A $\delta$-FIRST WHITEHEAD LEMMA

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Abstract. We prove that $\delta$-derivations of a simple finite-dimensional Lie algebra over a field of characteristic zero, with values in a finite-dimensional module, are either inner derivations, or, in the case of adjoint module, multiplications by a scalar, or some exceptional cases related to $sl(2)$. This can be viewed as an extension of the classical first Whitehead Lemma.

1. Introduction

Let $L$ be a Lie algebra, $V$ an $L$-module, with the module action denoted by $\bullet$, and $\delta$ is an element of the base field. Recall that a $\delta$-derivation of $L$ with values in $V$ is a linear map $D : L \to V$ such that

$$D([x, y]) = -\delta y \bullet D(x) + \delta x \bullet D(y)$$

for any $x, y \in L$. The set of all such maps for a fixed $\delta$ forms a vector space which will be denoted by $\text{Der}_\delta(L, V)$.

In the case $V = L$, the adjoint module, we speak about just $\delta$-derivation of $L$. The latter notion generalizes simultaneously the notions of derivation (ordinary derivations are just 1-derivations) and of centroid (any element of the centroid is, obviously, a $\frac{1}{2}$-derivation).

$\delta$-derivations of Lie and other classes of algebras were a subject of an intensive study (see, for example, [Fi1]–[Fi3], [H], [Z], [LL] and references therein; the latter paper is devoted to a more general notion of so-called quasiderivations which we do not discuss here). As a rule, algebras from “nice” classes (simple, prime, Kac–Moody, Lie algebras of vector fields, etc.) possess a very “few” nontrivial $\delta$-derivations. On the other hand, there are very few results, if at all, about $\delta$-derivations with values in modules.

The aim of this paper is to prove that $\delta$-derivations of simple finite-dimensional Lie algebras of characteristic zero, with values in finite-dimensional modules, are, as a rule, just inner 1-derivations. The exceptional cases are identity maps with values in adjoint modules (which are $\frac{1}{2}$-derivations), or are related to $sl(2)$ which, unlike all other simple Lie algebras, possesses nontrivial $\delta$-derivations. (Note the occurrence of the new exceptional values of $\delta$, in addition to the exceptional values $\delta = -1, \frac{1}{2}, 1$ previously known from the literature). The exact statement runs as follows.

Main Theorem. Let $g$ be a semisimple finite-dimensional Lie algebra over an algebraically closed field $K$ of characteristic 0, $V$ a finite-dimensional $g$-module, and $\delta \in K$. Then $\text{Der}_\delta(g, V)$ is nonzero if and only if one of the following holds:

(i) $\delta = 1$, in which case $\text{Der}_1(g, V) \cong V$ and consists of inner derivations of the form $x \mapsto x \bullet v$ for some $v \in V$.

(ii) $\delta = -\frac{2}{n}$ for some integer $n \geq 1$, or $\delta = \frac{2}{m+2}$ for some integer $n \geq 3$, or $\delta = \frac{1}{2}$, and $V$ is decomposable into the direct sum of irreducible $g$-modules in such a way that each direct summand of $V$ is a nontrivial irreducible module over exactly one of the simple direct summands of $g$, and a trivial module over the rest of them.

In the latter case, decomposing $g$ into the direct sum of simple algebras: $g = g_1 \oplus \cdots \oplus g_m$, and writing

$$V = V_1^1 \oplus \cdots \oplus V_1^{k_1} \oplus V_2^1 \oplus \cdots \oplus V_2^{k_2} \oplus \cdots \oplus V_m^1 \oplus \cdots \oplus V_m^{k_m},$$

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where \( V^j_i \) is an irreducible module over \( g_i \), and a trivial module over \( g_\ell \), \( \ell \neq i \), we have:
\[
\text{Der}_{\frac{2}{n}}(g, V) \cong \bigoplus_{g_i = sl(2)} \bigoplus_{V^j_i \cong V(n)} \mathcal{D}^j_i,
\]
where \( n \geq 1 \), \( V(n) \) is the \((n+1)\)-dimensional irreducible \( sl(2) \)-module, and \( \mathcal{D}^j_i \) is the \((n+3)\)-dimensional vector space of \((-\frac{2}{n})\)-derivations of \( g_i \cong sl(2) \) with values in \( V^j_i \cong V(n) \), as described in Lemma 9(ii).

\[
\text{Der}_{\frac{2}{n+2}}(g, V) \cong \bigoplus_{g_i = sl(2)} \bigoplus_{V^j_i \cong V(n)} \mathcal{E}^j_i,
\]
where \( n \geq 3 \), \( \mathcal{E}^j_i \) is the \((n-1)\)-dimensional vector space of \(\frac{2}{n+2}\)-derivations of \( g_i \cong sl(2) \) with values in \( V^j_i \cong V(n) \), as described in Lemma 9(iii).

\[
\text{Der}_{\frac{2}{2}}(g, V) \cong \bigoplus_{i=1}^{m} \bigoplus_{V^j_i \cong g_i} K^j_i,
\]
where the inner summation is carried over all occurrences of \( V^j_i \) being isomorphic to the adjoint module \( g_i \), and \( K^j_i \) is the one-dimensional vector space spanned by this isomorphism, considered as a map \( g_i \to V^j_i \).

Since \( \delta \)-derivations do not change under field extensions, this theorem essentially describes \( \delta \)-derivations of a semisimple finite-dimensional Lie algebra with values in a finite-dimensional module, over an arbitrary field of characteristic zero. However, the formulation in the case of an arbitrary field would involve forms of algebras and modules in the \( sl(2) \)-related cases, and elements of centroid instead of identity maps in the case \( \delta = \frac{1}{2} \), and would be even more cumbersome, so we confine ourselves with the present formulation.

The classical first Whitehead Lemma states that for \( g \) and \( V \) as in the statement of the main theorem, the first cohomology vanishes: \( H^1(g, V) = 0 \). As the first cohomology is interpreted as the quotient of derivations of \( g \) with values in \( V \) modulo inner derivations, and ordinary derivations are just \( 1 \)-derivations, this theorem can be viewed as an extension of the first Whitehead Lemma. The standard proof of the first Whitehead Lemma involves the Casimir operator (see, for example, [J] Chapter III, §7, Lemma 3) and will not work in the case \( \delta \neq 1 \). Moreover, taken verbatim, the first Whitehead Lemma is not true for arbitrary \( \delta \)-derivations, as the exceptional cases related to \( sl(2) \), and to the value \( \delta = \frac{1}{2} \) show. Therefore we employ a different approach, which, however, amounts to mere straightforward manipulations with the \( \delta \)-derivation equation (1), and utilizing standard facts about semisimple Lie algebras and their representations (as exposed, for example, in the classical treatises [Bo] and [J]).

On the other hand, this theorem is a generalization of the result saying that all nontrivial \( \delta \)-derivations of simple finite-dimensional Lie algebras of characteristic zero are either ordinary derivations (\( \delta = 1 \)), or multiple of the identity map (\( \delta = \frac{1}{2} \)), or some special family of \((-1)\)-derivations in the case of \( sl(2) \) (see [Fi2] Corollary 3, [Fi3] Theorem 2 and Corollary 1, or [LL] Corollary 4.6).

Our initial interest in such sort of results stems from [ZZ], where we computed \( \delta \)-derivations of certain nonassociative algebras which are of interest in physics (what, in its turn, helped to determine symmetric associative forms on these algebras). These algebras have some classical Lie algebras like \( sl(n) \) and \( so(n) \) as subalgebras, and considering restriction of \( \delta \)-derivations to these subalgebras, and employing the theorem above, would allow to streamline some of the proofs in [ZZ].

2. AUXILIARY LEMMAS

The proof of the main theorem consists of a series of simple lemmas. The ground field \( K \) is assumed to be arbitrary, unless stated otherwise.
Lemma 1. Let $L$ be a Lie algebra, and let an $L$-module $V$ be decomposable into a direct sum of submodules: $V = \bigoplus_i V_i$. Then for any $\delta \in K$,

$$\text{Der}_\delta(L, V) \simeq \bigoplus_i \text{Der}_\delta(L, V_i).$$

(Here and below the direct sum $\oplus$ is understood in an appropriate category: either vector spaces, or Lie algebra modules, or Lie algebras, what should be clear from the context).

*Proof.* The proof is trivial, and repeats the proof of the similar statement for ordinary derivations. Each direct summand $\text{Der}_\delta(L, V_i)$ is obtained by composition of $\delta$-derivations from $\text{Der}_\delta(L, V)$ with the canonical projection $V \to V_i$. \(\square\)

Lemma 2. Let $L_1$ and $L_2$ be Lie algebras, and $V$ is simultaneously an $L_1$- and an $L_2$-module. Then for any nonzero $\delta \in K$,

$$\text{Der}_\delta(L_1 \oplus L_2, V) \simeq \{(D_1, D_2) \in \text{Der}_\delta(L_1, V) \oplus \text{Der}_\delta(L_2, V) \mid x_1 \cdot D_2(x_2) = x_2 \cdot D_1(x_1) \text{ for any } x_1 \in L_1, x_2 \in L_2\}.$$ 

(Here and below, $\cdot$ denotes the action of a Lie algebra on its module).

*Proof.* Let $D$ be a $\delta$-derivation of $L_1 \oplus L_2$ with values in $V$. Its restrictions $D_1$ and $D_2$, on $L_1$ and $L_2$ respectively, are, obviously, $\delta$-derivations with values in $V$, and the condition (1) written for arbitrary pair $x_1 \in L_1$ and $x_2 \in L_2$, is equivalent to the equality

$$x_1 \bullet D_2(x_2) = x_2 \bullet D_1(x_1).$$

\(\square\)

Recall that a Lie algebra $L$ is called *perfect* if it coincides with its own commutant: $[L, L] = L$.

Lemma 3. Let $L_i$ be a perfect Lie algebra, and $V_i$ an $L_i$-module, $i = 1, \ldots, n$. Then for any $\delta \neq 1$,

$$\text{Der}_\delta(L_1 \oplus \cdots \oplus L_n, V_1 \otimes \cdots \otimes V_n) \simeq \bigoplus_{i=1}^n \left(V_1^{L_i} \otimes \cdots \otimes V_n^{L_i} \otimes \text{Der}_\delta(L_i, V_i) \otimes V_i^{L_i+1} \otimes \cdots \otimes V_n^{L_n}\right).$$

Here the action of $L_1 \oplus \cdots \oplus L_n$ on the tensor product $V_1 \otimes \cdots \otimes V_n$ is assembled, as usual, from the actions of $L_i$'s on $V_1 \otimes \cdots \otimes V_n$, where $L_i$ acts on the tensor factor $V_i$, leaving all other factors intact; $V_i^L$ denotes the submodule of invariants of an $L_i$-module $V$.

*Proof.* As the claim is trivial in the case $\delta = 0$, we may assume $\delta \neq 0, 1$. We will prove the case $n = 2$, the general case easily follows by induction.

Since $V_1 \otimes V_2$, as an $L_1$-module, is isomorphic to a direct sum of a number of copies of $V_1$, parametrized by $V_2$, by Lemma 1 we have

$$\text{Der}_\delta(L_1, V_1 \otimes V_2) \simeq \text{Der}_\delta(L_1, V_1) \otimes V_2,$$

and similarly,

$$\text{Der}_\delta(L_2, V_1 \otimes V_2) \simeq V_1 \otimes \text{Der}_\delta(L_2, V_2).$$

Apply Lemma 2, assuming $V = V_1 \otimes V_2$. The condition (1), written for

$$\sum_{i \in I} D_i^1 \otimes v_2^i \in \text{Der}_\delta(L_1, V_1) \otimes V_2,$$

and

$$\sum_{i \in J} v_1^i \otimes D_2^i \in V_1 \otimes \text{Der}_\delta(L_2, V_2),$$

where $D_1^i \in \text{Der}_\delta(L_1, V_1), D_2^i \in \text{Der}_\delta(L_2, V_2), v_1^i \in V_1, v_2^i \in V_2$, is equivalent to

$$\sum_{i \in I} (x_1 \bullet v_2^i) \otimes D_2^i(x_2) = \sum_{i \in I} D_1^i(x_1) \otimes (x_2 \bullet v_2^i).$$

It follows that one of the following holds:
\( i \) \( D^1_i(x_1) = x_1 \cdot u_1^i \) for some \( u_1^i \in V_1 \), any \( x_1 \in L_1 \) and any \( i \in \mathbb{I} \), and \( D^2_i(x_2) = x_2 \cdot u_2^i \) for some \( u_2^i \in V_2 \), any \( x_2 \in L_2 \) and any \( i \in \mathbb{I} \).

\( ii \) \( v_1^i \in V_1^{L_1} \) for any \( i \in \mathbb{I} \), and \( v_2^i \in V_2^{L_2} \) for any \( i \in \mathbb{I} \).

In the case \( (i) \), we have that \( D^1_i \) is simultaneously a \( \delta \)-derivation and an 1-derivation, what implies \((\delta - 1)D^1_i([L,L]) = 0\), and hence \( D^1_i \) vanishes for any \( i \in \mathbb{I} \). Similarly, \( D^2_i \) vanishes for any \( i \in \mathbb{I} \). Consequently, we are in case \( (ii) \), and

\[
\text{Der}_\delta(L_1 \oplus L_2, V_1 \otimes V_2) \simeq \left( \text{Der}_\delta(L_1, V_1) \otimes \text{Der}(L_2, V_2) \right) \oplus \left( V_1^{L_1} \otimes \text{Der}_\delta(L_2, V_2) \right),
\]

what is the particular case of formula \((3)\) for \( n = 2 \).

**Lemma 4.** In the conditions of Lemma \((3)\) assume additionally that each \( V_i \) is irreducible. Then \( \text{Der}_\delta(L_1 \oplus \cdots \oplus L_n, V_1 \otimes \cdots \otimes V_n) \) is:

(1) zero, if either \( V_i = K \) for all \( i \), or there are at least two different \( i \)'s such that \( V_i \neq K \);

(2) isomorphic to \( \text{Der}(L_i, V_i) \), if there is exactly one \( i \) such that \( V_i \neq K \).

**Proof.** Since each \( V_i \) is irreducible, either \( V_i^{L_i} = 0 \), or \( V_i = K \). Since \( \text{Der}_\delta(L_i, K) = 0 \), the claim follows from formula \((3)\).

**Lemma 5.** Let \( L = \bigoplus_{\alpha \in G} L_\alpha \) be a Lie algebra graded by an abelian group \( G \), and \( V = \bigoplus_{\alpha \in G} V_\alpha \) a graded \( L \)-module (i.e., \( L_\alpha \cdot V_\beta \subseteq V_{\alpha + \beta} \) for any \( \alpha, \beta \in G \)). Then for any \( \delta \in K \),

\[
\text{Der}_\delta(L, V) = \bigoplus_{\alpha \in G} \text{Der}^\alpha(L, V),
\]

where

\[
\text{Der}^\alpha(L, V) = \{ D \in \text{Der}_\delta(L, V) | D(L_\beta) \subseteq V_{\beta-\alpha} \text{ for any } \beta \in G \}.
\]

**Proof.** Exactly the same simple arguments as in the case of ordinary derivations (see, for example, [Fa, Proposition 1.1]).

\( \delta \)-derivations from the space \( \text{Der}^\alpha(L, V) \) are said to be of weight \( \alpha \).

**Lemma 6.** Let \( L \) be a simple Lie algebra, \( V \) an irreducible \( L \)-module, and \( D : L \to V \) a nonzero linear map such that \( D([x,y]) = x \cdot D(y) \) for any \( x, y \in L \). Then \( V \) is isomorphic to the adjoint module.

**Proof.** By definition, \( D \) is a homomorphism of \( L \)-modules from the adjoint module \( L \) to \( V \). Obviously, \( \text{Ker}D \) is an ideal in \( L \), thus \( D \) is an injection. On the other hand, \( \text{Im}D \) is an \( L \)-submodule of \( V \), thus \( D \) is a surjection.

**Lemma 7.** Any \( \delta \)-derivation, where \( \delta \neq 0 \), of an abelian Lie algebra \( H \) with values in a semisimple \( H \)-module \( V \), is of the form \( x \mapsto \varphi(x) + x \cdot v \), where \( x \in H \), for some linear map \( \varphi : H \to V \), and some \( v \in V \).

**Proof.** By extending the ground field, we may assume that \( K \) is algebraically closed. Let \( D \) be such a \( \delta \)-derivation. Suppose first that \( V \) is a one-dimensional nontrivial \( H \)-module, linearly spanned by a single element \( v \). We may write \( x \cdot v = \lambda(x)v \) and \( D(x) = \mu(x)v \) for any \( x \in H \) and some linear maps \( \lambda, \mu : H \to K \). Then the condition \((1)\) is equivalent to

\[
\lambda(x)\mu(y) = \lambda(y)\mu(x)
\]

for any \( x, y \in H \). If \( \text{Ker}\lambda \neq \text{Ker}\mu \), then taking \( x \in H \) belonging to \( \text{Ker}\lambda \) and not belonging to \( \text{Ker}\mu \), we get from \((3)\) that \( y \in \text{Ker}\lambda \) for any \( y \in H \), thus \( V \) is the trivial module, a contradiction. Hence \( \text{Ker}\lambda = \text{Ker}\mu \), and \( \lambda \) and \( \mu \) are proportional to each other, say, \( \mu = \alpha \lambda \) for some \( \alpha \in K \). Then \( D(x) = x \cdot (\alpha v) \), and the assertion of lemma in this case follows.

In the general case \( V \) can be represented as the direct sum of the trivial module \( V^H \) and a number of one-dimensional nontrivial \( H \)-modules: say, \( V = V^H \oplus \bigoplus_i K_{V_i} \). By Lemma \((1)\)

\[
\text{Der}_\delta(H, V) \simeq \text{Der}_\delta(H, V^H) \oplus \bigoplus_i \text{Der}_\delta(H, K_{V_i}).
\]
The space $\text{Der}_\delta(H,V^H)$, obviously, coincides with the space of all linear maps $\varphi : H \to V^H$, and by the just proved one-dimensional case, we may assume that any element of $\text{Der}_\delta(H,Kv_i)$ is of the form $x \mapsto x \cdot v_i$. Then for any $x \in H$, 
\[D(x) = \varphi(x) + \sum_i (x \cdot v_i) = \varphi(x) + x \cdot (\sum_i v_i),\]
and we are done. 

In what follows, for a finite-dimensional simple Lie algebra $g$ over an algebraically closed field $K$ of characteristic zero, we fix once and for all a Cartan subalgebra $h$, and the corresponding root space decomposition $g = h \oplus \bigoplus_{\alpha \in R} K e_{\alpha}$.

**Lemma 8.** Let $D$ be a nonzero $\delta$-derivation of a finite-dimensional simple Lie algebra $g$ over an algebraically closed field of characteristic zero, with values in a finite-dimensional irreducible $g$-module $V$. Assume that there is nonzero $v \in V$, and $\beta \in R$ such that $e_{\alpha} \cdot v = 0$ and $D(e_{\alpha}) = 0$ for any $\alpha \in R$ such that $\alpha \neq \beta$. Then $g \simeq \mathfrak{sl}(2)$ and $\dim V \leq 3$.

**Proof.** Decomposing, if necessary, $v$ into the sum of nonzero elements belonging to weight spaces, we may assume that $v$ belongs to some weight space $V_\lambda$. Replacing, if necessary, $\beta$ by $-\beta$ and $\lambda$ by $-\lambda$, we get that $\lambda$ is the highest weight. Since $V$ is irreducible, it is generated, as a module over the universal enveloping algebra $U(g)$, by a single element $v$. By the Poincaré–Birkhoff–Witt theorem, each element of $U(g)$ is a sum of elements of the form 
\[e_{\beta}^{k_1} h_1^{k_1} \cdots h_n^{k_n} e_{\alpha_1}^{\ell_1} \cdots e_{\alpha_m}^{\ell_m}\]
for some $h_1, \ldots, h_n \in h$, $\alpha_1, \ldots, \alpha_m \in R$, each $\alpha_i \neq \beta$, and some nonnegative integers $k$, $k_1, \ldots, k_n$, $\ell_1, \ldots, \ell_m$. If such an element really contains $e_{\alpha}$'s (that is, at least one of $\ell$'s is positive), then it acts on $v$ trivially, and since $h$'s act on $v$ by multiplying it by a scalar, we get that $U(g)v$ is linearly spanned by elements of the form $e_{\beta}^k \cdot v$, $k = 0, 1, 2, \ldots$. But since $\lambda$ is the highest weight, $e_{\beta}^k \cdot v = 0$ if $k \geq 3$, thus $V$ is at most 3-dimensional, linearly spanned by elements $v, e_{\beta} \cdot v, e_{\beta}^2 \cdot v$.

At this point, the dimension considerations imply that $\dim g \leq 9$ and hence $g$ is isomorphic either to $\mathfrak{sl}(2)$, or to $\mathfrak{sl}(3)$, but this obvious remark is superseded by the reasoning below.

Since $\lambda$ is the highest weight, we may assume $\beta = -\rho$, where $\rho$ is the highest root, and if rank of $g$ is $> 1$, then $\beta = \alpha + \alpha'$ for some $\alpha, \alpha' \in R$. Then the equation (1), written for $x = e_{\alpha}, y = e_{\alpha'}$, yields $D(e_{\beta}) = 0$, a contradiction. Hence $g$ is of rank 1, i.e. $g \simeq \mathfrak{sl}(2)$. 

\[\square\]

3. THE CASE OF $\mathfrak{sl}(2)$

In this section we shall prove the main theorem in the case of $\mathfrak{sl}(2)$. Let the characteristic of the ground field be zero, $\{e_-, h, e_+\}$ be the standard basis of $\mathfrak{sl}(2)$ with multiplication table 
\[[h, e_-] = -2e_-, \quad [h, e_+] = 2e_+, \quad [e_+, e_-] = h.\]
The algebra $\mathfrak{sl}(2)$ is $\mathbb{Z}$-graded. We assign to elements of the standard basis the weights $1, 0, -1$, respectively.

Let $V(n)$ denote the irreducible $(n+1)$-dimensional (i.e., of the highest weight $n$) $\mathfrak{sl}(2)$-module with the standard basis $\{v_0, v_1, \ldots, v_n\}$. The action is given as follows: 
\[e_- \cdot v_i = (i+1) v_{i+1}, \quad h \cdot v_i = (n-2i) v_i, \quad e_+ \cdot v_i = (n-i+1) v_{i-1}\]
(Customarily, here and below we assume $v_i = 0$ if $i$ is out of range $0, \ldots, n$). This is a graded module, with element $v_i$ having weight $i$. (Note that our assignment of weights in $\mathfrak{sl}(2)$ and $V(n)$ is not a standard one, but is slightly more convenient for keeping track of indices in computations below. Note also that $V(2) \simeq \mathfrak{sl}(2)$, the adjoint module.)
Lemma 9. Der$_{\delta}(\mathfrak{sl}(2), V(n)) = 0$ except for the following cases (in all these cases, $n \geq 1$):

(i) $\delta = 1$; the space Der$_1(\mathfrak{sl}(2), V(n))$ is $(n+1)$-dimensional, consisting of inner derivations.

(ii) $\delta = -\frac{2}{n}$; the space Der$_{-\frac{2}{n}}(\mathfrak{sl}(2), V(n))$ is $(n+3)$-dimensional, with a basis

\[
\begin{align*}
& e_- \mapsto 0 \quad e_- \mapsto 0 \quad e_- \mapsto -v_{k+1} \quad e_- \mapsto -v_1 \quad e_- \mapsto v_0 \\
& h \mapsto 0 \quad h \mapsto 2v_n \quad h \mapsto 2v_k \quad h \mapsto 2v_0 \quad h \mapsto 0 \\
& e_+ \mapsto v_n \quad e_+ \mapsto v_{n-1} \quad e_+ \mapsto v_{k-1} \quad e_+ \mapsto 0 \quad e_+ \mapsto 0
\end{align*}
\]

consisting of $(-\frac{2}{n})$-derivations of weight $-n-1$, $-n$, $-k$, $0$, and $1$, respectively, where $1 \leq k \leq n-1$.

(iii) $\delta = \frac{2}{n+2}$ and $n \geq 2$; the space Der$_{\frac{2}{n+2}}(\mathfrak{sl}(2), V(n))$ is $(n-1)$-dimensional, with a basis

\[
\begin{align*}
& e_- \mapsto k(k+1)v_{k+1} \\
& h \mapsto 2k(n-k)v_k \\
& e_+ \mapsto -(n-k)(n-k+1)v_{k-1}
\end{align*}
\]

consisting of $\frac{2}{n+2}$-derivations of weight $-k$, where $1 \leq k \leq n-1$.

As $V(2) \simeq \mathfrak{sl}(2)$, for $n = 2$ we are dealing with just $\delta$-derivations of $\mathfrak{sl}(2)$, and this particular case of Lemma 9 was known. Indeed, it follows from the results of [12] and [13] mentioned in the introduction, that $\mathfrak{sl}(2)$ has nonzero $\delta$-derivations only in the cases $\delta = 1$ (the ordinary inner derivations), $\delta = \frac{1}{2}$ (scalar multiples of the identity map, a particular case of (iii)), and $\delta = -1$; and the 5-dimensional space Der$_{-1}(\mathfrak{sl}(2), \mathfrak{sl}(2))$, a particular case of (ii), was described in [H] Example 1.5 and [Fi1] Example in §3.

Proof. As we are dealing with a $\mathbb{Z}$-graded module over a $\mathbb{Z}$-graded Lie algebra, by Lemma 5 it is sufficient to consider $\delta$-derivations of a fixed weight $\alpha \in \mathbb{Z}$. So, let $D$ be a nonzero map lying in Der$_{\delta}(\mathfrak{sl}(2), V(n))$. Note that $\delta \neq 0$ and $n > 0$. Also, the case $\delta = 1$ corresponds to the usual derivations ($= 1$-cocycles), case (i), so when encountered in the computations below, it can be readily discarded.

We have

\[
D(e_-) = \lambda v_{1-\alpha} \\
D(h) = \mu v_{-\alpha} \\
D(e_+) = \eta v_{-1-\alpha}
\]

for some $\lambda, \mu, \eta \in K$. To ensure that at least one of the indices is in the range $0, \ldots, n$, we have $-n-1 \leq \alpha \leq 1$. Writing the equation (ii) for all possible 3 pairs of the basis elements $(h,e_-)$, $(e_+,e_-)$, and $(h,e_+)$, we get respectively:

\[
\begin{align*}
& \left( -2\lambda + \delta \mu (1-\alpha) - \delta \lambda (n-2+2\alpha) \right) v_{1-\alpha} = 0 \\
& \left( \mu - \delta \eta \alpha - \delta \lambda (n+\alpha) \right) v_{-\alpha} = 0 \\
& \left( 2\eta + \delta \mu (n+\alpha + 1) - \delta \eta (n+2+2\alpha) \right) v_{-1-\alpha} = 0
\end{align*}
\]

Case 1. $\alpha = -n-1$. The first and the second equations in (5) give nothing, and the third one is equivalent to $\eta(2+n\delta) = 0$. Since $D$ is not zero, we may normalize it by assuming $\eta = 1$, what gives $\delta = -\frac{2}{n}$. This is the first $(-\frac{2}{n})$-derivation in case (ii).

Case 2. $\alpha = -n$. The first equation in (5) gives nothing, and the second and the third one are equivalent to:

\[
\begin{align*}
\mu + n\delta \eta &= 0 \\
2\eta + \delta \mu (n-2)\delta \eta &= 0,
\end{align*}
\]
respectively. If \( \eta = 0 \), then \( \mu = 0 \), a contradiction. Hence we may normalize \( D \) by assuming \( \eta = 1 \), and resolving the quadratic equation in \( \delta \) occurring from equations (6–7), we get that either \( \delta = 1 \), or \( \delta = -\frac{2}{n} \) and \( \mu = 2 \), and we are getting the second \( (-\frac{2}{n}) \)-derivation in case (ii).

**Case 3.** \( -n + 1 \leq \alpha \leq -1 \). Note that this implies \( n \geq 2 \). All the indices of \( v \)'s occurring in (5) are within the allowed range, thus all the coefficients of \( v \)'s vanish. Consider these vanishing conditions as a system of 3 homogeneous linear equations in \( \lambda, \mu, \eta \). The determinant of this system is a cubic equation in \( \delta \), whose roots are not dependent on \( \alpha \), and are equal to \( -\frac{2}{n}, \frac{2}{n+2} \), and 1.

**Case 3.1.** \( \delta = -\frac{2}{n} \). The space of solutions of the homogeneous system is 1-dimensional, linearly spanned by \( (\lambda = -1, \mu = 2, \eta = 1) \). This is the family of \( (-\frac{2}{n}) \)-derivations in case (ii), with \( k = -\alpha \).

**Case 3.2.** \( \delta = \frac{2}{n+2} \). The space of solutions of the homogeneous system is 1-dimensional, linearly spanned by

\[
\lambda = \alpha(\alpha - 1), \mu = -2\alpha(n + \alpha), \eta = -(n + \alpha)(n + \alpha + 1).
\]

These are exactly \( \frac{2}{n+2} \)-derivations in case (iii), with \( k = -\alpha \).

**Case 4.** \( \alpha = 0 \). The third equation in (5) gives nothing, and the first and the second are equivalent to

\[
-2\lambda + \delta \mu - (n - 2)\delta \lambda = 0
\]

and

\[
\mu - n\delta \lambda = 0,
\]

respectively. If \( \lambda = 0 \), then (9) implies \( \mu = 0 \), a contradiction. Hence we may normalize \( D \) by assuming \( \lambda = -1 \), and then the quadratic equation in \( \delta \) occurring from (8–9) gives that either \( \delta = 1 \), or \( \delta = -\frac{2}{n} \) and \( \mu = 2 \). In the last case we get the fourth \( (-\frac{2}{n}) \)-derivation in case (ii).

**Case 5.** \( \alpha = 1 \). The second and the third equations in (5) give nothing, and the first equation is equivalent to \( \lambda (2 + n\delta) = 0 \). Since \( D \) is nonzero, we may normalize it by assuming \( \lambda = 1 \), and then \( \delta = -\frac{2}{n} \), what gives the last, fifth, \( (-\frac{2}{n}) \)-derivation in case (ii).

\[
\square
\]

**4. The Case of \( g \not\simeq sl(2) \)**

In the previous section we proved the main theorem in the case of \( sl(2) \) and an irreducible \( sl(2) \)-module. Now we are ready to handle the case of any other simple Lie algebra \( g \) and an irreducible \( g \)-module.

**Lemma 10.** Let \( D \) be a nonzero \( \delta \)-derivation, \( \delta \neq 1 \), of a simple finite-dimensional Lie algebra \( g \) over an algebraically closed field of characteristic zero, \( g \not\simeq sl(2) \), with values in a finite-dimensional irreducible \( g \)-module \( V \). Then \( \delta = \frac{1}{2} \). \( V \) is isomorphic to the adjoint module, and \( D \) is a multiple of the identity map.

**Proof.** If \( V \) is trivial, then the condition (1) implies \( D([x,y]) = 0 \) for any \( x, y \in g \), thus \( D = 0 \), a contradiction; so we may assume that \( V \) is nontrivial, i.e. \( \dim V \geq 2 \).

The Cartan subalgebra \( \mathfrak{h} \) acts on \( V \) semisimply, with the corresponding weight space decomposition \( V = \bigoplus_{\beta \in \Phi} V_{\beta} \). By Lemma[7] we can write \( D(h) = \varphi(h) + h \cdot v \) for any \( h \in \mathfrak{h} \), some linear map \( \varphi : \mathfrak{h} \to V^{h} = V_{0} \), and some \( v \in V \). Writing the equality (11) for \( x = h \in \mathfrak{h} \) and \( y = e_{\alpha}, \alpha \in R \), we get

\[
(10) \quad \alpha(h)D(e_{\alpha}) = -\delta e_{\alpha} \cdot (\varphi(h) + h \cdot v) + \delta h \cdot D(e_{\alpha}).
\]

By Lemma[5] it is enough to consider \( \delta \)-derivations of some weight \( \gamma \in \langle R \rangle \), where \( \langle R \rangle \) is the abelian group generated by \( R \) (note that \( \Phi \subset \langle R \rangle \)). We consider several cases depending on the weight of \( D \).

**Case 1.** \( D \) is of zero weight. In this case \( D(h) \subseteq V_{0} \) and \( D(e_{\alpha}) \subseteq V_{\alpha} \) for any \( \alpha \in R \) (here and below we assume \( V_{\lambda} = 0 \) if \( \lambda \not\in \Phi \)). Then \( v \in V_{0} \), thus \( h \cdot v = 0 \), \( D = \varphi, h \cdot D(e_{\alpha}) = \alpha(h)D(e_{\alpha}) \), and the equality (10) reduces to

\[
(11) \quad \alpha(h)D(e_{\alpha}) = \frac{\delta}{\delta - 1} e_{\alpha} \cdot D(h).
\]
Writing the equality (11) for \( x = e_α \) and \( y = e_β \), where \( α, β ∈ R, α + β ≠ 0 \), and taking into account (11) and the fact that \([e_α, e_β] ∈ Ke_α + β\), we get

\[-((δ - 1)β(h) + δα(h))e_α • D(e_β) + ((δ - 1)α(h) + δβ(h))e_β • D(e_α) = 0\]

for any \( h ∈ h \). Assuming the rank of \( g \) is \( > 1 \), \( α ≠ β \), and picking \( h \) such that \( α(h) ≠ 0 \) and \( β(h) = 0 \), the last equality is reduced to

\[(12) \quad δe_α • D(e_β) + (1 - δ)e_β • D(e_α) = 0.\]

Assuming \( δ ≠ \frac{1}{2} \), and interchanging here \( α \) and \( β \), we get \( e_β • D(e_α) = 0 \) for any \( α, β ∈ R \) such that \( α + β ≠ 0 \). Now reasoning as in the first half of the proof of Lemma 8, we get that \( dim V = 3 \), and \( V \) is spanned by elements of weight \( α, 0, -α \). Since the rank of \( g \) is \( > 1 \), we can pick two linearly independent roots \( α \) and \( β \), thus getting different sets of weights of elements of \( V \), a contradiction.

We are left with the case \( δ = \frac{1}{2} \). In this case, (11) can be rewritten as

\[(13) \quad h • D(e_α) + e_α • D(h) = 0\]

for any \( h ∈ h \) and \( α ∈ R \), and (12) is equivalent to

\[(14) \quad e_α • D(e_β) + e_β • D(e_α) = 0\]

for any \( α, β ∈ R, α + β ≠ 0 \). Moreover, since \([e_α, e_β] ∈ h \) acts on \( D(h) \) trivially for any \( h ∈ h \), we have

\[e_α • (e_β • D(h)) - e_β • (e_α • D(h)) = 0,\]

what, taking into account (11) and assuming \( α(h) ≠ 0 \), yields

\[(15) \quad e_α • D(e_β) + e_β • D(e_α) = 0\]

for any \( α ∈ R \).

The equalities (13), (14), and (15) show that for any \( x, y ∈ g \), \( x • D(y) + y • D(x) = 0 \), and hence \( D([x, y]) = x • D(y) \). By Lemma 6, \( V \) is isomorphic to the adjoint module, and thus \( D \) is an element of the centroid of \( g \). But since \( g \) is simple (and hence central), \( D \) is a multiple of the identity map.

**Case 2.** \( D \) is of nonzero weight \( \gamma ∈ R \). In this case \( D(h) = 0 \) and \( D(e_α) ∈ V_α − γ \) for any \( α ∈ R \), and (10) reduces to

\[α(h)D(e_α) = δh • D(e_α).\]

This means that for any \( α ∈ R \), either \( D(e_α) = 0 \), or \( D(e_α) ∈ V_\frac{δ}{δ - 1}α \). The latter condition implies \( α - γ = \frac{1}{δ}α \), and \( α = \frac{δ}{δ - 1}γ \). Writing the equality (11) for \( x = e_α, α ≠ \frac{δ}{δ - 1}γ \), and \( y = e_\frac{δ}{δ - 1}γ \), we get

\[e_α • D(e_\frac{δ}{δ - 1}γ) = 0.\]

Note that \( D(e_\frac{δ}{δ - 1}γ) ≠ 0 \), otherwise \( D \) is the zero map. Therefore the conditions of Lemma 8 are satisfied, with \( v = D(e_\frac{δ}{δ - 1}γ) \) and \( β = \frac{δ}{δ - 1}γ \), and hence \( g ≃ sl(2) \), a contradiction.

**Case 3.** \( D \) is of nonzero weight \( β ∈ R \). In this case \( D(h) ⊆ V_β \) and \( D(e_α) ∈ V_α - β \) for any \( α ∈ R \). Then \( φ = 0 \) and \( v ∈ V_β \), thus \( D(h) = h • v = -β(h)v \), and the equality (10) reduces to

\[(16) \quad α(h)D(e_α) = δβ(h)e_α • v + δh • D(e_α)\]

for any \( α ∈ R \). Assume here \( α - β ∈ Φ \) and \( α ≠ β \). Since \( D(e_α) ∈ V_α - β \), we have

\[h • D(e_α) = (α(h) - β(h))D(e_α),\]

and the equality (16) in this case is equivalent to

\[\left((1 - δ)α(h) + δβ(h)\right)D(e_α) = δβ(h)e_α • v.\]

Picking \( h \) such that \( β(h) = 0 \) and \( α(h) ≠ 0 \), we get \( D(e_α) = 0 \). Since the latter equality is true also for any \( α ∈ R \) such that \( α - β ≠ Φ \), we have \( D(e_α) = 0 \) for any \( α ∈ R, α ≠ β \). It follows then from (16) that \( e_α • v = 0 \) for any \( α ≠ β \). Therefore, again, we are in the conditions of Lemma 8, thus \( g ≃ sl(2) \), a contradiction. □
Remark. A slight shortcut in part of the proof of Lemma 4.4 can be achieved by invoking [Z, Lemma 4.4] which implies that if a simple Lie algebra \( L \) has a nontrivial \( \delta \)-derivation with values in an \( L \)-module, then either \( L \) satisfies the standard identity of degree 5, or \( \delta = 1 \) or \( \frac{1}{2} \). But [Z, Lemma 4.4] is just a slight upgrade of [Fi3, Theorem 1] which involves sophisticated manipulations with identities, and, in its turn, is based on highly nontrivial results of Razmyslov about identities in simple Lie algebras. And still, we will have to handle the \( \delta = \frac{1}{2} \) part in Case 1 of the proof, and Cases 2 and 3 entirely. Thus we prefer a more direct approach, based on relatively trivial computations with root systems in simple Lie algebras.

Another possibility would be to consider subalgebras of \( \mathfrak{g} \) isomorphic to \( \mathfrak{sl}(2) \), and invoke Lemma 9, what would restrict our considerations to the exceptional cases of \( \delta \) described there, which still had to be handled separately, more or less along the lines of the given proof.

5. Completion of the proof of the main theorem

Lemmas 9 and 10 together establish the main theorem stated in the introduction, in the case of a simple \( \mathfrak{g} \) and an irreducible \( \mathfrak{g} \)-module. Now, basing of this, we complete the proof for the general case of a semisimple \( \mathfrak{g} \) and arbitrary \( \mathfrak{g} \)-module.

If \( \delta = 1 \), the statement reduces to the ordinary derivations, and, as noted above, is equivalent to the first Whitehead Lemma about triviality of the first Chevalley–Eilenberg cohomology \( H^1(\mathfrak{g}, V) \). So we may assume \( \delta \neq 1 \). Also, obviously, \( \delta \neq 0 \).

Let \( \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \) be the direct sum decomposition into simple ideals \( \mathfrak{g}_i \). Assume first that \( V \) is irreducible. Then \( V \simeq V_1 \oplus \cdots \oplus V_m \), where \( V_i \) is an irreducible \( \mathfrak{g}_i \)-module. By Lemma 4, exactly one of the modules \( V_1, \ldots, V_m \), say, \( V_k \), is different from the trivial module, and the rest \( m - 1 \) modules are trivial. Therefore \( V \simeq V_k \) is an irreducible nontrivial \( \mathfrak{g}_k \)-module, and \( \delta \) acts on \( V \) trivially if \( i \neq k \). The restriction of any \( \delta \)-derivation \( D : \mathfrak{g} \to V \) on \( \mathfrak{g}_k \) is a \( \delta \)-derivation of \( \mathfrak{g}_k \) with values in \( V_k \), \( D \) acts trivially on the rest of \( \mathfrak{g}_i \), \( i \neq k \), and \( \text{Der}_\delta(\mathfrak{g}, V) \simeq \text{Der}_\delta(\mathfrak{g}_k, V_k) \). By Lemmas 9 and 10, one of the following holds:

(i) \( \delta = -\frac{2}{n}, n \geq 1, \mathfrak{g}_k \simeq \mathfrak{sl}(2), V_k \simeq V(n) \), and \( \text{Der}_\delta(\mathfrak{g}_k, V_k) \) is as in Lemma 9(ii)

(ii) \( \delta = -\frac{2}{n+2}, n \geq 2, \mathfrak{g}_k \simeq \mathfrak{sl}(2), V_k \simeq V(n) \), and \( \text{Der}_\delta(\mathfrak{g}_k, V_k) \) is as in Lemma 9(iii)

(iii) \( \delta = \frac{1}{2}, V_k \simeq \mathfrak{g}_k \) (the adjoint module), and \( \text{Der}_\delta(\mathfrak{g}_k, V_k) \simeq K\text{id}_{\mathfrak{g}_k} \).

Note that the cases (ii) and (iii) overlap if \( n = 2 \) and \( \mathfrak{g}_k \simeq \mathfrak{sl}(2) \).

The proof is finished by the remark that in the general case \( V \) is decomposed into the direct sum of irreducible \( \mathfrak{g} \)-modules, and application of Lemma 1.

6. Open questions: positive characteristic and infinite-dimensional modules

What happens in positive characteristic? As it is commonly known, the general situation in this case is much more complicated. The first Whitehead Lemma does not hold (in a sense, the opposite is true: as proved in [J, Chapter VI, §3, Theorem 2]), any finite-dimensional Lie algebra over a field of positive characteristic has a finite-dimensional module with first nonzero cohomology), so there is no point to conjecture that (most of) \( \delta \)-derivations should be inner derivations. However, this does not preclude the possibility that “most”, in some sense, \( \delta \)-derivations can be ordinary derivations.

Conjecture. A nonzero \( \delta \)-derivation of a finite-dimensional semisimple Lie algebra \( L \) over a field of characteristic \( \neq 2, 3 \) with values in a finite-dimensional \( L \)-module is, excluding the small number of exceptional cases, either an ordinary derivation \( \delta = 1 \), or is a multiple of the identity map on a simple component of \( L \) with coefficients in the adjoint module \( \delta = \frac{1}{2} \).

To tackle this conjecture, one should overcome a number of difficulties specific to positive characteristic. First, in addition to \( \mathfrak{sl}(2) \), there is another class of simple Lie algebras of rank one having nontrivial \( \delta \)-derivations with \( \delta \neq 1 \) – namely, Zassenhaus algebras having nontrivial \( \frac{1}{2} \)-derivations (see [Fi3, §3] and [Z, §2]); this suggests that the number of exceptional cases will be (much) higher than in characteristic zero. Second, the description of irreducible representations of simple Lie algebras – except for the simplest cases of algebras of low rank – is lacking (and, quite possibly, such full description
is hopeless). Third, arbitrary representations of simple Lie algebras are, generally, not direct sums of irreducibles. And fourth, semisimple Lie algebras are, generally, not direct sums of simples.

The last difficulty, however, could be overcome using the classical Block theorem which says that in positive characteristic, semisimple Lie algebras are sitting between the direct sum of Lie algebras of the form $S \otimes D$, where $S$ is a simple Lie algebra, and $D$ is a divided powers algebra, and their derivation algebras. This, basically, reduces questions about semisimple algebras to semidirect sums of the form

$$S \otimes D,$$

where $D$ is a derivation algebra of $O$. It is proved in [Z] §1 that the space of $\delta$-derivations of the Lie algebra of the form $L \otimes A$, where $L$ is a Lie algebra, and $A$ an associative commutative algebra, is, essentially, reduced to the space Der$_{\delta}(L) \otimes A$, and an extension of this result to $\delta$-derivations of Lie algebras of the form (17) with values in more-or-less arbitrary modules would, hopefully, allow to reduce Conjecture from the semisimple case to the simple one.

Going back to characteristic zero, it would also be interesting to compute $\delta$-derivations of a (semi)simple finite-dimensional Lie algebra $g$ with values in infinite-dimensional $g$-modules. The first cohomology of $g$ in this case does not necessarily vanish: this follows from abstract nonsense (otherwise $g$ would be of cohomological dimension 1), and for concrete examples of infinite-dimensional $g$-modules with nonvanishing (first) cohomology see [Ba], [W], and references therein.

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**REFERENCES**

[Ba] V. Bavula, *Computation of $H^\ast(sl(2),M)$ with coefficients in a simple $sl(2)$-module*, Funkts. Anal. Prilozh. **26** (1992), no.1, 57–58 (in Russian); Funct. Anal. Appl. **26** (1992), no.1, 45–46 (English translation).

[Bo] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters 7-9*, Springer, 2005 (translated from the original French editions: Hermann, 1975 and Masson, 1982).

[Fa] R. Farnsteiner, *Derivations and central extensions of finitely generated graded Lie algebras*. J. Algebra **118** (1988), no.1, 33–45.

[Fi1] V.T. Filippov, *Lie algebras satisfying identities of degree 5*, Algebra Logika **34** (1995), no.6, 681–705 (in Russian); Algebra and Logic **34** (1995), no.6, 379–394 (English translation).

[Fi2] V.T. Filippov, *On $\delta$-derivations of Lie algebras*, Sibirsk. Mat. Zh. **39** (1998), no.6, 1409–1422 (in Russian); Siber. Math. J. **39** (1998), no.6, 1218–1230 (English translation).

[Fi3] V.T. Filippov, *$\delta$-derivations of prime Lie algebras*, Sibirsk. Mat. Zh. **40** (1999), no.1, 201–213 (in Russian); Siber. Math. J. **40** (1999), no.1, 174–184 (English translation).

[H] N.C. Hopkins, *Generalized derivations of nonassociative algebras*, Nova J. Math. Game Theory Algebra **5** (1996), no.3, 215–224.

[J] N. Jacobson, *Lie Algebras*, Interscience Publ., 1962; reprinted by Dover, 1979.

[LL] G.F. Leger and E.M. Luks, *Generalized derivations of Lie algebras*, J. Algebra **228** (2000), no.1, 165–203.

[W] F.L. Williams, *Lie algebra cohomology of infinite-dimensional modules*, Adv. Math. **35** (1980), no.1, 19–29.

[ZZ] A. Zohrabi and P. Zusmanovich, *On Hermitian and skew-Hermitian matrix algebras over octonions*, J. Nonlin. Math. Phys., to appear.

[Z] P. Zusmanovich, *On $\delta$-derivations of Lie algebras and superalgebras*, J. Algebra **324** (2010), no.12, 3470–3486; Erratum: **410** (2014), 545–546.

**SOFTWARE**

[G] GAP – *Groups, Algorithms, and Programming*, Version 4.10.2, 2019; https://www.gap-system.org/

[M] Maxima, a Computer Algebra System, Version 5.41.0, 2017; http://maxima.sourceforge.net/

†Using a Perl wrapper for GAP for solving linear equations on (nonassociative) algebras, available at https://web.osu.cz/~Zusmanovich/soft/lineq/
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