Sharp pointwise-in-time error estimate of L1 scheme for nonlinear subdiffusion equations

Dongfang Li (李东方)

joint work with

Hongyu Qin (秦红玉), Jiwei Zhang (张继伟)

School of Mathematics and Statistics, Huazhong University of Science and Technology

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Outline

1 Introduction
Outline

1. Introduction
2. Numerical Method
Outline

1. Introduction
2. Numerical Method
3. Main results
Outline

1. Introduction
2. Numerical Method
3. Main results
4. Numerical examples
Outline

1 Introduction
2 Numerical Method
3 Main results
4 Numerical examples
5 Conclusion
nonlinear subdiffusion equations

$$\partial_t^\alpha u - \Delta u = f(u), \quad x \in \Omega \times (0, T],$$

where

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{1}{(t - s)^\alpha} ds, \quad 0 < \alpha < 1.$$
Regularity of the solutions

- If the initial condition \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, the solution satisfies

\[
\| \partial_t u(t) \|_{L^2(\Omega)} \leq C t^{\alpha-1}
\]

(B. Jin, B. Li, Z. Zhou, SIAM J. Numer. Anal, 2018.)

- If \( u_0 \in \dot{H}^\nu(\Omega) \) with \( \nu \in (0, 2] \) and \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, the solution satisfies

\[
\| \partial_t u(t) \|_{L^2(\Omega)} \leq C t^{\nu \alpha/2 - 1}
\]

(M. Maskari, S. Karaa, SIAM J. Numer. Anal, 2019.)

- smooth initial conditions + \( f + \) compatibility conditions: smooth solutions
Under the assumption

\[ \| \partial_t u(t) \|_{L^2(\Omega)} \leq C t^{\alpha-1}, \]

many works indicate that the errors of many schemes in the maximum norm are \( O(\tau^\alpha) \)

- L1 schemes on the uniform meshes
- L2 schemes on the uniform meshes
- Convolution quadrature Euler method
- Convolution quadrature BDF methods

Numerical simulations show an interesting phenomenon that the convergence order of the schemes is \( O(\tau^\alpha) \) as \( t \) tends to 0, and high-order accurate at the final time \( t = T \).
Present the pointwise error estimates

Linear problem

\[ \partial_t^\alpha u = \Delta u \quad x \in \Omega \times (0, T], \]

- L1 scheme on the uniform meshes, \( \tau t_n^{\alpha-1} \), Gracia-ORiordan-Stynes (CMAM 2018)
- A modified L1 scheme, \( \tau t_n^{\alpha-1} \), Yan-Khan-Ford (SINUM 2018)
- L1 and L2 scheme on graded and uniform meshes, N. Kopteva, X. Meng, (MC 2019, 2021)
- Correction Convolution quadrature BDF method, \( \tau^k t_n^{\alpha-k} \), Jin-Li-Zhou (SISC 2017)
- Grünwald – Letnikov scheme, \( \tau t_n^{\alpha-1} \), Chen-Holland-Stynes, (APNUM 2019)
- …
Present the pointwise error estimates

Semi-linear problem

\[ \partial_t^\alpha u = \Delta u + f(u) \quad x \in \Omega \times (0, T], \]

- Convolution quadrature Euler method and assumption
  \[ \| \partial_t u(t) \|_{L^2(\Omega)} \leq C t^{\nu \alpha/2 - 1}, \text{Maskari-Karaa (SINUM 2019)} \]
  a generalized variant of the standard discrete Gronwall’s inequality

- L1 scheme on the graded meshes and assumption
  \[ \| \partial_t u(t) \|_{L^2(\Omega)} \leq C' t^{\alpha - 1}, \text{N. Kopteva, (SINUM 2020)} \]
  Some assumption on the nonlinear function \( f \) are needed. The proposed conditions on the nonlinearity can guarantee the exact solutions have the upper and lower bounds, and a method of upper and lower solutions is introduced to address that the numerical solutions lie within a certain range.

- Convolution quadrature BDF methods and assumption
  \[ u_0 \in H^1_0(\Omega) \cap C^2(\bar{\Omega}) \]
Semi-linear problem

$$\partial_t^\alpha u = \Delta u + f(u) \quad x \in \Omega \times (0, T],$$

Our motivation

- a more generalized nonlinear function $f$?
- a more generalized assumption on the regularity of the solutions?
- L1 scheme, Convolution quadrature BDF methods, Grünwald–Letnikov scheme and the other scheme?

A framework of pointwise-in-time error estimates of numerical schemes
Assumption

- We assume that

\[ \| \partial_t^m u \|_{L^2(\Omega)} \leq C t^{\sigma - m}, \quad \text{for } m = 1, 2, \text{ and } \sigma \in (0, 1) \cup (1, 2]. \]

- \( f(u) \in C^2(\mathbb{R}) \)

Remark: The assumption \( \sigma \in (0, \alpha] \) is reasonable and widely accepted. Suppose that the solution is smoother, i.e., \( \sigma > \alpha \), some additional hypothesis should be added. In this work, we assume that \( \sigma \in (0, 1) \cup (1, 2) \) in order to make the current analysis extendable.
Linearized numerical methods

- central finite difference method

\[
\sum_{j=1}^{d} u^n_{x_jx_j} = \frac{1}{h^2} \sum_{j=1}^{d} (u^n_{i_1,\ldots,x_{j-1},\ldots,i_d} - 2u^n_{i_1,\ldots,i_d} + u^n_{i_1,\ldots,x_{j+1},\ldots,i_d}) + R^n_{i_1,\ldots,i_d}
\]

\[
:= \sum_{j=1}^{d} \delta^2_{x_j} u^n_{i_1,\ldots,i_d} + R^n_{i_1,\ldots,i_d}, \tag{2.1}
\]

- standard L1-approximation

\[
\partial_{tn}^\alpha u = \frac{1}{\Gamma(1 - \alpha)} \sum_{j=1}^{n} \frac{u^n_j - u^{j-1}_{i_1,\ldots,i_d}}{\tau} \int_{t_{j-1}}^{t_j} \frac{1}{(t_n - s)\alpha} ds + r^n_{i_1,\ldots,i_d}
\]

\[
= \frac{\tau^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{j=1}^{n} a_{n-j} (u^n_{i_1,\ldots,i_d} - u^{j-1}_{i_1,\ldots,i_d}) + r^n_{i_1,\ldots,i_d}
\]

\[
:= D_{\tau}^\alpha u^n_{i_1,\ldots,i_d} + r^n_{i_1,\ldots,i_d}, \tag{2.2}
\]
**Newton linearized method**

\[
\begin{align*}
    f(u^n_{i_1,\ldots,i_d}) &= f(u^{n-1}_{i_1,\ldots,i_d}) + f_1(u^{n-1}_{i_1,\ldots,i_d})(u^n_{i_1,\ldots,i_d} - u^{n-1}_{i_1,\ldots,i_d}) \\
    &+ \tilde{r}^n_{i_1,\ldots,i_d}
\end{align*}
\]

(2.3)

**Fully-discrete and linearized scheme**

\[
D_\tau^\alpha U^n_{i_1,\ldots,i_d} = \sum_{j=1}^{d} \delta^2_{x_j} U^n_{i_1,\ldots,i_d} + f(U^{n-1}_{i_1,\ldots,i_d}) \\
+ f_1(U^{n-1}_{i_1,\ldots,i_d})(U^n_{i_1,\ldots,i_d} - U^{n-1}_{i_1,\ldots,i_d})
\]

with \( U^0_{i_1,\ldots,i_d} = u^0_{i_1,\ldots,i_d} \).
Convergence

Suppose that the system has a unique solution satisfying the assumptions. Then, there exists a positive constant \( \tau_0 \), such that when \( \tau \leq \tau_0 \), the full finite difference system admits a unique solution \( U^n \), it holds for \( n = 1, \cdots, N \) that

\[
\|u^n - U^n\|_\infty \lesssim \begin{cases} 
\tau^{\sigma+1-\alpha}t_n^{\alpha-1} + t_n^\alpha h^2, & 0 < \sigma < 1, \\
\tau^{2-\alpha}t_n^{\alpha+\sigma-2} + t_n^\alpha h^2, & 1 < \sigma \leq 2,
\end{cases}
\]

where

\[
 u^n = \begin{bmatrix} u_{1,1}, \cdots, 1, u_{2,1}, \cdots, 1, \cdots, u_{M-1,1}, \cdots, 1, \cdots, u_{1,1}, \cdots, M-1, \\
u_{2,1}, \cdots, M-1, \cdots, u_{M-1,1}, \cdots, M-1 \end{bmatrix}^T,
\]

\[
 U^n = \begin{bmatrix} U_{1,1}, \cdots, 1, U_{2,1}, \cdots, 1, \cdots, U_{M-1,1}, \cdots, 1, \cdots, U_{1,1}, \cdots, M-1, \\
U_{2,1}, \cdots, M-1, \cdots, U_{M-1,1}, \cdots, M-1 \end{bmatrix}^T.
\]
Some remarks

Remark 1

The convergence results imply that, when $t_n \to 0$, it holds that

$$\max_{1 \leq n \leq N} \| u^n - U^n \|_{\infty} \lesssim \tau^\sigma + \tau^\alpha h^2.$$ 

When $t$ is far away from 0, it holds that

$$\| u^n - U^n \|_{\infty} \lesssim \begin{cases} \tau^{\sigma+1-\alpha} + h^2, & \sigma \in (0, 1), \\ \tau^{2-\alpha} + h^2, & \sigma \in (1, 2]. \end{cases}$$
Remark 2

- If we consider the maximum error in the whole domain $\Omega \times [0, T]$, we have

$$\max_{1 \leq n \leq N} \|u^n - U^n\|_{\infty} \lesssim \begin{cases} \tau^\sigma + h^2, & \sigma \in (0, 1) \cup (1, 2 - \alpha), \\ \tau^{2-\alpha} + h^2, & \sigma \in [2 - \alpha, 2]. \end{cases}$$

- This convergent results not only consistent with the previous pointwise error estimates for L1-scheme under $\sigma = \alpha$, but also give the new pointwise error estimate for $\alpha \neq \sigma$. 
Proof of our main results

- truncation errors of nonlinear item

Suppose that $u$ satisfies the solution regularity. Then, it holds that

$$|r_n| := |D^\alpha u^n - \partial_t^\alpha u| \lesssim \begin{cases} \tau^{\sigma - \alpha} n^{\min(1+\alpha,2-\sigma)}, & 0 < \sigma < 1, \\ \tau^{\sigma - \alpha} n^{-2+\sigma}, & 1 < \sigma < 2. \end{cases}$$
Proof of our main results

- Discrete fractional Grönwall inequality
  Suppose $0 < \alpha < 1$ and $\tau > 0$. Let $y_i$, $0 \leq i \leq N$, be a sequence of non-negative real numbers satisfying

  \[ D^\alpha_{\tau} y_n \leq \lambda y_n + \lambda y_{n-1} + \mu_1 n^{-\sigma_1} + \mu_2 n^{-\sigma_2} + \eta, \text{ for } n = 1, \ldots, N, \]

  where $\sigma_1 > 1, \sigma_2 < 1, \lambda > 0$, and $\mu_1, \mu_2 \geq 0$. Then there exists a constant $\tau^* = \sqrt{\frac{1}{2\Gamma(2-\alpha)\lambda}}$ such that $\tau < \tau^*$, it holds

  \[ y_n \leq 2\Gamma(2-\alpha)C^*(1 + E_{\alpha,1}(2\Gamma(2-\alpha)\lambda t^\alpha_n))\mu_1 \tau t_n^{\alpha-1} \]

  \[ + \Gamma(1-\sigma_2)2\Gamma(2-\alpha)C^*(1 + E_{\alpha,1-\sigma_2+\alpha}(2\Gamma(2-\alpha)\lambda t^\alpha_n))\mu_2 \tau^{\sigma_2} t_n^{\alpha-\sigma_2} \]

  \[ + 2\Gamma(2-\alpha)C^*(1 + E_{\alpha,1+\alpha}(2\Gamma(2-\alpha)\lambda t^\alpha_n))(4y_0 + \eta)t^\alpha_n, \]

  where $\lambda_1 = \frac{(3-2\alpha)}{2-2\alpha} \lambda$, $t_n = n\tau$ and $C^*$ is a constant.
Proof of our main results

- Set $\|e^n\|_\infty = e_{i_0}^n$ and $e^n_i = u^n_i - U^n_i$. The error at the grid point $(x_{i_0}, t_n)$ satisfies

$$D_\tau^\alpha e^n_{i_0} = \delta^2_x e^n_{i_0} + (R_f)^n_{i_0} + r^n_{i_0} + \bar{r}^n_{i_0} + R^n_{i_0}.$$  

- Using the fact that $a_i > a_{i+1}$, we further have

$$\left(\frac{\tau^{-\alpha}}{\Gamma(2 - \alpha) a_0}\right)|e^n_{i_0}| \leq \frac{\tau^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) |e^j_{i_0}| + |(R_f)^n_{i_0}|.$$  

The above can be rewritten as

$$D_\tau^\alpha |e^n_{i_0}| \leq |(R_f)^n_{i_0}| + |r^n_{i_0}| + |\bar{r}^n_{i_0}| + |R^n_{i_0}|. \quad (3.1)$$  

- Using the previous lemmas, we finally get the convergence results.
Example 1

One-dimensional nonlinear subdiffusion problems

\[ \partial_t^{\alpha} u = u_{xx} + \sqrt{1 + u^2} + g(x, t), \quad (x, t) \in (0, \pi) \times (0, t_N], \]

where the initial condition and \( g(x, t) \) are specially chosen such that the problem admits an exact solution in the form of

\[ u(x, t) = t^\sigma \sin(x). \]
### Numerical result

- **Maximum errors at** $t = 1$ **and convergence orders in temporal direction with** $M = 1000$ **for Example 1**

| $\alpha$ | $\sigma$ | 10  | 20  | 40  | 80  | 160 | Rate | expected order       |
|----------|----------|-----|-----|-----|-----|-----|------|----------------------|
| 0.4      | 0.1      | 3.50e-2 | 2.05e-2 | 1.22e-2 | 7.37e-3 | 4.46e-3 | 0.73 | $\sigma + 1 - \alpha$ |
| 0.4      | 0.4      | 1.12e-2 | 5.21e-3 | 2.46e-3 | 1.18e-3 | 5.71e-4 | 1.07 | $\sigma + 1 - \alpha$ |
| 0.6      | 0.4      | 5.13e-3 | 2.10e-3 | 8.70e-4 | 3.63e-4 | 1.53e-4 | 1.25 | $\sigma + 1 - \alpha$ |
| 1.2      | 0.4      | 3.98e-3 | 1.57e-3 | 6.17e-4 | 2.41e-4 | 9.40e-5 | 1.36 | $2 - \alpha$         |
| 1.8      | 0.4      | 1.30e-2 | 5.03e-3 | 1.93e-3 | 7.38e-4 | 2.81e-4 | 1.39 | $2 - \alpha$         |
| 0.6      | 0.4      | 3.27e-2 | 1.78e-2 | 9.89e-3 | 5.54e-3 | 3.13e-3 | 0.84 | $\sigma + 1 - \alpha$ |
| 0.6      | 0.6      | 1.51e-2 | 7.28e-3 | 3.54e-3 | 1.73e-3 | 8.51e-4 | 1.03 | $\sigma + 1 - \alpha$ |
| 0.8      | 0.6      | 5.68e-3 | 2.49e-3 | 1.09e-3 | 4.76e-4 | 2.08e-4 | 1.19 | $\sigma + 1 - \alpha$ |
| 1.2      | 0.8      | 3.98e-3 | 1.57e-3 | 6.17e-4 | 2.41e-4 | 9.40e-5 | 1.36 | $2 - \alpha$         |
| 1.8      | 0.8      | 1.30e-2 | 5.03e-3 | 1.93e-3 | 7.38e-4 | 2.81e-4 | 1.39 | $2 - \alpha$         |
Maximum errors as $t \to 0$ and convergence orders in temporal direction with $M = 1000$ and $N = 10$ for Example 1

| $\alpha$ | $\sigma$ \( t_N \) | $1e - 3$ | $1e - 4$ | $1e - 5$ | $1e - 6$ | $1e - 7$ | Rate | expected order |
|----------|----------------|----------|----------|----------|----------|----------|-------|----------------|
| 0.4      | 0.1            | 2.90e-2  | 2.39e-2  | 1.92e-2  | 1.54e-2  | 1.23e-2  | 0.09  | $\sigma$       |
| 0.4      | 0.1            | 1.20e-3  | 5.02e-4  | 2.05e-4  | 8.22e-5  | 3.28e-5  | 0.40  | $\sigma$       |
| 0.6      | 0.1            | 1.38e-4  | 3.67e-5  | 9.43e-6  | 2.39e-6  | 6.03e-7  | 0.59  | $\sigma$       |
| 1.2      | 0.1            | 5.61e-7  | 3.70e-8  | 2.38e-9  | 1.51e-10 | 9.57e-12 | 1.19  | $\sigma$       |
| 1.8      | 0.1            | 2.87e-8  | 4.71e-10 | 7.58e-12 | 1.21e-13 | 1.92e-15 | 1.80  | $\sigma$       |
| 0.6      | 0.4            | 3.67e-3  | 1.49e-3  | 5.94e-4  | 2.37e-4  | 9.42e-5  | 0.40  | $\sigma$       |
| 0.6      | 0.4            | 4.29e-4  | 1.09e-4  | 2.76e-5  | 6.95e-6  | 1.75e-6  | 0.60  | $\sigma$       |
| 0.8      | 0.4            | 3.99e-5  | 6.43e-6  | 1.02e-6  | 1.62e-7  | 2.57e-8  | 0.80  | $\sigma$       |
| 1.2      | 0.4            | 1.64e-6  | 1.05e-7  | 6.64e-9  | 4.49e-10 | 2.55e-11 | 1.20  | $\sigma$       |
| 1.8      | 0.4            | 7.58e-7  | 1.21e-9  | 1.92e-11 | 3.05e-13 | 4.85e-14 | 1.80  | $\sigma$       |
Maximum errors at $t = 1$ and convergence orders in spatial direction with $N = 1000$ for Example 1

| $\alpha$ | $\sigma$ | $M$ | 8    | 16    | 24    | 32    | 40    | Rate | expected order |
|---------|---------|-----|------|-------|-------|-------|-------|------|----------------|
| 0.4     | 0.4     |     | 9.25e-3 | 2.30e-3 | 1.02e-3 | 5.67e-4 | 3.60e-4 | 2.02 | 2              |
| 1.2     |         |     | 7.68e-3 | 1.92E-3 | 8.51e-4 | 4.79e-4 | 3.06e-4 | 2.00 | 2              |
| 0.6     | 0.6     |     | 8.20e-3 | 2.03e-3 | 8.96e-4 | 4.98e-4 | 3.14e-4 | 2.03 | 2              |
| 0.2     |         |     | 6.79e-3 | 1.70e-3 | 7.54e-4 | 4.24e-4 | 2.71e-4 | 2.00 | 2              |
Example 2

Consider the nonlinear subdiffusion problems

$$\partial_t^\alpha u - \Delta u = f(u), \quad x \in \Omega \times (0, T]$$

with the following conditions

(a) $d = 1$, $f(u) = \sqrt{1 + u^2}$ and $u_0(x) = x(1 - x)$, $\Omega = (0, 1)$,
(b) $d = 1$, $f(u) = u - u^3$ and $u_0(x) = \sin(\pi x)$, $\Omega = (0, 1)$,
(c) $d = 2$, $f(u) = \sqrt{1 + u^2}$ and $u_0(x) = \sin(\pi x) \sin(\pi y)$, $\Omega = (0, 1)^2$,
(d) $d = 2$, $f(u) = u - u^3$ and $u_0(x) = x(1 - x)y(1 - y)$, $\Omega = (0, \pi)^2$.

The reference solutions are obtained by using small temporal stepsizes.
## Numerical result

- **Maximum errors at** $t = 1$ **and convergence orders in temporal direction for Example 2**

| $\alpha$ | case \( N \) | 10       | 20       | 40       | 80       | 160      | Rate | expected order |
|----------|---------------|----------|----------|----------|----------|----------|------|----------------|
| 0.4      | (a)           | 1.74e-4  | 8.36e-5  | 4.04e-5  | 1.94e-5  | 8.99e-6 | 1.06 | 1              |
|          | (b)           | 1.49e-3  | 7.14e-4  | 3.46e-4  | 1.66e-4  | 7.69e-5 | 1.06 | 1              |
|          | (c)           | 9.63e-5  | 4.77e-5  | 2.36e-5  | 1.16e-5  | 5.61e-6 | 1.03 | 1              |
|          | (d)           | 7.20e-6  | 3.57e-6  | 1.76e-6  | 8.67e-7  | 4.19e-7 | 1.02 | 1              |
| 0.6      | (a)           | 2.15e-4  | 1.01e-4  | 4.84e-5  | 2.30e-5  | 1.06e-5 | 1.08 | 1              |
|          | (b)           | 1.88e-3  | 8.84e-4  | 4.23e-4  | 2.01e-4  | 9.27e-5 | 1.08 | 1              |
|          | (c)           | 1.06e-4  | 5.21e-5  | 2.57e-5  | 1.26e-5  | 6.09e-6 | 1.03 | 1              |
|          | (d)           | 7.94e-6  | 3.92e-6  | 1.93e-6  | 9.47e-7  | 4.57e-7 | 1.03 | 1              |
| 0.8      | (a)           | 2.13e-4  | 9.53e-5  | 4.44e-5  | 2.08e-5  | 9.48e-6 | 1.10 | 1              |
|          | (b)           | 1.96e-3  | 8.74e-4  | 4.07e-5  | 1.90e-5  | 8.66e-6 | 1.12 | 1              |
|          | (c)           | 7.91e-5  | 3.88e-5  | 1.91e-5  | 9.31e-6  | 4.47e-6 | 1.04 | 1              |
|          | (d)           | 6.02e-6  | 2.95e-6  | 1.45e-6  | 7.06e-7  | 3.40e-7 | 1.03 | 1              |
### Numerical result

- **Maximum errors as** $t \to 0$ **and convergence orders in temporal direction for Example 2**

| $\alpha$ | case\$t_N$ | 1e-4  | 1e-5  | 1e-6  | 1e-7  | 1e-8  | *Rate* | expected order |
|-----------|-------------|-------|-------|-------|-------|-------|--------|----------------|
| 0.4       | (a)         | 4.70e-4 | 2.21e-5 | 9.00e-5 | 3.61e-5 | 1.45e-5 | 0.38   | $\alpha$       |
|           | (b)         | 3.65e-3 | 1.87e-3 | 8.36e-4 | 3.50e-4 | 1.43e-4 | 0.35   | $\alpha$       |
|           | (c)         | 4.96e-3 | 3.05e-3 | 1.47e-3 | 6.34e-4 | 2.61e-4 | 0.32   | $\alpha$       |
|           | (d)         | 3.31e-4 | 1.87e-4 | 8.26e-5 | 3.39e-5 | 1.37e-5 | 0.35   | $\alpha$       |
| 0.6       | (a)         | 1.20e-4 | 3.05e-5 | 7.70e-6 | 1.94e-6 | 4.87e-7 | 0.60   | $\alpha$       |
|           | (b)         | 1.14e-3 | 2.99e-4 | 7.59e-5 | 1.91e-5 | 4.81e-6 | 0.59   | $\alpha$       |
|           | (c)         | 2.03e-3 | 5.46e-4 | 1.39e-4 | 3.52e-5 | 8.84e-6 | 0.59   | $\alpha$       |
|           | (d)         | 1.11e-4 | 2.86e-5 | 7.23e-6 | 1.82e-6 | 4.57e-7 | 0.60   | $\alpha$       |
| 0.8       | (a)         | 1.73e-5 | 2.75e-6 | 4.35e-7 | 6.86e-8 | 1.09e-8 | 0.80   | $\alpha$       |
|           | (b)         | 1.63e-4 | 2.59e-5 | 4.10e-6 | 6.51e-7 | 1.03e-8 | 0.80   | $\alpha$       |
|           | (c)         | 3.06e-4 | 4.77e-5 | 7.56e-6 | 1.20e-6 | 1.90e-7 | 0.80   | $\alpha$       |
|           | (d)         | 1.55e-5 | 2.46e-6 | 3.90e-7 | 6.18e-8 | 9.80e-9 | 0.80   | $\alpha$       |
Our result

- establish a refine DFGI, take the weak regularity of solution into account.
- At $t = T$ the convergence order is $O(\tau^{\sigma+1-\alpha})$, while $0 < \sigma < 1$.
- At $t = T$ the convergence order is $O(\tau^{2-\alpha})$, while $\sigma > 1$.

Some open problems

- The pointwise error estimate on general nonuniform steps, under general assumption on nonlinearity or $f(u) = \kappa u$ with $\kappa > 0$.
- A refined and quantitative analysis of DCC kernels on general nonuniform steps.
- ...
Our recent results can be found in

- Dongfang Li, Hongyu Qin, Jiwei Zhang, Sharp pointwise-in-time error estimate of L1 scheme for nonlinear subdiffusion equations, J. Comput. Math, accepted.
- Dongfang Li, Mianfu She, Hai-wei Sun, A Novel Discrete Fractional Grönwall-Type Inequality and Its Application in Pointwise-in-Time Error Estimates, J. Sci. Comput. 2022.
Thank you!

Dongfang Li (李东方)
Email: dfli@hust.edu.cn