An integrable semi-discrete Degasperis–Procesi equation

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Abstract
Based on our previous work on the Degasperis–Procesi equation (Feng et al J. Phys. A: Math. Theor. 46 045205) and the integrable semi-discrete analogue of its short wave limit (Feng et al J. Phys. A: Math. Theor. 48 135203), we derive an integrable semi-discrete Degasperis–Procesi equation by Hirota’s bilinear method. Furthermore, N-soliton solution to the semi-discrete Degasperis–Procesi equation is constructed. It is shown that both the proposed semi-discrete Degasperis–Procesi equation, and its N-soliton solution converge to ones of the original Degasperis–Procesi equation in the continuum limit.

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1. Introduction

In this paper, we are concerned with integrable discretization of the Degasperis–Procesi (DP) equation [1, 2]

\[ m_t + 3mu_x + m_xu = 0, \quad m = -a + u - u_{xxx}, \]  

or

\[ u_t - 3au_x - u_{xxx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}. \]  

Prior to the DP equation, the Camassa–Holm (CH) equation firstly appeared in a mathematical search for recursion operators connected with the integrable partial differential
equations [3] and then has attracted considerable attention since it was derived as a model equation for shallow water waves [4]. The CH and the DP equations are the only two integrable equations among the b-family equations [5]

\[ m_t + bmu_x + m_2u = 0, \quad m = -a + u - u_{xxx}, \]

(1.3)
or alternatively

\[ u_t - bau_x - u_{xxx} + (b + 1)u_{xx} = bu_xu_{xx} + uu_{xxx}, \]

(1.4)

with \( b = 2 \) and \( b = 3 \), respectively.

It is shown in [6] that both the CH and the DP equations can be used as models for the propagation of shallow water waves over a flat bed, which accommodate wave breaking phenomena. These two equations are very similar, only differing by coefficients. They also share some common properties, for example, when \( a = 0 \), both the CH and DP equations have multi-peakon solutions [7–9] and admit wave breaking phenomena [10, 11].

On the other hand, although the DP equation has an apparent similarity to the CH equation, there are major structural differences between these two equations such as the Lax pair, blow up phenomena and the solutions. The isospectral problem in the Lax pair for the DP equation is the third-order equation [2], while the isospectral problem for the CH equation is the second order equation [4]. Therefore, the DP equation is much more complicated in term of the bi-Hamiltonian structures [12], the multi-peakon solution [13], the inverse scattering transform [14], the Riemann–Hilbert problem [15], as well as its bilinear equations and its multi-soliton solutions [16–18].

It is known that Hirota’s bilinear method is very useful in finding multi-soliton solutions and constructing integrable discretizations to soliton equations [19]. Following this direction, the authors have succeeded in constructing integrable discretizations to a class of soliton equations with hodograph transformation. The examples include the short pulse equation and its multi-component generalizations [20–23], the CH equation and its short wave limit (the Hunter–Saxton equation) [24, 25], the short wave limit of the DP equation (the reduced Ostrovsky equation or the Vakhnenko equation) [26–29].

The goal of this paper is to construct integrable semi-discretization of the DP equation. It is a challenging problem compared with the CH equation. In the present paper, we propose an integrable semi-discrete DP equation based on our previous results regarding the bilinear structure of the DP equation [18] and the integrable discretizations of its short wave limit [29]. The remainder of the present paper is organized as follows. In section 2, we give a brief review of the relevant bilinear equations and their connection with the negative flow of the CKP hierarchy. In section 3, semi-discrete integrable analogues of the bilinear equations of the DP equation are constructed. Then, we propose an integrable semi-discrete DP equation in section 4. The paper is concluded in section 5 by some comments and further topics.

### 2. Review of the bilinear equations of the Degasperis–Procesi equations and its N-soliton solution

In this section, in order to make this paper self-contained, we will give a brief review to the bilinear equations of the Degasperis–Procesi equation, as well as its \( N \)-soliton solution as presented in [18].

We start with a sequence of \( 2N \times 2N \) Gram-type determinants [30]

\[ F_n = \det_{1 \leq i, j \leq 2N} (m_{ij}(n)), \]

(2.1)
where the entries of the determinant are defined by

\[
m_{ij}(n) = \delta_{i,2N+1-j} \frac{a^2}{2N+1} \frac{\epsilon_{ij}(p_i - p_j)}{1 + ap_i} + \frac{1}{p_i + p_j} \left( \frac{p_i}{p_j} \right)^a \phi(0) \phi(j)(0),
\]

with

\[
\phi(j)(k) = \left( \frac{1 + ap_j}{1 - ap_j} \right)^k e^{\xi_j}, \quad \xi_i = p_i^{-1}s + p_iy + \xi_{i0}, \quad \epsilon_{ij} = \begin{cases} 1 & i < j, \\ -1 & i > j. \end{cases}
\]

Meanwhile a sequence of pfaffians which belongs to the BKP hierarchy [31, 32] can be defined by

\[
\tau_k = \text{Pf}(1, 2, \ldots, 2N)_k
\]
whose \((i, j)\) elements are

\[
(i, j)_k = \delta_{i,2N+1-j} \epsilon_{ij} + \frac{p_i - p_j}{p_i + p_j} \phi(j)(k).\]

Imposing a reduction condition

\[
p_i^2 - p_i p_{2N+1-i} + p_i^2 = \frac{1}{a^2},
\]
and setting

\[
f = \tau_0, \quad g = \tau_1, \quad F = F_0, \quad G = F_1,
\]
as shown in [18], the following relations hold between the determinants and pfaffians

\[
\left( D_jD_y - aD_y - \frac{1}{a} \right) \cdot f = 0,
\]

\[
gf = cG,
\]

\[
(-aD_y + 1)g \cdot f = cF,
\]

\[
\left( \frac{1}{2} D_jD_y - 1 \right) F \cdot F = -G^2,
\]

where

\[
c = \prod_{j=1}^{2N} \frac{1 + ap_j}{1 - ap_j},
\]

and \(D_yD_y\) is the Hirota \(D\)-operator defined by

\[
D_y^mD_y^n f(y, s) \cdot g(y, s) = \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial s'} \right)^n f(y, s)g(y', s')|_{y = y' = s = s'}.\]

We comment here that the above four equations (2.4)–(2.7) are basically equivalent to equations (2.20)–(2.24) in [18] although they seem slightly different. In what follows, we will show briefly how the equations (2.4)–(2.7) yield the DP equation through a dependent variable transformation.
\[ u = \left( \ln \frac{g}{f} \right)_s, \]  
(2.8)

and a hodograph transformation

\[ x = -\frac{1}{a} y + \ln \frac{g}{f}, \quad t = s. \]  
(2.9)

Dividing \( gf \) on both sides, the first three equations (2.4)–(2.6) can be rewritten as

\[
(\ln gf)_y + \left( \ln \frac{g}{f} \right)_y - \frac{1}{a} \left( \ln \frac{g}{f} \right)_s - a = 0, 
\]  
(2.10)

\[ 1 = \frac{eG}{gf}, \]  
(2.11)

\[ -a\left( \ln \frac{g}{f} \right)_y + 1 = \frac{eF}{gf}. \]  
(2.12)

While, by dividing \( F^2 \) on both sides, the bilinear equation (2.7) becomes

\[
(\ln F)_y - 1 = -\frac{G^2}{F^2}. 
\]  
(2.13)

With the use of (2.11), (2.12) becomes

\[ -a\left( \ln \frac{g}{f} \right)_y + 1 = \frac{F}{G}. \]  
(2.14)

Subtracting (2.13) from (2.10), one obtains

\[
\left( \ln \frac{G}{F} \right)_y + \left( \ln \frac{g}{f} \right)_y - \frac{1}{a} \left( \ln \frac{g}{f} \right)_s - a = \frac{G^2}{F^2} 
\]  
(2.15)

by referring to (2.11).

Introducing an intermediate variable \( \rho = GF \), one can calculate that

\[
\frac{\partial x}{\partial y} = -\frac{1}{a} + \left( \ln \frac{g}{f} \right)_y = -\frac{1}{a\rho}, \quad \frac{\partial x}{\partial x} = \left( \ln \frac{g}{f} \right)_s = u, \]
based on the transformations (2.8) and (2.9), which yields a conversion formula

\[ \partial_y = -\frac{1}{a\rho} \partial_x, \quad \partial_x = \partial_t + u \partial_y. \]  
(2.16)

Substituting (2.14) into (2.15), one obtains

\[
(\ln \rho)_y - \frac{1}{a\rho} (u - a) = \rho^2, \]  
(2.17)

which can be rewritten as

\[ ((\ln \rho)_t)_x + u - a = -a\rho^3. \]  
(2.18)
On the other hand, differentiating (2.14) with respect to \( s \), one yields
\[
\left( \frac{1}{\rho} \right)_s + au_s = 0,
\] (2.19)
which, in turn, becomes
\[
(\ln \rho)_s = -u_x
\] (2.20)
by using the conversion formula (2.16).

In the last, eliminating \( \rho \) from (2.18)–(2.20), one obtains
\[
(\partial_t + u\partial_x) \ln(a - u_{xx} - u) = -3u_x,
\] (2.21)
which is nothing but the Degasperis–Procesi equation (1.2).

### 3. Semi-discrete analogue of equations (2.4)–(2.7)

Based on the results mentioned in the previous section, we attempt to construct an integrable semi-discrete analogue of the DP equation (1.2) by using Hirota’s bilinear method. The key point is how to obtain discrete analogues of the bilinear equations (2.4)–(2.7).

Keeping in mind that the Degasperis–Procesi equation is derived from a pseudo 3-reduction of the CKP hierarchy, which means a reduction relating \( x_3 \)-dependence to the dependence of lower degree variables, for example, \( D_{\alpha_3} = \alpha D_{\omega_3} + \beta D_{\alpha_2} \) in bilinear equations where \( \alpha \) and \( \beta \) are constants [33]. By the way, a 3-reduction is the reduction killing the \( x_3 \)-dependence, i.e. \( D_{\alpha_3} = 0 \) in bilinear equations.

We start with the Gram-type determinants, which are soliton solutions of the CKP hierarchy,
\[
F_{k,l} = \det_{1 \leq i,j \leq 2N} (m_{ij}(k,l)), \quad G_{k,l} = \det_{1 \leq i,j \leq 2N} (m'_{ij}(k,l)),
\]
where
\[
m_{ij}(k,l) = C_{ij} + \frac{1}{p_i + p_j} \varphi_{i}^{(0)}(k,l) \varphi_{j}^{(0)}(k,l),
\]
\[
m'_{ij}(k,l) = C_{ij} + \left( \frac{p_i}{p_i + p_j} \right) \left( 1 + \frac{bp_j}{p_j} \right) \varphi_{i}^{(0)}(k,l) \varphi_{j}^{(0)}(k,l),
\]
with
\[
C_{ij} = C_{ji}, \quad \varphi_{i}^{(0)}(k,l) = p_i^s \left( \frac{1 + ap_j}{1 - ap_j} \right)^{\xi_i} \left( \frac{1 + bp_j}{1 - bp_j} \right)^{\xi_i}, \quad \xi_i = p_i^{-1}s + \xi_{0i}.
\]
Here \( 2b \) (not \( b \)) is the mesh size in \( y \)-direction. The relation between \( F_{k,l} \) and \( G_{k,l} \) is given by the following lemma.

**Lemma 3.1.**
\[
(D_s - 2b)F_{k,l+1} \cdot F_{k,l} = -2bG_{k,l}^2.
\] (3.1)

**Proof.** It can be easily verified that
\[
\partial_s m_{ij}(k,l) = \varphi_{i}^{(-1)}(k,l) \varphi_{j}^{(-1)}(k,l),
\]
\[ m_i(k, l + 1) = m_i(k, l) + \frac{2b}{(1 - bp_i)(1 - bp_j)} \varphi_i^{(0)}(k, l)\varphi_j^{(0)}(k, l), \]

and

\[ m_i'(k, l) = m_i(k, l) - \frac{1}{1 - bp_j} \varphi_i^{(-1)}(k, l)\varphi_j^{(0)}(k, l). \]

Then we have

\[
\begin{align*}
\partial_t F_{k,l} &= \begin{vmatrix}
m_i(k, l) & \varphi_i^{(-1)}(k, l) \\
-\varphi_j^{(-1)}(k, l) & 0
\end{vmatrix}, \\
F_{k,l+1} &= \begin{vmatrix}
m_i(k, l) & \frac{2b}{1 - bp_i} \varphi_i^{(0)}(k, l) \\
\frac{1}{1 - bp_j} \varphi_j^{(0)}(k, l) & 1
\end{vmatrix}, \\
G_{k,l} &= \begin{vmatrix}
m_i(k, l) & \varphi_i^{(-1)}(k, l) \\
\frac{1}{1 - bp_j} \varphi_j^{(0)}(k, l) & 1
\end{vmatrix} = \begin{vmatrix}
m_i(k, l) & \frac{1}{1 - bp_i} \varphi_i^{(0)}(k, l) \\
\varphi_j^{(-1)}(k, l) & 1
\end{vmatrix}, \\
(\partial_t - 2b) F_{k,l+1} &= \begin{vmatrix}
m_i(k, l) & \varphi_i^{(-1)}(k, l) & \frac{2b}{1 - bp_i} \varphi_i^{(0)}(k, l) \\
0 & -\varphi_j^{(-1)}(k, l) & -2b \\
-\frac{1}{1 - bp_j} \varphi_j^{(0)}(k, l) & -1 & 1
\end{vmatrix}.
\end{align*}
\]

By the Jacobi identity of determinants, we obtain

\[(\partial_t - 2b) F_{k,l+1} \times F_{k,l} = F_{k,l+1} \times \partial_t F_{k,l} - (-2b G_{k,l}) \times (-G_{k,l}),\]

which is exactly the bilinear equation (3.1).

Next, we perform reductions similar to the pseudo 3-reduction of the CKP hierarchy in the continuous case. To this end, we take

\[ c_y = \delta_{y, 2N+1} - c_i, \quad c_{2N+1-i} = c_i, \]

and further assume

\[ c_y = 2p_iC_y \frac{1 + ap_i}{1 - ap_i} \frac{1 - bp_j}{1 + bp_j}. \]

By imposing a reduction condition

\[ \frac{(1 - a^2 p_{2N+1-i}^2)(1 - b^2 p_i^2)}{p_i} + \frac{(1 - a^2 p_i^2)(1 - b^2 p_{2N+1-i}^2)}{p_{2N+1-i}} = 0, \]

By the Jacobi identity of determinants, we obtain

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and further assume

\[ c_y = 2p_iC_y \frac{1 + ap_i}{1 - ap_i} \frac{1 - bp_j}{1 + bp_j}. \]

By imposing a reduction condition

\[ \frac{(1 - a^2 p_{2N+1-i}^2)(1 - b^2 p_i^2)}{p_i} + \frac{(1 - a^2 p_i^2)(1 - b^2 p_{2N+1-i}^2)}{p_{2N+1-i}} = 0, \]
or, equivalently,
\[
\frac{p_i (1 + p_i) (1 - b_{2N+1-i})}{(1 - p_{2N+1-i})(1 + b_i)} = - \frac{p_{2N+1-i} (1 + p_{2N+1-i}) (1 - b_i)}{(1 - p_i)(1 + b_{2N+1-i})},
\]
(3.5)

it then follows that
\[
c_{ij} = \delta_{j,2N+1-i} \cdot \epsilon_{2N+1-i} \cdot \frac{2p_i (1 + p_i)}{1 - p_{2N+1-i}} \cdot \frac{1 - b_{2N+1-i}}{1 + b_i} = - \delta_{i,2N+1-j} \cdot \epsilon_{2N+1-j} \cdot \frac{2p_{2N+1-i} (1 + p_{2N+1-i})}{1 - p_i} \cdot \frac{1 - b_i}{1 + b_{2N+1-i}} = - c_{ji}.
\]

Therefore, we can define a pfaffian of the form
\[
f_{kl} = \text{Pf}(1, 2, \cdots, 2N)_{kl},
\]
whose \((i, j)_{kl}\) elements defined by
\[
(i, j)_{kl} = c_{ij} + \frac{p_i - p_j}{p_i + p_j} \varphi^{(0)}_i(k, l) \varphi^{(0)}_j(k, l).
\]

Then the following lemma provides a bilinear equation satisfied by the pfaffian \(f_{kl}\).

**Lemma 3.2.**
\[
\frac{1}{a + b} D_s - 1)_{k+1,l+1} \cdot f_{kl} = \left( \frac{1}{a - b} D_s - 1 \right)_{k+1,l+1} \cdot f_{k+1,l+1}.
\]
(3.6)

**Proof.** Letting
\[
(i, d_a)_{kl} = \varphi^{(a)}_i(k, l), \quad (d_m, d_a)_{kl} = 0,
\]
\[
(i, d^b_i)_{kl} = \varphi^{(b)}_i(k + 1, l), \quad (d_0, d^b)_{kl} = 1, \quad (d^b, d^b)_{kl} = -a,
\]
\[
(i, d^d_i)_{kl} = \varphi^{(d)}_i(k, l + 1), \quad (d_0, d^d)_{kl} = 1, \quad (d^d, d^d)_{kl} = -b,
\]

and
\[
(d^b_i, d^d_i)_{kl} = \frac{a - b}{a + b},
\]
it is shown in the Appendix that
\[
\partial_{d_k} f_{kl} = (1, 2, \cdots, 2N, d_{-1}, d_0)_{kl},
\]
(3.7)
\[
f_{k+1,l} = (1, 2, \cdots, 2N, d_0, d^b)_{kl},
\]
(3.8)
\[
f_{k,l+1} = (1, 2, \cdots, 2N, d_0, d^d)_{kl},
\]
(3.9)
\[
(\partial_a - a) f_{k+1,l} = (1, 2, \cdots, 2N, d_{-1}, d^d)_{kl},
\]
(3.10)
\[
(\partial_b - b) f_{k,l+1} = (1, 2, \cdots, 2N, d_{-1}, d^b)_{kl},
\]
(3.11)
\[
\frac{a - b}{a + b} f_{k+1,l+1} = (1, 2, \cdots, 2N, d^b, d^d)_{kl},
\]
(3.12)
\[
(\partial_a - a) b \frac{a - b}{a + b} f_{k+1,l+1} = (1, 2, \cdots, 2N, d_{-1}, d_0, d^d, d^d)_{kl}.
\]
(3.13)
Therefore, an algebraic identity of pfaffian [19],

\[
\text{Pf}(\cdots, d_{-1}, d_0, d'_1, d'_2)\text{Pf}(\cdots) = \text{Pf}(\cdots, d_{-1}, d_0)\text{Pf}(\cdots, d'_1, d'_2) - \text{Pf}(\cdots, d_{-1}, d'_1)\text{Pf}(\cdots, d_0, d'_2) + \text{Pf}(\cdots, d_{-1}, d'_2)\text{Pf}(\cdots, d_0, d'_1),
\]

implies

\[
(\partial_h - a - b)\frac{a - b}{a + b} f_{k+1,l+1} \times f_{kl} = \frac{a - b}{a + b} f_{k+1,l+1} \times \partial_h f_{kl} - f_{k+1,l} \times (\partial_h - a - b)f_{k,l+1} + (\partial_h - a)f_{k+1,l} \times f_{k,l+1},
\]

which is nothing but the bilinear equation (3.6).

The lemma below states the relations between pfaffians and determinants defined previously.

**Lemma 3.3.**

\[
f_{k+1,l} f_{kl} = c^* G_{kl}, \tag{3.14}
\]

\[
(a - b)f_{k+1,l+1} f_{kl} = (a + b)f_{k+1,l} f_{kl+1} = -2bc^* F_{k,l+1}, \tag{3.15}
\]

**Proof.** By an identity of pfaffian [18]

\[
f_{k+1,l} f_{kl} = \text{Pf} \left( \begin{array}{c} (i,j)_{kl} \ 
\phi^{(0)}_i(k,l) \ 
\phi^{(0)}_j(k+1,l) \ 
1 \end{array} \right) \times \text{Pf}(i,j)_{kl}
\]

\[
= \begin{vmatrix} (i,j)_{kl} & \phi^{(0)}_i(k,l) \\ \phi^{(0)}_j(k+1,l) & 1 \end{vmatrix}
\]

\[
= \det_{i,j \in N} (\langle(i,j)_{kl} - \phi^{(0)}_i(k,l)\phi^{(0)}_j(k+1,l)\rangle)
\]

\[
= \det \left( c_{ij} + \frac{p_i - p_j}{p_i + p_j} \phi^{(0)}_i(k,l)\phi^{(0)}_j(k,l) - \phi^{(0)}_i(k,l)\phi^{(0)}_j(k+1,l) \right)
\]

\[
= \det \left( c_{ij} + \frac{-2p_i}{p_i + p_j} \phi^{(0)}_i(k,l)\phi^{(0)}_j(k,l) \right)
\]

\[
= c' \det \left( c_{ij} + \frac{1}{2p_i} \frac{1 - ap_j}{1 + ap_j} \phi^{(0)}_i(k,l)\phi^{(0)}_j(k,l) \right)
\]

\[
= c' \det \left( c_{ij} + \frac{1}{p_i + p_j} \phi^{(0)}_i(k,l)\phi^{(0)}_j(k,l) \right)
\]

Thus, equation (3.14) is proved. Next, we proceed to the proof of equation (3.15). Firstly, by the same identity as above, the products of pfaffians can be rewritten into determinants.
\[
\begin{align*}
\frac{a - b}{a + b} f_{k+1,l+1} = & \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l + 1) \varphi_i(0)(k + 1, l) \right) \\
= & \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l + 1) \frac{a - b}{a + b} \right) \times \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l) \varphi_i(0)(k + 1, l) \right)
\end{align*}
\]

Consequently,
\[
\begin{align*}
\frac{a - b}{a + b} f_{k+1,l+1} - f_{k+1,l} = & \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l + 1) \varphi_i(0)(k, l) \varphi_i(0)(k + 1, l) \right) \\
= & \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l + 1) \varphi_i(0)(k, l) \varphi_i(0)(k + 1, l) \right) \\
= & \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l) \varphi_i(0)(k + 1, l) \right) \\
= & \text{ Pr} \left( (i,j)_{kl} \varphi_i(0)(k, l) \varphi_i(0)(k + 1, l) \right)
\end{align*}
\]
Multiplying both sides by \((a + b)\), we arrive at (3.15).

Summarizing what we have discussed and letting \(f_i = f_{0i},\ g_i = f_{1i},\ F_i = F_{0i},\ G_i = G_{0i}\), we arrive at the following four equations

\[
\left(\frac{1}{a + b} D_a - 1\right) g_{i+1} \cdot f_i = \left(\frac{1}{a - b} D_a - 1\right) g_i \cdot f_{i+1},
\]

\[g_{i+1} = c' G_i,\]

\[(D_a - 2b) f_{i+1} \cdot F_i = -2b G_i^2;\]

\[(a - b) g_{i+1} f_i - (a + b) g_i f_{i+1} = -2bc' F_{i+1}.\]

In fact, equations (3.20)–(3.23) are integrable semi-discrete analogues of equations (2.4)–(2.7). In other words, in the limit \(b \to 0\), equations (3.20)–(3.23) converge to equations (2.4)–(2.7), respectively. Meanwhile, the pfaffian and determinant solutions satisfying equations (3.20)–(3.23) also converge to the pfaffian and determinant solutions satisfying equations (2.4)–(2.7).

Note that \(2b\) is the mesh size. In the limit of \(b \to 0\), we have \(c' \to c\),

\[f_i \to f,\quad g_i \to g,\quad f_{i+1} \to f + 2bf_i,\quad g_{i+1} \to g + 2bg_i,
\]

and similar relations for the determinants \(F_i, G_i, F_{i+1}\) and \(G_{i+1}\). Obviously, (3.21) converges to (2.5) as \(b \to 0\). It can be easily shown that

\[
\frac{1}{2b} D_a F_{i+1} \cdot F_i \to \frac{1}{2} D_a D_a F \cdot F,
\]

and

\[
\frac{1}{2b} (g_{i+1} f_i - g_i f_{i+1}) \to D_a g \cdot f,\quad \frac{1}{2} (g_{i+1} f_i + g_i f_{i+1}) \to gf.
\]

Therefore, by dividing \(2b\) on both sides, (3.22) and (3.23) converge to (2.6) and (2.7), respectively, as \(b \to 0\). Now we show the convergence of the first bilinear equation. It is obvious by noting that

\[
\frac{a}{2b} \left(\frac{1}{a + b} D_a g_{i+1} \cdot f_i - \frac{1}{a - b} D_a g_i \cdot f_{i+1}\right)
\]

\[\to \frac{1}{2b} \left(1 - \frac{b}{a}\right) D_a g_{i+1} \cdot f_i - \left(1 + \frac{b}{a}\right) D_a g_i \cdot f_{i+1}\]

\[\to \frac{1}{2b} D_a (g_{i+1} \cdot f_i - g_i \cdot f_{i+1}) - \frac{1}{2a} D_a (g_{i+1} \cdot f_i + g_i \cdot f_{i+1})
\]

\[\to \frac{1}{2} D_a D_a g \cdot f - \frac{1}{a} gf,
\]

and

\[
\frac{a}{2b} (g_{i+1} f_i - g_i f_{i+1})
\]

\[\to aD_a g \cdot f.
\]
4. Semi-discrete Degasperis–Procesi equation

Now that we have constructed integrable semi-discrete analogues (3.20)–(3.23) of the set of equations (2.4)–(2.7) which produce the Degasperis–Procesi equation, we proceed to construct an integrable semi-discrete Degasperis–Procesi equation based on Hirota’s bilinear method. First, let us work on the bilinear equation (3.20), which can be recast into

\[
2g_{t+1}f_t \left( (a - b) \left( \ln \frac{g_{t+1}}{f_t} \right)_x - a^2 + b^2 \right) - 2g_{t+1}f_t \left( (a + b) \left( \ln \frac{g_t}{f_t} \right)_x - a^2 + b^2 \right) = 0,
\]

by multiplying \(2(a^2 - b^2)\) on both sides, or,

\[
((a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t) \left( \ln \frac{g_{t+1}f_{t+1}}{g_t f_t} \right)_x - 2b((a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t) \left( \ln \frac{g_{t+1}g_t}{f_{t+1}f_t} \right)_x - 2(a^2 - b^2)(g_{t+1}f_t - g_t f_{t+1}) = 0.
\]

Rearranging the terms, one obtains

\[
((a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t) \left( \ln \frac{g_{t+1}f_{t+1}}{g_t f_t} \right)_x - 2b((a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t) \left( \ln \frac{g_{t+1}g_t}{f_{t+1}f_t} \right)_x - 2a = 0.
\]

(4.1)

Dividing \((a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t\) on both sides of (4.1), we have

\[
\left( \ln \frac{g_{t+1}f_{t+1}}{g_t f_t} \right)_x + \frac{(a - b)g_{t+1}f_t - (a + b)g_{t+1}f_t}{(a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t} \left( \ln \frac{g_{t+1}g_t}{f_{t+1}f_t} \right)_x - 2a = 0.
\]

(4.2)

Secondly, equations (3.22) and (3.23) can be easily rewritten as

\[
\frac{(a - b)g_{t+1}f_t - (a + b)g_{t+1}f_t}{(a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t} = \frac{-2bcF_{t+1}}{(a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t},
\]

(4.3)

and

\[
\left( \ln \frac{F_{t+1}}{F_t} \right)_x - 2b = -2b\frac{G^2}{F_{t+1}F_t},
\]

(4.4)

respectively. Subtracting equations (4.4) from (4.2), we get

\[
\left( \ln \frac{G_{t+1}F_t}{G_t F_{t+1}} \right)_x + \frac{(a - b)g_{t+1}f_t - (a + b)g_{t+1}f_t}{(a - b)g_{t+1}f_t + (a + b)g_{t+1}f_t} \left( \ln \frac{G_{t+1}g_t}{f_{t+1}f_t} \right)_x - 2\left( -2b\frac{G^2}{F_{t+1}F_t} \right) = 0.
\]

(4.5)

by referring to equation (3.21).

Introducing variable transformations
\[ u_l = \left( \ln \frac{F_l}{f_l} \right), \quad r_l = \frac{G_l}{F_l}, \]  
\hspace{1cm} (4.6)

and a discrete hodograph transformation
\[
\delta_l = -\frac{4bc'F_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_lf_{l+1}}, \quad t = s, 
\]  
\hspace{1cm} (4.7)

where \( \delta_l = x_{l+1} - x_l \), we then have
\[
\frac{(a-b)g_{l+1}f_l - (a+b)g_lf_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_lf_{l+1}} = \frac{\delta_l}{2} 
\]  
\hspace{1cm} (4.8)

from equation (4.3). A substitution of equation (4.8) into equation (4.5) leads to
\[
\left( \ln \frac{r_{l+1}}{r_l} \right) + \delta_l \left( \frac{u_{l+1} + u_l}{2} - a \right) = 2b \frac{F_l}{F_{l+1}} r_l^2 
\]  
\hspace{1cm} (4.9)

by using variable transformations.

Furthermore, based on (4.3) and (4.7), we obtain
\[
\frac{a - b}{b} \frac{g_{l+1}f_l}{c'F_{l+1}} = \frac{2}{\delta_l} + 1, \quad \frac{a + b}{b} \frac{g_lf_{l+1}}{c'F_l} = \frac{2}{\delta_l} - 1. 
\]  
\hspace{1cm} (4.10)

Multiplying above two equations leads to
\[
\frac{a^2 - b^2}{b^2} \frac{F_l}{F_{l+1}} r_{l+1} r_l = \frac{4}{\delta_l^2} - 1, 
\]  
\hspace{1cm} (4.11)

while dividing them yields
\[
\frac{a - b}{a + b} \frac{g_{l+1}f_l}{f_{l+1}g_l} = \frac{2 + \delta_l}{2 - \delta_l}. 
\]  
\hspace{1cm} (4.12)

Taking logarithmic differentiation of equations (4.11) and (4.12) with respect to \( s \), one obtains
\[
(\ln r_{l+1} r_l) - \left( \ln \frac{F_{l+1}}{F_l} \right) = \frac{-8}{(4 - \delta_l^2)\delta_l} \frac{d}{d s}, 
\]  
\hspace{1cm} (4.13)

and
\[
\frac{u_{l+1} - u_l}{4} = \frac{d}{d s} \frac{\delta_l}{4 - \delta_l^2} \frac{d}{d s}, 
\]  
\hspace{1cm} (4.14)

respectively. Equation (4.14) can be rewritten as
\[
\frac{d}{d s} \frac{\delta_l}{d s} = \left( 1 - \frac{\delta_l^2}{4} \right) (u_{l+1} - u_l) 
\]  
\hspace{1cm} (4.15)

which constitutes part of the semi-discrete DP equation, describing the time evolution of the nonuniform mesh. Eliminating \( d\delta_l/ds \) from equations (4.13) and (4.14), one obtains
\[
\frac{u_{i+1} - u_i}{\delta_t} = -\frac{1}{2} \left( \ln r_{i+1} - r_i \right) + \frac{1}{2} \left( \ln \frac{F_{i+1}}{F_i} \right)_t \]
\[
= -\frac{1}{2} \left( \ln r_{i+1} - r_i \right) + b - b \frac{G_i^2}{F_i^2} \frac{r_i}{F_i + r_i} \]
\[
= -\frac{1}{2} \left( \ln r_{i+1} - r_i \right) + b - b \frac{F_i^2}{F_i + r_i} \]
\[
= -\frac{1}{2} \left( \ln r_{i+1} - r_i \right) + b - \frac{1}{2} \left( \ln \frac{r_i}{r_{i+1}} \right)_t - \delta_t \left( \frac{u_{i+1} + u_i}{2} - a \right) \]
\[
= -\left( \ln r_{i+1} \right)_t + b - \delta_t \left( \frac{u_{i+1} + u_i}{2} - a \right) \]
(4.16)

Here equations (4.4) and (4.9) have been used.

In summary, we propose the following which corresponds to an integrable semi-discrete Degasperis–Procesi equation:

\[
\frac{1}{\delta_t} \left( \frac{r_{i+1}}{r_i} \right)_t + \frac{u_{i+1} + u_i}{2} - a = \frac{2b}{\delta_t} \frac{r_i}{r_{i+1}} \frac{\frac{u_{i+1}}{2} - \frac{u_i}{2}}{r_i + \frac{u_{i+1}}{2} - \frac{u_i}{2} - 1}, \quad (4.17)
\]

\[
\left( \ln r_{i+1} \right)_t = -\frac{u_{i+1} - u_i}{\delta_t} + b - \delta_t \left( \frac{u_{i+1} + u_i}{2} - a \right), \quad (4.18)
\]

\[
\frac{d\delta_t}{ds} = \left( 1 - \frac{\delta_t^2}{4} \right) (u_{i+1} - u_i), \quad (4.19)
\]

where an intermediate variable \( r_i \) is used.

**Remark 4.1.** Due to the fact
\[
\frac{\delta_t}{2b} = \frac{x_{i+1} - x_i}{2b} \rightarrow \frac{\partial x}{\partial y} = -\frac{1}{ar},
\]
as \( b \rightarrow 0 \), it is obvious that equations (4.17) or (4.5) and (4.18) converge to equations (2.17) and (2.19), respectively.

In order to eliminate the intermediate variable \( r_i \), we substitute equations (4.18) into (4.17) and get

\[
\frac{u_{i+1} - u_{i-1}}{\delta_t} + \frac{\delta_{t-1}}{2} \left( \frac{u_{i+1} + u_{i-1}}{2} - a \right) = \frac{u_{i+1} - u_{i-1}}{\delta_t} + \delta_t \left( \frac{u_{i+1} + u_{i-1}}{2} - a \right) = 2b \frac{r_i}{r_{i+1}} \frac{\frac{u_{i+1}}{2} - \frac{u_i}{2}}{r_i + \frac{u_{i+1}}{2} - \frac{u_i}{2} - 1}. \quad (4.20)
\]

Defining
\[
m_i = \frac{2}{\delta_t + \delta_{t-1}} \left( \frac{u_{i+1} - u_i}{\delta_t} + \frac{u_i - u_{i-1}}{\delta_{t-1}} + \frac{\delta_t (u_{i+1} + u_i) + \delta_{t-1} (u_i + u_{i-1})}{4} \right) - a, \quad (4.21)
\]
and taking the logarithmic derivative on both sides of (4.20), we have
\[
\frac{d \ln m_l}{ds} = (\ln \rho_l)_h - (\ln \rho_{l+1})_h - \frac{8 \delta_i}{(4 - \delta_i^2)\delta_i} \frac{d\delta_i}{ds} \ln(\delta_i + \delta_{i-1}) \\
= (\ln \rho_l)_h - (\ln \rho_{l+1})_h - 2 \frac{\rho_{l+1} - \rho_l}{\delta_l} \ln(\delta_i + \delta_{i-1}) \quad \text{(by (4.19))}
\]
\[
= -\frac{\rho_{l+1} - \rho_l}{\delta_l} \left( \frac{\rho_{l+1} + \rho_{l-1}}{2} - \rho_l \right) + \frac{\rho_{l+1} - \rho_l}{\delta_l} \left( \frac{k_{l+1} + k_l}{2} - k_l \right) \\
- \frac{2(\rho_{l+1} - \rho_l)}{\delta_l} \ln(\delta_i + \delta_{i-1}) \quad \text{(by (4.18))}
\]
\[
= -\frac{\rho_{l+1} - \rho_l}{\delta_l} \left( \frac{\rho_{l+1} + \rho_{l-1}}{2} - \rho_l \right) + \frac{\rho_{l+1} - \rho_l}{\delta_l} \left( \frac{k_{l+1} + k_l}{2} - k_l \right) \\
- \frac{1}{\delta_l + \delta_{l-1}} \left( (\rho_{l+1} - \rho_l) - \delta_l^2 (\rho_{l+1} - \rho_l) + \delta_{l-1}^2 (\rho_{l-1} - \rho_l) \right) .
\]

(4.22)

By defining forward difference and average operators
\[
\Delta \rho_l = \frac{\rho_{l+1} - \rho_l}{\delta_l} , \quad M\rho_l = \frac{\rho_{l+1} + \rho_{l-1}}{2} ,
\]
we can summarize what we have deduced into the following theorem.

**Theorem 4.2.** The semi-discrete Degasperis–Procesi equation

\[
\begin{align*}
\frac{d \ln m_l}{ds} &= -2M \Delta \rho_l - \frac{M(\delta_l \Delta \rho_l)}{M\delta_l} + \frac{M(M\rho_{l+1} - a) - \delta_l - i(M\rho_l - a)}{2} + \frac{M(\delta_l^2 (\rho_{l+1} - \rho_l))}{4M\delta_l} , \\
\frac{d \delta_l}{ds} &= \left(1 - \frac{\delta_l^2}{4}\right) (\rho_{l+1} - \rho_l) , \\
m_l &= -\frac{\Delta \rho_l - \Delta \rho_{l-1}}{M\delta_l} + \frac{M(\delta_l (M\rho_l))}{M\delta_l} - a ,
\end{align*}
\]

is determined from the following bilinear equations

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{1}{a+b} D_a - 1 \right) g_{l+1} \cdot f_l = \left( \frac{1}{a-b} D_b - 1 \right) g_l \cdot f_{l+1}, \\
g_l f_l = e^c G_l , \\
(D_b - 2b) f_{l+1} \cdot f_l = -2b G_{l+1}^2 , \\
(a-b) g_{l+1} f_{l+1} - (a+b) g_l f_{l+1} = -2b e^{c} F_{l+1} .
\end{array} \right.
\end{align*}
\]

through a discrete hodograph transformation

\[
\delta_l = 2 \frac{(a-b) g_{l+1} f_l - (a+b) g_l f_{l+1}}{(a-b) g_{l+1} f_{l+1} + (a+b) g_l f_{l+1}} , \quad t = s
\]

and a dependent variable transformation

\[
u_l = \ln \frac{\delta_l f_l}{f_{l+1}} .
\]

Let us consider the continuum limit when \( b \to 0 \). The dependent variable \( u \) is a function of \( l \) and \( s \). Meanwhile, we regard it as a function of \( x \) and \( t \), where \( x \) is the space coordinate at \( l \)th lattice point and \( t \) is the time, defined by
Then in the continuous limit, \( b \to 0 (\delta_i \to 0) \), we have

\[
2M \Delta u_t = \frac{u_{i+1} - u_i}{\delta_i} + \frac{u_i - u_{i-1}}{\delta_{i-1}} \to 2u_t,
\]

\[
M(\delta_i \Delta u_t) = \frac{u_{i+1} - u_i}{\delta_i + \delta_{i-1}} \to u_t,
\]

\[
\frac{\delta_i}{2} (Mu_t - a) \to 0,
\]

\[
M(\delta_i^2 (u_{i+1} - u_i)) \to 0,
\]

\[
m_t = \frac{(\Delta u_t - \Delta u_{t-1})}{M\delta_i} + \frac{M(\delta_i (Mu_t))}{M\delta_i} - a \to m = u - u_{xx} - a.
\]

Moreover, since

\[
\partial_x = \partial_t + \frac{\partial_x}{\partial s} \partial_x = \partial_t + u \partial_x.
\]

Consequently, the third equation in (4.23) converges to \( m = u - u_{xx} - a \). Whereas the first equation in (4.23) converges to

\[
\partial_t + u \partial_x m = -3mu_x,
\]

which is exactly the Degasperis–Procesi equation (1.2). Based on the results in the previous section, we can provide \( N \)-soliton solution to the semi-discrete Degasperis–Procesi equation.

**Theorem 4.3.** The \( N \)-soliton solution of (4.23), the semi-discrete analogue of the Degasperis–Procesi equation, takes the following parametric form:

\[
u = \left( \ln \frac{g_i}{f_i} \right),
\]

\[\delta_i = 2\frac{(a - b)g_{i+1}f_i - (a + b)g_if_{i+1}}{(a - b)g_{i+1}f_i + (a + b)g_if_{i+1}},
\]

where \( g_i = f_{ik}, f_i = f_{ik} \) with pfaffian \( f_{ik} \) defined by

\[
f_{ik} = \text{Pf}(1, 2, \cdots, 2N)_{ik},
\]

whose elements are

\[(i, j)_{kl} = c_{ij} + \frac{p_i - p_j}{p_i + p_j} \phi_i^{(0)}(k, l) \phi_j^{(0)}(k, l),
\]

\[
\phi_i^{(0)}(k, l) = p_i^k \left( 1 + ap_i \right) \left( 1 + bp_i \right) \left( 1 - bp_i \right) \left( 1 - ap_i \right)
\]

under the reduction condition \( (i = 1, 2, \cdots, N) \)

\[
p_i(1 - a^2 p_i^2)(1 - b^2 p_{2N+1-i}^2) + p_{2N+1-i}(1 - a^2 p_{2N+1-i}^2)(1 - b^2 p_i^2) = 0.
\]

To conclude, we calculate the \( \tau \)-functions for one- and two-soliton solutions, and compared them with those of the DP equation (1.2).
4.1. One-soliton

For $N = 1$, we have

$$g_1 = \text{Pf}(1, 2)_{10} = c_{1,2} + \frac{p_1 - p_2}{p_1 + p_2} \varphi_1^{(0)}(1, I) \varphi_2^{(0)}(1, I)$$

(4.30)

$$\propto 1 + e^{\xi_1(I) + \xi_2(I) + \phi_1},$$

(4.31)

$$f_1 = \text{Pf}(1, 2)_{00} = c_{1,2} + \frac{p_1 - p_2}{p_1 + p_2} \varphi_1^{(0)}(0, I) \varphi_2^{(0)}(0, I)$$

(4.32)

$$\propto 1 + e^{\xi_1(I) + \xi_2(I) - \phi_1},$$

(4.33)

where

$$e^{\xi(I)} = \left( \frac{1 + bp_i}{1 - bp_i} \right)^{e_{1i} - e_{2i} \epsilon} (i = 1, 2), \quad e^{\phi_i} = \frac{(1 + ap_i)(1 + ap_2)}{(1 - ap_i)(1 - ap_2)}.$$  

Here $p_1, p_2 = p^*_i$ are two parameters related by a constraint

$$p_1(1 - a^2 p^2_1)(1 - b^2 p^2_2) + p_2(1 - a^2 p^2_2)(1 - b^2 p^2_1) = 0.$$  

(4.34)

Let $p_1 = A e^{i \theta},$ $p_2 = A e^{-i \theta},$ and $p_1 + p_2 = k_i$; we then have

$$1 - a^2 k_i^2 + (3a^2 - b^2)A_i^2 + a^2 b^2 A_i^4 = 0,$$  

(4.35)

from which $A_i^2$ can be found as

$$A_i^2 = \frac{\sqrt{3a^2 - b^2}^2 - 4a^2 b^2(1 - a^2 k_i^2) - (3a^2 - b^2)}{2a^2 b^2}.$$  

(4.36)

In the continuous limit of $b \to 0$ and $a = -1$, a simple calculation gives

$$A_i^2 \to \frac{k_i^2 - 1}{3},$$  

(4.37)

and

$$e^{\xi_1(I) + \xi_2(I)} \to e^{-2b(p_1 + p_2)(p^{-1}_1 + p^{-1}_2) + \eta_0} \to e^{\frac{3k_i^2}{k_i^2 - 1} + \eta_0}$$  

(4.38)

by letting $y = -2bl$. Therefore, the one-soliton solution of the semi-discrete DP equation converges to the one-soliton solution given in [16–18].

4.2. Two-soliton

For $N = 2$, by assuming $\varphi^{(0)}_{ij}(k, I) = \varphi^{(0)}_{i}(k, I) \varphi^{(0)}_{j}(k, I)$, we have

$$g_i = \text{Pf}(1, 2, 3, 4)_{10} = \text{Pf}(1, 2)_{10} \text{Pf}(3, 4)_{10} - \text{Pf}(1, 3)_{10} \text{Pf}(2, 4)_{10} + \text{Pf}(1, 4)_{10} \text{Pf}(2, 3)_{10}$$

$$= \frac{p_1 - p_2}{p_1 + p_2} \varphi_{12}^{(0)}(1, I) \times \frac{p_3 - p_4}{p_3 + p_4} \varphi_{34}^{(0)}(1, I) - \frac{p_1 - p_3}{p_1 + p_3} \varphi_{13}^{(0)}(1, I) \times \frac{p_2 - p_4}{p_2 + p_4} \varphi_{24}^{(0)}(1, I)$$

$$+ \left( c_{14} + \frac{p_1 - p_4}{p_1 + p_4} \varphi_{14}^{(0)}(1, I) \right) \left( c_{23} + \frac{p_2 - p_3}{p_2 + p_3} \varphi_{23}^{(0)}(1, I) \right)$$

$$\propto 1 + e^{\xi_1(I) + \xi_2(I) + \phi_1 + \gamma_1} + e^{\xi_3(I) + \xi_4(I) + \phi_2 + \gamma_2} + \sum_{j=1}^{4} \xi^{(0)}j + \phi_j + \gamma_j.$$  

(4.39)
\(f_i = \text{Pf}(1, 2, 3, 4)_{ij} \propto 1 + e^{\xi(l)+\zeta(l)} - \phi_i + \gamma_i + e^{\xi(l)+\zeta(l)} - \phi_2 + \gamma_2 + b_{12}e^{\xi(l)+\zeta(l)} - \phi_3 + \gamma_3 = \cdots,
\)

under the condition

\[p_i(1 - a^2 p_i^2)(1 - b^2 p_{2N+1-i}) + p_{2N+1-i}(1 - a^2 p_{2N+1-i}) = 0 \quad (i = 1, 2).
\]

Here \(c_{14} = c_{23} = 1, e^{\phi_i} = \frac{(1 + ap_i)(1 + ap_{2-N})}{(1 - ap_i)(1 - ap_{2-N})}, e^{\phi_j} = \frac{(1 + ap_j)(1 + ap_{2-N})}{(1 - ap_j)(1 - ap_{2-N})}, e^{\gamma_i} = \frac{p_i - p_0}{p_i + p_0}, e^{\gamma_2} = \frac{p_2 - p_1}{p_2 + p_1},
\]

and \(b_{12} = \frac{(p_1 - p_2)p_0 - (p_1 - p_2)p_0}{(p_1 + p_2)(p_1 + p_0 + p_2)(p_1 + p_2 + p_0)}\). In the continuous limit \(b \to 0\), we can show that the two-soliton solutions for semi-discrete DP equation converge to the two-soliton solutions of the DP equation found in [16–18] by letting \(p_1 + p_2 = k_1, p_2 + p_3 = k_2\).

5. Conclusion and further topics

In the present paper, we firstly reviewed a set of equations which lead to the DP equation through a dependent variable transformation and a hodograph transformation. Then by constructing the integrable semi-discrete analogues of these equations including bilinear equations and by defining a discrete hodograph transformation, an integrable semi-discrete DP equation was proposed.

We should point out here the semi-discrete DP equation is constructed for \(a \neq 0\). A natural question is: what about the semi-discrete analogue of the DP equation when \(a = 0\)? The semi-discrete DP equation proposed in the present paper is constructed based on the bilinear equations, as well as the \(\tau\)-functions, in the case of \(a \neq 0\). In the limit \(a \to 0\), the structure of solution space is changed completely. Similar situations happens to other soliton equations. For example, the dispersionless limit derives the dispersionless KP equation from the KP equation, but the limit does not preserve the soliton solution. The finite size Toda lattice, which is often called Toda molecule, admits a double Wronskian solution. However, if we take a limit of pushing the lattice size to infinite, then we are led to an infinite Toda lattice whose soliton solution can only be expressed in terms of two-directional Wronskian. These limit processes including \(a \to 0\) of the DP equations are singular limits which change the structure of solution space.

Consequently, the semi-discrete DP equation constructed here is based on the principle of keeping the structure of solution space of continuous equation. Since the solutions of the DP equation with \(a = 0\) have different structure with those of the DP equation with \(a \neq 0\), we should start from the solution of the DP equation with \(a = 0\) to construct a semi-discrete DP equation with \(a = 0\). This will be a future work. On the other hand, similar to what we have done for the CH equation [34], the short pulse equation [20] and a coupled short pulse equation [22], the problem of using the proposed semi-discrete DP equation as a self-adaptive moving mesh scheme for the numerical simulation of the DP equation deserves further exploration. Especially it is worth checking whether the semi-discrete DP equation we proposed still works as an accurate and stable numerical simulation or not.

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Appendix

\[ \frac{\partial f_{id}}{\partial z} = \frac{\partial}{\partial z} \begin{pmatrix} (i,j) & (i,j,k) \end{pmatrix} \]

\[ = \begin{pmatrix} (i,j) & (i,j,k) \end{pmatrix} \begin{pmatrix} \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) \\ 0 & 0 \end{pmatrix} \]

\[ = (1,2,\ldots,2N,d_{-1},d_0)_{id} \]

\[ f_{id}^{k+1} = \begin{pmatrix} \varphi_i^0(k+1,l) + \varphi_i^0(k,l) \varphi_j^0(k,l) - \varphi_i^0(k,l) \varphi_j^0(k+1,l) \end{pmatrix} \]

\[ = \begin{pmatrix} \varphi_i^0(k,l) & \varphi_j^0(k+1,l) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ = (1,2,\ldots,2N,d_0,d_k)_{id} \]

\[ (\partial-a)f_{id}^{k+1} = (\partial-a) \begin{pmatrix} (i,j) & (i,j,k) \end{pmatrix} \begin{pmatrix} \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} (i,j) & (i,j,k) \end{pmatrix} \begin{pmatrix} \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) & \varphi_j^0(k,l) & \varphi_j^0(k+1,l) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]

\[ = (1,2,\ldots,2N,d_{-1},d_0)_{id} \] (A.3)

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Similarly, we have

\[ f_{k,l+1} = (1, 2, \ldots, 2N, d_{a}, d_{b})_{kl}, \quad (A.4) \]

\[ \frac{a - b}{a + b} f_{k,l+1} = (1, 2, \ldots, 2N, d_{a}, d_{b})_{kl}. \quad (A.5) \]

\[
\begin{align*}
(a - b) f_{k,l+1} &= a - b \left( \begin{array}{c}
(i,j)_{kl} \quad \varphi_{ij}^{(0)}(k, l + 1) \quad - \varphi_{ij}^{(0)}(k, l)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
(i,j)_{kl} \quad \varphi_{ij}^{(0)}(k, l) \quad \varphi_{ij}^{(0)}(k, l + 1) \quad \varphi_{ij}^{(0)}(k + 1, l + 1)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k, l) \quad 0 \quad \varphi_{ij}^{(0)}(k + 1, l + 1) - \varphi_{ij}^{(0)}(k, l)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k, l) \quad 0 \quad 0 \quad 0
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k, l + 1) \quad \varphi_{ij}^{(0)}(k + 1, l + 1)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k + 1, l + 1) - \varphi_{ij}^{(0)}(k, l)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k, l + 1) \quad \varphi_{ij}^{(0)}(k + 1, l + 1) - \varphi_{ij}^{(0)}(k, l)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k + 1, l + 1) \\
\varphi_{ij}^{(0)}(k, l + 1)
\end{array} \right) \\
&= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k + 1, l + 1) \\
\varphi_{ij}^{(0)}(k + 1, l + 1)
\end{array} \right) \\
&= (1, 2, \ldots, 2N, d_{a}, d_{b})_{kl} \quad (A.6)
\end{align*}
\]

\[
(\partial_{s} - a - b) f_{k,l+1} = (\partial_{s} - a - b) \left( \begin{array}{c}
(i,j)_{kl} \quad \varphi_{ij}^{(0)}(k, l + 1) \quad \varphi_{ij}^{(0)}(k + 1, l)
\end{array} \right) \\
= (\partial_{s} - a - b) \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k, l + 1) \quad \varphi_{ij}^{(0)}(k + 1, l)
\end{array} \right) \\
= \frac{a - b}{a + b} \left( \begin{array}{c}
\varphi_{ij}^{(0)}(k, l + 1) \quad \varphi_{ij}^{(0)}(k + 1, l)
\end{array} \right) \\
= (1, 2, \ldots, 2N, d_{a}, d_{b})_{kl}
\]
\[
\left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & 0 & 0 & 0 & a-b \\
0 & 0 & 0 & a \end{array} \right)
\]

\[
+ \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & (\partial_b-a) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
\end{array} \right)
\]

\[
= \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & 0 & 0 & 0 & a-b \\
0 & 0 & 0 & a \end{array} \right)
\]

\[
+ \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) + b \varphi_i^0(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
\end{array} \right)
\]

\[
= \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & 0 & 0 & 0 & a-b \\
0 & 0 & 0 & a \\
\end{array} \right)
\]

\[
+ \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) + b \varphi_i^0(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
\end{array} \right)
\]

\[
= \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & 0 & 0 & 0 & a-b \\
0 & 0 & 0 & a \\
\end{array} \right)
\]

\[
+ \text{Pr} \left( \begin{array} {cccc}
(i,j)_{kl} & \varphi_i^{-1}(k,l) + b \varphi_i^0(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
0 & a-b & a+b & -b \\
\end{array} \right)
\]
$$\begin{pmatrix}
(i,j)_{kl} & \varphi_i^{-1}(k,l) & \varphi_i^0(k,l) & \varphi_i^0(k,l+1) & \varphi_i^0(k+1,l) \\
0 & -b & -a & 1 & 1 \\
& & & \frac{a-b}{a+b} & & \\
& & & & & \\
& & & & & \\
\end{pmatrix}$$

$$= \text{Pf}$$

$$= (1, 2, \cdots, 2N, d_{-l}, d_0, d^l)_{kl}$$

(A.7)

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