Spontaneous Particle-Hole Symmetry Breaking in the $\nu = 5/2$ Fractional Quantum Hall Effect

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The essence of the $\nu = 5/2$ fractional quantum Hall effect is believed to be captured by the Moore-Read Pfaffian (or anti-Pfaffian) description. However, a mystery regarding the formation of the Pfaffian state is the role of the three-body interaction Hamiltonian $H_3$ that produces it as an exact ground state and the concomitant particle-hole symmetry breaking. We show that a two-body interaction Hamiltonian $H_2$ constructed via particle-hole symmetrization of $H_3$ produces a ground state nearly exactly approximating the Pfaffian and anti-Pfaffian states, respectively, in the spherical geometry. Importantly, the ground state energy of $H_2$ exhibits a “Mexican-hat” structure as a function of particle number in the vicinity of half filling for a given flux indicating spontaneous particle-hole symmetry breaking. This signature is absent for the second Landau level Coulomb interaction at 5/2.

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The fractional quantum Hall effect (FQHE) at orbital Landau level (LL) filling factor $\nu = 5/2$ filling of the second LL (SLL)) is the subject of recent theoretical and experimental research. This is partly due to the Moore-Read Pfaffian state $\Psi$, the leading theoretical description of the $\nu = 5/2$ FQHE, possessing non-Abelian quasiparticle excitations with potential applications towards fault-tolerant topological quantum computation $\Psi$. Recent theoretical results along with previous work $\Psi$ provide compelling evidence that this non-abelian description is essentially correct.

However, a question remains regarding the Moore-Read Pfaffian (Pf) description best illustrated by contrasting it to the celebrated Laughlin state $\Psi$ for the FQHE at $\nu = 1/q$ ($q$ odd). When confined to a single LL, two-body interaction Hamiltonians can be parameterized by Haldane pseudopotentials $V_m$ for the energy of a pair of electrons in a state of relative angular momentum $m$ where only odd $m$ enters for spin-polarized electron systems. The Laughlin state is the exact ground state of a two-body Hamiltonian with only $V_1$ non-zero (the interaction is hard-core). Thus, through the pseudopotential description, the Laughlin state is shown to be continuously connected to the exact ground state of the Coulomb Hamiltonian at $\nu = 1/q$. The Pf wave function, by contrast, is an exact ground state of a repulsive three-body Hamiltonian $H_3$ for even number of electrons $N_e$ in a half-filled LL $\Psi$. There is no two-body Hamiltonian, and hence no exact pseudopotential description, for which the Pf is an exact eigenstate. So, as good as the physical description for the $\nu = 5/2$ FQHE state provided by the Pf may be, it is not continuously connected to the exact Coulomb ground state or, in fact, the ground state of any two-body Hamiltonian. This notion has been discussed in the literature $\Psi$ for over ten years, and recently questions have been raised $\Psi$ about the applicability of the Pf for the physical $\nu = 5/2$-state.

However, evidently some two-body Hamiltonians produce ground states that have nearly unity overlap ($\approx 0.99$) with the Pf. For example, Morf $\Psi$ showed in the spherical geometry (see below for details) that, for $N_e = 8$, the SLL Coulomb Hamiltonian has a nearly Pf ground state if $V_1 \rightarrow 1.1V_1$, i.e., $V_1$ is slightly increased from its SLL Coulomb value. In the torus geometry, Rezayi-Haldane $\Psi$ showed a nearly Pf ground state (although they compared to a particle-hole symmetrized Pf state) for an increase in $V_1$ and/or a decrease in $V_3$. A conceptual problem with these results is that no physical effect can produce an increased $V_1$—although a decreased $V_3$ is possible. Recently, it was shown $\Psi$ (sphere and torus) that the inclusion of finite-thickness effects inherent in realistic experimental quantum wells produces a nearly Pf ground state for well widths of about four magnetic lengths. Clearly, however, in realistic calculations all $V_m$’s change (not only $V_1, V_3$).

To make matters more interesting is the recently discussed fact $\Psi$ that the Pf is not particle-hole (PH) symmetric since it is the exact ground state of a three-body interaction Hamiltonian that explicitly breaks PH symmetry. Consequently, if correct, the Pf description of the $\nu = 5/2$ FQHE would require spontaneous PH symmetry breaking of the actual two-body Coulombic Hamiltonian—in the absence of inter-LL mixing which could break PH symmetry in real systems explicitly. This observation, in turn, leads to an identification of the PH conjugate state, the anti-Pfaffian $\Psi$, that is degenerate with the Pf in the PH symmetry respecting limit. An important question is whether PH symmetry is indeed broken spontaneously without LL mixing and addressing this question is a main goal of this work.
We begin our quantitative analysis by constructing a two-body interaction Hamiltonian $H_2$ which is PH symmetric and yet contains as much of the physics of $H_3$ as possible. The purpose is to use $H_2$ as a reference model Hamiltonian to which other more realistic Hamiltonians are compared. First note that, in the spherical geometry, $H_3 = \sum_{i<j<k} P_{ijk}(3N_\phi/2 - 3)$ with $P_{ijk}(L)$ projecting onto electron triplets with total angular momentum $L$ and $N_\phi$ the total magnetic flux piercing the surface. We consider a PH transformation of $H_3$ by taking $c_m^\dagger \rightarrow c_m^\dagger (c_m^\dagger)$ where $c_m^\dagger$ creates an electron in an angular momentum state $m$. We call this PH-conjugate Hamiltonian $\overline{H}_3$ and, naturally, the $\overline{Pf}$ is its ground state. Importantly, when normal ordered, $\overline{H}_3$ contains a three-body term exactly the minus of $H_3$ plus a two-body term, a one-body term (the chemical potential), and a constant $\overline{H}$. Adding $H_3$ and $\overline{H}_3$ simply eliminates the three-body interaction and simultaneously restores the PH symmetry suggesting an intriguing possibility: $H_2 \equiv H_3 + \overline{H}_3$.

Meanwhile, the relationship on the sphere between $N_\phi$ and $N_e$ for the Pf is $N_\phi = 2N_e - 3$ where $-3$ is the “shift”. Since $N_e$ is related to the number of holes $N_h$ through $N_h + N_e = N_\phi + 1$ the relationship between $N_e$ and $N_\phi$ for the Pf is $N_\phi = 2N_h - 3 = 2N_e + 1$ and thus the Pf and $\overline{H}_3$ have different “shifts”. On the torus, PH symmetrizing the Pf creates a new state with a significantly improved overlap with the exact Coulomb ground state in the SLL [8]. By contrast, on the sphere, such an attempt obviously fails since PH symmetrizing the Pf changes the particle number for a given flux. However, this dichotomy between the $(N_\phi, N_e)$ relationships for the Pf and $\overline{H}_3$ provides a convenient platform for addressing the issue of spontaneous PH symmetry breaking. The reason is as follows.

The Pf and $\overline{H}_3$ break PH symmetry differently. In principle, the difference can be parameterized in terms of an Ising-like “order parameter” measuring the “deviation” from a PH-symmetric ground state. Fortunately, on the sphere, the Pf and $\overline{H}_3$ belong to different $N_e$ sectors for a given flux and $N_e$ can be regarded as an order parameter; $N_e = (N_\phi + 3)/2$ for the Pf and $N_e = (N_\phi - 1)/2$ for the $\overline{H}_3$ with a PH-symmetric ground state obtained exactly in the middle. We conclude that the ground state energy can be regarded as the “Landau free energy” with $N_e$ being an order parameter (at least so long as $N_e$ does not deviate too far from half filling). The question then becomes whether this “Landau free energy” exhibits a Mexican-hat structure for $H_2$ which, if present, would indicate spontaneous PH symmetry breaking. Another important and experimentally relevant question is what happens in the case of the Coulomb interaction both with and without finite-thickness effects.

Before addressing the above questions we confirm that $H_2$ is indeed a good model Hamiltonian containing the physics of $H_3$. To this end, consider $H_2$ at the Pf value of $N_e$ for a given $N_\phi$, $N_{Pf} = (N_\phi + 3)/2$, and change the value of $N_e$ from $N_{Pf}$ to the $\overline{H}_3$ value, $N_{\overline{H}_3} = (N_\phi - 1)/2$. At $N_{Pf}$, $H_3$ produces energy spectra similar to those of the SLL $H_C$ (denoted only $H_C$ from this point) and generates the Pf as the ground state; see Fig. 1(a) and (c), where $N_\phi = 13$. In contrast, $\overline{H}_3$ creates $\overline{Pf}$ quasiparticles forming a degenerate ground state manifold shown in FIG. 1: Exact energy (arbitrary units) as a function of total angular momentum $L$ for a system with $N_e = 8 \ ([a]-(d)]$, $N_e = 7 \ ([e]-[h])$, and $N_e = 6 \ ([i]-(l])$ at $N_\phi = 13$ for the four considered Hamiltonians, $H_C$, $H_2$, $H_3$, and $\overline{H}_3$. Numbers inside the plots are the wave function overlaps between the indicated eigenstates and the lowest energy eigenstates (at that $L$) of $H_C$. The lines connecting the low energy states in (e) and (f) are a guide to the eye. In (g) two numbers are given at $L = 1.5$ and 2.5 for the two degenerate ground states at those angular momenta.
Fig. 2. (Color online) Contour plot of (a) $\langle \Psi_0^H | \Psi_0^H \rangle$ perturbing $\Delta V_1^H$ and $\Delta V_3^H$; (b) $\langle \Psi_0^H | \Psi_0^H \rangle$ perturbing $\Delta V_1^H$ and $\Delta V_3^H$, (c) $\langle \Psi_0^C | \Psi_0^H \rangle$ perturbing $\Delta V_1^H$ and $\Delta V_3^H$, and (d) $\langle \Psi_0^C | \Psi_0^C \rangle$ perturbing $\Delta V_1^H$ and $\Delta V_3^H$.

Fig. 1(d). Important for our purpose is that the low-energy spectra of $H_3$ (Fig. 1(b)) is essentially equivalent to that of $H_2$ in this number sector. At $N_{Pf}$, the role of the Pf and $\mathcal{F}$ are exactly reversed where $H_2$ generates almost the identical low-energy spectra as $H_3$ and also $H_C$; see Fig. 1(i)-(l).

Results between $N_{Pf}$ and $N_{Pf}$ are most intriguing. Due to the exact PH symmetry, $H_3$ and $H_3$ have precisely the same energy spectra containing a degenerate manifold of Pf quasiholes and $\mathcal{F}$ quasiparticles; see Fig. 1(g) and (h). A surprising fact is that $H_2$ produces an energy spectra, shown in Fig. 1(f), qualitatively similar to those of $H_C$ as a result of an intricate interaction between an equal mixture of Pf quasiholes and $\mathcal{F}$ quasiparticles. Quantitatively, the similarities can be investigated via the wave function overlaps calculated for the lowest branch of excitations ranging from 0.75 to 0.99. These are compared to those of $H_3$ (or $H_3$) which are usually below 0.9 and predominantly lower.

Since $H_2$ is a two-body interaction Hamiltonian, it is useful to calculate the Haldane pseudopotentials (on the sphere) through $H_2 = \sum_m V_m^2 \sum_{i<j} P_{ij} (N_0 - m)$, where $P_{ij} (L)$ projects onto states with relative pair angular momentum $L$. The pseudopotentials on the sphere are connected to those on the plane through the thermodynamic limit: $V_m = \lim_{N_0 \to \infty} V_m^{N_0}$. Explicit computation finds that, surprisingly, $V_m^{N_0}$ is non-zero only for $m = 1$ and 3 and the interaction is quite short-ranged and nearly hard-core compared to the Coulomb interaction. For the sake of further theoretical studies we provide the planar-geometry Haldane pseudopotentials: $V_1^H = 2.7119(10)$ and $V_3^H = 0.90173(73)$ where the numbers in parentheses represent the statistical error from taking the thermodynamic limit. (The superscript $H$ denotes the Hamiltonian from which the pseudopotentials $V^H$ are derived.)

To test the robustness of the Pfannian-like description of the ground state of $H_3$ we consider $N_e = 8$ electrons at $N_0 = 13$ and diagonalize $H_2$ allowing $V_1^H$ and $V_3^H$ to be perturbed from the original values. Fig. 2(a) shows the overlap between the Pf state $\langle \Psi_0^H | \Psi_0^H \rangle$ and the exact $H_2$ ground state $\langle \Psi_0^H | \Psi_0^H \rangle$ as a function of the variations in the $H_2$ pseudopotentials $\Delta V_1^H$ and $\Delta V_3^H$. $\langle \Psi_0^H | \Psi_0^H \rangle$ remains Pf-like to a large degree over a significant parameter range. A similar “phase” diagram can be obtained for the overlap between $\langle \Psi_0^H | \Psi_0^H \rangle$ and the $H_C$ ground state $\langle \Psi_0^H | \Psi_0^H \rangle$ where the $H_2$ pseudopotentials are allowed to vary, cf. Fig. 2(b). Again, there is a large region of the $\Delta V_1^H - \Delta V_3^H$ plane where $\langle \Psi_0^H | \Psi_0^H \rangle$ approximates $\langle \Psi_0^H | \Psi_0^H \rangle$ very accurately. Finally, we compute the overlap between $\langle \Psi_0^C | \Psi_0^H \rangle$ and $\langle \Psi_0^H | \Psi_0^H \rangle$ perturbing the first two pseudopotentials of $H_C$ shown in Fig. 2(c). Here the overlap “phase” diagram is nearly identical to that obtained between the Pf and the Coulomb ground state—the difference between the two [Figs. 2(c) and (d)] is always less than 2.5%. Note that, for positive $\Delta V_1^H$ and $\Delta V_3^H$, there is a region where the ground state of $H_C$ is well approximated by that of $H_2$ with the same being true for $H_3$.
We now present our main results. Figure 3 shows a comparison between the ground state energy of $H_C$ and $H_2$ as a function of electron number $N_e$ in the vicinity of half filling. The total magnetic flux piercing the sphere is $N_\phi = 2(l - 1)$ for $H_C$ and $2l$ for $H_2$. Note that an appropriate chemical potential energy is added in the case of $H_C$. 

Therefore, it is shown in our numerical studies that the PH symmetry is likely to be spontaneously broken in $H_2$, but not in $H_C$. We emphasize that the existence of the “Mexican-hat” structure in finite-size systems is usually a necessary condition for spontaneous PH symmetry breaking in the thermodynamic limit. Also, in our further numerical studies the “Mexican-hat” structure remains absent even when finite-thickness effects are incorporated into $H_C$ (or for the zero width LLL $H_C$). It is interesting that, despite the high overlaps between the ground states of $H_2$ and $H_C$ at the Pf ($\overline{\text{Pf}}$) sector, energy landscapes of the two Hamiltonians become qualitatively different in the vicinity of half filling.

Finally we mention experimental implications. Since the PH symmetry is not spontaneously broken in the case of the Coulomb interaction, with or without finite-thickness, it is likely that the true ground state is neither the pure Pfaffian nor anti-Pfaffian state in the absence of external PH-symmetry breaking terms such as those inherent with LL mixing. While it is possible that large LL mixing favors one state over another, a possible scenario is that the Pf and $\overline{\text{Pf}}$ are linearly superposed in the thermodynamic limit (where the number difference between the two states is infinitesimal compared to the total particle number) and form a new PH-symmetric ground state, in which case the edge-state behavior would be quite different from that of the pure Pf or $\overline{\text{Pf}}$. Another possibility is that local disorder could induce spatially random LL mixing producing spatially random local PH-symmetry breaking. This would yield random domains of Pf and $\overline{\text{Pf}}$ states in the two-dimensional plane. Either scenario is consistent with the previously mentioned fact that the overlap with the Coulomb ground state is significantly improved in the torus geometry when PH symmetrization is explicitly applied to the Pf. The other possibility that the “hat structure” is actually present in the Coulomb case, with an effect too small to been seen in numerical work, is unlikely in our opinion.

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FIG. 3: Comparison between the ground state energy of $H_C$ and $H_2$ as a function of electron number $N_e$ in the vicinity of half filling. The total magnetic flux piercing the sphere is $N_\phi = 2(l - 1)$ for $H_C$ and $2l$ for $H_2$. Note that an appropriate chemical potential energy is added in the case of $H_C$. 

We emphasize that the existence of the “Mexican-hat” structure whenever the Pf and $\overline{\text{Pf}}$ occur at even particle numbers given by the respective ($N_\phi, N_e$) relationships: see $l = 6.5$ and $8.5$ in Fig. 3. It is important to note that when nominal particle numbers for the Pf and $\overline{\text{Pf}}$ become odd at $l = 7.5$ the “Mexican-hat” structure disappears.

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We consider $(H_C + \overline{H}_C)/2$ (where $\overline{H}_C$ is the PH conjugate to $H_C$) to ensure that the energy spectra (not just eigenstates) for $N_e$ electrons is equivalent to $N_\phi + 1 - N_e$ electrons. This is equivalent to adding an appropriate $N_\phi$ and $N_e$ dependent (for finite systems) chemical potential to $H_C$.

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