Short-time behavior of the heat kernel and Weyl’s law on $\mathrm{RCD}^*(K, N)$ spaces

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Received: 22 January 2017 / Accepted: 20 July 2017 / Published online: 15 August 2017 © Springer Science+Business Media B.V. 2017

Abstract In this paper, we prove pointwise convergence of heat kernels for mGH-convergent sequences of $\mathrm{RCD}^*(K, N)$-spaces. We obtain as a corollary results on the short-time behavior of the heat kernel in $\mathrm{RCD}^*(K, N)$-spaces. We use then these results to initiate the study of Weyl’s law in the RCD setting.

Keywords Weyl’s law · Spectral analysis · RCD spaces

1 Introduction

More than a century ago, Weyl gave in [55] a nice description of the asymptotic behavior of the eigenvalues of the Laplacian on bounded domains of $\mathbb{R}^n$ for $n = 2, 3$ (his result was later on extended for any integer $n \geq 2$). More precisely, if $\Omega \subset \mathbb{R}^n$ is a bounded domain, it is well known that the spectrum of (minus) the Dirichlet Laplacian on $\Omega$ is a sequence of positive numbers $(\lambda_i)_{i \in \mathbb{N}}$ such that $\lambda_i \to +\infty$ as $i \to +\infty$. Weyl proved that

$$\lim_{\lambda \to +\infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{L}^n(\Omega)$$

where $N(\lambda) = \sharp \{ i \in \mathbb{N} : \lambda_i \leq \lambda \}$ (the eigenvalues being counted with multiplicity), $\omega_n$ is the volume of the $n$-dimensional Euclidean unit ball, and $\mathcal{L}^n(\Omega)$ is the $n$-dimensional Lebesgue measure of $\Omega$. This result is known as Weyl’s law. It has been widely used to tackle
some physical problems, and several refinements were found after Weyl’s first article. For a complete overview of the history of Weyl’s law and its refinements, we refer to [9].

Among the possible generalizations of Weyl’s law, one can replace the bounded domain $\Omega \subset \mathbb{R}^n$ by a $n$-dimensional compact closed manifold. The Laplacian is then replaced by the Laplace–Beltrami operator of the manifold, and the term $\mathcal{L}^n(\Omega)$ is replaced by $\mathcal{H}^n(M)$, where $\mathcal{H}^n$ denotes the $n$-dimensional Hausdorff measure. It has been proved by Levitan in [40] that Weyl’s law is still true in that case.

Another generalization concerns compact Riemannian manifolds $(M, g)$ equipped with the distance $d$ induced by the metric $g$ and a measure with positive smooth density $e^{-f}$ with respect to the volume measure $\mathcal{H}^n$. For such spaces $(M, d, e^{-f} \mathcal{H}^n)$, called weighted Riemannian manifolds, one has

$$\lim_{\lambda \to +\infty} \frac{N_{(M, d, e^{-f} \mathcal{H}^n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(M),$$

(1.1)

holds, where $N_{(M, d, e^{-f} \mathcal{H}^n)}(\lambda)$ denotes the counting function of the (weighted) Laplacian $\Delta^f := \Delta + \langle \nabla f, \nabla \cdot \rangle$ of $(M, d, e^{-f} \mathcal{H}^n)$. This result is a consequence of [29]. We stress that in the asymptotic behavior (1.1) the information of the weight, $e^{-f}$, disappears (as we obtain by different means in Example 4.9). This sounds surprising: the Hausdorff dimension is a purely metric notion, whereas the Laplace–Beltrami operator on weighted Riemannian manifolds and more generally on RCD$^*(K, N)$-spaces does depend on the reference measure.

In this paper, we focus on infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded from below, the so-called RCD-spaces. The curvature-dimension condition CD$(K, N)$ was independently formulated in terms of optimal transport by Sturm in [52, 53] and Lott–Villani in [41]. The CD condition extends to a non-smooth setting the Riemannian notion of Ricci curvature bounded below. Indeed, for given $K \in \mathbb{R}$ and $N \in [1, +\infty]$, a Riemannian manifold satisfies the CD$(K, N)$ condition if and only if it has Ricci curvature bounded below by $K$ and dimension bounded above by $N$. The CD condition is also stable under Gromov–Hausdorff convergence: any metric measure space obtained as a measured Gromov–Hausdorff limit of a sequence of Riemannian manifolds with Ricci curvature bounded below by $K$ and dimension bounded above by $N$ satisfies the CD$(K, N)$ condition. Such limit spaces are called Ricci limit spaces in the sequel.

In more recent times, two main requirements were added to that theory, namely the CD$^*$ condition introduced in [10] and the infinitesimal Hilbertianity introduced in [3], giving rise to the study of the so-called RCD (resp. RCD$^*$) spaces which are by definition infinitesimally Hilbertian spaces satisfying the CD (resp. CD$^*$) condition. All these notions are stable under Gromov–Hausdorff convergence. See also the papers [4, 6, 21], where the RCD$(K, \infty)$/RCD$^*(K, N)$ theories have been proved to be essentially equivalent to the Bakry–Émery theory. The latter, based on diffusion operators and Bochner’s inequality, for weighted Riemannian manifolds $(M, d, e^{-f} \mathcal{H}^n)$ reads as follows:

$$\text{Ric}_M + \text{Hess}_f - \frac{\nabla f \otimes \nabla f}{N - n} \geq Kg_M.$$

(1.2)

Let $(X, d, m)$ be a compact RCD$^*(K, N)$-space. The main result of this paper is a sharp criterion (see (1.3) below) for the validity of Weyl’s law on $(X, d, m)$. The authors do not know whether there exist RCD$^*(K, N)$-spaces which do not satisfy this criterion, since all known examples satisfy it.

As observed in (1.1) it is expected that the asymptotic behavior of the counting function $N_{(X, d, m)}(\lambda)$ is not related to the reference measure $m$, but rather to the Hausdorff measure $\mathcal{H}^\ell$, where $\ell$ is the Hausdorff dimension of $(X, d)$. 

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In order to introduce the precise statement of our criterion let us recall Mondino–Naber’s result [43]:

$$\mathfrak{m}\left( X \setminus \bigcup_{i=1}^{N} \mathcal{R}_i \right) = 0,$$

where $[N]$ is the integer part of $N$ and $\mathcal{R}_i$ is the $i$-dimensional regular set of $(X, \mathfrak{d}, \mathfrak{m})$. Recall that the $i$-dimensional regular set of $(X, \mathfrak{d})$ (Definition 3.4) is the set of points of $X$ admitting a unique tangent cone isometric to $(\mathbb{R}^i, d_{\text{eucl}}, c_i \mathcal{H}^i)$. More recently, building on [18], more than one group of authors have shown that $\mathfrak{m}$-almost all of $\mathcal{R}_i$ can be covered by bi-Lipschitz charts defined in subsets of $\mathbb{R}^i$, and that the restriction of the reference measure to each $\mathcal{R}_i$ is absolutely continuous w.r.t. $\mathcal{H}^i$ (Proposition 3.7 and [17, 27, 35]).

Let us define the maximal regular dimension $\dim_{d, \mathfrak{m}}(X)$ of $(X, \mathfrak{d}, \mathfrak{m})$ as the largest integer $k$ such that $\mathfrak{m}(\mathcal{R}_k) > 0$. We are now in a position to introduce our main result, Theorem 4.3: with $k = \dim_{d, \mathfrak{m}}(X)$, one has

$$\lim_{r \to 0^+} \int_X \frac{r^k}{\mathfrak{m}(\mathcal{B}_r(x))} \, d\mathfrak{m} = \int_X \lim_{r \to 0^+} \frac{r^k}{\mathfrak{m}(\mathcal{B}_r(x))} \, d\mathfrak{m} < \infty \quad (1.3)$$

if and only if

$$\lim_{\lambda \to +\infty} \frac{N_{(\mathfrak{d}, \mathfrak{m}, \lambda)}(\lambda)}{\lambda^{k/2}} = \frac{\omega_k}{(2\pi)^k} \mathcal{H}^k(\mathcal{R}_k^+) < \infty, \quad (1.4)$$

where $\mathcal{R}_k^+ \subset \mathcal{R}_k$ denotes a suitable reduced regular set (defined in Theorem 4.1) such that $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}_k^+) = 0$ and $\mathfrak{m}\mathcal{R}_k^+$ and $\mathcal{H}^k \mathcal{R}_k^+$ are mutually absolutely continuous (in particular $\mathcal{H}^k(\mathcal{R}_k^+) > 0$).

Note that with this criterion, the asymptotic behavior of $N_{(\mathfrak{d}, \mathfrak{m}, \lambda)}(\lambda)$ (including the growth order) is determined by the sole top-dimensional reduced regular set. As typical examples, thanks to the dominated convergence theorem, the criterion is automatically satisfied when $k = N$, or when the metric measure structure is Ahlfors regular. As a consequence, we obtain a new result, namely Weyl’s law for finite dimensional compact Alexandrov spaces (Corollaries 4.4 and 4.8). We can also provide examples (see (1.5) below) such that $k < N$ and Ahlfors regularity fails.

On the other hand, it is worth pointing out that from the viewpoint of RCD-theory the least number $N_{\text{min}}$ such that $(X, \mathfrak{d}, \mathfrak{m}) \in \text{RCD}^+(K', N_{\text{min}})$ for some $K' \in \mathbb{R}$ might be naturally regarded as another dimension of $(X, \mathfrak{d}, \mathfrak{m})$ (indeed, $N_{\text{min}} = n$ for weighted Riemannian manifolds $(M, \mathfrak{d}, e^{-f} \mathcal{H}^n)$). However, in general $N_{\text{min}}$ is not equal to the Hausdorff dimension of $(X, \mathfrak{d})$ and need not be related to the asymptotic behavior of $N_{(\mathfrak{d}, \mathfrak{m}, \lambda)}(\lambda)$, as the following example shows:

**Example 1.1** For $N \in (1, +\infty)$, let us consider the metric measure space

$$(X, \mathfrak{d}, \mathfrak{m}) := \left( [0, \pi], d_{[0,\pi]}, \sin^{N-1} t \, dt \right). \quad (1.5)$$

It is known that $(X, \mathfrak{d}, \mathfrak{m})$ is a RCD$^+(N - 1, N)$-space (see for instance [12]). Moreover, since $\mathfrak{m}(\mathcal{B}_r(x)) \sim r$ for $x \in (0, \pi)$ and $\mathfrak{m}(\mathcal{B}_r(x)) \sim r^\frac{N}{3}$ for $x \in [0, \pi]$ as $r \to 0^+$, for this metric measure structure one has $N_{\text{min}} = N$, because Bishop–Gromov inequality for RCD$^+(\hat{K}, \hat{N})$-spaces implies a positive lower bound on $\mathfrak{m}(\mathcal{B}_r(x))/r^\frac{N}{3} > 0$.

It turns out that our criterion can be applied to $(X, \mathfrak{d}, \mathfrak{m})$, as (1.3) holds by the dominated convergence theorem. Thus

$$\lim_{\lambda \to +\infty} \frac{N_{(\mathfrak{d}, \mathfrak{m}, \lambda)}(\lambda)}{\lambda^{1/2}} = \frac{\omega_1}{2\pi} \mathcal{H}^1((0, \pi)) = 1 \quad (1.6)$$
and the number $N$ does not appear in (1.6). Note that (1.6) is also new and that the same asymptotic behavior in (1.6) for $N = 1$ (i.e., the metric measure space is $([0, \pi], d_{[0,\pi]}, \sqrt{\pi})$) is well known as Weyl’s law on $[0, \pi]$ associated with homogeneous Neumann boundary conditions. See Example 4.5 for more details.

Let us conclude this part of the introduction, following the reviewer’s advice, by illustrating our strategy of proof. If $v = \sum_\lambda \delta_\lambda$ is the counting measure induced by the eigenvalues (counted with multiplicity), then the function $N(\lambda)$ can be interpreted as $v([0, \lambda])$; then, Karamata’s Tauberian theorem provides the equivalence

$$\lim_{s \to 0^+} s^{k/2} \hat{v}(s) = a \iff \lim_{\lambda \to +\infty} v([0, \lambda]) = a \frac{\omega_k}{\pi^{k/2}}.$$  

In turn, by the trace formula, it is easily seen that $\hat{v}(s) = \sum_\lambda e^{-\lambda s}$ can be represented by $\int p(x, x, s) \, dm(x)$, therefore one is led to study the asymptotic behavior of $s^{k/2} p(x, x, s)$ as $s \to 0$. Now, at $k$-regular points $x$, the pointwise convergence of heat kernels for Gromov–Hausdorff converging sequences of RCD$^*$ $(K, N)$ spaces (Theorem 3.3, which is a generalization of Ding’s Riemannian results [19]) provides the convergence

$$\lim_{s \to 0^+} m(B_{\sqrt{s}}(x)) p(x, x, s) = \frac{\omega_k}{(4\pi)^{k/2}}. \quad (1.7)$$

In this connection, a main advantage of our approach is the use of regularity theory of heat flows that avoids some technical difficulties of spectral theory. It remains to relate the power rate $s^{k/2}$ with the rate $m(B_{\sqrt{s}}(x))$ in (1.7); for $k = \dim_{d,m} X$, we are able to prove in Theorem 4.1 that the limit

$$\lim_{r \to 0^+} \frac{\omega_k r^k}{m(B_r(x))}$$

exists and coincides with $\chi_{R^*_1(x)}(x) \, d\mathcal{H}^k / dm(x)$ for m-a.e. $x \in X$. Finally, by integration w.r.t. $x$ of these convergence results, the equivalence between (1.3) and (1.4) follows.

The paper is organized as follows. In Sect. 2 we recall the notions on RCD$^*$ $(K, N)$ spaces that shall be used in the sequel and give some useful lemmas. Section 3 begins with the treatment of weak/strong $L^2$-convergence for sequences of functions defined on converging RCD$^*$ $(K, N)$ spaces. Then, using the Gaussian estimates (3.2) of the heat kernel on RCD$^*$ $(K, N)$-spaces established in [33], we prove the pointwise convergence of heat kernels defined on a converging sequence of RCD$^*$ $(K, N)$ spaces, and we deduce from this fact the short-time estimate of the heat kernel on regular sets of RCD$^*$ $(K, N)$ spaces. Section 4 is devoted to the study of Weyl’s law, first in a weak form (Theorem 4.2) and then strengthening the assumptions in a stronger and more classical form (Theorem 4.3). The rest of this section is dedicated to examples and applications (especially to compact Alexandrov spaces).

After completing our paper, we learned of an independent work by Zhang and Zhu [57] on Weyl’s law in the setting of RCD$^*$ $(K, N)$ spaces. The paper is based on a local analysis, along the same lines of [19], and provides sufficient conditions for the validity of Weyl’s law, different from our sharp criterion of Theorem 4.3.

2 Notation and preliminaries about RCD$^*$ $(K, N)$ spaces

Let us recall basic facts about Sobolev spaces and heat flow in metric measure spaces $(X, d, m)$, see [2] and [25] for a more systematic treatment of this topic. The so-called
Cheeger energy \( \text{Ch} : L^2(X, m) \to [0, +\infty] \) is the convex and \( L^2(X, m) \)-lower semicontinuous functional defined as follows:

\[
\text{Ch}(f) := \inf \left\{ \lim_{n \to \infty} \frac{1}{2} \int_X \text{Lip}_a^2 (f_n) \, dm : f_n \in \text{Lip}_b(X) \cap L^2(X, m), \| f_n - f \|_2 \to 0 \right\}.
\]

The original definition in \cite{[13]} involves generalized upper gradients of \( f_n \) in place of their asymptotic Lipschitz constant

\[
\text{Lip}_a(f) := \lim_{r \to 0^+} \text{Lip} (f, B_r(x))
\]

but many other pseudo gradients (upper gradients, or the slope \( \text{lip} (f) \leq \text{Lip}_a(f) \), which is a particular upper gradient) can be used and all of them lead to the same definition, see \cite{[1]} and the discussion in \cite{[2], Remark 5.12}).

The Sobolev space \( H^{1,2}(X, d, m) \) is simply defined as the finiteness domain of \( \text{Ch} \). When endowed with the norm

\[
\| f \|_{H^{1,2}} := \left( \| f \|_{L^2(X, m)}^2 + 2 \text{Ch}(f) \right)^{1/2}
\]

this space is Banach, and reflexive if \((X, d)\) is doubling (see \cite{[1]}). The Sobolev space is Hilbert if \( \text{Ch} \) is a quadratic form. We say that a metric measure space \((X, d, m)\) is infinestimally Hilbertian if \( \text{Ch} \) is a quadratic form.

By looking at minimal relaxed slopes and by a polarization procedure, one can then define a \textit{Carré du champ}

\[
\Gamma : H^{1,2}(X, d, m) \times H^{1,2}(X, d, m) \to L^1(X, m)
\]

playing in this abstract theory the role of the scalar product between gradients. In infinestimally Hilbertian metric measure spaces the \( \Gamma \) operator satisfies all natural symmetry, bilinearity, locality and chain rule properties, and provides integral representation to \( \text{Ch} \):

\[
2 \text{Ch}(f) = \int_X \Gamma (f, f) \, dm \quad \text{for all } f \in H^{1,2}(X, d, m).
\]

We can also define a densely defined operator \( \Delta : D(\Delta) \to L^2(X, m) \) by

\[
f \in D(\Delta) \iff \exists h := \Delta f \in L^2(X, m) \text{ s.t. } \int_X hg \, dm = -\int_X \Gamma (f, g) \, dm \quad \forall g \in H^{1,2}(X, d, m).
\]

Another object canonically associated to \( \text{Ch} \) and then to the metric measure structure is the heat flow \( h_t \), defined as the \( L^2(X, m) \) gradient flow of \( \text{Ch} \); even in general metric measure structures one can use the Brezis–Komura theory of gradient flows of lower semicontinuous functionals in Hilbert spaces to provide existence and uniqueness of this gradient flow. In the special case of infinestimally Hilbertian metric measure spaces, this provides a linear, continuous and self-adjoint contraction semigroup \( h_t \) in \( L^2(X, m) \) with the Markov property, characterized by: \( t \mapsto h_t f \) is locally absolutely continuous in \((0, +\infty)\) with values in \( L^2(X, m) \) and

\[
\frac{d}{dt} h_t f = \Delta h_t f \quad \text{for } L^1\text{-a.e. } t \in (0, +\infty)
\]

for all \( f \in L^2(X, m) \). Thanks to the Markov property, this semigroup has a unique \( L^p \) continuous extension from \( L^2 \cap L^p \) to \( L^p \), \( 1 \leq p < +\infty \), and by duality one defines also the \( w^*\)-continuous extension to \( L^\infty(X, m) \).

In order to introduce the class of RCD\((K, \infty)\) and RCD\(*(K, N)\) metric measure spaces we follow the \( \Gamma \)-calculus point of view, based on Bochner’s inequality, because this is the
point of view more relevant in the proof of heat kernel estimates, Li–Yau inequalities, etc. The equivalence with the Lagrangian point of view, based on the theory of optimal transport, is discussed in [4] (in the case \( N = \infty \)) and in [6, 21] (in the case \( N < \infty \)). The latter point of view does not play a role in this paper, but it plays indeed a key role in the proof of the results we need, mainly taken from [26] and [43].

**Definition 2.1 (RCD spaces)** Let \((X, d, m)\) be a metric measure space, with \((X, d)\) complete, satisfying

\[
m(B_r(x)) \leq c_1 e^{c_2 r^2} \quad \forall r > 0
\]  

(2.2)

for some \(c_1, c_2 > 0\) and \(x \in X\) and the so-called Sobolev to Lipschitz property: any \(f \in H^{1,2}(X, d, m) \cap L^\infty(X, m)\) with \(\Gamma(f) \leq 1\) m.a.e. in \(X\) has a representative in \(\tilde{f} \in \text{Lip}_b(X)\), with \(\text{Lip}(\tilde{f}) \leq 1\).

For \(K \in \mathbb{R}\), we say that \((X, d, m)\) is a \(\text{RCD}(K, \infty)\) metric measure space if, for all \(f \in H^{1,2}(X, d, m) \cap D(\Delta)\) with \(\Delta f \in H^{1,2}(X, d, m)\), Bochner’s inequality

\[
\frac{1}{2} \Delta \Gamma(f) \geq \Gamma(f, \Delta f) + K \Gamma(f)
\]

holds in the weak form

\[
\frac{1}{2} \int \Gamma(f) \Delta \varphi \, dm \geq \int \varphi (\Gamma(f, \Delta f) + K \Gamma(f)) \, dm \quad \forall \varphi \in D(\Delta) \text{ with } \varphi \geq 0, \ \Delta \varphi \in L^\infty(X, m).
\]

Analogously, for \(K \in \mathbb{R}\) and \(N > 1\), we say that \((X, d, m)\) is a \(\text{RCD}^*(K, N)\) metric measure space if, for all \(f \in H^{1,2}(X, d, m) \cap D(\Delta)\) with \(\Delta f \in H^{1,2}(X, d, m)\), Bochner’s inequality

\[
\frac{1}{2} \Delta \Gamma(f) \geq \Gamma(f, \Delta f) + \frac{1}{N} (\Delta f)^2 + K \Gamma(f)
\]

holds in the weak form

\[
\frac{1}{2} \int \Gamma(f) \Delta \varphi \, dm \geq \int \varphi (\Gamma(f, \Delta f) + \frac{1}{N} (\Delta f)^2 + K \Gamma(f)) \, dm
\]

for all \(\varphi \in D(\Delta)\) with \(\varphi \geq 0\) and \(\Delta \varphi \in L^\infty(X, m)\).

The assumption (2.2) is needed to ensure stochastic completeness, namely the property \(h_{t, 1} = 1\). For our purposes, it will be convenient not to add the assumption that \(X = \text{supp } m\), made in some other papers on this subject. Nevertheless, it is obvious that \((X, d, m)\) is \(\text{RCD}(K, \infty)\) (resp. \(\text{RCD}^*(K, N)\)) if and only if \((X, d, \text{supp } m)\) is \(\text{RCD}(K, \infty)\) (resp. \(\text{RCD}^*(K, N)\)).

For \(\text{RCD}(K, \infty)\) spaces it is proved in [3] that the dual semigroup \(\tilde{h}_t\), acting on the space \(\mathcal{P}_2(X)\) of probability measures with finite quadratic moments, is \(K\)-contractive and maps for all \(t > 0\) \(\mathcal{P}_2(X)\) into measures absolutely continuous w.r.t. \(m\), with finite logarithmic entropy. Setting then

\[
\tilde{h}_t \delta_x = p(x, \cdot, t) m \quad x \in X, \ t > 0
\]

this provides a version of the heat kernel \(p(x, y, t)\) in this class of spaces (defined for any \(x\) in \(\text{supp } m\), up to a \(m\)-negligible set of points \(y\)), so that

\[
h_t f(x) = \int_X p(x, y, t) f(y) \, dm \quad \forall f \in L^2(X, m).
\]
In RCD$^*(K, N)$ spaces with $N < \infty$, thanks to additional properties satisfied by the metric measure structure, one can find a version of $p$ continuous in $\text{supp} m \times \text{supp} m \times (0, +\infty)$, as illustrated in the next section.

**Definition 2.2 (Rectifiable sets)** Let $(X, d)$ be a metric space and let $k \geq 1$ be an integer. We say that $S \subset X$ is countably $k$-rectifiable if there exist at most countably many sets $B_i \subset \mathbb{R}^k$ and Lipschitz maps $f_i : B_i \to X$ such that $S \subset \bigcup_i f_i(B_i)$.

For a nonnegative Borel measure $\mu$ in $X$ (not necessarily $\sigma$-finite), we say that $S$ is $(\mu, k)$-rectifiable if there exists a countably $k$-rectifiable set $S' \subset S$ such that $\mu^*(S \setminus S') = 0$, namely $S \setminus S'$ is contained in a $\mu$-negligible Borel set.

In the next proposition we recall some basic differentiation properties of measures.

**Proposition 2.3** If $\mu$ is a locally finite and nonnegative Borel measure in $X$ and $S \subset X$ is a Borel set, one has

$$\mu(S) = 0 \implies \mu(B_r(x)) = o(r^k) \text{ for } \mathcal{H}^k\text{-a.e. } x \in S. \quad (2.3)$$

In addition,

$$\mu(S) = 0, \ S \subset \left\{ x : \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{r^k} > 0 \right\} \implies \mathcal{H}^k(S) = 0. \quad (2.4)$$

Finally, if $\mu = f\mathcal{H}^k\llcorner S$ with $S$ countably $k$-rectifiable, one has

$$\lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k} = f(x) \text{ for } \mathcal{H}^k\text{-a.e. } x \in S. \quad (2.5)$$

**Proof** The proof of (2.3) and (2.4) can be found for instance in [22, 2.10.19] in a much more general context. See also [8, Theorem 2.4.3] for more specific statements and proofs. The proof of (2.5) is given in [37] when $\mu = \mathcal{H}^k\llcorner S$, with $S$ countably $k$-rectifiable and having locally finite $\mathcal{H}^k$-measure (the proof uses the fact that for any $\epsilon > 0$ we can cover $\mathcal{H}^k$-almost all of $S$ by sets $S_i$ which are bi-Lipschitz deformations, with bi-Lipschitz constants smaller than $1 + \epsilon$, of $(\mathbb{R}^i, \| \cdot \|_i)$, for suitable norms $\| \cdot \|_i$). In the general case a simple comparison argument gives the result. \hfill $\Box$

We conclude this section with two auxiliary results.

**Lemma 2.4** Let $f_i, g_i, f, g \in L^1(X, m)$. Assume that $f_i, g_i \to f, g$ $m$-a.e., respectively, that $|f_i| \leq g_i$ $m$-a.e., and that $\lim_{i \to \infty} \|g_i\|_{L^1} = \|g\|_{L^1}$. Then $f_i \to f$ in $L^1(X, m)$.

**Proof** Obviously $|f| \leq g$ $m$-a.e. Applying Fatou’s lemma for $h_i := g_i + g - |f_i - f| \geq 0$ yields

$$\int_X \liminf_{i \to \infty} h_i \, dm \leq \liminf_{i \to \infty} \int_X h_i \, dm.$$ 

Then by assumption the left-hand side is equal to $2\|g\|_{L^1}$, and the right-hand side is equal to $2\|g\|_{L^1} - \limsup_i \|f_i - f\|_{L^1}$. It follows that $\limsup_i \|f_i - f\|_{L^1} = 0$, which completes the proof.

The proof of the next classical result can be found, for instance, in [23, Sec. XIII.5, Theorem 2].
**Theorem 2.5** (Karamata’s Tauberian theorem) Let \( \nu \) be a nonnegative and locally finite measure in \([0, +\infty)\) and set
\[
\hat{v}(t) := \int_{[0, +\infty)} e^{-\lambda t} d\nu(\lambda) \quad t > 0.
\]
Then, for all \( \gamma > 0 \) and \( a \in [0, +\infty) \) one has
\[
\lim_{t \to 0^+} t^\gamma \hat{v}(t) = a \iff \lim_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} = \frac{a}{\Gamma(\gamma + 1)}.
\]
In particular, if \( \gamma = k/2 \) with \( k \) integer, the limit in the right-hand side can be written as \( ao_k / \pi^{k/2} \).

**Remark 2.6** (One-sided versions) More generally we shall prove in the last section of the paper the so-called Abelian one-sided implications and inequalities:
\[
\liminf_{t \to 0^+} t^\gamma \hat{v}(t) \geq \Gamma(\gamma + 1) \liminf_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma}, \tag{2.6}
\]
\[
\limsup_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} < +\infty \implies \limsup_{t \to 0^+} t^\gamma \hat{v}(t) \leq \Gamma(\gamma + 1) \limsup_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} \tag{2.7}
\]
as well as the so-called Tauberian one-sided implications and inequalities
\[
\limsup_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} \leq e \limsup_{t \to 0^+} t^\gamma \hat{v}(t), \tag{2.8}
\]
\[
\liminf_{t \to 0^+} t^\gamma \hat{v}(t) > 0, \limsup_{t \to 0^+} t^\gamma \hat{v}(t) < +\infty \implies \liminf_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} > 0. \tag{2.9}
\]

Notice that (2.9) is not quantitative, and requires both bounds on the lim inf and the lim sup, see Remark 5.4 for an additional discussion.

### 3 Pointwise convergence of heat kernels

From now on, \( K \in \mathbb{R} \) and \( N \in [1, +\infty) \). Let us fix a pointed measured Gromov–Hausdorff (mGH for short in the sequel) convergent sequence \((X_i, d_i, x_i, m_i)^{mGH} \to (X, d, x, m)\) of \( \text{RCD}^*(K, N) \)-spaces. This means that there exist sequences of positive numbers \( \epsilon_i \to 0 \), \( R_i \uparrow \infty \), and of Borel maps \( \varphi_i : B_{R_i}(x_i) \to X \) such that:

(a) \( |d_i(x, y) - d(\varphi_i(x), \varphi_i(y))| < \epsilon_i \) for any \( i \) and all \( x, y \in B_{R_i}(x_i) \), so that \( B_{R_i - \epsilon_i}(\varphi_i(x_i)) \subset B_{\epsilon_i}(\varphi_i(B_{R_i}(x_i))) \);

(b) \( \varphi_i(x_i) \to x \) in \( X \) as \( i \to \infty \) (we denote it by \( x_i^{GH} \to x \) for short);

(c) \( (\varphi_i)_{m_i} \overset{C_{bs}(X)}{\longrightarrow} m \).

In statement (c) we have denoted by \( C_{bs}(X) \) the space of continuous functions with bounded support, and by \( f_p \) the push forward operator between measures induced by a Borel map \( f \). We shall use this notation also in the sequel and we call weak convergence the convergence in duality with \( C_{bs}(X) \).

Since \( m_i \) are uniformly doubling (it follows directly from Bishop–Gromov inequality, known to be true even in the \( \text{CD}^*(K, N) \) case), the mGH-convergence is equivalent to the pointed measured Gromov (pmG for short) convergence introduced in [26]. Recall that
“(X_i, d_i, x_i, m_i) pmG-converges to (X, d, x, m)” means that there exist a doubling and complete metric space X and isometric embeddings \( \psi_i : X_i \hookrightarrow X \), \( \psi : X \hookrightarrow X \) such that \( \psi_i(x_i) \to \psi(x) \) in X as \( i \to \infty \) (we also write \( x_i \to^G H x \) for short) and such that \( (\psi_i)_\sharp m_i \overset{C_{bs}(X)}{\to} (\psi)_\sharp m \). See [26, Theorem 3.15] for the proof of the equivalence.

Since all objects we are dealing with are invariant under isometric and measure-preserving embeddings, we identify in the sequel \( \{ \psi_i \} \) with its image by \( \psi_i \), i.e., \( (X_i, d_i, x_i, m_i) = (\psi_i(X_i), d, \psi_i(x_i), (\psi_i)_\sharp m_i) \). So, in the sequel the complete and doubling space \((X, d)\) will be fixed (playing the role of X), and we denote by \( X_i \subset X \) the supports of the measures \( m_i \), weakly convergent in X to \( m \). Because of this, we also use the simpler notation \( y_i \to y \) for \( y_i \to^G H y \). We recall that complete and doubling spaces are proper (i.e., bounded closed sets are compact), hence separable.

Under this notation let us recall the definition of \( L^2 \)-strong/weak convergence of functions with respect to the mGH-convergence. The following formulation is due to [26] and [7], which fits the pmG-convergence well. Other good formulations of \( L^2 \)-convergence, in connection with mGH-convergence, can be found in [30,39]. However, in our setting these formulations are equivalent by the volume doubling condition (e.g., [31, Proposition 3.3]).

**Definition 3.1** \( (L^2\text{-convergence of functions with respect to variable measures}) \)

1. \( (L^2\text{-weak convergence}) \) We say that \( f_i \in L^2(X_i, m_i) \) \( L^2 \)-weakly converge to \( f \in L^2(X, m) \) if \( \sup_i \| f_i \|_{L^2} < \infty \) and \( f_i m_i \rightharpoonup f m \). Moreover, we say that \( f_i \in L^2(X_i, m_i) \) \( L^2 \)-strongly converge to \( f \in L^2(X, m) \) if \( f_i \) \( L^2 \)-weakly converge to \( f \) with \( \lim_{i \to \infty} \| f_i \|_{L^2} \leq \| f \|_{L^2} \).

2. \( (L^2_{loc}\text{-weak/strong convergence}) \) We say that \( f_i \in L^2_{loc}(X_i, m_i) \) \( L^2_{loc} \)-weakly/strongly converge to \( f \in L^2_{loc}(X, m) \) if \( \zeta f_i \overset{L^2_{loc}}{\to} \zeta f \) for any \( \zeta \in C_{bs}(X) \).

**Proposition 3.2** Let \( f_i \in C^0(X_i) \) and \( f \in C^0(X) \), with X proper and

\[
\sup_i \sup_{X_i \cap B_R(x_i)} |f_i| < +\infty \quad \forall R > 0.
\]

Assume that \( \{ f_i \} \) is locally equi-continuous, i.e., for any \( \epsilon > 0 \) and any \( R > 0 \) there exists \( \delta > 0 \) independent of \( i \) such that

\[
(y, z) \in (X_i \cap B_R(x_i)) \quad d(y, z) < \delta \implies |f_i(y) - f_i(z)| < \epsilon. \tag{3.1}
\]

Then the following are equivalent:

1. \( \lim_{k \to \infty} f_i(i(k)) = f(y) \) whenever \( y \in \text{supp } m \), \( i(k) \to \infty \) and \( y_{i(k)} \in X_{i(k)} \to y \),
2. \( f_i \overset{L^2_{loc}}{\to} f \),
3. \( f_i \overset{L^2_{loc}}{\to} f \).

**Proof** We prove the implication from (1) to (3) and from (2) to (1), since the implication from (3) to (2) is trivial.

Assume that (2) holds, let \( \epsilon > 0 \) and let \( y_i \to y \). Take \( \zeta \) nonnegative, with support contained in \( B_\delta(y) \) and with \( \int \zeta \, dm = 1 \). Thanks to (3.1) and the continuity of \( f \), for \( \delta \) sufficiently small we have

\[
(f_i(y_i) - \epsilon) \int \zeta \, dm_i \leq \int \zeta f_i \, dm_i \leq (f_i(y_i) + \epsilon) \int \zeta \, dm_i \quad f(y) - \epsilon \leq \int \zeta f \, dm \leq f(y) + \epsilon.
\]
Since \( \int \xi f_i \, dm_i \to \int \xi f \, dm \) and \( \int \xi \, dm_i \to \int \xi \, dm = 1 \), from the arbitrariness of \( \epsilon \) we obtain that \( f_i(y_i) \to f(y) \). A similar argument, for arbitrary subsequences, gives (1).

In order to prove the implication from (1) to (3) we prove the implication from (1) to (2). Assuming with no loss of generality that \( f_i \) and \( f \) are nonnegative, for any \( \xi \in C_{bs}(X) \) nonnegative, (1) and the compactness of the support of \( \xi \) give that for any \( \epsilon > 0 \) and any \( s > 0 \) the set \( X \setminus \{ f_i \xi > s \} \) is contained in the \( \epsilon \)-neighborhood of \( \{ f \xi > s \} \) for \( i \) large enough, so that

\[
\lim_{i \to \infty} \sup \mathcal{H}_i(\{ f_i \xi > s \}) \leq \mathcal{H}(\{ f \xi > s \}).
\]

Analogously, any open set \( A \subseteq \{ f \xi > s \} \) is contained for \( i \) large enough in the set \( \{ f_i \xi > s \} \cup (X \setminus X_i) \), so that

\[
\lim_{i \to \infty} \inf \mathcal{H}_i(\{ f_i \xi > s \}) \geq \mathcal{H}(\{ f \xi > s \}).
\]

Combining these two informations, Cavalieri’s formula and the dominated convergence theorem provide \( \int_X f_i \xi \, dm_i \to \int_X f \xi \, dm \) and then, since \( \xi \) is arbitrary, (2).

Now we can prove the implication from (1) to (3). Thanks to the equi-boundedness assumption, the sequence \( g_i := f_i^2 \) is locally equi-continuous as well and \( g_i \) pointwise converge to \( g := f^2 \) in the sense of (1), applying the implication from (1) to (2) for \( g_i \) gives

\[
\lim_{i \to \infty} \int_{X_i} \xi^2 f_i^2 \, dm_i = \int_X \xi^2 f^2 \, dm \quad \forall \xi \in C_{bs}(X),
\]

which yields (3).

Let us recall the regularity of the heat kernel \( p_{(X,\mathfrak{d},\mathfrak{m})} := p(x,y,t) \) of a RCD\(^*(K,\mathcal{N})\)-space \((X, \mathfrak{d}, \mathfrak{m})\) we need, where \( N \in [1, \infty) \) and \( K \leq 0 \). The general theory of Dirichlet forms [50], together with the doubling and Poincaré properties ensure that we can find a locally Hölder continuous representative of \( p \) in \( X \times X \times (0, + \infty) \), which satisfies Gaussian bounds. See [48, Theorem 4], [49, Proposition 2.3], [50, Sections 3 and 4].

On RCD\(^*(K,\mathcal{N})\)-spaces, finer properties of the heat kernel are known, as follows. It was proven in [33] that for any \( \epsilon \in (0,1) \) there exist \( C_i := C_i(\epsilon, K, N) > 1 \) (i.e., for any \( \epsilon, K, N \)) such that

\[
\frac{1}{C_1 \mathfrak{m}(B_{1/2}(x))} \exp\left( - \frac{d^2(x, y)}{4(1 - \epsilon)t} - C_2t \right) \leq p(x,y,t) \leq \frac{C_1}{\mathfrak{m}(B_{1/2}(x))} \exp\left( - \frac{d^2(x, y)}{4(1 + \epsilon)t} + C_2t \right)
\]

for all \( x, y \in \text{supp} \, \mathfrak{m} \) and any \( t > 0 \). This, combined with the Li–Yau inequality [24,32] gives a gradient estimate

\[
\| \nabla p(x,y,t) \| \leq \frac{C_3}{t^{1/2} \mathfrak{m}(B_{1/2}(x))} \exp\left( - \frac{d^2(x, y)}{(4 + \epsilon)t} + C_4t \right)
\]

for any \( t > 0, y \in \text{supp} \, \mathfrak{m} \) and m-a.e. \( x \in X \), where \( C_i := C_i(\epsilon, K, N) > 1 \) (i.e., for any \( \epsilon, K, N \)). In particular one obtains a quantitative local Lipschitz bound on \( p \), i.e., for any \( z \in X \), any \( R > 0 \) and any \( 0 < t_0 \leq t_1 < \infty \) there exists \( C := C(K, N, R, t_0, t_1) > 0 \) such that

\[
\left| p(x, y, t) - p(\hat{x}, \hat{y}, t) \right| \leq \frac{C}{\mathfrak{m}(B_{1/2}(z))} \mathfrak{d}\left( (x, y), (\hat{x}, \hat{y}) \right)
\]

for all \( x, \hat{x}, \hat{y} \in B_R(z) \cap \text{supp} \, \mathfrak{m} \) and any \( t \in [t_0, t_1] \). See [33, Theorem 1.2, Corollary 1.2] (see also [24,43]).

The following is a generalization/refinement of Ding’s result [19, Theorems 2.6, 5.54 and 5.58] from the Ricci limit setting to our setting, via a different approach.
Theorem 3.3 (Pointwise convergence of heat kernels) The heat kernels $p_i$ of $(X_i, d_i, m_i)$ satisfy
\[ \lim_{i \to \infty} p_i(x_i, y_i, t_i) = p(x, y, t) \]
whenever $(x_i, y_i, t_i) \in X_i \times X_i \times (0, +\infty) \to (x, y, t) \in \text{supp } m \times \text{supp } m \times (0, +\infty)$.

Proof By rescaling $d \to (t/t_i)^{1/2}d$, without any loss of generality we can assume that $t_i \equiv t$. Let $f \in C_b(X)$ and recall that, viewing $f$ as an element of $L^2 \cap L^\infty(X_i, m_i)$, $h^i_t f$ $L^2$-strongly converge to $h_t f$ [26, Theorem 6.11]. By the Bakry–Émery estimate (see for instance [3, Theorem 6.5]), we have $I_0(t) = t$ and $I_\xi(t) := (e^{S_t} - 1)/S$ for $S \neq 0$.

\[ \sqrt{2I_{2k}(t)\text{Lip}(h_t f, \text{supp } m)} \leq \|f\|_{L^\infty(X, m)}, \]
valid in all RCD$(K, \infty)$ spaces, we see that $h^i_t f$ are equi-Lipschitz on $X_i$. Thus, applying Proposition 3.2 yields $h^i_t f(y_i) \to h_t f(y)$.

On the other hand, the Gaussian estimate (3.2) shows that $\sup_t \|p_i(\cdot, y_i, t)\|_{L^\infty} < \infty$. By definition, since

\[ h_t f(y_i) = \int_{X_i} p_i(z, y_i, t) f(z) dm_i(z), \quad h_t f(y) = \int_X p(z, y, t) f(z) dm(z), \]
we see that $p_i(\cdot, y_i, t) L^2_{\text{loc}}$-weakly converge to $p(\cdot, y, t)$. Moreover, since thanks to (3.4) the functions $p_i(\cdot, y_i, t)$ are locally equi-Lipschitz continuous, choosing any continuous extension of $p(\cdot, y, t)$ to the whole of $X$ and applying Proposition 3.2 once more to $p_i(\cdot, y_i, t)$ we obtain $p_i(x_i, y_i, t)$ converge to $p(x, y, t)$, which completes the proof.

Definition 3.4 ($k$-dimensional regular sets $\mathcal{R}_k$ and maximal dimension $\text{dim}_{d, m}(X)$) Recall that the $k$-dimensional regular set $\mathcal{R}_k$ of a RCD$^*(K, N)$-space $(X, d, m)$ in the sense of Mondino–Naber [43] is, by definition, the set of points $x \in \text{supp } m$ such that

\[ (X, r^{-1}d, m^x_r, x) \overset{m^{GH}}{\longrightarrow} ([\mathbb{R}^k, d_{\mathbb{R}^k}, c_k \mathcal{H}^k, 0_k]) \]
as $r \to 0^+$, where $c_k$ is the normalization constant such that $\int_{B_1(0_k)} (1 - |x|) d(c_k \mathcal{H}^k) = 1$, and

\[ m^x_r := \left( \int_{B_r(x)} \left( 1 - \frac{d(x, \cdot)}{r} \right) dm \right)^{-1} m. \]

We denote by $\text{dim}_{d, m}(X)$ the largest integer $k$ such that $\mathcal{R}_k$ has positive $m$-measure.

By the Bishop–Gromov inequality, it is easily seen that $\mathcal{R}_k = \emptyset$ if $k > [N]$. It is conjectured that RCD$^*$ $(K, N)$ spaces cannot be made of pieces of different dimensions, i.e., there exists only one integer $k$ such that $m(\mathcal{R}_k) > 0$. This property is known to be true for Ricci limit spaces, see [16].

Remark 3.5 By the $L^2_{\text{loc}}$-strong convergence of $d(y_i, \cdot)$ to $d(y, \cdot)$ for any mGH-convergent sequence $(Y_i, d_i, v_i, y_i) \overset{GH}{\longrightarrow} (Y, d, v, y)$ of RCD$^*(K, N)$-spaces, it is easy to check that $x \in X$ is a $k$-dimensional regular point if and only if

\[ \left( X, r^{-1}d, \frac{m}{m(B_{r}(x))}, x \right) \overset{m^{GH}}{\longrightarrow} \left( \mathbb{R}^k, d_{\mathbb{R}^k}, \frac{\mathcal{H}^k}{\omega_k}, 0_k \right), \]
where recall that $\omega_k$ denotes the volume of a unit ball in the $k$-dimensional Euclidean space.
Corollary 3.6 (Short-time diagonal behavior of heat kernel on the regular set) Let \((X, d, m)\) be a \(RCD^*(K, N)\)-space with \(K \in \mathbb{R}\) and \(N \in (1, +\infty)\). Then
\[
\lim_{t \to 0^+} m(B_{1/2}(x)) p(x, x, t) = \frac{\omega_k}{(4\pi)^{k/2}} \tag{3.5}
\]
for any \(k\)-dimensional regular point \(x\) of \((X, d, m)\).

**Proof** Let us recall that for any \(r > 0\) and any \(C > 0\) the heat kernel \(\hat{p}(x, y, t)\) of the rescaled \(RCD^*(r^2 K, N)\)-space \((X, r^{-1} d, C m)\) is given by \(\hat{p}(x, y, t) = C^{-1} p(x, y, r^2 t)\). Applying this for \(r := t^{1/2}, C := \frac{1}{m(B_t(x))}\) with Theorem 3.3 and Remark 3.5 shows
\[
\lim_{t \to 0^+} \frac{m(B_{1/2}(x))}{m(B_{1/2}(x))} p(x, x, t) = \lim_{t \to 0^+} \frac{p'(x, x, 1) = p_{\mathbb{R}^k}(0, 0, 1)}{\omega_k (4\pi)^{k/2}},
\]
where \(p', p_{\mathbb{R}^k}\) denote the heat kernels of \((X, t^{-1/2} d, m(B_{1/2}(x)))\), \((\mathbb{R}^k, d_{\mathbb{R}^k}, \frac{\omega_k}{\omega_k})\), respectively. \(\Box\)

In the proof of Weyl’s law, in the next section, the following finer properties of \(\mathcal{R}_k\) will be needed.

**Theorem 3.7** Let \((X, d, m)\) be a \(RCD^*(K, N)\) space with \(K \in \mathbb{R}\) and \(N \in (1, +\infty)\). For all \(k\) the set \(\mathcal{R}_k\) is \((m, k)\)-rectifiable and
\[
m \left( \bigcup_{k=1}^{[N]} X \setminus \mathcal{R}_k \right) = 0.
\]

In addition, \(m \ll \mathcal{R}_k \ll \mathcal{H}^k\).

**Proof** See [43] for the proof of the first two statements (more precisely, it has been proved the stronger property that \(m\)-almost all of \(\mathcal{R}_k\) can be covered by bi-Lipschitz charts with bi-Lipschitz constant arbitrarily close to 1, defined in subsets of the \(k\)-dimensional Euclidean space). See [17,35] and [27] for the proof of the absolute continuity statement.

4 Weyl’s law

In a metric measure space \((X, d, m)\) the sequence of eigenvalues can be defined appealing to Courant’s min-max procedure:
\[
\lambda_i := \min \left\{ \max_{f \in S, \|f\|_{L^2} = 1} \text{Ch}(f) : S \subset H^{1,2}(X, d, m), \dim(S) = i \right\} \quad i \geq 1. \tag{4.1}
\]
We then define
\[
N_{(X,d,m)}(\lambda) := \# \{ i \geq 1 : \lambda_i \leq \lambda \}
\]
as the “inverse” function of \(i \mapsto \lambda_i\). Notice that the formula makes sense even though \(\text{Ch}\) is not quadratic, and that the formula shows that the growth rate of \(N_{(X,d,m)}\) does not change if we replace the distance \(d\) by a bi-Lipschitz equivalent distance, or perturb the measure \(m\) by a factor uniformly bounded away from 0 and \(+\infty\). Notice also that if \((X, d)\) is doubling we can always find a Dirichlet form \(\mathcal{E}\) with \(C^{-1} \mathcal{E} \leq \text{Ch} \leq C \mathcal{E}\), with \(C\) depending only on the metric doubling constant, see [1] (a result previously proved in [13] for metric measure...
spaces whose measure is doubling and satisfies Poincaré inequality). Thus, the replacement of $\mathcal{Ch}$ with $\mathcal{E}$ makes the standard tools of Linear Algebra applicable.

Let us come now to Weyl’s law on RCD-spaces. It is not known yet to what extent the restriction of the measure of a RCD space (or even of a Ricci limit space) to a regular set is quantitatively comparable to the Hausdorff measure of the corresponding dimension. As the behavior of the Hausdorff measure on the regular sets turns out to be related to the asymptotic behavior of the eigenvalues of the Laplacian, this lack of knowledge seems to be a significant difficulty to establish Weyl’s law in the RCD context in full generality. However, we can bypass this difficulty for a significant class of spaces (including the class of compact Alexandrov spaces, see Corollary 4.8), by noticing that these spaces satisfy a suitable criterion which implies Weyl’s law (we provide the implication in Theorem 4.3). Let us point out that all known examples of compact RCD-spaces satisfy this criterion.

Let us start this section by giving the following, which is to some extent a generalization of [15, Theorem 4.6] to the RCD-setting:

**Theorem 4.1** (Weak Ahlfors regularity) Let $(X, d, m)$ be a RCD$^*$ $(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$ and set

$$\mathcal{R}_k^* := \left\{ x \in \mathcal{R}_k : \exists \lim_{r \to 0^+} \frac{m(B_r(x))}{\omega_k r^k} \in (0, +\infty) \right\}.$$  

(4.2)

Then $m(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0$, $\mathcal{H}^N(\mathcal{R}_N \setminus \mathcal{R}_N^*) = 0$ if $N$ is an integer, $m \ll \mathcal{R}_k^*$ and $\mathcal{H}^k \ll \mathcal{R}_k^*$ are mutually absolutely continuous and

$$\lim_{r \to 0^+} \frac{m(B_r(x))}{\omega_k r^k} = \frac{dm}{d\mathcal{H}^k \ll \mathcal{R}_k^*}(x) \quad \text{for m-a.e. } x \in \mathcal{R}_k^*.$$  

(4.3)

Finally, if $k_0 = \dim d_m(X)$, one has

$$\lim_{r \to 0^+} \frac{\omega_k r^{k_0}}{m(B_r(x))} = \chi_{\mathcal{R}_k^*}(x) \frac{d\mathcal{H}^{k_0} \ll \mathcal{R}_k^*}{dm \ll \mathcal{R}_k^*}(x) \quad \text{for m-a.e. } x \in X.$$  

(4.4)

**Proof** Let $S_k$ be a countably $k$-rectifiable subset of $\mathcal{R}_k$ with $m(\mathcal{R}_k \setminus S_k) = 0$. From (2.4) we obtain that the set $\mathcal{R}_k^* \setminus S_k$ is $\mathcal{H}^k$-negligible, hence $\mathcal{R}_k^*$ is $(\mathcal{H}^k, k)$-rectifiable. We denote $m_k = m \ll \mathcal{R}_k$ and recall that, thanks to Theorem 3.7, $m_k \ll \mathcal{H}^k$ and $m = \sum_k m_k$. We denote by $f : X \to [0, +\infty)$ a Borel function such that $m_k = f \mathcal{H}^k \ll \mathcal{R}_k^*$ (whose existence is ensured by the Radon–Nikodym theorem, being $\mathcal{R}_k^*$ $\sigma$-finite w.r.t. $\mathcal{H}^k$) and recall that (2.5) gives

$$\exists \lim_{r \to 0} \frac{m_k(B_r(x))}{\omega_k r^k} = f(x) \quad \text{for } \mathcal{H}^k \text{-a.e. } x \in \mathcal{R}_k^*.$$  

(4.5)

Now, in (4.5) we can replace $m_k$ by $m$ for $\mathcal{H}^k$-a.e. $x \in \mathcal{R}_k^*$; this is a direct consequence of (2.3) with $\mu = m - m_k$ and $S = \mathcal{R}_k^*$.

Calling then $N_k$ the $\mathcal{H}^k$-negligible (and then $m_k$-negligible) subset of $\mathcal{R}_k^*$ where the equality

$$\lim_{r \to 0} \frac{m(B_r(x))}{\omega_k r^k} = f(x)$$

fails, we obtain existence and finiteness of the limit on $\mathcal{R}_k^* \setminus N_k$: since $f$ is a density, it is also obvious that the limit is positive $m_k$-a.e., and that $\mathcal{H}^k \ll \mathcal{R}_k^* \cap \{ f > 0 \}$ is absolutely continuous w.r.t. $m_k$. 
This proves that \( m(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0 \) and that \( m \sqcap \mathcal{R}_k^* \) and \( \mathcal{H}^k \sqcap \mathcal{R}_k^* \) are mutually absolutely continuous. In the special case \( k = N \) a suitable density lower bound \( (m(B_r(x))/r^N \geq m(X)/(\text{diam}(X))^N \) in the case \( K \geq 0 \), a more complex lower bound involving the comparison spaces also holds when \( K \leq 0 \) coming from the Bishop–Gromov inequality gives that \( H^N \ll m \), hence \( \mathcal{R}_N \setminus \mathcal{R}_N^* \) is also \( H^N \)-negligible. The last statement \((4.4)\) follows by the fact that \( r^{k_0} = o(m(B_r(x))) \) for m-a.e. \( x \in \mathcal{R}_k \), \( k < k_0 \), since this property holds on the sets \( \mathcal{R}_k^* \).

\[ \square \]

Recall that, as direct consequence of standard arguments from spectral theory and elliptic regularity theory, for a compact \( \text{RCD}^*(K, N) \) space \((X, d, m)\) the heat kernel \( p \) can be expressed by eigenfunctions:

\[ p(x, y, t) = \sum_i e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \quad (4.6) \]

for any \( x, y \in \text{supp} \ m \) and any \( t > 0 \), where \( \lambda_i \) is the \( i \)-th eigenvalue of the Laplacian (counting with multiplicities) and \( \varphi_i \) is a corresponding eigenfunction, with \( \| \varphi_i \|_{L^2} = 1 \).

More precisely, in \((4.6)\) one choose the Hölder continuous representative of \( \varphi_i \), whose Hölder norm grows linearly w.r.t. \( \lambda \), so that the series in \((4.6)\) is locally Hölder continuous in \( \text{supp} \ m \times \text{supp} \ m \times (0, +\infty) \).

We are now in a position to introduce our first criterion. We always have \( \mathcal{H}^k (\mathcal{R}_k^*) > 0 \) and, if an assumption slightly stronger than the finiteness of \( k \)-dimensional Hausdorff measure holds, we obtain Weyl’s law in the weak asymptotic form. For simplicity, we use the notation; \( f(\lambda) \sim g(\lambda) \) for the existence of \( C > 1 \) satisfying \( C^{-1} f(\lambda) \leq g(\lambda) \leq C f(\lambda) \) for sufficiently large \( \lambda \).

**Theorem 4.2** Let \((X, d, m)\) be a compact \( \text{RCD}^*(K, N) \)-space with \( K \in \mathbb{R} \) and \( N \in (1, +\infty) \), let \( k = \text{dim}_{d,m}(X) \) and let \( \mathcal{R}_k^* \) be as in \((4.2)\) of Theorem 4.1. Then we have

\[ \lim_{t \to 0^+} \left( k^{k/2} \sum_i e^{-\lambda_i t} \right) \geq \frac{1}{(4\pi)^{k/2}} \mathcal{H}^k (\mathcal{R}_k^*) > 0. \quad (4.7) \]

In particular, if \( N_{(X,d,m)}(\lambda_0) \sim \lambda^i \) as \( \lambda \to +\infty \) for some \( i \), then Remark 2.6 gives \( i \geq k/2 \).

In addition

\[ \limsup_{s \to 0^+} \int_X \frac{s^k}{m(B_s(x))} \text{d}m(x) < +\infty \iff N_{(X,d,m)}(\lambda_0) \sim \Lambda^{k/2} (\lambda \to +\infty). \quad (4.8) \]

**Proof** In order to prove \((4.7)\), we first notice that the combination of \((3.5)\) and \((4.4)\) gives

\[ \lim_{t \to 0^+} \left( k^{k/2} \sum_i e^{-\lambda_i t} \right) \geq \frac{1}{(4\pi)^{k/2}} \int_X \frac{\text{d}H^k \sqcap \mathcal{R}_k^*(x)}{m \sqcap \mathcal{R}_k^*(x)} \quad \text{for m-a.e. } x \in X. \]

Using the identity \( t^{k/2} \sum_i e^{-\lambda_i t} = \int_X t^{k/2} p(x, x, t) \text{d}m(x) \) and Fatou’s lemma, we obtain

\[ \liminf_{t \to 0} \left( k^{k/2} \sum_i e^{-\lambda_i t} \right) \geq \frac{1}{(4\pi)^{k/2}} \int_{\mathcal{R}_k^*} \frac{\text{d}H^k \sqcap \mathcal{R}_k^*}{m \sqcap \mathcal{R}_k^*} = \frac{1}{(4\pi)^{k/2}} \mathcal{H}^k (\mathcal{R}_k^*). \]

The heat kernel estimate \((3.2)\) shows

\[ C^{-1} \frac{t^{k/2}}{m(B_{t^{1/2}}(x))} \leq t^{k/2} p(x, x, t) \leq C \frac{t^{k/2}}{m(B_{t^{1/2}}(x))} \quad (4.9) \]
for some $C > 1$, which is independent of $t$ and $x$. Thus the upper bound on $p$ gives
\[
\limsup_{t \to 0^+} t^{k/2} \int_X p(x, x, t) \, d\mu(x) \leq C \limsup_{s \to 0^+} \int_X \frac{s^k}{m(B_s(x))} \, d\mu(x) < +\infty.
\]
We can now invoke Remark 2.6 to obtain the implication $\Rightarrow$ in (4.8). The proof of the converse implication is similar and uses the lower bound in (4.9).

Under the stronger assumption (4.10) (notice that both the finiteness of the limit and the equality of the integrals are part of the assumption), we can recover Weyl’s law in the stronger form.

**Theorem 4.3** Let $(X, d, m)$ be a compact $\text{RCD}^*(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$, and let $k = \dim_{d, m}(X)$. Then
\[
\lim_{s \to 0^+} \int_X \frac{s^k}{m(B_s(x))} \, d\mu(x) = \int_X \lim_{s \to 0^+} \frac{s^k}{m(B_s(x))} \, d\mu(x) < +\infty \tag{4.10}
\]
if and only if
\[
\lim_{\lambda \to +\infty} \frac{N(X, d, m)(\lambda)}{\lambda^{k/2}} = \frac{\omega_k}{(2\pi)^k} \mathcal{H}^k(\mathcal{R}_k^*) < +\infty. \tag{4.11}
\]
**Proof** We first assume that (4.10) holds. Taking (4.4) and (4.9) into account, we can apply Lemma 2.4 with $f_t(x) = t^{k/2} p(x, x, t)$ and $g_t(x) = Ct^{k/2}/m(B_{t^{1/2}}(x))$ to get
\[
\lim_{t \to 0^+} t^{k/2} \int_X p(x, x, t) \, d\mu(x) = \int_X \lim_{t \to 0^+} t^{k/2} p(x, x, t) \, d\mu(x)
\]
\[
= \int_{\mathcal{R}_k^*} \frac{1}{(4\pi)^{k/2}} \, d\mathcal{H}^k \mathcal{L} \mathcal{R}_k^* \, d\mu
\]
\[
= \frac{1}{(4\pi)^{k/2}} \mathcal{H}^k(\mathcal{R}_k^*)
\]
which shows (4.11) by Karamata’s Tauberian theorem.

Next we assume that (4.11) holds. Then by (4.4) and Karamata’s Tauberian theorem again, (4.11) is equivalent to
\[
\lim_{t \to 0^+} t^{k/2} \int_X p(x, x, t) \, d\mu(x) = \int_X \lim_{t \to 0^+} t^{k/2} p(x, x, t) \, d\mu(x) < +\infty. \tag{4.12}
\]
Let $f_t(x) := t^{k/2}/m(B_{t^{1/2}}(x))$. Then the heat kernel estimate (4.9) shows that we can apply Lemma 2.4 with $g_t(x) = Ct^{k/2}/p(x, x, t)$ to get (4.10).

By the stability of $\text{RCD}$-conditions with respect to mGH-convergence and [14, Theorem 5.1], noncollapsed Ricci limit spaces give typical examples of $\text{RCD}^*(K, N)$-spaces $(X, d, m)$ with $\dim_{d, m} X = N$. For such metric measure spaces, Weyl’s law was proven in [19] by Ding. Thus the following corollary also recovers his result.

**Corollary 4.4** Let $(X, d, m)$ be a compact $\text{RCD}^*(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$, and let $k = \dim_{d, m} X$. Assume that either $k = N$, or that for any integer $i$ such that $m(\mathcal{R}_i) > 0$ there exists $g_i \in L^1(\mathcal{R}_i^*, \mathcal{H}^i)$ such that
\[
g_i(x, t) := \frac{t^k}{m(B_t(x))} \frac{d\mathcal{L} \mathcal{R}_i^*}{d\mathcal{H}^i \mathcal{L} \mathcal{R}_i^*}(x) \leq g_i(x) \quad \forall t \in (0, 1)
\]
for $\mathcal{H}^i$-a.e. $x \in \mathcal{R}_i^*$. Then (4.11) holds.
Proof If the functions $g_l$ exist, the proof follows by the dominated convergence theorem in conjunction with Theorem 4.3. When $k = N$ the existence of the functions $g_l$ follows directly from the Bishop–Gromov inequality, since $m(B_r(x))/r^k$ is bounded from below by a positive constant. \hfill \square

Example 4.5 Let us apply Theorem 4.3 to the following RCD$^\ast(N - 1, N)$-space:

$$(X, d, m) := \left([0, \pi], d_{[0,\pi]}, \sin^{N-1} r dr\right)$$

for $N \in (1, \infty)$ (note that this is a Ricci limit space if $N$ is an integer, see for instance [5]). Then we can apply Theorem 4.3 with $k = 1$ and $\mathcal{R}_1^+ = \mathcal{R}_1 = (0, \pi)$, because of the sup$_{t<1} \|g_1(\cdot, t)\|_{L^\infty} < \infty$, where $g_1$ is as in Corollary 4.4. Thus we have Weyl’s law:

$$\lim_{\lambda \to +\infty} \frac{N(X,d,m)(\lambda)}{\lambda^{1/2}} = \frac{\omega_1}{2\pi} \mathcal{H}^1((0, \pi)) = 1.$$

Example 4.6 (Iterated suspensions) Let us apply now Theorem 4.3 to iterated suspensions of $(X, d, m)$ as in Example 4.5:

$$(X_1, d_1, m_1) := \left([0, \pi], d_{[0,\pi]}, \sin^{N-1} r dr\right),$$

$$(X_{n+1}, d_{n+1}, m_{n+1}) := \left([0, \pi], d_{[0,\pi]}, \sin r dr\right) \times (X_n, d_n, m_n).$$

Recall that the spherical suspension $([0, \pi], d_{[0,\pi]}, \sin r dr) \times (X, d, m)$ of a metric measure space $(X, d, m)$ is the quotient of the product $[0, \pi] \times X$ by the identification of every point of $[0] \times X$ and $[\pi] \times X$ into two distinct points, equipped with the product measure $d\mu := \sin r dr \times m$ and with the distance $d_{\text{sup}}$ defined by

$$\cos d_{\text{sup}}((t, x), (s, y)) = \cos t \cos s + \sin t \sin s \cos (\min\{t(x), y\}).$$

Note that $(X_n, d_n, m_n)$ is a RCD$^\ast(N+n-2, N+n-1)$-space (see [36]) and that $(X_n, d_n)$ are isometric to a hemisphere of the $n$-dimensional unit sphere $S^n(1)$ as metric spaces.

Then we can apply Theorem 4.3 because an elementary calculation similar to the one of Example 4.5 shows that $\sup_{t<1} \|g_n(\cdot, t)\|_{L^\infty} < \infty$. Thus Weyl’s law follows:

$$\lim_{\lambda \to +\infty} \frac{N(X_n,d_{n},m_{n})(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(X_n) = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(S^n(1)).$$

Example 4.7 (Gaussian spaces) For noncompact RCD$(K, \infty)$-spaces the behavior of the spectrum is different and requires a more delicate analysis. For instance (see [42, (2.2)]) the $n$-dimensional Gaussian space $(X, d, m) := (\mathbb{R}^n, d_{\mathbb{R}^n}, \gamma_n)$ satisfies

$$\lim_{\lambda \to +\infty} \frac{N(X,d,m)(\lambda)}{\lambda^n} = \frac{1}{\Gamma(n+1)}.$$

Corollary 4.8 (Weyl’s law on compact Ahlfors regular RCD$^\ast(K, N)$-spaces—especially Alexandrov spaces) Let $(X, d, m)$ be a compact RCD$^\ast(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$. Assume that $(X, d, m)$ is Ahlfors $n$-regular for some $n \in \mathbb{N}$, i.e., there exists $C > 1$ such that

$$C^{-1}r^n \leq m(B_r(x)) \leq Cr^n \quad \forall x \in X, \ r \in (0, 1).$$

Then we have Weyl’s law:

$$\lim_{\lambda \to +\infty} \frac{N(X,d,m)(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(X).$$

(4.13)
In particular, this holds if \((X, d, m)\) is an n-dimensional compact Alexandrov space.

Proof Note that by the Ahlfors n-regularity of \((X, d, m)\), any tangent cone at x also satisfies the Ahlfors n-regularity, which implies that \(R_i = \emptyset\) for any \(i \neq n\). In particular since \(\mathcal{H}^n \ll m \ll \mathcal{H}^n\), we have
\[
m(X \backslash R_n) = \mathcal{H}^n(X \backslash R_n) = 0.
\]
Then Theorem 4.3 can be applied with \(g_k \equiv c\) for some \(c > 0\), which proves (4.13) by (4.14). The final statement follows from the compatibility between Alexandrov spaces and RCD-spaces [45,56]. □

Example 4.9 Let us discuss the simplest case we can apply Corollary 4.8; let \(M\) be a compact n-dimensional manifold and let \(f \in C^2(M)\). Then, thanks to (1.2), for any \(N \in (n, \infty)\) there exists \(K \in \mathbb{R}\) such that \((M, d, e^{-f} \mathcal{H}^n)\) is a RCD\(^*(K, N)\)-space. Moreover since \((M, d, e^{-f} \mathcal{H}^n)\) is Ahlfors n-regular, Corollary 4.8 yields Weyl’s law:
\[
\lim_{N \to +\infty} \frac{N(M, d, e^{-f} \mathcal{H}^n)(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(M).
\]

In order to give another application of Weyl’s law on compact finite dimensional Alexandrov spaces, let us recall that two compact finite dimensional Alexandrov spaces are said to be isospectral if the spectrums of their Laplacians coincide. See for instance [20,53] for constructions of isospectral manifolds and of isospectral Alexandrov spaces (see also [38] for analysis on Alexandrov spaces).

It is also well known as a direct consequence of Perelman’s stability theorem [44] (see also [34]) that for fixed \(n \in \mathbb{N}\), \(K \in \mathbb{R}\) and \(d, v > 0\) the isometry class of n-dimensional compact Alexandrov spaces \(X\) of sectional curvature bounded below by \(K\) with \(\text{diam } X \leq d\) and \(\mathcal{H}^n(X) \geq v\) has only finitely many topological types. By using this and Weyl’s law, we have the following which is a generalization of topological finiteness results for isospectral spaces proven in [11,28,47] to Alexandrov spaces.

**Corollary 4.10** (Topological finiteness theorem for isospectral Alexandrov spaces) Let \(\chi := \{(X_u, d_u, \mathcal{H}^n_u)\}_{u \in U}\) be a class of compact finite dimensional Alexandrov spaces with a uniform sectional curvature bound from below. Assume that there exists \(C > 1\) such that
\[
\limsup_{\lambda \to +\infty} \frac{N(X_u, d_u, \mathcal{H}^n_u)(\lambda)}{N(X_v, d_v, \mathcal{H}^n_v)(\lambda)} \leq C
\]
for all \(u, v \in U\). Then \(\chi\) has only finitely many topological types.

In particular, any class of isospectral compact finite dimensional Alexandrov spaces with a uniform sectional curvature bound from below has only finitely many members up to homeomorphism.

Proof By an argument similar to the proof of [11, Corollary 1.2] (or [47, Proposition 7.4]) with [54, Corollary 1]
there exists \(d > 0\) such that \(\text{diam } X_u \leq d\) for any \(u \in U\). Since Weyl’s law (4.13) with (4.15) implies that there exist \(n \in \mathbb{N}\) and \(v > 0\) such that \(\text{dim } X_u \equiv n\) and \(\mathcal{H}^n(X_u) \geq v\) for any \(u \in U\), the topological finiteness result stated above completes the proof.

Acknowledgements The first and third author acknowledge the support of the PRIN2015 MIUR Project “Calcolo delle Variazioni”. The second author acknowledges the support of the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers, the Grant-in-Aid for Young Scientists (B) 16K17585 and the Scuola Normale Superiore for warm hospitality.
5 Appendix: refinements of Karamata’s theorem

In this section we prove Theorem 2.5 and its one-sided versions mentioned in Remark 2.6. We follow the proofs in Theorems 10.2 and 10.3 of [46], borrowing also the terminology “Abelian”, “Tauberian” from there.

Throughout this section \( \nu \) is a nonnegative and \( \sigma \)-finite Borel measure on \([0, +\infty)\). The results will then be applied to the case when \( \nu := \sum_i \delta_{\lambda_i} \).

**Lemma 5.1** For all \( t > 0 \) one has
\[
\int_{[0, +\infty)} e^{-tx} \, d\nu(x) = \int_0^\infty t \nu([0, y)) e^{-ty} \, dy. \tag{5.1}
\]

**Proof** By Cavalieri’s formula and the change of variables \( r = e^{-ty} \) we get
\[
\int_{[0, +\infty)} e^{-tx} \, d\nu(x) = \int_0^1 \nu\left(\{x : e^{-tx} \geq r\}\right) \, dr = \int_0^\infty t e^{-ty} \nu\left(\{x : e^{-tx} \geq e^{-ty}\}\right) \, dy
\]
and we conclude, since \( \{x : e^{-tx} \geq e^{-ty}\} = [0, y] \). \( \Box \)

We start with the Abelian case, easier when compared to the Tauberian one.

**Theorem 5.2** (Abelian theorem) Assume that there exist \( \gamma \in [0, +\infty) \) and \( C \in [0, +\infty) \) such that
\[
\lim_{a \to +\infty} \frac{\nu([0, a])}{a^{\gamma}} = C. \tag{5.2}
\]

Then
\[
\lim_{t \to 0^+} t^{\gamma} \int_{[0, +\infty)} e^{-tx} \, d\nu(x) = C \Gamma(\gamma + 1). \tag{5.3}
\]

More generally,
\[
\limsup_{a \to +\infty} \frac{\nu([0, a])}{a^{\gamma}} \leq C < +\infty \implies \limsup_{t \to 0^+} t^{\gamma} \int_{[0, +\infty)} e^{-tx} \, d\nu(x) \leq C \Gamma(\gamma + 1) \tag{5.4}
\]

and
\[
\liminf_{a \to +\infty} \frac{\nu([0, a])}{a^{\gamma}} \geq c \implies \liminf_{t \to 0^+} t^{\gamma} \int_{[0, +\infty)} e^{-tx} \, d\nu(x) \geq c \Gamma(\gamma + 1). \tag{5.5}
\]

**Proof** Let \( F(a) := \nu([0, a]) \) and \( G(a) := (a + 1)^{-\gamma} F(a) \). Then (5.2) yields
\[
\lim_{a \to +\infty} G(a) = C. \tag{5.6}
\]

In particular \( \sup_a G(a) < \infty \). Then Lemma 5.1 gives
\[
t^{\gamma} \int_{[0, +\infty)} e^{-tx} \, d\nu(x) = t^{\gamma + 1} \int_0^\infty e^{-tx} (x + 1)^{\gamma} G(x) \, dx = \int_0^\infty e^{-\gamma(y + t)^{\gamma}} G(y/t) \, dy. \tag{5.7}
\]

Since for any \( t \in (0, 1] \)
\[
e^{-\gamma(y + t)^{\gamma}} G(y/t) \leq e^{-\gamma(y + 1)^{\gamma}} \sup_a G(a) \in L^1([0, +\infty)), \tag{5.8}
\]
applying the dominated convergence theorem to (5.7) as \( t \downarrow 0 \) shows (5.3) because \( G(y/t) \to C \) as \( t \downarrow 0 \) by (5.6).
The one-sided versions (5.4), (5.5) follow by an analogous argument, using Fatou’s lemma and noticing that in the lim sup case the functions in (5.8) are dominated as $t \to 0^+$ by an integrable function.

Now we deal with the Tauberian case.

**Theorem 5.3** (Tauberian theorem) Assume that there exist $\gamma \in [0, +\infty)$ and $D \in [0, +\infty)$ such that

$$\lim_{t \to 0^+} t^\gamma \int_{[0, +\infty)} e^{-tx} \, d\nu(x) = D. \quad (5.9)$$

Then

$$\lim_{a \to +\infty} \frac{\nu([0, a))}{a^\gamma} = \frac{D}{\Gamma(\gamma + 1)}. \quad (5.10)$$

**Proof** If $\gamma = 0$, then applying the monotone convergence theorem to (5.9) shows (5.10), hence we can assume $\gamma > 0$. For any $t \in (0, 1]$ let $\nu_t, \mu$ be Borel measures on $[0, +\infty)$ be, respectively, defined by

$$\nu_t(A) := t^\gamma \nu([0, t^{-1}A]), \quad \mu(A) := \int_A x^{\gamma-1} \, dx \quad (5.11)$$

for any Borel subset $A$. Then (5.10) is equivalent to

$$\lim_{t \to 0^+} \nu_t([0, 1)) = \frac{D}{\Gamma(\gamma)} \mu([0, 1)) \quad (5.12)$$

because

$$\nu_t([0, 1)) = t^\gamma \nu([0, t^{-1}]) \quad \text{and} \quad \mu([0, 1)) = \int_0^1 x^{\gamma-1} \, dx = \frac{1}{\gamma} = \frac{\Gamma(\gamma)}{\Gamma(\gamma + 1)}. \quad (5.13)$$

In order to prove (5.12), we will show

$$\lim_{t \to 0^+} \int f(x) \, d\nu_t(x) = \frac{D}{\Gamma(\gamma)} \int f(x) \, d\mu(x) \quad (5.14)$$

for any $f \in C_c([0, +\infty))$ as follows.

Let $\hat{\nu}_t := e^{-x} \, d\nu_t(x)$ and $\hat{\mu} := e^{-x} \, d\mu(x)$ be the corresponding weighted measures on $[0, +\infty)$. Then (5.9) with Lemma 5.1 yields

$$\lim_{t \to 0^+} \int e^{-tx} \, d\nu_t(x) = \lim_{t \to 0^+} t^\gamma \int e^{-tx} \, d\nu_t(x) = \frac{D}{\Gamma(\gamma)} \hat{\mu}([0, +\infty)). \quad (5.15)$$

In particular

$$\sup_{t < 1} \hat{\nu}_t([0, +\infty)) < +\infty. \quad (5.16)$$

More strongly, (5.9) yields

$$\lim_{t \to 0^+} \int g(x) \, d\hat{\nu}_t(x) = \frac{D}{\Gamma(\gamma)} \int g(x) \, d\hat{\mu}(x) \quad (5.17)$$
for any polynomial \( g(x) \) in \( e^{-x} \) (i.e., \( g(x) = \sum_{i=1}^{N} a_i e^{-i x} \)). Because

\[
\lim_{t \to 0^+} \int e^{-kx} d\nu_t(x) = \lim_{t \downarrow 0} \int e^{-(k+1)x} d\nu_t(x)
= \lim_{t \to 0^+} \int e^{-(k+1)x} t^\prime d\nu_t(x)
= \frac{D}{(k+1)^\gamma} \int e^{-kx} d\mu(x).
\]

Let \( C_0([0, +\infty)) \) be the set of continuous functions \( f \) on \([0, +\infty) \) such that \( f(x) \to 0 \) as \( x \to +\infty \). Then since the set of polynomials in \( e^{-x} \) is dense in \( C_0([0, +\infty)) \) with respect to the norm \( \sup |f| \), applying the Stone–Weierstrass theorem to \( (C_0([0, +\infty)), \sup |f|) \) with (5.16) shows that (5.17) is satisfied for any \( g \in C_0([0, +\infty)) \), which implies (5.14).

We are now in a position to prove (5.12) by using (5.14). Indeed, it is well known that the weak convergence implies \( \nu_t(E) \to D\mu(E)/\Gamma(\gamma) \) for any compact set \( E \subset [0, +\infty) \) with \( \mu(\partial E) = 0 \). Choosing \( E = [0, 1] \) we obtain (5.14). \( \square \)

**Remark 5.4** The difficulty to obtain a one-sided version out of the previous proof, as we did for the Abelian case, can also be explained as follows: if we consider the push forward \( \sigma_t \) of the measures \( \nu_t \) under the map \( x \to e^{-x} \), the argument above shows that all moments of all weak limit points of \( \sigma_t \) are uniquely determined. Hence, since a finite Borel measure in \([0, 1]\) is uniquely determined by its moments, uniqueness follows. If we replace the assumption (5.9) by a bound on the \( \lim \inf \) or the \( \lim \sup \), we find only an inequality between the moments of the measures, which does not seem to imply, in general, the corresponding inequality for the measures.

**Proposition 5.5** Assume that for some \( \gamma \in [0, +\infty) \) one has

\[
\lim_{t \to 0^+} t^{\gamma} \int_{[0, +\infty)} e^{-st} d\nu(s) \leq C_0 < +\infty. \tag{5.18}
\]

Then

\[
\lim_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} \leq eC_0. \tag{5.19}
\]

**Proof** Note that for any \( \lambda > 0 \) and any \( t > 0 \)

\[
\nu([0, \lambda]) \leq e^{\lambda t} \int_{[0, \lambda]} e^{-st} d\nu(s) \leq e^{\lambda t} \int_{[0, +\infty)} e^{-st} d\nu(s). \tag{5.20}
\]

By (5.18), for any \( \epsilon > 0 \) there exists \( t_0 > 0 \) such that \( \int_{[0, +\infty)} e^{-st} d\nu(s) \leq (C_0 + \epsilon) t^{-\gamma} \) for any \( t < t_0 \). Thus (5.20) yields \( \nu([0, \lambda]) \leq e^{\lambda t}(C_0 + \epsilon) t^{-\gamma} \) for any \( \lambda > 0 \) and any \( t < t_0 \). Letting \( \lambda := t^{-1} \) and then letting \( t \downarrow 0 \) shows (5.19). \( \square \)

**Proposition 5.6** Assume that for some \( \gamma \in [0, +\infty) \) one has

\[
\lim_{t \to 0^+} t^{\gamma} \int_{[0, +\infty)} e^{-st} d\nu(s) > 0, \quad \lim_{t \to 0^+} t^{\gamma} \int_{[0, +\infty)} e^{-st} d\nu(s) < +\infty. \tag{5.21}
\]

Then

\[
\lim_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} > 0. \tag{5.22}
\]
Proof Call $C_0 > 0$ the lim inf and $C_1 < +\infty$ the lim sup in (5.21). Note that for any $\lambda > 0$ and any $t > 0$

$$
\int_{[0, +\infty)} e^{-st} d\nu(s) = \int_{[0, \lambda]} e^{-st} d\nu(s) + \sum_{\ell=1}^{\infty} \int_{(\ell\lambda, (\ell+1)\lambda]} e^{-st} d\nu(s)
$$

$$
\leq \nu([0, \lambda]) + \sum_{\ell=1}^{\infty} e^{-\ell t} \nu([0, (\ell+1)\lambda])
$$

$$
= \sum_{\ell=0}^{\infty} e^{-\ell t} \nu([0, (\ell+1)\lambda]).
$$

In particular, letting $\lambda := t^{-1}$ yields

$$
t^n \int_{[0, +\infty)} e^{-st} d\nu(s) \leq t^n \sum_{\ell=0}^{\infty} e^{-\ell t} \nu \left( \left[ 0, \frac{\ell+1}{t} \right] \right). \tag{5.23}
$$

Thus there exists $t_0 > 0$ such that for any $t < t_0$

$$
0 < \frac{C_0}{2} \leq t^n \sum_{\ell=0}^{\infty} e^{-\ell t} \nu \left( \left[ 0, \frac{\ell+1}{t} \right] \right). \tag{5.24}
$$

Next let us discuss the right-hand side of (5.24). By (5.21) and Proposition 5.5 there exists $\hat{\lambda} > 0$ such that $\nu([0, \lambda]) \leq (eC_1 + 1)\lambda^\gamma$ for any $\lambda \geq \hat{\lambda}$. Thus for any $t > 0$ with $t^{-1} \geq \hat{\lambda}$ we get

$$
\nu \left( \left[ 0, \frac{\ell+1}{t} \right] \right) \leq (eC_1 + 1) \frac{(\ell+1)^\gamma}{t^\gamma}.
$$

In particular

$$
t^n \sum_{\ell=k}^{\infty} e^{-\ell t} \nu \left( \left[ 0, \frac{\ell+1}{t} \right] \right) \leq (eC_1 + 1) \sum_{\ell=k}^{\infty} e^{-\ell t} (\ell+1)^\gamma \tag{5.25}
$$

for any $k \in \mathbb{N}$ and any $t > 0$ with $t^{-1} \geq \hat{\lambda}$.

For any $\delta > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sum_{\ell=k_0+1}^{\infty} e^{-\ell t} (\ell+1)^\gamma < \delta$. Then, combining (5.24) with (5.25) yields

$$
0 < \frac{C_0}{2} < t^n \sum_{\ell=0}^{k_0} e^{-\ell t} \nu \left( \left[ 0, \frac{\ell+1}{t} \right] \right) + (eC_1 + 1) \delta \tag{5.26}
$$

for any $t > 0$ with $t < t_0$ and $t^{-1} \geq \hat{\lambda}$, which easily shows (5.22) choosing $\delta > 0$ so small that $(eC_1 + 1)\delta < C_0/2$.

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