Tricritical point and solid/liquid/gas phase transition of higher dimensional AdS black hole in massive gravity

Bo Liu$^{1,2,4}$, Zhan-Ying Yang$^{1,3}$, Rui-Hong Yue$^{4*}$

$^1$School of Physics, Northwest University, Xi’an, 710127, PR China
$^2$School of Arts and Sciences, Shaanxi University of Science and Technology, Xi’an, 710021, PR China
$^3$Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi’an 710127, PR China
$^4$Center for Gravity and Cosmology, College of physical science and technology, Yangzhou University, Yangzhou, 225009, PR China

(Dated: October 19, 2018)

By considering the fifth order term of the interaction potential in massive gravity theory, we study the $P-V$ critical behaviors of AdS black hole in $d \geq 7$ dimensional space-time, and find the tricritical point and the solid/liquid/gas phase transition in addition to the Van der Waals-like phase and the reentrant phase transition of the system. The critical phenomena of black holes depend crucially on the number $n$ of interaction potential terms.

PACS numbers: 04.50.Kd, 04.70.Dy, 04.20.Jb

I. INTRODUCTION

Thermodynamics of black hole revealing the relationship between the gravity and thermodynamics has always been one of the hot topics in theoretical physics since the Hawking radiation was discovered in 1975 [1]. In recent two decades, more and more attentions are paid on the thermodynamics of black hole in anti-de Sitter (AdS) space. One of the reason is that the AdS black hole can be in thermodynamical stable equilibrium with a positive specific heat, and has Hawking-Page (HP) phase transition between thermal gas and black hole in AdS space [2], which is absent in asymptotically flat or de Sitter space. Another reason is that the AdS/CFT correspondence [3–6], connecting gravity theory in bulk with a conformal field theory on the boundary of AdS space, is regarded as a powerful tool for understanding the strongly correlated system by investigating the classical gravity theory. In this view, HP phase transition could correspond to the confinement/deconfinement phase transition in gauge field theory [7].

In 1999, it was discovered [8, 9] that the first order phase transition between large black hole and small black hole(LBH/SBH) in charged Reissner-Nordström-AdS (RN-AdS) black displayed

* Email:rhyue@yzu.edu.cn
in many respects similar to the liquid-gas phase transition in Van der Waals fluid. In fact, it was found that the critical exponents analogous to those of the Van der Waals fluid do not depend on the number of dimensions of the RN-AdS black hole \[10\], and the identification of these features might be just a mathematical analogy rather than physical one.

Moreover, according to the Smarr relation of AdS black hole obtained by the scaling argument \[11, 12\], the cosmological constant \(\Lambda\) should be treated as a variable rather than a fixed external parameter. Once the variation of \(\Lambda\) is included in the first law, \(\Lambda\) should be identified with thermodynamical pressure in the geometric units \(G_N = \hbar = c = k = 1\) as \[13\]

\[
P = -\frac{\Lambda}{8\pi} = \frac{(d - 1)(d - 2)}{16\pi l^2},
\]

where \(d\) denotes the dimension of space-time and \(l\) is the radius of AdS space-time. In this case, the black hole mass \(M\) should be treated as enthalpy rather than the internal energy, and the thermodynamical volume conjugate with the pressure of black hole can be calculated by using the standard thermodynamic identities \[14–18\]. So the phase space in this point of view is usually called as the extended phase space, different from the earlier one of black hole.

A physical first-order (LBH/SBH) phase transition \[19\] was discovered in the extended phase space of charged AdS black hole, where the critical behaviors and exponents are precisely identical with the Van der Waals liquid-gas system. Then, more interesting new critical phenomena were shown in higher-dimensional rotating AdS black hole. For example, in singly spinning rotating case \[20\], there exists a zero-order reentrant phase transition between intermediate black hole and small black hole along with LBH/SBH phase transition, which is a phase transition from large black holes to small ones and then back to large one again with increasing the temperature, a phenomenon also seen for Born-Infeld black hole \[21\], while in multi-spinning case \[22\], the system has a small/intermediate/large black hole (S/I/L BH) phase transition with one tricritical (or triple critical) point reminiscent of the "water-like" solid/liquid/gas phase transition. More discussions for other kinds of AdS black hole can also be found in \[23–44\].

The Einstein’s general relativity (GR) is universally accepted as a beautiful and accurate theory describing the force of gravity. From the perspective of modern particle physics, GR can be treated as a unique theory of a massless spin-2 graviton \[45, 46\]. Despite many successes agreement with observations, GR might be searched for alternatives due to the open questions, such as the cosmological constant problem \[47\] that the cosmological constant of astronomical observations is many orders of magnitude smaller than estimated in modern theories of elementary particles, and the origin of acceleration of our universe indicated from the supernova data \[48, 49\], and so
on. Massive gravity is a straightforward and natural modification by simply giving a mass to the graviton, which can date back to 1939 when Fierz and Pauli \[50\] constructed a linear theory of massive gravity. Whereas, an elementary problem is that the massive theory in non-linear level is always plagued with the Boulware-Deser ghost \[51, 52\]. Fortunately, a ghost-free massive theory was proposed in references \[53–56\], known as dGRT massive gravity. A class of charged black holes \[57, 58\] and their corresponding thermodynamics \[59\] in asymptotically AdS space-time were investigated in ghost-free massive gravity. Then other black holes were also studied in massive gravity, such as rotating black hole \[60, 61\], Schwarzschild-dS black hole \[62\]. The extensions of the solutions in Born-Infeld-massive gravity and Gauss-Bonnet-Massive Gravity were also constructed in \[63–66\]. Recently, the holographical and thermodynamical aspects of black holes in massive gravity were also investigated in \[67, 68\]. Moreover, the Van der Waals-like phase was discovered in the extended phase space of the charged AdS black hole in massive gravity \[69, 70\]. The reference \[71\] showed the relationship between this Van der Waals-like phase transition and the behavior of quasi-normal modes of charged AdS black hole in massive gravity. Particularly, the reentrant phase transition with a triple critical point was discovered in 6-dimensional AdS black hole in massive gravity \[72\].

In this paper, we will concentrate on the critical behaviors of \(d \geq 7\) dimensional AdS black hole, and report the finding of the more richer critical phenomena, including Van der Waals-like phase transition, reentrant phase transition, water-like solid/liquid/gas phase transition, and triple critical point. The organization of this paper is as follows. In Sec. II, considering the fifth order term of interaction of massive gravity in \(d \geq 7\) dimensional space-time, we present the thermodynamics in extended phase space of higher-dimensional AdS black hole. In Sec. III, we study the behaviors of \(d \geq 7\) dimensional AdS black hole, and reveal the richer critical phenomena in the context of \(P − V\) criticality and phase diagram. Finally, a brief discussion is presented in Sec. IV.

## II. EXTENDED PHASE SPACE THERMODYNAMICS OF HIGHER-DIMENSIONAL ADS BLACK HOLE IN MASSIVE GRAVITY

Let us start with the action for \(d\)-dimensional massive gravity \[55, 56\]

\[
I = \frac{1}{16\pi} \int d^d x \sqrt{-g} \left[ R + \Lambda + m^2 \sum_{i=1}^{n} c_i U_i(g, f) \right],
\]

where the last term denotes general form of the interaction potential with graviton mass \(m\), and \(n \leq d − 2\) the number of dimensionless coupling coefficients \(c_i\). Moreover, \(f\) is a fixed rank-
2 symmetric tensor, and $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the $d \times d$ matrix

$$K_{\mu}^\nu = \sqrt{g^{\alpha\beta}f_{\alpha\beta}},$$

and satisfying the following recursion relation

$$\mathcal{U}_i = - \sum_{j=1}^{i} (-1)^j \frac{(i-1)!}{(i-j)!} [K^j] \mathcal{U}_{i-j}. \quad (3)$$

Obviously, the first few terms can be read as

$$\mathcal{U}_1 = [K],$$
$$\mathcal{U}_2 = [K]^2 - [K^2],$$
$$\mathcal{U}_3 = [K]^3 - 3[K][K^2] + 2[K^3], \quad (4)$$
$$\mathcal{U}_4 = [K]^4 - 6[K^2][K^2] + 8[K^3][K] + 3[K^2]^2 - 6[K^4],$$
$$\mathcal{U}_5 = [K]^5 - 10[K^2][K]^3 + 15[K][K^2]^2 + 20[K][K^3]^2 - 20[K^2][K^3] - 30[K][K^4] + 24[K^5],$$

\[ \ldots \]

where the square brackets denote traces, i.e. $[K] = K_{\mu}^\mu$.

A static black hole solution of d-dimensional space-time is given as

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2h_{ij}dx^i dx^j, \quad (5)$$

in which $h_{ij}dx^i dx^j$ is the line element for an Einstein space with constant curvature $(d-2)(d-3)k$, and $k = 1, 0, -1$ correspond respectively to a spherical, Ricci flat, and hyperbolic topology subspace. Considering the following reference metric

$$f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij}), \quad (6)$$

the interaction potential Eq. (4) changes into

$$\mathcal{U}_j = \left( \prod_{k=2}^{j+1} d_k \right) c_0^j r^{-j} \quad (7)$$

with positive constant $c_0$ and the notation $d_k = (d-k)$. It is worth to note that the $c_5m^2$ term only appears in the action for $d \geq 7$, so we just consider the $d \geq 7$ dimensional black hole and $n = 5$ in this paper.

Then, the metric function is calculated as

$$f(r) = k + \frac{16\pi P}{d_1 d_2} r^2 - \frac{16\pi M}{d_2 V_{d-2} r^{d-3}} + \frac{c_0 c_1 m^2}{d_2} r + c_0^2 c_2 m^2 \quad + \frac{d_3 c_0^3 c_3 m^2}{r} + \frac{d_3 d_4 c_0^4 c_4 m^2}{r^2} + \frac{d_3 d_4 d_5 c_0^5 c_5 m^2}{r^3}, \quad (8)$$
we can get the Smarr relation as follow using the scaling method 

\[ M = \frac{d_2 V_{d-2} r_h^{d-3}}{16\pi} \left[ k + \frac{16\pi P}{d_1 d_2} r_h^2 + \frac{c_0 c_1 m^2}{d_2} r_h + c_0^2 c_2 m^2 + \frac{d_3 c_0^3 c_3 m^2}{r_h} \right] + \frac{d_3 d_4 c_0^4 c_4 m^2}{r_h^2} + \frac{d_3 d_4 d_5 c_0^5 c_5 m^2}{r_h^3}. \] 

(9)

And the Hawking temperature \( T \) and the entropy \( S \) of black hole can be obtained as

\[ T = \frac{1}{4\pi r_h} \left[ d_3 k + \frac{16\pi P}{d_2} r_h^2 + c_0 c_1 m^2 r_h + d_3 c_0^2 c_2 m^2 + \frac{d_3 d_4 c_0^3 c_3 m^2}{r_h} \right] + \frac{d_3 d_4 d_5 c_0^4 c_4 m^2}{r_h^2} + \frac{d_3 d_4 d_5 c_0^5 c_5 m^2}{r_h^3} \]

\[ S = \frac{V_{d-2}}{4} r_h^{d-2}. \]

(10)

Due to the mass of black hole corresponding to the enthalpy of an AdS gravitational system, we can get the Smarr relation as follow using the scaling method

\[ (d - 3) M = (d - 2) TS - 2 PV - \frac{c_0 c_1 m^2 V_{d-2} r_h^{d-2}}{16\pi} + \frac{d_2 d_3 c_0^3 c_3 m^2 V_{d-2} r_h^{d-4}}{16\pi} + \frac{d_2 d_3 d_4 c_0^4 c_4 m^2 V_{d-2} r_h^{d-5}}{8\pi} + \frac{3d_2 d_3 d_4 d_5 c_0^5 c_5 m^2 V_{d-2} r_h^{d-6}}{16\pi}, \]

(11)

where \( V = \frac{V_{d-2}}{d-1} r_h^{d-1} \) denotes the thermodynamic volume conjugate with the pressure \( P \) in extended phase space. Moreover, the first law of black hole thermodynamics can be written as the following differential relation

\[ dM = T dS + V dP + \frac{c_0 c_1 m^2 V_{d-2} r_h^{d-2}}{16\pi} dc_1 + \frac{d_2 d_3 c_0^3 c_3 m^2 V_{d-2} r_h^{d-4}}{16\pi} dc_2 + \frac{d_2 d_3 d_4 c_0^4 c_4 m^2 V_{d-2} r_h^{d-5}}{16\pi} dc_3 + \frac{d_2 d_3 d_4 d_5 c_0^5 c_5 m^2 V_{d-2} r_h^{d-6}}{16\pi} dc_4. \]

(12)

Obeying the thermodynamical formulas, the Gibbs free energy can be written as

\[ G = M - TS = - \frac{V_{d-2} r_h^{d-3}}{2} \left[ \frac{2 P r_h^2}{d_1 d_2} + \omega_2 + \frac{2 d_3 \omega_3}{r_h} + \frac{3 d_3 d_4 \omega_4}{r_h^2} + \frac{4 d_3 d_4 d_5 \omega_5}{r_h^3} \right]. \]

(13)
III. CRITICAL BEHAVIORS OF HIGHER-DIMENSIONAL BLACK HOLE

A. Equation of state

With the help of Eq. (10), the equation of state of black hole \( P(V, T) \) can be written as

\[
P = \frac{d_2}{4r_h} \left[ T - \frac{d_3 k}{4\pi r_h} - \frac{c_0 c_1 m^2}{4\pi} - \frac{d_3 c_0^2 c_2 m^2}{4\pi r_h} - \frac{d_3 d_4 c_0^3 c_3 m^2}{4\pi r_h^3} \right].
\]

(14)

According to equation of state\[14\], we can find that the value \( v = \frac{4r_h}{d_2} \propto r_h \) is regarded as special volume comparing to the Van der Waals fluid equation. For the further convenience, we introduce the following denotations

\[\hat{T} = T - \frac{c_0 c_1 m^2}{4\pi}; \quad \omega_2 = -\frac{k + c_0^2 c_2 m^2}{8\pi}; \]
\[\omega_3 = -\frac{c_0^3 c_3 m^2}{8\pi}; \quad \omega_4 = -\frac{c_0^4 c_4 m^2}{8\pi}; \quad \omega_5 = -\frac{c_0^5 c_5 m^2}{8\pi}\]

(15)

where \( \hat{T} \) is called shifted temperature and could be negative value.

Using the condition of inflection point in Van der Walls system

\[
\frac{\partial P}{\partial r_h} \bigg|_{T=T_c,r_h=r_c} = \frac{\partial^2 P}{\partial r_h^2} \bigg|_{T=T_c,r_h=r_c} = 0,
\]

(16)

we can receive the critical shifted temperature and the equation of critical radius of black hole as

\[
\hat{T}_c = -\frac{2d_3}{r_c} \left[ 2\omega_2 + \frac{3d_4 \omega_3}{r_c} + \frac{4d_4 d_5 \omega_4}{r_c^2} + \frac{5d_4 d_5 d_6 \omega_5}{r_c^3} \right]
\]

(17)

\[
\omega_2 r_c^3 + 3d_4 \omega_3 r_c^2 + 6d_4 d_5 \omega_4 r_c + 10d_4 d_5 d_6 \omega_5 = 0
\]

(18)

In the case of \( \omega_2 = 0 \), or \( \omega_5 = 0 \), one can easily find that the Eq. (18) has at most two positive real roots corresponding to the critical radii of black hole, the critical behaviors are similar to the ones in reference[72]. We neglected it and focus only on the case of \( \omega_2 \neq 0, \omega_5 \neq 0 \), where exists probably three critical radii.

For the simplicity, denoting

\[
\alpha = 3d_4 \omega_3/\omega_2, \quad \beta = 6d_4 d_5 \omega_4/\omega_2, \quad \gamma = 10d_4 d_5 d_6 \omega_5/\omega_2,
\]

(19)

the critical temperature and pressure become into

\[
\hat{T}_c = \frac{2d_3 \omega_2}{r_c} \left[ 2 + \frac{\alpha}{r_c} + \frac{2\beta}{3r_c^2} + \frac{\gamma}{2r_c^3} \right],
\]

\[
\hat{P}_c = \frac{2d_2 d_3 \omega_2}{r_c^2} \left[ 2 + \frac{5\alpha}{3r_c} + \frac{\beta}{r_c^2} + \frac{4\gamma}{5r_c^3} \right].
\]

(20)
which just depends on the dimension of space-time by a factor $d_2d_3 (d_3)$ for given parameters $(\omega_2, \alpha, \beta, \gamma)$.

It is easy to find that the Eq.(18) has at most two positive roots in the case $\alpha \geq 0$. On the other hand, there would exist three real roots if $\alpha < 0$. Since the value of $\alpha$ could be set to be one by re-scaling $r_c$, we will assume $\alpha = -1$ in following discussion. Based on the requirement of three real roots of the Eq.(18), we can obtain the relation between the parameters and the number of positive real roots in TABLE. I.

| parameters | $\alpha = -1$ |
|-----------|----------------|
| $\beta$   | $1/4 < \beta < 1/3$ | $0 < \beta \leq 1/4$ | $\beta \leq 0$ |
| $\gamma$  | $\gamma_- < \gamma < \gamma_+$ | $0 < \gamma < \gamma_+$ | $0 < \gamma < \gamma_+$ | $\gamma_- < \gamma < 0$ |
| number of $r_c$ | 3 | 3 | 2 | 2 | 1 |

TABLE I: The behaviors of the critical radii for different values of $\beta$ and $\gamma$ with $\gamma \pm = \frac{2-9\beta \pm 2(1-3\beta)^{3/2}}{27}$.

Due to the positive pressures, when all of the pressures $P_{c1,2,3}$ corresponding to the critical radii above are positive, we can obtain three critical points in the P-V process, and when two of $P_{c1,2,3}$ are positive, we will get two critical points, and so on. For example, taking $d=7$ and letting $\omega_2 = \pm 1$ we will get three positive corresponding pressures while $\omega_2 = -1$, $2/9 < \beta < 1/4$, $\gamma_p < \gamma < 0$, or $\omega_2 = -1$, $1/4 < \beta < 1/3$, $\gamma_p < \gamma < 0$, where $\gamma_p$ must be determined numerically.

According to the Gibbs free energy Eq.(13) and the equation of state Eq.(14), let us now study the possible phase transitions of this system.

**B. Van der Waals-like phase transition**

![FIG. 1: The $P - r_h$ and $G - T$ diagrams in $d=7$ with $\omega_2 = -1$, $\alpha = -1$, $\beta = -1$, $\gamma = -0.1$.](image-url)
In the case of one critical radius of Eq.(18) in TABLE.I, the critical behaviors of black hole are analogous to that of the standard Van der Waals-like system as displayed in FIG.1.

In FIG.1(a) there is only one critical isotherm (red dashed line) when $\hat{T} = \hat{T}_c \approx 5.51733$, and exits an inflection point in the isotherm (the green solid line) when $\hat{T} < \hat{T}_c$. Moreover, in FIG.1(b) displaying the behaviors of $G$ with a critical isobar (red dashed line) when $P = P_c \approx 1.48178$, the isobar corresponding to $P < P_c$ is depicted with a "swallowtail", which implies a first-order LBH/SBH phase transition.

C. Reentrant phase transition

There are two ranges of parameters for two critical radii in TABLE.II, both of which display the similar thermodynamical processes with a reentrant phase transition, analogous to that in references [20, 21, 72]. In one case, by setting $\beta = 0.245$ and $\gamma = 0.0001$, the critical behaviors are investigated as follow.

In the $P - r_h$ processes indicated in FIG.2(a), there exit two critical isotherms, $\hat{T} = \hat{T}_{c1} \approx 10.50671$ and $\hat{T} = \hat{T}_{c2} \approx 10.38895$, corresponding respectively to the red and black dashed lines. Moreover, there are two inflection points located in each isotherm of the branch corresponding to $\hat{T}_{c2} < \hat{T} < \hat{T}_{c1}$ as displayed by the blue solid isotherm.

The behaviors of Gibbs free energy $G$ are displayed in FIG.2(b), where the radius of black hole increases from right to left along each isobar, and two critical isobars, the red and black dashed lines, correspond respectively to the critical pressures $P = P_{c1} \approx 6.39165$ and $P = P_{c2} \approx 6.09163$. The magenta solid isobar corresponds to the tricritical point, $(\hat{T}_{tr} \approx 10.44195, P_{tr} \approx 6.25396)$, while the blue one to $(\hat{T}_z \approx 10.44797, P_z \approx 6.27301)$. As the temperature decreases from right
to left, in the range \( P \in (P_z, P_{c1}) \) and \( \hat{T} \in (\hat{T}_z, \hat{T}_{c1}) \), there is a "swallowtail" indicated by the green solid isobar signifying a Van der Waals-like phase transition. In the range \( P \in (P_{c2}, P_{tr}) \) and \( \hat{T} \in (\hat{T}_{c2}, \hat{T}_{tr}) \), a "swallowtail" displayed by the purple isobar, does not correspond to the Van der Waals-like phase transition, because it is not a "physical" process due to the global minimum of the Gibbs free energy, so that \((\hat{T}_{c2}, P_{c2})\) is not a "really and physically" critical point.

Especially in the ranges \( P \in (P_{tr}, P_z) \) and \( \hat{T} \in (\hat{T}_{tr}, \hat{T}_z) \), the global minimum of \( G \) is discontinuous as the temperature decreasing. Take, the isobar \( P = 6.26 \in (P_{tr}, P_z) \) displayed in FIG.3(a), for example, the value of \( G \) experiences a finite jump at \( \hat{T} = \hat{T}_0 \approx 10.44382 \in (\hat{T}_{tr}, \hat{T}_z) \) due to the global minimum of the Gibbs free energy, which signifies the zero-order reentrant phase transition between small and intermediate black holes.

![Figure 3](image)

FIG. 3: The reentrant phase transition for \( P = 6.26 \) and the \( P - \hat{T} \) phase diagram in \( d=7 \) with \( \omega_2 = -1, \alpha = -1, \beta = 0.245, \gamma = 0.0001 \).

It is obviously found in \( P - \hat{T} \) phase diagram in FIG.3(b) that the red line in the inset, initiating from the triple critical point \((\hat{T}_{tr}, P_{tr})\) and terminating at \((\hat{T}_z, P_z)\), corresponds to the coexistence line between IBH and SBH, while the blue one, initiating from \((\hat{T}_{tr}, P_{tr})\) and terminating at \((\hat{T}_{c1}, P_{c1})\), to the coexistence line of SBH and LBH. It is worth to state that, in the range \( P < P_{tr} \) and \( \hat{T} < \hat{T}_{tr} \), there is no phase transition between SBH (or IBH) and LBH so that the \((\hat{T}_{tr}, P_{tr})\) is unlike the common one of water.

In another case, by setting \( \beta = -1 \) and \( \gamma = 0.85 \), the critical behaviors are analogous to those in preceding case ( \( \beta = 0.245 \) and \( \gamma = 0.0001 \)), since the \( P - \hat{T} \) phase diagram in FIG.4 is obviously similar to the one in FIG.3(b). In FIG.4 the "physical" critical point is \((\hat{T}_{c1} \approx 6.34216, P_{c1} \approx 2.05297)\), the tricritical point is \((\hat{T}_{tr} \approx 5.75374, P_{tr} \approx 1.57488)\), and the termination point of reentrant phase transition is \((\hat{T}_z \approx 5.76867, P_z \approx 1.60544)\). The blue line corresponds to the
coexistence line between SBH and LBH, while the red one in the inset to the one between SBH and IBH.

FIG. 4: The $P - \hat{T}$ phase diagram in $d=7$ with $\omega_2 = -1, \alpha = -1, \beta = -1, \gamma = 0.85$.

D. Triple critical point and solid/liquid/gas phase transition

In the case of three critical radii in TABLE II there are also two cases for different parameters, both of which display the analogous thermodynamical processes as a solid/liquid/gas phase transition with a tricritical (triple critical) point. Therefore, we just focus on one of them by setting $\beta = 0.3, \gamma = -0.027$.

FIG. 5: The $P - r_h$ diagram in $d=7$ with $\omega_2 = -1, \alpha = -1, \beta = 0.3, \gamma = -0.027$.

In FIG. 5, the temperature of isotherm increases from lower left to upper right in the $P - r_h$ diagram. Obviously, the $P - r_h$ diagram, which is more complex than that of the standard Van der Waals system, has three critical isotherms, $\hat{T} = \hat{T}_{c1} \approx 13.65904, \hat{T} = \hat{T}_{c2} \approx 11.08718, \hat{T} = \hat{T}_{c3} \approx 10.37037$, corresponding respectively to purple, black, cyan dashed lines. Particularly, there are
three inflection points of each isotherm of the branch in the range \((\hat{T}_{c3} < \hat{T} < \hat{T}_{c2})\), as indicated by the blue one, which maybe suggests the existence of the more complex phase structures.

The behaviors of Gibbs free energy \(G\) of black hole are depicted in Fig.6 where the three critical isobars, \(P_{c1} \approx 24.39363\), \(P_{c2} \approx 7.24789\), \(P_{c3} \approx 4.938276\), are labeled by the purple, black, and cyan dashed lines respectively. The blue isobar corresponds to the tricritical point \((P_{tr} \approx 6.64714, \hat{T}_{tr} \approx 10.80263)\). The pressure of isobar increases from left to right, and the black hole radius \(r_h\) also increases along each isobar from left to right. As the temperature decreases from right to left, for \(P > P_{c1}\), there is no critical behavior as shown by the green isobar. For \(P_{c2} < P < P_{c1}\) and \(\hat{T}_{c2} < \hat{T} < \hat{T}_{c1}\), the pink isobar has one swallowtail, implying a first-order Van der Waals-like phase transition. For \(P_{tr} < P < P_{c2}\) and \(\hat{T}_{tr} < \hat{T} < \hat{T}_{c2}\), the behaviors are displayed by the red isobar in FIG.6(b), which shows two swallowtails, corresponding to the coexistence of two first-order phase transitions—SBH/IBH and IBH/LBH phase transitions. Until at \(P = P_{tr}\) and \(\hat{T} = \hat{T}_{tr}\) in blue isobar in FIG.6(b), the two swallowtails merge with each other, corresponding to the tricritical point of the small, intermediate and large black holes. As the temperature continuously decreases, for \(P_{c3} < P < P_{tr}\) and \(\hat{T}_{c3} < \hat{T} < \hat{T}_{tr}\), the magenta isobar describing the behaviors is also with two swallowtails, but the "small" swallowtail on the upper-right side is not a "really and physical" critical process because of the globally minimization of the Gibbs free energy \(G\), similar to the situation of the reentrant phase transition in Subsec. III C. So only the "big" one left corresponds to the first-order phase transition. Finally, there is also only one swallowtail occurring in the case \(P < P_{tr}\), so that the critical point \((P_{c3}, \hat{T}_{c3})\) is not a physically critical one. Therefore, there exist two physically critical points \((P_{c1}, \hat{T}_{c1})\) and \((P_{c2}, \hat{T}_{c2})\) and one tricritical point \((P_{tr}, \hat{T}_{tr})\).

![G vs T](a)

![Magnification of G vs T](b)

**FIG. 6:** The behaviors of \(G\) in \(d=7\) with \(\omega_2 = -1, \alpha = -1, \beta = 0.3, \gamma = -0.027\). A magnification of the blue, red, and black isobars is displayed in (b).

This tricritical behavior of S/I/L BH phase transition is obviously depicted by \(P - \hat{T}\) phase
diagrams in Fig. These diagrams are analogous to that of the water-like solid/liquid/gas phase transition with a triple critical point. The SBH/LBH coexistence line, denoted in green color and terminating at the tricritical point $(\hat{T}_{tr}, P_{tr})$, corresponds to the solid/gas coexistence line of water. The LBH/IBH and IBH/SBH coexistence lines, depicted respectively by the red and blue ones, do not extend to infinity and terminate respectively at the critical points 1 $(\hat{T}_{c1}, P_{c1})$ and 2 $(\hat{T}_{c2}, P_{c2})$, corresponding respectively to the solid/liquid and liquid/gas coexistence ones of water. Moreover, the join point (or the tricritical point) of the the three coexistence lines corresponds to the state of coexistence of small/intermediate/large black holes with a special critical value of temperature and pressure. It is necessary to note that this triple critical point is a more "common" one than that in reentrant phase transition, because there still exists a coexistence line between SBH and LBH when $0 < \hat{T} < \hat{T}_{tr}$ and $0 < P_{c1} < P_{tr}$, which is not seen in the reentrant phase transition in FIG.3(b) and FIG.4.

FIG. 7: The $P - \hat{T}$ diagrams in d=7 with $\omega_2 = -1$, $\alpha = -1$, $\beta = 0.3$, $\gamma = -0.027$.

FIG. 8: The $P - r_h$ and $G - \hat{T}$ diagrams in d=7 with $\omega_2 = -1$, $\alpha = -1$, $\beta = 0.245$, $\gamma = -0.009$. 
By setting $\beta = 0.245$, $\gamma = -0.009$ in another case of three critical radii, the "P-V" processes and critical behaviors of $G$ of BH indicated in FIG.8 are respectively similar to those in FIG.5 and FIG.6. In FIG.8(a), the black, magenta, and purple dashed isotherms correspond respectively to the critical temperatures $T_{c1} \approx 1967.34429$, $T_{c2} \approx 10.23876$, and $T_{c3} \approx 8.31659$. Because $T_{c1} \gg T_{c2}, T_{c3}$, the pressures of the solid pink, dashed black, and solid green lines are reduced by 2500 times, while the radii enlarged by 9 times, so that all the isotherms can be displayed in one diagram. In FIG.8(b) the magenta and purple dashed isobars correspond respectively to $P_{c2} \approx 5.94913$, $P_{c3} \approx 0.29619$. The blue isobar has two swallowtails (big one displayed partially in Fig.8(b)), which implies existence of SBH/IBH and IBH/LBH phase transitions. The red isobar denotes to the tricritical point ($T_{tr} \approx 10.07367$, $P_{tr} \approx 5.64492$). Due to $P_{c1} \approx 35484.97609 \gg P_{c2}, P_{c3}$, the isobars for $P \geq P_{c1}$ are not displayed in this diagram. It is more obvious that the magnification of $P - \hat{T}$ phase diagram near tricritical point in FIG.9(b) is analogous to the one in FIG.7(b).

**FIG. 9:** The $P - \hat{T}$ diagrams in $d=7$ with $\omega_2 = -1$, $\alpha = -1$, $\beta = 0.245$, $\gamma = -0.009$.

**IV. DISCUSSION**

In this paper, we have shown that, treating the cosmological constant as pressure and interpreting the corresponding conjugate quantity as thermodynamic volume, the higher-dimensional ($d \geq 7$) AdS black holes in massive gravity exhibit more rich and interesting thermodynamical behaviors, including Van der Waals-like phase transition, reentrant phase transition, solid/liquid/gas phase transition and triple critical point, and so on. At first, introducing the fifth term $c_5U_5$ of interaction potential in the action of massive gravity theory in the higher dimensional ($d \geq 7$) space-time, we have obtained a class of solutions of AdS black holes and constructed the thermodynamics of
black hole in the extend phase space. According to the conditions of the inflection point, we then get the equations of critical temperature and radius basing on the state equation of this system. Meanwhile, we paid more attention on the case of three real roots of the equation of the critical radius because of the more complex phase construction, and obtained the conditions of parameters corresponding to the number of positive roots or critical radii. Finally, we studied the critical behaviors of the Gibbs free energy of black hole. In the case of only one critical radius, the critical behaviors of black hole is analogous to those of standard Van der Waals-like system. In the case of two critical radii, it exists a zero-order reentrant phase transition between IBH and SBH due to the global minimum of $G$. Particularly in the case of three ones, there is a water-like solid/liquid/gas phase transition among SBH, IBH and LBH with a ”common” triple critical point where three phase of black hole can coexists with the same values of temperature, pressure and the Gibbs free energy.

It is necessary to state that the critical behaviors of AdS black holes in massive gravity are dependent crucially on the dimension of the space-time through the parameter $n$, because whether the $c_i$’s term appears in the action of Eq.(2) is constrained by $n \leq d - 2\ [55]$. For $d = 5$ ($n \leq 3$) case in the reference[70], setting $\beta = \gamma = 0$ (or $\omega_4 = \omega_5 = 0$) due to $U_4 = U_5 = 0$, and find that there is at most one critical radius in Eq.(18), thus it has only one-order Van der Waals-like phase transition similar to that in Fig.1. For $d = 6$ ($n \leq 4$) case in reference[72], corresponding to $\gamma = 0$, the Eq.(18) has at best two critical radii, so the zero-order reentrant phase transition occurs analogous to that indicated in FIG.3 or FIG.4. Only for $d \geq 7$ could $\gamma$ not vanish, so that the solid/liquid/gase phase transition can appear in the case of three critical radii of the $P - r_h$ process. Specifying to 7 dimensions, in the case of three critical radii, the solid/liquid/gase phase transition with a common triple critical point can occur as displayed in FIG.7, where the S/I/L BH can merge into a single kind with the same critical value of temperature and pressure. For $n = 5$ and $d > 7$ case, the thermodynamical behaviors is similar to ones in $d = 7$. Moreover, the black hole will have more rich thermodynamical structure in high dimensional space-time since much interaction terms appears in the action, which is valued to be investigated in future.

**Acknowledgements**

We would like to thank Dr. Decheng Zou and Dr. Ming Zhang for many discussions. This work was supported by the National Natural Science Foundation of China under Grant No.11675139,
No.11435006 and No.11875220.

[1] S.W.Hawking, Particle creation by black holes Commun. Math. Phys. 43, 199(1975).
[2] S. Hawking and D.N. Page, Thermodynamics of black holes in Anti-de Sitter space, Commun. Math. Phys. 87, 577(1983).
[3] J. M.Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. 2, 231(1998).
[4] J. M.Maldacena, The large-N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. 38, 1113 (1999).
[5] S. S.Gubser, I. R.Klebanov, and A. M.Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B 428, 105(1998).
[6] E. Witten, Anti De Sitter Space And Holography, Adv. Theor. Math. Phys. 2,253 (1998).
[7] E. Witten, Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories, Adv. Theor. Math. Phys. 2, 505(1998).
[8] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, Charged AdS black holes and catastrophic holography, Phys. Rev. D 60, 064018(1999).
[9] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, Holography, thermodynamics and fluctuations of charged AdS black holes, Phys. Rev. D 60, 104026 (1999).
[10] C.Niu,Y.Tian and X.N. Wu, Critical phenomena and thermodynamic geometry of RN-AdS black holes, Phys. Rev. D 85, 024017(2012).
[11] N. Breton, Smarrs formula for black holes with non-linear electrodynamics, Gen. Rel. Grav. 37, 643(2005).
[12] Y. Sekiwa, Thermodynamics of de Sitter black holes: Thermal cosmological constant, Phys. Rev. D 73, 084009 (2006).
[13] D. Kastor, S. Ray and J. Traschen, Enthalpy and the mechanics of AdS black holes, Class. Quant. Grav. 26, 195011 (2009).
[14] M.K.Parikh, Volume of black holes, Phys. Rev. D 73, 124021 (2006).
[15] B. P.Dolan, The cosmological constant and black-hole thermodynamic potentials, Class. Quant. Grav. 28, 125020 (2011).
[16] B. P.Dolan, Pressure and volume in the first law of black hole thermodynamics, Class. Quant. Grav. 28, 235017 (2011).
[17] B. P. Dolan, Compressibility of rotating black holes, Phys. Rev. D 84, 127503 (2011).
[18] W.Ballik, and K.Lake, Vector volume and black holes, Phys. Rev. D 88, 104038 (2013).
[19] D.Kubiznák, and R.B. Mann, P-V criticality of charged AdS black holes J. High Energy Phys. 2012, 33(2012).
[20] N. Altamirano, D. Kubizňák, and R. B. Mann, Reentrant phase transitions in rotating anti-de Sitter black holes, Phys. Rev. D 88, 101502 (2013).
[21] S. Gunasekaran, D. Kubizňák, and R. B. Mann, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, J. High Energy Phys. 2012, 110 (2012).
[22] N. Altamirano, D. Kubizňák, R. B. Mann, and Z. Sherkatghanad, Kerr-AdS analogue of triple point and solid/liquid/gas phase transition, Class. Quant. Grav. 31, 042001 (2014).
[23] S. H. Hendi, and M. H. Vahidinia, Extended phase space thermodynamics and P-V criticality of black holes with a nonlinear source, Phys. Rev. D 88, 084045 (2013).
[24] D. Hansen, Kubizňák, and R. B. Mann, Universality of P-V criticality in horizon thermodynamics J. High Energy Phys. 2017, 47 (2017).
[25] R. G. Cai, L. M. Cao, L. Li, R. Q. Yang, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, J. High Energy Phys. 1309, 005 (2013).
[26] S. Dutta, A. Jain, R. Soni, Dyonic Black Hole and Holography, J. High Energy Phys. 2013, 60 (2013).
[27] W. Xu, H. Xu, L. Zhao, Gauss-Bonnet coupling constant as a free thermodynamical variable and the associated criticality, Eur. Phys. J. C 74, 2970 (2014).
[28] D. C. Zou, S. J. Zhang, B. Wang, Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics, Phys. Rev. D 89, 044002 (2014).
[29] H. H. Zhao, L. C. Zhang, M. S. Ma, R. Zhao, P-V criticality of higher dimensional charged topological dilaton de Sitter black holes, Phys. Rev. D 90, 064018 (2014).
[30] N. Altamirano, D. Kubiznak, R. B. Mann, Z. Sherkatghanad, Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume, Galaxies 2, 89 (2014).
[31] J. X. Mo, W. B. Liu, P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity, Eur. Phys. J. C 74, 2836 (2014).
[32] D. C. Zou, Y. Liu, B. Wang, Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble Phys. Rev. D 90, 044063 (2014).
[33] H. Xu, W. Xu, L. Zhao, Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions, Eur. Phys. J. C 74, 3074 (2014).
[34] W. Xu, L. Zhao, Critical phenomena of static charged AdS black holes in conformal gravity, Phys. Lett. B 736, 214 (2014).
[35] M. H. Dehghani, S. Kamrani, and A. Sheykhi, P-V criticality of charged dilatonic black holes. Phys. Rev. D 90, 104020 (2014).
[36] C. O. Lee, The Extended Thermodynamic Properties of Taub-NUT/Bolt-AdS spaces, Phys. Lett. B 738, 294 (2014).
[37] J. L. Zhang, R. G. Cai, H. Yu, Phase transition and thermodynamical geometry for Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime, J. High Energy Phys. 1502, 143 (2015).
[38] M. Zhang, Z. Y. Yang, D. C. Zou, W. Xu, R. H. Yue, P-V criticality of AdS black hole in the Einstein-Maxwell-power-Yang-Mills gravity, Gen. Relativ. Gravit. 47, 14 (2015).
[39] J.L. Zhang, R.G. Cai, H. Yu, Phase transition and Thermodynamical geometry of Reissner-Nordström-AdS Black Holes in Extended Phase Space, Phys. Rev. D 91, 044028 (2015).

[40] S.H. Hendi, S. Panahiyan, B. Eslam Panah, M.Faizal, M. Momennia, Critical behavior of charged black holes in Gauss-Bonnet gravity’s rainbow, Phys. Rev. D 94, 024028 (2016).

[41] D. Kubiznak, R.B. Mann, M. Teo, Black hole chemistry: thermodynamics with Lambda, Class. Quant. Grav. 34, 063001 (2017).

[42] M.Zhang, D. C.Zou, and R. H.Yue, Reentrant phase transitions and triple points of topological AdS black holes in Born-Infeld-massive gravity, Advances in High Energy Physics, 2017(2017).

[43] X.M. Kuang, O. Miskovic, Thermal phase transitions of dimensionally continued AdS black holes, Phys. Rev. D 95, 046009 (2017).

[44] Y.G. Miao, Y.M. Wu, Thermodynamics of the Schwarzschild-AdS black hole with a minimal length, Adv. High Energy Phys. 2017, 1095217 (2017).

[45] S. Weinberg, Photons and gravitons in perturbation theory: Derivation of Maxwell’s and Einstein’s equations, Phys. Rev. B 138, 988(1965).

[46] D. G.Boulware, and S. Deser, Classical general relativity derived from quantum gravity, Annals of Physics 89, 193(1975).

[47] S. Weinberg, The cosmological constant problem, Rev. mod. phys. 61, 1(1989).

[48] A. G.Riess, et al.(Supernova Search Team), Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J 116, 1009(1998).

[49] Perlmutter:1999, Perlmutter,  et al.(Surpernova Cosmoly Project), Measurements of Ω and Λ from 42 high-redshift supernovae, Astrophys. J 517, 565(1999).

[50] M.Fierz, and W.Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, Proc. R. Soc. Lond. A 173, 211(1939).

[51] D. G.Boulware, and S.Desser, Inconsistency of finite range gravitation, Phys. Lett. B 40, 227(1972).

[52] D. G.Boulware, and S.Deser, Can gravitation have a finite range?, Phys. Rev. D 6, 3368(1972).

[53] C.de Rham, and G.Gabadadze, Generalization of the Fierz-Pauli action, Phys. Rev. D 82(4), 044020(2010).

[54] C.de Rham, G.Gabadadze, and A. J.Tolley, Resummation of massive gravity, Phys. Rev. Lett. 106, 231101(2011).

[55] K. Hinterbichler, Theoretical aspects of massive gravity, Rev. Mod. Phys. 84, 671 (2012)

[56] C.de Rham, Massive gravity, Living Rev. Relativity 17, 7(2014).

[57] D. Vegh, Holography without translational symmetry, arXiv: 1301.0537 [hep-th].

[58] E.Babichev, and A.Fabbri, A class of charged black hole solutions in massive (bi) gravity, J High Energy Phys. 2014, 16(2014).

[59] R. G.Cai, Y. P.Hu, Q. Y.Pan, and Y. L.Zhang, Thermodynamics of black holes in massive gravity, Phys. Rev. D 91, 024032(2015).

[60] E.Babichev, and A.Fabbri, Rotating black holes in massive gravity, Phys. Rev. D 90, 084019 (2014).
[61] A. Aceña, E. López, and M. Llerena, Isoperimetric surfaces and area-angular momentum inequality in a rotating black hole in new massive gravity, Phys. Rev. D 97, 064043(2018).

[62] H. Kodama, and I. Arraut, Stability of the Schwarzschild-de Sitter black hole in the dRGT massive gravity theory, Progress of Theoretical and Experimental Physics 2014, 023E02(2014).

[63] S. H. Hendi, B. E. Panah, and S. Panahiyan, Einstein-Born-Infeld-massive gravity: adS-black hole solutions and their thermodynamical properties, J High Energy Phys. 2015, 157(2015).

[64] S. H. Hendi, G. Q. Li, J. X. Mo, S. Panahiyan, and B. E. Panah, New perspective for black hole thermodynamics in Gauss-Bonnet-Born-Infeld massive gravity, Euro. Phys. J C 76, 571(2016).

[65] S. H. Hendi, and M. Momennia, Thermodynamic description of (a)dS black holes in Born-Infeld massive gravity with a non-abelian hair, arXiv: 1801.07906.

[66] S. H. Hendi, B. Eslam Panah, and S. Panahiyan, Black Hole Solutions in Gauss-Bonnet-Massive Gravity in the Presence of Power-Maxwell Field, Fortschritte der Physik 66, 1800005(2018).

[67] X. X. Zeng, H. Zhang, and L. F. Li, Phase transition of holographic entanglement entropy in massive gravity, Phys. Lett. B 756, 170(2016).

[68] S. H. Hendi, N. Riazi, and S. Panahiyan, Holographical aspects of dyonic black holes: massive gravity generalization, Annalen der Physik 530, 1700211(2018).

[69] S. H. Hendi, R. B. Mann, S. Panahiyan, and B. E. Panah, Van der Waals like behavior of topological AdS black holes in massive gravity, Phys. Rev. D 95, 021501 (2017).

[70] J. Xu, L. M. Cao, and Y. P. Hu, P-V criticality in the extended phase space of black holes in massive gravity, Phys. Rev. D 95, 021501 (2017).

[71] D. C. Zou, Y. Q. Liu, R. H. Yue, Behavior of quasinormal modes and Van der Waals-like phase transition of charged AdS black holes in massive gravity, Eur. Phys. J. C 77, 365(2017).

[72] D. C. Zou, R. H. Yue, M. Zhang, Reentrant phase transitions of higher-dimensional AdS black holes in dRGT massive gravity, Eur. Phys. J. C 77, 256(2017).