Self-organised active carpets drive replenishing nutrient currents

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(Dated: April 30, 2018)

We demonstrate that collective self-organisation in a biofilm or carpet of self-propelled colloids generates coherent solvent currents towards the substrate, and propose that these flows provide efficient pathways to replenish fuel supply or nutrients that feed back into activity. A full theory is developed to express this long-ranged advection in terms of gradients in the active matter density and velocity fields. This approach is applied to bacterial clusters, topological defects of cell orientation, to vortex arrays and turbulent films. Attractive currents with complex spatiotemporal patterns are obtained, which are tuneable by controlling the carpet structure through confinement. This finding shows that diversity in biological and synthetic film architecture is essential to maintain their function.

The collective motion of microorganisms and active colloids has sparked great interest, as biological functions can emerge from the self-organisation of local power injection [1–7]. To sustain these processes, self-propelled particles increase nutrient uptake [8] and redistribute oxygen [9] by hydrodynamically enhanced mixing [10] and particle entrainment [11, 12]. The vast majority of these flow-driving swimmers accumulate at surfaces [13], and thus form ‘active carpets’. Unlike ciliary arrays [14], grafted cells [15] or developed biofilms [16], these carpets are initially without wall attachment and roam freely. However, this crowding drains reserves rapidly, and their renewal is restricted by the boundary [17], so self-propulsion is curtailed. Moreover, swimmer-generated flows cancel each other out in the case of homogeneous coverage, by symmetry, so supply of nutrients is limited by diffusion. A steady advection changes this situation changes radically; it opens effective pathways for resource replenishment and functional feedback.

In this paper, we show that such coherent transport arises from inhomogeneous carpet organisation, which emerges naturally from the long-ranged order in collective behaviour [18], such as in bacterial vortex arrays [19–21], bacterial turbulence [22–24], and giant density fluctuations [4, 25, 26]. Topology and geometry play a crucial role in these living fluids [2, 27, 28], providing a bridge with material sciences and cell biology [7, 29]. We focus on bacteria as a concrete example, but this theory applies to the broader class of active carpets to which no external forces and torques are applied.

First, we demonstrate that clusters of randomly oriented cells create a non-diffusive drift towards the surface. These currents stem from swimmer density gradients, and thus hold regardless of cluster shape. Second, in uniform-density carpets, gradients in swimmer orientation produce flows instead. We derive and implement...
these to topological defects commonly found in living fluids. Combining these fundamental ingredients, the nutrient transport by vortex arrays and bacterial turbulence are evaluated, and the spatiotemporal correlations of the flows compared to the collective dynamics.

RESULTS

Individual swimmer flows

We consider a colony of microswimmers with balanced propulsion and drag forces. These active particles, with orientation $\mathbf{p}$ and location $r_s$, travel parallel to a solid surface that is fixed at $z = 0$ in standard Cartesian coordinates. Each swimmer generates a flow $\mathbf{u}(r)$ that can displace nutrients, represented here by a tracer particle located at $r$. At low Reynolds numbers, and for distances $d = |r - r_s|$ larger than a few body lengths, this individual flow is well described by a Stokes dipole aligned with the swimming direction $[1]$, given by

$$\mathbf{u}(r, r_s, p) = \kappa (p \cdot \nabla_s) \mathcal{B}(r, r_s) \cdot p,$$

where the dipole strength is $|\kappa| \sim 3v_o a_s^2/4$ in terms of the swimmer’s speed and size $[12]$, and where $\nabla_s$ acts on the swimmer position. The no-slip condition at the wall is accounted for using the Blake tensor $\mathcal{B}(r, r_s)$ formalism [see Supplementary Information (SI) §1]. Throughout this paper, as an example, we use swimmer height $h = z_s = 1 \, \mu m$ and dipole moment $\kappa = 30 \, \mu m^3/s$ for the pusher E. coli $[30]$.

Figure 1(a) shows the resulting flow driven by a single bacterium. Liquid is attracted towards the surface directly above the swimmer (blue regions), but pushed upwards in front of and behind the cell (red regions). The net flux across any plane in $z$ vanishes due to the incompressibility of the liquid. $\int \mathbf{u} \, dx \, dy = 0$, but across a plane recirculating vortices can emerge (green stream lines). For pullers, $\kappa < 0$, the flow direction is inverted.

Taken together, the average flow velocity due to all swimmers on the surface combined is

$$\langle \mathbf{v}(r) \rangle = \int \mathbf{u}(r, r_s, p) f(r_s, p) \, dr_s \, dp,$$

where $f$ is the probability density of finding a swimmer at position $r_s$ and orientation $\mathbf{p}$. The convergence of this integral is guaranteed if $z > h$ because the dipolar flow decays as $1/d^3$ in the presence of a wall $[17]$.

Clusters & density gradients

We examine a cluster of $N$ self-propelled particles that assemble around a chemoattractant [Movie S1 and SI §5E]. Remarkably, this active carpet generates a steady current that brings nutrients down towards the surface. To analyse this, we first imagine a circular cluster of radius $R$ centred at the origin with constant density, $n = N/(\pi R^2)$, and uniformly distributed swimmer orientations in the plane. The total flow, derived in SI §2A and shown in Fig. 1(b), is found by inserting this profile, $f \propto \frac{\mu}{z}$, into Eq. (2). As in the movie, this yields a downwelling region for all lateral distances $\rho < R$ and all heights $z > h$, where $\rho = \sqrt{x^2 + y^2}$, despite the random swimmer orientations. Subsequently, the nutrients move from the centre to the edge of the cluster, to $\rho > R$, where incompressibility demands that liquid be transported back up, causing a large toroidal recirculation. Directly above the cluster, along the $z$ axis and in the limit $h \ll z$, the result simplifies to the mean drift velocity

$$\langle \mathbf{v}(z) \rangle = -12\pi n h \kappa \frac{z^2 R^2}{(R^2 + z^2)^{3/2}} \hat{z}.$$

This current has an optimal strength at $z^* = \sqrt{2/3} R$, apparent as the minimum in Fig. 1(c). Hence, for a typical bacterial density, $n \sim 0.1/\mu m^2$ $[31]$ and cluster size $R \sim 10 \mu m$ we expect significant nutrient transport up to $\langle \mathbf{v}(z^*) \rangle \sim 2.1 \mu m/s$, i.e. orders of magnitude larger than sedimentation velocities of submicron particles. Moreover, for more realistic clusters with a Gaussian profile we find the flows are a factor $\sim 2$ stronger [SI §2B].

Counterintuitively, large uniform clusters do not transport nutrients faster. To be precise, in the thermodynamic limit where $R, N \to \infty$ with constant $n$, the individual swimmer flows cancel each other out, on average, so the surface attraction vanishes. Indeed, the mean flow (Eq. 3) decays as $1/R^3$ in this limit [Fig. 1(d)], as the density gradient at the cluster edge moves away.

More generally, all gradients in swimmer surface density can drive currents. To see this we simulate a cluster...
FIG. 3. Defects in the director field generate strong flows because of large orientation gradients. Swimmers are arranged in a dense uniform lattice with orientation \( \phi = \phi_0 + m \theta \) (grey lines). Upper panels: Colours indicate vertical flows in \( \mu m/s \), simulated for the plane \( z = 5 \mu m \), and green arrows are stream lines, also for the planes \( x, y = -50 \mu m \). (a) Vortex defect with \((m, \phi_0) = (1, \pi/2)\). (b) Aster defect with \((1, 0)\). (c) Plus half defect with \((1/2, 0)\). (d) Minus half defect with \((-1/2, 0)\). (e) Saddle defect with \((-1, 0)\). Lower panels: Flows in \( \mu m/s \) for the plane \( y = 0 \), obtained numerically (markers) and analytically (lines).

with a linearly decreasing density \([\text{SI } \S5C]\). As shown in Fig. 1(e), this generates a horizontal flow in the direction along the gradient, like in Fig. 1(b). Mass conservation again causes recirculation, with downwelling at the high end of the gradient and upwelling at the lower end. Fluctuations due to the random orientations occur, but a mean current emerges clearly.

Using this information, one can predict transport driven by clusters of a more complex morphology. Figure 1(f) depicts flows generated by bacteria arranged in a branching pattern \([\text{SI } \S5D]\). In agreement with the previous simplified cases, flows move downwards to the high-density regions, the branches, and upwards from the sparse areas. This configuration is of course arbitrary, but serves to emphasize the robustness with respect to cluster shape.

**Orientation gradients**

In the previous scenario with random orientations, the mean flows vanish in the absence of gradients in activity. Furthermore, if all swimmers are oriented in the same direction, through collective motion or alignment interactions, say \( \vec{x} \), then the currents also cancel in the thermodynamic limit \([\text{SI } \S3A]\). However, gradients in swimmer orientation give rise to a second source of flow generation.

To classify the relevant orientation derivatives, it is important to note that the swimmer flow (Eq. 1) is nematic symmetry [Fig. 1(a)], i.e. invariant under \( p \rightarrow -p \). Hence, the only first-order derivatives that obey this symmetry in a 2D active carpet are, expressed in liquid crystal terminology, the ‘bend’ and ‘splay’ contributions,

\[
B = (p \times (\nabla \phi \times p))^2, \quad S = (\nabla \phi \cdot p)^2.
\]

The effect of these basic gradients is illustrated in Fig. 2. We consider active particles that swim collectively (a) in concentric circles, \( \phi = \theta + \frac{\pi}{2} \), or (b) towards a chemotactic source, \( \phi = \theta \), where \( \phi = \arctan(p_y/p_x) \) and \( \theta = \arctan(y/x) \), and they are spread out uniformly in space to minimise swimmer density gradients \([\text{SI } \S5F]\). In both cases the orientation gradients decay with distance from the centre quadratically; for (a) we have \( B(\rho) = 1/\rho^2 \) and \( S(\rho) = 0 \), and vice-versa for (b). Then, a strong correlation is observed between bend gradients and liquid moving downwards and outwards. Conversely, splay gradients drive flows inwards and upwards.

To make analytical progress we realise that it is not always possible to find a general formula \( \chi(r) \) for the local flow, \( \langle v(r) \rangle = \chi(B, S) \), because the velocity is generated by a region of swimmers in which the gradients vary. These variations increase for larger \( z \) values as the number of equidistant swimmers, i.e. this region of influence, grows. However, the gradients are approximately constant when \( z \ll \rho \), far from the circle centre in the above examples [Fig. 2(a,b)]. Therefore, we can couple the gradients and flows in that area. We first compute (Eq. 2) in the thermodynamic limit, and then expand the horizontal and vertical flows in terms of \( 1/\rho \) \([\text{SI } \S3B, \S3C]\).

To first order, this leads to the flows due to pure bend and splay gradients,

\[
\langle v_x \rangle \approx 8\pi n \Delta k \left( \sqrt{B(\rho)} - \sqrt{S(\rho)} \right), \quad (6)
\]

\[
\langle v_z \rangle \approx -8\pi n \Delta k \zeta z^2 \left[ (B(\rho))^{3/2} - (S(\rho))^{3/2} \right]. \quad (7)
\]

This approximation, shown in Fig. 2(c,d), offers a good agreement with its numerical counterpart. It also follows that for weak gradients, the horizontal flows are stronger than the vertical transport.

**Topological defects**

Like we saw for density gradients, it is now possible to interpret more complex carpet designs in terms of the
fundamental ingredients, bend and splay. The first non-trivial orientation patterns with significant orientation gradients are the lowest-order topological defects [Fig. 3]. Their director fields are defined as φ = φ0 + mθ, where m = ±1, ±1, ±3, ..., is the topological charge. Because these defect arrangements are well characterised mathematically, it is possible to find analytical solutions for the flows they generate [SI §4].

Swimmers with polar order feature integer-charge defects. For m = 1 [Fig. 3(a,b)], there is a continuous transition from nutrient attraction near ‘vortex’ defects (φ0 = πm), via no flow ‘spiral’ defects (φ0 = ±π/2), to repulsion near ‘aster’ defects (φ0 = 0).

$$
\langle v_z \rangle_{m=1} = 8 \pi n h \kappa \frac{z^2 \cos(2\phi_0)}{(\rho^2 + z^2)^{3/2}}.
$$

Active particles with nematic order feature half-integer charges. Near an m = ±1/2 defect [Fig. 3(c)], cooperation between bend and splay gradients drive horizontal currents, outwards from the bend curvature, in the −z direction here. The flows in z follow from recirculation, down towards the defect and back up again, with extrema at $z = \sqrt{2}$. Also near m = −1/2 defects the horizontal flows move towards the convex side of the bends and to the regions of converging splay [Fig. 3(d)]. This carpet features a 3-fold symmetry, with extrema in v_z at $\rho = 1.637z$ (polynomial root). Lastly, near a ‘saddle’ defect, m = −1, the bend and splay gradients once more govern the flows [Fig. 3(e)] with extrema in $v_z$ at $\rho = 2.111z$ and a 4-fold symmetry. In all cases, the calculated flows [SI §4] agree well with the simulated ones [Fig. 3, lower panels].

An important observation is that splay gradients (divergence of $\rho$ in Eq. 5) and density gradients are coupled in time, via motility. Specifically, bacteria can accumulate or deplete from defects, as observed in liquid crystals [32]. Therefore, vortex defects [Fig. 3a] can remain stable over time trivially, but steady states of aster defects [Fig. 3b] must feature more complex dynamics, such as defect ordering [33] or ejection of swimmers from the carpet into the bulk. Otherwise the defects can be motile, with time-dependent flows, as we discuss below for bacterial turbulence.

**DISCUSSION**

**Vortex arrays**

The topological building blocks can be used to comprehend the currents created by active carpets featuring collective motion. Particularly common in nature, and microfluidically controllable, are vortex patterns that bacteria or spermatozoa at high surface densities can self-organise into [19–21]. We first consider a Taylor-Green Vortex (TGV) carpet, which periodically features ‘vortex’ and ‘saddle’ defects (m = ±1) at the centre and corners of the unit cell, respectively [Fig. 4a, SI §5G]. Nutrients are attracted down to the vortex centres (locally described by Eq. 8), and recirculated upwards with 4-fold symmetry at the face centres of the unit cell, in agreement with the individual defect flows [Fig. 3a,e]. Changing the vortex size with confinement can therefore tune the flows.

**Bacterial turbulence**

Similarly, we consider the more complex patterns generated by bacterial turbulence [22–24]. Their collective dynamics are simulated using the Self-Propelled Rod (SPR) model [24] to determine swimmer positions and orientation [Movie S2, SI §6A,B]. Because of the high volume fraction, density gradients remain negligible but orientation gradients are abundant. Hence, recirculatory currents are generated, as shown in Fig. 4b. Weak flows occur in the regions where swimmers are aligned with each other [SI §3A], but defects give rise to strong bend and splay gradients and thus nutrient transport.

Movies S3 - S5 show how these currents develop during the onset of turbulence, giving top views at $z = 10, 25 \mu m$, respectively, and a side view for the cross section $y = 0$. Interestingly, further from the active carpet the downwelling and upwelling regions are slower but larger. We quantify this by computing the temporal and spatial correlation functions, $c_{v_z}(t)$ and $g_{v_z}(\rho)$, for different heights.
Hence, we obtain the correlation time \( t_\ast(z) \) and length \( \rho_\ast(z) \) from their fits [Fig. 4c,d]. At short timescales the nutrient transport is ballistic but, of course, after this memory time it is diffusive. Far from the carpet this memory is set by the decorrelation of swimmer orientations (dashed black), but nearby \( t_\ast \) reduces to the mean free time between collisions with individual swimmers. Conversely, the correlation length grows linearly with \( z \), and it is not bound by the correlation length of swimmer orientations because the region of influence by more equidistant bacteria grows beyond the turbulent swirl radius. Indeed, the renormalised correlations \( g_v(\rho/z) \) collapse onto one another [SI Fig. 5], highlighting the scaling relation of the flow’s long-rangedness.

**CONCLUSIONS**

We studied the establishment of large-scale recirculation by a carpet of self-propelled particles, with focus on bacteria, that leads to direct attraction of nutrients and other resources like oxygen, microbot fuel and fresh water. Exact solutions are derived in a few basic instances, including cells in colonies and topological defects. We show how replenishment emerges in terms of gradients in the swimmer density and orientation fields, that arise from self-organisation. This mathematical foundation provides understanding for transport by more complex-shaped clusters, bacterial turbulence, or biofilm architecture.

In nature, stable density gradients or clustering can arise by self-assembly [4, 6, 26] and numerous taxis (chemo-, thermo-, phototaxis). Orientation gradients can form through obstacles, individual actuation, or collective instabilities [18, 34]. Experimental realisations may also be achieved by light-controlled coordination [35, 36], defect ordering [33], lithographic surface patterning and rectification [37, 38]. Hence, currents may be kept steady by controlling the swimmer density and orientations, which could be employed to drive active flow networks or feedback into activity. Topological constraints are essential in stabilising this transport, for example by vortices inside drops or on spherical manifolds [39, 40].

**SUPPLEMENTARY MATERIALS**

The following Supplementary Materials are available at [URL to be inserted]:

- Supplementary Information (SI). Complete description of all derivations and simulation techniques.
- Movie S1. Downward attraction of tracer particles above a dynamic bacterial cluster.
- Movie S2. SPR model: Turbulent dynamics of 10,000 bacteria, where twenty cells are labelled in colour to trace their motion.
- Movie S3. Top view of flows generated by bacterial turbulence, at the horizontal plane \( z = 10 \mu m \).
- Movie S4. Top view of flows generated by bacterial turbulence, at the horizontal plane \( z = 25 \mu m \).
- Movie S5. Side view of flows generated by bacterial turbulence, at the vertical plane \( y = 0 \).

**ACKNOWLEDGEMENTS**

We would like to thank Manu Prakash and Deepak Krishnamurthy for helpful discussions. AM acknowledges funding from the Human Frontier Science Program (Fellowship LT001670/2017). FGL acknowledges Millennium Nucleus “Physics of active matter” of the Millennium Scientific Initiative of the Ministry of Economy, Development and Tourism, Chile. HL acknowledges support from the Deutsche Forschungsgemeinschaft, DFG project SPP 1726.

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SI §1. MODEL

The average flow \( \langle \mathbf{v}(\mathbf{r}) \rangle \) due to a distribution of swimmers \( f(\mathbf{r}_s, \mathbf{p}) \) at positions \( \mathbf{r}_s = (x_s, y_s, z_s) \) and orientations \( \mathbf{p} = (p_x, p_y, p_z) \), all in standard Cartesian coordinates and evaluated at position \( \mathbf{r} = (x, y, z) \), is

\[
\langle \mathbf{v}(\mathbf{r}) \rangle = \int \mathbf{u}(\mathbf{r}, \mathbf{r}_s, \mathbf{p}) f(\mathbf{r}_s, \mathbf{p}) d\mathbf{r}_s d\mathbf{p},
\]

(S1)

where \( \mathbf{u}(\mathbf{r}, \mathbf{r}_s, \mathbf{p}) \) is the flow due to an individual swimmer,

\[
\mathbf{u}(\mathbf{r}, \mathbf{r}_s, \mathbf{p}) = \kappa \left[ (\mathbf{p} \cdot \nabla) B(\mathbf{r}, \mathbf{r}_s) \right] \cdot \mathbf{p},
\]

(S2)

where \( \kappa \) is the dipole coefficient and the Blake tensor \([41]\) expressed in terms of the Oseen tensor (Stokeslet) \([42]\) is

\[
B_{ij}(\mathbf{r}, \mathbf{r}_s) = (-\delta_{jk} + 2h\delta_{k3}(\delta_{i} - \frac{\hat{x}_i}{|d|}) + h^2 M_{jk}\nabla^2_s) J_{ik}(\mathbf{r}, \mathbf{r}_s),
\]

(S3)

\[
J_{ij}(\mathbf{r}, \mathbf{r}_s) = \frac{1}{8\pi \eta} \left( \frac{\delta_{ij}}{|d|} + \frac{d_i d_j}{|d|^3} \right),
\]

(S4)

with indices \( i, j, k \in \{1, 2, 3\} \), mirror matrix \( M_{jk} = \text{diag}(1, 1, -1) \), distance \( d = \mathbf{r} - \mathbf{r}_s \), and all derivatives of the Oseen tensor \( J_{ij} \) are with respect to the swimmer position \( \mathbf{r}_s \). Combining these equations, the currents due to the active carpets can be derived by integrating Eq. S1 analytically. These exact results are given in SI §2 and SI §3 and SI §4 below.

Equivalently, these solutions can be verified by computing the flows numerically. Simulations are performed by summing the (exact) flows due to \( N \) individual swimmers,

\[
\langle \mathbf{v}(\mathbf{r}) \rangle = \sum_{i=1}^{N} \mathbf{u}_i(\mathbf{r}, \mathbf{r}_i, \mathbf{p}_i),
\]

(S5)

where the positions \( \mathbf{r}_i \) and orientations \( \mathbf{p}_i \) are distributed such that they satisfy the probability density \( f(\mathbf{r}_s, \mathbf{p}) \). All simulations were carried out using Wolfram Mathematica (version 10.0.1.0) on a desktop PC (operating on Windows 10). To expedite the evaluation of currents due large active carpets, \( N \sim 250,000 \), the flows generated by an individual swimmer (Eq. S2) were derived, maximally simplified and compiled with the built-in Compile[] function, and summed over in parallel with the ParallelSum[] function. Detailed descriptions of all simulation procedures are provided in SI §5 and SI §6 below.

SI §2. FLOWS DUE TO DENSITY GRADIENTS

A. Uniform cluster

Consider a cluster of \( N \) bacteria swimming at height \( z_s = h \) over a surface, located at \( z = 0 \). They are uniformly distributed within a disk of radius \( R \), so that the surface concentration is constant, \( n = N/(\pi R^2) \). The bacteria are oriented randomly, parallel to the surface, \( p_z = 0 \), according to a uniform distribution, \( \phi_s \in [-\pi, \pi] \), where \( \phi_s = \arctan(p_y/p_x) \). Without loss of generality, we take the cluster to be centred at the origin. The carpet distribution is then given by

\[
f(\mathbf{r}_s, \mathbf{p}) = \frac{\frac{n}{2\pi} \Theta(R - \rho_s) \delta(|\mathbf{p}| - 1) \delta(\mathbf{p} \cdot \hat{z})}{\pi R^2},
\]

(S6)

where \( \Theta(x) \) and \( \delta(x) \) are the Heaviside and Dirac delta functions, and cylindrical symmetry about the \( z \) axis gives \( \rho_s^2 = x_s^2 + y_s^2 \) and \( \theta_s = \arctan(y_s/x_s) \). For \( \rho_s \in [0, R] \), \( \theta_s \in [-\pi, \pi] \) and \( \phi_s \in [-\pi, \pi] \) this profile simplifies to \( f(\rho_s, \theta_s, \phi_s) = n/(2\pi) \), and it is normalised so that the integrated distribution gives the number of swimmers,

\[
\int f(\mathbf{r}_s, \mathbf{p}) d\mathbf{r}_s d\mathbf{p} = \int_0^R \int_{-\pi}^\pi \int_{-\pi}^\pi \frac{N}{2\pi} \frac{1}{\pi R^2} d\phi_s d\theta_s \rho_s d\rho_s = N.
\]

(S7)

To find the overall flow due to the cluster, we substitute this distribution (S6) into (S1) and evaluate the integral. In general this is not trivial, but progress can be made by considering flows far from the surface, \( z_s = \epsilon z \) with \( \epsilon \ll 1 \).
We Taylor-expand the individual swimmer flows (S2) to first order in \( \epsilon \),

\[
\mathbf{u}(r, \theta, p) = \frac{\partial \mathbf{u}}{\partial \epsilon} \bigg|_{\epsilon=0} \epsilon + \mathcal{O}(\epsilon^2)
\]

\[
= \hat{u}(r, \theta, p) + \mathcal{O}\left( \frac{z_s}{z} \right)^2.
\]

(S8)

(S9)

To be explicit, on the \( z \) axis this gives the individual flows

\[
\hat{u}_z|_{\rho=0} = \frac{3\kappa \rho \xi^2 \varepsilon \left(-5\rho^2 \cos(3\theta_s - 2\phi_s) + (2z_s^2 - 3\rho^2) \cos(\theta_s - 2\phi_s) + \cos(\theta_s) \left( 6z_s^2 - 4\rho^2 \right) \right)}{(\rho^2 + z^2)^{7/2}},
\]

(S10)

\[
\hat{u}_y|_{\rho=0} = \frac{-3\kappa \rho \xi^2 \varepsilon \left(5\rho^2 \sin(3\theta_s - 2\phi_s) + (2z_s^2 - 3\rho^2) \sin(\theta_s - 2\phi_s) + \sin(\theta_s) \left( 4\rho^2 - 6z_s^2 \right) \right)}{(\rho^2 + z^2)^{7/2}},
\]

(S11)

\[
\hat{u}_z|_{\rho=0} = \frac{-6\kappa \xi^2 \varepsilon \left(-5\rho^2 \cos(2(\theta_s - \phi_s)) - 3\rho^2 + 2z_s^2 \right)}{(\rho^2 + z^2)^{7/2}} + \mathcal{O}(\epsilon^2).
\]

(S12)

Inserting these into (S1) gives the average flows directly above a bacterial cluster,

\[
\langle v(r) \rangle|_{\rho=0} = \int \hat{u}(\rho=0)f(r, \theta, p)dr dp
\]

\[
= \int_0^\pi \int_{-\pi}^\pi \hat{u}(z, \rho, \theta, \phi)|_{\rho=0} \frac{N}{\pi R^2} \frac{1}{2\pi} d\phi d\theta d\rho
\]

\[
= -12\pi \varepsilon \kappa h \frac{z}{(R^2 + z^2)^{5/2}}
\]

which corresponds to Eq. (3) of the Main Text.

For all other positions, \( \rho \neq 0 \), the integral can be performed to give the complete cluster flow

\[
\langle v_z \rangle = \frac{4\kappa \xi z^2 h}{((R - \rho)^2 + z^2)^{3/2} ((R + \rho)^2 + z^2)^{3/2}} \left(2\rho z^2 (R - \rho) + (R - \rho)^3 (R + \rho) - z^4\right)
\]

\[
\langle v_\rho \rangle = \frac{4\kappa \xi z h}{\rho ((R - \rho)^2 + z^2)^{3/2} ((R + \rho)^2 + z^2)^{3/2}} \left(\left((R - \rho)^2 + z^2\right) \left(3z^2 (R^2 + \rho^2) + 2 (R^2 - \rho^2)^2 + z^4\right) - \frac{4R\rho}{(R - \rho)^2 + z^2} - \frac{4R\rho}{(R + \rho)^2 + z^2} \right).
\]

(S16)

(S17)

Here the complete elliptic integrals of the first and second kind are defined as

\[
K[z] = \int_0^{\pi/2} \frac{1}{\sqrt{1 - z \sin^2 \theta}} d\theta, \quad E[z] = \int_0^{\pi/2} \sqrt{1 - z \sin^2 \theta} d\theta
\]

(S18)

It is convenient to notice that these functions obey the following identities,

\[
K\left[\frac{-4\zeta}{(1+\zeta)^2 + \xi^2}\right] = K\left[\frac{+4\zeta}{(1+\zeta)^2 + \xi^2}\right] = \sqrt{(1+\zeta)^2 + \xi^2},
\]

\[
E\left[\frac{-4\zeta}{(1+\zeta)^2 + \xi^2}\right] = E\left[\frac{+4\zeta}{(1+\zeta)^2 + \xi^2}\right] = \sqrt{(1+\zeta)^2 + \xi^2},
\]

(S19)

(S20)

where \( \xi \) and \( \zeta \) are real variables. As a verification, note that we recover (S15) when evaluating (S16,S17) in the limit \( \rho \to 0 \). This result is plotted in Fig. 1(b-d) of the main text.
B. Gaussian density profile

Instead of a cluster with a ‘sharp’ density gradient at the cluster edge, the Heaviside function in (S6), we next consider a Gaussian density profile,

\[ f(r_s, p) = N \frac{\delta(z_s - h)}{2\pi R^2} \exp \left( - \frac{\rho_s^2}{2R^2} \right) \delta( |p| - 1 ) \delta(p \cdot \hat{z}), \]  

(S21)

which is normalised to the number of swimmers as in (S7). Inserting this profile into (S1) gives the average flows directly above a bacterial cluster,

\[ \langle v(r) \rangle |_{\rho = 0} = \int_0^R \int_{-\pi}^\pi \int_{-\pi}^\pi \tilde{u}(z, \rho_s, \theta_s, \phi_s; \rho = 0) \frac{N}{\pi R^2} \delta(\phi_s) d\phi_s d\theta_s d\rho_s \]  

(S22)

\[ = -\frac{12}{\pi h \kappa} N \left( \frac{z^2}{R^2 + z^2} \right) \hat{z}. \]  

(S24)

This expression is a little more complicated than (S15), but has exactly the same features:

- The function is always negative for pushers, \( \kappa > 0 \), representing attraction of nutrients towards the cluster,
- It has a minimum around \( z \sim R \), and
- It has the same decay with distance from the active carpet, \( v_z(z) \sim -12\pi h \kappa R^2 / z^3 \).

However, this flow is a factor of \( \sim 2 \) stronger because the swimmer density gradient is already present for small \( \rho_s \) values.

SI §3. FLOWS DUE TO ORIENTATION GRADIENTS

A. Laning swimmers

In the absence of density gradients, the currents cancel on average if all swimmers are oriented in the same direction, i.e. ‘laning’. Using the profile

\[ f(r_s, p) = N \frac{\delta(z_s - h)}{\pi R^2} \Theta(R - \rho_s) \delta^3(p - \hat{x}), \]  

(S25)

we obtain the average flow

\[ \langle v(r) \rangle |_{\rho = 0} = \int_0^R \int_{-\pi}^\pi \int_{-\pi}^\pi \tilde{u}(z, \rho_s, \theta_s, \phi_s; \rho = 0) \frac{N}{\pi R^2} \delta(\phi_s) d\phi_s d\theta_s d\rho_s \]  

(S26)

\[ = -12 h \kappa N \left( \frac{z^2}{R^2 + z^2} \right) \hat{z}, \]  

(S27)

which vanishes in the thermodynamic limit, where \( N, R \to \infty \) with constant \( n = N/(\pi R^2) \).

B. Bend gradients

To understand the effect of bend gradients, we consider swimmer orientations \( p(r_s) \) that are arranged along circle tangents, \( \phi_s = \theta + \pi/2 \), which gives the carpet profile

\[ f(\rho_s, \theta_s, \phi_s) = \frac{N}{\pi R^2} \delta(\phi_s - \theta_s - \pi/2). \]  

(S28)
This corresponds to a bend gradient of

\[ B(\rho) = (p \times (\nabla_s \times p))^2 = \frac{1}{\rho^2}. \]  

(S29)

In the thermodynamic limit this yields the average flow

\[ \langle v(\mathbf{r}) \rangle = \int_0^\infty \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} n \, \delta(\phi_s - \theta_s - \pi/2) \, \hat{\mathbf{u}}(\rho, z, \rho_s, \theta_s, \phi_s) \, d\phi_s \, d\theta_s \, d\rho_s \]  

\[ = -\frac{8\pi \hbar \kappa}{(\rho^2 + z^2)^{3/2}} \left( z^2 \hat{z} + \frac{\left( z^2 \left( z - \sqrt{\rho^2 + z^2} \right) \right) - \rho^2 \left( \sqrt{\rho^2 + z^2} - 2z \right)}{\rho} \right) \hat{\rho}. \]  

(S30)

The vertical component is always negative for bend gradients, but incompressibility demands that the horizontal component switches sign at \( \rho/z = \sqrt{(1 + \sqrt{5})/2} \), with outward flows for small \( z \).

In the limit \( \rho \gg z \) the bend gradients are approximately constant, and therefore it is possible to write the flow as

\[ \langle v_\rho \rangle = \frac{8\pi \hbar \kappa \rho}{\rho} + O \left( \frac{1}{\rho^2} \right) \approx 8\pi \hbar \kappa \sqrt{B(\rho)}, \]  

(S32)

\[ \langle v_z \rangle = \frac{8\pi \hbar \kappa z^2}{\rho^3} + O \left( \frac{1}{\rho^4} \right) \approx -8\pi \hbar \kappa z^2 [B(\rho)]^{3/2}. \]  

(S33)

C. Splay gradients

To understand the effect of splay gradients, we consider swimmer orientations \( \mathbf{p}(\mathbf{r}_s) \) that are arranged along circle radii, \( \phi_s = \theta \), which gives the carpet profile

\[ f(\rho_s, \theta_s, \phi_s) = \frac{N}{\pi R^2} \delta(\phi_s - \theta_s). \]  

(S34)

This corresponds to a splay gradient of

\[ S(\rho) = (\nabla_s \cdot \mathbf{p})^2 = \frac{1}{\rho^2}. \]  

(S35)

In the thermodynamic limit this yields the average flow

\[ \langle v(\mathbf{r}) \rangle = \int_0^\infty \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} n \, \delta(\phi_s - \theta_s - \pi/2) \, \hat{\mathbf{u}}(\rho, z, \rho_s, \theta_s, \phi_s) \, d\phi_s \, d\theta_s \, d\rho_s \]  

\[ = +\frac{8\pi \hbar \kappa}{(\rho^2 + z^2)^{3/2}} \left( z^2 \hat{z} + \frac{\left( z^2 \left( z - \sqrt{\rho^2 + z^2} \right) \right) - \rho^2 \left( \sqrt{\rho^2 + z^2} - 2z \right)}{\rho} \right) \hat{\rho}. \]  

(S36)

As opposed to bend gradients, the vertical component is always positive for splay gradients, and the horizontal component the still switches sign at \( \rho/z = \sqrt{(1 + \sqrt{5})/2} \), with inward flows for small \( z \).

As before, in the limit \( \rho \gg z \) the splay gradients are approximately constant, and therefore it is possible to write the flow as

\[ \langle v_\rho \rangle = -\frac{8\pi \hbar \kappa}{\rho} + O \left( \frac{1}{\rho^2} \right) \approx -8\pi \hbar \kappa \sqrt{S(\rho)}, \]  

(S38)

\[ \langle v_z \rangle = \frac{8\pi \hbar \kappa z^2}{\rho^3} + O \left( \frac{1}{\rho^4} \right) \approx 8\pi \hbar \kappa z^2 [S(\rho)]^{3/2}. \]  

(S39)

Combining the equations (S32-S39) yields an expression for \( \langle \mathbf{v}(\mathbf{r}) \rangle = g(B, S) \), which corresponds to Eqs. (6,7) in the Main Text.
Here we consider the flows due to swimmers arranged with a disinclination or defect at the origin, \( x = y = 0 \), defined as

\[
\phi_s = m\theta_s + \phi_0, \tag{S40}
\]

where \( m \) is the topological charge, \( \phi_s = \arctan(y_s/x_s) \) and \( \theta_s = \arctan(p_y/p_x) \).

### A. Vortex defect

This is the same calculation as the one for bend gradients above, equation (S31).

### B. Aster defect

This is the same calculation as the one for splay gradients above, equation (S37).

### C. +1/2 defect

The cylindrical symmetry that we employed earlier can no longer be used in the case of defects with \( m \neq 1 \). Therefore we revert to standard Cartesian coordinates, with swimmer positions \((x_s, y_s, z_s = h)\). We consider a +1/2 topological defect in the swimmer orientations along the \( x \) axis, so that \( \phi_s = \frac{1}{2}(\theta_s + \pi) \). Note that the offset \( \phi_0 \) in this case only contributes to a rotation of the defect about the origin (S40), and we choose \( \phi_0 = \pi/2 \) so that the convex end points towards the positive \( x \) direction.

Hence, using the carpet profile,

\[
f(\rho_s, \theta_s, \phi_s) = \frac{N}{\pi R^2} \delta \left( \phi_s - \frac{\theta_s + \pi}{2} \right), \tag{S41}
\]

we find the flow in the plane along the +1/2 defect direction

\[
\langle \mathbf{v}(\mathbf{r}) \rangle |_{y=0} = \int_0^\infty \int_0^{\pi} \int_0^{\pi} n \delta \left( \phi_s - \frac{\theta_s + \pi}{2} \right) \hat{u}(x, y = 0, z, \rho_s, \theta_s, \phi_s) d\phi_s d\theta_s \rho_s d\rho_s \tag{S42}
\]

\[
= \frac{4\pi nh\kappa}{x^2 (x^2 + z^2)^2 \sqrt{2x (x - \sqrt{x^2 + z^2}) + z^2}} \left( x^6 - 2x^2 z^4 + x^2 z (x^2 + z^2) \sqrt{2x (x - \sqrt{x^2 + z^2}) + z^2} 
+ x z^4 \sqrt{x^2 + z^2} + z^3 (x^2 + z^2) \sqrt{2x (x - \sqrt{x^2 + z^2}) + z^2} 
- x^5 \sqrt{x^2 + z^2} + x^3 z^2 \sqrt{x^2 + z^2 - z^6} \hat{x} 
+ \left( \frac{4\pi nh\kappa x \sqrt{x^2 + z^2 + x} \sqrt{2x (x - \sqrt{x^2 + z^2}) + z^2}}{z (x^2 + z^2)^{3/2}} \right) \hat{z}. \tag{S43}
\]

Directly above the defect this simplifies to a purely longitudinal flow,

\[
\langle \mathbf{v}(\mathbf{r}) \rangle |_{x=y=0} = \frac{2\pi nh\kappa}{z} \hat{x}. \tag{S44}
\]
D. $-1/2$ defect

Similarly as for its positive counterpart, we use the carpet profile

$$f(\rho_s, \theta_s, \phi_s) = \frac{N}{\pi R^2} \delta \left( \phi_s - \frac{\pi - \theta_s}{2} \right),$$

(S45)

to compute the flows in the plane along the $-1/2$ defect direction,

$$\langle v(r) \rangle |_{y=0} = \int_0^\infty \int_{-\pi}^\pi \int_{-\pi}^\pi n \delta \left( \phi_s - \frac{\pi - \theta_s}{2} \right) \tilde{u}(x, y = 0, z, \rho_s, \theta_s, \phi_s) d\phi_s d\theta_s d\rho_s$$

(S46)

$$= -\frac{4\pi \hbar \kappa}{x^4 (x^2 + z^2)^{3/2}} \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \left( -x^7 + 3x^5 z^2 + 17x^3 z^4 \right)$$

$$+ 12x^5 \left( \sqrt{(x^2 + z^2)} \left( 2x \left( x - \sqrt{x^2 + z^2} \right) + z^2 \right) - z \sqrt{x^2 + z^2} \right)$$

$$+ x^2 z^3 \left( 11 \sqrt{(x^2 + z^2)} \left( 2x \left( x - \sqrt{x^2 + z^2} \right) + z^2 \right) - 17z \sqrt{x^2 + z^2} \right)$$

$$+ x^6 \sqrt{x^2 + z^2} - x^4 z \left( 3z \sqrt{x^2 + z^2} + \sqrt{(x^2 + z^2)} \left( 2x \left( x - \sqrt{x^2 + z^2} \right) + z^2 \right) \right)$$

$$+ 12x^6 \hat{x}$$

$$+ \frac{4\pi \hbar \kappa z}{x^3 (x^2 + z^2)^{3/2}} \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \left( 3x^5 + 12x^3 z^2 \right)$$

$$+ 4x^2 z \left( 2 \sqrt{(x^2 + z^2)} \left( 2x \left( x - \sqrt{x^2 + z^2} \right) + z^2 \right) - 3z \sqrt{x^2 + z^2} \right)$$

$$+ 8z^3 \left( \sqrt{(x^2 + z^2)} \left( 2x \left( x - \sqrt{x^2 + z^2} \right) + z^2 \right) - z \sqrt{x^2 + z^2} \right)$$

$$- 3x^4 \sqrt{x^2 + z^2} + 8x^4 z \right) \hat{z}.$$ (S47)

Directly above the defect this simplifies to

$$\langle v(r) \rangle |_{x=y=0} = 0.$$ (S48)

E. Saddle defect

Lastly, using the carpet profile

$$f(\rho_s, \theta_s, \phi_s) = \frac{N}{\pi R^2} \delta \left( \phi_s + \theta_s \right),$$

(S49)
we find the flows for a saddle defect, 

\[ \langle v(r) \rangle |_{y=0} = \int_0^\infty \int_{-\pi}^\pi \int_{-\pi}^\pi \frac{\delta(\phi_s + \theta_s)}{n} \tilde{u}(x, y = 0, z, \rho_s, \theta_s, \phi_s) \, d\phi_s \, d\theta_s \, \rho_s \, d\rho_s \]  

(S50)

\[ = -\frac{8\pi n h \kappa}{x^5 (x^2 + z^2)^{3/2}} \frac{1}{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \left( -8 + 32z \right) \left( z - \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \right) \]

\[ -32xz^6 \sqrt{x^2 + z^2} + 4x^2 z^5 \left( 17z - 13 \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \right) \]

\[ + x^7 \sqrt{x^2 + z^2} + 2x^6 z \left( \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} + z \right) \]

\[ -3x^5 z^2 \sqrt{x^2 + z^2} + x^4 z^3 \left( 39z - 17 \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \right) \]

\[ -36x^3 z^4 \sqrt{x^2 + z^2} \right) \hat{x} \]

\[ -\frac{8\pi n h \kappa z}{x^4 (x^2 + z^2)^{3/2}} \frac{1}{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} \left( -4x^6 + 24z \right) \left( \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} - z \right) \]

\[ + 24xz^4 \sqrt{x^2 + z^2} + 4x^2 z^3 \left( 10 \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} - 13z \right) \]

\[ + 4x^5 \sqrt{x^2 + z^2} + x^4 z \left( 15 \sqrt{2x \left( x - \sqrt{x^2 + z^2} \right) + z^2} - 32z \right) \]

\[ + 28xz^2 \sqrt{x^2 + z^2} \right) \hat{z}. \]  

(S51)

Again, directly above the defect this simplifies to

\[ \langle v(r) \rangle |_{x=y=0} = 0. \]  

(S52)

SI §5. SIMULATIONS OF ACTIVE CARPET FLOWS

Next to analytical integration, the flows due to an active carpet may also be approximated in simulations. To determine the average flow we place \( N \) swimmers on a surface and compute the sum

\[ \langle v(r) \rangle = \sum_{i=1}^{N} u_i(r, r_i, p_i), \]  

(S53)

where the positions \( r_i \) and orientations \( p_i \) are found numerically via inverse transform sampling (Smirnov transform) in order to satisfy the probability distribution \( f(r_s, p) \).

A. Uniform cluster

To see this explicitly, we first consider the case of a uniform cluster profile (S6), for which we aim to sample the random variates \( \rho_s \in [0, R], \theta_s \in [-\pi, \pi], \phi_s \in [-\pi, \pi] \) in terms of three random variates, \( w_i \in [0,1] \) with
\(i \in [1, 2, 3]\), drawn from the standard uniform distribution. This profile (S6) is separable in the three variables, \(f(\rho_s, \theta_s, \phi_s) = N f_{\rho_s} f_\theta f_{\phi_s}\), with angular distributions \(f_\theta = f_{\phi_s} = 1/(2\pi)\) and the radial distribution \(f_{\rho_s} = 2/R^2\). Therefore we immediately find that the angles \(\theta_s, \phi_s\) can be sampled by taking

\[
\theta_s = -\pi + 2\pi w_1, \\
\phi_s = -\pi + 2\pi w_2.
\]  
(S54, S55)

To sample the distance \(\rho_s\), we compute the cumulative distribution function (CDF),

\[
F_{\rho_s}(\rho_s) = \int_0^{\rho_s} f_{\rho_s}(\tau) d\tau = \frac{\rho_s^2}{R^2}.
\]  
(S56)

Solving the inverse transform, \(w_3 = F_{\rho_s}(\rho_s)\), then gives the sampling

\[
\rho_s = R \sqrt{w_3}.
\]  
(S57)

**B. Gaussian cluster**

Similarly, a cluster with a Gaussian cluster profile (S21) can be integrate to get the CDF

\[
F_{\rho_s}(\rho_s) = \int_0^{\rho_s} \frac{1}{R^2} \exp \left( -\frac{\tau^2}{2R^2} \right) \tau d\tau = 1 - \exp \left( -\frac{\rho_s^2}{2R^2} \right) = w_3.
\]  
(S58)

This is inverted directly to obtain

\[
\rho_s = R \sqrt{2 \log \left( \frac{1}{1 - w} \right)}.
\]  
(S59)

**C. Density gradients**

To model a linear density gradient we consider the profile

\[
f(r_s, p) = \frac{3N(1 - \rho_s/R)\Theta(R - \rho_s)\delta(z_s - h)\delta(|p| - 1)\delta(p \cdot \hat{z})}{\pi R^2},
\]  
(S60)

which for the random variates \(\rho_s \in [0, R], \theta_s \in [-\pi, \pi], \phi_s \in [-\pi, \pi]\) simplifies to

\[
f(\rho_s, \theta_s, \phi_s) = N f_{\rho_s} f_\theta f_{\phi_s} = N \frac{6(1 - \rho_s/R)}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi},
\]  
(S61)

and which is again normalised with respect to the number of swimmers,

\[
\int f(r_s, p) dr_s dp = \int_0^R \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} N \frac{6(1 - \rho_s/R)}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} d\phi_s d\theta_s d\rho_s = N.
\]  
(S62)

Hence, it follows that the angular variates are still given by (S54,S55), and for radial variate we have the CDF

\[
F_{\rho_s}(\rho_s) = \int_0^{\rho_s} f_{\rho_s}(\tau) d\tau = \frac{\rho_s^2(3R - 2\rho_s)}{R^3}
\]  
(S63)

The radial variate is then given by the cubic expression

\[
\rho_s = F^{-1}_{\rho_s}(w_3),
\]  
(S64)

which can be solved numerically or by using the Cardano formula. In Fig. 1e of the main text we show the flows generated by \(N = 12,500\) swimmers sampled in this manner, with \(R = 200\mu m\) so that \(N/(\pi R^2) = 0.1/\mu m^2\), as computed using Eq. S53. We focus on the area in the middle of the gradient, around \(x = 100\).
D. Branching pattern

The flows that attract nutrients down towards a colony do not depend strongly on the cluster morphology. To demonstrate this we manually arrange $N = 1800$ bacteria in the shape of a branching pattern. These swimmer positions are visualised in the enlarged SI Fig. S1. In order to minimise orientation gradients and add focus on the density gradients at the edges of the cluster, the swimmers are given uniformly distributed orientations, $\phi_s \in [-\pi, \pi]$. The flows are then computed using Eq. S53.
E. Movie of dynamic cluster

We consider a dynamic cluster with \( N = 125 \) swimmers that move around a chemoattractant source located at the origin. This is modelled as a bias in the swimming direction (\( \phi_s \)) towards the origin direction (\( \theta_s + \pi \)), encoded in the Langevin equations

\[
\begin{align*}
\dot{\phi}_s &= A \sin(\phi_s - \theta_s) + \xi_r(t), \\
\dot{\mathbf{r}}_s &= v_s \mathbf{p}_s, \\
\dot{\mathbf{r}} &= \sum_{i=1}^{N} \mathbf{u}_i(\mathbf{r}, \mathbf{r}_i, p_i) + \zeta(t),
\end{align*}
\]

(S65) (S66) (S67)

where the bias \( A = 5\text{rad}/\text{s} \), the swimming speed \( v_s = 20\mu\text{m}/\text{s} \), and the individual swimmer flow is extensile (‘pusher’) with \( \kappa = 30\mu\text{m}^3/\text{s} \). The swimmer rotational noise obeys \( \langle \xi_r(t) \rangle = 0 \), \( \langle \xi_r(t) \xi_r(t') \rangle = 2D_r \delta(t - t') \) with \( D_r = 1\text{rad}^2/\text{s} \), and the tracer noise obeys \( \langle \zeta(t) \rangle = 0 \), \( \langle \zeta(t) \zeta(t') \rangle = 2D \delta(t - t') \) with \( D = 1\mu\text{m}^2/\text{s} \).

Swimmers are initiated with random positions and orientations, distributed as a Gaussian cluster (see Eq. S21) with radius \( R = 20\mu\text{m} \). As time evolves this swimmer distribution \( f(\mathbf{r}, \mathbf{p}) \) tends to a steady state when the attraction and fluctuations balance one another. This steady state is also a cluster of radius \( R \sim 20\mu\text{m} \), so that the number density remains constant at \( n \sim 0.1/\mu\text{m}^2 \). Tracer particles (\( N_T = 50 \)) are initiated with random positions, \( x \in [-25, 25]\mu\text{m}, \ y \in [-2, 2]\mu\text{m} \) and \( z \in [15, 25]\mu\text{m} \). Then, the swimmer and tracers dynamic are integrated numerically with time step \( \delta t = 10\text{ms} \) to \( t = 10\text{s} \). Note that the tracers are not allowed to pass the plane \( z = 2\mu\text{m} \) to avoid contact with the near-field swimmer flows.

The resulting Movie S1 shows that the tracer particles directly above the cluster are attracted downwards. Then they move sideways, down the swimmer concentration gradient, and finally back up again to complete the recirculation.

F. Bend & splay gradients, topological defects

To simulate the flows due to bend and splay gradients [Main text Fig. 2], and topological defects [Main text Fig. 3], we employ the carpet profile

\[
f(\rho, \theta, \phi) = \frac{N}{\pi R^2} \Theta(R - \rho) \delta(\phi - \theta - \phi_0).
\]

(S68)

In order to avoid density gradients, we place the swimmers on a dense regular lattice,

\[
x_s = (1 + 2i)\mu\text{m}, \quad y_s = (1 + 2j)\mu\text{m}, \quad i, j \in [-250, \ldots, 0, \ldots 249],
\]

(S69)

so that \( n = 0.25/\mu\text{m}^2 \) and the offset is chosen to keep symmetry about the \( x \) and \( y \) axes and to avoid conflicts at the origin. Next, the swimmers outside the radius \( R \) are removed to retain axial symmetry, and \( N = 195,502 \) swimmers remain. Once the positions are set, the orientations are set by \( \phi_s = \theta_s + \phi_0 \). The flows are then computed using Eq. S53.

G. Vortex array

The currents due to a bacterial vortex array is modeled using the Taylor-Green Vortex (TGV) model. As for the bend & splay gradients, the swimmers positions are determined by a dense regular mesh,

\[
x_s = (1 + 2i)\mu\text{m}, \quad y_s = (1 + 2j)\mu\text{m}, \quad i, j \in [-250, \ldots, 0, \ldots 249],
\]

(S70)

so that the density is uniform with \( n = 0.25/\mu\text{m}^2 \). No swimmers are removed for symmetry reasons, so \( N = 250,000 \) swimmers.

Next, the swimmer orientations are set by the TGV director field,

\[
p_x = -\cos \left( \frac{\pi x_s}{\lambda} \right) \sin \left( \frac{\pi y_s}{\lambda} \right), \quad p_y = \sin \left( \frac{\pi x_s}{\lambda} \right) \cos \left( \frac{\pi y_s}{\lambda} \right),
\]

(S71)

where the unit cell size (half-wavelength) is \( \lambda = 33\mu\text{m} \). The flows are then computed using Eq. S53, as shown enlarged in SI Fig. S2.
FIG. S2. Enlargement of Fig. 4a of the Main Text. Flows above a bacterial vortex array, simulated as a Taylor-Green pattern with unit cell size $\lambda = 33\mu m$, and uniform swimmer density $n = 0.25/\mu m^2$. Colours indicate vertical flows in $\mu m/s$, simulated for $z = 10\mu m$ and side views at $x = -33\mu m$ and $y = 50\mu m$. Green arrows are stream lines and black arrows the swimmer orientations.

SI §6. BACTERIAL TURBULENCE

A. Self-propelled rod (SPR) model

We model the bacterial bath in two spatial dimensions by $N$ rod-like self-propelled units [Fig. S3]. For a detailed description, also see Ref. [43]. Each rod has an aspect ratio $\Gamma = \ell / \lambda_r = 5$ which is chosen in order to model Bacillus subtilis suspensions, as considered in experiments dealing with bacterial turbulence. Rods of length $\ell$ and width $\lambda_r$ are discretized into $n = 6$ spherical segments equidistantly positioned, with a displacement $s = 0.85\lambda_r$, along the main rod axis $\mathbf{p} = (\cos \varphi, \sin \varphi)$. Between the segments of different rods a repulsive Yukawa potential is imposed [44]. The resulting pair potential of a rod pair $\alpha, \beta$ is given by

$$U_{\alpha\beta} = \sum_{i=1}^{n} \sum_{j=1}^{n} U_i U_j \exp\left[-r_{ij}^{\alpha\beta} / \lambda_r \right] / r_{ij}^{\alpha\beta},$$

where $\lambda_r$ is the screening length and $r_{ij}^{\alpha\beta} = |\mathbf{r}_i^\alpha - \mathbf{r}_j^\beta|$ the distance between segment $i$ of rod $\alpha$ and segment $j$ of rod $\beta$ ($\alpha \neq \beta$). Any overlap of particles is prohibited by choosing a large interaction strength $U_j^2 = 2.5 F_0 \ell$. Here $F_0$ is an
FIG. S3. Diagram of a pair-wise interaction in the self-propelled rod (SPR) model. Rods of aspect ratio \( \Gamma = \ell/\lambda_c \) are composed of \( n = 5 \) repulsive Yukawa segments. Self-propulsion arises from a constant force \( F \) acting along the rod axis, indicated by the unit vector \( \hat{p} \). The overall pair-wise interaction is obtained by summing the Yukawa potentials over all segment pairs with separation \( \mathbf{r}_{ij} \), which decays rapidly with the centre-of-mass separation \( \Delta \mathbf{r} \).

The effective self-propulsion force directed along the main rod axis and leading to a constant propulsion velocity \( v_0 \). We do not resolve details of the actual propulsion mechanism or hydrodynamics interactions.

Micro-swimmers move in the low Reynolds number regime. The corresponding overdamped equations of motion for the positions \( \mathbf{r}_\alpha \) and orientations \( \mathbf{p}_\alpha \) are

\[
\begin{align*}
\mathbf{f}_T \cdot \partial_t \mathbf{r}_\alpha(t) &= -\nabla_{\mathbf{r}_\alpha} U(t) + F_0 \mathbf{p}_\alpha(t), \\
\mathbf{f}_R \cdot \partial_t \mathbf{p}_\alpha(t) &= -\nabla_{\mathbf{p}_\alpha} U(t),
\end{align*}
\]

(S73, S74)

in terms of the total potential energy \( U = (1/2) \sum_{\alpha,\beta(\alpha\neq\beta)} U_{\alpha\beta} + \sum_{\alpha,\gamma} U_{\alpha\gamma} \) with \( U_{\alpha\gamma} \) the potential energy of rod \( \alpha \) with the carrier \( \gamma \). The one-body translational and rotational friction tensors for the rods \( \mathbf{f}_T \) and \( \mathbf{f}_R \) can be decomposed into parallel \( f_{||} \), perpendicular \( f_{\perp} \) and rotational \( f_R \) contributions which depend solely on the aspect ratio \( \Gamma = \ell/\lambda_c \) [45],

\[
\begin{align*}
\frac{2\pi}{f_{||}} &= \ln p - 0.207 + 0.980p^{-1} - 0.133p^{-2}, \\
\frac{4\pi}{f_{\perp}} &= \ln p + 0.839 + 0.185p^{-1} + 0.233p^{-2}, \\
\frac{\pi a^2}{3f_R} &= \ln p - 0.662 + 0.917p^{-1} - 0.050p^{-2}.
\end{align*}
\]

(S75, S76, S77)

Accordingly, the propulsion velocity is given by \( v_0 = F_0/f_{||} \) and sets the characteristic time unit \( \tau = \ell/v_0 \).

The total number of rods is \( N_0 = 10,000 \) and we use a quadratic simulation domain of size \( L = 200\mu m \) with periodic boundary conditions, \( (x_0, y_0) \in [-L/2, L/2] \mu m \), to establish a uniform swimmer density of \( n = 0.25 \text{bact.}/\mu m^2 \). The dimensionless packing fraction \( \Phi = \lambda_c \ell N_0/L^2 \) is fixed to \( \Phi = 0.7 \) to achieve a turbulent bacterial bath [24]. The initial swimmer configuration is a smectic lattice of rods, where the rods are randomly orientated up- and downwards. We then simulate 301 time steps of \( \delta t = 0.01 \) seconds, so the dimensional simulation times are \( t = [0, \delta t, 2\delta t, \ldots, 3s] \). Movie S2 shows the resulting dynamics, with some swimmers coloured so they can be identified throughout the turbulent motion.

B. Movies of flow due to bacterial turbulence

To compute the long-ranged flows, and to avoid edge effects, we enlarge the carpet by duplicating the periodic swimmer positions,

\[
\begin{align*}
x_s &= x_0 + 200\mu m \times i, & y_s &= y_0 + 200\mu m \times j, & i, j = [-2, -1, 0, 1, 2],
\end{align*}
\]

(S78)
FIG. S4. Enlargement of Fig. 4b of the Main Text. Flows above a carpet of bacterial turbulence, simulated with the SPR model with aspect ratio $\Gamma = 5$, packing fraction $\Phi = 0.7$ and swimmer density $n = 0.25/\mu m^2$. Colours indicate vertical flows in $\mu m/s$, simulated for $z = 10\mu m$, and side views at $x = -50\mu m$ and $y = 50\mu m$. Green arrows are stream lines. Black arrows show the individual swimmer positions and orientations.

so that the total number of swimmers is $N = 5 \times 5 \times N_0 = 250,000$, in the domain $(x_s, y_s) \in [-500, 500] \mu m$. The flows are then computed using Eq. S53, as shown enlarged in SI Fig. S4.

Movie S3 shows the flows due to bacterial turbulence in the plane $z = 10\mu m$, for the local area $(x, y) \in [-50, 50] \mu m$. Colours indicate vertical flows, $v_z \in [-4, 4] \mu m/s$. Green arrows are stream lines of the lateral flows, and black arrows show the individual swimmer positions and orientations.

Movie S4 shows these flows in the plane $z = 25\mu m$, again for the local area $(x, y) \in [-50, 50] \mu m$. Colours indicate vertical flows, $v_z \in [-1, 1] \mu m/s$. Green arrows are stream lines of the lateral flows, and black arrows show the individual swimmer positions and orientations.

Movie S5 shows a side view of these flows, for the cross section $y = 0$, with lateral position $x \in [-50, 50] \mu m$ and heights $z \in [2, 25] \mu m$. Colours indicate vertical flows, $v_z \in [-3, 3] \mu m/s$. Green arrows are stream lines in the plane.
FIG. S5. (a) Spatial correlation functions of vertical flows generated by a carpet of bacterial turbulence, $g_v(\rho)$, for heights $z \in [2, 20] \mu m$ (blue-red). Same as Fig. 4d of the Main Text. (b) Collapse of these spatial correlation functions onto one curve, when rescaling the lateral distance with the distance from the wall, $g_{v}(\rho/z)$.

C. Temporal correlation functions

The bacterial turbulence flows are computed as in the previous section, with $N = 250,000$ swimmers in the domain $(x_s, y_s) \in [-500, 500] \mu m$. Next, the temporal correlation function of the vertical flows is defined as

$$c_{v}(t) = \frac{\langle v_z(t_1)v_z(t_2) \rangle}{\sqrt{\langle v_z^2(t_1) \rangle \langle v_z^2(t_2) \rangle}}; \quad t = |t_1 - t_2|,$$

where the average is over lateral space. This is implemented numerically by sampling the flow with Eq. S53 at $N_q = 200$ points with positions uniformly distributed over the carpet, $(x_q, y_q) \in [-100, 100] \mu m$, and fixed height $z$ for each correlation function. The average is then taken over all points but only for the last 251 time steps, $t \in [0.5, 3]$ s, in order to exclude the initial phase where the swimmers still develop turbulent motion after initiation.

Equivalently, the temporal correlation function of the swimmer orientations is defined as

$$c_{p}(t) = \langle p(t_1) \cdot p(t_2) \rangle; \quad t = |t_1 - t_2|,$$

where the average is performed over all $N_0$ swimmers, and again only for times $t \in [0.5, 3]$.

To determine the typical correlation time, $t_*(z)$, the resulting temporal correlation functions are fitted to exponentials,

$$c(t) \to \exp \left(-\frac{t}{t_*}\right),$$

which is the most elementary function with one parameter that well describes the data. We tried other functions (Gaussian, $1/t$ decay) and these give similar results. Hence, the fitted correlation times are plotted against $z$ in Fig. 4c of the Main Text.

D. Spatial correlation functions

The equal-time spatial correlation function of the swimmer orientations is defined as

$$g_p(\rho) = \langle p(r_1) \cdot p(r_2) \rangle; \quad |\rho - \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}| < \epsilon,$$

where the average is performed over all $N_{ps}$ pairs of swimmers between the $N_0$ that are separated a distance $\rho$ in the simulation with a margin of $\epsilon = 0.5 \mu m$. The number of pairs grows with $\rho$, and is still fairly large; $N_{ps} \sim 100$ for $\rho = 10 \mu m$. A larger value of $\epsilon$ increases the number of sample pairs but reduces the resolution of the correlation function.
The equal-time spatial correlation of the vertical flows is defined as

\[ g_{v_z}(\rho) = \frac{\langle v_z(r_1)v_z(r_2) \rangle}{\sqrt{\langle v_z^2(r_1) \rangle \langle v_z^2(r_2) \rangle}}; \quad \rho = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \quad (S83) \]

This is implemented numerically by sampling the flow with Eq. S53 for \( N_q = 200 \) pairs of points. These are sampled by selecting a midpoint \( r_m \) with a position uniformly distributed over the carpet, \( (x_m, y_m) \in [-100, 100] \mu m \), and fixed height \( z \). Next, a random orientation \( \theta_m \in [-\pi, \pi] \) is sampled, so that the pair is given by \( r_{1,2} = r_m \pm \frac{\rho}{2} (\cos \theta_m, \sin \theta_m, 0) \). Hence, the correlation functions are found by averaging over all the pairs, with the fixed time \( t = 1 s \).

To determine the typical correlation length, \( \rho_*(z) \), the resulting spatial correlation functions are fitted to Gaussians,

\[ g(\rho) \rightarrow \exp \left( \frac{-\rho^2}{2\rho_*^2} \right), \quad (S84) \]

which is the most elementary function with one parameter that well describes the data. We tried other functions (exponential, \( 1/t \) decay) and these give similar results.

Figure S5(a) shows these spatial flow correlations, \( g_{v_z}(\rho) \), the same as in Fig. 4d of the main text. Figure S5(b) shows a collapse of the correlation functions when rescaling with respect to the distance from the surface, \( g_{v_z}(\rho/z) \).