Higgs Potential from Weyl Conformal Gravity

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Abstract

We consider Weyl’s conformal gravity coupled to a complex matter field in Weyl geometry. It is shown that a Higgs potential naturally arises from a $\tilde{R}^2$ term in moving from the Jordan frame to the Einstein frame. A massless Nambu-Goldstone boson, which stems from spontaneous symmetry breakdown of the Weyl gauge invariance, is absorbed into the Weyl gauge field, thereby the gauge field becoming massive. We present a model where the gravitational interaction generates a Higgs potential whose form is a perfect square. Finally, we show that a theory in the Jordan frame is gauge-equivalent to the corresponding theory in the Einstein frame via the BRST formalism.

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1 Introduction

One of the most important purposes of modern particle physics is to find symmetries hidden in nature and study their breaking mechanism. It is somewhat surprising that there are not so many symmetries which are useful in clarifying the fundamental laws existing in nature owing to Coleman and Mandula’s no-go theorem [1] which states that no Lorentz non-scalar charges other than Poincare generators can be S-matrix symmetries in Poincare-invariant theories.

As one of such important symmetries which are not against the Coleman-Mandula theorem, we have a conformal symmetry. In particular, a scale symmetry in the conformal symmetry is a mysterious symmetry and occupies the special position in the sense that it always appears as an approximate symmetry even if it is ubiquitous from particle physics to cosmology [2]. The reason is that the scale symmetry prohibits definite mass or length scales but nature is full of various kinds of scales and consequently the scale symmetry is broken spontaneously or explicitly. Understanding of the origin of different mass scales could be a key step toward the resolution of the hierarchy problems such as the gauge hierarchy problem and the elusive cosmological constant problem.

In this article, we would like to study the theory with a global or a local scale symmetry, especially from the viewpoint of its spontaneous symmetry breakdown and emergence of the Higgs potential. As is well-known, a natural avenue of developments in quantum field theories is to extend a global scale symmetry to a local Weyl symmetry. This is in particular true of gravitational theories owing to no-hair theorems of black holes [3]. After a painful retreat due to the second clock problem [4], we have recently watched a revival of interests in Weyl’s conformal gravity [5]-[32] since this gravitational theory provides us with a playground for not only treating the Weyl symmetry but also supplying us with a candidate of dark matter, which is a massive Weyl gauge field.

This article is organized as follows: In Section 2, we review a Higgs potential emerging from an $R^2$ term in theories with a restricted Weyl symmetry.

\footnote{We sometimes refer to either a global or a local scale symmetry as a scale symmetry or a Weyl symmetry in this article.}
and a global scale symmetry and point out its problems. In Section 3, we show that a massless dilaton is absorbed into a Weyl gauge field and therefore the problems of cosmology [37, 17] and the fifth force [38], which are associated with the massless dilaton, are solved in Weyl’s conformal gravity. Furthermore, we present a new model where a Higgs potential arises from the gravity and discuss how we can obtain the electroweak scale from the Planck scale by selecting the parameters belonging to the gravitational sector. A peculiar feature of this Higgs potential is that it has the form of a perfect square so the cosmological constant at the minimum is identically vanishing. In Section 4, we rederive the Lagrangian density obtained by a change of variables in Section 3 through the BRST formalism, and show that the Lagrangian density in the Jordan frame is gauge-equivalent to that in the Einstein frame. The final section is devoted to conclusion.

2 Review of Higgs potential from $R^2$ term

We begin with a review of emergence of a Higgs potential from an $R^2$ term in a gravitational theory coupled to a $U(1)$ gauge theory with a complex scalar field whose Lagrangian density is given by [36]:

$$L = \sqrt{-g} \left( \xi_1^2 R^2 + \xi_2 R |\Phi|^2 - |D_\mu \Phi|^2 - \lambda |\Phi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),$$

where $\xi_1, \xi_2, \lambda$ are dimensionless coupling constants, $A_\mu, F_{\mu\nu}, D_\mu \Phi$ are respectively a $U(1)$ gauge field, its field strength defined as $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, a complex scalar field, and its covariant derivative defined as $D_\mu \Phi \equiv (\partial_\mu - i e A_\mu) \Phi$. The parameter $\xi_1^2$ is positive to avoid tachyons [39] and the gravitational coupling constant corresponds to $1/\xi_1^2$. Finally, note that the complex scalar field couples to the scalar curvature via a nonminimal coupling term $\xi_2 R |\Phi|^2$.

For simplicity of writing, in this section we drop the gauge field $A_\mu$ and we work with the following Lagrangian density:

$$L = \sqrt{-g} \left( \xi_2^2 R^2 + \xi_2 R |\Phi|^2 - |\partial_\mu \Phi|^2 - \lambda |\Phi|^4 \right).$$

We follow the conventions and notation of the MTW textbook [3].
This Lagrangian density is invariant under both a global scale transformation \((\Omega = \text{constant})\) and a restricted Weyl transformation

\[ g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \Phi \rightarrow \Phi' = \Omega^{-1}(x)\Phi, \quad (3) \]

where the gauge parameter obeys a constraint \(\Box \Omega = 0\) \([33]-[36]\). In order to prove the restricted Weyl invariance, we need to use the following transformation of the scalar curvature under (3):

\[ R \rightarrow R' = \Omega^{-2}(R - 6\Omega^{-1}\Box \Omega). \quad (4) \]

Now we are ready to show that a Higgs potential emerges for the Higgs field \(\Phi\) in addition to a scale-invariant potential term \(\lambda|\Phi|^4\) in (2). The first key observation is that the \(\xi R^2\) term can be cast to the form of the scalar-tensor gravity, \(\varphi R - \frac{1}{4\xi_1}\varphi^2\), where \(\varphi\) is a scalar field with the dimension of mass squared. Thus, the Lagrangian density (2) reads

\[ \mathcal{L} = \sqrt{-g} \left( \varphi R - \frac{1}{4\xi_1}\varphi^2 + \xi_2 R|\Phi|^2 - |\partial_\mu \Phi|^2 - \lambda|\Phi|^4 \right). \quad (5) \]

Next, let us move from the Jordan frame to the Einstein frame. To do so, we will do a change of variables, which takes the same form as a local conformal transformation

\[ g_{\mu\nu} \rightarrow g_{*\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \Phi \rightarrow \Phi_* = \Omega^{-1}(x)\Phi, \quad (6) \]

except the scalar field \(\varphi\).\(^5\) Henceforth, we express quantities in the Einstein frame by putting the symbol \(*\) on them.

Under this transformation (6) we have formulae \([40]\)

\[ \sqrt{-g} = \Omega^{-4}\sqrt{-g_*}, \quad R = \Omega^2(R_* + 6\Box_* F - 6g_*^{\mu\nu}F_\mu F_\nu), \quad (7) \]

\(^4\)In the conventional approach, we consider a conformal transformation of only the metric but it usually yields non-canonical kinetic terms and non-polynomial potentials. To avoid such a situation, we also consider a conformal transformation of the matter field.

\(^5\)As can be understood in the next model, we could also consider the conformal transformation of \(\varphi, \varphi \rightarrow \varphi_* = \Omega^{-2}(x)\varphi\), without changing the final result, but the method in hand is more useful in seeing the role of a conformal factor.
where we have defined

\[ F \equiv \log \Omega, \quad \Box_* F \equiv \frac{1}{\sqrt{-g_*}} \partial_{\mu}(\sqrt{-g_*} g_*^{\mu \nu} \partial_{\nu} F) , \quad F_{\mu} \equiv \partial_{\mu} F = \frac{\partial_{\mu} \Omega}{\Omega}. \tag{8} \]

Then, the Lagrangian density (5) is rewritten as

\[
\mathcal{L} = \sqrt{-g_*} \left[ (\varphi \Omega^{-2} + \xi_2 |\Phi_*|^2) (R_* - 6 \Box_* F - 6 g_*^{\mu \nu} F_{\mu} F_{\nu}) - \frac{1}{4 \xi_1^2} \varphi^2 \Omega^{-4} \right. \\
- \left. \Omega^{-2} g_*^{\mu \nu} \partial_{\mu}(\Omega \Phi^\dagger_*) \partial_{\nu}(\Omega \Phi_*) - \lambda |\Phi_*|^4 \right]. \tag{9} 
\]

To reach the Einstein frame, we have to choose a conformal factor \( \Omega(x) \) to satisfy a relation

\[ \varphi \Omega^{-2} = -\xi_2 |\Phi_*|^2 + \frac{M_{P_l}^2}{2}, \tag{10} \]

where \( M_{P_l} \) is the reduced Planck mass. As a result, with the redefinition \( \omega(x) \equiv \sqrt{6} M_{P_l} F(x) \), we obtain a Lagrangian density in the Einstein frame:

\[
\mathcal{L} = \sqrt{-g_*} \left[ \frac{M_{P_l}^2}{2} R_* - \frac{1}{2} g_*^{\mu \nu} \partial_{\mu} \omega \partial_{\nu} \omega - |\partial_{\mu} \Phi_*|^2 - \frac{1}{16 \xi_1^2} M_{P_l}^4 + \frac{\xi_2}{4 \xi_1^2} M_{P_l}^2 |\Phi_*|^2 \right. \\
- \left. \left( \lambda + \frac{\xi_2}{4 \xi_1^2} \right) |\Phi_*|^4 + \left( \frac{1}{\sqrt{6} M_{P_l}} \Box_* \omega - \frac{1}{6 M_{P_l}^2} g_*^{\mu \nu} \partial_{\mu} \omega \partial_{\nu} \omega \right) |\Phi_*|^2 \right]. \tag{11} \]

It is worthwhile to notice that spontaneous symmetry breakdown of a scale invariance has occurred and consequently we have a massless Nambu-Goldstone boson \( \omega(x) \), which is often called “dilaton”. Note that the kinetic term for the dilaton comes from \( g_*^{\mu \nu} F_{\mu} F_{\nu} \) in Eq. (9).

Let us now explain the reason why the spontaneous symmetry breakdown of a scale invariance has occurred in this model. (This reasoning can be also applied for the other models considered in this article with a suitable modification.) From the Lagrangian density (5), the full potential of the two scalar fields is given by

\[
V(\varphi, |\Phi|) = \sqrt{-g} \left( -\varphi R + \frac{1}{4 \xi_1^2} \varphi^2 - \xi_2 R |\Phi|^2 + \lambda |\Phi|^4 \right). \tag{12} \]

\(^6\)This redefinition implies that we use \( \omega(x) \) instead of \( \varphi(x) \) as a dynamical degree of freedom.
In order to understand a common ground state of the two scalar fields, we have to find (local) minima of the potential (12) in the both variables. For the minima, the gradient of the potential, \((\partial_{\phi}V, \partial_{\Phi}|V|)\), must vanish. The solution is given by

\[
\langle |\Phi|^2 \rangle = \frac{\xi_2}{4\lambda \xi_1^2} \langle \phi \rangle, \quad \langle \phi \rangle = 2\xi_1^2 \langle R \rangle,
\]

(13)

where \(\langle A \rangle\) denotes a vacuum expectation value for a generic field \(A\). We can also verify that these extremal values are indeed local minima by evaluating the Hessian provided that \(\xi_2 \langle R \rangle\) is positive. In the above theory, among the degenerate minima, we have chosen a specific configuration given by

\[
\langle \phi \rangle = \frac{2\lambda \xi_1^2}{\xi_2 + 4\lambda \xi_1^2} M_{Pl}^2,
\]

(14)

by which the spontaneous symmetry breakdown of a scale symmetry has been triggered. Note that since in this theory the vacuum expectation value of the scalar curvature is given by

\[
\langle R \rangle = \frac{\lambda}{\xi_2 + 4\lambda \xi_1^2} M_{Pl}^2,
\]

(15)

it is positive definite due to \(\lambda > 0\).

A remarkable thing in the theory under consideration is that a Higgs potential is generated from the \(R^2\) term together with the \(R|\Phi|^2\). Actually, a gauge symmetry is spontaneously broken if we choose the parameters to be

\[
\frac{\xi_2}{4\xi_1^2} > 0, \quad \lambda + \frac{\xi_2^2}{4\xi_1^2} > 0.
\]

(16)

Of course, we can construct similar models where spontaneous symmetry breakdown of a scale symmetry yields a Higgs potential. For comparison, let us present such a model whose Lagrangian density is composed of a scalar-tensor gravity with scale-invariant potentials made out of a real scalar field \(\phi\) and a complex scalar field \(\Phi\). To simplify the argument, we consider the case where only the real scalar \(\phi\) couples to a scalar curvature as follows \([26, 30]\):

\[
\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} \xi \phi^2 R - \frac{1}{2} (\partial_{\mu} \phi)^2 - |\partial_{\mu} \Phi|^2 - \lambda_1 \phi^4 - \lambda_2 \phi^2 |\Phi|^2 - \lambda_3 |\Phi|^4 \right],
\]

(17)
where we assume $\xi \neq -\frac{1}{6}$ to avoid the case of a local scale invariance, and $\lambda_i (i = 1, 2, 3)$ are dimensionless coupling constants. Following the same line of the argument as before, we do a change of variables in Eq. (6) as well as $\phi \rightarrow \phi_* = \Omega^{-1}(x)\phi$. Moving to the Einstein frame requires us to choose a conformal factor $\Omega(x)$ to be

$$\xi \phi^2 = \Omega^2 M_{Pl}^2,$$

or equivalently,

$$\xi \phi_*^2 = M_{Pl}^2.$$

Consequently, we can obtain the final expression

$$\mathcal{L} = \sqrt{-g_*} \left[ \frac{M_{Pl}^2}{2} R_* - \frac{1}{2} g_*^{\mu \nu} \partial_\mu \hat{\omega} \partial_\nu \hat{\omega} - |\mathcal{D}_\mu \Phi_*|^2 - \frac{\lambda_1}{\xi^2} M_{Pl}^4 \right. \\
- \left. \frac{\lambda_2}{\xi} M_{Pl}^2 |\Phi_*|^2 - \lambda_3 |\Phi_*|^4 \right],$$

(20)

where we have defined

$$\hat{\omega} \equiv \sqrt{\frac{6\xi + 1}{\xi}} M_{Pl} F, \quad \mathcal{D}_\mu \Phi_* \equiv \left( \partial_\mu + \sqrt{\frac{\xi}{6\xi + 1}} \frac{1}{M_{Pl}} \partial_\mu \hat{\omega} \right) \Phi_*.$$

(21)

Let us note that even in this simple model a Higgs potential emerges when a scale symmetry is spontaneously broken and as a result a Nambu-Goldstone boson $\hat{\omega}$ appears in the mass spectrum. However, there is a big difference: In (11), the Higgs potential arises from an $R^2$ term with the help of the nonminimal term while in (20) it comes from the scale-invariant potentials which already existed in the classical action. In this sense, we can state that the Higgs potential in (11) is of the gravitational origin.

To close this section, we should pick up two issues which will be clarified in Sections 3 and 4. First, it is known that the presence of a massless dilaton causes cosmological problems through the gravity at large scales since the dilaton couples to any fields in a universal manner [37, 17]. In addition, if the massless dilaton couples to the matter directly at the level of an action, the weak equivalence principle is violated, thereby yielding the fifth force,
but at present there is no such a force by experiments in the solar system [38].

Thus, the dilaton should either have a mass anyway or be absorbed into a
gauge field. Indeed, the dilaton could acquire a small mass via trace anomaly [40]. In the next section, we will pursue an alternative possibility that the
massless dilaton is absorbed into a Weyl gauge field, thereby the Weyl gauge
field becoming massive and at the same the massless dilaton disappearing
from the mass spectrum. Moreover, we will present a model where a Higgs
potential whose form is a perfect square, is generated by the gravitational
interaction.

As a second issue, there is an ongoing debate with a long history on th e
equivalence between the Jordan frame and the Einstein frame [41]. In the
derivation of the Higgs potentials, we have heavily relied on the equivalence
between the Jordan frame and the Einstein frame, so it would be more de-
sirable if we could derive the Higgs potentials via a different but more sound
method. To this end, in Section 4 we will use the BRST formalism and
prove the quantum equivalence of the theory between the Jordan frame and
the Einstein one. This proof is one of advantages in our formalism in the
sense that the existence of a local scale invariance makes it possible to show
a gauge-equivalence between the two frames.

3 Weyl’s conformal gravity and Higgs poten-
tial

To remove a massless Nambu-Goldstone boson, i.e., the dilaton, from the
mass spectrum, we make use of Weyl’s conformal gravity where a global scale
invariance is promoted to a local scale one. When we consider a $U(1)$ gauge
theory with a complex scalar field coupled to the gravity in Weyl geometry,
the most general Lagrangian density, which is invariant under Weyl gauge
transformation (discussed shortly), is given by $^7$

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2\xi_0} \tilde{C}_{\mu\nu\rho\sigma} \tilde{C}^{\mu\nu\rho\sigma} + \xi_1 \tilde{R}^2 + \xi_2 \tilde{R} |\Phi|^2 - \frac{1}{4} H_{\mu\nu} H^{\mu\nu}\right)$$

\(^7\text{We have used the conventions and notation in Ref. [26] for the Weyl geometry.}\)
\[ - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - |D_\mu \Phi|^2 - \lambda |\Phi|^4 \], (22)\]

where \( \tilde{C}_{\mu \nu \rho \sigma} \), \( \tilde{R} \) are respectively the conformal tensor and scalar curvature in the Weyl geometry. For instance, \( \tilde{R} \) is defined as

\[ \tilde{R} \equiv g^{\mu \nu} \tilde{R}_{\mu \nu} = R - 6 f \nabla_\mu S^\mu - 6 f^2 S_\mu S^\mu, \] (23)

where \( R, S_\mu \), and \( f \) are the scalar curvature in the Riemann geometry, the Weyl gauge field, and the coupling constant for a noncompact Abelian group, respectively. Moreover, \( H_{\mu \nu} \) is the field strength of the Weyl gauge field and \( D_\mu \Phi \) is a covariant derivative, which are defined as

\[ H_{\mu \nu} = \partial_\mu S_\nu - \partial_\nu S_\mu, \quad D_\mu \Phi = (\partial_\mu - f S_\mu - ie A_\mu) \Phi. \] (24)

For simplicity of writing, we will put \( \xi_0^{-2} = \lambda_\mu = 0 \) and work with the following Lagrangian density:

\[ \mathcal{L} = \sqrt{-g} \left( \xi_1^2 \tilde{R}^2 + \xi_2 \tilde{R} |\Phi|^2 - \frac{1}{4} H_{\mu \nu} H^{\mu \nu} - |D_\mu \Phi|^2 - \lambda |\Phi|^4 \right), \] (25)

where we now have \( D_\mu \Phi = (\partial_\mu - f S_\mu) \Phi \) since we have dropped the \( U(1) \) gauge field \( A_\mu \) in Eq. (24). The Weyl gauge transformation reads

\[ g_{\mu \nu} \rightarrow g'_{\mu \nu} = \Omega^2(x) g_{\mu \nu}, \quad \Phi \rightarrow \Phi' = \Omega^{-1}(x) \Phi, \quad S_\mu \rightarrow S'_\mu = S_\mu - \frac{1}{f} \partial_\mu \log \Omega. \] (26)

The argument proceeds in the same fashion as that in Section 2. By introducing a scalar field \( \varphi \), we can rewrite (25) into the form

\[ \mathcal{L} = \sqrt{-g} \left( \left( \varphi + \xi_1 |\Phi|^2 \right) \tilde{R} - \frac{1}{4 \xi_1^2} \varphi^2 - \frac{1}{4} H_{\mu \nu} H^{\mu \nu} - |D_\mu \Phi|^2 - \lambda |\Phi|^4 \right). \] (27)

To move from the Jordan frame to the Einstein frame, we will do a change of variables, which is the Weyl gauge transformation except the scalar field \( \varphi \):

\[ g_{\mu \nu} \rightarrow g_{*\mu \nu} = \Omega^2(x) g_{\mu \nu}, \quad \Phi \rightarrow \Phi_* = \Omega^{-1}(x) \Phi, \]

\[ S_\mu \rightarrow S_{*\mu} = S_\mu - \frac{1}{f} \partial_\mu \log \Omega \equiv S_\mu - \frac{1}{f} \partial_\mu F. \] (28)
As a result, we find that

\[
L = \sqrt{-g} \left( \varphi \Omega^{-2} + \xi_2 |\Phi_*|^2 \right) (R_* - 6 f g_*^{\mu\nu} \nabla_{\mu\nu} S_{\mu\nu} - 6 f^2 g_*^{\mu\nu} S_{\mu\nu} S_{\mu\nu}) \\
- \frac{1}{4 \xi_1^2} g_*^{\mu\nu} g_*^{\alpha\beta} H_{\mu\alpha} H_{\nu\beta} - g_*^{\mu\nu} D_{\mu\nu} \Phi^\dagger D_{\mu\nu} \Phi_* - \lambda |\Phi_*|^4 \right),
\]

where we have defined

\[
\nabla_{\mu\nu} S_{\mu\nu} \equiv \partial_{\mu} S_{\nu\mu} - \Gamma^\lambda_{\mu\nu\lambda} S_{\lambda}, \quad D_{\mu\nu} \Phi_* \equiv (\partial_{\mu} - f S_{\mu\nu}) \Phi_*.
\]

Fixing a conformal factor as in (10) leads to the final expression

\[
L = \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R_* - \frac{1}{2} m_s^2 g_*^{\mu\nu} S_{\mu\nu} S_{\mu\nu} - \frac{1}{4} g_*^{\mu\nu} g_*^{\alpha\beta} H_{\mu\alpha} H_{\nu\beta} - |D_{\mu\nu} \Phi_*|^2 \right. \\
- \left. \lambda |\Phi_*|^4 + \frac{1}{4 \xi_1^2} \left( \xi_2 |\Phi_*|^2 - \frac{M_{Pl}^2}{2} \right)^2 \right],
\]

where the mass of the Weyl gauge field is \( m_s = \sqrt{6} f M_{Pl} \) and \( H_{\mu\nu} = \partial_{\mu} S_{\nu\mu} - \partial_{\nu} S_{\mu\nu} \). Comparing with the case of a global scale symmetry, in this case \( F_{\mu} \) is replaced with \( S_{\mu\nu} \), and as seen in (28) the Nambu-Goldstone boson \( F \) is absorbed into the Weyl gauge field \( S_{\mu\nu} \). In this way, we have shown in an explicit manner that the Nambu-Goldstone boson, which comes from spontaneous symmetry breakdown of the Weyl gauge symmetry, is absorbed into the Weyl gauge field, thereby the Weyl gauge having the mass \( m_s \). Since we have no more a massless dilaton, we are now free from problems of both cosmology and the fifth force associated with the dilaton.

Next, let us notice that after spontaneous symmetry breakdown of the Weyl gauge symmetry, not only the massive Weyl gauge field but also a new Higgs potential have appeared in Eq. (31). The Higgs potential can be read as

\[
V(\Phi_*) = \lambda |\Phi_*|^4 + \frac{1}{4 \xi_1^2} \left( \xi_2 |\Phi_*|^2 - \frac{M_{Pl}^2}{2} \right)^2 \\
= \left( \lambda + \frac{\xi_2^2}{4 \xi_1^2} \right) |\Phi_*|^2 - \frac{\xi_2}{2(\xi_2^2 + 4 \lambda \xi_1^2)} M_{Pl}^2 \\
+ \frac{\lambda}{4(\xi_2^2 + 4 \lambda \xi_1^2)} M_{Pl}^4.
\]
Then, we obtain a minimum of the potential and a cosmological constant

\[ \langle |\Phi|^2 \rangle = \frac{\xi_2}{2(\xi_2^2 + 4\lambda \xi_1^2)} M_{Pl}^2 \equiv \frac{v^2}{2}, \quad \Lambda = \frac{\lambda}{4(\xi_2^2 + 4\lambda \xi_1^2)} M_{Pl}^2, \]  

(33)

where \( v \approx 250 \text{GeV} \).

The first equality in Eq. (33) yields a relation

\[ \frac{\xi_2}{\xi_2^2 + 4\lambda \xi_1^2} = \left( \frac{v}{M_{Pl}} \right)^2 \simeq 10^{-32}. \]  

(34)

Next, using (34), the second equality in Eq. (33) produces a value of the cosmological constant

\[ \Lambda = \frac{\lambda}{4 \xi_2} v^2. \]  

(35)

A key observation here is that the parameters \( \xi_1 \) and \( \xi_2 \) are not limited to satisfy the various experimental constraints owing to the absence of the Einstein-Hilbert term in (25) [39]. Though we can choose any values of them, an interesting choice for \( \xi_2 \) might be \( \xi_2 \approx 10^{5-6} \) which comes from the Higgs inflation [42]. In any case, assuming that \( \lambda \simeq 1, \xi_2 \ll \xi_1 \), we choose \( \xi_1 \) and \( \xi_2 \) to satisfy

\[ \frac{\xi_2}{\xi_1^2} \approx \left( \frac{v}{M_{Pl}} \right)^2 \simeq 10^{-32}. \]  

(36)

Since \( \frac{1}{\xi_1} \) corresponds to the gravitational coupling, Eq. (36) means a very weak coupling constant. Furthermore, as seen in Eq. (35), it is true that the cosmological constant is not so small compared to the magnitude that the cosmological observation implies, but it is very small compared to the Planck mass squared and becomes smaller for the larger \( \xi_2 \).

To close this section, it is of interest to imagine that the Higgs potential entirely comes from the gravity by putting \( \lambda = 0 \). Under such a situation, the Higgs potential in (32) is a perfect square with a positive coefficient. This form of the Higgs potential implies two important facts; the automatic stability of the ground state and no cosmological constant at the minimum. In addition, this potential has emerged from the requirement that the Einstein-Hilbert term should appear. Otherwise, we could never have general relativity.
at low energies, which is against our world. To put it differently, the appearance of general relativity at low energies naturally leads to spontaneous symmetry breaking of the gauge symmetry to occur. This situation should be contrasted to the conventional situation in the standard model: In the standard model, the renormalizability of the theory requires that the Higgs potential takes a rather simple form

\[ V(\Phi) = m^2|\Phi|^2 + \lambda|\Phi|^4, \]  

(37)

to radiative corrections. For spontaneous symmetry breaking to occur, the renormalized value of the parameter \( m^2 \) should be negative. But even the qualitative prediction that the symmetry is broken is not a prediction of the model. The parameter \( m^2 \) could have either sign; there is no logic that prefers one sign to the other. On the other hand, in our theory, the existence of general relativity at low energies naturally leads to the spontaneous symmetry breakdown of the gauge symmetry.

4 Derivation from BRST formalism

We wish to understand the quantum equivalence between (25) and (31). In deriving (31) in the Einstein frame from (25) in the Jordan frame, we have heavily used the change of variables in Eq. (28). Usually in quantum field theory, the change of variables does not modify the physical content, but there has been a prolonged controversy about the quantum equivalence of the theory between the two frames [41].

In this section, to clarify this issue, we wish to derive (31) in the Einstein frame by beginning with (25) in the Jordan frame through a different but more sound formalism, that is, the BRST formalism. The key idea is to show that the Lagrangian density in the Jordan frame is gauge-equivalent to that in the Einstein frame by taking a suitable gauge fixing condition for the Weyl symmetry.

To do that, we fix the Weyl symmetry in such a way that a gauge fixing condition breaks only the Weyl invariance but leaves the general coordinate invariance unbroken. Then, a suitable gauge condition is

\[ \tilde{R} + a|\Phi|^2 = b, \]  

(38)
where $a, b$ are constants. This gauge choice certainly breaks only the Weyl invariance since the LHS has a Weyl weight $-2$ whereas the RHS does a vanishing Weyl weight, and the both sides are invariant under the general coordinate transformation. Incidentally, we cannot put $b = 0$ ("Landau gauge") since the quantities on the LHS of the gauge condition (38) transform only by an overall scale factor $\Omega^{-2}(x)$ under the Weyl gauge transformation.

Let us recall that the BRST transformation for the Weyl symmetry is given by [28, 29]

\[
\begin{align*}
\delta_B g_{\mu\nu} &= 2cg_{\mu\nu}, \quad \delta_B \sqrt{-g} = 4c\sqrt{-g}, \quad \delta_B \tilde{R} = -2c\tilde{R}, \\
\delta_B \Phi &= -c\Phi, \quad \delta_B \bar{c} = iB, \quad \delta_B c = \delta_B B = 0.
\end{align*}
\] (39)

Then, we find that a Lagrangian density for the gauge condition and the FP ghost reads [43]

\[
\mathcal{L}_{GF+FP} = -i\delta_B \left[ \sqrt{-g} \bar{c} \left( \tilde{R} + a|\Phi|^2 - b + \frac{\alpha}{2}B \right) \right] \\
= \sqrt{-g} \left[ \tilde{B} \left( \tilde{R} + a|\Phi|^2 - b \right) + \frac{\alpha}{2} \tilde{B}^2 - 2ib\bar{c}c \right] \\
= \sqrt{-g} \left[ \frac{1}{2\alpha} \left( \tilde{R} + a|\Phi|^2 - b \right)^2 - 2ib\bar{c}c \right] \\
= -\sqrt{-g} \frac{1}{2\alpha} \left( \tilde{R} + a|\Phi|^2 - b \right)^2 \\
- i\hbar\delta^4(0) \log \left( b\sqrt{-g}(x) \right),
\] (40)

where we have defined $\tilde{B} \equiv B + 2i\bar{c}c$, we performed the path integral over the auxiliary field $\tilde{B}$, and in the last step we have done the integration over the FP ghosts [28, 29]. The last term proportional to $\delta^4(0)$ has also appeared in Ref. [13], which we can neglect when we use the dimensional regularization.

Now, adding the gauge-fixing term (40) to the Lagrangian density (25) in the Jordan frame, we can obtain a gauge-fixed and BRST-invariant Lagrangian density given by

\[
\mathcal{L} = \sqrt{-g} \left[ \left( \xi_1^2 - \frac{1}{2\alpha} \right) \tilde{R}^2 + \left( \xi_2 - \frac{a}{\alpha} \right) \tilde{R}|\Phi|^2 - \left( \lambda + \frac{a^2}{2\alpha} \right)|\Phi|^4 \\
- \frac{1}{4} H^{\mu\nu} H_{\mu\nu} - |D_\mu \Phi|^2 - \frac{b^2}{2\alpha} + \frac{b}{\alpha} \tilde{R} + \frac{ab}{\alpha}|\Phi|^2 \right].
\] (41)
It turns out that this Lagrangian density (41) coincides with the Lagrangian density (31) in the Einstein frame when the parameters satisfy the following relations

$$\alpha = \frac{1}{2\xi_1^2}, \quad a = \frac{\xi_2^2}{2\xi_1^2}, \quad b = \frac{1}{4\xi_1^2}M_{Pl}^2.$$  \hfill (42)

Actually, under the conditions (42), Eq. (41) is reduced to (31) (without the symbol * which simply means the Einstein frame, but is irrelevant to the present context) as follows:

$$L = \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} \hat{R} - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} - |D_\mu \Phi|^2 \right. $$

$$\quad \left. - \frac{1}{16\xi_1^2} M_{Pl}^4 + \frac{\xi_2^2}{4\xi_1^2} M_{Pl}^2 |\Phi|^2 - \left( \lambda + \frac{\xi_2^2}{4\xi_1^2} \right) |\Phi|^4 \right]$$

$$= \sqrt{-g} \left[ M_{Pl}^2 \frac{R}{2} - \frac{1}{2} m_s^2 S_\mu S^\mu - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} - |D_\mu \Phi|^2 \right.$$

$$\quad \left. - \lambda |\Phi|^4 - \frac{1}{4\xi_1^2} \left( \xi_2 |\Phi|^2 - \frac{M_{Pl}^2}{2} \right)^2 \right].$$  \hfill (43)

Thus, we have derived the Lagrangian density in the Einstein frame by starting with that in the Jordan frame in the framework of the BRST formalism. The approach based on the BRST formalism is free from the problem associated with the functional measure and provides a rather reliable method which demonstrates the equivalence between the Jordan frame and the Einstein frame at the quantum level.

### 5 Conclusion

In this article, we have investigated a possibility that the Higgs potential is generated from the gravity. To make this idea be more realistic, we have to solve two problems, one of which is related to the presence of a massless dilaton and the other is the quantum equivalence of the theory between the Jordan frame and the Einstein frame.

We have solved the former problem by extending a global scale invariance to a local scale one where it turns out that the Weyl geometry provides a
natural arena for formulating the local scale invariance as the Weyl gauge invariance. In our theory, the Weyl gauge field becomes massive by eating the massless dilaton and its magnitude of the mass is of order of the Planck mass with the Abelian coupling constant $f \simeq 1$, so the Weyl gauge field might be a candidate for dark matter.

As a resolution of the latter problem, an ongoing debate on the equivalence of the theory between the Jordan frame and the Einstein frame, we have used the BRST formalism which does not depend on the definition of the functional measure, and we have shown that up to a factor $\delta^4(0)$ which can be ignored in the dimensional regularization procedure, the theory in the both frames is gauge-equivalent.

It is remarkable that an $R^2$ term with the nonminimal coupling term $R|\Phi|^2$ gives us a Higgs potential of a perfect square, by which the problem of the negative tachyonic mass in the Higgs potential and the cosmological constant problem are solved. Boldly speaking, a complete resolution of the origin of the Higgs potential in the standard model amounts to the problem of why the bare quartic interaction $\lambda|\Phi|^4$ is zero.

Finally, we wish to mention some future problems. It is straightforward to extend the present theory to the standard model and the grand unified models by extending the gauge group and the definition of the covariant derivatives. Another interesting question is to verify that the massive Weyl gauge field could be really a candidate of dark matter by the explicit calculation. The other problem is to introduce a manifestly scale-invariant regularization technique to treat with the Weyl gauge symmetry without anomalies. We wish to return these problems in future.

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