Stochastic dominance-constrained Markov decision processes

William B. Haskell\textsuperscript{*} and Rahul Jain\textsuperscript{†}

May 29, 2021

Abstract

We are interested in risk constraints for infinite horizon discrete time Markov decision processes (MDPs). Starting with average reward MDPs, we show that increasing concave stochastic dominance constraints on the empirical distribution of reward lead to linear constraints on occupation measures. The optimal policy for the resulting class of dominance-constrained MDPs is obtained by solving a linear program. We compute the dual of this linear program to obtain average dynamic programming optimality equations that reflect the dominance constraint. In particular, a new pricing term appears in the optimality equations corresponding to the dominance constraint. We show that many types of stochastic orders can be used in place of the increasing concave stochastic order. We also carry out a parallel development for discounted reward MDPs with stochastic dominance constraints. The paper concludes with a portfolio optimization example.

1 Introduction

Markov decision processes (MDPs) are a natural and powerful framework for stochastic control problems. In the present paper, we take up the issue of risk constraints in MDPs. Convex analytic methods for MDPs have been successful at handling many types of constraints. Our specific goal is to find and study risk constraints for MDPs that are amenable to convex analytic formulation. It turns out that stochastic dominance constraints are natural risk constraints for MDPs.

Convex analytic methods are well studied for Markov decision processes. The linear programming approach for MDPs is pioneered in \cite{puterman1994markov}, and an early survey is found in \cite{shaked1986stochastic}. The main idea is that some MDPs can be written as convex optimization problems in terms of appropriate occupation measures. \cite{shaked1986stochastic,shaked1990stochastic,shaked1985stochastic,shaked1984stochastic} discuss a rigorous theory of convex optimization for MDPs with general Borel state and action spaces. Detailed monographs on Markov decision processes are found in \cite{puterman2009markov,puterman1994markov,puterman1995markov,puterman1994markov}. Constrained MDPs can naturally be embedded in this framework. Constrained discounted MDPs are explored in \cite{shaked1988stochastic,shaked1988stochastic,shaked1988stochastic} is a substantial monograph on constrained MDPs. Constrained discounted MDPs in Borel spaces are analyzed in \cite{shaked1988stochastic}, and constrained average cost MDPs in Borel spaces are developed in \cite{shaked1988stochastic}. Infinite dimensional linear programming plays a fundamental role in both \cite{shaked1988stochastic,shaked1988stochastic}, and the theory of infinite dimensional linear programming is developed in \cite{shaked1988stochastic}. The special case of constraints on expected utility in discounted MDPs is considered in \cite{shaked1988stochastic}. MDPs with expected constraints and pathwise constraints, also called hard constraints, are considered in \cite{shaked1988stochastic} using convex analytic methods. An inventory system is detailed to motivate the theoretical results.

Policies in MDPs induce Markov chains. Typically, policies are evaluated with respect to some measure of expected reward, such as long-run average reward or discounted reward. The variation/spread/dispersion of policies is also critical to their evaluation. Given two policies with equal expected performance, we would prefer the one with smaller variation in some sense. Consider a discounted portfolio optimization problem, for example. The expected discounted reward of an investment policy is a key performance measure; the downside variation of an investment policy is also a key performance measure. When rewards and costs are involved, the variation of a policy can also be called its risk.

\textsuperscript{*}William Haskell is a visiting assistant professor in the Department of Industrial and Systems Engineering at the University of Southern California.

\textsuperscript{†}Rahul Jain is an assistant professor in the Departments of Electrical Engineering and Industrial and Systems Engineering at the University of Southern California. This research is supported by the Air Force Office of Scientific Research and the Office of Naval Research.
Risk management for MDPs has been considered from many perspectives in the literature. [20] includes penalties for the variance of rewards in MDPs. The optimal policy is obtained by solving a nonlinear programming problem in occupation measures. In [37], the mean-variance trade-off in MDPs is further explored in a Pareto-optimality sense. The conditional value-at-risk of the total cost in a finite horizon MDPs is constrained in [4]. It is argued that convex analytic methods do not apply to this problem type and an offline iterative algorithm is employed to solve for the optimal policy. [35] develops Markov risk measures for finite horizon and infinite horizon discounted MDPs. Dynamic programming equations are derived that reflect the risk aversion, and policy iteration is shown to solve the infinite horizon problem.

Our notion of risk constrained MDPs differs from this literature survey. We are interested in the empirical distribution of reward, rather than in its expectation, variance, or other summary statistics. Our approach is based on stochastic orders, which are partial orders on the space of random variables, see [33, 36] for extensive monographs on stochastic orders. [9, 10] use the increasing concave stochastic order to define stochastic dominance constraints in single stage stochastic optimization. The increasing concave stochastic order is notable for its connection to risk-averse decision makers, i.e. it captures the preferences of all risk-averse decision makers. A benchmark random variable is introduced, and a concave random variable-valued mapping is constrained to dominate the benchmark in the increasing concave stochastic order. It is shown that increasing concave functions are the Lagrange multipliers of the dominance constraints. The dual problem is a search over a certain class of increasing concave functions, interpreted as utility functions, and strong duality is established. Stochastic dominance constraints are applied to finite horizon stochastic programming problems with linear system dynamics in [12]. Specifically, a stochastic dominance constraint is placed on a vector of state and action dependent reward functions across the finite planning horizon. The Lagrange multipliers of this dynamic stochastic dominance constraint are again determined to be increasing concave functions, and strong duality holds. In contrast, we place a stochastic dominance constraint on the empirical distribution of reward in infinite horizon MDPs. We argue that this type of constraint comprehensively accounts for the variation in policies in MDPs.

We make two main contributions in this paper. First, we show how to formulate stochastic dominance constraints for long-run average reward maximizing MDPs. More immediately, we show that stochastic dominance constrained MDPs can be solved via linear programming over occupation measures. Our model is more general than [12] because it allows for an arbitrary transition kernel and is also infinite horizon. Also, our model is more computationally tractable than the stochastic programming model in [12] because it leads to linear programs. Second, we apply infinite-dimensional linear programming duality to gain more insight: the resulting duals are similar to the linear programming form of the average reward dynamic programming optimality equations. However, new decision variables corresponding to the stochastic dominance constraint appear in an intuitive way. Specifically, the new decision variables are increasing concave functions that price rewards. This observation parallels the results in [9, 10, 13] and is natural because our stochastic dominance constraints are defined in terms of increasing concave functions. The upcoming dual problems are themselves linear programs, unlike the dual problems in [9, 10, 13] which are general infinite-dimensional convex optimization problems.

This paper is organized as follows. In section 2, we consider stochastic dominance constraints for long-run average reward maximizing MDPs. In section 3 we formulate this problem as a static optimization problem, in fact a linear programming problem, in a space of occupation measures. Section 4 develops the dual for this problem using infinite dimensional linear programming duality, and reveals the form of the Lagrange multipliers. In section 5, we discuss a number of immediate variations and extensions, especially the drastically simpler development on finite state and action spaces. We illustrate our method in section 6 with a portfolio optimization example, and then conclude the paper in section 7.

2 MDPs and stochastic dominance

The first subsection presents a general model for average reward MDPs, and the second explains how to apply stochastic dominance constraints.
2.1 Average reward MDPs

A typical representation of a discrete time MDP is the 5-tuple

\[(S, A, \{A(s) : s \in S\}, Q, r)\].

The state space \(S\) and the action space \(A\) are Borel spaces, subsets of complete and separable metric spaces, with corresponding Borel \(\sigma\)-algebras \(\mathcal{B}(S)\) and \(\mathcal{B}(A)\). We define \(\mathcal{P}(S)\) to be the space of probability measures over \(S\) with respect to \(\mathcal{B}(S)\), and we define \(\mathcal{P}(A)\) analogously. For each state \(s \in S\), the set \(A(s) \subset A\) is a measurable set in \(\mathcal{B}(A)\) and indicates the set of feasible actions available in state \(s\). The set of feasible state-action pairs is written

\[K = \{(s, a) \in S \times A : a \in A(s)\},\]

and \(K\) is assumed to be closed in \(S \times A\). The transition law \(Q\) governs the system evolution. Explicitly, \(Q(B \mid s, a)\) for \(B \in \mathcal{B}(S)\) is the probability of visiting the set \(B\) given the state-action pair \((s, a)\). Finally, \(r : K \rightarrow \mathbb{R}\) is a measurable reward function that depends on state-action pairs.

We now describe two classes of policies for MDPs. Let \(H_t\) be the set of histories at time \(t\), \(H_0 = S\), \(H_1 = K \times S\), and \(H_t = K^t \times S\) for all \(t \geq 2\). A specific history \(h_t \in H_t\) records the state-action pairs visited at times \(0, 1, \ldots, t - 1\) and the current state \(s_t\). Define \(\Pi\) to be the set of all history-dependent randomized policies: collections of mappings \(\pi_t : H_t \rightarrow \mathcal{P}(A)\) for all \(t \geq 0\). Given a history \(h_t \in H_t\) and a set \(B \in \mathcal{B}(A)\), \(\pi(B \mid h_t)\) is the probability of selecting an action in \(B\). Define \(\Phi\) to be the class of stationary randomized Markov policies: mappings \(\phi : S \rightarrow \mathcal{P}(A)\) which only depend on history through the current state. For a given state \(s \in S\) and a set \(B \in \mathcal{B}(A)\), \(\phi(B \mid s)\) is the probability of choosing an action in \(B\). The class \(\Phi\) will be viewed as a subset of \(\Pi\). We explicitly assume that both \(\Pi\) and \(\Phi\) only include feasible policies that respect the constraints \(K\).

The state and action at time \(t\) are denoted \(s_t\) and \(a_t\), respectively. Any policy \(\pi \in \Pi\) and initial distribution \(\nu \in \mathcal{P}(S)\) determines a probability measure \(P^\pi_\nu\) and stochastic process \(\{(s_t, a_t), t \geq 0\}\) defined on a measurable space \((\Omega, \mathcal{F})\). The expectation operator with respect to \(P^\pi_\nu\) is denoted \(\mathbb{E}^\pi_\nu[]\). Consider the long-run expected average reward

\[R(\pi, \nu) = \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}^\pi_\nu \left[ \sum_{t=0}^{T-1} r(s_t, a_t) \right].\]

The classic long-run expected average reward maximization problem is

\[\sup_{\pi \in \Pi} R(\pi, \nu), \quad \text{s.t. } \pi \in \Pi. \tag{2.1} \]

It is known that a stationary policy in \(\Phi\) is optimal for problem (2.1) - (2.2) under suitable conditions (this result is found in [34] for finite and countable state spaces, and [26, 27] for general Borel state and action spaces).

2.2 Stochastic dominance

Now we will motivate and formalize stochastic dominance constraints for problem (2.1) - (2.2). To begin, let \(z : K \rightarrow \mathbb{R}\) be another measurable reward function, possibly different from \(r\). A risk-averse decision maker with an increasing concave utility function \(u : \mathbb{R} \rightarrow \mathbb{R}\) would be interested in maximizing his long-run average expected utility

\[\liminf_{T \to \infty} \frac{1}{T} \mathbb{E}^\pi_\nu \left[ \sum_{t=0}^{T-1} u(z(s_t, a_t)) \right].\]

However, it is difficult to choose one utility function to represent a risk-averse decision maker without considerable information. We will use the increasing concave order to express a continuum of risk preferences in MDPs.
Definition 2.1. For random variables \(X, Y \in \mathbb{R}\), \(X\) dominates \(Y\) in the increasing concave stochastic order, written \(X \succeq_{ivc} Y\), if \(\mathbb{E} [u(X)] \geq \mathbb{E} [u(Y)]\) for all increasing concave functions \(u : \mathbb{R} \to \mathbb{R}\) such that both expectations exist.

Let \(\mathcal{C}(\mathbb{R})\) be the set of all continuous functions \(f : \mathbb{R} \to \mathbb{R}\). Let \(\mathcal{U}(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})\) be the set of all increasing concave functions \(u : \mathbb{R} \to \mathbb{R}\) such that
\[
\lim_{x \to \infty} u(x) = 0
\]
and
\[
u(x) = u(x_0) + \nu(x_0)
\]
for all \(x \leq x_0\) for some \(\kappa > 0\) and \(x_0 \in \mathbb{R}\) (the choices of \(\kappa\) and \(x_0\) differ among \(u\)). The second condition just means that all \(u \in \mathcal{U}(\mathbb{R})\) become linear as \(x \to -\infty\). By construction, functions \(u \in \mathcal{U}(\mathbb{R})\) are bounded from above by zero. We will use the set \(\mathcal{U}(\mathbb{R})\) to characterize \(X \succeq_{ivc} Y\).

Now define \((x)_- = \min \{x, 0\}\). We note that any function in \(\mathcal{U}(\mathbb{R})\) can be written in terms of the family \(\{(x - \eta)_- : \eta \in \mathbb{R}\}\). To understand this result, choose \(u \in \mathcal{U}(\mathbb{R})\) and a finite set of points \(\{x_1, \ldots, x_j\}\). By concavity, there exist \(a_i \in \mathbb{R}\) such that \(a_i (x - x_i) + u(x_i) \geq u(x)\) for all \(x \in \mathbb{R}\) and for all \(i = 1, \ldots, j\). Each linear function \(a_i (x - x_i) + u(x_i)\) is a global over-estimator of \(u\). The piecewise linear increasing concave function
\[
\min_{i=1,\ldots,j} \{a_i (x - x_i) + u(x_i)\}
\]
is also a global over-estimator of \(u\), and certainly
\[
u(x) \leq \min_{i=1,\ldots,j} \{a_i (x - x_i) + u(x_i)\} \leq a_i (x - x_i) + u(x_i)
\]
for all \(i = 1, \ldots, j\) and \(x \in \mathbb{R}\). As the number of sample points \(j\) increases, the polyhedral concave function \(\min_{i=1,\ldots,j} \{a_i (x - x_i) + u(x_i)\}\) becomes a better approximation of \(u\). We realize that the function \(\min_{i=1,\ldots,j} \{a_i (x - x_i) + u(x_i)\}\) is equal to a finite sum of nonnegative scalar multiples of functions from \(\{(x - \eta)_- : \eta \in \mathbb{R}\}\). It follows that the relation \(X \succeq_{ivc} Y\) is equivalent to \(\mathbb{E} [(X - \eta)_-] \geq \mathbb{E} [(Y - \eta)_-]\) for all \(\eta \in \mathbb{R}\). When the support of \(Y\) is contained in a compact interval \([a, b]\), the condition \(\mathbb{E} [(X - \eta)_-] \geq \mathbb{E} [(Y - \eta)_-]\) for all \(\eta \in [a, b]\) is sufficient for \(X \succeq_{ivc} Y\).

From now on, let \(Y\) be a fixed reference random variable on \(\mathbb{R}\) to benchmark the empirical distribution of reward \(z\). We assume that \(Y\) has support in a compact interval \([a, b]\) throughout the rest of this paper. Define
\[
Z_\eta(\pi, \nu) \triangleq \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_\nu \left[ \sum_{t=0}^{T-1} (z(s_t, a_t) - \eta)_- \right]
\]
to be the long-run expected average shortfall in \(z\) at level \(\eta\). We propose the class of stochastic dominance-constrained MDPs:

\[
\sup \quad R(\pi, \nu)
\]
s.t. \[
Z_\eta(\pi, \nu) \geq \mathbb{E} [(Y - \eta)_-], \quad \forall \eta \in [a, b],
\]
\[
\pi \in \Pi.
\]

For emphasis, we index \(\eta\) over the compact set \([a, b]\) in (2.4). Allowing \(\eta\) to range over all \(\mathbb{R}\) would lead to major technical difficulties, as first observed in [9] [10].

Constraint (2.4) is a continuum of constraints on the long-run expected average shortfall of the policy \(\pi\) for all \(\eta \in [a, b]\). We will approach problem (2.3) - (2.5) by casting it in the space of long-run average occupation measures. Then we will see that constraint (2.4) is equivalent to a stochastic dominance constraint on the empirical distribution of rewards \(z\), namely
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} z(s_t, a_t) \succeq_{ivc} Y.
\]
To be clear, \( \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} z(s_t, a_t) \) indicates a random variable on \( \mathbb{R} \), not the long-run average of \( z(s_t, a_t) \).

We can denote the feasible region of problem (2.3) - (2.5) succinctly as

\[
\Delta \triangleq \{(\pi, \nu) \in \Pi \times \mathcal{P}(S) : R(\pi, \nu) > -\infty \text{ and } Z_\eta(\pi, \nu) \geq \mathbb{E}[Y - \eta] \} \quad \text{for all } \eta \in [a, b],
\]

allowing problem (2.3) - (2.5) to be written as

\[
\rho^* \triangleq \sup \{R(\pi, \nu) : (\pi, \nu) \in \Delta\},
\]

where \( \rho^* \) is the optimal value.

**Remark 2.2.** We focus on the average reward case in this paper. The extension to the average cost case is immediate. Let \( c : S \times A \to \mathbb{R} \) be a measurable cost function. The long-run expected average cost is

\[
C(\pi, \nu) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^\pi_T \left[ \sum_{t=0}^{T-1} c(s_t, a_t) \right].
\]

Similarly, let \( z : S \times A \to \mathbb{R} \) be another measurable cost function that possibly differs from \( c \). Since \( z \) represents costs, we want the empirical distribution of \( z \) to be “small” in a stochastic sense. For costs, it is logical to use the increasing convex order rather than the increasing concave order. For random variables \( X, Y \in \mathbb{R} \), \( X \) dominates \( Y \) in the increasing convex stochastic order, written \( X \geq_{icx} Y \), if \( \mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)] \) for all increasing convex functions \( f : \mathbb{R} \to \mathbb{R} \) such that both expectations exist. Define \( (x)_+ \triangleq \max\{x, 0\} \), and recall that the relation \( X \geq_{icx} Y \) is equivalent to \( \mathbb{E}[(X - \eta)_+] \geq \mathbb{E}[(Y - \eta)_+] \) for all \( \eta \in \mathbb{R} \). When the support of \( Y \) is contained in an interval \( [a, b] \), the relation \( X \geq_{icx} Y \) is equivalent to \( \mathbb{E}[(X - \eta)_+] \geq \mathbb{E}[(Y - \eta)_+] \) for all \( \eta \in [a, b] \).

Momentarily, let \( Y \) be a benchmark random variable that we require to dominate the empirical distribution of \( z \). Define

\[
Z_\eta(\pi, \nu) \triangleq \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^\nu_T \left[ \sum_{t=0}^{T-1} (z(s_t, a_t) - \eta)_+ \right]
\]

for all \( \eta \in [a, b] \). We obtain the cost minimization problem

\[
\inf_{\pi \in \Pi} C(\pi, \nu)
\]

s.t. \( Z_\eta(\pi, \nu) \leq \mathbb{E}[Y - \eta], \quad \forall \eta \in [a, b], \)

The upcoming results of this paper all have immediate analogs for the average cost case.

### 3 A linear programming formulation

This section develops problem (2.3) - (2.5) as an infinite dimensional linear program. First, we discuss occupation measures on the set \( K \). Occupation measures on \( K \) can be interpreted as the long-run average expected number of visits of a stochastic process \( \{(s_t, a_t), t \geq 0\} \) to each state-action pair. Next, we argue that a stationary policy in \( \Phi \) is optimal for problem (2.3) - (2.5). It will follow that the functions \( R(\phi, \nu) \) and \( Z_\eta(\phi, \nu) \) can be written as linear functions of the occupation measure corresponding to \( \phi \) and \( \nu \). These linear functions give us the desired linear program.

To proceed, we recall several well known results in convex analytic methods for MDPs. We will use \( \mu \) to denote probability measures on \( K \), and the set of all probability measures on \( K \) is denoted \( \mathcal{P}(K) \). Probability measures on \( K \) can be equivalently viewed as probability measures on all of \( S \times A \) with all mass concentrated on \( K \), \( \mu(K) = 1 \). For any \( \mu \in \mathcal{P}(K) \), the marginal of \( \mu \) on \( S \) is the probability measure \( \hat{\mu} \in \mathcal{P}(S) \) defined by \( \hat{\mu}(B) = \mu(B \times A) \) for all \( B \in \mathcal{B}(S) \).

The following two well known facts are ubiquitous in the literature on convex analytic methods for MDPs (see [15] for example). First, if \( \mu \) is a probability measure on \( K \), then there exists a stationary randomized
Markov policy $\phi \in \Phi$ such that $\mu$ can be *disintegrated* as $\mu = \hat{\mu} \cdot \phi$ where $\hat{\mu}$ is the marginal of $\mu$. Specifically, $\mu = \hat{\mu} \cdot \phi$ is defined by

$$\mu (B \times C) = \int_B \phi (C \mid s) \hat{\mu} (ds)$$

for all $B \in \mathcal{B} (S)$ and $C \in \mathcal{B} (A)$. Second, for each $\phi \in \Phi$ and $\nu \in \mathcal{P} (S)$, the probability measure $\mu = \nu \cdot \phi$ on $S \times A$ satisfies $\mu (K) = 1$ and $\hat{\mu} = \nu$. Specifically, $\mu = \nu \cdot \phi$ is defined by

$$\mu (B \times C) = \int_B \phi (C \mid s) \nu (ds)$$

for all $B \in \mathcal{B} (S)$ and $C \in \mathcal{B} (A)$.

We can integrate measurable functions $f$ on $K$ with respect to measures $\mu \in \mathcal{P} (K)$. Define

$$\langle \mu, f \rangle \overset{\triangle}{=} \int_K f (s, a) \mu (d (s, a))$$

as the integral of $f$ over state-action pairs $(s, a) \in K$ with respect to $\mu$. Then

$$\langle \mu, r \rangle = \int_K r (s, a) \mu (d (s, a))$$

is the expected reward with respect to the probability measure $\mu$ and

$$\langle \mu, (z - \eta)_- \rangle = \int_K (z (s, a) - \eta)_- \mu (d (s, a))$$

is the expected shortfall in $z$ at level $\eta$ with respect to the probability measure $\mu$.

We need to restrict to a certain class of probability measures. For notational convenience, define $r (s, \phi) \overset{\triangle}{=} \int_A r (s, a) \phi (da \mid s)$ and $Q (\cdot \mid s, \phi) \overset{\triangle}{=} \int_A Q (\cdot \mid s, a) \phi (da \mid s)$.

**Definition 3.1.** [23] Definition 3.4] A probability measure $\mu = \hat{\mu} \cdot \phi$ is called *stable* if

$$\langle \mu, r \rangle = \int r (s, a) \mu (d (s, a)) > -\infty$$

and the marginal $\hat{\mu}$ is invariant with respect to $Q (\cdot \mid \cdot, \phi)$, i.e. $\hat{\mu} (B) = \int_S Q (B \mid s, \phi) \hat{\mu} (ds)$ for all $B \in \mathcal{B} (S)$.

When $\mu$ is stable, the long-run expected average cost $R (\phi, \hat{\mu})$ is

$$R (\phi, \hat{\mu}) = \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_{\hat{\mu}} \left[ \sum_{t=0}^{T-1} r (s_t, a_t) \right] = \langle \mu, r \rangle,$$

by the individual ergodic theorem [33] Page 388, Theorem 6]. Then for stable $\mu = \hat{\mu} \cdot \phi \in \mathcal{P} (K)$, it follows that

$$R (\phi, \hat{\mu}) = \langle \mu, r \rangle = \int_S r (s, \phi) \hat{\mu} (ds).$$

Similarly, for stable $\mu = \hat{\mu} \cdot \phi$, it is true that

$$Z_\eta (\phi, \hat{\mu}) = \langle \mu, (z - \eta)_- \rangle = \int_S \left[ \int_A (z (s, a) - \eta)_- \phi (da \mid s) \right] \hat{\mu} (ds)$$

for all $\eta \in [a, b]$.

To see the connection between problem (2.3) - (2.5) and stable policies, let $I_\Gamma$ be the indicator function of a set $\Gamma$ in $\mathcal{B} (K)$. Define the *occupation measure* $\mu$ on $K$ via

$$\mu_{\nu, T}^\pi (\Gamma) = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_\nu ^\pi \{ I_\Gamma (s_t, a_t) \} = \frac{1}{T} \sum_{t=0}^{T-1} P_{\nu, \pi} ^\Gamma \{(s_t, a_t) \in \Gamma \}.$$
for all $\Gamma \in \mathcal{B}(K)$. Then,

$$R(\pi, \nu) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_\nu^\pi \left[ \sum_{t=0}^{T-1} r(s_t, a_t) \right] = \liminf_{T \to \infty} \langle \mu^T_T, r \rangle$$

and

$$Z_\eta(\phi, \hat{\mu}) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_\nu^\pi \left[ \sum_{t=0}^{T-1} (z(s_t, a_t) - \eta)_- \right] = \liminf_{T \to \infty} \langle \mu^T_T, (z - \eta)_- \rangle$$

for all $\eta \in [a, b]$.

To continue, we introduce some technical assumptions for the rest of the paper. Let $\mathcal{C}_b(K)$ be the space of continuous and bounded functions on $K$. The transition law $Q$ is defined to be weakly continuous when $\int h(\xi) Q(d\xi \mid \cdot)$ is in $\mathcal{C}_b(K)$ for all $h \in \mathcal{C}_b(K)$.

**Assumption 3.2.** (a) Problem (2.3) - (2.5) is consistent, i.e. the set $\Delta$ is nonempty.

(b) The reward function $r$ is nonpositive, and for any $\epsilon \geq 0$ the set $\{ (s, a) \in S \times A : r(s, a) \geq -\epsilon \}$ is compact.

c) The function $z(s, a)$ is bounded and upper semi-continuous on $S \times A$.

d) The transition law $Q$ is weakly continuous.

A function $f$ on $K$ is called a moment if there exists a nondecreasing sequence of compact sets $K_n \uparrow K$ such that

$$\lim_{n \to \infty} \inf_{(s, a) \notin K_n} f(s, a) = \infty,$$

see [26] Definition E.7. When $K$ is compact, then any function on $K$ is a moment. Assumption 3.2(b) implies that $-r$ is a moment. By construction, all of the functions $(z(s, a) - \eta)_-$ are bounded above by zero on $S \times A$ for all $\eta \in [a, b]$.

The next lemma reduces the search for optimal policies to stable policies. We define

$$\Delta_s \triangleq \{ \mu \in \mathcal{P}(K) : \mu \text{ is stable, } \mu = \hat{\mu} \cdot \phi \text{ and } (\phi, \hat{\mu}) \in \Delta \}$$

to be the set of all stable probability measures $\mu$ that are feasible for problem (2.3) - (2.5).

**Lemma 3.3.** Suppose assumption 3.2 holds. For each feasible pair $(\pi, \nu) \in \Delta$, there exists a stable probability measure $\mu = \hat{\mu} \cdot \phi$ such that $(\phi, \hat{\mu}) \in \Delta$ and $R(\pi, \nu) \leq R(\phi, \hat{\mu}) = \langle \mu, r \rangle$.

**Proof.** For any $(\pi, \nu) \in \Delta$, there exists a stable policy $\mu = \hat{\mu} \cdot \phi$ such that

$$R(\pi, \nu) \leq R(\phi, \hat{\mu}) = \langle \mu, r \rangle$$

by [26] Lemma 5.7.10. By the same reasoning,

$$\mathbb{E}[(Y - \eta)_-] \leq Z_\eta(\pi, \nu) \leq Z_\eta(\phi, \hat{\mu}) = \langle \mu, (z - \eta)_- \rangle$$

for all $\eta \in [a, b]$ so that $\mu = \hat{\mu} \cdot \phi$ is feasible.

Problem (2.3) - (2.5) is solvable if there exists a pair $(\pi^*, \nu^*) \in \Delta$ with $R(\pi^*, \nu^*) = \rho^*$, i.e. the optimal value is attained. When an optimization problem is solvable, we can replace 'sup' and 'inf' with 'max' and 'min'. We use the preceding lemma to show that problem (2.3) - (2.5) is solvable.

**Theorem 3.4.** Problem (2.3) - (2.5) is solvable.

**Proof.** By lemma 3.3

$$\rho^* = \sup \{ \langle \mu, r \rangle : \mu \in \Delta_s \}.$$ 

Now apply the proof of [26] Theorem 5.7.9]. Let $\{ \epsilon_n \}$ be a sequence with $\epsilon_n \downarrow 0$ and $\epsilon_n \leq 1$. For any $\epsilon_n$, there is a pair $(\pi^n, \nu^n) \in \Delta$ with $R(\pi^n, \nu^n) \geq \rho^* - \epsilon_n$ by the definition of $\rho^*$. Again, by lemma 3.3 for each $(\pi^n, \nu^n) \in \Delta$ there is a pair $(\phi^n, \hat{\mu}^n) \in \Delta$ such that $\mu^n = \hat{\mu}^n \cdot \phi^n$ is stable and $R(\pi^n, \nu^n) \leq R(\phi^n, \hat{\mu}^n) = \langle \mu^n, r \rangle$. 

7
By construction, $\langle \mu^n, r \rangle \geq \rho^* - \epsilon_n$ and $\epsilon_n \in (0, 1)$ for all $n$, so $\inf_n \langle \mu^n, r \rangle \geq \rho^* - 1$. It follows that $\sup_n (\mu^n, -r) \leq 1 - \rho^*$. Since $-r$ is a moment, the preceding inequality along with [26] Proposition E.8 and [26] Proposition E.6 imply that there exists a subsequence of measures $\{\mu^n\}$ converging weakly to a measure $\mu$ on $K$. Now

$$\rho^* \leq \limsup_{i \to \infty} \langle \mu^n, r \rangle$$

holds since $\langle \mu^n, r \rangle \geq \rho^* - \epsilon_n$ for all $n$ and $\epsilon_n \downarrow 0$. By [26] Proposition E.2,

$$\limsup_{i \to \infty} \langle \mu^n, r \rangle \leq \langle \mu, r \rangle,$$

so we obtain

$$\rho^* \leq \langle \mu, r \rangle.$$

Since $\langle \mu, r \rangle \leq \rho^*$ must hold by definition of $\rho^*$, the preceding inequality shows that $\langle \mu, r \rangle = \rho^*$, i.e. $\mu$ attains the optimal value $\rho^*$ and is stable. By a similar argument,

$$\mathbb{E} [(Y - \eta)_-] \leq \limsup_{i \to \infty} \langle \mu^{ni}, (z - \eta)_- \rangle \leq \langle \mu, (z - \eta)_- \rangle$$

since each $\langle \mu^{ni}, (z - \eta)_- \rangle \geq \mathbb{E} [(Y - \eta)_-]$ for all $i$ and all $\eta \in [a, b]$. Thus, $\mu$ is feasible.

Let $\mu^*$ be the optimal stable measure just guaranteed, and disintegrate to obtain $\mu^* = \hat{\mu}^* \cdot \phi^*$. The pair $(\phi^*, \hat{\mu}^*)$ is then optimal for problem (2.3) - (2.5) since

$$R (\phi^*, \hat{\mu}^*) = \langle \mu^*, r \rangle = \rho^*,$$

and

$$Z_\eta (\phi^*, \hat{\mu}^*) = \langle \mu^*, (z - \eta)_- \rangle \geq \mathbb{E} [(Y - \eta)_-]$$

for all $\eta \in [a, b]$. \hfill \Box

From the preceding theorem, we can now write maximization instead of supremum in the objective of problem (2.3) - (2.5),

$$\rho^* \triangleq \max \{ R (\pi, \nu) : (\pi, \nu) \in \Delta_s \}.$$

We are now ready to formalize problem (2.3) - (2.5) as a linear program. Introduce the weight function

$$w (s, a) = 1 - r (s, a)$$

on $K$. Under our assumption that $r$ is nonpositive, $w$ is bounded from below by one. The space of signed Borel measures on $K$ is denoted $\mathcal{M} (K)$. With the preceding weight function, define $\mathcal{M}_w (K)$ to be the space of signed measures $\mu$ on $K$ such that

$$\| \mu \|_{\mathcal{M}_w (K)} \triangleq \int_K w (s, a) \mu (d (s, a)) < \infty.$$

We can identify elements in $\mathcal{M}_w (K)$ with stable policies, and vice versa. First, observe that the space $\mathcal{M}_w (K)$ is contained in the set of stable probability measures. If $\| \mu \|_{\mathcal{M}_w (K)} < \infty$, then certainly

$$\langle \mu, r \rangle = \int_K r (s, a) \mu (d (s, a)) > -\infty$$

since $1 - r = w$. Conversely, if $\mu$ is a stable probability measure, then it is an element of $\mathcal{M}_w (K)$ since

$$\int_K w (s, a) |\mu| (d (s, a)) = \int_K (1 - r (s, a)) \mu (d (s, a)) = \mu (K) - \langle \mu, r \rangle < \infty.$$

Also define the weight function

$$\hat{w} (s) = 1 - \sup_{a \in A (s)} r (s, a)$$

8
on $S$ which is also bounded from below by one. The space $\mathcal{M}_w(S)$ is defined analogously with $\hat{w}$ and $S$ in place of $w$ and $S \times A$.

The topological dual of $\mathcal{M}_w(K)$ is $\mathcal{F}_w(K)$, the vector space of measurable functions $h : K \to \mathbb{R}$ such that
\[
\|h\|_{\mathcal{F}_w(K)} \triangleq \sup_{(s,a) \in K} \frac{|h(s,a)|}{w(s,a)} < \infty.
\]
Certainly, $r \in \mathcal{F}_w(K)$ by definition of $w$ since
\[
\|r\|_{\mathcal{F}_w(K)} = \sup_{(s,a) \in K} \frac{|r(s,a)|}{w(s,a)} = \sup_{(s,a) \in K} \frac{|r(s,a)|}{1 + |r(s,a)|} \leq 1.
\]
Every element $h \in \mathcal{F}_w(K)$ induces a continuous linear functional on $\mathcal{M}_w(K)$ defined by
\[
\langle \mu, h \rangle \triangleq \int_K h(s,a) \mu(d(s,a)).
\]
The two spaces $(\mathcal{M}_w(K), \mathcal{F}_w(K))$ are called a dual pair, and the duality pairing is the bilinear form $\langle u, h \rangle : \mathcal{M}_w(K) \times \mathcal{F}_w(K) \to \mathbb{R}$ just defined. The topological dual of $\mathcal{M}_w(S)$ is $\mathcal{F}_w(S)$, which is defined analogously with $S$ and $\hat{w}$ in place of $K$ and $w$.

We can now make some additional technical assumptions.

**Assumption 3.5.** (a) The function $(z - \eta)_-$ is an element of $\mathcal{F}_w(K)$ for all $\eta \in [a,b]$.

(b) The function $\int_S \hat{w}(\xi) Q(d\xi \mid s,a) : S \times A \to \mathbb{R}$ is an element of $\mathcal{F}_w(K)$.

Notice that assumption (3.5a) is satisfied if $z \in \mathcal{F}_w(K)$. To see this fact, reason that
\[
\|z\|_{\mathcal{F}_w(K)} \leq \|z - \eta\|_{\mathcal{F}_w(K)} \leq \|z\|_{\mathcal{F}_w(K)} + \|\eta\|_{\mathcal{F}_w(K)},
\]
where the first inequality follows from $|z - \eta| \leq |z - \eta|$. The constant function $f(x) = \eta$ on $K$ is in $\mathcal{F}_w(K)$ since
\[
\|\eta\|_{\mathcal{F}_w(K)} = \sup_{(s,a) \in K} \frac{|\eta|}{w(s,a)} \leq |\eta|.
\]

The linear mapping $L_0 : \mathcal{M}_w(K) \to \mathcal{M}_w(S)$ defined by
\[
[L_0\mu](B) \triangleq \hat{\mu}(B) - \int_K \phi(B \mid s,a) \mu(d(s,a)) \quad \forall B \in \mathcal{B}(S),
\]
(3.1)
is used to verify that $\mu$ is an invariant probability measure on $K$ with respect to $Q$. The mapping $\hat{\mu}$ appears in all work on convex analytic methods for long-run average reward/cost MDPs. When $L_0\mu(B) = 0$, it means that the long-run proportion of time in state $B$ is equal to the rate at which the system transitions to state $B$ from all state-action pairs $(s,a) \in K$.

**Lemma 3.6.** The condition $\mu \in \Delta_s$ is equivalent to $\langle \mu, r \rangle > -\infty$ and
\[
L_0\mu = 0,
\]
\[
\langle \mu, 1 \rangle = 1,
\]
\[
\langle \mu, (z - \eta)_- \rangle \geq \mathbb{E}[(Y - \eta)_-, \forall \eta \in [a,b],
\]
\[
\mu \geq 0.
\]

**Proof.** The linear constraints $\langle \mu, 1 \rangle = \int_K \mu(d(s,a)) = 1$ and $\mu \geq 0$ just ensure that $\mu$ is a probability measure on $K$. The condition $L_0\mu = 0$ is equivalent to invariance of $\mu$ with respect to $Q$. For stable $\mu = \hat{\mu} \cdot \phi$, $R(\phi, \hat{\mu}) = \langle \mu, r \rangle > -\infty$ and $Z_0(\phi, \hat{\mu}) = \langle \mu, (z - \eta)_- \rangle$. Since $Z_\eta(\phi, \hat{\mu}) \geq \mathbb{E}[(Y - \eta)_-, \forall \eta \in [a,b]$ for all $\eta \in [a,b]$, the conclusion follows.

Next we continue with the representation of the dominance constraints (2.3). We would like to express the constraints $\langle \mu, (z - \eta)_- \rangle \geq \mathbb{E}[(Y - \eta)_-, \forall \eta \in [a,b]$ through a single linear operator.
Lemma 3.7. For any \( \mu \in \mathcal{P}(K) \), \( \langle \mu, (z - \eta)_- \rangle \) is uniformly continuous in \( \eta \) on \([a,b]\).

Proof. Write \( \langle \mu, (z - \eta)_- \rangle = \int_K (z(s,a) - \eta)_- \mu(d(s,a)) \). Certainly, each function \( (z(s,a) - \eta)_- \) is continuous in \( \eta \) for fixed \( s \times a \). Choose \( \epsilon > 0 \) and \( |\eta' - \eta| < \epsilon \). Then

\[
| (z(s,a) - \eta')_- - (z(s,a) - \eta)_- | \\
\leq |z(s,a) - \eta' - z(s,a) + \eta| \\
\leq \epsilon,
\]

by definition of \((x)_-\). It follows that

\[
| \int_{S \times A} (z(s,a) - \eta')_- \mu(d(s,a)) - \int_K (z(s,a) - \eta)_- \mu(d(s,a)) | \\
\leq | \int_K \epsilon \mu(d(s,a)) | \\
= \epsilon,
\]

since \( \mu \) is a probability measure. \( \square \)

The preceding lemma allows us to write the dominance constraints (2.4) as a linear operator in the space of continuous functions. Recall that we have assumed \([a,b]\) to be a compact set. Let \( C([a,b]) \) be the space of continuous functions on \([a,b]\) in the supremum norm,

\[
\|f\|_{C([a,b])} = \sup_{a \leq x \leq b} |f(x)|
\]

for \( f \in C([a,b]) \). The topological dual of \( C([a,b]) \) is \( M([a,b]) \), the space of finite signed Borel measures on \([a,b]\). Every measure \( \Lambda \in M([a,b]) \) induces a continuous linear functional on \( C([a,b]) \) through the bilinear form

\[
\langle \Lambda, f \rangle = \int_a^b f(\eta) \Lambda(d\eta).
\]

Define the linear operator \( L_1 : M_w(K) \rightarrow C([a,b]) \) by

\[
[L_1 \mu](\eta) \triangleq \langle \mu, (z - \eta)_- \rangle, \quad \forall \eta \in [a,b].
\]

(3.2)

Also define the continuous function \( y \in C([a,b]) \) where \( y(\eta) = E[(Y - \eta)_-] \) is the shortfall in \( Y \) at level \( \eta \) for all \( \eta \in [a,b] \). The dominance constraints are then equivalent to \([L_1 \mu](\eta) \geq y(\eta) \) for all \( \eta \in [a,b] \), which can be written as the single inequality \( L_1 \mu \geq y \) in \( C([a,b]) \).

The linear programming form of problem (2.3) - (2.5) is

\[
\max \quad \langle \mu, r \rangle \\
\text{s.t.} \quad L_0 \mu = 0, \\
\quad \langle \mu, 1 \rangle = 1, \\
\quad L_1 \mu \geq y, \\
\quad \mu \in M_w(K), \mu \geq 0.
\]

(3.3) - (3.7)

Since \( \rho^* \triangleq \max \{ R(\pi, \nu) : (\pi, \nu) \in \Delta_s \} \), and stable probability measures on \( K \) can be identified as elements of \( M_w(K) \), problem (2.3) - (2.5) is equivalent to problem (3.3) - (3.7).
4 Establishing strong duality

In this section we apply infinite-dimensional linear programming duality to obtain the strong dual to problem (3.3) - (3.7). The development in [2] is behind our duality development, and the duality theory for linear programming for MDPs on Borel spaces in general.

We will introduce Lagrange multipliers for constraints (3.4), (3.5), and (3.6), each Lagrange multiplier is drawn from the appropriate topological dual space. Introduce Lagrange multipliers \( h \in \mathcal{F}_w(S) \) for constraint (3.4). The constraint \( \langle \mu, \eta \rangle = 1 \) is an equality in \( \mathbb{R} \), so introduce Lagrange multipliers \( \beta \in \mathbb{R} \) for constraint (3.5). Finally, introduce Lagrange multipliers \( \Lambda \in \mathcal{M}([a, b]) \) for constraints (3.6). The Lagrange multipliers \((h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b])\) will be the decision variables in the upcoming dual to problem (3.3) - (3.7).

To proceed with duality, we compute the adjoints of \( L_0 \) and \( L_1 \). The adjoint is analogous to the transpose for linear operators in Euclidean spaces.

**Lemma 4.1.** (a) The adjoint of \( L_0 \) is \( L_0^*: \mathcal{F}_w(S) \rightarrow \mathcal{F}_w(K) \) where

\[
[L_0^* h](s, a) \triangleq h(s) - \int_S h(\xi) Q(d\xi | s, a)
\]

for all \((s, a) \in K\).

(b) The adjoint of \( L_1 \) is \( L_1^*: \mathcal{M}([a, b]) \rightarrow \mathcal{F}_w(K) \) where

\[
[L_1^* \Lambda](s, a) = \int_a^b (z(s, a) - \eta)_- \Lambda(d(s, a)).
\]

**Proof.** (a) This result is well known, see [26, 27].

(b) Write

\[
\langle \Lambda, L_1 \mu \rangle = \int_a^b \langle \mu, (z - \eta)_- \rangle \Lambda(d\eta)
\]

\[
= \int_a^b \left( \int_K (z(s, a) - \eta)_- \mu(d(s, a)) \right) \Lambda(d\eta).
\]

When \( z \) is bounded on \( S \times A \), then

\[
\left| \int_K (z(s, a) - \eta)_- (\mu \times \Lambda)(d((s, a) \times \eta)) \right| = \int_K \frac{(z(s, a) - \eta)_-}{w(s, a)} w(s, a) (\mu \times \Lambda)(d((s, a) \times \eta))
\]

\[
\leq \|z - \eta\_\|_{\mathcal{F}_w(K)} \|\mu\|_{\mathcal{M}_w(K)} \|\Lambda\|_{\mathcal{M}([a, b])} < \infty,
\]

since \( \|\mu\|_{\mathcal{M}(K)} = 1 \) and \( \|\Lambda\|_{\mathcal{M}([a, b])} < \infty \). The Fubini theorem applies to justify interchange of the order of integration,

\[
\langle \Lambda, L_1 x \rangle = \int_a^b \left( \int_K (z(s, a) - \eta)_- \mu(d(s, a)) \right) \Lambda(d\eta)
\]

\[
= \int_K \int_a^b (z(s, a) - \eta)_- \Lambda(d\eta) \mu(d(s, a))
\]

\[
= \int_K \langle \Lambda, (z(s, a) - \eta)_- \rangle \mu(d(s, a)),
\]

revealing \( L_1^*: \mathcal{M}([a, b]) \rightarrow \mathcal{F}_w(K) \).

We obtain the dual to problem (3.3) - (3.7) in the next theorem.
Theorem 4.2. The dual to problem (3.3) - (3.7) is

\[
\inf_{\beta} \quad \beta - \langle \Lambda, y \rangle \\
\text{s.t.} \quad r + L^*_0 h - \beta 1 + L^*_1 \Lambda \leq 0, \\
(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b]), \ \Lambda \geq 0. \tag{4.1}
\]

\[
(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b]). \tag{4.2}
\]

\[
(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b]), \ \Lambda \geq 0. \tag{4.3}
\]

Proof. The Lagrangian for problem (3.3) - (3.7) is

\[
\vartheta (\mu, h, \beta, \Lambda) \equiv \langle \mu, r \rangle + \langle h, L_0 \mu \rangle + \beta ((\mu, 1) - 1) + \langle \Lambda, L_1 \mu - y \rangle,
\]

allowing problem (3.3) - (3.7) to be expressed as

\[
\max_{\mu \in \mathcal{M}_u(K)} \left\{ \inf_{(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b])} \{ \vartheta (\mu, h, \beta, \Lambda) : \Lambda \geq 0 \} : \mu \geq 0 \right\}. \tag{4.5}
\]

We rearrange the Lagrangian to obtain

\[
\vartheta (\mu, h, \beta, \Lambda) = \langle \mu, r \rangle + \langle h, L_0 \mu \rangle + \beta ((\mu, 1) - 1) + \langle \Lambda, L_1 \mu - y \rangle
\]

\[
= \langle \mu, r \rangle + \langle L^*_0 h, \mu \rangle + \langle \mu, \beta 1 \rangle - \beta + \langle L^*_1 \Lambda, \mu \rangle - \langle \Lambda, y \rangle
\]

\[
= \langle \mu, r + L^*_0 h + \beta 1 + L^*_1 \Lambda \rangle - \beta - \langle \Lambda, y \rangle.
\]

The dual to problem (3.3) - (3.7) is then

\[
\inf_{(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b])} \left\{ \max_{\mu \in \mathcal{M}_u(K)} \{ \vartheta (\mu, h, \beta, \Lambda) : \mu \geq 0 \} : \Lambda \geq 0 \right\}. \tag{4.4}
\]

Since \( \mu \geq 0 \), the constraint \( r + L^*_0 h + \beta 1 + L^*_1 \Lambda \leq 0 \) is implied. Since \( \beta \) is unrestricted, take \( \beta = -\beta \) to get the desired form.

We write problem (4.4) - (4.6) with the infimum objective rather than the minimization objective because we must verify that the optimal value is attained. The dual problem (4.4) - (4.6) is explicitly

\[
\inf_{\beta} \quad \beta - \int_a^b \mathbb{E} \left[ (Y - \eta)_{-} \right] \Lambda (d\eta) \tag{4.7}
\]

\[
\text{s.t.} \quad r (s, a) + \int_a^b (z (s, a) - \eta)_{-} \Lambda (d\eta) \leq \beta + \int_s^\Lambda h (\xi) Q (d\xi | s, a), \quad \forall (s, a) \in K, \tag{4.8}
\]

\[
(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a, b]), \ \Lambda \geq 0. \tag{4.9}
\]

Since \( r \leq 0 \), problem (4.7) - (4.9) is readily seen to be consistent by choosing \( h = 0, \beta = 0, \) and \( \Lambda = 0. \)

Problem (4.4) - (4.6) has another, more intuitive form. In [9] [10] [12], it is recognized that the Lagrange multipliers of stochastic dominance constraints are utility functions. This result is true in our case as well. Using the family \( \{(x - \eta)_{-} : \eta \in [a, b]\} \), any measure \( \Lambda \in \mathcal{M}([a, b]) \) induces an increasing concave function

\[
u (x) = \int_a^b (x - \eta)_{-} \Lambda (d\eta)
\]

for all \( x \in \mathbb{R} \). In fact, the above definition of \( u \) gives a function in \( \mathcal{C} (\mathbb{R}) \) as well. Define

\[
\mathcal{U} ([a, b]) = \text{cl cone} \left\{ (x - \eta)_{-} : \eta \in [a, b] \right\}
\]

\[
= \left\{ u (x) = \int_a^b (x - \eta)_{-} \Lambda (d\eta) : \Lambda \in \mathcal{M}([a, b]), \ \Lambda \geq 0 \right\}
\]

to be the closure of the cone generated by the family \( \{(x - \eta)_{-} : \eta \in [a, b]\} \). The set \( \mathcal{U} ([a, b]) \subset \mathcal{U} (\mathbb{R}) \) is the set of all utility functions that can be constructed by limits of sums of scalar multiplies of functions in \( \{(x - \eta)_{-} : \eta \in [a, b]\} \).
Corollary 4.3. Problem (4.4) - (4.6) is equivalent to
\[
\begin{align*}
\inf & \quad \beta - \mathbb{E}[u(Y)] \\
\text{s.t.} & \quad r(s,a) + u(z(s,a)) \leq \beta + h(s) - \int_S h(\xi) Q(d\xi | s, a), \quad \forall (s,a) \in K, \\
& \quad (h, \beta, u) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{U}([a,b]).
\end{align*}
\]

Proof. Notice that the function
\[u(x) = \int_a^b (x - \eta) \Lambda(d\eta)\]
is an increasing concave function in \(x\) for any \(\Lambda \in \mathcal{M}([a,b])\) with \(\Lambda \geq 0\). By using this definition of \(u\), we see that for each state-action pair \((s,a)\),
\[\langle \Lambda, (z(s,a) - \eta)_- \rangle = \int_a^b (z(s,a) - \eta) \Lambda(d\eta) = u(z(s,a)).\]
Further, we can apply the Fubini theorem again to obtain
\[\langle \Lambda, y \rangle = \int_a^b \mathbb{E}[(Y - \eta)_-] \Lambda(d\eta) = \mathbb{E}\left[\int_a^b (Y - \eta)_- \Lambda(d\eta)\right] = \mathbb{E}[u(Y)].\]

Next we verify that there is no duality gap between the primal problem (3.3) - (3.7) and its dual (4.1) - (4.3). All three dual problems (4.1) - (4.3), (4.4) - (4.6), and (4.7) - (4.9) are equivalent so the upcoming results apply to all of them.

The following result states that the optimal values of problems (3.3) - (3.7) and (4.1) - (4.3) are equal.

Theorem 4.4. The optimal values of problems (3.3) - (3.7) and (4.1) - (4.3) are equal,
\[\rho^* = \max \{ R(\pi, \nu) : (\pi, \nu) \in \Delta \} = \inf \{ \beta - \langle \Lambda, y \rangle : (h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}([a,b]), \Lambda \geq 0 \} .\]

Proof. Apply [27, Theorem 12.3.4], which in turn follows from [2, Theorem 3.9]. Introduce slack variables \(\alpha \in \mathcal{C}([a,b])\) for the dominance constraints \(L_{11} \mu \geq y\). We must show that the set
\[H \triangleq \{(L_{00}\mu, \langle \mu, 1 \rangle, L_{1} x - \alpha, \langle \mu, r \rangle - \zeta) : \mu \geq 0, \alpha \geq 0, \zeta \geq 0\}\]
is weakly closed (closed in the weak topology). Let \((D, \leq)\) be a directed (partially ordered) set, and consider a net \[\{(\mu_\kappa, \alpha_\kappa, \zeta_\kappa) : \kappa \in D\}\]
where \(\mu_\kappa \geq 0, \alpha_\kappa \geq 0, \text{ and } \zeta_\kappa \geq 0\) in \(\mathcal{M}_w(K) \times \mathbb{R} \times \mathcal{C}([a,b])\) such that
\[\left(L_{00}\mu_\kappa, \langle \mu_\kappa, 1 \rangle, L_{1} \mu_\kappa - \alpha_\kappa, \langle \mu_\kappa, r \rangle - \zeta_\kappa\right)\]
has weak limit \((\nu^*, \gamma^*, f^*, \rho^*) \in \mathcal{M}_w(S) \times \mathbb{R} \times \mathcal{C}([a,b]) \times \mathbb{R}\). Specifically,
\[\langle \mu_\kappa, 1 \rangle \to \gamma^*\]
and
\[\langle \mu_\kappa, r \rangle - \zeta_\kappa \to \rho^*,\]
since weak convergence on \(\mathbb{R}\) is equivalent to the usual notion of convergence,
\[\langle L_{00}\mu_\kappa, g \rangle \to \langle \nu^*, g \rangle\]
for all $g \in \mathcal{F}_{\hat{w}}(S)$, and
\[
(L_{1}\mu_{\kappa} - \alpha_{\kappa}, \Lambda) \to (f^{*}, \Lambda)
\]
for all $\Lambda \in \mathcal{M}([a, b])$. We must show that $(\nu^{*}, \gamma^{*}, f^{*}, \rho^{*}) \in H$ under these conditions, i.e. that there exist $x \geq 0$, $\alpha \geq 0$, and $\zeta \geq 0$ such that
\[
\nu^{*} = L_{0}\mu_{\kappa}, \gamma^{*} = \langle \mu, 1 \rangle, f^{*} = L_{1}\mu_{\kappa} - \alpha, \rho^{*} = \langle \mu, r \rangle - \zeta.
\]
The fact that there exist $\mu \geq 0$ and $\zeta \geq 0$ such that
\[
\nu^{*} = L_{0}\mu_{\kappa}, \gamma^{*} = \langle \mu, 1 \rangle, \rho^{*} = \langle \mu, r \rangle - \zeta,
\]
is already established in [27, Theorem 12.3.4], and applies to our setting without modification.

It remains to verify that there exists $\alpha \in \mathcal{C}([a, b])$ with $\alpha \geq 0$ and $f^{*} = L_{1}\mu_{\kappa} - \alpha$. Choose $\Lambda = \delta_{\eta}$ for the Dirac delta function at $\eta \in [a, b]$ to see that
\[
[L_{1}\mu_{\kappa}](\eta) - \alpha_{\kappa}(\eta) \to f^{*}(\eta)
\]
for all $\eta \in [a, b]$, establishing pointwise convergence. Pointwise convergence on a compact set implies uniform convergence, so in fact
\[
L_{1}\mu_{\kappa} - \alpha_{\kappa} \to f^{*}
\]
in the supremum norm topology on $\mathcal{C}([a, b])$. Since $L_{1}\mu_{\kappa} \in \mathcal{C}([a, b])$ and $f^{*} \in \mathcal{C}([a, b])$, it follows that $L_{1}\mu_{\kappa} - f^{*} \in \mathcal{C}([a, b])$ for any $\kappa$. Define $\alpha_{\kappa} = L_{1}\mu_{\kappa} - f^{*}$ and $\alpha = L_{1}\mu_{\kappa} - f^{*}$, and notice that $\alpha \geq 0$ necessarily.

The next theorem shows that the dual problem (4.1) - (4.3) is solvable, i.e. there exists $(h^{*}, \beta^{*}, \Lambda^{*})$ satisfying $r + L_{0}h^{*} - \beta^{*}1 + L_{1}\Lambda^{*} \leq 0$ that attain the optimal value
\[
\beta^{*} - \langle \Lambda^{*}, y \rangle = \rho^{*}.
\]

When problem (4.1) - (4.3) is solvable, we are justified in saying that strong duality holds: the optimal values of both problems (3.3) - (3.7) and (4.1) - (4.3) are equal and both problems attain their optimal value.

To continue we make some assumptions in line with [23].

**Assumption 4.5.** There exists a minimizing sequence $(h^{n}, \beta^{n}, \Lambda^{n})$ in problem (4.1) - (4.3) such that
\[
(a) \{\beta^{n}\} \text{ is bounded in } \mathbb{R},
\]
\[
(b) \{h^{n}\} \text{ is bounded in } \mathcal{F}_{\hat{w}}(S), \text{ and}
\]
\[
(c) \{\Lambda^{n}\} \text{ is bounded in the weak* topology on } \mathcal{M}([a, b]).
\]

We establish strong duality next. To reiterate, strong duality holds when the optimal values of problems (3.3) - (3.7) and (4.1) - (4.3) are equal, and both problems are solvable.

**Theorem 4.6.** Suppose assumption 4.5 holds. Strong duality holds between problem (3.3) - (3.7) and problem (4.1) - (4.3).

**Proof.** Let $(h^{n}, \beta^{n}, \Lambda^{n}) \in \mathcal{F}_{\hat{w}}(S) \times \mathbb{R} \times \mathcal{M}([a, b])$ for $n \geq 0$ be a minimizing sequence of triples given in the preceding assumption 4.5.

\[
r(s,a) + \int_{a}^{b} (z(s,a) - \eta)_{-} \Lambda^{n}(d\eta) \leq \beta^{n} + h^{n}(s) - \int_{S} h^{n}(\xi) Q(\text{d}\xi \mid s, a), \quad \forall (s,a) \in K,
\]
for all $n \geq 0$ and
\[
\beta^{n} - \int_{a}^{b} \mathbb{E} \left[(Y - \eta)_{-}\right] \Lambda^{n}(d\eta) \downarrow \rho^{*}.
\]
Since the sequence $\{\beta^{n}\}$ is bounded, it has a convergent subsequence with $\lim_{n \to \infty} \beta^{n} = \beta^{*}$. 

14
Now \( \{ \Lambda^n \} \) is bounded in \( M([a,b]) \) in the weak* topology induced by \( C([a,b]) \) by assumption. Since \( \{ \Lambda^n \} \) is bounded, the sequence can be scaled to lie in the closed unit ball of \( M([a,b]) \) in the weak* topology. Since \( C([a,b]) \) is separable (there exists a countable dense set, i.e., the polynomials with rational coefficients), the weak* topology on \( M([a,b]) \) is metrizable. By the Banach-Alaoglu theorem, it follows that \( \{ \Lambda^n \} \) has a subsequence that converges to some \( \Lambda^* \) in the weak* topology, i.e.,

\[
\langle \Lambda^n, f \rangle \to \langle \Lambda^*, f \rangle
\]

for all \( f \in C([a,b]) \). In particular, since \( E \left[ (Y - \eta)_{-} \right] \) and \( (z(s, a) - \eta)_{-} \) are continuous functions on \([a,b]\) for all \((s, a) \in K\), it follows that

\[
\lim_{n \to \infty} \int_a^b E \left[ (Y - \eta)_{-} \right] \Lambda^n(d\eta) = \int_a^b E \left[ (Y - \eta)_{-} \right] \Lambda^*(d\eta)
\]

and

\[
\lim_{n \to \infty} \int_a^b (z(s, a) - \eta)_{-} \Lambda^n(d\eta) = \int_a^b (z(s, a) - \eta)_{-} \Lambda^*(d\eta).
\]

Finally, since \( \{ h^n \} \) is bounded in \( F_{\tilde{w}}^*(S) \) we can define

\[
h^*(s) \triangleq \lim_{n \to \infty} h^n(s)
\]

for all \( s \in S \). Then the function \( h^*(s) \) is bounded in \( F_{\tilde{w}}^*(S) \), and

\[
\lim_{n \to \infty} \int_S h^n(\xi) Q(d\xi | s, a) \geq \int_S h^*(\xi) Q(d\xi | s, a)
\]

by Fatou’s lemma. Taking the limit, it follows that \( (h^*, \beta^*, \Lambda^*) \) is an optimal solution to the dual problem. \( \square \)

The role of the utility function \( u \) in problem \((4.7) - (4.9)\) is fairly intuitive. The function \( u \) serves as an additional pricing variable for the performance function \( z(s, a) \), and the total reward is treated as if it were \( r(s, a) + u(z(s, a)) \). Problem \((4.7) - (4.9)\) leads to a new version of the optimality equations for average reward based on infinite-dimensional linear programming complementary slackness.

**Theorem 4.7.** Let \( \mu^* = \hat{\mu}^* \cdot \phi^* \) be an optimal solution to problem \((3.3) - (3.7)\), and \( (h^*, \beta^*, u^*) \) be an optimal solution to problem \((4.7) - (4.9)\). Then

\[
\langle \mu^*, u^*(z) \rangle = E[u^*(Y)],
\]

and

\[
\beta^* + h^*(s) = \sup_{a \in A(s)} \left\{ r(s, a) + u^*(z(s, a)) + \int_S h^*(\xi) Q(d\xi | s, a) \right\}
\]

for \( \hat{\mu}^* \)—almost all \( s \in S \).

**Proof.** There is a corresponding optimal solution \( (h^*, \beta^*, \Lambda^*) \) to problem \((4.1) - (4.3)\). Complementary slackness between problems \((3.3) - (3.7)\) and \((4.1) - (4.3)\) gives \( (\Lambda^*, L_1 \mu^* - y) = 0 \), where \( (h^*, \beta^*, u^*) \) is a corresponding optimal solution of problem \((4.1) - (4.3)\). Then

\[
\langle \Lambda^*, L_1 \mu^* \rangle = \langle L_1^* \Lambda^*, \mu^* \rangle = \langle \mu^*, u^*(z) \rangle
\]

and \( \langle \Lambda^*, y \rangle = E[u^*(Y)] \).

Complementary slackness also gives

\[
\langle r + L_0^* h^* - \beta^* 1 + L_1^* \Lambda^*, \mu^* \rangle = 0,
\]

which yields the second statement since \( \mu^* \geq 0 \) and \( r + L_0^* h^* - \beta^* 1 + L_1^* \Lambda^* \leq 0 \). \( \square \)
5 Variations and extensions

5.1 Multivariate integral stochastic orders

We extend our repertoire in this section to include some additional stochastic orders. Integral stochastic orders (see [23]) refer to stochastic orders that are defined in terms of families of functions. The increasing concave stochastic order is an example of an integral stochastic order, because it is defined in terms of the family of increasing concave functions. We now give attention to some multivariate integral stochastic orders. So far, we have considered a the family of increasing concave functions. We now give attention to some multivariate integral stochastic orders. For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution

For any fixed $s, a$.

For example, $z(s, a)$ may represent the service rate to $n$ customers in a wireless network. The empirical distribution
\[ [L_1 x](\xi) \triangleq \langle x, u(z; \xi) \rangle, \quad \xi \in \Xi. \] (5.4)

Also define the continuous function \( y \in \mathcal{C}(\Xi) \) by \( y(\xi) = \mathbb{E}[u(Y; \xi)] \) for all \( \xi \in \Xi \) to represent the benchmark.

The steady-state version of problem (5.1) - (5.3) is the modified linear program:

\[
\begin{aligned}
\text{max} & \quad \langle \mu, r \rangle \\
\text{s.t.} & \quad L_0 \mu = 0, \\
& \quad \langle \mu, 1 \rangle = 1, \\
& \quad L_1 \mu \geq y, \\
& \quad \mu \in \mathcal{M}_w(K), \; \mu \geq 0.
\end{aligned}
\] (5.5) - (5.9)

Problem (5.5) - (5.9) is almost the same as problem (3.3) - (3.7), except that now \( L_1 \mu \) is an element in \( \mathcal{C}(\Xi) \) to reflect the multivariate dominance constraint.

We now compute the adjoint of \( L_1 \), which depends on the choice of family \( \{ u(\cdot; \xi) : \xi \in \Xi \} \). The parametrization \( u(\cdot; \xi) \) will appear explicitly in this computation.

**Lemma 5.2.** The adjoint of \( L_1 \) is \( L_1^* : \mathcal{M}(\Xi) \to \mathcal{F}_w(K) \) where

\[
[L_1^* \Lambda](s,a) \triangleq \int_{\Xi} u(z(s,a); \xi) \Lambda(d\xi).
\]

**Proof.** Write

\[
\langle \Lambda, L_1 \mu \rangle = \int_{\Xi} \langle \mu, u(z; \xi) \rangle \Lambda(d\xi)
\]

\[
= \int_{\Xi} \left[ \int_{K} u(z(s,a); \xi) \mu(d(s,a)) \right] \Lambda(d\xi).
\]

When \( z \) is bounded on \( S \times A \), then

\[
| \int_{\Xi} u(z; \xi)(\mu \times \Lambda)(d((s,a) \times \xi)) | \leq \| u(z(\cdot; \xi) \|_{\mathcal{F}_w(K)} \| \mu \|_{\mathcal{M}_w(K)} \| \Lambda \|_{\mathcal{M}([a,b])} < \infty.
\]

The Fubini theorem applies to justify interchange of the order of integration,

\[
\langle \Lambda, L_1 \mu \rangle = \int_{K} \left[ \int_{\Xi} u(z(s,a); \xi) \Lambda(d\xi) \right] \mu(d(s,a))
\]

\[
= \int_{K} \langle \Lambda, u(z(s,a); \xi) \rangle \mu(d(s,a)).
\]

The dual to problem (5.5) - (5.9) looks identical to problem (4.1) - (4.3) and is now explicitly

\[
\inf \beta - \int_{\Xi} \mathbb{E}[u(Y; \xi)] \Lambda(d\xi)
\]

\[
\text{s.t.} \quad r(s,a) + \int_{\Xi} u(z(s,a); \xi) \Lambda(d\xi) \leq \beta + h(s) - \int_{S} h(\xi) Q(d\xi | s,a), \quad \forall (s,a) \in K,
\]

\[
(h, \beta, \Lambda) \in \mathcal{F}_w(S) \times \mathbb{R} \times \mathcal{M}(\Xi), \; \Lambda \geq 0.
\] (5.10) - (5.12)

Define

\[
\mathcal{U}(\Xi) = \text{cl cone} \{ u(x; \xi) : \xi \in \Xi \}
\]

\[
= \left\{ u(x) = \int_{\Xi} u(x; \xi) \Lambda(d\xi) \; \text{for} \; \Lambda \in \mathcal{M}(\Xi), \; \Lambda \geq 0 \right\}
\]

17
to be the closure of the cone of functions generated by \( \{ u(x; \xi) : \xi \in \Xi \} \). In this case \( U(\Xi) \) is a family of functions in \( C(\mathbb{R}^n) \), the space of continuous functions \( f : \mathbb{R}^n \to \mathbb{R} \). We see immediately that problem (5.10) - (5.12) is equivalent to

\[
\inf \quad \beta - \mathbb{E}[u(Y)] \\
\text{s.t.} \quad r(s, a) + u(z(s, a)) \leq \beta + h(s) - \int_{S} h(\xi) Q(d\xi | s, a), \quad \forall (s, a) \in K, \\
h(\beta, u) \in \mathcal{F}_w(S) \times \mathbb{R} \times U(\Xi).
\]

The variables \( u \in U(\Xi) \) in problem (5.13) - (5.15) are now pricing variables for the vector \( z \). When our earlier assumptions are suitably adapted, then strong duality holds between problem (5.5) - (5.9) and problem (5.13) - (5.15).

**Theorem 5.3.** The optimal values of problems (5.5) - (5.9) and (5.10) - (5.12) are equal. Further, the dual problem (5.10) - (5.12) is solvable and strong duality holds between problems (5.5) - (5.9) and (5.10) - (5.12).

### 5.2 Discounted reward

We briefly sketch the development for discounted reward, it is mostly similar. Discounted cost MDPs in Borel spaces with finitely many constraints are considered in [22]. Introduce the discount factor \( \delta \in (0, 1) \) and consider the long-run expected discounted reward

\[
R(\pi, \nu) = \mathbb{E}_{\nu}^\pi \left[ \sum_{t=0}^{\infty} \delta^t r(s_t, a_t) \right].
\]

We are interested in the distribution of discounted reward \( z \),

\[
\sum_{t=0}^{\infty} \delta^t z(s_t, a_t).
\]

Define

\[
Z_\eta(\pi, \nu) \triangleq \mathbb{E}_{\nu}^\pi \left[ \sum_{t=0}^{\infty} \delta^t (z(s_t, a_t) - \eta) \right].
\]

We propose the dominance-constrained MDP:

\[
\sup \quad R(\pi, \nu) \\
\text{s.t.} \quad Z_\eta(\pi, \nu) \geq \mathbb{E}[ (Y - \eta) ] \quad \forall \eta \in [a, b] , \quad \pi \in \Pi.
\]

We work with the \( \delta \)-discounted expected occupation measure

\[
\mu_\nu^\pi(\Gamma) \triangleq \sum_{t=0}^{\infty} \delta^t P_\nu^\pi((s_t, a_t) \in \Gamma)
\]

for all \( \Gamma \in \mathcal{B}(S \times A) \). Now let

\[
[L_0 \mu](B) \triangleq \hat{\mu}(B) - \delta \int_{S \times A} Q(B | s, a) \mu(d(s, a)), \quad \forall B \in \mathcal{B}(S),
\]

and

\[
[L_1 \mu](\eta) \triangleq (\mu, (z - \eta)_\Gamma), \quad \forall \eta \in [a, b].
\]

18
Also continue to define \( y \in \mathcal{C}([a,b]) \) by \( y(\eta) = \mathbb{E}[(Y - \eta)_-] \) for all \( \eta \in [a,b] \). Problem (5.16) - (5.18) is then equivalent to the linear program

\[
\begin{align*}
\max & \quad \langle \mu, r \rangle \\ 
s.t. & \quad L_0 \mu = \nu, \quad (5.21) \\
& \quad L_1 \mu \geq y, \quad (5.22) \\
& \quad \mu \in \mathcal{M}(K), \mu \geq 0. \quad (5.23)
\end{align*}
\]

Introduce Lagrange multipliers \( h \in \mathcal{F}_w(S) \) for constraint \( L_0 \mu = \nu \) and multipliers \( \Lambda \in \mathcal{M}([a,b]) \) for constraint \( L_1 \mu \geq y \), the Lagrangian is then

\[
\vartheta(\mu, h, \Lambda) = \langle \mu, r \rangle + \langle h, L_0 \mu - \nu \rangle + \langle \Lambda, L_1 \mu - y \rangle.
\]

The adjoint of \( L_0 \) is \( L_0^* : \mathcal{F}_w(S) \to \mathcal{F}_w(S \times A) \) defined by

\[
[L_0^* h](s,a) \triangleq h(s) - \delta \int_S h(\xi) Q(\xi | s, a).
\]

The adjoint of \( L_1 \) is still \( L_1^* : \mathcal{M}([a,b]) \to \mathcal{F}_w(S \times A) \) where

\[
[L_1^* \Lambda](s,a) \triangleq \int_a^b (z(s,a) - \eta)_- \Lambda(d\eta).
\]

The form of the dual follows.

**Theorem 5.4.** The dual to problem (5.21) - (5.24) is

\[
\begin{align*}
\min & \quad \langle h, \nu \rangle - \langle \Lambda, y \rangle \\ 
s.t. & \quad r + L_0^* h + L_1^* \Lambda \geq 0, \\
& \quad h \in \mathcal{F}_w(K), \Lambda \in \mathcal{M}([a,b]), \Lambda \geq 0. \quad (5.27)
\end{align*}
\]

The optimal values of problems (5.21) - (5.24) and (5.26) - (5.27) are equal, and problem (5.25) - (5.27) is solvable.

This dual is explicitly

\[
\begin{align*}
\min & \quad \langle h, \nu \rangle - \mathbb{E}[u(Y)] \\
\text{s.t.} & \quad r(s,a) + u(z(s,a)) \leq h(s) - \delta \int_S h(\xi) Q(\xi | s, a), \quad \forall (s,a) \in K, \\
& \quad h \in \mathcal{F}_w(K), u \in \mathcal{U}([a,b]).
\end{align*}
\]

Problem (5.28) - (5.30) leads to a modified set of optimality equations for the infinite horizon discounted reward case, namely

\[
h(s) = \max_{a \in A(s)} \left\{ r(s,a) + u(z(s,a)) + \delta \int_S h(\xi) Q(\xi | s, a) \right\}
\]

for all \( s \in S \).

### 5.3 Approximate linear programming

Various approaches have been put forward for solving infinite-dimensional LPs with sequences of finite-dimensional LPs, such as in [24 31]. Approximate linear programming (ALP) has been put forward as an approach to the curse of dimensionality, and it can be applied to our present setting. The average reward linear program (3.3) - (3.7) and the discounted reward linear program (5.21) - (5.24) generally have uncountably many variables and constraints.
ALP for average cost dynamic programming is developed in [8]. Previous work on ALP for dynamic programming has focused on approximating the cost-to-go function \( h \) rather than the steady-state occupation measure \( \mu \). It is more intuitive to design basis functions for the cost-to-go function than the occupation measure. For problem (5.31) - (5.33), we approximate the cost-to-go function \( h \in \mathcal{F}_\psi (S) \) with the basis functions \( \{ \phi_1, \ldots, \phi_m \} \subset \mathcal{F}_\psi (S) \). We approximate the pricing variable \( u \in \mathcal{U} ([a, b]) \) with basis functions \( \{ u_1, \ldots, u_n \} \subset \mathcal{U} ([a, b]) \). The resulting approximate linear program is

\[
\min \beta - \mathbb{E} \left[ \sum_{i=1}^{n} \alpha_i u_i (Y) \right] \\
\text{s.t.} \quad r(s, a) + \sum_{i=1}^{n} \alpha_i u_i (z(s, a)) \leq \beta + \sum_{j=1}^{m} \gamma_j h_j (s) - \int_{S} \sum_{j=1}^{m} \gamma_j h_j (\xi) Q (d\xi \mid s, a), \quad \forall (s, a) \in K, (5.32)
\]

\[
(\gamma, \beta, \alpha) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n.
\] (5.33)

We are justified in writing minimization instead of infimum in problem (5.31) - (5.33) because there are only finitely many decision variables. ALP has been studied extensively for the linear programming representation of the optimality equations for discounted infinite horizon dynamic programming (see [16, 17, 14]). The discounted approximate linear program is

\[
\min \langle h, \nu \rangle - \mathbb{E} \left[ \sum_{i=1}^{n} \alpha_i u_i (Y) \right] \\
\text{s.t.} \quad r(s, a) + \sum_{i=1}^{n} \alpha_i u_i (z(s, a)) \leq \sum_{j=1}^{m} \gamma_j h_j (s) - \delta \int_{S} \sum_{j=1}^{m} \gamma_j h_j (\xi) Q (d\xi \mid s, a), \quad \forall (s, a) \in K, (5.35)
\]

\[
(\gamma, \alpha) \in \mathbb{R}^m \times \mathbb{R}^n.
\] (5.36)

Both problems (5.31) - (5.33) and (5.35) - (5.36) are restrictions of the corresponding problems (4.7) - (4.9) and (5.28) - (5.30).

Problems (5.31) - (5.33) and (5.35) - (5.36) have a manageable number of decision variables but an intractable number of constraints. Constraint sampling has been a prominent tool in ALP, and we cite a relevant result now. Let

\[
\langle \gamma_z, r \rangle + \kappa_z \geq 0, \quad \forall z \in \mathcal{L},
\] (5.37)

be a set of linear inequalities in the variables \( r \in \mathbb{R}^k \) indexed by an arbitrary set \( \mathcal{L} \). Let \( \psi \) be a probability distribution on \( \mathcal{L} \), we would like to take i.i.d. samples from \( \mathcal{L} \) to construct a set \( \mathcal{W} \subseteq \mathcal{L} \) with

\[
\sup \left\{ r \mid \langle \gamma_z, r \rangle + \kappa_z \geq 0, \forall z \in \mathcal{W} \right\} \psi \left( \{ y : \langle \gamma_y, r \rangle + \kappa_y < 0 \} \right) \leq \epsilon.
\]

**Theorem 5.5.** [17] Theorem 2.1] For any \( \delta \in (0, 1) \) and \( \epsilon \in (0, 1) \), and

\[
m \geq \frac{4}{\epsilon} \left( k \ln \frac{12}{\epsilon} + \ln \frac{2}{\delta} \right),
\]

a set \( \mathcal{W} \) of \( m \) i.i.d. samples drawn from \( \mathcal{L} \) according to distribution \( \psi \), satisfies

\[
\sup \left\{ r \mid \langle \gamma_z, r \rangle + \kappa_z \geq 0, \forall z \in \mathcal{W} \right\} \psi \left( \{ y : \langle \gamma_y, r \rangle + \kappa_y < 0 \} \right) \leq \epsilon
\]

with probability at least \( 1 - \delta \).

Thus, we can sample state-action pairs from any distribution \( \psi \) on \( K \) to obtain tractable relaxations of problems (5.31) - (5.33) and (5.35) - (5.36) with probabilistic feasibility guarantees. Note that the number of samples required is \( O \left( \frac{1}{\epsilon^2} \frac{1}{\ln \frac{1}{\epsilon}} \right) \).
5.4 Finite state and action spaces

The development for finite state and action spaces is much simpler. Now both problems (3.3) - (3.7) and (4.1) - (4.3) are usual linear programming problems with finitely many variables and constraints. The usual linear programming duality theory applies immediately to establish strong duality between these two problems.

For this section, let $x$ denote an occupation measure on $K$ to emphasize that it is finite-dimensional.

Also suppose the benchmark $Y$ has finite support $\text{supp } Y = \{\eta_1, \ldots, \eta_q\} \subset \mathbb{R}$, so that constraint (2.5) is equivalent to

$$E_x \left[ (z(s,a) - \eta)_- \right] \geq E \left[ (Y - \eta)_- \right], \quad \forall \eta \in \text{supp } Y,$$

by [9, Proposition 3.2]. Each expectation

$$E_x \left[ (z(s,a) - \eta)_- \right] = \sum_{(s,a) \in K} x(s,a) (z(s,a) - \eta)_-$$

is a linear function of $x$.

For finite state and action spaces, the steady-state version of problem (2.3) - (2.5) is:

$$\max \sum_{(s,a) \in K} r(s,a) x(s,a) \quad (5.39)$$

subject to

$$\sum_{a \in A_s} x(j,a) - \sum_{s \in S} \sum_{a \in A(s)} P(j \mid s,a) x(s,a) = 0, \quad \forall j \in S, \quad (5.40)$$

$$\sum_{(s,a) \in \Psi} x(s,a) = 1, \quad (5.41)$$

$$E_x \left[ (z(s,a) - \eta)_- \right] \geq E \left[ (Y - \eta)_- \right], \quad \forall \eta \in \text{supp } Y, \quad (5.42)$$

$$x \geq 0. \quad (5.43)$$

Duality for problem (3.3) - (3.7) is immediate from linear programming duality. As discussed in [34, Chapter 8], the dual of the linear programming problem without the dominance constraints is

$$\min g$$

subject to

$$g + h(s) - \sum_{j \in S} P(j \mid s,a) h(j) \geq r(s,a), \quad \forall (s,a) \in K,$$

$$g \in \mathbb{R}, \ h \in \mathbb{R}^{|Y|}.$$

The vector $h$ is interpreted as the average cost-to-go function. To proceed with the dual for problem (3.3) - (3.7), let $\lambda \in \mathbb{R}^{|Y|}$ with $\lambda \geq 0$ and consider the piecewise linear increasing concave function

$$u(\xi) = \sum_{\eta \in \mathcal{Y}} \lambda(\eta) (\xi - \eta)_-$$

with breakpoints at $\eta \in \mathcal{Y}$. The above function $u(\xi)$ can be interpreted as a utility function for a risk-averse decision maker. We define

$$\mathcal{U}(\mathcal{Y}) = \text{cl cone} \left\{ (x - \eta)_- : \eta \in \mathcal{Y} \right\}$$

$$= \left\{ u(x) = \sum_{\eta \in \mathcal{Y}} \lambda(\eta) (x - \eta)_- \text{ for } \lambda \in \mathbb{R}^{|Y|}, \lambda \geq 0 \right\}$$

to be the set of all such functions. Since $\mathcal{Y}$ is assumed to be finite, $\mathcal{U}(\mathcal{Y})$ is a finite dimensional set.
**Theorem 5.6.** The dual to problem (5.39) - (5.43) is

\[
\begin{align*}
\min & \quad g - \mathbb{E} [u (Y)] \\
\text{s.t.} & \quad r(s, a) + u(z(s, a)) \leq g + h(s) - \sum_{j \in S} P(j | s, a) h(j), \quad \forall (s, a) \in K, \\
& \quad g \in \mathbb{R}, \; h \in \mathbb{R}^{|S|}, \; u \in \mathcal{U}(\mathcal{Y}).
\end{align*}
\] 

(5.44)

(5.45)

(5.46)

Strong duality holds between problem (5.39) - (5.43) and problem (5.44) - (5.46).

**Proof.** Introduce the Lagrangian

\[
L(x, g, h, \lambda) \triangleq \sum_{(s,a) \in K} r(s, a) x(s, a) + g \left[ \sum_{(s,a) \in K} x(s, a) - 1 \right] \\
+ \sum_{j \in S} h(j) \left[ \sum_{a \in A(s)} x(j, a) - \sum_{s \in S} \sum_{a \in A(s)} P(j | s, a) x(s, a) \right] \\
+ \sum_{\eta \in \mathcal{Y}} \lambda(\eta) \left[ \left( \sum_{(s,a) \in K} x(s, a) (z(s, a) - \eta) - \mathbb{E} [u(Y)] \right) \right].
\]

Define the increasing concave function

\[
u(\xi) = \sum_{\eta \in \mathcal{Y}} \lambda(\eta) (\xi - \eta)
\]

then

\[
\sum_{\eta \in \mathcal{Y}} \lambda(\eta) \left[ \left( \sum_{(s,a) \in K} x(s, a) (z(s, a) - \eta) - \mathbb{E} [u(Y)] \right) \right] \\
= \sum_{(s,a) \in K} x(s, a) u(z(s, a)) - \mathbb{E} [u(Y)]
\]

by interchanging finite sums. So, the Lagrangian could also be written as

\[
L(x, g, h, u) = \sum_{(s,a) \in K} r(s, a) x(s, a) + g \left[ \sum_{(s,a) \in K} x(s, a) - 1 \right] \\
+ \sum_{j \in S} h(j) \left[ \sum_{a \in A(s)} x(j, a) - \sum_{s \in S} \sum_{a \in A(s)} P(j | s, a) x(s, a) \right] \\
+ \sum_{(s,a) \in K} x(s, a) u(z(s, a)) - \mathbb{E} [u(Y)],
\]

for \(u \in \mathcal{U}\). The dual to problem (5.43) - (5.42) is defined as

\[
\min_{g \in \mathbb{R}, \; h \in \mathbb{R}^{|S|}, \; u \in \mathcal{U}(\mathcal{Y})} \left\{ \max_{x \geq 0} L(x, g, h, u) \right\}.
\]

Rearranging the Lagrangian gives
\[ L(x, g, h, u) = \sum_{(s,a) \in K} x(s,a) \left[ r(s,a) + g + h(s) - \sum_{j \in S} P(j | s, a) h(j) + u(z(s,a)) \right] \]

so that the dual to problem (5.43) - (5.42) is

\[
\begin{align*}
\min & \quad -g - \mathbb{E}[u(Y)] \\
\text{s.t.} & \quad r(s,a) + g + h(s) - \sum_{j \in S} P(j | s, a) h(j) + u(z(s,a)) \leq 0, \quad \forall (s,a) \in K, \\
& \quad g \in \mathbb{R}, h \in \mathbb{R}^{|S|}, u \in U(Y).
\end{align*}
\]

We used linear programming duality in the preceding proof for illustration. Alternatively, we could have just applied our general strong duality result from earlier. It is immediate that problem (5.45) - (5.46) is the finite-dimensional version of problem (4.7) - (4.9).

There is no difficulty with the Slater condition for problems (5.42) - (5.43) and (5.45) - (5.46) as there is in [9, 10]. In [9, 10], the decision variable in a stochastic program is a random variable so stochastic dominance constraints are nonlinear. In our case, the decision variable \(x\) is in the space of measures and the dominance constraints are linear. Linear programming duality does not depend on the Slater condition.

We compute the dual to problem (5.47) - (5.50) in the next theorem using the space of utility functions \(U\) from earlier.

**Theorem 5.7.** The dual to problem (5.47) - (5.50) is

\[
\begin{align*}
\min & \quad \sum_{j \in S} \alpha(j) v(j) - \mathbb{E}[u(Y)] \\
\text{s.t.} & \quad v(s) - \sum_{s \in S} \sum_{a \in A(s)} \gamma P(j | s, a) v(j) \geq r(s,a) + u(z(s,a)), \quad \forall (s,a) \in K, \\
& \quad v \in \mathbb{R}^{|S|}, u \in U(Y).
\end{align*}
\]

Strong duality holds between problem (5.47) - (5.50) and problem (5.51) - (5.53).

6 Portfolio optimization

We use an infinite horizon discounted portfolio optimization problem to illustrate our ideas in this section. A single period portfolio optimization with stochastic dominance constraints is analyzed in [11]. Specifically,
the model in [11] puts a stochastic dominance constraint on the return rate of a portfolio allocation. We use this model as our motivation for the dynamic setting and put a stochastic dominance constraint on the discounted infinite horizon return rate.

Suppose there are $n$ assets whose prices evolve according to a discrete time Markov chain. We can include a risk-less asset with a constant return rate in this set. The asset prices at time $t$ are

$$ p_t = (p_t(1), \ldots, p_t(n)) \in \mathbb{R}^n, $$

where $p_t(i)$ is the price per share of asset $i$ at time $t$. The portfolio at time $t$ is captured by

$$ x_t = (x_t(1), \ldots, x_t(n)) \in \mathbb{R}^n, $$

where $x_t(i)$ is the quantity of shares held of asset $i$ at time $t$. For a cleaner model, we just treat each $x_t(i)$ as a continuous decision variable. We require $\sum_{i=1}^n x_t(i) = 1$ and $x_t \geq 0$ for all $t \geq 0$, there is no shorting.

The total wealth at time $t$ is then $\langle p_t, x_t \rangle$.

At each time $t \geq 0$, the investor observes the current prices of the assets and then updates portfolio positions subject to transaction costs before new prices are realized. Let $a_t \subset \mathbb{R}^n$ be the buying and selling decisions at time $t$, where $a_t(i)$ is the total change in the number of shares held of asset $i$. Define

$$ A(p, x) \triangleq \{ a \in \mathbb{R}^n : x(i) + a(i) \geq 0 \text{ for all } i = 1, \ldots, n, \ \
\sum_{i=1}^n p(i) a(i) = 0 \}, $$

to be the set of feasible reallocations given prices and holdings $x$. The constraint $\sum_{i=1}^n p(i) a(i) = 0$ requires the total change in wealth from buying and selling decisions to be zero in any period. The system dynamic for portfolio positions is then

$$ x_t(t+1) = x_t(i) + a_t(i), \quad i = 1, \ldots, n, t \geq 0. \quad (6.1) $$

The transaction costs $c : A \to \mathbb{R}$ are defined to be

$$ c(a) \triangleq \sum_{i=1}^n a_t(i)^2, $$

this cost function is a moment on $S \times A$.

The overall return rate between time $t$ and $t+1$ is

$$ z(p_t, x_t; p_{t+1}, x_{t+1}) \triangleq \frac{\langle p_{t+1}, x_{t+1} \rangle - \langle p_t, x_t \rangle}{\langle p_t, x_t \rangle}. $$

We make the reasonable assumption that $z(p_t, x_t; p_{t+1}, x_{t+1})$ is bounded for this example.

We want to minimize discounted transaction costs

$$ C(\pi, \nu) \triangleq \mathbb{E}_\nu^\pi \left[ \sum_{t=0}^\infty \delta^t c(a_t) \right] $$

subject to a stochastic dominance constraint on the discounted return rate. Define

$$ Z_\eta(\pi, \nu) \triangleq \mathbb{E}_\nu^\pi \left[ \sum_{t=0}^\infty \delta^t (z(p_t, x_t; p_{t+1}, x_{t+1}) - \eta)_- \right] $$

to be the expected discounted shortfall in relative returns at level $\eta$. We introduce a benchmark $Y$ for the discounted return rate, and we suppose the support of $Y$ is bounded within $[a, b]$. In this example, the benchmark can be taken as any market index.
We absorb the system dynamic (6.1) into a transition kernel $Q$. Our resulting portfolio optimization problem is then

$$\max_{\pi \in \Pi} -C(\pi, \nu)$$

subject to

$$Z_\eta(\pi, \nu) \geq \mathbb{E}[(Y - \eta)], \quad \eta \in [a, b].$$

In the linear programming formulation of (6.2) - (6.3), we simply augment the state space and consider occupation measures over sequences

$$(p_t, x_t, a_t; p_{t+1}, x_{t+1}, a_{t+1})$$

to correctly compute $z$.

7 Conclusion

We have shown how to use stochastic dominance constraints in infinite horizon MDPs. Convex analytic methods establish that stochastic dominance constrained MDPs can be solved via linear programming, and have corresponding dual linear programming problems. Conditions are given for strong duality to hold between these two linear programs. Utility functions appear in the dual as pricing variables corresponding to the stochastic dominance constraints. This result has intuitive appeal, since our stochastic dominance constraints are defined in terms of utility functions, and parallels earlier results [9, 10, 12]. Our results are shown to be extendable to many types of stochastic dominance constraints, particularly multivariate ones.

There are three main directions for our future work. First, we will consider efficient strategies for computing the optimal policy to stochastic dominance constrained MDPs. Second, we would like explore other methods for modeling risk in MDPs using convex analytic methods. Specifically, we are interested in solving MDPs with convex risk measures and chance constraints with “static” optimization problems as we have done here. Third, as suggested by the portfolio example, we will consider online data-driven optimization for the stochastic dominance-constrained MDPs in this paper. The transition probabilities of underlying MDPs are not known in practice and must be learned online.

References

[1] Eitan Altman. Constrained Markov Decision Processes. Chapman & Hall/CRC, 1999.

[2] Edward J. Anderson and Peter Nash. Linear Programming in Infinite-Dimensional Spaces. John Wiley & Sons, 1987.

[3] Aristotle Arapostathis, Vivek S. Borkar, Emmanuel Fernández-Gaucherand, Mrinal K. Ghosh, and Steven I. Marcus. Discrete-time controlled markov processes with average cost criterion: a survey. *SIAM J. Control Optim.*, 31(2):282–344, March 1993.

[4] V. Borkar and R. Jain. Risk-constrained Markov Decision Processes. In *Proc. of the IEEE Control and Decision Conference*, December 2010.

[5] Vivek S. Borkar. A Convex Analytic Approach to Markov Decision Processes. *Probability Theory and Related Fields*, 78(4):583–602, 1988.

[6] Vivek S. Borkar. Convex Analytic Methods in Markov Decision Processes. In E. A. Feinberg, A. Shwartz, and F. S. Hillier, editors, *Handbook of Markov Decision Processes*, volume 40 of *International Series in Operations Research & Management Science*, pages 347–375. Springer US, 2002.

[7] E. M. Bronshtein. Extremal convex functions. *Sibirskii Matematicheskii Zhurnal*, 19(1):10–18, 1978.

[8] Daniela Pucci de Farias and Benjamin Van Roy. A cost-shaping linear program for average-cost approximate dynamic programming with performance guarantees. *Math. Oper. Res.*, 31(3):597–620, August 2006.
[9] Darinka Dentcheva and Andrzej Ruszczyński. Optimization with stochastic dominance constraints. *SIAM Journal of Optimization*, 14(2):548–566, 2003.

[10] Darinka Dentcheva and Andrzej Ruszczyński. Optimality and duality theory for stochastic optimization problems with nonlinear dominance constraints. *Mathematical Programming*, 99:329–350, 2004.

[11] Darinka Dentcheva and Andrzej Ruszczyński. Portfolio optimization with stochastic dominance constraints. *Journal of Banking and Finance*, 30(2):433 – 451, 2006.

[12] Darinka Dentcheva and Andrzej Ruszczyński. Stochastic dynamic optimization with discounted stochastic dominance constraints. *SIAM Journal of Control and Optimization*, 47(5):2540–2556, 2008.

[13] Darinka Dentcheva and Andrzej Ruszczyński. Optimization with multivariate stochastic dominance constraints. *Mathematical Programming*, 117:111–127, 2009.

[14] V. V. Desai, V. F. Farias, and C. C. Moallemi. Approximate Dynamic Programming via a Smoothed Linear Program. *ArXiv e-prints*, August 2009.

[15] E. B. Dynkin and A. A. Yushkevich. *Controlled Markov Processes*. Springer-Verlag, Berlin, 1979.

[16] D. P. De Farias and B. Van Roy. The linear programming approach to approximate dynamic programming. *Operations Research*, 51(6):850–865, 11 2003.

[17] Daniela Pucci de Farias and Benjamin Van Roy. On constraint sampling in the linear programming approach to approximate dynamic programming. *Mathematics of Operations Research*, 29(3):462–478, 08 2004.

[18] Eugene A. Feinberg and Adam Shwartz. Constrained Markov decision models with weighted discounted rewards. *Mathematics of Operations Research*, 20(2):302–320, 05 1995.

[19] Eugene A. Feinberg and Adam Shwartz. Constrained discounted dynamic programming. *Mathematics of Operations Research*, 21(4):922–945, 11 1996.

[20] Jerzy A. Filar, L. C. M. Kallenberg, and Huey-Miin Lee. Variance-penalized Markov decision processes. *Mathematics of Operations Research*, 14(1):147–161, 1989.

[21] Onesimo Hernandez-Lerma and Juan Gonzalez-Hernandez. Infinite linear programming and multichain Markov control processes in uncountable spaces. *SIAM Journal on Control and Optimization*, 36(1):313–335, 1998.

[22] Onésimo Hernández-Lerma and Juan González-Hernández. Constrained Markov control processes in Borel spaces: the discounted case. *Mathematical Methods of Operations Research*, 52:271–285, 2000. 10.1007/s001860000071.

[23] Onésimo Hernández-Lerma, Juan González-Hernández, and Raquiel R. López-Martínez. Constrained average cost Markov control processes in Borel spaces. *SIAM J. Control Optim.*, 42(2):442–468, February 2003.

[24] Onesimo Hernandez-Lerma and Jean B. Lasserre. Approximation schemes for infinite linear programs. *SIAM Journal on Optimization*, 8(4):973–988, 1998.

[25] Onésimo Hernández-Lerma, Jean B. Lasserre, Eugene A. Feinberg, and Adam Shwartz. *The Linear Programming Approach*, volume 40, pages 377–407. Springer US, 2002.

[26] Onesimo Hernandez-Lerma and Jean Bernard Lasserre. *Discrete-Time Markov Control Processes: Basic Optimality Criteria*. Springer-Verlag New York, Inc., 1996.

[27] Onesimo Hernandez-Lerma and Jean Bernard Lasserre. *Further Topics On Discrete-Time Markov Control Processes*. Springer-Verlag New York, Inc., 1999.

[28] Soren Johansen. The extremal convex functions. *Math. Scand.*, 34:61–68, 1974.
[29] Yoshinobu Kadota, Masami Kurano, and Masami Yasuda. Discounted Markov decision processes with utility constraints. *Computers & Mathematics with Applications*, 51(2):279–284, 2006.

[30] L. C. M. Kallenberg. Linear programming and finite Markovian control problems. *Mathematisch Centre Tracts*, 148:1–245, 1983.

[31] Marta Susana Mendiondo and Richard H. Stockbridge. Approximation of infinite-dimensional linear programming problems which arise in stochastic control. *SIAM Journal on Control and Optimization*, 36(4):1448–1472, 1998.

[32] Armando Mendoza-Pérez and Onésimo Hernández-Lerma. Markov control processes with pathwise constraints. *Math. Methods of Operations Research*, 71(3):477–502, 2010.

[33] Alfred Muller and Dietrich Stoyan. *Comparison Methods for Stochastic Models and Risks*. John Wiley and Sons, Inc., 2002.

[34] Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, 2005.

[35] Andrzej Ruszczyński. Risk-averse dynamic programming for Markov decision processes. *Mathematical Programming*, 125:235–261, 2010.

[36] Moshe Shaked and J. George Shanthikumar. *Stochastic Orders*. Springer, 2007.

[37] Matthew J. Sobel. Mean-variance tradeoffs in an undiscounted MDP. *Operations Research*, 42(1):175–183, 1994.

[38] Kosaku Yosida. *Functional Analysis*. Springer Berlin / Heidelberg, 1980.