The consistency strength of projective uniformization, revisited

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Consider the following assumptions, whose conjunction we denote by $(RP)$:

(1) every projective set of reals is Lebesgue measurable and has the property of Baire, and

(2) every projective subset of the plane has a projective uniformization.

Woodin had asked, in [4], whether $(RP)$ implies Projective Determinacy. This is not the case, by a recent observation of Steel:

Theorem 0.1 (Woodin, Steel) Suppose $V = K$, where $K$ is Steel’s core model. If there are $\kappa_0 < \kappa_1 < \ldots$ with supremum $\lambda$ such that for all $n < \omega \kappa_n$ is $< \lambda^+$ strong [i.e., for all $x \in H_{\lambda^+}$ there is $\pi: V \to M$ with critical point $\kappa_n$ and $M$ transitive such that $x \in M$] then $(RP)$ holds in a generic extension.

We here show that this is best possible:

Theorem 0.2 If $ZFC + (RP)$ holds and Steel’s $K$ exists then $J_{\omega_1}^K \models$ there are infinitely many strong cardinals.

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Proof. Suppose not. Let $n < \omega$ be the number of strongs in $J_{\omega_1}^K$. We work towards a contradiction.

Case 1. $\omega_1$ is a successor in $K$.

Then Corollary 2.2 of [2] gives that $J_{\omega_1}^K$ is (boldface) $\Delta^1_{n+4}$ (in the codes). But then we get a projective sequence of distinct reals of length $\omega_1$, contradicting [4].

Case 2. $\omega_1$ is inaccessible in $K$.

Let $\Phi_m(M)$ denote the following statement, for $m \geq n$:

$M$ is a countable $m$-full mouse, $M \models$ there are $\leq m$ many strongs, and for all countable $m$-full $N$, if $M$, $N$ simply coiterate to $M^*$, $N^*$ with iteration maps $i : M \to M^*$ and $j : N \to N^*$ such that $M^*$ is an initial segment of $N^*$ then $i''M \subset j''N$.

The concept of $m$-fullness was defined in [2] where we showed that $\Phi_m(J_{\kappa}^K)$ holds for all $\kappa \leq$ the $(m+1)^{st}$ strong cardinal of $J_{\omega_1}^K$ which is either a double successor or an inaccessible in $K$.

It is also shown in [2] that if $\omega_1$ is inaccessible in $K$ and there are $\leq m$ strong cardinals in $J_{\omega_1}^K$ then $\Phi_m(M)$ characterizes (in a $\Pi^1_{m+4}$ way) (cofinally many of) the proper initial segments of $J_{\omega_1}^K$. (Cf. [2] Theorem 2.1. This gives a (lightface) $\Delta^1_{m+5}$ definition of $J_{\omega_1}^K$.)

In particular, for all $m \geq n$ the following holds, abbreviated by $\Psi^m_n$:

For any two $M$, $M'$, if $\Phi_m(M)$ and $\Phi_m(M')$ both hold then $M$ and $M'$ are lined up and if $\tilde{M}$ is the "union" of all $M'$s with $\Phi_m(M)$ then $On \cap \tilde{M} = \omega_1$ and $\tilde{M} \models$ there are exactly $n$ strong cardinals.

Notice that $\Psi^m_n$ is $\Pi^1_{m+5}$.

By [3], there is a model $P = L_{\omega_1}[X] \models ZFC$ for some $X \subset \omega_1$, such that

(a) $P[g]$ is $\Sigma^1_{n+1000}$ correct in $P[g][h]$ whenever $g$ is set-generic over $P$ and $h$ is set-generic over $P[g]$, and

(b) $P$ is $\Sigma^1_{n+1000}$ correct in $V$. 

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Now $P \models \Psi_n^{n+94}$ by (b) and the fact that $\Psi_n^{n+94}$ holds in $V$. Moreover, $P$ is closed under the dagger operator by (a), so Steel’s $K$ exists in $P$, denoted by $K^P$, and $K^P \models$ there are $> n$ strong cardinals, by [1] and (a). We may pick $\mathcal{g} \mathcal{C}ol(\mu, \omega)$-generic over $P$ for some appropriate $\mu$ such that in $P[g]$, $J_{\omega_1}^K \models$ there are $> n$ strongs. By (a), $\Psi_n^{n+94}$ still holds in $P[g]$.

From this we now derive a contradiction, by working in $P[g]$ for the rest of this proof. So let us assume that (in $V$) $\Psi_n^{n+94}$ holds, Steel’s $K$ exists, and $J_{\omega_1}^K \models$ there are $> n$ strongs.

By $\Psi_n^{n+96}$, there is a J-model $\tilde{M}$ of height $\omega_1$ such that $\tilde{M} \models ZF^- + \text{there are exactly } n \text{ strong cardinals, and } \Phi_{n+96}(M) \text{ holds for every proper initial segment } M \text{ of } \tilde{M}$. By [2], there is a universal weasel $W$ end-extending $\tilde{M}$ such that for all countable (in $V$) $\kappa$ which are cardinals in $W$ and such that $J_{\kappa}^{\tilde{M}} \models$ there are $< n + 94$ strong cardinals, $W$ has the definability property at $\kappa$. [This follows from the fact that cofinally many proper initial segments of $\tilde{M}$ are $n + 94$ full].

Because $W$ is universal, there is some $\sigma: K \to W$ given by the coiteration of $K$ with $W$. Let $\kappa$ denote the $(n+1)^{st}$ strong cardinal of $J_{\omega_1}^K$. By a remark above, $J_{\kappa}^K$ is an initial segment of $W$. But this implies that the critical point of $\sigma$ is $> \kappa$. [This follows from the fact that if $\mu$ is strong in $J_{\kappa}^K$, or $\mu = \kappa$, then $K$ as well as $W$ has the definability property at $\mu$.] But now, using $\sigma$, $\tilde{M} = J_{\omega_1}^W \models$ there are at least $n + 1$ many strong cardinals. Contradiction!

□

References

[1] Hauser, Kai, The consistency strength of projective absoluteness.
[2] Schindler, Ralf, The projectiveness of $K \cap HC$, handwritten notes.
[3] Steel, John, The core model iterability problem.
[4] Woodin, Hugh, The consistency strength of projective uniformization.