Robust $H_\infty$ Coherent-Classical Estimation

Shibdas Roy* and Ian R. Petersen

Abstract—We study robust $H_\infty$ coherent-classical estimation for a class of physically realizable linear quantum systems with parameter uncertainties. Such a robust coherent-classical estimator, with or without coherent feedback, can yield better disturbance-to-error performance than the corresponding robust purely-classical estimator for an uncertain plant. Moreover, coherent feedback allows for such a robust coherent-classical estimator to be more robust to uncertainty in comparison to the robust classical-only estimator.

I. INTRODUCTION

It has been of significant interest recently to study estimation and control problems for quantum systems [1]–[5]. Linear quantum systems [1]–[4], [6], [7] are an important class of quantum systems, and allow for describing quantum optical devices such as finite bandwidth squeezers [7], optical cavities [6], and linear quantum amplifiers [7]. Coherent feedback control for linear quantum systems, where the feedback controller is also a quantum system, has been more recently studied [3]–[5]. A related coherent-classical estimation problem has been considered by the authors in [8]–[10], where the estimator consists of a classical part, which produces the desired final estimate and a quantum part, which may also involve coherent feedback. A quantum observer, as constructed in [2], is a purely quantum system, that produces a quantum estimate of a variable for the quantum plant. In contrast, a coherent-classical estimator is a mixed quantum-classical system, that produces a classical estimate of a variable for the quantum plant.

The authors have previously studied robust $H_\infty$ classical estimation for an uncertain linear quantum system [11]. In this paper, we apply and extend such a robust $H_\infty$ estimator to the problem of coherent-classical estimation of an uncertain quantum plant. We note that for a suitable choice of the coherent controller, a robust $H_\infty$ coherent-classical estimator may yield improved disturbance attenuation when compared to the classical-only estimation scheme of [11]. Furthermore, we observe that with the addition of coherent feedback from the controller to the plant, such a robust $H_\infty$ coherent-classical estimator exhibits superior robustness to uncertainty than the purely-classical robust $H_\infty$ estimator.

II. ROBUST PURELY-CLASSICAL ESTIMATION

A schematic diagram of a classical estimation scheme is provided in Fig. 1. The quantum plant is defined as linear quantum stochastic differential equations (QSDEs) [9]:

\[
\begin{align*}
\frac{da(t)}{dt} &= A \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} + B \begin{bmatrix} dA(t) \\ dA(t)^* \end{bmatrix}, \\
\frac{dY(t)}{dt} &= C \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} + D \begin{bmatrix} dA(t) \\ dA(t)^* \end{bmatrix}, \\
\dot{z} &= L \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix},
\end{align*}
\]

where \(A = \Delta(A_1, A_2)\), \(B = \Delta(B_1, B_2)\), \(C = \Delta(C_1, C_2)\), and \(D = \Delta(D_1, D_2)\).

Here, \(a(t) = [a_1(t), \ldots, a_n(t)]^T\) is a vector of annihilation operators. The vector \(A(t) = [A_1(t), \ldots, A_m(t)]^T\) represents a collection of external independent quantum field operators and the vector \(Y(t)\) represents the corresponding vector of output field operators. Also, \(\Delta\) denotes a scalar operator on the underlying Hilbert space and represents the quantity to be estimated. The notation \(\Delta(A_1, A_2)\) denotes the matrix \(\begin{bmatrix} A_1 & A_2 \\ A_2^* & A_1^* \end{bmatrix}\). Moreover, \(A_1, A_2 \in \mathbb{C}^{n \times n}\), \(B_1, B_2 \in \mathbb{C}^{n \times m}\), \(C_1, C_2 \in \mathbb{C}^{m \times n}\), and \(D_1, D_2 \in \mathbb{C}^{m \times m}\). Furthermore, \(^*\) denotes the adjoint of a vector of operators or the complex conjugate of a complex matrix, and \(^\dagger\) denotes the adjoint transpose of a vector of operators or the complex conjugate transpose of a complex matrix.

A linear quantum system of the form (1) should satisfy certain physical realizability conditions (see [8]–[10]) to represent an actual physical system. A quadrature of each output field operator may yield a corresponding classical signal \(y_i\) [8]:

\[
dy_1 = \cos(\theta_1) dY_1 + \sin(\theta_1) dY_1^*, \\
\vdots \\
dy_m = \cos(\theta_m) dY_m + \sin(\theta_m) dY_m^*.
\]

Here, the angles \(\theta_1, \ldots, \theta_m\) determine the quadrature measured by each homodyne detector. The vector of classical signals \(y = [y_1^T, \ldots, y_m^T]\) is then input to a classical estimator.

Corresponding to the system described by (1), (3), we define our uncertain system modelled as follows [11]:

\[
\begin{align*}
\dot{x}(t) &= [A + \Delta A(t)]x(t) + [B + \Delta B(t)]w(t), \\
z(t) &= Lx(t), \\
y(t) &= Sc(t) + Sdw(t),
\end{align*}
\]

Fig. 1. Schematic diagram of classical estimation for a quantum plant.
where $x(t) := \begin{bmatrix} a(t) & a(t)^\# \end{bmatrix}^T$ is the state, $w(t)$ is the disturbance input, $z(t)$ is a linear combination of the state variables to be estimated, $y'(t)$ is the measured output, $L \in \mathbb{C}^{p \times 2n}$, $SC \in \mathbb{C}^{m \times 2n}$, $SD \in \mathbb{C}^{m \times 2n}$, $S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$,

$$S_1 = \begin{bmatrix} \cos(\theta_1) & 0 & \cdots & 0 \\ 0 & \cos(\theta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cos(\theta_m) \end{bmatrix},$$

$$S_2 = \begin{bmatrix} \sin(\theta_1) & 0 & \cdots & 0 \\ 0 & \sin(\theta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sin(\theta_m) \end{bmatrix},$$

and $\Delta A(\cdot)$, $\Delta B(\cdot)$ and $\Delta C(\cdot)$ denote the time-varying parameter uncertainties, that have the following structure:

$$\begin{bmatrix} \Delta A(t) \\ \Delta C(t) \end{bmatrix} = \begin{bmatrix} H_1 & H_3 \\ H_2 \end{bmatrix} F_1(t) E, \quad \Delta B(t) = H_2 F_2(t) G,$$

where $H_1$, $H_2$, $H_3$, $E$ and $G$ are known complex constant matrices with appropriate dimensions, and the unknown constant matrices $F_1(\cdot)$ and $F_2(\cdot)$ satisfy the following bounds:

$$F_1^\dagger(t) F_1(t) \leq I, \quad F_2^\dagger(t) F_2(t) \leq I, \quad \forall t.$$

In addition, the uncertainties $\Delta A(t)$, $\Delta B(t)$ and $\Delta C(t)$ should satisfy certain constraints for the uncertain system to be physically realizable. (See [11]). The robust $H_\infty$ estimation problem for the uncertain system can be converted into a scaled $H_\infty$ control problem, similar to [12], by introducing the following parameterized linear time-invariant system corresponding to [11]:

$$\dot{\tilde{x}}(t) = A \tilde{x}(t) + \begin{bmatrix} B & \tilde{z}_1 H_1 & \tilde{z}_2 H_2 \end{bmatrix} \tilde{w}(t),$$

$$\tilde{z}(t) = \begin{bmatrix} \epsilon_1 E \\ 0 \\ L \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} \tilde{w}(t) + \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix} u(t),$$

$$y'(t) = SC x(t) + \begin{bmatrix} SD & \gamma \tilde{w} H_3 & 0 \end{bmatrix} \tilde{w}(t).$$

Here, $u(t)$ is the control input, $\tilde{z}(t)$ is the controlled output, $\epsilon_1$, $\epsilon_2 > 0$ are suitably chosen scaling parameters and $\gamma > 0$ is the desired level of disturbance attenuation for the robust $H_\infty$ estimation problem. We also have the augmented disturbance $\tilde{w}(t) := [w(t) \quad \tilde{\eta}(t) \quad \tilde{\xi}(t)]^T$, where $\eta(t) := F_1(t) E x(t)$, and $\xi(t) := F_2(t) G w(t)$.

**Theorem 1:** (See [11]) Consider the robust $H_\infty$ estimation problem for the uncertain system converted to a scaled $H_\infty$ control problem for the system. Given a prescribed level of disturbance attenuation $\gamma > 0$, a robust $H_\infty$ estimator for the uncertain system can be constructed, for some $\epsilon_1$, $\epsilon_2 > 0$, by solving the following two algebraic Riccati equations (AREs):

$$\overline{A} X + X \overline{A}^+ + X (\gamma^{-2} \overline{B}_1 \overline{B}_1^+) X + \overline{C}_1^\dagger (I - \overline{D}_{12} \overline{B}_1^+ \overline{D}_{12}^+) \overline{C}_1 = 0.$$  

A quadrature of each component of $\tilde{y}(t)$ is homodyne detected to produce a corresponding classical signal $\tilde{y}_1$ [9]:

$$d\tilde{y}_1 = \cos(\tilde{\theta}_1) d\tilde{y}_1 + \sin(\tilde{\theta}_1) d\tilde{y}_1^\ast,$$

$$d\tilde{y}_m = \cos(\tilde{\theta}_m) d\tilde{y}_m + \sin(\tilde{\theta}_m) d\tilde{y}_m^\ast.$$
where \( \tilde{y} = [\tilde{y}_1 \ldots \tilde{y}_n]^T \) is then used as the input to a robust \( H_\infty \) classical estimator of the form in the previous section.

The quantum plant \( \{1\} \) augmented with the coherent controller \( \{16\} \) is defined by the QSDEs \( \{9\} \):

\[
\begin{bmatrix}
da(t) \\
da(t) \\
da(t)
\end{bmatrix} = \begin{bmatrix} A & 0 & A_c \\ B & C & A_c \\
D & C & 0
\end{bmatrix}
\begin{bmatrix}
a(t) \\
a(t) \\
a(t)
\end{bmatrix} dt + \begin{bmatrix} B \\ B & D \\
D & C
\end{bmatrix}
\begin{bmatrix}
dA(t) \\
dA(t) \\
dA(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
d\tilde{y}(t) \\
d\tilde{y}(t) \\
d\tilde{y}(t)
\end{bmatrix} = \begin{bmatrix} D & C \\ D & C \\
D & C
\end{bmatrix}
\begin{bmatrix}
a(t) \\
a(t) \\
a(t)
\end{bmatrix} dt + \begin{bmatrix} D & D \\ D & D \\
D & D
\end{bmatrix}
\begin{bmatrix}
dA(t) \\
dA(t) \\
dA(t)
\end{bmatrix}.
\]  

The system \( \{18\} \) along with \( \{17\} \) is of the form:

\[
\begin{align*}
\dot{x}_a(t) &= A_a x_a(t) + B_a w(t), \\
\dot{y}(t) &= S_a C_a x_a(t) + S_a D_a w(t),
\end{align*}
\]

where \( x_a(t) = [a(t) \, a(t)^\# \, a_c(t) \, a_c(t)^\#]^T \), \( \tilde{y}(t) \) is the measured output and

\[
A_a = \begin{bmatrix} A & 0 & 0 & 0 \\ B & C & 0 & 0 \\ D & D & C & 0 \\ D & D & 0 & D
\end{bmatrix},
B_a = \begin{bmatrix} B & 0 \\ B & D \\ D & C & 0 \\ D & C & 0
\end{bmatrix},
C_a = \begin{bmatrix} D & C & 0 & 0 \\ D & C & 0 & 0 \\ D & D & C & 0 \\ D & D & 0 & D
\end{bmatrix},
S_a = \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 \\ \tilde{S}_1 & \tilde{S}_2 \\ \tilde{S}_1 & \tilde{S}_2 \\ \tilde{S}_1 & \tilde{S}_2
\end{bmatrix}.
\]

Let us now consider an uncertain plant of the form \( \{4\} \). Then the \( H_\infty \) estimation problem is for the following uncertain augmented system:

\[
\begin{align*}
\dot{x}_a(t) &= [A_a + \Delta A_a(t)]x_a(t) + [B_a + \Delta B_a(t)]w(t), \\
\dot{z}(t) &= L_a x_a(t), \\
\tilde{y}(t) &= S_a [C_a + \Delta C_a(t)]x_a(t) + S_a D_a w(t),
\end{align*}
\]

where we take \( \Delta A_a(t) = H_{a1} F_1(t) E_a \), \( \Delta B_a(t) = H_{a2} F_2(t) G_a \) and \( \Delta C_a(t) = H_{a3} F_1(t) E_a \).

One can verify that these matrices can be expressed as:

\[
H_{a1} = \begin{bmatrix} H_1 & H_2 \\ B & H_3
\end{bmatrix},
H_{a2} = \begin{bmatrix} H_2 \\ 0
\end{bmatrix},
H_{a3} = D_a H_3,
E_a = \begin{bmatrix} E & 0
\end{bmatrix},
G_a = G, L_a = \begin{bmatrix} L & 0
\end{bmatrix}.
\]

The robust \( H_\infty \) estimator for the coherent-classical system is obtained by solving the following two AREs:

\[
\begin{align*}
\bar{A}_a X_a + X_a \bar{A}_a + X_a (\gamma^{-2} \bar{B}_a \bar{B}_a^\dagger) X_a \\
&+ \bar{C}_a ((I - D_{a12} E_{a1} \bar{D}_{a12}) \bar{C}_a) = 0,
\end{align*}
\]

\[
\begin{align*}
\bar{A}_a Y_a + Y_a \bar{A}_a + Y_a \bar{C}_a \bar{C}_a^\dagger Y_a + \gamma^{-2} \bar{B}_a \bar{B}_a^\dagger \\
&+ \bar{C}_a ((I - D_{a12} E_{a1} \bar{D}_{a12} + \gamma Y_a \bar{C}_a) \bar{C}_a) \\
&\times \bar{S}_a \bar{D}_a^{-2} \bar{S}_a (\gamma^{-1} \bar{B}_a \bar{D}_{a21} + \gamma Y_a \bar{C}_a) \bar{D}_{a21} = 0,
\end{align*}
\]

which are of the forms \( \{9\} \) and \( \{10\} \), respectively.

Here, we have

\[
\begin{align*}
\bar{A}_a &= A_a, \\
\bar{B}_{a1} &= B_a(I - \epsilon_2^2 G_a G_a)^{-1/2} \frac{\epsilon_1}{\epsilon_2} H_{a1}, \\
\bar{C}_a &= \begin{bmatrix} \epsilon_1 E_a \\ 0 \end{bmatrix}, \\
\bar{D}_{a12} &= \begin{bmatrix} 0 \\ I \end{bmatrix}.
\end{align*}
\]

Then, a suitable robust estimator is given by

\[
\begin{align*}
\hat{x}_a(t) &= A_{aK} \hat{x}_a(t) + B_{aK} \hat{y}(t), \\
\hat{z}(t) &= C_{aK} \hat{x}_a(t),
\end{align*}
\]

where

\[
A_{aK} = A_a - B_{aK} S_a C_{a2} + \gamma^{-2} (\bar{B}_{a1} - B_{a1} S_a \bar{D}_{a21}) \bar{D}_{a12} X_a,
B_{aK} = \gamma^2 (I - Y_a X_a)^{-1} (Y_a \bar{D}_{a12} S_a + \gamma^{-2} \bar{B}_{a1} \bar{D}_{a21} \bar{S}_a) \bar{D}_{a21},
C_{aK} = -E_{a1} \bar{D}_{a12} C_{a1}.
\]

Note that the matrices in \( \{25\} \) of the robust coherent-classical estimator can be expressed in terms of the corresponding matrices in \( \{11\} \) of the robust purely-classical estimator as follows:

\[
\begin{align*}
\bar{A}_a &= \begin{bmatrix} \tilde{A} & 0 \\ B & C
\end{bmatrix}, \\
\bar{B}_{a1} &= \begin{bmatrix} \tilde{B}_1 \\ B & \tilde{B}_{21}
\end{bmatrix}, \\
\bar{C}_a &= \begin{bmatrix} \tilde{C}_1 \\ C_a
\end{bmatrix}, \\
\bar{D}_{a12} &= \begin{bmatrix} D_a C_a \\ D_a \end{bmatrix}, \\
\bar{D}_{a21} &= \begin{bmatrix} D_a \tilde{D}_{21} \\ \tilde{D}_{21}
\end{bmatrix}.
\end{align*}
\]

IV. NUMERICAL EXAMPLE

We now present a numerical example. A linear quantum system arising quantum optics is a dynamic squeezer - an optical cavity with a non-linear active medium inside. Let the plant be a dynamic squeezer, described by \( \{9\} \):

\[
\begin{align*}
\begin{bmatrix} da \\ da^*
\end{bmatrix} &= \begin{bmatrix} -\gamma & -\chi^* \\ -\chi & -\gamma^*
\end{bmatrix} \begin{bmatrix} a \\ a^*
\end{bmatrix} dt + \sqrt{\kappa} \begin{bmatrix} da \\ da^*
\end{bmatrix}, \\
\begin{bmatrix} d\tilde{y} \\ d\tilde{y}^*
\end{bmatrix} &= \sqrt{\kappa} \begin{bmatrix} a \\ a^*
\end{bmatrix} dt + \begin{bmatrix} da \\ da^*
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
z(t) &= \begin{bmatrix} 0.1 \\ -0.1
\end{bmatrix} \begin{bmatrix} a \\ a^*
\end{bmatrix},
\end{align*}
\]

where \( \beta > 0 \) is the overall cavity loss, \( \kappa > 0 \) determines the loss owing to the cavity mirrors, \( \chi \in \mathbb{C} \) quantifies the non-linearity of the active medium, and \( a \) is a single annihilation operator of the cavity mode.

Here, we choose \( \beta = 4, \kappa = 4, \) and \( \chi = 0.5. \) Then, the above quantum system is physically realizable, since we have
\[ \beta = \kappa. \] Moreover, we fix the homodyne detection angle at 36^\circ. Thus, the matrices in (1) may be obtained.

We introduce uncertainty in the parameter \( \alpha := \sqrt{\varepsilon} \) as follows: \( \alpha \rightarrow \alpha + \mu \delta(t) \alpha \), where \( |\delta(t)| \leq 1 \) is an uncertain parameter and \( \mu \in [0, 1] \) is the level of uncertainty. Then,

\[
\Delta A = \begin{bmatrix} -\alpha^2 \mu \delta - \frac{\alpha^2 \mu^2 \delta^2}{2} & 0 \\ 0 & -\alpha^2 \mu \delta - \frac{\alpha^2 \mu^2 \delta^2}{2} \end{bmatrix}, \quad \Delta B = \begin{bmatrix} -\mu \delta \alpha & 0 \\ 0 & -\mu \delta \alpha \end{bmatrix}, \quad \Delta C = \begin{bmatrix} \mu \delta \alpha & 0 \\ 0 & \mu \delta \alpha \end{bmatrix}. \tag{30}
\]

We define the relevant matrices as follows:

\[
F_1(t) = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \end{bmatrix}, \quad F_2(t) = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \end{bmatrix},
\]

\[
E = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{31}
\]

\[
H_1 = \begin{bmatrix} 2\mu \alpha^2 & 0 & \mu^2 \alpha^2 & 0 \\ 0 & 2\mu \alpha^2 & 0 & \mu^2 \alpha^2 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -\mu \alpha & 0 \\ 0 & -\mu \alpha \end{bmatrix}, \quad H_3 = \begin{bmatrix} -2\mu \alpha & 0 \\ 0 & -2\mu \alpha \end{bmatrix}.
\]

One can then verify that we have \( \Delta A(t) = H_1 F_1(t) E, \quad \Delta B(t) = H_2 F_2(t) G \) and \( \Delta C(t) = H_3 F_1(t) E \), as required in (6). We choose a fixed value of \( \delta = -1 \), such that \( \mu \) is satisfied. Moreover, we set the uncertainty level to \( \mu = 0.1 \).

We now solve the associated robust purely-classical \( H_\infty \) estimation problem using Theorem 1. We choose the desired disturbance attenuation level to be \( \gamma = 0.62 \). Also, the scaling parameters are suitably chosen to be \( \epsilon_1 = 0.19, \epsilon_2 = 0.185 \). Then, an estimator is obtained as in (12) with:

\[
A_K = \begin{bmatrix} -1.2156 & -0.0769 \\ 3.8704 & 1.3760 \end{bmatrix}, \quad B_K = \begin{bmatrix} -0.4036 \\ -2.8075 \end{bmatrix}, \quad C_K = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}. \tag{32}
\]

Now, let the controller be another dynamic squeezer [9]:

\[
\begin{bmatrix} \frac{d a_c}{d \tau c} \\ \frac{d a^*_c}{d \tau c} \end{bmatrix} = \begin{bmatrix} -\frac{d}{d \tau c} & -\chi_c \\ -\chi^*_c & -\frac{d}{d \tau c} \end{bmatrix} \begin{bmatrix} a_c \\ a^*_c \end{bmatrix} dt - \sqrt{\kappa_c} \begin{bmatrix} \frac{d Y}{d \tau^c} \\ \frac{d \tau^c}{d \tau c} \end{bmatrix},
\]

\[
\begin{bmatrix} \frac{d Y}{d \tau^c} \\ \frac{d \tau^c}{d \tau c} \end{bmatrix} = \sqrt{\kappa_c} \begin{bmatrix} a_c \\ a^*_c \end{bmatrix} dt + \begin{bmatrix} \frac{d Y}{d \tau^c} \\ \frac{d Y^*}{d \tau^c} \end{bmatrix}, \tag{33}
\]

where we choose \( \beta_c = 4, \kappa_c = 4, \) and \( \chi_c = -0.5 \). Then, the above quantum system is physically realizable, since we have \( \beta_c = \kappa_c \). The matrices in (16) may be obtained.

We now consider the plant to be uncertain as in (30), (31). Also, we fix \( \delta \) at \(-1\), the homodyne detection angle at \( 36^\circ \) and uncertainty level at \( \mu = 0.1 \). We again set \( \gamma = 0.62 \), and further suitably choose \( \epsilon_1 = 0.19, \epsilon_2 = 0.185 \). A robust coherent-classical estimator is then obtained as in (26) with:

\[
A_{\alpha K} = \begin{bmatrix} -1.7302 & -0.4532 & 0.1231 & 0.0894 \\ -0.7771 & -2.0226 & -0.2435 & -0.1769 \\ -5.5052 & -1.2451 & -3.6985 & -0.7341 \\ -6.1111 & -4.2742 & -0.0892 & -2.4281 \end{bmatrix},
\]

\[
B_{\alpha K} = \begin{bmatrix} -0.0761 \\ 0.1505 \\ 1.0497 \\ 0.3642 \end{bmatrix}, \quad C_{\alpha K} = \begin{bmatrix} 0.1 & -0.1 & 0 & 0 \end{bmatrix}. \tag{34}
\]

Fig. 3 shows a comparison of the error spectra (bode magnitude plots of the disturbance \( A \) to error \( e \) transfer function only from (15) for augmented plant-controller system and plant alone respectively) of the robust coherent-classical filter and the robust purely-classical filter. Clearly, the robust \( H_\infty \) coherent-classical filter provides better disturbance attenuation compared to the robust \( H_\infty \) purely-classical filter. Fig. 4 shows a comparison of the \( H_\infty \) norm of the disturbance-to-error transfer functions as a function of \( \delta \in [-1, 1] \) for the two robust filters. Clearly, the robust coherent-classical filter provides higher disturbance attenuation than the robust classical-only filter across the entire uncertainty window.

V. COHERENT FEEDBACK CASE

Here, we consider the case where there is quantum feedback from the controller to the plant [8], [10]. For this

\[
\begin{align*}
\hat{\mathcal{U}} & \quad \text{Quantum Plant} \\
\hat{\mathcal{Y}} & \quad \text{Homodyne Detector} \\
\hat{Y} & \quad \text{Classical Estimator} \\
\end{align*}
\]

Fig. 5. Modified schematic diagram of purely-classical estimation.
Fig. 6. Schematic of coherent-classical estimation with coherent feedback.

The purpose, the plant is assumed to have an additional control input $U$ (See Fig. 5). The plant \((1)\) then is of the form \([10]\):

\[
\begin{bmatrix}
  \frac{da(t)}{dt} \\
  \frac{da(t)^*}{dt}
\end{bmatrix} = A \begin{bmatrix}
  a(t) \\
  a(t)^*
\end{bmatrix} dt + \begin{bmatrix}
  B_1 \\
  B_2
\end{bmatrix} \begin{bmatrix}
  dA(t) \\
  dA(t)^*
\end{bmatrix} dt,
\]

\[
\begin{bmatrix}
  \frac{d\tilde{Y}(t)}{dt} \\
  \frac{d\tilde{Y}(t)^*}{dt}
\end{bmatrix} = C \begin{bmatrix}
  a(t) \\
  a(t)^*
\end{bmatrix} dt + \begin{bmatrix}
  D
\end{bmatrix} \begin{bmatrix}
  dA(t) \\
  dA(t)^*
\end{bmatrix} dt,
\]

\[z = L \begin{bmatrix}
  a(t) \\
  a(t)^*
\end{bmatrix}.	ag{35}\]

The uncertain plant along with \((3)\) is then modelled as:

\[
\dot{x}(t) = [A + \Delta A(t)]x(t) + \begin{bmatrix}
  B_1 + \Delta B_1(t) \\
  B_2
\end{bmatrix} \bar{w}(t),
\]

\[z(t) = Lx(t),
\]

\[y'(t) = S[C + \Delta C(t)]x(t) + S \begin{bmatrix}
  D
\end{bmatrix} \bar{w}(t),\tag{36}\]

where $\bar{w}(t) := \begin{bmatrix}
  w(t) \\
  u_c(t)
\end{bmatrix}^T$, and $u_c(t)$ is the spare control input. Also, we have $\Delta A(t) = H_2 F_1(t) E, \Delta B_1(t) = H_2 F_2(t) G, \Delta C(t) = H_3 F_2(t) E$. The robust purely-casual estimator is then obtained from Theorem 1 where we have:

\[
\overline{A} = A, \overline{C}_2 = C, \overline{S} = S,
\]

\[
\overline{B}_1 = \begin{bmatrix}
  B_1(I - \epsilon^T \hat{G}^T \hat{G})^{-1/2} \\
  B_2
\end{bmatrix} \overline{\hat{H}_1}, \overline{\hat{H}_2},
\]

\[
\overline{C}_1 = \begin{bmatrix}
  \epsilon_1 E \\
  0
\end{bmatrix}, \overline{B}_{12} = \begin{bmatrix}
  \epsilon_2 D \\
  -I
\end{bmatrix}, \overline{D}_{21} = \begin{bmatrix}
  D(I - \epsilon_2 \hat{G}^T \hat{G})^{-1/2} \\
  0
\end{bmatrix} \overline{\hat{H}_3}.	ag{37}\]

The controller here would have an additional output that is fed back to the control input of the plant (See Fig. 6). The controller is defined as \([8], [10]\):

\[
\begin{bmatrix}
  \frac{da_{c}(t)}{dt} \\
  \frac{da_{c}(t)^*}{dt}
\end{bmatrix} = A_c \begin{bmatrix}
  a_{c}(t) \\
  a_{c}(t)^*
\end{bmatrix} dt + \begin{bmatrix}
  B_{c1} \\
  B_{c2}
\end{bmatrix} \begin{bmatrix}
  dA(t) \\
  dA(t)^*
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \frac{d\tilde{Y}(t)}{dt} \\
  \frac{d\tilde{Y}(t)^*}{dt}
\end{bmatrix} = \begin{bmatrix}
  \hat{C}_c \\
  C_c
\end{bmatrix} \begin{bmatrix}
  a_{c}(t) \\
  a_{c}(t)^*
\end{bmatrix} dt + \begin{bmatrix}
  \hat{D}_{c1} \\
  \hat{D}_{c2}
\end{bmatrix} \begin{bmatrix}
  dA(t) \\
  dA(t)^*
\end{bmatrix},	ag{38}\]

The plant \((35)\) and the controller \((38)\) can be combined to yield an augmented system \([8], [10]\):

\[
\begin{bmatrix}
  \frac{da(t)}{dt} \\
  \frac{da(t)^*}{dt}
\end{bmatrix} = \begin{bmatrix}
  A + B_2 D_2 C \\
  B_2 C_c
\end{bmatrix} \begin{bmatrix}
  a(t) \\
  a(t)^*
\end{bmatrix} dt + \begin{bmatrix}
  B_1 + B_2 D_2 D \\
  B_2 D_1
\end{bmatrix} \begin{bmatrix}
  dA(t) \\
  dA(t)^*
\end{bmatrix}.
\]

The augmented system \((39)\) along with \((17)\) is of the form \((19), (20)\), with $w(t)$ replaced by $w'(t) := \begin{bmatrix}
  dA(t) \\
  dA(t)^*
\end{bmatrix}^T$, and with

\[
\begin{align*}
A_a &= \begin{bmatrix}
  A + B_2 D_2 C \\
  B_2 C_c
\end{bmatrix}, \\
B_a &= \begin{bmatrix}
  B_1 + B_2 D_2 D \\
  B_2 D_1
\end{bmatrix}, \\
C_a &= \begin{bmatrix}
  \hat{D}_{c2} C \\
  D_c
\end{bmatrix}, D_a = \begin{bmatrix}
  \hat{D}_{c2} D \\
  \hat{D}_{c1}
\end{bmatrix}.
\end{align*}	ag{40}\]

Let us now consider an uncertain plant of the form \((36)\). Then the $H_{\infty}$ estimation problem is considered for the following uncertain augmented system:

\[
\begin{bmatrix}
  \dot{x}_a(t) \\
  \dot{y}'(t)
\end{bmatrix} = \begin{bmatrix}
  A_a + \Delta A_a(t) & [B_a + \Delta B_a(t)]w'(t), \\
  S[C_a + \Delta C_a(t)]x_a(t) + S D_a w'(t)
\end{bmatrix},
\]

\[z(t) = 0, \quad y'(t) = S_a[C_a + \Delta C_a(t)]x_a(t) + S_a D_a w'(t),\tag{41}\]

where we take $\hat{A}_a(t) = H_2 F_1(t) E_a, \hat{B}_a(t) = H_2 F_2(t) G_a$ and $\hat{C}_a(t) = H_3 F_2(t) E_a$. One can verify that these matrices can be expressed as follows:

\[
H_{a1} = \begin{bmatrix}
  H_1 + B_2 D_2 H_3 & 0 \\
  B_2 H_3
\end{bmatrix}, H_{a2} = \begin{bmatrix}
  H_2 \\
  0
\end{bmatrix}, H_{a3} = \hat{D}_{c2} H_3, E_{a} = \begin{bmatrix}
  E \\
  0
\end{bmatrix}, G_{a} = \begin{bmatrix}
  G \\
  0
\end{bmatrix}, L_{a} = \begin{bmatrix}
  L \\
  0
\end{bmatrix}.	ag{42}\]

The robust $H_{\infty}$ estimator for the coherent-classical system here is then obtained by solving two AREs of the form \((23)\) and \((24)\) with the relevant matrices as defined in \((25)\). A suitable estimator is as defined in \((26), (27)\).

Consider an example with the plant given by \([8], [10]\):

\[
\begin{bmatrix}
  \frac{da_{a}}{dt} \\
  \frac{da_{a}^*}{dt}
\end{bmatrix} = \begin{bmatrix}
  a_{a} \\
  a_{a}^*
\end{bmatrix} dt + \begin{bmatrix}
  a_{a} \\
  a_{a}^*
\end{bmatrix} \begin{bmatrix}
  dA \\
  dA^*
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \frac{d\hat{Y}}{dt} \\
  \frac{d\hat{Y}^*}{dt}
\end{bmatrix} = \begin{bmatrix}
  a_{a} \\
  a_{a}^*
\end{bmatrix} dt + \begin{bmatrix}
  a_{a} \\
  a_{a}^*
\end{bmatrix} \begin{bmatrix}
  dA \\
  dA^*
\end{bmatrix},	ag{43}\]

\[z(t) = \begin{bmatrix}
  0.1 \\
  0.1
\end{bmatrix} \begin{bmatrix}
  a_{a} \\
  a_{a}^*
\end{bmatrix}.\]
Here, we choose $\beta = 4$, $\kappa_1 = \kappa_2 = 2$ and $\chi = 0.5$. Note that this system is physically realizable, since $\beta = \kappa_1 + \kappa_2$. The matrices in (35) may then be obtained.

The coherent controller (33) is of the form [8], [10]:

$$\begin{bmatrix}
d a_c \\ d a_c^* 
\end{bmatrix} = \begin{bmatrix}
-\frac{\beta}{\kappa_1} - \frac{\chi}{\kappa_1} & \frac{\alpha}{\kappa_1} \\
-\frac{\chi}{\kappa_1} & \frac{\alpha}{\kappa_1} 
\end{bmatrix} dt - \sqrt{\kappa_{c1}} \begin{bmatrix}
d \hat{A} \\ d \hat{A}^* 
\end{bmatrix} - \sqrt{\kappa_{c2}} \begin{bmatrix}
d \hat{Y} \\ d \hat{Y}^* 
\end{bmatrix},$$

(44)

$$\begin{bmatrix}
d \hat{Y} \\ d \hat{Y}^* 
\end{bmatrix} = \begin{bmatrix}
\frac{\alpha}{\kappa_{c1}} & \frac{\alpha}{\kappa_{c1}} \\
\frac{\alpha}{\kappa_{c1}} & \frac{\alpha}{\kappa_{c1}} 
\end{bmatrix} dt + \begin{bmatrix}
d \hat{A} \\ d \hat{A}^* 
\end{bmatrix},$$

$$\begin{bmatrix}
d \hat{A} \\ d \hat{A}^* 
\end{bmatrix} = \sqrt{\kappa_{c2}} \begin{bmatrix}
a_c \\ a_c^* 
\end{bmatrix} dt + \begin{bmatrix}
d \hat{Y} \\ d \hat{Y}^* 
\end{bmatrix}. $$

Here, we choose $\beta = 4$, $\kappa_{c1} = \kappa_{c2} = 2$ and $\chi = -0.5$. Note that this system is physically realizable since $\beta = \kappa_{c1} + \kappa_{c2}$. Then the matrices in (38) may be obtained.

We now consider the plant to be uncertain as in (36), (31) here. We again set $\delta = -1$, $\mu = 0.1$ and fix the homodyne detection angle at $36^\circ$. Also, we choose $\gamma = 0.62$, $\epsilon_1 = 0.19$, $\epsilon_2 = 0.185$ again. A robust $H_\infty$ purely-classical estimator is obtained as in (12), (37) with:

$$A_K = \begin{bmatrix}
1.1028 & 1.6875 \\ 2.8045 & 0.5026 
\end{bmatrix},$$

$$B_K = \begin{bmatrix}
-2.6976 \\ -2.9640 
\end{bmatrix},$$

$$C_K = \begin{bmatrix}
0.1 \\ 0.1 
\end{bmatrix}. $$

Also, a robust $H_\infty$ coherent-classical estimator is obtained as in (26) with:

$$A_{aK} = \begin{bmatrix}
-3.9134 & -0.5213 & -3.6245 & -1.1481 \\ -0.5211 & -3.9100 & -1.1087 & -2.8424 \\ -1.9802 & -0.0208 & -5.9350 & -2.3458 \\ -0.0255 & -1.9756 & -3.4443 & -4.8740 
\end{bmatrix},$$

(46)

$$B_{aK} = \begin{bmatrix}
1.4003 \\ 0.9819 \\ 3.4535 \\ 3.4726 
\end{bmatrix},$$

$$C_{aK} = \begin{bmatrix}
0.1 \\ 0.1 \\ 0 \\ 0 
\end{bmatrix}. $$

Fig. 7 illustrates that the robust $H_\infty$ coherent-classical filter provides with better disturbance attenuation compared to the robust $H_\infty$ purely-classical filter for the uncertain plant. Fig. 8 shows that this holds for all values of $\delta$. Moreover, we can see that the $H_\infty$ norm for the robust coherent-classical estimator is uniform across the uncertainty window unlike the robust purely-classical estimator. That is, our robust coherent-classical estimator exhibits improved robustness to uncertainty as compared to the robust classically-only estimator. This is due to the coherent feedback involved here as opposed to the case of Fig. 5.

Note that we here considered uncertainty in $B_1$ only in (50). This is because any uncertainty in $B_1$ (and not $B_2$) in (35) will also cause $C$ (besides $A$) to be accordingly uncertain owing to physical realizability constraints for our example (43). However, uncertainty in $B_2$ only and/or uncertainties in both $B_1$ and $B_2$ can be treated as well.

VI. CONCLUSION

In this paper, we studied robust $H_\infty$ coherent-classical estimation, with and without coherent feedback, and compared with robust $H_\infty$ purely-classical estimation for an uncertain linear quantum plant. We observed that our robust coherent-classical filter, whether or not involving coherent feedback, can provide better disturbance attenuation than the purely-classical filter. Additionally, with coherent feedback, our robust coherent-classical filter provides superior robustness to uncertainty compared to the classical-only filter.

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