The geodesic form of light-ray trace in the inhomogeneous media

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Abstract: The canonical equations of the optical cloaking proposed by Shurig, Pendry and Smith has been proved to be equivalent to the geodesic in a 3-dimensional curved space. Carrying out the argument we extend to the 4-dimensional Riemannian space where the extra time item appears as the potential term in the canonical equations. The physical meaning of the results is interpreted.

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1. Introduction

Recently, the theory of electromagnetic cloaking has been developed based on coordinate transformation to achieve invisible cloaking, which makes objects in the cloak cannot be seen from outside [11]. This effect has been proved by ray tracing method assuming geometrical optics
Meanwhile, a conformal mapping approach has also been used to design a medium that can make invisible cloaking work in short-wavelength geometrical limit. After that, more specific works have been done such as full-wave simulations of the cloaking structures, studying on the dynamical process of dispersive cloaking and analytical solutions on the sensitivity of small perturbation to the cloaking. More excitedly, the experiment of cloaking effect has come true at microwave frequencies by the method of metamaterial.

One of the original ideas about cloaking comes from the analogy between a static gravitational field and a medium whose permittivity and permeability are tensors. For gravity in general relativity, 4-dimensional Riemannian geometry should work. Meanwhile, based on the equivalent principle, the trace of the light is also a geodesic line in the curved 4-dimensional space-time. When the mixed time-space item of the space-time metric equals to zero and the time item equals to 1 in the synchronous reference system, we can treat 3-dimensional spatial part and time separately. In this case, light propagates along a 3-dimensional geodesic line. Meanwhile in the ray tracing treatment, Schurig, Pendry and Smith used the canonical equations of a Hamiltonian system to describe the light propagation in an inhomogeneous media. If the analogy between curved space and inhomogeneous medium is valid, the propagation of light can also be described by the geodesic equation. As all the processes mentioned above, many theoretical works are restricted to transformations under which time is invariant. Thus, the situations where a cloak or an observer has an acceleration cannot be involved. Consequently, we need to use the geodesic description.

In this letter, we develop an equivalent treatment that describes the light-ray traces with the geodesic equation, which is solved numerically and used to draw exactly the same pictures as drawn in Ref. Then, with a proper conformal transformation, the light-ray in the media will show a similar action as in the static gravitational field. However, it can make an imperfect cloaking effect which shows the light-ray extending outside and is not a really invisible cloak. Furthermore, we generalize the geodesic equation in 4-dimensional space-time, which the spatial part and time part are transformed separately. In this case, a simple example is considered as the time item of the metric changes along the distance of the moving direction. By choosing an appropriate parameter, the light-ray trace recurves apparently and makes the cloaking effect imperfect similarly, which is interpreted by non-inertial relative motion.

### 2. Relation between canonical and geodesic forms of light-ray

Before going to our discussion, it is useful to distinguish two kinds of explanations of light propagation, the material form and the geometric form. If we interpret the light propagation in a medium, the permittivity and permeability of the material determine the way how light propagates in the flat space. On the other hand, it is also valid to interpret that light propagates along a geodesic line in a curved vacuum space with the spacial metric

\[ \gamma_{ij} = \frac{\gamma_{ij}}{\sqrt{\gamma}} \]

where the permittivity and permeability are equivalent, and \( \gamma = \det(\gamma_{ij}) \). For notation here, Latin indices run from 1 to 3 for spatial part and Greek indices run from 0 to 3 for both spatial and time parts. In the following we denote the spatial coordinate by \( x^i \) and the space-time coordinate by \( x^\mu \), where \( x^0 = ct \) describes time measured in spatial units. The sign of the curved space-time metric takes the form \((+,-,-,-)\). The spacial metric \( \gamma_{ij} \) has the form

\[ \gamma_{ij} = -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \]

and its inverse is

\[ \gamma^i_j = -g^{ij} \]
Because the reference system considered by Schurig et al[1,2] is the synchronous reference system which satisfies $g_{00} = 1$ and $g_{\mu\nu} = 0$. Thus, in the condition (1), we can write
\[ \gamma = \det(\gamma_{jk}) = -\det(g_{\mu\nu})/g_{00} = -g. \]

2.1. Material interpretation with canonical form

With the assumption (1), the eikonal equation [2] of light-ray in the inhomogeneous media in the inertial reference can be written as
\[ n^{ij}k_i k_j - \det(n^{ij}) = 0. \] (4)

Consequently, Schurig et al [2] describe the light-ray in the material interpretation with the Hamiltonian:
\[ H = f(x) \left[ n^{ij}k_i k_j - \det(n^{ij}) \right], \] (5)
where $f(x)$ is an arbitrary function of position coordinates $x$, and $k_i$ is wave vector. Then the equations of motion can be given as following canonical equations:
\[ \frac{dx^i}{d\tau} = \frac{\partial H}{\partial k_i}, \] (6a)
\[ \frac{dk_i}{d\tau} = -\frac{\partial H}{\partial x^i}, \] (6b)
where $\tau$ is an arbitrary parameter describing the path. Thus a light propagation in the cloaking material can be determined by this procedure.

2.2. Geometric interpretation with geodesic form

Likewise we can also use the geometric interpretation to portray light propagation. Actually by doing this it has following merits: (i) Light propagating along geodesic in a curved space gives us a vivid picture of the traces, which is easily handled by using preliminary Riemannian geometry; (ii) With this interpretation, a variety of techniques accumulated in the field of general relativity may provide us potential candidates for designing cloaking materials; (iii) Establishing this interpretation is fundamentally important for the integrity of a theory. Based on the equivalent principle a light ray propagates along the geodesic line and thereby a geodesic equation should be derived from Eqs. (6). We will show this in Section 2.3.

Firstly, the eikonal equation in the 4-dimensional space-time is
\[ k^\mu k_\mu = 0. \] (7)

So the Hamiltonian of the light-ray can be written as
\[ H = f'(x)g^{\mu\nu}k_\mu k_\nu, \] (8)
where $f'(x)$ is another arbitrary function of position coordinates $x$. Considering the synchronous reference system, which $g_{0\nu} = 0$ and $g_{00} = 1$, the eikonal equation is
\[ \gamma^{ij}k_i k_j - \left( \frac{\omega}{c} \right)^2 = 0, \] (9)
where $k_0 = \omega/c$, $\omega$ is the angular frequency of the light and $c$ is the speed of light. Substituting it into the geometric Hamiltonian [3] with the unit $\omega = c = 1$, we get
\[ H = f'(x) \left[ \gamma^{ij}k_i k_j - \left( \frac{\omega}{c} \right)^2 \right]. \]
Comparing it with the material Hamiltonian (5), we have

\[ f'(x) = f(x) \frac{\partial}{c} \sqrt{\gamma}, \]

and the refractive index tensor has the form

\[ \varepsilon^{ij} = \mu^{ij} = n^{ij}(x) = \frac{\omega}{c} \sqrt{\gamma^{ij}(x)}. \] (10)

So the Hamiltonian of the light ray in the synchronous reference has the form

\[ H = f(x) \frac{\omega}{c} \sqrt{\gamma} \left[ \gamma^{ij} k_i k_j - \left( \frac{\omega}{c} \right)^2 \right], \] (11)

which satisfies the material Hamiltonian (5) of Schurig et al \[2\]. Our goal is, from the Hamiltonian (11) and the canonical equations (6), to derive the geodesic equation of light

\[ \frac{dk^i}{d\lambda} + \Gamma^i_{jl} k^j k^l = 0, \] (12)

where \( k^i = dx^i/d\lambda \); \( \lambda \) is a parameter varying along the light traces which is also determined by Eq. (12), and \( \Gamma^i_{jl} = \frac{1}{2} \gamma^{im} (\gamma_{mj,l} + \gamma_{ml,j} - \gamma_{jl,m}) \) is the Christoffel symbol in 3D curved space.

2.3. The proof process

a) The arbitrary parameter \( \tau \) (in Eq. (6)) and the determining parameter \( \lambda \) (in Eq. (12)) are not same. We assume that \( \lambda \) and \( \tau \) are connected by a function \( f_A(x) \) with arbitrary character

\[ d\lambda = f_A(x) d\tau. \] (13)

b) With using the canonical equation (6a), we get (calculating details in Appendix (A.1)):

\[ k^i = \frac{dx^i}{d\lambda} = \frac{dx^i}{d\tau} \cdot \frac{d\tau}{d\lambda} = \frac{2\omega}{c} f(x) \frac{f_A(x)}{f_A(x)} \sqrt{\gamma^{ij} k_i}. \] (14)

If we let \( f_A(x) = \frac{2\omega}{c} f(x) \sqrt{\gamma} \). Eq. (14) satisfies \( k^i = \gamma^{ij} k_j \), and Hamiltonian (11) changes into

\[ H = \frac{f_A(x)}{2} \left[ \gamma^{ij} k_i k_j - \left( \frac{\omega}{c} \right)^2 \right]. \] (15)

c) With calculating \( dk^i/d\lambda \), we change it into the form

\[ \frac{dk^i}{d\lambda} = \frac{d(\gamma^{ij} k_j)}{d\tau} \cdot \frac{d\tau}{d\lambda} = \frac{1}{f_A(x)} \left( \gamma^{ij} \frac{dk_j}{d\tau} + \frac{\partial \gamma^{ij}}{\partial x^m} \frac{dx^m}{d\tau} \cdot k_j \right). \]

Considering the canonical equations (6), eikonal equation (9) and the metric property \( \gamma_{il} \gamma_{jm} + \gamma_{il} \gamma_{jm} = 0 \), we can finally get the geodesic equation (12). (See more calculating details in Appendix (A.2).)

Therefore, the equivalence between the canonical form and the geodesic form shows the following statements in the viewpoint of mathematics. The solution of optic cloaking is the inverse problem which is to find the metric \( \gamma_{ij} \) on the given boundary conditions as light-ray traces.
2.4. Example of spherical cloaking

Now we can take spherical cloaking as an example to check our calculation. The metric inside the cloak derived from the coordinate transformation

\[ r' = \frac{b-a}{b} r + a. \]  

(16)

Correspondingly, the metric is

\[ ds^2 = \left( \frac{b}{b-a} \right)^2 dr'^2 + \left( \frac{b}{b-a} \right)^2 (r' - a)^2 d\theta^2 + \left( \frac{b}{b-a} \right)^2 (r' - a)^2 \sin^2 \theta d\phi^2. \]  

(17)

So we have the material properties

\[ \varepsilon_{rr'}' = \frac{b}{b-a} (r' - a)^2 \sin \theta, \quad \varepsilon_{\theta\theta'}' = \frac{b}{b-a} \sin \theta, \quad \varepsilon_{\phi\phi'}' = \frac{b}{b-a} \cdot \frac{1}{\sin \theta}. \]  

(18)

If the original material items

\[ \varepsilon_{0rr'} = r^2 \sin \theta, \quad \varepsilon_{0\theta\theta'} = \sin \theta, \quad \varepsilon_{0\phi\phi'} = \frac{1}{\sin \theta} \]

are eliminated, we get

\[ \frac{\varepsilon_{rr'}'}{\varepsilon_{0rr'}} = \frac{b}{b-a} \left( \frac{r' - a}{r} \right)^2, \quad \frac{\varepsilon_{\theta\theta'}'}{\varepsilon_{0\theta\theta'}} = \frac{b}{b-a}, \quad \frac{\varepsilon_{\phi\phi'}'}{\varepsilon_{0\phi\phi'}} = \frac{b}{b-a}. \]  

(19)

which are the same as the results in Ref. [1].

For the calculation, the boundary conditions are given in Ref. [2]

\[ (k_1 - k_2) \times n = 0, \]  

(20a)

\[ H(k_2) = 0, \]  

(20b)

where \( k_1 \) is the wave vector outside of the cloak boundary; \( k_2 \) is inside; and \( n \) is the unit

Fig. 1. The direct calculation result of the geodesic by Eq. (12), which gives the same spherical cloaking as in Ref. [2].
normal to the boundary. By choosing $\omega/c = 1$, we compute the geodesic equation (12) with NDSlove of Mathematica and draw pictures. The Fig. 1 shows the result. The two concentric circles and homocentric spheres are boundaries of the cloak, and lines drawn in the pictures are light-ray traces. We can see clearly that light-ray traces are bent around the inside cloaking boundary, which they can still propagate regularly out of the outside boundary. These pictures are the same as those in Ref. [2].

3. The effect of conformal transformation on 3-dimensional metric

3.1. A trial on 4-dimensional extending

In the previous section, we establish the geodesic description of the light propagation in an invisible cloaking. In this section, we want to extend the analogy between the inhomogeneous media $\epsilon^{ij} = \mu^{ij} = n^{ij}$ and the curved space from three dimensions spacial space to four dimensions space-time. As in the Ref. [9] Landau and Lifshitz proved a static gravitational field plays the role of a medium with permittivity and permeability which satisfy $\epsilon^{ij} = \epsilon_0 \gamma^{ij} / \sqrt{g_{00}}$, $\mu^{ij} = \mu_0 \gamma^{ij} / \sqrt{g_{00}}$ as follows.

For Maxwell’s equations in the curved 4-dimensional space-time

$$\frac{1}{\sqrt{\gamma}} \partial_t (\sqrt{\gamma} B^i) = 0,$$

$$\frac{\partial B^i}{\partial t} + \frac{1}{\sqrt{\gamma}} \epsilon^{ijk} \partial_j E_k = 0,$$

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} (\sqrt{\gamma} D^\alpha) = \rho,$$

$$\frac{\partial D^i}{\partial t} + \frac{1}{\sqrt{\gamma}} \epsilon^{ijk} \partial_j H_k = J^i,$$

and their apparently covariant form

$$\partial_\mu F_{\nu \mu} + \partial_\nu F_{\mu \mu} + \partial_\mu F_{\nu \mu} = 0,$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} G_{\mu \nu}) = j^\nu.$$

Thus, comparing with two forms of Maxwell’s equations (21,22), we get

$$F_{ij} = -c \sqrt{\gamma} \epsilon_{ijk} B^k$$

$$F_{0i} = E_i$$

$$G^{0i} = \frac{c}{\sqrt{g_{00}}} D^i$$

$$G^{ij} = -\frac{1}{\sqrt{-g}} \epsilon^{ijk} H_k$$

$$B^i = -\frac{1}{2c^{\sqrt{\gamma}}} \epsilon^{ijk} F_{jk}$$

$$E_i = F_{0i}$$

$$D^i = \frac{\sqrt{g_{00}}}{c} G^{0i}$$

$$H_i = -\frac{1}{2} \epsilon_{ijk} G^{jk}$$

where $\epsilon^{ijk}$ and $\epsilon_{ijk}$ are the Levi-Civita symbol. Considering the vacuum media in the curved space-time, we have the relation

$$G^{\mu \nu} = g^{\alpha \mu} g^{\beta \nu} F_{\mu \nu}.$$

Thus, we have

$$D^i = \left[ \epsilon_0 \sqrt{g_{00}} (g^{0i} g^{00} - g^{ij} g^{00}) E_j + \frac{c \epsilon_0 \sqrt{g_{00}}}{2} \epsilon^{ijk} \epsilon^{kl} \right] B^k,$$

$$B^i = \frac{c^2}{\sqrt{-g}} g_{ij} g_{kl} \epsilon^{ijk} D^l + \frac{\epsilon_{ijkl} \epsilon^{jlmn}}{2g_{kl}} H_n.$$
where we set \( \varepsilon_0 = \mu_0 = 1/c \). If the curved space-time satisfies \( g_{0i} = 0 \) and \( g_{ij} = -\gamma_{ij} \), then we have
\[
D^i = \varepsilon_0 \frac{\gamma^j}{\sqrt{\gamma_{00}}} E_j, \quad B^i = \mu_0 \frac{\gamma^j}{\sqrt{\gamma_{00}}} H_j.
\]  
(24)
Comparing it with the inhomogeneous media in the flat space-time,
\[
D^i = \varepsilon^{ij} E_j, \quad B^i = \mu^{ij} H_j,
\]  
(25)
we have the result
\[
\varepsilon^{ij} = \varepsilon_0 \frac{\gamma^{ij}}{\sqrt{\gamma_{00}}} ; \quad \mu^{ij} = \mu_0 \frac{\gamma^{ij}}{\sqrt{\gamma_{00}}}.
\]  
(26)

3.2. Conformal transformation
Thus, we will show how to design a media satisfies the condition (26). Our first trial is to make a conformal transformation
\[
\gamma^{ij} \rightarrow \gamma'^{ij} = \sigma(x) \gamma^{ij},
\]  
(27)
Substituting it into the eikonal equation (9), we get the Hamiltonian
\[
H = f'(x) \left[ \sigma(x) \gamma^{ij} k_i k_j - \left( \frac{\omega}{c} \right)^2 \right].
\]
Similarly, comparing it with the material Hamiltonian (5), we have
\[
f'(x) = f(x) \frac{\omega}{c} \sqrt{\gamma},
\]
and the refractive index tensor has the form
\[
\varepsilon^{ij} = \mu^{ij} = n^{ij} = \frac{\omega}{c} \sqrt{\gamma} \gamma^{ij} = \frac{\omega}{c} \sqrt{\frac{\gamma}{\sigma(x)}} \gamma^{ij}.
\]  
(28)
Thus the Hamiltonian of the light ray in the synchronous reference with a conformal transformation (27) has the form
\[
H = f(x) \frac{\omega}{c} \sqrt{\frac{\gamma}{\sigma}} \left[ \sigma(x) \gamma^{ij} k_i k_j - \left( \frac{\omega}{c} \right)^2 \right].
\]
If we want the definition of \( k^i = dx^i/d\lambda \) satisfying \( k^i = \gamma^{ij} k_j = \sigma(x) \gamma^{ij} k_j \), it should be
\[
f_{\lambda}(x) = \frac{2\omega}{c \sigma(x)} \sqrt{\frac{\gamma}{\sigma(x)}} f(x).
\]
So the Hamiltonian (5) changes to
\[
H = \frac{f_{\lambda}(x)}{2} \left[ \sigma(x) \gamma^{ij} k_i k_j - \left( \frac{\omega}{c} \right)^2 \right].
\]  
(29)
Calculating \( dk^i/d\lambda \) with the canonical Eqs. (6) and the Hamiltonian (29), we get the new geodesic equation in 3D space
\[
\frac{dk^i}{d\lambda} + \Gamma^i_{st} k^s k^t = 0, \quad \Gamma^i_{st} = \Gamma^i_{ts} + \left( \frac{\partial}{\partial x^j} \sqrt{\sigma(x)} \right) \gamma^{ij} k_j - \delta^i_{s} \frac{\partial}{\partial x^s} \ln \sigma(x) \right).}
\]  
(30)
The details are in Appendix (B).
3.3. Physical interpretation

Comparing the similarity between the curved space-time (26) with the refractive index \(n^{ij}\) and conformal transformation (28), we get the analog

\[
\sigma(x) = -g(x) = g_{00} \cdot \gamma.
\]  

(31)

So the media with the condition (28) can describe an artificial static gravitational field for the light-ray. Meanwhile, we can also make the analog

\[
\sigma(x) = g_{00}
\]  

(32)

for the next section with different physical meaning. It is similar to the 4-D coordinate transformation but totally different, which will be discussed later.

Considering the Hamiltonian (29), if the function with arbitrary property \(f_A(x)\) satisfies

\[
f_A(x) = \frac{1}{m\sigma(x)} \quad (m = \hbar \omega / c^2),
\]  

(33)

the Hamiltonian (29) recasts to

\[
H = K(k) + V(x) = \frac{\gamma^{ij} k_i k_j}{2m} - \frac{1}{2\sigma(x)} \left( \frac{c}{\hbar} \right)^2 m.
\]  

(34)

Here, the photon in the media is massless, but the speed is less than \(c\). So we can treat it as a “non-relative particle” with an equivalent mass \(m\). The Hamiltonian (34) has a kinetic energy \(K(k)\) with an inertia mass \(m\) and a potential \(V(x)\) with a “gravitational” mass \(m\), where the inertia mass and the “gravitational” mass equal to each other. Taking account of the principle of equivalence, it can be viewed as a particle \(m\) accelerating in a curved 3D space with spatial metric \(\gamma^{ij}\). The “Newtonian equation” is

\[
v = \frac{1}{2\sigma(x)^2} \left( \frac{c}{\hbar} \right)^2 \frac{\partial \sigma(x)}{\partial x} - \frac{k_i k_j}{2m^2} \left( \frac{\partial \gamma^{ij}(x)}{\partial x} \right),
\]  

(35)

where the first term is generated by the potential \(V(x)\) and the second is contributed by kinetic energy in the curved space.

Consequently, with the given Hamiltonian (34), we can write the Poisson brackets

\[
\{F, G\} = \frac{\partial F}{\partial k_a} \frac{\partial G}{\partial x^a} - \frac{\partial F}{\partial x^a} \frac{\partial G}{\partial k_a},
\]

and the following results

\[
\{x^i, k_j\} = -\delta^i_j, \quad (36a)
\]
\[
\{x^i, k^j\} = -\gamma^{ij}, \quad (36b)
\]
\[
\{H, x^i\} = \frac{\partial H}{\partial k_a} = \frac{\gamma^{ij} k_i}{m}, \quad (36c)
\]
\[
\{H, k_i\} = -\frac{\partial H}{\partial x^i} = -\frac{k_i k_j}{2m} \left( \frac{\partial \gamma^{ij}}{\partial x^i} \right) - \frac{m}{2\sigma(x)} \left( \frac{c}{\hbar} \right)^2 \frac{\partial \sigma(x)}{\partial x^i} \left( \frac{c}{\hbar} \right)^2 \frac{\partial \sigma(x)}{\partial x^i}. \quad (36d)
\]

It may be viewed as a classical model for the light propagation in the medium with \(\epsilon^{ij} = \mu^{ij} = n^{ij}\). 

Here, we need to notice that the conformal transformation with analog (32) cannot give a real 4-dimensional curved coordinates. The gravitational “red-shift” effect does not occur and the light-ray trace is not the geodesic line in the 4-dimensional curved space-time. The conformal transformation in 3-dimensional can’t be given by the coordinate transformation. The real 4-dimensional coordinate transformation will be given in the next section. So the analog (32) cannot show a real 4-dimensional transformation as the gravitational field or any accelerating systems.

3.4. Example of spherical cloaking with conformal transformation

If we still take the coordinate transformation (16) and choose $\sigma(x)$ properly, we can make sure that the light can not go into the cloak region $r < a$.

By choosing $\sigma(x) = 1 + \frac{b-a}{2b(r-a)}$ and calculating Eq. (30), we draw Fig. 2. From the picture it can be seen that the light ray is bent towards outside. Here we choose a proper $\sigma(x)$. Thus, the cloak can still be preserved to some extend. In general relativity [9], sometimes $g_{00}$ has a direct relationship with Newtonian potential. So $\sigma(x) = g_{00}(x) = 1 + \frac{b-a}{2b(r-a)}$ represents a field having repulsion force, which causes the light ray extending outside.

![Fig. 2. Adding $\sigma(x)$ cloak: the adding $\sigma(x)$ term bends the light-rays and can make sure the light does not go into the cloaked region $r < a$.](image)

4. The cloaking system with 4-dimensional coordinate transformation

At the beginning, we discuss the problem in the synchronous reference, which $g_{00} = 1$ and $g_{0i} = 0$. Like Ref. [2], the analogy between the inhomogeneous media and the 3D curved space has been shown and the time $x^0 = ct$ has not been transformed. Additionally, we try to give a trial on 4-dimensional cloaking system by conformal transformation. However, it is not a real 4-dimensional cloaking system. More generally, we will discuss how to make the time item $g_{00} = g_{00}(x)$ a function with variant spatial coordinate $x$, which extends a real analogy in Ref. [2]. Meanwhile, it can be interpreted as the cloaking system or an observer has an accelerating motion.

4.1. The light-ray’s change under time transformation

For the eikonal equation (7) given previously, we have its form under the condition $g_{00}$ changes to

$$\gamma^{ij}k_ik_j - g^{(0)}(k_0)^2 = 0.$$
So we have its Hamiltonian

\[ H = f'(x) \left[ \gamma_{ij} k_i k_j - g^{00}(k_0)^2 \right], \quad k_0 = \frac{\omega_0}{c}. \]  

(37)

With the same process of demonstration, we can prove the light-ray traces are consistent with the geodesic lines in 4-dimensional curved space with \( g_{0i} = 0 \) (calculating details in Appendix C.1)

\[
\frac{dk^i}{d\lambda} + \Gamma^i_{jl} k^j + \Gamma^i_{00} (k_0)^2 = \frac{dk^i}{d\lambda} + \left( \Gamma^i_{jl} + \gamma_{jl} \gamma_{im} \frac{\partial \ln \sqrt{g_{00}}}{\partial x^m} \right) k^j k^l = 0, \quad (38)
\]

where the third item on the left side of the equation (38) above recurves the geodesic line in 3-dimensional space, and whatever \( \sigma(x) \) is, the equations are totally different from the equations (30) (see the discussion in Appendix C.2). The Christoffel symbol \( \Gamma^i_{jl} \) and \( \Gamma^i_{00} \) are components of the 4-D Christoffel symbol \( \Gamma^\mu_{\rho\lambda} \). However, in the condition \( g_{0i} = 0 \), \( \Gamma^i_{jl} \) are also 3-D Christoffel symbol.

Here, we need to notice the material Hamiltonian (5) does not fit to the condition here. Because the Hamiltonian (5) describes the light-ray in the media in a synchronous reference system. The reference here, however, is not synchronous anymore, which time item \( g_{00} \) changes along the path in the spacial space. In consequence of this, we need to redesign the material property for cloaking optic device.

4.2. Cloaking design and “red-shift”

Now, we consider how the refractive index changes and how the changes will affect the cloaking system under the time transformation. As in the synchronous reference, we take time and space transformation independently. So

\[ \Lambda^\alpha_{\alpha'} = \frac{\partial x^\alpha}{\partial x'^\alpha} \quad \text{and} \quad \Lambda^i_{0'} = 0, \quad \Lambda^0_0 = \psi(x), \]

which shows the time part transformation only depend on the spatial coordinate. Thus, we get

\[ dt' = \psi(x) dt \quad \text{and} \quad t' = \psi(x)t. \]

Because of \( g_{00} dt'^2 = 1 \cdot dt^2 \), we can write

\[ \psi(x) = \frac{1}{\sqrt{g_{00}}}. \]

(39)

Now, we take Faraday’s law of induction as an example. With \( B'(x,t) = \mu^{ij}(x) H_j(x)e^{i\omega t} \), the equation has the form

\[ \varepsilon^{ijk} \frac{\partial E_k(x)}{\partial x^j} + \mu^{ij}(x) H_j \frac{\partial e^{i\omega t}}{\partial t} = 0. \]

(40)

For the 4-dimensional space-time transformation, the equation turns out to be

\[ \varepsilon^{ij'} j' \left[ \frac{E_k'(x')}{\partial x^{j'}} \right] + \left[ \frac{1}{\det(\Lambda^i_j) \sqrt{g_{00}}} \Lambda^i_j \Lambda^j_i \mu^{ij}(x) \right] H_j'(x') \frac{\partial e^{i\omega t' \sqrt{g_{00}}} }{\partial t'} = 0, \]

(41)

Comparing Eqs. (40) with (41), we can see that \( \mu^{ij} \) transforms to

\[ \mu'^{ij}(x') = \frac{1}{\det(\Lambda^i_j) \sqrt{g_{00}}} \Lambda^i_j \Lambda^j_i \mu^{ij}(x). \]

(42)
and $\omega$ transforms to
\[ \omega' = \omega \sqrt{g_{00}}. \] (43)

The Ampère’s circuital law without electric current has the same transformation properties. Thus, we design the cloaking device which has the property of
\[ \varepsilon^{ij} = \mu^{ij} = n^{ij} = \sqrt{\gamma} \gamma^{ij} \] (44)
and the light for the cloaking effect will show a “red-shift” like (43).

4.3. Physical Interpretation

As a physical example, we consider the following situation. Assuming the cloak moves in a constant acceleration $A$ relative to an observer and due to the equivalence principle, it is equal to say there is a constant gravity in the space-time whose metric is the so-called Rindler metric [12]
\[ ds^2 = (1 + \frac{\mathcal{A}z}{c^2})^2 dt^2 - dx^2 - dy^2 - dz^2. \] (45)

Based on this time transformation, we see that in this reference $\varepsilon^{ij}$ and $\mu^{ij}$ in the cloak change to be $\varepsilon^{ij} = \mu^{ij} = \sqrt{\gamma} \gamma^{ij} / \sqrt{g_{00}}$, where $g_{00} = \left(1 + \frac{\mathcal{A}z}{c^2}\right)^2$. The Hamiltonian can be written as
\[ H = \frac{1}{2} \left[ \gamma_{ij} k^i k^j - \left(1 + \frac{\mathcal{A}z}{c^2}\right)^2 \left( \frac{\omega'}{c} \right)^2 \right]. \] (46)

Fig. 3. Cloak with a time transformation: The acceleration we choose to draw the picture is very large $\mathcal{A} = 0.11 \times (3 \times 10^8)^2 m/s^2$, when $\mathcal{A}$ decreases to one tenth, the light-rays observed in the acceleration reference will be analogous to the light-rays in the original inertia reference.

In the new metric (45), we still take the transformation (16). Parameters for the calculation are $a = 1m$, $b = 2m$, $\mathcal{A} = 0.11 \times (3 \times 10^8)^2 m/s^2$, and the results are shown in Fig. (3). If $\mathcal{A}$ decreases to one tenth, the light-rays observed in the acceleration reference will be analogous to the light-rays in the original inertia reference, but meanwhile $\mathcal{A}$ is still very large. So if we are in a reference which has an acceleration that is the same as the acceleration of a body on the
earth surface, it is impossible for us to see the changes of the light through the cloak compared to the original cloak in an inertia reference.

The curvature of this space-time is not zero. Actually the scalar Ricci curvature reads

\[ R = \frac{4a\alpha\cos\theta(-b+a)^2}{(-r+a)^2b^2(1+\alpha r\cos\theta)}. \]

However, the curvature of space-time is zero when we just consider 3D space coordinate transformations.

5. Conclusion

In this paper, we establish the relationship between the canonical equations and the geodesic equation describing the light propagation, and use the latter form to obtain the light-ray trace in the cloaking system. In the discussion, it is easy to show that equations of both forms can derive the same light-ray traces in an invisible cloaking system. Then we generalize the geodesic equation with a 3D conformal transformation in metric. By choosing a specific 3D coordinate transformation and the conformal transforation item \( \sigma(x) \) properly, we can also make sure that the light-ray do not go into cloak region \( r < a \). In addition, we give the physical interpretation of the conformal transformation that the photon in the inhomogeneous media can be viewed as a particle with an equivalent mass \( m \) accelerating in a curved 3D space. Finally, we make a 4D coordinate transformation which the time item \( g_{00} \) is variable and the “red-shift” effect and accelerating cloak can be seen.

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A. The demonstration of equivalence between canonical equations and geodesic

A.1. The equivalence of \( k^i \)'s definition

Firstly, we have the definition of wave vector with upper index

\[ k^i = \frac{dx^i}{d\lambda}, \]

where \( \lambda \) is determined by the geodesic Eq. \[12\] \[9\]. And we also know that the \( k^i \), as a contravariant vector, satisfies

\[ k^i = \chi^l k_l. \]

We need these two forms to be equivalent.

So by using Eq. \[47\] we have

\[ k^i = \frac{dx^i}{d\tau} \cdot \frac{d\tau}{d\lambda} = \frac{\partial H}{\partial k_i} \frac{1}{f_A(x)} \frac{2\omega f(x)}{c f_A(x)} \sqrt{\gamma} \chi^l k_l. \]

If we let the arbitrary functions \( f_A(x) \) and \( f(x) \) satisfy the relation

\[ f_A(x) = \frac{2\omega c}{\sqrt{\gamma} f(x)}, \]

two forms of \( k^i \) \[48\] \[47\] are equivalent.
A.2. The deduction from canonical equations to geodesic equation

With the canonical Eqs. (6) and the Eq. (48), we have

\[
\frac{dk^i}{d\lambda} = \frac{d}{d\tau} \left( \gamma^j k_j \right)
\]

\[
= \left( \frac{d\gamma^j}{d\tau} k_j + \gamma^j \frac{dk_j}{d\tau} \right) \cdot \frac{1}{f_A(x)}
\]

\[
= \left( \frac{\partial \gamma^j}{\partial x^s} \frac{dx^s}{d\tau} k_j - \gamma^j \frac{\partial H}{\partial x^j} \right) \cdot \frac{1}{f_A(x)}
\]

\[
= \frac{\partial \gamma^j}{\partial x^i} k^i k_j - \gamma^j \frac{\partial \gamma}{\partial x^j} k_j k_m - \gamma^j \frac{1}{2} \frac{df_A(x)}{dx^j} \left[ \gamma^m k_i k_m - \left( \frac{\omega}{c} \right)^2 \right] \cdot \frac{1}{f_A(x)}
\]

combining with

\[
\gamma^m k_i k_m - \left( \frac{\omega}{c} \right)^2 = 0
\]

\[
= \gamma^m \frac{\partial \gamma^i}{\partial x^m} k^i k^j - \frac{1}{2} \partial \gamma^m \frac{\partial g_{lm}}{\partial x^j} k^i k^j - 0
\]

\[
= \left( \frac{\partial}{\partial x^i} \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^i} \right) k^k k^l - \frac{1}{2} \partial g_{lm} \frac{\partial g_{ij}}{\partial x^j} k^k k^l
\]

\[
= - \gamma^j \left( \frac{1}{2} \frac{\partial \gamma}{\partial x^j} k_i k_j + \frac{1}{2} \frac{\partial \gamma}{\partial x^j} - \frac{1}{2} \frac{\partial \gamma}{\partial x^j} k^k k^j \right)
\]

\[
= - \Gamma^j_i k^k k^j.
\]

So we get the geodesic Eq. (12).

B. The change of geodesic equation under 3-dimensional conformal transformation

Under the conformal transformation (27), we can derive another geodesic equation (30) from the canonical equations (6).

\[
\frac{dk^i}{d\lambda} = \frac{d}{d\tau} \left( \sigma(x) \gamma^j k_j \right)
\]

\[
= \sigma(x) \frac{\partial \gamma^j}{\partial x^i} \frac{dk^i}{d\lambda} + \gamma^j \frac{dk_j}{d\lambda} \cdot \frac{d\tau}{d\lambda} + \gamma^j \frac{\partial \gamma}{\partial x^j} k_j k_m \cdot \frac{d\tau}{d\lambda} + \gamma^j \frac{\partial H}{\partial x^j} \frac{d\tau}{d\lambda}
\]

\[
= \sigma(x) \frac{dx^i}{d\lambda} \gamma^j k_j + \sigma(x) \frac{\partial \gamma^j}{\partial x^i} \frac{dk^i}{d\lambda} + \frac{1}{2} \frac{\partial (\sigma(x) \gamma^m)}{\partial x^j} k^m
\]

\[
= - \sigma(x) \gamma^j \frac{1}{2} \partial f_A(x) \left[ \sigma(x) \gamma^j k_j - \frac{\omega^2}{c^2} \right] \cdot \frac{1}{f_A(x)}
\]

combining with

\[
\sigma(x) \gamma^j k_j = \frac{\omega^2}{c^2} = 0 \quad \text{and} \quad k_j = \frac{\gamma^j}{\sigma(x)} \frac{k^j}{k^j} \quad k_i = \frac{\gamma^i}{\sigma(x)} \frac{k^i}{k^i} \quad k_m = \frac{\gamma^m}{\sigma(x)} \frac{k^m}{k^m}
\]

\[
= \gamma^j \gamma^i \frac{1}{\sigma(x)} \frac{d\sigma(x)}{dx^i} k^k k^l + \gamma^j \frac{\partial \gamma^j}{\partial x^i} k^k k^l - \frac{1}{2} \gamma^m \frac{\gamma^i}{\sigma(x)} \frac{\partial (\sigma(x) \gamma^m)}{\partial x^j} k^k k^j.
\]
Under this situation, derive that the other terms are given by the conformation transformation (27).

\[
\begin{align*}
\mathbf{d} = & -\frac{1}{2}\gamma^{ij} \gamma^{jm} \left( \frac{\partial (\gamma_{ji}/\gamma^{jm})}{\partial x^j} - \gamma^{jm} \frac{\partial (\gamma_{ji}/\gamma^{jm})}{\partial x^0} \right) \mathbf{k}^i \\
\end{align*}
\]

We can define \( \mathbf{27} \).

\[
\begin{align*}
\mathbf{d} = & -\frac{1}{2}\gamma^{ij} \gamma^{jm} \left( \frac{\partial (\gamma_{ji}/\gamma^{jm})}{\partial x^j} - \gamma^{jm} \frac{\partial (\gamma_{ji}/\gamma^{jm})}{\partial x^0} \right) \mathbf{k}^i \\
\end{align*}
\]

So we get the new geodesic Eq. \( \mathbf{50} \), where \( \Gamma'_{ij} \) is given by the coordinate transformation and the other terms are given by the conformation transformation \( \mathbf{27} \).

C. **Four dimensional transformation**

C.1. The proof process

We have the Hamiltonian \( \mathbf{57} \) and the canonical equations \( \mathbf{60} \). For the \( k^i \)’s definition, we can derive that

\[
\begin{align*}
k^i = & \frac{dx^i}{d\lambda} = \frac{dx^i}{d\tau} \frac{d\tau}{d\lambda} = \frac{\partial H}{\partial \dot{k}^i} \cdot \frac{1}{f_A(x)} = 2 \frac{f'(x)}{f_A} \gamma^i k_i. \\
\end{align*}
\]

This is as previous calculation, we can define \( f'(x) = f_A(x)/2 \). Then the Hamiltonian \( \mathbf{57} \) can be written as

\[
H = \frac{f_A(x)}{2} (\gamma^i k_i - g_{00} k_0). 
\]

Under this situation, \( \gamma^i \) is independent of \( \mathbf{00} \), by using the specialized Hamiltonian \( \mathbf{50} \) and the following canonical Eq. \( \mathbf{63} \), we can derive

\[
\begin{align*}
dk^i &= \frac{d(\gamma^i k_i)}{d\lambda} = \frac{d\lambda}{d\tau} \frac{d\tau}{d\lambda} = \frac{\partial H}{\partial k^i} \cdot \frac{1}{f_A(x)} = \frac{2 f'(x)}{f_A} \gamma^i k_i. \\
\end{align*}
\]
\[\begin{align*}
&= -\frac{1}{2} \gamma^j_i \left( \frac{\partial \gamma_{jk}^s}{\partial x^s} + \frac{\partial \gamma_{ks}^j}{\partial x^t} - \frac{\partial \gamma_{st}^k}{\partial x^j} \right) k' k' - \frac{1}{2} \gamma^j_i \frac{\partial g_{00}}{\partial x^j} (k^0)^2
\end{align*}\]

because \(\Gamma^i_{jk} = \frac{1}{2} g^{i\mu} (g_{jk,\mu} + g_{k\mu,j} - g_{jk,\mu}) = \frac{1}{2} \gamma^j_i \left( \frac{\partial \gamma_{jk}^s}{\partial x^s} + \frac{\partial \gamma_{ks}^j}{\partial x^t} - \frac{\partial \gamma_{st}^k}{\partial x^j} \right)\)

and \(\Gamma^i_{00} = \frac{1}{2} g^{ij} (g_{j0,0} + g_{j0,0} - g_{00,j}) = \frac{1}{2} \gamma^j_i g_{00,j}\)

and \(\Gamma^i_{00} = \frac{1}{2} g^{ij} (g_{j0,0} + g_{j0,0} - g_{00,j}) = 0\)

So we finally get

\[
\frac{d k_i}{d \lambda} = -\Gamma_{00}^i k^0 k^0 - \Gamma_{jk}^i k^j k^0 - \Gamma_{jk}^i k^j k^0 = -\Gamma_{\mu\nu}^i k^\mu k^\nu
\]

which is the geodesic equation in the 4-dimensional space-time.

**C.2. Comparing with conformal transformation**

Furthermore, because

\[
ds^2 = g_{00} k^0 k^0 + g_{00} k^0 k^0 = 0,
\]

we have

\[
g_{00} k^0 k^0 = -g_{00} k^0 k^0 = \gamma_{00} k^0 k^0.
\]

Thus

\[
\frac{d k_i}{d \lambda} + \left( \Gamma_{st}^i k^s k^t + \frac{1}{2} \gamma^j_i g_{00,j} k^0 \right) = 0
\]

\[
\frac{d k_i}{d \lambda} + \left( \Gamma_{st}^i k^s k^t + \frac{1}{2} \gamma^j_i \frac{\partial g_{00}}{\partial x^j} \gamma_{00} k^k k^t \right) = 0
\]

\[
\frac{d k_i}{d \lambda} + \left( \Gamma_{st}^i + \frac{\partial \ln \sqrt{g_{00}}}{\partial x^j} \gamma^j_i \gamma_{00} \right) k^k k^t = 0.
\]

(51)

If we make \(\sigma(x) = g_{00}\) in the equation (50)

\[
\frac{d k_i}{d \lambda} + \left( \Gamma_{st}^i + \frac{\partial \ln \sqrt{g_{00}}}{\partial x^j} \gamma^j_i \gamma_{00} - \partial_i \frac{\partial \ln |g_{00}|}{\partial x^k} \right) k^k k^t = 0,
\]

(52)

and compare it with the equation (51), there is one more item \(-\delta^j_i \frac{\partial \ln |g_{00}|}{\partial x^k} \) in the equation (52), which shows different geodesic lines for two geodesic equations deriving from two kinds of transformations.