Efficient Construction, Update and Downdate Of The Coefficients Of Interpolants Based On Polynomials Satisfying A Three-Term Recurrence Relation

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Abstract
In this paper, we consider methods to compute the coefficients of interpolants relative to a basis of polynomials satisfying a three-term recurrence relation. Two new algorithms are presented: the first constructs the coefficients of the interpolation incrementally and can be used to update the coefficients whenever a node is added to or removed from the interpolation. The second algorithm, which constructs the interpolation coefficients by decomposing the Vandermonde-like matrix iteratively, cannot be used to update or downdate an interpolation, yet is more numerically stable than the first algorithm and is more efficient when the coefficients of multiple interpolations are to be computed over the same set of nodes.

1 Introduction
In many applications, we are interested in computing the coefficients of a polynomial interpolation of discrete data relative to a basis of polynomials $p_k(x)$ of increasing degree $k = 0 \ldots n$. In practical terms, this means that given $n + 1$ function values $f_i$ at the $n + 1$ nodes $x_i$, $i = 0 \ldots n$, we want to compute the coefficients $c_k$, $k = 0 \ldots n$ of a polynomial $g_n(x)$ of degree $n$ such that

$$g_n(x_i) = \sum_{k=0}^{n} c_k p_k(x) = f_i, \quad i = 0 \ldots n,$$

That is, the polynomial $g_n(x)$ interpolates the $n + 1$ function values $f_i$ at the nodes $x_i$.

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If we are only interested in evaluating \( g_n(x) \) at different \( x \), then the method of choice is Barycentric Lagrange Interpolation (Berrut and Trefethen 2004), which avoids representing the interpolant in any specific base. Such coefficient-based representations are useful, however, if we are interested in computing other quantities such as the integral or derivative of the interpolant, its \( L_2 \)-norm and/or performing other operations on it such as transforming it to another interval. Additionally, we may also be interested in updating the coefficients when new data is added or existing data is removed. In Gonnet (2010), such a representation is used in an adaptive quadrature routine for just these purposes.

In the following, we will assume that the polynomials \( p_i(x) \) of degree \( i \) can be constructed using a three-term recurrence relation, which we will write as

\[
\alpha_k p_{k+1}(x) = xp_k(x) + \beta_k p_k(x) - \gamma_k p_{k-1}(x),
\]

with \( p_0(x) = 1, \ p_{-1}(x) = 0 \).

Examples of such polynomials are the Legendre polynomials \( P_k(x) \) with

\[
\alpha_k = \frac{k}{2k-1}, \quad \beta_k = 0, \quad \gamma_k = \frac{k-1}{2k-1}
\]

or the Chebyshev polynomials \( T_k(x) \) with

\[
\alpha_0 = 1, \quad \alpha_k = \frac{1}{2}, \quad \beta_k = 0, \quad \gamma_k = \frac{1}{2}
\]

The coefficients \( c_i \) of Equation (1) can be computed solving the system of linear equations

\[
\begin{pmatrix}
  p_0(x_0) & p_1(x_0) & \ldots & p_n(x_0) \\
p_0(x_1) & p_1(x_1) & \ldots & p_n(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
p_0(x_n) & p_1(x_n) & \ldots & p_n(x_n)
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n
\end{pmatrix}
= \begin{pmatrix}
f_0 \\
f_1 \\
\vdots \\
f_n
\end{pmatrix}
\]

which can be written as

\[
P^{(n)} c^{(n)} = f^{(n)}.
\]

The matrix \( P^{(n)} \) is a Vandermonde-like matrix and the system of equations can be solved in \( O(n^3) \) using Gaussian elimination. As with the computation of the monomial coefficients, the matrix may be ill-conditioned (Gautschi 1983).

Björck and Pereyra (1970) present an algorithm to compute the monomial coefficients of an interpolation without the expensive and potentially unstable solution of a Vandermonde system using Gaussian elimination, by computing first the coefficients of a Newton interpolation and then converting these to monomial coefficients. This approach was later extended by Higham (1988)
to compute the coefficients relative to any polynomial basis satisfying a three-term recurrence relation. Both algorithms compute the coefficients in \( \mathcal{O}(n^2) \) operations and in Higham (1990), both methods are shown to be numerically stable when a proper ordering of the nodes \( x_i, i = 0 \ldots n \) is used.

In Section 2 we will re-formulate the algorithms of Björck and Pereyra and of Higham and extend them to update the coefficients after a *downdate*, i.e. the removal of a node, of an interpolation. In Section 3 we present a new algorithm for the construction of interpolations of the type of Equation (1) based on a successive decomposition of the Vandermonde-like matrix in Equation (5). Finally, in Section 4 we will present some results regarding the efficiency and stability of both algorithms.

## 2 A Modification of Björck and Pereyra’s and of Higham’s Algorithms Allowing Downdates

Björck and Pereyra (1970) present an algorithm which exploits the recursive definition of the Newton polynomials

\[
\pi_k(x) = (x - x_{k-1})\pi_{k-1}(x).
\]

They note that given the Newton interpolation coefficients \( a_i \), the interpolation polynomial can be constructed using Horner’s scheme:

\[
q_n(x) = a_n, \quad q_k(x) = (x - x_k)q_{k+1}(x) + a_k, \quad k = n - 1 \ldots 0
\]

where the interpolation polynomial is \( g_n(x) = q_0(x) \).

They also note that given a monomial representation for \( q_k(x) \), such as

\[
q_k(x) = \sum_{i=0}^{n-k} b^{(k)}_i x^i
\]

then the polynomial \( q_{k-1}(x) \) can be constructed, following the recursion in Equation (5), as

\[
q_{k-1}(x) = (x - x_{k-1})q_k(x) + a_{k-1}
\]

\[
= (x - x_{k-1}) \sum_{i=0}^{n-k} b^{(k)}_i x^i + a_{k-1}
\]

\[
= \sum_{i=1}^{n-k+1} b^{(k)}_{i-1} x^i - x_{k-1} \sum_{i=0}^{n-k} b^{(k)}_i x^i + a_{k-1}
\]

\[
= b^{(k)}_{n-k} x^{n-k+1} + \sum_{i=1}^{n-k} (b^{(k)}_{i-1} - x_{k-1} b^{(k)}_i) x^i + b^{(k)}_0 x_{k-1} + a_{k-1}.
\]
From Equation (10) we can then extract the new coefficients $b_i^{(k-1)}$:

$$
b_i^{(k-1)} = \begin{cases} 
b_i^{(k)} - \frac{n-k}{k-1} b_i^{(k)}, & i = n-k+1, \\
b_i^{(k)} - x_{k-1} b_i^{(k)}, & 1 \leq i \leq n-k, \\
-b_0^{(k)} x_{k-1} + a_{k-1}, & i = 0.
\end{cases}
$$

(10)

Higham (1988) uses the same approach, yet represents the Newton polynomials as a linear combination of polynomials satisfying a three-term recurrence relation. Using such a representation he computes $q_{k-1}(x)$ by expanding the recursion in Equation (8) using the representation Equation (11):

$$q_k(x) = \sum_{i=0}^{n-k} c_i^{(k)} p_i(x)$$

(11)

Expanding Equation (12) for the individual $p_k(x)$, and keeping in mind that $p_{-1}(x) = 0$, we obtain

$$q_{k-1}(x) = \sum_{i=1}^{n-k+1} c_i^{(k)} x_{i-1} p_i(x) - \sum_{i=0}^{n-k} c_i^{(k)} (x_{k-1} + \beta_i) p_i(x)$$

$$+ \sum_{i=0}^{n-k-1} c_i^{(k)} \gamma_{i+1} p_i(x) + a_{k-1}.$$  

(13)

By shifting the sums in Equation (13) and re-grouping around the individual $p_k(x)$ we finally obtain

$$q_{k-1}(x) = c_{n-k}^{(k)} x_{n-k} p_{n-k+1}(x) + \left( c_{n-k-1}^{(k)} x_{n-k-1} + \beta_{n-k} \right) p_{n-k}(x)$$

$$+ \sum_{i=1}^{n-k-1} \left( c_{i-1}^{(k)} x_{k-1} + \beta_i \right) p_i(x)$$

$$- c_0^{(k)} (x_{k-1} + \beta_0) + c_1^{(k)} \gamma_1 + a_{k-1}.$$  

(14)
Higham then extracts the new coefficients $c_i^{(k-1)}$ from Equation (14) as:

$$
c_i^{(k-1)} = \begin{cases} 
c_i^{(k)} \alpha_{i-1}, & i = n - k + 1, \\
c_i^{(k)} \alpha_{i-1} - c_i^{(k)} (x_{k-1} + \beta_i), & i = n - k, \\
c_i^{(k)} \alpha_{i-1} - c_i^{(k)} (x_{k-1} + \beta_i) + c_{i+1}^{(k)} \gamma_{i+1}, & 1 \leq i < n - k, \\
c_0^{(k)} (x_{k-1} + \beta_0) + c_1^{(k)} \gamma_1 + a_{k-1}, & i = 0 \end{cases} 
$$

(15)

In both algorithms, the interpolating polynomial is constructed by first computing the divided differences

$$a_i = f[x_0, \ldots, x_i], \quad i = 0 \ldots n
$$

(16) and, starting with $q_n(x) = a_n$, and hence $c_0^{(n)} = a_n$ or $b_0^{(n)} = a_n$, successively updating the coefficients per Equation (14) or Equation (11) respectively.

Alternatively, we could use the same approach to compute the coefficients of the Newton polynomials themselves

$$\pi_k(x) = \sum_{i=0}^{k} \eta_i^{(k)} p_i(x).
$$

Expanding the recurrence relation in Equation (7) analogously to Equation (14), we get

$$\pi_{k+1}(x) = \eta_i^{(k)} \alpha_k p_{k+1}(x) + \left( \eta_i^{(k)} \alpha_{k-1} - \eta_i^{(k)} (x_k + \beta_k) \right) p_k(x)
$$

$$+ \sum_{i=1}^{k-1} \left( \eta_i^{(k)} \alpha_{i-1} - \eta_i^{(k)} (x_k + \beta_i) + \eta_i^{(k)} \gamma_{i+1} \right) p_i(x)
$$

$$- \eta_0^{(k)} (x_k + \beta_0) + \eta_1^{(k)} \gamma_1.
$$

(17)

We initialize with $\eta_0^{(0)} = 1$ and use

$$
\eta_i^{(k+1)} = \begin{cases} 
\eta_i^{(k)} \alpha_{i-1}, & i = k + 1, \\
\eta_i^{(k)} \alpha_{i-1} - \eta_i^{(k)} (x_k + \beta_i), & i = k, \\
\eta_i^{(k)} \alpha_{i-1} - \eta_i^{(k)} (x_k + \beta_i) + \eta_i^{(k)} \gamma_{i+1}, & 1 \leq i < k, \\
-\eta_0^{(k)} (x_k + \beta_0) + \eta_1^{(k)} \gamma_1, & i = 0,
\end{cases}
$$

(18) to compute the coefficients for $\pi_k(x)$, $k = 1 \ldots n$. Alongside this computation, we can also compute the coefficients of a sequence of polynomials $g_k(x)$ of increasing degree $k$

$$g_k(x) = \sum_{i=0}^{k} c_i^{(k)} p_i(x)
$$

initializing with $c_0^{(0)} = a_0$, where the $a_i$ are still the Newton coefficients computed and used above. The subsequent coefficients $c_i^{(k)}$, $k = 1 \ldots n$ are computed using

$$c_i^{(k)} = \begin{cases} 
c_i^{(k)} a_k, & i = k, \\
c_i^{(k-1)} + \eta_i^{(k)} a_k, & 0 \leq i < k.
\end{cases}
$$

(19)
This incremental construction of the coefficients, which is equivalent to effecting the summation of the weighted Newton polynomials and is referred to by Björck and Pereyra as the “progressive algorithm”, can be used to efficiently update an interpolation. If the coefficients \( \eta^{(n)}_{i} \) and \( c^{(n)}_{i} \) are stored and a new node \( x_{n+1} \) and function value \( f_{n+1} \) are added to the data, a new coefficient \( a_{n+1} \) can be computed per Equation (16), the coefficients \( \eta^{(n+1)}_{i} \) computed per Equation (18) and, finally, the \( c^{(n)}_{i} \) updated per Equation (19), resulting in the coefficients \( c^{(n+1)}_{i} \) for the updated interpolation polynomial \( g_{n+1}(x) \).

**Algorithm 1 Incremental construction of \( g_{n}(x) \)**

1: \( c^{(0)}_{0} \leftarrow f_{0} \) \hspace{1cm} \text{(init} \( c^{(0)} \))
2: \( \eta^{(1)}_{0} \leftarrow -x_{0} - \beta_{0}, \eta^{(1)}_{1} \leftarrow \alpha_{0} \) \hspace{1cm} \text{(init} \( \eta^{(1)} \))
3: \( \text{for} \ k = 1 \ldots n \ \text{do} \)
4: \( v_{0} \leftarrow 0, v_{1} \leftarrow x_{k} \) \hspace{1cm} \text{(init} \( v \))
5: \( \text{for} \ i = 2 \ldots k \ \text{do} \)
6: \( v_{i} \leftarrow ((x_{k} + \beta_{i-1})v_{i-1} - \gamma_{i-1}v_{i-2})/\alpha_{i-1} \) \hspace{1cm} \text{(compute the} \( v_{i} \))
7: \( \text{end for} \)
8: \( g_{k} \leftarrow v(0 : k - 1)^{T}c^{(k-1)} \) \hspace{1cm} \text{(compute} \( g_{k-1}(x_{k}) \))
9: \( \pi_{k} \leftarrow v^{T}\eta^{(k)} \) \hspace{1cm} \text{(compute} \( \pi_{k}(x_{k}) \))
10: \( a_{k} \leftarrow (f_{k} - g_{k})/\pi_{k} \) \hspace{1cm} \text{(compute} \( a_{k}, \text{Equation (21)} \))
11: \( c^{(k)} \leftarrow [c^{(k-1)}; 0] + a_{k}\eta^{(k)} \) \hspace{1cm} \text{(compute the new} \( c^{(k)} \), \text{Equation (14)} \))
12: \( \eta^{(k+1)} \leftarrow [0; \mathbf{2}(0 : k); \ast \eta^{(k)}] - [(x_{k} + \beta(0 : k)); \ast \eta^{(k)}; 0] + \gamma_{(1 : k); \ast \eta^{(k)}(1 : k); 0; 0} \) \hspace{1cm} \text{(compute the new} \( \eta^{(k+1)} \), \text{Equation (18)} \))
13: \( \text{end for} \)

We can re-write the recursion for the coefficients \( \eta_{i}^{(k)} \) of the Newton polynomials in matrix-vector notation as

\[
\begin{pmatrix}
T^{(k+1)} - L_{0}x_{k}
\end{pmatrix}
\eta^{(k)} = \eta^{(k+1)}
\]

where \( T^{(k+1)} \) is the \((k + 2) \times (k + 1)\) tri-diagonal matrix

\[
T^{(k+1)} =
\begin{pmatrix}
-\beta_{0} & \gamma_{1} & \gamma_{2} & \cdots & \gamma_{k} \\
\alpha_{0} & -\beta_{1} & \gamma_{2} & \cdots & \gamma_{k} \\
\alpha_{k-2} & -\beta_{k-1} & \gamma_{k} & \cdots \\
\alpha_{k-1} & -\beta_{k} & \gamma_{k} & \cdots \\
\alpha_{k} & \gamma_{k} & \cdots
\end{pmatrix}
\]

and \( L_{0}x_{k} \) is a \((k + 2) \times (k + 1)\) matrix with \( x_{k} \) in the diagonal and zeros elsewhere

\[
L_{0}x_{k} =
\begin{pmatrix}
x_{k} & x_{k} & \cdots & x_{k} \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]
The vectors \( \eta^{(k)} = (\eta^{(k)}_0, \eta^{(k)}_1, \ldots, \eta^{(k)}_k)^T \) and \( \eta^{(k+1)} = (\eta^{(k+1)}_0, \eta^{(k+1)}_1, \ldots, \eta^{(k+1)}_{k+1})^T \) contain the coefficients of the \( k \)th and \((k+1)\)st Newton polynomial respectively.

Given the vector of coefficients \( c^{(n)} = (c^{(n)}_0, c^{(n)}_1, \ldots, c^{(n)}_n)^T \) of an interpolation polynomial \( g_n(x) \) of degree \( n \) and the vector of coefficients \( \eta^{(n+1)} \) of the \((n+1)\)st Newton polynomial over the \( n+1 \) nodes, we can update the interpolation for a new node \( x_{n+1} \) and function value \( f_{n+1} \) as follows: Instead of computing the new Newton interpolation coefficient \( a_{n+1} \) using the divided differences as in Equation (16), we choose \( a_{n+1} \) such that the new interpolation constraint

\[
g_{n+1}(x_{n+1}) = g_n(x_{n+1}) + a_{n+1} \pi_{n+1}(x_{n+1}) = f_{n+1}
\]
is satisfied, resulting in

\[
a_{n+1} = \frac{f_{n+1} - g_n(x_{n+1})}{\pi_{n+1}(x_{n+1})}
\]

which can be computed by evaluating \( g_n(x_{n+1}) \) and \( \pi_{n+1}(x_{n+1}) \). Note that since \( \pi_{n+1}(x_i) = 0 \) for \( i = 0 \ldots n \), the addition of any multiple of \( \pi_{n+1}(x) \) to \( g_n(x) \) does not affect the interpolation at the other nodes at all. This expression for \( a_{n+1} \) is used instead of the divided difference since we have not explicitly stored the previous \( a_i, i = 0 \ldots n \), which are needed for the recursive computation of the latter.

We then update the coefficients of the interpolating polynomial using

\[
c^{(n+1)} = \begin{pmatrix} c^{(n)} \\ 0 \end{pmatrix} + a_{n+1} \eta^{(n+1)}
\]

and then the coefficients of the Newton polynomial using

\[
\eta^{(n+2)} = \left( T^{(n+2)} - L_n x_{n+1} \right) \eta^{(n+1)}
\]
such that it is ready for further updates. Starting with \( \eta^{(0)}_0 = 1 \) and \( n = 0 \), this update can be used to construct \( g_n(x) \) by adding each \( x_i \) and \( f_i \), \( i = 0 \ldots n \), successively.

The complete algorithm doing just that is shown in Algorithm [1]. The addition of each \( n \)th node requires \( O(n) \) operations, resulting in a total of \( O(n^2) \) operations for the construction of an \( n \)-node interpolation.

This is essentially the progressive algorithm of Björck and Pereyra, yet instead of storing the Newton coefficients \( a_i \), we store the coefficients \( \eta_i^{(n+1)} \) of the last Newton polynomial. This new representation offers no obvious advantage for the update, other than that it can be easily reversed.

Given an interpolation over a set of \( n+1 \) nodes \( x_i \) and function values \( f_i \), \( i = 0 \ldots n \) defined by the coefficients \( c_i^{(n)} \) and given the coefficients \( \eta_i^{(n+1)} \) of the \((n+1)\)st Newton polynomial over the same nodes, we will downdate the interpolation by removing the function value \( f_j \) at the node \( x_j \). The resulting polynomial of degree \( n-1 \) will still interpolate the remaining \( n \) nodes.
We start by removing the root $x_j$ from the $(n+1)$st Newton polynomial by solving

$$
(\mathbf{T}^{(n+1)} - \mathbf{L}_j x_j) \eta^{(n)} = \eta^{(n+1)}
$$

for the vector of coefficients $\eta^{(n)}$. Since $x_j$ is a root of $\pi_{n+1}(x)$, the system is over-determined yet has a unique solution. We can therefore remove the first row of $(\mathbf{T}^{(n+1)} - \mathbf{L}_j x_j)$ and the first entry of $\eta^{(n+1)}$, resulting in the upper-tridiagonal system of linear equations

$$
\begin{pmatrix}
\alpha_0 & - (x_j + \beta_1) & \gamma_2 & \ddots & \\
& \alpha_{n-2} & - (x_j + \beta_{n-1}) & \gamma_{n-1} & \\
& & \alpha_{n-1} & - (x_j + \beta_n) \end{pmatrix}
\begin{pmatrix}
\eta_0^{(n)} \\
\eta_1^{(n)} \\
\vdots \\
\eta_n^{(n)}
\end{pmatrix}
= 
\begin{pmatrix}
\eta_0^{(n+1)} \\
\eta_1^{(n+1)} \\
\vdots \\
\eta_n^{(n+1)}
\end{pmatrix}
$$

which can be conveniently solved in $O(n)$ using back-substitution.

Once we have our downdated $\eta^{(n)}$, and thus the downdated Newton polynomial $\pi_n(x)$, we can downdate the coefficients of $g_n(x)$ by computing

$$
g_{n-1}(x) = g_n(x) - a_j^* \pi_n(x)
$$

where the Newton coefficient $a_j^*$ would need to be re-computed from the divided difference over all nodes except $x_j$. We can avoid this computation by noting that $g_{n-1}(x)$ has to be of degree $n - 1$ and therefore the highest coefficient of $g_n(x)$, $c_n^{(n)}$, must disappear. This is the case when

$$
c_n^{(n-1)} = c_n^{(n)} - a_j^* \eta_n^{(n)} = 0
$$

and therefore

$$
a_j^* = \frac{c_n^{(n)}}{\eta_n^{(n)}}.
$$

Using this $a_j^*$, we can compute the coefficients of $g_{n-1}(x)$ as

$$
c_i^{(n-1)} = c_i^{(n)} - \frac{c_n^{(n)}}{\eta_n^{(n)}} \eta_i^{(n)}, \quad i = 1 \ldots n - 1.
$$

The whole process is shown in Algorithm\textsuperscript{2} The downdate of an $n$-node interpolation requires $O(n)$ operations.

\textsuperscript{2}Note that the $n \times (n+1)$ matrix $(\mathbf{T}^{(n+1)} - \mathbf{L}_j x_j)^\top$ has rank $n$ and the null space $\mathbf{p}(x_j) = (p_0(x_j), p_1(x_j), \ldots, p_n(x_j))$ since for $\mathbf{v} = (\mathbf{T}^{(n+1)} - \mathbf{L}_j x_j)^\top \mathbf{p}(x_j)$, $v_i = \alpha_i p_{i+1}(x_j) - (x_j + \beta_i) p_i(x_j) + \gamma_i p_{i-1}(x_j) = 0$ by the definition in Equation\textsuperscript{2} and the right-hand side $\eta^{(n+1)}$ is consistent.
Algorithm 2 Remove a function value $f_j$ at the node $x_j$ from the interpolation given by the coefficients $c^{(n)}$

1: $\eta_n^{(n)} \leftarrow \frac{\eta_{n+1}^{(n+1)}}{\alpha_n}$ (compute $\eta_n^{(n)}$ from $\eta^{(n+1)}$ using back-substitution)
2: $\eta_{n-1}^{(n)} \leftarrow \left(\eta_n^{(n+1)} + (x_j + \beta_n)\eta_{n+1}^{(n)}\right)/\alpha_{n-1}$
3: for $i = n-2 \ldots 0$ do
4: $\eta_i^{(n)} \leftarrow \left(\eta_{i+1}^{(n+1)} + (x_j + \beta_{i+1})\eta_{i+1}^{(n)} - \gamma_i + 2\eta_{i+2}^{(n)}\right)/\alpha_i$
5: end for
6: $a_j \leftarrow c_n^{(n)}/\eta_n^{(n)}$ (compute the coefficient $a_j$)
7: $c^{(n-1)} \leftarrow c^{(n)} - a_j\eta^{(n)}$ (compute the new coefficients $c^{(n-1)}$)

3 A New Algorithm for the Construction of Interpolations

Returning to the representation in Equation (5), we can try to solve the Vandermonde-like system of linear equations directly. The matrix has some special characteristics which we can exploit to achieve better performance and stability than when using Gaussian elimination or even the algorithms of Björck and Pereyra, Higham or the one described in the previous section.

We start by de-composing the $(n+1) \times (n+1)$ Vandermonde-like matrix $P^{(n)}$ as follows:

$$P^{(n)} = \begin{pmatrix} P^{(n-1)} & P^{(n)} \\ \mathbf{q}^T & p_n(x_n) \end{pmatrix}.$$ 

The sub-matrix $P^{(n-1)}$ is a Vandermonde-like matrix analogous to $P^{(n)}$. The column $p^{(n)}$ contains the $n$th polynomial evaluated at the nodes $x_i, i = 0 \ldots n-1$

$$p^{(n)} = \begin{pmatrix} p_n(x_0) \\ p_n(x_1) \\ \vdots \\ p_n(x_{n-1}) \end{pmatrix}$$

and the vector $\mathbf{q}^T$ contains the values of the first $0 \ldots n-1$ polynomials at the node $x_n$

$$\mathbf{q}^T = (p_0(x_n), p_1(x_n), \ldots, p_{n-1}(x_n)) .$$
Inserting this into the product in Equation (23), we obtain
\[
\begin{pmatrix}
\mathbf{P}^{(n-1)} & \mathbf{p}^{(n)} \\
\mathbf{q}^T & p_n(x_n)
\end{pmatrix}
\begin{pmatrix}
\mathbf{c}^{(n-1)} \\
\mathbf{c}_n
\end{pmatrix}
= 
\begin{pmatrix}
\mathbf{f}^{(n-1)}
\end{pmatrix}
\]
which, when effected, results in the pair of equations
\[
\begin{align}
\mathbf{P}^{(n-1)}\mathbf{c}^{(n-1)} + \mathbf{p}^{(n)}\mathbf{c}_n &= \mathbf{f}^{(n-1)} \\
\mathbf{q}^T\mathbf{c}^{(n-1)} + p_n(x_n)\mathbf{c}_n &= f_n,
\end{align}
\]  
(23)
where the vectors \(\mathbf{c}^{(n-1)} = (c_0, c_1, \ldots, c_{n-1})^T\) and \(\mathbf{f}^{(n-1)} = (f_0, f_1, \ldots, f_{n-1})^T\) contain the first \(n\) coefficients or function values respectively.

Before trying to solve Equation (23), we note that the columns of the matrix \(\mathbf{P}^{(n-1)}\) contain the first \(0\ldots n-1\) polynomials evaluated at the same \(n\) nodes each. Similarly, \(\mathbf{q}^T\) contains the same polynomials evaluated at the node \(x_n\). Since the polynomials in the columns are of degree \(< n\) and they are evaluated at \(n\) points, \(\mathbf{P}^{(n-1)}\) actually contains enough data to extrapolate the values of these polynomials at \(x_n\). Using Lagrange interpolation we can write
\[
q_i = \sum_{j=0}^{n-1} \ell_j^{(n)}(x_n)P_j^{(n-1)}
\]  
(24)
where the
\[
\ell_j^{(n)}(x) = \prod_{j=0, j\neq i}^{n-1} \frac{x - x_j}{x_i - x_j}
\]  
(25)
are the Lagrange polynomials over the first \(n\) nodes \(x_i, i = 0\ldots n-1\). We can write Equation (24) as
\[
\mathbf{q}^T = \ell^{(n)}\mathbf{P}^{(n-1)}
\]  
(26)
where the entries of the \(1 \times n\) vector \(\ell^{(n)}\) are
\[
\ell_i^{(n)} = \ell_i^{(n)}(x_n).
\]

The entries of \(\ell^{(n)}\) can be computed recursively. Using the definition in Equation (25), we define
\[
w_n = \prod_{j=0}^{n-1} (x_n - x_j),
\]
and re-write \(\ell_i^{(n)}\) as
\[
\ell_i^{(n)} = \frac{w_n}{x_n - x_i} \left[ \prod_{j=0, j\neq i}^{n-1} (x_i - x_j) \right]^{-1}.
\]  
(27)
Using the previous $\ell_i^{(n-1)}$ and $w_{n-1}$, we can re-write Equation (27) as
\[
\ell_i^{(n-1)} = \frac{\ell_i^{(n-1)}}{w_{n-1}} (x_{n-1} - x_i)
\]
which, re-inserted into Equation (27), gives
\[
\ell_i^{(n)} = \frac{w_n}{x_n - x_i} \frac{\ell_i^{(n-1)}}{w_{n-1}} \frac{x_{n-1} - x_i}{x_i - x_{n-1}} = \frac{w_n}{x_n - x_i} w_{n-1}
\]
for all $i < n - 1$. We then compute the last entry $i = n - 1$ using
\[
\ell_i^{(n)} = \frac{w_n}{w_{n-1}(x_n - x_{n-1})}.
\]
Therefore, starting with $\ell_0^{(1)} = 1$, we can construct all the $\ell^{(k)}$, $k = 2 \ldots n$ successively. Since, given $\ell^{(k)}$, $k = 1 \ldots n$ the construction of each additional $\ell^{(n+1)}$ requires $O(n)$ operations, the construction of all the $\ell^{(k)}$, $k = 1 \ldots n$ requires a total of $O(n^2)$ operations.

Returning to the Vandermonde-like matrix, inserting Equation (26) into Equation (23), we obtain
\[
\left[ P^{(n-1)} c^{(n-1)} + p^{(n)} c_n \right] = f^{(n-1)}
\]
\[
\ell^{(n)} p^{(n)} c_n + p_n(x_n) c_n = f_n.
\]
(30)

Multiplying the first line in Equation (30) with $\ell^{(n)}$ from the left and subtracting the bottom equation from the top one we obtain
\[
\ell^{(n)} p^{(n)} c_n - p_n(x_n) c_n = \ell^{(n)} f^{(n-1)} - f_n
\]
from which we can finally isolate the coefficient $c_n$:
\[
c_n = \frac{\ell^{(n)} f^{(n-1)} - f_n}{\ell^{(n)} p^{(n)} - p_n(x_n)}.
\]
(31)

Having computed $c_n$, we can now re-insert it into the first Equation in Equation (30), resulting in the new system
\[
P^{(n-1)} c^{(n-1)} = f^{(n-1)} - p^{(n)} c_n
\]
(32)
in the remaining coefficients $c^{(n-1)}$.

Applying this computation recursively to $P^{(n)}$, $P^{(n-1)}$, \ldots, $P^{(1)}$ we can compute the interpolation coefficients $c^{(n)}$. The final coefficient $c_0$ can be computed as
\[
c_0 = f_0 / P_{1,1}^{(0)}.
\]

The complete algorithm is shown in Algorithm 3. Since the construction of the $\ell^{(k)}$, $k = 1 \ldots n$ requires $O(n^2)$ operations (for each $\ell^{(k)}$, $O(k)$ operations are
Algorithm 3 Direct construction of \( g_n(x) \)

1: \( p^{(0)}(0) \leftarrow 0, \ p^{(1)} \leftarrow x \) (init \( P^{(n)} \))
2: for \( i = 2 \ldots n \) do
3: \( p^{(i)} \leftarrow \left( (x + \beta_{i-1}) \cdot p^{(i-1)} - \gamma_{i-1}p^{(i-2)} \right) / \alpha_{i-1} \) (fill \( P^{(n)} \))
4: end for
5: \( \ell_0^{(1)} \leftarrow 1, \ w_1 \leftarrow x_1 - x_0 \) (init \( \ell_0^{(1)} \) and \( w_1 \))
6: for \( i = 2 \ldots n \) do
7: \( w_i \leftarrow 1 \) (construct \( w_i \))
8: for \( j = 0 \ldots i - 1 \) do
9: \( w_i \leftarrow w_i(x_i - x_j) \)
10: end for
11: for \( j = 0 \ldots i - 2 \) do
12: \( \ell_j^{(i)} \leftarrow -\frac{w_i}{w_{i-1}(x_i - x_{i-1})} \) (compute the \( \ell_j^{(i)} \), Equation (28))
13: end for
14: \( \ell_{i-1}^{(i)} \leftarrow -\frac{w_i}{w_{i-1}(x_i - x_{i-1})} \) (compute \( \ell_{i-1}^{(i)} \), Equation (29))
15: end for
16: for \( i = n \ldots 1 \) do
17: \( c_i \leftarrow \frac{\ell_{i-1}^{(i)}(0:1) - f_i}{p^{(i)}(0:1) - p_i(x_i)} \) (compute coefficient \( c_i \), Equation (31))
18: \( f \leftarrow f - c_i p^{(i)} \) (update the right-hand side \( f \))
19: end for
20: \( c_0 \leftarrow f_0 / p^{(0)}(0) \) (compute the final \( c_0 \))

required to compute \( w_k \) and \( O(k) \) are required to compute the new entries) and the evaluation of Equation (31) requires \( O(k) \) operations for each \( c_k, k = n \ldots 0 \), the total cost of the algorithm is in \( O(n^2) \) operations.

Note that, as opposed to the algorithm presented in Section 2, this algorithm can not be extended to update or downdate an interpolation. It has an advantage, however, when multiple right-hand sides, i.e., interpolations over the same set of nodes, are to be computed. In such a case, the vectors \( \ell^{(k)}, k = 1 \ldots n \) need to be computed only once (Lines 6 to 17 of Algorithm 3). For any new vector \( f \), only the Lines 16 to 20 need to be re-evaluated.

4 Results

To assess the stability of the two new interpolation routines described herein, we will follow the methodology used by Higham (1988). Higham defines a set of interpolations consisting of all combinations of the nodes

- A1: \( x_i = -\cos(i\pi/n) \), (extrema of \( T_n(x) \))
- A2: \( x_i = -\cos \left( \left( i + \frac{1}{2} \right)\pi/(n + 1) \right) \), (zeros of \( T_{n+1}(x) \))
- A3: \( x_i = -1 + 2i/n \), (equidistant on \([-1, 1])
- A4: \( x_i = i/n \), (equidistant on \([0, 1])

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with the right-hand sides

\[
\begin{align*}
F_1: & \quad f_i = (-1)^i, \\
F_2: & \quad f = (1, 0, \ldots, 0)^T, \\
F_3: & \quad f_i = 1/(1 + 25x_i^2).
\end{align*}
\]

for \( i = 0 \ldots n \).

To avoid instabilities due to unfortunate orderings of the nodes \( x_i \), the nodes and corresponding function values were re-ordered according to the same permutation that would be produced by Gaussian elimination with partial pivoting applied to the Vandermonde-like matrix, as described in (Higham 1990). This ordering is optimal for the Björck-Pereyra and Higham algorithms and produces good results for the two new algorithms described herein.

For each combination of nodes and right-hand sides, we compute, following Higham, the coefficients \( c \) for the Chebyshev base (see Equation (4)) for \( n = 5, 10, 20 \) and \( 30 \) and compute the quantities

\[
\text{ERR} = \frac{\|c - c^*\|_2}{u\|c^*\|_2}, \quad \text{RES} = \frac{\|f - Pc\|_2}{u\|c^*\|_2},
\]

where \( c^* \) is the exact solution and \( u \) is the unit roundoff as defined by Golub and Van Loan (1996, Section 2.4.2). Note that for the special case of A1 or A2 using the Chebyshev base, the coefficients can also be computed efficiently and reliably using the Fast Fourier Transform (Battles and Trefethen 2004).

Results were computed using Gaussian elimination (GE), Higham’s extension of the algorithm of Björck and Pereyra (BP/H), the incremental Algorithm 1 (INCR) and the direct Algorithm 3 (DIRECT). The exact values were computed in Maple (Char, Geddes and Gonnet 1983) with 50 decimal digits of precision using the \texttt{interp} function therein.

Results were also computed for the interpolation downdate (DEL) described in Algorithm 2. Starting from \( c^* \) and \( \eta^* \), the exact coefficients for the interpolation \( g_n(x) \) and the Newton polynomial \( \pi_{n+1}(x) \) respectively, we compute the coefficients \( c^{(n-1)} \) and \( \eta^{(n)} \) for \( g_{n-1}(x) \) and \( \pi_n(x) \), resulting from the removal of the rightmost function value \( f_k \) at \( x_k \), \( k = \arg \max_i x_i \). The exact coefficients \( \hat{c}^* \) after deletion were computed and used to compute the quantities ERR and RES.

The results are shown in Tables 1 to 12. For each \( n \), the largest values for ERR and RES are highlighted. For the problem sets over the nodes A1 and A2 (Tables 1 to 6), the condition of the Vandermonde-like matrix is always \( \leq 2 \) (Gautschi 1983), resulting in very small errors for Gaussian elimination. The Björck-Pereyra/Higham algorithm generates slightly larger residuals than both the incremental and direct algorithms for both sets of nodes. The values for

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3 All results were computed using IEEE 754 double-precision arithmetic and hence \( u \approx 2.2 \times 10^{-16} \).

4 For the tests in this section, Matlab’s backslash-operator, which uses partial pivoting, was used. In cases where the matrix is rank-deficient, a minimum-norm solution is returned.

5 Algorithm 1 in (Higham 1988) was implemented in Matlab.
| n  | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|----|---------|---------|---------|---------|---------|---------|
| 5  | 0.00    | 0.50    | 2.00    | 3.94    | 3.20    | 4.30    |
| 10 | 3.20    | 5.32    | 1.20e1  | 2.79e1  | 7.76    | 2.21e1  |
| 20 | 7.28    | 2.16e1  | 1.61e2  | 5.27e2  | 8.92    | 3.64e1  |
| 30 | 2.61    | 1.03e1  | 6.72e2  | 2.65e3  | 2.08e1  | 1.11e2  |

Table 1: Results for problem A1/F1.

| n  | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|----|---------|---------|---------|---------|---------|---------|
| 5  | 0.97    | 2.40    | 1.82    | 3.19    | 0.93    | 2.07    |
| 10 | 2.22    | 4.19    | 1.37e1  | 3.77e1  | 1.94    | 3.12e1  |
| 20 | 2.11e1  | 8.96    | 9.30e1  | 3.47e2  | 2.24e1  | 4.80e1  |
| 30 | 3.63e1  | 1.36e1  | 1.27e2  | 4.84e2  | 5.55e1  | 2.27e2  |

Table 2: Results for problem A1/F2.

| n  | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|----|---------|---------|---------|---------|---------|---------|
| 5  | 1.26    | 2.67    | 1.16    | 2.27    | 1.26    | 2.70    |
| 10 | 2.12    | 5.30    | 7.27    | 1.66e1  | 1.83    | 3.63    |
| 20 | 1.13    | 4.81    | 8.28    | 2.68e1  | 1.78    | 6.74    |
| 30 | 1.99    | 7.35    | 6.33    | 2.53e1  | 1.14    | 5.86    |

Table 3: Results for problem A1/F3.

| n  | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|----|---------|---------|---------|---------|---------|---------|
| 5  | 3.55    | 0.70    | 5.07    | 7.31    | 3.79    | 1.58    |
| 10 | 1.19e1  | 3.65    | 2.03e1  | 3.31e1  | 8.34    | 1.08e1  |
| 20 | 1.66e1  | 1.23e1  | 4.64e1  | 1.63e2  | 5.67e1  | 1.41e2  |
| 30 | 6.48e1  | 2.28e1  | 1.24e2  | 4.36e2  | 4.54e1  | 2.29e2  |

Table 4: Results for problem A2/F1.

| n  | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|----|---------|---------|---------|---------|---------|---------|
| 5  | 2.57    | 3.29    | 2.88    | 4.52    | 2.35    | 0.97    |
| 10 | 5.27    | 3.89    | 1.67e1  | 4.59e1  | 4.94    | 5.23    |
| 20 | 8.40    | 8.80    | 5.09e1  | 1.63e2  | 3.89e1  | 9.85e1  |
| 30 | 3.39e1  | 1.95e1  | 1.17e2  | 4.31e2  | 3.00e1  | 1.99e1  |

Table 5: Results for problem A2/F2.
Table 6: Results for problem A2/F3.

| n  | GE   | BP/H | INCR  | DIRECT | DEL  |
|----|------|------|-------|--------|------|
| 5  | 1.44 | 0.11 | 1.40  | 1.71   | 1.17 | 1.45 | 1.12 | 1.67 | 0.00 | 2.76 |
| 10 | 2.73 | 3.22 | 6.06  | 1.14e1 | 2.86 | 3.75 | 3.64 | 6.34 | 6.56 | 7.47 |
| 20 | 1.52 | 5.59 | 7.34  | 2.42e1 | 2.06 | 6.34 | 3.92 | 1.34e1| 1.27e1| 1.81e1|
| 30 | 2.81 | 1.19e1| 7.24  | 3.08e1 | 1.65 | 6.07 | 2.79 | 9.46 | 1.41e1| 2.94e1|

Table 7: Results for problem A3/F1.

| n  | GE   | BP/H | INCR  | DIRECT | DEL  |
|----|------|------|-------|--------|------|
| 5  | 1.41 | 0.58 | 6.12  | 1.01e1 | 2.04 | 0.97 | 1.48 | 0.73 | 0.60 | 0.78 |
| 10 | 2.99 | 2.50 | 1.18e1| 2.00e1 | 2.16 | 1.52 | 2.62 | 3.11 | 0.50 | 1.42 |
| 20 | 2.26e3| 4.85 | 2.01e1| 7.19e1 | 3.58e1| 1.02e1| 6.23e1| 3.97 | 0.55 | 2.77 |
| 30 | 1.39e6| 9.51 | 3.90e1| 1.34e2 | 5.42e4| 2.60e1| 3.07e2| 7.33 | 0.55 | 2.20 |

Table 8: Results for problem A3/F2.

| n  | GE   | BP/H | INCR  | DIRECT | DEL  |
|----|------|------|-------|--------|------|
| 5  | 1.02 | 1.98 | 2.19  | 3.53   | 0.69 | 0.80 | 0.73 | 1.40 | 0.40 | 0.86 |
| 10 | 4.91 | 7.58 | 3.83  | 1.03e1 | 1.05 | 1.82 | 1.61 | 2.46 | 0.47 | 0.85 |
| 20 | 6.65e3| 1.07e1| 8.32  | 2.54e1 | 5.73e2| 2.94 | 1.29 | 4.25 | 0.63 | 2.26 |
| 30 | 4.02e5| 9.93 | 3.23e1| 1.09e2 | 1.35e5| 1.25e1| 4.98 | 7.32 | 0.65 | 3.48 |

Table 9: Results for problem A3/F3.

| n  | GE   | BP/H | INCR  | DIRECT | DEL  |
|----|------|------|-------|--------|------|
| 5  | 1.48 | 0.86 | 1.87  | 1.98   | 1.78 | 1.89 | 1.38 | 0.36 | 0.00 | 2.23 |
| 10 | 2.31 | 3.28 | 8.96  | 2.76e1 | 2.00 | 2.97 | 3.27 | 3.80 | 0.45 | 1.12 |
| 20 | 2.30e3| 5.53 | 3.81e1| 7.38e1 | 1.12e2| 1.24e1| 3.13e1| 5.11 | 0.47 | 3.39 |
| 30 | 1.39e6| 1.10e1| 2.28e2| 1.88e2 | 5.29e4| 2.69e1| 2.41e2| 4.78 | 0.49 | 2.29 |

Table 10: Results for problem A4/F1.
Table 11: Results for problem A4/F2.

| n   | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|-----|---------|---------|---------|---------|---------|
| 5   | 1.82e2  | 0.48    | 0.55    | 0.65    | 3.64e2  |
| 10  | 1.35e5  | 0.63    | 0.40    | 0.85    | 3.77e6  |
| 20  | 2.81e15 | 3.64    | 0.71    | 1.20    | 6.57e14 |
| 30  | 4.50e15 | 0.00    | 0.40    | 1.27    | 1.37e11 |

Table 12: Results for problem A4/F3.

| n   | ERR RES | ERR RES | ERR RES | ERR RES | ERR RES |
|-----|---------|---------|---------|---------|---------|
| 5   | 2.41e2  | 0.23    | 3.96e1  | 0.72    | 8.24e2  |
| 10  | 3.79e5  | 1.38    | 1.55e3  | 0.88    | 4.94e6  |
| 20  | 2.81e15 | 3.74    | 4.84e6  | 1.21    | 9.48e14 |
| 30  | 4.50e15 | 0.00    | 1.02e11 | 1.47    | 1.18e11 |

ERR, however, are usually within the same order of magnitude for the three algorithms.

For the nodes A3 (Tables 7 to 9), the condition number of the Vandermonde-like matrix is 5.11e6 for n = 30, resulting in the errors of approximately that magnitude when Gaussian elimination is used. In general, both the Björck-Pereyra/Higham and the direct algorithm generate smaller errors and residues than Gaussian elimination. The errors for the incremental algorithm are due to cancellation while evaluating $g_n(x_{n+1})$ for Equation (20) since the intermediate coefficients $c^{(k)}$ are several orders of magnitude larger than the result.

Finally, the condition number of the Vandermonde-like matrix for the nodes A4 is 4.26e16 for n = 30, making it numerically singular and thus resulting in the complete failure of Gaussian elimination. Note that since in such cases Matlab’s backslash-operator computes the minimum norm solution, the resulting residual error RES is quite small. For the first two right-hand sides F1 and F2, the Björck-Pereyra/Higham algorithm performs significantly better than the two new algorithms, since the magnitude of the intermediate coefficients does not vary significantly. For the right-hand side F3, however, the errors are larger, caused by truncation in computing the Newton coefficients $a_i$. The incremental algorithm fails completely for all right-hand sides since the intermediate and final coefficients $c^{(k)}$, $k \leq n$, are more than ten orders of magnitude larger than the function value, and the numerical condition of $g_n(x_{n+1})$ in Equation (20).

6 In (Higham 1988), Higham shows that the coefficients can be written as the weighted sum of any of the intermediate coefficients $c^{(n)}_i = \sum_j \mu_j c^{(k)}_j$, where the $\mu_j$ depend only on the nodes and the coefficients of the three-term recurrence relation. If the $\mu_j$ are $O(1)$ and the intermediate $c^{(k)}_j$ are much larger than the $c^{(n)}_i$, then cancellation is likely to occur in the above sum.

7 $\|c^*\| = 2.23e13$ for F3 and n = 30.
exceeds machine precision, resulting in numerical overflow. These relatively large coefficients also cause problems for the direct algorithm when evaluating the right-hand side of Equation (32), where the original function values are clobbered by the subtraction of the much larger $p^{(n)}c_n$.

The errors and residuals for the downdate algorithm are shown in the rightmost columns of Tables 1 to 12. In general the errors of the downdate are relatively small for all test cases. The larger residues, *e.g.* for A2/F2, are due to cancellation in the final subtraction in Algorithm 2 Line 7.

5 Conclusions

We have presented here two new algorithms for the construction of polynomial interpolations. The first algorithm (Algorithm 1) offers no substantial improvement over that of Björck-Pereyra/Higham except that it can be easily downdated. The second algorithm, which does not allow for updates nor downdates, is slightly more stable than the other algorithms tested and is more efficient when multiple right-hand sides need to be computed over the same set of nodes.

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