PHOTON SPLITTING AND PHOTON-PLASMON
INTERACTION IN A DENSE MEDIUM

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Three-photon vertex in a dense degenerated plasma is calculated. It is discovered the polarization tensor has the longitudinal part which makes possible an interaction between transverse and longitudinal modes in the medium. Using dispersion relations it is shown that the only kinematically permitted type of the photon splitting is the decay of the transverse photon to two plasmons.

1. Introduction

The photon splitting is the process with minimal rank of nonlinearity in the electromagnetic interactions. This kind of reaction is to be realized in various astrophysical objects (white dwarves, neutron stars), the early Universe and in the heavy ion collision experiments. In all these cases Furry’s theorem is violated because of the presence of the environments such as external fields or dense media. The photon splitting due to a magnetic field has been rather intensively investigated by many authors [1]. This process was also considered in the rest plasma [2] and the uniformly moving medium [3]. In the latter article the three-photon vertex has been calculated in one-loop approximation for the most general case but the final expressions represented in terms of complicate integrals, that makes difficult the detailed analysis of the corresponding amplitude. Besides, there were some calculation peculiarities that have not been taken

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into account and therefore must be pointed out. First it is the use of Feyn-
man’s parametrization. As it was proved by Weldon [4], the employing the 
parametrization at finite temperature and chemical potential could lead 
to errors for some ratios of the momentum components $p_0/p$. The second 
problem arises because of the covariant decomposition the authors elabo-
rated. This decomposition allows to investigate some general properties of 
the polarization tensor in moving medium but the basis chosen automatic-
ically excludes a longitudinal part which tends not to be zero in general 
case, as it will be shown below.

Taking all this in mind, we carry out another investigation of the three-
photon interaction in dense cold medium. In Sect.2 we present straightfor-
ward calculation (without Feynman’s parametrization) of the three-photon 
vertex in one-loop approximation for arbitrary relations between the mo-
momenta and the frequencies of the photons. We do not use the covariant 
decomposition and perform the calculations separately for the temporal, 
the spatial and the mixed components of the polarization tensor. For the 
most important range of momenta and frequency for the high density limit 
($\mu \gg m, \omega, |\vec{k}|$) the final results have been obtained in terms of elementary 
functions. Therefore the analytical properties of the vertex can easily be 
analysed. One of the most interesting features discovered is the longitu-
dinal part of the polarization tensor which becomes zero when $\omega = |\vec{k}|$.
Obviously, the existence of the longitudinal part makes it possible the in-
teraction between the transversal excitations (photons) in medium and the 
longitudinal ones (plasmons) via the three-photon vertex.

In Sect.3 we consider the photon splitting process for real particles. 
To investigate that the corresponding dispersion relations were obtained. 
Basing on these relations the kinematic analysis of the splitting has been 
done and it was shown that the only type of the permitted process is the 
splitting of photon to two plasmons. Section 4 contains the conclusions. 
In Appendix the expressions for the component of the polarization tensor 
are presented.

3. Three-photon vertex in dense medium
The neutron stars and cores of white dwarves are the examples of degenerate QED plasma where $T \ll \mu$ and therefore Fermi distribution function can be well approximated by step function. For such environment the three-photon tensor can be expressed in Euclidean space as follows

$$\Pi_{\mu\nu\lambda}(k, k', k'') \delta(k + k' + k'') = \Pi^{(1)}_{\mu\nu\lambda} + \Pi^{(2)}_{\mu\nu\lambda} =$$

$$= \delta(k + k' + k'') \frac{e^3}{(2\pi)^4} Tr \int d^4 p [\gamma_\mu G(p^* + k) \gamma_\nu G(p^* - k') \gamma_\lambda G(p^*) +$$

$$+ \gamma_\mu G(p^*) \gamma_\lambda G(p^* + k') \gamma_\nu G(p^* - k)],$$

where $G(p^*)$ - fermion Green function

$$G(p^*) = \frac{-ip^* + m}{p^2 + m^2};$$

and the notations are introduced

$$p^*_\mu = \begin{cases} p_\mu, & \mu = 1, 2, 3 \\ p_4 - i\mu, & \mu = 4. \end{cases}$$

Corresponding Feynman diagrams are pictured in Fig.1.

Because of the reasons mentioned in the introduction we in our calculations will avoid either the Feynman parametrization and the covariant decomposition. Instead, we will perform straightforward integration for all the tensor components.

While integrating over spatial part of the momentum space one faces the integrals like

$$\int \vec{p} \, f(p, k, k') \, d^3 p,$$
where \( f(p, k, k') \) are certain scalar functions.

Here we represent \( \vec{p} \) as a sum of two components that are transversal and parallel to the plane created by \( \vec{k}, \vec{k}', \vec{k}'' \)

\[
\vec{p} = p_{\perp} + p_{\parallel}
\]  (5)

Taking into account the symmetry properties of \( f(p, k, k') \) one can easily show that only parallel component \( p_{\parallel} \) contributes to integrals (4). So for further calculations we need basis consisted only of two independent vectors within reaction plane. We chose them as

\[
\vec{a} = \vec{k} + \vec{k}'
\]

\[
\vec{b} = - (\vec{k} \times \vec{k}') \times (\vec{k} + \vec{k}') = \vec{k}((\vec{k} \vec{k}') + \vec{k}'^2) - \vec{k}'((\vec{k} \vec{k}') + \vec{k}^2)
\]  (7)

Expanding \( p_{\parallel} \) over the normalized basis gives

\[
p_{\parallel} = \frac{a}{\vec{a}^2}(p_{\parallel} \vec{a}) + \frac{b}{\vec{b}^2}(p_{\parallel} \vec{b}) = \frac{1}{s^2}[(p_{\parallel} \vec{k})\vec{Q}(k, k') + (p_{\parallel} \vec{k}')\vec{Q}(k', k)],
\]  (8)

where \( s \equiv |\vec{k} \times \vec{k}'| \) and

\[
\vec{Q}(k, k') = \vec{k}k'^2 - \vec{k}'(\vec{k} \vec{k}')
\]

\[
\vec{Q}(k', k) = \vec{k}'k^2 - \vec{k}(\vec{k} \vec{k}').
\]  (9)

The integration over ”hard” momenta \( (\omega, |\vec{k}| \gg \mu) \) makes small contribution \( \sim \mu \frac{\mu}{|\vec{k}|} \) which can be neglected. Integrating over the region \( (\omega, |\vec{k}| \ll \mu) \) we obtain final expressions for high density limit \( (\mu >> m, \omega, |\vec{k}|) \). These expressions are bulky and placed in Appendix.

Having the results for the polarization tensor in analytical form, one can see that its components are proportional to chemical potential. It means that dense medium may amplify processes corresponding to the three-photon interaction.

Next we can note, all the components contain \( \theta \)-function with four-momentum squared in its argument. Such a dependence restricts the participation in three-photon interaction in dense medium admitting only time-like excitations.
One of the most important features of the three-photon vertex can be discovered if to calculate the contraction $k_{\mu}\Pi_{\mu\nu\lambda}$. One can convinced that

$$k_{\mu}\Pi_{\mu\nu\lambda} \sim k^2. \quad (11)$$

So, the polarization tensor has a non-zero longitudinal part which becomes zero on the mass-shell of free photons. As for physical meaning of this result, one can say the existence of the longitudinal part makes possible an interaction between transverse (photons) and longitudinal excitations (plasmons) via tree-photon vertex in dense medium.

3. Dispersion relations and photon-plasmon interaction

All the results in previous part has been made for arbitrary relations between momenta and frequencies of external photons. For studying the processes with the real particles in medium we need the corresponding dispersion relations. This type of calculation has been made by several authors for different cases [5]-[7].

The relations mostly match our purposes have been obtained in Ref. [6] and in our notations look as follows

$$\omega^2 - \vec{k}^2 = \text{Re} \Pi(\omega, \vec{k}), \quad (12)$$

where for transverse excitations we have

$$\Pi_T(\omega) = \frac{e^2 k_F^2}{2\pi^2} \frac{\mu \omega}{\mu} \left[ \left( \frac{\mu \omega}{k_F |\vec{k}|} \right)^2 + \frac{1}{2} \frac{\mu \omega}{k_F |\vec{k}|} \left( 1 - \left( \frac{\mu \omega}{k_F |\vec{k}|} \right)^2 \right) \ln \frac{\mu \omega + k_F |\vec{k}|}{\mu \omega - k_F |\vec{k}|} \right], \quad (13)$$

and for longitudinal ones

$$\Pi_L(\omega, \vec{k}) = \frac{e^2}{\pi^2} k_F \mu \left( 1 - \frac{\omega^2}{k^2} \right) \left( 1 - \frac{1}{2} \frac{\mu \omega}{k_F |\vec{k}|} \ln \frac{\mu \omega + k_F |\vec{k}|}{\mu \omega - k_F |\vec{k}|} \right). \quad (14)$$

In these formulae $\mu = \sqrt{m^2 + k_F^2}$.

For high density limit ($\mu \gg m, \omega, |\vec{k}|$) the expressions (15), (14) give

$$\Pi_T(\omega, \vec{k}) = \frac{e^2}{2\pi^2} \mu^2 \left[ \left( \frac{\omega}{|\vec{k}|} \right)^2 + \frac{1}{2} \frac{\omega}{|\vec{k}|} \left( 1 - \left( \frac{\omega}{|\vec{k}|} \right)^2 \right) \ln \frac{\omega + |\vec{k}|}{\omega - |\vec{k}|} \right], \quad (15)$$

$$\Pi_L(\omega, \vec{k}) = \frac{e^2}{\pi^2} \mu^2 \left( 1 - \frac{\omega^2}{k^2} \right) \left( 1 - \frac{1}{2} \frac{\omega}{|\vec{k}|} \ln \frac{\omega + |\vec{k}|}{\omega - |\vec{k}|} \right). \quad (15)$$
Substituting these in Eq. (12) we obtain dispersion relations for both modes:

\[ f_T(\omega, \vec{k}) = \frac{\omega_0^2}{k^2} \left( \lambda^2 + \frac{\lambda}{2} \left( 1 - \lambda^2 \right) \ln \left( \frac{\lambda + 1}{\lambda - 1} \right) \right) - \lambda^2 + 1 = 0, \]

\[ f_L(\omega, \vec{k}) = \frac{2\omega_0^2}{k^2} \left( \frac{\lambda}{2} \ln \left[ \frac{\lambda + 1}{\lambda - 1} \right] - 1 \right) - 1 = 0, \]

where \( \lambda = \frac{\omega}{|\vec{k}|}, \omega_0 = \frac{e}{\pi \mu}. \)

The dispersion relations obtained are plotted in Fig. 2, where the solid line represents the plasmon mode and the dashed line shows the photon mode. As it is seen from the plot, both excitations are time-like and therefore can interact via the three-photon vertex.

Having the dispersion relations we are able to analyse the kinematics of splitting of the real photon in dense media. Let us consider symmetric decay of photon with frequency \( \omega \) and momentum \( \vec{k} \) to two photons with frequencies \( \omega/2 \), angle between their momenta \( \alpha \) and modulus of momenta...
For this case we have

\[
\begin{align*}
&\left\{ \begin{array}{l}
f_T(\omega, |\vec{k}|) = 0, \\
f_T\left(\frac{\omega}{2}, \frac{|\vec{k}|}{2 \cos \frac{\alpha}{2}} \right) = 0.
\end{array} \right.
\end{align*}
\]

Solving this system numerically we find it has no solutions. Therefore, the photon splitting to two photons is kinematically forbidden.

Making the similar consideration for other cases we obtain that the only permitted by kinematics process is the decay of the photon to two plasmons. The solving of the corresponding equations

\[
\begin{align*}
&\left\{ \begin{array}{l}
f_T(\omega, |\vec{k}|) = 0, \\
f_L\left(\frac{\omega}{2}, \frac{|\vec{k}|}{2 \cos \frac{\alpha}{2}} \right) = 0
\end{array} \right.
\end{align*}
\]

gives two following results.

First, the photon splitting becomes possible if only its energy exceeds the minimal value \(\approx 2.7\omega_0\). When the energy of the initial photon exactly equals to minimal value the angle between the momenta of plasmons is zero.

Besides, if energy of photon becomes greater, the angle increases but its value limited approximately by \(11^\circ\).

4. Conclusions

In this paper we have calculated the tree-photon vertex in one-loop approximation for dense degenerated medium. The calculations has been carried out without Feynman’s parametrization and the covariant decomposition. The results are represented in terms of elementary functions that allows the detailed analysis of the analytical properties of the polarization tensor.

The most important property discovered is the existence of the longitudinal part of the vertex. This part makes possible the interaction between photons and plasmons in dense medium.

Basing on dispersion relations it was shown that all the types of the three-photon interaction of real particles are kinematically forbidden but
the splitting of the photon to two plasmons. This process becomes possible if energy of the initial photon is larger than a certain minimal value.

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Appendix

In this appendix we present the final expressions for the components of three-photon polarization tensor

\[
\Pi_{444} = -\frac{ie^3\mu}{2\pi} \left\{ \frac{J_1(k, k')}{|k||k'|} (k^2 + k'^2 + 4\omega\omega') + \right.
\]
\[
+ \frac{J_2(k, k'')}{|k||k''|} (k^2 + k''^2 + 4\omega\omega'') + \right.
\]
\[
+ \frac{J_3(k', k'')}{|k'||k''|} (k'^2 + k''^2 + 4\omega'\omega'') + \right.
\]
\[
+ 2\omega I(k) \frac{2\omega' I(k')}{|k|} + 2\omega'' I(k'' \frac{2\omega'' I(k'')}{|k''|} \right\},
\]

\[
\Pi_{i44} = -\frac{e^3\mu}{2\pi} \left\{ \frac{J_1(k, k')}{|k||k'|} [2k_i\omega' + \right.
\]
\[
+ \frac{(k'^2 + 2\omega\omega')}{s^2} (\omega Q_i(k, k') + \omega' Q_i(k', k)) \right\} + \right.
\]
\[
+ \frac{J_2(k, k'')}{|k||k''|} [2k_i\omega'' + \right.
\]
\[
+ \frac{(k''^2 + 2\omega\omega'')}{s^2} (\omega Q_i(k, k'') + \omega'' Q_i(k'', k)) \right\} + \right.
\]
\[
+ \frac{J_3(k', k'')}{|k'||k''|} [2k'\omega'' + 2k''\omega'] + \frac{2k_i I(k)}{|k|} + \right.
\]
\[
+ \frac{(k'^2 + 2\omega\omega')}{s^2} I(k') \frac{Q_i(k, k')}{|k'|} + \frac{I(k)}{|k|} Q_i(k', k) \right\} + \right.
\]
\[
\Pi_{i j k} = \frac{i e^3 \mu}{2 \pi} \left\{ \frac{J_1(k', k'')}{|k||k'|} \left[ \frac{2 k_i k'_j - \delta_{i j} k^2}{s^2} \right] (\omega Q_j(k', k') + \omega' Q_j(k', k)) + \frac{(2 k_i k'_j - \delta_{i j} k^2)}{s^2} (\omega Q_i(k', k') + \omega' Q_i(k', k)) \right\} + \frac{J_2(k, k'')}{|k||k'|} \left[ \frac{2 k_j k''_j - \delta_{j j} k^2}{s^2} \right] (\omega Q_i(k', k''') + \omega' Q_i(k', k''') + \omega'' Q_i(k', k'')) + \frac{J_1(k, k'')}{|k||k'|} \left[ \frac{2 k_i k''_j - \delta_{i j} k^2}{s^2} \right] (\omega Q_j(k', k'') + \omega' Q_j(k', k'')) + \frac{J_1(k, k')}{|k|} \left[ \frac{2 k_i k''_j - \delta_{i j} k^2}{s^2} \right] (\omega Q_j(k', k) + \omega' Q_j(k', k)) + \frac{J_2(k, k'')}{|k||k'|} \left[ \frac{2 k_j k''_j - \delta_{j j} k^2}{s^2} \right] (\omega Q_i(k', k'') + \omega' Q_i(k', k'') + \omega'' Q_i(k', k'')) + \frac{J_2(k', k'')}{|k'| |k''|} \left[ \frac{2 k_i k''_j - \delta_{i j} k^2}{s^2} \right] (\omega Q_j(k', k''') + \omega' Q_j(k', k''')) + \frac{J_1(k', k'')}{|k'| |k''|} \left[ \frac{2 k_i k''_j - \delta_{i j} k^2}{s^2} \right] (\omega Q_j(k', k'') + \omega' Q_j(k', k'') + \omega'' Q_j(k', k'') + \omega'' Q_j(k', k'')) + \right\}
\]
\[ \frac{(2k''_i k'_j - \delta_{ij}k'^2)}{s^2} (\omega'Q_{k}(k', k'') + \omega''Q_{k}(k'', k')) - \]

\[ \frac{2k_i I(k)}{|k|} \delta_{jk} - \frac{2k'_j I(k')}{|k'|} \delta_{ik} - \frac{2k''_k I(k'')}{|k''|} \delta_{ij} + \]

\[ \frac{(2k_i k'_j - \delta_{ij}k'^2)}{s^2} \left[ I(k') \frac{Q_j(k, k')}{|k'|} + \frac{I(k)}{|k|} Q_j(k, k) + \right. \]

\[ \left. \frac{(2k_i k''_j - \delta_{ij}k'^2)}{s^2} \left( I(k') \frac{Q_i(k, k')}{|k'|} + \frac{I(k)}{|k|} Q_i(k', k) + \right. \]

\[ \left. \frac{(2k_j k''_k - \delta_{ij}k'^2)}{s^2} \left( I(k'') \frac{Q_i(k', k'')}{|k''|} + \frac{I(k)}{|k|} Q_i(k'', k) + \right. \]

\[ \left. \frac{(2k_j k''_k - \delta_{ij}k'^2)}{s^2} \left( I(k'') \frac{Q_j(k', k'')}{|k''|} + \frac{I(k)}{|k|} Q_j(k'', k') \right) \right\}, \]

where \( s \equiv |\vec{k} \times \vec{k}'|, k^2 = \omega^2 - \vec{k} \cdot \vec{k}'. \)

The functions \( I, J_1, J_2, J_3 \) defined as follows

\[ I(k) = \theta(k^2) \ln \left[ \frac{\lambda + 1}{\lambda - 1} \right], \quad (20) \]

\[ J_1(k, k') = \frac{\theta(k^2) \theta(k'^2)}{B(k, k')} \left\{ 2 \ln \left[ \frac{\lambda'' + 1}{\lambda'' - 1} \right] + \right. \]

\[ + \ln \left[ \frac{\lambda - \cos \beta - \frac{1}{|k'|} \frac{1}{B(k, k')} ((1 - \lambda \cos \beta)(1 + \lambda'') - (\lambda - \cos \beta)^2)}{\lambda + \cos \beta - \frac{1}{|k'|} \frac{1}{B(k, k')} ((1 + \lambda \cos \beta)(1 - \lambda'') - (\lambda + \cos \beta)^2)} \right] + \]

\[ + \ln \left[ \frac{\lambda' - \cos \alpha - \frac{1}{|k'|} \frac{1}{B(k, k')} ((1 - \lambda' \cos \alpha)(1 + \lambda'') - (\lambda' - \cos \alpha)^2)}{\lambda' + \cos \alpha - \frac{1}{|k'|} \frac{1}{B(k, k')} ((1 + \lambda' \cos \alpha)(1 - \lambda'') - (\lambda' + \cos \alpha)^2)} \right] \right\}, \]

where \( B(k, k') = \sqrt{\lambda^2 + \lambda'^2 - 2\lambda \lambda' \cos \gamma - \sin^2 \gamma}, \)

\( \lambda = \frac{\omega}{|k|}, \lambda' = \frac{\omega'}{|k'|}, \lambda'' = \frac{\omega''}{|k''|}, \)

\( \alpha = \frac{k' k''}{|k|}, \beta = \frac{k' k''}{|k'|}, \gamma = \frac{k' k''}{|k''|}. \)

The functions \( J_2 \) and \( J_3 \) can be derived from \( J_1 \) by following correspondent substitutions in its argument \((k' \leftrightarrow k'', \gamma \leftrightarrow \beta), (k \leftrightarrow k'', \gamma \leftrightarrow \alpha)\).
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