Abstract. Dimensional reduction of theories involving (super-)gravity gives rise to sigma models on coset spaces of the form $G/H$, with $G$ a non-compact group, and $H$ its maximal compact subgroup. The reverse process, called oxidation, is the reconstruction of the possible higher dimensional theories, given the lower dimensional theory. In 3 dimensions, all degrees of freedom can be dualized to scalars. Given the group $G$ for a 3 dimensional sigma model on the coset $G/H$, we demonstrate an efficient method for recovering the higher dimensional theories, essentially by decomposition into subgroups. The equations of motion, Bianchi identities, Kaluza-Klein modifications and Chern-Simons terms are easily extracted from the root lattice of the group $G$. We briefly discuss some aspects of oxidation from the $E_{8(8)}/SO(16)$ coset, and demonstrate that our formalism reproduces the Chern-Simons term of 11-d supergravity, knows about the T-duality of IIA and IIB theory, and easily deals with self-dual tensors, like the 5-tensor of IIB supergravity.

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1. Introduction

If one takes the idea of extra dimensions seriously, as is usually done in Kaluza-Klein-, string- and M-theory, the immediate question one faces is to guess the higher dimensional theory and geometry from a lower dimensional one. The process of constructing the higher dimensional theory from the lower dimensional one is called “oxidation”. An interesting aspect of oxidation is that it is not unique; there may be multiple theories giving rise to the same low dimensional theory. This is the phenomenon of duality: Different higher dimensional theories lead to the same low energy physics, and we should not discriminate between the various higher dimensional formulations.

In this paper we discuss classical field theories only, and their dimensional reduction (toroidal compactification and subsequent truncation to the massless sector). For theories involving (super-)gravity, this scheme leads to theories with scalar sectors formulated as sigma models on symmetric spaces \( G/H \), with \( G \) a non-compact group, and \( G \) its maximal compact subgroup. Reducing to 2+1 dimensions, all bosonic degrees of freedom are scalars (or can be dualized to scalars). We suppose that these live on a coset space \( G/H \), and ask the question: Which higher dimensional theories lead to these theories, upon dimensional reduction?

Earlier studies of this question focussed on oxidation to 4 dimensions [1], or to all dimensions, but restricted to cosets for which \( G \) is split [2]. One approach is to postulate the higher dimensional theory, and then derive that it indeed results in the wanted low dimensional theory. This is slightly unsatisfactory in view of the non-uniqueness of the oxidation process. A more systematic approach is presented in [3]: Building on developments from [4] and suggestive conjectures from [5] they developed a geometric scheme based on Del Pezzo-surfaces. A drawback is however that at present, this method only applies to groups that are split, and sub-groups of \( E_{8(8)} \).

In [6] we presented a scheme that allows one to attack the problem for so-called split forms; the extension to generic non-compact groups is discussed in [7]. The method is based on decomposition of \( G \) into subgroups; due to its mathematical nature it allows an exhaustive analysis. As a bonus we recover all equations (equations of motion and Bianchi identities) for all theories in the oxidation chain, and are able to furnish proofs of some observations in the literature. As the method concentrates on equations rather than actions, we have no difficulty in handling theories with (anti-)self-dual tensors.

2. The idea

An important reference for the ideas expressed in this section is [8]. For an extended discussion, and more references we refer the reader to [9].

In theories of gravity the concept of a vielbein is an important one, and generalizations of the concept play an important role in coset theories. A vielbein represents an element of \( GL(D)/SO(1, D-1) \). The quotient factor is the Lorentz group. It is standard lore that upon dimensional reduction, the number of (massless) degrees of
freedom stays constant; these massless degrees organize in representations of the helicity group $SO(D-2)$. Motivated by this, we turn to the cosets $SL(D-2)/SO(D-2)$. We might consider $GL(D-2)$ instead of $SL(D-2)$, but the determinant of the $GL(D)$ element represents a scale factor that plays no role in the following. In 3 dimensions all degrees of freedom can be represented by scalars (after appropriate dualizations), and indeed, the dimensional reduction of general relativity in $D$ dimensions to 3 dimensions gives rise to a sigma model on $SL(D-2)/SO(D-2)$.

We now consider 3 dimensional sigma models on coset spaces of the form $G/H$, with $G$ a non-compact group, and $H$ its maximal compact subgroup. A famous example is the 3-d coset on $E_8(8)/SO(16)$, resulting from reducing the bosonic sector of a maximal supergravity to 3 dimensions. This example immediately suggests the inverse problem: Given the coset $G/H$ can this be interpreted as a dimensional reduction of a higher dimensional theory and if so, which one(s)? The result that general relativity gives rise to a $SL(D-2)/SO(D-2)$ coset suggests that if the answer is affirmative, we should be able to identify an $SL(D-2)$ subgroup in $G$. Suppose we have such a subgroup, then we should decompose $G$ in $SL(D-2)$ irreducible representations (irreps). Applying the “vielbein” we can convert $SL(D-2)$ irreps into $SO(D-2)$ irreps, which in view of the previous discussion, we would like to interpret as matter fields.

The group $G$ is represented by its adjoint representation, which is a self-conjugate irrep, and decomposition will lead to self-conjugate representations. These come in two kinds: pairs of conjugate irreps; and self-conjugate irreps. We want to interpret these as massless fields, i.e. one graviton, forms and scalar fields. The pairs of conjugate irreps represent forms and their duals; self-conjugate irreps are the adjoint of $SL(D-2)$, representing the graviton, singlets (scalars), or self-conjugate forms. The latter are important in theories with (anti)self-dual tensors.

The centralizer of $SL(D-2)$ in $G$ acts as a symmetry group on the $SL(D-2)$ representations. We call the centralizer the $U$-duality group, even though most cosets do not have a direct relation with string theory.

We have assumed thus far that upon decomposition of $G$ into $SL(D-2)$ irreps, we find irreps that can be interpreted as one (and no more) graviton, and some forms and scalars. In other words, we want only one adjoint irrep in the decomposition, and the rest must be anti-symmetric tensors and singlets. Requiring this is equivalent to a constraint on the $SL(D-2)$-subgroup in question; it has to be an index 1 subgroup of $G$ (see [6] for a derivation of this fact).

Index 1 subgroups are special. They must be regular, which means that that the root lattice of the subgroup can be chosen to be a sublattice of the lattice of the original group. All regular subalgebra’s of a given (complexified) algebra can be found by a procedure described by Dynkin [10]. One still has to verify that the algebra describes the real form that we want, $SL(D-2,\mathbb{R})$. This can be guaranteed by choosing a particular realization of the real form, and some technology described in [6, 7].

If the group is simply laced, all regular embeddings have index 1. If the group is non-simply laced, then regular subgroups involving only short roots are possible. Such
subgroups have an index bigger than one, and hence are excluded.

The conclusion is that regular $SL(D - 2, \mathbb{R})$ subgroups of long roots are the appropriate mathematical structure for encoding the graviton in the theories under study. Regularity of subgroups is a criterion that has been observed before $[8]$, and is implicit in many discussions on algebraic aspects of compactified gravity.

3. Coset sigma models

We use a basis for the Lie algebra of $G$ consisting of Cartan subalgebra generators $H_i$ and ladder operators $E_{\alpha}$ (labelled by the roots of $G$), obeying:

\[ [H_i, H_j] = 0; \quad [H_i, E_{\alpha}] = \alpha^i E_{\alpha}; \quad [E_{\alpha}, E_{-\alpha}] = \alpha^i H_i; \quad [E_{\alpha}, E_{\beta}] = N_{\alpha,\beta} E_{\alpha + \beta}. \]  \hspace{1cm} (1)

Our discussion of sigma models is based on the action ($\mathcal{V} \in G$)

\[ L_{G/H} = -e \text{ tr} \left( (\partial \mathcal{V})\mathcal{V}^{-1} \frac{1}{2} (1 + T)(\partial \mathcal{V})\mathcal{V}^{-1} \right). \]  \hspace{1cm} (2)

Here $T$ is an operator that encodes the action of the Cartan involution on the algebra. As $T$ represents an involution, $\frac{1}{2}(1 + T)$ is a projection operator, projecting out the generators of the compact subgroup $H$. Although the analysis can be done for any Cartan involution $[7]$, we follow $[6]$, and restrict to $G$'s that are so called split real forms. These are generated by linear combinations of the $H_i$’s and $E_{\alpha}$’s with real coefficients.

The 1-form $(d\mathcal{V})\mathcal{V}^{-1}$ is an element of the algebra of $G$. It can be expanded as

\[ (d\mathcal{V})\mathcal{V}^{-1} = \frac{1}{2} \sum_{i=1}^r d\phi^i H^i + \sum_{\alpha \in \Delta^+} e^{\frac{1}{2}(\alpha, \phi)} F_{(1)\alpha} E_{\alpha}. \]  \hspace{1cm} (3)

The second sum runs only over the set of positive roots of $G$, $\Delta^+$, and $r$ denotes the (real) rank. This expansion assumes a particular choice of gauge, the so-called positive root gauge, that is possible by virtue of the Iwasawa decomposition.

The $F_{(1)\alpha}$ are one forms. To get more detail on these, we take the derivative $d((d\mathcal{V})\mathcal{V}^{-1}) = ((d\mathcal{V})\mathcal{V}^{-1}) \wedge ((d\mathcal{V})\mathcal{V}^{-1})$ which translates into

\[ dF_{(1)\gamma} = \frac{1}{2} \sum_{\alpha,\beta} N_{\alpha,\beta} F_{(1)\alpha} \wedge F_{(1)\beta} \quad * = \begin{cases} \alpha, \beta, \gamma \in \Delta^+ \\ \alpha + \beta = \gamma \end{cases}. \]  \hspace{1cm} (4)

Note the appearance of the structure constants $N_{\alpha,\beta}$. The structure becomes even nicer when also considering the equations of motion for the scalar coset. We recover these by regarding the $F_{(1)\alpha}$ as independent fields, and enforcing (4) by Lagrange multipliers (which will be $(D - 2)$-forms). Therefore we add to the Lagrangian

\[ L_{\text{Bianchi}} = \sum_{\alpha \in \Delta^+} \left( dF_{(1)\alpha} - \sum_{\alpha = \beta + \gamma} N_{\beta,\gamma} F_{(1)\beta} \wedge F_{(1)\gamma} \right) \wedge A_{(D - 2) - \alpha}. \]  \hspace{1cm} (5)

The labels $-\alpha$ appearing on the $(D - 2)$-forms turn out meaningful. Varying with respect to $F_{(1)\gamma}$ we find the equations

\[ F_{(D - 1) - \gamma} \equiv e^{(\gamma, \phi)} \ast F_{(1)\gamma} = dA_{(D - 2) - \gamma} - \sum_{\beta - \alpha = -\gamma} N_{\beta, -\alpha} F_{(1)\beta} \wedge A_{(D - 2) - \alpha}. \]  \hspace{1cm} (6)
We defined $F_{(D-1)-\alpha}$, and used a group theoretical identity for the structure constants. We are not interested in the dual potential $A_{(D-2)-\alpha}$, but in the field strengths $F_{(D-1)-\alpha}$. Taking the derivatives of these (note that in general $dF_{(1)\beta} \neq 0$), one finds \[ dF_{(D-1)\gamma} = \sum_{\*=\;} N_{\alpha,\beta} F_{(1)\alpha} \wedge F_{(D-1)\beta} \] (7)

In terms of $F_{(D-1)-\alpha}$, the Lagrangian can be rewritten to

$$ L_{G/H} = -\langle *d\phi, d\phi \rangle - \frac{1}{2} \sum_{\alpha \in \Delta^+} F_{(D-1)-\alpha} \wedge F_{(1)\alpha}, $$

from which one finds the equation of motion for $\phi$

$$ 2d(*d\phi^i) = \sum_{\alpha \in \Delta^+} \alpha^i F_{(D-1)-\alpha} \wedge F_{(1)\alpha}. $$

Note that also the $\alpha^i$ are structure constants of the algebra.

With (4),(7) and (9) we have established a one-to-one relation between the conventional basis of the Lie-algebra of $G$, and the equations relevant to the coset sigma model. For every $E_\alpha$ where $\alpha$ is a positive root, we have a Bianchi identity from (4), when $\alpha$ is a negative root we have an equation of motion from (7), while the Cartan subalgebra corresponds to equations (9) for $\phi$. The Cartan subalgebra gives only rank $G$ equations, because the “Bianchi identity” for the potential for $\phi$, $d^2 \phi = 0$ is trivial.

Coupling to other forms $F_{(n)}$ can be done by adding quadratic terms to the action. If the $F_{(n)}$ form a non-trivial representation of U-duality, we contract on an internal “metric” to render the action U-duality invariant. This schematically takes the form

$$ L_m = \frac{1}{2} *F_{(n)} \wedge M F_{(n)}, $$

with $M$ in an appropriate representation. The possibility of Chern-Simons terms will turn up naturally later.

The equation of motion, and Bianchi identity for $F_{(n)}$ can be “covariantized” to reflect the local $H$-invariance, and then become

$$ (d + T(dVV^{-1})) V \ast F_{(n)} = 0 \quad \text{(equation of motion)}; \quad (d - (dVV^{-1})) VF_{(n)} = 0 \quad \text{(Bianchi identity)}. $$

If the fields $\ast F_{(n)}$ and $F_{(n)}$ represent the same degrees of freedom, we cannot allow them to transform differently. Hence, only local transformations $V(x) \rightarrow U(x)V(x)$ for which $\frac{1}{2}(1 + T)(dUU^{-1}) = 0$ are allowed, but this is precisely the restriction to the compact subgroup that was imposed already.

Both $VF_{(n)}$ and $V \ast F_{(n)}$ represent a full $G$ multiplet of fields. The components of a multiplet can be labelled by their weights, and we decompose by writing

$$ VF_{(n)} \equiv \sum_{\lambda \in \Lambda} e^{\frac{1}{2}(\lambda, \phi)} F_{(n)\lambda}, \quad V \ast F_{(n)} \equiv \sum_{-\lambda \in \Lambda} e^{-\frac{1}{2}(\lambda, \phi)} F_{(D-n)-\lambda}. $$

(13)
The sum runs over the weights $\lambda (-\lambda)$ on the weight lattice $\Lambda (\overline{\Lambda})$ of the representation. $\mathcal{V} \ast F_{(n)}$ transforms in the representation conjugate that of $\mathcal{V} F_{(n)}$, and the weights differ by a minus sign. The forms of degree $D - n$ are not the duals to $F_{(n)\lambda}$. Rather,

$$e^{-\frac{1}{2}(\lambda,\phi)} F_{(D-n)-\lambda} = e^{\frac{1}{2}(\lambda,\phi)} F_{(n)\lambda}. \quad (14)$$

Inserting $(d\mathcal{V}) V^{-1}$ from (3) in the Bianchi identity (12), one obtains

$$dF_{(n)\lambda'} = \sum N_{\alpha,\lambda} F_{(1)\alpha} \wedge F_{(n)\lambda} \quad * = \begin{cases} \lambda, \lambda' \in \Lambda \\ \alpha \in \Delta^+ \\ \alpha + \lambda = \lambda' \end{cases} \quad (15)$$

The constants $N_{\alpha,\lambda}$ can be computed but we will not need them explicitly; when finding expressions like (15) in the future, the constants are determined in the derivation.

The equation of motion (11) can be rewritten similarly, and becomes

$$dF_{(D-n)-\lambda'} = \sum N_{\alpha,-\lambda} F_{(1)\alpha} \wedge F_{(D-n)-\lambda} \quad * = \begin{cases} \lambda, \lambda' \in \Lambda \\ \alpha \in \Delta^+ \\ \alpha - \lambda = -\lambda' \end{cases} \quad (16)$$

It can happen that form and dual form transform in a self-conjugate representation; in theories with self-dual tensors they must be in such a representation. In that case the equation of motion (16) and Bianchi identity (15) are essentially the same equation, and we can consistently impose self-duality.

The Lagrangian (10) for coupled matter becomes

$$L_m = -\frac{1}{2} \sum_{\lambda \in \Lambda} F_{(D-n)-\lambda} \wedge F_{(n)\lambda}. \quad (17)$$

The sum over $\lambda$ indicates a sum over the weights of the representation. Note that again there is one equation for every weight.

With extra matter, the equation of motion for the dilatonic scalars (11) becomes

$$2d(*d\phi^i) = \sum_{\alpha \in \Delta^+} \alpha^i F_{(D-1)-\alpha} \wedge F_{(1)\alpha} + \sum_{\lambda \in \Lambda} \lambda^i F_{(D-n)-\lambda} \wedge F_{(n)\lambda}, \quad (18)$$

while (7) is not modified. Note that singlet representations of U-duality do not couple to $\phi$ (as for these, $\lambda = 0$).

For the non-dilatonic fields we have used roots and weights as labels. In equations we find that left and right hand side have the same degrees, and their labels sum up to the same vector. The rule on addition of forms is implied by Lorentz symmetry in the non-compact directions. The additivity of the vector and weight labels follows from the U-duality group. The theory is invariant under $\mathcal{V} \rightarrow U\mathcal{V}$, with $U$ a constant element of $G$. Most of these symmetries were eliminated by gauge fixing, but when $U$ is an element obtained by exponentiating an element of the Cartan subalgebra, we have

$$\phi \rightarrow \phi + \zeta; \quad F_{(n)\xi} \rightarrow e^{-\frac{1}{2}(\xi,\zeta)} F_{(n)\xi}. \quad (19)$$

regardless of whether $\xi$ is a weight or a root. Covariance of the equations of motion and Bianchi identities requires the labels of various terms to add up on both sides. The exponential factors included in the definitions of fields labelled by negative roots (3) and conjugate weights (14) ensure the transformation behavior implied by their labels.
4. Oxidation

The equations governing the axions (4) (7), and form fields (15) (16) have a very similar common structure. Together with the considerations on the group structure, they suggest the following “oxidation recipe”:

- We start from a 3 dimensional sigma model on $G/H$.
- To oxidize to $D$ dimensions, decompose $G$ into $SL(D-2) \times U_D$ irreps, with $U_D$ the U-duality group, and $SL(D-2)$ an index 1 subgroup. Of course, in the absence of such a group, we cannot oxidize. The irreps of $SL(D-2)$ are $n$-forms, singlets and one adjoint. The conjugate irrep of an $n$-form is a $(D-2-n)$-form.
- Separate the roots of $G$ in $SL(D-2)$-weights (by projecting onto the subspace containing the $SL(D-2)$ lattice), and assign to each complete set of weights for a form representation a field $F_{(n+1)\alpha'}$, with $n$ the degree of the form, and $\alpha'$ the projection of the root $\alpha$ onto the subspace orthogonal to the $SL(D-2)$ sublattice.
- Some of the $SL(D-2)$ singlets are labelled by the roots of $U_D$ (these correspond to axions). Make a positive root decomposition for these, and assign a form $F_{(1)\alpha}$ to positive roots, and $F_{(D-1)-\alpha}$ for negative roots.
- All (!) equations for the forms, and axions in the oxidized theories are given by

$$dF_{(n)\alpha'} = \frac{1}{2} \sum \eta_{l,\beta;m,\gamma} N_{\beta,\gamma} F_{(l)\beta'} \wedge F_{(m)\gamma'} \quad * = \left\{ \begin{array}{c} l + m = n + 1 \\ \alpha' + \beta' = \gamma' \end{array} \right. \quad (20)$$

The $N_{\beta,\gamma}$ are structure constants inherited from $G$. We have included sign factors $\eta_{l,\beta;m,\gamma} = \pm 1$, because the structure constants are antisymmetric, but the product of two forms is not necessarily, and the sign factors prevent terms from vanishing pairwise. The computation of $\eta_{l,\beta;m,\gamma}$ is straightforward, and there is freedom in the choice of signs [6]. For many purposes it is not necessary to know them explicitly.

- The remaining equations are the dilaton equation (18), and the Einstein equation. The terms on the right hand side of (18) and the matter fields to which the Einstein tensor couples are governed by the decomposition of $G$ in $SL(D-2)$ irreps.

It can be demonstrated [3] that this elegant and simple procedure, is exactly the inverse to the systematic dimensional reduction procedure as developed in [11, 9, 2]. Note that, whereas the equations (4), (7), (15) and (16) always involve a 1 form in the terms on the right side, we have made no such restriction in (20). Such terms are known to arise due to modified Bianchi identities, and Chern-Simons terms in the action.

5. Examples

Maximal supergravity in 3 dimensions gives rise to a sigma model on $E_{8(8)}/SO(16)$. The group $E_{8(8)}$ has an index 1 subgroup $SL(9)$, under which the 248-dimensional adjoint decomposes into the adjoint of $SL(9)$, a 3-form, and a 6-form. The oxidation recipe
Oxidation = group theory

gives rise to an 11 dimensional theory, with equations:
\[
dF(7) = \frac{1}{2} F(4) \wedge F(4) \quad dF(4) = 0
\quad (21)
\]
The first equation is a consequence of the Chern-Simons term of 11-d supergravity.

There are two non-equivalent embeddings of \( SL(8) \) into \( E_8(8) \); one results in an \( SL(8) \times \mathbb{R} \) subgroup, and leads to the bosonic sector of IIA supergravity; the other decomposition is \( SL(8) \times SL(2) \), giving the bosonic sector of IIB supergravity. All \( SL(7) \) embeddings are equivalent, and can be embedded in the \( SL(8) \) of IIA theory or the one for IIB theory; this is the statement of T-duality of IIA and IIB theory.

Decomposing \( E_8(8) \) into \( SL(8) \times SL(2) \) we find the adjoints of \( SL(8) \) and \( SL(2) \) (giving the graviton and a sigma model), an \( SL(2) \)-doublet of \( SL(8) \) 2-tensors, and an \( SL(8) \) 4-tensor, invariant under \( SL(2) \). Equation (6) and (20) result in
\[
*F(5)_0 = F(5)_0 \quad dF(5)_0 = F(3) - \frac{1}{\sqrt{2}} \wedge F(3) + \frac{1}{\sqrt{2}}
\quad (22)
\]
demonstrating that the formalism easily deals with self-dual tensors.

Our formalism covers a wealth of other theories, among which many that allow a supersymmetric extension. More examples, and a discussion on how the graphical language of Dynkin and Satake diagrams facilitates the analysis can be found in [6, 7].

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