The Carlitz Algebras

V. Bavula

Abstract

The Carlitz $\mathbb{F}_q$-algebra $C = C_\nu$, $\nu \in \mathbb{N}$, is generated by an algebraically closed field $K$ (which contains a non-discrete locally compact field of positive characteristic $p > 0$, i.e. $K \simeq \mathbb{F}_q[[x, x^{-1}]]$; $q = p^\nu$), by the (power of the) Frobenius map $X = X_\nu : f \mapsto f^q$, and by the Carlitz derivative $Y = Y_\nu$. It is proved that the Krull and global dimensions of $C$ are 2, a classification of simple $C$-modules and ideals are given, there are only countably many ideals, they commute ($IJ = JI$), and each ideal is a unique product of maximal ones. It is a remarkable fact that any simple $C$-module is a sum of eigenspaces of the element $YX$ (the set of eigenvalues for $YX$ is given explicitly for each simple $C$-module). This fact is crucial in finding the group $\text{Aut}_{\mathbb{F}_q}(C)$ of $\mathbb{F}_q$-algebra automorphisms of $C$ and in proving that two distinct Carlitz rings are not isomorphic ($C_\nu \not\simeq C_\mu$ if $\nu \neq \mu$). The centre of $C$ is found explicitly, it is a UFD that contains countably many elements.

Mathematics subject classification 2000: 16G99, 16D30, 16P40, 16U70

1 Introduction

In this paper, module means a left module. Recall that $A_1(F) = F\langle X, \partial | \partial X - X\partial = 1 \rangle$ is the first Weyl algebra over a field $F$. Let $k$ and $l = l_p$ be fields of characteristic zero and $p > 0$ respectively. The first Weyl algebras $A_1(k)$ and $A(l)$ have very distinctive properties, name just a few: $A_1(k)$ is a simple algebra but $A(l)$ is not (there are finite and infinite dimensional factor algebras of $A(l)$); $A_1(k)$ has a small centre (equal to $k$) but $A(l)$ has a big centre (equal to a polynomial algebra $l[X^p, \partial^p]$ in two variables); the Krull and global dimensions $\text{Kdim}(A_1(k)) = \text{gl.dim}(A_1(k)) = 1$ but $\text{Kdim}(A_1(l)) = \text{gl.dim}(A_1(l)) = 2$; the classical Krull dimension $\text{cl.Kdim}(A_1(k)) = 0$ but $\text{cl.Kdim}(A_1(l)) = 2$; the algebra $A_1(k)$ has no nonzero simple finite dimensional modules but all the simple modules over the algebra $A_1(l)$ are finite dimensional; only a tiny bit of simple $A_1(k)$-modules are weight modules but all simple $A_1(l)$-modules are weight and finite dimensional over $l$.

The Carlitz algebras $C_\nu$ exhibit properties that are “in between” of that of $A_1(k)$ and $A_1(l)$: the centre of $C_\nu$ is very small (it contains countably many elements), $\text{Kdim}(C_\nu) = \text{gl.dim}(C_\nu) = 2$ and $\text{cl.Kdim}(C_\nu) = 1$, all simple $C_\nu$-modules are weight but there are plenty of simple infinite dimensional and finite dimensional $C_\nu$-modules over $K$, every factor ring of $C$ is infinite dimensional over $K$. 

1
The paper has the following structure: In Section 2, we recall the definition of the Carlitz algebras and prove that they are generalized Weyl algebras. Using this fact, in Section 3, a classification of simple $C$-modules is given using some results of [2] and [4] (Theorems 3.1 and 3.5). The Carlitz algebras have pleasant representation theory due to the fact that every simple module is weight (i.e. a direct sum of eigenspaces for the element $YX$, Corollary 3.4). Any nonunit element of $C$ has finite dimensional kernel and cokernel over $\mathcal{K}$ on any simple $C$-module (Theorem 3.6).

In Section 4, the ideals of $C$ are explicitly described (Theorem 4.1), name just a few (rather peculiar) results: ideals commute ($IJ =JI$), and there are countably many of them, every ideal is a unique product of maximal ones (as in the case of the ring of integers), each simple factor ring is infinite dimensional over $\mathcal{K}$ and has Krull and global dimension 1. The centre of $C$ is found (Lemma 4.2.(3)), it is a UFD and contains countably many elements.

In Section 5, it is proved that two distinct Carlitz algebras are not isomorphic (Theorem 5.1) and the groups Aut($C$) and Aut$_K$(C) of ring automorphisms and of $K$-automorphisms are found explicitly (Theorem 5.2).

In Section 6, it is proved that the ring $C$ has Krull and global dimension 2.

2 The Carlitz algebras are generalized Weyl algebras

Generalized Weyl algebras. Let $D$ be a ring with an automorphism $\sigma$ and a central element $a$. A generalized Weyl algebra (a GWA, for short) $A = D(\sigma, a)$ of degree 1, is the ring generated by $D$ and two indeterminates $X$ and $Y$ subject to the relations [1]:

$$X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y,$$

for all $\alpha \in D$, $YX = a$ and $XY = \sigma(a)$.

The algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a $\mathbb{Z}$-graded algebra where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), $v_0 = 1$. It follows from the above relations that

$$v_nv_m = (n,m)v_{n+m} = v_{n+m} < n,m >$$

for some $(n,m) = \sigma^{-n-m}(<n,m>) \in D$. If $n > 0$ and $m > 0$ then

$$n \geq m : (n,-m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \quad (n,-m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$

$$n \leq m : \quad (n,-m) = \sigma^n(a) \cdots \sigma(a), \quad (n,-m) = \sigma^{-n+1}(a) \cdots a,$$

in other cases $(n,m) = 1$.

Let $K[H]$ be a polynomial ring in one variable $H$ over the field $K$, $\sigma : H \rightarrow H - 1$ be the $K$-automorphism of the algebra $K[H]$ and $a = H$. The first Weyl algebra $A_1$ is isomorphic to the generalized Weyl algebra

$$A_1 \simeq K[H](\sigma,H), \quad X \leftrightarrow X, \quad \partial \leftrightarrow Y, \quad \partial X \leftrightarrow H.$$

The Carlitz algebras. Any non-discrete locally compact field of positive characteristic $p > 0$ is isomorphic to the field $K = \mathbb{F}_q[[x,x^{-1}]]$ of Laurent series with coefficients from
the Galois field \( \mathbb{F}_q \) that contains \( q := p^r \) elements \( \nu \in \mathbb{N} \). A typical nonzero element \( \lambda \) of \( K \) is a series \( \sum_{i=m}^{\infty} \lambda_i x^i \) where \( m \in \mathbb{Z} \), \( \lambda_i \in \mathbb{F}_q \), \( \lambda_m \neq 0 \). The field \( K \) is equipped with a \textit{non-archimedian absolute value} \( |\lambda| := q^{-m} \) which can be extended onto the the completion of an algebraic closure \( \overline{K} \) of the field \( K \). Let \( \mathcal{K} \) be an algebraically closed field extension of \( K \). We denote by \( F : a \mapsto F(a) := a^q \) the \textit{Frobenious map} of \( \mathcal{K} \) (\( K \) or \( \mathbb{F}_q \)), and let \( \sigma = \sigma_\nu := F^\nu \) where \( q := p^\nu \). The fields \( \mathcal{K}, K \) and \( \mathbb{F}_q \) are topological fields with respect to the \( p \)-adic filtration induced by \( | \cdot | \), and the maps \( F \) and \( \sigma \) are continuous. The action of \( F \) and \( \sigma \) on \( \mathcal{K} \) can be extended to the action on the set of maps \( \mathcal{F} \) from \( K \) to \( \mathcal{K} \) by the rule \( F : f \mapsto f^p \) and \( X : f \mapsto f^q \) where \( X = X_\nu \) denotes the extension of the \( \sigma \) (we have introduced another letter in order to avoid confusion later). There are two distinguished \( \mathcal{F} \) and \( \mathcal{Y} \) from \( \mathcal{F} \) to \( \mathcal{F} \) \( [1] \):

\[
\text{the difference operator} : \quad \Delta u(t) := u(xt) - xu(t), \quad u(t) \in \mathcal{F},
\]

\[
\text{the Carlitz derivative} : \quad Y := \sqrt[\nu]{\Delta}.
\]

The \textbf{Carlitz algebra} \( C = C_\nu \) is the subalgebra of \( \text{End}_{\mathbb{F}_q}(\mathcal{F}) \) generated by the isomorphic copy of the field \( K \) in the endomorphism algebra \( \text{End}_{\mathbb{F}_q}(\mathcal{F}) \) (\( K \to \text{End}_{\mathbb{F}_q}(\mathcal{F}) \), \( \lambda \mapsto (u \mapsto \lambda u) \)) and the maps \( X = X_\nu \) and \( Y = Y_\nu \).

The elements \( X \) and \( Y \) satisfy the following relations \( [7] \):

\[
YX - XY = [1]^\frac{1}{\nu}, \quad X\lambda = \lambda^q X, \quad Y\lambda = \lambda^\frac{1}{\nu} Y \quad (\lambda \in \mathcal{K}),
\]

where \([1] := -x + x^q\). The algebra \( C \) is a Noetherian domain and any element of the algebra \( C \) is a unique sum \( \sum \lambda_{ij} X^i Y^j \) where \( \lambda_{ij} \in \mathcal{K} \) \([3] \). It follows immediately that the map

\[
C \to \mathcal{K}[H](\sigma, H), \quad X \mapsto X, \quad Y \mapsto Y, \quad YX \mapsto H, \quad \lambda \mapsto \lambda \quad (\lambda \in \mathcal{K}),
\]

is an \( \mathbb{F}_q \)-algebra isomorphism (as the GWA \( \mathcal{K}[H](\sigma, H) = \oplus_{i \in \mathbb{Z}} \mathcal{K}[H]v_i \) where

\[
\sigma \in \text{Aut}(\mathcal{K}[H]) : H \mapsto H - \lambda_1, \quad \lambda \mapsto \lambda^q \quad (\lambda \in \mathcal{K}),
\]

and \( \lambda_1 := [1]^\frac{1}{\nu} = -x^\frac{1}{\nu} + x \). So, the Carlitz algebra is the generalized Weyl algebra \( C = D(\sigma, H) \) with coefficients from the polynomial algebra \( D := \mathcal{K}[H] \) with coefficients from the field \( \mathcal{K} \). Since the polynomial algebra \( D \) is a Noetherian domain then so is the GWA \( C \), by \([2] \), Proposition 1.3. Note that the Carlitz algebra is an algebra over the \textit{finite} field \( \mathbb{F}_q \) and is \textit{not} an algebra over \( \mathcal{K} \) or even over \( \mathbb{F}_q', i \geq 2 \).

\section{3 A classification of simple modules over the Carlitz algebras}

The representation theory of GWAs are well understood, see for example \([2] \) and \([4] \). We use some of the results of these two papers to give a classification of simple modules over the Carlitz algebras. A surprising feature is that all simple \( C \)-modules are weight modules (that is \( H \) acts as a semi-simple linear map).
Let $\mathcal{K}(H) = S^{-1}D$ the field of fractions of $D$ where $S = D \setminus \{0\}$. The localization $B = S^{-1}C$ of the ring $C$ at $S$ is the skew Laurent polynomial ring $B = \mathcal{K}(H)[X, X^{-1}; \sigma]$ with coefficients from the field $\mathcal{K}(H)$. We may identify the ring $C$ with a subring of $B$ via the ring monomorphism

$$C \to B, \ X \mapsto X, \ Y \mapsto HX^{-1}, \ d \mapsto d \ (d \in D).$$

The ring $B$ is an Euclidean ring (a left and right division algorithm with remainder holds), hence a principal left and right ideal domain. We have the partition of the set $\hat{C}$ of isoclasses of simple $C$-modules

$$\hat{C} = \hat{C}(D - \text{torsion}) \cup \hat{C}(D - \text{torsionfree}), \quad (1)$$

where a simple $C$-module $M$ is $D$-torsion (respectively, $D$-torsionfree) if $S^{-1}M = 0$ (respectively, $S^{-1}M \neq 0$). We will see shortly that $\hat{C}(D - \text{torsionfree}) = \emptyset$ (Corollary 3.4).

**Max**$(D)$. Let $G = \langle \sigma \rangle$ be the subgroup of the group of ring automorphisms $\text{Aut}(D)$ of $D$ generated by the element $\sigma$. The group $G$ acts in the obvious way on the set of maximal ideals of the algebra $D$,

$$\text{Max}(D) := \{D(H - \lambda) \mid \lambda \in \mathcal{K}\} \cong \mathcal{K}, \quad D(H - \lambda) \leftrightarrow \lambda.$$

So, the orbit $\mathcal{O}$ of an element $p \in \text{Max}(D)$ is $\mathcal{O}(p) = \{\sigma^i(p), i \in \mathbb{Z}\}$. In more detail, if $p = D(H - \lambda)$ for some $\lambda \in \mathcal{K}$ then an easy induction gives that for each $\lambda \in \mathcal{K}$ and each natural number $n \geq 1$:

$$\sigma^n(H - \lambda) = H - \lambda_1 - \lambda_1^2 - \cdots - \lambda_1^{n-1} - \lambda^n, \quad (2)$$

$$\sigma^{-n}(H - \lambda) = H + \lambda_1 + \lambda_1^2 + \cdots + \lambda_1^{\frac{1}{n}} - \lambda^\frac{1}{n}. \quad (3)$$

An orbit is called cyclic of length $n$ (respectively, linear) if it contains a finite (respectively, infinite) number $n = |\mathcal{O}|$ of elements. The set of cyclic (resp. linear) orbits is denoted by $\text{Cyc}$ (resp. $\text{Lin}$). An orbit $\mathcal{O}$ is called degenerate, if it contains a maximal ideal $p$ such that $H \in p$ (such ideals are called marked). We denote by $\text{Cycd}$ and $\text{Lind}$ (resp. $\text{Cycn}$ and $\text{Linn}$) the set of all degenerate (resp. non-degenerate) cyclic and linear orbits, respectively.

Each linear orbit $\mathcal{O}(p)$, via the map $\mathcal{O}(p) \to \mathbb{Z}, \sigma^i(p) \to i$, may be identified with the set of integers $\mathbb{Z}$. Therefore, for $\mathcal{O}(p) \in \text{Lin}$ we may use, without mentioning it explicitly, all the definitions and notations which are employed for $\mathbb{Z}$ (such as the order, the segment, the interval, etc.). For example, $\sigma^i(p) \leq \sigma^j(p)$ iff $i \leq j$; $(-\infty, p] := \{\sigma^i(p), i \leq 0\}$, etc. Marked ideals $p_1 < \cdots < p_s$ of a degenerate linear orbit $\mathcal{O}$ divide it into $s + 1$ parts,

$$\Gamma_1 = (-\infty, p_1], \quad \Gamma_2 = (p_1, p_2], \ldots, \Gamma_{s+1} = (p_s, \infty).$$

We say that maximal ideals $p$ and $q$ from a linear orbit are equivalent ($p \sim q$) if they belong either to a non-degenerate orbit or to some $\Gamma_i$. 


**Weight C-modules.** An $C$-module $V$ is *weight* if $D V$ is semi-simple, i.e.

$$V = \bigoplus_{p \in \text{Max}(D)} V_p$$

where $V_p = \{v \in V : pv = 0\} = \{\text{the sum of simple } D\text{-submodules which are isomorphic to } D(D/p)\}$, the *component* of $V$ of weight $p$. The *support* $\text{Supp}(V)$ of the weight module $V$ is the set of maximal ideals $p$ such that $V_p \neq 0$.

Since

$$X V_p \subseteq V_{\sigma(p)} \text{ and } Y V_p \subseteq V_{\sigma^{-1}(p)},$$

each weight $C$-module $V$ decomposes into the direct sum of $C$-submodules (the *orbit decomposition*)

$$V = \bigoplus \{ V_O | O \text{ is an orbit } \},$$

where $V_O = \bigoplus \{ V_p | p \in O \}$. Obviously, for each maximal ideal $p$ of $D$ the module

$$C(p) := C/Cp \simeq C \otimes D/p = \bigoplus_{i \in \mathbb{Z}} v_i \otimes D/p$$

is weight $(D(v_i \otimes D/p) \simeq D/\sigma^i(p))$ with support $O(p)$.

Denote by $\hat{C}(\text{weight})$ the set of isoclasses of simple weight $C$-modules. Each simple weight $C$-module and each simple $D$-torsion $C$-module is a homomorphic image of $C(p)$ for some $p \in \text{Max}(D)$, and so by (4),

$$\hat{C}(D - \text{torsion}) = \hat{C}(D - \text{weight}).$$

We denote by $\hat{C}(\text{weight, linear})$ and $\hat{C}(\text{weight, cyclic})$ the sets of isoclasses of simple weight $C$-modules with support from a *linear* and a *cyclic* orbit respectively. Then the set of isoclasses of simple weight $C$-modules is the following disjoint union:

$$\hat{C}(\text{weight}) = \hat{C}(\text{weight, linear}) \cup \hat{C}(\text{weight, cyclic}).$$

The ideal $(H)$ of the polynomial algebra $D := K[H]$ generated by the element $H$ is a maximal ideal of $D$. By [2], $\lambda_i^n \neq \lambda_1$ for all $n \geq 1$ ($\lambda_1 := [1]^\frac{1}{n} = x - x^2$), therefore the orbit $O(H)$ of the maximal ideal $(H)$ is an infinite orbit. This is the only degenerate orbit, this orbit is infinite (i.e. linear), and there are only two equivalence classes in $O(H)$: $\Gamma_- := (\infty, (H))$ and $\Gamma_+ := ((H), \infty)$.

Let $\text{Lin}/ \sim$ denote the set of equivalence classes in $\text{Lin}$, that is

$$\text{Lin}/ \sim = \{ \Gamma_-, \Gamma_+, O(p) \} \text{ where } O(p) \text{ is a non-degenerate linear orbit} \}.$$

**Theorem 3.1** The map

$$\text{Lin}/ \sim \rightarrow \hat{C}(D - \text{weight, linear}), \Gamma \rightarrow [L(\Gamma)],$$

is a bijection with inverse $[L] \rightarrow \text{Supp } L \text{ (in particular, } \text{Supp } L(\Gamma) = \Gamma \text{) where}$
1. if $\Gamma \in \text{Linn}$ then $L(\Gamma) = C/Cp$, for any $p \in \Gamma$,

2. if $\Gamma = (-\infty, (H)] \in$ then $L_+ := L(\Gamma) = C/(CH + CX)$,

3. if $\Gamma = ((H), \infty) \in$ then $L_+ := L(\Gamma) = C/(C\sigma(H) + CY)$.

$\dim_K(L(\Gamma)) = |\Gamma| = \infty$ for each $[L(\Gamma)] \in \hat{C}(D - \text{weight, linear})$.

Proof. This is a special case of [4], Corollary 4.1. $\square$

Let us give more detail about the simple modules just described.

The simple weight $C$-module $L_+ = \bigoplus_{i \geq 0} K\bar{X}^i$, $\bar{X}^i := X^i + CH + CX$, where the action of the generators for the algebra $C$ is given by the rule:

$$
H\bar{X}^0 = 0, \quad H\bar{X}^i = -(\lambda_1 + \lambda_1^q + \cdots + \lambda_1^{q^{i-1}})\bar{X}^i, \quad i \geq 1,
$$

$$
X\bar{X}^0 = 0, \quad X\bar{X}^i = -(\lambda_1 + \lambda_1^q + \cdots + \lambda_1^{q^{i-1}})\bar{X}^{i-1}, \quad i \geq 1,
$$

$$
Y\bar{X}^i = \bar{X}^{i+1}, \quad i \geq 0.
$$

The simple weight $C$-module $L_+ = \bigoplus_{i \geq 0} K\bar{X}^i$, $\bar{X}^i := X^i + C\sigma(H) + CY$, where the action of the generators for the algebra $C$ is given by the rule:

$$
H\bar{X}^0 = \lambda_1\bar{X}^0, \quad H\bar{X}^i = (2\lambda_1 + \lambda_1^1 + \cdots + \lambda_1^{q^{i-1}})\bar{X}^i, \quad i \geq 2,
$$

$$
Y\bar{X}^0 = 0, \quad Y\bar{X} = \lambda_1\bar{X}^0, \quad Y\bar{X}^i = (2\lambda_1 + \lambda_1^1 + \cdots + \lambda_1^{q^{i-1}})\bar{X}^{i-1}, \quad i \geq 2,
$$

$$
X\bar{X}^i = \bar{X}^{i+1}, \quad Y\bar{X} = \lambda\bar{X}.
$$

The simple weight $C$-module

$$
L = L(\Gamma) = C/Cp = \left(\bigoplus_{i \geq 1} K\bar{Y}^i\right) \bigoplus K\bar{T} \bigoplus \left(\bigoplus_{i \geq 1} K\bar{X}^i\right)
$$

where $\bar{\omega} := u + Cp$, $\bar{T} := \bar{Y}^0 = \bar{X}^0$ and $p = (H - \lambda) \in \Gamma \in \text{Linn},$

$$
X\bar{X}^i = \bar{X}^{i+1}, \quad Y\bar{X} = \bar{X}^{i+1}, \quad i \geq 0,
$$

$$
X\bar{Y}^i = (\lambda - \lambda_1 - \lambda_1^q - \cdots - \lambda_1^{q^{i-1}})\bar{Y}^{i-1}, \quad i \geq 1,
$$

$$
Y\bar{X}^i = (\lambda + \lambda_1^1 + \cdots + \lambda_1^{q^{i-1}})\bar{X}^{i-1}, \quad i \geq 2,
$$

$$
Y\bar{X} = \lambda\bar{T}.
$$

The formulas above can be simplified taking into account that, for $n \geq 2$, $x$

$$
\lambda_1 + \lambda_1^q + \cdots + \lambda_1^{q^n} = -x^\frac{1}{q} + x^{q^n},
$$

$$
\lambda_1 + \lambda_1^q + \cdots + \lambda_1^{q^n} = -x^\frac{1}{q^{n-1}} + x.
$$
Lemma 3.2 For λ ∈ K and a natural number n, \( \sigma^n(D(H - \lambda)) = D(H - \lambda) \) iff λ ∈ \(-x^{\frac{1}{q}} + \mathbb{F}_{q^n}\).

Proof. If \( n = 1 \) then \( \sigma(D(H - \lambda)) = D(H - \lambda) \) iff \( f_1 = 0 \) where \( f_1 := \lambda^q - \lambda + \lambda_1 = \lambda^q - \lambda - x^{1/3} + x \). The polynomial \( f_1 \) of degree \( q \) (in \( \lambda \)) has \( q \) distinct roots since its derivative \( \frac{df_1}{d\lambda} = -1 \neq 0 \). It follows from the equality \( f_1(\lambda + \mu) = f_1(\lambda) + \mu^q - \mu \) that if \( \lambda \) is a root of the polynomial \( f_1 \) then so is \( \lambda + \mu \) for each \( \mu \in \mathbb{F}_q \). Clearly, \(-x^{1/3}\) is a root of the polynomial \( f_1 \). Therefore, \(-x^{1/3} + \mathbb{F}_q\) are the roots of \( f_1 \).

Similarly, if \( n \geq 2 \) then, by (2), \( \sigma^n(D(H - \lambda)) = D(H - \lambda) \) iff \( f_n = 0 \) where

\[
f_n := \lambda^{q^n} - \lambda + \lambda_1 + \lambda_1^q + \cdots + \lambda_1^{q^{n-1}} = \lambda^{q^n} - \lambda - x^{1/3} + x^{q^n - 1}.
\]

The polynomial \( f_n \) of degree \( q^n \) (in \( \lambda \)) has \( q^n \) distinct roots since its derivative \( \frac{df_n}{d\lambda} = -1 \neq 0 \). It follows from the equality \( f_n(\lambda + \mu) = f_n(\lambda) + \mu^{q^n} - \mu \) that if \( \lambda \) is a root of the polynomial \( f_n \) then so is \( \lambda + \mu \) for each \( \mu \in \mathbb{F}_{q^n} \). Clearly, \(-x^{1/3}\) is a root of the polynomial \( f_n \). Therefore, \(-x^{1/3} + \mathbb{F}_{q^n}\) are the roots of \( f_n \). \( \square \)

Recall that the Möbius function \( \mu : \mathbb{N} \to \{0, \pm 1\} \) is given by the rule: \( \mu(1) = 1 \), \( \mu(p_1 \cdots p_r) = (-1)^r \) where \( p_1, \ldots, p_r \) are distinct primes, and \( \mu(n) = 0 \), otherwise. Given a function \( f \) on \( \mathbb{N} \) and a second function \( g \) defined by the formula \( g(n) = \sum_{d|n} f(d) \). Then (it is well-known)

\[
f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right). \quad (6)
\]

The Euler function \( \varphi \) on \( \mathbb{N} \) is defined by \( \varphi(1) = 1 \) and, for \( n > 1 \), \( \varphi(n) \) is the number of natural numbers \( m \) that are co-prime to \( n \) and \( 1 \leq m < n \).

Theorem 3.3 (Classification of finite orbits) Let \( \mathcal{O} = \mathcal{O}_\lambda (\lambda \in K) \) be the orbit of the maximal ideal \( D(H - \lambda) \) of the polynomial algebra \( D := K[H] \) under the action of the cyclic group \( \langle \sigma \rangle \). Then

1. the orbit \( \mathcal{O}_\lambda \) contains a single element iff \( \lambda \in -x^{1/3} + \mathbb{F}_q \). So, there are exactly \( q \) distinct maximal \( \sigma \)-invariant ideals of the algebra \( D \),

2. the orbit \( \mathcal{O}_\lambda \) contains exactly \( n \geq 2 \) elements iff \( \lambda \in -x^{1/3} + (\mathbb{F}_{q^n} \setminus \cup_{m|n, m \neq n} \mathbb{F}_{q^m}) \). So, there are exactly \( \alpha_n := n^{-1} \sum_{d|n} \mu(d) q^{\frac{n}{d}} = n^{-1} \varphi(q^n - 1) \) distinct orbits that contain exactly \( n \geq 2 \) elements; and \( \alpha_n \geq n^{-1} q^n (1 - \frac{1}{q-1}) > 0 \).

Proof. 1. This evident (see Lemma 3.2).

2. By Lemma 3.2 the orbit \( \mathcal{O}_\lambda \) contains exactly \( n \geq 2 \) elements iff \( \lambda \in -x^{1/3} + (\mathbb{F}_{q^n} \setminus \cup_{m|n, m \neq n} \mathbb{F}_{q^m}) \). Let \( \alpha_n \) be the number of distinct orbits that contain exactly \( n \) elements, then \( f(n) = \alpha_n \) is the number of all maximal ideals of \( D \) that lie on all the orbits that contain exactly \( n \) elements. By Lemma 3.2 the function \( g(n) := \sum_{d|n} f(d) \) is equal to
| − \frac{1}{3} \cdot x^\frac{1}{3} + \mathbb{F}_q^n | = | \mathbb{F}_q^n | = q^n where in this proof |S| means the number of elements in a set S. Therefore, by (3), we have \( f(n) = n^{-1} \sum_{d|n} \mu(d) q^{\frac{n}{d}} \). Clearly,

\[
\begin{align*}
\text{no}_n & \geq | − \frac{1}{3} \cdot x^\frac{1}{3} + \mathbb{F}_q^n | − | − \frac{1}{3} \cdot x^\frac{1}{3} + \mathbb{F}_q^{n-1} | − \cdots − | − \frac{1}{3} \cdot x^\frac{1}{3} + \mathbb{F}_q | \\
& = q^n − q^{n-1} − \cdots − q = q^n − q^{n-1} − \frac{1}{q − 1} \\
& > q^n − \frac{q^n}{q − 1} = q^n(1 − \frac{1}{q − 1}) \geq 0.
\end{align*}
\]

Clearly, \( o_n = n^{-1} | \mathbb{F}_q^n \setminus \bigcup_{m|n, m \neq n} \mathbb{F}_q^m | = n^{-1} \varphi(q^n − 1). \)

**Corollary 3.4** \( \hat{C}(D − \text{torsionfree}) = \emptyset. \)

**Proof.** By \[4\], Theorem 5.14, \( \hat{C}(D − \text{torsionfree}) = \emptyset \) iff there are infinitely many cyclic orbits, and the result follows from Theorem 3.3. (2). \( \square \)

\( \hat{C} \)\( (\text{weight, cyclic}), \text{simple finite dimensional (over } \mathcal{K}) \) \( C \)-modules. For a natural number \( n \), set

\[
C_{[n]} = \bigoplus_{i \in \mathbb{Z}} C_{in},
\]

the Veronese subring of \( C \). The ring \( C \) considered as a left (or right ) \( C_{[n]} \)-module is \((\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z})\)-graded,

\[
C = \bigoplus_{i \in \mathbb{Z}_n} C_i, \text{ where } C_i = \bigoplus_{j \in \mathbb{Z}} C_{i + nj}, \ i = i + n\mathbb{Z}.
\]

It means that \( C_{[n]} C_i \subseteq C_i \) for all \( i \in \mathbb{Z}_n \). Moreover, it is a \( \mathbb{Z}_n \)-graded ring (i.e. \( C_i C_j \subseteq C_{i+j} \) for all \( i, j \in \mathbb{Z}_n \)).

Let \( \mathcal{O} \) be a cyclic orbit which contains \( n = | \mathcal{O} | \) elements. For a maximal ideal \( p \in \mathcal{O} \) \((\sigma^n(p) = p)\), let us consider the factor ring

\[
C_{[n],p} := C_{[n]}/(p)
\]
of \( C_{[n]} \) modulo the ideal \( (p) = \bigoplus_{i \in \mathbb{Z}} (pv_{in} = v_{in}p) \) of \( C_{[n]} \) generated by \( p \). Since every cyclic orbit is a non-degenerate one, the algebra \( C_{[n],p} = \mathcal{K}[X^n, X^{-n}; \sigma^n] \) is the skew Laurent extension with coefficients from the field \( \mathcal{K} \). Since \( \mathcal{K} \) is a field and \( \sigma^n \) is an automorphism of the field \( \mathcal{K} \), the ring \( C_{[n],p} \) is a left principal ideal domain and a right principal ideal domain. This means that every left (and right) ideal of the ring \( C_{[n],p} \) is generated by a single element. The reason for this is that in the ring \( C_{[n],p} \) the left (and right) division algorithm with remainder holds. So, any left simple \( C_{[n],p} \)-module has the form \( C_{[n],p}/C_{[n],p}b \) where \( b \) is an irreducible element of \( C_{[n],p} \).

Let \( M \) be a simple \( C \)-module with support from a cyclic orbit, say \( \mathcal{O} \), that contains \( n = | \mathcal{O} | \) elements. It follows from the weight decomposition

\[
M = \bigoplus_{p \in \mathcal{O}} M_p
\]
that \( M_p \) is a simple \( C_{[n],p} \)-module for each \( p \in \text{Supp}(M) \), and
\[
C_i M_p \subseteq M_{\sigma^i(p)} \quad \text{for all} \quad i \in \mathbb{Z}_n.
\]

It follows that \( M \cong C \otimes_{C_{[n],p}} M_p \) for each \( p \in \text{Supp}(M) \), and so we have the following theorem.

**Theorem 3.5**

1. \( \hat{C}(\text{weight, cyclic}) = \bigcup_{n \geq 1} \bigcup_{\sigma \in \text{Cyc}_n} \hat{C}(O) \)
   where \( \hat{C}(O) \) is the set of isoclasses of simple weight \( C \)-modules with support from the orbit \( O \), and \( \text{Cyc}_n \) is the set of all the cyclic orbits that contain exactly \( n \) elements.

2. For each orbit \( O \in \text{Cyc}_n \) and a fixed element \( p \in O \), the map
   \[
   \hat{C}_{[n],p} \rightarrow \hat{C}(O), \quad [N] \mapsto [C \otimes_{C_{[n],p}} N],
   \]
   is a bijection where \( \hat{C}_{[n],p} \) is the set of isoclasses of simple \( \hat{C}_{[n],p} \)-modules.

Theorems 3.1 and 3.5 classify the simple \( C \)-modules.

**Finite dimensionality of kernels and cokernels.** In [9], McConnel and Robson proved that for a simple module \( M \) over the first Weyl algebra \( A_1(F) \) over a field \( F \) of characteristic zero and for any non-scalar element \( u \) of \( A_1(F) \), \( \dim_F(\ker u_M) < \infty \) and \( \dim_F(\text{coker} u_M) < \infty \) where \( u_M : M \rightarrow M, \ m \mapsto um \). This result is also true for some GWAs see [2], and for the Carlitz algebras.

**Theorem 3.6** Let \( M \) be a simple \( C \)-module and \( u \in C \setminus K \). Then \( \dim_K(\ker u_M) < \infty \) and \( \dim_K(\text{coker} u_M) < \infty \).

**Proof.** The result is obvious if the module \( M \) is finite dimensional over \( K \). So, let us assume that the module \( M \) is infinite dimensional over \( K \), that is \( M \) is from Theorem 3.1. The \( C \)-module \( M \) is a \( \mathbb{Z} \)-graded module, each graded component is a 1-dimensional vector space over the field \( K \) (see the description of modules after Theorem 3.1).

If \( u \in D \) then the kernel and cokernel of the map \( u_M \) coincide, and their common dimension over \( K \) does not exceed the number of distinct roots of the polynomial \( u \in D := K[H] \).

If \( u = \alpha X^i \) or \( u = \alpha Y^i \) for some \( i \geq 1 \) and \( 0 \neq \alpha \in D \) then the dimension of the kernel and cokernel of the map \( u_M \) over \( K \) does not exceed \( i+1 \) the number of distinct roots of the polynomial \( \alpha \).

Finally, if \( u = \alpha v_n + \beta v_{n-1} + \cdots + \gamma v_m, \ n > m, \ \alpha \) and \( \gamma \) are nonzero polynomials of \( D \). Since
\[
\ker_K u_M \subseteq \ker_K (\alpha v_n)_M + \ker_K (\gamma v_m)_M,
\]
we have \( \dim_K(\ker u_M) < \infty \).

Note that \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) is a \( \mathbb{Z} \)-graded \( C \)-module where \( M_i \) is a weight component of dimension 1 over \( K \). Let \( k = |n| + |m| \) and \( M = \bigcup_{j \geq 0} M^j \) where \( M^j := \bigoplus_{-kj \leq i \leq kj} M_i \).
Since $\dim_K(\ker u_M) < \infty$, $uM^j \subseteq M^{j+1}$ for all $j \geq 0$, and $\dim_K(M^{j+1}) - \dim_K(M^j) = 2k = \text{const}$, we have, for all $j \geq 0$,

$$
\dim_K(M^{j+1}/uM^j) = \dim_K(M^{j+1}) - \dim_K(uM^j) \\
\leq \dim_K(M^{j+1}) - (\dim_K(M^{j+1}) - \dim_K(\ker u_M)) \\
= \dim_K(M^{j+1}) - \dim_K(M^j) + \dim_K(\ker u_M) \\
= \color{red}{c = \text{const} < \infty}.
$$

Therefore, $\dim_K(\coker u_M) < c < \infty$. □

4 The prime spectrum and the ideal structure of the Carlitz algebras

In this section, the following theorem will be proved that completely describes the ideal structure of the Carlitz algebra.

Recall that $\text{Cyc}_n$ denote the set of all the finite orbits that contain exactly $n$ elements.

By Theorem 3.3.(2), $0 < |\text{Cyc}_n| < \infty$. For each orbit $O \in \text{Cyc}_n$ and any $\lambda$ such that $D(H - \lambda) \in O$, let

$$
\alpha_O := \prod_{i=0}^{n-1} \sigma^i(H - \lambda).
$$

(7)

Clearly, $\alpha_O$ is a central element of the algebra $C$ (since $\sigma^n(H - \lambda) = H - f_n(\lambda) - \lambda = H - \lambda$, see the proof of Lemma 3.2), $\alpha_O$ is a monic polynomial of $K[H]$ which is does not depend on the choice of $\lambda$. The polynomial $\alpha_O$ is the only monic polynomial which is a generator for the ideal $\prod_{p \in O} p$ of the algebra $D$.

Theorem 4.1

1. Every nonzero prime ideal of the Carlitz algebra $C$ is a maximal ideal.

2. Maximal ideals of $C$ commute, $mn = nm$.

3. Each nonzero ideal $I$ of $C$ is a unique finite product of maximal ideals of $C$: $I = \prod_{m \in \text{Max}(C)} m^{n(m)}$ for some $n(m) \geq 0$ all but finitely many $n(m) = 0$ and if $I = \prod_{m \in \text{Max}(C)} m^{n(m)} = \prod_{m \in \text{Max}(C)} m^{l(m)}$ then $n(m) = l(m)$ for all $m$. So, all ideals commute.

4. The map $\text{Cyc} \rightarrow \text{Max}(C)$, $\mathcal{O} \mapsto m_\mathcal{O} := C \prod_{p \in \mathcal{O}} p = C\alpha_\mathcal{O}$, is a bijection with inverse $m \mapsto \text{Supp}(C/m)$.

5. For each $\mathcal{O} \in \text{Cyc}$, the factor algebra $C/m_\mathcal{O} \cong M_n(K[t, t^{-1}; \sigma^n])$ is the $n \times n$ matrix algebra with coefficients from the skew Laurent extension $K[t, t^{-1}; \sigma^n]$ where $n = |\mathcal{O}|$, the number of element in the orbit $\mathcal{O}$. The centre $Z(C/m_\mathcal{O}) \cong \mathbb{F}_{q^n}$ and $K\dim(C/m_\mathcal{O}) = \text{gl.dim}(C/m_\mathcal{O}) = 1$. 

10
6. The factor algebra \( C/\mathfrak{m}_O \) is a domain iff \( O = \{ \mathfrak{p} \} \) where \( \mathfrak{p} \) is a \( \sigma \)-invariant maximal ideal of the algebra \( D \). There are exactly \( q \) such ideals (Theorem 3.3).

7. The factor algebras \( C/\mathfrak{m}_O \) and \( C/\mathfrak{m}_O' \) are isomorphic over \( \mathbb{F}_p \) iff \( |O| = |O'| \).

In order to prove Theorem 4.1 we first establish some preliminary results that are interesting on their own.

The next result describes the centre of the algebra \( C \) and its localization \( B \).

**Lemma 4.2**

1. For each natural number \( n \), the skew Laurent extension \( \mathcal{K}[t, t^{-1}; \sigma^n] \) is a simple \( \mathbb{F}_q \)-algebra with centre \( \mathbb{F}_q \).

2. The algebra \( B = S^{-1}C = \mathcal{K}(H)[X, X^{-1}; \sigma] \) is a simple algebra with centre \( Z(B) = \{ \mathbb{F}_q \prod_{O \in Cyc} \alpha_O^{n(O)} \mid n(O) \in \mathbb{Z} \text{ and all but finitely many } n(O) = 0 \} \) that contains countably many elements.

3. The centre \( Z(C) = \{ \mathbb{F}_q \prod_{O \in Cyc} \alpha_O^{n(O)} \mid n(O) \geq 0 \text{ and all but finitely many } n(O) = 0 \} \) is a unique factorization domain that contains countably many elements.

**Proof.**

1. For each natural \( i \), \( \mathcal{K}^{\sigma^i} := \{ \lambda \in \mathcal{K} \mid \lambda = \sigma^i(\lambda) \lambda^{\sigma^i} \} = \mathbb{F}_q \). We claim that the centre \( Z \) of the \( \mathbb{F}_p \)-algebra \( R := \mathcal{K}[t, t^{-1}; \sigma^n] \) belongs to the field \( \mathcal{K} \) since otherwise we would have a nonzero central element of the form \( z = \lambda t^m + \cdots \) for some \( 0 \neq m \in \mathbb{Z} \) and \( 0 \neq \lambda \in \mathcal{K} \) where the three dots mean elements of strictly higher or strictly lower degree in \( t \). For any \( \mu \in \mathcal{K} \setminus \mathbb{F}_q \), \( \mu z - z \mu = (\mu - \sigma^m(\mu)) \lambda t^m + \cdots \neq 0 \), a contradiction. An element \( \lambda \in \mathcal{K} \) commutes with \( t \) iff \( \lambda \in \mathbb{F}_q \). Therefore, \( Z = \mathbb{F}_q \).

2. Similarly, by [10], 1.8.5, \( B \) is a simple ring. By exactly the same reason as in the first case, \( Z(B) \subseteq \mathcal{K}(H) \). Clearly, \( \mathbb{F}_q \prod_{O \in Cyc} \alpha_O^{n(O)} \subseteq Z(B) \). We have to prove that any nonzero element \( z \in Z(B) \) can be written in this form. The rational function \( z \) is equal to \( \gamma F \) where \( f, g \in \mathcal{K}[H] \) are co-prime monic polynomials and \( \gamma \in \mathcal{K} \).

If \( z = \gamma \) then \( \gamma = X \gamma X^{-1} = \sigma(\gamma) \), and so \( \gamma \in \mathbb{F}_q \), and we are done.

So, suppose that \( z \neq \gamma \). Since \( z = X^n z X^{-n} = \sigma^n(z) \) for all \( n \in \mathbb{Z} \), and since \( f \) and \( g \) are co-prime we see that \( f \) and \( g \) are equal to finite products of the form \( \prod \alpha_O^{n(O)} \) with \( n(O) \geq 0 \). Then \( \gamma \in Z(B) \), and so \( \gamma \in \mathbb{F}_q \).

3. Clearly, \( Z(C) \subseteq Z(B) \), and the result follows from statement 2. \( \square \)

Recall that \( Cyc_n = \{ O \in Cyc \mid |O| = n \} \).

**Lemma 4.3**

For each \( O \in Cyc_n \),

1. the ideal \( C\alpha_O = \alpha_O C \) of \( C \) generated by the central element \( \alpha_O \) of \( C \) is a maximal ideal, and

2. the factor ring \( C/C\alpha_O \simeq M_n(\mathcal{K}[t, t^{-1}; \sigma^n]) \), the matrix algebra with entries from the skew Laurent extension \( \mathcal{K}[t, t^{-1}; \sigma^n] \).
3. The centre \( Z(C/C_{\alpha O}) \simeq F_q^n \).

**Proof.** There are obvious \( C \)-module isomorphisms (where \( p \in O \)):

\[
C/C_{\alpha O} \simeq C \otimes_D D/\alpha O \simeq C \otimes_D (\oplus_{i=1}^n D/\sigma^i(p)) \simeq \oplus_{i=1}^n C \otimes_D D/\sigma^i(p)
\]

\[
\simeq \oplus_{i=1}^n C \otimes_D D/p \simeq (C \otimes_D D/p)^n.
\]

Now, there are obvious \( \mathbb{F}_q \)-algebra isomorphisms:

\[
C/C_{\alpha O} \simeq \text{End}_{C/C_{\alpha O}}(C/C_{\alpha O}) \simeq \text{End}_C((C \otimes_D D/p)^n) \simeq M_n(\text{End}_C(C \otimes_D D/p))
\]

\[
\simeq M_n(\text{End}_{C/O}(C_{[n],p}) \simeq M_n(\text{End}_{C_{[n],p}}(C_{[n],p}) \simeq M_n(C_{[n],p}) \simeq M_n(K[t, t^{-1}; \sigma^n]).
\]

This proves the second statement. By Lemma 4.2, \( C/C_{\alpha O} \) is a simple algebra, hence \( C_{\alpha O} \) is a maximal ideal. The third statement follows from Lemma 4.2.1,

\[
Z(C/C_{\alpha O}) \simeq Z(M_n(K[t, t^{-1}; \sigma^n])) \simeq Z(K[t, t^{-1}; \sigma^n]) \simeq \mathbb{F}_q^n. \quad \square
\]

The next corollary describes the annihilators of simple \( C \)-modules.

**Corollary 4.4**

1. \( \text{ann}_C(L) = 0 \) for any \( [L] \in \hat{C} \) (weight, linear).

2. \( \text{ann}_C(L) = C_{\alpha \text{Supp}(L)} \) for any \( [L] \in \hat{C} \) (weight, cyclic).

**Proof.**

1. This follows immediately from Theorem 3.1 and the description of the modules that follows Theorem 3.1.

2. Let \( O = \text{Supp}(L) \). Then clearly, \( C_{\alpha O}L = 0 \), and by maximality of the ideal \( C_{\alpha O} \) (Lemma 4.3.1) we must have \( \text{ann}_C(L) = C_{\alpha O} \). \( \square \)

**Proof of Theorem 4.1**

1. Let \( I \) be a proper ideal of the algebra \( C \). Since the algebra \( B = S^{-1}C \) is a simple algebra (Lemma 4.1.2), we must have \( S^{-1}I = B \), and so the \( C \)-module \( \overline{C} := C/I \) is a torsion one \( (S^{-1}\overline{C} = S^{-1}(C/I) \simeq S^{-1}C/S^{-1}I = 0) \). Its orbit decomposition

\[
\overline{C} = \bigoplus O \overline{C} O, \quad \overline{C} O := \text{ann}_\overline{C}(\alpha^O),
\]

contains only finitely many summands (as \( \overline{C} \) is a Noetherian module) and all the \( O \) must be cyclic (since otherwise, there exists \( O \in \text{Lin} \), then \( \overline{C} O \) contains a simple submodule, say \( L \), with support from an equivalence class of \( O \), then \( 0 \neq I = \text{ann}_C(\overline{C}) \subseteq \text{ann}_C(L) = 0 \), by Corollary 4.1 a contradiction). Suppose that \( O_1, \ldots, O_s \) are the cyclic orbits involved in the orbit decomposition above and \( \alpha_1 := \alpha_{O_1}, \ldots, \alpha_s := \alpha_{O_s} \) are the corresponding central polynomials. There exists a natural number \( n \) such that \( \alpha_i^n \overline{C} O_i = 0 \) for all \( i \). Therefore, \( C \prod_{i=1}^s \alpha_i^n \subseteq I \). If \( I \) is a prime ideal then \( C\alpha_i \subseteq I \) for some \( i \), and then \( I = C\alpha_i \) since \( C\alpha_i \) is a maximal ideal (Lemma 4.3.1).

2. We have just proved that any maximal ideal \( \mathfrak{m} \) of the ring \( C \) is equal to the ideal generated by a central element \( \alpha_O \). Now, it is clear that maximal ideals commute.
4. Clearly, the map \( Cyc \to \text{Max}(C) \), \( \mathcal{O} \mapsto \mathfrak{m}_\mathcal{O} := C\alpha_\mathcal{O} \) is a bijection with inverse \( \mathfrak{m} \mapsto \text{Supp}(C/\mathfrak{m}) \).

5 and 6. Lemma 4.3, the (left and right) global dimension \( \text{gl.dim}(C/\mathfrak{m}_\mathcal{O}) = 1 \) (by [10], 7.9.18) and the (left and right) Krull dimension \( \text{Kdim}(C/\mathfrak{m}_\mathcal{O}) = 1 \) (by [10], 6.5.4).

7. If \( C/\mathfrak{m}_\mathcal{O} \simeq C/\mathfrak{m}_\mathcal{O}' \) then, by Lemma 4.3, (where \( n = |\mathcal{O}| \) and \( n' = |\mathcal{O}'| \))

\[
\mathbb{F}_{q^n} \simeq Z(C/\mathfrak{m}_\mathcal{O}) \simeq Z(C/\mathfrak{m}_\mathcal{O}') \simeq \mathbb{F}_{q^{n'}},
\]

and so \( n = n' \).

If \( n = n' \) then by Lemma 4.3

\[
C/\mathfrak{m}_\mathcal{O} \simeq M_n(K[t, t^{-1}; \sigma^n]) \simeq C/\mathfrak{m}_\mathcal{O}'.
\]

3. Let \( I \) be a proper ideal of the ring \( C \), we keep the notation of the proof of statement 1. The component \( \overline{C}^\mathcal{O} \) is equal to the union \( \bigcup_{i \geq 1} \text{ann}_C(\alpha_i^\mathcal{O}) \), and so \( \overline{C}^\mathcal{O} \) is a two-sided ideal of the algebra \( C \). Since \( \overline{C}^\mathcal{O} \) is a Noetherian \( C \)-module the chain

\[
\text{ann}_C(\alpha_i^\mathcal{O}) \subseteq \text{ann}_C(\alpha_i^{n_2}) \subseteq \cdots
\]

must terminate, say on \( n_\mathcal{O} \) step, that is \( \overline{C}^\mathcal{O} = \text{ann}_C(\alpha_i^{n_\mathcal{O}}) \). Since the polynomial \( \alpha_\mathcal{O} \in D \) has only simple roots, it follows that for the \( C \)-bimodule \( C/C\alpha_i^\mathcal{O} \) is a uniserial bimodule of finite length, and the bimodule structure is given by the descending chain of \( C \)-bimodules:

\[
C \supset C\alpha_\mathcal{O} \supset C\alpha_\mathcal{O}^2 \supset \cdots \supset C\alpha_\mathcal{O}^{i-1} \supset C\alpha_\mathcal{O}^i
\]

and each subfactor \( C\alpha_\mathcal{O}^j/C\alpha_\mathcal{O}^{j+1} \simeq C/C\alpha_\mathcal{O} \simeq K[t, t^{-1}; \sigma^n] \) (\( n = |\mathcal{O}| \)) is a simple algebra = a simple \( C \)-bimodule (Lemma 4.3). Now, there exist unique numbers \( n_i \) such that

\[
\overline{C} = \bigoplus_{i=1}^s \overline{C}^{\mathcal{O}_{i}} = \bigoplus_{i=1}^s C/C\alpha_i^{n_i} = C/C \prod_{i=1}^s \alpha_i^{n_i}.
\]

Taking the annihilator we have

\[
I = \text{ann}_C(\overline{C}) = \text{ann}_C(C/C \prod_{i=1}^s \alpha_i^{n_i}) = C \prod_{i=1}^s \alpha_i^{n_i} = \prod_{i=1}^s \mathfrak{m}_{\mathcal{O}_i}^{n_i}.
\]

Note that \( I \cap D = D \prod_{i=1}^s \alpha_i^{n_i} \) and the uniqueness of the \( n_i \) now is obvious. Clearly, \( I = C(I \cap D) = (I \cap D)C = C(I \cap Z(C)) \). □

In fact we have proved the following corollary.

**Corollary 4.5**

1. If \( I \) is an ideal of \( C \) then \( I = C(I \cap D) = (I \cap D)C = C(I \cap Z(C)) \).

2. If \( I \) and \( J \) are ideals of \( C \) then \( I = J \) iff \( I \cap D = J \cap D \) iff \( I \cap Z(C) = J \cap Z(C) \).
5 The group of automorphisms \( \text{Aut}(C) \) and the isomorphism problem for the Carlitz algebras

Theorem 5.1 Two distinct Carlitz rings are not isomorphic.

Proof. Given two distinct Carlitz rings \( C_\nu \) and \( C_\mu \) with \( \nu < \mu \), and so \( p^\nu < p^\mu \). Then, by Lemma 4.3 (3) and Theorem 4.1, the isomorphism invariant numbers

\[
\begin{align*}
\varepsilon &= \min\{|Z(C_\nu/C_\nu m_\mathcal{O})| = p^\nu|\mathcal{O}| : \mathcal{O} \in C_y c\}, \\
p^\mu &= \min\{|Z(C_\mu/C_\mu m_\mathcal{O}')| = p^\mu|\mathcal{O}'| : \mathcal{O}' \in C_y c\},
\end{align*}
\]

are distinct. Therefore, the rings \( C_\nu \) and \( C_\mu \) cannot be isomorphic. \( \square \)

For a domain \( R \), let \( R^* := R\setminus\{0\} \). In particular, \( \mathcal{K}^* \) is a multiplicative group.

Theorem 5.2

1. The group of ring isomorphisms of the ring \( C \), \( \text{Aut}(C) = \{ \tau = \tau_{\alpha,\gamma,\delta,\omega} : X \mapsto \alpha X, \ Y \mapsto \gamma\sigma^{-1}(\alpha^{-1})Y \mid \alpha \in \mathcal{K}^*, \ \gamma \in \mathbb{F}_p^*, \ \delta \in \mathbb{F}_{p^\nu}, \ \omega \in \text{Aut}(\mathcal{K}) \} \) such that \( \omega(x) = \gamma x + \delta \).

2. The group of \( \mathcal{K} \)-isomorphisms of \( C = C_\nu \), \( \text{Aut}_{\mathcal{K}}(C) = \{ \tau = \tau_{\alpha} : X \mapsto \alpha X, \ Y \mapsto \sigma^{-1}(\alpha^{-1})Y \mid \alpha \in \mathcal{K}^* \} \) \( \cong \mathcal{K}^* \).

Proof. 1. Given \( \tau \in \text{Aut}(C) \). Then \( \tau \) induces the ring isomorphism of the centre of \( C \). By Lemma 4.2 (3) and Theorem 5.3, we must have \( \tau(H) = \gamma(H + \varepsilon) \) for some \( \gamma \in \mathbb{F}_q^* \) and \( \varepsilon \in \mathbb{F}_q \). The algebra \( C \) is the GWA, and so \( C = \bigoplus_{i \in \mathbb{Z}} C_i \) is a \( \mathbb{Z} \)-graded algebra where \( C_i = Dv_i \) is an eigenspace of the inner derivation \( \text{ad}(H) : C \to C, \ c \mapsto [H,c] := Hc - cH \), that corresponds to the eigenvalue \( e_i \) given by the rule

\[
\begin{align*}
\lambda_1 + \lambda_2 q + \cdots + \lambda_i^{q^i-1} &= -x^\frac{1}{q^i} + x^{\frac{1}{q^i}-1}, \quad \text{if} \ i \geq 1, \\
-\lambda_1 - \lambda_2 q - \cdots - \lambda_i^{q^i-1} &= -x + x^\frac{1}{q^i}, \quad \text{if} \ i \leq -1, \\
0 &= \quad \text{if} \ i = 0,
\end{align*}
\]

that is, \( C_i = \{ c \in C \mid [H,c] = e_i c \} \). Note that all the \( e_i \) are distinct. Since \( C_i = C_1 \) and \( C_{-i} = C_{-1} \) for all \( i \geq 1 \). We must have either \( \tau(C_{\pm 1}) = C_{\pm 1} \) or, otherwise, \( \tau(C_1) = C_{-1} \) and \( \tau(c_{-1}) = C_1 \).

In the first case, \( \tau(X) = \alpha X \) and \( \tau(Y) = \beta Y \) for some \( \alpha, \beta \in D^* \). Applying \( \tau \) to the identity \( YX = H \) we have the identity \( \beta\sigma^{-1}(\alpha)H = \gamma(H + \varepsilon) \), therefore \( \beta = \gamma\sigma^{-1}(\alpha^{-1}) \), \( \alpha, \beta \in \mathcal{K}^* \), and \( \varepsilon = 0 \). Applying \( \tau \) to the identity \( YX - XY = \lambda_1 \) and then dividing by \( \gamma \) we get \( YX - XY = \gamma^{-1}\tau(\lambda_1) \), therefore, \( \tau(\lambda_1) = \gamma\lambda_1 \), equivalently, \( (\tau(x) - \gamma x)^q = \tau(x) - \gamma x \), and so \( \tau(x) = \gamma x + \delta \) for some \( \delta \in \mathbb{F}_q^* \). The units of the ring \( C \) form the set \( \mathcal{K}^* \), therefore the \( \tau \) induces an automorphism of the field \( \mathcal{K} \), say \( \omega := \tau|_{\mathcal{K}} \). Thus \( \tau = \tau_{\alpha,\gamma,\delta,\omega} \). One can easily verify that \( \tau_{\alpha,\gamma,\delta,\omega} \in \text{Aut}(C) \).

Let us show that the second case is impossible. In the second case, \( \tau(X) = \alpha Y \) and \( \tau(Y) = \beta X \) for some \( \alpha, \beta \in D^* \). Applying \( \tau \) to the identity \( YX = H \) we have...
\(\gamma(H + \varepsilon) = \beta \sigma(\alpha) XY = \beta \sigma(\alpha)(H - \lambda_1)\) which is impossible since \(\varepsilon \in \mathbb{F}_q\) and \(\lambda_1 \notin \mathbb{F}_q\). So, the second case is vacuous.

2. Since \(\omega = \tau|_K\) is the identity map and \(x \in K^*\), we must have \(\gamma = 1\) and \(\delta = 0\), and the result follows from the first statement. \(\square\)

6 The Krull and the global dimensions of the Carlitz algebras

The global dimension of the Carlitz algebra. The global dimension, \(\text{gl.dim}\), means the left or right global dimension.

**Theorem 6.1** ([3], Theorem 1.6) Let \(A = D(\sigma, a)\) be a GWA where \(D\) is a commutative Dedekind domain, \(Da = p_1^{n_1} \cdots p_s^{n_s}\) be a product of distinct maximal ideals of \(D\). Then the global dimension of the algebra \(A\) is equal to

\[
\text{gl.dim}(A) = \begin{cases} 
\infty, & \text{if } a = 0 \text{ or } \exists n_i \geq 2, \\
2, & \text{if either } a \neq 0, n_1 = \cdots = n_s = 1, s \geq 1, \text{ or } a \text{ is invertible; and either } \\
\text{Cyc} \neq \emptyset \text{ or } \exists i \neq j \text{ s.t. } \sigma^k(p_i) = p_j \text{ for some } k \geq 1, \\
1, & \text{otherwise.}
\end{cases}
\]

**Corollary 6.2** The global dimension of the Carlitz algebra is 2.

**Proof.** In the case of the Carlitz algebra \(C\), we have \(D = K[H]\) and \(a = H\), and \(\text{Cyc} \neq \emptyset\). Therefore, by Theorem 6.1, \(\text{gl.dim}(C) = 2\). \(\square\)

The Krull dimension of the Carlitz algebra. The Krull dimension, \(\text{Kdim}\), means the left or right Krull dimension.

Let \(R\) be a commutative Noetherian ring and \(\sigma \in \text{Aut}(R)\). A prime ideal \(p\) of \(R\) is called \(\sigma\)-semistable if \(\sigma^n(p) = p\) for some natural number \(n \geq 1\). If there is no such \(n\) the ideal \(p\) is called \(\sigma\)-unstable. \(\text{ht}\) stands for the height of the prime ideal \(p\).

**Theorem 6.3** ([5], Theorem 1.2) Let \(R\) be a commutative Noetherian ring with Krull dimension \(\text{Kdim}(R) < \infty\) and \(A = R(\sigma, a)\) be a GWA. Then its Krull dimension \(\text{Kdim}(A) = \max\{\text{Kdim}(R), \text{ht} p + 1, \text{ht} q + 1 \mid p\text{ is a } \sigma\text{-unstable prime ideal of } R \text{ for which there exist infinitely many } i \text{ with } a \in \sigma^i(p); q\text{ is a } \sigma\text{-semistable prime ideal of } R\}\).

**Corollary 6.4** The Krull dimension of the Carlitz algebra is 2.

**Proof.** Note that \(\text{Kdim}(D) = 1\) and so, by Theorem 6.3, \(\text{Kdim}(C) \leq 2\). There exists a maximal ideal \(p\) of the algebra \(D\) such that \(\sigma(p) = p\). Then, by Theorem 6.3 we must have \(\text{Kdim}(C) = 2\) since \(\text{ht} p = 1\). \(\square\)
References

[1] V. V. Bavula, Finite-dimensionality of Ext^n and Tor_n of simple modules over a class of algebras. Funktsional. Anal. i Prilozhen. 25 (1991), no. 3, 80–82.

[2] V. V. Bavula, Generalized Weyl algebras and their representations. (Russian) Algebra i Analiz 4 (1992), no. 1, 75–97; translation in St. Petersburg Math. J. 4 (1993), no. 1, 71–92.

[3] V. V. Bavula, Tensor homological minimal algebras, global dimension of the tensor product of algebras and of generalized Weyl algebras. Bull. Sci. Math. 120 (1996), no. 3, 293–335.

[4] V. V. Bavula and F. van Oystaeyen, The simple modules of certain generalized crossed products. J. Algebra 194 (1997), no. 2, 521–566.

[5] V. V. Bavula and F. van Oystaeyen, Krull dimension of generalized Weyl algebras and iterated skew polynomial rings: commutative coefficients. J. Algebra 208 (1998), no. 1, 1–34.

[6] L. Carlitz, On certain functions connected with polynomials in a Galois field, Duke Math. J. 1 (1935), 137–168.

[7] A. N. Kochubei, Harmonic oscillator in characteristic p. Lett. Math. Phys. 45 (1998), no. 1, 11–20.

[8] A. N. Kochubei, Differential equations for \( F_q \)-linear functions. J. Number Theory 83 (2000), no. 1, 137–154.

[9] J. C. McConnell and J. C. Robson, Homomorphisms and extensions of modules over certain differential polynomial rings. J. Algebra, 26 (1973), 319–342.

[10] J. C. McConnell and J. C. Robson, Noncommutative Noetherian rings. With the co-operation of L. W. Small. Revised edition. Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI, 2001.

Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk