Quantum phase transition of the static extremal black hole coupled to a Stückelberg scalar

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Abstract. We show that a static extremal black hole in 4D gravity, non-linearly coupled to a massive Stückelberg scalar, can exhibit a quantum phase transition due to electric charge variations. The critical point exists only if the spacetime possesses cosmological constant (positive or negative) and the mass of the scalar is non-vanishing. In that case, around the critical point, two solutions coexist: Reissner-Nordström (A)dS black hole and the hairy (A)dS black hole. A near-critical expansion reveals that the entropy and its derivative are continuous functions at the critical point. Obtained results are analytical and based on the entropy function formalism.

1. Introduction

An equilibrium state of a thermodynamic system corresponds to a minimum of the internal energy in the energy representation of states, or a maximum of the entropy in the entropy representation of states. The latter one is suitable for studying equilibria of extremal black holes because they have zero temperature and are not described by conventional thermodynamics. The entropy of extremal black hole arises due to a degenerate quantum ground state. This phenomenon is known in the physics of condensed matter (spin glasses).

The macroscopic entropy of extremal black hole can be calculated using the entropy function formalism [1, 2]. It is based on a variational principle applied to a generic class of entropy functions of the charges, scalar fields and the parameters of the near-horizon geometry, \( \text{AdS}_2 \times S^{D-2} \). The formalism can be generalized to other extremal geometries, such as warped ones [3]. Extremization of the entropy function determines all the near-horizon parameters, enabling to extract the information about the black holes without knowledge of a particular solution.

On the other hand, phase transitions due to thermal or, in our case, quantum fluctuations, are related to instabilities of the system around a critical point. In particular, it has been noticed that a massless scalar field produces an instability at the horizon of an extreme Reissner-Nordström black hole [4] and that the axisymmetric extremal horizons are unstable under linear scalar perturbations [5]. Similar instability also occurs for a massive scalar field [6]. Recently, these instabilities were studied from the extremal limit point of view through analysis of charged scalar perturbations for Reissner-Nordström and Kerr solutions [7].

We are interested in studying quantum phase transitions of a 4D charged black hole in General Relativity, when it is non-minimally coupled to a Stückelberg scalar [8]. Namely, non-linear Stückelberg interaction has been known to describe both first and second order thermal

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phase transitions [9, 10]. A question we address is whether a similar change would also occur at the zero temperature.

2. Entropy of the extremal black hole

The extremal black hole is the smallest mass black hole for a given electric charge and angular momentum. Geometrically, it has at least two horizons that overlap, what implies that its near-horizon geometry has the topology $H \simeq \text{AdS}_2 \times S^2$ in four space-time dimensions, where AdS$_2$ is two-dimensional anti-de Sitter space and $S^2$ is a two-sphere. Let $v_1$ and $v_2$ be the radii of the AdS$_2$ and $S^2$-sphere, respectively. Then the horizon of the black hole is described by the following metric in the spherical coordinates $x^\mu = (t, r, \theta, \varphi)$,

$$
\left. ds^2 \right|_H = g_{\mu\nu}(x) dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right),
$$

(1)

where $r$ is a radial distance from the horizon.

The metric (1) corresponds to a near-horizon limit of some static black hole coupled to the electromagnetic field $A_\mu(x)$ and the scalar field $\phi(x)$, described by the action

$$
I = \int d^4x \sqrt{-g} \mathcal{L}(g, A, \phi).
$$

(2)

Near the horizon, the metric behaves as (1), the electric field is $e$ and the scalar field, due to the attractor mechanism, depends only on its asymptotic value $u$. Thus, on the horizon, the fields are represented by a set of parameters

$$
\mathbb{H} : \quad g_{\mu\nu} \rightarrow (v_1, v_2), \quad A_\mu \rightarrow e, \quad \phi \rightarrow u.
$$

(3)

The action (2) evaluated on the horizon is given by an auxiliary function

$$
f(v, e, u) = \int_{\mathbb{H}} d\theta d\phi \sqrt{-g} \mathcal{L}(v, e, u),
$$

(4)

which satisfies the action principle, that is, has an extremum on the equations of motion, for given boundary conditions. The entropy function $\mathcal{E}(v, e, u)$ is a Legendre transformation of the function $f$ with respect to the electric field,

$$
\mathcal{E}(v, e, u) = 2\pi [eq - f(v, e, u)],
$$

(5)

where $q$ is the asymptotic electric charge. The parameters near the horizon are calculated as an extremum of the entropy function [1, 2],

$$
\frac{\partial \mathcal{E}}{\partial v_i} = 0 \quad (i = 1, 2), \quad \frac{\partial \mathcal{E}}{\partial u} = 0, \quad \frac{\partial \mathcal{E}}{\partial e} = 0,
$$

(6)

and the black hole entropy $S$ is its extremal value,

$$
S = \mathcal{E}_{\text{extr}}.
$$

(7)

Therefore, finding the entropy function $\mathcal{E}(v, e, u)$ and its maximum, one can calculate the entropy, electric field and AdS$_2$ and $S^2$ radii of the extremal black hole, independently on a particular static solution considered.
3. Black hole coupled to a St"uckelberg scalar

Consider General Relativity with a cosmological constant $\Lambda$ coupled to electromagnetic and scalar fields,

\[ I = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G_N} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left[ (\partial \phi)^2 + m^2 \phi^2 + F(\phi)(\partial \sigma - A^2) \right] \right\}, \]  

where $R = g^{\mu\nu} R_{\mu\nu}^\alpha$ is the scalar curvature and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the EM field strength. St"uckelberg complex scalar $\phi(x) = \phi(x) e^{i\sigma(x)}$ (with $\phi$ and $\sigma$ real) is minimally coupled when the St"uckelber function is quadratic, $F(\phi) = \phi^2$. Non-minimal interaction has to satisfy $F(\phi) > 0$, $\phi \neq 0$, and $F(0) = 0$ [8]. Such couplings preserve an $U(1)$ invariance and lead to first and second order phase transitions in non-extremal cases [9, 10]. In the extremal case, non-linear terms in $F$ might produce instabilities at the horizon [4], so we choose

\[ F(\phi) = \phi^2 - \frac{a}{4} \phi^4 \geq 0, \]

where $a$ is some coupling constant of dimension $(length)^2$ in natural units.

On the horizon $\mathbb{H}$, spherically symmetric static field configurations are replaced by four parameters $(v_1, v_2, e, u)$, where the electric field $F_{et} = e$ can be obtained from the electric potential $A_\mu = \delta_\mu^t e r$. The Lagrangian density on $\mathbb{H}$ reads

\[ \mathcal{L} = \frac{1}{8\pi G_N} \left( \frac{1}{v_2} - \frac{1}{v_1} - \Lambda \right) + \frac{e^2}{2v_1^2} - \frac{1}{2} \left( m^2 u^2 - \frac{e^2}{v_1} F(u) \right), \]

where for the scalar field we assumed that its value on the horizon is $\phi = u$, $\partial_\mu \phi = 0$. Furthermore, the equation of motion of the field $\sigma(x)$ is not independent on other equations due to a $U(1)$ gauge freedom, so it can be gauge-fixed to $\sigma = 0$.

Gathering all the terms together, the auxiliary function $f = \int d\theta d\phi v_1 v_2 \sin \theta \mathcal{L}$ is obtained using Eq.(10) and the entropy function (5) reads

\[ \mathcal{E} = 2\pi eq - 8\pi^2 v_1 v_2 \left[ \frac{1}{8\pi G_N} \left( \frac{1}{v_2} - \frac{1}{v_1} - \Lambda \right) + \frac{e^2}{2v_1^2} - \frac{1}{2} \left( m^2 u^2 - \frac{e^2}{v_1} F(u) \right) \right]. \]

An extremum of the above function gives rise to the following equations of motion,

\[ 0 = \frac{\partial \mathcal{E}}{\partial v_1} \Rightarrow 1 - \Lambda v_2 = v_2 \left( \frac{e^2}{v_1^2} + m^2 u^2 \right), \]
\[ 0 = \frac{\partial \mathcal{E}}{\partial v_2} \Rightarrow 1 + \Lambda v_1 = \frac{e^2}{v_1} - v_1 m^2 u^2 + e^2 F(u), \]
\[ 0 = \frac{\partial \mathcal{E}}{\partial e} \Rightarrow Q = v_2 e \left( \frac{1}{v_1} + F(u) \right), \]
\[ 0 = \frac{\partial \mathcal{E}}{\partial u} \Rightarrow 0 = v_2 \left( 2v_1 m^2 u - e^2 \frac{dF}{du} \right). \]  

We set $4\pi G_N = 1$ for the sake of simplicity and also denote $Q = \frac{e}{4\pi}$. Eqs.(12) are invariant under the reflection $(e, Q) \rightarrow (-e, -Q)$, and $e$ and $Q$ have the same sign (see third equation) so we can choose, without loss of generality, $e, Q > 0$.

Equations (12) have different branches of solutions that have to be discussed independently.
4. Entropy of a non-hairy black hole

\( u = 0 \) is always a particular solution of the scalar equation in (12), thus we first focus to that case. A general solution for fixed \( Q \) exists provided \( \Delta = 1 - 4\Lambda Q^2 > 0 \), and it reads

\[
\begin{align*}
    v_1^\pm (Q) &= \frac{2Q^2}{1 - 4\Lambda Q^2 \pm \sqrt{1 - 4\Lambda Q^2}}, \\
    v_2^\pm (Q) &= \frac{2Q^2}{1 \pm \sqrt{1 - 4\Lambda Q^2}}, \\
    e^\pm (Q) &= \frac{Q}{1 - 4\Lambda Q^2 \pm \sqrt{1 - 4\Lambda Q^2}}.
\end{align*}
\]

(13)

Other inequalities that have to be fulfilled are

\[
\frac{1}{v_1} + \Lambda > 0, \quad 2\Lambda + \frac{1}{v_1} > 0,
\]

what is satisfied if \( 1 \pm \sqrt{\Delta} > 0 \). We conclude:

(i) When \( \Lambda < \frac{1}{4Q^2} \), there is only one black hole solution, given by Eqs.(13) with the sign ‘+’;
(ii) When \( 0 < \Lambda < \frac{1}{4Q^2} \), there are two solutions, given by Eqs.(13) with the signs ‘+’ and ‘−’.
(iii) When \( \Lambda = 1/4Q^2 \), there is no a finite solution for \( v_1 \).
(iv) The case \( \Lambda = 0 \) can be reproduced from the limit \( \Lambda \to 0 \) of the positive branch of the solution (13).

Under these conditions, the extremum of the entropy function (11) for the fixed charge and the values (13) of the black hole parameters is

\[
S^\pm (Q) = \frac{8\pi^2 Q^2}{1 \pm \sqrt{1 - 4\Lambda Q^2}}.
\]

(15)

Note that \( S^+_\pm (Q) > 0 \) is always fulfilled while, for the negative branch, \( S^-_\pm (Q) > 0 \) is true only for the negative values of the cosmological constant. Thus, spaces with positive cosmological constant have only one solution, corresponding to the positive branch in (13).

When \( \Lambda = 0 \), then the positive branch reproduces the well-known result

\[
S^+_\pm (Q)|_{\Lambda=0} = 4\pi^2 Q^2 = \frac{q^2}{4}.
\]

(16)

5. Hairy extremal black hole and the critical point

For the study of phase transitions, the most interesting cases involve non-linear St"uckelberg interaction \( F \) of the form (9). The last equation in (12) implies that, for the minimal coupling \( a = 0 \), the only solution for the scalar field is \( u = 0 \), for which the black hole entropy was calculated in the previous section. When \( a \neq 0 \), then there are three solutions for the scalar parameter,

\[
u = 0, \quad u = \pm \sqrt{\frac{2}{a} \left( 1 - \frac{v_1 m^2}{e^2} \right)}.
\]

(17)

The case \( u = 0 \) was analyzed in Section 4. When \( u \neq 0 \), the equations of motion (17) are invariant under the replacement \( u \to -u \), so we can chose \( u > 0 \).
Now we address the following question: Is there a critical point for which two solutions $u = 0$ and $u \geq 0$ co-exist? If the answer is yes, this would be a possible branch point where a quantum phase transition from one configuration to another might occur.

To answer this question, let us analyze the critical point limit $u \to 0$ of the non-trivial solution in (17). The critical point exists only for the massive fields ($m \neq 0$). Then Eq.(17) implies

$$v_{1c} = \frac{e_c^2}{m^2}.$$  \hfill (18)

First two equations of (12) give that the critical point exist only for $\Lambda \neq 0$ and $m^2 \neq 1, \frac{1}{2}$, leading to

$$v_{1c} = \frac{m^2 - 1}{\Lambda},$$

$$e_c = \sqrt{\frac{m^2 (m^2 - 1)}{\Lambda}},$$

$$v_{2c} = \frac{m^2 - 1}{\Lambda (2m^2 - 1)}.$$  \hfill (19)

The result does not depend on the strength of the scalar coupling $a \neq 0$. Non-vanishing and positive $v_{1c}$ and $e_c$ exist only if

$$m^2 > \frac{1}{2}, \quad \frac{m^2 - 1}{\Lambda} > 0.$$  \hfill (20)

Note that the above conditions have to be satisfied for both positive and negative values of the cosmological constant. In addition, in case of negative cosmological constant, the Breitenlohner-Freedman bound that ensures the stability of scalar field in AdS$_4$ imposes $m^2 > \frac{3\Lambda}{4}$ to the scalar mass, what is weaker than the inequalities (20), so it is always satisfied.

The last unsolved equation (third in Eq.(12)) shows that the critical values of the parameters can be reached only for the critical charge

$$Q_c = \frac{v_{2c} e_c}{v_{1c}} = \frac{1}{2m^2 - 1} \sqrt{\frac{m^2 (m^2 - 1)}{\Lambda}}.$$  \hfill (21)

Compared to the original two $u = 0$ solutions given by Eqs.(13) where $Q$ is replaced by $Q_c$ we find that, for the positive branch ‘+’, we reproduce the known critical results of the parameters,

$$v_i^+(Q_c) = v_{1c}, \quad e^+(Q_c) = e_c,$$  \hfill (22)

whereas the branch with the sign ‘−’ is inconsistent because it gives negative values of parameters. The critical entropy reads

$$S_c = \frac{4\pi^2 (m^2 - 1)}{\Lambda (2m^2 - 1)} > 0,$$  \hfill (23)

and it is always positive. Thus, the entropy is a continuous function at $Q_c$ because

$$S_+(Q_c) = S_c.$$  \hfill (24)

When $\Lambda = 0$, then $m^2 = 1$ (and vice versa), and the solution is $v_{1c} = v_{2c} = Q^2$, $e_c = Q$ with the charge $Q$ which remains arbitrary. The case $m^2 = \frac{1}{2}$ does not have solutions.

We will discuss only the spacetimes with $\Lambda \neq 0$ when the critical point is well-defined and analyze a behavior of the hairy black hole in its vicinity. We will also explore whether the quantum phase transition would occur in this point using the second thermodynamics law ($dS > 0$).
6. Near-critical behaviour of the entropy

Knowing the critical point given by Eqs.(19) and (21), we study a near-critical behavior of the solution by introducing a small parameter

$$ \epsilon = Q - Q_c, \quad (25) $$

where \( \epsilon \) can be either positive or negative. We assume that \( u \to 0 \) when \( Q \to Q_c \) so that, near \( Q_c \), it behaves as

$$ u^2 = A \epsilon^\beta + \cdots > 0, \quad \beta > 0, \quad (26) $$

where \( \beta \) is a critical exponent. We also seek for the expansions

$$ e = e_c + B \epsilon^\beta + \cdots, $$

$$ v_2 = v_{2c} + C \epsilon^\beta + \cdots, $$

$$ v_1 = v_{1c} + V \epsilon^\beta + \cdots. \quad (27) $$

Expanding the third equation in (12), all finite terms cancel out since they correspond to the critical values and, at the lowest order of \( \epsilon \), we obtain

$$ \epsilon = \left( e_c v_{2c} A + \frac{v_{2c}}{v_{1c}} B + \frac{e_c}{v_{1c}} C - \frac{e_c v_{2c}}{v_{1c}^2} V \right) \epsilon^\beta + \cdots, \quad (28) $$

leading to the critical exponent \( \beta = 1 \). Thus, the order parameter shows a typical mean-field behavior near the critical point, namely

$$ u = \sqrt{A(Q - Q_c)} + \mathcal{O}((Q - Q_c)^2). \quad (29) $$

With this result at hand, all equations at the linear order become

$$ 0 = v_{1c} v_{2c} e_c^2 A + 2 e_c v_{2c} B + (\Lambda v_{1c} + m^2) v_{1c} C - 2 m^2 v_{2c} V, $$

$$ 0 = 2 e_c B - (\Lambda + m^2) V, $$

$$ v_{1c}^2 = e_c v_{1c} v_{2c} A + v_{1c} v_{2c} B + v_{1c} e_c C - e_c v_{2c} V, $$

$$ 0 = a e_c^2 A - 4 e_c B + 2 m^2 V, \quad (30) $$

where Eq.(18) was used to cancel out some terms. The above system is algebraic in \( A, B, C \) and \( V \), with a general solution

$$ A = \sqrt{\frac{m^2 - 1}{4(m^2 - 1)^3 + \Lambda a}}, $$

$$ B = \frac{\Lambda a (2m^2 - 1)^3}{4(m^2 - 1)^3 + \Lambda a}, $$

$$ C = \sqrt{\frac{m^2 (m^2 - 1)^2}{\Lambda a}} \left( -\frac{2}{(m^2 - 1)^2} + \Lambda a \right), $$

$$ V = \sqrt{\frac{m^2 (m^2 - 1)^2}{\Lambda a}} \frac{2 \Lambda a (2m^2 - 1)^2}{4(m^2 - 1)^3 + \Lambda a}. \quad (31) $$

Replacing the above solutions in the expression for the entropy function (11), we obtain the entropy near the critical point in the form

$$ S|_{\epsilon \approx 0} = S_c + 8 \pi^2 e_c (Q - Q_c) + \mathcal{O}((Q - Q_c)^2). \quad (32) $$
We conclude that, near the critical value of the charge $Q_c$, there are two extremal black hole solutions – the hairy one whose entropy is given by Eq.(32), and a non-hairy one (Reissner-Nordström (A)dS) with the entropy given by $S_+(Q)$ in Eq.(15). When expanded around the critical point, the non-hairy solution becomes

$$S|_{u=0} = S_+(Q) = S_c + \left. \frac{dS_+}{dQ} \right|_{Q_c} (Q - Q_c) + \mathcal{O}((Q - Q_c)^2)$$

$$= S_c + 8\pi^2 e_c (Q - Q_c) + \mathcal{O}((Q - Q_c)^2),$$

(33)

what exactly matches the expression (32) up to quadratic terms. Surprisingly, two different solutions that extremize the entropy, $u = 0$ and $u > 0$, glue almost smoothly at the critical point such that both $S$ and its first derivative are continuous functions around $Q_c$. This indicates that the hair develops in the black hole in such a way that there is a quantum phase transition of the order higher than two. Higher-power terms in the expansions (32) and (33) are needed to determine the exact order of the phase transition.

7. Conclusions

We studied quantum phase transitions of black holes in 4D at zero temperature that arise due to variations of electric charge. We showed analytically that the static extremal black hole coupled to the Stückelberg scalar possesses two different solutions, one with scalar hair and another without it, which indicates that a phase transition might be possible at some critical electric charge. If so, a change in the phases is such that $S$ and $\partial S/\partial Q$ are continuous at the critical point, so the discontinuity should appear at higher order. This type of a phase transition would be possible only for spacetimes with non-vanishing cosmological constant and non-vanishing mass of the scalar, which interact non-linearly with gravitational and electromagnetic fields. Higher-power terms in the expansions of two branches of solutions are necessary to gain more insight in the process.

One should also include in this study the AdS black holes with planar and hyperbolic horizons, magnetic charge with the magnetic field and ungauged U(1) freedom through the field $\sigma(x)$. Another open question is how to embed the model in the framework of supergravity.

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