ORIGINAL PROOFS OF STIRLING’S SERIES FOR \( \log(n!) \)

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Abstract. This is a transcription into modern notations of the derivation by Stirling and De Moivre of an asymptotic series for \( \log(n!) \), usually called Stirling’s series. The previous discovery by Wallis of an infinite product for \( \pi \), and later results on the divergence of the series are also presented. We conclude that James Stirling has priority over Abraham de Moivre for Stirling’s formula and Stirling’s series.

1. Origin of Stirling’s series

In order to evaluate the central binomial coefficient \( \binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}} \) for his research in probability theory \([9]\), Abraham De Moivre needed a formula for \( \log n! \) to replace the table of 14 figure logarithms of factorials from 10! to 900! by differences of 10 printed in his new book “Miscellanea Analytica” \([10]\). A “few days” after publication in 1730, James Stirling sent him a letter pointing out numerical errors in that table past the fifth decimal making it “unsuitable for research”, and stated without proof a series for \( \log n! \), involving the variable \( z = n + 1/2 \), the constant \( \log \sqrt{2\pi} \), and certain rational coefficients defined by a triangular system of linear equations:

\[
\log n! = z \log z - z + \log \sqrt{2\pi} - \frac{1}{2 \times 12z} + \frac{7}{8 \times 360z^3} - \frac{31}{32 \times 1260z^5} + \ldots
\]

De Moivre saw immediately as factors in these coefficients the Bernoulli numbers \( B_{2k} \) that he knew \((12 = 1 \cdot 2 \cdot 6, 360 = 3 \cdot 4 \cdot 30, 1260 = 5 \cdot 6 \cdot 42)\) and proceeded to derive independently a closely related series involving more simply the variable \( n \) itself:

\[
\log n! = \left( n + \frac{1}{2} \right) \log n - n + 1 + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left( \frac{1}{n^{2k-1}} - 1 \right).
\]

Stirling gave the proof of (1) in his book “Methodus Differentialis” that same year \([32]\). De Moivre later published the very different proof of (2) through a 22 page “Supplementum” \([11]\) to “Miscellanea Analytica”, giving due credit to Stirling by quoting explicitly (1), and even appending it in 1756 to his book “Doctrine of chances” \([9]\), along with a corrected table of logarithms (see below).

Both series are usually called Stirling’s series, but the original one by Stirling has an explicit constant and can be used, while the simpler modified one by De Moivre involves a constant expression whose value must still be determined by other means, for example from Wallis infinite product.

In summary, the priority for the discovery of the series and the determination of the constant \( \log \sqrt{2\pi} \) was given to Stirling by De Moivre himself, while the modifications in its expression and in its proof from the recognition of Bernoulli numbers must be attributed to De Moivre.

Unfortunately, about half of the known copies of the “Miscellanea Analytica” lack the “Supplementum” (translated in \([19]\)), and some publications or Wikipedia web pages unaware of its contents attribute incorrectly to De Moivre the discovery of the series, leaving only the determination of the constant to Stirling’s credit \([23, 7]\). However, this does not diminish the importance of De Moivre in the early history of statistics, which is now well documented and accepted \([27]\).

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2. Derivation of the simpler series by De Moivre

First, \( \log n! \) is cleverly expressed as a linear combination of \( n \) simpler logarithms.

\[
\log n! = \log n + \log (n^{n-1}) + \log(n-1)! - \log (n^{n-1})
\]

\[
= \log n + (n-1) \log n + \log \prod_{k=1}^{n-1} \left( \frac{n-k}{n} \right)
\]

\[
= n \log n + \sum_{k=1}^{n-1} \log \left( 1 - \frac{k}{n} \right).
\]

Next, each logarithm in the finite sum of \( n-1 \) logarithms is replaced by an infinite Taylor series.

\[
- \log \left( 1 - \frac{k}{n} \right) = \sum_{p=1}^{\infty} \frac{1}{p} \left( \frac{k}{n} \right)^p = \sum_{p=1}^{\infty} \frac{1}{p} \frac{n^p}{p^p}. \]

Summing this on \( k \) produces a double summation involving, after permutation, sums of powers of the first integers for which James Bernoulli had published a nice explicit formula earlier in 1713 involving the factors \( B_k = 1, \frac{1}{2}, \frac{1}{6}, -\frac{1}{30}, 0, -\frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{1}{66}, \ldots \) that De Moivre called “Bernoulli numbers”. After extracting the first two terms, De Moivre inverted the order of summation yet again and derived as examples the first four terms of the infinite series. We also set with him \( x = (n-1)/n = 1 - 1/n \) but, after inserting one term, modify one index in order to proceed along diagonals and thus obtain the general expression.

\[
\sum_{p=1}^{\infty} \frac{1}{p} \sum_{k=1}^{n-1} k^p = \sum_{p=1}^{\infty} \frac{1}{p} \sum_{k=1}^{n-1} \left( \frac{p+1}{k} \right) B_k(n-1)^{p+1-k}
\]

\[
= \sum_{p=1}^{\infty} \left( \frac{n x^{p+1}}{p(p+1)} + \frac{x^p}{2p} \right) + \sum_{p=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{p(p+1)} \left( \frac{p+1}{k} \right) B_k \frac{x^{p+1-k}}{n^{k-1}}
\]

\[
= \int_0^x \sum_{p=1}^{\infty} \left( \frac{n x^p}{p} + \frac{x^{p-1}}{2} \right) dt + \sum_{k=2}^{\infty} B_k \frac{x^k}{n^{k-1}} \left[ \sum_{r=0}^{\infty} \frac{(r+k)^{r+1}}{k+1} - \frac{1}{k(k-1)} \right]
\]

\[
= n - 1 - \frac{1}{2} \log n + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left( 1 - \frac{1}{n^{2k-1}} \right), \quad (n = 1, 2, 3, \ldots).
\]

Our last step uses the following elementary summation formula for a recurring series,

\[
\sum_{r=0}^{\infty} \frac{(r+k)!}{(r+k)(r+k-1)} x^r = \frac{1}{k!} \sum_{r=0}^{\infty} \frac{(r+k-2)!}{r!} x^r = \frac{1}{k!} D^{k-2} \sum_{r=0}^{\infty} x^{r+k-2}
\]

\[
= \frac{1}{k!} D^{k-2} \frac{x^{k-2}}{1-x} = \frac{1}{k!} \frac{x^{k-2} - 1}{1-x} + \frac{1}{1-x}
\]

\[
= \frac{(k-2)!}{k!} \frac{1}{(1-x)^{k-1}} = \frac{n^{k-1}}{k(k-1)}. \quad (k \geq 2).
\]

After regrouping terms we finally obtain with De Moivre, for \( n = 1, 2, 3, \ldots \),

\[
\log n! = n \log n - \left\{ n - 1 - \frac{1}{2} \log n + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left( 1 - \frac{1}{n^k} \right) \right\}
\]

\[
= \left( n + \frac{1}{2} \right) \log n - n + \left[ 1 - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \right] + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \frac{1}{n^{2k-1}}.
\]

\[\text{The sum up to } n - 1 \text{ has } B_1 = +\frac{1}{2} \text{ in terms of } n - 1, \text{ instead of the usual } B_1 = -\frac{1}{2} \text{ in terms of } n.\]
3. Quadrature of the circle by Wallis

In order to determine the value of the constant expression $1 - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)}$ in the previous equation \[3\], we will use an infinite product for $\pi$ discovered via an ingenious induction by John Wallis in 1656, before the invention of the infinitesimal calculus.

Taking limits in our now familiar Riemann sums, Wallis first computes $\int_{0}^{1} x^r \, dx$ for a few rational values of $r = p/q$, where $p$ and $q$ are small positive integers, then gives the general expression of the integral as $1/(r + 1)$. From this, he can deduce with the integral binomial theorem a few values of $\int_{0}^{1} (x^p \pm x^q)^r \, dx$ where $p, q$ are rational and $r$ is a small positive integer, and then he generalizes.

A good number of known geometrical applications are given for parabolas, hyperbolas, spirals and solids of revolution, but the real target is the “quadrature of the circle”, or the evaluation of

\[
\frac{\pi}{4} = \int_{0}^{1} (1 - x^2)^{1/2} \, dx.
\]

Without the general binomial theorem (found later by Newton after reading this \[25\]), Wallis is forced to use interpolation between other integrals already computed.

In the symmetric table of the reciprocals of $\int_{0}^{1} (1 - x^{1/q})^p \, dx$ where $p, q = 0, 1, 2, 3, 4, 5$, he detects the figurate numbers \[8\] whose values \([p+q]\) had been found earlier by Harriot and Fermat. What he is looking for corresponds to $p = q = 1/2$, so he inserts “odd” lines and columns in the table for $p, q = -1/2, 1/2, 3/2, \ldots$. Their missing entries are then found by applying multiplication rules relating the values of figurate numbers in the other “even” lines and columns: Wallis knows intuitively that some general rules are valid for all lines and columns which indeed contain $\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)}$ \[2\] p. 298.

The expanded table can thus be completely filled \[25\] \[26\], leaving a single unknown factor in the entries of the odd lines and odd columns, the ratio of the diameter squared to the circle area which is denoted by the small square “□”.

Next, Wallis looks at the entries that he had found in the line corresponding for $q = 1/2$ to the successive multiples of one half, $p = -1/2, 0/2, 1/2, 2/2, 3/2, 4/2, 5/2, 6/2, \ldots$:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
1/2 & 1 & 3/2 & 4/3 & 3 \times 5 & 4 \times 6 & 3 \times 5 \times 7 & 2 \times 4 \times 6 \\
\hline
\end{array}
\]

The ratios of an odd/even column entry to the previous odd/even column entry obviously decrease as $(2/1, 3/2, 4/3, 5/4, \ldots)$, and so should the ratios of one column to the previous one, giving

\[
\frac{1}{1+2} > \frac{1}{1} > \frac{3}{2} \quad \Rightarrow \quad \sqrt{1 + \frac{1}{1}} > \frac{3}{2} > \sqrt{1 + \frac{1}{1}}.
\]

The other columns give similar inequalities, thus completing the quadrature of the circle \[4\] :

\[
\begin{align*}
\theta_n &= \frac{4}{\pi} \cdot \frac{3 \cdot 5 \cdot 5 \cdot \cdots (2n - 1)(2n - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots (2n - 2)(2n)} \theta_n, \\
\left(1 + \frac{1}{2n-1}\right)^{1/2} &> \theta_n > \sqrt{1 + \frac{1}{2n}} n \geq 1.
\end{align*}
\]

Wallis shows in passing how to get the integer power sum formulas from the figurate numbers values via Vandermonde’s identity \[31\] p. 140, but fails to notice in them the pattern of recurring rational numbers that James Bernoulli would find fifty years later, using the same method.

Sharp criticisms from some scientists of Wallis’ time were directed at his inductive method based on informal analogies with arithmetic progressions, explained thus in Proposition 1 \[31\] p. 13 :

“The simplest method of investigation, in this and various problems that follow, is to exhibit the thing to a certain extent and to observe the ratios produced and to compare them to each other; so that at length a general proposition may become known by induction.”

He responded that this was appropriate during discovery, and we adopted his concept of limit (“the space will be less than any assigned quantity, and may therefore be taken as nothing”) \[31\] p. 145.
Now the finite partial product can be expressed by factorials [5]

\[ \frac{\pi}{2} \theta_n = \frac{1}{2n} \left[ \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \ldots \times \frac{2n}{2n-1} \right]^2 = \frac{2^{4n}}{2n} \left( \frac{(n!)^2}{(2n)!} \right)^2 \]

and we obtain by taking logarithms

(5) \[ 4 \log n! - 2 \log(2n)! = \log 2n - 4n \log 2 + \log \left( \frac{\pi}{2} \theta_n \right). \]

We also rewrite De Moivre’s formula (2) by introducing a constant \( C \) and a correction term \( \delta_n \):

(6) \[ \log n! = (n + \frac{1}{2}) \log n - n + C + \delta_n, \]

(7) \[ \log(2n)! = (2n + \frac{1}{2}) \log 2n - 2n + C + \delta_{2n}. \]

Comparing (5) to the corresponding (4, −2) linear combination of the last two equations (6) and (7) finally yields

\[ C = \lim_{n \to \infty} \left( \log \sqrt{2\pi \theta_n + \delta_{2n} - 2\delta_n} \right) = \log \sqrt{2\pi}, \]

assuming, as may be easily shown [5, 29], that the correction term \( \delta_n \) vanishes as \( n \to \infty \).

4. Stirling’s derivation of the original series

While the proof by de Moivre could be purely analytical since he knew the formula, the original proof by Stirling looks more experimental, like a sophisticated version for series [33] of the “inductive” method of pattern discovery used earlier by John Wallis for the quadrature of the circle.

The problem considered in Proposition XXVIII of “Methodus Differentialis” is the evaluation of a finite sum of logarithms in arithmetic progression of step \( 2h \),

\[ S = \log(x + h) + \log(x + 3h) + \ldots + \log(z - h), \]

which Stirling expresses as a telescoping sum of differences of a “finite integral” \( F(z) \) (see [28]),

\[ S = [F(x + 2h) - F(x)] + [F(x + 4h) - F(x + 2h)] + \ldots + [F(z) - F(z - 2h)]. \]

Obviously, it is sufficient to choose the function so that \( F(z) - F(z - 2h) = \log(z - h) \), and Stirling simply indicates that this is satisfied by

\[ F(z) = \frac{z \log z - z}{2h} - \frac{1}{12} \frac{h}{z} + \frac{7}{360} \frac{h^3}{z^3} - \frac{31}{1260} \frac{h^5}{z^5} + \frac{127}{1680} \frac{h^7}{z^7} - \frac{511}{1188} \frac{h^9}{z^9} + \ldots . \]

The constants in the series must be computed from linear equations formed by alternate binomial coefficients of odd order:

\[ -\frac{1}{1 \times 3} = a_1, \]
\[ -\frac{1}{5 \times 8} = a_1 + 3a_3, \]
\[ -\frac{1}{7 \times 12} = a_1 + 10a_3 + 5a_5, \]
\[ -\frac{1}{9 \times 16} = a_1 + 21a_3 + 35a_5 + 7a_7, \]
\[ -\frac{1}{11 \times 20} = a_1 + 36a_3 + 126a_5 + 84a_7 + 9a_9. \]

This ends the short proof, to which a hint to a connection with an “area correction” is added.

Next, in the very important example II, Stirling indicates that the logarithm of \( n! \) can be approximated by computing three or four terms of the series in \( F(n + 1/2) \), then adding half the logarithm of the circumference of a unit circle, equal to 0.399089934179 in base 10. This constant \( \log \sqrt{2\pi} \) comes from the Wallis product that he used earlier to evaluate \( (2n)_n \) in Proposition XXIII.
Indeed, we can verify the previous equations as follows. First, since the sum of logarithms is a Riemann sum, a leading term \( \int \log z \, dz = z \log z - z \) is expected, and a correcting series with undetermined coefficients \( a_k \) can be added, giving as candidate for the telescoping finite integral

\[
F(z) = \frac{z \log z - z}{2h} + \sum_{k=0}^{\infty} a_k \frac{h^k}{z^k}.
\]

Secondly, we set \( z = x + h \) and \( t = h/x \), which eliminates \( x \) in

\[
F(z) - F(z - 2h) - \log(z - h) = F(x + h) - F(x - h) - \log x
\]

\[
= (x + h) \frac{\log x + \log(1 + h/x) - 1}{2h} + \sum_{k=0}^{\infty} a_k \frac{h^k}{(x + h)^k}
\]

\[
- (x - h) \frac{\log x + \log(1 - h/x) - 1}{2h} - \sum_{k=0}^{\infty} a_k \frac{h^k}{(x - h)^k} - \log x
\]

\[
= \frac{1}{2t} \log \frac{1 + t}{1 - t} + \frac{1}{2} \log(1 - t^2) - 1 + \sum_{k=1}^{\infty} a_k t^k \left[ \frac{1}{(1 + t)^k} - \frac{1}{(1 - t)^k} \right]
\]

\[
= \sum_{j=1}^{\infty} \left[ \frac{1}{2} - \frac{(-1)^j}{j} \right] t^j - \sum_{j=1}^{\infty} \frac{t^{2j}}{2j} + \sum_{k=1}^{\infty} a_k t^k \frac{D^{k-1}}{(k-1)!} \left[ \frac{(-1)^{k-1}}{1 + t} - \frac{1}{1 - t} \right]
\]

\[
= \sum_{j=1}^{\infty} \left[ \frac{t^{2j}}{2j + 1} - \frac{t^{2j}}{2j} \right] + \sum_{k=1}^{\infty} a_k \frac{t^k}{(k-1)!} \sum_{j=k-1}^{\infty} \frac{j!}{(j-k+1)!} \left[ (-1)^{j-k+1} - 1 \right]
\]

The right hand side will vanish identically if (for \( j = 2n \) in the second sum)

\[
0 = \sum_{k=1}^{2n} \binom{2n}{k} a_{k+1} \left[ (-1)^{2n-k-1} - 1 \right] = -2 \sum_{k=0}^{n-1} \binom{2n}{2k+1} a_{2k+2};
\]

and also if (for \( j = n \) in the first sum and \( j = 2n - 1 \) in the second sum)

\[
- \frac{1}{2n(2n+1)} = - \sum_{k=0}^{2n-1} \binom{2n-1}{k} a_{k+1} \left[ (-1)^{2n-k-1} - 1 \right] = 2 \sum_{k=0}^{n-1} \binom{2n-1}{2k} a_{2k+1}, \quad (n = 1, 2, \ldots).
\]

Clearly, as indicated by Stirling, the \( a_{2k} \) must vanish for \( k > 0 \) while the \( a_{2k+1} \) must satisfy

\[
- \frac{1}{(2n + 1)(4n)} = \sum_{k=0}^{n-1} \binom{2n-1}{2k} a_{2k+1}, \quad (n = 1, 2, \ldots).
\]

The last recurrence relation is equivalent to

\[
\frac{1}{2} + \sum_{k=1}^{n} \binom{2n+1}{2k} B_{2k} (2k - 1) = 0, \quad (n = 1, 2, \ldots).
\]

Now, as stated by James Bernoulli, with \( B_0 = 1, B_1 = -1/2, \) and \( B_{2k+1} = 0 \) for \( k > 0, \)

\[
1 - \frac{1}{2} + \sum_{k=1}^{n} \binom{2n+1}{2k} B_{2k} = \sum_{k=0}^{n} \binom{2n+1}{k} B_k = 0, \quad (n = 1, 2, \ldots).
\]

Thus, as seen by De Moivre while reading the letter of Stirling,

\[
a_{2k} = \frac{B_{2k}}{2k(2k-1)}, \quad (k = 1, 2, \ldots).
\]

The equivalence of De Moivre’s and Stirling’s versions of the series can now be shown easily.
5. Semi-convergence

De Moivre had only the six $B_{2k}$ numbers that we listed above, from his copy of Ars Conjectandi. Although he reformulated the recurrence rule given by James Bernoulli to determine more $B_{2k}$, he never computed enough terms to notice the divergence of the correcting series for $\log n!$. This was proven later in 1763 by Thomas Bayes in a posthumous letter published by the Royal Society in London:

“It has been asserted by some eminent mathematicians, that the sum of the logarithms of the numbers $1 \cdot 2 \cdot 3 \cdot 4 \&c.$ to $n$, is equal to

$$\frac{1}{2} \log C + \left( n + \frac{1}{2} \right) \log n$$

lessened by the series

$$n - \frac{1}{12n} + \frac{1}{360n^3} - \frac{1}{1260n^5} + \frac{1}{1680n^7} - \frac{1}{1188n^9} + \&c.$$ if $C$ demote the circumference of a circle whose radius is unity. (.........)

Wherefore at length the subsequent terms of this series are greater than the preceding ones, and increase in infinitum, and therefore the whole series can have no ultimate value whatsoever. Much less can that series have any ultimate value, which is deduced from it by taking $n = 1$, and is supposed to be equal to the logarithm of the square root of the periphery of a circle whose radius is unity.\(^2\)

The Reverend Bayes was careful in using the term “whole series”, since he had noted that the first terms of the series gave a good approximation, in particular when $n$ is large.

In fact the best number of terms to use has been determined to be near the integer part of $\pi n$, after which terms stop decreasing in absolute value \([21, 30, 21]\). Moreover, the rest of the series is bounded in absolute value by the first neglected term and has the same sign, which alternates from term to term following the sign of the Bernoulli numbers $B_{2k}$ \([4]\). This is an “enveloping series” whose successive partial sums encompass a specific value which can be assigned in a certain sense to the whole series. For example, George Boole writes in 1860 \(3\) p. 87:

“Hence for large values of $n$ we may assume

$$1 \cdot 2 \cdot \ldots \cdot n = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n ,$$

the ratio of the two members tending to unity as $n$ tends to infinity. And speaking generally it is with the ratios, not the actual values of functions of large numbers, that we are concerned.”

In order to obtain a formula valid according to our modern standards, it is necessary to replace the divergent infinite series by its partial sum, and find an expression for the neglected rest of the series. This is usually done first for the Euler-Maclaurin formula, and then applied generally to $\log \Gamma(z)$, where $z$ is a complex number outside the negative real axis \(18\). But various shortcuts have been found when $z$ is restricted to the positive real axis, using properties of the $\Gamma$ function and infinite series \(15\) \(16\) \(9\) \(34\) \(17\) \(24\). In \(30\), Schaar replaced the series involving the Bernoulli numbers by an integral, using the Euler partial fraction decomposition of the cotangent function,

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2x}{x^2 + 4k^2\pi^2},$$

deduced from the partial fractions of $1/(t^{2n} - 1)$ where $t = 1 + x/2n$ and $e^x = \lim(1 + x/n)^n \[1\]$. From clever manipulations of the integral for $\Gamma(a + 1)$, he finally obtains, for $a > 0$ and $m \geq 0$,

$$\log \Gamma(a + 1) = \log \sqrt{2\pi a} + a(\log a - 1) + \sum_{k=1}^{m} \frac{B_{2k}}{2k(2k - 1)} \frac{1}{a^{2k-1}}$$

$$- (-1)^m \frac{1}{\pi} \int_{0}^{\infty} \frac{x^{2m}}{1 + x^2} \log(1 - e^{-2\pi ax}) \, dx.$$\(^{2}\)

\(^{2}\)In 1755, Euler had stated the divergence in $1 - \frac{B_2}{2} - \frac{B_4}{4} - \cdots = \log \sqrt{2\pi}$ \(19\) C. VI, §157, p. 465.\]
6. Errors in the table of logarithms of \( n! \) by Abraham de Moivre

The 1756 edition of the first De Moivre book \([9]\) contains in Appendix IV on page 333 “A correct Table of the Sums of Logarithms, from the Author’s Supplement to his Miscellanea Analytica” that we compare to the original table of \( \log(10n!) \) for \( 1 \leq n \leq 90 \) on pages 103–104 in the 1730 “Miscellanea Analytica”. Spaces between digits in the following table section highlight differences which are also represented in the last column.

| \( n \) | \( \log(n!) \) (1730) | \( \log(n!) \) (1756) | \( (1730) - (1756) \) |
|---|---|---|---|
| 10 | 6.55977 303287678 | 6.55976 303287678 | +10^-9 |
| 20 | 18.38613 461687770 | 18.38612 461687770 | +10^-5 |
| 30 | 32.42367 007492572 | 32.42366 007492572 | +10^-5 |
| 40 | 47.91165 506815591 | 47.91164 506815591 | +10^-5 |
| 50 | 64.48308 487247209 | 64.48307 487247209 | +10^-5 |
| 60 | 81.92018 484939024 | 81.92017 484939024 | +10^-5 |
| 70 | 100.07841 503568004 | 100.07840 503568004 | +10^-5 |
| 80 | 118.85473 71 2249966 | 118.85472 77 2249966 | +10^-5 - 6 \cdot 10^{-7} |
| 90 | 138.17194 51 9001086 | 138.17193 57 9001086 | +10^-5 - 6 \cdot 10^{-7} |
| 100 | 157.97001 30 5471585 | 157.97000 36 5471585 | +10^-5 - 6 \cdot 10^{-7} |
| 110 | 178.20092 70 4487008 | 178.20091 76 4487008 | +10^-5 - 6 \cdot 10^{-7} |
| 120 | 198.82540 32 4721977 | 198.82539 38 4721977 | +10^-5 - 6 \cdot 10^{-7} |
| 130 | 219.81070 25 5614815 | 219.81069 31 5614815 | +10^-5 - 6 \cdot 10^{-7} |
| 140 | 241.12911 93 8698689 | 241.12910 99 8698689 | +10^-5 - 6 \cdot 10^{-7} |
| 150 | 262.75690 28 1092616 | 262.75689 34 1092616 | +10^-5 - 6 \cdot 10^{-7} |
| 160 | 284.67346 56 4068298 | 284.67345 62 4068298 | +10^-5 - 6 \cdot 10^{-7} |
| 170 | 306.66079 13 9492847 | 306.66078 19 9492847 | +10^-5 - 6 \cdot 10^{-7} |
| 180 | 329.30298 08 238 9393 | 329.30297 14 247 9393 | +10^-5 - 6 \cdot 10^{-7} - 9 \cdot 10^{-9} |
| 190 | 351.98589 92 366 3535 | 351.98588 98 339 3535 | +10^-5 - 6 \cdot 10^{-7} + 27 \cdot 10^{-10} |
| 200 | 374.89689 80 427 4044 | 374.89688 86 400 4044 | +10^-5 - 6 \cdot 10^{-7} + 27 \cdot 10^{-10} |

Clearly, the systematic errors which recur from one line to the next are not typographic errors. One can see that there was in 1730 an error of +1 unit in the fifth decimal of \( \log(10!) \), which affected all other entries. And for \( \log(80!) \), ”1” replaced a ”7” in the seventh decimal, also affecting all subsequent entries below. Then a most probable inversion of the ninth and tenth decimals (”74” instead of ”47”) in \( \log(180!) \) was also propagated to the rest of the table.

Based on this data and its title, one can speculate that De Moivre simply added the logarithms of natural numbers in sequence to compute the logarithms of the factorials, without using as a check an approximating formula, another order of additions, or sums of digits modulo 9 (“casting out the nines” \([22,6,8]\), which does not detect digit permutations).

7. Stirling’s series in “The Doctrine of Chances” by Abraham de Moivre, 1756

The 1756 edition of the first De Moivre book \([9]\) also contains on page 334 the following presentation of the original Stirling series after the corrected table of logarithms.

“If we would examine these numbers, or continue the Table farther on, we have that excellent rule communicated to the Author by Mr. James Stirling; published in his Supplement to the “Miscellanea Analytica” and by Mr. Stirling himself in his “Methodis Differentialis”, Prop. XXVIII.

Let \( z - 1/2 \) be the last term of any Series of the Natural Numbers 1, 2, 3, 4, \ldots, \( z - 1/2 \); \( a = .34328448190325 \) the reciprocal of Neper’s Logarithm of 10: Then three or four terms of this series

\[
z \log z - az - \frac{a}{2 \cdot 12 \cdot z} + \frac{7a}{8 \cdot 360 \cdot z^3} - \frac{31a}{32 \cdot 1260 \cdot z^5} + \frac{127a}{128 \cdot 1680 \cdot z^7} - &c.
\]

added to 0.399089934179 which is half the Logarithm of a Circumference whose Radius is Unity, will be the Sum of the Logarithms of the given Series; or the Logarithm of the Product \( 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times (z - 1/2) \).
The coefficients of all the terms after the first two being formed as follows. Put

\[
\begin{align*}
-\frac{1}{3 \cdot 4} &= A \\
-\frac{1}{5 \cdot 8} &= A + 3B \\
-\frac{1}{7 \cdot 12} &= A + 10B + 5C \\
-\frac{1}{9 \cdot 15} &= A + 21B + 35C + 7D \\
-\frac{1}{11 \cdot 20} &= A + 36B + 126C + 84D + 9E.
\end{align*}
\]

In which the Numbers 1, 1, 1, &c., 3, 10, 21, 36, &c., 5, 35, 126, &c. that multiply A, B, C, &c. are the alternate Unciae of the odd Powers of a Binomial.

Then the Coefficients of the several Terms will be \((1/2) \times A = -1/(2 \cdot 12), (1/2)^3 \times B = -7/(8 \cdot 360), (1/2)^5 \times C = -31/(32 \cdot 1260), &c.\) 

See the general Theorem and Demonstration in Mr. Stirling’s Proposition quoted above.

8. Conclusion

The errors in the table of logarithms were in fact a godsend, since they prompted the letter of Stirling; and by chance De Moivre then worked not only on correcting his additions but also on finding a new proof of the original formula (1). His simpler version (2) of the series is now universally adopted because so much more properties are known for the Bernoulli numbers than for the coefficients of Stirling, after the appearance in 1755 of the revolutionary “Institutiones Calculi Differentialis” by Euler [13] – another lucky break for De Moivre.

Stirling himself admitted later gracefully in a September 1730 letter to Cramer that De Moivre’s version was simpler than his own [12]. He also suggested in 1738 the formula (which must then be De Moivre’s version) to Euler as an example for his new summation method using differential quotients and Bernoulli numbers, while making him aware of the existence of the similar Maclaurin’s summation method [14].

Now De Moivre first discovered, proved geometrically, and published the trigonometric formula \((\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\). While Euler’s reformulation and proof via his own relation \(e^{i\theta} = \cos \theta + i \sin \theta\) is now universally known and preferred, nobody (certainly not Euler!) would suggest switching priority from De Moivre to Euler for that new proof and new notation.

Likewise indeed, James Stirling must keep his indisputable priority for the discovery, proof, and publication of Stirling’s series [1], including of course the explicit constant \(\log \sqrt{2\pi}\).

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