DECAY RATES FOR THE MOORE-GIBSON-THOMPSON EQUATION WITH MEMORY

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Abstract. The main goal of this paper is to investigate the existence and stability of the solutions for the Moore–Gibson–Thompson equation (MGT) with a memory term in the whole spaces \( \mathbb{R}^N \). The MGT equation arises from modeling high-frequency ultrasound waves as an alternative model to the well-known Kuznetsov’s equation. First, following [8] and [26], we show that the problem is well-posed under an appropriate assumption on the coefficients of the system. Then, we built some Lyapunov functionals by using the energy method in Fourier space. These functionals allows us to get control estimates on the Fourier image of the solution. These estimates of the Fourier image together with some integral inequalities lead to the decay rate of the \( L^2 \)-norm of the solution. We use two types of memory term here: type I memory term and type III memory term. Decay rates are obtained in both types. More precisely, decay rates of the solution are obtained depending on the exponential or polynomial decay of the memory kernel. More importantly, we show stability of the solution in both cases: a subcritical range of the parameters and a critical range. However for the type I memory we show in the critical case that the solution has the regularity-loss property.

1. Introduction. The Kokhlov–Zabolotskaya–Kuznetsov equation, the Kuznetsov equation and the Westervelt equation are all classical model equations in nonlinear acoustics, these models are of second order in time and characterized by the presence of a viscoelastic damping of the form \(-\Delta u_t\). The most popular model is Kuznetsov’s equation:

\[
    u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (u_t) + |\nabla u|^2 \right). \tag{1}
\]

The derivation of equation (1) (see [5], [15] and [30]) can be obtained from the general equations of fluid mechanics by means of some asymptotic expansions in powers of small parameters. Here \( u \) represents the acoustic velocity potential, \( c^2 > 2020 \) Mathematics Subject Classification. Primary: 35L80, 35B40, 74D05; Secondary: 49K20, 93D30, 93D20.

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0, \( b \geq 0 \) and \( B/A \) are the speed of sound, the diffusivity of sound and the parameter of nonlinearity, respectively. In the derivation of equation (1), the Fourier law of heat conduction has been used in the equation of the conservation of energy (first law of thermodynamics or entropy balance). The classical Fourier law of heat conduction assumes that the heat flux is proportional to the gradient of the temperature at the same time, that is
\[
q(x, t) = -k \nabla \theta(x, t),
\]  
where \( k \) is the thermal conductivity of the material and \( \theta \) is the absolute temperature. It is known that, by using Fourier’s law we obtain an infinite signal speed paradox of the energy propagation. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. The drawback of the infinite propagation velocity of heat in the Fourier law becomes unacceptable in some physical experiments. For instance the Fourier law does not agree with experimental data on the propagation of sound waves in dilute gases at high wave frequency. Consequently in order to avoid this paradox, other equations were considered to model the heat transfer. The Cattaneo (or the Maxwell–Cattaneo) law is the most obvious constitutive equation that leads to a finite speed of propagation
\[
\tau q_t(x, t) + q(x, t) = -k \nabla \theta(x, t),
\]  
with \( \tau \) is a small parameter known as the relaxation time of the heat flux. It is obvious that (3) can be obtained from (2) by allowing a time lag between the gradient of temperature and the heat flux. That is if we consider the equation
\[
q(x, t + \tau) = -\nabla \theta(x, t),
\]  
and applying the first order Taylor expansion we get (3). Now, the use of the Cattaneo law combined with the equations of fluid mechanics lead to the nonlinear third order “in time” partial differential equation model:
\[
\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right).
\]  
This nonlinear equation is known as the Jordan–Moore–Gibson–Thompson equation (JMGT). For a brief overview on nonlinear acoustics and the derivation of the JMGT equation see the recent papers [11], [12] and [14] and references therein. In what follows we shall first review the linearized version of equation (4), which is known as the Moore–Gibson–Thompson equation (MGT), in its more general form:
\[
\tau u_{ttt} + \alpha u_{tt} + c^2 A u + b A u_t = 0,
\]  
where \( A \) is a positive self-adjoint operator. Recently many authors have been interested to the equation (5) mainly in bounded domains. In [12], the authors investigated the problem (5) and they defined a critical parameter
\[
\chi = \alpha - \frac{c^2 \tau}{b},
\]  
that controls the behavior of the solution. More precisely, they showed that when \( b = 0 \), there is a lack of existence of a semigroup associated with the linear dynamics. In addition, for \( b > 0 \), they proved that the problem is well-posed and its solution is exponentially stable in the subcritical case \( \chi > 0 \). On the other hand for the critical case \( \chi = 0 \), the energy remains constant. They confirmed also the theoretical results by numerical calculations. Similar results were obtained in [9] and in [21], where the authors used the analysis of the spectrum of the operator. The exponential decay rate results in [21] are completed in [25], where the obtention of an explicit
solutions in the case $N=1$ data, by using the energy method. Then they provided some decay rates for the linear equation in the Fourier space to show that under the assumption $\chi > 0$ the $L^2$-norm of the vector $V = (u_t + \tau u_{ttt}, \nabla (u + \tau u_t), \nabla u_t)$, and of its higher-order derivatives decay as
\[
\|\nabla^3 V(t)\|_{L^2(\mathbb{R}^N)} \leq C(1 + t)^{-N/4-j/2}\|V_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^3 V_0\|_{L^2(\mathbb{R}^N)}.
\] (7)
In addition, they proved the optimality of the decay rate in (7) by the eigenvalues expansion method. In the recent paper [27], the authors showed that the condition $\chi > 0$ is also a necessary condition for stability.

For the nonlinear model (4), the first result seems to be the one of Kaltenbacher et al. in [13]. They investigated the equation (4) in its abstract form and proved that the problem is well-posed for the linear equation with variable viscosity and positive diffusivity and showed the exponential decay of solutions of the equation (5). After that they gave local and global well-posedness and exponential decay for the nonlinear equation in a certain range of the parameters and for small initial data. Racke and Said–Houari in [29] considered the JMGT equation (4) in the whole space $\mathbb{R}^N$ and showed a local existence result in appropriate function spaces by using the contraction mapping theorem and a global existence result for small data, by using the energy method. Then they provided some decay rates for the solution in the case $N \geq 2$.

The MGT equation with a memory term of the form:
\[
\tau u_{ttt}(t) + \alpha u_t(t) - \beta \Delta u_t(t) - \gamma \Delta u(t) + \int_0^t g(s)\Delta u(t-s)ds = 0,
\] (8)
where $\alpha, \beta, \gamma, \tau$ are positive constants and $g > 0$ is the relaxation memory kernel, has recently caught the attention of several authors. Here the memory kernel $g$ directly relates to whether or how the energy decays and satisfies some assumptions that will be mentioned later on.

For $\tau = 0$ and $\beta = 0$, the equation (8) reduces to the following classical viscoelastic equation
\[
u_{tt} - \Delta u + \int_0^t g(s)\Delta u(t-s)ds = 0.
\] (9)
Conti et al. in [4] studied (9) on the whole space $\mathbb{R}^N$ and obtained some results concerning the decay of the energy as time goes to infinity when $g$ decays either exponentially or polynomially. Some improvements of the decay rates were also given in [28]. Dell’Oro and Pata in [7], discussed the relationship between the MGT equation (5) and the linear viscoelastic equation (9) for an exponentially decaying kernel.

For $\tau > 0$ and $\beta > 0$, Lasiecka and Wang in [17] investigated (8) (with $-\Delta u = A$) and instead of the convolution term $\int_0^t g(s)Au(t-s)ds$, they studied $\int_0^t g(s)A\omega(t-s)ds$, by considering $\omega$ to be one of the three case: $\omega = u$ (type I memory), $\omega = u_t$ (type II memory) or $\omega = u + u_t$ (type III memory), where the memory kernel $g$ is decaying exponentially. The authors studied the effect of the memory on the decay of the energy in the subcritical case $\chi > 0$ and showed how memory induced damping mechanism that leads to an exponential decay of the solution. Lasiecka
and Wang [18] generalized their work in [17] and allowed the memory kernel to satisfy a more general decay estimate of the form
\[ g'(t) + H(g(t)) \leq 0, \quad \text{for all } t > 0, \]
with \( H(0) = 0 \). They proved that the decay rate of the memory kernel is transferred to a decay rate of the solution when \( \chi > 0 \). The authors in [1] proved the uniform stability of the MGT model encompassing the three different types of memory introduced by Lasiecka and Wang in [18] in a history space setting and they showed some stability results, using the linear semigroup theory. In [19], the authors proved new general decay results for the solution in the subcritical case \( \chi > 0 \), by allowing a more general class of functions \( g \) than those in (10). That is, they allowed the kernel \( g \) to satisfy
\[ g'(t) \leq -\zeta(t)H(g(t)) \]
for some positive function \( \zeta(t) \). Their proof is based on the perturbed energy method and on some properties of convex function inequalities.

The critical case \( \chi = 0 \) has been investigated by Dell’Oro et al. [8] in a bounded domain and proved that system (8) is exponential stable if and only if \( A \) is a bounded operator. In addition, in the case of an unbounded operator \( A \), they only proved that the corresponding energy decays polynomially with the rate \( 1/t \), at least for regular initial data.

The above mentioned results of the MGT equation with memory were obtained in bounded domains where Poincaré’s inequality is available. Our main goal is to discuss the well-posedness and the decay rate of the solution of the MGT equation with memory in \( \mathbb{R}^N \), where \( g \) satisfies the following differential inequality
\[ g'(t) \leq -\delta g^{\frac{p+1}{p}}(t), \quad \text{for all } t \geq 0, \quad p > 1. \]
We prove that the problem (8) is well-posed in some appropriate functional spaces using the semigroup approach. By applying the energy method in Fourier space we build appropriate Lyapunov functionals that are used to prove the asymptotic stability and to give the decay rate of an energy norm for the solution of the problem (8). More precisely, for the asymptotic behaviour we prove the following:

- First, for the subcritical case \( \chi > 0 \) and when \( g \) decays either exponentially or polynomially, we give the decay rate of the solution and its higher-order derivatives (see Theorem 3.1 below). For instance if \( g \) decays exponentially, we prove that the solution decays exactly as in (7).
- Second, in the critical case \( \chi = 0 \), we show that the decay results obtained are slightly different from the previous ones. In fact, the decay rates is of regularity-loss type, (see Theorem 3.2 below). This somehow agrees with the result obtained in [8] in bounded domains.

In this critical case we discuss the stability of (8) with a memory term of the form:
\[ \int_0^t g(s)\Delta z(t-s)ds \]
with \( z = u \) (type I memory) or \( z = \alpha u + \tau u_t \) (type III memory). In the type I memory we showed that if \( g \) decays exponentially, the solution is stable and decays to the steady state and enjoys the regularity-loss property. If \( g \) decays polynomially, the stability problem for the type I memory is left open. For the type III memory, we proved the decay rate in both case, when \( g \) decays exponentially and polynomially (Theorem 3.3). The main idea that
we applied here is that for the critical case \(\chi = 0\), we succeeded to rewrite (8) as a second order in “time” wave equation for the unknown \(\tau u_t + \alpha u\) with a “nice” memory term that allows us to apply the energy method. Our result in this case agrees with those in [4]. For the type II memory (i.e., \(z = u_t\)) and if \(g\) decays exponentially, then it seems that a decay result under the new assumption
\[
\alpha\beta - \tau\gamma - \alpha(\gamma - \ell) > 0
\]
is possible. However, the case where \(g\) is decaying polynomially seems more challenging. We do not discuss the type II memory here since it requires some new energy functionals.

The remaining part of this paper is organized as follows. In Section 2, we state some preliminaries and assumptions, then we prove the existence and uniqueness of the solutions of the MGT equation (8) using the semigroup method. In Section 3, we present our main results. Sections 4 and 5 are devoted to build the appropriate Lyapunov functionals in Fourier space in both cases when \(\chi > 0\) and when \(\chi = 0\), respectively, for \(p^* < p \leq \infty\). In Section 6 we give the decay rate of \(L^2\)-norm of the solution by applying Plancherel’s theorem. In Section 7, we discuss the type III memory.

2. Preliminaries and wellposedness of the problem. In this section, we state some preliminaries and assumptions, then we prove the existence and uniqueness of the solutions of the Moore–Gibson–Thompson equation (8). Indeed we use the following standard \(H^1(\mathbb{R}^N)\) space, the scalar product and norm are denoted by

\[
\langle u, v \rangle_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} uv dx + \int_{\mathbb{R}^N} \nabla u \nabla v dx, \quad \|u\|_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx.
\]

We use the notations \(\langle u, v \rangle_1 = \langle u, v \rangle_{H^1(\mathbb{R}^N)}\), \(\|u\|_1^2 = \|u\|_{H^1(\mathbb{R}^N)}^2\) and \(\langle x, \bar{y} \rangle = x\bar{y}\).

We impose now the basic assumptions on \(g\) as follows:

(G1): \(g : \mathbb{R}_+ \to \mathbb{R}_+\) is a nonincreasing twice differentiable function, such that
\[
g(0) > 0, \quad \gamma - \int_0^\infty g(s) ds = \ell > 0.
\]

and \(g'(s) \leq 0\) for every \(s > 0\).

(G2): For some \(\delta > 0\) the function \(g\) satisfies the following differential inequality:
\[
g'(t) \leq -\delta g^{p+\frac{1}{p}}(t), \quad \text{for all} \quad t \in (0, \infty) \quad \text{and} \quad p^* < p \leq \infty,
\]
where \(p^* = \frac{1 + \sqrt{5}}{2}\).

(G3): \(g'' \geq 0\) almost everywhere.

The following two technical lemmas will be needed in the proof of the main result. We refer the reader to the proof in [4].

Lemma 2.1. Let \(k \geq 1, t \geq 0\). Then the following estimate holds:
\[
\int_0^1 r^{k-1} e^{-r^2 t} dr \leq c(k)(1 + t)^{-k/2}.
\]

Lemma 2.2. Let \(k \geq 1, m > 0\) and \(t \geq 0\). Then
\[
\int_0^1 \frac{r^{k-1}}{(1 + rt)^m} dr \leq \begin{cases} 
  c(m, k)(1 + t)^{-m + \min(m, k)}, & \text{if} \quad m \neq k, \\
  c(k)(1 + t)^{-k} \log(2 + t), & \text{if} \quad m = k
\end{cases}
\]
We immediately adopt the following lemma from [20] without proof which will be used in the sequel.

**Lemma 2.3.** Assume that the assumption (G2) is satisfied. Then for any $\nu \in (1/p,1)$ there exists a positive constant $c_0 = c_0(p, \nu, \gamma, \ell)$ such that

$$\int_0^\infty [g(s)]^\nu ds \leq c_0 < \infty.$$  

To prove the well-posedness of the MGT equation with memory in $\mathbb{R}^N$, we supplement (8) with the following initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x). \quad (14)$$

Now, we use a change of variables from Dafermos [6] and extend the solution of (8) for all times, by setting $u(x,t) = 0$ when $t < 0$ and considering for $t \geq 0$ the auxiliary past history variable $\eta(t,s) = \eta^t(s)$, defined as:

$$\eta^t(s) = u(t) - u(t-s), \quad t \geq 0, \quad s \in \mathbb{R}^+.$$  

Consequently, the problem (8) together with (14) read as:

$$\begin{cases}
\tau u_{ttt}(t) + \alpha u_t(t) - \beta \Delta u(t) - \ell \Delta u(t) - \int_0^\infty g(s) \Delta \eta^t(s) ds = 0, \\
\eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t),
\end{cases} \quad (16)$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x), \quad \eta^0(x,s) = \eta^0(s). \quad (17)$$

Introducing now the $(-g')$-weighted $L^2$-space

$$\mathcal{M} = L^2_{-g'}(\mathbb{R}^+, H^1(\mathbb{R}^N)),$$

endowed with the inner product

$$\langle \eta^t, \tilde{\eta}^t \rangle_{\mathcal{M}} = \int_0^\infty -g'(s) \langle \nabla \eta^t(s), \nabla \tilde{\eta}^t(s) \rangle_{L^2(\mathbb{R}^N)} ds$$

and with the following associated norm

$$\|\eta^t\|_{\mathcal{M}}^2 = \int_0^\infty -g'(s) \|\nabla \eta^t(s)\|_{L^2(\mathbb{R}^N)}^2 ds,$$

for all $\eta^t, \tilde{\eta}^t \in \mathcal{M}$. In addition, we consider the infinitesimal generator of the right-translation $C_0$-semigroup on $\mathcal{M}$, i.e., the linear operator $T$ given by

$$T\eta^t = -(\eta^t)' \quad \text{with} \quad D(T) = \{ \eta^t \in \mathcal{M} : (\eta^t)' \in \mathcal{M}, \eta^t(0) = 0 \},$$

where the prime stands for the distributional derivative with respect to the variable $s > 0$.

In order to write the problem (16) as a first-order “in time” evolution equation, we introduce the change of variables:

$$v = u_t, \quad \text{and} \quad w = u_{tt},$$

and hence, we rewrite (16) as a first order system of the form:

$$\begin{cases}
u_t = v, \\
v_t = w, \\
\tau w_t = -\alpha \nu + \beta \Delta v + \ell \Delta u + \int_0^\infty g(s) \Delta \eta^t(s) ds, \\
\eta^t_t = -v - \eta^t.'
\end{cases} \quad (18)$$
Now, the problem (18) with initial data (17) can be reduced to

\[
\begin{align*}
\frac{d}{dt} U(t) &= AU(t), \quad t \in (0, +\infty), \\
U(x,0) &= U_0.
\end{align*}
\] (19)

where \(U(t) = (u(t), v(t), w(t), \eta^t), U_0 = (u_0, v_0, w_0, \eta^0)\), and \(A\) is the linear operator given by

\[
AU = A \begin{pmatrix}
u \\
v \\
w \\
\eta^t
\end{pmatrix} = \begin{pmatrix}
-\frac{\alpha}{\tau} w + \frac{\beta}{\tau} \Delta v + \frac{\ell}{\tau} \Delta u + \frac{1}{\tau} \int_0^\infty g(s) \Delta \eta^t(s) ds
\end{pmatrix}.
\] (20)

Inspired by [8], we define the Hilbert space

\[\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times \mathcal{M},\]

with the following inner product

\[
(U, \tilde{U})_\mathcal{H} = \frac{\ell}{\alpha} \langle (\alpha u + \tau v), (\alpha \tilde{u} + \tau \tilde{v}) \rangle_{\mathcal{M}} + \frac{\tau}{\alpha} \langle (\alpha \beta - \tau \ell) v, \tilde{v} \rangle_{\mathcal{M}}
\]

\[
+ \langle (\alpha v + \tau u), (\alpha \tilde{v} + \tau \tilde{w}) \rangle_{L^2(\mathbb{R}^N)} + \tau \langle \eta^t(s), \tilde{\eta}^t(s) \rangle_{\mathcal{M}}
\]

\[
+ \alpha \int_0^\infty g(s) \langle \nabla \eta^t(s), \nabla \tilde{\eta}^t(s) \rangle_{L^2(\mathbb{R}^N)} ds
\]

\[
+ \tau \int_0^\infty g(s) \left[ \langle \nabla \eta^t(s), \nabla^t(s) \rangle_{L^2(\mathbb{R}^N)} + \langle \nabla \tilde{v}, \nabla \tilde{\eta}^t(s) \rangle_{L^2(\mathbb{R}^N)} \right] ds
\]

and the corresponding norm

\[
\|U\|_\mathcal{H}^2 = \frac{\ell}{\alpha} \| (\alpha u + \tau v) \|_1^2 + \frac{\tau}{\alpha} \| (\alpha \beta - \tau \ell) v \|_1^2 + \| \alpha v + \tau u \|_{L^2(\mathbb{R}^N)}^2 + \| \eta^t(s) \|_{\mathcal{M}}^2
\]

\[
+ \alpha \int_0^\infty g(s) \| \nabla \eta^t(s) \|_{L^2(\mathbb{R}^N)}^2 ds + 2\tau \int_0^\infty g(s) \langle \nabla \eta^t(s), \nabla \tilde{\eta}^t(s) \rangle_{L^2(\mathbb{R}^N)} ds.
\]

For all vectors \(U = (u, v, w, \eta^t)\) and \(\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\eta}^t)\) in \(\mathcal{H}\). We consider (19) in the Hilbert space \(\mathcal{H}\), with the following domain of the operator \(A\):

\[
D(A) = \left\{ (u, v, w, \eta) \in \mathcal{H} \left| \begin{array}{c}
\beta v + \frac{\ell}{\tau} u + \frac{1}{\tau} \int_0^\infty g(s) \eta^t(s) ds \\
w \in H^1(\mathbb{R}^N)
\end{array} \right. \right\}
\]

**Theorem 2.4.** Assume that (G1)-(G3) are satisfied. Then under the assumption \(\alpha \beta - \tau \gamma \geq 0\), the linear operator \(A\) is the infinitesimal generator of a \(C_0\)-semigroup \(S(t) = e^{tA}\) of contraction on \(\mathcal{H}\).

**Proof.** Instead of considering our problem (19), we follow [26] and consider the perturbed problem

\[
\begin{align*}
\frac{d}{dt} U(t) &= A_B U(t), \quad t \in (0, +\infty), \\
U(x,0) &= U_0,
\end{align*}
\] (21)
where $A_B$ is given by
\[
A_B \begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} = (A + B) \begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{\alpha}{\tau} w + \frac{\beta}{\tau} \Delta v + \frac{\ell}{\tau} \Delta u + \frac{1}{\tau} \int_0^\infty g(s) \Delta \eta^t(s) ds - \ell u - \frac{\tau \ell}{\alpha} v - (\beta - \frac{\tau \ell}{\alpha}) v \\ v \\ w \\ v + T \eta^t \end{pmatrix}.
\]

**Remark 1.** Here, due to the standard semigroup theory, when we prove that $A_B$ generates a $C_0$–semigroup of contractions on $\mathcal{H}$, we can say that $A$ generates a $C_0$–semigroup on $\mathcal{H}$, because we have $A_B$ is a bounded perturbation of $A$, (see [24], Theorem 1.1 in Chap. 3).

The first thing to notice is that, as we are in a Hilbert space, the operator is densely defined. Hence, we only need to prove that the operator $A_B$ is dissipative and 0 belongs to the resolvent set of $A_B$, denoted by $\rho(A)$. Then our conclusion will follow using the well known Lumer–Phillips theorem [24]. Following the same steps as in [2], Proof of Proposition 2.1 and using our new inner product, we can see that we have by direct calculations
\[
\langle A_B U, U \rangle_{\mathcal{H}} = \frac{\tau}{\alpha} (\alpha \beta - \tau \ell) \int_{\mathbb{R}^N} \nabla w \cdot \nabla v dx + \frac{\ell}{\alpha} \int_{\mathbb{R}^N} (\alpha \nabla v + \tau \nabla w) \cdot (\alpha \nabla u + \tau \nabla v) dx
\]
\[
+ \frac{\tau}{\alpha} (\alpha \beta - \tau \ell) \int_{\mathbb{R}^N} w \cdot v dx + \frac{\ell}{\alpha} \int_{\mathbb{R}^N} (\alpha v + \tau w) \cdot (\alpha u + \tau v) dx
\]
\[
- \int_{\mathbb{R}^N} \left( \beta \Delta v + \tau v + \int_0^\infty g(s) \Delta \eta^t(s) ds \right) \cdot (\alpha \nabla v + \tau \nabla w) dx + \langle v + T \eta^t, \eta^t \rangle_{\mathcal{M}}
\]
\[
= - (\alpha \beta - \tau \gamma) \Vert \nabla v \Vert_{L^2(\mathbb{R}^N)}^2 + \frac{\alpha}{2} \Vert \nabla \eta^t \Vert_{H^1(\mathbb{R}^N)}^2 - \frac{\tau}{2} \int_0^\infty g''(s) \Vert \nabla \eta^t(s) \Vert_{L^2(\mathbb{R}^N)}^2 ds \leq 0.
\]

Therefore, according to the assumption (G3) and since $\alpha \beta - \tau \gamma \geq 0$, then the operator $A$ is dissipative.

Next, we show that for all $\mathcal{F} = (f, g, h, p) \in \mathcal{H}$, there exists a unique solution $U = (u, v, w, \eta^t) \in D(A)$ to the equation
\[
U - A_B U = \mathcal{F},
\]

such that in terms of its components, we have
\[
u - v = f,
\]
\[
v - w = g,
\]
\[(\tau + \alpha) w - \beta \Delta v - \ell \Delta u - \int_0^\infty g(s) \Delta \eta^t(s) ds + \ell u + \frac{\tau \ell}{\alpha} v + (\beta - \frac{\tau \ell}{\alpha}) v = \tau h,
\]
\[
\eta^t - v - T \eta^t = p.
\]

Integrating (26) with the condition $\eta^t(0) = 0$, we obtain
\[
\eta^t(s) = (1 - e^{-s}) v + \int_0^s e^{-(s-y)} p(y) dy.
\]

Plugging (23), (24) and (27) into (25), we arrive at an elliptic problem of the form
\[
- \kappa \Delta v + \sigma v = q,
\]
such that,
\[ \kappa = \beta + \ell + \int_0^\infty g(s)(1 - e^{-s})ds, \]
and
\[ q = \ell(\Delta f + f) + (\tau + \alpha)g + \tau h + \int_0^\infty g(s) \left[ \int_0^s e^{-(s-y)}\Delta p(y)dy \right] ds. \]

The constant \( \kappa > 0 \) and the functional \( q \) belongs to \( H^{-1}(\mathbb{R}^N) \). Thus, in view of the Lax-Milgram Theorem, problem (28) has a unique solution \( v \in H^1(\mathbb{R}^N) \). From (23) and (24) we see that \( u, w \in H^1(\mathbb{R}^N) \). Furthermore, from (27), one can also show that \( \eta^t \in \mathcal{M} \), and checking that \( T\eta^t = \eta^t - p - v \in \mathcal{M} \) as well. Finally, going back to (25), we conclude that
\[ \beta v + \ell u + \int_0^\infty g(s)\eta^t(s)ds = (-\Delta)^{-1}(\tau h - (\tau + \alpha)w - \ell u - \beta v) \in H^2(\mathbb{R}^N), \]
which implies that \( U = (u, v, w, \eta^t) \in D(A) \) is the unique solution of (22). As we said, as \( A \) is a bounded perturbation of \( A_B \), we therefore obtain that \( A \) is the generator of a \( C_0 \)-semigroup on \( \mathcal{H} \). This finishes the proof.

**Corollary 1.** Under the dissipativity condition \( \alpha \beta > \tau \gamma \) with the results obtained above and for any \( U_0 \in D(A) \), the problem (16),(17) has a unique classical solution in the class
\[ (u, u_t, u_{tt}, \eta^t) \in C^1([0, \infty); \mathcal{H}) \cap C([0, \infty); D(A)). \]

3. **Main results.** In this section, we state the main results of this paper. First, we state the decay results for the subcritical case \( \alpha \beta > \tau \gamma \).

**Theorem 3.1.** Let \( u \) be the solution of (8), (14). Assume that \( \alpha \beta > \tau \gamma \). Let \( U = (\alpha u_t + \tau u_{tt}, \nabla(\alpha u + \tau u_t), \nabla u_t) \) and assume in addition that \( U_0 \in L^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \).
Then, for all \( 0 \leq j \leq s \), we have:

(i): for \( p = \infty \):
\[ \|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \leq C(1 + t)^{-N/4-j/2}\|U_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^j U_0\|_{L^2(\mathbb{R}^N)}, \quad (29) \]

(ii): for \( p' < p < \infty \):
\[ \|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \leq CI_0^\delta \begin{cases} (1 + t)^{-\min(\frac{N}{2p'}, \frac{N}{2} + \frac{1}{2})}, & \text{if } 2j + N \neq \frac{3}{2}, \\ (1 + t)^{-\frac{N}{2p'}} \log(2 + t), & \text{if } 2j + N = \frac{3}{2}, \end{cases}, \quad (30) \]

where \( C, c \) are two positive constants independent of \( t \) and \( U_0 \) and \( I_0 = \|\nabla^j U_0\|_{L^2(\mathbb{R}^N)} + \|U_0\|_{L^1(\mathbb{R}^N)} \).

The proof of the above theorem will be given in Section 6 and it is based on Proposition 1 below.

As we have said in the introduction, in the absence of the memory term, that is for \( g = 0 \) the assumption \( \alpha \beta > \tau \gamma \) has been proved in [27] to be a necessary condition for the stability result. Here, we show that in the presence of the memory damping term, then we can push the stability result even to the critical case \( \alpha \beta = \tau \gamma \). Hence, we have the following theorem. Our second main result reads as follows.
Theorem 3.2. Let $u$ be the solution of (8), (14). Assume that $\alpha \beta = \tau \gamma$. Let $U = (\alpha u + \tau u_t, \nabla (\alpha u + \tau u_t), \nabla u_t)$ and assume in addition that $U_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Then $p = \infty$, we have
\[
\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \leq C(1 + t)^{-N/4-j/2}\|U_0\|_{L^1(\mathbb{R}^N)} + C(1 + t)^{-k/2}\|\nabla^{j+k} U_0\|_{L^2(\mathbb{R}^N)}, \tag{31}
\]
for all $0 \leq j \leq s - k$ where $C, c$ are two positive constants independent of $U_0$ and $t$.

The proof of Theorem 3.2 is based on Proposition 2. We have $\frac{|\xi|^2}{(1 + |\xi|^2)^2} \sim c|\xi|^{-2}$ for $|\xi| \to \infty$, this behavior in high frequency region yields the slow decay $(1 + t)^{-k/2}$ of the high frequency part of the solution. In addition, to get this decay rate, we need to pay $k + j$ derivative of the initial data.

Here, we present the main result of the energy decay in the critical case for the memory type III (the subcritical case is easier). We can easily see that for $\alpha \beta = \tau \gamma$, we can rewrite the problem (8) with the type III memory as a second order in time wave equation with a memory term of the form
\[\tau z_{tt} - \beta \Delta z + \tau \int_0^t g(s)\Delta z(t - s)ds = 0, \tag{32}\]
where $z = \alpha u + \tau u_t$. Now, we state the main result of the decay rate of the $L^2$-norm of the energy
\[E(t, \nabla z, z_t) = \frac{1}{2} \left[ \|\nabla z\|^2_{L^2} + \tau \|z_t\|^2_{L^2} + \tau \int_0^\infty g(s)\|\nabla \mu^t(s)\|^2_{L^2}ds \right]. \tag{33}\]
where $z(x, t)$ is the solution of (32) and $\mu^t(s) = -\mu^t_s(s) + z_t(t)$.

Theorem 3.3. Let $z$ be the solution of (32). Assume that $Z(0) = (z_t(0), \nabla z(0)) \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ for all $0 \leq j \leq s$. Then, we have for $Z(t) = (z(t), \nabla z(t))$

(i): for $p = \infty$:
\[\|\nabla^j Z(t)\|_{L^2} \leq C(1 + t)^{-N/4-j/2}\|Z_0\|_{L^1(\mathbb{R}^N)} + C e^{-ct}\|\nabla^j Z_0\|_{L^2(\mathbb{R}^N)}, \tag{34}\]

(ii): for $p^* < p < \infty$:
\[\|\nabla^j Z(t)\|_{L^2} \leq C \frac{J_0^j}{p} \left\{ (1 + t)^{-\min(1/2p,N/4+j/2)}, \text{ if } 2j + N \neq \frac{p}{2}, \right. \tag{35}\]
\[\left. (1 + t)^{-1/2p \log(2 + t)}, \text{ if } 2j + N = \frac{p}{2}, \right\}
\]
where
\[J_0^j = \|\nabla^j Z_0\|_{L^2(\mathbb{R}^N)} + \|Z_0\|_{L^1(\mathbb{R}^N)},\]
$C$ and $c$ are two positive constants independent of $t$ and the initial data.

The proof of Theorem 3.3 is a result of the point-wise estimates in Proposition 3.

4. Decay estimates—the subcritical case $\alpha \beta > \tau \gamma$. In this section, we apply the energy method in the Fourier space to get some point-wise estimate of $\hat{U}(\xi, t)$ with $U = (\alpha u + \tau u_t, \nabla (\alpha u + \tau u_t), \nabla u_t)$, where $u(x, t)$ is the solution of (8), (14). First, taking the Fourier transform of (18), we obtain
\[
\begin{cases}
\hat{u}_t = \hat{v}, \\
\hat{v}_t = \hat{w}, \\
\tau \hat{w}_t = -\alpha \hat{w} - \beta |\xi|^2 \hat{v} - \ell |\xi|^2 \hat{u} - |\xi|^2 \int_0^\infty g(s)\eta^s(s)ds, \\
\hat{\eta}_t = \hat{v} - \hat{\eta}_s.
\end{cases} \tag{36}
\]
with
\[ \hat{U}(\xi, 0) = (\hat{u}_0, \hat{v}_0, \hat{\omega}_0, \hat{\eta}_0)(\xi). \] (37)

Our main goal is to build an appropriate Lyapunov functional \( \hat{L}(\xi, t) \) which is equivalent to the energy functional associated to (36):
\[
\hat{E}(\xi, t) = \frac{1}{2} \left[ \frac{\ell}{\alpha} |\xi|^2 |\alpha \hat{u} + \tau \hat{v}|^2 + \frac{\tau}{\alpha} (\alpha \beta - \tau \gamma) |\xi|^2 |\hat{v}|^2 + |\alpha \hat{v} + \tau \hat{\omega}|^2 + \tau \|\hat{\eta}\|^2_L \right.
\]
\[ + \alpha |\xi|^2 \int_0^\infty g(s)|\hat{\eta}'(s)|^2 ds + 2\tau |\xi|^2 \Re \left( \int_0^\infty g(s) \langle \hat{\eta}'(s), \hat{v} \rangle ds \right) \] (38)
in the sense that there exist two positive constants \( d_1 \) and \( d_2 \) such that
\[ d_1 \hat{E}(\xi, t) \leq \hat{L}(\xi, t) \leq d_2 \hat{E}(\xi, t), \]
for all \( \xi \in \mathbb{R}^N \) and for all \( t \geq 0 \) and also satisfies (65). Hence, we have the following result.

**Proposition 1.** Let \( \hat{U}(\xi, t) = (\hat{u}, \hat{v}, \hat{\omega}, \hat{\eta})(\xi, t) \) be the solution of (36)-(37). Assume that (G1)-(G3) hold and \( \alpha \beta > \tau \gamma \). Then, \( \hat{U}(\xi, t) \) satisfies the following estimates:
\[ |\hat{U}(\xi, t)|^2 \leq C|\hat{U}(\xi, 0)|^2 e^{-\rho(\xi)t}, \quad \text{for } p = \infty, \] (39)
\[ |\hat{U}(\xi, t)|^2 \leq C|\hat{U}(\xi, 0)|^2 \left( 1 + \rho(\xi)t \right)^{-\frac{1}{p}}, \quad \text{for } p^* < p < \infty, \] (40)
for all \( t > 0 \), where
\[ \rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}. \] (41)

Proposition (1) is main ingredient in proving the decay estimates in Theorem 3.1. Its proof will be given through several lemmas. First, we have the following lemma.

**Lemma 4.1.** Under the condition \( \alpha \beta \geq \tau \gamma \), the energy functional defined in (38) satisfies for all \( t \geq 0 \) and for all \( \xi \in \mathbb{R}^N \) the inequality:
\[ \frac{d}{dt} \hat{E}(\xi, t) \leq - (\alpha \beta - \tau \gamma) |\xi|^2 |\hat{v}|^2 - \frac{\alpha}{2} \|\hat{\eta}\|^2_L \leq 0. \] (42)

**Proof.** In the same spirit of [26], multiplying the first equation in (36) by \( \alpha \) and the second by \( \tau \), adding the results to get:
\[ (\alpha \hat{u} + \tau \hat{\omega})_t = (\alpha \hat{v} + \tau \hat{\omega}). \] (43)
Multiplying (43) by \( \ell (\alpha \hat{u} + \tau \hat{\omega}) \) and taking the real part, we get
\[ \frac{1}{2} \frac{d}{dt} |\alpha \hat{u} + \tau \hat{\omega}|^2 = \alpha \ell |\hat{v}|^2 + \ell \alpha^2 \Re(\hat{\omega}) + \tau^2 \ell \Re(\hat{\omega}) + \alpha \tau \ell \Re(\hat{w}). \] (44)
Next, we multiply the second equation in (36) by \( \tau (\alpha \beta - \tau \ell) \hat{v} \) and taking the real part, we get
\[ \frac{1}{2} \tau (\alpha \beta - \tau \ell) \frac{d}{dt} |\hat{v}|^2 = \tau (\alpha \beta - \tau \ell) \Re(\hat{w}). \] (45)
Furthermore, multiplying the second equation in (36) by \( \alpha \) and adding the result to the third equation we obtain
\[ (\alpha \hat{v} + \tau \hat{w})_t = - |\xi|^2 \beta \hat{v} - \ell |\xi|^2 \hat{u} - |\xi|^2 \int_0^\infty g(s) \eta^2(s) ds. \] (46)
Multiplying (46) by $\alpha(\alpha\bar{v} + \tau\tilde{w})$ and taking the real part, we get

$$
\frac{1}{2} \frac{d}{dt} \alpha|\alpha\bar{v} + \tau\tilde{w}|^2 = -\alpha^2|\dot{\xi}|^2 - \tau\alpha|\dot{\xi}|^2 \text{Re}(\dot{\bar{v}}\bar{w}) - \ell\alpha^2|\dot{\xi}|^2 \text{Re}(\dot{\tilde{w}})
$$

$$
- \ell\alpha|\dot{\xi}|^2 \text{Re}(\dot{\tilde{w}}) - \alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \bar{v} \rangle \, ds \right)
$$

$$
- \tau\alpha|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \tilde{w} \rangle \, ds \right).
$$

Using the fact that $\dot{\eta}^t + \dot{\eta}^{t_0} = \dot{v}$, then the term

$$
-\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \bar{v} \rangle \, ds \right)
$$

can be rewritten as:

$$
-\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \bar{v} \rangle \, ds \right) = -\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \tilde{\eta}^{t_0}(s) \rangle \, ds \right)
$$

$$
-\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \tilde{\eta}^{t_0}(s) \rangle \, ds \right).
$$

Integrating by parts with respect to $s$ (the boundary terms vanish see [23]), then we obtain

$$
-\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \bar{v} \rangle \, ds \right) = -\frac{1}{2} \frac{d}{dt} \alpha^2|\dot{\xi}|^2 \int_0^\infty g(s)|\dot{\eta}^t(s)|^2 \, ds
$$

$$
+ \frac{\alpha^2}{2} \int_0^\infty g(s)|\dot{\eta}^t(s)|^2 \, ds.
$$

Hence, we get

$$
-\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \bar{v} \rangle \, ds \right) = -\frac{1}{2} \frac{d}{dt} \alpha^2|\dot{\xi}|^2 \int_0^\infty g(s)|\dot{\eta}^t(s)|^2 \, ds
$$

$$
- \frac{\alpha^2}{2} \|\dot{\eta}^t\|^2_{\lambda t}.
$$

Applying the same computation to the term

$$
-\tau\alpha|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \tilde{w} \rangle \, ds \right)
$$

and using the equation $\dot{\eta}^{t_0} + \dot{\eta}^{t_0} = \tilde{w}$, to obtain:

$$
-\alpha^2|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \bar{v} \rangle \, ds \right) = -\tau\alpha|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \tilde{\eta}^{t_0}(s) \rangle \, ds \right)
$$

$$
- \tau\alpha|\dot{\xi}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{\eta}^t(s), \tilde{\eta}^{t_0}(s) \rangle \, ds \right).
$$
Then, we have
\[-\tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right) = -\tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right)
+ \tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right)
- \tau\alpha|\xi|^2 \frac{d}{dt} \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right)
+ \tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right).
\]

Moreover, we obtain
\[-\tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right) = -\tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right)
+ \tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right).
\]

Inserting the estimates above into (47), we get
\[
\frac{1}{2} \frac{d}{dt} I_1 = -\alpha^2 |\xi|^2 |\hat{\eta}^\ell|^2 - \tau\alpha|\xi|^2 \text{Re}(\hat{\nu}\hat{\nu}) - \tau\alpha|\xi|^2 \text{Re}(\hat{\nu}\hat{\nu})
- \alpha \|\hat{\eta}^\ell\|_{\mathcal{M}}^2 + \tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right),
\]

where
\[
I_1 = \alpha|\alpha \hat{\nu} + \tau\hat{\nu}|^2 + \alpha^2 |\xi|^2 \int_0^\infty g(s)|\hat{\eta}^\ell(s)|^2 \, ds
+ 2\tau\alpha|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right).
\]

Now, computing (48)+|\xi|^2 (44)+|\xi|^2 (45) and dividing the result by \(\alpha\), we obtain
\[
\frac{1}{2} \frac{d}{dt} I_2 = -\frac{1}{\alpha} (\alpha\beta - \tau\ell)|\xi|^2 |\hat{\nu}|^2 - \frac{\alpha}{2} \|\hat{\eta}^\ell\|_{\mathcal{M}}^2
+ \tau|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}^\ell(s), \hat{\xi} \rangle \, ds \right),
\]

with
\[
I_2 = \frac{I_1}{\alpha} + \frac{\tau}{\alpha} (\alpha\beta - \tau\ell)|\xi|^2 |\hat{\nu}|^2 + \frac{\ell}{\alpha} |\xi|^2 |\alpha \hat{\nu} + \tau\hat{\nu}|^2.
\]

Here, substituting \(\hat{\eta}^\ell\) by \(\hat{\nu} - \hat{\xi}^\ell\), we get
\[
\frac{1}{2} \frac{d}{dt} I_2 = -\frac{1}{\alpha} (\alpha\beta - \tau\ell)|\xi|^2 |\hat{\nu}|^2 - \frac{\alpha}{2} \|\hat{\eta}^\ell\|_{\mathcal{M}}^2 + \tau|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\nu} - \hat{\xi}^\ell(s), \hat{\xi} \rangle \, ds \right).
\]

Hence, we have
\[
\frac{1}{2} \frac{d}{dt} I_2 = -\frac{1}{\alpha} (\alpha\beta - \tau\ell)|\xi|^2 |\hat{\nu}|^2 - \frac{\alpha}{2} \|\hat{\eta}^\ell\|_{\mathcal{M}}^2 + \tau(\gamma - \ell)|\xi|^2 |\hat{\nu}|^2
+ \tau|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\nu}^\ell(s), \hat{\xi} \rangle \, ds \right).
\]
Thus, we fix $\epsilon > 0$. Again substituting $\tilde{v}$ by $\tilde{\eta}_l^l + \tilde{\eta}_s^s$, then we find

$$
\frac{1}{2} \frac{d}{dt} I_2 = -(\alpha \beta - \tau \gamma)|\xi|^2|\tilde{v}|^2 - \frac{\alpha}{2} \|\tilde{\eta}^l\|^2_M + \tau |\xi|^2 \text{Re} \left( \int_0^\infty g'(s) \langle \tilde{\eta}^l(s), \tilde{\eta}_l^l(s) \rangle \, ds \right) + \tau |\xi|^2 \text{Re} \left( \int_0^\infty g'(s) \langle \tilde{\eta}^l(s), \tilde{\eta}_s^s(s) \rangle \, ds \right).
$$

Integrating by parts once again with respect to $s$, then we obtain

$$
\frac{1}{2} \frac{d}{dt} \hat{E}(\xi, t) = -(\alpha \beta - \tau \gamma)|\xi|^2|\tilde{v}|^2 - \frac{\alpha}{2} \|\tilde{\eta}^l\|^2_M - \frac{\tau}{2} |\xi|^2 \int_0^\infty g'(s)|\tilde{\eta}^l(s)|^2 \, ds,
$$

with

$$
\hat{E}(\xi, t) = I_2 + \tau \|\tilde{\eta}^l\|^2_M.
$$

Consequently, we obtain the energy (38) and the use of assumptions (G2) with the condition $\alpha \beta - \tau \gamma \geq 0$, we obtain the desired result (42).

Now, we define for all $t \geq 0$,

$$
|\hat{V}(\xi, t)|^2 = |\xi|^2|\alpha \hat{u} + \tau \hat{v}|^2 + |\xi|^2|\tilde{\eta}|^2 + |\alpha \hat{v} + \tau \hat{w}|^2 + \|\tilde{\eta}^l\|^2_M + |\xi|^2 \int_0^\infty g(s)|\tilde{\eta}^l(s)|^2 \, ds.
$$

Hence, we have the following result.

**Lemma 4.2.** Assume that $\alpha \beta > \tau \gamma$, then there exist two positive constants $C_1$ and $C_2$ such that

$$
C_1|\hat{V}(\xi, t)|^2 \leq \hat{E}(\xi, t) \leq C_2|\hat{V}(\xi, t)|^2,
$$

for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$.

**Proof.** To show (51), we infer from Young’s inequality that for every $\epsilon > 0$,

$$
2\tau |\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \tilde{\eta}^l(s), \hat{v} \rangle \, ds \right) \leq \frac{2\tau^2(\gamma - \ell)}{\alpha \epsilon} |\xi|^2|\tilde{\eta}|^2 + \frac{\alpha \epsilon}{2} |\xi|^2 \int_0^\infty g(s)|\tilde{\eta}^l(s)|^2 \, ds.
$$

Hence, we obtain from (38)

$$
\hat{E}(\xi, t) \geq \frac{1}{2} \left[ \frac{\ell}{\alpha} |\xi|^2|\alpha \hat{u} + \tau \hat{v}|^2 + |\alpha \hat{v} + \tau \hat{w}|^2 + \tau \|\tilde{\eta}^l\|^2_M \right.
$$

$$
+ \left( \frac{\tau (\alpha \beta - \tau \ell)}{\alpha} - \frac{2\tau^2(\gamma - \ell)}{\alpha \epsilon} \right) |\xi|^2|\tilde{\eta}|^2
$$

$$
+ \left( \alpha - \frac{\alpha \epsilon}{2} \right) |\xi|^2 \int_0^\infty g(s)|\tilde{\eta}^l(s)|^2 \, ds.
$$

Recall that the assumption $\alpha \beta > \tau \gamma$ implies that $\alpha \beta > \tau \ell$. This yields

$$
\frac{\tau (\gamma - \ell)}{\alpha \beta - \tau \ell} < 1.
$$

Thus, we fix $\epsilon > 0$, such that

$$
\frac{\tau (\gamma - \ell)}{\alpha \beta - \tau \ell} < \frac{\epsilon}{2} < 1.
$$
Consequently, the left-hand side inequality in (51) holds. For the other side, we have
\[
\begin{align*}
\hat{E}(\xi, t) &\leq \frac{1}{2} \left[ \frac{\ell}{\alpha} |\xi|^2 |\alpha \hat{u} + \tau \hat{v}|^2 + |\alpha \hat{v} + \tau \hat{w}|^2 + \tau |\hat{\eta}'|^2 \right] \\
&\quad + \left( \frac{\tau (\alpha \beta - \tau \ell)}{\alpha} + \frac{2\tau^2 (\gamma - \ell)}{\alpha \epsilon} \right) |\xi|^2 |\hat{v}|^2 \\
&\quad + \left( \alpha + \frac{\alpha \epsilon}{2} \right) |\xi|^2 \int_0^\infty g(s) |\hat{\eta}'(s)|^2 ds.
\end{align*}
\]
(53)
This yields the right-hand side inequality in (51). \qed

Now, following [26] we define the functional \(\hat{F}_1(\xi, t)\) as:
\[
\hat{F}_1(\xi, t) = \text{Re} \left\{ \alpha (\alpha \hat{u} + \tau \hat{v}) (\alpha \hat{v} + \tau \hat{w}) \right\}.
\]
(54)
Then, we have the following lemma.

**Lemma 4.3.** For any \(\epsilon_0, \epsilon_1 > 0\), we have
\[
\frac{d}{dt} \hat{F}_1(\xi, t) + (\ell - \epsilon_0 - (\gamma - \ell) \epsilon_1) |\xi|^2 |\alpha \hat{u} + \tau \hat{v}|^2 \\
\leq \alpha |\alpha \hat{v} + \tau \hat{w}|^2 + C(\epsilon_0) |\xi|^2 |\hat{v}|^2 + C(\epsilon_1) |\xi|^2 \int_0^\infty g(s) |\hat{\eta}'(s)|^2 ds.
\]
(55)

**Proof.** Multiplying (46) by \(\alpha (\alpha \hat{u} + \tau \hat{v})\) and (43) by \(\alpha (\alpha \hat{v} + \tau \hat{w})\) we obtain, respectively,
\[
\alpha (\alpha \hat{v} + \tau \hat{w})_t (\alpha \hat{u} + \tau \hat{v}) = \left( -|\xi|^2 \alpha \beta \hat{v} - \alpha \ell |\xi|^2 \hat{u} - \alpha |\xi|^2 \int_0^\infty g(s) \hat{\eta}'(s) ds \right) (\alpha \hat{u} + \tau \hat{v}) \\
= \left( -|\xi|^2 \alpha \beta \hat{v} - \alpha \ell |\xi|^2 \hat{u} + \tau \ell |\xi|^2 \hat{v} - \tau \ell |\xi|^2 \hat{v} \\
- \alpha |\xi|^2 \int_0^\infty g(s) \hat{\eta}'(s) ds \right) (\alpha \hat{u} + \tau \hat{v})
\]
and
\[
\alpha (\alpha \hat{u} + \tau \hat{v})_t (\alpha \hat{v} + \tau \hat{w}) = \alpha (\alpha \hat{v} + \tau \hat{w}) (\alpha \hat{v} + \tau \hat{w}).
\]
Summing up the above equations and taking the real part, we obtain
\[
\frac{d}{dt} \hat{F}_1(\xi, t) + \ell |\xi|^2 |\alpha \hat{u} + \tau \hat{v}|^2 - \alpha |\alpha \hat{v} + \tau \hat{w}|^2 \\
= |\xi|^2 (\tau \ell - \alpha \beta) \text{Re}(\hat{v}(\alpha \hat{u} + \tau \hat{v}) - \alpha |\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}'(s), \alpha \hat{u} + \tau \hat{v} \rangle ds \right).
\]
Applying Young’s inequality, we obtain the estimate (55) for any \(\epsilon_0, \epsilon_1 > 0\). \qed

Next, we set the functional \(\hat{F}_2(\xi, t)\) as
\[
\hat{F}_2(\xi, t) = -\tau \alpha \text{Re}(\hat{v}(\alpha \hat{v} + \tau \hat{w})).
\]
(56)

**Lemma 4.4.** For any \(\epsilon_2, \epsilon_3 > 0\), we have
\[
\frac{d}{dt} \hat{F}_2(\xi, t) + (\alpha - \epsilon_3)|\alpha \hat{v} + \tau \hat{w}|^2 \leq \epsilon_2 |\xi|^2 |\alpha \hat{u} + \tau \hat{v}|^2 + C(\epsilon_2, \epsilon_3) (1 + |\xi|^2) |\hat{v}|^2 \\
+ \frac{1}{2} |\xi|^2 \int_0^\infty g(s) |\hat{\eta}'(s)|^2 ds.
\]
(57)
Proof. Multiplying the second equation in (36) by \( -\tau\alpha(\alpha\hat{v} + \tau\hat{w}) \) and the equation (46) by \( -\tau\beta\), then we have, respectively,

\[
-\alpha\tau(\alpha\hat{v} + \tau\hat{w})\hat{v}_t = -\tau\alpha\hat{w}(\alpha\hat{v} + \tau\hat{w})
\]

and

\[
-\tau\alpha\hat{v}(\alpha\hat{v} + \tau\hat{w})_t = \left( |\xi|^2\alpha\beta\tau\hat{v} + \alpha\ell\tau|\xi|^2\hat{u} + \tau\alpha|\xi|^2\int_0^\infty g(s)\hat{\eta}'(s)ds \right)\hat{v}
\]

\[
= (|\xi|^2\alpha\beta\tau\hat{v} + \alpha\ell\tau|\xi|^2\hat{u} + \tau^2\ell|\xi|^2\hat{v} - \tau^2\ell|\xi|^2\hat{v}
\]

\[
+ \alpha^2(\alpha\hat{v} + \tau\hat{w}) - \alpha^2(\alpha\hat{v} + \tau\hat{w}) + \tau\alpha|\xi|^2\int_0^\infty g(s)\hat{\eta}'(s)ds \right)\hat{v}.
\]

Summing up the above two equations and taking the real parts, we obtain

\[
\frac{d}{dt}\hat{F}_2(\xi, t) + \alpha|\alpha\hat{v} + \tau\hat{w}|^2 - \tau(\alpha\beta - \tau\ell)|\xi|^2|\hat{v}|^2
\]

\[
= \tau\ell|\xi|^2\Re(\hat{v}(\alpha\hat{u} + \tau\hat{v})) + \alpha^2\Re(\hat{v}(\alpha\hat{v} + \tau\hat{w}))
\]

\[
+ \alpha\tau|\xi|^2\Re\left( \int_0^\infty g(s)\langle \hat{\eta}'(s), \hat{v} \rangle \, ds \right).
\]

Applying Young’s inequality, we obtain the estimate (57) for any \( \epsilon_2, \epsilon_3 > 0 \). \( \square \)

4.1. Proof of Proposition 1. We define the Lyapunov functional \( \hat{L}_1(\xi, t) \) as:

\[
\hat{L}_1(\xi, t) = N_0\hat{E}(\xi, t) + \rho(\xi)\hat{F}_1(\xi, t) + N_1\rho(\xi)\hat{F}_2(\xi, t),
\]

where \( N_0 \) and \( N_1 \) are positive constants that will be fixed later on. Taking the derivative of (58) with respect to \( t \) and making use of (42), (55) and (57), we obtain

\[
\frac{d}{dt}\hat{L}_1(\xi, t) + \left[ N_0(\alpha\beta - \tau\gamma) - C(\epsilon_0) - N_1C(\epsilon_2, \epsilon_3) \right]|\xi|^2|\hat{v}|^2
\]

\[
+ \frac{\alpha N_0}{2}|\xi|^2\int_0^\infty -g'(s)|\hat{\eta}'(s)|^2\, ds
\]

\[
+ \left[ (\ell - \epsilon_0 - (\gamma - \ell)\epsilon_1) - \epsilon_2 N_1 \right] \rho(\xi)|\xi|^2|\alpha\hat{u} + \tau\hat{v}|^2
\]

\[
+ \left[ N_1(\alpha - \epsilon_3) - \alpha \right] \rho(\xi)|\alpha\hat{v} + \tau\hat{w}|^2
\]

\[
- \left[ C(\epsilon_1) + \frac{N_1}{2} \right] \rho(\xi)|\xi|^2\int_0^\infty g(s)|\hat{\eta}'(s)|^2\, ds \leq 0,
\]

where we used the fact that \( \rho(\xi) \leq 1 \). Now, we discuss the case where \( g \) is decaying exponentially (i.e., \( p = \infty \)) and then the case where \( g \) is decaying polynomially (i.e., \( p^* < p < \infty \)) since the proof is slightly different.

4.1.1. Exponentially decaying kernel. Taking \( p = \infty \), then the assumption (G2) can be rewritten as follows:

\[
g'(t) \leq -\delta g(t).
\]

Therefore, making use of (60), we can now estimate the last term in (59) as:

\[
\left[ C(\epsilon_1) + \frac{N_1}{2} \right] \int_0^\infty g(s)|\hat{\eta}'(s)|^2\, ds \leq \left[ \frac{C(\epsilon_1)}{\delta} + \frac{N_1}{2\delta} \right] \int_0^\infty -g'(s)|\hat{\eta}'(s)|^2\, ds.
\]
Plugging (61) into (59), we get
\[
\frac{d}{dt} \hat{L}_1(\xi, t) + \left[ N_0(\alpha\beta - \tau\ell) - C(\epsilon_0) - N_1 C(\epsilon_2, \epsilon_3) \right] \rho(\xi) |\xi|^2 |\hat{v}|^2 \\
+ \left[ \frac{\alpha N_0}{2} - \frac{C(\epsilon_1)}{\delta} - \frac{N_1}{20} \right] \rho(\xi) |\xi|^2 \int_0^\infty (-g')(s) |\eta'(s)|^2 ds \\
+ \left[ (\ell - \epsilon_0 - (\gamma - \ell)\epsilon_1) - \epsilon_2 N_1 \right] \rho(\xi) |\xi|^2 |\alpha \hat{u} + \tau \hat{w}|^2 \\
+ \left[ N_1(\alpha - \epsilon_3) - \alpha \right] \rho(\xi) |\alpha \hat{u} + \tau \hat{w}|^2 \leq 0, \quad \forall t \geq 0. \tag{62}
\]

In the above estimate, we can fix our constants in such a way that the coefficients in (62) are positive. This can be achieved as follows: we pick \( \epsilon_3 < \alpha \) and \( \epsilon_0 = \epsilon_1 \), then we can select \( \epsilon_0 \) small enough such that
\[
\epsilon_0 < \frac{\ell}{1 + (\gamma - \ell)}.
\]
After that, we take \( N_1 \) large enough such that
\[
N_1 > \frac{\alpha}{\alpha - \epsilon_3}.
\]
Once \( N_1 \) and \( \epsilon_0 = \epsilon_1 \) are fixed, we select \( \epsilon_2 \) small enough such that
\[
\epsilon_2 < \frac{\ell - \epsilon_0(1 + (\gamma - \ell))}{N_1}.
\]
Finally, keeping in mind the assumption \( \tau\gamma < \alpha\beta \), we take \( N_0 \) large enough such that
\[
N_0 = \max \left\{ \frac{C(\epsilon_0) + N_1 C(\epsilon_2, \epsilon_3)}{\alpha\beta - \tau\gamma}, \frac{2C(\epsilon_1) + N_1}{\alpha\delta} \right\}.
\]
Consequently, from above and (51), we deduce that there exists a positive constant \( \Lambda_0 \) such that for all \( t \geq 0 \),
\[
\frac{d}{dt} \hat{L}_1(\xi, t) + \Lambda_0 \rho(\xi) \hat{E}(\xi, t) \leq 0. \tag{63}
\]
It is not difficult to see that from (51), (54), (56), (58) and for \( N_0 \) large enough, there exist two positive constants \( d_1 \) and \( d_2 \), such that for all \( \xi \in \mathbb{R}^N \) and for all \( t \geq 0 \), it holds that
\[
d_1 \hat{E}(\xi, t) \leq \hat{L}_1(\xi, t) \leq d_2 \hat{E}(\xi, t). \tag{64}
\]
Consequently, the last estimate together with (63), leads to
\[
\frac{d}{dt} \hat{L}_1(\xi, t) + \Lambda_1 \rho(\xi) \hat{E}_1(\xi, t) \leq 0, \quad \text{for all} \quad t \geq 0, \tag{65}
\]
and for some \( \Lambda_1 > 0 \). A simple application of Gronwall’s lemma to the estimate (65) yields
\[
\hat{E}_1(\xi, t) \leq \hat{E}_1(\xi, 0) e^{-\rho(\xi)t}, \quad \text{for all} \quad t \geq 0. \tag{66}
\]
Once again, (64) and (66) yield for some \( \Lambda_2 > 0 \),
\[
\hat{E}(\xi, t) \leq \Lambda_2 \hat{E}(\xi, 0) e^{-\rho(\xi)t}, \quad \text{for all} \quad t \geq 0. \tag{67}
\]
On the other hand, from (51), we have
\[
\hat{E}(\xi, t) \geq C_3 |\hat{U}(\xi, t)|^2
\]
and the use of \( \eta^0(s) = \hat{u}_0 \), yields
\[
\hat{E}(\xi, 0) \leq C_4 |\hat{U}(\xi, 0)|^2. \tag{69}
\]
This leads to the estimate (39).

4.1.2. Polynomially decaying kernel. Now, we assume that $g$ is decaying polynomially and we try to absorb the integral term $(\int_0^\infty g(s)\eta'(s)^2ds)$ in (59) by the term $(\int_0^\infty -g'(s)\eta'(s)^2ds)$ for $p^* < p < \infty$.

As we did above in Section 4.1.1, we take the same coefficients of (59) which are already fixed in such a way they are positive except the coefficient of $(\int_0^\infty g(s)\eta'(s)^2ds)$. We therefore deduce that for all $t \geq 0$, (59) can be rewritten as:

$$\frac{d}{dt}\tilde{\varphi}(\xi, t) \leq -\alpha_1\rho(\xi)\tilde{E}(\xi, t) + \alpha_2\rho(\xi)|\xi|^2 \int_0^\infty g(s)|\eta'(s)|^2ds,$$

(70)

where

$$\alpha_2 = C(\epsilon_1) + \frac{N_1}{2} + \alpha_1\alpha$$

and for some $\alpha_1 > 0$. Our main goal now is to estimate the last term in (70). Indeed, we have by using Hölder’s inequality,

$$\left(\int_0^\infty g(s)|\eta'(s)|^2ds\right) \leq \int_0^\infty \left[g(s)\right]^\frac{p}{p+1} |\eta'(s)|^\frac{2}{p+1} g(s)^{1-\frac{p}{p+1}}|\eta'(s)|^{2\left(1-\frac{1}{p+1}\right)}ds,$$

where

The use of (15) leads to

$$\int_0^\infty |\xi|^2 \int_0^\infty g(s)\left[\int_0^\infty |\eta'(s)|^2ds\right]^{p+1\left(1-p\right)} |\eta'(s)|^2ds \leq \int_0^\infty \left[g(s)\right]^{\frac{p+1\left(1-p\right)}{p^*}} |\xi|^2|\hat{u}(t) - \hat{u}(t-s)|^2ds$$

$$\leq 2 \int_0^\infty \left[g(s)\right]^{\frac{(p+1)(p-1)}{p^*}} |\xi|^2|\hat{u}(t)|^2ds$$

$$+ 2 \int_0^\infty \left[g(s)\right]^{\frac{(p+1)(p-1)}{p^*}} |\xi|^2|\hat{u}(t-s)|^2ds$$

Now, using the fact that (see (52)),

$$\tilde{E}(\xi, t) \geq \frac{1}{2\beta}(\alpha\beta - \ell\tau)|\hat{u}|^2$$

then we obtain from above

$$\int_0^\infty |\xi|^2 \int_0^\infty g(s)\left[\int_0^\infty |\eta'(s)|^2ds\right]^{p+1\left(1-p\right)} |\eta'(s)|^2ds \leq \frac{4\beta}{\ell(\alpha\beta - \tau\ell)} \int_0^\infty \left[g(s)\right]^{\frac{(p+1)(p-1)}{p^*}} \tilde{E}(\xi, t)ds$$

$$+ \frac{4\beta}{\ell(\alpha\beta - \tau\ell)} \int_0^t \left[g(s)\right]^{\frac{(p+1)(p-1)}{p^*}} \tilde{E}(\xi, t-s)ds$$

$$\leq \frac{8\beta}{\ell(\alpha\beta - \tau\ell)} \left[\int_0^\infty \left[g(s)\right]^{\frac{(p+1)(p-1)}{p^*}} \tilde{E}(\xi, 0)\right] ds.$$
Let \( \nu = \frac{(p+1)(p-1)}{p^2} \), it is clear that the assumption \( p > p^* \) implies that \( \nu \in (\frac{1}{p}, 1) \). Hence, due to (42) and \((G2)\), we have

\[
\left( |\xi|^2 \int_0^{\infty} g(s)|\hat{\eta}(s)|^2 ds \right) \leq d_3 \left( \hat{E}(\xi, 0) \right)^{\frac{p^*}{p+1}} \left( -\frac{d}{dt} \hat{E}(\xi, t) \right)^{\frac{1}{p+1}}, \tag{71}
\]

where

\[ d_3 = 2c_1 \left( \frac{1}{\delta_\alpha} \right)^{\frac{1}{p+1}}. \]

Combining now (71) and (70), we get

\[
\frac{d}{dt} \hat{L}_1(\xi, t) \leq -\alpha_1 \rho(\xi) \hat{E}(\xi, t) + d_4 \rho(\xi) \left( \hat{E}(\xi, 0) \right)^{\frac{p^*}{p+1}} \left( -\frac{d}{dt} \hat{E}(\xi, t) \right)^{\frac{1}{p+1}},
\]

with \( d_4 = \alpha_2 d_3 \). In the same spirit of [22], multiplying the last inequality by \( \hat{E}^p(\xi, t) \), gives

\[
\hat{E}^p(\xi, t) \left( \frac{d}{dt} \hat{L}_1(\xi, t) \right) \leq -\alpha_1 \rho(\xi) \hat{E}^{p+1}(\xi, t) + d_4 \rho(\xi) \hat{E}^p(\xi, t) \left( \hat{E}(\xi, 0) \right)^{\frac{p^*}{p+1}} \left( -\frac{d}{dt} \hat{E}(\xi, t) \right)^{\frac{1}{p+1}}.
\]

The use of Young’s inequality, yields

\[
\hat{E}^p(\xi, t) \left( \frac{d}{dt} \hat{L}_1(\xi, t) \right) \leq -\alpha_1 \rho(\xi) \hat{E}^{p+1}(\xi, t) + d_4 \rho(\xi) \left[ \epsilon \hat{E}^{p+1}(\xi, t) - C(\epsilon) \hat{E}^p(\xi, 0) \frac{d}{dt} \hat{E}(\xi, t) \right]
\]

\[
= -\left( \alpha_1 - \epsilon d_4 \right) \rho(\xi) \hat{E}^{p+1}(\xi, t) - d_5 \rho(\xi) \hat{E}^p(\xi, 0) \frac{d}{dt} \hat{E}(\xi, t).
\]

Now, choosing \( \epsilon < \frac{\alpha_1}{d_4} \) and recalling that \( \frac{d}{dt} \hat{E}(\xi, t) \leq 0 \) and using the fact that \( \rho(\xi) \leq 1 \), to get

\[
\frac{d}{dt} \left[ \hat{E}^p(\xi, t) \hat{L}_1(\xi, t) \right] \leq \hat{E}^p(\xi, t) \frac{d}{dt} \hat{L}_1(\xi, t)
\]

\[
\leq -d_6 \rho(\xi) \hat{E}^{p+1}(\xi, t) - \frac{d}{dt} \left[ d_5 \hat{E}^p(\xi, 0) \hat{E}(\xi, t) \right],
\]

for some \( d_6 > 0 \). This implies

\[
\frac{d}{dt} \left[ \hat{E}^p(\xi, t) \hat{L}_1(\xi, t) + d_5 \hat{E}^p(\xi, 0) \hat{E}(\xi, t) \right] \leq -d_6 \rho(\xi) \hat{E}^{p+1}(\xi, t). \tag{72}
\]

Now, we define

\[
\hat{E}(\xi, t) = \hat{E}^p(\xi, t) \hat{L}_1(\xi, t) + d_5 \hat{E}^p(\xi, 0) \hat{E}(\xi, t),
\]

which satisfies, by using (64)

\[
d_5 \hat{E}^p(\xi, 0) \hat{E}(\xi, t) \leq \hat{E}(\xi, t) \leq d_7 \hat{E}^p(\xi, 0) \hat{E}(\xi, t), \tag{73}
\]

for some \( d_7 > 0 \). Therefore, we obtain from (72) the estimate

\[
\frac{d}{dt} \hat{E}(\xi, t) \leq -d_8 \rho(\xi) \left( \frac{\hat{E}(\xi, t)}{\hat{E}^p(\xi, 0)} \right)^{p+1},
\]
for some \( d_k > 0 \). A simple application of Gronwall’s lemma leads to the estimate:

\[
\dot{E}(\xi, t) \leq \Lambda_3 \dot{E}^{p+1}(\xi,0) \left(1 + \rho(\xi)t\right)^{-\frac{1}{p}},
\]

for \( \Lambda_3 > 0 \). Using (73) we obtain

\[
\dot{E}(\xi, t) \leq \Lambda_4 \dot{E}(\xi,0) \left(1 + \rho(\xi)t\right)^{-\frac{1}{p}},
\]

for \( \Lambda_4 > 0 \). This leads to the desired estimate (40) using (68) and (69). Thus the proof of Proposition 1 is finished.

5. **Decay estimates—the critical case** \( \alpha\beta = \tau\gamma \). In this section, we assume that \( \alpha\beta = \tau\gamma \) and derive the point-wise estimate (76) for the Fourier transform of the solution. This estimate is the key ingredient in proving the decay rate stated in Theorem 3.2.

**Proposition 2.** Let \( \hat{U}(\xi, t) \) be the solution of (36)-(37). Assume that (G1)-(G3) and \( \alpha\beta = \tau\gamma \) hold. Then, \( \hat{U}(\xi, t) \) satisfies the following estimate

\[
|\hat{U}(\xi, t)|^2 \leq C|\hat{U}(\xi,0)|^2 e^{-\tilde{\rho}(\xi)t}, \quad \text{for} \quad p = \infty,
\]

for all \( t > 0 \), where

\[
\tilde{\rho}(\xi) = \frac{|\xi|^2}{(1 + |\xi|^2)^2}.
\]

**Remark 2.** It is clear from the form of \( \frac{|\xi|^2}{(1 + |\xi|^2)^2} \) obtained in the estimate (76) that \( \frac{|\xi|^2}{(1 + |\xi|^2)^2} \sim |\xi|^{-2} \) for \( |\xi| \to \infty \). This leads to the decay rate of regularity-loss type shown in Theorem 3.2. On the other hand, for \( |\xi| \to 0 \), we have \( \frac{|\xi|^2}{(1 + |\xi|^2)^2} \sim |\xi|^2 \), this means that the dissipation is still effective for low frequencies and yields the decay rate \((1 + t)^{-N/4}\) of the \( L^2 \)-norm of the solution.

To prove Proposition 2, we need to define another functional \( \hat{F}_3(\xi, t) \) (which is unnecessary in the first case) in such a way to recover the dissipation of \( |\hat{v}|^2 \), since for \( \alpha\beta = \tau\gamma \), we lose the dissipation of \( |\hat{v}|^2 \) in (42). In this case (42) becomes:

\[
\frac{d}{dt} \hat{E}(\xi, t) \leq -\frac{\alpha}{2} \|\hat{\eta}(s)\|^2_M, \quad \forall t \geq 0.
\]

Now, we show that if \( g \) decays exponentially, that is

\[
g'(t) \leq -\delta g(t),
\]

which holds for \( p = \infty \) in the assumption (G2), then the vector \( |\hat{V}(\xi, t)|^2 \) defined in (50) is equivalent to \( \hat{E}(\xi, t) \), as we did in Lemma 4.2. Thus we have the following lemma.

**Lemma 5.1.** Let \( \alpha\beta = \tau\gamma \) and assume that (79) holds. Then there exist two positive constants \( C_5 \) and \( C_6 \), such that

\[
C_5|\hat{V}(\xi, t)|^2 \leq \hat{E}(\xi, t) \leq C_6|\hat{V}(\xi, t)|^2,
\]

for all \( \xi \in \mathbb{R}^N \) and for all \( t \geq 0 \).
Proof. The right-hand side in (80) is estimated just in the same way as in (53). For the left-hand side, we follow [8, Lemma 3.1] to have from Young’s inequality the estimate:

\[ 2\tau|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}'(s), \hat{\nu} \rangle \, ds \right) \leq \frac{\tau^2}{2\epsilon_0} |\xi|^2 |\hat{\nu}|^2 + 2\epsilon_0 (\gamma - \ell) |\xi|^2 \int_0^\infty g(s) |\hat{\eta}'(s)|^2 \, ds. \]

Taking \( 2\epsilon_0 = \frac{\alpha(\epsilon_1 + 1)}{\gamma - \ell} \), yields

\[ |2\tau|\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}'(s), \hat{\nu} \rangle \, ds \right) \leq \frac{\tau^2}{\alpha(\epsilon_1 + 1)} \left( \frac{\alpha(\epsilon_1 + 1)}{\gamma - \ell} \right) |\xi|^2 |\hat{\nu}|^2 + \alpha(\epsilon_1 + 1) \int_0^\infty g(s) |\hat{\eta}'(s)|^2 \, ds. \]

Hence, we have, as in [8]

\[ \alpha |\xi|^2 \int_0^\infty g(s) |\hat{\eta}'(s)|^2 \, ds + 2\tau |\xi|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\eta}'(s), \hat{\nu} \rangle \, ds \right) \]

\[ \geq - \frac{\tau^2}{\alpha(\epsilon_1 + 1)} |\xi|^2 |\hat{\nu}|^2 - \frac{\alpha\epsilon_1}{\delta} |\xi|^2 \int_0^\infty g(s) |\hat{\eta}'(s)|^2 \, ds \]

\[ \geq - \frac{\tau^2}{\alpha(\epsilon_1 + 1)} |\xi|^2 |\hat{\nu}|^2 - \frac{\alpha\epsilon_1}{\delta} |\xi|^2 |\eta''|^2 \|_{\mathcal{M}}. \]

Consequently, from (38) and the above estimate, we obtain

\[ \dot{E}(\xi, t) \geq \frac{1}{2} \left[ \frac{\tau}{\alpha} |\xi|^2 |\hat{\nu}|^2 + |\alpha \hat{\eta} + \tau \hat{\nu}|^2 + \frac{\epsilon_1 \tau^2 (\gamma - \ell)}{\alpha(\epsilon_1 + 1)} |\xi|^2 |\hat{\nu}|^2 + (\tau - \frac{\alpha\epsilon_1}{\delta}) |\eta''|^2 \|_{\mathcal{M}} \right]. \]

It is clear that for \( \epsilon_1 \) small enough, that is for \( \epsilon_1 < \frac{\delta\tau}{\alpha} \), we get the left-hand side of (80). This finishes the proof of Lemma 5.1. \( \square \)

Now, we define the functional \( \hat{F}_3(\xi, t) \) as

\[ \hat{F}_3(\xi, t) = -\tau |\xi|^2 \text{Re} \left\{ \int_0^\infty g(s) \langle \hat{\eta}'(s), \hat{\nu} \rangle \, ds \right\}. \]  

(81)

Then, we have the following result.

**Lemma 5.2.** For any \( \epsilon_4, \epsilon_5, \epsilon_6 > 0 \), it holds that

\[ \frac{d}{dt} \hat{F}_3(\xi, t) + (\tau (\gamma - \ell) - \epsilon_4 g_0 - \epsilon_6 (\gamma - \ell)) |\xi|^2 |\hat{\nu}|^2 \]

\[ \leq \epsilon_5 \frac{\xi^2}{1 + |\xi|^2} (\gamma - \ell) |\alpha \hat{\nu} + \tau \hat{\eta}|^2 + C(\epsilon_4) |\xi|^2 \int_0^\infty -g(s) |\hat{\eta}'(s)|^2 \, ds \]

\[ + C(\epsilon_5, \epsilon_6) |\xi|^2 (1 + |\xi|^2) \int_0^\infty g(s) |\hat{\eta}'(s)|^2 \, ds. \]  

(82)

**Proof.** Multiplying the second equation in (36) by \( \tau |\xi|^2 \int_0^\infty g(s) \hat{\eta}'(s) \, ds \), we get

\[ |\xi|^2 \int_0^\infty g(s) \langle \tau \hat{\nu}_t, \hat{\eta}'(s) \rangle \, ds = |\xi|^2 \int_0^\infty g(s) \langle \tau \hat{\nu}, \hat{\eta}'(s) \rangle \, ds. \]

Consequently,

\[ \frac{d}{dt} |\xi|^2 \int_0^\infty g(s) \langle \tau \hat{\nu}, \hat{\eta}'(s) \rangle \, ds - |\xi|^2 \int_0^\infty g(s) \langle \tau \hat{\nu}, \hat{\eta}'(s) \rangle \, ds = |\xi|^2 \int_0^\infty g(s) \langle \tau \hat{\nu}, \hat{\eta}'(s) \rangle \, ds. \]
Using the fact that $\hat{\eta}^i_t + \hat{\eta}^j_s = \hat{v}$ and integrating by parts with respect to $s$ and taking the real part we obtain
\[
\frac{d}{dt} \hat{F}_3(\xi, t) + \tau(\gamma - \ell)|\xi|^2|\hat{v}|^2 = \tau|\xi|^2 \text{Re} \left\{ \int_0^\infty -g'(s) \langle \hat{v}, \hat{\eta}^i(s) \rangle \, ds \right\} - |\xi|^2 \text{Re} \left\{ \int_0^\infty g(s) \langle \tau \hat{\dot{w}} + \alpha \hat{v}, \hat{\eta}^i(s) \rangle \, ds \right\} + |\xi|^2 \text{Re} \left\{ \int_0^\infty g(s) \langle \alpha \hat{v}, \hat{\eta}^i(s) \rangle \, ds \right\}.
\]

Applying Youngs inequality, we obtain the estimate (82) for any $\epsilon_4, \epsilon_5, \epsilon_6 > 0$. \hfill \Box

5.1. **Proof of Proposition 2.** Now, we define the new Lyapunov functional $\hat{L}_2(\xi, t)$ associated to the critical case as follows:
\[
\hat{L}_2(\xi, t) = N \hat{E}(\xi, t) + \frac{|\xi|^2}{(1 + |\xi|^2)^2} \hat{F}_1(\xi, t) + 2 \frac{|\xi|^2}{(1 + |\xi|^2)^2} \hat{F}_2(\xi, t) + M \frac{1}{1 + |\xi|^2} \hat{F}_3(\xi, t),
\]
for some positive constants $N$ and $M$ that have to be chosen later. Taking the derivative of (83) with respect to $t$, and using (55), (57), (78) and (82), Since $g$ decays exponentially (i.e., satisfies (79)), we get
\[
\frac{d}{dt} \hat{L}_2(\xi, t) + \frac{|\xi|^2}{(1 + |\xi|^2)^2} \left[ (\ell - \epsilon_0 - (\gamma - \ell)\epsilon_1) - 2\epsilon_2 \right] |\xi|^2 |\alpha \hat{\hat{v}} + \tau \hat{v}|^2 + \frac{1}{1 + |\xi|^2} \left[ M(\gamma - \ell) - \epsilon_4 g_0 - \epsilon_6 (\gamma - \ell) \right] |\xi|^2 |\hat{v}|^2 + \frac{|\xi|^2}{(1 + |\xi|^2)^2} \left[ 2(\alpha - \epsilon_3) - \alpha - M \epsilon_5 (\gamma - \ell) \right] |\alpha \hat{\hat{v}} + \tau \hat{v}|^2 + (N - \Lambda)|\xi|^2 \int_0^\infty -g'(s)|\hat{\eta}^i(s)|^2 \, ds \leq 0,
\]
where for the coefficient of $|\hat{v}|^2$ in $F_1$, we used the fact that $\frac{|\xi|^4}{(1 + |\xi|^2)^2} \leq \frac{|\xi|^2}{1 + |\xi|^2}$. Here $\Lambda$ is a constant that depends on all the other constants and also on $\delta$.

In the above estimate, we can fix our constants in such a way that the previous coefficients are positive. We take $\epsilon_0 = \epsilon_1$, then we pick $\epsilon_0$ small enough such that
\[
\epsilon_0 < \frac{\ell}{1 + (\gamma - \ell)}.
\]
Once $\epsilon_0, \epsilon_1$ are fixed, we select $\epsilon_2$ small enough such that
\[
\epsilon_2 < \frac{\ell - \epsilon_0 (1 + (\gamma - \ell))}{2}.
\]
Now, we pick $\epsilon_4 = \epsilon_6$, and select
\[
\epsilon_4 < \frac{\tau(\gamma - \ell)}{g_0 + (\gamma - \ell)}.
\]
Next, we pick $\epsilon_3 < \frac{\alpha}{4}$ and we take $M$ large enough such that
\[
M > \frac{C(\epsilon_0) + 2C(\epsilon_2, \epsilon_3)}{\tau(\gamma - \ell) - \epsilon_4(g_0 + (\gamma - \ell))}.
\]
Then, we can select $\epsilon_5$ small enough such that
\[ \epsilon_5 < \frac{2(\alpha - \epsilon_3) - \alpha}{M(\gamma - \ell)}. \]

Finally, we take
\[ N > \Lambda. \]
Consequently, we deduce that there exists a constant $R_1 > 0$ such that for all $t > 0$
\[ \frac{d}{dt} \hat{L}_2(\xi, t) + R_1 \hat{\rho}(\xi) \hat{V}(\xi, t) \leq 0. \]  
(85)

Thanks to (80), there exists a constant $R_2 > 0$ such that for all $t > 0$, we deduce
\[ \frac{d}{dt} \hat{L}_2(\xi, t) + R_2 \hat{\rho}(\xi) \hat{E}(\xi, t) \leq 0. \]  
(86)

On the other hand, for $N$ large enough, there exist two positive constants $R_3, R_4$ such that
\[ R_3 \hat{E}(\xi, t) \leq \hat{L}_2(\xi, t) \leq R_4 \hat{E}(\xi, t). \]  
(87)
Consequently, (86) with (87) lead to
\[ \frac{d}{dt} \hat{L}_2(\xi, t) + R_5 \hat{\rho}(\xi) \hat{L}_2(\xi, t) \leq 0. \]  
(88)

For some $R_5 > 0$. Integrating (65) with respect to $t$ yields
\[ \hat{L}_2(\xi, t) \leq \hat{L}_2(\xi, 0)e^{-c\hat{\rho}(\xi)t}, \]  
(89)
This last estimate leads for $R_6 > 0$, to
\[ \hat{E}(\xi, t) \leq R_6 \hat{E}(\xi, 0)e^{-c\hat{\rho}(\xi)t}, \]  
(90)
Therefore, by (80), (68) and (69) we get the desired result (76).

6. **Proof of the decay estimates of Theorems 3.1 and 3.2.** In this section, we prove the decay rate both in the subcritical case (Theorem 3.1) and in the critical case (Theorem 3.2), for $p = \infty$ and $p^* < p < \infty$.

6.1. **Proof of Theorem 3.1.** We proceed with the proof of Theorem 3.1, starting with the first case when $p = \infty$. To show (29) we have by Plancherel’s theorem and the estimate (39) that (the constant $C$ here is a generic positive constant that may take different values in different places)
\[ \|\nabla U(t)\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi, t)|^2d\xi \leq C \int_{\mathbb{R}^N} |\xi|^{2j} e^{-c\hat{\rho}(\xi)t} |\hat{U}(\xi, 0)|^2d\xi. \]  
(91)
It is obvious that the term on the right-hand side of (91) depends on the behavior of the function (41). Since
\[ \hat{\rho}(\xi) \geq \begin{cases} \frac{1}{2}|\xi|^2, & \text{if } |\xi| \leq 1, \\ \frac{1}{2}, & \text{if } |\xi| \geq 1. \end{cases} \]  
(92)
Then we write the integral on the right-hand side of (91) as
\[ \int_{\mathbb{R}^N} |\xi|^{2j} e^{-c\hat{\rho}(\xi)t} |\hat{U}(\xi, 0)|^2d\xi = \int_{|\xi| \leq 1} |\xi|^{2j} e^{-c\hat{\rho}(\xi)t} |\hat{U}(\xi, 0)|^2d\xi \\
+ \int_{|\xi| \geq 1} |\xi|^{2j} e^{-c\hat{\rho}(\xi)t} |\hat{U}(\xi, 0)|^2d\xi := I_1 + I_2. \]  
(93)
Concerning the integral $I_1$, we have by exploiting (12) that
\[
I_1 \leq \|\hat{U}_0\|_{L^\infty(\mathbb{R}^N)}^2 \int_{|\xi| \leq 1} |\xi|^{2j} e^{-\frac{\xi^2}{2}} d\xi \leq C(1 + t)^{-N/2-j} \|U_0\|_{L^1(\mathbb{R}^N)}^2.
\] (94)

On the other hand, in the high frequency region ($|\xi| \geq 1$), we have
\[
I_2 \leq e^{-\frac{\xi^2}{2}} \int_{|\xi| \geq 1} |\xi|^{2j} |\hat{U}(\xi,0)|^2 d\xi \leq e^{-\frac{\xi^2}{2}} \|\nabla^j U_0\|_{L^2(\mathbb{R}^N)}^2.
\] (95)

Collecting the above two estimates give the desired decay estimate (29).

Now, we show the decay rate of the vector $U$ in the case of polynomially decaying kernel as follows: to establish (30), we have by Plancherel's theorem and the estimate (40) that
\[
\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi,t)|^2 d\xi
\leq C \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi,0)|^2 (1 + \rho(\xi)t)^{-\frac{p}{2}} d\xi
\leq C(1 + t)^{-\frac{j}{p}} \int_{|\xi| \geq 1} |\xi|^{2j} |\hat{U}(\xi,0)|^2 d\xi
+ C \|\hat{U}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\xi| \leq 1} |\xi|^{2j} \left(1 + |\xi|^2t\right)^{-\frac{p}{2}} d\xi.
\] (96)

According to (92), the integral was split into two parts. For the last term above, the use of polar coordinates leads to
\[
\int_{|\xi| \leq 1} |\xi|^{2j} \left(1 + |\xi|^2t\right)^{-\frac{p}{2}} d\xi \leq \int_0^1 \frac{r^{j+p-\frac{p}{2}+1}}{(1 + rt)^{j+p}} dr.
\] (97)

Plugging (97) into (96), then exploiting Lemma 2.2, we deduce the desired result (30). This completes the proof of Theorem 3.1.

6.2. **Proof of Theorem 3.2.** To show (31), we have from (77) that
\[
\frac{|\xi|^2}{(1 + |\xi|^2)^2} \geq c \begin{cases} |\xi|^2, & \text{if } |\xi| \leq 1, \\ |\xi|^{-2}, & \text{if } |\xi| \geq 1. \end{cases}
\] (98)

By Plancherel's theorem, we obtain
\[
\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi,t)|^2 d\xi
\leq C \int_{\mathbb{R}^N} |\xi|^{2j} e^{-\frac{\xi^2}{2(1 + |\xi|^2)^2}} |\hat{U}(\xi,0)|^2 d\xi
= C \int_{|\xi| \leq 1} |\xi|^{2j} e^{-\frac{\xi^2}{2(1 + |\xi|^2)^2}} |\hat{U}(\xi,0)|^2 d\xi + C \int_{|\xi| \geq 1} |\xi|^{2j} e^{-\frac{\xi^2}{2(1 + |\xi|^2)^2}} |\hat{U}(\xi,0)|^2 d\xi
= J_1 + J_2.
\] (99)

Concerning the integral $J_1$, we have by using (98)
\[
J_1 \leq C \|\hat{U}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\xi| \leq 1} |\xi|^{2j} e^{-\frac{\xi^2}{2}} d\xi \leq C(1 + t)^{-N/2-j} \|U_0\|_{L^1(\mathbb{R}^N)}^2.
\] (100)

Using (98) and the estimate
\[
\sup_{|\xi| \geq 1} \left\{\frac{|\xi|^{-2k} e^{-c|\xi|^{-2}}}{|\xi|^{-2k}}\right\} \leq C(1 + t)^{-k}.
\]
Then, the high frequency part \( J_2 \) is estimated as follow
\[
J_2 \leq \sup_{|\xi| \geq 1} \left\{|\xi|^{-2k} e^{-c|\xi|^{-2}}\right\} \int_{|\xi| \geq 1} |\xi|^{2(k+1)} |\hat{U}(\xi,0)|^2 d\xi \leq C(1 + t)^{-k} \|
abla^j + k U_0 \|_{L^2(\mathbb{R}^N)}^2.
\]
Collecting the above two estimates yields (31).

7. Decay rates for the type III memory-- the critical case \( \alpha \beta = \tau \gamma \). As in the previous sections, we first, need to find the decay rate of the Fourier image of the solution. Taking Fourier’s transform of (32) for all \( t \geq 0 \) and for all \( \xi \in \mathbb{R}^N \), we get
\[
\tau \hat{z}_{tt} + \beta |\xi|^2 \hat{z} - \tau |\xi|^2 \int_0^t g(s) \hat{z}(t-s) ds = 0.
\]
In this case the second assumption in (G1), is replaced by \( (G4): g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a nonincreasing twice differentiable function, such that
\[
g(0) > 0, \quad \beta - \tau \int_0^\infty g(s) ds = \ell > 0.
\]
and \( g'(s) \leq 0 \) for every \( s > 0 \).

As we have seen before in Section 2, we extend the solution of (102) for all times, by setting \( z(x, t) = 0 \) when \( t < 0 \) and considering for \( t \geq 0 \) the auxiliary past history variable \( \mu^t(s) \), defined as:
\[
\mu^t(s) = z(t) - z(t-s), \quad t \geq 0, \quad s \in \mathbb{R}^+.
\]
Consequently, problem (102) can be recast as follows:
\[
\begin{cases}
\tau \hat{z}_{tt} + \ell |\xi|^2 \hat{z} + \tau |\xi|^2 \int_0^\infty g(s) \hat{\mu}^t(s) ds = 0 \\
\hat{\mu}^t(s) + \hat{\mu}^t(s) = \hat{z}_t.
\end{cases}
\]
Hence, we have the following result.

**Proposition 3.** Let \( \hat{z}(\xi, t) \) be the solution of (104). Assume that (G2) and (G4) hold. Then, \( \hat{\phi}(\xi, t) \) satisfies the following estimates:
\[
\hat{\phi}(\xi, t) \leq C |\hat{Z}(\xi, 0)|^2 e^{-c_0 t}, \quad \text{for} \quad p = \infty,
\]
\[
\hat{\phi}(\xi, t) \leq C |\hat{Z}(\xi, 0)|^2 \left(1 + \rho(\xi) t\right)^{-\frac{1}{2}}, \quad \text{for} \quad p^* < p < \infty,
\]
for all \( t > 0 \), where
\[
\hat{\phi}(\xi, t) = \frac{1}{2} \left[|\xi|^2 |\hat{z}|^2 + \tau |\hat{z}_t|^2 + \tau |\xi|^2 \int_0^\infty g(s) |\hat{\mu}^t(s)|^2 ds \right].
\]
The proof of Proposition 3 will be given through several lemmas. It is clear that \( \hat{\phi}(\xi, t) \) satisfies for all \( t \geq 0 \) and for all \( \xi \in \mathbb{R}^N \),
\[
\frac{d}{dt} \hat{\phi}(\xi, t) = \frac{\tau}{2} |\xi|^2 \int_0^\infty g'(s) |\hat{\mu}^t(s)|^2 ds.
\]
Since \( g(s) \) is nonincreasing, then \( \hat{\phi}(\xi, t) \) is nonincreasing and dissipative.
Lemma 7.1. For all \( t \geq 0 \) and for all \( \xi \in \mathbb{R}^N \), the functional
\[
\hat{\Phi}_1(\xi, t) = \tau \text{Re}(\bar{z} \dot{z})
\]
satisfies, for any \( \lambda_1 > 0 \),
\[
\frac{d}{dt} \hat{\Phi}_1(\xi, t) + \left[ \bar{\ell} - \lambda_1(\beta - \bar{\ell}) \right] |\xi|^2 |\dot{z}|^2 \leq \tau |\dot{z}|^2 + C(\lambda_1) |\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds. \tag{109}
\]

Proof. Multiplying (104) by \( \ddot{z} \) and taking the real part, we obtain
\[
\tau \bar{\ell} \ddot{z}_t + \ell |\dot{z}|^2 |\ddot{z}|^2 + \tau |\dot{z}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\mu}(s), \ddot{z} \rangle ds \right) = 0.
\]

Hence,
\[
\tau \frac{d}{dt} \hat{\Phi}_1(\xi, t) - \tau |\dot{z}|^2 + \ell |\ddot{z}|^2 |\dot{z}|^2 + \tau |\dot{z}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \hat{\mu}(s), \ddot{z} \rangle ds \right) = 0.
\]

Then by applying Young’s inequality, we get the estimate (109) for any \( \lambda_1 > 0 \).

\[
\square
\]

Lemma 7.2. For all \( t \geq 0 \) and for all \( \xi \in \mathbb{R}^N \), the functional
\[
\hat{\Phi}_2(\xi, t) = -\tau \text{Re} \left( \int_0^\infty g(s) \langle \dot{z}_t, \hat{\mu}^t(s) \rangle ds \right)
\]
satisfies for any \( \lambda_2, \lambda_3 > 0 \), the following estimate
\[
\frac{d}{dt} \hat{\Phi}_2(\xi, t) + \left( (\beta - \bar{\ell}) - \lambda_3 g(0) \right) |\dot{z}_t|^2 \leq \lambda_2 \frac{(\beta - \bar{\ell})}{\tau} |\xi|^2 |\dot{z}|^2 + C(\lambda_3) |\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds
\]
\[
+ \left( C(\lambda_2) + (\beta - \bar{\ell}) \right) |\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds.
\]

Proof. Multiplying (104) by \( \int_0^\infty g(s)\hat{\mu}^t(s)ds \) and taking the real part, we obtain
\[
\tau \text{Re} \left( \int_0^\infty g(s) \langle \ddot{z}_t, \hat{\mu}^t(s) \rangle ds \right) + \ell |\ddot{z}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \dot{z}, \hat{\mu}^t(s) \rangle ds \right) + \tau |\dot{z}|^2 \left( \int_0^\infty g(s)\hat{\mu}(s)ds \right)^2 = 0.
\]

Hence, we have
\[
\tau \frac{d}{dt} \text{Re} \left( \int_0^\infty g(s) \langle \dot{z}_t, \hat{\mu}^t(s) \rangle ds \right) - \tau \text{Re} \left( \int_0^\infty g(s) \langle \dot{z}, \hat{\mu}^t(s) \rangle ds \right)
\]
\[
+ \ell |\dot{z}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \ddot{z}, \hat{\mu}^t(s) \rangle ds \right) + \tau |\dot{z}|^2 \left( \int_0^\infty g(s)\hat{\mu}(s)ds \right)^2 = 0.
\]

Using the fact that \( \hat{\mu}^t_1 + \hat{\mu}^t_* = \dot{z}_t \) and integrating by parts with respect to \( s \), we obtain
\[
\frac{d}{dt} \hat{\Phi}_2(\xi, t) = - (\beta - \bar{\ell}) |\dot{z}_t|^2 + \tau \text{Re} \left( \int_0^\infty -g'(s) \langle \ddot{z}_t, \hat{\mu}^t(s) \rangle ds \right)
\]
\[
+ \ell |\dot{z}|^2 \text{Re} \left( \int_0^\infty g(s) \langle \ddot{z}, \hat{\mu}^t(s) \rangle ds \right) + \tau |\dot{z}|^2 \left( \int_0^\infty g(s)\hat{\mu}(s)ds \right)^2 = 0.
\]

Then, to estimate the last three terms we apply Young’s inequality and H"older’s inequality, we get the estimate (110) for any \( \lambda_2, \lambda_3 > 0 \).

\[
\square
\]
Now, we define the Lyapunov functional $\hat{L}_3(\xi, t)$ as:

$$
\hat{L}_3(\xi, t) = N_2 \hat{E}(\xi, t) + \rho(\xi) \hat{\Phi}_1(\xi, t) + N_3 \rho(\xi) \hat{\Phi}_2(\xi, t),
$$

where $N_2$ and $N_3$ are positive constants that will be fixed later on. Taking the derivative of (111) with respect to $t$ and making use of (108), (109) and (110), we obtain

$$
\frac{d}{dt} \hat{L}_3(\xi, t) + \left[ (\tilde{\ell} - \lambda_3(\beta - \tilde{\ell})) - N_3 \lambda_3 \frac{(\beta - \tilde{\ell})}{\tau} \right] |\xi|^2 \rho(\xi) |\ddot{z}|^2 + \left[ N_3 ((\beta - \tilde{\ell}) - \lambda_3 g(0)) - \tau \right] |\xi|^2 \rho(\xi) |\dot{z}|^2
$$

$$
- \left[ \tau C(\lambda_1) + N_3 \left( C(\lambda_2) + (\beta - \tilde{\ell}) \right) \right] |\xi|^2 \rho(\xi) \int_0^\infty g(s)|\dot{\mu}^t(s)|^2 ds + \left[ N_2 \frac{\tau}{2} - N_3 C(\lambda_3) \right] |\xi|^2 \rho(\xi) \int_0^\infty -g'(s)|\dot{\mu}^t(s)|^2 ds \leq 0.
$$

Now, we discuss the case where $g$ is decaying exponentially (i.e., $p = \infty$) and then the case where $g$ is decaying polynomially (i.e., $p^* < p < \infty$).

### 7.1. Exponentially decaying kernel

Taking $p = \infty$, according to the assumption (G2), (112) can be written as

$$
\frac{d}{dt} \hat{L}_3(\xi, t) + \left[ (\tilde{\ell} - \lambda_3(\beta - \tilde{\ell})) - N_3 \lambda_3 \frac{(\beta - \tilde{\ell})}{\tau} \right] |\xi|^2 \rho(\xi) |\ddot{z}|^2 + \left[ N_3 ((\beta - \tilde{\ell}) - \lambda_3 g(0)) - \tau \right] |\xi|^2 \rho(\xi) |\dot{z}|^2
$$

$$
+ \left[ N_2 \frac{\tau}{2} - N_3 C(\lambda_3) - \frac{1}{\delta} \left( C(\lambda_1) + N_3 \left( C(\lambda_2) + (\beta - \tilde{\ell}) \right) \right) \right] |\xi|^2 \rho(\xi) \int_0^\infty -g'(s)|\dot{\mu}^t(s)|^2 ds \leq 0, \quad t \geq 0.
$$

In the above estimate, we can fix our constants in such a way that the coefficients in (7.1) are positive. This can be achieved as follows:

$$
\lambda_1 < \frac{\tilde{\ell}}{\beta - \tilde{\ell}}
$$

and we fix $\lambda_3$ small enough such that

$$
\lambda_3 < \frac{\beta - \tilde{\ell}}{g(0)}.
$$

Then, we can select $N_3$ large enough

$$
N_3 > \frac{\tau}{(\beta - \tilde{\ell}) - \lambda_3 g(0)}.
$$

Once $N_3$ is fixed, we select $\lambda_2$ small enough such that

$$
\lambda_2 < \frac{\tau (\tilde{\ell} - \lambda_3(\beta - \tilde{\ell}))}{N_3 (\beta - \tilde{\ell})}.
$$

Finally, we take $N_2$ large enough such that

$$
N_2 > \frac{2}{\tau} \left( N_3 C(\lambda_3) + \frac{1}{\delta} \left( C(\lambda_1) + N_3 \left( C(\lambda_2) + (\beta - \tilde{\ell}) \right) \right) \right).
$$
Consequently, from above we deduce that there exists a positive constant Λ\(_5\) such that for all \(t \geq 0\) and for all \(\xi \in \mathbb{R}^N\),
\[
\frac{d}{dt} \hat{L}_3(\xi, t) + \Lambda_5 \rho(\xi) \hat{E}(\xi, t) \leq 0.
\] (114)

It is clear that for \(N_2\) large enough, there exist two positives constants \(d_9\) and \(d_{10}\), for all \(t \geq 0\) and for all \(\xi \in \mathbb{R}^N\), such that
\[
d_9 \hat{E}(\xi, t) \leq \hat{L}_3(\xi, t) \leq d_{10} \hat{E}(\xi, t).
\] (115)

Consequently, the last estimate lead to
\[
\frac{d}{dt} \hat{L}_3(\xi, t) + \Lambda_6 \rho(\xi) \hat{L}_3(\xi, t) \leq 0,
\] (116)

and for some \(\Lambda_6 > 0\). A simple application of Gronwall’s lemma to the estimate (116) yields
\[
\hat{L}_3(\xi, t) \leq \hat{L}_3(\xi, 0) e^{-\eta \rho(\xi) t}, \quad \text{for all} \quad t \geq 0.
\] (117)

Once again, (115) and (117) yield for some \(\Lambda_7 > 0\),
\[
\hat{E}(\xi, t) \leq \Lambda_7 \hat{E}(\xi, 0) e^{-\eta \rho(\xi) t}, \quad \text{for all} \quad t \geq 0.
\] (118)

The use of \(\hat{\mu}^0(s) = \hat{z}_0\), yields the desired estimate (105).

7.2. **Polynomially decaying kernel.** In this section, we assume that \(g\) is decaying polynomially for \(p^* < p < \infty\). As we did above, taking the same coefficients of (112) which are already fixed in such a way they are positive except the coefficient of \((\int_0^\infty g(s)|\hat{\mu}|^2 ds)\). Then for all \(t \geq 0\), and \(\xi \in \mathbb{R}^N\), the inequality (112) can be rewritten as:
\[
\frac{d}{dt} \hat{L}_3(\xi, t) \leq -m_1 \rho(\xi) \hat{E}(\xi, t) + m_2 \rho(\xi)|\xi|^2 \int_0^\infty g(s)|\hat{\mu}|^2 ds,
\] (119)

with
\[
m_2 = \left[ C(\lambda_1) + N_3 \left( C(\lambda_2) + (\beta - \tilde{\ell}) \right) \right] + \tau m_1
\]

and for some \(m_1 > 0\). To establish the result (106), we apply the same techniques used in Section 4. We estimate the integral term of (119) by using Hölder inequality, Lemma 2.3, (107), (108) and applying the method used in Section 4.1.2, we obtain the desired result. We omit the details. This finishes the proof of Proposition 3.

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