Chaotic behavior in a perturbed soliton system

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Abstract. The influence of an external perturbation on a soliton system is considered, the generalized KdV equation is used as an example to investigate numerically the chaotic behavior of the system with a periodic force. Different routes to chaos such as period doubling, and the shapes of strange attractors are observed by using bifurcation diagrams, the largest Lyapunov exponents, phase projections and Poincare maps.

1. Introduction

During the last decades the soliton systems such as KdV equation, Burgers equation and sine-Gordon equation have received much attention [1-3]. These nonlinear evolution equations arise from many physical fields, and many of them are completely integrable. It is known that a completely integrable nonlinear system possesses some nice properties such as the Lax pair, N-soliton solutions, infinite conservation laws, Painleve property and bi-Hamiltonian structure.

However there exist often various perturbations or excitations in many real physical processes. The addition of a perturbing or forcing term to an integrable equation can lead to chaotic dynamics, while deterministic chaos is one of the most interesting nonlinear phenomena [4,5]. For example, chaos was found in the perturbed sine-Gordon equation, and in the cubic nonlinear Schrodinger equation [6]. A route to chaos by a period-doubling sequence was investigated in a perturbed sine-Gordon system [7]. Recently a particular attention is paid to the study of soliton equations under external perturbations [8,9].

In this paper a generalized form of KdV (GKdV) equation in the form of

\[
u_t + au^p u_x + bu_{xxx} + \nu u_{xx} = f_x, \quad p > 0,
\]

with nonlinear, dispersive, damping and external forcing terms is considered, where \(\nu\) is a damping parameter, \(f\) is the forcing term, \(a\) and \(b\) are real parameters. The GKdV equation with zero damping, arising from fluid mechanics and other fields of physics, is known to possess traveling wave solutions in the form of solitary or periodic solutions. For Eq.(1), the initial value problem was considered by using Sobolev space and finite element method [10], and the harmonic solutions of non-resonance and primary resonance were obtained with the perturbed method [11]. Though the chaotic behavior under an harmonic excitation has been predicted by the Melnikov theory in Ref.[11], the bifurcation and route to chaos in the system have not been studied yet.

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In order to investigate the possible chaotic behavior of system (1), we still assume that the forcing term \( f \) is a periodic function. Let \( u(x,t) = \varphi(\xi) \) with \( \xi = x - ct \) be a traveling wave solution for Eq.(1). Substituting \( \varphi(\xi) \) into Eq.(1) yields

\[
-c\varphi_\xi + a\varphi^p \varphi_\xi + v\varphi_{\xi\xi} + b\varphi_{\xi\xi\xi\xi} = f_\xi,
\]

where \( \varphi_\xi = d\varphi / d\xi \). Taking \( f \) as \( g_0 \cos(\omega \xi) \), and then integrating Eq.(2) once leads to

\[
\varphi_{\xi\xi} + \gamma\varphi_\xi - \alpha\varphi + \frac{\beta}{p+1}\varphi^{p+1} = f_\xi \cos(\omega \xi),
\]

where \( \gamma = v / b \), \( \alpha = c / b \), \( \beta = a / b \) and \( f_\xi = g_0 / b \). Corresponding to the different values of the parameter \( p \), the system (3) can describe some different physical or mechanical modes. For \( p = 1 \) and \( p = 2 \), Eq.(3) reduces to well known Helmholtz oscillator and Duffing oscillator, respectively. The oscillator equations especially Duffing oscillator have been found to play an important role in the understanding of some physical problems. These equations, which exhibit very rich nonlinear dynamics, have been studied from a theoretical, numerical, experimental and control point of view. But to our knowledge, the system (3) with \( p \geq 3 \) was not deeply investigated in the past.

Let \( \varphi = X \), \( X' = Y \) in (3), then it follows that

\[
X' = Y,
\]

\[
Y' = f_\xi \cos(\omega \xi) - \gamma Y + \alpha X - \frac{\beta}{p+1} X^{p+1},
\]

where prime denotes the derivative with respect to \( \xi \).

System (4) involves five parameters: \( f_\xi, \gamma, \alpha, \beta \) and \( \omega \). Due to the relatively large number of parameters, the detailed influence of each parameter on dynamics of the system is not presented here. But it is of interest to analyze the influence of some parameter on the perturbed system. To simplify the analysis, all parameters are kept constants except one to be varied. Here we will choose \( \alpha \) or \( \omega \) as bifurcation parameters, respectively.

In this work we numerically investigated dynamical behaviors of the system (4) for \( p = 3, 4, 5 \) and 6 by using Maple software in some detail, and found that the system will exhibit the complex chaotic dynamics as the bifurcation parameter changes.

Fig.1(a)-(d) show the bifurcation diagrams of (4) with \( p = 3, 4, 5 \) and 6 respectively. The concrete parameters are: (a) \( p = 3, f_\xi = 1.07, \gamma = 0.7, \beta = 1.668 \) and \( \omega = 1.75 \); (b) \( p = 4, f_\xi = 0.95, \gamma = 0.85, \beta = 2.05 \) and \( \omega = 1.5 \); (c) \( p = 5, f_\xi = 0.8, \gamma = 0.7, \beta = 2.76 \) and \( \omega = 1.8 \); (d) \( p = 6, f_\xi = 0.75, \gamma = 0.85, \beta = 3.542 \) and \( \omega = 1.5 \). From Fig.1 we may observe that the system enters chaotic states usually via periodic doubling bifurcation, while period doubling is at present the most commonly known route to chaos.

Fig.2(a) further illustrate the dynamical responses of system (4) for \( p = 4 \) varying with \( \omega \), which the system parameters are taken as \( p = 4, f_\xi = 0.5, \gamma = 0.3, \alpha = 0.2 \) and \( \beta = 2.0 \). Fig.2(b) gives the corresponding largest Lyapunov exponents so that confirm the existence of the chaotic regions and periodic orbits. We can observe, from Fig.2(a), that the nonlinear dynamical system (4) exhibits periodic and chaotic behaviors as \( \omega \) varies. It is clear that the system returns to regular motions from chaos usually through a sequence of inverse period-doubling bifurcations. We can also see an other
type of route leading to chaos in Fig.2(a): intermittency. For example, the response of system (4) suddenly jumps to chaotic region from simple period orbit via an intermittent bifurcation at $\omega = 0.987$. When $\omega$ increases from about 1.27, the system gets back to period-6, period-3 motions by an inverse period doubling route, then enters a broad chaotic region from period-3 orbit through intermittency route at $\omega = 1.393$.

Figure 1: Bifurcation in ($\alpha - X$) plane for (4): (a) $p = 3$; (b) $p = 4$; (c) $p = 5$ and (d) $p = 6$.

Figure 2: (a) Bifurcation diagram in ($\omega - Y$) plane for (4), with $p = 4, f_0 = 0.35, \gamma = 0.4, \alpha = 0.5$ and $\beta = 2.75$; (b) the largest Lyapunov exponent corresponding to (a).
It was noted that there is an interior crisis taking place at $\omega=1.61$ in bifurcation diagram, where the shapes of chaotic attractors will suddenly change. For clarity, Fig. 3 is used to demonstrate two different shapes of Poincare sections of the chaotic attractors where interior crisis appears. Fig. 3(a) depicts the phase portrait of the chaotic state at $\omega=1.60$, and the Poincare sections of the strange attractors at $\omega=1.60$ and $\omega=1.61$ are shown in Fig. 3(b) and (c), respectively.

To obtain some better insight of chaos feature, Fig. 4 displays the chaotic attractor in the phase portrait and Poincare section at $\omega=1.005$ in Fig. 2(a), which the corresponding largest Lyapunov exponents is 0.178. According to the definition of Kaplan and Yorke [12], we can obtain the Lyapunov dimensions of the attractor. The calculated fractal dimension of this strange attractor shown in Fig. 4(b) is 2.342.

With the aid of Maple software, a lot of numerical calculation for perturbed system (4) is carried out. The regular and chaotic motions in the generalized KdV equation under a harmonic excitation for $p=3, 4, 5$ and $6$ are investigated respectively through computer simulations. It is shown that dynamical chaos can occur when we choose appropriately the system parameters and initial conditions.

**Figure 3**: Phase projections and Poincare maps of chaotic states in Fig. 2(a): (a,b) $\omega=1.60$; (c) $\omega=1.61$.

**Figure 4**: (a) Phase projection of a chaotic state in Fig. 1(a), with $\omega=1.005$; (b) the Poincare map corresponding to (a).

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**References**
[1] Lou S Y 1998 *Phys. Rev. Lett.* 80 5027
[2] Yu J and Lou S Y 2000 *Science in China (Series A)* 43 655
[3] He J H 2006 *Int. J. Mod. Phys. B* 20 1141
[4] Ge Z M, Hsiao C L, Chen Y S, et al, 2007 *Int. J. Nonlinear Sci.* 8 89
[5] Yu J, Pan W Z, Zhang R B, et al, 2006 *Int. J. Nonlinear Sci.* 7 365
[6] Moon H T 1990 *Phys. Rev. Lett.* 64 412
[7] Zheng D J, Yeh W J and Symko O G 1989 *Phys. Lett. A* 140 225
[8] Yu J, Zhang W J and Gao X M 2006 *Commun. Theor. Phys.* 46 1
[9] Yu J, Zhang W J and Gao X M 2007 *Chaos Soliton. Fract.* 33 1307
[10] Lai X, Cao Q and Twizell EH 2005 *Chaos Soliton. Fract.* 23 1613
[11] Cao Q, Djidjeli K, Price W G and Twizell E H 1999 *Physica D* 125 201
[12] Kaplan H and Yorke J A 1979 *Lecture note in mathematics* (Berlin: Springer-Verlag)