Numerical Integration over Ellipsoid by transforming into 10-noded Tetrahedral Elements

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Abstract. In this paper we try to obtain the numerical integration formulas to evaluate volume integrals over an ellipsoid by transforming it into a 10-noded standard tetrahedral element. We first transform the ellipsoid to a sphere of radius one. A sphere of radius one in the first octant is divided into six tetrahedral elements (with three straight edges and three curved edges) by choosing a point P on the surface of the sphere. Later we consider each curved tetrahedral element to be 10-noded elements and transform them to standard tetrahedral elements (10-noded) with straight edges. Then we evaluate numerical integral values of some integrands by applying these transformations over the ellipsoid using MATHEMATICA-software. The performance of the proposed method with that of the generated meshes over ellipsoid is analyzed using some example problems. We observe that the ellipsoid has been discretized into 48 standard 10-noded tetrahedral elements and the results are converging to the exact integral values with minimum computational time.

1. Introduction

Many applications in science and engineering require the solution of 3D integrals. In physics, triple integral arises in the computation of mass, volume, moment of inertia and force on a three dimensional object. The calculation of the volume of ellipsoid and evaluation of some integrands over ellipsoid surface is a very important problem in geodesics, cartography, shell structures, electromagnetic theory and many engineering problems. Higher order finite elements ‘as in [22]’, which are well known for the faster rates of convergence in terms of computational efficiency, can provide an effective approach to perform large-scale simulations. When applying higher order finite element method to 3D curved domains, the elements must be properly curved to maintain the rate of convergence. Curved elements have been widely used in the context of FE models to obtain the advantages of using higher-order approximations and lack of high-order curved mesh generators for complex geometries is recognized to be an obstacle preventing the widespread application of high-order methods. The finite element method(FEM) on tetrahedron have wide range of applications in numerical integration, in the field of computer graphics for color maps, sampling pixel images, flow through porous media and generating

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points on surfaces. Linear tetrahedral elements admit only homogeneous deformation and well known for their stiff behavior and volumetric locking, hence, 10-noded tetrahedral elements are of great importance. Also straight-sided(straight edges, flat faces) element is characterized by constant metric namely Jacobian determinant of global- local coordinate is independent of natural coordinates, hence analytical integration result is simple, easy to implement expressions. For a curved-sided element, Jacobian matrix is linear and the metric is cubic w.r.t coordinates. Analytical integration result in exact mass matrix, which is used later in numerical study as reference values for error computation for approximate schemes.

In this paper we consider an ellipsoid, transform it into a sphere of radius one. We mesh the first octant of a sphere with six 10-noded tetrahedral elements with (three-straight edges and three- curved edges). We consider each tetrahedral element transform into a standard tetrahedral element with straight edges and then perform numerical integration over the ellipsoid with these meshed regions, to check for the convergence rate and computational time involved in these transformations.

2. Volume Integration over an ellipsoid by transforming into a standard arbitrary tetrahedron

An exact, closed-form expression for the surface area of an arbitrary ellipsoid ‘as in [10]’ has been known (at least) since the time of Legendre (1752-1833). Legendre’s expression for the surface area of an arbitrary ellipsoid was presented below from *Exercices de calcul integral ‘as in [10]’ namely,

$$S(a,b,c) = 2\pi c^2 + \frac{2\pi ab}{\sin \nu} \left[ \frac{c^2}{a^2} F(\nu, b') + \frac{a^2 - c^2}{a^2} E(\nu, b') \right]$$

Here $S(a,b,c)$ represents the total surface area of an ellipsoid with ordered variables $(a, b, c)$ such that $a > b > c > 0$, $\cos \nu = \frac{c}{a}$, $b'^2 = \frac{b^2 - c^2}{b^2 \sin^2 \nu}$, and $F(b, \nu')$, $E(b, \nu')$ are incomplete elliptic integrals of the first and second kind, respectively, with argument $\nu$ and modulus $b'$. Note that Richard has reversed the order of argument and modulus for the incomplete elliptic integrals. In Integrals and Series - 2, Legendre gave expressions for the surface area of an oblate spheroid ‘as in equation (1)’,

$$S(a,a,c) = 2\pi a^2 + \frac{\pi c^2}{\sin \nu} \ln \left[ \frac{1 + \sin \nu}{1 - \sin \nu} \right], \text{where } a = b > c > 0,$$

and for a prolate spheroid ‘as in equation (2)’,

$$S(a,a,c) = \frac{2\pi ab}{\sin \nu} \left[ \nu + \sin \nu \cos \nu \right], \text{where } a > b = c > 0.$$
references `as in [7-15]`, together with some additional comments intended to clarify the literature. It is important to note `as in [22]` that the expression for the surface area of an arbitrary ellipsoid has not always been published, nor clearly represented when published, even in very recent standard references. Of the many references that provide an exact, closed-form expression for the surface area of an arbitrary ellipsoid, researchers are most closely following the work of Bowman `as in [15]`.

George Coman and Teodora Catinas, `as in [21]`, have constructed Lagrange, Hermite and Birkhoff-type operators, which interpolate a given function and some of its derivatives on the border of a tetrahedron with three straight edges and three curved edges; they consider some of their product and boolean sum operators and study the interpolation properties, the order of accuracy of the constructed operators. Finally the authors give some applications and numerical examples.

In references, `as in [2, 19-20]`, Rathod et al. have evaluated the numerical integration of arbitrary functions applying Gauss Legendre quadrature rules, over the $p^3$ tetrahedral regions using one of the parametric transformations to transform a standard tetrahedron into a standard $1$-cube, by discretization of standard tetrahedral region $V_i$ into $p^3$ tetrahedral regions each of which has volume equal to $\frac{1}{6p^3}$.

Mamatha T M and Venkatesh B, `as in [18]`, have evaluated the numerical integration of arbitrary functions over a standard tetrahedral element $T(0,1)$ by decomposing into four hexahedral elements $H(-1,1)$ using Gauss quadrature rules. With minimum number of divisions of the tetrahedral region they have shown the convergence of integral values to exact solutions, and also the number of computations and errors are reduced drastically.

The problem now is to evaluate the volume integral of arbitrary integrands over an ellipsoid by discretizing it with curved tetrahedral elements `as in equation (3)`,

$$ I = \iiint_E f(x, y, z)dzdydx $$

(3)

2.1 Discretization of an Ellipsoid using arbitrary 10-noded Tetrahedron

Linear tetrahedral elements admit only homogeneous deformation and well known for their stiff behavior and volumetric locking, hence, 10-noded tetrahedral elements are of great importance. Also straight-sided(straight edges, flat faces) element is characterized by constant metric namely Jacobian determinant of global- local coordinate is independent of natural coordinates, hence analytical integration result is simple, easy to implement expressions. For a curved-sided element, Jacobian matrix is linear and the metric is cubic w.r.t coordinates. Analytical integration result in exact mass matrix, which is used later in numerical study as reference values for error computation for approximate schemes.

2.1.1 Transformation of an Ellipsoid into a unit Sphere

In this paper, we consider the ellipsoid `as in equation (4)`, transform into a sphere using the transformation `as in equation (5)`.

$$ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, $$

(4)
Transformation-I:

\[
X = \frac{x}{a}; \quad Y = \frac{y}{b}; \quad Z = \frac{z}{c}
\]

where the Jacobian of the transformation are given ‘as in equation (6)’

\[
|J_1| = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{vmatrix}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z}
\end{vmatrix} = abc.
\]

The volume integral over an ellipsoid, ‘as in equation (3)’ is transformed into an integral over a unit sphere ‘as in equation (7)’,

\[
I = \iiint f(x, y, z)dzdydx = \iiint f(aX, bY, cZ)J_1dXdYdZ = abc\iiint g(X, Y, Z)dXdYdZ
\]

2.1.2 Discretization of a Unit Sphere into standard 10-noded tetrahedral elements (with curved edges)

The volume of the unit sphere in the first octant is meshed into 6-sub volumes (elements) that are curved tetrahedral elements (with three curved edges and three straight edges) by considering a point P on the surface of the sphere. Each curved tetrahedral elements are disoriented, so we try to bring each element to the standard orientation by using a transformation and obtain a common Jacobian for all the six curved tetrahedral elements using the transformation ‘as in equation (8)’.

\[
|J_2| = \frac{\partial(X, Y, Z)}{\partial(X^*, Y^*, Z^*)} = \begin{vmatrix}
\frac{\partial X}{\partial X^*} & \frac{\partial X}{\partial Y^*} & \frac{\partial X}{\partial Z^*} \\
\frac{\partial Y}{\partial X^*} & \frac{\partial Y}{\partial Y^*} & \frac{\partial Y}{\partial Z^*} \\
\frac{\partial Z}{\partial X^*} & \frac{\partial Z}{\partial Y^*} & \frac{\partial Z}{\partial Z^*}
\end{vmatrix} = \frac{1}{\sqrt{6}}.
\]

Thus we transform the volume integral over a sphere into the volume integral over standard tetrahedral elements with curved edges ‘as in equation(9)’.

\[
I = abc\iiint g(X, Y, Z)dXdYdZ = abc\sum_{V_{int}}^{6}\iiint h(X^*, Y^*, Z^*)|J_2|dX^*dY^*dZ^*
\]

\[
= \frac{abc}{\sqrt{6}}\sum_{V_{int}}^{6}\iiint h(X^*, Y^*, Z^*)dX^*dY^*dZ^*
\]

2.1.3 Transformation of arbitrary 10-noded tetrahedral elements (with curved edges) into standard 10-node tetrahedral elements with straight edges

Later we consider each sub-volume (element) as a 10-noded standard tetrahedral element with three curved edges and three straight edges; transform it into 10-noded standard tetrahedral element with straight edges by using the following transformation ‘as in equation (10)’ and obtained a common Jacobian for all the six elements ‘as in equation (11)’.

Transformation-III

\[ X^* = \xi + 2[2 X^*_s - 1] \eta + 2[2 X^*_{10} - 1] \zeta \]

\[ Y^* = \eta + 2[2 Y^*_s - 1] \xi + 2[2 Y^*_{10} - 1] \zeta \]

\[ Z^* = \zeta + 2[2 Z^*_s - 1] \eta + 2[2 Z^*_{10} - 1] \xi \]

\[ |J_s| = \left| \begin{matrix} \frac{\partial X^*}{\partial \xi} & \frac{\partial X^*}{\partial \eta} & \frac{\partial X^*}{\partial \zeta} \\ \frac{\partial Y^*}{\partial \xi} & \frac{\partial Y^*}{\partial \eta} & \frac{\partial Y^*}{\partial \zeta} \\ \frac{\partial Z^*}{\partial \xi} & \frac{\partial Z^*}{\partial \eta} & \frac{\partial Z^*}{\partial \zeta} \end{matrix} \right| \tag{10} \]

\[ I = abc \iiint_T g(X,Y,Z) dX dY dZ = \frac{abc}{\sqrt{6}} \iiint_T h(X^*,Y^*,Z^*) dX^* dY^* dZ^* \]

\[ = \frac{abc}{\sqrt{6}} \sum_{i=0}^{6} \int_{1-\xi-\eta}^{1-\xi-\eta} \int_{0}^{1} \int_{0}^{1} F(\xi,\eta,\zeta) |J_s| d\xi d\eta d\zeta \tag{12} \]

The effectiveness of the transformations of the proposed method ‘as in equation (12)’ is tested by considering different types of integrands as typical examples. The integrands are selected based on availability of analytical solutions and some integrands are chosen such that their analytical integration is not possible due to singularity.

3. Numerical integration

We now evaluate the following integrals over an arbitrary Ellipsoid by transforming into the summation of volume integrals over a standard tetrahedral element (10-noded), applying the transformations in ‘as in equation (12)’. We use MATHEMATICA software (version-7) ‘as in [16]’ to evaluate the integrals, whose results are given in ‘Table 1’. We can observe that the computed results are closer to the exact integral values with minimum relative errors. We have also evaluated the numerical solution to the integrals with the singular functions.

The integral of the integrand \[ f(x, y, z) = 1 - x - y - z \] is taken form reference ‘as in [21]’ (with reference to ellipsoid instead of a sphere) and evaluated over an arbitrary Ellipsoid by transforming into the summation of volume integrals over a standard tetrahedral element (10-noded), applying the transformations. The computed results are very close to the exact values with relative error (percentage error) equal to 0.538414.
Table 1. Numerical Integration values over Ellipsoid ($a=2$, $b=3$, $c=1$).

| Sl. No. | Functions | Exact Value     | 10Noded Tetrahedral elements (Computed Value) | Timing (Seconds) |
|---------|-----------|-----------------|---------------------------------------------|------------------|
| 1       | 1         | 3.141592653589790 | 3.121590000000000 | 4.56500         |
| 2       | Xyz       | 0.750000000000000 | 0.752748000000000 | 28.57900        |
| 3       | x^2 y z   | 0.6857142857142857 | 0.691402000000000 | 12.35400        |
| 4       | Sqrt(x+y+z) | NInt=4.63966000000 | 4.610840000000000 | 0.09300         |
| 5       | 1/Sqrt(x+y+z) | NInt=2.21468000000 | 2.200840000000000 | 0.45200         |
| 6       | x^2 y^2   | 3.231350000000000 | 3.277390000000000 | 17.16100        |
| 7       | (1+x+y+z)^-4 | 0.058077100000000 | 0.057852200000000 | 1.27900         |
| 8       | Sin(x+2y+4z) | Singularity and highly oscillatory function | -0.892748000000000 | 1.91900         |
| 9       | (1 + x + y^2 + (x^2*y) + x^4 + y^5 + (x^2*y^2+z^2)) | 65.889300000000000 | 65.904400000000000 | 34.92800        |
| 10      | 1-x-y-z   | 3.9269908169872414 | -3.905847367294324 | 0.217           |

4. Conclusions

In this paper we obtain the numerical integration formulas over an ellipsoid by transforming into a 10-noded standard tetrahedral element. We first transform the ellipsoid to a sphere of radius one. A sphere of radius one in the first octant is divided into six tetrahedral elements with three straight edges and three curved edges by choosing a point P on the surface of the sphere. We consider each curved tetrahedral element to be 10-noded elements and transform them to standard tetrahedral elements (10-noded) with straight edges. Then we evaluate numerical integral values of some integrands by applying these transformations over the ellipsoid using MATHEMATICA-software. The performance of the proposed method, with the quality of the generated meshes over ellipsoid is analyzed using some example problems. We observe that the ellipsoid has been discretized into 48 standard 10-noded tetrahedral elements and the results are converging to the exact integral values with minimum computational time.
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