Homotopy types of $SU(n)$-gauge groups over non-spin 4-manifolds

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Abstract
Let $M$ be an orientable, simply-connected, closed, non-spin 4-manifold and let $G_k(M)$ be the gauge group of the principal $G$-bundle over $M$ with second Chern class $k \in \mathbb{Z}$. It is known that the homotopy type of $G_k(M)$ is determined by the homotopy type of $G_k(\mathbb{C}P^2)$. In this paper we investigate properties of $G_k(\mathbb{C}P^2)$ when $G = SU(n)$ that partly classify the homotopy types of the gauge groups.

Keywords
Gauge groups · Homotopy type · Non-spin 4-manifolds

Mathematics Subject Classification
Primary 55P15; Secondary 54C35 · 81T13

1 Introduction
Let $G$ be a simple, simply-connected, compact Lie group and let $M$ be an orientable, simply-connected, closed 4-manifold. Then the isomorphism class of a principal $G$-bundle $P$ over $M$ is classified by its second Chern class $k \in \mathbb{Z}$. In particular, if $k = 0$, then $P$ is a trivial $G$-bundle. The associated gauge group $G_k(M)$ is the topological group of $G$-equivariant automorphisms of $P$ which fix $M$.

A simply-connected 4-manifold is spin if and only if its intersection form is even. In the case of simply-connected 4-manifolds, the spin condition is equivalent to all cup product squares being trivial in mod 2 cohomology. In this paper, we consider the homotopy types of gauge groups $G_k(M)$, where $M$ is a non-spin 4-manifold such as $\mathbb{C}P^2$. When $M$ is a spin 4-manifold, topologists have been studying the homotopy types of gauge groups over $M$ extensively over the last twenty years. On the one hand, Theriault showed in [16] that there is a homotopy equivalence

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$$G_k(M) \simeq G_k(S^4) \times \prod_{i=1}^{d} \Omega^2 G,$$

where \(d\) is the second Betti number of \(M\). Therefore to study the homotopy type of \(G_k(M)\) it suffices to study \(G_k(S^4)\). On the other hand, many cases of homotopy types of \(G_k(S^4)\)’s are known. For examples, there are 6 distinct homotopy types of \(G_k(S^4)\)’s for \(G = SU(2)\) \([11]\), and 8 distinct homotopy types for \(G = SU(3)\) \([5]\). When localized rationally or at any prime, there are 16 distinct homotopy types for \(G = SU(5)\) \([19]\) and 8 distinct homotopy types for \(G = Sp(2)\) \([17]\).

When \(M\) is a non-spin 4-manifold, the author in \([14]\) showed that there is a homotopy equivalence

$$G_k(M) \simeq G_k(\mathbb{C}P^2) \times \prod_{i=1}^{d-1} \Omega^2 G,$$

so the homotopy type of \(G_k(M)\) depends on the special case \(G_k(\mathbb{C}P^2)\). Compared to the extensive work on \(G_k(S^4)\), only two cases of \(G_k(\mathbb{C}P^2)\) have been studied, which are the \(SU(2)\)- and \(SU(3)\)-cases \([12,18]\). As a sequel to \([14]\), this paper investigates the homotopy types of \(G_k(\mathbb{C}P^2)\)’s in order to explore gauge groups over non-spin 4-manifolds.

A common approach to classifying the homotopy types of gauge groups is as follows. Atiyah, Bott and Gottlieb \([1,3]\) showed that the classifying space \(BG_k(M)\) is homotopy equivalent to the connected component \(\text{Map}_k(M, BG)\) of the mapping space \(\text{Map}(M, BG)\) containing the map \(k \alpha \circ q\), where \(q : M \to S^4\) is the quotient map and \(\alpha\) is a generator of \(\pi_4(BG) \cong \mathbb{Z}\). The evaluation map \(ev : BG_k(M) \to BG\) induces a fibration sequence

$$G_k(M) \longrightarrow G \longrightarrow \text{Map}_k^*(M, BG) \longrightarrow BG_k(M) \longrightarrow ev : BG_k(M) \to BG,$$  \hspace{1cm} (1)

where \(\partial_k : G \to \text{Map}_k^*(M, BG)\) is the boundary map. The action of \(\pi_4(BG) \cong \mathbb{Z}\) on \(\text{Map}_k^*(M, BG)\) induces a homotopy equivalence \(\text{Map}_k^*(M, BG) \simeq \text{Map}_k^*(M, BG)\).

Denote the composition \(G \longrightarrow \text{Map}_k^*(M, BG) \simeq \text{Map}_k^*(M, BG)\) also by \(\partial_k\) for convenience. For \(M = S^4\), \(\text{Map}_k^*(M, BG) \simeq \Omega_0^3 G\) is an H-group so \([G, \Omega_0^3 G]\) is a group. The order of \(\partial_1 : G \to \Omega_0^3 G\) is important for distinguishing the homotopy types of \(G_k(S^4)\).

**Theorem 1.1** (Theriault, \([17]\)) Let \(m\) be the order of \(\partial_1\). If \((m, k) = (m, l)\), then \(G_k(S^4)\) is homotopy equivalent to \(G_l(S^4)\) when localized rationally or at any prime.

For most cases of \(G\), the exact value of the order of \(\partial_1\) is difficult to compute. When \(G = SU(n)\), the exact value or a partial result of the order of \(\partial_1\) was worked out for certain cases. For any number \(a = p'q\) where \(q\) is coprime to \(p\), the \(p\)-component of \(a\) is \(p'\) and is denoted by \(v_p(a)\).
Theorem 1.2 ([2,5,9,11,19,20]) Let $G$ be $SU(n)$ and let $m$ be the order of $\partial_1$. Then

- $m = 12$ for $n = 2$
- $m = 24$ for $n = 3$
- $m = 120$ for $n = 5$
- $m = 60$ or $120$ for $n = 4$
- $v_p(m) = v_p(n(n^2 - 1))$ for $n < (p - 1)^2 + 1$.

In Theorem 1.1, the g.c.d condition $(m, k) = (m, l)$ gives a sufficient condition for the homotopy equivalence $\mathcal{G}_k(S^4) \simeq \mathcal{G}_l(S^4)$. Conversely, there is a partial necessary condition for certain cases of $G = SU(n)$.

Theorem 1.3 (Hamanaka and Kono [5]; Kishimoto, Kono and Tsutaya [9]) Let $G$ be $SU(n)$ and let $p$ be an odd prime. If $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_l(S^4)$, then

- $(n(n^2 - 1), k) = (n(n^2 - 1), l)$ for $n$ odd,
- $v_p(n(n^2 - 1), k) = v_p(n(n^2 - 1), l)$ for $n$ less than $(p - 1)^2 + 1$.

In this paper we consider gauge groups over $\mathbb{C}P^2$. Take $M = \mathbb{C}P^2$ in (1) and denote the boundary map by $\partial'_k : G \to \text{Map}_0^*(\mathbb{C}P^2, BG)$. Since $\text{Map}_0^*(\mathbb{C}P^2, BG)$ is not an H-space, $[G, \text{Map}_0^*(\mathbb{C}P^2, BG)]$ is not a group so the order of $\partial'_k$ makes no sense. However, we can still define an “order” of $\partial'_k$ [18], which will be described in Sect. 2. We show that the “order” of $\partial'_1$ helps distinguish the homotopy type of $\mathcal{G}_k(\mathbb{C}P^2)$ as in Theorem 1.1.

Theorem 1.4 Let $m'$ be the “order” of $\partial'_1$. If $(m', k) = (m', l)$, then $\mathcal{G}_k(\mathbb{C}P^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{C}P^2)$ when localized rationally or at any prime.

We study the $SU(n)$-gauge groups over $\mathbb{C}P^2$ and use unstable $K$-theory to give a lower bound on the “order” of $\partial'_1$ that is in the spirit of Theorem 1.2.

Theorem 1.5 When $G$ is $SU(n)$, the “order” of $\partial'_1$ is at least $\frac{1}{2}n(n^2 - 1)$ for $n$ odd, and $n(n^2 - 1)$ for $n$ even.

Localized rationally or at an odd prime, we have $\mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_k(S^4) \times \Omega^2 G$ [16]. The homotopy types of $\mathcal{G}_k(\mathbb{C}P^2)$ are then completely determined by that of $\mathcal{G}_k(S^4)$, which have been investigated in many cases when the localizing prime is relatively large [6,7,9,10,20]. A large part of the remaining cases can be understood by studying the 2-localized order of $\partial'_1$, on which Theorem 1.5 gives bounds for the $SU(n)$ case. For example, combining Theorem 1.5 with Lemma 2.2 implies the order of $\partial'_1$ is either 120 or 60 for $G = SU(5)$. Furthermore, when $G = SU(4)$ since the order of $\partial_1$ is either 120 or 60, the order of $\partial'_1$ is either 60 or 120.

Finally we prove a necessary condition for the homotopy equivalence $\mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_l(\mathbb{C}P^2)$ similar to Theorem 1.3.

Theorem 1.6 Let $G$ be $SU(n)$. If $\mathcal{G}_k(\mathbb{C}P^2)$ is homotopy equivalent to $\mathcal{G}_l(\mathbb{C}P^2)$, then

- $(\frac{1}{2}n(n^2 - 1), k) = (\frac{1}{2}n(n^2 - 1), l)$ for $n$ odd,
- $(n(n^2 - 1), k) = (n(n^2 - 1), l)$ for $n$ even.
2 Some facts about boundary map $\partial'_k$

Take $M$ to be $S^4$ and $\mathbb{C}P^2$ respectively in fibration (1) to obtain fibration sequences

$$
\begin{align*}
G_k(S^4) & \longrightarrow G \overset{\partial_k}{\longrightarrow} \Omega^3_0 G \longrightarrow BG_k(S^4) \overset{ev}{\longrightarrow} BG \\
G_k(\mathbb{C}P^2) & \longrightarrow G \overset{\partial'_k}{\longrightarrow} Map^*_0(\mathbb{C}P^2, BG) \longrightarrow BG_k(\mathbb{C}P^2) \overset{ev}{\longrightarrow} BG.
\end{align*}
$$

There is also a cofibration sequence

$$
S^3 \overset{\eta}{\longrightarrow} S^2 \longrightarrow \mathbb{C}P^2 \overset{q}{\longrightarrow} S^4,
$$

where $\eta$ is Hopf map and $q$ is the quotient map. Due to the naturality of $q^*$, we combine fibrations (2) and (3) to obtain a commutative diagram of fibration sequences

$$
\begin{array}{ccc}
G_k(S^4) & \overset{q^*}{\longrightarrow} & G_k(\mathbb{C}P^2) \\
\downarrow \partial_k & & \downarrow \partial'_k \\
\Omega^3_0 G & \overset{q^*}{\longrightarrow} & Map^*_0(\mathbb{C}P^2, BG) \\
\downarrow & & \downarrow \\
BG_k(S^4) & \longrightarrow & BG_k(\mathbb{C}P^2) \\
\end{array}
$$

It is known, [13], that $\partial_k$ is triple adjoint to Samelson product

$$
\langle k_1, 1 \rangle : S^3 \wedge G \overset{\partial_k}{\longrightarrow} G \wedge G \overset{(1, 1)}{\longrightarrow} G,
$$

where $\iota : S^3 \to SU(n)$ is the inclusion of the bottom cell and $\langle 1, 1 \rangle$ is the Samelson product of the identity on $G$ with itself. The order of $\partial_k$ is its multiplicative order in the group $[G, \Omega^3_0 G]$.

Unlike $\Omega^3_0 G, Map^*_0(\mathbb{C}P^2, BG)$ is not an H-space, so $\partial'_k$ has no order. In [18], Theriault defined the “order” of $\partial'_k$ to be the smallest number $m'$ such that the composition

$$
G \overset{\partial_k}{\longrightarrow} \Omega^3_0 G \overset{m'}{\longrightarrow} \Omega^3_0 G \overset{q^*}{\longrightarrow} Map^*_0(\mathbb{C}P^2, BG)
$$

is null homotopic. In the following, we interpret the “order” of $\partial'_k$ as its multiplicative order in a group contained in $[\mathbb{C}P^2 \wedge G, BG]$.

Apply $[- \wedge G, BG]$ to cofibration (4) to obtain an exact sequence of sets

$$
[S^3 G, BG] \overset{(\Sigma \eta)^*}{\longrightarrow} [\Sigma^4 G, BG] \overset{q^*}{\longrightarrow} [\mathbb{C}P^2 \wedge G, BG].
$$

All terms except $[\mathbb{C}P^2 \wedge G, BG]$ are groups and $(\Sigma \eta)^*$ is a group homomorphism since $\Sigma \eta$ is a suspension. We want to refine this exact sequence so that the last term is replaced by a group. Observe that $\mathbb{C}P^2$ is the cofiber of $\eta$ and so there is a coaction $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$. We show that the coaction gives a group structure on $Im(q^*)$.
Lemma 2.1  Let \( Y \) be a space and let \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) be a cofibration sequence. If \( \Sigma A \) is homotopy cocommutative, then \( \text{Im}(h^*) \) is an abelian group and

\[
\begin{align*}
[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} \text{Im}(h^*) \xrightarrow{} 0
\end{align*}
\]

is an exact sequence of groups and group homomorphisms.

**Proof**  Apply \([- , Y]\) to the cofibration to get an exact sequence of sets

\[
[\Sigma B, Y] \xrightarrow{(\Sigma f)^*} [\Sigma A, Y] \xrightarrow{h^*} [C, Y].
\]

Note that \([\Sigma B, Y]\) and \([\Sigma A, Y]\) are groups, and \((\Sigma f)^*\) is a group homomorphism. We will replace \([C, Y]\) by \(\text{Im}(h^*)\) and define a group structure on it such that \(h^* : [\Sigma A, Y] \to \text{Im}(h^*)\) is a group homomorphism.

For any \(\alpha, \beta\) in \([\Sigma A, Y]\), we define a binary operator \(\boxdot\) on \(\text{Im}(h^*)\) by

\[
h^*\alpha \boxdot h^*\beta = h^*(\alpha + \beta).
\]

To check this is well-defined we need to show \(h^*(\alpha + \beta) \simeq h^*(\alpha' + \beta) \simeq h^*(\alpha + \beta')\) for any \(\alpha, \alpha', \beta, \beta'\) satisfying \(h^*\alpha \simeq h^*\alpha'\) and \(h^*\beta \simeq h^*\beta'\).

First we show \(h^*(\alpha + \beta) \simeq h^*(\alpha' + \beta)\). By definition, we have

\[
h^*(\alpha + \beta) = (\alpha + \beta) \circ h = \nabla \circ (\alpha \lor \beta) \circ \sigma \circ h,
\]

where \(\sigma : \Sigma A \to \Sigma A \lor \Sigma A\) is the comultiplication and \(\nabla : Y \lor Y \to Y\) is the folding map. Since \(C\) is a cofiber, there is a coaction \(\psi : C \to C \lor \Sigma A\) such that \(\sigma \circ h \simeq (h \lor \mathbb{1}) \circ \psi\).

Then we obtain a string of equivalences

\[
h^*(\alpha + \beta) = \nabla \circ (\alpha \lor \beta) \circ \sigma \circ h
\]

\[
\simeq \nabla \circ (\alpha \lor \beta) \circ (h \lor \mathbb{1}) \circ \psi
\]

\[
\simeq \nabla \circ (\alpha' \lor \beta) \circ (h \lor \mathbb{1}) \circ \psi
\]

\[
\simeq \nabla \circ (\alpha' \lor \beta) \circ \sigma \circ h
\]

\[
= h^*(\alpha' + \beta)
\]

The third line is due to the assumption \(h^*\alpha \simeq h^*\alpha'\). Therefore we have \(h^*(\alpha + \beta) \simeq h^*(\alpha' + \beta)\). Since \(\Sigma A\) is cocommutative, \([\Sigma A, Y]\) is abelian and \(h^*(\alpha + \beta) \simeq h^*(\beta + \alpha)\). Then we have
\[ h^*(\alpha + \beta) \simeq h^*(\beta + \alpha) \simeq h^*(\beta' + \alpha) \simeq h^*(\alpha + \beta'). \]

This implies \( \boxtimes \) is well-defined.

Due to the associativity of \( + \) in \([\Sigma A, Y]\), \( \boxtimes \) is associative since

\[
(h^*\alpha \boxtimes h^*\beta) \boxtimes h^*\gamma = h^*(\alpha + \beta) \boxtimes h^*\gamma \\
= h^*((\alpha + \beta) + \gamma) \\
= h^*(\alpha + (\beta + \gamma)) \\
= h^*\alpha \boxtimes h^*(\beta + \gamma) \\
= h^*\alpha \boxtimes (h^*\beta \boxtimes h^*\gamma).
\]

Clearly the trivial map \( * : C \to Y \) is the identity of \( \boxtimes \) and \( h^*(-\alpha) \) is the inverse of \( h^*\alpha \). Therefore \( \boxtimes \) is indeed a group multiplication.

By definition of \( \boxtimes \), \( h^* : [\Sigma A, Y] \to \text{Im}(h^*) \) is a group homomorphism, and hence an epimorphism. Since \([\Sigma A, Y]\) is abelian, so is \( \text{Im}(h^*) \). We replace \([C, Y]\) by \( \text{Im}(h^*) \) in (6) to obtain a sequence of groups and group homomorphisms

\[
[S\Sigma B, Y]^{(\Sigma f)^*} \to [\Sigma A, Y] \xrightarrow{h^*} \text{Im}(h^*) \to 0.
\]

The exactness of (6) implies \( \text{ker}(h^*) = \text{Im}(\Sigma f)^* \), so the sequence is exact. \( \square \)

Applying Lemma 2.1 to cofibration \( \Sigma^3 G \to \Sigma^2 G \to \mathbb{C}P^2 \wedge G \) and the space \( Y = BG \), we obtain an exact sequence of abelian groups

\[
[S\Sigma^3 G, BG]^{(\Sigma \eta)^*} \to [\Sigma^4 G, BG] \xrightarrow{q^*} \text{Im}(q^*) \to 0. \tag{7}
\]

In the middle square of (5) \( \partial_k' \simeq q^*\partial_k \), so \( \partial_k' \) is in \( \text{Im}(q^*) \). For any number \( m \), \( q^*(m\partial_k) = mq^*\partial_k \), so the “order” of \( \partial_k' \) defined in [18] coincides with the multiplicative order of \( \partial_k' \) in \( \text{Im}(q^*) \). The exact sequence (7) allows us to compare the orders of \( \partial_1 \) and \( \partial_1' \).

**Lemma 2.2** Let \( m \) be the order of \( \partial_1 \) and let \( m' \) be the order of \( \partial_1' \). Then \( m \) is \( m' \) or \( 2m' \).

**Proof** By exactness of (7), there is some \( f \in [\Sigma^3 G, BG] \) such that \( (\Sigma \eta)^*f \simeq m'\partial_1' \). Since \( \Sigma \eta \) has order 2, \( 2m'\partial_1 \) is null homotopic. It follows that \( 2m' \) is a multiple of \( m \).

Since \( m \) is greater than or equal to \( m' \), \( m \) is either \( m' \) or \( 2m' \). \( \square \)

When \( G = SU(2) \), the order \( m \) of \( \partial_1 \) is 12 and the order \( m' \) of \( \partial_1' \) is 6 [12]. When \( G = SU(3) \), \( m = 24 \) and \( m' = 12 \) [18]. When \( G = Sp(2) \), \( m = 40 \) and \( m' = 20 \) [15]. It is natural to ask whether \( m = 2m' \) for all \( G \).

In the \( S^4 \) case, Theorem 1.1 gives a sufficient condition for \( G_k(S^4) \simeq G_l(S^4) \) when localized rationally or at any prime. In the \( \mathbb{C}P^2 \) case, Theriault showed a similar counting statement, in which the sufficient condition depends on the order of \( \partial_1 \) instead of \( \partial_1' \).
Let \( m \) be the order of \( \partial_1 \). If \((m, k) = (m, l)\), then \( G_k(\mathbb{CP}^2) \) is homotopy equivalent to \( G_l(\mathbb{CP}^2) \) when localized rationally or at any prime.

Lemma 2.2 can be used to improve the sufficient condition of Theorem 2.3.

**Theorem 2.4** Let \( m' \) be the order of \( \partial_1' \). If \((m', k) = (m', l)\), then \( G_k(\mathbb{CP}^2) \) is homotopy equivalent to \( G_l(\mathbb{CP}^2) \) when localized rationally or at any prime.

**Proof** By Lemma 2.2, \( m \) is either \( m' \) or \( 2m' \). If \( m = m' \), then the statement is same as Theorem 2.3. Assume \( m = 2m' \). Localize at an odd prime \( p \). Let \( p' \) be the \( p \)-component of \( m \), that is \( m = p' \cdot q \) where \( q \) is coprime to \( p \). Observe that \( m \circ \partial_1 \simeq (p' \cdot q) \circ \partial_1 \simeq p' \circ \partial_1 \) since the power map \( q : \Omega_0^3 G \to \Omega_0^3 G \) is a homotopy equivalence. Therefore \( p' \) is the order of \( \partial_1 \) after localization. The hypothesis \((m', k) = (m', l)\) implies \((p', k) = (p', l)\), so a homotopy equivalence \( G_k(\mathbb{CP}^2) \simeq G_l(\mathbb{CP}^2) \) follows by Theorem 2.3. A similar argument works for rational localization. Now it remains to consider the case where \( m = 2m' \) when localized at 2.

Assume \( m = 2^n \) and \( m' = 2^{n-1} \). For any \( k \), \((2^{n-1}, k) = 2^i \) where \( i \) is an integer such that \( 0 \leq i \leq n - 1 \). If \( i \leq n - 2 \), then \( k = 2^t \) for some odd number \( t \) and \((2^{n-1}, k) = 2^t \). The sufficient condition \((2^{n-1}, k) = (2^{n-1}, l)\) is equivalent to \((2^n, k) = (2^n, l)\). Again the homotopy equivalence \( G_k(\mathbb{CP}^2) \simeq G_l(\mathbb{CP}^2) \) follows by Theorem 2.3. If \( i = n - 1 \), then \((2^n, k) \) is either \( 2^n \) or \( 2^{n-1} \). We claim that \( G_k(\mathbb{CP}^2) \) has the same homotopy type for both \((2^n, k) = 2^n \) or \((2^n, k) = 2^{n-1} \).

Consider fibration (3)

\[
\text{Map}_0^*(\mathbb{CP}^2, G) \longrightarrow G_k(\mathbb{CP}^2) \longrightarrow G \longrightarrow \text{Map}_0^*(\mathbb{CP}^2, BG).
\]

If \((2^n, k) = 2^n \), then \( k = 2^t \) for some number \( t \). By linearity of Samelson products, \( \partial_k \simeq k \partial_1 \). Since \( \partial_k' \simeq q^* k \partial_1 \simeq q^* 2^n t \partial_1 \) and \( \partial_1 \) has order \( 2^n \), \( \partial_k' \) is null homotopic and we have

\[
G_k(\mathbb{CP}^2) \simeq G \times \text{Map}_0^*(\mathbb{CP}^2, G).
\]

If \((2^n, k) = 2^{n-1} \), then \( k = 2^{n-1} t \) for some odd number \( t \). Writing \( t = 2s + 1 \) gives \( k = 2^s 2^{n-1} \). Since \( \partial_k' \simeq q^* k \partial_1 \simeq q^* 2^n s \partial_1 \simeq q^* 2^{n-1} \partial_1 \) and \( \partial_1 \) has order \( 2^{n-1} \), \( \partial_k' \) is null homotopic and we have

\[
G_k(\mathbb{CP}^2) \simeq G \times \text{Map}_0^*(\mathbb{CP}^2, G).
\]

The same is true for \( G_l(\mathbb{CP}^2) \) and hence \( G_k(\mathbb{CP}^2) \simeq G_l(\mathbb{CP}^2) \).

**3 Plan for the proofs of Theorems 1.5 and 1.6**

From this section onward, we will focus on \( SU(n) \)-gauge groups over \( \mathbb{CP}^2 \). There is a fibration

\[
SU(n) \longrightarrow SU(\infty) \xrightarrow{p} W_n,
\]

\( \otimes \) Springer
where \( p : SU(\infty) \to W_n \) is the projection and \( W_n \) is the symmetric space \( SU(\infty)/SU(n) \). Then we have

\[
\begin{align*}
\tilde{H}^*(SU(\infty)) &= \Lambda(x_3, \ldots, x_{2n-1}, \ldots), \\
\tilde{H}^*(SU(n)) &= \Lambda(x_3, \ldots, x_{2n-1}), \\
\tilde{H}^*(BSU(n)) &= \mathbb{Z}[c_2, \ldots, c_n], \\
\tilde{H}^*(W_n) &= \Lambda(\bar{x}_{2n+1}, \bar{x}_{2n+3}, \ldots),
\end{align*}
\]

where \( x_{2n+1} \) has degree \( 2n + 1 \), \( c_i \) is the \( i \)th universal Chern class and \( x_{2i+1} = \sigma(c_{i+1}) \) is the image of \( c_{i+1} \) under the cohomology suspension \( \sigma \), and \( \tilde{p}^*(\bar{x}_{2i+1}) = x_{2i+1} \). Furthermore, \( H^{2n}(\Omega W_n) \cong \mathbb{Z} \) and \( H^{2n+2}(\Omega W_n) \cong \mathbb{Z} \) are generated by \( a_{2n} \) and \( a_{2n+2} \), where \( a_{2i} \) is the transgression of \( x_{2i+1} \).

The \((2n + 4)\)-skeleton of \( W_n \) is \( \Sigma^{2n-1}\mathbb{C}P^2 \) for \( n \) odd, and is \( S^{2n+3} \cup S^{2n+1} \) for \( n \) even, so its homotopy groups are as follows:

| \( i \)       | \( \pi_i(W_n) \) |
|-------------|----------------|
| \( n \) odd | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |
| \( n \) even | \( \mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) |

The canonical map \( \epsilon : \Sigma\mathbb{C}P^{n-1} \to SU(n) \) induces the inclusion \( \epsilon_* : H_*(\Sigma\mathbb{C}P^{n-1}) \to H_*(SU(n)) \) of the generating set. Let \( C \) be the quotient \( \mathbb{C}P^{n-1}/\mathbb{C}P^{n-3} \) and let \( \tilde{q} : \Sigma\mathbb{C}P^{n-1} \to C \) be the quotient map. Then there is a diagram

\[
\begin{array}{ccc}
[\Sigma, SU(n)] & \xrightarrow{(\partial'_k)_*} & [\Sigma, \text{Map}^*(\mathbb{C}P^2, BSU(n))] \\
\downarrow \tilde{q}^* & & \downarrow \tilde{q}^* \\
[\Sigma\mathbb{C}P^{n-1}, SU(n)] & \xrightarrow{(\partial'_k)_*} & [\Sigma\mathbb{C}P^{n-1}, \text{Map}^*(\mathbb{C}P^2, BSU(n))] \\
\end{array}
\]

where \( (\partial'_k)_* \) sends \( f \) to \( \partial'_k \circ f \) and the rows are induced by fibration (3). In particular, in the second row the map \( \epsilon : \Sigma\mathbb{C}P^{n-1} \to SU(n) \) is sent to \( (\partial'_k)_*(\epsilon) = \partial'_k \circ \epsilon \). In Sect. 4, we use unstable K-theory to calculate the order of \( \partial'_k \circ \epsilon \), giving a lower bound on the order of \( \partial'_k \). Furthermore, in [5] Hamanaka and Kono considered an exact sequence similar to the first row to give a necessary condition for \( G_k(S^4) \cong G_k(S^3) \). In Sect. 5 we follow the same approach and use the first row to give a necessary condition for \( G_k(\mathbb{C}P^2) \cong G_k(\mathbb{C}P^3) \).

We remark that it is difficult to use only one of the two rows to prove both Theorems 1.5 and 1.6. On the one hand, \( \partial'_k \circ \epsilon \) factors through a map \( \tilde{\epsilon} : \Sigma C \to \text{Map}^*(\mathbb{C}P^2, BSU(n)) \). There is no obvious method to show that \( \tilde{\epsilon} \) and \( \partial'_k \circ \epsilon \) have the same orders except direct calculation. Therefore we cannot compare the orders of \( \tilde{\epsilon} \) and \( \partial'_k \) to prove Theorem 1.5 without calculating the order of \( \partial'_k \circ \epsilon \). On the other hand, applying the method used in Sect. 5 to the second row gives a much weaker

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conclusion than Theorem 1.6. This is because \([\Sigma C, BG_k(\mathbb{C}P^2)]\) is a much smaller group than \([\Sigma \mathbb{C}P^{n-1}, BG_k(\mathbb{C}P^2)]\) and much information is lost by the map \(\tilde{q}^*\).

4 A lower bound on the order of \(\partial_1\)

The restriction of \(\partial_1\) to \(\Sigma \mathbb{C}P^n\) is \(\partial_1 \circ \epsilon\), which is the triple adjoint of the composition

\[
(i, \epsilon) : S^3 \wedge \Sigma \mathbb{C}P^n \xrightarrow{t \wedge \epsilon} SU(n) \wedge SU(n) \xrightarrow{(1,1)} SU(n).
\]

Since \(SU(n) \simeq \Omega BSU(n)\), we can further take its adjoint and get

\[
\rho : \Sigma S^3 \wedge \Sigma \mathbb{C}P^n \xrightarrow{\Sigma t \wedge \epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev, ev]} BSU(n),
\]

where \([ev, ev]\) is the Whitehead product of the evaluation map

\[
ev : \Sigma SU(n) \simeq \Sigma \Omega BSU(n) \rightarrow BSU(n)
\]

with itself. Similarly, the restriction \(\partial_1' \circ \epsilon\) is adjoint to the composition

\[
\rho' : \mathbb{C}P^2 \wedge \Sigma \mathbb{C}P^n \xrightarrow{q \wedge 1} S^4 \wedge \Sigma \mathbb{C}P^n \xrightarrow{\Sigma t \wedge \epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev, ev]} BSU(n).
\]

Since we will frequently refer to the facts established in \([4,5]\), it is easier to follow their setting and consider its adjoint

\[
\gamma = \tau(\rho' \circ T) : \mathbb{C}P^2 \wedge \mathbb{C}P^n \rightarrow SU(n),
\]

where \(T : \Sigma \mathbb{C}P^2 \wedge \mathbb{C}P^n \rightarrow \mathbb{C}P^2 \wedge \Sigma \mathbb{C}P^n\) is the swapping map and \(\tau : [\Sigma \mathbb{C}P^2 \wedge \mathbb{C}P^n, BSU(n)] \rightarrow [\mathbb{C}P^2 \wedge \mathbb{C}P^n, SU(n)]\) is the adjunction. By adjunction, the orders of \(\partial_1' \circ \epsilon, \rho'\) and \(\gamma\) are the same. We will calculate the order of \(\gamma\) using unstable \(K\)-theory to prove Theorem 1.5.

Apply \([\mathbb{C}P^2 \wedge \mathbb{C}P^n, -]\) to fibration (8) to obtain the exact sequence

\[
\tilde{K}_0(\mathbb{C}P^2 \wedge \mathbb{C}P^n) \xrightarrow{P_*} [\mathbb{C}P^2 \wedge \mathbb{C}P^n, \Omega W_n] \rightarrow [\mathbb{C}P^2 \wedge \mathbb{C}P^n, SU(n)] \rightarrow 0.
\]

Since \(\mathbb{C}P^2 \wedge \mathbb{C}P^n\) is a CW-complex with even dimensional cells, the last item \([\mathbb{C}P^2 \wedge \mathbb{C}P^n, SU(\infty)] \simeq \tilde{K}_1(\mathbb{C}P^2 \wedge \mathbb{C}P^n)\) is zero. First we identify the term \([\mathbb{C}P^2 \wedge \mathbb{C}P^n, \Omega W_n]\).

Lemma 4.1 We have the following:

- \([\Sigma^{2n-4}\mathbb{C}P^2, \Omega W_n]\) \(\cong \mathbb{Z}\);
- \([\Sigma^{2n-3}\mathbb{C}P^2, \Omega W_n]\) \(= 0\) for \(n\) odd;
- \([\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_n]\) \(\cong \mathbb{Z} \oplus \mathbb{Z}\).
Proof First, apply \([\Sigma^{2n-4}-, \Omega W_n]\) to cofibration (4) to obtain the exact sequence

\[
\pi_{2n}(W_n) \longrightarrow \pi_{2n+1}(W_n) \longrightarrow [\Sigma^{2n-4}\mathbb{C}P^2, \Omega W_n] \longrightarrow \pi_{2n-1}(W_n).
\]

We refer to Table (9) freely for the homotopy groups of \(W_n\). Since \(\pi_{2n-1}(W_n)\) and \(\pi_{2n}(W_n)\) are zero, \([\Sigma^{2n-4}\mathbb{C}P^n-1, \Omega W_n]\) is isomorphic to \(\pi_{2n+1}(W_n) \cong \mathbb{Z}\).

Second, apply \([\Sigma^{2n-3}-, \Omega W_n]\) to (4) to obtain

\[
\pi_{2n+2}(W_n) \longrightarrow [\Sigma^{2n-3}\mathbb{C}P^2, \Omega W_n] \longrightarrow \pi_{2n}(W_n).
\]

Since \(\pi_{2n}(W_n)\) and \(\pi_{2n+2}(W_n)\) are zero for \(n\) odd, so is \([\Sigma^{2n-3}\mathbb{C}P^2, \Omega W_n]\).

Third, apply \([\Sigma^{2n-2}-, \Omega W_n]\) to (4) to obtain

\[
\pi_{2n+2}(W_n) \xrightarrow{\eta_1} \pi_{2n+3}(W_n) \longrightarrow [\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_n] \xrightarrow{j} \pi_{2n+1}(W_n) \xrightarrow{\eta_2} \pi_{2n+2}(W_n),
\]

where \(\eta_1\) and \(\eta_2\) are induced by Hopf maps \(\Sigma^{2n}\eta : S^{2n+3} \to S^{2n+2} \to S^{2n+1}\), and \(j\) is induced by the inclusion \(S^{2n+1} \hookrightarrow \Sigma^{2n-2}\mathbb{C}P^2\) of the bottom cell. When \(n\) is odd, \(\pi_{2n+2}(W_n)\) is zero and \(\pi_{2n+1}(W_n)\) and \(\pi_{2n+3}(W_n)\) are \(\mathbb{Z}\), so \([\Sigma^{2n-2}\mathbb{C}P^n-1, \Omega W_n]\) is \(\mathbb{Z} \oplus \mathbb{Z}\). When \(n\) is even, the \((2n+4)\)-skeleton of \(W_n\) is \(S^{2n+1} \sqcup S^{2n+3}\). The inclusions

\[
i_1 : S^{2n+1} \to S^{2n+1} \vee S^{2n+3} \quad \text{and} \quad i_2 : S^{2n+3} \to S^{2n+1} \vee S^{2n+3}
\]

generate \(\pi_{2n+1}(W_n)\) and the \(\mathbb{Z}\)-summand of \(\pi_{2n+3}(W_n)\), and the compositions

\[
j_1 : S^{2n+2} \xrightarrow{\Sigma^{2n+1}\eta} S^{2n+1} \xrightarrow{i_1} W_n \quad \text{and} \quad j_2 : S^{2n+3} \xrightarrow{\Sigma^{2n+3}\eta} S^{2n+2} \xrightarrow{\Sigma^{2n-1}\eta} S^{2n+1} \xrightarrow{i_1} W_n
\]

generate \(\pi_{2n+2}(W_n)\) and the \(\mathbb{Z}/2\mathbb{Z}\)-summand of \(\pi_{2n+3}(W_n)\) respectively. Since \(\eta_1\) sends \(j_1\) to \(j_2\), the cokernel of \(\eta_1\) is \(\mathbb{Z}\). Similarly, \(\eta_2\) sends \(i_1\) to \(j_1\), so \(\eta_2 : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}\) is surjective. This implies the preimage of \(j\) is a \(\mathbb{Z}\)-summand. Therefore \([\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_n] \cong \mathbb{Z} \oplus \mathbb{Z}\). \(\square\)

Let \(C\) be the quotient \(\mathbb{C}P^{n-1}/\mathbb{C}P^{n-3}\). Since \(\Omega W_n\) is \((2n-1)\)-connected, \([\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, \Omega W_n]\) is isomorphic to \([\mathbb{C}P^2 \wedge C, \Omega W_n]\) which is easier to determine.

Lemma 4.2 The group \([\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, \Omega W_n] \cong [\mathbb{C}P^2 \wedge C, \Omega W_n]\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}^3\).

Proof When \(n\) is even, \(C\) is \(S^{2n-2} \vee S^{2n-4}\). By Lemma 4.1, \([\mathbb{C}P^2 \wedge C, \Omega W_n]\) is \([\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_n] \oplus [\Sigma^{2n-4}\mathbb{C}P^2, \Omega W_n] \cong \mathbb{Z} \oplus \mathbb{Z}^3\).

When \(n\) is odd, \(C\) is \(S^{2n-6}\mathbb{C}P^2\). Apply \([\Sigma^{2n-6}\mathbb{C}P^2 \wedge -, \Omega W_n]\) to cofibration (4) to obtain the exact sequence

\[
[\Sigma^{2n-3}\mathbb{C}P^2, \Omega W_n] \longrightarrow [\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_n] \longrightarrow [\Sigma^{2n-6}\mathbb{C}P^2 \wedge \mathbb{C}P^2, \Omega W_n] \longrightarrow [\Sigma^{2n-4}\mathbb{C}P^2, \Omega W_n] \longrightarrow [\Sigma^{2n-3}\mathbb{C}P^2, \Omega W_n]
\]
By Lemma 4.1, the first and the last terms $[\Sigma^{2n-3}\mathbb{C}P^2, \Omega W_n]$ are zero, while the second term $[\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_n]$ is $\mathbb{Z} \oplus \mathbb{Z}$ and the fourth $[\Sigma^{2n-4}\mathbb{C}P^2, \Omega W_n]$ is $\mathbb{Z}$. Therefore $[\mathbb{C}P^2 \wedge C, \Omega W_n]$ is $\mathbb{Z}^\oplus 3$.

\[\square\]

Define $a : [\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, \Omega W_n] \to H^{2n}(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}) \oplus H^{2n+2}(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1})$ to be a map sending $f \in [\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, \Omega W_n]$ to $a(f) = f^*(a_{2n}) \oplus f^*(a_{2n+2})$. The cohomology class $\tilde{x}_{2n+1}$ represents a map $\tilde{x}_{2n+1} : W_n \to K(\mathbb{Z}, 2n + 1)$ and $a_{2n} = \sigma(\tilde{x}_{2n+1})$ represents its loop $\Omega \tilde{x}_{2n+1} : \Omega W_n \to \Omega K(\mathbb{Z}, 2n + 1)$. Similarly $a_{2n+2} = \sigma(\tilde{x}_{2n+3})$ represents a loop map. This implies $a$ is a group homomorphism. Furthermore, $a_{2n}$ and $a_{2n+2}$ induce isomorphisms between $H^i(\Omega W_n)$ and $H^i(K(2n, \mathbb{Z}) \times K(2n + 2, \mathbb{Z}))$ for $i = 2n$ and $2n + 2$. Since $[\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, \Omega W_n]$ is a free $\mathbb{Z}$-module by Lemma 4.2, $a$ is a monomorphism. Consider the diagram

\[\begin{array}{ccc}
\tilde{K}^0(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}) & \xrightarrow{p^*} & [\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, \Omega W_n] \\
\Phi & \xrightarrow{a} & [\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, SU(n)] \\
\Phi \oplus \bigoplus_{i=0,2} H^{2n+i}(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}) & \xrightarrow{\psi} & Coker(\Phi) \\
\end{array}\]

In the left square, $\Phi$ is defined to be $a \circ p^*$. In the right square, $\psi$ is the quotient map and $b$ is defined as follows. Any $f \in [\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, SU(n)]$ has a preimage $\tilde{f}$ and $b(f)$ is defined to be $\psi(a(\tilde{f}))$. An easy diagram chase shows that $b$ is well-defined and injective. Since $b$ is injective, the order of $\gamma \in [\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}, SU(n)]$ equals the order of $b(\gamma) \in Coker(\Phi)$. In [4], Hamanaka and Kono gave an explicit formula for $\Phi$.

**Theorem 4.3** (Hamanaka and Kono [4]) Let $Y$ be a CW-complex. For any $f \in \tilde{K}^0(Y)$ we have

$$\Phi(f) = n!ch_{2n}(f) \oplus (n+1)!ch_{2n+2}(f),$$

where $ch_{2i}(f)$ is the $2i$th part of $ch(f)$.

Let $u$ and $v$ be the generators of $H^2(\mathbb{C}P^2)$ and $H^2(\mathbb{C}P^{n-1})$. For $1 \leq i \leq n - 1$, denote $L_i$ and $L_i'$ as the generators of $\tilde{K}^0(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1})$ with Chern characters $ch(L_i) = u^2(e^v - 1)^i$ and $ch(L_i') = (u + \frac{1}{2}u^2) \cdot (e^v - 1)^i$. By Theorem 4.3 we have

$$\Phi(L_i) = n(n-1)A_iu^2v^{n-2} + n(n+1)B_iu^2v^{n-1},$$

$$\Phi(L_i') = \frac{n(n-1)}{2}A_iu^2v^{n-2} + nB_iuv^{n-1} + \frac{n(n+1)}{2}B_iu^2v^{n-1},$$

where

$$A_i = \sum_{j=1}^{i}(-1)^{j+i}\left(\begin{array}{c}i \\ j\end{array}\right)j^{n-2} \quad \text{and} \quad B_i = \sum_{j=1}^{i}(-1)^{j+i}\left(\begin{array}{c}i \\ j\end{array}\right)j^{n-1}.$$

Write an element $xu^2v^{n-2} + yuv^{n-1} + zu^2v^{n-1} \in H^{2n}(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1}) \oplus H^{2n+2}(\mathbb{C}P^2 \wedge \mathbb{C}P^{n-1})$ as $(x, y, z)$. Then the coordinates of $\Phi(L_i)$ and $\Phi(L_i')$ are $(n(n-1)A_i, 0, n(n+1)B_i)$ and $(\frac{n(n-1)}{2}A_i, nB_i, \frac{n(n+1)}{2}B_i)$ respectively.
Lemma 4.4 For $n \geq 3$, $\text{Im}(\Phi)$ is spanned by $\left(\frac{n(n+1)}{2}, n, \frac{n(n-1)}{2}\right)$, $(n(n-1), 0, 0)$ and $(0, 2n, 0)$.

**Proof** By definition, $\text{Im}(\Phi) = \text{span}\{\Phi(L_i), \Phi(L'_i)\}_{i=1}^{n-1}$. For $i = 1$, $A_1 = B_1 = 1$. Then

$$\Phi(L_1) = (n(n-1), 0, n(n+1))$$

$$= 2 \left(\frac{1}{2} n(n-1), n, \frac{1}{2} n(n+1)\right) - (0, 2n, 0)$$

$$= 2\Phi(L'_1) - (0, 2n, 0)$$

Equivalently $(0, 2n, 0) = 2\Phi(L'_1) - \Phi(L_1)$, so $\text{span}\{\Phi(L_1), \Phi(L'_1)\} = \text{span}\{\Phi(L'_1), (0, 2n, 0)\}$. For other $i$'s,

$$\Phi(L_i) = (n(n-1)A_i, 0, n(n+1)B_i)$$

$$= 2 \left(\frac{1}{2} n(n-1)A_i, nB_i, \frac{1}{2} n(n+1)B_i\right) - (0, 2nB_i, 0)$$

$$= 2\Phi(L'_i) - B_i(0, 2n, 0)$$

is a linear combination of $\Phi(L'_i)$ and $(0, 2n, 0)$, so

$$\text{Im}(\Phi) = \text{span}\{\Phi(L'_1), \ldots, \Phi(L'_{n-1}), (0, 2n, 0)\}.$$ 

We claim that $\text{span}\{\Phi(L'_i)\}_{i=1}^{n-1} = \text{span}\{\Phi(L'_1), (n(n-1), 0, 0)\}$. Observe that

$$\Phi(L'_i) = \left(\frac{n(n-1)}{2} A_i, nB_i, \frac{n(n+1)}{2} B_i\right)$$

$$= \left(\frac{n(n-1)}{2} B_i, nB_i, \frac{n(n+1)}{2} B_i\right) + \left(\frac{n(n-1)}{2} (A_i - B_i), 0, 0\right)$$

$$= B_i\Phi(L'_1) + \frac{A_i - B_i}{2} \cdot (n(n-1), 0, 0).$$

The difference

$$A_i - B_i = \sum_{j=1}^{i} (-1)^{i+j} \binom{i}{j} j^{n-2} - \sum_{j=1}^{i} (-1)^{i+j} \binom{i}{j} j^{n-1}$$

$$= \sum_{j=1}^{i} (-1)^{i+j+1} \binom{i}{j} (j^{n-1} - j^{n-2})$$

$$= \sum_{j=1}^{i} (-1)^{i+j+1} \binom{i}{j} (j - 1) j^{n-2}$$

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is even since each term \((j - 1)j^{n-2}\) is even and \(n \geq 3\). Therefore \(\frac{A_i - B_i}{2}\) is an integer and \(\Phi(L'_i)\) is a linear combination of \(\Phi(L'_1)\) and \((n(n - 1), 0, 0)\).

Furthermore,

\[
\Phi(L'_2) = B_2\Phi(L'_1) + (A_2 - B_2)\left(\frac{n(n - 1)}{2}, 0, 0\right) = B_2\Phi(L'_1) - 2^{n-3}(n(n - 1), 0, 0)
\]

and

\[
\Phi(L'_3) = B_3\Phi(L'_1) + (A_3 - B_3)\left(\frac{n(n - 1)}{2}, 0, 0\right) = B_3\Phi(L'_1) - (3^{n-2} - 3 \cdot 2^{n-3})(n(n - 1), 0, 0).
\]

For \(n = 3\), \(B_2 = 2\) and \(\Phi(L'_2) = 2\Phi(L'_1) - (n(n - 1), 0, 0)\), so we have

\[
\text{span}\{\Phi(L'_i)\}_{i=1}^{n-1} = \text{span}\{\Phi(L'_1), \Phi(L'_2)\} = \text{span}\{\Phi(L'_1), (n(n - 1), 0, 0)\}.
\]

For \(n \geq 4\), since \(2^{n-3}\) and \(3^{n-2} - 3 \cdot 2^{n-3}\) are coprime to each other, there exist integers \(s, t\) such that \(2^{n-3} + (3^{n-2} - 3 \cdot 2^{n-3})t = 1\) and

\[
(n(n - 1), 0, 0) = (sB_2 + tB_3)\Phi(L'_1) - s\Phi(L'_2) - t\Phi(L'_3).
\]

Therefore \((n(n - 1), 0, 0)\) is a linear combination of \(\Phi(L'_1), \Phi(L'_2)\) and \(\Phi(L'_3)\). This implies \(\text{span}\{\Phi(L'_i), (n(n - 1), 0, 0)\} = \text{span}\{\Phi(L'_i)\}_{i=1}^{n-1}\).

Combine all these together to obtain

\[
\text{Im}(\Phi) = \text{span}\{\Phi(L_i), \Phi(L'_i)\}_{i=1}^{n-1} = \text{span}\{\Phi(L'_1), (n(n - 1), 0, 0), (0, 2n, 0)\} = \text{span}\left\{\left(\frac{n(n - 1)}{2}, n, \frac{n(n + 1)}{2}\right), (n(n - 1), 0, 0), (0, 2n, 0)\right\}.
\]

\(\square\)

Back to diagram (10). The map \(\gamma\) has a lift \(\tilde{\gamma} : \mathbb{CP}^2 \wedge \mathbb{CP}^{n-1} \rightarrow \Omega W_n\). By exactness, the order of \(\gamma\) equals the minimum number \(m\) such that \(m\tilde{\gamma}\) is contained in \(\text{Im}(p_\alpha)\). Since \(a\) and \(b\) are injective, the order of \(\gamma\) equals the minimum number \(m'\) such that \(m'\alpha(\tilde{\gamma})\) is contained in \(\text{Im}(\Phi)\).

**Lemma 4.5** Let \(\alpha : \Sigma X \rightarrow SU(n)\) be a map for some space \(X\). If \(\alpha' : \mathbb{CP}^2 \wedge X \rightarrow SU(n)\) is the adjoint of the composition

\[
\mathbb{CP}^2 \wedge \Sigma X \xrightarrow{q \wedge \text{Id}} \Sigma S^3 \wedge \Sigma X \xrightarrow{\Sigma L \wedge \alpha} \Sigma SU(n) \wedge SU(n) [ev, ev] BSU(n),
\]

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then there is a lift $\tilde{\alpha}$ of $\alpha'$ such that $\tilde{\alpha}^*(a_{2i}) = u^2 \otimes \Sigma^{-1} \alpha^*(x_{2i-3})$, where $\Sigma$ is the cohomology suspension isomorphism.

\[ \begin{array}{c}
\Omega W_n \\
\downarrow \alpha' \rightarrow SU(n)
\end{array} \]

**Proof** In [4,5], Hamanaka and Kono constructed a lift $\Gamma : \Sigma SU(n) \wedge SU(n) \to W_n$ of $[ev, ev]$ such that $\Gamma^*(\tilde{x}_{2i+1}) = \sum_{j+k=i-1} \Sigma x_{2j+1} \otimes x_{2k+1}$. Let $\tilde{\Gamma}$ be the composition

$\tilde{\Gamma} : \mathbb{C}P^2 \wedge \Sigma X \xrightarrow{q^\wedge 1} \Sigma S^3 \wedge \Sigma X \xrightarrow{\Sigma i \wedge \alpha} \Sigma SU(n) \wedge SU(n) \xrightarrow{\Gamma} W_n.$

Then we have

\[
\tilde{\Gamma}^*(\tilde{x}_{2i+1}) = (q \wedge 1)^*(\Sigma i \wedge \alpha)^* \Gamma^*(\tilde{x}_{2i+1}) \]

\[
= (q \wedge 1)^*(\Sigma i \wedge \alpha)^* \left( \sum_{j+k=i-1} \Sigma x_{2j+1} \otimes x_{2k+1} \right) \]

\[
= (q \wedge 1)^*(\Sigma u_3 \otimes \alpha^*(x_{2i-3})) \]

\[
= u^2 \otimes \alpha^*(x_{2i-3}),
\]

where $u_3$ is the generator of $H^3(S^3)$.

Let $T : \Sigma \mathbb{C}P^2 \wedge X \to \mathbb{C}P^2 \wedge \Sigma X$ be the swapping map and let $\tau : [\Sigma \mathbb{C}P^2 \wedge X, W_n] \to [\mathbb{C}P^2 \wedge X, \Omega W_n]$ be the adjunction. Take $\tilde{\alpha} : \mathbb{C}P^2 \wedge X \to \Omega W_n$ to be the adjoint of $\tilde{\Gamma}$, that is $\tilde{\alpha} = \tau(\tilde{\Gamma} \circ T)$. Then $\tilde{\alpha}$ is a lift of $\alpha'$. Since

\[(\tilde{\Gamma} \circ T)^*(\tilde{x}_{2i+1}) = T^* \circ \tilde{\Gamma}^*(\tilde{x}_{2i+1}) = T^*(u^2 \otimes \alpha^*(x_{2i-3})) = \Sigma u^2 \otimes \Sigma^{-1} \alpha^*(x_{2i-3}),\]

we have $\tilde{\alpha}^*(a_{2i}) = u^2 \otimes \Sigma^{-1} \alpha^*(x_{2i-3}).$ \hfill $\square$

**Lemma 4.6** In diagram (10), $\gamma$ has a lift $\tilde{\gamma}$ such that $a(\tilde{\gamma}) = u^2 v^{n-2} \oplus u^2 v^{n-1}$.

**Proof** Recall that $\gamma$ is the adjoint of the composition

$\rho' : \mathbb{C}P^2 \wedge \Sigma \mathbb{C}P^{n-1} \xrightarrow{q^\wedge 1} \Sigma S^3 \wedge \Sigma \mathbb{C}P^{n-1} \xrightarrow{\Sigma i \wedge \epsilon} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev, ev]} BSU(n).$

Now we use Lemma 4.5 and take $\alpha$ to be $\epsilon : \Sigma \mathbb{C}P^{n-1} \to SU(n)$. Then $\gamma$ has a lift $\tilde{\gamma}$ such that $\tilde{\gamma}^*(a_{2i}) = u^2 \otimes \Sigma^{-1} \epsilon^*(x_{2i-3}) = u^2 \otimes v^{i-2}$. This implies

\[ a(\tilde{\gamma}) = \tilde{\gamma}^*(a_{2n}) \oplus \tilde{\gamma}^*(a_{2n+2}) = u^2 v^{n-2} \oplus u^2 v^{n-1}. \] \hfill $\square$
Now we can calculate the order of $\partial_1' \circ \epsilon$, which gives a lower bound on the order of $\partial_1'$.

**Theorem 4.7** When $n \geq 3$, the order of $\partial_1' \circ \epsilon$ is $\frac{1}{2} n(n^2 - 1)$ for $n$ odd and $n(n^2 - 1)$ for $n$ even.

**Proof** Since $\partial_1' \circ \epsilon$ is adjoint to $\gamma$, it suffices to calculate the order of $\gamma$. By Lemma 4.4, $\text{Im}(\Phi)$ is spanned by $(\frac{1}{2} n(n - 1), n, \frac{1}{2} n(n + 1)), (n(n - 1), 0, 0)$ and $(0, 2n, 0)$. By Lemma 4.6, $a(\gamma)$ has coordinates $(1, 0, 1)$. Let $m$ be a number such that $ma(\gamma)$ is contained in $\text{Im}(\Phi)$. Then

$$m(1, 0, 1) = s \left( \frac{1}{2} n(n - 1), n, \frac{1}{2} n(n + 1) \right) + t(n(n - 1), 0, 0) + r(0, 2n, 0)$$

for some integers $s, t$ and $r$. Solve this to get

$$m = \frac{1}{2} t n(n^2 - 1), s = -2r, s = t(n - 1).$$

Since $s = -2r$ is even, the smallest positive value of $t$ satisfying $s = t(n - 1)$ is $1$ for $n$ odd and $2$ for $n$ even. Therefore $m$ is $\frac{1}{2} n(n^2 - 1)$ for $n$ odd and $n(n^2 - 1)$ for $n$ even. \hfill $\Box$

For $SU(n)$-gauge groups over $S^4$, the order $m$ of $\partial_1$ has the form $m = n(n^2 - 1)$ for $n = 3$ and $5 [5, 19]$. If $p$ is an odd prime and $n < (p - 1)^2 + 1$, then $m$ and $n(n^2 - 1)$ have the same $p$-components $[9, 20]$. These facts suggest it may be the case that $m = n(n^2 - 1)$ for any $n > 2$. In fact, one can follow the method Hamanaka and Kono used in [5] and calculate the order of $\partial \circ \epsilon$ to obtain a lower bound $n(n^2 - 1)$ for $n$ odd. However, it does not work for the $n$ even case since $[S^4 \wedge \mathbb{C}P^{n-1}, \Omega W_n]$ is not a free $\mathbb{Z}$-module. An interesting corollary of Theorem 4.7 is to give a lower bound on the order of $\partial_1$ for $n$ even.

**Corollary 4.8** When $n$ is even and greater than $2$, the order of $\partial_1$ is at least $n(n^2 - 1)$.

**Proof** The order of $\partial_1' \circ \epsilon$ is a lower bound on the order of $\partial_1'$, which is either the same as or half of the order of $\partial_1$ by Lemma 2.2. The corollary follows from Theorem 4.7. \hfill $\Box$

## 5 A necessary condition for $G_k(\mathbb{C}P^2) \simeq G_1(\mathbb{C}P^2)$

In this section we follow the approach in [5] to prove Theorem 1.6. The techniques used are similar to that in Sect. 4, except we are working with the quotient $\Sigma C = \Sigma \mathbb{C}P^{n-1}/\Sigma \mathbb{C}P^{n-1}$ instead of $\Sigma \mathbb{C}P^{n-1}$. When $n$ is odd, $C$ is $\Sigma 2n - 6 \mathbb{C}P^2$, and when $n$ is even, $C$ is $S^{2n-2} / S^{2n-4}$. Apply $[\Sigma C, -]$ to fibration (3) to obtain the exact sequence

$$[\Sigma C, SU(n)] \xrightarrow{(\partial_1')_*} [\Sigma C, \text{Map}_0(\mathbb{C}P^2, BSU(n))]$$

$$\quad \quad \quad \quad \rightarrow [\Sigma C, BG_k(\mathbb{C}P^2)] \rightarrow [\Sigma C, BSU(n)].$$
where \((\partial_k')_*\) sends \(f \in [\Sigma C, SU(n)]\) to \(\partial_k' \circ f \in [\Sigma C, Map_0^*(\mathbb{CP}^2, BSU(n))]\). Since \(BSU(n) \rightarrow BSU(\infty)\) is a \(2n\)-equivalence and \(\Sigma C\) has dimension \(2n - 1\), \([\Sigma C, BSU(n)]\) is \(\tilde{K}^0(\Sigma C)\) which is zero. Similarly, \([\Sigma C, SU(n)] \cong [\Sigma^2 C, BSU(n)]\) is \(\tilde{K}^0(\Sigma^2 C) \cong \mathbb{Z} \oplus \mathbb{Z}\). Furthermore, by adjunction we have \([\Sigma C, Map_0^*(\mathbb{CP}^2, BSU(n))] \cong [\Sigma C \wedge \mathbb{CP}^2, BSU(n)]\). The exact sequence becomes

\[
\tilde{K}^0(\Sigma^2 C) \xrightarrow{(\partial_k')_*} [\Sigma C \wedge \mathbb{CP}^2, BSU(n)] \longrightarrow [\Sigma C, BG_k(\mathbb{CP}^2)] \longrightarrow 0. \tag{11}
\]

This implies \([\Sigma C, BG_k(\mathbb{CP}^2)] \cong [C, \mathcal{G}_k(\mathbb{CP}^2)]\) is \(Coker(\partial_k')_*\). Also, apply \([\mathbb{CP}^2 \wedge C, -]\) to fibration (8) to obtain the exact sequence

\[
[\mathbb{CP}^2 \wedge C, \Omega SU(\infty)] \xrightarrow{p_*} [\mathbb{CP}^2 \wedge C, \Omega W_n] \longrightarrow [\mathbb{CP}^2 \wedge C, SU(n)] \longrightarrow [\mathbb{CP}^2 \wedge C, SU(\infty)]. \tag{12}
\]

Observe that \([\mathbb{CP}^2 \wedge C, \Omega SU(\infty)] \cong \tilde{K}^0(\mathbb{CP}^2 \wedge C)\) is \(\mathbb{Z}^\oplus 4\) and \([\mathbb{CP}^2 \wedge C, SU(\infty)] \cong \tilde{K}^1(\mathbb{CP}^2 \wedge C)\) is zero. Combine exact sequences (11) and (12) to obtain the diagram

\[
\begin{array}{ccc}
\tilde{K}^0(\mathbb{CP}^2 \wedge C) & \xrightarrow{p_*} & [\mathbb{CP}^2 \wedge C, \Omega W_n] \xrightarrow{a} H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C) \\
& \phi \downarrow & \downarrow \\
[\mathbb{CP}^2 \wedge C, SU(n)] & \xrightarrow{\tilde{K}^0(\Sigma^2 C, (\partial_k')_*} & [\Sigma C, BG_k(\mathbb{CP}^2)] \longrightarrow 0
\end{array}
\]

where \(a(f) = f^*(a_{2n}) \oplus f^*(a_{2n+2})\) for any \(f \in [\mathbb{CP}^2 \wedge C, \Omega W_n]\), and \(\Phi\) is defined to be \(a \circ p_*\). By Lemma 4.2 \([\mathbb{CP}^2 \wedge C, \Omega W_n]\) is free. Following the same argument in Sect. 4 implies the injectivity of \(a\).

Our strategy to prove Theorem 1.6 is as follows. If \(\mathcal{G}_k(\mathbb{CP}^2)\) is homotopy equivalent to \(\mathcal{G}_l(\mathbb{CP}^2)\), then \([C, \mathcal{G}_k(\mathbb{CP}^2)] \cong [C, \mathcal{G}_l(\mathbb{CP}^2)]\) and exactness in (12) implies that \(Im(\partial_k')_*\) and \(Im(\partial_l')_*\) have the same order in \([\mathbb{CP}^2 \wedge C, SU(\infty)]\) resulting in a necessary condition for a homotopy equivalence \(\mathcal{G}_k(\mathbb{CP}^2) \simeq \mathcal{G}_l(\mathbb{CP}^2)\). To calculate the order of \(Im(\partial_k')_*\), we will find a preimage \(\tilde{\partial}_k\) of \(Im(\partial_k')_*\) in \([\mathbb{CP}^2 \wedge C, \Omega W_n]\). Since \(a\) is injective, we can embed \(\tilde{\partial}_k\) into \(H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C)\) and work out the order of \(Im(\partial_k')_*\) there.

Let \(u, v_{2n-4}\) and \(v_{2n-2}\) be generators of \(H^2(\mathbb{CP}^2), H^{2n-4}(C)\) and \(H^{2n-2}(C)\). Then we write an element \(xu^2v_{2n-4} + yu v_{2n-2} + z^2 v_{2n-2} \in H^{2n}(\mathbb{CP}^2 \wedge C) \oplus H^{2n+2}(\mathbb{CP}^2 \wedge C)\) as \((x, y, z)\). First we need to find the submodule \(Im(a)\).

**Lemma 5.1** For \(n\) odd, \(Im(a) = \{(x, y, z) | x + y \equiv z \pmod{2}\}\), and for \(n\) even, \(Im(a) = \{(x, y, z) | y \equiv 0 \pmod{2}\}\).
When $n$ is odd, $C$ is $\Sigma^{2n-6}\mathbb{C}\mathbb{P}^2$ and the $(2n+3)$-skeleton of $\Omega W_n$ is $\Sigma^{2n-2}\mathbb{C}\mathbb{P}^2$. To say $(x, y, z) \in Im(a)$ means there exists $f \in [\mathbb{C}\mathbb{P}^2 \wedge C, \Omega W_n]$ such that

$$f^*(a_{2n}) = xu^2v_{2n-4} + yuv_{2n-2} \quad \text{and} \quad f^*(a_{2n+2}) = zu^2v_{2n-2}. \quad (13)$$

Reducing to homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, we have

$$Sq^2(u) = u^2, \quad Sq^2(v_{2n-4}) = v_{2n-2}, \quad Sq^2(a_{2n}) = a_{2n+2}.$$

Apply $Sq^2$ to $(13)$ to get $x + y \equiv z \pmod{2}$. Therefore $Im(a)$ is contained in $\{(x, y, z) | x + y \equiv z \pmod{2}\}$. To show that they are equal, we need to show that $(1, 0, 1), (0, 1, 1)$ and $(0, 0, 2)$ are in $Im(a)$. Consider maps

$$f_1 : \mathbb{C}\mathbb{P}^2 \wedge C \xrightarrow{q_1} S^4 \wedge C \simeq \Sigma^{2n-2}\mathbb{C}\mathbb{P}^2 \hookrightarrow \Omega W_n$$
$$f_2 : \mathbb{C}\mathbb{P}^2 \wedge C \xrightarrow{q_2} \mathbb{C}\mathbb{P}^2 \wedge S^{2n-2} \hookrightarrow \Omega W_n$$
$$f_3 : \mathbb{C}\mathbb{P}^2 \wedge C \xrightarrow{q_3} S^{2n+2} \xrightarrow{\theta} \Omega W_n$$

where $q_1, q_2$ and $q_3$ are quotient maps and $\theta$ is the generator of $\pi_{2n+3}(W_n)$. Their images are

$$a(f_1) = (1, 0, 1) \quad a(f_2) = (0, 1, 1) \quad a(f_3) = (0, 0, 2)$$

respectively, so $Im(a) = \{(x, y, z) | x + y \equiv z \pmod{2}\}$.

When $n$ is even, $C$ is $S^{2n-2} \vee S^{2n-4}$ and the $(2n+3)$-skeleton of $\Omega W_n$ is $S^{2n+2} \vee S^{2n}$. Reducing to homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, $Sq^2(v_{2n-4}) = 0$ and $Sq^2(a_{2n}) = 0$. Apply $Sq^2$ to $(13)$ to get $y \equiv 0 \pmod{2}$. Therefore $Im(a)$ is contained in $\{(x, y, z) | y \equiv 0 \pmod{2}\}$. To show that they are equal, we need to show that $(1, 0, 0), (0, 2, 0)$ and $(0, 0, 1)$ are in $Im(a)$. The maps

$$f_1' : \mathbb{C}\mathbb{P}^2 \wedge C \xrightarrow{q_1'} S^4 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{p_1} S^4 \wedge S^{2n-4} \hookrightarrow \Omega W_n$$
$$f_2' : \mathbb{C}\mathbb{P}^2 \wedge C \xrightarrow{q_2'} S^4 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{p_2} S^4 \wedge S^{2n-2} \hookrightarrow \Omega W_n$$

where $q_1'$ and $q_2'$ are quotient maps and $p_1$ and $p_2$ are pinch maps, have images $a(f_1') = (1, 0, 0)$ and $a(f_2') = (0, 0, 1)$. To find $(0, 2, 0)$, apply $[- \wedge S^{2n-2}, \Omega W_n]$ to cofibration $(4)$ to obtain the exact sequence

$$\pi_{2n+3}(W_n) \longrightarrow [\mathbb{C}\mathbb{P}^2 \wedge S^{2n-2}, \Omega W_n] \xrightarrow{i^*} \pi_{2n+1}(W_n) \xrightarrow{\eta^*} \pi_{2n+2}(W_n)$$

where $i^*$ is induced by the inclusion $i : S^2 \hookrightarrow \mathbb{C}\mathbb{P}^2$ and $\eta^*$ is induced by Hopf map $\eta$. The third term $\pi_{2n+1}(W_n) \cong \mathbb{Z}$ is generated by $i' : S^{2n+1} \rightarrow W_n$, the inclusion of the bottom cell, and the fourth term $\pi_{2n+2}(W_n) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $i' \circ \eta$, so $\eta^* : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjection. By exactness $[\mathbb{C}\mathbb{P}^2 \wedge S^{2n-2}, \Omega W_n]$ has a $\mathbb{Z}$-summand
with the property that $i^*$ sends its generator $g$ to $2i'$. Therefore the composition

$$f_3' : \mathbb{C}P^2 \wedge (S^{2n-2} \vee S^{2n-4}) \xrightarrow{\text{pinch}} \mathbb{C}P^2 \wedge S^{2n-2} \xrightarrow{g} \Omega W_n$$

has image $(0, 2, 0)$. It follows that $Im(a) = \{(x, y, z) | y \equiv 0 \pmod{2}\}$. □

Now we split into the $n$ odd and $n$ even cases to calculate the order of $Im(\partial_k')$.

### 5.1 The order of $Im(\partial_k')$ for $n$ odd

When $n$ is odd, $C$ is $\mathbb{C}^{2n-6} \mathbb{C}P^2$. First we find $Im(\Phi)$ in $Im(a)$. For $1 \leq i \leq 4$, let $L_i$ be the generators of $\tilde{k}^0(\mathbb{C}P^2 \wedge C) \cong \mathbb{Z}^4$ with Chern characters

$$ch(L_1) = (u + \frac{1}{2}u^2) \cdot (v_{2n-4} + \frac{1}{2}v_{2n-2}) \quad ch(L_2) = (u + \frac{1}{2}u^2) \cdot v_{2n-2}$$

$$ch(L_3) = u^2 \cdot (v_{2n-4} + \frac{1}{2}v_{2n-2}) \quad ch(L_4) = u^2 \cdot v_{2n-2}.$$ 

By Theorem 4.3, we have

$$\Phi(L_1) = \frac{n!}{2}u^2 v_{2n-4} + \frac{n!}{2}u v_{2n-2} + \frac{(n + 1)!}{4} u^2 v_{2n-2}$$

$$\Phi(L_2) = n!u v_{2n-2} + \frac{(n + 1)!}{2} u^2 v_{2n-2}$$

$$\Phi(L_3) = n!u^2 v_{2n-4} + \frac{(n + 1)!}{2} u^2 v_{2n-2}$$

$$\Phi(L_4) = (n + 1)!u^2 v_{2n-2}.$$ 

By Lemma 5.1, $Im(a)$ is spanned by $(1, 0, 1), (0, 1, 1)$ and $(0, 0, 2)$. Under this basis, the coordinates of the $\Phi(L_i)$’s are

$$\Phi(L_1) = \left(\frac{n!}{2}, \frac{n!}{2}, \frac{(n-3)n!}{8}\right), \quad \Phi(L_2) = (0, n!, \frac{(n-1)n!}{4}),$$

$$\Phi(L_3) = (n!, 0, \frac{(n-1)n!}{4}), \quad \Phi(L_4) = (0, 0, \frac{(n+1)!}{2}).$$

We represent their coordinates by the matrix

$$M_\Phi = L \begin{pmatrix}
\frac{n(n-1)}{2} & \frac{n(n-1)}{2} & \frac{n(n-1)(n-3)}{8} \\
0 & n(n-1) & \frac{n(n-1)^2}{4} \\
n(n-1) & 0 & \frac{n(n-1)^2}{3} \\
0 & 0 & \frac{n(n-1)^2}{2}
\end{pmatrix},$$

where $L = (n - 2)!$. Then $Im(\Phi)$ is spanned by the row vectors of $M_\Phi$. 

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Next, we find a preimage of \( \text{Im}(\partial_k) \) in \([\mathbb{CP}^2 \land C, \Omega W_n]\). In exact sequence (11) \( \tilde{K}^0(\Sigma^2 C) \) is \( \mathbb{Z} \oplus \mathbb{Z} \). Let \( \alpha_1 \) and \( \alpha_2 \) be its generators with Chern classes

\[
\begin{align*}
c_{n-1}(\alpha_1) &= (n-2)! \Sigma^2 v_{2n-4} \\
c_{n-1}(\alpha_2) &= 0 \\
c_n(\alpha_1) &= \frac{(n-1)!}{2} \Sigma^2 v_{2n-2} \\
c_n(\alpha_2) &= (n-1)! \Sigma^2 v_{2n-2}.
\end{align*}
\]

Lemma 5.2 For \( i = 1, 2 \), \((\partial_k')_*(\alpha_i)\) has a lift \( \tilde{\alpha}_{i,k} : \mathbb{CP}^2 \land C \to \Omega W_n \) such that

\[
a(\tilde{\alpha}_{i,k}) = ku^2 \otimes \Sigma^{-2} c_{n-1}(\alpha_i) \oplus ku^2 \otimes \Sigma^{-2} c_n(\alpha_i).
\]

Proof For dimension and connectivity reasons, \( \alpha_i : \Sigma^2 C \to BSU(\infty) \) lifts through \( BSU(n) \to BSU(\infty) \). Label the lift \( \Sigma^2 C \to BSU(n) \) by \( \alpha_i \) as well. Let \( \alpha'_i : \Sigma C \to SU(n) \) be the adjoint of \( \alpha_i \). Then \((\partial_k')_*(\alpha_i)\) is the adjoint of the composition

\[
\begin{array}{cccc}
\mathbb{CP}^2 \land C & \xrightarrow{q^\land 1} & \Sigma S^3 & \xrightarrow{\Sigma k_l \land \alpha'_i} & \Sigma SU(n) \land SU(n) & \xrightarrow{[ev, ev]} & BSU(n).
\end{array}
\]

By Lemma 4.5, \((\partial_k')_*(\alpha_i)\) has a lift \( \tilde{\alpha}_{i,k} \) such that \( \tilde{\alpha}^*_{i,k}(a_{2j}) = ku^2 \otimes \Sigma^{-1}(\alpha'_i)^*(x_{2j-3}) \). Since \( \sigma(c_{j-1}) = x_{2j-3} \), we have \( \tilde{\alpha}^*_{i,k}(a_{2j}) = ku^2 \otimes \Sigma^{-2} c_{j-1}(\alpha_i) \) and

\[
a(\tilde{\alpha}_{i,k}) = ku^2 \otimes \Sigma^{-2} c_{n-1}(\alpha_i) \oplus ku^2 \otimes \Sigma^{-2} c_n(\alpha_i).
\]

By Lemma 5.2, \((\partial_k')_*(\alpha_1)\) and \((\partial_k')_*(\alpha_2)\) have lifts

\[
\tilde{\alpha}_{1,k} = (n-2)! ku^2 v_{2n-4} + \frac{(n-1)!}{2} ku^2 v_{2n-2} \quad \text{and} \quad \tilde{\alpha}_{2,k} = (n-1)! ku^2 v_{2n-2}.
\]

We represent their coordinates by the matrix

\[
M_\beta = kL \begin{pmatrix} 1 & 0 & \frac{n-3}{n^2} \\ 0 & 0 & \frac{n-1}{n^2} \end{pmatrix}.
\]

Let \( \tilde{\delta}_k = \text{span}(\tilde{\alpha}_{1,k}, \tilde{\alpha}_{2,k}) \) be the preimage of \( \text{Im}(\partial_k') \) in \([\mathbb{CP}^2 \land C, \Omega W_n]\). Then \( \tilde{\delta}_k \) is spanned by the row vectors of \( M_\beta \).

Lemma 5.3 When \( n \) is odd, the order of \( \text{Im}(\partial_k') \) is

\[
|\text{Im}(\partial_k')| = \frac{\frac{1}{2} n(n^2 - 1)}{\frac{1}{2} n(n^2 - 1), k} \cdot \frac{n}{(n,k)}.
\]

Proof Suppose \( n = 4m + 3 \) for some integer \( m \). Then

\[
M_\Phi = (4m+3)L \begin{pmatrix} 2m+1 & 2m+1 & 2m^2 + m \\ 0 & 4m+2 & 4m^2 + 4m + 1 \\ 4m+2 & 0 & 8m^2 + 12m + 4 \end{pmatrix}.
\]
and

\[ M_\partial = kL \begin{pmatrix} 1 & 0 & m \\ 0 & 0 & 2m + 1 \end{pmatrix}. \]

Transform \( M_\Phi \) into Smith normal form

\[
A \cdot M_\Phi \cdot B = (4m + 3)L \begin{pmatrix} (2m + 1) & (2m + 1) & (2m + 1)(4m + 4) \\ 0 & (4m + 2)(2m + 1) & 0 \end{pmatrix},
\]

where

\[
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 4m + 2 & 1 & -(2m + 1) & 0 \\ 4 & -2 & -(2m + 1) & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -m & -(2m + 1) \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The matrix \( B \) represents a basis change in \( Im(a) \) and \( A \) represents a basis change in \( Im(\Phi) \). Therefore \([\mathbb{CP}^2 \wedge C, SU(n)]\) is isomorphic to

\[
\mathbb{Z} \frac{1}{2} (4m + 3)! \mathbb{Z} \oplus \mathbb{Z} \frac{1}{2} (4m + 3)! \mathbb{Z} \oplus \mathbb{Z} \frac{1}{2} (4m + 4)! \mathbb{Z}.
\]

We need to find the representation of \( \tilde{\partial}_k \) under the new basis represented by \( B \). The new coordinates of \( \tilde{\alpha}_{1,k} \) and \( \tilde{\alpha}_{2,k} \) are the row vectors of the matrix

\[
M_\partial \cdot \begin{pmatrix} 1 & -m & -(2m + 1) \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} kL & 0 & -kL \\ 0 & (2m + 1)kL & (4m + 2)kL \end{pmatrix}.
\]

Apply row operations to get

\[
\begin{pmatrix} 1 & 0 & 0 \\ 4m + 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} kL & 0 & -kL \\ 0 & (2m + 1)kL & (4m + 2)kL \end{pmatrix} = \begin{pmatrix} kL & 0 & -kL \\ 0 & (4m + 2)kL & (2m + 1)kL \end{pmatrix}.
\]

Let \( \mu = (kL, 0, -kL) \) and \( v = ((4m + 2)kL, (2m + 1)kL, 0) \). Then

\[
\tilde{\partial}_k = \{ x\mu + yv \in [\mathbb{CP}^2 \wedge C, \Omega W_n] | x, y \in \mathbb{Z} \}.
\]

If \( x\mu + yv \) and \( x'\mu + y'v \) are the same modulo \( Im(\Phi) \), then we have

\[
\begin{align*}
(xkL + (4m + 2)ykL) & \equiv (x'kL + (4m + 2)y'kL) \pmod{(2m + 1)(4m + 3)L} \\
(2m + 1)ykL & \equiv (2m + 1)y'kL \pmod{(2m + 1)(4m + 3)L} \\
xkL & \equiv x'kL \pmod{(2m + 1)(4m + 3)(4m + 4)L}
\end{align*}
\]

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These conditions are equivalent to
\[
\begin{align*}
  xk &\equiv x'k \pmod{(2m+2)(4m+3)(4m+2)} \\
yk &\equiv y'k \pmod{(4m+3)}
\end{align*}
\]

This implies that there are \(\frac{(2m+2)(4m+3)(4m+2)}{(2m+2)(4m+3)(4m+2), k}\) distinct values of \(x\) and \(\frac{4m+3}{(4m+3, k)}\) distinct values of \(y\), so we have
\[
|\text{Im}(\partial'_{k})_*| = \frac{(2m+2)(4m+3)(4m+2)}{(2m+2)(4m+3)(4m+2), k} \cdot \frac{4m+3}{(4m+3, k)}.
\]

When \(n = 4m + 1\), we can repeat the calculation above to obtain
\[
|\text{Im}(\partial'_{k})_*| = \frac{2m(4m+2)(4m+1)}{(2m(4m+2)(4m+1), k)} \cdot \frac{4m+1}{(4m+1, k)}.
\]

\[\blacksquare\]

5.2 The order of \(\text{Im}(\partial'_{k})_*\) for \(n\) even

When \(n\) is even, \(C\) is \(S^{2n-2} \lor S^{2n-4}\). For \(1 \leq i \leq 4\), let \(L_i\) be the generators of \(K^0(\mathbb{CP}^2 \land C) \cong \mathbb{Z}^{\oplus 4}\) with Chern characters
\[
\begin{align*}
  ch(L_1) &= (u + \frac{1}{2}u^2) v_{2n-4} \\
  ch(L_2) &= u^2 v_{2n-4} \\
  ch(L_3) &= (u + \frac{1}{2}u^2) v_{2n-2} \\
  ch(L_4) &= u^2 v_{2n-2}.
\end{align*}
\]

By Theorem 4.3, we have
\[
\begin{align*}
  \Phi(L_1) &= \frac{n!}{2} u^2 v_{2n-4} \\
  \Phi(L_2) &= n! u^2 v_{2n-4} \\
  \Phi(L_3) &= n! u v_{2n-2} + \frac{(n+1)!}{2} u^2 v_{2n-2} \\
  \Phi(L_4) &= (n+1)! u^2 v_{2n-2}.
\end{align*}
\]

By Lemma 5.1, \(\text{Im}(a)\) is spanned by \((1, 0, 0), (0, 2, 0)\) and \((0, 0, 1)\). Under this basis, the coordinates of the \(\Phi(L_i)\)'s are
\[
\begin{align*}
  \Phi(L_1) &= \left(\frac{n!}{2}, 0, 0\right), & \Phi(L_2) &= (n!, 0, 0), \\
  \Phi(L_3) &= \left(0, \frac{n!}{2}, \frac{(n+1)!}{2}\right), & \Phi(L_4) &= (0, 0, (n+1)!).
\end{align*}
\]
We represent the coordinates of $\Phi(L_i)$’s by the matrix

$$M_\Phi = \frac{n(n-1)}{2} L \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & n+1 \\ 0 & 0 & 2n+2 \end{pmatrix}$$

Then $\text{Im}(\Phi)$ is spanned by the row vectors of $M_\Phi$.

In exact sequence (11) $\tilde{K}^0(\Sigma^2 C)$ is $\mathbb{Z} \oplus \mathbb{Z}$. Let $\alpha_1$ and $\alpha_2$ be its generators with Chern classes

$$c_{n-1}(\alpha_1) = (n - 2)!\Sigma^2 v_{2n-4} \quad c_n(\alpha_1) = 0$$
$$c_{n-1}(\alpha_2) = 0 \quad c_n(\alpha_2) = (n - 1)!\Sigma^2 v_{2n-2}.$$ 

By Lemma 5.2, $(\partial_k')_*(\alpha_1)$ and $(\partial_k')_*(\alpha_2)$ have lifts

$$\tilde{\alpha}_{1,k} = (n - 2)!k u^2 v_{2n-4} \quad \text{and} \quad \tilde{\alpha}_{2,k} = (n - 1)!k u^2 v_{2n-2}.$$ 

We represent their coordinates by a matrix

$$M_\partial = k L \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & n-1 \end{pmatrix}.$$ 

Then the preimage $\tilde{\partial}_k = \text{span}\{\tilde{\alpha}_{1,k}, \tilde{\alpha}_{2,k}\}$ of $\text{Im}(\partial_k')_*$ is spanned by the row vectors of $M_\partial$. We calculate as in the proof of Lemma 5.3 to obtain the following lemma.

**Lemma 5.4** When $n$ is even, the order of $\text{Im}(\partial_k')_*$ is

$$|\text{Im}(\partial_k')_*| = \frac{\frac{1}{2} n(n-1)}{\binom{\frac{1}{2} n(n-1), k}{(n+1), k}} \cdot \frac{n(n+1)}{(n+1), k}.$$ 

### 5.3 Proof of Theorem 1.6

Before comparing the orders of $\text{Im}(\partial_k')_*$ and $\text{Im}(\partial_k')_*$, we prove a preliminary lemma.

**Lemma 5.5** Let $n$ be an even number and let $p$ be a prime. Denote the $p$-component of $t$ by $v_p(t)$. If there are integers $k$ and $l$ such that

$$v_p\left(\frac{1}{2} n, k\right) \cdot v_p(n, k) = v_p\left(\frac{1}{2} n, l\right) \cdot v_p(n, l),$$

then $v_p(n, k) = v_p(n, l)$.

**Proof** Suppose $p$ is odd. If $p$ does not divide $n$, then $v_p(n, k) = v_p(n, l) = 1$, so the lemma holds. If $p$ divides $n$, then $v_p\left(\frac{1}{2} n, k\right) = v_p(n, k)$. The hypothesis becomes $v_p(n, k)^2 = v_p(n, l)^2$, implying that $v_p(n, k) = v_p(n, l)$. 

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Suppose \( p = 2 \). Let \( \nu_2(n) = 2^r \), \( \nu_2(k) = 2^s \) and \( \nu_2(l) = 2^t \). Then the hypothesis implies

\[
\min(r - 1, t) + \min(r, t) = \min(r - 1, s) + \min(r, s).
\]

(14)

To show \( \nu_2(n, k) = \nu_2(n, l) \), we need to show \( \min(r, t) = \min(r, s) \). Consider the following cases: (1) \( t, s \geq r \), (2) \( t, s \leq r - 1 \), (3) \( t \leq r - 1, s \geq r \) and (4) \( s \leq r - 1, t \geq r \).

Case (1) obviously gives \( \min(r, t) = \min(r, s) \). In case (2), when \( t, s \leq r - 1 \), equation (14) implies \( 2t = 2s \). Therefore \( t = s \) and \( \min(r, t) = \min(r, s) \).

It remains to show cases (3) and (4). For case (3) with \( t \leq r - 1, s \geq r \), equation (14) implies

\[
2t = \min(r - 1, s) + r.
\]

Since \( s \geq r \), \( \min(r - 1, s) = r - 1 \) and the right hand side is \( 2r - 1 \) which is odd. However, the left hand side is even, leading to a contradiction. This implies that this case does not satisfy the hypothesis. Case (4) is similar. Therefore \( \nu_2(n, k) = \nu_2(n, l) \) and the asserted statement follows. \( \square \)

**Proof of Theorem 1.6** In exact sequence (11), \([C, \mathcal{G}_k(\mathbb{C}P^2)]\) is \( \text{Coker}(\partial_k^\prime)_* \). By hypothesis, \( \mathcal{G}_k(\mathbb{C}P^2) \) is homotopy equivalent to \( \mathcal{G}_l(\mathbb{C}P^2) \), so \( |\text{Im}(\partial_k^\prime)_*| = |\text{Im}(\partial_k^\prime)_*| \). The \( n \) odd and \( n \) even cases are proved similarly, but the even case is harder.

When \( n \) is even, by Lemma 5.4 the order of \( \text{Im}(\partial_k^\prime)_* \) is

\[
|\text{Im}(\partial_k^\prime)_*| = \frac{1}{2} n(n-1) \cdot \frac{n(n+1)}{\frac{1}{2} n(n-1), k} \cdot \frac{n(n+1)}{n(n+1), k},
\]

so we have

\[
\left(\frac{1}{2} n(n-1), k\right) \cdot (n(n+1), k) = \left(\frac{1}{2} n(n-1), l\right) \cdot (n(n+1), l).
\]

(15)

We need to show that

\[
\nu_p(n(n^2-1), k) = \nu_p(n(n^2-1), l)
\]

(16)

for all primes \( p \). Suppose \( p \) does not divide \( \frac{1}{2} n(n^2-1) \). Equation (16) holds since both sides are 1. Suppose \( p \) divides \( \frac{1}{2} n(n^2-1) \). Since \( n-1, n \) and \( n+1 \) are coprime, \( p \) divides only one of them. If \( p \) divides \( n - 1 \), then \( \nu_p\left(\frac{1}{2} n, k\right) = \nu_p(n, k) = \nu_p(n + 1, k) = 1 \).

Equation (15) implies \( \nu_p(n-1, k) = \nu_p(n-1, l) \). Since

\[
\nu_p(n(n^2-1), k) = \nu_p(n-1, k) \cdot \nu_p(n, k) \cdot \nu_p(n + 1, k),
\]
this implies equation (16) holds. If $p$ divides $n + 1$, then equation (16) follows from a similar argument. If $p$ divides $n$, then equation (15) implies $v_p\left(\frac{1}{2}n, k\right) \cdot v_p(n, k) = v_p\left(\frac{1}{2}n, l\right) \cdot v_p(n, l)$. By Lemma 5.5 $v_p(n, k) = v_p(n, l)$, so equation (16) holds.

When $n$ is odd, by Lemma 5.3 the order of $Im(\partial'_k)_*$ is

\[ |Im(\partial'_k)_*| = \frac{\frac{1}{2}n(n^2 - 1)}{\left(\frac{1}{2}n(n^2 - 1), k\right) \cdot (n, k)}, \]

so we have

\[ \left(\frac{1}{2}n(n^2 - 1), k\right) \cdot (n, k) = \left(\frac{1}{2}n(n^2 - 1), l\right) \cdot (n, l). \]

We can argue as above to show that for all primes $p$,

\[ v_p\left(\frac{1}{2}n(n^2 - 1), k\right) = v_p\left(\frac{1}{2}n(n^2 - 1), l\right). \]

\[ \square \]

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