Agent swarms: cooperation and coordination under stringent communications constraint

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Appendix: The continuum model steady state

We can normalise the dynamical equation by scaling with respect to $\gamma$, so that with $s = \gamma t$, we have a dynamical equation for any accuracy, which is

$$\frac{d}{ds} \Phi_a^b = - (\Phi_a^b - \Phi_m) + \frac{\alpha_a^b L_{ba} \Phi_b^b}{\gamma} (1 - \Phi_a^b) \tag{20}$$

and

$$= - (\Phi_a^b - \Phi_m) + \frac{1}{R_{ab}} \Phi_b^b (1 - \Phi_a^b), \tag{21}$$

where $R_{ab} = \gamma/\alpha_a^b L_{ba}$.

Thus the steady state un-normalised (& normalised) accuracy $\Phi_a^b$ is

$$\Phi_a^b = \frac{\gamma \Phi_m + \alpha_a^b L_{ba} \Phi_a^b}{\gamma + \alpha_a^b L_{ba} \Phi_a^b} = \frac{\Phi_m + R^{-1}_{ab} \Phi_a^b}{1 + R^{-1}_{ab} \Phi_a^b}, \tag{22}$$

and we can substitute either of these expressions into itself (with with reversed $a, b$ indices), to get a polynomial for $\Phi_a^b$.

We create a forward loss rate $r = \gamma/\alpha^b_a L_{ba}$ and a backward rate $r' = \gamma/\alpha^a_b L_{ab}$, so that we have

$$\Phi_a^b = \frac{r \Phi_m + \Phi_a^b}{r + \Phi_a^b} \tag{23}$$

and

$$= \frac{r \Phi_m + \frac{r' \Phi_m + \Phi_a^b}{r + \Phi_a^b}}{r + \frac{r' \Phi_m + \Phi_a^b}{r + \Phi_a^b}} \tag{24}$$

$$\Phi_a^b \left[ r \left[ r' + \Phi_a^b \right] + \left[ r' \Phi_m + \Phi_a^b \right] \right] = r \Phi_m \left[ r' + \Phi_a^b \right] + \left[ r' \Phi_m + \Phi_a^b \right] \tag{25}$$

$$r' \left[ r' + \Phi_m \right] \Phi_a^b + \left[ r + 1 \right] \left[ \Phi_a^b \right]^2 = r' \Phi_m \left[ r + 1 \right] + \left[ r \Phi_m + 1 \right] \Phi_a^b \tag{26}$$

$$\left[ r' + r \Phi_m \right] \Phi_a^b + \left[ r + 1 \right] \left[ \Phi_a^b \right]^2 = r' \Phi_m \left[ r + 1 \right] + \Phi_a^b. \tag{27}$$

Hence

$$\left[ r + 1 \right] \left[ \Phi_a^b \right]^2 + \left[ r' r + \left( r' - r \right) \Phi_m \right] - \left[ \Phi_a^b \right] - r' \Phi_m \left[ r + 1 \right] = 0 \tag{28}$$

And if $r = r'$, which would be reasonable for a symmetric environment $L_{ab} = L_{ba}$, where both agent $a$ and $b$ were transmitting at the same default rate, we have the simpler form

$$\left[ \Phi_a^b \right]^2 + \left[ r' - 1 \right] \Phi_a^b - r \Phi_m = 0, \tag{29}$$

an expression which has just two free parameters, the rate ratio $r$ and the minimum information $\Phi_m$. 

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Thus

\[ \Phi^a_b = \frac{1 - r}{2} \pm \frac{1}{2} \sqrt{r^2 - 2r + 1 - 4(-r)\Phi_m}, \]  

(30)

\[ = \frac{1}{2} - \frac{r}{2} \pm \frac{1}{2} \sqrt{r^2 - 2r (1 - 2\Phi_m) + 1}. \]  

(31)

If \( \Phi_m = 0 \), then

\[ \Phi^a_b = \frac{1 - r}{2} \pm \frac{1}{2} (r - 1), \]  

(32)

which has two solutions; firstly the zero accuracy case \( \Phi^a_b = 0 \), and secondly the finite-accuracy case \( \Phi^a_b = 1 - r \). However, if \( \Phi_m > 0 \), the “zero accuracy” solution (sign choice “+”) is pushed negative so that only the finite-accuracy one remains.