Dynamical-Space Regular-Time Lattice
and Induced Gravity

by

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ABSTRACT

It is proposed that gravity may arise in the low energy limit of a model of matter fields defined on a special kind of a dynamical random lattice. Time is discretized into regular intervals, whereas the discretization of space is random and dynamical. A triangulation is associated to each distribution of the spacetime points using the flat metric of the embedding space. We introduce a diffeomorphism invariant, bilinear scalar action, but no “pure gravity” action.

Evidence for the existence of a non-trivial continuum limit is provided by showing that the zero momentum scalar excitation has a finite energy in the limit of vanishing lattice spacing. Assuming the existence of localized low energy states which are described by a natural set of observables, we show that an effective curved metric will be induced dynamically. The components of the metric tensor are identified with quasi-local averages of certain microscopic properties of the quantum spacetime. The Planck scale is identified with the highest mass scale of the matter sector.
1. INTRODUCTION

One of the fundamental problems in high energy physics is the construction of a consistent quantum theory of gravity that will reduce at large distances to General Relativity. At the same time, such a quantum theory is expected to be free of the diseases that arise when a straightforward quantization of the classical theory is attempted.

A natural approach to the definition of Quantum Gravity [1-4] is to consider models which are based on some discretization of spacetime (see ref. [1] for a recent review). Over the last several years, important progress has been made in two dimensions where, using the Dynamical Triangulation (DT) approach [3, 4], it was possible to prove the existence of a continuum limit which reproduces known results on 2D gravity [5].

The basic idea of the DT approach, is to approximate $d$-dimensional Riemannian manifolds by triangulations made of equilateral $d$-simplices. The euclidean partition function is defined as a sum over all such $d$-dimensional triangulations weighted by some probability measure. Typically, the probability measure is written as the exponential of (minus) an action. The sum over triangulations is sometimes restricted to a fixed topology. We comment, however, that a complete topological classification in four dimensions is still an open problem.

The advantage of the DT approach is that general coordinate invariance is automatically satisfied, because the triangulations are defined without any reference to a coordinate system, and a unique discrete Riemannian structure can be assigned to each triangulation. Initially it was hoped that the successes of the DT approach in two dimensions could be repeated in four dimensions [4]. But gradually it was realized that there are great difficulties with the DT approach in four dimensions, which are related to the rapid growth in the number of triangulations as a function of $N_4$, the number of four-simplices.

In fact, recent results [6] suggest that the number of distinct four dimensional triangulations with the topology of a four-sphere is not bounded by an exponential function of $N_4$. This implies that the grand canonical partition function which involves a sum over $N_4$, does not exists for conventional probability measures even if the sum is restricted to triangulations with the topology of a four-sphere. The inclusion of an arbitrarily large bare cosmological constant in the action is not sufficient to compensate for the rapid growth in the number of triangulations. Another result which indicates that the space of four-triangulation is too big, is the algorithmic unrecognizability of four-triangulations discussed in [7].
The program of defining four dimensional Quantum Gravity is therefore still struggling with the basic issue of finding a consistent measure, and proving the existence of some critical point which can be identified with a sensible continuum limit.

If we ignore cosmological issues such as the expansion of the universe for the time being, the observed ground state of the universe is, to an extremely good accuracy, flat space. Moreover, the observed curved metric is essentially classical, and one cannot detect any effect which can be attributed to quantum fluctuations of the metric.

These simple observational facts should provide a guideline in the search of a consistent measure for Quantum Gravity. If the lack of progress with conventional measures is due to the fact that the space of four-triangulations is so huge, then what is needed is to truncate that huge space into a much smaller one. One would like a measure that enhances the contribution of smooth triangulations over the contribution of irregular ones. Moreover, to account for the classical character of the observed metric, the dominant configurations should resemble only a single smooth manifold, which would then be identified with the ground state.

Suppose that a measure with these good properties can be found. How unique would such a measure be? From our experience with quantum field theories on a regular lattice, we are used to the concept of universality. Namely, many lattice actions which differ in details have the same continuum limit, and the continuum limit can be characterized by a small number of renormalized parameters such as coupling constants and masses.

But logically there could be other possibilities. Since the space of all four-dimensional triangulations is so huge, the case could be that many different “good” measures can be defined on it. Each “good” measure would correspond to a different truncation of the space of four-triangulations, and would define a different, but consistent, theory of Quantum Gravity through its continuum limit. In particular, each measure would in general give rise to a different smooth manifold as the ground state.

In fact, we cannot exclude the possibility that the space of all consistent measures may be as big as the space of smooth manifolds which are candidates for the ground state. If this is true, then the ground state cannot be considered as a prediction of Quantum Gravity. Rather, under these circumstances, the desired ground state would have to be prescribed, and would have to be supplied as an additional information in some way, when selecting the appropriate measure for a phenomenologically viable model of Quantum Gravity.

There is a simple way to enhance the contribution of triangulations that resemble a prescribed smooth four-manifold which we henceforth denote as the target manifold.
One can allow only for those triangulations which can be embedded in the target manifold. This can be realized by demanding that all the vertices of the triangulation should belong to the target manifold, and that linking should be performed using some prescription which makes use of the geodesic distance between points in the target manifold. The partition function’s measure is then defined as an integral over the positions of all vertices in the target manifold.

As long as the action is diffeomorphism invariant, in the sense that it depends only on the connectivity of the resulting triangulations (but not on the distances between vertices in the target manifold) the partition function will define an effective measure on the space of triangulations and, hence, some theory of gravity. Apart from restricting the sum to a single topology, the virtue of this prescription is that it assigns to each admissible triangulation an entropy factor which significantly enhances the contribution of regular triangulations. By this we mean that, as in classical statistical mechanics, one can define thermodynamical fields such as the local density of points, and on the dominant triangulations these fields will fluctuate very little around their mean values throughout the entire target space.

Moreover, if the target manifold is a symmetric space, then there is a good chance that the ground state of the quantum theory will reproduce that same space. Consider for example flat space as the target manifold. A partition function constructed alone the lines described above will be invariant under continuous global translations. If this global symmetry remains unbroken (or if it is broken spontaneously leaving a sufficiently large unbroken discrete subgroup) the ground state found by low energy observers will be flat space.

In this paper we consider a realization of the above ideas, taking the target manifold to be flat space. (To define the finite volume partition function we compactify flat space to a flat four-torus). The measure of the euclidean partition function is an integral over the positions of a fixed number of points in the four-torus, where in the continuum limit the density of points should tend to infinity. To each distribution of points we associate a triangulation using a prescription which involves the flat norm on the four-torus.

The plan of this paper is as follows. In sect. 2 we define the model and establish some of its basic properties. The first guideline for the definition of the model has already been discussed above. This is the requirement that the discretized spacetime structure is embedded in a fixed target manifold which we take to be a flat four-torus.

Our second requirement is that any matter action that we introduce should depend only on the connectivity of the spacetime triangulations. Moreover, we do not introduce any “pure gravity” action. The reason is that our spacetime measure already
induces an effective pure gravity action.

The last feature is that we treat time in a different way from space. Time is discretized into regular intervals, and only the discretization of space is random and dynamical. We denote the resulting framework as Dynamical-Space Regular-Time lattice (or DSRT lattice for short). The special role of time allows us to prove the existence of a euclidean transfer matrix and, hence, to prove unitarity of the time evolution in the continuum limit. The reasons for the uneven treatment of space and time will be clarified later. In short, we will argue that if one expects to recover General Relativity at sufficiently low energies, then the special role of time is not only a sufficient condition, but actually a necessary condition for consistency of the continuum limit.

In this paper we discuss only the simplest possible matter action. This action describes a scalar field without self-interaction. By deriving the Ward identity that corresponds to the broken shift symmetry of the scalar action, we prove that the zero momentum scalar excitation has a finite energy in the limit of vanishingly small lattice spacing. This provides a first indication that the model has a non-trivial continuum limit. Strictly speaking, based on this result alone one cannot rule out the possibility that the continuum limit might describe a system with a finite number of states. But we believe that similar techniques, together with variational estimates, can be used to show that the zero momentum state is the end point of a continuous spectrum.

In sect. 3 we investigate the feasibility that the continuum limit of the DSRT model contains the familiar gravitational interaction. A preliminary requisite is the construction of an appropriate set of observables. There are well known difficulties associated with the definition of local observables in gravitational theories, which we will not repeat here.

In our approach, the embedding in the target manifold provides an additional structure, which can be used in the definition of local observables. The quantities which parametrize multi-local observables are taken to be the coordinates of the \( n \) points in the target manifold. The classical continuum limit of the matter action has a natural interpretation for such observables, and it allows us to identify the (inverse) curved metric tensor with thermodynamical fields that describe quasi-local averages of certain microscopic properties of spacetime.

Our definition of observables and the ensuing identification of the curved metric tensor with thermodynamical fields has important advantages, which are discussed in subsect. 3.2. The price is that we introduce dependence on a coordinate system. We argue that, by introducing an integration over a small subset of the diffeomorphism group, one recovers general coordinate invariance of the equation of motion in the
continuum limit.

The rest of sect. 3 is devoted to a discussion of the resulting dynamics of the gravitation field. In particular, we argue that the Planck scale should be identified with the largest mass scale of the matter sector. Other issues that we address include the nature of the effective theory above the Planck scale, and recovery of Lorentz covariance in the low energy limit.

Some concluding remarks are given in sect. 4. In particular, we offer several research directions to test key features of the DSRT model.

2. THE DYNAMICAL-SPACE REGULAR-TIME LATTICE

The formulation of quantum field theories on a Dynamical-Space Regular-Time (DSRT) lattice is developed in this section. The random lattice points are taken from a fixed underlying flat space compactified to a four torus. Time is discretized into regular intervals, whereas the discretization of space is random and dynamical. We assume a diffeomorphism invariant matter action. In this paper we limit ourselves to the simplest case where the matter action is bilinear in a set of scalar fields.

The advantage of the DSRT lattice compared to the alternative approach in which space and time are treated on equal footing, in that one can establish rigorously certain fundamental properties which are expected to hold in a consistent quantum field theory. The special role of time allows us, using standard methods [8], to define a Hilbert space of observables and to construct a bounded, positive definite transfer matrix. This guarantees unitarity of the time evolution in the continuum limit. The price paid is that restoration of full Lorentz covariance is not automatic.

We discuss the properties of the ground state. A simple entropy argument provides evidence that the ground state should be homogeneous when probed on a sufficiently large distance scales, and that the ground state found by a low energy observer is flat space. We next discuss the properties of the zero momentum scalar excitation. We prove that this is a stable excitation, whose energy is exactly equal to the mass parameter in the scalar action. This provides a first indication for the existence of a non-trivial continuum limit. Finally, we show that spacetime fluctuations induce an attractive interaction among matter fields.

2.1 The partition function

In order to avoid excessive notation we define here the partition function for a DSRT lattice model containing a single scalar field. (The generalization to several scalar fields is trivial). We will assume the absence of scalar self-interactions.
The canonical partition function depends on the following parameters: $N$ is the total number of lattice points on every time slice and $L^3$ is the space volume which we take to be a large cube. The discrete time coordinate $t$ takes the values $t = -N_0, -N_0 + 1, \ldots, N_0$, where the time slices $t = \pm N_0$ are identified. Periodic boundary conditions are assumed in the space directions too. The time interval between two neighbouring time slices is $a_0$. Finally, $M$ is the mass of the scalar field. We define the mean distance between space points through the relation
\[ \bar{a} = L/N^\frac{3}{4}, \]
\[ i.e. \ 1/\bar{a}^3 \text{ is the average spatial density of lattice points.} \]

The partition function is
\[ Z = Z(N_0, a_0, N, L, M) = \int_{n, t} \mathcal{D}y \int_{n, t} \mathcal{D}\Phi e^{-S}. \]
Our notation conventions will be as follows. The subscripts $n, m = 1, \ldots, N$ label the space points on every time slice, whereas the superscripts $i, j, k, l = 1, 2, 3$ denote space coordinates. Thus, $y_{n,t}^i$ is the $i$-th space coordinates of the $n$-th point on the time slice $t$, and $\Phi_{n,t}$ is the value of the real scalar field residing at this point. In order to avoid cumbersome expressions we use the shorthands
\[ \int_{n,t} \mathcal{D}y \equiv \prod_{t = -N_0}^{N_0 - 1} \int_{n} \mathcal{D}y_{t}, \]
\[ \int_{n} \mathcal{D}y_{t} \equiv L^{-3N} \prod_{n=1}^{N} \int_{-L/2}^{L/2} d^3 y_{n,t}, \]
and
\[ \int_{n,t} \mathcal{D}\Phi \equiv \prod_{t = -N_0}^{N_0 - 1} \int_{n} \mathcal{D}\Phi_{t}, \]
\[ \int_{n} \mathcal{D}\Phi_{t} \equiv \left( \frac{a_0 \bar{a}^3 M^2}{2\pi} \right)^\frac{N}{2} \prod_{n=1}^{N} \int_{-\infty}^{\infty} d\Phi_{n,t}. \]
The spacetime measure has been normalized to unity $\int_{n,t} \mathcal{D}y = 1$. The choice of normalization for the scalar measure is a matter of convenience. Using the action (2.5) below, one has $0 < Z < 1$.

For every time slice (i.e. for every copy of space) the measure describes an integration over the locations of $N$ points in a flat three-torus. The three-torus is the cube $L^3$ with opposite faces identified, and it is equipped with the flat norm inherited from $R^{(3)}$. The location of each point is defined by its flat coordinates $y_{n,t}^i$ and the distance
between any pair of points in the flat three-torus norm can easily be determined from these coordinates.

Because of the special role of time, the triangulations we build are not made of four-simplices. Instead, on every time slice we build a triangulation made of tetrahedra, whereas the temporal links provide a one-to-one mapping between the points on two neighbouring time slices.

In more detail, for a given set of space points \( \{y_1^t, \ldots, y_N^t\} \) a triangulation is constructed on every time slice using the “canonical” prescription described for example in ref. [9]. According to this prescription, four space points are considered the vertices of a tetrahedron, and every pair of them is joined by a link, provided there are no other points inside their circumscribed sphere. This prescription assigns a unique triangulation to a given set of space points, except for degenerate cases that have zero measure.

In addition, every point on the time slice \( t \) is connected to a single point on each of the two neighbouring time slices \( t \pm 1 \). The linking between the time slices \( t \) and \( t+1 \) is described by a one-to-one mapping \( \tau: (n, t) \to (m, t+1) \), where \( \tau((n, t)) = (m, t+1) \) if and only if the two points \( y_{n,t} \) and \( y_{m,t+1} \) are connected by a link. The linking is determined by requiring that the mapping \( \tau \) minimize the average displacement squared

\[
N^{-1} \sum_{n=1}^{N} \left| y_{\tau(n,t)} - y_{n,t} \right|^2.
\]

(2.3)

On the r.h.s. of eq. (2.3) distances are measured in the flat three-torus norm, and we have made use of the natural identification between the time slices \( t \) and \( t+1 \).

Successive applications of the mapping \( \tau \) define a “world line” for each space point. We will describe the (imaginary) time evolution of a given point by

\[
Y_{n,t}^i = y_{\tau(n,t)}^i = y_{m,t+\tau t + N_0(n,-N_0)}^i.
\]

(2.4)

We denote by \( \Phi(Y_{n,t}) \) the value of the scalar field at the point \( Y_{n,t} \). Hence, \( \Phi(Y_{n,t}) = \Phi_{m,t} \) if \( \tau t + N_0(n,-N_0) = (m,t) \). Notice that, in spite of the use of periodic boundary conditions, \( \tau 2N_0(n,-N_0) \) will in general be different from \( (n,-N_0) \).

While the prescription (2.3) is clearly intended to prevent the occurrence of very big displacements in the continuum limit, this prescription is non-local in space, in the sense that it relies on information about the entire distribution of points on the two neighbouring time slices. By contrast, the triangulation prescription on every time slice is strictly local.

The non-locality in space of the time linking prescription is in fact unavoidable, if we want to prevent the occurrence of arbitrarily big displacements. It arises because
the mapping between the points of two time slices is one-to-one. To see this, imagine that we take a pair of points which are connected by a time link, and we move one of them to a new location which is very far away. In order to prevent the occurrence of macroscopically big displacements in the new mapping, relinking has to take place over a connected region which covers both the old location and the new one.

Whether or not the non-locality of the time linking prescription leaves a detectable trace in the continuum limit, is a question that will have to be investigated in the future. We insist on this prescription because, as argued in subsect. 3.1 below, consistency of our identification of the dynamical curved metric tensor requires that the fluctuations of world lines should obey a controlled bound.

Having completed the construction of the \textit{spacetime} triangulation that corresponds to given sets of space points on every time slice, we can now define the matter action. The scalar action is given by

\[ S = a_0 \sum_{t=-N_0}^{N_0-1} \mathcal{L}_t, \quad (2.5) \]

where, suppressing the \( t \)-dependence,

\[ \mathcal{L} = \frac{\bar{a}^3}{2} \sum_n \Phi_n^2 + \frac{\bar{a}}{2} \sum_{(mn)} (\Phi_m - \Phi_n)^2 + \frac{M^2}{2} \sum_n \Phi_n^2. \quad (2.6) \]

In eq. (2.6) the second term is a sum over the spatial links of the triangulation, and the third term is a sum over its vertices. The first term is a sum over the temporal links, and it is the only one which depends on a knowledge of the world line \( Y_{n,t} \). For the coordinates of the space points, we define the discrete time derivative as

\[ \dot{Y}_{n,t}^i = a_0^{-1} (Y_{n,t+1}^i - Y_{n,t}^i), \quad (2.7) \]

and for the scalar field we let

\[ \dot{\Phi}_n = a_0^{-1} (\Phi(Y_{n,t+1}) - \Phi(Y_{n,t})). \quad (2.8) \]

Notice that, while each triangulation is constructed using the flat norm of the target space, the matter action is diffeomorphism invariant. Namely, it depends only on the connectivity of the triangulations, but not on the metric properties of their embedding in the target space.
2.2 The Hilbert space of observables and the transfer matrix

In the previous subsection we defined the partition function of scalar matter on a DSRT lattice. Starting from the partition function $Z$ we now want to reconstruct a \textit{Hilbert space of observables} and a \textit{bounded, symmetric and positive definite transfer matrix}.

Our definition of local observables relies on the embedding of the discrete space-time structure in $L^3 \times Z^{(2N_0)}$. The time $t$ observables are defined as

$$O(y_{n,t}, \Phi_{n,t}; f, x^i) = f(\Phi_t(\epsilon, x^i)),$$

where $f(\Phi)$ is a real polynomial and

$$\Phi_t(\epsilon, x^i) = \frac{3a^3}{4\pi \epsilon^3} \sum_{n \in D(x; \epsilon)} \Phi_{n,t}.$$  

(2.10)

In eq. (2.10), $D(x; \epsilon)$ is the ball of radius $\epsilon$ centered at $x^i \in R^3$, and $n \in D(x; \epsilon)$ is a shorthand for summation over all the time $t$ space points that belong to $D(x; \epsilon)$.

The above equations define observables which are local in both space and time. In a similar way one can define multi-local observables, as well as observables that involve the discrete time derivative of the scalar field (2.8). One can also consider observables that depend explicitly on $y_{n,t}$.

We define the Hilbert space of observables to be the linear space of all observables which depend only on dynamical variables with $0 \leq t \leq N_0$. In this Hilbert space we define a real inner product by

$$\langle O | O' \rangle \equiv \int_{n,t} D y \int_{n,t} D \Phi e^{-S(\Theta O)} O',$$

(2.11)

where

$$\Theta O(y_{n,t}, \Phi_{n,t}; f; \epsilon_1, x_1; \epsilon_2, x_2; \ldots) = O(y_{n,-t}, \Phi_{n,-t}; f; \epsilon_1, x_1; \epsilon_2, x_2; \ldots).$$

(2.12)

In order to verify that the induced norm is strictly positive we have to demonstrate reflection positivity around the $t = 0$ plane. Namely, we have to prove that the r.h.s. of eq. (2.11) is strictly positive for $O' = O$. This property will be established below.

We next turn to the construction of the transfer matrix. Let $\hat{H}$ be the Time Zero Hilbert space, defined as the linear space of all time zero observables

$$O(y_n, \Phi_n; f; \epsilon_1, x_1; \epsilon_2, x_2; \ldots)$$

where now

$$f(\Phi(\epsilon_1, x_1), \Phi(\epsilon_2, x_2), \ldots)$$
is a square integrable function with respect to the real inner product

\[
\langle \mathcal{O} | \mathcal{O}' \rangle_0 = \int_n \mathcal{D}y \int_n \mathcal{D}\Phi \mathcal{O}(y_n, \Phi_n) \mathcal{O}'(y_n, \Phi_n).
\] (2.13)

The transfer matrix is a mapping of \( \hat{\mathcal{H}} \) into itself. For a given element of \( \hat{\mathcal{H}} \), we define

\[
(\mathcal{T} \mathcal{O})(y'_m, \Phi'_m) = \int_n \mathcal{D}y \int_n \mathcal{D}\Phi \mathcal{K}(y'_m, \Phi'_m; y_n, \Phi_n) \mathcal{O}(y_n, \Phi_n).
\] (2.14)

The integral kernel is given by

\[
\mathcal{K}(y'_m, \Phi'_m; y_n, \Phi_n) = \mathcal{K}_1(y'_m, \Phi'_m) \mathcal{K}_2(y'_m, \Phi'_m; y_n, \Phi_n) \mathcal{K}_1(y_n, \Phi_n),
\] (2.15)

where

\[
\mathcal{K}_1(y_n, \Phi_n) = \exp \left\{ -\frac{\bar{a}a_0}{4} \sum_{(mn)} (\Phi_m - \Phi_n)^2 - \frac{M^2\bar{a}^3a_0}{4} \sum_{n} \Phi_n^2 \right\},
\] (2.16)

and

\[
\mathcal{K}_2(y'_m, \Phi'_m; y_n, \Phi_n) = \exp \left\{ -\frac{\bar{a}^3}{2a_0} \sum_{m,n} (\Phi'_m - \Phi_n)^2 \delta_{n,T(m)} \right\}.
\] (2.17)

In eq. (2.17) the mapping \( \tau \) between the primed and unprimed space points is defined analogous to eq. (2.3). We have explicitly denoted the dependence of the observable \( \mathcal{O} \) on the dynamical variables. Notice that the complete set of space points has to be included in the definition of the observable, otherwise the mapping \( \tau \) which enters the integral kernel \( \mathcal{K}_2 \) is undefined. This is a manifestation of the non-locality in space of the time linking prescription.

The transfer matrix is a manifestly bounded operator. It is also symmetric, because the minimization prescription (2.3) which defines the mapping \( \tau \) is symmetric with respect to the two time slices. However, the integral kernel \( \mathcal{K}_2 \) is symmetric only up to a permutation of one of the indices \( m \) or \( n \). Since this permutation depends on the coordinates of the space points, we are unable to “take the square root of \( \mathcal{K}_2 \)”, and so we are unable to prove the positivity of the one step transfer matrix.

Instead, we will now show that the two steps transfer matrix \( \mathcal{T}^2 \) is strictly positive. The same reasoning will also prove reflection positivity around the \( t = 0 \) hyperplane. In order to prove the positivity of \( \mathcal{T}^2 \), it is sufficient to prove

\[
\langle \mathcal{O} | \mathcal{T}^2 \mathcal{O} \rangle_0 > 0,
\] (2.18)

for all non-zero operators \( \mathcal{O} \in \hat{\mathcal{H}} \). The validity of inequality (2.18) follows trivially by using \( \mathcal{T}^\perp = \mathcal{T} \). For the reader who is worried that this compact notation could be hiding some subtlety, we now explain in detail how the positivity of \( \mathcal{T}^2 \) can be established directly from its integral representation.
Thinking of the variables \((y_n, \Phi_n)\) and \((y'_m, \Phi'_m)\) as living on the Time Zero and Time One slices respectively, let us introduce a third set of variables \((y''_p, \Phi''_p)\) that lives on the Time Two slice. \(\mathcal{T}^2\) is represented by integrating over both \((y_n, \Phi_n)\) and \((y'_m, \Phi'_m)\) using the integral kernel. Now, for the action of \(\mathcal{T}\) between the Time Zero and Time One slices we simply use eq. (2.17) for the integral kernel \(\mathcal{K}_2(y'_m, \Phi'_m; y_n, \Phi_n)\). This integral kernel contains a factor \(\delta_{n,\tau(m)}\). For the action of \(\mathcal{T}\) between the Time One and Time Two slices we have \(\mathcal{K}_2(y''_p, \Phi''_p; y'_m, \Phi'_m)\) but with a factor \(\delta_{m,\bar{\tau}(p)}\). We distinguish between the permutations \(\tau\) and \(\bar{\tau}\). Notice that \(m = \bar{\tau}(p)\) is the mapping from the Time Two slice to the Time One slice, while \(n = \tau(m)\) is the mapping from the Time One to the Time Zero slice. Rewriting \(\delta_{m,\bar{\tau}(p)} = \delta_{p,\bar{\tau}^{-1}(m)}\), we observe that \(p = \bar{\tau}^{-1}(m)\) is the mapping from the Time One to the Time Two slice. The permutation \(\tau = \tau(y_n, y'_m)\) is determined by the coordinates of the Time Zero and Time One points. The permutation \(\bar{\tau}^{-1} = \bar{\tau}^{-1}(y''_p, y'_m)\) is determined by the coordinates of the Time Two and Time One points. We thus see that \(\tau\) and \(\bar{\tau}^{-1}\) are the same function of their ordered set of arguments. As a result, the integral representation of \(((\mathcal{O}|\mathcal{T}^2\mathcal{O})_0\) is equal to the integral representation of \(((\mathcal{O}|\mathcal{T}^2\mathcal{O})_0\). This proves explicitly both the symmetry of the operator \(\mathcal{T}\) and the positivity of \(\mathcal{T}^2\).

As expected, the partition function is \(Z = \text{Tr} \mathcal{T}^{2N_0}\), and in the continuous time limit we can defined the hamiltonian through \(H = \lim_{a_0 \to 0} :H(a_0):\) where
\[
H(a_0) = -1/(2a_0) \log \mathcal{T}^2.
\] (2.19)
The normal ordering symbol stands for a subtraction of the bare vacuum energy density.

2.3 The ground state

We now proceed to discuss the properties of ground state. Let us first examine the global spacetime symmetries of the DSRT lattice. Thanks to the choice of periodic boundary conditions, the partition function \(Z\) is manifestly invariant under discrete time and continuous space translations. In addition, the finite volume partition function is invariant under a discrete subgroup of space rotations. (In the infinite volume limit full rotation symmetry is recovered). If none of these symmetries are spontaneously broken, then the ground state will be flat space.

In order to tell whether or not translation invariance is broken in some region of the phase diagram of the model, one has to carry out a detailed dynamical calculation. But at this stage we do not really care whether the ground state is exactly translationally invariant. What we care about is whether the ground state as probed by a low energy measurement cannot be distinguished from a homogeneous, continuous flat space. For example, an interesting possibility is that solidification may take
place [11], resulting in a regular crystal-like structure of the ground state. But if the cell of that crystal is of Planck size, then low energy excitations will be blind to the existence of that microscopic structure, and the ground state of the low energy effective theory will be ordinary flat space.

What we should exclude is the possibility that all space points form an aggregate which occupies only a small fraction of the available volume. But this possibility can be ruled out by a standard entropy argument. (The same argument explains why the density of gas molecules is constant throughout the container which holds them). Suppose that ground state were an aggregate whose size is smaller than $L/2$. The relative phase space for such configurations is $2^{3/2}N$, where the numerator counts the eight different regions in which the aggregate may be localized. Moreover, since the matter action is diffeomorphism invariant, the change in the matter free energy that arises from confining all space points to a small region will be negligible. The relevant triangulations will be distinguishable from ones which occupy evenly the entire volume only by surface effects which, in turn, will be suppressed in the limit of large $N$. The leading $N$-dependence of the effective measure for all such configurations therefore goes like $2^{-N}$, and so they are completely suppressed in the continuum limit.

A more exotic possibility is that some structure may form which occupies the entire volume, but whose unit cell has, say, a fivefold symmetry. In this case, although the density of points will be approximately constant, the ground state would not be invariant under a discrete translation group. We are unable to make further comments on this situation at the moment, except to express hope that this will not turn out to be the generic situation.

Finally, we note that even if none of the spacetime symmetries of the DSRT lattice are spontaneously broken, this does not yet guarantee recovery of full Lorentz covariance. This is the price that we have to pay for the special role of time. We return to this issue in Sect. 3.

2.4 The zero momentum excitation

Being a lattice model, the continuum limit of the partition function eq. (2.2) should correspond to $a_0 M \to 0$ and $\bar{a} M \to 0$, with the product $N_0 a_0$ held fixed. The precise behaviour of the ratio $\bar{a}/a_0$ in the continuum limit will be discussed in Sect. 3. We will now prove the following statement. In the limit $a_0 M \to 0$ and regardless of the value of $\bar{a} M$, the model defined by the partition function (2.2) and the action (2.5) has a stable zero momentum one particle excitation, whose energy is given by $E = M$ exactly.

The proof is based on the bilinearity of the lagrangian (2.6) and the fact that,
for $M = 0$, the scalar action is invariant under a global shift symmetry

$$
\Phi_{n,t} \rightarrow \Phi_{n,t} + \alpha,
$$

(2.20)

where $\alpha$ is a constant. Notice that the measure is invariant under the local shift

$$
\Phi_{n,t} \rightarrow \Phi_{n,t} + \alpha_{n,t}.
$$

(2.21)

We proceed by writing down the Ward identity that corresponds to the broken shift symmetry. Starting from the expectation value of $\Phi_t(\epsilon, x^i)$ eq. (2.10), we perform the local change of variables (2.21). The resulting Ward identity is

$$
\langle \delta S \Phi_t(\epsilon, x^i) \rangle = \langle \delta \Phi_t(\epsilon, x^i) \rangle,
$$

(2.22)

where

$$
\delta \Phi_t(\epsilon, x^i) = \frac{3\bar{a}^3}{4\pi \epsilon^3} \sum_{n \in D(x, \epsilon)} \alpha_{n,t}.
$$

(2.23)

The local variation of the action is

$$
\delta S = a_0 \sum_{t=-N_0}^{N_0-1} \delta L_t,
$$

(2.24)

where

$$
\delta L = \bar{a}^3 \sum_n \hat{\alpha}_n \hat{\Phi}_n \\
+ \bar{a} \sum_{\langle mn \rangle} (\alpha_m - \alpha_n) (\Phi_m - \Phi_n) \\
+ M^2 \bar{a}^3 \sum_n \alpha_n \Phi_n.
$$

(2.25)

The discrete time derivative $\hat{\alpha}_n$ is defined analogous to eq. (2.8).

The Ward identity (2.22) is not directly useful, because in order to evaluate it one has to specify the values of the local function $\alpha_{n,t}$ in some reasonable way, and to carry out the integrations over all dynamical variables. But there is one case where the computation simplifies considerably, and all integrations can be carried out exactly.

We now assume that $\alpha_{n,t} = \alpha_t$ is a function of $t$ only. Moreover, we replace the local operator $\Phi_t(\epsilon, x^i)$ by $\hat{\Phi}_0$, where $\hat{\Phi}_t$ is the zero momentum projection

$$
\hat{\Phi}_t = N^{-1} \sum_n \Phi_{n,t},
$$

(2.26)

evaluated at time $t$. For this particular choice, the second term on the r.h.s. of eq. (2.25) vanishes, and the Ward identity simplifies to

$$
\sum_{t=-N_0}^{N_0-1} \left\{ \hat{\alpha}_t (G_{t+1} - G_t) + a_0 M^2 \alpha_t G_t \right\} = \alpha_0.
$$

(2.27)
Here $G_t$ is the correlator

\[ G_t = L^3 \langle \Phi_t \Phi_0 \rangle . \]  

(2.28)

We now let $\alpha_t = \exp(-i\omega a_0 t)$, where $\omega$ is one of the allowed frequencies of the time lattice. We find

\[ \left( \frac{4}{a_0^2} \sin^2(a_0 \omega/2) + M^2 \right) G(\omega) = 1 , \]  

(2.29)

where

\[ G(\omega) = a_0 \sum_{t=-N_0}^{N_0-1} e^{-i\omega a_0 t} G_t . \]  

(2.30)

Eq. (2.29) is an exact expression for the zero momentum correlator. Taking the limit $a_0 \to 0$ and analytically continuing to Minkowski space we find that the zero momentum propagator has a pole at $E = M$. We have thus arrived at the remarkable result, that the parameter $M$ in the lagrangian (2.6) is the exact energy of the zero momentum scalar excitation. We also notice that this is a stable excitation, because the action is invariant under the discrete transformation $\Phi \to -\Phi$, and so the number of $\Phi$-particles is conserved modulo two.

This result provides the first indication that the DSRT model possesses a non-trivial continuum limit. More detailed analysis will be needed in order to prove that the zero momentum excitation is the end point of a continuous spectrum, and not just an isolated state. This might be achieved by the application of the above Ward identity to localized states, together with the use of some variational estimates.

We would like to stress, however, that even the present result contains some non-trivial information about the dynamics of the model. The relation $E = M$ implies that the zero momentum component of the scalar field decouples from spacetime dynamics. Showing that the zero momentum state is the end point of a continuous spectrum amounts to showing that this decoupling occurs gradually as the scalar state becomes more and more delocalized. But apriori it is not at all obvious that even the zero momentum state should decouple from the dynamics of the random lattice.

To illustrate how things could be different let us consider, instead of the diffeomorphism invariant lagrangian (2.6), an action of the kind studied in ref. [9]. There, the aim was to provide a non-perturbative definition of quantum field theories in flat space by using a dynamical random lattice. The guiding principle in the construction of the action, was that its classical continuum limit should coincide with the corresponding flat space continuum action. Such an action must depend explicitly on the embedding of the triangulations in the target space. For example, one has to account for the length of each link and for the volume of cells in the dual lattice. Thus, while
the action is still bilinear in the scalar field, it has terms that depend in a complicated non-polynomial way on the coordinates of the spacetime points. Since that action is not bilinear in the dynamical degrees of freedom, one cannot prove the existence of a non-trivial continuum limit using the present approach. In fact, it is not clear at all that a non-trivial continuum limit exists in that model.

2.5 The spacetime-induced interaction

Before we turn to discuss how gravity may show up in the continuum limit of the DSRT model, there is one more interesting result which can be proved rigorously. To this end, we compare the partition function (2.2) of the one scalar model, with the partition function of a two scalars model

\[
Z(2) = \int_{n,t} \mathcal{D}y \int_{n,t} \mathcal{D}\Phi \int_{n,t} \mathcal{D}\Phi' e^{-S(\Phi) - S(\Phi')}.
\]  

(2.31)

The action \(S\) is still given by eqs. (2.5) and (2.6). The partition function (2.31) describes two scalar fields which are uncoupled accept possibly through the spacetime dynamics. We now claim that, in fact, spacetime fluctuation induce an attractive interaction among the scalar excitations.

The partition function (2.2) defines an average of the matter free energy

\[
e^{-F(y_{n,t})} = \int_{n,t} \mathcal{D}\Phi e^{-S(\Phi)},
\]  

(2.32)

with respect to the normalized spacetime measure \(\int_{n,t} \mathcal{D}y\). Let us denote this average by

\[
Z = \langle e^{-F} \rangle.
\]  

(2.33)

It is easy to see that the two fields partition function is

\[
Z(2) = \langle e^{-2F} \rangle.
\]  

(2.34)

Since the free energy has some dependence on the positions of the spacetime points, the averaging is non-trivial and we obtain

\[
\langle e^{-2F} \rangle > \langle e^{-F} \rangle^2.
\]  

(2.35)

Let us introduce a dimensionless vacuum energy density through

\[
\mathcal{E} = -1/(NN_0) \log Z,
\]  

(2.36)

Analogous definition applies to \(\mathcal{E}(2)\), the vacuum energy density of the two scalars model. Eq. (2.35) implies

\[
\mathcal{E}(2) < 2\mathcal{E}.
\]  

(2.37)
Eq. (2.37) is the main result of this subsection. If spacetime fluctuations did not induce any interaction among the matter fields, we would expect that $\mathcal{E}(2)$ should be equal to $2\mathcal{E}$, as in the case of two uncoupled scalar fields on a regular lattice. From the fact that $\mathcal{E}(2)$ is actually lower than $2\mathcal{E}$, we learn that spacetime fluctuations induce an attractive interaction among matter excitation.

This result is in fact very general. The one condition needed to establish inequality (2.35) is that the action should depend only on translationally invariant properties of the spacetime triangulations. Thus, it applies to a much more general class of actions and not only to diffeomorphism invariant ones. While it is encouraging to find that the generic spacetime-induced interaction is attractive, one expects that the details of that attractive interaction should be model dependent. One needs a more quantitative information about that interaction in order to tell for which models (if any) it gives rise to a long range force in the continuum limit.

3. INDUCED GRAVITY

In this section we investigate the feasibility of obtaining the familiar gravitational interaction in the continuum limit of the DSRT model. We begin in subsec. 3.1 by calculating the classical continuum limit of the scalar action. To facilitate this calculation, we assume that the following relation

$$\Phi^{\lambda}_{n,t} = \Phi^{\lambda'}(x, t)\big|_{x=y_{n,t}}, \quad \lambda \ll 1,$$  \hspace{1cm} (3.1)

holds between low energy eigenstates $\Phi^{\lambda}_{n,t}$ of the discrete action, and smooth continuum wave functions $\Phi^{\lambda'}(x, t)$. Here $\lambda$ and $\lambda'$ denote respectively the eigenvalues of the discrete and continuum actions, and what will be described below can be thought of as a self-consistent procedure to determine $\Phi^{\lambda'}(x, t)$.

In order to describe the dynamics of the model, we introduced in sect. 2 a set of observables that depend on the embedding of the discrete spacetime structure in the target flat space. If this is an adequate set of observables, then it should be possible to extract the dynamical curved metric felt by low energy observers by enforcing the correspondence (3.1) on the discrete matter action.

Taking the classical continuum limit of the matter action necessitates the introduction of two thermodynamical fields, a symmetric tensor and a field that describes the local density of points. We recover the continuum action of a scalar field in a gravitational background, where the symmetric tensor plays the role of the inverse curved metric, and provided one can identify the density field with the spatial volume element of the curved metric. Whether or not the fluctuations of the local point
density coincide with the fluctuations of the volume element of the curved metric tensor, is a dynamical question. Later we will argue that this should be the case in the continuum limit.

Because of the preferred role of time, only inverse metrics with $g^{00} = 1$ are attainable in the DSRT model. From the point of view of the curved space effective theory at low energies, this can be interpreted as partial gauge fixing. Another property which can be interpreted that way, is the local conservation law of space points (see subsect. 3.4).

The partition function of the DSRT model defines an effective measure for the gravitational (and density) fields. In subsect. 3.2 we introduce an improved spacetime measure which involves an integration over a small subset of the diffeomorphism group. We then argue that, in the continuum limit, the effective gravitational measure reduces to the product of a gauge fixing condition which is natural for the model, times the exponential of an effective action which is invariant under a subgroup of infinitesimal general coordinate transformations (eq. (3.17) below). This invariance is sufficient to ensure the masslessness of the graviton.

The rest of this section is devoted to a discussion of the expected properties of the effective gravitational action. While the discussion is rather heuristic at this stage, it indicates that satisfactory solutions to key issues in Quantum Gravity are feasible in the DSRT model. Of course, only detailed investigations will ultimately tell whether or not the physical picture that we present here is correct. But one cannot carry out any detailed study of the model without first having in mind some idea of what one is looking for.

In subsect. 3.3 we show that the effective action cannot contain a cosmological term. Next, we argue that the Planck scale should be identified with the largest mass of the matter sector. We propose the name pregraviton to denote any ordinary particle of Planckian mass. In our model, gravitons are excitations of the quantum spacetime, whose size is determined by the Compton wave length of the pregraviton. We discuss the possibility that perturbative processes above the Planck scale, but still much below the cutoff scale, may be described by an asymptotically free, curvature squared continuum theory [11]. Finally, in subsect. 3.4 we discuss in what ways, and under what circumstances, the physics of the DSRT model may differ from General Relativity.
3.1 The classical continuum limit of the matter lagrangian

Because time is discretized into regular intervals, taking the continuous time limit of the summation on the r.h.s. of eq. (2.5) is trivial. What we have to do, is to calculate the classical continuum limit of the lagrangian (2.6) by enforcing the correspondence (3.1) on the scalar field’s configurations. On the r.h.s. of eq. (3.1), the coordinates $x$ and $t$ take values in the flat four-torus $L^3 \times [0, 2a_0 N_0]$. As we will see, an effective curved metric is induced dynamically in that flat space.

In deriving the classical continuum limit we introduce a coarse graining scale $l$. We will assume that $\Phi^\lambda(x, t)$ is slowly varying inside a ball of radius $l$. To justify a thermodynamical treatment of the spacetime triangulations, we demand that $l \gg \max\{a_0, \bar{a}\}$. Since $l$ is arbitrary, we want that a change in $l$ will amount to finite renormalization of the parameters of the effective low energy theory. To make this possible, we assume that $l$ is much smaller then any physical distance scale in the theory.

In the continuum limit both $a_0M$ and $\bar{a}M$ should tend to zero. As we will see, requiring consistency of the analytic continuation to Minkowski space, implies that in the continuum limit the ratio $\bar{a}/a_0$ should tend to zero too.

We begin with the mass term which is the simplest one to consider. We make use of the identity

$$1 = \frac{3}{4\pi l^3} \int d^3 x \, \theta(x - y; l),$$

where

$$\theta(z; l) = \begin{cases} 1, & |z| \leq l, \\ 0, & |z| > l. \end{cases}$$

Substituting the identity (3.2) in the mass term of the lagrangian (2.6) and interchanging the order of summation and integration, we find (suppressing the time dependence of the dynamical variables)

$$\mathcal{L}_m = \frac{1}{2} M^2 \frac{3\bar{a}^3}{4\pi l^3} \sum_n \int d^3 x \, \theta(x - y_n; l) \Phi_n^2$$

$$= \frac{1}{2} M^2 \frac{3\bar{a}^3}{4\pi l^3} \int d^3 x \sum_{n \in \mathcal{D}(x,t)} \Phi_n^2,$$

where $\mathcal{D}(x,t) \equiv \mathcal{D}(x,t;l)$ is the ball in the time slice $t$ with center at $x$ and radius $l$. In eq. (3.4) we do not worry about points that happen to sit on the boundary of $\mathcal{D}(x,t)$. Also, when evaluating the hopping terms below, we will not worry about links that cross that boundary. The prescription chosen for handling these cases in unimportant, as long as it respects the translation and rotation invariance of the model. A simple prescription would be for example to include in the sum only points or links which lie entirely inside a given ball.
Approximating $\Phi_n$ by its value at the center of $\mathcal{D}(x, t)$ we obtain

$$\mathcal{L}_m = \frac{1}{2} M^2 \int d^3x \rho(x, t) \Phi^2(x, t),$$

(3.5)

where

$$\rho(x, t) = \frac{3\bar{a}^3}{4\pi l^3} n(x, t).$$

(3.6)

In the above equations, $n(x, t)$ is the number of lattice points inside the ball $\mathcal{D}(x, t)$, and $\rho(x, t)$ is a rescaled, dimensionless density of points.

We next turn to the gradient term (the second term on the r.h.s. of eq. (2.6)). For nearest neighbour sites inside the ball $\mathcal{D}(x, t)$ on a given time slice, we make the approximation

$$\Phi_m - \Phi_n = \left(y^i_m - y^i_n\right) \partial_i \Phi(x, t).$$

(3.7)

Proceeding along the same lines as before we find

$$\mathcal{L}_\text{grad} = \frac{3\bar{a}}{8\pi l^3} \int d^3x \sum_{\langle mn \rangle \in \mathcal{D}(x, t)} (y^i_m - y^i_n)(y^j_m - y^j_n) \partial_i \Phi(x, t) \partial_j \Phi(x, t).$$

(3.8)

Introducing the dimensionless tensor field

$$^{(3)}g^{ij}(x, t) = \frac{1}{\bar{a}^2 n(x, t)} \sum_{\langle mn \rangle \in \mathcal{D}(x, t)} (y^i_m - y^i_n)(y^j_m - y^j_n),$$

(3.9)

we finally obtain

$$\mathcal{L}_\text{grad} = \frac{1}{2} \int d^3x \rho^{(3)}g^{ij} \partial_i \Phi \partial_j \Phi.$$ 

(3.10)

As suggested by eq. (3.10), $^{(3)}g^{ij}(x, t)$ will be identified with the inverse of the space-space part of the metric tensor.

We last turn to the kinetic term. The discrete time derivative (2.8) involves the world line $Y_{n,t}$. In calculating its continuum analog, we have to take into account both the explicit time dependence of $\Phi_{n,t}$ and its implicit time dependence through the motion of $Y_{n,t}$. We thus arrive at

$$\overset{\circ}{\Phi}_{n,t} = \left(\overset{\circ}{\Phi}(x, t) + \overset{\circ}{Y}_{n,t}^i \partial_i \Phi(x, t)\right)_{x=Y_{n,t}}.$$ 

(3.11)

In the above equation we have substituted $\overset{\circ}{Y}_{n,t}$ for $\overset{\circ}{Y}_{n,t}$. This is not really an additional approximation. To justify eq. (3.11) already for $\overset{\circ}{Y}_{n,t}$ the displacement $a_0 \overset{\circ}{Y}_{n,t}$ should be small compared to the scale of variation of $\Phi(x, t)$ and, in any event, it should not be allowed to become arbitrarily large.

There are two effects that govern the world line $Y_{n,t}$. One is the statistical fluctuation due to the random nature of the spacetime measure. The other is a collective motion that may be induced, say, by a macroscopic matter distribution. Thanks to
our time linking prescription, the fluctuating part of $\dot{Y}_{n,t}$ is $O(\bar{a}/a_0)$. In order to justified eq. (3.11) we demand that the continuum limit should correspond to $\bar{a}/a_0 \to 0$. In this limit, only the smooth, collective component of each world line survives, and so in this limit the world line is differentiable.

By contrast, had we taken the opposite limit, we would have found that after $\bar{a}/a_0$ time slices (which would be a very big number), the world line has been carried away by brownian motion to a macroscopically large distance compared to $\bar{a}$. Hence, this limit is unacceptable. For example, it is likely that in this limit, localized states may dissipate into completely delocalized ones in a finite time.

Substituting eq. (3.11) in the kinetic term we now find

$$L_k = \frac{1}{2} \int d^3 x \rho \ (t)g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi .$$

(3.12)

The components of the symmetric tensor $(t)g^{\mu\nu}(x,t)$ are

$$(t)g^{00}(x,t) = 1 ,$$

(3.13a)

$$(t)g^{0i}(x,t) = n^{-1}(x,t) \sum_{n \in D(x,t)} \dot{Y}^i_n ,$$

(3.13b)

$$(t)g^{ij}(x,t) = n^{-1}(x,t) \sum_{n \in D(x,t)} \dot{Y}^i_n \dot{Y}^j_n .$$

(3.13c)

Notice that $(t)g^{0i}(x,t)$ describes a collective drift velocity of the spacetime medium. Because of the coherent character of that motion, one has $(t)g^{ij} = (t)g^{0i} (t)g^{0j}$. Hence-forth we will assume this relation to hold except under extreme circumstances (see subsect. 3.4).

The last step is to introduce the symmetric tensor $(4)g^{\mu\nu}(x,t)$ which is identified with the inverse of the curved four-metric. Its components are

$$(4)g^{00}(x,t) = 1 ,$$

(3.14a)

$$(4)g^{0i}(x,t) = (t)g^{0i}(x,t) ,$$

(3.14b)

$$(4)g^{ij}(x,t) = (3)g^{ij}(x,t) + (t)g^{ij}(x,t) .$$

(3.14c)

Eq. (3.14) resembles the ADM parametrization of the inverse metric [12], with the lapse function set equal to one. But, in the DSRT model eq. (3.14) is more than a parametrization: it defines the gravitational field in terms of the microscopic, quantum spacetime structure. The metric tensor $(4)g_{\mu\nu}(x,t)$ itself is

$$(4)g_{00}(x,t) = 1 + (t)g^{0i} (t)g^{0i} ,$$

(3.15a)

$$(4)g_{0i}(x,t) = - (t)g^{0i}(x,t) ,$$

(3.15b)

$$(4)g_{ij}(x,t) = (3)g_{ij}(x,t) .$$

(3.15c)
On the r.h.s. of eq. (3.15) \(^{(3)}g_{ij}\) is the inverse of \(^{(3)}g^{ij}\), and space indices are lowered and raised using the three-metric. (The position of the index 0 in \(^{(3)}g\) is unimportant).

A formal analytic continuation of the curved metric to Minkowski space is facilitated by substituting \(-1\) instead of 1 on the r.h.s. of eq. (3.15a). As we have explained earlier, our continuum limit forces the time-space components \(^{(0)}g^{0i}\) to describe a collective motion of the space points. Under most circumstances, we expect this motion to vanish or to be negligibly small. We now see that both the collective character and the smallness of that motion are indeed necessary, in order that the effective curved metric will not become singular and will have the correct signature after the analytic continuation.

Expressed in terms of the gravitation and density fields, the continuum action is

\[
S_{cont} = \int dt \mathcal{L}_{cont} = \frac{1}{2} \int dt d^3x \rho \left\{ g^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + M^2 \Phi^2 \right\}.
\]

This is recognized as the continuum action for a massive scalar field in the background of an external gravitational field, provided we can identify the density field \(\rho(x, t)\) with the curved volume element. Notice that, since \(g^{00} = 1\), the local four-volume element is equal to the local three-volume element [12].

From the point of view of the low energy effective curved space, \(g^{00} = 1\) looks like a partial gauge fixing of general coordinate invariance. There is a large subgroup of coordinate transformations which leave the component \(g^{00}\) invariant. These are given by

\[
t' = t, \\
x' = x'(x, t).
\]

The coordinate transformation (3.17) will play an important role below.

Whether or not one can identify the density field \(\rho(x, t)\) with \(\sqrt{g(x, t)}\) is a dynamical question. It is convenient to examine this question by first making the field redefinition

\[
\rho(x, t) = e^{2\phi(x, t)} \sqrt{g(x, t)}.
\]

If eq. (3.18) is substituted in eq. (3.16), we obtain the generally covariant continuum action of a scalar field in a dilaton-gravity background, where \(\phi(x, t)\) is the “dilaton” field. This provides a convenient parametrization to study the dynamical questions. Namely, does the gravitation field as defined above propagates massless excitations, and does the “dilaton” field propagates any low energy excitations. These questions are addressed in the following subsection.
We will argue that, with a certain improvement of the spacetime measure, the curve metric that we have introduced should have massless spin two excitations in the continuum limit. As for the "dilaton" field, we find that this field is completely frozen. As a result, the dynamics of the DSRT model justifies the identification of $\rho(x, t)$ with $\sqrt{g(x, t)}$.

We close with a comment on the expectation values of the thermodynamical fields. In subsect. 2.3 we argued that the ground state as probed by low energy observers, should be Poincaré invariant. Neglecting possible microscopic crystal-like structure, which is irrelevant for most of the subsequent discussion, we thus have constant expectation values for the thermodynamical fields. The expectation value of the density field is trivially found to be $\langle \rho(x, t) \rangle = 1$. The expectation value of the metric tensor is diagonal, and by making some redefinition of parameters we may assume that it takes the canonical value $\langle g^{\mu\nu}(x, t) \rangle = \delta^{\mu\nu}$.

### 3.2 An improved spacetime measure

In subsect. 3.1 we identified the components of the curved metric tensor with quasi-local averages of certain microscopic properties of the discrete spacetime structure. If this is a valid identification, then the thermodynamical curved metric field should be governed by Einstein’s equation in the appropriate limit.

In order to investigate our dynamical curved metric we define an effective measure $\Omega(g^{\mu\nu}, \phi)$ as follows

$$\Omega(g^{\mu\nu}, \phi) = \langle e^{-F} \rangle_{g^{\mu\nu}, \phi}.$$  (3.19)

The expectation value on the r.h.s. of eq. (3.19) is a partial ensemble average in the spacetime measure $\int_{n,t} Dy$, which is taken over all triangulations that correspond to given $g^{\mu\nu}$ and $\phi$ fields. $F$ is the matter free energy. If we are interested in an energy range below some physical scale, then we have to include in $F$ only the contribution of matter excitations with energies comparable to or larger than that scale. The infra-red cutoff of the matter’s functional integration can be imposed self-consistently, because low energy matter eigenstates should depend only on the thermodynamical fields (and not on the microscopic details) and their energies are invariant with respect to the coordinate system chosen to describe the effective low energy curved space.

The space of metrics $g^{\mu\nu}(x, t)$ over the four-torus has a natural fiber bundle structure, where each fiber is an equivalence class of all metrics which are obtained from some initial metric by a general coordinate transformation. The gravitation field will have the desired dynamics if, in the continuum limit, the effective measure takes the form

$$\Omega(g^{\mu\nu}, \phi) = \delta(C(g^{\mu\nu}, \phi)) e^{-S_{eff}(g^{\mu\nu}, \phi)}.$$  (3.20)
In eq. (3.20) $S_{\text{eff}}(g^{\mu\nu}, \phi)$ is a generally covariant effective action, and $\delta(C(g^{\mu\nu}, \phi))$ stands for a generic (possibly partial) gauge fixing condition.

In other words, there is no need that the support of the effective measure will cover all representatives of each metric. The crucial requirement is that, inside each fiber, the effective measure should be constant over its support. The saddle point of the effective measure will then be a solution of a generally covariant equation of motion.

Our definition of the curved metric depends on the flat coordinates on the four-torus. Because of the statistical character of our $g^{\mu\nu}(x, t)$, we expect that in each fiber, the effective gravitational measure will be highly peaked around those representatives where $g^{\mu\nu}(x, t)$ is as close as possible to a constant function. The reason is simply that the phase space for such configurations is maximal. Moreover, the dynamical geodesic distance inferred from these representatives should reduce to the coordinate distance in the underlying flat space, provided one is sufficiently far away from matter concentrations. This is a good feature that we should not spoil unnecessarily.

In this subsection we introduce an improved spacetime measure, which incorporates an integration over a small subset of the diffeomorphism group. We will argue that, with the improved measure, eq. (3.20) should hold in the continuum limit, where $S_{\text{eff}}(g^{\mu\nu}, \phi)$ is invariant under infinitesimal general coordinate transformations of the form (3.17). While we will not bother to calculate the details of the effective gauge fixing condition, we know that it selects representatives with the qualitative properties described above.

In order to define the improved spacetime measure we introduce an auxiliary cubic lattice which is overlayed with the flat target space from which the points on every time slice are taken. The infrared scale $L$ is assumed to be an integer multiple of $a'$, the lattice spacing of the cubic lattice. Like $a_0$ and $\bar{a}$, this is a cutoff parameter which has to be sent to zero in the continuum limit. Its physical role (see below) suggests that we should take $a'/\bar{a} \gg 1$. We already have one cutoff parameter with this property, namely $a_0$. The natural choice is to take $a' = a_0$. Henceforth we will denote the common value of these two parameters by $a$.

The improved spacetime measure contains a new degree of freedom, a spatial vector field $\xi^i_{x,t}$ that resides on the sites of the cubic lattice. We will use the convention that when the coordinate $x$ appears as a subscript, it takes values on the sites of the auxiliary regular lattice. Finally, there is also an additional dimensionless parameter $\delta$ which serves to determine the subset of the diffeomorphism group over which we are integrating. As we will see, $\delta$ can be sent to zero in the continuum limit, but rather slowly compared to other cutoff parameters.
The new partition function is defined by

\[ Z_D = Z(N_0, N, L, a, \delta, M) = \int_{n,t} D\gamma \int_{n,t} D\Phi \int_{x,t} D\xi e^{-S}, \]  

(3.21)

where

\[ \int_{x,t} D\xi \equiv N(a\delta)^{-6N_0(L/a)^3} (a\delta/L)^3 \prod_{x,t} d^3\xi_{x,t} F_\delta(\xi). \]

Notice that 2\(N_0(L/a)^3\) is the number of sites of the auxiliary lattice. The name \(Z_D\) is chosen to remind us that the improved measure has certain built in diffeomorphism invariance. The function \(F_\delta(\xi)\) determines the range of the vector field \(\xi_{x,t}\) and, hence, the subset of the diffeomorphism group over which will are integrating. We take

\[ F_\delta(\xi) = \prod_{x,t,k} \theta\left(|(x, t)' - (x + \hat{k}, t)'| - a; a\delta\right) \]
\[ \times \prod_{x,t} \theta\left(|(x, t)' - (x, t + 1)'| - a; a\delta\right), \]

(3.22)

where \(\theta\) has been defined in eq. (3.3), and

\[ (x^i, t)' = (x^i + \xi_x^i, t). \]

(3.23)

Notice that eq. (3.23) is a discretization of the coordinate transformation (3.17), thought of as an actual deformation of spacetime. \(x + \hat{k}\) is the neighbouring site of \(x\) in the positive \(k\) direction. The meaning of these definitions is that under the (as yet discrete) transformation (3.23), the fractional change in the distance between the images of two neighbouring sites should not exceed \(\delta\).

In spite of the introduction of the regular lattice, the new partition function \(Z_D\) has the same global spacetime symmetry as the old one. Continuous translation invariance is restored by the integration over the diffeomorphism’s translational zero mode. Likewise, in the new partition function, rotation invariance is broken only by the infrared boundary conditions. The normalization of the \(\int_{x,t} D\xi\) measure takes into account that the basic range of variation of \(\xi_x^i\) is \(O(a\delta)\), and the factor \((a\delta/L)^3\) compensates for the unrestricted range of the zero mode of \(\xi_x^i\). The normalization constant \(\mathcal{N}\) is therefore \(O(1)\). We will not need the numerical value of \(\mathcal{N}\) and, for most purposes we can simply drop it from the definition of \(Z_D\). Notice that, to derive inequality (2.37) for the improved measure we only have to know that \(\mathcal{N}\) exists.

The triangulations are built in two steps. We first connect a given set of spacetime points \(\{y_{n,t}^i\}\) according to the prescription of Sect. 2. The embedding of the triangulation in the target flat space is then determined by the vector field \(\xi_x^i\).
To proceed, each vector field configuration $\xi^i_{x,t}$ is first extended to a smooth, continuous vector field $\xi^i(x, t)$. We require that the vector field $\xi^i(x, t)$ should coincide with the original vector field configuration $\xi^i_{x,t}$ when $(x, t)$ is the coordinate of a site on the auxiliary lattice. The diffeomorphism generated by $\xi^i(x, t)$ is then applied to the entire triangulation. Namely, without changing the connectivity of the triangulation, we move the lattice points to new locations given by

$$y^i_{n,t} \rightarrow (y')^i_{n,t} = y^i_{n,t} + \xi^i(y_{n,t}, t).$$

(3.24)

What is described above is a family of embeddings, or deformations, of the same triangulation. These deformations leave the matter action invariant, and so the improved partition function really defines the same quantum theory as the original one. In fact, with the normalization constant $N$ included, one has $Z_D = Z$. The advantage of the improved measure is that the relevant observables (the ones defined in analogy to subsect. 2.2) are local. The local observables of the improved partition function are related to a set of non-local observables of the original partition function.

We will now show that the embeddings described by eq. (3.24) generate a curved metric tensor and a density field which are related to the original ones at $\xi^i_{x,t} = 0$ by a general coordinate transformation of the special type (3.17).

To find how the new metric looks we have to apply the transformation (3.24) in eqs. (3.9) and (3.13). Let us start with the spatial part $^{(3)}g^{ij}$ (eq. (3.9)). Suppressing the $t$-dependence, we make the following approximation for any two points which are connected by a space link

$$ (y')^i_n - (y')^i_m = (y^i_n - y^i_m) + (\xi^i(y_n) - \xi^i(y_m)) $$

$$ = (y^i_n - y^i_m)(\delta^i_j + \partial_j \xi^i). $$

(3.25)

Substituting in eq. (3.9) we find

$$ ^{(3)}g^{ij}(x', t) = (\delta^i_k + \partial_k \xi^i)(\delta^j_l + \partial_l \xi^j) \times $$

$$ \frac{1}{\bar{a}^2 n'(x,t)} \sum_{(mn) \in D'(x,t)} (y^k_m - y^k_n)(y^l_m - y^l_n). $$

(3.26)

The r.h.s. of eq. (2.26) is evaluated on the original triangulation. But this is not yet an expression for the transformed metric in terms of the old one. The reason is that the summation is not carried over the links that belong to $D(x,t)$. Rather, it is carried over the links that belong to $D'(x,t)$, the pullback of $D(x',t)$. Similarly, $n'(x,t)$ is the number of points in $D'(x,t)$. The pullback $D'(x,t)$ is defined as

$$ D'(x,t) = \{(z,t) \mid (z',t) \in D(x',t)\}. $$

(3.27)
The typical deformation of the pullback $\mathcal{D}'(x, t)$ compared to $\mathcal{D}(x, t)$ is of order $a\delta$. In the continuum limit we let $\bar{a}/a \to 0$ and so we expect that, already on the scale $a$, the thermodynamical fields will fluctuate very little around their mean values. Moreover, as long as we perturb the ground state only by smooth, extended sources, we anticipate these mean values to be slowly varying. We can therefore self-consistently replace the primed quantities on the r.h.s. of eq. (3.26) by unprimed ones. We thus obtain

$$(g'')^{ij}(x', t) = (\delta^i_k + \partial_k \xi^i)(\delta^j_l + \partial_l \xi^j) (g^{kl}(x, t)). \quad (3.28)$$

Making a similar analysis for the transformation of $(g')^{\mu\nu}$ we get

$$(g')^{\mu\nu}(x', t) = (\delta^\mu_\sigma + \partial_\sigma \xi^\mu)(\delta^\nu_\tau + \partial_\tau \xi^\nu) g^{\sigma\tau}(x, t). \quad (3.29)$$

In order to write this compact transformation law we have introduced a vanishing zeroth component field $\xi^0(x, t) = 0$. Finally, one can check that the density field $\rho(x, t)$ transforms as a scalar density, namely

$$\rho'(x', t) = \det^{-1}(\delta^i_k + \partial_k \xi^i) \rho(x, t). \quad (3.30)$$

This completes the demonstration that the effect of the $\xi$-transformations on the gravitation and density field is equivalent to the effect of a general coordinate transformation.

To see how the integration over the $\xi$-field enforces the form (3.20) in the continuum limit, let us consider the width of $\Omega(g^{\mu\nu}, \phi)$ inside each fiber, as derived from the original partition function eq. (2.2). The gravitation field is obtained by a statistical average over a region of size $l$. The lower bound on the size of that region is $a$, and therefore the number of points involved in this average is at least $n_0 \approx (a/\bar{a})^3$. In the continuum limit $n_0$ tends to infinity. We thus expect $\Omega(g^{\mu\nu}, \phi)$ to be highly peaked around a mean value, with a statistical uncertainty which is governed by $n_0^{-\gamma}$ for some calculable $\gamma > 0$. In the continuum limit, $\Omega(g^{\mu\nu}, \phi)$ will rapidly tend to a delta function.

Turning to the effective measure $\Omega_D(g^{\mu\nu}, \phi)$ of the improved partition function, we now see that, inside each fiber, is it obtained by a convolution of $\Omega(g^{\mu\nu}, \phi)$ over some domain whose size is $O(\delta)$. If we approach the continuum limit in a controlled way, such that $\delta$ is kept very big relative to $n_0^{-\gamma}$, then the improved effective measure will be almost constant over its support inside each fiber. As we move inside a given fiber, the width of $\Omega(g^{\mu\nu}, \phi)$ will now govern only the width of the tiny boundary region over which $\Omega_D(g^{\mu\nu}, \phi)$ abruptly changes from its constant positive value to zero.
In the continuum limit, while $\delta$ tends slowly to zero, $\frac{\delta}{n_0^\gamma}$ tends to infinity, and so the limiting effective measure inside each fiber becomes exactly constant over its support. This gives rise to the desired form (3.20) for the effective gravitational measure, where the effective action is invariant under infinitesimal coordinate transformations (3.17), and the effective gauge fixing condition has the desirable features described in the beginning of this subsection.

Invariance of the effective action under infinitesimal transformations of type (3.17) is sufficient to ensure the masslessness of the graviton. The argument is a standard one. We define $g^{\mu\nu}(x, t) = \delta^{\mu\nu} - h^{\mu\nu}(x, t)$. (Notice the slight modification compared to the usual definition, which we do because in the DSRT model $g^{00} = 1$ identically, i.e. according to our definition $h^{00} = 0$). Let us now write down all possible mass terms for $h^{\mu\nu}$. We have to allow time and space components to appear in an asymmetric way, because time and space are not treated symmetrically in the DSRT lattice. In fact, we know that the spin two part is contained in the space-space components of the curved metric, but we might as well check for possible mass terms for the time-space components. The only possible mass terms are $(h_{0k})^2$, $h^2$ and $(h_{kl} - \frac{1}{3}\delta^{kl}h)^2$, where $h = h^{kk}$. The reader can easily verify that no linear combination of these terms is invariant under infinitesimal coordinate transformations (3.17). Hence, the curved metric field should have massless spin two excitations.

As for the “dilaton” field, substituting the defining eq. (3.18) in the continuum action (3.16) would suggest the existence of a global shift symmetry for $\phi(x)$ which is not a part of the coordinate transformation group. But the spacetime measure (with or without improvement) does not have an additional symmetry which acts that way on the $\phi$ field. In particular, the diffeomorphisms (3.24) leave the $\phi$ field invariant. As a result, the fluctuations of the $\phi$ field are suppressed in the continuum limit, and the $\phi$ field is frozen to its expectation value. Returning to eq. (3.18) we thus see that, while a priori we had to introduce the density field as an independent variable, the dynamics forces the density field to coincide with the volume element of the curved metric tensor.

We conclude with a technical comment. Since $g^{00} = 1$, the curved metric tensor as defined in subsect. 3.1 contains only nine algebraically independent degrees of freedom. The density field contains one degree of freedom. Altogether we have ten degrees of freedom, as the metric tensor should normally have. It is therefore suggestive to try to make a field redefinition that will replace the constrained metric tensor and the density field by an unconstrained metric tensor.

If we focus on the kinetic term in the continuum action (3.16), then the desired
field redefinition is easily found to be

\[ \bar{g}^{\mu\nu} = \frac{\sqrt{g}}{\rho} g^{\mu\nu}. \]  

(3.31)

The above definition implies the equality of \( \rho g^{\mu\nu} \) and \( \sqrt{\bar{g}} \bar{g}^{\mu\nu} \). Therefore, this field redefinition brings the kinetic term to the standard form without having to rely on the dynamics. However, there is also a mass term, and that term does not take the standard form in terms of the new metric tensor. Thus, we are unable to trade algebraically the independent density field with a tenth component of the metric tensor, because the continuum scalar action is not conformally invariant.

This consideration suggests that a similar trick might work for gauge fields. But the dynamics forces the proportionality factor \( \sqrt{\bar{g}}/\rho \) in eq. (3.31) to be equal to unity anyway. Hence, the final result will be the same. Even if we can work with the modified metric \( \bar{g}^{\mu\nu} \), the effective action will be invariant only under the restricted coordinate transformations (3.17). Therefore, there is no symmetry that prevents the appearance of a mass term with a divergent coefficient for \( \bar{h}^{00} \). So again we arrive at the same conclusions, namely, that the component \( \bar{g}^{00} \) is completely frozen, and that the dynamical density of points is given by the curved volume element.

### 3.3 The effective gravitational action

So far, we provided evidence that the ground state of the DSRT model as probed by low energy observers, is flat space. We also argued that, with the improved space-time measure of subsect. 3.2, the set of observables defined in Sect. 2 should exhibit the presence of a long range spacetime-induced interaction. If both of these assertions are correct, the distance to proving the emergence of the familiar gravitational interaction is very short.

The relevance of the DSRT model to Quantum Gravity therefore depends on one’s ability to prove the validity of the above assertions. We will not attempt to provide the detailed proofs in this paper. But if the DSRT model provides a consistent model of Quantum Gravity, then it should be possible to obtain satisfactory solutions to numerous well known problems. Our aim in this subsection is to indicate that such answers may indeed exist. The following discussion is also important in order to decide what is the best way to proceed with a detailed study of the properties of the DSRT model.

The gravitational effective action defined in the previous subsection may not, in fact, be the best tool to investigate the model. When we speak about the gravitational action, we have in mind an approximation where one keeps a finite number of local terms in a derivative expansion of the exact, necessarily non-local effective action.
Further complications arise because of the need to disentangle the effective action from the effective gauge fixing condition (see eq. (3.20)). But the effective action provides a convenient and familiar terminology, and so we will continue to use it in this subsection.

In order that a local effective action will be a useful tool in the investigation of the model, a minimal requirement is that the true ground state should appear as a (possibly local) minimum of that effective action. In subsect. 2.3 we argued that the ground state should be homogeneous, and this provides a potential explanation for the absence of a cosmological term in the effective action. Moreover, one can interpret instability problems such as the unboundedness from below of the Einstein-Hilbert action, as an artifact of the truncation leading from the full non-local effective action to the local approximate one.

Two central issues which can be addressed by computing an approximate gravitational effective action, are the identification of the Planck scale with some physical scale of the DSRT model, and the nature of the effective theory at shorter-than-Planck-scale distances.

The explicit physical mass scales in the partition function (3.21) include one mass parameter for each scalar field. Let us assume that a particular mass parameter, namely $M$, is much bigger than the others. If we are interested in physics much below the scale $M$ then we should integrate out the mass-$M$ scalar field. The scalar action is diffeomorphism invariant, and it should clearly give rise to a generally covariant contribution to the effective gravitational action.

One can get an idea of the outcome by considering the calculation of the scalar free energy using a continuum regularization technique which respects general coordinate invariance, such as dimensional regularization. This computation was done long ago by ‘tHooft and Veltman [13]. Here we are not interested in the details of their computation but only in the qualitative features of their result. The bottom line is that the local part of the scalar free energy may contain all generally covariant local operators with dimension up to four. Operators of lower dimensions are multiplied by the appropriate power of $M$.

The infrared leading term in this expansion is a cosmological term. As we discussed above, we do not expect any cosmological term to arise in the complete effective gravitational action of the DSRT model. In the present terminology, this means that a compensating, negative cosmological constant should arise from the dynamics of the spacetime points. This may turn out to be the entropy factor of subsect. 2.3 in disguise. The anticipated cancellation signals that the presence of matter should have an important effect on the infrared limit of the spacetime dynamics.
In the absence of a cosmological term, the leading infrared term is the Einstein-Hilbert action. For this term, we do not see any reason why the spacetime dynamics should conspire to cancel a contribution coming from integrating out the heavy scalar field. On dimensional grounds, the dimension two coefficient of the Einstein-Hilbert action should be $O(M^2)$. Thus, the Planck scale is identified with the highest mass scale of the matter sector. We believe that this should be a general relation which is valid also for other choices of the matter sector. For example, in a model containing several asymptotically free non-abelian groups, the Planck scale may correspond to the highest confinement scale. We comment that the sign of the coefficient of the Einstein-Hilbert action, as inferred from the result of ref. [13], comes out right. This is an important consistency check. Having the right sign means the absence of perturbative instabilities in the euclidean region, which should clearly be the case if we are expanding around the correct ground state.

Having identified the Planck scale with the highest mass scale of the matter sector, the next question is whether the DSRT model can provide a consistent description of the gravitational dynamics at short (i.e. shorter than Planckian) distances. As long as we do not get too close to the lattice scale(s), there should exist a scaling region where perturbative correlation functions may depend on dimensionful parameters only through logarithms.

An interesting possibility, and perhaps the only consistent one, is that the scaling region is governed by an asymptotically free curvature squared [11, 14] effective action. As shown by Fradkin and Tseytlin, the continuum curvature squared theory is asymptotically free in all essential coupling constants [11]. If one demands that the euclidean action be bounded from below, then the curvature squared theory in not unitary due to the presence of negative norm states. But the question of unitarity is really a non-perturbative one [11, 14, 15]. In order to conclude that negative norm states are indeed present, one has to analytically continue to Minkowski space and to go on shell. There are many ways in which non-perturbative effects could invalidate this troublesome prediction of perturbation theory.

We conjecture that the perturbative, short distance behaviour of the DSRT model should be governed a curvature squared theory with a positive definite euclidean action. This effective short distance theory should arise from the “pure gravity” sector of the model. By this we mean that, even in the absence of any matter fields, the improved spacetime measure already defines some effective gravitational action through eq. (3.19). We expect this “pure gravity” effective action to be of the kind described above.

One may consider the possibility that, being asymptotically free, the pure gravity
theory could generate a scale dynamically as in QCD. It is hard to imagine what would be the physics of a model in which the metric tensor is “confined”. In any event, in our universe the gravitational field, like the electromagnetic field, is clearly not confined in the sense of the gluon field in QCD. We consider this as an indication that a matter sector must be introduced in order to generate the Planck scale. An interesting question is whether there is a relation between the way matter fields should provide the scale which characterize the gravitational interaction, and the Higgs mechanism which provides the scale of the Electro-Weak interactions [15].

3.4 Beyond General Relativity

In this paper we discussed the possibility that the DSRT model provides a consistent model of Quantum Gravity. If the considerations we made are correct, the dynamics of the DSRT model should reduce at sufficiently low energies to solutions of Einstein’s equation coupled self-consistently to quantum matter. But under certain circumstances and, in particular, where General Relativity gives rise to singular solutions, the physics of the DSRT model should be different and consistent.

Deviations of the DSRT model from General Relativity may arise from two sources. The first is the thermodynamical character of the metric tensor in the DSRT model. Under certain circumstances we anticipate that dissipative processes will take place. This is expected to occur mainly in the vicinity of singular points of classical solutions, and so it should be relevant to the physics of black holes (see below). Under ordinary circumstances, however, dissipation should be negligible, otherwise one may run into conflict with observation. A case where modified dynamics may account for an apparent discrepancy between observation and theory is discussed in ref. [16].

A second source for deviations from General Relativity is the preferred role of time in the DSRT model. In the domain of validity of General Relativity, it should be possible to interpret all non-relativistic features of the DSRT model as non-covariant gauge fixings. But outside the scope of General Relativity, constraints such as \( g^{00} = 1 \) contain relevant dynamical information.

Another feature of a similar nature is the existence of a local conservation law for the number of space points. The point density field \( \rho(x,t) \) has been introduced earlier. The current density of space points is (see eq. (3.13b))

\[
J^k = \rho g^{0k}.
\]  

Point number conservation implies the odd-looking conservation equation

\[
\partial_t \rho + \partial_k (\rho g^{0k}) = 0.
\]

(3.33a)
As we argued earlier, $\rho = \sqrt{g}$ dynamically. Therefore, from the point of view of the low energy curved space, eq. (3.33a) looks like a gauge fixing condition too. Although this may be an unconventional gauge fixing, enforcing it cannot change the physics of General Relativity.

We can turn around the argument and try to learn under what circumstances General Relativity may not work. To this end, we rewrite the number conservation equation in the following form

$$\partial_t(\log \rho) + g^{0k} \partial_k(\log \rho) + \partial_k g^{0k} = 0.$$  

(3.33b)

the fact that eq. (3.33b) depends on $\log \rho$ suggests that point number conservation should be relevant, but only if the point density varies by many orders of magnitude. If this is the case for some physical process, then General Relativity might not account accurately for that process.

Again, in most astronomical systems this is in not the case. Whenever one has a stationary gravitational system, it should be possible to find a coordinate system where both $g^{00} = 1$ and $g^{0k} = 0$. Under these circumstances, point number conservation holds trivially, and so the coordinate system may coincide with the preferred coordinate system of the DSRT lattice. Other coordinate systems exist in which $g^{0k}$ does not vanish. As long as both $g^{00} = 1$ and eq. (3.33) hold, the metric is in principle attainable in the DSRT spacetime measure, and so these coordinate systems could coincide with the DSRT frame too.

The above examples show how important it is to identify the preferred frame of the DSRT lattice with some coordinate system suggested by the physical process in question. An important case where such an identification can be made, is if we consider the evolution of entire universe. It is natural to identify the cosmological time coordinate defined by the expansion of the universe and the rest frame of the microwave background radiation, with the fundamental time coordinate of the DSRT lattice. Although at this stage we do not have a clear understanding of how the DSRT model can account naturally for the expansion of the universe, we hope that this issue will be clarified in the future. (One can of course enforce the expansion by hand, if one imposes appropriate initial conditions).

We conclude with a few comments on the subject of black hole formation and evaporation. This subject has attracted a lot of attention recently [17-21]. In the DSRT preferred frame, the components of the metric tensor carry genuine information about the microscopic properties of spacetime. The difficulty is to determine how the DSRT frame is related to curved coordinate systems used to describe the Schwarzschild metric.
The Schwarzschild and Kruskal coordinates cannot coincide with the DSRT frame because they do not satisfy $g^{00} = 1$. A coordinate system which does satisfy this condition and also $g^{0k} = 0$ is the Novikov coordinate system \[22\]. However, the Schwarzschild metric has genuine time dependence, which might show up as non-trivial time dependence of $\rho$ and $g^{0k}$ when expressed in the DSRT frame. Thus, the relation between the DSRT frame and familiar coordinate systems is not clear at the moment.

Finding how the Schwarzschild metric looks in the DSRT frame is in fact of crucial importance. If the metric tensor turns out to be singular only at the curvature singularity, then this supports scenarios \[17, 19\] in which the evaporation of a black hole leaves behind a massive, infinitely degenerate stable remnant. We will not enter at this stage the discussion of whether or not that stable remnant can be interpreted as a new asymptotic region. On the other hand, if the metric tensor has a “coordinate singularity” in the DSRT frame (say, near the horizon) then this singularity would not be spurious, and this would support scenarios in which the horizon acts as a physical membrane \[20, 21\].

Let us return for a moment to the space-space components of the metric tensor and discuss their physical significance in the DSRT frame. The spatial metric becomes singular if \((3)g_{ij}\) diverges in some region of spacetime, which is the same as having a zero for \((3)g^{ij}\). The field \((3)g^{ij}\) measures the local link length squared relative to its global average value, times the number of links per vertex. The number of links per vertex is a positive integer. Thus, the vanishing of \((3)g^{ij}\) can arise only from the vanishing of the local link length, which means that the local point density blows up.

A singularity in $g_{ij}$ therefore implies that a nearby volume of a tiny radius in the DSRT frame, will support an enormous number of matter states compared to the “normal” number of states it supports in the vacuum. This means that the DSRT model can potentially allow for the existence of massive stable remnants. Whether or not this is the way that the paradoxes associated to the evaporation of macroscopic black holes are resolved, is a question that must await a more detailed investigation.

4. CONCLUSIONS

The interplay between flat space and curved space has always been an important theme in the literature on General Relativity \[23\]. Also, there are numerous attempts to quantize the gravitational interaction which can be categorized under the title of Induced Gravity \[15, 24, 25, 26\]. We believe that the DSRT model goes beyond previous attempts, and that it provides a unique non-perturbative realization of the
above themes.

The DSRT model is characterized by the embedding of the spacetime triangulations in a target flat space, and by the uneven treatment of time and space. Thanks to the special role of time, the DSRT model is unitary. In subsect. 3.1 we argued that, on top of the discretization of time into regular intervals, one has to let $\bar{a}/a \rightarrow 0$ in order to ensure the consistency of the analytic continuation to Minkowski space. As usual, the price that we pay for having a manifestly unitary theory, is that restoration of Lorentz covariance is not automatic. Indeed, we anticipate that under those rare circumstances where General Relativity fails, the DSRT model may show genuine non-relativistic features.

One could consider an alternative approach in which time and space are treated on equal footing. This is in fact a simpler and less elaborate framework. The spacetime points are allowed to move inside the entire four dimensional volume and are not constrained to time slices, while the triangulations are made out of four-simplices. In this approach, which we denote as the spacetime symmetric approach, one has the full rotational invariance of four dimension euclidean space, but the theory is not unitary for finite mean lattice spacing.

One may hope that, in this approach, unitarity will be recovered in the continuum limit. A first signal that this may not be the case is provided by the discussion of subsect. 3.1. There is no way to impose the analog of the limit $\bar{a}/a \rightarrow 0$ in the spacetime symmetric approach, and so we suspect that the analytic continuation may not be consistent in this case. Also, many of the problems of continuum gravitational theories boil down to lack of unitarity (see subsect. 3.3) and so there is a serious danger that they will show up in the spacetime symmetric approach.

A pessimistic point of view is that, regardless of which option is used, the embedding of triangulations in a target flat space does not provide a consistent definition of Quantum Gravity, and that the only difference between the two options is where the failure occurs. We believe that it is much too early to draw that negative conclusion. But we do have a clear cut argument that only the DSRT lattice can possibly give rise to a consistent theory of Quantum Gravity.

The argument is based on the Weinberg-Witten (WW) theorem [27]. This theorem asserts that if one has an energy-momentum tensor which satisfies an ordinary conservation equation and which transforms as a true Lorentz covariant tensor, then the Fock space of that theory cannot contain massless spin two states. In General Relativity one avoids the WW theorem because the true energy-momentum tensor is covariantly conserved. In an asymptotically flat space one can define another energy-momentum tensor which satisfies an ordinary conservation equation, but this object
is really a pseudo-tensor.

In the case of the spacetime symmetric approach, one has an energy-momentum tensor which satisfies all the assumptions of the WW theorem already for finite lattice spacing. Hence, the analytic continuation to Minkowski space cannot contain gravity and be consistent simultaneously. The DSRT lattice avoids the WW theorem because it is not a Lorentz covariant framework. If the DSRT lattice provides a consistent theory of Quantum Gravity, then relativistic covariance at low energies should be considered as a consequence of the fact that the underlying dynamics reduces to solutions of the generally covariant Einstein’s equation (albeit in a special non-covariant gauge).

The DSRT model has very rich physics, and progress in understanding any aspect of its physical properties may be important in telling whether or not it contains gravity. We conclude with a list of possible lines of future investigation.

One way is to start directly from the infrared end of the model. In subsect. 2.4 we proved that the zero momentum scalar state has a finite energy in the limit of vanishing lattice spacing. It may be possible to generalized this result to localized states by making use of variational bounds. Sufficiently detailed information about localized states may also provide a window to investigate the Planck scale physics of the model.

Another possibility is to start from the ultraviolet end. One can investigate two key features in a simplified version of the model. These are the method of enforcing general covariance in the continuum limit (subsect. 3.2) and the conjecture that short distance physics is governed by an asymptotically free curvature squared theory (subsect. 3.3).

A nice way to investigate these features is to consider a “pure measure” model. This model does not contain any matter fields. We can furthermore start from a spacetime symmetric model. While in this model we anticipate difficulties with the analytic continuation to Minkowski space, we believe that its short distance correlation functions in the euclidean region may be very similar to those of the DSRT model.

The spacetime symmetric, pure measure model is defined by the partition function

\[
Z' = Z(N, L, a, \delta) = \int_n \mathcal{D}y \int_\mathcal{D}\xi ,
\]

where

\[
\int_n \mathcal{D}y \equiv L^{-4N} \prod_{n=1}^N \int_{-L/2}^{L/2} d^4y_n ,
\]

(3.34)
\[
\int_{x^\mu} D\xi \equiv (a\delta)^{-4(L/a)^4} \left(a\delta/L\right)^4 \prod_{x^\mu} \int d^4\xi_{x^\mu} \mathcal{F}_\delta(\xi),
\]
and \(\mathcal{F}_\delta(\xi)\) is the spacetime symmetric analog of eq. (3.22). Here \(N\) is the number of spacetime points which are embedded in a four-torus of circumference \(L\) in each direction, and \(a\) and \(\delta\) have the same significance as in subsect. 3.2.

The pure measure partition function defines a non-trivial effective gravitational measure through eq. (3.19). Apart from the infrared and ultraviolet cutoffs, it depends on two dimensionless parameters \(\delta\) and \(\bar{a}/a\) where now \(\bar{a} = L/N^{\frac{1}{4}}\). One combination of these parameters should be sent to zero in order to recover general coordinate invariance in the sense of eq. (3.20). The other linearly independent combination is a genuine free parameter.

If our conjecture concerning the relation between the pure (improved) measure model and a continuum curvature squared theory is correct, then this second dimensionless combination should be related to a coupling constant of the continuum theory. There are in fact two distinct asymptotically free continuum theories [11] which could be relevant. One is the classically scale invariant Weyl theory [14, 15]. The other is a theory which contains the two linearly independent curvature squared terms.

The Weyl theory has one coupling constant. An interesting observation is that the other curvature squared theory also has only one truly independent coupling constant which governs the deep euclidean region. Although perturbatively one has two coupling constants corresponding to the two linearly independent curvature squared terms, at the fixed point of the RG flow, the ratio of the two coupling constants approaches a calculable constant [11]. The ultraviolet behaviour of both curvature squared theories is therefore governed by a single free parameter. It would be interesting to see whether the pure measure model can be related to one of these continuum theories.

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