On feedback classification of control-affine systems with one and two-dimensional inputs

Andrei Agrachev∗ Igor Zelenko †

Abstract

The paper is devoted to the local classification of generic control-affine systems on an $n$-dimensional manifold with scalar input for any $n \geq 4$ or with two inputs for $n = 4$ and $n = 5$, up to state-feedback transformations, preserving the affine structure (in $C^\infty$ category for $n = 4$ and $C^\omega$ category for $n \geq 5$). First using the Poincare series of moduli numbers we introduce the intrinsic numbers of functional moduli of each prescribed number of variables on which a classification problem depends. In order to classify affine systems with scalar input we associate with such a system the canonical frame by normalizing some structural functions in a commutative relation of the vector fields, which define our control system. Then, using this canonical frame, we introduce the canonical coordinates and find a complete system of state-feedback invariants of the system. It also gives automatically the micro-local (i.e. local in state-input space) classification of the generic non-affine $n$-dimensional control system with scalar input for $n \geq 3$ (in $C^\infty$ category for $n = 3$ and in $C^\omega$ category for $n \geq 4$). Further we show how the problem of feedback-equivalence of affine systems with two-dimensional input in state space of dimensions 4 and 5 can be reduced to the same problem for affine systems with scalar input. In order to make this reduction we distinguish the subsystem of our control system, consisting of the directions of all extremals in dimension 4 and all abnormal extremals in dimension 5 of the time optimal problem, defined by the original control system. In each classification problem under consideration we find the intrinsic numbers of functional moduli of each prescribed number of variables according to its Poincare series.

Key words: State-feedback equivalence, control-affine systems, Poincare series, extremals.

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1 Introduction

In the paper for the convenience of the presentation all objects are $C^\infty$ without special mentioning, although all constructions and some statements remain valid in an obvious way also in $C^k$ category for an appropriate finite $k$. On the other hand, some of the statements are known to be true only in the real analytic category, which will be indicated explicitly.

∗S.I.S.S.A., Via Beirut 2-4, 34014 Trieste Italy and Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow Russia; email: agrachev@sissa.it
†S.I.S.S.A., Via Beirut 2-4, 34014 Trieste Italy; email: zelenko@sissa.it
Let $M$ be an $n$-dimensional manifold, $f_0, f_1, \ldots, f_r$ be vector fields on $M$, $r < n$. Consider the following control-affine system with $r$ inputs on $M$

$$\dot{q} = f_0(q) + \sum_{k=1}^{r} u_k f_r(q), \quad q \in M, \quad u_1 \ldots u_r \in \mathbb{R}. \quad (1.1)$$

We say that a system of the type (1.1) is an $(r, n)$ control-affine system. We also assume that at given point $q_0$

$$\dim \text{span}(f_0(q_0), f_1(q_0), \ldots, f_r(q_0)) = r + 1. \quad (1.2)$$

Consider the group $FB_{q_0}$ of state-feedback transformations, preserving an affine structure and the point $q_0$, i.e. transformations of the type

$$\begin{cases}
q = \Phi(\tilde{q}) \\
u = B(\tilde{q})\tilde{u} + A(\tilde{q}) \\
q_0 = \Phi(q_0),
\end{cases} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}, \quad (1.3)$$

where $\Phi$ is a diffeomorphism in a neighborhood of $q_0$, $A(q) \in \mathbb{R}^r$, $B(q)$ is a $r \times r$-matrix, $\det B(q_0) \neq 0$. This group of transformations acts naturally on the set of germs of systems of the type (1.1) and defines the equivalence relation, called state-feedback equivalence. The natural question is when two germs of the systems of the type (1.1) are state-feedback equivalent.

Let us roughly estimate the ”number of parameters” in the considered equivalence problem. The set of $r$-dimensional affine subspaces in $\mathbb{R}^n$ forms $(r + 1)(n - r)$-dimensional manifold. Therefore, if the coordinates on $M$ are fixed, then the control system of the type (1.1) can be defined by $(r + 1)(n - r)$ functions of $n$ variables. The group of the coordinate changes on $M$ is parameterized by $n$ functions of $n$ variables. So, by a coordinate change one can ”normalize”, in general, only $n$ functions among those $(r + 1)(n - r)$ functions, defining our control system. Thus we may expect that the set of orbits of generic germs of systems (1.1) at $q_0 \in M$ w.r.t. the action of the group of transformations of the type (1.3) can be parameterized by $(r + 1)(n - r) - n = r(n - r - 1)$ arbitrary germs of functions of $n$ variables and a number of germs of functions, depending on less than $n$ variables (see the next section for the discussion about the number of these additional functional invariants).

According to the last estimate the only case when the functional parameters (also called functional moduli) are not expected in the parameterization of the set of orbits of generic germs of systems (1.1) is the case $r = n - 1$ (i.e., corank 1 control-affine systems). In this case, under assumption (1.2) there is the natural one-to-one correspondence between the set of control-affine systems, up to feedback transformations, and the set of differential 1-forms on the ambient manifold: to any affine system (1.1) one can assign a unique differential 1-form $\omega$ such that $\omega(f_i) = 0$ for $i = 1, \ldots n - 1$ and $\omega(f_0) = 1$. So, the state-feedback classification of corank 1 control-affine systems satisfying (1.2) is equivalent to the well-known classification of differential 1-forms w.r.t. the action of the group of diffeomorphisms (see, for example, [8], section 3 and Appendix C there ). In particular, all germs of (1.1) such that the underlying vector distribution $\text{span}(f_1, \ldots, f_{n-1})$ is contact for odd $n$ or quasi-contact for even $n$ (which is generic assumption) are state-feedback equivalent to the control-affine system, corresponding to the classical Darboux model (note also that in [8] normal forms for codimension 1 singularities are given too). In the context of corank 1 control-affine systems it is not worse also to mention the works [3], [6], [9], where the case $f(q_0) \in \text{span}(f_1(q_0), \ldots, f_{n-1}(q_0))$ was treated.

The case $r = 1$, $n = 3$ (the smallest dimensions, when the functional parameters appear) was treated in [1] (section 3, Proposition 3.2 there). In particular, it was shown that the set of
orbits of generic germs of the systems w.r.t. the action of the group of transformations of the type \( (1.3) \) can be parametrized by one arbitrary function of 3 variables, 2 arbitrary functions of two variable , and the discrete invariant from the set \( \{-1,1\} \).

**Remark 1** Actually, in Proposition 3.2 of \([1]\) the two functions of two variables satisfy certain conditions on coordinate subspaces of some special coordinates, which are canonical up to some reflections, but instead of these functions one can take their appropriate partial derivatives, which are already arbitrary. The functions of the parameterization are state-feedback invariants up to some reflections in the coordinates. \( \square \)

In the present paper we make a classification of generic germs of systems of the type \( (1.1) \), up to state-feedback equivalence, in the following cases

1. \( r = 1, \; n = 4 \);
2. \( r = 1, \; n \geq 5 \) in the real analytic category;
3. \( r = 2, \; n = 4 \);
4. \( r = 2, \; n = 5 \) in the real analytic category.

In general, statements of the kind "the classification problem depends on the tuple of functional invariants, consisting of certain number of functions of each number of variables" need to be clarified: although the number of functional invariants, depending on the maximal number of variables, could be found in the more or less rigorous way by counting of dimensions arguments (as was done above), it seems that the numbers of functions of each number of variables less than the maximal one depend on the way of normalization of the objects in the considered classification problem. Besides, these numbers could be changed rather arbitrary by mixing, combining or separating the formal Taylor series of this functional invariants without losing any information (at least if we work in the category of formal Taylor series or in the real analytic category). In \([2]\) (section 1 there) it was proposed to use the so-called *Poincare series of the moduli numbers of the classification problem* in order to determine intrinsically the number of functions of each prescribed number of variables, on which some classification problem depends. In section 2 below, using the Poincare series, we give a canonical selection of these numbers. The way the canonical parameterization is obtained indicates the presence of an interesting algebraic structure on the set of all tuples of fundamental invariants parameterizing given classification problem. For the moment, this algebraic structure remains hidden and needs further research.

In the case of scalar input our method of the classification is similar to the procedure, used in \([1]\) for the case \( n = 3 \) and it is described in section 3. It consists basically of the following two steps: first for any control system, satisfying some genericity assumptions, we construct the canonical frame by normalizing some structural functions in a commutative relation of the vector fields, which define our control-affine systems; then, using this canonical frame, we introduce the canonical coordinates and find the complete system of state-feedback invariants of the system.

Besides, to any control system

\[
\dot{y} = F(y,v), \quad y \in S, v \in V
\]  

(1.4)

on an \( m \)-dimensional manifold \( S \) (the state space) with one-dimensional control space \( V \) one can assign the following control-affine system on the \((m+1)\)-dimensional state-space \( S \times V \).

\[
\begin{cases}
\dot{y} = F(y,v) \\
\dot{v} = u
\end{cases}, \quad v \in \mathbb{R}
\]  

(1.5)
(here we look on \( u \) as on a new state variable, \( v \) is the new control, \( f_0 = (F(y,v),0)^T \) and \( f_1 = (0,1)^T \) in the notations of (1.1)). It turns out that having the local classification of generic \( m+1 \)-dimensional control-affine systems with scalar input, one gets also the micro-local (i.e. local in state-input space) classification of the generic \( m \)-dimensional control system (see, Remark 4 at the end of section 3).

Further, in section 4, we show that the problem of state-feedback classification of the control-affine systems with two-dimensional input in dimensions 4 and 5 can be reduced to the previous problem for the control-affine systems with scalar input in the same dimensions. In order to make this reduction we distinguish the subsystem, consisting of the directions of all extremals in dimension 4 and all abnormal extremals in dimension 5 of the time optimal problem, defined by the original control system.

In each classification problem under consideration we find the intrinsic numbers of functional moduli of each prescribed number of variables according to its Poincare series.

Finally note that the problem, considered here, is different from one, considered in the paper [7], which has the similar title. In the mentioned paper the authors study germs of \( n \)-dimensional control-affine system with scalar input at an equilibrium point \( q_0 \), i.e. when \( f_0(q_0) \in \{\mathbb{R}f_1(q_0)\} \). Their method is a generalization of technique, developed in [5] and [4], which is similar to classical Poincare-Dulac procedure for normalization of vector fields near a stationary point. Therefore in the method of [7] it is crucial that \( q_0 \) is an equilibrium point. Here we classify the control-affine systems with scalar control near non-equilibrium point, which seems very natural in view of the fact that this classification, except the case \( n = 2 \), a priori contains functional moduli. Besides, the feedback invariants, constructed here for generic germs could be used also for the problem with equilibrium points by passing to the limit.

2 Poincare series and the intrinsic number of functional invariants.

We start with some terminology. Let \( M \) be a smooth manifold. Fix a point \( q_0 \in M \). Consider a set \( \mathcal{O} \) of germs at \( q_0 \) of smooth objects on \( M \) (for example, Riemanian metrics, vector distributions, control-affine systems) such that the group of local diffeomorphisms \( \text{Diff}_{q_0} \), preserving the point \( q_0 \), acts naturally on it. This action defines the equivalence relation on \( \mathcal{O} \).

Denote by \( J^k(\mathcal{O}) \) the space of all \( k \)-jets at \( q_0 \) of objects from the set \( \mathcal{O} \). We say that the set \( \tilde{\mathcal{O}} \subset \mathcal{O} \) is a generic subset of \( \mathcal{O} \) if there exists an integer \( k \geq 0 \) and a Zariski open set \( U \) in \( J^k(\mathcal{O}) \) such that
\[
\tilde{\mathcal{O}} = \{ b \in \mathcal{O} : k\text{-jet of } b \text{ belongs to } U \}.
\]

By classification problem on \( \mathcal{O} \) we mean the problem to find a system of fundamental invariants for objects from some generic subset of \( \mathcal{O} \) such that two generic objects are equivalent if and only if they have the same systems of fundamental invariants. Let \( \tilde{\mathcal{O}} \) be a generic subset of \( \mathcal{O} \), which is invariant w.r.t. the action of the group \( \text{Diff}_{q_0} \).

**Definition 1** A mapping \( F \) from the set \( \tilde{\mathcal{O}} \) to the set \( C_0^\infty(\mathbb{R}^l, \mathbb{R}) \) of germs at 0 of smooth functions in \( \mathbb{R}^l \), which is invariant w.r.t. the action of the group \( \text{Diff}_{q_0} \) on \( \tilde{\mathcal{O}} \), is called a functional invariant of \( l \) variables of a generic subset of objects from \( \mathcal{O} \).

When the object \( b \in \tilde{\mathcal{O}} \) is fixed, we will mean by the functional invariant also the value of the mapping \( F \) at \( b \). We will denote this germ of function by the same letter \( F \) without special mentioning.
Now let us describe the Poincare series of the moduli numbers of the classification problem. The action of the group $\text{Diff}_{\mathbb{R}}$ induces the action $\mathcal{A}_k$ of some finite dimensional Lie group $G_k$ on the space $J^k(\mathcal{O})$ for any integer $k \geq 0$. So, $\mathcal{A}_k$ is a mapping from $G_k \times J^k(\mathcal{O})$ to $J^k(\mathcal{O})$. Given any $b \in J^k(\mathcal{O})$ let $\mathcal{A}_k^b$ be the mapping from $G_k$ to $J^k(\mathcal{O})$ such that $\mathcal{A}_k^b(\cdot) = \mathcal{A}_k(\cdot, b)$. Let $e_k$ be the identity of the group $G_k$. Set

$$m(k) = \dim J^k(\mathcal{O}) - \max_{b \in J^k(\mathcal{O})} \text{rank} \, d\mathcal{A}_k^b(e_k) = \min_{b \in J^k(\mathcal{O})} \text{corank} \, d\mathcal{A}_k^b(e_k)$$

(2.1)

(here $\text{rank} \, d\mathcal{A}_k^b(e_k)$ and $\text{corank} \, d\mathcal{A}_k^b(e_k)$ are the rank and the corank of the differential of $\mathcal{A}_k^b$ at $e_k$ respectively). Roughly speaking $m(k)$ is the dimension of the space of orbits w.r.t. the last action $\mathcal{A}_k$. The number $m(k)$ is called the moduli number of the $k$-jets. The Poincare series of the moduli numbers of the classification problem (or shortly the Poincare series of the classification problem) is by definition the following function:

$$M(t) = \sum_{k=0}^{\infty} m(k) t^k$$

(2.2)

**Remark 2** Since the integer-valued function $b \mapsto \text{rank} \, d\mathcal{A}_k^b(e_k)$ takes its maximal value at a Zarisky open set, in (2.1) we can replace $\mathcal{O}$ by any its generic subset $\tilde{\mathcal{O}}$. □

The Poincare series could be useful in evaluating of the number of the functional invariants of the given number of variables, on which the given classification problem depends, because of the following well-known fact: if one denotes by $j_i(k)$ the dimension of the space $J^k(\mathbb{R}^l, \mathbb{R})$ of $k$-jets of functions of $l$-variables, then the corresponding Poincare series of numbers $j_i(k)$ satisfies

$$\sum_{k=0}^{\infty} j_i(k) t^k = \frac{1}{(1-t)^{l+1}}$$

(2.3)

(here one uses that $j_i(k) = \frac{(l+k)!}{l!k!}$). So, if, for example, the Poincare series of some classification problem is equal to

$$M(t) = w \sum_{i=1}^{n} \frac{p_i}{(1-t)^{i+1}}$$

(2.4)

where all $p_i$ are nonnegative integers, then it is natural to conclude that this problem depends on the tuple, consisting of $p_i$ functional invariants of $i$ variables for each $1 \leq i \leq n$, while the parameter $w$ (i.e. the order of zero of the Poincare series $M(t)$ at $t = 0$) is equal to the minimal $k \geq 0$ such that the action of the group $G_k$ on the space of $J^k(\mathcal{O})$ is non-transitive.

Till now nothing is known about the form of the functions $M(t)$ for general classification problems. For example, the following open question is stated in [2]: Is it true that the Poincare series of moduli numbers are rational functions in the most classification problems? In the next sections we will show by direct computations that in all classification problems 1-4 listed in Introduction it is true and moreover the function $M(t)$ has a unique pole at $t = 1$. On the other hand, in all considered cases (except the cases $r = 1$, $n = 3$ or 5) the Poincare series has no a representation of the type (2.4) with nonnegative $p_i$. Below we give an algorithm to extract the number of functional invariants from Poincare series also in these cases.

From now on we will suppose that the Poincare series $M(t)$ of the classification problem is a rational function with a unique pole at $t = 1$. Let $w_0$ be the order of zero of the function $M(t)$ at $t = 0$. 

5
Lemma 1 For any integers \( w \geq w_0 \) and \( l \) there exist a unique polynomial \( R(t) \) with

\[
\deg R(t) < w - w_0
\]  

and a unique rational function \( Q(t) \) with the unique pole at \( t = 1 \) such that

\[
M(t) = \frac{t^{w_0}R(t)}{(1-t)^{l+1}} + t^wQ(t).
\]  

Proof. Let us fix \( l \in \mathbb{Z} \) and prove the existence of a representation of the type \((2.6)\) for any \( w \geq w_0 \) by induction in \( w \).

If \( w = w_0 \), then from the condition \((2.5)\) it follows that \( R(t) \equiv 0 \). Then by definition of order of zero the function \( Q(t) = \frac{1}{t^{w_0}}M(t) \) is rational with the unique pole at \( t = 1 \), which implies \((2.6)\).

Now suppose that a representation of the type \((2.6)\) exists for some \( w = \bar{w} \geq w_0 \), and prove its existence for \( w = \bar{w} + 1 \). For this let \( Q(t) \) and \( R(t) \) be as in the representation \((2.6)\) for \( w = \bar{w} \). Denote by

\[
Q_1(t) = \frac{1}{t} \left( Q(t) - \frac{Q(0)}{(1-t)^{l+1}} \right).
\]  

Then by construction \( Q_1 \) is also the rational function with the unique pole at \( t = 1 \). Expressing \( Q(t) \) from \((2.7)\) and substituting it into \((2.6)\), one has

\[
M(t) = \frac{t^{w_0}(R(t) + Q(0)t^{ar{w} - w_0})}{(1-t)^{l+1}} + t^{ar{w}+1}Q_1(t).
\]  

Since \( \deg (R(t) + Q(0)t^{ar{w} - w_0}) < \bar{w} - w_0 + 1 \) it implies the existence of a representation \((2.6)\) also for \( w = \bar{w} + 1 \). This completes the proof by induction of the existence part of the lemma.

Now let us prove the uniqueness part. If there exists another representation of \( M(t) \) of the type \((2.6)\) with a polynomial \( \bar{R}(t) \), \( \deg \bar{R}(t) < w - w_0 \), and a rational function \( \bar{Q}(t) \) instead of \( R(t) \) and \( Q(t) \), then we have the following identity

\[
\bar{R}(t) - R(t) = t^{w-w_0}(1-t)^{l+1} \left( Q(t) - \bar{Q}(t) \right).
\]  

It implies that the polynomial \( \bar{R}(t) - R(t) \) has zero of order not less than \( w - w_0 \). On the other hand, by assumptions \( \deg (R(t) - \bar{R}(t)) < w - w_0 \), which implies that \( R(t) \equiv \bar{R}(t) \) and then also \( Q(t) \equiv \bar{Q}(t) \). \( \square \)

We will call the representation \((2.6)\) (with \( R(t) \) satisfying \((2.5)\)) the \((w, l)\)–representation of the function \( M(t) \). Let \( N \) be the order of pole of \((1-t)M(t)\) at \( t = 1 \).

Definition 2 The \((w, l)\)-representation \((2.6)\) of \( M(t) \) with \( R(t) \) and \( Q(t) \) satisfying

\[
R(t) = \sum_{i=0}^{w-w_0-1} r_it^i, \quad Q(t) = \sum_{j=l_1}^{N} \frac{q_j}{(1-t)^{j+1}}, \quad q_{l_1} \neq 0
\]  

is called nice, if \( 1 \leq l \leq N, \ l_1 \geq l, \) and all coefficients \( r_i, q_j \) in \((2.6)\) are nonnegative integers.
Of course, in general a rational function \( M(t) = \frac{t^{w_0}Z(t)}{(1-t)^{N}}, \) where \( Z(t) \) is a polynomial (even with integer coefficients), may not have any nice \((w,l)-\)representation. But if the function \( M(t) \) is the Poincare series of a classification problem, which can be parameterized by functional invariants in some reasonable way, then \( M(t) \) has at least one nice representation.

To be more precise and to explain why the nice representation of the Poincare series are interesting let us introduce an additional terminology. Let \( F \) be a functional invariant of \( l \) variables of a generic subset \( \mathcal{O} \) of objects from \( \mathcal{O} \). Denote by \( \pi_k \) the natural projection from the set \( \mathcal{O} \) to the space \( J^k(\mathcal{O}) \). Let \( (x_1, \ldots, x_l) \) be the standard coordinates in \( \mathbb{R}^l \). Let \( C^\infty_0(\mathbb{R}^l, \mathbb{R}) \) be the set of germs at 0 of smooth functions in \( \mathbb{R}^l \). Given a tuple \((i_1, \ldots, i_l)\) of nonnegative integers denote by \( D(i_1, \ldots, i_l) \) the mapping from \( C^\infty_0(\mathbb{R}^l, \mathbb{R}) \) to \( \mathbb{R} \), which assigns to any function \( f: \mathbb{R}^l \to \mathbb{R} \) the value of its partial derivative \( \frac{\partial^{i_1+\ldots+i_l}f}{\partial x_1^{i_1}\ldots\partial x_l^{i_l}} \) at 0.

**Definition 3** We say that the functional invariant \( F \) of \( l \) variables of a generic subset \( \mathcal{O} \subset \mathcal{O} \) has the weight \( w \) if for any integer \( k \geq w \) the following condition holds: for any tuple \((i_1, \ldots, i_l)\) of nonnegative integers with \( \sum_{s=1}^{l} i_s \leq k - w \) there exists a mapping \( \mathcal{G}_{(i_1, \ldots, i_l)}: J^k(\mathcal{O}) \to \mathbb{R} \) such that the following diagram

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{F} & C^\infty_0(\mathbb{R}^l, \mathbb{R}) \\
\pi_k \downarrow & & \downarrow D(i_1, \ldots, i_l) \\
J^k(\mathcal{O}) & \xrightarrow{\mathcal{G}_{(i_1, \ldots, i_l)}} & \mathbb{R}
\end{array}
\]

is commutative, while for any tuple \((i_1, \ldots, i_l)\) of nonnegative integers with \( \sum_{s=1}^{l} i_s > k - w \) such mapping \( \mathcal{G}_{i_1, \ldots, i_l} \) does not exist.

Essentially the weight of the functional invariant \( F \) is the integer \( w \geq 0 \) such that for any \( k \geq w \) the \( k \)-jets of our objects depend on all derivatives of \( F \) of order not greater than \( k - w \) but do not depend on derivatives of \( F \) of order greater than \( k - w \).

The classification problem with functional moduli will be called **regular**, if there exist integers \( w_1, w_2, n_1, \) and \( n_2 \) (where \( 0 \leq w_1 \leq w_2, 1 \leq n_1 \leq n_2 \)) and a \((n_2 - n_1 + 1) \times (w_2 - w_1 + 1)\) matrix \( P \) with nonnegative entries \( p_{ij} \) such that the set of orbits of generic germs of systems \( (\mathcal{L}_i) \) w.r.t. the group \( \text{Diff}_{\mathcal{O}_0} \) can be parameterized by the set of functional invariants, consisting of \( p_{ij} \) functional invariants of \( i + n_1 - 1 \) variables and weight \( j + w_1 - 1 \) for any \( 1 \leq i \leq n_2 - n_1 + 1 \) and \( 1 \leq j \leq w_2 - w_1 + 1 \). Without loss of generality we suppose also that on the first and the last column and row of the matrix \( P \) there is at least one nonzero entry. Directly from the definition of weight, formula \( (2.2) \), and Remark \( 2 \) it follows that the Poincare series of such classification problem satisfies

\[
M(t) = \frac{t^{w_1-1}}{(1-t)^{n_1}} \sum_{j=1}^{w_2-w_1+1} \frac{t^j}{\sum_{i=1}^{n_2-n_1+1} \frac{p_{ij}}{(1-t)^{i+1}}}.
\]

(2.10)

Note that \( w_1 = w_0 \) and \( n_2 = N \), where as before \( w_0 \) is the order of zero of \( M(t) \) at \( t = 0 \) and \( N \) is the order of pole of \((1-t)M(t)\) at \( t = 1 \), but all other parameters, appearing in \( (2.10) \), could not be uniquely recovered from the Poincare series \( M(t) \). In the considered situation we will say that the classification problem admits a regular \((w_2, n_1)\)-parameterization with parameterization matrix \( P \).
Given some \((w_2, n_1)\)-parameterization of the classification problem one can easily build new \((w_2, n_1)\)-parameterization with another parameterizations matrices. Indeed, take some functional invariant \(F\) of the weight \(j_0\), depending on \(i_0\) variables, say \(x_1, x_2, \ldots, x_{i_0}\), where \(2 \leq i_0 \leq n_2 - n_1 + 1\) and \(1 \leq j_0 \leq w_2 - w_1\). Let \(G(x_1, \ldots, x_{i_0})\) be the function such that

\[
F(x_1, \ldots, x_{i_0}) = F(x_1, \ldots, x_{i_0-1}, 0) + x_{i_0}G(x_1, \ldots, x_{i_0}).
\]

(2.11)

Then we can obtain the new parameterization of the classification problem by replacing the functional invariant \(F(x_1, \ldots, x_{i_0})\) by two functional invariants \(F(x_1, \ldots, x_{i_0-1}, 0)\) and \(G(x_1, \ldots, x_{i_0})\). Obviously, the first invariant has weight \(j_0\) and depends on \(i_0 - 1\) variables, while the second one has weight \(j_0 + 1\) and depends on \(i_0\) invariants. The matrix of the new parameterization is obtained from the original one by decreasing the \((j_0, j_0)\)-entry by 1 and increasing both \((i_0 - 1, j_0)\)-entry and \((i_0, j_0 + 1)\) entry by 1. Such transformation on the set of \((N - n_1 + 1) \times (w_2 - w_0 + 1)\) matrices will be called an elementary transformation. Conversely, given two functional invariants \(G_1\) and \(G_2\) such that \(G_1\) depends on \(i_0 - 1\) variables say \(x_1, \ldots, x_{i_0-1}\) and has the weight \(j_0\), while \(G_2\) depends on \(i_0\) variables say \(x_1, \ldots, x_{i_0}\) and has the weight \(j_0 + 1\) (here again \(2 \leq i_0 \leq n_2 - n_1 + 1\) and \(1 \leq j_0 \leq w_2 - w_1\)) one can build the new parameterization by replacing the invariants \(G_1\) and \(G_2\) by one invariant \(G_1 + x_{i_0}G_2\), which depends on \(i_0\) variables and has the weight \(j_0\). Of course, in this case the matrix of the new parameterization is obtain from the original one by the transformation, which is inverse to the elementary one.

Now for convenience denote

\[
K_1 = N - n_1 + 1, \quad K_2 = w_2 - w_0 + 1.
\]

(2.12)

Note that among all matrices, which can be obtained from the given \(K_1 \times K_2\) matrix \(P\) with integer entries by a composition of a finite number of elementary transformations and their inverses, there exists a unique matrix, denoted by \(\text{Norm}(P)\), such that all its entries, except those lying on the first row and the last column, are equal to zero. To prove the existence of \(\text{Norm}(P)\) one can vanish the entries of the matrix \(P\) by a composition of elementary transformations and their inverses step by step, starting from the entry in the left-lower corner, going along the first column from the bottom to the top till the entry on the second row, then passing to the bottom of the second column, going along it from the bottom to the top till the entry on the second row and so on till the column before the last one. The uniqueness follows from the fact that if we put the entries of the matrix \(\text{Norm}(P)\) instead of the entries of \(P\) in the representation (2.10), then we obtain the \((w_2, n_1)\)-representation of the Poincare function \(M(t)\). This representation is unique according to Lemma 1 and the matrix \(\text{Norm}(P)\) is obviously uniquely recovered from it. Also it is not difficult to express all nontrivial entries of \(\text{Norm}(P)\) by the entries of \(P\):

\[
(\text{Norm}(P))_{1j} = p_{1j} + \sum_{l=0}^{j-1} \sum_{k=1}^{K_1-1} \binom{k + l - 1}{l} p_{k+1,j-l}, \quad 1 \leq j \leq K_2 - 1,
\]

(2.13a)

\[
(\text{Norm}(P))_{i,K_2} = p_{i,K_2} + \sum_{l=0}^{K_2-1} \sum_{k=1}^{K_1-1} \binom{K_2 - k + l - 1}{l} p_{i+l,k}, \quad 2 \leq i \leq K_1,
\]

(2.13b)

\[
(\text{Norm}(P))_{1,K_2} = p_{1,K_2}
\]

(2.13c)

The last relation can be proved, for example, using the procedure of passing from \(P\) to \(\text{Norm}(P)\), described above, and the following well-known combinatorial identity:

\[
\sum_{i=1}^{n} \binom{i + k - 1}{k} = \binom{n + k}{k+1}.
\]
If the matrix $P$ has only nonnegative integer entries, then the matrix $\text{Norm}(P)$ is obtained from $P$ by a finite composition of elementary transformations (without using their inverses) and has also only nonnegative integer entries (which follows also from relations (2.13)). Moreover, if we put the entries of the matrix $\text{Norm}(P)$ instead of the entries of $P$ in the representation (2.10), then we obtain the nice $(w_2,n_1)$-representation of the Poincare function $M(t)$ of our classification problem. We also say that this nice representation corresponds to the matrix $\text{Norm}(P)$. We can summarize all above in the following

**Proposition 1** If the classification problem admits a regular $(w_2,n_1)$-parameterization with parameterization matrix $P$, then it admits a regular $(w_2,n_1)$-parameterization with parameterization matrix $\text{Norm}(P)$ and its Poincare series has the nice $(w_2,n_1)$-representation, which corresponds to the matrix $\text{Norm}(P)$.

The last proposition indicates that the nice representations of the Poincare series (if they exist) may be used in the definition of the intrinsic number of functional invariants of each number of variables and weight, on which the given classification problem depends. Suppose that the Poincare series $M(t)$ has the nice representation for some $(w,l)$. So, the set

$$\text{NS}(M(t)) \overset{\text{def}}{=} \{(w,l) : \text{the } (w,l)\text{-representation of } M(t) \text{ is nice}\} \quad (2.14)$$

is not empty. The natural question is what pair to choose from $\text{NS}(M(t))$? To answer this question we propose to introduce the order $<$ on the set of ordered pairs $(w,l)$ in the following way: $(w,l) < (\bar{w},\bar{l})$ if and only if $w < \bar{w}$ or $w = \bar{w}$, but $l > \bar{l}$. By Definition 2

$$\text{NS}(M(t)) \subset \{(w,l) : w \geq w_0, l \leq N\}, \quad (2.15)$$

which implies immediately that the set $\text{NS}(M(t))$ contains the minimal element w.r.t. the introduced order $<$. This minimal element will be called the characteristic pair of the classification problem. Denote it by $(\bar{w},\bar{l})$. Let $C$ be the $(N-l+1) \times (\bar{w}-w_0+1)$ matrix such that the $(\bar{w},\bar{l})$-representation of $M(t)$ corresponds to the matrix $C$.

**Definition 4** The $(ij)$-entry of the matrix $C$ is called the intrinsic number of the functional invariants of $i+l-1$ variables and the weight $j+w_0-1$ of the considered classification problem. The matrix $C$ is called the characteristic matrix of the classification problem. Any regular $(\bar{w},\bar{l})$-parameterization of the problem (if it exists) with the parameterization matrix $C$ is called the characteristic regular parameterization.

In general, it is better to have a parameterization, consisting of invariants, which have minimal possible weight and depend on maximal possible number of variables. Our definition of characteristic parameterization is in accordance with this goal. Actually the maximal weight of invariants, appearing in characteristic regular parameterization is not greater than the maximal weight of invariants, appearing in any other regular parameterization. Besides the minimal number of variables in invariants of regular parameterization is not less than the minimal number of variables in invariants of any other regular parameterization, having the same maximal weight of invariants as a characteristic one.

One can improve the formula (2.15) for the localization of the set $\text{NS}(M(t))$. Indeed let $d$ be the degree of the rational function $M(t)$ at infinity. Namely, if $M(t) = \frac{Q_1(t)}{Q_2(t)}$, where $Q_1(t)$ and $Q_2(t)$ are polynomials, then $d = \deg Q_1(t) - \deg Q_2(t)$. Then

$$\text{NS}(M(t)) \subset \{(w,l) : w \geq w_0, 1 \leq l \leq \min (w-d-1, N)\}. \quad (2.16)$$

9
To prove \(2.16\) we actually have to prove that if the pair \((w, l) \in \text{NS}(M(t))\) then \(l \leq w - d - 1\). Indeed, from \(2.16\) and \(2.19\) it follows that \(d = \max(w - l_1 - 1, w_0 + \deg R - l_1 - 1)\). But first, since \(l_1 > l\), we have \(w - l_1 - 1 \leq w - l - 1\) and secondly, since \(\deg R < w - w_0\), we have \(w_0 + \deg R - l_1 - 1 < w - l - 1\). Therefore \(d \leq w - l - 1\), Q.E.D.

From \(2.16\) it follows also that

\[
\text{NS}(M(t)) \subset \{(w, l) : w \geq \max(w_0, d + 2)\}. \tag{2.17}
\]

The relations \(2.16\) and \(2.17\) may be useful in searching for the characteristic pair of the classification problem. Another useful property of the set \(\text{NS}(M(t))\) can be formulated as follows:

**Lemma 2** Assume that the function \(M(t)\) has the nice \((w, l)\)-representation \(2.6\), the functions \(R(t)\), \(Q(t)\), and the number \(l_1\) are as in \(2.2\), and \(l_1 = l\) (or, equivalent, \(q_l > 0\)), then \((w - 1, l - 1) \notin \text{NS}(M(t))\).

**Proof.** Let \(S(t)\) be the polynomial such that \(M(t) = \frac{S(t)}{(1 - t)^{w - l}}\). Then, using the assumption \(l = l_1\), it is easy to get

\[
\deg S(t) = w + N - l. \tag{2.18}
\]

Moreover, directly from \(2.6\) and \(2.9\) one can obtain that

\[
\frac{d^{w + N - l}}{dt^{w + N - l}} S = (-1)^{N - l}(w + N - l)! q_l. \tag{2.19}
\]

On the other hand, if the \((w - 1, l - 1)\)-representation of \(M(t)\) has the form

\[
M(t) = \sum_{i=0}^{w - w_0 - 2} \tilde{r}_i t^i + t^{w - 1} \sum_{j=l_2}^{N} \tilde{q}_j (1 - t)^{j + 1}, \quad \tilde{q}_{l_2} \neq 0, \tag{2.20}
\]

then \(\deg S(t) = \max(w - 1 + N - l_2, w + N - l - 1)\). Comparing this with \(2.18\) one gets easily that \(l_2 = l - 1\). But then by analogy with \(2.19\) (applied for \((w - 1, l - 1)\)-representation instead of \((w, l)\)-representation) one has

\[
\frac{d^{w + N - l}}{dt^{w + N - l}} S = (-1)^{N - l + 1}(w + N - l)! \tilde{q}_{l - 1}. \tag{2.21}
\]

Comparing \(2.19\) and \(2.21\), we obtain that \(\tilde{q}_{l - 1} = -q_l\). Hence \(\tilde{q}_{l - 1} < 0\) and the \((w - 1, l - 1)\)-representation \(2.20\) is not nice. \(\Box\)

As a direct consequence of Proposition \(2.2\), the previous lemma, and the relation \(2.16\) one has the following

**Corollary 1** If the classification problem with the Poincare series \(M(t)\) admits the regular \((w, l)\)-parameterization with the parameterization matrix \(P\) such that the entry in the right-upper corner of \(P\) is positive (i.e., in the previous notations \(p_{1, w - w_0 + 1} > 0\)), then \((w - 1, l - 1) \notin \text{NS}(M(t))\).

In the sequel we will show that all classification problems 1-4 listed in Introduction are regular. For each of this problems we will describe explicitly some its regular \((w, l)\)-parameterization such that \((w, l)\) is the characteristic pair of the problem and find its characteristic matrix, which also gives the way to obtain the characteristic parameterization.
Remark 3 Sometimes (as in the case of (1,3) control-affine systems) in the parameterization of a classification problem appear invariants from the finite set in addition to the functional parameters. Also these functional parameters may be not invariants by themselves, but up to some finite group of transformation of coordinates. We call such classification problems quasiregular. Obviously, the additional invariants from the finite set and the additional freedom in choosing the coordinates, up to the finite group of transformation, do not affect on the Poincare series. Therefore the identity (2.10) for the Poincare series is true also for quasiregular problems, where, as before, $p_{ij}$ is the number of functional invariants of $i + n_1 - 1$ variables and weight $j + w_1 - 1$ for any $1 \leq i \leq n_2 - n_1 + 1$ and $1 \leq j \leq w_2 - w_1 + 1$, and we can extend to the quasiregular case the same terminology and the constructions, as in the regular one. In particular, according to Proposition 3.2 of [1] (see also Remark 1 and the paragraph before it in Introduction), the state-feedback classification problem for (1,3) control-affine system admits quasiregular $(2,2)$-parameterization with the following $2 \times 2$ parameterization matrix $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Besides, $w_0 = 1$ and $N = 3$. By the inverse to the elementary transformation one can transform $P$ to the matrix $\tilde{P} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The Poincare series $M(t)$ of the problem satisfies

$$M(t) = t \left( \frac{1}{(1-t)^3} + \frac{1}{(1-t)^4} \right).$$

It is not difficult to see that by erasing the last column of the matrix $\tilde{P}$ one obtains the characteristic matrix $C = (1,1)^T$ of the considered classification problem and the characteristic pair is equal to $(1,2)$. So, the characteristic quasiregular parameterization consists of one function of 3 variables and the weight 1, one function of 2 variables and the weight 1, and the discrete invariant from the set $\{-1,1\}$. This parameterization is obtained from the original one by a rearrangement of the invariants, which corresponds to the inverse to elementary transformation, transforming the matrix $P$ to the matrix $\tilde{P}$ (such rearrangements were described in the paragraph after the formula (2.11)). $\square$

3 Classification of $(1, n)$ control-affine systems for $n \geq 4$

For $r = 1$ the system [1.1] has the following form

$$\dot{q} = f_0(q) + u f_1(q), \quad q \in M, \quad u \in \mathbb{R}.$$  \hfill (3.1)

Our genericity assumptions are

$$\dim \text{span}(f_0, f_1, [f_1, f_0], \ldots, (\text{ad} f_1)^{n-2} f_0) = n,$$  \hfill (3.2)

and for $n \geq 5$ also

$$\dim \text{span}(f_0, f_1, [f_0, [f_0, f_1]], [f_1, f_0], \ldots, (\text{ad} f_1)^{n-3} f_0) = n.$$  \hfill (3.4)

The group of feedback transformations

$$u = \beta(q) \dot{u} + \alpha(q), \beta(q) \neq 0$$  \hfill (3.5)
acts naturally on the set of pairs of vector fields \((f_0, f_1)\). The orbit w.r.t. this action is
\[
\mathcal{O}_{(f_0, f_1)} = \{(f_0 + \alpha f_1, \beta f_1) : \alpha, \beta : M \to \mathbb{R} \text{ are functions, } \beta \neq 0\}. \tag{3.6}
\]
The first observation is given by the following

**Proposition 2** If the pair \((f_0, f_1)\) satisfies conditions (3.2) and (3.3), then there exists a unique pair \((F_0, F_1) \in \mathcal{O}_{(f_0, f_1)}\) such that
\[
[F_0, [F_0, F_1]] = F_0 + I_1 F_1 + I_2 [F_1, F_0] + \sum_{k=3}^{n-2} I_k (\text{ad} f_1)^k F_0 \tag{3.7}
\]

**Proof.** By assumptions (3.2) the vector fields \(f_0, f_1, [f_1, f_0], \ldots, (\text{ad} f_1)^{n-2} f_0\) constitute the frame on \(M\). Therefore there are functions \(E, N, J_1, \ldots, J_{n-2}\) such that
\[
[f_0, [f_0, f_1]] = E f_0 + J_1 f_1 + J_2 [f_1, f_0] + Z [f_1, [f_1, f_0]] + \sum_{k=3}^{n-2} J_k (\text{ad} f_1)^k f_0 \tag{3.8}
\]
Take some pair \((\tilde{f}_0, \tilde{f}_1) \in \mathcal{O}_{(f_0, f_1)}\),
\[
\tilde{f}_0 = f_0 + \alpha f_1, \quad \tilde{f}_1 = \beta f_1. \tag{3.9}
\]
Suppose that
\[
[f_0, [\tilde{f}_0, \tilde{f}_1]] = E \tilde{f}_0 + J_1 \tilde{f}_1 + J_2 [\tilde{f}_1, \tilde{f}_0] + Z [\tilde{f}_1, [\tilde{f}_1, \tilde{f}_0]] + \sum_{k=3}^{n-2} J_k (\text{ad} \tilde{f}_1)^k \tilde{f}_0 \tag{3.10}
\]
First note that
\[
\tilde{E} = \beta E. \tag{3.11}
\]
It follows immediately from (3.9) and the following relations
\[
[f_0, [\tilde{f}_0, \tilde{f}_1]] \equiv \beta f_0 + \alpha \beta f_1 + \alpha (\text{ad} f_1)^k f_0 \mod \text{span}(f_1, [f_0, f_1]), \tag{3.12}
\]
\[
(\text{ad} f_1)^k f_0 \in \text{span}(f_1, \ldots, (\text{ad} f_1)^k f_0), \quad k \in \mathbb{N}. \tag{3.13}
\]
From assumption (3.3) it follows that \(E \neq 0\). Therefore, taking \(\beta = \frac{1}{E}\), we make
\[
\tilde{E} = 1. \tag{3.14}
\]
Let us denote by \(\overline{\mathcal{O}}_{(f_0, f_1)}\) the set of all pairs \((\tilde{f}_0, \tilde{f}_1)\), satisfying (3.11). We can assume from the beginning that the original pair \((f_0, f_1)\) belongs to \(\overline{\mathcal{O}}_{(f_0, f_1)}\), i.e. \(E = 1\) (we make this assumption just in order to avoid extra notations). If \((\tilde{f}_0, \tilde{f}_1) \in \overline{\mathcal{O}}_{(f_0, f_1)}\), then also \(\tilde{E} = 1\). Hence \(\beta = 1\) or equivalently \(f_1 = \tilde{f}_1\). In other words, condition (3.14) normalizes the vector field \(f_1\) or the direction defining the straight line in the set of admissible velocities of the system (3.6) at any point.

Further, from (3.9), taking into account that \(\beta = 1\), it follows easily that
\[
(\text{ad} f_1)^k f_0 \equiv (\text{ad} \tilde{f}_1)^k \tilde{f}_0 \mod \text{span}(f_1), \quad k \in \mathbb{N}. \tag{3.15}
\]
This and relation (3.12) imply that
\[
\tilde{Z} = Z - \alpha. \tag{3.15}
\]
Setting \(\alpha = Z\), we make \(\tilde{Z} = 0\), which normalizes the drift \(\tilde{f}_0\). So, we have proved that there is a unique \((\tilde{f}_0, \tilde{f}_1) \in \overline{\mathcal{O}}_{(f_0, f_1)}\) such that \(\tilde{E} = 1\) and \(\tilde{Z} = 0\), which completes the proof of the proposition. □
Remark 4 The mappings $I_1, \ldots, I_{n-2}$ from $M$ to $\mathbb{R}$, defined by identity (3.7), are state-feedback invariants of the control system (3.1). $\square$

The vector field $F_0$ and the pair of vector fields $(F_0, F_1)$ from Proposition 2 are called the canonical drift and the canonical pair of the system (3.1) respectively.

Remark 5 Actually, in the case $n = 4$, the vector $F_0(q)$ is the velocity of the unique abnormal extremal starting at $q$ of the time optimal problem defined by system (3.1). $\square$

Now fix some point $q_0 \in M$. Denote by $e^f$ the flow, generated by the vector field $f$ and $q \circ e^f$ the image of the point $q$ w.r.t. this flow. Let $\Phi_n : \mathbb{R}^n \rightarrow M$ be the following mapping

$$
\Phi_4(x_1, x_2, x_3, x_4) = q_0 \circ e^{x_4[F_1, F_0]} \circ e^{x_3[F_1, F_0]} \circ e^{x_2 F_1} \circ e^{x_1 F_0},
$$

(3.16)

$$
\Phi_n(x_1, \ldots, x_n) = q_0 \circ e^{x_n(ad F_1)^{n-3} F_0} \circ \ldots \circ e^{x_3[F_1, F_0]} \circ e^{x_2 F_1} \circ e^{x_1 F_0},
$$

(3.17)

From assumption (3.2) in the case $n = 4$ or assumption (3.4) in the case $n \geq 5$ it follows that $\Phi'_n(0)$ is bijective. Hence $\Phi_n^{-1}$ defines the canonical coordinates in a neighborhood of $q_0$ (or shortly the canonical coordinates at $q_0$). Denote

$$
\mathcal{I}_k = I_k \circ \Phi_n, \quad k = 1, \ldots, n - 2.
$$

(3.18)

Assigning to any generic germ at $q_0$ of control-affine systems (3.1) the function $\mathcal{I}_k$ we obtain the functional invariant of $n$ variables of this set of objects in the sense of Definition 1 for any $1 \leq k \leq n - 2$.

Now let us consider the cases $n = 4$ and $n \geq 5$ separately:

a) The case $n = 4$. By (3.16) and (3.7), in the canonical coordinates the vector fields $F_0$ and $F_1$ have the following form:

$$
F_0 = \frac{\partial}{\partial x_1}, \quad F_1 = \sum_{k=1}^{4} a_k \frac{\partial}{\partial x_k},
$$

(3.19)

where the components of $F_1$ satisfy the following second order linear ordinary differential equations w.r.t. the variable $x_1$

$$
\frac{\partial^2 a_k}{\partial x_1^2} + \mathcal{I}_2 \frac{\partial a_k}{\partial x_1} - \mathcal{I}_1 a_k - \delta_{1,k} = 0 \quad k = 1, 2, 3, 4,
$$

(3.20)

with the following restrictions on the initial conditions for any $k = 1, 2, 3, 4$

$$
a_k(0, x_2, x_3, x_4) \equiv \delta_{2k},
$$

(3.21a)

$$
\frac{\partial a_k}{\partial x_1}(0, 0, x_3, x_4) \equiv -\delta_{3k},
$$

(3.21b)

$$
\frac{\partial^2 a_k}{\partial x_1 \partial x_2}(0, 0, 0, x_4) \equiv -\delta_{4k},
$$

(3.21c)
where $\delta_{ij}$ is the Kronecker symbol. Let for any $k = 1, 2, 3, 4$

$$
\beta_k(x_2, x_3, x_4) \stackrel{\text{def}}{=} \frac{\partial^3 a_k}{\partial x_1 \partial x_2}(0, x_2, x_3, x_4),
$$
\tag{3.22a}

$$
\psi_k(x_3, x_4) \stackrel{\text{def}}{=} \frac{\partial^3 a_k}{\partial x_1 \partial x_2 \partial x_3}(0, 0, x_3, x_4).
$$
\tag{3.22b}

So, with any germ at $q_0$ of a four-dimensional affine system \([3.1]\), satisfying genericity assumptions \([3.2]\) and \([3.3]\), one can associate the ordered tuple

$$(\mathcal{I}_1, \mathcal{I}_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4),
$$
\tag{3.23}

of state-feedback functional invariants, consisting of two germs $\mathcal{I}_1$ and $\mathcal{I}_2$ of functions of four variables at 0, four germs $\beta_1, \beta_2, \beta_3, \beta_4$ of functions of three variables at 0, and four germs $\psi_1, \psi_2, \psi_3, \psi_4$ of functions of three variables at 0. We call it the tuple of the primary invariants of the \((1, 4)\) control-affine system \([3.1]\) at the point $q_0$. Note that by \([3.22]\), the functional invariants $\beta_k$ and $\psi_k$ have the weight 3 for any $1 \leq k \leq 4$, while by \([3.20]\) and \([3.21]\), the functional invariants $\mathcal{I}_1$ and $\mathcal{I}_2$ have the weight 2.

Further, fixing $\beta_k$ and $\psi_k$ and using \([3.21]\) and \([3.21c]\), one can find $\frac{\partial a_k}{\partial x_1}(0, x_2, x_3, x_4)$ for any $1 \leq k \leq 4$ by the appropriate integrations (see \([3.31]\) below). If in turn one fixes also $\mathcal{I}_1$ and $\mathcal{I}_2$, then from the knowledge of $\frac{\partial a_k}{\partial x_1}(0, x_2, x_3, x_4)$, condition \([3.21a]\), and differential equation \([3.20]\) we can recover the functions $a_k(x_2, x_3, x_4)$ and therefore our control-affine system itself, just using the standard existence and uniqueness results from the theory of ordinary differential equations. We summarize all above in the following:

**Theorem 1** Given two arbitrary germs $\mathcal{I}_1$ and $\mathcal{I}_2$ of functions of four variables at 0, four arbitrary germs $\beta_1, \beta_2, \beta_3, \beta_4$ of functions of three variables at 0, and four arbitrary germs $\psi_1, \psi_2, \psi_3, \psi_4$ of functions of two variables at 0 there exists a unique, up to state-feedback transformation of the type \([3.4]\), four-dimensional control-affine system with scalar input, satisfying genericity assumptions \([3.2]\) and \([3.3]\), such that the tuple $(\mathcal{I}_1, \mathcal{I}_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4)$ is its tuple of the primary invariants at the given point $q_0$. In other words, the tuples of the primary invariants give the regular \((3, 2)\)-parameterization of the considered classification problem with the following $3 \times 2$ parameterization matrix $P$:

$$
P = \begin{pmatrix}
0 & 4 \\
0 & 4 \\
2 & 0
\end{pmatrix}.
$$
\tag{3.24}

The Poincare series $M(t)$ of the considered classification problem satisfies

$$
M(t) = \frac{2t^2}{(1-t)^2} + t^3 \left( \frac{4}{(1-t)^4} + \frac{4}{(1-t)^3} \right).
$$
\tag{3.25}

It turns out that \((3, 2)\) is the characteristic pair of the considered classification problem. Indeed, let as before $w_0$ be the order of zero of $M(t)$ at $t = 0$, $N$ is the order of pole of $(1-t)M(t)$ at $t = 1$, and $d$ is the degree of $M(t)$ (at infinity). Then from \([3.25]\) it follows that $w_0 = 2$, $N = 4$, and $d = 0$. Hence from \([2.15]\) or \([2.17]\) it follows that

$$
\text{NS}(M(t)) \subset \{(w, l) : w \geq 2\}.
$$
\tag{3.26}
Further, using (2.16), one has
\[ \text{NS}(M(t)) \cap \{(w,l): w = 2\} \subset \{(2,1)\}, \quad \text{NS}(M(t)) \cap \{(w,l): w = 3\} \subset \{(3,1),(3,2)\}. \] (3.27)

But by the previous theorem our classification problem admits (3,2)-parameterization such that its parameterization metric has a positive entry in the right-upper corner. Therefore from Corollary 1 it follows that (2,1) \( \notin \) NS(M(t)). Since (3,2) \( \prec \) (3,1) we can conclude from (3.28) and (3.27) that (3,2) is the minimal element of NS(M(t)). In other words, (3,2) is the characteristic pair of our classification problem. Also, it implies that the characteristic matrix \( C \) of the problem is equal to Norm(\( P \)), which can be found easily by the series of elementary transformations. Namely,
\[ C = \text{Norm}(P) = \begin{pmatrix} 2 & 4 \\ 0 & 6 \\ 0 & 2 \end{pmatrix}. \] (3.28)

**Conclusion 1** The characteristic parameterization of (1,4) control-affine systems, up to state-feedback transformations, consists of 2 functional invariants of 4 variables and the weight 3, 6 functional invariants of 3 variables and the weight 3, 2 functional invariants of 2 variables and the weight 2, and 4 functional invariants of 2 variables and the weight 3.

In order to obtain a characteristic parameterization from the parameterization by the tuple of the primary invariants one can implement some series of rearrangement of the primary invariants according to the series of elementary transformations from the matrix \( P \) to Norm(\( P \)), as was described in the previous section (see, for example, formula (2.11) and the paragraph after it).

We finish the treatment of the case of (1,4)-affine control systems by writing the local normal form of such system, up to state-feedback transformation, in terms of the tuple of their primary invariants: Let \( N \) be the solution of the following non-homogeneous second order linear ordinary differential equation w.r.t. the variable \( x_1 \) with prescribed initial values
\[
\begin{aligned}
\frac{\partial^2 N}{\partial x_1^2} + \mathcal{I}_2 \frac{\partial N}{\partial x_1} - \mathcal{I}_1 N - 1 &= 0; \\
N(0,x_2,x_3,x_4) &= 0, \quad \frac{\partial N}{\partial x_1}(x_1,x_2,x_3,x_4) \bigg|_{x_1=0} = 0,
\end{aligned}
\] (3.29)

and the functions \( \rho_1, \rho_2 \) be the solution of the following homogeneous second order linear ordinary differential equations w.r.t. the variable \( x_1 \) with prescribed initial values
\[
\begin{aligned}
\frac{\partial^2 \rho_i}{\partial x_1^2} + \mathcal{I}_2 \frac{\partial \rho_i}{\partial x_1} - \mathcal{I}_1 \rho_i &= 0, \quad i = 1,2, \\
\left( \frac{\partial}{\partial x_1} \rho_1(0,x_2,x_3,x_4) \right) &= \left( \frac{\partial}{\partial x_1} \rho_2(0,x_2,x_3,x_4) \right) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
\] (3.30)

Let also
\[ B_k(x_2,x_3,x_4) = -\delta_{3k} + x_2 \left( -\delta_{4k} + \int_0^{x_3} \psi_k(y,x_4) \, dy \right) + \int_0^{x_2} (x_2 - y)\beta_k(y,x_3,x_4) \, dy \] (3.31)
for \( 1 \leq k \leq 4 \) (actually \( B_k(x_2,x_3,x_4) = \frac{\partial a_k}{\partial x_1}(0,x_2,x_3,x_4) \), where the functions \( a_k \) are as in (3.19)). Then a four-dimensional control-affine system (3.1) with the tuple of the primary
Let us introduce the following functions for any \( \delta \), where \( q \):

\[
\begin{aligned}
\dot{x}_1 &= 1 + (N + B_1 \rho_2)u, \\
\dot{x}_2 &= (\rho_1 + B_2 \rho_2)u, \\
\dot{x}_i &= B_i \rho_2 u, \quad i = 3, 4
\end{aligned}
\]

(3.32)

b) The case \( n \geq 5 \). By (3.17) and (3.17), in the canonical coordinates the vector fields \( F_0 \) and \( F_1 \) have the following form:

\[
F_0 = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k}, \quad F_1 = \frac{\partial}{\partial x_1},
\]

(3.33)

where the components \( a_k \) of \( F_0 \) satisfy the following system of partial differential equations

\[
I_{n-2} \frac{\partial^{n-2} a_k}{\partial x_1^{n-2}} + \sum_{j=3}^{n-3} I_j \frac{\partial^j a_k}{\partial x_1^j} + \sum_{i=1}^{n} \left( a_i \frac{\partial^2 a_k}{\partial x_1^2} - \frac{\partial a_i}{\partial x_1} \frac{\partial a_k}{\partial x_1} \right) + I_2 \frac{\partial a_k}{\partial x_1} + a_k + I_1 \delta_{1,k} = 0,
\]

(3.34)

\( k = 1, \ldots, n \),

with the following restrictions on the boundary conditions for any \( 1 \leq k \leq n \)

\[
a_k(0, x_2, \ldots, x_n) \equiv \delta_{2k},
\]

(3.35a)

\[
\frac{\partial a_k}{\partial x_1}(0, 0, x_3, \ldots, x_n) \equiv \delta_{3k},
\]

(3.35b)

\[
\frac{\partial^2 a_k}{\partial x_1^2}(0, 0, 0, x_4, \ldots, x_n) \equiv -\delta_{4k},
\]

(3.35c)

\[
\frac{\partial^j a_k}{\partial x_1^j}(0, \ldots, 0, x_{j+1}, \ldots, x_n) \equiv \delta_{j+3,k}, \quad 2 \leq j \leq n-3,
\]

(3.35d)

where \( \delta_{ij} \) is the Kronecker symbol. Note also that the genericity assumption (3.4) implies that

\[
I_{n-2} \neq 0.
\]

(3.36)

Let us introduce the following functions for any \( 1 \leq k \leq n \):

\[
\begin{aligned}
\beta_k(x_2, \ldots, x_n) &\equiv \frac{\partial^3 a_k}{\partial x_1 \partial x_2^2}(0, x_2, \ldots, x_n), \\
\psi_k(x_3, \ldots, x_n) &\equiv \frac{\partial^3 a_k}{\partial x_1 \partial x_2 \partial x_3}(0, 0, x_3, \ldots, x_n), \\
\phi_{jk}(x_l, \ldots, x_n) &\equiv \frac{\partial^{j+1} a_k}{\partial x_1^j \partial x_l}(0, 0, 0, x_l, \ldots, x_n), \quad 2 \leq j \leq n-3, \quad 2 \leq l \leq j + 2.
\end{aligned}
\]

(3.37)

So, with any germ at \( q_0 \) of an \( n \)-dimensional affine system (3.1), satisfying genericity assumptions (3.2) and (3.3), one can associate the ordered tuple

\[
\left( \{ I_s(x_1, \ldots, x_n) \}_{s=1}^{n-2}, \{ \beta_k(x_2, \ldots, x_n) \}_{k=1}^{n}, \{ \psi_k(x_3, \ldots, x_n) \}_{k=1}^{n}, \{ \phi_{jk}(x_l, \ldots, x_n) : 1 \leq k \leq n, 2 \leq j \leq n-3, 2 \leq l \leq j + 2 \} \right)
\]

(3.38)
of state-feedback invariants. of functions of \( n - 1 \) variables at 0. We call it the tuple of the primary invariants of the \((1,n)\)-affine control system \( (3.38) \) with \( n > 4 \) at the point \( q_0 \). Note that by \( (3.22) \) for any \( 1 \leq k \leq n \) the functional invariants \( \beta_k \) and \( \psi_k \) have the weight 3, the functional invariants \( \phi_{kj} \) have the weight \( j + 1 \), while by \( (3.20) \) and \( (3.21) \) for any \( 1 \leq s \leq n - 2 \) the functional invariants \( \mathcal{I}_s \) have the weight \( n - 2 \).

Further, fixing \( \beta_k \) and \( \psi_k \) and using \( (3.35a) \) and \( (3.35b) \), one can find \( \frac{\partial a_k}{\partial x_1}(0,x_2,\ldots,x_n) \) for any \( 1 \leq k \leq n \) by the appropriate integrations. Similarly, fixing \( \{\phi_{kj}\}_{t=2}^n \) for given \( j \), \( 2 \leq j \leq n - 3 \), and using \( (3.35d) \), one can find \( \frac{\partial a_k}{\partial x_1}(0,x_2,\ldots,x_n) \) for any \( 1 \leq k \leq n \) by the appropriate integrations. Finally, if we suppose that all functions \( \beta_k, \psi_k \), and \( \phi_{kj} \) are real analytic and fix also real analytic \( \{\mathcal{I}_s\}_{s=1}^{n-2} \), then from the knowledge of \( \frac{\partial a_k}{\partial x_1}(0,x_2,\ldots,x_n) \) for all \( 1 \leq j \leq n - 3 \), condition \( (3.35a) \), and differential equation \( (3.34) \) we can recover the functions \( a_k(x_1,\ldots,x_n) \) and therefore our affine control system itself, just using the classical Cauchy-Kowalewsky theorem for system \( (3.34) \). We summarize all above in the following:

**Theorem 2** If \( n \geq 5 \), then given an arbitrary tuple \( (3.38) \) of real analytic functions \( n - 2 \) germs \( \{\mathcal{I}_j\}_{j=1}^{n-2} \) of real analytic functions there exists a unique, up to state-feedback real analytic transformation of the type \( (1.3) \), \( n \)-dimensional real analytic control-affine system with scalar input, satisfying genericity assumptions \( (3.2), (3.3), \) and \( (3.4) \), such that the tuple \( (3.38) \) is its tuple of the primary invariants at the given point \( q_0 \). In other words, the tuples of the primary invariants give the regular \( (n - 2,2) \)-parameterization of the considered classification problem in the real analytic category with the \((n - 1) \times (n - 4) \) parameterization matrix \( P \) such that for \( n = 5 \)

\[
P = (5, 10, 10, 3)^T \tag{3.39}
\]

and for \( n > 5 \)

\[
P = \begin{pmatrix}
0 & \cdots & 0 & n \\
\vdots & \ddots & \vdots & \vdots \\
0 & n & \cdots & n \\
2n & n & \cdots & n \\
2n & n & \cdots & n \\
0 & \cdots & 0 & n - 2
\end{pmatrix} \tag{3.40}
\]

(in the matrix \( P \) all entries in the triangle with vertices in the \((1,1)\)-entry, the \((1,n-5)\)-entry, and the \((n-5,1)\)-entry are equal to 0, while all entries in the triangle with vertices in the \((1,n-4)\)-entry, the \((n-4,1)\)-entry, and the \((n-4,n-4)\)-entry are equal to \( n \)). The Poincare series \( M(t) \) of the considered classification problem satisfies

\[
M(t) = nt^2 \left( \frac{2}{(1-t)^n} + \frac{2}{(1-t)^{n-1}} + \frac{1}{(1-t)^{n-2}} \right) + \sum_{i=4}^{n-2} t^i \sum_{j=n-i}^{n-1} \frac{1}{(1-t)^{j+1}} + \frac{(n-2)t^{n-2}}{(1-t)^{n+1}}. \tag{3.41}
\]

It turns out that \((n-2,2)\) is the characteristic pair of the considered classification problem. Indeed, let as before \( w_0 \) be the order of zero of \( M(t) \) at \( t = 0 \), \( N \) is the order of pole of \((1-t)M(t)\)
at $t = 1$, and $d$ is the degree of $M(t)$ (at infinity). Then from (3.25) it follows that $w_0 = 3$, $N = n$, and $d = n - 5$. Hence from (2.17) it follows that
\[
\text{NS}(M(t)) \subset \{(w, l) : w \geq n - 3\}. \tag{3.42}
\]
Further, using (2.16), one has
\[
\text{NS}(M(t)) \cap \{(w, l) : w = 2\} \subset \{(n - 3, 1)\},
\]
\[
\text{NS}(M(t)) \cap \{(w, l) : w = n - 2\} \subset \{(n - 2, 1), (n - 2, 2)\}. \tag{3.43}
\]
But by the previous theorem our classification problem admits $(n - 2, 2)$-parameterization such that its parameterization metric has a positive entry in the right-upper corner. Therefore from Corollary it follows that $(n - 3, 1) \notin \text{NS}(M(t))$. Since $(n - 2, 2) < (n - 2, 1)$ we can conclude from (3.42) and (3.43) that $(n - 2, 2)$ is the minimal element of $\text{NS}(M(t))$. In other words, $(n - 2, 2)$ is the characteristic pair of our classification problem. Also, it implies that the characteristic matrix $C$ of the problem is equal to $\text{Norm}(P)$, where $P$ is as in Theorem 2. If $n = 5$ then obviously $C = \text{Norm}(P) = P$. For $n > 5$ one can calculate all nontrivial entries of $\text{Norm}(P)$, using identities (2.13). It gives all nontrivial entries of the characteristic matrix $C:
\begin{align*}
C_{n-1,n-4} &= n - 2, \\
C_{n-2,n-4} &= n(n - 3), \\
C_{i,n-4} &= n \left(1 + \sum_{l=1}^{n-3-i} (i+2l-3) + (2(n-9-i)) + 2(2n-8-i) + (2n-7-i)\right), \quad 2 \leq i \leq n - 3, \\
C_{1,n-4} &= n, \\
C_{ij} &= n \left((n+4+j) + 2(n+5+j) + 2(n+6+j) + (n+7+j) - 1 - \sum_{l=1}^{j-1} (n+2l-7-j)\right), \quad 1 \leq j \leq n - 5.
\end{align*} \tag{3.44}

Recall that $C_{ij}$ is the intrinsic number of the functional invariants of $i + 1$ variables and the weight $j + 2$. In order to obtain a characteristic parameterization from the parameterization by the tuple of the primary invariants one can implement some series of rearrangement of the primary invariants according to the series of elementary transformations from the matrix $P$ to $\text{Norm}(P)$, as was described in the previous section (see, for example, the formula (2.11) and the paragraph after it).

**Remark 6** Two control systems $\dot{y} = F(y, v)$ and $\tilde{\dot{y}} = \tilde{F}(\tilde{y}, \tilde{v})$, with $m$-dimensional state-space $S$ and one dimensional control space $V$ are called micro-locally state-feedback equivalent at the point $(y_0, v_0) \in S \times V$, if there exist the state-feedback transformation
\[
\begin{align*}
\tilde{y} &= \Phi_1(y) \\
\tilde{v} &= \Phi_2(y, v) \\
y_0 &= \Phi_1(y_0), \quad v_0 = \Phi_2(y_0, v_0)
\end{align*}
\]
such that in the neighborhood of $(y_0, v_0)$ in $S \times V$ the following identity holds:
\[
\tilde{F}(\Phi_1(y), \Phi_2(y, v)) = d\Phi_1F(y, v).
\]
The affine $(1, m + 1)$ control system (1.5) will be called the affine extension of the control system (1.4). It is not difficult to show that two control systems with scalar input are micro-locally
state-feedback equivalent at some point \((y_0, v_0) \in S \times V\) if and only if their affine extensions are locally equivalent w.r.t. the state-feedback transformations of the type \((\ref{eq:affine})\) at the same point. Note also that the affine extensions of generic \(m\)-dimensional control systems with scalar input are generic in the set of all \((1, m+1)\)-affine systems. Using this fact and Theorems \([1,2]\) one obtains the micro-local parameterization of non-affine \(m\)-dimension control systems with scalar inputs and \(m \geq 3\) by the tuples of the primary invariants of their affine extensions (in \(C^\infty\) category for \(m = 3\) and \(C^\omega\) category for \(m \geq 4\)). Obviously, the Poincaré series, the characteristic pair and the characteristic matrix of the micro-local state-feedback classification problem for \(m\)-dimensional control systems with scalar input are exactly the same as in the case of the state-feedback classification problem of \((1, m+1)\) control-affine systems. Besides, since in our method of normalization of \((1, n)\) control-affine system with \(n \geq 4\) we rectify the vector field \(f_1\), then in the case \(m \geq 4\) a generic \(m\)-dimensional real analytic control system with the prescribed tuple \((\ref{eq:primary})\) of the primary invariants of its affine extension is micro-locally state-feedback equivalent in the real analytic category to the system

\[
\dot{y}_s = f_s(y_1, \ldots, y_m, v), \quad 1 \leq s \leq m,
\]

such that \(f_s(x_2, x_3, \ldots, x_m, x_1) = a_{s+1}(x_1, \ldots, x_{m+1})\), where the tuple \(\{a_k(x_1, \ldots, x_m)\}_{k=1}^{m+1}\) is the solution of the system of partial differential equations \((\ref{eq:pde})\) with boundary conditions, which can be expressed by the primary invariants \(\beta_k, \psi_k, \phi_{kjl}\), using \((\ref{eq:primary})\).

4 Reduction of control-affine systems with two-dimensional input to the scalar input case in dimensions four and five

For \(r = 2\) the system \((\ref{eq:system})\) has the following form

\[
\dot{q} = f_0(q) + u_1 f_1(q) + u_2 f_2(q), \quad q \in M, \quad u_1, u_2 \in \mathbb{R}.
\]

(4.1)

Our aim is to assign to the system \((\ref{eq:system})\) in a canonical way an affine subsystem with scalar input\(^1\). It turns out that in the case \(n = 4\) the original system can be recovered from it uniquely up to a feedback transformation, while in the case \(n = 5\) such unique recovering is possible after introducing an additional invariant function of \(n\) variables (which is natural in view of the estimates for the number of functional parameters, given in the Introduction).

4.1 Preliminaries. Let us look on \((\ref{eq:system})\), as on the time optimal control problem and find its extremals. First we introduce some notations. Let \(T^*M\) be the cotangent bundle of \(M\) with canonical symplectic form \(\sigma\). Denote by \(h_i, 0 \leq i \leq 2\), the following functions on \(T^*M\):

\[
h_i(\lambda) = p \cdot f_i(q), \quad \lambda = (p, q), \quad q \in M, \quad p \in T^*_q M.
\]

(4.2)

For a given function \(G : T^*M \rightarrow \mathbb{R}\) denote by \(\vec{G}\) the corresponding Hamiltonian vector field defined by the relation \(\sigma(\vec{G}, \cdot) = -dG(\cdot)\). For a given vector distribution \(D\) on \(M\) (i.e., a subbundle of the tangent bundle), define the \(l\)th power \(D^l\) by the recursive relation

\[
D^l = D^{l-1} + [D, D^{l-1}], \quad D^1 = D,
\]

and denote by \((D^l)^\perp \subset T^*M\) the annihilator of \(D^l\), namely

\[
(D^l)^\perp = \{(p, q) \in T^*M : p \cdot v = 0 \quad \forall v \in D^l(q)\}.
\]

\(^1\)the meaning of the word “subsystem” is that at any point \(q\) the set of its admissible velocities is a subset of the set of the admissible velocities of the original system at \(q\).
Let \( \gamma \) be an extremal of \((4.1)\) with extremal control functions \( \tilde{u}_1(t) \) and \( \tilde{u}_2(t) \). Then
\[
\dot{\gamma}(t) = \tilde{h}_0(\gamma(t)) + \tilde{u}_1(t)\tilde{h}_1(\gamma(t)) + \tilde{u}_2(t)\tilde{h}_2(\gamma(t))
\] (4.4)
and from the maximality condition for \( H \) it follows that
\[
\gamma(\cdot) \subseteq \{ \lambda \in T^*M : h_1(\lambda) = h_2(\lambda) = 0 \}. \tag{4.5}
\]
If we denote \( D_2 = \text{span}(f_1, f_2) \), then \((4.3)\) is equivalent to \( \gamma(\cdot) \subseteq (D_2)^\perp \). Combining \((4.4)\) and \((4.5)\), we obtain
\[
d_{\gamma(t)}h_i(\dot{\gamma}(t)) = 0, \quad i = 1, 2. \tag{4.6}
\]
Then from \((4.4)\) and \((4.6)\) it follows
\[
\{ h_0, h_1 \}(\gamma(t)) + \tilde{u}_2(t)\{ h_2, h_1 \}(\gamma(t)) = 0,
\{ h_0, h_2 \}(\gamma(t)) + \tilde{u}_1(t)\{ h_1, h_2 \}(\gamma(t)) = 0
\] (4.7)
(here \( \{ h_i, h_j \} \) are Poisson brackets of the Hamiltonians \( h_i \) and \( h_j \): \( \{ h_i, h_j \} = dh_j(\delta h_i) \)). Now suppose that
\[
\dim D_2^2 = 3
\] (4.8)
Then relations \((4.7)\) implies that the extremals of \((4.1)\), lying in \((D_2)^\perp \), are exactly the integral curves of the vector field
\[
\bar{X} = \bar{h}_0 + \frac{\{ h_0, h_2 \}}{\{ h_2, h_1 \}}\bar{h}_1 + \frac{\{ h_1, h_0 \}}{\{ h_2, h_1 \}}\bar{h}_2 \tag{4.9}
\]
(which is the Hamiltonian vector field, corresponding to the Hamiltonian \( X = h_0 + \frac{\{ h_0, h_2 \}}{\{ h_2, h_1 \}}h_1 + \frac{\{ h_1, h_0 \}}{\{ h_2, h_1 \}}h_2 \)). Denote by \( V \) the affine subbundle of \( TM \), defined by system \((4.1)\) and \( V(q) \) be the set of all admissible velocities of the system \((4.1)\) at the point \( q \),
\[
V(q) = \{ f_0(q) + u_1f_1(q) + u_2f_2(q) : u_1, u_2 \in \mathbb{R} \}.
\]
Let \( \pi : T^*M \rightarrow M \) be the canonical projection. The set
\[
\text{Ext}(q) = \{ \pi_*\bar{X}(\lambda) : \lambda \in T^*_qM \cap (D_2)^\perp \}, \quad q \in M \tag{4.10}
\]
is the subset of \( V(q) \), consisting of the velocities of all extremal trajectories starting at \( q \) and having a lift in \((D_2)^\perp \). Among all extremals on \((D_2)^\perp \), one can distinguish so-called abnormal extremals, i.e., the extremals lying on the zero level set of the Hamiltonian \( X \). Denote \( D_3 = \text{span}(f_0, f_1, f_2) \) and suppose that
\[
\dim(D_2^2 + D_3) = 4 \tag{4.11}
\]
The set
\[
\text{Abn}(q) = \{ \pi_*\bar{X}(\lambda) : \lambda \in T^*_qM \cap (D_3)^\perp \}, \quad q \in M \tag{4.12}
\]
is the subset of $\text{Ext}(q)$, consisting of the velocities of all abnormal extremal trajectories starting at $q$ and having a lift in $(D_2)^\perp \setminus (D_2^2)^\perp$. One can show that for generic affine systems of the type \eqref{eq:4.1} $\text{Ext}(q) = V(q)$ in the case $n \geq 5$ and $\text{Abn}(q) = V(q)$ in the case $n \geq 6$. But in the case $n = 4$ and $n = 5$ either $\text{Ext}(q)$ or $\text{Abn}(q)$ (or both of them) define the proper subsystem of the original system \eqref{eq:4.1}. Moreover, it turns out that these subsystems are affine with scalar input, so one can apply the theory of the previous section. Now let us consider the cases $n = 4$ and $n = 5$ separately.

4.2 The case $n = 4$. Let
\begin{equation}
[V,D_2](q) = \{[X,Y](q) : X \in V, Y \in D_2, \text{ are vector fields}\}.\end{equation}
It is not difficult to show that $[V,D_2](q)$ is a linear space and
\begin{equation}
[V,D_2](q) = \text{span}(f_1(q), f_2(q), [f_1,f_2](q), [f_0,f_1](q), [f_0,f_2](q))\end{equation}
The crucial observation is formulated in the following

**Proposition 3** The set $\text{Ext}(q)$ is an affine line, provided that \eqref{eq:4.8} holds and \eqref{eq:4.14}
\begin{equation}
\dim [V,D_2](q) = 4
\end{equation}

**Proof.** Take some vector field $f_3$ such that the tuple $(f_0,f_1,f_2,f_3)$ constitutes the frame on $M$. Denote by $c^k_{ji}$ the structural functions of this frame, i.e., the functions, satisfying
\begin{equation}
[f_i,f_j] = \sum_{k=0}^3 c^k_{ji} f_k.\end{equation}
Using the following well-known property of the Poison brackets
\begin{equation}
\{h_i,h_j\}(p,q) = p \cdot [f_i,f_j](q), \quad q \in M, p \in T^*_q M
\end{equation}
and \eqref{eq:4.9}, one can easily obtain that
\begin{equation}
\text{Ext}(q) = \Pi(q) \cap V(q),
\end{equation}
where
\begin{equation}
\Pi(q) = \{(c^0_{12}(q) \nu + c^3_{12}(q) \mu)f_0(q) + (c^0_{20}(q) \nu + c^3_{20}(q) \mu)f_1(q) + (c^0_{01}(q) \nu + c^3_{01}(q) \mu)f_2(q) : \mu, \nu \in \mathbb{R}\}
\end{equation}
From assumption \eqref{eq:4.14} and identity \eqref{eq:4.13} it follows that $\Pi(q)$ is a plane. Assumption \eqref{eq:4.8} implies that the plane $\Pi(q)$ is not parallel to the plane $V(q)$. Note also both $\Pi(q)$ and $V(q)$ belong to $D_3(q)$. Hence by \eqref{eq:4.17} the set $\text{Ext}(q)$ is an affine line. □

Consider the control system such that $\text{Ext}(q)$ is its set of the admissible velocities at $q$. By Proposition 3 it is an affine system with scalar input. We call this system the reduction of the four-dimensional control-affine system \eqref{eq:4.1}. The following proposition gives another characterization of the reduction of the system \eqref{eq:4.1}:
Proposition 4 Assume that the four-dimensional control-affine system (4.1) satisfies the conditions (4.8) and (4.14). Then the subsystem
\[ \dot{q} = g_0 + u g_1, \] (4.19)
of (4.1) is its reduction if and only if
\[ [g_0, g_1] \in D_2. \] (4.20)

Proof. By definition, the system (4.19) is the reduction of (4.1) if and only if
\[ \text{Ext}(q) = \{ g_0(q) + t g_1(q) : t \in \mathbb{R} \}. \] (4.21)
On the other hand, one can take from the beginning \( f_0 = g_0 \) and \( f_1 = g_1 \). Then comparing (4.21) with (4.17) and (4.18) we obtain that the system (4.19) is the reduction of (4.1) if and only if \( c_{01}^0 = c_{01}^3 = 0 \), which is equivalent to \([g_0, g_1] \in \text{span}(g_1, f_2) = D_2. \] □

Corollary 2 Assume that a four-dimensional affine control system (4.19) satisfies
\[ \dim \text{span}(g_1, [g_1, g_0], [g_1, [g_1, g_0]], [g_0, [g_1, g_0]]) = 4 \] (4.22)
Then any four-dimensional control-affine system with two-dimensional input, having the system (4.19) as its reduction, is feedback equivalent to the system
\[ \dot{q} = g_0 + u_1 g_1 + u_2 [g_0, g_1], \quad u_1, u_2 \in \mathbb{R}. \] (4.23)

Proof. First by assumption (4.22) and relation (4.13) the system (4.19) satisfies conditions (4.8) and (4.14) (where \( f_0, f_1, \) and \( f_2 \) are replaced by \( g_0, g_1 \) and \([g_0, g_1]) \). Hence, by Proposition 3 the system (4.19) admits the reduction and by Proposition 4 this reduction is the system (4.19). On the other hand, suppose that some system (4.1) has the reduction (4.19). Then from the previous proposition \([g_0, g_1] \in \text{span}(g_1, f_2) \). According to (4.22), \( g_1 \) and \([g_0, g_1] \) are linearly independent. Hence the system (4.1) is feedback equivalent to (4.23). □

According to the previous proposition, a generic four-dimensional control-affine system with two-dimensional input can be uniquely, up to a feedback transformation, recovered from its reduction. Suppose that the reduction (4.19) of the system (4.1) satisfies (4.24) and
\[ \dim \text{span}(g_0, g_1, [g_1, g_0], [g_1, [g_1, g_0]]) = 4. \] (4.24)
Then we can apply to the system (4.19) all constructions of section 2. In particular, one can construct the tuple of the primary invariants of (4.19) at a given point, which are also feedback invariants of the original system (4.1). Note that the set of germs of systems of the type (4.1) having the reductions, which satisfies conditions (4.22) and (4.24), is generic. Combining Theorem 1 Corollary 2 and normal form (3.32), we obtain the following classification of generic germs of systems of the type (4.1) in terms of the tuple of the primary invariants of their reductions:

Theorem 3 Given two arbitrary germs \( I_1 \) and \( I_2 \) of functions of four variables at 0, four arbitrary germs \( \beta_1, \beta_2, \beta_3, \beta_4 \) of functions of three variables at 0, and four arbitrary germs \( \psi_1, \psi_2, \psi_3, \psi_4 \) of functions of two variables at 0 there exists a unique, up to state-feedback transformation of the type (1.3), four-dimensional control-affine system with two-dimensional input such
that its reduction satisfies genericity assumptions (4.22) and (4.24) and \((I_1, I_2, \beta_1, \beta_2, \beta_3, \beta_4, \psi_1, \psi_2, \psi_3, \psi_4)\) is the tuple of the primary invariants of the reduction at the given point \(q_0\). This control system is state-feedback equivalent to the following one:

\[
\begin{align*}
\dot{x}_1 &= 1 + (N + B_1 \rho_2)u_1 + \left(\frac{\partial N}{\partial x_1} + \beta_1 \frac{\partial \rho_2}{\partial x_1}\right)u_2 \\
\dot{x}_2 &= (\rho_1 + B_2 \rho_2)u_1 + \left(\frac{\partial \rho_1}{\partial x_1} + B_2 \frac{\partial \rho_2}{\partial x_1}\right)u_2, \quad u_1, u_2 \in \mathbb{R}, \\
\dot{x}_i &= B_i \rho_2 u_1 + \beta_i \frac{\partial \rho_2}{\partial x_1}u_2, \quad i = 3, 4
\end{align*}
\]

where \(N\) is the solution of (3.20), \(\rho_i, i = 1, 2\) are the solutions of (3.21), and \(B_k, 1 \leq k \leq 4\), are as in (3.31). The Poincaré series, the characteristic pair and the characteristic matrix of the classification problem are exactly the same as in the case of (1,4) control-affine systems.

**Remark 7** It is easy to show that in the case \(n = 4\) the set \(\text{Abn}(q)\) consists of one vector provided that (4.11) holds. Besides, if the system (4.19) is the reduction of the system (4.1) and it satisfies (4.22) and (4.24), then \(\text{Abn}(q)\) is exactly its canonical drift. □

**Remark 8** Actually, there is another intrinsic way to assign to the system (4.1), satisfying (4.11), an affine subsystem with scalar input: As a drift one can take again \(\text{Ab}(q)\). It remains to define canonically the direction of the affine line of the reduction. For this note first that the distribution \(D_2\) satisfies (4.8) because of assumption (4.11). Therefore through any point of \(M\) the unique (unparameterized) abnormal extremal trajectory of the rank 2 distribution \(D_2\) passes: the line subdistribution \(L\) of \(D_2\), tangent to the abnormal extremal trajectories at any point is characterized by the relation \([L, D^2] \subseteq D^2\). The direction of the affine line of the reduction can be taken parallel to \(L\). The direction of \(L\) is different in general from the direction of the affine line in the first reduction. But this new reduction is worse than the previous one, because the original system (4.1) is not uniquely recovered from it: if \((\bar{g}_0, \bar{g}_1)\) is the canonical pair of the new reduction (by construction and the previous remark \(g_0 = \text{Abn}\)), then the field \(f_2\) can be taken in the form \(f_2 = \alpha \bar{g}_0 + [\bar{g}_1, \bar{g}_0]\), where the function \(\alpha\) satisfies some second order ordinary differential equation along each integral curve of \(\bar{g}_1\). Note that the direction \(\bar{g}_1\) depends on the second jet of the original system (4.1), while the direction of the affine line in the first reduction depends only on the first jet. This could be the reason for the loss of some information about the original system during the reduction described in the present remark. □

### 4.3 The case \(n = 5\)

In this case by analogy with Proposition 3 we have

**Proposition 5** The set \(\text{Abn}(q)\) is an affine line, provided that (4.11) holds and

\[
\dim D_3^2 = 5.
\]

**Proof.** Take some vector fields \(f_3\) and \(f_4\) such that the tuple \((f_0, f_1, f_2, f_3, f_4)\) constitutes the frame on \(M\). By analogy with (4.13), let \(c_{ji}^k, 0 \leq i, j, k \leq 4\), be the structural functions of this frame. From (4.9) and (4.12), using (4.16) and the fact that \(h_0 = 0\) on \((D_3)^\perp\), one can easily obtain that

\[
\text{Abn}(q) = \Pi_1(q) \cap V(q),
\]

where

\[
\Pi_1(q) = \{(c_{12}^1(q)\nu + c_{12}^4(q)\mu) f_0(q) + (c_{20}^3(q)\nu + c_{20}^4(q)\mu) f_1(q) + (c_{01}^3(q)\nu + c_{01}^4(q)\mu) f_2(q) : \mu, \nu \in \mathbb{R}\}
\]
From assumption (4.26) and identity (4.13) it follows that \( \Pi_1(q) \) is a plane. Assumption (4.11) implies that the plane \( \Pi_1(q) \) is not parallel to the plane \( V(q) \). Note also both \( \Pi_1(q) \) and \( V(q) \) belong to \( D_3(q) \). Hence by (4.27) the set \( Abn(q) \) is an affine line. □

Consider the control system such that \( Abn(q) \) is its set of the admissible velocities at \( q \). By Proposition 5 it is an affine system with scalar input. We call this system the reduction of the five-dimensional control-affine system (4.1). The following proposition gives another characterization of the reduction of the system (4.1):

**Proposition 6** Assume that the five-dimensional control-affine system (4.1) satisfies the conditions (4.26) and (4.11). Then the subsystem

\[
\dot{q} = g_0 + u g_1,
\]

of (4.1) is its reduction if and only if

\[
[g_0, g_1] \in D_3.
\]

**Proof.** By definition, the system (4.19) is the reduction of (4.1) if and only if

\[
Abn(q) = \{g_0(q) + t g_1(q) : t \in \mathbb{R}\}.
\]

On the other hand, one can take from the beginning \( f_0 = g_0 \) and \( f_1 = g_1 \). Then comparing (4.31) with (4.27) and (4.23) we obtain that the system (4.29) is the reduction of (4.1) if and only if \( c_{01}^3 = c_{01}^4 = 0 \), which is equivalent to \( [g_0, g_1] \in \text{span}(g_0, g_1, f_2) = D_3 \). □

**Corollary 3** Assume that a five-dimensional affine control system (4.19) satisfies

\[
\text{dim span}\left(g_0, g_1, [g_1, g_0], [g_1, [g_1, g_0]], [g_0, [g_1, g_0]]\right) = 5
\]

Then a five-dimensional control-affine system with two-dimensional input, has the system (4.19) as its reduction if and only if it is feedback equivalent to the system

\[
\dot{q} = g_0 + u_1 g_1 + u_2 (\alpha g_0 + [g_0, g_1]), \quad u_1, u_2 \in \mathbb{R},
\]

where \( \alpha \) is some function.

**Proof.** First by assumption (4.32) the system (4.33) satisfies conditions (4.26) and (4.11). Hence, by Proposition 5 the system (4.33) admits the reduction and by Proposition 6 this reduction is the system (4.19). On the other hand, suppose that some system (4.1) has the reduction (4.19). Then from the previous proposition \( [g_0, g_1] \in \text{span}(g_0, g_1, f_2) \). According to (4.22), \( g_0, g_1 \) and \( [g_0, g_1] \) are linearly independent. Hence, \( f_2 = \xi_0 g_0 + \xi_1 g_1 + \xi_2 [g_0, g_1] \), where \( \xi_2 \neq 0 \). Hence by a feedback transformation we can replace \( f_2 \) with \( \alpha g_0 + [g_0, g_1] \). □

So, in contrast to the case \( n = 4 \), five-dimensional control-affine systems with two-dimensional input cannot be recovered from its reduction only. Suppose that the system (4.11) has the reduction (4.29) satisfying condition (4.32) and also

\[
\text{dim span}\left(g_0, g_1, [g_1, g_0], [g_1, [g_1, g_0]], [g_1, [g_1, g_0]]\right) = 5,
\]

\[
\text{dim span}\left(g_1, [g_0, g_0, g_1], [g_1, g_0], [g_1, [g_1, g_0]], [g_1, [g_1, g_0]]\right) = 5.
\]
Then we can apply to the system (4.29) all constructions of section 2. In particular, let \((G_0, G_1)\) be the canonical pair of the system (4.29). As before, let \(V\) be the affine subbundle of \(TM\), defined by system (4.1). Then by the same arguments, as in the proof of Corollary 3, there exist a unique vector field \(G_2 \in V\) such that
\[
G_2 = RG_0 + [G_0, G_1].
\]
(4.36)

By construction, the function \(R\) is a feedback invariant of system (4.1). Moreover, the system (4.1) can be uniquely, up to a feedback transformation, recovered from its reduction and the function \(R\).

Fix some point \(q_0\) in \(M\). Let \(\Phi_5\) be as in (3.17). Denote
\[
R = R \circ \Phi_5.
\]
(4.37)

\(R\) is the germ of a function of five variables at 0. We call it the recovering invariant of (4.1) at the point \(q_0\). By construction, it is invariant of the weight 0. Note that the set of germs of systems of the type (4.1) having the reductions, which satisfies conditions (4.32), (4.34), and (4.35), is generic. Using Theorem 2 in the case \(n = 5\) and the definition of the recovering invariant, we obtain the following classification of generic real analytic germs of systems of the type (4.1) in terms of the tuple of the primary invariants of their reductions and their recovering invariant:

**Theorem 4**

Given 4 germs \(I_1, I_2, I_3,\) and \(R\) of real analytic functions of 5 variables at 0 such that \(I_3(0) \neq 0\), 10 germs \(\{\beta_k\}_{k=1}^5\) and \(\{\phi_{k2l}\}_{k=1}^5\) of real analytic functions of 4 variables at 0, 10 germs \(\{\psi_k\}_{k=1}^5\) and \(\{\phi_{k23}\}_{k=1}^5\) of real analytic functions of 3 variables at 0, and 5 germs \(\{\phi_{k24}\}_{k=1}^5\) of real analytic functions of 2 variables at 0 there exists a unique, up to state-feedback real analytic transformation of the type (1.3), five-dimensional real analytic control-affine system with two input such that first its reduction satisfies genericity assumptions (4.32), (4.34), (4.35), secondly \(\{I_j\}_{j=1}^3, \{\beta_k\}_{k=1}^5, \{\psi_k\}_{k=1}^5, \{\phi_{k2l} : 1 \leq k \leq 5, 2 \leq l \leq 4\}\) is its tuple of the primary invariants, and finally \(R\) is its recovering invariant at the given point \(q_0\). In other words, the tuples of the primary invariants give the regular (3, 2)-parameterization of the considered classification problem in the real analytic category with the following \(4 \times 4\) parameterization matrix \(P\):

\[
P = \begin{pmatrix}
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 10 \\
0 & 0 & 0 & 10 \\
1 & 0 & 0 & 3
\end{pmatrix}.
\]
(4.38)

The Poincare series \(M(t)\) of the considered classification problem satisfies
\[
M(t) = \frac{1}{(1-t)^6} + t^3 \left( \frac{5}{(1-t)^3} + \frac{10}{(1-t)^4} + \frac{10}{(1-t)^5} + \frac{3}{(1-t)^6} \right).
\]
(4.39)

By the same arguments, as in the case of \((1, n)\)-affine control system with \(n \geq 5\), treated in section 3, one can show that \((3, 2)\) is the characteristic pair of the considered classification problem (just use formulas (2.16), (2.17), and Corollary 10). The characteristic matrix \(C\) of the problem is equal to \(\text{Norm}(P)\), which can be found easily by the series of elementary transformations. Namely,
\[
C = \text{Norm}(P) = \begin{pmatrix}
1 & 3 & 6 & 5 \\
0 & 0 & 0 & 16 \\
0 & 0 & 0 & 13 \\
0 & 0 & 0 & 4
\end{pmatrix}.
\]
(4.40)
Conclusion 2  The characteristic parameterization of (2,5) control-affine systems, up to state-feedback transformations, consists of 4 functional invariants of 5 variables and the weight 3, 13 functional invariants of 4 variables and the weight 3, 16 functional invariants of 3 variables and the weight 3, 1 functional invariants of 2 variables and the weight 0, 3 functional invariants of 2 variables and the weight 1, 6 functional invariants of 2 variables and the weight 2, and 5 functional invariants of 2 variables and the weight 3.

In order to obtain a characteristic parameterization from the parameterization by the tuple of the primary invariants and the recovering invariant one can implement some series of rearrangement of the primary invariants according to the series of elementary transformations from the matrix $P$ to $\text{Norm}(P)$, as was described in the previous section (see, for example, formula (2.11) and the paragraph after it).

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