Deep Equilibrium Assisted Block Sparse Coding of Inter-dependent Signals: Application to Hyperspectral Imaging

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Abstract—In this study, the problem of computing a sparse representation for datasets of inter-dependent signals, given a fixed dictionary, is considered. A dataset of inter-dependent signals is defined as a matrix whose columns demonstrate strong dependencies. A computational efficient sparse coding optimization problem is derived by employing regularization terms that are adapted to the properties of the signals of interest. Exploiting the merits of the learnable regularization techniques, a neural network is employed to act as structure prior and reveal the underlying signal inter-dependencies. To solve the optimization problem Deep unrolling and Deep equilibrium based algorithms are developed, forming highly interpretable and concise deep-learning-based architectures, that process the input dataset in a block-by-block fashion. Extensive simulation results, in the context of hyperspectral image denoising, are provided, that demonstrate that the proposed algorithms outperform significantly other sparse coding approaches and exhibit superior performance against recent state-of-the-art deep-learning-based denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models. In a wider perspective, our work provides a unique bridge between a classic approach, that is the sparse denoising models.

Index Terms—Sparse coding, Deep Equilibrium models, Deep Unrolling methods, Inter-dependent signals, Hyperspectral imaging.

I. INTRODUCTION

O VER the past years, the sparse representation theory has evolved into a mature and highly influential mathematical modeling framework, which has led to remarkable results in a wide variety of applications across numerous disciplines e.g., signal processing, image processing [1] and machine learning [2]. A plethora of works have utilized the sparse representation framework as an effective prior to model signals encountered in various problems, ranging from image denoising [3], inpainting [4] and spatial/spectral super-resolution [5], [6] to hyperspectral unmixing [7], magnetic resonance imaging (MRI) [8], classification [9] and compression [10]. Building upon its elegant theoretical foundation, sparse representation modeling seeks to discover the inherent sparsity structure that exists in many natural signals [1]. In greater detail, in its most usual form, this model aims to approximate a signal, represented by a vector, as a linear combination of a limited number of columns, termed atoms, from a given redundant matrix, known as a dictionary [11], [12]. In other words, each signal is effectively approximated as the product of the dictionary with a proper sparse vector. The task of computing such a sparse vector constitutes the sparse coding problem.

Nevertheless, the process of computing the optimal sparse representation vector requires solving an NP-hard optimization problem, suggesting that only approximate solutions can be computed in polynomial time [13]. In general, sparse coding algorithms can be divided into two main categories, that is, greedy methods and convex relaxation based approaches. Greedy methods try to minimize the \( l_0 \) pseudo-norm, that captures the sparsity of the solution, in a greedy fashion. They include prominent algorithms such as the orthogonal matching pursuit (OMP) [14], the batch-OMP [15] and the compressive sampling matching pursuit (CoSaMP) [16]. On the other hand, convex relaxation based approaches, such as the basis pursuit (Lasso) [18], replace the \( l_0 \) pseudo-norm with the \( l_1 \) norm, thus forming a convex problem which is much easier to solve. However, with the exception of very few works [6], [19], [17], most existing sparse coding algorithms do not take into account the fact that, in many applications, the signal vectors that populate the data matrix exhibit strong dependencies among each other. One of the key points in this work is that the proposed algorithms take into account such inter-dependencies among signals that appear at nearby columns inside the data matrix. In the remaining of this work, we refer to this property, that is usually present in datasets of signals, as the dependency along the data matrix, to emphasize the fact that signals that appear at different columns inside a data matrix exhibit strong dependencies. Also, in the following, we use the term dependency across the data matrix to refer to the structure that is present in each individual signal vector, and which of course also includes sparsity.

Signals that exhibit dependencies both along and across the corresponding data matrix often appear in various engineering disciplines, e.g., image processing, remote sensing, control systems and environmental monitoring [20]. A typical case of signals that exhibit such inter-dependencies is hyperspectral images (HSI), where each individual hyper-pixel/spectrum has internal structure (e.g., it admits a sparse representation) but also hyper-pixels located at neighbouring spatial locations demonstrate strong dependencies [7], [21] (See also Fig. 1). Additionally, in numerous scenarios and settings these signals are corrupted by noise and/or interference. For instance, in remote sensing applications the quality of hyperspectral images can be degraded by several factors e.g., atmospheric turbulence, extreme temperatures and sensor imperfections [22]. Under the assumption that the considered signals exhibit
dependencies both along and across the corresponding data matrix, this *a-priori* information can be utilized to derive sparse coding algorithms offering enhanced signal restoration performance, but also reduced computational complexity.

To capture the inter-dependencies present in a data matrix, in this work, we explore the potential of an emerging body of studies which employs regularization terms properly learnt from the data via the use of suitable neural networks \([23]–[28]\). In particular, these regularizers can be learnt effectively from a collection of training data, thus enabling them to capture the inherent structure of the data, and in the sequel, they can be properly used in an optimization problem to promote the properties of the signals of interest. Convolutional Neural Networks (CNNs) constitute effective models for learning the structure of data and deriving proper regularization terms, due to their powerful representation capacity \([24], [29]\). Based on these remarks and different from the above-mentioned studies that focus on inverse imaging problems, in this study, we investigate the idea of combining the learnable regularizer (i.e., a convolutional neural network) in conjunction with the sparsity promoting \(l_1\) norm and a data-consistency term, to form a novel cost function for the sparse coding problem, that is able not only to reveal the sparsity nature of the signals but also to model their inter-dependencies. Employing variable splitting techniques, and in particular the half quadratic splitting (HQS) approach \([30]\), it is possible to decouple the data-consistency term, the \(l_1\) norm and the CNN prior, thus forming a novel iterative sparse coding solver.

Having derived an iterative solver for the problem at hand, in the sequel, we leverage our results by employing the recently developed model-based deep learning theory, and in particular, Deep Unrolling (DU) \([31]\) and Deep Equilibrium (DEQ) \([32]\) approaches. In particular, by unrolling a small number of iterations of the proposed solver, a deep learning architecture is formed where each layer corresponds to an iteration of the sparse coding solver. The forward pass of this network is equivalent to iterating the considered algorithm a fixed number of iterations \([31]\). However, the deep unrolling techniques are characterised by several limitations, including stability, memory and numerical issues during the training process, thereby the number of the unrolling iteration must be kept quite small \([33], [31]\). To surmount the key limitations introduced by the DU method, a second sparse coding approach is proposed utilizing the efficiency of the Deep Equilibrium models. The DEQ approach aims to express the entire deep learning architecture derived from the DU method as an equilibrium (fixed-point) computation, which corresponds to an efficient network with an infinite number of layers.

Adopting a wider perspective, our study aligns with the work in \([34]\), in the sense that it provides an attractive connection between a classical approach, that is the sparse representation theory, and modern representation tools that are based on deep learning modeling. Indeed, following the above-mentioned remarks, the proposed sparse coding algorithm can be effectively formulated into deep networks based on the deep unrolling and deep equilibrium strategies, and be trained via end-to-end supervised learning. A great benefit that stems from such an approach is the fact that the architecture of the proposed deep networks is highly interpretable, since the network parameters (i.e., the weights of the CNN prior and the regularization coefficients) derive directly from the parameters of the proposed sparse coding iterative optimization scheme. Along with their high interpretability, the derived deep unrolling and deep equilibrium models are more concise, requiring less training data. This derives from the fact that they are designed via modeling of the physical processes of the proposed sparse coding problem and by utilizing prior domain knowledge, in the form of correlation structure and sparsity priors.

The key contributions of this work are summarized as follows:

- Considering the potentials of the learnable regularization techniques, a novel sparse coding optimization problem is proposed, involving a sparsity promoting \(l_1\) norm, a learnable regularizer (using a convolutional neural network) and a data-consistency term. The combination of the \(l_1\) norm and the CNN module empowers the proposed scheme to reveal and utilize both sparsity and structural priors adapted to the properties of the signals, thus offering significant performance gains, especially when the signals are corrupted by severe noise.

- Building upon the proposed sparse coding optimization framework, two novel deep learning-based methods are derived for solving the sparse coding problem. The first method is based on the deep unrolling paradigm, which unrolls a fixed number of iterations of the proposed sparse coding optimization scheme, and thus creating a highly interpretable deep learning architecture. The second sparse coding method utilizes the deep equilibrium models, thus expressing effectively the above deep unrolling architecture as an equilibrium computation, which corresponds to a deep network with infinite number of layers. Unlike the deep unrolling architecture, in which the number of layers must be kept relatively small due to stability issues, the deep equilibrium model is able to provide more accurate results, overcoming efficiently many of the limitations introduced by the deep unrolling model. The proposed DEQ framework constitutes a unique bridge between the sparse representation theory and the deep equilibrium models. In both proposed deep learning frameworks, the network parameters (i.e., CNN prior and the regularization coefficients derived from the sparse coding optimization scheme) can be optimized via end-to-end supervised learning.

- Based on the findings on our previous work \([19]\), the proposed sparse coding deep networks can be significantly enhanced in terms of computational complexity, without sacrificing accuracy. In more detail, under the assumption that neighboring signals can be represented using the same support set from a given dictionary, two additional approximate sparse coding networks are derived that utilise the deep unrolling and deep equilibrium models, respectively.

The remainder of this paper is organized as follows. In
Section II provides a detailed literature review of related works in the context of hyperspectral image denoising, that demonstrate the efficacy of the new algorithms. Finally, Section VII concludes the paper.

II. RELATED WORKS

A. Sparse coding algorithms

When signals that exhibit inter-dependencies are to be modelled (i.e., signals that demonstrate dependencies both across and along their data matrix), existing sparse coding algorithms such as the OMP, LASSO or their accelerated learnable versions (i.e., the learnable OMP [35] and the learnable LISTA [36]) are incapable of capturing the dependencies along the data matrix, since they treat each signal vector independently. In literature, only few studies have explored the design of sparse coding algorithms by incorporating priors regarding also the dependencies of the signals along the data matrix. Among them, the most prominent algorithm, called SunSAL-TV [7] combined the $l_1$ norm with a total variation regularizer [37], achieving state of the art results in the hyperspectral unmixing problem. However, this algorithm is excessively expensive in computational terms inducing its limited applicability in real-time and/or high-dimensional applications with large data-sets.

Additionally, in our previous work [19], a block-wise sparse coding strategy was employed under the assumption that a block of nearby signals (e.g., a small spatial patch in a hyperspectral image) can be described by the same atoms from a given dictionary (same support set). Having identified the proper support set for the signals, a total variation regularized optimization problem was proposed to compute the optimal representation coefficients. However, the total variation regularizer has a tendency to over-smooth the reconstructed signals.

In contrast to the above approaches, in this study we argue that more accurate and computationally efficient sparse coding methods can be derived by employing regularization terms that are adapted to the properties of the signals of interest. Exploiting the merits of the learnable regularization techniques, a convolutional neural network is employed to act as structure prior and reveal the underlying signal inter-dependencies, thus forming a novel sparse coding problem. This optimization problem can be properly handled in the context of deep unrolling and deep equilibrium approaches.

Our study aligns with the above works only with this unfolding strategy, since here we focus on designing sparse representation frameworks able to capture the sparsity and the inherent dependencies of signals that exhibit a specific structure (e.g., hyperspectral images that consists of locally smooth patches). Note that, as we mentioned in Section II-A, the learnable versions of the OMP [35] and lasso [36], although they follow the deep unrolling paradigm, however they focus only on accelerating the sparse coding procedure, without offering any additional performance accuracy. In our study, we explore deep architectures that both ameliorate the accuracy and the computational complexity of the sparse coding process.

C. Deep Equilibrium Models

Deep equilibrium (DEQ) models have recently appeared in literature, proposing an appealing framework for employing an infinite-depth networks by expressing the entire deep architecture as an equilibrium computation [32]. In this study, we explore the potentials of the above idea to alleviate some of the drawbacks associated with the deep unrolling models. To better explain the deep equilibrium approaches, let us proceed with a short introduction. Consider a generic K-layer deep feedforward model expressed by the following recursion

$$g^{(k+1)} = f^g_{\theta}(g^{(k)}; y) \text{ for } k = 0, 1 \ldots K - 1,$$

where $k$ is the layer index, $g^{(k)}$ denotes the output of the $k$-th layer, $y$ is the input and $f^g_{\theta}$ stands for some nonlinear transformation (defined iteration map as well). Interestingly, recent studies that impose the same transformation $f^g_{\theta}$ in each layer, namely $f^g_{\theta} = f^g_{\theta}$ were able to yield competitive results against other state-of-the-art methods [48]–[50]. Under this weight tying practice [51] (i.e., the same transformation in each layer), authors in [52] proposed a deep equilibrium model aiming to efficiently find the fixed point $g^*$ where the further application of the nonlinear transformation or iteration map $f^g_{\theta}$ does not alter its value. Particularly, the fixed point derives from the solution of the following system

$$g^* = f^g_{\theta}(g^*; y).$$

Note that the above solution can be interpreted as an infinite depth network. However, instead of computing this point by repeating the transformation $f^g_{\theta}$, the DEQ method employs more efficient ways to obtain the equilibrium or fixed point, such as root finding techniques. Importantly, the weights ($\theta$) of the network can be obtained via implicit differentiation using only constant memory.

In literature only two preprints [33, 52] have explored the applicability of this methodology to solve generic inverse problems concerning the image reconstruction, such as MRI reconstruction or image deblurring. Different from these approaches, we extend the potentials of the deep equilibrium framework by providing a novel bridge between the sparse coding problem and this new deep learning method, that is the DEQ models. In more detail, the proposed sparse coding problem can be considered as a fixed-point network that is able to surpass all the limitations introduced by the corresponding...
deep unrolling sparse coding model. To the best of the authors’ knowledge, this is the first time that some study considers connections between the sparse representation theory and deep equilibrium models.

III. PROBLEM FORMULATION

Consider a data matrix \( Y' \) that comprises of a number of \( p \) blocks, \( Y_i, i = 1, \ldots, n \) as

\[
Y' = \begin{bmatrix}
Y_1 & \cdots & Y_i & \cdots & Y_p
\end{bmatrix},
\]

where each \( Y_i \in \mathbb{R}^{d \times N} \) is termed as a block of signals. Consider also that each block of signals is given as

\[
Y_i = X_i + W_i,
\]

where \( W_i \in \mathbb{R}^{d \times N} \) denotes a zero-mean noise term and \( X_i \) denotes a clean block of signals. We focus in this work in the case where each block \( X_i \) exhibits dependencies both across and along its dimensions. To be more specific, we consider that each individual column of each block \( X_i \) has some structure (e.g., it admits some sparse representation), and we term this property as dependency across the data matrix/block. Furthermore, we also consider that all the columns in each \( X_i \) have some dependencies, in the sense that the knowledge of a subset of columns gives us some information regarding the other columns, and we term this property as dependency along the data matrix/block. As an example, consider that block \( X_i \) may correspond to a patch of neighboring hyper-pixels of some hyperspectral image. Figure (1) exemplifies the strong similarity relations of hyper-pixels in this hyperspectral image patch.

In such a setting, given an overcomplete dictionary \( D \in \mathbb{R}^{d \times M} \) and focusing on only one block of data, our scope is to compute a sparse representation matrix \( G_i \in \mathbb{R}^{M \times N} \), so that

\[
X_i \approx DG_i,
\]

taking into account the fact that \( X_i \) exhibits dependencies both along and across its dimensions. In solving this problem, we also utilize knowledge about the assumed structure of \( X_i \). In this work, we also consider the problem of learning such structure given suitable training data.

In the following, since we only focus on the sparse coding of each block of data, we use the symbols \( Y \) and \( X \) to refer to one block of data, in order to simplify our notation. At some points, where necessary, we resort to the notation \( Y_i \) and \( X_i \), for the blocks of the data, but their use is clear from the context.

IV. DEEP ARCHITECTURES FOR SPARSE CODING

Considering the underlying structure of the noisy data block \( Y \) (i.e., dependencies both along and across its dimensions), the studied sparse coding problem can be formulated into the following regularized optimization form

\[
\arg \min_G \frac{1}{2} \| Y - DG \|_F^2 + \mu \| G \|_{1,1} + \lambda R(DG),
\]

where the solution minimizes an energy function consisting of a data consistency term \( \| Y - DG \|_F^2 \), a sparsity promoting \( l_1 \)-norm, and some regularization term aiming to promote the inherent structure of the reconstructed clean estimate \( DG \).

In more detail, the term \( R(DG) \) should obtain small values when the estimate of the clean block \( DG \) agrees with our knowledge about the structure of \( X \), and big values otherwise. Furthermore, \( \mu \) and \( \lambda \) are positive scalar constants controlling the relative importance of the sparsity level and the \( R(\cdot) \) prior, respectively.

To efficiently solve (6), an alternating optimization methodology (AO) is employed in order to decouple the data fidelity term, the sparsity term and the regularization term into three individuals sub-problems. To this end, the above optimization problem can be solved efficiently by employing the Half Quadratic Splitting (HQS) methodology [30]. By introducing two auxiliary variables, namely \( V \) and \( Z \), the problem in (6) can be reformulated as follows,

\[
\arg \min_G \frac{1}{2} \| Y - DG \|_F^2 + \mu \| V \|_{1,1} + \lambda R(Z) \]

s.t. \( V = G \), \( Z = DG \).

The corresponding augmented Lagrangian function is given by

\[
\mathcal{L}(G, V, Z) = \frac{1}{2} \| Y - DG \|_F^2 + \mu \| V \|_{1,1} + \lambda R(Z) + \frac{b}{2} \| V - G \|_F^2 + \frac{b}{2} \| Z - DG \|_F^2,
\]

where \( b > 0 \) denotes a user-defined penalty parameter. Thus, a sequence of individual subproblems emerges, given by

\[
G^{(k+1)} = \arg \min \mathcal{L}(G, V^{(k)}, Z^{(k)}),
\]

\[
V^{(k+1)} = \arg \min \mathcal{L}(G^{(k+1)}, V, Z^{(k)}),
\]

\[
Z^{(k+1)} = \arg \min \mathcal{L}(G^{(k+1)}, V^{(k+1)}, Z),
\]
The solutions of the subproblems for $G$, $V$ and $Z$ derive from

$$
G^{(k+1)} = (D^T D + bD^T D + I)^{-1} (D^T Y + bV^{(k)} + bD^T Z^{(k)})
$$

$$
V^{(k+1)} = \text{soft}(G^{(k+1)}, \mu/b)
$$

$$
Z^{(k+1)} = \text{prox}_{\frac{1}{2}R} (DG^{(k+1)}),
$$

where the \text{soft}(\cdot, \tau) stands for the soft-thresholding operator $x = \text{sign}(x) \max(|x| - \tau, 0)$.

Additionally, leveraging upon the studies in [24], [27], the proximal operator involved in (10c) can be interpreted as a Gaussian denoiser in a MAP sense based on the Bayesian probability. Hence, we consider to replace $\text{prox}_{\frac{1}{2}R}(\cdot)$ with a neural network $N_\theta(\cdot) : \mathbb{R}^{d\times N} \rightarrow \mathbb{R}^{d\times N}$, whose weights, denoted as $\theta$, can be learned from training data. In the case of interest in this work, a convolutional neural network is employed to capture the underlying structure priors and the dependencies of the inter-dependent signals (e.g., hyperspectral images). A great merit of the above modeling procedure is the fact that, the explicit prior (regularizer) $R(\cdot)$ can be unknown in relation [5] designed with properties adapted to the under-examined signals and trained from.

Based on above findings the proposed iterative updated rules of the half quadratic splitting are summarized as follows

$$
G^{(k+1)} = (D^T D + bD^T D + I)^{-1} (D^T Y + bV^{(k)} + bD^T Z^{(k)})
$$

$$
V^{(k+1)} = \text{soft}(G^{(k+1)}, \mu/b)
$$

$$
Z^{(k+1)} = N_\theta(DG^{(k+1)}),
$$

Note that the neural network $N_\theta$ is pre-trained according to the following cost function

$$
I(\theta) = \sum_{i=1}^{p} \left\|N_\theta(Y_i; \theta) - X_i\right\|_F^2,
$$

where $\theta$ denotes the weights of the CNN prior and $\{X_i, Y_i\}$ represent $p$ pairs of the training blocks of inter-dependent (dependencies along the block) signals and their corresponding noisy versions, as for example pairs of ground-truth and their corresponding noisy hyperspectral image patches. After the training, a simple alternative is to plug the learned CNN into the proposed iterative strategy defined in (11) and then execute it until convergence is reached. Although, this plug-and-play approach provides empirically sufficient results, the procedure can be considered piecemeal and sub-optimal, since the CNN module is learnt independently from the forward model (i.e., the dictionary $D$) associated with the proposed sparse coding problem.

A. Deep Unrolling full sparse coding method

Considering the limitations of the plug-and-play methodology, in this section we propose to efficiently tackle these issues by utilizing a form of training called end-to-end. In more details, instead of learning the CNN prior $N_\theta$ offline, we unroll a small number of iterations, say $K$, of HQS scheme (11) and we treat them as a deep learning network, where each iteration is considered a unique layer of the proposed model. Thus, a $K$-layer deep learning architecture is formed, where its depth and parameters are highly interpretable due to the fact that the modeling of the network is based on the physical process underlying the examined problem. Figure 2 illustrates the architecture of a $K$-layer deep unrolling network.

By denoting that $G^{(K)}$ is the output of the proposed approach, an end-to-end training is employed seeking to minimize some loss function with respect to CNN weights $\theta$. As loss function is employed the following

$$
l(\theta) = \sum_{i=1}^{n} \left\|DG_i^{(K)} - X_i\right\|_F^2,
$$

where $X_i$ corresponds to the $i^{th}$ target block of inter-dependent signals (e.g., hyperspectral image patch). Note that the CNN prior $N_\theta$ is learnt based on the quality of the estimate $X = DG^{(K)}$, which strongly depends on the dictionary. Above, it is assumed that all CNN priors $N_\theta(\cdot)$ of each layer (iteration) have identical weights $\theta$ (weight tying practice), thus reducing the learnable parameters and simplifying the architecture. Another noteworthy merit of the deep unrolling approach is the fact that the penalty parameters $\lambda, \mu$ and $b$ introduced in relation (8) can be treated also as network parameters to be learned via the end-to-end training scheme. It should be highlighted that an accurate estimate of $X$ implies that the corresponding sparse coding matrix $G^{(K)}$ is enforced to capture the inherent structure (inter-dependencies) of the data.

We should highlight that the number of layers in our deep unrolling approach must be kept relatively small ($K$ ranges from 5 to 15). This is attributed to the fact that training the deep unrolling network with many layers requires high GPU memory resources, since the calculation of the back-propagation scales with the number of layers (unrolled iterations) [33]. In view of this, in the following section a novel deep equilibrium model is proposed corresponding to an efficient network with infinite number of layers, without demanding high computational resources as the deep unrolling method.

B. Deep equilibrium full sparse coding method

Assuming that all instances of the CNN prior $N_\theta$ have identical weights $\theta$ at each layer of the proposed deep unrolling method, it can be easily observed that the same layer is repeated $K$ times. In other words, the same transformation or iteration map, say $f_\theta$ is applied to the noisy input $Y$ in order to obtain an accurate denoised estimate $\hat{X}$ of the input and thus an accurate sparse coding matrix $G$ capturing the underlying structure of the noisy data.

In light of this, utilizing the deep equilibrium modeling an implicit infinite-depth network (infinite number of iterations) can be efficiently developed. Thus, the goal is to develop a suitable iteration map $f_\theta(\cdot, y)$ based on the equations in (11).

In particular, by substituting the update of $Z^{(k+1)}$ and $V^{(k+1)}$
directly into the update rule of $G^{(k+1)}$, a single and more concrete update expression is derived

$$
G^{(k+1)} = \left(D^T D + bD^T D + I\right)^{-1}\left(D^TY + bD^TV + bD^T Z\right).
$$

Note that the update of $G^{(k+1)}$ depends only on the previous iterate $G^{(k)}$, thus formulating a fixed point iteration on the sparse coding matrix $G$. Hence, the iteration map $f_\theta(\cdot, Y)$ is given by relation (14) satisfying

$$
G^{(k+1)} = f_\theta(G^{(k)}, Y)
$$

Based on (15), the estimate of the target block of inter-dependent signals $X$, denoted as $X^*$, derives from

$$
X^* = DG^*.
$$

where $G^*$ is the fixed point of the iteration map $f_\theta(\cdot, Y)$. Figure 3 provides an illustration of the designed iteration map $f_\theta$.

Having effectively designed the iteration map $f_\theta(\cdot, Y)$, the following challenging procedures emerge. First, given a noisy block of signals $Y$ and the weights $\theta$ of the CNN prior, a fixed point is required to be computed during the forward pass. Second, given pairs of ground-truth blocks of signals exhibiting dependencies along and across the block and their corresponding noisy versions $\{X_i, Y_i\}_{i=1}^p$, the parameters of the network need to be efficiently obtained during the training process. Without loss of generality, to simplify the notations and calculations below a single pair of training examples, namely $X, Y$ is considered. Additionally, we use the vectorized versions of the matrices $Y, X, G$ denoted as $y, x, g$, respectively.

1) Forward pass - Calculating fixed points: The "forward" pass during training and inference (testing) in the proposed DEQ model requires computing the fixed point

$$
g^* = f_\theta(g^*, y)
$$

for the iteration map $f_\theta(\cdot, y)$ defined in relation (15). A simple strategy to estimate this point is to employ fixed point iterations, i.e., iterating the following recursive scheme

$$
g^{(k+1)} = f_\theta(g^{(k)}, y)
$$

However, this approach may be a time-demanding procedure. To this end, the Anderson acceleration strategy (54) is deployed in order to efficiently accelerate the process of the fixed-point iterations. In particular, instead of using only the previous iterate (i.e., $g^{(k)}$) to compute the next proper point to move, the Anderson acceleration procedure utilizes the $m$ previous iterates formulating the following update rule

$$
g^{(k+1)} = (1-\beta)\sum_{i=0}^{m-1}\alpha_i g^{(k-i)} + \beta\sum_{i=0}^{m-1}\alpha_if_\theta(g^{(k-i)}, y),
$$

for $\beta > 0$. The vector $\alpha \in \mathbb{R}^m$ derives from solving the following optimization problem

$$
\arg\min_{\alpha} \|U\alpha\|_2^2, \text{ s.t. } 1^T\alpha = 1
$$

where $U = [f_\theta(g^k, y) - g^{(k)}, \ldots, f_\theta(g^{k-m+1}, y) - g^{(k-m+1)}]$ is a matrix containing the $m$ past residuals. Note that, when $m$ is kept small (e.g., $m=5$) reduces significantly the computational complexity of optimization problem (20).

2) Backward pass - Calculating the Gradient: In this section, a brief review is provided about the implicit back-propagation employed during the training process to obtain the optimal weights $\theta$ of the proposed DEQ model. Based on the findings in (32), (55), the goal is to efficiently train the
network without requiring to backpropagate though a large number of fixed point iterations.

Let \( g^* = f_\theta(g^*, y) \) be the fixed point derived from the forward pass and \( y \) the noisy block of signals such as a hyperspectral image patch corresponding to the original (noisy-free) patch, denoted as \( x \). Let \( l(\cdot) \) be a loss function such as the mean-squared error (MSE) loss that computes

\[
l(\theta) = \frac{1}{2} \| Dg^* - x \|^2_2 \tag{21}
\]

The loss gradient with respect to the network parameters \( \theta \) is given by

\[
\frac{\partial l}{\partial \theta} = \frac{\partial g^*}{\partial \theta} \frac{\partial l}{\partial g^*} = \frac{\partial g^*}{\partial \theta} D^T(Dg^* - x), \tag{22}
\]

where the first term is the Jacobian of \( g^* \) w.r.t. \( \theta \) and the second term is the gradient of loss function.

The goal is to calculate the Jacobian of \( g^* \) w.r.t. \( \theta \). By implicitly differentiating both sides of the fixed point equation \( g^* = f_\theta(g^*, y) \), and solving for \( \frac{\partial g^*}{\partial \theta} \), an explicit expression for the Jacobian is derived

\[
\frac{\partial g^*}{\partial \theta} = \left( I - \frac{\partial f_\theta(g^*, y)}{\partial g^*} \right)^{-1} \frac{\partial f_\theta(g^*, y)}{\partial \theta} \tag{23}
\]

Using equation (23), relation (22) is reformulated as follows

\[
\frac{\partial l}{\partial \theta} = f_\theta(g^*, y)^T \left( I - \frac{\partial f_\theta(g^*, y)}{\partial g^*} \right)^{-T} D^T(Dg^* - x). \tag{24}
\]

In view of this, instead of backpropagating though a large number of fixed point iterations, we need to calculate only a memory- and computation-efficient Jacobian-vector product. Based on (23), we define the following vector in order to compute this Jacobian-vector product (via conventional automatic differentiation tools)

\[
\gamma = \left( I - \frac{\partial f_\theta(z^*, y)}{\partial g^*} \right)^{-T} D^T(Dg^* - x) \Rightarrow
\]

\[
\gamma = \left( \frac{\partial f_\theta(g^*, y)}{\partial g^*} \right)^T \gamma + D^T(Dg^* - x). \tag{25}
\]

Expression (25) is also a fixed point equation. Thus, solving this fixed point equation and computing the fixed point \( \gamma^* \), the gradient in (24) can be written as

\[
\frac{\partial l}{\partial \theta} = f_\theta(g^*, y)^T \gamma^*. \tag{26}
\]

V. DEEP FAST ARCHITECTURES FOR SPARSE CODING

The scheme proposed in the previous section successfully captures the inter-dependencies that exist in each block of data, and performs regularized sparse coding at the cost of increased computational complexity. We note that the main reason for this complexity is due to the requirement for a sparse matrix \( G \), which is promoted via the use of the \( \ell_1 \) norm term that is present in the cost function. Furthermore, the inter-dependency property employed, suggests that the signal vectors in a block may have some sort of similarity, as the signals in Figure 1. Taking into account these thoughts, in this section we explore a more specific notion of inter-dependency, that suggests that the signal vectors in a block can be represented using the same set of atoms from the dictionary. By adopting such a model, it is possible to drastically reduce the computational complexity of the scheme, since only one set of atoms, i.e., support set, must be determined. In the following we adopt a two step approach, where in the first step a proper support set \( S \) is determined for the whole block of signals, and in the second step, an optimization problem is solved for the computation of the coefficients of the sparse coding matrix, given the support set determined.

At the first step of the proposed approach, the set of atoms that will be used for the representation of the whole block of signals must be determined. For this task, we propose to first compute the average/centroid signal \( y_c = \frac{1}{N} \sum_{i=1}^N y_i \), and then employ some sparse coding algorithm (e.g., the OMP, the LASSO) to sparsely encode the vector \( y_c \) using the given dictionary. The required support \( S \) is finally given as the set of atoms used in the representation of the vector \( y_c \).

Having identified a proper support set \( S \), it is possible to formulate an optimization problem that is significantly less computationally intensive as compared to the problem in (9), for the computation of the entries of the sparse coding matrix. In particular, we consider the problem

\[
\arg \min_{G_S} \| Y - D_S G_S \|_F^2 + \lambda R_\theta(D_S G_S), \tag{27}
\]

where \( D_S \) is the matrix that results from the dictionary \( D \) after keeping only the columns/atoms indexed by the set \( S \), and \( G_S \) is the corresponding matrix of representation weights. It should be noted that matrix \( G_S \) is not sparse, since it contains only the nonzero elements of the sparse coding matrix \( G \).

To sum up, given a block of signals \( Y \), the following procedure is employed:

1) Compute the centroid signal \( y_c \) of the block \( Y \).
2) Determine the support set \( S \) of the centroid signal \( y_c \), using some sparse coding algorithm,
3) Based on the support of the centroid signal, compute the corresponding representation coefficients for all the signals in \( Y \) by solving problem (27).

The details of efficiently solving problem (27) are provided in the following sub-section.

A. Optimization via HQS

To efficiently solve the proposed optimization problem in (27), the HQS methodology is employed again, hence reformulating the considered problem into the following constrained optimization form,

\[
\arg \min_{G_S} \frac{1}{2} \| Y - D_S G_S \|_F^2 + \lambda R(Z) \tag{28}
\]

\[
s.t. \quad Z - D_S G_S = 0 .
\]

The corresponding augmented Lagrangian function is given by

\[
\mathcal{L}(G_S, Z) = \frac{1}{2} \| Y - D_S G_S \|_F^2 + \lambda R(Z) \tag{29}
\]

\[
+ \frac{b}{2} \| Z - D_S G_S \|_F^2 ,
\]

where \( b > 0 \) denotes a penalty parameter.
Similarly with the proposed methods in section IV, we derive the following update rules

\[
G^{(k+1)}_S = (D_S^T D_S + b D_S^T D_S)^{-1} (D_S^T Y + b D_S^T Z^{(k)})
\]

\[
Z^{(k+1)} = \text{prox}_{\frac{1}{b} \mathcal{R}} (D_S G^{(k+1)}_S) = N_\theta (D_S G^{(k+1)}_S)
\]  

(30)

Thus, equation (32) can be considered as a fixed point iteration of the variable \( G_S \), where the iteration map satisfies the following recursive scheme

\[
G^{(k+1)}_S = f_\theta (G^{(k)}_S, Y)
\]  

(33)

The corresponding estimate of the target block of interdependent signals \( X \), denoted as \( Z^* \), is given by

\[
Z^* = D_S G^*_S,
\]  

(34)

where \( G^* \) is the fixed point of the iteration map \( f_\theta (\cdot, Y) \). Figure 5 provides an illustration of the proposed DEQ approach. Details concerning the calculation of fixed points and the training procedure are given in the previous Sections IV-B1 and IV-B2, accordingly.

VI. EXPERIMENTAL PART

In this section, extensive numerical experiments are presented, in the context of hyperspectral imaging, to validate the efficacy and applicability of the proposed sparse coding schemes. In more detail, the problem of hyperspectral image (HSI) denoising is considered, where the proposed deep unrolling and deep equilibrium models are compared against various sparse coding algorithms and state-of-the-art HSI denoising approaches. Our principal goals are to highlight that the proposed sparse coding approaches:

- Perform remarkably better in denoising applications, as compared to classical sparse coding based algorithms
- Exhibit competitive performance against state-of-the-art deep-learning based approaches

A. Dataset

To underscore the merits of the proposed models, we used a publicly available hyperspectral image dataset, pertaining to a variety of natural scenes. In more detail, hyperspectral images from the iCVL dataset [53] were employed to train and validate our models. From this dataset, 70 hyperspectral images were employed to generate the training set, whereas another 50 images were used to generate the testing set. The examined hyperspectral images constitute 1300 x 1300 x 31 dimensional cubes, where the last dimension \( d = 31 \) corresponds to the spectral dimension (i.e., 31 spectral bands in the 400 – 700 nm spectrum).
\[ Y = [y_1, y_2, \ldots, y_N] \]

\[ y_c = \frac{1}{N} \sum_{i=1}^{N} y_i \]

\[ \text{sparse coding} \]

\[ f_\theta (G_S, Y) \]

\[ D_S \]

\[ G_S \]

\[ N \theta (D_S G_S) \]

\[ N \theta (D_S G_S) \]

\[ (D_S^T D_S + b D_S^T D_S)^{-1} (D_S^T Y + b D_S^T Z) \]

\[ f_\theta (G_S, Y) \]

\[ G_S \]

\[ N \theta (D_S G_S) \]

\[ Z \]

\[ (D_S^T D_S + b D_S^T D_S)^{-1} (D_S^T Y + b D_S^T Z) \]

\[ f_\theta (G_S, Y) \]

Fig. 5. An illustration of the proposed DEQ-fast-sc approach (Section V-C) for solving the sparse coding problem, for one block \( Y \). The proposed architecture consists of two main stages. First, based on the centroid signal of the block \( Y \) the most suitable atoms from the dictionary, denoted as \( D_S \), are selected to represent the signals in \( Y \). After that, the proposed Deep Equilibrium method is deployed in order to accurately compute the corresponding representation coefficients \( G_S \) for each signal in the block \( Y \). The output matrix \( G_S \) is the fixed point of the iteration map \( f_\theta (G_S, Y) \) defined in (32).

B. Implementation Details

1) Block processing: The training as well as the testing datasets used in our experiments contain patches collected by the hyperspectral images. Each such patch has a size of \( n \times n \times d \). The parameter \( n \), that defines the spatial size of each patch, plays an important role in the accuracy as well as in the computational complexity of the proposed approaches. In order to obtain a proper value for the parameter \( n \), several experiments with different values for \( n \) and noise levels were conducted. Figure 6 presents the resulting hyperspectral image denoising performance, in terms of the Peak Signal to Noise Ratio (PSNR), the Structural Similarity Index (SSIM) [56] metrics and the Spectral Angle Mapper (SAM) [57].

2) Training and Testing setting: During the training phase, the hyperspectral images are corrupted by i.i.d. Gaussian noise corresponding to three different noise levels of \( \sigma = 30, 50, 70 \). After that we split both ground-truth and noisy images into non-overlapping patches as described above and then we randomly sample pairs of hyperspectral patches (blocks of signals, which exhibit dependencies along and across the block) and their corresponding noisy versions, denoted as \( \{X_i, Y_i\}_{i=1}^{N} \).

During the testing phase, the following procedure was employed. Initially, the image was corrupted with a specified level of Gaussian noise, and then it was processed into \( n \times n \times d \) non-overlapping patches. For each noisy patch (block), say \( Y \), the corresponding sparse representation matrix was computed using the proposed schemes. To validate the accuracy of the sparse coding models we compared the reconstructed hyperspectral image with the corresponding original image in terms of the Peak Signal to Noise Ratio (PSNR), the Structural Similarity Index (SSIM) [56] metrics and the Spectral Angle Mapper (SAM) [57].

3) CNN architecture: In our proposed models, a relatively small CNN module \( N_\theta(\cdot) \) was employed with 4 layers. In more detail, each convolutional layer comprised of 64 filters with size \( 3 \times 3 \) followed by a non-linear activation function ReLU. Following the remarks of [27] the spectral normalization [58] was imposed to all layers, thus guaranteeing that each layer has Lipschitz constant (no more than) 1. The Regularizing Lipschitz continuity was employed not only during the pre-training of the CNN network but also during the training phase of proposed sparse coding models, hence providing training stabilization and improved performance.

4) Dictionary: The analysis that has been conducted assumes that the dictionary \( D \in \mathbb{R}^{d \times M} (d = 31) \) is known. Thus, in all experiments a fixed dictionary was employed that had been learnt using the KSVD algorithm [1] over a collection of clean hyperspectral images different from the images considered during the testing phase. The dictionary
considered in our experiments contained $M = 512$ atoms.

5) Hyperparameters setting: Based on the training dataset, the CNN module $\mathcal{N}_\theta(\cdot)$ was pre-trained utilizing the ADAM optimizer. In particular, the number of epochs was set to 150, the learning rate was set to $1e-03$ and the batch size was 16. After that the pre-trained $\mathcal{N}_\theta$ module was used for the end-to-end training procedure of the proposed models.

Regarding the deep unrolling methods described in Section (IV-A) and (IV-B) (i.e., DU-full-sc and DU-fast-sc) the number of unrolling iterations (or the number of the layers) was set to $K = 10$. To proceed further, concerning the deep equilibrium methods defined in Section (IV-B) and (V-C) (i.e., DEQ-full-sc and DEQ-fast-sc) the Anderson acceleration procedure [54] was deployed during the training phase for the forward and backward pass fixed-point iterations. Specifically, the number of fixed-point iteration during the forward and backward pass was set to 20. Finally, during the end-to-end training phase of the deep unrolling and deep equilibrium models, the ADAM optimizer was deployed to update the network parameters. In this case, the number of epochs was set to 100, the learning rate was set to $1e-04$ and the batch size to 16.

C. Denoising performance against various sparse coding algorithms

Table I summarizes the average quantitative results of our proposed methods, namely the DU-full-sc (Section IV-A), the DEQ-full-sc (Section IV-B), the DU-fast-sc (Section IV-B) and the DEQ-fast-sc (Section V-C) in comparison with various well-established sparse coding algorithms. It is evident that the proposed approaches is markedly better than the other methodologies. In particular, even for high level of noise, the proposed sparse coding algorithms are able to maintain high PSNR and SSIM values, while the SAM values remain remarkably low. Since the hyperspectral images are a great example of signals containing blocks with strong inter-dependencies, the incorporation of a learnable regularizer (CNN module) and the transformation of the sparse coding optimization schemes into a meaningful and highly interpretable deep learning architectures enable the proposed methods to model effectively and accurately the underlying structure of the noisy signals, thus offering great denoising properties.

Focusing on the proposed methods, the Deep Equilibrium-based models, namely the DEQ-full-sc and DEQ-fast-sc consistently outperform the Deep Unrolling approaches (i.e., the DU-full-SC and the DU-fast-sc) in all cases. Furthermore, Figure 7 illustrates another great merit of the proposed Deep Equilibrium-based approaches. In more detail, the Deep Unrolling approaches, both the full and fast versions are optimized for a fixed number of iterations/layers during training, and hence increasing the number of iteration during inference deteriorates the reconstruction accuracy. On the other, the proposed deep equilibrium approaches are able to maintain or increase their performance for a wide range of iteration, providing a balance between the desired computational complexity and accuracy.

D. Denoising performance against state-of-the-art

To fully demonstrate the merits of the proposed models, in this Section we compare our methods with some traditional-baselines approaches including dictionary learning (TDL [59]) and tensor-based (LLRT [60]) approaches and some recent state-of-the-art deep learning-based denoising competitors: QRU3D [61], MemNet [62] and HSID-CNN [63]. Table II summarizes the main reconstruction accuracy comparison. As can be clearly seen from the above table, the proposed Deep Unrolling and Deep Equilibrium-based models exhibit better performance results as compared to the other approaches.

Figure 8 shades light on the above results by presenting the number of learnable parameters of the models, which are involved in this experiment. It is notable that our proposed methods not only achieve better performance but also require considerably less parameters compared to the two
best performing deep learning-based approaches, i.e., QRU3D and MemNet. The above interesting remark can be justified taking into account that the proposed models have well-justified architecture derived from modeling of the physical processes of the sparse coding problem and utilizing prior domain knowledge of the examined problem, in the form of correlation structure and sparsity priors. Thus, the proposed methods enjoy both the modeling capacity of the deep-learning methods, and the concise structure of the sparse representation theory.

**E. Discussion and Future Work**

Apart from the superior performance of the proposed models, their major difference/advantage over the other deep learning approaches e.g., QRU3D [61], MemNet [62] and HSID-CNN [63] is more fundamental. In more detail, the above mentioned techniques are designed to tackle a specific task that is the HSI denoising problem. On the other hand, the primary goal of this work is to present a novel bridge between the sparse representation theory and the deep-learning models, providing sparse coding architectures that can be deployed to various hyperspectral imaging problems such as unmixing, spatial and spectral super-resolution and hyperspectral deconvolution, as the proposed methods preserve the generic modeling capacity of the classical sparse representation algorithms.

Furthermore, it should be highlighted that the proposed approaches can be extended to consider learnable regularizers other than the simple CNN model in relation (10c). In particular, any state-of-the-art HSI denoiser, such as QRU3D [61], can be employed. The resulting scheme is expected to exhibit good performance in a wide range of applications, other than denoising, such as in HSI image unmixing or in super-resolution approaches.

Another appealing direction, is to incorporate the dictionary matrix into the proposed sparse coding approaches and treat it as a learnable parameter, thus providing novel deep unrolling/equilibrium dictionary learning algorithms. However, this procedure entails some challenges and is left for future work.

**VII. Conclusion**

The problem of computing a sparse representation for datasets of inter-dependent signals was considered. A regularized optimization approach was proposed, where the considered inter-dependencies are captured using regularization terms which was properly learnt from the data. Deep equilibrium and deep unrolling based algorithms were developed for the considered problem. Extensive simulation results, in the context of hyperspectral image denoising, were provided, that demonstrated some very promising results in comparison to various sparse coding algorithms and several recent state-of-the-art denoising models.

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TABLE II
QUANTITATIVE RESULTS OF DIFFERENT METHODS UNDER SEVERAL NOISE LEVELS ON ICVL DATASET.

| Sigma | Metrics | Noisy | TDL [59] | LLRT [60] | HSID-CNN [63] | MemNet [62] | QRUD [61] | DU-full-sc Section [IV-A] | DEQ-full-sc Section [IV-B] | DU-fast-sc Section [V-B] | DEQ-fast-sc Section [V-C] |
|-------|---------|-------|---------|---------|-------------|-------------|---------|------------------------|------------------------|------------------------|------------------------|
| 30    | PSNR    | 18.54 | 40.58   | 41.99   | 38.70       | 41.45       | 42.28   | 42.60                  | 42.80                  | 42.75                  | 42.87                  |
|       | SSIM    | 0.112 | 0.957   | 0.967   | 0.949       | 0.972       | 0.973   | 0.972                  | 0.973                  | 0.974                  | 0.975                  |
|       | SAM     | 0.807 | 0.062   | 0.056   | 0.103       | 0.065       | 0.061   | 0.045                  | 0.042                  | 0.042                  | 0.041                  |
| 50    | PSNR    | 14.12 | 38.01   | 38.99   | 36.17       | 39.76       | 40.23   | 40.22                  | 40.75                  | 40.62                  | 40.77                  |
|       | SSIM    | 0.043 | 0.932   | 0.945   | 0.919       | 0.960       | 0.961   | 0.956                  | 0.962                  | 0.961                  | 0.963                  |
|       | SAM     | 0.993 | 0.085   | 0.075   | 0.134       | 0.076       | 0.072   | 0.054                  | 0.047                  | 0.050                  | 0.048                  |
| 70    | PSNR    | 11.21 | 36.36   | 37.36   | 34.31       | 38.57       | 39.14   | 39.27                  | 39.01                  | 39.25                  | 39.50                  |
|       | SSIM    | 0.024 | 0.909   | 0.930   | 0.886       | 0.946       | 0.947   | 0.947                  | 0.949                  | 0.948                  | 0.950                  |
|       | SAM     | 1.102 | 0.105   | 0.087   | 0.161       | 0.088       | 0.087   | 0.056                  | 0.054                  | 0.056                  | 0.055                  |

TABLE III
AVERAGE RUNTIME (IN SECONDS) OF THE PROPOSED DEEP UNROLLING AND DEEP EQUILIBRIUM APPROACHES TO RECONSTRUCT A HYPERSONTICAL IMAGE OF SIZE 1300 × 1300 × 31.

| Method | DU-full-sc time[sec] | DEQ-full-sc time[sec] | DU-fast-sc 2 time[sec] | DEQ-fast-sc time[sec] |
|--------|----------------------|-----------------------|------------------------|-----------------------|
|        | [IV-A]               | [IV-B]                | [V-B]                  | [V-C]                  |
|        | 13.12                | 25.03                 | 4.70                   | 7.45                  |
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