Proximal Causal Inference with Hidden Mediators: Front-Door and Related Mediation Problems

AmirEmad Ghassami\textsuperscript{1}, Ilya Shpitser\textsuperscript{1}, and Eric Tchetgen Tchetgen\textsuperscript{2}

\textsuperscript{1}Department of Computer Science, Johns Hopkins University
\textsuperscript{2}Department of Statistics, The Wharton School, University of Pennsylvania

Abstract

Proximal causal inference was recently proposed as a framework to identify causal effects from observational data in the presence of hidden confounders for which proxies are available. In this paper, we extend the proximal causal approach to settings where identification of causal effects hinges upon a set of mediators which unfortunately are not directly observed, however proxies of the hidden mediators are measured. Specifically, we establish (i) a new hidden front-door criterion which extends the classical front-door result to allow for hidden mediators for which proxies are available; (ii) We extend causal mediation analysis to identify direct and indirect causal effects under unconfoundedness conditions in a setting where the mediator in view is hidden, but error prone proxies of the latter are available. We view (i) and (ii) as important steps towards the practical application of front-door criteria and mediation analysis as mediators are almost always error prone and thus, the most one can hope for in practice is that our measurements are at best proxies of mediating mechanisms. Finally, we show that identification of certain causal effects remains possible even in settings where challenges in (i) and (ii) might co-exist.

Keywords: Proximal Causal Inference, Mediation Analysis, Front-Door Model, Direct and Indirect Effects, Confounding.

1 Introduction

Majority of the work in the literature on causal inference from observational data posit the strong assumption that there are no unmeasured confounders of the treatment and the outcome variables in the system. The recently proposed framework of proximal causal inference (Miao et al., 2018; Tchetgen Tchetgen et al., 2020; Cui et al., 2020) has established a step forward towards relaxing the no unconfoundedness assumption. More specifically, this framework allows latent confounders of the treatment and outcome, yet requires that two proxies of the latent confounder, which satisfy certain conditional independence conditions, should be available.

In this paper we demonstrate that the power of proximal causal inference framework goes beyond controlling for unmeasured confounders. We consider two setups in which there exists a (possibly multivariate) mediator in the system, observations of which is needed for identification of a certain causal quantity of interest in the existing methods. We demonstrate that if instead we have access to proxies of the mediator which satisfy certain properties, the causal quantities of interest are still identifiable. An important example of such proxies is when we have measurements with error from the variable of interest. Hence, the contribution could be considered as theoretical characterization of sufficiently descriptive measurements of a variable of interest which renders identification possible.
Two key causal models we consider are (i) Front-Door Model for which we extend Pearl’s original identification result to allow for hidden mediators, in a setting where confounding of the treatment effects on the outcome is intractable; (ii) Identification of direct and indirect causal effects with respect to a set of hidden mediators for which imperfect proxy measurements are available under standard no unmeasured confounding conditions. Furthermore, we demonstrate that important causal effects remain identified in settings where challenges present in (i) and (ii) might co-exist. Specifically, a corollary of our results is proximal identification of an indirect causal effect mediated by hidden intermediate factors for which error prone proxies are available, in a setting where treatment-outcome confounding is intractable. Notably, unlike in front-door case, this latter setting does not rule out a direct effect of the treatment on the outcome through a pathway not mediated by hidden intermediate variables for which proxies are available, thus effectively extending the generalized front-door result of Fulcher et al. (2020) to the hidden mediator setting.

For the case of dealing with unmeasured confounders, the proximal causal inference framework requires estimating nuisance functions, called bridge functions, which are solutions to Fredholm integral equations of the first kind. We demonstrate that the same challenge exists in the case of dealing with unmeasured mediators. We present a non-parametric estimation approach for estimating the bridge functions. The estimation technique is based on an adversarial learning method recently introduced by [Dikkala et al. (2020)](Dikkala et al., 2020), which was adopted to the original proximal causal inference framework for handling latent confounders in [Ghassami et al. (2021)](Ghassami et al., 2021). We extend the application of this approach for estimating our causal parameters of interest in mediation, front-door, and generalized front-door setups in the presence of unmeasured mediators.

## 2 Mediation and Front-Door Models

We consider the following two models.

### 2.1 Mediation Model

We consider a setup, comprised of a treatment variable $A$, an outcome variable $Y$, a mediator variable $M$, and observed pre-treatment covariates $X$, where the variables satisfy the following conditions:

- **Positivity:**

  \[ p(m \mid A, X) > 0 \text{ with probability } 1 \ \forall m, \]
  \[ p(a \mid X) > 0 \text{ with probability } 1 \ \forall a. \]

- **Consistency:**

  \[ M^{(a)} = M \text{ if } A = a, \]
  \[ Y^{(a,m)} = Y \text{ if } A = a, M = m. \]

- **Sequential Exchangeability:** For any $a, a' \in \mathcal{X}_A$ and $m \in \mathcal{X}_M$,

  \[ \{Y^{(a,m)}, M^{(a')}\} \perp \perp A \mid X, \]
  \[ Y^{(a,m)} \perp \perp M^{(a')} \mid X. \]

Figure 1 demonstrates a graphical model which satisfies the conditions of the mediation model.
Our parameter of interest is $\psi_1 = \mathbb{E}[Y^{(a',M^{(a)})}]$, which is used for defining natural direct and indirect effects (Pearl 2001). In the case that the mediator is observed, parameter $\psi_1$ can be identified as follows.

$$
\psi_1 = \mathbb{E}[Y^{(a',M^{(a)})}] \\
= \sum_{m,y,x} yp(y^{(a',m)} = y \mid x)p(M^{(a)} = m \mid x)p(x) \\
= \sum_{m,y,x} yp(y \mid a', m, x)p(m \mid a, x)p(x).
$$

### 2.2 Front-Door Model

We consider a setup, comprised of a treatment variable $A$, an outcome variable $Y$, a mediator variable $M$, and observed pre-treatment covariates $X$, where the variables satisfy the following conditions:

- **Positivity:**
  
  $p(m \mid A, X) > 0$ with probability 1 $\forall m$,  
  $p(a \mid X) > 0$ with probability 1 $\forall a$.

- **Consistency:**
  
  $M^{(a)} = M$ if $A = a$,  
  $Y^{(a,m)} = Y$ if $A = a, M = m$.

- **Exchangeability:** For any $a \in \mathcal{X}_A$ and $m \in \mathcal{X}_M$,  
  
  $M^{(a)} \perp \!\!\!\!\perp A \mid X$,  
  $Y^{(m)} \perp \!\!\!\!\perp M \mid A, X$.

- **Exclusion restriction:** For any $a \in \mathcal{X}_A$ and $m \in \mathcal{X}_M$,  
  
  $Y^{(a,m)} = Y^{(m)}$.

Figure 3 demonstrates a graphical model which satisfies the conditions of the front-door model.
Our parameter of interest is $\psi_2 = \mathbb{E}[Y^{(a)}]$, which in the case that the mediator is observed, can be identified as follows.

$$
\psi_2 = \mathbb{E}[Y^{(a)}] = \mathbb{E}[Y^{(a,M^{(a)})}]
= \sum_{m,y,x} yp(Y^{(a,m)} = y \mid x)p(M^{(a)} = m \mid x)p(x)
= \sum_{a',m,y,x} yp(y \mid a',m,x)p(m \mid a,x)p(a' \mid x)p(x).
$$

### 3 Handling Hidden Mediators Using Proxies

Now, suppose the mediator variable $M$ is not observed, yet we have access to proxies $W$ and $Z$ of this variable, satisfying the following conditions.

**Assumption 1.** There exists Proxies $Z$ and $W$ of the mediator $M$ which satisfy

- $Y \perp \perp Z \mid \{A, M, X\}$,
- $W \perp \perp \{A, Z\} \mid \{M, X\}$.

Figures 2 and 4 demonstrate examples of graphical models which satisfy the proxy variable conditions for the mediation model and front-door model, respectively. In these figures, dashed edges can be present or absent.

In order to obtain identifiability, we consider the following assumptions.

**Assumption 2.**

(i) For any $a$ and $x$, if $\mathbb{E}[g(M) \mid Z, a, x] = 0$ almost surely, then $g(M) = 0$ almost surely.

(ii) There exists an outcome bridge function $h(w, a, x)$ that solves the following integral equation

$$
\mathbb{E}[Y \mid Z, A, X] = \mathbb{E}[h(W, A, X) \mid Z, A, X].
$$

We have the following non-parametric identification result.
Theorem 1. Under Assumptions 1 and 2, parameters $\psi_1$ and $\psi_2$ are non-parametrically identified by
\[
\psi_1 = \sum_{w,x} h(w, a', x)p(w \mid a, x)p(x), \\
\psi_2 = \sum_{w,a',x} h(w, a', x)p(w \mid a, x)p(a' \mid x)p(x). \tag{2}
\]

3.1 An Alternative Identification Method

Next, we establish an alternative proximal identification result based on the following counterpart of Assumption 2.

Assumption 3.
(i) For any $a$ and $x$, if $E[g(M) \mid W, a, x] = 0$ almost surely, then $g(M) = 0$ almost surely.
(ii) For any given target value $A = a$, there exists a treatment bridge function $q_a(z, a', x)$ that solves the following integral equation
\[
E[q_a(Z, A, X) \mid W, A = a', X] = \frac{p(W \mid A = a, X)}{p(W \mid A = a', X)}. \tag{3}
\]

Proposition 1. The function $q_a$ can be estimated using the following conditional moment equation:
\[
E\left[\frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) - \frac{I(A = a)}{p(A = a \mid X)} \mid W, X\right] = 0.
\]

Next we provide a result regarding semiparametric estimation of the function $q_a$.

Theorem 2. Assume $q_a$ is parametrically identified by parameter $\theta$. Then all influence functions of regular and asymptotically linear estimators of $\theta$ are of the form
\[
- \left\{E[\partial \theta, q_a(Z, A, X; \theta)g(W, A, X)]\right\}^{-1} \left\{q_a(Z, A, X)g(W, A, X) - \sum_w g(w, A, X)p(w \mid a, X) - \frac{I(A = a)}{p(a \mid X)} \left\{\sum_{a'} g(W, a', X)p(a' \mid X)p(w \mid a, X)\right\}\right\},
\]
for some function $g$ of same dimension as $\theta$.

We have the following non-parametric identification result.

Theorem 3. Under Assumptions 1 and 2, parameters $\psi_1$ and $\psi_2$ are non-parametrically identified by
\[
\psi_1 = \mathbb{E}\left[\frac{I(A = a')}{p(A = a' \mid X)} Yq_a(Z, A, X)\right], \\
\psi_2 = \mathbb{E}\left[Yq_a(Z, A, X)\right]. \tag{4}
\]

4 Estimating the Bridge Functions

In this section we present a non-parametric estimation approach for the bridge functions $h$ and $q_a$. The estimator is based on an adversarial learning method recently introduced by Dikkala et al. (2020). See Ghassami et al. (2021) for adoption of this approach to the original proximal causal inference framework for handling latent confounders.
Proposition 2. The bridge functions $h$ and $q_a$ can be estimated by the following Tikhonov regularization-based optimizations.

$$
\hat{h} = \arg \min_{h \in \mathcal{H}} \sup_{q \in \mathcal{Q}} \mathbb{E}\left[ \{h(W, A, X) - Y\}q(Z, A, X) - q^2(Z, A, X) \right] - \lambda_Q^2 \|q\|_{\mathcal{Q}}^2 + \lambda^2_{\mathcal{H}} \|h\|_{\mathcal{H}}^2.
$$

(5)

$$
\hat{q}_a = \arg \min_{q \in \mathcal{Q}} \sup_{h \in \mathcal{H}} \mathbb{E}\left[ \left\{ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) - \frac{I(A = a)}{p(A = a \mid X)} \right\}h(W, X) - h^2(W, X) \right] - \lambda_Q^2 \|h\|_{\mathcal{H}}^2 + \lambda^2_{\mathcal{Q}} \|q\|_{\mathcal{Q}}^2,
$$

(6)

where any function classes such as reproducing kernel Hilbert spaces or a class of neural networks can be used for $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{Q}$.

Proof is similar to [Dikkala et al., 2020; Ghassami et al., 2021] and hence omitted.

5 Semiparametric Inference

Theorem 4. Under Assumptions [13] an influence function of the parameters $\psi_1$ and $\psi_2$ is given by

$$
IF_{\psi_1}(O) = \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \{Y - h(W, A, X)\}
$$

$$
+ \frac{I(A = a)}{p(A = a \mid X)} \{h(W, a', X) - \eta_1(X, a', a)\} + \eta_1(X, a', a) - \psi_1,
$$

$$
IF_{\psi_2}(O) = q_a(Z, A, X) \{Y - h(W, A, X)\}
$$

$$
+ \frac{I(A = a)}{p(A = a \mid X)} (\hat{h}(W, X) - \eta_2(X, a)) + \eta_1(X, A, a) - \psi_2,
$$

where

$$
\eta_1(x, a', a) := \mathbb{E}[h(W, a', X) \mid A = a, X = x]
$$

$$
= \sum_w h(w, a', x)p(w \mid a, x)
$$

$$
= \sum_{w, z} \frac{1}{p(a' \mid x)} h(w, a', x)q_a(z, a', x)p(z, w, a' \mid x)
$$

$$
= \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} \hat{h}(W, A, X)q_a(Z, A, X) \mid X \right],
$$

$$
\hat{h}(W, X) := \sum_{a'} h(W, a', X)p(a' \mid X),
$$

and

$$
\eta_2(x, a) := \sum_{a'} \eta_1(x, a', a)p(a' \mid x)
$$

$$
= \mathbb{E}[\hat{h}(W, X) \mid A = a, X = x]
$$

$$
= \sum_{w, a'} h(w, a', x)p(a' \mid x)p(w \mid a, x)
$$

$$
= \sum_{w, z, a'} h(w, a', x)q_a(z, a', x)p(z, w, a' \mid x)
$$

$$
= \mathbb{E}[\hat{h}(W, A, X)q_a(Z, A, X) \mid X = x].
$$
Based on the influence functions of the parameters $\psi_1$ and $\psi_2$, we propose the following non-parametric identification result.

**Corollary 1.** Under Assumptions 1-3, parameters $\psi_1$ and $\psi_2$ are non-parametrically identified by

$$
\psi_1 = \mathbb{E} \left[ \frac{I(A = a')}{\bar{p}(A = a' \mid X)} \bar{q}_a(Z, A, X) \{ Y - \bar{h}(W, A, X) \} + I(A = a) \bar{p}(A = a \mid X) \{ \bar{h}(W, A, X) - \bar{\eta}_1(X, a', a) \} + \eta_1(X, a', a) \right],
$$

$$
\psi_2 = \mathbb{E} \left[ q_a(Z, A, X) \{ Y - h(W, A, X) \} + I(A = a) \{ \bar{h}(W, X) - \eta_2(X, a) \} + \eta_1(X, a, a) \right],
$$

where $\bar{h}$, $\eta_1$, and $\eta_2$ are defined in the statement of Theorem 4.

For two given functions $\hat{h}$ and $\hat{q}_a$, we say $\hat{h}$ is correctly specified if it satisfies equation (1); we say $\hat{q}_a$ is correctly specified if it satisfies equation (3). We have the following result regarding the robustness of the identification formulas in Corollary 1.

**Theorem 5.** Identification formulas in Corollary 1 are multiple-robust in the sense that

- If at least one of the following pairs is correctly specified,
  - $\{ \hat{h}, \bar{p}(W \mid A, X) \}$
  - $\{ \hat{h}, \bar{p}(A \mid X) \}$
  - $\{ \bar{q}_a, \bar{p}(A \mid X) \}$

  then

$$
\psi_1 = \mathbb{E} \left[ \frac{I(A = a')}{\bar{p}(A = a' \mid X)} \bar{q}_a(Z, A, X) \{ Y - \hat{h}(W, A, X) \} + I(A = a) \bar{p}(A = a \mid X) \{ \hat{h}(W, A, X) - \hat{\eta}_1(X, a', a) \} + \hat{\eta}_1(X, a', a) \right],
$$

where $\hat{\eta}_1(X, a', a) := \sum_w \hat{h}(w, a', X) \bar{p}(w \mid a, X)$.

- If at least one of the following pairs is correctly specified,
  - $\{ \hat{h}, \bar{p}(W \mid A, X) \}$
  - $\{ \hat{h}, \bar{p}(A \mid X) \}$
  - $\{ \bar{q}_a, \bar{p}(W \mid A, X) \}$
  - $\{ \bar{q}_a, \bar{p}(A \mid X) \}$

  then

$$
\psi_2 = \mathbb{E} \left[ \bar{q}_a(Z, A, X) \{ Y - \hat{h}(W, A, X) \} + I(A = a) \bar{p}(A = a \mid X) \{ \hat{h}_1(W, X) - \hat{\eta}_2(X, a) \} + \hat{\eta}_1(X, A, a) \right],
$$

where $\hat{\eta}_1(X, A, a) := \sum_w \hat{h}(w, A, X) \bar{p}(w \mid a, X)$, $\hat{h}_1(W, X) := \sum_{w'} \hat{h}(W, A', X) \bar{p}(a' \mid X)$, and $\hat{\eta}_2(X, a) := \sum_{w,a'} \hat{h}(w, a', X) \bar{p}(w \mid a, X) \bar{p}(a' \mid X)$.
6 Estimation Strategies

Based on Theorems 1 and 3 and Corollary 1 we propose the following estimation strategies for parameters $\psi_1$ and $\psi_2$.

6.1 Estimation Strategies for $\psi_1$

Strategy 1.

In Theorem 1 we observed that

$$\psi_1 = \sum_{w,x} h(w, a', x)p(w \mid a, x)p(x).$$

Therefore estimating nuisance functions $\hat{h}$ and $\hat{p}(w \mid a, x)$, we can estimate $\psi_1$ as

$$\hat{\psi}_1^{(h,w)} = \mathbb{P}_n \left[ \sum_w \hat{h}(w, a', X)\hat{p}(w \mid a, X) \right].$$

Strategy 2.

In Theorem 3 we observed that

$$\psi_1 = \mathbb{E} \left[ \frac{I(A = a')}{p(A = a' \mid X)}Yq_a(Z, A, X) \right].$$

Therefore estimating nuisance functions $\hat{q}_a$ and $\hat{p}(A = a' \mid x)$, we can estimate $\psi_1$ as

$$\hat{\psi}_1^{(q,a)} = \mathbb{P}_n \left[ \frac{I(A = a')}{\hat{p}(A = a' \mid X)}Y\hat{q}_a(Z, A, X) \right].$$

Strategy 3.

In Theorem 1 we observed that

$$\psi_1 = \sum_{w,x} h(w, a', x)p(w \mid a, x)p(x).$$

Therefore,

$$\psi_1 = \sum_{w,\tilde{a},x} I(\tilde{a} = a) \frac{I(A = a')}{p(A = a' \mid x)}h(w, a', x)p(w, \tilde{a}, x)$$

Therefore estimating nuisance functions $\hat{h}$ and $\hat{p}(A = a \mid x)$, we can estimate $\psi_1$ as

$$\hat{\psi}_1^{(h,a)} = \mathbb{P}_n \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)}\hat{h}(W, A, X) \right].$$
Strategy 4.

In Theorem 1, we observed that
\[ \psi_1 = \sum_{w,x} h(w, a', x)p(w \mid a, x)p(x). \]

Therefore,
\[ \psi_1 = \sum_{w,x} h(w, a', x)\frac{p(w \mid a, x)}{p(w \mid a', x)}p(w \mid a', x)p(x) \]
\[ = \sum_{w,z,x} I(\tilde{a} = a') \frac{I(\tilde{a} = a')}{p(A = a' \mid x)} h(w, \tilde{a}, x)q_{a}(z, \tilde{a}, x)p(z, w, a', x). \]

Therefore estimating nuisance functions \( \hat{h}, \hat{q}_a, \) and \( \hat{p}(A = a' \mid x), \) we can estimate \( \psi_1 \) as
\[ \hat{\psi}_1^{(h,q,a)} = \frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{h}(W, A, X)\hat{q}_a(Z, A, X) \]

Strategy 5.

The final estimation strategy is the following multiple-robust estimator based on Corollary 1. Estimating nuisance functions \( \hat{h}, \hat{q}_a, \) and \( \hat{p}(A = a' \mid x), \) and \( \hat{p}(w \mid a, x), \) we can estimate \( \psi_1 \) as
\[ \psi_1 = \sum_{w,x} h(w, a', x)\frac{p(w \mid a, x)}{p(w \mid a', x)}p(w \mid a', x)p(x) \]
\[ = \sum_{w,z,a,x} I(\tilde{a} = a') \frac{I(\tilde{a} = a')}{p(A = a' \mid x)} h(w, \tilde{a}, x)q_{a}(z, \tilde{a}, x)p(z, w, a', x). \]

Therefore estimating nuisance functions \( \hat{h}, \hat{q}_a, \) and \( \hat{p}(A = a' \mid x), \) and \( \hat{p}(w \mid a, x), \) we can estimate \( \psi_1 \) as
\[ \hat{\psi}_1^{MR} = \frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{h}(W, A, X)\hat{q}_a(Z, A, X) \]

6.2 Estimation Strategies for \( \psi_2 \)

Strategy 1.

In Theorem 1, we observed that
\[ \psi_2 = \sum_{w,a',x} h(w, a', x)p(w \mid a, x)p(a' \mid x)p(x). \]

Therefore estimating nuisance functions \( \hat{h} \) and \( \hat{p}(w \mid a, x), \) we can estimate \( \psi_2 \) as
\[ \hat{\psi}_2^{(h,w)} = \frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{h}(W, A, X)\hat{p}(w \mid a, X) \]
Strategy 2.

In Theorem 3, we observed that
\[ \psi_2 = \mathbb{E}[Y_{q_0}(Z, A, X)]. \]

Therefore estimating nuisance function \( \hat{q}_a \), we can estimate \( \psi_2 \) as
\[ \hat{\psi}_2^{(q)} = \mathbb{P}_n \left[ Y_{\hat{q}_0}(Z, A, X) \right]. \]

Strategy 3.

In Theorem 1, we observed that
\[ \psi_2 = \sum_{w, a', x} h(w, a', x) p(w \mid a, x) p(a' \mid x) p(x). \]

Therefore,
\[ \psi_2 = \sum_{w, a', x} \frac{1}{p(a \mid x)} h(w, a', x) p(a' \mid x) p(w \mid a, x) p(a \mid x) p(x) \]
\[ = \sum_{w, a', x} \frac{I(\hat{a} = a)}{p(a \mid x)} \sum_{a'} h(w, a', x) p(a' \mid x) p(w, \hat{a}, x). \]

Therefore estimating nuisance functions \( \hat{h} \) and \( \hat{p}(a \mid x) \), we can estimate \( \psi_2 \) as
\[ \hat{\psi}_2^{(h, a)} = \mathbb{P}_n \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \sum_{a'} \hat{h}(W, a', X) \hat{p}(a' \mid X) \right]. \]

Strategy 4.

In Theorem 1, we observed that
\[ \psi_2 = \sum_{w, a', x} h(w, a', x) p(w \mid a, x) p(a' \mid x) p(x). \]

Therefore,
\[ \psi_2 = \sum_{w, a', x} h(w, a', x) \frac{p(w \mid a, x)}{p(w \mid a', x)} p(w, a', x) \]
\[ = \sum_{w, z, a', x} h(w, a', x) q_a(z, a', x) p(z, w, a', x). \]

Therefore estimating nuisance functions \( \hat{h} \) and \( \hat{q}_a \), we can estimate \( \psi_2 \) as
\[ \hat{\psi}_2^{(h, a)} = \mathbb{P}_n \left[ \hat{h}(W, A, X) \hat{q}_a(Z, A, X) \right]. \]
Strategy 5.

The final estimation strategy is the following multiple-robust estimator based on Corollary 1. Estimating nuisance functions \( \hat{h}, \hat{q}_{a}, \hat{p}(a \mid x), \) and \( \hat{p}(w \mid a, x) \), we can estimate \( \psi_{1} \) as

\[
\hat{\psi}_{MR}^{2} = n^{-1} \left[ \hat{q}_{a}(Z, A, X) \{ Y - \hat{h}(W, A, X) \} + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{ \sum_{a'} \hat{h}(W, a', X) \hat{p}(a' \mid X) - \sum_{w,a'} \hat{h}(w, a', X) \hat{p}(w \mid a, X) \hat{p}(a' \mid X) \} + \sum_{w} \hat{h}(w, A, X) \hat{p}(w \mid a, X) \} \right].
\]

7 Generalized Front-Door Model

In this section, we consider the identification of a third parameter in a setting where challenges of the mediation and front-door models co-exist. We consider a setup comprised of a treatment variable \( A \), an outcome variable \( Y \), a mediator variable \( M \), and observed pre-treatment covariates \( X \), where the variables satisfy the following conditions:

- **Positivity:**
  
  \[ p(m \mid A, X) > 0 \text{ with probability 1} \forall m, \]
  \[ p(a \mid X) > 0 \text{ with probability 1} \forall a. \]

- **Consistency:**
  
  \[ M^{(a)} = M \text{ if } A = a, \]
  \[ Y^{(a,m)} = Y \text{ if } A = a, M = m. \]

- For any \( a, a' \in X_{A} \) and \( m \in X_{M} \),
  
  \[ M^{(a')} \perp \perp A \mid X, \]
  \[ Y^{(a,m)} \perp \perp M^{(a')} \mid X, A. \]

Note that compared to the mediation model, the condition of \( Y^{(a,m)} \perp \perp A \mid X \) is relaxed. That is, there can be an unmeasured confounder of the treatment and outcome variables in the system. Also, compared to the front-door model, a direct causal effect from \( A \) to \( Y \) can be present in the system. Figure 5 demonstrates a graphical model which satisfies the conditions of the generalized front-door model.

Our parameter of interest is \( \psi_{3} = \mathbb{E} \left[ Y^{(M^{(a)})} \right] \). This parameter is the counterfactual part of the population intervention indirect effect, which is a measure of indirect effect introduced by Fulcher et al. (2020) corresponding to the effect of an intervention which changes the mediator from its natural value to the value that it would have had under exposure value \( a \).
In the case that the mediator is observed, parameter $\psi_3$ can be identified as follows.

$$\psi_3 = E[Y^{(M^{(\alpha)})}]$$
$$= E[Y^{(A,M^{(\alpha)})}]$$
$$= \sum_{a',m,y,x} yp(Y^{(a',m)} = y \mid M^{(a)} = m, A = a', x)p(M^{(a)} = m \mid x)p(a' \mid x)p(x)$$
$$= \sum_{a',m,y,x} yp(Y^{(a',m)} = y \mid M^{(a')} = m, A = a', x)p(M^{(a)} = m \mid x)p(a' \mid x)p(x)$$
$$= \sum_{a',m,y,x} yp(y \mid a', m, x)p(m \mid a, x)p(a' \mid x)p(x).$$

Interestingly, the identification formula for $\psi_3$ is the same as the one for $\psi_2$. A fact that was noted in (Fulcher et al., 2020). Now, suppose the mediator variable $M$ is not observed, yet we have access to proxies $W$ and $Z$ of this variable, satisfying the conditions presented Section 3. Figure 6 demonstrates an example of graphical models which satisfy the proxy variable conditions for the generalized front-door model. In this figure, dashed edges can be present or absent. Since the identification formula for $\psi_3$ and $\psi_2$ are the same, a corollary of our results is proximal identification of the parameter $\psi_3$ when the mediator is unobserved. Specifically, all identification and estimation results proposed for $\psi_2$ for the case that the mediator is unobserved in the proximal setup can also be applied for identification and estimation of the parameter $\psi_3$. This effectively extends the results of Fulcher et al. (2020) to the hidden mediator setting.

References

Cui, Y., Pu, H., Shi, X., Miao, W., and Tchetgen Tchetgen, E. (2020). Semiparametric proximal causal inference. arXiv preprint arXiv:2011.08411.

Dikkala, N., Lewis, G., Mackey, L., and Syrgkanis, V. (2020). Minimax estimation of conditional moment models. arXiv preprint arXiv:2006.07201.

Fulcher, I. R., Shpitser, I., Marealle, S., and Tchetgen Tchetgen, E. J. (2020). Robust inference on population indirect causal effects: the generalized front door criterion. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 82(1):199–214.

Ghassami, A., Ying, A., Shpitser, I., and Tchetgen, E. T. (2021). Minimax kernel machine learning for a class of doubly robust functionals. arXiv preprint arXiv:2104.02929.

Miao, W., Geng, Z., and Tchetgen Tchetgen, E. J. (2018). Identifying causal effects with proxy variables of an unmeasured confounder. Biometrika, 105(4):987–993.
Appendices

A Proofs

Proof of Theorem 1

By Assumption 2 $(ii)$ for specific choice of $Z = z, A = a', X = x$, we have

$$\sum_y yp(y \mid z, a', x) = \sum_w h(w, a', x)p(w \mid z, a', x)$$

$$\Rightarrow \sum_{y,m} yp(y, m \mid z, a', x) = \sum_{w,m} h(w, a', x)p(w, m \mid z, a', x)$$

$$(a) \sum_{y,m} yp(y \mid m, a', x)p(m \mid z, a', x) = \sum_{w,m} h(w, a', x)p(w \mid m, x)p(m \mid z, a', x)$$

$$(b) \sum_{y} yp(y \mid m, a', x) = \sum_{w} h(w, a', x)p(w \mid m, x) \text{ a.s.}$$

$$(c) \sum_{y} yp(y \mid m, a', x) = \sum_{w} h(w, a', x)p(w \mid m, a, x) \text{ a.s.}$$

$$\Rightarrow \sum_{y,m,x} yp(y \mid m, a', x)p(m \mid a, x)p(x) = \sum_{w,m,x} h(w, a', x)p(w \mid m, a, x)p(m \mid a, x)p(x)$$

$$\Rightarrow \sum_{y,m,x} yp(y \mid m, a', x)p(m \mid a, x)p(x) = \sum_{w,m,x} h(w, a', x)p(w \mid a, x)p(x),$$

where $(a)$ follows from $Y \perp \perp Z \mid \{A, M, X\}$ and $W \perp \perp \{A, Z\} \mid \{M, X\}$, $(b)$ is due to Assumption 2 $(i)$, and $(c)$ follows from $W \perp \perp \{A, Z\} \mid \{M, X\}$.

Therefore,

$$\psi_1 = \sum_{y,m,x} yp(y \mid m, a', x)p(m \mid a, x)p(x)$$

$$= \sum_{w,x} h(w, a', x)p(w \mid a, x)p(x).$$

and

$$\psi_2 = \sum_{a',y,m,x} yp(y \mid m, a', x)p(m \mid a, x)p(a' \mid x)p(x)$$

$$= \sum_{w,a',x} h(w, a', x)p(w \mid a, x)p(a' \mid x)p(x).$$
Proof of Proposition 1

\[E[q_a(Z, a', X)|W, A = a', X] = \frac{p(W | A = a, X)}{p(W | A = a', X)}\]

\[\Leftrightarrow E[q_a(Z, a', X)|W, A = a', X] = \frac{p(A = a | W, X)p(A = a | X)}{p(A = a' | W, X)p(A = a | X)}\]

\[\Leftrightarrow \frac{p(A = a' | W, X)}{p(A = a' | X)}E[q_a(Z, a', X)|W, A = a', X] = \frac{E[I(A = a)]}{p(A = a | X)} | W, X\]

\[\Leftrightarrow E[\frac{I(A = a')}{p(A = a' | X)} q_a(Z, A, X) - \frac{I(A = a)}{p(A = a | X)}] | W, X] = 0.\]

Proof of Theorem 2

\[E[q_a(Z, A, X)|W, A, X] = \frac{p(W | A = a, X)}{p(W | A, X)}\]

\[\Rightarrow E[q_a(Z, A, X) - \frac{p(W | A = a, X)}{p(W | A, X)} | W, A, X] = 0\]

\[\Rightarrow E[q_a(Z, A, X)h(W, A, X) - \frac{p(W | A = a, X)}{p(W | A, X)}h(W, A, X) = 0 \quad \forall h\]

\[\Rightarrow \partial_t E_t[q_a(Z, A, X; \theta_t)h(W, A, X) - \frac{p_t(W | A = a, X)}{p_t(W | A, X)}h(W, A, X)] = 0 \quad \forall h\]

\[\Rightarrow -E[\partial_t q_a(Z, A, X; \theta_t)h(W, A, X)]\partial_t \theta_t = -E\left[\partial_t \frac{p_t(W | A = a, X)}{p_t(W | A, X)}h(W, A, X)\right]\]

\[+ E[q_a(Z, A, X)h(W, A, X)S(O)]\]

\[E\left[p(W | A = a, X) \frac{p_t(W | A = a, X)}{p_t(W | A, X)}h(W, A, X)S(O)\right] = 0 \quad \forall h\]

\[\Rightarrow -E[\partial_t q_a(Z, A, X; \theta_t)h(W, A, X)]\partial_t \theta_t = -E\left[\partial_t \frac{p_t(W | A = a, X)}{p_t(W | A, X)}h(W, A, X)\right]\]

\[+ E[q_a(Z, A, X)h(W, A, X)S(O)]\]

\[E\left[p(W | A = a, X) \frac{p_t(W | A = a, X)}{p_t(W | A, X)}h(W, A, X)S(O)\right] = 0 \quad \forall h\]
\[
\mathbb{E}\left[ \frac{p(W \mid a, X)}{p(W \mid A, X)^2} \frac{\partial p_h(W \mid A, X)}{p(W \mid A, X)} \right] = \mathbb{E}\left[ \frac{p(W \mid A, X)^2}{p(W \mid A, X)} h(W, A, X) \right] = \mathbb{E}\left[ \frac{p(W \mid a, X)}{p(W \mid A, X)} h(W, A, X) - \mathbb{E}\left[ \frac{p(W \mid a, X)}{p(W \mid A, X)} h(W, A, X) \right] S(W \mid A, X) \right]
\]

\[
\mathbb{E}\left[ \frac{p(W \mid a, X)}{p(W \mid A, X)} h(W, A, X) S(O) \right] = \mathbb{E}\left[ \sum_w h(w, A, X) p(w \mid a, X) S(O) \right]
\]
Therefore,
\[-E[\partial_{\theta_i} q_{a}(Z, A; \theta_1) h(W, A, X)] \partial_{\theta_i} = -E \left[ \sum_w h(w, A, X)p(w | a, X)S(O) \right] \]
\[-E \left[ \frac{I(A = a)}{p(a | X)} \left\{ \sum_{a'} h(W, a', X)p(a' | X) \right\} S(O) \right] \]
\[+ E \left[ \frac{I(A = a)}{p(a | X)} \left\{ \sum_{w,a'} h(W, a', X)p(a' | X)p(w | a, X) \right\} S(O) \right] \]
\[+ E \left[ q_{a}(Z, A, X)h(W, A, X)S(O) \right] \]

**Proof of Theorem 3**

By Assumption 3 (ii) for specific choice of $W = w, A = a', X = x$, we have
\[
E[q_{a}(Z, A, X)|W = w, A = a', X = x] = \frac{p(w | a, x)}{p(w | a', x)}
\]
\[
\Rightarrow \sum_z q_{a}(z, a', x)p(z | w, a', x) = \frac{p(w | a, x)}{p(w | a', x)}
\]
\[
\Rightarrow \sum_z q_{a}(z, a', x)p(z | w, a', x) = \frac{p(w | a, x)}{p(w | a', x)} \sum_m p(m | a, w, x)
\]
\[
\Rightarrow \sum_z q_{a}(z, a', x)p(z | w, a', x) = \frac{p(m | a, x)p(w | a, m, x)}{p(m | a', x)p(w | a', m, x)}p(m | a', w, x)
\]
\[
\Rightarrow \sum_z q_{a}(z, a', x)p(z | w, a', x) = \frac{p(m | a, x)}{p(m | a', x)}p(m | a', w, x)
\]
\[\overset{(a)}{\Rightarrow} \sum_z q_{a}(z, a', x)p(z | w, a', x) = \sum_m \frac{p(m | a, x)}{p(m | a', x)}p(m | a', w, x)
\]
\[
\Rightarrow \sum_{m,z} q_{a}(z, a', x)p(m, z | w, a', x) = \sum_m \frac{p(m | a, x)}{p(m | a', x)}p(m | a', w, x)
\]
\[\overset{(b)}{\Rightarrow} \sum_{m,z} q_{a}(z, a', x)p(z | m, a', x)p(m | a', w, x) = \sum_m \frac{p(m | a, x)}{p(m | a', x)}p(m | a', w, x)
\]
\[\overset{(c)}{\Rightarrow} \sum_z q_{a}(z, a', x)p(z | m, a', x) = \frac{p(m | a, x)}{p(m | a', x)}.
\]

where (a) and (b) follow from $W \perp \perp \{A, Z\} | \{M, X\}$, and (c) is due to Assumption 3 (i).
This implies that

\[ \sum_{y,m} y p(y \mid m, a', x) p(m \mid a, x) = \sum_{y,m} y p(y \mid m, a', x) p(m \mid a', x) \frac{p(m \mid a, x)}{p(m \mid a', x)} \]

\[ \overset{(d)}{=} \sum_{y,m} y p(y \mid m, a', x) p(m \mid a', x) \sum_z q_a(z, a', x) p(z \mid m, a', x) \]

\[ = \sum_{y,m,z} y q_a(z, a', x) p(z \mid m, a', x) p(y \mid m, a', x) \]

\[ \overset{(e)}{=} \sum_{y,m,z} y q_a(z, a', x) p(y, z \mid m, a', x) p(m \mid a', x) \]

\[ = \sum_{y,m,z} y q_a(z, a', x) p(y, z, m \mid a', x), \]

where \((d)\) is due to \([3]\), and \((e)\) follows from \(Y \perp \perp Z \mid \{A, M, X\}\).

Therefore,

\[ \psi_1 = \sum_{y,m,x} y p(y \mid m, a', x) p(m \mid a, x) p(x) \]

\[ = \sum_{y,m,z,x} \frac{1}{p(A = a' \mid x)} y q_a(z, a', x) p(y, z, m, a', x) \]

\[ = \sum_{y,m,z,a} f(\hat{a} = a') \frac{p(A = \hat{a} \mid x)}{p(A = a' \mid x)} y q_a(z, a, x) p(y, z, m, a, x) \]

\[ = E \left[ \frac{f(A = a')}{p(A = a' \mid X)} Y q_a(Z, A, X) \right], \]

and

\[ \psi_2 = \sum_{a',y,m,x} y p(y \mid m, a', x) p(m \mid a, x) a' \]

\[ = \sum_{y,m,z,a',x} y q_a(z, a', x) p(y, z, m, a', x) \]

\[ = E \left[ Y q_a(Z, A, X) \right]. \]

**Proof of Theorem 4**

We use the notation \(\partial_t f(t)\) to denote \(\frac{\partial f(t)}{\partial t}\) \(\mid_{t=0}\). Also, for parameter \(\psi \in \{\psi_1, \psi_2\}\), let \(\psi_t\) be the parameter of interest under a regular parametric sub-model indexed by \(t\), that includes the ground-truth model at \(t = 0\).

In order to obtain an influence function of \(\psi_1\), we need to find a random variable \(G\) with mean zero, that satisfies

\[ \partial_t \psi_{1t} = E[GS(O)], \]

where \(S(O) = \partial_t \log p_t(O)\).
Therefore,

\[ \partial_t \psi_t = \partial_t \sum_{w, x} h_t(w, a', x)p_t(w \mid a, x)p_t(x) \]

\[ = \sum_{w, x} \partial_t h_t(w, a', x)p(w \mid a, x)p(x) \]

\[ + \sum_{w, x} h(w, a', x)\partial_t p_t(w \mid a, x)p(x) \]

\[ + \sum_{w, x} h(w, a', x)p(w \mid a, x)\partial_t p_t(x). \]  \hfill (10)

For the first term in (10), we have

\[ \sum_{w, x} \partial_t h_t(w, a', x)p(w \mid a, x)p(x) = \sum_{w, x} \partial_t h_t(w, a', x) \frac{p(w \mid a, x)}{p(w \mid a', x)}p(w \mid a', x)p(x) \]

\[ = \sum_{w, x} \frac{1}{p(A = a' \mid x)} \partial_t h_t(w, a', x) \frac{p(w \mid a, x)}{p(w \mid a', x)}p(w, a', x) \]

\[ = \sum_{w, x} \frac{1}{p(A = a' \mid x)} \partial_t h_t(w, a', x)q_a(z, a', x)p(w, z, a', x) \]  \hfill (11)

\[ = \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \partial_t h_t(W, A, X) \right] \]

\[ = \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \mathbb{E}[\partial_t h_t(W, A, X) \mid Z, A, X] \right]. \]

Note that by Assumption (ii),

\[ \mathbb{E}[Y - h(W, A, X) \mid Z, A, X] = 0 \]

\[ \Rightarrow \partial_t \mathbb{E}[Y - h_t(W, A, X) \mid Z, A, X] = 0 \]

\[ \Rightarrow \mathbb{E}[\partial_t \{Y - h_t(W, A, X)\} \mid Z, A, X] + \mathbb{E}[\{Y - h(W, A, X)\}S(W, Y \mid Z, A, X) \mid Z, A, X] = 0 \]

\[ \Rightarrow \mathbb{E}[\partial_t h_t(W, A, X) \mid Z, A, X] = \mathbb{E}[\{Y - h(W, A, X)\}S(W, Y \mid Z, A, X) \mid Z, A, X]. \]

Therefore,

\[ \sum_{w, x} \partial_t h_t(w, a', x)p(w \mid a, x)p(x) \]

\[ = \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \mathbb{E}[\{Y - h(W, A, X)\}S(W, Y \mid Z, A, X) \mid Z, A, X] \right] \]  \hfill (12)

\[ = \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \{Y - h(W, A, X)\}S(W, Y \mid Z, A, X) \right]. \]

Also, note that

\[ \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \{Y - h(W, A, X)\}S(Z, A, X) \right] = 0. \]  \hfill (13)

Therefore, (12) and (13) imply that

\[ \sum_{w, x} \partial_t h_t(w, a', x)p(w \mid a, x)p(x) = \mathbb{E}\left[ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \{Y - h(W, A, X)\}S(O) \right]. \]  \hfill (14)
For the second term in (10), we have
\[
\sum_{w,x} h(w, a', x) \partial_t p_t(w | a, x)p(x) = \sum_{w,x} h(w, a', x) \frac{\partial_t p_t(w | a, x)}{p(w | a, x)} p(w | a, x)p(x)
\]
\[
= \sum_{w,x} h(w, a', x) S(w | a, x)p(w | a, x)p(x)
\]
\[
= \sum_{w,x} \{h(w, a', x) - \eta_1(x, a', a)\} S(w | a, x)p(w | a, x)p(x)
\]
\[
= \sum_{w,x} \frac{1}{p(A = a | x)} \{h(w, a', x) - \eta_1(x, a', a)\} S(w | a, x)p(w, a, x)
\] (15)
\[
= \sum_{w,a,x} \frac{I(\hat{a} = a)}{p(A = a | x)} \{h(w, a', x) - \eta_1(x, a', \hat{a})\} S(w | \hat{a}, x)p(w, \hat{a}, x)
\]
\[
= \mathbb{E} \left[ \frac{I(A = a)}{p(A = a | X)} \{h(W, a', X) - \eta_1(X, a', A)\} S(W | A, X) \right],
\]
where
\[
\eta_1(x, a', a) := \mathbb{E}[h(W, a', X) | A = a, X = x]
\]
\[
= \sum_w h(w, a', x)p(w | a, x)
\]
\[
= \sum_w h(w, a', x) \frac{p(w | a, x)}{p(w | a', x)} \frac{p(a' | x)}{p(a' | x)} p(w | a', x)
\]
\[
= \sum_{w,z} \frac{1}{p(a' | x)} h(w, a', x)q_a(z, a', x)p(z, w, a' | x)
\]
\[
= \mathbb{E} \left[ \frac{I(A = a')}{p(A = a' | X)} h(W, A, X)q_a(Z, A, X) | X = x \right].
\]

Note that
\[
\mathbb{E} \left[ \frac{I(A = a)}{p(A = a | X)} \{h(W, a', X) - \eta_1(X, a', A)\} S(A, X) \right] = 0. \] (16)

Therefore, (15) and (16) imply that
\[
\sum_{w,x} h(w, a', x) \partial_t p_t(w | a, x)p(x) = \mathbb{E} \left[ \frac{I(A = a)}{p(A = a | X)} \{h(W, a', X) - \eta_1(X, a', a)\} S(W, A, X) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(A = a)}{p(A = a | X)} \{h(W, a', X) - \eta_1(X, a', a)\} S(O) \right]. \] (17)

For the third term in (10), we have
\[
\sum_{w,x} h(w, a', x)p(w | a, x)\partial_t p_t(x) = \sum_x \sum_w h(w, a', x)p(w | a, x) \frac{\partial_t p_t(x)}{p(x)} p(x)
\]
\[
= \mathbb{E}[\eta_1(X, a', a) S(X)]
\]
\[
= \mathbb{E}[(\eta_1(X, a', a) - \mathbb{E}[\eta_1(X, a', a)]) S(X)]
\]
\[
= \mathbb{E}[(\eta_1(X, a', a) - \psi_1) S(X)]
\]
\[
= \mathbb{E}[(\eta_1(X, a', a) - \psi_1) S(O)]. \] (18)
Combining (14), (17), and (18) concludes that
\[
\partial_t \psi_{1,t} = \mathbb{E} \left[ \left\{ \frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \{ Y - h(W, A, X) \} + \frac{I(A = a)}{p(A = a \mid X)} \{ h(W, a', X) - \eta_1(X, a', a) \} \right\} S(O) \right].
\]

Therefore,
\[
\frac{I(A = a')}{p(A = a' \mid X)} q_a(Z, A, X) \{ Y - h(W, A, X) \} + \frac{I(A = a)}{p(A = a \mid X)} \{ h(W, a', X) - \eta_1(X, a', a) \} + \eta_1(X, a', a) - \psi_1
\]
is an influence function of $\psi_1$.

In order to obtain an influence function of $\psi_2$, we need to find a random variable $G$ with mean zero, that satisfies
\[
\partial_t \psi_{2,t} = \mathbb{E} [GS(O)],
\]

Note that
\[
\partial_t \psi_{2,t} = \partial_t \sum_{w, a', x} h_t(w, a', x) p_t(w \mid a, x) p_t(a', x)
= \sum_{w, a', x} \partial_t h_t(w, a', x) p(w \mid a, x) p(a', x)
+ \sum_{w, a', x} h(w, a', x) \partial_t p_t(w \mid a, x) p(a', x)
+ \sum_{w, a', x} h(w, a', x) p(w \mid a, x) \partial_t p_t(a', x).
\]

For the first term in (19), we have
\[
\sum_{w, a', x} \partial_t h_t(w, a', x) p(w \mid a, x) p(a' \mid x) p(x) = \sum_{w, a', x} \partial_t h_t(w, a', x) \frac{p(w \mid a, x)}{p(w \mid a', x)} p(w \mid a', x)
= \sum_{w, a', x} \partial_t h_t(w, a', x) q_{a}(z, a', x) p(w, z, a', x)
= \mathbb{E} [q_{a}(Z, A, X) \partial_t h_t(W, A, X)]
= \mathbb{E} [q_{a}(Z, A, X) \mathbb{E} [\partial_t h_t(W, A, X) \mid Z, A, X]].
\]

Note that by Assumption (ii),
\[
\mathbb{E}[Y - h(W, A, X) \mid Z, A, X] = 0
\Rightarrow \partial_t \mathbb{E} [Y - h_t(W, A, X) \mid Z, A, X] = 0
\Rightarrow \mathbb{E}[\partial_t \{Y - h_t(W, A, X)\} \mid Z, A, X] + \mathbb{E} \{Y - h(W, A, X)\} S(W, Y \mid Z, A, X) = 0
\Rightarrow \mathbb{E}[\partial_t h_t(W, A, X) \mid Z, A, X] = \mathbb{E} \{Y - h(W, A, X)\} S(W, Y \mid Z, A, X) = 0.
\]

Therefore,
\[
\sum_{w, a', x} \partial_t h_t(w, a', x) p(w \mid a, x) p(a' \mid x) p(x)
= \mathbb{E} [q_{a}(Z, A, X) \mathbb{E} \{Y - h(W, A, X)\} S(W, Y \mid Z, A, X)]
= \mathbb{E} [q_{a}(Z, A, X) \{Y - h(W, A, X)\} S(W, Y \mid Z, A, X)].
\]
Also, note that
\[
\mathbb{E}[q_a(Z, A, X) \{ Y - h(W, A, X) \}] S(Z, A, X) = 0.
\]  
(22)

Therefore, (21) and (22) imply that
\[
\sum_{w,a',x} \partial_i h_i(w, a', x) p(w | a, x)p(a' | x)p(x) = \mathbb{E}[q_a(Z, A, X) \{ Y - h(W, A, X) \}] S(O).
\]  
(23)

For the second term in (19), we have
\[
\sum_{w,a',x} h(w, a', x) \partial_i p_i(w | a, x)p(a' | x)p(x) = \sum_{w,a',x} h(w, a', x) \frac{\partial w_p w}{p(w | a, x)} p(w | a, x)p(a' | x)p(x)
\]
\[
= \sum_{w,a',x} h(w, a', x) S(w | a, x)p(w | a, x)p(a' | x)p(x)
\]
\[
= \sum_{w,a',x} \left\{ \sum_{a'} h(w, a', x)p(a' | x) - \eta_2(x, a) \right\} S(w | a, x)p(w | a, x)p(x)
\]
\[
= \sum_{w,a,x} \frac{1}{p(A = a | x)} \{ \bar{h}(w, x) - \eta_2(x, a) \} S(w | a, x)p(w, a, x)
\]
\[
= \sum_{w,a,x} \frac{I(\bar{a} = a)}{p(A = \bar{a} | x)} \{ \bar{h}(w, x) - \eta_2(x, \bar{a}) \} S(w | \bar{a}, x)p(w, \bar{a}, x)
\]
\[
= \mathbb{E}\left[ \frac{I(A = a)}{p(A | X)} \{ \bar{h}(W, X) - \eta_2(X, a) \} S(W | A, X) \right],
\]  
(24)

where,
\[
\bar{h}(W, X) := \sum_{a'} h(W, a', X)p(a' | X),
\]

and
\[
\eta_2(x, a) := \mathbb{E}[\bar{h}(W, X) | A = a, X = x]
\]
\[
= \sum_{w,a',x} h(w, a', x)p(a' | x)p(w | a, x)
\]
\[
= \sum_{w,a',x} h(w, a', x) \frac{p(w | a, x)}{p(w | a', x)} p(w, a' | x)
\]
\[
= \sum_{w,a',x} h(w, a', x)q_a(z, a', x)p(z, w, a' | x)
\]
\[
= \mathbb{E}[h(W, A, X) q_a(Z, A, X) | X = x].
\]

Note that
\[
\eta_2(X, a) = \sum_{a'} \eta_1(X, a', a)p(a' | X) = \mathbb{E}[\eta_1(X, A, a) | X].
\]  
(25)

Note that
\[
\mathbb{E}\left[ \frac{I(A = a)}{p(A = a | X)} \{ \bar{h}(W, X) - \eta_2(X, a) \} S(A, X) \right] = 0.
\]  
(26)

Therefore, (24) and (26) imply that
\[
\sum_{w,a',x} h(w, a', x) \partial_i p_i(w | a, x)p(a' | x)p(x) = \mathbb{E}\left[ \frac{I(A = a)}{p(A = a | X)} \{ \bar{h}(W, X) - \eta_2(X, a) \} S(W, A, X) \right]
\]
\[
= \mathbb{E}\left[ \frac{I(A = a)}{p(A = a | X)} \{ \bar{h}(W, X) - \eta_2(X, a) \} S(O) \right].
\]  
(27)
For the third term in (19), we have
\[
\sum_{w,a',x} h(w, a', x)p(w \mid a, x)\partial_{q}p_{1}(a', x) = \sum_{x,a'} \sum_{w} h(w, a', x)p(w \mid a, x) \frac{\partial_{q}p_{1}(a', x)}{p(a', x)}p(a', x)
\]
\[
= \mathbb{E}[\eta_{1}(X, a)S(A, X)]
\]
\[
= \mathbb{E}\{\eta_{1}(X, a) - \mathbb{E}[\eta_{1}(X, a, a)]\}S(A, X)
\]
\[
= \mathbb{E}\{\eta_{1}(X, a) - \psi_{2}\}S(A, X)
\]
\[
= \mathbb{E}\{\eta_{1}(X, a) - \psi_{2}\}S(O).
\]
Combining (23), (27), and (28) concludes that
\[
\partial_{a}\psi_{2} = \mathbb{E}\left\{\left. q_{a}(Z, A, X)\{Y - h(W, A, X)\} + \frac{I(A = a)}{p(A = a \mid X)}\{\tilde{h}(W, X) - \eta_{2}(X, a)\} + \eta_{1}(X, a) - \psi_{2}\right\}S(O)\right\}.
\]
Therefore,
\[
q_{a}(Z, A, X)\{Y - h(W, A, X)\} + \frac{I(A = a)}{p(A = a \mid X)}\{\tilde{h}(W, X) - \eta_{2}(X, a)\} + \eta_{1}(X, a) - \psi_{2}
\]
is an influence function of $\psi_{2}$.

**Proof of Theorem 5**

**Multiple-robustness of the identification formula of $\psi_{1}$:**
First, suppose the pair \{\hat{h}, \hat{p}(W \mid A, X)\} is correctly specified. We have,
\[
\mathbb{E}\left[\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_{a}(Z, A, X)\{Y - \hat{h}(W, A, X)\}
\right.
\]
\[
\left. + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{\tilde{h}(W, a', X) - \hat{\eta}_{1}(X, a', a)\} + \hat{\eta}_{1}(X, a', a)\right]\n\]
\[
= \mathbb{E}\left[\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_{a}(Z, A, X)\mathbb{E}[Y - \hat{h}(W, A, X) \mid Z, A, X]
\right.
\]
\[
\left. = 0\right] \frac{\tilde{h}(W, a', X) - \hat{\eta}_{1}(X, a', a)}{\hat{\eta}_{1}(X, a', a)} + \hat{\eta}_{1}(X, a', a)
\]
\[
= \mathbb{E}\left[\frac{I(A = a)}{\hat{p}(A = a \mid X)} \tilde{h}(W, a', X) - \hat{\eta}_{1}(X, a', a)\right] + \hat{\eta}_{1}(X, a', a)
\]
\[
= \mathbb{E}\left[\frac{I(A = a)}{\hat{p}(A = a \mid X)} \mathbb{E}[\tilde{h}(W, a', X) \mid A, X] - \hat{\eta}_{1}(X, a', a)\right] + \hat{\eta}_{1}(X, a', a)
\]
\[
= \mathbb{E}\left[\hat{\eta}_{1}(X, a', a)\right]
\]
\[
= \mathbb{E}\left[\mathbb{E}[\tilde{h}(W, a', X) \mid A = a, X]\right]
\]
\[
= \psi_{1},
\]
where the last equality is estimation strategy 1 for $\psi_{1}$.
Second, suppose the pair \{\hat{h}, \hat{p}(A \mid X)\} is correctly specified. We have,

\[
\mathbb{E}
\left[
\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_a(Z, A, X)\{Y - \hat{h}(W, A, X)\} + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{\hat{h}(W, a', X) - \hat{q}_1(X, a', a)\} + \hat{q}_1(X, a', a)\right]
\]

\[
= \mathbb{E}
\left[
\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_a(Z, A, X)\{Y - \hat{h}(W, A, X)\} + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}(W, a', X)\right]
\]

\[
= \mathbb{E}
\left[
\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_a(Z, A, X)\left(\mathbb{E}[Y - \hat{h}(W, A, X) \mid Z, A, X] + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}(W, a', X)\right)\right]
\]

\[
= \mathbb{E}
\left[
\frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}(W, a', X)\right]
\]

= \psi_1.

where the last equality is estimation strategy 3 for \psi_1.

Third, suppose the pair \{\hat{q}_a, \hat{p}(A \mid X)\} is correctly specified. We have,

\[
\mathbb{E}
\left[
\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_a(Z, A, X)\{Y - \hat{h}(W, A, X)\} + I(A = a)\right]
\]

\[
= \mathbb{E}
\left[
\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_a(Z, A, X)\{Y - \hat{h}(W, A, X)\} + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}(W, a', X)\right]
\]

\[
= \mathbb{E}
\left[
\frac{I(A = a')}{\hat{p}(A = a' \mid X)} \hat{q}_a(Z, A, X)\{Y - \hat{h}(W, A, X)\} + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}(W, a', X)\right]
\]

\[
= \psi_1.
\]
where the last equality is estimation strategy 2 for $\psi_1$, and (a) is due to the fact that

$$
\mathbb{E} \left[ \frac{I(A = a')}{p(A = a' \mid X)} \hat{q}_a(Z, A, X) \hat{h}(W, A, X) \right] \\
= \sum_{w, x} \frac{1}{p(a' \mid x)} \hat{q}_a(z, a', x) \hat{h}(w, a', x) p(w, z, a', x) \\
= \sum_{w, x} \frac{1}{p(a' \mid x)} \left\{ \sum_z \hat{q}_a(z, a', x) p(z \mid w, a', x) \right\} \hat{h}(w, a', x) p(w, a', x) \\
= \sum_{w, x} \frac{1}{p(a' \mid x)} \hat{h}(w, a', x) p(w \mid a, x) p(x) \\
= \sum_{w, x} \frac{1}{p(a \mid x)} \hat{h}(w, a', x) p(w, a, x) \\
= \mathbb{E} \left[ \frac{I(A = a)}{p(A = a \mid X)} \hat{h}(W, a', X) \right].
$$

**Multiple-robustness of the identification formula of $\psi_2$:**

First, suppose the pair $\{\hat{h}, \hat{p}(W \mid A, X)\}$ is correctly specified. We have,

$$
\mathbb{E} \left[ \hat{q}_a(Z, A, X) \{Y - \hat{h}(W, A, X)\} \\
+ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{\hat{h}_1(W, X) - \hat{h}_2(X, a)\} + \hat{\eta}_1(X, A, a) \right] \\
= \mathbb{E} \left[ \hat{q}_a(Z, A, X) \mathbb{E}[Y - \hat{h}(W, A, X) \mid Z, A, X] \right] \\
+ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{\hat{h}_1(W, X) - \hat{h}_2(X, a)\} + \hat{\eta}_1(X, A, a) \\
= \mathbb{E} \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{\hat{h}_1(W, X) - \hat{h}_2(X, a)\} + \hat{\eta}_1(X, A, a) \right] \\
= \mathbb{E} \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} (\mathbb{E}[\hat{h}_1(W, X) \mid X, A] - \hat{h}_2(X, a)\} + \hat{\eta}_1(X, A, a) \right] \\
\overset{(a)}{=} \mathbb{E} \left[ \hat{\eta}_1(X, A, a) \right] \\
= \psi_2,
$$

where the last equality is estimation strategy 1 for $\psi_2$, and (a) is due to
Second, suppose the pair \( \{ \hat{h}, \hat{p}(A \mid X) \} \) is correctly specified. We have,

\[
\mathbb{E} \left[ \hat{q}_a(Z, A, X) \{ Y - \hat{h}(W, A, X) \} \right. \\
+ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \hat{h}_1(W, X) - \hat{\eta}_2(X, a) \right) + \hat{\eta}_1(X, a) \\
= \mathbb{E} \left[ \hat{q}_a(Z, A, X) \mathbb{E} [ Y - \hat{h}(W, A, X) \mid Z, A, X ] \right. \\
+ \left. \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \hat{h}_1(W, X) - \hat{\eta}_2(X, a) + \hat{\eta}_1(X, A, a) \right) \right] \\
= \mathbb{E} \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}_1(W, X) \right] \\
= \psi_2,
\]

where the last equality is estimation strategy 3 for \( \psi_2 \), and (a) is due to

\[ \eta_2(X, a) = \sum_{a'} \eta_1(X, a', a)p(a' \mid X) = \mathbb{E}[\eta_1(X, A, a) \mid X]. \]

Third, suppose the pair \( \{ \hat{q}_a, \hat{p}(W \mid A, X) \} \) is correctly specified. We have,

\[
\mathbb{E} \left[ \hat{q}_a(Z, A, X) \{ Y - \hat{h}(W, A, X) \} \right. \\
+ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \hat{h}_1(W, X) - \hat{\eta}_2(X, a) \right) + \hat{\eta}_1(X, a) \\
= \mathbb{E} \left[ \hat{q}_a(Z, A, X)Y - \hat{q}_a(Z, A, X)\hat{h}(W, A, X) \right. \\
+ \left. \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \mathbb{E}[\hat{h}_1(W, X) \mid X, A] - \hat{\eta}_2(X, a) \right) + \hat{\eta}_1(X, A, a) \right] \\
= \mathbb{E} \left[ \hat{q}_a(Z, A, X)Y - \hat{q}_a(Z, A, X)\hat{h}(W, A, X) + \hat{\eta}_1(X, A, a) \right] \\
= \mathbb{E} \left[ \hat{q}_a(Z, A, X)Y \right] \\
= \psi_2,
\]

where the last equality is estimation strategy 2 for \( \psi_2 \), (a) is due to

\[
\mathbb{E} \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \mathbb{E}[\hat{h}_1(W, X) \mid X, A] - \hat{\eta}_2(X, a) \right) \right] \\
= \mathbb{E} \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \mathbb{E}[\hat{h}_1(W, X) \mid X, A] - \hat{\eta}_2(X, a) \right) \right] \\
= \mathbb{E} \left[ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \left( \sum_{a'} \hat{h}(w, a', X)\hat{p}(a' \mid X) \mid X, A = a \right) - \mathbb{E} \left[ \sum_{a'} \hat{h}(w, a', X)\hat{p}(a' \mid X) \mid X, A = a \right] \right] \\
= 0,
\]

25
and (b) is due to

\[
E \left[ \hat{q}_a (Z, A, X) \hat{h}(W, A, X) \right] \\
= \sum_{w, z, a', x} \hat{q}_a (z, a', x) \hat{h}(w, a', x) p(w, z, a', x) \\
= \sum_{w, a', x} \frac{p(w \mid a, x)}{p(w \mid a', x)} \hat{h}(w, a', x) p(w, a', x) \\
= \sum_{a', x} \sum_w \hat{h}(w, a', x) p(w \mid a, x) p(a', x) \\
= E \left[ \sum_w \hat{h}(w, A, X) p(w \mid a, X) \right] \\
= E \left[ \hat{\eta}_1 (X, A, a) \right].
\]

Fourth, suppose the pair \( \{ \hat{q}_a, \hat{p}(A \mid X) \} \) is correctly specified. We have,

\[
E \left[ \hat{q}_a (Z, A, X) \{ Y - \hat{h}(W, A, X) \} \right] \\
+ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \{ \hat{h}_1 (W, X) - \hat{\eta}_2(X, a) \} + \hat{\eta}_1 (X, A, a) \\
= E \left[ \hat{q}_a (Z, A, X) Y - \hat{q}_a (Z, A, X) \hat{h}(W, A, X) \right] \\
+ \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}_1 (W, X) - E[I(A = a) \mid X] \hat{\eta}_2(X, a) + \hat{\eta}_1 (X, A, a) \\
\overset{(a)}{=} E \left[ \hat{q}_a (Z, A, X) Y - \hat{q}_a (Z, A, X) \hat{h}(W, A, X) + \frac{I(A = a)}{\hat{p}(A = a \mid X)} \hat{h}_1 (W, X) \right] \\
\overset{(b)}{=} E \left[ \hat{q}_a (Z, A, X) Y \right] \\
= \psi_2,
\]

where the last equality is estimation strategy 2 for \( \psi_2 \), (a) is due to

\[
\eta_2(X, a) = \sum_{a'} \eta_1(X, a', a) p(a' \mid X) = E[\eta_1 (X, A, a) \mid X],
\]

\(26\)
and (b) is due to

\[ E \left[ \hat{q}_a(Z, A, X) \hat{h}(W, A, X) \right] \]
\[ = \sum_{w, z, a', x} \hat{q}_a(z, a', x) \hat{h}(w, a', x) p(w, z, a', x) \]
\[ = \sum_{w, a', x} \frac{p(w | a, x)}{p(w | a', x)} \hat{h}(w, a', x) p(w, a', x) \]
\[ = \sum_{w, a', x} \hat{h}(w, a', x) p(w | a, x) p(a', x) \]
\[ = \sum_{w, x} \sum_{a'} \hat{h}(w, a', x) p(a' | x) p(w | a, x) p(x) \]
\[ = \sum_{w, x} \hat{h}(w, x) p(w | a, x) p(x) \]
\[ = \sum_{w, a', x} \frac{I(\tilde{a} = a)}{p(A = a | X)} \hat{h}(w, x) p(w, a, x) \]
\[ = E \left[ \frac{I(A = a)}{p(A = a | X)} \hat{h}_1(W, X) \right]. \]