The Second Cohomology of Regular Semisimple Hessenberg Varieties from GKM Theory

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Received March 18, 2022; revised June 1, 2022; accepted June 21, 2022

Abstract—We describe the second cohomology of a regular semisimple Hessenberg variety by generators and relations explicitly in terms of GKM theory. The cohomology of a regular semisimple Hessenberg variety becomes a module of a symmetric group $S_n$ by the dot action introduced by Tymoczko. As an application of our explicit description, we give a formula describing the isomorphism class of the second cohomology as an $S_n$-module. Our formula is not exactly the same as the known formula by Chow or Cho, Hong, and Lee, but they are equivalent. We also discuss its higher degree generalization.

DOI: 10.1134/S0081543822020018

1. INTRODUCTION

Let $\text{Fl}(n)$ denote the variety of all complete flags in $\mathbb{C}^n$. A regular semisimple Hessenberg variety $\text{Hess}(S,h)$ is a smooth subvariety of $\text{Fl}(n)$. It is determined by a square matrix $S$ of order $n$ with distinct eigenvalues and by a function $h$ (called a Hessenberg function) from the set of integers $[n] = \{1, \ldots, n\}$ to itself satisfying the conditions

$$h(1) \leq h(2) \leq \ldots \leq h(n) \quad \text{and} \quad h(j) \geq j \quad \forall j \in [n].$$

The topology of $\text{Hess}(S,h)$ depends only on $h$; i.e., it does not depend on the choice of $S$. The maximal $\mathbb{C}^*$-torus $T$ in the general linear group $\text{GL}_n(\mathbb{C})$ that commutes with $S$ acts naturally on $\text{Hess}(S,h)$. Using this $T$-action, one can study the cohomology $H^*(\text{Hess}(S,h))$. Specifically, Tymoczko \cite{18} constructed an $S_n$-action (called the dot action) on $H^*(\text{Hess}(S,h))$ making it an $S_n$-module.

A theorem of Brosnan and Chow \cite{5} (solution of the Shareshian–Wachs conjecture \cite{15}) says that the graded $S_n$-module $H^*(\text{Hess}(S,h))$ is equivalent to the (graded) chromatic symmetric function $X_{G_h}(x,t)$ of a graph $G_h$ associated to $h$, where $X_{G_h}(x,t)$ is a polynomial in $t$ with symmetric functions in infinitely many variables $x = (x_1, x_2, \ldots)$ as coefficients. Moreover, it is shown in \cite{13} that the Stanley–Stembridge conjecture on the $e$-positivity of $(3+1)$-incomparability graphs is reduced to showing the $e$-positivity of the graph $G_h$ for any Hessenberg function $h$. Thus, we are led to the study of $H^*(\text{Hess}(S,h))$.

In this paper, we investigate the second cohomology $H^2(\text{Hess}(S,h))$ using GKM theory \cite{12} when $\text{Hess}(S,h)$ is connected. We exhibit its generators explicitly in terms of GKM theory and give a formula describing the isomorphism class of the $S_n$-module in terms of $h$. Prior to our work, Chow \cite{7} gave a formula for the coefficient of $t$ in $X_{G_h}(x,t)$ using $P$-tableaux. Through the

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Brosnan–Chow theorem mentioned above, Chow’s formula is equivalent to ours. After Chow’s work, Cho, Hong, and Lee [6] exhibited generators of \( H^2(\text{Hess}(S,h)) \) geometrically using the Bialynicki-Birula decomposition of \( \text{Hess}(S,h) \) and gave a formula describing the isomorphism class of the \( \mathfrak{S}_n \)-module \( H^2(\text{Hess}(S,h)) \). Their formula is also equivalent to ours. However, the methods are different and the relation between their generators of \( H^2(\text{Hess}(S,h)) \) and ours is unclear.

Since we obtain explicit generators of \( H^2(\text{Hess}(S,h)) \) in this paper, we are able to characterize the Hessenberg functions \( h \) for which the whole cohomology ring \( H^*(\text{Hess}(S,h)) \) is generated in degree 2 as a graded ring. It turns out that the graph \( G_h \) associated to such \( h \) is what is called a (double) lollipop [8, 14]. We will discuss this subject in a forthcoming paper.¹

As mentioned above, we describe \( H^2(\text{Hess}(S,h)) \) in terms of explicit generators and relations when \( \text{Hess}(S,h) \) is connected. It is known that \( \text{Hess}(S,h) \) is connected, in other words, the restriction map \( \iota^*: H^0(\text{Fl}(n)) \rightarrow H^0(\text{Hess}(S,h)) \) is an isomorphism, if and only if \( h(j) \geq j + 1 \) for any \( j \in [n-1] \). As a generalization of this setting, we consider the case where \( h(j) \geq j + d \) for any \( j \in [n-d] \), where \( d \geq 2 \) is an integer. In this case, we show that the restriction map \( \iota^*: H^{2p}(\text{Fl}(n)) \rightarrow H^{2p}(\text{Hess}(S,h)) \) is an isomorphism for \( p < d \) and describe \( H^{2d}(\text{Hess}(S,h)) \) in terms of explicit generators and relations.

The paper is organized as follows. In Section 2, we review necessary facts on regular semisimple Hessenberg varieties and labeled graphs associated to them. Section 3 discusses an inductive formula for computing the Poincaré polynomial of \( \text{Hess}(S,h) \) and a formula for the second Betti number of \( \text{Hess}(S,h) \) in terms of \( h \). In Section 4, we provide three types of elements in the \( T \)-equivariant cohomology \( H^*_T(\text{Hess}(S,h)) \) using GKM theory, discuss relations among them, and observe the dot action of \( \mathfrak{S}_n \) on them. We prove that these three types of elements together with \( H^2(BT) \) generate \( H^2_T(\text{Hess}(S,h)) \). In Section 5, we explicitly describe \( H^2(\text{Hess}(S,h)) \) in terms of generators and relations, and give a formula describing the isomorphism class of the \( \mathfrak{S}_n \)-module \( H^2(\text{Hess}(S,h)) \) in terms of \( h \). In Section 6 we discuss the generalization of the result on the second cohomology to the higher degree cohomology mentioned above.

2. REGULAR SEMISIMPLE HESSENBERG VARIETIES

2.1. Hessenberg variety. The flag variety \( \text{Fl}(n) \) is defined as the set of nested linear subspaces of \( \mathbb{C}^n \):

\[
\text{Fl}(n) = \{ V_i = (V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \ \forall i \in [n] \}.
\]

Given a square matrix \( A \) of order \( n \) and a function \( h: [n] \rightarrow [n] \) (called a Hessenberg function) satisfying

\[
h(1) \leq h(2) \leq \ldots \leq h(n) \quad \text{and} \quad h(j) \geq j \ \forall j \in [n],
\]

the Hessenberg variety \( \text{Hess}(A,h) \) is defined by

\[
\text{Hess}(A,h) := \{ V_i \in \text{Fl}(n) \mid A(V_j) \subset V_{h(j)} \ \forall j \in [n] \},
\]

where the matrix \( A \) is regarded as a linear transformation on \( \mathbb{C}^n \). We often express the Hessenberg function \( h \) as a vector \( (h(1), \ldots, h(n)) \) by listing the values of \( h \). When \( h = (n, \ldots, n) \), it is obvious from the definition that \( \text{Hess}(A,h) \) is the flag variety \( \text{Fl}(n) \) regardless of the choice of \( A \).

As illustrated in the following example, we can visualize a Hessenberg function \( h \) by drawing a configuration of shaded boxes on a square grid of size \( n \times n \), which consists of boxes in the \( i \)th row and the \( j \)th column satisfying \( i \leq h(j) \). Since it is assumed that \( j \leq h(j) \) for any \( j \in [n] \), the essential part is the shaded boxes below the diagonal.

¹A. Ayzenberg, M. Masuda, and T. Sato, “Regular semisimple Hessenberg varieties with cohomology generated in degree two” (in preparation).
Example 2.1. Let \( n = 5 \). The Hessenberg function \( h = (3, 3, 4, 5, 5) \) corresponds to the configuration of shaded boxes drawn in Fig. 1a, and the essential shaded boxes (i.e., those below the diagonal) are drawn in Fig. 1b.

The Hessenberg variety \( \text{Hess}(S, h) \) for a square matrix \( S \) of order \( n \) with distinct eigenvalues is said to be regular semisimple.

Theorem 2.2 [9]. The following holds:
(1) \( \text{Hess}(S, h) \) is a nonsingular variety;
(2) \( \dim \mathbb{C} \text{Hess}(S, h) = \sum_{j=1}^{n} (h(j) - j) \);
(3) \( \text{Hess}(S, h) \) is connected if and only if \( h(j) \geq j + 1 \) for all \( j \in [n - 1] \);
(4) \( H^{odd}(\text{Hess}(S, h)) = 0 \) and the \( 2k \)-th Betti number of \( \text{Hess}(S, h) \) is given by

\[
\# \{ w \in \mathfrak{S}_n \mid \ell_h(w) = k \}
\]

where

\[
\ell_h(w) = \# \{ 1 \leq j < i \leq n \mid w(j) > w(i), \ i \leq h(j) \}. \tag{2.1}
\]

Since \( S \) commutes with a maximal torus \( T \) of \( \text{GL}_n(\mathbb{C}) \), the restricted action of \( T \) on \( \text{Fl}(n) \) leaves \( \text{Hess}(S, h) \) invariant. One sees that \( \text{Hess}(S, h)^T = \text{Fl}(n)^T = \mathfrak{S}_n \).

2.2. Equivariant cohomology. We will briefly review equivariant cohomology. For a \( T \)-space \( X \), the equivariant cohomology \( H^*_T(X) \) is defined as

\[
H^*_T(X) := H^*(ET \times_T X)
\]

where \( ET \to BT \) is the universal principal \( T \)-bundle and \( ET \times_T X \) is the orbit space of \( ET \times X \) by the diagonal \( T \)-action. Since \( T \) is isomorphic to \( (\mathbb{C}^*)^n \), \( BT \) is homeomorphic to \( (\mathbb{C}P^\infty)^n \) and hence \( H^*(BT) \) is a polynomial ring in \( n \) elements of \( H^2(BT) \) which form a basis of \( H^2(BT) \). Note that for a one-point space \( pt \), we have

\[
H^*_T(pt) = H^*(BT).
\]

Since the \( T \)-action on \( ET \) is free, the projection \( ET \times X \to ET \) on the first factor induces a fibration

\[
X \xrightarrow{\iota} ET \times_T X \xrightarrow{\pi} ET/T = BT.
\]

The equivariant cohomology \( H^*_T(X) \) is not only a ring but also an algebra over \( H^*(BT) \) through \( \pi^*: H^*(BT) \to H^*_T(X) \). As is easily seen, the restriction map \( \iota^*: H^*_T(X) \to H^*(X) \) sends \( H^2(BT) \) to zero. Therefore, it induces a ring homomorphism

\[
H^*_T(X)/(H^2(BT)) \to H^*(X), \tag{2.3}
\]

where \( (H^2(BT)) \) denotes the ideal generated by \( \pi^*(H^2(BT)) \). If \( H^{odd}(X) = 0 \) (this is the case when \( X = \text{Hess}(S, h) \)), then the map (2.3) above is an isomorphism.
2.3. GKM theory and labeled graph. We choose and fix a basis of \( H^2(BT) \) and denote its elements by \( t_1, \ldots, t_n \); they correspond to the choice of coordinates in the ambient space \( \mathbb{C}^n \) of flags. Since \( H^\text{odd}(\text{Hess}(S, h)) = 0 \), the torus action is cohomologically equivariantly formal [12]. Therefore, the inclusion \( \text{Hess}(S, h)^T \to \text{Hess}(S, h) \) induces the homomorphism
\[
H^*_T(\text{Hess}(S, h)) \to H^*_T(\text{Hess}(S, h)^T) = \bigoplus_{w \in \mathfrak{S}_n} H^*_T(w) = \bigoplus_{w \in \mathfrak{S}_n} \mathbb{Z}[t_1, \ldots, t_n] = \text{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1, \ldots, t_n]),
\]
which is injective, where \( \text{Map}(P, Q) \) denotes the set of all maps from \( P \) to \( Q \). Since the restriction map above is injective, we think of \( H^*_T(\text{Hess}(S, h)) \) as a subset of \( \text{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1, \ldots, t_n]) \).

**Proposition 2.3** [18]. An element \( f \in \text{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1, \ldots, t_n]) \) is in \( H^*_T(\text{Hess}(S, h)) \) if and only if
\[
f(v) \equiv f(w) \pmod{t_{w(i)} - t_{w(j)}} \quad \text{whenever} \quad v = w \cdot (i, j) \quad \text{for} \quad j < i \leq h(j),
\]
where \((i, j)\) denotes the transposition interchanging \( i \) and \( j \).

To a Hessenberg function \( h \), one associates a graph \((V, E)\) with a label on the edge set \( E \),
\[
\alpha : E \to H^2(BT) \setminus \{0\},
\]
where
\begin{enumerate}
\item \( V = \mathfrak{S}_n \);
\item \( E = \{\{v, w\} \mid v, w \in V, \ v = w \cdot (i, j) \text{ for some } j < i \leq h(j)\} \);
\item \( \alpha(\{v, w\}) = t_{w(i)} - t_{w(j)} \) up to sign for \( v = w \cdot (i, j) \).
\end{enumerate}

We denote the triple \((V, E, \alpha)\) by \( \Gamma(h) \) and call it a labeled graph associated to \( h \). This is a slight variant of the notion of GKM graph. The set of elements in \( \text{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1, \ldots, t_n]) \) satisfying the congruence relation in Proposition 2.3 is sometimes called the graph cohomology of \( \Gamma(h) \).

Proposition 2.3 says that \( H^*_T(\text{Hess}(S, h)) \) agrees with the graph cohomology of \( \Gamma(h) \). In particular, \( H^*_T(\text{Hess}(S, h)) \) is independent of the choice of the regular semisimple matrix \( S \). Notice that even the equivariant diffeomorphism type of \( \text{Hess}(S, h) \) is independent of \( S \) (see [2]).

We often think of \( t_i \) as an element of \( \text{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1, \ldots, t_n]) \) by regarding it as a constant map. Obviously, \( t_i \) satisfies the congruence relation in Proposition 2.3, so it is in \( H^2_T(\text{Hess}(S, h)) \). Then
\[
H^*(\text{Hess}(S, h)) = H^*_T(\text{Hess}(S, h))/(t_1, \ldots, t_n),
\]
because \( H^\text{odd}(\text{Hess}(S, h)) = 0 \), where \( (t_1, \ldots, t_n) \) is the ideal generated by \( t_1, \ldots, t_n \).

**Example 2.4.** Let \( n = 3 \). For \( h = (2, 3, 3) \) and \( h' = (3, 3, 3) \), the corresponding labeled graphs \( \Gamma(h) \) and \( \Gamma(h') \) are depicted in Fig. 2, where we use the one-line notation for each vertex.

Both graphs \( \Gamma(h) \) and \( \Gamma(h') \) in Example 2.4 are connected. In general, it is not difficult to see that \( \Gamma(h) \) is connected if and only if \( h(j) \geq j + 1 \) for any \( j \in [n - 1] \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{The labeled graphs \( \Gamma(h) \) and \( \Gamma(h') \).}
\end{figure}
3. The Second Betti Number of Hess($S, h$)

In this section, we give an inductive formula for computing the Poincaré polynomial of Hess($S, h$) using Theorem 2.2(4) and apply it to obtain an explicit formula for the second Betti number of Hess($S, h$) in terms of the Hessenberg function $h$.

For a space $X$ such that $H^{\text{odd}}(X) = 0$ and the rank of $H^*(X)$ over $\mathbb{Z}$ is finite, we define

$$\text{Poin}(X, \sqrt{q}) := \sum_{r=0}^{\infty} b_{2r}(X)q^r$$

where $b_{2r}(X)$ denotes the $2r$th Betti number over $\mathbb{Z}$. Let $h^j$ be the Hessenberg function obtained by removing all the boxes in the $j$th row and all the boxes in the $j$th column (Fig. 3). To be precise,

$$h^j(i) = \begin{cases} h(i) & \text{if } i < j, h(i) < j, \\ h(i) - 1 & \text{if } i < j, h(i) \geq j, \\ h(i + 1) - 1 & \text{if } i \geq j. \end{cases}$$

**Proposition 3.1.** With the above notation, we have

$$\text{Poin}(\text{Hess}(S, h), \sqrt{q}) = \sum_{j=1}^{n} q^{h(j) - j} \text{Poin}(\text{Hess}(S', h^j), \sqrt{q})$$

where $S'$ denotes a matrix of order $n-1$ with distinct eigenvalues.

**Proof.** It follows from Theorem 2.2(4) that

$$\text{Poin}(\text{Hess}(S, h), \sqrt{q}) = \sum_{w \in S_n} q^{\ell_h(w)}$$

where

$$\ell_h(w) = \# \{ 1 \leq j < i \leq n \mid w(j) > w(i), \ i \leq h(j) \}.$$ 

For $j \in [n]$ we set

$$\mathcal{S}_n^j := \{ w \in S_n \mid w(j) = n \}$$

and consider the following decomposition of $S_n$:

$$S_n = \mathcal{S}_n^1 \sqcup \mathcal{S}_n^2 \sqcup \ldots \sqcup \mathcal{S}_n^n.$$ 

If $w$ is in $\mathcal{S}_n^j$, then $w(j) > w(i)$ for any $j < i \leq h(j)$ since $w(j) = n$. Therefore, if we denote by $w^j$ the permutation on $[n-1]$ obtained by removing $w(j) = n$ from $w$, then we have

$$\ell_h(w) = h(j) - j + \ell_{h^j}(w^j).$$

This together with (3.1) implies the formula in the proposition. □

![Fig. 3. The configuration corresponding to $h^j$.](image)
For a Hessenberg function $h$, we consider the following two sets:

$$\perp(h) := \{ j \in [n-1] \mid h(j-1) = h(j) = j+1 \},$$

$$L(h) := \{ j \in [n-1] \mid h(j-1) = j \text{ and } h(j) = j+1 \},$$

where we set $h(0) = 1$.

**Lemma 3.2.** Suppose that $h(j) \geq j + 1$ for $j \in [n-1]$. Then the second Betti number $b_2(\text{Hess}(S, h))$ of Hess($S, h$) can be expressed as

$$b_2(\text{Hess}(S, h)) = \sum_{j \in L(h)} \binom{n}{j} + (n-1)|\perp(h)| - |L(h)|.$$  \hspace{1cm} (3.2)

**Example 3.3.** For the Hessenberg function $h = (3, 3, 4, 5, 5)$ in Example 2.1, we have $\perp(h) = \{2\}$ and $L(h) = \{3, 4\}$. Therefore,

$$b_2(\text{Hess}(S, h)) = \binom{5}{3} + \binom{5}{4} + (5-1) \cdot 1 - 2 = 17.$$  \hspace{1cm} (3.3)

**Proof of Lemma 3.2.** We prove the lemma by induction on $n$. When $n = 2$, $h = (2, 2)$ since $h(j) \geq j + 1$ for $j \in [n-1]$ by assumption. Therefore, Hess($S,h$) is CP$^1$ while $\perp(h) = \emptyset$ and $L(h) = \{1\}$. This shows that the lemma holds when $n = 2$.

Suppose that $n \geq 3$. It follows from Proposition 3.1 that

$$b_2(\text{Hess}(S, h)) = \sum_{h(j) = j+1} b_0(\text{Hess}(S', h^j)) + b_2(\text{Hess}(S', h^n)).$$  \hspace{1cm} (3.4)

Here, Hess($S', h^j$) is connected if $h(j-1) = j + 1$ and has $\binom{n-1}{j}$ connected components if $h(j-1) = j$ (a detailed description of the connected components can be found in [17]). This means that

$$b_0(\text{Hess}(S', h^j)) = \begin{cases} 1 & \text{if } j \in \perp(h), \\ \binom{n-1}{j-1} & \text{if } j \in L(h), \end{cases}$$  \hspace{1cm} (3.5)

since $h(j) = j + 1$ for $j$ in the sum of (3.3).

On the other hand, by the induction assumption, we have

$$b_2(\text{Hess}(S', h^n)) = \sum_{j \in L(h^n)} \binom{n-1}{j} + (n-2)|\perp(h^n)| - |L(h^n)|.$$  \hspace{1cm} (3.6)

Here, by looking at the $(n-2, n-1)$ and $(n-1, n)$ boxes in the configuration associated to $h$, one sees that

$$\perp(h^n) = \begin{cases} \perp(h) & \text{if } h(n-2) = n-1, \\ \perp(h) \setminus \{n-1\} & \text{if } h(n-2) = n, h(n-3) = n-2, \\ (\perp(h) \setminus \{n-1\}) \cup \{n-2\} & \text{if } h(n-2) = n, h(n-3) \geq n-1, \end{cases} \hspace{1cm} (3.7)$$

$$L(h^n) = \begin{cases} L(h) \setminus \{n-1\} & \text{if } h(n-2) = n-1, \\ L(h) \cup \{n-2\} & \text{if } h(n-2) = n, h(n-3) = n-2, \\ L(h) & \text{if } h(n-2) = n, h(n-3) \geq n-1. \end{cases}$$  \hspace{1cm} (3.8)

We consider three cases according to the cases above and plug (3.6) and (3.7) into the right-hand side of (3.5) in each case. Then, this together with (3.4) will show that the right-hand side of (3.3)
agrees with the right-hand side of the identity in the lemma. For instance, when \( h(n - 2) = n - 1 \), it follows from (3.4)–(3.7) that the right-hand side of (3.3) turns into

\[
\sum_{j \in L(h)} \binom{n - 1}{j - 1} + \left| \downarrow(h) \right| + \sum_{j \in L(h) \setminus \{n-1\}} \binom{n - 1}{j} + (n - 2)\left| \downarrow(h) \right| - |L(h) \setminus \{n-1\}|
\]

\[
= \sum_{j \in L(h)} \left( \binom{n - 1}{j - 1} + \binom{n - 1}{j} \right) - \binom{n - 1}{n - 1} + (n - 1)\left| \downarrow(h) \right| - |L(h)| + 1
\]

\[
= \sum_{j \in L(h)} \binom{n}{j} + (n - 1)\left| \downarrow(h) \right| - |L(h)|,
\]

which coincides with the right-hand side of the identity in the lemma. The other two cases can be proved similarly, so we omit the proof. \( \Box \)

4. GENERATORS OF \( H^2(\text{Hess}(S, h)) \)

4.1. Three types of elements in \( H^2(\text{Hess}(S, h)) \). The following three types of elements play a role in our argument.

**Lemma 4.1.** The elements \( x_i, y_{j,k} \), and \( \tau_A \) in \( \text{Map}(\mathfrak{g}_n, \mathbb{Z}[t_1, \ldots, t_n]) \) defined below are in \( H^2_{\mathbb{T}}(\text{Hess}(S, h)) \):

1. for \( i \in [n] \), \( x_i(w) := t_{w(i)} \);
2. for \( j \in \downarrow(h) \) and \( k \in [n] \),

\[
y_{j,k}(w) := \begin{cases} t_k - t_{w(j+1)} & \text{if } k \in \{w(1), \ldots, w(j)\}, \\ 0 & \text{otherwise}; \end{cases}
\]

3. for \( A \subset [n] \) with \( |A| \in L(h) \),

\[
\tau_A(w) := \begin{cases} t_{w(j)} - t_{w(j+1)} & \text{if } \{w(1), \ldots, w(j)\} = A, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** What we have to do is to check that the elements in the lemma satisfy the congruence relation in Proposition 2.3. The check for \( x_i \) is straightforward and that for \( y_{j,k} \) is done in [1, Lemma 10.2] in a more general setting. Therefore, we will check the congruence relation for \( \tau_A \). We set \( j = |A| \).

Suppose that \( v = w \cdot (r, s) \) for \( r < s \leq h(r) \). We consider three cases.

If \( r = j \), then \( s = j + 1 \) because \( h(r) = h(j) = j + 1 \), so that \( t_{w(r)} - t_{w(s)} = t_{w(j)} - t_{w(j+1)} \). Therefore, looking at the definition of \( \tau_A \), we see that the congruence relation is obviously satisfied in this case.

If \( r \leq j - 1 \), then \( s \leq h(r) \leq h(j - 1) = j \), where \( h(j - 1) = j \) is because \( j \in L(h) \). Therefore, \( \{w(1), \ldots, w(j)\} = \{v(1), \ldots, v(j)\} \) and hence \( \tau_A(w) = \tau_A(v) \); thereby the congruence relation is trivially satisfied.

If \( r \geq j + 1 \), then \( s \geq j + 2 \) as \( s > r \). Therefore, \( \{w(1), \ldots, w(j)\} = \{v(1), \ldots, v(j)\} \) in this case, too. Hence the congruence relation is satisfied. \( \Box \)

**Remark 4.2.** 1. The elements \( x_i \) lie in \( H^2_{\mathbb{T}}(\text{Hess}(S, h)) \) for any Hessenberg function. Indeed, when \( h = (n, \ldots, n) \), \( \text{Hess}(S, h) \) is the flag variety \( \text{Fl}(n) \) and \( x_i \) is the equivariant first Chern class of the line bundle \( E_i/E_{i-1} \) where \( E_i \) is the ith tautological vector bundle over the flag variety \( \text{Fl}(n) \):

\[
E_i := \{(V_*, v) \in \text{Fl}(n) \times \mathbb{C}^n \mid v \in V_i\}.
\]
2. The element $y_{j,k}$ exists in $H_T^2(\text{Hess}(S, h))$ for any $j$ with $h(j) = j + 1$ by the same definition as above, but for our purpose it suffices to consider $y_{j,k}$ for $j \in \perp(h)$.

**Lemma 4.3.** For $x_i$, $y_{j,k}$, and $\tau_A$ in Lemma 4.1, the following holds:

1. $\sum_{i=1}^n x_i = \sum_{i=1}^n t_i$;
2. $\sum_{k=1}^n y_{j,k} = \sum_{i=1}^j i(x_i - x_{i+1}) = x_1 + \ldots + x_j - jx_{j+1}$;
3. $\sum_{|A|=j} \tau_A = x_j - x_{j+1}$;
4. let $m$ be the maximum element in $\perp(h)$ if $\perp(h) \neq \emptyset$ and $m = 0$ otherwise; then
   $$y_{m,k} + \sum_{k \in A, m<|A|\leq n-1} \tau_A = t_k - x_n$$
   for any $k \in [n]$.

   where we set $y_{0,k} = 0$ and the sum is 0 when $m = n - 1$.

**Proof.** For each identity, we check that the left- and right-hand sides take the same value at every $w \in \mathfrak{S}_n$. The check for the identities in (1)-(3) is straightforward, so we leave it to the reader and will check the identity in (4).

It follows from the definition of $\tau_A$ in Lemma 4.1 that

$$\left(\sum_{k \in A, |A|=j} \tau_A\right)(w) = \begin{cases} t_w(j) - t_w(j+1) & \text{if } k \in \{w(1), \ldots, w(j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\left(\sum_{k \in A, m<|A|\leq n-1} \tau_A\right)(w) = \begin{cases} t_w(m+1) - t_w(n) & \text{if } k \in \{w(1), \ldots, w(m)\}, \\ t_k - t_w(n) & \text{if } k \in \{w(m+1), \ldots, w(n-1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

This, together with the definition of $y_{m,k}$, that is,

$$y_{m,k}(w) = \begin{cases} t_k - t_w(m+1) & \text{if } k \in \{w(1), \ldots, w(m)\}, \\ 0 & \text{otherwise,} \end{cases}$$

shows that the left-hand side of the identity in (4) evaluated at $w$ agrees with $t_k - t_w(n)$, proving the desired identity. \qed

### 4.2. Dot action.

We consider an action of $\sigma \in \mathfrak{S}_n$ on $\mathbb{Z}[t_1, \ldots, t_n]$ sending $t_i$ to $t_{\sigma(i)}$ for $i \in [n]$ and define an action of $\sigma \in \mathfrak{S}_n$ on $f \in \text{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1, \ldots, t_n])$ by

$$(\sigma \cdot f)(w) := \sigma(f(\sigma^{-1}w)).$$

One can check that if $f$ is in $H_T^*(\text{Hess}(S, h))$, then so is $\sigma \cdot f$. The action of $\mathfrak{S}_n$ on $H_T^*(\text{Hess}(S, h))$ preserves the ideal $(t_1, \ldots, t_n)$ generated by $t_1, \ldots, t_n$, so the action descends to an action of $\mathfrak{S}_n$ on

$$H^*(\text{Hess}(S, h)) = H_T^*(\text{Hess}(S, h))/(t_1, \ldots, t_n).$$

This action, called the **dot action**, was introduced by Tymoczko [18].

**Lemma 4.4.** Let $x_i$, $y_{j,k}$, and $\tau_A$ be as in Lemma 4.1. Then, for $\sigma \in \mathfrak{S}_n$, we have

$$\sigma \cdot x_i = x_i, \quad \sigma \cdot y_{j,k} = y_{j,\sigma(k)}, \quad \sigma \cdot \tau_A = \tau_{\sigma(A)}.$$

**Proof.** The proof is straightforward. Indeed, we have

$$(\sigma \cdot x_i)(w) = \sigma(x_i(\sigma^{-1}w)) = \sigma(t_{\sigma^{-1}w(i)}) = t_{\sigma^{-1}w(i)} = t_{w(i)}.$$
proving \( \sigma \cdot x_i = x_i \). As for \( y_{j,k} \), we have

\[
(\sigma \cdot y_{j,k})(w) = \sigma(y_{j,k}(\sigma^{-1}w)) = \begin{cases} 
\sigma(t_k - t_{\sigma^{-1}w(k+1)}) & \text{if } k \in \{\sigma^{-1}w(1), \ldots, \sigma^{-1}w(j)\}, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\tau_A(w) = \begin{cases} 
\tau_A(\sigma^{-1}w) & \text{if } \{\sigma^{-1}w(1), \ldots, \sigma^{-1}w(j)\} = A, \\
0 & \text{otherwise}
\end{cases}
\]

proving \( \sigma \cdot y_{j,k} = y_{j,\sigma(k)} \). Similarly, for \( \tau_A \) with \( |A| = j \), we have

\[
(\sigma \cdot \tau_A)(w) = \sigma(\tau_A(\sigma^{-1}w)) = \begin{cases} 
\sigma(t_{\sigma^{-1}w(j)} - t_{\sigma^{-1}w(j+1)}) & \text{if } \{\sigma^{-1}w(1), \ldots, \sigma^{-1}w(j)\} = A, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\tau_A(w) = \begin{cases} 
t_w(j) - t_{w(j+1)} & \text{if } \{w(1), \ldots, w(j)\} = \sigma(A), \\
0 & \text{otherwise}
\end{cases}
\]

proving \( \sigma \cdot \tau_A = \tau_{\sigma(A)} \).

4.3. Generators of \( H^2_\Gamma(\text{Hess}(S,h)) \). Recall that \( \perp(h) \) and \( L(h) \) were defined in (3.2), \( y_{j,k} \) is defined for \( j \in \perp(h) \), and \( \tau_A \) is defined for \( A \subset [n] \) with \( |A| \in L(h) \).

**Proposition 4.5.** Suppose that \( h(j) \geq j + 1 \) for any \( j \in [n-1] \). Then \( H^2_\Gamma(\text{Hess}(S,h)) \) is generated by \( t_i, x_i \) (\( i \in [n] \)), \( y_{j,k} \) (\( j \in \perp(h), k \in [n] \)), and \( \tau_A \) (\( |A| \in L(h) \)).

**Proof.** Recall that \( \Gamma(h) \) is the labeled graph introduced at the end of Section 3. As before, we consider the decomposition of \( \mathfrak{G}_n \),

\[
\mathfrak{G}_n = \mathfrak{G}_n^1 \sqcup \mathfrak{G}_n^2 \sqcup \ldots \sqcup \mathfrak{G}_n^n, \quad \text{where } \mathfrak{G}_n^j := \{w \in \mathfrak{G}_n \mid w(j) = n\},
\]

and the Hessenberg function \( h^j \) obtained by removing all the boxes in the \( j \)th row and all the boxes in the \( j \)th column from the configuration corresponding to \( h \). Note that the full subgraph of \( \Gamma(h) \) with vertices \( \mathfrak{G}_n^j = \Gamma(h^j) \).

We prove the proposition by induction on \( n \) following the idea developed in [10]. The idea is also used in [3]. Let \( z \) be an arbitrary element of \( H^2_\Gamma(\text{Hess}(S,h)) \).

**Step 1.** Since \( \mathfrak{G}_n^j \) is isomorphic to \( \mathfrak{G}_{n-1} \), it follows from the inductive assumption that \( H^2_\Gamma(\text{Hess}(S',h^n)) \) is generated by \( t_i \)'s and the elements corresponding to \( x_\ast, y_\ast, \) and \( \tau_\ast \), where \( S' \) is a square matrix of order \( n - 1 \) with distinct eigenvalues. Indeed, those elements in \( H^2_\Gamma(\text{Hess}(S',h^n)) \), denoted with \( (n) \) as superscript, are defined as follows:

\[
x_i^{(n)}(w) := x_i(w) \quad \text{for } i \in [n-1],
\]

\[
y_{j,k}^{(n)}(w) := y_{j,k}(w) \quad \text{for } j \in \perp(h^n),
\]

\[
\tau_B^{(n)}(w) := \tau_B(w) \quad \text{for } B \subset [n-1], |B| \in L(h^n),
\]

where \( w \in \mathfrak{G}_n^j \). This shows that any element of \( H^2_\Gamma(\text{Hess}(S',h^n)) \) is the restriction image of a linear combination of \( t_\ast, x_\ast, y_\ast, \) and \( \tau_\ast \) to \( H^2_\Gamma(\text{Hess}(S',h^n)) \). Therefore, we may assume that \( z = 0 \) on \( \mathfrak{G}_n^j \) by subtracting an appropriate linear combination of \( t_\ast, x_\ast, y_\ast, \) and \( \tau_\ast \) from \( z \).

**Step 2.** Suppose that \( z = 0 \) on \( \mathfrak{G}_n^{r+1} \sqcup \ldots \sqcup \mathfrak{G}_n^n \) for some \( 1 \leq r \leq n - 1 \). Then we will show that \( z \) minus an appropriate linear combination of \( t_\ast, x_\ast, y_\ast, \) and \( \tau_\ast \), vanishes on \( \mathfrak{G}_n^{r+1} \sqcup \ldots \sqcup \mathfrak{G}_n^n \). We consider two cases.
Case 1: \( h(r) \geq r + 2 \) (so \( r < n - 1 \)). In this case, the \( (r + 1, r) \) and \( (r + 2, r) \) boxes are shaded in the configuration associated to \( h \). This means that at each vertex \( w \in \mathcal{G}_n^r \), there are edges in \( \Gamma(h) \) emanating from \( w \) to \( w \cdot (r, r + 1) \in \mathcal{G}_n^{r+1} \) and \( w \cdot (r, r + 2) \in \mathcal{G}_n^{r+2} \), where the labels on those edges are respectively \( t_w(r) - t_w(r+1) \) and \( t_w(r) - t_w(r+2) \) up to sign. Since \( z = 0 \) on \( \mathcal{G}_n^{r+1} \sqcup \ldots \sqcup \mathcal{G}_n^n \), \( z(w) \) must be divisible by these linear polynomials. However, since the cohomological degree of \( z \) is 2, this implies that \( z \) must be 0 on \( \mathcal{G}_n^r \).

Case 2: \( h(r) = r + 1 \). In this case, each vertex \( w \in \mathcal{G}_n^r \) is joined by an edge to a vertex \( w \cdot (r, r + 1) \) of \( \mathcal{G}_n^{r+1} \). Since the label on the edge is \( t_w(r) - t_w(r+1) = t_n - t_{w(r+1)} \) up to sign and \( z = 0 \) on \( \mathcal{G}_n^{r+1} \), \( z(w) \) for \( w \in \mathcal{G}_n^r \) is a constant multiple of \( t_n - t_{w(r+1)} \). Note that \( h(r - 1) \leq h(r) = r + 1 \) and

\[
t_n - t_{w(r+1)} = \begin{cases} y_{r,n}(w) & \text{when } h(r - 1) = r + 1, \\ \tau_{(w(1),\ldots,w(r))}(w) & \text{when } h(r - 1) = r.\end{cases}
\]

We distinguish two cases according to the above.

(i) Let \( h(r - 1) = r + 1 \). In this case, \( h'(j) \geq j + 1 \) for any \( j \in [n - 2] \), so the graph \( \Gamma(h') \) is connected. By the observation above, we have \( z(w) = c_w y_{r,n}(w) \) for \( w \in \mathcal{G}_n^r \) with some integer \( c_w \).

Claim. \( c_w = c_v \) for any \( w, v \in \mathcal{G}_n^r \).

Proof. Since \( \Gamma(h') \) is connected, it suffices to prove the identity for a pair of \( w \) and \( v \) joined by an edge of \( \Gamma(h') \). Then, \( v = w \cdot (p, q) \) for some \( p, q \in [n] \setminus \{r\} \) and

\[
z(w) - z(v) = c_w y_{r,n}(w) - c_v y_{r,n}(v)
\]

\[
= c_w(t_n - t_{w(r+1)}) - c_v(t_n - t_{v(r+1)})
\]

\[
= (c_w - c_v) t_n - c_w t_{w(r+1)} + c_v t_{v(r+1)},
\]

which must be divisible by \( t_{w(p)} - t_{w(q)} = t_{v(q)} - t_{v(p)} \). Here, \( n = w(r) = v(r) \), \( p \neq r \), and \( q \neq r \), so the coefficient \( c_w - c_v \) of \( t_n \) in (4.2) must be zero. □

By the claim, we may write \( c_w \) as \( c \) and \( z - cy_{r,n} = 0 \) on \( \mathcal{G}_n^r \). Furthermore, \( y_{r,n} = 0 \) on \( \mathcal{G}_n^{r+1} \sqcup \ldots \sqcup \mathcal{G}_n^n \) because \( y_{r,n}(v) = 0 \) for \( v \in \mathcal{G}_n^r \) with \( n \in \{v(r + 1),\ldots,v(n)\} \) by the definition of \( y_{r,n} \) in Lemma 4.1(2). Therefore, \( z - cy_{r,n} = 0 \) on \( \mathcal{G}_n^r \sqcup \ldots \sqcup \mathcal{G}_n^n \).

(ii) Let \( h(r - 1) = r \). In this case, the graph \( \Gamma(h') \) is disconnected since \( h'(r - 1) = r - 1 \). Indeed, there are \( \binom{n}{r-1} \) connected components because \( h'(j) \geq j + 1 \) for any \( 1 \leq j \leq n - 2 \) with \( j \neq r - 1 \). The vertex set of a connected component of \( \Gamma(h') \) is

\[
\mathcal{G}_n^r(A) := \{ w \in \mathcal{G}_n^r \mid \{w(1),\ldots,w(r)\} = A \}
\]

for some \( n \in A \subset [n] \) with \( |A| = r \). On this connected component, \( z(w) = c_A \tau_A(w) \), \( w \in \mathcal{G}_n^r(A) \), with some integer \( c_A \), where \( c_A \) is independent of \( w \in \mathcal{G}_n^r(A) \) by an argument similar to the claim above, and \( \tau_A \) vanishes on \( \bigcup_{B \neq A} \mathcal{G}_n^r(B) = \mathcal{G}_n^r \setminus \mathcal{G}_n^r(A) \). Therefore,

\[
z - \sum_{n \in A \subset [n], |A| = r} c_A \tau_A = 0 \quad \text{on } \mathcal{G}_n^r(A).
\]

(4.3)

Furthermore, since \( n \in A \) and \( |A| = r \), \( \tau_A \) in (4.3) vanishes on \( \mathcal{G}_n^{r+1} \sqcup \ldots \sqcup \mathcal{G}_n^n \) by the definition of \( \tau_A \) in Lemma 4.1(3). Therefore, the left-hand side in (4.3) vanishes on \( \mathcal{G}_n^r \sqcup \ldots \sqcup \mathcal{G}_n^n \).

The inductive argument developed above shows that one can change \( z \) to 0 on the whole set \( \mathcal{G}_n \) by subtracting an appropriate linear combination of \( t_*, x_*, y_* \), and \( \tau_* \). Since \( z \) is an arbitrary element of \( H^2_T(\text{Hess}(S, h)) \), this proves the proposition. □
5. STRUCTURE OF $H^2(\text{Hess}(S, h))$

5.1. Explicit presentation of $H^2(\text{Hess}(S, h))$. The generators of $H^2(\text{Hess}(S, h))$ in Proposition 4.5 can be reduced. Indeed, it follows from Lemma 4.3(4) that we can drop $y_{n-1,k}$ when $n - 1 \in \perp(h)$ and $\tau_A$ with $|A| = n - 1$ when $n - 1 \in L(h)$; i.e., it suffices to consider $\perp(h) \setminus \{n - 1\}$ and $L(h) \setminus \{n - 1\}$ as index sets of $y_{j,k}$ and $\tau_A$. This is also true for the ordinary cohomology $H^2(\text{Hess}(S, h))$ because

$$H^2(\text{Hess}(S, h)) = H^2(\text{Hess}(S, h))/\mathbb{Z}\langle t_1, \ldots, t_n \rangle$$

where $\mathbb{Z}\langle t_1, \ldots, t_n \rangle$ denotes the module generated by $t_1, \ldots, t_n$ over $\mathbb{Z}$. Thus, we have a surjective homomorphism

$$\Phi: \mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle \rightarrow H^2(\text{Hess}(S, h))$$

sending $X_i$ to $x_i$, $Y_{j,k}$ to $y_{j,k}$, and $T_A$ to $\tau_A$, where

1. $i \in [n]$;
2. $j \in \perp(h) \setminus \{n - 1\}$ and $k \in [n]$;
3. $A \subset [n]$, $|A| \in L(h) \setminus \{n - 1\}$.

Let $U$ be the submodule of the free module $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle$ generated by the following elements:

(R1) $X_1 + \ldots + X_n$;
(R2) $\sum_{k=1}^n Y_{j,k} - (X_1 + \ldots + X_j - jX_{j+1})$ for $j \in \perp(h) \setminus \{n - 1\}$;
(R3) $\sum_{|A|=j} T_A - (X_j - X_{j+1})$ for $j \in L(h) \setminus \{n - 1\}$.

By Lemma 4.3, the submodule $U$ is mapped to zero by $\Phi$. Therefore, the map $\Phi$ induces a surjective homomorphism

$$\Phi: \mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle/ U \rightarrow H^2(\text{Hess}(S, h)).$$

With this understanding, we have the following.

Theorem 5.1. Suppose that $h(j) \geq j + 1$ for any $j \in [n - 1]$. Then, the map $\Phi$ in (5.1) is an isomorphism.

Proof. First we note that the quotient module $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle/ U$ is free. Indeed, let $V$ be the submodule of $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle$ generated by

1. $X_i$, $i \neq n$;
2. $Y_{j,k}$, $k \neq n$;
3. $T_A$, $A \neq |A|$.

Then $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle = U \oplus V$. Therefore, $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle/ U \cong V$, which is free.

The rank of the free module $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle$ is

$$n + n|\perp(h) \setminus \{n - 1\}| + \sum_{j \in L(h) \setminus \{n - 1\}} \binom{n}{j} = n|\perp(h)| + \sum_{j \in L(h)} \binom{n}{j},$$

while the elements in (R1)–(R3) are linearly independent, so that the rank of $U$ is

$$1 + |\perp(h) \setminus \{n - 1\}| + |L(h) \setminus \{n - 1\}| = |\perp(h)| + |L(h)|.$$

Therefore, the rank of the source module $\mathbb{Z}\langle X_i, Y_{j,k}, T_A \rangle/ U$ of $\Phi$ is

$$(n - 1)|\perp(h)| + \sum_{j \in L(h)} \binom{n}{j} - |L(h)|,$$
which agrees with the second Betti number of Hess$(S, h)$ by Lemma 3.2. This implies that $\overline{\Phi}$ is an isomorphism because $\overline{\Phi}$ is surjective and both the source and target modules of $\overline{\Phi}$ are free. □

Following Lemma 4.4, we define an action of $\mathfrak{S}_n$ on the variables $X_i, Y_{j,k}$, and $T_A$ by

$$\sigma \cdot X_i := X_i, \quad \sigma \cdot Y_{j,k} := Y_{j,\sigma(k)}, \quad \sigma \cdot T_A := T_{\sigma(A)}$$

and extend the action to the free module $\mathbb{Z}(X_i, Y_{j,k}, T_A)$ linearly. Then, $\mathfrak{S}_n$ acts on the submodule $U$ trivially, so that the $\mathfrak{S}_n$-action on $\mathbb{Z}(X_i, Y_{j,k}, T_A)$ descends to an $\mathfrak{S}_n$-action on the quotient $\mathbb{Z}(X_i, Y_{j,k}, T_A)/U$ and the isomorphism $\overline{\Phi}$ becomes $\mathfrak{S}_n$-equivariant.

### 5.2. $\mathfrak{S}_n$-module structure on $H^2(\text{Hess}(S, h))$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of $n$, denoted by $\lambda \vdash n$, and let $\mathfrak{S}_{\lambda} = \mathfrak{S}_{\lambda_1} \times \ldots \times \mathfrak{S}_{\lambda_\ell}$ be the Young subgroup of $\mathfrak{S}_n$ associated to $\lambda$. We set

$$M^\lambda = \mathbb{C}[(\mathfrak{S}_n) \otimes \mathbb{C}[\mathfrak{S}_\lambda]]$$

where $\mathbb{C}[G]$ denotes the group ring of a finite group $G$ over $\mathbb{C}$. As is well-known, $\{M^\lambda \mid \lambda \vdash n\}$ forms an additive basis of the complex representation ring $R(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ (see [11, § 7.3]). Therefore, any $\mathfrak{S}_n$-module over $\mathbb{C}$ can be expressed uniquely as a linear combination of $M^\lambda$’s over $\mathbb{Z}$.

**Theorem 5.2.** Suppose that $h(j) \geq j + 1$, and for $1 \leq j \leq n - 2$, let

$$\beta_j := \begin{cases} 
(n - j, j) & \text{if } h(j - 1) = j, \ h(j) = j + 1, \\
(n - 1, 1) & \text{if } h(j - 1) = h(j) = j + 1, \\
(n) & \text{otherwise,}
\end{cases}$$

where $h(0) = 1$. Then

$$H^2(\text{Hess}(S, h)) \otimes \mathbb{C} = \sum_{j=1}^{n-2} M^{\beta_j} + M^{(n)} \quad \text{in } R(\mathfrak{S}_n).$$

**Proof.** Since the action of $\mathfrak{S}_n$ on $X_i, Y_{j,k}$, and $T_A$ is given by (5.2), $\mathbb{C}(X_i, Y_{j,k}, T_A)$ decomposes into a direct sum

$$\mathbb{C}(X_i \mid i \in [n]) \oplus \mathbb{C}(Y_{j,k} \mid j \in \perp(h) \setminus \{n - 1\}, \ k \in [n]) \oplus \mathbb{C}(T_A \mid A \subset [n], \ |A| \in L(h) \setminus \{n - 1\})$$

as an $\mathfrak{S}_n$-module and the submodule $U \otimes \mathbb{C}$ of $\mathbb{C}(X_i, Y_{j,k}, T_A)$ is trivial as an $\mathfrak{S}_n$-module. The space $\mathbb{C}(Y_{j,k} \mid k \in [n])$ is isomorphic to $M^{(n-1,1)}$ for $j \in \perp(h)$ while the space $\mathbb{C}(T_A \mid |A| = j)$ is isomorphic to $M^{(n-j,j)}$ for $j \in L(h)$. There are no other nontrivial $\mathfrak{S}_n$-modules in $\mathbb{C}(X_i, Y_{j,k}, T_A)/(U \otimes \mathbb{C})$ and one can see that the dimension of the complementary module is the number of $j \in [n - 2] \setminus (\perp(h) \cup \perp(L(h)))$ plus 1 (this 1 comes from $j = n - 1$ and corresponds to $M^{(n)}$ in the last part of the right-hand side of (5.3)). □

**Example 5.3** (cf. [6, Example 6.2]). Let $n = 8$ and $h = (2, 3, 6, 6, 6, 7, 8, 8)$. Then

$$\beta_1 = (7, 1), \quad \beta_2 = (6, 2), \quad \beta_3 = (8), \quad \beta_4 = (8), \quad \beta_5 = (7, 1), \quad \beta_6 = (2, 6).$$

Therefore,

$$H^2(\text{Hess}(S, h)) \otimes \mathbb{C} = 3M^{(8)} + 2M^{(7,1)} + 2M^{(6,2)} \quad \text{in } R(\mathfrak{S}_8).$$

### 6. A GENERALIZATION

In the previous sections, we studied $H^2(\text{Hess}(S, h))$ under the condition that $h(j) \geq j + 1$ for any $j \in [n - 1]$. In this section, we will study $H^{2d}(\text{Hess}(S, h))$ under the condition that $h(j) \geq j + d$ for any $j \in [n - d]$, where $d \geq 2$. 

**PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS**

Vol. 317 2022
For $j \in [n]$ and $k \in [n]$, there is an element $y_{j,k} \in H^{2(h(j)−j)}_T(\text{Hess}(S,h))$ defined by

$$y_{j,k}(w) := \begin{cases} h(j) \prod_{\ell=j+1}^k (t_k − t_{w(\ell)}) & \text{if } k \in \{w(1), \ldots, w(j)\}, \\ 0 & \text{otherwise.} \end{cases} \quad (6.1)$$

The element $y_{j,k}$ is introduced in [1] and proved to be in $H^*_T(\text{Hess}(S,h))$ (see [1, Lemma 10.2]). Note that when $d = 1$ and $h(j) = j + 1$, the element $y_{j,k}$ above agrees with the $y_{j,k}$ in Lemma 4.1(2) (see also Remark 4.2). We use the same notation $y_{j,k}$ for its image in the ordinary cohomology $H^{2d}(\text{Hess}(S,h))$. The dot action on $y_{j,k}$ is the same as before, i.e.,

$$\sigma \cdot y_{j,k} = y_{j,\sigma(k)} \quad \text{for } \sigma \in \mathfrak{S}_n. \quad (6.2)$$

Therefore, $\sum_{k=1}^n y_{j,k}$ is $\mathfrak{S}_n$-invariant. In fact, the sum has the following expression:

$$\sum_{k=1}^n y_{j,k} = \sum_{i=1}^j \prod_{\ell=j+1}^k (x_i − x_\ell), \quad (6.3)$$

which can be proved by checking that both sides take the same value at each $w \in \mathfrak{S}_n$.

We set

$$\Lambda_d(h) := \{j \in [n-d] \mid h(j) = j + d\}.$$

Our main result in this section is the following.

**Theorem 6.1.** Suppose that $d \geq 2$ and $h(j) \geq j + d$ for any $j \in [n-d]$. Then, the restriction map

$$\iota^*: H^{2p}(\text{Fl}(n)) \to H^{2p}(\text{Hess}(S,h))$$

is an isomorphism for $p < d$. For $p = d$, the restriction map is injective and we have an isomorphism

$$H^{2d}(\text{Hess}(S,h)) \cong \left(\iota^*(H^{2d}(\text{Fl}(n))) \oplus \mathbb{Z}\langle Y_{j,k} \mid j \in \Lambda_d(h), k \in [n] \rangle\right) / \mathbb{Z}\left(\sum_{k=1}^n Y_{j,k} - y_j \mid j \in \Lambda_d(h)\right)$$

where $Y_{j,k}$ corresponds to $y_{j,k}$ in $H^*(\text{Hess}(S,h))$ and

$$y_j = \sum_{i=1}^j \prod_{\ell=j+1}^k (x_i − x_\ell).$$

**Remark 6.2.** 1. Theorem 2.2(3) says that $\text{Hess}(S,h)$ is connected, in other words, the restriction map

$$\iota: H^0(\text{Fl}(n)) \to H^0(\text{Hess}(S,h))$$

is an isomorphism, if and only if $h(j) \geq j + 1$ for any $j \in [n-1]$. Therefore, Theorem 6.1 can be regarded as a generalization of Theorem 2.2(3) and Theorem 5.1.

2. When $d = 1$, the elements $\tau_A$ appear in $H^2(\text{Hess}(S,h))$ as shown in Theorem 5.1, but such type of elements does not appear when $d \geq 2$.

Since the action of $\mathfrak{S}_n$ on $x_i$'s is trivial, so is that on $\iota^*(H^{2d}(\text{Fl}(n)))$ while that on $y_{j,k}$ is given by (6.2). Therefore, we obtain the following corollary from Theorem 6.1.

**Corollary 6.3.** Under the hypotheses of Theorem 6.1, the action of $\mathfrak{S}_n$ on $H^{2p}(\text{Hess}(S,h))$ is trivial while

$$H^{2d}(\text{Hess}(S,h)) \otimes \mathbb{C} = m_d M^{(n)} + |\Lambda_d(h)| M^{(n-1,1)} \quad \text{in } R(\mathfrak{S}_n)$$

where $m_d = b_{2d}(\text{Fl}(n)) - |\Lambda_d(h)|$. 

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 317 2022
Remark 6.4. Since

\[ \text{Poin}(\text{Fl}(n), \sqrt{q}) = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q}, \]

we have \( b_{2d}(\text{Fl}(n)) \geq n - 1 \) while \( |A_d(h)| \leq n - 2 \). Therefore, \( m_d \) in the corollary is positive.

The idea of the proof of Theorem 6.1 is the same as before. We compute the Betti numbers of \( \text{Hess}(S, h) \) up to degree \( 2d \) using Proposition 3.1 and observe that \( H^*(\text{Hess}(S, h)) \) is generated by \( x_i \)'s and \( y_{j,k} \)'s as a graded ring up to degree \( 2d \).

For the first part of Theorem 6.1, we have a homotopical version which is proved easier and may be of independent interest.

Proposition 6.5. Suppose that \( d \geq 1 \) and \( h(j) \geq j + d \) for any \( j \in [n - d] \). Then, the natural inclusion \( \iota : \text{Hess}(S, h) \to \text{Fl}(n) \) induces isomorphisms of homotopy groups \( \pi_q \) in degrees \( q \leq 2d - 1 \).

To prove this, it is sufficient to construct cellular structures on \( \text{Hess}(S, h) \) and \( \text{Fl}(n) \) which are consistent under \( \iota \) and coincide in small dimensions. Instead of cellular structures, one can use affine pavings [4]. There is an affine paving of \( \text{Fl}(n) \) by even-dimensional Bruhat cells indexed by permutations \( w \in S_n \). Intersecting a Bruhat cell \( C_w \) of \( \text{Fl}(n) \) with the subvariety \( \text{Hess}(S, h) \) gives an affine cell \( C_w' \) of \( \text{Hess}(S, h) \). The dimensions of the cells \( C_w \) and \( C_w' \) can be computed from the Białynicki-Birula theory (see [9]). The dimension \( \dim C_w \) equals the number \( \ell(w) \) of inversions of \( w \), while \( \dim C_w' \) equals \( \ell_h(w) \), the number of inversions \( j < i, w(j) > w(i) \) satisfying \( i \leq h(j) \) (see (2.1)).

Lemma 6.6. Let \( h(j) \geq j + d \) for any \( j \in [n - d] \). Then for any permutation \( w \), each of the conditions \( \ell_h(w) < d \) or \( \ell(w) < d \) implies \( \ell_h(w) = \ell(w) \).

Proof. Since \( \ell_h(w) \leq \ell(w) \), it suffices to prove \( \ell_h(w) = \ell(w) \) when \( \ell_h(w) < d \). Suppose that \( \ell_h(w) \neq \ell(w) \). Then there is an inversion \( \{ j, i \} (j < i) \) in \( w \) which contributes to \( \ell(w) \) but does not contribute to \( \ell_h(w) \). This means \( j + d \leq h(j) < i \).

We assume that the difference \( i - j \) is minimum among those inversions. Any number \( j' \) such that \( j < j' \leq j + d \) either produces an inversion \( \{ j, j' \} \) or \( \{ j', i \} \). Obviously, \( j' - j \leq d \) and if \( \{ j', i \} \) is an inversion, then \( i - j' \leq d \), which follows from the minimality of \( i - j \). In any case, each \( j' \) produces an inversion which contributes to \( \ell_h(w) \). Since there are \( d \) many such \( j' \), we have \( \ell_h(w) \geq d \). However, this contradicts the condition \( \ell_h(w) < d \). Therefore, \( \ell_h(w) = \ell(w) \).

Proof of Proposition 6.5. Since the cell \( C_w' \) of \( \text{Hess}(S, h) \) is the intersection of the Bruhat cell \( C_w \) of \( \text{Fl}(n) \) with \( \text{Hess}(S, h) \), Lemma 6.6 implies that the spaces \( \text{Fl}(n) \) and \( \text{Hess}(S, h) \) have the same \( (2d - 1) \)-skeletons and that any \( 2d \)-dimensional cell \( C_w \) of \( \text{Fl}(n) \) agrees with the cell \( C_w' \) of \( \text{Hess}(S, h) \). Therefore,

\[ \iota_* : H_q(\text{Hess}(S, h)) \to H_q(\text{Fl}(n)) \]

is an isomorphism for \( q \leq 2d - 1 \) and an epimorphism for \( q = 2d \). Moreover, both \( \text{Hess}(S, h) \) and \( \text{Fl}(n) \) are simply connected. Therefore, the proposition follows from the Whitehead theorem (see [16, p. 399]).

We now proceed with the more detailed analysis of homology needed to prove Theorem 6.1.

6.1. Betti numbers. The Betti numbers \( b_{2i}(\text{Hess}(S, h)) \) of \( \text{Hess}(S, h) \) for \( i \leq d \) are given as follows.

Lemma 6.7. Suppose that \( d \geq 2 \) and \( h(j) \geq j + d \) for any \( j \in [n - d] \). Then

\[ b_{2i}(\text{Hess}(S, h)) = \begin{cases} b_{2i}(\text{Fl}(n)) & \text{if } i < d, \\ b_{2i}(\text{Fl}(n)) + (n - 1)|A_d(h)| & \text{if } i = d. \end{cases} \]
**Proof.** For a polynomial $f(q)$ in $q$, we denote by $f(q)^{\leq d}$ the polynomial obtained from $f(q)$ by truncating the terms of degree $> d$. Then, the lemma is equivalent to

$$
Poin\left(\text{Hess}(S, h), \sqrt{q}\right)^{\leq d} = Poin\left(\text{Fl}(n), \sqrt{q}\right)^{\leq d} + (n-1)|\Lambda_d(h)|q^d. \tag{6.4}
$$

We prove identity (6.4) by induction on $n + d$ where $d \geq 2$. Since $d \geq 2$ and $h(j) \geq j + d$ for any $j \in [n-d]$ by assumption, we have $n \geq 3$, and when $(n, d) = (3, 2)$, $h$ must be $(3, 3, 3)$. In this case, $\text{Hess}(S, h) = \text{Fl}(3)$, $\Lambda_d(h) = \emptyset$, and hence the lemma holds.

Suppose that $n + d \geq 6$ and the lemma holds for any pair $(n', d')$ such that $n' + d' < n + d$. It follows from Proposition 3.1 that we have

$$
Poin\left(\text{Hess}(S, h), \sqrt{q}\right)^{\leq d} = \sum_{j=1}^{n-d-1} (q^{h(j)-j} Poin\left(\text{Hess}(S', h^j), \sqrt{q}\right))^{\leq d} + \sum_{j=n-d}^{n} (q^{h(j)-j} Poin\left(\text{Hess}(S', h^j), \sqrt{q}\right))^{\leq d}
$$

$$
= |\Lambda_d(h)|q^d + \sum_{j=n-d}^{n} (q^{n-j} Poin\left(\text{Hess}(S', h^j), \sqrt{q}\right))^{\leq d}, \tag{6.5}
$$

because $h(j) \geq j + d$ for any $j \in [n-d]$ and $h(j) = j + d < n$ if and only if $j \in \Lambda_d(h)$. Applying the inductive assumption (and Lemma 3.2 when $d = 2$) to the last sum in (6.5), we obtain

$$
\sum_{j=n-d}^{n} (q^{n-j} Poin\left(\text{Hess}(S', h^j), \sqrt{q}\right))^{\leq d}
$$

$$
= \sum_{j=n-d}^{n-2} (q^{n-j} Poin\left(\text{Fl}(n-1), \sqrt{q}\right))^{\leq d} + \sum_{j=n-d}^{n} (q^{n-j} Poin\left(\text{Fl}(n-1), \sqrt{q}\right))^{\leq d} + (n-2)|\Lambda_{d-1}(h^{n-1})|q^{d-1})^{\leq d}
$$

$$
+ (\text{Poin}\left(\text{Fl}(n-1), \sqrt{q}\right))^{\leq d} + (n-2)|\Lambda_d(h^n)|q^d
$$

$$
= \sum_{j=n-d}^{n} (q^{n-j} Poin\left(\text{Fl}(n-1), \sqrt{q}\right))^{\leq d} + (n-2)(|\Lambda_{d-1}(h^{n-1})| + |\Lambda_d(h^n)|)q^d
$$

$$
= \text{Poin}\left(\text{Fl}(n), \sqrt{q}\right)^{\leq d} + (n-2)(|\Lambda_{d-1}(h^{n-1})| + |\Lambda_d(h^n)|)q^d, \tag{6.6}
$$

where we can see the last identity above by applying Proposition 3.1 to the flag variety $\text{Fl}(n)$. Thus, if we prove

$$
|\Lambda_{d-1}(h^{n-1})| + |\Lambda_d(h^n)| = |\Lambda_d(h)|, \tag{6.7}
$$

then identity (6.4) follows from (6.5) and (6.6). However, one can easily see that

$$
(|\Lambda_{d-1}(h^{n-1})|, |\Lambda_d(h^n)|) = \begin{cases} (0, |\Lambda_d(h)|) & \text{if } h(n-d-1) = n, \\ (1, |\Lambda_d(h)| - 1) & \text{if } h(n-d-1) = n - 1,
\end{cases}
$$

and this implies (6.7). □

**6.2. Complementary elements.** We introduce complementary elements $y_{i,k}^*$ which will make our argument prospective. The element $y_{j,k}$ in (6.1) is defined by looking at the $j$th column of the configuration associated to the Hessenberg function $h$. Similarly, one can define an element $y_{i,k}^*$ of $H^*_T(\text{Hess}(S, h))$ by looking at the $i$th row of the configuration as follows. For $1 < i \leq n$, we define

$$
h^*(i) := \min\{j \in [n] \mid h(j) \geq i\}.
$$
Similarly to the proof of Proposition 4.5, we consider the decomposition
\[ w \rightarrow \bigcap_{i} t_{w(i)} \]  
looking at those shaded boxes, we define
\[ y_{i,k}(w) := \begin{cases} 
\prod_{\ell=h^*(i)}^{i-1} (t_k - t_{w(\ell)}) & \text{if } k \in \{w(i), \ldots, w(n)\}, \\
0 & \text{otherwise.}
\end{cases} \] (6.8)

One can see that \( y_{i,k}^* \) is in \( H^*_T(\text{Hess}(S, h)) \) similarly to \( y_{j,k} \).

Note that
(1) \( h(j) \geq j + d \) for all \( j \in [n-d] \) if and only if \( h^*(i) \leq i - d \) for all \( d + 1 \leq i \leq n \); 
(2) under the assumption that \( h(j) \geq j + d \) for any \( j \in [n-d] \), we have
\[ h(j) = j + d < n \quad \text{(i.e., } j \in \Lambda_d(h) \text{)} \quad \Leftrightarrow \quad h^*(j + 1 + d) = j + 1. \]

Based on this observation, we define
\[ \Lambda^*(h) = \{ i \mid d + 2 \leq i \leq n, \ h^*(i) = i - d \}. \]

Then, (2) above can be restated as
\[ j \in \Lambda_d(h) \quad \Leftrightarrow \quad j + 1 + d \in \Lambda_d^*(h). \]

The elements \( y_{i,k}^*, i \in \Lambda_d^*(h) \), do not provide new elements, as is seen from the following lemma.

**Lemma 6.8.** Suppose that \( h(j) \geq j + d \) for any \( j \in [n-d] \). Then, for \( j \in \Lambda_d(h) \) we have
\[ y_{j,k} + y_{j+1+d,k} = \prod_{\ell=j+1}^{j+d} (t_k - x_\ell). \]

**Proof.** It immediately follows from the definitions of \( y_{j,k}, y_{j,k}^* \), and \( x_i \) that both sides in the lemma take the same value at every \( w \in \mathcal{S}_n \). \( \square \)

### 6.3. Generators of \( H^*_T(\text{Hess}(S, h)) \)

We show that the elements \( t_i, x_i \), and \( y_{j,k} \) generate \( H^*_T(\text{Hess}(S, h)) \) up to \( \ast \leq 2d \) as a graded ring under our assumption.

**Lemma 6.9.** Suppose that \( h(j) \geq j + d \) for any \( j \in [n-d] \) and \( d \geq 2 \). Then \( H^*_T(\text{Hess}(S, h)) \) is generated by \( t_i, x_i \ (i \in [n]) \), and \( y_{j,k} \ (j \in \Lambda_d(h), k \in [n]) \) up to \( \ast \leq 2d \) as a graded ring, where the degree of \( x_i \) is 2 while that of \( y_{j,k} \) is 2d.

**Proof.** We prove the lemma by induction on \( n + d \) similarly to the proof of Lemma 6.7. When \( (n, d) = (3, 2) \), \( \text{Hess}(S, h) = \text{Fl}(3) \) and the lemma holds since \( H^*_T(\text{Fl}(n)) \) is generated by \( t_1 \) and \( x_i \ (i \in [n]) \).

Suppose that \( n + d \geq 6 \) and the lemma holds for any pair \((n', d')\) such that \( n' + d' < n + d \).

Similarly to the proof of Proposition 4.5, we consider the decomposition
\[ \mathcal{S}_n = \mathcal{S}^1_n \sqcup \mathcal{S}^2_n \sqcup \ldots \sqcup \mathcal{S}^n_n, \quad \text{where} \quad \mathcal{S}^j_n = \{ w \in \mathcal{S}_n \mid w(j) = n \}. \]

Let \( z \) be an arbitrary element of \( H^*_T(\text{Hess}(S, h)) \) for \( p \leq d \).

**Step 1.** By the same reasoning as at step 1 in the proof of Proposition 4.5, we may assume \( z = 0 \) on \( \mathcal{S}^n_n \) by subtracting an appropriate polynomial in \( t_1, x_1 \), and \( y_1 \).

**Step 2.** Suppose that \( z = 0 \) on \( \mathcal{S}^n_n \). We will show that \( z \) minus an appropriate polynomial of \( t_1, x_1 \), and \( y_1 \) vanishes on \( \mathcal{S}^n_1 \sqcup \mathcal{S}^n_0 \).

Since \( h(n-1) = n \), each vertex \( w \in \mathcal{S}^n_{n-1} \) is connected to the vertex \( w \cdot (n, n-1) \) of \( \mathcal{S}^n_0 \) by an edge of the labeled graph \( \Gamma(h) \). Since \( z = 0 \) on \( \mathcal{S}^n_1 \), \( z(w) \) must be divisible by the label
Therefore, it follows from (6.9) that
\[ z(w) = (t_{w(n)} - t_n)g(w) \quad \text{for} \quad w \in S^{n-1}_n. \]  
(6.9)

We express
\[ g(w) = \sum_{\ell=0}^{p-1} g_{\ell}(w)t_{n}^\ell \]  
(6.10)

with homogeneous polynomial \( g_{\ell}(w) \) in \( \mathbb{Z}[t_1, \ldots, t_{n-1}] \) of degree \( 2(p-1-\ell) \).

**Claim.** If \( v, w \in S^{n-1}_n \) are joined by an edge of the labeled graph \( \Gamma(h^{n-1}) \), i.e., \( v = w \cdot (i, j) \) for some transposition \((i, j)\) with \( j < i \leq n \), \( j \neq n-1 \), and \( i \neq n-1 \), then
\[ g_{\ell}(v) \equiv g_{\ell}(w) \pmod{t_{w(i)} - t_{w(j)}}. \]

**Proof.** Since \( z \) is an element of \( H^*_T(\text{Hess}(S', h)) \), it satisfies the congruence relation
\[ z(v) \equiv z(w) \pmod{t_{w(i)} - t_{w(j)}}. \]  
(6.11)

Since \( v = w \cdot (i, j) \), we have \( v(i) = w(j), v(j) = w(i) \), and \( v(s) = w(s) \) for \( s \neq i, j \). Moreover, \( w(i) \) and \( w(j) \) are not equal to \( n \) because \( i \) and \( j \) are not equal to \( n-1 \) and \( w \in S^{n-1}_n \). Therefore,
\[ t_{w(n)} - t_n \equiv t_{w(n)} - t_n \neq 0 \pmod{t_{w(i)} - t_{w(j)}}. \]

This together with (6.9)–(6.11) implies the congruence relation in the claim. \( \square \)

By the claim above, each \( g_{\ell} \) is an element of \( H^*_T(\text{Hess}(S', h^{n-1})) \) by Proposition 2.3. Since \( \ell \geq 0 \) and \( p \leq d \), we have
\[ \deg g_{\ell} = 2(p-1-\ell) \leq 2(d-1). \]  
(6.12)

We note that \( h(n-d-1) = n \) or \( n-1 \) by the assumption \( h(j) \geq j + d \) for any \( j \in [n-d] \). Now we take two cases according to the value of \( h(n-d-1) \).

**Case 1:** \( h(n-d-1) = n \). In this case, \( h^{n-1}(j) \geq j + d \) for any \( j \in [n-d-1] \). Therefore, by the inductive assumption and (6.12), any \( g_{\ell} \) can be written as a polynomial in \( t_i \) and \( x_i^{(n-1)} \) (\( i \in [n-1] \)), where for \( w \in S^{n-1}_n \),
\[ x_i^{(n-1)}(w) = \begin{cases} x_i(w) & \text{if } i \leq n-2, \\ x_n(w) & \text{if } i = n-1. \end{cases} \]

This shows that there is a polynomial \( G_{\ell} \) in \( t_i \) and \( x_i^{(n-1)} \) whose restriction to \( S^{n-1}_n \) agrees with \( g_{\ell} \). Therefore, it follows from (6.9) that
\[ z = (x_n - t_n) \sum_{\ell=0}^{p-1} G_{\ell} t_n^\ell \quad \text{on} \quad S^{n-1}_n. \]

Both sides above vanish on \( S_n^n \), so they agree on \( S^{n-1}_n \sqcup S_n^n \). Therefore, subtracting the right-hand side above from \( z \), we may assume \( z = 0 \) on \( S^{n-1}_n \sqcup S_n^n \).

**Case 2:** \( h(n-d-1) = n-1 \). In this case, \( h^{n-1}(j) \geq j + (d-1) \) for any \( j \in [n-1-(d-1)] \) and \( \Lambda_{d-1}(h^{n-1}) = \{n-d-1\} \). Therefore, by the induction assumption and (6.12), any \( g_{\ell} \) can be written as a polynomial in \( t_i, x_i^{(n-1)}, \) and \( y_i^{(n-1)}(n-d-1,k) \) where \( k \in [n-1] \) and
\[ y_i^{(n-1)}(n-d-1,k)(w) = \frac{y_{n-d-1,k}(w)}{t_k - t_{w(n-1)}} \quad \text{for} \quad w \in S^{n-1}_n. \]
By Lemma 6.8, we may use $y_{n-1,k}^{*(n-1)}$ instead of $y_{n-d-1,k}^{(n-1)}$, where

$$y_{n-1,k}^{*(n-1)} = \frac{y_{n,k}^*}{t_k - t_{w(n-1)}}.$$

We note that since

$$\deg g_{\ell} = 2(p - 1 - \ell), \quad \deg y_{n-1,k}^{*(n-1)} = 2(d - 1), \quad p \leq d,$$

$y_{n-1,k}^{*(n-1)}$ does not appear in the polynomial expression of $g_{\ell}$ unless $p = d$ and $\ell = 0$. Therefore, it follows from (6.9) and (6.10) that we can write

$$z(w) = (t_{w(n)} - t_n)(\sum_{k=1}^{n-1} c_k y_{n-1,k}^{*(n-1)}(w) + \sum_{\ell=0}^{p-1} f_{\ell}(w)t_n^{\ell})$$

$$= \sum_{k=1}^{n-1} c_k(t_{w(n)} - t_n)y_{n-1,k}^{*(n-1)}(w) + (t_{w(n)} - t_n)\sum_{\ell=0}^{p-1} f_{\ell}(w)t_n^{\ell} \quad (6.13)$$

where $c_k \in \mathbb{Z}$ and $f_{\ell}$ is a polynomial in $t_i$ and $x_i^{(n-1)}$ ($i \in [n - 1]$).

Similarly to case 1, $f_{\ell} \in H^+ (\text{Hess} (S', h^{n-1}))$ in (6.13) is the image of some $F_{\ell} \in H^+(\text{Hess}(S, h))$. Although $y_{n-1,k}^{*(n-1)}$ may not be in the image of the restriction map, we have

$$(t_{w(n)} - t_n)y_{n-1,k}^{*(n-1)}(w) = \begin{cases} \prod_{\ell=\text{h}^*(n)}^{n} (t_k - t_{w(\ell)}) & \text{if } k = w(n) \\ 1 & \text{otherwise} \end{cases} = y_{n,k}^*(w)$$

for $w \in \mathcal{S}_{n-1}$ (so $t_n = t_{w(n-1)}$), where $h^*(n) = n - d$ because $h(n - d - 1) = n - 1$ and $h(j) = n$ for $j \geq n - d$. This observation and (6.13) show that

$$z = \sum_{k=1}^{n-1} c_k y_{n,k}^* + (x_n - t_n)\sum_{\ell=0}^{p-1} F_{\ell} t_n^{\ell} \quad \text{on } \mathcal{S}_{n-1}^{n-1}. \quad (6.14)$$

Here, $z = 0$ on $\mathcal{S}_{n}^{r}$ by assumption and the right-hand side above also vanishes on $\mathcal{S}_{n}^{r}$. Indeed, since $w(n) = n$ for $w \in \mathcal{S}_{n}^{r}$, we have $y_{n,k}^*(w) = 0$ for $k \neq n$ and $(x_n - t_n)(w) = t_{w(n)} - t_n = 0$. Thus, identity (6.14) holds on $\mathcal{S}_{n-1}^{n-1} \cup \mathcal{S}_{n}^{r}$. This together with Lemm 6.8 shows that $z$ minus an appropriate polynomial in $t_{\bullet}, x_{\bullet}$, and $y_{\bullet}$ vanishes on $\mathcal{S}_{n-1}^{n-1} \cup \mathcal{S}_{n}^{r}$.

**Step 3.** Suppose that $z = 0$ on $\mathcal{S}_{n}^{r+1} \cup \ldots \cup \mathcal{S}_{n}^{n}$ for some $r$ with $n - d \leq r \leq n - 2$. Then, since $h(r) = n$, there are shaded boxes at the positions $(r + 1, r), (r + 2, r), \ldots, (n, r)$ in the configuration associated to $h$. This means that each vertex $w \in \mathcal{S}_{n}^{r}$ is connected to $\mathcal{S}_{n}^{r}$ for $r + 1 \leq \ell \leq n$ by an edge with label $t_{w(\ell)} - t_{w(r)} = t_{w(\ell)} - t_n$. Since $z = 0$ on $\mathcal{S}_{n}^{r+1} \cup \ldots \cup \mathcal{S}_{n}^{n}$, it follows that $z(w)$ for $w \in \mathcal{S}_{n}^{r}$ must be divisible by $\prod_{\ell=r+1}^{n}(t_{w(\ell)} - t_n)$. Therefore, there is a homogeneous element $g(w) \in \mathbb{Z}[t_1, \ldots, t_n]$ such that

$$z(w) = \left(\prod_{\ell=r+1}^{n}(t_{w(\ell)} - t_n)\right) g(w) \quad \text{for } w \in \mathcal{S}_{n}^{r}. \quad (6.15)$$

The same argument as in the claim in step 2 shows that the $g$ in (6.15) satisfies the congruence relation for the labeled graph $\Gamma(h^r)$. Since $p \leq d$ and $n - r \geq 2$, we have

$$\deg g = 2(p - n + r) \leq 2(d - 2).$$

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 317 2022
Moreover, $h^r(j) \geq j + (d - 1)$ for any $j \in [n - 1 - (d - 1)]$. Therefore, by the induction assumption, $g$ can be expressed as a polynomial in $t_i$ and $x_i^{(r)}$ where

$$x_i^{(r)}(w) := \begin{cases} x_i(w) & \text{if } i < r, \\ x_{i+1}(w) & \text{if } r \leq i, \end{cases}$$

for $w \in S_n^r$. Since $x_i^{(r)}$ is in the image of the restriction map from $H^*(\text{Hess}(S, h))$ to $H^*(\text{Hess}(S', h))$, there is an element $G \in H^*(\text{Hess}(S, h))$ whose restriction to $H^*(\text{Hess}(S', h^r))$ agrees with $g$. It follows from (6.15) that

$$z = \left( \prod_{\ell=r+1}^n (x_\ell - t_n) \right) G \quad \text{on } S_n^r. \quad (6.16)$$

Here, $z = 0$ on $S_n^{r+1} \cup \ldots \cup S_n^n$ by assumption, and the right-hand side above also vanishes on $S_n^{r+1} \cup \ldots \cup S_n^n$. Indeed, since $x_\ell(w) = t_{w(\ell)} = t_n$ for $w \in S_n^\ell$, $\prod_{\ell=r+1}^n (x_\ell - t_n)(w) = 0$ for $w \in S_n^{r+1} \cup \ldots \cup S_n^n$. Thus, identity (6.16) holds on $S_n^r \cup \ldots \cup S_n^n$, so that $z$ minus an appropriate polynomial in $t_\ast, x_\ast$, and $y_\ast$ vanishes on $S_n^r \cup \ldots \cup S_n^n$.

**Step 4.** Suppose that $z = 0$ on $S_n^{r+1} \cup \ldots \cup S_n^n$ for some $r$ with $1 \leq r \leq n - d - 1$. Then, similarly to step 3, $z(w)$ for $w \in S_n^r$ must be divisible by $\prod_{\ell=r+1}^{h(r)} (t_n - t_{w(\ell)})$. Here, the degree of $z$ is $2p \leq 2d$ and $h(r) - r \geq d$, so $z(w) = 0$ unless $p = d$ and $h(r) = r + d$. When $p = d$ and $h(r) = r + d$, we have

$$z(w) = c \left( \prod_{\ell=r+1}^{r+d} (t_n - t_{w(\ell)}) \right) \quad \text{for } w \in S_n^r, \quad (6.17)$$

where $c \in \mathbb{Z}$.

Since $r \leq n - d - 1$ by assumption, $h(r) = r + d < n$. Therefore, $r \in \Lambda_d(h)$ so that we have an element $y_{r,k} \in H_2^{2d}(\text{Hess}(S, h))$ for any $k \in [n]$. We take $k = n$. Since $h(r) = r + d$, we have

$$y_{r,n}(w) = \begin{cases} \prod_{\ell=r+1}^{r+d} (t_n - t_{w(\ell)}) & \text{if } n \in \{w(1), \ldots, w(r)\}, \\ 0 & \text{otherwise} \end{cases}$$

by definition. This together with (6.17) shows that

$$z = cy_{r,n} \quad \text{on } S_n^r.$$

Here, $z = 0$ on $S_n^{r+1} \cup \ldots \cup S_n^n$ by assumption and the right-hand side above also vanishes because $y_{r,n}(w) = 0$ if $n \in \{w(r+1), \ldots, w(n)\}$ by definition. Therefore, $z$ minus $cy_{r,n}$ vanishes on the set $S_n^r \cup \ldots \cup S_n^n$.

This completes the induction step, and the lemma is proved. \(\square\)

### 6.4. Proof of Theorem 6.1.

Under these preparations, we prove Theorem 6.1. When $p < d$, the restriction map

$$i^*: H^{2p}(\text{Fl}(n)) \to H^{2p}(\text{Hess}(S, h))$$

is surjective by Lemma 6.9 and is indeed an isomorphism by Lemma 6.6 (or Lemma 6.7).

When $p = d$, we consider the homomorphism

$$\Phi: i^*(H^{2d}(\text{Fl}(n))) \oplus \mathbb{Z}\langle Y_{j,k} \mid j \in \Lambda_d(h), k \in [n] \rangle \to H^{2d}(\text{Hess}(S, h)).$$
sending $Y_{j,k}$ to $y_{j,k}$. The map $\Phi$ is surjective by Lemma 6.9, and $\sum_{k=1}^n Y_{j,k} - y_j$ for $j \in \Lambda_d(h)$ are in the kernel of $\Phi$ by (6.3). Therefore, the map $\Phi$ induces a surjective homomorphism

$$\overline{\Phi} : (i^*(H^{2d}(\text{Fl}(n))) \oplus \mathbb{Z}\langle Y_{j,k} \mid j \in \Lambda_d(h), k \in [n] \rangle) / \mathbb{Z}\langle \sum_{k=1}^n Y_{j,k} - y_j \rangle \rightarrow H^{2d}(\text{Hess}(S,h)).$$

Here, the rank of the source module is at most the rank of the target module by Lemma 6.7. Moreover, one can easily see that the source module is torsion-free and we know that $H^*(\text{Hess}(S,h))$ is also torsion-free. Thus, the surjective homomorphism $\overline{\Phi}$ must be an isomorphism, proving the theorem.

**FUNDING**

The work of the first and second authors was performed within the framework of the HSE University Basic Research Program.

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This article was submitted by the authors simultaneously in Russian and English.