On the Convergence Analysis of Asynchronous SGD for Solving Consistent Linear Systems

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Abstract

In the realm of big data and machine learning, data-parallel, distributed stochastic algorithms have drawn significant attention in the present days. While the synchronous versions of these algorithms are well understood in terms of their convergence, the convergence analyses of their asynchronous counterparts are not widely studied. In this paper, we propose and analyze a distributed, asynchronous parallel SGD in light of solving an arbitrary consistent linear system by reformulating the system into a stochastic optimization problem as studied by Richtárik and Takáč in [35]. We compare the convergence rates of our asynchronous SGD algorithm with the synchronous parallel algorithm proposed by Richtárik and Takáč in [35] under different choices of the hyperparameters—the stepsize, the damping factor, the number of processors, and the delay factor. We show that our asynchronous parallel SGD algorithm also enjoys a global linear convergence rate, similar to the basic method and the synchronous parallel method in [35] for solving any arbitrary consistent linear system via stochastic reformulation. We also show that our asynchronous parallel SGD improves upon the basic method with a better convergence rate when the number of processors is larger than four. We further show that this asynchronous approach performs asymptotically better than its synchronous counterpart for certain linear systems. Moreover, for certain linear systems, we compute the minimum number of processors required for which our asynchronous parallel SGD is better, and find that this number can be as low as two for some ill-conditioned problems.

Keywords: Linear systems, distributed optimization, stochastic optimization, asynchronous communication, parallel algorithms, iterative methods.

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1 Introduction

In the era of big data and artificial intelligence, optimization problems have become increasingly complex in nature. Although the computers are now more powerful and inexpensive, the problems have grown continuously larger in size and they are difficult to be maneuvered by a single processor. Owing to the nature of the high-volume of the data, an emerging interest is to device and analyze scalable, parallel, and distributed algorithms that can handle the data more efficiently as compared to the traditional optimization algorithms designed to run on a single processor. To deal with the large-scale data, these new class of algorithms can take advantage of a multi-processor system where each processor has access to its own data-partition and it processes the data-partitions in mini-batches. Shalev-Shwartz et al. [37] in 2007 and Gimpel et al. [15] in 2010, explored the idea of mini-batches for stochastic algorithms in both the serial and parallel settings. In 2011, Dekel et al. [9] proposed a distributed mini-batch algorithm (for online predictions)—a method that converts many serial gradient-based online prediction algorithms into distributed algorithms with an asymptotically optimal regret bound. However, in a distributed environment, the synchronous parallel algorithms tend to slow down due to unpredictable communication faults, significant network latency, and processors with different processing speeds.

To overcome the above issues posed by the synchronous parallel algorithms in a distributed environment, there has been a recent focus on developing and analyzing asynchronous algorithms. In asynchronous algorithms, processors with different storage capacity and processing speeds perform updates without synchronizing with others.
Asynchronous algorithms were first introduced by Chazan and Miranker on chaotic relaxation in 1969 \[6\] (also, see Frommer and Szyld \[14\] and \[3\]). However, not only the inherent dynamics of asynchronous algorithms are challenging compared to their synchronous counterparts, but also their convergence analyses are much more mathematically involved. Historically, in the literature, the comparisons between the convergence rates of the asynchronous algorithms and their synchronous counterparts are also not vastly present and less understood. In this paper, our goal is to understand the convergence rates of the asynchronous and synchronous SGD in a fairly simple set-up. However, before discussing problem formulation, set-up, and contribution, we start with a brief overview of stochastic optimization.

Stochastic optimization

In machine learning and data-fitting applications, stochastic optimization is a broadly studied field. Consider the stochastic optimization problem:

\[
\min_{x \in \mathbb{R}^n} f(x) = \min_{x \in \mathbb{R}^n} \mathbb{E}_{S \sim \mathcal{D}}[f_S(x)],
\]

where \(\mathcal{D}\) is an user inferred distribution and \(S\) is a random sample drawn from that distribution. In supervised machine learning or deep learning, the above problem is known as empirical risk minimization (ERM) problem:

\[
\min_{x \in \mathbb{R}^n} \left[ f(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{S_i \sim \mathcal{D}_i}[f_{S_i}(x)] \right],
\]

where \(f_{S_i}(x)\)’s are instantiated by different distributions \(\mathcal{D}_i\) and \(S_i\) is sampled from \(\mathcal{D}_i\). In the distributed setting, \(n\) in the ERM problem denotes number of processors/workers. Therefore, \(2\) is an important problem from the deep learning perspective as it captures data-parallelism (distributed over \(n\) processors (GPUs/CPUs etc.)). One of the most popular algorithms for solving \(1\) is the stochastic gradient descent (SGD) \[36\]. For a given sequence of stepsize parameters \(\{\omega_k\}\) with \(\omega_k > 0\) and the sequence of iterates \(\{x_k\}\), the updates of SGD take the form:

\[
x_{k+1} = x_k - \omega_k \nabla f_S(x_k),
\]

where \(\nabla f_S(x_k)\) is the stochastic gradient arising from the sample \(S_k \sim \mathcal{D}\) drawn afresh in each iteration and is an unbiased estimator of the gradient of \(f\). A natural next direction in solving \(1\) is to design a synchronized parallel update by using SGD \[45\]. If more than one processors are available, then they can work simultaneously and each one of them can calculate a stochastic gradient independent of the other processors. At the end, the user can average the stochastic gradients from all processors to obtain the update as:

\[
x_{k+1} = x_k - \frac{\omega_k}{\tau} \sum_{i=1}^{\tau} \nabla f_{S_{ki}}(x_k),
\]

where \(\nabla f_{S_{ki}}(x_k)\) is the stochastic gradient arising from the independent sample \(S_{ki} \sim \mathcal{D}\) from each of the \(\tau\) processors and \(\omega_k > 0\) is a uniform stepsize across \(k\)th iteration. Note that, if \(\tau = 1\), then the update scheme in \(4\) is \(3\). In this paper, we compare the convergence rates of the asynchronous and synchronous algorithms in a fairly simple set-up of solving any arbitrary consistent linear systems by reformulating them into stochastic optimization problems. Before explaining our main contributions, we introduce the stochastic reformulation of a linear system.

1.1 Stochastic reformulation of a linear system

In \[35\], Richtárik and Takáč reformulated any arbitrary consistent linear system into a stochastic optimization problem (see details in \[35\] \[17\]; also see \[16\]). Consider a linear system:

\[
A x = b,
\]

where \(A \in \mathbb{R}^{m \times n}\) and \(A \neq 0\). Let the set of solutions \(\mathcal{L} := \{x : Ax = b\}\) be non-empty. Simply put, we consider a consistent linear system that has a solution but the solution is not necessarily unique. Richtárik and Takáč, reformulated
into different stochastic problems and showed that for any arbitrary consistent linear system, the stochastic reformulations of (5) are exact. In other words, the set of the solutions of any of those equivalent stochastic formulations is exactly the same as $L$ — which they formally defined as the exactness assumption (see Assumption 1) in Section 2. To motivate further, we will now introduce some technicalities. For a symmetric positive definite matrix $B$, denote $\langle \cdot, \cdot \rangle_B$ as the $B$-inner product and let $\|x\|_B = \sqrt{x^\top B x}$ be the (semi)-norm induced by it. Therefore, by using the idea proposed in [35], one can define a stochastic function $f_S(x) := \frac{1}{2}\|Ax - b\|_H^2$, where $H = S(S^\top A B^{-1} A^\top S)^{-1} S^\top$ is a random, symmetric, and positive definite matrix. Indeed minimizing (1) with $f_S(x) = \frac{1}{2}\|Ax - b\|_H^2$ solves (5). At this end, Richtárik and Takáč in [35] proposed a set of simple and easy-to-implement stochastic optimization algorithms—the basic method, the parallel or minibatch method, and an accelerated method to solve the stochastic reformulations of the linear system (5). The basic method is the primary algorithm whose iterative updates can be seen as SGD steps (as given in (3) [36]) applied to solve the problem (1) with a fixed stepsize parameter. Therefore, to solve (5), minimize the stochastic objective function $f_S(x) = \frac{1}{2}\|Ax - b\|_H^2$, and the iterates of the basic method takes the following form:

$$x_{k+1} = x_k - \omega B^{-1} A^\top S_k (S_k^\top A B^{-1} A^\top S_k)^{-1} S_k^\top (Ax_k - b),$$

where $S_k$ is sampled from the distribution $D$ in each iteration and $\omega > 0$ is a fixed stepsize parameter. As mentioned before, the parallel/minibatch method is a natural extension of the basic method applied to a synchronized system of $\tau$ processors such that each one of them can perform an SGD step. At the end the user averages the total yield to perform the iterative update step. We mention the parallel method formally in Section 2.1

1.2 Contribution

In this paper, we solve (1) via a stochastic reformulation of the linear system (5) in a distributed asynchronous set-up. We consider an ensemble with a central master server and say, $\tau$ independent workers, where the master obtains the gradients from the workers with a delay. Although, we follow the framework of the basic method proposed by Richtárik and Takáč in [35] to design our asynchronous SGD, our algorithm is closely related to Hogwild! of Recht et al. [32] and inspired by the delayed proximal gradient algorithm of Feyzmahdavian et al. [13]. In a shared-memory model with $\tau$ independent workers, our iterative scheme updates the vector $x$ that is accessible to all workers. Each worker can contribute an update to the vector $x$, although, they can be of different processing speeds. Therefore, whenever a worker computes a stochastic gradient at $x_k$, it performs a SGD step at that point and communicates the update to the master processor. The master eventually experiences a delay and updates the final iterate by using a convex combination of its current instance of the vector $x$ and the SGD update that was communicated with a delay. We explain this process formally in Section 3. The following are our main contributions in this paper:

- Inspired by Richtárik and Takáč’s parallel basic method in [35], we design an asynchronous parallel SGD to solve a consistent linear system. See Algorithm 1 in Section 3.

- We propose a detailed convergence analysis of our algorithm in Section 4. We compare the convergence rates of our asynchronous algorithm with the basic method and the synchronous parallel algorithm proposed by Richtárik and Takáč in [35]. At this end, we consider different choices of the stepsize parameter ($\omega$), the damping factor ($\theta$), number of processors ($\tau$), and the delay factor ($d$) to analyze our results. We refer the readers to Section 3 for details about this parameters. Moreover, our convergence analysis does not require any stronger assumption such as sparsity as in [24] and Leblond et al. [20]. We refer the readers to Table 1 for a quick overview of these results.

- We compare the iteration complexity of the asynchronous and synchronous method. We find that asymptotically as the number of processors, $\tau$, approaches to $\infty$, asynchronous approach has a better iteration complexity than its synchronous counterpart for certain linear systems. Moreover, for such linear systems, we also compute the minimum number of processors such that the asynchronous method has a better iteration complexity than the synchronous method, and find that this number can be as low as 2 in some cases, even for highly ill-conditioned problems. We show them in Table 2 and Table 3.

\footnote{In order to solve (5) via minimizing (1), one only needs local information of the stochastic function $f_S(x)$, for example, the stochastic gradient $\nabla f_S(x)$ without any explicit access to the function, its gradient, or the Hessian.}
1.3 Centralized algorithms—Related work

As the digital data and computing power of the processors are increasing, in the past decade, there has been a strong focus on developing the parallel versions of stochastic algorithms. Based on the communication protocol, these parallel algorithms can be broadly classified as centralized (that follows a master-worker architecture) and decentralized (for example, Allreduce communication strategy, see [10, 41]). In this paper, we will follow the centralized set-up, where a master node coordinates with all the worker nodes. Depending on the update rule, the centralized algorithms can be further categorized into two categories—synchronous and asynchronous. In this scope, for completeness, we will quote a few representatives of each of those categories. While Zinkevich et al. [45] proposed and analyzed synchronous parallel SGD, Richtárik and Takáč in [34] showed by parallelizing, randomized block coordinate descent methods can be accelerated. Yang [42] proposed a distributed stochastic dual coordinate ascent algorithm in a star-shaped distributed network, and analyzed the trade-off between computation and communication. In a similar spirit, Jaggi et al. [18] proposed Communication-efficient distributed dual Coordinate Ascent or COCOA that uses an arbitrary dual optimization method on the local data on each computing node in parallel and reduces communication (we refer to the references in [18, 40] for distributed primal-dual methods; additionally, see [27] for application to distributed MPC). Fercoq and Richtárik in [12] proposed APPROX or Accelerated Parallel PROXimal method—a union of three ideas, that is, acceleration, parallelization, and proximal method. In [33], Richtárik and Takáč proposed and analyzed a hybrid coordinate descent method known as HYDRA that partitions the coordinates over the nodes, independently from the other nodes, and applies updates to the selected coordinates in parallel (we also refer to [4, 25] and HYDRA-2 [11] for more insights). In a similar line of work, Shamir et al. [39] proposed a distributed approximate Newton-type method or DANE. Synchronous stochastic algorithms are well explored regarding their convergence rates, acceleration, and parallelization [38, 19]. However, they may suffer from the memory locking, that is, the processors or the computing nodes need to wait for the update from the slowest node. Hogwild! by Recht et al. [32] is one of the prime examples of asynchronous stochastic algorithms and first one of its kind which do not use the memory locking protocol and as a result, the computing nodes can modify the parameters at the same time. De Sa et al. [8] proposed Buckwild! which is a low-precision asynchronous SGD. Additionally, they analyzed Hogwild! type algorithms with relaxed assumptions and analyzed asynchronous SGD algorithms for (non-convex) matrix-completion type problems (also see [28]). Chaturapruet et al. [5] showed that for convex problems, under similar conditions as regular SGD, asynchronous SGD achieves similar asymptotic convergence rate. However, as in [5] the perturbed iterate analysis of Mania et al. [24] and Leblond et al. [20] for proving the convergence of asynchronous SGD use rigorous sparsity assumption. Noel et al. [29] proposed Dogwild! that is distributed hogwild for CPU and GPU. In the advent of the deep neural networks, asynchronous parallel SGD type algorithms are highly deployed in practice. Recently, Lian et al. [22] showed that in their setting, proposed asynchronous parallel algorithms can achieve a linear speedup if the number of workers are bounded by the square root of the total number of iterations. In 2017, Zheng et al. [44] proposed an algorithm called delay compensated asynchronous SGD (DC-ASGD) for training deep neural networks and to compensate the delayed gradient update by the local workers to the global model. With experimental validity on deep neural networks, Zheng et al. claimed that their DC-ASGD outperforms both synchronous SGD and asynchronous SGD, and nearly approaches the performance of sequential SGD. Among the others, asynchronous algorithms by Aytekin et. al [11], distributed SDCA by Ma et. al [23], distributed SVRG by Lee et. al [21] and Zhao and Li [43], asynchronous parallel SAGA by Leblond et al. [20], proximal asynchronous SAGA by Pedregosa et al. [30], are to name a few. We refer the readers to [7, 2] for an in-depth understanding.

Notation

We provide a table of the most frequently used notation in this paper for convenience (See Appendix B). Here we include some basic notations. We write the matrices in bold uppercase letters and denote vectors and scalars by simple lowercase letters. We define the range space and null space of a matrix \( A \in \mathbb{R}^{m \times n} \) as \( \text{Im}(A) := \{ y \in \mathbb{R}^m : y = Ax \} \) and \( \text{N}(A) := \{ x \in \mathbb{R}^n : Ax = 0 \} \), respectively. We further define the Euclidean inner product as \( \langle \cdot, \cdot \rangle \) and for a symmetric positive definite matrix \( B \), we denote \( \langle \cdot, \cdot \rangle_B \) as the \( B \)-inner product and define \( \| x \|_B = \sqrt{x^\top B x} \) as the (semi)-norm induced by it.
### Assumption 1. \cite{35} Let $\mathcal{X} = \arg \min_{x \in \mathbb{R}^n} f(x) = \{x : f(x) = 0\} = \{x : \nabla f(x) = 0\}$. Then $\mathcal{X} = \mathcal{L}$.

By $x_* = \Pi_{\mathcal{L}}(x_0)$ we denote $x_*$ to be the projection of the initial iterate $x_0$ onto the set $\mathcal{L}$ in B-norm and quote the following results.

### Organization

The paper is organized as follows. In Section 2, we review some key results related to the stochastic reformulation of linear systems and describe the synchronous parallel method by Richtárik and Takáč, in \cite{35}. Next in Section 3, we present our asynchronous parallel SGD. We compare the convergence rates of the asynchronous SGD with the synchronous parallel SGD, we define, $\theta := \frac{1}{\sqrt{1+c(2-k)-1/\sqrt{(2-k)}}} / \sqrt{1+c(2-k)-1/\sqrt{(2-k)}}$. For parallel SGD, we define $\xi \lambda = \frac{1}{\sqrt{1+c(2-k)-1/\sqrt{(2-k)}}} / \sqrt{1+c(2-k)-1/\sqrt{(2-k)}}$. Table 1: Iteration complexities of asynchronous parallel SGD (APSGD) and parallel SGD or parallel basic method. For asynchronous SGD, we define, $\xi(1, \tau) := \frac{1}{4} + \frac{1+\sqrt{1+2c(1-\lambda_{\max})}}{4c\tau}$, $\xi(\omega, \tau) := \frac{3\lambda_{\max}^+ + \lambda_{\max}}{4} + \frac{1+\sqrt{1+2c(1-\lambda_{\max})}}{4c\tau}$, where $k = \lambda_{\min}^+ + \lambda_{\max}$ and $c \geq 1$. We also define, $\theta_1 := \frac{1}{\sqrt{1+c(2-k)-1/\sqrt{(2-k)}}} / \sqrt{1+c(2-k)-1/\sqrt{(2-k)}}$.

| Algorithm     | Quantity | Case      | $\omega$ | $\theta$ | $\tau$ | Complexity                                                                 | Reference |
|---------------|----------|-----------|----------|----------|--------|-----------------------------------------------------------------------------|-----------|
| APSGD         | $E[\|x_t - x_*\|_B^2]$ | $\omega^* \leq 2$ | 1        | $\theta_1$ | $\tau$ | $\xi(1, \tau)$                                                          | This paper |
| APSGD         | $E[\|x_t - x_*\|_B^2]$ | $\omega^* \geq 2$ | 1        | $\theta_1$ | $\tau$ | $\xi(1, \tau)$                                                          | This paper |
| Parallel SGD  | $E[\|x_t - x_*\|_B^2]$ | $\omega \in (0, 2/\xi_\tau(\tau))$ | 1        | $\frac{1}{\xi_\tau}$ | $\tau$ | $\xi_\tau(\tau)$                                                      | \cite{35} |
|               |          |           |          | $1/\lambda_{\max}$ | $\tau$ | $\lambda_{\min}$                                                      | \cite{35} |
Lemma 1 (Lemma 4.7 in [35]). For all \( x \in \mathbb{R}^n, x_* \in \mathcal{L} \) and for a given \( S \) we have

\[
\|x - x_* - \omega \nabla f_S(x)\|_B^2 = \|(I - \omega B^{-1}Z)(x - x_*)\|_B^2 = \|x - x_*\|_B^2 - 2\omega(2 - \omega) f_S(x). \tag{7}
\]

Lemma 2 (Lemma 4.2 in [35]). For all \( x \in \mathbb{R}^n \) and \( x_* = \Pi^B_\mathcal{L}(x) \) we have

\[
\frac{\lambda_{\text{max}}}{2} \|x - x_*\|_B^2 \leq f(x) \leq \frac{\lambda_{\text{max}}}{2} \|x - x_*\|_B^2. \tag{8}
\]

Lemma 3 (Lemma 4.5 in [35]). Consider any \( x \in \mathbb{R}^n \) and \( x_* = \Pi^B_\mathcal{L}(x) \). If \( \lambda_i = 0 \) then we have \( u_i^\top B^{1/2}(x - x_*) = 0 \).

### 2.1 Parallel method

In this section, we describe the parallel basic method proposed by Richtárik and Takáč in [35]. This is also referred to as the minibatch method. Let there be \( \tau \) processors working independently. Starting from a given iterate \( x_k \), the parallel or minibatch method performs one step of the basic method independently on each of the \( \tau \) processors and finally averages the results. This leads to the update rule of parallel basic method (see [35]) as follows:

\[
x_{k+1} = \frac{1}{\tau} \sum_{i=1}^\tau z_{k+1,i}, \tag{9}
\]

where \( z_{k+1,i} = x_k - \omega B^{-1}A^\top S_{ki}(S_{ki}^\top A B^{-1}A^\top S_{ki})^{-1} S_{ki}^\top (Ax_k - b) \). Note that, in each iterate an independent sample \( S_{ki} \sim \mathcal{D} \) drawn afresh for each of the \( \tau \) processors and \( \omega > 0 \) is a fixed stepsize parameter. One can see the parallel basic method as a synchronous parallel method. In fact, the parallel basic method enjoys a linear convergence rate under the exactness assumption.

Theorem 1 (Convergence of parallel basic method [35]). Let the exactness assumption hold and \( x_* = \Pi^B_\mathcal{L}(x_0) \). Let \( \{x_k\}_{k \geq 0} \) be the sequence of random iterates produced by the parallel method (see [8]) where the stepsize \( \omega \in (0, 2/\xi_s(\tau)) \), such that \( \xi_s(\tau) = \frac{1}{\tau} + (1 - \frac{1}{\tau})\lambda_{\text{max}} \). Then

\[
\mathbb{E}[\|x_{k+1} - x_*\|_B^2] \leq \rho_s(\omega, \tau) \frac{\lambda_{\text{max}}}{2} \|x_0 - x_*\|_B^2, \tag{10}
\]

where

\[
\rho_s(\omega, \tau) = 1 - \omega(2 - \omega \xi_s(\tau))\lambda_{\text{max}}^+. \tag{11}
\]

[35] further showed that \( \rho_s(\omega, \tau) \) is minimized for \( \omega(\tau) = 1/\xi_s(\tau) \) and the optimal \( \rho_{s_{\text{opt}}}(\omega(\tau), \tau) \) is

\[
\rho_{s_{\text{opt}}}(\omega(\tau), \tau) = 1 - \frac{\lambda_{\text{min}}^+}{\frac{1}{\tau} + (1 - \frac{1}{\tau})\lambda_{\text{max}}}. \tag{12}
\]

Let \( \chi_{s_{\text{opt}}}(\tau) \) be the best iteration complexity of the synchronous parallel method. Then as in [35], we find:

\[
\chi_{s_{\text{opt}}}(\tau) = \mathcal{O} \left( \frac{\frac{1}{\tau} + (1 - \frac{1}{\tau})\lambda_{\text{max}}}{\lambda_{\text{min}}^+} \right) = \mathcal{O} \left( \frac{\xi_s(\tau)}{\lambda_{\text{min}}^+} \right). \tag{13}
\]

which is achieved at \( \omega = \frac{1}{\xi_s(\tau)} \).

Remark 2. The best strong convergence rate for the basic method is achieved when the stepsize parameter \( \omega = 1 \). We will define the strong convergence in Section 4.2.
3 Asynchronous parallel SGD

Let there be \( \tau \) independent processors or workers and a central server or the master (as described in Section 1) and we perform the iterative updates in an asynchronous manner. That is, whenever a worker computes a stochastic gradient at a given point, it performs a SGD step at that point and communicates the update to the master processor. After that, the master generates a new update by using a convex combination of its current update and the update reported to it by the worker, and communicates back the resulting update to the worker to perform the next SGD step, and this process continues. Regardless of their processing speeds, whenever a worker communicates its latest update to the master, the master makes a convex combination of the latest iterate with it and assigns it to the worker to perform an SGD step. Therefore, no worker stays idle. Each of them performs and communicates the task they are assigned to the master, albeit in an asynchronous way. This iterative protocol was introduced in [13] and our proposed update rule is inspired by it.

Let \( t \) denote the iteration count with respect to the master’s frame. Let \( \delta \geq 0 \) be a factor representing delay between the iterates of the master and the workers, which indeed varies in each iteration. Let \( \delta_a \) be the maximum delay throughout the execution of the algorithm. Note that the delay \( \delta \) is functions of \( t \). But for brevity, we will write it as \( \delta \) in this paper. Let \( \theta \in [0, 1] \) represent a damping factor. Choose \( x_0 \in \mathbb{R}^n \) and consider an iterative method defined for \( t \geq 1 \) as:

\[
\begin{align*}
y_t &= x_{t-\delta} - \omega \nabla f_{S_{t-\delta}}(x_{t-\delta}), \\
x_{t+1} &= (1 - \theta)x_t + \theta y_t.
\end{align*}
\]

(13)

(14)

Recall from Remark [1] that \( \nabla f_S(x) = B^{-1}Z(x - x_\star) \), where \( x_\star \) is any solution of the system \( Ax = b \). Therefore, we can rewrite the update rule as:

\[
\begin{align*}
x_{t+1} - x_\star &= (1 - \theta)(x_t - x_\star) + \theta(I - \omega B^{-1}Z_{t-\delta})(x_{t-\delta} - x_\star),
\end{align*}
\]

(15)

where we denote \( Z_t = Z_{S_t} \) and \( S_t \) is sampled independently from \( D \).

Algorithm 1: Distributed Asynchronous SGD in master-worker architecture (time \( t \) in master’s frame)

| System | 1 Master and \( K \) workers, \( k = 1, \ldots, K \); |
| repeat | |
| | Master |
| 2 | |
| 3 | Receive \( y_t \) from a worker \( k \); |
| 4 | Update current iterate \( x_t \) by using equation (14); |
| 5 | Send updated iterate \( x_{t+1} \) to worker \( k \); |
| Workers | |
| 6 | |
| 7 | Receive iterate \( x_t \) from master; |
| 8 | Compute \( y_{t+\delta} \) via equation (13); |
| 9 | Send \( y_{t+\delta} \) to the master; |
| until convergence | |

4 Convergence analysis: Comparison with the synchronous parallel method

Denote a unit time interval as the time in which the slowest worker performs an update. By slowest, we mean having the maximum delay between an iterate sent to the worker by the master, to its SGD update being considered by the master. The slowness can be attributed due to less processing power, or/and due to processing a “tougher” job in terms of time complexity, or due to slower communication between this worker and the master. For the sake of simplicity, we assume the number of updates by the asynchronous parallel SGD in a unit-time interval to be a constant throughout the execution of the algorithm. Thus, in a unit-time interval, we assume exactly \( \delta_a \) updates by the master. Let \( S_t \) be the
number of updates the $i^{th}$ processor performs in this time interval. Thus, $\delta_i = \sum_i S_i$. We note that in this unit time interval, the synchronous parallel method performs one step of the basic method independently $\tau$ times and averages the results, where $\delta_i \geq \tau$. The last inequality is obvious because of the fundamental structure of the asynchronous method. Thus, the asynchronous algorithm leverages the idle time of the faster worker to perform more updates in a unit-time interval in comparison to its synchronous counterpart, but in an asynchronous manner. Let $c \geq 1$ be such that $\delta_i = c\tau$. Then $\rho_a(\theta, \omega)$ be the rate of convergence of the asynchronous parallel SGD in a unit-time interval, with step-size $\omega$ and damping parameter $\theta$.

## 4.1 Key results used for convergence

In this section we propose a few key results that are necessary for our convergence analysis in Section 4.2. As in [35], we do not formally provide the weak convergence analysis of our asynchronous parallel method (see (15)). The convergence results in Section 4.2 is strong convergence (and that is how the algorithm is expected to behave in practice). Moreover, the weak convergence can be directly implied from the strong convergence result. First, in Theorem[2] we establish a recurrence relation that can lead to the weak convergence of our algorithm.

**Theorem 2.** Denote $P_1 = U^T B^{1/2} E[x_t - x_*]$. Then from (15) we obtain the following:

$$U^T B^{1/2} E[x_{t+1} - x_*] = (1 - \theta) U^T B^{1/2} E[x_t - x_*] + \theta (I - \omega A) U^T B^{1/2} E[x_{t-\delta} - x_*],$$

(16)

**Proof.** See Appendix A

**Remark 3.** One can split (16) coordinate-wise. Thus for each $1 \leq i \leq n$, (16) can be written as

$$P^i_{t+1} = (1 - \theta) P^i_t + \theta (1 - \omega \lambda^i) P^i_{t-\delta},$$

(17)

where $P^i_t$ is the $i^{th}$ coordinate of $P_t$. Therefore, we are left to understand the convergence of the following recurrence relation:

$$p_{t+1} = a p_t + b p_{t-\delta},$$

(18)

where $a = 1 - \theta > 0$, $b = \theta (1 - \omega \lambda_i) > 0$, and $a + b < 1$.

The state transition matrix for the recurrence relation in (18) is

$$\begin{pmatrix}
 p_t \\
 \vdots \\
 p_{t-\delta}
\end{pmatrix} = A
\begin{pmatrix}
 p_{t-1} \\
 \vdots \\
 p_{t-\delta-1}
\end{pmatrix},$$

(19)

where $A = \begin{pmatrix} a & 0_{n-2}^T \\ 0_{n-1} & b \end{pmatrix}$ and $0_n$ is a vector in $\mathbb{R}^n$ with all zeros. One can analyze the convergence of (16) by analyzing the spectral radius, $\rho(A)$, of the state transition matrix $A$. Note that, the characteristic equation of the matrix $A$ is:

$$\gamma^{\delta+1} - a \gamma^\delta - b = 0.$$

(20)

Calculating the spectral radius of $A$ is equivalent to finding the magnitude of the highest root of the polynomial in (20). We quote two classic results: one is on the bounds of the root of a polynomial and the second one is on the spectral radius of a non-negative matrix. At the core they are the same results and complement each other.

**Theorem 3.** [Cauchy [31]] Let $f(x) = x^n - \sum_{i=1}^{n} b_i x^{n-i}$ be a polynomial, where all the coefficients $b_i$‘s are non-negative and at least one of them is nonzero. The polynomial $f(x)$ has a unique (simple) positive root $p$ and the absolute values of the other roots do not exceed $p$.

$^2$A sequence of random vectors $\{x_k\}_{k \geq 0}$ (the iterates) converge to $x_*$ weakly if $\|E[x_k - x_*]\|_\infty \to 0$ as $k \to \infty$. 

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Theorem 4. \cite{Perron-Frobenius} Let \( M = (m_{ij})_{n \times n} \) be a square irreducible non-negative matrix, that is \( m_{ij} \geq 0 \) for \( 1 \leq i, j \leq n \). Then there exists a positive real number \( r \) such that \( r \) is a simple eigenvalue of the matrix \( M \) and the magnitude of all other eigenvalues of \( M \) is strictly smaller than \( r \). Therefore \( \rho(M) = r \). Moreover, \( r \) satisfies
\[
\min_j m_{ij} \leq r \leq \max_j \sum_i m_{ij}.
\]

Theorem 3 was proposed by Cauchy and Theorem 4 is the famous Perron-Frobenius Theorem on non-negative matrices. In our case, Cauchy’s theorem is directly related to the Perron-Frobenius theorem. The positive real number \( r \) in Theorem 4 is same as \( p \) in Theorem 3 and is known as the Perron root or Perron-Frobenius eigenvalue. It is easy to check that the associated digraph for the matrix \( A \) is strongly connected and hence \( A \) is an irreducible non-negative matrix. By Theorem 3 (or Theorem 4) we can conclude the spectral radius of \( A \) is the unique positive real root of (20).

4.2 Strong convergence

To demonstrate the convergence of our asynchronous parallel method, we follow the convention used by \cite{35} and refer to this convergence analysis as strong convergence analysis. We recall that Richtárik and Takáč in \cite{35} defined the strong convergence as follows: A sequence of random vectors \( \{x_k\}_{k \geq 0} \) (the iterates) converge to \( x_* \) strongly if \( \mathbb{E}[\|x_k - x_*\|_B^2] \to 0 \) as \( k \to \infty \). We start with the following Lemma.

Lemma 4. Assume that \( x_0 \in \text{Im}(B^{-1}A^\top) \). Then \( x_t \in \text{Im}(B^{-1}A^\top) \) for all \( t \). It follows that \( \Pi_{B}^{\mathbb{P}}(x_t) = \Pi_{B}^{\mathbb{P}}(x_0) \) for all \( t \).

Proof. See Appendix A.

4.2.1 Two scenarios based on the relative position of \( \omega^* \) and 2

Let \( \omega^* = \frac{\lambda_{\min}^+ + \lambda_{\max}}{2} \). According to the problem, there may be two scenarios based on the relative position of \( \omega^* \) with respect to 2 as follows:

\begin{itemize}
  \item \textbf{Case 1:} \( \omega^* \leq 2 \) i.e. \( \lambda_{\min}^+ + \lambda_{\max} \geq 1 \).
  \item \textbf{Case 2:} \( \omega^* \geq 2 \) i.e. \( \lambda_{\min}^+ + \lambda_{\max} \leq 1 \).
\end{itemize}

Definition 1. Define \( \alpha(\omega) := \max_i,\lambda_i > 0 |1 - \omega \lambda_i| \). It is easy to see that
\[
\alpha(\omega) = \begin{cases} 
1 - \omega \lambda_{\min}^+, & \text{if } 0 \leq \omega \leq \omega^* \\
\omega \lambda_{\max} - 1, & \text{if } \omega \geq \omega^*.
\end{cases}
\]

4.2.2 Recurrence relation

The following theorem establishes a recurrence relation that we need to analyze the strong convergence of our asynchronous parallel method.

\textbf{Theorem 5 (Recurrence).} Assume exactness. Assume that \( x_0, \ldots, x_{\delta_a} \in \text{Im}(B^{-1}A^\top) \) and let \( x_* = \Pi_{B}^{P}(x_0) \). Let \( \{x_t\} \) be the sequence of random iterates produced by the asynchronous method (via (15)) with delay \( \delta \geq 0 \), stepsize \( \omega \geq 0 \), and damping parameter \( \theta \in [0, 1] \). Let \( r_t = B^{1/2}\|x_t - x_*\| \) and \( \alpha(\omega) := \max_i,\lambda_i > 0 |1 - \omega \lambda_i| \). Then
\[
\mathbb{E}[\|r_{t+1}\|^2] \leq K_1(\theta, \omega)\mathbb{E}[\|r_t\|^2] + K_2(\theta, \omega)\mathbb{E}[\|r_{t-\delta}\|^2],
\]
and we have the following estimates of \( K_1(\theta, \omega) \) and \( K_2(\theta, \omega) \):

\begin{itemize}
  \item \textbf{Case 1:}
\end{itemize}
(i) For $\omega \leq \omega^* \leq 2$ and $\alpha(\omega) = 1 - \omega \lambda_{\text{min}}^+$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\text{min}}^+) + (1 - \theta)\alpha(\omega)).$$  \hfill (22)

(ii) For $\omega^* \leq \omega \leq 2$ and $\alpha(\omega) = \omega \lambda_{\text{max}} - 1$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\text{min}}^+) + (1 - \theta)\alpha(\omega)).$$

(iii) For $\omega \geq 2$ and $\alpha(\omega) = \omega \lambda_{\text{max}} - 1$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\text{max}}) + (1 - \theta)\alpha(\omega)).$$

**Case 2:**

(i) For $2 \leq \omega \leq \omega^*$ and $\alpha(\omega) = 1 - \omega \lambda_{\text{min}}^+$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\text{min}}^+) + (1 - \theta)\alpha(\omega)).$$  \hfill (23)

(ii) For $2 \leq \omega \leq \omega^*$ and $\alpha(\omega) = 1 - \omega \lambda_{\text{min}}^+$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\text{min}}^+) + (1 - \theta)\alpha(\omega)).$$

(iii) For $\omega \geq \omega^*$ and $\alpha(\omega) = \omega \lambda_{\text{max}} - 1$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\text{max}}) + (1 - \theta)\alpha(\omega)).$$

**Proof.** See Appendix A. \hfill \Box

**Remark 4.** To analyze the convergence of the recurrence relation in (21) we can replace the inequality in (21) with equality and write

$$q_{t+1} = K_1(\theta, \omega)q_t + K_2(\theta, \omega)q_{t-\delta},$$

where we initialize the process by setting $q_t = \mathbb{E}[\|x_t - x_*\|_B^2]$ for $t \in \{0, \ldots, \tau\}$. It can be easily seen by using induction that $\mathbb{E}[\|x_t - x_*\|_B^2] \leq q_t$ for all $t$, and hence we can claim that the rate of convergence of $\mathbb{E}[\|x_t - x_*\|_B^2]$ will not be slower than $q_t$. 

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Remark 5. Note that (25) is similar to (18). Therefore, one can obtain a recurrence relation that involves a state transition matrix similar to that in (19). To analyze the strong convergence of our method and to compare it against the parallel method, we examine the characteristic equation of the state transition matrix in (25).

4.2.3 Characteristic polynomial

The characteristic polynomial of the state transition matrix in recurrence (25) is

\[ p_\delta(\gamma) := \gamma^{\delta+1} - K_1(\theta, \omega)\gamma^\delta - K_2(\theta, \omega). \] (26)

For convenience, we denote \( K_1 = K_1(\theta, \omega) \) and \( K_2 = K_2(\theta, \omega) \), from now on. By using Theorem 3 (or Theorem 4), we can conclude that the positive root of (26) is the spectral radius of (26), and therefore, we can obtain the convergence rate of (25). This motivates us to propose the following theorem.

Theorem 6. Denote \( \varrho_A(\theta, \omega, \delta) \) as the spectral radius of (25). Then \( \varrho_A(\theta, \omega, \delta) \) is the convergence factor of (25).

Remark 6. We see that the spectral radius is smaller than 1 only when \( K_1 + K_2 < 1 \), as \( p_\delta(1) \) should be positive for the root to be smaller than 1. If \( K_1 + K_2 \geq 1 \), then either the spectral radius is not a good bound for the one-step rate of convergence or the algorithm itself does not converge.

Lemma 5. For all the updates, the delay factor \( \delta \leq \delta_a \), we have for all \( \delta 
\[ \varrho_A(\theta, \omega, \delta) \leq \varrho_A(\theta, \omega, \delta_a). \] (27)

Proof. See Appendix A.

Lemma 6. Let \( \delta_i \) denote the delay factor for the \( i^{th} \) update in a unit time interval. Then, we have the following relation for unit time rate of convergence \( \rho_a(\theta, \omega) \)

\[ \rho_a(\theta, \omega) \leq \prod_{i=1}^{\delta_a} \varrho(\theta, \omega, \delta_i) \leq \varrho(\theta, \omega, \delta_a)^{\delta_a} \] (28)

Proof. See Appendix A.

4.2.4 Bounding polynomial for the characteristic polynomial

It is hard to find a closed form analytic expression for the unique positive root (or, the spectral radius) of the polynomial. Therefore, we find a polynomial that bounds \( p_\delta(\gamma) \) in the interval \([0, 1]\). This will provide us with a bound for the spectral radius and we can comment on the convergence factor of the asynchronous parallel method.

Lemma 7. The polynomial

\[ g_\delta(\gamma) := \left( 1 + \frac{1}{\delta} - K_1 \right) \gamma^\delta - \left( K_2 + \frac{1}{\delta} \right) \] (29)

bounds the characteristic polynomial \( p_\delta(\gamma) \) from below on \([0, 1]\) and its root

\[ u(\theta, \omega, \delta) = \left( \frac{K_2 + \frac{1}{\delta}}{1 - K_1 + \frac{1}{\delta}} \right)^{\frac{1}{\delta}} \] (30)

is an upper bound to the unique positive root of \( p_\delta(\gamma) \).

Proof. See Appendix A.
4.2.5 The convergence factor $\rho_a(\theta, \omega)$ for a unit time interval

**Lemma 8.** Recall the unit-time interval is the time in which the slowest processor (or the processor which takes the maximum time to perform an update) preforms an update. Then

$$\rho_a(\theta, \omega) \leq u(\theta, \omega, \delta_a).$$

(31)

**Proof.** The proof follows from Lemma 6 and Lemma 7.

From (30) we have

$$\rho_a(\theta, \omega) \leq \left( \frac{K_2 + \frac{1}{s_a}}{1 - K_1 + \frac{1}{s_a}} \right).$$

(32)

Let $\rho_{a,\text{opt}}$ denote the optimal rate of convergence for the asynchronous SGD algorithm. Then

$$\rho_{a,\text{opt}} \leq \min_{\theta, \omega} \left( \frac{K_2 + \frac{1}{s_a}}{1 - K_1 + \frac{1}{s_a}} \right).$$

(33)

4.2.6 The iteration complexity

Let $\chi_a(\theta, \omega, \delta_a)$ denote the iteration complexity of the asynchronous SGD for some choice of $\theta$ and $\omega$, where 1 iteration denotes one unit time interval. This ensures a fair comparison between synchronous and asynchronous SGD as synchronous parallel SGD performs single update of the parameter $x_t$ in a unit time interval.

**Remark 7.** For simplicity, we are showing the iteration complexity results without the “big Oh” ($O$) notation in this and all the subsequent sections.

Then

$$\chi_a(\theta, \omega, \delta_a) = \frac{1}{1 - \rho_a(\theta, \omega)}.$$

Notice that, for iteration complexity, the above expression holds only when $\rho_a(\theta, \omega) < 1$. Let $\chi_{a,\text{opt}}(\delta_a)$ denote the best possible iteration complexity of the asynchronous algorithm, that is,

$$\chi_{a,\text{opt}}(\delta_a) = \min_{\theta, \omega} \chi_a(\theta, \omega, \delta_a)$$

(34)

which implies

$$\chi_{a,\text{opt}}(\delta_a) = \frac{1}{1 - \rho_{a,\text{opt}}}.$$  

(35)

From (33) we have

$$\chi_{a,\text{opt}}(\delta_a) \leq \min_{\theta, \omega} \left( \frac{1 - K_1 + \frac{1}{s_a}}{1 - K_1 - K_2} \right).$$

(36)

Denote

$$U(\theta, \omega) = \frac{1 - K_1 + \frac{1}{s_a}}{1 - K_1 - K_2}.$$  

(37)

For an arbitrary choice of $\theta$ and $\omega$, $U(\theta, \omega)$ denotes an upper bound on the iteration complexity of the asynchronous SGD with $\theta$ and $\omega$ set as the parameters of the algorithm. One should again note that the above expression for $U(\theta, \omega)$ is feasible only when $K_1 + K_2 < 1$. In the next section, we show that we can find optimal $(\theta, \omega)$ for Case 1, and a good combination of $(\theta, \omega)$ for Case 2. Therefore,

$$\chi_{a,\text{opt}}(\delta_a) \leq \min_{\theta, \omega} U(\theta, \omega).$$

(38)

**Remark 8.** In this paper, our goal is to compare between $\chi_{s,\text{opt}}(\tau)$ (as defined in Theorem 4) and $\chi_{a,\text{opt}}(\delta_a)$.
4.2.7 The optimal $\omega$

**Lemma 9.** Assume that we have two choices of $\omega$, say, $\omega_1$ and $\omega_2$, such that for all $\theta \in [0, 1]$, both $K_1$ and $K_2$ are smaller for $\omega_1$ than $\omega_2$. Then $\omega_1$ is a better choice than $\omega_2$.

*Proof.* See Appendix A.

**Lemma 10.** The optimal $\omega$ for all $\theta$ in $[0, 1]$ in both the cases (See Theorem 5) lies in the range $[1, \omega^*]$.

*Proof.* See Appendix A.

We now try to find the minimum value of the upper bound $U(\theta, \omega)$ for $\theta \in [0, 1]$ and $\omega \in [1, \omega^*]$.

### 4.2.8 Upper bounding the condition number for Case 1

As proved in Lemma 10, we only need to focus in the region $\omega \in [1, \omega^*]$. Please see Figure 1(a). From (22) we have:

$$K_1(\theta, \omega) := (1-\theta)^2 + \theta(1-\theta)(1-\omega \lambda^+_{\min}), \quad K_2(\theta, \omega) := \theta^2 (1 - \omega(2 - \omega) \lambda^+_{\min} + \theta(1-\theta)(1-\omega \lambda^+_{\min}).$$

Substituting the values of $K_1$ and $K_2$ in (37), we have

$$U(\theta, \omega) = \frac{\theta (1 + (1-\theta) \omega \lambda^+_{\min}) + \frac{1}{\delta_n}}{\theta \omega (2 - \theta \omega) \lambda^+_{\min}}.$$  \hfill (40)

Now we compute $\nabla U$ and find:

$$\frac{\partial U}{\partial \theta} = \frac{\theta^2 \omega \delta_n (1 + (\omega - 2) \lambda^+_{\min}) + 2 (\theta \omega - 1)}{\theta^2 \omega \delta_n (2 - \theta \omega)^2 \lambda^+_{\min}}. \hfill (41)$$

Setting $\frac{\partial U}{\partial \theta} = 0$ we have

$$\theta^2 \omega \delta_n (1 + (\omega - 2) \lambda^+_{\min}) + 2 (\theta \omega - 1) = 0, \hfill (42)$$

which implies

$$\theta^2 (\omega \delta_n + \omega^2 \delta_n \lambda^+_{\min} - 2 \omega \delta_n \lambda^+_{\min}) + \theta (2\omega) - 2 = 0. \hfill (43)$$
Similarly, 
\[
\frac{\partial U}{\partial \omega} = \frac{2\delta a \theta (\theta \omega - 1) + \delta a \theta^2 \omega^2 (1 - \theta) \lambda_{\min}^+ + 2(\theta \omega - 1)}{\theta \omega^2 \delta a (2 - \theta \omega)^2 \lambda_{\min}^+},
\]
(44)
and setting \(\frac{\partial U}{\partial \omega} = 0\) gives
\[
2\delta a \theta (\theta \omega - 1) + \delta a \theta^2 \omega^2 (1 - \theta) \lambda_{\min}^+ + 2(\theta \omega - 1) = 0,
\]
(45)
which implies
\[
\theta^3 \left( -\omega^2 \delta a \lambda_{\min}^+ \right) + \theta^2 \left( \omega^2 \delta a \lambda_{\min}^+ + 2 \omega \delta a \right) + \theta \left( 2 \omega - 2 \delta a \right) - 2 = 0.
\]
(46)
For the gradient \(\nabla U\) to vanish inside the region of interest \((\omega \in (1, \omega^*)\) and \(\theta \in (0, 1)\)), we want both equations (43) as well as (46) to hold. We try to find out the solutions of: \(\ref{eq:43} - \ref{eq:46}=0\) as any solution satisfying both equations \(\ref{eq:43}\) and \(\ref{eq:46}\) should also satisfy \(\ref{eq:43} - \ref{eq:46}=0\). Therefore, we have
\[
\begin{align*}
\ref{eq:43} - \ref{eq:46} &= \theta^3 \left( \omega^2 \delta a \lambda_{\min}^+ \right) + \theta^2 \left( -2 \omega \delta a \lambda_{\min}^+ - \omega \delta a \right) + 2 \theta \delta a \\
&= \delta a \theta \left( \lambda_{\min}^+ \theta^2 \omega^2 - (2 \lambda_{\min}^+ + 1) \theta \omega + 2 \right). \\
\end{align*}
\]
(47)
A quadratic in \(\theta \omega\)

One solution of \(\ref{eq:43} - \ref{eq:46}=0\) from above is \(\theta = 0\), which does not lie inside the region of interest. Next we focus on the quadratic:
\[
\lambda_{\min}^+ \theta^2 \omega^2 - (2 \lambda_{\min}^+ + 1) \theta \omega + 2 = 0,
\]
(49)
that is,
\[
\lambda_{\min}^+ \left( \theta \omega - \frac{1}{\lambda_{\min}^+} \right) (\theta \omega - 2) = 0,
\]
(50)
which gives the solution \(\theta \omega = \frac{1}{\lambda_{\min}^+}\) and \(\theta \omega = 2\). The maximum value of \(\theta \omega\) attainable in the region of interest is \(\omega^*\), for \(\theta = 1\) and \(\omega = \omega^*\). Hence \(\theta \omega = \frac{1}{\lambda_{\min}^+}\) is not attainable inside the region of interest. Also, as we are dealing with Case 1, \(\omega^* \leq 2\), hence \(\theta \omega = 2\) is also not attainable inside the region of interest. Thus, \(\nabla U\) does not vanish inside the region of interest. Therefore, the minima lies on the boundary of the region. We now discuss following four boundary cases:

**4.2.9** \(\theta = 0\)

\(U\) is not defined.

**4.2.10** \(\theta = 1\)

We have
\[
U(1, \omega) = \frac{1 + \frac{1}{\lambda_{\min}^+}}{\omega(2 - \omega) \lambda_{\min}^+} \implies U(1, 1) = \frac{1 + \frac{1}{\lambda_{\min}^+}}{\lambda_{\min}^+}.
\]

We note that \(U(1, \omega) > \frac{1}{\lambda_{\min}^+}\) for all \(\delta a\).
4.2.11 $\omega = 1$

Let $\theta_1 = \arg\min_{\theta} U(\theta, 1)$. Then $\theta_1$ is the solution of equation (43) at $\omega = 1$. Substituting $\omega = 1$ in (43) we get the following quadratic in $\theta$

$$\theta^2\delta_a(1 - \lambda_{\min}^+) + 2\theta - 2 = 0.$$  

(51)

As we want $\theta$ in $[0, 1]$ and find $\theta_1 = \sqrt{1 + \lambda_{\min}^+ / \delta_a (1 - \lambda_{\min}^+)}$.

Therefore, $U(\theta_1, 1) = \frac{3\lambda_{\min}^+ + \lambda_{\max}}{4}$.

(52)

Note that for $\delta_a \geq 4$, $U(\theta_1, 1) \leq \frac{1}{\lambda_{\min}}$. Thus, when $c \tau \geq 4$, the asynchronous parallel method performs better than the basic method for Case 1.

4.2.12 $\omega = \omega^*$

Let $\theta_{\omega^*} = \arg\min_{\theta} U(\theta, \omega^*)$. Then we get $\theta_{\omega^*}$ as:

$$\theta_{\omega^*} = \frac{k(\sqrt{1 + \delta_a(2 - k)} - 1)}{\delta_a(2 - k)} \quad (k = \lambda_{\min}^+ + \lambda_{\max}).$$  

(54)

Therefore,

$$U(\theta_{\omega^*}, \omega^*) = \frac{k^2 + \lambda_{\min}^2}{2 \delta_a(2 - k)} + \frac{3\lambda_{\min}^2 + \lambda_{\max}}{4} + \frac{3\lambda_{\min}^2 + \lambda_{\max}}{4} \frac{\sqrt{1 + \delta_a(2 - k) + 2(\delta_a + 1 + \delta_a(1 - k)\lambda_{\min}^+)}}{2\delta_a \sqrt{1 + \delta_a(2 - k)}}.$$  

(55)

(56)

Lemma 11. For Case 1, we get the iteration complexity

$$\chi_{a, op}(\delta_a) \leq \min_{\theta} (U(\theta, 1), U(\theta_{\omega^*}, \omega^*)).$$  

(57)

4.2.13 Upper bounding the condition number for Case 2

We split the interval into two parts: $\omega \in [1, 2]$ and $\omega \in [2, \omega^*]$. Please see Figure 1b.

4.2.14 Part 1: $\omega \in [1, 2]$

$K_1(\theta, \omega)$ and $K_1(\theta, \omega)$ are the same as described in equation (39). Similar to the Case 1, we find that $\nabla U$ does not vanish inside the region $\theta \in (0, 1)$ and $\omega \in (1, 2)$. Therefore, the minima lies on the boundary of the region. We now discuss four boundary cases:

4.2.14.1 $\theta = 0$

$U$ is not defined.
4.2.14.2 \( \theta = 1 \)

We have
\[
U(1, \omega) = \frac{1 + \frac{1}{\delta_a}}{\omega(2 - \omega)} \lambda_{\min}^{\lambda^+} \quad \Rightarrow \quad U(1, 1) = \frac{1 + \frac{1}{\delta_a}}{\lambda_{\min}^{\lambda^+}}.
\]

We note that \( U(1, \omega) > \frac{1}{\lambda_{\min}^{\lambda^+}} \) for all \( \delta_a \).

4.2.14.3 \( \omega = 1 \)

Let \( \theta_1 = \arg\min_{\theta} U(\theta, 1) \). Then we get \( \theta_1 \) as:
\[
\theta_1 = \sqrt{1 + 2\delta_a(1 - \lambda_{\min}^{\lambda^+})} - 1 \quad \frac{\delta_a(1 - \lambda_{\min}^{\lambda^+})}{\delta_a(1 - \lambda_{\min}^{\lambda^+})}.
\]

Therefore,
\[
U(\theta_1, 1) = \frac{3 + \sqrt{1 + 2\delta_a(1 - \lambda_{\min}^{\lambda^+})}}{\lambda_{\min}^{\lambda^+}}.
\]

We note that for \( \delta_a \geq 4, U(\theta_1, 1) \leq \frac{1}{\lambda_{\min}^{\lambda^+}} \).

4.2.14.4 \( \omega = 2 \)

Let \( \theta_2 = \arg\min_{\theta} U(\theta, 2) \). Then we get \( \theta_2 \) as:
\[
\theta_2 = \sqrt{\delta_a + 1} - 1 \quad \frac{\delta_a + 1}{\delta_a}.
\]

Therefore,
\[
U(\theta_2, 2) = \frac{1 + \lambda_{\min}^{\lambda^+}}{\lambda_{\min}^{\lambda^+}} + \frac{1 + \sqrt{\delta_a + 1}}{2\delta_a}.
\]

We note that for \( \delta_a \geq 3, U(\theta_2, 2) \leq \frac{1}{\lambda_{\min}^{\lambda^+}} \). Thus, for \( \omega \geq 3 \), the asynchronous parallel method performs better than the basic method for Case 2.

Lemma 12. For Case 2,
\[
U(\theta_2, 2) \leq U(\theta_1, 1) \leq U(1, 1) \quad \forall \delta_a \geq 4.
\]

Proof. See Appendix A. \( \square \)

Thus, in Case 2, the optimal stepsize is \( \omega = 2 \) for \( \omega \in [1, 2] \). This leads us to the following result:

Lemma 13. For Case 2, the iteration complexity
\[
\chi_{a, \omega^*}(\delta_a) \leq \frac{1 + \lambda_{\min}^{\lambda^+}}{\lambda_{\min}^{\lambda^+}} + \frac{1 + \sqrt{\delta_a + 1}}{2\delta_a}.
\]

4.2.15 Part 2: \( \omega \in [2, \omega^*] \)

We were not able to find the optimal stepsize \( \theta \) for Case 2 in \( \omega \in [2, \omega^*] \). However, the iteration complexity for \( \omega = 2 \) is a significant result.
\[ k = \lambda_{\min}^+ + \lambda_{\max} \geq 1. \]

Minimum number of processors \( \tau \) required for asynchronous SGD to have better iteration complexity than synchronous SGD.

### 4.3 Comparing the iteration complexity with the synchronous parallel method

We know from (12) the best iteration complexity \( \chi_{\text{s}_{\text{opt}}} (\tau) \) of the synchronous parallel method is

\[ \chi_{\text{s}_{\text{opt}}} (\tau) = \frac{\frac{1}{2} + \left(1 - \frac{1}{\tau}\right) \lambda_{\max}}{\lambda_{\min}^+}. \]

Rearranging the above equation we get

\[ \chi_{\text{s}_{\text{opt}}} (\tau) = \frac{\lambda_{\max} + \frac{1 - \lambda_{\max}}{\tau}}{\lambda_{\min}^+}. \quad (64) \]

Therefore,

\[ \lim_{\tau \to \infty} \chi_{\text{s}_{\text{opt}}} (\tau) = \frac{\lambda_{\max}}{\lambda_{\min}^+}. \quad (65) \]

#### 4.3.1 Case 1

Substituting \( \delta_a = c\tau \) in equation (56) we get

\[ \chi_a (\theta_{\omega^*}, \omega^*, \tau, c) = \frac{3\lambda_{\min}^+ + \lambda_{\max}}{4} + \frac{\sqrt{1 + c\tau(2 - k) + 2(c\tau + 1 + c\tau(1 - k)\lambda_{\min}^+)}}{2c\tau \sqrt{1 + c\tau(2 - k)}}. \quad (k = \lambda_{\min}^+ + \lambda_{\max}). \quad (66) \]

Therefore,

\[ \lim_{\tau \to \infty} \chi_a (\theta_{\omega^*}, \omega^*, \tau, c) = \frac{\lambda_{\min}^+ + \lambda_{\max}}{4}. \quad (67) \]

Thus, comparing equations (65) and (67), we find out that asymptotically, asynchronous SGD has a tighter iteration complexity than the synchronous parallel method in Case 1 (when \( (\lambda_{\min}^+ + \lambda_{\max}) \in [1, 2] \)).
4.3.2 Case 2

Similarly for Case 2, from (61) we get

$$\chi_a(\theta_2, 2, \tau, c) = \frac{1}{4} + \frac{\lambda_{\min}^+}{2} + \frac{1 + \sqrt{c\tau + 1}}{2\tau}. \tag{68}$$

Therefore,

$$\lim_{\tau \to \infty} \chi_a(\theta_2, 2, \tau, c) = \frac{1}{4} + \frac{\lambda_{\min}^+}{2}. \tag{69}$$

Finally, comparing equations (64) and (69), we find out that asymptotically, asynchronous SGD enjoys a tighter iteration complexity than synchronous parallel method Case 2 (when $\lambda_{\min}^+ + \lambda_{\max} \in [0, 1]$ and $\frac{1}{4} + \frac{\lambda_{\min}^+}{2} \leq \lambda_{\max}$).

4.4 Non asymptotic comparison

As the above comparisons are asymptotic, we want to find out when exactly the asynchronous algorithm is better to use compared to its synchronous counterpart. For given combinations of $\lambda_{\min}^+$, $\lambda_{\max}$ and $c$, we computed the minimum number of processors, $\tau$, for which the asynchronous SGD algorithm has better iteration complexity (equation (12)) than its synchronous counterpart (equation (12)). We have considered $c \in \{1, 1.5, 2\}$, and various values of interest for $\lambda_{\min}^+$ and $\lambda_{\max}$. Denote the condition number, $\kappa = \frac{2\lambda_{\max}}{\lambda_{\min}}$.

4.4.1 Case 1

Table 2 refers to the first case, when $k = \lambda_{\min}^+ + \lambda_{\max} \geq 1$. We notice that the asynchronous SGD performs better than synchronous for small minimum number of processors $\tau$ even for highly ill conditioned problems in Case 1.

4.4.2 Case 2

Table 3 refers to the second case, when $k = \lambda_{\min}^+ + \lambda_{\max} \leq 1$. We note that the synchronous method is always better when $\frac{1}{4} + \frac{\lambda_{\min}^+}{2} > \lambda_{\max}$. Thus, in order to consider other cases, all the linear systems below satisfy $\frac{1}{4} + \frac{\lambda_{\min}^+}{2} \leq \lambda_{\max}$. We vary $\lambda_{\min}^+$ in log-scale, $\lambda_{\max} \in \{0.4, 0.3, 0.27, 0.26\}$, and $c \in \{1, 1.5, 2\}$.

- **Effect of varying $\lambda_{\max}$**: We observe that asynchronous SGD can perform better than the synchronous SGD in a reasonable number of processors, $\tau$, if $\lambda_{\max}$ is reasonably larger than $\frac{1}{4} + \frac{\lambda_{\min}^+}{2}$. For example, when $\lambda_{\max} = 0.4$, the minimum number of processors, $\tau = 2$ for $c = 2$ and $\tau = 11$ for $c = 1$ even for highly ill-conditioned problems (for both cases, $\kappa = 40000$). Fixing $c = 2$, for each value of $\lambda_{\min}^+$, we observe that the minimum number of processors, $\tau$ increases from 2 for $\lambda_{\max} = 0.4$ to values in thousands for $\lambda_{\max} = 0.26$.

- **Effect of varying $\lambda_{\min}^+$**: We observe that for the same $\lambda_{\max}$, the smaller the $\lambda_{\min}^+$ is, the smaller the minimum number of processors, $\tau$, is required. In other words, for the same $\lambda_{\max}$, asynchronous SGD requires less minimum number of processors, $\tau$ to beat its synchronous counterpart for more ill-conditioned problems. For example, for $\lambda_{\max} = 0.27$ and $c = 2$, the minimum number of processors, $\tau = 491$ when $\lambda_{\min}^+ = 10^{-2}$ (in this case, $\kappa = 27$) and $\tau = 264$ when $\lambda_{\min}^+ = 10^{-5}$ (in this case $\kappa = 27000$).

- **Effect of varying $c$**: We also observe that increasing $c$ from 1 to 2 significantly brings down the minimum number of processors, $\tau$. This intuitively makes sense and in accordance to our set-up—as $c$ increases, the asynchronous parallel SGD performs more updates in a unit time interval.
\[ k = \lambda_{\text{min}}^+ + \lambda_{\text{max}} \leq 1 \]

| \( \lambda_{\text{min}}^+ \) | \( \lambda_{\text{max}} \) | \( c \) | \( k = \lambda_{\text{min}}^+ + \lambda_{\text{max}} \) | \( \kappa \) | \( \tau \) |
|-----------------|-----------------|---|-----------------|---|---|
| \( 10^{-2} \) | 0.4 | 1 | 0.41 | 40 | 12 |
| \( 10^{-2} \) | 0.3 | 1 | 0.31 | 30 | 116 |
| \( 10^{-2} \) | 0.27 | 1 | 0.28 | 27 | 1082 |
| \( 10^{-2} \) | 0.26 | 1 | 0.27 | 26 | 9905 |
| \( 10^{-3} \) | 0.4 | 1 | 0.401 | 400 | 11 |
| \( 10^{-3} \) | 0.3 | 1 | 0.301 | 300 | 95 |
| \( 10^{-5} \) | 0.27 | 1 | 0.271 | 270 | 635 |
| \( 10^{-5} \) | 0.26 | 1 | 0.261 | 260 | 2721 |
| \( 10^{-5} \) | 0.4 | 1 | 0.4001 | 4000 | 11 |
| \( 10^{-5} \) | 0.3 | 1 | 0.3001 | 30000 | 11 |
| \( 10^{-5} \) | 0.27 | 1 | 0.2701 | 27000 | 604 |
| \( 10^{-5} \) | 0.26 | 1 | 0.2601 | 26000 | 2456 |
| \( 10^{-2} \) | 0.4 | 1.5 | 0.41 | 40 | 5 |
| \( 10^{-2} \) | 0.3 | 1.5 | 0.31 | 30 | 66 |
| \( 10^{-2} \) | 0.27 | 1.5 | 0.28 | 27 | 688 |
| \( 10^{-2} \) | 0.26 | 1.5 | 0.27 | 26 | 6504 |
| \( 10^{-3} \) | 0.4 | 1.5 | 0.401 | 400 | 5 |
| \( 10^{-3} \) | 0.3 | 1.5 | 0.301 | 300 | 54 |
| \( 10^{-3} \) | 0.27 | 1.5 | 0.271 | 270 | 398 |
| \( 10^{-3} \) | 0.26 | 1.5 | 0.261 | 260 | 1761 |
| \( 10^{-4} \) | 0.4 | 1.5 | 0.4001 | 4000 | 5 |
| \( 10^{-4} \) | 0.3 | 1.5 | 0.3001 | 3000 | 52 |
| \( 10^{-4} \) | 0.27 | 1.5 | 0.2701 | 2700 | 379 |
| \( 10^{-4} \) | 0.26 | 1.5 | 0.2601 | 2600 | 1602 |
| \( 10^{-5} \) | 0.4 | 1.5 | 0.40001 | 40000 | 5 |
| \( 10^{-5} \) | 0.3 | 1.5 | 0.30001 | 30000 | 52 |
| \( 10^{-5} \) | 0.27 | 1.5 | 0.27001 | 27000 | 377 |
| \( 10^{-5} \) | 0.26 | 1.5 | 0.26001 | 26000 | 1587 |
| \( 10^{-2} \) | 0.4 | 2 | 0.41 | 40 | 2 |
| \( 10^{-2} \) | 0.3 | 2 | 0.31 | 30 | 40 |
| \( 10^{-2} \) | 0.27 | 2 | 0.28 | 27 | 491 |
| \( 10^{-2} \) | 0.26 | 2 | 0.27 | 26 | 4803 |
| \( 10^{-3} \) | 0.4 | 2 | 0.401 | 400 | 2 |
| \( 10^{-3} \) | 0.3 | 2 | 0.301 | 300 | 31 |
| \( 10^{-3} \) | 0.27 | 2 | 0.271 | 270 | 278 |
| \( 10^{-3} \) | 0.26 | 2 | 0.261 | 260 | 1281 |
| \( 10^{-4} \) | 0.4 | 2 | 0.4001 | 4000 | 2 |
| \( 10^{-4} \) | 0.3 | 2 | 0.3001 | 3000 | 31 |
| \( 10^{-4} \) | 0.27 | 2 | 0.2701 | 2700 | 265 |
| \( 10^{-4} \) | 0.26 | 2 | 0.2601 | 2600 | 1163 |
| \( 10^{-5} \) | 0.4 | 2 | 0.40001 | 40000 | 2 |
| \( 10^{-5} \) | 0.3 | 2 | 0.30001 | 30000 | 31 |
| \( 10^{-5} \) | 0.27 | 2 | 0.27001 | 27000 | 264 |
| \( 10^{-5} \) | 0.26 | 2 | 0.26001 | 26000 | 1152 |

Table 3: \( k = \lambda_{\text{min}}^+ + \lambda_{\text{max}} \leq 1 \). Minimum number of processors \( \tau \) required for asynchronous SGD to have better iteration complexity than synchronous SGD.
A   Proofs of the Theorems and Lemmas

Proof of Theorem 2

Note that $S_{t-\delta}$ (and hence $Z_{t-\delta}$) is independent of both $x_{t-\delta}$ and $x_t$. Thus, taking expectation in (15) with respect to $S_{t-\delta}$ we get

$$
E_{S_{t-\delta}\sim D}[x_{t+1} - x_*] = (1 - \theta)(x_t - x_*) + \theta(I - \omega B^{-1}E_{S_{t-\delta}\sim D}[Z_{t-\delta}]) (x_{t-\delta} - x_*)
$$

Taking expectation again, and by using the tower property, we get

$$
E[x_{t+1} - x_*] = (1 - \theta)E[x_t - x_*] + \theta(I - \omega B^{-1}E[Z])E[x_{t-\delta} - x_*].
$$

(70)

Pre-multiplying both sides of (70) by $B^{1/2}$ we find

$$
B^{1/2}E[x_{t+1} - x_*] = (1 - \theta)B^{1/2}E[x_t - x_*] + \theta(I - \omega B^{-1/2}E[Z]B^{-1/2})B^{1/2}E[x_{t-\delta} - x_*].
$$

Recalling that $B^{-1/2}E[Z]B^{-1/2} = W$ has an eigenvalue decomposition $W = U\Lambda U^\top$ we obtain the desired result. □

Proof of Lemma 4

The first part follows from (15). The second part follows from the observation that $\text{Im}(B^{-1}A^\top)$ is the $B$-orthogonal complement of the nullspace of $A$. □

Proof of Theorem 5

We hereby prove the proof for $\omega \in [0, 2]$. The proof for $\omega \geq 2$ follows similarly, by using the upper bound of the inequality of Lemma 2. By taking norms on both sides of (15) and then applying Lemma 1, we get

$$
\|x_{t+1} - x_*\|_B^2 = (1 - \theta)^2\|x_t - x_*\|_B^2 + \theta^2\|E[I - \omega B^{-1}Z_{t-\delta}] (x_{t-\delta} - x_*)\|_B^2
$$

 Lemma 1

$$
\|x_{t+1} - x_*\|_B^2 = (1 - \theta)^2\|x_t - x_*\|_B^2 + \theta^2\|x_{t-\delta} - x_*\|_B^2 + 2(1 - \theta)\theta (x_t - x_*, (I - \omega B^{-1}Z_{t-\delta})(x_{t-\delta} - x_*))_B
$$

The above identity can be written as

$$
\|x_{t+1} - x_*\|_B^2 = (1 - \theta)^2 \|x_t\|_B^2 + \theta^2 \|x_{t-\delta}\|_B^2 - 2\omega(2 - \omega)\theta^2 f_{S_{t-\delta}}(x_{t-\delta}) + 2(1 - \theta)\theta (r_t, (I - \omega B^{-1/2}Z_{t-\delta}B^{-1/2})r_{t-\delta})
$$

Conditioning on $x_t, \ldots, x_0$, the only free random variable is $S_{t-\delta}$. Therefore, in view of Lemma 2 and using the eigenvalue decomposition $B^{-1/2}E[Z]B^{-1/2} = U\Lambda U^\top$, we get the following bound on $C := E[\|r_{t+1}\|_B^2 | x_t, \ldots, x_0]$

$$
C \leq (1 - \theta)^2 \|r_t\|_B^2 + \theta^2(1 - \omega(2 - \omega)\lambda_{\text{min}}^+)) \|r_{t-\delta}\|_B^2 + 2(1 - \theta)\theta (r_t, (I - \omega U\Lambda U^\top) r_{t-\delta})
$$

D
Further, we have
\[
D = r_t^T (I - \omega U \Lambda U^T) r_{t-\delta} = (U^T r_t \Lambda U^T r_{t-\delta}) = \sum_{i}(1 - \omega \lambda_i) u_i^T r_{t} u_i^T r_{t-\delta}
\]

Therefore, for all \( \gamma \in \varrho \), we have
\[
\sum_{i: \lambda_i \neq 0} (1 - \omega \lambda_i) u_i^T r_{t} u_i^T r_{t-\delta}
\leq \sum_{i: \lambda_i \neq 0} |1 - \omega \lambda_i| u_i^T r_{t} u_i^T r_{t-\delta} |u_i^T r_{t} u_i^T r_{t-\delta}|
\leq \alpha(\omega) \sum_{i: \lambda_i \neq 0} |u_i^T r_{t} u_i^T r_{t-\delta}|
\leq \alpha(\omega) \|r_{t \delta}\| \|r_{t-\delta}\|
\]

(Cauchy-Schwarz)
\[
\leq \alpha(\omega) \frac{\|r_{t \delta}\|^2 + \|r_{t-\delta}\|^2}{2}.
\]

Combining the bounds on \( C \) and \( D \), we get
\[
C \leq (1 - \theta)^2 \|r_{t \delta}\|^2 + \theta^2 (1 - \omega (2 - \omega) \lambda_{\min}^+) \|r_{t-\delta}\|^2 + (1 - \theta) \theta \alpha(\omega) \left( \|r_{t \delta}\|^2 + \|r_{t-\delta}\|^2 \right)
\]
\[
= \left[ (1 - \theta)^2 + (1 - \theta) \theta \alpha(\omega) \right] \|r_{t \delta}\|^2 + \left[ \theta^2 (1 - \omega (2 - \omega) \lambda_{\min}^+) + (1 - \theta) \theta \alpha(\omega) \right] \|r_{t-\delta}\|^2.
\]

(71)

The final result is obtained after we take full expectation and apply tower property.

Proof of Lemma 5
We have
\[
p_{\bar{\delta}_a}(\gamma) - p_{\delta}(\gamma) = \gamma^\delta (\gamma - K_1) (\gamma^\delta - 1) \leq 0 \quad \forall \gamma \in [K_1, 1] \text{ and } \delta \in [0, \delta_a].
\]

Thus, \( p_{\bar{\delta}_a}(\gamma) \) bounds \( p_{\delta}(\gamma) \) from below in \( \gamma \in [K_1, 1] \), and hence its positive root \( \varrho_A(\theta, \omega, \delta) \) is greater than or equal to \( \varrho_A(\theta, \omega, \bar{\delta}_a) \).

Proof of Lemma 6
The inequality \( p_{\bar{\delta}}(\theta, \omega) \leq \prod_{i=1}^{\bar{\delta}_a} \varrho(\theta, \omega, \delta_i) \) follows from Theorem 6, and \( \prod_{i=1}^{\bar{\delta}_a} \varrho(\theta, \omega, \delta_i) \leq \varrho(\theta, \omega, \bar{\delta}_a)^{\bar{\delta}_a} \) follows from Lemma 5. Together, we have (28).

Proof of Lemma 7
We have
\[
p_{\bar{\delta}}(0) \geq g_{\bar{\delta}}(0) \quad \text{and} \quad p_{\delta}(1) = g_{\delta}(1).
\]
Let \( g_{\bar{\delta}}(\gamma) = p_{\delta}(\gamma) - g_{\bar{\delta}}(\gamma) \) then \( g_{\bar{\delta}}(\gamma) = \gamma^{\delta+1} - (1 + \frac{1}{\delta}) \gamma^\delta + \frac{1}{\delta} \), and \( g_{\delta}(\gamma) = (\delta + 1) (\gamma - 1) \gamma^\delta - 1 \leq 0 \) for all \( \gamma \in [0, 1] \). Therefore, for all \( \gamma \in [0, 1] \), we have \( g_{\bar{\delta}}(\gamma) \geq g_{\bar{\delta}}(1) \), which implies \( g_{\delta}(\gamma) \geq 0 \), and finally, \( p_{\delta}(\gamma) - g_{\delta}(\gamma) \geq 0 \) for all \( \gamma \in [0, 1] \). Therefore, the root \( u(\theta, \omega, \delta) = \left( \frac{K_3 + \frac{1}{\delta}}{1 - K_1 + \frac{1}{\delta}} \right)^{\frac{1}{\delta}} \) of \( g_{\delta}(\gamma) \) is an upper bound to the unique positive root of the characteristic polynomial \( p_{\delta}(\gamma) \).
Proof of Lemma 9

We have for all $\theta$

$$K_1(\theta, \omega_1) \leq K_1(\theta, \omega_2)$$

and

$$K_2(\theta, \omega_1) \leq K_2(\theta, \omega_2).$$

Then by equation (25), we have $\rho_a(\theta, \omega_1) \leq \rho_a(\theta, \omega_2)$ for all $\theta \in [0, 1]$. Similarly, we see that by the definition of $U(\theta, \omega)$ in (37) (provided $K_1 + K_2 < 1$ in both choices of parameters) $U(\theta, \omega_1) \leq U(\theta, \omega_2)$ for all $\theta \in [0, 1]$. Hence the result. \qed

Proof of Lemma 10

Now we show how $K_1$ and $K_2$ behave for both the cases in the range $\omega \in [0, 1]$ and $\omega \in [\omega^*, \infty)$. Define $s(\omega) := (1 - \omega(2 - \omega)\lambda_{\min}^+)$ and $t(\omega) := (1 - \omega(2 - \omega)\lambda_{\max})$.

Case 1:

(i) For $0 \leq \omega \leq 1$ and $\alpha(\omega) = 1 - \omega\lambda_{\min}^+$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta\alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\min}^+)) + (1 - \theta)\alpha(\omega)). \quad (73)$$

We see that both $\alpha(\omega)$ and $s(\omega)$ are monotonically decreasing in the interval $\omega \in [0, 1]$. Hence, both $K_1$ and $K_2$ are monotonically decreasing in $\omega \in [0, 1]$.

(ii) For $\omega^* \leq \omega \leq 2$ and $\alpha(\omega) = \omega\lambda_{\max} - 1$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta\alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\min}^+)) + (1 - \theta)\alpha(\omega)).$$

Again, both $\alpha(\omega)$ and $s(\omega)$ are monotonically increasing in the interval $\omega \in [\omega^*, 2]$. Hence, both $K_1$ and $K_2$ are monotonically increasing in $\omega \in [\omega^*, 2]$.

(iii) For $\omega \geq 2$ and $\alpha(\omega) = \omega\lambda_{\max} - 1$:

$$K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta\alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\max})) + (1 - \theta)\alpha(\omega)).$$

Again, both $\alpha(\omega)$ and $t(\omega)$ are monotonically increasing in the interval $\omega \in [2, \infty)$. Hence, both $K_1$ and $K_2$ are monotonically increasing in $\omega \in [2, \infty)$.

Case 2:
(i) For $0 \leq \omega \leq 1$ and $\alpha(\omega) = 1 - \omega \lambda^+_{\min}$:

$$
K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda^+_{\min}) + (1 - \theta)\alpha(\omega)).
$$

(74)

Similar to case 1, both $\alpha(\omega)$ and $s(\omega)$ are monotonically decreasing in the interval $\omega \in [0, 1]$. Hence, both $K_1$ and $K_2$ are monotonically decreasing in $\omega \in [0, 1]$.

(ii) For $\omega \geq \omega^*$ and $\alpha(\omega) = \omega \lambda_{\max} - 1$:

$$
K_1(\theta, \omega) := (1 - \theta)(1 - \theta + \theta \alpha(\omega)), \quad K_2(\theta, \omega) := \theta(\theta(1 - \omega(2 - \omega)\lambda_{\max}) + (1 - \theta)\alpha(\omega)).
$$

Again, both $\alpha(\omega)$ and $t(\omega)$ are monotonically increasing in the interval $\omega \in [\omega^*, \infty)$. Hence, both $K_1$ and $K_2$ are monotonically increasing in $\omega \in [\omega^*, \infty)$.

Now, for both the cases, for all $\theta \in [0, 1]$ we have the following:

$\omega \leq 1$ : Both $K_1$ and $K_2$ monotonically decrease till $\omega = 1$, then from Lemma[9] 1 is the optimal $\omega$ in $[0, 1]$. (Note that $\omega^*$ is always greater than or equal to 1)

$\omega \geq \omega^*$ : Both $K_1$ and $K_2$ monotonically increase after $\omega = \omega^*$, then from remark[9] $\omega^*$ is the optimal $\omega$ in $[\omega^*, \infty)$.

Thus, the optimal $\omega$ for all $\theta \in [0, 1]$ in both the cases lies in the range $[1, \omega^*]$. □

**Proof of Lemma[12]**

The first inequality follows from the fact that $\lambda^+_{\min} \leq \frac{1}{2}$ for Case 2 (as $\lambda_{\min} + \lambda_{\max} \leq 1$). The second follows from the fact that $U(1, \omega)$ is always greater than $\frac{1}{\lambda_{\min}}$ for all $\delta_\alpha$, whereas $U(\theta_1, 1)$ is smaller than $\frac{1}{\lambda_{\min}}$ for $\delta_\alpha \geq 4$. □

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## B Notation Glossary

### The Basics

| Symbol | Description |
|--------|-------------|
| \(A, b\) | \(m \times n\) matrix and \(m \times 1\) vector defining the system \(Ax = b\) |
| \(L\) | \(\{x : Ax = b\}\) (solution set of the linear system) |
| \(B\) | \(n \times n\) symmetric positive definite matrix |
| \(\langle x, y \rangle_B\) | \(x^\top B y\) (B-inner product) |
| \(\|x\|_B\) | \(\sqrt{\langle x, x \rangle_B}\) (Norm induced by B-inner product) |
| \(M^\dagger\) | Moore-Penrose pseudoinverse of matrix \(M\) |
| \(S\) | A random real matrix with \(m\) rows |
| \(D\) | Distribution from which matrix \(S\) is drawn |
| \(H\) | \(H = S(S^\top A B^{-1} A^\top S)^\dagger S^\top\) |
| \(Z\) | \(A^\top HA\) |
| \(\text{Im}(M)\) | Image (range) space of matrix \(M\) |
| \(\text{Null}(M)\) | Null space of matrix \(M\) |
| \(\mathbb{E}[:]\) | Expectation |

### Projections

| Symbol | Description |
|--------|-------------|
| \(\Pi_B^\dagger(x)\) | Projection of \(x\) onto the set \(L\) in the B-norm |
| \(B^{-1}Z\) | Projection matrix in the B-norm onto \(\text{Im}(B^{-1}A^\top S)\) |

### Optimization

| Symbol | Description |
|--------|-------------|
| \(x_\star\) | A solution of the linear system \(Ax = b\) |
| \(f_S, \nabla f_S, \nabla^2 f_S\) | Stochastic function, its gradient and Hessian, respectively |
| \(\mathcal{L}_S\) | \(\{x : S^\top Ax = S^\top b\}\) (set of minimizers of \(f_S\)) |
| \(f\) | \(\mathbb{E}[f_S]\) |
| \(\nabla f\) | Gradient of \(f\) with respect to the B-inner product |
| \(\nabla^2 f\) | B\(^{-1}\mathbb{E}[Z]\) (Hessian of \(f\) in the B-inner product) |

### Eigenvalues

| Symbol | Description |
|--------|-------------|
| \(\lambda_1, \ldots, \lambda_n\) | Eigenvalues of \(W\) |
| \(\Lambda\) | Diag(\(\lambda_1, \ldots, \lambda_n\)) (diagonal matrix of eigenvalues) |
| \(U\) | \([u_1, \ldots, u_n]\) (eigenvectors of \(W\)) |
| \(U\Lambda U^\top\) | Eigenvalue decomposition of \(W\) |
| \(\lambda_{\text{max}}, \lambda_{\text{min}}\) | Largest and smallest nonzero eigenvalues of \(W\) |

### Algorithms

| Symbol | Description |
|--------|-------------|
| \(\theta\) | Damping parameter |
| \(\omega\) | Stepsize / relaxation parameter |
| \(\omega^*\) | \(\frac{2}{(\lambda_{\text{min}} + \lambda_{\text{max}})}\) |
| \(\tau\) | Number of processors in the assembly |
| \(\delta\) | Delay between the master and a particular worker processor |
| \(\delta_a(=c\tau)\) | Number of updates by the asynchronous parallel assembly in a unit time interval a.k.a the delay for the slowest worker processor (assumed to be constant) |
| \(\alpha(\omega)\) | For particular values of \(\omega\) and \(\delta\): |
| \(K_1(\theta, \omega), K_2(\theta, \omega)\) | Coefficient of \(\mathbb{E}[\|r_t\|^2]\) and \(\mathbb{E}[\|r_{t-\delta}\|^2]\) respectively in Theorem 5 |
| \(\rho_\alpha(\theta, \omega)\) | Rate of convergence of the asynchronous parallel SGD in a unit time interval iteration complexity of the asynchronous parallel SGD |
| \(\chi_a(\theta, \omega, \delta_a)\) | |

### Related to

- Stochastic reformulation of the linear system (see [5])