Freeness Conditions for Crossed Squares and Squared Complexes.

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Abstract

Following Ellis, we investigate the notion of totally free crossed square and related squared complexes. It is shown how to interpret the information in a free simplicial group given with a choice of CW-basis, in terms of the data for a totally free crossed square. Results of Ellis then apply to give a description in terms of tensor products of crossed modules. The paper ends with a purely algebraic derivation of a result of Brown and Loday.

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Introduction

Crossed squares were introduced by Loday and Guin-Walery in [12]. They arose in various problems of relative algebraic K-theory. Loday later showed in [14] that these quite simple algebraic gadgets modelled all homotopy 3-types. More generally his notion of cat^n-group and the related crossed n-cubes of Ellis and Steiner were shown by Loday to model all connected (n + 1)-types. The possibilities of calculation with these models was enhanced by the development with R.Brown of a van Kampen type theorem for these structures [3].

A link between simplicial groups and crossed n-cubes was used by Porter, [21] to give an algebraic form of Loday’s result and in particular to give a functor from the category of simplicial groups to that of crossed n-cubes realising the equivalence.

In 1993, Ellis introduced a notion of free crossed square and showed how to assign a free crossed square to a CW-complex. As there was an established notion of free simplicial group, it seemed important to investigate the extent to which the two notions of freeness are related. That was the initial motivation for this paper. The two notions were intimately related and moreover combining this with Ellis’ alternative description of free crossed squares in terms of the Brown-Loday non-abelian tensor product of groups and coproducts of crossed modules, gives a new purely algebraic derivation
of Brown and Loday’s result describing the homotopy 3-type of the suspension of an Eilenberg-Mac Lane space. This success raises our hopes that this method of attack can yield new results in higher dimensions.

1 Preliminaries

In this paper we will concentrate on the reduced case and hence on simplicial groups rather than simplicial groupoids. This is for ease of exposition only and all the results do go through for simplicially enriched groupoids.

Notation: If $X$ is a set, $F(X)$ will denote the free group on $X$. If $Y$ is a subset of $F(X)$, $\langle Y \rangle$ will denote the normal subgroup generated by $Y$ within $F(X)$.

1.1 Simplicial groups and groupoids

Denoting the usual category of finite ordinals by $\Delta$, we obtain for each $k \geq 0$, a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of $\Delta$ with $j \leq k$. A simplicial group is a functor from the opposite category $\Delta^{op}$ to $\mathcal{Gp}$; a $k$-truncated simplicial group is a functor from $\Delta_{\leq k}^{op}$ to $\mathcal{Gp}$. We will denote the category of simplicial groups by $\Simp\mathcal{Gp}$ and the category of $k$-truncated simplicial groups by $\Tr_k\Simp\mathcal{Gp}$. By a $k$-truncation of a simplicial group, we mean a $k$-truncated simplicial group $\operatorname{tr}_k G$ obtained by forgetting dimensions of order $> k$ in a simplicial group $G$, that is restricting $G$ to $\Delta_{\leq k}^{op}$. This gives a truncation functor $\operatorname{tr}_k : \Simp\mathcal{Gp} \rightarrow \Tr_k\Simp\mathcal{Gp}$ which admits a right adjoint $\cosk_k : \Tr_k\Simp\mathcal{Gp} \rightarrow \Simp\mathcal{Gp}$ called the $k$-coskeleton functor, and a left adjoint $\sk_k : \Tr_k\Simp\mathcal{Gp} \rightarrow \Simp\mathcal{Gp}$, called the $k$-skeleton functor. For explicit constructions of these see [11]. We will say that a simplicial group $G$ is $k$-skeletal if the natural morphism $\sk_k G \rightarrow G$ is an isomorphism.

Recall that given a simplicial group $G$, the Moore complex $(NG, \partial)$ of $G$ is the normal chain complex defined by

$$(NG)_n = \bigcap_{i=0}^{n-1} \text{Ker} d_i^n$$

with $\partial_n : NG_n \rightarrow NG_{n-1}$ induced from $d_i^n$ by restriction. There is an alternative form of Moore complex given by the convention of taking

$$\bigcap_{i=1}^{n} \text{Ker} d_i^n$$

and using $d_0$ instead of $d_n$ as the boundary. One convention is used by Curtis [6] (the $d_0$ convention) and the other by May [13] (the $d_n$ convention). They lead to equivalent theories.
The $n^{th}$ homotopy group $\pi_n(G)$ of $G$ is the $n^{th}$ homology of the Moore complex of $G$, i.e.

$$\pi_n(G) \cong H_n(NG, \partial)$$

$$= \bigcap_{i=0}^n \text{Ker} d_i^n / d_{n+1}^n \bigcap_{i=0}^n \text{Ker} d_i^{n+1}.$$ 

We say that the Moore complex $NG$ of a simplicial group is of length $k$ if $NG_n = 1$ for all $n \geq k + 1$, so that a Moore complex of length $k$ is also of length $l$ for $l \geq k$. For example, if $G$ has Moore complex of length 1, then $(NG_1, NG_0, \partial_1)$ is a crossed module and conversely. If $NG$ is of length 2, the corresponding Moore complex gives a 2-crossed module in the sense of Conduché, [5], cf. the companion paper to this, [20].

### 1.2 Free Simplicial Groups

Recall from [8] and [13] the definitions of free simplicial group and of a $CW$–basis for a free simplicial group.

**Definition**

A simplicial group $F$ is called free if

(a) $F_n$ is a free group with a given basis, for every integer $n \geq 0$,

(b) The bases are stable under all degeneracy operators, i.e., for every pair of integers $(i, n)$ with $0 \leq i \leq n$ and every basic generator $x \in F_n$ the element $s_i(x)$ is a basic generator of $F_{n+1}$.

**Definition**

Let $F$ be a free simplicial group (as above). A subset $\mathcal{F} \subset F$ will be called a $CW$–basis of $F$ if

(a) $\mathcal{F}_n = \mathcal{F} \cap F_n$ freely generates $F_n$ for all $n \geq 0$,

(b) $\mathcal{F}$ is closed under degeneracies, i.e. $x \in \mathcal{F}_n$ implies $s_i(x) \in \mathcal{F}_{n+1}$ for all $0 \leq i \leq n$,

(c) if $x \in \mathcal{F}_n$ is non-degenerate, then $d_i(x) = e_{n-1}$, the identity element of $F_n$, for all $0 \leq i < n$.

As explained earlier, we have restricted attention so far to simplicial groups and hence to connected homotopy types. This is traditional but a bit unnatural as all the results and definitions so far extend with little or no trouble to simplicial groupoids in the sense of Dwyer and Kan [7] and hence to non-connected homotopy types. It should be noted that such simplicial groupoids have a fixed and constant simplicial set of objects and so are not merely simplicial objects in the category of groupoids. In this context if $G$ is a simplicial groupoid with set of objects $O$, the natural form of the Moore complex $NG$ is given by the same formula as in the reduced case, interpreting $\text{Ker} d_i^m$ as being the subgroupoid of elements in $G_n$ whose $i^{th}$ face is an identity of $G_{n-1}$. Of course if $n \geq 1$, the resulting $NG_n$ is a disjoint union of groups, so $NG$ is a disjoint union of the Moore complexes of the vertex simplicial groups of $G$ together with the groupoid $G_0$ providing
elements that allow conjugation between (some of) these vertex complexes (cf. Ehlers and Porter [8]).

Crossed modules of, or over, groupoids are well known from the work of Brown and Higgins. The only changes from the definition for groups (cf. [14]) is that one has to handle the conjugation operation slightly more carefully:

A crossed module is a morphism of groupoids \( \partial : M \to N \) where \( N \) is a groupoid with object set \( O \) say and \( M \) is a family of groups, \( M = \{ M(a) : a \in O \} \), together with an action of \( N \) on \( M \) satisfying (i) if \( m \in M(a) \) and \( n \in N(a, b) \) for \( a, b, \in O \), the result of \( n \) acting on \( m \) is \( ^nm \in M(b) \); (ii) \( \partial(^nm) = n\partial(m)n^{-1} \) and (iii) \( \partial(m)m' = mm'n^{-1} \) for all \( m, m' \in M \), \( n \in N \). For the weaker notion in which condition (iii) is not required, the models are called precrossed modules.

The definition of a CW-basis likewise generalises with each \( \mathcal{F} \) a subgraph of the corresponding free simplicial groupoid.

2 Crossed Squares and Simplicial Groups

Although we will be mainly concerned with crossed squares in this paper, many of the arguments either clearly apply or would seem to apply in the more general case of crossed \( n \)-cubes and \( n \)-cube complexes. We therefore give some background in this more general setting.

Again although we give the definitions and results for groups, the adaptation to handle groupoids over a fixed base is routine.

The following definition is due to Ellis and Steiner [10]. Let \( < n > \) denote the set \( \{1, ..., n\} \).

Definition

A crossed \( n \)-cube of groups is a family \( \{ \mathcal{M}_A : A \subseteq < n > \} \) of groups, together with homomorphisms \( \mu_i : \mathcal{M}_A \to \mathcal{M}_{A\setminus\{i\}} \) for \( i \in < n > \) and functions

\[ h : \mathcal{M}_A \times \mathcal{M}_B \to \mathcal{M}_{A \cup B} \]

for \( A, B \subseteq < n > \), such that if \( ab \) denotes \( h(a, b)b \) for \( a \in \mathcal{M}_A \) and \( b \in \mathcal{M}_B \) with \( A \subseteq B \), then for all \( a, a' \in \mathcal{M}_A \) and \( b, b' \in \mathcal{M}_B, c \in \mathcal{M}_C \) and \( i, j \in < n > \),
the following hold:

1) $\mu_i a = a$ if $i \not\in A$,
2) $\mu_i\mu_j a = \mu_j\mu_i a$,
3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$,  
4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$,
5) $h(a, a') = [a, a']$,  
6) $h(a, b) = h(b, a)^{-1}$,  
7) $h(a, b) = 1$ if $a = 1$ or $b = 1$,  
8) $h(aa', b) = h(a', b)h(a, b)$,  
9) $h(a, bb') = h(a, b)h(a, b')$,  
10) $a h(b, c) = h(a b, a c)$ if $A \subseteq B \cap C$,  
11) $a h(h(a^{-1}, b), c) c h(h(c^{-1}, a), b) b h(h(b^{-1}, c), a) = 1$.

A morphism of crossed $n$-cubes is defined in the obvious way: It is a family of group homomorphisms, for $A \subseteq \langle n \rangle$, $f_A : \mathcal{M}_A \to \mathcal{M}_A'$ commuting with the $\mu_i$'s and $h$'s. We thus obtain a category of crossed $n$-cubes which will be denoted by $\text{Crs}^n$, cf. Ellis and Steiner [10]. Again there is an obvious variant of this definition for groupoids over a fixed set of objects, $O$.

**Remark:** Crossed squares, that is the case $n = 2$, were introduced by Loday and Guin-Walery, [12], but with an apparently different definition. The two notions are however equivalent.

**Example 1:** For $n = 1$, a crossed 1-cube is the same as a crossed module.

For $n = 2$, one has a crossed 2-cube is a crossed square:

$$\begin{array}{cccc}
\mathcal{M}_{<2>} & \mu_2 & \mathcal{M}_{\{1\}} \\
\mu_1 & \mu_1 & \\
\mathcal{M}_{\{2\}} & \mu_2 & \mathcal{M}_\emptyset.
\end{array}$$

Each $\mu_i$ is a crossed module, as is $\mu_1\mu_2$. The $h$-functions give actions and a function

$$h : \mathcal{M}_{\{1\}} \times \mathcal{M}_{\{2\}} \to \mathcal{M}_{<2>}.$$ 

The maps $\mu_2$ also define a map of crossed modules from $(\mathcal{M}_{<2>}, \mathcal{M}_{\{2\}}, \mu_1)$ to $(\mathcal{M}_{<1>}, \mathcal{M}_\emptyset, \mu_1)$. In fact a crossed square can be thought of as a crossed module in the category of crossed modules.

**Example 2:** Let $N_1, N_2$ be normal subgroups of a group $G$. The commutative square diagram of inclusions:

$$\begin{array}{cccc}
N_1 \cap N_2 & \text{inc.} & N_2 \\
\text{inc.} & \text{inc.} & \\
N_1 & \text{inc.} & G
\end{array}$$
naturally comes together with actions of $G$ on $N_1, N_2$ and $N_1 \cap N_2$ given by conjugation and functions

$$h : N_A \times N_B \rightarrow N_A \cap N_B = N_{A \cup B}$$

$$(n_1, n_2) \mapsto [n_1, n_2].$$

That this is a crossed square is easily checked.

The following proposition is noted by the second author in [21].

**Proposition 2.1** [21] Let $G$ be a simplicial group with simplicial normal subgroups $N_1$ and $N_2$. Then the square

$$\begin{array}{ccc}
N_1 \cap N_2 & \longrightarrow & N_2 \\
\downarrow & & \downarrow \\
N_1 & \longrightarrow & G
\end{array}$$

induces a crossed square

$$\begin{array}{ccc}
\pi_0(N_1 \cap N_2) & \longrightarrow & \pi_0(N_2) \\
\downarrow & & \downarrow \\
\pi_0(N_1) & \longrightarrow & \pi_0(G)
\end{array}$$

**Proof:** The $h$-function

$$h : \pi_0(N_1) \times \pi_0(N_2) \rightarrow \pi_0(N_1 \cap N_2)$$

is given by

$$h(\bar{n}_1, \bar{n}_2) = [n_1, n_2]$$

for all $\bar{n}_1 \in \pi_0(N_1), \bar{n}_2 \in \pi_0(N_2)$. It is then simple, cf. [21], to see that the second diagram above is a crossed square.

In fact up to isomorphism all crossed squares arise in this way, cf. [14] and [21].

**Example 3:** Let $G$ be a simplicial group. Let $\mathcal{M}(G, 2)$ denote the following diagram

$$\begin{array}{ccc}
NG_2/\partial_3NG_3 & \longrightarrow & NG_1 \\
\downarrow & & \downarrow \\
NG_1 & \longrightarrow & G_1
\end{array}$$

Then this is the underlying square of a crossed square. The extra structure is given as follows: $NG_1 = \ker d_0^1$ and $\overline{NG_1} = \ker d_1^1$. Since $G_1$ acts on $NG_2/\partial_3NG_3$, $\overline{NG_1}$ and $NG_1$, there are actions of $\overline{NG_1}$ on $NG_2/\partial_3NG_3$ and $NG_1$ via $\mu'$, and $NG_1$ acts on $NG_2/\partial_3NG_3$ and $\overline{NG_1}$ via $\mu$. Both $\mu$
and \( \mu' \) are inclusions, and all actions are given by conjugation. The \( h \)-map is

\[
NG_1 \times \overline{NG_1} \rightarrow NG_2/\partial_3 NG_3
\]

\((x, y) \mapsto h(x, y) = [s_1 x, s_1 y s_0 y^{-1}] \partial_3 NG_3.\]

Here \( x \) and \( y \) are in \( NG_1 \) as there is a bijection between \( NG_1 \) and \( \overline{NG_1} \). We leave the verification of the axioms of a crossed square to the reader.

This example is clearly functorial and we denote by

\[
\mathcal{M}(-, 2) : \text{SimpGrp} \rightarrow \text{Crs}^2,
\]

the resulting functor. This is the case \( n = 2 \) of a general construction of a crossed \( n \)-cube from a simplicial group given by the second author in [21] based on some ideas of Loday.

**Examples 2 and 3 revisited:** Let \( G \) be a group with normal subgroups \( N_1, \ldots, N_n \) of \( G \). Let

\[
\mathcal{M}_A = \bigcap \{ N_i : i \in A \} \quad \text{and} \quad \mathcal{M}_\emptyset = G
\]

with \( A \subseteq \langle n \rangle \). For \( i \in \langle n \rangle \), \( \mathcal{M}_A \) is a normal subgroup of \( \mathcal{M}_{A-\{i\}} \). Define

\[
\mu_i : \mathcal{M}_A \rightarrow \mathcal{M}_{A-\{i\}}
\]

to be the inclusion. If \( A, B \subseteq \langle n \rangle \), then \( \mathcal{M}_{A \cup B} = \mathcal{M}_A \cap \mathcal{M}_B \), let

\[
h : \mathcal{M}_A \times \mathcal{M}_B \rightarrow \mathcal{M}_{A \cup B}
\]

\((a, b) \mapsto [a, b]\)

as \( [\mathcal{M}_A, \mathcal{M}_B] \subseteq \mathcal{M}_A \cap \mathcal{M}_B \), where \( a \in \mathcal{M}_A \), \( b \in \mathcal{M}_B \). Then

\[
\{ \mathcal{M}_A : A \subseteq \langle n \rangle , \mu_i, h \}
\]

is a crossed \( n \)-cube, called the *inclusion crossed \( n \)-cube* given by the normal \( n \)-ad of groups \((G; N_1, \ldots, N_n)\).

**Proposition 2.2** Let \((G; N_1, \ldots, N_n)\) be a simplicial normal \( n \)-ad of subgroups of groups and define for \( A \subseteq \langle n \rangle \)

\[
\mathcal{M}_A = \pi_0(\bigcap_{i \in A} N_i)
\]

with homomorphisms \( \mu_i : \mathcal{M}_A \rightarrow \mathcal{M}_{A-\{i\}} \) and \( h \)-maps induced by the corresponding maps in the simplicial inclusion crossed \( n \)-cube, constructed by applying the previous example to each level. Then \( \{ \mathcal{M}_A : A \subseteq \langle n \rangle , \mu_i, h \} \) is a crossed \( n \)-cube.

**Proof:** See [21]. \( \square \)

This describes a functor, [21], from the category of simplicial groups to that of crossed \( n \)-cubes of groups.
Theorem 2.3 If $G$ is a simplicial group, then the crossed $n$-cube $M(G, n)$ is determined by:

(i) for $A \subseteq < n >$, 

$$M(G, n)_A = \frac{\bigcap_{j \in A} \text{Ker}_{d_{j-1}^n}}{d_{n+1}(\text{Ker}_{d_{0}^{n+1}} \cap \{\bigcap_{j \in A} \text{Ker}_{d_{j}^{n+1}}\})},$$

(ii) the inclusion

$$\bigcap_{j \in A} \text{Ker}_{d_{j-1}^n} \rightarrow \bigcap_{j \in A \setminus \{i\}} \text{Ker}_{d_{j-1}^n}$$

induces the morphism

$$\mu_i : M(G, n)_A \rightarrow M(G, n)_{A \setminus \{i\}};$$

(iii) the functions, for $A, B \subseteq < n >$, 

$$h : M(G, n)_A \times M(G, n)_B \rightarrow M(G, n)_{A \cup B}$$

are given by

$$h(\bar{x}, \bar{y}) = [x, y],$$

where an element of $M(G, n)_A$ is denoted by $\bar{x}$ with $x \in \bigcap_{j \in A} \text{Ker}_{d_{j-1}^n}$.

Some simplification is possible, again see [21] for the details.

Proposition 2.4 If $G$ is a simplicial group, then

i) for $A \subseteq < n >$, $A \neq < n >$, 

$$M(G, n)_A \cong \bigcap_{i \in A} \text{Ker}_{d_{i-1}^{n-1}}$$

so that in particular, $M(G, n)_\emptyset \cong G_{n-1}$; in every case the isomorphism is induced by $d_0$.

ii) if $A \neq < n >$ and $i \in < n >$,

$$\mu_i : M(G, n)_A \rightarrow M(G, n)_{A \setminus \{i\}}$$

is the inclusion of a normal simplicial subgroup,

iii) for $j \in < n >$,

$$\mu_j : M(G, n)_{<n>} \rightarrow \bigcap_{i \neq j} \text{Ker}_{d_{i}^{n+1}}$$

is induced by $d_n$. 

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Expanding this data out for low values of $n$ gives:

1) For $n = 0$, 
\[ \mathfrak{M}(G, 0) = G_0/d_1(\text{Ker}d_0), \]
\[ \cong \pi_0(G), \]
\[ = H_0(NG). \]

2) For $n = 1$, $\mathfrak{M}(G, 1)$ is the crossed module 
\[ \mu_1 : \text{Ker}d_0^1/d_2^1(NG_2) \to G_1/d_2^1(\text{Ker}d_0^1). \]

3) For $n = 2$, $\mathfrak{M}(G, 2)$ is 
\[ \text{Ker}d_0^2 \cap \text{Ker}d_1^2/d_3^2(\text{Ker}d_0^2 \cap \text{Ker}d_1^2 \cap \text{Ker}d_2^2) \xrightarrow{\mu_2} \text{Ker}d_0^2/d_3^2(\text{Ker}d_0^2 \cap \text{Ker}d_1^2). \]

By Proposition 2.4, this is isomorphic to 
\[ NG_2/d_3^2(NG_3) \xrightarrow{\mu_2} \text{Ker}d_0^1 \]
\[ \text{Ker}d_1^1 \xrightarrow{\mu_2} G_1, \]
that is 
\[ \mathfrak{M}(G, 2) \cong \left( \begin{array}{c}
    \begin{array}{c}
        \text{Ker}d_0^1 \xrightarrow{\mu_1} \\
        \xrightarrow{\mu_1} \\
        \xrightarrow{\mu_1} \\
    \end{array}
    \begin{array}{c}
        NG_2/\partial_3(NG_3) \xrightarrow{\mu_2} \text{Ker}d_0^1 \\
        \xrightarrow{\mu_2} \text{Ker}d_1^1 \xrightarrow{\mu_2} G_1
    \end{array}
\end{array} \right) \]
is a crossed square. Here the $h$-map is 
\[ h : \text{Ker}d_0^1 \times \text{Ker}d_1^1 \to NG_2/d_3^2(NG_3) \]
given by $h(x, y) = [s_1x, s_1y s_0y^{-1}] \partial_3NG_3$, as before.

Note if we consider the above crossed square as a vertical morphism of crossed modules, we can take its kernel and cokernel within the category of crossed modules. In the above, the morphisms in the top left hand corner are induced from $d_2$ so 
\[ \text{Ker} \left( \mu_1 : \frac{NG_2}{\partial_3NG_3} \to \text{Ker}d_1^1 \right) = \frac{NG_2 \cap \text{Ker}d_2}{\partial_3NG_3} \cong \pi_2(G) \]
whilst the other map labelled $\mu_1$ is an inclusion so has trivial kernel. Hence the kernel of this morphism of crossed modules is 
\[ \pi_2(G) \to 1. \]
The image of $\mu_2$ is closed and normal in both the groups on the bottom line and as $\text{Ker}d_0 = NG_1$ with the corresponding $\text{Im}\mu_1$ being $d_2NG_2$, the cokernel is $NG_1/\partial_2NG_2$, whilst $G_1/\text{Ker}d_0 \cong G_0$, i.e., the cokernel of $\mu_1$ is $\mathfrak{M}(G, 1)$.

In fact of course $\mu_1$ is not only a morphism of crossed modules, it is a crossed module. This means that $\pi_2(G) \to 1$ is in some sense a $\mathfrak{M}(G, 1)$-module and that $\mathfrak{M}(G, 2)$ can be thought of as a crossed extension of $\mathfrak{M}(G, 1)$ by $\pi_2(G)$.

3 Free Crossed Squares

3.1 Definitions

G. Ellis, [9], in 1993 presented the notion of a free crossed square. In this section, we recall his definition and give a construction of free crossed squares by using the second dimensional Peiffer elements and the 2-skeleton of a ‘step-by-step’ construction of a free simplicial group with given $CW$-basis. We firstly recall the definition of a free crossed square on a pair of functions $(f_2, f_3)$, as given by Ellis. We will call these crossed squares totally free.

Let $B_1, B_2$ and $B_3$ be sets. Take $F(B_1)$ to be the free group on $B_1$. Suppose given a function $f_2 : B_2 \to F(B_1)$. Let $\partial : M \to F(B_1)$ be the free pre-crossed module on $f_2$. Using the action of $F(B_1)$ on $M$ we can form the semi-direct product $M \rtimes F(B_1)$. The canonical inclusion $\mu : M \to M \rtimes F(B_1)$ given by $m \mapsto (m, 1)$ allows us to consider $M$ as a normal subgroup of $M \rtimes F(B_1)$. (Recall that any normal inclusion is a crossed module with action given by conjugation.) There is a second normal subgroup of $M \rtimes F(B_1)$ arising from $M$, namely

$$N = \{(m, \partial m^{-1}) : m \in M\} \subset M \rtimes F(B_1)$$

with inclusion denoted $\mu' : N \to M \times F(B_1)$. For $m \in M$, we let $m'$ denote the element $(m^{-1}, \partial m)$ in $N$.

Assume given a function $f_3 : B_3 \to M$, whose image lies in the kernel of the homomorphism $\partial : M \to F(B_1)$. There is then a corresponding function $f_3' : B_3 \to N$ given by $y \mapsto (f_3(y), 1)$.

Definition [9]

A crossed square,

$$\begin{array}{c}
L \rightarrow^L M \\
\downarrow^{\partial_2} \\
\downarrow^{\delta_2} \\
N \rightarrow^\mu M \rtimes F(B_1),
\end{array}$$

is totally free on the pair of functions $(f_2, f_3)$ if

(i) $(M, F(B_1), \partial)$ is the free pre-crossed module on $f_2$;
(ii) $B_3$ is a subset of $L$ with $f_3$ and $f'_3$ the restrictions of $\partial_2$ and $\partial'_2$ respectively;

(iii) for any crossed square

```
\[
\begin{array}{ccc}
L' & \xrightarrow{\tau} & M \\
\downarrow{\tau'} & & \downarrow{\mu} \\
N & \xrightarrow{\mu'} & M \rtimes F(B_1),
\end{array}
\]
```

and any function $\nu : B_3 \to L'$ satisfying $\tau \nu = f_3$, there is a unique morphism $\Phi = (\phi, 1, 1, 1)$ of crossed squares:

```
\[
\begin{array}{ccc}
L & \xrightarrow{\partial_2} & M \\
\downarrow{\phi} & & \downarrow{\mu} \\
L' & \xrightarrow{\tau} & M \\
\downarrow{\tau'} & & \downarrow{\mu} \\
N & \xrightarrow{\mu'} & M \rtimes F(B_1),
\end{array}
\]
```

such that $\phi \nu' = \nu$, where $\nu' : B_3 \to L$ is the inclusion.

We denote such a totally free crossed square by $(L, M, N, M \rtimes F(B_1))$ omitting the structural morphisms from the notation when there is no danger of confusion.

We know the free pre-crossed module on $f_2 : B_2 \to F(B_1)$ is $\partial : \langle B_2 \rangle \to F(B_1)$, where $\langle B_2 \rangle$ denotes the normal closure of $B_2$ in the free group $F(B_2 \cup s_0(B_1))$, so the function $f_3 : B_3 \to M$ ($= \langle B_2 \rangle$) is precisely the data $(B_3, f_3)$ for 2-dimensional construction data in the simplicial context, cf. [19].

We thus need to recall the 2-dimensional construction for a free simplicial group. This 2-dimensional form can be summarised by the diagram

```
\[
\begin{array}{llll}
\emptyset^{(2)} : & \dots & F(s_1s_0(B_1) \cup s_0(B_2) \cup s_1(B_2) \cup B_3) & \xrightarrow{d_0,d_1,d_2} \\
& & F(s_0(B_1) \cup B_2) & \xrightarrow{d_1,d_0} F(B_1)
\end{array}
\]
```

with the simplicial morphisms given as in [19].

### 3.2 Free crossed squares exist.

**Theorem 3.1** A totally free crossed square $(L, M, N, M \rtimes F(B_1))$ exists on the 2-dimensional construction data and is given by $\mathfrak{M}(F^{(2)}, 2)$ where $F^{(2)}$ is the 2-skeletal free simplicial group defined by the construction data.
\textbf{Proof:} Suppose given the 2-dimensional construction data for a free simplicial group, $F$, which we will take as above as the data for a totally free crossed square. We will not assume detailed knowledge of [19] so we start with $F(B_1)$ and $f_2 : B_2 \rightarrow F(B_1)$ and form $M = \langle B_2 \rangle$. This gives $\partial_1 : (B_2) \rightarrow F(X_0)$ as the free pre-crossed module on $f_2$. The semidirect product gives 

$$F(s_0(B_1) \cup B_2) \cong M \rtimes F(B_1)$$

and we can identify this with $F_1^{(2)}$. This identification also makes 

$$M \cong \text{Ker} d_0^1$$

for the $d_0^1$ of $F^{(2)}$.

Next form $N = \{(m, \partial m^{-1}) \in M \rtimes F(B_1) : m \in M \}$. As $m \in \langle B_2 \rangle$, it is a product of conjugates of elements of $B_2$ and their inverses, so writing $m = \prod (m_{\alpha_i})t_{\alpha_i}^\varepsilon(m_{\alpha_i})^{-1}$ for indices $\alpha_i$, and $\varepsilon_i = \pm 1$, we get $\partial m = \prod m_{\alpha_i}t_{\alpha_i}^\varepsilon m_{\alpha_i}^{-1}$ where $t_i = f_2(y_i)$, which is also $d_0^1(y_i)$. Thus we can identify $N$ with $\langle \{y_{s_1}d_0^1(y)^{-1} : y \in B_2 \} \rangle$, which is exactly $\text{Ker} d_1^1$.

Now $f_3 : B_3 \rightarrow \text{Ker} \partial_1 = \text{Ker}(\partial : NF_1^{(2)} \rightarrow NF_0^{(2)}) \subset \langle B_2 \rangle$. We know that this allows us to construct $F_2^{(2)}$ and hence $F_n^{(2)}$ for $n \geq 3$, and in addition that taking 

$$L = NF_2^{(2)}/\partial_3(NF_3^{(2)})$$

gives a crossed square

$$\begin{array}{ccc}
L & \stackrel{\partial}{\longrightarrow} & M \\
\downarrow & & \downarrow \mu \\
N & \stackrel{\mu'}{\longrightarrow} & F_1^{(2)}
\end{array}$$

which is $\mathfrak{M}(F^{(2)}, 2)$. We claim this is the totally free crossed square on the construction data.

At this stage it is worth noting that there seems to be no simple adjointness statement between $\mathfrak{M}(-, 2)$ and some functor that would give a quick proof of freeness. The problem is that $\mathfrak{M}(-, 2)$ seems to be an adjoint only up to some sort of coherent homotopy. To avoid this difficulty we use a more combinatorial approach involving the higher dimension Peiffer elements and the explicit description of $L$.

In [18], we analysed in general the structure of groups of boundaries such as $\partial_3(NF_3^{(2)})$. There we showed that $NF_3^{(2)}$ is normally generated by elements of the following forms:-

(i) For all $x \in NF_1^{(2)}, y \in NF_2^{(2)}$,

$$\begin{align*}
&f_{(1,0)}^{(2)}(x,y) = [s_1s_0(x), s_2(y)][s_2(y), s_2s_0(x)], \\
&f_{(2,0)}^{(1)}(x,y) = [s_2s_0(x), s_1(y)][s_1(y), s_2s_1(x)][s_2s_1(x), s_2(y)][s_2(y), s_2s_0(x)];
\end{align*}$$

(ii) For all $x \in NF_1^{(2)}, y \in NF_2^{(2)}$.
(ii) for all \( y \in NF_2^{(2)}, x \in NF_1^{(2)}, \)
\[
f_{(0),(1)}(x, y) = \langle s_0(x), s_2s_1(y) \rangle \langle s_2s_1(y), s_1(x) \rangle \langle s_2(x), s_2s_1(y) \rangle,
\]
and (iii) for all \( x, y \in NF_2^{(2)}, \)
\[
f_{(0),(2)}(x, y) = \langle s_0(x), s_1(y) \rangle \langle s_1(y), s_1(x) \rangle \langle s_2(x), s_2(y) \rangle,
\]
\[
f_{(1),(2)}(x, y) = \langle s_1(x), s_2(y) \rangle \langle s_1(y), s_2(y) \rangle \langle s_2(x), s_2(y) \rangle.
\]

Given our description of \( NF^{(2)} \) in low dimensions, it is routine to calculate normal generators of the various groups involved here in terms of \( B_1 \) and \( B_2 \). We set
\[
Z = \{ s_1(y)^{-1}s_0(y) : y \in B_2 \}.
\]
The above diagram can then be realised as
\[
\begin{array}{ccc}
J & \xrightarrow{\partial_2} & \langle B_2 \rangle \\
\downarrow{\partial_2} & & \downarrow{\mu} \\
\langle Z \rangle & \xrightarrow{\mu'} & \langle B_2 \rangle \rtimes F(B_1)
\end{array}
\]
Here \( J \) is \( (\langle s_1(B_2) \cup B_3 \rangle \cap \langle Z \cup B_3 \rangle)/P_2 \), \( P_2 \) being the second dimensional Peiffer normal subgroup, which is in fact just \( \partial_3(NF_2^{(2)}) \), and which is a subgroup of \( \langle s_1(B_2) \cup B_3 \rangle \cap \langle Z \cup B_3 \rangle \).

Given any crossed square \((L', M, N, M \rtimes F(B_1))\) and a function \( \nu : B_3 \rightarrow L' \), there then exists a unique morphism
\[
\phi : (L, M, N, M \rtimes F(B_1)) \rightarrow (L', M, N, M \rtimes F(B_1))
\]
given by
\[
\phi(y_i^\prime P_2) = \nu(y_i^\prime)
\]
such that \( \phi \nu' = \nu \). The existence of \( \phi \) follows by using the freeness property of the group \( NF_2^{(2)} \) and then restricting to \( \langle s_1(B_2) \cup B_3 \rangle \cap \langle Z \cup B_3 \rangle \). The normal generating elements of \( P_2 \) are then easily shown to have trivial image in \( L' \) as that group is part of the second crossed square.

Thus the diagram is the desired totally free crossed square on the 2-dimensional construction data. The crossed square properties of \((L, M, N, M \rtimes F(B_1))\) may be easily verified or derived from the fact that this is exactly \( \mathfrak{F}(F^{(2)}, 2) \).

\( \square \)

**Remark:**

At this stage, it is important to note that nowhere in the argument was use made of the freeness of the 1-skeleton. If \( G \) is any 1-skeletal simplicial group and we form a new simplicial group \( H \) by adding in a set \( B_3 \) of new
generators in dimension 2, so that for instance, $H_2 = G_2 \ast F(B_3)$, then we can use $M = NG_1 = \text{Ker}d_0^{G,1}$ as before even though it need not be free. The corresponding $N$ is then isomorphic to $\text{Ker}d_1^{G,1}$ with the bottom right hand corner being $G_1$. The ‘construction data’ is now replaced by data for killing some elements of $\pi_1(G)$, specified by $f_3 : B_3 \to M$. Although slightly at variance with the terminology used by Ellis, [9], we felt it sensible to introduce the term “totally free crossed square” for the type of free crossed square constructed in the above theorem, using “free crossed square” for the more general situation in which $(M, G, \partial)$ and $f_3$ are specified and no requirement on $(M, G, \partial)$ to be a free precrossed module is made.

3.3 The $n$-type of the $k$-skeleton

As in the other papers in this series, we will use the ‘step-by-step’ construction of a free simplicial group to observe the way in which the models react to the various steps of the construction.

In a ‘step-by-step’ construction of a free simplicial group, there are simplicial inclusions

$$F^{(0)} \subseteq F^{(1)} \subseteq F^{(2)} \ldots$$

In general, considering the functor, $\mathcal{M}(\cdot, n)$, from the category of simplicial groups to that of crossed $n$-cubes, gives the corresponding morphisms

$$\mathcal{M}(F^{(0)}, n) \to \mathcal{M}(F^{(1)}, n) \to \mathcal{M}(F^{(2)}, n) \to \ldots \to \mathcal{M}(F, n).$$

We will investigate $\mathcal{M}(F^{(i)}, n)$, for $n = 0, 1, 2$, and varying $i$.

Firstly look at $\mathcal{M}(F^{(0)}, n)$, where the 0-skeleton $F^{(0)}$ can be thought of as simplifying to

$$F^{(0)} : \ldots \to F(B_1) \to F(B_1) \to F(B_1)$$

with the $d_i^n = s_j^n = \text{identity homomorphism on } F(B_1)$.

For $n = 0$, there is an equality

$$\mathcal{M}(F^{(0)}, 0) = F_0^{(0)}/d_2F_2^{(0)} = F(B_1),$$

and so $\mathcal{M}(F^{(0)}, 0)$ is just the free group of 0-simplices of $F$.

For $n = 1$, $\mathcal{M}(F^{(0)}, 1)$ is $NF_1^{(0)}/\partial_2NF_2^{(0)} \to F_0$. It is easy to show that $NF_1^{(0)}/\partial_2NF_2^{(0)}$ is trivial and hence

$$\mathcal{M}(F^{(0)}, 1) \cong (1 \to F(B_1)).$$
For \( n = 2 \), \( \mathfrak{M}(F^{[0]}, 2) \) is the trivial crossed square

\[
\begin{array}{ccc}
NF_2/d_3^2(NF_3) & \longrightarrow & \text{Ker } d_0^1 \\
\downarrow & & \downarrow \\
\text{Ker } d_1^1 & \longrightarrow & F_1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
\]

Next look at \( \mathfrak{M}(F^{(1)}, n) \) and recall that the 1-skeleton \( F^{(1)} \) is

\[
F^{(1)} : \quad \ldots F(s_1 s_0(B_1) \cup s_0(B_2) \cup s_1(B_2)) \xrightarrow{d_0, d_1, d_2} F(s_0(B_1) \cup B_2) \xrightarrow{d_1, d_0} F(B_1).
\]

For \( n = 0 \), \( \mathfrak{M}(F^{(1)}, 0) \) is \( F^{(1)} / d_1(\text{Ker } d_0) \cong F(B_1) / \partial_1 NF_1 \), which is \( \pi_0(F^{(1)}) \cong \pi_0(F) \).

For \( n = 1 \), we have that

\[
\mathfrak{M}(F^{(1)}, 1) = (NF_1 / \partial_2 NF_2 \longrightarrow F_0),
\]

\[
= \langle B_2 \rangle / P_1 \longrightarrow F(B_1),
\]

which is a free crossed module. In fact this is the free crossed module on the (generalised) presentation \( (B_1; B_2, f_2) \). As pointed out in [2], it is often convenient to generalise the notion of a presentation \( (X, R) \) with

\[ R \subset F(X) \]

to one with the map \( R \to F(X) \) specified and not necessarily monic. Thus if \( f_2 \) is injective, this is just a presentation \( P \) of \( \pi_1(F) \). The kernel of this crossed module is then the module of identities of \( P \), again see [2].

For \( n = 2 \), \( NF_2^{(1)} = \langle s_1(B_2) \rangle \cap \langle Z \rangle \), so \( \mathfrak{M}(F^{(1)}, 2) \) simplifies to give (up to isomorphism),

\[
\begin{array}{ccc}
NF_2/d_3^3(NF_3) & \longrightarrow & \text{Ker } d_0^1 \\
\downarrow & & \downarrow \\
\text{Ker } d_1^1 & \longrightarrow & G_1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
\]

\[
\begin{array}{ccc}
J & \longrightarrow & \langle B_2 \rangle \\
\downarrow & & \downarrow \\
\langle Z \rangle & \longrightarrow & F(s_0(B_1) \cup B_2)
\end{array}
\]

which is a crossed square with \( J = (\langle s_1(B_2) \rangle \cap \langle Z \rangle) / P_2 \).

Next look at \( \mathfrak{M}(F^{(2)}, n) \). Recall the 2-skeleton \( F^{(2)} \) is

\[
F^{(2)} : \quad \ldots F(s_1 s_0(B_1) \cup s_0(B_2) \cup s_1(B_2) \cup B_3) \xrightarrow{d_0, d_1, d_2} F(s_0(B_1) \cup B_2) \xrightarrow{d_1, d_0} F(B_1).
\]
The following can be easily obtained by direct calculation:

for \( n = 0 \),

\[
\mathcal{M}(F(2), 0) = F_0/d_1(Ker d_0) \cong \pi_0(F(2)) = \mathcal{M}(F(1), 0);
\]

for \( n = 1 \),

\[
\mathcal{M}(F(2), 1) \cong \langle B_2 \rangle / P_1 \rightarrow F(B_1).
\]

Finally, let \( n = 2 \). By an earlier result of this section, \( \mathcal{M}(F(2), 2) \) corresponds to the free crossed square,

\[
NF_2/d_3^3(NF_3) \rightarrow Ker d_0^1 \rightarrow F_1 \rightarrow \langle Z_2 \rangle \rightarrow F(s_0(B_1) \cup B_2)
\]

where \( J \) is now \( (\langle s_1(B_2) \cup B_3 \rangle \cap \langle Z \cup B_3 \rangle)/P_2 \) and \( \langle Z_2 \rangle \) is \( \langle Z \cup B_3 \rangle \), so this reduces to the earlier case if \( B_3 \) is empty. Thus we have the following relations

\[
\mathcal{M}(F(2), 0) = \mathcal{M}(F(1), 0), \quad \mathcal{M}(F(2), 1) = \mathcal{M}(F(1), 1)
\]

but \( \mathcal{M}(F(2), 2) \) and \( \mathcal{M}(F(3), 2) \) need not be the same due to the additional influence of \( B_3 \). Of course it is clear that, in general:

\[
\mathcal{M}(F(i), n) = \mathcal{M}(F(i+1), n) \quad \text{if} \quad i \geq n + 1.
\]

4 Squared Complexes

The authors and Z. Arvasi have defined \( n \)-crossed complexes in [4]. In this paper, we will only need the case \( n = 2 \), which had already been defined by Ellis in [3]. We shall follow him in calling these \textit{squared complexes}. A squared complex consists of a diagram of group homomorphisms

\[
\cdots \rightarrow C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} L \xrightarrow{\lambda} M \xrightarrow{\mu} P \xrightarrow{\lambda'} N \xrightarrow{\mu'} \cdots
\]

together with actions of \( P \) on \( L, N, M \) and \( C_i \) for \( i \geq 3 \), and a function \( h : M \times N \rightarrow L \). The following axioms need to be satisfied.
(i) The square \[
\begin{pmatrix}
\lambda & N \\
M & \mu' & P
\end{pmatrix}
\] is a crossed square;

(ii) The group \(C_n\) is abelian for \(n \geq 3\);

(iii) The boundary homomorphisms satisfy \(\partial_n \partial_{n+1} = 1\) for \(n \geq 3\), and \(\partial_3(C_3)\) lies in the intersection \(\ker \lambda \cap \ker \lambda'\);

(iv) The action of \(P\) on \(C_n\) for \(n \geq 3\) is such that \(\mu M\) and \(\mu' N\) act trivially. Thus each \(C_n\) is a \(\pi_0\)-module with \(\pi_0 = P/\mu M\mu' N\);

(v) The homomorphisms \(\partial_n\) are \(\pi_0\)-module homomorphisms for \(n \geq 3\).

This last condition does make sense since the axioms for crossed squares imply that \(\ker \mu' \cap \ker \mu\) is a \(\pi_0\)-module.

A morphism of squared complexes 

\[
\Phi : \left(C_\ast, \begin{pmatrix}
\lambda & N \\
M & \mu' & P
\end{pmatrix}\right) \longrightarrow \left(C'_\ast, \begin{pmatrix}
\lambda' & N' \\
M' & \mu' & P'
\end{pmatrix}\right)
\]

consists of a morphism of crossed squares \((\Phi_L, \Phi_N, \Phi_M, \Phi_P)\), together with a family of equivariant homomorphisms \(\Phi_n\) for \(n \geq 3\) satisfying \(\Phi_L \partial_3 = \partial_3' \Phi_L\) and \(\Phi_{n-1} \partial_n = \partial_n' \Phi_n\) for \(n \geq 4\). There is clearly a category \(\mathcal{SqComp}\) of squared complexes. This exists in both group and groupoid based versions.

By a (totally) free squared complex, we will mean one in which the crossed square is (totally) free, and in which each \(C_n\) is free as a \(\pi_0\)-module for \(i \geq 3\).

**Proposition 4.1** There is a functor 

\[
\mathcal{C}(\cdot, 2) : \mathcal{SimpGrp} \longrightarrow \mathcal{SqComp}
\]

such that free simplicial groups are sent to totally free squared complexes.

**Proof:**

Let \(G\) be a simplicial group or groupoid. We will define a squared complex \(\mathcal{C}(G, 2)\) by specifying \(\mathcal{C}(G, 2)_A\) for each \(A \subseteq \langle 2 \rangle\) and for \(n \geq 3\), \(\mathcal{C}(G, 2)_n\). As usual, (cf. the other papers in this series, [17, 18, 19, 20]), we will denote by \(D_n\) the subgroup or subgroupoid of \(NG_n\) generated by the degenerate elements.

For \(A \subseteq \langle 2 \rangle\), we define 

\[
\mathcal{C}(G, 2)_A = \mathfrak{M}(\mathfrak{s}^2 G, 2)_A = \frac{\cap \{\text{Ker} d^2_i : i \in A\}}{d_3(\text{Ker} d^3_0 \cap \cap \{\text{Ker} d^3_{i+1} : i \in A\} \cap D_3)}.
\]

We do not need to define \(\mu_i\) and the \(h\)-maps relative to these groups as they are already defined in the crossed square \(\mathfrak{M}(\mathfrak{s}^2 G, 2)\).

For \(n \geq 3\), we set 

\[
\mathcal{C}(G, 2)_n = \frac{NG_n}{(NG_n \cap D_n) d_{n+1}(NG_{n+1} \cap D_{n+1})}.
\]
As this is part of the crossed complex associated to \( G \), we can take the structure maps to be those of that crossed complex, cf. [8, 19]. The terms are all modules over the corresponding \( \pi_0 \) as is easily checked. The final missing piece, \( \partial_3 \), of the structure is induced by the differential \( \partial_3 \) of \( NG \).

The axioms for a squared complex can now be verified using the known results for crossed squares and for crossed complexes with a direct verification of those axioms relating to the interaction of the two parts of the structure, much as in [8] and [19].

Now suppose the simplicial group is free. The proof above of the freeness of \( M(\text{sk}_2 G, 2) \) together with the freeness of the crossed complex of a free simplicial group, [19], now completes the proof. \( \square \)

Suppose that \( \rho \) is a general squared complex. The homotopy groups \( \pi_n(\rho) \), \( n \geq 0 \) of \( \rho \) are defined cf. [9], to be the homology groups of the complex

\[
\cdots \rightarrow C_4 \xrightarrow{\partial_3} C_3 \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M \times N \xrightarrow{\partial_0} P \rightarrow 1
\]

with \( \partial_2(l) = (\lambda l^{-1}, \lambda l) \) and \( \partial_1(m, n) = \mu(m)\mu'(n) \). The axioms of a crossed square guarantee that \( \partial_2 \) and \( \partial_1 \) are homomorphisms with \( \partial_3(C_3) \) normal in \( \text{Ker}(\partial_2) \), \( \partial_2(L) \) normal in \( \text{Ker}(\partial_1) \), and \( \partial_1(M \times N) \) normal in \( P \).

**Proposition 4.2** The homotopy groups of \( C(G, 2) \) are isomorphic to those of \( G \) itself.

**Proof:** Again this is a consequence of well-known results on the two parts of the structure. \( \square \)

### 5 Alternative Descriptions of Freeness.

In the context of CW-complexes, Ellis, [9] gave a neat description of the top group \( L \) in a (totally) free crossed square derived from that data. A simplicial group with a given CW-basis is the algebraic analogue of a CW-complex so one would expect a similar result to hold in that setting. Ellis uses the generalised van Kampen theorem of Brown and Loday, [3]. In the algebraic setting no such tool is available, but in fact its use is not needed.

Ellis’ description is in terms of tensor products and coproducts. For completeness we recall the background definitions of these constructions.

#### 5.1 Tensor Products

Suppose that \( \mu : M \rightarrow P \) and \( \nu : N \rightarrow P \) are crossed modules over \( P \). The groups \( M \) and \( N \) act on each other, and themselves, via the action of \( P \).
The tensor product $M \otimes N$ is the group generated by the symbols $m \otimes n$ for $m \in M$, $n \in N$ subject to the relations

\[ mm' \otimes n = (^mn' \otimes m)(m \otimes n), \]
\[ m \otimes nn' = (m \otimes n)(^mnn'), \]

for $m, m' \in M$, $n, n' \in N$. There are homomorphisms $\lambda : M \otimes N \to M$, $\lambda' : M \otimes N \to N$ defined on generators by $\lambda(m \otimes n) = m(^mn)^{-1}$ and $\lambda'(m \otimes n) = (^mn)n^{-1}$. The group $P$ acts on $M \otimes N$ by $^p(m \otimes n) = (^pm \otimes ^pn)$, and there is a function $h : M \times N \to M \otimes N$, $(m, n) \mapsto m \otimes n$. In [3], it is verified that this structure gives a crossed square

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\lambda} & N \\
\downarrow{\lambda'} & & \downarrow{\nu} \\
M & \xrightarrow{\mu} & P
\end{array}
\]

with the universal property of extending the corner

\[
\begin{array}{ccc}
N & & \\
\downarrow{\nu} & & \\
M & \xrightarrow{\mu} & P
\end{array}
\]

### 5.2 Coproducts

Let $(M, P, \partial_1), (N, P, \partial_2)$ be $P$-crossed modules. Then $N$ acts on $M$, and $M$ acts on $N$, via the given actions of $P$. Let $M \rtimes N$ denote the semidirect product with the multiplication given by

\[(m, n)(m', n') = (mm', ^mn'n')\]

and injections

\[
i' : M \to M \rtimes N \quad \text{and} \quad j' : N \to M \rtimes N
m \mapsto (m, 1) \quad \text{and} \quad n \mapsto (1, n).
\]

We define the pre-crossed module

\[
\delta : M \rtimes N \to P
(m, n) \mapsto \partial_1(m)\partial_2(n).
\]

Let \{M, N\} be the subgroup of $M \rtimes N$ generated by the elements of the form

\[(^mm^{-1}, ^nn^{-1})\]
for all $m \in M$, $n \in N$, thus we are able to form the quotient group $M \rtimes N/{M, N}$ and obtain an induced morphism

$$\partial : M \rtimes N/{M, N} \to P$$

given by

$$\partial(m, n){M, N} = \partial_1(m)\partial_2(n).$$

Let $q : M \ltimes N \to M \rtimes N/{M, N}$ be projection and let $i = q_i'$, $j = q_j'$. Then $M \circ N = (M \times N)/\{M, N\}$ with the morphisms $i, j$, is a coproduct of $(M, P, \partial_1)$ and $(N, P, \partial_2)$ in the category of $P$-crossed modules.

**Proposition 5.1** [9] Let $(L, M, \bar{M}, M \rtimes F)$ be a (totally) free crossed square on the 2-dimensional construction data or on functions $(f_2, f_3)$ as described above. Let $\partial : C \to M \rtimes F$ be the free crossed module on the function $B_3 \to M \rtimes F$ given by $y \mapsto (f_3y, 1)$. From the crossed module $M \otimes \bar{M} \to M \rtimes F$, then $L$ is isomorphic to the coproduct $(M \otimes \bar{M}) \circ C$ factored by the relations

1. $i(\partial c \otimes \bar{m}) = j(c)j^{(m,c^{-1})}$
2. $i(m \otimes \partial c) = j^{(m,c)}j^{(c^{-1})}$

for $c \in C$, $m \in M$ and $\bar{m} \in \bar{M}$.

The homomorphisms $L \to M$, $L \to \bar{M}$ are given by the homomorphisms

$$\lambda : M \otimes M \to M \quad \text{and} \quad \lambda' : M \otimes \bar{M} \to \bar{M}$$

and $\partial : C \to M \cap \bar{M}$. The $h$-map of the crossed square is given by

$$h(m, \bar{n}) = i(m \otimes \bar{n})$$

for $m, n \in M$.

**Proof:** This comes by direct verification using the universal properties of tensors and coproducts.

**Remark:** For future applications it is again important to note that the result is not dependent on the crossed square being totally free, although this is the form proved and used by Ellis, [9]. If $M \to F$ is any pre-crossed module, one can form the ‘corner’

$$\xymatrix{ M \ar[d] & \\
\bar{M} \ar[r] & M \rtimes F,}$$

complete it to a crossed square via $M \otimes \bar{M}$ and then add in $B_3 \to M$. Nowhere does this use freeness of $M \to F$. 

20
Corollary 5.2 Let \( G^{(1)} \) be the 1-skeleton of a simplicial group. Then in the free crossed square \( \mathcal{M}(G^{(1)}, 2) \) described above,
\[
NG_2^{(1)}/\partial_3NG_3^{(1)} \cong \text{Kerd}_1 \otimes \text{Kerd}_1.
\]

Proof: This is clear from the previous proposition. \( \square \)

Remark: If we set \( M = \text{Kerd}_0 \), then the identification given by the Corollary gives
\[
NG_2^{(1)}/\partial_3NG_3^{(1)} \cong M \otimes \bar{M}.
\]
This uses the fact that \( \text{Kerd}_0 \) and \( \text{Kerd}_1 \) are linked via the map sending \( m \) to \( ms_0d_1m^{-1} \) for \( m \in \text{Kerd}_0 \). The h-map \( h : M \times M \rightarrow NG_2^{(1)}/\partial_3NG_3^{(1)} \) is
\[
h(x, y) = [s_1x, s_1ys_0y^{-1}]d_3^2NG_3^{(1)},
\]
but this is also \( h(x, y) = x \otimes y \). Thus
\[
x \otimes y = [s_1x, s_1ys_0y^{-1}]d_3^2NG_3^{(1)}
\]
under the identification via the isomorphism of 5.2.

This explains the ‘mysterious’ formula of [17] in the discussion before Proposition 4.6 of that paper.

5.3 Applications to 2-crossed complexes.

Of course there are similar results for free squared complexes. What is less obvious is the way in which these results can be applied to the situation that we studied in our earlier paper, [20]. There we considered the alternative model for 3-types given by Conduché’s 2-crossed modules and also looked at the corresponding 2-crossed complexes. We will not repeat all that discussion here but note the definition:

Definition: A 2-crossed complex of group(oid)s is a sequence of group(oid)s
\[
C : \ldots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \ldots C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0
\]
in which
(i) \( C_n \) is abelian for \( n \geq 3 \);
(ii) \( C_0 \) acts on \( C_n, n \geq 1 \), the action of \( \partial C_1 \) being trivial on \( C_n \) for \( n \geq 3 \);
(iii) each \( \partial_n \) is a \( C_0 \)-group(oid) homomorphism and \( \partial_i\partial_{i+1} = 1 \) for all \( i \geq 1 \); and
(iv) \( C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \) is a 2-crossed module.

We refer the reader to [5] or [20] for the exact meaning of 2-crossed module.

Given a simplicial group or groupoid, \( G \), define
\[
C_n = \begin{cases} 
NG_n & \text{for } n = 0, 1 \\
NG_2/d_3(NG_3 \cap D_3) & \text{for } n = 2 \\
NG_n/(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1}) & \text{for } n \geq 3 
\end{cases}
\]
with $\partial_n$ induced by the differential of $NG$. Note that the bottom three terms (for $n = 0, 1,$ and 2) form a 2-crossed module considered in [11] or [20] and that for $n \geq 3$, the groups are all $\pi_0(G)$-modules, since in these dimensions $C_n$ is the same as the corresponding crossed complex term (cf. Ehlers and Porter [8] for instance).

**Proposition 5.3** [20]

With the above structure $(C_n, \partial_n)$ is a 2-crossed complex, which will be denoted $C(G)$.

Here we note in particular that the term $C_2$ is $NG_2/d_3(NG_3 \cap D_3)$ and so is the same as $C(G, 2)_{<2>}$. Thus if $G$ is a simplicial group, we obtain gratis:

**Corollary 5.4** Let $G^{(1)}$ be the 1-skeleton of a simplicial group. The 2-crossed complex of $G^{(1)}$ satisfies

$$C(G^{(1)})_2 \cong \text{Ker}d_1 \otimes \text{Ker}d_0.$$

We also get in general a description of $C(G^{(2)})_2$ as a quotient of the form $(\text{Ker}d_1 \otimes \text{Ker}d_0 \circ C)/ \sim$ where as in Proposition 5.1, this $C$ is a free crossed module on the ‘new cells’ in dimension 2.

### 5.4 The suspension of a $K(\pi, 1)$.

As was mentioned in [19], Brown and Loday used their generalised van Kampen Theorem, [3], to calculate $\pi_3\Sigma K(\pi, 1)$ for $\pi$ a group, as the kernel of the commutator map from $\pi \otimes \pi$ to $\pi$. Jie Wu, ([22] Theorem 5.9), for any group $\pi$ and set of generators $\{x_\alpha | \alpha \in J\}$ for $\pi$, gives a presentation of $\pi_n\Sigma K(\pi, 1)$ in terms of higher commutators, but does not manage to get the Brown-Loday result explicitly although his result is clearly linked to theirs.

Wu’s methods use a study of simplicial groups and a construction he ascribes to Carlsson, [4]. This gives a simplicial group $F^\pi(S^1)$ that has $\pi_{n+2}\Sigma K(\pi, 1) \cong \Omega \Sigma K(\pi, 1) \cong \pi^{n+1}F^\pi(S^1)$. As we pointed out in [17], $F^\pi(S^1)$ is a pointed analogue of the ‘tensorisation’ of $K(\pi, 0)$, the constant simplicial group on $\pi$, with the simplicial circle $S^1$. In general if $G$ is a simplicial group and $K$ a pointed simplicial set, $G \wedge K$ will denote the simplicial group with group of $n$-simplices given by

$$\prod_{x \in K_n} (G_n)_x/(G_n)_*.$$

If $x \in K_n$, we denote the $x$-indexed copy of $g \in G_n$ within $(G \wedge K)_n$ by $g \wedge x$. The face and degeneracy maps of $G \wedge K$ are induced by the componentwise application of the corresponding morphisms of $G$ and $K$

$$d_i(g \wedge x) = d_i^G g \wedge d_i^K x,$$
s_i(g\wedge x) = s_i^G g\wedge s_i^K x.

Of course if d^K_i x = * then d_i(g\wedge x) = 1.

The case of interest to us is G = K(\pi, 0), K = S^1 and we will adopt the notation for simplices in S^1 used by us in [17]. We write S^1_0 = \{*\} and will take * to denote the corresponding degenerate n-simplex basing S^1_n in all dimensions; S^1_1 = \{\sigma, *\}, S^1_2 = \{x_0, x_1, *\}, where x_0 = s_1\sigma, x_1 = s_0\sigma and in general S^1_{n+1} = \{x_0, \ldots, x_n, *\}, where x_i = s_n \ldots s_{i+1}s_{i-1} \ldots s_0\sigma, 0 \leq i \leq n.

We write G = K(\pi, 0) for simplicity and will usually make no distinction between simplices in different dimensions unless confusion might arise. We have

\[(G\wedge S^1)_0 = 1, \quad \text{the trivial group,}\]
\[(G\wedge S^1)_1 \cong \pi, \quad \text{the free product of two copies of } \pi, \text{ and so on.}\]

The group (G\wedge S^1)_n is a free product of n-copies of \pi, \coprod{\{(\pi)_x : x \in S^1_n \setminus \{\ast\}\}}, and writing as above g\wedge x for the x-indexed copy of g \in \pi in this, we note that (g\wedge x)(g'\wedge x) = (gg'\wedge x) for g, g' \in \pi. As g\wedge x^{(n+1)}_i = s_n(g\wedge x^{(n)}_i) holds in all dimensions, n \geq 2 and for all 0 \leq i \leq n, it is clear that N(G\wedge S^1)_n = D_n, that is, it is generated by degenerate elements in all dimensions n \geq 2, we can therefore apply Corollary 5.2. As N(G\wedge S^1)_0 is trivial, Ker d^1_0 = Ker d^1_1 = (G\wedge S^1)_1 \cong \pi, so we get:

For \(H = G\wedge S^1\),

\(NH_2/\partial_3 NH_3 \cong \pi \otimes \pi.\)

We have by [21] that the algebraic 2-type of H is completely modelled by the crossed square \(\mathfrak{M}(H, 2)\), that is by

\[
\pi \otimes \pi \xrightarrow{\mu_2} \pi \]
\[
\pi \xrightarrow{\mu_1} \pi \]

where \(\mu_1\) and \(\mu_2\) are the commutator maps.

As a consequence we have:

**Corollary 5.5** The 3-type of \(\Sigma K(\pi, 1)\) is completely specified by the above crossed square. In particular there is an isomorphism

\[\pi_3(\Sigma K(\pi, 1)) \cong \text{Ker}(\mu : \pi \otimes \pi \to \pi).\]

\(\square\)

This result was first found by Brown and Loday [3]. Their proof was an illustration of the use of their generalised van Kampen Theorem. Jie Wu, [22], gives some methods that shed light on the higher homotopy groups, but although they yield a description of \(\pi_4\), they do not analyse the 4-type itself.

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The model $G\bar{\Lambda}S^1$ is 1-skeletal and one might expect that a triple tensor $\pi \otimes \pi \otimes \pi$ may be involved in any model of its 4-type. Of course $\mathfrak{M}(H, 3)$ gives a complete model, but the individual terms involved in that model are not as easy to analyse as in $\mathfrak{M}(H, 2)$. An amalgam of Wu's methods and the methods developed in the earlier papers of this series, [17, 18, 19, 20], might provide insight into this. This problem is not of itself that important, but it does seem to provide an excellent testbed for the development of methods to aid in calculation with low dimensional algebraic models of homotopy types.

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