AN ELECTROMAGNETIC TRINITY FROM
“NEGATIVE PERMITTIVITY” AND
“NEGATIVE PERMEABILITY”

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An electromagnetic trinity comprising vacuum, anti–vacuum, and nihility is postulated — after making use of materials with “negative permittivity” and “negative permeability” — to illuminate the structure of electromagnetic theory, at least insofar as the relationship of phase velocity with Poynting vector is concerned.

KEYWORDS: Anti–vacuum, Negative permeability, Negative permittivity, Vacuum, Nihility

1 Introduction

Illuminating normally one of the two relevant faces of a wedge by a 10 GHz beam, Shelby et al. [1] observed that the angle of emergence of the beam from the other face changed sign when the teflon wedge was replaced by one made of the so–called left–handed material (LHM) [2, 3]. Assuming that their LHM is homogeneous and isotropic, and virtually disregarding dissipation, they concluded that it possesses a negative index of refraction.

Never mind that the so–called LHM does not actually possess handedness [4], and therefore is mis–named. Never mind that this material is actually a particulate composite [4] containing particles that are not electrically small — especially when the fiber–glass content of the LHM is factored in — and therefore is not homogenizable into a medium with local response properties [5]. Never mind also that the LHM is actually anisotropic and must have some scattering losses.
A negative index of refraction is not needed *per se*, because all that it connotes is that the phase velocity is directed opposite to the velocity of energy transport — as shown in the Appendix. But the phase velocity is an inessential concept, whether in vacuum or in materials. What really matter are the solutions of Maxwell equations that propagate energy on a desired trajectory, which is chosen epistemically.

Anyway, virtually all publications on *LHMs* that I have come across deal substantively only with non–dissipative materials, with just one exception. Perhaps, in time to come, samples of these materials with proven negligible dissipation at some frequency will emerge. In the meanwhile, they permit me to postulate an electromagnetic trinity comprising (i) vacuum, (ii) anti–vacuum, and (iii) nihility. This trinity illuminates the structure of electromagnetic theory, at least insofar as the relationship of phase velocity with Poynting vector is concerned.

## 2 Vacuum

Vacuum, of course, is well–known. Also called free space, it is a matter–free medium — the substrate on which microscopic electromagnetics is formulated and later homogenized for macroscopic research. The frequency–domain constitutive relations of vacuum are specified as

\[
\begin{align*}
D(x, y, z, \omega) &= \epsilon_0 \mathbf{E}(x, y, z, \omega) \\
B(x, y, z, \omega) &= \mu_0 \mathbf{H}(x, y, z, \omega)
\end{align*}
\]

where \(x, y, z\) are the cartesian components of the position vector; \(\omega\) is the angular frequency; \(\epsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1}\) and \(\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}\). The phase velocity and the wavenumber of a plane wave in vacuum are co–parallel.

## 3 Anti–vacuum

The constitutive relations of anti–vacuum are postulated as

\[
\begin{align*}
D(x, y, z, \omega) &= -\epsilon_0 \mathbf{E}(x, y, z, \omega) \\
B(x, y, z, \omega) &= -\mu_0 \mathbf{H}(x, y, z, \omega)
\end{align*}
\]
Among the more significant properties of anti–vacuum is that the phase velocity of a plane wave therein is opposite in direction to the wavevector. Hence, anti–vacuum is complementary to vacuum.

Clearly, anti–vacuum does not exist, but it could be simulated at a particular spatial frequency as follows: Suppose one could fabricate a \( LHM \) whose relative permittivity scalar \( \epsilon_r(\omega) \) is purely real–valued at a certain angular frequency \( \tilde{\omega} \), and whose relative permittivity scalar \( \mu_r(\omega) \) is such that \( \mu_r(\tilde{\omega}) = \epsilon_r(\tilde{\omega}) \). Let electrically small spheres of this \( LHM \) be homogeneously but randomly dispersed in vacuum. As both mediums would be impedance–matched, the effective constitutive parameters of the particulate composite thus formed would have to satisfy the relation \( \mu_{eff}(\tilde{\omega}) = \epsilon_{eff}(\tilde{\omega}) \). Then, \( \epsilon_{eff}(\tilde{\omega}) = -1 \) according to the Bruggeman formalism \([13]\), provided the volume fraction of the \( LHM \) is

\[
f_{LHM} = \frac{2}{3} \frac{\epsilon_r(\tilde{\omega}) - 2}{\epsilon_r(\tilde{\omega}) - 1}, \quad 0 \leq f_{LHM} \leq 1. \tag{3}
\]

As an example, \( f_{LHM} = 0.8 \) if \( \epsilon_r(\tilde{\omega}) = -4 \). Hence, this composite would mimic anti–vacuum at \( \omega = \tilde{\omega} \).

4 Nihility

Nihility is the electromagnetic nilpotent, with the following constitutive relations

\[
\begin{align*}
D(x, y, z, \omega) &= 0 \quad \text{and} \\
B(x, y, z, \omega) &= 0
\end{align*}
\tag{4}
\]

Wave propagation cannot occur in nihility, because \( \nabla \times E(x, y, z, \omega) = 0 \) and \( \nabla \times H(x, y, z, \omega) = 0 \) in the absence of sources therein \([14, 15]\). The directionality of the phase velocity relative to the wavevector in nihility is thus a non–issue.

The Bruggeman formalism is unable to predict the realization of nihility as a particulate composite either by mixing anti–vacuum and vacuum, or by mixing an \( LHM \) with an isoimpedant commonplace material (such as teflon). This is because the polarizability and the magnetizability (per unit volume) of an isotropic dielectric–magnetic sphere embedded in nihility do not depend on the constitutive parameters of the medium that the sphere is made of.

The Maxwell Garnett formalism \([13]\), however, predicts that a homogeneous dispersion of electrically small, anti–vacuum spheres in vacuum is effectively
equivalent to nihility, provided the volume fraction of anti–vacuum is 0.25. Conversely, if matter were scooped out of anti–vacuum in the form of electrically small spheres, the resulting Swiss–cheese composite would also mimic nihility, if the volume fraction of anti–vacuum were 0.75. Thus, nihility too could be mimicked by particulate composites at $\omega = \tilde{\omega}$.

## 5 Concluding Remarks

Whereas vacuum and anti–vacuum are mutually complementary in regard to the directionality of the phase velocity with respect to the Poynting vector, that issue does not arise for nihility because waves cannot propagate inside it. Of the three, vacuum alone is matter–free. Both anti–vacuum and nihility would have to be simulated — at a specific angular frequency — as particulate composites which would ultimately require the fabrication of virtually non–dissipative LHMs that are impedance–matched to vacuum. When would this become possible at all, let alone routinely, is anybody’s guess.

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Appendix

Let us consider an isotropic, dielectric–magnetic, homogeneous medium with relative permittivity \( \varepsilon_r = -a(1 - i \tan \delta_\varepsilon) \) and relative permeability \( \mu_r = -b(1 - i \tan \delta_\mu) \) at a certain angular frequency \( \omega \). Let \( a, b, m = \pm \sqrt{ab} \), \( \tan \delta_\varepsilon \) and \( \tan \delta_\mu \) all be positive real, so that \( \varepsilon_r \) and \( \mu_r \) are in accord with the Lorentz model \([1, 16]\), with \( \omega \) significantly larger than the resonant angular frequencies of the medium. The \( \exp(-i\omega t) \) time–dependence is implicit. To ensure that dissipation is moderate, let \( \tan \delta_\varepsilon < 0 \) and \( \tan \delta_\mu < 0 \).

The electromagnetic phasors of a planewave traveling along the \(+z\) axis in the chosen medium are given by

\[
\begin{align*}
\mathbf{E}(z, \omega) &= A \mathbf{u}_x \exp(i k_0 n z)
\mathbf{H}(z, \omega) &= \frac{n}{\mu_r \eta_0} A \mathbf{u}_y \exp(i k_0 n z)
\end{align*}
\]

where \( A \) is a complex–valued amplitude, \( k_0 \) is the vacuum wavenumber, \( \eta_0 \) is the intrinsic impedance of vacuum, and \( n^2 = \varepsilon_r \mu_r \). Writing \( n = n' + in'' \), we see that the time–averaged Poynting vector is given as

\[
P(z, \omega) = \mathbf{u}_z \text{Re}\left\{ \frac{n}{\mu_r} \frac{|A|^2}{2\eta_0} \exp(-2k_0 n'' z) \right\} = \mathbf{u}_z P_z(z, \omega),
\]

the velocity of energy transport is co–parallel with \( \mathbf{P}(z, \omega) \), and the phase velocity \( v_{ph} = \omega/k_0 n' \). Correct to first order in both \( \tan \delta_\varepsilon \) and \( \tan \delta_\mu \), we get \( n = \pm m [1 - i(\tan \delta_\varepsilon + \tan \delta_\mu)/2] \) and \( \text{Re}\left\{ \frac{n}{\mu_r} \right\} = \mp m/b \).

For energy to flow along the \(+z\) axis, we must have \( P_z > 0 \). Furthermore, the directions of energy attenuation and energy flow must coincide (as befits a passive medium), so that \( n'' > 0 \). Both of these criterions lead to \( n' < 0 \), which means that the phase velocity is pointed exactly opposite to the common direction of energy attenuation and flow.