The controllability of linear descriptor systems using direct methods

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Abstract. In this paper, the controllability of linear descriptor systems is investigated by using a direct approach. This approach is based on viewing the system as a whole. In general method, the system is first decomposed into two subsystems in the canonical form. Then, these subsystems are dealt with separately. However, in this paper, we developed the system generally not viewed separately. Then, we derived the structure of controllable subspaces for linear descriptor systems. From these results, we can state the definition of a controllable matrix and then we investigated the properties of the linear descriptor systems related to the controllability matrices by using geometric term.

1. Introduction

The descriptor systems also known as singular systems, semi-state systems, differential algebraic equations (DAEs) are general enough to provide a complete understanding of inner dynamics for underlying physical problems. The descriptor systems have been one of the major research fields of control theory. During the past three decades, a large amount of attention has been paid to the structural properties of linear descriptor systems. Many approaches have been developed to study descriptor systems. These approaches are mainly dedicated to descriptor systems in standard form for which many structural properties such as controllability, observability, input-output grouped coupling (see [1, 2, 3], and stability are studied. Effectively, descriptor systems are more complicated to study than classical linear systems in standard form. Descriptor systems result from a convenient and natural process modeling [4] and their applications can be found in various fields [5] such as robotics, electrical circuit networks, biologic and economic systems.

Controllability for linear descriptor systems have also been studied in the continuous-time case for example in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. They have taken a different approach which utilizes the standard canonical form. The system is first decomposed into two subsystems in the canonical form. Then, these subsystems are dealt with separately. In view of this result, several attempts have been made to describe the characterization of linear descriptor system in terms of the coefficient matrices of the whole system, for example in [17,17]. Based on viewing the system as a whole, in this paper, we develop a direct approach for the study of controllability for linear descriptor system by using geometric method. Then, we derived the structure of controllable subspaces for linear descriptor systems. From these results, we can derive the definition of a controllable matrix and then we investigated the properties of the linear descriptor systems related to the controllability subspace for a whole system.
In general, the method used in this paper is different from the method used in many literature to investigate the controllability problem of linear descriptor systems.

2. Preliminaries

In this part, we provide some definitions and results to the controllability of linear descriptor systems which are useful in developing the main results in thenext sections. For simplicity, the definitions and some theorems are extracted from [18, 19].

Consider the linear descriptor systems of the form

\[
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{aligned}
\]  

(1)

where \(x \in \mathbb{R}^n\) is the state vector, \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^p\) present the input and output vectors, respectively, \(E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\) are constant matrices and assumed that \(E\) is singular. It is well-known that the existence and uniqueness of solution (1) are guaranteed if the system (1) is regular, i.e., \(\det(sE - A) \neq 0\) for some \(s \in \mathbb{C}\).

**Theorem 1.** [19]. Given the linear descriptor system (1) with \(E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\), and \((E, A)\) is regular, then there exist two nonsingular matrices \(Q\) and \(P\) such that

\[
(E, A, B, C) \overset{Q,P}{\rightarrow} (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})
\]

with

\[
\begin{aligned}
\tilde{E} &= QEP = \text{diag}(I_{n_1}, N) \\
\tilde{A} &= QAP = \text{diag}(A_{l_1}, I_{n_2}) \\
\tilde{B} &= QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
\tilde{C} &= CP = [C_1 \ C_2]
\end{aligned}
\]

(2)

where \(n_1 + n_2 = n\), and the involved partitions are compatible. Furthermore, the matrix \(N \in \mathbb{R}^{n_2 \times n_2}\) is nilpotent.

According to Theorem 1, the linear descriptor system (1) can be decomposed into the following systems:

\[
\begin{aligned}
\dot{x}_1(t) &= A_1x_1(t) + B_1u(t) \\
y_1(t) &= C_1x_1(t),
\end{aligned}
\]

(3)

\[
\begin{aligned}
N\dot{x}_2(t) &= x_2(t) + B_2u(t) \\
y_2(t) &= C_2x_2(t),
\end{aligned}
\]

(4)

and the output systems is given by

\[
y(t) = y_1(t) + y_2(t) = C_1x_1(t) + C_2x_2(t).
\]

(5)

The system represented by (3)–(5) is called the standard decomposition form. Especially, the subsystems (3) and (4) are called the slow and fast subsystems, respectively. Furthermore, the linear descriptor systems (1) is called a system with index \(h\) if the nilpotent matrix \(N\) in (4) has index \(h\) (i.e. \(N^h = 0\) and \(N^{h-1} \neq 0\)).

The solution of slow and fast subsystem (3) and (4) are given by \(x_1\) and \(x_2\) of the form:

\[
x_1(t) = e^{A_1t}x_1(0) + \int_0^te^{A_1(t-\tau)}B_1u(\tau)\ d\tau
\]

(6)
In this subsection, the definition of controllability for linear descriptor systems is given. Then we give the characterization of controllability in terms of slow and fast subsystems.

**Definition 1.** [18, 19]. The linear descriptor system (1) is called controllable if, for any \( t_1 > 0 \), \( x_0 \in \mathbb{R}^n \) and \( w \in \mathbb{R}^m \), there exists a control input \( u(t) \in \mathbb{R}^m \) such that the response of the system (1) starting from the initial condition \( x(0) = x_0 \) satisfies \( x(t_1, u, x_0) = w \).

This definition states that under controllability assumption, for any initial condition \( x(0) = x_0 \), we may always choose a control input such that the state response starting from \( x(0) \) to any prescribed position in \( \mathbb{R}^n \) in any specified time period. It is easy to see that the definition is a natural generalization of the controllability for classical linear systems. In the controllable term, we introduce for any matrix pair \( (A, B) \) with \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). As in the classical linear system case, we define the controllability matrix pair \( (A, B) \) as

\[
C(A, B) = [B \ AB \ \cdots \ A^{n-1}B].
\]

The controllability matrix of linear descriptor system (1) related to the slow and fast subsystems are given by \( C(A_1, B_1) \) and \( C(N, B_2) \), respectively. For the matrix pair \( (N, B_2) \) of fast subsystem, we have the controllable matrix,

\[
C(N, B_2) = [B_2 \ NB_2 \ \cdots \ N^{n_2-1}B_2].
\]

Since

\[
[B_2 \ NB_2 \ \cdots \ N^{n_2-1}B_2] = [B_2 \ NB_2 \ \cdots \ N^{h-1}B_2 \ 0].
\]

Thus, we can convert the controllability matrix \( C(N, B_2) = [B_2 \ NB_2 \ \cdots \ N^{h-1}B_2] \). Then we have the following theorem.

**Theorem 1.** [19]. The linear descriptor system (1) is controllable if and only if

\[
\text{Im } C(A_1, B_1) = \mathbb{R}^{n_1}; \ \text{Im } C(N, B_2) = \mathbb{R}^{n_2}.
\]

In view of the controllability of linear descriptor systems (1) as well as its slow and fast subsystems (3) and (4), we have the following theorem.

**Theorem 2.** [18, 19]. Consider the linear descriptor systems (1) and its slow and fast subsystems (3) and (4). Then the following condition is satisfied:

- The slow subsystem (3) is controllable if and only if \( \text{rank } C(A_1, B_1) = n_1 \) or equivalently \( \text{rank } [sl - A_1 \ B_1] = n_1, \forall s \in \mathbb{C}, s \text{ finite.} \)
- The fast subsystem (4) is controllable if and only if \( \text{rank } C(N, B_2) = n_2 \) or equivalently \( \text{rank } [N \ B_2] = n_2 \)
- The linear descriptor system (1) is controllable if and only if both its slow and fast sub systems (3) and (4) are controllable.

Thus we can say that the concept of controllability is only related to the terminal behavior for the system, but in the state response of the linear descriptor system, there are impulse terms that are determined either by initial conditions or by the possibility of jump behavior in control inputs and its derivatives. Therefore it is necessary to analyze the effect of control on the impulse term in the state response.
It is known that the state response of linear descriptor system, there are no impulse term in $x_1$ as a solution of slow subsystems when $u(t) \in \mathbb{C}^{h-1}_p$ but in the solutions of fast subsystems, it have the impulse term. It is determined by

$$x_{2i}(t) = -\sum_{l=1}^{h-1} N^l \delta^{(i-1)}(t) \left( x_{20} + \sum_{j=0}^{h-1} N^j B_2 u^{(j)}(0) \right). \quad (8)$$

This fact resulted by the initial values of the state vector and the control vector and its derivatives. Therefore, to eliminate the impulse in the state response by control that means we can choose the suitable control $u(t)$ such that the state response of the system at $t = 0$ is a limited number. Therefore, in view of this condition, we can give the following definition for $I$-controllability for linear descriptor systems.

**Definition 2.** [19]. The linear descriptor system (1) is called $I$-controllable, if for any vector $x_2(0) \in \mathbb{R}^{n_2}$ there exists an admissible control input $u(t) \in \mathbb{C}^{h-1}_p$ such that impulse part in the response of the fast subsystem given by (8) is identically zero, that is,

$$x_{2i}(t) = 0, \quad \forall t \geq 0.$$

Based on this definition and condition (8) that the following basic fact holds for linear descriptor systems (1).

**Proposition 1.** [19]. The linear descriptor system (1) is $I$-controllable if and only if for any vector $x_2(0) \in \mathbb{R}^{n_2}$ there exists an admissible control input $u(t) \in \mathbb{C}^{h-1}_p$ such that

$$Nx_2(0) + \left[ NB_2 \quad N^2 B_2 \quad \ldots \quad N^{h-1} B_2 \right] \left[ \begin{array}{c} u(0) \\ \ddots \\ u^{(h-2)}(0) \end{array} \right] = 0.$$

From this fact, we can say that the impulse controllability is important for the necessity to eliminate the impulse portions in a system in which impulse terms are generally not expected to appear. Otherwise, strong impulse behavior may stop the system from working or even destroy the system, (Dai, [18]).

We have the following theorem as a consequence of the above definition.

**Theorem 2.** [19]. The linear descriptor system (1) with its slow subsystem is always $I$-controllable.

From this theorem, the impulse controllability of linear descriptor system must be completely determined by the impulse controllability of its fast subsystem (4). This fact can be state in the following theorem.

**Theorem 3.** [18, 19]. Consider the linear descriptor systems (1) with its slow and fast subsystems (3) and (4) of the linear descriptor system (1). Then we have the following statement:

- The linear descriptor system (1) is $I$-controllable if and only if its fast subsystem (4) is $I$-controllable.
- The fast subsystem (3) is $I$-controllable if and only if one of the following conditions is holds.
  - (a). $Image \hat{N} = Image[ NB_2 \quad N^2 B_2 \quad \ldots \quad N^{h-1} B_2 ]$.
  - (b). $Ker \hat{N} + Im \hat{C}(N, B_2) = \mathbb{R}^{n_2}$.
  - (c). $Image \hat{N} + Ker \hat{N} + Im B_2 = \mathbb{R}^{n_2}$.
3. Main Results
In this section, the structural properties and definitions of the controllability for the linear descriptor system (1) are given without requiring that the system be viewed separately. Consider the linear descriptor systems (1) of the form
\[
\begin{align*}
&E\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t),
\end{align*}
\]
(9)

Let \( X = \mathbb{R}^n, U = \mathbb{R}^m \) and \( Y = \mathbb{R}^p \), where \( E, A : X \rightarrow X, B : U \rightarrow X \) and \( C : X \rightarrow Y \) are linear maps. We assumed \( E \) is singular and \( \det(sE - A) \neq 0 \). In this part, let the degree of \( \det(sE - A) \) is \( n_2 := r \). Thus, the state space \( \mathbb{R}^n \) can be decomposed into two subspaces \( X_1 \) and \( X_2 \):
\[
\mathbb{R}^n = X_1 \oplus X_2
\]
where \( \dim X_1 = r \). These subspaces \( X_1 \) and \( X_2 \) can be decomposed into the standard decomposition form of slow subsystem and fast subsystem of the form:
\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\
N\dot{x}_2(t) &= x_2(t) + B_2 u(t) \\
y(t) &= C_1 x_1(t) + C_2 x_2(t)
\end{align*}
\]
(10) (11) (12)
where \( x_1 \in X_1, x_2 \in X_2 \) and \( N \) is nilpotent of index \( h \), (Cobb, [8]).

In the linear descriptor system (9), let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) denote the distinct roots of the polynomial \( \det(sE - A) \) and let \( m_i \) is the multiplicity of \( \lambda_i \). Suppose that the \( \det(sE - A) \) can be stated as
\[
\det(sE - A) = \phi_0(s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \cdots (s - \lambda_k)^{m_k} = \phi_0 \prod_{i=1}^{k} (s - \lambda_i)^{m_i},
\]
where \( \phi_0 \neq 0 \) and \( \lambda_i \neq \lambda_j \) for \( i \neq j \). If we define \( \sigma(E, A) = \{ \lambda_1, \lambda_2, \ldots, \lambda_k \} \) and let \( \alpha \notin \sigma(E, A) \), then we have
\[
\det(\alpha E - A) = \phi_0 \prod_{i=1}^{k} (\alpha - \lambda_i)^{m_i} \neq 0.
\]
Therefore \( \alpha E - A \) is nonsingular.

Let \( \alpha \) be any real number such that \( \det(\alpha E - A) \neq 0 \). Multiply System (9) on the left side by \( (\alpha E - A)^{-1} \) then we get
\[
(\alpha E - A)^{-1} E \dot{x}(t) = (\alpha E - A)^{-1} Ax(t) + (\alpha E - A)^{-1} Bu(t).
\]
(13)
Let
\[
det(sI - (\alpha E - A)^{-1}E)) = s^{n-r} \prod_{i=1}^{k} (s - \frac{1}{\alpha - \lambda_i})^{m_i},
\]
where \( \sum_{i=1}^{k} m_i = r \) and \( n - r \) is the multiplicity of zero eigenvalue of \( (\alpha E - A)^{-1}E \). Defined \( X_1 \) and \( X_2 \) are a spectral subspace of \( (\alpha E - A)^{-1}E \) associated with its non-zero and zero eigenvalues, respectively:
\[
X_1 = \bigoplus_{i=1}^{k} \text{Ker} \left( (\alpha E - A)^{-1}E - \frac{1}{\alpha - \lambda_i}I \right)^{m_i}.
\]
\[
X_2 = \text{Ker} \left( (\alpha E - A)^{-1}E \right)^{n-r}.
\]
The subspaces $X_1$ and $X_2$ are invariant under $(aE - A)^{-1}E$. Let $T_1$ and $T_2$ be the restrictions of $(aE - A)^{-1}E$ to $X_1$ and $X_2$, respectively:

$$T_1 = (aE - A)^{-1}E|X_1$$
$$T_2 = (aE - A)^{-1}E|X_2$$

Since the $(aE - A)^{-1}A$ can be rewritten as $(aE - A)^{-1}A = a(aE - A)^{-1}E - I$, then $X_1$ and $X_2$ are invariant under $(aE - A)^{-1}A$. From this fact, we can make the restriction of $(aE - A)^{-1}A$ to $X_1$ and $X_2$ i.e.

$$(aE - A)^{-1}A|X_1 = aT_1 - I_1$$
$$(aE - A)^{-1}A|X_2 = aT_2 - I_2,$$

where $I_1$ and $I_2$ are the identity matrices. Furthermore, since $T_1$ has nonzero eigenvalue and $T_2$ is nilpotent, then $T_1$ and $aT_2 - I_2$ are nonsingular.

Let $V$ denote the projection onto $X_1$ along $X_2$ and $I - V$ denote the projection onto $X_2$ along $X_1$. Define $M_1 = V(aE - A)^{-1}B$ and $M_2 = (I - V)(aE - A)^{-1}B$ and let $K_1 = C|X_1$ and $K_2 = C|X_2$ denote the restrictions of $C$ to $X_1$ and $X_2$, respectively. From this result, we get the system (9) is equivalent to the decomposed system in the following:

$$T_1\dot{x}_1(t) = (aT_1 - I_1)x_1(t) + M_1u(t)$$
$$T_2\dot{x}_2(t) = (aT_2 - I_2)x_2(t) + M_2u(t)$$
$$y(t) = K_1x_1(t) + K_2x_2(t).$$

Since $T_1$ and $(aT_2 - I_2)$ are nonsingular, then the linear descriptor system (9) will be equivalent to the decomposition form as follows:

$$\dot{x}_1(t) = (aI_1 - T_1^{-1})x_1(t) + T_1^{-1}M_1u(t)$$
$$\dot{x}_2(t) = (aT_2 - I_2)x_2(t) + (aT_2 - I_2)^{-1}M_2u(t)$$
$$y(t) = K_1x_1(t) + K_2x_2(t).$$

Equation (17)-(19) is of the form (10)-(12).

In this subsection, we will be shown that the pair $((aE - A)^{-1}E,(aE - A)^{-1}B)$ plays an important role in the controllability for linear descriptor systems (9). Defined the subspaces for slow and fast subsystem (10)-(11) as follows:

$$S_1 := \sum_{k=0}^{h-1} \text{Im} A_1^k B_1; \quad S_2 := \sum_{k=0}^{h-1} \text{Im} N^k B_2,$$

and define subspace $S$ as a direct sum of subspaces of $S_1$ and $S_2$, we have

$$S := S_1 \oplus S_2.$$

Furthermore, is called the controllable subspace of linear descriptor systems (9).

The following result gives a characterization of $S$ which does not require that the linear descriptor systems (9) can be decomposed into the slow and fast subsystem (10) - (12). By extracting the result from Cobb, [16] and Zhou, et.al.[17], we have the following theorem.

**Theorem 3.** [17]. Given the linear descriptor systems (9) then the controllable subspace of systems (9) is equal to
\[ \sum_{k=0}^{n-1} \text{Im} \left( (\alpha E - A)^{-1} E \right)^k (\alpha E - A)^{-1} B, \]

where \( \alpha \) is any real number that satisfies \( \det(\alpha E - A) \neq 0. \)

**Proof:** Based on the system (17) it is known that \( S_1 \) is the smallest subspace which invariant to \( (\alpha l_1 - l_1^{-1}) \) contains \( \text{Im} \ T_1^{-1} M_1. \) Thus we have

\[ S_1 := \sum_{k=0}^{n-1} \text{Im} \left( (\alpha l_1 - l_1^{-1}) \right)^k T_1^{-1} M_1. \]

Since \( (\alpha l_1 - l_1^{-1}) \) can be rewritten as \( (\alpha l_1 - l_1^{-1}) = T_1^{-1} (l_1 - l_1) \), and using the exponential nature of the free response system (9) which determined by the eigenvalues of \( T_1 = (\alpha E - A)^{-1} E | X_1. \)

Therefore, this fact equivalent to \( S_1 \) is the smallest \( T_1 \)-invariant subspace which contains \( \text{Im} \ M_1. \) Thus we have

\[ S_1 = \sum_{k=0}^{n-1} \text{Im} \ T_1^k M_1. \]

Similarly for (18) it is obtained that

\[ S_2 = \sum_{k=0}^{n-1} \text{Im} \left( (\alpha T_2 - l_2)^{-1} T_2 \right)^k (\alpha T_2 - l_2)^{-1} M_2. \]

Since \( T_2 \) has the same Jordan form as \( (\alpha T_2 - l_2)^{-1} T_2 \), this implies that the impulse of the system (18) is determined by the eigen structure of \( T_2 = (\alpha E - A)^{-1} E | X_2. \) So it is obtained

\[ S_2 = \sum_{k=0}^{n-1} \text{Im} \ T_2^k M_2. \]

Using the properties of nil potency of \( T_2 \) and nonsingularity of \( T_1 \), then we have

\[
\begin{align*}
\sum_{k=0}^{n-1} \text{Im} \left( (\alpha E - A)^{-1} E \right)^k (\alpha E - A)^{-1} B &= \sum_{k=0}^{n-1} \text{Im} \left( T_1^k M_1 + T_2^k M_2 \right) \\
&= \sum_{k=0}^{n-1} \text{Im} \left( T_1^k M_1 + T_2^k M_2 \right) + \sum_{k=n-r}^{n-1} \text{Im} \ T_1^k M_1 \\
&= \sum_{k=0}^{n-1} \text{Im} \left( T_1^k M_1 + T_2^k M_2 \right) + \sum_{k=0}^{r-1} \text{Im} \ T_1^k M_1 \\
&= \sum_{k=0}^{n-1} \text{Im} \ T_2^k M_2 \bigoplus \sum_{k=0}^{r-1} \text{Im} \ T_1^k M_1 \\
&= S_2 \bigoplus S_1 = S.
\end{align*}
\]

This proves that the controllable subspace of linear descriptor system (9) is given by

\[ S = \sum_{k=0}^{n-1} \text{Im} \left( (\alpha E - A)^{-1} E \right)^k (\alpha E - A)^{-1} B, \]

where \( \alpha \) is any real number that satisfies \( \det(\alpha E - A) \neq 0. \)

According to [8], that the linear descriptor system (9) is controllable if and only if the controllable subspace equals to \( \mathbb{R}^n \), that is \( S = \mathbb{R}^n \). The following result is obtained as a direct consequence of Theorem 3.

**Corollary 1.** Given the linear descriptor systems (9). This system is controllable if and only if

\[ \sum_{k=0}^{n-1} \text{Im} \left( (\alpha E - A)^{-1} E \right)^k (\alpha E - A)^{-1} B = \mathbb{R}^n, \]

where \( \alpha \) is any real number that satisfies \( \det(\alpha E - A) \neq 0. \)
It is known that the controllability matrix of linear descriptor system (9) related to the slow and fast subsystems which are given by $C(A_1, B_1)$ and $C(N, B_2)$, respectively. However, in this result we give the definition of controllability matrix for linear descriptor system (9) without requiring that the system be decomposed into slow and fast subsystems.

**Definition 3.** Let the linear descriptor systems (9). The controllability matrix of system (9) is given by

\[
C(E, A, C) := \left[ (aE - A)^{-1}B, (aE - A)^{-1}E(aE - A)^{-1}B, ((aE - A)^{-1}E)^2(aE - A)^{-1}B, \ldots, ((aE - A)^{-1}E)^{n-1}(aE - A)^{-1}B \right]
\]  

The controllable subspace $S$ is a subspace of $\mathbb{R}^n$ generated by the columns of $C(E, A, C)$. Directly from Theorem 3 that the controllable subspace $S$ can be stated as:

\[
S = \text{Im } C(E, A, B) = \text{Im } \left[ (aE - A)^{-1}B, (aE - A)^{-1}E(aE - A)^{-1}B, ((aE - A)^{-1}E)^2(aE - A)^{-1}B, \ldots, ((aE - A)^{-1}E)^{n-1}(aE - A)^{-1}B \right] = \sum_{k=0}^{n-1} \text{Im } ((aE - A)^{-1}E)^k(aE - A)^{-1}B.
\]  

In the classical linear systems, the controllability of system can be determined from its rank. However, in the linear descriptor system, we have the rank test of controllability which it analogous to those for classical linear system. Therefore we have the following theorem as a consequence of this fact.

**Theorem 4.** The linear descriptor systems (9) is controllable if and only if the rank of controllability matrix $C(E, A, C)$ equals to the dimension of subspace $\mathbb{R}^n$, i.e.

\[
\text{rank } C(E, A, C) = n.
\]

### 4. Conclusion

The controllability of linear descriptor systems has been discussed by using a direct approach. Then, we have derived the properties of controllable subspaces for linear descriptor systems in geometric term. The controllable matrix has been defined without requiring that the system can be decomposed into slow and fast subsystems. In particular, we have the rank test of controllability which it analogous to those for classical linear system.

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