Constraining conformal field theories with a slightly broken higher spin symmetry

Juan Maldacena\(^1\) and Alexander Zhiboedov\(^2\)

\(^1\) School of Natural Sciences, Institute for Advanced Study, Princeton, NJ, USA
\(^2\) Department of Physics, Princeton University, Princeton, NJ, USA

E-mail: malda@ias.edu

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Abstract

We consider three-dimensional conformal field theories that have a higher spin symmetry that is slightly broken. The theories have a large-\(N\) limit in the sense that the operators separate into single trace and multitrace and obey the usual large-\(N\) factorization properties. We assume that the spectrum of single trace operators is similar to the one that one obtains in the Vasiliev theories. Namely the only single trace operators are the higher spin currents plus an additional scalar. The anomalous dimensions of the higher spin currents are of the order \(1/N\). Using the slightly broken higher spin symmetry, we constrain the three-point functions of the theories to a leading order in \(N\). We show that there are two families of solutions. One family can be realized as a theory of \(N\) fermions with an \(O(N)\) Chern–Simons gauge field, the other as an \(N\) bosons plus the Chern–Simons gauge field. The family of solutions is parametrized by the \('t\) Hooft coupling. At special parity preserving points, we obtain the critical \(O(N)\) models: the Wilson–Fisher one and the Gross–Neveu one. Our analysis also fixes the on-shell three-point functions of Vasiliev’s theory on AdS\(_4\) or dS\(_4\).

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1. Introduction

In this paper, we study a special class of three-dimensional conformal field theories that have a weakly broken higher spin symmetry. The theories have a structure similar to what we expect for the CFT dual to a weakly coupled four-dimensional higher spin gravity theory in AdS\(_4\) [1–9]. We compute the leading-order three-point functions of the higher spin operators. We use current algebra methods. Our only assumption is that the correlation functions defined on the boundary of AdS\(_4\) obey all the properties that a boundary CFT would obey. But we will not need any details regarding this theory other than some general features which follow from natural expectations for a weakly coupled bulk dual. This seems a reasonable assumption for Vasiliev’s theory since Vasiliev’s theory appears to be local on distances much larger than the
AdS radius. This would imply that the usual definition of boundary correlators is possible [9–11]. In order to apply our analysis to Vasiliev’s theory, we need to make the assumption that these boundary correlators can be defined and that they obey the general properties of a CFT. Thus, we are assuming AdS/CFT, but we are not specifying the precise definition of the boundary CFT. Since our assumptions are very general, they also apply to theories involving \( N \) scalar or fermion fields coupled to \( O(N) \) or \( SU(N) \) Chern–Simons gauge fields [12, 13]. So our methods are also useful for computing three-point functions in these theories as well.

Our assumptions are the following3. We have a CFT with a unique stress tensor and has a large parameter \( \tilde{N} \). In the Chern–Simons examples, \( \tilde{N} \sim N \). In the Vasiliev gravity theories, \( 1/\tilde{N} \sim \hbar \) sets the bulk coupling constant of the theory. We then assume that the spectrum of operators has the structure of an approximate Fock space, with single particle states and multiparticle states. The dimensions of the multiparticle states are given by the sum of the dimensions of their single particle constituents up to small \( 1/\tilde{N} \) corrections. This Fock space should not be confused with the Fock space of a free theory in three dimensions. We should think of this Fock space as the Fock space of the weakly coupled four-dimensional gravity theory. In order to avoid this confusion, we will call the single particle states ‘single trace’ and the multiparticle states ‘multiple trace’. In the Chern–Simons gauge theories, this is indeed the case. We also assume that the theory has the following spectrum of single trace states. It has a single spin-2 conserved current. In addition, it has a sequence of approximately conserved currents \( J_s \), with \( s = 4, 6, 8, \ldots \). These currents are approximately conserved, so that their twist differs from one by a small amount of order \( 1/\tilde{N} \):

\[
\tau_s = \Delta_s - s = 1 + O \left( \frac{1}{\tilde{N}} \right).
\]

In addition, we have one single trace scalar operator. All connected correlators of the single trace operators scale as \( \tilde{N} \). This includes the two-point function of the stress tensor. We also assume that the spectrum of single trace operators is such that the higher spin symmetry can be broken only by double trace operators via effects of order \( 1/\tilde{N} \). In particular, we assume that there are no twist-3 single trace operators in the theory.

With these assumptions, we will find that the three-point functions in these theories, to leading order in \( \tilde{N} \), are constrained to lie on a one-parameter family

\[
\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \tilde{N} \left[ \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{bos}} + \frac{1}{1 + \tilde{\lambda}^2} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{fer}} + \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}} \right]
\]

(1.2)

where the subindices ‘bos’ and ‘fer’ indicate the results in the theory of a single real boson or a single Majorana fermion. The subindex ‘odd’ denotes an odd structure which will be defined more clearly below. Here, \( \tilde{\lambda} \) is a parameter labeling the family of solutions of the current algebra constraints. More precise statements will be made below, including the precise normalization of the currents.

The class of theories for which our assumptions apply includes Vasiliev higher spin theories in AdS4 with higher spin symmetry broken by the boundary conditions [1]. It also applies for theories containing \( N \) fermions [12] or \( N \) bosons [13] interacting with an \( SO(N) \) or \( U(N) \) Chern–Simons gauge field. We call these theories quasi-fermion and quasi-boson theories, respectively. In such theories, \( 1/\tilde{N} \propto 1/N \) is the small parameter. In addition, \( \tilde{\lambda} \sim \lambda = N/k \) is an effective ‘t Hooft coupling in these theories. We emphasize that the analysis here is only based on the symmetries and it covers both types of theories, independent of any conjectured dualities between them. Of course, the results we obtain are consistent

3 These assumptions are not independent of each other, but we will not give the minimal set.
with the proposed dualities between these Chern–Simons theories and Vasiliev’s theories [7, 6, 9, 12, 13].

We can take the limit of large $\tilde{\lambda}$ in (1.2) and find that the correlators of the quasi-fermion theory go over to those of the critical $O(N)$ theory. We have a similar statement for the quasi-boson theory.

Our analysis is centered on studying the spin-4 single trace operator $J_4$. We write the most general form for its divergence, or lack of conservation. With our assumptions, this takes the schematic form

$$\partial \cdot J_4 = a_2 J^f + a_3 J^f J^{''},$$

where on the right-hand side, we have products of two or three single trace operators, together with derivatives sprinkled on the right-hand side. The coefficients $a_2$ and $a_3$ are small quantities of order $1/N$ and $1/N^2$, respectively. We will be able to use this approximate conservation law in the expression for the three-point function in order to obtain (1.2). Note that in the case that $J_4$ is exactly conserved, we simply have free-boson or free-fermion correlators [14].

We should emphasize that our discussion applies only to the special theories in [12, 13], but not to more general large-$N$ Chern–Simons matter theories. The special feature that we are using is the lack of single trace operators of twist 3. Such operators can appear in the divergence of the spin-4 current (1.3). This can give rise to an anomalous dimension for the higher spin currents already at the level of the classical theory (or large-$\tilde{N}$ approximation). In the theories in [12, 13], we do not have a single trace operator that can appear on the right-hand side of (1.3). In the language of the bulk theory, we have the pure higher spin theory without extra matter4. In particular, the bulk theory lacks the matter fields that could give a mass to the higher spin gauge fields via the Higgs mechanism already at the classical level. Thus, the Higgsing is occurring via quantum effects involving two (or three) particle states [15].

Our analysis can also be viewed as an on-shell analysis of the Vasiliev theory with AdS$_4$ asymptotic boundary conditions. If the higher spin symmetry is unbroken, then we can use [14] to compute all correlators, just from the symmetry. In this paper, we also use an on-shell analysis, but for the case that the higher spin symmetry is broken. As has often been emphasized, on-shell results in gauge theories can be simpler than what the fully covariant formalisms would suggest.

For a more general motivational introduction, see appendix G.

This paper is organized as follows.

In section 2, we discuss the most general form of the divergence of the spin-4 current.

In section 3, we present several facts about three-point functions which are necessary for the later analysis.

In section 4, we explain how one can use the slightly broken higher spin symmetry to fix the three-point functions.

In section 5, we present the results and explain their relation to known microscopic theories.

In section 6, we present conclusions and discussions.

Several appendices contain technical details used in the main body of this paper.

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4 By ‘matter’, we mean extra multiplets under the higher spin symmetry. The scalar field of the Vasiliev theory is a part of the pure higher spin theory.
2. Possible divergence of the spin-4 operator

2.1. Spectrum of the theory

We consider theories with a large-\(\tilde{N}\) expansion. We do not assume that \(\tilde{N}\) is an integer. We assume that the set of operators develops the structure of a Fock space for large \(\tilde{N}\). Namely we can talk about single particle operators and multiparticle operators. In the case of the Chern–Simons matter theories discussed in [12, 13], these correspond to single sum and multiple sum operators (sometimes called single trace or multitrace operators). The spectrum of single trace operators includes a conserved spin-2 current, the stress tensor \(J_{\mu\nu}\). We will often suppress the spacetime indices and denote operators with spin simply by \(J_s\).

We have approximately conserved single trace operators \(J_s\), with \(s = 4, 6, 8, \ldots\). These operators have twists \(\tau = 1 + O(1/\tilde{N})\). In addition, we have a single scalar operator. We will see that the dimension of this operator has to be either 1 or 2. We denote the first possibility as \(j_0\) and the second as \(\tilde{j}_0\). The theory that contains \(\tilde{j}_0\) is called the quasi-fermion theory. The theory with \(j_0\) is called the quasi-boson theory. The theory might also contain single particle operators with odd spins. For simplicity, let us assume that these are not present, but we will later allow their presence and explain that the correlators of even spin currents are unchanged.

An example of a theory that obeys these properties is a theory with \(N\) massless Majorana fermions interacting with an \(O(N)\) Chern–Simons gauge field at level \(k\) [12]. In this theory, the scalar is \(j_0 = \sum_i \psi_i \psi_i\), which has dimension 2, at leading order in \(N\) for any \(\lambda = N/k\). The name quasi-fermion was inspired by this theory since we start from fermions and the Chern–Simons interactions turn them into non-Abelian anyons which for large \(k\) are very close to ordinary fermions. Our discussion is valid for any theory whose single particle spectrum was described above. We are just calling ‘quasi-fermion’ the case where the spectrum includes a scalar with dimension 2 (\(\tilde{j}_0\)).

A second example is a theory with \(N\) massless real scalars, again interacting with an \(O(N)\) Chern–Simons gauge field at level \(k\) [13]. This theory also allows the presence of a \((\phi \bar{\phi})^3\) potential while preserving conformal symmetry, at least to leading order in \(N\). As higher orders in the \(1/N\) expansion are taken into account, this coefficient is fixed, if we want to preserve the conformal symmetry [13]. Here, we will only do computations to leading order in \(N\); thus, we have two parameters \(N/k\) and the coupling of the \((\phi \bar{\phi})^3\) potential. We call this case the quasi-boson theory. Again, we will not use any of the microscopic details of its definition. For us, the property that defines it is that the scalar has dimension 1 (\(j_0\)).

A third example is the critical \(O(N)\) theory (as well as interacting UV Gross–Neveu fixed point). Namely we can have \(N\) free scalars, perturbed by a potential of the form \((\phi \bar{\phi})^2\), which flows in the IR to a new conformal field theory (after adjusting the coefficient of the mass term to criticality). This is just the usual large-\(N\) limit of the Wilson–Fisher fixed point. This theory has no free parameters. Here, the scalar operator \(j_0 \sim \phi \bar{\phi}\) has dimension 2. It starts with dimension 1 in the UV but has dimension 2 in the IR. The UV theory has a higher spin symmetry. In the IR CFT, this symmetry is broken by \(1/N\) effects. This theory is in the family of what we are calling the quasi-fermion case.

A fourth example is the Vasiliev theory in AdS_4 (or dS_4) with general boundary conditions which would generically break the higher spin symmetry. Here, the bulk coupling is \(h \sim 1/\tilde{N}\). Depending on whether the scalar has dimensions 1 and 2, we would have a quasi-fermion or quasi-boson case.

We should emphasize that the theories we call quasi-fermion or quasi-boson are not specific microscopic theories. They are any theory that obeys our assumptions, where the scalar has dimension 2 or 1, respectively.
2.2. Divergence of the spin-4 current

Let us consider the spin-4 current \( J_4 \). We consider the divergence of this current. If it is zero, then we have a conserved higher spin current and all correlators of the currents are as in a theory of either free bosons or free fermions [14]. Here, we consider the case that this divergence is non-zero. Our assumptions are that the current is conserved in the large-\( \tilde{N} \) limit. This means that in this limit, \( J_4 \) belongs to a smaller multiplet than at finite \( \tilde{N} \). At finite \( \tilde{N} \), \( J_4 \) combines with another operator to form a full massive multiplet. More precisely, it combines with the operator that appears on the right-hand side of \( \nabla J_4 = \partial_\mu J_\mu^{(\eta v_8 v_9)} \). In other words, \( \nabla J_4 \) should be a conformal primary operator in the large-\( \tilde{N} \) limit [16, 15]; see appendix A. \( \nabla J_4 \) should be a twist-3, spin-3 primary operator. According to our assumptions, there are no single particle operators of this kind. Note that in general matter Chern–Simons theories, such as theories with adjoint fields, we can certainly have single trace operators with twist 3 and spin 3. So, in this respect, the theories we are considering are very special. In our case, we can only have two-particle or three-particle states with these quantum numbers.

Let us choose the metric
\[
\text{d}x^2 = \text{d}x^+ \text{d}x^- + \text{d}y^2
\]
and denote the indices of a vector by \( v_k, v_l \).

Let us see what is the most general expression we can write down for \( (\nabla J)_{-\cdots} \). Since the total twist is 3, and the total spin is 3, we can only make this operator out of the stress tensor \( J_2 \) and the scalar field. Any attempt to include a higher spin field would have to raise the twist by more than 3. A scalar field can only appear if its twist is 1 or 2. Note that we cannot have two stress tensors. The reason is that we cannot make a twist-3, spin-3 primary out of the stress \( \text{t} \)ensor.5 Naively, we could imagine an expression like \( J_2 \cdot J_2 \cdots \). But this cannot be promoted into a covariant structure, even if we use the \( \epsilon \) tensor5.

Let us consider first the case that the scalar has twist 2 (\( \tilde{j}_0 \)). The most general operator that we can write down is
\[
\partial_\mu J^\mu_{- \cdots} = a_5 (\partial_- \tilde{j}_0 \partial_- j_2 - \frac{2}{5} \tilde{j}_0 \partial_- j_2).
\]
(2.2)

Here, we are denoting by \( j_k = J_k \cdots \), the all-minus components of \( J_k \). In (2.2), we have used the fact that the right-hand side should be a conformal primary in order to fix the relative coefficient. If we started out from the free-fermion theory, then this structure would break parity since \( \tilde{j}_0 \) is parity odd. Then, we have \( a_5 \propto N/k \), at least for large \( k \). In general, \( \tilde{j}_0 \) does not have well-defined parity and the theory breaks parity. If we had the critical \( O(N) \) theory, then (2.2) is perfectly consistent with parity since in that case, \( \tilde{j}_0 \) is parity even.

Let us now consider the case where we have a scalar of twist 1 (\( j_0 \)). Now there are more conformal primaries that we can write down
\[
\partial_\mu j^\mu_{- \cdots} = a_2 \epsilon_{\mu \nu} [8 \partial^\mu \tilde{j}_0 \partial^\nu j_- - 6 \partial^\mu j_0 \partial_- j_- - 5 \partial_- \partial^\mu j_0 j_- - j_0 \partial_- \partial^\mu j_-] + a_3 [\tilde{j}_0 j_0 \partial^3 j_0 - 9 j_0 \partial_\nu j_0 \partial^2 j_0 + 12 \partial_\nu j_0 \partial j_0 \partial j_0] + a_4 [\partial_\nu j_0 j_0 j_0 - 5 j_2 \partial j_0 j_0].
\]
(2.3)

We have only written combinations that are conformal primaries. We have also denoted \( \partial \equiv \partial_- \). The analysis of the broken charge conservation identities will relate \( a_3 \) and \( a_4 \), leaving us with only two parameters (besides \( \tilde{N} \)). This agrees with the two parameters in the large-\( \tilde{N} \) limit in the boson plus Chern–Simons theories of [13].

Here, we have concentrated on the case of \( J_4 \). Let us briefly discuss the situation for higher spin currents. We focus, as usual, on the all-minus component of the current \( \partial_\mu J^\mu_{- \cdots \cdots} \). This

5 We could have promoted it if we had two different spin-2 currents: \( \epsilon_{\mu \nu} j^\mu_{-} j^\nu_{-} \).
operator has twist $\tau = \Delta - S = 3$ and spin $s = 1$. Let us examine possible double particle operators that can appear. The minimum twist of a double trace operator is $2 = 1 + 1$. We should make up the twist by considering other components, or derivatives other than $\partial_\nu$, which has twist 0. All of these should arise from a rotationally invariant structure involving the flat space metric or the $\epsilon$ tensor. The only structure that can raise the twist by 1 is $\epsilon_{\mu\nu\rho}$. For the quasi-fermion theory, we can also use the scalar operator of twist 2, $\hat{j}_0$, and one of the twist-1 currents.

Matching the scaling dimensions in $\partial_\mu J_0^0 \propto J_{s_1} J_{s_2}$ (with derivatives sprinkled on the right-hand side) with all-minus indices leads to

$$ s + 2 = (s_1 + 1) + (s_2 + 1) + n_0, $$

where $n_0 \geq 0$ is the number of derivatives which raise the dimension. Thus, we obtain an inequality

$$ s \geq s_1 + s_2, \quad s > s_1, s_2, \quad \text{double trace}, $$

where we show that $s > s_1, s_2$ as follows. For $s_1 = s$, the only operator with the right twist would be $\hat{j}_0 J_{s-\ldots}$, but this is not really a spin $s-1$ operator, namely it does not come from any covariant structure. Equation (2.5) is a constraint on the spins of the operators that can appear in the divergence of a current of spin $s$.

At the level of triple trace operators, the product of three operators has already twist 3. So the only structure which is allowed is $\partial_\nu$ by the twist counting. Matching the dimensions in $\partial_\mu J_0^0 \propto J_{s_1} J_{s_2} J_{s_3}$, we obtain

$$ s + 2 = (s_1 + 1) + (s_2 + 1) + (s_3 + 1) + n_3 $$

and the constraint

$$ s \geq s_1 + s_2 + s_3 + 1, \quad \text{triple trace}. $$

Twist counting prohibits having the product of more than three operators.

Now let us comment on the scaling of the coefficients in (1.3) with $\tilde{N}$. Let us normalize the scaling of single particle operators so that their connected $n$-point functions scale like $\tilde{N}$. Then, if we consider a three-point correlator of a given current with the two currents that appear on the right-hand side of its divergence, we obtain

$$ \tilde{N} \sim \partial_\mu \{ J_0^0(x) J_{s_1}(x_1) J_{s_2}(x_2) \} = a_2 \langle J_{s_1}(x_1) J_{s_2}(x_2) \rangle \langle J_{s_1}(x) J_{s_2}(x_2) \rangle \sim a_2 \tilde{N}^2 $$

with derivatives sprinkled on the right-hand side. Thus, we obtain that $a_2 \propto \frac{1}{\tilde{N}}$. For $a_3$, the same argument leads to $a_3 \propto \frac{1}{\tilde{N}^2}$. This scaling is the only one that is consistent with the $\tilde{N}$ counting and is such that it leads to non-zero terms in the leading contribution.

3. Structures for the three-point functions

In this section, we constrain the structure of three-point functions. When we have exactly conserved currents, the possible three-point functions were found in [17] (see also [18, 14]). They were found by imposing conformal symmetry and current conservation. The three-point functions were given by three possible structures. One structure arises in the free-fermion theory and another arises in the free-boson theory. We call these the fermion and boson structures, respectively. Finally, there is a third ‘odd’ structure which does not arise in a free theory. For twist-1 fields, this structure is parity odd (it involves an epsilon tensor). However, for correlators of the form $\langle \hat{j}_0 J_{s_1} J_{s_2} \rangle$, the fermion structure is parity odd (it involves an epsilon tensor) and the ‘odd’ structure is parity even. This is due to the fact that $\hat{j}_0$ is parity odd in the free-fermion theory. Alternatively, if we assign parity minus to $\hat{j}_0$ and parity plus to all
the twist-1 operators, then the ‘odd’ structures always violate parity\(^6\). The reader should think that when we denote a structure as ‘odd’, we simply mean ‘strange’ in the sense that it does not arise in a theory of a free boson or free fermion.

In our case, the currents are not conserved, so we need to revisit these constraints. For example, the divergence of a current can produce a double trace operator. If these contract with the two remaining operators, as in (2.8), we obtain a term that is of the same order in the \(1/N\) expansion as the original three-point function. Note that only double trace operators can contribute in this manner to the current non-conservation of a three-point function. We emphasize that we are computing these three-point functions to leading order in the \(1/N\) expansion, where we can set their twist to be 1. All statements we make in this section are about the structure of correlators to leading order in the \(1/N\) expansion.

We will show below that even correlation functions (fermion and boson) stay the same and all new structures appear in the odd piece. Consider the three-point function of twist-1 operators \(\langle J_1 J_2 J_3 \rangle\), with \(s_i \geq 2\). Let us say that \(s_1\) is larger than or equal to the other two spins. Then, the \(J_1\) and \(J_3\) currents are conserved inside this three-point function since the spins appearing in the divergence of a current are always strictly less than those of the current itself (2.5). On the other hand, in order to obtain a non-zero contribution, we would need to contract \(J_2\) with one of the two currents that appears on the right-hand side of the divergence of \(J_1\) or \(J_3\). Thus, we can impose current conservation on \(J_1\) and \(J_3\) for this three-point function. Let us consider the parity even structures first. As we discussed in [14], for two operators of the same twist and one conserved current, say \(j_s\), we have the most general even structure

\[
\langle O_i O_2 J_s \rangle \sim \frac{1}{[x_{12}]^{[2s_1-1][2s_2][2s_3]}|x_{13}|} \sum_{l=0}^{\min\{s_j, s_k\}} P^{2l}_{5} \{\langle j_{s_1-l} j_{s_2-l} j_s \rangle_{\text{bos}} + \langle j_{s_1-l} j_{s_2-l} j_s \rangle_{\text{fer}}\}
\]

(3.1)

where \(P_l\) are as in [17]. We obtain this result by considering the light cone limit between \(O_i O_2\) and imposing the conservation of \(J_{s_1}\) [14]. We then take light cone limit \(j_{s_1}, j_{s_2}\) and impose conservation of \(J_{s_1}\). Then, we take the light cone limit \(j_{s_1} j_{s_0}\) and impose conservation of \(j_{s_2}\). From these two operations, we would conclude that

\[
\langle j_{s_1} j_{s_2} j_{s_0} \rangle = \frac{1}{[x_{12}]^{[2s_1-1][2s_2][2s_3]}[x_{13}]} \sum_{l=0}^{\min\{s_j, s_k\}} P^{2l}_{5} \{j_{s_1-l} j_{s_2-l} j_{s_0}\}_{\text{bos}} + j_{s_1-l} j_{s_2-l} j_{s_0}\}_{\text{fer}}\}
\]

(3.2)

The only consistent solution is \(b_0 = \tilde{b}_0\), \(f_0 = \tilde{f}_0\) and \(b_l\) and \(f_l\) with \(l \neq 0\) are equal to zero. This can be seen by taking the light cone limit in \(x_{12}\) first, which sets to zero all terms of the form \(P_l\), with \(l > 0\), as well as the fermion terms. In the second line, only the boson structures survive, but only the \(l = 0\) structure is the same as the one surviving in the first line. This shows that all \(\tilde{b}_l = 0\) for \(l > 0\). Repeating this argument, we can show it for the other cases.

We can now consider also the case when one of the particles has spin zero, or is \(j_0\). Then, any of the expressions in (3.2) only allows the \(l = 0\) term. Thus, the even structures with only one \(j_0\) are the same as in the free-boson theory (the free-fermion ones are zero).

For the odd structure, the situation is more tricky. Inside the triangle, \(s_i \leq s_{i+1} + s_{i-1}\) for \(i = 1, 2, 3\), we have the structures that we had before since (2.5) does not allow any of the three currents to have a non-zero divergence. However, outside the triangle, we have new structures that obey (2.8) with a non-zero double trace term. Precisely, the existence of these

\[^6\text{This is not always the natural parity assignment. For example, in the critical }O(N)\text{ theory, }\tilde{j}_0\text{ has parity plus and the theory preserves parity. In this theory, we have only the ‘odd’ structure for the correlators of the form }\langle j_0 j_{s_1} j_{s_2} \rangle.\]
new structures makes the whole setup consistent. The current non-conservation identity has the form of the current conservation one but with a non-zero term on the right-hand side. Since outside the triangle we had no solutions of the homogeneous equations, this guarantees that the solutions are uniquely fixed in terms of the operators that appear on the right-hand side of the conservation laws. Thus, we have unique solutions for these structures.

One interesting example of this phenomenon is the correlator

\[ \langle J_4 J_2 J_0 \rangle_{\text{odd, nc}} \propto a_2 \sum_{i=1}^{3} \left[ O_i^2 + 4 P_i^2 \right] \]

where \( P_i \) and \( Q_i \) are defined in [17]. This odd correlator would be zero if all currents were conserved. However, using the lack of conservation of the \( J_4 \) current, (2.3), we can derive (3.3). Clearly, only \( a_2 \) contributes to it.

As another example, consider \( \langle J_4 J_2 J_0 \rangle \). Here, we can have structures that are parity odd and parity even, the fermion and the ‘odd’ structure, respectively. Due to the form of the current non-conservation of \( J_4 \), (2.2), we obtain

\[ \langle J_4 J_2 J_0 \rangle_{\text{odd, nc}} \propto a_2 \sum_{i=1}^{3} \left[ P_i^2 - 10 P_i^2 Q_i^2 Q_0^2 \right] \]

This ‘odd’ structure would vanish if \( J_4 \) were exactly conserved. Of course, we also have the structure that we obtain in the free-fermion theory for these cases, which is parity odd (while (3.4) is parity even). In fact, any correlator of two twist-1 currents and one \( J_0 \) which is parity odd will have the same structure as in the free-fermion theory since (2.2) (or its higher spin versions) will not modify it. On the other hand, the parity even ones can be modified. Again, we find that a structure that was forced to be zero when the current is exactly conserved can become non-zero when the current is not conserved.

Finally, we should mention that any correlator that involves a current and two scalars is uniquely determined by conformal symmetry. In this case, the current is automatically conserved. Of course, the three-point function of three scalars is also unique.

In summary, we constrained the possible structures for various three-point functions. These are the boson, fermion and odd structures. When the operator \( J_0 \) is involved, we can only have fermion or odd structures. In the next section, we will constrain the relative coefficients of all these three-point functions.

4. Charge non-conservation identities

4.1. General story

We use the following technique to constrain the three-point function. We start from a three-point function \( \langle O_1 O_2 O_3 \rangle \). We then insert a \( J_0 \) current and take its divergence, which gives us an identity of the form

\[ \langle \nabla J_4(x) O_1 O_2 O_3 \rangle = a_2 \langle J^f O_1 O_2 O_3 \rangle + a_3 \langle J J^f O_1 O_2 O_3 \rangle \]

\[ \sim a_2 \langle J O_1 \rangle \langle J^f O_2 O_3 \rangle + a_2 \langle J O_1 \rangle \langle J^f O_2 \rangle \langle J^f O_3 \rangle \text{ + permutations.} \]

This equation is schematic since we dropped derivatives that should be sprinkled on the right-hand side. The two-point functions on the right-hand side are non-zero only if \( J \) or \( J^f \) is the same as one of the operators \( O_i \). Thus, the right-hand side is non-zero only when any of the operators \( O_i \) is the same as one of the currents that appears on the right-hand side of the divergence of \( J_4 \) (1.3), (2.2), (2.3). We have only considered disconnected contributions on the right-hand side because those are the only ones that survive to leading order in \( 1/N \). Here, we used the scaling of the coefficients \( a_2 \) and \( a_3 \) with \( N \) given in (2.8).
Given this equation, we can now integrate over $x$ on the left- and right-hand sides. We integrate over a region which includes the whole space except for little spheres, $S_i$, around each point $x_i$ where the operators $O_i$ are inserted. The left-hand side of (4.1) contributes only with a boundary term of the form

$$\sum_{i=1}^3 \left( \int_{S_i} n^J J_{j=-} O_1 O_2 O_3 \right),$$

(4.2)

where the integral is over the surface of the little spheres or radii $r_i$ around each point and $n^J$ is the normal vector to the spheres.

If the current $J_4$ were exactly conserved, these integrals would give the charge acting on each operator. In our case, the charges are not conserved and the integrals may depend on the radius of the little spheres. This dependence can give rise to divergent terms going like inverse powers of the radius of the spheres. These terms diverge when $r_i \to 0$. These divergent terms should precisely match similar divergent terms that arise in the integral on the right-hand side of (4.1). After matching all the divergent terms, we are left with the finite terms in the $r_i \to 0$ limit. These also have to match between the left- and right-hand sides. Demanding that they match, we will obtain interesting constraints. Note that, at the order we are working, we do not obtain any logarithms of $r_i$ since the anomalous dimensions of operators start at higher order of the $1/\tilde{N}$ expansion. Thus, the separation of the finite and the divergent terms is always unambiguous.

Thus, we define a pseudo-charge $Q$ that acts on the operators by selecting the finite part of the above integrals in the small radius limit:

$$[Q, O(0)] = \int_{|x|=r} n^J J_{j=-} O(0) \bigg|_{\text{finite as } r \to 0}. $$

(4.3)

This action of this pseudo-charge on single trace operators is determined by the three-point functions we discussed above. It is also constrained by twist and spin conservation to have a similar structure to the one we had for absolutely conserved currents. For example, on the twist-1 operators, we have

$$[Q, j_s] = \sum_{r=0}^{r+3} c_{s,r} \eta^{r-s+3} j_r. $$

(4.4)

In concrete computations, we found it useful to work in the metric (2.1) and to cut out little ‘slabs’ of width $\Delta x^+$ around the operators, instead of cutting out little spheres.

The advantage is that the integral involves the all-minus component of the current. In addition, minus and $y$-derivatives can be integrated by parts or be pulled out of the integral

$$Q_s(x^+) = \int dx^- dy j_s(x^-, x^+, y). $$

(4.5)
The action of the pseudo-charge on $O_i$ comes from three-point functions of the form $\langle J_i O_i O_j \rangle$. As mentioned, the structures that arise in the boson or fermion theories will continue to produce three-point functions where the charges are conserved. The odd correlators can give us something new. The odd structures involving all twist-1 fields, such as $\langle j_4 j_2 j_1 \rangle$, are parity odd and do not contribute to the action of $Q$. This can be seen by setting $y_2 = y_3 = 0$. Then, the fact that the three-point function is odd under the parity $y_1 \rightarrow -y_1$ implies that the integral over $y$ in (4.5) must vanish. This implies that commutator must vanish also for arbitrary $y_2 \neq y_3$ since the two-point function structures that could possibly contribute do not vanish for $y_2 = y_3$. When one of the operators $O_i$ has twist 2, we can obtain non-vanishing contributions to the action of the pseudo-charge from odd structures.

In conclusion, after integrating (4.1), we obtain an expression of the form

$$
\langle [Q, O_1] O_2 O_3 \rangle + \text{cyclic} = \int \! d^3x \{ a_1 \langle J(x) O_1 \rangle \langle J'(x) O_2 O_3 \rangle
$$

$$+ a_1 \langle J(x) O_1 \rangle \langle J'(x) O_2 \rangle \langle J'(x) O_3 \rangle \rangle + \text{permutations} \}_{\text{finite}}
$$

(4.6)

where the operators $O_i$ are evaluated at $x_i$. The integral is over the full $R^3$ after subtracting all the divergent terms that can arise around each point $x_i$. This is the main identity that we will use to relate the various three-point functions to each other. We can call it a pseudo-conservation of the pseudo-charges.

4.2. Constraints on three-point functions with non-zero even spins

In this section, we will consider the constraints that arise on three-point functions of operators with spins $s_j \geq 2$. We will consider all their indices to be minus. So we take the operators $O_i$ in (4.1) to be $j_{s_j}$, with $s_j \geq 2$.

In this case, the action of $Q$ can only produce other twist-1 fields, which are only single particle states. As we mentioned above, the action of $Q$ is determined by three-point functions of the form $\langle j_4 j_2 j_1 \rangle$ or $\langle j_4 j_0 j_0 \rangle$. Only the even structures contribute to the charges, and these are the same (up to overall coefficients) as in the free theories; thus, the action of the pseudo-charge is well defined and produces

$$
\langle [Q, j_1] = c_{s,s-2} \partial \partial j_{s-2} + c_{s,s} \partial^2 j_s + c_{s,s+2} \partial j_{s+2},
$$

(4.7)

where $\partial = \partial_{+}$. We can always choose the following normalization conditions:

$$
c_{2,0} = 1, \quad c_{2,4} = 1, \quad c_{4,4} = 1, \quad c_{4,6} = 1, \quad c_{s,s+2} = 1, \quad s \geq 2.
$$

(4.8)

This can be done in all cases, pure fermions, pure bosons, or the interesting theory we are considering. We are not setting the normalization of two-point function to one. The two-point functions are not used for the time being. The stress tensor is normalized in the canonical way. Note that, since the normalization of stress tensor is fixed, $c_{2,4}$ does not depend on the normalization of $j_4$ and is fixed by conformal invariance to 3 (recall $[Q, j_2] = (s-1) j_2$ [14]). Thus, we fix it to one by the rescaling freedom in the definition of $Q_i \rightarrow 1/Q_i$. Then, we fix $c_{4,4} = 1$ by rescaling $j_4$ itself. $c_{s,s+2} = 1$ by rescaling $j_{s+2}$.

Now, in our case, it is clear that if all $s_j \geq 4$, then the right-hand side of (4.6) vanishes. In addition, if one or more of $s_j$ is equal to 2, then the following occurs. In that case, the two current terms in (2.2) and (2.3) could contribute since we can contract $J_2$ in the divergence of
Figure 2. Instead of spheres as in figure 1, we can cut out little slabs of width $\Delta x^+$ around the insertion point of every operator. The charge is given by integrating current over $x^+$ and $y$ at the edges of these slabs. This simplifies some computations compared to figure 1.

$J_4$ with $s_1$ that is equal to 2. It is useful to recall the two-point functions of various components of the stress tensor

$$\langle J_4^-(x)J_4^-(0) \rangle \propto \frac{(x^+)^4}{(x^+x^-+y^2)^5} = \frac{1}{4!} \hat{a}^2 \frac{1}{x^+x^-+y^2};$$

$$\langle J_4^-(x)J_4^-(0) \rangle \propto 2 \frac{(x^+)^3 y}{(x^+x^-+y^2)^5} = \frac{1}{4!} \hat{a}^2 \partial_x \frac{1}{x^+x^-+y^2};$$

$$\langle J_4^-(x)J_4^-(0) \rangle \propto -\frac{(x^+)^2 (x^+x^-+y^2)}{(x^+x^-+y^2)^5} = \frac{1}{4!} \left( \hat{a}^2 \hat{a}^2 \frac{1}{x^+x^-+y^2} - 2 \hat{a}^2 \frac{1}{(x^+x^-+y^2)^2} \right), \quad (4.9)$$

where the equations are true up to a normalization factor common to all three equations.

If we look at the first term on the right-hand side of (4.6) in the case that $O_1 = J_2$, then we can use the above two-point function (4.9). We can integrate by parts all derivatives so that they act on the two-point function. Then, we can write these as derivatives acting on $x_i$ and pull them out of the integral.\footnote{If we work with the ‘slabs’ described in figure 2, together with the usual $i\epsilon$ prescription, it is clear that these operations do not produce boundary terms since we only have $\partial_-$ or $\partial_\phi$ derivatives.}

Now the result depends on whether we are dealing with the quasi-fermion or quasi-boson cases. In the quasi-boson case, it is possible to check that the particular combination of currents that appear in the two current terms in (2.3) is, after integrating by parts, $\partial_x J_2^-=\partial_x J_2^+$. When this is contracted with $j_2 = J_2^-$, we obtain zero after using (4.9). Thus, the right-hand side of (4.6) vanishes in the quasi-boson case.

In the quasi-fermion case, we end up having to compute (the $\hat{a}^2$ derivative) of an integral of the form

$$\int d^3x \frac{1}{|x-x_1|^2} \langle \tilde{j}_0(x)O_2(x_2)O_3(x_3) \rangle \propto \langle j_0^\text{eff}(x_1)O_2(x_2)O_3(x_3) \rangle. \quad (4.10)$$

The factor $\frac{1}{|x-x_1|^2}$ is exactly the one that makes the integral conformal-covariant. It gives a result that effectively transforms as the three-point function with the insertion of an operator of dimension $\Delta = 1$ at $x_1$. We have denoted this in terms of an effective operator $J_0^\text{eff}$ of dimension
Thus, we can treat these two cases in parallel, after we remember that equations arising from the pseudo-charge conservation are that this is likely to be true for all cases.

The solution is definitely unique in the sense that all relative coefficients are fixed. We think definitely exists! For low values of the spins, we have explicitly analyzed the equations and we have not proved this for the odd case. However, given the existence of the Chern–Simons matter theories [12, 13], we know that at least one solution exists.

Focusing first on the equations that constrain the boson structures, we obtain that the pseudo-conservation identities acting on \( j_1, j_2, j_3 \) becomes identical to a charge conservation identity for \( \hat{Q} \). Note also that after adding \( j_0^{\text{eff}} \) in (4.11), the action of \( \hat{Q} \) is essentially the same as the action of \( Q \) in the quasi-boson theory. Thus, we can treat these two cases in parallel, after we remember that \( j_0^{\text{eff}} \) is not a real operator but just the integral in (4.10).

Next, we write all twist-1 three-point functions as

\[
\langle j_1, j_2, j_3 \rangle = \alpha_{1,t_1,\epsilon_1} \langle j_1, j_2, j_3 \rangle_{\text{bos}} + \beta_{t_1,\epsilon_1,\epsilon_2} \langle j_1, j_2, j_3 \rangle_{\text{ferm}} + \gamma_{t_1,\epsilon_1,\epsilon_2} \langle j_1, j_2, j_3 \rangle_{\text{odd}} \tag{4.12}
\]

Here, the boson and fermion ones are the three-point functions for a single real boson and a single Majorana fermion, in the normalization of the currents set by (4.8). The normalization of the odd piece is fixed so that the identities we describe below are true. In the quasi-fermion theory, we are also including in (4.12) the case with \( \langle j_0^{\text{eff}}, j_1, j_3 \rangle \), where only the boson and odd structures are non-zero.

We now consider the charge conservation identities for various cases. An important property of these identities is that each identity separates into three independent equations which relate only the boson structures to each other, the fermion structures to each other and the odd structures to each other. Each equation involves sums of objects of the form \( \partial_{\mu} \langle j_1, j_2, j_3 \rangle_{\text{str}} \), where \( \text{str} \) runs over boson, fermion and odd. These equations are such that the coefficient in front of each term is fixed relative to all the other coefficients. In other words, only an overall constant is left undetermined. See appendix B. This was proven for the boson and fermion terms in [14]. We have not proved this for the odd case. However, given the existence of the Chern–Simons matter theories [12, 13], we know that at least one solution definitely exists! For low values of the spins, we have explicitly analyzed the equations and the solution is definitely unique in the sense that all relative coefficients are fixed. We think that this is likely to be true for all cases.

Focusing first on the equations that constrain the boson structures, we obtain that the equations arising from the pseudo-charge conservation are

\[
\begin{align*}
\hat{Q}(j_1, j_2, j_3) & : \tilde{c}_{2,022} = \alpha_{224} = \alpha_{022} \\
\hat{Q}(j_1, j_2, j_3) & : \tilde{c}_{4,024} = \alpha_{224} = \alpha_{044} = \tilde{c}_{4,222} = \alpha_{226} = \alpha_{024} \\
\hat{Q}(j_1, j_2, j_3) & : \tilde{c}_{6,026} = \alpha_{226} = \alpha_{044} = \tilde{c}_{6,424} = \alpha_{228} = \alpha_{026}
\end{align*}
\tag{4.13}
\]

and we can continue in this way. We are defining \( \tilde{c}_{t,s} \) to be the ratio

\[
\tilde{c}_{t,s} = \frac{c_{t,s}}{c_{t,s}^{\text{free boson}}}
\tag{4.14}
\]

with both in the normalization (4.8).
We now can start solving (4.13). We see that equations (4.13) fix all \( \alpha \)'s in terms of \( \alpha_{222} \) and \( cs \). In addition, we obtain multiple equations for the same \( \alpha \)'s. This fixes \( \tilde{c}s \). For example, we start obtaining things like

\[
\begin{align*}
\alpha_{224} & = \tilde{c}_{2,222}, \\
\alpha_{244} & = \tilde{c}_{4,222}, \\
\alpha_{444} & = \tilde{c}_{4,222},
\end{align*}
\]

where each line in (4.15) comes from the corresponding line in (4.13). Using the fact that \( \tilde{c}_{4,2} = 1 \), we see that \( \tilde{c}_{4,2} = 1 \), and also \( \tilde{c}_{4,2} = 1 \), and so on. So all \( \tilde{c}s = 1 \). We also find that all \( \alpha \)'s are also fixed to be equal to \( \alpha_{222} \).

If we did the same for the free fermions, we would also obtain the same pattern if we define

\[
\tilde{c}_{s,s'} = \frac{c_{s,s'}}{c_{s,s'}^{\text{free fermion}}}. \tag{4.16}
\]

Then, we obtain that all \( \beta_{s,s',s'} = \beta_{222} \) and all \( \tilde{c}_{s,s'} = 1 \). One subtlety is that we have defined \( \tilde{c}s \) differently for the bosons than for the fermions. So we can only hope to get both structures present only if

\[
\begin{align*}
\tilde{c}_{s,s'}^{\text{free boson}} & = \tilde{c}_{s,s'}^{\text{free fermion}},
\end{align*}
\]

This can be checked by using form factors (see appendix C). Of course, the mere existence of the Chern–Simons matter theories implies that this is true.

The conclusion from this analysis is that all \( \alpha \)'s are equal to \( \alpha_{222} \). Analogously, all \( \beta \)'s are equal to \( \beta_{222} \).

We will also need the coefficients of the two-point functions \( n_s \). In a theory of a single free boson or single free fermion, with the normalization conditions (4.8), these are given by \( n_s^\text{bos} \) and \( n_s^\text{fer} \), respectively. We do not need their explicit forms, but they can be computed in the free theories. We can now determine the two-point functions in the full theory by demanding that the stress tensor Ward identities are obeyed. In other words, from the previous discussion, we know that \( \alpha_{2ss} = \alpha_{222} \) and \( \beta_{2ss} = \beta_{222} \). In addition, the Ward identity of the stress tensor relates this to the two-point function. More precisely,

\[
n_s = \alpha_{222} n_s^\text{bos} + \beta_{222} n_s^\text{fer}. \tag{4.18}
\]

Note that \( n_s^\text{bos} = n_s^\text{fer} \) according to formulas (5.1) and (5.7) of [19]\(^8\). Thus, we find

\[
\tilde{n}_s \equiv \frac{n_s^\text{fer}}{n_s^\text{bos}} = \alpha_{222} + \beta_{222}. \tag{4.19}
\]

Note that the analysis so far is equivalently valid for the theories of quasi-bosons and the theory of quasi-fermions. Of course, in the latter case, whenever \( j_0 \) appeared, it should be interpreted as \( j_0^{\text{fer}} \).

Thus, so far, we have written all correlators in terms of three undetermined coefficients \( \alpha_{222}, \beta_{222} \) and \( \gamma_{222} \). As an aside, we should note that when we solve the equations for \( \gamma_{s_1s_2s_3} \), we need to use some of the odd correlators that are outside the triangle. These are possible thanks to the non-conservation of the currents.

### 4.3. Closing the chain: the quasi-fermion case

In this section, we consider current conservation identities in the case that one of the operators is a twist-2 field such as \( j_0 \) or \( J_{2,-2} \).

\(^{8}\) There, the formula for the Dirac fermion is written. Here, we consider a Majorana one.
Let us start by discussing the possible action of the pseudo-charge $Q$ on $J_{-\gamma}$. All single trace operators that could appear are already fixed by the correspondent three-point functions to be the same as in the free-fermion theory. The most general expression involving double trace terms takes the form

$$[Q, J_{-\gamma}] = \partial^4 j_0 + \partial^3 J_{-\gamma} + \partial J_{4-\gamma-y} + \frac{\chi}{N} j_2 j_2. \quad (4.20)$$

The double trace term comes with a $\frac{1}{N}$ factor because it enters in the pseudo-conservation identity with the $(jj)(jj) \sim N^2$ factor.

We have normalized $j_0$ by setting the first coefficient to $1^9$. One can check that $\chi$ should be set to zero by considering $(j_2 j_2 J_{-\gamma})$ pseudo-conservation identity.$^{10}$

In addition, we write the possible correlators of $j_0$ in terms of free-fermion correlators, introducing $\beta_0^{\text{eff}}$. This is the coefficient that multiplies $(j_0 j_0 j_0)_{\text{bet}}$, which is the correlator in a theory of a single Majorana fermion in the normalization of $j_0$ in (4.20). Note that these free-fermion correlators are parity odd. We also introduce $\gamma_0^{\text{eff}}$, which multiply the ‘odd’ structures, which are parity even structures involving $j_0$. These structures are more subtle since they can be affected by the violation of current conservation, as in (3.4).

An additional issue we should discuss is the type of contribution we expect from the right-hand side of (4.6) when the operator is $J_{-\gamma}$. The same reasoning we used around (4.10) together with (4.9) tells us that we also effectively produce $\beta_0^{\text{eff}}$. Thus, the net result is that we should simply add terms involving $\beta_0^{\text{eff}}$ in (4.20) and treat the charge as conserved. This is necessary for getting all the identities to work. We then find that $\beta_0^{\text{eff}} = \beta_{222}$ and $\gamma_0^{\text{eff}} = \gamma_{222}$.

Interestingly, in this case, in order to satisfy the pseudo-charge conservation identities for the odd part, we need both $j_0$ and $j_0^{\text{eff}}$.

Let us consider the three-point function $(j_0^{\text{eff}} j_2 j_2)$. We see that its definition via (4.10) involves a three-point function given by $\beta_{022}$ and $\gamma_{022}$. But we have just fixed these coefficients. Thus, going back to the first line (4.13), we note that $\alpha_{022}$ is really the coefficient of $(j_0^{\text{eff}} j_2 j_2)_{\text{bet}}$. We obtain this structure by calculating the integral (4.10) with the ‘odd’ structure for the three-point function. Using that $\alpha_{022}$ was fixed to $\alpha_{222}$, we obtain the equation

$$\alpha_{222} = x_2 a_2 n_2 \gamma_{022} = x_2 a_2 (\alpha_{222} + \beta_{222}) \gamma_{222}. \quad (4.21)$$

Similarly, the odd structure gives

$$\gamma_{222} = x_1 a_2 n_2 \beta_{022} = x_1 a_2 (\alpha_{222} + \beta_{222}) \beta_{222}, \quad (4.22)$$

where $x_1$ and $x_2$ are two calculable numerical coefficients, given by calculating the integral in (4.10), etc. We used (4.19) and $\bar{x}_i = x_i / h_0^{\text{eff}}$.

These are two equations for four unknowns ($\alpha_{222}$, $\beta_{222}$, $\gamma_{222}$, $\alpha_2$). So we have two undetermined coefficients which are $\tilde{N}$ and $\tilde{\lambda}$. We can define $\tilde{N} = n_2 = \alpha_{222} + \beta_{222}$ and $\tilde{\lambda} = a_2 \tilde{N} / \sqrt{x_1 x_2}$. With these definitions, we obtain

$$\alpha_{s_1 s_2 s_3} = \tilde{N} \frac{\tilde{\lambda}^2}{(1 + \tilde{\lambda}^2)}, \quad \beta_{s_1 s_2 s_3} = \tilde{N} \frac{1}{(1 + \tilde{\lambda}^2)}, \quad \gamma_{s_1 s_2 s_3} = \tilde{N} \frac{\tilde{\lambda}}{(1 + \tilde{\lambda}^2)}. \quad (4.23)$$

Here, we are considering three twist-l fields. Similar equations are true for correlators involving one $j_0$ field

$$\beta_{s_1 s_2 0} = \tilde{N} \frac{1}{(1 + \tilde{\lambda}^2)}, \quad \gamma_{s_1 s_2 0} = \tilde{N} \frac{\tilde{\lambda}}{(1 + \tilde{\lambda}^2)}. \quad (4.24)$$

$^9$ This is not possible for the critical $O(N)$ theory since in that case, parity prevents $j_0$ on the right-hand side. We will obtain this case as a limit. It can also be analyzed directly by considering the action of $Q$ on $j_0$, etc.

$^{10}$ This double trace structure can arise in a covariant way as $[Q, J_{2 \mu}] = \epsilon_{-\mu \nu} F_{J_{-\gamma}} + \epsilon_{-\nu \mu} F_{J_{-\gamma}}$. 
We will discuss the correlation functions that involve more than one \( j_0 \) separately. In both
equations, (4.23) and (4.24), we can choose the (\( \tilde{\lambda} \)-independent) numerical normalization of
the odd correlators so that we can replace \( \sim \) by an equality.

The final conclusion is that we expressed all the correlators of the currents in terms of just
two parameters: \( \tilde{N} \) and \( \tilde{\lambda} \).

Since the fermions plus Chern–Simons theory has precisely two parameters, \( N \) and \( k \),
we conclude that we exhausted all the constraints. Our analysis was based only on general
symmetry consideration and does not allow us to find the precise relation between the
parameters. However, in the ’t Hooft limit, we expect
\[
\tilde{N} = N f(\lambda), \quad \tilde{\lambda} = h(\lambda) = d_1 \lambda + d_3 \lambda^3 + \cdots, \tag{4.25}
\]
where \( f(\lambda) = f_0 + f_2 \lambda^2 + \cdots \). We used the symmetry of the theory under \( \lambda \rightarrow -\lambda \) (or
\( k \rightarrow -k \)), together with parity. Note that the function \( f \) encodes how the two-point function
of the stress tensor depends on \( \lambda \).

4.4. Closing the chain: the quasi-boson case

In the quasi-boson theory, we will again consider the charge non-conservation identity on
\( \langle j_2 j_2 \rangle \). Again, we need to write the most general action of \( \tilde{Q} \) on \( J \) involving double trace
terms. It takes the form
\[
[\tilde{Q}, J_{\cdots}] = \partial^2 \partial_j j_0 + \partial^2 J_{\cdots} + \partial J_{\cdots} + \frac{x_1}{\tilde{N}} j_1 j_2 + \frac{x_2}{\tilde{N}} \partial^2 j_0 j_2 + \frac{x_3}{\tilde{N}} \partial^4 j_0 j_0. \tag{4.26}
\]
We have normalized \( j_0 \) by setting the first coefficient to 1.\(^{12}\)

Note also that \( x_1 \sim O(\frac{1}{\tilde{N}}) \) to contribute at leading order in \( \tilde{N} \). We fix \( x_1 = 0 \) as in the case of
fermions. The presence or absence of \( x_{2,3} \) is not important for any of the arguments.

In addition, we need to consider the contribution of \( J \) on the right-hand side of (4.6). In
other words, we will need to consider the right-hand side of (4.6) when the operator \( Q_1 \) is \( J \).

Using (4.9), we find (derivatives of) an integral of the form
\[
\int d^3 x \frac{1}{|x - x_1|^4} \langle j_0(x) O_2(x_2) O_3(x_3) \rangle \sim \langle \tilde{j}_0^{\text{eff}}(x_1) O_2(x_2) O_3(x_3) \rangle. \tag{4.27}
\]
Here, we used that, again, this is a conformal integral which behaves as a correlator with a
scalar of weight \( \Delta = 2 \) at \( x_1 \). We have denoted this by introducing a fictitious operator \( \tilde{j}_0^{\text{eff}} \).
This is not an operator that exists in the theory, but it is appearing in three-point functions in
the same way as an operator of this form. Thus, the net effect of the action of \( \tilde{Q} \) on \( J \) also
includes the operator \( \tilde{j}_0^{\text{eff}} \) on the right-hand side of (4.20).

With all these features, we now have a situation which is rather similar to the one we had
in the quasi-fermion case and we can relate the different coefficients
\[
\beta_{222} = y_1 a_2 n_2 \gamma_{222} = \tilde{y}_1 \tilde{a}_2 (\beta_{222} + \gamma_{222}), \quad \gamma_{222} = y_2 a_2 n_2 \alpha_{222} = \tilde{y}_2 \tilde{a}_2 (\alpha_{222} + \beta_{222} \tilde{a}_{222}). \tag{4.28}
\]
Again, \( y_1 \) and \( y_2 \) are some numbers. Defining again \( \tilde{N} = \tilde{n}_2 = (\alpha_{222} + \beta_{222}) \) and
\( \tilde{\lambda} = a_2 \tilde{N} / \sqrt{y_1 y_2} \), we obtain
\[
\alpha_{32} \tilde{a}_{32} = \tilde{N} \frac{1}{(1 + \tilde{\lambda}^2)}, \quad \beta_{32} \tilde{a}_{32} = \tilde{N} \frac{\tilde{\lambda}}{(1 + \tilde{\lambda}^2)}, \quad \gamma_{32} \tilde{a}_{32} = \tilde{N} \frac{\tilde{\lambda}}{(1 + \tilde{\lambda}^2)}. \tag{4.29}
\]
Here, at least two of the spins should be bigger than zero, \( s_i \gtrsim 2 \). We will discuss the correlation
functions that involve more than one \( j_0 \) separately.

\(^{11}\) For the theory considered in [12], \( f_0 = 2 \) and \( d_1 = 0 \).

\(^{12}\) Again, this is not possible in the critical \( O(N) \) Gross–Neveu theory. But we will obtain this case as a limit. It can
also be analyzed directly through a slightly longer route.
4.5. Three-point functions involving scalars: the quasi-fermion case

Fixing the three-point functions in the scalar sector involves several new subtleties that were absent before. We describe them in detail in appendix D and here sketch the method and present the results.

We have already discussed how to fix three-point functions which include one scalar operator. We used the pseudo-conservation identity on $\langle J_{-y} s_1 s_2 \rangle$ to obtain (4.24).

To proceed, it is necessary to specify how $Q$ acts on $\tilde{j}_0$. As explained in appendix D, the result is

$$[Q, \tilde{j}_0] = \partial^3 \tilde{j}_0 + \frac{1}{1 + \lambda^2} \partial_3 \partial_3 \partial_3 \tilde{j}_0; \quad (4.30)$$

here, the interesting new ingredient is a $1/(1 + \lambda^2)$ factor. This is obtained by inserting arbitrary coefficients and fixing them by analyzing the $\langle j_4 j_2 \tilde{j}_0 \rangle$ three-point function.

To fix the correlators with two scalars, we consider the pseudo-charge conservation identity that is generated by acting with $Q$ on $\langle \tilde{j}_0 \tilde{j}_0 \rangle$. This identity involves tricky relations (see appendix D) between different three-point functions. After the dust settles, we obtain that

$$\beta_{\tilde{j}_0 \tilde{j}_0} = \beta_{222}, \quad \gamma_{\tilde{j}_0 \tilde{j}_0} = \gamma_{222}. \quad (4.31)$$

The last step is to consider the action of $Q$ on $\langle \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \rangle$. We then obtain

$$\gamma_{\tilde{j}_0 \tilde{j}_0 \tilde{j}_0} = 0. \quad (4.32)$$

Note that $\beta_{\tilde{j}_0 \tilde{j}_0 \tilde{j}_0} = 0$ by definition since this correlator vanishes in the free-fermion theory (due to parity).

From these three-point functions, it is also possible to extract the normalization of the two-point function $\langle \tilde{j}_0 \tilde{j}_0 \rangle$. This two-point function is related by a Ward identity to $\langle j_2 \tilde{j}_0 \tilde{j}_0 \rangle$. We then obtain

$$\tilde{n}_0 \equiv \frac{n_{\tilde{j}_0}}{n_{\text{free fermion}}} = \beta_{\tilde{j}_0 \tilde{j}_0} = \beta_{222} = \gamma_{222} = \frac{1}{1 + \lambda^2}. \quad (4.33)$$

This can be used, together with (2.2), to compute the anomalous dimension for the spin-4 current, as explained in appendix A.

4.6. Three-point functions involving scalars: the quasi-boson case

For the quasi-boson sector, the story is almost identical. We put details of the analysis in appendix E and here again sketch the general idea and present the results.

We have already discussed above how to fix three-point functions which include one scalar operator. Then, the charge conservation identity on $\langle j_2 \tilde{j}_0 \tilde{j}_0 \rangle$ fixes

$$\alpha_{01} = \alpha_{222}, \quad \gamma_{01} = \gamma_{222}. \quad (4.34)$$

For the action of the charge on the scalar, we obtain

$$[Q, j_0] = \partial^3 j_0 + \frac{1}{1 + \lambda^2} \partial_3 j_0. \quad (4.35)$$

By considering the pseudo-conservation of $\langle j_0 \tilde{j}_0 j_0 \rangle$ when $s > 2$, we obtain

$$\alpha_0 = \beta_{222}, \quad \gamma_0 = \gamma_{222}. \quad (4.36)$$

We now need to consider the pseudo-conservation identities for $\langle j_2 j_0 j_0 \rangle$ and $\langle \tilde{j}_0 \tilde{j}_0 j_0 \rangle$. A new feature of these two cases is that the triple trace terms in (2.3) contribute. Analyzing these, we obtain that (4.36) is also true for $s = 2$. The triple trace terms contribute as follows. Let us
first consider the pseudo-conservation identity for $\langle j_0 j_0 j_0 \rangle$. The triple trace non-conservation term takes the form
\[
\alpha_3 n_0^3 \sum_{i=1}^{3} \tilde{a}_i^0 \langle j_0 (x_1) j_0 (x_2) j_0 (x_3) \rangle, \tag{4.37}
\]
where $n_0$ is the coefficient in the two-point function for the scalar. Thus, we obtain the equation
\[
\alpha_{000} = \frac{1}{1 + \lambda^2} \alpha_{222} + z_1 \alpha_3 n_0^3,
\tag{4.38}
\]
where $z_1$ is a numerical constant and $b_3$ is the coefficient in (2.3). Note that the double trace deformation does not influence this computation. This fact is established in appendix C.

The triple trace term in the pseudo-conservation identity for $\langle j_2 j_0 j_0 \rangle$ is
\[
\alpha_3' n_0^2 \tilde{a}_0^2 \langle j_0 (x_1) j_0 (x_2) j_0 (x_3) \rangle, \tag{4.39}
\]
leading to
\[
\alpha_{000} = \frac{1}{1 + \lambda^2} \alpha_{222} + z_2 \alpha_3' n_0^2 + \cdots, \tag{4.40}
\]
with $z_2$ being a numerical constant and the dots denote the contribution of the double trace non-conservation piece whose coefficients are already known.

Importantly, we conclude that two triple trace deformations are not independent. In other words, $\alpha_3$ and $\alpha_3'$ are related by equating (4.40) and (4.38). Thus, we recover the known counting of marginal deformation of free boson in $d = 3$, namely there are two parameters. Microscopically, one corresponds to the Chern–Simons coupling and the second one to adding a $(\phi, \tilde{\phi})^3$ operator.

On the Vasiliev theory side, the freedom to add this $\phi^6$ deformation translates into the fact that we can choose nonlinear boundary conditions for the scalar which preserves the conformal symmetry. These were discussed in a similar situation in [20]. In our context, we have a scalar of mass $(mR_{AdS})^2 = -2$ which at infinity decays as $\phi = \alpha / r + \beta / r^2$. Then, the boundary condition that corresponds to adding the $\lambda_6 \phi^6$ deformation is $\beta = \lambda_6 \alpha^2$ [20].

Also, the whole effect of the presence of the triple trace deformations, at the level of three-point functions, is to change $\langle j_0 j_0 j_0 \rangle$.

Finally, the two-point function of $j_0$ can be fixed by using the usual stress tensor Ward identity via $\langle j_2 j_0 j_0 \rangle$. We obtain
\[
\tilde{n}_0 \equiv \frac{n_0}{n_{\text{free boson}}} = \alpha_{222} = \tilde{N} \frac{1}{1 + \lambda^2}. \tag{4.41}
\]
Recall that the normalization of $j_0$ was given by setting $c_{2,0} = 1$ in (4.8).

4.7. Comments about higher point correlation functions

We can wonder whether we can determine higher point correlation functions. It seems possible to use the same logic, namely inserting $\nabla \cdot j_4$ into an $n$-point function and then integrating as in (4.6). This relates the action of $Q$ on an $n$-point function to integrals of disconnected correlators. These integrals also involve $n$-point functions. (Recall that for three-point functions, the integrals involved other three-point functions.) When the charge is conserved, this is expected to fix the connected correlation uniquely to that of the free theory. This was done explicitly in [14] for the action of $Q$ on $\langle j_0 j_0 j_0 j_0 \rangle$. Now that the right-hand side is non-zero, we still expect this to fix uniquely the correlator, though we have not tried to carry this out explicitly. It is not totally obvious that this will fix the correlators because the integral terms also involve $n$-point
functions. But it seems reasonable to conjecture that this procedure would fix the leading-order connected correlator for all \( n \)-point functions of single particle operators. It would be interesting to see whether this is indeed true!

## 5. Final results

In this section, we summarize the results for the three-point functions.

The normalization of the stress tensor is the canonical one. The normalization of the charge is \( Q = \frac{1}{2} \int j_4 \). This sets \( c_{2,4} = 1 \) in (4.7). The normalization of \( j_4 \) is fixed by setting \( c_{4,4} = 1 \). Then, all other operators are normalized by setting \( c_{s,s+2} = 1 \). The operator \( j_0 \) is normalized by setting \( c_{s,0} = 1 \) in (4.20).

We have the two-point functions

\[
\langle j_s(x_1) j_s(x_2) \rangle = n_s \left( \frac{x_{12}^+}{|x_{12}|} \right)^{2s},
\]

(5.1)

For three-point functions

\[
\langle j_s(x_1) j_s(x_2) j_s(x_3) \rangle = \alpha_{s,s,s} \langle j_s(x_1) j_s(x_2) j_s(x_3) \rangle_{\text{free boson}} + \beta_{s,s,s} \langle j_s(x_1) j_s(x_2) j_s(x_3) \rangle_{\text{free fermion}} + \gamma_{s,s,s} \langle j_s(x_1) j_s(x_2) j_s(x_3) \rangle_{\text{odd}},
\]

(5.2)

where ‘bos’ and ‘fer’ denote the three-point functions in the theory of free boson and free fermion in the normalization of currents described above. Their functional form can be found in [17]. The odd generating functional for the spins inside the triangle can be found in appendix B of [14]. Outside the triangle, the odd correlation functions are the ones that satisfy the double trace deformed non-conservation equations. We do not know their explicit form in general but nevertheless we know that they exist and know how the dependence on the coupling will enter. We fix the numerical normalization of the odd pieces to be such that the pseudo-charge conservation identities are obeyed. In a similar fashion, we define the correlators involving \( j_0 \) (\( j_0 \)) operator, except that there is no free-boson (fermion) structure.

### 5.1. Quasi-fermion theory

The interacting theory two-point functions are given by

\[
\begin{align*}
n_s &= \tilde{N} \frac{\lambda^2}{1 + \lambda^2} n_s^{\text{free boson}} + \tilde{N} \frac{1}{1 + \lambda^2} n_s^{\text{free fermion}} = \tilde{N} n_s^{\text{free boson}}, \quad s \geq 2, \\
n_0 &= \tilde{N} \frac{1}{1 + \lambda^2} n_0^{\text{free fermion}},
\end{align*}
\]

(5.3)

where \( n_s^{\text{free boson}} \) and \( n_s^{\text{free fermion}} \) are two-point functions computed in the theory of single free boson or single fermion with the normalization of operators such that (4.8) and (4.20) holds. We also used in the first line the fact that \( n_s^{\text{free boson}} = n_s^{\text{free fermion}} \) in the normalization that we adopted. This is explained in appendix C.

The three-point functions in the interacting theory are then given by

\[
\begin{align*}
\alpha_{s,s,s} &= \tilde{N} \frac{\lambda^2}{1 + \lambda^2}, \\
\beta_{s,s,s} &= \tilde{N} \frac{1}{1 + \lambda^2}, \\
\gamma_{s,s,s} &= \tilde{N} \frac{\lambda}{1 + \lambda^2}, \\
\beta_{s,s,0} &= \tilde{N} \frac{1}{1 + \lambda^2}, \\
\gamma_{s,0,0} &= 0.
\end{align*}
\]

(5.4)
All coefficients not explicitly written do not appear because there is no corresponding structure. The two parameters are defined as follows. We take the stress tensor to have a canonical normalization. We then set
\[ \tilde{N} = \frac{n_2}{n_2^{\text{free boson}}} , \]
\[ \tilde{\lambda}^2 = \frac{\alpha_{222}}{\beta_{222}} . \]

From the bounds on energy correlators discussed in [14], it follows that \( \tilde{\lambda}^2 \geq 0 \). Note that one would then find that \( \tilde{\lambda} \propto a_2 \tilde{N} \), with \( a_2 \) defined in (2.2). We have also computed the anomalous dimension of the spin-4 current (see appendix A):
\[ \tau_4 - 1 = \frac{32}{21 \pi^2} \frac{1}{\tilde{N}} \frac{\tilde{\lambda}^2}{(1 + \tilde{\lambda}^2)} . \]

5.2. Quasi-boson theory

The interacting theory two-point functions are given by
\[ n_s = \tilde{N} \frac{1}{1 + \tilde{\lambda}^2} n_s^{\text{free boson}} + \tilde{N} \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} n_s^{\text{free fermion}} = \tilde{N} n_s^{\text{free boson}} , \quad s \geq 2 , \]
\[ n_0 = \tilde{N} \frac{1}{1 + \tilde{\lambda}^2} n_0^{\text{free boson}} , \]
and the three-point functions are
\[ \alpha_{s_1 s_2 s_3} = \tilde{N} \frac{1}{1 + \tilde{\lambda}^2} , \quad \beta_{s_1 s_2 s_3} = \tilde{N} \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} , \quad \gamma_{s_1 s_2 s_3} = \tilde{N} \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} , \]
\[ \alpha_{s_1 s_0} = \tilde{N} \frac{1}{1 + \tilde{\lambda}^2} , \quad \gamma_{s_1 s_2 s_3} = \tilde{N} \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} , \]
\[ \alpha_{s_0 s_0} = \tilde{N} \frac{1}{1 + \tilde{\lambda}^2} + z_1 a_3 n_0^3 , \]
where we have separated out the correlators involving the scalar. Two of the parameters are defined by (taking into account that the stress tensor has a canonical normalization)
\[ \tilde{N} = \frac{n_2}{n_2^{\text{free boson}}} , \]
\[ \tilde{\lambda}^2 = \frac{\beta_{222}}{\alpha_{222}} . \]

Again, one can see that \( \tilde{\lambda}^2 \geq 0 \) from the bounds on energy correlators discussed in [14]. A third parameter can be introduced which is the combination involving \( a_3 \) that shifts the three-scalar correlator in (5.8). Again, we find that \( \tilde{\lambda} \propto a_2 \tilde{N} \) in (2.3). Thus, we find a three-parameter family of solutions, as expected to leading order in the large-\( N \) limit.

5.3. The critical point of \( O(N) \)

The critical point of \( O(N) \) is a theory that we obtain from the free-boson theory by adding a \( j^2_0 \) interaction and flowing to the IR, while tuning the mass of the scalars to criticality. Then,
the operator $j_0$ in the IR should have dimension 2, for large $N$. Thus, we can view it as a $j_0$ scalar operator of dimension 2. This is a theory that fits into the quasi-fermion case. Namely the divergence of $J_4$ is given by (2.2). By the way, if we assume that the IR limit has a scalar operator of dimension different from 1, then since (2.2) is the only expression we can write down for the divergence of the current, we conclude that the operator has to have dimension 2. Note that in this case, $j_0$ is parity even and (2.2) is consistent with parity. Thus, we could in principle do the same analysis as above. The only point where something different occurs is at (4.20) where $j_0$ cannot appear on the right-hand side since it is inconsistent with the parity of the theory. In addition, all parity odd correlators should be set to zero. In this case, parity implies that

$$[Q, \tilde{j}_0] = \partial^3 \tilde{j}_0. \quad (5.10)$$

In principle, we have an arbitrary coefficient in this equation but the coefficient is then fixed by considering various charge conservation identities we mention below. When we write the conservation identity for $\langle j_0 j_2 j_2 \rangle$, we will obtain the correlator $\langle j_0 j_2 j_2 \rangle$ as the integral term on the right-hand side. Again, considering the Ward identity on this last one will require $\langle j_0 j_0 j_2 \rangle$ on the right-hand side. In this fashion, one can determine the solution. Note that in this case, there is only one parameter which is $\tilde{N}$.

Interestingly, we can obtain these correlators by taking the large-$\tilde{\lambda}$ limit of (5.3), (5.4). At the level of three-point functions of currents, the limit is simple to take, namely we see that for any $s \geq 2$,

$$\langle s_1 s_2 s_3 \rangle \rightarrow \langle s_1 s_2 s_3 \rangle_{\text{bos}}, \quad (5.11)$$

so that all three-point functions become purely boson ones. For $\tilde{j}_0$, due to (5.3), it is necessary to rescale the operator and define a new operator $\tilde{j}_0 = \tilde{\lambda} j_0$. The two-point function of $\tilde{j}_0$ remains finite. This also has the nice feature of removing $j_0$ from the right-hand side of (4.20). The three-point functions of the form $\langle \tilde{j}_0 j_2 j_2 \rangle$ loose their $\beta$ structure and remain only with the parity preserving $\gamma_{000}$ structure. The three-point structures involving two scalars survive through a $\beta$ structure, after rescaling the operator. $\gamma_{000}$ stays being equal to zero, which is consistent with the large-$N$ limit of the critical $O(N)$ theory. This three-point function becomes non-zero at higher orders in the $1/N$ expansion.

Note that at $\tilde{\lambda} = \infty$, the three-point functions become parity-invariant. However, the parity of the operator $\tilde{j}_0$ got flipped compared to the one at $\tilde{\lambda} = 0$.

This suggests that the large-$\tilde{\lambda}$ limit of the fermions plus Chern–Simons matter theory should agree with the critical $O(N)$ theory, at least in the large-$N$ limit. This was conjectured in [12]. But this conjecture appears to require a funny relation between $N$’s and $k$’s of both theories. In other words, if we start with the quasi-fermion theory with $N$ and $k$ and take the limit where $\lambda$, defined in [12], tends to 1, then the conjecture would say that this should be the same as the critical $O(N)$ theory with some $k'$ in the limit that $k' \rightarrow \infty$. But the behavior of the free energy in [12] would require that

$$N' \propto N(\lambda - 1)[-\log(\lambda - 1)]^3 \quad (5.12)$$

as $\lambda \rightarrow 1$. Here, the proportionality is just a numerical constant. This formula is derived as follows. First, note that Giombi et al [12] derived a formula for the free energy, in the large-$N$ limit, for a theory of $N$ fermions with a Chern–Simons coupling $k$. The conjecture is that this matches a critical bosonic $O(N)$ theory perturbed by a Chern–Simons coupling $k'$. When $\lambda \rightarrow 1$, we expect that $k' \rightarrow \infty$. So we can compute the free energy of the critical bosonic $O(N)$ ignoring the Chern–Simons coupling. The free energy of the $O(N)$ theory with no Chern–Simons coupling goes like $N'$, for large $N'$ [21]. Matching the two expressions for the
free energy, we obtain the relation (5.12). At first sight, (5.12) seems incompatible with the fact that $N'$ should be an integer. However, we should recall that (5.12) is only supposed to be true in the large-$N$ (and $N'$) limit. Thus, it could be that there is an integer-valued function of $N$ and $\lambda$ which reduces to the right-hand side of (5.12) in the large-$N$ limit. Note that if this were true, we would also find the same function in the two-point function of the stress tensor, the function $f$ discussed in (4.25). Of course, here we are assuming that our $\tilde{\lambda} \to \lambda$ limit is the same as the $\lambda \to 1$ limit in \[12\].

5.4. The critical $O(N)$ fermion theory

Starting from $N$ free fermions, we can add a perturbation $\hat{j}_0$. This is the three-dimensional Gross–Neveu model [22]. This is an irrelevant perturbation. One can wonder whether there is a UV fixed point that leads in the IR to the free fermion plus this perturbation. In the large-$N$ limit, it is easy to see that such a fixed point exists and it is given by a theory where the operator $\hat{j}_0$ in the UV has dimension 1 [23]. Thus, it has the properties of the quasi-boson theory. Again, this is a theory that is parity symmetric. In [23], this theory was argued to be renormalizable to all orders in the $1/N$ expansion.

We can now take the large-$g$ limit of the quasi-boson results (5.7), (5.8). Again, we need to rescale $\hat{j}_0 = \tilde{\lambda} \hat{j}_0$. Only $\beta_0, \alpha_3, \alpha_4$ survive in this limit. With one scalar operator, we obtain only the $\gamma_{000}$ structure surviving, which is consistent with parity since now $\hat{j}_0$ is parity odd.

Note that, after the rescaling of the scalar operator, $\alpha_{\bar{a}\bar{a}} \bar{a}$ still tends to zero if we hold $a_3$ fixed. This is necessary because the three-point function of $\langle \hat{j}_0 \hat{j}_0 \hat{j}_0 \rangle$ should be zero by parity.

In principle, we could also introduce, in the large-$N$ limit, a parity breaking interaction of the form $\hat{j}_3$ which would lead to a non-zero three-point function. This could be obtained by rescaling $a_3$ in (5.8), so that a finite term remains.

5.5. Relation to other Chern–Simons matter theories

Throughout this paper, we have focused on theories where the only single trace operators are given by even spin currents, plus a scalar operator. This is definitely the case for theories with $N$ bosons or $N$ fermions coupled to an $O(N)$ Chern–Simons theory.

If instead we consider a theory of $N$ complex fermions coupled to a $U(N)$ or $SU(N)$ Chern–Simons gauge field, then we have additional single trace operators. We still have the same spectrum for even spins, but we also have additional odd spin currents. However, the theory has a charge conjugation symmetry under which the odd spin currents obtain a minus sign. Thus, the odd spin currents cannot appear on the right-hand side of $[Q, j_s]$ where $s$ is even. These odd spin currents can (and do) appear on the right-hand side of $\nabla \cdot J_4$ in bilinear combinations. However, since we only considered insertions of $J_4$ into correlators involving operators with even spin, these extra terms do not contribute. Thus, the whole analysis in this paper goes through, for these Chern–Simons theories. Our results give the three-point functions of even spin currents\[13\]. Presumably, a similar analysis can be done for correlators of odd spin currents, but we will not do this here. It is also worth mentioning that these theories do not have a single trace twist-3 operator which could appear on the right-hand side of $\nabla \cdot J_4$. Thus, the higher spin symmetry breaking only happens through double trace operators.

\[13\] The result for the anomalous dimension (5.6) would be changed by the presence of the odd spin currents.
5.6. Comments about higher dimensions

We can consider the extension of the small breaking ansatz that we explored for $d = 3$ to higher dimensions. For simplicity, we limit ourself to the case of almost conserved currents which are symmetric traceless tensors. We assume that the presence of a conserved $J_4$ will again fix all three-point functions$^{14}$. Thus, we are interested in the vector-like scenarios when the conservation of $J_4$ is broken at $\frac{1}{N}$.

To analyze this possibility, we consider $\partial_\mu J^\mu$ in an arbitrary number of dimensions. This operator has twist $d$, while the conserved currents has twist $d - 2$. Matching also the spin, we get that we can write the following equation:

$$\partial_\mu J^\mu = a \partial J^\mu + b \partial J^\mu,$$

where $\frac{d}{2}$ is fixed by the condition that the right-hand side is a primary operator. The scalar operator $O$ has the scaling dimension $\Delta_1 = 2$ by matching the quantum numbers.

First, note that the unitarity bound for the scalar operators is $d - 2$. Thus, if we restrict our attention to unitary theories, equation (5.13) can be only valid in $d \leq 6$.

We also need to check that there exists a three-point function $\langle J_4 J_2 O \rangle$ that reproduces (5.13). Imposing the conservation of $J_2$ leads to the result (5.13) as long as $\Delta_O \neq d - 2$. When $d = 4$, and $\Delta_O = 2$, the correlator $\langle J_4 J_2 O \rangle$ obeys $J_4$ current conservation automatically once we impose the $J_2$ current conservation$^{15}$.

Thus, we conclude that the scenario that we considered in $d = 3$ is impossible to realize in $d = 4$. In $d = 5$, it seems possible to realize the scenario via the UV fixed point of a $-(\bar{\phi} \phi)^2$ theory. This is a sick theory because the potential is negative, but one would probably not see the problem in $1/N$ perturbation theory. In $d = 6$, we do not know whether there is any example.

5.7. A comment on higher order parity violating terms

The parity breaking terms in Vasiliev’s theories are characterized by a function $\theta(X) = \theta_0 + \theta_2 X^2 + \cdots$ [1]. The computations we have done are sensitive to $\theta_0$. At a tree level, the term proportional to $\theta_2$ would start contributing to a planar five-point function, but not to lower point functions [12]. However, if we start with $\theta = 0$ and try to add $\theta_2$ (keeping $\theta_0 = 0$), we run into the following issues. For $\theta_0 = 0$, we can choose boundary conditions that preserve the higher spin symmetry. That symmetry fixes all $n$-point functions$^{14}$. Thus, if turning on $\theta_2$ modifies a five-point function, then it must be breaking the higher spin symmetry. But if we break the higher spin symmetry, we need to modify a three- or four-point function at the same order. (To see this, we can just consider the correlator of $J_4$ together with the currents that appear on the right-hand side of (1.3.) However, $\theta_2$ would only contribute to the five-point function. Thus, our conclusion is that the function $\theta(X)$ is either constrained to be constant or its non-constant part can be removed by a field redefinition. In this argument, we have assumed that these deformations of Vasiliev’s theory in AdS$_4$ also lead to boundary correlators that obey all the properties of a CFT.

$^{14}$ Although we have not proved this, the discussion in [14] looks almost identical in higher dimensions for symmetric traceless tensors.

$^{15}$ We thank David Poland for providing us with the Mathematica code to analyze the relevant three-point functions in the case of higher dimensions.

$^{16}$ Note that in $d = 4$, we can write also $j_-, j_-, \ldots$. However, in this case, $J_4$ is automatically conserved as soon as we impose conservation of $J_2$ and $J_1$. 

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In this paper, we have computed the three-point correlation functions in conformal field theories with a large-\(\tilde{N}\) approximation, where the higher spin symmetry is broken by \(1/\tilde{N}\) effects. Equivalently, we have performed an on-shell analysis of Vasiliev theories on AdS\(_4\), constraining the boundary three-point correlators. These constraints also apply to the dS\(_4\) case, where they can be viewed as computing possible non-Gaussianities in Vasiliev’s theory. From the dS\(_4\) point of view, we are computing the leading non-Gaussianities of the de Sitter wavefunction\(^{17}\).

We restricted the theories to contain a single trace spectrum with only one spin-2 current (the stress tensor) and only one scalar. The scalar can only have dimensions either 1 or 2 at leading order in \(\tilde{N}\). This defines two classes of correlators which we called quasi-fermion (scalar of dimension 2) and quasi-bosons, where the scalar has dimension 1. These are the only possible dimensions that enable the \(1/\tilde{N}\) breaking of the higher spin symmetry. The final three-point functions depend on an overall constant \(\tilde{N}\) which is also the two-point function of the stress tensor. Thus, we can view \(1/\tilde{N}\) as the coupling of the bulk theory. In addition, they depend on an extra parameter which selects the relative weights of the three possible structures in the correlators. The final results are given in (5.4) and (5.8). These results apply, in particular, to theories of \(N\) bosons or \(N\) fermions coupled to an SO\((N)\) Chern–Simons gauge field. Here, \(\tilde{N}\) scales with \(N\) and the extra parameter \(\tilde{\lambda}\) is a function of \(N/k\), where \(k\) is the Chern–Simons level; see (4.25).

These results also apply to Vasiliev’s theories which have parity breaking terms in the Lagrangian [1]. It also applies to Vasiliev’s theories with boundary conditions that break the higher spin symmetry, but preserve conformal invariance.

At strong coupling, the quasi-fermion three-point functions go into the three-point functions of the critical \(O(N)\) theory. This suggests that there should be a duality between the small \(k\) limit of the Chern–Simons theories and the critical \(O(N)\) theory [12]. However, the thermal partition function computed in [12] suggests that if this duality is true, the connection between the values of \(N\) and \(k\) of the two theories is rather intricate; see (5.12). It would be nice to further understand this issue, and to find the correct duality, if there is one. Naively, if our methods actually fix all \(n\)-point functions, then all leading-order correlators would be consistent with the duality. This duality would be a three-dimensional version of bosonization.

Our method was based on starting with the simplest possible even spin higher spin current \(J_4\) and writing the most general form for its divergence. We then noted that the violation of conservation of the \(J_4\) current leads to constraints on three-point functions, which we solved. These equations were mostly identical to the ones one would obtain in the conserved case, except when acting on correlators with spin 2 or lower. In these cases, we obtained some relations which eliminated some of the free parameters and left only two parameters in the quasi-fermion theory and three parameters in the quasi-boson case. These parameters match with the free parameters that we have in large-\(N\) Chern–Simons models.

Since we only considered the current \(J_4\), one can wonder whether it is possible to add other deformation parameters which affect current conservation for higher spin currents but not \(J_4\). We argue in appendix F that this is not possible.

An interesting extension of these results would be to carry out this procedure for higher point functions. This is in principle conceptually straightforward, but it seems

\(^{17}\) Of course, for gravitons, these match the particular structures discussed in [19], but with particular coefficients.
Figure 3. The analysis of three-point functions can be summarized by this picture. The quasi-fermion theory is the top line and the quasi-boson is the bottom line. At the two end points, we have the free boson or fermion theory on one side and the interacting $O(N)$ theories on another. This is a statement about the three-point functions. It would be interesting to understand whether we have a full duality between the two theories. Note that the duality would relate a theory of fermions with a theory with scalars. This duality would be a form of bosonization in three dimensions. We also expect an RG flow connecting the quasi-boson theory on the top line and the quasi-fermion theory on the bottom line for general values of the Chern–Simons coupling.

computationally difficult. Note that this analysis amounts to an on-shell study of the Vasiliev theory. We study the physical, gauge-invariant observables of this theory with AdS$_4$ or dS$_4$ boundary conditions. As has often been emphasized, the on-shell analysis of gauge theories can be simpler than doing computations in a fully Lorentz-invariant formulation.

The methods discussed in this paper could be viewed as on-shell methods to compute correlation functions in certain matter plus Chern–Simons theories. Note that we did not have any gauge-fixing issues, since we never considered gauge non-invariant quantities. These methods apply only to the special class of theories that do not contain a twist-3, spin-3 single trace operator that can directly Higgs the $J_4$ higher spin symmetry already at leading order in $N$. It would also be interesting to consider cases where this Higgsing can occur already for single trace operators. If the mixing with this other operator is small, which occurs in weakly coupled theories, we can probably generalize the discussion of this paper. We would only need to add the twist-3 single trace operator on the right-hand side of the divergence of the current $\nabla J_4$. This would lead to an on-shell method for computing correlators. Something in this spirit was discussed in [24, 25]. This Higgsing mechanism, named ‘La Grande Bouffe’ in [26] is also important for understanding the emergence of a more ordinary looking string theory in AdS from the higher spin system.

Our analysis also works for higher spin theories on de Sitter space. In that case, we are constraining the wavefunction of the universe. It is valid for the proposed examples of dS-CFT [27, 28], as well as further examples that one might propose by looking at the parity violating versions of the Vasiliev theory. To consider these cases, one should set $\tilde{N} < 0$ in our formulas. Of course, the constraints hold whether we know the CFT dual or not!

We have restricted our analysis to the case of AdS$_d$/CFT$_{d-1}$. Recently, examples of AdS$_3$/CFT$_2$ theories with higher spin symmetry have been considered. See, for example, [29–34]. In lower dimensions, the higher spin symmetry appears less restrictive, so one would need a more sophisticated analysis than the one presented in this paper. On the other hand, in higher dimensions, the higher spin symmetry is more constraining. The higher dimensional case will hopefully be discussed separately.

We have used conformal symmetry in an important way; it would also be interesting to study non-conformal cases. For example, we can expect to constrain the flows between the top and bottom lines of figure 3.
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Appendix A. Why the divergence of the currents should be conformal primaries

In this appendix, we show that the divergence of an operator with spin becomes a conformal primary when $\tau - 1 \to 0$. What happens is that the large representation with twist $\tau$ and spin $s$ is splitting into a representation with $\tau = 1$ and spin $s$ plus a representation with $\tau = 3$ and spin $s - 1$. The second representation is what appears on the right-hand side of the divergence of $J_s$ as $\tau \to 1$. This is a well-known fact that was used in the past in [16, 24, 25, 15]. Here, we recall its proof for completeness.

Let us normalize the current to 1, $\langle J_s | J_s \rangle$. Let us define

$$\partial J_s = \alpha O_{s-1}, \quad (A.1)$$

where $O_{s-1}$ is also normalized to 1. Then, we have

$$\langle \partial J | \partial J \rangle \propto (\tau - 1) \langle J | J \rangle = \tau - 1$$

$$\langle \partial J | \partial J \rangle = \alpha^2, \quad (A.2)$$

where in the first line, we converted the derivative in the bra into a special conformal generator in the ket, and then commuted it through the derivative in the ket using the conformal algebra. We ignored numerical factors. In the second line, we used (A.1). The conclusion is that

$$\tau - 1 \propto \alpha^2. \quad (A.3)$$

We now act on both sides of (A.1) with the special conformal generator $K_\nu$. On the left-hand side, we use the conformal algebra to evaluate the answer. We then obtain that $\alpha K_\mu O_{s-1} \propto (\tau - 1) J_\mu \propto \alpha^2 J_\mu$. The conclusion is that $\langle K_\mu O_{s-1} | K_\mu O_{s-1} \rangle \propto \alpha^2$. Thus, we see that, to leading order in $\alpha$, the operator $O_{s-1}$ is a conformal primary. Of course, this is true regardless of whether $O_{s-1}$ is a single trace or multitrace operator, as long as $\tau - 1$ is very small.

Finally, note that if we know $\alpha$ appearing in (A.1), then we can compute the anomalous dimension $\tau - 1$ of the current via the same formulas.

In particular, in the quasi-fermion theory, we have argued around (4.23) that $a_2 \sim \tilde{\lambda}/\tilde{N}$. This together with the value of the $j_0$ two-point function (4.33) implies that for the spin-4 current, we have

$$\tau_4 - 1 = \frac{32}{21\pi^2} \frac{1}{\tilde{N} 1 + \tilde{\lambda}^2}. \quad (A.4)$$

This formula should be applied to $O(N)$ theory.

For general group and arbitrary spin $s$, we expect the following formula to be true:

$$\tau_s - 1 = a_s \frac{1}{\tilde{N} 1 + \tilde{\lambda}^2} + b_s \frac{1}{\tilde{N} (1 + \tilde{\lambda}^2)^2}, \quad (A.5)$$

where $a_s$ and $b_s$ are some fixed numbers.

In principle, these arguments allow us to fix the overall numerical coefficient. However, as a shortcut, we have used the formula for the anomalous dimensions for the critical $O(N)$ theory given in equation (2.20) of [35]. Thus, we fixed the overall coefficient in (A.4) so that the $\lambda \to \infty$ limit matches [35]. This also fixes $a_s = \frac{16}{3\pi^2} \frac{\tau^2}{2s-1}$.
Appendix B. Structure of the pseudo-charge conservation identities

In this appendix, we recall how the various coefficients that appear in the charge conservation identity are fixed. Assume that we consider only twist-1 three-point functions, and that the spins are all non-zero. Then, we can use (4.7), perhaps including a $j^{\text{eff}}_0$ operator when it acts on $j_2$. Then, the action of $Q$ on $\langle j_1 j_2 j_3 \rangle$ gives an expression of the form

$$\sum_{i=0,\pm 2} r_{\text{type},a_i} j^{3-i} (j_1 + j_2 + j_3) \text{bos} + \text{cyclic} + [\text{bos} \rightarrow \text{fer}] + [\text{bos} \rightarrow \text{odd}].$$

(B.1)

Here, we have the set of coefficients $r_{\text{type},a_i}$. Here, type goes over bos, fer, odd. $a_i$ labels the point and runs over 1,2,3. Finally, $i$ runs over $\pm 2, 0$. In total, there are up to $3^3 = 27$ coefficients. These coefficients result from multiplying $c_{s,s'}$ in (4.7) and $\alpha, \beta$ and $\gamma$ in (4.12).

These equations split into three sets of equations, one for each type. In each set of equations, the coefficients in front of different three-point functions are all fixed up to an overall constant, except in the cases where the corresponding three-point structure vanishes automatically, where, obviously, the corresponding coefficient is not fixed.

For the boson and fermion types, this follows from the discussion in appendix J of [14]. One simply needs to take successive light cone limits of the three possible pairs of particles to argue that the coefficients are all uniquely fixed.

For the odd structure, the situation is more subtle since some of the equations do not have any non-zero solutions if we restrict to structures inside the triangle rule $s_i \leq s_{i+1} + s_{i-1}$. However, there are non-zero solutions once we take into account that the current non-conservation allows solutions outside the triangle. We have checked this in some cases, and we think that it is likely to be true in general, but we did not prove it. We know that there is at least one solution with non-zero coefficients, which is the one that the Chern–Simons construction would produce. Thus, in order to show that the solution is unique, we would need to show that there is no solution after we set one of the coefficients to zero. We leave this problem for the future.

Appendix C. Compendium of normalizations

Here, we would like to present more details on the normalization convention that we chose for the currents. Noted that we have set them by the choice (4.8). Here, we will check that with (4.8), the coefficients of all the terms in (4.7) are the same for the single free-boson and single free-fermion theory. This is related to the fact that the higher spin algebra is the same for bosons and fermions in three dimensions (see, for example, [8]).

In this appendix, we do computations in the free theories. Then, it is convenient to consider the matrix elements of the currents in Fourier space. As explained in appendix J of [14], it is convenient to introduce a combination of the two momenta that appear in the on-shell matrix elements of a current with two on-shell fermions or bosons. The idea is roughly to change

$$\partial^m \psi_1 \rightarrow (z + \bar{z})^{2m+1}, \partial^k \psi_2 \rightarrow (-1)^k (z - \bar{z})^{2k+1},$$

where the indices 1 and 2 denote the two fermion fields that make up the current. Then, currents take a form

$$j_z = \alpha_z [z^{2s} - \bar{z}^{2s}].$$

(C.1)

where we introduced the normalization factor $\alpha_z$ explicitly.

The charge generated by $j_4$ that we consider in the main text is then given by

$$Q = \alpha_Q [z + \bar{z}]^{6} - (z - \bar{z})^{6} \propto \left[ \partial_{t_1}^3 + \partial_{t_2}^3 \right].$$

(C.2)
where we again introduced the normalization factor for the charge. We have written the action of the charge on the two free fields that make the current. Also, note that \( \alpha_2 \) is fixed in a canonical way.

Now we can start fixing the normalizations. We start from \( c_{2,4} = 1 \). Using \([Q, j_2]\), we obtain

\[
\alpha_Q = \frac{\alpha_4}{12\alpha_2},
\]

then we consider \([Q, j_4]\) to fix \( c_{4,4} = 1, c_{4,6} = 1 \). It gives

\[
\begin{align*}
\alpha_4 &= \frac{3}{10} \alpha_2 \\
\alpha_6 &= \left( \frac{17}{10} \right)^2 \alpha_2.
\end{align*}
\]

A simple analysis further shows that

\[
\alpha_s = \left( \frac{3}{10} \right) \alpha_{s-2}.
\]

This follows from formula (J.12) in [14] which can be written as

\[
[Q_r, j_r] = (2s - 2)! \sum_{r=0}^{s-2} \bar{\alpha}_r(s, s') \partial^{s+s'-r} j_r,
\]

\[
\bar{\alpha}_r(s, s') = \left[ 1 + (-1)^{s+s'+r} \right] \frac{1}{\Gamma(r+s-s') \Gamma(s+s'-r) \Gamma(s-s'-r)}.
\]

Note that in this normalization, \( c_{2,2} \) is also fixed to 1 automatically. Here, the \( \pm \) sign corresponds to the boson or fermion case.

The case of bosons is almost identical. It is important that the term with \( \pm \) in (C.6) is zero for all terms \( \bar{\alpha}_r(4, s') \) with \( r, s' \geq 2 \). These are all the cases where we can compare the normalization between fermions and bosons. This implies that all constants \( c_{\alpha,s} \) and \( c_{\alpha,s,\pm 2} \) that appear in (4.7) are the same in the free-boson or free-fermion theories.

The case when one of the spins is zero corresponds to the appearance of the scalar operator of twist-1 \( j_0 \), which is only present in the free-boson theory.

Moreover, using the formulas above and the results in [36], one can check that in the normalization, we are using \( n_{j_2}^{\text{free boson}} = n_{j_2}^{\text{free fermion}} \). More precisely, one can first relate the normalization used in [36] to the normalization here and then use the results in [36] for two-point functions to obtain that

\[
\begin{align*}
n_{j_2}^{\text{free boson}} &= n_{j_2}^{\text{free fermion}} \\
\alpha_2 &= \frac{1}{16}, \\
\alpha_4 &= \left( \frac{3}{10} \right) \alpha_{s-2}.
\end{align*}
\]

We used this result in the main text.

After we fix all normalizations of currents in this way, we can compute all three-point functions in the free-fermion theory \( \langle j_{i_1} j_{i_2} j_{i_3} \rangle_{\text{fer}} \) (as well as in the theory of free boson to obtain \( \langle j_{i_1} j_{i_2} j_{i_3} \rangle_{\text{bos}} \)). These are the solutions that we use in the main text, for example, in (4.12).

**Appendix D. Exploring the scalar sector for the quasi-fermion**

In this appendix, we explain how the scalar sector correlation functions could be recovered. We start from writing the general form of the variation of the scalar operator \( j_0 \) under the action of \( Q \)

\[
\begin{align*}
[Q, j_0] &= \tilde{c}_{0,0} \partial \tilde{j}_0 + \tilde{c}_{0,2} \partial_0 j_{0,2} \\
[Q, \bar{j}_0] &= \tilde{c}_{0,0} \partial_0 \bar{j}_0 + \tilde{c}_{0,2} \partial_2 j_{0,2}.
\end{align*}
\]
where remember coefficients \( c \) are the ones we would obtain in the theory of free fermion with the normalization of the operators that we chose. The interesting part is a deviation from the free theory which we denoted by \( \tilde{c} \) following the notations used in the main text. In the second line, we introduce a shortened notation where we replaced \( j_\lambda \rightarrow s \) and in addition, we also introduced the symbol \( \tilde{\sigma}^{\pm} \) to denote the combination of derivatives and indices of \( J_\lambda \) that appear in the first line.

To fix some of the coefficients, we use the \((420)\) three-point function. It has two different structures: fermion and odd ones and we have already determined their coefficients. It is easy to see that it leads to the identities

\[
\psi_{420} \propto \tilde{\lambda} \eta_0
\]

(\ref{eq:420})

In the first line, we took the divergence of the equation. Note that the equations contain terms with various \( \gamma \) dependences. First, we match \( \tilde{\lambda} \) dependence in this equation then fixes

\[
\tilde{c}_{0,2} = \frac{1}{1 + \lambda^2}.
\]

(D.3)

**D.1. Fixing \( \tilde{c}_{0,0} \) and nontrivial consistency check**

Here, we fix \( \tilde{c}_{0,0} \) by considering the pseudo-conservation identities for \( \langle \tilde{0} s_1 s_2 \rangle \) where both \( s_1 \) and \( s_2 \) are larger than 2 for simplicity. We will encounter a rather intricate structure for this identity.

We schematically write relevant terms in pseudo-conservation identities with their scaling

\[
\tilde{c}_{0,0} \left( \frac{1}{1 + \lambda^2} \langle \partial^3 \tilde{0} s_1 s_2 \rangle_{\text{fer}} \right) + \tilde{c}_{0,0} \left( \frac{\tilde{\lambda}}{1 + \lambda^2} \langle \partial^3 \tilde{0} s_1 s_2 \rangle_{\text{odd}} + \frac{1}{1 + \lambda^2} \right)
\]

\[
\times \left( \frac{1}{1 + \lambda^2} \langle \partial^2 \tilde{s}_1 s_2 \rangle_{\text{fer}} + \frac{\tilde{\lambda}}{1 + \lambda^2} \langle \partial^2 \tilde{s}_1 s_2 \rangle_{\text{odd}} + \frac{\tilde{\lambda}^2}{1 + \lambda^2} \langle \partial^2 \tilde{s}_1 s_2 \rangle_{\text{bos}} \right)
\]

\[
+ \frac{1}{1 + \lambda^2} \langle \tilde{0} \text{ standard terms} \rangle_{\text{fer}} + \frac{\tilde{\lambda}}{1 + \lambda^2} \langle \tilde{0} \text{ standard terms} \rangle_{\text{odd}}
\]

\[
= \frac{\tilde{\lambda}}{1 + \lambda^2} \partial_{\tilde{\gamma}_1} \int \frac{1}{|x - x_1|^4} \left( \frac{1}{1 + \lambda^2} \langle 2s_1 s_2 \rangle_{\text{fer}} + \frac{\tilde{\lambda}}{1 + \lambda^2} \langle 2s_1 s_2 \rangle_{\text{odd}} + \frac{\tilde{\lambda}^2}{1 + \lambda^2} \langle 2s_1 s_2 \rangle_{\text{bos}} \right).
\]

(D.4)

Here, except for the terms involving \( \partial^{2\pm 2} \), all components of the currents are minus, as are all derivatives. Now let us start looking for a solution. All \( \tilde{\lambda} \) dependence is explicit in this equation. Note that the equations contain terms with various \( \tilde{\lambda} \) dependences. First, we match the double poles at \( \tilde{\lambda}^2 = -1 \). This requires the following two identities:

\[
\langle \partial^{2\pm 2} \rangle_{\text{odd}} = \partial_{\tilde{\gamma}_1} \int \frac{1}{|x - x_1|^4} \left( \langle 2s_1 s_2 \rangle_{\text{fer}} - \langle 2s_1 s_2 \rangle_{\text{bos}} \right)
\]

\[
\partial_{\tilde{\gamma}_1} \int \frac{1}{|x - x_1|^4} \langle 2s_1 s_2 \rangle_{\text{odd}} = - \langle \partial^{2\pm 2} \rangle_{\text{fer}} + \langle \partial^{2\pm 2} \rangle_{\text{bos}}.
\]

(D.5)

The second line can also be used to make all terms of the ‘fer’ terms work. This would work nicely if we set \( \tilde{c}_{0,0} = 1 \), which is what we wanted to argue. In fact, replacing the second line,
we obtain all the parity odd pieces in the full equation work out. By parity odd, we mean the terms that are odd under $y \to -y$.

We are then left with all terms that are even under $y \to -y$. After using the first equation in (D.5), we are left with the condition

$$\langle \delta^3 \tilde{0} s_1 s_2 \rangle_{\text{odd}} + \langle 0 \text{ standard terms} \rangle_{\text{odd}} = \partial \langle s_1 \tilde{c}_0 \rangle_{\text{bos}} \int \frac{1}{|x - x_1|^4} \langle 2 s_1 s_2 \rangle_{\text{bos}}. \quad (D.6)$$

To summarize, in this subsection, we fixed $\tilde{c}_{0,0} = 1$ and presented the self-consistency relations (D.5), (D.6) that are necessary for the whole construction to work. We checked one of the relation (D.5) in the light cone limit and it indeed works. It would be nice to check these identities more fully.

D.2. Fixing $\langle \tilde{s}_0 \tilde{s}_0 \rangle$

To fix these three-point functions, we consider pseudo-conservation identities that we obtain from $\langle \tilde{s}_0 \tilde{s}_0 \rangle$. Note that we already know that $\langle 2 \tilde{0} \tilde{0} \rangle$ and $\langle 0 \tilde{0} 0 \rangle$ functions are given by $\beta_{222}$. The first follows from the stress tensor Ward identity and the two-point function. The second follows from the action of the charge $Q$ on $\tilde{J}_0$, with $\tilde{c}_{0,0} = 1$ that we derived above.

Starting from this, we can build the induction. Consider first pseudo-conservation identity for $\langle 0 \tilde{0} 0 \rangle$. We obtain schematically

$$\beta_{\tilde{0}\tilde{0}\tilde{0}}(\partial \tilde{0} \tilde{0} \tilde{0})_{\text{fer}} + \frac{1}{1 + \lambda^2} (\langle \delta^3 \tilde{0} \tilde{0} \tilde{0} \rangle_{\text{fer}} + \langle \delta^3 \tilde{0} \tilde{0} \tilde{0} \rangle_{\text{fer}}) + \frac{1}{1 + \lambda^2} \times \left( \frac{1}{1 + \lambda^2} (4 \delta^2 \tilde{0} \tilde{0})_{\text{fer}} + \frac{\lambda}{1 + \lambda^2} (4 \delta^2 \tilde{0} \tilde{0})_{\text{odd}} \right)
+ \lambda \leftrightarrow x_3 \right)

\begin{equation}
\frac{1}{1 + \lambda^2} \partial \tilde{\lambda} \int \frac{1}{|x - x_1|^4} \left( \frac{1}{1 + \lambda^2} (4 \delta^2 \tilde{0})_{\text{fer}} + \frac{\lambda}{1 + \lambda^2} (4 \delta^2 \tilde{0})_{\text{odd}} + [x_2 \leftrightarrow x_1] \right) \right).
\end{equation}

(D.7)

Again, matching the double poles at $\tilde{\lambda}^2 = -1$ requires

$$\langle 4 \delta^2 \tilde{0} \tilde{0} \rangle_{\text{odd}} = \partial \tilde{\lambda} \int \frac{1}{|x - x_1|^4} \langle 4 \delta \tilde{0} \tilde{0} \rangle_{\text{fer}}, \quad (D.8)$$

Then, replacing these identities in the equations implies that the odd piece cancels and the fermion piece works if

$$\beta_{\tilde{0}\tilde{0}\tilde{0}} = \beta_{222} = \frac{\tilde{N}}{1 + \tilde{\lambda}^2}. \quad (D.9)$$

Then, we proceed by induction. Considering pseudo-charge conservation identities for $\langle \tilde{s}_0 \tilde{s}_0 \rangle$, we obtain a similar story. Double pole matching leads to

$$\langle \delta^2 \tilde{0} \tilde{0} \rangle_{\text{odd}} = \partial \tilde{\lambda} \int \frac{1}{|x - x_1|^4} \langle \delta \tilde{0} \tilde{0} \rangle_{\text{fer}}, \quad (D.10)$$

and the fermion piece fixes

$$\beta_{s\tilde{0}\tilde{0}} = \beta_{222} = \frac{\tilde{N}}{1 + \tilde{\lambda}^2}. \quad (D.11)$$

This fixes the three-point function involving two $\tilde{J}_0$ operators.
D.3. Fixing $\gamma_{000}^0$

To fix the last three-point function, we consider the WI (000). We obtain schematically the equation

$$\gamma_{000}^0 (\partial_1^3 + \partial_2^3 + \partial_3^3) (000)_{\text{odd}} = \frac{\tilde{\lambda} n_0}{N} \beta_{200} \sum_{\text{perm}} \partial_{x_i} \int \frac{1}{|x - x_1|^2 |x - x_2|^2 |x - x_3|^2} (200)_{\text{fer}}. \quad (D.12)$$

The right-hand side of this equation can be computed explicitly using the star-triangle identity (see, for example, [37]). More precisely, each term on the right-hand side of (D.12) can be rewritten (thanks to our sandwich geometry as we can pull out the derivatives) as

$$\frac{1}{x_{23}^2} \partial_{x_i} \left( \partial_2^2 + \partial_3^2 - 6 \partial_{x_1} \partial_{x_i} \right) \int d^3x \frac{1}{|x - x_1|^2 |x - x_2|^2 |x - x_3|^2} + \text{perm.} \quad (D.13)$$

The integral is finite and is given by $\frac{n_0}{x_{12}^2 x_{23}^2}$. By taking the derivatives and summing over permutations which come from different contractions in (D.12), one can check that the sum is actually zero. Thus, we are forced to set

$$\gamma_{000}^0 = 0. \quad (D.14)$$

This is nicely consisted with the critical $O(N)$ limit.

Appendix E. Exploring the scalar sector for the quasi-boson

Now we would like to go through the same analysis but for the quasi-boson theory.

We start by writing the general form of the variation of the scalar operator $j_0$ under the action of $Q$

$$[Q, j_0] = \tilde{c}_{0,0} c_{0,0} \partial^3 j_0 + \tilde{c}_{0,2} c_{0,2} \partial j_2. \quad (E.1)$$

Recall that coefficients $c$ are the ones we would obtain in the theory of free boson with the normalization of the operators that we chose. The interesting part is a deviation from the free theory which we denoted by $\tilde{c}$ following the notations used in the main text.

First, we consider the (420) three-point function. It has two different structures: boson and odd ones. This leads to the identities

$$\gamma_{420} \propto \tilde{\lambda} n_0$$

$$\tilde{c}_{0,2} c_{0,2} n_2 = \alpha_{420} = \alpha_{222}. \quad (E.2)$$

In the first line, we simply have taken the divergence of $J_4$ and used (2.3). In the second line, we have integrated $J_4$ around $J_0$. From the first line, we obtain that $n_0 = \frac{1}{1 + \tilde{\lambda}^2}$ where we used the fact that at $\tilde{\lambda} = 0$, it should be equal to 1. The second equation then fixes

$$\tilde{c}_{0,2} = \frac{1}{1 + \tilde{\lambda}^2}. \quad (E.3)$$

E.1. Fixing $\tilde{c}_{0,0}$

To fix $\tilde{c}_{0,0}$, we are analogously considering the pseudo-conservation identities for $(0s_1 s_2)$, where both $s_1$ and $s_2$ are larger than 2 for simplicity.

We schematically write relevant terms in pseudo-conservation identities with their $\tilde{\lambda}$ scaling

$$\tilde{c}_{0,0} \frac{1}{1 + \tilde{\lambda}^2} \langle \partial^3 0 s_1 s_2 \rangle_{\text{bos}} + \tilde{c}_{0,0} \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} \langle \partial^3 0 s_1 s_2 \rangle_{\text{odd}}$$

$$+ \frac{1}{1 + \tilde{\lambda}^2} \left( \frac{1}{1 + \tilde{\lambda}^2} \langle \partial 2 s_1 s_2 \rangle_{\text{bos}} + \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} \langle \partial 2 s_1 s_2 \rangle_{\text{odd}} + \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} \langle \partial 2 s_1 s_2 \rangle_{\text{fer}} \right)$$
Again, we match the double poles at $\tilde{\lambda}^2 = -1$ to obtain
\begin{align*}
\langle \partial^2 s_1 s_2 \rangle_{\text{odd}} &= \int \frac{1}{|x - x_1|^2} \left( \langle \partial^2 \pm 2 s_1 s_2 \rangle_{\text{bos}} - \langle \partial^2 \pm 2 s_1 s_2 \rangle_{\text{fer}} \right) \\
-\langle \partial s_1 s_2 \rangle_{\text{bos}} + \langle \partial s_1 s_2 \rangle_{\text{fer}} &= \int \frac{1}{|x - x_1|^2} \langle \partial^2 \pm 2 s_1 s_2 \rangle_{\text{odd}}.
\end{align*}
\tag{E.5}

Now using these equations, we find that all the even pieces under $y \to -y$ work properly only if $c_{0,0} = 1$. Then, the odd piece reduces to the condition
\begin{align*}
\langle \partial^3 0 s_1 s_2 \rangle_{\text{odd}} + (0 \text{ standard terms})_{\text{odd}} &= \int \frac{1}{|x - x_1|^2} \langle \partial^2 \pm 2 s_1 s_2 \rangle_{\text{fer}}.
\end{align*}
\tag{E.6}

We have not checked explicitly whether (E.6) and (E.5) are true, but they should be for consistency.

E.2. Fixing $\langle 000 \rangle$

To fix these three-point functions, we consider pseudo-conservation identities that we obtain from $\langle 000 \rangle$. Note that we already know that $\langle 200 \rangle$ and $\langle 400 \rangle$ functions are given by $\alpha_{222}$. This follows from the stress tensor Ward identity and from the action of $Q$ on $j_0$, together with the normalization of $j_0$.

Starting from this, we can build the induction. Consider first pseudo-conservation identity for $\langle 400 \rangle$. We obtain schematically
\begin{align*}
\alpha_{400} \langle \partial 400 \rangle_{\text{bos}} + \frac{1}{1 + \tilde{\lambda}^2} \left( \langle \partial^5 200 \rangle_{\text{bos}} + \langle \partial^3 400 \rangle_{\text{bos}} \right) &+ \frac{1}{1 + \tilde{\lambda}^2} \left( \langle \partial^3 200 \rangle_{\text{bos}} + \langle \partial^3 400 \rangle_{\text{odd}} \right) \\
+ \frac{1}{1 + \tilde{\lambda}^2} \langle \partial^3 000 \rangle_{\text{bos}} + \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} \langle \partial^3 000 \rangle_{\text{odd}} + [x_2 \leftrightarrow x_1] &+ \int \frac{1}{|x - x_1|^2} \left( \frac{1}{1 + \tilde{\lambda}^2} \langle \partial^2 \pm 2 20 \rangle_{\text{bos}} + \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} \langle \partial^2 \pm 2 20 \rangle_{\text{odd}} + [x_2 \leftrightarrow x_1] \right).
\end{align*}
\tag{E.7}

Again looking at the double pole at $\tilde{\lambda}^2 = -1$, we obtain the equations
\begin{align*}
\langle \partial 20 \rangle_{\text{bos}} &= - \int \frac{1}{|x - x_1|^2} \langle \partial^2 \pm 2 20 \rangle_{\text{odd}}, \\
\langle \partial 20 \rangle_{\text{odd}} &= \int \frac{1}{|x - x_1|^2} \langle \partial^2 \pm 2 20 \rangle_{\text{bos}}.
\end{align*}
\tag{E.8}

This implies that $\alpha_{400} = \alpha_{222}$.

By induction through the identities
\begin{align*}
\langle x \partial 20 \rangle_{\text{bos}} &= - \int \frac{1}{|x - x_1|^2} \langle x \partial^2 \pm 2 20 \rangle_{\text{odd}}, \\
\langle x \partial 20 \rangle_{\text{odd}} &= \int \frac{1}{|x - x_1|^2} \langle x \partial^2 \pm 2 20 \rangle_{\text{bos}}.
\end{align*}
\tag{E.9}
we obtain that
\[ \alpha_{00} = \alpha_{22}. \]  
(E.10)

Again, we did not verify the identities (E.9) but they are necessary for consistency.

### E.3. Vanishing of the double trace term in the ⟨000⟩ identity

When we studied the pseudo-charge conservation identity for the triple scalar correlator \( \langle j_0 j_0 j_0 \rangle \) in the main text, we said that the double trace term in the divergence of \( J^4 \), (2.3), vanishes. Here, we prove that assertion.

We start from the following integral:
\[ I(x_1, x_2, x_3) = a_2 \int d^3x (\partial_0 \partial_0 - \partial_2 \partial_2) \int d^3y \frac{1}{|x_1 - y| |x_2 - y| |x_3 - y|}. \]  
where the operator \( j_0(x) \partial^2 J_j(x) \) comes from an insertion of the double trace term in (2.3). After some algebra, one can re-express this as the following integral:
\[ I(x_1, x_2, x_3) \sim \partial_1 (\partial_1 - \partial_2) \int d^3x \frac{1}{|x_1 - x_2| |x_1 - x_3|}. \]  
written in this way it is manifestly finite. However, we rewrite it again as follows:
\[ I(x_1, x_2, x_3) \sim \partial_1 (\partial_1 - \partial_2) \left[ y_{23} \partial_1 J(x_1, x_2, x_3) - \frac{1}{2} x_{23} \partial_1 J(x_1, x_2, x_3) \right]. \]  
where
\[ J(x_1, x_2, x_3) = \int d^3x \frac{1}{|x_1 - x_2| |x_1 - x_3|}. \]  
is the conformally invariant integral. It is divergent; however, the difference in (E.13) is, of course, finite. We can take this integral using the well-known star-triangle identity. We regularize the integral as
\[ J_\delta(x_1, x_2, x_3) = \int d^{3+\delta}x \frac{1}{|x_1 - x_2| |x_1 - x_3|}. \]  
using the star-triangle formula; expanding in \( \delta \) and plugging into (E.12), we see that the divergent piece cancels and the finite piece is given by
\[ I(x_1, x_2, x_3) \sim \frac{1}{x_{23}^2} \left( \partial_1 (\partial_1 - \partial_2) \right) \frac{x_{12}^2 y_{23} + x_1 y_{31} + x_2 y_{12}}{|x_1 - x_2| |x_1 - x_3| |x_2 - x_3|}. \]  
The total contribution to the pseudo-charge conservation identity is given by the sum of three terms which is zero
\[ I(x_1, x_2, x_3) + I(x_2, x_3, x_1) + I(x_3, x_1, x_2) = 0. \]  
(E.17)

Thus, the double trace non-conservation does not contribute to the ⟨000⟩ pseudo-charge conservation identity.

### Appendix F. Impossibility to add any further double trace deformations

Our whole analysis was based on studying the possible terms that appear in the divergence of the spin-4 current \( J_4 \). We could wonder whether we can add further double terms to the divergence of higher spin currents which are not fixed by the analysis we have already done. In other words, these would be terms that appear for higher spin currents but not for the spin-4 current. This would only be possible via the odd terms, which are the only ones that could have a non-conserved current. We suspect that all odd terms are fixed by the \( J_4 \) pseudo-conservation.
identities, but we did not prove it. Therefore, we will do a separate analysis to argue that we cannot continuously deform the divergence of the higher spin currents once we have fixed the $J_d$ one.

Let us imagine that it is possible to introduce an additional parameter for the breaking of some higher spin current. We will focus first on possible double trace terms. Consider the lowest spin $s$ at which the new term enters\(^{18}\)

$$\nabla J_s = q J_s J_s + \text{rest}. \quad \text{(F.1)}$$

By assumption, $s > 4$. The rest denotes terms that are required by $J_d$ non-conservation. This extra term contributes to the $(J_s J_s, J_s J_s)$ odd three-point function. Consider then the pseudo-conservation identity for the $J_s$ current on $(j_{-2}, j_s, j_s)$. For $s_1, s_2 < s - 2$, we obtain

$$0 = q \langle j_s j_s, j_s > + \text{rest} = q \langle j_s j_s, >, \quad \text{(F.2)}$$

where by ‘rest’, we denote the terms that were present when $q = 0$. These sum up to zero by construction so we have to conclude that $q = 0$.

The argument slightly changes when one of the spins is equal to $s - 2$. Below, we discuss this case separately for the quasi-boson and quasi-fermion.

### F.1. Case of quasi-fermion

In this case if, say, $s_1 = s - 2$, we have two possibilities for $s_2$: 2 or $\tilde{0}$. So we write schematically

$$\nabla J_s = q(\epsilon J_{s-2} J_2 + \partial J_{s-2} J_0) + \text{rest}, \quad \text{(F.3)}$$

where $\epsilon$ is the three-dimensional Levi-Civita tensor. This modifies the correlators $(J_s J_{s-2} J_2)$ odd and $(J_s J_{s-2} J_0)$ odd. Consider now pseudo-conservation of the $J_s$ current for $(j_{-2}, j_{s-2}, j_2)$, we obtain

$$q (\partial_1 \langle j_s j_{s-2}, j_2 > + \partial_2 \langle j_{s-2}, j_2, >) = 0. \quad \text{(F.4)}$$

To check this identity, it is convenient to introduce in the formula above the dependence on the insertion of $j_2(x)$ and consider the integral

$$\partial_j \int d^3 x \frac{1}{|x - x_3|^2} \left( \partial_1 \langle s s - 2 2(x) \rangle + \partial_2 \langle s - 2 s 2(x) \rangle \right) = 0. \quad \text{(F.5)}$$

Using formula (D.5), we obtain that it is equivalent to the identity

$$\partial_1 \langle s s - 2 2^1+2 \rangle + \partial_2 \langle s - 2 s 2^1+2 \rangle = 0. \quad \text{(F.6)}$$

We now take a light cone limit $(s_1, s_2) \tilde{0} \pm 2$. More specifically, we take the limits $\lim_{y \to 0} y \lim_{y \to 0} y^2 \lim_{y \to 0} 12$, which pick out the boson and fermion pieces, respectively. Then, one can check that (F.6) does not have a solution.

The next case to consider is $(s - 2 s 2 0)$. In this case, we have

$$\partial_1 \langle s s - 2 0 \rangle + \partial_2 \langle s - 2 s 0 \rangle = 0. \quad \text{(F.7)}$$

Again, we introduce the explicit dependence on the insertion point $\tilde{0}(x)$ and integrate to obtain

$$q \int d^3 x \frac{1}{|x - x_3|^2} \partial_1 \langle s s - 2 \tilde{0}(x) \rangle + \partial_2 \langle s - 2 s \tilde{0}(x) \rangle = 0. \quad \text{(F.8)}$$

Using (4.10), this becomes

$$q [\partial_1 \langle s s - 2 0 \rangle + \partial_2 \langle s - 2 s 0 \rangle] = 0. \quad \text{(F.9)}$$

This identity does not hold and we conclude that $q = 0$.

\(^{18}\)This expression is schematic. We require and tensor on the right-hand side if the two operators $J_{s_1}$ and $J_{s_2}$ have twist 1.
F.2. Case of quasi-boson

In this case if, say, $s_1 = s - 2$, we have two possibilities for $s_2$: 2 or 0. So we write schematically

$$\nabla J_s = q (\epsilon J_{s-2} J_2 + \epsilon \partial^2 J_{s-2} J_0) + \text{rest.}$$  \hspace{1cm} (F.10)

We modify the correlators $\langle J_{s-2} J_2 \rangle_{\text{odd}}$ and $\langle J_{s-2} J_0 \rangle_{\text{odd}}$. Consider now the $J_4$ pseudo-conservation identity for $\langle J_{s-2} J_j \rangle_{\text{odd}}$. The argument is then identical to the fermion one. We conclude that the first term in (F.10) is not possible. Then, we consider the $J_4$ pseudo-conservation on $\langle J_{s-2} J_{s-2} J_0 \rangle$. In this case, we have

$$q (\partial_1 \langle J_{s-2} J_0 \rangle_{\text{odd}} + \partial_2 \langle J_{s-2} J_j \rangle_{\text{odd}}) = 0.$$  \hspace{1cm} (F.11)

We introduce the explicit dependence on the insertion point 0$(x)$ and integrate to obtain

$$\int \frac{d^3 x}{|x - x_0|^4} (\partial_1 \langle s s - 2 0(x) \rangle_{\text{odd}} + \partial_2 \langle s - 2 s 0(x) \rangle_{\text{odd}}) = 0.$$  \hspace{1cm} (F.12)

Using (4.27), we find

$$\partial_1 \langle s s - 2 0 \rangle_{\text{fer}} + \partial_2 \langle s - 2 s 0 \rangle_{\text{fer}} = 0.$$  \hspace{1cm} (F.13)

This identity does not hold and we conclude that $q = 0$.

F.3. Triple trace deformation

Here, we would like to analyze the new triple trace deformations.

$$\nabla J_s = q J_s J_s J_s + \text{rest.}$$  \hspace{1cm} (F.14)

First note that this deformation does not affect any of the three-point functions. Thus, it does not affect any of the identities that we obtained using $J_4$. Thus, if our assumption that the identities for the three-point functions fix the odd pieces completely is correct, then any of these terms will not affect any of our conclusions.

Applying arguments similar to the above ones for pseudo-conservation of $J_4$ on $\langle J_{s-2} J_0, J_2, J_0 \rangle$, we expect to find that we cannot add the $q$ deformation. However, we did not perform a complete analysis.

Appendix G. Motivational introduction

In this appendix, we give a longer motivational introduction for the study of the constraints imposed by the higher spin symmetry.

As well known, the structure of theories with massless particles with spin is highly constrained. For example, massless particles of spin 1 lead to the Yang–Mills theory, at leading order in derivatives. Similarly, massless particles of spin 2 lead to general relativity. We also need the assumption that the leading-order interaction at low energies is such that the particles are charged under the gauge symmetries or that gravitons couple to energy in the usual way.

Now for massive spin-1 particles, what can we say? If we assume that we have a weakly interacting theory for energies much bigger than the mass of the particle, and we assume a local bulk Lagrangian without other higher spin particles, then we find that the theory should contain at least a Yang–Mills field plus a Higgs particle. We can then view the theory as having a spontaneously broken gauge symmetry. Note that we are assuming that the theory is weakly coupled.

Now consider a theory that has massless particles with higher spin, $s > 2$. If we are in flat space and the $S$-matrix can be defined, then we expect that the Coleman–Mandula theorem
should forbid any interaction [38, 39]19. Here, we are assuming that the couplings of the higher spin fields are such that particles are charged under the higher spin transformations. This is the case if we have a graviton in the theory; the same vertex that makes sure that the graviton couples to the energy of the higher spin particle also implies that the particles transform under a higher spin charge.

Now we can consider theories that contain massive particles with spin \( s > 2 \). We assume that we have a weakly coupled theory at all energies, even at very high energies. In other words, we assume a suitable decay of the amplitudes at high energy (such as the one we have in string theory). Then, it is likely that the theory in question is a full string theory. In other words, we can propose the following conjecture:

A Lorentz-invariant theory in three or more dimensions, which is weakly coupled20, contains a massive particle with spin \( s > 2 \), and has amplitudes with suitably bounded behavior at high energies, should be a string theory 21.

Of course, there are many string compactifications, so we do not expect this to fix the theory uniquely. This is precisely the problem that the old dual model literature tried to solve, and the string theory was found as a particular solution. The above conjecture is just that it is the only solution. As an analogy, in gauge theories with a spontaneously broken symmetry, we can have many realizations of the Higgs mechanism. However, the paradigm of spontaneously broken gauge symmetry constrains the theories in an important way. In string theories, or theories with higher spin massive particles, we also expect that their structure is constrained by the spontaneous breaking of the higher spin symmetry. For example, we expect that once we have a higher spin particle, we have infinitely many of them. It is notoriously difficult to study infinite-dimensional symmetries. It is even more difficult if there is no unbroken phase that is easy to study. These ideas were discussed in [40, 41] (see [42] for a recent discussion containing many further references).

Here, the AdS case appears a bit simpler. In AdS space, as opposed to flat space, it is possible to have a theory which is interacting (in AdS) but that nevertheless realizes the unbroken higher spin symmetry [1–3]. The fact that these theories exist can be understood from AdS/CFT; they are duals to free theories [4–9]. In any example of AdS/CFT that has a coupling constant on the boundary, we can take the zero coupling limit. In this limit, the boundary theory has single trace operators sitting at the twist bound. These are all states that are given by bilinears in the fields. Furthermore, this forms a closed subsector under the OPE [43]. Thus, we can always find a Vasiliev-type theory as a consistent truncation of the zero coupling limit of the full theory. The Vasiliev-type theories contain only the higher spin fields (perhaps plus a scalar). They are analogous to the Yang–Mills theory without an elementary Higgs particle. For example, if we take the zero \( \lambda \) limit of the \( N = 4 \) super Yang–Mills theory, we obtain a certain theory with higher spin symmetry. Besides the higher spin currents, this theory contains lots of other states that are given by single trace operators which contain more than two field insertions, say four six, fields in the trace. Here, the Vasiliev-like theory is the restriction to the bilinears.

19 The Coleman–Mandula theorem assumes that we have a finite number of states below a certain mass shell. For the purpose of this discussion, we assume that it still applies.
20 The weak coupling assumption is important. In fact, the reader might think of the following apparent counter example. Consider the scattering of higher spin excited states of hydrogen atoms. These are higher spin states, but they are not strings! However, these states are not weakly coupled to each other in the particle physics sense.
21 One probably needs to assume some interactions between the massive spin particles which makes sure that the particles are ‘charged’ under the higher spin symmetry. This would be automatically true if we also include an \( m = 0 \), \( s = 2 \) graviton. This should also apply to theories like large-\( N \) QCD, which have a conserved stress tensor leading to massless graviton in the bulk.
In AdS₄, we can realize the dream of constraining the theory from its symmetries. If the higher spin symmetry is unbroken, we can determine all the correlation functions on the AdS boundary. This can be done not only at the tree level, but for all values of the bulk coupling constant (1/N). This is one way of reading the results in [14]. Here, we are assuming that the AdS/CFT correspondence.

In Vasiliev-type theories, the higher spin symmetry can be broken only by two (or three) particle states, which is what we study in this paper.

Finally, one would like to study theories that, besides the higher spin fields, also contain enough extra fields that can Higgs the higher spin symmetry at the classical level. In such theories, the mass of the higher spin particles would be non-zero and finite even for very small bulk coupling. First, one needs to understand what kind of ‘matter’ can be added to the Vasiliev theory and still preserves the higher spin symmetry. Since the higher spin symmetry is highly constraining, it is likely that the only ‘matter’ that we can add is what results from large-N gauge theories. As far as we know, this is an unproven speculation, and it is closely related to the conjecture above. Note that such gauge theories have a Hagedorn density of states, so that the bulk theories would look more like ordinary string theories. Once this problem is understood, one could consider a case where the Higgs mechanism can be introduced with a small parameter (as in N = 4 SYM, for example). Here, we will have single trace terms in the divergence of the higher spin currents. This breaking mechanism will be constrained by the higher spin symmetry. Understanding how it is constrained when the coupling is small will probably give us clues for how the mechanism works for larger values of the coupling. In addition, it could give us a way to do perturbation theory for correlators in gauge theories in a completely on-shell fashion.

Whether this idea is feasible or not, it remains to be seen.

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