PERIOD INTEGRALS OF HYPERSURFACES VIA TROPICAL GEOMETRY

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Abstract. Let \( \{Z_t\}_t \) be a one-parameter family of complex toric hypersurfaces of dimension \( d \geq 1 \). We compute asymptotics of period integrals for \( \{Z_t\}_t \) by applying the method of Abouzaid–Ganatra–Iritani–Sheridan, which uses tropical geometry. As integrands, we consider Poincaré residues of meromorphic \((d+1)\)-forms on the ambient toric variety, which have poles along the hypersurface \( Z_t \). The cycles over which we integrate them are spheres and tori which correspond to tropical \((0,d)\)-cycles and \((d,0)\)-cycles on the tropicalization of \( \{Z_t\}_t \), respectively. In the case of \( d = 1 \), we explicitly write down the polarized logarithmic Hodge structure of Kato–Usui at the limit as a corollary.

1. Introduction

Abouzaid–Ganatra–Iritani–Sheridan [AGIS20] computed asymptotics of integrations of holomorphic volume forms on toric Calabi–Yau hypersurfaces over Lagrangian sections of SYZ fibrations by using tropical geometry. They gave a new proof of the gamma conjecture (an asymptotic version of Hosono’s conjecture [Hos96, Conjecture 2.2]) for ambient line bundles on Batyrev [Bat94] pairs of mirror Calabi–Yau hypersurfaces. Their work also gave us a new perspective from Strominger–Yau–Zaslow conjecture [SYZ96] and tropical geometry to the gamma conjecture. In particular, they observed that the Riemann zeta values (the effect of gamma classes) appearing in the subleading terms of periods are contributions from the discriminants of SYZ fibrations. The goal of this article is to generalize their work to integrations of Poincaré residues of meromorphic forms on the ambient toric varieties, which have poles along the hypersurfaces, for toric hypersurfaces which are not necessarily Calabi–Yau hypersurfaces.

1.1. Main result. Let \( K \) be the convergent Puiseux series field over \( \mathbb{C} \), i.e., the field of formal series \( \sum_{j \in \mathbb{Z}} c_j x^{j/n} \) \( (c_j \in \mathbb{C}, n \in \mathbb{Z}_{\geq 1}) \) that have only finitely many coefficients with negative index and whose positive part is convergent in a neighborhood of \( x = 0 \). It has the standard non-archimedean valuation

\[
\text{val} : K \rightarrow \mathbb{Q} \cup \{ \infty \}, \quad \text{min} \left\{ j/n \mid c_j \neq 0 \right\}.
\]

Let \( d \) be a positive integer. Consider a free \( \mathbb{Z} \)-module \( N \) of rank \( d + 1 \), and its dual \( M := \mathrm{Hom}(N, \mathbb{Z}) \). We write \( N_Q := N \otimes \mathbb{Q}, M_Q := M \otimes \mathbb{Q} \) for a \( \mathbb{Z} \)-module \( Q \). Let \( \Delta \subset M_{\mathbb{R}} \) be a lattice polytope of dimension \( d + 1 \) such that \( W := \mathrm{Int}(\Delta) \cap M \neq \emptyset \), where \( \mathrm{Int}(\Delta) \) denotes the interior of \( \Delta \). We consider a Laurent polynomial \( f = \sum_{m \in A} k_m z^m \in K[M] \) \( (k_m \neq 0, \forall m \in A := \Delta \cap M) \) over \( K \) such that the function

\[
\lambda : A \rightarrow \mathbb{Q}, \quad m \mapsto \text{val}(k_m)
\]

extends to a strictly-convex piecewise affine function on a unimodular triangulation \( \mathcal{F} \) of \( \Delta \), i.e., a triangulation consisting only of \( (d+1)\)-dimensional simplices of the minimal volume \( 1/(d+1)! \) and their faces. (In this article, we say that a function \( h : \Delta \rightarrow \mathbb{R} \) is convex if it satisfies \( h(tm_1 + (1-t)m_2) \leq th(m_1) + (1-t)h(m_2) \) for any \( t \in [0,1] \) and \( m_1, m_2 \in \Delta \).) We also take a rational simplicial fan \( \Sigma \) in \( N_{\mathbb{R}} \) that is a refinement of the normal fan of \( \Delta \).

Let \( t \in \mathbb{R}_{>0} \) be an element in a small neighborhood of \( 0 \), where all coefficients \( k_m \) of the Laurent polynomial \( f \) converge. We consider the Laurent polynomial \( f_t \in \mathbb{C}[M] \) over \( \mathbb{C} \) obtained by substituting \( t \) to the indeterminate \( x \) in \( f \). We set

\[
\tilde{Z}_t := \{ z \in N_{\mathbb{C}^*} \mid f_t(z) = 0 \}
\]

and let \( Z_t \) be the closure of \( \tilde{Z}_t \) in the toric variety \( Y_{\Sigma} \supset N_{\mathbb{C}^*} \) associated with the fan \( \Sigma \).

Let further \( l \geq 1 \) be an integer. We consider the polytope \( l \cdot \Delta := \{ l \cdot m \in M_{\mathbb{R}} \mid m \in \Delta \} \), and its triangulation \( l \cdot \mathcal{F} := \{ l \cdot \tau \mid \tau \in \mathcal{F} \} \). We set \( V_l := M \cap \mathrm{Int}(l \cdot \Delta) \), and take an element \( v \in V_l \). Let \( \tau_v \in \mathcal{F} \) be the minimal cell such that \( v \in l \cdot \tau_v \). One can uniquely write \( v = \sum_{m \in A \cap \tau_v} p_m \cdot m \) with \( p_m \in \mathbb{Z} \cap (0,l] \) such that \( \sum_{m \in A \cap \tau_v} p_m = l \). We consider

\[
\omega^{l,v} := \left( \bigwedge_{i=0}^d \frac{dz_i}{z_i} \right) \frac{1}{(f_t)^d} \prod_{m \in A \cap \tau_v} (k_m, t \cdot z^m)^{p_m},
\]

where
where \((z_0, \cdots, z_d)\) are \(\mathbb{C}^*\)-coordinates on \(N_{\mathbb{C}^*} \cong (\mathbb{C}^*)^{d+1}\), and \(k_{m,t} \in \mathbb{C}^*\) is the number obtained by substituting \(t\) to the indeterminate \(x\) in \(k_m\). This extends to a meromorphic \((d+1)\)-form on \(Y_\Sigma\) that has a pole along \(Z_t\), and such forms generate \(H^0 \left( Y_\Sigma, \Omega^{d+1}(l \cdot Z_t) \right)\) (cf. Section 2). We write the image of \(\tilde{\omega}_t^{l,w}\) by the Poincaré residue map

\[
\text{Res}: \quad H^0 \left( Y_\Sigma, \Omega^{d+1}(l \cdot Z_t) \right) \to H^d (Z_t, \mathbb{C})
\]
as \(\Omega_t^{l,w} \in H^d (Z_t, \mathbb{C})\). This is the form which we consider as an integrand in this article. Note that the cohomology group of a toric hypersurface decomposes into the residual part (the image of the Poincaré residue map (1.5)) and the ambient part which consists of cohomology classes coming from the ambient toric variety, and therefore the former part is essential when it comes to Hodge structure of toric hypersurfaces (cf. [Mav03, Section 5]).

For \(w \in W\), we rewrite the equation \(f = 0\) as \(- (f - k_w z^w) / k_w z^w = 1\). Let \(f_t^w \in \mathbb{C}[M]\) denote the Laurent polynomial obtained by substituting \(t\) to \(x\) in \(- (f - k_w z^w) / k_w z^w\). Then \(Z_t = \{ z \in N_{\mathbb{C}^*} \mid f_t^w(z) = 1 \}\). The hypersurface \(Z_t\) is a member of the family of complex toric hypersurfaces defined by the second projection

\[
\left\{ \left( z, \{ a_m \}_{m \in A \setminus \{ w \}} \right) \in Y_\Sigma \times (\mathbb{C}^*)^{A \setminus \{ w \}} \left| \sum_{m \in A \setminus \{ w \}} a_m z^m = 1 \right. \right\} \to (\mathbb{C}^*)^{A \setminus \{ w \}}.
\]

For \(m \in A\), we set \(\lambda_m := \text{val}(k_m)\). Let \(c_m \in \mathbb{C}^*\) denote the coefficient of \(x^{\lambda_m}\) in \(k_m \in K\), and we write its absolute value \(|c_m|\) as \(r_m \in \mathbb{R}_{>0}\). We define

\[
\tilde{f}_t^w := \sum_{m \in A \setminus \{ w \}} r_m \lambda_m - \lambda_w z^{m-w},
\]

\[
\tilde{Z}_t^w := \{ z \in Y_\Sigma \mid \tilde{f}_t^w(z) = 1 \}.
\]

The complex hypersurface \(\tilde{Z}_t^w\) is also a member of the family (1.6). The positive real locus \(\tilde{Z}_t^w \cap N_{\mathbb{R}_{>0}}\) is homeomorphic to a \(d\)-sphere. Let \(C_t^w \subset Z_t\) be the parallel transport of \(\tilde{Z}_t^w \cap N_{\mathbb{R}_{>0}}\) in (1.6) as we vary the complex coefficients of the polynomial defining the hypersurface from \(f_t^w\) to \(f_t^w\). Here we choose a branch of the argument \(\text{arg} (-c_m/c_w)\) for every \(m \in A \setminus \{ w \}\), and vary the complex coefficients so that the argument \(\text{arg} (r_m/r_w) = 0\) of every coefficient in \(\tilde{f}_t^w\) changes to the argument \(\text{arg} (-k_{m,t}/k_{w,t}) \sim \text{arg} (-c_m/c_w)\) (as \(t \to +0\)) of the coefficient in \(f_t^w\) continuously. (The cycle \(C_t^w \subset Z_t\) depends on the choices of branches of \(\text{arg} (-c_m/c_w)\).) We will construct the cycle \(C_t^w\) concretely in Section 3 by the method of [AGIS20].

The tropicalization \(\text{trop}(f)\) of \(f\) is the piecewise affine function \(\text{trop}(f) : N_{\mathbb{R}} \to \mathbb{R}\) defined by

\[
\text{trop}(f)(n) := \min_{m \in A} \{ \text{val}(k_m) + \langle m, n \rangle \},
\]

and the tropical hypersurface \(X(\text{trop}(f)) \subset N_{\mathbb{R}}\) defined by \(\text{trop}(f)\) is the corner locus of \(\text{trop}(f)\). The tropical hypersurface \(X(\text{trop}(f))\) is a polyhedral complex of dimension \(d\), and gives a polyhedral decomposition of \(N_{\mathbb{R}}\), which is dual to \(\mathcal{T}\) (cf. e.g. [MS15, Proposition 3.1.6]). Recall that it is also the limit of the image of \(Z_t \subset N_{\mathbb{C}^*}\) by the map \(\log_t : N_{\mathbb{C}^*} \to N_{\mathbb{R}}\) defined by

\[
\log_t(z_0, \cdots, z_d) := (\log_t |z_0|, \cdots, \log_t |z_d|)
\]
as \(t \to +0\) [Rul01, Mik04]. Let \(\sigma \in \mathcal{P}\) be a cell of dimension \(d\) contained in \(X(\text{trop}(f))\) whose dual edge in \(\mathcal{T}\) contains an element in \(W\) as its vertex. We take a point \(n_0 \in \text{Int}(\sigma)\). There is a submanifold that is diffeomorphic to a \(d\)-torus in the inverse image of a small neighborhood of \(n_0\) in \(N_{\mathbb{R}}\) by the restriction of the map \(\log_t\) to \(Z_t \subset N_{\mathbb{C}^*}\). (See Section 3 for the detail.) We write it as \(T_\sigma^t \subset Z_t\). According to [Mik04], the hypersurface \(Z_t\) admits a stratified torus fibration over the tropical hypersurface \(X(\text{trop}(f))\). The torus \(T_\sigma^t\) is a fiber of it. (The cycles \(C_t^w, T_\sigma^t\) actually can be taken so that they are Lagrangian submanifolds in the stratified torus fibration. See [Mik04, Theorem 3].) We will compute the period integrals over these cycles \(C_t^w, T_\sigma^t\).

For \(w \in W\), we set

\[
A_w := \{ m \in A \setminus \{ w \} \mid \text{conv} \{ \{m, w\} \} \in \mathcal{P} \},
\]

where \(\text{conv}(\bullet)\) denotes the convex hull. Let further \(Y_w\) be the toric variety associated with the fan \(\Sigma_w := \{ \mathbb{R}_{\geq 0} \cdot (\tau - w) \mid \tau \in \mathcal{T}, \tau \ni w \}\), and \(D_m^w (m \in A_w)\) be the toric divisor on \(Y_w\) associated with the
1-dimensional cone $\mathbb{R}_{\geq 0} \cdot (m - w) \in \Sigma_w$. We also set

$$\omega_{\lambda}^w := \sum_{m \in A_w} (\lambda_m - \lambda_w) D_{m}^w, \quad \sigma^w := \sum_{m \in A_w} D_{m}^w$$

and

$$E_{v,w} := \left( \prod_{m \in A_w \cap v} p_{m}^{-1} \prod_{i = 0}^{p_{m} - 1} (D_{m}^w + i) \prod_{i = 0}^{p_{w} - 1} (\sigma^w - i) \right), \quad \hat{\Gamma}_w := \frac{\prod_{m \in A_w} \Gamma (1 + D_{m}^w)}{\Gamma (1 + \sigma^w)} ,$$

where $p_{m}$ are the integers that we used to write $v = \sum_{m \in A \cap v} p_{m} \cdot m$, and if $w \notin \tau_v$, we set $p_{w} := 0$ and the last product in $E_{v,w}$ means the empty product. $\Gamma (1 + \bullet)$ means the power series expansion of the gamma function $\Gamma (1 + x)$:

$$\Gamma (1 + x) = \exp \left( -\gamma x + \sum_{k = 2}^{\infty} \frac{(-1)^k}{k} \zeta (k) x^k \right) ,$$

where $\gamma$ is the Euler constant, and $\zeta (k)$ is the Riemann zeta value. Note that the restriction of the class $\hat{\Gamma}_w$ to an anticanonical hypersurface of the toric variety $Y_w$ is the gamma class of the hypersurface. See [AGIS20 Section 4.1]. One can also find a more explicit expression of $\hat{\Gamma}_w$ at (15) in loc.cit. The following is the main theorem of this article.

**Theorem 1.1.** We have

$$\int_{C_t^w} \Omega_{t}^{w} = \begin{cases} \left( \frac{(-1)^d}{(t^{d})} \right) \int_{Y_w} t^{-w_{\lambda}} \cdot \prod_{m \in A_w} \left( -\frac{c_m}{c_w} \right) \cdot E_{v,w} \cdot \hat{\Gamma}_w + O (t^{\epsilon}) & \text{conv} (\{ w \} \cup \tau_v) \in \mathcal{F} \\ \frac{2 \pi \sqrt{-1}}{O (t^{\epsilon})} t^{\epsilon} \tau_v \text{ is an element in } W \text{ and } \tau_v = \text{val}(k_{\tau_v}) + \langle \tau_v, \bullet \rangle \text{ on } \sigma & \text{otherwise} \end{cases}$$

as $t \to +0$, for some constant $\epsilon > 0$.

In (1.15), the term $(-c_m / c_w)^{-D_{m}^w}$ is defined to be

$$\exp \left( -D_{m}^w \log \left( -\frac{c_m}{c_w} \right) \right) = \exp \left( -D_{m}^w \left( \log \left| \frac{c_m}{c_w} \right| + \sqrt{-1} \arg \left( \frac{c_m}{c_w} \right) \right) \right) ,$$

where the branch of $\arg (-c_m / c_w)$ is the one we chose when we constructed the cycle $C_t^w$. Needless to say, the signs of the results of integrations depend on the choices of orientations of the cycles $C_t^w, T_{t}^\sigma$. Concerning these choices of orientations, see Remark 5.1.

One can see from (1.15) that Riemann zeta values (the effect of the gamma class) appear in the principal part of $\int_{C_t^w} \Omega_{t}^{w}$ in general as in the case of Calabi–Yau manifolds. We can also see from (1.15) that the affine volumes of bounded cells in the tropical hypersurface $X (\text{trop} (f))$ appear in the leading terms of the periods $\int_{C_t^w} \Omega_{t}^{w}$. This is discussed in Section 6. We also give a concrete example illustrating these in Section 7.

The main results of [AGIS20] correspond to (1.15) with assumptions that

- the polytope $\Delta$ is reflexive,
- $l = 1, v = w (= 0), k_w = -1$, and
- the coefficients $k_{m} (m \in A \setminus \{ w \})$ can be written as $t^{\lambda_{m}}$ or $t^{\lambda_{m}} e^{\sqrt{-1} \theta}$ with $\theta \in \mathbb{R}$.

(Notice that the orientations of cycles used in loc.cit. are different from that used in this article.) Our purpose of working over the convergent Puiseux series field over $\mathbb{C}$ rather than simply setting $k_{m} = t^{\lambda_{m}}$ or $k_{m} = t^{\lambda_{m}} e^{\sqrt{-1} \theta}$ as in [AGIS20] is to see which informations of limits of Hodge structure are encoded by tropical hypersurfaces or not. Indeed we can see from (1.15) that the asymptotics of the period depends also on the complex coefficients $c_{m}$ (not only their arguments but also their absolute values) which are thrown away by tropicalization.

When the polytope $\Delta$ is reflexive, one can also obtain (1.15) and (1.16) (as well as the main results of [AGIS20]) by taking (the derivatives of) the asymptotic expansions of the result of [Grill11 Theorem 1.1]. For the correspondence between the cycles $C_t^w$ and the ambient line bundles on the mirror Calabi–Yau hypersurfaces, we refer the reader to [AGIS20 Remark 1.3].
1.2. Logarithmic Hodge theory. We replace the field $K$ with the convergent Laurent series field, i.e., the field of Laurent series $\sum_{j \in \mathbb{Z}} c_j x^j$ that have only finitely many coefficients with negative index and whose positive part is convergent in a neighborhood of $x = 0$. For a polynomial $f = \sum_{m \in A} k_m z^m$ over it, let $\rho > 0$ be a real number that is smaller than the radius of convergence of every coefficient $k_m$ of $f$. We also set $D_\rho^* := \{ z \in \mathbb{C}^* \mid |z| < \rho \}$. By substituting elements $q \in D_\rho^*$ to the indeterminate $x$ in $f$, we obtain a family $\{\mathbb{Z}_q\}$ of complex toric hypersurfaces over the punctured disc $D_\rho^*$. It defines a variation of polarized Hodge structure over $D_\rho^*$, which extends to a logarithmic variation of polarized Hodge structure (LVPH) of Kato–Usui [Ku09] over the whole disc $D_\rho := \{ z \in \mathbb{C} \mid |z| < \rho \}$. When $d = 1$, we can explicitly write down its restriction to the limit $D = 0$, by Theorem [1.1]. In the following, we assume that $d = 1$ and the polynomial $f = \sum_{m \in A} k_m z^m$ satisfies the same assumptions as those in the previous subsection.

Let $t \in \mathbb{R}_{>0}$ a number such that $t < \rho$. Since $d = 1$, the hypersurface $Z_t$ is a Riemann surface, and the tropical hypersurface $X(\text{trop}(f))$ is a tropical curve. Let $\beta_w \ (w \in W)$ be the cycle class in $H_1(Z_t, \mathbb{Z})$ represented by $C_t^w$. One can take the cycles $C_t^w$ so that we have $\langle \beta_{w_1}, \beta_{w_1} \rangle = 0$ for any $w_0, w_1 \in W$, where $\langle \cdot, \cdot \rangle$ denotes the intersection pairing (Lemma [4.3]). We can also take a basis $\{\alpha_w\}_{w \in W}$ of the subspace of $H_1(Z_t, \mathbb{Z})$ generated by $\{T^w\}$ so that

\[
(1.18) \quad \langle \alpha_{w_0}, \alpha_{w_1} \rangle = 0, \quad \langle \alpha_{w_0}, \beta_{w_1} \rangle = \delta_{w_0, w_1},
\]

where $\delta_{w_0, w_1}$ is the Kronecker delta (i.e., $\{\alpha_w, \beta_w \mid w \in W\}$ is a symplectic basis of $H_1(Z_t, \mathbb{Z})$). Let $\alpha_w^\ast, \beta_w^\ast \ (w \in W)$ denote the dual basis of $H^1(Z_t, \mathbb{Z})$, and define the nilpotent endomorphism $N : H^1(Z_t, \mathbb{Z}) \rightarrow H^1(Z_t, \mathbb{Z})$ by

\[
(1.19) \quad N(\beta_w^\ast) = 0, \quad N(\alpha_w^\ast) = \sum_{w' \in \{w\} \cup (A_w \cap W)} (-1)^{1+\delta_{w, w'}} \cdot l(w, w') \beta_{w'}^\ast,
\]

where $l(w, w') \in \mathbb{R}_{>0}$ denotes the affine length of the 1-cell in $\mathcal{P}$ that is dual to $\text{conv}(\{w, w'\}) \in \mathcal{I}$ when $w \neq w'$, and the affine length of the boundary of the 2-cell in $\mathcal{P}$ that is dual to $w \in \mathcal{P}$ when $w = w'$. The data $\{l(w, w')\}_{w, w' \in W}$ is exactly the tropical periods of the tropical curve $X(\text{trop}(f))$ introduced in [MZ08]. We also set

\[
(1.20) \quad P(v, w) := \begin{cases} -\sum_{m \in A_w} \left( f_{Y_w} D_m^w \cdot \sigma_w \right) \log \left( -\frac{c_m}{c_w} \right) & v = w \\ \sum_{m \in A_w} \left( f_{Y_w} D_m^w \cdot D_m^w \right) \log \left( -\frac{c_m}{c_w} \right) & v \in A_w \\ 0 & \text{otherwise} \end{cases}
\]

for $v, w \in W$.

The one parameter family $\{Z_t\}_{t \in D_\rho}$ of complex hypersurfaces (curves) defines the variation of polarized Hodge structure over the punctured disk $D_\rho^*$, which extends to the LVPH on the whole disk $D_\rho$.

Corollary 1.2. The inverse image of the above LVPH by the inclusion $\{0\} \hookrightarrow D_\rho$ is isomorphic to the polarized logarithmic Hodge structure $(H_\mathbb{Z}, Q, \mathcal{P})$ at the standard log point $\{0\}$ given as follows:

- $H_\mathbb{Z}$ is the locally constant sheaf on $\{0\}^{\log} \cong S^1$ whose stalk is isomorphic to $H^1(Z_t, \mathbb{Z}) \cong \bigoplus_{w \in W} \mathbb{Z} \alpha_w^\ast \oplus \mathbb{Z} \beta_w^\ast$

and the monodromy is given by $\exp(N) = \text{id} + N$,

- $Q : H_\mathbb{Z} \times H_\mathbb{Z} \rightarrow \mathbb{Z}$ is the pairing given by the cup product $\langle \cdot, \cdot \rangle$ of $H^1(Z_t, \mathbb{Z})$, and

- $\mathcal{P} = \{\mathcal{P}^p\}_{p=0}^2$ is the decreasing filtration of $\mathcal{O}^{\log}_{\{0\}} \otimes_{\mathbb{Z}} H_\mathbb{Z} \cong \mathcal{O}^{\log}_{\{0\}} \otimes_{\mathbb{Z}} H^1(Z_t, \mathbb{Z})$ defined by

\[
(1.22) \quad \mathcal{P}^p := \mathcal{O}^{\log}_{\{0\}} \otimes_{\mathbb{Z}} F^p
\]

with

\[
(1.23) \quad F^p := \begin{cases} H^1(Z_t, \mathbb{Z}) & p = 0 \\ \bigoplus_{v \in W} \mathbb{C} \cdot (-2\pi \sqrt{-1} \alpha_v^\ast + \sum_{w \in W} P(v, w) \beta_w^\ast) & p = 1 \\ \{0\} & p = 2. \end{cases}
\]

Here $\{0\}^{\log}, \mathcal{O}^{\log}_{\{0\}}$ is the Kato–Nakayama space associated with the standard log point $\{0\}$ [KN99]. We refer the reader to [Ku09, Section 2] or [Yam22, Section 5.1] for the definition of polarized logarithmic Hodge structure.
1.3. Related work. The Hodge structure of toric hypersurfaces have been studied in many papers (e.g., [Gri99, BC94, Mav00, Mav03]). In the case where $\Delta$ is reflexive (the case of Calabi–Yau hypersurfaces), the image of the Poincaré residue map (1.5) has the residual $B$-model Hodge structure introduced by Iritani [Iri11]. It is known to be isomorphic to the ambient $A$-model Hodge structure of the mirror Calabi–Yau hypersurface via the mirror map (cf. [Iri11] Theorem 6.9). The asymptotics of the periods in this case can be written down by using Givental’s $I$-function, and one can see that the polarized logarithmic Hodge structure at the limit can be described in terms of the tropical Calabi–Yau hypersurface obtained by tropicalization [Yam22].

There is a homology theory for tropical varieties, which was introduced in [MZ14, IKMZ19]. A cycle representing a class of a tropical homology group is called a tropical cycle, and it has a bigrade $(p, q)$ $(p, q \in \mathbb{Z}_{\geq 0})$. It is expected that classical cycles and tropical $(p, q)$-cycles correspond in such a way that a classical cycle admits a torus fibration structure over a tropical cycle. The integers $p, q$ are the dimension of its fiber and the dimension of the tropical cycle respectively. The cycles $C_{t}^{\nu}$ and $T_{t}^{\sigma}$ which we consider in this article correspond to a tropical $(0, d)$-cycle and $(d, 0)$-cycle on the tropical hypersurface $X(trop(f))$ respectively.

Concerning period integrals over cycles corresponding to tropical cycles of other degree, there is work by Ruddat and Siebert [RS20, Rud21]. They computed integrals of holomorphic volume forms over cycles corresponding to tropical cycles of dimension 1 for toric degenerations constructed from wall structures. Their technique was also used by Wang [Wan20] to confirm the gamma conjecture for the canonical bundle of a smooth complete toric surface and its mirror toric degeneration constructed in [GS14]. (Notice that tropical cycles in these work are on integral affine manifolds with singularities rather than on tropical varieties. Relations between tropical cycles of these two sorts of tropical spaces are studied in [Yam21].)

It is also known that tropical (co)homology groups correspond to the grade pieces of the limiting mixed Hodge structure of the corresponding degenerating family, and the residues of the logarithmic extensions of the Gauss–Manin connections correspond to the cup products of the invariants of tropical spaces called radiance obstructions or eigenwaves [GS10, MZ14, IKMZ19]. These invariants closely relate to the affine volumes of (bounded cells in) tropical spaces.

1.4. Organization of this article. In Section 2 we review a description of meromorphic forms on toric varieties, which have poles along hypersurfaces. In Section 3 we construct the sphere cycle $C_{t}^{\nu}$. We will prove (1.15) and (1.10) of Theorem 1.1 in Section 4 and Section 5 respectively. In Section 6 we will see that the affine volumes of bounded cells in tropical hypersurfaces appear in the leading terms of period integrals. In Section 7 we give a concrete example illustrating (1.15). Lastly, in Section 8 we discuss polarized logarithmic Hodge structure at the limit in the case of $d = 1$. Corollary 1.2 is proved in this section.

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2. Forms having poles along hypersurfaces

We recall a description of meromorphic forms on toric varieties, which have poles along hypersurfaces. In particular, we will see that forms of (1.1) generate $H^{0}(Y_{\Sigma}, \Omega^{d+1}(l \cdot Z_{l}))$. We basically follow [BC94].

Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope of dimension $d + 1$, and $\Sigma$ be a rational simplicial fan in $M_{\mathbb{R}}$, which is a refinement of the normal fan of $\Delta$. We consider the toric variety $Y_{\Sigma}$ over $\mathbb{C}$ associated with the fan $\Sigma$. Recall that the class group $\text{Cl}(Y_{\Sigma})$ of the toric variety $Y_{\Sigma}$ is isomorphic to the cokernel of the map

$$M \rightarrow \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_{\rho}, \quad m \mapsto \sum_{\rho \in \Sigma(1)} \langle m, n_{\rho} \rangle D_{\rho},$$

where $\Sigma(1)$ is the set of 1-dimensional cones in the fan $\Sigma$, $D_{\rho}$ is the toric divisor on $Y_{\Sigma}$ corresponding to $\rho \in \Sigma(1)$, and $n_{\rho} \in \mathbb{N}$ is the primitive generator of $\rho \in \Sigma(1)$. The polynomial ring $S := \mathbb{C}[y_{\rho} : \rho \in \Sigma(1)]$ together with the natural grading by $\text{Cl}(Y_{\Sigma})$, which is defined by

$$\deg \left( \prod_{\rho \in \Sigma(1)} y_{\rho}^{a_{\rho}} \right) := \sum_{\rho \in \Sigma(1)} a_{\rho}D_{\rho} \in \text{Cl}(Y_{\Sigma})$$

is called the homogeneous coordinate ring of the toric variety $Y_{\Sigma}$ [Cox95, Section 1]. For a class $\mu \in \text{Cl}(Y_{\Sigma})$, let $S_{\mu} \subset S$ denote the corresponding graded piece of $S$. A homogeneous polynomial $F \in S_{\mu}$ defines a
hypersurface $Z_F$ in the toric variety $Y_\Sigma$ (cf. [BC94, Section 3]). We write 1-dimensional cones in $\Sigma$ as $\Sigma(1) = \{ \rho_0, \ldots, \rho_r \}$. For a subset $I = \{ i_0, \ldots, i_d \} \subset \{ 0, \ldots, r \}$ consisting of $d + 1$ elements, we set
\begin{equation}
\det (e_I) := \det \left( \langle e_j, n_{i_k} \rangle_{0 \leq j, k \leq d} \right),
\end{equation}
where $e_0, \ldots, e_d$ are a basis of the lattice $M$. We define the $(d + 1)$-form $\Omega_0$ by
\begin{equation}
\Omega_0 := \sum_{|I| = d + 1} \det (e_I) \hat{y}_I dy_I,
\end{equation}
where $\hat{y}_I := \prod_{i \in I} y_{i \rho}$ and $dy_I := \wedge_{i \in I} dy_{i \rho}$ ([BC94, Definition 9.3]).

**Theorem 2.1.** ([BC94, Theorem 9.7]) One has
\begin{equation}
H^0 \left( Y_\Sigma, \Omega^{d+1}(Z_F) \right) = \left\{ \frac{A \Omega_0}{F} \mid A \in S_{\mu - \mu_0} \right\},
\end{equation}
where $\mu_0 := \sum_{\rho \in \Sigma(1)} \deg (y_\rho) \in \text{Cl}(Y_\Sigma)$.

Since the fan $\Sigma$ is a refinement of the normal fan of the lattice polytope $\Delta$, one can write
\begin{equation}
\Delta = \{ m \in M_\mathbb{R} \mid \langle m, n_\rho \rangle + k_\rho \geq 0, \forall \rho \in \Sigma(1) \},
\end{equation}
where $k_\rho$ is the integer defined by
\begin{equation}
k_\rho := - \inf_{m \in \Delta} \langle m, n_\rho \rangle.
\end{equation}
Hence, one also has
\begin{equation}
l \cdot \Delta = \{ m \in M_\mathbb{R} \mid \langle m, n_\rho \rangle + lk_\rho \geq 0, \forall \rho \in \Sigma(1) \}.
\end{equation}

Let $f \in \mathbb{C}[M]$ be a Laurent polynomial over $\mathbb{C}$ whose Newton polytope is $\Delta$. One can write the polynomial $f^l$ ($l \in \mathbb{Z}_{>0}$) in the homogeneous coordinates by replacing every monomial $z^m$ ($m \in M \cap l \cdot \Delta$) in $f^l$ with $\prod_{\rho \in \Sigma(1)} y_{i \rho}^{\langle m, n_\rho \rangle}$ and multiplying $\prod_{\rho \in \Sigma(1)} y_{i \rho}^{k_\rho}$ so that we get an element in $S$. We will write it as $F_l \in S$ in the following. Similarly, the form $\wedge_{i=0}^d dz_i/z_i = \wedge_{i=0}^d d \log z_i$ with $z_i := x^{e_i}$ can be written in the homogeneous coordinates as
\begin{equation}
\wedge_{i=0}^d \left( \sum_{\rho \in \Sigma(1)} \langle e_i, n_\rho \rangle \cdot d \log y_\rho \right) = \sum_{|I| = d + 1} \det (e_I) \prod_{i \in I} \frac{dy_I}{\prod_{\rho \in \Sigma(1)} y_\rho} = \Omega_0.
\end{equation}
We apply Theorem 2.1 to the homogeneous polynomial $F_l$. Since $\deg (F_l) = \deg \left( \prod_{\rho \in \Sigma(1)} y_{i \rho}^{k_\rho} \right)$, the cohomology group $H^0 \left( Y_\Sigma, \Omega^{d+1}(Z_{F_l}) \right)$ for $F_l$ is generated by elements
\begin{equation}
\left( \prod_{\rho \in \Sigma(1)} y_{i \rho}^{l-1} \right) \left( \prod_{\rho \in \Sigma(1)} y_{i \rho}^{\langle m, n_\rho \rangle} \right) \frac{\Omega_0}{F_l}
\end{equation}
with $m \in M$ such that $\left( \prod_{\rho \in \Sigma(1)} y_{i \rho}^{l-1} \right) \left( \prod_{\rho \in \Sigma(1)} y_{i \rho}^{\langle m, n_\rho \rangle} \right) \in S$. This condition for $m \in M$ holds if and only if $\langle m, n_\rho \rangle + lk_\rho - 1 \geq 0$ for all $\rho \in \Sigma(1)$. One can see from (2.8) that this is equivalent to $m \in M \cap \text{Int} (l \cdot \Delta)$. Furthermore, we can see by (2.9) that (2.10) is written in the affine coordinates as
\begin{equation}
z^m \frac{d}{f^l} \wedge_{i=0}^d \frac{dz_i}{z_i}.
\end{equation}
We can conclude that (2.11) with $m \in M \cap \text{Int} (l \cdot \Delta)$ define elements in $H^0 \left( Y_\Sigma, \Omega^{d+1}(Z_{F_l}) \right)$ and generate $H^0 \left( Y_\Sigma, \Omega^{d+1}(Z_{F_l}) \right)$. Also in the setup of Section 1 we can see that (1.4) defines an element in $H^0 \left( Y_\Sigma, \Omega^{d+1}(l \cdot Z_l) \right)$ and such forms generate $H^0 \left( Y_\Sigma, \Omega^{d+1}(l \cdot Z_l) \right)$ since we have
\begin{equation}
\prod_{m \in A \cap \tau_v} (k_{m,t} \cdot z^m)^{p_m} = z^v \prod_{m \in A \cap \tau_v} (k_{m,t})^{p_m}
\end{equation}
and $v \in V_l := M \cap \text{Int} (l \cdot \Delta)$.

**Remark 2.2.** Forms of the form (2.11) were originally considered in [Bat93] in the case of hypersurfaces in algebraic tori $N_{\mathbb{C}^+}$. It is known that such forms with $m \in M \cap \text{Int} (l \cdot \Delta)$ generate the lowest weight component of the middle cohomology group of the complement of the hypersurface ([Bat93, Theorem 8.2]).
3. Construction of sphere cycles

In this section, we construct the sphere cycle $C^w_t \ (w \in W)$ in Section 1.1 by using the technique in [AGIS20, Section 5.2]. We work under the same assumptions and use the same notations as in Section 1.1.

Recall that the period of the Poincaré residue $\Omega^w_t$ of $\omega^w_t$ is given by

$$\int_C \Omega^w_t = \frac{1}{2\pi \sqrt{-1}} \int_{T(C)} \omega^w_t$$

for any $d$-cycle $C \subset Z_t$, where $T(C)$ denotes a tube over the cycle $C$ (the boundary of a small tubular neighborhood of $C$, which sits in $Y \setminus Z_t$). In order to compute period integrals over $C^w_t$, we will also construct a tube over $C^w_t$ in addition to $C^w_t$ itself.

For $m \in A$, we set

$$\mu_m : N_C \to \mathbb{R}, \ n \mapsto \lambda_m + \langle m, n \rangle,$$

where $\lambda_m := \text{val}(k_m)$. Recall that the polyhedral decomposition $\mathcal{P}$ of $N_\mathbb{R}$ induced by the tropical hypersurface $X$ (trop$(f)$) $\subset N_\mathbb{R}$ is dual to the triangulation $\mathcal{T}$ of $\Delta$ induced by [12] (cf. e.g. [MS15, Proposition 3.1.6]). The correspondence is given by

$$\mathcal{P} \to \mathcal{T}, \ \sigma \mapsto \text{conv} (\{m \in A \mid \text{trop}(f)(n) = \mu_m(n), \forall n \in \sigma\}).$$

We fix $w \in W$ for which we construct the cycle $C^w_t$, and define

$$\nabla^w := \{n \in N_\mathbb{R} \mid \mu_m(n) \geq \mu_w(n), \forall m \in A\}.$$

This is the element in $\mathcal{P}$ dual to $\{w\} \in \mathcal{I}$. The normal fan of $\nabla^w$ is $\Sigma_w$ defined just after [1.1]. Due to the assumption that $\mathcal{T}$ is unimodular, the normal fan $\Sigma_w$ is also unimodular. For a real constant $\kappa > 0$, we set

$$N^w_{\kappa} := \left\{ n \in N_\mathbb{R} \mid \mu_w(n) - \kappa \leq \min_{m \in A \setminus \{w\}} \mu_m(n) \leq \mu_w(n) + \kappa \right\}.$$

This is a neighborhood of $\partial \nabla^w$. Notice that this set $N^w_{\kappa}$ is different from $N_{\kappa}(\Delta_\lambda)$ considered in [AGIS20, Section 5.2]. For an element $n \in N_\mathbb{R}$, we also set

$$K^n_{\kappa} := \left\{ k \in A \setminus \{w\} \mid k_m(n) \leq \mu_w(n) + \kappa \right\},$$

$$L^n_{\kappa} := \left\{ k \in A \setminus \{w\} \mid k_m(n) \leq \min_{m \in A \setminus \{w\}} \mu_m(n) + \kappa \right\}.$$

Lemma 3.1. If the constant $\kappa > 0$ is sufficiently small, the following hold:

1. For any $n \in N^w_{\kappa}$, the convex hull of $\{w\} \cup K^n_{\kappa}$ is in the triangulation $\mathcal{T}$ of $\Delta$.
2. For any $n \in N^w_{2\kappa}$, the convex hull of $\{w\} \cup L^n_{\kappa}$ is also in $\mathcal{T}$.
3. For any coefficient $k_m = \sum_{j \in \mathbb{Z}} c_j e^{j/n} \in K (\in A)$ of the polynomial $f$, one has $2\kappa < 1/n$.

Proof. The last condition (3) obviously holds since $A$ is a finite set. We discuss (1) and (2). Take a small constant $k_0 > 0$ such that for any $\{c_m \in [0, k_0]\}_{m \in A}$; even if we replace each $\lambda_m$ with $\lambda_m - c_m$, the induced triangulation $\mathcal{T}$ of $\Delta$ does not change. For an element $n \in N_\mathbb{R}$, we set

$$J^n_{\kappa} := \left\{ k \in A \mid k_m(n) \leq \text{trop}(f)(n) + \kappa \right\}.$$

When $\kappa \leq k_0$, we have $\text{conv}(J^n_{\kappa}) \in \mathcal{T}$ for any $n \in N_\mathbb{R}$.

Suppose $\kappa < k_0/3$. For any $n \in N^w_{\kappa}$ and $k \in K^n_{\kappa}$, we have

$$\text{trop}(f)(n) = \min \left\{ \mu_w(n), \min_{m \in A \setminus \{w\}} \mu_m(n) \right\} \geq \mu_w(n) - \kappa \geq \mu_k(n) - 2\kappa.$$

From this, we can see that $\{w\} \cup K^n_{\kappa}$ is a subset of $J^n_{2\kappa}$. Since $2\kappa < k_0$ and $\mathcal{T}$ is a triangulation, we have

$$\text{conv}(\{w\} \cup K^n_{\kappa}) \subset \text{conv}(J^n_{2\kappa}) \in \mathcal{T}.$$

Similarly, for any $n \in N^w_{2\kappa}$ and $k \in L^n_{\kappa}$, we have

$$\text{trop}(f)(n) = \min \left\{ \mu_w(n), \min_{m \in A \setminus \{w\}} \mu_m(n) \right\} \geq \min_{m \in A \setminus \{w\}} \mu_m(n) - 2\kappa \geq \mu_k(n) - 3\kappa.$$

We also have $\text{trop}(f)(n) \geq \mu_w(n) - 2\kappa$. From these, we can see that $\{w\} \cup L^n_{\kappa}$ is a subset of $J^n_{3\kappa}$. Again, since $3\kappa < k_0$ and $\mathcal{T}$ is a triangulation, we have

$$\text{conv}(\{w\} \cup L^n_{\kappa}) \subset \text{conv}(J^n_{3\kappa}) \in \mathcal{T}.$$

We obtained the claim. \qed
Remark 3.2. In [AGIS20 Section 5.2], they also take a constant \( k > 0 \) that plays the same role. Concerning the above (2), they take a constant \( k > 0 \) so that \( L^n_w \) is linearly independent for any \( n \in N_R \). To be precise, this is not possible in general also in the setup of [AGIS20]. For instance, consider the case where \( d = 1, w = 0 \), and

\[
\Delta := \text{conv} \{ \pm e_1, \pm e_2 \} \subset M_R \cong \mathbb{R}^2, \quad \lambda(m) = \begin{cases} 0 & m = 0 \\ 1 & m = \pm e_1 \\ 2 & m = \pm e_2, \end{cases}
\]

where \( \{e_1, e_2\} \) is a basis of \( M \cong \mathbb{Z}^2 \). The set \( L^n_w \) with \( n \in \{ re^*_r \in N_R \cong \mathbb{R}^2 \mid -1 \leq r \leq 1 \} \) contains \( \pm e_1 \) for any \( k \geq 0 \), and is not linearly independent. This affects [AGIS20 Lemma 5.6] which corresponds to Lemma 3.6 of this article. In this article, we impose the condition (2) only for elements in \( N^w_{2\kappa} \) so that we are able to take such a constant \( k > 0 \). This is actually sufficient for our purpose.

We fix \( k > 0 \) satisfying the conditions of Lemma 3.1. For an element \( n \in N^w_{\kappa} \), we have \( K^n_\kappa \neq \emptyset \) since an element \( k \in A \setminus \{ w \} \) such that \( \mu_k(n) = \min_{m \in A \setminus \{ w \}} \mu_m(n) \) is contained in \( K^n_\kappa \). For every cone \( C \in \Sigma_w \setminus \{0\} \), we consider the subset

\[
N^w_w(C) := \{ n \in N^w_\kappa \mid \mathbb{R} \geq 0 \cdot (\text{conv} \{ w \} \cup K^n_\kappa) - w = C \}.
\]

By (1) of Lemma 3.1, the subsets \( \{ N^w_w(C) \mid C \in \Sigma_w \setminus \{0\} \} \) cover the whole \( N^w_\kappa \). Notice also that \( N^w_w(C) \) is contained in a small neighborhood of the cell in \( \mathcal{P} \) dual to \( \text{conv} \{ w \} \cup K^n_\kappa \in \mathcal{T} \) with \( n \in N^w_\kappa(C) \).

Let \( \varepsilon > 0 \) be another small real number, and set \( D_\varepsilon := \{ x \in \mathbb{C} \mid |x| < \varepsilon \} \). (Notice that we distinguish this \( \varepsilon \) and \( \varepsilon \) appearing in Theorem 1.1.) We choose a smooth function \( \phi \colon N^w_\kappa \times D_\varepsilon \to N_R \) such that

\[
\langle m - w, \phi(n, x) \rangle = \text{Arg} (1 + x) - \arg \left( \frac{-c_m}{c_w} \right)
\]

for any \( n \in N^w_\kappa, x \in D_\varepsilon \), and \( m \in K^n_\kappa \). Here \( \text{Arg} \) denotes the principal value of the argument, i.e., the value of the argument in \( (-\pi, \pi] \). The branch of \( \text{Arg} (\frac{-c_m}{c_w}) \) is the one we choose in order to consider the parallel transport of the real locus \( \mathbb{R}^w \cap N^w_\kappa \) in (1.9). Such a function \( \phi \) is called a phase-shifting function in [AGIS20 Section 5.2]. We refer the reader to Example 5.4 in loc.cit. for an example of a function \( \phi \) (with \( w = 0, x = 0, c_w = -1 \)).

The function \( \phi \) can be constructed as follows: We take a Euclidean metric on \( N_R \cong \mathbb{R}^{d+1} \), and identify \( N_R \cong M_R \). For \( x \in D_\varepsilon \), we consider the piecewise linear function on the normal fan \( \Sigma_w \) of \( \nabla_w \), which takes the value \( \text{Arg} (1 + x) - \arg (\frac{-c_m}{c_w}) \) at the primitive generator \( (m - w) \) of the 1-dimensional cone \( \mathbb{R} \geq 0 \cdot (m - w) \in \Sigma_w \). By varying \( x \in D_\varepsilon \), we obtain a function on \( M_R \times D_\varepsilon \cong N_R \times D_\varepsilon \). We further take the gradient of a smoothing of it, and compose it with the projection \( T(N_R \times D_\varepsilon) \to T(N_R) = N_R \), where \( T \) denotes the tangent bundle. By taking its translation such that the origin \( 0 \in N_R \) moves to a point in \( \text{Int} (\nabla_w) \) and the subset \( N^w_w(C) \) of (3.14) for any \( C \in \Sigma_w \setminus \{0\} \) is contained in the union of cones in \( \Sigma_w \) containing \( C \), we obtain the function \( \phi \).

We set

\[
N^w_{\kappa, \mathcal{C}} := \{ n \in N_\mathcal{C} \mid \Re(n) \in N^w_\kappa \}.
\]

We also consider the maps

\[
\Phi_t \colon N^w_\kappa \times D_\varepsilon \to N^w_{\kappa, \mathcal{C}}, \quad (n, x) \mapsto n + \sqrt{-1} \cdot \frac{\phi(n, x)}{\log t},
\]

and

\[
i_t \colon N_\mathcal{C} \to N_{\mathcal{C}^*},
\]

induced by \( \mathbb{C} \to \mathbb{C}^*, x \mapsto t^x \), where \( t > 0 \). We set

\[
R_t := \left\{ n \in N_R \mid \frac{1}{2} \leq \tilde{f}^w_t(i_t(n)) \leq \frac{3}{2} \right\}.
\]

Lemma 3.3. One has \( R_t \subset N^w_\kappa \) for sufficiently small \( t > 0 \).

Proof. Let \( n \in N_R \setminus N^w_\kappa \) be an element. We will show \( n \not\in R_t \). We have either \( \mu_w(n) - \kappa < \min_{m \in A \setminus \{ w \}} \mu_m(n) \) or \( \min_{m \in A \setminus \{ w \}} \mu_m(n) > \mu_w(n) + \kappa \). In the former case, we have

\[
\tilde{f}^w_t(i_t(n)) \geq \frac{r_{m_0}}{r_w} \cdot t^{(\mu_{m_0} - \mu_w)(n)} \geq \frac{r_{m_0}}{r_w} \cdot t^{-\kappa} \geq C \cdot t^{-\kappa},
\]
where $m_0 \in A \setminus \{w\}$ is an element such that $\mu_{m_0}(n) = \min_{m \in A \setminus \{w\}} \mu_m(n)$, and $C := \min_{m \in A \setminus \{w\}} r_m/r_w > 0$. When $t > 0$ is sufficiently small, we have $C \cdot t^{-\kappa} > 3/2$, which implies $n \notin R_t$. Also in the latter case, we have

$$
(3.21) \quad \tilde{f}_{t}^w(i_t(n)) = \sum_{m \in A \setminus \{w\}} \frac{r_m}{r_w} t^{(\mu_m - \mu_w)(n)} \leq C' \cdot t^\kappa,
$$

where $C' := |A \setminus \{w\}| \cdot \max_{m \in A \setminus \{w\}} r_m/r_w > 0$. When $t > 0$ is sufficiently small, we have $C \cdot t^\kappa < 1/2$, which implies $n \notin R_t$. We obtained the claim. \hfill \Box

**Proposition 3.4.** (cf. [AGIS20 Proposition 5.3]) For sufficiently small $t > 0$, there exists a smooth map $\delta_t: R_t \times D_\varepsilon \to N_C$ satisfying the following conditions:

1. For all $(n, x) \in R_t \times D_\varepsilon$, one has

$$
(3.22) \quad \frac{1}{1 + x} \cdot \tilde{f}_{t}^w(i_t(\Phi_t(n, x) + \delta_t(n, x))) = \frac{1}{|1 + x|} \cdot \tilde{f}_{t}^w(i_t(n)).
$$

2. $||\delta_t||_{C^1} = O(t^\kappa)$, where $|| \bullet ||_{C^1}$ denotes the $C^1$-norm over $R_t \times D_\varepsilon$.

We define

$$
(3.23) \quad \tilde{\Phi}_t: R_t \times D_\varepsilon \to N_C \times D_\varepsilon, \quad (n, x) \mapsto (\Phi_t(n, x) + \delta_t(n, x), x)
$$

and set

$$
(3.24) \quad B_{t}^w := \left\{ n \in R_t \mid \tilde{f}_{t}^w(i_t(n)) = 1 \right\},
$$

$$
(3.25) \quad C_{t}^w := i_t \circ \pi_1 \left( \tilde{\Phi}_t(B_{t}^w \times \{0\}) \right),
$$

where $\pi_1: N_C \times D_\varepsilon \to N_C$ is the first projection. By (3.22) with $x = 0$, we have $C_{t}^w \subset Z_t$. In the limit $t \to +0$, the set $B_{t}^w$ converges to the boundary of the polytope $\nabla^w \in \mathcal{P}$. In particular, we can see that the sets $B_{t}^w$ and $C_{t}^w$ are $d$-spheres when $t > 0$ is sufficiently small. The set $C_{t}^w$ is the cycle over which we integrate forms. We further take a real number $\varepsilon_0$ such that $0 < \varepsilon_0 < \varepsilon$. We set $S_{t, \varepsilon_0} := \{ x \in \mathbb{C} \mid |x| = \varepsilon_0 \} \subset D_\varepsilon$ and

$$
(3.26) \quad S_{t}^w := \left\{ (n, x) \in R_t \times S_{t, \varepsilon_0} \mid \tilde{f}_{t}^w(i_t(n)) = |1 + x| \right\},
$$

$$
(3.27) \quad T_{t}^w := i_t \circ \pi_1 \left( \tilde{\Phi}_t(S_{t}^w) \right).
$$

From (3.22) again, we can see that $T_{t}^w \subset N_C \setminus Z_t$ is a tube over the cycle $C_{t}^w$. The rest of this section is devoted to prove Proposition 3.4.

**Proof of Proposition 3.4.** We define a holomorphic function $g_{t}^w: N_C \times D_\varepsilon \to \mathbb{C}$ by

$$
(3.28) \quad g_{t}^w(n, x) := \frac{1}{1 + x} f_{t}^w(i_t(n)) = \frac{1}{1 + x} \sum_{m \in A \setminus \{w\}} \left( -\frac{k_{m,t}}{k_{w,t}} \right) t^{(m-w, n)},
$$

and the function $\xi_{t}^w: N_k^w \times D_\varepsilon \to \mathbb{C}$ by

$$
(3.29) \quad \xi_{t}^w(n, x) := \frac{1}{|1 + x|} \tilde{f}_{t}^w(i_t(n)) - g_{t}^w(\Phi_t(n, x), x).
$$

**Lemma 3.5.** (cf. [AGIS20 Lemma 5.2]) There is some constant $C > 0$ such that for any $(n, x) \in N_k^w \times D_\varepsilon$, one has

$$
(3.30) \quad |\xi_{t}^w(n, x)| \leq C \cdot t^\kappa.
$$

**Proof.** By (3.15), one has

$$
(3.31) \quad g_{t}^w(\Phi_t(n, x), x) = \frac{1}{1 + x} \sum_{m \in A \setminus \{w\}} \left( -\frac{k_{m,t}}{k_{w,t}} \right) t^{(m-w, n - \sqrt{\log t})} = O(t^\kappa) + \frac{1}{|1 + x|} \sum_{m \in K^w \setminus \{w\}} \frac{r_m}{r_w} (1 + O(t^{2\kappa})) t^{(\mu_m - \mu_w)(n)},
$$

$$
(3.32) \quad = O(t^\kappa) + \frac{1}{|1 + x|} \sum_{m \in K^w \setminus \{w\}} \frac{r_m}{r_w} (1 + O(t^{2\kappa})) t^{(\mu_m - \mu_w)(n)}.
$$





where we also used (3) of Lemma 3.1. One also has

\[
\begin{align*}
\frac{1}{1 + x} \tilde{f}_t^w (i_t(n)) &= \frac{1}{1 + x} \sum_{m \in A \setminus \{w\}} \frac{r_m}{t} (\mu_m - \mu_w (n)) \\
&= O(t^\kappa) + \frac{1}{1 + x} \sum_{m \in K_n} \frac{r_m}{t} (\mu_m - \mu_w (n)).
\end{align*}
\]

By combining these, we get

\[
\xi_t^w (n, x) = O(t^\kappa) + \frac{1}{1 + x} \sum_{m \in K_n} O(t^{2\kappa}) \frac{r_m}{t} (\mu_m - \mu_w (n)).
\]

Since we have \(\min_{m \in A \setminus \{w\}} \mu_m (n) - \mu_w (n) \geq -\kappa\) for \(n \in N_w\), this is \(O(t^\kappa)\). We obtained the claim. \(\square\)

We consider the gradient vector field of \(g_t^w (\bullet, x)\) with fixed \(x \in D_\varepsilon\) on \(\mathbb{C} \times \{x\}\)

\[
\text{grad} g_t^w (n, x) := \left( \frac{\partial g_t^w}{\partial n_0}, \ldots, \frac{\partial g_t^w}{\partial n_d} \right),
\]

where \((n_0, \ldots, n_d)\) are \(\mathbb{C}\)-coordinates on \(\mathbb{C}^{d+1}\).

**Lemma 3.6.** (cf. AGIS20 Lemma 5.6) When \(t > 0\) is sufficiently small, one has \(|\text{grad} g_t^w (n, x)| \neq 0\) on \(N_{2\kappa, \mathbb{C}} \times D_\varepsilon\). Furthermore, there exist constants \(C_1, C_2 > 0\) such that for any \((n, x) \in N_{2\kappa, \mathbb{C}} \times D_\varepsilon\) satisfying \(g_t^w (n, x) \neq 0\), we have

\[
\frac{|\text{grad} g_t^w (n, x)|}{g_t^w (n, x)} \geq (-\log t) (C_1 - C_2 t^\kappa).
\]

**Proof.** We have

\[
\text{grad} g_t^w (n, x) = (-\log t) \sum_{m \in A \setminus \{w\}} \frac{1}{1 + x} \left( \frac{k_m t}{k_{w,t}} \right) t^{m \cdot w, n} \cdot (m - w).
\]

We set \(\tilde{\mu}_w (n) := \min_{m \in A \setminus \{w\}} (\mu_m - \mu_w) (\Re(n))\). The index set \((3.7)\) for \(\Re(n) \in N_{2\kappa}\) is

\[
L_{\kappa}^\Re(n) = \{ m \in A \setminus \{w\} \mid (\mu_m - \mu_w) (\Re(n)) \leq \tilde{\mu}_w (n) + \kappa \}.
\]

By (2) of Lemma 3.1 and the assumption that the triangulation \(T\) is unimodular, the set \((L_{\kappa}^\Re(n) - w)\) is a subset of a basis of the lattice \(M\). Hence, we can see that (3.38) is greater than or equal to

\[
(-\log t) t^{\tilde{\mu}_w (n)} \left( \sum_{m \in L_{\kappa}^\Re(n)} \frac{1}{1 + x} \left( \frac{k_m t}{k_{w,t}} \right) t^{m \cdot w, n - \tilde{\mu}_w (n)} \cdot (m - w) - \rho_2 t^\kappa \right)
\]

\[
\geq (-\log t) t^{\tilde{\mu}_w (n)} \left\{ \rho_1 \left( \sum_{m \in L_{\kappa}^\Re(n)} t^{(\mu_m - \mu_w - \tilde{\mu}_w) (n)} \right) - \rho_2 t^\kappa \right\}
\]

for some constants \(\rho_1, \rho_2 > 0\), where we also used the fact that all norms on a finite dimensional vector space are equivalent. The former claim of the lemma follows from this and

\[
\sum_{m \in L_{\kappa}^\Re(n)} t^{(\mu_m - \mu_w - \tilde{\mu}_w) (n)} \geq 1
\]

which holds since there exists \(m_0 \in L_{\kappa}^\Re(n)\) such that \((\mu_{m_0} - \mu_w) (\Re(n)) = \tilde{\mu}_w (n)\), and \(t^{(\mu_{m_0} - \mu_w - \tilde{\mu}_w) (n)} = 1\). Similarly we also have

\[
|g_t^w (n, x)| \leq t^{\tilde{\mu}_w (n)} \left\{ \rho_3 \left( \sum_{m \in L_{\kappa}^\Re(n)} t^{(\mu_m - \mu_w - \tilde{\mu}_w) (n)} \right) + \rho_4 t^\kappa \right\}
\]
for some constants $\rho_3, \rho_4 > 0$. By combining these inequalities, we obtain

$$
(3.43) \quad \frac{\| \nabla g_t^w(n, x) \|}{|g_t^w(n, x)|} \geq (- \log t) \left( \frac{\rho_1}{\rho_3} \left( \sum_{m \in L^R_{\nu(n)}} |t(\mu_m - \mu_w - \bar{\mu}_w(n))| \right) - \rho_2 t^\kappa \right)
$$

$$
\geq (- \log t) \left( \frac{\rho_1}{\rho_3} \left( \frac{\rho_1 \rho_4}{(\rho_3)^2} + \frac{\rho_2}{\rho_3} \right) \sum_{m \in L^R_{\nu(n)}} |t(\mu_m - \mu_w - \bar{\mu}_w(n))| \right)

(3.44)
$$

We can get (3.37) by setting $C_1 := \rho_1 / \rho_3, C_2 := \rho_1 \rho_4 / (\rho_3)^2 + \rho_2 / \rho_3$ and using (3.31) again. $\square$

We consider the differential equation for an unknown function $c: R_t \times D_\varepsilon \times [0, 2] \rightarrow N_\kappa^\nu$. We will show that there exists a solution $c(n, x, s) \in N_\kappa^\nu$ for (3.45).

We fix $(n, x) \in R_t \times D_\varepsilon$. Let $s(n, x) \in [0, 2]$ be the supremum of $s' \in [0, 2]$ such that there exists a solution $c(n, x, s)$ of (3.45) on the interval $[0, s')$ and $c(n, x, s) \in N_\kappa^\nu$ for all $s \in [0, s')$. For any $s_0 \in [0, s(n, x))$, we have

$$
\int_0^{s_0} \nabla g_t^w(c(n, x, s), x) \cdot \frac{d}{ds} c(n, x, s) \, ds = \int_0^{s_0} \xi_t^w(n, x) \, ds = s_0 \cdot \xi_t^w(n, x).
$$

This is also equal to $g_t^w(c(n, x, s_0), x) - g_t^w(c(n, x, 0), x)$. Hence, we have

$$
(3.47) \quad g_t^w(c(n, x, s_0), x) = g_t^w(\Phi_t(n, x), x) + s_0 \cdot \xi_t^w(n, x).
$$

By this and (3.29), one can get

$$
(3.48) \quad |g_t^w(c(n, x, s_0), x)| \geq |g_t^w(\Phi_t(n, x), x)| - s_0 \cdot |\xi_t^w(n, x)|
$$

$$
(3.49) \quad \geq \frac{1}{1 + x} \cdot \tilde{g}_t^w(\tilde{i}(n)) - (1 + s_0) |\xi_t^w(n, x)|
$$

$$
(3.50) \quad \geq \frac{1}{2(1 + x)} - (1 + s_0) C \cdot t^\kappa,
$$

where we used Lemma 3.3 and $1/2 \leq \tilde{g}_t^w(\tilde{i}(n))$ for $n \in R_t$ in the last inequality. This is greater than, for instance, 1/3 when $\varepsilon, t > 0$ are sufficiently small. Thus by Lemma 3.6, we have

$$
(3.51) \quad \frac{\| \nabla g_t^w(c(n, x, s_0), x) \|}{|g_t^w(c(n, x, s_0), x)|} \geq (- \log t) \rho_0
$$

with some constant $\rho_0 > 0$ for sufficiently small $t > 0$. Therefore, (3.45) implies

$$
(3.52) \quad \frac{d}{ds} c(n, x, s) = \frac{\xi_t^w(n, x)}{\nabla g_t^w(c(n, x, s), x)} \leq \frac{|\xi_t^w(n, x)|}{(- \log t) \rho_0} \leq C \rho_0^{-1}(- \log t)^{-1} t^\kappa
$$

for $s \in [0, s(n, x))$. This implies the limit $\lim_{s \rightarrow s(n, x) - 0} c(n, x, s)$ exists. Suppose $s(n, x) < 2$. Then the solution for (3.45) can be extended to a larger interval $[0, s(n, x) + \varepsilon_1)$ with some small $\varepsilon_1 > 0$. For any $s_0 \in [0, s(n, x) + \varepsilon_1)$, one has

$$
(3.53) \quad |c(n, x, s_0) - c(n, x, 0)| \leq \int_0^{s_0} \frac{d}{ds} c(n, x, s) \, ds \leq 2C \rho_0^{-1}(- \log t)^{-1} t^\kappa
$$

Since $c(n, x, 0) = \Phi_t(n, x) \in N_\kappa^\nu$, one can get

$$
(3.54) \quad \left| \left( \min_{m \in A_1(w)} \mu_m - \mu_w \right) (\Re(c(n, x, s_0))) \right| = \left| \left( \min_{m \in A_1(w)} \mu_m - \mu_w \right) (\Re(c(n, x, s_0) - c(n, x, 0)) + \Re(c(n, x, 0))) \right| \leq C'(- \log t)^{-1} t^\kappa + \kappa
$$

$$
(3.55) \quad \leq 2\kappa
$$

when $t > 0$ is sufficiently small. Here $C' > 0$ is some constant. Hence, one has $c(n, x, s_0) \in N_{\kappa, 2\kappa}$ for any $s_0 \in [0, s(n, x) + \varepsilon_1)$. This contradicts the original assumption on $s(n, x)$. We conclude that $s(n, x) = 2$ and the solution $c(n, x, s) \in N_{\kappa, 2\kappa}$ exists on the interval $[0, 2]$.

We define the map $\delta_t: R_t \times D_\varepsilon \rightarrow N_\kappa$ by

$$
(3.57) \quad \delta_t(n, x) := c(n, x, 1) - \Phi_t(n, x).
$$
Then we get
\begin{equation}
\frac{1}{1 + x} f^w_t (i_t (\Phi_{1} (n, x) + \delta_{t} (n, x))) = \frac{1}{1 + x} f^w_t (i_t (c(n, x, 1))) = g^w_{i_t} (c(n, x, 1), x) = \frac{1}{|1 + x|} \tilde{f}^w_{i_t} (i_t (n)) ,
\end{equation}
where we use (3.47) with \( s_0 = 1 \) for the last equality. This is (3.22).

Next we show that the \( C^1 \)-norm of \( \delta_{t} \) is of order \( t^\kappa \). From (3.53) which holds also for \( s_0 = 1 \), we obtain
\begin{equation}
|c(n, x, 1) - c(n, x, 0)| \leq C \rho_0^{-1} ( - \log t )^{-1} t^\kappa .
\end{equation}
This gives a \( C^0 \)-bound for \( \delta_{t} (n, x) = c(n, x, 1) - c(n, x, 0) \).

In order to obtain a \( C^1 \)-bound for \( \delta_{t} \), we consider the differential equations
\begin{align}
\frac{d}{ds} c(n, x, s) &= \frac{\partial \xi^w_{t} (n, x)}{\partial n_i} \cdot \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)} + \xi^w_{t} (n, x) \cdot \left( F(c, x) \frac{\partial c(n, x, s)}{\partial n_i} + \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)} \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)} \right) \\
\frac{d}{ds} c(n, x, s) &= \frac{\partial \xi^w_{t} (n, x)}{\partial x} \cdot \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)} + \xi^w_{t} (n, x) \cdot \left( F(c, x) \frac{\partial c(n, x, s)}{\partial x} + \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)} \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)} \right)
\end{align}
and (3.61) whose derivative \( \partial / \partial x \) is replaced with \( \partial / \partial \xi \), where \( c = c(n, x, s) \), and \( F(c, x), G(c, x) \) are the square matrices of size \( d + 1 \) whose \((j, k)\)-entries are given by
\begin{equation}
F(c, x)_{jk} := \frac{\partial}{\partial c_k} \left( \frac{\partial g^w_{i_t}}{\partial c_j} \right), \quad G(c, x)_{jk} := \frac{\partial}{\partial \xi_k} \left( \frac{\partial g^w_{i_t}}{\partial c_j} \right).
\end{equation}
We use [AGIS20] Lemma 5.7 for \( \frac{\partial c(n, x, s)}{\partial n_i}, \frac{\partial c(n, x, s)}{\partial x}, \frac{\partial c(n, x, s)}{\partial \xi} \) and these differential equations. It turns out that it suffices to show
\begin{align}
|\frac{\partial \xi^w_{t} (n, x)}{\partial n_i} \cdot \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)}| &\leq C t^\kappa \\
|\frac{\partial \xi^w_{t} (n, x)}{\partial x} \cdot \frac{\text{grad} g^w_{i_t}}{\text{grad} g^w_{i_t}^\sharp (c, x)}| &\leq C t^\kappa \\
|\frac{\partial c(n, x, 0)}{\partial n_i} - \frac{\partial c(n, x, 0)}{\partial x} - \frac{\partial c(n, x, 0)}{\partial \xi}| &\leq C
\end{align}
for sufficiently small \( t > 0 \), where \( C > 0 \) is some constant that is independent of \( n, x, s, t \).

The last inequality follows from the initial condition \( c(n, x, 0) = \Phi_{1} (n, x) = n + \sqrt{-1} \cdot \phi(n, x) / \log t \).

Concerning the first inequality, one can see from the equations in the proof of Lemma 5.3 that we have
\begin{equation}
\xi^w_{t} (n, x) = \sum_{m \in A \setminus \{w\} \cup K^\omega} t^{(\mu_m - \mu_w)(n)} \left( \frac{1}{1 + x} \frac{r_m}{r_w} + \frac{1 + O(t^{2\kappa})}{1 + x} c_m (1 + O(t^{2\kappa})) e^{\sqrt{-1}(m-w, \phi(n, x))} \right) \\
+ \frac{1}{|1 + x|} \sum_{m \in K^\omega} O(t^{2\kappa}) t^{(\mu_m - \mu_w)(n)} .
\end{equation}
From this, we can get
\begin{equation}
|\frac{\partial \xi^w_{t} (n, x)}{\partial n_i}| \leq \rho_1 (- \log t) t^\kappa , \quad |\frac{\partial \xi^w_{t} (n, x)}{\partial x}| \leq \rho_2 t^\kappa , \quad |\frac{\partial \xi^w_{t} (n, x)}{\partial \xi}| \leq \rho_2 t^\kappa ,
\end{equation}
for some constants \( \rho_1, \rho_2 > 0 \). From these and (3.51), we get (3.58).
Lastly, we show the inequalities (3.64) and (3.65). One has

\[ F(c, x)_{jk} = \frac{1}{|\nabla g|^4} \sum_{i=0}^{d} \frac{\partial^2 g^w}{\partial c_i \partial c_j} \]

\[ G(c, x)_{jk} = \frac{1}{|\nabla g|^4} \sum_{i=0}^{d} \frac{\partial^2 g^w}{\partial c_i \partial c_j} \]

Since \( n \in R_t \), we have

\[ \frac{r_m}{r_w} (\mu_m - \mu_w)(n) \leq \tilde{f}_t(w) (i_t(n)) \leq \frac{3}{2} \]

for any \( m \in A \setminus \{w\} \). Hence, one has \( (\mu_m - \mu_w)(n) \geq \log t \rho_3 \) for any \( m \in A \setminus \{w\} \), if we set \( \rho_3 := \max_{m \in A \setminus \{w\}} 2r_m/3r_w > 0 \). We also have

\[ \Re \left( (\mu_m - \mu_w) (c(n, x, 0)) \right) = \Re \left( (\mu_m - \mu_w) (\Phi_t(n, x)) \right) \]

\[ = \Re \left( (\mu_m - \mu_w) \left( n + \sqrt{-1} \cdot \frac{\phi(n, x)}{\log t} \right) \right) \]

\[ = (\mu_m - \mu_w)(n) . \]

From these and (3.33), one can get

\[ \Re \left( (\mu_m - \mu_w) (c(n, x, s)) \right) \geq \Re \left( (\mu_m - \mu_w) (c(n, x, 0)) \right) - |m - w| \cdot C \rho_0^{-1} (\log t)^{-1} t^\kappa \]

\[ \geq \log t \rho_3 - \rho_4 (\log t)^{-1} t^\kappa \]

for any \( m \in A \setminus \{w\} \). Here \( \rho_4 > 0 \) is also some constant. Therefore we have

\[ \frac{|\partial g^w_{ij}(c, x)|}{\partial c_i} = (-\log t) \sum_{m \in A \setminus \{w\}} \frac{1}{1 + x} \left( \frac{c_w}{c_m} \right) (1 + O(t^\kappa))(m - w)_i \cdot t^{(\mu_m - \mu_w)(c)} \]

\[ \leq (-\log t) \sum_{m \in A \setminus \{w\}} \frac{1}{1 + x} \left( \frac{c_w}{c_m} \right) (1 + O(t^\kappa))(m - w)_i \cdot t^{(\mu_m - \mu_w)(c)} \]

\[ \leq (-\log t) \sum_{m \in A \setminus \{w\}} \frac{1}{1 + x} \left( \frac{c_w}{c_m} \right) (1 + O(t^\kappa))(m - w)_i \cdot \rho_3 e^{\rho_4 t^\kappa} \]

\[ \leq \rho_5 (-\log t) \]

for some constant \( \rho_5 > 0 \). Similarly one can also show

\[ \frac{|\partial^2 g^w_{ij}(c, x)|}{\partial c_i \partial c_j} \leq \rho_6 (-\log t)^2, \quad \frac{|\partial^2 g^w_{ij}(c, x)|}{\partial c_i \partial x} \leq \rho_7 (-\log t) \]

for some constants \( \rho_6, \rho_7 > 0 \). These estimates together with (3.51) and Lemma 3.5 imply (3.64) and (3.65). We proved the proposition. \[ \Box \]

4. Proof of (1.15)

We keep the same assumptions and notations as in Section 1.1 and Section 3. We fix elements \( w \in W \) and \( v \in V_l (l \geq 1) \). We will compute the asymptotics of the period \( \int_{C^w} \Omega^w_l \). Let \( \kappa > 0 \) be a small real number satisfying the conditions of Lemma 3.1. We take a small constant \( \epsilon > 0 \) so that \( \epsilon < \kappa/2 \). For each pair of an element \( q \in A \setminus \{w\} \) and a subset \( K \subset A \setminus \{w, q\} \), we set

\[ S^w_{l, q, K} := \left\{ (n, x) \in S^w_l \mid \begin{array}{l} \mu_k(n) - \mu_q(n) \in [0, \epsilon], \forall k \in \{q\} \cup K \\ \mu(n) - \mu_q(n) \geq \epsilon, \forall m \in A \setminus \{w, q \} \end{array} \right\} \]
When $\epsilon > 0$ is sufficiently small, one has $S_{t}^{w,q,K} \neq \emptyset$ if and only if the convex hull of $\{w,q\} \cup K$ is contained in the triangulation $\mathcal{T}$ of $\Delta$. We replace $\epsilon$ with a smaller one if necessary so that this holds. Then one has
\begin{equation}
\int_{C_{t}^{\epsilon}} \Omega_{t}^{w} = \frac{1}{2\pi \sqrt{-1}} \int_{T_{t}^{\epsilon}} \omega_{t}^{w} = \frac{1}{2\pi \sqrt{-1}} \int_{T_{t}^{w}} \Phi_{t}^{*} \pi_{1}^{*} \Omega_{t}^{w} = \frac{1}{2\pi \sqrt{-1}} \sum_{q,K} \int_{C_{t}^{w,q,K}} \Phi_{t}^{*} \pi_{1}^{*} \Omega_{t}^{w},
\end{equation}
where the sum is taken over $\{q\} \cup K \subset A \setminus \{w\}$ such that the convex hull of $\{w,q\} \cup K$ is contained in $\mathcal{T}$. For such a pair $(q, K)$, we define the integral affine functions $\alpha, \beta_{k} : N_{C} \to \mathbb{C}$ $(k \in K)$ as
\begin{equation}
\alpha := \mu_{q} - \mu_{w}, \quad \beta_{k} := \mu_{k} - \mu_{q}.
\end{equation}
Since the triangulation $\mathcal{T}$ is unimodular, we can take a collection of integral linear functions $\{\gamma_{j}\}_{j \in J}$ so that $\{\alpha, \beta_{k}, \gamma_{j} | k \in K, j \in J\}$ forms an integral affine coordinate system on $N_{C}$.

**Lemma 4.1.** When $t > 0$ is sufficiently small, one has
\begin{equation}
\log_{\epsilon} C_{1} \leq \alpha \leq \log_{\epsilon} C_{2}
\end{equation}
on $S_{t}^{w,q,K}$ for some constants $C_{1}, C_{2} > 0$. In particular, we have $-\epsilon \leq \alpha \leq \epsilon$, when $t > 0$ is sufficiently small.

**Proof.** On $S_{t}^{w,q,K}$, we have
\begin{equation}
\frac{r_{q}}{r_{w}} t^{(\mu_{q} - \mu_{w})(n)} \leq \sum_{m \in A \setminus \{w\}} \frac{r_{m}}{r_{w}} \frac{r_{m} t^{(\mu_{m} - \mu_{w})(n)}}{r_{w}} = |1 + x| \leq 2,
\end{equation}
from which we get the former inequality. We also have
\begin{equation}
\frac{1}{2} \leq |1 + x| = \sum_{m \in A \setminus \{w\}} \frac{r_{m}}{r_{w}} \frac{r_{m} t^{(\mu_{m} - \mu_{w})(n)}}{r_{w}} \leq C \cdot t^{(\mu_{q} - \mu_{w})(n)}
\end{equation}
for $C := |A \setminus \{w\}| \cdot \max_{m \in A \setminus \{w\}} r_{m}/r_{w} > 0$. The latter inequality follows from this. \hfill \Box

On $S_{t}^{w,q,K}$, we have
\begin{equation}
|1 + x| = \sum_{m \in A \setminus \{w\}} \frac{r_{m}}{r_{w}} \frac{r_{m} \mu_{m} - \mu_{q}}{r_{w}} = t^{\alpha} \left( \frac{r_{q}}{r_{w}} + \sum_{k \in K} \frac{r_{k} \beta_{k}}{r_{w}} + \sum_{m \in A \setminus \{w,q\} \cup K} \frac{r_{m} \mu_{m} - \mu_{q}}{r_{w}} \right).
\end{equation}
The last term $\sum_{m \in A \setminus \{w,q\} \cup K} \mu_{m} - \mu_{q}$ is $O(t)$, and this equation can be used to write $\alpha$ as a function $\alpha_{w,q,K}(\beta, \gamma, x)$ of the variables $\beta := \{\beta_{k}\}_{k}, \gamma := \{\gamma_{j}\}_{j}$, and $x$. We set
\begin{equation}
\alpha'_{w,q,K}(\beta, x) := \log_{t} |1 + x| - \log_{t} \left( \frac{r_{q}}{r_{w}} + \sum_{k \in K} \frac{r_{k} \beta_{k}}{r_{w}} \right).
\end{equation}

**Lemma 4.2.** On $S_{t}^{w,q,K}$, one has
\begin{equation}
\alpha_{w,q,K}(\beta, \gamma, x) = \alpha'_{w,q,K}(\beta, x) + O(t)
\end{equation}
\begin{equation}
\frac{\partial \alpha_{w,q,K}}{\partial \gamma_{j}}(\beta, \gamma, x) = O(t).
\end{equation}

**Proof.** \begin{equation} \end{equation} implies
\begin{equation}
\alpha_{w,q,K}(\beta, \gamma, x) = \log_{t} |1 + x| - \log_{t} \left( \frac{r_{q}}{r_{w}} + \sum_{k \in K} \frac{r_{k} \beta_{k}}{r_{w}} + O(t) \right)
\end{equation}
\begin{equation}
= \alpha'_{w,q,K}(\beta, x) - \log_{t} (1 + O(t))
\end{equation}
\begin{equation}
= \alpha'_{w,q,K}(\beta, x) + O(t).
\end{equation}
We got the first estimate. Next, we will show the second estimate. By differentiating (4.3) with respect to the variable $\gamma_{j}$, one can get
\begin{equation}
\frac{\partial \alpha_{w,q,K}}{\partial \gamma_{j}}(\beta, \gamma, x)|1 + x| + t^{\alpha} \sum_{m \in A \setminus \{w,q\} \cup K} \frac{r_{m} \mu_{m} - \mu_{q}}{r_{w}} \left( \frac{\partial (\mu_{m} - \mu_{q})}{\partial \alpha} \frac{\partial \alpha_{w,q,K}}{\partial \gamma_{j}}(\beta, \gamma, x) + \frac{\partial (\mu_{m} - \mu_{q})}{\partial \gamma_{j}} \right) = 0,
\end{equation}
where we regard $\mu_m = \mu_m(\alpha, \beta, \gamma)$ as an affine function of variables $\alpha, \beta, \gamma$. This is equivalent to

$$\frac{\partial \alpha_{w,q,K}}{\partial \gamma_j} \left[1 + x \right] + t^\alpha \sum_{m \in A \setminus \{w, q \cup K\}} \frac{r_m \mu_m - \mu_w \partial (\mu_m - \mu_q)}{r_w} \partial \gamma_j \right] = -t^\alpha \sum_{m \in A \setminus \{w, q \cup K\}} \frac{r_m \mu_m - \mu_w \partial (\mu_m - \mu_q)}{r_w}. $$

By Lemma 4.1, we also have $t^\alpha \cdot \mu_{m,-w} = O(t^\epsilon)$ for $m \in A \setminus \{w, q \cup K\}$. The second estimate follows from these.

**Lemma 4.3.** On $S^w_{t^q,K}$, we have

$$\Phi^*_t \beta_k = \beta_k + \frac{\sqrt{-1}}{\log t} \left( \arg \left( \frac{-c_q}{c_w} \right) - \arg \left( \frac{-c_k}{c_w} \right) \right),$$

where $\Phi_t: N^w_{\kappa} \to N^w_{\kappa^0}$ is the map defined in (3.15).

**Proof.** Let $(n, x)$ be an element in $S^w_{t^q,K}$. By Lemma 4.1, we have $(\mu_q - \mu_w)(n) = \alpha(n) \leq \epsilon < \kappa$. For any element $k \in K$, we also have $(\mu_k - \mu_w)(n) = (\alpha + \beta_k)(n) \leq 2\epsilon < \kappa$. Thus we have $\{q\} \cup K \subset K^n$. This and (3.15) imply

$$\begin{align*}
(\Phi^*_t \beta_k)(n, x) &= \beta_k \left( n + \frac{\sqrt{-1}}{\log t} \cdot \phi(n, x) \right) \\
&= \beta_k(n) + \frac{\sqrt{-1}}{\log t} \cdot ((k - w) - (q - w) \cdot \phi(n, x)) \\
&= \beta_k(n) + \frac{\sqrt{-1}}{\log t} \cdot \left( \arg \left( \frac{-c_q}{c_w} \right) - \arg \left( \frac{-c_k}{c_w} \right) \right).
\end{align*}$$

We obtained the claim.

On $\tilde{\Phi}_t \left( S^w_{t^q,K} \right)$, we write

$$1 + x = f^w_t(i_t(n)) = -\frac{c_q}{c_w} t^\alpha \left( 1 + \sum_{k \in K} \frac{c_k}{c_q} \beta_k + h_t(n) \right)$$

with a function $h_t: N_C \to C$. It follows from the $C^0$-estimate for $\delta_t$ in Proposition 3.4 that the function $h_t$ satisfies the uniform estimates

$$h_t = O(t^\epsilon), \quad \frac{\partial h_t}{\partial n_i} = O((-\log t) t^\epsilon),$$

where $(n_0, \cdots, n_d)$ are $C$-coordinates on $N_C \cong C^{d+1}$. By (4.21), we can write

$$\alpha = \log_t \left\{ -\frac{c_q}{c_w} (1 + x) \left( 1 + \sum_{k \in K} \frac{c_k}{c_q} \beta_k + h_t(n) \right)^{-1} \right\}.$$ 

The standard volume form on $N_\mathbb{R}$ is given by $d\alpha \wedge_{k \in K} d\beta_k \wedge_{j \in J} d\gamma_j$. We set $d\nu_{q,K} := \wedge_{j \in J} d\gamma_j$. By (4.22), we get

$$\begin{align*}
d\alpha \wedge_{k \in K} d\beta_k \cdot d\nu_{q,K} &= \left( 1 + \left( 1 + \sum_{k \in K} \frac{c_k}{c_q} \beta_k + h_t(n) \right)^{-1} \frac{\partial h_t}{\partial \alpha} \frac{1}{\log t} \right)^{-1} \frac{1}{1 + x \log t} \wedge_{k \in K} d\beta_k \cdot d\nu_{q,K} \\
&= \left( 1 + t^\alpha \left( -\frac{c_q}{c_w} (1 + x) \frac{1}{\log t} \right)^{-1} \frac{1}{1 + x \log t} \right)^{-1} \frac{1}{1 + x \log t} \wedge_{k \in K} d\beta_k \cdot d\nu_{q,K}
\end{align*}$$

on $\tilde{\Phi}_t \left( S^w_{t^q,K} \right)$. From this, (4.22), Lemma 4.1 and the $C^0$-estimate for $\delta_t$, we obtain

$$\begin{align*}
d\alpha \wedge_{k \in K} d\beta_k \cdot d\nu_{q,K} &= (1 + O(t^\epsilon)) \frac{1}{1 + x \log t} \wedge_{k \in K} d\beta_k \cdot d\nu_{q,K}.
\end{align*}$$
On the other hand, we have

\begin{equation}
\Phi_t^{\pi_1^* \pi_1^* \omega_1^{lw}} = \Phi_t^{\pi_1^* \pi_1^*} \left( \prod_{z_i} \left( \frac{d^d z_i}{z_i} \right) \prod_{m \in A \cap \tau_v} \left( -k_{w,t} z_i^w (f_t^w - 1) \right)^{\frac{1}{m}} \prod_{k \in K} \left( k_{m,t} \cdot z_m^{pm} \right) \right)
\end{equation}

\begin{equation}
= (\log t)^{d+1} \frac{1}{x^l} \Phi_t^{\pi_1^*} \left( \prod_{m \in A \cap \tau_v} \left( -k_{m,t} \right)^{pm} \cdot t^{v-lw} \cdot d\alpha \sum_{k \in K} d\beta_k \cdot dvol_{q,K} \right)
\end{equation}

\begin{equation}
= (\log t)^{d+1} \frac{t^{\lambda_t - \lambda_w} \cdot \Phi_t^{\pi_1^*} \left( \prod_{m \in A \cap \tau_v} \left( 1 + O(\tau^2) \right) \right)}{x^l} \cdot \frac{-e_m}{e_w} \cdot t^{v-lw} \cdot d\alpha \sum_{k \in K} d\beta_k \cdot dvol_{q,K},
\end{equation}

where \( \lambda_t := \sum_{m \in A \cap \tau_v} p_m \lambda_m \). Therefore, we have

\begin{equation}
\frac{d}{dx} \int_{S_t^{w,q,K}} \Phi_t^{\pi_1^* \pi_1^* \omega_1^{lw}} = (1 + O(t^e)) \left( \log t \right)^d \int_{S_t^{w,q,K}} \frac{dx}{x^l(1 + x)} \Phi_t^{\pi_1^* \pi_1^*}(n, x) \cdot \Phi_t^{\pi_1^*} \left( \sum_{k \in K} d\beta_k \cdot dvol_{q,K} \right),
\end{equation}

where

\begin{equation}
P_t^{v,w}(n, x) := t^{\lambda_t - \lambda_w} \cdot \Phi_t^{\pi_1^*} \left( t^{v-lw} \right) \prod_{m \in A \cap \tau_v} \left( \frac{e_m}{e_w} \right)^{pm}.
\end{equation}

**Lemma 4.4.** Let \( \tau_{w,q,K} \in \mathcal{T} \) be the convex hull of \( \{w, q\} \cup K \). Then we have

\begin{equation}
P_t^{v,w}(n, x) = \begin{cases} (1 + O(t^e)) \cdot (1 - p_w) \cdot (1 + x)^{-p_w} \cdot \frac{e_q^p \prod_{k \in \tau \cap K} (r_k \cdot t^{\beta_k})^{p_w}}{(1 + \sum_{k \in K} r_k \cdot t^{\beta_k})^{-p_w}} \cdot \tau_v & \quad \text{if } \tau_v \subset \tau_{w,q,K} \\text{and } t \notin \tau_{w,q,K} \end{cases}
\end{equation}

\begin{equation}
on \text{on } \mathcal{S}_{t^{w,q,K}}, \text{ where } p_w := 0 \text{ if } w \notin \tau_v, \text{ and } p_q := 0 \text{ if } q \notin \tau_v.
\end{equation}

Proof. First, suppose \( \tau_v \subset \tau_{w,q,K} \). Since we have

\begin{equation}
\lambda_{t,v} - \lambda_w + v - lw = \sum_{m \in A \cap \tau_v} p_m \{ (\lambda_m - \lambda_w) + (m - w) \}
\end{equation}

\begin{equation}
= p_q \alpha + \sum_{k \in K \cap \tau_v} p_k (\alpha + \beta_k)
\end{equation}

\begin{equation}
= (l - p_w) \alpha + \sum_{k \in K \cap \tau_v} p_k \beta_k,
\end{equation}

one can see from [4.21] that we have

\begin{equation}
t^{\lambda_{t,v} - \lambda_w} \cdot \Phi_t^{\pi_1^*} \left( t^{v-lw} \right) = \Phi_t^{\pi_1^*} \left( t^{l-p_w} \alpha + \sum_{k \in \tau \cap K} p_k \beta_k \right)
\end{equation}

\begin{equation}
= \Phi_t^{\pi_1^*} \left( \frac{-e_q}{e_q} \right)^{l-p_w} \left( 1 + \sum_{k \in K} \frac{e_q}{e_q} \right)^{-l+p_w} \left( 1 + x \right)^{l-p_w} \left( \sum_{k \in \tau \cap K} p_k \Phi_t^{\beta_k} \right).
\end{equation}

By the estimate

\begin{equation}
- \log t = O \left( t^{-\delta} \right)
\end{equation}

that holds for any \( \delta > 0 \), and the \( C^0 \)-estimate for \( \delta_t \) in Proposition 3.4, we also have

\begin{equation}
t^{\lambda_t} \leq t^{\lambda_t} = \exp (C t^\kappa \log t) = 1 + O (t^{\kappa} \log t) = 1 + O(t^e)
\end{equation}

for \( m \in M_R \), where \( C > 0 \) is some constant. Hence, by this and [4.22], the above [4.37] is equal to

\begin{equation}
(1 + O(t^e)) \cdot \left( \frac{-e_q}{e_q} \right)^{l-p_w} \left( 1 + \sum_{k \in K} \frac{e_q}{e_q} \right)^{-l+p_w} \left( 1 + x \right)^{l-p_w} \left( \sum_{k \in \tau \cap K} p_k \Phi_t^{\beta_k} \right).
\end{equation}

By this and Lemma 4.3, one can get [4.32] for the case where \( \tau_v \subset \tau_{w,q,K} \).
Next, suppose \( \tau_v \not\subset \tau_{w,q,K} \). By (4.39) again, we have
\[
(4.41) \quad t^{b_n - l_{\delta v}} \cdot \Phi_t^s \left( t^{l_{-l-w}} \right) = \Phi_t^s \left( \prod_{m \in A \cap \tau_v} t^{p_m} \right) = (1 + O \left( t^\epsilon \right)) \cdot \Phi_t^s \left( \prod_{m \in A \cap \tau_v} t^{p_m} \right).
\]
We also have
\[
(4.42) \quad \left| \Phi_t^s \left( \prod_{m \in A \cap \tau_v} t^{p_m} \right) \right| = \prod_{m \in (A \cap \tau_v) \setminus \{w\}} t^{p_m} \cdot \prod_{m \in (A \cap \tau_v) \setminus \{w\}} t^{p_m} = C_1 \cdot \prod_{m \in (A \cap \tau_v) \setminus \{w\}} t^{p_m}.
\]
for some constant \( C_1 > 0 \). We used Lemma 4.11 in the last inequality. Let \( m_0 \in A \cap \tau_v \) be an element that is not in \( \tau_{w,q,K} \). Then we have
\[
(4.45) \quad \prod_{m \in (A \cap \tau_v) \setminus \{w\}} t^{p_m} \leq t^{p_m} \cdot \prod_{m \in (A \cap \tau_v) \setminus \{w, m_0\}} t^{p_m} = O \left( t^\epsilon \right).
\]
The claim of the lemma in the case of \( \tau_v \not\subset \tau_{w,q,K} \) follows from these. \( \square \)

We can see from Lemma 4.4 that \( \frac{1}{t^{l_{-l-w}}} P_t^{v,w} (n, x) \) is uniformly bounded. It is also obvious from the construction of \( \Phi_t \) that the \( C^1 \)-norm of \( \Phi_t \) (and hence of \( \Phi_t |_{S^w} \)) is also bounded. Furthermore, when we define the \( C^1 \)-norm of \( \delta_t |_{S^w} \) using the Riemannian metric induced from the Euclidean metric on the ambient space \( N_{\mathbb{R}} \times D_\epsilon \), we have \( \| \delta_t |_{S^w} \|_{C^1} \sim O \left( t^\epsilon \right) \) by Proposition 5.4. From these facts and Lemma 4.11 which we will prove later, we can see that (4.30) is equal to
\[
(4.46) \quad (1 + O \left( t^\epsilon \right)) \left( \log t \right)^d \left( \int_{S^w_{\tau_v}} \frac{dx}{x^d (1 + x)} \right) \Phi_t^{v,w} \left( \bigwedge_{k \in K} d\beta_k \cdot d\varepsilon_{q,K} \right) + O \left( t^\epsilon \right).
\]
From Lemma 4.3 and Lemma 4.4, we can see that (4.46) is equal to
\[
(4.47) \quad (1 + O \left( t^\epsilon \right)) \left( -1 \right)^{p_w} \left( \log t \right)^d \left( \int_{S_{\tau_v}^q} \frac{(1 + x)^{l_{-l-w} - 1}}{x^d} \right) \Phi_t^{v,w} \left( \bigwedge_{k \in K} d\beta_k \cdot d\varepsilon_{q,K} \right) \Phi_t^{\gamma,w} \left( \bigwedge_{k \in K} d\beta_k \cdot d\varepsilon_{q,K} \right) + O \left( t^\epsilon \right),
\]
where \( b := \{b_k\}_{k \in K} \) is the coordinate system on \( [0, \epsilon]^K \) and \( \Phi_t^{\gamma,w} \) is a function on \( [0, \epsilon]^K \) defined by
\[
(4.48) \quad \Phi_t^{\gamma,w} (b) := \left\{ \begin{array}{ll}
\frac{1}{t^{l_{-l-w}}} \int_{S_{\tau_v}^q} \frac{dx}{x^d} & \tau_v \subset \tau_{w,q,K} \\
0 & \tau_v \not\subset \tau_{w,q,K}
\end{array} \right.,
\]
where \( \tau_{w,q,K} \) is a subset of \( \tau_v \) of the form \( \{ \beta = b, x = \hat{x} \} \) for some \( \hat{x} \). We can see that (4.47) is equal to
\[
(4.49) \quad \Phi_t^{\gamma,w} (d\varepsilon_{q,K}) = \int_{F_{q,K} (b, \hat{x})} (\Phi_t \circ s_{b,\hat{x}})^* (d\varepsilon_{q,K}),
\]
where \( F_{q,K} (b, \hat{x}) \) is the projection of \( S_{\tau_v}^{w,q,K} \cap \{ \beta = b, x = \hat{x} \} \) to the \( \gamma \)-plane, and \( s_{b,\hat{x}} \) is the map from the \( \gamma \)-plane to \( S_{\tau_v}^{w,q,K} \cap \{ \beta = b, x = \hat{x} \} \) given by \( \gamma \mapsto (\alpha_{w,q,K} (b, \gamma, \hat{x}), b, \gamma, \hat{x}) \). The subset \( F_{q,K} (b, \hat{x}) \) is defined in the \( \gamma \)-plane by
\[
(4.50) \quad \mu_m (\alpha_{w,q,K} (b, \gamma, \hat{x}), b, \gamma) - \mu_q (\alpha_{w,q,K} (b, \gamma, \hat{x}), b, \gamma) \geq \epsilon, \forall m \in A \setminus \{w, q\} \sqcup K.
\]
By Lemma 4.2, one can see
\[
(4.51) \quad \int_{F_{q,K} (b, \hat{x})} (\Phi_t \circ s_{b,\hat{x}})^* (d\varepsilon_{q,K}) = \int_{F_{q,K} (b, \hat{x})} (\Phi_t \circ s_{b,\hat{x}})^* (d\varepsilon_{q,K}) + O \left( t^\epsilon \right)
\]
and
\[
(4.52) \quad \int_{F_{q,K} (b, \hat{x})} (\Phi_t \circ s_{b,\hat{x}})^* (d\varepsilon_{q,K}) = \int_{F_{q,K} (b, \hat{x})} (\Phi_t \circ s_{b,\hat{x}})^* (d\varepsilon_{q,K}) + O \left( t^\epsilon \right),
\]
Lemma 4.5. \[ (4.58) \]

Let \( s_{b,x}' \) be the map from the \( \gamma \)-plane to \( N \times S^1_{x_0} \) given by \( \gamma \mapsto \left( \alpha'_{w,q,K}(b,x), b, \gamma, x \right) \), and \( F'_{q,K}(b,x) \) is the polytope in the \( \gamma \)-plane defined by

\[ \mu_m \left( \alpha'_{w,q,K}(b,x), b, \gamma \right) - \mu_q \left( \alpha'_{w,q,K}(b,x), b, \gamma \right) \geq \epsilon, \forall m \in A \setminus \{ \{w,q\} \cup K \} . \]

For a subset \( J \subset A \setminus \{w,q\} \) containing \( K \), we consider the polytope \( E_{q,J} \left( \{b_j\}_{j \in J}, x \right) \) in the \( \gamma \)-plane defined by

\[ \mu_j \left( \alpha'_{w,q,K}(b,x), b, \gamma \right) - \mu_q \left( \alpha'_{w,q,K}(b,x), b, \gamma \right) = b_j, \forall j \in J \setminus K \]

\[ \mu_m \left( \alpha'_{w,q,K}(b,x), b, \gamma \right) - \mu_q \left( \alpha'_{w,q,K}(b,x), b, \gamma \right) \geq 0, \forall m \in A \setminus \{ \{w,q\} \cup J \} . \]

Then one has

\[ E'_{q,K}(b,x) = E_{q,K}(b,x) \setminus \bigcup_{j \notin \{w,q\} \cup K} \bigcup_{b_j \in [0,\epsilon]} E_{q,K \cup \{j\}}(b, b_j, x) , \]

and

\[ \bigcap_{j \in I} \left( \bigcup_{b_j \in [0,\epsilon]} E_{q,K \cup \{j\}}(b, b_j, x) \right) = \bigcup_{b' \in [0,\epsilon]^I} E_{q,K \cup I}(b, b', x) \]

for a subset \( I \subset A \setminus \{ \{w,q\} \cup K \} \). By using these and the inclusion-exclusion principle, one can get

\[ \int_{F'_{q,K}(b,x)} (\Phi_t \circ s_{b,x}')^* (d\text{vol}_q) = \sum_{J \subset A \setminus \{w,q\}, J \supset K} (-1)^{|J|} \int_{D_{q,J}(b,x)} (\Phi_t \circ s_{b,x}')^* (d\text{vol}_q) , \]

where \( D_{q,J}(b,x) := \bigcup_{b' \in [0,\epsilon]^J \cap K} E_{q,J}(b, b', x) \).

Lemma 4.5. On the image of \( D_{q,J}(b,x) \) by the map \( s_{b,x}' \), we have

\[ \Phi_t \alpha \equiv \alpha'_{w,q,K}(b,x) + \frac{\sqrt{-1}}{\log t} \cdot \left( \arg(1 + x) - \arg \left( -\frac{c_q}{c_w} \right) \right) \]

\[ \Phi_t \beta_k \equiv b_k + \frac{\sqrt{-1}}{\log t} \cdot \left( \arg \left( -\frac{c_q}{c_w} \right) - \arg \left( -\frac{c_j}{c_w} \right) \right) \quad (k \in K) \]

\[ \Phi_t \beta_j' = \beta_j' + \frac{\sqrt{-1}}{\log t} \cdot \left( \arg \left( -\frac{c_q}{c_w} \right) - \arg \left( -\frac{c_j}{c_w} \right) \right) \quad (j \in J \setminus K) , \]

when \( t > 0 \) is sufficiently small. Here \( \beta_j' := \mu_j - \mu_q \) for \( j \in J \setminus K \).

Proof. Let \( (n, x) \) be an element of the image of \( D_{q,J}(b,x) \) by the map \( s_{b,x}' \). By Lemma 4.2 and Lemma 4.1, we have

\[ (\mu_q - \mu_w)(n) = \alpha'_{w,q,K}(b,x) = \alpha_{w,q,K}(b, \gamma, x) + O(t^\epsilon) \leq \epsilon + O(t^\epsilon) \leq \kappa \]

for sufficiently small \( t > 0 \). For any element \( j \in J \), we also have

\[(\mu_j - \mu_w)(n) = (\mu_j - \mu_q)(n) + (\mu_q - \mu_w)(n) = b_j + (\mu_q - \mu_w)(n) \leq 2\epsilon + O(t^\epsilon) \leq \kappa \]

for sufficiently small \( t > 0 \). Thus we have \( \{q,j\} \subset K^* \). By (4.13) and the similar equations as (4.18)-(4.20), we can check the claim. \( \square \)

Lemma 4.6. (cf. [AGIS20, Lemma 3.2]) The polytope \( E_{q,J} \left( \{b_j\}_{j \in J}, x \right) \) is non-empty if and only if the convex hull of \( \{w,q\} \cup J \) is contained in \( \mathcal{T} \), and its volume is given by

\[ \int_{Y_w} \exp \left( \omega^w_X - \sum_{j \in J} b_j D^w_j - \alpha'_{w,q,K}(b,x)\sigma^w \right) \cdot D^w_q \prod_{j \in J} D^w_j , \]

where \( Y_w, D^w_j \) are the toric variety and its toric divisor defined in Section 7 and \( \omega^w_X, \sigma^w \) are the ones defined in (4.12).
Proof. We identify the polytope $E_{q,J} \left\{ \{b_j\}_{j \in J}, x \right\}$ with its image by the map $s'_{b,x}$. Then it is defined in $N_{\mathbb{R}}$ by

\begin{align}
\mu_q &= \mu_w + \alpha'_{w,q,K}(b,x), \\
\mu_j - b_j &= \mu_w + \alpha'_{w,q,K}(b,x), \quad j \in J, \\
\mu_m &\geq \mu_w + \alpha'_{w,q,K}(b,x), \quad m \in A \setminus \left\{ \{w, q \} \sqcup J \right\}.
\end{align}

This is exactly the face of the tropical hypersurface in $N_{\mathbb{R}}$ defined by the tropical polynomial

\begin{equation}
f'(n) := \min_{m \in A} \left\{ \lambda'_m + \langle m, n \rangle \right\}
\end{equation}

with

\begin{equation}
\lambda'_m = \begin{cases} 
\lambda_w + \alpha'_{w,q,K}(b,x) & m = w \\
\lambda_j - b_j & m = j \in J \\
\lambda_m & m \in A \setminus \left\{ \{w, q \} \sqcup J \right\},
\end{cases}
\end{equation}

on which all the tropical monomials corresponding to $\{w, q \} \sqcup J$ attain the minimum of $f'$. Since we have $b_j \in [0, \epsilon]$ and $\alpha'_{w,q,K}(b,x) = \alpha_{w,q,K}(b, \gamma, x) + O(t') \in [-\epsilon, \epsilon] + O(t')$ by Lemma 4.2 and Lemma 4.1 and these are small, the combinatorial type of the tropical hypersurface is the same as that of $X$ (trop($f$)). The former claim of the lemma follows from this.

When the polytope $E_{q,J} \left\{ \{b_j\}_{j \in J}, x \right\}$ is non-empty, it is a face of the polytope $P_w \subset N_{\mathbb{R}}$ defined by

\begin{equation}
P_w := \left\{ n \in N_{\mathbb{R}} \mid \lambda'_w + \langle w, n \rangle \leq \lambda'_m + \langle m, n \rangle, \forall m \in A_w \right\}.
\end{equation}

The normal fan of $P_w$ coincides with the fan $\Sigma_w$, defined in Section 5.3. By [AGIS20, Theorem 2.10], it turns out that the volume of $E_{q,J} \left\{ \{b_j\}_{j \in J}, x \right\}$ is equal to the symplectic volume of the stratum $D^w_q \cap \bigcap_{j \in J} D^w_j$ in the toric variety $Y_w$ with respect to a symplectic form whose cohomology class is

\begin{equation}
\omega^w = \sum_{j \in J} b_j D^w_j - \alpha'_{w,q,K}(b,x) \sigma^w.
\end{equation}

Thus we obtain the latter claim of the lemma. Notice that when the convex hull of $\{w, q \} \sqcup J$ is not contained in $\mathcal{T}$, \eqref{eq:4.64} is zero, since $D^w_q \cap \bigcap_{j \in J} D^w_j$ is empty. \hfill \Box

It turns out by Lemma 4.2 that in \eqref{eq:4.68}, it suffices to take the sum only over $J$ such that the convex hull of $\{w, q \} \sqcup J$ is contained in $\mathcal{T}$. By Lemma 4.3 we also have $\Phi^t_j \left( d\beta'_j \right) = d\beta'_j$ for $j \in J \setminus K$. When we identify the polytope $E_{q,J} \left\{ \{b_j\}_{j \in J}, x \right\}$ with its image by the map $s'_{b,x}$, \eqref{eq:4.68} can be written as

\begin{equation}
\int_{E_{q,J}(b,x)} \left( \Phi_t \circ s'_{b,x} \right)^* (d\text{vol}_{q,K}) = \sum_{J \subset A \setminus \{w, q \}, J \cap K} (-1)^{J \cap K} \int_{\{0, \epsilon\}^{J \cap K}} \int_{\bigcap_{j \in J \setminus K} \mathbb{R} \setminus \{0, \epsilon\}} \int_{E_{q,J}(b, b', x)} \Phi_t^* (d\text{vol}_{q,J}),
\end{equation}

where $b' := \left\{ b'_j \mid j \in J \setminus K \right\}$ is the coordinate system on $[0, \epsilon]^{J \cap K}$, and $\text{vol}_{q,J}$ is the holomorphic form such that $\text{vol}_{q,J} = \bigwedge_{j \in J \setminus K} d\beta'_j \wedge \text{vol}_{q,J}$.

Here we consider the complex volumes of polytopes which were introduced in [AGIS20, Section 5.3]. By Lemma 4.3 we can see that the image of $E_{q,J}(b, b', x)$ by the map $\Phi_t$ is contained in the complex affine subspace of $N_{\mathbb{C}}$ defined by

\begin{align}
\mu_q - \mu_w &= \alpha'_{w,q,K}(b,x) + \frac{\sqrt{-1}}{\log t} \cdot \left( \text{Arg} \left( 1 + x \right) - \text{Arg} \left( -\frac{c^w}{c_w} \right) \right), \\
\mu_j - \mu_q &= b_j + \frac{\sqrt{-1}}{\log t} \cdot \left( \text{Arg} \left( -\frac{c^w}{c_w} \right) - \text{Arg} \left( -\frac{c^{m_0}}{c_w} \right) \right), \quad j \in J,
\end{align}

which we let $S_{q,J}(b, b', x)$ denote. Each facet of $E_{q,J}(b, b', x)$ is given by $E_{q,J}(b, b', x) \cap \{ \mu_{m_0} - \mu_q = 0 \}$ for some $m_0 \in A_w \setminus \{ \{q \} \sqcup J \}$. For any point $n$ in the facet, we have $n \in K_n^{m_0}$. We can see from \eqref{eq:3.17} that the facet is mapped by the map $\Phi_t$ to the complex affine hyperplane in $S_{q,J}(b, b', x)$ given by

\begin{equation}
\mu_{m_0} - \mu_q = \frac{\sqrt{-1}}{\log t} \cdot \left( \text{Arg} \left( -\frac{c^w}{c_w} \right) - \text{Arg} \left( -\frac{c^{m_0}}{c_w} \right) \right).
\end{equation}

The complex volume of the pair $(E_{q,J}(b, b', x), \Phi_t)$ with respect to the form $d\text{vol}_{q,J}$ in the sense of [AGIS20, Section 5.3] is defined to be $\int_{E_{q,J}(b, b', x)} \Phi_t^* (d\text{vol}_{q,J})$. By [AGIS20, Lemma 5.8], we can see that it is a
polynomial function of the constant terms of affine equations (4.73)-(4.75). By analytic continuation of (4.64), it turns out that\[ I_{\ell_t}(b,b',x,D) \Phi^*_{\ell_t}(d\nu_{q,J}) \]is equal to

\[ \int_{Y_w} E_{\ell_t}^{q,K,J}(b,b',x,D) \cdot D_w^m \prod_{j \in J} D_j^w, \]

where

\[ E_{\ell_t}^{q,K,J}(b,b',x,D) := \exp \left( \omega_w^m + \sqrt{\frac{-t}{\log t}} \sum_{m \in A_w} \left( \arg \left( -\frac{c_m}{c_w} \right) - \arg (1+x) \right) D_m^w - \sum_{j \in J} b_j D_j^w - \alpha_{w,q,K}(b,x) \sigma^w \right). \]

This is obtained by substituting \( \lambda_m + \sqrt{\frac{-t}{\log t}} \arg \left( -\frac{c_m}{c_w} \right) \) to \( \lambda_m \) for all \( m \in A_w \), and \( \lambda_w + \sqrt{\frac{-t}{\log t}} \arg (1+x) \) to \( \lambda_w \) in (4.64).

By (4.47), (4.49), (4.52), (4.72), and substituting (4.76) to \( I_{\ell_t}(b,b',x,D) \Phi^*_{\ell_t}(d\nu_{q,J}) \), we obtain

\[ \int_{S_t^w} \Phi^*_{\ell_t} \Phi^*_{\ell_t} \omega^m_{\ell_t} = (1 + O(t^p))(-1)^{p_w} \cdot \left( \int_{Y_t} \int_{S_t^1} \left( 1 + x \right)^{-p_w-1} P_t(x,D) dx + O(t^p) \right) , \]

where \( P_t(x,D)_{d+1} \) denotes the part of \( P_t(x,D) \) in degree \( 2(d+1) \), which is defined by

\[ P_t(x,D) := (\log t)^d \sum_{K \subseteq J \subseteq A_w \setminus \{q\}} (-1)^{|J|} \int_{[0,\epsilon]} \Phi_{\ell_t}^{q,w}(b) \cdot E_t^{q,K,J}(b,b',x,D) \cdot D_q^w \prod_{j \in J} D_j^w db db' , \]

where the sum is taken over all pairs \( (q,K,J) \) such that the convex hull of \( \{q\} \cup J \) is in \( \mathcal{T} \). If we replace \( \omega_w^m, D_m^w \) with \( \omega_w^m,\sigma_w^m \), then \( E_t^{q,K,J}(b,b',x,D) \cdot D_q^w \prod_{j \in J} D_j^w \) in \( P_t(x,D) \) changes to

\[ F_t(x,D) \cdot \exp \left( \sum_{j \in J} (-\log t) b_j D_j^w - \sigma^w \log \left( \frac{r_w}{r_w + \sum_{k \in K} s_k} \right) \right) \cdot (\log t)^{|J|+1} \cdot D_q^w \prod_{j \in J} D_j^w , \]

where

\[ F_t(x,D) := t^{-\omega_w^m} \cdot \exp \left( \sqrt{\frac{-t}{\log t}} \sum_{m \in A_w} \left( \arg \left( -\frac{c_m}{c_w} \right) - \arg (1+x) \right) D_m^w \right) . \]

Here we used (4.19). The degree \( 2(d+1) \)-part of what we get by replacing \( \omega_w^m, D_m^w \) in \( P_t(x,D) \) with \( \omega_w^m,\sigma_w^m \) is equal to \( (\log t)^{d+1} P_t(D)_{d+1} \). Hence, if we change the coordinates from \( b_j \) to \( s_j := -\log t \cdot b_j (j \in J) \), and set

\[ Q_t(x,D) := (-1)^d \sum_{K \subseteq J \subseteq A_w \setminus \{q\}} (-1)^{|J|} F_t(x,D) \cdot G_t^{q,K,J}(x,D) , \]

where

\[ G_t^{q,K,J}(x,D) := \int_{[0,-\log t]} \Phi_{\ell_t}^{q,w}(s) \cdot \exp \left( \sum_{j \in J} s_j D_j^w - \sigma^w \log \left( \frac{r_{q,w}}{r_{q,w} + \sum_{k \in K} s_k} \right) \right) ds \cdot D_q^w \prod_{j \in J} D_j^w , \]

and \( \Phi_{\ell_t}^{q,w}(s) \) is the function on \( (\mathbb{R}_{\geq 0})^K \) defined by

\[ \Phi_{\ell_t}^{q,w}(s) := \begin{cases} \prod_{k \in K} (r_k e^{-s_k})^{p_k} \prod_{k \in K} (r_k e^{-s_k})^{p_k} & \text{all vertices of } \tau_v \text{ are contained in } \{w,q\} \cup K \vspace{1em} \\ 0 & \text{otherwise,} \end{cases} \]

then one has \( P_t(x,D)_{d+1} = Q_t(x,D)_{d+1} \). Although we considered the sum only over pairs \( (q,K,J) \) such that the convex hull of \( \{q\} \cup J \) is contained in \( \mathcal{T} \) in (4.73), we do not impose this restriction for the sum in
Notice that the summand for \((q, K, J)\) in (4.82) vanishes if the convex hull of \(\{q\} \sqcup J\) is not contained in \(\mathcal{F}\) due to \(D^w_q \prod_{j \in J} D^w_j\) in \(G^{q,K,J}(x,D)\). We will compute

\[
(4.85) \int_{S_{1,t}^\circ} \frac{(1+x)^{l-p_w-1}}{x^l} Q_L(x,D)_{d+1} dx.
\]

The part of (4.85), in which the variable \(x\) appears

\[
(4.86) \int_{S_{1,t}^\circ} \frac{(1+x)^{l-p_w-1}}{x^l} \exp \left( \sqrt{-1} \sum_{m \in A_w} \text{Arg}(1+x) D^w_m \right) \exp (\sigma^w \log |1+x|) dx
\]
is equal to

\[
(4.87) \int_{S_{1,t}^\circ} \frac{(1+x)^{l-p_w-1}}{x^l} \exp (\sigma^w \log(1+x)) dx = \int_{S_{1,t}^\circ} \frac{(1+x)^{l-p_w-1}}{x^l} (1+x)^{\sigma^w} dx
\]

\[
(4.88) = (2\pi \sqrt{-1}) \cdot \left( \frac{\sigma^w + l - p_w - 1}{l - 1} \right),
\]

where \(\text{Log}(1+x) := \log |1+x| + \sqrt{-1} \text{Arg}(1+x)\) is the principal value of the complex logarithmic function, and the last term is the binomial coefficient, i.e.,

\[
(4.89) \left( \frac{\sigma^w + l - p_w - 1}{l - 1} \right) = \frac{1}{(l-1)!} \prod_{i=0}^{l-2} (\sigma^w + l - p_w - 1 - i).
\]

Next, we compute the part of (4.85), in which the variables \(s_j\) appear

\[
(4.90) \sum_{q \in A_w, K \subset J \subset A_w \setminus \{q\}} (-1)^{|J\setminus K|} \int_{[0,-\epsilon \log l]^J} \varphi_{q,K}^w(s) \exp \left( - \sum_{j \in J} s_j D^w_j - \sigma^w \log \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right) \right) ds \cdot D^w_q \prod_{j \in J} D^w_j.
\]

When we expand the exponential, this is equal to

\[
(4.91) I_0 + \sum_{q \in A_w} \sum_{K \subset J \subset A_w \setminus \{q\}} \frac{(-1)^{|K|}}{h! \prod_{j \in J} m_j} \left\{ \int_{[0,-\epsilon \log l]^J} \left( \prod_{j \in J} (s_j)^{m_j} \right) \log^h \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right) \varphi_{q,K}^w(s) ds \right\} \cdot (\sigma^w)^h \cdot D^w_q \prod_{j \in J} (-D^w_j)^{m_j+1},
\]

where

\[
(4.92) I_0 := \begin{cases} \sum_{q \in A_w} \exp \left( -\sigma^w \log \left( \frac{r_q}{r_w} \right) \right) D^w_q & \tau_v = w \\ 0 & \text{either } \tau_v = q \text{ or } \tau_v = \text{conv } \{\{w,q\}\} \text{ for some } q \in A_w \\ \text{otherwise.} \end{cases}
\]

The first term \(I_0\) arises from the case of \(J = K = \emptyset\). By taking the sum concerning \(K\) first, we write (4.91) as

\[
(4.93) I_0 + \sum_{\{q\} \sqcup J \subset A_w, J \neq \emptyset} \sum_{h \in \mathbb{Z}_{\geq 0}} \frac{t_{r_q,h,m}^w(t)}{h! \prod_{j \in J} m_j} \cdot (\sigma^w)^h \cdot D^w_q \prod_{j \in J} (-D^w_j)^{m_j+1},
\]

where

\[
(4.94) t_{r_q,h,m}^w(t) := \int_{[0,-\epsilon \log l]^J} \prod_{j \in J} (s_j)^{m_j} \sum_{K \subset J} (-1)^{|K|} \log^h \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right) \varphi_{q,K}^w(s) ds.
\]

We also define

\[
(4.95) I_{h,m} := \int_{[0,\infty]^J} \prod_{j \in J} (s_j)^{m_j} \sum_{K \subset J} (-1)^{|K|} \log^h \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right) \varphi_{q,K}^w(s) ds.
\]
Lemma 4.7. (cf. [AGIS20] Section 1.4, Lemma 3.4) The integral (4.95) converges, and one has
\begin{equation}
I_{h, \tilde{m}}^q(t) = I_{h, \tilde{m}}^q + O \left( (-\log t)^{|\tilde{m}|} t^\epsilon \right),
\end{equation}
where $|\tilde{m}| := \sum_{j \in J} m_j$.

Proof. First, we show the former claim. If we change the coordinates from $s_j$ to $X_j := e^{-s_j}$ ($j \in J$), then (4.95) is equal to
\begin{equation}
\int_{[0,1]^J} g_{h, J}^q(X) \cdot \prod_{j \in J} (-\log X_j)^{m_j} \frac{dX_j}{X_j},
\end{equation}
where $X = \{X_j\}_{j \in J}$ and
\begin{equation}
g_{h, J}^q(X) := \sum_{K \subset J} (-1)^{|K|} \log^h \left( \frac{r_q w + \sum_{k \in K} r_k X_k}{r_q} \right) \prod_{j \in K \cap \tau_v} (r_k X_k)^{p_k} \left( r_q + \sum_{k \in K} r_k X_k \right)^{-l + p_w}.
\end{equation}
The first sum in (4.98) is taken over all $K \subset J$ such that all vertices of $\tau_v$ are contained in $\{w, q\} \cup K$. The function $g_{h, J}^q(X)$ is analytic in a neighborhood of $[0,1]^J$, and obviously vanishes along the coordinate hyperplane $\{X_j = 0\}$ for $k \in (A_w \cap \tau_v) \setminus \{q\}$. Furthermore, it vanishes also along the coordinate hyperplane $\{X_j = 0\}$ for $j \in J \setminus \tau_v$, since the terms of $K = K_0$ and $K_0 \cup \{j\}$ cancel each other out for any $K_0 \subset J$ such that $j \notin K_0$ and $\{w, q\} \cup K_0$ contains all vertices of $\tau_v$. By these and $[(A_w \cap \tau_v) \setminus \{q\}] \cup (J \setminus \tau_v) = J$, we have
\begin{equation}
g_{h, J}^q(X) \leq C \cdot \prod_{j \in J} X_j
\end{equation}
on $[0,1]^J$, where $C > 0$ is some constant. From this and (4.97), we can see that the integral (4.95) converges.

By using the fact $\int_{-\infty}^{\infty} e^{-s} s^m ds = O (\log^{|\tilde{m}|} |\tilde{m}|)$ and (4.99), we can also obtain
\begin{equation}
\begin{split}
\left| I_{h, \tilde{m}}^q(t) - I_{h, \tilde{m}}^q(t) \right| &\leq C \int_{(-\log t, \infty)}^\infty \left( \prod_{j \in J} e^{-s_j} \right) \cdot \prod_{j \in J} s_j^{m_j} ds_j = O ((-\log t)^{|\tilde{m}|} t^\epsilon).
\end{split}
\end{equation}

We set
\begin{equation}
\hat{G} := I_0 + \sum_{\{q\} \cup J \subset A_w} \sum_{h \in E_0} \frac{I_{h, \tilde{m}}^q}{h! \prod_{j \in J} m_j!} (-\sigma^w)^h \cdot D_q^w \cdot \prod_{j \in J} (-D_j^w)^{m_j + 1}.
\end{equation}
This is the one obtained by replacing $I_{h, \tilde{m}}^q(t)$ in (4.93) with $I_{h, \tilde{m}}^q$. From the above computations and Lemma 4.7, we can see that (4.2) is equal to
\begin{equation}
(1 + O (t^\epsilon)) (-1)^{d + p_w} \left\{ \int_{Y_w} t^{-\omega^w} \exp \left( -\sqrt{1} \sum_{m \in A_w} \arg \left( -\frac{c_m}{c_w} \right) D_m^w \right) \left( \sigma^w + l - p_w - 1 \right) \hat{G} + O (\log^{|\tilde{m}|} |\tilde{m}|) \right\}.
\end{equation}
By using (4.38) and reducing the constant $\epsilon > 0$, we get
\begin{equation}
\int_{Y_w} \Omega_{t}^{\omega^w} = (-1)^{d + p_w} \int_{Y_w} t^{-\omega^w} \exp \left( -\sqrt{1} \sum_{m \in A_w} \arg \left( -\frac{c_m}{c_w} \right) D_m^w \right) \left( \sigma^w + l - p_w - 1 \right) \hat{G} + O (t^\epsilon).
\end{equation}
If $\conv (\{w\} \cup \tau_v) \notin \mathcal{F}$, then there is a vertex of $\tau_v$ which is not contained in $A_w \cup \{w\}$. This implies $I_0, I_{h, \tilde{m}}^q = 0$, and $\hat{G} = 0$. Therefore, in this case, we have $\int_{\mathcal{F}} \Omega_{t}^{\omega^w} = O (t^\epsilon)$ which is one of the claims of Theorem 4.1. In the following, we assume $\conv (\{w\} \cup \tau_v) \in \mathcal{F}$.

We regard $D = \left\{ D_j^w \right\}_{j \in A_w}$ as positive real numbers and define the function $G : (\mathbb{R}_{>0})^{A_w} \to \mathbb{R}$ by
\begin{equation}
G(D) := \sum_{\{q\} \cup J \subset A_w} D_q^w \prod_{j \in J} (-D_j^w) \int_{(0, \infty)^J} \exp \left( -\sum_{j \in J} s_j D_j^w \right) \left\{ \sum_{K \subset J} (-1)^K \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right)^{-\sigma^w} \varphi_{q, K}^w (s) \right\} ds^J,
\end{equation}
where \( \sigma^w := \sum_{v \in A_w} D^w_j \). Notice that if the Taylor expansion of the integrand could be exchanged with the integral in (4.101), then the result would be the formal power series \( \hat{G} \) of (4.101). We also define the function \( I: (\mathbb{R}_{>0})^{A_w} \to \mathbb{R} \) by

\[
I(D) := \prod_{j \in A_w} D^w_j \prod_{j \in A_w} \left( \frac{r_w}{r_j} \right) D^w_j \prod_{j \in A_w} \frac{\Gamma \left( D^w_j \right) \prod_{j \in A_w \cap \tau_v} \Gamma \left( D^w_j + p_j \right)}{\Gamma \left( \sigma^w + l - p_w \right)},
\]

where \( \Gamma(\bullet) \) is the gamma function.

**Lemma 4.8.** (cf. [AGIS20 Section 4.2]) One has \( G(D) = I(D) \) as functions on \((\mathbb{R}_{>0})^{A_w}\).

**Proof.** In (4.101), one can interchange the integration and summation concerning \( K \) because of the factor \( \exp \left( -\sum_{j \in J} s_j D^w_j \right) \). By doing it and integrating with respect to \( s_j \) \((j \in J \setminus K)\), it turns out that (4.101) is equal to

\[
\sum_{\{q\} \cup J \subseteq A_w} (-1)^{|J|/|K|} D^w \prod_{k \in K} \left( \int_{0, \infty} e^{-\sum_{k \in \tau_v} s_k D^w_k} \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right)^{-\sigma^w} \varphi^w_{q,K}(s) ds \right).
\]

For a subset \( K \subseteq A_w \setminus \{q\} \) such that \( \{w, q\} \cup K \) contains all vertices of \( \tau_v \), one has

\[
\sum_{K \subseteq J \subseteq A_w \setminus \{q\}} (-1)^{|J|/|K|} = \begin{cases} 1 & K = A_w \setminus \{q\} \\ 0 & \text{otherwise} \end{cases},
\]

where the sum is taken over \( J \). Hence, (4.106) is equal to

\[
\prod_{k \in A_w} D^w_k \prod_{\{q\} \cup J = A_w} \int_{0, \infty} e^{-\sum_{k \in \tau_v} s_k D^w_k} \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right)^{-\sigma^w} \varphi^w_{q,K}(s) ds \cdot H(D).
\]

One can write

\[
G(D) = \frac{\prod_{k \in \tau_v \cap \tau_q} \frac{j^p_k}{(r_w)^{l-p_w}} \prod_{k \in A_w} D^w_k}{\prod_{k \in A_w} D^w_k} \cdot H(D),
\]

with

\[
H(D) := \sum_{\{q\} \cup K = A_w} \int_{0, \infty} e^{-\sum_{k \in \tau_v} s_k D^w_k} \cdot e^{-\sum_{k \in \tau_v \cap \tau_q} s_k (D^w_k + p_k)} \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} e^{-s_k} \right)^{-\sigma^w - l + p_w} ds K.
\]

In order to compute \( H(D) \), we consider the tropical projective space \( TP^{A_w|w|-1} \) defined by

\[
TP^{A_w|w|-1} := \left( (\mathbb{R}_{\geq 0})^{A_w \setminus \{0\}} \right) / \mathbb{R}_{>0},
\]

where \( \mathbb{R}_{>0} \) acts on \((\mathbb{R}_{\geq 0})^{A_w \setminus \{0\}}\) diagonally by scalar multiplication. Let \( \{u_j\}_{j \in A_w} \) denote its homogeneous coordinates. We also consider the forms

\[
\prod_{j \in A_w \setminus \{q\}} d \log \frac{u_j}{u_q} = \prod_{j \in A_w \setminus \{q\}} d \log t_j
\]

with \( q \in A_w \), each of which is defined on the affine chart

\[
TP^{A_w|w|-1} \setminus \{u_q = 0\} \xrightarrow{\cong} (\mathbb{R}_{\geq 0})^{A_w|w|-1}, \quad \{u_j\}_{j \in A_w} \mapsto \left\{ t_j := \frac{u_j}{u_q} \right\}_{j \in A_w \setminus \{q\}}.
\]

They agree with each other on the overlap, and define a volume form on \( TP^{A_w|w|-1} \), which will be denoted by \( d\text{vol} \) (cf. [AGIS20 Section 4.2]).

**Lemma 4.9.** (cf. [AGIS20 Lemma 4.2]) One has

\[
H(D) = \int_{TP^{A_w|w|-1}} \frac{\left( \prod_{j \in A_w} u_j^p \right) \cdot \left( \prod_{j \in A_w \cap \tau_q} u_j^p \right)}{(\sum_{j \in A_w} \frac{r_j}{r_w} u_j)^{2\sigma^w + l - p_w}} d\text{vol}.
\]
Proof. When we change the variables \( s_k \) to \( t_k := e^{-s_k} (k \in K) \), we have

\[
H(D) = \sum_{\{q\} \cup K = A_w} \int_{[0,1]^K} \left( \prod_{k \in K \setminus \tau_v} \frac{D^w_i}{t_k^w} \right) \cdot \left( \prod_{k \in K \cap \tau_v} \frac{D^{w + p_k}}{t_k} \right) \prod_{k \in K} \frac{dt_k}{t_k}. \tag{4.115}
\]

By (4.113), we write the integrals over \([0,1]^K \subset (\mathbb{R}_{\geq 0})^{|A_w| - 1}\) in (4.115) as integrals on \(T^{|A_w| - 1}\) in the homogeneous coordinates \(\{u_j\}_{j \in A_w}\). Then we get

\[
H(D) = \sum_{\{q\} \cup K = A_w} \int_{R_q} \left( \prod_{k \in K \setminus \tau_v} \frac{D^w_i}{u_k^w} \right) \cdot \left( \prod_{k \in K \cap \tau_v} \frac{D^{w + p_k}}{u_k} \right) \prod_{k \in K} \frac{u_k^{p_k}}{dvol}, \tag{4.116}
\]

where \(R_q := \left\{ \{u_j\}_{j \in A_w} \in T^{|A_w| - 1} \mid u_q = \max \{u_j \mid j \in A_w\} \right\}\) is the preimage of \([0,1]^K\) by the map (4.113). This is equal to

\[
\sum_{\{q\} \cup K = A_w} \int_{R_q} \left( \prod_{k \in K \setminus \tau_v} \frac{D^w_i}{u_k^w} \right) \cdot \left( \prod_{k \in K \cap \tau_v} \frac{D^{w + p_k}}{u_k} \right) \prod_{k \in K} \frac{u_k^{p_k}}{dvol} + \sum_{\{q\} \cup K = A_w} \int_{R_q} \left( \prod_{k \in K \setminus \tau_v} \frac{D^w_i}{u_k^w} \right) \cdot \left( \prod_{k \in K \cap \tau_v} \frac{D^{w + p_k}}{u_k} \right) \prod_{k \in K} \frac{u_k^{p_k}}{dvol}. \tag{4.117}
\]

This coincides with the right hand side of (4.113). \(\Box\)

We rewrite the right hand side of (4.114) as an integral over the simplex

\[
\nabla' := \left\{ \{u_j\}_{j \in A_w} \in (\mathbb{R}_{\geq 0})^{|A_w| - 1} \mid \sum_{j \in A_w} \frac{p_j}{r_j} u_j = 1 \right\},
\]

which is a slice of the diagonal action on \((\mathbb{R}_{\geq 0})^{|A_w| - 1}\). On the simplex \(\nabla'\), we have

\[
\frac{du_q}{u_q} = -\frac{\sum_{j \in A_w \setminus \{q\}} r_j du_j}{r_w - \sum_{j \in A_w \setminus \{q\}} r_j u_j},
\]

and

\[
dvol = \prod_{j \in A_w \setminus \{q\}} \left( \frac{du_j}{u_j} - \frac{du_q}{u_q} \right) = \left( 1 + \frac{\sum_{j \in A_w \setminus \{q\}} r_j u_j}{r_w - \sum_{j \in A_w \setminus \{q\}} r_j u_j} \right) \prod_{j \in A_w \setminus \{q\}} \frac{du_j}{u_j} = \frac{r_w}{r_q} \prod_{j \in A_w \setminus \{q\}} \frac{du_j}{u_j},
\]

where \(q \in A_w\). Hence, we get

\[
\int_{T^{|A_w| - 1}} \left( \prod_{j \in A_w \setminus \{q\}} \frac{D^w_i}{u_j^w} \right) \cdot \left( \prod_{j \in A_w \cap \tau_v} \frac{D^{w + p_j}}{u_j} \right) \phi_{\nabla'} \prod_{j \in A_w \setminus \{q\}} du_j = \frac{r_w}{r_q} \int_{\nabla'} \left( \prod_{j \in A_w \setminus \tau_v} \frac{D^{w - 1}}{u_j} \right) \left( \prod_{j \in A_w \cap \tau_v} \frac{u_j^{p_j}}{u_j} \right) \prod_{j \in A_w \setminus \{q\}} du_j. \tag{4.121}
\]

If we change the variables to \(v_j := \frac{r_w}{r_j} u_j\), then this is equal to

\[
\prod_{j \in A_w} \left( \frac{r_w}{r_j} \right)^{D^w_j} \prod_{j \in A_w \cap \tau_v} \left( \frac{r_w}{r_j} \right)^{p_j} \int_{\nabla} \left( \prod_{j \in A_w \setminus \tau_v} \frac{D^{w - 1}}{v_j} \right) \left( \prod_{j \in A_w \cap \tau_v} \frac{v_j^{p_j}}{v_j} \right) \prod_{j \in A_w \setminus \{q\}} dv_j, \tag{4.122}
\]

where \(\nabla := \left\{ \{v_j\}_{j \in A_w} \in (\mathbb{R}_{\geq 0})^{|A_w| - 1} \mid \sum_{j \in A_w} v_j = 1 \right\}\). By using the Dirichlet integral [LD39], we obtain

\[
H(D) = \prod_{j \in A_w} \left( \frac{r_w}{r_j} \right)^{D^w_j} \prod_{j \in A_w \cap \tau_v} \left( \frac{r_w}{r_j} \right)^{p_j} \prod_{j \in A_w \setminus \tau_v} \Gamma \left( \frac{D^w_j}{\Gamma} \right) \prod_{j \in A_w \cap \tau_v} \Gamma \left( \frac{D^w_j + p_j}{\Gamma} \right). \tag{4.123}
\]
By this and \((4.109)\), we get Lemma 4.8.

**Lemma 4.10.** (cf. AGIS20, Lemma 4.3) For a fixed \(D \in (\mathbb{R}_{>0})^{A_w}\), we have the asymptotic expansion

\[
G(yD) \sim \hat{G} \bigg|_{D_j \to yD_j} \quad (y \to +0),
\]

where \(\hat{G} \bigg|_{D_j \to yD_j}\) means the substitution of \(yD_j\) to \(D_j\) in the formal power series \(\hat{G}\) defined in \((4.101)\).

**Proof.** By changing the variables \(X_j := e^{-s_j}\), we have

\[
G(yD) = \sum_{\{q\} \cup J \subset A_w} y D^w_j \prod_{j \in J} (-y D^w_j) \int_{[0,1]^J} \prod_{j \in J} X_j^{-1} \cdot g_j(X, y) \, dX,
\]

where

\[
g_j(X, y) := \prod_{j \in J} X_j^{y D^w_j} \cdot h_j(X, y)
\]

\[
h_j(X, y) := \sum_{K \subset J} (-1)^K \left( \frac{r_q}{r_w} + \sum_{k \in K} \frac{r_k}{r_w} X_k \right)^{-y r_w} \prod_{k \in \tau \cap K} (r_k X_k)^{p_k} \prod_{k \in \tau \cap K} (r_k X_k)^{l-p_k}.
\]

The first sum in \((4.127)\) is taken over all \(K \subset J\) such that all vertices of \(\tau\) are contained in \(\{w, q\} \cup K\). Recall that if the Taylor expansion of the integrand could be exchanged with the integral in \((4.104)\), the result would be the formal power series \(\hat{G}\) of \((4.101)\). Therefore, it suffices to show that we can exchange the Taylor expansion of \(\prod_{j \in J} X_j^{-1} \cdot g_j(X, y)\) in \(y\) with the integral over \([0,1]^J\) to get the asymptotic expansion. By Taylor’s theorem, we have

\[
g_j(X, y) = \sum_{i_0=0}^{i_0-1} \frac{1}{i!} \frac{\partial^i g_j}{\partial y^i} (X, 0) \cdot y^i + \frac{1}{i_0!} \frac{\partial^i g_j}{\partial y^{i_0}} (X, \xi(X)) \cdot y^{i_0}, \quad \exists \xi(X) \in [0, y]
\]

for any \(i_0 \in \mathbb{Z}_{>0}\). It suffices to show

- the function \(\prod_{j \in J} X_j^{-1} \cdot \frac{1}{i_0!} \frac{\partial^i g_j}{\partial y^i} (X, 0)\) is integrable on \([0,1]^J\) for any \(i \in \mathbb{Z}_{>0}\), and
- \(\prod_{j \in J} X_j^{-1} \cdot \frac{1}{i_0!} \frac{\partial^i g_j}{\partial y^{i_0}} (X, y)\) is bounded by an integrable function on \([0,1]^J\), which is independent of \(y \in [0, 1]\) for any \(i_0 \in \mathbb{Z}_{>0}\).

We will show these two claims.

Concerning the former claim, by a straightforward computation, one can see that each term \(\frac{\partial^i g_j}{\partial y^i} (X, 0)\) is a linear combination of products of \(\log^h X_j (h \in \mathbb{Z}_{>0})\) and \(g^w_{h,j}(X)\) of \((4.98)\). Since \(\log^h X_j\) is integrable on \([0,1]\), we can get the former claim by \((4.99)\).

The latter claim can be shown as follows: The function \(h_j(X, y)\) is analytic in a neighborhood of \([0,1]^J \times [0,1]\), and vanishes along the coordinate hyperplanes \(\{X_j = 0\} (j \in J)\) by the same reason as \(g^w_{h,j}(X)\) (see the proof of Lemma 4.7). Hence, the function \(\prod_{j \in J} X_j^{-1} \cdot h_j(X, y)\) is a smooth function in a neighborhood of \([0,1]^J \times [0,1]\). We can see that there exist smooth functions \(f_i(X, y)\) in a neighborhood of \([0,1]^J \times [0,1]\) such that

\[
\frac{\partial^i}{\partial y^{i_0}} \left( \prod_{j \in J} X_j^{y D^w_j} \cdot \prod_{j \in J} X_j^{-1} \cdot h_j(X, y) \right) = \sum_{i_0=0}^{i_0} f_i(X, y) \cdot \left( \sum_{j \in J} D^w_j \log X_j \right)^i \prod_{j \in J} X_j^{y D^w_j}.
\]

Thus we have

\[
\left| \prod_{j \in J} X_j^{-1} \cdot \frac{1}{i_0!} \frac{\partial^i g_j}{\partial y^{i_0}} (X, y) \right| \leq C \cdot \sum_{i_0=0}^{i_0} \left| \sum_{j \in J} D^w_j \log X_j \right|^i
\]

for \((X, y) \in (0,1]^J \times (0,1]\), where \(C > 0\) is some constant. The right hand side is integrable on \([0,1]^J\), and the lemma follows. \(\square\)
By using the identity $\Gamma(x + m) = \Gamma(x) \cdot \prod_{i=0}^{m-1}(x + i)$ for $x \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$, we can see that (4.105) is equal to

$$
\prod_{j \in A_w} \left( \frac{r_w}{p_j} \right)^{D_j^w} \left( \prod_{j \in A_w \cap \tau_w} \prod_{i=0}^{p_j-1} (D_j^w + i) \right) \left( \prod_{i=1}^{l-p_w-1} \frac{1}{\sigma^w + i} \right) \prod_{j \in A_w} \frac{\Gamma(D_j^w + 1)}{\Gamma(\sigma^w + 1)}.
$$

Let $\widehat{I}$ denote the formal power series in the variables $\{ D_j^w \}_{j \in A_w}$, which one can get by applying the power series expansion (1.14) of $\Gamma(1 + x)$ to (4.131). For a fixed $D \in (\mathbb{R}_{>0})^{A_w}$, we have the asymptotic expansion (Taylor expansion)

$$I(yD) \sim \left. \widehat{I} \right|_{D_j \rightarrow yD_j} \quad (y \rightarrow +0),$$

where $\left. \widehat{I} \right|_{D_j \rightarrow yD_j}$ also means the substitution of $yD_j$ to $D_j$ in $\widehat{I}$. By combining this, Lemma 4.8 and Lemma 4.10 one can obtain $\widehat{G} = \widehat{I}$ as formal power series in $\{ D_j^w \}_{j \in A_w}$. By substituting $\widehat{I}$ to $\widehat{G}$ in (4.103), we obtain (1.15) in the case of $\operatorname{conv} (\{ w \} \cup \tau_w) \in \mathcal{T}$.

Lastly, we show the following lemma that we used in the above proof.

**Lemma 4.11.** (cf. [AGLS20 Section 5.4]) The volume of $S_t^w$ with respect to the Riemannian metric induced from the Euclidean metric on the ambient space $\mathbb{R}_x \times D_e$ is bounded as $t \rightarrow +0$.

**Proof.** We define the function $\tilde{f}_{t, w, \varepsilon^0} : N_{\mathbb{R}} \times S_{\varepsilon^0}^1 \rightarrow \mathbb{R}$ by

$$
\tilde{f}_{t, w, \varepsilon^0} (n, \varepsilon_0 e^{-t\theta}) := \tilde{f}_{t} (i_t(n)) - \left| 1 + \varepsilon_0 e^{-t\theta} \right|.
$$

Then the volume form on $S_t^w$ is given by

$$
\left| d\tilde{f}_{t, w, \varepsilon^0} \right| \frac{\Lambda_{i=0}^d dn_i \wedge \varepsilon_0 d\theta}{d\tilde{f}_{t, w, \varepsilon^0}},
$$

where $(n_0, \ldots, n_d)$ are $\mathbb{R}$-coordinates on $N_{\mathbb{R}} \cong \mathbb{R}^{d+1}$. We can easily see

$$
\frac{\partial \tilde{f}_{t, w, \varepsilon^0}}{\partial n_i} = O (- \log t), \quad \frac{\partial \tilde{f}_{t, w, \varepsilon^0}}{\partial \theta} = O (1).
$$

Hence, we have

$$
\int_{S_t^w} \left| d\tilde{f}_{t, w, \varepsilon^0} \right| \frac{\Lambda_{i=0}^d dn_i \wedge \varepsilon_0 d\theta}{d\tilde{f}_{t, w, \varepsilon^0}} \leq C (- \log t) \int_{S_t^w} \frac{\Lambda_{i=0}^d dn_i \wedge d\theta}{d\tilde{f}_{t, w, \varepsilon^0}} = C (- \log t) \int_0^{2\pi} d\theta \int_{S_{t, w, \varepsilon^0}^\theta} \frac{\Lambda_{i=0}^d dn_i}{d \tilde{f}_{t, w, \varepsilon^0} (i_t)}
$$

for some constant $C > 0$, where $S_{t, w, \varepsilon^0}^\theta := \{ n \in N_{\mathbb{R}} | \tilde{f}_{t, w, \varepsilon^0} (i_t(n)) = \left| 1 + \varepsilon_0 e^{-t\theta} \right| \}$.

We fix $\theta \in S^1$ and apply Theorem 1.1 to the polynomial $g_{t}^\theta$ defined by

$$
g_{t}^\theta := \frac{1}{1 + \varepsilon_0 e^{-t\theta}} \tilde{f}_{t} - 1.
$$

If we set $l = 1$, $v = w = 0$, then

$$
\omega_{t}^{1, 0} = - \left( \prod_{i=0}^d dz_i \right) \frac{1}{g_{t}^\theta}, \quad C_{t}^{0} = i_t \left( S_{t, w, \varepsilon^0}^\theta \right), \quad \int_{C_{t}^{0}} \Omega_{t}^{1, 0} = O \left( (- \log t)^d \right).
$$

Hence, we get

$$
\int_{C_{t}^{0}} \Omega_{t}^{1, 0} = - (\log t)^{d+1} \int_{S_{t, w, \varepsilon^0}^\theta} \frac{\Lambda_{i=0}^d dn_i}{d \left( g_{t}^\theta \circ i_t \right)} = O \left( (- \log t)^d \right).
$$

From this and $d \left( \tilde{f}_{t, w, \varepsilon^0} \circ i_t \right) = \left| 1 + \varepsilon_0 e^{-t\theta} \right| \cdot d \left( g_{t}^\theta \circ i_t \right)$, we can see that (4.136) is bounded by a constant. We obtained the lemma.

Indeed we used this lemma in order to prove Theorem 1.1 when we ignore the perturbation $\delta$ of the cycle in (1.49). However, the perturbation is unnecessary for the above integral $\int_{C_{t}^{0}} \Omega_{t}^{1, 0}$, since the cycle $C_{t}^{0}$ is given as the positive real locus of $\{ g_{t}^\theta = 0 \}$, and we do not need to consider its parallel transport in (1.6) for constructing $C_{t}^{0}$. Therefore, there is no circular reasoning.
5. Proof of (1.46)

Let $\sigma \in \mathcal{P}$ be a cell of dimension $d$ in the tropical hypersurface $X$ (trop$(f)$) whose dual edge $\tau_\sigma$ in $\mathcal{I}$ contains an element in $W$ as its vertex. Let $w$ denote the vertex of $\tau_\sigma$ in $W$, and $m_0 \in A$ denote the other vertex. (The element $m_0$ may also be in $W$.) They satisfy trop$(f) = \mu_w = \mu_{m_0}$ on $\sigma$. As before, we define $\alpha := \mu_{m_0} - \mu_w$, and take a collection of integral linear functions $\{\gamma_j\}_{j \in J}$ so that $\{\alpha, \gamma_j \mid j \in J\}$ forms an integral affine coordinate system on $N_C$. We use the corresponding coordinate system on $N_{C*}$ given by

\begin{align*}
y := t^{\lambda_{m_0} - \lambda_w} z^{m_0 - w}, \quad x_j := z^{\gamma_j} \quad (j \in J).
\end{align*}

Take a point $n_0 \in \text{Int}(\sigma)$ and its small neighborhood $U$ in $N_\mathbb{R}$ such that $\mu_m(n) - \mu_m(n_0) \geq \epsilon$ for any $n \in U$ and $m \in A \setminus \{w, m_0\}$. We set

\begin{align*}
\tilde{Z}_t := \{(z, x) \in N_{C*} \times D_\varepsilon \mid f_t^w(z) = 1 + x\},
\end{align*}

and consider the map $\text{Log}_t : N_{C*} \rightarrow N_\mathbb{R}$ of (1.40). On $\tilde{Z}_t \cap (\text{Log}_t^{-1}(U) \times D_\varepsilon)$, we have

\begin{align*}
1 + x = \frac{\epsilon_m}{c_w} y(1 + h_t(z)),
\end{align*}

where $h_t(z) = O(t^c)$. This equation can be used to write $y$ as a function $y(x_j, x)$ of the variables $x_j, x$. We consider the map

\begin{align*}
\iota_t : (S^1)^d \times D_\varepsilon \rightarrow \tilde{Z}_t \cap (\text{Log}_t^{-1}(U) \times D_\varepsilon), \quad (e^{\sqrt{-1} \theta_1}, \ldots, e^{\sqrt{-1} \theta_d}, x) \mapsto (y(x_j, x), x_j, x),
\end{align*}

where $x_j = t^{\gamma_j} e^{\sqrt{-1} \theta_j}$. We set

\begin{align*}
S_t^0 := \iota_t \left((S^1)^d \times S^1_{\epsilon_0}\right) \subset (\text{Log}_t^{-1}(U) \setminus Z_t) \times S^1_{\epsilon_0},
\end{align*}

\begin{align*}
T_t^0 := \iota_t \left((S^1)^d \times \{0\}\right) \subset Z_t \times \{0\}.
\end{align*}

$T_t^0$ is the cycle mentioned in Section (1.1) and $S_t^0$ is a tube over it.

Suppose that $\tau_\sigma \in \mathcal{I}$ is either $\tau_\sigma$ or one of the vertices $w, m_0$. One can write $v = p_w w + p_m m_0$ with $p_w, p_m \in \mathbb{Z} \cap [0, l]$ such that $p_w + p_m = l$. By (5.3), one can get

\begin{align*}
\omega_t^{l,v} &= (1 + O(t^c)) \cdot (-1)^l \cdot \left(\frac{\epsilon_m}{c_w}\right)^{p_m} \cdot \frac{1}{f_t^w = 1} \cdot dy \wedge dx_j \quad (j \in J),
\end{align*}

on $\tilde{Z}_t \cap (\text{Log}_t^{-1}(U) \times D_\varepsilon)$. Hence, we get

\begin{align*}
\int_{T_t^0} \Omega_t^{l,v} = \frac{1}{2\sqrt{-1}} \int_{S_t^0} \omega_t^{l,v} = (2\pi\sqrt{-1})^d \cdot (-1)^{p_m} \cdot \left(\frac{p_m - 1}{l - 1}\right) + O(t^c)
\end{align*}

\begin{align*}
\iota_t : (S^1)^d \times D_\varepsilon \rightarrow \tilde{Z}_t \cap (\text{Log}_t^{-1}(U) \times D_\varepsilon), \quad \begin{cases} 
- \left(2\pi\sqrt{-1}\right)^d + O(t^c) & \tau_v = w \\left(2\pi\sqrt{-1}\right)^d + O(t^c) & \tau_v = m_0 \\
O(t^c) & \text{otherwise}.
\end{cases}
\end{align*}

Next, suppose that $\tau_v \in \mathcal{I}$ is neither $\tau_\sigma$ nor the vertices $w, m_0$. One can write $v = \sum_{m \in A \cap \tau_v} p_m m$ with $p_m \in \mathbb{Z} \cap [0, l]$ such that $\sum_{m \in A \cap \tau_v} p_m = l$. Again by (5.3), one can get

\begin{align*}
\omega_t^{l,v} &= (1 + O(t^c)) \cdot \prod_{m \in A \cap \tau_v} \left(-\frac{c_m}{c_w}\right)^{p_m} \cdot \frac{t^{\lambda_{l,v} z^w}}{(t^{\lambda_{l,v} z^w} - 1)} \cdot \frac{1}{f_t^w = 1} \cdot dy \wedge dx_j \quad (j \in J),
\end{align*}

on $\tilde{Z}_t \cap (\text{Log}_t^{-1}(U) \times D_\varepsilon)$, where $\lambda_{l,v} := \sum_{m \in A \cap \tau_v} p_m \lambda_m$. We get

\begin{align*}
\int_{T_t^0} \Omega_t^{l,v} = \frac{1}{2\sqrt{-1}} \int_{S_t^0} \omega_t^{l,v} = O(t^c).
\end{align*}

If $m_0 \notin W$, then the second case $\tau_v = m_0$ in (5.10) can not happen since $v$ is in the interior of $l \cdot \Delta$. Therefore, $\int_{T_t^0} \Omega_t^{l,v} = O(t^c)$ unless $\tau_v \in W$ and trop$(f) = \mu_{\tau_v}$ on $\sigma$. If $\tau_v \in W$ and trop$(f) = \mu_{\tau_v}$ on $\sigma$, 

\[ \int_{T_t^0} \Omega_t^{l,v} = O(t^c). \]
then \( \tau_v \) is a vertex of \( \tau_{\sigma} \), and we get \( \int_{T^w} \Omega_{t,w} = -(2\pi \sqrt{-1})^d + O(t^e) \) when we choose \( \tau_v \) as the above vertex \( w \in W \). We obtained (1.19).

**Remark 5.1.** The orientation of \( T^w \) that we used is the one determined by the above ordered coordinates \( \{ \theta_j \}_{j \in J} \). In the above computation, if \( m_0 \in W \) and we swap \( w \) and \( m_0 \), then the orientation of \( T^w \) also switches. The orientation of \( B^w \) that we used when we compute the integral over \( C_t^w \) is the one defined by the interior product of the standard volume form on \( N_{k} \). We choose orientations of the cycles in these ways, the intersection number of \( \tau_v \in \Sigma \) by \( (6.3) \) is \( \Omega_{t,w} \). Notice that since \( \log D_{j,v} \) \( \in \mathbb{R} \), the standard orientation of \( Z_t \cap \text{Log}_t^{-1}(U) \) is given by \( \frac{\langle \wedge_{j \in J} \partial \theta_j \rangle \wedge \langle \wedge_{j \in J} \partial \gamma_j \rangle}{\wedge_{j \in J} (\partial \theta_j \wedge \partial \gamma_j)} = (-1)^{d(d-1)/2} \).

Notice that since \( d(t^e) = t^e \log t \cdot d\gamma_j \) and \( t^e \log t < 0 \), the standard orientation of \( Z_t \) \( \in \mathbb{R} \) is \( \Omega_{t,w} \).

6. **LEADING TERMS OF PERIODS**

From Theorem 1.1, we can see that the affine volumes of bounded cells in the tropical hypersurface \( X(trop(f)) \) appear in the leading terms of periods \( \int_{C_t^w} \Omega_{t,w} \). Suppose \( \{ w \} \cup \tau_v \in \mathcal{T} \). For \( m \in A_w \), let \( \sigma_m \in \mathcal{P} \) be the cell of dimension \( d \) in the tropical hypersurface \( X(trop(f)) \), which is dual to \( \text{conv}(\{ w, m \}) \in \mathcal{T} \). We also let \( \sigma_v \in \mathcal{P} \) denote the cell that is dual to \( \text{conv}(\{ w \} \cup \tau_v) \in \mathcal{T} \).

**Corollary 6.1.** The leading term of \( \int_{C_t^w} \Omega_{t,w} \) is given by

\[
(6.1) \quad (-1)^{d+1} (-\log t)^d \cdot \sum_{m \in A_w} \text{vol}(\sigma_m)
\]

when \( \tau_v = w \), and by

\[
(6.2) \quad (-1)^d (-\log t)^{\dim \sigma_v} \cdot \prod_{j \in A_w \cap \tau_v} (p_j - 1)! \cdot \text{vol}(\sigma_v)
\]

when \( \tau_v \notin w \), where \( \text{vol} \) denotes the affine volume of the polytope.

**Proof.** When \( \tau_v = w \), we have \( p_w = l \) and \( E_{v,w} = \prod_{i=0}^{l-1} (\sigma^w - i) \). From (1.15), we can see that the leading term of \( \int_{C_t^w} \Omega_{t,w} \) is given by

\[
(6.3) \quad \frac{(-1)^{d+1}}{(l-1)!} \prod_{i=1}^{l} (-i) \cdot \int_{Y_w} \exp((-\log t)\omega^w \cdot \sigma^w = (-1)^{d+1} \sum_{m \in A_w} \int_{Y_w} \exp((-\log t)\omega^w \cdot D^w).
\]

When \( \tau_v \notin w \), we have \( p_w = 0 \) and \( E_{v,w} = \prod_{j \in A_w \cap \tau_v} (p_j - 1)! \cdot \prod_{j \in A_w \cap \tau_v} D^w_j \). The leading term of \( \int_{C_t^w} \Omega_{t,w} \) is

\[
(6.4) \quad \frac{(-1)^d}{(l-1)!} \prod_{j \in A_w \cap \tau_v} (p_j - 1)! \cdot \int_{Y_w} \exp((-\log t)\omega^w \cdot \sigma^w).
\]

The cells \( \sigma_m \) and \( \sigma_v \) are contained in the polytope \( \nabla^w \) of (3.4) as its faces. The normal fan of \( \nabla^w \) is \( \Sigma^w \), and the faces \( \sigma_m \) and \( \sigma_v \) correspond to the strata \( D^w_m \) and \( \nabla_{j \in A_w \cap \tau_v} \) in the toric variety \( Y_w \) respectively. By [Gui94, Theorem 2.10] again, one can get

\[
(6.5) \quad \int_{Y_w} \exp(\omega^w \cdot D^w_m) = \text{vol}(\sigma_m), \quad \int_{Y_w} \exp(\omega^w \cdot \prod_{j \in A_w \cap \tau_v} D^w_j = \text{vol}(\sigma_v).
\]

By using these, we obtain the claim.

**Remark 6.2.** When \( d = 1, l = 1 \), the leading terms (6.1) and (6.2) are written as \( (-\log t) \cdot l(w,w) \) and \( -(-\log t) \cdot l(v,w) \) respectively, where \( l(w,w), l(v,w) \) are the tropical periods of the tropical curve \( X(trop(f)) \), which we recalled in Section 1.2. The fact that the leading terms of the periods of a degenerating family of (plane) curves are given by the tropical periods of the tropical curve obtained by tropicalization was first observed by Iwao [Iwa10]. Lang [Lan20] also studied the leading terms of the periods of a degenerating family of curves under the tropical limit in a more general setup. See Remark 3.7 in loc. cit. It is also known that the valuation of the \( j \)-invariant of an elliptic curve over a non-archimedean valuation field coincides with the affine length of the cycle in the tropical elliptic curve obtained by tropicalization [KMM08, KMM09].
7. Example

Let $d = 2$, and fix a basis $\{e_1, e_2, e_3\}$ of the lattice $M \cong \mathbb{Z}^3$. As an example, we consider the polynomial

$$f_t(z) := -t^{\lambda_0} z^0 + \sum_{m \in (\Delta \cap M) \setminus \{0\}} t^{\lambda_m} z^m = -1 + \sum_{m \in (\Delta \cap M) \setminus \{0\}} t^{\lambda_m} z^m$$

with

$$\Delta := \text{conv} \{\{2e_1, -e_1, e_2, -e_2, e_3, -e_3\}\} \subset M_{\mathbb{R}}, \quad \lambda_m := \begin{cases} 3 & m = 2e_1 \\ 0 & m = 0 \\ 1 & m \in (\Delta \cap M) \setminus \{0, 2e_1\}. \end{cases}$$

As before, let $\mathcal{T}$ denote the triangulation of $\Delta$ induced by $\{\lambda_m\}_{m}$, and $\mathcal{P}$ denote the polyhedral decomposition of $N_{\mathbb{R}}$ induced by the tropical hypersurface obtained by the tropicalization of (7.1). Let us consider the integral of the residue $\Omega^v_t$ with $t = 1, v = e_1$ over the cycle $C^0_t$ with $w = 0$. The cycle $C^0_t$ is given as the positive real locus of $\{f_t(z) = 0\}$, and $p_w = 0, p_{e_1} = 1$. According to Theorem 1.1, we have

$$\int_{C^0_t} \Omega_{t_1}^{1,e_1} = \int_{Y_0} t^{-\omega^0} \cdot D_{e_1}^0 \cdot \hat{\Gamma}_0 + O \left( t^\epsilon \right),$$

where $Y_0 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, D_{e_1}^0 = \{0\} \times \mathbb{P}^1 \times \mathbb{P}^1$, and

$$\omega^0 = \sum_{m \in A_0} D_m^0 = \sigma^0, \quad \hat{\Gamma}_0 = \exp \left( \sum_{k \geq 2} \left( \sum_{m \in A_0} (D_m^0)^k - (\sigma^0)^k \right) \right).$$

Here $A_0$ denotes $A_w$ of (1.11) with $w = 0$. It turns out that we have

$$\int_{C^0_t} \Omega_{t_1}^{1,e_1} = (-\log t)^2 \vol (\sigma_1) - 4\zeta(2) + O \left( t^\epsilon \right)$$

$$= 4 \left( -\log t^2 - 4\zeta(2) + O \left( t^\epsilon \right) \right),$$

where $\sigma_1 \in \mathcal{P}$ is the 2-cell dual to $\text{conv} \{\{0, e_1\}\} \in \mathcal{T}$. Here we used (6.3). In order to illustrate (1.15), we give a simpler computation for getting this than that given in the proof of (1.15) in the following.

We fix small $\epsilon > 0$. For each pair of an element $q \in (\Delta \cap M) \setminus \{0\}$ and a subset $K \subset (\Delta \cap M) \setminus \{0, q\}$, we set

$$B_{t, q, K}^0 := \left\{ n \in B_t^0 \mid \mu_k(n) - \mu_q(n) \in [0, \epsilon], \forall k \in \{q\} \cup K, \mu_m(n) - \mu_q(n) \geq \epsilon, \forall m \in (\Delta \cap M) \setminus \{(0, q) \cup K\} \right\},$$

where $B_t^0 := \{n \in N_{\mathbb{R}} \mid f_t(i_t(n)) = 0\}$ and $\mu_m(n) := \lambda_m + \langle m, n \rangle \ (m \in (\Delta \cap M) \setminus \{0\})$ as before. One has $B_{t, q, K}^0 \neq \emptyset$ if and only if the convex hull of $\{0, q\} \cup K$ is contained in $\mathcal{T}$. These sets $B_{t, q, K}^0$ give a decomposition of the set $B_t^0$, which is shown in Figure 7.1.

![Figure 7.1. The decomposition of the set $B_t^0$](image)

Let $\alpha, \beta_k, \gamma_j$ be the coordinates that we considered in (4.3), and set

$$x_q := t^{\lambda_q - \lambda_0} q^0 = t^{\lambda_q} z^q, \quad x_k := t^{\lambda_k - \lambda_0} z_k^k \ (k \in K), \quad x_j := z^j \ (j \in J).$$
On $i_t \left( B_{t}^{0,w,K} \right)$, one can rewrite $f_t(z) = 0$ as

$$1 = x_q \left( 1 + \sum_{k \in K} x_k + h_t(x) \right),$$

with a function $h_t(x)$ satisfying

$$h_t = O \left( t^\epsilon \right), \quad z^m \frac{\partial h_t}{\partial z^m} = O \left( t^\epsilon \right),$$

where $z^m$ is a $C^*$-coordinate of $N_{C^*}$. Concerning the form

$$\Omega^{1,e_t}_t = t^{-\epsilon_1} \frac{\sum_{i=0}^m \log z_i}{df_t} = t^{-\epsilon_1} \frac{\int_{k \in K} d \log x_k \int_{j \in J} d \log x_j}{x_q \frac{\partial h_t}{\partial x_q}},$$

we have

$$x_q \frac{\partial f_t}{\partial x_q} = x_q \left( 1 + \sum_{k \in K} x_k + h_t(x) \right) + x_q^2 \frac{\partial h_t}{\partial x_q} = 1 + O \left( t^\epsilon \right),$$

where we used (7.9), (7.10), and the fact that $0 < x_q < 1$ on the positive real locus $C_t^0$. Hence, we get

$$\Omega^{1,e_t}_t = (1 + O \left( t^\epsilon \right)) t^{-\epsilon_1} \int_{k \in K} d \log x_k \int_{j \in J} d \log x_j$$

and

$$\int_{C_t^0} \Omega^{1,e_t}_t = \int_{B_t^0} \int_{B_t^{0,q,K}} i_t^* \Omega^{1,e_t}_t = \sum_{q,K} \int_{B_t^{0,q,K}} i_t^* \Omega^{1,e_t}_t = (1 + O \left( t^\epsilon \right)) \left( -\log t \right)^2 \sum_{q,K} \int_{B_t^{0,q,K}} i_t^* \left( t^{\epsilon_1} \right) \int_{j \in J} d \beta_k \int_{j \in J} d \gamma_j,$$

where the sum is taken over pairs $(q, K)$ such that the convex hull of $(0, q) \cup K$ is contained in $T$. This is $O \left( t^\epsilon \right)$ unless $e_1 \in \{q \} \cup K$ since otherwise $t^{\epsilon_1} = O \left( t^\epsilon \right)$. We will compute the integral

$$\int_{B_t^{0,q,K}} i_t^* \left( t^{\epsilon_1} \right) \int_{k \in K} d \beta_k \int_{j \in J} d \gamma_j$$

for $(q, K)$ such that $e_1 \in \{q \} \cup K$.

First, consider the case where $\{q \} \cup K$ consists of a single element. In this case, we have $q = e_1, K = \emptyset$, and the set $B_t^{0,e_1,0}$ is the green region of Figure 7.1. We have $\alpha = \mu_{e_1}$, and one can take $\{\gamma_j \} \in J = \{\langle e_2, \bullet \rangle, \langle e_3, \bullet \rangle\}$. We can see that (7.15) is equal to

$$\int_{B_t^{0,q,K}} i_t^* \left( x_q \right) \int_{j \in J} d \gamma_j = \int_{B_t^{0,q,K}} \frac{1}{1 + i_t^* h_t} \int_{j \in J} d \gamma_j = (1 + O \left( t^\epsilon \right)) \int_{B_t^{0,q,K}} \int_{j \in J} d \gamma_j.$$

The set $B_t^{0,e_1,0}$ is defined by $\mu_{\pm e_2} - \mu_q \geq \epsilon (i = 2, 3)$ in $B_t^0$. Since we have

$$\mu_q = -\log t \left( 1 + i_t^* h_t \right) = O \left( t^\epsilon \right),$$

we can see that (7.16) is equal to

$$(1 + O \left( t^\epsilon \right)) \cdot (2 - 2\epsilon + O \left( t^\epsilon \right))^2 = (2 - 2\epsilon)^2 + O \left( t^\epsilon \right).$$

Next, we consider the case where $\{q \} \cup K$ consists of two elements. One of them is $e_1$ and the other one is either $\pm e_2$ or $\pm e_3$. We consider the case $\{q \} \cup K = \{e_1, e_2\}$, i.e., $q = e_1, K = \{e_2\}$ or $q = e_2, K = \{e_1\}$. (We can compute the integrals also for the other cases in the same way, and obtain the same results.) The sets $B_t^{0,q,K}$ of this case are the blue regions of Figure 7.1. When $q = e_1, K = \{e_2\}$, we have $\alpha = \mu_{e_1}$, $\beta_k = \langle e_2 - e_1, \bullet \rangle$, and one can take $\gamma_j := \langle e_3, \bullet \rangle$. (7.17) is equal to

$$\int_{B_t^{0,q,K}} i_t^* \left( x_q \right) d \beta_k \wedge d \gamma_j = \int_{B_t^{0,q,K}} \frac{1}{1 + i_t^* h_t} d \beta_k \wedge d \gamma_j = (1 + O \left( t^\epsilon \right)) \int_{B_t^{0,q,K}} \frac{1}{1 + i_t^* h_t} d \beta_k \wedge d \gamma_j.$$

The set $B_t^{0,e_1,\langle e_2 \rangle}$ is defined by $\beta_k \in [0, \epsilon]$ and $\mu_{\pm e_2} - \mu_q \geq \epsilon$ in $B_t^0$. Since we have

$$(1 + O \left( t^\epsilon \right)) \cdot (2 - 2\epsilon + O \left( t^\epsilon \right))^2 = (2 - 2\epsilon)^2 + O \left( t^\epsilon \right).$$
we can see that (7.19) is equal to

\[(7.21) \quad (1 + O(t^\epsilon)) \int_{0}^{t} \frac{1}{1 + t^\epsilon} \left(2 - 2\epsilon + 2 \log(1 + t^\epsilon) + O(t^\epsilon)\right) db,\]

where \(b\) is a coordinate of \([0, \epsilon]\). Similarly, when \(q = e_2, K = \{e_1\}\), we can see that (7.15) is equal to

\[(7.22) \quad \int_{B^q_{0,K}} \sum_{j \in K} \frac{1}{1 + \sum_{k \in K} i^*_k x_k + i^*_k h_k t} d\beta_k = (1 + O(t^\epsilon)) \int_{[0, \epsilon]^2} \frac{1}{1 + t^{b_1} + t^{b_2}} db_1 db_2,\]

where \((b_1, b_2)\) are coordinates of \([0, \epsilon]^2\). Similarly, when \(q = e_2, K = \{e_1, e_3\}\) and when \(q = -e_3, K = \{e_1, e_2\}\), (7.15) is equal to

\[(7.23) \quad (1 + O(t^\epsilon)) \int_{[0, \epsilon]^2} \frac{t^{b_1}}{1 + t^{b_1} + t^{b_2}} db_1 db_2, \quad (1 + O(t^\epsilon)) \int_{[0, \epsilon]^2} \frac{t^{b_2}}{1 + t^{b_1} + t^{b_2}} db_1 db_2\]

respectively. By combining these, we obtain \(\epsilon^2 + O(t^\epsilon)\).

By summing up all the above integrals, we get

\[(7.24) \quad \Omega_{\epsilon}^{1,e_1} = (1 + O(t^\epsilon)) (-\log t)^2 \left(2 - 2\epsilon\right)^2 + 4 \left(\epsilon(2 - 2\epsilon) - (-\log t)^{-2}\right) + 4\epsilon^2 + O(t^\epsilon)\]

\[(7.25) \quad = 4 (-\log t)^2 - 4\epsilon^2 + O(t^\epsilon),\]

which coincides with (7.6).

8. Polarized logarithmic Hodge structure of curves

We work under the same assumptions and use the same notations as in Section 1.2. In particular, we assume \(d = 1\). We will prove Corollary 1.2. We fix a basis \(\{e_1, e_2\}\) of the lattice \(M \cong \mathbb{Z}^2\). We set

\[(8.1) \quad A := \{(m_0, m_1) \in A \times A \mid m_0 \neq m_1\}\]

\[(8.2) \quad B := \{(m_0, m_1) \in A \mid \text{conv} \{\{m_0, m_1\}\} \in \mathcal{F}\},\]

where \(A := \Delta \cap M\).

**Lemma 8.1.** One can simultaneously choose branches of \(\arg(-c_{m_1}/c_{m_0}) \in \mathbb{R}\) for all pairs \((m_0, m_1) \in A\) so that we have

\[(8.3) \quad \arg \left(-\frac{c_{m_1}}{c_{m_0}}\right) = -\arg \left(-\frac{c_{m_0}}{c_{m_1}}\right)\]

for any pair \((m_0, m_1) \in B\) and its reversed pair \((m_1, m_0) \in B\), and

\[(8.4) \quad \arg \left(-\frac{c_{m_1}}{c_{m_0}}\right) - \arg \left(-\frac{c_{m_2}}{c_{m_0}}\right) = \arg \left(-\frac{c_{m_1}}{c_{m_2}}\right) + \frac{(m_1 - m_0) \wedge (m_2 - m_0)}{e_1 \wedge e_2} \cdot \pi\]

for any (ordered) pair \((m_0, m_1, m_2) \in A \times A \times A\) such that \(\text{conv} \{\{m_0, m_1, m_2\}\} = \text{a 2-cell in } \mathcal{F}\).
Proof. First, note the following: The former condition (8.3) implies that if we choose a branch of \( \arg (-c_{m_1}/c_{m_0}) \) for a pair \((m_0, m_1) \in B\), then that of \( \arg (-c_{m_0}/c_{m_1}) \) for the reversed pair \((m_1, m_0) \in B\) is automatically determined. The latter condition (8.4) implies that for a 2-cell \((m_0, m_1, m_2) \in T\), if we choose branches of \( \arg (-c_{m_1}/c_{m_0}) \) and \( \arg (-c_{m_2}/c_{m_0}) \), then the branch of \( \arg (-c_{m_1}/c_{m_2}) \) is also automatically determined. One can easily check that even if we use (8.4) for the same elements \( m_0, m_1, m_2 \) but with a different order, the argument \( \arg (-c_{m_1}/c_{m_2}) = -\arg (-c_{m_2}/c_{m_1}) \) determined by the choices of \( \arg (-c_{m_1}/c_{m_0}) \) and \( \arg (-c_{m_2}/c_{m_0}) \) is the same.

We first choose branches of \( \arg (-c_{m_1}/c_{m_0}) \) for elements \((m_0, m_1) \in B\). By adding a real constant whose absolute value is small to the value of the function \( \lambda : A \to \mathbb{Q} \) at each \( m \in A \) if necessary, one can make a new function \( \lambda' : A \to \mathbb{R} \) such that it takes different values at all points in \( A \), and still extends to a strictly-convex piecewise affine function on the same triangulation \( T \) of \( A \) as well as the original function \( \lambda \). For \( k \in \mathbb{Z} \) which is less than or equal to the number of elements in \( A \), let \( m_k \in A \) denote the unique element at which the function \( \lambda' : A \to \mathbb{R} \) takes the \( k \)-th smallest value, and we set

\[
A(k) := \{ m_1, \ldots, m_k \}, \quad \hat{A}(k) := \{ m \in A(k) | \text{conv} \{ m, m_{k+1} \} \in T \}.
\]

Claim 8.2. One has \( \hat{A}(k) \neq \emptyset \).

Proof. Suppose \( \hat{A}(k) = \emptyset \). This implies that the values of the function \( \lambda' \) at points in

\[
\{ m \in A \setminus \{ m_{k+1} \} | \text{conv} \{ m, m_{k+1} \} \in T \}
\]

are greater than \( \lambda'(m_{k+1}) \). Take the line segment whose endpoints are \( m_k, m_{k+1} \). It intersects with a 1-cell in \( T \) whose endpoints are contained in \( \{ m_0, m_1 \} \), since the point \( m_{k+1} \) is surrounded by such 1-cells. Let \( p \in \Delta \) denote the intersection point. The value at the point \( p \) of the extension of the function \( \lambda' \) to \( \Delta \) is greater than \( \lambda'(m_{k+1}) > \lambda'(m_k) \). This contradicts the convexity of the extension of the function \( \lambda' \) to \( \Delta \). Therefore, we have \( \hat{A}(k) \neq \emptyset \).

In particular, we have \( \hat{A}(1) \neq \emptyset \), i.e., \( \text{conv} \{ \{ m_1, m_2 \} \} \in T \). We first choose a branch of \( \arg (-c_{m_2}/c_{m_1}) = -\arg (-c_{m_1}/c_{m_2}) \) arbitrarily. We continue to choose branches of \( \arg \) inductively as follows: Suppose that we have chosen branches of \( \arg (-c_{m_j}/c_{m_0}) \) for all pairs \((m_i, m_j) \in B \) such that \( m_i, m_j \in A(k) \) so that all of them satisfy the conditions (8.3) and (8.4). There exists at least one element \( m \in \hat{A}(k) \) satisfying the following:

Condition 8.3. The number of 2-cells in \( T \), which contains \( m, m_{k+1} \), and another element in \( A(k) \) is at most one.

(Otherwise elements in \( A \) which are adjacent to \( m_{k+1} \) in \( T \) are all contained in \( A(k) \), and the element \( m_{k+1} \) is contained in \( \text{Int}(\Delta) \). Since the values of \( \lambda' \) at points in \( A(k) \) are less than \( \lambda'(m_{k+1}) \), we get contradiction with the convexity of \( \lambda' \) again.) We fix such an element \( m \in \hat{A}(k) \). We choose a branch of \( \arg (-c_{m}/c_{m_{k+1}}) = -\arg (-c_{m_{k+1}}/c_{m}) \) for every \( m \in \hat{A}(k) \) one by one counterclockwise or clockwise, starting from choosing for the element \( m \in \hat{A}(k) \). We choose either counterclockwise or clockwise so that the element in \( \hat{A}(k) \) for which we choose a branch of \( \arg \) after \( m \) is contained in the 2-cell of Condition 8.3 if that 2-cell exists. If the 2-cell does not exist, then one may choose either counterclockwise or clockwise arbitrarily.

For \( m \in \hat{A}(k) \), one may choose a branch arbitrarily. Between choosing a branch for \( m \) and \( m' \) in \( \hat{A}(k) \), if there is a 2-cell in \( T \), which contains \( m, m', m_{k+1} \), then the branch of \( \arg (-c_{m'}/c_{m_{k+1}}) = -\arg (-c_{m_{k+1}}/c_{m'}) \) is automatically determined, once we choose that of \( \arg (-c_{m}/c_{m_{k+1}}) = -\arg (-c_{m_{k+1}}/c_{m}) \) by (8.4). If there is not such a 2-cell, then we may choose a branch of \( \arg (-c_{m'}/c_{m_{k+1}}) = -\arg (-c_{m_{k+1}}/c_{m'}) \) arbitrarily again. In this way, we are able to choose branches of \( \arg (-c_{m}/c_{m_{k+1}}) = -\arg (-c_{m_{k+1}}/c_{m}) \) for all \( m \in \hat{A}(k) \) so that they satisfy (8.3) and (8.4). What might be worried about is if there is a 2-cell in \( T \) containing \( m, m_{k+1} \), and the element in \( \hat{A}(k) \) for which we lastly chose a branch of \( \arg \), and (8.4) fails for that 2-cell. However, such a 2-cell does not exist due to Condition 8.3 for \( m \) and the way to choose counterclockwise/clockwise.

By continuing this process, we can choose branches of \( \arg (-c_{m}/c_{w}) \) for all elements \((m_0, m_1) \in B \) as the claim of the lemma requires. For elements \((m_0, m_1) \in A \setminus B \), one may choose branches of \( \arg (-c_{m}/c_{w}) \) arbitrarily, since no conditions are required for them.

Recall that when we constructed the cycle \( C^w \), we chose branches of \( \arg (-c_{m}/c_{w}) \) for all \( m \in A \setminus \{ w \} \).
Lemma 8.4. We choose branches of \( \arg (-c_{m_1}/c_{n_0}) \in \mathbb{R} \) for all \( (m_0, m_1) \in A \) as in Lemma 8.7. If we construct the cycles \( C_t^w (w \in W) \) with these choices of branches, then the intersection number of the cycles \( C_{t_0}^w \) and \( C_{t_1}^w \) is zero for any \( w_0, w_1 \in W \).

Proof. The set \( B_t^w \) converges to the boundary of the cell in \( \mathcal{P} \) dual to \( \{w\} \in \mathcal{T} \) as \( t \to +0 \). Hence, it is obvious that the cycles \( C_{t_0}^w \) and \( C_{t_1}^w \) can intersect with each other only when the boundaries of the cells in \( \mathcal{P} \) that are dual to \( \{w_0\}, \{w_1\} \in \mathcal{T} \) intersect. This happens if and only if \( \text{conv} \{(w_0, w_1)\} \) is a 1-cell in \( \mathcal{T} \). We can also see that even when \( \text{conv} \{(w_0, w_1)\} \) is a 1-cell in \( \mathcal{T} \), the cycles \( C_{t_0}^w \) and \( C_{t_1}^w \) can intersect only in the subset \( \text{Log}^{-1}_t(U) \subset \mathbb{C}_+ \), where \( U \subset \mathbb{R} \) is a small neighborhood of the edge in \( \mathcal{P} \) dual to \( \text{conv} \{(w_0, w_1)\} \in \mathcal{T} \).

Suppose that \( \text{conv} \{(w_0, w_1)\} \) is a 1-cell in \( \mathcal{T} \) in the following. Let \( e \in \mathcal{P} \) be the edge dual to \( \text{conv} \{(w_0, w_1)\} \in \mathcal{T} \), and \( v_0, v_1 \in \mathcal{P} \) be its vertices. Let further \( m_0, m_1 \in A \setminus \{w_0, w_1\} \) denote the elements such that \( \mu_{m_i} = \text{trop}(f) \) at \( v_i \) \( (i = 0, 1) \) respectively. We take small neighborhoods \( U_{v_0}, U_{v_1} \subset \mathbb{R} \) of the points \( v_0, v_1 \). When these neighborhoods are sufficiently small, one has

\[
K^n_k (w_i) = \{w_j, m_k\}, \forall n \in U_{v_n} \quad ((i, j) = (0, 1), (1, 0), \; k = 0, 1),
\]

where \( K^n_k (w_i) \) denotes the set \( K^n_k \) of (8.6) for \( w = w_i \). Let \( n \in U_{v_0} \) be a point. By (8.7) and (3.15), we have

\[
\langle m_0 - w_0, \phi^{w_0} (n, 0) \rangle = - \arg \left( \frac{c_{m_0}}{c_{w_0}} \right),
\]

\[
\langle m_0 - w_1, \phi^{w_1} (n, 0) \rangle = - \arg \left( \frac{c_{m_0}}{c_{w_1}} \right), \quad \langle w_0 - w_1, \phi^{w_1} (n, 0) \rangle = - \arg \left( \frac{c_{w_0}}{c_{w_1}} \right),
\]

\[
\text{where } \phi^{w_i} : N^{w_i} \times D_e \to \mathbb{R} \text{ denotes the function taken in (3.15) for } w = w_i \quad (i = 0, 1). \] By (8.9), (8.4), and (3.3), we get

\[
\langle m_0 - w_0, \phi^{w_1} (n, 0) \rangle = - \arg \left( \frac{c_{m_0}}{c_{w_1}} \right) - \arg \left( \frac{c_{w_0}}{c_{w_1}} \right) = - \arg \left( \frac{c_{m_0}}{c_{w_1}} \right) + \frac{(w_0 - w_1) \land (m_0 - w_1)}{e_1 \land e_2} \cdot \pi.
\]

Since this differs from (8.8) by \( \pm \pi \), we can see that the cycles \( C_{t_0}^w \) and \( C_{t_1}^w \) do not intersect around \( \text{Log}^{-1}_t (v_0) \). By the same argument, we can also see that this is the case also around \( \text{Log}^{-1}_t (v_1) \).

Take points \( n_i \in B^w_t \cap U_{v_i} \quad (i = 0, 1) \). By (8.7) and (3.15) again, we have

\[
\langle m_1 - w_0, \phi^{w_0} (n_0, 0) \rangle = - \arg \left( \frac{c_{m_1}}{c_{w_0}} \right), \quad \langle w_1 - w_0, \phi^{w_0} (n_0, 0) \rangle = - \arg \left( \frac{c_{w_1}}{c_{w_0}} \right),
\]

\[
\text{where } \phi^{w_i} : N^{w_i} \times D_e \to \mathbb{R} \text{ denotes the function taken in (3.15) for } w = w_i \quad (i = 0, 1). \] By (8.11) and (8.12), we can also get

\[
\langle m_1 - w_1, \phi^{w_1} (n_1, 0) \rangle = - \arg \left( \frac{c_{m_1}}{c_{w_1}} \right), \quad \langle w_1 - w_1, \phi^{w_1} (n_1, 0) \rangle = - \arg \left( \frac{c_{w_1}}{c_{w_1}} \right).
\]

We can write \( m_0 - w_0 = k_1 (w_1 - w_0) + k_2 (m_1 - w_0) \) with \( k_1, k_2 \in \mathbb{R} \). By taking \( \bullet \land (w_1 - w_0) / e_1 \land e_2 \) for the both sides of this, we obtain

\[
k_2 = \frac{(m_0 - w_0) \land (w_1 - w_0)}{(m_1 - w_0) \land (w_1 - w_0)}. \]

By (8.11) and (8.12), we can also get

\[
\langle m_0 - w_0, \phi^{w_0} (n_0, 0) \rangle = - k_1 \arg \left( \frac{c_{w_1}}{c_{w_0}} \right) - k_2 \arg \left( \frac{c_{m_1}}{c_{w_0}} \right),
\]

\[
\langle m_0 - w_0, \phi^{w_1} (n_1, 0) \rangle = (k_1 + k_2) \arg \left( \frac{c_{w_1}}{c_{w_1}} \right) - k_2 \arg \left( \frac{c_{m_1}}{c_{w_1}} \right).
\]

The difference between (8.14) and (8.8) and the difference between (8.15) and (8.10) are

\[
-k_1 \arg \left( \frac{c_{w_1}}{c_{w_0}} \right) - k_2 \arg \left( \frac{c_{m_1}}{c_{w_0}} \right) + \arg \left( \frac{c_{m_0}}{c_{w_0}} \right),
\]

\[
(1 + k_2) \arg \left( \frac{c_{w_1}}{c_{w_1}} \right) - k_2 \arg \left( \frac{c_{m_1}}{c_{w_1}} \right) - \frac{(w_0 - w_1) \land (m_0 - w_1)}{e_1 \land e_2} \cdot \pi
\]
respectively, and the difference between these two is
\begin{equation}
-k_2 \arg \left( \frac{c_{w_1}}{c_{w_0}} \right) - k_2 \arg \left( \frac{c_{m_1}}{c_{w_1}} \right) + k_2 \arg \left( \frac{c_{m_1}}{c_{w_0}} \right) - \left( \frac{(w_0 - w_1) \wedge (m_0 - w_1)}{e_1 \wedge e_2} \right) \cdot \pi
= k_2 \left( (m_1 - w_0) \wedge (w_1 - w_0) \right) \cdot \pi - \left( \frac{(w_0 - w_1) \wedge (m_0 - w_1)}{e_1 \wedge e_2} \right) \cdot \pi = 0,
\end{equation}
where we used (8.23), (8.3), and (8.13). This means that both $C_{t_1}^{w_0}$ and $C_{t_1}^{w_1}$ wind around the cylinder in the Riemann surface $Z_t$, which corresponds to the edge $e \in \mathcal{P}$ by the same argument. Thus we can conclude that the intersection number of these is zero.

In the following, we use the cycles $C_t^w$ ($w \in W$) considered in Lemma 8.4. Let $\beta_w$ ($w \in W$) be the cycle class in $H_1(Z_t, \mathbb{Z})$ represented by $C_t^w$. The torus cycle $T_t^w$ intersects with the sphere cycle $C_t^w$ if and only if $\text{trop}(f) = \mu_w$ on $\sigma$. In that case, they intersect positively (see Remark 5.1). Hence, we can see from (1.16) that for any cycle class $\alpha$ in the subspace of $H_1(Z_t, \mathbb{Z})$ generated by $\{T_t^w\}_\sigma$, one has
\begin{equation}
\int_{\alpha} \Omega_1^{1,v} = -2\pi \sqrt{-1} \langle \alpha, \beta_v \rangle + O(t^4)
\end{equation}
for $v \in W$. Therefore, if we take a basis $\{\alpha_w\}_{w \in W}$ of the subspace of $H_1(Z_t, \mathbb{Z})$ generated by $\{T_t^w\}_\sigma$ so that we have (1.18), then the cohomology class represented by $\Omega_1^{1,v}$ ($v \in W$) in $H^1(Z_t, \mathbb{Z})$ can be written as
\begin{equation}
\left[ \Omega_1^{1,v} \right] = (-2\pi \sqrt{-1} + O(t^4)) \alpha_v^* + \sum_{w \in W \setminus \{v\}} O(t^4) \alpha_w^* + \sum_{w \in W} \left( \int_{C_t^w} \Omega_1^{1,v} \right) \beta_w^*.
\end{equation}

Consider the exact sequence of sheaves
\begin{equation}
0 \to \Omega^2_{Y_S} \to \Omega^2_{Y_S}(Z_t) \to \Omega^1_{Z_t} \to 0.
\end{equation}
Since we have $H^i(Y_S, \Omega^2) = 0$ for $i = 0, 1$ (cf. e.g. [Dan78, Corollary 12.7]), we can see that the Poincaré residue map (1.5) with $l = 1$ is an isomorphism onto $H^0(Z_t, \Omega^1) \subset H^1(Z_t, \mathbb{C})$. Therefore, the Hodge structure of the Riemann surface $Z_t$ is given by
\begin{equation}
F^0 = H^1(Z_t, \mathbb{C}) , \quad F^1 = \bigoplus_{v \in W} \mathbb{C} \left[ \Omega_1^{1,v} \right] , \quad F^2 = 0.
\end{equation}

Proof of Corollary 1.2. We refer the reader to [KL09, Section 2.5.15] or [Yam22, Section 5.2] for how variations of polarized Hodge structure on a punctured disk extend to logarithmic variations of polarized Hodge structure on the whole disk. What we need to prove are the following: For the one-parameter family $\{Z_t\}_{t \in D^*_\rho}$ of complex curves,

- the monodromy around $q = 0$ for (1.21) is given by $\exp(N) = \text{id} + N$, where $N$ is the one defined in (1.19), and
- the limit Hodge structure is given by (1.23).

We first compute the monodromy. We write the monodromy around $q = 0$ for the homology group as $M : H_1(Z_t, \mathbb{Z}) \to H_1(Z_t, \mathbb{Z})$. We substitute $q = te^{\sqrt{-1} \theta}$ ($\theta \in [0, 2\pi]$) to the indeterminate $x$ in $f$, and consider the limit $t \to +0$ of the periods. By Theorem 1.1 we get
\begin{align}
\int_{\beta_w} \Omega_1^{1,v} &= \begin{cases} 
\int_{Y_w} t^{-\omega_w} \cdot \prod_{m \in A_w} \left( -\frac{c_{e^w v} T_{m w}^\theta}{c_{e^w v} T_{m w}^\theta} \right) - D_{m_1}^w \cdot \sigma_w + O(t^4) & v = w \\
- \int_{Y_w} t^{-\omega_w} \cdot \prod_{m \in A_w} \left( -\frac{c_{e^w v} T_{m w}^\theta}{c_{e^w v} T_{m w}^\theta} \right) - D_{m_1}^w \cdot \sigma_{v w} + O(t^4) & v \neq w
\end{cases} \\
\int_{\alpha_w} \Omega_1^{1,v} &= \begin{cases} 
-2\pi \sqrt{-1} + O(t^4) & v = w \\
O(t^4) & v \neq w
\end{cases}
\end{align}
for $v, w \in W$. The leading terms of (8.23) and (8.24) do not depend on $\theta$. We can see that we have $M(\alpha_w) = \alpha_w$, and can write
\begin{equation}
M(\beta_w) = \beta_w + \sum_{w' \in W} M(w, w') \alpha_{w'}
\end{equation}
for $w' \in W$. The leading terms of (8.23) and (8.24) do not depend on $\theta$. We can see that we have $M(\alpha_w) = \alpha_w$, and can write
\begin{equation}
M(\beta_w) = \beta_w + \sum_{w' \in W} M(w, w') \alpha_{w'}
\end{equation}
with $M(w, w') \in \mathbb{Z}$. We also have

\begin{equation}
(8.26) \quad \prod_{m \in A_w} \left( -\frac{c_m e^{-\sqrt{-1} \frac{\lambda_m}{\theta}}}{c_w e^{-\sqrt{-1} \frac{\lambda_w}{\theta}}} \right)^{-D_m^w} = \prod_{m \in A_w} \left( -\frac{c_m}{c_w} \right)^{-D_m^w} \exp \left( -\sqrt{-1} \frac{\theta}{1} (\lambda_m - \lambda_w) D_m^w \right)
\end{equation}

\begin{equation}
(8.27) \quad = \exp \left( -\sqrt{-1} \frac{\theta}{1} \omega^w \right) \prod_{m \in A_w} \left( -\frac{c_m}{c_w} \right)^{-D_m^w}
\end{equation}

and

\begin{equation}
(8.28) \quad \int_{M(\beta_w)} \Omega_{1,v}^w = \int_{C^v_{\mu}} \Omega_{1,v}^w - 2\pi \sqrt{-1} M(w, v) + O(t').
\end{equation}

From these, (8.23) with $\theta = 2\pi$, and (6.5), we can get

\begin{equation}
(8.29) \quad M(w, v) = \begin{cases} (-1)^{1+\delta_{w,v}} l(w, v) & v \in A_w \cup \{w\} \\ 0 & \text{otherwise} \end{cases}
\end{equation}

We computed the monodromy $M$ for the homology group. By taking the dual of $M$ and subtracting id, we obtain $N$ of (1.19).

Next we compute the limit Hodge structure. It is the filtration $\{F^p_{\infty}\}_{0 \leq p \leq 2}$ determined by

\begin{equation}
(8.30) \quad F^p_{\infty} = \lim_{t \to +0} \exp \left( \frac{1}{2\pi \sqrt{-1}} (-\log t) N \right) \cdot \left( \bigoplus_{v \in W} \mathbb{C} \left[ \Omega_{1,v}^w \right] \right)
\end{equation}

and $F^0_{\infty} = H^1(Z_t, \mathbb{Z})$, $F^2_{\infty} = \{0\}$. By (8.20) and (1.19), the cohomology class $\exp \left( \frac{1}{2\pi \sqrt{-1}} (-\log t) N \right) \cdot \left[ \Omega_{1,v}^w \right]$ is equal to

\begin{equation}
(8.31) \quad \left[ \Omega_{1,v}^w \right] - (-\log t) \left( \sum_{w \in \{v\} \cup A_w \cap W} (-1)^{1+\delta_{w,v}} \cdot l(v, w) \beta^*_w \right) + \sum_{w \in W} O((-\log t) t') \beta^*_w.
\end{equation}

By (8.23) with $\theta = 0$, we can see that this is equal to

\begin{equation}
(8.32) \quad -2\pi \sqrt{-1} \alpha^*_v + \left( \int_{Y_v} \prod_{m \in A_v} \left( -\frac{c_m}{c_v} \right)^{-D_m^v} \sigma^v \right) \beta^*_v - \sum_{w \in A_v \cap W} \left( \int_{Y_w} \prod_{m \in A_w} \left( -\frac{c_m}{c_w} \right)^{-D_m^w} D_m^w \right) \beta^*_w
\end{equation}

plus $\sum_{w \in W} O(t') \alpha^*_w + \sum_{w \in W} O((-\log t) t') \beta^*_w$. By taking the limit $t \to +0$, we obtain

\begin{equation}
(8.33) \quad F^1_{\infty} = \bigoplus_{v \in W} \mathbb{C} \cdot \left( -2\pi \sqrt{-1} \alpha^*_v + \sum_{w \in W} P(v, w) \beta^*_w \right)
\end{equation}

where $P(v, w)$ is the one defined in (1.20). We proved the claim. \hfill \Box

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