NON LOCAL POINCARÉ INEQUALITIES ON LIE GROUPS WITH POLYNOMIAL VOLUME GROWTH

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Abstract. Let $G$ be a real connected Lie group with polynomial volume growth, endowed with its Haar measure $dx$. Given a $C^2$ positive function $M$ on $G$, we give a sufficient condition for an $L^2$ Poincaré inequality with respect to the measure $M(x)dx$ to hold on $G$. We then establish a non-local Poincaré inequality on $G$ with respect to $M(x)dx$.

Contents

1. Introduction 1
2. A proof of the Poincaré inequality for $d\mu_M$ 4
3. Proof of Theorem 1.4 8
  3.1. Rewriting the improved Poincaré inequality 8
  3.2. Off-diagonal $L^2$ estimates for the resolvent of $L_M$ 9
  3.3. Control of $\|L^\alpha/4_M f\|_{L^2(G,d\mu_M)}$ and conclusion of the proof of Theorem 1.4 10
4. Appendix A: Technical lemma 16
5. Appendix B: Estimates for $g_j'$ 16
References 17

1. Introduction

Let $G$ be a unimodular connected Lie group endowed with a measure $M(x)dx$ where $M \in L^1(G)$ and $dx$ stands for the Haar measure on $G$. By “unimodular”, we mean that the Haar measure is left and right-invariant. We always assume that $M = e^{-v}$ where $v$ is a $C^2$ function on $G$. If we denote by $\mathcal{G}$ the Lie algebra of $G$, we consider a family

$X = \{X_1,\ldots,X_k\}$

of left-invariant vector fields on $G$ satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the $X'_i$s. A standard metric on $G$, called the Carnot-Caratheodory metric, is naturally associated with $X$.
and is defined as follows: let \( \ell : [0, 1] \to G \) be an absolutely continuous path. We say that \( \ell \) is admissible if there exist measurable functions \( a_1, \ldots, a_k : [0, 1] \to \mathbb{C} \) such that, for almost every \( t \in [0, 1] \), one has
\[
\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).
\]
If \( \ell \) is admissible, its length is defined by
\[
|\ell| = \int_{0}^{1} \left( \sum_{i=1}^{k} |a_i(t)|^2 dt \right)^{\frac{1}{2}}.
\]

For all \( x, y \in G \), define \( d(x, y) \) as the infimum of the lengths of all admissible paths joining \( x \) to \( y \) (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by \(|x|\) the distance between \( e \), the neutral element of the group and \( x \), so that the distance from \( x \) to \( y \) is equal to \(|y^{-1}x|\).

For all \( r > 0 \), denote by \( B(x, r) \) the open ball in \( G \) with respect to the Carnot-Carathéodory distance and by \( V(r) \) the Haar measure of any ball. There exists \( d \in \mathbb{N}^* \) (called the local dimension of \((G, X)\)) and \( 0 < c < C \) such that, for all \( r \in (0, 1) \),
\[
cr^d \leq V(r) \leq Cr^d,
\]
see [NSW85]. When \( r > 1 \), two situations may occur (see [Gui73]):

- Either there exist \( c, C, D > 0 \) such that, for all \( r > 1 \),
  \[
  cr^D \leq V(r) \leq Cr^D
  \]
  where \( D \) is called the dimension at infinity of the group (note that, contrary to \( d \), \( D \) does not depend on \( X \)). The group is said to have polynomial volume growth.

- Or there exist \( c_1, c_2, C_1, C_2 > 0 \) such that, for all \( r > 1 \),
  \[
  c_1e^{c_2r} \leq V(r) \leq C_1e^{c_2r}
  \]
  and the group is said to have exponential volume growth.

When \( G \) has polynomial volume growth, it is plain to see that there exists \( C > 0 \) such that, for all \( r > 0 \),
\[
V(2r) \leq CV(r),
\]
which implies that there exist \( C > 0 \) and \( \kappa > 0 \) such that, for all \( r > 0 \) and all \( \theta > 1 \),
\[
V(\theta r) \leq C\theta^\kappa V(r).
\]

(1.1) \[ V(2r) \leq CV(r), \]

(1.2) \[ V(\theta r) \leq C\theta^\kappa V(r). \]
Denote by $H^1(G, d\mu_M)$ the Sobolev space of functions $f \in L^2(G, d\mu_M)$ such that $X_i f \in L^2(G, d\mu_M)$ for all $1 \leq i \leq k$. We are interested in $L^2$ Poincaré inequalities for the measure $d\mu_M$. In order to state sufficient conditions for such an inequality to hold, we introduce the operator

$$L_M f = -M^{-1} \sum_{i=1}^k X_i \{ M X_i f \}$$

for all $f$ such that

$$f \in \mathcal{D}(L_M) := \left\{ g \in H^1(G, d\mu_M); \frac{1}{\sqrt{M}} X_i \{ M X_i f \} \in L^2(G, dx), \forall 1 \leq i \leq k \right\}.$$

One therefore has, for all $f \in \mathcal{D}(L_M)$ and $g \in H^1(G, d\mu_M)$,

$$\int_G L_M f(x) g(x) d\mu_M(x) = \sum_{i=1}^k \int_G X_i f(x) \cdot X_i g(x) d\mu_M(x).$$

In particular, the operator $L_M$ is symmetric on $L^2(G, d\mu_M)$.

Following [BBCG08], say that a $C^2$ function $W : G \to \mathbb{R}$ is a Lyapunov function if $W(x) \geq 1$ for all $x \in G$ and there exist constants $\theta > 0$, $b \geq 0$ and $R > 0$ such that, for all $x \in G$,

$$-L_M W(x) \leq -\theta W(x) + b 1_{B(e,R)}(x),$$

where, for all $A \subset G$, $1_A$ denotes the characteristic function of $A$. We first claim:

**Theorem 1.1.** Assume that $G$ is unimodular and that there exists a Lyapunov function $W$ on $G$. Then, $d\mu_M$ satisfies the following $L^2$ Poincaré inequality: there exists $C > 0$ such that, for all function $f \in H^1(G, d\mu_M)$ with $\int_G f(x) d\mu_M(x) = 0$,

$$\int_G |f(x)|^2 d\mu_M(x) \leq C \sum_{i=1}^k \int_G |X_i f(x)|^2 d\mu_M(x). \quad (1.4)$$

Let us give, as a corollary, a sufficient condition on $v$ for (1.4) to hold:

**Corollary 1.2.** Assume that $G$ is unimodular and there exist constants $a \in (0,1)$, $c > 0$ and $R > 0$ such that, for all $x \in G$ with $|x| > R$,

$$a \sum_{i=1}^k |X_i v(x)|^2 - \sum_{i=1}^k X_i^2 v(x) \geq c. \quad (1.5)$$

Then (1.4) holds.
Notice that, if (1.5) holds with \( a \in (0, \frac{1}{2}) \), then the Poincaré inequality (1.4) has the following self-improvement:

**Proposition 1.3.** Assume that \( G \) is unimodular and that there exist constants \( c > 0, R > 0 \) and \( \varepsilon \in (0, 1) \) such that, for all \( x \in G \),

\[
\frac{1 - \varepsilon}{2} \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c \quad \text{whenever} \quad |x| > R.
\]

Then there exists \( C > 0 \) such that, for all function \( f \in H^1(G, d\mu_M) \) such that \( \int_G f(x) d\mu_M(x) = 0 \):

\[
\sum_{i=1}^{k} \int_G |X_i f(x)|^2 d\mu_M(x) \geq C \int_G |f(x)|^2 \left( 1 + \sum_{i=1}^{k} |X_i v(x)|^2 \right) d\mu_M(x)
\]

We finally obtain a Poincaré inequality for \( d\mu_M \) involving a non-local term:

**Theorem 1.4.** Let \( G \) be a unimodular Lie group with polynomial growth. Let \( d\mu_M = M dx \) be a measure absolutely continuous with respect to the Haar measure on \( G \) where \( M = e^{-v} \in L^1(G) \) and \( v \in C^2(G) \). Assume that there exist constants \( c > 0, R > 0 \) and \( \varepsilon \in (0, 1) \) such that (1.6) holds. Let \( \alpha \in (0, 2) \). Then there exists \( \lambda_\alpha(M) > 0 \) such that, for any function \( f \in \mathcal{D}(G) \) satisfying \( \int_G f(x) d\mu_M(x) = 0 \),

\[
\int\int_{G \times G} \frac{|f(x) - f(y)|^2}{V((|y|^{-1} |x|)|y|^{-1} |x|^{\alpha})} \, dx \, d\mu_M(y) \geq \lambda_\alpha(M)
\]

\[
\int_{\mathbb{R}^n} |f(x)|^2 \left( 1 + \sum_{i=1}^{k} |X_i v(x)|^2 \right) d\mu_M(x).
\]

Note that (1.8) is an improvement of (1.7) in terms of fractional non-local quantities. The proof follows the same line as the paper [MRS09] but we concentrate here on a more geometric context.

In order to prove Theorem 1.4, we need to introduce fractional powers of \( L_M \). This is the object of the following developments. Since the operator \( L_M \) is symmetric and non-negative on \( L^2(G, d\mu_M) \), we can define the usual power \( L^\beta \) for any \( \beta \in (0, 1) \) by means of spectral theory.

Section 2 is devoted to the proof of Theorem 1.1 and Corollary 1.2. Then, in Section 3, we check \( L^2 \) “off-diagonal” estimates for the resolvent of \( L_M \) and use them to establish Theorem 1.4.
2. A proof of the Poincaré inequality for $d\mu_M$

We follow closely the approach of [BBCG08]. Recall first that the following $L^2$ local Poincaré inequality holds on $G$ for the measure $dx$: for all $R > 0$, there exists $C_R > 0$ such that, for all $x \in G$, all $r \in (0, R)$, all ball $B := B(x, r)$ and all function $f \in C^\infty(B)$,

\begin{equation}
\int_B |f(x) - f_B|^2 \, dx \leq C_R r^2 \sum_{i=1}^k \int_B |X_i f(x)|^2 \, dx,
\end{equation}

where $f_B := \frac{1}{V(r)} \int_B f(x) \, dx$. In the Euclidean context, Poincaré inequalities for vector-fields satisfying Hörmander conditions were obtained by Jerison in [Jer86]. A proof of (2.9) in the case of unimodular Lie groups can be found in [SC95], but the idea goes back to [Var87]. A nice survey on this topic can be found in [HK00]. Notice that no global growth assumption on the volume of balls is required for (2.9) to hold.

The proof of (1.4) relies on the following inequality:

**Lemma 2.1.** For all function $f \in H^1(G, d\mu_M)$ on $G$,

\begin{equation}
\int_G \frac{L_M W(x) f^2(x)}{W(x)} d\mu_M(x) \leq \sum_{i=1}^k \int_G |X_i f(x)|^2 d\mu_M(x).
\end{equation}

**Proof:** Assume first that $f$ is compactly supported on $G$. Using the definition of $L_M$, one has

\[
\int_G \frac{L_M W(x) f^2(x)}{W(x)} d\mu_M(x) = \sum_{i=1}^k \int_G \left( \frac{f^2}{W} \right) (x) \cdot X_i W(x) d\mu_M(x)
\]

\[
= 2 \sum_{i=1}^k \int_G \frac{f}{W} (x) X_i f(x) \cdot X_i W(x) d\mu_M(x)
\]

\[
- \sum_{i=1}^k \int_G \frac{f^2}{W^2} (x) |X_i W(x)|^2 d\mu_M(x)
\]

\[
= \sum_{i=1}^k \int_G |X_i f(x)|^2 d\mu_M(x)
\]

\[
- \sum_{i=1}^k \int_G \left| X_i f - \frac{f}{W} X_i W \right|^2 (x) d\mu_M(x)
\]

\[
\leq \sum_{i=1}^k \int_G |X_i f(x)|^2 d\mu_M(x).
\]
Notice that all the previous integrals are finite because of the support condition on $f$. Now, if $f$ is as in Lemma 2.1 consider a nondecreasing sequence of smooth compactly supported functions $\chi_n$ satisfying
\[ 1_{B(e,nR)} \leq \chi_n \leq 1 \text{ and } |X_i\chi_n| \leq 1 \text{ for all } 1 \leq i \leq k. \]
Applying (2.10) to $f\chi_n$ and letting $n$ go to $+\infty$ yields the desired conclusion, by use of the monotone convergence theorem in the left-hand side and the dominated convergence theorem in the right-hand side.

Let us now establish (1.4). Let $g$ be a smooth function on $G$ and let $f := g - c$ on $G$ where $c$ is a constant to be chosen. By assumption (1.3),
\[ \int_G f^2(x) d\mu_M(x) \leq \int_G f^2(x) \frac{L_M W}{\theta W}(x) d\mu_M(x) + \int_{B(e,R)} f^2(x) \frac{b}{\theta W}(x) d\mu_M(x). \]
Lemma 2.1 shows that (2.10) holds. Let us now turn to the second term in the right-hand side of (2.11). Fix $c$ such that $\int_{B(e,R)} f(x) d\mu_M(x) = 0$. By (2.9) applied to $f$ on $B(e,R)$ and the fact that $M$ is bounded from above and below on $B(e,R)$, one has
\[ \int_{B(e,R)} f^2(x) d\mu_M(x) \leq CR^2 \sum_{i=1}^{k} \int_{B(e,R)} |X_i f(x)|^2 d\mu_M(x) \]
where the constant $C$ depends on $R$ and $M$. Therefore, using the fact that $W \geq 1$ on $G$,
\[ \int_{B(e,R)} f^2(x) d\mu_M(x) \leq CR^2 \sum_{i=1}^{k} \int_{B(e,R)} |X_i f(x)|^2 d\mu_M(x) \]
where the constant $C$ depends on $R, M, \theta$ and $b$. Gathering (2.11), (2.10) and (2.12) yields
\[ \int_G (g(x) - c)^2 d\mu_M(x) \leq C \sum_{i=1}^{k} \int_G |X_i g(x)|^2 d\mu_M(x), \]
which easily implies (1.4) for the function $g$ (and the same dependence for the constant $C$).

**Proof of Corollary 1.2**: according to Theorem 1.1 it is enough to find a Lyapunov function $W$. Define
\[ W(x) := e^{\gamma (v(x) - \inf_G v)} \]
where $\gamma > 0$ will be chosen later. Since

$$-L MW(x) = \gamma \left( \sum_{i=1}^{k} X_i^2 v(x) - (1 - \gamma) \sum_{i=1}^{k} |X_i v(x)|^2 \right) W(x),$$

$W$ is a Lyapunov function for $\gamma := 1 - a$ because of the assumption on $v$. Indeed, one can take $\theta = c \gamma$ and $b = \max_{B(e,R)} \left\{ -LMW + \theta W \right\}$ (recall that $M$ is a $C^2$ function).

Let us now prove Proposition 1.3. Observe first that, since $v$ is $C^2$ on $G$ and (1.6) holds, there exists $\alpha \in \mathbb{R}$ such that, for all $x \in G$,

$$(2.13) \quad \frac{1 - \varepsilon}{2} \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq \alpha.$$  

Let $f$ be as in the statement of Proposition 1.3 and let $g := f M^{\frac{1}{2}}$. Since, for all $1 \leq i \leq k$,

$$X_i f = M^{-\frac{1}{2}} X_i g - \frac{1}{2} g M^{-\frac{3}{2}} X_i M.$$  

Assumption (2.13) yields two positive constants $\beta, \gamma$ such that

$$(2.14) \quad \sum_{i=1}^{k} \int_G |X_i f(x)|^2 \, d\mu_M(x) =$$

$$\sum_{i=1}^{k} \int_G \left( |X_i g(x)|^2 + \frac{1}{4} g^2(x) |X_i v(x)|^2 + g(x) X_i g(x) X_i v(x) \right) dx$$

$$= \sum_{i=1}^{k} \int_G \left( |X_i g(x)|^2 + \frac{1}{4} g^2(x) |X_i v(x)|^2 + \frac{1}{2} X_i \left( g^2 \right)(x) X_i v(x) \right) dx$$

$$\geq \sum_{i=1}^{k} \int_G g^2(x) \left( \frac{1}{4} |X_i v(x)|^2 - \frac{1}{2} X_i^2 v(x) \right) dx$$

$$\geq \sum_{i=1}^{k} \int_G f^2(x) \left( \beta |X_i v(x)|^2 - \gamma \right) d\mu_M(x).$$

The conjunction of (1.4), which holds because of (1.6), and (2.14) yields the desired conclusion. \qed

3. Proof of Theorem 1.4

We divide the proof into several steps.
3.1. **Rewriting the improved Poincaré inequality.** By the definition of $L_M$, the conclusion of Proposition 1.3 means, in terms of operators in $L^2(G, d\mu_M)$, that, for some $\lambda > 0$,

\[(3.15)\quad L_M \geq \lambda \mu,\]

where $\mu$ is the multiplication operator by $1 + \sum_{i=1}^k |X_i v|^2$. Using a functional calculus argument (see [Dav80], p. 110), one deduces from (3.15) that, for any $\alpha \in (0, 2)$,

$$L_M^{\alpha/2} \geq \lambda^{\alpha/2} \mu^{\alpha/2}$$

which implies, thanks to the fact $L_M^{\alpha/2} = (L_M^{\alpha/4})^2$ and the symmetry of $L_M^{\alpha/4}$ on $L^2(G, d\mu_M)$, that

$$\int_G |f(x)|^2 \left(1 + \sum_{i=1}^k |X_i v(x)|^2\right)^{\alpha/2} d\mu_M(x) \leq \frac{C}{t^{\alpha/4}} \int_G \left| \left(I + t L_M\right)^{-1} f \right|^2 d\mu_M(x) = C \left\| L_M^{\alpha/4} f \right\|_{L^2(G, d\mu_M)}^2.$$}

The conclusion of Theorem 1.4 will follow by estimating the quantity $\left\| L_M^{\alpha/4} f \right\|_{L^2(G, d\mu_M)}^2$.

3.2. **Off-diagonal $L^2$ estimates for the resolvent of $L_M$.** The crucial estimates to derive the desired inequality are some $L^2$ “off-diagonal” estimates for the resolvent of $L_M$, in the spirit of [Gal59]. This is the object of the following lemma.

**Lemma 3.1.** There exists $C$ with the following property: for all closed disjoint subsets $E, F \subset G$ with $d(E, F) := d > 0$, all function $f \in L^2(G, d\mu_M)$ supported in $E$ and all $t > 0$,

$$\left\| \left(I + t L_M\right)^{-1} f \right\|_{L^2(F, d\mu_M)} + \left\| t L_M \left(I + t L_M\right)^{-1} f \right\|_{L^2(F, d\mu_M)} \leq 8 e^{-C d^{\frac{a}{\alpha}}} \left\| f \right\|_{L^2(E, d\mu_M)}.$$}

**Proof.** We argue as in [AHL+02], Lemma 1.1. From the fact that $L_M$ is self-adjoint on $L^2(G, d\mu_M)$ we have

$$\left\| (L_M - \mu)^{-1} \right\|_{L^2(G, d\mu_M)} \leq \frac{1}{\text{dist}(\mu, \Sigma(L_M))}$$

where $\Sigma(L_M)$ denotes the spectrum of $L_M$, and $\mu \notin \Sigma(L_M)$. Then we deduce that $(I + t L_M)^{-1}$ is bounded with norm less than 1 for all $t > 0$, and it is clearly enough to argue when $0 < t < d$. 


In the following computations, we will make explicit the dependence of the measure $d\mu_M$ in terms of $M$ for sake of clarity. Define $u_t = (I + t L_M)^{-1} f$, so that, for all function $v \in H^1(G, d\mu_M)$,

\begin{equation}
(3.16) \quad \int_G u_t(x) v(x) M(x) \, dx + t \sum_{i=1}^k \int_G X_i u_t(x) \cdot X_i v(x) M(x) \, dx = \int_G f(x) v(x) M(x) \, dx.
\end{equation}

Fix now a nonnegative function $\eta \in D(G)$ vanishing on $E$. Since $f$ is supported in $E$, applying (3.16) with $v = \eta^2 u_t$ (remember that $u_t \in H^1(G, d\mu_M)$) yields

\begin{equation}
\int_G \eta^2(x) |u_t(x)|^2 M(x) \, dx + t \sum_{i=1}^k \int_G X_i u_t(x) \cdot X_i (\eta^2 u_t) M(x) \, dx = 0,
\end{equation}

which implies

\begin{align*}
&\int_G \eta^2(x) |u_t(x)|^2 M(x) \, dx + t \int_G \eta^2(x) \sum_{i=1}^k |X_i u_t(x)|^2 M(x) \, dx \\
&= -2 t \sum_{i=1}^k \int_G \eta(x) u_t(x) X_i \eta(x) \cdot X_i u_t(x) M(x) \, dx \\
&\leq t \int_G |u_t(x)|^2 \sum_{i=1}^k |X_i \eta(x)|^2 M(x) \, dx + t \int_G \eta^2(x) \sum_{i=1}^k |X_i u_t(x)|^2 M(x) \, dx,
\end{align*}

hence

\begin{equation}
(3.17) \quad \int_G \eta^2(x) |u_t(x)|^2 M(x) \, dx \leq t \int_G |u_t(x)|^2 \sum_{i=1}^k |X_i \eta(x)|^2 M(x) \, dx.
\end{equation}

Let $\zeta$ be a nonnegative smooth function on $G$ such that $\zeta = 0$ on $E$, so that $\eta := e^{\alpha \zeta} - 1 \geq 0$ and $\eta$ vanishes on $E$ for some $\alpha > 0$ to be chosen. Choosing this particular $\eta$ in (3.17) with $\alpha > 0$ gives

\begin{equation}
\int_G |e^{\alpha \zeta(x)} - 1|^2 |u_t(x)|^2 M(x) \, dx \leq
\end{equation}
\[ \alpha^2 t \int_G |u_t(x)|^2 \sum_{i=1}^k |X_i \zeta(x)|^2 \ e^{2 \alpha \zeta(x)} M(x) \, dx. \]

Taking \( \alpha = 1/(2 \sqrt{t} \ max \|X_i \zeta\|_\infty) \), one obtains
\[ \int_G \left| e^{\alpha \zeta(x)} - 1 \right|^2 |u_t(x)|^2 M(x) \, dx \leq \frac{1}{4} \int_G |u_t(x)|^2 \ e^{2 \alpha \zeta(x)} M(x) \, dx. \]

Using the fact that the norm of \((I+tL_M)^{-1}\) is bounded by 1 uniformly in \( t > 0 \), this gives
\[
\|e^{\alpha \zeta} u_t\|_{L^2(G,d\mu_M)} \leq \|(e^{\alpha \zeta} - 1) u_t\|_{L^2(G,d\mu_M)} + \|u_t\|_{L^2(G,d\mu_M)} 
\leq \frac{1}{2} \|e^{\alpha \zeta} u_t\|_{L^2(G,d\mu_M)} + \|f\|_{L^2(G,d\mu_M)},
\]
therefore
\[ \int_G \left| e^{\alpha \zeta(x)} \right|^2 |u_t(x)|^2 M(x) \, dx \leq 4 \int_G |f(x)|^2 M(x) \, dx. \]

We choose now \( \zeta \) such that \( \zeta = 0 \) on \( E \) as before and additionally that \( \zeta = 1 \) on \( F \). It can furthermore be chosen with max\( i=1,...,k \|X_i \zeta\|_\infty \leq C/d \), which yields the desired conclusion for the \( L^2 \) norm of \((I+tL_M)^{-1}f\) with a factor 4 in the right-hand side. Since \( t L_M (1+t L_M)^{-1} f = f - (I+t L_M)^{-1} f \), the desired inequality with a factor 8 readily follows. \( \Box \)

3.3. Control of \( \|L_M^{\alpha/4} f\|_{L^2(G,d\mu_M)} \) and conclusion of the proof of Theorem 1.4. This is now the heart of the proof to reach the conclusion of Theorem 1.4. The following first lemma is a standard quadratic estimate on powers of subelliptic operators. It is based on spectral theory.

**Lemma 3.2.** Let \( \alpha \in (0,2) \). There exists \( C > 0 \) such that, for all \( f \in \mathcal{D}(L_M) \),
\begin{equation}
\left\| L_M^{\alpha/4} f \right\|_{L^2(G,d\mu_M)} \leq C_3 \int_0^{+\infty} t^{-1-\alpha/2} \left\| t L_M (I+t L_M)^{-1} f \right\|_{L^2(G,d\mu_M)}^2 \, dt.
\end{equation}

We now come to the desired estimate.

**Lemma 3.3.** Let \( \alpha \in (0,2) \). There exists \( C > 0 \) such that, for all \( f \in \mathcal{D}(G) \),
\[
\int_0^{\infty} t^{-1-\alpha/2} \left\| t L_M (I+t L_M)^{-1} f \right\|_{L^2(G,d\mu_M)}^2 \, dt \leq C \int_{G \times G} \frac{|f(x) - f(y)|^2}{V(\|y^{-1} x\| \|y^{-1} x\|)} M(x) \, dx \, dy.
\]
Proof. Fix \( t \in (0, +\infty) \). Following Lemma 3.2 we give an upper bound of

\[
\| t L_M (I + t L_M)^{-1} f \|_{L^2(G, d\mu_M)}^2
\]

involving first order differences for \( f \). Using (1.1), one can pick up a countable family \( x_j^t, j \in \mathbb{N} \), such that the balls \( B(x_j^t, \sqrt{t}) \) are pairwise disjoint and

\[
G = \bigcup_{j \in \mathbb{N}} B(x_j^t, 2\sqrt{t}).
\]

By Lemma 4.1 in Appendix A, there exists a constant \( \tilde{C} > 0 \) such that for all \( \theta > 1 \) and all \( x \in G \), there are at most \( \tilde{C} \theta^2 \kappa \) indexes \( j \) such that \(|x^{-1}x_j^t| \leq \theta \sqrt{t} \) where \( \kappa \) is given by (1.2).

For fixed \( j \), one has

\[
t L_M (I + t L_M)^{-1} f = t L_M (I + t L_M)^{-1} g_j^t,
\]

where, for all \( x \in G \),

\[
g_j^t(x) := f(x) - m_j^t
\]

and \( m_j^t \) is defined by

\[
m_j^t := \frac{1}{V(2\sqrt{t})} \int_{B(x_j^t, 2\sqrt{t})} f(y) dy.
\]

Note that, here, the mean value of \( f \) is computed with respect to the Haar measure on \( G \). Since (3.19) holds, one clearly has

\[
\| t L_M (I + t L_M)^{-1} f \|_{L^2(G, d\mu_M)}^2 \leq \sum_{j \in \mathbb{N}} \| t L_M (I + t L_M)^{-1} f \|_{L^2(B(x_j^t, 2\sqrt{t}), d\mu_M)}^2
\]

and we are left with the task of estimating

\[
\| t L_M (I + t L_M)^{-1} g_j^t \|_{L^2(B(x_j^t, 2\sqrt{t}), d\mu_M)}^2.
\]

To that purpose, set

\[
C_j^0 = B(x_j^t, 4\sqrt{t}) \quad \text{and} \quad C_k^j = B(x_j^t, 2^{k+2}\sqrt{t}) \setminus B(x_j^t, 2^{k+1}\sqrt{t}), \quad \forall k \geq 1,
\]

and \( g_k^j := g_j^t 1_{C_k^j} \), \( k \geq 0 \), where, for any subset \( A \subset G \), \( 1_A \) is the usual characteristic function of \( A \). Since \( g_j^t = \sum_{k \geq 0} g_k^j \) one has

\[
\| t L_M (I + t L_M)^{-1} g_j^t \|_{L^2(B(x_j^t, 2\sqrt{t}), d\mu_M)} \leq \sum_{k \geq 0} \| t L_M (I + t L_M)^{-1} g_k^j \|_{L^2(B(x_j^t, 2\sqrt{t}), d\mu_M)}.
\]

(3.20)
and, using Lemma 3.1, one obtains (for some constants $C, c > 0$

\[ (3.21) \quad \left\| t L_M (I + t L_M)^{-1} g^{j,t} \right\|_{L^2(B(x_j^+, 2\sqrt{t}), \mu_M)} \leq C \left( \left\| g_0^{j,t} \right\|_{L^2(C_0^{j,t}, \mu_M)} + \sum_{k \geq 1} e^{-c2^k} \left\| g_k^{j,t} \right\|_{L^2(C_k^{j,t}, \mu_M)} \right). \]

By Cauchy-Schwarz’s inequality, we deduce (for another constant $C' > 0$

\[ (3.22) \quad \left\| t L_M (I + t L_M)^{-1} g^{j,t} \right\|_{L^2(B(x_j^+, 2\sqrt{t}), \mu_M)} \leq C' \left( \left\| g_0^{j,t} \right\|_{L^2(C_0^{j,t}, \mu_M)}^2 + \sum_{k \geq 1} e^{-c2^k} \sum_{j \geq 0} \frac{\left\| g_k^{j,t} \right\|_{L^2(C_k^{j,t}, \mu_M)}^2}{2} \right). \]

As a consequence, we have

\[ (3.23) \quad \int_0^\infty t^{-1-\alpha/2} \left\| t L_M (I + t L_M)^{-1} f \right\|_{L^2(G, \mu_M)}^2 dt \leq C' \int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \left\| g_0^{j,t} \right\|_{L^2(C_0^{j,t}, \mu_M)}^2 dt + C' \int_0^\infty t^{-1-\alpha/2} \sum_{k \geq 1} e^{-c2^k} \sum_{j \geq 0} \frac{\left\| g_k^{j,t} \right\|_{L^2(C_k^{j,t}, \mu_M)}^2}{2} dt. \]

We claim that, and we postpone the proof into Appendix B:

**Lemma 3.4.** There exists $\bar{C} > 0$ such that, for all $t > 0$ and all $j \in \mathbb{N}$:

A. For the first term:

\[ \left\| g_0^{j,t} \right\|_{L^2(C_0^{j,t}, \mu_M)}^2 \leq \frac{\bar{C}}{V(\sqrt{t})} \int_{B(x_j^+, A\sqrt{t})} \int_{B(x_j^+, A\sqrt{t})} |f(x) - f(y)|^2 d\mu_M(x) dy. \]

B. For all $k \geq 1$,

\[ \left\| g_k^{j,t} \right\|_{L^2(C_k^{j,t}, \mu_M)}^2 \leq \frac{\bar{C}}{V(2^k\sqrt{t})} \int_{x \in B(x_j^+, 2^{k+2}\sqrt{t})} \int_{y \in B(x_j^+, 2^{k+2}\sqrt{t})} |f(x) - f(y)|^2 d\mu_M(x) dy. \]
We finish the proof of the theorem. Using Assertion A in Lemma 3.4 summing up on \( j \geq 0 \) and integrating over \((0, \infty)\), we get

\[
\int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t}, \rho_M)} dt = \sum_{j \geq 0} \int_0^\infty t^{-1-\alpha/2} \left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t}, \rho_M)} dt
\]

\[
\leq \bar{C} \sum_{j \geq 0} \int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} \left( \int_{B(x_j^t, 4\sqrt{t})} \int_{B(x_j^t, 4\sqrt{t})} |f(x) - f(y)|^2 d\mu_M(x) dy \right) dt
\]

\[
\leq \bar{C} \sum_{j \geq 0} \iint_{(x,y) \in G \times G} |f(x) - f(y)|^2 M(x) \times
\]

\[
\left( \int_{t \geq \max \left\{ \frac{|x_1^{-1} x_j^t|^2}{16}; \frac{|y_1^{-1} x_j^t|^2}{16} \right\}} t^{-1-\frac{\alpha}{2}} \frac{1}{V(\sqrt{t})} dt \right) dx dy.
\]

The Fubini theorem now shows

\[
\sum_{j \geq 0} \int_{t \geq \max \left\{ \frac{|x_1^{-1} x_j^t|^2}{16}; \frac{|y_1^{-1} x_j^t|^2}{16} \right\}} t^{-1-\frac{\alpha}{2}} \frac{1}{V(\sqrt{t})} dt =
\]

\[
\int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} \sum_{j \geq 0} 1 \left( \max \left\{ \frac{|x_1^{-1} x_j^t|^2}{16}; \frac{|y_1^{-1} x_j^t|^2}{16} \right\}, +\infty \right)(t) dt.
\]

Observe that, by Lemma 4.1, there is a constant \( N \in \mathbb{N} \) such that, for all \( t > 0 \), there are at most \( N \) indexes \( j \) such that \( |x_1^{-1} x_j^t|^2 < 16 t \) and \( |y_1^{-1} x_j^t|^2 < 16 t \), and for these indexes \( j \), one has \( |x_1^{-1} y_1| < 8\sqrt{t} \). It therefore follows that

\[
\sum_{j \geq 0} 1 \left( \max \left\{ \frac{|x_1^{-1} x_j^t|^2}{16}; \frac{|y_1^{-1} x_j^t|^2}{16} \right\}, +\infty \right)(t) \leq N 1_{[|x_1^{-1} y_1|^2/64, +\infty)}(t),
\]

so that, by 111,

\[
(3.24) \int_0^\infty t^{-1-\alpha/2} \sum_j \left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t}, \rho_M)} dt
\]

\[
\leq \bar{C} N \int_{G \times G} |f(x) - f(y)|^2 M(x) \left( \int_{|x_1^{-1} y_1|^2/64}^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} dt \right) dx dy
\]

\[
\leq \bar{C} N \int_{G \times G} \frac{|f(x) - f(y)|^2}{V(|x_1^{-1} y_1| |x_1^{-1} y_1|^{\alpha})} d\mu_M(x) dy.
\]

Using now Assertion B in Lemma 3.4 we obtain, for all \( j \geq 0 \) and all \( k \geq 1 \),
\[
\int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \| g_{j,t}^k \|_2^2 \, dt \\
\leq \tilde{C} \sum_{j \geq 0} \int_0^\infty \frac{t^{-1-\alpha/2}}{V(2^k \sqrt{t})} \left( \int \int_{B(x_j^t, 2^k \sqrt{t})} |f(x) - f(y)|^2 M(x) \, dx \, dy \right) \, dt \\
\leq \tilde{C} \sum_{j \geq 0} \int_{x,y \in G} |f(x) - f(y)|^2 M(x) \times \\
\left( \int_0^\infty \frac{t^{-1-\alpha/2}}{V(2^k \sqrt{t})} \frac{1}{\left( \max \left\{ \frac{|x^{-1}x_j^t|^2}{4^{k+2}}, \frac{|y^{-1}x_j^t|^2}{4^{k+2}} \right\}, +\infty \right)} (t) \, dt \right) \, dx \, dy.
\]

But, given \( t > 0 \), \( x, y \in G \), by Lemma 4.1 again, there exist at most \( \tilde{C} 2^{2k\kappa} \) indexes \( j \) such that

\[
|x^{-1}x_j^t| \leq 2^{k+2} \sqrt{t} \quad \text{and} \quad |y^{-1}x_j^t| \leq 2^{k+2} \sqrt{t},
\]

and for these indexes \( j \), \( |x^{-1}y| \leq 2^{k+3} \sqrt{t} \). As a consequence,

\[
\int_0^\infty \frac{t^{-1-\alpha/2}}{V(2^k \sqrt{t})} \sum_{j \geq 0} \frac{1}{\left( \max \left\{ \frac{|x^{-1}x_j^t|^2}{4^{k+2}}, \frac{|y^{-1}x_j^t|^2}{4^{k+2}} \right\}, +\infty \right)} (t) \, dt \leq \\
\tilde{C} 2^{2k\kappa} \int_{t \geq \frac{|x^{-1}y|^2}{2^{(2\kappa+\alpha)}}} \frac{t^{-1-\alpha/2}}{V(2^k \sqrt{t})} \, dt \leq \\
\tilde{C}' \frac{V(|x^{-1}y|)}{|x^{-1}y|^{\alpha}},
\]

for some other constant \( \tilde{C}' > 0 \), and therefore

\[
\int_0^\infty \frac{t^{-1-\alpha/2}}{V(2^k \sqrt{t})} \sum_{j} \| g_{j,t}^k \|_2^2 \, dt \leq \\
\tilde{C} \tilde{C}' \frac{V(|x^{-1}y|)}{|x^{-1}y|^{\alpha}} \int_{G \times G} \frac{|f(x) - f(y)|^2}{V(|x^{-1}y|)} M(x) \, dx \, dy.
\]
We can now conclude the proof of Lemma 3.3, using Lemma 3.2, (3.21), (3.24) and (3.25). We have proved, by reconsidering (3.23):

\[
\int_0^\infty t^{-1-\alpha/2} \left\| t L_M (I + t L_M)^{-1} f \right\|_{L^2(G,d\mu_M)}^2 dt \leq C' \bar{C} N \int \int_{G \times G} \frac{|f(x) - f(y)|^2}{V(|x-1y|)|x-y|^\alpha} M(x) \, dx \, dy
\]

\[+ \sum_{k \geq 1} C' \bar{C} e^{2k(\kappa+\alpha)} e^{-c2^k} \int \int_{G \times G} \frac{|f(x) - f(y)|^2}{V(|x-1y|)|x-y|^\alpha} M(x) \, dx \, dy \]

and we deduce that

\[
\int_0^\infty t^{-1-\alpha/2} \left\| t L_M (I + t L_M)^{-1} f \right\|_{L^2(G,d\mu_M)}^2 dt \leq C \int \int_{G \times G} \frac{|f(x) - f(y)|^2}{V(|x-1y|)|x-y|^\alpha} d\mu_M(x) \, dy
\]

for some constant $C$ as claimed in the statement. \(\Box\)

**Remark 3.5.** In the Euclidean context, Strichartz proved in ([Str67]) that, when $0 < \alpha < 2$, for all $p \in (1, +\infty)$,

\[
\left\| (-\Delta)^{\alpha/4} f \right\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,p} \| S_{\alpha} f \|_{L^p(\mathbb{R}^n)}
\]

where

\[
S_{\alpha} f(x) = \left( \int_0^{+\infty} \left( \int_B |f(x + ry) - f(x)| \, dy \right)^2 \frac{dr}{r^{1+\alpha}} \right)^{\frac{1}{2}},
\]

and also ([Ste61])

\[
\left\| (-\Delta)^{\alpha/4} f \right\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,p} \| D_{\alpha} f \|_{L^p(\mathbb{R}^n)}
\]

where

\[
D_{\alpha} f(x) = \left( \int_{\mathbb{R}^n} \frac{|f(x + y) - f(x)|^2}{|y|^{\alpha+n}} \, dy \right)^{\frac{1}{2}}.
\]

In [CRTN01], these inequalities were extended to the setting of a unimodular Lie group endowed with a sub-Laplacian $\Delta$, relying on semigroups techniques and Littlewood-Paley-Stein functionals. In particular, in [CRTN01], the authors use pointwise estimates of the kernel of the semigroup generated by $\Delta$. In the present paper, we deal with the operator $L_M$ for which these pointwise estimates are not available, but it turns out that $L^2$ off-diagonal estimates are enough for our purpose. Note that we do not obtain $L^p$ inequalities here.
4. Appendix A: Technical lemma

We prove the following lemma.

Lemma 4.1. Let $G$ and the $x^t_j$ be as in the proof of Lemma 3.3. Then there exists a constant $\tilde{C} > 0$ with the following property: for all $\theta > 1$ and all $x \in G$, there are at most $\tilde{C} \theta^2 \kappa$ indexes $j$ such that $|x^{-1}x^t_j| \leq \theta \sqrt{t}$.

Proof of Lemma 4.1. The argument is very simple (see [Kan85]) and we give it for the sake of completeness. Let $x \in G$ and denote

$$ I(x) := \left\{ j \in \mathbb{N} ; |x^{-1}x^t_j| \leq \theta \sqrt{t} \right\}. $$

Since, for all $j \in I(x)$

$$ B \left( x^t_j, \sqrt{t} \right) \subset B \left( x, (1 + \theta) \sqrt{t} \right), $$

and

$$ B \left( x, \sqrt{t} \right) \subset B \left( x^t_j, (1 + \theta) \sqrt{t} \right), $$

one has by (1.2) and the fact that the balls $B \left( x^t_j, \sqrt{t} \right)$ are pairwise disjoint,

$$ |I(x)| V \left( x, \sqrt{t} \right) \leq \sum_{j \in I(x)} V \left( x^t_j, (1 + \theta) \sqrt{t} \right) \leq C (1 + \theta)^\kappa \sum_{j \in I(x)} V \left( x^t_j, \sqrt{t} \right) \leq C (1 + \theta)^\kappa V \left( x, (1 + \theta) \sqrt{t} \right) \leq C (1 + \theta)^{2\kappa} V \left( x, \sqrt{t} \right) $$

and we get the desired conclusion.

5. Appendix B: Estimates for $g^t_j$

We prove Lemma 3.4. For all $x \in G$,

$$ g^t_0(x) = f(x) - \frac{1}{V(2\sqrt{t})} \int_{B(x^t_j, 2\sqrt{t})} f(y) \, dy = \frac{1}{V(2\sqrt{t})} \int_{B(x^t_j, 2\sqrt{t})} (f(x) - f(y)) \, dy. $$

By Cauchy-Schwarz inequality and (1.1), it follows that

$$ |g^t_0(x)|^2 \leq \frac{C}{V(\sqrt{t})} \int_{B(x^t_j, 4\sqrt{t})} |f(x) - f(y)|^2 \, dy. $$
Therefore,
\[ \|g_{0,t}^j\|_{L^2(C_0^{j,t},M)}^2 \leq \frac{C}{V(\sqrt{t})} \int_{B(x_j^t,4\sqrt{t})} \int_{B(x_j^t,4\sqrt{t})} |f(x) - f(y)|^2 \, d\mu_M(x) \, dy, \]
which shows Assertion A. We argue similarly for Assertion B and obtain
\[ \|g_{k,t}^j\|_{L^2(C_k^{j,t},M)}^2 \leq \frac{C}{V(2^{k\sqrt{t}})} \int_{x \in B(x_j^t,2^{k+2}\sqrt{t})} \int_{y \in B(x_j^t,2^{k+2}\sqrt{t})} |f(x) - f(y)|^2 \, d\mu_M(x) \, dy, \]
which ends the proof.

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