Coordinate noncommutativity in strong non-uniform magnetic fields

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Noncommuting spatial coordinates are studied in connection with the motion of a charged particle in a strong generic magnetic field. We derive a relation involving the commutators of the coordinates, which generalizes the one realized in a strong constant magnetic field. As an application, we discuss the coordinate noncommutativity in a slowly varying magnetic field.

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I. INTRODUCTION

Noncommutativity of space coordinates has been much studied from various points of view [1, 2, 3]. It arises naturally in string theory, where it is related to the presence of a strong background magnetic-like field. If this is constant, one obtains the more familiar case where the coordinate noncommutativity \([x^i, x^j]\) is a constant antisymmetric quantity. However, if the background field depends on the spatial coordinates, one would expect the coordinate noncommutativity to be a local function. This possibility has been recently studied in the context of noncommutative gauge field theories [4].

On the other hand, coordinate noncommutativity may also arise in more physical situations involving the motion of electric charges in a strong external magnetic field [5, 6]. When a charged \((e)\) and massive \((m)\) particle moves on a plane \((x, y)\) in the presence of a strong constant magnetic field \(B\) pointing along the \(z\)-axis, it has been shown that the noncommutativity of space coordinates is of order of the inverse of the magnetic field:

\[\lbrack x, y \rbrack = -\frac{i\hbar c}{eB}. \quad (1)\]

An interesting discussion of this behavior, which is related to the fact that the large \(B\) limit corresponds to small \(m\), has been recently given by Jackiw [7].

Motivated by the above observations, we study in this note the motion of a charged particle in a strong non-uniform magnetic field \(B(x)\). Then, we argue that the relation (1) can be generalized to the rotationally symmetric form:

\[\lbrack x^i, x^j \rbrack = -\frac{i\hbar c}{e} \epsilon^{ijk} \frac{B_k(x)}{B^2(x)}, \quad (i, j, k = 1, 2, 3) \quad (2)\]

which shows that the coordinate noncommutativity is in this case a local function.

This result for the coordinate noncommutativity in non-uniform magnetic fields is derived in section 2. As an application, we study in section 3 the behavior of noncommuting coordinates in a slowly varying magnetic field which is present, for example, in a magnetic mirror.

II. NONCOMMUTING COORDINATES

In order to derive the relation (2), we consider the equation of motion of a charged particle in a static external magnetic field:

\[m\ddot{x} = \frac{e}{c} \dot{x} \times B(x) + f(x), \quad (3)\]

where \(f(x)\) represents additional static forces which may be derived from a potential \(V\): \(f = -\nabla V\). In the presence of a strong magnetic field, the Lorentz force term can dominate the kinetic term \(m\ddot{x}\), which therefore may be dropped in first approximation. The resulting equation, however, cannot determine all components of the velocity \(\dot{x}\), since the projection of \(\dot{x}\) along \(B\) is not specified in (3). This is reflected in the equation:

\[(\dot{x} \times B)_k + \frac{c}{e} f_k = \epsilon_{kij} \dot{x}^i B^j + \frac{c}{e} f_k = 0, \quad (4)\]

in that the antisymmetric matrix \((\epsilon_k)_{ij}\) does not have an inverse. Multiplying (4) by \(B^k\), we obtain the consistency condition:

\[B^k f_k = 0. \quad (5)\]
This relation ensures that the net force in the direction of $\mathbf{B}$ vanishes, which represents a condition necessary to obtain, in the limit $m \to 0$, a consistent set of equations of motion. In fact, since the Lorentz force is orthogonal to the magnetic field, this condition allows us to set the projection of $m\ddot{x}$ along $\mathbf{B}$ equal to zero. The configuration described by equation (5) may be achieved provided the magnetic field is perpendicular to some two-dimensional manifold $\mathcal{M}$. Then, if we take the potential $V$ to be a function defined on $\mathcal{M}$, $\mathbf{f} = -\nabla V$ will be tangential to this manifold, so that the condition (5) can be satisfied.

One can see in a simple way that the form (2) for the coordinate noncommutativity is consistent with the equation of motion (4). To this end, let us consider the reduced Hamiltonian:

$$H_0 = V(x)$$

which is obtained in the limit $m \to 0$, by setting the kinematical momentum $m\dot{x}$ equal to zero. Then, taking the Poisson bracket of $x^i$ with $H_0$ and using the relation:

$$\dot{x}^i = \{x^i, H_0\} = f_j \{x^j, x^i\}$$

one can verify that the equation of motion (4) is satisfied when the brackets which describe noncommuting coordinates are given by the relation (2).

We shall now give a canonical derivation of noncommutativity in the limit $m \to 0$, which is based on the Hamiltonian:

$$H = \frac{\pi^2}{2m} + V(x) = \frac{1}{2m}(p - \frac{e}{c}A)^2 + V(x)$$

where $\pi$ is the kinematical momentum, $p$ is the canonical momentum and $A$ denotes the vector potential in the Maxwell theory. In order to be able to set $m = 0$ in (5), we must impose $\pi = 0$ as a constraint. This can be implemented using Dirac’s method for dealing with constrained systems (for an alternative approach, see reference [10]). Using this method, we consider the constraints:

$$\pi^i = p^i - \frac{e}{c}A^i \approx 0 \quad (i = 1, 2, 3)$$

and evaluate their time evolution using the relation:

$$\dot{\pi}^i = \{\pi^i, H + \lambda_j \pi^j\} = \{\pi^i, V + \lambda_j \pi^j\} = 0$$

where $\lambda_j$ represent the Lagrange multipliers in the constrained theory. Using the canonical Poisson brackets, together with the relation:

$$\{\pi^i, \pi^j\} = \frac{e}{c}(\partial^i A^j - \partial^j A^i) = \frac{e}{c}\epsilon^{ijk} B_k,$$

we obtain from (10) the following set of equations involving these multipliers:

$$\epsilon^{ijk} \lambda^j B^i - \frac{e}{c} \frac{\partial V}{\partial x^k} = (\lambda \times B)_k + \frac{e}{c} f_k = 0.$$  

(12)

This has the same structure as the one of the equation (11), so that we may apply similar considerations as before. Namely, although this system leads to a consistent relation among the Lagrangian multipliers which implies the condition (5), it cannot determine all the $\lambda^i$ since the projection of $\lambda$ along $\mathbf{B}$ is not specified. One can check this in more detail by writing $\lambda$ in terms of a linear combination, with arbitrary coefficients, of the orthogonal vectors $\mathbf{B}, \mathbf{f}$ and $\mathbf{B} \times \mathbf{f}$. Then, from equation (12) it follows that $\lambda$ must actually have the form:

$$\lambda = \alpha \mathbf{B} - \frac{e}{c} \frac{\mathbf{B} \times \mathbf{f}}{B^2},$$

(13)

so that the coefficient $\alpha$ remains undetermined. Using this result, the total Hamiltonian in equation (10) can be written in the form:

$$H_t = V + \alpha \mathbf{B} \cdot \pi + \frac{e}{c} \frac{(\mathbf{f} \times \mathbf{B})}{B^2} \cdot \pi.$$  

(14)

We note here that $e(f \times B)/eB^2$ represents the drift velocity of the particle due to the force $f$. A well-known example of the particle drift is the $E \times B$ drift which arises in a static electric field $E$. 


Since the coefficient $\alpha$ in the Hamiltonian (14) is arbitrary, one may expect that:

$$\phi = B \cdot \pi$$

would be a first class constraint [8], which commutes with all constraints $\pi^i$. This is indeed the case, as one can easily check with the help of equation (11).

Consequently, out of the three constraints $\pi^i$, we will be left over with just two second-class constraints, which do not commute. We may take these to be given by the following linear combinations of the $\pi^i$:

$$\chi^1 = f \cdot \pi; \quad \chi^2 = (B \times f) \cdot \pi$$

(16)

There is no loss of generality by this choice, since the vectors $B$, $f$ and $B \times f$ are linearly independent.

To proceed with the canonical formalism, we now introduce the Dirac brackets:

$$\{x^i, x^j\}_D = \{x^i, x^j\} - \{x^i, \chi^k\} C_{kl} \{\chi^l, x^j\}$$

(17)

where the matrix $C_{kl}$ is defined by:

$$C_{kl} \{\chi^l, \chi^i\} = \delta^{ik}.$$ (18)

From equations (11) and (16) one can check, using the canonical Poisson brackets, that:

$$\{x^i, \chi^1\} = f^i; \quad \{x^i, \chi^2\} = \epsilon^{ijk} B_j f_k; \quad \{\chi^1, \chi^2\} = \frac{e}{c} B^2 f^2.$$ (19)

Then, with the help of equations (18) and (19), one finds that the Dirac bracket (17) takes the form:

$$\{x^i, x^j\}_D = -\frac{e}{c} \epsilon^{ijk} B_k(x) \frac{\nabla \times (B B^2)}{B^2(x)}.$$ (20)

One may pass over to the quantum theory, by taking the commutation relations to correspond to $i\hbar$ times the Dirac bracket relations. Then, from (20), one can verify the result given in equation (2).

Examples of this type emerge on any 2D (co)adjoint orbit $M$ (see [11, 12, 13]), e.g. for a unit sphere $S^2$ with magnetic monopole in its centre. The monopole magnetic field $B = B x$, with $x^2 = 1$, gives the Dirac brackets (20) in the form $\{x^i, x^j\}_D = -\frac{e}{c} \epsilon^{ijk} x_k$. For discrete values of $B = \pm \frac{e}{c} \sqrt{s(s+1)}$, $s$ half-integer, the quantization leads to the well-known fuzzy sphere.

III. DISCUSSION

The solution (2), which is symmetric under rotations in three dimensions, describes noncommuting spatial coordinates in a generic magnetic field. For consistency, such a noncommutative algebra must satisfy the Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$ (21)

In order to show that this identity is satisfied, we note that the Jacobi identity requires the condition:

$$\epsilon_{kij} \{x^i, x^j\} \partial_k [x^l, x^k] = 0.$$ (22)

Then, using the expression (2), we may write this condition in the form:

$$B \cdot \nabla \times \left( \frac{B}{B^2} \right) = 0.$$ (23)

With the help of the Maxwell equation $\nabla \times B = 0$ for the static external magnetic field, we can see that the above equation is indeed satisfied.

As an application of the result (2), let us consider the case of a slowly varying magnetic field in the $z$-direction. Such a field occurs in a magnetic mirror [14] which confines the particle’s motion in the $z$-direction. It may be written in cylindrical coordinates in the form:

$$B = \frac{1}{r} \frac{\partial B_z(z)}{\partial z} \hat{\phi} + B_0(z) \hat{z}.$$ (24)
where $\rho B_z' << B_z$. Then, the solution (2) implies the following relations among the noncommuting coordinates:

$$[x, y] = -i\hbar \frac{c}{e B_z} B_z' ; \quad [y, z] = i\hbar \frac{c}{e B_z} \frac{x B_z'}{B_z^2} ; \quad [z, x] = i\hbar \frac{c}{2e B_z} y B_z' .$$  \hspace{1cm} (25)

We see that in this case the strongest coordinate noncommutativity occurs in the $(x, y)$ plane and that the noncommutativity in the $(x, z)$ and $(y, z)$ plane is weaker by a factor of order $\rho B_z'/B_z << 1$.

As is well known [15], in the presence of a constant magnetic field along the $z$-direction, the quantum energy levels of a charged particle are given by:

$$E_{n,l} = \frac{e B_z}{2mc} \hbar (2n + |l| - l + 1) + \frac{\hbar^2 k^2}{2m} ,$$  \hspace{1cm} (26)

where $n = 0, 1, 2...$ and $\hbar l$ gives the projection of the angular momentum on the $z$-axis. The first term in (26) is associated with the motion in the $(x, y)$ plane, and describes the Landau levels which are infinitely degenerate. The second term gives the translational energy of the particle associated with its motion in the $z$-direction.

One can show that the relation (26) may also provide a good approximation for the quantum energy levels of a charged particle in a magnetic mirror, where $B_z$ is a slowly varying function of $z$. In this case, one can see that as the particle drifts along the $z$-axis, there will occur a gradual shift of the Landau levels. This shift will be compensated by a corresponding change in the translational energy of the particle, so that its total energy remains conserved. We note that, since the separation between the Landau levels is given by $\hbar e B_z/mc$, in a strong magnetic field only the lowest Landau level is relevant. Furthermore, the large $B_z$ limit is asymptotically equivalent to the limit $m \to 0$.

Hence, we may interpret the coordinate noncommutativity (25) as arising in consequence of the fact that our system is constrained to lie in the lowest Landau level.

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