Theoretical study of boundary instability due to internal medium perturbations

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Abstract. The ignition of laser-initiated inertial confined fusion (ICF) targets is still not reached, despite considerable efforts taken in this direction. The exact physical reasons for this failure are still questionable, but there are significant hints for the significant role of uncontrollable hydrodynamics instabilities, that prevent ignition. Many papers are devoted to such instabilities, special attention is paid to the Rayleigh–Taylor and Richtmeyer–Meshkov instabilities of accelerated perturbed interfaces. In this paper we consider an alternative variant, when the interface is initially smooth and perturbations are given inside of one medium. During acceleration the perturbations are carried from medium interior to the interface even if conditions for the Rayleigh–Taylor instability are not fulfilled. This initiates the instability and mediums mixing. The effect may be important for ICF targets, where it is crucial to support target symmetry and purity of each layer during compression.

1. Introduction
The development of hydrodynamics instabilities is governed by flow parameters (its gradients, velocities, etc.) and medium properties (equation of state, viscosity, etc.). Each instability type has its own criterion that shows whether it grows or is stable. An instability is seeded by small flow perturbations. The initial amplitudes of these perturbations specify time for an instability to develop: the smaller initial amplitude is – the later instability affects the flow. In case the flow is always unstable, the effect of an instability could be partially diminished by minimizing initial amplitudes of perturbations, what can be done during production.

The instability of accelerated interfaces are broadly discussed in publications, see e.g. [1, 2]. Most papers consider Rayleigh–Taylor instability [3, 4], gravitational surface waves [7], Richtmyer–Meshkov instability [5, 6], etc. Conventionally in these studies, the seeds are initial perturbations of interface itself. In this paper we consider an alternative setup: mediums interface is initially smooth and perturbations are given inside of one medium.

Such intrinsic perturbations could initiate instability development, what was numerically studied in several papers. See, e.g. [8–12], where the influence of different initial perturbations (interface and internal) on the hot–spot shape in NIF targets is considered. The paper has the following structure. Section 2 presents the problem setup and approximations in details. Section 3 contain analytical analysis of internal perturbations development. In Section 4 we compare analytics with numerical simulations.

2. Problem setup
Two semi-infinite mediums fill the whole space and have a plain interface (see Fig. 1). Near the interface mediums have densities $\rho_1, \rho_2$. The whole system is accelerated with $g$ at some moment. Accidental acceleration initiate dynamical processes that form the counter-pressure. We assume that characteristic time of counter-pressure formation is much smaller than instability times. So our analysis starts from the moment when mediums are at rest and in equilibrium: in the interface frame the noninertial force is compensated by the gradient of pressure. The exact equilibrium is possible only for stratified density, the appearance of multidimensional perturbations makes the system unstable.
We will use ideal gas equation of state for both mediums:

\[ P = \Gamma \rho \zeta, \]  

where \( \Gamma = (\gamma - 1) \), \( \gamma \) – adiabatics constant, \( \zeta \) – specific thermal energy. Such equation of state suits well for the high–temperature plasma, when the ionization state have small variations. For simplicity we consider the identical equations of state for mediums 1 and 2.

Hydrodynamics equations describe evolution of the system:

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0, \quad \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \nabla) \mathbf{v} = -\nabla P + \mathbf{g}, \quad \frac{dE}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} = 0.  
\]

Where \( d/dt = \partial/\partial t + (\mathbf{v} \nabla) \) – the Lagrangian derivative. The boundary condition should be set at the interface and because of continuity we have:

\[ P_1 = P_2, \quad v_{n,1} = v_{n,2}. \]

The presence of external acceleration force leads to mediums stratification. The hydrodynamic equations give the following equilibrium density distribution in the presence of external acceleration force for each medium (for uniform temperature):

\[ \rho = \rho_{CD} \exp \{-gz/\Gamma \zeta\}, \]

where \( \rho_{CD} \) – density at the interface.

The instability is initiated by multidimensional perturbations. Suppose, one medium (medium 1) initially has density perturbations \( \rho_1 + \delta \rho_1 \):

\[ \delta \rho_1 = A_{\delta \rho_1} \exp \{ik_0z + ik_0y\}. \]
3. Linear perturbation analysis

Linearized equations (2) for medium \( j \) \((j = 1, 2)\) are

\[
\begin{align*}
\frac{\partial \delta \rho_j}{\partial t} + \delta v_{z,j} \frac{\partial \rho_j}{\partial z} + \rho_j \left( \frac{\partial \delta v_{z,j}}{\partial z} + \frac{\partial \delta v_{y,j}}{\partial y} \right) &= 0, \\
\rho_j \frac{\partial \delta v_{z,j}}{\partial t} + \frac{\partial \delta P_j}{\partial z} + \delta \rho_j g &= 0, \\
\rho_j \frac{\partial \delta v_{y,j}}{\partial t} + \frac{\partial \delta P_j}{\partial y} &= 0,
\end{align*}
\]

(6) (7) (8)

From (7) we have

\[
\rho_j \frac{\partial \delta \zeta_j}{\partial t} + \delta v_{z,j} \frac{\partial \delta \zeta_j}{\partial z} - \frac{\Gamma \delta \rho_j}{\rho} \left( \frac{\partial \delta v_{z,j}}{\partial t} + \delta v_{z,j} \frac{\partial \rho_j}{\partial z} \right) = 0,
\]

(9)

\[
\delta P_j - \frac{\partial P_j(\rho_j, E_j)}{\partial \rho_j} \delta \rho_j - \frac{\partial P_j(\rho_j, E_j)}{\partial E_j} \delta E_j = 0.
\]

(10)

Eqns. (6)–(8) give

\[
\frac{\partial^2 \delta \rho_j}{\partial t^2} - \left( \frac{\partial^2 \delta P_j}{\partial z^2} + \frac{\partial^2 \delta P_j}{\partial y^2} \right) - \frac{\partial \delta \rho_j}{\partial z} g = 0.
\]

(11)

From (9) and (10) we have

\[
\frac{\partial^2 \delta P_j}{\partial t^2} - c_j^2 \frac{\partial^2 \delta \rho_j}{\partial t^2} - \frac{c_j^2}{g} \left( \frac{\partial \delta P_j}{\partial z} + \delta \rho_j g \right) N_j^2 = 0,
\]

(12)

where \( N_j^2 = -g \left( g/c_j^2 + 1/\rho_j \partial \rho_j/\partial z \right) \) is Brunt–Väisälä frequency, the general parameter of intrinsic gravitational waves \([13]\).

Eqns. (11)–(12) are valid for the general equation of state. With (1) one could write (12) as

\[
\frac{\partial^2 \delta P_j}{\partial t^2} - c_j^2 \frac{\partial^2 \delta \rho_j}{\partial t^2} - \Gamma_j g \left( \frac{\partial \delta P_j}{\partial z} + \delta \rho_j g \right) = 0.
\]

(13)

3.1. Laplace transform analysis

Equations (11) and (13) can be solved using Laplace transform \([14]\): \( \hat{f}(s) = \int_0^{\infty} e^{-st} f(t) dt \), where \( s = s' + is'' \) is a complex variable. The inverse transform: \( f(t) = 1/2\pi i \int_{\ell-i\infty}^{\ell+i\infty} e^{st} \hat{f}(s) ds \), with \( \ell \) — some real constant value.

Eqns. (11) and (13) for \( j \)-medium give

\[
\hat{s}^2 \delta \hat{P}_j - \hat{s} e^{i k_0 y} + i k_0 \hat{\rho}_j - \frac{\partial^2 \delta \hat{P}_j}{\partial z^2} + \frac{i k_0 \delta \rho_j}{\partial z} = 0,
\]

(14)

\[
\Delta \hat{s}^2 \delta \hat{P}_j - \hat{s} \left( \hat{s} \delta \hat{\rho}_j - e^{i k_0 y} + i k_0 \hat{\rho}_j \right) - \Delta \left( \frac{\partial \delta \hat{P}_j}{\partial z} + \delta \hat{\rho}_j \right) = 0.
\]

(15)

Where the following non-dimensional variables are defined \( \hat{s} = s \sqrt{g}/z, \hat{\zeta} = \delta \rho_j g/\beta, \hat{P}_j = \delta P_j g/\beta^3 \delta \rho_1(+0), \Delta = \Gamma/\Gamma_1, \beta = \Gamma \zeta_1 \).

From (7) we have

\[
\delta \hat{v}_{z,j} = -\left( \frac{\partial \hat{\rho}_j}{\partial \hat{z}} + \frac{\Delta}{\Gamma_j} \hat{\rho}_j \right) \hat{s} + 1 \delta \hat{\rho}_1(+0),
\]

(16)
where \( \delta \tilde{v}_{z,j} = \delta v_{z,j}g/\beta, \delta \tilde{p}_1(+) = \delta \rho_1(+) / \rho_1 \).

According to [15] the velocity near the interface is

\[
\delta \tilde{v}_{z,1}(t) = \int_{\ell-i\infty}^{\ell+i\infty} \frac{\delta \tilde{v}_{z,1}(s)e^{i\tilde{s}i}}{2\pi i} ds + \frac{A\delta \rho_1}{2\rho_1} \sum_{\bar{\sigma}_i} \frac{\left( \frac{\bar{\sigma}_i^2}{\Gamma + 1} + \tilde{k}_{0y}^2 \right) \exp \{ \bar{\sigma}_i t \}}{i\tilde{k}_{0z} - (\tilde{k}_{0z}^2 + \tilde{k}_{0y}^2) \frac{2\bar{\sigma}_i^2}{\Gamma + 1}} e^{i\tilde{k}_{0y}y} \tag{17}
\]

with pole residues

\[
\bar{\sigma}_i = \pm \sqrt{\frac{\Gamma}{2\Delta}} \left( i\tilde{k}_{0z} - \tilde{k}_{0z}^2 - \tilde{k}_{0y}^2 \right) \pm \sqrt{\left( \tilde{k}_{0z}^2 + \tilde{k}_{0y}^2 - i\tilde{k}_{0z} \right)^2 - 4\tilde{k}_{0y}^2 \Delta},
\]

and

\[
\delta \tilde{v}_{z,1}^{(1)} = - \frac{\sqrt{\frac{s^2}{4} + \frac{\Delta s^4 + \tilde{k}_{0y}^2 (s^2 + \Delta)}{\Gamma}}}{s^2 + \Delta} \delta \tilde{p}_1(+) e^{i\tilde{k}_{0y}y},
\]

\[
C_{1,1}(\tilde{s}) = \left[ \frac{\sqrt{s^2 + \Delta} \left( \tilde{k}_{0z}^2 + \tilde{k}_{0y}^2 - i\tilde{k}_{0z} \right) s^2 + \Delta + \tilde{k}_{0z}^2}{s^2 + \Delta} \right] - 1 \times \frac{1}{\sqrt{s^2 + \Delta} \frac{s^2}{4} + \frac{\Delta s^4 + \tilde{k}_{0y}^2 (s^2 + \Delta)}{\Gamma}} \eta
\]

The result is derived under assumptions that EOS constants for both mediums are identical, density of both mediums are almost equal \( \rho_1 / \rho_2 \sim (1 + \eta), \) and \( \eta \ll 1, \) the same for temperatures. Integral in Eq. (17) is evaluated over integration path that takes into account poles distribution and points of branching. Integrals over cut edges are negligible, so finally

\[
\delta \tilde{v}_{z,1}^{(1)}(t) = \sum_{s=\bar{\sigma}_i} \frac{(s - \bar{\sigma}_i) \left( i\tilde{k}_{0z} + \frac{\Delta}{s} \right) (\tilde{s} - \bar{\sigma}_1)(\tilde{s} - \bar{\sigma}_2)(\tilde{s} - \bar{\sigma}_3)(\tilde{s} - \bar{\sigma}_4)}{\rho_1} A\delta \rho_1 e^{i\tilde{s}i}.
\]

Amplitude of perturbation can be calculated for any moment, when linear approximation is valid:

\[
\tilde{a}(\tilde{t}) = \int_0^{\tilde{t}} \delta \tilde{v}_{z,1}(\tilde{t}) d\tilde{t}.
\]

**4. Numerical simulations**

To check analytical results we study the effect numerically. It could be important for ICF targets dynamics, as they have multi-layer structure: e.g. the target proposed for the Russian laser megajoule facility [17] have structure similar to NIF’s targets [16] and contain 3 layers: DT gas, DT ice and a plastic ablator. In such design the gas–ice interface is known to be stable.
at acceleration phase. Though initially gas–ice interface is smooth, internal perturbations of ice medium could emerge due to nonuniform target irradiation or external layers perturbations: the ideal conditions for the instability studied above. Consequently the interface could be unstable. This effect for given targets is not crucial, because it leads to the mix of identical substances. The potential impact on ignition efficiency may be related to the penetration of cold dense ice pieces into the hotspot, what increases its heterogeneity. Nevertheless, we are interested in the very fact of instability as it could have more evident effect on another target designs.

According to hydrodynamic simulations of target [17] produced by code FRONT [18] the typical plasma parameters during acceleration phase on the ice–ablator interface are: $P = 300$ GPa, $\rho_1 = 0.0144$ g/cm$^3$ (ice), $\rho_2 = 0.014$ g/cm$^3$ (gas). In simulations we set uniform density for each medium. Because of hydrodynamic equilibrium, gas and ice densities are close to each other for significant part of the acceleration stage. The average acceleration value is $\sim 5 \times 10^4$ m/s$^2$. We suppose that the ice density has single–mode perturbation: $A\delta\rho_1/\rho_1 = 0.01$ and wavelengths $\lambda_{0y} = 100$ µm and $\lambda_{0z} = 300$ µm.

Figs. 2 and 3 shows the initial density distribution and the result of 10 ns evolution. Initially plain interface is curved, though densities of two mediums at the interface are equal (the conditions for the RT instability are not fulfilled). In numerical simulations the interface is defined as a surface where local density parts of each medium are equal. Fig. 4 compares the time dependence of maximum amplitude value obtained in simulations with analytical result calculated using Eq. (18). First of all, the increasing rate of amplitude growth means the instability regime. The linear theory describes the early stage of the instability well ($t < 2$ ns). The final amplitude at $t = 10$ ns is $\sim 17$ µm for simulations and 6 µm for linear theory, what shows that for large amplitudes instability grows faster. The difference could be due to stratification that is taken into account in analytical analysis. Also, the asymmetry of perturbation that is observed in Fig. 3 indicates the significant role of nonlinear processes, what is not included in the theory. The goal of the study is to show the presence of the effect and check its linear stage.

5. Conclusions
The paper considers the instability of accelerated interface of two mediums in case when the interface is initially smooth and perturbations are present only inside one medium. It appears...
Figure 4. Time dependence of gas–ice interface amplitude from simulations (“FRONT”) and analytical result (“Theory”).

appears even if conditions for the Rayleigh–Taylor instability development are not fulfilled. In the paper the case of equal densities of two mediums is analyzed. The linear stage of instability is well described with the elaborated analytical model. At nonlinear stage the instability grows faster than predicts analytical model. The effect could be important for ICF targets as it leads to the instability of interfaces that are usually stable at acceleration stage.

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