We will change our reparameterization slightly by setting

\[
\partial_t \sigma_t = K_t, \quad \tilde{K}_t = K_{t \wedge \tau}.
\]

This will have no affect on times \( t \leq \tau \) which is all that is important for us. The advantage of this change is that we can write

\[
c \leq \tilde{K}_t \leq C, \quad |\tilde{K}_t - \tilde{K}_s| \leq u(t - s),
\]

for a continuous function \( u \) with \( u(0^+) = 0 \). Since we are considering only \( t \leq \tau \), we will write \( K_t \) rather than \( \tilde{K}_t \).

In order to prove (51), we consider \( \tilde{L}_t = L_{\sigma_t} \). This is a supermartingale, and as noted in Section 3.1, we can write

\[
\tilde{\Theta}_t = \lim_{n \to \infty} \tilde{\Theta}_{t,n}, \tag{52}
\]
Arguing as in Proposition 3.9, we get
\[ \hat{\Theta}_{t,n} = \sum_{j < t^{2n}} \tilde{L} \left( \frac{j - 1}{2^n}, \frac{j}{2^n} \right), \quad \tilde{L}(r, s) = \mathbb{E} \left[ \tilde{L}_r - \tilde{L}_s \mid \tilde{F}_r \right]. \]

In this case, the Doob-Meyer theorem only gives a weak-\( L^1 \) limit in (52), but this is all that we need. We will show that
\[ \mathbb{E} \left[ |\hat{\Theta}_{t \land \tau,n} - \hat{\Theta}_{t \land \tau,n}| \right] \to 0. \tag{53} \]

Since weak limits are unique, this implies that \( \hat{\Theta}_{t \land \tau} = \hat{\Theta}_{t \land \tau} \). To establish (53), we claim that it suffices to find for fixed \( t \) and \( \tau \), a uniformly bounded sequence of random variables \( J_n \) with \( J_n \to 0 \) with probability one and
\[ |\hat{\Theta}_{t \land \tau,n} - \hat{\Theta}_{t \land \tau,n}| \leq J_n \hat{\Theta}_{t \land \tau,n}. \]

Indeed, since \( \hat{\Theta}_{t,n} \to \hat{\Theta}_t \) in \( L^1 \), the random variables \( \{\hat{\Theta}_{t,n} : n = 1, 2, \ldots\} \) are uniformly integrable. Using this and the fact that the \( J_n \) are uniformly bounded with \( J_n \to 0 \), we see that \( \mathbb{E}[J_n \hat{\Theta}_{t,n}] \to 0 \).

If we let \( u_n = u(2^{-n}) \), \( K_{j,n} = K_{(j-1)2^{-n}} \), then if \( \tau > (j - 1)2^{-n} \),
\[ L \left( \sigma_{j-1,n}, \sigma_{j-1,n} + \frac{K_{j,n} - u_n}{2^n} \right) \leq \tilde{L} \left( \frac{j - 1}{2^n}, \frac{j}{2^n} \right) \leq L \left( \sigma_{j-1,n}, \sigma_{j-1,n} + \frac{K_{j,n} + u_n}{2^n} \right). \]

Using Lemma 3.8, we can conclude that
\[ \tilde{L} \left( \frac{j - 1}{2^n}, \frac{j}{2^n} \right) = L \left( \sigma_{j-1,n}, \sigma_{j-1,n} + \frac{K_{j,n}}{2^n} \right) [1 + O(u_n)]. \]

Using (52), we see that
\[ \hat{\Theta}_t = \lim_{n \to \infty} \sum_{j < t^{2n}} L \left( \sigma_{j-1,n}, \sigma_{j-1,n} + \frac{K_{j,n}}{2^n} \right). \]

We recall that
\[ L \left( \sigma_{j-1,n}, \sigma_{j-1,n} + \frac{K_{j,n}}{2^n} \right) = \mathbb{E} [\Theta_{t+r^{2^{-n}}} - \Theta_t \mid \mathcal{F}_t] \]
\[ = \int \mathbb{E} [\tilde{f}'(z)]^d G^{2^{-n}}(z) \, dA(z), \]
with \( t = \sigma_{j-1,n} \) and \( r^2 = K_{j,n} \). Using (27), this can also be written as
\[ r_{j-1,n}^d \int_{\mathbb{H}} |\tilde{f}_{j-1,n}(r_{j-1,n} z)|^d G^{2^{-n}}(z) \, dA(z), \quad \tilde{f}_{j,n} = \tilde{f}_{j2^{-n}}, \quad r_{j,n} = \tilde{r}_{j2^{-n}}(0). \]

Arguing as in Proposition 3.9, we get
\[ \hat{\Theta}_t = \lim_{n \to \infty} \sum_{j < t^{2n}} r_{j-1,n}^d \int_{\mathbb{H}_n} |\tilde{f}_{j-1,n}(r_{j-1,n} z)|^d G^{2^{-n}}(z) \, dA(z), \tag{54} \]
where as before $\mathbb{H}_n = \{ z \in \mathbb{H} : |z| \leq n 2^{-n/2} \}$.

By comparing (50) and (54), we see that it suffices to prove the following. For each $m$, there exists a sequence $u_n \downarrow 0$, such that if $t \leq \tau_m$ and $r = \hat{F}_t'(0)$, then for $z \in \mathbb{H}_n$,

$$\left| r |\tilde{f}_t'(rz)| - |F'(\eta(t))| \right| |\hat{f}_t'(z)| \leq u_n r |\tilde{f}_t'(rz)|.$$

The sequence $u_n$ can depend on the path but the estimate must hold uniformly for all $t \leq \tau_m$ and $z \in \mathbb{H}_n$. We write this shorthand as

$$r |\tilde{f}_t'(rz)| = |F'(\eta(t))| |\hat{f}_t'(z)| [1 + o(1)],$$

with the above uniformly implied. We claim that

$$|F'(\hat{f}_t(z))| = |F'(\eta(t))| [1 + o(1)].$$

Since $\eta(t) = \hat{f}_t(0)$, this follows from distortion estimates on $F$ provided that we have uniform bounds on $|\hat{f}_t(z) - \hat{f}_t(0)|$. But these are provided by Lemma 3.10. Distortion estimates also imply

$$|\hat{F}_t'(z)| = r [1 + o(1)],$$

$$rz = \hat{F}_t(z) [1 + o(1)],$$

$$|\tilde{f}_t'(rz)| = |\tilde{f}_t'(F_t(z))| [1 + o(1)].$$

Therefore, (55) becomes

$$|\tilde{f}_t'(\hat{F}_t(z))| |\hat{F}_t'(z)| = |F'(\hat{f}_t(z))| |\hat{f}_t'(z)| [1 + o(1)].$$

But $\hat{f}_t \circ \hat{F}_t = F \circ \hat{f}_t$, so the chain rule implies that $|\tilde{f}_t'(\hat{F}_t(z))| |\hat{F}_t'(z)| = |F'(\hat{f}_t(z))| |\hat{f}_t'(z)|$.

### 3.5 A particular case

If $\gamma$ is an $S\mathcal{L}E_\kappa$ curve from 0 to $\infty$ in $\mathbb{H}$ and $r > 0$, then

$$\gamma^{(r)}(t) = \gamma(t + r),$$

is an $S\mathcal{L}E_\kappa$ curve from $\gamma(r)$ to $\infty$ in $H_r$. We would like to say that the natural parametrization in $H_t$ is the same as that in $\mathbb{H}$, and we make this precise here.

Suppose $\gamma(t)$ has driving function $U_t = -B_t$. We recall that we can define $\Theta_t$ by first defining it for dyadic rational $t$ by the $L^1$ limit

$$\Theta_t = \lim_{n \to \infty} \Theta_t^{(n)},$$

and for other $t$ it is defined by continuity. Here we use the fact that the natural parametrization is continuous with respect to the capacity parametrization. A key observation, is that
given the Brownian motion $B_t$, there is a well defined natural parametrization $\Theta_t$ for all $t \geq 0$, up to a single null event.

If $\tau$ is a stopping time, let

$$\gamma^\tau(t) = \gamma(\tau + t), \quad \eta^\tau(t) = g_\tau(\gamma^\tau(t)) - U_\tau,$$

and note that

$$\gamma^\tau(t) = f_\tau(\eta^\tau(t)).$$

The strong Markov property implies that $\eta^\tau$ is an $SLE_\kappa$ curve from 0 to $\infty$ with driving function $U^\tau_t = -B^\tau_t$ where $B^\tau$ is the Brownian motion,

$$B^\tau_t = B_{t+\tau} - B_\tau.$$

Let $\Theta^\tau_\tau$ denote the corresponding version of the natural parametrization defined as in the previous paragraph. Since $f_\tau : \mathbb{H} \to H_\tau$ is a conformal transformation, we can view $\gamma^\tau$ as an $SLE_\kappa$ curve from $\gamma(\tau)$ to $\infty$. The natural length of $\gamma^\tau(0, t]$ considered as a curve in $H_\tau$ is

$$\int_0^t |f^\prime_{\tau}(\eta^\tau(r))|^d \, d\Theta^\tau_r.$$

**Lemma 3.12.** Suppose $r > 0$, and $\tau$ is a stopping time with $\tau \leq r$. Let

$$Z_n(s, t) = \sum L(\tau + j2^{-n}, \tau + (j + 1)2^{-n}),$$

where the sum is over all $j$ with $s \leq j/2^n \leq t$. Then with probability one for all $0 < s < t$,

$$\lim_{n \to \infty} Z_n(s, t) = \Theta_{\tau+t} - \Theta_{\tau+s}.$$

**Proof.** By continuity it suffices to prove the result for fixed $s < t$ that are dyadic rationals. Apply the same proof as for Theorem 3.4 to the supermartingale

$$\Psi^\tau_t = \Psi_{t+\tau} - \Psi_\tau.$$

**Proposition 3.13.** If $\tau$ is a stopping time with $\mathbb{P}\{\tau < \infty\} = 1$, then with probability one, for all $0 < s < t$,

$$\Theta_{t+\tau} - \Theta_{s+\tau} = \int_s^t |f^\prime_{\tau}(\eta^\tau(r))|^d \, d\Theta^\tau_r. \quad (56)$$

**Proof.** Without loss of generality, we may assume $\tau$ is a bounded stopping time (otherwise, apply the proposition to $\tau \wedge k$ and let $k \to \infty$). We note that for each $s$, the natural parametrization after time $s$ is supported on the curves in $H_s$. Since both sides of (56) are
continuous in $t$ with probability one, it suffices to show that for positive dyadic rationals $s < t$ such that $\gamma[s, t] \subset H_\tau$

$$\Theta_{t+\tau} - \Theta_{s+\tau} = \int_s^t |f'_\tau(\eta^\tau(r))|^d \, d\Theta^\tau_r.$$  

Let $t_{j,n} = \tau + j2^{-n}$. By the lemma and Proposition 3.9

$$\Theta_{t+\tau} - \Theta_{s+\tau} = \lim_{n \to \infty} \sum_{s+\tau \leq t_{j,n} < t+\tau} \int_{\mathbb{H}_n} |f'_{t_{j,n}}(z)|^d G^{2^{-n}}(z) \, dA(z).$$

In Proposition 3.9 we have a $L^1$ limit, but we can assume it is an almost sure limit by taking a subsequence if necessary (for notational ease, we will assume it is an almost sure limit, but the remainder of this proof could be done along a subsequence if necessary). Take $\delta_n$ in the definition of $H_n$ such that $\text{diam}(H_n) \to 0$ as $n \to \infty$.

Define $\tilde{f}_{j,n}$ by $f_{t_{j,n}} = f_\tau \circ \tilde{f}_{j,n}$. Then by Proposition 3.9

$$\Theta^\tau_r = \lim_{n \to \infty} \sum_{t_{j,n} \leq r} \int_{\mathbb{H}_n} |f'_{t_{j,n}}(z)|^d G^{2^{-n}}(z) \, dA(z). \quad (57)$$

By continuity of natural length, we can write the right-hand side of (56) as

$$\lim_{n \to \infty} \sum_{s+\tau \leq t_{j,n} \leq t+\tau} \int_{\mathbb{H}_n} |f'_\tau(\eta^\tau(t_{j,n} - \tau))|^d |\tilde{f}'_{t_{j,n}}(z)|^d G^{2^{-n}}(z) \, dA(z).$$

Hence, the difference of the two sides of (56) is

$$\lim_{n \to \infty} \sum_{s+\tau \leq t_{j,n} \leq t+\tau} \int_{\mathbb{H}_n} \left[ |f'_\tau(\eta^\tau(t_{j,n} - \tau))|^d - |f'_\tau(\tilde{f}_{t_{j,n}}(z))|^d \right] |\tilde{f}'_{t_{j,n}}(z)|^d G^{2^{-n}}(z) \, dA(z). \quad (58)$$

By (57), we see that this is bounded above by

$$[\Theta^\tau_r - \Theta^\tau_s] \limsup_{n \to \infty} K_n(\gamma),$$

where

$$K_n(\gamma) = \max_{j, z \in \mathbb{H}_n} \{||f'_\tau(\eta^\tau(t_{j,n} - \tau))|^d - |f'_\tau(\tilde{f}_{t_{j,n}}(z))|^d||\}.$$  

Hence it suffices to show that $K_n(\gamma) \to 0$ with probability one.

Note that because it is an almost sure expression we can show it goes to zero for a given curve $\gamma$ so our constants may depend on it. We have $\tilde{f}_{j,n}(0) = \eta^\tau(t_{j,n} - \tau)$ so because $z \in \mathbb{H}_n$ and $\text{diam}(\mathbb{H}_n) \to 0$ this is equivalent to the fact that $f'_\tau$ is continuous in a neighborhood of $\eta^\tau[s, t]$. Using the assumption $\gamma[s, t] \subset H_\tau$, we see that $\eta^\tau[s, t] \subset \{\text{Im}(z) > \epsilon\}$ for some $\epsilon > 0$ which depends on $\gamma$. Continuity of $f'_\tau$ on $\text{Im}(z) > \epsilon$ is a consequence of Cauchy integral formula and we are done.
4 Bounds on the two-point Green’s function

4.1 Lower bound for the time-dependent Green’s function

In this section we prove Lemma 2.12. This is a generalization of a result in [11] where it is shown that there exists \( c < \infty \) such that for all \( z, w \in \mathbb{H} \),

\[
G(z) G(w) \leq c \left[ \hat{G}(z, w) + \hat{G}(w, z) \right].
\]

Examination of that proof shows that the same argument establishes the existence of \( c < \infty \) such that for \( |z|, |w| \leq 1 \),

\[
G(z) G(w) \leq c \left[ \hat{G}_{c,c}(z, w) + \hat{G}_{c,c}(w, z) \right].
\]  (59)

We will assume (59).

We write \( z = x_z + iy_z, w = x_w + iy_w \). We assume

\[
y_z^2 \leq 2as, \quad y_w^2 \leq 2at,
\]  (60)

for otherwise the left-hand side of (31) is 0. All constants in this proof are independent of \( z, w, s, t \), but may depend on \( a \). We will first handle the case when \( t \ll y_z^2 \) or \( s \ll y_w^2 \).

- **Claim 1**: There exist \( \varepsilon > 0, c < \infty \) such that

\[
G^s(z) G^t(w) \leq c \hat{G}^{ct,cs}(w, z) \quad \text{if} \quad t \leq \varepsilon y_z^2,
\]

and

\[
G^s(z) G^t(w) \leq c \hat{G}^{cs,ct}(z, w) \quad \text{if} \quad s \leq \varepsilon y_w^2.
\]

By scaling, we may assume that \( 1 = |z| \leq |w| \). We will prove the first inequality which is the harder of the two; the second can be proved similarly. We will assume that \( t \leq y_z^2/(200a) \); we will later choose \( \varepsilon < 1/(200a) \). Using (60) we see that \( t \leq s/100 \), and \( y_z \leq 1 \) implies that \( t \leq 1/(200a) \leq 1/50 \). Using (60) again, we see that \( y_w \leq 1/10 \), and hence \( |w| \asymp |x_w| \geq \sqrt{99}/10 \). Let \( T = T_w \) and \( \gamma = \gamma[0, T] \). By Corollary 2.9, there exists \( c_1 < \infty \) such that

\[
\mathbb{P}_w^s \{ \text{diam}(\gamma) \leq c_1 |w|, T \leq 2t \} \geq \frac{1}{2}.
\]

Let

\[
E = \{ \text{diam}(\gamma) \leq c_1 |w|, T \leq 2t \},
\]

and note that \( \mathbb{P}_w^s(E) \geq \mathbb{P}_w^s\{ T \leq 2t \}/2 \). On the event \( E \), we can use the Loewner equation (4) and conformal invariance arguments to estimate \( Z_r(z) \) and \( g_r(z) \) for \( 0 \leq r \leq T \). We list the estimates here omitting the straightforward proofs. First, there exists \( c_2 < \infty \) such that

\[
|U_r| \leq c_2 |w|, \quad 0 \leq r \leq T.
\]
Also,

\[ |Z_r(z)| \geq Y_r(z) \geq \frac{y_z}{2}, \quad 0 \leq r \leq T, \]

\[ |\partial_r g_r(z)| \leq \frac{a}{|Z_r(z)|} \leq \frac{2a}{y_z}, \quad 0 \leq r \leq T, \]

\[ |g_T(z) - z| \leq \frac{4at}{y_z} \leq \frac{y_z}{50} \leq \frac{1}{50}. \]

\[ |Z_T(z)| \leq |g_T(z)| + |U_T(z)| \leq \frac{51}{50} + c_1|w| \leq c_2|w|, \]

\[ |\partial_r \log g'_r(z)| \leq \frac{a}{|Z_r(z)|^2} \leq \frac{4a}{y_z^2}, \]

\[ |\log |g'_T(z)|| \leq \frac{8at}{y_z^2} \leq \frac{1}{25}. \]

If \( S_T(z) = \sin \arg Z_T(z) \), then

\[ S_T(z) = \frac{Y_T(z)}{|Z_T(z)|} \geq \frac{c_3 y_z}{|w|}. \]

\[ |g'_T(z)|^{2-d} G(Z_T(z)) = |g'_T(z)|^{2-d} Y_T(z)^{1-4a} S_T(z)^{4a-1} \geq c_4 |w|^{1-4a} G(z). \]

Using Lemma 2.7 we can find \( c_5, \beta_1 \) such that

\[ |g'_T(z)|^{2-d} G^s(Z_T(z)) \geq c_5 e^{-\beta_1|w|^2/s} G(z). \]

Therefore, we get

\[ G^{2s,2t}(z, w) \geq c_6 e^{-\beta_1|w|^2/s} G^s(z) G^{2t}(w). \]

Using the last inequality in Lemma 2.7, we can see that there exists \( \delta \) such that

\[ G^{2s,2t}(z, w) \geq c_7 G^s(z) G^{\delta t}(w). \]

If we let \( \hat{t} = \delta t \), then we can rewrite this as

\[ G^{2s,(2/\delta)\hat{t}} \geq c_7 G^s(z) G^{\hat{t}}(w), \]

which is now valid if \( \hat{t} \leq \delta y_z^2/(200a) \). This establishes the claim with \( \epsilon = \delta/(200a) \).

For the remainder of the proof we fix \( 0 < \epsilon_0 < 1/2a \) such that Claim 1 holds, and we assume that \( t \geq \epsilon_0 y_z^2, s \geq \epsilon_0 y_w^2 \), and hence

\[ s \geq \frac{y_z^2}{2a}, \quad t \geq \frac{y_w^2}{2a}, \quad s \land t \geq \epsilon_0 (y_z \lor y_w)^2 \] (61)
• **Claim 2.** There exist \( l, \rho > 0 \) such that the following holds. Suppose \( z = x + iy \) with \( |z| \leq 1 \) and \( y^2/a \leq s \). Let \( V \) denote the event

\[
V = V_{z,l} = \{ \gamma[0, T_z] \subset \{ z' : |z'| \leq l \} \}.
\]

Then

\[
\mathbb{P}^*_z(V \mid T_z \leq s) \geq \rho.
\] (62)

This was proved in Corollary 2.9. We fix \( l \geq 8 \) and \( \rho > 0 \) such that (62) holds.

• **Claim 3.** There exist \( c_1, \beta_1 < \infty \) such that if \( |w| \geq 2l|z| \), then

\[
G_s(z) G_t(w) \leq c_1 \check{G}_{\beta_1 s, \beta_1 t}(z, w).
\]

By scaling we may assume \( |z| = 1 \). Let

\[
u = \min \left\{ 4s, \frac{2t}{\alpha \epsilon_0}, 8 \right\},
\]

and note that \( y_z^2 \leq au/2 \). Let \( T = T_z, \gamma = \gamma[0, T] \) and let \( V \) be as above. Then

\[
\mathbb{P}^*_z(V \mid T \leq \nu) \geq \rho.
\]

On the event \( V \), since \( T \leq 8 \),

\[
\gamma \subset \{ x + iy : 0 < y \leq \sqrt{4a}, \ -l \leq x \leq l \}.
\]

Standard conformal mapping estimates imply that

\[
|X_T(w)| \asymp |x_w|, \quad Y_T(w) \asymp y_w, \quad |g_T'(w)| \asymp 1.
\]

So if \( u \in \{ 4s, 8 \} \) because we get \( G^u(z) > cG^s(z) \) the argument proceeds in the same way (actually, somewhat more easily) as Claim 1.

If \( u = \frac{2t}{\alpha \epsilon_0} \) then if we have \( \beta_1 \) then we can find \( \beta_2 \) such that

\[
cG_{\beta_1 s, \beta_1 t}(z, w) \geq cG_{\beta_2 t, 2\beta_2 t}(z, w) \geq cG(z) \mathbb{E}_z^* [M_{\beta_2 t}(w) \mid T \leq \beta_2 t, V] \mathbb{P}[T \leq \beta_2 t]
\]

Then again we have same estimates on \( |X_T(w)|, Y_T(w) \) and \( |g_T'(w)| \) as above, so by lemma 2.7 the last term is bigger than \( cG(z) G_{\beta_2 t}(w) e^{-\frac{-5|w|^2}{\delta \beta_2}} \) for some \( \delta > 0 \) fixed. Again using Lemma 2.7 we need to show existence of \( \beta_2 \) such that \( \frac{-5|w|^2}{\delta \beta_2} - \frac{5}{\beta_2} \geq \frac{-|w|^2}{8} \) which we can do by \( |w| > 2l \) so we are done.
• **Claim 4.** There exists $c < \infty$ such that if $|z|, |w| \leq 4l$ and $t \geq (y_z \vee y_w)^2/(2a)$, then

$$G(z) G(w) \leq c e^{\beta/t} G^{\text{ext,ct}}(z, w).$$

It suffices to prove the result for $t$ sufficiently small for otherwise we can use $59$. For the moment we assume $t \leq 1/(100a)$. Using Claim 3, we see that it suffices to prove the estimate for $z = 1 + iy_z, w = x_w + iy_w$ with $y_z, y_w \leq 1/10$ and $1 \leq |x_w| \leq 4l$. In this case, $G(z) \asymp y_z^{4a-1}, G(w) \asymp y_w^{4a-1}$. Let

$$\sigma = \inf\{t : \text{Re}[\gamma(t)] = 1 - \sqrt{t}\},$$

and let $E$ be the event

$$E = E_t = \{\sigma \leq t, \quad \gamma_\sigma \subset \{\text{Re}[z'] \geq -1/2\}\}.$$

Arguing as in Lemma 2.7, we can see that

$$\mathbb{P}\{\sigma \leq t, \text{Re}[\gamma(s)] \geq -1/2 \text{ for } 0 \leq s \leq t\} \geq c_1 e^{-\beta_1/t}. \quad (63)$$

Since $\text{hc}(\gamma_\sigma) = a\sigma$, we see that on the event $E$,

$$\gamma_\sigma \subset V := \{x + iy : -1/2 \leq x \leq 1 - \sqrt{t}, 0 \leq y \leq \sqrt{2at}\}.$$

By the strong Markov property, for every $s > 0$,

$$G^{s+t, s+t}(z, w) \geq \mathbb{E}\left[|g'_\sigma(z)|^{2-d} |g'_\sigma(w)|^{2-d} G^{s, s}(Z_\sigma(z), Z_\sigma(w)) 1_E\right].$$

We now list some deterministic estimates that hold on the event $E$. The constants $0 < c_1 < c_2 < \infty$ can be chosen uniformly over every curve $\gamma$ with $\gamma_\sigma \subset V$ and $\text{Re}[\gamma(\sigma)] = 1 - \sqrt{t}$. We follow the statement of each estimate with a brief justification. We write $g = g_\sigma$.

• $c_1 y_z \leq Y_\sigma(z) \leq y_z, \quad c_1 y_w \leq Y_\sigma(w) \leq y_w$. The argument is the same for $z$ and $w$. By conformal invariance and the fact that $g(z') \sim z'$ as $z' \to \infty$,

$$Y_\sigma(z) = \lim_{R \to \infty} R \mathbb{P}^z \{\text{Im}[B_r] = R\},$$

where $B$ is a complex Brownian motion and $\tau = \tau_{R, \gamma}$ is the first time $r$ that $B_r \in \mathbb{R} \cup \gamma_\sigma \cup \{\text{Im}(w') = R\}$. The probability that a Brownian motion starting at $z = 1 + iy_z$ reaches $\{\text{Im}(w') = 2\sqrt{at}\}$ before time $\tau$ is bounded below by $c y_z / \sqrt{t}$ and given this the probability that $\text{Im}(B_\tau) = R$ is greater than $c \sqrt{t}/R$.

• $c_1 \leq |g'(z)|, |g'(w)| \leq c_2$.

It follows from the previous estimate that $c_1 \leq g'(1), g'(x_w) \leq 1$. We can then use the Schwarz reflection and distortion theorem.
• $c_1 \sqrt{t} \leq |Z_\sigma(z)| \leq c_2 \sqrt{t}$.

For the lower bound consider $g$ (extended by Schwarz reflection) on the disk of radius $\sqrt{t}$ about $z$. The image of this disk contains a disk of radius $|g'(z)|\sqrt{t}/4$. Therefore, $|Z_\sigma(z)| \geq |g'(z)|\sqrt{t}/4 \geq c_1 \sqrt{t}$. For the upper bound consider $\zeta = 1 + i\sqrt{t}$. Using conformal invariance, it is easy to see that $S_\sigma(\zeta) \geq c$ and hence $|Z_\sigma(\zeta)| \approx Y_\sigma(\zeta) \leq \sqrt{t}$. Since $|g'|$ is uniformly bounded on the line segment $[1, 1+i\sqrt{t}]$, we get the upper bound.

• $G(Z_\sigma(z)) \geq c_1 G(z)$, $G(Z_\sigma(w)) \geq c_1 G(w)$.

This is immediate from the estimates we have already established.

Since $|g'_\sigma(z)| \approx |g'_\sigma(w)| \approx 1$, (63) implies for all $\beta > 0$,

$$G^{(\beta+1)t,(\beta+1)t}(z, w) \geq c \mathbb{E} \left[ G^{\beta t, \beta t}(Z_\sigma(z), Z_\sigma(w)) 1_V \right] \geq c e^{-\beta/t} \mathbb{E} \left[ G^{\beta t, \beta t}(Z_\sigma(z), Z_\sigma(w)) \right].$$

We now consider two cases. First, if $|Z_\sigma(w)| \geq 2t|Z_\sigma(z)|$, then Claim 3 gives the existence of $c, \beta$ such that

$$G(Z_t(z)) G(Z_t(w)) \leq c G^{\beta t, \beta t}(Z_t(z), Z_t(w)).$$

If $|Z_\sigma(w)| \leq 2t|Z_\sigma(z)|$, then distortion estimates imply that $|Z_\sigma(w)| \approx \sqrt{t}$. Scaling gives

$$G^{\beta t, \beta t}(Z_t(z), Z_t(w)) = t^{2-d} G^{\beta, \beta}(Z_t(z)/\sqrt{t}, Z_t(w)/\sqrt{t}).$$

If we choose $\beta$ sufficiently large,

$$G^{\beta, \beta}(Z_t(z)/\sqrt{t}, Z_t(w)/\sqrt{t}) \geq c G(Z_t(z)/\sqrt{t}) G(Z_t(w)/\sqrt{t}) \geq c t^{d-2} G(z) G(w).$$

Hence, we can find $c, \beta$ such that

$$G^{\beta t, \beta t}(z, w) \geq c e^{-\beta_1/t} G(z) G(w).$$

Claim 5 finishes the proof of the lemma.

• **Claim 5.** There exists $c < \infty$ such that if $|z|, |w| \leq 4l$, and $s, t$ satisfy (61), then

$$G_s(z) G_t(w) \leq c G^{cs, ct}(z, w).$$

By symmetry we may assume $t \leq s$. By (61), we can find $r$ such that $rt \geq (y_z \vee y_w)^2/(2a)$.

From Claim 4 we can find (different) $c, \beta$ such that

$$G(z) G(w) \leq c e^{\beta_1/t} G^{ct, ct}(z, w).$$

We also know that there exist $c_1, \beta_2$ such that $G'(w) \leq c e^{-\beta_2/t} G(w)$. By choosing $c$ even larger in the last displayed formula, we can guarantee that

$$G(z) G(w) \leq c e^{\beta_2/t} G^{ct, ct}(z, w).$$

Therefore,

$$G^s(z) G^t(w) \leq G(z) G'(w) \leq c e^{-\beta_2/t} G(z) G(w) \leq c G^{ct, ct}(z, w) \leq c G^{cs, ct}(z, w).$$
4.2 Proof of Theorem 2.11

In [11] it is shown that there exists \( c < \infty \) such that for all \( z, w \in \mathbb{H} \),

\[
G(z) G(w) \leq c G(z, w).
\]

In [10], a bound in the other direction was given: if \( |w| = 1 \) and \( |z - w| \leq 1/2 \), then

\[
G(z, w) \leq c |z - w|^{d-2}.
\]

However, these papers did not give precise estimates in the case where \( S(z), S(w) \) are small. Our proof will use the ideas in [10]. Throughout this section, \( \gamma \) will denote an \( SLE_\kappa \) curve and

\[
\gamma_t = \gamma(0, t), \quad \Delta_t(z) = \text{dist}(z, \gamma_t), \quad \Delta(z) = \Delta_\infty(z).
\]

We write \( \asymp \) to indicate that quantities are bounded by constants where the constants depend only on \( \kappa \). We recall that in [10] it is shown that for each \( z, w \), there exist \( \epsilon_z, \delta_w \) such that if \( \epsilon < \epsilon_z, \delta < \delta_w \),

\[
\mathbb{P}\{\Delta(z) \leq \epsilon\} \asymp G(z) \epsilon^{2-d}, \quad \mathbb{P}\{\Delta(w) \leq \delta\} \asymp G(w) \delta^{2-d}, \quad (64)
\]

\[
\mathbb{P}\{\Delta(z) \leq \epsilon, \Delta(w) \leq \delta\} \asymp G(z, w) \epsilon^{2-d} \delta^{2-d}. \quad (65)
\]

When estimating \( \mathbb{P}\{\Delta(z) \leq \epsilon\} \) there are two regimes. The interior or bulk regime, where \( \epsilon \leq \text{Im}(z) \) can be estimated using Proposition 2.2 since in this case \( \Delta(z) \asymp \Upsilon(z) \). However for the boundary regime \( \epsilon > \text{Im}(z) \), one needs a different result.

**Lemma 4.1.** There exists \( 0 < c_1 < c_2 < \infty \) such that if \( 0 < y \leq 1/4 \) and \( \sigma = \inf\{t : |\gamma(t) - 1| \leq 2y\} \), then

\[
c_1 y^{4a-1} \leq \mathbb{P}\{\sigma < \infty, S_\sigma(1 + iy) \geq 1/10\} \leq \mathbb{P}\{\sigma < \infty\} \leq c_2 y^{4a-1}.
\]

**Proof.** The bound \( \mathbb{P}\{\sigma < \infty\} \asymp y^{4a-1} \) can be found in a number of places. A proof which includes a proof of the first inequality can be found in [11]. The first inequality is Lemma 2.10 of that paper. \( \square \)

We will prove Theorem 2.11 in a sequence of propositions. We assume \( |z| \leq |w| \) and let

\[
q = |w - z|, \quad \beta = (4a - 1) - (2 - d) = 4a + \frac{1}{4a} - 2 > 0.
\]

It will be useful to define a quantity that allows us to consider the boundary and interior cases simultaneously. Let

\[
\Phi_t(z) = \Delta_t(z)^{4a-1} \quad \text{if} \quad \Delta_t(z) \geq \text{Im}(z),
\]

\[
\Phi_t(z) = \text{Im}(z)^{4a-1} \left[ \frac{\Delta_t(z)}{\text{Im}(z)} \right]^{2-d} \quad \text{if} \quad \Delta_t(z) \leq \text{Im}(z),
\]

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and let \( \Phi(z) = \Phi_\infty(z) \). Note that \( \Phi_0(z) = |z|^{4a-1} \), and scaling implies that the distribution of \( \Phi(rz) \) is the same as that of \( r^{4a-1}\Phi(z) \). Since \( 4a - 1 > 2 - d \), we see that
\[
\Delta_t(z)^{4a-1} \leq \Phi_t(z).
\]

The next lemma combines the interior and boundary estimates into one estimate.

**Lemma 4.2.** There exist \( 0 < c_1 < c_2 < \infty \) such that for all \( z \in \mathbb{H} \) and \( 0 < \epsilon \leq 1 \),
\[
c_1 \epsilon \leq \mathbb{P}\{\Phi(z) \leq \epsilon \Phi_0(z)\} \leq c_2 \epsilon.
\]

**Proof.** Let \( z = x + iy \). By scaling we may assume that \( |z| = 1 \) and hence \( \Phi_0(z) = 1, S(z) = y \).

Let \( \Delta = \Delta_\infty(z), \Phi = \Phi_\infty(z) \). Proposition 2.2 and Lemma 4.1 imply that
\[
\mathbb{P}\{\Delta \leq \epsilon\} \asymp \epsilon^{4a-1}, \quad \epsilon \geq y,
\]
\[
\mathbb{P}\{\Delta \leq \epsilon\} \asymp y^{4a-1}[\epsilon/y]^{2-d}, \quad \epsilon \leq y.
\]

If \( \epsilon \geq y \), then
\[
\mathbb{P}\{\Phi \leq \epsilon^{4a-1}\} = \mathbb{P}\{\Delta \leq \epsilon\} \asymp \epsilon^{4a-1}.
\]

If \( \epsilon \leq y \), then if \( u = (4a - 1)/(2 - d) \),
\[
\mathbb{P}\{\Phi \leq \epsilon^{4a-1}\} = \mathbb{P}\{y(\Delta/y)^{2-d/(4a-1)} \leq \epsilon\} = \mathbb{P}\{\Delta \leq y(\epsilon/y)^u\} \asymp y^{4a-1}[\epsilon/y]^u = \epsilon^{4a-1}.
\]

The next proposition establishes an upper bound for the probability that an SLE path gets close to a point and subsequently returns to a given crosscut. It is a generalization of Lemmas 4.10 and 4.11 of [10], and we use ideas from those proofs. Suppose \( \eta : (0,1) \to \mathbb{H} \) is a simple curve with \( \eta(0+) = 0, \eta(1-) > 0 \) and write \( \eta = \eta(0,1) \). Let \( V_1, V_2 \) denote respectively the bounded and unbounded components of \( \mathbb{H} \setminus \eta \) and assume that \( z = x_z + iy_z \in V_1, w = x_w + iy_w \in V_2 \). Given the SLE curve \( \gamma \), one can show (see [10, Appendix A]) that there is a collection of open subarcs \( \{I_t : t < T_z \wedge T_w\} \) of \( \eta \) with the following properties. Recall that \( H_t \) is the unbounded component of \( \mathbb{H} \setminus \gamma_t \).

- \( I_0 = \eta \).
- \( I_t \subset H_t \). Moreover, \( H_t \setminus I_t \) has two connected components, one containing \( z \) and the other containing \( w \).
- If \( s < t \), then \( I_t \subset I_s \). Moreover, if \( \gamma(s,t) \cap I_z = \emptyset \), then \( I_t = I_s \).

If \( \zeta \in \{z,w\} \), define stopping times \( \sigma_k, \sigma, \tau \) depending on \( \zeta \) by
\[
\sigma_k = \inf\{t : \Phi_t(\zeta) = 2^{-k}\Phi_0(\zeta)\}, \quad \sigma = \sigma_1, \quad \tau = \inf\{t \geq \sigma : \gamma(t) \in T_x\}.
\]

If \( \tau < \infty \), let
\[
J = \frac{\Phi_\tau(\zeta)}{\Phi_0(\zeta)}.
\]
Proposition 4.3. There exists $c < \infty$ such that under the setup above if $0 < \epsilon \leq 1/2$ and $\alpha = 2a - \frac{1}{2} > 0$,

\[
\mathbb{P}\{\tau < \infty, J \leq \epsilon\} \leq c \epsilon, \quad \text{if } \zeta = z,
\]

\[
\mathbb{P}\{\tau < \infty, J \leq \epsilon\} \leq c \epsilon \left[\frac{\text{diam}(\eta)}{|w|}\right]^\alpha, \quad \text{if } \zeta = w.
\]

Proof. The first inequality follows immediately from (66), as does the second if $|w| \leq 4 \text{diam}(\eta)$. Therefore, using scaling, we may assume that $\text{diam}(\eta) = 1$, $|w| \geq 4$, $\zeta = w$. Let $C$ denote the half-circle of radius $\sqrt{|w|}$ in $\mathbb{H}$ centered at the origin. Let $k_0$ be the largest integer such that $2^{-k_0} \geq S(w) = \text{Im}(w)/|w|$. Let $\rho$ be the first time $t$ that $w$ is not in the unbounded component of $H_t \setminus C$. Note that if $\rho < T_w$, then $\gamma(\rho) \in C$. Let

\[
\hat{j} = \frac{\Phi_\rho(w)}{\Phi_0(w)}.
\]

Then, if $k$ is a positive integer and $\hat{\sigma} = \sigma_k$,

\[
\mathbb{P}\{\tau < \infty, J \leq 2^{-k}\} \leq \mathbb{P}\{\hat{\sigma} < \rho \wedge \tau, \tau < \infty\} + \sum_{j=1}^{k} \mathbb{P}\{\rho < \hat{\sigma} < \infty, 2^{-j} < \hat{j} \leq 2^{-j+1}\}.
\]

We will now show that

\[
\mathbb{P}\{\hat{\sigma} < \rho \wedge \tau, \tau < \infty\} \leq c 2^{-k} |w|^{-\alpha}.
\]  \hspace{1cm} (67)

Let $H = H_\sigma$, $I = I_\sigma$, $g = g_\sigma$, $U = U_\sigma$. By (66),

\[
\mathbb{P}\{\hat{\sigma} < \rho \wedge \tau\} \leq \mathbb{P}\{\hat{\sigma} < \infty\} \leq c 2^{-k}.
\]

Let $H^*$ be the component of $H \setminus C$ containing $w$. On the event $\hat{\sigma} < \rho$, $H^*$ is unbounded. Using simple connectedness of $H$, we can see that there is a subarc $l \in \partial H^* \cap C$ that is a crosscut of $H$ and that separates $w$ from $I$ in $H$. Since $l$ does not separate $w$ from $\infty$, $g(l)$ is a crosscut of $\mathbb{H}$ that does not separate $U$ from $\infty$; for ease let us assume that its endpoints are on $(-\infty, U]$. Since $l$ separates $w$ from $I$, $l$ also separates $I$ from infinity in $H$. Therefore $g(l)$ separates $g(I)$ from $U$ and infinity in $\mathbb{H}$. We use excursion measure (see [10] 4.1 for definitions and similar estimates) to estimate the probability that $\gamma(\hat{\sigma}, \infty)$ returns to $I$. The excursion measure between $g(I)$ and $[U, \infty)$ in $\mathbb{H} \setminus g(I)$ is bounded above by the excursion measure between $g(I)$ and $g(l)$ in $\mathbb{H} \setminus (g(I) \cup g(l))$ which by conformal invariance equals the excursion measure between $I$ and $l$ in $H \setminus (I \cup l)$. This in turn is bounded above by the excursion measure between $C$ and $\partial L$ in $\{\zeta \in \mathbb{H} : 1 < |\zeta| < \sqrt{|w|}\}$ which is $O(1/\sqrt{|w|})$. Given this, we can use the estimate from Lemma 4.4 to see that the probability that an $SLE_\kappa$ path from $U$ to $\infty$ in $\mathbb{H}$ hits $g(I)$ is $O(|w|^{-(4a-1)/2})$. Using conformal invariance, we conclude that

\[
\mathbb{P}\{\tau < \infty \mid \hat{\sigma} < \rho \wedge \tau\} \leq c |w|^{(1-4a)/2}
\]
which gives (67).

We noted above that if $j \leq k_0$, then
$$\mathbb{P}\{\rho < \hat{\sigma} < \infty\} = 0.$$ We will now show that if $j > k_0$,
$$\mathbb{P}\{\rho < \hat{\sigma} < \infty, 2^{-j} < \hat{J} \leq 2^{-j+1}\} \leq c 2^{-k} 2^{-j\alpha} |w|^{-\alpha}. \tag{68}$$
The proposition then follows by summing over $j$. Consider the event
$$E_j = \{\rho < \infty, 2^{-j} < \hat{J} \leq 2^{-j+1}\}.$$ Using (66), we see that
$$\mathbb{P}(E_j) \leq c 2^{-j}. \tag{69}$$
Let $H = H_\rho$. On the event $E_j$, there is a subarc $l$ of $H \cap C$ that is a crosscut of $H$ with one endpoint equal to $\gamma(\rho)$ such that $l$ disconnects $w$ from infinity in $H$. Using this and the relationship between $S$ and harmonic measure we see that $S_\rho(w)$ is bounded above by the probability that a Brownian motion starting at $w$ reaches $C$ without leaving $H$. Using the Beurling estimate, we see that the probability that it reaches distance $|w|/2$ from $w$ without leaving $H$ is $O(2^{-j/2})$. Given this, the probability that it reaches $C$ without leaving $H$ is bounded above by $O(1/|\sqrt{|w|}|)$. Therefore, on the event $E_j$,
$$S_\rho(w) \leq c 2^{-j/2} |w|^{-1/2}.$$ Using the strong Markov property and (66), we see that
$$\mathbb{P}\{\hat{\sigma} < \infty \mid E_j\} \leq c 2^{-j\alpha} |w|^{-\alpha} 2^{-(k-j)},$$ which combined with (69) gives (68).

In this section, we will consider two-point correlations. The next proposition, which is the hardest to prove, shows that if $z,w$ are separated, then the events are independent up to a multiplicative constant.

**Proposition 4.4.** There exists $c < \infty$ such that if $|z| \leq 4|w|$, and $0 < \epsilon_z, \epsilon_w \leq 1$, then
$$\mathbb{P}\{\Phi(z) \leq \epsilon_z \Phi_0(z), \Phi(w) \leq \epsilon_w \Phi_0(w)\} \leq c \epsilon_z \epsilon_w,$$

**Proof.** By scaling, we may assume that $|z| \leq 1/2, |w| = 2$. As before we consider a decreasing collection of arcs $\{I_t : t < T_z \cap T_w\}$ with the following properties.

- $I_0 = \{\zeta \in \mathbb{H} : |\zeta| = 1\}$.
- For each $t$, $I_t$ is a crosscut of $H_t$ that separates $z$ from $w$ in $H_t$.  

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• If \( t > s \), then \( I_t \subset I_s \). Moreover, if \( \gamma(s,t] \cap T_s = \emptyset \), then \( I_t = I_s \).

We define a sequence of stopping times as follows:

\[
\begin{align*}
\sigma_0 &= 0, \\
\tau_0 &= \inf\{t : |\gamma(t)| = 1\} = \inf\{t : \gamma(t) \in \overline{T_{\sigma_0}}\}.
\end{align*}
\]

Recursively, if \( \tau_k < \infty \),

\[
\sigma_{k+1} = \inf\left\{ t > \tau_k : \Phi_t(w) = \frac{1}{2} \Phi_{\tau_k}(w) \text{ or } \Phi_t(z) = \frac{1}{2} \Phi_{\tau_k}(z) \right\},
\]

and if \( \sigma_{k+1} < \infty \),

\[
\tau_{k+1} = \inf\{t \geq \sigma_{k+1} : \gamma(t) \in \overline{T_{\sigma_{k+1}}}\}.
\]

If one of the stopping times takes on the value infinity, then all the subsequent ones are set equal to infinity. If \( \sigma_{k+1} < \infty \), we set \( R_k = z \) if \( \Phi_{\sigma_{k+1}}(z) = \Phi_{\tau_k}(z)/2 \). Note that in this case,

\[
\Delta_{\sigma_{k+1}}(z) \leq q \Delta_{\tau_k}(z) \text{ where } q = 2^{-1/4} < 1,
\]

and \( \Phi_t(w) > \Phi_{\tau_k}(w)/2 \) for all \( t \leq \tau_{k+1} \). Likewise, we set \( R_k = w \) if \( \Phi_{\sigma_{k+1}}(w) = \Phi_{\tau_k}(w)/2 \).

It follows immediately from (66), that for \( r \leq 1/2 \),

\[
\mathbb{P}\{\Phi_{\tau_0}(z) \leq r \Phi_0(z)\} \leq c r,
\]

and for \( r \) sufficiently small

\[
\mathbb{P}\{\Phi_{\tau_0}(w) \leq r \Phi_0(w)\} = 0.
\]

The key estimate, which we now establish, is the following.

• There exists \( c, \alpha \) such that if \( \tau_k < \infty, 0 < r \leq 1/2 \) and \( \zeta = x + iy \in \{z, w\} \), then

\[
\mathbb{P}\{\tau_{k+1} < \infty, R_k = \zeta, \Phi_{\tau_{k+1}}(\zeta) \leq r \Phi_{\tau_k}(\zeta) \mid \gamma_{\tau_k}\} \leq c r \Phi_{\tau_k}(\zeta)^\alpha.
\]

Let \( H = H_{\tau_k}, I = I_{\tau_k}, \hat{g} = g_{\tau_k} - U_{\tau_k}, \hat{\zeta} = \hat{g}(\zeta), \Delta = \Delta_{\tau_k}(\zeta), \Phi = \Phi_{\tau_k}(\zeta), \lambda = |g'(\zeta)|. \)

Recall that \( \Delta^{4a-1} \leq \Phi. \) If \( \Phi_t(\zeta) = r\Phi \) then \( |\zeta - \gamma(t)| = \theta \Delta \) where

\[
\theta = \left[ \frac{y \wedge \Delta \vee r}{\Delta} \right]^{\frac{4a-1}{4a}} \left[ \frac{r \Delta}{y \wedge \Delta \wedge 1} \right]^{\frac{1}{2a}}.
\]

Note that if \( r \leq 1/2 \) then \( \theta \leq 2^{-\frac{1}{4a-1}} < 1. \)

Let \( V \) denote the closed disk of radius \( 2^{-\frac{1}{4a-1}} \Delta \) about \( \zeta, y_* = y \vee (\theta \Delta/2) \) and \( \zeta_* = x + y_* i \in V \). Note that \( g \) is a conformal transformation defined on the open disk of radius \( \Delta \) about \( \zeta \) (if \( y < \Delta \), then we extend \( g \) by Schwarz reflection). Hence by the distortion theorem, there exist \( 0 < c_1 < c_2 < \infty \) such that if \( \zeta_1 \in V \),

\[
c_1 \lambda \leq |\hat{g}'(\zeta_1)| \leq c_2 \lambda,
\]

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In particular, 
\[ c_1 \lambda y \leq \text{Im} \hat{\zeta} \leq c_2 \lambda y. \]

Note that \( \hat{I} \) is a crosscut of \( \mathbb{H} \) with one endpoint equal to zero. We consider separately the cases where \( \hat{\zeta} \) is in the bounded or unbounded component of \( \mathbb{H} \setminus \hat{I} \).

Let \( E_1 \) denote the event that \( \hat{\zeta} \) is in the bounded component. We claim that there exists \( c < \infty \), such that for all \( \hat{\zeta}' = \hat{g}(\zeta') \in \hat{g}(V) \),

\[ S(\hat{\zeta}') = \frac{\text{Im}(\hat{\zeta}')}{|\hat{\zeta}'|} \leq c \Delta^{1/2} \]

(71)

To see this, assume for ease that \( \text{Re}[\hat{\zeta}'] \geq 0 \) and let \( \Theta = \arg \hat{\zeta}' \). Then \( \text{Im}(\hat{\zeta}')/|\hat{\zeta}'| = \sin \Theta \leq \Theta \) and \( \Theta/\pi \) is the probability that a Brownian motion starting at \( \hat{\zeta}' \) hits \( (-\infty, 0] \) before leaving \( \mathbb{H} \). This is bounded above by the probability that a Brownian motion starting at \( \hat{\zeta}' \) hits \( \hat{I} \) before leaving \( \mathbb{H} \). By conformal invariance, this last probability is the same as the probability that a Brownian motion starting at \( \zeta' \) hits \( I \) before leaving \( H \). The Beurling estimate implies that this is bounded above by \( c \Delta^{1/2} \). This gives (71). Therefore, there exists \( c \) such that if \( |\zeta - \gamma(t)| = \theta \Delta \), then

\[ \Phi(\hat{g}(\gamma(t))) \leq c \Delta^{(4a-1)/2} \frac{r}{|\gamma(t)|} \leq c \sqrt{\Phi} r |\gamma(t)|. \]

Using (66), we see that

\[ \mathbb{P}\{ \Phi(\zeta) \leq r \Phi_{\tau_k}(\zeta), E_1 \mid \gamma_{\tau_k} \} \leq c \sqrt{\Phi} r. \]

We now suppose that \( \hat{\zeta} \) is in the unbounded component. By the same argument, for every \( \hat{\zeta}' := \hat{g}(\zeta') \in \hat{g}(V) \), the probability that a Brownian motion starting at \( \hat{\zeta}' := \hat{g}(\zeta') \) hits \( \hat{I} \) before leaving \( \mathbb{H} \) is bounded above by \( c \Delta^{1/2} \). We will split into two subcases. We first assume that

\[ \text{Im}(\hat{\zeta}') \leq \Delta^{1/4} |\hat{\zeta}'|, \quad \zeta' \in V. \]

In this case, we an argue as in the previous paragraph to see that the probability \( SLE_\kappa \) in \( \mathbb{H} \) hits \( \hat{g}(V) \) is bounded above by \( c \Phi^{1/4} \). For the other case we assume that \( \text{Im}(\hat{\zeta}') \geq \Delta^{1/4} |\hat{\zeta}'| \) for some \( \hat{\zeta}' \in \hat{g}(V) \). Using the Poisson kernel in \( \mathbb{H} \), we can see that the probability that a Brownian motion starting at \( \hat{\zeta}' \) hits \( \hat{I} \) before leaving \( \mathbb{H} \) is bounded below by a constant times

\[ \frac{\text{diam}(\hat{I})}{\Delta^{1/4} |\hat{\zeta}'|} \]

From this we conclude that

\[ \text{diam}(\hat{I}) \leq c \Delta^{1/4} |\hat{\zeta}'|. \]

We appeal to Proposition 4.3 to say that the probability that \( SLE_\kappa \) in \( \mathbb{H} \) hits \( \hat{g}(V) \) and then returns to \( \hat{I} \) is bounded above by a constant times

\[ r\left[ \text{diam}(\hat{I})/|\hat{\zeta}'| \right]^{(4a-1)/2} \leq c \sqrt{\Phi} r^{1/8}. \]
Given (70), the remainder of the proof proceeds in the same way as [10] Section 4.4] so we omit this.

\[\]

**Proposition 4.5.** There exist \(0 < c_1 < c_2 < \infty\) such that if \(|z| \leq |w|/4\),
\[
c_1 G(z) G(w) \leq G(z, w) \leq c_2 G(z) G(w).
\]

**Proof.** The bound \(G(z, w) \geq c G(z) G(w)\) was proved in [11] so we need only show the other inequality. Proposition 4.4 implies that for \(\epsilon\) sufficiently small
\[
\mathbb{P}\{\Delta(z) \leq \epsilon, \Delta(w) \leq \epsilon\} \leq c \mathbb{P}\{\Delta(z) \leq \epsilon\} \mathbb{P}\{\Delta(w) \leq \epsilon\}.
\]
Hence (64) and (65) imply that \(G(z, w) \leq c G(z) G(w)\).

The next estimate will be important even though it is not a very sharp bound for large \(|z|, |w|\).

**Proposition 4.6.** For every \(\epsilon > 0\), there exists \(c < \infty\) such that if \(|z|, |w| \geq \epsilon\) and \(|z-w| \geq \epsilon\), then
\[
G(z, w) \leq c \text{Im}(z)^{4a-1} \text{Im}(w)^{4a-1}.
\]

**Proof.** By scaling it suffices to prove the result when \(\epsilon = 1\). This can be done as the proof of the previous proposition, so we omit the details. The key step is to choose an appropriate splitting curve \(I_0\). We can choose \(I_0\) either to be a half-circle with endpoints on \(\mathbb{R}\) or a vertical line. We choose \(I_0\) so that \(I_0\) separates \(z\) and \(w\) and dist\((z, I_0), \text{dist}(w, I_0) \geq 1/4\).

We will now prove Theorem 2.11. By scaling, we may assume that \(|w| = 1\) and hence \(q = |w-z|\). If \(q \geq 1/10\), the conclusion is
\[
G(z, w) \asymp G(z) G(w).
\]
The bound \(G(z, w) \geq c G(z) G(w)\) was done in [11]. The other inequality can be deduced from Propositions 4.5 and 4.6 respectively, for \(|z| \leq 1/4\) and \(|z| \geq 1/4\). Here we use the fact that \(G(z) \geq \text{Im}(z)^{4a-1}\) for \(|z| \leq 1\).

For the remainder of the proof we assume \(q \leq 1/10\), and hence \(9/10 \leq |z| \leq 1\). Let \(z = x_z + i y_z, w = x_w + i y_w, \text{ and } \zeta = x_i + i(y_w \vee q).\) Note that \(G(w) \asymp y_w^{4a-1}, G(z) \asymp y_z^{4a-1}\). Let \(\sigma = \inf\{t : |\gamma(t) - w| = 2q\}\), and on the event \(\{\sigma < \infty\}\), let \(h = \lambda [g_{\sigma} - U_{\sigma}]\) where the constant \(\lambda\) is chosen so that \(\text{Im}[h(\zeta)] = 1\). We write
\[
h(\zeta) = \hat{\zeta} + i, \quad h(z) = \hat{z} = \hat{x} + i \hat{y}_z, \quad h(w) = \hat{w} = \hat{x}_w + i \hat{y}_w.
\]
Then
\[
G(z, w) = \mathbb{E} \left[ |g_{\sigma}(z)|^{2-d} |g_{\sigma}(w)|^{2-d} G(Z_{\sigma}(z), Z_{\sigma}(w)); \sigma < \infty \right] \\
= \mathbb{E} \left[ |g_{\sigma}(z)|^{2-d} |g_{\sigma}(w)|^{2-d} \lambda^{2(2-d)} G(\lambda Z_{\sigma}(z), \lambda Z_{\sigma}(w)); \sigma < \infty \right] \\
= \mathbb{E} \left[ |h'(z)|^{2-d} |h'(w)|^{2-d} G(\hat{z}, \hat{w}); \sigma < \infty \right].
\]
The Koebe (1/4)-theorem implies that $|h'(\zeta)| \asymp q^{-1}$. Distortion estimates (using Schwarz reflection if $y_w \leq 2q$) imply that

$$|h'(z)| \asymp |h'(w)| \asymp |h'(\zeta)| \asymp q^{-1},$$

$$|\hat{z} - \hat{w}| \asymp 1,$$

$$|\hat{z}|, |\hat{w}| \geq c,$$

$$\hat{y}_z \asymp (y_z \wedge q) q^{-1}, \quad \hat{y}_w \asymp (y_w \wedge q) q^{-1}.$$

These estimates hold regardless of the value of $S(\hat{\zeta})$. If we also know that $S(\hat{\zeta}) \geq 1/10$, then

$$|\hat{\zeta}| \asymp |\hat{z}| \asymp |\hat{w}| \asymp 1.$$

Hence, by Proposition 4.6 we see that

$$G(\hat{z}, \hat{w}) \leq c \left[ \frac{(y_z \wedge q) (y_w \wedge q)}{q^2} \right]^{4a-1},$$

$$G(\hat{z}, \hat{w}) \geq c' \left[ \frac{(y_z \wedge q) (y_w \wedge q)}{q^2} \right]^{4a-1}, \quad \text{if } S(\hat{\zeta}) \geq 1/10.$$

Lemma 4.1 implies that

$$\mathbb{P}\{\sigma < \infty\} \asymp \mathbb{P}\{\sigma < \infty, S(\hat{\zeta}) \geq 1/10\} \asymp \begin{cases} y_w^{4a-1} (q/y_w)^{2-d}, & y_w \geq q \\ q^{4a-1}, & y_w \leq q. \end{cases}$$

Therefore,

$$G(z, w) \asymp y_w^{4a-1} (q/y_w)^{2-d} q^{2(d-2)} \left[ \frac{(y_z \wedge q) q}{q^2} \right]^{4a-1}, \quad y_w \geq q,$$

$$G(z, w) \asymp q^{4a-1} q^{2(d-2)} \left[ \frac{(y_z \wedge q) y_w}{q^2} \right]^{4a-1}, \quad y_w \leq q.$$

If $q \leq y_w \leq 2q$ we can use either expression. If $y_w \leq 2q$, then $y_w \wedge q \asymp y_w, y_z \wedge q \asymp y_z, S(w) \vee q \asymp q$ and we can write

$$G(z, w) \asymp q^{2(d-2)} q^{1-4a} y_z^{4a-1} y_w^{4a-1} \asymp q^{d-2} [S(w) \vee q]^{-\beta} G(z) G(w).$$

If $y_w \geq 2q$, then $y_z \asymp y_w, y_z \wedge q \asymp q, S(w) \vee q \asymp y_w$, and we can write

$$G(z, w) \asymp y_w^{4a-1} q^{d-2} y_w^{2(d-2)} \asymp q^{d-2} y_w^{-\beta} y_w^{2(4a-1)} \asymp q^{d-2} [S(w) \vee q]^{-\beta} G(z) G(w).$$
5 Proof of Theorem 2.3

By scaling and translation invariance, we may assume that $z = 0$ and $\text{dist}(z, \partial D) = 1$. We first consider the case $D = D$, $w_1 = 1$, and $w_2 = w = e^{2\theta i}$. Note that

$$G_D(0; 1, e^{2\theta i}) = S_D(0; 1, e^{2\theta i})^{4a-1} = \sin^{4a-1}\theta.$$ 

Let $D_t = e^{-t}D$. Let $\gamma$ be a chordal $\text{SLE}_\kappa$ path from $0$ to $w = e^{2\theta i}$ in $D$ and

$$q(t, \theta) = [\sin \theta]^{1-4a} \mathbb{P}\{\text{dist}(0, \gamma) \leq e^{-t}\}.$$ 

We will use the radial parametrization normalized so that at time $t$, $|g_t'(0)| = e^t$. Here $g_t$ denotes the conformal transformation of (the connected component containing the origin) of $D \setminus \gamma_t$ with $g_t(0) = 0, g_t(\gamma(t)) = 1$. We define $\theta_t$ by $g_t(w) = e^{2\theta_t i}$. Under this parametrization, the local martingale is

$$M_t = e^{(2-d)t} [\sin \theta_t]^{4a-1}.$$ 

This parameterization ends at the time $T$ which is the first time that $w$ is disconnected from $0$ by the curve $\gamma_T$; if $d \leq 4$, then $T$ is the time at which $\gamma(T) = w$. Note that $\text{dist}(0, \gamma) = \text{dist}(0, \gamma_T)$.

We will also consider two-sided radial $\text{SLE}$ which is the measure obtained by tilting by the local martingale $M_t$. We recall some facts (see [10, Lemmas 2.8 and 2.9]). The invariant probability density is $f(\theta) = c \sin^{4a} \theta$. Moreover, there exists $\alpha > 0$ such that if $f_t(\theta) = f_{t, \theta}(\theta')$ denotes the density at time $t$, then for $t \geq 1$,

$$f_t(\theta') = f(\theta') [1 + O(e^{-\alpha t})],$$ 

where the error term is bounded uniformly over the starting angle $\theta$. Here $\alpha > 0$ is a constant that could be determined, but we will not need its exact value. Let

$$q(t) = \int_0^{\pi} q(t, \theta) f(\theta) d\theta.$$ 

Proposition 5.1. There exists $c < \infty$ such that for all $\theta$ and all $t \geq 1, s \geq 2$,

$$q(s + ce^{-s}) [1 - ce^{-\alpha t}] \leq e^{t(2-d)} q(t + s, \theta) \leq q(s - ce^{-s}) [1 + ce^{-\alpha t}].$$

In particular,

$$q(s + ce^{-s}) [1 - ce^{-\alpha t}] \leq e^{t(2-d)} q(t + s) \leq q(s - ce^{-s}) [1 + ce^{-\alpha t}].$$

Proof. The Koebe $(1/4)$-theorem implies that the domain of $g_t$ includes $\mathbb{D}_{t+\log 4}$. Using the distortion theorem, we see that if $z \in \mathbb{D}_{t+2}$, then

$$|g_t(z)| = e^t |z|[1 + O(e^t|z|)].$$

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In particular, if $|z| = e^{-(s+t)}$, then

$$|g_t(z)| = e^{-s} [1 + O(e^{-s})] = \exp \{-s + O(e^{-s})\}.$$  

Therefore, there exists $c_1$ such that on the event $\{T > t\}$,

$$\mathbb{D}_{s+c_1e^{-s}} \subset g_t(\mathbb{D}_{s+t}) \subset \mathbb{D}_{s-c_1e^{-s}}.$$  

(73)

Let $\xi = \xi_{s+t} = \inf\{r : |\gamma(r)| = e^{-(t+s)}\}$. Let

$$Y(t, s) = \mathbb{P}\{\xi < T \mid \gamma_{t\wedge T}\}.$$  

Since $s \geq 2$, the Koebe $(1/4)$-theorem implies that $Y(t, s) = 0$ if $T \leq t$. The domain Markov property implies that

$$q(t + s, \theta) = \sin^4 \theta \mathbb{E}[Y(t, s)] = \sin^4 \theta \mathbb{E}[Y(t, s); T > t],$$

where here and below $\mathbb{E}$ denotes expectation with respect to chordal $SLE_\kappa$ from 0 to $w$. If $S_t = \sin \theta_t$, we know that

$$M_t = e^{(2-d)t} S_t^{1-4a}$$

is a local martingale with $\mathbb{P}\{M_T = 0\} = 1$. Therefore,

$$\mathbb{E}[Y(t, s); T > t] = e^{(d-2)t} \mathbb{E}[Y(t, s) M_t S_t^{1-4a}; T > t] = e^{(d-2)t} S_t^{1-4a} \mathbb{E}^* [Y(t, s) S_t^{1-4a}].$$

Here $\mathbb{E}^*$ denotes the measure obtained by tilting by $M$ which is the same as the two-sided radial measure. Again the initial $\theta$ is implicit in the notation. Hence,

$$q(t + s, \theta) = e^{(d-2)t} \mathbb{E}^* [Y(t, s) S_t^{1-4a}].$$

By the strong Markov property, $Y(t, s)$ is the probability that a chordal $SLE$ from 0 to $e^{2i\theta t}$ enters $g_t(\mathbb{D}_{t+s})$. Using (73), we see that for $s \geq 2$,

$$q(s + c_1e^{-s}, \theta_t) \leq S_t^{1-4a} Y(t, s) \leq q(s - c_1e^{-s}, \theta_t).$$

From (72) we see that

$$\mathbb{E}^* [Y(t, s) S_t^{1-4a}] \leq q(s - c_3e^{-s}) [1 + O(e^{-\alpha t})],$$

$$\mathbb{E}^* [Y(t, s) S_t^{1-4a}] \geq q(s + c_3e^{-s}) [1 - O(e^{-\alpha t})].$$

This completes the proof. 

The next proposition finishes the proof of Theorem 2.3 in the case $D = \mathbb{D}$ with $u = \alpha/2$.

**Proposition 5.2.** There exist $\hat{c}, c$ such that for all $\theta$ and all $t \geq 2$,

$$|e^{t(2-d)} q(t, \theta) - \hat{c}| \leq c e^{-t\alpha/2}.$$  

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Proof. By choosing \( t' = t \pm c' e^{-s} \) for large \( c' \) in the last proposition, we see that
\[
q(t + s, \theta) = e^{t(d-2)} q(s) \left[ 1 + O(e^{-\alpha t}) + O(e^{-s}) \right].
\]
If \( L(t) = \log[e^{t(2-d)} q(s)] \), then this implies that
\[
|L(t + s) - L(t)| \leq c[e^{-s} + e^{-\alpha t}].
\]
If \( t \geq 2 \) and \( s \geq 0 \), we have
\[
|L(t + s) - L(t)| = |L(t + s) - L(t/2) + L(t/2) - L(t)| \leq ce^{-\alpha t/2}.
\]
This implies that \( \lim_{t \to \infty} L(t) = L_{\infty} \in (-\infty, \infty) \) exists and
\[
L(t) = L_{\infty} + O(e^{-\alpha t/2}).
\]
This gives the result with \( \hat{c} = e^{L_{\infty}} \). \( \square \)

To finish the proof of Theorem 2.3 for general \( D \) with \( z = 0, \text{dist}(z, \partial D) = 1 \), let \( F : \mathbb{D} \to D \) be a conformal transformation with \( F(0) = 0, F(1) = w_1, F(e^{2\theta i}) = w_2 \). The \( \theta \) depends on \( D, w_1, w_2 \), but conformal invariance implies that
\[
S_D(0; w_1, w_2) = \sin \theta,
\]
and hence
\[
G_D(0; w_1, w_2) = F'(0)^{2-d} [\sin \theta]^{4a-1} = F'(0)^{2-d} G_\mathbb{D}(0; 1, e^{2\theta i}).
\]
The distortion theorem implies that there exists \( c < \infty \) such that
\[
[\lambda \epsilon - ce^2] \mathbb{D} \subset F^{-1}(\epsilon \mathbb{D}) \leq [\lambda \epsilon + ce^2] \mathbb{D}, \quad \lambda = \frac{1}{F'(0)} \in [1/4, 1].
\]
Therefore, by conformal invariance,
\[
P\{\text{dist}(\gamma, 0) \leq \epsilon\} = \hat{c} [\sin \theta]^{4a-1} (\lambda \epsilon)^{2-d} [1 + O(\epsilon^u)] = \hat{c} G_D(0; w_1, w_2) \epsilon^{2-d} [1 + O(\epsilon^u)].
\]

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