Estimates of gradient of $\mathcal{L}$-harmonic functions for nonlocal operators with order $\alpha > 1$.

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Abstract

We obtain Grönwall type estimates for the gradient of the harmonic functions for a Lévy operator with order strictly larger than 1 and minimal assumptions of its Lévy measure.

1 Introduction

We consider a Lévy operator on $\mathbb{R}^d$, $d \in \mathbb{N}$,

$$\mathcal{L} f(x) = \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - 1_{|z| < 1} (z \cdot \nabla f(x)) \right) \nu(z) \, dz, \quad f \in C^2_b(\mathbb{R}^d),$$

(1.1)

where $\nu(z)dz$ is a Lévy measure, i.e. $\nu(z) \geq 0$ and $\int (1 \wedge |z|^2) \nu(z) \, dz < \infty$. We assume that $\int_{\mathbb{R}} \nu(z) \, dz = \infty$, $\nu$ is symmetric and comparable with radial and nonincreasing function. That is, there exist a nonincreasing function $g : (0, \infty) \rightarrow [0, \infty)$ and a constant $c > 0$ such that

$$c^{-1} g(|x|) \leq \nu(x) \leq cg(|x|), \quad x \in \mathbb{R}^d \setminus \{0\}.$$ (1.2)

Following [22], we define

$$h(r) = \int_{\mathbb{R}^d} \left( 1 \wedge \frac{|x|^2}{r^2} \right) g(|x|) \, dx, \quad r > 0.$$ 

We focus on the operator with order strictly larger than 1 in the sense of the lower weak scaling conditions for its Fourier multiplier

$$\psi(x) = \int_{\mathbb{R}^d} (1 - \cos(x \cdot z)) \nu(dz), \quad x \in \mathbb{R}^d.$$ 

One of the primary example of a mentioned class of operators is the fractional Laplacian $\Delta^{\alpha/2}$, for $\alpha > 1$.

We note that there exists a nonincreasing function $g^*$ such that $g(r) \leq g^*(r)$ and $g^*(r+1) \approx g^*(r)$, $r > r_0$ for some $r_0 > 2$. We introduce this function because we will need the majorant of

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\( \nu \), which does not decrease very fast. Generally the function \( g \) may not satisfy this property, e.g. \( g(r) = e^{-\gamma r^{d-\alpha}} \), with \( \alpha \in (0, 2) \). Let
\[
Y = \{ u \in L^1_{\text{loc}}(\mathbb{R}^d) : u \in L^1(\mathbb{R}^d, 1 \wedge g^*(|x|)dx) \}
\]
and denote \( \|u\|_Y = \|u\|_{L^1(\mathbb{R}^d, 1 \wedge g^*(|x|)dx)} \).

**Definition 1.** We say that \( u \in Y \) is \( \mathcal{L} \)-harmonic in \( D \) if

\[
(u, \mathcal{L}\varphi) = 0, \quad \varphi \in C_c^\infty(D).
\]

We note that if \( g(r) = 0 \) for \( r > r_0 \), for \( \alpha \) sufficiently small, \( \mathcal{L} \)-harmonic functions may not be differentiable (see e.g. [21, Example 7.5]) and do not satisfy Harnack inequality (see e.g. [14, Example 5.5]). In the first example \( \alpha < 1/2 \), while in the second one \( \alpha = 0 \). We say that a function \( f \) satisfies the doubling condition if there is a constant \( c \) such that

\[
c^{-1}f(2r) \leq f(r) \leq cf(2r), \quad r \in (0, 1].
\]

Our main result is the following theorem

**Theorem 1.1.** Let \( d \in \mathbb{N}, \alpha \in (1, 2) \) and \( D \subset \mathbb{R}^d \) be an open set. Let \( \psi \in \text{WLSC}(\alpha, \theta, c_\alpha) \), where \( \theta = 0 \) for \( d = 1 \) and \( \theta = 1 \) for \( d \geq 2 \). Furthermore, we assume that \( g \) satisfies doubling condition \( (1.3) \). Let \( f \in Y \) be a non-negative function, \( \mathcal{L} \)-harmonic in \( D \). There exists a constant \( C = C(d, \psi) \) such that

\[
|\nabla_x f(x)| \leq C \frac{f(x)}{\delta^D_x \wedge 1}, \quad x \in D,
\]

where \( \delta^D_x \) is the distance of \( x \) to the boundary of \( D \). For the definitions of \( \text{WLSC}(\alpha, \theta, c) \) we refer to the Section 2, Definition 2.

**Remark 1.** The condition \( \psi \in \text{WLSC}(\alpha, \theta, c_\alpha) \) in Theorem 1.1 means that \( h(r)/r^\alpha \) is almost decreasing on \((0, 1/\theta)\), i.e. \( h(\lambda r) \geq c_\alpha \lambda^\alpha h(r) \) for \( \lambda \leq 1 \) and \( 0 < r < 1/\theta \) (see (2.2) and (2.3)).

**Remark 2.** If \( \psi \in \text{WLSC}(\alpha, 0, c_\alpha) \), then \((1.3)\) may be strengthen to \( |\nabla_x f(x)| \leq C f(x)/\delta^D_x \).

In fact, in Theorem 3.3 we show \((1.4)\) for the larger class of functions satisfying the mean value property (abbreviated MVP, see Definition 4). In Appendix we present some relations between \( \mathcal{L} \)-harmonic and MVP functions, in particular for \( f \in Y \) both definitions are equivalent.

The assertion of Theorem 3.3 is known for harmonic functions with respect to several types of operators and under various assumptions, see e.g. [11, 8, 20, 11]. In the paper [21], Kulczycki and Ryznar proved \((1.4)\) under certain additional assumptions on the density of the Lévy measure. They assumed that \( \nu(x) \) is positive and radial, with an absolutely continuous and decreasing profile \( g \) such that \( -g'(r)/r \) is nonincreasing. Furthermore \( g \) had the doubling property \((1.3)\) and \( g(r) \approx g(r + 1) \) for large \( r \). These assumptions are satisfied, e.g., for a large class of subordinated Brownian motions. In our paper \( \nu \) is only comparable with the decreasing function which is bounded by \( g^* \) possessing similar properties to the function \( g \) from [21].

A very natural approach to prove estimates \((1.4)\) is to use the formula \((2.17)\) and differentiate the Poisson kernel \( P_{B(0,r)}(x,z) \) under the integral. If the exact form of the Poisson kernel is known, \((1.4)\) follows easily from the estimates of \( \nabla_x P_{B(0,r)}(x,z) \) (see [5]). However, it is rather a rare situation and usually we do not know the formula for the Poisson kernel. Another method uses a construction of the difference process developed in [21]. With the help of this process the authors get the estimates of the Green function of the ball in the form \((1.5)\). By using
these estimates together with the Harnack inequality they finally prove (1.4). We note that the unimodality of the Lévy measure is crucial in this approach.

One of the tools used both in our approach and in [21] are the estimates of the gradient of the fundamental solution $p_t(x)$ of the operator $L$. They are needed to get the proper bounds for the potential kernel of $L$. If $L = -\varphi(-\Delta)$, where $\varphi$ is a Bernstein function (i.e. $L$ is a generator of the subordinated Brownian motion) such estimates are given by the transference property expressing $\nabla p_t(x)$ in terms of the transition density of the process in dimension $d + 2$ (see e.g. [7]). This transference property was generalized in [21] to the large class of Lévy process and was used in the construction of the mentioned difference process. We refer also to [19] and [25] for other developments in this direction. In our paper, we use the estimates of $\nabla p_t(x)$ from [17] proven for pure jump Lévy process with $\nu$ satisfying (1.2) and $\psi \in WLSC(\alpha, \theta, c)$.

We emphasize that in our approach we do not need the estimates of the Green function near the boundary of the domain as in [21]. The main technical tool we use in the proof of Theorem 3.4 are the $L^1$-type estimates of the gradient of the Green function of the ball obtained in Lemma 3.1. In order to get Lemma 3.1 we only need the estimates of the gradient of the fundamental solution (see the proof of Lemma 2.1), which yield some preliminary estimates of $G_{B(0,r)}(x)$ and $\nabla G_{B(0,r)}(x)$ given in Lemmas 2.6 and 2.7.

Let us note that in [21] Corollaries 1.2 and 1.3 stated below were important tools in the proof of (1.4), while in our approach they are simple consequences of Theorem 3.4. Namely, by the Grönwall lemma and Theorem 3.4 we get the scale invariant Harnack inequality.

**Corollary 1.2.** Let $d \in \mathbb{N}$, $\alpha \in (1, 2)$ and $D \subset \mathbb{R}^d$ be an open set. Let $\psi \in WLSC(\alpha, \theta, c_\alpha)$, where $\theta = 0$ for $d = 1$ and $\theta = 1$ for $d \geq 2$. Furthermore, we assume that $g$ satisfies doubling condition (1.3). There exists a constant $C$ such that for any $x_0 \in \mathbb{R}^d$, $r \in (0, 1]$, and any function $f$ nonnegative on $\mathbb{R}^d$ and satisfying MVP in a ball $B(x_0, r)$,

$$\sup_{x \in B(x_0, r/2)} f(x) \leq C \inf_{x \in B(x_0, r/2)} f(x).$$

Furthermore, since the Green function $G_D(\cdot, y)$ satisfies MVP property inside $D \setminus \{y\}$, Theorem 3.4 yields the following estimates for the gradient of the Green function.

**Corollary 1.3.** Suppose the assumptions of Corollary 1.2 holds. If $\int_{\mathbb{R}^d} \frac{1}{\psi(\xi)} \text{d}\xi = \infty$, we additionally assume that $D$ is bounded. Then, there exists a constant $C$ such that

$$|\nabla_x G_D(x, y)| \leq C \frac{G_D(x, y)}{\delta_D \wedge |x - y| \wedge 1}, \quad x, y \in D. \quad (1.5)$$

We note that if $\int_{\mathbb{R}^d} \frac{1}{\psi(\xi)} \text{d}\xi = \infty$ and $D$ is unbounded, the Green function may not exist.

The paper is organized as follows. In Section 2, we provide the necessary definitions and prove auxiliary results on the Green function. In Section 3 we state and prove Theorem 3.4. In Appendix we show some relations between functions satisfying MVP property and $L$-harmonic functions, which yield Theorem 1.1.
functions in \( \mathbb{R}^d \). As usual, we write \( a \land b = \min(a, b) \) and \( a \lor b = \max(a, b) \). For \( r > 0 \), we let \( B(x, r) = \{ y \in \mathbb{R}^d : |x - y| < r \} \). We denote \( B_r = B(0, r) \). For the arbitrary set \( A \subset \mathbb{R} \), the distance to the boundary of \( A \) will be denoted by

\[
\delta^A_x = \text{dist}(x, \partial A).
\]

To simplify the notation, while referring to the set \( D \), we will omit the superscript, i.e.,

\[
\delta_x = \delta^D_x = \text{dist}(x, \partial D).
\]

When we write \( f(x) \approx g(x) \), we mean that there is a number \( 0 < C < \infty \) independent of \( x \), i.e., a constant, such that for every \( x \) we have \( C^{-1}f(x) \leq g(x) \leq Cf(x) \). The notation \( C = C(a, b, \ldots, c) \) means that \( C \) is a constant which depends only on \( a, b, \ldots, c \). We use a convention that constants denoted by capital letters do not change throughout the paper. For a radial function \( f : \mathbb{R}^d \to [0, \infty) \) we shall often write \( f(r) = f(x) \) for any \( x \in \mathbb{R}^d \) with \( |x| = r \).

### 2.2 Fundamental solution for \( L \)

We define

\[
K(r) = \frac{1}{r^2} \int_{|x|<r} |x|^2 g(|x|) \, dx, \quad r > 0.
\]

Clearly \( K(r) \leq h(r) \). Let us notice that

\[
h(\lambda r) \leq h(r) \leq \lambda^2 h(\lambda r), \quad \lambda > 1.
\]

We define the function \( V \) as follows,

\[
V(0) = 0 \quad \text{and} \quad V(r) = 1/\sqrt{h(r)}, \quad r > 0.
\]

Since \( h(r) \) is non-increasing to 0, \( V \) is non-decreasing and unbounded. We have

\[
V(r) \leq V(\lambda r) \leq \lambda V(r), \quad r \geq 0, \lambda > 1.
\]

Let \( \psi^*(r) = \sup_{|x| \leq r} \psi(x) \). Since \( g \) in (1.2) is nonincreasing, by \([\text{III} \text{ Lemma 1 and (6)}]\), for certain \( C_1 \)

\[
2^{-1} \psi(\xi) \leq 2^{-1} \psi^*(|\xi|) \leq h(1/|\xi|) \leq C_1 \psi^*(|\xi|) \leq \pi^2 C_1 \psi(\xi), \quad \xi \in \mathbb{R}^d.
\]

**Definition 2.** Let \( \theta \in [0, \infty) \) and \( \phi \) be a non-negative non-zero function on \( (0, \infty) \). We say that \( \phi \) satisfies the **weak lower scaling condition** (at infinity) if there are numbers \( \alpha > 0 \) and \( c \in (0, 1] \) such that

\[
\phi(\lambda r) \geq c \lambda^\alpha \phi(r) \quad \text{for} \quad \lambda \geq 1, \quad r > \theta.
\]

In short, we say that \( \phi \) satisfies WLS\( C(\alpha, \theta, c) \) and write \( \phi \in \text{WLSC}(\alpha, \theta, c) \). If \( \phi \in \text{WLSC}(\alpha, 0, c) \), then we say that \( \phi \) satisfies the **global weak lower scaling condition**.

Notice that if \( \psi^* \in \text{WLSC}(\beta, \theta, C) \) there is a constant \( c_1 = c_1(C, C_1) \) such that

\[
\frac{V(\eta r)}{V(r)} \leq c_1 \eta^{\beta/2}
\]
for $r < 1/\theta$ and $\eta < 1$. Moreover, if $\theta = 0$

$$\frac{V(\lambda r)}{V(r)} \geq c_1^{-1}\lambda^{\beta/2}, \quad r > 0, \lambda > 1.$$  \hfill (2.5)

By similar argumentation as in [1] Remark 4 in regard to (2.4), we have for $r < V(1/\theta)$ and $\eta < 1$

$$c_2\eta^{2/\alpha_1} \leq \frac{V^{-1}(\eta r)}{V^{-1}(r)}$$  \hfill (2.6)

where $c_3 = c_3(\alpha_1, c_1, C_1)$.

Remark 3. We note that the range of $r$ in (2.4) and (2.6) may be increased to any interval $(0, M)$ in the expense of the constants $c_1$ and $c_2$. The dependence of constants on $\alpha, c_\alpha$ and $M$ will be denoted by $\sigma$, i.e. $\sigma = \sigma(\alpha, c_\alpha, M)$

If $\psi^* \in WLSC(\alpha, 1, c)$ with $\alpha > 0$ the operator $\mathcal{L}$ possesses the heat kernel $p$, where

$$p(t, x, y) = p_t(y - x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} \cos((x - y) \cdot \xi) d\xi, \quad x, y \in \mathbb{R}^d.$$  \hfill (2.7)

Furthermore $p_t$ is smooth. We use the next lemma to show existence of the fundamental solution for $\mathcal{L}$.

Lemma 2.1. Assume that $\psi^* \in WLSC(\alpha, 1, c)$ with $\alpha > 1/2$. Then there exists a constant $C_2 > 0$ such that, for any $x \in \mathbb{R}^d$,

$$\int_0^\infty |\nabla_x p_t(x)| dt \leq C_2 \frac{|x| V^2(|x| \wedge 1)}{(|x| \wedge 1)^{d+2}}.$$  

Proof. By [17] Theorem 5.2] we have, for $t \leq V^2(1)$ and $x \in \mathbb{R}^d$,

$$|\nabla_x p_t(x)| \leq c_1 \frac{t}{V^{-1}(\sqrt{t})} \frac{K(|x|)}{|x|^d}.$$  

Hence, for $T = V^2(|x| \wedge 1)$,

$$\int_0^T |\nabla_x p_t(x)| dt \leq \int_0^T c_1 \frac{t}{V^{-1}(\sqrt{t})} \frac{K(|x|)}{|x|^d} dt \leq c_2 \frac{K(|x|)}{|x|^d} \frac{T^{1/\alpha}}{V^{-1}(\sqrt{T})} \int_0^T t^{1-1/\alpha} dt = c_3 \frac{K(|x|)}{|x|^d} \frac{T^2}{V^{-1}(\sqrt{T})} \leq c_3 \frac{V^2(|x| \wedge 1)}{|x|^d (|x| \wedge 1)}.$$  \hfill (2.8)

where in the last line we used $K(r) \leq h(r)$. Let us observe that (2.7) implies

$$|\nabla_x p_t(x)| \leq |x|(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} |\xi|^2 d\xi, \quad x \in \mathbb{R}^d.$$
Since \( V(s) \leq V(|x|)(1 + s/|x|) \), for \( s \geq 0 \),

\[
\int_0^\infty |\nabla_x p_t(x)| \, dt \leq |x|(2\pi)^{-d} \int_\mathbb{R}^d e^{-\psi(\xi)} |\xi|^2 \xi \, d\xi \, dt
\]

\[
|x|(2\pi)^{-d} \int_\mathbb{R}^d e^{-\psi(\xi)} |\xi|^2 \psi(\xi) \, d\xi \leq c_4 |x| \int_\mathbb{R}^d e^{-\psi(\xi)} |\xi|^2 V^2(1/|\xi|) \, d\xi
\]

\[
\leq c_4 |x| V^2(|x| \land 1) \int_\mathbb{R}^d e^{-\psi(\xi)} (|\xi|^2 + (|x| \land 1)^{-2}) \, d\xi
\]

\[
\leq c_3 |x| V^2(|x| \land 1) \left( \frac{1}{|V^{-1}(\sqrt{T})|^d + 2} + \frac{1}{(|x| \land 1)^2 |V^{-1}(\sqrt{T})|^d} \right)
\]

\[
= 2c_3 |x| V^2(|x| \land 1) \frac{1}{|x| \land 1}^{d+2},
\]

where the last inequality is a consequence of [16, Proposition 3.6 and Theorem 3.1].

**Corollary 2.2.** Let \( \mathbf{1} = (1, 0, \ldots, 0) \). For \( x \in \mathbb{R}^d \setminus \{0\} \) we have

\[
\int_0^\infty |p_t(x) - p_t(\mathbf{1})| \, dt < \infty.
\]

**Proof.** Let \( x \neq 0 \). By the symmetry of \( p_t \) we may and do assume that the first coordinate of \( x \) is nonnegative. Due to Lemma 2.1 and mean value theorem we have

\[
\int_0^\infty |p_t(x) - p_t(\mathbf{1})| \, dt \leq \int_0^\infty |(x - \mathbf{1}) \cdot \nabla p_t(\mathbf{1} + s(x - \mathbf{1}))| \, ds \, dt
\]

\[
\leq C_2 |x - \mathbf{1}| \int_0^1 \frac{|1 + s(x - \mathbf{1})| V^2(|1 + s(x - \mathbf{1})| \land 1)}{(|1 + s(x - \mathbf{1})| \land 1)^{d+2}} \, ds
\]

\[
\leq C_2 |x - \mathbf{1}| \frac{(|x| + 1) V^2(|x| \land 1)}{(|x| \land 1)^{d+2}}.
\]

Corollary 2.2 together with the symmetry of \( p_t \) let us define a fundamental solution \( G(x) \) of \( \mathcal{L} \). Namely, let \( \mathbf{1} = (1, 0, \ldots, 0) \) and define

\[
G(x) = \begin{cases} 
\int_0^\infty p_t(x) \, dt, & \text{if it is finite almost everywhere}, \\
\int_0^\infty (p_t(x) - p_t(\mathbf{1})) \, dt, & \text{otherwise}.
\end{cases}
\]

The first integral above is finite if and only if \( \int_{B(0,1)} \frac{1}{\psi(\xi)} \, d\xi < \infty \) (see [23, Theorem 37.5]). In particular it happens when \( d \geq 3 \) (see [23, Theorem 37.8]). Note that if \( \int_0^\infty p_t(x) \, dt = \infty \) on the set \( A \) of the positive Lebesgue measure, by the Chapman-Kolmogorov equation it is infinite everywhere. Indeed,

\[
\int_0^\infty p_t(y) \, dt \geq \int_1^\infty \int_A p_t(y - x)p_{t-1}(x) \, dx \, dt = \infty, \quad y \in \mathbb{R}^d.
\]
Lemma 2.3. Assume that \( \psi^* \in \text{WLSC}(\alpha, 1, c) \) with \( \alpha > 1/2 \). Then, \( G \) is differentiable on \( \mathbb{R}^d \setminus \{0\} \) and there exists \( C_3 \) such that, for \( |x| \leq 1 \),
\[
|\nabla G(x)| \leq C_3 \frac{V^2(|x|)}{|x|^{d+1}}.
\]
Furthermore if \( d \geq 2 \) or \( \psi^* \in \text{WLSC}(\alpha, 0, \beta) \) for some \( \beta > 0 \),
\[
|\nabla G(x)| \leq C_3 \frac{V^2(|x| \wedge 1)}{(|x| \wedge 1)^{d+1}}, \quad x \in \mathbb{R}^d.
\]

Proof. Let \( x \neq 0 \) and \( |h| < |x|/2 \). Observe that Lemma 2.1 and (2.1) allow us the Fubini theorem and get
\[
G(x + he_i) - G(x) = \int_0^\infty \int_0^h \partial_i p_t(x + u e_i) \, du \, dt = \int_0^h \int_0^\infty \partial_i p_t(x + u e_i) \, dt \, du.
\]
Hence,
\[
\nabla G(x) = \int_0^\infty \nabla_x p_t(x) \, dt.
\]
Another application of Lemma 2.1 gives us the upper bound for \( |\nabla G| \) if \( |x| \leq 1 \). Now we consider \( |x| \geq 1 \). Let us observe that (2.7) implies
\[
|\nabla_x p_t(x)| \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} |\xi| \, d\xi, \quad x \in \mathbb{R}^d.
\]
Now the same line of reasoning as in the proof of Lemma 2.1 gives us for \( d \geq 2 \),
\[
\int_0^\infty |\nabla_x p_t(x)| \, dt \leq c V^2(1) \int_{\mathbb{R}^d} e^{-V^2(1)\psi(\xi)} (|\xi|^2 + |\xi|^{-1}) \, d\xi < \infty.
\]
This together with (2.3) yield the boundedness of \( |\nabla G| \) on \( B^c(0, 1) \) in this case.

Under global lower scaling condition, by [16, Proposition 3.6 and Theorem 3.1] and (2.1) we obtain
\[
\int_{V^2(|x|)}^\infty |\nabla_x p_t(x)| \, dt \leq c |x| V^2(|x|) \int_{\mathbb{R}^d} e^{-V^2(|x|)\psi(\xi)} (|\xi|^2 + |x|^{-2}) \, d\xi \leq c \frac{V^2(|x|)}{|x|^{d+1}} \leq c V^2(1)
\]
Let us notice that, since \( V(\lambda r) \leq \lambda V(r) \), for \( \lambda \geq 1, \, r > 0 \) we have \( \lambda V^{-1}(r) \leq V^{-1}(\lambda r) \). Hence
\[
\int_{V^2(|x|)}^{V^2(|x|)} |\nabla_x p_t(x)| \, dt \leq c \frac{K(|x|)}{|x|^d} \int_{V^2(1)}^{V^2(|x|)} t \frac{V^{-1}(V(1))}{V^{-1}(\sqrt{t})} \, dt
\]
\[
\leq c \frac{K(|x|)}{|x|^d} \int_{V^2(1)}^{V^2(|x|)} \frac{V(1)}{\sqrt{t}} \, dt
\]
\[
\leq c V(1) \frac{V(|x|)}{|x|^d} \leq c V^2(1).
\]
\( \Box \)
2.3 Green function

Let \( p(t, x, y) = p_t(y - x) \). We consider the time-homogeneous transition probabilities

\[
P_t(x, A) = \int_A p(t, x, y) \, dy, \quad t > 0, x \in \mathbb{R}, A \subset \mathbb{R}^d.
\]

By the Kolmogorov and Dynkin-Kinney theorems the transition probability \( P_t \) define in the usual way Markov probability measures \( \{P^x, x \in \mathbb{R}^d\} \) on the space \( \Omega \) of the right-continuous and left-limited functions \( \omega : [0, \infty) \to \mathbb{R} \). We let \( \mathbb{E}^x \) be the corresponding expectations. We will denote by \( X = \{X_t\}_{t \geq 0} \) the canonical process on \( \Omega \), \( X_t(\omega) = \omega(t) \). Hence,

\[
P(X_t \in B) = \int_B p(t, x, y) \, dy.
\]

and \( X_t \) is a pure-jump symmetric Lévy process on \( \mathbb{R}^d \) with the Lévy-Khinchine exponent \( \psi \).

For any open set \( D \), we define the first exit time of the process \( X_t \) from \( D \),

\[
\tau_D = \inf\{t > 0 : X_t \notin D\}.
\]

Now, by the usual Hunt’s formula, we define the transition density of the process killed when leaving \( D \) : \( p_D(t, x, y) = p(t, x, y) - \mathbb{E}^y[\tau_D < t; p(t - \tau_D, x, X_{\tau_D})], \quad t > 0, x, y \in \mathbb{R}^d. \) (2.11)

We briefly recall some well known properties of \( p_D \) (see [5]). The function \( p_D \) satisfies the Chapman-Kolmogorov equations

\[
\int_{\mathbb{R}^d} p_D(s, x, z)p_D(t, z, y) \, dz = p_D(s + t, x, y), \quad s, t > 0, x, y \in \mathbb{R}^d.
\]

Furthermore, \( p_D \) is jointly continuous when \( t \neq 0 \), and we have

\[
0 \leq p_D(t, x, y) = p_D(t, y, x) \leq p(t, x, y).
\] (2.12)

In particular,

\[
\int_{\mathbb{R}} p_D(t, x, y) \, dy \leq 1. \quad (2.13)
\]

If \( D \) is a ball, by the Blumenthal 0-1 law, symmetry of \( p_t \), we have \( \mathbb{P}^x(\tau_D = 0) = 1 \) for every \( x \in D^c \). In particular, \( p_D(t, x, y) = 0 \) if \( x \in D^c \) or \( y \in D^c \).

We define the Green function of \( X_t \) for \( D \),

\[
G_D(x, y) = \int_0^\infty p_D(t, x, y) \, dt, \quad x, y \in \mathbb{R}^d \quad (2.14)
\]

**Lemma 2.4.** Let \( D \) be an open set. Additionally, we assume that \( D \) is bounded if \( \int_0^\infty p(t, x, y) \, dt = \infty \). If \( (x, y) \notin D^c \times D^c \), then

\[
G_D(x, y) = G(x - y) - \mathbb{E}^yG(x - X_{\tau_D}). \quad (2.15)
\]

If \( D \) has the outer cone property, (2.15) holds for every \( x, y \in \mathbb{R}^d \).
Proof. If \( \int_0^\infty p_t(x) \, dt < \infty \) almost everywhere, (2.15) follows directly by the Hunt formula (2.11). Now, suppose that \( \int_0^\infty p_t(x) \, dt = \infty \) everywhere. For \( \lambda > 0 \), we define

\[
U_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x) \, dt.
\]

By the Hunt formula (2.11), we have

\[
\int_0^\infty e^{-\lambda t} p_D(t, x, y) \, dt = U_\lambda(y - x) - \mathbb{E}^y \left[ e^{-\lambda \tau_D} (U_\lambda(X_{\tau_D} - x)) \right]
= U_\lambda(y - x) - U_\lambda(1) - \mathbb{E}^y \left[ e^{-\lambda \tau_D} (U_\lambda(X_{\tau_D} - x) - U_\lambda(1)) \right]
+ U_\lambda(1) \mathbb{E}^y[1 - e^{-\lambda \tau_D}]
\]

Note that \( \lim_{\lambda \to 0} \lambda U_\lambda(1) = 0 \). Indeed, due to (2.9) if there would be a sequence \( \lambda_k \to 0 \) such that \( \lim_{k \to \infty} \lambda_k U_{\lambda_k}(1) > 0 \), then we would have \( \lim_{k \to \infty} \lambda_k U_{\lambda_k}(x) > 0 \), for \( x \neq 0 \). Therefore

\[
\lim_{\lambda \to 0} \lambda \int_{B_1} U_\lambda(x) \, dx > 0,
\]

which gives a contradiction with [13] the proof of Lemma A.1. Since \( 1 - e^{-\lambda \tau_D} \leq \lambda \tau_D \) and \( \mathbb{E}^\tau_D < \infty \), we get

\[
\lim_{\lambda \to 0} U_\lambda(1) \mathbb{E}^y[1 - e^{-\lambda \tau_D}] = 0.
\]

Therefore, due to (2.9) and the symmetry of \( p_D \), by taking \( \lambda \to 0 \), (2.15) holds for \( (x, y) \not\in D^c \times D^c \). If \( D \) has the outer cone property, then \( \mathbb{P}^y(\tau_D = 0) = 1 \) for \( y \in D^c \) and (2.15) holds for every \( x, y \in \mathbb{R}^d \).

\ \ \ \ \Box

**Definition 3.** We say that a non-empty open \( D \subset \mathbb{R}^d \) is of class \( C^{1,1} \) at scale \( r > 0 \) if for every \( Q \in \partial D \) there are balls \( B(x', r) \subset D \) and \( B(x'', r) \subset D^c \) tangent at \( Q \).

We note that if the boundary of \( D \) is sufficiently regular (e.g. if \( D \) is \( C^{1,1} \) set), by the Ikeda-Watanabe formula, the \( \mathbb{P}^x \) distribution of \( X_{\tau_D} \) is absolutely continuous with respect to Lebesgue measure in \( \mathbb{R}^d \), [26]. Its density \( P_D(x, z) \) is called the Poisson kernel and it is given by the Ikeda-Watanabe formula ([18, Theorem 1])

\[
P_D(x, z) = \int_D G_D(x, z) \nu(z - y) \, dz, \quad x \in D, \ z \in \overline{D}^c.
\] (2.16)

Denote \( P_D[f](x) = \mathbb{E}^x f(X_{\tau_D}) \). If \( D \) is \( C^{1,1} \) domain (or more generally if \( \mathbb{P}^x(X_{\tau_D} \in \partial D) = 0 \)), we have

\[
P_D[f](x) = \int_{D^c} P_D(x, y) f(y) \, dy.
\]

**Definition 4.** We say that \( u \) satisfies mean value property (MVP) in \( D \) if \( P_D[u](x) < \infty \) and \( u(x) = P_D[u](x) \). If \( u \) satisfies MVP in every open set relatively compact in \( D \) then we say that \( u \) satisfies MVP inside \( D \).

Hence, if \( f \) satisfies MVP inside the open set \( D \), then for any \( r < \delta_x \),

\[
f(x) = \mathbb{E}^x(f(X_{\tau_D(x,r)})) = \int_{B(x,r)^c} P_{B(x,r)}(x, z) f(z) \, dz.
\] (2.17)
Sometimes, in the literature functions satisfying MVP are called harmonic. We use this terminology to distinguish the mean value property from $L$-harmonicity. Both properties are closely related as it is shown in Appendix. Note that by (2.11) and (2.14),

$$
\int_D G_D(x, y) \, dy = \mathbb{E}^\tau_D.
$$

Let us observe that the Pruitt bounds [22, see p. 954, Theorem 1 and (3.2) ibid.] imply existence a constant $c = c(d)$ such that for any bounded open set $D$

$$
c^{-1}V^2(\delta_x) \leq \mathbb{E}^\tau_D \leq cV^2(\text{diam } D).
$$

We note that if $D$ is a ball, better estimates for $\mathbb{E}^\tau_D$ may be derived (see Lemma A.3 in Appendix). Since $g$ is nonincreasing this implies, due to (2.16) and (2.18) that there exists a constant $C$ such that for any bounded open $D$

$$
C^{-1}V^2(\delta_x)g(\delta_x + \text{diam } D) \leq P_D(x, z) \leq CV^2(\text{diam } D)g(\delta_x), \quad x \in D, \ z \in \overline{D}
$$

**Lemma 2.5.** Let $D$ be a $C^{1,1}$ domain. For every $x \in D$, $G_D(x, y)$ satisfies MVP inside $D \setminus \{x\}$ and satisfies MVP in $D \setminus B(x, \varepsilon)$, where $0 < \varepsilon < \delta_x$.

**Proof.** Let $B$ be any $C^{1,1}$ bounded domain. Since $G_B(x, y) = 0$ if $x \in \overline{B}$ we have by (2.15) that

$$
G(x - y) = \mathbb{E}^yG(x - X_{\tau_B}), \quad x \in B.
$$

That is $G$ satisfies MVP inside $\mathbb{R}^d \setminus \{0\}$. This and the strong Markov property implies that $G_D(x, \cdot)$ satisfies MVP inside in $D \setminus \{x\}$. By the strong Markov property and the MVP property of $G$, $y \mapsto \mathbb{E}^y(G(x - X_{\tau_D}))$ satisfies MVP in $D \setminus B(x, \varepsilon)$. It yields that $y \mapsto G_D(x, y)$ satisfies MVP in $D \setminus B(x, \varepsilon)$. \qed

**Lemma 2.6.** Assume (2.10) holds. Let $0 < a < 1$. Then, there is a constant $C$ such that, for $r \leq 1$ and $|x| < ar$,

$$
|\nabla_x G_B(x, y)| \leq C \frac{V^2(|x - y|)}{|x - y|^{d+1}}.
$$

**Proof.** By Lemma 2.3 and the dominated convergence theorem

$$
\nabla_x G_B(x, y) = \nabla_x G(x - y) - \mathbb{E}^y \nabla_x G(x - X_{\tau_B}).
$$

Furthermore, Lemma 2.3 (2.1) and the monotonicity of $s \mapsto V^2(s)/s^{d+1}$ implies

$$
|\nabla_x G_B(x, y)| \leq C_3 \frac{V^2(|x - y| \wedge 1)}{(|x - y| \wedge 1)^{d+1}} + C_3 \mathbb{E}^y \frac{V^2(|x - X_{\tau_B} \wedge 1|)}{(|x - X_{\tau_B} \wedge 1|)^{d+1}}
$$

$$
\leq C_3 \frac{V^2(|x - y|)}{|x - y|^{d+1}} + \frac{C_3}{(1 - a)r} \frac{V^2((1 - a)r)}{(1 - a)r)^{d+1}}
$$

$$
\leq C_3 \frac{V^2(|x - y|)}{|x - y|^{d+1}}.
$$

\qed

**Lemma 2.7.** Assume that $\psi^* \in WLSC(\alpha, 1, c)$ with $\alpha > 0$. Then there exists $C > 0$ and $0 < \kappa < 1$ such that, for any $r < 1$,

$$
G_B(x, y) \geq C \frac{V^2(|x - y|)}{|x - y|^2}, \quad |x|, |y| \leq kr.
$$
Remark 4. If the measure $\nu$ is unimodal, i.e. its density is radial and nonincreasing, due to [14, Theorem 1.3] and [16, Lemma 2.3] the above inequality is equivalent to the lower scaling condition.

Proof of Lemma 2.7. Theorem 5.2] gives us, for $t < 1$ and $x \in \mathbb{R}^d$,

$$ p_t(x) \leq c_1 t \frac{K(|x|)}{|x|^d}. $$

Furthermore, by [16, Corollary 5.5], for $t \leq 1$,

$$ p_t(x) \geq \frac{c_2}{[V^{-1}(\sqrt{t})]^d}, \quad V^2(|x|) \leq t. $$

Let $\lambda = \frac{2c_1}{2c_2} < 1$. By (2.11), monotonicity of $s \mapsto \frac{K(s)}{s^d}$ and the inequalities above, for $V^2(|x - y|) \leq t \leq \lambda V^2((\delta_x \lor \delta_y)$ we get

$$ p_{B_r}(t, x, y) \geq \frac{c_2}{[V^{-1}(\sqrt{t})]^d} - c_1 t \frac{K(\delta_x \lor \delta_y)}{(\delta_x \lor \delta_y)^d} \geq \frac{c_2}{[V^{-1}(\sqrt{t})]^d} - c_1 t \frac{K(V^{-1}(\sqrt{t}/\lambda))}{(V^{-1}(\sqrt{t}/\lambda))^d} \geq \frac{c_2}{[V^{-1}(\sqrt{t})]^d} - c_1 t \frac{V^2(1)}{(V^{-1}(\sqrt{t}/\lambda))^d} \geq \frac{c_2}{2[V^{-1}(\sqrt{t})]^d}.$$

Let $a < b < 1$. Then, for $|x - y| \leq ar$ and $|x|, |y| \leq (1 - b)r$, we have

$$ G_{B_r}(x, y) \geq \int_{V^2(ar)}^{V^2(br)} \frac{c_2}{2[V^{-1}(\sqrt{t})]^d} dt \geq \frac{c_2(V^2(br) - V^2(ar))}{2(br)^d}. $$

Now it is enough to take $\frac{a}{b}$ small enough, such that $\frac{V^2(br)}{V^2(ar)} \geq (2\lambda)^{-1}$ to get

$$ G_{B_r}(x, y) \geq \frac{c}{b^d} \frac{V^2(br)}{r^d} \geq \frac{c}{b^d} V^2(r). \quad (2.21) $$

Now, for given $x, y \in B_{\epsilon r}$ we put $r_0 = |x - y|/2$. Eventually, by (2.11), we get

$$ G_{B_r}(x, y) \geq G_{B_{r_0}}(x, y) \geq \frac{c}{b^d + 2} \frac{V^2(r_0)}{r_0^d} \geq \frac{c 2^{d-1} V^2(|x - y|)}{|x - y|^d}. $$

Lemma 2.8. Assume (2.10). Let $r \leq 1$. Then, for $f \in L_{\infty}(B_r)$,

$$ \nabla_x \int_{B_r} G_{B_r}(x, z) f(z) \, dz = \int_{B_r} \nabla_x G_{B_r}(x, z) f(z) \, dz, \quad |x| < r. \quad (2.22) $$

Proof. First, note that by (2.21) for any $0 < \rho \leq M$, we have

$$ \int_{B_{\rho}} \frac{V^2(|y|)}{|y|^{d+1}} \, dy = c_1 \int_0^\rho \frac{V^2(s)}{s^2} \, ds \leq c_2 \frac{V^2(\rho)}{\rho}. \quad (2.23) $$
for some constant $c_2$ depending on $M$. Denote by $e_1, \ldots, e_d$ the standard basis in $\mathbb{R}^d$. We fix $x \in B_r$ and let $0 < h < \delta_x/2$ and $h_i = he_i \in \mathbb{R}^d$. By Lemma 2.6 with $a = (r + |x|)/2$

$$\left| \frac{G_{B_r}(x + h_i, z) - G_{B_r}(x, z)}{h} \right| = \frac{1}{h} \left| \int_0^1 \frac{d}{ds} G_{B_r}(x + sh_i, z) \, ds \right|$$

$$= \left| \int_0^1 \frac{\partial}{\partial x_i} G_{B_r}(x + sh_i, z) \, ds \right| \leq c \int_0^1 V^2(|x + sh_i - z|) \frac{|x + sh_i - z|^{d+1}}{|x + sh_i - z|} \, ds.$$

Since $f$ is bounded and $r \mapsto \frac{V^2(r)}{r^{d+1}}$ is nonincreasing, the rearrangement inequality implies uniformly in $h$ integrability of $z \mapsto \frac{G_D(x+h_i,z) - G_D(x,z)}{h} f(z)$ on $B_r$, thus by (2.23) we get the assertion.

As in Lemma 2.5 we prove the MVP property for $\nabla_x G_D(x, \cdot)$.

**Lemma 2.9.** Let $D$ be a $C^{1,1}$ domain. For every $x \in D$, $\nabla G_D(x, y)$ satisfies MVP inside $D \setminus \{x\}$ and satisfies MVP in $D \setminus B(x, \varepsilon)$, where $0 < \varepsilon < \delta_x/2$.

**Lemma 2.10.** Let $0 < a < b < 1$, $x \in B_{ar}$, $y \in B_r \setminus B_{br}$. Then,

$$\nabla_x G_{B_r}(x, y) = \int_{B_r \setminus B_{br}} \nabla_x G_{B_r}(x, z) P_{B_r \setminus B_{br}}(y, z) \, dz.$$

**Proof.** Let $A = B_r \setminus B_{br}$. By Lemma 2.5 and (2.16), we have

$$G_{B_r}(x, y) = \int_{B_r} G_{B_r}(x, z) P_A(y, z) \, dz, \quad x \in B_r, \ y \in A.$$

Let us fix $y \in A$. For $z \in B_{br}$ we define $P_1(y, z) = P_A(y, z) 1_{B_r \setminus B_{br}}(z)$ and $P_2(y, z) = P_A(y, z) - P_1(y, z)$. By (2.20), $P_1(y, \cdot)$ is bounded. Hence, by Lemma 2.10 we have

$$\nabla_x \int_{B_{br}} G_{B_r}(x, z) P_1(y, z) \, dz = \int_{B_{br}} \nabla_x G_{B_r}(x, z) P_1(y, z) \, dz.$$

Denote by $e_1, \ldots, e_d$ the standard basis in $\mathbb{R}^d$. For $i = 1, \ldots, d$ and $h \in \mathbb{R} \setminus \{0\}$ we denote $h_i = he_i$. By Lemma 2.6 we see $\partial_{x_i} G_D(x, z)$ is bounded on the support of the function $P_2(y, \cdot)$. By this, mean value theorem, and dominated convergence theorem we have

$$\lim_{h \to 0} \frac{1}{h} \left( \int_{B_{br}} G_{B_r}(x + h_i, z) P_2(y, z) \, dz - \int_{B_{br}} G_{B_r}(x, z) P_2(y, z) \, dz \right)$$

$$= \lim_{h \to 0} \int_{B_{br}} \frac{G_{B_r}(x + h_i, z) - G_{B_r}(x, z)}{h} P_2(y, z) \, dz = \int_{B_{br}} \partial_{x_i} G_{B_r}(x, z) P_2(y, z) \, dz,$$

which completes the proof.

3 Proof of Theorem 1.1

In this chapter we focus on the gradient of functions satisfying MVP property. The main aim is to prove that for such functions inequality (1.4) holds.
Let $D$ be an open set in $\mathbb{R}^d$, $d \in \mathbb{N}$ and $f$ be a nonnegative function in $\mathbb{R}^d$ satisfying MVP inside $D$. We fix $x \in D$ and $r = (\delta_x \wedge 1)/2$. In order to prove (1.4), without a loss of generality we may and do assume $x = 0$. By Lemma 2.6 for every $a \in (0, 1)$ there is a constant $c$ such that
\[
|\nabla x G_{B_r}(0, y)| \leq c \frac{V^2(|y|)}{|y|^{d+1}}, \quad y \in \mathbb{R}^d.
\] (3.1)

Similarly, by Lemmas 2.6 and 2.7 there is $\kappa \in (0, 1)$ such that
\[
|\nabla x G_{B_r}(0, y)| \leq c \frac{G_{B_r}(0, y)}{|y|}, \quad |y| \leq \kappa r.
\] (3.2)

Let us fix $\kappa \in (0, 1)$ such that (3.2) holds.

**Lemma 3.1.** There exist $C_4 = C_4(d, \sigma)$ such that
\[
\int_{B_{\rho}} |\nabla x G_{B_{\rho}}(0, y)| \, dy \leq C_4 \frac{V^2(\rho)}{\rho}, \quad 0 < \rho < M.
\]

**Proof.** By (3.1) and (2.23), for $0 < \rho < M$ we get
\[
\int_{B_{\rho}} |\nabla x G_{B_{\rho}}(0, y)| \, dy \leq c_0 \int_{B_{\rho}} \frac{V^2(|y|)}{|y|^{d+1}} \, dy \leq c_1 \frac{V^2(\rho)}{\rho}.
\]

Since $f$ satisfies MVP inside $D$, by the Ikeda-Watanabe formula (2.16) we have
\[
f(0) = \int_{B'_r} P_{B_r}(0, w) f(w) \, dw = \int_{B_r} G_{B_r}(0, y) \int_{B'_r} f(w) \nu(w - y) \, dw \, dy.
\] (3.3)

**Lemma 3.2.** Let $C_r = B_{2r} \setminus B_r$ and suppose $g$ satisfies the doubling condition (1.3). There exists a constant $C_5 = C_5(d, \sigma, \kappa)$ such that
\[
\int_{B_{\kappa r}} |\nabla x G_{B_{r}}(0, y)| \int_{C_r} f(w) \nu(w - y) \, dw \, dy \leq C_5 \frac{f(0)}{r}.
\]

**Proof.** By Lemma 3.1 and (2.21), we have
\[
\int_{B_{\kappa r}} |\nabla x G_{B_{r}}(0, y)| \, dy \leq c \frac{V^2(r)}{r} \leq \frac{c_1}{r} \int_{B_{\kappa r}} G_{B_r}(0, y) \, dy.
\] (3.4)

Note that there exists a constant $c_2 = c_2(d, \sigma, \kappa)$ such that
\[
c_2^{-1} \nu(w) \leq \nu(w - y) \leq c_2 \nu(w), \quad r < |w| < 2r, \quad |y| < \kappa r.
\] (3.5)

Indeed, since $\frac{2}{1-r}|w - y| > 2r > |w| > r > |w - y|/3$, by (1.2) and (1.3) we get (3.5). Now, by (3.5), (3.4) and (3.3)
\[
\int_{B_{\kappa r}} |\nabla x G_{B_{r}}(0, y)| \int_{C_r} f(w) \nu(w - y) \, dw \, dy \leq c_2 \int_{B_{\kappa r}} |\nabla x G_{B_{r}}(0, y)| \int_{C_r} f(w) \nu(w) \, dw \, dy \\
\leq \frac{c_3}{r} \int_{B_{\kappa r}} G_{B_r}(0, y) \, dy \int_{C_r} f(w) \nu(w) \, dw \leq \frac{c_4}{r} \int_{B_{\kappa r}} G_{B_r}(0, y) \, dy \int_{C_r} f(w) \nu(w - y) \, dw \, dy \\
\leq c_5 \frac{f(0)}{r}.
\]

\]
Lemma 3.3. Let $D_r = B_2^c$. There exists a constant $C_6 = C_6(d, \sigma, \kappa)$ such that

$$
\int_{B_{\kappa r}} |\nabla_x G_{B_r}(0, y)| \int_{D_r} f(w)\nu(w - y) \, dw \, dy \leq C_6 \frac{f(0)}{r}.
$$

Proof. Let $\rho = 7r/4$ and $a = 6\kappa/7$. Note that $\rho < 1$. For a given $w \in D_r$ we let $w' \in \partial D_r$ be such that $|w - w'| = \text{dist}(w, \partial D_r)$. We consider a one-sided circular cone $\gamma^w$ with apex in $0$ and axis going through the point $w$. The cone $\gamma^w$ is chosen in such a way that its boundary contains $\partial B(0, ar') \cap \partial B(w', (2 - \kappa)r)$. Denote $\gamma^w_1 := (B_{\kappa \rho} \setminus B_{a \rho}) \cap \gamma^w$. Now, by Lemma 3.1 (2.1) and (2.21), we have

$$
\int_{B_{\kappa r}} |\nabla_x G_{B_r}(0, y)| \, dy \leq c_1 \frac{V^2(r)}{r} \leq \frac{c_2}{r} \int_{\gamma^w_1} \frac{V^2(r)}{r^d} \, dy \leq \frac{c_3}{r} \int_{\gamma^w_1} G_{B_r}(0, y) \, dy,
$$

(3.6)

where the constant $c_3$ does not depend on $w$. We observe also that (see Figure 1)

$$
|w| - \kappa r > |w - y|, \quad y \in \gamma^w_1, \quad (3.7)
$$

$$
|w| - \kappa r < |w - y|, \quad y \in \kappa r. \quad (3.8)
$$
Therefore, by (1.2), monotonicity of $g$, (3.7), (3.6) (3.8) and (3.3), we get
\[
\int_{B_{kr}} |\nabla_x G_{B_r}(0, y)| \int_{D_r} f(w) \nu(w - y) \, dw \, dy \\
\leq c_4 \int_{B_{kr}} |\nabla_x G_{B_r}(0, y)| \int_{D_r} f(w) \nu(|w| - kr) \, dw \, dy \\
\leq \frac{c_4 c_4}{r} \int_{D_r} \int_{\gamma_{\kappa r}^n} G_{B_r}(0, y) f(w) \nu(|w| - kr) \, dy \, dw \\
\leq \frac{c_5}{r} \int_{D_r} \int_{\gamma_{\kappa r}^n} G_{B_r}(0, y) f(w) \nu(w - y) \, dy \, dw \\
\leq \frac{c_6}{r} \int_{B_{kr}} G_{B_r}(0, y) \int_{D_r} f(w) \nu(w - y) \, dw \, dy \leq c_7 \frac{f(0)}{r}.
\]

Now, we are ready to prove the main result.

**Theorem 3.4.** Let $d \in \mathbb{N}$, $\alpha \in (1, 2)$ and $D \subset \mathbb{R}^d$ be an open set. Let $\psi \in \text{WLSC}(\alpha, \theta, c_\alpha)$, where $\theta = 0$ for $d = 1$ and $\theta = 1$ for $d \geq 2$. Furthermore, we assume that $g$ satisfies doubling condition (1.3). Let $f$ be a function satisfying MVP inside $D$. There exists a constant $C$ such that
\[
|\nabla_x f(x)| \leq C \frac{f(x)}{\delta^\alpha D^\alpha \Lambda_1}, \quad x \in D,
\]

**Proof.** Assume that $\alpha > 1$. By Lemmas 3.2 and 3.3 for $r \leq 1/2$, we have
\[
\int_{B_{kr}} |\nabla_x G_{B_r}(0, y)| \int_{B_r^c} f(w) \nu(w - y) \, dw \, dy \leq C \frac{f(0)}{r}. \tag{3.9}
\]
As before we may and do assume $x = 0$. Let $r = (\delta_r \wedge 1)/2$. Denote $A_r = B_r \setminus B_{kr}$. By Lemmas 2.9 and 2.10, we have
\[
\int_{A_r} |\nabla_x G_{B_r}(0, y)| \int_{B_r^c} f(w) \nu(w - y) \, dw \, dy \\
\leq \int_{A_r} \int_{B_{kr}} |\nabla_x G_{B_r}(0, u)| P_A(y, u) \nu(w - y) \, dw \, dy \\
= \int_{C_r} |\nabla_x G_{B_r}(0, u)| W(u) \, du + \int_{D_r} |\nabla_x G_{B_r}(0, u)| W(u) \, du = I_1 + I_2,
\]
where $C_r = B_{\kappa r}, D_r = B_{kr} \setminus B_{\kappa r}$ and
\[
W(u) = \int_{A_r} P_A(y, u) \int_{B_r^c} f(w) \nu(w - y) \, dw \, dy.
\]
We need to prove that $I_1$ and $I_2$ are bounded by $c f(0)/r$. By Lemma 2.5, $G_{B_r}(0, y) = \int_{\kappa r} G_{B_r}(0, u) P_A(y, u) \, du$ for $y \in A_r$. Hence, by (3.2) we get
\[
I_2 \leq \frac{C}{r} \int_{D_r} \int_{A_r} G_{B_r}(0, u) P_A(y, u) \int_{B_r^c} f(w) \nu(w - y) \, dw \, dy \, du \\
\leq \frac{C}{r} \int_{A_r} G_{B_r}(0, y) \int_{B_r^c} f(w) \nu(w - y) \, dw \, dy \leq C \frac{f(0)}{r}.
\]
Next, since \( f \) satisfies MVP, by \((2.10)\) we have

\[
W(u) = \int_{A_r} \int_{A_r} G_{A_r}(y, z) \nu(u - z) \int_{B_{\xi}} f(w) \nu(w - y) \, dw \, dy \\
\leq \int_{A_r} \nu(u - z) \int_{A_r} f(w) \int_{A_r} G_{A_r}(y, z) \nu(w - y) \, dy \, dw \\
= \int_{A_r} \nu(u - z) \int_{A_r} f(w) P_{A_r}(z, w) \, dw \\
= \int_{A_r} \nu(u - z) f(z) \, dz.
\]

Hence, by \((3.9)\)

\[
I_1 \leq \int_{C_r} |\nabla_x G_{B_r}(\theta, u)| \int_{B_{\xi_r}} \nu(u - z) f(z) \, dz \, du \leq Cf(0) \frac{1}{r}.
\]

**Proof of Theorem 7.7.** If \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, 1 \wedge g^*(|x|)dx) \) is \( \mathcal{L} \)-harmonic, then by Lemma A.9 \( f \) satisfies MVP. Now \((1.4)\) follows from Theorem 3.4. \( \square \)

### A Appendix

In this section we will prove relation between \( \mathcal{L} \)-harmonic functions and functions satisfying MVP. Analogous results were proved for the fractional Laplacian in [3] and operators \( \mathcal{L} \) with twice differentiable Lévy density ([13]). We will work with slightly different assumptions. We will not assume WLSC property for \( \psi \), but we will still keep an assumption that \( \int_{\mathbb{R}^d} g(z) \, dz = \infty \) (generally it follows from WLSC condition). However, to simplify the discussion, we will assume that there is \( R \in (0, \infty] \) such that a function \( y \mapsto G_{B_R}(0, y) \) is continuous on \( B_R \setminus \{0\} \). Let us notice that by the Hunt formula \((2.11)\) and the strong Markov property we have

\[
G_{B_r}(0, y) = G_{B_R}(0, y) - P_{B_r}[G_{B_R}(0, \cdot)](y), \quad y \neq 0, \; 0 < r < R. \tag{A.1}
\]

In particular, by the proof of Lemma 2.3 \( G_{B_r}(0, \cdot) \) satisfies MVP in \( B_r \setminus \overline{B}_\rho \) for every \( \rho > 0 \).

By \( \mathcal{U} \) we denote the Dynkin operator that is (see [24] Chapter 7)

\[
\mathcal{U}[f](x) = \lim_{r \to 0} \frac{P_{B(x,r)}[f](x) - f(x)}{\mathbb{E}^x_{\tau_{B(x,r)}}}.
\]

By the Markov property if \( u \) satisfies MVP in an open set \( D \) then \( \mathcal{U}[u](x) = 0 \) on \( D \). Let \( g^* \) be such that \( g(s) \leq g^*(s), \; s > 0 \), \( g^* \) satisfies doubling and \( g^*(r) \approx g^*(r + 1) \) for large \( r \). Notice that \( g^*(r) \) can vanish for large \( r \). Recall that

\[
Y = \{ u \in L^1_{\text{loc}}(\mathbb{R}^d) : u \in L^1(\mathbb{R}^d, 1 \wedge g^*(|x|)dx) \}
\]

and \( \|u\|_Y = \|u\|_{L^1(\mathbb{R}^d, 1 \wedge g^*(|x|)dx)} \). Observe that if \( g^* \) is positive, for \( \varphi \in C^\infty_c(D) \) we have

\[
|\mathcal{L}[\varphi](x)| \leq C(1 \wedge g^*(|x|)).
\]

If \( g^* \) vanishes for large arguments, then \( |\mathcal{L}[\varphi](x)| \leq C, \; x \in \mathbb{R}^d \) and \( |\mathcal{L}[\varphi](x)| = 0 \) for \( |x| \) sufficiently large.
Lemma A.1. Assume that \( u \in Y \). If \( u \) satisfies MVP in an open set \( D \) then \( u \) is \( \mathcal{L} \)-harmonic in \( D \).

**Proof.** Let \( \phi_\epsilon, \epsilon > 0 \) be a standard mollifier. Since \( \phi_\epsilon * u \in C^2 \) we have

\[
(\phi_\epsilon * u, \mathcal{L}\varphi) = (\mathcal{L}(\phi_\epsilon * u), \varphi).
\]

Since \( U \) is an extension of \( \mathcal{L} \) (see [24] Theorem 7.26) and it is a translation invariant, we conclude

\[
\mathcal{L}[\phi_\epsilon * u](x) = U[\phi_\epsilon * u](x) = \phi_\epsilon * U[u](x) = 0
\]
on \( D_\epsilon = \{ x \in D : \text{dist}(x, \partial D) > \epsilon \} \). That is \( \phi_\epsilon * u \) is \( \mathcal{L} \)-harmonic in \( D_\epsilon \). Since \( u \in Y \) the distribution \( \mathcal{L}u \) is properly define. Passing with \( \epsilon \to 0 \) we obtain (we use \( \mathcal{L}(\phi_\epsilon * u) = \phi_\epsilon * \mathcal{L}u \) in distributional sense),

\[
(u, \mathcal{L}\varphi) = 0, \quad \varphi \in C_0^\infty(D_\delta),
\]
for every \( \delta > 0 \) which ends the proof. \( \square \)

We will need the following maximum principle for \( U \).

Lemma A.2. Let \( D_1 \) be a bounded domain and \( h \) a continuous function on \( D_1 \). Suppose \( x_0 \in D_1 \) is such that \( h(x_0) = \sup_{x \in D_1} h(x) > 0 \). Then either \( h \) is constant on \( \text{int}(\text{supp}(\nu) + x_0) \) or \( U[h](x_0) < 0 \).

**Proof.** Let \( x \in \text{int}(\text{supp}(\nu) + x_0) \). If \( h(x) < h(x_0) \) there exists \( r > 0 \) such that \( h(x_0) - h(z) \geq (h(x_0) - h(x))/2 \) for \( z \in B(x, 2r) \subset \text{int}(\text{supp}(\nu) + x_0) \). For \( s < r \), by the Ikeda-Watanabe formula and [2,18] we have

\[
h(x_0) - P_{B(x_0,s)}[h](x_0) = \int_{B(x,r)} P_{B(x_0,s)}(x_0,z)(h(x_0) - h(z)) \, dz
\]

\[
\geq (h(x_0) - h(x))/2 \int_{B(x,r)} P_{B(x_0,s)}(x_0,z) \, dz
\]

\[
\geq c(h(x_0) - h(x)) \int_{B(x,r)} g(|z - x_0| + s) \, dz \mathbb{E}^{x_0} \tau_{B(x_0,s)}
\]

\[
\geq c(h(x_0) - h(x)) |B(x,r)| g(|x - x_0| + 2r) \mathbb{E}^{x_0} \tau_{B(x_0,s)}.
\]

This implies

\[
-U[u](x_0) \geq c(h(x_0) - h(x)) |B(x,r)| g(|x - x_0| + 2r) > 0.
\]

\( \square \)

For \( r > 0 \), denote \( s_r(x) = \mathbb{E}^x \tau_{B_r} \).

Lemma A.3. There exists a constant \( c \) such that, for any \( r > 0 \)

\[
c^{-1}V(r - |x|)^2 \leq s_r(x) \leq cV(r) V(r - |x|), \quad |x| < r.
\]

(A.2)

Furthermore, \( s_r \in C_0(B_r) \).
Proof. By the slight modification of the proof of [6] Lemma 2.9 we obtain \( s_r \in C_0(B_r) \). For \( x = 0 \) (A.2) follows by (2.19). So, let \( x \neq 0 \). Again, by (2.19), we have

\[
\begin{align*}
s_r(x) & \geq \mathbb{E}^x(\tau_{B(x,|x|-r)}) \geq cV^2(|x| - r).
\end{align*}
\]

To get the upper bound let \( \theta = x/|x| \) and \( H_r = \{ y \in \mathbb{R}^d : -r < y \cdot \theta < r \} \). Then, \( s_r(x) \leq \mathbb{E}^x \tau_{H_r}(x) = \mathbb{E}^{|x|}(\tau^Z(x)), \) where \( \tau^Z \) is the first exit time of a Lévy process \( Z = \theta \cdot X \) with characteristic exponent \( \psi_Z(y) = \psi(\theta y), \ y \in \mathbb{R} \). Now, by [15] Proposition 3.5 we get \( \mathbb{E}^x \tau_{H_r}(x) \leq cV^2 \mathbb{E}^2(r)V_Z(r - |x|), \) where \( V_Z \) is a function constructed for the process \( Z \) in the same way as \( V \) is constructed for \( X \). Now, by [6] Proposition 2.4 \( V_Z \approx V \) and we get the desired result. \( \square \)

**Lemma A.4.** For \( 0 < r < R \) the function \( y \mapsto G_{B_r}(0, y) \) is continuous on \( \mathbb{R}^d \setminus \{0\} \).

**Proof.** First, let us observe that by (A.1) it remains to prove that \( P_{B_r}[G_{B_R}(0, \cdot)] \) is continuous. We note that since \( G_{B_R}(0, \cdot) \) is continuous, it is bounded on \( \mathbb{R}^d \setminus B_r \). Therefore, the remaining part can be proved by repeating the proof of [10] Theorem 1.3, where instead of Proposition 1.19 therein one should use [9] Proposition 4.4.1. \( \square \)

By Lemma A.4 \( G_{B_r}(0, \cdot) \) is continuous on \( \partial B_r \). We will use it to obtain the following decay of the Green function near the boundary.

**Lemma A.5.** Let \( 0 < r < R \). Then,

\[
G_{B_r}(0, y) \leq CV(r - |y|), \quad |y| > 3r/4,
\]

where \( C = c(d)V(r) \left( r^{-d} + V^{-2}(r) \sup_{|z| \geq r/2} G_{B_r}(0, z) \right) \).

**Proof.** Fix \( r > 0 \) and \( w(y) = G_{B_r}(0, y) \). Observe that by the strong Markov property \( U[s_r](x) = -1, \ x \in B_r \). Denote \( w_n(y) = w(y) \wedge n \). We take \( n \) so large that \( w_n(y) = w(y) \) on \( B_r \setminus B_{r/2} \) (it is possible by the continuity of \( G_{B_R}(0, \cdot) \)). Since \( w \) satisfies MVP in \( B_r \setminus \overline{B}_{3r/4} \), by (2.18) and (A.2) we have

\[
U[w_n](v) = U[w_n - w](v) = L[w_n - w](v) = \int_{B_{r/2}} (w_n(z) - w(z))\nu(z - v) \ dz \geq -cg(r/4) \int_B w(z) \ dz \geq -c_1g(r/4)V^2(r), \quad 3r/4 \leq |v| < r.
\]

That is for \( a > c_1g(r/4)V^2(r), \)

\[
U[as_r - w_n](v) < 0, \quad 3r/4 \leq |v| < r.
\]

Furthermore, by (A.2)

\[
s_r(v) \geq cV^2(r - |v|) \geq c_2V^2(r), \quad |v| < 3r/4.
\]

Hence, \( as_r(v) - w_n(v) > 0 \) for \( |v| < 3r/4 \) if \( a > n/(c_2V^2(r)) \).

Since \( w_n = s_r = 0 \) on \( B_r \), by the maximum principle for \( U \) (Lemma A.2) and the continuity of \( as_r - w_n \) on \( \mathbb{R}^d \setminus \{0\} \), by Lemmas A.3 and A.4 we obtain that \( w_n \leq as_r \) for \( a = c_1g(r/4)V^2(r) + n/(c_2V^2(r)) \). Since \( w_n(y) = w(y) \) on \( B_r \setminus B_{r/2} \) and \( g(r) \leq c(d)r^{-d}V^{-2}(r) \) we get

\[
G_{B_r}(0, y) \leq CS_r(y), \quad |y| > 3r/4,
\]

where \( C = c(d) \left( r^{-d} + V^{-2}(r) \sup_{|z| \geq r/2} G_{B_r}(0, z) \right) \). Now, the assertion follows by (A.2). \( \square \)
We define
\[ \tilde{P}_r(z) = 4r^{-1} \int_{r/4}^{r/2} P_{B_s}(0, -z) ds, \quad z \in \mathbb{R}^d. \]

The function \(\tilde{P}_r\) is called a regularized Poisson kernel. In the next lemma we show that like \(P_{B_0}(0, z)\), it reproduces functions satisfying MVP but in contrary to the latter, \(\tilde{P}_r\) is bounded.

**Lemma A.6.** Let \(0 < r \leq 1\). Then, \(\tilde{P}_r\) is bounded and for every \(u\) satisfying MVP in \(B(x_0, r)\) we have
\[ u(x) = \tilde{P}_r(u(x), |x - x_0| < r/2. \]

Furthermore, if \(g^*\) is positive
\[ \tilde{P}_r(z) \leq cg^*(|z|)1_{|z| > r/4}, \quad z \in \mathbb{R}^d. \] (A.3)

If \(g\) is only nonnegative, (A.3) holds only for large \(|z|\).

**Proof.** Fix \(x_0 \in \mathbb{R}^d\) and let \(u_0(x) = u(x + x_0)\). Then, \(u_0\) satisfies MVP on \(B_r\). Note that \(P_{B_0}(0, -z) = 0\) for \(|z| \leq s\) and \(P_{B_s}[u] < \infty\), \(s < r\). For \(|x - x_0| < r\), we have
\[ \frac{r}{4} \tilde{P}_r(u(x)) = \frac{r}{4} \tilde{P}_r(u(x) - x_0) \int_{r/4}^{r/2} P_{B_s}(0, z - x_0) u_0(z) dz ds \]
\[ = \int_{r/4}^{r/2} \int_{\mathbb{R}^d} P_{B_s}(x - x_0, z) u_0(z) dz ds = \int_{r/4}^{r/2} u_0(x - x_0) dz ds = \frac{r}{4} u(x). \]

It remains to prove bounds for \(\tilde{P}_r\). For \(|z| < r/4\) we have \(\tilde{P}_r(z) = 0\). Let \(r/4 < |z| < r\). Observe that for \(r/4 < s < r/2\), by Lemmas A.5, A.3, and A.4 we have
\[ P_{B_s}(0, z) \leq cg(r/16) \int_{|y| < 3s/4, |z - y| < r/16} V(s - |y|) g(z - y) dy \]
\[ \leq c_1 g(r/16) V^2(r) + c_2 \int_{|y| > 3s/4, |z - y| < r/16} V(s - |y|) g(z - y) dy, \]
where \(c_2 = c(d)V(r) (r^{-d} + V^{-2}(r) \sup_{|z| > r/8} G_{B_s}(0, z)) < \infty\). Hence,
\[ \frac{r}{4} \tilde{P}_r(z) \leq c_1 g(r/16) V^2(r)/4 + c_2 \int_{r/4}^{r/2} \int_{3s/4 < |y| < s} 1_{B(z, r/16)}(y)V(s - |y|) g(z - y) dy ds \]
\[ = c g(r/16) V^2(r) + c_2 \int_{3r/16 < |y| < r/2} \int_{|y|}^{|y|/3 \vee r/2} V(s - |y|) ds 1_{B(z, r/16)}(y) g(z - y) dy \]
\[ \leq c g(r/16) V^2(r) + c_2 \int_{3r/16 < |y| < r/2} V(s - |y|) ds 1_{B(z, r/16)}(y) g(z - y) dy \]
\[ \leq c g(r/16) V^2(r) + c_2 \int_{|y| < r/16} V(|y|) V(|y|) g(z - y) dy \]

By (6, Proposition 3.4) we obtain
\[ \tilde{P}_r(z) \leq cg(r/16) V^2(r) + c_3 \frac{1}{V(r)} < \infty, \quad |z| < r. \]

For \(|z| \geq r\), by (2.20) we have \(P_{B_s}(0, z) \leq c V^2(s) g(|z| - s)\). If \(g^*\) is positive, \(g(|z| - s) \leq cg^*(|z|)\), which gives (A.3) in this case. If \(g^*(u) = 0\), then \(\tilde{P}_r(z) = 0\), for \(|z| \geq u + r/2. \)

\[ \square \]
Lemma A.7. Let \( u \in Y \). If \( u \) satisfies MVP in an open set \( D \), then \( u \in C(D) \).

**Proof.** Let \( \varepsilon > 0 \). Since \( u \in Y \), there is \( R > 0 \) such that \( \|u1_{B_R^\varepsilon}\|_Y < \varepsilon \). Fix \( x_0 \in D \) and let \( \rho = 1 \wedge \operatorname{dist}(x_0, D^C) \) and \( B = B(x_0, \rho) \). Let \( u_1 = u1_{B_R^\varepsilon} \) and \( u_2 = u - u_1 \). By Lemma [A.6], we have

\[
    u(x) = \tilde{P}_\rho \ast u(x) = \tilde{P}_\rho \ast u_1(x) + \tilde{P}_\rho \ast u_2(x), \quad |x - x_0| < \rho/2.
\]

Since \( \tilde{P}_\rho \) is bounded and \( u_1 \in L^1(\mathbb{R}^d) \), \( \tilde{P}_\rho \ast u_1 \in C(\mathbb{R}^d) \).

If there is \( r_0 \) such that \( g^r(r) = 0 \) for \( r > r_0 \), then \( \tilde{P}_\rho(z) = 0 \) for \( |z| > \rho + r_0 \). Thus, by taking sufficiently large \( R \), we get \( \tilde{P} \ast u_2(x) = 0 \) on \( B \). Now, suppose \( g^r > 0 \) on \( (0, \infty) \). By Lemma [A.6] \( \tilde{P}_\rho(z) \leq c(1 \wedge g^r(|z|)) \). Furthermore, there is a constant \( c_1 = c_1(B) \) such that \( 1 \wedge g^r(|y - x|) \leq c_1(1 \wedge g^r(|y|)) \) for \( x \in B \). Then,

\[
    |\tilde{P}_\rho \ast u_2(x)| \leq \int_{B_R^\varepsilon} \tilde{P}_\rho(x - y)|u(y)| \, dy \leq c_1 c_2 \int_{B_R^\varepsilon} |u(y)|(1 \wedge g^r(y)) \, dy \leq c_1 c_2 \varepsilon, \quad x \in B.
\]

Hence, for \( x \in B \) we have

\[
    |u(x_0) - u(x)| \leq 2c_1 c_2 \varepsilon + |\tilde{P}_\rho \ast u_1(x_0) - \tilde{P}_\rho \ast u_1(x)|,
\]

which gives the continuity of \( u \) at \( x_0 \). \( \square \)

Lemma A.8. Assume \( u \in Y \cap C^2(D) \). If \( u \) is \( \mathcal{L} \)-harmonic in \( D \), it satisfies MPV inside \( D \).

**Proof.** Fix \( u \in Y \cap C^2(D) \), so \( \mathcal{L}u(x) \) can be calculated pointwise for \( x \in D \) and consequently, \( \mathcal{L}u(x) = 0 \) for \( x \in D \). Fix an open set \( D_1 \) relatively bounded in \( D \) and define \( \tilde{u}(x) = P_{D_1}[u](x) \), \( x \in \mathbb{R}^d \). We need too prove that \( \tilde{u} = u \). Since every open set may be approximated by bounded smooth domains (see e.g. [12] Proposition 8.2.1), by the strong Markov property we may and do assume that \( D_1 \) is a \( C^{1,1} \) domain.

We claim that \( \tilde{u} \) has the mean-value property in \( D_1 \). To prove it, we only need to show \( P_{D_1}[[u]](x) < \infty \). Let \( D_2 \) be an open set relatively compact in \( D \) such that \( \overline{D_1} \subset D_2 \). There exist continuous functions \( u_1, u_2 \) on \( D_1^c \) such that \( u = u_1 + u_2 \), \( u_1 \) is bounded on \( D_1^c \) and \( u_2 \equiv 0 \) in \( D_2 \). We have

\[
    \tilde{u}(x) = P_{D_1}[u_1](x) + P_{D_1}[u_2](x), \quad x \in \mathbb{R}^d.
\]

Note that \( u_1 = P_{D_1}[u_1] \) and \( u_2 = P_{D_1}[u_2] \) on \( D_1^c \). Since \( P_{D_1}[u_1] \) satisfy MVP in \( D_1 \), by Lemma A.7 it is continuous in \( D_1 \). Note that \( P_{D_1}[u_1] \) is also continuous as a function of \( x \) in \( D_1 \). Indeed, let \( x_0 \in \partial D \). For \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
    \left| \int_{D_1^c} P_{D_1}(x, z)u_1(z) \, dz - u_1(x_0) \right| \leq \varepsilon + 2\|u_1\|_{\infty, \mathbb{P}^x} \left( \left| X_{\tau_{D_1}} - x_0 \right| > \delta \right).
\]

Since the second term goes to 0 as \( x \to x_0 \) (see [6] Lemmas 2.1 and 2.9), by arbitrary choice of \( \varepsilon \) we get the continuity on \( D_1 \).

Furthermore, from monotonicity of \( 1 \wedge g^r(|x|) \), the Ikeda-Watanabe formula and (2.18) we obtain

\[
    P_{D_1}(x, z) \leq (1 \wedge g^r(\operatorname{dist}(z, D_1)))\mathbb{E}^x_{\tau_{D_1}}, \quad x \in D_1, \ z \in D_2^c.
\]

(A.4)
Since \( u \in Y \), and the fact that \( g^*(r + 1) \approx g^*(r) \), we get \( P_{D_1}[u_2] < \infty \). Hence \( P_{D_1}[u_2] \) satisfies MVP and by Lemma A.7, \( P_{D_1}[u_2] \) is continuous in \( D_1 \). Since \( E^x \tau_{D_1} \in C_0(D_1) \) and \( u_2 = 0 \) on \( \partial D_1 \subset D_2 \), by Lemma A.8 \( P_{D_1}[u_2] \) is continuous in \( D_1 \) as well. Hence, \( \tilde{u} \) is continuous and has the mean-value property in \( D_1 \). Note that \( \tilde{u} = u \) on \( D_1^c \), since \( D_1 \) is a \( C^{1,1} \) domain.

Let \( h = \tilde{u} - u \). We now verify that \( h \equiv 0 \) so that \( u = \tilde{u} \) has the mean-value property in \( D_1 \). Observe that \( h \) is continuous and compactly supported. Suppose \( x_0 \in D_1 \) is such that \( h(x_0) = \sup_{x \in D_1} h(x) > 0 \). Since \( u \in C^2(D) \) we have \( 0 = \mathcal{L} u(x_0) = \mathcal{U}[u](x_0) \). Hence, by Lemma A.2, \( h \) is constant on \( \text{supp}(\nu) + x_0 \). If \( D_1 \subset \text{supp}(\nu) + x_0 \) we get that \( h > 0 \) on \( D_1 \), which is a contradiction due to continuity of \( h \) and the fact that \( h \equiv 0 \) on \( \partial D_1 \). Thus, \( h \leq 0 \). If this is not the case, then we can use the chain rule to get for any \( n \in \mathbb{N} \) that \( h \) is constant on \( n\text{supp}(\nu) + x_0 \) and eventually get a contradiction. Similarly, \( h \) must be non-negative. \( \square \)

**Lemma A.9.** Let \( u \in Y \) be a solution of \( \mathcal{L} u = 0 \) in \( D \) in distributional sense. Then \( u \) satisfies MPV inside \( D \).

*Proof.* Let \( \Omega \subset \mathring{\Omega} \subset D \) be a bounded \( C^{1,1} \) domain. Define \( \rho = (1 \wedge \text{dist}(\Omega, D^c))/2 \) and let \( V = \Omega + B_\rho \). For \( \epsilon < \rho/4 \) we consider standard mollifiers \( \phi_\epsilon \). Since \( \mathcal{L} \) is translation-invariant we have that \( \mathcal{L}(\phi_\epsilon \ast u) = \mathcal{L} u \ast \phi_\epsilon = 0 \) in \( V_\epsilon = \{x \in D : \text{dist}(x, V^c) > \epsilon \} \) in distributional sense. By Lemma A.8 we obtain

\[
\phi_\epsilon \ast u(x) = P_{\Omega}[\phi_\epsilon \ast u](x), \quad x \in \Omega.
\]

Note that by [13, Lemma 2.9], for \( u \in Y \), \( \phi_\epsilon \ast u \rightarrow u \) in \( Y \) as \( \epsilon \rightarrow 0^+ \). Hence, up to the subsequence,

\[
\lim_{\epsilon \rightarrow 0^+} \phi_\epsilon \ast u(x) = u(x) \quad \text{a.e.}
\]

Furthermore, since \( \phi_\epsilon \ast u \) satisfies MVP in \( V_{\rho/4} \), by Lemma A.6

\[
\phi_\epsilon \ast u(x) = \phi_\epsilon \ast u \ast \bar{P}_r(x), \quad x \in V_{\rho/2}
\]

for \( r = \rho/8 \). Hence, for any \( E \subset \Omega^c \),

\[
P_{\Omega}[\phi_\epsilon \ast u|1_{V_{\rho/2} \cap E}](x) = \int_{V_{\rho/2} \cap \Omega \cap E} |\phi_\epsilon \ast u(z)| P_{\Omega}(x, z) \, dz
\]

\[
= \int_{V_{\rho/2} \cap \Omega \cap E} |\phi_\epsilon \ast u \ast \bar{P}_r(z)| P_{\Omega}(x, z) \, dz
\]

\[
\leq \int_{B_\epsilon} \phi_\epsilon(v) \int_{\mathbb{R}^d} |u(y)| \int_{V_{\rho/2} \cap \Omega \cap E} \bar{P}_r(z - y - s) P_{\Omega}(x, z) \, dz \, dy \, dv.
\]

Let \( R > 0 \). Then from boundedness of \( \bar{P}_r \) and local integrability of \( u \) we get

\[
\int_{|y| \leq R} |u(y)| \int_{V_{\rho/2} \cap \Omega \cap E} \bar{P}_r(z - y - s) P_{\Omega}(x, z) \, dz \, dy \leq c \int_{|y| \leq c} |u(y)| \int_E P_{\Omega}(x, z) \, dz
\]

\[
\leq c \|u\|_Y \int_E P_{\Omega}(x, z) \, dz.
\]

Furthermore, for \( |y| > R \) and sufficiently large \( R \), we have \( \bar{P}_r(z - y - v) \leq c(1 \wedge g^* (|y|)) \). Thus,

\[
\int_{|y| > R} |u(y)| \int_{V_{\rho/2} \cap \Omega \cap E} \bar{P}_r(z - y - s) P_{\Omega}(x, z) \, dz \, dy \leq c \|u\|_Y \int_E P_{\Omega}(x, z) \, dz.
\]
Since $\int_E P_\Omega(x, z) \, dz \to 0$ as measure of $E$ tends to 0, $\phi_\epsilon \ast u$ are uniformly integrable in $V_{\rho/2}$ with respect to the measure $P_\Omega(x, z) \, dz$. By the Vitali convergence theorem,

$$\lim_{\epsilon \to 0^+} P_\Omega[(\phi_\epsilon \ast u)1_{V_{\rho/2}}](x) = P_\Omega[u1_{V_{\rho/2}}](x).$$

It remains to show that $\lim_{\epsilon \to 0^+} P_\Omega[(\phi_\epsilon \ast u)1_{V_{\rho/2}}] = P_\Omega[u1_{V_{\rho/2}}]$. Since $\text{dist}(\Omega, V_{\rho/2}) = \rho/2$, we obtain, by (2.20), $P_\Omega(x, z) \leq c(1 \wedge g^*(z))$. Due to $\lim_{\epsilon \to 0^+} \phi_\epsilon \ast u = u$ in $\mathcal{Y}$ we obtain the result.

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