CONSTRUCTIVE COMPARISON IN BIDDING COMBINATORIAL GAMES

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ABSTRACT. A class of discrete Bidding Combinatorial Games that generalize alternating normal play was introduced by Kant, Larsson, Rai, and Upasany (2022). The major questions concerning optimal outcomes were resolved. By generalizing standard game comparison techniques from alternating normal play, we propose an algorithmic play-solution to the problem of game comparison for bidding games. We demonstrate some consequences of this result that generalize classical results in alternating play (from Winning Ways 1982 and On Numbers and Games 1976). In particular, integers, dyadics and numbers have many nice properties, such as group structures, but on the other hand the game $^* = \{0|0\}$ is non-invertible. We state a couple of thrilling conjectures and open problems for readers to dive into this promising path of bidding combinatorial games.

1. Introduction

In a recent paper [4], we introduce Discrete Bidding Combinatorial Games that generalize alternating normal play. All essential structure problems regarding outcomes were resolved. In this paper we study the same families of games, by generalizing standard game comparison techniques from alternating normal play [1, 2, 8]: we propose a constructive i.e. algorithmic solution to the problem of game comparison.

Our bidding is similar to Richman auctions; for further references see [5, 3, 7, 6]. Let us review the basic definitions and main results of our main reference [4]. A game form $G$ is recursively defined, with $G = \{G^L | G^R\}$, where $G^L$ is the set of all Left options and $G^R$ is the set of all Right options. If $G^L = \emptyset$ or $G^R = \emptyset$ then $G$ is Left terminal or Right terminal respectively, i.e., the current player, Left or Right, cannot move. In the case when $G^L = G^R = \emptyset$, then $G = 0$ is terminal irrespective of move order. A typical Left (Right) option of a game form $G$ is written $G^L (G^R)$. Game forms are finite and contain no cycles, i.e. each game form has finitely many options, and the birthday (rank of game tree) is finite, implying that each play sequence is finite. The game form $H$ is a subgame/follower of a game form $G$ if there exists a path of moves, perhaps empty, (in any order of play) from $G$ to $H$.

For the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider a total budget $TB \in \mathbb{N}_0$. A bidding game is a game form $G$ together with the total budget $TB$ and is denoted by $(TB, G)$. An instance of a bidding game is a triple $(TB, G, \tilde{p})$, where we take a note of Left’s part of the budget, $\tilde{p} \in B = \{0, \ldots, TB, \hat{0}, \ldots, \hat{TB}\}$. Here $\hat{\cdot}$ indicates that Left holds the tie-breaking marker.

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1The formal birthday is the rank of the literal form game tree. Since we do not yet study reduction techniques of games, we will not distinguish between birthday and formal birthday.
If TB is understood, we write \((G, \tilde{p})\). The word “game” is used interchangeably where each surrounding context decides its precise meaning.

There is a bidding round at a terminal position, and a player who wins a bid but cannot move, loses. The player that does not hold the marker can pass in any such situation, by bidding zero.

Consider a non-terminal position \((G, \tilde{p})\) (Left holds the marker). There are four cases to establish the next game (in case there is one). An instance of bidding is an ordered pair, \((\ell, r)\), where Left bids \(\ell\) and Right bids \(r\).

(i) \((G, \tilde{p}) \rightarrow (G_L, p - \ell)\); Left outbids Right, i.e. \(\ell > r\).
(ii) \((G, \tilde{p}) \rightarrow (G_L, p - \ell)\); Left outbids Right, while including the marker, \(\ell \geq r\).
(iii) \((G, \tilde{p}) \rightarrow (G_L, p - \ell)\); Left wins a tie, i.e. \(\ell = r\).
(iv) \((G, \tilde{p}) \rightarrow (G_R, p + r)\); Right outbids Left, i.e. \(\ell < r\).

Note that the third item is included in the second; but from a recreational play point of view, those bids are distinguished. Observe that in case of a tie, the marker shifts owner. This automatic rule is to the core of our generalization of alternating play. Namely TB = 0 corresponds to alternating normal play rules.

**Theorem 1** (First Fundamental Theorem, [4]). Consider the bidding convention where the tie-breaking marker may be included in a bid. For any game \((TB, G, \tilde{p})\) there is a pure strategy subgame perfect equilibrium, computed by standard backward induction.

An important consequence of our auction definition (above item (ii)) is that “the last move wins” is equivalent with “a player who cannot move loses”. A player who has the larger part of the budget, or half the budget together with the marker, is called the (currently) dominating player. A player is strictly dominating if they have a strictly larger budget than their opponent.

**Corollary 2** (Last Move Wins, [4]). If a dominating player has an option from which the other player cannot move, then the dominating player wins the game.

The pure subgame perfect equilibrium of a game \((TB, G, \tilde{p})\) is the partial outcome \(o(G, \tilde{p}) \in \{R, L\}\), where by convention the total order of the results is \(L > R\), i.e. ‘Left wins’ > ‘Right wins’. The outcome of the bidding game \((TB, G)\) is \(o(G) = \sigma_{TB}(G)\), defined via the \(2(TB + 1)\) tuple of partial outcomes as

\[ o(G) = \left( o(G, \tilde{TB}), \ldots, o(G, \tilde{0}), o(G, TB), \ldots, o(G, 0) \right) . \]

**Definition 3** (Outcome Relation). Consider a fixed TB and the set of all budgets \(B\). Then for any games \(G\) and \(H\), \(o(G) \geq o(H)\) if, \(\forall \tilde{p} \in B, o(G, \tilde{p}) \geq o(H, \tilde{p})\).

Note that the outcomes inherit a partial order from the total order of the partial outcomes.

The next two results have good use in this paper, and together will be referred as “MMW” (Monotonicity and Marker Worth).

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2If Left does not want to hold the marker in the next round, then she should of course include it to the current bid; if she prefers to keep the marker in case she wins the bid, she would act differently.

3It is easy to check that the outcome relation is reflexive, antisymmetric and transitive. Hence the set of all outcomes together with this relation is a poset.
**Lemma 4** (Outcome Monotonicity, [4]). Consider a fixed \( \bar{p} \in B \), with \( p < TB \). Then, for all games \( G \), \( o(G, \bar{p}) \leq o(G, \bar{p} + 1) \).

**Lemma 5** (Marker Worth, [4]). Consider TB \( \in N_0 \). Then, for all games \( G \), \( o(G, \bar{p}) \leq o(G, p + 1) \).

The main result of paper [4] will not be used here, but it is worthwhile restating it. An outcome is feasible if it satisfies MMW.

**Theorem 6** ([4]). For any total budget, for each feasible outcome \( \omega \), there is a game \( G \) such that \( o(G) = \omega \).

In Section 2 we review the definitions of disjunctive sum and the partial order of games. The main result of this paper is a constructive approach for game comparison, which is discussed in Section 4; this is a generalization of comparison of number games from Section 3. In Section 5 we show that, when TB > 0, the game \( * = \{0|0\} \) does not have an inverse. Thus, the set of all bidding games is not a group structure if TB > 0. In Section 6 we study the influence of invertibility in game comparison. The next parts of the paper, Sections 7, 8, 9 focus on games that are numbers, integers and dyadic rationals respectively. In Section 10 we study infinitesimals and zero games, and in Section 11 we provide some interesting conjectures and open problems.

## 2. SUM, PARTIAL ORDER AND NUMBERS

Let us recall the definitions of disjunctive sum of game forms and partial order of games [1, 8]. The disjunctive sum of the game forms \( G \) and \( H \) is defined recursively as:

\[
G + H = \{ G + H^L, G^L + H | G + H^R, G^R + H \},
\]

where \( G + H^L = \{ G + H^L : H^L \in H^L \} \), in case \( H^L \neq \emptyset \); and otherwise the set is not defined and omitted. The *conjugate* of a game form \( G \) is the game form where players have swapped positions, and is recursively defined as \( \overline{G} = \{ G^R | G^L \} \).

The *partial order* of games is defined as usual. Consider games \( G, H \). Then \( G \succeq H \) if, for all games \( X \), \( o(G + X) \geq o(H + X) \). Game equality satisfies \( G = H \) if \( G \succeq H \) and \( H \succeq G \). The games \( G > H \) (\( G \) is greater than \( H \)) if \( G \succeq H \) but \( H \nless G \). The games \( G \) and \( H \) are confused if \( G \not\succeq H \) and \( H \not\succeq G \); this is denoted \( G \parallel H \). The games \( G \prec H \) (\( G \) is less than or confused with \( H \)) if either \( G < H \) or \( G \parallel H \). The game \( G \) is invertible if there exist a game \( G' \) such that \( G + G' = 0 \).

A game form \( G \) is a number if for all \( G^L \) and for all \( G^R \), \( G^L < G < G^R \), and all options are numbers [4]. We will later prove that all numbers are invertible, and the inverse is the conjugate form. And we will prove that numbers are closed under addition, so they are a subgroup of the monoid of all bidding games. Since we observe infinitely many invertible elements, we also have infinitely many 0s.

The empty game satisfies: for all \( X \), \( o(\emptyset \cup \emptyset + X) = o(X) \). Namely, independent of play sequence, the final auction in a follower of \( \emptyset \cup \emptyset + X \) will appear at the game \( \emptyset \cup \emptyset + \emptyset \cup \emptyset = \emptyset \cup \emptyset \) (an idempotent). We already introduced the name \( 0 = \emptyset \cup \emptyset \). We argue that this is the natural name: if a game \( H = 0 \), then we may omit it in a sum of games. That is, for all games \( G \), if \( H = 0 \), then

\footnote{For alternating play the definition of game \( G \) being number is \( H^L < H^R \) for every subposition \( H \) of \( G \) and every \( H^L \) and \( H^R \). However the direct implication of this is \( G^L < G < G^R \) [8].}
$G + H = G$. In fact, the proof is immediate by definition (and does not depend on the specifics of the auction).

**Theorem 7.** If the game $H = 0$, then, for any game $G$, $H + G = G$.

**Proof.** We must demonstrate that for all $Y$, $o(G + H + Y) = o(G + Y)$. Since $H = 0$, we have that for all $X$, $o(H + X) = o(X)$. Take $X = G + Y$.

By uniqueness, we mean with respect to equivalence class / game value (as opposed to ‘game form’).

**Theorem 8** (Uniqueness of Identity). Consider games $G$ and $H$. Suppose $G$ has an inverse $G'$; i.e. $G + G' = 0$. Then $G + H = G$ implies $H = 0$.

**Proof.** Let $H$ be such that $G + H = G$. Then for all $(X, p)$, $o(G + H + X, p) = o(G + X, p)$. Now, for any game $Y$, taking $X = G' + Y$, gives the result.  

3. A CASE STUDY FOR NUMBERS

Left and Right are going to play a intricate bidding game $X$. Just before start, Right offers Left to include a game $G$ to play in a disjunctive sum with $X$. Now the question is, should she accept this kind offer? To make any decision, she inquires whether the game $G$ is a number. If he affirms, then she plays the game $G$ without budget or marker. If she wins it then she accepts the offer and otherwise she rejects. However if he does not reveal anything about $G$, then she has to wait until the next section.

Our first result states that, Left weakly prefers a number game form before the neutral element if and only if she wins playing without the marker.

**Theorem 9** (Number Comparison). Consider any total budget and a number game $G$. Then $G \geq 0$ if and only if $o(G, 0) = L$.

**Proof.** By definition, $G \geq 0$ if and only if $\forall X \in B$, $o(G + X, \bar{p}) \geq o(X, \bar{p})$. That is, $G \geq 0$ if and only if $\forall X \in B$, $o(X, \bar{p}) = L \implies o(G + X, \bar{p}) = L$.

$\implies$: Take $X = 0$, and $\bar{p} = 0$. Then since $o(0, 0) = L$, so $G \geq 0$ implies $o(G, 0) = L$.

$\Leftarrow$: Let us assume $o(G, 0) = L$ and $o(X, \bar{p}) = L$. We must prove $o(G + X, \bar{p}) = L$.

**Case 1:** $o(X, \bar{p}) = L$.

Suppose Left wins $(X, \bar{p})$ by bidding $\bar{\ell}$. We claim that Left wins $(G + X, \bar{p})$, by bidding the same $\bar{\ell}$.

To prove this, suppose first Right bids $r < \ell$. In this case if Left’s optimal bid in $(X, \bar{p})$ is $\bar{\ell}$, then she plays to $(G + X^L, p - \ell)$, and if it is $\ell$, then she plays to $(G + X^L, p - \ell)$. There is a Left option $X^L$, in $(X, \bar{p})$ since Right can bid 0. Left wins $(G + X^L, p - \ell)$ by induction, and she wins $(G + X^L, p - \ell)$ by applying induction, using Case 2.

Now, if Right bids $r = \ell$, then Left wins the bid and plays to $(G + X^L, p - \ell)$. Left wins this by applying induction using Case 2.

Next assume that Right outbids Left with $r > \ell$. He plays to either $(G^R + X, p + r)$ for some $G^R \in G^R$ or to $(G + X^R, p + r)$ for some $X^R \in X^R$; otherwise we are done. Using induction Left wins $(G + X^R, p + r)$ because $\forall X^R \in X^R$ Left
wins \((X^R, \overline{p} + \overline{r})\). However for \((G^R + X, \overline{p} + \overline{r})\), since \(G\) is a number, \(G^R > G\). Thus \(o(G, 0) = L \implies o(G^R, 0) = L\), which implies \(o(G^R + X, \overline{p} + \overline{r}) = L\), by induction and budget monotonicity.

**Case 2:** \(o(X, p) = L\).

Suppose Left wins \((X, p)\) by bidding \(\ell\). We claim that she wins \((G + X, p)\), by bidding the same \(\ell\).

To prove this, suppose first Right bids \(r < \ell\) in game \((G + X, p)\). Left wins the bid and plays to \((G + X^L, p - \ell)\). There is a Left option \(X^L\), since by assumption Left wins \((X, p)\) by bidding \(\ell\); the case \(\ell = 0\) will not arise because \(r < \ell\). She wins \((G + X^L, p - \ell)\) using induction.

Suppose next Right bids \(r = \ell\). Then he wins the bid and plays to \((G^R + X, p + \overline{r})\) for some \(G^R \in G^R\) or to \((G + X^R, \overline{p} + r)\) for some \(X^R \in X^R\); otherwise we are done. By using induction on Case 1, Left wins \((G + X^R, \overline{p} + r)\). For \((G^R + X, p + \overline{r})\), we do the analysis for one more step in which Left bids \(\ell\). Then for Right’s bid \(r_2\) the following subcases arise:

i) If \(r_2 < \ell\), then Left wins the bid and plays to \((G^R + X^L, p + r - \ell)\), i.e. \((G^R + X^L, p)\). From first paragraph of this case Left wins \((G + X^L, p - \ell)\). Using budget monotonicity and property of number game \(G\), Left wins \((G + X^L, p)\).

ii) If \(r_2 = \ell\) then Left wins the bid and plays to \((G^{RL} + X, p)\). There is a Left option \(G^{RL}\), since by assumption she wins \((G, 0)\). So, after Right winning the first bid and playing to \((G^R, 0)\), she has a defense, by playing to some \(G^{RL}\).

Here, Left wins \((G^{RL} + X, p)\) using induction.

iii) If \(r_2 > \ell\) then Right wins the bid and plays to either \((G^R + X^R, \overline{p} + r + r_2)\) or \((G^{RR} + X, \overline{p} + r + r_2)\). For \((G^R + X^R, \overline{p} + r + r_2)\), using the property of number game \(G\) we have \(o(G^R + X^R, \overline{p} + r + r_2) \geq o(G + X^R, \overline{p} + r + r_2)\). Thus Left wins \((G^R + X^R, \overline{p} + r + r_2)\) using budget monotonicity and induction. Next using the property of number game \(G^R\) we have \(o(G^{RR} + X, \overline{p} + r + r_2) \geq o(G^R + X, \overline{p} + r + r_2)\). Note that \(r_2 > 0\). Thus, by repeating the analysis eventually we reach case i) or case ii).

Next assume that Right bids \(\overline{r}\) with \(r > \ell\). He wins the bid and plays to either \((G^R + X, \overline{p} + \overline{r})\) or \((G^{RR} + X, \overline{p} + \overline{r})\), for some \(G^R \in G^R\), or to \((G + X^R, \overline{p} + \overline{r})\) or \((G + X^R, p + r)\), for some \(X^R \in X^R\). From the analysis of the previous paragraph Left wins \((G + X^R, \overline{p} + \overline{r})\) and \((G^R + X, \overline{p} + \overline{r})\). Further she wins \((G + X^R, p + r)\) by using induction. And for \((G^R + X, p + r)\), since \(G\) is a number, \(G^R > G\). Thus \(o(G, 0) = L \implies o(G^R, 0) = L\), which implies \(o(G^R + X, p + r) = L\), by induction and budget monotonicity.

Given a game \(G\), Theorem 3 is applicable only if it is a number. And verifying that itself involves checking \(H^L < H < H^R\) for every subposition \(H\) of \(G\). Furthermore, even if we are provided with a number game \(G\), the computation of \(o(G, 0)\) remains a necessary step. To overcome these challenges, we will next present a more efficient approach.

### 4. A constructive main theorem

In alternating play, it is well known and easy to prove that if Left wins both \(G\) and \(H\) playing second, then she wins \(G + H\) playing second. If she wins \(G\) playing
second and \( H \) playing first, then she wins \( G + H \) playing first. The following lemma generalizes these results, to a particular bidding setting.

Consider a game \((G, \bar{p})\). Left plays a 0-bid strategy if she bids 0 at each follower of \( G \). Left has an optimal 0-bid strategy if the 0-bid is optimal at each follower of \( G \). Note that if a player has an optimal 0-bid strategy, they are not necessarily winning.

**Lemma 10** (Additive Property). Consider any total budget \( \text{TB} \). Suppose, for fixed marker owner \( \bar{\gamma} \), \( o(G, p) = o(H, \bar{q}) = L \). If Left has an optimal 0-bid strategy in \((G, p)\) and \( p + q \leq \text{TB} \), then \( o(G + H, \bar{p} + q) = L \). Suppose \( o(G, \bar{p}) = o(H, q) = L \). If Left has an optimal 0-bid strategy in \((G, \bar{p})\) and \( p + q \leq \text{TB} \), then \( o(G + H, \bar{p} + q) = L \).

**Proof.** We must prove,

(i) if \( o(G, p) = o(H, q) = L \), then \( o(G + H, p + q) = L \);
(ii) if \( o(G, p) = o(H, \bar{q}) = L \), then \( o(G + H, \bar{p} + q) = L \);
(iii) if \( o(G, \bar{p}) = o(H, q) = L \), then \( o(G + H, \bar{p} + q) = L \);

where, in each case, we assume that Left has an optimal 0-bid strategy in \((G, \bar{p})\), and \( p + q \leq \text{TB} \). The proof is by induction.

Case (i): Let \( \ell \) be an optimal Left bid in \((H, q)\). If \( \ell > 0 \), then there exists a Left option in \( H \) (since Right can pass). If \( \ell = 0 \) then Right will win the bid (and a Left option in \( H \) might not exist).

Suppose that Left bids \( \ell \) in \((G + H, p + q)\). If Left wins the bid, she plays to \((G + H^L, p + q - \ell)\), and wins by induction.

If Right ties Left’s \( \ell \)-bid, then he plays to

\[
\left( G^R + H, \bar{p} + q + \bar{\ell} \right) \text{ or } \left( G + H^R, \bar{p} + q + \bar{\ell} \right)
\]

(note that in this case \( p + q \leq \text{TB} - \ell \)). But, by monotonicity, since Left has an optimal 0-bid in \( G \), then \( o(G^R, \bar{p} + \bar{\ell}) = o(H, q) = L \). Also since \( \ell \) is an optimal Left bid in \( H \), we get \( o(G, p) = o(H^R, \bar{q} + \bar{\ell}) = L \). Hence by induction, using (iii) and (ii),

\[
o(G^R + H, \bar{p} + q + \bar{\ell}) = o(G + H^R, \bar{p} + q + \bar{\ell}) = L.
\]

If Right outbids Left by not using the marker, then he plays to \((G^R + H, p + q + \ell + 1)\) or \((G + H^R, p + q + \ell + 1)\) (note that in this case \( p + q \leq \text{TB} - \ell - 1 \)). In this case using Marker Worth, Right is not better than the previous situation where he ties the bid.

Altogether \( o(G + H, p + q) = L \).

Case (ii). Let \( \bar{\ell} \) be an optimal Left bid in \((H, \bar{q})\). A winning Left option \( H^L \) exists, because she holds the marker.

She bids the same in the game \((G + H, \bar{p} + \bar{q})\). If she wins the bid and \( \bar{\ell} = \ell \) she plays to \((G + H^L, p + q - \ell)\) and otherwise she plays to \((G + H^L, \bar{p} + q - \bar{\ell})\). In the first case, she wins by induction, since \( o(G, p) = o(H^L, q - \ell) = L \). In the second case, she wins by induction, since \( o(G, p) = o(H^L, \bar{q} - \bar{\ell}) = L \).
If Right outbids Left, by monotonicity \( r = \ell + 1 \), he plays to \((G^R + H, p + q + \ell + 1)\) or \((G + H^R, p + q + \ell + 1)\). But, since Left has an optimal 0-bid in \((G, p)\) by monotonicity, \(o(G^R, p + \ell + 1) = o(H, \tilde{q}) = L\). And moreover, Left has a defence to a Right move in \(H\), so \(o(G, p) = o(H^R, q + \ell + 1) = L\). Hence, by induction,

\[
o(G^R + H, p + q + \ell + 1) = o(G + H^R, p + q + \ell + 1) = L
\]

Altogether \(o(G + H, p + q) = L\).

Case (iii). Let \(\ell\) be Left’s optimal bid in \((H, q)\). She bids the same in \((G + H, p + q)\) and if she wins the bid, she plays to \((G + H^L, p + q - \ell)\) and wins by induction. Note that \(H^L\) exists if \(\ell > 0\), and otherwise she cannot win the bid in \(H\), because Right owns the marker. But in \((G + H, p + q)\) she might win the bid if \(\ell = 0\) and Right passes. In this case, she has a winning move in \((G, \tilde{p})\) to \((G^L, p)\), and so, by using (i), by induction \(o(G^L + H, p + q) = L\).

Suppose Right wins the bid. Since Left owns the marker, we assume by monotonicity that he bids \(r = \ell + 1\), and plays to \((G^R + H, p + q + \ell + 1)\) or \((G + H^R, p + q + \ell + 1)\).

Observe that \(o(G^R, p + 1) = o(H, q) = L\). Therefore induction together with monotonicity gives \(o(G^R + H, p + q + \ell + 1) = L\) and \(o(G + H^R, p + q + \ell + 1) = L\) by induction, since by assumption \(o(G, \tilde{p}) = o(H^R, q + \ell + 1) = L\).

Altogether \(o(G + H, p + q) = L\).

One of the most celebrated results in alternating play theory states that Left wins \(G\) playing second if and only if \(G \geq 0\). We generalize this constructive-, algorithmic-, recursive-, play-, local-comparison to bidding play. For constructive comparison it is required to satisfy another recursive test: “Does Left have an optimal 0-bid strategy?”

**Theorem 11 (Main Theorem).** Consider a game form \(G\) and any total budget. Suppose that Left has an optimal 0-bid strategy in \((G, 0)\). Then \(G \geq 0\) if and only if \(o(G, 0) = L\).

**Proof.** By definition, \(G \geq 0\) if for all \(X\) and for all \(\tilde{p}\), \(o(X, \tilde{p}) = L\) implies \(o(G + X, \tilde{p}) = L\).

\(\Rightarrow:\) Assume \(G \geq 0\). Take \(X = 0\) and \(\tilde{p} = 0\). By \(o(0, 0) = L\), the implication gives \(o(G, 0) = L\).

\(\Leftarrow:\)

**Case 1:** Suppose \(o(G, 0) = L\) and \(o(X, p) = L\), where Left has an optimal 0-bid strategy in \((G, 0)\). We must show that \(o(G + X, p) = L\).

Consider a Left optimal bid in \((X, p)\), say \(\ell\). Assume that she bids \(\ell\) in \((G + X, p)\).
If she wins the bid, she has a move to \((G + X^L, p - \ell)\), which she wins by induction.

If Right ties the bid and plays in the \(X\)-component, Left wins by induction, using Case 2 below. Hence assume he ties and plays to
\[
(G_R + X, p + \ell) \quad (1)
\]

Now, since by assumption, Left has an optimal 0-bid strategy in \(G_R\), Lemma 10 applies; since \(o(G_R, 0) = o(X, p) = L\), by monotonicity Left wins (1).

If Right outbids Left and includes the marker, then Monotonicity implies that his result cannot be better than in the previous paragraph; if he does not include the marker, then Marker Worth implies the same.

**Case 2:** Suppose next that \(o(G, 0) = L\) and \(o(X, \hat{p}) = L\), where Left has an optimal 0-bid strategy in \((G, 0)\). We must show that \(o(G + X, \hat{p}) = L\). Note that by the assumption \(o(X, \hat{p}) = L\), Left has a (winning) move in \(X\) (Right can bid 0 and force Left to move).

Consider an optimal Left bid \(\bar{\ell}\) in \((X, \hat{p})\), and assume that Left bids \(\bar{\ell}\) in \((G + X, \hat{p})\). If she wins the bid by including the marker, she has a move to \((G + X^L, p - \ell)\), and otherwise she has a move to \((G + X^L, p - \ell)\). In both cases Left wins by induction, since, by assumption, \(o(X^L, p - \ell) = L\) or \(o(X^L, p - \ell) = L\). Similarly, if Right outbids Left and plays in the \(X\)-component, Left wins by induction.

Hence suppose Right outbids Left and plays to
\[
(G_R + X, p + \ell + \bar{\ell}) \quad (2)
\]
Left bids \(\ell\), and, unless Right outbids or ties Left’s bid, she plays in the \(G\)-component to \((G^{RL} + X, \hat{p} + 1)\) such that \(o(G^{RL}, 0) = L\). She wins by induction, since she has an optimal 0-bid strategy in \(G^{RL}\) and by Monotonicity. Assume that Right outbids Left and plays to
\[
(G_R + X^R, \hat{p} + 2\ell + 1) \quad (3)
\]

Then we use Marker Worth together with Lemma 10. Namely \(o(G_R, \hat{p}) = L\) implies \(o(G_R, 1) = L\), and Left has an optimal 0-bid strategy in \(G_R\). Combine this with \(o(X^R, \hat{p} + 2\ell + 1) = L\), to see that Left wins (3) (by using also Monotonicity).

If Right ties Left’s bid then she wins the bid by giving away the marker and playing to \((G^R + X^L, p + 1)\). We have \(o(G^R, 1) = L\) with Left’s optimal 0-bid strategy in \(G^R\). Also, since \(o(X, \hat{p}) = L\), we have \(o(X^L, p - \ell) = L\). Hence, by using Monotonicity and Lemma 10 we get \(o(G^R + X^L, p + 1) = L\).

If Right outbids Left and instead plays to
\[
(G^{RR} + X, \hat{p} + 2\ell + 2) \quad (4)
\]
then observe that, since Left has an optimal 0-bid strategy in \((G^R, \hat{p})\), then \(o(G^{RR}, \hat{p}) = L\). And so, by Marker Worth, \(o(G^{RR}, 2) = o(X, \hat{p}) = L\). Therefore, since Left has an optimal 0-bid strategy in \(G^{RR}\), Lemma 10 together with Monotonicity implies
\[
o(G^{RR} + X, \hat{p} + 2\ell + 2) = L.
\]
\[\square\]
We summarize the result in terms of how it is used in applications.

**Corollary 12** (Constructive Comparison Tests). Consider any bidding game \((TB, G)\).

1. If \(o(G, 0) = L\), with a Left 0-bid strategy, then \(G \geq 0\).
2. If \(o(G, TB) = R\), with a Right 0-bid strategy, then \(G \leq 0\).
3. If \(o(G, 0) = L\), with a Left 0-bid strategy and \(o(G, TB) = L\), then \(G > 0\).
4. If \(o(G, 0) = R\), with a Right 0-bid strategy, and \(o(G, 0) = R\), then \(G < 0\).
5. If \(o(G, 0) = R\) and \(o(G, TB) = L\), then \(G = 0\).
6. If \(o(G, TB) = L\) and either \(o(G, 0) = R\) or \(o(G, 0) = L\), with a Left 0-bid strategy, then \(G \nleq 0\).
7. If \(o(G, 0) = R\) and either \(o(G, TB) = L\) or \(o(G, TB) = R\), with a Right 0-bid strategy, then \(G \nleq 0\).

**Proof.** This follows from Theorem 11. \(\square\)

The statement of the Main Theorem (Theorem 11) raises some questions, especially with respect to the similar theorem for games that are numbers in Section 3.

**Problem 13.** Concerning the similarities of Theorems 9 and 11 some questions arise:

(i) Is it true that “Left has a winning 0-bid strategy in \((G, 0)\)” if and only if “for every Right option \(H^R\), \(H\) a follower of \(G\), there is an answer by Left, \(H^{RL}\)”?

(ii) Is it true that the second property in (i) holds if and only if \(G\) is a non-negative number?

(iii) Is it true that if \(G\) is a number then each player optimally plays a 0-bid strategy?

(iv) May we put the Left-bidding-0 strategy inside the equivalence in Theorem 11, i.e. is it true that “\(G \geq 0\) if and only if \(o(G, 0) = L\) and Left has an optimal 0-bid strategy in \(G\)”?

Item (iv) does not seem to be valid, since the first implication in the proof does not give a 0-bidding strategy. But the reverse direction holds. We observe that we may not remove the proviso of a Left winning 0-bid strategy in the statement of Theorem 11.

**Observation 14** (No Generic 0-optimal Strategy). We demonstrate that, for any non-zero total budget there is a game form \(G\) such that \(o(G, 0) = L\), but Left does not optimally bid 0 at each follower. Assume that Right knows that Left bids 0 at each follower of some game \(G\), that is yet to be constructed. Set \(G^L = G^{RL} = \cdots = G^{R\cdots RL} = 0\). He bids $1 at each such round, for otherwise he loses. After \(x > TB/2\) rounds, the game has reached \((G^{R\cdots R}, x)\). Let \(H = G^{R\cdots R} = *\). Left must outbid Right at this penultimate stage \(H\), because “last move wins”. In optimal play she must bid $x > 0 to win. For example, for \(TB = 4\), the game \(G = \{0\{0\uparrow}\}\) suffices, and is depicted in Figure 1. Here the game \(\uparrow = \{0\}*\). For \(TB = 2\), we take instead the game \(G = \{0\uparrow\}\). Observe that our game is not a number, since \(*\) is a follower, and \(*\) is not a number.

Problem 13 (i) and (ii) motivates (iii), for which we have a solution.

**Proposition 15** (Zero-bids at Numbers). If \(G\) is a number, then both optimally play 0-bid strategies.
Proof. Consider a number game $G$. If Left wins the bid, then she plays to $G^L < G$, which does not improve her situation. And by Monotonicity she cannot gain by bidding $\ell > 0$. Right faces the analogous situation. Since each follower of a number game $G$ is also a number, the result follows. 

Hence Theorem 9 is a consequence of Theorem 11.

5. No Group Structure

In alternating play, game families are groups. In bidding play games are closed with respect to addition, and there is a neutral element namely the empty game 0. But there exist non-invertible elements when $TB > 0$.

Theorem 16 (No Group Structure). Consider $TB > 0$. There is no game $G$ such that $* + G = 0$.

Proof. The proof is by way of contradiction. The inequality $* + G \geq 0$ implies that Left wins $* + G$ if Right holds the marker (take $X = 0$). This implies that there exists $G^L$ such that

(5) \[ o\left(G^L, 0\right) = L. \]

Namely, Right begins in $*$, and after that Left plays her winning response in $G$; Left cannot win the first bid, Right can play in $*$; if Right passes the second bid, then Left gets to play and wins.

On the other hand, she also wins if Right starts by playing in $*$, and then outbids Left and plays to $(G^R, \hat{1})$. That is, for any $G^R$,

(6) \[ o\left(G^R, \hat{1}\right) = L. \]

Observe, by symmetry, (with $* + G \leq 0$) \[ o\left(G^R, \hat{TB}\right) = R. \] But then, by Monotonicity, for the same $G^R$, \( o(G^R, \hat{1}) = R, \) contradicting (6).

Thus, if $TB > 0$, then bidding games do not have a group structure. In general, we conjecture (Conjecture 69) that if a game has an inverse, then this game is its conjugate.
6. Comparative Analysis

In this section, we continue exploring comparison of bidding games. The existence of inverse becomes crucial.

**Observation 17** (Constructive Game Comparison). Suppose that $H$ has an inverse, say $H'$. Then $G \geq H$ if Left has a winning 0-bid strategy in $(G + H', 0)$.

Games preserve order with respect to addition.

**Theorem 18** (Order Preservation). Consider games $G, H$ and $J$. If $J \geq H$, then $G + J \geq G + H$.

Proof. This is immediate by definition of $J \geq H$.

For the converse of order preservation, the inverse is required.

**Theorem 19**. Consider games $G, H$ and $J$. Suppose $G$ has an inverse $G'$. Then $G + J \geq G + H$ implies $J \geq H$.

Proof. By $G + J \geq G + H$, for all $(X, \tilde{p})$, $o(G + J + X, \tilde{p}) \geq o(G + H + X, \tilde{p})$. Now, for any game $Y$, taking $X = G' + Y$, gives the result.

Further, order preservation with respect to addition under strict inequality requires inverse.

**Theorem 20**. Consider games $G, H$ and $J$. Suppose $G$ has an inverse $G'$. Then $H < J$ if and only if $G + H < G + J$.

Proof. Let us assume that $H < J$. Then there exists $(Y, \bar{q})$ such that

\begin{equation}
\label{eq:7}
o(H + Y, \bar{q}) < o(J + Y, \bar{q})
\end{equation}

Since $H < J$, we also have $H \leq J$. Thus using Theorem 18 we have $G + H \leq G + J$. Next, to prove $G + H < G + J$ we need to show there exists $(X, \tilde{p})$ such that $o(G + J + X, \tilde{p}) = L$ but $o(G + H + X, \tilde{p}) = R$. On the contrary, let us assume that there does not exist any such $(X, \tilde{p})$. Then, for all $(X, \tilde{p})$, $o(G + J + X, \tilde{p}) = o(G + H + X, \tilde{p})$. By taking $X = G' + Y$ and $\tilde{p} = \bar{q}$ we get $o(J + Y, \bar{q}) = o(H + Y, \bar{q})$. This contradicts \eqref{eq:7}. Hence $G + H < G + J$. The other direction follows similarly.

**Corollary 21**. Suppose a game $G$ has an inverse $G'$. Then $G > 0$ if and only if $G' < 0$.

Proof. This follows from Theorem 20.

The following theorem shows transitivity of order.

**Theorem 22**. Consider games $G, H$ and $J$. If $G \geq H$ and $H \geq J$ then $G \geq J$.

Proof. By $G \geq H$, for all $(X, \tilde{p})$, $o(G + X, \tilde{p}) \geq o(H + X, \tilde{p})$. Also since $H \geq J$, therefore, for all $(X, \tilde{p})$, $o(H + X, \tilde{p}) \geq o(J + X, \tilde{p})$. Thus, for all $(X, \tilde{p})$, we have $o(G + X, \tilde{p}) \geq o(J + X, \tilde{p})$. Hence $G \geq J$.

Transitivity of order also holds with respect to strict inequality.

**Theorem 23**. Consider games $G, H$ and $J$. If $G > H$ and $H > J$, then $G > J$.
Proof. Suppose $G > H$ and $H > J$. This implies $G \geq H$ and $H \geq J$, respectively. By using Theorem 22 we get $G \geq J$. Further since $G > H$, there exists a $(X, \tilde{p})$ such that $o(G + X, \tilde{p}) = L$ but $o(H + X, \tilde{p}) = R$. Since $H > J$, for the same $(X, \tilde{p})$, we get that $o(H + X, \tilde{p}) = R$ implies $o(J + X, \tilde{p}) = R$. Hence there exist $(X, \tilde{p})$ such that $o(G + X, \tilde{p}) = L$ but $o(J + X, \tilde{p}) = R$. □

In alternating play, if $G > 0$, then Left wins irrespective of who starts the game. A similar result is also true for the bidding games.

**Theorem 24.** If the bidding game $G > 0$, then for any budget partition $\tilde{p} \in B$, $o(G, \tilde{p}) = L$.

**Proof.** From Monotonicity it is sufficient to prove the following:

i) $o(G, 0) = L$ and

ii) $o(G, \tilde{0}) = L$.

*Proof of i).* Since $G > 0$, thus $o(G, 0) \geq o(0, 0) = L$.

*Proof of ii).* On the contrary let us assume that $o(G, \tilde{0}) = R$. Since $G > 0$, there exists a $(X, \tilde{p})$ such that (8) $o(G + X, \tilde{p}) = L$ but $o(X, \tilde{p}) = R$

From Lemma 10, $o(G, \tilde{0}) = R$ and $o(X, \tilde{p}) = R$ gives $o(G + X, \tilde{p}) = R$, which contradicts (8). Thus $o(G, \tilde{0}) = L$. □

**Corollary 25.** If the bidding game $G < 0$ then, for any budget partition $\tilde{p} \in B$, $o(G, \tilde{p}) = R$.

**Proof.** By symmetry, this follows from Theorem 24. □

We demonstrated the influence of “inverse” in game comparison. Sometimes, we can ensure the existence of an inverse. Towards this end, we next study number games.

7. Numbers

Recall that a game form $G$ is a *number* if for all $G^L$, for all $G^R$, $G^L < G < G^R$, and all options are numbers. In alternating play, numbers are a subgroup of all game forms. In this section, we prove that the same is true in the bidding set up.

**Theorem 26 (Numbers are Invertible).** Consider a game $G$. If $G$ is a number, then $G + \overline{G} = 0$.

**Proof.** By Corollary 12 we must show:

1) $o(G + \overline{G}, 0) = L$, with a Left 0-bid strategy and

2) $o(G + \overline{G}, \overline{0}) = R$, with a Right 0-bid strategy.

By symmetry, it suffices to prove item (1).

Consider the game $(G + \overline{G}, 0)$. Left starts by bidding 0. Right will win this bid, so without loss of generality suppose he also bids 0 (bidding 1 to keep the marker cannot benefit him because of Marker Worth Lemma). Right plays to either $(G^R + \overline{G}, \tilde{0})$ or $(G + \overline{G}^L, \tilde{0})$, and we must prove that Left wins both cases.

**Case 1:** In the game $(G^R + \overline{G}, \tilde{0})$ she will again bid 0, and if she wins the bid, she plays to $(G^R + \overline{G}^R, 0)$. The outcome is $L$ by induction.
Further, in the game \((G^R + \overline{G}, \hat{1})\), if Right outbids Left then he can play to either \((G^{RR} + \overline{G}, \hat{1})\) or \((G^R + \overline{G^L}, \hat{1})\). Since \(G^R\) is a number, we have \(G^R < G^{RR}\). Thus \(o(G^R + \overline{G}, \hat{1}) \leq o(G^{RR} + \overline{G}, \hat{1})\). By induction \(o(G^R + \overline{G}, \hat{1}) = L\), and therefore \(o(G^{RR} + \overline{G}, \hat{1}) = L\). Similarly, since \(G\) is a number, we have \(G < G^L\). Thus \(o(G^R + \overline{G^L}, \hat{1}) \leq o(G^R + \overline{G^L}, \hat{1})\). By induction \(o(G^R + \overline{G^L}, \hat{1}) = L\), and therefore \(o(G^R + \overline{G^L}, \hat{1}) = L\).

**Case 2:** Similar to the previous case, we get \(o(G + \overline{G^L}, \hat{0}) = L\). □

**Theorem 27 (Uniqueness of Number Inverse).** For any number game \(G\), the unique inverse is \(\overline{G}\).

**Proof.** Consider a number \(G\). Let apart from \(\overline{G}\), \(G'\) is another inverse of \(G\). We need to show, \(G = G'\).

\[
\overline{G} = G' \tag{9}
\]

\(\text{i.e. } o(G + \overline{Y}, \overline{p}) = o(G' + \overline{Y}, \overline{p}) \quad \forall \text{ game } Y, \text{ and } \forall \overline{p} \in B
\]

Since \(\overline{G}\) is inverse of \(G\), we have \(G + \overline{G} = 0\). Then

\[
o(G + \overline{G} + \overline{X}, \overline{p}) = o(X, \overline{p}) \quad \forall \text{ game } X, \text{ and } \forall \overline{p} \in B \tag{10}
\]

For any game \(Y\) in (9), taking \(X = G' + Y\) in (10) gives the result. □

**Theorem 28 (Numbers are Additive).** For any total budget, if \(G\) and \(H\) are numbers, then \(G + H\) is a number.

**Proof.** Study the Left option \(G^L + H\). Since \(G^L < G\), then, by Theorems 20 and 26, \(G^L + H < G + H\). □

Thus, the submonoid of all games that are numbers is a group.

8. Integer Games

In alternating play, integer games is a subgroup of games that are numbers. We will prove that this continues to hold when \(TB > 0\).

Can Right ever win the game \(1 := \{0|\emptyset\}\)? For Right to win, Left must win two consecutive bids. This is not possible in optimal play since Left can bid 0 twice, and either lose the first or second bid depending on who starts with the marker.

**Observation 29.** Given any total budget and any budget partition \(\overline{p} \in B\), Left wins the game \(1 = \{0|\emptyset\}\). Notice that she wins by using a 0-bid strategy.

In fact, the game form \(1 > 0\). For all \(n \in \mathbb{N}\), define the positive integer game form \(n\) by \(n = \{n-1|\emptyset\}\).

**Proposition 30.** Consider any \(TB\) and any budget partition \(\overline{p} \in B\). For any \(n \in \mathbb{N}\), Left wins the game form \(n\) with a 0-bid strategy.

**Proof.** From Monotonicity it is sufficient to prove the following with Left 0-bid strategy:

i) \(o(n, 0) = L\)

ii) \(o(n, \hat{0}) = L\).
Proof of i). Consider the game $(n, 0)$. Right wins the first bid, but does not have any move to play. Thus Left wins with her 0-bid strategy.

Proof of ii). Consider the game $(n, 0)$. If Right outbids Left then he immediately loses, as he cannot play. Thus assume that Left wins the first bid and plays to $(n - 1, 0)$. Left wins using part i) or by playing the last move. □

Observation 31. Consider any $n \in \mathbb{N}$. A symmetric result to Proposition 30 gives that, for any TB and any budget partition $\bar{p} \in \mathcal{B}$, Right wins the game form $\bar{\pi}$ with a 0-bid strategy.

Observation 32. From Proposition 30, Observation 31 and Corollary 12, we have: for any $n \in \mathbb{N}$, $n > 0$ and $n < 0$.

Using Observation 32 in the proof of Theorem 16, shows that switches, i.e. games such as $(1 \mid -1)$, do not have inverses.

Theorem 33. Consider $n \in \mathbb{N}$. Then the disjunctive sum $n + \bar{\pi} = 0$.

Proof. Similar to the proof of Theorem 26, it is sufficient to show $o(n + \bar{\pi}, 0) = L$ with a Left 0-bid strategy. Consider the game $(n + \bar{\pi}, 0)$. By Left’s 0-bid strategy, Right will win the first bid, so (by MMW) suppose that he also bids 0. Right plays to $(n + \bar{\pi} - 1, 0)$. In this position, Left bids 0 again and if she wins the bid, she plays to $(n - 1 + \bar{\pi} - 1, 0)$. Here Left wins by induction. However if Right outbids Left, then he must play to $(n + \bar{\pi} - 2, 1)$ (if $n = 1$, then Right immediately loses). If Left wins the next bid, she plays to $(n - 1 + \bar{\pi} - 2, 1)$. She wins this using induction and Monotonicity. If Right wins the bid, he plays to $(n + \bar{\pi} - 3, 2)$ (if possible, and otherwise loses immediately).

Since both $n$ and TB are finite, he cannot continue winning the bid indefinitely while having options to play. Therefore, he will eventually end up with no move or he loses the bid, and when this happens, Left wins with a 0-bid strategy by induction and Monotonicity. □

Observe that from Theorem 33, the inverse of $n$ is $\bar{n}$. The set $\{n, \bar{n} : n \in \mathbb{N}_0\}$ is the set of all integer game forms. The set $\{\bar{n} : n \in \mathbb{N}\}$ is the set of all negative integer game forms. Hence note that the inverse of a positive integer game form is a negative integer game form and vice versa. In the coming, we use interchangeably $-n = \{\varnothing \mid -n + 1\}$. Also note that $-n = \{\varnothing \mid n + 1\}$.

Theorem 34. Suppose that $(n)$ and $(m)$ are integer game forms. Then the disjunctive sum $(n) + (m)$ is an integer game form that equals the integer game form $(n + m)^6$.

Proof. If any of the integers $n$ or $m$ equals 0, then by Theorem 7 there is nothing to prove.

Case i) Let us first assume $n, m > 0$, that is $(n) = \{(n - 1) \mid \varnothing\}$ and $(m) = \{(m - 1) \mid \varnothing\}$. Right cannot move in the sum $(n) + (m)$. Left moves to $(n - 1) + (m)$ or $(n) + (m - 1)$. Let $(k) = \{(n + m - 1) \mid \varnothing\}$. By induction, regarded as games, $(n + m - 1) = (n - 1) + (m)$ and $(n + m - 1) = (n) + (m - 1)$. Hence, Left has the same options in $(k)$ as in $(n) + (m)$. Hence the games are the same.

Case ii) The case when both $n, m < 0$ is symmetric.

Case iii) Assume $n > 0$ and $m < 0$.

---

6Here, and in Theorem 45 it is convenient to use the notation $(n)$ for the game form $n$. 


Subase i) \( n < -m \)
\[
(n) + (m) = (n) + (m + n - n)
\]
\[
= (n) + (-n) + (m + n) \quad ; (-n) < 0 , (m + n) < 0
\]
\[
= (m + n) \quad ; \text{By using Theorem 33}
\]

Subcase ii) \( n > -m \)
\[
(n) + (m) = (n + m) + (m)
\]
\[
= (n + m) + (-m) + (m) \quad ; -m > 0 , (n + m) > 0
\]
\[
= (n + m)
\]

Subcase iii) \( n = -m \)
The equality \((n) + (m) = (n + m)\) follows by Theorem 33.

\[\square\]

**Theorem 35.** Consider any total budget \( TB \). If the integers \( n > m \), then the game forms \((n) > (m)\).

**Proof.** By using Theorems 20 and 33, it suffices to prove that \((n + m) > 0\). According to Corollary 12 we prove the following:

i) \( o((n + m), 0) = L \) with Left 0-bid strategy and

ii) \( o((n + m), TB) = L \).

**Case 1.** Consider \( m > 0 \).

**Proof of i.** Consider the game \(((n + m), 0)\). Right wins the first bid, so (by MMW) he plays to \(((n + m - 1), 0)\). Now Left again bids 0; if she wins this bid then she plays to \(((n - 1 + m - 1), 0)\). Left wins this by induction. However if Right outbids her then he plays to \(((n + m - 2), 1)\) (if \( m = 1 \), then Right immediately loses, however if \( m > 1 \) then only this case arises). Left will again bid 0. If she wins the bid, she plays to \(((n - 1 + m - 2), 1)\). She wins this using induction and Monotonicity. If Right wins the bid, he plays to \(((n + m - 3), 2)\) (if possible, and otherwise loses immediately).

Since both \( m \) and \( TB \) are finite, he cannot continue winning the bid indefinitely while having options to play. Therefore, he will eventually end up with no move or she loses the bid, and when this happens, Right wins with a 0-bid strategy by induction and Corollary 47.

Since \( m \) is finite, he cannot keep winning the bid indefinitely and have option to play. Thus either Right will lose by winning the bid and have no option to play or he loses the bid and when this happens, Left wins by induction and Monotonicity. **Proof of ii.** Left wins the first bid and plays to \(((n - 1 + m), TB)\). She will again bid 0, which forces Right to play. He moves to \(((n - 1 + m - 1), TB)\). Thus Left wins by induction.

**Case 2.** In case \( n = 0 \) or \( m = 0 \), the result follows from Observation 32.

**Case 3.** Consider \( n > 0 \) but \( m < 0 \). Then \((n) > 0\) and \((m) > 0\). Thus from Theorem 34 we have \((n + m) > 0\).

**Case 4.** Consider \( n < 0 \). This case is symmetric to Case 1, so by analogy \((n + m) > 0\).

From Theorem 35, we have that the integer game forms follow the natural total order. We conclude this part by showing that, for a given total budget, the integer
game forms form a subgroup of all bidding games. First, by using previous results in this section, we demonstrate a “simplicity” lemma for integers.

**Lemma 36.** For all integers $n$, $n = \{n - 1 | n + 1\}$.

**Proof.** Since the inverse of $n$ is $-n$, it suffices to demonstrate that $\{n - 1 | n + 1\} - n = 0$.

**Case i).** Consider $n > 0$. By Corollary 12, we must prove the following:

i) $o(\{n - 1 | n + 1\} - n, \widehat{TB}) = R$ with Right 0-bid strategy.

ii) $o(\{n - 1 | n + 1\} - n, 0) = L$ with Left 0-bid strategy.

Proof of i). Consider the game $(\{n - 1 | n + 1\} - n, \widehat{TB})$. Left wins the first bid and plays to $(n - 1 - n, TB)$. From Theorems 33 and 34 it is equivalent to $(-1, TB)$. Right wins this using 0-bid strategy from Observation 31.

Proof of ii). Consider the game $(\{n - 1 | n + 1\} - n, 0)$. By Left’s 0-bid strategy, Right will win the first bid. He plays to either $(n + 1 - n, \hat{0})$ or $(\{n - 1 | n + 1\} - n + 1, \hat{0})$.

From Theorem 33 and 34 $(n + 1 - n, \hat{0})$ is equivalent to $(1, \hat{0})$. Left wins this using a 0-bid strategy from Proposition 30. Hence playing in first component is losing for Right. So, we may assume Right will not play in the first component if similar situation arises.

Next, in the game $(\{n - 1 | n + 1\} - n + 1, \hat{0})$, Left will again bid 0. If she wins the bid then she plays to $(n - 1 - n + 1, 0)$. From Theorem 33 and 34 it is equivalent to $(0, 0)$, which Left wins because she made the last move. If he outbids her, then he plays to $(\{n - 1 | n + 1\} + 2, \hat{1})$, because of the reason in the previous paragraph. Since both $n$ and TB are finite, he cannot continue winning the bid indefinitely while having options to play. Therefore, he will eventually end up with no move or he loses the bid, and when this happens, Left wins with a 0-bid strategy by Theorem 33, 34 and Proposition 30.

**Case ii).** Consider $n = 0$. By Corollary 12, we must prove the following:

i) $o(\{-1 | 1\}, \widehat{TB}) = R$ with Right 0-bid strategy.

ii) $o(\{-1 | 1\}, 0) = L$ with Left 0-bid strategy.

In the game $(\{-1 | 1\}, \widehat{TB})$, Left will win the first bid and plays to $(-1, TB)$. This Right wins from Observation 31 with his 0-bid strategy. Symmetrically Left wins $(\{-1 | 1\}, 0)$ with her 0-bid strategy.

**Case iii).** Consider $n < 0$. This case is symmetric to Case i, hence have the same result.  

**Corollary 37.** Integers are numbers.

**Proof.** It follows from Lemma 36 and Theorem 35. 

**Theorem 38.** The set of all integer game forms is a subgroup of the numbers.

**Proof.** It follows from Corollary 37, Theorems 33 and 34. 

Perhaps the most central remaining problem of this section is as follows.

**Problem 39.** Suppose that the birthday of $G$ is smaller than the integer $n > 0$. Is it true that $n > G$? Does Right require an optimal 0-bid strategy in the game $G - n$?
9. Dyadic rational games

The bidding integers are thus well understood. The obvious next question concerns bidding dyadic rationals. The first few questions would be:

Problem 40. Is the bidding game \(1/2 = \{0|1\} > \emptyset\)?

Problem 41. Is the bidding game \(1/2 + 1/2 = \emptyset\)?

Problem 42. Is the bidding game \(1/2 + 1/2 = 1\)?

These types of problems will be resolved in a general form. Let the game \(1/2^0\) be the game 1. For \(k \in \mathbb{N}\), we define the positive dyadic rational game forms as 
\[\frac{1}{2^k} = \{0|1/2^{k-1}\}.\]
For \(n \in \mathbb{N}_0\), \(n/2^k\) is the disjunctive sum of \(n\) copies of \(1/2^k\).

Proposition 43. Consider any TB and any budget partition \(\tilde{p} \in B\). For any \(k \in \mathbb{N}\), Left wins the game \((1/2^k, \tilde{p})\) with a 0-bid strategy.

Proof. By Monotonicity it is sufficient to prove that, with a Left 0-bid strategy:

i) \(o(1/2^k, \tilde{0}) = L\), and
ii) \(o(1/2^k, 0) = L\).

Proof of i). Consider the game \((1/2^k, \tilde{0})\) together with a Left 0-bid strategy. If she wins the bid, she will make the last move and win. If Right outbids her, he must play to \((1/2^{k-1}, \tilde{1})\), which she wins by induction.

Proof of ii). Consider the game \((1/2^k, 0)\) together with a Left 0-bid strategy. Right wins the first bid and moves to \((1/2^{k-1}, 0)\) (by MMW). Left wins this from the first part of the proof. \(\square\)

Corollary 44. Consider any TB and any budget partition \(\tilde{p} \in B\). For any \(k \in \mathbb{N}\), Right wins \((1/2^k, \tilde{p})\) with a 0-bid strategy.

Proof. By symmetry, this follows from Proposition 43. \(\square\)

Corollary 45. For any \(k \in \mathbb{N}\), \(1/2^k > 0\) and \(1/2^k < 0\).

Proof. Apply Corollary 42 to Proposition 43 and Corollary 44. \(\square\)

Theorem 46. For all integers \(k' > k \geq 0\) and for any budget partition \(\tilde{p} \in B\), Left wins \((1/2^k + 1/2^{k'}, \tilde{p})\) with a 0-bid strategy.

Proof. Consider integers \(k' > k \geq 0\). From monotonicity it is sufficient to show

i) \(o(1/2^k + 1/2^{k'}, 0) = L\) with Left’s 0-bid strategy, and
ii) \(o(1/2^k + 1/2^{k'}, \tilde{0}) = L\) with Left’s 0-bid strategy.

Consider \(k = 0\). Right wins the first bid in the game \((1 + 1/2^{k'}, 0)\), and, by MMW, plays to \((1, \tilde{0})\). By Proposition 39, Left wins this game using a 0-bid strategy.

Next, Left bids \(0\) in the game \((1 + 1/2^{k'}, 0)\). If she wins the bid then she wins the game with a 0-bid strategy using induction by playing to \((1 + 1/2^{k'-1}, 0)\). However if Right outbids her, then by MMW, he plays to \((1, \tilde{1})\). By Proposition 39, Left wins this with a 0-bid strategy.

Consider \(k > 0\). Right wins the first bid in the game \((1/2^k + 1/2^{k'}, 0)\), and, by MMW, plays to either \((1/2^k, 0)\) or \((1/2^{k-1} + 1/2^{k'}, 0)\). By Proposition 43, Left wins \((1/2^k, 0)\) with a 0-bid strategy. Hence playing in the second component is losing for Right. Therefore we may assume that Right will not play in the second component
if a similar situation arise. In the other game \((1/2^{k-1} + 1/2^k, 0)\), Left continues to use her 0-bid strategy. If she wins the bid, then she plays to \((1/2^{k-1} + 1/2^k - 1, 0)\) and wins with 0-bid strategy by induction. However, if Right outbids her, then by MMW, he plays to \((1/2^{k-2} + 1/2^k, 1)\) (if \(k = 1\), he loses immediately). Since both \(k\) and TB are finite, he cannot continue winning the bid indefinitely while having options to play. Therefore, he will eventually end up with no move or he loses the bid, and when this happens, Left wins by induction.

Next, in the game \((1/2^k + 1/2^k, 0)\), by Left’s 0-bid strategy, if she wins the bid, then she plays to \((1/2^k + 1/2^{k-1}, 0)\). Left bids 0 again. Right wins the bid and, by MMW, plays to \((1/2^{k-1} + 1/2^k, 0)\). Left wins this using induction. However, in the game \((1/2^k + 1/2^k, 0)\), if Right outbids her, then by MMW, he plays to either \((1/2^k, 1)\) or to \((1/2^{k-1} + 1/2^k, 1)\). From Proposition 43 and from previous paragraph Left wins both using her 0-bid strategy.

**Corollary 47.** For all integers \(k' > k \geq 0\) and for any budget partition \(\tilde{p} \in \mathcal{B}\), Right wins \((1/2^{k'} + 1/2^k, \tilde{p})\) with a 0-bid strategy.

*Proof.* By symmetry, this follows from Theorem 46.

**Corollary 48.** For all integers \(k' > k \geq 0\), the disjunctive sum \(1/2^k + 1/2^{k'} > 0\) and \(1/2^k + 1/2^{k'} < 0\).

*Proof.* This follows by applying Corollary 12 to Theorem 46 and Corollary 47.

We prove that dyadic rational games have their conjugates as inverses.

**Theorem 49.** Consider \(k \in \mathbb{N}\). Then the disjunctive sum \(1/2^k + 1/2^k = 0\).

*Proof.* Similar to the proof of Theorem 26 it is sufficient to show \(o(1/2^k + 1/2^k, 0) = L\) with a Left 0-bid strategy. Consider the game \((1/2^k + 1/2^k, 0)\). By Left’s 0-bid strategy, Right wins the first bid. By MMW, he plays to either \((1/2^{k-1} + 1/2^k, 0)\) or \((1/2^k, 0)\). By Proposition 43 Left wins \((1/2^k, 0)\) with a 0-bid strategy. Thus, playing in the second component is losing for Right. Therefore we may assume that Right will not play in the second component when a similar situation arise.

Next, in the game \((1/2^{k-1} + 1/2^k, 0)\), Left will again bid 0. If she wins the bid she plays to \((1/2^{k-1} + 1/2^k, 0)\) and wins by induction using a 0-bid strategy. However, if Right outbids her, then he plays to \((1/2^{k-2} + 1/2^k, 1)\), because of the reason in the previous paragraph (if \(k = 1\) he loses immediately). Since both \(k\) and TB are finite, he cannot continue winning the bid indefinitely while having options to play. Therefore, he will eventually end up with no move or he loses the bid, and when this happens, he wins with a 0-bid strategy by Proposition 43 or Theorem 46.

**Corollary 50.** Consider \(n, k \in \mathbb{N}\). Then the disjunctive sum \(n/2^k + \overline{n/2^k} = 0\).

*Proof.* This follows from Theorem 49.

By Corollary 50, the inverse of \(n/2^k\) is \(\overline{n/2^k}\) and vice versa. The set \(\mathbb{D} = \{n/2^k : n \in \mathbb{Z}, k \in \mathbb{N}\}\), represents the set of all dyadic rational game forms. The set \(\{n/2^k : n \in \mathbb{N}, k \in \mathbb{N}\}\), constitutes the set of all negative dyadic rational game forms. In the coming, we use interchangeably \(-n/2^k = n/2^k\).
Theorem 51. For any \( n, k \in \mathbb{N} \), \( n/2^k > 0 \) and \(-n/2^k < 0\).

Proof. By Corollary 45 for any \( k \in \mathbb{N} \), \( 1/2^k > 0 \). Since, by Theorems 49 \( 1/2^k \) is invertible, therefore, by Theorems 20 we get \( 2/2^k > 1/2^k \). Consequently, by Theorems 23 we get \( 2/2^k > 0 \). By recursively applying the above steps, for any \( n \in \mathbb{N} \), we get \( n/2^k > 0 \).

Furthermore, since \(-n/2^k\) is inverse of \( n/2^k \) and \( n/2^k > 0 \), therefore, by Corollary 21 it follows that \(-n/2^k < 0\). \( \square \)

Lemma 52. For any \( k \in \mathbb{N} \), \( 1/2^k = \{0\} \cup \{1/2^k - 1\} \) is a number.

Proof. By Corollary 43 \( 1/2^k - 1/2^k - 1 < 0 \). Since \( 1/2^k \) is invertible, therefore, by Theorem 23 \( 1/2^k < 1/2^k - 1 \). Further, by Corollary 43 \( 1/2^k > 0 \). Hence, \( 1/2^k \) is a number. \( \square \)

We now show that the set of dyadic rational game forms is closed under disjunctive sum.

Lemma 53. For any \( k \in \mathbb{N} \), the disjunctive sum \( 1/2^k + 1/2^k = 1/2^k - 1 \).

Proof. Consider \( k \in \mathbb{N} \). By Theorem 13 and 49 we show \( 1/2^k + 1/2^k - 1/2^k - 1 = 0 \). Further, by Corollary 12 it is sufficient to show

i) \( o(1/2^k + 1/2^k - 1/2^k - 1, 0) = L \) with a Left’s 0-bid strategy, and

ii) \( o(1/2^k + 1/2^k - 1/2^k - 1, \overline{TB}) = R \) with a Right 0-bid strategy.

Proof of i). Consider the game \((1/2^k + 1/2^k - 1/2^k - 1, 0)\). By Left’s 0-bid strategy, Right wins the first bid. By MMW, he plays to either \((1/2^k - 1/2^k - 1, 0)\) or \((1/2^k + 1/2^k, \overline{0})\). By Theorem 49 \((1/2^k - 1/2^k - 1, \overline{0})\) is the same as \((1/2^k, 0)\), which Left wins with a 0-bid strategy (see Proposition 43). By Theorems 51 \( 1/2^k + 1/2^k > 0 \). Thus, by Theorems 21 Left wins \((2/2^k, 0)\). Further, by Lemma 52 \( 1/2^k \) is a number and number games form a group, thus, \( 2/2^k \) is a number. Therefore, Proposition 15 ensures an optimal 0-bid strategy of Left in \((2/2^k, 0)\).

Proof of ii). Consider the game \((1/2^k + 1/2^k - 1/2^k - 1, \overline{TB})\). We must prove that Right wins this game with a 0-bid strategy. By Right’s 0-bid strategy, Left wins the first bid. By MMW, she plays to either \((1/2^k - 1/2^k - 1, TB)\) or \((1/2^k + 1/2^k - 1/2^k - 1, TB)\) (if \(k = 1\), she only gets to play in the first two components).

By Corollary 47 Right wins \((1/2^k - 1/2^k - 1, TB)\) using a 0-bid strategy. Thus, playing in the first two components are losing for Left. Therefore we may assume that she will not play in those components if similar situations arise.

For \((1/2^k + 1/2^k - 1/2^k - 1, TB)\), if Right wins the bid by his 0-bid strategy, then he plays to \((1/2^k + 1/2^k - 1/2^k - 1, \overline{TB})\). Using induction, it is equivalent to \((1/2^k - 1/2^k - 1/2^k - 1, \overline{TB})\), which, by Theorem 49 is equal to \((1/2^k - 1/2^k - 1, \overline{TB})\). By Corollary 47 Right wins using a 0-bid strategy. However, if Left outbids him in the game \((1/2^k + 1/2^k - 1/2^k - 1, TB)\), then she plays to \((1/2^k + 1/2^k - 1/2^k - 3, TB - 1)\), because of the reason in the previous paragraph (if \(k = 2\) she loses immediately). Since both \(k\) and \(TB\) are finite, she cannot continue winning the bid indefinitely while having options to play. Therefore, she will eventually end up with no move or she loses the bid, and when this happens, Right wins with a 0-bid strategy by induction and Corollary 47. \( \square \)
Theorem 54. Let \( n \) and \( m \) be odd integers. For any total budget, the disjunctive sum \( n/2^k + m/2^{k-j} \) is a dyadic rational, namely \((n + 2^j m)/2^k\).

Proof. By definition, \( m/2^{k-j} \) is the disjunctive sum of \( m \) copies of \( 1/2^{k-j} \) and by Lemma 53, \( 1/2^{k-j} \) is the disjunctive sum of \( 1/2^{k-j+1} \) and \( 1/2^{k-j+1} \). Consequently, \( m/2^{k-j} \) is the disjunctive sum of \( 2m \) copies of \( 1/2^{k-j+1} \). Proceeding in a similar way, we get \( m/2^{k-j} \) is the disjunctive sum of \( 2^m \) copies of \( 1/2^k \). Therefore the disjunctive sum of \( n \) copies of \( 1/2^k \) and \( 2^m \) copies of \( 1/2^k \) is \((n + 2^m) \) copies of \( 1/2^k \). \( \square \)

Lemma 55. For any \( m \in \mathbb{N} \), \( m/2^k = \{(m-1)/2^k \mid (m+1)/2^k \} \).

Proof. By the definition of disjunctive sum and by using Lemma 53,

\[
m/2^k = \{(m-1)/2^k \mid (m-1)/2^k + 1/2^{k-1} \};
\]

\[
= \{(m-1)/2^k \mid (m-1)/2^k + 2/2^k \};
\]

\[
= \{(m-1)/2^k \mid (m+1)/2^k \}.
\]

\( \square \)

Theorem 56. The dyadic rational game forms are numbers.

Proof. For any \( m \in \mathbb{Z} \) and \( k \in \mathbb{N} \), consider a dyadic rational game \( G = m/2^k \). From Lemma 53, \( m/2^k = \{(m-1)/2^k \mid (m+1)/2^k \} \). By Corollary 15 and Theorem 19, \( G^R = (m+1)/2^k > G \) and \( G^L = (m-1)/2^k < G \). Hence, dyadic rational game forms are numbers. \( \square \)

Corollary 57. The set of all integer game forms is a subgroup of the numbers.

Proof. This follows from Theorems 56 54 and Corollary 50. \( \square \)

10. INFINITESIMALS AND 0-GAMES

Having understood the numbers and its subgroups, next we look into infinitesimals.

Definition 58 (Infinitesimal). Consider a total budget \( TB \). A game form \( G \) is infinitesimal if for all \( k \in \mathbb{N} \), \( 1/2^k < G < 1/2^k \).

We have seen in Section 5 that the game \( * \) does not have an inverse if \( TB > 0 \), whereas in alternating play \( * + * = 0 \). However, another interesting property of the game \( * \) continues to hold, it is infinitesimal for any \( TB \geq 0 \).

Theorem 59. Consider any total budget. For any \( k \in \mathbb{N} \), \(-1/2^k < * < 1/2^k\).

Proof. Suppose \( k \in \mathbb{N} \). By symmetry, it is sufficient to prove \( * < 1/2^k \). By Theorem 20, this is equivalent to proving \( * - 1/2^k < 0 \). To prove this, by Corollary 12 we prove the following:

i) \( o(* - 1/2^k, TB) = R \), with a Right 0-bid strategy, and

ii) \( o(* - 1/2^k, 0) = R \).

Proof of i). Consider the game \( (* - 1/2^k, TB) \). By Right’s 0-bid strategy, Left wins the first bid. By MMW, she plays to either \( (* - 1/2^{k-1}, TB) \) or \(- 1/2^k, TB) \). By Corollary 14 Right wins \(- 1/2^k \) with a 0-bid strategy. Thus, playing in the first component is losing for Left. Therefore we may assume that Left will not play in the first component if a similar situation arises.
Next, in the game \((* - 1/2^{k-1}, TB)\), Right will again bid 0. If he wins the bid then he plays to \((-1/2^{k-1}, TB)\) and wins by Corollary 44 with a 0-bid strategy. However, if Left outbids him, she plays to \((* - 1/2^{k-2}, TB - 1)\), because of the reason in the previous paragraph (if \(k = 1\) she loses immediately).

Since both \(k\) and TB are finite, she cannot continue winning the bid indefinitely while having options to play. Therefore, she will eventually end up with no move or she loses the bid, and when this happens, Right plays in first component and wins with a 0-bid strategy by Corollary 44.

Proof of ii). Consider the game \((* - 1/2^{k}, 0)\). Right wins the first bid and plays to \((-1/2^{k}, 0 \uparrow)\), which he wins by Corollary 44. □

Recall the games \(\uparrow = \{0|\} \) and \(\downarrow = \{\}|0\).

Corollary 60. The games \(\uparrow\) and \(\downarrow\) are infinitesimals.

Proof. This is similar to the proof of Theorem 59. □

In alternating play, it holds that \({\ast}|{\ast}\} = 0\). But it is not true if TB > 0, because a player with budget \(TB\) can win two consecutive bids and get the last move. But, since numbers are invertible, we already have seen a huge class of 0s. In alternating play, of course \(\uparrow + \downarrow = 0\). This does not hold in strict bidding play.

Observation 61. Consider TB > 0. We demonstrate that \(\downarrow + \uparrow \neq 0\). This is obviously true if TB \(\geq 3\), since Left can win three consecutive moves if she owns all budget and the marker. Consider the game \(\downarrow + \uparrow\) with TB = 1. Now let’s assume Left holds the marker as well as all budget. Then Left will start by bidding 0. Since Right doesn’t have any other choice than bidding 0, Left will win the bid by giving away the marker and will play to \(\downarrow\), and the game will reach \(\uparrow + \ast\). Again Left will bid 0. This time Right holds the marker, so he will win the bid and the game goes to either \((\uparrow, \hat{1})\) or \((\ast + \ast, \hat{1})\). If the game reaches \((\uparrow, \hat{1})\), then Left will again bid 0 and since she has the marker the game will reach \((0, 1)\), which Left will win. Next, if the game will reach \((\ast + \ast, \hat{1})\), then Left has sufficient budget to win two consecutive moves and take the game to 0, with Right holding the marker. Thus \(o(\downarrow + \uparrow, \hat{1}) = L\), which contradicts \(o(0, \hat{1}) = R\). Hence it is not true that \(\downarrow + \uparrow = 0\). A similar argument works for TB = 2. Hence \(\downarrow\) is the inverse of \(\uparrow\) if and only if TB = 0.

Next follows another infinite class of 0s.

Theorem 62. Suppose Right has an optimal 0-bid strategy in \(G < 0\) and Left has an optimal 0-bid strategy in \(H > 0\). Then \(\{G|H\} = 0\).

Proof. By symmetry and Corollary 12 it is sufficient to show that Left wins \((\{G|H\}, 0)\) with a 0-bid strategy. In the game \((\{G|H\}, 0)\), Right wins the first bid. By MMW, he must play to \((H, \hat{0})\). By assumption, Left wins this with a 0-bid strategy. □

Corollary 63. For all positive integers \(k\), \({-k}|{k}\} = 0\) and \({-1/2^{\hat{k}}|1/2^{\hat{k}}}\} = 0\).

Proof. This follows by Theorem 62. □
11. Open Problems

In view of the uniqueness of identity (Theorem 8), we have the following problem.

**Problem 64.** Consider $TB > 0$. Does there exist any non-trivial identity element, for any non-invertible $(TB, G)^T$?

**Conjecture 65.** For all total budgets, the equivalence class of 0s is the unique idempotent (i.e. $0 + 0 = 0$).

By Theorem 11 if $o(G, 0) = L$ and Left has an optimal 0-bid strategy in $(G, 0)$, then $G \geq 0$. Moreover, by Observation 14 for any non-zero total budget, we have the example of a game $G$ such that $o(G, 0) = L$, but Left does not have a winning 0-bid strategy.

**Problem 66.** Is there a game $G$ such that $o(G, 0) = L$, without a winning 0-bid strategy for Left, but where $G \not\geq 0$?

Observe that if $TB > 0$, ↑ is confused with 0. This raises the following problem.

**Problem 67.** Are there positive infinitesimals if $TB > 0$?

In view of the difference between the definition of a number game used in this paper and the one used in [8] for alternating play, we believe that they are equivalent.

**Conjecture 68.** For any game $G$, the following are equivalent:

1) $H_L < H_R$ for every subposition $H$ of $G$ and every $H^L$ and $H^R$.
2) $H_L < H < H_R$ for every subposition $H$ of $G$ and every $H^L$ and $H^R$.

We believe that each invertible game is a number. The following conjecture would be an immediate consequence (by Theorem 26).

**Conjecture 69.** If a bidding game $G$ has an inverse, $H$, then $H = \overline{G}$.

Since defined numbers behave well in bidding play, we believe the celebrated number avoidance result for alternating play continues to hold.

**Conjecture 70** (Number Avoidance/Translation). For any total budget, if $G$ is a number and $H$ is not a number, then $G + H = \{G + H^L | G + H^R \}$.

The next conjecture holds in alternating play.

**Conjecture 71** (Number Simplicity). For any total budget, each number is a dyadic rational.

**Problem 72.** In alternating play Hackenbush there is no $N$-position. What outcomes do we lose in bidding play?

**Conjecture 73.** All numbers are Hackenbush positions, and vice versa.

If this conjecture does not hold, then we ask Problem 74 for numbers.

\[\text{Footnote: For example in alternating play, in a restriction of impartial misère nim, } \ast 2 + \ast 2 \text{ can “act like a 0”, in that } \ast 2 = \ast 2 + \ast 2 + \ast 2 \text{ [8].}\]
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