Approximation by Genuine $q$-Bernstein-Durrmeyer Polynomials in Compact Disks in the Case $q > 1$

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This paper deals with approximating properties of the newly defined $q$-generalization of the genuine Bernstein-Durrmeyer polynomials in the case $q > 1$, which are no longer positive linear operators on $C[0,1]$. Quantitative estimates of the convergence, the Voronovskaja-type theorem, and saturation of convergence for complex genuine $q$-Bernstein-Durrmeyer polynomials attached to analytic functions in compact disks are given. In particular, it is proved that, for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the genuine $q$-Bernstein-Durrmeyer polynomials ($q > 1$) is of order $q^{-n}$ versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials. We give explicit formulas of Voronovskaja type for the genuine $q$-Bernstein-Durrmeyer for $q > 1$. This paper represents an answer to the open problem initiated by Gal in 2013, page 115.

1. Introduction

In several recent papers, convergence properties of complex $q$-Bernstein polynomials, proposed by Phillips [1], attached to an analytic function $f$ in closed disks, were intensively studied. Ostrovska [2, 3] and Wang and Wu [4, 5] have investigated convergence properties of $B_{n,q}$ in the case $q > 1$. In the case $q > 1$, the $q$-Bernstein polynomials are no longer positive operators; however, for a function analytic in a disc $D_R := \{z \in \mathbb{C} : |z| < R\}$, $R > q$, it was proved in [2] that the rate of convergence of $\{B_{n,q}(f;z)\}$ to $f(z)$ has the order $q^{-n}$ (versus $1/n$ for the classical Bernstein polynomials). Moreover, Ostrovska [3] obtained Voronovskaya-type theorem for monomials. If $q \geq 1$, then qualitative Voronovskaja-type theorem and saturation results for complex $q$-Bernstein polynomials were obtained by Wang and Wu [4]. Wu [5] studied saturation of convergence on the interval $[0,1]$ for the $q$-Bernstein polynomials of a continuous function $f$ for arbitrary fixed $q > 1$.

Genuine Bernstein-Durrmeyer operators were first considered by Chen [6] and Goodman and Sharma [7] around 1987. In recent years, the genuine Bernstein-Durrmeyer operators have been investigated intensively by a number of authors. Among the many papers written on the genuine Bernstein-Durrmeyer operators, we mention here only the ones by Gonska et al. [8], Parvanov and Popov [9], Sauer [10], Waldron [11], and the book of Păltănea [12].

On the other hand, Gal [13] obtained quantitative estimates of the convergence and of the Voronovskaja-type theorem in compact disks, for the complex genuine Bernstein-Durrmeyer polynomials attached to analytic functions. Besides, in other very recent papers, similar studies were done for complex Bernstein-Durrmeyer operators in Anastassiou and Gal [14], for complex Bernstein-Durrmeyer operators based on Jacobi weights in Gal [15], for complex genuine $q$-Bernstein-Durrmeyer operators ($0 < q < 1$) by Mahmudov [16], and for other kinds of complex Durrmeyer operators in Mahmudov [17] and Gal et al. [18]. It should be stressed out that study of $q$-Durrmeyer-type operators ($0 < q < 1$) in the real case was first initiated by Derriennic [19].

Also, for the case $q > 1$, exact quantitative estimates and quantitative Voronovskaja-type results for complex $q$-Lorentz polynomials, $q$-Stancu polynomials [20], $q$-Stancu-Faber polynomials, $q$-Bernstein-Faber polynomials, $q$-Kantorovich polynomials [21], $q$-Szász-Mirakjan operators [22] obtained by different researchers are collected in the recent book of Gal [23]. In this book the definition and study of complex $q$-Durrmeyer-kind operators for $q > 1$ presented an open
problem. This paper presents a positive solution to this problem.

In this paper we define the genuine \( q \)-Bernstein-Durrmeyer polynomials for \( q > 1 \). Note that similar to the \( q \)-Bernstein operators the genuine \( q \)-Bernstein-Durrmeyer operators in the case \( q > 1 \) are not positive operators on \( C[0,1] \). The lack of positivity makes the investigation of convergence in the case \( q > 1 \) essentially more difficult than that for \( 0 < q < 1 \). We present upper estimates in approximation and we prove the Voronovskaja-type convergence theorem in compact disks in \( C \), centered at origin, with quantitative estimate of this convergence. These results allow us to obtain the exact degrees of approximation by complex genuine \( q \)-Bernstein-Durrmeyer polynomials. Our results show that approximation properties of the complex genuine \( q \)-Bernstein-Durrmeyer polynomials are better than approximation properties of the complex Bernstein-Durrmeyer polynomials considered in [13].

2. Main Results

We begin with some notions and definitions of \( q \)-calculus; see, for example, [24, 25]. Let \( q > 0 \). For any \( n \in \mathbb{N} \cup \{0\} \), the \( q \)-integer \([n]_q\) is defined by

\[
[n]_q := 1 + q + \cdots + q^{n-1}, \quad [0]_q := 0; \tag{1}
\]

and the \( q \)-factorial \([n]_q!\) is defined by

\[
[n]_q! := [1]_q [2]_q \cdots [n]_q, \quad [0]_q! := 1. \tag{2}
\]

For integers \( 0 \leq k \leq n \), the \( q \)-binomial is defined by

\[
\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}. \tag{3}
\]

For \( q = 1 \) we obviously get \([n]_q = n, [n]_q! = n!\), and \([\binom{n}{k}_1] = \binom{n}{k}\). Moreover

\[
(1 - z)_q^n := \prod_{s=0}^{n-1} (1 - q^s z),
\]

\[
p_{n,k}(q; z) := \binom{n}{k}_q z^k (1 - q z)^{n-k}, \quad z \in \mathbb{C}. \tag{4}
\]

For fixed \( q > 0, q \neq 1 \), we denote the \( q \)-derivative \( D_q f(z) \) of \( f \) by

\[
D_q f(z) = \begin{cases} 
\frac{f(qz) - f(z)}{(q-1) z}, & z \neq 0, \\
\frac{f'(0)}{q}, & z = 0. 
\end{cases} \tag{5}
\]

The \( q \)-analogue of integration in the interval \([0, A]\) (see [24]) is defined by

\[
\int_0^A f(t) \, d_q t := A (1 - q) \sum_{n=0}^\infty f(A q^n) q^n, \quad 0 < q < 1. \tag{6}
\]

Let \( D_R \) be a disc \( D_R := \{ z \in \mathbb{C} : |z| < R \} \) in the complex plane \( C \). Denote by \( H(D_R) \) the space of all analytic functions on \( D_R \). For \( f \in H(D_R) \) we assume that \( f(z) = \sum_{m=0}^\infty a_m z^m \) for all \( z \in D_R \). The norm \( \|f\|_r := \max \{|f(z)| : |z| \leq r\} \). We denote \( e_m(z) = z^m \) for all \( m \in \mathbb{N} \cup \{0\} \).

Definition 1. For \( f : [0,1] \rightarrow \mathbb{C} \), the genuine \( q \)-Bernstein-Durrmeyer operator is defined as follows:

\[
U_{n,q}(f;z) := \begin{cases} 
0, & 0 \leq z < q, \\
\displaystyle \frac{1}{[n]_q} \sum_{k=0}^{n-1} p_{n,k}(q; z) f(q^k), & z \geq q. 
\end{cases}
\]

where for \( n = 1 \) the sum is empty; that is, it is equal to 0.

\[
U_{n,q}(f;z) \text{ are linear operators reproducing linear functions and interpolating every function } f \in C[0,1] \text{ at 0 and 1. The genuine } q \text{-Bernstein-Durrmeyer operators are positive operators on } C[0,1] \text{ for } 0 < q \leq 1, \text{ and they are not positive for } q > 1. \text{ As a consequence, the cases } 0 < q \leq 1 \text{ and } q > 1 \text{ are not similar to each other regarding the convergence. For } q \rightarrow 1^- \text{ and } q \rightarrow 1^+ \text{ we recapture the classical } (q = 1) \text{ genuine Bernstein-Durrmeyer polynomials.}
\]

We start with the following quantitative estimates of the convergence for complex \( q \)-Bernstein-Durrmeyer polynomials attached to an analytic function in a disk of radius \( R > 1 \) and center 0.

Theorem 2. Let \( f \in H(D_R), 1 \leq r < R/q, \) and \( q > 1 \). Then for all \( |z| \leq r \) one has

\[
|U_{n,q}(f;z) - f(z)| \leq \frac{r(1 + r)}{[n+1]_q} \sum_{m=2}^\infty |a_m| (m-1) q^{m-2} r^{m-2}. \tag{8}
\]

Theorem 2 says that, for functions analytic in \( D_R, R > q \), the rate of approximation by the genuine \( q \)-Bernstein-Durrmeyer polynomials \( (q > 1) \) is of order \( q^n \) versus \( 1/n \) for the classical genuine Bernstein-Durrmeyer polynomials; see [13].
The Voronovskaya theorem for the real case with a quan-
titative estimate is obtained by Gonska et al. [26] in the
following form:

\[ \left| U_n (f; x) - f (x) - \frac{x (1 - x)}{n + 1} f'' (z) \right| \leq \frac{x (1 - x)}{n + 1} \omega \left( f''; \frac{2}{3n + 3} \right). \] (9)

and, for all \( n \in \mathbb{N}, 0 \leq x \leq 1 \). For the complex genuine q-
Bernstein-Durrmeyer (0 < q ≤ 1) a quantitative estimate is
obtained by Gal [13] (q = 1) and Mahmudov [16] (0 < q < 1)
in the following form:

\[ \left| U_{n,q} (f; z) - f (z) - \frac{z (1 - z)}{n + 1} q f'' (z) \right| \leq \frac{M_{r,f}^2}{|n|^2}, \quad 0 < q \leq 1, \] (10)

and, for all \( n \in \mathbb{N}, |z| \leq r \).

To formulate and prove the Voronovskaya-type theorem
with a quantitative estimate in the case \( q > 1 \) we introduce
a function \( L_q (f; z) \).

Let \( R > q \geq 1 \) and let \( f \in H(D_R) \). For \( |z| < R/q^2 \), we
define

\[ L_q (f; z) := \frac{(1 - z) q (D_q f (z) - D_q^{0} f (z))}{q - 1} \] for \( q > 1 \).

And, for \( 0 < q \leq 1 \),

\[ L_q (f; z) = L_1 (f; z) := f'' (z) (1 - z) \] (12)

The next theorem gives Voronovskaya-type result in com-
 pact disks; for complex \( q \)-Bernstein-Durrmeyer polynomials
attached to an analytic function in \( D_R, R > q^2 > 1 \) and center
0 in terms of the function \( L_q (f; z) \).

Theorem 3. Let \( f \in H(D_R), \ 1 \leq r \leq R/q^2, \) and \( q > 1 \). The
following Voronovskaya-type result holds:

\[ \left| U_{n,q} (f; z) - f (z) - \frac{1}{[n + 1]_q} L_q (f; z) \right| \leq \frac{4r^2(1 + r)^2}{[n + 1]_q} \sum_{m=3}^{\infty} |a_m| (m - 1)^2 (m - 2)^2 (q^2 r)^{m-2}. \] (13)

For all \( n \in \mathbb{N}, |z| \leq r \).

Now we are in position to prove that the order of approxi-
mation in Theorem 2 is exactly \( q^n \) versus \( 1/n \) for the classical
genuine Bernstein-Durrmeyer polynomials; see [13].

Theorem 4. Let \( 1 < q < R, 1 \leq r < R/q^2, \) and \( f \in H(D_R) \). If
\( f \) is not a polynomial of degree \( \leq k, \) the estimate,

\[ \left\| U_{n,q} (f) - f \right\|_r \geq \frac{1}{[n + 1]_q} C_{r,q} (f), \quad n \in \mathbb{N}, \] (14)

holds, where the constant \( C_{r,q} (f) \) depends on \( f, q, \) and \( r \) but is
independent of \( n \).

From Theorem 3 we conclude that, for \( q > 1, [n + 1]_q \) \( U_{n,q} (f; z) - f (z) \rightarrow L_q (f; z) \) in \( H(D_{R/q}) \) and
therefore \( L_q (f; z) \in H(D_{R/q}) \). Furthermore, we have
the following saturation of convergence for the genuine \( q-
Bernstein-Durrmeyer polynomials for fixed \( q > 1 \).

Theorem 5. Let \( 1 < q < R, 1 \leq r < R/q^2. \) If a function \( f \)
is analytic in the disc \( D_{R/q}, \) then \( |U_{n,q} (f; z) - f (z)| = o(q^{-n}) \)
for infinite number of points having an accumulation point on
\( D_{R/q} \) if and only if \( f \) is linear.

The next theorem shows that \( L_q (f; z), q \geq 1, \) is continuous
in the parameter \( q \) for \( f \in H(D_Q), R > 1 \).

Theorem 6. Let \( R > 1 \) and \( f \in H(D_R). \) Then, for any \( r, \ 0 <
q < R, \)

\[ \lim_{q \rightarrow 1^+} L_q (f; z) = L_1 (f; z) \] (15)
uniformly on \( D_R \).

3. Auxiliary Results

The \( q \)-analogue of beta function for \( 0 < q < 1 \) (see [24]) is
defined as

\[ B_q (m, n) = \int_0^1 t^{m-1} (1 - qt)^{n-1} \, dq, \quad m, n > 0, \ 0 < q < 1. \] (16)

Since we consider the case \( q > 1 \), we need to use \( B_{q^{-1}} (m, n) \) as
follows:

\[ B_{q^{-1}} (m, n) = \int_0^{1/q} t^{m-1} (1 - q^{-1} t)^{n-1} \, dt, \quad m, n > 0, \ 0 < q^{-1} < 1. \] (17)

Also, it is known that

\[ B_{q^{-1}} (m, n) = \frac{[m - 1]_{q^{-1}}! [n - 1]_{q^{-1}}!}{[m + n - 1]_{q^{-1}}!}, \quad 0 < q^{-1} < 1. \] (18)

For \( m = 0, 1, \ldots, \) we have

\[ [n - 1]_{q^{-1}} q^{k-1} \int_0^1 t^m p_{n-2k-1} (q^{-1} t) \, dt \]
\[ = [n - 1]_{q^{-1}} [n - 2]_{q^{-1}} \frac{q^{m(k-n)}}{k-1}_{q^{-1}} \]
\[ \times \int_0^1 t^{m-1} (1 - q^{-1} t)^{n-1} \, dt \]
\[ = q^{m(k-n)} [n - 1]_{q^{-1}}! [n - k - 1]_{q^{-1}}! B_{q^{-1}} (k + m, n - k) \]
\[
q^{m(k-n)} \frac{[n-1]_q}{[k-1]_q} = \frac{[k+m-1]_q}{[k+m-n-k-1]_q} [n-k]_q \frac{[n-1]_q}{[k]_q} [k-1]_q [n+m-1]_q [n]_q.
\]

(19)

Thus, we get the following formula for \(U_{n,q}(e_m; z)\):

\[
U_{n,q}(e_m; z) = f(0) P_{n,0}(q; z) + f(1) P_{n,n}(q; z)
+ [n-1]_q \sum_{k=1}^{n-1} P_{nk}(q; z)
\times \int_q^1 p_{n-2,k-1}(q^{-1}; t) f(q^{k-1} t) d_q t
= z^n + \sum_{k=1}^{n-1} P_{nk}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}.
\]

(20)

Note that, for \(m = 0, 1, 2\), we have

\[
U_{n,q}(e_0; z) = 1, \quad U_{n,q}(e_1; z) = z,
U_{n,q}(e_2; z) = z^2 + \frac{(1+q) z (1-z)}{[n+1]}.
\]

(21)

Lemma 7. \(U_{n,q}(e_m; z)\) is a polynomial of degree less than or equal to \(\min(m,n)\) and

\[
U_{n,q}(e_m; z) = \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q B_{n,q}(e_s; z).
\]

(22)

Proof. From (20) it follows that

\[
U_{n,q}(e_m; z) = \sum_{k=1}^{n} P_{nk}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}
= \frac{[n-1]_q}{[n+m-1]_q} \sum_{k=1}^{n} [k]_q [k+1]_q \cdots [k+m-1]_q P_{nk}(q; z).
\]

(23)

Now using

\[
[k]_q [k+1]_q \cdots [k+m-1]_q
= \prod_{s=0}^{m-1} (q^s[k]_q + [s]_q) = \sum_{s=1}^{m} S_q(m,s) [k]_q^s
= \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s B_{n,q}(e_s; z).
\]

(24)

where \(S_q(m, s) > 0, s = 1, 2, \ldots, m\), are the constants independent of \(k\), we get

\[
U_{n,q}(e_m; z) = \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s B_{n,q}(e_s; z).
\]

(25)

Since \(U_{n,q}(e_s; z)\) is a polynomial of degree less than or equal to \(\min(s,n)\) and \(S_q(m,s) > 0, s = 1, 2, \ldots, m\), it follows that \(U_{n,q}(e_m; z)\) is a polynomial of degree less than or equal to \(\min(m,n)\).

Lemma 8. The numbers \(S_q(m, s), (m, s) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})\), given by (24), enjoy the following properties:

\[
S_q(0,0) = 1, \quad S_q(m,0) = 0, \quad m \in \mathbb{N},
S_q(m+1,1) = [m]_q S_q(m,s) + q^m S_q(m,s-1), \quad m \in \mathbb{N}_0, \quad s \in \mathbb{N},
S_q(m+1,m+1) = q^{m} S_q(m,m),
S_q(m,s) = 0 \quad \text{for} \quad s > m.
\]

(26)

Lemma 9. For all \(m, n \in \mathbb{N}\) the identity

\[
\frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s = 1,
\]

(27)

holds.

Proof. It follows from the end points interpolation property of \(U_{n,q}(e_m; z)\) and \(B_{n,q}(e_s; z)\). Indeed

\[
1 = U_{n,q}(e_m; 1) = \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s B_{n,q}(e_s; 1)
= \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s B_{n,q}(e_s; z). \quad \square
\]

Lemma 9 implies that for all \(m, n \in \mathbb{N}\) and \(|z| \leq r\) we have

\[
|U_{n,q}(e_m; z)| \leq \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s |B_{n,q}(e_s; z)| \leq \frac{[n-1]_q}{[n+m-1]_q} \sum_{s=1}^{m} S_q(m,s) [n]_q^s r^m \leq r^m.
\]

(29)

For our purpose first we need a recurrence formula for \(U_{n,q}(e_m; z)\).
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Lemma 10. For all \( m, n \in \mathbb{N} \cup \{0\} \) and \( z \in \mathbb{C} \) one has

\[
U_{n,q}(e_m; z) = \frac{q^n z (1 - z)}{[n + m]_q} D_q U_{n,q}(e_m; z) + \frac{q^n [n]_q z + [m]_q}{[n + m]_q} U_{n,q}(e_m; z).
\] (30)

Proof. By simple calculation we obtain (see [27])

\[
z(1 - z) D_q (p_{n,k}(q; z)) = ([k]_q - [n]_q z) p_{n,k}(q; z),
\]

\[
U_{n,q}(e_m; z) = z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m},
\] (31)

\[
I_{k,m} := \frac{[k + m - 1]_q \cdots [k]_q [n + m - 1]_q \cdots [n]_q}{[n + m - 1]_q \cdots [n]_q}.
\]

It follows that

\[
z(1 - z) D_q U_{n,q}(e_m; z)
\]

\[
= [n]_q z(1 - z) z^{n-1} + \sum_{k=1}^{n-1} ([k]_q - [n]_q z) p_{n,k}(q; z) I_{k,m}
\]

\[
= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) I_{k,m} - [n]_q z^{n+1}
\]

\[
= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) I_{k,m} - [n]_q z^{n+1}
\]

\[
= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) (q^n [k]_q + [m]_q - [m]_q) I_{k,m}
\]

\[
- z [n]_q U_{n,q}(e_m; z)
\]

\[
= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) (q^n [k]_q + [m]_q - [m]_q) I_{k,m}
\]

\[
- z [n]_q U_{n,q}(e_m; z)
\]

\[
= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) (q^n [k]_q + [m]_q - [m]_q) I_{k,m}
\]

\[
- z [n]_q U_{n,q}(e_m; z)
\]

\[
= q^n [n + 1]_q U_{n,q}(e_m+1; z) - q^n [m]_q U_{n,q}(e_m; z)
\]

\[
- z [n]_q U_{n,q}(e_m; z),
\] (32)

which implies the recurrence in the statement. \(\square\)

Let

\[
\Theta_{n,m}(q; z) := U_{n,q}(e_m; z) - z^m - \frac{1}{[n + 1]_q}
\]

\[
x \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q \right) z^{m-1} (1 - z).
\] (33)

Using the recurrence formula (30) we prove two more recurrence formulas.

Lemma 11. For all \( m, n \in \mathbb{N} \) and \( z \in \mathbb{C} \) one has

\[
U_{n,q}(e_m; z) - z^m
\]

\[
= \frac{q^{m-1} z (1 - z)}{[n + m - 1]_q} D_q U_{n,q}(e_{m-1}; z)
\]

\[
+ \frac{q^{m-1} [n]_q z + [m - 1]_q}{[n + m - 1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1})
\]

\[
+ \frac{[m - 1]_q}{[n + m - 1]_q} (1 - z) z^{m-1},
\] (34)

\[
\Theta_{n,m}(q; z)
\]

\[
= \frac{q^{m-1} z (1 - z)}{[n + m - 1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1})
\]

\[
+ \frac{q^{m-1} [n]_q z + [m - 1]_q}{[n + m - 1]_q} \Theta_{n,m-1}(q; z) + R_{n,m}(q; z),
\] (35)

where

\[
R_{n,m}(q; z)
\]

\[
= \frac{[m - 1]_q}{[n + m - 1]_q[n + 1]_q}
\]

\[
\times \left( 1 + q^{m-1} + \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q \right) (z + 1) \right)\]

\[
\times z^{m-2} (1 - z).
\] (36)
Proof. From the recurrence formula in Lemma 10, for all \( m \geq 2 \), we get

\[
U_{n,q}(e_m; z) - z^m
= \frac{q^{m-1}z(1-z)}{[n + m - 1]_q} D_q U_{n,q}(e_{m-1}; z)
+ \frac{q^{m-1}[n]z + [m - 1]_q}{[n + m - 1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1})
+ \frac{q^{m-1}[n]z + [m - 1]_q}{[n + m - 1]_q} z^{m-1} - z^m
= \frac{q^{m-1}z(1-z)}{[n + m - 1]_q} D_q U_{n,q}(e_{m-1}; z)
+ \frac{q^{m-1}[n]z + [m - 1]_q}{[n + m - 1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1})
+ \frac{[m - 1]_q}{[n + m - 1]_q} (1-z) z^{m-1},
\]

\[
U_{n,q}(e_m; z) - z^m
= \frac{1}{[n + 1]_q} \left( \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right) z^{m-1} (1-z)
= \frac{q^{m-1}z(1-z)}{[n + m - 1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1})
+ \frac{q^{m-1}[n]z + [m - 1]_q}{[n + m - 1]_q}
\times \left( U_{n,q}(e_{m-1}; z) - z^{m-1} \right)
\times \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right) z^{m-2} (1-z)
+ R_{n,m}(q;z),
\]

(37)

where

\[
R_{n,m}(q;z)
= \frac{[m - 1]_q}{[n + m - 1]_q} (1-z) z^{m-1}
- \frac{1}{[n + 1]_q} \left( \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right) z^{m-1} (1-z)
+ \frac{q^{m-1}[n]_q}{[n + m - 1]_q} (1-z) z^{m-1}
\]

Again by simple calculation we obtain

\[
T_{n,m}(q)
= \frac{[m - 1]_q}{[n + m - 1]_q} - \frac{1}{[n + 1]_q} \left( \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right)
+ \frac{q^{m-1}[n]_q}{[n + m - 1]_q} \frac{1}{[n + m - 1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right)
\] 
\begin{align*}
&= \left( \frac{[m - 1]_q}{[n + m - 1]_q} - \frac{1}{[n + 1]_q} \left( \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right) \right) \frac{q^{m-1}[n]_q}{[n + m - 1]_q} \\
&\quad \times \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right) \\
&\quad - \frac{q^{m-1}[n]_q}{[n + m - 1]_q} \frac{1}{[n + 1]_q} \left( q[m - 1]_q + [m - 1]_q^{-1} \right) \\
&\quad \times \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right) \\
&\quad \times \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right) \left( q[m - 1]_q + [m - 1]_q^{-1} \right)
\times \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right)
\end{align*}
\]

(38)

\[
T_{n,m}^1(q) + T_{n,m}^2(q),
\]

(39)

where \( T_{n,m}^1(q) \) and \( T_{n,m}^2(q) \) can be simplified as follows:

\[
T_{n,m}^1(q) = \left( 1 - \frac{q^{m-1}[n]_q}{[n + m - 1]_q} \right) \frac{1}{[n + 1]_q} \\
\times \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right)
\] 
\begin{align*}
&= \frac{[m - 1]_q}{[n + m - 1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right) \\
&\quad \times \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q^{-1} \right)
\end{align*}


\[ T_{n,m}^1(q) = \frac{[m-1]_q}{[n + m - 1]_q} + \frac{q^{m-1}[m-1]_q}{[n + m - 1]_q} \]

\[ = \frac{q^{m-1}[n]_q - 1}{[n + m - 1]_q[n + 1]_q} \times (q[m-1]_q + [m-1]_q^{-1}) \]

\[ = [m-1]_q \left( \frac{1}{[n + m - 1]_q} - \frac{q^{m-1}[n]_q}{[n + m - 1]_q[n + 1]_q} \right) \]

\[ = [m-1]_q \left( 1 + q^{m-1} \right)[n + [m-1]_q[n + 1]_q] \]

\[ = [m-1]_q \left( \frac{1 + q^{m-1}}{[n + m - 1]_q[n + 1]_q} \right) \]

\[ = \frac{[m-1]_q}{[n + m - 1]_q[n + 1]_q}. \tag{40} \]

**Lemma 12.** Let \( q > 1 \) and \( f \in H(D_R) \). The function \( L_q(f;z) \) has the following representation:

\[ L_q(f;z) = \sum_{m=2}^{\infty} a_m \left( q \sum_{i=0}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right) z^{m-1} (1 - z), \quad z \in D_R. \tag{41} \]

**Proof.** Using the following identity:

\[ [m]_q - m \]

\[ = 1 + q + q^2 + \cdots + q^{m-1} - m \]

\[ = (1 - 1) + (q - 1) + (q^2 - 1) + \cdots + (q^{m-1} - 1) \]

\[ = (q-1) [1]_q + (q-1) [2]_q + \cdots + (q-1) [m-1]_q \]

\[ = (q-1) ([1]_q + \cdots + [m-1]_q) = (q-1) \sum_{i=1}^{m-1} [i]_q. \tag{42} \]

we get

\[ L_q(f;z) \]

\[ = \sum_{m=2}^{\infty} a_m \left( q \left( \frac{[m]_q - [m]_q^{-1}}{q-1} \right) z^{m-1} (1 - z) \right) \]

\[ = \sum_{m=2}^{\infty} a_m \left( q \left( \frac{[m]_q - m}{q-1} + \frac{[m]_q - m}{q^{m-1} - 1} \right) z^{m-1} (1 - z) \right) \]

\[ = \sum_{m=2}^{\infty} a_m \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_q^{-1} \right) z^{m-1} (1 - z), \tag{43} \]

where \( f(z) = \sum_{m=0}^{\infty} a_m z^m \).

**4. Proofs of the Main Results**

Firstly we prove that \( U_{n,q}(f;z) = \sum_{m=0}^{\infty} a_m U_{n,q}(e_m,z) \). Indeed denoting \( f_k(z) = \sum_{j=0}^{k} a_j z^j \), \( |z| \leq r \) with \( m \in \mathbb{N} \), by the linearity of \( U_{n,q} \), we have

\[ U_{n,q}(f_k,z) = \sum_{m=0}^{\infty} a_m U_{n,q}(e_m,z), \tag{44} \]

and it is sufficient to show that, for any fixed \( n \in \mathbb{N} \) and \( |z| \leq r \) with \( r \geq 1 \), we have \( \lim_{k \to \infty} U_{n,q}(f_k,z) = U_{n,q}(f;z) \). But this is immediate from \( \lim_{k \to \infty} \|f_k - f\|_r = 0 \), the norm being defined as \( \|f\|_r = \max \{|f(z)| : |z| \leq r\} \), and from the inequality

\[ \left| U_{n,q}(f_k,z) - U_{n,q}(f,z) \right| \]

\[ \leq |f_k(0) - f(0)| \cdot (1 - z)^n + |f_k(1) - f(1)| \cdot |z^n| \]

\[ + [n + 1]_q^{-1} \sum_{i=1}^{n-1} |p_{n,i}(q;z)| q^{i-1} \]

\[ \times \int_0^1 p_{n-2,i-1}(q^{-1} t) \left| f_k(t) - f(t) \right| d_q^{-1} t \]

\[ \leq C_{r,n} \|f_k - f\|_r, \quad \text{valid for all } |z| \leq r, \]

where

\[ C_{r,n} = (1 + r)^n + r^n + [n + 1]_q^{-1} \]

\[ \times \sum_{j=1}^{n} \frac{[j]_q}{[n]_q} (1 + r)^{n-j} r^{j-1} \]

\[ \times \int_0^1 p_{n-2,j-1}(q^{-1} t) d_q^{-1} t \]

\[ = (1 + r)^n + r^n \]

\[ + \sum_{j=1}^{n-1} \frac{[j]_q}{[n]_q} (1 + q^{n-j} r)^{n-j} r^{j-1} q^{-j}. \tag{46} \]
Therefore we get
\[ |U_{n,q}(f; z) - f(z)| \leq \sum_{m=0}^{\infty} |a_m| |U_{n,q}(e_m; z) - e_m(z)| = \sum_{m=2}^\infty |a_m| \] (47)
\[ \times |U_{n,q}(e_m; z) - e_m(z)|, \]
as \( U_{n,q}(e_0; z) = e_0(z) \) and \( U_{n,q}(e_1; z) = e_1(z) \).

**Proof of Theorem 2.** From the recurrence formula (34) and the inequality (29) for \( m \geq 2 \) we get
\[ |U_{n,q}(e_m; z) - z^m| \leq \sum_{m=0}^{\infty} |a_m| \left| U_{n,q}(e_m; z) - z^m \right| \]
\[ \times \left| U_{n,q}(e_m; z) - z^m \right|^2, \] (48)
\[ \leq r (1 + r) \sum_{m=2}^\infty |a_m| m (m - 1) q^{m-2} r^{m-2}. \] (52)
It follows that
\[ |U_{n,q}(f; z) - f(z)| \leq \sum_{m=2}^\infty |a_m| |U_{n,q}(e_m; z) - z^m| \]
\[ \leq r (1 + r) \sum_{m=2}^\infty |a_m| m (m - 1) q^{m-2} r^{m-2}. \] (53)

The second main result of the paper is the Voronovskaja-type theorem with a quantitative estimate for the complex version of genuine \( q \)-Bernstein-Durrmeyer polynomials.

**Proof of Theorem 3.** By Lemma 11 we have
\[ \Theta_{n,m}(q; z) = \frac{q^{m-1}}{[m + 1]_q} \frac{1 - z}{q} \left| D_q U_{n,q}(e_{m-1}; z) - z^{m-1} \right| \]
\[ + \frac{q^{m-1}}{[m + 1]_q} \left| [m - 1]_q (1 + r) r^{m-1} \right| \]
\[ \leq \frac{q^{m-1}}{[m + 1]_q} \frac{1 - z}{q} \left| D_q U_{n,q}(e_{m-1}; z) - z^{m-1} \right| \]
\[ + \frac{q^{m-1}}{[m + 1]_q} \left| [m - 1]_q (1 + r) r^{m-1} \right| \]
\[ \leq 2q (m - 1) r (1 + r) (qr)^{m-2} \]
\[ \times (1 + q^{m-1}) + \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_q \right) (z + 1) \]
\[ \times z^{m-2} (1 - z). \] (55)
It follows that
\[
|R_{n,m}(q;z)| \\
\leq \frac{|m-1|_q}{|n+1|_q^2} \\
\times \left((1 + q^{m-1}) + (q (m-1) + (m-2)^2)\right) \\
\times (1 + r)^2 r^{m-2} \\
\leq \frac{|m-1|_q}{|n+1|_q^2} \\
\times \left((1 + q^{m-1}) + (q (m-2) [m-2]_q + (m-2)^2)\right) \\
\times (1 + r)^2 r^{m-2} \\
= \frac{q^{m-2}[m-1]_q}{|n+1|_q} \times (\frac{1}{q^{m-2} + q} + (m-2) [m-2]_q^{m-1} + \frac{1}{q^{m-2}}(m-2)^2) \\
\times (1 + r)^2 r^{m-2} \\
\leq \frac{3}{|n+1|_q^2} (m-1) (m-2)^2 (1 + r)^2 (q^2 r)^{m-2}
\]

for all $m \geq 2$, $n \in \mathbb{N}$, and $z \in \mathbb{C}$. Equation (54) implies that for $|z| \leq r$

\[
|\Theta_{n,m}(q;z)| \\
\leq r |\Theta_{n,m-1}(q;z)| + \frac{q^{m-1}r (1 + r) m - 1}{|n+1|_q^2} \times \left((1 + q^{m-1}) + (q (m-2) [m-2]_q + (m-2)^2)\right) \\
\times (1 + r)^2 r^{m-2} \\
\leq r |\Theta_{n,m-1}(q;z)| + \frac{q^{m-1}r (1 + r) m - 1}{|n+1|_q^2} \\
\times (m-1)^2 (m-2)^2 (q^2 r)^{m-2} \\
+ \frac{3}{|n+1|_q^2} (m-1) (m-2)^2 (1 + r)^2 (q^2 r)^{m-2} \\
\leq r |\Theta_{n,m-1}(q;z)| + \frac{4r^2(1 + r)^2}{|n+1|_q^2} \\
\times (m-1)^2 (m-2)^2 (q^2 r)^{m-2} \\
\leq r |\Theta_{n,m-1}(q;z)| + \frac{4r^2(1 + r)^2}{|n+1|_q^2} \\
\times (m-1)^2 (m-2)^2 (q^2 r)^{m-2}
\]

By writing the last inequality for $m = 3, 4, \ldots$, we easily obtain, step by step, the following:

\[
\left|U_{n,q}(f;z) - f(z) - \frac{1}{|n+1|_q} L_q(f;z)\right| \\
\leq 4r^2(1 + r)^2 \sum_{m=2}^\infty |a_m|(q^2 r)^{m-2} \\
\times \sum_{j=2}^m (j - 1)^2 (j - 2) \leq 4r^2(1 + r)^2 \sum_{m=2}^\infty |a_m|(m-1)^2(m-2)^2(q^2 r)^{m-2}.
\]

**Proof of Theorem 4.** For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$ we get

\[
U_{n,q}(f;z) - f(z) = \frac{1}{|n+1|_q} \left( L_q(f;z) + |n+1|_q \right) \\
\times \left( U_{n,q}(f;z) - f(z) - \frac{1}{|n+1|_q} L_q(f;z) \right).
\]

It follows that

\[
\left\| U_{n,q}(f) - f \right\|_r \\
\geq \frac{1}{|n+1|_q} \left( \left\| L_q(f;z) \right\|_r - |n+1|_q \right) \\
\times \left\| U_{n,q}(f) - f - \frac{1}{|n+1|_q} L_q(f;z) \right\|.
\]

Because by hypothesis $f$ is not a polynomial of degree less than 1 in $\mathbb{D}_r$, it follows $\|L_q(f;z)\|_r > 0$. Indeed, assuming the contrary it follows that $L_q(f;z) = 0$ for all $z \in \overline{\mathbb{D}}$; that is, $D_qf(z) = D_q\cdot f(z)$ for all $z \in \overline{\mathbb{D}}$. Thus $a_m = 0$, $m = 2, 3, \ldots$ and $f$ is linear, which is a contradiction with the hypothesis.

Now, by Theorem 3, we have

\[
|n+1|_q \left\| U_{n,q}(f;z) - f(z) - \frac{1}{|n+1|_q} L_q(f;z) \right\| \\
\leq \frac{4r^2(1 + r)^2}{|n+1|_q} \sum_{m=2}^\infty |a_m|(m-1)^2(m-2)^2(q^2 r)^{m-2} \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Consequently, there exists $n_1$ (depending only on $f$ and $r$) such that for all $n \geq n_1$ we have

\[
\left\| L_q(f;z) \right\|_r - |n+1|_q \left\| U_{n,q}(f) - f - \frac{1}{|n+1|_q} L_q(f;z) \right\|_r \\
\geq \frac{1}{2} \left\| L_q(f;z) \right\|_r.
\]
which implies that
\[ \| U_{n,q}(f) - f \|_r \geq \frac{1}{2^{n+1}q} \| L_q(f;z) \|_r, \] \forall n \geq n_1. \quad (63)

For 1 \leq n \leq n_1 - 1 we have
\[ \| U_{n,q}(f) - f \|_r \geq \frac{1}{n+1} \| U_{n,q}(f) - f \|_r \]
\[ = \frac{1}{n+1} M_{r,n,q}(f) > 0, \] \quad (64)
which finally implies that
\[ \| U_{n,q}(f) - f \|_r \geq \frac{1}{n+1} C_{r,q}(f), \] \quad (65)
for all \( n \), with \( C_{r,q}(f) = \min\{M_{r,1,q}(f), \ldots, M_{r,n-1,q}(f), \}
\((1/2)\|L_q(f;z)\|_r\}\), which ends the proof.

**Proof of Theorem 6.** Let \( 1 \leq r < R, 1 < q_0 < R/r \) be fixed. Then, by Lemma 12 for any \( 1 \leq q \leq q_0 \) and \( |z| \leq r \), we have
\[ L_q(f;z) = \sum_{m=2}^{\infty} a_m \left( q \sum_{i=1}^{m-1} |[i]_q + \sum_{i=1}^{m-1} |[i]_q^{-1} \right) z^{m-1}(1 - z), \]
\[ L_1(f;z) = \sum_{m=2}^{\infty} a_m m (m - 1) z^{m-1}(1 - z). \] \quad (66)

Using the inequality
\[ \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m - 1)}{2} \right| \]
\[ = q \sum_{i=2}^{m-1} ([i]_q - i) + (q - 1) \frac{m(m - 1)}{2} \]
\[ = q (q - 1) \sum_{i=2}^{m-1} [i]_q + (q - 1) \frac{m(m - 1)}{2} \]
\[ \leq q (q - 1) \frac{m(m - 1)}{2} \left( [m-1]_q + 1 \right) \]
\[ = (q - 1) \frac{m(m - 1)}{2} \left( q[m-1]_q + 1 \right) \]
\[ \leq (q - 1) q^{m-1} m^2 (m - 1) \]
\[ \left| \sum_{i=1}^{m-1} [i]_q^{-1} - \frac{m(m - 1)}{2} \right| \]
\[ = \sum_{i=2}^{m-1} \left( i - [i]_q^{-1} \right) \]
\[ = (1 - q^{-1}) \sum_{i=2}^{m-1} [i]_q^{-1} \]
\[ \leq (1 - q^{-1}) \frac{m(m - 1)^2}{2}, \] \quad (67)
we get, for \( 1 \leq q \leq q_0 \) and \( |z| \leq r \),
\[ \left| L_q(f;z) - L_1(f;z) \right| \]
\[ \leq \sum_{m=2}^{N-1} a_m \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m - 1)}{2} \right| z^{m-1} - z^m \]
\[ + \sum_{m=N}^{\infty} a_m \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m - 1)}{2} \right| z^{m-1} - z^m \]
\[ + \sum_{m=2}^{N-1} a_m \left| \sum_{i=1}^{m-1} [i]_q^{-1} - \frac{m(m - 1)}{2} \right| z^{m-1} - z^m \] \quad (68)
\[ \leq (q - 1) \sum_{m=2}^{N-1} a_m m^2 (m - 1) q_0^{m-1} r^m \]
\[ + 4 \sum_{m=N}^{\infty} a_m (m - 1)^2 q_0^m r^m \]
\[ + (1 - q^{-1}) \sum_{m=2}^{N-1} a_m (m - 1)^2 r^m \]
\[ \leq (1 - q^{-1}) \sum_{m=2}^{N-1} a_m (m - 1)^2 r^m \]
\[ + 2 \sum_{m=N}^{\infty} a_m |m(m - 1)| r^m. \]
Since \( f \in H(\mathbb{D}_R) \), we can find that \( N = N_\varepsilon \in \mathbb{N} \) such that
\[ 4 \sum_{m=N}^{\infty} a_m |m(m - 1)|^2 q_0^m r^m + 2 \sum_{m=N}^{\infty} a_m |m(m - 1)| r^m < \frac{\varepsilon}{2}. \] \quad (69)
Thus, for \( q \) sufficiently close to \( 1 \) from the right, we conclude that
\[ \lim_{q \rightarrow 1^+} L_q(f;z) = L_1(f;z) \] \quad (70)
uniformly on \( \mathbb{D}_r \). The proof is finished. \[ \square \]

**Proof of Theorem 5.** Then, by Theorem 3, we get \( L_q(f;z) = \lim_{n \rightarrow \infty} [n + 1]_q(U_{n,q}(f;z) - f(z)) = 0 \) for infinite number of points having an accumulation point on \( \mathbb{D}_{R/q^2} \). Since \( L_q(f;z) \in H(\mathbb{D}_{R/q^2}) \), by the unicity Theorem for analytic functions, we get \( L_q(f;z) = 0 \) in \( \mathbb{D}_{R/q^2} \), and, therefore, by (11), \( a_m = 0, m = 2, 3, \ldots \). Thus, \( f \) is linear. Theorem 5 is proved. \[ \square \]

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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