Naked Singularity of the Vaidya-deSitter Spacetime and Cosmic Censorship Conjecture

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Abstract

We investigate the formation of a locally naked singularity in the collapse of radiation shells in an expanding Vaidya-deSitter background. This is achieved by considering the behaviour of non-spacelike and radial geodesics originating at the singularity. A specific condition is determined for the existence of radially outgoing, null geodesics originating at the singularity which, when this condition is satisfied, becomes locally naked. This condition turns out to be the same as that in the collapse of radiation shells in an asymptotically flat background. Therefore, we have, at least for the case considered here, established that the asymptotic flatness of the spacetime is not essential for the development of a locally naked singularity. Our result then unequivocally supports the view that no special role be given to asymptotic observers (or, for that matter, any set of observers) in the formulation of the Cosmic Censorship Hypothesis.

Keywords: gravitational collapse – naked singularity – cosmic censorship

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1 Introduction

Recently a detailed examination of several gravitational collapse scenarios has shown [1] the development of \textit{locally} naked singularities in a variety of cases such as the collapse of radiation shells, spherically symmetric self-similar collapse of perfect fluid, collapse of spherical inhomogeneous dust cloud [2], spherical collapse of a massless scalar field [3] and other physically relevant situations. It is indeed remarkable that in all these cases families of non-spacelike geodesics emerge from the naked singularity; consequently these cases can be considered to be serious examples of locally naked singularity of strong curvature type as can be verified in each individual case separately. Such studies are expected to lead us to a proper formulation of the Cosmic Censorship Hypothesis.

Note that all the scenarios considered so far (see [1] for details) are spherically symmetric and asymptotically flat, and that the singularity obtained is \textit{locally naked}. We may then ask if the occurrence of a locally naked singularity in these cases is an artefact of the special symmetry. Or, since the real universe has no genuine asymptotically flat objects, whether the local nakedness of the singularity in these cases is, in some way, a manifestation of the asymptotic flatness of the solutions considered.

The question of special symmetry playing any crucial role in these situations is a hard one to settle and this possibility cannot be ruled out easily. However, the question of asymptotic flatness playing any special role in the development of a locally naked singularity, at least in the collapse of radiation shells, is an easy one to settle since the Vaidya metric in an expanding background is already known [4].

It is the purpose of this paper to investigate the collapse of radiation shells in an expanding deSitter background to find out if the locally naked singularity occurs in this situation and to compare any difference with the similar collapse in the asymptotically flat case. We refer the reader to [1] for the details of the latter situation and also for references pertaining to it. We should point out, for the benefit of those interested in the end result, that our conclusion is that the locally naked singularity of the Vaidya-deSitter metric is the same as that obtained in the asymptotically flat case. Therefore, asymptotic flatness of the solutions considered so far does not manifest itself in the nakedness of the singularity arising in these situations. This result then supports the view that the asymptotic observer be not given any special role in the formulation of the cosmic censorship hypothesis [5] as will be discussed later.
2 Outgoing Radial Null Geodesics of the Vaidya-deSitter Metric

The Vaidya-deSitter metric, or the Vaidya metric in a deSitter background, is [4]

\[ ds^2 = - \left[ 1 - \frac{2m(v)}{r} - \Lambda \frac{r^2}{3} \right] dv^2 + 2 dv dr + r^2 d\Omega^2 \]  

(1)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), \( v \) is the advanced time coordinate as is appropriate for the collapse situation, \( \Lambda \) is the cosmological constant and \( m(v) \) is called the mass function. In this form the metric (1) describes the collapse of radiation. The radiation collapses at the origin \( r = 0 \).

As is well-known, the energy-momentum tensor for the radial influx of radiation is:

\[ T_{\alpha \beta} = \rho U_\alpha U_\beta \]

\[ = \frac{1}{4\pi r^2} \frac{dm}{dv} U_\alpha U_\beta \]  

(2)

where the null 4-vector \( U_\alpha \) satisfies

\[ U_\alpha = -\delta_\alpha^v, \quad U_\mu U^\mu = 0 \]

and represents the radial inflow of radiation, in the optic limit, along the world-lines \( v = \text{constant} \). Clearly, for the weak energy condition \( (T_{\alpha \beta} U^\alpha U^\beta \geq 0) \) we require

\[ \frac{dm}{dv} \geq 0 \]  

(3)

to be satisfied.

Now, let us consider the situation of radially injected flow of radiation in an initially empty region of the deSitter universe. The radiation is injected into the spacetime at \( v = 0 \) and, hence, we have \( m(v) = 0 \) for \( v < 0 \) and the metric is that of a pure deSitter universe. [Therefore, the inside of the radiation shells, to begin with, is an empty region of the deSitter metric and not the flat Minkowski metric.] The metric for \( v = 0 \) to \( v = T \) is the Vaidya-deSitter metric representing a Schwarzschild field.
of growing mass $m(v)$ embedded in a deSitter background. The first radiation shell collapses at $r = 0$ at time $v = 0$. The subsequent shells collapse at $r = 0$ successively till $v = T$ when, finally, there is a singularity of total mass $m(T) = m_o$ at $r = 0$. For $v > T$, all the radiation is assumed to have collapsed and the spacetime to have settled to the Schwarzschild field of constant mass $m(T) = m_o$ embedded in a deSitter background [6].

To simplify the calculations, we choose $m(v)$ as a linear function

$$2m(v) = \lambda v, \quad \lambda > 0 \quad (4)$$

This linear mass-function was introduced by Papapetrou [7] in the asymptotically flat case of the Vaidya metric. Hence, in our case, the Vaidya-Papapetrou-deSitter spacetime is described by the following mass function for the metric (1):

$$m(v) = 0 \quad v < 0 \quad \text{pure deSitter}$$

$$2m(v) = \lambda v \quad 0 < v < T \quad \text{Vaidya-deSitter} \quad (5)$$

$$m(v) = m_o \quad v > T \quad \text{Schwarzschild-deSitter}$$

We note at the outset that the Vaidya-deSitter spacetime for linear mass-function as in (5) is not homothetically Killing unlike the asymptotically flat Vaidya metric. In fact, the line element (1) does not admit any proper conformal Killing symmetries.

Consider the geodesic equations of motion for the Vaidya-deSitter metric as in (1). Let the tangent vector of a geodesic be

$$K^\alpha = \frac{dx^\alpha}{dk} \equiv \left( \dot{v}, \dot{r}, \dot{\theta}, \dot{\phi} \right) \quad (6)$$

or, equivalently,

$$K_\alpha \equiv g_{\alpha\beta}K^\beta = \left( K_v, K_r, K_\theta, K_\phi \right)$$

$$\equiv \left( g_{vv}K^v + K^r, K^v, r^2K^\theta, r^2\sin^2\theta K^\phi \right)$$

Then, the geodesic equations can be obtained from the Lagrangian

$$2\mathcal{L} = K_\alpha K^\alpha \quad (7)$$
For our purpose here it is sufficient to consider only the radially outgoing, future-directed, null geodesics originating at the singularity. Such geodesics can be obtained directly from the above Lagrangian as the following equation:

\[
\frac{dv}{dr} = \frac{2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}
\]  

which, for the linear mass function as in (5), is:

\[
\frac{dv}{dr} = \frac{2}{1 - \lambda \frac{v}{r} - \frac{\Lambda r^2}{3}}
\]  

Now for the geodetic tangent to be uniquely defined and to exist at the singular point, \( r = 0, \ v = 0 \), of equation (9) the following must hold

\[
\lim_{v \to 0} \frac{v}{r} = \lim_{v \to 0} \frac{dv}{dr} = X_0
\]  

say, and when the limit exists, \( X_0 \) is real and positive. In this last situation, we obtain a future-directed, non-spacelike geodesic originating from the singularity \( r = 0, v = 0 \) if we further demand that \( 2\mathcal{L} \leq 0 \). Then, the singularity will, at least, be \textit{locally naked}. On the other hand, if there is no real and positive \( X_0 \), then there is no non-spacelike geodesic from the singularity to any observer and, hence, the singularity is not visible to any observer. Then, we may show that the singularity is covered by a null hypersurface (the horizon) and the spacetime is a black hole spacetime.

Then as we approach the singular point of the differential equation (9) we have, using equations (9) and (10),

\[
2 - X_0 + \lambda X_0^2 = 0
\]  

after suitable rearrangement of the terms. Thus for the real values of the tangent to a radially outgoing, null, future-directed geodesic originating in the singularity we obtain

\[
X_0 = a_\pm = \frac{1 \pm \sqrt{1 - 8\lambda}}{2\lambda}
\]  

Clearly, we require

\[
\lambda \leq \frac{1}{8}
\]
for \( X_o = \lim_{v \to 0} r \to 0 v/r \) to be real in the situation considered.

Note that the equation (10) is the same as that obtained by Dwivedi & Joshi [8] when the metric (1) is asymptotically flat i.e., \( \Lambda = 0 \). Consequently the values \( a_{\pm} \) for the geodetic tangent and the condition (11) for these values to be real are the same as those obtained for the asymptotically flat situation when the mass function \( m(v) \) is linear in \( v \) as in equation (4).

3 Discussion

The present-day picture of the gravitational collapse imagines that a sufficiently massive body compressed in too small a volume undergoes an unavoidable collapse leading to a singularity in the very structure of the spacetime. Of course, the deduction that a singularity will form as a result of such collapse tacitly assumes that we disregard those principles of the still-ellusive quantum theory of gravity which alter the nature of the spacetime from that given by the classical theory of gravitation - the general theory of relativity.

Within the limits of applicability of the classical general relativity, we characterize such unavoidable collapse by demanding the existence of a point or of a hypersurface, called the \textit{trapped surface}, whose future lightcone begins to reconverge in every direction along the cone. The deduction that a spacetime singularity will form is then obtained from the well-known Hawking-Penrose Singularity Theorems [9]. These theorems require further physically reasonable assumptions such as the positivity of energy and total pressure, the absence of closed timelike curves and some notion of the genericity of the collapse situation. However, note that the existence of a trapped surface does not imply the absence of a naked singularity or its absence does not imply the presence of a naked singularity. The assumption of a trapped surface (or some other equivalent assumption) is, however, required to infer the occurrence of the spacetime singularity. [ See [9] for further details on this and other related issues. ]

Now, our notion of the classical black hole situation is that of a spacetime singularity completely covered by an absolute event horizon. Unfortunately, the chronology of the developments related to now-famous black hole solutions emphasized the observers at future null infinity, \( I^+ \), in earlier ideas of the cosmic censor. We note that there is no theory concerning what happens as a result of the appearance of a spacetime
singularity. And, hence, the observer witnessing any such singularity will not be able to account for the observed physical behaviour of processes involving the singularity in any manner whatsoever. The cosmic censorship is then necessary to avoid precisely such situations. The black hole solutions, while emphasizing the role of observers at the future null infinity, led us into demanding that in the region between the absolute event horizon - the boundary $\partial I^+(\mathcal{I}^+)$ of the past of $\mathcal{I}^+$ - and the set of observers at infinity, $\mathcal{I}^+$, no spacetime singularity occurs.

However, it is not hard to imagine a situation in which an observer and a collapsing body, both, are within a larger trapped surface. Thus, no information reaches $\mathcal{I}^+$ from this region. But, that trapped observer would be able to witness the forming spacetime singularity. We are, in essence, discussing here the case of a locally naked singularity. For such an observer, however, it would be impossible to account for the physical behaviour of systems involving the singularity since there is no theory for that. The purpose of a cosmic censor, being that of avoiding precisely such unpredictable physical situations for legitimate observers, is then lost on its formulation in terms of the observers at infinity since any such formulation cannot help the above observer.

It is for avoiding such situations that we require some reasonable formulation of the cosmic censorship which does not single out the set of observers at infinity. One such formulation is that of Strong Cosmic Censorship as given by Penrose [5].

Since our main interest here is to explore the role of asymptotic flatness in the development of a naked singularity in the situation of collapsing radiation shells, further analysis than that presented in Section 2 is not necessary to draw definite conclusions about it. The very fact that we have obtained a condition for the occurrence of a naked singularity in the collapse of radiation shells in an expanding background which is the same as that obtained when the background is non-expanding and asymptotically flat establishes that it is not the asymptotic flatness of the solutions considered that manifests, in some sense, in the development of a locally naked singularity. In other words, whether the spacetime is asymptotically flat or not does not make any difference to the occurrence of a locally naked singularity. This is evident in at least the situation of collapsing radiation shells as considered here.

Furthermore the example considered above shows that the asymptotic observer has no role to play in the occurrence or non-occurrence of a naked singularity.
in the collapse of radiation shells. This means that the same asymptotic observer cannot have any special role to play in the formulation of the Cosmic Censorship Hypothesis which is being envisaged as a basic principle of nature, a physical law. Also, the above result is then consistent with the viewpoint that if the cosmic censorship is to be any basic principle of nature then it has to operate at a local level. Hence, no special role can be given to any set of observers in the formulation of such a basic principle; since the general theory of relativity as a theory of gravitation provides no fundamental length scale. Then, the present result unequivocally supports Penrose’s [5] Strong Cosmic Censorship Hypothesis which, in essence, states that singularities should not be visible to any observer or, equivalently, no observer sees a singularity unless and until it is actually encountered.
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