Kolmogorov’s Theory of Turbulence and Inviscid Limit of the Navier-Stokes Equations in $\mathbb{R}^3$

Gui-Qiang Chen$^{1,2,3}$, James Glimm$^4$

1 Mathematical Institute, University of Oxford, Oxford OX1 3LB, UK.
E-mail: chengq@maths.ox.ac.uk
2 School of Mathematical Sciences, Fudan University, Shanghai 200433, China
3 Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA
4 Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY 11794-3600, USA. E-mail: glimm@ams.sunysb.edu

Received: 31 August 2010 / Accepted: 10 August 2011
Published online: 10 January 2012 – © Springer-Verlag 2012

Abstract: We are concerned with the inviscid limit of the Navier-Stokes equations to the Euler equations in $\mathbb{R}^3$. We first observe that a pathwise Kolmogorov hypothesis implies the uniform boundedness of the $\alpha$th-order fractional derivatives of the velocity for some $\alpha > 0$ in the space variables in $L^2$, which is independent of the viscosity $\mu > 0$. Then it is shown that this key observation yields the $L^2$-equicontinuity in the time variable and the uniform bound in $L^q$, for some $q > 2$, of the velocity independent of $\mu > 0$. These results lead to the strong convergence of solutions of the Navier-Stokes equations to a solution of the Euler equations in $\mathbb{R}^3$. We also consider passive scalars coupled to the incompressible Navier-Stokes equations and, in this case, find the weak-star convergence for the passive scalars with a limit in the form of a Young measure (pdf depending on space and time). Not only do we offer a framework for mathematical existence theories, but also we offer a framework for the interpretation of numerical solutions through the identification of a function space in which convergence should take place, with the bounds that are independent of $\mu > 0$, that is in the high Reynolds number limit.

1. Introduction

Consider the incompressible Navier-Stokes equations in $\mathbb{R}^3$:

\[
\begin{aligned}
\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p &= \mu \Delta \mathbf{u} + \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0,
\end{aligned}
\]

(1.1)

with Cauchy data:

\[
\mathbf{u}|_{t=0} = \mathbf{u}_0(x),
\]

(1.2)

where $\mathbf{u}$ is the fluid velocity, $p$ is the pressure, $\mu > 0$ is the viscosity, $\nabla$ is the gradient with respect to the space variable $x \in \mathbb{R}^3$, $\mathbf{u} \otimes \mathbf{u} = (u_i u_j)$ is the $3 \times 3$ matrix for $\mathbf{u} = (u_1, u_2, u_3)$, and $\mathbf{f} = f(t, x)$ is a given external force.
The global existence theory for the Cauchy problem (1.1)–(1.2) was first established by J. Leray [14–16]; also see Hopf [11], Temam [27], J.-L. Lions [19], P.-L. Lions [20], and the references cited therein. For clarity of presentation, we focus on periodic solutions with period $T_p = [-P/2, P/2]^3 \subset \mathbb{R}^3$, $P > 0$, that is,

$$u^\mu(t, x + P e_i) = u^\mu(t, x)$$

with $(e_i)_{1 \leq i \leq 3}$ the canonical basis in $\mathbb{R}^3$. Other cases can be analyzed correspondingly. We always assume that $f \in L^1_{loc}(0, \infty; L^2(T_p))$ is periodic in $x$ with period $T_p$.

**Theorem 1.1.** Let $u_0 \in L^2(T_p)$ be periodic in $x \in \mathbb{R}^3$ with period $T_p$. Then, for any $T > 0$, there exists a periodic weak solution $u^\mu = u^\mu(t, x)$ with period $T_p$, along with a corresponding periodic pressure field $p^\mu(t, x)$, of (1.1)–(1.2) such that the equations in (1.1) hold in the sense of distributions in $\mathbb{R}^3_T := [0, T) \times \mathbb{R}^3$, and the following properties hold:

$$u^\mu \in L^2(0, T; H^1) \cap C([0, T]; L^2_w) \cap C([0, T]; L^5_4),$$

$$\partial_t u^\mu \in L^2(0, T; H^{-1}) + (L^{s_2}(0, T; W^{-1, 3s_2/2}) \cap L^q(0, T; L^r)), $$

$$p^\mu \in L^2((0, T) \times T_p) + L^{s_2}(0, T; L^{3s_2/2}),$$

$$\nabla p^\mu \in L^2(0, T; H^{-1}) + L^q(0, T; L^r),$$

where $1 \leq s_1 < 2$, $1 \leq s_2 < \infty$, $1 \leq q < 2$, and $r = \frac{3q}{2(2q-1)}$; and, in addition,

$$\partial_t \left(\frac{1}{2} |u^\mu|^2\right) + \nabla \cdot \left( u^\mu \left(\frac{1}{2} |u^\mu|^2 + p^\mu\right) \right) + \mu |\nabla u^\mu|^2 - \mu \Delta \left( \frac{|u^\mu|^2}{2} \right) \leq f \cdot u^\mu$$

in the sense of distributions in $\mathbb{R}^3_T$.

In Theorem 1.1, $v \in C([0, T]; L^2_w(T_p))$ means that $v \in L^\infty(0, T; L^2(T_p))$ and $v$ is continuous in $t$ with values in $L^2(T_p)$ endowed with the weak topology. Some further a priori estimates and properties of solutions to the Navier-Stokes equations (1.1) can be found in [3,9,20] and the references cited therein.

By contrast, less is known regarding the existence theory for the Euler equations (defined with $\mu = 0$ in (1.1)). For the compressible case, the analysis of [10,22] gives convergence subsequences, but to a very weak limit as a measure-valued vector function. Moreover, this limit is not shown to satisfy the original equations, in that the interchange of limits with nonlinear terms in the equations is not justified in this analysis. On this basis, we state that the existence of solutions for the Euler equations in $\mathbb{R}^3$ is open as is the convergence of the inviscid limit from the Navier-Stokes to the Euler equations. On the other hand, the Euler equations are fundamental for turbulence; see Constantin [4] and the references cited therein.

The purpose of this paper is to establish a framework for the existence theory for the Euler equations. We introduce a physically well accepted pathwise hypothesis by Kolmogorov [12,13], Assumption (K41). One of our main contributions in this paper is our key observation that Assumption (K41), even a weaker version Assumption (K41w), assures sufficient regularity of the solutions to the Navier-Stokes equations with an external force that convergence through a subsequence to solutions of the Euler equations is guaranteed.
As is well-known, there are two types of turbulence: driven turbulence by a forcing function and transient turbulence by (strong) initial conditions. The same flow (in a turbulent wind tunnel or turbulent flow in a pipe) could be of either type depending on how the system is modeled: If the pipe is considered in isolation, the flow might be forced; if the force is a flow connection from a reservoir to the wind tunnel or pipe, and the reservoir is part of the model, then the turbulence arises from initial conditions, and the turbulence will die out when the reservoir is exhausted and no longer drives the flow. Thus, the distinction between the two (transient and driven turbulence) is to some extent a matter of points of view and of modeling convenience. In this paper, our framework is focused on transient turbulence, though the forced turbulence is also included. For some recent developments in the mathematical study of energy dissipation in body-forced turbulence, see Constantin-Doering [5], Doering-Foias [8], and the references cited therein.

Whenever a mathematically precise formulation of a physically precise idea is attempted, ambiguities may arise. In the present case, the assumptions concern the rate $\epsilon$ of dissipation of kinetic energy, with a fundamental definition which depends on the viscosity, as the ultimate source of energy dissipation. The essence of Assumption (K41), as stated in a physical language, is that in the inertial range, the energy is transferred from larger to smaller length scales (eddies) in a manner which is independent of viscosity, because all aspects of the inertial range are assumed to be independent of viscosity. The energy dissipation occurs only at smaller scales, i.e., below the Kolmogorov scale, and even there, it is limited by the energy arriving at these scales through the energy cascade. Given that the energy dissipation rate is supposed to be independent of viscosity, the question remains as to which statistical ensemble or in what norm or function space to express this property. Here the physical literature is not a good guide, and we introduce the assumptions sufficient to allow our proofs to go forward. The distinction among time averages, spatial averages at a fixed time, space-time averages and ensemble averages relates to the well known ergodic hypothesis and is out of the scope of the present paper.

Mathematically, our result has the status of an informed conjecture and mathematically rigorous consequences of this conjecture. Numerical analysts may find the framework useful, in view of the many difficulties involved in assessing convergence of numerical simulations of turbulent and turbulent mixing flows. We expect that many physicists will probably accept the conclusions as being correct, even if unproven mathematically. There has been some discussion regarding the exponent $5/3$ which occurs in Assumption (K41). We note that the main results (if not the detailed estimates) are not sensitive to this specific number, and corrections (as conventionally understood) to it due to intermittency should not affect our result. In fact, our rigorous argument works for an even weaker version, Assumption (K41w), for any $\beta > 0$.

Not only do we offer a framework for mathematical existence theories, but also we offer a framework for the interpretation of numerical solutions of (1.1). Only for very modest problems and with the largest computers can converged solutions of (1.1) be achieved. These solutions are called direct numerical solutions (DNS). For most solutions of interest to science or engineering, the large eddy simulations (LES) or Reynolds Averaged Navier-Stokes (RANS) simulations are required. We discuss here the more accurate LES methodology. Briefly, it employs a numerical grid which will resolve some but not all of the turbulent eddies. The smallest of those, below the level of the grid spacing, are not resolved. However, either (a) the equations in (1.1) are modified with additional subgrid scale (SGS) terms to model the influence of the unresolved scales on those that are being computed or (b) the numerical algorithm is modified in some
manner to accomplish this effect in some other way. The present article contributes to this analysis through the introduction of a function space in which convergence should take place, with bounds that are independent of $\mu$, that is in the high Reynolds number limit.

Because of the common occurrence of high Reynolds numbers in flows of practical and scientific interest and the need to perform LES simulations to achieve scientific understanding and engineering designs, we observe that existence theories for the Euler equations are relevant to the mathematical theories of numerical analysis.

2. The Kolmogorov Hypotheses (1941)

By the definition of weak solutions, we know that, for any $T > 0$, there exists $C_T > 0$ independent of $\mu$ such that

$$
\|u^\mu - \bar{u}\|^2_{L^\infty(0,T;L^2(T_P))} + \|\sqrt{\mu} \nabla u^\mu\|^2_{L^2([0,T]\times T_P)} 
\leq C_T (\|u_0 - \bar{u}\|^2_{L^2(T_P)} + \|f - \bar{f}\|^2_{L^2([0,T]\times T_P)}),
$$

(2.1)

where

$$
\bar{u}(t) = \frac{1}{|T_P|} \int_{T_P} u^\mu(t, x) dx = \frac{1}{|T_P|} \int_{T_P} u_0(x) dx + \int_0^t \bar{f}(s) ds
$$

and

$$
\bar{f}(t) = \frac{1}{|T_P|} \int_{T_P} f(t, x) dx
$$

are independent of $\mu$. Without loss of generality, we assume that the mean velocity and pressure are zero, and interpret $u^\mu$ as the fluctuating velocity. Then the total energy $E(t)$ per unit mass at time $t$ for isotropic turbulence is:

$$
E(t) = \frac{1}{2|T_P|} \int_{T_P} |u^\mu(t, x)|^2 dx = \sum_{k \geq 0} E(t, k) = \sum_{k \geq 0} 4\pi q(t, k) k^2.
$$

(2.2)

Here $E(t, k) = |k|$, is the energy wavenumber spectrum, $q(t, k)$ can be interpreted as the density of contributions in wavenumber space to the total energy, which is sometimes called the spectral density, and $k = (k_1, k_2, k_3) = \frac{2\pi}{P} (n_1, n_2, n_3) \in \mathbb{R}^3$, with $n_j = 0, \pm 1, \pm 2, \ldots$, and $j = 1, 2, 3$, is the discrete wavevector in the Fourier transform:

$$
\hat{u}(t, k) = \frac{1}{|T_P|} \int_{T_P} u(t, x) e^{-ik \cdot x} dx
$$

of the velocity $u(t, x)$ in the $x$-variable. Then

$$
u(t, x) = \sum_k \hat{u}(t, k) e^{ik \cdot x}.
$$

Kolmogorov’s two assumptions in his description of isotropic turbulence in Kolmogorov [12,13] (also see McComb [21]) are:
(i) At sufficiently high wavenumbers, the energy spectrum $E(t, k)$, can depend only on the fluid viscosity $\mu$, the dissipation rate $\varepsilon$, and the wavenumber $k$ itself.

(ii) $E(t, k)$ should become independent of the viscosity as the Reynolds number tends to infinity:

$$E(t, k) \approx \alpha \varepsilon^{2/3} k^{-5/3}$$

in the limit of infinite Reynolds number, where $\alpha$ may depend on $t$, but is independent of $k$ and $\varepsilon$.

For general turbulence, the energy wavenumber spectrum $E(t, k)$ in (2.3) may be replaced by $E(t, k) = E(t, k, \phi, \theta)$ in the spherical coordinates $(k, \phi, \theta), 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$, in the $k$-space, but it should be in the same asymptotics as in (2.3) for sufficiently high wavenumber $k = |k|$.

For the remainder of this paper, we assume these two hypotheses, which we interpret in mathematical terms as a pathwise Kolmogorov hypothesis:

**Assumption (K41).** For any $T > 0$, there exist $C_T > 0$ and $k_*$ (sufficiently large) depending on $u_0$ and $f$ but independent of the viscosity $\mu$ such that, for $k = |k| \geq k_*$,

$$\int_0^T E(t, k) dt \leq C_T k^{-5/3}.$$  

We should point out that the constant $C_T = C_T(u_0, f)$ depends on the maximum time $T$, the initial data $u_0$, and the external force $f$, but is independent of the viscosity $\mu > 0$. For given maximum time $T$, the dependence on $u_0$ and $f$ of the constant $C_T$ is from the control of the upper bound of $\int_0^T \varepsilon^{2/3} \, ds$, with $\varepsilon = \varepsilon(t)$ being the rate of energy dissipation, by the initial data and the external force, more specifically by $\|u_0\|_{L^2(\mathbb{T}_P)}$ and $\|f\|_{L^2([0,T] \times \mathbb{T}_P)}$. The dimensionless version of the bound follows from (2.3).

For our analysis, the following weaker version of Assumption (K41) is sufficient:

**Assumption (K41w).** For any $T > 0$, there exist $C_T = C_T(u_0, f)$ and $k_*$ (sufficiently large) depending on $u_0$ and $f$ but independent of the viscosity $\mu$ such that, for $k = |k| \geq k_*$,

$$\sup_{k \geq k_*} \left( \frac{1}{|k|^{3+\beta}} \int_0^T |\tilde{u}(t, k)|^2 dt \right) \leq C_T \quad \text{for some } \beta > 0.$$  

A mathematical proof of Assumption (K41) may well depend on developing a mathematical version of the renormalization group. The renormalization group has proved to be very powerful in theoretical physics calculations. The basic idea is to integrate differentially a segment of the problem, say from $k$ to $k - dk$, following by a rescaling of all variables, so that all variables are redefined to be integrated through $k$ only. The integration is quite a messy operation, but in the rescaling, it is important to examine the scaling dimensions of all terms. Most of them get smaller under rescaling and are called inessential. The few that preserve their magnitude are called essential and serve to define the key parameters of the problem. These must be reset (renormalized) to their observed (interaction) values after the integration. Since the method is generally applied to self-similar problems, the result of integration does not change the form of the equations. That is, it looks the same at $k$ as at $k - dk$ rescaled back to $k$. Thus, one has a kind of group operation, if the steps are discrete, and a differential equation, if the steps, as the present notation suggests, are infinitesimal. In either case, the desired solution is a fixed point.
3. $L^2$–Equicontinuity of the Velocity in the Space Variables, Independent of the Viscosity

In this section we show that the pathwise Kolmogorov hypothesis, Assumption (K41w), implies a uniform bound of the velocity $u^\mu(t, x)$ in $L^2(0, T; H^\alpha(\mathbb{T}_P))$ for any $\alpha \in (0, \beta/2)$, especially the uniform equicontinuity of the velocity $u^\mu(t, x)$ in the space variables in $L^2([0, T] \times \mathbb{T}_P)$, independent of $\mu > 0$.

**Proposition 3.1.** Under Assumption (K41w), for any $T \in (0, \infty)$, there exists $M_T > 0$ depending on $u_0, f$, and $T > 0$, but independent of $\mu > 0$, such that

$$
\|u^\mu\|_{L^2(0,T;H^\alpha(\mathbb{T}_P))} \leq M_T < \infty,
$$

where $\alpha \in (0, \beta/2)$.

**Proof.** Using the definition of fractional derivatives via the Fourier transform, the Parseval identity, and Assumption (K41w), i.e., (2.5), we have

$$
\int_0^T \int_{\mathbb{T}_P} |D^\alpha_x u^\mu(t, x)|^2 dx dt \\
\leq C_1 \int_0^T \left( \sum_{|k| \leq k_*} |k|^{2\alpha} |\hat{u}^\mu(t, k)|^2 \right) dt \\
= C_1 \int_0^T \left( \sum_{0 \leq |k| \leq k_*} |k|^{2\alpha} |\hat{u}^\mu(t, k)|^2 \right) dt + C_1 \int_0^T \left( \sum_{|k| > k_*} |k|^{2\alpha} |\hat{u}^\mu(t, k)|^2 \right) dt \\
\leq C_1 k_*^{2\alpha} \int_0^T \left( \sum_{0 \leq |k| \leq k_*} |\hat{u}^\mu(t, k)|^2 \right) dt + C_2 \sum_{|k| \geq k_*} |k|^{2\alpha - 3 - \beta} \\
\leq C_1 k_*^{2\alpha} \int_0^T \left( \sum_{k} |\hat{u}^\mu(t, k)|^2 \right) dt + C_2 k_*^{2\alpha - 1 - \beta} \\
\leq C_1 k_*^{2\alpha} \int_0^T \int_{\mathbb{T}_P} |u^\mu(t, x)|^2 dx dt + C_2 k_*^{2\alpha - \beta} \\
\leq C_3 k_*^{2\alpha} \int_0^T \int_{\mathbb{T}_P} |u^\mu(t, x)|^2 dx dt + C_4 k_*^{2\alpha - \beta} \\
\leq C_5 k_*^{2\alpha} T \left( \int_{\mathbb{T}_P} |u_0(x)|^2 dx + \int_0^T \int_{\mathbb{T}_P} |f(t, x)|^2 dx dt \right) + C_4 k_*^{2\alpha - \beta} \\
\leq M(T, k_*, \alpha) < \infty,
$$

since $\alpha < \beta/2$, which $C_j, j = 1, \ldots, 5$, are the constants independent of $\mu$. This completes the proof. □

Proposition 3.1 directly yields the uniform equicontinuity of $u^\mu(t, x)$ in $x$ in $L^2([0, T] \times \mathbb{T}_P)$ independent of $\mu > 0$.

4. $L^2$–Equicontinuity of the Velocity in the Time Variable, Independent of the Viscosity

In this section, we show that Proposition 3.1 implies the uniform equicontinuity of the velocity in the time variable $t > 0$ in $L^2$, independent of $\mu > 0$. 

Proposition 4.1. For any \( T > 0 \), there exists \( M_T > 0 \) depending on \( u_0, f, \) and \( T > 0 \), but independent of \( \mu > 0 \), such that, for all small \( \Delta t > 0 \),

\[
\int_0^{T-\Delta t} \int_{T_P} |u^\mu(t + \Delta t, x) - u^\mu(t, x)|^2 \, dx \, dt \leq M_T(\Delta t)^{\frac{2\alpha}{2\alpha + 2}} \rightarrow 0 \quad \text{as} \quad \Delta t \rightarrow 0.
\]

(4.1)

Proof. For simplicity, we drop the superscript \( \mu > 0 \) of \( u^\mu \) in the proof.

Fix \( \Delta t > 0 \). For \( t \in [0, T - \Delta t] \), set

\[
w(t, \cdot) = u(t + \Delta t, \cdot) - u(t, \cdot).
\]

Then, for any \( \varphi(t, x) \in C^\infty([0, T) \times T_P) \) that is periodic in \( x \in \mathbb{R}^3 \) with period \( T_P \), we have

\[
\int_{T_P} w(t, x) \cdot \varphi(t, x) \, dx
\]

\[
= \int_t^{t+\Delta t} \int_{T_P} \partial_s u(s, x) \cdot \varphi(t, x) \, dx \, ds
\]

\[
= \int_t^{t+\Delta t} \int_{T_P} (u \otimes u)(s, x) : \nabla \varphi(t, x) \, dx \, ds
\]

\[
+ \int_t^{t+\Delta t} \int_{T_P} p(s, x) \nabla \cdot \varphi(t, x) \, dx \, ds
\]

\[
- \mu \int_t^{t+\Delta t} \int_{T_P} \nabla u(s, x) : \nabla \varphi(t, x) \, dx \, ds
\]

\[
+ \int_t^{t+\Delta t} \int_{T_P} f(s, x) \cdot \varphi(t, x) \, dx \, ds,
\]

(4.2)

where \( \nabla \varphi = (\partial_j \varphi_i) \) is the \( 3 \times 3 \) matrix, and \( A : B \) is the matrix product \( \sum_{i,j} a_{ij} b_{ij} \) for \( A = (a_{ij}) \) and \( B = (b_{ij}) \).

By approximation, equality (4.2) still holds for

\[
\varphi \in L^\infty(0, T; H^1(T_P)) \cap C([0, T]; L^2(T_P)) \cap L^\infty([0, T) \times T_P).
\]

Choose

\[
\varphi = \varphi^\delta(t, x) := (j_\delta * w)(t, x) = \int j_\delta(y) w(t, x - y) \, dy \in \mathbb{R}^3
\]

for \( \delta > 0 \), which is periodic in \( x \in \mathbb{R}^3 \) with period \( T_P \), where \( j_\delta(x) = \frac{1}{\delta^3} j(\frac{x}{\delta}) \geq 0 \) is a standard mollifier with \( j \in C_0^\infty(\mathbb{R}^3) \) and \( \int j(x) \, dx = 1 \). Then

\[
\nabla \cdot \varphi^\delta(t, x) = 0,
\]

(4.3)

since \( \nabla \cdot w(t, x) = 0 \).
Integrating (4.2) in $t$ over $[0, T - \Delta t]$ with $\varphi = \varphi^\delta(t, x)$ and using (4.3), we have

$$
\int_0^{T-\Delta t} \int_{T_p} |w(t, x)|^2 dx dt
$$

$$
= \int_0^{T-\Delta t} \int_{T_p} \int_t^{t+\Delta t} (u \otimes u)(s, x) : \nabla \varphi^\delta(t, x) dxdst
$$

$$
- \mu \int_0^{T-\Delta t} \int_{T_p} \int_t^{t+\Delta t} \nabla u(s, x) : \nabla \varphi^\delta(t, x) dxdst
$$

$$
+ \int_0^{T-\Delta t} \int_{T_p} w(t, x) \cdot (w(t, x) - \varphi^\delta(t, x)) dxdt
$$

$$
+ \int_0^{T-\Delta t} \int_{T_p} f(s, x) \cdot \varphi^\delta(t, x) dxdst
$$

$$
=: J_1^\delta + J_2^\delta + J_3^\delta + J_4^\delta. \quad (4.4)
$$

Notice that, for any $x \in T_p$,

$$
|\nabla \varphi^\delta(t, x)| \leq \frac{1}{\delta} \left| \int_{|x-y| \leq \delta} j^\delta_s(x-y) w(t, y) dy \right|
$$

$$
\leq \frac{C}{\delta^{5/2}} \left( \int |j^\delta_s(y)|^2 dy \right)^{1/2} \|w(t, \cdot)\|_{L^2(T_p)}
$$

$$
\leq \frac{C}{\delta^{5/2}}.
$$

Here and hereafter, we use $C > 0$ as a universal constant independent of $\mu > 0$.

Then

$$
|J_1^\delta| \leq \frac{CT \Delta t}{\delta^{5/2}} \|u(t, \cdot)\|_{L^2(T_p)}^2 \leq \frac{CT \Delta t}{\delta^{5/2}} \|u_0\|_{L^2(T_p)}^2 \leq \frac{C \Delta t}{\delta^{5/2}}. \quad (4.5)
$$

Furthermore, we have

$$
|J_2^\delta| \leq \int_0^{T-\Delta t} \int_{T_p} \int_t^{t+\Delta t} \mu |\nabla u(s, x)| |\nabla \varphi^\delta(t, x)| dxdst
$$

$$
\leq \int_0^{T-\Delta t} \int_{T_p} \left( \int_t^{t+\Delta t} \sqrt{\mu} |\nabla u(s, x)| ds \right) \sqrt{\mu} |\nabla \varphi^\delta(t, x)| dxdt
$$

$$
\leq \left( \int_0^{T-\Delta t} \int_{T_p} \left( \int_t^{t+\Delta t} \sqrt{\mu} |\nabla u(s, x)| ds \right)^2 dxdt \right)^{1/2}
$$

$$
\times \left( \int_0^{T-\Delta t} \int_{T_p} \mu |\nabla \varphi^\delta(t, x)|^2 dxdt \right)^{1/2}
$$

$$
\leq (\Delta t)^{1/2} \left( \int_0^{T-\Delta t} \int_{T_p} \int_t^{t+\Delta t} \mu |\nabla \varphi^\delta(t, x)|^2 dxdst \right)^{1/2}
$$

$$
\times \left( \int_0^{T-\Delta t} \int_{T_p} \mu \left( \int_{|y| \leq \delta} j^\delta_s(y) |\nabla w(t, x-y)| dy \right)^2 dxdt \right)^{1/2}
$$
\[
\leq C \Delta t \left( \mu \int_{\mathbb{R}^3} j_\delta(y)^2 \, dy \int_{|y| \leq \delta} \int_0^{T-\Delta t} \int_{T_p} \mu \nabla w(t, x - y)^2 \, dx \, dt \, dy \right)^{1/2} \\
\leq C \Delta t, 
\]
(4.6)

where we have used
\[\| \sqrt{\mu \nabla u} \|_{L^2([0,T] \times T_p)}^2 \leq M_T \]
from (1.7) for the weak solutions in Theorem 1.1 with \( M_T > 0 \) independent of \( \mu \).

On the other hand, for
\[ J_3^\delta := \int_0^{T-\Delta t} \int_{T_p} w(t, x) (w(t, x) - \varphi_\delta(t, x)) \, dx \, dt, \]
we find
\[
|J_3^\delta| = \left| \int_0^{T-\Delta t} \int_{T_p} w(t, x) \left( \int_{\mathbb{R}^3} j_\delta(x - y) (w(t, x) - w(t, y)) \, dy \right) \, dx \, dt \right| \\
\leq \int_0^{T-\Delta t} \int_{T_p} \int_{|y| \leq 1} j(y)|w(t, x)||w(t, x) - w(t, x - \delta y)| \, dy \, dx \, dt \\
\leq \int_{|y| \leq 1} j(y) \left( \int_0^T \int_{T_p} |w(t, x)|^2 \, dx \, dt \right)^{1/2} \, dy \\
\leq C \delta^\alpha \int_{\mathbb{R}^3} j(y)|y|^{\alpha} \, dy \\
\leq C_3 \delta^\alpha. 
\]
(4.7)

Here we have used the fact that \( \|w(t, \cdot)\|_{L^2(T_p)} \leq C \) and
\[
\int_0^T \int_{T_p} |w(t, x) - w(t, x - \delta y)|^2 \, dx \, dt \\
= \int_0^T \left( \sum_k |\hat{w}(t, k)|^2 \left( 1 - e^{i\delta k \cdot y} \right)^2 \right) dt \\
\leq C \delta^{2\alpha} |y|^{2\alpha} \int_0^T \left( \sum_k |k|^{2\alpha} |\hat{w}(t, k)|^2 \right) dt \\
\leq C \delta^{2\alpha} |y|^{2\alpha} \int_0^T \int_{T_p} |D^\alpha_{\delta}(t, x)|^2 \, dx \, dt \\
\leq C \delta^{2\alpha} |y|^{2\alpha}.
\]

Moreover, we have
\[
|J_4^\delta| = C \Delta t \|f\|_{L^2([0,T] \times T_p)} \|\varphi_\delta\|_{L^2((0,T-\Delta t) \times T_p)} \\
\leq C \Delta t \|w\|_{L^2([0,T-\Delta t] \times T_p)} \\
\leq C \Delta t \|u\|_{L^2((0,T) \times T_p)} \\
\leq C_4 \Delta t. 
\]
(4.8)
Combining (4.4)–(4.7) with (4.8), we have
\[
\int_0^{T-\Delta t} \int_{\mathbb{T}_p} |w(t, x)|^2 \, dx \, dt \leq \inf_{\delta > 0} \{ C_1 \frac{\Delta t}{\delta^{5/2}} + (C_2 + C_4) \Delta t + C_3 \delta^\alpha \}
\leq \inf_{\delta > 0} \{ C_5 \frac{\Delta t}{\delta^{5/2}} + C_3 \delta^\alpha \}.
\]
Choose
\[
\delta = \left( \frac{5C_5}{2\alpha C_3} \right)^{2/(2\alpha)} (\Delta t)^{2/(2\alpha)}.
\]
We conclude
\[
\int_0^{T-\Delta t} \int_{\mathbb{T}_p} |w(t, x)|^2 \, dx \, dt \leq C (\Delta t)^{\frac{2\alpha}{5+2\alpha}}. \tag{4.9}
\]
This completes the proof. \(\square\)

As a direct corollary, we have

**Proposition 4.2.** Under Assumption (K41w), for any \(T \in (0, \infty)\), there exists \(M_T > 0\) depending on \(u_0, f\), and \(T > 0\), but independent of \(\mu > 0\), such that
\[
\int_0^{T} \int_{\mathbb{T}_p} |D^{\alpha} u^\mu(t, x)|^2 \, dt \, dx \leq M_T < \infty, \tag{4.10}
\]
for some \(\alpha > 0\).

Combining Proposition 3.1 with Proposition 4.2, we have
\[
\|u^\mu\|_{H^\alpha([0,T] \times \mathbb{T}_p)} \leq M_T < \infty \tag{4.11}
\]
for some \(\alpha > 0\). Then, by the Sobolev imbedding theorem, we have

**Proposition 4.3.** Under Assumption (K41w), for any \(T \in (0, \infty)\), there exists \(M_T > 0\) depending on \(u_0, f\), and \(T > 0\), but independent of \(\mu > 0\), such that
\[
\|u^\mu\|_{L^2 \cap L^q([0,T] \times \mathbb{T}_p)} \leq M_T < \infty \tag{4.12}
\]
for some \(q > 2\).

### 5. Inviscid Limit

In this section we show the inviscid limit from the Navier-Stokes to the Euler equations under Assumption (K41w).
Theorem 5.1. The pathwise Kolmogorov hypothesis, Assumption (K41w), implies the strong compactness in $L^2 \cap L^q ([0, T) \times \mathbb{T}_p)$, for some $q > 2$, of the solutions $u^\mu (t, x)$ of the Navier-Stokes equations in $\mathbb{R}^3$ when the viscosity $\mu$ tends to zero. That is, there exist a subsequence (still denoted) $u^\mu (t, x)$ and a function $u \in L^2 \cap L^q ([0, T) \times \mathbb{T}_p)$ with a corresponding pressure function $p$ such that

$$u^\mu (t, x) \rightarrow u(t, x) \quad \text{a.e. as } \mu \rightarrow 0,$$

and $u(t, x)$ is a weak solution of the incompressible Euler equations with Cauchy data $u_0(x)$ along with the corresponding pressure $p$. Furthermore,

$$\int_{\mathbb{T}_p} |u(t, x)|^2 dx \leq \int_{\mathbb{T}_p} |u_0(x)|^2 dx + 2 \int_0^T \int_{\mathbb{T}_p} (u^\mu \cdot f(s, x)) dx ds. \quad (5.1)$$

Proof. Propositions 3.1 and 4.1 imply the $L^2$-equicontinuity of $u^\mu (t, x)$ in $(t, x) \in [0, T) \times \mathbb{T}_p$, independent of $\mu > 0$. This yields that there exist both a subsequence (still denoted) $u^\mu (t, x)$ and a function $u \in L^2$ such that

$$u^\mu (t, x) \rightarrow u(t, x) \quad \text{in } L^2 \quad \text{as } \mu \rightarrow 0,$$

which implies that

$$u^\mu (t, x) \rightarrow u(t, x) \quad \text{a.e. as } \mu \rightarrow 0, \quad (5.2)$$

and

$$\nabla \cdot u(t, x) = 0 \quad (5.3)$$

in the sense of distributions in $\mathbb{R}^3_T$.

From Proposition 4.3, we have

$$u \in L^2 \cap L^q ([0, T) \times \mathbb{T}_p),$$

and

$$u^\mu (t, x) \rightarrow u(t, x) \quad \text{in } L^2 \cap L^q ([0, T) \times \mathbb{T}_p) \quad \text{as } \mu \rightarrow 0$$

for some $q > 2$.

Furthermore, for any $\varphi \in C_0^\infty (\mathbb{R}^3_T)$ with $\nabla \cdot \varphi = 0$, we multiply both sides (1.1) by $\varphi$ and integrate over $\mathbb{R}^3_T$ to obtain

$$\int_0^T \int_{\mathbb{R}^3} (\nabla \cdot u^\mu \varphi_t + (u^\mu \otimes u^\mu) : \nabla \varphi + u^\mu \cdot f \varphi) dx dt + \int_{\mathbb{R}^3} u_0(x) \varphi (0, x) dx = -\mu \int_0^T \int_{\mathbb{R}^3} u^\mu \cdot \Delta \varphi dx dt.$$

Letting $\mu \rightarrow 0$, we conclude that

$$\int_0^T \int_{\mathbb{R}^3} (\nabla \varphi_t + (u \otimes u) : \nabla \varphi + (u \cdot f) \varphi) dx dt + \int_{\mathbb{R}^3} u_0(x) \varphi (0, x) dx = 0. \quad (5.4)$$
Combining (5.4) with (5.2) yields that \( \mathbf{u}(t, \mathbf{x}) \) is a weak solution of the Euler equations (1.1) (\( \mu = 0 \)) with Cauchy data (1.2) along with the corresponding pressure \( p \).

Integrating (1.7) over \([0, T] \times \mathbb{T}_p\) yields

\[
\int_{\mathbb{T}_p} |\mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x} \leq \int_{\mathbb{T}_p} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + 2 \int_0^T \int_{\mathbb{T}_p} \mathbf{u}(s, \mathbf{x}) \cdot \mathbf{f}(s, \mathbf{x}) d\mathbf{x} ds.
\]

Letting \( \mu \to 0 \), we have

\[
\int_{\mathbb{T}_p} |\mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x} \leq \int_{\mathbb{T}_p} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + 2 \int_0^T \int_{\mathbb{T}_p} \mathbf{u}(s, \mathbf{x}) \cdot \mathbf{f}(s, \mathbf{x}) d\mathbf{x} ds.
\]

This completes the proof. \( \square \)

**Remark 5.1.** The question whether the energy inequality in (5.1) becomes identical is exactly Onsager’s conjecture [24]. It states that weak solutions of the Euler equations for incompressible fluids in \( \mathbb{R}^3 \) conserve energy only if they have a certain minimal smoothness of the order of \( 1/3 \) fractional derivatives in \( \mathbf{x} \in \mathbb{R}^3 \) and that they dissipate energy if they are rougher. Indeed, in Cheskidov-Constantin-Friedlander-Shvydkoy [2], it is proved that energy is conserved when \( \mathbf{u}(t, \mathbf{x}) \) is in the Besov space of tempered distributions \( B^{1/3}_{2, \infty} \). This is a space in which the Hölder exponent is exactly \( 1/3 \), but the slightly better regularity is encoded in the summability condition. We show in Proposition 3.1 that Assumption (K41w) implies the uniform boundedness in the norm of \( L^2(0, T; H^\alpha(\mathbb{T}_p)) \) for some \( \alpha > 0 \) for the solutions to the Navier-Stokes equations, independent of the viscosity \( \mu > 0 \). From the point of view of physics, strengthening the Kolmogorov exponent 5/3 (i.e. \( \beta = 2/3 \)), as would be needed to obtain a stronger \( \alpha = 1/3 \) bound, may result from some theory of intermittency.

### 6. Incompressible Navier-Stokes Equations with Passive Scalar Fields

Consider the incompressible Navier-Stokes equations with passive scalar fields in \( \mathbb{R}^3 \):

\[
\begin{align*}
\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p &= \mu_0 \Delta \mathbf{u} + \mathbf{f}, \\
\partial_t \chi_i + \nabla \cdot (\chi_i \mathbf{u}) &= \mu_i \Delta \chi_i, \quad i = 1, \ldots, I, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

(6.1)

with Cauchy functions:

\[
(\mathbf{u}, \chi_1, \ldots, \chi_I)|_{t=0} = (\mathbf{u}_0, \chi_{10}, \ldots, \chi_{I0})(\mathbf{x})
\]

(6.2)

that are periodic in \( \mathbf{x} \in \mathbb{R}^3 \) with period \( \mathbb{T}_p \), where \( \chi_{I0}(\mathbf{x}) \geq 0 \) with \( \sum_i \chi_{i0}(\mathbf{x}) = 1 \), and \( \mathbf{f} \in L^2_{\text{loc}}(0, \infty; L^2(\mathbb{T}_p)) \) is a given external force.

These equations are interpreted as describing concentrations \( \chi_i, i < I \), of minority species, such as dilute chemical reagents in a majority carrier species \( \chi_I \), in an approximation that the minority species do not influence the bulk fluid density nor the bulk fluid viscosity. Thus, they do not interact with (influence) the bulk fluid, but are passively transported by it; hence named passive scalars. The new physics introduced by \( \chi_i \) is mixing, with new dimensionless parameters, the species Schmidt numbers \( \mu_0/\mu_i \), and the associated length scales for diffusion, the Batchelor scales. See Monin-Yaglom [23] for further information.
Similarly to Theorem 1.1, for each fixed \( \mathbf{\mu} = (\mu_0, \mu_1, \ldots, \mu_I), \mu_j > 0, j = 0, 1, \ldots, I, \) there exist a global periodic solution \((u^\mu, \chi_i^\mu, \ldots, \chi_I^\mu)(t, x)\) in \( x \in \mathbb{R}^3 \) with period \( \mathbb{T}_p \) and a corresponding pressure function \( p^\mu(t, x) \) of the Cauchy problem (6.1)–(6.2) such that the equations in (6.1) hold in the sense of distributions, and the following properties hold:

\[
\begin{align*}
\mathbf{u}^\mu &\in L^2((0, T); H^1) \cap C([0, T]; L^2) \cap C([0, T]; L^{s_1}), \\
\partial_t \mathbf{u}^\mu &\in L^2((0, T); H^{-1}) + (L^{s_2}((0, T); W^{-1,\frac{3r_2}{3r_2-2}}) \cap L^q(0, T; L^r)), \\
p^\mu &\in L^2((0, T) \times \mathbb{T}_p) + L^{s_2}(0, T; \mathbb{L}^{3s_2})^2, \\
\nabla p^\mu &\in L^2((0, T); H^{-1}) + L^q(0, T; L^r), \\
0 &\leq \chi_i^\mu(t, x) \leq 1, \quad i = 1, 2, \ldots, I, \\
\chi_i^\mu &\in L^2(0, T; H^1) \cap C([0, T]; L^{s_2}), \\
\partial_i \chi_i^\mu &\in L^2((0, T); H^{-1}) + (L^{s_2}(0, T; W^{-1,\frac{6r_2}{3r_2-2}}) \cap L^q(0, T; L^{2r})), 
\end{align*}
\]

where \( 1 \leq s_1 < 2, 1 \leq s_2 < \infty, 1 \leq q \leq 2, \) and \( r = \frac{3q}{2(2q-1)} \); and, in addition,

\[
\begin{align*}
\partial_t \left( \frac{1}{2} |\mathbf{u}^\mu|^2 \right) + \nabla \cdot (\mathbf{u}^\mu \left( \frac{1}{2} |\mathbf{u}^\mu|^2 + p^\mu \right)) + \mu_0 |\nabla \mathbf{u}^\mu|^2 - \mu_0 \Delta \left( \frac{|\mathbf{u}^\mu|^2}{2} \right) &\leq f \cdot \mathbf{u}^\mu, \\
\partial_t \beta(\chi_i^\mu) + \nabla \cdot (\mathbf{u}^\mu \beta(\chi_i^\mu)) &\leq \mu_i \Delta \beta(\chi_i^\mu)
\end{align*}
\]

in the sense of distributions, for any \( C^2 \)-function \( \beta(\chi), \beta''(\chi) \geq 0. \)

**Theorem 6.1.** The pathwise Kolmogorov hypothesis, Assumption (K41w), implies the strong compactness in \( L^2 \cap L^q \) of the velocity sequence \( \mathbf{u}^\mu(t, x) \), for some \( q > 2 \), when the viscosity \( \mu \) tends to zero. That is, there exist a function \( u \in L^2 \cap L^q((0, T) \times \mathbb{T}_p) \) and a subsequence (still denoted) \( \mathbf{u}^\mu(t, x) \) such that

\[
\mathbf{u}^\mu(t, x) \rightarrow u(t, x) \quad \text{in} \quad L^2 \cap L^q((0, T) \times \mathbb{T}_p) \quad \text{as} \quad \mu \rightarrow 0.
\]

Furthermore,

\[
0 \leq \chi_i^\mu(x) \leq 1, \quad i = 1, \ldots, I,
\]

which implies the weak-star convergence subsequentially (still denoted) \( \chi_i^\mu(t, x) \) in \( L^\infty \) to some functions \( \chi_i(t, x), 0 \leq \chi_i(t, x) \leq 1, \) as \( \mu \rightarrow 0. \) Moreover, the limit function \((u, \chi_1, \ldots, \chi_I)(t, x)\) is a weak solution of (6.1)–(6.2) and in addition,

\[
\int_{\mathbb{T}_p} |\mathbf{u}(t, x)|^2 dx \leq \int_{\mathbb{T}_p} |\mathbf{u}_0(x)|^2 dx + 2 \int_0^t \int_{\mathbb{T}_p} \mathbf{u}(s, x) \cdot \mathbf{f}(s, x) dx ds. \quad (6.3)
\]

Denoting the weak-star limit in terms of Young measures, with \( \nu_i^\mu \) the Young measure corresponding to the uniformly bounded sequence \( \chi_i^\mu \) for each \( i = 1, \ldots, I \), we have

\[
\beta(\chi_i) = (\nu_i^\mu, \beta(\chi_i)) = u_* - \lim_{\mu \rightarrow 0} \beta(\chi_i^\mu)
\]

for any \( C^2 \)-function \( \beta(\chi) \). If, in addition, \( \beta''(\chi) \geq 0 \), then

\[
\partial_t \beta(\chi_i) + \nabla \cdot (\beta(\chi_i) \mathbf{u}) \leq 0 \quad (6.4)
\]

in the sense of distributions.
Proof. The strong convergence of \( u^\mu(t, x) \) can be obtained by following the same argument as for Theorem 5.1, which implies
\[
\begin{align*}
  u^\mu(t, x) & \rightarrow u(t, x) \quad \text{in } L^2 \cap L^q([0, T) \times \mathbb{T}_p), \quad q > 2.
\end{align*}
\]
Since \( 0 \leq \chi_i^\mu \leq 1 \), there exist both a further subsequence (still denoted) \( \chi_i^\mu \) and an \( L^\infty \)–function \( \chi_i \) with \( 0 \leq \chi_i \leq 1 \) such that
\[
\chi_i^\mu \rightharpoonup^* \chi_i \quad \text{as } \mu \rightarrow 0.
\]
Then the strong convergence of \( u^\mu(t, x) \) implies that
\[
\chi_i^\mu u^\mu(t, x) \rightharpoonup \chi_i u(t, x)
\]
in the sense of distributions as \( \mu \rightarrow 0 \).
This implies
\[
\partial_t \chi_i + \nabla \cdot (\chi_i u) = 0, \quad i = 1, \ldots, I,
\]
in the sense of distributions.
Furthermore, for any \( C^2 \)–function \( \beta(\chi) \), \( \beta(\chi_i^\mu) \) is uniformly bounded. By the representation theorem of weak limit by Young measures (cf. [26]; also see [1]), we conclude
\[
\beta(\chi_i^\mu) \rightharpoonup^* \langle v_i^{j, x}, \beta(\chi_i) \rangle \quad \text{as } \mu \rightarrow 0,
\]
and, using the strong convergence of \( u^\mu(t, x) \), we further have
\[
u^\mu(t, x) \beta(\chi_i^\mu) \rightharpoonup u(t, x) \langle v_i^{j, x}, \beta(\chi_i) \rangle
\]
in the sense of distributions as \( \mu \rightarrow 0 \).
We now assume \( \beta''(\chi) \geq 0 \). Since
\[
\partial_t \beta(\chi_i^\mu) + \nabla \cdot (u^\mu \beta(\chi_i^\mu)) \leq \mu_i \Delta \beta(\chi_i^\mu),
\]
we take the limit \( \mu \rightarrow 0 \) above to conclude
\[
\partial_t \langle v_i^{j, x}, \beta(\chi_i) \rangle + \nabla \cdot (u \langle v_i^{j, x}, \beta(\chi_i) \rangle) \leq 0
\]
in the sense of distributions. \( \square \)

7. Implications to Numerical Computations

We discuss what can be expected in terms of numerical convergence, on the basis of the convergence framework developed in this paper. Because the framework depends on compactness and subsequences, there is an implicit hint that uniqueness will be a problem. In fact, nonunique solutions for the Euler equations has been known for over a decade; also see Scheffer [25] and De Lellis-Székelyhidi [6,7]. For numerical solutions, this means that LES solutions could depend in an essential manner on the SGS models or the equivalent in the form of modified numerical algorithms. Stated more directly, the turbulent Schmidt and Prandtl numbers of the models or the numerical Schmidt and Prandtl numbers of the algorithms could influence the selection of the numerical LES solution. Distinct but apparently converged solutions from distinct solutions for the identical problem are thus a possibility. Moreover, the LES solution regime is very
sensitive to parameters (other than turbulent or numerical fluid transport) which introduce a regularizing length scale, and to the dimensionless ratios of these length scales. The literature for multiphase flow, turbulent mixing and turbulent chemistry has many dimensionless parameters. Those that are sensitive (experimentally, for example) in a particular problem can be expected to be also sensitive in the definition of the SGS models and in their numerical manifestations as numerical analogues of the SGS models. Our optimistic hope is that a well designed LES algorithm should employ the SGS terms which eliminate further nonuniqueness due to numerical or algorithmic issues, and thus should yield a unique solution.

Assuming the uniqueness of solutions has been achieved, within an LES regime, what does convergence mean? Convergence of the velocity is strong (norm) in an $L^q$ space. This conventional notion of convergence requires no further discussion. For the passive scalars, the convergence is weak-star, meaning that convergence holds for any test function (observable) in the dual space $(L^\infty)^*$. This dual space is a space of signed measures. The measures and the abstract space on which they are defined can be realized concretely in the form of the space-time dependent Young’s measures. In language more familiar to physicists and applied mathematicians, the Young measures represent the space-time dependent probability distribution function (pdf) of limiting values of $\chi_i$. Although the limit also yields a classical weak solution, we expect that the pdf description will prove to be more useful. The reason for this claim is that nonlinear functions of the solution can pass to the limit in the pdf–Young measure formulation but not in the weak star limit formulation. The classical weak solution is the mean value of the Young measure solution, both regarded as space-time dependent distributions. The classical weak solution has far less information than the Young measure. Thus we state that the limiting concentrations $\chi_i$ are pdfs. Clearly, this is a modification of standard ways of judging convergence, but now with a basis in mathematical theory.

Is the Young measure limit a result of a weak mathematical technology, to be improved at some future date? Within the generality of Theorem 6.1, this seems unlikely. That theorem allows arbitrary and even infinite Schmidt numbers, meaning that sharp concentration discontinuities can persist for all time. Reasoning theoretically, the essence of the Kolmogorov theory is that smaller and smaller vortices persist at all length scales, in the absence of viscosity. Thus the sharp interfaces must become ever more convoluted. Numerical evidence [17,18] supports this view, in which unregularized simulations display turbulent mixing concentration interfaces proportional to the available mesh degrees of freedom, uniformly under mesh refinement. More useful is to ask what can be expected in the case of finite, bounded Schmidt numbers. According to an analysis in [23], the species diffusion has a logarithmic singularity at high Schmidt number. But this argument omits time scales. A theoretical model [18], which does include time scales as well as SGS terms to modify the equations, suggests a rapid or instantaneous smoothing of species concentration discontinuities, but at length scales that still may lead to Young measure solutions, i.e., convergence to a pdf. In view of the importance of the nature of this convergence, further studies, both numerical and theoretical, are called for.

Finally, we note that our convergence framework stops at the level of passive scalar fields. Compressible Euler existence theories beyond this level are known only at the level of measure-valued solutions and even there at the level of convergent subsequences, with no proof that the limit is actually a solution of the original equations. Because the concentrations are already described by this measure-valued framework, it is plausible that such a radical change in the convergence framework is required to move from passive scalars to true multiphase flow. Beyond the physical issues involved in the extension
of Assumption \((K41w)\) to variable and species dependent viscosity and density, the mathematical issue is the proof of uniform estimates for partial differential equations whose coefficients are converging to Young measures (pdfs).

Acknowledgements. The authors thank Peter Constantin for helpful discussions and suggestions. Gui-Qiang Chen’s research was supported in part by the National Science Foundation under Grants DMS-0935967, DMS-0807551, the Natural Science Foundation of China under Grant NSFC-10728101, the Royal Society-Wolfson Research Merit Award (UK), and the UK EPSRC Science and Innovation Award to the Oxford Centre for Non-linear PDE (EP/E035027/1). James Glimm’s research was supported in part by the U.S. Department of Energy grants DE-FC02-06ER25770, DE-FG07-07ID14889, DE-FG52-08NA28614, and DE-AC07-05ID14517 and by the Army Research Organization grant W911NF0910306. This manuscript has been co-authored by Brookhaven Science Associates, LLC, under Contract No. DE-AC02-98CH1-886 with the U.S. Department of Energy. The United States Government retains, and the publisher, by accepting this article for publication, acknowledges, a world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for the United States Government purposes.

References
1. Ball, J.: A version of the fundamental theorem of Young measures. In: PDEs and Continuum Models of Phase Transitions, Lecture Notes of Physics, Vol. 344, pp. 207–215. Springer, Berlin-Heidelberg-New York, 1989
2. Cheskidov, A., Constantin, P., Friedlander, S., Shvydkoy, R.: Energy conservation and Onsager’s conjecture for the Euler equations. Nonlinearity 21, 1233–1252 (2008)
3. Constantin, P.: Navier-Stokes equations and area of interfaces. Commun. Math. Phys. 129, 241–266 (1990)
4. Constantin, P.: On the Euler equations of incompressible fluids. Bull. Amer. Math. Soc. 44, 603–621 (2007)
5. Constantin, P., Doering, C.: Variational bounds on dissipative systems. Physica D 82, 221–228 (1995), Variational bounds on energy dissipation in incompressible flows: II. Channel flow. Phys. Rev. E 51, 3192–3198 (1995)
6. De Leliss, C., Székelyhidi, L. Jr.: The Euler equations as a differential inclusion. Ann. of Math. 170(2), 1417–1436 (2009)
7. De Leliss, C., Székelyhidi, L. Jr.: On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal. 195, 225–260 (2010)
8. Doering, C.R., Foias, C.: Energy dissipation in body-forced turbulence. J. Fluid Mech. 467, 289–306 (2002)
9. Foias, C., Manley, O., Rosa, R., Temam, R.: Navier-Stokes Equations and Turbulence. Cambridge: Cambridge University Press, 2001
10. Gangbo, W., Westdickenberg, M.: Optimal transport for the system of isentropic Euler equations. Comm. Partial Differential Equations 34, 1041–1073 (2009)
11. Hopf, E.: Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nachr. 4, 213–231 (1951)
12. Kolmogorov, A.N.: The local structure of turbulence in incompressible viscous fluid for very large Reynolds’s numbers. C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 301–305 (1941)
13. Kolmogorov, A.N.: Dissipation of energy in the locally isotropic turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 16–18 (1941)
14. Leray, J.: Etude de diverses équations intégrales nonlinéaires et de queques problèmes que pose l’hydrodynamique. J. Math. Pures Appl. 12, 1–82 (1933)
15. Leray, J.: Essai sur les mouvements plans d’un liquide visqueux que limitent des parois. J. Math. Pures Appl. 13, 331–418 (1934)
16. Leray, J.: Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 63, 193–248 (1934)
17. Lim, H., Yu, Y., Glimm, J., Li, X.-L., Sharp, D.H.: Chaos, Transport, and Mesh Convergence for Fluid Mixing. Acta Math. App. Sinica 24, 355–368 (2008)
18. Lim, H., Yu, Y., Glimm, J., Li, X.L., Sharp, D.H.: Subgrid Models for Mass and Thermal Diffusion in Turbulent Mixing. Physica Scripta T142, 014014 (2010)
19. Lions, J.-L.: Quelques Méthodes de Résolution des Problèmes aux Limites Nonlineaire eaires. Paris: Dunod, 1969
20. Lions, P.-L.: Mathematical Topics in Fluid Mechanics. Vol. 1: Incompressible Models. Oxford: Oxford Sci. Pub., 1996
21. McComb, W.D.: The Physics of Fluid Turbulence. Oxford: Oxford University Press, 1990
22. Neustupa, J.: Measure-valued solutions of the Euler and Navier-Stokes equations for compressible barotropic fluids. Math. Nachr. 163, 217–227 (1993)
23. Monin, A.S., Yaglom, A.M.: Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. I–II. Translated from the 1965 Russian original. Edited and with a preface by John L. Lumley. English edition updated, augmented and revised by the authors. Mineola, NY: Dover Publications, Inc., 2007
24. Onsager, L.: Statistical hydrodynamics. Nuovo Cimento (Suppl.) 6, 279–287 (1949)
25. Scheffer, V.: An inviscid flow with compact support in space-time. J. Geom. Anal. 3, 343–401 (1993)
26. Tartar, L.: Compensated compactness and applications to partial differential equations. Research Notes in Mathematics, Nonlinear Analysis and Mechanics, Herriot-Watt Symposium, Vol. 4, Knops R.J. ed., Bath: Pitman Press, 1979
27. Temam, R.: Navier-Stokes Equations. Amsterdam: North-Holland, 1977

Communicated by P. Constantin