Nonsingular potentials from excited state factorization of a quantum system with position-dependent mass

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Abstract
The modified factorization technique of a quantum system characterized by a position-dependent mass Hamiltonian is presented. It has been shown that the singular superpotential defined in terms of a mass function and an excited state wavefunction of a given position-dependent mass Hamiltonian can be used to construct non-singular isospectral Hamiltonians. The method has been illustrated with the help of a few examples.

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1. Introduction

The one-dimensional time-independent position-dependent mass Schrödinger equation (PDMSE) [1]

\[ H \psi(x) = \left[ \frac{1}{2m_0} \left( \hat{p} - \frac{1}{m(x)} \hat{p} \right) + V(x) \right] \psi = E \psi, \]  

\[ \hat{p} = -i\hbar \frac{d}{dx} \] being the momentum operator and \( m(x) \) the dimensionless form of mass function \( M(x) = m_0 m(x) \), has attracted a lot of attention during the past few years due to its applications in describing the dynamics of electrons in many condensed-matter systems [2]. The concept of position-dependent mass comes from the effective mass approximation [3] which is a useful tool for studying the motion of carrier electrons in pure crystals and also for the virtual-crystal approximation in the treatment of homogeneous alloys as well as in graded mixed semiconductors. This field has also arisen due to the impressive development in crystallographic growth techniques [4] which allow the production of a non-uniform semiconductor specimen with abrupt heterojunctions (see e.g. [5] for a review). In different areas of possible applications of low-dimensional structures as already mentioned, there is a need to have Hamiltonians with a predetermined energy spectrum. For example, in the quantum well profile optimization [6], isospectral potentials are generated through supersymmetric quantum mechanics. This is necessary because intersubband optical transitions in a quantum well may be grossly enhanced by achieving the resonance conditions, e.g. appropriate spacings.
between the most relevant states and also by tailoring the wavefunctions so that the matrix elements relevant for this particular effect are maximized [7]. Consequently, there has been a growing interest in obtaining exactly solvable PDM Hamiltonians through the usage of supersymmetric quantum mechanics (SUSYQM) [8–11], point canonical transformation [12–14], Lie algebra [15], the Heun equation [16] and the intertwining method [17, 18] (and references cited therein). However, despite being studied widely, the number of exactly solvable PDM Hamiltonians is still rather few.

Fairly recently, the standard factorization technique [19–21] has been modified [22] so that the excited state wavefunction of a given nonsingular constant mass Hamiltonian can be used to construct a nonsingular isospectral partner Hamiltonian. In general, the superpotential defined by an excited state wavefunction gives rise to the singular partner Hamiltonians with negative energy eigenvalues and usual degeneracy of the energy spectrum is lost [23, 24]. In [22], the nonuniqueness of the factorization of a constant mass Hamiltonian is used to make it possible to construct acceptable (nonsingular) isospectral Hamiltonians. This algorithm is independent of the choice of wavefunctions used to define the superpotential. Since exactly solvable PDM Hamiltonians are few in number, it should be interesting, therefore, to extend the modified factorization technique to handle cases with position-dependent mass. It may be noted that because of the presence of a non-constant mass the algorithm in the case of the PDMSE is different from the constant mass case and here our objective is to examine whether or not factorization based on excited states can be used to search for physically acceptable Hamiltonians within the context of the PDMSE.

In order to make this report self-contained, in section 2.1 we have briefly recalled the notion of the usual factorization technique of a quantum system with position-dependent mass. In section 2.2, we have generalized the usual factorization technique so that excited state wavefunctions can be used to construct new exactly solvable PDM Hamiltonians. Furthermore, as a useful application of the proposal, in section 3 we have considered two examples. We summarize in section 4. Throughout this paper we consider \( \hbar = 2m_0 = 1 \).

2. Factorization in the quantum system with position-dependent mass

2.1. Usual factorization

The minimal version of the PDM SUSYQM is based on the ground state wavefunction and the charge operators are represented by \( Q = \sigma_1 \sigma_- + \sigma_0 \sigma_+ \) and \( Q^\dagger = \sigma_0 \sigma_- \sigma_+ \), where \( \sigma_{\pm} = \sigma_1 \pm i\sigma_2 \) are the combinations of the Pauli matrices \( \sigma_1,2 \) and the associated two first-order linear operators \( A_0^\pm \) are given by [8]

\[
A_0^\pm = \pm \frac{1}{\sqrt{m}} \frac{d}{dx} - \left( \frac{1}{2} \frac{1}{\sqrt{m}} \right) \sigma_- + W_0, \tag{2}
\]

where ‘prime’ denotes the differentiation with respect to \( x \). The components of the supersymmetric Hamiltonian \( H = \text{diag}(H^-, H^+) \) can be factorized in terms of \( A_0^- \) and \( A_0^+ \) to obtain

\[
H_0^\pm = A_0^\mp A_0^\pm = -\frac{1}{m} \frac{d^2}{dx^2} + \frac{m'}{m''} \frac{d}{dx} + V_0^\mp(x), \tag{3}
\]

where without loss of generality, we have considered the zero ground state energy of \( H_0^- \), and the two partner potentials \( V_0^\pm \) can be written in terms of the superpotential \( W_0 \) and mass
function as
\[ V_0^±(x) = W_0^2 ± \left( \frac{W_0}{\sqrt{m}} \right)' \mp \left( 1 \mp \frac{1}{2} \right) \left[ \frac{W_0m'}{m^2} - \frac{1}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \right)' \right]. \]  
(4)

Clearly, \( V_0^+ = V_0^- + \frac{2W_0}{\sqrt{m}} - \frac{1}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \right)'' \). The ground state wavefunction \( \psi_0^- \) of the Hamiltonian \( H_0^- \) determines the superpotential \( W_0(x) = -\frac{\psi_0^-}{\sqrt{m}\psi_0^-} \),

which implies that for a physically acceptable mass function\(^1\) the superpotential and hence two partner potentials \( V_0^\pm \) are free from singularity (provided the given potential \( V_0^- \) is without any singularity). It is well established that when SUSY is unbroken, the spectrum of the two PDM partner Hamiltonians \( H^\pm \) is degenerate except for the lowest energy of \( H_0^- \). Moreover, the energy eigenvalues and normalized eigenfunctions of the SUSY partner Hamiltonians \( H_0^\pm \) are related by \[ E_n^{(0)+} = E_{n+1}^{(0)-}, \quad E_0^{(0)-} = 0 \]
\[ \psi_n^+ = (E_n^{(0)+} - iA_{n0}^+) \psi_{n+1}, \quad \psi_{n+1}^- = (E_n^{(0)+} - iA_{n0}^-) \psi_n, \quad n = 0, 1, 2, \ldots \]  
(6)

Summing up, for a well-behaved mass function, the usual factorization technique for the PDMSE allows us to construct nonsingular isospectral potentials of a given nonsingular potential using equations (5) and (4). It should be noted here that the factorization technique mentioned above is not unique and can be extended to use excited state wavefunctions. This generalization is mathematically straightforward but physically very nontrivial since it yields singular partner potentials. In the following we show that the excited state of a given position-dependent mass Hamiltonian can be used to construct nonsingular isospectral potentials.

2.2. Excited state factorization and its modification

In order to generalize the usual factorization technique to excited states we define the superpotential as
\[ W_n(x) = -\frac{\psi_n^-}{\sqrt{m}\psi_n^-}, \quad n > 0, \]  
(7)

where \( \psi_n^- \) is the \( n \)th excited state wavefunction of an exactly solvable nonsingular Hamiltonian \( H_0^- \). It is clear that for a well-behaved mass function the superpotential \( W_n \) is a singular function but for a suitably chosen singular mass function this becomes nonsingular. Since singular mass functions are physically absurd we cannot consider this type of mass function. Now subtracting the energy \( E_n^{(0)-} \) of the excited state from the Hamiltonian so that the resulting Hamiltonian can be factorized, we have the form of the two partner potentials
\[ V_n^-(x) = W_n^2 - \left( \frac{W_n}{\sqrt{m}} \right)' = V_0^- (x) - E_n^{(0)-} \]
\[ V_n^+(x) = V_n^- (x) + \frac{2W_n'}{\sqrt{m}} - \frac{1}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \right)'' \]  
(8)

The singular superpotential \( W_n \) contributes nothing new to \( V_n^- \) except a constant energy shift to \( V_0^- \). In this case, energy eigenvalues of the Hamiltonians \( H_n^- \) and \( H_n^+ \) are related by \( E_k^{(n)-} = E_k^{(0)-} - E_n^{(0)-}, k = 0, 1, 2, \ldots \) (\( k \) denotes the energy level and \( n \) refers to the \( n \)th

\(^1\) The mass function should be positive and without any singularity throughout its domain of definition.
eigenfunction of $H_n^-$ used in factorization). But the partner potential $V_{n}^+$ becomes singular at the node(s) of the wavefunction $\psi_n^-$. This singularity is responsible for both the destruction of degeneracy of the spectrum and creation of negative energy state(s). The singularity breaks the real axis into more than one disjoint intervals and imposes additional boundary conditions on the wavefunctions. In this case, the supersymmetry operators map square integrable functions to states outside the Hilbert space. As a result, some or all of the wavefunctions of the SUSY partner Hamiltonians may not belong to the same Hilbert space of square integrable functions and the usual proof of degeneracy between the excited states of the partner Hamiltonians does not hold [23, 24].

To overcome this, we require the modification of this excited state factorization technique which can be achieved by deforming the operators $A_{n}^\pm$ as

$$\tilde{A}_{n}^\pm = \pm \frac{1}{\sqrt{m}} \frac{d}{dx} \left( 1 \mp \frac{1}{2} \right) \left( \frac{1}{\sqrt{m}} \right)' + W_n + f_n,$$

where $W_n$ is given by equation (7) and the unknown function $f_n(x)$ will be evaluated shortly. Using this operators $\tilde{A}_{n}^\pm$ we first obtain

$$\tilde{H}_{n}^+ = \tilde{A}_{n}^+ \tilde{A}_{n}^- = -\frac{1}{m} \frac{d^2}{dx^2} + \frac{m'}{m^2} \frac{d}{dx} + \tilde{V}_{n}^+ (x),$$

where

$$\tilde{V}_{n}^+ = V_{n}^+ + \frac{f_n'}{\sqrt{m}} + \left( \frac{W_n}{\sqrt{m}} \right)' f_n + f_n^2.$$

At this point we assume

$$f_n' \sqrt{m} + \left( 2W_n + \frac{m'}{2m^2} \right) f_n + f_n^2 = \beta, \quad \beta \in \mathbb{R},$$

so that $\tilde{H}_{n}^+ = H_{n}^+ + \beta$ is solvable. This assumption also ensures the non-uniqueness of the factorization namely

$$A_{n}^+ A_{n}^- + \beta = \tilde{A}_{n}^+ \tilde{A}_{n}^-.$$

For $\beta = 0$, the Riccati equation (12) reduces to the Bernoulli equation whose solution is given by

$$f_n(x) = \frac{(\psi_n^-)^2}{\sqrt{m}(\lambda + \int (\psi_n^-)^2 dx)}, \quad \lambda \in \mathbb{R}.$$  

For nonzero $\beta$, the Riccati equation (12) is not always solvable but it can be transformed into the following second-order linear differential equation:

$$-\frac{1}{m} \psi''^+ + \frac{m'}{m^2} \psi'^+ + \left[ W_{n}^2 + \left( \frac{W_n}{\sqrt{m}} \right)' \right] \psi^+ = \beta \psi^+,$$

with the help of the following transformation:

$$f_n(x) = \frac{1}{\sqrt{m}} (\log \chi_n)', \quad \chi_n = e^{-\int \sqrt{m} \psi^+ dx}.$$  

Using relation (8) it is clear that equation (15) is (nearly) isospectral to the following solvable equation:

$$-\frac{1}{m} \psi''^- + \frac{m'}{m^2} \psi'^- + V_{0}^- (x) \psi^- = (E_{n}^{(0)} - \beta) \psi^-,$$

$^2 A_{n}^\pm$ are obtained from equation (2) after replacing $W_0$ by $W_n$. 


where the solutions \( \psi^\pm(x) \) corresponding to two equations (15) and (17) are related by
\[
\psi^+ \sim \tilde{A}_n^+ \psi^-.
\] (18)
By changing the order of operations between \( \tilde{A}_n^+ \) and \( \tilde{A}_n^- \) and using relations (8) and (12) we obtain the partner Hamiltonian \( \tilde{H}_n^- \) of \( \tilde{H}_n^+ \) as
\[
\tilde{H}_n^- = \tilde{A}_n^- \tilde{A}_n^+ = -\frac{d^2}{m \, dx^2} + \frac{m'}{m} \frac{d}{dx} + \tilde{V}_n^-(x),
\] (19)
where
\[
\tilde{V}_n^- = V_n^- - \frac{2f_n^\prime}{\sqrt{m}} + \beta = V_0^- - \frac{2f_n^\prime}{\sqrt{m}} - E_n^{(0)-} + \beta.
\] (20)
At this point a few comments are worth mentioning. For a given nonsingular potential \( V_0^- \) and a mass function the new potential \( \tilde{V}_n^- = V_n^- + \beta \) is singular but the potential \( \tilde{V}_n^- \) has no singularity if \( f_n \) is nonsingular in which case the corresponding spectrum of the two isospectral Hamiltonians \( \tilde{H}_n^- \) and \( H_n^- \) is related by
\[
\tilde{E}_k^{(n)-} = E_k^{(n)-} + \beta, \quad k = 0, 1, 2, \ldots
\] (21)
For \( \beta = 0 \), the singularity of \( f_n \) given in (14) can be controlled by the arbitrary constant \( \lambda \) and in all other cases it depends on the nature of solution of equation (17). Moreover, the eigenfunctions of the two nonsingular Hamiltonians \( H_n^- \) and \( \tilde{H}_n^- \) are related by
\[
\tilde{\psi}_k^- \sim \tilde{A}_k^- A_k^+ \psi_k^- \quad \text{and} \quad \psi_k^- \sim A_k^- \tilde{A}_k^+ \tilde{\psi}_k^-
\] (22)
except for the zero energy state of \( \tilde{H}_n^- \). The proof of these results is as follows: we start with
\[
\tilde{H}_n^- (\tilde{A}_n^- A_n^+ \psi_k^-) = \tilde{A}_n^- (\tilde{A}_n^+ \tilde{A}_n^-) A_n^+ \psi_k^-
= \tilde{A}_n^- (A_n^+ A_n^- + \beta) A_n^+ \psi_k^-
= \tilde{A}_n^- A_n^+ (A_n^+ A_n^- + \beta) \psi_k^- + \beta \tilde{A}_n^- A_n^+ \psi_k^-
= \tilde{A}_n^- A_n^+ (H_n^- \psi_k^-) + \beta \tilde{A}_n^- A_n^+ \psi_k^- = (E_k^{(n)-} + \beta) \tilde{A}_n^- A_n^+ \psi_k^-.
\] (23)
In the second step we have used the non-uniqueness of the factorization of \( H_n^+ \) and in the last step we have used the PDMSHE \( H_n^- \tilde{\psi}_k^- = E_k^{(n)-} \tilde{\psi}_k^- \). Hence, if \( \tilde{\psi}_k^- \) is an eigenfunction of \( H_n^- \) corresponding to the eigenvalue \( E_k^{(n)-} \), then \( \tilde{A}_n^- A_n^+ \tilde{\psi}_k^- \) is an eigenfunction of \( H_n^- \) with eigenvalue \( E_k^{(n)-} + \beta \). The second relation of equation (22) can be proved in an analogous way. The wavefunction \( \tilde{\psi}_k^- \) (say) corresponding to zero energy of the Hamiltonian \( \tilde{H}_n^- \) can be obtained by directly solving \( \tilde{A}_n^- \tilde{A}_n^+ \tilde{\psi}_k^- = 0 \).

Summing up, the excited state wavefunction of a given position-dependent mass Hamiltonian can be used to construct a nonsingular partner Hamiltonian \( \tilde{H}_n^- \) using equation (20) provided the function \( f_n \) is nonsingular. This has been made possible in two-step factorization. In the first step, an intermediate singular Hamiltonian \( \tilde{H}_n^+ \) has been created and in the second step non-uniqueness of the factorization technique has been used to remove the singularities. The operator \( -\tilde{A}_n^- A_n^+ \) is similar to the second-order reducible intertwining operator \( \mathcal{L} = \frac{1}{m(x)} \frac{d^2}{dx^2} + \tilde{\eta}(x) \frac{d}{dx} + \tilde{\gamma}(x) \) mentioned in [17] which intertwines two Hamiltonians \( H_n^- \) and \( \tilde{H}_n^- \):
\[
\tilde{H}_n^- \tilde{A}_n^- A_n^+ = \tilde{A}_n^- A_n^+ H_n^-,
\] (24)
if one identifies \( -\tilde{\eta}(x) = \frac{\tilde{m}_0}{m^2} + \frac{\tilde{m}_1}{m} \) and \( -\tilde{\gamma}(x) = V_n^- + f_n W_n \). In this regard, the present algorithm is equivalent to the confluent second-order SUSY transformation [25] in the constant mass case but the relation between the two isospectral partner potentials is different in the present approach due to the position dependence of the mass function. Moreover, the energy
levels of the two nonsingular isospectral partner Hamiltonians $H^-_n$ and $\tilde{H}^-_n$ are degenerate. In figure 1, we have shown schematically the combined action of the operators $\tilde{A}^-_n A^+_n$ and $A^-_n \tilde{A}^+_n$ on the wavefunction of two isospectral partner Hamiltonians $H^-_n$ and $\tilde{H}^-_n$, respectively. The operators $A^+_n$ or $\tilde{A}^+_n$ destroy a node but $A^-_n$ or $\tilde{A}^-_n$ create the same in the eigenfunctions so that the overall number of nodes remains the same.

3. Illustration

It is worth mentioning here that the proposed technique mentioned in the preceding section is more general and can be applied to any solvable PDM Hamiltonian to obtain some acceptable isospectral Hamiltonians which might be useful in various fields of condensed matter physics. For the purposes of illustration we are considering here two exactly solvable position-dependent mass Hamiltonians. In example 1, we have considered $\beta = 0$ while non-zero $\beta$ has been considered in example 2.

Example 1 ($\beta = 0$). We consider the following mass function and potential possessing harmonic-oscillator-like spectra [26]:

$$m(x) = \frac{1}{1 + \alpha^2 x^2}$$

$$V^-(x) = \left(\sinh^{-1}(\alpha x)\right)^2 - \frac{\alpha^2}{4} \left(\frac{2 + \alpha^2 x^2}{1 + \alpha^2 x^2}\right), \quad x \in (-\infty, \infty).$$

The bound state solution is given by

$$\psi_k = \sqrt{\frac{1}{2^{2k}k!}} \left(\frac{1}{\pi}\right)^{1/4} e^{-\frac{1}{2} \left(\sinh^{-1}(\alpha x)\right)^2} \left(\sinh^{-1}(\alpha x)\right)^{1/4} H_k \left(\sinh^{-1}(\alpha x)\right)$$

$$E^{(0)}_k = 2k + 1, \quad k = 0, 1, 2, \ldots,$$
where $H_1(x)$ denotes the Hermite polynomial. Now we consider here the first excited state 

$$
\psi_1^- = 2 \sinh^{-1} x \left( \frac{1}{2} + \frac{x^2}{1 + x^2} \right) - 3 
$$

respectively. The energy eigenvalues of the Hamiltonian $\tilde{H}_1^-$ are obtained using equations (8), (14) and (20) as 

$$
V_1^- = (\sinh^{-1}(x))^2 - \frac{1}{4} \left( \frac{2 + x^2}{1 + x^2} \right) - 3 
$$

The normalized first excited state wavefunction corresponding to the zero energy of $\tilde{H}_1^-$ is evaluated by solving $\hat{A}_1^+ \psi_1^- = 0$ as 

$$
\psi_1^- = \frac{\pi^{\frac{1}{2}} \sqrt{8} \lambda^2 - 2 e^{\sinh^{-1} x}}{(1 + x^2)^{\frac{1}{2}} [2 \sqrt{\pi} \lambda e^{\sinh^{-1} x} - 2 \sinh^{-1} x + \sqrt{\pi} e^{\sinh^{-1} x} \text{Erf}(\sinh^{-1} x)]}. 
$$

Hence, the new potential $\tilde{V}_1^{(1)-}$ is nonsingular for $\lambda > 1/2$. The lowest energy wavefunction corresponding to the negative energy $-2$ is obtained through $\hat{A}_1^+ \psi_1^- = 0$ as 

$$
\psi_0^- = \frac{\pi^{\frac{1}{2}} [2 \lambda + \text{Erf}(\sinh^{-1} x)] e^{\sinh^{-1} x}}{(1 + x^2)^{\frac{1}{2}} [2 \sqrt{\pi} \lambda e^{\sinh^{-1} x} - 2 \sinh^{-1} x + \sqrt{\pi} \text{Erf}(\sinh^{-1} x) e^{\sinh^{-1} x}]} 
$$

In figure 2(a), we have plotted the potential $V_1^- (x)$, its isospectral partner $\tilde{V}_1^-$ for two parameter values $\lambda = 1, .7$ and the mass function. In figure 2(b), lowest and first excited state wavefunctions of the Hamiltonian $\tilde{H}_1^-$ for $\lambda = 1$ have been plotted.

**Example 2** ($\beta \neq 0$). Here we consider the following mass function and the potential:

$$
m(x) = \frac{1}{4} \text{sech}^2 \left( \frac{x}{2} \right) 
$$

$$
V_0(x) = \frac{(a + b - c)^2 - 1}{4} e^x + \frac{c(c - 2)}{4} e^{-x}, \quad c > \frac{1}{2}, \quad a + b - c + \frac{1}{2} > 0, \quad x \in (-\infty, \infty), 
$$

**Figure 2.** (a) Plots of the given potentials $V_1^-$ (solid line) and $\tilde{V}_1^-$ for $\lambda = 1$ (dashed line), $\lambda = .7$ (dotted line) and the mass function $m(x)$. (b) The lowest (solid line) and first excited state (dashed line) wavefunctions of the Hamiltonian $\tilde{H}_1^-$ for $\lambda = 1$. 
for which one linearly independent solution of equation (17) is known to be (see the appendix of [17])

\[ \psi^{-}(x) = C_{1}e^{x^{2}/2}(1 + e^{x})^{2F_{1}}(\frac{a + b + P}{2}, -a - b + 2 \epsilon + P, c, e^{-x}) , \]  

(35)

where

\[ P^{2} = (a + b)^{2} - 2c(a + b - c + 1) + 4(E_{n}^{(0)} - \beta) \]

\[ E_{n}^{(0)} = n^{2} + n(a + b) + \frac{c(a + b - c + 1)}{2} . \]

(36)

The bound state wavefunctions of the Hamiltonian \( H_{0}^{-} \) are given [17], in terms of the Jacobi polynomial \( P_{n}^{(a,b)}(x) \), as

\[ \psi_{n}^{-} \sim \frac{e^{x^{2}/2}}{(1 + e^{x})^{(a+b+1)/2}P_{n}^{(a-1,a+b-c)}(\frac{1 - e^{x}}{1 + e^{x}})} , \quad n = 0, 1, 2 \ldots \]  

(37)

Now we consider the first excited state, i.e. \( n = 1 \), for which the function \( f_{n} \) can be obtained using equations (16) and (18) as

\[ f_{n} = \frac{1}{\sqrt{m}} \frac{d}{dx} \left( \log[A_{1}^{+}\psi^{-}(x)] \right) \]

\[ + \frac{2c}{4} \left( \frac{a + b + 3}{(a + b - c + 1)e^{x} - c} + \frac{a + b - 3}{1 + e^{x}} + c - a - b - 1 \right) + \sinh \left( \frac{x}{2} \right) , \]  

(38)

where \( A_{1}^{+} = \frac{1}{\sqrt{m}} \left( \frac{d}{dx} - \frac{\psi_{1}^{-}}{\psi_{1}^{-}} \right) \), \( m(x) \), \( \psi^{-} \) and \( \psi_{1}^{-} \) are given by equations (33), (35) and (37), respectively. In order to obtain a new nonsingular potential \( \tilde{V}_{1}^{-} \) we have to choose \( a, b, c \) such that \( f_{n} \) has no singularity. The analytical expression of \( f_{n} \) is too involved so it is very difficult to find the range of parameter values for which \( f_{n} \) is nonsingular. In figure 3(a), we have plotted the potential \( V_{1}^{-}(x) = V_{0}^{-} - E_{1}^{(0)} \) and its nonsingular partner \( \tilde{V}_{1}^{-} = \tilde{V}_{1}^{-} - 2f_{n}/\sqrt{m} - E_{1}^{(0)} + \beta \) for particular values of \( a = 1, b = 5, c = 4 \) and \( \beta = 1 \). For the same set of parameter values, the isospectral partner potential \( \tilde{V}_{2}^{-} \) has been drawn in figure 3(b) by considering second excited state factorization, i.e. \( n = 2 \). The energy eigenvalues of the new Hamiltonians \( H_{1}^{-} \) and \( H_{2}^{-} \) are given by \( \tilde{E}_{2}^{(1)} = k^{2} + 6k - 6 \) and \( \tilde{E}_{2}^{(2)} = k^{2} + 6k - 15 \), respectively. In both cases the wavefunctions can be obtained using equations (22) and (37).
4. Summary

To conclude, we have generalized the modified factorization technique of a Schrödinger Hamiltonian suggested by Berger and Ussembayev [22] to a quantum system characterized by a position-dependent mass Hamiltonian. This generalization is done in such a way that a given mass function and the excited state wavefunction of a PDM Hamiltonian can be used to generate new solvable physical Hamiltonians. The method discussed here is applied to a number of exactly solvable PDM Hamiltonians which leads to nontrivial isospectral Hamiltonians. In the constant mass limit of the present algorithm, all the recently published results [22] on modified factorization can be recovered.

It would be interesting to study the modified factorization technique for a quantum system characterized by broken supersymmetry.

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