Jacobi-Maupertius metric and Kepler equation

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Abstract

This article studies the application of the Jacobi-Eisenhart lift, Jacobi metric and Maupertius transformation to the Kepler system. We start by reviewing fundamentals and the Jacobi metric. Then we study various ways to apply the lift to Kepler related systems: first as conformal description and Bohlin transformation of Hooke’s oscillator, second in contact geometry, third in Houri’s transformation \cite{13}, coupled with Milnor’s construction \cite{21} with eccentric anomaly.

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1 Introduction

The Kepler system, derived by Johannes Kepler in 1609, as interpreted by Newton is a 3-dimensional integrable system for an inverse square law force describing elliptic trajectories [9, 19]. It is related to the oscillator system via a canonical transformation known as the Bohlin transformation, resulting in many properties of the two systems being interrelated. It has many integrals of motion such as the angular momentum, the Hamiltonian and the Runge-Lenz vector. The last two translate into the equivalent conserved quantities known as the Fradkin tensors for the oscillator system under Bohlin’s transformation. Recently Kepler problem has been studied on noncommutative $\kappa$-spacetime and corresponding Bohlin-Arnold duality [11]. In particular, regularization of the Kepler problem on $\kappa$-spacetime in several different ways [12]. Regularization is a mathematical procedure to cure this singularity. A nice clear treatment of regularizing the Kepler problem was done by Moser in his 1970 paper [23], the treatment of Moser relates the Kepler flow for a fixed negative energy level to the geodesic flow on the sphere $S^n$. A lucid analysis of the geometrical aspects of Kepler problem can be found in Milnor [21]. Belbruno extended the cases of positive energy to negative energy, in correspondence to the 3-hyperboloid $H^3$, and zero energy which corresponds to 3-dimensional Euclidean space [2].

The Jacobi-Maupertius (JM) metric is a projection of the action functional onto a fixed energy surface, reducing the problem to a spatial geodesic [28]. In other words, the Jacobi-Maupertius metric reformulates Newton’s equations as geodesic equations for a Riemannian metric which degenerate at the Hill boundary [22]. An important application to gravity was shown [26] by Ong who studied the curvature of the the Jacobi metric for the Newtonian $n$-body problem. For $n = 2$, the problem reduces to the Kepler’s problem of the relative motion and the relevant Jacobi metric is up to an unimportant overall constant factor. Recently, one of us [10] showed that free motion of massive particles in static spacetimes is given by geodesics of an energy-dependent Riemannian metric on the spatial sections analogous to Jacobi’s metric in classical dynamics. Recently this result has been extended [7] to explore the Jacobi metrics for various stationary metrics. In particular, the Jacobi-Maupertius metric is formulated for time-dependent metrics by including the Eisenhart-Duval lift, known as the Jacobi-Eisenhart metric.

This results in geodesic trajectory reparameterization, redefining the Hamiltonian and effectively making it a canonical transformation of the extended phase space to a conformal theory. All other conserved quantities of a system are preserved under this lift. The Bohlin transformation is possibly itself a Jacobi-Maupertius lift of the oscillator metric. The JM metric plays an important role in statistical mechanics [3, 27]. Krylov [17] suggests that viewing $n$-body dynamics as a geodesic flow on an appropriate manifold may provide a universal tool for discussing relaxation processes.
An $n$-dimensional system is Liouville integrable if it admits $n$ first-integrals in involution which the Lagrangian submanifold depends upon. This means that integrability is a geometric property that is independent of choice of parameterisation, as seen when the same conserved quantities remain unchanged under reparameterisation. The only consequence of a different choice of parameterisation would be to produce a new integrable models. [8] have claimed that the harmonic oscillator, when it is reformulated in terms of JM geodesics, has positive Lyapunov exponents.

In this article, we shall first introduce some preliminaries and the basic formulation of the Jacobi-Maupertuis metric [13, 32] and show that conserved quantities are preserved and the equation of motion reduces to the geodesic equation [28] under reparameterisation by examining the metric from the Lagrangian and Hamiltonian perspectives. Following that, the Kepler system will be shown to be geodesic flow on constant curvature surfaces. Here, we shall demonstrate how such a projection to a fixed energy surface following a canonical transformation is the Bohlin’s transformation [29] that converts the oscillator system into the Kepler system. This will be followed by a discussion on application in Houri’s canonical transformation [13]. First we shall couple it with Milnor’s construction to study the preservation of the form of geodesic flows under such canonical transformations.

Organization of the paper is as follows. Section 2 outlines the basic aspects of the Jacobi-Maupertuis metric, Maupertuis transformation and integrable metrics. We also describe Jacobi-Eisenhart lift and Lagrange-Hamiltonian formulation in this section. We apply all these toolkits to Kepler equation in section 3.

## 2 Preliminaries

The Maupertuis form of the action defines the differential form of the action along the geodesic. From this form we are able to formulate the Hamilton’s equations of motion, as well as the canonical transformations that are possible.

$$S = \int_1^2 d\tau L(x^{\mu}, \dot{x}^{\mu}) = \int_1^2 dt \left( p_i \dot{x}^i - H(x, p) \right).$$

(2.1)

Here, we deal with the extended phase-space which treats time as another co-ordinate $q^{n+1} = t$ and the Hamiltonian as its conjugate momentum $p_{n+1} = -H(x, p)$. According to Maupertuis, the dynamical path solution from the extremal of the action $S$ coincides with that of the reduced action $S_0$ for a fixed energy $H(x, p) = E$, given by:

$$S_0 = \int_1^2 dt \ p_i \dot{x}^i = \int_1^2 dt \ \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i.$$

(2.2)

This reduced action is independent of any time evolution parameter, resulting in loss of information since we cannot restore the Hamiltonian function. The Jacobi-Eisenhart lift is one such process for dimensionally reducing geodesics. Such trajectories can be seen as geodesics of a corresponding configuration space or its enlargement under some constraints. Upon parametrizeing as $\tau = t$, the time quadratic action term provides the potential. Since such Hamiltonians arise from Lagrangians with a metric origin, the Jacobi-Eisenhart lift reduces the dimensions of the geodesic.
Given a metric on $n + 1$ dimensional space-time $ds^2 = g_{\mu
u}^{n+1}(x) \, dx^\mu dx^\nu$, it is simple to formulate the Lagrangian describing dynamics on the $n$ dimensional sub-space with a potential $U(x)$. Let $M$ be a manifold with local co-ordinates $x = (x^i)$, $i = 1, \ldots, n$, with $x(t) \in M \subseteq \mathbb{R}^n$, $t \in [0, T]$ being a curve. Define the velocity as $\dot{x}(t) \in T_x M \subseteq \mathbb{R}^n$ and the momenta as $p(t) \in T^*x M \subseteq \mathbb{R}^n$, $T_x M$, $T^*x M$ being tangent and co-tangent spaces respectively at $x = x(t)$.

If $L = T - U; T_x M \rightarrow \mathbb{R}$ is a natural Lagrangian, then $T$ is a non-degenerate quadratic form and $V$ is constant on each $T_x M$. Such dynamical systems under affine parametrization $\tau = x^0 = t$ are defined by the Lagrangian:

$$L(x, \dot{x}) = \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j - U(x), \quad (2.3)$$

where $g_{\mu\nu}(x)$ is a Riemannian metric. The Euler-Lagrange equation is given by:

$$\ddot{x}^i = -\sum_{jk} g^{ij}(x) \dot{x}^j \dot{x}^k - \sum_i g^{il}(x) \partial_l U(x). \quad (2.4)$$

The time-independent Hamiltonian $H : T^*M \rightarrow \mathbb{R}$ is a conserved quantity given by a Legendre transformation that maps the dynamics from the tangent to the co-tangent space $F_L : TM \rightarrow T^*M; (x, \dot{x}) \mapsto (x, p) = \left(x, \frac{\partial L}{\partial \dot{x}}\right)$.

$$H(x, p) = \sum_{i=1}^n p_i \dot{x}^i - L(x, \dot{x}), \quad p_i = \frac{\partial L}{\partial x^i} = g_{ij}(x) \dot{x}^j. \quad (2.5)$$

If the Lagrangian has a natural form given by (2.3), then will the Hamiltonian. The natural Hamiltonian for an autonomous system is a conserved quantity. As seen in (2.1) it acts as the generator for time-evolution of the geodesic action given by:

$$H(x, p) = \frac{1}{2m} g^{ij}(x) p_i p_j + U(x) \equiv T + U = E, \quad (2.6)$$

where the Hamilton’s dynamical equations are elaborated as:

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \frac{g^{ij}(x)}{m} p_j \quad \dot{p}_i = \frac{\partial H}{\partial x^i} = \frac{1}{2m} \frac{\partial g^{ij}(x)}{\partial x^i} p_i p_j + \frac{\partial U}{\partial x^i}. \quad (2.7)$$

We have so far dealt with the action on a space-time manifold for a general system with no fixed value for the Hamiltonian as a function on the cotangent bundle. For an autonomous (time-independent) system, we will have a fixed energy level defining the hypersurface on which motion takes place. We will study the reduced action on this hypersurface, by employing the Jacobi metric arising from the Jacobi-Maupertius lift.

### 2.1 Jacobi-Maupertuis metric and Maupertuis principle

It is possible to derive a metric which is given by the kinetic energy itself. Let us consider a conservative system with $n$ degrees of freedom whose Lagrangian is given by (2.3). The
kinetic energy $T$ is a homogeneous function of degree 2, hence Euler theorem implies $2T = \dot{x}^i \partial \mathcal{L} / \partial \dot{x}^i$, thus Maupertuis principle becomes

$$\delta S = \delta \int_{t_1}^{t_2} 2T \, dt = 0, \quad \text{where} \quad T = \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j.$$  

Since the total energy of the conservative system is constant $E = T + U$, then substituting $U$ in the Lagrangian we find $\mathcal{L} = 2T - E$. Substituting in (2.1) we obtain

$$\delta \int_{t_1}^{t_2} (2T - E) \, dt = \delta \int_{t_1}^{t_2} 2T \, dt - \delta \int_{t_1}^{t_2} E \, dt = \delta \int_{t_1}^{t_2} 2T \, dt.$$  

(2.1.1)

If we take the kinetic energy to be diagonal and all masses are equal, i.e., $a_{ij} = \delta_{ij}$, then equation (2.1.1) can be re-written as

$$\delta \int_{t_1}^{t_2} 2T \, dt = \delta \int_{t_1}^{t_2} ds = \delta \int_{t_1}^{t_2} (\overline{g}_{ij}(x) \dot{x}^i \dot{x}^j)^{1/2} \, dt.$$  

so that natural motion is geodesic of a configuration space $M$ and $ds$ is the differential arc length. The metric on $M$ is referred to as the Jacobi metric, given by

$$ds^2 = \overline{g}_{ij}(x) dx^i dx^j = 4(E - U(x)) T dt^2.$$  

(2.1.2)

Alternatively this expression can be obtained straight away by squaring $ds = 2T \, dt = 2(E - U(x))dt$. If we substitute $2T = mg_{ij}(x) \dot{x}^i \dot{x}^j$ into (2.1.2) we find

$$ds^2 = 2m(E - U(x)) g_{ij}(x) \dot{x}^i \dot{x}^j \, dt^2 = 2m(E - U(x)) \overline{g}_{ij}(x) dx^i dx^j.$$  

Thus we obtain

$$ds^2 = \overline{g}_{ij}(x) dx^i dx^j, \quad \text{where} \quad \overline{g}_{ij}(x) = 2m(E - U(x)) g_{ij}(x).$$  

(2.1.3)

Thus the above arguments imply that a physical path of energy value $E$ is a geodesic with respect to the metric (2.1.3). So we can define the JM-meter $\overline{g}_{jm}$ corresponding to an energy value $E$ of simple mechanical system $(M, g, U)$ as

$$\overline{g}_{jm} := 2m(E - U)g(x).$$

This is the metric that defines the geodesic on the hypersurface of energy $E$ within space-time. As we can see, the potential $U(x)$ has been merged into the metric components, giving the appearance of a potential-free space for a free article.

### 2.2 Jacobi-Maupertuis transform and integrable metrics

Let $M$ be a compact smooth $n$-dimensional Riemannian manifold with metric $g_{ij}(x)$. The cotangent bundle $T^* M$ is a smooth symplectic manifold with standard 2-form $\omega = \sum_{i=1}^n dp_i \wedge dx^i$. One can find other integrable metrics using the Jacobi-Maupertuis transformation. Let us consider the natural mechanical systems with Hamiltonian given by $H = \frac{1}{2m} \sum_{i,j} g^{ij}(x)p_i p_j + U(x)$. It is said that a Hamiltonian system on a $2n$-dimensional symplectic manifold is Liouville integrable if the $n$ first integrals are in
involution and functionally independent everywhere. Integrable geodesic flows play a very important role in geometry, mechanics and integrable systems [4, 5].

By the Maupertuis principle, for sufficiently large energy \( E \), greater than \( \max U(x) \), on a fixed \((2n - 1)\)-dimensional smooth level surface \( H(x, p) = E \), the integral trajectories of the vector field \( X_H \) coincide with the trajectories of the another vector field \( \tilde{X}_{\tilde{H}} \) corresponding to a new Hamiltonian \( \tilde{H} \) given by the formula

\[
\tilde{H}(x, p) = \frac{1}{2m} \sum_{i,j=1}^{n} g^{ij}(x) \frac{E - U(x)}{p_i p_j}, \tag{2.2.1}
\]

The Maupertuis transformation \( X_H \rightarrow \tilde{X}_{\tilde{H}} \) relates two vector field on \( M \). If \( t \) and \( \sigma \) are time along trajectories of the vector fields \( X_H \) and \( \tilde{X}_{\tilde{H}} \), then

\[
d\sigma = (E - U(x)) dt \tag{2.2.2}
\]

The distinguished role of the time \( t \) is not desirable in the general case of non-autonomous Hamiltonian systems. We therefore introduce an evolution parameter \( s \) to parameterize time evolution of the system. In the extended formalism, time \( t \) is treated as an ordinary canonical function \( t(s) \equiv x^0(s) \) of an evolution parameter \( s \). We may conceive a ‘new’ momentum coordinate \( p_0(s) \) in conjunction with the time as an additional pair of canonically conjugate coordinates. The extended Hamiltonian \( \mathcal{H}(x^0, p_0, x^i, p_i) \) is defined as a differentiable function on the cotangent bundle \( T^*Q = T^*(\mathbb{R} \times M) \) endowed with a chart \((p_0, p_i) \in T^*_{x_0, x_i}Q \) with \( \frac{\partial \mathcal{H}}{\partial s} = 0 \). It is given by \( \mathcal{H}(x^0, p_0, x^i, p_i) = H(x^i, p_i, x^0) + p_0 \), where \( x^0 \) and \( p_0 \) are conjugate variables and \( p_0 = -\dot{H} \). The extended phase space admits a Liouville form (or integral invariant of Poincaré-Cartan)

\[
\theta_{\mathcal{H}} = p_0 dt + p_i dx^i \tag{2.2.3}
\]

and the Hamiltonian flow is completely determined by the conditions:

\[
\langle X_{\mathcal{H}}, dt \rangle = 1 \quad \text{and} \quad X_{\mathcal{H}} \cdot d\theta_{\mathcal{H}} = 0,
\]

where

\[
X_{\mathcal{H}} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}_\mu \frac{\partial}{\partial p_\mu} \tag{2.2.4}
\]

Invoking Hamilton’s equations of motion, and keeping in mind that \( \dot{t} = 1, p_0 = -q(t) \) and the Maupertuis form of action, we have the extended Hamiltonian given below

Maupertuis form of action: \( \mathcal{L}(x^\mu, \dot{x}^\mu) = \sum_{\mu=0}^{n} p_\mu \dot{x}^\mu = \sum_{i=1}^{n} p_i \dot{x}^i + p_0 \dot{t} \)

\[
\mathcal{H}(x^i, p_i, t) = \sum_{\mu=0}^{n} p_\mu \dot{x}^\mu - \mathcal{L}(x^\mu, \dot{x}^\mu) = \left[ \sum_{i=1}^{n} p_i \dot{x}^i - \mathcal{L}(x^i, \dot{x}^i, t) \right] + p_0 \dot{t} = 0
\]

\[
\mathcal{H}(x^i, p_i, t) = H(x^i, p_i, t) - q(t) = 0
\]
Thus, the extended Hamiltonian vector field is given by
\[ X_H = \sum_\mu \left( \frac{\partial H}{\partial p_\mu} \frac{\partial}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu} \frac{\partial}{\partial p_\mu} \right) = \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} \right) + \frac{\partial H}{\partial H} \frac{\partial}{\partial t} - \frac{\partial H}{\partial t} \frac{\partial}{\partial H} \right) \]

Here, we apply some rules:
\[ \frac{\partial H}{\partial x^i} = \frac{\partial H}{\partial x^i}, \quad \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial H} = 1, \quad \frac{\partial H}{\partial t} = 0 \]

\[ X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \] (2.2.5)
is the time-dependent Hamiltonian vector field. The vector field \( X_H \) lies in the kernel of \( d\theta_H \), so the bicharacteristic of \( \theta_H \) is a path through the extended phase space such that the tangent vector to the path at any point is parallel to \( X_H \).

It is clear that the Poincaré-Cartan two form associated to (2.2.3)
\[ \omega = \sum_i dp_i \wedge dx^i - dH \wedge dt \] (2.2.6)
is invariant under Jacobi-Maupertuis transformation. This reveals that the JM transformation is the \textit{time-dependent canonical transformation}.

Consider the time-dependent canonical transformations of the extended phase space,
\[ t \to \sigma \quad d\sigma = \Lambda(x,p) dt \]
\[ H \to \tilde{H} \quad \tilde{H} = \Lambda^{-1}(x,p) H \] (2.2.7)
where \( \Lambda(x,p) = (E - U(x)) \). This changes the initial equations of motion
\[ \frac{dx^i}{d\sigma} = \Lambda^{-1}(x,p) \left( \frac{dx^i}{dt} - \tilde{H} \frac{\partial \Lambda}{\partial p_i} \right), \quad \frac{dp_i}{d\sigma} = \Lambda^{-1}(x,p) \left( \frac{dp_i}{dt} + \tilde{H} \frac{\partial \Lambda}{\partial x^i} \right). \]

This preserves the canonical form of the Hamilton-Jacobi equation given by
\[ \frac{\partial S}{\partial \sigma} + \tilde{H} = \frac{\partial S}{\partial t} \frac{dt}{d\sigma} + \Lambda^{-1} H = \Lambda^{-1} \left( \frac{\partial S}{\partial t} + H \right) = 0. \]
In other words, \( S \) satisfies
\[ S = \int (p_i dx^i - H dt) = \int \left( p_i dx^i - \tilde{H} d\sigma \right). \]

Integral trajectories have two parametric forms \( X_H \) and \( X_{\tilde{H}} \) corresponding to the Hamiltonians \( H \) and \( \tilde{H} = \Lambda^{-1}(x,p) H \) respectively. The transformation \( X_H \to X_{\tilde{H}} \) is the Maupertuis transformation. If \( \sigma \) be the time along trajectories of the vector \( X_{\tilde{H}} \), then the Maupertuis transformation gives the Jacobi transformation \( d\sigma = (E - U(x)) dt \).

Thus, the reparameterization can be seen as part of the canonical transformation [30] [31] to counter the changes in the form of the equation of motion. This maps the geodesic onto another geodesic while preserving integrability.
2.3 Jacobi-Eisenhart lift and Jacobi-Maupertuis metric

The Jacobi-Eisenhart lift eliminates the potential in a $n$ dimensional Hamiltonian system, reducing it into a spatial $n$-dimensional free particle geodesic. The result is the Jacobi metric which, for time-independent Hamiltonian systems, projects the original geodesic onto a constant energy surface as a spatial geodesic describing a free particle with no potentials.

So, for a fixed energy $H(x,p) = E$, the Jacobi Hamiltonian $\tilde{H}$ is:

$$g^{ij}(x)p_i p_j = 2m \left[ E - U(x) \right] \Rightarrow \tilde{H} = \frac{1}{2m}g^{ij}(x)p_i p_j = \frac{T}{E - U(x)} = 1 \quad \text{(2.3.1)}$$

This means that the metric and its inverse transform into their Jacobi-Maupertius equivalent as shown below:

$$\tilde{g}^{ij}(x)p_i p_j = 1$$

$$\tilde{g}^{ij}(x) = \frac{g^{ij}(x)}{2m \left[ E - U(x) \right]} \Rightarrow \tilde{g}_{ij}(x) = 2m \left[ E - U(x) \right] g_{ij}(x) \quad \text{(2.3.2)}$$

Naturally, for a transformed Hamiltonian, the dynamical description should also change to match the new generator of time translations. This essentially means that the geodesic must be reparameterized to keep the form of Hamilton’s equations invariant. Furthermore, from the lifted Hamiltonian, using (2.7) and (2.3.1) gives the momentum and reparameterization factor:

$$\frac{dx^i}{d\sigma} = \frac{\partial \tilde{H}}{\partial p_i} = 2\tilde{g}^{ij}(x)p_j = \frac{1}{m \left[ E - U(x) \right]} \left( g^{ij}(x)p_j \right) = \frac{dt}{d\sigma} \dot{x}^i$$

$$p_i = \frac{1}{2} \tilde{g}_{ij}(x) \frac{dx^j}{d\sigma} \quad \Lambda(x,p) = \frac{d\sigma}{dt} = \left| E - U(x) \right| \quad \text{(2.3.3)}$$

Thus, according to (2.2.7), the new Hamiltonian can be said to be:

$$\mathcal{H} = \frac{H}{\left| E - U(x) \right|} \quad \text{(2.3.4)}$$

Using (2.3.3) for the Jacobi Hamiltonian, we can say that the reduced Lagrangian is

$$\tilde{\mathcal{L}} = \tilde{g}_{ij}(x) \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} = 4\tilde{g}^{ij}(x) \left( \frac{1}{2} \tilde{g}_{ik}(x) \frac{dx^k}{d\sigma} \right) \left( \frac{1}{2} \tilde{g}_{jl}(x) \frac{dx^l}{d\sigma} \right) = 4\tilde{g}^{ij}(x)p_ip_j = 4\tilde{H} = 4$$

$$\therefore \quad \tilde{\mathcal{L}} = \left[ E - U(x) \right] \mathcal{L} = \tilde{g}_{ij}(x) \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} = 4 \quad \text{(2.3.5)}$$

Liouville integrability of an $n$-dimensional geodesic flow is defined to imply that:

a. $n$ functionally independent first-integrals of motion $I_n$ exist almost everywhere.

b. Such integrals are in involution: $\{I_j, I_k\} = 0$ for all $1 < j, k < n$.

Restricting the geodesic flow onto any non-zero fixed energy level surfaces are smoothly equivalent to the trajectory. Consequently, we may redefine the condition of integrability to imply the existence of $n-1$ functionally independent first integrals in involution almost everywhere on the unit covector bundle $\{H(x,p) = \tilde{g}^{ij}(x)p_ip_j = 1\} \subset T^\ast M^n$. 


2.4 Lagrangian and Hamiltonian perspective

Mechanics has been historically studied from two approaches: Lagrange’s and Hamilton’s. This results in two different, yet equivalent formulations of the equations of motion to describe geodesics. Since we have shown how to formulate the lifted Hamiltonian and Lagrangian, it is natural to explore how the equations of motion take shape under such formulations, and the effect on conserved quantities.

Starting with the Hamiltonian in (2.3.1), we shall write the dynamical equations with respect to a new parameter \( \sigma \) as shown in [28]

\[
\begin{align*}
\frac{dx^i}{d\sigma} &= \frac{\partial H}{\partial p_i} = \frac{g^{ij}(x)}{m(E - U(x))} p_j \\
\frac{dp_i}{d\sigma} &= -\frac{\partial H}{\partial x^i} = \frac{1}{2m} \frac{\partial g^{mn}(x)}{\partial x^i} p_m p_n + \frac{\partial U}{\partial x^i}
\end{align*}
\]  

(2.4.1)

**Theorem 2.1.** Let \( T : TQ \rightarrow \mathbb{R} \) be a smooth pseudo-Riemannian metric, \( U : Q \rightarrow \mathbb{R} \) be a smooth potential energy function, and \( t \mapsto q(t), I \rightarrow Q \) be a curve in \( Q \) such that \( E(q(t), \dot{q}(t)) = E \in \mathbb{R} \) and \( U(q(t)) \neq E \forall t \mapsto \tilde{t}(t), I \rightarrow \mathbb{R} \) defined by

\[
\sigma(t) = \int_0^t d\tau (E - U(q(\tau)))
\]

is a diffeomorphism into its image \( J : s \mapsto t(s), J \rightarrow I \). Moreover, \( t \mapsto q(t) \) in \( Q \) is a solution to the Euler-Lagrange equation \( EL(L) = 0 \) iff the curve \( s \mapsto x(t(s)), J \rightarrow Q \) is a geodesic of the Jacobi metric \( \tilde{L} = (E - U)L \).

**Proof.** So long as we have

\[
\frac{d\sigma(t)}{dt} = E - U(x(t)) \neq 0
\]

the inverse function theorem guarantees that \( t \mapsto s(t) \) is a diffeomorphism onto its image \( Q \), reparameterizing the the curve as \( s \mapsto x(t(s)) = x(t(s)). \) Thus, the velocity upon differentiation wrt \( t \):

\[
\frac{dx^i}{dt} = \frac{dx^i}{d\sigma} \frac{d\sigma}{dt} = (E - U(x)) \frac{dx^i}{d\sigma}
\]

(2.4.2)

and the acceleration from (2.4) can be re-written as:

\[
\ddot{x}^i = \frac{d\sigma}{dt} \frac{d}{d\sigma} \left[ \frac{d\sigma}{dx^i} \frac{dx^i}{dt} \right] = (E - U(x)) \frac{d^2 x^i}{d\sigma^2} - (E - U(x)) \frac{\partial_j U(x)}{\partial x^i} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}
\]

(2.4.3)

and the Euler-Lagrange equation (2.4) transforms as:

\[
(E - U(x)) \frac{d^2 x^i}{d\sigma^2} - \partial_j U(x) \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} = - \sum_{jk} (E - U(x)) \Gamma^i_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} - \sum_l \tilde{g}^{il} \partial_l U(x)
\]

where

\[
\Gamma^i_{jk} = \frac{1}{2} g^{im} (\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk})
\]

\[
\Gamma^i_{jk} = \left[ \frac{1}{2(E - U(x))} \left( \partial_j U(x) \delta^i_k + \partial_k U(x) \delta^i_j - \tilde{g}^{im} \partial_m U(x) \tilde{g}_{jk} \right) + \tilde{\Gamma}^i_{jk} \right]
\]
\[(E - U(x)) \frac{d^2x^i}{d\sigma^2} - \partial_t U(x) \frac{dx^i}{d\sigma} = - \sum_{jkl} \left[ (E - U(x)) \Gamma^i_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} + \tilde{g}^{il} \partial_l U(x) \right] \]

\[= - \sum_{jkl} \partial_t U(x) \frac{dx^j}{d\sigma} \frac{dx^i}{d\sigma} - \tilde{g}^{im} \partial_m U(x) \left( \frac{1}{2} \tilde{g}^{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} \right) + \tilde{g}^{il} \partial_l U(x) \]

\[= - \sum_{jkl} (E - U(x)) \Gamma^i_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} \]

\[\frac{d^2x^i}{d\sigma^2} = - \Gamma^i_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} \quad (2.4.4)\]

Thus, the Euler-Lagrange equation has been mapped to a regular geodesic equation for the Jacobi metric \([2.1]\). The Jacobi-Maupertius principle holds for any system with non-zero kinetic energy.

Also, for any conserved quantity \(K = K^{(2)ij} p_i p_j + K^{(0)}\), we can say:

\[\frac{dK}{dt} = \{K, \tilde{H}\} = \frac{dt}{dt} \frac{dK}{dt} = \frac{1}{E - U(x)} \{K, H\} \quad (2.4.5)\]

\[\therefore \quad \{K, \tilde{H}\} = 0 \quad \Rightarrow \quad \{K, H\} = 0 \quad (2.4.6)\]

In \([13]\), T. Houri describes \(\tilde{K} = K^{(2)ij} p_i p_j + K^{(0)} \tilde{H}\) where according to \([2.3.1]\):

\[\tilde{K} = K^{(2)ij} p_i p_j + K^{(0)} \tilde{H} = K^{(2)ij} p_i p_j + K^{(0)} = K \quad \therefore \quad \tilde{H} = 1 \quad (2.4.7)\]

Thus, showing that the conserved quantities remain the same for the Jacobi metric. This is not surprising given that the Jacobi -Eisenhart lift was just a reparameterization that left position and momenta unaltered. Since all conserved quantities or first integrals in Hamiltonian mechanics are polynomials of position and momenta, they should also be unchanged under such a transformation, unless a canonical transformation is involved.

We shall now proceed to apply the the Jacobi metric to the Kepler problem.

### 3 Application to Kepler problem

We now consider the Kepler problem of orbital motion in the presence of a central potential \(U(r) = -\frac{\alpha}{r}\). Since this is a problem involving spherical symmetry, we have the spatial part of the metric as the conformally flat polar metric. We shall only consider two dimensional motion because of angular momentum conservation in a radial potential.

Thus, the Jacobi-Kepler metric is given as a conformally flat metric:

\[ds^2 = (E - U(r)) (dr^2 + r^2 d\theta^2) = f^2(r) (dr^2 + r^2 d\theta^2) \quad (3.1)\]
Here, the Gaussian curvature is given by:

\[ e^r = f(r) \, dr \quad e^\theta = r f(r) \, d\theta \]

\[ de^\theta = (rf(r))' \, dr \wedge d\theta \quad \Rightarrow \quad \omega^\theta_r = \frac{rf(r)}{f(r)} \, d\theta \]

\[ d\omega^\theta_r = \left( \frac{(rf(r))'}{f(r)} \right)' \, dr \wedge d\theta \quad \Rightarrow \quad R^\theta_r = -\frac{1}{rf^2(r)} \left( \frac{(rf(r))'}{f(r)} \right)' \]

\[ \therefore \quad K_G = R^\theta_r = -\frac{1}{rf^2(r)} \left( \frac{1}{f(r)} \frac{d}{dr} (rf(r)) \right) \quad (3.2) \]

Thus, for \( f^2(r) = E - U(r) \), the Gaussian curvature (3.2) in this case is given as:

\[ K_G = \frac{(rU'(r))'(E - U(r)) + r(U'(r))^2}{2r(E - U(r))^3} \quad (3.3) \]

If \( h \) is a regular value of \( U(r) \) on the boundary ring, ie. \( U(r) = h; x \in \partial M \) we have by continuity

\[ (rU'(r))'(E - U(r)) + r(U'(r))^2 > 0, \quad K_G \to \infty \quad (3.4) \]

In case of the Kepler problem, we have \( U(r) = -\frac{\alpha}{r} \), so the Gaussian curve \( K_G \) is:

\[ K_G = -\frac{\alpha E}{2(rE + \alpha)^3}. \quad (3.5) \]

Thus, we can see that the curvature is classified as:

\[ \forall \quad E > -\frac{\alpha}{r} \quad \begin{cases} \quad E < 0 \quad \Rightarrow \quad K_G > 0 \quad ; \quad \text{ellipse} \\ \quad E = 0 \quad \Rightarrow \quad K_G = 0 \quad ; \quad \text{parabola} \\ \quad E > 0 \quad \Rightarrow \quad K_G < 0 \quad ; \quad \text{hyperbola} \end{cases} \quad (3.6) \]

Thus, for the Kepler problem, for negative energies in the range \(-\frac{\alpha}{r} < E < 0\), we will have positive curvature, and thus closed periodic orbits described by the Jacobi-Kepler metric. What motivates us to connect this theory with the Kepler problem is that it describes \( \tilde{H} = 1 \) geodesic flow on \( T^*S^3 \), \( K_G = 1 \) energy surface.

The Hamiltonian flow along a geodesic is given by the Hamiltonian vector field operator, which for the Kepler equation, essentially becomes:

\[ X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} = p_i \frac{\partial}{\partial x^i} - \frac{\alpha x^i}{r^3} \frac{\partial}{\partial p_i} \quad (3.7) \]

Thus, under circumstances of constant curvature, the radial equation of motion is:

\[ \ddot{r} - \frac{\dot{r}^2}{r^3} = -U'(r) \quad (3.8) \]
Thus, for constant vanishing Gaussian curvature, we will have the Kepler potential, and thus, the Kepler equations of motion. However, if we consider the Jacobi metric and Hamiltonian, we will have:

\[
\frac{dr}{d\sigma} = \frac{1}{E - U(r)} p_r \quad \Rightarrow \quad p_r = \frac{rE + \alpha}{r} \frac{dr}{d\sigma} \quad (3.9)
\]

\[
\frac{dp_r}{d\sigma} = -\frac{r}{rE + \alpha} \left[ -\frac{p^2_r}{r^3} + \frac{\alpha}{r^2} \right] \quad (3.10)
\]

\[
-\frac{\alpha}{r^2} \left( \frac{dr}{d\sigma} \right)^2 + \frac{rE + \alpha}{r} \frac{d^2r}{d\sigma^2} = -\frac{r}{rE + \alpha} \left[ -\frac{p^2_r}{r^3} + \frac{\alpha}{r^2} \right]
\]

\[
\therefore \quad \frac{d^2r}{d\sigma^2} = -\frac{E p^2_\alpha}{(rE + \alpha)^3} + \frac{\alpha}{(rE + \alpha)^2} \quad (3.11)
\]

If one wishes to verify, it can be confirmed in (3.11) that:

\[
\bar{\Gamma}^r_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} = \frac{E p^2_\alpha}{(rE + \alpha)^3} + \frac{\alpha}{(rE + \alpha)^2} \quad (3.12)
\]

showing that the RHS of (3.11) matches that of (2.3.4), and our analysis is consistent.

### 3.1 Bohlin transformation and duality

The Bohlin transformation is a canonically converts the dynamics of the oscillator system into that of the Kepler system and vice versa. We shall see how the Jacobi metric for a fixed energy following a canonical transformation demonstrates this as shown in [32].

The transformation rule involves expressing the co-ordinates as a complex variable:

\[
r = q_1 + iq_2 \quad (3.1.1)
\]

The canonical transformation we shall use as shown in [29] is:

\[
r \rightarrow z = \frac{r^2}{2} = \left( \frac{q_1^2 - q_2^2}{2} \right) + i(q_1 q_2) = x + iy \quad \begin{cases} x = \frac{q_1^2 - q_2^2}{2} \\ y = q_1 q_2 \end{cases} \quad (3.1.2)
\]

\[
x^2 + y^2 = \left( \frac{q_1^2 + q_2^2}{2} \right)^2, \quad \text{or} \quad 2 \sqrt{x^2 + y^2} = q_1^2 + q_2^2. \quad (3.1.3)
\]

For the covariant momentum, in accordance with Bohlin’s transformation rule:

\[
\begin{align*}
p_1 &= \frac{\partial x}{\partial q_1} p_x + \frac{\partial y}{\partial q_1} p_y = q_1 p_x + q_2 p_y \\
p_2 &= \frac{\partial x}{\partial q_2} p_x + \frac{\partial y}{\partial q_2} p_y = -q_2 p_x + q_1 p_y \quad \begin{cases} p = p_1 + ip_2 = (q_1 - iq_2)(p_x + ip_y) \end{cases} \quad (3.1.4)
\end{align*}
\]
This transformation can also be written in matrix form as:

\[
\begin{pmatrix}
  p_x \\ p_y
\end{pmatrix} = \frac{1}{q_1^2 + q_2^2} \begin{pmatrix}
  q_1 & -q_2 \\ q_2 & q_1
\end{pmatrix} \begin{pmatrix}
  p_1 \\ p_2
\end{pmatrix}
\] (3.1.5)

Thus we obtain

\[
\frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} = p_x^2 + p_y^2.
\] (3.1.6)

Let \(H(p,q)\) be any Hamiltonian and fix the energy \(E\). Let us consider flow by the reparametrization \(\frac{dt}{d\tau} = f(q,p)\) This immediately yields

\[
\tilde{H}(p,q) = f(p,q)(H(p,q) - E),
\]

which retains the zero energy surface on the level set of \(H\) to the energy \(E\)

\[
H^{-1}(E) = \{(p,q)|H(p,q) = E\}.
\]

If the oscillator Hamiltonian is given as

\[
H_{osc}(q_i, p_i) = \frac{1}{2} (p_1^2 + p_2^2) + \frac{a}{2} (q_1^2 + q_2^2) - b
\] (3.1.7)

The transformation (3.14) maps the Hamiltonian of the oscillator equation to that of Kepler,

\[
H_{kepler}(x, p) = p_x^2 + p_y^2 - \frac{b}{2\sqrt{x^2 + y^2}} + a,
\] (3.1.8)

thus (3.14) can be considered to be the Bohlin transformed co-ordinates and for the time being we assume \(r = \sqrt{x^2 + y^2} \neq 0\). This clearly yields the transformation of the oscillator hamiltonian into the Kepler Hamiltonian.

If \(a\) and \(b\) are treated as new momenta then the null lift of the (3.1.8), given by

\[
\tilde{H}(x, p) = p_x^2 + p_y^2 - \frac{p_z^2}{\sqrt{x^2 + y^2}} + p_a^2,
\] (3.1.9)

where we have added two new conjugate variables \((z, p_z), (a, p_a)\) and corresponding momenta being conserved. Recently, Cariglia \[6\] made a fine observation to connect all the energy (positive, null and negative) regimes of Kepler orbit by introducing an additional conjugate pair. This one can be done if we replace \((a, p_a)\) pair by two additional conjugate pair \((\alpha, p_\alpha)\) and \((\gamma, p_\gamma)\) and Hamiltonian \(\tilde{H}(x, p)\) is replaced by

\[
\tilde{H}(x, p) = p_x^2 + p_y^2 - \frac{p_z^2}{\sqrt{x^2 + y^2}} - p_\alpha^2 + p_\gamma^2,
\]
3.2 Contact method, reparametrization and regularization

A contact form $\alpha$ on a $(2n + 1)$-dimensional manifold $M$ is a Pfaffian form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. The contact distribution is given by $\mathcal{C}|_U = \text{Ker} \alpha|_U$, where $U$ is the open set in $M$. Given a contact form $\alpha$, the Reeb vector field $Z$ is a vector field uniquely defined by

$$i_Z \alpha = 1, \quad i_Z d\alpha = 0. \tag{3.2.1}$$

Here we are interested in problem of closed Hamiltonian trajectories on a fixed energy $H = E$ surface, so we follow Weinstein’s method. Let $P^{2n}$ be the total space of the principle $\mathbb{R}^*$-bundle $\pi : P \to M$, whose fibers are non-zero covectors $(q, p)$ that vanish on the contact element $\mathcal{C}(x)$ in $M$. The symplectization $P$ has a canonical 1-form $\alpha$, restriction of Liouville 1-form, and the symplectic form is given by $\omega = d\alpha$. Consider the multiplicative $\mathbb{R}^*$ action on $(P, \omega)$, from the nongeneracy of $\omega$, there exist a unique vector field $Y$, called the Liouville vector field, which satisfies the following identities:

$$i_Y \omega = \alpha, \quad \alpha(Y) = 0, \quad L_Y \omega = \omega. \tag{3.2.2}$$

Since the Reeb vector field $Z$ is a section of $\text{Ker} d\alpha|_M = 0$, hence it is proportional to $X_{H|_M}$. $Z$ can be manifested as a flow of $X_{H|_M}$ after a time reparametrization $dt = f(q, p) d\tau$ introduced earlier. Thus we obtain

$$Z(x) = \frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = f(x)X_H(x), \quad x = (q, p).$$

Claim 3.1. The Reeb vector field $Z$ is

$$Z = \frac{X_H}{Y(H)}, \quad \text{where} \quad f(x) = \frac{1}{Y(H)}. \tag{3.2.3}$$

**Proof:** By definition we know $\omega(Y, \cdot) = \alpha$ and $\alpha(Z) = 1$. Thus we obtain

$$1 = \alpha(Z) = \omega(Y, f(x)X_H) = f(x)\omega(Y, X_H) = f(x)dH(Y) = f(x)Y(H).$$

The function $H_0 = H - E/Y(H)$ is defined on $M$ as an invariant surface. Then the vector field $X_{H_0|_M}$ is equal to the Reeb vector field $Z$.

3.2.1 Application to Kepler equation

Consider a special symplectic transformation $(p, q) \to (-q, p)$. It is easy to check that this transformation leaves the symplectic form:

$$\omega = d\alpha = \sum_{i=1}^{n} dp_i \wedge dq_i = \sum_{i=1}^{n} -dq_i \wedge dp_i = \sum_{i=1}^{n} d(-q_i dp_i) = d\tilde{\alpha}.$$

The associated Liouville vector field is $Y = \sum_{i=1}^{n} q_i \partial Y = \sum_{i=1}^{n} (q_i)^2 \frac{\beta}{|q|^3} = f(x)Y(H)$.

$$Y(H) = \sum_{i=1}^{n} q_i \frac{\partial H}{\partial q_i} = \sum_{i=1}^{n} (q_i)^2 \frac{\beta}{|q|^3}.$$  

1This transformation appears in Moser’s work on regularization of Kepler orbit
Thus on isoenergetic surface we obtain
\[
\frac{H - E}{Y(H)} = \frac{1}{2} \left( \frac{1}{\beta} |q|^2 - \beta \right) = \left( |p|^2 - 2E \right) \frac{|q|}{\beta} - 1 = H_0.
\]

Consider a smooth function
\[
F = \left( H_0 + \frac{1}{2} \right)^2 = \left( \frac{|p|^2 - 2E}{8\beta^2} \right) |q|^2.
\]

On the fixed energy surface \( H = E \), \( F \) becomes \( F|_{M_E} = \frac{1}{2} \). The trajectories of the Hamiltonian flow of \( F \) on the isoenergetic surface are governed by the reparametrized time \( \tau \). The Hamiltonian vector fields of \( F \) and \( H_0 \) coincide on the level hypersurface \( F = 1/2 \) or equivalently \( H_0 = 0 \). One can easily check
\[
X_F = \frac{q_i}{\beta} \frac{\partial}{\partial q_i} - \frac{p_i}{\beta} \frac{\partial}{\partial p_i} = \left( \frac{\beta q_i}{|q|^3} \frac{\partial}{\partial p_i} - \frac{p_i}{|q|^2} \frac{\partial}{\partial q_i} \right) = X_H / Y(H).
\]

Thus we establish regularization theorem due to Moser.

**Theorem 3.1.** On the isoenergetic surface \( F = 1/2 \) the trajectories of the Hamiltonian flow of the function \( F = \left( \frac{|p|^2 - 2E}{8\beta^2} \right) |q|^2 \) traversed in time \( \tau \) equal to trajectories of the Hamiltonian flow of the function \( H = \frac{1}{2} |p|^2 - \frac{\beta}{|q|} \) traversed in real time \( t \), and these two times are connected by
\[
\frac{d\tau}{dt} = \frac{\beta}{|q|}.
\]

### 3.3 Houri’s canonical transformation

Another canonical transformation that can be applied to the Kepler problem, as performed by Tsuyoshi Houri in [13], involves swapping the position and momentum phase-space co-ordinates.

\[
\tilde{x}^i = p_i, \quad \tilde{p}_i = x^i
\]

Thus, the Kepler hamiltonian will transform as:

\[
H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \frac{\alpha}{r} \longrightarrow \frac{1}{2} \left( (\tilde{x}^1)^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2 \right) + \frac{\alpha}{\sqrt{\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2}} \quad (3.3.2)
\]

As a result, if we choose a fixed energy surface \( H = E \) we can further say:

\[
\tilde{H} = \left[ E - \frac{\tilde{r}^2}{2} \right] (\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2) = \alpha^2 \quad \tilde{r}^2 = (\tilde{x}^1)^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2 \quad (3.3.3)
\]

Thus, the related metric with constant curvature \( 4E \) on a fixed energy surface is:

\[
\tilde{g}^{ij}(x) = \left[ E - \frac{\tilde{r}^2}{2} \right] \delta^{ij} \quad \tilde{g}_{ij}(x) = \left[ E - \frac{\tilde{r}^2}{2} \right]^{-2} \delta_{ij} \quad (3.3.4)
\]
So the metric is given by
\[ ds^2 = \left( E - \frac{|x|^2}{2} \right)^2 \, dx^2 \]  
(3.3.5)

If we set the energy to be \( E = -\frac{k^2}{2} \), we obtain
\[ ds^2 = 4(k + |x|^2)^{-2} \, dx^2 \]  
(3.3.6)

Let \( M_k \) be the space of constant curvature manifold. It is known that the Kepler phase space geodesically incomplete, since in the collision orbits, the particle arrives to the attractive center with infinite velocity in a finite time, hence does not admit a transitive group of motion. The mapping of the inversion
\[ I_k : M_k/\{0\} \to \tilde{M}_k/\{0\}, \]
and \( x \to \frac{x}{|x|^2} \) realizes isometry between its source metric \( g \) and the target metric \( \hat{g} \).

Suppose
\[ (I_k)_* : p \mapsto \frac{p}{|x|^2} - 2 \frac{x}{|x|^3} (x, p), \]
then one can easily check that
\[ \hat{g}_{I(q)}(I_*x, I_*x) = \frac{4}{(1 + k|x|^2)^2} (I_*x, I_*x) \]
\[ = \frac{4}{(1 + k\frac{1}{|x|^2})^2} \frac{|p|^2}{|x|^4} = 4(\frac{|x|^2}{1 + k\frac{1}{|x|^2}})^2 = g_q(x, x). \]

This describes another conformally flat metric. The question that arises here is; How to connect with the Milnor construction?

If we set the energy to be \( E = -\frac{k^2}{2} \), then we will have
\[ \tilde{H} = 4 \left( k^2 + \tilde{r}^2 \right)^2 \left( \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 \right) = \alpha^2 \]  
(3.3.7)

If we choose the reparameterization as:
\[ \frac{dt}{d\tau} = \frac{r}{k} \]  
(3.3.8)

Then we will have the new Hamiltonian as:
\[ \mathcal{H} = \frac{r}{k} \left( H + \frac{k^2}{2} \right) = \frac{r}{k} \left( \frac{|p|^2}{2} - \frac{\alpha}{r} + \frac{k^2}{2} \right) = \frac{r}{2k} (|p|^2 + k^2) - \frac{\alpha}{k} \]
\[ \mathcal{H} = k\mathcal{H} + \alpha = \frac{r}{2} (|p|^2 + k^2) \]  
(3.3.9)

However, Houri’s approach does not preserve the form of equations of the motion or geodesic flow operator. That requires another step with Milnor’s construction [21].
3.4 Milnor’s construction

We shall now separately formulate the Kepler problem under Milnor’s construction [21], which essentially involves a momentum inversion. From this formulation we shall write the metric and the trajectory equation in terms of inverse momentum.

The Kepler equation implies:
\[
\frac{dp}{dt} = -\alpha \frac{x}{r^3} \quad \Rightarrow \quad \frac{dp}{dt} = \frac{\alpha}{r^2}
\] (3.4.1)

Levi-Civita showed that it is possible to simplify Kepler solutions by introducing a fictitious parameter \(\sigma\) such that:
\[
\frac{d\sigma}{dt} = \frac{1}{r}
\] (3.4.2)

This makes the reparameterized Kepler equation of motion:
\[
\frac{dp}{d\sigma} = -\alpha \frac{x}{r^2} = \left( E - \frac{|p|^2}{2} \right) \frac{x}{r} \quad \Rightarrow \quad \frac{dp}{d\sigma} = \frac{\alpha}{r} = \frac{|p|^2}{2} - E
\] (3.4.3)

\[
\therefore \quad ds^2 = 4 \left[ 2E - |p|^2 \right]^{-2} |dp|^2
\] (3.4.4)

Thus, there is one and only one metric on \(M_E\) that satisfies our condition. Comparing (3.4.4) result with the Houri’s formulation (3.3.5), we can see that they are identical, except for a swap between momentum and co-ordinate. To describe events in the neighbourhood of infinity, we shall work with the inverted momentum co-ordinate.

\[
w = \frac{p}{|p|^2}, \quad |w|^2 = \frac{1}{|p|^2}, \quad 2E|w|^2 < 1
\] (3.4.5)

\[
\therefore \quad p = \frac{w}{|w|^2}, \quad dp = \frac{dw}{|w|^2} - 2 \left( \frac{w \cdot dw}{|w|^4} \right) w \quad |dp|^2 = \frac{|dw|^2}{|w|^4}
\] (3.4.6)

Using (3.4.3), (3.4.5) and (3.4.6) and defining \(\gamma = \frac{dt}{d\sigma}\), we will get:

\[
p' = \left( E - \frac{|p|^2}{2} \right) \frac{x}{r} = \frac{2E|w|^2}{2|w|^2} - 1 \frac{x}{r}
\]

\[
\Rightarrow \quad \frac{w'}{|w|^2} - 2 \left( \frac{w \cdot w'}{|w|^4} \right) w = \frac{2E|w|^2}{2|w|^2} - 1 \frac{x}{r}
\] (3.4.7)

and
\[
\frac{|w'|^2}{|w|^4} = \left( \frac{2E|w|^2}{2|w|^2} - 1 \right)^2 \quad \Rightarrow \quad 4|w'|^2 = \left( 2E|w|^2 - 1 \right)^2
\] (3.4.8)

If we now substitute the fixed energy level \(E = -\frac{k^2}{2}\) in (3.4.8), then we will have the metric in terms of the inverse momentum given as:
\[
ds^2 = 4 \left( 1 + k^2|w|^2 \right)^{-2} |dw|^2
\] (3.4.9)
which is the inverse-momentum version of (3.4.4) and a constant mean-curvature metric. From (3.4.7), we get the trajectory equation in terms of inverse momentum as:

\[ x = \frac{|w|^2 w' - 2(w \cdot w') w}{|w|^2 (2E|w|^2 - 1)} = 2\alpha \frac{2(w \cdot w') w - |w|^2 w'}{(1 - 2E|w|^2)^2} \quad (3.4.10) \]

Thus, \( x \) can be expressed as a smooth function of the parameter \( \sigma \). If we use \( t \) in place of \( \sigma \), the function stops being smooth only at the point \( x = 0 \).

### 3.5 Geodesic flow

Now we will see if the form of geodesic flow is preserved after using momentum inversion upon Houri’s canonical transformation. The Hamiltonian (3.3.7) describing geodesics on such spaces under a momentum inversion for \( E = -k \) \([15]\) is given by

\[ \tilde{H} = \frac{1}{4} (1 + k|x|^2)^2 |p|^2 \quad (3.5.1) \]

From this Hamiltonian, setting we can derive the Hamiltonian flow vector field

\[ \tilde{X}_{\tilde{H}} = \frac{\partial \tilde{H}}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial \tilde{H}}{\partial x^i} \frac{\partial}{\partial p_i} \]

\[ = \frac{1}{2} (1 + k|x|^2)^2 |p|^2 \frac{\partial}{\partial x^i} - (1 + k|x|^2) |p|^2 k x_i \frac{\partial}{\partial p_i} \]

\[ = 2\tilde{H} \frac{p^i}{|p|^2} \frac{\partial}{\partial x^i} - 2k \tilde{H}^2 |p| x_i \frac{\partial}{\partial p_i} = \frac{2\tilde{H}^2 |p|}{|p|^3} \frac{\partial}{\partial x^i} - k x_i \frac{\partial}{\partial p_i} \]

Thus we finally obtain

\[ \therefore (2\tilde{H}^2 |p|)^{-1} \tilde{X}_{\tilde{H}} = \frac{p^i}{|p|^3} \frac{\partial}{\partial x^i} - x_i \frac{\partial}{\partial p_i} \quad (3.5.2) \]

Comparing the flow operator above with the geodesic flow in (3.7), we obtain the quasi-Hamiltonian vector field of Kepler equation in momentum space

\[ X^{\text{mom}}_{\text{Kepler}} = (2k^2 \tilde{H}^2 |p|)^{-1} \tilde{X}_{\tilde{H}} \quad (3.5.3) \]

Thus, we can see that combining Houri’s transformation with Milnor’s momentum inversion preserves the form of the geodesic flow, aside from a momentum factor. A vector field \( X \) on a symplectic manifold \((M, \omega)\) is quasi-Hamiltonian if there exists a (nowhere-vanishing) function \( \Lambda \) such that \( X \) is a Hamiltonian vector field \( \Lambda X \in \mathfrak{X}_H(M) \), thus \( i_X(\Lambda \omega) = d\Lambda \). This condition can alternatively be written as as \( i_X(\Lambda \omega) = d\Lambda \), but the point is that the 2-form \( \Lambda \omega \) is not closed in the general case.

By applying the special canonical transformation that interchanges \( x \) and \( p \), the Kepler equation on momentum space transforms to the usual Kepler equation with the Hamiltonian

\[ H = \frac{1}{4} (k + |p|^2)^2 |x|^2. \]

Finally, we will explore the results of parameterizing the JM metric and Kepler equation with the eccentric anomaly.
3.6 JM metric and Kepler equation parametrized by eccentric anomaly

The Kepler Hamiltonian is given as

\[ H = \frac{1}{2} \sum_{n=1}^{3} (p_n)^2 - \frac{\alpha}{r} \]

\[ r^2 = \sum_{n=1}^{3} (x^n)^2 \]  \hspace{1cm} (3.6.1)

Let us perform the following canonical transformation:

\[ x^i \leftrightarrow p_i : \quad (x^i, p_i) \rightarrow (\tilde{p}_i, \tilde{x}^i) \]  \hspace{1cm} (3.6.2)

Setting \( H = E \), this transformation allows us to write a new Hamiltonian \( \tilde{H} \) as

\[ \tilde{H} = E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 - \frac{\alpha}{\sqrt{\sum_{n=1}^{3} \tilde{p}_n^2}} \]

\[ \Rightarrow \quad \tilde{H} = \left[ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right] \left[ \sum_{n=1}^{3} \tilde{p}_n^2 = \alpha^2 \right] \]

The Hamilton’s equations for this canonically transformed system (3.3.7) are:

\[ \dot{\tilde{x}}^i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i} = 2 \left[ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right] \tilde{p}_i \]

\[ \dot{\tilde{p}}_i = \frac{\partial \tilde{H}}{\partial \tilde{x}^i} = 2 \left[ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right] \left( \sum_{n=1}^{3} \tilde{p}_n^2 \right) \tilde{x}^i \]  \hspace{1cm} (3.6.3)

To proceed to equations of motion, we shall use (3.6.3) to write:

\[ \ddot{x}^i = 2 \frac{d}{dt} \left[ \left\{ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right\} \right] \tilde{p}_i \]

\[ = -4 \left\{ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right\} \left( \sum_{k=1}^{3} \tilde{x}^k \tilde{x}^k \right) \tilde{p}_i + 2 \left\{ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right\} \tilde{p}_i \]

\[ = -2 \left\{ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right\} \left( \sum_{n=1}^{3} \tilde{p}_n^2 \right) \tilde{x}^i + 4 \left\{ E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2 \right\} \tilde{H} \tilde{x}^i \]

\[ = -2 \left( \frac{\tilde{x} \cdot \dot{\tilde{x}}}{\Lambda} \right) \dot{\tilde{x}} + 4 \Lambda \tilde{H} \tilde{x} \]

where \( \Lambda = (E - \frac{1}{2} \sum_{n=1}^{3} (\tilde{x}^n)^2) \). Let us write \( \tilde{x} \) as \( x \), hence we obtain

\[ \ddot{x} = -2 \frac{\left( x \cdot \dot{x} \right)}{\Lambda} \dot{x} + 4 \Lambda \tilde{H} x. \]  \hspace{1cm} (3.6.4)

It is known that the Laplace Lenz Runge vector

\[ A(x, \dot{x}) = \frac{1}{\mu} \left( 2H + \frac{\mu}{|x|} x - \frac{1}{\mu} (x \cdot \dot{x}) \dot{x} \right) \]  \hspace{1cm} (3.6.5)
is a conserved quantity for the Kepler flow, we can re-write this equation using \( A(x, \dot{x}) \). Using the Laplace Lenz Runge vector we obtain

\[
\frac{2\mu}{\Lambda} (A(x, \dot{x}) - \frac{x}{|x|}) = -2 \frac{(x \cdot \dot{x}) \cdot \dot{x}}{\Lambda} + 4\Lambda \tilde{H} x,
\]

where \( E = \tilde{H}\Lambda^2 \). Thus equation (3.6.4) can be written as

\[
\ddot{x} + \frac{2\mu}{\Lambda} \frac{x}{|x|} = \frac{2\mu A}{\Lambda}.
\tag{3.6.6}
\]

### 3.6.1 Kepler equation parameterizing the eccentric anomaly

An advantage of the eccentric anomaly is that it is well suited to describe Kepler motion in position space. Therefore we derive the equation of motion w.r.t. this parameter.

Let us reparametrize the time as

\[
dt = \frac{|x|}{\epsilon} ds.
\tag{3.6.7}
\]

Thus we obtain

\[
\frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{\dot{x}}{\epsilon} |x|.
\]

The second derivative yields

\[
\frac{d^2 x}{ds^2} = \frac{1}{|x|^2} \left( x \cdot \frac{dx}{ds} \right) \frac{dx}{ds} + \frac{|x|^2}{\epsilon^2} \ddot{x} = \frac{1}{|x|^2} \left( x \cdot \frac{dx}{ds} \right) \frac{dx}{ds} - \frac{\mu x}{\epsilon^2 |x|} x,
\]

where we have used the Kepler equation \( \ddot{x} = -\frac{\mu x}{|x|^3} \). The Laplace Lenz Runge vector

\[
A(x, \dot{x}) = \frac{1}{\mu} \left( 2E + \frac{\mu}{|x|} \right) x - \frac{1}{\mu} (x \cdot \dot{x}) \dot{x} = -\frac{\epsilon^2}{\mu} \left[ \frac{1}{|x|^2} \left( x \cdot \frac{dx}{ds} \right) \frac{dx}{ds} - \frac{\mu x}{\epsilon^2 |x|} + x \right].
\]

Here we consider the case of negative energy, i.e. bounded orbits. Therefore we obtain

\[
\frac{d^2 x}{ds^2} + x = -\frac{\mu}{\epsilon^2} A.
\tag{3.6.8}
\]

Let us start with the Hamiltonian

\[
\tilde{H} = \alpha^2 = \left[ E - \frac{\epsilon^2}{2} \right]^2 \left( \sum_{n=1}^{3} \frac{\tilde{p}_n^2}{\tilde{m}_n} \right) = \frac{1}{4} \left( \epsilon^2 + |x|^2 \right)^2 |p|^2,
\]

where we have used \( 2E = -\epsilon^2 \).

Define

\[
G(x, p) = \tilde{H}^{1/2} = \frac{1}{2} (\epsilon^2 + |x|^2) |p|.
\]
We now consider regularized Kepler Hamiltonian system. The system of the Hamiltonian obtained from
\[ \tilde{G}(x, p) = \frac{1}{2\epsilon} (\epsilon^2 + |x|^2) |p| - \frac{\mu}{\epsilon}, \quad \epsilon \neq 0, \] (3.6.9)
is given by
\[ \dot{p} = |p|x \quad \dot{x} = -\frac{1}{|p|^2} (\tilde{G}(x, p) + \frac{\mu}{\epsilon}) \]
By the first equation, \( x = \frac{\epsilon}{|p|} \dot{p} \), we obtain
\[ \ddot{p} = \frac{1}{|p|^2} (p \cdot \dot{p}) \dot{p} - \frac{1}{\epsilon^2 |p|^2} (\tilde{G}(x, p) + \mu) p. \]
Its restriction to the level set \( \{(x, p) | \tilde{G}(x, p) = 0\} \) is flow of the Kepler problem in the momentum space parametrized by the eccentric anomaly.

4 Conclusion

So far, we have seen that the Jacobi metric transforms the dynamics from both time-independent Lagrangian and Hamiltonian perspective from a space with potential functions to an equivalent free particle geodesic of lower dimension. All aspects of integrability and the first integrals are preserved under such lifts. The Hamiltonian and Lagrangian of such metrics possess a conformal factor and equate to unity. Such a procedure can cast the TeVeS theory into the form of a Kaluza-Klein construction [18].

When applied to the Kepler problem, this holds true. Such a transformation for a particular energy level combined with Bohlin’s canonical transformation converts the isotropic oscillator problem to the Kepler problem. Houri’s canonical transformation is found to be incomplete without Milnor’s momentum inversion map, which preserves the form of geodesic flows as identical to that of the Kepler problem.

There are quite a few areas of Jacobi-Maupertuis have been less studied, for example, the Maupertuis principle can be used in the construction of the theory of many-valued functionals, which arises naturally in the study of the motion of charged particle in a scalar potential field in the presence of magnetic field [25]. It would be interesting to extended this project to the study of integrable magnetic geodesic flows [30, 31]. Recently this has been extended in [5] to present a modern outlook to describe the mechanism of the Maupertuis principle using classical integrable dynamical systems. This mechanism yields integrable geodesic flows and integrable system associated to curved spaces. In fact other related topics like the formulation of the Jacobi metric for time-like geodesics and its application to curved space-time [10], applications of geodesic instabilities for the planar gravitational three-body problem [16] should get more attention. The application of this analysis to the generalized MICZ-Kepler problem would be fascinating.

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