NON-ASSOCIATIVE MAGNETIC TRANSLATIONS FROM
PARALLEL TRANSPORT IN PROJECTIVE HILBERT BUNDLES

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Abstract. The non-associativity of translations in a quantum system with magnetic field background has received renewed interest in association with topologically trivial gerbes over $\mathbb{R}^n$. The non-associativity is described by a 3-cocycle of the group $\mathbb{R}^n$ with values in the unit circle $S^1$. The gerbes over a space $M$ are topologically classified by the Dixmier-Douady class which is an element of $H^3(M, \mathbb{Z})$. However, there is a finer description in terms of local differential forms of degrees $d = 0, 1, 2, 3$ and the case of the magnetic translations for $n = 3$ the 2-form part is the magnetic field $B$ with non zero divergence.

In this paper we study a quantum field theoretic construction in terms of $n$-component fermions on a circle. The non associativity arises when trying to lift the translation group action on the 1-particle system to the second quantized system.

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1. Introduction

The motivation for the present short note is to understand the recent paper by Bunk, Müller and Szabo [2] in terms of quantization of Dirac operators on a real line or on the circle coupled to an abelian vector potential with gauge group $\mathbb{R}^n$ or the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. The central topic in [2] is a 3-cocycle on $\mathbb{R}^n$ arising from composing certain functors coming from translations acting on differential data of a topologically trivial gerbe on $\mathbb{R}^n$. The non-associativity in the case of a magnetic field with sources in the case $n = 3$ was suggested already long ago in [7]. An interpretation of the 3-cocycle in terms of representations of canonical anticommutator algebras was then proposed in [3].

In this paper we interpret the magnetic translations as (non periodic) $\mathbb{R}^n$ valued gauge transformations on the unit interval $[0,1]$ acting on fermions with $n$ complex components. They actually define true operators on the level of 1-particle Dirac operators. However, they cannot be lifted to unitary operators in the fermionic Fock space; if they could, there would be no 3-cocycle since the composition of linear operators is associative. Nevertheless, these gauge transformations define functors acting on certain categories of representations of canonical anticommutation relations. The composition of functors respects the group law in $\mathbb{R}^n$ only modulo the action of automorphisms in the Fock space; these automorphisms come from a projective representation of an abelian gauge group.

The calculation of the group 3-cocycle is based on the interpretation of the (non periodic) gauge transformations as parallel transport in a projective Hilbert bundle of fermionic Fock spaces parametrized by a simply connected Lie group $G$; in the
As a reference to CAR representations and their automorphisms see [1].

Let \( G \) be a simply connected Lie group. Let \( H \) be the space of square integrable periodic functions on the unit interval \([0, 1]\) taking values in the complex vector space \( \mathbb{C}^n \) with a unitary \( G \) action through a representation \( \rho \) of \( G \). For \( g \in G \) let \( P_g \) be the category (actually a Frechet manifold) of smooth paths \( f(t) \) with \( 0 \leq t \leq 1 \) in \( G \) starting from the identity \( e \) in \( G \) and with the end point at \( g \), with the additional condition that \( A = f^{-1} df \) is smooth and periodic. Then the set \( P = \{ P_g | g \in G \} \) defines a principal bundle over \( G \) with fibers diffeomorphic to the based loop group \( \Omega G \).

Each \( f \in P \) defines an 1-dimensional antihermitean Dirac operator \( D_f = \frac{d}{dt} + A \) with domain \( H_0 \subset H \) consisting of smooth periodic functions. Since \( H \) can be naturally identified as the space of square integrable functions on the unit circle we can define a polarization \( H = H_+ \oplus H_- \) to positive and non-positive Fourier modes. This choice defines a fermionic Fock space \( F \) carrying an irreducible representation of the canonical anticommutation relations (CAR) algebra; this algebra is generated by elements \( a^*(v), a(v) \) with \( v \in H \) such that the annihilation operators \( a(v) \) depend antilinearly on the argument whereas the creation operators depend linearly on \( v \). The only nonzero anticommutators are

\[
a^*(v)a(u) + a(u)a^*(v) = \langle u, v \rangle 1
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product in the complex Hilbert space. The representation in \( F \) is characterized by the existence of a cyclic vector \( \psi_0 \), the vacuum vector, such that

\[
a(u)\psi_0 = 0 = a^*(v)\psi_0, \quad \text{for } u \in H_+ \text{ and } v \in H_-.
\]

As a reference to CAR representations and their automorphisms see [1].

The category of the Dirac operators \( D_f \) can be quantized (using the standard normal ordering associated to the Fourier modes in \( H \)) to define quantum Dirac operators \( \hat{D}_f \) acting on the dense subspace of the Fock space consisting of polynomials of the creation and annihilation operators acting on the vacuum vector. The based loop group \( \Omega G \) acts in \( F \) through a central extension \( \hat{\Omega} G \). The Lie algebra \( \hat{\mathfrak{g}} \) of the central extension is defined by the 2-cocycle

\[
(1.1) \quad c_2(X, Y) = \frac{1}{2\pi i} \int \text{tr} XdY
\]

with values in the group of purely imaginary numbers, where the trace is evaluated in the representation of \( G \) in \( \mathbb{C}^n \). The 2-cocycle defines a left invariant form on the loop group with periods in \( 2\pi \mathbb{Z} \). Elements \( h \in \hat{\Omega} G \) define automorphisms of the category of Dirac operators \( \hat{D}_f \), parametrized by paths \( f \) with \( f(1) = g \), through right multiplication \( f \mapsto fh \) which corresponds to the gauge transformation \( A \mapsto h^{-1}Ah + h^{-1}dh \). Using the action of the central extension of the loop group in \( F \) this action can be lifted to the quantized Dirac operators through \( \hat{D}_f \hat{h} = \hat{D}_{f\hat{h}} \). The projective representation of \( \hat{\Omega} G \) can be viewed as a homomorphism of \( \hat{\Omega} G \) to the projective unitary group \( PU(F) \) of the Hilbert space \( F \), so the principal bundle \( P \) defines a \( PU(F) \) bundle over \( G \). Projective Hilbert bundles are classified by their Dixmier-Douady class \( \omega \in \mathbb{H}^3(G, \mathbb{Z}) \). The class \( \omega \) in this case depends on the representation of \( G \) in \( \mathbb{C}^n \). In the case of \( G = SU(n) \) in its
defining representation this class is the generator in \(Z = H^3(G, \mathbb{Z})\). For \(G = SU(n)\) an explicit representation of the generator as a closed 3-form is given as

\[
\frac{1}{24\pi^2} \text{tr}(g^{-1}dg)^3
\]

with the trace evaluated in the defining representation of \(SU(n)\). See \cite{1, 2} for more information on gauge group cocycles in relation to quantized Dirac operators.

2. Transgression from the Dixmier-Douady class and the 3-cocycle

For a given \(g\) the elements \(f \in P_g\) define a category of 1-particle Dirac operators associated to potentials \(A = f^{-1}df\) for \(f \in P_g\) and the quantized Dirac operators \(\hat{D}_f\). If \(g, g' \in G\) is a pair of different group elements and \(f \in P_g, f' \in P_g\) then formally \(D_f\) is equivalent to \(D_{f'}\) through a gauge transformation \(D_f \mapsto D_{f'}\) by \(f \mapsto fh\) with \(f : [0, 1] \to G, h(t) = f^{-1}f'\). But \(h\) is nonperiodic and therefore it does not preserve the domain of the Dirac operators. A consequence of this is that there is no quantization \(\hat{h}\) acting in the Fock space which would take \(\hat{D}_f\) to \(\hat{D}_{f'}\) by conjugation. Thus we can only say that the path \(h\) with end point \(g^{-1}g'\) takes one category of Dirac operators (parametrized by \(g\)) to another category (parametrized by \(g'\)).

Next fix a connection in the principal bundle \(P \to G\). A based loop \(\gamma\) at \(e \in G\) defines through parallel transport an element \(h(\gamma)\) in the model fiber \(\Omega G\) and thus an element in \(PU(F)\). We have now a map \(\phi : \Omega G \to PU(F)\). The canonical circle bundle \(S^1 \to U(F) \to PU(F)\) with curvature \(F\) pulls back to a circle bundle over \(\Omega G\) and the Chern class of this circle bundle is represented by a closed 2-form \(\theta = \phi^*F\).

**Theorem 2.1.** The Chern class \(\theta\) on \(\Omega G\) is equal to the transgression of the Dixmier-Douady class \(\omega\) on \(G\), that is,

\[
\theta(X,Y) = \int_{\gamma} \omega(h^{-1}dh, X, Y).
\]

In particular, since \(H^3(G, \mathbb{Z}) = \mathbb{Z}\) when \(G\) is a simple compact Lie group, we get

\[
\theta(X,Y) = \frac{k}{8\pi^2} \int_{\gamma} \text{tr}h^{-1}dh[X,Y].
\]

The form \(\theta\) is cohomologous to the form \(\omega_c\), namely \(\omega_c = \theta + d\xi\) with \(\xi(X) = \frac{1}{8\pi^2} \int_{\gamma} \text{tr}h^{-1}dhX\).

**Proof.** Recall that there is a map \(\tau : P^{[2]} \to \Omega G\) defined by \(p_1\tau(p_1, p_2) = p_2\). The induced \(PU(F)\) bundle associated to \(P\) by the homomorphism \(\phi\) gives rise to a lifting bundle gerbe \((Q, P)\) over \(G\) where \(Q\) is the pullback to \(P^{[2]}\) of the bundle \(U(F) \to PU(F)\) by \(\phi \circ \tau : P^{[2]} \to PU(F)\). See \cite{13} for further details such as the bundle gerbe multiplication. This bundle gerbe has Dixmier-Douady class \(\omega\) and we can choose a bundle gerbe connection on \(Q\) with curvature \(R \in \Omega^2(P^{[2]})\) and curving \(B \in \Omega^3(P)\) such that \(R = \pi_1^*(B - \pi_2^*(B))\) and \(dB = \pi^*(\omega)\). From the second equation we see that the transgression of \(\omega\) is \(B|_{\Omega G}\) where \(\Omega G \subset P\) as the space of loops.

Let \(P_0 \subset P^{[2]}\) be defined by \(P_0 = \{(e, \gamma) \mid \gamma \in \Omega G\}\) where \(e\) is the constant identity loop. Clearly \(P_0\) is diffeomorphic to \(\Omega G\) and \(\pi_1 : P^{[2]} \to P\) restricted to \(P_0\)
induces the identity map from $\Omega G$ to the copy of $\Omega G$ in $P$. On the other hand $\pi_2$ restricted to $P_0$ is the constant map to $e \in P$. Notice also that $\tau$ restricted to $P_0$ is the identity. Note that in general the pullback connection by $\phi \circ \tau$ might not be a bundle gerbe connection as it might fail to respect the bundle gerbe multiplication but certainly the Chern class of the pullback bundle is the pullback of the Chern class by naturality. So the class of $\theta$ is the class of $R = \pi_1^*(B) - \pi_2^*(B)$ restricted to $P_0 \simeq \Omega G$ which is the class of $B_{\Omega G}$ as required.

Let $B$ be a local potential of the Dixmier-Douady form, $\omega = dB$; this can be defined in an open neighborhood of the unit in $G$. In the special case $G = SU(2)$ it is well-defined in the open set not containing the matrix $-1$. In the same way as the curvature on the loop space is a transgression of $\omega$, we may choose a local connection $A$ in the loop space as a transgression of $B$. On the other hand, since the curvature in the loop space is a pull-back of $\phi^*F$ we can set $A = \phi^*\eta$ where $\eta$ is a local connection form $d\eta = F$, with respect to a local trivialization of $U(\mathcal{F}) \to PU(\mathcal{F})$. With respect to this local trivialization we can fix a phase of the lift of $\phi(\gamma) \in PU$ to the unitary group $U$ by selecting a path $\gamma_s$ ($0 \leq s \leq 1$) in the loop space with $\gamma_0$ the constant loop at $e \in G$ and $\gamma_1 = \gamma$.

Next choose a triple $g_1, g_2, g_3$ of elements in $G$. For each $g_i$ choose a path $g_i(t)$ with $g_i(0) = e$ and $g_i(1) = g_i$. We have now four closed 1-simplices as follows: The first is the loop $\ell(g_1, g_2)$ defined as the composition of $g_1(t), g_1g_2(t)$ and $-(g_1g_2(t)) = (g_1g_2)(1 - t)$. (The minus sign means the opposite orientation as compared to the path $(g_1g_2)(t)$. The second and third are $\ell(g_2, g_3)$ and $\ell(g_1, g_3)$. Finally the fourth is the left translated simplex $g_1\ell(g_2, g_3)$. Then it is easy to check that as a 1-simplex

\begin{equation}
\ell(g_1, g_2) - g_1 \cdot \ell(g_2, g_3) + \ell(g_1g_2, g_3) - \ell(g_1, g_2g_3) = 0.
\end{equation}

For each of the 1-simplices above choose a singular 2-simplex $s(g_1, g_2) \ldots$ such that $\delta s(g_1, g_2) = \ell(g_1, g_2)$ and so on; this is possible since $G$ is simply connected. The sum of these 2-simplices is closed by the equation above and we may choose a 3-simplex $\Delta_3(g_1, g_2, g_3)$ such that its boundary is the sum $S(g_1, g_2, g_3)$ of the 2-simplices $s$.

Given four elements $g_i \in G$ with $1 \leq i \leq 4$ we can set

\begin{equation}
V(g_1, \ldots, g_4) = \Delta_3(g_1, g_2, g_3) + \Delta_3(g_1g_2, g_3, g_4) + \Delta_3(g_1, g_2g_3, g_4) + \Delta_3(g_1g_2, g_3g_4) + g_1 \cdot \Delta_3(g_2, g_3, g_4)
\end{equation}

and then $\delta V(g_1, g_2, g_3, g_4) = 0$ since the boundaries of the $\Delta$’s cancel pairwise. From this follows that

$$C_3(g_1, g_2, g_3) = e^{2\pi i \int_{\Delta_3(g_1, g_2, g_3)} \omega}$$

is a 3-cocycle in the sense that

$$C_3(g_1, g_2, g_3)C_3(g_1g_2, g_3, g_4)^{-1}C_3(g_1, g_2g_3, g_4)C_3(g_1, g_2, g_3g_4)^{-1}C_3(g_2, g_3, g_4) = 1.$$

We have used the fact that $\omega$ is an integral form, so its integral $I$ over the 3-simplex without boundary is in $\mathbb{Z}$ and so $\exp(2\pi i I) = 1$.

In a local trivialization of the pull-back of the bundle $U(\mathcal{F}) \to PU(\mathcal{F})$ to the loop group $\Omega G$ the phase of the unitaries can be fixed by a parallel transport along a path $\ell_s$ in $\Omega G$ connecting the identity $\ell_0 = 1$ to a given loop $\ell = \ell_1$. In
particular, the phases of the parallel transports around the 1-simplices $\ell(g_i, g_j)$ are determined as $\exp 2\pi i \int_{s(g_i, g_j)} B$ since the connection in the loop space is the transgression of the local form $B$. The phases of the parallel transports along the faces of the 3-simplex $\Delta_3(g_1, g_2, g_3)$ add up to $\exp(2\pi i \int_{\partial \Delta_3(g_1, g_2, g_3)} B)$ which is equal to $\exp(2\pi i \int_{\Delta_3(g_1, g_2, g_3)} \omega)$ by Stokes’ theorem. Thus we can state

**Theorem 2.2.** The obstruction to lifting the 2-cocycle $\ell$ to the central extension $U(F)$ of $PU(F)$ is the class of 3-cocycle $C_3$. Changing the cocycle $C_3$ by a coboundary $\delta b$ is the same as changing the lifts of the loops $\ell$ to the central extension by phases $b(g_i, g_j)$.

An equivalent way to arrive at the 3-cocycle $C_3$ is as follows. Denoting the parallel transports along the edge loops $\ell$ by the same symbols we have then the cocycle property (which follows from 2.1)

$$\ell(g_1, g_2)\ell(g_2, g_3) = \ell(g_2, g_3)^{g_1}\ell(g_1, g_2, g_3)$$

where the first factor on the right denotes the parallel transport from $g_1$ to $g_1g_2$ and continuing via $g_1g_2g_3$ back to the vertex $g_1$.

Next we fix a lift $\hat{\ell}$ to the central extension $\hat{\Omega}G$ acting in the Fock space $F_0$. The lift is fixed by a parallel transport from the constant loop to the loop $\ell$ as explained previously. The cocycle property above does not hold anymore for the lifts but there is a correction factor,

$$\hat{\ell}(g_1, g_2)\hat{\ell}(g_1g_2, g_3) = \hat{\ell}(g_2, g_3)^{g_1}\hat{\ell}(g_1, g_2g_3)C_3(g_1, g_2, g_3),$$

with the $S^1$ valued 3-cocycle $C_3$.

The cocycle $C_3$ depends on the choices of the lifts $\hat{\ell}$ modulo a coboundary. However, since the loops involved come from the boundaries of the faces in a tetraed we can use this to fix the phases. The central extension $\hat{\Omega}G$ comes equipped with a canonical connection: The tangent bundle of a Lie group is trivial and the tangent spaces can be identified as the Lie algebra of the group. The Lie algebra of the central extension is as a vector space the direct sum of the Lie algebra of $\Omega G$ and the 1-dimensional Lie algebra $i\mathbb{R}$. The projection onto the center defines the connection 1-form $\xi$ on $\hat{\Omega}G$. This connection is actually the connection coming from the map $\Omega G \rightarrow PU(F)$ through canonical quantization [8]. The face maps to $G$ define a contraction of the parallel transport along each boundary loop to a constant loop at $e$. The contraction defines a path in $\Omega G$ starting from the constant loop and using the canonical connection this defines an element in $\hat{\Omega}G$. If $\delta, \delta'$ are a pair of face maps with a common boundary 1-simplex then the phases differ by a factor $\psi(\delta, \delta') \in S^1 = \mathbb{R}/\mathbb{Z}$. Since $H^2(G, \mathbb{Z}) = 0$ the union $\delta \cup \delta'$ is a boundary of some volume $V \subset G$ and then $\psi(\delta, \delta') = \exp(2\pi i \int_V \omega)$.

Returning to the calculation of the cocycle $C_3(g_1, g_2, g_3)$: The phases of the parallel transports along the edges of the tetraed are fixed by the faces. The difference of the phases is the given by

$$C_3(g_1, g_2, g_3) = \exp(2\pi i \int_{\Delta_3(g_1, g_2, g_3)} \omega).$$

As stated before in Section 1, a parallel transport from a point $g$ to a point $g'$ can be viewed as a functor in the 1-particle Hilbert space $H$.
edges of the faces of $\Delta_3$ define projective operators and mismatch of the phases of the composed lifted unitary operators give rise to the $S^1$ valued 3-cocycle.

In the case of a topologically trivial gerbe, $\omega = dB$ for some globally defined 2-form on $G$, the parallel transport in the loop space is defined by the transgression of $B$ giving the potential in the loop space as

$$A(f; X) = \int_{S^1} B(f^{-1}df, X)$$

for a loop $f$ and a tangent vector $X$ at $f$.

That gives the parallel transport from the constant loop at $e$ to the boundary loop of the 2-simplex $s(g_1, g_2)$ (with vertices $e, g_1, g_1g_2$) as

$$d(g_1, g_2) = \exp(2\pi i \int_{s_2} B).$$

In this case the 3-cocycle $C_3$ is trivial,

$$C_3(g_1, g_2, g_3) = d(g_1, g_2)d(g_1g_2, g_3)d(g_1, g_2g_3)^{-1}g_1 \cdot d(g_2, g_3)^{-1}$$

where in the last factor the integration is around the face on the 3-simplex with vertices at $g_1, g_1g_2, g_1g_2g_3$.

3. The case of $G = \mathbb{R}^3$

The group $G = \mathbb{R}^3$ has an unitary representation in the Hilbert space $H$ of square integrable periodic functions on $[0, 1]$ with values in $\mathbb{C}^3$ through multiplication $z_k \mapsto e^{2\pi i x_k} z_k$ for $k = 1, 2, 3$. This defines also an action of the loop group $LG$ in $H$ through point-wise multiplication by the phase $e^{2\pi i x_k(t)}$.

Through the canonical quantization in a fermionic Fock space the Lie algebra valued loops $X, Y : S^1 \to i\mathbb{R}^3$ are represented projectively with the 2-cocycle

$$c_2(X, Y) = \frac{1}{2\pi i} \int_{S^1} <X, dY>.$$

This 2-cocycle is trivial in the cohomology with coefficients in the space of Frechet smooth functions on $\Omega G$, $c_2 = \delta b$ with $b_f(X) = \frac{1}{4\pi i} \int_{S^1} <d\log(f), X >$. Let $c'_2$ be another trivial cocycle,

$$c'_2(f; X, Y) = \frac{1}{2\pi i} \int_{S^1} d\log f \wedge X \wedge Y.$$

One can check that $c'_2$ is the transgression of of the 3-form $\omega(X, Y, Z) = X \wedge Y \wedge Z$ on $G$. As a 3-form $\omega$ is exact, $\omega = dB$ with $B_w(X, Y) = w \wedge X \wedge Y$ for $w \in \mathbb{R}^3$ and $X, Y$ tangent vectors at $w$. We can take $\omega$ as the Dixmier-Douady form of topologically trivial $PU$ bundle over $\mathbb{R}^3$. In this case the argument in the previous section lead to a 3-cocycle of $\mathbb{R}^3$,

$$C_3(X, Y, Z) = \exp(2\pi i \int_{\Delta_3} \omega) = \exp\left(\frac{2\pi i}{6} X \wedge Y \wedge Z\right)$$

where $\Delta_3$ is the tetraed with vertices at the points $0, X, X + Y, X + Y + Z$. This cocycle is a coboundary of the 2-cochain $R_z$ with coefficients in the group $G = \mathbb{R}^3$,

$$R_z(X, Y) = \exp(\pi i z \wedge X \wedge Y).$$

This is again a consequence of the fact that the Dixmier-Douady form $\omega$ is trivial in cohomology.
We can easily generalize the discussion above to the case $G = \mathbb{R}^n$ for $n \geq 3$. Each antisymmetric real valued tensor $a_{ijk}$ on $\mathbb{R}^n$ defines a closed form $\omega = \sum_{i,j,k} a_{ijk} dx_i \wedge dx_j \wedge dx_k$ which can be taken as the Dixmier-Douady form of a trivial gerbe on $G$. This closed form is exact, $\omega = dB$ with $B = \sum_{i,j,k} a_{ijk} x_i dx_j \wedge dx_k$.

The cocycle $C_3$ is recovered by the parallel transport argument as before. In this case we can take the path connecting the identity $0 \in \mathbb{R}^n$ to a vector $X$ as the straight line $X(t) = tX$. The loops $\ell$ are now triangles connecting the vectors $0, X, X + Y, 0, X, X + Y + Z, 0, X + Y, X + Y + Z$ and $X, X + Y, X + Y + Z$ and the cocycle $C_3(X,Y,Z)$ is again given by the integral of the 3-form $\Omega$ over the 3-simplex with vertices at the points $0, X, X + Y, X + Y + Z$,

$$C_3(X,Y,Z) = e^{2\pi i \sum_{ijk} \frac{1}{6} a_{ijk} X_i Y_j Z_k}. \quad (3.1)$$

Although this cocycle for (nonzero $a$) is nontrivial as a group cocycle it is however trivial as a transformation groupoid cocycle: The group $\mathbb{R}^n$ acts on itself by translations and $C = \delta b$ for the the 2-cochain $b(u; X,Y) = C_3(u,X,Y)$ with

$$(\delta b)(X,Y,Z) = b(u; X,Y)^{-1} b(u; X + Y,Z)^{-1} b(u; X,Y + Z) b(u + X; Y,Z).$$

In the present setting the loops take values in $\mathbb{R}^n$ and $\rho(x) z_k = e^{ix_k z_k}$ for $k = 1, 2, \ldots, n$. Each component defines a circle value function $e^{i f(t)}$ acting as a multiplication operator in the 1-particle Hilbert space $H = L_2([0,1], \mathbb{C}^n)$. The cocycle $c_2$ is nontrivial on the abelian loop group. However, in the case of a family of Dirac operators $D_A = i \frac{d}{dt} + A$ coupled to an abelian vector potential $A$ (with values in $\mathbb{R}^n$) the relevant cohomology is with coefficients in the space of Frechet smooth functions of the vector potential. In this cohomology the cocycle becomes trivial: we have $c_2 = \delta b_1$ where

$$b_1(A; X) = \frac{1}{4\pi i} \int \sum_k A_k X_k dt$$

and the loop algebra element $X$ acts on $A$ through the gauge transformation $A \rightarrow A + dX$. For this reason the bundle of Fock spaces parametrized by the vector potentials becomes equivariant with respect to the gauge action and can be pushed forward to a bundle over the flat moduli space $\mathbb{R}^n = A/G$ of gauge potentials; here $G$ is the group of periodic functions $[0,1] \rightarrow \mathbb{R}^n$ acting on potentials as $A \rightarrow A + df$.

4. THE CASE OF A TORUS

If we replace the gauge group $\mathbb{R}^n$ by the torus $T^n$ the situation becomes different. All the maps $f : [0,1] \rightarrow \mathbb{R}^n$ which are periodic modulo $\mathbb{Z}^n$ satisfy the Hilbert-Schmidt condition on off-diagonal blocks with respect to the energy polarization; again, a function $f$ defines a multiplication operator in the one-particle space through $z_k \mapsto e^{2\pi i f_k z_k}$. These functions $f$ can be viewed as loops $S^1 \rightarrow T^n$. Now the group of gauge transformations $G$ factorizes as a product of the group $G_0$ contractible maps to $T^n$ (represented as loops on $\mathbb{R}^n$) and a group $\mathbb{Z}^n$ of maps of the form $f(t) = 0$ for $t \leq 0$, $f(t) = tv$ for $0 \leq t \leq 1$ with $v \in \mathbb{Z}^n$ and $f(t) = v$ for $t \geq 1$.

The moduli space of gauge potentials $A/G$ is now the torus $T^n$; we have $A/G_0 = \mathbb{R}^n$ and the second factor in $G$ is isomorphic to the subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$. In the case of $\mathbb{R}^n$ there was no restriction on the normalization of the 3-cocycle [as a group
cocycle or as a 3-form on \( \mathbb{R}^n \) but in the case of the torus the 3-cocycle must satisfy an integrality constraint in order that the gerbe over \( T^n \) is well-defined.

As explained in [6], [11] (see also [12] Section 7) the 1-particle Dirac hamiltonians can be twisted in such a way that their K-theory class over the moduli space \( T^n \) is nontrivial: the Chern character has a nonzero component \( \omega_3 \) in \( H^3(\mathbb{T}^n, \mathbb{Z}) \). The basis in \( H^3(\mathbb{T}^n, \mathbb{Z}) \) is given by the 3-forms \( \omega = \sum a_{ijk} dx_i \wedge dx_j \wedge dx_k \) where the \( a \)'s form a basis of totally antisymmetric tensors of rank 3 with integral coefficients.

The pull-back with respect to the projection \( \mathbb{R}^n \to \mathbb{T}^n \) is the form \( d \sum a_{ijk} x_i dx_j \wedge dx_k \). The quantum field theoretic construction of a gerbe over the torus from a non-zero class \([\omega]\) is recalled in the Appendix.

The 3-form part \( \omega \) of the Chern character is the Dixmier-Douady class of the projective vector bundle over \( T^n \) obtained by canonical quantization of the family of 1-particle Dirac operators. The pull-back of this bundle over \( \mathbb{R}^n \) comes by projectivization of a vector bundle (the bundle of fermionic Fock spaces). The group \( \mathbb{Z}^n \) acts through an abelian extension of the Fock spaces. The extension is defined by the 2-cocycle

\[
c_2(u; x, y) = e^{2\pi i \sum a_{ijk} u_i x_j y_k}
\]

where \( x, y \in \mathbb{Z}^n \) and \( u \in \mathbb{R}^n \) and \( \mathbb{Z}^n \) acts on the functions of the vector \( u \) as translations.

For integral coefficients \( a_{ijk} \) the 3-cocycle (3.1) is identically \( = 1 \) when the arguments are in \( \mathbb{Z}^n \subset \mathbb{R}^n \) in conformity with the (projective) action of \( \mathbb{Z}^n \) on the Fock spaces. The 3-cocycle is the obstruction to an extension of the \( \mathbb{Z}^n \) action to a \( \mathbb{R}^n \) action on the bundle of Fock spaces.

**Remark** The cocycle \( c_2 \) is also a group cocycle even in the case of constant coefficients (no group action on \( u \)) but since the coboundary operator is different the cohomology with variable coefficients is different from the cohomology with constant coefficients.

5. **Appendix**

In this appendix we briefly recall the quantum field theoretic construction of a gerbe over the torus using a twisted family of CAR algebra representations, [6], [11].

A hermitean complex line bundle \( L \) over the torus \( T^n \) is characterized by a class \( \omega \) in \( H^2(\mathbb{T}^n, \mathbb{Z}) \). Parametrizing the circles in the torus by the interval \([0, 1]\) the 2-cohomology is spanned by antisymmetric bilinear forms on \( \mathbb{R}^n \) such that \( \omega(x, y) \in \mathbb{Z} \) for \( x, y \in \mathbb{Z}^n \). The pull-back of \( L \) over \( \mathbb{R}^n \) is trivial and the sections of that line bundle are complex valued functions \( \psi \) such that

\[
\psi(x + z) = \psi(x)e^{2\pi i \omega(x, z)}
\]

for \( z \in \mathbb{Z}^n \).

Next we construct a family of fermionic Fock spaces parametrized by vectors in \( \mathbb{R}^n \). For each \( k = 1, 2, \ldots, n \) and \( u, v \in H \) let \( a_k(v), a^*_k(u) \) be generators of a CAR algebra with nonzero anticommutators

\[
a^*_k(u)a_k(v) + a_k(v)a^*_k(u) = 2 < u, v >.
\]

The generators for different lower indices are assumed to commute. It will be convenient to compactify the real line to the unit circle so we can take \( H = \)
We twist the CAR algebra by the line bundle $L$. This means that the families of creation and annihilation operators are sections of the tensor products of $L$ or its dual and the CAR algebra. The sections are $\mathbb{Z}^n$ equivariant functions on $\mathbb{R}^n$, that is, for $x \in \mathbb{R}^n$ and for $z \in \mathbb{Z}^n$

$$a_k^*(u, x + z) = a_k^*(u, x)e^{2\pi i \omega(x, z)}, \quad a_k(u, x + z) = a_k(u, x)e^{-2\pi i \omega(x, z)}.$$ 

The right-hand-side of the canonical anticommutation relations, when evaluated at a point $x \in \mathbb{R}^n$, is multiplied by the pairing of sections of $L, L^*$ involved in the construction of $a_k^*(u, x) = a_k^*(u) \otimes \psi(x)$ and of $a_k(v) \otimes \xi$.

The Fock vacuum is again annihilated by $a^*(u, x)$ and $a(v, x)$ for $u \in H_-$ and $v \in H_+$. (One could generalize this construction by allowing the modes for different lower index $k$ be twisted by different line bundles.)

Thus the states with net particle number $N$ in the Fock space are twisted by the $N$th tensor power of $L$.

The group $\mathbb{Z}^n$ acts as automorphisms of the twisted CAR algebra by

$$g(p)a^*(u, x)g(p)^{-1} = a^*(p \cdot u, x + p) = e^{2\pi i \omega(x, p)}a^*(p \cdot u, x)$$

where $p$ acts on a function $u(\zeta)$ by multiplication by a phase, $u_k \mapsto e^{2\pi i \zeta x u_k}$, that is, the Fourier modes are shifted by $p$ units. Likewise, for the annihilation operators

$$g(p)a(u, x)g(p)^{-1} = e^{-2\pi i \omega(x, p)}a((-p) \cdot u, x).$$

The action of $g(p)$ in the Fock spaces parametrized by $x$ is now completely defined by fixing the action on the vacuum vector $\psi$. This is easiest done thinking the vectors as elements in the semi-infinite cohomology (in physics terms, the ‘Dirac sea’). For $n = 1$ the vacuum is symbolically the semi-infinite product

$$\psi = a_0^*a_{-1}^*a_{-2}^*\cdots$$

where the lower index refers to the Fourier modes in $L_2(S^1)$. For general $n$ the vacuum is defined in a similar way inserting the non-negative Fourier modes for each of the $n$ components. The CAR generators are labelled by a double index $(k, j)$ with $k \in \mathbb{Z}$ and $j = 1, 2, \ldots n$. The action of $g(p)$ on the vacuum is now defined as a shift operator: The index $k$ of the element $a_{k, j}$ is shifted by the integer $p_j$ for $j = 1, 2, \ldots, n$, $k \mapsto k + p_j$.

Because of the phase shifts when the CAR algebra generators are conjugated by $g(p)$ the product $g(p)g(q)$ is not equal to $g(p + q)$ but they differ by an $x$ dependent phase,

$$g(p)g(q) = C(x; p, q) = e^{2\pi i N \omega(x, p)}g(p + q)$$

where $N$ is the particle number of the state $g(q)\psi$, that is, $N = \sum_{j=1}^n q_j$.

**Remark** The projective vector bundles over $T^n$ are classified by elements of $H^3(T^n, \mathbb{Z})$. Representatives of these elements can be written as de Rham forms $\Omega = \sum a_{ijk}dx_i \wedge dx_j \wedge dx_k$ where the coefficients $a_{ijk}$ are integers. The pull-back of $\Omega$ with respect to the projection $\pi : \mathbb{R}^n \rightarrow T^n$ is $\pi^*\Omega = d\theta = d\sum a_{ijk}x_i dx_j \wedge dx_k$. Evaluating $\theta$ for tangent vectors $u, v$ in the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and exponentiating gives the 2-group cocycle

$$C'(x; u, v) = e^{2\pi i \sum a_{ijk}x_i u_j v_k}$$
where the group $\mathbb{Z}^n$ acts on the vector $x$ by $x \mapsto x + u$. According to the discussion in [12], Section 7.1, there is a 1-1 correspondence between the group cohomology $H^2_{\text{grp}}(\mathbb{Z}^n, A)$ and the de Rham cohomology $H^3(T^n, \mathbb{Z})$ where $A$ is the $\mathbb{Z}^n$ module of (smooth) functions $T^n \to S^1$. The above map $\{c_{ijk}\} \to C'$ realizes this isomorphism.

**Example:** When $n = 3$ the cocycle $C$ is equivalent to $C'$ for the choice $a_{ijk} = \alpha \epsilon_{ijk}$ where $\epsilon$ is totally antisymmetric tensor with $\epsilon_{123} = 1$ and $\alpha = 2(\omega_{12} + \omega_{23} + \omega_{31})$ where $\omega = \omega_{12} dx_1 \wedge dx_2 + \omega_{31} dx_3 \wedge dx_1 + \omega_{23} dx_2 \wedge dx_3$. This is seen by projecting the exponent in $C$ to its totally antisymmetric component.

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