The Convolution Series Solution of Divergent Navier-Stokes Equations

Yan Yimin

Abstract
This paper gives out the solution of divergent Navier-Stokes equations, and shows that in this case, under a physical acceptable condition, the solution would be smooth.

Keywords: NSE, Divergent, Explicit solution

1. Introduction
Any physical system, the behavior of local state is actually a result of global interaction, which is probably to be interpreted as the convolution of two objects:

\[ f(x) * g(x) = \int_{\mathbb{R}^n} f(x-y) \cdot g(y) \, dy \]

for instance:

1. Brownian motion of free electric charges: if \( q(x) \) is initial charge distribution of the system, \( K(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{x^2}{4t}} \) is the heat conduction kernel, which means the probability of a charge move to a distance \( x \) over a period of time \( t \), then \( K(x, t) * q(x) \) means the charge distribution after time \( t \). Because "probability" always makes us uneasy in understanding physical phenomenon, we give out another example.

2. An electric system, if \( q(x) \) is the charge distribution, \( E(x) \) is the electric field, and the interaction between the field and the charge is merely depended on the vector distant \( x \), then \( E(x) * q(x) \) means the total external force act on each position.

It is not doubt that convolution is of crucial important in describing physical phenomenons. To illustrate such idea, we provide a convolution expression for Divergent Navier-Stokes Equations.

2. Main Result

**Theorem 2.1.** For the Navier-Stokes Equations with divergent condition:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= \Delta u + \nabla (f - p) \\
u(x, t)|_{t=0} &= u_0(x) = \nabla \varphi(x)
\end{align*}
\]

(1)

where \( \nabla f(x, t) \) is the external force, \( p(x, t) \) is the pressure, and \( \varphi(x) \) is the initial potential.

If

1. \( f(x, t) - p(x, t) \in L^\infty(\mathbb{R}^n \times [0, T]) \);
2. \( u_0 \in L^\infty(\mathbb{R}^n) \), such as \( \|u_0\|_\infty \leq \text{light speed } c \), and integrable in any bounded closure of \( \mathbb{R}^n \);
3. \( \exists x_0 \in \mathbb{R}^n \), s.t. \( \varphi(x_0) = a \) is bounded.
then

1. the solution of such equations is

\[ u(x, t) = -2 \frac{\nabla G(x, t)}{G(x, t)} \] (2)

where

\[
G(x, t) = K(x, t) * e^{-\frac{i}{2} p(x)} + \int_0^t K(x, t-s) * \left[ F(x, s)[K(x, s) * e^{-\frac{i}{2} p(x)}] \right] ds
\]

\[
+ \int_0^t K(x, t-s) * \left[ F(x, s) \right] \left\{ \int_0^\tau K(x, s-\tau) * \left[ F(x, \tau)[K(x, \tau) * e^{-\frac{i}{2} p(x)}] \right] d\tau \right\} ds
\]

\[
+ \int_0^t K(x, t-s) * \left[ F(x, s) \right] \left\{ \int_0^\tau K(x, s-\tau) * \left[ F(x, \tau)[K(x, \tau) * e^{-\frac{i}{2} p(x)}] \right] d\theta \right\} d\tau \right\} ds
\]

\[
+ \cdots
\]

\[
K(x, t) = \frac{1}{(4\pi t)^n} e^{-\frac{|x|^2}{4t}}, \quad F(x, t) = \frac{1}{2} [p(x, t) - f(x, t)]
\] (3)

2. If \( u_0(x), F(x, t) \in C^0(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T]) \), then \( u(x, t) \in C^\infty(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T]) \).

**Proof.** 2.1. **Deduce the Expression of** \( u(x, t) \)

It goes without saying that, if excluding the nonlinear terms, Eq.(1) turns out to be a heat conduction equation. So, take an undefined Kernel \( G(x, t) \) into consideration:

\[
G \partial_t u + G (u \cdot \nabla) u = G \Delta u + G \nabla (f - p)
\] (4)

1. Seeing that

\[
\partial_{xx}(G u) = \partial_{xx} G \cdot u + G \partial_{xx} u + 2 \partial_x G \cdot \partial_x u
\]

so,

\[
\Delta (G u) = \Delta G \cdot u + G \Delta u + 2 \left[ \sum_{k=1}^n \partial_{x_k} G \cdot \partial_{x_k} u \right]
\]

\[
= \Delta G \cdot u + G \Delta u + 2 [(\partial_{x_1} G, \cdots, \partial_{x_n} G) \cdot \nabla] u
\]

\[
= \Delta G \cdot u + G \Delta u + 2 [\nabla G \cdot \nabla] u
\]

therefore, Eq.(4) is equivalent to

\[
\partial_t (G u) - \partial_t G \cdot u + (G u \cdot \nabla) u
\]

\[
= \Delta (G u) - \Delta G \cdot u - 2 [\nabla G \cdot \nabla] u + G \nabla (f - p)
\] (5)

2. Now, assume that \( G, u \) satisfy that

\[
\begin{aligned}
\partial_t G &= \Delta G + Q(x, t) \\
G u &= -2 \nabla G
\end{aligned}
\] (6)

Then the answer occur to us that Eq.(5) leaves out to be

\[
-2 \nabla Q - Q u = G \nabla (f - p)
\] (7)

According to Eq.(6),

\[
u = -2 \frac{\nabla G}{G}
\] (8)

so,

\[
\nabla (f - p) = -2 \frac{\nabla Q}{G} - \frac{Q u}{G} = -2 \frac{\nabla Q}{G} + 2 \frac{Q \nabla G}{G} = -2 \frac{\nabla Q}{G}
\] (9)
Hence, it is clearly reasonable and compatible, if we set

\[ f - p = -2 \frac{\Omega}{G} + C \]  

(10)

where \( C = C(t) \). As a result, \( G \) is the solution of

\[ \partial_t G = \Delta G + \frac{1}{2} \left[ C + p(x, t) - f(x, t) \right] G \]  

(11)

It has not escaped our attention that \( C \) is not an essential variable, since if we let \( G = e^{\frac{1}{2} \int_0^t C(t) \cdot W(x, t) \, dt} \), then

\[
\begin{aligned}
\begin{cases}
\partial_t W = \Delta W + \frac{1}{2} [p(x, t) - f(x, t)] W \\
G(x, t) = G_0(x)
\end{cases}
\end{aligned}
\]

So, let \( C = 0 \).

3. And let

\[ F(x, t) = \frac{1}{2} [p(x, t) - f(x, t)] \]  

(12)

and consider the controlled heat conduction equation

\[
\begin{aligned}
\begin{cases}
\partial_t G(x, t) = \Delta G(x, t) + F(x, t) G(x, t) \\
G(x, 0) = G_0(x)
\end{cases}
\end{aligned}
\]

(13)

which has the solution

\[
\begin{aligned}
\begin{cases}
G(x, t) = K(x, t) * G_0(x) + \int_0^t K(x, t - s) * (F(x, s) [K(x, s) * G_0(x)]) \, ds \\
+ \int_0^t K(x, t - s) \left[ F(x, s) \int_0^s K(x, s - \tau) \left[ F(x, \tau) [K(x, \tau) * G_0(x)] \right] d\tau \right] \, ds \\
+ \int_0^t K(x, t - s) \left[ F(x, s) \int_0^s K(x, s - \tau) \left( F(x, \tau) \int_0^\tau K(x, \tau - \theta) \left[ F(x, \theta) [K(x, \theta) * G_0(x)] \right] d\theta \right] d\tau \right] \, ds \\
+ \ldots \\
K(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-
\end{cases}
\end{aligned}
\]

(14)

Briefly verification could be set up by following steps:

(a) It is clear that Eq.(14) is equivalent to

\[ G(x, t) = K(x, t) * G_0(x) + \int_0^t K(x, t - s) * \left( F(x, s) * G(x, s) \right) ds \]  

(15)

then

\[
\begin{aligned}
\begin{cases}
\partial_t G(x, t) = \partial_t K(x, t) * G_0(x) + \partial_t \int_0^t K(x, t - s) * \left( F(x, s) * G(x, s) \right) ds \\
= \partial_t K(x, t) * G_0(x) + \int_0^t \partial_t K(x, t - s) * \left( F(x, s) * G(x, s) \right) ds + \lim_{s \to t} K(x, t - s) \left( F(x, s) * G(x, s) \right) \\
= \partial_t K(x, t) * G_0(x) + \int_0^t \Delta K(x, t - s) * \left( F(x, s) * G(x, s) \right) ds + \lim_{s \to t} \Delta K(x, t - s) * \left( F(x, s) * G(x, s) \right) \\
= \Delta K(x, t) * G_0(x) + \Delta \int_0^t K(x, t - s) * \left( F(x, s) * G(x, s) \right) ds + \lim_{s \to t} \Delta K(x, t - s) * \left( F(x, s) * G(x, s) \right) \\
= \Delta K(x, t) * G_0(x) + \Delta \int_0^t K(x, t - s) * \left( F(x, s) * G(x, s) \right) ds + \lim_{s \to t} \Delta K(x, t - s) * \left( F(x, s) * G(x, s) \right) \\
= \Delta G(x, t) + F(x, t) G(x, t)
\end{cases}
\end{aligned}
\]
4. Firstly, 

\[ G_0(x) = e^{-\frac{\psi(x)}{2}} \]

because according to the Assumption(6):

\[ u = -2 \frac{\nabla G}{G} = -2 \nabla \ln G \]

it follows 

\[ -2 \Delta \ln G_0 = \text{div} u_0 = \text{div} \nabla \varphi(x) = \Delta \varphi(x) \]

(b) Beside, let’s show that \( G(x, t) \) is absolute convergence for all fixed point \((x, t) \in \mathbb{R}^n \times [0, T]\).

If \( F(x, t) \) is bounded, i.e. 

\[ \exists M > 0, \quad \text{s.t.} \quad \|F(x, t)\|_{L^\infty(\mathbb{R}^n \times [0, T])} \leq M. \]

then, 

\[
\begin{align*}
|G(x, t)| & \leq \left| K(x, t) * G_0(x) \right| + \left| \int_0^t K(x, t-s) * (F(x, s)[K(x, s) * G_0(x)]) \, ds \right| \\
& + \left| \int_0^t K(x, t-s) * \left[ F(x, s) \int_0^\tau K(x, \tau-s) * (F(x, \tau)[K(x, \tau) * G_0(x)]) \, d\tau \right] \, ds \right| \\
& + \left| \int_0^t K(x, t-s) * \left[ F(x, s) \int_0^\tau K(x, \tau-s) * (F(x, \tau) \int_0^{\tau-\theta} K(x, \tau-\theta) * (F(x, \theta)[K(x, \theta) * G_0(x)]) \, d\theta \right] \, d\tau \right| \, ds \\
& + \cdots .
\end{align*}
\]

\[
\leq K(x, t) * |G_0(x)| + \int_0^t K(x, t-s) * \left( MK(x, s) * |G_0(x)| \right) \, ds \\
+ \left| \int_0^t K(x, t-s) * \left[ M \int_0^\tau K(x, \tau-s) * \left( MK(x, \tau) * |G_0(x)| \right) \, d\tau \right] \, ds \right| \\
+ \left| \int_0^t K(x, t-s) * \left[ M \int_0^\tau K(x, \tau-s) * \left( M \int_0^{\tau-\theta} K(x, \tau-\theta) * \left( MK(x, \theta) * |G_0(x)| \right) \, d\theta \right] \, d\tau \right| \, ds + \cdots .
\]

\[ \overset{\text{def}}{=} V(x, t) \]

Clearly, \( V(x, t) \) is the solution of 

\[
\begin{align*}
\partial_t V(x, t) &= \Delta V(x, t) + M V(x, t) \\
V(x, 0) &= |G_0(x)|
\end{align*}
\]

so, let \( V(x, t) = e^{Mt} \cdot W(x, t) \), we get 

\[
\begin{align*}
\partial_t W(x, t) &= \Delta W(x, t) \\
W(x, 0) &= |G_0(x)|
\end{align*}
\]

i.e. 

\[ W(x, t) = K(x, t) * |G_0(x)| \]

Therefore, we get a fundamental conclusion 

\[
\left| G(x, t) \right| \leq e^{Mt} \cdot K(x, t) * |G_0(x)|
\]

(17)

So, if \( G_0(x) \) are bounded, such as \( |G_0(x)| \leq M \), it follows 

\[
\left| G(x, t) \right| \leq e^{Mt} \cdot K(x, t) * M = Me^{Mt}
\]

But such restriction is too narrow. In the next step, we will consider a general case, so that \( G(x, t) \) is still absolute convergence at any fixed point \((x, t) \in \mathbb{R}^n \times [0, T]\).
or

\[-2\ln G_0 = \varphi(x) + g(x), \quad \triangle g(x) = 0\]

So, if we add an additional restriction that:

\[\lim_{|x| \to \infty} G_0(x) \quad \text{and} \quad \lim_{|x| \to \infty} \partial_i G_0(x) \text{ are possible bounded}\]

we get

\[\ln G_0 = -\frac{1}{2} \varphi(x) \quad \text{or} \quad G_0 = e^{-\frac{1}{2} \varphi(x)}, \quad g(x) = 0\]

Secondly, if the initial condition meets the **Restricted Conditions**

- **Bounded Restriction**: \(u_0\) is bounded, such as \(|u_0| \leq \text{light speed } c\), and integrable in any bounded closure of \(\mathbb{R}^n\)

- **Non-trivial Restriction**: \(\exists x_0 \in \mathbb{R}^n\), s.t. \(\varphi(x_0) = a\) is bounded; otherwise, \(\forall x \in \mathbb{R}^n, \varphi(x) = \infty \) or \(-\infty\) or non-existing, which is a trivial case we won’t consider. Without lose of generality, assume \(x_0 = 0\), i.e. \(\varphi(0) = a\) is bounded.

Then it is clear that, in the non-rotational fluid, the integral along any curve \(L : x_0 \to x\)

\[\varphi(x) = \int_L u_0 \, dx + a\]

is independent of the path.

So

\[e^{-\frac{1}{2} \varphi(x)} = e^{-\frac{1}{2} \left( \int u_0 \, dx + a \right)} \leq e^{-\frac{1}{2} \int c \, dr} = e^{-\frac{1}{2} \int c \, dr} \quad (r = |x - x_0|)\]  

(20)

Consider the spherical coordinates \((r, \theta, \phi)\):

\[
\begin{align*}
    x &= r \cos \theta \sin \phi \\
    y &= r \sin \theta \sin \phi \\
    z &= r \cos \phi
\end{align*}
\]

\(r \in [0, \infty), \ \theta \in [0, 2\pi], \ \phi \in [0, \pi]\)

then

\[K(x, t) = e^{-\frac{1}{2} \varphi(x)} = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} \, dy = \int_0^{2\pi} d\theta \int_0^\infty dr \int_0^{\infty} d\phi \left( r^2 \sin \theta \sin \phi \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{x^2}{2t} - \frac{y^2}{2t}} e^{-\frac{1}{2} \varphi(y)} \right)\]

where

\[b = (\cos \theta \sin \phi) \cdot (\cos \theta_1 \sin \phi_1) + (\sin \theta \sin \phi) \cdot (\sin \theta_1 \sin \phi_1) + (\cos \phi) \cdot (\cos \phi_1)\]

\[\leq \left( (\cos \theta \sin \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \phi)^2 \right)^{\frac{1}{2}} \cdot \left( (\cos \theta_1 \sin \phi_1)^2 + (\sin \theta_1 \sin \phi_1)^2 + (\cos \phi_1)^2 \right)^{\frac{1}{2}} = 1\]
Therefore
\[
K(x, t) = e^{-\frac{1}{4}\varphi(x)} \leq \int_0^{2\pi} d\theta \int_0^\pi d\phi \int_0^{\infty} dp \left( \frac{1}{4\pi t^2} e^{-\frac{(u^2 + v^2)}{4\pi t}} e^{-\frac{w}{2\pi}} \right) = 4\pi \int_0^{\infty} \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{u^2 + v^2}{4\pi t}} e^{-\frac{w}{2\pi}} dp
\]
\[
e^{-\frac{(u^2 + v^2)}{4\pi t}} 4\pi \int_0^{\infty} \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{w}{2\pi}} dp \quad (B = r + ct)
\]
\[
e^{-\frac{(u^2 + v^2)}{4\pi t}} 4\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi d\eta = e^{-\frac{r^2 + c^2}{4\pi t}} \int_{-\infty}^{\infty} \left( \frac{r + ct}{2\sqrt{\pi}} \right) \left( \frac{r - ct}{2\sqrt{\pi}} \right)
\]
\[
e^{-\frac{(u^2 + v^2)}{4\pi t}} \cdot 4\left( \frac{r + ct}{2\sqrt{\pi}} \right)^2 = e^{-\frac{(u^2 + v^2)}{4\pi t}} \left( \frac{r + ct}{2\sqrt{\pi}} \right)^2 + \frac{1}{2}
\]
which is clearly bounded at any fixed point \((x, t) \in \mathbb{R}^3 \times [0, T]\), so it is
\[
|G(x, t)| \leq e^{\frac{1}{3t}} \cdot K(x, t) \cdot e^{-\frac{1}{4}\varphi(x)}
\]
(21)

As a result, we get the desired expression
\[
\begin{aligned}
\mathbf{u}(x, t) &= -2 \frac{\nabla G(x, t)}{G(x, t)} \\
G(x, t) &= K(x, t) \cdot e^{-\frac{1}{4}\varphi(x)} + \int_0^t \left( F(x, s) \cdot \left\{ K(x, s) \cdot e^{-\frac{1}{4}\varphi(x)} \right\} ds \\
&+ \int_0^s K(x, t - s) \cdot \left[ F(x, s) \cdot \int_0^\infty K(x, s - \tau) \cdot \left( F(x, \tau) \cdot \left[ K(x, \tau) \cdot e^{-\frac{1}{4}\varphi(x)} \right] \right) d\tau \right] ds \\
&+ \int_0^t K(x, t - s) \cdot \left[ F(x, s) \cdot \int_0^\infty K(x, s - \tau) \cdot \left( F(x, \tau) \cdot \int_0^\infty K(x, \tau - \theta) \cdot \left( F(x, \theta) \cdot \left[ K(x, \theta) \cdot e^{-\frac{1}{4}\varphi(x)} \right] \right) d\theta \right] d\tau \right] ds \\
&+ \ldots.
\end{aligned}
\]
(23)

2.2. \(u (x, t) \text{ in } C^\infty(\mathbb{R}^n \times \{0, T\})\)

In order to obtain \(\mathbf{u}(x, t) = -2 \frac{\nabla G(x, t)}{G(x, t)} \in C^\infty(\mathbb{R}^n \times \{0, T\})\), all we need to prove are

1. \(\forall (x, t) \in \mathbb{R}^n \times [0, T], \exists \epsilon_{x,t} > 0, \text{ s.t. } G(x, t) > \epsilon_{x,t}\)
2. \(G(x, t) \in C^\infty(\mathbb{R}^n \times \{0, T\})\)

so,

2.2.1. Floor Estimation of \(G (x, t)\)

For \(G(x, t)\), its bound could be estimated as follows:
1. If \( F(x, t) \geq 0 \),
\[
G(x, t) = K(x, t) + e^{-\frac{1}{2} \varphi(x)} + \int_0^t K(x, t - s) \left( F(x, s) [K(x, s) + e^{-\frac{1}{2} \varphi(x)}] \right) ds
\]
\[
+ \int_0^t K(x, t - s) \left[ F(x, s) \int_0^t K(x, s - \tau) \left( F(x, \tau) [K(x, \tau) + e^{-\frac{1}{2} \varphi(x)}] \right) d\tau \right] ds
\]
\[
+ \int_0^t K(x, t - s) \left[ F(x, s) \int_0^t (K(x, s - \tau) - K(x, \tau)) \left( F(x, \tau) [K(x, \tau) + e^{-\frac{1}{2} \varphi(x)}] \right) d\tau \right] ds
\]
\[
\geq K(x, t) + e^{-\frac{1}{2} \varphi(x)}
\]

2. For the general case: \( F(x, t) \) is no always non-negative, but
\[
\inf_{x \in \mathbb{R}^n} [F(x, t)]
\]
is existing and integrable in \([0, T] \)

Considering the transformation
\[
G(x, t) = e^{\int_0^t \inf_{x \in \mathbb{R}^n} [F(x, s)] ds} \cdot W(x, t)
\]
so
\[
\begin{cases}
\partial_t G = \Delta G + F(x, t) G \\
G(x, 0) = K(x, t) + e^{-\frac{1}{2} \varphi(x)}
\end{cases}
\]
turns out to be
\[
\begin{cases}
\partial_t W = \Delta W + [F(x, t) - \inf_{x \in \mathbb{R}^n} [F(x, t)]] W \\
W(x, 0) = G(x, 0) = K(x, t) + e^{-\frac{1}{2} \varphi(x)}
\end{cases}
\]

According to Case (1), we get
\[
W(x, t) \geq K(x, t) + e^{-\frac{1}{2} \varphi(x)}
\]

Therefore,
\[
G(x, t) = e^{\int_0^t \inf_{x \in \mathbb{R}^n} [F(x, s)] ds} \cdot W(x, t) \geq e^{\int_0^t \inf_{x \in \mathbb{R}^n} [F(x, s)] ds} \cdot \left[ K(x, t) + e^{-\frac{1}{2} \varphi(x)} \right]
\]

Similarly, we can obtain the upper estimation:
\[
G(x, t) \leq e^{\int_0^t \inf_{x \in \mathbb{R}^n} [F(x, s)] ds} \cdot \left[ K(x, t) + e^{-\frac{1}{2} \varphi(x)} \right] \leq e^{\int_0^t \sup_{x \in \mathbb{R}^n} [F(x, s)] ds} \cdot \left[ K(x, t) + e^{-\frac{1}{2} \varphi(x)} \right]
\]

So, we can get a simple estimation:

1. according to our Restriction 4, \( \forall \) bounded closure \( \mathcal{U}, \varphi(x) = \int_0^1 u_0 d x + a \) is bounded in \( \mathcal{U} \), i.e. \( \exists M_0 > 0 \), s.t.
\( e^{-\frac{1}{2} \varphi(x)} \geq M_0 \) in \( \mathcal{U} \). Then, \( \forall (x, t) \in \mathbb{R}^n \times [0, T], \ \exists \delta_{st} > 0 \), s.t.
\[
K(x, t) + e^{-\frac{1}{2} \varphi(x)} = \int_{\mathbb{R}^n} K(x - y, y) \cdot e^{-\frac{1}{2} \varphi(y)} dy \geq \int_0^t K(x - y, y) \cdot e^{-\frac{1}{2} \varphi(y)} dy \geq \int_{\mathcal{U}} K(x - y, y) \cdot M_0 dy \stackrel{def}{=} \delta_{st}
\]

2. Since \( F(x, t) \) are bounded, i.e.
\[
\exists M_1 > 0, \ s.t. \ \inf_{x \in \mathbb{R}^n} [F(x, t)] \geq -M_1
\]
then, \( \exists \varepsilon_{st} > 0 \), s.t.
\[
G(x, t) \geq e^{\int_0^t -M_1 ds} \cdot K(x, t) + e^{-\frac{1}{2} \varphi(x)} \geq e^{-M_1 t} \cdot \delta_{st} \stackrel{def}{=} \varepsilon_{st} \quad \left( (x, t) \in \mathbb{R}^n \times [0, T] \right)
\]

In all, \( \forall (x, t) \in \mathbb{R}^n \times [0, T], \ \exists \varepsilon_{st} > 0, \ s.t. \ G(x, t) > \varepsilon_{st} \).
2.2.2. $G(x,t)$ is Smooth with Respect to $x$

Recalling that:

$$G(x,t) = K(x,t) * e^{-\frac{|x|^2}{4t}} + \int_0^t K(x,t-s) * [F(x,s)G(x,s)]ds$$

Now one can assert that:

$$G(x,t) \in C^{\infty}(\mathbb{R}^n \times [0, T])$$

Recalling that:

$$\forall \alpha \in \mathbb{N}^n \text{ w.r.t. } x:$$

1. $\partial^\alpha G(x,t)$ bounded and exists.

By Mathematical induction, we can assume $\partial^\alpha G(x,t)$ is bounded ($\forall |\beta| < |\alpha|$), then

$$\partial^\beta (FG) \leq F \cdot \partial^\beta G + 1 \leq 1 + |\partial^\beta G|$$

So,

$$|\partial^\beta G| \leq 1 + \int_0^t K(x,t-s) * |\partial^\beta G|ds \leq 1 + \int_0^t K(x,t-s) * 1 ds + \int_0^t K(x, s - \tau) * |\partial^\beta G| d\tau ds$$

$$\leq \cdots \leq 1 + \int_0^t K(x,t-s) * 1 ds + \int_0^t K(x,t-s) * \int_0^s K(x, s - \tau) * 1 d\tau ds$$

$$+ \int_0^t K(x,t-s) * \int_0^s K(x, s - \tau) * 1 d\tau ds + \cdots$$

$$= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t$$

Therefore, $\partial^\beta G(x,t)$ bounded and exists.

2. $\partial^\beta G(x,t)$ is continuous.

Therefore, $G(x,t) \in C^{\infty}(\mathbb{R}^n \cap L^\infty(\mathbb{R}^n))$. Further, $G(x,t) \in C^{\infty}(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T])$.

Now, it is all clearly: for $u(x,t) = -\frac{\nabla G(x,t)}{G(x,t)}$,

$$\forall (x,t) \in \mathbb{R}^n \times [0, T], \quad u(x,t) = -\frac{\nabla G(x,t)}{G(x,t)} \in C^{\infty}(\mathbb{R}^n \times [0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T])$$

(27)

\square

3. Solution of One Dimension Parabolic Equation

The convolution series plays a crucial important role in solving PDEs. Here we give another example as illustration. Our following discussion bases on N.H. Ibragimov’s work[1] on one dimension parabolic equations:

$$u_t + A(t,x) \cdot u_{xx} + a(t,x) \cdot u_x + c(t,x) \cdot u + f(t,x) = 0 \quad (A(t,x) \leq 0)$$

(28)

Let

$$\tau = \phi(t), \quad y = \psi(t,x)$$

then it follows

$$\begin{cases}
    u_t = \phi_t u_t + \psi_t u_y \\
    u_x = \psi_x u_t \\
    u_{xx} = \psi_{xx} u_y + \psi_{xy} u_y
\end{cases}$$
so Eq.(28) becomes

\[ \phi_t u_t + A\psi^2 u_{yy} + (\psi_t + A\psi_{xx} + a\psi)u_y + cu + f = 0 \]

Therefore, by choosing \( \phi, \psi, s.t. \)

\[ \phi_t = 1, \quad A \cdot \psi^2 = -1 \]

i.e.

\[ \tau = t, \quad \psi_x = \sqrt{-\frac{1}{A}} \]

then we get

\[ u_t - u_{yy} + P \cdot u_y + cu + f = 0 \quad (P = \psi_t + A\psi_{xx} + a\psi_x) \] (29)

Employ the transformation

\[ v = e^{-\rho(y)} \cdot u \]

it follows that

\[ u_t - u_{yy} + P \cdot u_y + cu + f = \left[ v_t - v_{yy} + (P + 2\rho_y) \cdot v_y + (\rho_{yy} - \rho_y^2 - \rho_t - P \cdot \rho_y + c)v \right] \cdot e^{-\rho(y)} + f \]

Clearly, if

\[ P + 2\rho_y = 0, \quad \text{or} \quad \rho = -\frac{1}{2} \int_0^y P(t,z)dz \]

Eq.(29) turns out to be

\[ v_t - v_{yy} + Q \cdot v + g = 0 \quad (30) \]

where

\[ Q = \rho_{yy} - \rho_y^2 - \rho_t - P \cdot \rho_y + c = -\frac{1}{2} P_y + \frac{1}{4} P^2 + \frac{1}{2} \int_0^y P_s dy + c, \quad g = \frac{1}{2} \int_0^y P(t,z)dz \]

Now, it is clearly that Eq.(30) is equitant to

\[ v(x,t) = K * v_0 - \int_0^y K(x,t-s) \cdot \left[ Q(x,s) \cdot v(x,s) + g(x,s) \right] ds \] (31)

whose solution could be easily obtained by irritating.

Besides, it is convenient to give the notion

\[ v(x,t) := G + Lv(x,t) = K * v_0 - \int_0^y K(x,t-s) \cdot g(x,s)ds \]

so that we have the expression

\[ v = G + Lv \]

\[ = G + L[G + Lv] = G + LG + LLv \]

\[ = G + LG + LLG + LLLv \]

\[ \cdots \cdots \]

\[ = G + LG + LLLG + LLLLL + \cdots \cdots \] (32)

References

[1] N.H. Ibragimov, Extension of Euler’s method to parabolic equations, Commun Nonlinear Sci Numer Simulat, 1157-1168, 2009.