A new point-weighting finite-difference modelling for the frequency-domain wave equation

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Abstract. In this paper, we propose a new 21 point finite-difference scheme for modelling the frequency-domain wave equation in 2-dimensional domain. To discretize the second derivative term, we develop a modified central difference scheme in which each of the gridpoints is replaced with a linear combination of it and its neighbouring gridpoints along one direction. For the discretization of the zeroth-order term, we use the weighted average of all the 21 points. The combination coefficients and the weight parameters are determined by minimizing the numerical dispersion. In comparison with the scheme of a derivative-weighting one, the new scheme has a much better performance in reducing the numerical dispersion when the step sizes are not equal in different directions. Numerical experiments are presented to illustrate the effectiveness of the new scheme.

1. Introduction
The frequency-domain wave equation is extensively applied in many fields of science and engineering, for instance, geophysics, marine technology, petroleum exploration. Frequency-domain modelling has many advantages over the time-domain modelling. For example, it is very convenient to manipulate a single frequency, and easy to implement attenuation. Moreover, as each frequency can be computed independently, it is very favourable for parallel computing.

For the modeling frequency-domain wave equation, the finite difference method is preferred, since it is easily implemented and has less computational complexity. For the classical central difference, it leads to a poor numerical dispersion, which is not able to be eliminated [1]. To reduce numerical dispersion, the rotated 9-point difference scheme was proposed in [2], which combined the classical Cartesian coordinate system and its rotated systems to discretize the equation. In [3], the rotated difference scheme was extended to the 25-point formula. However, the rotated difference scheme is not robust. On the one hand, it fails when directional sampling intervals are different, that is, the step sizes are not equal in different directions. On the other hand, it is not pointwise consistent, which is an important property for the convergence. To improve the robustness, [4] proposed an average-derivative 9-point scheme and [5] developed a point-weighting 9-point scheme. To obtain a consistent scheme, [6] and [7] constructed two types of derivative-weighting schemes, in which a weighted derivative and linear combination of the gridpoints are employed to discretize the Laplacian and the zeroth-order term respectively. Nevertheless, the derivative-weighting schemes still cannot handle the...
problem of non-equidistant sampling. High order schemes [8-10] are also proposed to improve the accuracy, however, they are demanding with the smoothness of the right-hand side.

In this paper, we propose a new 21 point-weighting finite-difference scheme, which is pointwise consistent and suitable for the situation that the step sizes are not equal in different direction. To discretize the Laplacian, the new scheme employs a modified central difference scheme in which each of the gridpoints is replaced with a linear combination of it and its neighbouring gridpoints along one direction. The discretization of the zeroth-order term adopts weighted averages of all the 21 points. The combination coefficients and the weight parameters are obtained by minimizing the numerical dispersion. The new 21 point-weighting scheme is more flexible compared to the derivative-weighting scheme. Furthermore, it also outperforms the 9-point schemes in reducing the numerical dispersion. Finally, numerical simulations are given to demonstrate the efficiency of the new scheme.

2. A new point-weight finite-difference scheme

In this section, we present the construction of the new finite-difference scheme. Consider the 2-dimensional Helmholtz equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - k^2 u = 0, \]

where \( u \) is the displacement, \( k = \frac{2\pi f}{v} \) is the wavenumber with \( f \) and \( v \) being the frequency and velocity respectively. To discretizing equation (1), we uses the 21-point finite difference stencils which are presented in figure 1 with \((0,0)\) representing the central point, while the others denoting the neighboring points of \((0,0)\). Let \( u \big|_{mn} = u(x_0 + m\Delta x, y_0 + n\Delta y) \) denote the discretization of \( u \) at location \((x_0 + m\Delta x, y_0 + n\Delta y)\), where \( \Delta x, \Delta y \) are the step sizes in the horizontal and vertical directions respectively, and \((x_0, y_0)\) is a given initial point. Here, let \( \Delta x = h, \Delta y = rh \) with \( r \) being a positive number. Then, \( x_m = x_0 + mh, y_n = y_0 + nrh \).

![Finite difference stencils](image)

Figure 1. Finite difference stencils. (a),(b) are used for discretizing \( \frac{\partial^2 u}{\partial x^2} \), and (c),(d) are for \( \frac{\partial^2 u}{\partial y^2} \).

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\[ T_{k,n} u \big|_{m,n} := \frac{1}{h^2} \left( \tilde{u}_{m-1,n} - 2 \tilde{u}_{m,n} + \tilde{u}_{m+1,n} \right), \]

\[ T_{2h,n} u \big|_{m,n} := \frac{1}{(2h)^2} \left( \hat{u}_{m-2,n} - 2 \hat{u}_{m,n} + \hat{u}_{m+2,n} \right), \]

where

\[ \tilde{u}_{m-1,n} = a_1 u_{m-1,n} + \frac{a_2}{2} \left( u_{m-1,n-1} + u_{m-1,n+1} \right) + \frac{a_3}{2} \left( u_{m-2,n-1} + u_{m-2,n+1} \right), \]

\[ \tilde{u}_{m,n} = a_1 u_{m,n} + \frac{a_2}{2} \left( u_{m,n-1} + u_{m,n+1} \right) + \frac{a_3}{2} \left( u_{m,n-2} + u_{m,n+2} \right), \]

\[ \tilde{u}_{m+1,n} = a_1 u_{m+1,n} + \frac{a_2}{2} \left( u_{m+1,n-1} + u_{m+1,n+1} \right) + \frac{a_3}{2} \left( u_{m+2,n-1} + u_{m+2,n+1} \right), \]

\[ \hat{u}_{m-2,n} = a_4 u_{m-2,n} + \frac{a_5}{2} \left( u_{m-2,n-1} + u_{m-2,n+1} \right), \]

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\[ \hat{u}_{m+2,n} = a_4 u_{m+2,n} + \frac{a_5}{2} \left( u_{m+2,n-1} + u_{m+2,n+1} \right). \]

Here, parameters \( a_1, a_2, \ldots, a_5 \) satisfy \( \sum_{j=1}^{5} a_j = 1 \), which guarantees that the approximations of \( \frac{\partial^2}{\partial x^2} \) is second order in accuracy. Similarly, define \( T_{k,y} u \big|_{m,n} \), \( T_{2h,y} u \big|_{m,n} \) as follows

\[ T_{k,y} u \big|_{m,n} := \frac{1}{h^2} \left( \tilde{u}_{m,n-1} - 2 \tilde{u}_{m,n} + \tilde{u}_{m,n+1} \right), \]

\[ T_{2h,y} u \big|_{m,n} := \frac{1}{(2h)^2} \left( \hat{u}_{m,n-2} - 2 \hat{u}_{m,n} + \hat{u}_{m,n+2} \right), \]

with

\[ \tilde{u}_{m,n-1} = a_1 u_{m,n-1} + \frac{a_2}{2} \left( u_{m,n-1} + u_{m,n+1} \right) + \frac{a_3}{2} \left( u_{m-1,n-1} + u_{m-1,n+1} \right), \]

\[ \tilde{u}_{m,n} = a_1 u_{m,n} + \frac{a_2}{2} \left( u_{m,n-1} + u_{m,n+1} \right) + \frac{a_3}{2} \left( u_{m-1,n-1} + u_{m-1,n+1} \right), \]

\[ \tilde{u}_{m,n+1} = a_1 u_{m,n+1} + \frac{a_2}{2} \left( u_{m,n-1} + u_{m,n+1} \right) + \frac{a_3}{2} \left( u_{m-1,n-1} + u_{m-1,n+1} \right), \]

\[ \hat{u}_{m,n-2} = a_4 u_{m,n-2} + \frac{a_5}{2} \left( u_{m,n-2} + u_{m,n+2} \right), \]

\[ \hat{u}_{m,n} = a_4 u_{m,n} + \frac{a_5}{2} \left( u_{m,n-1} + u_{m,n+1} \right), \]

\[ \hat{u}_{m,n+2} = a_4 u_{m,n+2} + \frac{a_5}{2} \left( u_{m,n-1} + u_{m,n+1} \right). \]

Then, the second partial derivatives \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial y^2} \) at \( (m,n) \) are approximated respectively by

\[ T_x u \big|_{m,n} := T_{k,x} u \big|_{m,n} + T_{2h,x} u \big|_{m,n}, \quad T_y u \big|_{m,n} := T_{k,y} u \big|_{m,n} + T_{2h,y} u \big|_{m,n}. \]
Moreover, the zeroth term $k^2u$ is approximated by weighted averages of all the 21 points as follows.

\[
I(k^2u)_{|_{mn}} = b_1(k^2u)_{|_{mn}} + \frac{1}{4} \sum_{j=1}^{S} b_{j+1} I_j(k^2u)_{|_{mn}},
\]

where

\[
\begin{align*}
I_1(k^2u)_{|_{mn}} &= (k^2u)_{|_{mn}} + (k^2u)_{|_{m+1,n}} + (k^2u)_{|_{m,n+1}} + (k^2u)_{|_{m-1,n}}, \\
I_2(k^2u)_{|_{mn}} &= (k^2u)_{|_{m+1,n}} + (k^2u)_{|_{m-1,n}} + (k^2u)_{|_{m,n+2}} + (k^2u)_{|_{m,n-2}}, \\
I_3(k^2u)_{|_{mn}} &= (k^2u)_{|_{m+1,n+1}} + (k^2u)_{|_{m-1,n+1}} + (k^2u)_{|_{m,n+2}} + (k^2u)_{|_{m,n-2}}, \\
I_4(k^2u)_{|_{mn}} &= (k^2u)_{|_{m-1,n+1}} + (k^2u)_{|_{m+1,n-1}} + (k^2u)_{|_{m-1,n+2}} + (k^2u)_{|_{m+1,n-2}}, \\
I_5(k^2u)_{|_{mn}} &= (k^2u)_{|_{m,n+2}} + (k^2u)_{|_{m,n+1}} + (k^2u)_{|_{m,n+2}} + (k^2u)_{|_{m,n-2}}, \\
I_6(k^2u)_{|_{mn}} &= (k^2u)_{|_{m,n+1}} + (k^2u)_{|_{m,n+1}} + (k^2u)_{|_{m,n+2}} + (k^2u)_{|_{m,n-2}},
\end{align*}
\]

Here, parameters $b_1, b_2, \ldots, b_6$ satisfy $\sum_{j=1}^{S} b_j = 1$, which guarantees the approximation of $k^2u$ is second order in accuracy. Finally, we obtain the new difference scheme for equation (1) as follow

\[
-T_u_{|_{mn}} - T_{u|_{m,n}} = -I(k^2u)_{|_{mn}} = 0.
\]

Substituting equation (2)-(6) into (7), then the 21 points difference scheme is given by

\[
\begin{align*}
T_1 U_{m-1,n-2} + T_2 U_{m,n+2} + T_3 U_{m-1,n+2} + T_4 U_{m,n-1} + T_5 U_{m,n+1} + T_6 U_{m-1,n-1} + \\
T_7 U_{m+1,n-2} + T_8 U_{m,n+1} + T_9 U_{m,n+2} + T_{10} U_{m+1,n+2} + T_{11} U_{m+1,n-1} +
\end{align*}
\]

\[
= -T_{u|_{m,n}} - T_{u|_{m,n}} + I(k^2u)_{|_{mn}} = 0,
\]

where $U_{m-1,n-1}$ $(j,l = -2, -1, 0, 1, 2)$ denote the unknowns, and

\[
\begin{align*}
T_1 &= \frac{a_1}{2h^2} + \frac{a_2}{8r^2h^2} + \frac{b_3}{4} k^2, & T_2 &= -\frac{a_1}{h^2} + \frac{a_2}{4r^2h^2} + \frac{b_3}{4} k^2, & T_3 &= -\frac{a_1}{2h^2} + \frac{a_2}{8r^2h^2} + \frac{b_3}{4} k^2, \\
T_4 &= \frac{a_1}{h^2} + \frac{a_2}{2r^2h^2} + \frac{b_3}{4} k^2, & T_5 &= -\frac{a_1}{2h^2} + \frac{a_2}{4r^2h^2} + \frac{b_3}{4} k^2, & T_6 &= -\frac{a_1}{4h^2} + \frac{a_2}{4r^2h^2} + \frac{b_3}{4} k^2, \\
T_7 &= \frac{a_1}{h^2} + \frac{a_2}{2r^2h^2} + \frac{b_3}{4} k^2, & T_8 &= \frac{a_1}{2h^2} + \frac{a_2}{4r^2h^2} + \frac{b_3}{4} k^2, & T_9 &= -\frac{a_1}{4h^2} + \frac{a_2}{4r^2h^2} + \frac{b_3}{4} k^2, \\
T_{10} &= \frac{2a_1}{h^2} - \frac{a_2}{2r^2h^2} - \frac{b_3}{2} k^2.
\end{align*}
\]

3. Numerical dispersion analysis and determination of the weight parameters

In this section, we perform the classical dispersion analysis, and obtain the weight parameters by minimizing the numerical dispersion.

For equation (1), the classical plane-wave solution is $U(x,y) := e^{-i(kx + y \sin \theta)}$, where $\theta$ is the propagation angle from the y-axis, and the wavenumber $k = \frac{2\pi}{\lambda}$ is assumed to be a constant. Let $\lambda$ be the wavelength and $G$ be the number of gridpoints per wavelength, that is, $G = \frac{\lambda}{h}$. Since $\lambda = \frac{2\pi}{k}$, we have $kh = \frac{2\pi}{\lambda}$. Then, substituting the discrete plane-wave solution $U_{m-1,n+1} := e^{-i(kx_{m,n} + y \sin \theta)}$ into equation (8) and applying the Euler formula $e^{-i\theta} = \cos \theta + i \sin \theta$, we obtain
\[ T_1 R_1 + T_2 P_2 + T_4 R_2 + 2T_3 P_1 Q_1 + T_6 S_1 + T_8 Q_2 + T_9 Q_1 + \frac{1}{2} T_{10} = 0, \quad (9) \]

where

\[ p_1 = \cos \left( \frac{2\pi}{G} r \sin \theta \right), Q_1 = \cos \left( \frac{2\pi}{G} \cos \theta \right), S_1 = \cos \left( \frac{2\pi}{G} (2 \cos \theta - r \sin \theta) \right), R_1 = \cos \left( \frac{2\pi}{G} (\cos \theta + 2 r \sin \theta) \right), \]

\[ p_2 = \cos \left( \frac{4\pi}{G} r \sin \theta \right), Q_2 = \cos \left( \frac{4\pi}{G} \cos \theta \right), S_2 = \cos \left( \frac{4\pi}{G} (\cos \theta - 2 r \sin \theta) \right), R_2 = \cos \left( \frac{4\pi}{G} (2 \cos \theta + r \sin \theta) \right). \]

Let \( k_N \) denote the numerical wavenumber. Then, replacing \( k \) with \( k_N \) in \( T_j (j = 1, 2, \ldots, 10) \) yields

\[ k_N^3 h^2 L = R, \quad (10) \]

with

\[ L = \frac{1}{2} \left[ b_1 + \frac{b_2}{2} (P_1 + Q_1) + b_2 P_2 Q_2 + \frac{b_2}{2} (P_2 + Q_2) + \frac{b_2}{2} (R_1 + S_1) + \frac{b_2}{2} (R_2 + S_2) \right], \]

\[ R = a_1 \left[ \frac{P_1}{r^2} + Q_1 - \left( 1 + \frac{1}{r^2} \right) \right] + a_2 \left[ \left( 1 + \frac{1}{r^2} \right) P_2 Q_2 - Q_2 \right] + a_3 \left[ \frac{P_1}{r^2} + S_1 - \left( 1 + \frac{1}{r^2} \right) \right] + a_4 \left[ \frac{R_1}{r^2} + S_1 - \left( 1 + \frac{1}{r^2} \right) \right] + a_5 \left[ \frac{P_1}{r^2} + Q_1 - \left( 1 + \frac{1}{r^2} \right) \right] + a_6 \left[ \frac{R_1}{r^2} + S_1 - \left( 1 + \frac{1}{r^2} \right) \right]. \]

It follows from \( kh = \frac{2\pi}{G} \) that

\[ \frac{k_N^3}{k} = \frac{G}{2\pi} \sqrt{\frac{r}{L}}. \quad (11) \]

Equation (11) is the so-called dispersion relation formula. To obtain a good difference scheme with small numerical dispersion, it requires that value of equation (11) should be close to 1, that is, \( k_N \) is close to \( k \). Consequently, parameters \( a_j, b_j \ (j = 1, 2, \ldots, 5, l = 1, 2, \ldots, 6) \) can be determined by

\[ (a_1, \ldots, a_5, b_1, \ldots, b_6) = \arg \min \left\{ \| J(a_1, \ldots, a_5, b_1, \ldots, b_6) \|_{\ell_0, \ell_2} \right\}, \quad (12) \]

where

\[ J(a_1, \ldots, a_5, b_1, \ldots, b_6) := \frac{k_N^3}{k} - 1 = \frac{G}{2\pi} \sqrt{\frac{r}{L}} - 1, \]

with \( \sum_{j=1}^5 a_j = 1, \sum_{j=1}^6 b_j = 1, \theta \in \left[ 0, \frac{\pi}{2} \right], G \in [G_{\text{min}}, G_{\text{max}}] \). The Nyquist sampling limit requires \( G_{\text{min}} \geq 2 \).

We solve numerically the optimization problem (12) by the least-squares method [5-6], and finally the parameters \( a_j, b_j \ (j = 1, 2, \ldots, 5, l = 1, 2, \ldots, 6) \) are obtained.

**4. Simulation experiments**

In this section, we present simulation examples to demonstrate the efficiency of the new difference scheme. Firstly, we use the least-squares method to solve the optimization problem (12) with setting \( G_{\text{min}} = 4 \) and \( G_{\text{max}} = 400 \). Then, the parameters obtained are \( a_1 = 0.2065, a_2 = 0.2000, a_3 = 0.0164, a_4 = 0.3871, a_5 = 0.1900, b_1 = 0.3398, b_2 = 0.4792, b_3 = 0.1490, b_4 = 0.0266, b_5 = 0.0027 \).

We next plot the normalized phase velocity curves for new difference scheme in Figure 2, where the horizontal coordinate is \( G^{-1} \), and the vertical coordinate is \( k_N^3 k^{-1} \). The propagation angles are
chosen to be 0°, 15°, 30°, 45°. As a comparison, we also present normalized phase velocity curves for the derivative-weighting scheme in [6]. As is observed in Figure 2, the new scheme is efficiency in reducing the numerical dispersion, and performs much better than the scheme in [6], especially for the situations of non-equidistant sampling. Specifically, in Figure 2(c) and (d), set $r = 0.5$, that is, $\Delta x = h$, $\Delta y = rh = 0.5h$. It is seen that the new scheme performs robustly, however, the scheme in [6] fails. For $r = 2$ in Figure 2 (e) and (f), we have the similar results. Finally, we use the new difference scheme to model the frequency-domain wave equation (1) with a point source placed at the center of the 2D computational domain $(0,1) \times (0,1)$. In Figure 3, we present the real part of the simulation result (numerical solution) for $f = 5, 20$ respectively, which confirms the effect of the new point-weighting difference scheme.

Figure 2. Normalized phase velocity curves for the new point-weighting difference scheme and the derivative-weighting scheme. (a),(c),(e): new point-weighting scheme with $r = 1, 0.5, 2$. (b), (d), (f): derivative-weighting scheme with $r = 1, 0.5, 2$. 
5. Conclusion

In this paper, a new point-weighting finite-difference scheme is proposed for modeling the frequency-domain wave equation in 2-dimensional domain. The new scheme uses a modified central difference formula to discretize the second derivative term. Specifically, in the modified central difference formula, each of the gridpoints is replaced with a linear combination of it and its neighbouring gridpoints along one direction. Additionally, the weighted average of all the 21 points is used to discretize the zeroth-order term. The new scheme is second order in accuracy and is pointwise consistent. Simulation experiments illustrate the efficiency of the new scheme. It outperforms the 9-point derivative-weighting scheme not only in reducing the numerical dispersion, but also in tackling the problems of non-equidistant samplings.

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