A Theorem of Siemons and Wagner

Paul Bradley
Jacobs CH2M
Birmingham, U.K.

paulmbradley82@gmail.com

October 29, 2021

Acknowledgements

Abstract

In [7] Siemons and Wagner described a relationship between the lengths of $G$-orbits on subsets of a $G$-set $\Omega$. They highlighted the situation where $\Delta \subset \Omega$ with $|\Delta| = k$, and $|\Delta^G| > |\Sigma^G|$ for all $(k+1)$-subsets, $\Sigma$ of $\Omega$ where $\Delta \subset \Sigma$. They went on to classify all primitive groups with this property for $k = 2$. Here we address some questions about primitive permutation groups satisfying this property when $k = 3$ and list all 3-homogeneous groups satisfying this condition.

1 Introduction

Let $G$ be a permutation group acting on finite set $\Omega$ then $G$ has an induced action on $\Omega_k$ the set of subsets of $\Omega$ of size $k$. The number of orbits has been the subject of numerous papers, not least of these is the paper of Livingstone and Wagner [4]. However, similar results for $G$-orbit lengths are much less common. The main contributions coming from Siemons and Wagner [7], [8] and Mnukhin [5]. We note that we are interested in the finite case and acknowledge much work has been done on the infinite case.

In [7] Siemons and Wagner describe a relationship between the lengths of $G$-orbits on subsets of a $G$-set $\Omega$. They proved the following;
Theorem 1.1 (Siemons and Wagner). Let $G$ be a transitive permutation group on a finite set $\Omega$ and let $\Delta$ be a subset of $\Omega$ of cardinality $k$ such that $|\Delta^G| > |\Sigma^G|$ for every subset $\Sigma$ containing $\Delta$ of cardinality $k + 1$. Then

$$k + 1 \geq |\Delta^G\Sigma| > |\Sigma^G\Delta| \geq 1.$$ 

Furthermore, if $k \geq 2$ then either

1. every 2-element subset of $\Omega$ will appear in some $G$ image of $\Delta$ or
2. $G$ is imprimitive with blocks of imprimitivity $\mathcal{B}_1, \ldots, \mathcal{B}_r$ ($1 < |\mathcal{B}_i| < |\Omega|$) each intersecting $\Delta$ in at most 1 point such that every 2-element subset of the form $\{\alpha_i, \alpha_j\}$ with $\alpha_i \in \mathcal{B}_i \neq \mathcal{B}_j \ni \alpha_j$ is contained in some $G$ image of $\Delta$.

They also proved that if $G$ is primitive and $k = 2$ then $G \cong L_2(5)$ with $G$ acting on 6 points. The aim of this paper is to answer some questions regarding primitive groups with the Siemons Wagner property and more specifically the case $k = 3$.

The Siemons Wagner property appears to be rare amongst primitive groups, indeed we can identify all cases where $\Omega$ has cardinality $n \leq 24$.

Proposition 1.2. Let $G$ be a primitive permutation group acting on a set $\Omega$ of degree $n < 25$. If there exists a $k$-subset $\Delta \subset \Omega$ such that $|\Delta^G| > |\Sigma^G|$ for any $k + 1$-subset $\Sigma \subset \Omega$ where $\Delta \subset \Sigma$, then $G$ is in Table 1.

Proof. We can compile Table 1 by direct calculation with the database of primitive groups in Magma. The code used to test for the Siemons Wagner property is given at the end of this paper.  

Remark 1. We note that the database of primitive groups used in the Magma calculations was produced by Roney-Dougal in [6] and Sims [9].

Definition 1.3. Let $G$ be a group acting on a set $\Omega$ with cardinality $n$. Denote by $\Omega_k$ the set of $k$-subsets of $\Omega$. We note that $|\Omega_k| = \binom{n}{k}$ and that $G$ has an induced action on $\Omega_k$. If $G$ is transitive on $\Omega_k$ we call $G$ $k$-homogeneous.

If $G$ is a permutation group on a finite set $\Omega$ and $\Delta$ is a subset of $\Omega$ of cardinality 3 such that $|\Delta^G| > |\Sigma^G|$ for every subset $\Sigma$ containing $\Delta$ of cardinality 4. Then we say that $G$ satisfies condition $\star$. 

\[2\]
| Group        | Degree | k |
|--------------|--------|---|
| $L_2(5)$     | 6      | 2 |
| $L_2(7)$     | 8      | 3 |
| $PGL(2,7)$   | 8      | 3 |
| $L_2(9)$     | 10     | 4 |
| $Sym(6)$     | 10     | 4 |
| $L_2(11)$    | 12     | 5 |
| $PGL(2,11)$  | 12     | 5 |
| $L_2(13)$    | 14     | 6 |
| $Alt(7)$     | 15     | 6 |
| $ASL(2,4)$   | 16     | 6 |
| $Alt(7) \ltimes (\mathbb{Z}_2)^4$ | 16 | 7 |
| $L_2(16)$    | 17     | 5 |
| $L_3(4)$     | 21     | 6 |
| $M(22)$      | 22     | 10|
| $M(23)$      | 23     | 10|
| $M(24)$      | 24     | 11|

Table 1: Primitive permutation groups satisfying Siemons Wagner property

2 General Results

Here we present a few preliminary results looking at the case when $k = 3$ and take some steps towards classifying primitive groups with this property for $k = 3$.

In general we will make use of the following simple but effective lemma.

**Lemma 2.1.** Let $G$ be a permutation group acting on $n$ points, let $\Sigma^G$ be an $G$-orbit of a $(k+1)$-subset, and $\Delta^G$ and $G$-orbit of a $k$-subset $\Delta$. If $\Sigma \supset \Delta$ then letting $d = |\{\alpha \in \Sigma \mid \Sigma \setminus \{\alpha\} \in \Delta^G\}|$ and $u = |\{\beta \in \Omega \mid \Delta \cup \{\beta\} \in \Sigma^G\}|$ then

$$d|\Sigma^G| = u|\Delta^G|.$$

**Proof.** We form a graph with vertex set the elements of $\Sigma^G$ and $\Delta^G$, then we draw edge $(s, t)$ if and only if $t \subset s$. Then the number of edges is equal to $d|\Sigma^G|$ and $u|\Delta^G|$. \qed

We now are able to prove the following initial results regarding primitive permutation groups satisfying * without any further restrictions.

**Proposition 2.2.** Let $G$ be a primitive permutation group acting on a set $\Omega$ of cardinality $n \geq 8$. Let $\Delta$ be a 3-subset of $\Omega$ such that $|\Delta^G| > |\Sigma^G|$ for
all 4-subsets $\Sigma$ containing $\Delta$. Then $\Delta^G$ is of maximal length of any orbit on 3-subsets.

Proof of Proposition 2.2. Let $\Delta = \Delta_1$. If $G$ is 3-homogeneous we are done, so assume there exists a 3-subset $\Delta_2$ which is not in $\Delta^G_1$. We wish to show that $|\Delta_1^G| \geq |\Delta_2^G|$. Theorem 1.1 tells us that every 2-subset of $\Omega$ appears in the $G$-image of $\Delta_1$. Hence we may assume that $|\Delta_1 \cap \Delta_2| = 2$. We may now set $\Delta_1 = \{1, 2, 3\}$ and $\Delta_2 = \{1, 2, 4\}$. We may now choose $\Sigma = \{1, 2, 3, 4\}$ and note that $G_\Sigma$ is not transitive on $\Sigma$ and that $|\Sigma^G| < |\Delta_1^G|$. We now consider the possible structure of $G_\Sigma$ as a subgroup of $Sym(4)$. Theorem 1.1 tells us that $4 \geq |\Delta_1^{G_\Sigma}| > |\Sigma^G| > 1$, and moreover $|\Delta_1^{G_\Sigma}| = 2$ or 3.

By applying Lemma 2.1 and noting that $|\Delta_1^G| > |\Sigma^G|$ we have the following equations.

$$u_1 |\Delta_1^G| = d_1 |\Sigma^G|,$$
$$u_2 |\Delta_2^G| = d_2 |\Sigma^G|.$$

We also note that $1 \leq u_1 < d_1 \leq 3$ and $u_1, u_2, d_1, d_2 > 0$.

If $|\Delta_1^{G_\Sigma}| = 3$ then $d_1 = 3$ giving us $u_1 |\Delta_1^G| = 3 |\Sigma^G|$. This implies that $d_2 = 1$. Combining these gives

$$|\Delta_2^G| \leq u_2 |\Delta_2^G| = |\Sigma^G| = \frac{u_1 |\Delta_1^G|}{3} < |\Delta_1^G|.$$

Next we consider the case when $|\Delta_1^{G_\Sigma}| = 2$. Here we have $d_1 = 2$ and so $2 |\Sigma^G| = u_1 |\Delta_1^G|$. It immediately follows that $u_1 = 1$. Moreover $d_2 |\Sigma^G| = u_2 |\Delta_2^G|$ where $1 \leq d_2 \leq 2$. Now

$$u_2 |\Delta_2^G| = d_2 |\Sigma^G|,$$
$$u_2 |\Delta_2^G| = \frac{d_2}{2} |\Delta_1^G|,$$
$$\frac{2u_2}{d_2} |\Delta_2^G| = |\Delta_1^G|,$$

where $2u_2 \geq d_2$ and so $|\Delta_1^G| \geq |\Delta_2^G|$ as required.
Proposition 2.3. Let $G$ be a primitive permutation group satisfying $\star$. Let $\Delta$ be a representative of the large orbit on $\Omega_3$ and let $G_\Delta$, the set-wise stabilizer of $\Delta$ in $G$, be transitive on $\Delta$. Then $G$ is 3-homogeneous.

Proof. We assume $G$ is not 3-homogeneous, then we let $\Delta_1$ and $\Delta_2$ be a representative from a second $G$-orbit on $\Omega_3$. By Theorem 1.1 we can assume that $\Delta_1 = \{1, 2, 3\}$ and $\Delta_2 = \{1, 2, 4\}$. We also let $\Sigma = \{1, 2, 3, 4\}$. As $G_{\Delta_1}$ is transitive on $\Delta_1$ there exists $g \in G_{\Delta_1}$ such that $g = \pi_1 \pi_2 \pi_3$ where $\pi_1 = (1, 2, 3)$, $\pi_2 = (4, ...,)$ (the cycle containing 4) and $\pi_3$ is a permutation on the remaining points of $\Omega$. Now as $\Delta_1^{G_{\Delta_1}} > 1$ there exists a subset of $\Sigma$ of the form $\{\alpha, \beta, 4\}$ which is in $\Delta_1^G$. If $\pi_2 = (4)$ then $\Delta_2 \in \Delta_1^{(g)}$ which is a contradiction. Without loss then we can assume $g$ is such that $\pi_2$ is a cycle with length some power of 3. We then only need consider two cases.

1. $g = (1, 2, 3)(4, 5, 6)...$ or
2. $g = (1)(2)(3)(4, 5, 6)...$

In both cases we have that $\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}$ are all in $\Sigma^{(g)}$. Using Lemma 2.1 we have $u|\Delta_1^G| = d|\Sigma^G|$ with $u \geq 3$ giving us that $d = 4$ which is only possible if $\Delta_2 \in \Delta_1^G$ which is a contradiction. Hence $G$ is 3-homogeneous as required.

3 The case that $G$ is 3-homogeneous

The list of Primitive groups stored in MAGMA and the procedure used in Proposition 1.2 were used to check for any instances of the Siemons Wagner property for $k = 3$ in all primitive groups of degree $n \leq 200$. The only two groups found are given in Proposition 1.2. As both of these groups are 3-homogeneous the obvious question is are there any more.

Theorem 3.1. Let $G$ be a 3-homogeneous permutation group acting on a $G$-set, $\Omega$, of cardinality $n \geq 8$. If the $G$-orbit $\Omega_3$ has length strictly greater than the $G$-orbit of any 4-subset of $\Omega$ then $G \cong L_2(7)$ or $G \cong PGL(2, 7)$.

Before we prove this we mention that the following notation for $PG(q)$ is used.

Definition 3.2. The projective line over $\mathbb{F}_q$ contains $q + 1$ points and we denote it by $PG(q)$. The composition of these points and their representatives for $\mathbb{F}_q$ is as given in Table 2.
Proof of Theorem 3.1. As $G$ is 3-homogeneous we note that any group hoping to have the property we are searching for cannot have any regular orbits on $\Omega_4$ and furthermore cannot be 4-homogeneous.

We begin by compiling a list of possible candidates using results from Kantor [3] and Cameron [2]. This gives the following possibilities $M_{11}, M_{22}, AGL(1,8), AGammaL(1,8), AGammaL(1,32), L_2(q)$ and a family of groups $L_2(q) \leq G \leq PGammaL(2,q)$, where $q \equiv 3 \mod 4$.

Initially we can compute the groups $M_{11}, M_{22}, AGL(1,8), AGammaL(1,8)$ and $AGammaL(1,32)$ and see that these do not satisfy the condition of having such a 3-subset of their respective $G$-sets.

We now eliminate the possibilities for $q$ in the remaining families of groups. We begin by noting that all the groups which remain have $L_2(q)$ as their respective socles. As the $G$-sets of these groups are the same as for their socles, we need only show that $L_2(q)$ does not satisfy the condition of having all 4-subsets with stabilizers of order greater than 2. This follows as the size of the orbit on 3-subsets being as large as possible but less than $\frac{|G|}{2}$ and so any over group cannot increase the length of this 3-orbit but could fuse two or more orbits on $\Omega_4$.

Next we consider a specific subset of the projective line on which these remaining groups act.

We let $\omega$ be a generator for the multiplicative field of $q$ elements, that is $\omega^{q-1} = 1$. We choose $\Sigma = \{0, 1, \infty, \omega^a\}$ where $\omega^a \not\in \{-1, 2, 2^{-1}\}$ and, $\omega^{2a} - \omega^a + 1 \neq 0$.

We can now make use of a result in [1] which states that for such a set the stabilizer in $PGL(2,q)$ has order 4. We now show that this set must have a stabilizer in $L_2(q)$ with order less than 4 by giving an element of $PGL(2,q)\Sigma$ which is not in $L_2(q)$.

$$A = \begin{bmatrix} 1 & \omega^a \\ -1 & -1 \end{bmatrix}.$$  

It is clear that $A$ will act on $\Sigma$ with the cycles $(0, \omega^a)(1, \infty)$ and so $A \in$
$PGL(2,q)_\Sigma$. The determinant of $A$ is equal to $\omega^n - 1$, moreover $\omega^n \neq 2$ by choice and so $A \not\in L_2(q)$ as required. Hence the stabilizer of $\Sigma$ in $L_2(q)$ must have order 1 or 2 and so the $L_2(q)$ orbit of $\Sigma$ is greater than the total number of 3-subsets of $PG(q)$ hence such groups cannot have a large enough 3-subset orbit to satisfy our condition.

This has now reduced our problem to finding fields for which no such element $\omega^n$ exists. We also note that we are interested in $q \geq 7$. In fact $q = 7$ is the only such field in our range without such an element as a simple counting argument shows that any field with more than 8 elements must satisfy this requirement.

Finally we note that $L_2(7) < PGL(2, 7) = PΓL(2, 7)$ and that Proposition 1.2 shows both of these groups satisfy the condition.

We leave this section with the following Conjecture.

**Conjecture 1.** Let $G$ be a primitive permutation group acting on a $G$-set, $\Omega$, of cardinality $n \geq 8$. If there exists a 3-subset $\Delta \subset \Omega$ such that $|\Delta^G| > |\Sigma^G|$ for any 4-subset $\Sigma$ containing $\Delta$, then $G \cong PSL(2,7)$ or $G \cong PGL(2,7)$.

### 4 Non-primitive group examples

For the reader’s consideration we now turn briefly to non-primitive but transitive examples of groups with the Siemons Wagner property when $k = 3$. These are far more common than imprimitive examples. We present here three such groups and give their generators. In all three of the following examples we keep $\Delta = \{1, 2, 3\}$ as a representative of the large orbit on $\Omega_3$.

For a given permutation group $G$ acting on a set $\Omega$, we denote the number of $G$-orbits on $\Omega_k$ by $\sigma_k$.

Example 1 is a subgroup of $Sym(8)$ where

$$G_1 \cong \langle (4,6), (1,2,5,3)(4,8)(6,7), (1,8)(4,6), (3,4,6), (1,7,8), (2,3)(4,6), (2,4)(3,6), (1,5)(7,8), (1,7)(5,8) \rangle.$$  

Here $|G_1| = 1152$ and $|\Delta^{G_1}| = 48$. This group satisfies the condition that every 2-subset appears in some $G_1$-image of $\Delta$. The system of imprimitivity for $G_1$ is the set

$$\{\{1,5,7,8\}, \{2,3,4,6\}\}.$$  

The $\sigma_k$ values are $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 2$ and $\sigma_4 = 3$. 

7
It is also clear that there exists a 4-subset for which no $G_1$-image contains $\Delta$ as a subset (the system of imprimitivity is a single orbit).

Example 2 is a subgroup of $Sym(9)$ where

$$G_2 \cong \langle (4,7)(5,9)(6,1), (8,9,5)(2,7,4)(3,1,6), (4,5,6)(7,1,9), (8,3,2)(7,1,9) \rangle.$$ 

Here $|G_2| = 54$ and $|\Delta G_2| = 54$. This group satisfies the condition that every 2-subset appears in some $G_2$-image of $\Delta$. The system of imprimitivity for $G_2$ is the set

$$\{\{1,7,9\}, \{2,3,8\}, \{4,5,6\}\}.$$ 

The $\sigma_k$ values are $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 5$ and $\sigma_4 = 5$. In this case $\Delta$ appears as a subset of an element of every $G_2$-orbit on subsets of size 4.

Example 3 is a subgroup of $Sym(16)$ where

$$G_3 \cong \langle (1,12)(7,3)(11,8)(4,2)(5,10)(6,9)(13,15)(14,16), (1,8,6,14)(7,2,5,13)(11,9,16,12)(4,10,15,3), (1,14)(7,13)(11,12)(4,3)(5,2)(6,8)(9,16)(10,15), (1,6)(7,5)(11,15)(4,16)(2,14)(8,13)(9,12)(10,3), (1,16)(7,15)(11,6)(4,5)(2,3)(8,12)(9,14)(10,13), (7,8)(9,10)(3,12)(13,14), (11,4)(9,10)(3,12)(15,16), (1,7)(11,4)(5,6)(2,8)(9,10)(3,12)(13,14)(15,16) \rangle.$$ 

Here $|G_3| = 256$ and $|\Delta G_3| = 256$. This group does not satisfy the condition of every 2-subset being contained in some $G_3$-image of $\Delta$. The $\sigma_k$ values for $G_3$ are $\sigma_1 = 1$, $\sigma_2 = 6$, $\sigma_3 = 11$, $\sigma_4 = 35$, $\sigma_5 = 48$, $\sigma_6 = 91$, $\sigma_7 = 100$ and $\sigma_8 = 132$. Here we have three systems of imprimitivity

$$\{\{1,5\}, \{2,14\}, \{3,9\}, \{4,16\}, \{6,7\}, \{8,13\}, \{10,12\}, \{11,15\}\},$$

$$\{\{3,9,10,12\}, \{1,5,6,7\}, \{4,11,15,16\}, \{2,8,13,14\}\},$$

and

$$\{\{1,3,5,6,7,9,10,12\}, \{2,4,8,11,13,14,15,16\}\}.$$

It is also clear that there exists a 4-subset for which no $G_3$-image contains $\Delta$ as a subset (the system of imprimitivity is a single orbit).
5 Magma Code for Proposition 1.2

We finish by giving the MAGMA implementation for the search for Primitive groups with the Siemons Wagner property used to compile Table 1.

\[ Z := \text{Integers();} \]

\[ \text{SizeofOrbsPRIMk := procedure(G, k, a);} \]
\[ S := \{} ; K := \{} ; D := \{1..\text{Degree(G)}\} ; \]
\[ kD := \text{Subsets(D, k); a := \{} ; \]
\[ \Omega := \text{GSet(G, kD)} ; \]
\[ O := \text{Orbits(G, Omega)} ; \]
\[ \text{for Orbs in O do} \]
\[ T := \text{Random(Orbs)} ; \]
\[ \text{Include(}^{\sim} K, T) ; \]
\[ \text{end for} ; \]
\[ V := \{} ; \]
\[ \text{for } T \text{ in } K \text{ do} ; \]
\[ N := Z!(\text{#G/\#Stabilizer(G, T)}); \]
\[ P := D \text{ diff } T ; \]
\[ \text{for } b \text{ in } P \text{ do} ; \]
\[ \text{Include(}^{\sim} V, \text{#Stabilizer(G, T} \text{ join\{b\}}) ; \]
\[ \text{end for} ; \]
\[ S := \text{Min(V)} ; L := Z!(\text{#G/S}) ; \]
\[ \text{if } N \text{ gt } L \text{ then Include(}^{\sim} a, N, T) ; \]
\[ \text{end if} ; \]
\[ \text{end for} ; \]
\[ \text{end procedure} ; \]

Letting \( D \) be the degree.

\[ \text{for } k \text{ in } [3..Z!(\text{Floor(D/2)-1})] \text{ do} \]
\[ \text{for I := 1 to (Z!(NumberOfPrimitiveGroups(D)-2)) do} \]
\[ \text{SizeofOrbsPRIMk(PrimitiveGroup(D, I), k, }^{\sim} T) ; \]
\[ \text{if } \#T \text{ ge } 1 \text{ then } <D, I, T> ; \]
\[ \text{end if} ; \]
\[ \text{end for} ; \]
\[ \text{end for} ; \]
References

[1] T. Beth, D. Jungnickel and H. Lenz. Design theory. Vol. I. Second edition. Encyclopedia of Mathematics and its Applications, 69. (Cambridge University Press, 1999).

[2] P. J. Cameron. Finite permutation groups and finite simple groups. Bull. London Math. Soc. 13 (1981), no. 1, 1–22.

[3] W. M. Kantor. $k$-homogeneous groups. Math. Z. 124 (1972), 261–265.

[4] D. Livingstone and A. Wagner. Transitivity of finite permutation groups on unordered sets. Math. Z. 90 (1965) 393-403.

[5] V. B. Mnukhin. Some relations for the lengths of orbits on $k$-sets and $(k - 1)$-sets. Arch. Math. (Basel) 69, no. 4 (1997) 275–278.

[6] C. M. Roney-Dougal. The primitive permutation groups of degree less than 2500. J. Algebra 292 no. 1, (2005) 154—183.

[7] J. Siemons and A. Wagner. On the relationship between the lengths of orbits on $k$ -sets and $(k + 1)$ -sets. Abh. Math. Sem. Univ. Hamburg 58 (1988), 267—274.

[8] J. Siemons and A. Wagner. On finite permutation groups with the same orbits on unordered sets. Arch. Math. (Basel) 45 , no. 6 (1985), 492–500.

[9] C.C. Sims. Computational methods in the study of permutation groups. Computational problems in abstract algebra, (Oxford - Pergamon, 1970) pp. 169–183.