A 64-dimensional two-distance counterexample to Borsuk’s conjecture

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2013-08-25

1 Abstract and introduction

In [1] (1933) Karol Borsuk asked whether each bounded set in the $n$-dimensional Euclidean space can be divided into $n+1$ parts of smaller diameter. The diameter of a set is defined as the supremum (least upper bound) of the distances of contained points. Although not explicitly stated in Borsuk’s question the diameter of the whole set is assumed to be positive.

The question became famous under the (inaccurate) name Borsuk’s conjecture. From 1993 to 2003 several authors have proved that in high dimensions such a division is not generally possible.

In [2] (2013) Andriy V. Bondarenko constructed a 65-dimensional two-distance set of 416 vectors that cannot be divided into less than 84 parts of smaller diameter. That was not just the first known two-distance counterexample to Borsuk’s conjecture but also a considerable reduction of the lowest known dimension the conjecture fails in in general.

This article presents a 64-dimensional subset of the vector set mentioned above that cannot be divided into less than 71 parts of smaller diameter, that way delivering a two-distance counterexample to Borsuk’s conjecture in dimension 64.

The contained proof relies on the results of some (combinatorial) calculations. The additionally (in the source package) provided small computer program G24CHK needs about one second for that task on a 1 GHz Intel PIII.

2 Euclidean representations of strongly regular graphs

The following is mainly an essence of the content of the corresponding section in [2], giving just the needed basic facts. Detailed information can be found e.g. in [3].

Saying $G$ is a srg($v, k, \lambda, \mu$), where srg abbreviates “strongly regular graph”, means that there is a set $V$ of $v$ elements (vertices) such that $G \subset V \times V$, for all $i \in V \neg G(i, i) \wedge |\{j : G(i,j)\}| = k$, and for all different $i, j \in V$

$$G(i,j)=G(j,i)$$

and

$$|\{p \in V : G(p, i) \wedge G(p, j)\}| = \begin{cases} \lambda & \text{if } G(i,j) \\ \mu & \text{otherwise} \end{cases}$$

The adjacency matrix $A$ of $G$ has exactly 3 different eigenvalues: $k$ of multiplicity 1, one positive eigenvalue of multiplicity

$$f = \frac{1}{2} \left( v - 1 - \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right),$$

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and one negative eigenvalue

\[ s = \frac{1}{2} \left( \lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right) \]

In the remaining part of this article we use these notations: \( I \) is the identity matrix of size \( v \), \( y \) is \( A - sI \), \( y_i \), where \( i \in V \), are the columns of \( y \), and \( y_{i,j} \), where \( i, j \in V \), are the entries of \( y \).
The given properties of the eigenvalues imply \( \text{dim}\{y_i : i \in V\} \leq f \).

3 The \( G_2(4) \) graph

Let \( G \) be isomorphic to a graph known as \( G_2(4) \) graph, a \( \text{sr}(416,100,36,20) \). The value of \( s \) is \(-4\), thus for \( i, j \in V \)

\[ y_{i,j} = \begin{cases} 4 & \text{if } i = j \\ 1 & \text{if } G(i,j) \\ 0 & \text{otherwise} \end{cases} \]

For \( i \in V \), \( y_i \) consists of one 4 (at position \( i \)), 100 1’s, and 315 0’s. For different \( i, j \in V \)

\[ \|y_i - y_j\|^2 = \begin{cases} 2 \times (100 - 36 - 1 + (4 - 1)^2) = 144 & \text{if } G(i,j) \\ 2 \times (100 - 20 + 4^2) = 192 & \text{otherwise} \end{cases} \]

The value of \( f \) is 65 and therefore \( \text{dim}\{y_i : i \in V\} \leq 65 \).

For each \( W \subseteq V : \{y_i : i \in W\}\) has a smaller diameter than \( \{y_i : i \in V\} \) iff all members of \( W \) are pairwise adjacent.

By [2] the \( G_2(4) \) graph does not contain a clique (subgraph of pairwise adjacent vertices) with more than 5 vertices, and because there are 416 vertices (and corresponding vectors), \( \{y_i : i \in V\} \) cannot be divided into less than 84 subsets of smaller diameter.

4 A construction of the \( G_2(4) \) graph

The following algorithm is an extract of the description given in [4] and [5], there originated to [6]. See also [7]. Notice: The points mentioned here are strictly to distinct from the points in the \( n \)-dimensional Euclidean space.

“Consider the projective plane \( \text{PG}(2,16) \) provided with a nondegenerate Hermitean form. It has 273 points, 65 isotropic and 208 nonisotropic. There are 416 = 208 \cdot 12 \cdot 1/6 orthogonal bases. These are the vertices of \( G \). [...] Associated with a basis \{a,b,c\} is the triangle consisting of the 15 isotropic points on the three lines \( ab \), \( ac \), and \( bc \). [...] \( G \) can be described as the graph on the 416 triangles, adjacent when they have 3 points in common.”

5 A division of the vertices of the \( G_2(4) \) graph

The construction given in the previous section assigns a set of 15 different isotropic points to each of the 416 vertices/triangles. Assume that integer numbers from 1 to 65 are given to the isotropic points and consider the sets of the numbers of the isotropic points, here shortly named iso-sets. Thus two vertices are adjacent if the cardinality of the intersection of their iso-sets is 3. Now we start to divide the set of the 416 vertices: First we define \( B \) to be the set of all vertices in \( V \) containing the number 1 in their iso-set and \( C \) to be the set of the remaining vertices. The edges of \( G \) connecting two vertices in \( B \) constitute a subgraph of \( G \). We divide \( B \) into non-empty subsets \( B_h \), \( h \) positive integer, such that two different vertices are in the same subset iff \( G \) contains a path between them not leaving \( B \).
6 The computer program G24CHK

The computer program G24CHK, provided in the source package of this article and described in more detail in the last sections before the references, implements the construction of the iso-sets of the members of \( V \) and its just defined subsets. Additionally, it checks that \( B \) consists of 3 pairwise disjunctive subsets \((B_1, B_2, \text{ and } B_3)\), each containing exactly 32 vertices. Consequently, \(|B| = 96 < |C| = 416 - 96 = 320\).

Finally, G24CHK is able to check this claim:

(1) \( \forall i \in V, h \in \{1, 2, 3\} \)

\[ |\{j \in B_h : G(i, j)\}| = \begin{cases} 20 & \text{if } i \in B_h \\ 0 & \text{if } i \in B \setminus B_h \\ 8 & \text{otherwise} \end{cases} \]

The second case is already implied by the construction of the \( B_h \). The number and cardinalities of the unconnected subsets of \( B \) and the first two cases in (1) correspond with this slightly cut quotation from [5], section Subgraphs:

“b) Three copies of the 2-coclique extension of the Clebsch graph.
The 65520 nonedges of \( G \) [...] fall into 1365 sets of 48, where each set induces a subgraph of size 96 that is the disjoint union of three copies of the 2-coclique extension of the Clebsch graph (the halved 5-cube), and the 48 nonedges are the 2-cocliques.”

So actually just a proof of the third case in (1) is needed to make the actual construction and counting superfluous, possibly already done or easy to achieve by someone more familiar with strongly regular graphs.

However, in the following two sections the claims concerning \( G \) and the subsets of \( V \) are taken as facts.

7 The 64-dimensional counterexample

Recall that for \( i, j \in V \)

\[ y_{i,j} = \begin{cases} 4 & \text{if } i = j \\ 1 & \text{if } G(i, j) \\ 0 & \text{otherwise} \end{cases} \]

Therefore statement (1) implies

(2) \( \forall i \in V, h \in \{1, 2, 3\} \)

\[ \sum_{j \in B_h} y_{i,j} = \begin{cases} 20 \times 1 + 1 \times 4 = 24 & \text{if } i \in B_h \\ 0 & \text{if } i \in B \setminus B_h \\ 8 \times 1 = 8 & \text{otherwise} \end{cases} \]

We define a vector \( p \) in the same 416-dimensional space as the \( y_{i,j}, i \in V \), consisting of entries \( p_j, j \in V \):

\[ p_j = \begin{cases} 1 & \text{if } j \in B_2 \\ -1 & \text{if } j \in B_3 \\ 0 & \text{otherwise} \end{cases} \]

Combined with (2) this definition implies \( \forall i \in V \)

\[ \langle p, y_i \rangle = \sum_{j \in B_2} y_{i,j} - \sum_{j \in B_3} y_{i,j} = \begin{cases} 0 - 0 = 0 & \text{if } i \in B_1 \\ 24 - 0 = 24 & \text{if } i \in B_2 \\ 0 - 24 = -24 & \text{if } i \in B_3 \\ 8 - 8 = 0 & \text{otherwise} \end{cases} \]
The essence is that $p$ is orthogonal to $\{y_i : i \in C \cup B_1\}$, but not to $\{y_i : i \in V\}$. Therefore $\dim \{y_i : i \in C \cup B_1\} \leq \dim \{y_i : i \in V\} - 1$. Because $\dim \{y_i : i \in V\} \leq 65$ is known, we have $\dim \{y_i : i \in C \cup B_1\} \leq 64$.

The proofs are not included here, but one can show (e.g., by actual vector calculations) that these statements stay valid if we replace $\leq$ by $=$.

Because $\{y_i : i \in C \cup B_1\}$ contains 352 vectors and a subset of smaller diameter contains at most 5 vectors, a division into less than 71 parts of smaller diameter is impossible. Because $71 > 64 + 1$, the answer to Borsuk’s question for $n = 64$ is negative.

8 A 63-dimensional almost-counterexample

We define a vector $q$ in the same 416-dimensional space as the $y_i, i \in V$, consisting of entries $q_j, j \in V$:

$$q_j = \begin{cases} 2 & \text{if } j \in B_1 \\ -1 & \text{if } j \in B_2 \cup B_3 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\langle p, q \rangle = 0 - 32 + 32 = 0$.

Combined with (2) the definition of $q$ implies $\forall i \in V$

$$\langle q, y_i \rangle = 2 \times \sum_{j \in B_1} y_{i,j} - \sum_{j \in B_2} y_{i,j} - \sum_{j \in B_3} y_{i,j} = \begin{cases} 48 - 0 - 0 = 48 & \text{if } i \in B_1 \\ 0 - 24 - 0 = -24 & \text{if } i \in B_2 \\ 0 - 0 - 24 = -24 & \text{if } i \in B_3 \\ 16 - 8 - 8 = 0 & \text{otherwise} \end{cases}$$

The essence is that $q$ is orthogonal to $\{y_i : i \in C\}$, but not to $\{y_i : i \in C \cup B_1\}$. Therefore $\dim \{y_i : i \in C\} \leq \dim \{y_i : i \in C \cup B_1\} - 1$. Because $\dim \{y_i : i \in C \cup B_1\} \leq 64$ has been derived, we have $\dim \{y_i : i \in C\} \leq 63$.

Again, these statements stay valid if we replace $\leq$ by $=$.

But $\{y_i : i \in C\}$ contains just 320 vectors. The proof is not included here, but one can divide $C$ into 64 5-cliques and therefore $\{y_i : i \in C\}$ into 64 parts of smaller diameter.

9 Mathematical foundations of the implementation in G24CHK

The $G_2(4)$ graph construction algorithm given above is very concise. Sufficient information on projective planes can be found in [8]. The implementation of the algorithm uses elements of the three-dimensional vector space over the finite field GF(16) to constitute and represent the objects in the projective plane PG(2,16). Enough information on finite fields can be found in [9], using just GF(16) as example. Unfortunately, the calculation of the list of the polynomials representing the elements of GF(16) contains an error, resulting especially in the impossible relation $x^{10} = x^{13}$.

The applied Hermitean form takes three-dimensional vectors $a$ and $b$ over GF(16) and returns $a_1b_3 + a_2b_2 + a_3b_1$, where addition, multiplication, and conjugation operate in GF(16).

Finally, [10] contains enough information on isotropic objects.

10 Compiling and executing G24CHK

To compile the provided source code file G24CHK.PAS you will need a PASCAL compiler compatible with Turbo Pascal 4.0. Even Turbo Pascal versions 3.02 and 1.0 are sufficient after removing the
right half of the first occurrence of the sequence (**), this way transforming the source up to the next occurrence of *) into pure comment.

The compilers listed below have successfully compiled the program on a 1 GHz Intel PIII running MS Windows 98 SE and partly on a 500 MHz AMD K6-II running MS DOS 5.0; the respective execution times, roughly measured with some overhead utilizing the PC clock (resolution: 0.055 s), are given.

- Turbo Pascal 1.0 : 1.373 s / 2.527 s
- Turbo Pascal 3.02 : 1.428 s / 2.636 s
- Turbo Pascal 5.5 : 2.636 s / 2.856 s
- Turbo Pascal 7.01 : 0.879 s / 0.989 s
- Borland Delphi 4.0 build 5.37 : 0.824 s / -
- Virtual Pascal 2.1 build 279 : 0.989 s / -
- Free Pascal 2.4.4 i386-Win32 : 0.769 s / -

To avoid a compilation result depending on the settings you could (not in the case of Turbo Pascal versions 1.0 and 3.02) use the command line versions of the compilers (TPC for Turbo Pascal, BPC for Borland Pascal 7, DCC32 for Borland Delphi (32 bit versions; do not misst to use the -CC option in order to generate a console executable), VPC for Virtual Pascal, FPC for Free Pascal) instead of the compilers integrated in the IDEs.

The program uses the heap and pointers in general just to store the vectors constituting the adjacency matrix. It does it in a way reducing the readability as little as possible while still allowing the use of compilers storing each Boolean value in a separate byte and unable to handle memory block sizes reaching or exceeding 64 KB.

If you don’t change the respective compiler directives, range checks and (if the compiler is compatible with Turbo Pascal 4.0) stack overflow checks are generated. So the resulting executable will be very safe. It is also quite small and needs less then 1 KB for the stack. Turbo Pascal versions before 4.0 do not know conditional compiling and interpret some compiler options differently. For enabling or disabling the generation of stack overflow checks, the relevant letter is K (instead of S). So you could insert the sequence \{\$K+\} (e.g. after \{\$R+\}) in order to generate those checks.

The program ignores any command line parameters or inputs other than pressing Ctrl-C to cancel the execution.

It writes only to the standard output device. In the default case that will be the monitor screen. But if the used compiler was compatible with Turbo Pascal 4.0 you can redirect the output to a file. That way the lines below enclosed in <<< and >>> were generated. You could use them to compare your results with.

<<<
=== Constructing and checking a G2(4) graph representation ===
=== Version 1 Copyright (c) 2013-07-31 Thomas Jenrich ===
Search for orthogonal bases ... OK
Allocating dynamic memory ... OK
Filling the adjacency matrix ... OK
Separating subsets B_1, B_2, B_3, C ... OK
Checking the counts of adjacencies in B_1, B_2, B_3 ... OK
OK
== Regular program stop ==
>>>

5
References

[1] Karol Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math., 20 (1933), 177-190.

[2] Andriy V. Bondarenko, *On Borsuk’s conjecture for two-distance sets*, arXiv:math.MG/1305.2584 (version 2), http://front.math.ucdavis.edu/1305.2584v2

[3] Andries E. Brouwer, Willem H. Haemers, *Spectra of graphs*, Monograph, February 1, 2011, Springer

[4] Andries E. Brouwer, *A construction of the Suzuki Graph*, 2008-07-08, http://www.win.tue.nl/~aeb/preprints/Suz.pdf

[5] Andries E. Brouwer, *G2(4) graph*, http://www.win.tue.nl/~aeb/graphics/G24.html, retrieved 2013-05-22

[6] N. Horiguchi, M. Kitazume, H. Nakasora, *A construction of the sporadic Suzuki graph from U_3(4)*, preprint, 2008

[7] Dean Crnković, Vedrana Mikulić, *BLOCK DESIGNS AND STRONGLY REGULAR GRAPHS CONSTRUCTED FROM THE GROUP U(3, 4)*, GLASNIK MATEMATIČKI, Vol. 41(61)(2006), 189-194

[8] William Cherowitzo, *A Short Introduction to Projective Geometry*, http://www-math.ucdenver.edu/~wcherowi/courses/m5410/pgintro.pdf

[9] Robert Campbell, *Lectures 12 & 13: Finite Fields*, (Course) Number Theory, Math 413, 26 Jan 2003 http://userpages.umbc.edu/~rcampbel/Math413Spr05/Notes/12-13_Finite_Fields.html

[10] Namik Ciblak, Harvey Lipkin, *ORTHONORMAL ISOTROPIC VECTOR BASES*, Proceedings of the 1998 ASME Design Engineering Technical Conference, September, 13-16, 1998, Atlanta, Georgia

[11] *Turbo Pascal* versions 1.0, 3.02, and 5.5 (binaries only) http://edn.embarcadero.com/museum/antiquesoftware#
    For downloading one has to register or sign-in.

[12] *Virtual Pascal* (Closed Source freeware)
    One ZIP-file including binaries and documentation for Win32, OS/2, and Linux
    Official forum:
    http://vpascal.ning.com/
    Forum entry *Where can I download VP?* :
    http://vpascal.ning.com/forum/topic/show?id=854411%3ATopic%3A9

[13] *Free Pascal* (Open Source freeware)
    Sources, documentation, and binaries for several systems
    http://www.freepascal.org

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