MULTIPLICATION AND CONVOLUTION TOPOLOGICAL ALGEBRAS IN SPACES OF \(\omega\)-ULTRADIFFERENTIABLE FUNCTIONS OF BEURLING TYPE

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Dedicated to Prof. F. Altomare on the occasion of his 70th birthday

ABSTRACT. We determine multiplication and convolution topological algebras for classes of \(\omega\)-ultradifferentiable functions of Beurling type. Hypocontinuity and discontinuity of the multiplication and convolution mappings are also investigated.

1. Introduction

Schwartz started in 1966 the study of multipliers and convolutors of the space \(S(\mathbb{R})\) of rapidly decreasing functions. The interest lies in the importance of their application to the study of partial differential equations. Since then many authors introduced and studied particular aspects of the spaces of multipliers and of convolutors for ultradifferentiable classes of rapidly decreasing functions of Beurling or Roumieu type in the sense of Komatsu [12] (see [8, 9, 10, 11, 12] for recent results in this setting). In the last years the attention has focused on the study of the space \(S_\omega(\mathbb{R}^N)\) of the ultradifferentiable rapidly decreasing functions of Beurling type, as introduced by Björck [3] (see [4, 5, 6, 9], for instance, and the references therein). Inspired by this line of research and by the previous work, in [1, 2] the authors introduced and studied the space \(O_{M,\omega}(\mathbb{R}^N)\) of the slowly increasing functions of Beurling type in the setting of ultradifferentiable function spaces of Beurling type, showing that it is the space of the multipliers of the space \(S_\omega(\mathbb{R}^N)\) and of its dual \(S'_\omega(\mathbb{R}^N)\), and the space \(O_{C,\omega}(\mathbb{R}^N)\) of the very slowly increasing functions of Beurling type, whose strong dual \(O'_{C,\omega}(\mathbb{R}^N)\) is the space of the convolutors of the space \(S_\omega(\mathbb{R}^N)\) and of its dual \(S'_{\omega}(\mathbb{R}^N)\). Their rich topological structure led us to determine deeply results concerning regularity, equivalent systems of seminorms, representations of the ultradistributions and the action of the Fourier transform, that is a topological isomorphism from \(O_{M,\omega}(\mathbb{R}^N)\) to \(O'_{C,\omega}(\mathbb{R}^N)\).

In this paper the authors continue the study of these spaces to complete their description. The aim is to establish that \((O_{M,\omega}(\mathbb{R}^N), \cdot)\), \((S_\omega(\mathbb{R}^N), \cdot)\) are multiplication topological algebras, while \((S_\omega(\mathbb{R}^N), \star)\) and \((O'_{C,\omega}(\mathbb{R}^N), \star)\) are convolution topological algebras. We also determine the spaces of multipliers and of convolutors of the spaces \(O_{M,\omega}(\mathbb{R}^N), O_{C,\omega}(\mathbb{R}^N)\) and their duals. Furthermore, we analyze the continuity of the multiplication and convolution bilinear mappings on some pairs between the

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spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, $\mathcal{S}_\omega(\mathbb{R}^N)$ and their duals, studied classically by Schwartz [17] (see also Larcher [16] and the references therein). This approach is done not only treating the hypocontinuity, but also describing and investigating the continuity properties of these mappings.

The paper is organized as follows. Section 2 is devoted to recalling some definitions and properties of the weights $\omega$ and of the $\omega$-ultradifferentiable functions, that we use in the following. In Section 3 we prove the results about the multiplication and convolution (topological) algebras and determine the spaces of multipliers and of convolutors of the spaces under consideration. Finally, in the last section, we analyze the multiplication and convolution bilinear mappings on some pairs between the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, $\mathcal{S}_\omega(\mathbb{R}^N)$ and their duals, proving when are hypocontinuous or discontinuous.

2. Definitions and preliminary results

We first give the definition of non-quasianalytic weight function in the sense of Braun, Meise and Taylor [7] suitable for the Beurling case, i.e., we also consider the logarithm as a weight function.

**Definition 2.1.** A non-quasianalytic weight function is a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ satisfying the following properties:

- (a) there exists $K \geq 1$ such that $\omega(2t) \leq K(1 + \omega(t))$ for every $t \geq 0$;
- (b) $\int_1^\infty \frac{\omega(t)}{t^2} \, dt < \infty$;
- (c) there exist $a \in \mathbb{R}$, $b > 0$ such that $\omega(t) \geq a + b \log(1 + t)$, for every $t \geq 0$;
- (d) $\varphi_\omega(t) = \omega \circ \exp(t)$ is a convex function.

We recall some known properties of the weight functions that shall be useful in the following (the proofs can be found in the literature):

1. Condition (a) implies that $\omega(t_1 + t_2) \leq K(1 + \omega(t_1) + \omega(t_2))$, $\forall t_1, t_2 \geq 0$. \hspace{1cm} (2.1)

Observe that this condition is weaker than subadditivity (i.e., $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$). The weight functions satisfying (a) are not necessarily subadditive in general.

2. Condition (a) implies that there exists $L \geq 1$ such that $\omega(et) \leq L(1 + \omega(t))$, $\forall t \geq 0$. \hspace{1cm} (2.2)

3. By condition (c) we have that $e^{-\lambda \omega(t)} \in L^p(\mathbb{R}^N)$, $\forall \lambda \geq \frac{N + 1}{bp}$. \hspace{1cm} (2.3)

Given a non-quasianalytic weight function $\omega$, we define the Young conjugate $\varphi_\omega^*$ of $\varphi_\omega$ as the function $\varphi_\omega^* : [0, \infty) \to [0, \infty)$ by

$$\varphi_\omega^*(s) := \sup_{t \geq 0} \{st - \varphi_\omega(t)\}, \quad s \geq 0.$$ 

There is no loss of generality to assume that $\omega$ vanishes on $[0, 1]$. Therefore, $\varphi_\omega^*$ is convex and increasing, $\varphi_\omega^*(0) = 0$ and $(\varphi_\omega^*)^* = \varphi_\omega$.

Further useful properties of $\varphi_\omega^*$ are listed below (see [7]):

(1) \( \varphi_t^* \) is an increasing function in \((0, \infty)\).

(2) For every \( s, t \geq 0 \) and \( \lambda > 0 \)
\[
2\lambda \varphi^*_\omega \left( \frac{s + t}{2\lambda} \right) \leq \lambda \varphi^*_\omega \left( \frac{s}{\lambda} \right) + \lambda \varphi^*_\omega \left( \frac{t}{\lambda} \right) \leq \lambda \varphi^*_\omega \left( \frac{s + t}{\lambda} \right),
\]
(2.4)

(3) For every \( t \geq 0 \) and \( \lambda > 0 \)
\[
\lambda L \varphi^*_\omega \left( \frac{t}{\lambda L} \right) + t \leq \lambda \varphi^*_\omega \left( \frac{t}{\lambda} \right) + \lambda L,
\]
where \( L \geq 1 \) is the constant appearing in formula (2.2).

(4) For all \( m, M \in \mathbb{N} \) with \( M \geq mL \), where \( L \) is the constant appearing in formula (2.2), and for every \( t \geq 0 \)
\[
2^t \exp \left( M \varphi^*_\omega \left( \frac{t}{M} \right) \right) \leq C \exp \left( m \varphi^*_\omega \left( \frac{t}{m} \right) \right),
\]
(2.5)

We now introduce the ultradifferentiable function space \( S_\omega(\mathbb{R}^N) \) in the sense of Björk [3].

**Definition 2.2.** Let \( \omega \) be a non-quasianalytic weight function. We denote by \( S_\omega(\mathbb{R}^N) \) the set of all functions \( f \in L^1(\mathbb{R}^N) \) such that \( f, \hat{f} \in C^\infty(\mathbb{R}^N) \) and for all \( \lambda > 0 \) and \( \alpha \in \mathbb{N}_0^N \) we have
\[
\| \exp(\lambda \omega) \partial^\alpha f \|_\infty < \infty \quad \text{and} \quad \| \exp(\lambda \omega) \partial^\alpha \hat{f} \|_\infty < \infty,
\]
where \( \hat{f} \) denotes the Fourier transform of \( f \). The elements of \( S_\omega(\mathbb{R}^N) \) are called \( \omega \)-ultradifferentiable rapidly decreasing functions of Beurling type. We denote by \( S'_\omega(\mathbb{R}^N) \) the dual of \( S_\omega(\mathbb{R}^N) \) endowed with its strong topology.

The space \( S_\omega(\mathbb{R}^N) \) is a Fréchet space with different equivalent systems of seminorms (see [5, Theorem 4.8] and [4, Theorem 2.6]). In the following, we will use the following system of norms generating the Fréchet topology of \( S_\omega(\mathbb{R}^N) \):
\[
q_{\lambda, \mu}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \| \exp(\mu \omega) \partial^\alpha f \|_\infty \exp \left( -\lambda \varphi^*_\omega \left( \frac{|\alpha|}{\lambda} \right) \right), \quad \lambda, \mu > 0, \ f \in S_\omega(\mathbb{R}^N),
\]
or equivalently, the sequence of norms \( \{ q_{m,n} \}_{m,n \in \mathbb{N}} \).

We point out that the space \( S_\omega(\mathbb{R}^N) \) is a nuclear Fréchet space, see, f.i., [6, Theorem 3.3] or [9, Theorem 1.1].

We refer to [7] for the definition and the main properties of the ultradifferentiable function spaces \( E_\omega(\Omega) \), \( D_\omega(\Omega) \) and their duals of Beurling type in the sense of Braun, Meise and Taylor. We only recall that for an open subset \( \Omega \) of \( \mathbb{R}^N \), the space \( E_\omega(\Omega) \) is defined as
\[
E_\omega(\Omega) := \{ f \in C^\infty(\Omega) : p_{K,m}(f) < \infty \ \forall K \Subset \Omega, \ m \in \mathbb{N} \},
\]
where
\[
p_{K,m}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |\partial^\alpha f(x)| \exp \left( -m \varphi^*_\omega \left( \frac{|\alpha|}{m} \right) \right).
\]
Definition 2.3. Let \( \omega \) be a non-quasianalytic weight function.

(a) The space \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) of \emph{slowly increasing functions of Beurling type} on \( \mathbb{R}^N \) is defined by

\[
\mathcal{O}_{M,\omega}(\mathbb{R}^N) := \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N),
\]

where

\[
\mathcal{O}_{n,\omega}^m(\mathbb{R}^N) := \left\{ f \in C^\infty(\mathbb{R}^N) : r_{m,n}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp \left( -n\omega(x) - m\varphi_\omega^*(\frac{|\alpha|}{m}) \right) < \infty \right\}
\]

endowed with the norm \( r_{m,n} \) is a Banach space for any \( m, n \in \mathbb{N} \). The space \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) is endowed with its natural lc-topology \( t \), i.e., \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) = \text{proj}_{\mathbb{N}} \text{ind}_{\mathbb{N}} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \) is the projective limit of the (LB)-spaces \( \mathcal{O}_{\omega}^m(\mathbb{R}^N) := \text{ind}_{\mathbb{N}} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \).

(c) The space \( \mathcal{O}_{C,\omega}(\mathbb{R}^N) \) of \emph{very slowly increasing functions of Beurling type} on \( \mathbb{R}^N \) is defined by

\[
\mathcal{O}_{C,\omega}(\mathbb{R}^N) := \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N).
\]

The space \( \mathcal{O}_{C,\omega}(\mathbb{R}^N) \) endowed with its natural lc-topology is an (LF)-space, i.e., \( \mathcal{O}_{C,\omega}(\mathbb{R}^N) = \text{ind}_{\mathbb{N}} \mathcal{O}_{n,\omega}(\mathbb{R}^N) \), where \( \mathcal{O}_{n,\omega}(\mathbb{R}^N) := \text{proj}_{\mathbb{N}} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \).

Let \( \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \) (\( \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \), resp.) denote the strong dual of \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) (of \( \mathcal{O}_{C,\omega}(\mathbb{R}^N) \), resp.) endowed with its strong topology.

The space \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) is the space of multipliers of \( \mathcal{S}_{\omega}(\mathbb{R}^N) \) and of its dual space \( \mathcal{S}'_{\omega}(\mathbb{R}^N) \) as proved in [1,8], i.e., \( f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) if, and only if, \( fg \in \mathcal{S}_{\omega}(\mathbb{R}^N) \) for all \( g \in \mathcal{S}_{\omega}(\mathbb{R}^N) \) if, and only if, \( fT \in \mathcal{S}'_{\omega}(\mathbb{R}^N) \) for all \( T \in \mathcal{S}_{\omega}(\mathbb{R}^N) \). Moreover, if \( f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N) \), then the linear operators \( M_f : \mathcal{S}_{\omega}(\mathbb{R}^N) \to \mathcal{S}_{\omega}(\mathbb{R}^N) \) defined by \( M_f(g) := fg \), for \( g \in \mathcal{S}_{\omega}(\mathbb{R}^N) \), and \( M_f : \mathcal{S}'_{\omega}(\mathbb{R}^N) \to \mathcal{S}'_{\omega}(\mathbb{R}^N) \) defined by \( M_f(T) := fT \), for \( T \in \mathcal{S}'_{\omega}(\mathbb{R}^N) \), are continuous. Furthermore, \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) is an ultrabornological space by [8] Theorem 5.3. Due to [8] Theorem 5.3 and the results in [1], we have that \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) is a closed subspace of \( \mathcal{L}_b(\mathcal{S}_{\omega}(\mathbb{R}^N)) \) (\( \mathcal{L}_b(\mathcal{S}_{\omega}(\mathbb{R}^N)) \) denotes the space of all continuous linear operators from \( \mathcal{S}_{\omega}(\mathbb{R}^N) \) into itself endowed with the topology of the uniform convergence on bounded subsets of \( \mathcal{S}_{\omega}(\mathbb{R}^N) \)). In particular, a fundamental system of continuous norms on \( \mathcal{O}_{M,\omega}(\mathbb{R}^N) \) is given by

\[
q_{m,\omega}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |g(x)||\partial^\alpha f(x)| \exp \left( -m\varphi_\omega^*(\frac{|\alpha|}{m}) \right), \quad g \in \mathcal{S}_{\omega}(\mathbb{R}^N), \quad m \in \mathbb{N},
\]

see [1] Proposition 5.6 and Theorem 5.9.
For any $\omega$ non-quasianalytic weight function satisfying the condition $\log(1 + t) = o(\omega(t))$ as $t \to \infty$, the space $\mathcal{O}_C(\omega)(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$ and of its dual space $\mathcal{S}'_\omega(\mathbb{R}^N)$ as proved in [2], i.e., $T \in \mathcal{O}_C(\omega)(\mathbb{R}^N)$ if, and only if, $T \ast f \in \mathcal{S}_\omega(\mathbb{R}^N)$ for all $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ if, and only if, $T \ast S \in \mathcal{S}'_\omega(\mathbb{R}^N)$ for all $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$. Moreover, if $T \in \mathcal{O}_C(\omega)(\mathbb{R}^N)$, then the linear operators $C_T : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $C_T(f) := T \ast f$, for $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, and $C_T : \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N)$ defined by $C_T(S) := T \ast S$, for $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, are continuous.

We also recall that the inclusions $D(\mathbb{R}^N) \hookrightarrow \mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}_C(\omega)(\mathbb{R}^N) \hookrightarrow \mathcal{O}_M(\omega)(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ are well-defined, continuous with dense range, see [1] Theorems 3.8, 3.9 and 5.2(1)]. Hence, the inclusions $E'_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}'_M(\omega)(\mathbb{R}^N) \hookrightarrow \mathcal{O}'_C(\omega)(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N) \hookrightarrow D'_\omega(\mathbb{R}^N)$ are also well-defined and continuous. On the other hand, $\mathcal{O}_M(\omega)(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ and $\mathcal{O}_C(\omega)(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ continuously, as it is easy to see.

In [2] the following spaces have been introduced.

**Definition 2.4.** Let $\omega$ be a non-quasianalytic weight function and $1 \leq p < \infty$.

(a) The space $\mathcal{O}_{\omega,p}(\mathbb{R}^N)$ is defined by $\mathcal{O}_{\omega,p}(\mathbb{R}^N) := \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$, where

$$\mathcal{O}_{n,\omega,p}(\mathbb{R}^N) := \left\{ f \in C^\infty(\mathbb{R}^N) : r_{m,n,p}^p(f) := \sum_{\alpha \in \mathbb{N}_0^n} \| \exp(-n\omega)\partial^\alpha f \|_p \| \exp \left( -m p \omega \left( \frac{|\alpha|}{m} \right) \right) \| < \infty \right\}$$

endowed with the norm $r_{m,n,p}$ is Banach space for any $m, n \in \mathbb{N}$. The space is endowed with its natural lc-topology, i.e., $\mathcal{O}_{\omega,p}(\mathbb{R}^N) = \text{proj}_{m \to \infty} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ is the projective limit of the (LB)-spaces $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N) := \text{ind}_{n \to \infty} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$.

(b) The space $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ is defined by $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) := \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$. $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ endowed with its natural lc-topology is an (LF)-space, i.e., $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) = \text{ind}_{n \to \infty} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$, where $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N) := \text{proj}_{m \to \infty} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$.

As shown in [2] Proposition 3.8] it holds true that $\mathcal{O}_{\omega}(\mathbb{R}^N) = \mathcal{O}_{\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{n,\omega}(\mathbb{R}^N) = \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ algebraically and topologically for any $1 \leq p < \infty$, thereby obtaining that $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ algebraically and topologically for any $1 \leq p < \infty$. Since the Banach spaces $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ are reflexive whenever $1 < p < \infty$, by a result of Vogt [19] Proposition 4.4] we can conclude that $\mathcal{O}_{\omega}(\mathbb{R}^N) = \mathcal{O}_{\omega,p}(\mathbb{R}^N)$ is a reflexive and complete (LB)-space, hence a regular (LB)-space, being it an inductive limit of reflexive Banach spaces. Recall that an (LF)-space $E = \text{ind}_{n \to \infty} E_n$ is said to be regular if every bounded subset in $E$ is contained and bounded in $E_n$ for some $n \in \mathbb{N}$ and that complete (LF)-spaces are always regular. An (LF)-space $E = \text{ind}_{n \to \infty} E_n$ is said to be sequentially retractive if for every null sequence in $E$ there exists $n \in \mathbb{N}$ such that the sequence is contained and converges to zero in $E_n$. We observe that sequential retractivity implies completeness, see [20] Corollary 2.8].

We end this section by showing that the (LB)-spaces $\mathcal{O}_{\omega}(\mathbb{R}^N)$, with $m \in \mathbb{N}$, are sequentially retractive and Montel. For this, we need the following lemma.
Lemma 2.5. Let $\omega$ be a non-quasianalytic weight function and let $m \in \mathbb{N}$. Then the spaces $\mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on the closed unit ball $\mathcal{B}_n^m$ of $\mathcal{O}_n^{m,\omega}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. The set $U := \{f \in \mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N) : r_{m,n+1}(f) < \varepsilon\}$ is a $0$-neighborhood of $\mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N)$. Now, let $M > 0$ such that $\exp(-\omega(x)) < \varepsilon$ for every $|x| \geq M$ and $V := \{f \in \mathcal{E}_\omega(\mathbb{R}^N) : p_{K,m}(f) < \varepsilon\}$, where $K := \{x \in \mathbb{R}^N : |x| \leq M\}$. Then $V \cap \mathcal{B}_n^m \subseteq U \cap \mathcal{B}_n^m$. Indeed, if $f \in V \cap \mathcal{B}_n^m$, then

$$|\partial^\alpha f(x)| \leq \exp\left(n\omega(x) + m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) < \varepsilon \exp\left((n+1)\omega(x) + m\varphi^*\left(\frac{|\alpha|}{m}\right)\right)$$

for every $\alpha \in \mathbb{N}_0^N$ and $|x| \geq M$. Moreover, $f \in V \cap \mathcal{B}_n^m$ also implies that

$$|\partial^\alpha f(x)| < \varepsilon \exp\left(m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) \leq \varepsilon \exp\left((n+1)\omega(x) + m\varphi^*\left(\frac{|\alpha|}{m}\right)\right)$$

for every $\alpha \in \mathbb{N}_0^N$ and $|x| \leq M$. It follows that $r_{m,n+1}(f) < \varepsilon$ and so, $f \in U \cap \mathcal{B}_n^m$. Since $\varepsilon > 0$ is arbitrary, we get the thesis. \qed

We can to show that the (LB)-spaces $\mathcal{O}_\omega^m(\mathbb{R}^N)$, with $m \in \mathbb{N}$, are sequentially retractive.

Theorem 2.6. Let $\omega$ be a non-quasianalytic weight function. For every $m \in \mathbb{N}$ the (LB)-space $\mathcal{O}_\omega^m(\mathbb{R}^N)$ is sequentially retractive.

Proof. Let $\{f_j\}_{j \in \mathbb{N}}$ be a null sequence of $\mathcal{O}_\omega^m(\mathbb{R}^N)$. Then $B := \{f_j : j \in \mathbb{N}\}$ is a bounded subset of $\mathcal{O}_\omega^m(\mathbb{R}^N)$. Since $\mathcal{O}_\omega^m(\mathbb{R}^N)$ is a regular (LB)-space, $B$ is contained and bounded in $\mathcal{O}_n^{m,\omega}(\mathbb{R}^N)$ for some $n \in \mathbb{N}$. On the other hand, $\{f_j\}_{j \in \mathbb{N}}$ converges to $0$ in $\mathcal{E}_\omega(\mathbb{R}^N)$, as $\mathcal{O}_\omega^m(\mathbb{R}^N)$ is continuously included in $\mathcal{E}_\omega(\mathbb{R}^N)$. Since $\mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on $B$ by Lemma 2.5, it follows that $\{f_j\}_{j \in \mathbb{N}}$ converges to $0$ in $\mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N)$. \qed

Theorem 2.7. Let $\omega$ be a non-quasianalytic weight function. Then the spaces $\mathcal{O}_\omega^m(\mathbb{R}^N)$, for $m \in \mathbb{N}$, and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ are complete Montel spaces.

Proof. The spaces $\mathcal{O}_\omega^m(\mathbb{R}^N)$, for $m \in \mathbb{N}$, are clearly barrelled as inductive limits of barrelled spaces. On the other hand, $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is ultrabornological by \cite{R}, hence barrelled. Hence, it remains to show that each bounded set in such spaces is relatively compact. This follows from Lemma 2.5. Indeed, fixed $m \in \mathbb{N}$ and a bounded subset $B$ of $\mathcal{O}_\omega^m(\mathbb{R}^N)$, from the regularity of $\mathcal{O}_\omega^m(\mathbb{R}^N)$ there exists $n \in \mathbb{N}$ such that $B$ is contained and bounded in $\mathcal{O}_n^{m,\omega}(\mathbb{R}^N)$. By Lemma 2.5 the spaces $\mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on $B$ and hence, $B$ is relatively compact in $\mathcal{O}_n^{m,n+1,\omega}(\mathbb{R}^N)$, after having observed that $\mathcal{E}_\omega(\mathbb{R}^N)$ is Montel. Therefore, $B$ is also relatively compact in $\mathcal{O}_\omega^m(\mathbb{R}^N)$.

Since $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the projective limit of the Montel spaces $\mathcal{O}_\omega^m(\mathbb{R}^N)$, each bounded subset of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is clearly relatively compact. The completeness of the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ follows from the fact that it is the projective limit of the complete spaces $\mathcal{O}_\omega^m(\mathbb{R}^N)$. \qed
Remark 2.8. We remark that the completeness of the space $O_{M,\omega}(\mathbb{R}^N)$ follows also from [1] Theorem 5.2(ii) combined with [3] Theorem 5.3.

3. The Algebras $(O_{M,\omega}(\mathbb{R}^N), \cdot)$, $(S_\omega(\mathbb{R}^N), \cdot)$, $(S_\omega(\mathbb{R}^N), \star)$ and $(O'_{C,\omega}(\mathbb{R}^N), \star)$

Let us recall that a bilinear mapping $b: E \times F \to G$ between locally convex Hausdorff spaces $E$, $F$, $G$ is continuous if, and only if, for every continuous seminorm $p_1$ on $G$ there exist continuous seminorms $p_2$ and $p_3$ on $E$ and $F$, respectively, such that the inequality

$$p_1(b(v, w)) \leq p_2(v)p_3(w)$$

holds for every pair $(v, w) \in E \times F$. A locally convex algebra (topological algebra, briefly) over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is a $\mathbb{K}$Hs $E$ together with a bilinear map $b: E \times E \to E$ that turns $E$ into an algebra over $\mathbb{K}$ with $b$ continuous.

The aim of this section is to establish that $(O_{M,\omega}(\mathbb{R}^N), \cdot)$, $(S_\omega(\mathbb{R}^N), \cdot)$ are multiplication topological algebras, while $(S_\omega(\mathbb{R}^N), \star)$ and $(O'_{C,\omega}(\mathbb{R}^N), \star)$ are convolution topological algebras.

We first consider the case $O_{M,\omega}(\mathbb{R}^N)$ and establish the following fact.

Lemma 3.1. Let $\omega$ be a non-quasianalytic weight function. Then for all $k \in S_\omega(\mathbb{R}^N)$ there exists $l \in S_\omega(\mathbb{R}^N)$ such that $|k(x)| \leq l^2(x)$ for every $x \in \mathbb{R}^N$.

Proof. Let $k \in S_\omega(\mathbb{R}^N)$ be fixed. The function $h(x) := \sqrt{|k(x)|}$ for $x \in \mathbb{R}^N$ is a non-negative function such that

$$\lim_{|x| \to \infty} \exp(\lambda \omega(x))h(x) = \lim_{|x| \to \infty} \sqrt{\exp \left( \frac{\lambda}{2} \omega(x) \right)} |k(x)| = 0$$

for all $\lambda > 0$. By [1] Lemma 5.8] there exists $l \in S_\omega(\mathbb{R}^N)$ for which $h(x) \leq l(x)$ for every $x \in \mathbb{R}^N$. Accordingly, $|k(x)| \leq l^2(x)$ for every $x \in \mathbb{R}^N$. \hfill $\square$

Theorem 3.2. Let $\omega$ be a non-quasianalytic weight function. Then $(O_{M,\omega}(\mathbb{R}^N), \cdot)$ is a multiplication topological algebra.

Proof. To see this, we fix $m \in \mathbb{N}$ and $k \in S_\omega(\mathbb{R}^N)$. We choose $m' \in \mathbb{N}$ such that $m' \geq Lm$, where $L$ is the constant appearing in (2.2), and $l \in S_\omega(\mathbb{R}^N)$ as in Lemma 3.1. Now, let $f, g \in O_{M,\omega}(\mathbb{R}^N)$. Then we have for every $x \in \mathbb{R}^N$ that

$$|k(x)||\partial^\alpha(fg)(x)| \leq \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) l(x)|\partial^\gamma f(x)||l(x)|\partial^\alpha-\gamma g(x)|$$

$$\leq q_{m', l}(f)q_{m', l}(g) \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) \exp \left( m' \varphi_\omega^*(\frac{|\gamma|}{m'}) \right) \exp \left( m \varphi_\omega^*(\frac{|\alpha - \gamma|}{m}) \right).$$

Applying (2.4) and (2.5), we obtain for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$|k(x)||\partial^\alpha(fg)(x)| \leq q_{m', l}(f)q_{m', l}(g)2^{|\alpha|} \exp \left( m' \varphi_\omega^*(\frac{|\alpha|}{m'}) \right)$$

$$\leq Cq_{m', l}(f)q_{m', l}(g) \exp \left( m \varphi_\omega^*(\frac{|\alpha|}{m}) \right),$$
where $C := e^{mL}$. Accordingly,

$$g_{m,k}(fg) \leq C q_{m',l}(f)q_{m',l}(g). \quad (3.1)$$

Since $f, g \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, $m \in \mathbb{N}$ and $k \in \mathcal{S}_\omega(\mathbb{R}^N)$ are arbitrary, we can conclude from (3.1) that the multiplication operator $M: \mathcal{O}_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}_{M,\omega}(\mathbb{R}^N) \to \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, $(f, g) \mapsto fg$, is well-defined and continuous, i.e., $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is a multiplication topological algebra.

Arguing in a similar way with simple changes (indeed, it suffices to consider $\exp(n\omega)$ instead of the function $k$ and to take $\exp\left(\frac{n}{2}\omega\right)$ instead of the function $l$), one shows the same result for $\mathcal{S}_\omega(\mathbb{R}^N)$.

**Theorem 3.3.** Let $\omega$ be a non-quasianalytic weight function. Then $(\mathcal{S}_\omega(\mathbb{R}^N), \cdot)$ is a multiplication topological algebra.

We now introduce the following definition.

**Definition 3.4.** Let $\omega$ be a non-quasianalytic weight function. Let $E$ be a lcHs of $\omega$-ultradifferentiable functions on $\mathbb{R}^N$ continuously included in $\mathcal{E}_\omega(\mathbb{R}^N)$ with dense range. We denote by $M(E)$ the space of all multipliers of $E$, i.e., the largest space of $\omega$-ultradifferentiable functions on $\mathbb{R}^N$ satisfying the following conditions:

1. the multiplication operator on $E \times M(E) \to \mathcal{E}_\omega(\mathbb{R}^N)$, $(f, g) \mapsto fg$, is well-defined and takes values in $E$;
2. for all $f \in M(E)$, the operator $M_f: E \to E$, $g \mapsto fg$ is continuous.

If $E'$ is the strong dual of $E$, we denote by $M(E')$ the space of all multipliers of $E'$, i.e., the largest space of $\omega$-ultradistributions on $\mathbb{R}^N$ for which the following conditions are satisfied:

1. for all $T \in E'$ and $f \in M(E')$ we have that $fT$ is well-defined on $E$ and belongs to $E'$;
2. for all $f \in M(E')$, the operator $M_f: E' \to E'$, $T \mapsto fT$ is continuous.

**Remark 3.5.** We first recall that $M(\mathcal{S}_\omega(\mathbb{R}^N)) = M(\mathcal{S}_\omega'(\mathbb{R}^N)) = \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

We now observe that if $E$ is a lcHs of $\omega$-ultradifferentiable functions on $\mathbb{R}^N$ continuously included in $\mathcal{E}_\omega(\mathbb{R}^N)$ with dense range and such that the constant functions belong to $E$, then

$$M(E) = \{1\} \cdot M(E) \subseteq E \cdot M(E) \subseteq E.$$  

Since $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ contain the constant functions, we get $M(\mathcal{O}_{M,\omega}(\mathbb{R}^N)) \subseteq \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $M(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) \subseteq \mathcal{O}_{C,\omega}(\mathbb{R}^N)$. But $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is a multiplication algebra as shown in Theorem 3.2. Therefore, we clearly have that $\mathcal{O}_{M,\omega}(\mathbb{R}^N) \subseteq M(\mathcal{O}_{M,\omega}(\mathbb{R}^N))$. Thus, $M(\mathcal{O}_{M,\omega}(\mathbb{R}^N)) = \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

It is also true that $M(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}_{C,\omega}(\mathbb{R}^N)$, as the following result shows.

**Proposition 3.6.** Let $\omega$ be a non-quasianalytic weight function. Then $M(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}_{C,\omega}(\mathbb{R}^N)$. Hence, $(\mathcal{O}_{C,\omega}(\mathbb{R}^N), \cdot)$ is a multiplication algebra.

**Proof.** By Remark 3.5 it suffices to show that $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \subseteq M(\mathcal{O}_{C,\omega}(\mathbb{R}^N))$. In order to do this, we fix $n, m \in \mathbb{N}$ with $n \geq 2$ and choose $m' \geq Lm$, with $L$ the constant appearing in (2.2). Let $n_1, n_2 \in \mathbb{N}$ be such that $n_1 + n_2 = n$. If $f \in \mathcal{O}_{n_1,\omega}(\mathbb{R}^N) = \mathcal{O}_{n_2,\omega}(\mathbb{R}^N)$ and $\omega$ be a non-quasianalytic weight function. Then $M(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}_{C,\omega}(\mathbb{R}^N)$.
\[ \cap_{r=1}^{\infty} O_{r,\omega}(\mathbb{R}^N) \text{ and } g \in O_{n,\omega}(\mathbb{R}^N) = \cap_{r=1}^{\infty} O_{r,\omega}(\mathbb{R}^N) \text{, then via (2.4) and (2.5) we have for every } \alpha \in \mathbb{N}_0^N \text{ and } x \in \mathbb{R}^N \text{ that} \]
\[
|\partial^\alpha (fg)(x)| \leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |\partial^\gamma f(x)||\partial^{\alpha - \gamma} g(x)| \\
\leq r_{m',n_1}(f)r_{m',n_2}(g) \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \exp \left( n_1\omega(x) + m' \varphi^* \left( \frac{|\gamma|}{m'} \right) \right) \times \\
\times \exp \left( n_2\omega(x) + m' \varphi^* \left( \frac{|\alpha - \gamma|}{m'} \right) \right) \\
\leq C r_{m',n_1}(f)r_{m',n_2}(g) \exp(n\omega(x)) \exp \left( m\varphi^* \left( \frac{|\alpha|}{m} \right) \right),
\]
where \( C := e^{mL} \). Accordingly, we have
\[
\tag{3.2}
r_{m,n}(fg) \leq C r_{m',n_1}(f)r_{m',n_2}(g).
\]

Since \( m \in \mathbb{N} \), \( f \in O_{n,\omega}(\mathbb{R}^N) \) and \( g \in O_{n,\omega}(\mathbb{R}^N) \) are arbitrary, from (3.2) it follows that the multiplication operator \( M: O_{n,\omega}(\mathbb{R}^N) \times O_{n,\omega}(\mathbb{R}^N) \rightarrow O_{n,\omega}(\mathbb{R}^N) \) is continuous. But also \( n \in \mathbb{N} \) is arbitrary and \( O_{C,\omega}(\mathbb{R}^N) \) is the inductive limit of the Fréchet spaces \( O_{n,\omega}(\mathbb{R}^N) \). So, we can conclude that the multiplication operator \( M: O_{C,\omega}(\mathbb{R}^N) \times O_{C,\omega}(\mathbb{R}^N) \rightarrow O_{C,\omega}(\mathbb{R}^N) \) is well-defined. Moreover, (3.2) also implies that the operator \( M_f: O_{C,\omega}(\mathbb{R}^N) \rightarrow O_{C,\omega}(\mathbb{R}^N), g \mapsto fg \), is continuous for all \( f \in O_{C,\omega}(\mathbb{R}^N) \). Therefore, \( O_{C,\omega}(\mathbb{R}^N) \subseteq M(O_{C,\omega}(\mathbb{R}^N)) \). This completes the proof. \( \square \)

Remark 3.7. For any non-quasianalytic weight function \( \omega \) such that \( \log(1 + t) = o(\omega(t)) \) as \( t \rightarrow \infty \)., the space \( (O_{C,\omega}(\mathbb{R}^N), \cdot) \) is not a multiplication topological algebra (see Section 4).

Corollary 3.8. Let \( \omega \) be a non-quasianalytic weight function. Then \( M(O'_{M,\omega}(\mathbb{R}^N)) = O_{M,\omega}(\mathbb{R}^N) \) and \( M(O'_{C,\omega}(\mathbb{R}^N)) = O_{C,\omega}(\mathbb{R}^N) \).

Proof. The result follows from Theorem 3.2 and Proposition 3.6 taking into account of the fact that the multiplication of \( \omega \)-ultradistributions with \( \omega \)-ultradifferentiable functions is defined by transposition. \( \square \)

We now pass to the case \( (S_{\omega}(\mathbb{R}^N), \cdot) \), for which the following result is true.

Theorem 3.9. Let \( \omega \) be a non-quasianalytic weight function. Then \( (S_{\omega}(\mathbb{R}^N), \cdot) \) is a convolution topological algebra.

Proof. To see this we have to show that the bilinear map \( \cdot: S_{\omega}(\mathbb{R}^N) \times S_{\omega}(\mathbb{R}^N) \rightarrow S_{\omega}(\mathbb{R}^N) \) is continuous. So, let \( n_0 \in \mathbb{N} \) be fixed such that \( n_0 \geq \frac{N+1}{2} \). Hence, \( \exp(-n_0\omega) \in L^1(\mathbb{R}^N) \) by (2.3). Fixed \( n \in \mathbb{N} \) and \( f, g \in S_{\omega}(\mathbb{R}^N) \), we have
\[
\partial^\alpha (f \ast g)(x) = \int_{\mathbb{R}^N} f(y)\partial^\alpha g(x-y) \, dy, \quad x \in \mathbb{R}^N, \quad \alpha \in \mathbb{N}_0^N.
\]
By \((2.1)\) \(\omega(x) = \omega(x - y + y) \leq K(1 + \omega(x - y) + \omega(y))\). Hence, we get for every \(x \in \mathbb{R}^N\) and \(\alpha \in \mathbb{N}_0^N\) that
\[
|\partial^\alpha (f * g)(x)| \exp(n \omega(x)) \leq e^{Kn} \int_{\mathbb{R}^N} |f(y)| \exp(Kn \omega(y)) |\partial^\alpha g(x - y)| \exp(Kn \omega(x - y)) \, dy
\]
\[
\leq e^{Kn} \| \exp(-n_0 \omega) \|_1 \| f \exp((Kn + n_0)\omega) \|_\infty \| \partial^\alpha g \exp(Kn \omega) \|_\infty.
\]
Therefore, for all \(m, n \in \mathbb{N}\)
\[
q_{m,n}(f * g) \leq e^{Kn} \| \exp(-n_0 \omega) \|_1 \| f \exp((Kn + n_0)\omega) \|_\infty q_{m,Kn}(g)
\]
\[
\leq e^{Kn} \| \exp(-n_0 \omega) \|_1 q_{m,Kn+n_0}(f) q_{m,Kn}(g)
\]
\[
\leq e^{Kn} \| \exp(-n_0 \omega) \|_1 q_{m,n'}(f) q_{m,n'}(g),
\]
with \(n' \in \mathbb{N}\) such that \(n' \geq Kn + n_0\).

Remark 3.10. We point out that the result in Theorem 3.9 can be achieved by applying Theorem 3.3 combined with the use of the Fourier transform, which is a topological isomorphism from \(S_\omega(\mathbb{R}^N)\) onto itself such that \(\hat{f} * \hat{g} = \hat{f} \hat{g}\), for all \(f, g \in S_\omega(\mathbb{R}^N)\), see [3].

The next aim is to show that \((O'_{C,\omega}(\mathbb{R}^N), \ast)\) is a convolution topological algebra. Hence, let us recall that in case \(T \in O'_{C,\omega}(\mathbb{R}^N)\) and \(S \in S'_\omega(\mathbb{R}^N)\) the convolution \(T \ast S\) is well defined on \(S_\omega(\mathbb{R}^N)\) and belongs to \(S'_\omega(\mathbb{R}^N)\). Indeed, for all \(f \in S_\omega(\mathbb{R}^N)\) we have \(T \ast f \in S_\omega(\mathbb{R}^N)\) and the operator \(C_T : S'_\omega(\mathbb{R}^N) \to S_\omega(\mathbb{R}^N), f \mapsto T \ast f\), is continuous, see, [2] Theorem 5.3 (the distribution \(\hat{T}\) is defined by \(f \mapsto \hat{T}(f) := T(\hat{f})\), with \(\hat{f}(x) := f(-x)\) for all \(x \in \mathbb{R}^N\). Furthermore, for \(R\) a distribution and \(f\) a function the convolution \(R \ast f\) is defined by \((R \ast f)(x) := \langle R_y, \tau_x f \rangle\), where the notation \(R_y\) means that the distribution \(R\) acts on a function \(\phi(x - y)\), when the latter is regarded as a function of the variable \(y\). Hence, the convolution \(T \ast S\) defined by \((T \ast S, f) = (S, T \ast f)\), for \(f \in S_\omega(\mathbb{R}^N)\), is clearly a well-defined element of \(S'_\omega(\mathbb{R}^N)\). Since \(C_T\) is a continuous operator from \(S_\omega(\mathbb{R}^N)\) into \(O'_{C,\omega}(\mathbb{R}^N)\) and the space \(S_\omega(\mathbb{R}^N)\) is dense in \(O'_{C,\omega}(\mathbb{R}^N)\), we also have that \(T \ast S \in O'_{C,\omega}(\mathbb{R}^N)\) (in the sense that \(T \ast S\) extends continuously on whole \(O'_{C,\omega}(\mathbb{R}^N)\)).

Moreover, for any \(\omega\) non-quasianalytic weight function satisfying the condition \(\log(1 + t) = o(\omega(t))\) as \(t \to \infty\), the Fourier transform \(\mathcal{F}\) is a topological isomorphism from the space \(O'_{C,\omega}(\mathbb{R}^N)\) onto the space \(O_{M,\omega}(\mathbb{R}^N)\) (see [2] Theorem 6.1). Accordingly, we have \(\mathcal{F}(O'_{C,\omega}(\mathbb{R}^N)) = O_{M,\omega}(\mathbb{R}^N)\) and \(\mathcal{F}(O_{M,\omega}(\mathbb{R}^N)) = O'_{C,\omega}(\mathbb{R}^N)\). In particular, for all \(T \in O'_{C,\omega}(\mathbb{R}^N)\) and \(S \in S'_\omega(\mathbb{R}^N)\) the convolution \(T \ast S\) satisfies the following property:
\[
\mathcal{F}(T \ast S) = \mathcal{F}(T) \mathcal{F}(S).
\]
We can now state the following result.

Theorem 3.11. Let \(\omega\) be a non-quasianalytic weight function such that \(\log(1 + t) = o(\omega(t))\) for \(t \to \infty\). Then \((O'_{C,\omega}(\mathbb{R}^N), \ast)\) is a convolution topological algebra.

Proof. We first observe that if \(S, T \in O'_{C,\omega}(\mathbb{R}^N) \subset S'_\omega(\mathbb{R}^N)\), then the convolutions \(T \ast S\) and \(S \ast T\) are well defined and belong to \(S'_\omega(\mathbb{R}^N)\). Since \(\mathcal{F}(S), \mathcal{F}(T) \in O_{M,\omega}(\mathbb{R}^N)\) and so, \(\mathcal{F}(S) \mathcal{F}(T) = \mathcal{F}(T) \mathcal{F}(S)\), by (3.3) it follows that \(S \ast T = T \ast S \in O'_{C,\omega}(\mathbb{R}^N)\).
This means that \((\mathcal{O}_{C,\omega}(\mathbb{R}^N), \ast)\) is a convolution algebra. Finally, since the Fourier transform is a topological isomorphism from \(\mathcal{O}_{C,\omega}(\mathbb{R}^N)\) onto \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\) satisfying equation (3.3), we can apply Theorem 3.2 to obtain that \((\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \ast)\) is a topological algebra.

As done in the case of multipliers of \(\text{lcHs}'\) of \(\omega\)-ultradifferentiable functions on \(\mathbb{R}^N\), we now introduce the space of convolutors.

**Definition 3.12.** Let \(E\) be a \(\text{lcHs}\) of \(\omega\)-ultradifferentiable functions on \(\mathbb{R}^N\). We denote by \(C(E)\) the space of all convolutors of \(E\), i.e., the largest space of \(\omega\)-ultradistributions on \(\mathbb{R}^N\) satisfying the following conditions:

- (1) the convolution operator on \(E \times C(E) \to \mathcal{E}_\omega(\mathbb{R})\), \((f, T) \mapsto T \ast f\), is well-defined and takes value in \(E\);
- (2) for all \(T \in C(E)\) the operator \(C_T: E \to E\), \(f \mapsto T \ast f\), is continuous.

We denote by \(C(E')\) the space of all convolutors of \(E'\), i.e., the largest space of \(\omega\)-ultradistributions on \(\mathbb{R}^N\) satisfying the following conditions:

- (1) for all \(S \in E'\) and \(T \in C(E')\) we have that \(T \ast S\) is well-defined on \(E\) and belongs to \(E'\);
- (2) for all \(T \in C(E')\) the operator \(C_T: E' \to E'\), \(S \mapsto T \ast S\), is continuous.

**Remark 3.13.** Let \(\omega\) be a non-quasianalytic weight function \(\omega\) such that \(\log(1 + t) = o(\omega(t))\) for \(t \to \infty\). Then \(C(S_{\omega}(\mathbb{R}^N)) = C(S'_{\omega}(\mathbb{R}^N)) = \mathcal{O}'_{C,\omega}(\mathbb{R}^N)\).

We point out that the equation (3.3) is also satisfied for any pairs \((T, S)\) of \(\omega\)-ultradistributions in \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N) \times \mathcal{O}_{C,\omega}(\mathbb{R}^N)\), \(\mathcal{O}'_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}_{M,\omega}(\mathbb{R}^N)\), \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N)\) and \(\mathcal{O}'_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N)\), because these spaces are continuously included in \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N) \times S_{\omega}(\mathbb{R}^N)\). On the other hand, by Remark 3.3 and Proposition 3.6 we have that \(M(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}_{C,\omega}(\mathbb{R}^N)\) and \(M(\mathcal{O}_{M,\omega}(\mathbb{R}^N)) = \mathcal{O}_{M,\omega}(\mathbb{R}^N)\). Recalling that the Fourier transform is a topological isomorphism from \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N)\) onto \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\), these facts yield that \(C(\mathcal{O}'_{M,\omega}(\mathbb{R}^N)) = \mathcal{O}'_{M,\omega}(\mathbb{R}^N)\) and \(C(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}'_{C,\omega}(\mathbb{R}^N)\). Furthermore, by Corollary 3.8 we have that \(M(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}_{C,\omega}(\mathbb{R}^N)\) and \(M(\mathcal{O}'_{M,\omega}(\mathbb{R}^N)) = \mathcal{O}'_{M,\omega}(\mathbb{R}^N)\). Thus, by the same arguments we obtain that \(C(\mathcal{O}_{M,\omega}(\mathbb{R}^N)) = \mathcal{O}'_{M,\omega}(\mathbb{R}^N)\) and \(C(\mathcal{O}_{C,\omega}(\mathbb{R}^N)) = \mathcal{O}'_{C,\omega}(\mathbb{R}^N)\).

We summarize our results in this simple table.

| \(E\)          | \(M(E)\)          | \(C(E)\)          |
|---------------|-------------------|-------------------|
| \(S_{\omega}(\mathbb{R}^N)\) | \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\) | \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N)\) |
| \(S'_{\omega}(\mathbb{R}^N)\) | \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\) | \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N)\) |
| \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\) | \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\) | \(\mathcal{O}_{M,\omega}(\mathbb{R}^N)\) |
| \(\mathcal{O}_{C,\omega}(\mathbb{R}^N)\) | \(\mathcal{O}_{C,\omega}(\mathbb{R}^N)\) | \(\mathcal{O}'_{C,\omega}(\mathbb{R}^N)\) |

From Remark 3.13 it follows this result.

**Corollary 3.14.** Let \(\omega\) be a non-quasianalytic weight function such that \(\log(1 + t) = o(\omega(t))\) for \(t \to \infty\). Then \(\mathcal{O}'_{M,\omega}(\mathbb{R}^N)\) is a convolution algebra.
Remark 3.15. For any non-quasianalytic weight function $\omega$ such that $\log(1 + t) = o(\omega(t))$ for $t \to \infty$, the space $(O^*_M(\mathbb{R}^N), \star)$ is not a convolution topological algebra (see Section 4).

4. Hypocontinuity and discontinuity

4.1. Hypocontinuity. In this final section we discuss the hypocontinuity of the multiplication mapping on some pairs between the spaces $O_{M,\omega}(\mathbb{R}^N)$, $O_{C,\omega}(\mathbb{R}^N)$, $S_\omega(\mathbb{R}^N)$ and their duals.

Let us recall that if $E$, $F$ and $G$ are topological vector spaces and $b: E \times F \to G$ is bilinear map, then $b$ is called hypocontinuous if the following holds:

1. For every bounded subset $A$ of $E$, the set $\{b(x, \cdot): x \in A\}$ is equicontinuous in $L(F, G)$;
2. For every bounded subset $B$ of $F$, the set $\{b(\cdot, y): y \in B\}$ is equicontinuous in $L(E, G)$.

Proposition 4.1. Let $\omega$ be a non-quasianalytic weight function. Then the multiplication operator $M: O_{M,\omega}(\mathbb{R}^N) \times S_\omega(\mathbb{R}^N) \to S_\omega(\mathbb{R}^N)$, $(f, g) \mapsto fg$, is separately continuous and hence, a hypocontinuous bilinear mapping.

Proof. By [1] Theorem 4.4 the operator $M_f := M(f, \cdot): S_\omega(\mathbb{R}^N) \to S_\omega(\mathbb{R}^N)$, is continuous for all $f \in O_{M,\omega}(\mathbb{R}^N)$.

Let $g \in S_\omega(\mathbb{R}^N)$ be fixed. We claim that the operator $M_g := M(\cdot, g): O_{M,\omega}(\mathbb{R}^N) \to S_\omega(\mathbb{R}^N)$, is continuous. To show the claim, we suppose that $\{f_i\}_i \subset O_{M,\omega}(\mathbb{R}^N)$ is any net such that $f_i \to f$ in $O_{M,\omega}(\mathbb{R}^N)$ and $M_g(f_i) = f_i g \to h$ in $S_\omega(\mathbb{R}^N)$. Then by [1] Theorem 5.2(1) and Proposition 5.6 we have that $f_i \to f$ in $E_\omega(\mathbb{R}^N)$ and hence, $M_g(f_i) = f_i g \to fg$ in $E_\omega(\mathbb{R}^N)$ too. But $M_g(f_i) = f_i g \to h$ also in $E_\omega(\mathbb{R}^N)$. Therefore, $h = fg = M_g(f)$. Since $\{f_i\}_i \subset O_{M,\omega}(\mathbb{R}^N)$ is arbitrary, this shows that the graph of the operator $M_g$ is closed. But, the space $O_{M,\omega}(\mathbb{R}^N)$ is ultrabornological by [B], hence barrelled, and $S_\omega(\mathbb{R}^N)$ is a Fréchet space, and so $M_g$ is necessarily continuous.

Finally, as the operator $M$ is separately continuous and $O_{M,\omega}(\mathbb{R}^N)$ and $S_\omega(\mathbb{R}^N)$ are barrelled spaces, applying [B] Theorem 41.2 we can conclude that $M$ is a hypocontinuous bilinear mapping.

Corollary 4.2. Let $\omega$ be a non-quasianalytic weight function. Then the multiplication operator $M: O_{C,\omega}(\mathbb{R}^N) \times S_\omega(\mathbb{R}^N) \to S_\omega(\mathbb{R}^N)$, $(f, g) \mapsto fg$, is separately continuous and hence, a hypocontinuous bilinear mapping.

Proof. By [1] Theorem 3.8(1) the space $O_{C,\omega}(\mathbb{R}^N)$ is continuously included in $O_{M,\omega}(\mathbb{R}^N)$. So, via Proposition 4.1 it follows that the multiplication operator $M$ is separately continuous and so hypocontinuous.

Proposition 4.3. Let $\omega$ be a non-quasianalytic weight function. Then the multiplication operator $M: O_{M,\omega}(\mathbb{R}^N) \times S'_\omega(\mathbb{R}^N) \to S'_\omega(\mathbb{R}^N)$, $(f, T) \mapsto fT$, is separately continuous and hence, a hypocontinuous bilinear mapping.

Proof. By [1] Theorem 4.6 the multiplication operator $M_f := M(f, \cdot): S'_\omega(\mathbb{R}^N) \to S'_\omega(\mathbb{R}^N)$ is continuous for all $f \in O_{M,\omega}(\mathbb{R}^N)$.

Let $T \in S'_\omega(\mathbb{R}^N)$ be fixed. We claim that the operator $M_T := M(\cdot, T): O_{M,\omega}(\mathbb{R}^N) \to S'_\omega(\mathbb{R}^N)$ is continuous. To show the claim, we fix a 0-neighborhood $V$ in $S'_\omega(\mathbb{R}^N)$. We
Proof. If \( m, n \in \mathbb{N} \) and \( c > 0 \) such that \(|\langle T, g \rangle| \leq c q_{m,n}(g)\) for all \( g \in S_\omega(\mathbb{R}^N) \). We now set \( U := \{ g \in S_\omega(\mathbb{R}^N) : q_{m,n}(g) \leq c^{-1} \varepsilon \} \) which is a 0-neighborhood in \( S_\omega(\mathbb{R}^N) \), and define the set
\[
W := \{ T \in L(S_\omega(\mathbb{R}^N)) : T(B) \subseteq U \}.
\]
Then \( W \) is a 0-neighborhood in \( L_b(S_\omega(\mathbb{R}^N)) \). Since \( O_{M,\omega}(\mathbb{R}^N) \) is a subspace of \( L_b(S_\omega(\mathbb{R}^N)) \), there exists a 0-neighborhood \( W_0 \) in \( O_{M,\omega}(\mathbb{R}^N) \) for which \( f g \in U \) for all \( f \in W_0 \) and \( g \in B \) (i.e., \( M_f(B) \subseteq U \) for all \( f \in W_0 \)). Accordingly, \( M_T(W_0) \subseteq V \).

Indeed, for a fixed \( f \in W_0 \), \( f g \in U \) for all \( g \in B \) and hence, \( q_{m,n}(f g) \leq c^{-1} \varepsilon \) for all \( g \in B \). This yields for all \( g \in B \) that
\[
|\langle M_T(f), g \rangle| = |\langle f T, g \rangle| = |\langle T, f g \rangle| \leq c q_{m,n}(f g) \leq \varepsilon.
\]
This means that \( M_T(f) \in V \). Since \( M_T(W_0) \subseteq V \), it is obvious that \( W_0 \subseteq M_T^{-1}(V) \).

Finally, since \( V \) is an arbitrary 0-neighborhood in \( S_\omega'(\mathbb{R}^N) \), we can conclude that the operator \( M_T \) is continuous.

We now observe that the space \( S_\omega(\mathbb{R}^N) \) is nuclear and hence distinguished, i.e., its strong dual \( S_\omega'(\mathbb{R}^N) \) is a barrelled lcHs. So, since \( M \) is separately continuous and \( O_{M,\omega}(\mathbb{R}^N) \) and \( S_\omega'(\mathbb{R}^N) \) are barrelled lcHs, applying [18, Theorem 41.2] we get that \( M \) is a hypocontinuous bilinear mapping. 

\begin{corollary}
Let \( \omega \) be a non-quasianalytic weight function satisfying the condition \( \log(1 + t) = o(\omega(t)) \) as \( t \to \infty \). Then the multiplication operator \( M : O_{M,\omega}(\mathbb{R}^N) \times O_{C,\omega}(\mathbb{R}^N) \to S_\omega'(\mathbb{R}^N), (f, T) \mapsto fT \), is separately continuous and hence, a hypocontinuous bilinear mapping.
\end{corollary}

\begin{proof}
By [1] Theorems 3.8(2) and Theorem 3.9] the space \( O_{C,\omega}(\mathbb{R}^N) \) is continuously included in \( S_\omega'(\mathbb{R}^N) \). So, via Proposition [3.3] it follows that the multiplication operator \( M \) is separately continuous and so hypocontinuous, being \( O_{C,\omega}(\mathbb{R}^N) \) topologically isomorphic to \( O_{M,\omega}(\mathbb{R}^N) \) via the Fourier transfom and hence barrelled. 
\end{proof}

\begin{proposition}
Let \( \omega \) be a non-quasianalytic weight function. Then the multiplication operator \( M : O_{M,\omega}(\mathbb{R}^N) \times O_{M,\omega}(\mathbb{R}^N) \to O_{M,\omega}(\mathbb{R}^N), (f, T) \mapsto fT \), is separately continuous and hence, a hypocontinuous bilinear mapping.
\end{proposition}

\begin{proof}
By Corollary [3.3] the multiplication operator \( M_f := M(f, \cdot) : O_{M,\omega}(\mathbb{R}^N) \to O_{M,\omega}(\mathbb{R}^N) \) is continuous for all \( f \in O_{M,\omega}(\mathbb{R}^N) \).

Let \( T \in O_{M,\omega}(\mathbb{R}^N) \) be fixed. We claim that \( M_T := M(\cdot, T) : O_{M,\omega}(\mathbb{R}^N) \to O_{M,\omega}(\mathbb{R}^N) \) is a continuous operator. To this end, we observe that there exist a function \( k \in S_\omega(\mathbb{R}^N), m \in \mathbb{N} \) and \( c > 0 \) such that for all \( g \in O_{M,\omega}(\mathbb{R}^N) \) we have
\[
|\langle T, g \rangle| \leq c q_{m,k}(g).
\]
If \( B \) is any bounded subset of \( O_{M,\omega}(\mathbb{R}^N) \), then from [1.1] it follows for all \( f \in O_{M,\omega}(\mathbb{R}^N) \) that
\[
\sup_{g \in B} |M_T(f)(g)| = \sup_{g \in B} |\langle T, f g \rangle| \leq c \sup_{g \in B} q_{m,k}(f g).
\]
Now, as it is shown in Theorem 3.2 there exist $l \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $m' \in N$ such that $q_{m,k}(uv) \leq C q_{m',l}(u)q_{m',l}(v)$ whenever $u, v \in \mathcal{S}_\omega(\mathbb{R}^N)$ and for $C = e^{mL}$. Accordingly, we obtain via (4.2) that

$$\sup_{g \in B} |M_T(f)(g)| \leq C \sup_{g \in B} q_{m',l}(g)q_{m',l}(f)$$

for all $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, where $D := \sup_{g \in B} q_{m',l}(g) < \infty$, being $B$ a bounded subset of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Since $B$ is an arbitrary bounded subset of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, this yields that the multiplication operator $M_T: \mathcal{O}_{M,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ is continuous.

Finally, since $M$ is separately continuous and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ are barrelled lcHs (the latter space is barrelled because it is reflexive as the strong dual of a Montel space), applying [18, Theorem 41.2] we get that $M$ is a hypocontinuous bilinear mapping.

**Proposition 4.6.** Let $\omega$ be a non-quasianalytic weight function satisfying the condition $\log(1 + t) = o(\omega(t))$ as $t \to \infty$. Then the multiplication operator $M: \mathcal{O}_{C,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, $(f, T) \mapsto fT$, is separately continuous and hence, a hypocontinuous bilinear mapping.

**Proof.** By Corollary 3.3 the multiplication operator $M_f := M(f, \cdot): \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is continuous for all $f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$.

Let $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ be fixed. We show that $M_T := M(\cdot, T): \mathcal{O}_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is a continuous operator. To see this, let $\{f_i\}_i \subset \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ be any net such that $f_i \to f$ in $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $M_T(f_i) = f_iT \to S$ in $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. Then $(f_iT)(g) = T(f_ig) \to S(g)$ for all $g \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. On the other hand, $f_ig \to fg$ in $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ for all $g \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ by Proposition 3.6. Therefore, $T(f_g) = S(g)$ for all $g \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$. This means that $S = fT = M_T(f)$. Since $\{f_i\}_i \subset \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is arbitrary, this shows that the graph of the operator $M_T$ is closed. But, the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is an (LF)-space and $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is a webbed space, and so $M_T$ is necessarily continuous (see [14]).

Finally, since $M$ is separately continuous and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ are barrelled lcHs, applying [18, Theorem 41.2] we get that $M$ is a hypocontinuous bilinear mapping.

**4.2. Discontinuity.** In this last part we show examples of multiplication and convolution mapping on some pairs between the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, $\mathcal{S}_\omega(\mathbb{R}^N)$ and their duals that are not continuous.

**Proposition 4.7.** Let $\omega$ be a non-quasianalytic weight function. Then the following assertions hold true:

(i) The multiplication operator $M: \mathcal{O}_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$, $(f, T) \mapsto fT$, is discontinuous.

(ii) The multiplication operator $M: \mathcal{O}_{C,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, $(f, T) \mapsto fT$, is discontinuous.

**Proof.** Since the proof is analogous in both the cases, we show only assertion (i).

Assume by contradiction that $M: \mathcal{O}_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$, $(f, T) \mapsto fT$, is continuous. Thus, the canonical bilinear form $\langle \cdot, \cdot \rangle: \mathcal{O}_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ →
Corollary 4.11. Let \( \omega \) be a non-quasianalytic weight function satisfying the condition \( \log(1 + t) = o(\omega(t)) \) as \( t \to \infty \). Then the convolution operator \( C : \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \), \( (T, S) \mapsto T \ast S \), is discontinuous.

Now, we prove that \( (\mathcal{O}'_{M,\omega}(\mathbb{R}^N), \ast) \) is not a topological algebra.
Proposition 4.12. Let $\omega$ be a non-quasianalytic weight function satisfying the condition $\log(1 + t) = o(\omega(t))$ as $t \to \infty$. Then the convolution operator $C : \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$, $(T, S) \mapsto T \ast S$, is discontinuous.

Proof. We first observe that by Remark 3.13 the convolution $C$ is well-defined. Now, assume by contradiction that $C : \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \times \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \to \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ is continuous. Since $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is continuously included in the space $\mathcal{O}_{M}(\mathbb{R}^N)$ of multipliers of $\mathcal{S}(\mathbb{R}^N)$ with dense range, $\mathcal{O}'_{M}(\mathbb{R}^N)$ is continuously included in $\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$. Therefore, we get that the map $C : \mathcal{O}'_{M}(\mathbb{R}^N) \times \mathcal{O}'_{M}(\mathbb{R}^N) \to \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ is continuous with range a subset of $\mathcal{O}'_{M}(\mathbb{R}^N)$. This yields that the map $\mathcal{O}'_{M}(\mathbb{R}^N) \times \mathcal{O}'_{M}(\mathbb{R}^N) \to \mathcal{O}'_{M}(\mathbb{R}^N)$ has closed graph, as it is easy to verify. By applying the closed graph theorem for (LF)-spaces (see, f.i., [14, Chap. 5, 5.4.1]) we obtain that $\mathcal{O}'_{M}(\mathbb{R}^N) \times \mathcal{O}'_{M}(\mathbb{R}^N) \to \mathcal{O}'_{M}(\mathbb{R}^N)$ is continuous, after having observed that $\mathcal{O}'_{M}(\mathbb{R}^N)$ in an (LF)-space. This is a contradiction with [16, Proposition 4].

Thanks to Proposition 4.12 and the properties of the Fourier transform we get that $(\mathcal{O}_{C,\omega}(\mathbb{R}^N), \cdot )$ is not a topological algebra.

Corollary 4.13. Let $\omega$ be a non-quasianalytic weight function satisfying the condition $\log(1 + t) = o(\omega(t))$ as $t \to \infty$. Then the multiplication operator $M : \mathcal{O}_{C,\omega}(\mathbb{R}^N) \times \mathcal{O}_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}_{C,\omega}(\mathbb{R}^N)$, $(f, g) \mapsto fg$, is discontinuous.

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