The Third Proof of Lovász’s Cathedral Theorem

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Abstract. This paper is on matching theory. A graph with perfect matching is called saturated if any addition of one complement edge creates a new perfect matching. Lovász gave a characterization of the saturated graphs called the cathedral theorem, and later Szigeti gave another proof. In this paper, we gave a new proof with our preceding works, which revealed the canonical structure of general graph with perfect matchings. Here, the cathedral theorem is derived in quite a natural way and shows that it holds independently from the Gallai-Edmonds structure theorem.

1 Introduction

This paper is on matching theory. For general accounts on matchings we refer to the book by Lovász and Plummer [1].

A graph with perfect matching is called factorizable. A factorizable graph is called saturated if any addition of an arbitrary complement edge creates a new perfect matching. There is a constructive characterization of the saturated graphs known as the cathedral theorem [1–4]. The cathedral theorem was originally given by Lovász [2] (see also [4]), with some applications to the problem on estimating the number of all the perfect matchings [1, 2]—a basic enumeration problem, with application to physical science [2]—, and later given another proof by Szigeti [3, 4]. Lovász’s proof is based on the Gallai-Edmonds structure theorem [1], one of the most powerful theorem in matching theory.

Any graph \(G\) has a partition of its vertices into three parts, some of which might be empty, so-called \(D(G), A(G),\) and \(C(G)\) [1], which we call in this paper the Gallai-Edmonds partition. The property that \(A(G)\) forms a barrier with certain special properties is called the Gallai-Edmonds structure theorem [1].

Szigeti’s proof is based on some results on the optimal ear-decompositions by Frank [5], which is also based on the Gallai-Edmonds structure theorem and is not a “matching-theory-closed” notion, while the cathedral theorem itself is so.

The cathedral theorem is outlined as follows:
- a constructive characterization of the saturated graphs with the operation cathedral construction,
- the preservation of the perfect matchings regarding cathedral construction, and
- the relation between cathedral construction and the Gallai-Edmonds partition.

In our preceding works [6–8], we introduced canonical structure theorems which tells non-trivial structures for general factorizable graphs. Based on these results, we provide yet another proof of the cathedral theorem in this paper. The features of the new proof is the following: First, the cathedral theorem, which is a structure theorem of the special class of factorizable graphs defined by a certain extremal condition i.e. “saturated”, can be derived in quite a natural way as an extremal consequence of the structure theorems for general factorizable graphs. Therefore, our proof reveals what lies essentially underneath the cathedral theorem, and provides a bit more refined or generalized statements from a point of view of the canonical structure of general factorizable graphs.

Second, it shows that the cathedral theorem holds independently from the Gallai-Edmonds structure theorem nor the notion of barriers, since the previous works, as well as the proofs presented in this paper, are also obtained without them. Even the portion of the statements of the cathedral theorem stating its relation to the Gallai-Edmonds partition can be obtained without them.

In Section 2 we introduce notations, definitions, and some preliminary facts on matchings used in this paper. Section 3 is to introduce our previous works [6, 7], the canonical structure theorems for general factorizable graphs. In Section 4, we introduce one of the new theorems, which is a generalized version of the part of the cathedral theorem regarding the Gallai-Edmonds partition. In Section 5, we complete the new proof of the cathedral theorem.

2 Preliminaries

2.1 Notations and Definitions

Here, we list some standard notations and definitions, observing mostly those given by Schrijver [9].

Hereafter for a while let $G$ be a graph and $X \subseteq V(G)$. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. $G - X$ means $G[V(G) \setminus X]$. We define the contraction of $G$ by $X$ as the graph arising from contracting each edge of $E(G[X])$ respectively into one vertex, and denote as $G/X$.

Let $\hat{G}$ be a super graph of $G$, and $e = xy \in E(\hat{G})$. Then, $G + e$ denotes the graph $(V(G) \cup \{x, y\}, E(G) \cup \{e\})$, and $G - e$ the graph $(V(G), E(G) \setminus \{e\})$. For $F = \{e_1, \ldots, e_k\} \subseteq E(\hat{G})$, $G + F := G + e_1 + \cdots + e_k$, and $G - F := G - e_1 - \cdots - e_k$.

We define the set of neighbors of $X$ as the vertices in $V(G) \setminus X$ that are joined to some vertices of $X$, and denote as $N_G(X)$. Given $Y, Z \subseteq V(G)$, $E_G[Y, Z]$ denotes the edges joining between $Y$ and $Z$, and $\delta_G(Y)$ denotes $E_G[Y, V(G) \setminus Y]$. 
A set of edges is called a matching if any two of them are disjoint. A matching of cardinality \(|V(G)|/2\) (resp. \(|V(G)|/2 - 1\)) is called a perfect matching (resp. a near-perfect matching). Hereafter for a while let \(M\) be a matching of a graph \(G\). We say \(M\) exposes a vertex \(v \in V(G)\) if \(\delta_G(v) \cap M = \emptyset\). For a matching \(M\) of \(G\) and \(u \in V(G)\), \(u'\) denote the vertex such that \(uu' \in M\), if it is not exposed.

In this paper, we treat paths and circuits as graphs. For a subgraph \(Q\) of \(G\), which is a path or circuit, we call \(Q\) \(M\)-alternating if \(E(Q) \setminus M\) is a matching of \(Q\), in other words, if edges of \(M\) and \(E \setminus M\) appear alternately in \(Q\). Let \(P\) be an \(M\)-alternating path of \(G\) with end vertices \(u\) and \(v\). If \(P\) has an even number of edges and \(M \cap E(P)\) is a near-perfect matching of \(P\) exposing only \(v\), we call it an \(M\)-balanced path from \(u\) to \(v\). We regard a trivial path, that is, a path composed of one vertex and no edges as an \(M\)-balanced path. If \(P\) has an odd number of edges and \(M \cap E(P)\) (resp. \(E(P) \setminus M\)) is a perfect matching of \(P\), we call it \(M\)-saturated (resp. \(M\)-exposed).

We say a path \(P\) of \(G\) is an ear relative to \(X\) if both end vertices of \(P\) are in \(X\) while internal vertices are not. So do we to a circuit if exactly one vertex of it is in \(X\). For simplicity, we call the vertices of \(P \cap X\) end vertices of \(P\), even if \(P\) is a circuit. For an ear \(P\) of \(G\) relative to \(X\), we call it an \(M\)-ear if \(P \setminus X\) is an \(M\)-saturated path.

A graph is called factor-critical if any deletion of its single vertex leaves a factorizable graph. For convenience, we regard a graph with only one vertex as factor-critical.

We sometimes regard a graph as the set of its vertices. For example, given a subgraph \(H\) of \(G\), \(N_G(H)\) denotes \(N_G(V(H))\). For simplicity, regarding the operations of the contraction or taking the union of graphs, we identify vertices, edges, and subgraphs of the newly created graph with those of old graphs that naturally correspond to them.

We call a graph factorizable if it has at least one perfect matching. Now let \(G\) be a factorizable graph. An edge \(e \in E(G)\) is called allowed if there is a perfect matching of \(G\) containing \(e\). For a factorizable graph \(G\), each connected component of the subgraph of \(G\) determined by the union of all the allowed edges is called an elementary component of \(G\). A factorizable graph which has exactly one elementary component is called elementary. For each elementary component \(H\), we call \(G[V(H)]\) a factor-connected component of \(G\), and denote the set of all the factor-connected components of \(G\) as \(G(G)\). Hence, a factorizable graphs is composed of its factor-connected components and additional edges joining between distinct factor-connected components.

We say \(X \subseteq V(G)\) is separating if any \(H \in G(G)\) satisfies \(V(H) \subseteq X\) or \(V(H) \cap X = \emptyset\).

### 2.2 Basic Properties

In this subsection we introduce some preliminary properties used later in this paper, which are easy to observe and some of which might be folklores. We denote the subgraph of \(G\) determined by \(F \subseteq E(G)\) as \(G.F\). We define the
symmetric difference of two set $A, B$ which have a common super set, as $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

**Proposition 1.** Let $M$ be a near-perfect matching of a graph $G$ that exposes $v \in V(G)$. Then, $G$ is factor-critical if and only if for any $u \in V(G)$ there exists an $M$-balanced path from $u$ to $v$.

**Proof.** Take $u \in V(G)$ arbitrarily. Since $G$ is factor-critical, there is a near-perfect matching $N$ of $G$ exposing only $u$. Then, $G.M \triangle N$ is a $M$-balanced path from $u$ to $v$, and the sufficiency part follows.

Now suppose there is an $M$-balanced path $P$ from $u$ to $v$. Then, $M \triangle E(P)$ is a near-perfect matching of $G$ exposing $u$. Hence, the necessity part follows. $\square$

**Proposition 2.** Let $G$ be a factorizable graph, $M$ be a perfect matching of $G$, and $e = xy \in E(G)$ be such that $e \notin M$. The following three properties are equivalent:

(i) $e$ is allowed in $G$.

(ii) There is an $M$-alternating circuit $C$ such that $e \in E(C)$.

(iii) There is an $M$-saturated path between $x$ and $y$.

**Proof.** We first show that (i) and (ii) are equivalent. Let $N$ be a perfect matching of $G$ such that $e \in N$. Then, $G.M \triangle N$ has a component which is an $M$-alternating circuit containing $e$. Hence, (1) yields (ii).

Now let $N := M \triangle E(C)$. Then, $N$ is a perfect matching of $G$ such that $e \in N$. Hence, (ii) yields (i) consequently, they are equivalent.

Since (ii) and (iii) are obviously equivalent, now we are done. $\square$

**Proposition 3.** Let $G$ be a factorizable graph, $M$ be a perfect matching of $G$, and $u, v \in V(G)$. Then, $G - u - v$ is factorizable if and only if there is an $M$-saturated path of $G$ between $u$ and $v$.

**Proof.** For the sufficiency part, let $N$ be a perfect matching of $G - u - v$. Then, $G.M \triangle N$ has an connected component which is an $M$-saturated path between $u$ and $v$. For the necessity part, let $P$ be an $M$-saturated path between $u$ and $v$. Then, $M \triangle E(P)$ is a perfect matching of $N$, and we are done. $\square$

The next proposition follows rather immediately by the definition, and yields Proposition 5 easily.

**Proposition 4.** Let $G$ be a factorizable graph, and $X \subseteq V(G)$ be a separating set. Then, the followings are equivalent.

(i) There exist $H_1, \ldots, H_k \in \mathcal{G}(G)$ such that $X = V(H_1) \cup \cdots \cup V(H_k)$.

(ii) For any perfect matching $M$ of $G$, $M$ contains a perfect matching of $G[X]$.

(iii) For any perfect matching $M$ of $G$, $\delta_G(X) \cap M = \emptyset$.

**Proposition 5.** Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Let $X \subseteq V(G)$ be a separating set, and $P$ be an $M$-saturated path. Then,

(i) each component of $P[X]$ is an $M$-saturated path, and

(ii) each component of $P - E(G[X])$ is, if it does not contain any end vertices of $P$, an $M$-ear relative to $X$. 
2.3 Preliminaries regarding the Gallai-Edmonds Partition

Given a graph $G$, we define $D(G)$ as the set of vertices that are respectively exposed by some maximum matchings, $A(G)$ as $N(D(G))$ and $C(G)$ as $V(G) \setminus (D(G) \cup A(G))$. We call in this paper this partition of $V(G) = D(G) \cup A(G) \cup C(G)$ into three parts the Gallai-Edmonds partition. It is known as the Gallai-Edmonds structure theorem that $A(G)$ forms a barrier with special properties, which plays key roles in matching theory. In this section, we introduce a yet more elementary and fundamental property the Gallai-Edmonds partition satisfies. This is a well-known fact that connects the Gallai-Edmonds structure theorem and Edmonds’ maximum matching algorithm [10, 11] and one can find it in literatures on them. We here present it with a proof to confirm that it can be obtained in an elementary way and holds independently from those two. Note that Proposition 6 itself is NOT the Gallai-Edmonds structure theorem.

**Proposition 6.** Let $G$ be a graph, $M$ be a maximum matching of $G$, and $S$ be the set of vertices that are exposed by $M$. Then, the followings hold:

(i) $u \in D(G)$ if and only if there exists $v \in S$ such that there is an $M$-balanced path from $u$ to $v$.

(ii) $u \in A(G)$ if and only if there is no $M$-balanced path from $u$ to any vertex of $S$, while there exists $v \in S$ such that there is an $M$-exposed path between $u$ and $v$.

(iii) $u \in C(G)$ if and only if for any $v \in S$ there is no $M$-balanced path from any $u$ to $v$ nor $M$-exposed path between $u$ and $v$.

**Proof.** For the necessity part of (i), let $P$ be the $M$-balanced path from $u$ to $v$. Then, $M \Delta E(P)$ is a maximum matching of $G$ that exposes $u$. Thus, $u \in D(G)$.

For the sufficiency part of (i), if $u \in D(G) \cap S$, the trivial $M$-balanced path $(\{u\}, \emptyset)$ satisfies the property. Otherwise, that is, if $u \in D(G) \setminus S$, by the definition of $D(G)$ there is a maximum matching $N$ of $G$ that exposes $u$. $G.M \Delta N$ has a component which is an $M$-balanced path from $u$ to some vertex in $S$, and we are done for (i).

For (ii) we first prove the necessity part. Let $P$ be the $M$-exposed path between $u$ and $v$, and $w \in V(P)$ be such that $uw \in E(P)$. Then, $P - u$ is an $M$-balanced path from $w$ to $v$, which means $w \in D(G)$ by (i). Then, we have $u \in A(G)$. Since the first part of the condition on $P$ yields $u \notin D(G)$ by (i).

Now we move on to the sufficiency part of (ii). Note that the first part of the conclusion follows by (i). By the definition of $A(G)$, there exists $w \in D(G)$ such that $wu \in E(G)$. By (i), there is an $M$-balanced path $Q$ from $w$ to a vertex $v \in S$. If $u \in V(Q)$, since $u \notin D(G)$, $vQu$ is an $M$-exposed path between $v$ and $u$ by (i) and the claim follows. Otherwise, that is, if $u \notin V(Q)$, then $Q + wu$ forms an $M$-exposed path between $v$ and $u$, and again the claim follows. Thus, we are done for (ii).

Since we obtain (i) and (ii) consequently (iii) follows.

The next proposition is easy to see.
Proposition 7. Let $G$ be a factorizable graph, $M$ be a perfect matching of $G$, and $x \in V(G)$. Then, $M \setminus \{xx'\}$ is a maximum matching of $G - x$, exposing only $x'$.

The next proposition is also known (see [11]), and easily obtained from Proposition 6.

Proposition 8. Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Then, for any $x \in V(G)$, the followings hold;

(i) $u \in D(G - x)$ if and only if there is an $M$-saturated path between $x$ and $u$.
(ii) $u \in A(G - x) \cup \{x\}$ if and only if there is no $M$-saturated path between $x$ and $u$, while there is an $M$-balanced path from $x$ to $u$.
(iii) $u \in C(G - x)$ if and only if there is no $M$-saturated path between $u$ and $x$ nor $M$-balanced path from $x$ to $u$.

Proof. Let $G' := G - x$ and $M' := M \setminus \{xx'\}$. By Propositions 6 and 7, $u \in D(G')$ if and only if there is an $M'$-balanced path from $u$ to $x'$. Additionally, the following apparently follows; there is an $M'$-balanced path from $u$ to $x'$ in $G'$ if and only if there is an $M$-saturated path between $u$ and $x$ in $G$. Thus, we obtain (i). The other claims, (ii) and (iii), also follow by similar arguments. 

3 Canonical Structures of Factorizable Graphs

In this section we introduce canonical structures of factorizable graphs, which later turn out to be the underlying structure of the cathedral theorem. They are composed mainly of three parts: a partial order on factor-connected components (Theorem 1), a generalization of the canonical partition (Theorem 2), and their correlation (Theorem 3).

Definition 1. Let $G$ be a factorizable graph, and $G_1, G_2 \in G(G)$. We say $G_1 \preceq G_2$ if there exists $X \subseteq V(G)$ such that

1. $X$ is separating,
2. $V(G_1) \cup V(G_2) \subseteq X$, and
3. $G[X]/V(G_1)$ is factor-critical.

Theorem 1 (Kita [6, 7]). For any factorizable graph $G$, $\preceq$ is a partial order on $G(G)$.

Definition 2. Let $G$ be a factorizable graph, and $u, v \in V(G)$. We say $u \sim_G v$ if $u$ and $v$ are contained in the same factor-connected component, and $u = v$ or $G - u - v$ is not factorizable.

Theorem 2 (Kita [6, 7]). For any factorizable graph $G$, $\sim_G$ is an equivalence relation on $V(G)$. 

Given a factorizable graph $G$, we call the family of equivalence classes by $\sim$ the 
 generalized canonical partition, and denote $\mathcal{P}(G) := V(G)/\sim$, since it coincides 
 to Kotzig’s canonical partition [1][2][14] if $G$ is elementary [6][7]. As you can see 
 by the definition of $\sim$, each member of $\mathcal{P}(G)$ is respectively contained in a 
 factor-connected component. Therefore, $\mathcal{P}_G(H) := \{S \in \mathcal{P}(G) : S \subseteq V(H)\}$ 
 forms a partition of $V(H)$ for each $H \in \mathcal{G}(G)$. Note also the followings, which 
 are also stated in [6][7]:

**Proposition 9.** Let $G$ be a factorizable graph, and $H \in \mathcal{G}(G)$. Then, $\mathcal{P}_G(H)$ is 
 a refinement of $\mathcal{P}(H) = \mathcal{P}_H(H)$. Namely, if $u, v \in V(H)$ satisfies $u \sim_H v$, then 
 $u \sim_G v$ holds.

**Proof.** We prove the contrapositive; Let $u, v \in V(H)$ be such that $u \not\sim_H v$, which 
 is equivalent to $u \neq v$ and $H - u - v$ is factorizable. Let $N$ be a perfect matching of 
 $H - u - v$. Since $G - V(H)$ is also factorizable, by letting $M'$ be a perfect 
 matching of it, we can construct a perfect matching of $G - u - v$, namely $N \cup M'$. 
 Therefore, $u \not\sim_G v$. 

**Proposition 10.** Let $G$ be a factorizable graph, and $M$ be a perfect matching of 
 $G$. Let $u, v \in V(G)$ be vertices contained in the same factor-connected component 
 of $G$. Then, $u \sim_G v$ if and only if there is no $M$-saturated path between $u$ and $v$.

**Proof.** This is immediate by Proposition 3.
4 Factorizable Graphs through the Gallai-Edmonds Partition

In this section, we investigate correlations between the Gallai-Edmonds partition and the canonical structures of factorizable graphs introduced in Section 3. As we later see in Section 5, it can be regarded as a generalization of a part of the statements of the cathedral theorem.

Let $G$ be a factorizable graph, $H \in \mathcal{G}(G)$, and $S \in \mathcal{P}_G(H)$. Given Theorem 3, we denote the set of all the strict upper bounds of $H$ “assigned” to $S$ as $\mathcal{U}_G(S)$. Namely, $H' \in \mathcal{U}_G(S)$ if and only if $H' \in \mathcal{U}(H)$ and there is a connected component $K$ of $G[U(H)]$ such that $V(H') \subseteq V(K)$ and $N_G(K) \cap V(H) \subseteq S$. We define $U_G(S) := \bigcup_{H' \in \mathcal{U}_G(S)} V(H')$, and $U_G^*(S) := U_G(S) \cup S$. We often omit the subscripts “$G$” if they are apparent from contexts. Note that $\mathcal{U}(H) = \bigcup_{S \in \mathcal{P}_G(H)} U(S)$.

Proposition 12 (Kita [6, 7]). If $H$ is an elementary graph, then for any $u, v \in V(H)$ there is an $M$-saturated path between $u$ and $v$, or there are $M$-balanced paths from $u$ to $v$, where $M$ is an arbitrary perfect matching of $G$.

Lemma 1 (Kita [8]). Let $G$ be a factorizable graph, $M$ be a perfect matching of $G$. Let $H \in \mathcal{G}(G), S \in \mathcal{P}_G(H)$, and $T \in \mathcal{P}_G(H) \setminus \{S\}$.

(i) For any $u \in U^*(S)$, there is an $M$-balanced path from $u$ to some vertex $v \in S$ whose vertices except $v$ are in $U(S)$.

(ii) For any $u \in S$ and $v \in U^*(T)$, there is an $M$-saturated path between $u$ and $v$ whose vertices are all contained in $U^*(H) \setminus U(S)$.

(iii) For any $u \in S$ and $v \in U(S)$, there are no $M$-saturated paths between $u$ and $v$ nor $M$-balanced paths from $u$ to $v$.

(iv) For any $u, v \in S$, there is no $M$-saturated path between $u$ and $v$, while there is an $M$-balanced path from $u$ to $v$.

Proof. (i), (ii), and (iii) are stated in [8]. (iii) is immediately obtained by combining Propositions 10 and 12. □

Proposition 13. Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, the followings hold;

(i) For any $u \in U(S)$ and $v \in U^*(T)$, where $T \in \mathcal{P}_G(H) \setminus \{S\}$, there is an $M$-saturated path between $u$ and $v$.

(ii) For any $u \in U(S)$ and $v \in S$, there is no $M$-saturated path between $u$ and $v$, however there is an $M$-balanced path from $u$ to $v$.

(iii) For any $w \in S$ and $v \in U^*(T)$, where $T \in \mathcal{P}_G(H) \setminus \{S\}$, there is an $M$-saturated path between $w$ and $v$.

(iv) For any $w, v \in S$, there is no $M$-saturated path between $w$ and $v$, however there is an $M$-balanced path from $w$ to $v$.

(v) For any $w \in S$ and $v \in U(S)$, there is no $M$-saturated path between $w$ and $v$ nor $M$-balanced path from $w$ to $v$. 
Proof. 

(iii) and (v) are immediate from respectively (ii) (iv) and (iii) of Lemma 1.

For (i) let $P_1$ be an $M$-balanced path from $u$ to some vertex $x \in S$ such that $V(P_1) \setminus \{x\} \subseteq U(S)$, given by (i) of Lemma 1. By (iii) of Lemma 1, there is an $M$-saturated path $P_2$ between $x$ and $v$ such that $V(P_2) \subseteq U^*(H) \setminus U(S)$. Hence, the path obtained by adding $P_1$ and $P_2$ forms an $M$-saturated path between $u$ and $v$, and (i) follows.

The first and latter halves of (ii) are restatements of respectively (iii) and (i) of Lemma 1.

\[ \square \]

**Proposition 14.** Let $G$ be a factorizable graph such that the poset $(\mathcal{G}(G), \preceq)$ has the minimum element $G_0$. Let $S \in \mathcal{P}_G(G_0)$.

(i) If $x \in U(S)$, then $D(G-x) \supseteq U^*(G_0) \setminus U^*(S)$, $A(G-x) \cup \{x\} \supseteq S$, and $C(G-x) \subseteq U(S)$.

(ii) If $x \in S$, then $D(G-x) = U^*(G_0) \setminus U^*(S)$, $A(G-x) \cup \{x\} = S$, and $C(G-x) = U(S)$.

Proof. The claims are all obtained by comparing the reachabilities of alternating paths regarding Propositions 8 and 13. Let $x \in U(S)$. By Proposition 8 (i) and Proposition 13 (i), $D(G-x) \supseteq U^*(G_0) \setminus U^*(S)$ follows. $A(G-x) \cup \{x\} \supseteq S$ also follows by a similar argument. Therefore, since $V(G) = D(G) \cup A(G) \cup C(G) = (U^*(G_0) \setminus U^*(S)) \cup S \cup U(S)$, $C(G-x) \subseteq U(S)$ follows, and we are done for (i).

(ii) also follows by similar arguments.

\[ \square \]

**Theorem 5.** Let $G$ be a factorizable graph such that the poset $(\mathcal{G}(G), \preceq)$ has the minimum element $G_0$. Then, $V(G_0)$ is exactly the set of vertices that is disjoint from $C(G-x)$ for any $x \in V(G)$.

Proof. Claim 1. For any $x \in V(G)$, $V(G_0) \cap C(G-x) = \emptyset$.

Proof. Let $u \in V(G_0)$ and let $S \in \mathcal{P}_C(G_0)$ be such that $u \in S$. By Proposition 14 if $x \in U^*(S)$ then $u \in A(G-x)$, and if $x \in U^*(G_0) \setminus U^*(S)$ then $u \in D(G-x)$. Thus, anyway we have $u \notin C(G-x)$. Namely, the claim follows.

Claim 2. For any $u \in V(G) \setminus V(G_0)$, there exists $x \in V(G)$ such that $u \in C(G-x)$.

Proof. Let $u \in V(G) \setminus V(G_0)$ and let $S \in \mathcal{P}_G(G_0)$ be such that $u \in U(S)$. Then, for any $x \in S$, $u \in C(G-x)$ by Proposition 14. Thus, we have the claim.

By Claims 1 and 2 we obtain the claim.

\[ \square \]

5 Another Proof of the Cathedral Theorem

A graph with perfect matchings is called saturated if any addition of an arbitrary complement edge creates a new perfect matching. The cathedral theorem is a
structure theorem of saturated graphs, originally given by Lovász \[1, 2\], and later given another proof by Szigeti \[3, 4\]. In this section, we give yet another proof as a consequence of the structure given in Section 3. For convenience, we treat empty graphs as factorizable and saturated.

**Definition 4 (Cathedral Construction).** Let $G_0$ be a saturated elementary graph and let $\{G_S\}_{S \in \mathcal{P}(G_0)}$ be a family of saturated graphs, some of which might be empty. For each $S \in \mathcal{P}(G_0)$, join every vertex in $S$ and every vertex of $G_S$. We call this operation cathedral construction. Here $G_0$ and $\{G_S\}_{S \in \mathcal{P}(G_0)}$ are called respectively the foundation and the towers.

**Theorem 6 (The Cathedral Theorem \[1, 2\]).** A factorizable graph $G$ is saturated if and only if it is constructed by iterating cathedral construction. Additionally, if $G$ is a saturated graph obtained by conducting cathedral construction to the foundation $G_0$ and the towers $T = \{G_S\}_{S \in \mathcal{P}(G_0)}$, then,

(i) $e \in E(G)$ is allowed if and only if it is an allowed edge of $G_0$ or $G_S$ for some $S \in \mathcal{P}(G_0)$,

(ii) Such $G_0$ uniquely exists, that is, if $G$ can be obtained by conducting cathedral construction to the foundation $G_0'$ and the towers $T'$, then $V(G_0) = V(G_0')$ holds.

(iii) $V(G_0)$ is exactly the set of vertices that is disjoint from $C(G - x)$ for any $x \in V(G)$.

Since the necessity of cathedral construction, the next proposition, is rather obvious (see \[1\]), we here present it without a proof.

**Proposition 15 (Lovász \[1, 2\]).** Let $G_0$ be an saturated elementary graph, and $T = \{G_S\}_{S \in \mathcal{P}(G_0)}$ be a family of saturated graphs. Then, the graph $G$ obtained by conducting cathedral construction to the foundation $G_0$ and the towers $T$ is saturated.

We give another proof for the other parts of Theorem \[6\] in the following. Theorem \[7\] states the other direction of the first claim of Theorem \[6\] and Theorem \[8\] together with Theorem \[5\] prove the other parts, referring to the special features of the poset and the canonical partition of saturated graphs. Hereafter the following property, which is immediate by Proposition \[2\], appears everywhere:

**Proposition 16.** Let $G$ be a factorizable graph, $M$ be a perfect matching, and $x, y \in V(G)$ be such that $xy \notin E(G)$. Then, the following properties are equivalent;

(i) The complement edge $xy$ creates a new perfect matching in $G + xy$.

(ii) $xy$ is allowed in $G + xy$.

(iii) There is an $M$-saturated path between $x$ and $y$ in $G$.

The next proposition is easy to see by Proposition \[10\] and is for Lemma \[2\]

**Proposition 17.** Let $G$ be a saturated graph, and $H \in \mathcal{G}(G)$. Then, for any $u, v \in V(H)$ with $u \sim_G v$, $uv \in E(G)$. 

The next lemma is for both of Theorem 7 and Theorem 8, stating that the canonical partition and the generalized one coincide in saturated graphs.

**Lemma 2.** Let $G$ be a saturated graph, and $G_0 \in \mathcal{G}(G)$. Then, $\mathcal{P}_G(G_0) = \mathcal{P}(G_0)$.

**Proof.** Since we know by Proposition 10 that $\mathcal{P}_G(G_0)$ is a refinement of $\mathcal{P}(G_0)$, it suffices to prove that $\mathcal{P}(G_0)$ is a refinement of $\mathcal{P}_G(G_0)$, that is, if $u \sim_{G_0} v$, then $u \sim_G v$. We prove the contraposition of this.

Let $u, v \in V(G_0)$ with $u \not\sim_G v$. Let $M$ be a perfect matching of $G$. By Proposition 10 there are $M$-saturated paths between $u$ and $v$. Let $P$ be the shortest one. Suppose $E(P) \setminus E(G_0) \neq \emptyset$, and let $Q$ be one of the components of $P - E(G_0)$, with end vertices $x$ and $y$. Since $Q$ is an $M$-ear relative to $G_0$ by Proposition 15, $x \sim_G y$ follows by Proposition 11. Therefore, $xy \in E(G)$ by Proposition 17, which means we can get a shorter $M$-saturated path between $u$ and $v$ by replacing $Q$ by $xy$ on $P$, a contradiction. Thus, we have $E(P) \setminus E(G_0) = \emptyset$, namely $P$ is a path of $G_0$. Accordingly, $u \not\sim_{G_0} v$ by Proposition 10.

The succeeding four properties are for Theorem 7.

**Lemma 3.** If a factorizable graph $G$ is saturated, then the poset $(\mathcal{G}(G), \preceq)$ has the minimum element.

**Proof.** Suppose the claim fails, that is, the poset has distinct minimal elements $G_1, G_2 \in \mathcal{G}(G)$. Then, by Theorem 3, there exist possibly identical complement edges $e, f$ joining between $V(G_1)$ and $V(G_2)$ such that $\mathcal{G}(G + e + f) = \mathcal{G}(G)$. This means that adding $e$ or $f$ to $G$ does not create any new perfect matchings, which contradicts $G$ being saturated.

**Lemma 4.** Let $G$ be a saturated graph, and $G_0$ be the minimum element of the poset $(\mathcal{G}(G), \preceq)$, and let $K$ be a component of $G - G_0$, whose neighbors are in $S \in \mathcal{P}_G(G_0)$. Then, for any $u \in V(K)$ and for any $v \in S$, uv $E(G)$.

**Proof.** Suppose the claim fails, that is, there are $u \in V(K)$ and $v \in S$ such that $uv \not\in E(G)$. Then, by Proposition 16, there is an $M$-saturated path between $u$ and $v$, where $M$ is an arbitrary perfect matching of $G$. By the definitions, $V(K) \subseteq U(S)$, therefore, $u \in U(S)$. Hence, this contradicts (iii) of Lemma 11 and we have the claim.

**Proposition 18.** If a factorizable graph $G$ is saturated, $G$ is connected.

**Proof.** Suppose the claim fails, that is, $G$ has two distinct connected components, $K$ and $L$. Let $u \in V(K)$ and $v \in V(L)$, and let $M$ be a perfect matching of $G$. By Proposition 16, there is an $M$-saturated path between $u$ and $v$, contradicting the hypothesis that $K$ and $L$ are distinct.

**Lemma 5.** Let $G$ be a saturated graph, and $G_0$ be the minimum element of the poset $(\mathcal{G}(G), \preceq)$. Then, $G_0$ and the components of $G - V(G_0)$ are each saturated. Additionally, for each $S \in \mathcal{P}_G(G_0)$, a connected component $K$ of $G - V(G_0)$ such that $N_G(K) \cap V(G_0) \subseteq S$ exists uniquely or does not exist.
Proof. We first prove that $G_0$ is saturated. Let $e = xy$ be a complement edge of $G_0$. By Proposition 17, $x \not\sim G_{0} y$, which means $x \not\sim G_{0} y$ by Proposition 9. Therefore, by Propositions 10 and 11, $e$ creates a new perfect matching of $G_0$ if it is added. Hence, $G_0$ is saturated.

Now we move on to the remained claims. Take $S \in \mathcal{P}_G(G_0)$ arbitrarily, and let $K_1, \ldots, K_l$ be the connected components of $G - V(G_0)$ which satisfy $N_G(K_i) \cap V(G_0) \subseteq S$ for each $i = 1, \ldots, l$. Let $\hat{K} := G[V(K_1) \cup \cdots \cup V(K_l)]$.

We are going to obtain the remained claims by showing that $\hat{K}$ is factor-critical. Now let $e = xy$ be a complement edge of $\hat{K}$, i.e., $x, y \in V(\hat{K})$. Let $M$ be a perfect matching of $\hat{K}$. Obviously by the definition, $N_G(\hat{K}) \subseteq S$. Therefore, if $E(P) \setminus E(\hat{K}) \neq \emptyset$, each connected component of $P - V(\hat{K})$ is an $M$-saturated path, both of whose end vertices are contained in $S$, by Proposition 9. This contradicts Proposition 11. Hence, $E(P) \subseteq E(\hat{K})$, which means $\hat{K}$ is itself saturated. Thus, by Proposition 15, $\hat{K}$ is connected, which is equivalent to $l = 1$. This completes the proof.

\begin{theorem}
If a factorizable graph $G$ is saturated, then the poset $(\mathcal{G}(G), \preceq)$ has the minimum element, say $G_0$, and $\mathcal{P}_G(G_0) = \mathcal{P}(G_0) =: \mathcal{P}_0$. Additionally, for each $S \in \mathcal{P}_0$, the connected component $G_S$ of $G - V(G_0)$ such that $N_G(G_S) \cap V(H) \subseteq S$ exists uniquely or is an empty graph, and $G$ is the graph obtained by conducting cathedral construction to the foundation $G_0$ and the towers $T := \{G_S\}_{S \in \mathcal{P}_0}$.
\end{theorem}

Proof. The claims in the first sentence are immediate from respectively from Lemma 3 and Lemma 2. So is the first half of the second sentence by Lemma 5. For the remained claim, first note that by Lemma 5, $G_0$ and $G_S$ are all saturated. Therefore, $G_0$ and $T$ are well-defined as the foundation and the towers of cathedral construction.

By the definition, for each $S \in \mathcal{P}_0$, $N_G(G_S) \subseteq S$, and by Lemma 4, every vertex of $V(G_S)$ and every vertex of $S$ are joined. Additionally, by Theorem 8, $V(G) = V(G_0) \cup \bigcup_{S \in \mathcal{P}_0} V(G_S)$. Thus, $G$ is the graph obtained by conducting cathedral construction to $G_0$ and $T$.

\begin{lemma}
Let $G$ be a saturated graph, obtained by conducting cathedral construction to the foundation $G_0$ and the towers $\{G_S\}_{S \in \mathcal{P}(G_0)}$. Then, $G' := G / V(G_0)$ is factor-critical.
\end{lemma}

Proof. Let $M^S$ be a perfect matching of $G_S$ for each $S \in \mathcal{P}(G_0)$, and let $M := \bigcup_{S \in \mathcal{P}(G_0)} M^S$. Then, $M$ forms a near-perfect matching of $G'$, exposing only the contracted vertex $g_0$ corresponding to $V(G_0)$. Take $u \in V(G') \setminus \{g_0\}$ arbitrarily. Since $uu' \in M \cap E(G')$ and $u'g_0 \in E(G') \setminus M$, there is an $M$-balanced path from $u$ to $g_0$ in $G'$, namely the one with edges $\{uu', u'g_0\}$. Thus, by Proposition 11, $G'$ is factor-critical.

\begin{theorem}
Let $G_0$ be a saturated elementary graph, and $T := \{G_S\}_{S \in \mathcal{P}(G_0)}$ be a family of saturated graphs. Let $G$ be the graph obtained by conducting cathedral construction to the foundation $G_0$ and the towers $T$. Then, $G$ is saturated, $G_0$
forms a factor-connected component of \( G \), that is, \( G[V(G_0)] \in \mathcal{G}(G) \), and it is the minimum element of the poset \((\mathcal{G}(G), \preceq)\).

**Proof.** Immediately by Proposition 15, \( G \) is saturated. Now that we have Lemma 6, in order to obtain the remained claims, it suffices to prove that \( G_0 \in \mathcal{G}(G) \). Let \( p \) be the number of non-empty graph in \( T \). We proceed by induction on \( p \). If \( p = 0 \), the claim obviously follows. Let \( p > 0 \) and suppose the claim is true for \( p - 1 \). Take a non-empty graph \( G \) from \( T \), and let \( G' = G - V(G_S) \). Then, \( G' \) is the graph obtained by conducting cathedral construction to \( G_0 \) and \( T \setminus \{G_S\} \cup \{H_S\} \), where \( H_S \) is an empty graph. Therefore, Proposition 15 yields that \( G' \) is saturated, and the induction hypothesis that \( G_0 \in \mathcal{G}(G') \) and \( G_0 \) is the minimum element of the poset \((\mathcal{G}(G'), \preceq)\). Thus, by Lemma 2, Claim 3.

\[ P_{G'}(G_0) = P(G_0). \]

Let \( M' \) be a perfect matching of \( G' \) and \( M_S \) be of \( G_S \), and construct a perfect matching \( M := M' \cup M_S \) of \( G \).

**Claim 4.** No edge of \( E_G[S, V(G_S)] \) is allowed in \( G \).

**Proof.** Suppose the claim fails, that is, an edge \( xy \in E_G[S, V(G_S)] \) is allowed in \( G \). Then, there is an \( M \)-saturated path \( Q \) between \( x \) and \( y \) by Proposition 2, and \( Q[V(G')] \) is an \( M \)-saturated path by Proposition 6. Moreover, since \( N_G(G_S) \cap V(G') \subseteq S \), we can say that \( Q[V(G')] \) is an \( M \)-saturated path of \( G' \) between two vertices in \( S \). With Proposition 10 this is a contradiction, because \( S \in P_{G'}(G_0) \) by Claim 3. Hence, we have the claim. \( \square \)

Given Claim 4, we can see that a set of edges is a perfect matching of \( G \) if and only if it is a disjoint union of a perfect matching of \( G' \) and \( G_S \). Thus, \( G_0 \) forms a factor-connected component of \( G \), and we are done. \( \square \)

**Proof (the cathedral theorem).** By Proposition 15 and Theorem 7, the first claim of Theorem 6 is proved. (i) is by Theorem 8 since it states that \( G_0 \in \mathcal{G}(G) \). (ii) is also by Theorem 8 since the poset \((\mathcal{G}(G), \preceq)\) is a canonical notion. (iii) is by combining Theorem 8 and Theorem 6. \( \square \)

**Remark 1.** Theorems 8 and 7 can be regarded as a refinement, and Theorem 5 as a generalization of Theorem 6, from the point of view of the canonical structures of Section 3.

Due to this, you can see how a factorizable graph leads to a saturated graph having the same family of perfect matchings by sequentially adding complement edges.

The poset \((\mathcal{G}(G), \preceq)\) and \( \mathcal{P}(G) \) can be computed in \( O(|V(G)| \cdot |E(G)|) \) time \( [6,7] \), where \( G \) is any factorizable graph. Therefore, given a saturated graph, you can also find how it is constructed by cathedral construction in the above time by computing the associated poset and so on.

**Remark 2.** The canonical structures of general factorizable graphs introduced in Section 8 can be obtained without the Gallai-Edmonds structure theorem nor
the notion of barriers. So do the other properties we cite in this paper to prove the cathedral theorem. Therefore, our proof shows that the cathedral theorem holds independently from them.

With the whole proof, we can conclude that the structures in Section 3 is what essentially underlie the cathedral theorem. Hence, it can be said that yet more would be found on the field of estimating the number of perfect matchings with the results in this paper and [6, 8].

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