EVALUATION OF FOUR CONVOLUTION SUMS AND REPRESENTATION OF INTEGERS BY CERTAIN QUADRATIC FORMS IN TWELVE VARIABLES

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Abstract. In this paper the convolution sums \( \sum_{6i+j=n} \sigma(l)\sigma_3(m) \), \( \sum_{2i+3j=n} \sigma(l)\sigma_3(m) \), \( \sum_{i+6j=n} \sigma(l)\sigma_3(m) \) and \( \sum_{3i+2j=n} \sigma(l)\sigma_3(m) \) are evaluated for all \( n \in \mathbb{N} \), and then their evaluations are used to determine the representation number formulae \( N(1,1,1,1,1,2;n) \), \( N(1,1,1,1,2,2;n) \) and \( N(1,1,1,2,2,2;n) \) where \( N(a_1,\ldots,a_6;n) \) denote the representation numbers of \( n \) by the form \( a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2) \).

Keywords: convolution sum; divisor function; Eisenstein series; quadratic forms; representation numbers.

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1. INTRODUCTION

Let \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{R} \), and \( \mathbb{C} \) denote the set of positive integers, integers, real numbers and complex numbers respectively and let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Throughout this paper \( q \in \mathbb{C} \) is taken to satisfy \( |q| < 1 \). For \( k,n \in \mathbb{N} \) we set

\[
\sigma_k(n) = \sum_{d|n} d^k,
\]
where $d$ runs through the positive divisors of $n$. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. For $a, b, r, s, n \in \mathbb{N}$, we define the convolution sum $W_{a,b}^{r,s}(n)$ by

$$W_{a,b}^{r,s}(n) := \sum_{(l,m) \in \mathbb{N}^2} \sigma_r(l) \sigma_s(m).$$

Ramanujan [1] evaluated $W_{1,1}^{1,3}(n)$ explicitly. He proved that

$$W_{1,1}^{1,3}(n) = \frac{7}{80} \sigma_5(n) + \frac{1}{2} \frac{3n}{24} \sigma_3(n) - \frac{1}{240} \sigma(n).$$

The following two sums are given by Huard, Ou, Spearman and Williams [2].

$$W_{1,2}^{1,3}(n) = \frac{1}{240} \left( \sigma_5(n) + 20 \sigma_5\left(\frac{n}{2}\right) + (10 - 30n) \sigma_3\left(\frac{n}{2}\right) - \sigma(n) \right)$$

and

$$W_{2,1}^{1,3}(n) = \frac{1}{240} \left( 5 \sigma_5(n) + 16 \sigma_5\left(\frac{n}{2}\right) + (10 - 15n) \sigma_3(n) - \sigma\left(\frac{n}{2}\right) \right).$$

The convolution sums $W_{1,4}^{1,3}(n)$, $W_{4,1}^{1,3}(n)$ have been evaluated by Cheng and Williams [3]. A linear equation for $W_{1,3}^{1,3}(n)$ and $W_{3,1}^{1,3}(n)$ was given before by Huard, Ou, Spearman and Williams [2] without individual determination. In a recent publication, Yao and Xia [4] completed the determination of $W_{1,3}^{1,3}(n)$ and $W_{3,1}^{1,3}(n)$. They proved that

$$W_{1,3}^{1,3}(n) = \frac{1}{1040} \sigma_5(n) + \frac{9}{104} \sigma_5\left(\frac{n}{3}\right) + \frac{1}{24} \frac{3n}{2} \sigma_3\left(\frac{n}{3}\right) - \frac{1}{240} \sigma(n) + \frac{1}{312} a(n)$$

and

$$W_{3,1}^{1,3}(n) = \frac{1}{104} \sigma_5(n) + \frac{81}{1040} \sigma_5\left(\frac{n}{3}\right) + \frac{1}{24} \frac{n}{2} \sigma_3(n) - \frac{1}{240} \sigma\left(\frac{n}{3}\right) - \frac{1}{104} a(n),$$

where $a(n)$ is defined by

$$\sum_{n=1}^{\infty} a(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6.$$

Convolution sum $W_{a,b}^{r,s}(n)$ involving the divisor function $\sigma(n)$ has been evaluated for certain values $a, b, r, s$ (see, for example, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]).
In this paper, motivated from the work of Yao and Xia[4] and using the \((p,k)\)-parametrization of Eisenstein series and theta functions given by Alaca, Alaca and Williams we determine the convolution sums \(W_{6,1}^{1,3}(n), W_{2,3}^{1,3}(n), W_{1,6}^{1,3}(n)\) and \(W_{3,2}^{1,3}(n)\). We’ve used 4 Eisenstein series \(N(q)\), \(N(q^2)\), \(N(q^3)\), \(N(q^6)\) and 3 eta products which form a basis of the Modular space \(M_6(\Gamma_0(6))\) (with dimension 7) and express the products \((6L(q^6) - L(q))M(q)\), \((3L(q^3) - 2L(q^2))M(q^3)\), \((2L(q^2) - L(q))M(q^6)\) and \((3L(q^2) - 2L(q^2))M(q^2)\) of Eisenstein Series (all of which clearly belongs to space) as a linear combination the mentioned Eisenstein series and eta products. One of the eta products which is used here appeared in literature[4]. Today, some other authors use the theory of quasimodular forms with softwares Magma or the database LMFDP of \(L\) functions and modular forms to obtain formulae for convolution sums.

As an application we use our evaluations to determine formulae for \(N(1,1,1,1,1,2;n)\), \(N(1,1,1,1,2,2;n)\) and \(N(1,1,1,2,2,2;n)\) where \(N(a_1,\ldots,a_6;n)\) denote the representation number of \(n\) by the form:

\[
\begin{align*}
(a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2),
\end{align*}
\]

that is

\[
N(a_1,\ldots,a_6;n) := \text{card}\left\{(x_1,\ldots,x_{12}) \in \mathbb{Z}^{12} : n = a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2)\right\}
\]

Formulae for \(N(1,1,1,1,1,1;n)\) is given recently by Yao and Xia [4]. Our main results are as follows.

**Theorem 1.** For \(n \in \mathbb{N}\), we have

\[
W_{6,1}^{1,3}(n) = \frac{5}{2184}\sigma_5(n) + \frac{2}{273}\sigma_5\left(\frac{n}{2}\right) + \frac{27}{1456}\sigma_5\left(\frac{n}{3}\right) + \frac{27}{455}\sigma_5\left(\frac{n}{6}\right) + \frac{2-n}{48}\sigma_3(n) - \frac{1}{240}\sigma\left(\frac{n}{6}\right) - \frac{1}{4368}u_1(n),
\]

(9)
\[ W_{2,3}^{1,3}(n) = \frac{1}{4368} \sigma_5(n) + \frac{1}{1365} \sigma_5\left(\frac{n}{2}\right) + \frac{15}{728} \sigma_5\left(\frac{n}{3}\right) + \frac{6}{91} \sigma_5\left(\frac{n}{6}\right) + \frac{2 - 3n}{48} \sigma_3\left(\frac{n}{3}\right) - \frac{1}{240} \sigma\left(\frac{n}{2}\right) + \frac{1}{8736} u_2(n), \]

\[ W_{1,6}^{1,3}(n) = \frac{1}{21840} \sigma_5(n) + \frac{1}{1092} \sigma_5\left(\frac{n}{2}\right) + \frac{3}{728} \sigma_5\left(\frac{n}{3}\right) + \frac{15}{182} \sigma_5\left(\frac{n}{6}\right) + \frac{1 - 3n}{24} \sigma_3\left(\frac{n}{6}\right) - \frac{1}{240} \sigma\left(\frac{n}{2}\right) + \frac{1}{156} a\left(\frac{n}{2}\right) + \frac{1}{4368} u_3(n), \]

and

\[ W_{3,2}^{1,3}(n) = \frac{1}{2184} \sigma_5(n) + \frac{5}{546} \sigma_5\left(\frac{n}{2}\right) + \frac{27}{7280} \sigma_5\left(\frac{n}{3}\right) + \frac{27}{364} \sigma_5\left(\frac{n}{6}\right) + \frac{1 - n}{24} \sigma_3\left(\frac{n}{2}\right) - \frac{1}{240} \sigma\left(\frac{n}{3}\right) - \frac{1}{8736} u_4(n), \]

where \( u_1(n), u_2(n), u_3(n) \) and \( u_4(n) \) are respectively defined by

\[ \sum_{n=1}^{\infty} u_1(n) q^n = -4q \prod_{n=1}^{\infty} (1 - q^n)^5(1 - q^{2n})^5(1 - q^{3n})(1 - q^{6n}) + 105q \prod_{n=1}^{\infty} (1 - q^n)^6(1 - q^{3n})^6 + 972q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5(1 - q^{6n})^5, \]

\[ \sum_{n=1}^{\infty} u_2(n) q^n = -72q \prod_{n=1}^{\infty} (1 - q^n)^5(1 - q^{2n})^5(1 - q^{3n})(1 - q^{6n}) + 70q \prod_{n=1}^{\infty} (1 - q^n)^6(1 - q^{3n})^6 + 24q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5(1 - q^{6n})^5, \]
Theorem 2. For $n \in \mathbb{N}$,

(i) \[ N(1, 1, 1, 1, 1, 2; n) = \frac{66}{7} \sigma_5(n) - \frac{192}{13} \sigma_5\left(\frac{n}{2}\right) + \frac{1782}{7} \sigma_5\left(\frac{n}{3}\right) - \frac{5184}{7} \sigma_5\left(\frac{n}{6}\right) \]
\[ + \frac{432}{13} a\left(\frac{n}{2}\right) + \frac{108}{91} u_3(n) - \frac{18}{91} u_4(n), \]

(ii) \[ N(1, 1, 1, 1, 2, 2; n) = \frac{60}{13} \sigma_5(n) + \frac{192}{13} \sigma_5\left(\frac{n}{2}\right) - \frac{1620}{13} \sigma_5\left(\frac{n}{3}\right) - \frac{5184}{13} \sigma_5\left(\frac{n}{6}\right) \]
\[ + \frac{18}{91} u_1(n) + \frac{27}{91} u_2(n), \]

and

(iii) \[ N(1, 1, 1, 2, 2; n) = \frac{18}{7} \sigma_5(n) - \frac{144}{7} \sigma_5\left(\frac{n}{2}\right) + \frac{486}{7} \sigma_5\left(\frac{n}{3}\right) - \frac{3888}{7} \sigma_5\left(\frac{n}{6}\right) \]
\[ + \frac{324}{13} a\left(\frac{n}{2}\right) + \frac{81}{91} u_3(n) - \frac{27}{91} u_4(n). \]
2. Proof of Theorem 1

In his second notebook [18] Ramanujan gives the definitions of Eisenstein series \(L(q), M(q)\) and \(N(q)\) by

\[
L(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},
\]

\[
M(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}
\]

and

\[
N(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}.
\]

It can be easily seen that

\[
L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma (n) q^n,
\]

\[
M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3 (n) q^n
\]

and

\[
N(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5 (n) q^n.
\]

The Jacobi theta function \(\varphi(q)\) is defined by

\[
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2},
\]

see for example ([20], p.92).

Alaca, Alaca, and Williams [19] defined \(p\) and \(k\) respectively by

\[
p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}
\]

and

\[
k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}.
\]
The \((p-k)\)-parametrization of \(2L(q^2) - L(q)\), \(6L(q^6) - L(q)\) and \(3L(q^3) - 2L(q^2)\) are given by the equations (3.84), (3.89) and (3.90) in [9] respectively as follows:

\[
(29) \quad 2L(q^2) - L(q) = (1 + 14p + 24p^2 + 14p^3 + p^4)k^2,
\]

\[
(30) \quad 6L(q^6) - L(q) = (5 + 22p + 36p^2 + 22p^3 + 5p^4)k^2
\]

and

\[
(31) \quad 3L(q^3) - 2L(q^2) = (1 + 2p + 12p^2 + 2p^3 + p^4)k^2.
\]

Formulae for the series \(M(q), M(q^2), M(q^3), M(q^6), N(q), N(q^2), N(q^3), N(q^6)\) in terms of \(p\) and \(k\) are determined by Alaca, Alaca, and Williams [5]. Equations (3.14)-(3.16), (3.18), (3.22)-(3.24) and (3.26) are as follows

\[
M(q) = (1 + 124p + 964p^2 + 2788p^3 + 3910p^4
\]
\[
+2788p^5 + 964p^6 + 124p^7 + p^8)k^4,
\]

\[
M(q^2) = (1 + 4p + 64p^2 + 178p^3 + 235p^4
\]
\[
+178p^5 + 64p^6 + 4p^7 + p^8)k^4,
\]

\[
M(q^3) = (1 + 4p + 4p^2 + 28p^3 + 70p^4
\]
\[
+28p^5 + 4p^6 + 4p^7 + p^8)k^4,
\]

\[
M(q^6) = (1 + 4p + 4p^2 - 2p^3 - 5p^4
\]
\[
-2p^5 + 4p^6 + 4p^7 + p^8)k^4,
\]

\[
N(q) = (1 - 246p - 5532p^2 - 38614p^3 - 135369p^4
\]
\[
-276084p^5 - 348024p^6 - 276084p^7 - 135369p^8
\]
\[
-38614p^9 - 5532p^{10} - 246p^{11} + p^{12})k^6,
\]

(36)
Using (40), (41), (42) and (43) we obtain

\[
N(q^2) = (1 + 6p - 114p^2 - 625p^3 - \frac{4059}{2}p^4
- 4302p^5 - 5556p^6 - 4302p^7 - \frac{4059}{2}p^8
- 625p^9 - 114p^{10} + 6p^{11} + p^{12})k^6,
\]

(37)

\[
N(q^3) = (1 + 6p + 12p^2 - 58p^3 - 297p^4 - 396p^5 - 264p^6
- 396p^7 - 297p^8 - 58p^9 + 12p^{10} + 6p^{11} + p^{12})k^6,
\]

(38)

and

\[
N(q^6) = (1 + 6p + 12p^2 + 5p^3 - \frac{27}{2}p^4 - 18p^5 - 12p^6
- 18p^7 - \frac{27}{2}p^8 + 5p^9 + 12p^{10} + 6p^{11} + p^{12})k^6.
\]

(39)

In that study, Alaca, Alaca and Williams also derived formulae for \(\prod_{n=1}^{\infty} (1 - q^n)\) and \(\prod_{n=1}^{\infty} (1 - q^{2n})\) in terms of \(p\) and \(k\). Equations (3.28)-(3.30) and (3.32) in [5] are

\[
\prod_{n=1}^{\infty} (1 - q^n) = q^{-\frac{1}{2}} \left(2^{-\frac{1}{2}} p^{\frac{1}{2}} (1 - p)^{\frac{1}{2}} (1 + p)^{\frac{1}{2}} (2 + p)^{\frac{1}{2}} k^{\frac{1}{2}}\right),
\]

(40)

\[
\prod_{n=1}^{\infty} (1 - q^{2n}) = q^{-\frac{1}{4}} 2^{-\frac{3}{4}} p^{\frac{1}{4}} (1 - p)^{\frac{1}{4}} (1 + p)^{\frac{1}{4}} (2 + p)^{\frac{1}{4}} k^{\frac{1}{4}},
\]

(41)

\[
\prod_{n=1}^{\infty} (1 - q^{3n}) = q^{-\frac{1}{6}} 2^{-\frac{1}{3}} p^{\frac{1}{3}} (1 - p)^{\frac{1}{3}} (1 + p)^{\frac{1}{3}} (2 + p)^{\frac{1}{3}} k^{\frac{1}{3}},
\]

(42)

and

\[
\prod_{n=1}^{\infty} (1 - q^{6n}) = q^{-\frac{1}{12}} 2^{-\frac{1}{6}} p^{\frac{1}{6}} (1 - p)^{\frac{1}{6}} (1 + p)^{\frac{1}{6}} (2 + p)^{\frac{1}{6}} k^{\frac{1}{6}}.
\]

(43)

Using (40), (41), (42) and (43) we obtain

\[
q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n}) (1 - q^{6n}) = \frac{1}{8} p(1 - p)^4(1 + p)^2(1 + 2p)^2(2 + p)^2k^6,
\]

(44)

\[
q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6 = \frac{1}{4} p(1 - p)^4(1 + p)^4(1 + 2p)(2 + p)k^6,
\]

(45)
Appealing to (13) and (25), we obtain

\[ q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5 = \frac{1}{8} p^2 (1 - p)^2 (1 + p)^4 (1 + 2p)(2 + p) k^6. \]

From (8) and (45) it is clear that the eta products in equations are same.

**Proof.** (i) From (30), (32), (36)-(39), (44), (45), and (46) we deduce that

\[
\left( 6L(q^6) - L(q) \right) M(q) = \frac{-537}{637} N(q) + \frac{320}{637} N(q^2) + \frac{810}{637} N(q^3) + \frac{2592}{637} N(q^6)
- \frac{2880}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n}) (1 - q^{6n})
+ \frac{10800}{13} q^2 \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6
+ \frac{699840}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5.
\]

By (23) and (24), we see that

\[
\left( 6L(q^6) - L(q) \right) M(q)
= (5 - 144 \sum_{n=1}^{\infty} \sigma(n) q^{6n} + 24 \sum_{n=1}^{\infty} \sigma(n) q^n)(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n)
= 5 - 144 \sum_{n=1}^{\infty} \sigma(n) q^{6n} + 24 \sum_{n=1}^{\infty} \sigma(n) q^n + 1200 \sum_{n=1}^{\infty} \sigma_3(n) q^n
- 34560 \sum_{n=1}^{\infty} \sigma(n) q^{6n} \sum_{n=1}^{\infty} \sigma_3(n) q^n
+ 5760 \sum_{n=1}^{\infty} \sigma(n) q^n \sum_{n=1}^{\infty} \sigma_3(n) q^n.
\]

Appealing to (13) and (25), we obtain

\[
\frac{-537}{637} N(q) + \frac{320}{637} N(q^2) + \frac{810}{637} N(q^3) + \frac{2592}{637} N(q^6)
- \frac{2880}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n}) (1 - q^{6n})
+ \frac{10800}{13} q^2 \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6
+ \frac{699840}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5
= \frac{-537}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) + \frac{320}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right).
\]
$$\sum_{n=1}^{\infty} \sigma(n) q^{6n} \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$= -\frac{179}{14560} \sum_{n=1}^{\infty} \sigma_5(n) q^n + \frac{2}{273} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} + \frac{27}{1456} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n}$$

$$+ \frac{27}{455} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} + \frac{5}{144} \sum_{n=1}^{\infty} \sigma_3(n) q^n + \frac{1}{1440} \sum_{n=1}^{\infty} \sigma(n) q^n$$

$$- \frac{1}{240} \sum_{n=1}^{\infty} \sigma(n) q^{6n} + \frac{1}{6} \sum_{n=1}^{\infty} \sigma(n) q^n \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$- \frac{1}{4368} \sum_{n=1}^{\infty} u_1(n) q^n. \quad (50)$$

For $n \in \mathbb{N}$, equating the coefficients of $q^n$ on both sides of (50) and using (2) we have

$$W_{6,1}^{1,3}(n) = -\frac{179}{14560} \sigma_5(n) + \frac{2}{273} \sigma_5\left(\frac{n}{2}\right) + \frac{27}{1456} \sigma_5\left(\frac{n}{3}\right) + \frac{27}{455} \sigma_5\left(\frac{n}{6}\right) + \frac{5}{144} \sigma_3(n)$$

$$+ \frac{1}{1440} \sigma(n) - \frac{1}{240} \sigma\left(\frac{n}{6}\right) + \frac{1}{6} W_{1,1}^{1,3}(n) - \frac{1}{4368} u_1(n). \quad (51)$$

Identity (9) follows from (3) and (51).

(ii) From (31), (34), (36)-(39), (44), (45), and (46) we obtain
(3L(q^3) - 2L(q^2)) M(q^3) = -\frac{10}{1911} N(q) - \frac{32}{1911} N(q^2) + \frac{1611}{637} N(q^3) - \frac{960}{637} N(q^6)
- \frac{8640}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n})(1 - q^{6n})
+ \frac{1200}{13} q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6
+ \frac{2880}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5.

(52)

From (23) and (24) we have

(3L(q^3) - 2L(q^2)) M(q^3)
= (1 - 72 \sum_{n=1}^{\infty} \sigma(n) q^{3n} + 48 \sum_{n=1}^{\infty} \sigma(n) q^{2n})(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{3n})
= 1 - 72 \sum_{n=1}^{\infty} \sigma(n) q^{3n} + 48 \sum_{n=1}^{\infty} \sigma(n) q^{2n} + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{3n}
- 17280 \sum_{n=1}^{\infty} \sigma(n) q^{3n} \sum_{n=1}^{\infty} \sigma_3(n) q^{3n}
+ 11520 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \sum_{n=1}^{\infty} \sigma_3(n) q^{3n}

(53)

It follows from (14) and (25) that

-\frac{10}{1911} N(q) - \frac{32}{1911} N(q^2) + \frac{1611}{637} N(q^3) - \frac{960}{637} N(q^6)
- \frac{8640}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n})(1 - q^{6n})
+ \frac{1200}{13} q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6
+ \frac{2880}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5
= -\frac{10}{1911} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) - \frac{32}{1911} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right)
\[
12 \frac{B}{\theta} \text{üleNT KÖLÜCE}
\]
\[
+ \frac{1611}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} \right) - \frac{960}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} \right)
\]
\[
+ \frac{120}{91} \sum_{n=1}^{\infty} u_2(n) q^n
\]
\[
= 1 + \frac{240}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^n + \frac{768}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}
\]
\[
- \frac{115992}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} + \frac{69120}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} + \frac{120}{91} \sum_{n=1}^{\infty} u_2(n) q^n.
\]
(54)

Combining (52), (53) and (54) we have

\[
\sum_{n=1}^{\infty} \sigma(n) q^{2n} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n}
\]
\[
= \frac{1}{4368} \sum_{n=1}^{\infty} \sigma_5(n) q^n + \frac{1}{1365} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} - \frac{1611}{14560} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n}
\]
\[
+ \frac{6}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} - \frac{1}{48} \sum_{n=1}^{\infty} \sigma_3(n) q^{3n} - \frac{1}{240} \sum_{n=1}^{\infty} \sigma(n) q^{2n}
\]
\[
+ \frac{1}{160} \sum_{n=1}^{\infty} \sigma(n) q^{3n} + \frac{3}{2} \sum_{n=1}^{\infty} \sigma(n) q^{3n} \sum_{n=1}^{\infty} \sigma_3(n) q^{3n}
\]
\[
+ \frac{1}{8736} \sum_{n=1}^{\infty} u_2(n) q^n.
\]
(55)

For \( n \in \mathbb{N} \), equating the coefficients of \( q^n \) on both sides of (55) and using (2) we have

\[
\]
\[
W^{1,3}_{2,3}(n) = \frac{1}{4368} \sigma_5(n) + \frac{1}{1365} \sigma_5 \left( \frac{n}{2} \right) - \frac{1611}{14560} \sigma_5 \left( \frac{n}{3} \right) + \frac{6}{91} \sigma_5 \left( \frac{n}{6} \right) - \frac{1}{48} \sigma_3 \left( \frac{n}{3} \right)
\]
\[
- \frac{1}{240} \sigma \left( \frac{n}{2} \right) + \frac{1}{160} \sigma \left( \frac{n}{3} \right) + \frac{3}{2} W^{1,3}_{2,1} \left( \frac{n}{3} \right) + \frac{1}{8736} u_2(n).
\]
(56)

Identity (10) follows from (3) and (56).

(iii) From (29), (35)-(39), (44), (45), and (46) we deduce that
\[(2L(q^2) - L(q)) M(q^6) = \frac{-1}{1911} N(q) + \frac{22}{1911} N(q^2) - \frac{30}{637} N(q^3) + \frac{660}{637} N(q^6) \]
\[+ \frac{480}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n})(1 - q^{6n}) \]
\[+ \frac{240}{13} q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6 \]
\[+ \frac{14400}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5. \]

(57)

By (23) and (24), we see that

\[(2L(q^2) - L(q)) M(q^6) \]
\[= (1 - 48 \sum_{n=1}^{\infty} \sigma(n) q^{2n} + 24 \sum_{n=1}^{\infty} \sigma(n) q^n (1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{6n}) \]
\[= 1 - 48 \sum_{n=1}^{\infty} \sigma(n) q^{2n} + 24 \sum_{n=1}^{\infty} \sigma(n) q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{6n} \]
\[= -11520 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \sum_{n=1}^{\infty} \sigma_3(n) q^{6n} + 5760 \sum_{n=1}^{\infty} \sigma(n) q^n \sum_{n=1}^{\infty} \sigma_3(n) q^{6n}. \]

(58)

Appealing to (15) and (25), we obtain

\[-\frac{1}{1911} N(q) + \frac{22}{1911} N(q^2) - \frac{30}{637} N(q^3) + \frac{660}{637} N(q^6) \]
\[+ \frac{480}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n})(1 - q^{6n}) \]
\[+ \frac{240}{13} q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6 \]
\[+ \frac{14400}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 - q^{3n})^5 (1 - q^{6n})^5 \]
\[= -\frac{1}{1911} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) + \frac{22}{1911} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) \]
\[- \frac{30}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} \right) + \frac{660}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} \right). \]
\[\begin{align*}
+ \frac{120}{91} \sum_{n=1}^{\infty} u_3(n) q^n.
\end{align*}\]

\[\begin{align*}
&= 1 + \frac{24}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^n - \frac{528}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \\
&\quad + \frac{2160}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} - \frac{47520}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} + \frac{120}{91} \sum_{n=1}^{\infty} u_3(n) q^n.
\end{align*}\]

Combining (57), (58) and (59) we have

\[\begin{align*}
\sum_{n=1}^{\infty} \sigma(n) q^n \sum_{n=1}^{\infty} \sigma_3(n) q^{6n} \\
&= \frac{1}{21840} \sum_{n=1}^{\infty} \sigma_5(n) q^n - \frac{11}{10920} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} + \frac{3}{728} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} \\
&\quad - \frac{33}{364} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} - \frac{1}{24} \sum_{n=1}^{\infty} \sigma_3(n) q^{6n} - \frac{1}{240} \sum_{n=1}^{\infty} \sigma(n) q^n \\
&\quad + \frac{1}{120} \sum_{n=1}^{\infty} \sigma(n) q^{2n} + 2 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \sum_{n=1}^{\infty} \sigma_3(n) q^{6n} \\
&\quad + \frac{1}{4368} \sum_{n=1}^{\infty} u_3(n) q^n.
\end{align*}\]

(60)

For \(n \in \mathbb{N}\), equating the coefficients of \(q^n\) on both sides of (60) and using (2) we have

\[\begin{align*}
W_{1,3}^{1,3}(n) &= \frac{1}{21840} \sigma_5(n) - \frac{11}{10920} \sigma_5\left(\frac{n}{2}\right) + \frac{3}{728} \sigma_5\left(\frac{n}{3}\right) - \frac{33}{364} \sigma_5\left(\frac{n}{6}\right) - \frac{1}{24} \sigma_3\left(\frac{n}{6}\right) \\
&\quad - \frac{1}{240} \sigma(n) + \frac{1}{120} \sigma\left(\frac{n}{2}\right) + 2W_{1,3}^{1,3}\left(\frac{n}{2}\right) + \frac{1}{4368} u_3(n).
\end{align*}\]

(61)

Identity (11) follows from (6) and (61).

(iv) From (31), (33), (36)-(39), (44), (45), and (46) we deduce that

\[\begin{align*}
(3L(q^2) - 2L(q^3)) M(q^3) &= \frac{10}{637} N(q) - \frac{1074}{637} N(q^2) + \frac{81}{637} N(q^3) + \frac{1620}{637} N(q^6) \\
&\quad - \frac{18180}{91} q \prod_{n=1}^{\infty} (1-q^n)^5 (1-q^{2n})^5 (1-q^{3n})(1-q^{6n}).
\end{align*}\]
By (23) and (24), we see that

\[
(3L(q^3) - 2L(q^2)) M(q^2)
\]

\[
= 1 - 72 \sum_{n=1}^{\infty} \sigma(n) q^{3n} + 48 \sum_{n=1}^{\infty} \sigma(n) q^{2n} (1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n})
\]

\[
= 1 - 72 \sum_{n=1}^{\infty} \sigma(n) q^{3n} + 48 \sum_{n=1}^{\infty} \sigma(n) q^{2n} + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}
\]

Appealing to (16) and (25), we obtain

\[
\frac{10}{637} N(q) - \frac{1074}{637} N(q^2) + \frac{81}{637} N(q^3) + \frac{1620}{637} N(q^6)
\]

\[-\frac{18180}{91} q \prod_{n=1}^{\infty} (1 - q^n)^5 (1 - q^{2n})^5 (1 - q^{3n})(1 - q^{6n})
\]

\[+ \frac{2700}{13} q \prod_{n=1}^{\infty} (1 - q^n)^6 (1 - q^{3n})^6
\]

\[-\frac{4860}{91} q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n}) (1 - q^{3n})^5 (1 - q^{6n})^5
\]

\[= \frac{10}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) - \frac{1074}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right)
\]

\[+ \frac{81}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} \right) + \frac{1620}{637} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} \right)
\]

\[+ \frac{180}{91} \sum_{n=1}^{\infty} \mu_4(n) q^n.
\]

\[= 1 - \frac{720}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^n + \frac{77328}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}
\]

\[-\frac{5832}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} - \frac{116640}{91} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} + \frac{180}{91} \sum_{n=1}^{\infty} \mu_4(n) q^n.
\]

Combining (62), (63) and (64) we have
\[
\sum_{n=1}^{\infty} \sigma(n) q^{6n} \sum_{n=1}^{\infty} \sigma_3(n) q^n = \frac{1}{2184} \sum_{n=1}^{\infty} \sigma_5(n) q^n - \frac{179}{3640} \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} + \frac{27}{7280} \sum_{n=1}^{\infty} \sigma_5(n) q^{3n} \\
+ \frac{27}{364} \sum_{n=1}^{\infty} \sigma_5(n) q^{6n} + \frac{1}{72} \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} + \frac{1}{360} \sum_{n=1}^{\infty} \sigma(n) q^{2n} \\
- \frac{1}{240} \sum_{n=1}^{\infty} \sigma(n) q^{3n} + \frac{2}{3} \sum_{n=1}^{\infty} \sigma(n) q^{2n} \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \\
- \frac{1}{8736} \sum_{n=1}^{\infty} u_4(n) q^n.
\]

(65)

For \( n \in \mathbb{N} \), equating the coefficients of \( q^n \) on both sides of (65) and using (2) we have

\[
W_{3,2}^{1,3}(n) = \frac{1}{2184} \sigma_5(n) - \frac{179}{3640} \sigma_5 \left( \frac{n}{2} \right) + \frac{27}{7280} \sigma_5 \left( \frac{n}{3} \right) + \frac{27}{364} \sigma_5 \left( \frac{n}{6} \right) + \frac{1}{72} \sigma_3 \left( \frac{n}{2} \right) \\
+ \frac{1}{360} \sigma \left( \frac{n}{2} \right) - \frac{1}{240} \sigma \left( \frac{n}{3} \right) + \frac{2}{3} W_{1,1}^{1,3} \left( \frac{n}{2} \right) - \frac{1}{8736} u_4(n).
\]

(66)

Identity (12) follows from (3) and (66).

\[\square\]

3. PROOF OF THEOREM 2

For \( l \in \mathbb{N}_0 \) we set,

\[
r_1(l) = \text{card } \{ (x_1, \ldots, x_4) \in \mathbb{Z}^4 : l = x_1^2 + x_1 x_2 + x_2^2 + x_3 x_4 + x_4^2 \},
\]

(67)

and let \( r_2(l), r_3(l) \) and \( r_4(l) \) be respectively \( N(1, 1, 1, 1) \), \( N(1, 1, 1, 2) \) and \( N(1, 2, 2, 2) \) where

\[
N(a_1, a_2, a_3, a_4; n) = \text{card } \left\{ (x_1, \ldots, x_8) \in \mathbb{Z}^8 : l = a_1(x_1^2 + x_1 x_2 + x_2^2) + a_2(x_3^2 + x_3 x_4 + x_4^2) + a_3(x_5^2 + x_5 x_6 + x_6^2) + a_4(x_7^2 + x_7 x_8 + x_8^2) \right\}
\]

(68)

Obviously \( r_1(0) = 1 \) for any \( i \in \{1, 2, 3, 4\} \). It is known (see for example [2]) that

\[
r_1(l) = 12 \sigma(l) - 36 \sigma \left( \frac{l}{3} \right), \quad l \in \mathbb{N}.
\]

(69)

Lomadze [21] proved that
The following two formulae was proved by Köklüce [22].

\[ r_2(l) = 24\sigma_3(l) + 216\sigma_3\left(\frac{l}{3}\right), \ l \in \mathbb{N}. \]

And

\[ r_3(l) = 18\sigma_3(l) - 48\sigma_3\left(\frac{l}{2}\right) - 162\sigma_3\left(\frac{l}{3}\right) + 432\sigma_3\left(\frac{l}{6}\right), \ l \in \mathbb{N}. \]

Proof. We just prove (ii) in detail. The remaining can be proved in a similar way.

It is clear that

\[ N(1, 1, 1, 1, 2, 2; n) = \sum_{l, m \in \mathbb{N}_0 \atop 2l + m = n} r_1(l)r_2(m) = r_1(0)r_2(n) + r_1(n)r_2(0) + \sum_{l, m \in \mathbb{N} \atop 2l + m = n} r_1(l)r_2(m). \]

Thus using (69) and (70) we have

\[
N(1, 1, 1, 1, 2, 2; n) - (24\sigma_3(n) + 216\sigma_3\left(\frac{n}{2}\right) + 12\sigma\left(\frac{n}{2}\right) - 36\sigma\left(\frac{n}{6}\right)) \\
= \sum_{l, m \in \mathbb{N} \atop 2l + m = n} (12\sigma(l) - 36\sigma\left(\frac{l}{3}\right))(24\sigma_3(m) + 216\sigma_3\left(\frac{m}{3}\right)) \\
= 288 \sum_{l, m \in \mathbb{N} \atop 2l + m = n} \sigma(l)\sigma_3(m) + 2592 \sum_{l, m \in \mathbb{N} \atop 2l + m = n} \sigma(l)\sigma_3\left(\frac{m}{3}\right) - 864 \sum_{l, m \in \mathbb{N} \atop 2l + m = n} \sigma\left(\frac{l}{3}\right)\sigma_3(m) \\
- 7776 \sum_{l, m \in \mathbb{N} \atop 2l + m = n} \sigma\left(\frac{l}{3}\right)\sigma_3\left(\frac{m}{3}\right) \\
= 288W_{2,1}^{1,3}(n) + 2592W_{2,3}^{1,3}(n) - 864W_{6,1}^{1,3}(n) - 7776W_{2,1}^{1,3}\left(\frac{n}{3}\right) \]

Appealing to (5), (9) and (10) and adding \(24\sigma_3(n) + 216\sigma_3\left(\frac{n}{2}\right) + 12\sigma\left(\frac{n}{2}\right) - 36\sigma\left(\frac{n}{6}\right)\) to both sides of the equation we obtain the desired result.

For (i) use \(N(1, 1, 1, 1, 1, 2; n) = \sum_{l, m \in \mathbb{N}_0 \atop l + m = n} r_1(l)r_3(m)\) and
for (iii) use \( N(1,1,1,2,2;n) = \sum_{l,m \in \mathbb{N}_0, l+2m=n} r_1(l)r_4(m) \).

Denoting the right hand side of (18) by \( S(1,1,1,1,2,2;n) \), we give the first ten values of \( N(1,1,1,1,2,2;n) \) and \( S(1,1,1,1,2,2;n) \) in Table 1 to illustrate the equations.

Table 1 The first ten values of \( S(1,1,1,1,2,2;n) \) and \( N(1,1,1,1,2,2;n) \)

| \( n \) | \( S(1,1,1,1,2,2;n) \) | \( \sigma_5(n) \) | \( u_1(n) \) | \( u_2(n) \) | \( N(1,1,1,1,2,2;n) \) |
|---|---|---|---|---|---|
| 1 | 24 | 1 | 101 | -2 | 24 |
| 2 | 228 | 33 | 362 | -36 | 228 |
| 3 | 1176 | 244 | -27 | 606 | 1176 |
| 4 | 4380 | 1057 | 1660 | -2216 | 4380 |
| 5 | 14544 | 3126 | -3138 | 2484 | 14544 |
| 6 | 36804 | 8052 | -486 | 2172 | 36804 |
| 7 | 76800 | 16808 | 7192 | -7408 | 76800 |
| 8 | 175692 | 33825 | 13352 | 4464 | 175692 |
| 9 | 244824 | 59293 | 8181 | -162 | 244824 |
| 10 | 522648 | 103158 | -12804 | 9768 | 522648 |

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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