AN EXPLICIT CORRESPONDENCE OF MODULAR CURVES

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Abstract. In this paper, we recall an alternative proof of Merel’s conjecture which asserts that a certain explicit correspondence gives the isogeny relation between the Jacobians associated to the normalizer of split and non-split Cartan subgroups. This alternative proof does not require extensive representation theory and can be formulated in terms of certain finite geometries modulo $\ell$.

Secondly, we generalize these arguments to exhibit an explicit correspondence which gives the isogeny relation between the Jacobians associated to split and non-split Cartan subgroups. An interesting feature is that the required explicit correspondence is considerably more complicated but can expressed as a certain linear combination of double coset operators whose coefficients we are able to make explicit.

1. Introduction

Modular curves, which are coarse moduli spaces for elliptic curves with prescribed level structure, appear in the study of Galois torsion structures on elliptic curves.

A well-known example is Mazur’s Theorem [5] which states that there are no rational $\ell$-isogenies between rational elliptic curves if $\ell > 163$. This result is proven by showing the modular curve $X_0(\ell)$ has no non-cuspidal rational point if $\ell > 163$. Mazur’s method is based on descent on the Jacobian of $X_0(\ell)$, but because of the rich arithmetic structure of these curves, the method is more powerful and efficient.

Let $\ell$ be a prime, and $\mathbb{Z}/\ell\mathbb{Z} = \mathbb{F}_\ell$ be a finite field of cardinality $\ell$.

For a subgroup $H$ of $\text{GL}_2(\mathbb{F}_\ell)$ which contains $-1$, it is possible to associate a modular curve $X_H := X/H$. In the case when $H$ is a non-split Cartan subgroup $C'$ or its normalizer $N'$, it is relevant from the point of view of Mazur’s method to understand the Jacobian of $X_H$. In [2], it was proven using the trace formula that $X_{N'}$ and $X_{C'}$ are related by an isogeny over $\mathbb{Q}$ to certain quotients of the Jacobian of the modular curves $X_0(\ell^2)$. Subsequently, a proof based on the representation theory of $\text{GL}_2(\mathbb{F}_\ell)$ was given in [3].

In [4], it was conjectured that the above isogeny relation between the Jacobian of $X_{N'}$ and the Jacobian of $X_0(\ell^2)$ was given by a certain explicit correspondence. This was proven in [1] using the representation theory of $\text{GL}_2(\mathbb{F}_\ell)$ and identities in finite double coset algebras.

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In this paper, we recall an alternative proof of Merel’s conjecture, which does not require extensive representation theory, based on arguments given by B. Birch and D. Zagier [6]. The proof can be formulated in terms of certain finite geometries over $\mathbb{F}_\ell$ and is largely elementary in its statement and proof, though some algebraic number theory is used.

Secondly, we generalize these arguments to exhibit an explicit correspondence which gives the isogeny relation between the Jacobians associated to split and non-split Cartan subgroups. An interesting feature is that the required explicit correspondence is considerably more complicated but can be expressed as a certain linear combination of double coset operators whose coefficients we are able to make explicit.

The precise statements of the theorems we prove are as follows.

- Let $\ell$ be an odd prime and $\epsilon$ a non-square in $\mathbb{F}_\ell^\times$.
- Let $G = \text{GL}_2(\mathbb{F}_\ell)$.
- Let $\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta$ denote the set of ordered pairs $(a, b)$ of distinct points in $\mathbb{P}^1(\mathbb{F}_\ell)$.
- Let $(\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta)/\sim$, where $(a, b) \sim (b, a)$, denote the set of unordered pairs $\{a, b\}$ of distinct points in $\mathbb{P}^1(\mathbb{F}_\ell)$.
- Let $\mathcal{C}_\ell = \{x + y\sqrt{\epsilon} : x \in \mathbb{F}_\ell, y \in \mathbb{F}_\ell^\times\}$.
- Let $\mathcal{H}_\ell = \mathcal{C}_\ell/\sim$, where $x + y\sqrt{\epsilon} \sim x - y\sqrt{\epsilon}$.
- When $S$ is a set, we denote by $\mathbb{Q}[S]$ the free $\mathbb{Q}$-vector space generated by the set $S$.
- For convenience, we write column vectors in the form $(x, y)^t$ for instance.

Given an unordered pair $\{a, b\}$ in $(\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta)/\sim$, we define in (2) a ‘geodesic’ $\gamma_{\{a, b\}}$ in $\mathcal{H}_\ell$ between $a$ and $b$.

**Theorem 1.** The map

$$
\psi^+ : \mathbb{Q}[(\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta)/\sim] \to \mathbb{Q}[\mathcal{H}_\ell]
$$

$$
\{a, b\} \mapsto \sum_{x \in \gamma_{\{a, b\}}} x
$$

is a surjective $\mathbb{Q}[G]$-module homomorphism.

Given an ordered pair $(a, b)$ in $\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta$ and a parameter $s \in \mathbb{F}_\ell^\times$, we define in (12) a ‘path’ $\gamma^s_{\{a, b\}}$ in $\mathcal{C}_\ell$ from $a$ to $b$.

**Theorem 2.** The map

$$
\psi : \mathbb{Q}[\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta] \to \mathbb{Q}[\mathcal{C}_\ell]
$$

$$
(a, b) \mapsto \sum_{s=1}^{\ell-1}(\alpha_s + \beta_s) \sum_{x \in \gamma^s_{\{a, b\}}} x
$$

is a surjective $\mathbb{Q}[G]$-module homomorphism, where $0 \leq \alpha_s, \beta_s \leq \ell - 1$ are integers satisfying $\alpha_s \equiv 1 \pmod{\ell}$ and $\beta_s \equiv s^{-1} \pmod{\ell}$ for $s \in \{1, \ldots, \ell - 1\}$. 
We explain in section 5 how Theorems 1 and 2 imply relations between the Jacobians of \( X_{N'} \) and \( X_{C'} \) and quotients of the Jacobians of the more standard modular curve \( X_0(\ell^2) \).

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2. Double coset operators

Lemma 3. Let \( G \) be a group, \( H \) and \( K \) be subgroups of \( G \), then

\[
HgK = \bigcup_{\alpha \in H/(H \cap gKg^{-1})} \alpha gK,
\]

where the union is disjoint. We call \([H : H \cap gKg^{-1}]\) the degree of \( HgK \). This is independent of the choice of \( g \) in the sense that \( \text{deg}(HgK) = \text{deg}(Hg'K) \) if \( HgK = Hg'K \).

Definition 4. Let \( G \) be a finite group with subgroups \( H \) and \( K \). Given a double coset \( HgK \) and a decomposition into disjoint cosets

\[
HgK = \bigcup_{\alpha \in \Omega} \alpha gK,
\]

we obtain a \( \mathbb{Z}[G] \)-module homomorphism \( \sigma = \sigma(HgK) \) given by

\[
\sigma : \mathbb{Z}[G/H] \to \mathbb{Z}[G/K]
\]

\[
xH \mapsto \sum_{\alpha \in \Omega} x \alpha gK.
\]

The \( \mathbb{Z}[G] \)-module homomorphism \( \sigma \) is called a double coset operator.

Let \( C \) (resp. \( C' \)) be the split (resp. non-split) Cartan subgroup of \( G \) given respectively by

\[
C = \left\{ \begin{pmatrix} \eta & 0 \\ 0 & \beta \end{pmatrix} : \eta, \beta \in \mathbb{F}_\ell^* \right\},
\]

\[
C' = \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} : (x, y) \neq (0, 0), x, y \in \mathbb{F}_\ell \right\}.
\]

Let \( N \) (resp. \( N' \)) be the normalizer in \( G \) of \( C \) (resp. \( C' \)) which is given respectively by

\[
N = \left\{ \begin{pmatrix} \eta & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & \eta \\ \beta & 0 \end{pmatrix} : \eta, \beta \in \mathbb{F}_\ell^* \right\},
\]

\[
N' = \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}, \begin{pmatrix} x & -\epsilon y \\ y & -x \end{pmatrix} : (x, y) \neq (0, 0), x, y \in \mathbb{F}_\ell \right\}.
\]
Lemma 5. The double coset operator $NN' : \mathbb{Z}[G/N] \to \mathbb{Z}[G/N']$ coincides with the map $\psi^+ : \mathbb{Z}[(\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta)/\sim] \to \mathbb{Z}[\mathfrak{N}_\ell]$ in (2) and is hence a $\mathbb{Z}[G]$-module homomorphism.

Proof. Since $N \cap N' = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \pm\alpha \end{pmatrix} : \alpha \in \mathbb{F}_\ell^\times \right\} \cup \left\{ \begin{pmatrix} 0 & \pm\varepsilon\alpha \\ \alpha & 0 \end{pmatrix} : \alpha \in \mathbb{F}_\ell^\times \right\}$, we have from Lemma 3 that $NN' = \bigcup_{\alpha \in \mathbb{F}_\ell^\times /\{\pm 1\}} \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix} N'$.

The $\mathbb{Z}[G]$-module homomorphism from $\mathbb{Z}[G/N] \to \mathbb{Z}[G/N']$ induced by $NN'$ from (1) is then seen to be the map $\psi^+$. □

Lemma 6. The double coset operator $C \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} C' : \mathbb{Z}[G/C] \to \mathbb{Z}[G/C']$ coincides with the map $H_s : \mathbb{Z}[\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta] \to \mathbb{Z}[\mathfrak{C}_\ell]$ in (12) and is hence a $\mathbb{Z}[G]$-module homomorphism.

Proof. For $g = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, we have that $C \cap gC'g^{-1} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in \mathbb{F}_\ell^\times \right\}$.

Thus, from Lemma 3 we have that $C \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} C' = \bigcup_{\alpha \in \mathbb{F}_\ell^\times} \begin{pmatrix} \alpha & \alpha s \\ 0 & 1 \end{pmatrix} C'$.

The $\mathbb{Z}[G]$-module homomorphism from $\mathbb{Z}[G/C] \to \mathbb{Z}[G/C']$ induced by $CC'$ from (1) is then seen to be the map $H_s$. □

3. Normalizer of Cartan subgroup case

In this section, we explain and give a detailed proof of Merel’s conjecture for normalizers of Cartan subgroups using methods in [6]. In this situation, the conjectural explicit intertwining operator is given by a single double coset operator.

Define $\gamma_{(0,\infty)} := \mathbb{F}_\ell^\times \sqrt{\varepsilon} \subseteq \mathfrak{S}_\ell$, which can be thought of as the geodesic in $\mathfrak{S}_\ell$ between 0 and $\infty$. Given an unordered pair $\{a,b\}$, there is a $g \in G$ such that $\{a,b\} = \{g(0), g(\infty)\}$, which is unique up to multiplication on the left by $N$. Thus, we may define

$$\gamma_{\{a,b\}} := g(\gamma_{(0,\infty)}),$$

which can be thought of as the geodesic in $\mathfrak{S}_\ell$ between $a$ and $b$. 
Lemma 7. A choice for the element $g$ above is given by

$$
\begin{pmatrix}
  b & a \\
  1 & 1
\end{pmatrix}.
$$

Proof. The point at infinity $\infty$ is denoted by $(1,0)^t$ and the point $0$ by $(0,1)^t$. We require a matrix $g$ such that $g \cdot 0 = (a,1)^t$ and $g \cdot \infty = (b,1)^t$, which is given by the above matrix. \hfill \Box

The finite field $\mathbb{F}_\ell \sqrt{\epsilon}$ is a vector space over $\mathbb{F}_\ell$ of dimension 2. The basis $\{1, \sqrt{\epsilon}\}$ gives us an identification $\mathbb{F}_{\ell^2} \cong \mathbb{F}_\ell + \sqrt{\epsilon} \mathbb{F}_\ell$. Thus, for every $z \in \mathbb{F}_{\ell^2}$, we can write $z = x + \sqrt{\epsilon} y$ for some $x, y \in \mathbb{F}_\ell$.

Lemma 8. The quadratic equation

$$
(x - \frac{a+b}{2})^2 - \epsilon y^2 = \left(\frac{b-a}{2}\right)^2.
$$

(3)

gives the geodesic $\gamma_{\{a,b\}}$ with coordinates (see Figure 1)

$$
x = \frac{a - \epsilon \lambda^2 b}{1 - \epsilon \lambda^2},
$$

$$
y = \lambda \left(\frac{a - b}{1 - \epsilon \lambda^2}\right).
$$
Proof. Writing \( g(\lambda \sqrt{\epsilon}, 1) \) as a fraction and then rationalizing it, we obtain:
\[
\frac{b\lambda \sqrt{\epsilon} + a}{\lambda \sqrt{\epsilon} + 1} = \frac{a - b\lambda^2 \epsilon}{1 - \epsilon \lambda^2} + \sqrt{\epsilon} \frac{\lambda(b - a)}{1 - \epsilon \lambda^2}.
\]
Therefore, as \( \gamma(a, b) \in \mathbb{F}_\ell^\times \) we conclude \( x, y \) from the above expression are given by (see Figure 1)
\[
x = \frac{a - \epsilon \lambda^2 b}{1 - \epsilon \lambda^2},
\]
\[
y = \lambda \left( \frac{a - b}{1 - \epsilon \lambda^2} \right).
\]

3.1. Coordinates for \( G/N \) and \( G/N' \).

We need a more convenient coordinate to represent elements in (a certain subset of) \((\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta) / \sim \) and \( \mathcal{S}_\ell \), where \((\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta) / \sim \) is in bijection with \( G/N \), and \( \mathcal{S}_\ell \) is in bijection with \( G/N' \).

Lemma 9. Let \( A_+ = (\mathbb{F}_\ell \times \mathbb{F}_\ell - \Delta) / \sim \) and \( B_+ = \{(t, n) : t^2 - 4n \neq 0 \text{ is a square in } \mathbb{F}_\ell\} \). Then there is a bijection between the sets \( A_+ \) and \( B_+ \) given by
\[
\{a, b\} \mapsto (a + b, ab).
\]

Proof. The inverse map is given by \((t, n) \mapsto \{a, b\} \), where \( \{a, b\} \) is the set of roots in \( \mathbb{F}_\ell \) of the polynomial \( x^2 - tx + n \).

Lemma 10. Let \( A'_+ = \mathcal{S}_\ell \) and \( B'_+ = \{(T, N) : T^2 - 4N \text{ is a non-square in } \mathbb{F}_\ell\} \). Then there is a bijection between the sets \( A'_+ \) and \( B'_+ \) given by
\[
\{z, \bar{z}\} \mapsto (z + \bar{z}, z\bar{z}).
\]

Proof. The inverse map is given by \((T, N) \mapsto \{z, \bar{z}\} \), where \( \{z, \bar{z}\} \) is the set of roots in \( \mathcal{S}_\ell \) of the polynomial \( x^2 - Tx + N \).

Lemma 11. Let
\[
B_+ = \{(t, n) : t^2 - 4n \neq 0 \text{ is a square in } \mathbb{F}_\ell\},
\]
\[
S_+ = \{(t, m) : m \text{ is a square in } \mathbb{F}_\ell\}.
\]
Then there is a bijection between the sets \( B_+ \) and \( S_+ \) given by:
\[
(t, n) \mapsto (t, m),
\]
where \( m = t^2 - 4n \).

Proof. The inverse map is given by \((t, m) \mapsto (t, \frac{t^2 - m}{4}) \).
Lemma 12. Let
\[ B'_+ = \{ (T, N) : T^2 - 4N \text{ is a non-square in } \mathbb{F}_\ell \} , \]
\[ S'_+ = \{ (T, M) : M \text{ is a non-square in } \mathbb{F}_\ell \} . \]
Then there is a bijection between the sets \( B'_+ \) and \( S'_+ \) given by:
\[ (T, N) \mapsto (T, M), \]
where \( M = T^2 - 4N \).

Proof. The inverse map is given by \((T, M) \mapsto (T, \frac{T^2 - M}{4})\). \( \square \)

3.2. Proof of Theorem 1

By Lemma 5 \( \psi^+ \) is a \( \mathbb{Q}[G] \)-module homomorphism. To prove Theorem 1 it suffices to prove that the restriction
\[ \psi^+_{|\mathbb{Q}[A_+] : \mathbb{Q}[A_+] \to \mathbb{Q}[S_+],} \]
is an isomorphism of \( \mathbb{Q} \)-vector spaces.

Using the bijections given by Lemmas 9-12 to prove (4) is equivalent to proving that
\[ \psi^+ : \mathbb{Q}[S_+] \to \mathbb{Q}[S'_+], \]
is an isomorphism of \( \mathbb{Q} \)-vector spaces, where \( \psi^+ \) is the same map as \( \psi^+_{|\mathbb{Q}[A]} \) under the identifications given by two bijections \( A_+ \leftrightarrow S_+ \) and \( S_\ell \leftrightarrow S'_+ \).

Recall the equation giving the geodesic between \( a \) and \( b \) is
\[ \left( x - \frac{a + b}{2} \right)^2 - \epsilon y^2 = \left( \frac{b - a}{2} \right)^2, \]
by Lemma 8. This equation becomes
\[ \left( x - \frac{a + b}{2} \right)^2 - \epsilon y^2 = \left( \frac{b - a}{2} \right)^2 \]
\[ \iff (T - t)^2 = l^2 + 4\epsilon y^2 = m + M, \]
in the new coordinates from Lemmas 11 and 12. Here, \( m \) and \( M \) satisfy \( (\frac{m}{\ell}) = 1 \) and \( (\frac{M}{\ell}) = -1 \), where \( (\cdot) \) is the Legendre symbol modulo \( \ell \).

Hence, the matrix of \( \psi^+_{|\mathbb{Q}[S_+] \to \mathbb{Q}[S'_+] \) with respect to the basis \( S_+ \) is given by
\[ a_{(T,M),(t,m)} = \begin{cases} 1 & \text{if } (T - t)^2 \equiv m + M \pmod{\ell}, \\ 0 & \text{otherwise}. \end{cases} \]

Thus, the above matrix is an \( \frac{\ell \pm 1}{2} \times \frac{\ell \pm 1}{2} \) matrix \( D_{m,M} \), with entries being the \( \ell \times \ell \) matrices given by
\[ (D_{m,M})_{t,T} := \begin{cases} 1 & \text{if } (T - t)^2 \equiv m + M \pmod{\ell}, \\ 0 & \text{otherwise}. \end{cases} \]
Let $D$ be the matrix obtained from the $\ell \times \ell$ identity matrix by permuting its columns according to the cycle $(1 \, 2 \, 3 \ldots \ell)$.

**Definition 13.** A circulant matrix is a matrix of the form

$$
\begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & a_{r-1} \\
a_{r-1} & a_0 & a_1 & \ldots & a_{r-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \ldots & a_{r-1} & a_0 
\end{pmatrix},
$$

that is, a matrix whose $i$-th row is obtained from the $(i-1)$-th row by cyclically shifting the entries one position to the right.

**Lemma 14.** $D_{m,M} = \sum_{x^2 \equiv m + M (\ell)} D^x$

**Proof.** If $m + M$ is not a square in $\mathbb{F}_\ell$, therefore $D_{m,M}$ is a zero matrix due to 0 entries, so $D_{m,M} = \sum_{x^2 \equiv m + M (\ell)} D^x = 0$.

If $m + M = x^2$ is a square in $\mathbb{F}_\ell$. Then $T - t = \pm x$ and $(D_{m,M})_{t,T} = \begin{cases} 1 & T = t \pm x, \\ 0 & \text{otherwise.} \end{cases}$

In this case, $D_{m,M}$ coincides with $\sum_{x^2 \equiv m + M (\ell)} D^x$. \hfill \Box

Let $\zeta$ be an $\ell$-th root of unity. The matrix $D_{m,M}$ has entries in $\mathbb{Q}[D]$, but we can replace the matrix $D$ by an element in the cyclotomic field $\mathbb{Q}(\zeta)$ in the following manner: the minimal polynomial of $D$ over $\mathbb{Q}$ is given by $m(x) = x^{\ell-1} + \cdots + x + 1$, so we have that

$$
\mathbb{Q}[D] \cong \frac{\mathbb{Q}[x]}{(m(x))} \cong \mathbb{Q}[\zeta] \cong \mathbb{Q}(\zeta),
$$

where $\zeta$ is a primitive $\ell$-th root of unity.

**Lemma 15.** Let $\mathfrak{p}$ be a prime of $\mathbb{Q}(\zeta)$ above $\ell$. Then $\zeta \equiv 1 \pmod{\mathfrak{p}}$.

**Proof.** [9, lemma 10.1]. \hfill \Box

From the above discussion, we see that $D_{m,M} = \sum_{x^2 \equiv m + M (\ell)} 1$ (after reduction modulo $\mathfrak{p}$). We label $m, M$ as $m = g^{2i}$ for $0 \leq i \leq r - 1$ and $M = \epsilon g^{2j}$ for $0 \leq j \leq r - 1$, where $r = \frac{\ell - 1}{2}$ and $g$ is a primitive root modulo $\ell$. This gives us a new matrix denoted by $D_{i,j}$:

$$
D_{i,j} = \sum_{x^2 \equiv g^{2i} + \epsilon g^{2j} \pmod{\ell}} 1.
$$
Proposition 16. The determinant of a circulant matrix is given by
\[
\prod_{k=0}^{r-1}(a_0 + a_1\omega_k + a_2\omega_k^2 + \ldots + a_{r-1}\omega_k^{r-1}) = \prod_{k=0}^{r-1} \left( \sum_{j=0}^{r-1} a_j\omega_k^j \right),
\]
where $\omega_k = e^{2\pi ik/r} = \omega^k, r \geq 1$ and $\omega = e^{2\pi i/r}$.

Proof. Suppose
\[
A = \begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & a_{r-1} \\
a_{r-1} & a_0 & a_1 & \ldots & a_{r-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \ldots & a_0
\end{pmatrix},
\]
is a circulant matrix. Let $\omega_k = e^{2\pi ik/r}$ for $0 \leq k \leq r - 1$. Now, consider the row vector $(1, \omega_k, \omega_k^2, \ldots, \omega_k^{r-1})$, whose transpose we denote by $\gamma_k \in \mathbb{C}^r$, and let $\sigma_k = a_0 + a_1\omega_k + a_2\omega_k^2 + \ldots + a_{r-1}\omega_k^{r-1}$. Then we get that
\[
\begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & a_{r-1} \\
a_{r-1} & a_0 & a_1 & \ldots & a_{r-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \ldots & a_0
\end{pmatrix} \begin{pmatrix} 1 \\ \omega_k \\ \vdots \\ \omega_k^{r-1} \end{pmatrix} = \sigma_k \begin{pmatrix} 1 \\ \omega_k \\ \vdots \\ \omega_k^{r-1} \end{pmatrix},
\]
which implies that $\sigma_k$ is an eigenvalue of $A$ with eigenvector $\gamma_k$. Furthermore, the set $\{\gamma_0, \gamma_1, \ldots, \gamma_{r-1}\}$ is a linearly independent set in $\mathbb{C}^r$, since the eigenvalues $\sigma_k$ are distinct. Therefore, a diagonal matrix with the corresponding eigenvalues is conjugate to $A$, and hence the determinant of $A$ is given by $\det(A) = \prod_{k=0}^{r-1} \sigma_k$. \hfill \Box

For later reference, we call each factor in the above formula an eigenvalue for $k$. We also let $r = \ell - 1 \over 2$ in this section.

Lemma 17. The matrix $D_{i,j}$ is an $r \times r$ circulant matrix.

Proof. This follows because
\[
D_{i,j} = \sum_{x^2 \equiv g^2j + \epsilon g^2j} 1 \equiv \sum_{x^2 \equiv g^2j + \epsilon g^2j} 1 = D_{i-1,j-1},
\]
where the indices are taken modulo $\ell$. \hfill \Box

Remark that $D_{0,j} = a_j$ is equal to the number of solutions of $x^2 \equiv 1 + \epsilon g^2j \pmod{\ell}$. To show that $D_{i,j}$ has non-zero determinant, it suffices to show that $D_{i,j}$ has non-zero determinant modulo $\ell$ in (7).

Using the above formula for the determinant of a circulant matrix, it suffices to show in $\mathbb{Z}[\omega]$ that we have
\[
a_0 + a_1\omega_k + a_2\omega_k^2 + \ldots + a_{r-1}\omega_k^{r-1} \neq 0(\vartheta)
\]
for every $0 \leq k \leq r - 1$, where $\vartheta$ is any prime above $\ell$ in $\mathbb{Z}[\omega]$, $\omega_k = \omega^k$ and $\omega = e^{2\pi i/r}$. 

Lemma 18. Let $\vartheta$ be a prime above $\ell$ in $\mathbb{Z}[\omega]$ where $\omega = e^{2\pi i/r}$. Then $\omega \equiv g^2(\vartheta)$, where $g$ is a primitive root modulo $\ell$.

Proof. Let $\mathcal{O} = \mathbb{Z}[\omega]$ be the maximal order of $\mathbb{Q}(\omega)$. The residue field of $\vartheta$ is $\mathcal{O}/\vartheta \cong \mathbb{F}_\ell$. Furthermore, since the polynomial $x^r - 1$ splits in $\mathcal{O}/\vartheta[x] \cong \mathbb{F}_\ell[x]$ with distinct roots $\omega_1 = \omega, \omega_2 = \omega^2, \ldots, \omega_r = \omega^r = 1$, we have that every root of $x^r - 1$ in $\mathbb{F}_\ell$ is a power of $\omega \in \mathcal{O}/\vartheta \cong \mathbb{F}_\ell$. Hence, $\omega \equiv g^2(\vartheta)$ for some primitive root $g$ modulo $\ell$. □

By the above lemma, to show (3), it suffices to show

Lemma 19.

\[ \sum_{j=0}^{r-1} a_j(g^{2k})^j \neq 0(\ell), \]

for every $0 \leq j, k \leq r - 1$.

Proof. The above sum can be calculated as:

\[ \sum_{j=0}^{r-1} D_{0,j}(g^{2k})^j \equiv \sum_{j=0}^{r-1} a_j(g^{2k})^j \equiv \sum_{j=0}^{r-1} \left( \sum_{\lambda \equiv 1+\epsilon g^2j \pmod{\ell}} 1 \right) (g^{2j})^k \]

\[ \equiv \sum_{j=0}^{r-1} (g^{2j})^k \equiv 2 \sum_{x=0}^{\ell-1} \left( \frac{x^2 - 1}{\epsilon} \right)^k. \]

Now, we need to show that (9) is non-zero modulo $\ell$ for every $0 \leq k \leq r - 1$.

We can rewrite \((x^2-1)/\epsilon\) as $y^2$ since $y = g^j$ for $0 \leq j \leq r - 1$. The conic $x^2 \equiv 1 + \epsilon y^2 \pmod{\ell}$ is parametrized by $x = \frac{a - \epsilon \lambda^2}{1 - \epsilon \lambda^2}, y = \frac{\lambda(a-b)}{1 - \epsilon \lambda^2}$ from (3). Here, we compute $D_{i,j}$ for $i = 0$ which corresponds to $m = 1 = (a - b)^2$. Thus, we can rewrite (9) as

\[ \sum_{\lambda=1}^{\ell-1} \left( \frac{\lambda}{1 - \epsilon \lambda^2} \right)^{2k} \equiv \sum_{\lambda=1}^{\ell-1} (\lambda^{-1} - \epsilon \lambda)^{-2k} \equiv \sum_{\lambda=1}^{\ell-1} (\lambda^{-1} - \epsilon \lambda)^{2k'}, \]

where $-2k' \equiv 2k \pmod{\ell - 1}$ and $0 \leq 2k' \leq \ell - 2$, hence $0 \leq k' \leq \frac{\ell - 3}{2}$. Here, we have to consider two cases.

The first case is $k \geq 1$. The sum of all terms except the constant terms will be zero modulo $\ell$. Therefore, we just have to compute the sum of the constant terms which is

\[ \sum_{\lambda=1}^{\ell-1} \frac{(2k')!}{k'!k'!} (-1)^{k'} \epsilon^{k'} \equiv \frac{(2k')!}{k'!k'!} (-1)^{k'+1} \epsilon^{k'} \pmod{\ell}, \]

which is non-zero modulo $\ell$, since $2k' < \ell$ for all values of $k'$, hence none of the terms of (10) is a multiple of $\ell$, therefore it is non-zero modulo $\ell$.

The second case is when $k = 0$, then the sum in (10) becomes $\sum_{\lambda=1}^{\ell-1} 1 \equiv -1 \neq 0 \pmod{\ell}$. □
This concludes the proof of Theorem 1.

4. Cartan subgroup case

In this section, we generalize Merel’s conjecture to Cartan subgroups and give a proof by generalizing the methods in Section 3. A new feature is that the conjectural explicit intertwining operator is now a linear combination of double coset operators (rather than a single double coset operator) whose coefficients we are able to make explicit.

Define $\gamma_{(0, \infty)} := \mathbb{F}_\ell^\times \sqrt{\epsilon} \subseteq \mathfrak{E}_\ell$, which can be thought of as a path in $\mathfrak{E}_\ell$ from 0 to $\infty$. Given an ordered pair $(a, b)$, there is a $g \in G$ such that $(a, b) = (g(0), g(\infty))$, which is unique up to multiplication on the left by $C$. Thus, we may define $\gamma_{(a, b)} := g(\gamma_{(0, \infty)})$, which can be thought as a path in $\mathfrak{E}_\ell$ from $a$ to $b$ (see Figure 2).

Here, for each $s = 1, \ldots, \ell - 1$, we define the linear operator $H_s$ by:

$$H_s : \mathbb{Q}[\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta] \longrightarrow \mathbb{Q}[\mathfrak{E}_\ell]$$

$$(a, b) \mapsto \sum_{x \in \gamma_{(a, b)}^s} x.$$

Definition 20. Define $\gamma_{(0, \infty)}^s$ to be $\{(\lambda s + \lambda \sqrt{\epsilon}, 1) : \lambda \in \mathbb{F}_\ell^\times, s \in \mathbb{F}_\ell^\times \} \subseteq \mathfrak{E}_\ell$. This is a path in $\mathfrak{E}_\ell$ which is a line with slope $s$.

By Lemma 7 we know that $g = \left( \begin{array}{cc} b & a \\ 1 & 1 \end{array} \right)$, hence the path in $\mathfrak{E}_\ell$ from $a$ to $b$ can be obtained as

$$\gamma_{(a, b)}^s = g(\gamma_{(0, \infty)}^s) = g(\lambda s + \lambda \sqrt{\epsilon}, 1)^t = (bs\lambda + b\lambda \sqrt{\epsilon} + a, \lambda s + \lambda \sqrt{\epsilon} + 1)^t,$$

which is represented by an equation defined by the next lemma.

Lemma 21. The quadratic equation

$$\left( x - \frac{a + b}{2} \right)^2 - \epsilon \left( y - \frac{s(b - a)}{2\epsilon} \right)^2 = \frac{(\epsilon - s^2)(a - b)^2}{4\epsilon}$$

gives the path $\gamma_{(a, b)}^s$ with coordinates (see Figure 2)

$$x = \frac{(b\lambda s + a)(\lambda s + 1) - b\lambda^2 \epsilon}{(\lambda s + 1)^2 - \lambda^2 \epsilon},$$

$$y = \frac{\lambda(b - a)}{(\lambda s + 1)^2 - \lambda^2 \epsilon}.$$

Proof. Writing $g(\lambda s + \lambda \sqrt{\epsilon}, 1)^t$ as a fraction and then rationalizing it, we obtain:

$$\frac{(b\lambda s + a) + b\lambda \sqrt{\epsilon}}{(\lambda s + 1) + \lambda \sqrt{\epsilon}} = \frac{(b\lambda s + a)(\lambda s + 1) - b\lambda^2 \epsilon}{(\lambda s + 1)^2 - \lambda^2 \epsilon} + \sqrt{\epsilon} \frac{\lambda(b - a)}{(\lambda s + 1)^2 - \lambda^2 \epsilon}.$$
Therefore, as $γ_{(a,b)} \in \mathbb{F}_ℓ^\times$ we conclude $x, y$ from the above expression are given by (see Figure 2)

$$x = \frac{(bλs + a)(λs + 1) - bλ^2ε}{(λs + 1)^2 - λ^2ε},$$

$$y = \frac{λ(b - a)}{(λs + 1)^2 - λ^2ε}.$$

4.1. Coordinates for $G/C$ and $G/C'$. 

We need more convenient coordinates to represent elements in (a certain subset of) $\mathbb{P}^1(\mathbb{F}_ℓ) \times \mathbb{P}^1(\mathbb{F}_ℓ) - Δ$ and $C_ℓ$, where $\mathbb{P}^1(\mathbb{F}_ℓ) \times \mathbb{P}^1(\mathbb{F}_ℓ) - Δ$ is in bijection with $G/C'$ and $C_ℓ$ is in bijection with $G/C'$.

**Lemma 22.** Let $A = \mathbb{F}_ℓ \times \mathbb{F}_ℓ - Δ$ and $S = \{(t, t') : t' \neq 0\}$. Then there is a bijection between the sets $A$ and $S$ given by:

$$(a, b) \mapsto (a + b, a - b).$$

**Proof.** The inverse map is given by $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$. \qed

**Figure 2.** The path $γ_{(a,b)}$ in $C_ℓ$
Lemma 23. Let $A' = \mathcal{C}_\ell$ and $S' = \{(T, T') : T' \neq 0\}$. Then there is a bijection between the sets $A'$ and $S'$ given by:

$$(z, \bar{z}) \mapsto (z + \bar{z}, z - \bar{z}).$$

**Proof.** The inverse map is given by $z = \frac{z + \bar{z}}{2} + \sqrt{\epsilon} \frac{z - \bar{z}}{2}$ and $\bar{z} = \frac{z + \bar{z}}{2} - \sqrt{\epsilon} \frac{z - \bar{z}}{2}$. □

4.2. **Proof of Theorem 2.**

By Lemma 6, $\psi$ is a $\mathbb{Q}[G]$-module homomorphism. To prove Theorem 2, it suffices to prove that the restriction

$$(14) \quad \psi|_{\mathbb{Q}[A]} : \mathbb{Q}[A] \to \mathbb{Q}[\mathcal{C}_\ell],$$

is an isomorphism of $\mathbb{Q}$-vector spaces.

Using the bijections given by Lemma 22 and Lemma 23, to prove (14) is equivalent to proving that

$$\psi : \mathbb{Q}[S] \to \mathbb{Q}[S'],$$

is an isomorphism of $\mathbb{Q}$-vector spaces, where $\psi$ is the same map as $\psi|_{\mathbb{Q}[A]}$ under identifications given by two bijections $A \leftrightarrow S$ and $\mathcal{C}_\ell \leftrightarrow S'$.

Recall the equation giving the path $\gamma_{s(a,b)}^s$ from $a$ to $b$ is

$$(x - \frac{a + b}{2})^2 - \epsilon \left( y - \frac{s(b - a)}{2\epsilon} \right)^2 = \frac{(\epsilon - s^2)(a - b)^2}{4\epsilon},$$

by Lemma 21. By the bijections $P \leftrightarrow E$ and $P' \leftrightarrow E'$, this equation becomes

$$(x - \frac{a + b}{2})^2 - \epsilon \left( y - \frac{s(b - a)}{2\epsilon} \right)^2 = \frac{(\epsilon - s^2)(a - b)^2}{4\epsilon},$$

(15) $\iff (T - t)^2 = (a - b)^2 + 4\epsilon y^2 + 4sy(b - a),$

in the new coordinates from Lemma 22 and Lemma 23. Hence, the matrix of $H_s$ restricted to $\mathbb{Q}[S]$ with respect to the basis $S$ is given by

$$(16) \quad a_{(t,t'),(T,T')}(s) = \begin{cases} 1 & \text{if } (T - t)^2 \equiv t'^2 + 4\epsilon T'^2 + 4sT't' \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

The above matrix is an $(\ell - 1) \times (\ell - 1)$ matrix $X_{t',T'}(s)$, with entries being the $\ell \times \ell$ matrices $(X_{t',T'})_{t,T}(s)$ given by

$$(X_{t',T'})_{t,T}(s) = \begin{cases} 1 & \text{if } (T - t)^2 \equiv t'^2 + 4\epsilon T'^2 + 4sT't' \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X$ be the matrix which permutes columns of the $\ell \times \ell$ identity matrix according to the cycle $(1\ 2\ 3 \cdots \ell)$.

**Lemma 24.** $X_{t',T'}(s) = \sum v^2 \equiv t'^2 + 4\epsilon T'^2 + 4sT't' \pmod{\ell} X^v$. 
Lemma 26. Let \( t^2 + 4ct'^2 + 4st't' \) be not square in \( \mathbb{F}_t \), then \( X_{t,T}(s) \) is a zero matrix due to 0 entries. Therefore, \( X_{t,T}(s) = \sum_{v^2 \equiv t^2 + 4ct'^2 + 4st't'}(t) X_v = 0 \).

If \( t^2 + 4ct'^2 + 4st't' = v^2 \) is a square in \( \mathbb{F}_t \), then \( T - t = \pm v \) and

\[
(X_{t',T'}),_{t,T}(s) = \begin{cases} 
1 & T = t + v, \\
0 & \text{otherwise.} 
\end{cases}
\]

In this case, \( X_{t',T'}(s) \) coincides with \( \sum_{v^2 \equiv t^2 + 4ct'^2 + 4st't'}(t) X_v \). \( \square \)

Proof. If \( t^2 + 4ct'^2 + 4st't' \) is not square in \( \mathbb{F}_t \), then \( X_{t,T}(s) \) is a zero matrix due to 0 entries. Therefore, \( X_{t,T}(s) = \sum_{v^2 \equiv t^2 + 4ct'^2 + 4st't'}(t) X_v = 0 \).

If \( t^2 + 4ct'^2 + 4st't' = v^2 \) is a square in \( \mathbb{F}_t \), then \( T - t = \pm v \) and

\[
(X_{t',T'}),_{t,T}(s) = \begin{cases} 
1 & T = t + v, \\
0 & \text{otherwise.} 
\end{cases}
\]

In this case, \( X_{t',T'}(s) \) coincides with \( \sum_{v^2 \equiv t^2 + 4ct'^2 + 4st't'}(t) X_v \). \( \square \)

Arguing similarly as in the discussion preceding Lemma 15, we obtain that \( X_{t',T'}(s) = \sum_{v^2 \equiv t^2 + 4ct'^2 + 4st't'}(t) 1 \). We label \( t', T' \) as \( t' = g^i \) and \( T' = g^j \) for \( 0 \leq i, j \leq \ell - 1 \), and \( (T - t)^2 = v^2 \). This gives us a new matrix denoted by \( X_{i,j}(s) \) which is given by

\[
X_{i,j}(s) = \sum_{v^2 \equiv g^{2j} - 4sg^{i+j}}1(t). 
\]

Lemma 25. The matrix \( X_{i,j}(s) \) is a \((\ell - 1) \times (\ell - 1)\) circulant matrix.

Proof. This follows since

\[
X_{i,j}(s) \equiv \sum_{v^2 \equiv g^{2j} - 4sg^{i+j}}1(t) \equiv \sum_{v^2 \equiv g^{2i-1} + 4sg^{2j-1} - 4sg^{i+j} - 2}(t) X_{i-1,j-1}(s) \equiv X_{i-1,j-1}(s) \quad (\ell),
\]

where the indices are taken modulo \( \ell \). Remark, \( X_{0,0}(s) = X_{0,j}(s) \) is equal to the number of solutions of \( v^2 \equiv 1 + 4g^{2j} - 4sg^j \) \( (\ell) \). \( \square \)

Let \( a_j(s) = X_{0,j}(s) = c_j \), and \( \omega = e^{\frac{2\pi i}{\ell}} \).

Lemma 26. Let \( \vartheta \) be a prime above \( \ell \) in \( \mathbb{Z}[\omega] \) where \( \omega = e^{\frac{2\pi i}{\ell}} \). Then \( \omega \equiv g \quad (\vartheta) \), where \( g \) is a primitive root modulo \( \ell \).

Proof. Let \( \mathcal{O} = \mathbb{Z}[\omega] \) be the maximal order of \( \mathbb{Q}(\omega) \). The residue field of \( \vartheta \) is \( \mathcal{O}/\vartheta \cong \mathbb{F}_t \) by [9] Proposition 10.3]. Furthermore, since the polynomial \( x^\ell - 1 \) splits in \( \mathcal{O}/\vartheta[x] \cong \mathbb{F}_t[x] \) with distinct roots \( \omega_1 = \omega, \omega_2 = \omega^2, \ldots, \omega_{\ell-1} = \omega^{\ell-1} = 1 \), we have that every root of \( x^\ell - 1 \) in \( \mathbb{F}_t \) is a power of \( \omega \in \mathcal{O}/\vartheta \cong \mathbb{F}_t \). Hence, \( \omega \equiv g \quad (\vartheta) \) for some positive root \( g \) modulo \( \ell \). \( \square \)

The eigenvalues of \( H_s \) modulo \( \vartheta \) can be calculated as

\[
\sum_{j=0}^{\ell-2} a_j(s) \omega^{kj} \equiv \sum_{j=0}^{\ell-2} a_j(s)(g^k)^j \equiv \sum_{j=0}^{\ell-2} \left( \sum_{v^2 \equiv 1 + 4g^{2j} - 4sg^j}1(t) \right)(g^k)^j
\]

(17) \equiv \sum_{j=0}^{\ell-2} \sum_{v^2 \equiv 1 + 4g^{2j} - 4sg^j} g^{kj} \equiv \sum_{\lambda=1}^\ell y(\lambda)^k \equiv \sum_{\lambda=1}^\ell \frac{\lambda^k(a - b)^k}{((\lambda s + 1)^2 - \lambda^2 \epsilon^2)^k} \quad (\ell).
\]

Here, \( a_j(s) = X_{0,j}(s) \), which corresponds to \( m = 1 = (a - b)^2 \).
We now consider a linear combination \( \sum_{s=1}^{\ell-1} \alpha_s H_s : \mathbb{Q}[\mathbb{P}^1(\mathbb{F}_\ell) \times \mathbb{P}^1(\mathbb{F}_\ell) - \Delta] \to \mathbb{Q}[\mathbb{C}_\ell] \) of the maps \( H_s \). Note that a linear combination of circulant matrices is circulant. The eigenvalue of \( \sum_{s=1}^{\ell-1} \alpha_s H_s \) is thus given by \( \sum_{j=0}^{\ell-2} b_j \omega^j \), where \( b_j = \sum_{s=1}^{\ell-1} \alpha_s a_j(s) \). Then, we have that

\[
\sum_{j=0}^{\ell-2} b_j \omega^j = \sum_{j=0}^{\ell-2} \left( \sum_{s=1}^{\ell-1} \alpha_s a_j(s) \right) \omega^j = \sum_{s=1}^{\ell-1} \alpha_s \sum_{j=0}^{\ell-2} a_j(s) \omega^j
\]

\[
\equiv \sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} y(\lambda, s)^k = \sum_{\lambda=1}^{\ell-1} \sum_{s=1}^{\ell-1} \alpha_s y(\lambda, s)^k
\]

\[
= \sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{\lambda}{(\lambda s + 1)^2 - \epsilon \lambda^2} \right)^k.
\]

**Lemma 27.** Let \( \alpha_s \equiv 1 \) \((\ell)\) for \( s \in \mathbb{F}_\ell^\times \), then the sum

\[
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{\lambda}{(\lambda s + 1)^2 - \epsilon \lambda^2} \right)^k, \tag{18}
\]

is non-zero modulo \( \ell \) for \( k \) even.

**Proof.** In the case \( k = 0 \), we cannot use a binomial expansion so we perform a direct computation:

\[
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{\lambda}{(\lambda s + 1)^2 - \epsilon \lambda^2} \right)^k \equiv \sum_{s=1}^{\ell-1} \alpha_s (\ell - 1)
\]

\[
\equiv (\ell - 1) \sum_{s=1}^{\ell-1} \alpha_s \equiv (\ell - 1) \sum_{s=1}^{\ell-1} 1 \equiv (\ell - 1)(\ell - 1) \equiv 1 \quad (\ell). \tag{19}
\]

If \( k > 0 \), then choose \( k' \in \mathbb{N} \) such that \( k \equiv -k' \) \((\ell - 1)\), and \( 1 \leq k' \leq \ell - 2 \). Then

\[
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{-k} 
\]

\[
\equiv \sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{k'} \quad (\ell),
\]
where
\[
\left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{k'} = \left( \frac{\lambda^2 s^2 + 2 \lambda s + 1 - \epsilon \lambda^2}{\lambda} \right)^{k'} = (\lambda s^2 + 2s + \lambda^{-1} - \epsilon \lambda)^{k'} = (\lambda(s^2 - \epsilon) + 2s + \lambda^{-1})^{k'}.
\]

Here, we just need the constant terms of \( \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{k'} \) as the other terms are powers of \( \lambda \), and the sum of these powers is zero modulo \( \ell \). Now, the constant term of \( (\lambda(s^2 - \epsilon) + 2s + \lambda^{-1})^{k'} \)
\[
\equiv \sum_{i=0}^{k'} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i \quad (\ell) \quad \text{for } k' \text{ even.}
\]

Thus, we obtain that
\[
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{-k} = \sum_{s=1}^{\ell-1} \alpha_s \sum_{i=0}^{k'} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i \quad (\ell).
\]

If \( \alpha_s \equiv 1 \quad (\ell) \) for \( s = 1, \ldots, \ell - 1 \), then for \( k' > 0 \) even, we have that
\[
\sum_{s=1}^{\ell-1} \alpha_s \sum_{i=0}^{k'} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i
\]
\[
\equiv \sum_{s=1}^{\ell-1} \alpha_s \sum_{i=0}^{k'} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i
\]
\[
= \sum_{i=0}^{k'} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i
\]
\[
= \epsilon \frac{k'}{\ell} \frac{k'!}{k'^2!} \neq 0 \quad (\ell).
\]

The last equality holds because the only power of \( s \) whose exponent is divisible by \( \ell - 1 \) happens when \( i = k'/2 \). This proves the lemma. \( \square \)
Lemma 28. Let $\alpha_s \equiv 1 \pmod{\ell}$ for $s \in \mathbb{F}_\ell^*$, then the sum

$$
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{\lambda}{(\lambda s + 1)^2 - \epsilon \lambda^2} \right)^k,
$$

is equal to zero modulo $\ell$ for $k$ odd.

Proof. We choose $k' \in \mathbb{N}$ such that $k \equiv -k' (\ell - 1)$, and $1 \leq k' \leq \ell - 2$. Then

$$
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^k
\equiv \sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{k'} (\ell).
$$

Hence, we get

$$
\left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{k'}
= \left( \frac{\lambda^2 s^2 + 2\lambda s + 1 - \epsilon \lambda^2}{\lambda} \right)^{k'}
= (\lambda^2 s^2 + 2s + \lambda^{-1} - \epsilon \lambda)^{k'}
= (\lambda(s^2 - \epsilon) + 2s + \lambda^{-1})^{k'}.
$$

Therefore, we have

$$
\text{constant term of } (\lambda(s^2 - \epsilon) + 2s + \lambda^{-1})^{k'} \equiv
\sum_{i=0}^{k'+1} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i} (s^2 - \epsilon)^i \pmod{\ell} \text{ for } k' \text{ odd}
$$

Thus,

$$
\sum_{s=1}^{\ell-1} \alpha_s \sum_{\lambda=1}^{\ell-1} \left( \frac{(\lambda s + 1)^2 - \epsilon \lambda^2}{\lambda} \right)^{-k} \equiv \sum_{s=1}^{\ell-1} \alpha_s \sum_{i=0}^{k'} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i} (s^2 - \epsilon)^i \pmod{\ell}.
$$
If \( \alpha_s \equiv 1 \) (\( \ell \)) for \( s = 1, \ldots, \ell - 1 \), then for \( k' > 0 \) odd, we have that
\[
\ell - 1 \sum_{s=1}^{\ell - 1} \alpha_s \sum_{i=0}^{\left\lfloor \frac{k'}{2} \right\rfloor} \frac{k'}{i!i!(k' - 2i)!} (2s)^{k' - 2i}(s^2 - \epsilon)^i
\equiv \sum_{s=1}^{\ell - 1} \left( \frac{k'}{i!i!(k' - 2i)!} (2s)^{k' - 2i}(s^2 - \epsilon)^i \right)_s
\equiv \sum_{i=0}^{\ell - 1} \sum_{s=1}^{\ell - 1} \frac{k'}{i!i!(k' - 2i)!} (2s)^{k' - 2i}(s^2 - \epsilon)^i
\equiv 0 \pmod{\ell}.
\]

The last equality holds because there are no powers of \( s \) whose exponent is divisible by \( \ell - 1 \), which proves the lemma.

**Lemma 29.** Let \( \beta_s \equiv s^{-1} \pmod{\ell} \) for \( s \in \mathbb{F}_\ell^\times \). Then, the sum
\[
(21) \sum_{s=1}^{\ell - 1} \beta_s \sum_{\lambda=1}^{\ell - 1} \left( \frac{\lambda}{(\lambda s + 1)^2 - \epsilon s^2} \right)^k,
\]
is zero modulo \( \ell \) for \( k \) even.

**Proof.** In the case \( k = 0 \), we cannot use a binomial expansion so we perform a direct computation:
\[
(22) \sum_{s=1}^{\ell - 1} s^{\ell - 2}(\ell - 1) \equiv (\ell - 1) \sum_{s=1}^{\ell - 1} s^{\ell - 2} \equiv 0 \pmod{\ell}.
\]

For \( k' > 0 \) even, we have that
\[
\ell - 1 \sum_{s=1}^{\ell - 1} \beta_s \sum_{i=0}^{\left\lfloor \frac{k'}{2} \right\rfloor} \frac{k'}{i!i!(k' - 2i)!} (2s)^{k' - 2i}(s^2 - \epsilon)^i
\equiv \sum_{i=0}^{\ell - 1} \sum_{s=1}^{\ell - 1} \frac{k'}{i!i!(k' - 2i)!} (2s)^{k' - 2i}(s^2 - \epsilon)^i \equiv 0 \pmod{\ell}.
\]
The last equality holds because there are no powers of $s$ whose exponent is divisible by $\ell - 1$, which proves the lemma.

**Lemma 30.** Let $\beta_s \equiv s^{-1} (\ell)$ for $s \in \mathbb{F}_\ell^\times$. Then the sum

$$
\sum_{s=1}^{\ell-1} \beta_s \sum_{\lambda=1}^{\ell-1} \left( \frac{\lambda}{(\lambda s + 1)^2 - \epsilon \lambda^2} \right)^k
$$

is non-zero modulo $\ell$ for $k$ odd.

**Proof.** For $k' > 0$ odd, we have that

$$
\sum_{s=1}^{\ell-1} \beta_s \sum_{i=0}^{\left\lfloor k'/2 \right\rfloor} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i
$$

is congruent to

$$
\sum_{s=1}^{\ell-1} s^{-1} \sum_{i=0}^{\left\lfloor k'/2 \right\rfloor} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i \pmod{\ell}
$$

and

$$
\sum_{i=0}^{\left\lfloor k'/2 \right\rfloor} s^{-1} \sum_{s=1}^{\ell-1} \frac{k'}{i!i!(k'-2i)!} (2s)^{k'-2i}(s^2 - \epsilon)^i
$$

is congruent to

$$
\epsilon^{k'-1} \frac{k'}{i!i!(k'-1)!} \neq 0 \pmod{\ell}.
$$

The last equality holds because the only power of $s$ whose exponent is divisible by $\ell - 1$ happens when $i = \frac{k'-1}{2}$, which proves the lemma.

**Corollary 31.** The operator $\sum_{s=1}^{\ell-1} (\alpha_s + \beta_s)H_s$ has non-zero eigenvalue modulo $\ell$ for all $k > 0$ in its circulant determinant formula.

**Proof.** Using (22), the operator $\sum_{s=1}^{\ell-1} (\alpha_s + \beta_s)H_s$ has non-zero eigenvalue for $k = 0$. Furthermore, by Lemmas 27, 28, 29, and 30 the eigenvalue of $\sum_{s=1}^{\ell-1} (\alpha_s + \beta_s)H_s$ is non-zero modulo $\vartheta$ for $k > 0$, since the eigenvalue of $\sum_{s=1}^{\ell-1} (\alpha_s + \beta_s)H_s$ for $k$ is the sum of the eigenvalues for $k$ of $\sum_{s=1}^{\ell-1} \alpha_s H_s$ and $\sum_{s=1}^{\ell-1} \beta_s H_s$.

The above corollary shows that determinant of $\sum_{s=1}^{\ell-1} (\alpha_s + \beta_s)H_s$ is non-zero modulo $\ell$, and is hence non-zero. This concludes the proof of Theorem 2.

5. **Relations between Jacobians of certain modular curves**

In this section, we summarize some applications of the main results of this paper to Jacobians of modular curves.

Let $X = X(\ell)$ denote the modular curve of full level $\ell$ structure which has the structure of a projective algebraic curve over $\mathbb{Q}$ for $p \geq 3$ (cf. [8, p.241] or [7]).
The group $G = \text{GL}_2(\mathbb{F}_\ell)$ acts on $X$ and the quotients $X_H := X/H$ by subgroups $H$ of $G$ (which contain $-1$) exist as projective algebraic curves over $\mathbb{Q}$ \cite[p.244]{8} and \cite{7}.

Let $J$ denote the Jacobian of $X$ and $J_H$ denote the Jacobian of $X_H$.

**Proposition 32.** Let $\sigma : \mathbb{Z}[G/H'] \to \mathbb{Z}[G/H]$ be a $\mathbb{Z}[G]$-module homomorphism. Then $\sigma$ induces a homomorphism of Jacobians $\sigma^* : J_H \to J_{H'}$.

*Proof.* This is proved in \cite[Lemma 3.3]{1}. \hfill $\square$

**Proposition 33.** Suppose a cochain complex of $\mathbb{Z}[G]$-modules

$$\cdots \to \mathbb{Z}[G/H_{i-1}] \to \mathbb{Z}[G/H_i] \to \mathbb{Z}[G/H_{i+1}] \to \cdots$$

has finite cohomology groups. Then the induced sequence of Jacobians by applying Proposition 32 yields a chain complex

$$\cdots \leftarrow J_{H_{i-1}} \leftarrow J_{H_i} \leftarrow J_{H_{i+1}} \leftarrow \cdots$$

with finite homology groups.

*Proof.* This is proved in \cite[Proposition 3.7]{1}. \hfill $\square$

Theorems 1 and 2 imply that

$$\begin{align*}
(24) & \quad \mathbb{Q}[G/N] \to \psi^+ \mathbb{Q}[G/N'] \to 0 \\
(25) & \quad \mathbb{Q}[G/C] \to \psi \mathbb{Q}[G/C'] \to 0
\end{align*}$$

are exact cochain complexes of $\mathbb{Q}[G]$-modules.

**Proposition 34.** The following are cochain complexes

$$\begin{align*}
(26) & \quad \mathbb{Z}[G/N] \to \psi^+ \mathbb{Z}[G/N'] \to 0 \\
(27) & \quad \mathbb{Z}[G/C] \to \psi \mathbb{Z}[G/C'] \to 0
\end{align*}$$

with finite cohomology groups.

*Proof.* This follows from tensoring the cochain complexes above by $\mathbb{Q}$. If the cohomology groups were not finite, this would contradict the exactness of the cochain complexes in (24)-(25). \hfill $\square$

Applying Proposition 33, we obtain:

**Corollary 35.** The following are chain complexes

$$\begin{align*}
(28) & \quad 0 \to J_{N'} \to \psi^+ J_N \\
(29) & \quad 0 \to J_{C'} \to \psi J_C
\end{align*}$$

with finite homology groups.
From [2], we have that

\begin{align}
J_N & \sim J_{N'} \times J_B \\
J_C & \sim J_{C'} \times J_B^2,
\end{align}

where \( \sim \) denotes the relation of isogeny over \( \mathbb{Q} \), and \( B \) is the subgroup of upper triangular matrices in \( G \). Hence, Corollary describes the main part of the well-known relations between \( J_N \) and \( J_{N'} \) (resp. \( J_C \) and \( J_{C'} \)) using explicit correspondences.

It is known that \( X_C \cong X_0(\ell^2) \) and \( X_N \cong X_0(\ell^2)/\langle w_\ell \rangle \), which are the more standard modular curves studied in the literature.

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