An unstructured grid, nonhydrostatic, generalized vertical coordinate ocean model

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Abstract

We present a method to simulate nonhydrostatic ocean flows on a horizontally-unstructured grid with a moving generalized vertical coordinate (GVC). The nonhydrostatic governing equations are transformed to a GVC system that can represent the well-known \(z\)-level, terrain-following, or isopycnal coordinates while also being able to employ a vertically-adaptive coordinate using \(r\)-adaptivity. Different vertical coordinates are accommodated with the arbitrary Lagrangian-Eulerian (ALE) approach in which the vertical coordinate lines translate vertically, and the layer heights are made consistent with the vertical grid velocities through a discrete layer-height equation. Vertical grid velocities are also accounted for in the discrete momentum and scalar transport equations. While momentum is approximately conserved, the mass, heat, and volume are conserved both locally and globally. The nonhydrostatic pressure is implemented using a pressure-correction method that enforces the transformed continuity equation. The proposed GVC framework is implemented in the SUNTANS (Fringer et al., 2006) ocean model. Nonhydrostatic internal solitary-like waves are simulated to demonstrate that isopycnal coordinates can represent similar dynamics as \(z\)-levels at a fraction of the computational cost. The nonhydrostatic lock-exchange is then simulated to demonstrate that adaptive vertical coordinates can improve the accuracy of the model by concentrating more grid layers in regions of higher vertical density gradients.

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1. Introduction

The vertical coordinate system is critical to the design of an ocean model (e.g. Griffies et al., 2000; Willebrand et al., 2001; Chassignet, 2011), and common vertical coordinate systems include (1) height or z-coordinates, (2) terrain-following or \( \sigma \)-coordinates, and (3) density-following (isopycnal) or \( \rho \)-coordinates. These vertical coordinates have been employed in different ocean models, e.g. MITgcm (Marshall et al., 1997) and SUNTANS (Fringer et al., 2006) for \( z \)-coordinates, POM (Blumberg and Mellor, 1987), ROMS (Shchepetkin and McWilliams, 2005) and FVCOM (Lai et al., 2010) for \( \sigma \)- and terrain-following coordinates, the hydrostatic MICOM model (Bleck et al., 1992) and the nonhydrostatic isopycnal model proposed by Vitousek and Fringer (2014) for \( \rho \)-coordinates.

Each vertical coordinate system provides advantages and disadvantages, particularly with regard to the representation of bathymetry and stratification (Griffies et al., 2000). For example, \( z \)-coordinates are straightforward to implement and are ideal for resolving surface mixed layer dynamics and horizontal pressure gradients. However, \( z \)-coordinates cannot accurately represent bottom geometry and bottom boundary layers due to the “stair-step” representation of bathymetry (Adcroft et al., 1997). On the other hand, \( \sigma \)- and terrain-following coordinates smoothly resolve bottom geometry and bottom boundary layers, although they often cannot accurately represent the horizontal pressure gradient particularly in the presence of steep bathymetry (Mellor et al., 1998). Isopycnal coordinates naturally discretize stratified flows and eliminate spurious diapycnal mixing (Griffies et al., 2000). However, isopycnal coordinates cannot be used for overturning and convective motions since the vertical grid mapping must be monotonic (Mandli, 2013).

Several studies have outlined efforts to overcome the shortcomings related to each coordinate system. For example, Adcroft et al. (1997) proposed a shaved-cell, \( z \)-coordinate method to follow bottom topography by modifying the bottom-most cells to align faces with the bed. Casulli (2009) introduced the subgrid bathymetry method for \( z \)-coordinates to resolve bottom geometry with high-resolution subgrid-scale bathymetry data. For \( \sigma \)- and terrain-following coordinates, several methods have been proposed to achieve a more
accurate pressure gradient discretization (e.g. Stelling and Van Kester, 1994; Song, 1998; Auclair et al., 2000), although it is impossible to completely eliminate such errors. Despite the benefits of these improvements, they do not eliminate all of the disadvantages associated with using a single traditional coordinate system.

As described by Griffies et al. (2000), the ideal vertical coordinate is a hybrid system that applies \( z \)-coordinates at the surface mixed layers, \( \sigma \)- or terrain-following coordinates at the bottom to resolve topography, and \( \rho \)-coordinates in the middle to resolve stratification and internal waves. This vertical coordinate system applies the three traditional coordinates at certain locations in order to leverage each of their advantages. The HYCOM model (Bleck, 2002) is a hybrid coordinate model that applies isopycnal coordinates in the open ocean interior, but makes smooth transitions to \( \sigma \)-coordinates in shallow coastal regions and to fixed \( z \)-coordinates in unstratified seas. HYCOM applies the arbitrary Lagrangian-Eulerian (ALE) technique (Hirt et al., 1974) to remap and maintain different vertical coordinates within the domain. Similar to HYCOM, many other ocean models including MPAS-Ocean (Ringler et al., 2013) and MOM (Adcroft et al., 2019) have applied the ALE technique to provide the flexibility to employ different vertical coordinates. These approaches rely on the hydrostatic approximation and thus cannot accurately resolve nonhydrostatic processes like overturning eddies and internal solitary waves. Several nonhydrostatic models have been developed on \( z \)-level grids including TRIM/UnTRIM (Casulli, 1999a,b), MITgcm (Marshall et al., 1997) and SUNTANS (Fringer et al., 2006) and terrain-following coordinates including CROCO (Auclair et al., 2018), PSOM (Mahaevan et al., 1996), FVCOM-NH (Lai et al., 2010), and SWASH (Zijlema et al., 2011). To our knowledge, only GETM (Burchard and Bolding, 2002) has both a generalized vertical coordinate through use of adaptive vertical grids (Hofmeister et al., 2010) and nonhydrostatic capability (Klingbeil and Burchard, 2013). That model is built upon a terrain-following framework on a horizontally-structured curvilinear grid.

In this paper, we present a finite-volume formulation of the nonhydrostatic governing equations in generalized vertical coordinates (Adcroft and Hallberg, 2006), and develop an ocean model that uses unstructured grids in the horizontal plane with the application of ALE approach for the vertical coordinate. The model is an extension of the nonhydrostatic, isopycnal-coordinate framework of Vitousek and Fringer (2014), but here we adapt the method for solution of the nonhydrostatic pressure to a generalized
vertical coordinate using the horizontally unstructured-grid framework of the SUNTANS model (Fringer et al., 2006). The GVC framework and the vertically-adaptive grid are based on the r-adaptive method of Koltakov and Fringer (2013). Application of generalized vertical coordinates enables the model to simulate overturning and convective motions while the horizontally-unstructured grid enables simulation in complex geometries. The remainder of this paper is laid out as follows. In Section 2, we transform the Reynolds-averaged Navier-Stokes (RANS) equations from a Cartesian to a generalized vertical coordinate system. The finite-volume implementation of the transformed governing equations is presented in Section 3. Application of the ALE approach for arbitrary layer heights is explained in Section 4. The model is benchmarked with test cases in Section 5, which demonstrate the effectiveness of our model for hydrostatic or nonhydrostatic problems. Conclusions are given in Section 6.

2. Model Formulation

2.1. Governing equations in Cartesian coordinates

The three-dimensional RANS equations with the Boussinesq approximation in a rotating frame are of interest in the present paper. In Cartesian coordinates, these equations read

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) + 2\Omega_j u_k \epsilon_{jki} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \nu_{jk}^T \frac{\partial u_i}{\partial x_k} \right) - \frac{g}{\rho_0} \rho \delta_{i3}, \quad (1)$$

subject to the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (2)$$

In Equations (1) and (2), $t$ denotes time, $u_i$ is the velocity vector corresponding to the $x_i$ Cartesian-coordinate directions, $\rho$ is the fluid density while $\rho_0$ is a constant reference, $g$ is the gravitational acceleration, and $\Omega_j$ is the angular velocity vector in the $f$-plane. In Equation (1) and hereafter, the Einstein summation convention is assumed unless otherwise indicated, where $\delta_{ij}$ is the Kronecker delta function and $\epsilon_{jki}$ is the Levi-Civita symbol. The anisotropic eddy-viscosity tensor $\nu_{jk}^T = 0$ if $j \neq k$, $\nu_{11}^T = \nu_{22}^T = \nu_{12}^H$ is the horizontal eddy-viscosity, and $\nu_{33}^T = \nu_{33}^V$ is the vertical eddy-viscosity.

Following the approach and notation used in the SUNTANS model (Fringer et al., 2006), the total pressure in Equation (1) is split into two components
as $p = p_h + q$, where $p_h = \rho_0 g (\eta - x_3 + r)$ is the hydrostatic part and $q$ is the nonhydrostatic part. The free-surface elevation is denoted as $\eta$, while the baroclinic pressure head $r$ is defined as

$$r = \frac{1}{\rho_0} \int_{x_3}^{\eta} (\rho - \rho_0) \, dx_3'. \tag{3}$$

Substitution of $p = \rho_0 g (\eta - x_3 + r) + q$ into Equation (1) gives the pressure-split form

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) + 2\Omega_j u_k \epsilon_{jki} = -g \frac{\partial}{\partial x_i} (\eta + r) - \frac{1}{\rho_0} \frac{\partial q}{\partial x_i} - \frac{g}{\rho_0} (\rho - \rho_0) \delta_{i3} + \frac{\partial}{\partial x_j} \left( \nu_j^{T} \frac{\partial u_i}{\partial x_k} \right). \tag{4}$$

To complete the equation set, governing equations for the free-surface elevation and density field are needed. These are obtained by integrating the continuity Equation (2) from the bottom ($z = -d$) to the free surface ($z = \eta$), and employing the corresponding free-surface and bottom kinematic boundary conditions to obtain the depth-integrated continuity equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_1} \int_{-d}^{\eta} u_1 \, dx_3 + \frac{\partial}{\partial x_2} \int_{-d}^{\eta} u_2 \, dx_3 = 0. \tag{5}$$

The density field is determined by an equation of state of the form $\rho = \rho(s, T)$, where $s$ and $T$ represent the salinity and temperature anomalies from their reference states, respectively. In the present study, the effects of temperature stratification are neglected and the linear equation of state (i.e. $\rho = \beta s$, where $\beta$ is a constant coefficient) is implemented for simplicity. If needed, the salinity and temperature fields are solved with the scalar transport equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x_j} (u_j \phi) = \frac{\partial}{\partial x_i} \left( \kappa_{ij}^{T} \frac{\partial \phi}{\partial x_j} \right), \tag{6}$$

where $\phi$ denotes either the salinity or temperature anomaly. Similar to $\nu_{ij}^{T}$, the anisotropic eddy-diffusivity tensor $\kappa_{ij}^{T} = 0$ if $i \neq j$, $\kappa_{11}^{T} = \kappa_{22}^{T} = \kappa_{H}^{T}$ is the horizontal eddy-diffusivity, and $\kappa_{33}^{T} = \kappa_{V}^{T}$ is the vertical eddy-diffusivity.
2.2. Governing equations in generalized vertical coordinates

To enable application to a broad suite of vertical coordinates (Griffies et al., 2020), the governing equations are transformed from physical to computational space with the algebraic mapping

\[
\begin{align*}
\xi_1 &= x_1, & \xi_2 &= x_2, & \xi_3 &= \xi_3(x_1, x_2, x_3, t), & \tau &= t,
\end{align*}
\]

where the generalized vertical coordinate \(\xi_3\) varies in both time and space. In terms of the Cartesian coordinates, derivatives in the transformed coordinate system are given by

\[
\frac{\partial}{\partial \xi_1} = \begin{bmatrix} 1 & 0 & \frac{\partial x_3}{\partial \xi_1} \end{bmatrix}, \quad \frac{\partial}{\partial \xi_2} = \begin{bmatrix} 0 & 1 & \frac{\partial x_3}{\partial \xi_2} \end{bmatrix}, \quad \frac{\partial}{\partial \xi_3} = \begin{bmatrix} 0 & 0 & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix}, \quad \frac{\partial}{\partial \tau} = \begin{bmatrix} \frac{\partial}{\partial t} \end{bmatrix},
\]

implying that the Jacobian of the coordinate transformation (7), or the layer height is

\[
J = \frac{\partial x_3}{\partial \xi_3}.
\]

Inverting the system (8) gives the derivatives in Cartesian coordinates in terms of those in the transformed coordinate system as

\[
\begin{align*}
\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial \xi_1} - \frac{1}{J} \frac{\partial x_3}{\partial \xi_1} \frac{\partial}{\partial x_1}, & \frac{\partial}{\partial x_2} &= \frac{\partial}{\partial \xi_2} - \frac{1}{J} \frac{\partial x_3}{\partial \xi_2} \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_3} &= \frac{\partial}{\partial \xi_3} - \frac{1}{J} \frac{\partial x_3}{\partial \xi_3} \frac{\partial}{\partial x_3}, & \frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} - \frac{1}{J} \frac{\partial x_3}{\partial \xi_3} \frac{\partial}{\partial \tau}.
\end{align*}
\]

Following Koltakov and Fringer (2013), substitution of the operators (10) into Equations (2), (4), and (6) gives the conservative momentum equations, scalar transport equation, and the layer-height continuity equation in transformed coordinates

\[
\begin{align*}
\frac{\partial}{\partial \tau} (Ju_i) + \frac{\partial}{\partial \xi_1} (Ju_1 u_i) + \frac{\partial}{\partial \xi_2} (Ju_2 u_i) + \frac{\partial}{\partial \xi_3} (W u_i) &= S_i^c, \quad \text{(11)}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial \tau} (J\phi) + \frac{\partial}{\partial \xi_1} (J u_1 \phi) + \frac{\partial}{\partial \xi_2} (J u_2 \phi) + \frac{\partial}{\partial \xi_3} (W \phi) &= S_\phi^c, \quad \text{(12)}
\end{align*}
\]

\[
\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \xi_1} (Ju_1) + \frac{\partial}{\partial \xi_2} (Ju_2) + \frac{\partial W}{\partial \xi_3} = 0, \quad \text{(13)}
\]


where $S^c_i$ and $S^c_{φ}$ are the corresponding source terms. Defining the vertical grid velocity $w_g = \partial x_3 / \partial τ$, the contravariant volume flux in a frame moving with the grid, or simply the cross-coordinate vertical velocity, is given by

$$W = U_3 - w_g = u_3 - \frac{\partial x_3}{\partial ξ_1} u_1 - \frac{\partial x_3}{\partial ξ_2} u_2 - w_g,$$

(14)

where $U_3$ is the contravariant volume flux in the $ξ_3$ direction. In addition to Equation (13) which governs the evolution of layer heights, the continuity Equation (2) can also be written in the form

$$\frac{\partial}{\partial ξ_1} (J u_1) + \frac{\partial}{\partial ξ_2} (J u_2) + \frac{\partial U_3}{\partial ξ_3} = 0,$$

(15)

which is enforced by the nonhydrostatic pressure in the nonhydrostatic correction step described in what follows. In our implementation, rather than solve the conservative momentum Equation (11), we solve its non-conservative counterpart

$$\frac{∂u_i}{∂τ} + u_1 \frac{∂u_i}{∂ξ_1} + u_2 \frac{∂u_i}{∂ξ_2} + \frac{W J}{∂\xi_3} = S_i,$$

(16)

where $S_i = S^c_i / J$ is the non-conservative momentum source term. As described in Section 3.2, we manipulate this form into one that can be solved with the existing SUNTANS framework of Fringer et al. (2006). After employing the mild-slope approximation (Vitousek and Fringer, 2014), the source
terms in the momentum and scalar transport equations are given by

\[ S_1 = fu_2 - bu_3 - g \left[ \frac{\partial}{\partial \xi_1} (\eta + r) + \frac{\rho - \rho_0}{\rho_0} \frac{\partial x_3}{\partial \xi_1} \right] - \frac{1}{\rho_0} \frac{\partial q}{\partial \xi_1} \]
\[ + \frac{\partial}{\partial \xi_1} \left( \nu_T \frac{\partial u_1}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \nu_T \frac{\partial u_1}{\partial \xi_2} \right) + \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_T}{J} \frac{\partial u_1}{\partial \xi_3} \right), \]  
(17)

\[ S_2 = -fu_1 - g \left[ \frac{\partial}{\partial \xi_2} (\eta + r) + \frac{\rho - \rho_0}{\rho_0} \frac{\partial x_3}{\partial \xi_2} \right] - \frac{1}{\rho_0} \frac{\partial q}{\partial \xi_2} \]
\[ + \frac{\partial}{\partial \xi_1} \left( \nu_T \frac{\partial u_2}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \nu_T \frac{\partial u_2}{\partial \xi_2} \right) + \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_T}{J} \frac{\partial u_2}{\partial \xi_3} \right), \]  
(18)

\[ S_3 = bu_1 - \frac{1}{\rho_0 J} \frac{\partial q}{\partial \xi_3} \]
\[ + \frac{\partial}{\partial \xi_1} \left( \nu_T \frac{\partial u_3}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \nu_T \frac{\partial u_3}{\partial \xi_2} \right) + \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_T}{J} \frac{\partial u_3}{\partial \xi_3} \right), \]  
(19)

\[ S^c_{\phi} = \frac{\partial}{\partial \xi_1} \left( \kappa_T \frac{\partial \phi}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \kappa_T \frac{\partial \phi}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_3} \left( \frac{\kappa_T}{J} \frac{\partial \phi}{\partial \xi_3} \right), \]  
(20)

where \( f = 2\omega_e \sin \psi \) and \( b = 2\omega_e \cos \psi \) are respectively the sine and cosine of latitude Coriolis terms, \( \psi \) is the latitude, and \( \omega_e \) is the angular velocity of the earth. We note that the mild-slope approximation limits the variation in layer heights for a smooth vertical grid, which is a reasonable approximation in most flows of interest. Nevertheless, it is possible to include all terms in the governing equation at the expense of an increased computational cost associated primarily with inverting the elliptic equation for the nonhydrostatic pressure (Vitousek and Fringer, 2014).

3. Numerical Discretization

3.1. Unstructured, finite-volume grid

In the vertical direction, the grid discretizations in physical and computational space are illustrated using vertically distributed layers as shown in Figure 1(a) and (b), respectively. In general, the layer heights in physical space, \( \Delta x_3 \), are not uniform in the horizontal direction. However, after the coordinate transformation, each layer is defined to be uniform with layer height \( \Delta \xi_3 = 1 \). This implies the transformation of the vertical coordinate in a layer \( k \) as

\[ \xi_{3(k)} = k - 1 + \frac{x_3 - \sum_{k'=1}^{k-1} \Delta x_{3(k')}}{\Delta x_{3(k)}}, \]  
(21)
Figure 1: The vertical coordinate system in (a) physical space and (b) computational space. In the left panel, $\Delta x_{3(i,k)}$ represents the layer thickness of cell $i$ in layer $k$ in physical space. In the right panel, $u_f(j,k)$ is the component of velocity normal to edge $j$ in layer $k$, $W_{(i,k+1/2)}$ is the relative velocity of the top face of layer $k$ in cell $i$, and $\Delta \xi_{3(k)} = 1$ is the thickness of layer $k$ in computational space.

where $\Delta x_{3(k)}$ is the thickness of layer $k$ in Cartesian coordinates and is a function of $(x_1, x_2, t)$. The corresponding Jacobian of the coordinate transformation defined by Equation (9) for layer $k$ is then

$$J(k) = \frac{\partial x_3}{\partial \xi_{3(k)}} = \Delta x_{3(k)}.$$ (22)

As previously mentioned, we take advantage of the existing framework provided by the nonhydrostatic SUNTANS model (Fringer et al., 2006), which employs unstructured orthogonal C-grids to discretize the governing equations in the horizontal plane. As shown in Figure 2(a), the cell center of a triangular cell is defined as the Voronoi point, and the Voronoi edges, or the lines connecting centers of two neighboring cells, are orthogonal to the Delaunay edges they intersect. The Delaunay edges are the edges connecting the vertices of the triangles. For the case of quadrilateral cells, the cell center is defined as the centroid. Although this can incur discretization errors associated with non-orthogonal grids, the non-orthogonality, or the deviation from a right angle between the Voronoi and Delaunay edges, is kept as small as possible when employing quadrilateral grids. The C-grid layout defines the temperature, salinity, density, eddy-viscosity, and scalar-diffusivity at the vertical centers of the cells, as shown in Figure 2(b). The nonhydrostatic pressure is also defined at the vertical cell centers. Although
nonhydrostatic surface gravity wave dispersion can be represented more efficiently with fewer layers when the nonhydrostatic pressure is defined at the top and bottom of the layers (Zijlema and Stelling, 2005; Zijlema et al., 2011), our approach suffices for internal flows such as internal gravity waves and the lock-exchange problem, as discussed below. The free-surface elevation is defined at the centers on the surface of the top-most cells, while the depth is defined at the same locations but at the bottom of the bottom-most cells. The component of the horizontal velocity normal to the grid edges is defined as \( u_f \) and stored at the vertical center of each grid edge, while the vertical velocity \( u_3 \) is defined at centers of the top and bottom surfaces of each layer (illustrated in Figure 1(b)).

Following the discussion in Fringer et al. (2006), each grid edge has a predefined normal direction \( \vec{n}_f = n_{f1}\vec{e}_1 + n_{f2}\vec{e}_2 \), and the component of velocity normal to that edge is given by \( u_f = u_1n_{f1} + u_2n_{f2} \). The indices of the two cells neighboring grid edge \( j \) are denoted by \( G_{2j} \) and \( G_{2j+1} \) (see Figure 2(b)). Thus, the components of the normal vector \( \vec{n}_{f(j)} \) are calculated with

\[
\begin{align*}
n_{f1(j)} &= \frac{x_{G_{2j}} - x_{G_{2j+1}}}{d_{f(j)}} \quad \text{and} \quad n_{f2(j)} = \frac{y_{G_{2j}} - y_{G_{2j+1}}}{d_{f(j)}},
\end{align*}
\]

Figure 2: Illustration of (a) horizontal unstructured, orthogonal C-grid and (b) the placement of variables. In the left panel, the dashed lines correspond to the Voronoi edges with length \( l_f \) that connect cell centers (Voronoi points, indicated by the \( \bullet \)) and are perpendicular to the Delaunay edges with length \( d_f \) that connect Delaunay points, or vertices. In the right panel, \( u_f \) is the component of velocity normal to each edge in the predefined direction \( \vec{n}_f \) that is normal to each edge, and \( \vec{n}_f^\perp \) is the outward-pointing normal vector at each grid edge. \( \Phi \) represents a variable defined at the cell centers.
where \( d_{f(j)} = [(x_{G2j} - x_{G2j+1})^2 + (y_{G2j} - y_{G2j+1})^2]^{1/2} \) is the distance between the two neighboring cells \( G_{2j} \) and \( G_{2j+1} \). With this notation, the edge-normal directional derivative of a cell-centered variable \( \varphi \) defined on edge \( j \) can be approximated as

\[
\left( \frac{\partial \varphi}{\partial n_f} \right)_{(j)} = \left( \nabla_H \varphi \right)_{(j)} \cdot \vec{n}_f(j) = \frac{\varphi_{G2j} - \varphi_{G2j+1}}{d_{f(j)}} + E_g,
\]

where \( \nabla_H = \tilde{e}_1 \partial/\partial \xi_1 + \tilde{e}_2 \partial/\partial \xi_2 \) is the horizontal gradient operator, and \( E_g \) is a small truncation error in terms of \( d_{f(j)} \). For equilateral triangles, \( E_g = O(d_{f(j)}^2) \) and first-order otherwise.

For edge \( j \) on grid cell \( i \), the corresponding outward-pointing normal \( \vec{n}_o_{(i,j)} \) is in the direction of the edge-normal \( \vec{n}_f(j) \) if \( G_{2j+1} = i \), while it is in the opposite direction when \( G_{2j} = i \). Rather than storing every component of the outward-pointing normals for a cell, we store the dot product of the cell-outward-normal \( \vec{n}_o_{(i,j)} \) with the edge-normal \( \vec{n}_f(j) \) as

\[
N_{(i,j)} = \vec{n}_o_{(i,j)} \cdot \vec{n}_f(j) = \pm 1.
\]

Combining Equations (24) and (25), the component of the gradient in the direction of the unit vector at edge \( j \) that points in the outward direction relative to cell \( i \) is given by

\[
\left( \frac{\partial \varphi}{\partial n^o} \right)_{(i,j)} = \left[ \left( \nabla_H \varphi \right)_{(j)} \cdot \vec{n}_f(j) \right] \vec{n}_o_{(i,j)} = \frac{\varphi_{G2j} - \varphi_{G2j+1}}{d_{f(j)}} N_{(i,j)}.
\]

On the aforementioned unstructured grids, a combination of finite-difference and finite-volume methods are used to discretize the governing Equations (12), (13), (15) and (16). The governing equation for the edge-centered horizontal velocity \( u_f \) is obtained by taking the dot product of the edge-normal vector \( \vec{n}_f \) with the horizontal components of the non-conservative momentum Equation (16) to give

\[
\frac{\partial u_f}{\partial \tau} = F_H - g \frac{\partial \eta}{\partial n_f} - \frac{1}{\rho_0} \frac{\partial q}{\partial n_f} + \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_T}{J} \frac{\partial u_f}{\partial \xi_3} \right).
\]

The vertical momentum equation is solved at the top and bottom of each cell and is given by

\[
\frac{\partial u_3}{\partial \tau} = F_V - \frac{1}{\rho_0 J} \frac{\partial q}{\partial \xi_3} + \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_T}{J} \frac{\partial u_3}{\partial \xi_3} \right).
\]
The terms $F_H$ and $F_V$ in Equations (27) and (28) contain the Coriolis, advection, baroclinic pressure gradient, and horizontal diffusion terms, which are given by

\[
F_H = (fu_2 - bu_3)n_{f1} - fu_1n_{f2} - A(u_f) + D_H(u_f)
- g \left( \frac{\partial r}{\partial n_f} + \frac{\rho - \rho_0}{\rho_0} \frac{\partial x_3}{\partial n_f} \right),
\]

(29)

\[
F_V = bu_1 - A(u_3) + D_H(u_3),
\]

(30)

where the advection and diffusion operators are given by, respectively,

\[
A(\varphi) = u_1 \frac{\partial \varphi}{\partial \xi_1} + u_2 \frac{\partial \varphi}{\partial \xi_2} + W \frac{\partial \varphi}{\partial \xi_3},
\]

(31)

\[
D_H(\varphi) = \frac{\partial}{\partial \xi_1} \left( \nu_T \frac{\partial \varphi}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \nu_T \frac{\partial \varphi}{\partial \xi_2} \right).
\]

(32)

As the SUNTANS model is designed for z-coordinates (Fringer et al., 2006) and the layer height in our computational domain is $\Delta \xi_3 = 1$, treatment in the original model can be directly used for the spatial discretization of the Coriolis, baroclinic pressure gradient, and diffusion terms. Discretization of the non-conservative advection operator given by Equation (31) is described in Section 3.2. For the vertical turbulent diffusion, boundary conditions include a specification of wind stress at the free surface and a quadratic drag law at the bed where a drag coefficient is obtained with a specification of bottom roughness, are described in Fringer et al. (2006).

### 3.2. Treatment of momentum advection

Using the continuity Equation (13), the non-conservative momentum advection operator defined in Equation (31) can be rewritten as

\[
A(\varphi) = \frac{1}{J} \left( \frac{\partial Ju_1}{\partial \xi_1} + \frac{\partial Ju_2}{\partial \xi_2} + \frac{\partial W}{\partial \xi_3} \right) + \frac{\varphi}{J} \frac{\partial J}{\partial \tau},
\]

(33)

where $A_c(\varphi)$ represents the component of advection driven by the divergence of momentum fluxes, while $A_g(\varphi)$ represents the change in momentum driven by the time-varying layer thickness. The advantage of using this form is that $A_c$ can be computed with the conservative momentum advection schemes in...
SUNTANS described by Fringer et al. (2006), in which the cell-centered velocity field is approximated with the reconstruction method of Perot (2000). This velocity field can then be interpolated onto the faces with a number of methods including first-order upwinding, central-differencing, or the total variation diminishing (TVD) schemes described by Casulli and Zanolli (2005). The resulting momentum advection scheme is applied to update the velocity $u_f$ on the faces in a conservative way on a stationary grid. The time-varying component is discretized explicitly at time-level $n$ at the cell centers with

$$A_g (\varphi^n) = \frac{\varphi^n J^n - J^{n-1}}{\Delta \tau}.$$  \hfill (34)

This is then interpolated onto the $u_f$ faces by averaging the cell-centered values. Although the resulting momentum advection scheme is not strictly conservative on a moving grid and the approximation in Equation (34) is first-order accurate in time, the effects of momentum advection due to the time-varying term are weak, leading to a robust momentum advection scheme that behaves similarly to the scheme in the original SUNTANS model.

### 3.3. Discrete momentum equations

To advanced the momentum Equations (27) and (28), we adapt the non-hydrostatic, pressure-correction scheme on isopycnal coordinates described by Vitousek and Fringer (2014) to the SUNTANS model (Fringer et al., 2006). The pressure-correction scheme is second-order accurate in time and has been shown to be much less dissipative than the projection scheme (Vitousek and Fringer, 2013). In the predictor step, the momentum equations are advanced forward in time from time-step $n$ using the nonhydrostatic pressure defined at time step $n - 1/2$, and the predictor hydrostatic velocity field
\( u^*_f(j,k) - u^n_f(j,k) \) is given by
\[
\frac{u^*_f(j,k) - u^n_f(j,k)}{\Delta \tau} = F_{xH(j,k)}^{ex} - g \frac{\partial \eta}{\partial n_f} \bigg|_{(j,k)}^{im*} - \frac{1}{\rho_0} \frac{\partial q}{\partial n_f} \bigg|_{(j,k)}^{n-\frac{1}{2}}
+ \left[ \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_T}{J} \frac{\partial u_f}{\partial \xi_3} \right) \right]^{im*} \bigg|_{(j,k)},
\]
\( u^*_3(i,k+\frac{1}{2}) - u^n_3(i,k+\frac{1}{2}) \) is given by
\[
\frac{u^*_3(i,k+\frac{1}{2}) - u^n_3(i,k+\frac{1}{2})}{\Delta \tau} = F_{xV(i,k+\frac{1}{2})}^{ex} - \frac{1}{\rho_0} \left( \frac{1}{J} \frac{\partial q}{\partial \xi_3} \right) \bigg|_{(i,k+\frac{1}{2})}^{n-\frac{1}{2}}
+ \left[ \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( \frac{\nu_{t,v}}{J} \frac{\partial u_3}{\partial \xi_3} \right) \right]^{im*} \bigg|_{(i,k+\frac{1}{2})}.
\]

The corrector step to obtain the divergence-free velocity field at time-step \( n + 1 \) will be discussed in Section 3.5.

The time-stepping schemes implemented in Vitousek and Fringer (2014) are used to discretize the terms on the right-hand side of Equations (35) and (36) in time. The semi-implicit discretization of a term \( \Phi \), denoted by the superscript \( \text{im} \), is given by
\[
\Phi^{im*} = \frac{1}{2} (c_{im} + 2\theta) \Phi^* + (1 - c_{im} - \theta) \Phi^n + \frac{c_{im}}{2} \Phi^{n-1}
= \alpha_1 \Phi^* + \alpha_2 \Phi^n + \alpha_3 \Phi^{n-1},
\]
where \( \Phi \) represents the free-surface gradient or vertical diffusion terms in Equations (35) and (36). The superscript \( \text{im}^* \) implies that the method is implicit with respect to the predictor step in terms of \( \Phi^* \) rather than \( \Phi^{n+1} \). The explicit terms \( F_{xH} \) in Equation (35) or \( F_{xV} \) in Equation (36), denoted by the superscript \( \text{ex} \), are discretized with
\[
\Phi^{ex} = \frac{1}{2} (3 + b_{ex}) \Phi^n - \frac{1}{2} (1 + 2b_{ex}) \Phi^{n-1} + \frac{b_{ex}}{2} \Phi^{n-2}
= \beta_1 \Phi^n + \beta_2 \Phi^{n-1} + \beta_3 \Phi^{n-2}.
\]
In these time-advancement schemes, the parameters \( \theta, c_{im} \) and \( b_{ex} \) dictate a particular time-stepping scheme. The implicit scheme (Equation (37)) with \( \theta = 1/2 \) and \( c_{im} = 1/2 \) represents the second-order accurate Adams-Moulton (AM2) method, while \( \theta = 1/2 \) and \( c_{im} = 3/2 \) represents the second-order accurate AI2* method described by Durran and Blossey (2012). If
\( c_{im} = 0 \), the implicit scheme reverts to the theta method of Casulli and Cattani (1994), which is second-order accurate in time if only \( \theta = 0.5 \). For the explicit scheme (Equation (38)), \( b_{ex} = 0 \) represents the second-order accurate Adams-Bashforth (AB2) method, \( b_{ex} = 5/6 \) represents the third-order accurate Adams-Bashforth (AB3) method, and \( b_{ex} = 1/2 \) corresponds to the AX2* method (Durran and Blossey, 2012). Stability of these methods is discussed in Section 3.9.

3.4. Discrete continuity equations

A semi-implicit, finite-volume discretization of the layer-height continuity Equation (13) is given by

\[
J^{n+1}_{i,k} - J^n_{i,k} \Delta \tau + \frac{1}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} u_{i,j}^{*} J_{f(j,k)} l_{f(j)} N_{i,j} + W_{i,k+1/2}^{im} - W_{i,k-1/2}^{im} = 0, \tag{39}
\]

where \( A_{p(i)} \) and \( N_{s(i)} \) are the planform area and number of cell edges, respectively. For a given grid cell \( i \) in layer \( k \), the Jacobian \( J_{i,k} \) represents the grid cell height with the relationship (22). For edge \( j \) on grid cell \( i \), \( l_{f(j)} \) is the edge length, \( u_{f(j,k)} \) is the edge-normal velocity, and \( J_{f(j,k)} \) is the flux-face height. To evaluate the flux-face height at the faces based on the cell-centered layer heights, the unstructured-grid, flux-limiting scheme of Casulli and Zanolli (2005) is employed to ensure non-negative values of \( J_f \). Kinematic boundary conditions at the top and bottom of the computational domain require \( W_{i,1/2} = W_{i,N_k(i)+1/2} = 0 \), where \( N_k(i) \) is the active number of layers in cell \( i \) and the bottom-most face is defined at \( k = 1/2 \).

The discrete, depth-integrated continuity equation for the free-surface elevation is obtained by summing the discrete continuity Equation (39) over the active layers to give

\[
\eta_{i}^{n+1} - \eta_{i}^{n} \Delta \tau + \frac{1}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} \sum_{k=1}^{N_k(i)} u_{f(j,k)}^{im} J_{f(j,k)} l_{f(j)} N_{i,j} = 0. \tag{40}
\]

The linear system associated with the implicit free-surface discretization is derived by substituting the predictor horizontal velocity \( u_f^* \) from Equation (35) into Equation (40). This results in a symmetric and positive-definite linear system for \( \eta^{n+1} \), which is solved efficiently with the preconditioned conjugate gradient algorithm (Casulli and Walters, 2000). After obtaining the
free-surface height at time-step \( n+1 \), the horizontal predictor velocity is obtained by solving Equation (35).

For a hydrostatic model, we assume that the horizontal velocity at time step \( n+1 \) is equal to the predictor velocity such that \( u_f^{n+1} = u_f^* \). Interestingly, the Eulerian vertical velocity \( u_3^{n+1} \) is never needed under the hydrostatic approximation, since momentum advection and scalar transport only require the cross-coordinate vertical velocity \( W \). With the assumption that \( W_{i,1/2} = 0 \) at the bed, the cross-coordinate vertical velocity can be obtained by manipulating the finite-volume form of the layer-height continuity Equation (39) to give

\[
W_{i,k+1/2}^{n+1} = W_{i,k-1/2}^{n+1} - \frac{1}{A_p(i)} \sum_{j=1}^{N_s(i)} u_{f(j,k)}^{n+1} J_{f(j,k)} I_f(j) N_{i,j} - \frac{J_{n+1}^{i,k} - J_n^{i,k}}{\Delta \tau}.
\]

(41)

For a general ALE approach (see Section 4), \( J^{n+1} \) is known a-priori, thereby giving all the terms needed on the right-hand side of Equation (41) to compute \( W_{i,k+1/2}^{n+1} \). The hydrostatic vertical predictor velocity is then obtained with Equation (37) to give

\[
W_{i,k+1/2}^{n+1} = W_{i,k+1/2}^{n+1} - \frac{1}{\alpha_1} \left( W_{i,k+1/2}^{n+1} - \alpha_2 W_{i,k+1/2}^{n} - \alpha_3 W_{i,k+1/2}^{n-1} \right),
\]

(42)

where \( W_{i,k+1/2}^{n+1} = W_{i,k+1/2}^{n+1} \) under the hydrostatic assumption.

3.5. Nonhydrostatic pressure correction step

For a nonhydrostatic model, after solving for the hydrostatic predictor velocities \( u_f^* \) and \( u_3^* \) with the predictor step based on Equations (35) and (36), the velocities at the new time step are obtained with the corrector step

\[
u_{f(j,k)}^{n+1} = u_{f(j,k)} - \Delta \tau \frac{\partial q_c}{\partial n_f} |_{(j,k)},
\]

(43)

\[
u_{3(i,k+1/2)}^{n+1} = u_{3(i,k+1/2)} - \Delta \tau \frac{\partial q_c}{\partial \xi_3} |_{(i,k+1/2)},
\]

(44)

where \( q_c \) denotes a correction to the nonhydrostatic pressure which is used to update the full nonhydrostatic pressure with

\[
q_{i,k}^{n+1/2} = q_{i,k}^{n-1/2} + q_c(i,k).
\]

(45)
The nonhydrostatic pressure is stored at half time steps to ensure the second-order temporal accuracy (Armfield and Street, 2000).

The governing equation for the pressure correction \( q_c \) is derived by enforcing the finite-volume form of the divergence-free constraint (15) at time-step \( n + 1 \), which is given by

\[
U^{n+1}_{3(i,k+\frac{1}{2})} - U^{n+1}_{3(i,k-\frac{1}{2})} + \frac{1}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} u^{n+1}_{f(j,k)} J_{f(j,k)} l_{f(j)} N_{(i,j)} = 0. \tag{46}
\]

Using Equations (43) and (44), the contravariant volume flux introduced in Equation (14) is approximated with

\[
U^{n+1}_{3(i,k+\frac{1}{2})} = u^{n+1}_{3(i,k+\frac{1}{2})} - \frac{\partial x_3}{\partial \xi_1} u^{n+1}_{1(i,k+\frac{1}{2})} - \frac{\partial x_3}{\partial \xi_2} u^{n+1}_{2(i,k+\frac{1}{2})} \tag{47}
\]

where the mild-slope approximation has been used to eliminate the non-orthogonal terms in the nonhydrostatic pressure gradient, and

\[
U^*_{3(i,k+\frac{1}{2})} = u^*_{3(i,k+\frac{1}{2})} - \frac{\partial x_3}{\partial \xi_1} u^*_{1(i,k+\frac{1}{2})} - \frac{\partial x_3}{\partial \xi_2} u^*_{2(i,k+\frac{1}{2})} \tag{48}
\]

After substituting \( u_f^{n+1} \) defined in Equation (43) and \( U^+_3 \) defined in Equation (47) into the discrete divergence-free constraint (46), the Poisson equation for the pressure correction \( q_c \) is given by

\[
L(q_c(i,k)) = S^*_q(i,k), \tag{49}
\]

where the Poisson operator is given by

\[
L(q_c(i,k)) = \frac{\Delta x}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} \frac{\partial q_c}{\partial n_f} J_{f(j,k)} l_{f(j)} N_{(i,j)} + \Delta x \left[ \frac{1}{J^+_{3(i,k+\frac{1}{2})}} \frac{\partial q_c}{\partial \xi_3} \bigg|_{(i,k+\frac{1}{2})} - \frac{1}{J^-_{3(i,k-\frac{1}{2})}} \frac{\partial q_c}{\partial \xi_3} \bigg|_{(i,k-\frac{1}{2})} \right]. \tag{50}
\]
and the source term is given by

\[ S_{q(i,k)}^* = U_{3(i,k+\frac{1}{2})}^* - U_{3(i,k-\frac{1}{2})}^* + \frac{1}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} u_{f(j,k)}^* J_{f(j,k)} N_{f(j)} N_{i,j}. \]  

The discrete Poisson Equation (49) represents a symmetric, positive-definite system of linear equations, which is solved efficiently with the preconditioned conjugate gradient method using a block-Jacobi preconditioner (Fringer et al., 2006).

After solving for \( q_c \), the horizontal component of the Eulerian velocity is corrected with Equation (43) to obtain \( u_{n+1}^f \). The cross-coordinate velocity is obtained with the discrete layer-height continuity Equation (41) evaluated with \( u_{im}^m \) and \( W_{im}^m \) instead of \( u_{im}^{im*} \) and \( W_{im}^{im*} \), where \( u_{im}^m = \alpha_1 u_{n+1}^f + \alpha_2 u_{f}^n + \alpha_3 u_{n-1}^f \) (Equation (38)) and \( W_{n+1} \) is obtained with Equation (42) after replacing \( W^* \) with \( W_{n+1}^* \). There is no need to correct the vertical velocity with Equation (44) since the contravariant volume flux \( U_{3}^{n+1} \) is never used in the calculation. Instead, only its predictor \( U_{3}^* \) (Equation (48)) is needed at the next time step to compute the right-hand side of the pressure Poisson Equation (51). The cross-coordinate velocity \( W \), in turn, is needed to compute advection of momentum with Equation (33) and scalars with Equation (52).

3.6. Discrete scalar transport

In contrast to the discrete momentum equation, scalar transport is discretized in a conservative manner since the scalar quantities are stored at cell centers. Following Gross et al. (2002) and Koltakov and Fringer (2013), in order to ensure consistency with the discrete layer-height continuity Equation (39) and hence guarantee local and global conservation of heat and mass, the predictor velocity field must be used instead of the corrected velocity defined in Equations (43) and (44). Therefore, the corresponding finite-volume discretization of Equation (12) that is consistent with the layer-height con-
continuity Equation (39) is given by

$$\frac{J_{n+1}}{(i,k)} \phi_{(i,k)} - \frac{J_n}{(i,k)} \phi_{(i,k)} = - \frac{1}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} u^*_f \phi_f(j,k) J_f(j,k) f(j) N(i,j)$$

$$- \left( W^*_{im} \phi_f^{im}_{(i,k+\frac{1}{2})} - W^*_{im} \phi_f^{im}_{(i,k-\frac{1}{2})} \right)$$

$$+ \kappa^T V_{(i,k+\frac{1}{2})} \left( \phi^{im}_{(i,k+1)} - \phi^{im}_{(i,k)} \right)$$

$$- \kappa^T V_{(i,k-\frac{1}{2})} \left( \phi^{im}_{(i,k)} - \phi^{im}_{(i,k-1)} \right)$$

$$+ D_H(\phi)|_{ex(i)}$$

(52)

where $\phi_f(j,k)$ denotes the scalar on face $j$ in layer $k$. The vertical advection and diffusion terms in Equation (12) have been discretized implicitly in Equation (52) to avoid the stability limitation associated with small layer heights. Boundary conditions at the free surface and bed for scalar transport are given by the no-flux conditions

$$\left( \kappa_T H \frac{\partial \phi}{\partial \xi_3} \right)_{(i,k_{top})} = \left( \kappa_T H \frac{\partial \phi}{\partial \xi_3} \right)_{(i,k_{bot})} = 0,$$

(53)

where $k_{top}$ and $k_{bot}$ denote indices of the top- and bottom-most active layers, respectively. Computation of the scalar values on the faces based on cell-centered quantities is performed with the unstructured-grid, flux-limiting scheme of Casulli and Zanolli (2005).

### 3.7. Solution procedure

The model system solved in the present study consists of Equations (12), (13), (15) and (16), which are discretized in Sections 3.3 to 3.6. The solution procedure to update the velocity, free-surface heights, grid, and scalars at each time step is summarized as follows:

1. Solve the linear system arising from Equation (40) for the elevation of the free surface $h^{n+1}$.
2. Compute the horizontal predictor velocity $u^*_f$ with Equation (35).
3. Update the layer height (or Jacobian) at the cell centers $J^{n+1}$ and the grid location $x_3^{n+1}$ using the methods outlined in Section 4. Then update the grid metrics including $\partial x_3/\partial \xi_1$ and $\partial x_3/\partial \xi_2$. 

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4. Compute the predictor cross-coordinate velocity $W^*$ with Equation (41). If isopycnal coordinates are desired, set $W^* = 0$.

5. Solve the discrete scalar transport Equation (52) for the salinity or temperature field using the predictor velocities $u^*_f$ and $W^*$. With the updated scalar field, compute the density field with an equation of state.

6. For a hydrostatic model, return to step 1 with $u^{n+1}_f = u^*_f$ and $W^{n+1} = W^*$. Otherwise, solve the vertical momentum Equation (36) for the predictor vertical velocity $u^*_3$. Then, compute the vertical contravariant volume flux $U^*_3$ using the predictor velocities $u^*_f$ and $u^*_3$ with Equation (48).

7. Solve the Poisson Equation (49) for the nonhydrostatic pressure correction $q_c$, and update $u^{n+1}_f$ with the corrector step Equation (43). Compute the cross-coordinate velocity $W^{n+1}$ using Equation (41) but with $W^{im}$ and $u^{im}_f$ instead of $W^*$ and $u^*_f$. Then update the nonhydrostatic pressure $q^{n+1/2}$ with Equation (45).

3.8. Discussion of the method

Several features of the method that require some justification are outlined extensively by Vitousek and Fringer (2014), which we summarize here. First, a common problem with the ALE approach is an inconsistency between the free-surface height and the equivalent height based on a vertical sum of the layers, particularly with mode-splitting (Hallberg and Adcroft, 2009). In principle, since the depth-integrated continuity Equation (40) for the free-surface is derived from a discrete vertical sum of the layer-height continuity Equation (39), the water depth should be exactly equal to the sum of the layer heights. However, small errors associated with solution of the linear system arising from the implicit discretization of the free-surface in Equation (40) lead to inconsistencies. To account for these, we apply a correction to ensure that the vertical sum of the layer heights is identically equal to the water depth, following Vitousek and Fringer (2014). Second, Adcroft and Hallberg (2006) suggested that the ALE approach to update the layer heights is inconsistent with the nonhydrostatic velocity field because the layer heights are updated with the hydrostatic velocity field. Indeed, the nonhydrostatic pressure does not affect the layer heights, the free surface, or the scalar transport during each time step. However, our justification follows that of Vitousek and Fringer (2014), in that this is a common feature of
moving-grid Navier-Stokes solvers which generally assume a fixed grid upon evaluating the nonhydrostatic pressure and correcting the velocity field (e.g. Chou and Fringer, 2010; Koltakov and Fringer, 2013). One could consider an iterative approach in which the corrected velocity is substituted back into the corrector step, and the free surface and layer heights are updated accordingly. When implemented for the nonhydrostatic pressure correction method in \( z \)-coordinates, this procedure convergences in a few iterations (Vitousek and Fringer, 2013). However, the added expense is not worth the effort given that omitting the nonhydrostatic effect from the layer-height and free-surface calculations does not impact the overall time accuracy of the time-stepping scheme (Armfield and Street, 2000; Vitousek and Fringer, 2013).

3.9. Accuracy and stability

The model guarantees conservation of volume and scalars both locally and globally, although momentum and kinetic energy are not conserved in a discrete sense. We conducted a series of test cases (not shown) following those in Vitousek and Fringer (2014) to demonstrate that the model is second-order accurate in both time and space on Cartesian meshes. This spatio-temporal accuracy degrades to first-order on unstructured meshes or in the presence of fronts or discontinuities in the velocity or scalar fields.

For model stability, the explicit discretization of momentum advection incurs a constraint on the Courant number \( C_U = u_{\text{max}} \Delta \tau/d_f + W_{\text{max}} \Delta \tau/J \), which reverts to \( C_U = C_u = u_{\text{max}} \Delta \tau/d_f \) for isopycnal coordinates for which \( W = 0 \). In practice, however, \( u_{\text{max}}/d_f \sim W_{\text{max}}/J \) in nonhydrostatic simulations, which is not the case for hydrostatic simulations, for which \( |W|/J \gg u_{\text{max}}/d_f \). Additionally, adaptive-grid simulations may incur small layer heights and lead to a situation in which \( |W|/J \gg u_{\text{max}}/d_f \) even for nonhydrostatic cases. In most cases, it suffices to constrain \( C_u = u_{\text{max}} \Delta \tau/d_f \) with the understanding that multiple dimensions or grid adaptivity may incur a stronger constraint on stability.

Since vertical advection of scalars is implicit, \( C_u \) rather than \( C_U \) is a better indicator of stability for scalar transport. \( C_u \) is also a good metric for stability of layer height advection which is restricted to the horizontal. Explicit discretization of horizontal diffusion incurs a stability restriction on the horizontal diffusion Courant number \( C_\nu = \max(\nu^T_H, \kappa^T_H) \Delta \tau/d_f^2 \). Finally, the explicit discretization of the baroclinic pressure gradient incurs a stability restriction on the horizontal internal wave Courant number \( C_i = c_1 \Delta \tau/d_f \), where \( c_1 \) is the speed of first-mode internal gravity wave.
Although it is difficult to determine the exact stability bounds in terms of \( C_U, C_\nu, \) and \( C_i \) on unstructured grids, the linear stability properties of our model are similar to those of the isopycnal coordinate model developed by Vitousek and Fringer (2014). These properties are dictated by different combinations of the coefficients \( c_{im}, \theta \) and \( b_{ex} \) as defined by the implicit (Equation (37)) and explicit (Equation (38)) temporal discretization schemes. Following the discussion by Durran and Blossey (2012), the maximum internal wave Courant number \( C_i \) is 0.76 and 0.72 for the AM2-AX2* and AI2*-AB3 schemes, respectively. In general, following the suggestion of Vitousek and Fringer (2014), for two-dimensional (x-z) simulations we choose a time step based on the most restrictive of \( C_i \leq 0.5, C_\nu \leq 0.25, \) and \( C_U \leq 1 \). For most practical applications, the internal wave speed \( c_1 > u_{max} \) and \( c_1 > \max(\nu_T^L, \kappa_T^L)/d_f \). Therefore, the time step is typically limited by the explicit discretization of internal gravity waves, and it must be reduced by an additional factor of two for three-dimensional simulations.

4. Updating the layer heights

The advantage of applying the ALE method is that the vertical coordinates can be updated at each time step rather arbitrarily, as long as the motion is small relative to the local layer height. The motion of the grid is accounted for naturally with the cross-coordinate velocity using Equation (41), which allows for cross-coordinate fluxes of momentum and scalars. Accordingly, we can specify the layer heights to represent the commonly used \( z, \sigma, \) or isopycnal coordinates, as described in Section 4.1. The layer heights can also be updated adaptively to resolve vertical density gradients as described in Section 4.2.

4.1. \( z, \sigma, \) or \( \rho \)-coordinates

Representation of \( z \)-levels is trivial with the ALE approach because it amounts to layer heights that are fixed in time and constant in the horizontal. With constant \( z \)-levels, terms associated with grid motion vanish, and the approach is identical to the SUNTANS model except for the application of higher-order time-discretization schemes in Equations (35) and (36).

To implement terrain-following or \( \sigma \)-coordinates, the layer heights are given by

\[
J_{(i,k)}^{n+1} = \frac{\eta_{(i)}^{n+1} + d_{(i)}}{N_{k(i)}} \Delta \sigma_{(i,k)}. \tag{54}
\]
If uniformly-spaced layers are required in each the water column, the height of sigma layers $\Delta \sigma_{(i,k)} = 1$. For general terrain-following coordinates in which finer resolution of top or bottom boundary layers is desired, $\Delta \sigma_{(i,k)}$ is not constant although it must satisfy $\sum_{k=1}^{N_k(i)} \Delta \sigma_{(i,k)} = N_k(i)$.

Finally, if isopycnal coordinates are desired, the layer heights are updated with the discrete continuity Equation (39) after assuming there is no cross-coordinate flux $W = 0$. The resulting discrete evolution equation for the layer heights is given by

$$\frac{J_{n+1}^{i,k} - J_n^{i,k}}{\Delta \tau} + \frac{1}{A_p(i)} \sum_{j=1}^{N_{s(i)}} u_{j}^{i,m^*} J_{f(j,k)}^n l_{f(j)}^n N_{(i,j)} = 0.$$  

Following Vitousek and Fringer (2014), we do not need to explicitly stabilize the isopycnal-coordinate approach to account for the possibility of Kelvin-Helmholtz instabilities. These are naturally damped by the time-stepping scheme and through regularization by the nonhydrostatic pressure.

4.2. Adaptive vertical coordinate

Although the isopycnal coordinates naturally follow density lines and eliminate vertical spurious numerical diffusion, they cannot be used in the presence of overturning or convective motions. An alternative is to update the grid adaptively based on density or velocity gradients. Hofmeister et al. (2010) developed an adaptive method in which the vertical grid is updated with a diffusion equation. The corresponding diffusion coefficients depend on the vertical stratification, shear and distance from the surface (to resolve near-surface gradients). In addition to vertical and horizontal diffusion steps, an isopycnal tendency step seeks to align the vertical coordinates with the isopycnals to reduce spurious vertical diffusion associated with vertical advection. Rather than employ a diffusion equation, we follow the $r$-adaptive approach described by Tang and Tang (2003) and Koltakov and Fringer (2013), in which the grid locations at each time step are given by solution of the Laplace equation

$$\frac{\partial}{\partial \xi_1} \left( M_{11} \frac{\partial x_3^{n+1}}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( M_{22} \frac{\partial x_3^{n+1}}{\partial \xi_2} \right) + \frac{1}{J} \frac{\partial}{\partial \xi_3} \left( M_{33} \frac{\partial x_3^{n+1}}{J \partial \xi_3} \right) = 0.$$  

(56)
The so-called monitor functions are given by

\[ M_{11} = \sqrt{1 + \alpha_H \left( \frac{\partial \rho}{\partial \xi_1} \right)^2}, \]  
\[ M_{22} = \sqrt{1 + \alpha_H \left( \frac{\partial \rho}{\partial \xi_2} \right)^2}, \]  
\[ M_{33} = \sqrt{1 + \alpha_V \left( \frac{1}{J} \frac{\partial \rho}{\partial \xi_3} \right)^2}, \]

where \( \alpha_H \) and \( \alpha_V \) are coefficients that dictate the degree of adaptivity, as discussed below. This is the general three-dimensional form of the one-dimensional Euler-Lagrange equation in the vertical derived by Burchard and Beckers (2004) and Hofmeister et al. (2010) that minimizes vertical gradients with respect to the vertical coordinate \( \xi_3 \) of some scalar \( f \), i.e. \( \partial f / \partial \xi_3 \). However, instead of solving the Euler-Lagrange equation, Hofmeister et al. (2010) updated the grid with a diffusion equation with horizontal and vertical diffusion coefficients that are analogous to the coefficients \( \alpha_H \) and \( \alpha_V \). In fact, Equation (56) is essentially the steady-state equivalent of the diffusion equation derived by Hofmeister et al. (2010). Although the two approaches are similar, the advantage of the \( r \)-adaptive approach is the ability to specify coefficients that enforce the desired constraints through solution of one equation rather than having to update the grid with several steps as in the approach of Hofmeister et al. (2010).

A finite-difference discretization of Equation (56) in cell \((i, k)\) gives the equation governing the vertical coordinates at the new time step, \( x_{3(i,k+1/2)}^{n+1} \), as a function of the monitor functions and grid quantities at the old time step, viz.

\[
\frac{J_{(i,k+1/2)}^{n}}{A_{p(i)}} \sum_{j=1}^{N_{s(i)}} \partial x_3 \bigg|_{(j,k+1/2)}^{n+1} M_{f(j,k+1/2)}^{n} f_{(j)} N_{(i,j)} + \\
M_{33}^{n} \bigg|_{(i,k+1)}^{n} \left( x_{3(i,k+3/2)}^{n+1} - x_{3(i,k+1/2)}^{n+1} \right) - M_{33}^{n} \bigg|_{(i,k)}^{n} \left( x_{3(i,k+3/2)}^{n+1} - x_{3(i,k-3/2)}^{n+1} \right) = 0,
\]

(60)

where the top and bottom boundary conditions are given by \( x_{3(i,1/2)} = -d_{(i)} \) and \( x_{3(i,N_{k(i)}+1/2)} = h_{(i)}^{n+1} \), and Neumann conditions are assumed on lateral
boundaries. On cell faces, we have made the approximation

\[ n_f M_{11} \frac{\partial x_3}{\partial \xi_1} + n_f M_{22} \frac{\partial z}{\partial \xi_2} \approx M_f \frac{\partial x_3}{\partial n_f}, \]  

(61)

where the face-centered monitor function is given by

\[ M_f = \sqrt{1 + \alpha_H \left( \frac{\partial \rho}{\partial n_f} \right)^2}. \]  

(62)

Following Koltakov and Fringer (2013), Equation (60) is solved with a line-by-line method. Numerical convergence to a small tolerance is not necessary because the grid is smoothed as the simulation evolves in time as well as during the course of each iteration. Therefore, we limit the number of iterations at each time step to three, which ensures minimal overhead while still producing a sufficiently adapted grid. Indeed, since Equation (60) is not solved exactly, the method can essentially be written as a diffusion update of the grid following Hofmeister et al. (2010). At the end of each iteration, layer heights are updated with

\[ J_{n+1}^{(i,k)} = x_{n+1}^{3(i,k+1/2)} - x_{n+1}^{3(i,k-1/2)}. \]  

Solution of Equation (60) induces a computational overhead of roughly 10%. This is less than the 30 – 40\% incurred for idealized test cases with the method of Hofmeister et al. (2010), although their method incurs an overhead of just 5 – 8\% when applied to realistic three-dimensional problems (Gräwe et al., 2015).

The essence of Equation (60) is that the terms involving \( M_{33} \) concentrate grid nodes in the vertical where density gradients are highest, while the term involving \( M_f \) ensures smooth horizontal variability of the vertical coordinate. We can approximate the behavior of Equation (60) through analysis of the simple case \( M_f = 0 \), which implies

\[ \frac{M_{33}}{J} \left| \frac{J_{n+1}^{(i,k)}}{J_{(i,k)}} \right| = f_i, \]

where, to ensure that the adaptivity does not change the water depth given by the sum of the layer heights,

\[ f_i = \frac{\eta_{(i)}^{n+1} + d_{(i)}}{N_{k(i)} \sum_{k=1}^{n} \eta_{(i)}^{n+1}}. \]
Therefore, $M_{33}^n(i,k)$ will dictate the new layer height according to $J_{n+1}^{i,k} = f_i J_n^{i,k} / M_{33}^n(i,k)$. For $\alpha_V \neq 0$, $M_{33}^n(i,k)$ is larger where density gradients are larger, thus giving smaller layer heights in those regions, while the magnitude of $\alpha_V$ dictates the smallest layer height in each water column. Because it is difficult to determine a value of $\alpha_V$ \textit{a-priori} that gives the desired minimum layer height, we prevent the minimum layer height at the new time step from being smaller than one-half of the layer height at the old time step by limiting the vertical monitor function with $\max\left(M_{33}^n(i,k)\right) = 2$. If there is no adaptivity, $\alpha_V = 0$ and $M_{33}^n(i,k) = 1$, giving $J_{n+1}^{i,k} = J_n^{i,k}$ and $f_i = 1$.

5. Numerical Experiments

5.1. Cartesian-coordinate, nonhydrostatic test cases

Vitousek and Fringer (2014) outlined numerous test cases to demonstrate the robustness of their nonhydrostatic, isopycnal-coordinate model on a one-dimensional, horizontally-Cartesian grid. To test model ability to reproduce linear, nonhydrostatic gravity wave dispersion, surface and internal gravity wave seiches were simulated. These test cases demonstrate the need for fewer layers (by up to one order of magnitude) to simulate hydrostatic internal wave propagation. However, more layers are needed for nonhydrostatic simulations in order to resolve the vertical variability associated with nonhydrostatic effects, particularly for short surface or internal gravity waves.

Vitousek and Fringer (2014) also demonstrated linear, nonhydrostatic dispersion through the simulation of internal wave beams generated by tidal flow over a small-amplitude Gaussian hill. The relative importance of nonhydrostatic effects is dictated by the ratio of the tidal frequency $\omega$ to the buoyancy frequency $N$. Most flows in the ocean are hydrostatic since $\omega \ll N$, and hence a hydrostatic model will accurately predict the slope of the internal wave beam. However, Vitousek and Fringer (2014) showed that roughly when $\omega = 0.3 N$, the hydrostatic model diverges from the nonhydrostatic model, producing internal wave beams that are not as steep as indicated by the nonhydrostatic dispersion relation. In the limit $\omega \to N$, the nonhydrostatic model correctly produces the limiting case of vertically-propagating beams, while the hydrostatic model produces beams that incorrectly propagate at an angle of 45°.

Because most ocean models (including the present model) discretize the baroclinic pressure gradient with a second-order accurate central differencing
operator in space, the result has a truncation error that produces numerical dispersion that mimics physical, nonhydrostatic dispersion (Vitousek and Fringer, 2011). To ensure that the numerical dispersion is smaller than the nonhydrostatic dispersion when simulating internal solitary-like waves, Vitousek and Fringer (2011) show that the horizontal grid resolution must satisfy \( d_f \leq h_e \), where \( h_e \) is the effective depth of the mixed layer that supports the internal solitary-like waves. This requirement was readily demonstrated with several test cases related to internal solitary-like waves by Vitousek and Fringer (2014), including the evolution of an internal Gaussian of depression into a train of internal solitary-like waves, the generation of internal solitary-like waves by tidal flow over an idealized deep-ocean ridge, and the degeneration of an internal seiche into trains of internal solitary-like waves.

We tested the present model using a one-dimensional array of quadrilaterals and showed that it reproduces the results of all test cases outlined in Vitousek and Fringer (2014). Therefore, we do not reproduce those results here and instead focus on test cases that accentuate the unique features of our approach, namely the unstructured grid through simulation of an internal solitary-like wave interacting with an isolated island (Section 5.2) and the adaptive grid through simulation of the lock exchange problem (Section 5.3).

5.2. Internal waves interacting with a circular island

We compare \( z \)-levels to isopycnal coordinates with simulations of internal solitary-like waves propagating past a circular island over the bathymetry shown in Figure 3(a). This test case is similar to the case discussed by Lynett and Liu (2002) which highlights internal wave refraction, reflection, diffraction, and wave-wave interactions around the idealized island. These features are evident in satellite imagery of internal waves interacting with the Dongsha Atoll in the South China Sea (Li et al., 2013).

As shown in Figure 3(a), the square domain has a length \( L_x = 120 \text{ km} \) and a width \( L_y = 120 \text{ km} \), while the depth ranges from 300 to 600 m. The circular island near the center of the domain has a diameter of 20 km. The density field is initialized with the approximate two-layer stratification

\[
\rho = \rho_0 + \frac{\Delta \rho}{2} \left\{ 1 - \tanh \left[ \frac{2 \tanh^{-1} \alpha_s}{\delta \rho} \left( z + h_1 - \eta \right) \right] \right\},
\]

(63)

where \( \rho_0 = 1000 \text{ kg m}^{-3} \) is the reference density, \( \Delta \rho = 10 \text{ kg m}^{-3} \) is the density difference between the two layers, \( \alpha_s = 99\% \) of the pycnocline has a
thickness of $\delta_{\rho} = 80\, \text{m}$, and $h_1 = 100\, \text{m}$ is the upper-layer depth. The initial wave of depression $\eta$ that evolves into a solitary-like internal wave is shown in Figure 3(b) and given by

$$\eta(x, y, t = 0) = -2\eta_0 \sech^2 \left[ \frac{2(L_0 - x)}{L_0} \right],$$

(64)

where $\eta_0 = 36\, \text{m}$ and $L_0 = 1309\, \text{m}$ are the approximate amplitude and length of the solitary wave of depression that propagates to the west from the eastern boundary.

In the horizontal plane, we employ the unstructured, triangular grid depicted in Figure 4(a). To resolve the leading-order nonhydrostatic effects, the grid resolution is dictated by the need for the ratio of the numerical to physical dispersion $\Gamma \equiv K\lambda^2 \ll 1$. As proposed by Vitousek and Fringer (2011), this requirement ensures the dominance of physical over numerical internal gravity wave dispersion. The constant $K$ depends on the numerical
discretization and $\lambda = d_f/h_e$ is the grid lepticity. Following Vitousek and Fringer (2011), the effective depth in a continuously stratified fluid of depth $d$ is given by

$$h_e = \sqrt{\frac{3 \int_{-d}^{0} \chi^2 dz}{\int_{-d}^{0} \left( \frac{\partial \chi}{\partial z} \right)^2 dz}},$$

(65)

where $\chi(z)$ is the first-mode eigenfunction associated with the stratification (63) (Fringer and Street, 2003). Using the JIGSAW mesh generator (Engwirda, 2018), an unstructured grid is generated with the mesh-size constraint $d_f \leq h_e \sqrt{\Gamma / K}$, where we set $\Gamma = 0.2$ which represents a good balance between computational cost and sufficient resolution of nonhydrostatic effects. Following Vitousek and Fringer (2011), we assume $K = 0.075$ since the numerical methods we employ are similar to those in the SUNTANS model. Indeed, this value appears appropriate given that the results indicate that the leading-order nonhydrostatic effects are sufficiently resolved. The resulting horizontal mesh depicted in Figure 4(a) has a total of 391,338 grid cells with an average mesh size of 282 m and minimum/maximum mesh size of 113 m and 488 m, respectively.

Elimination of vertical spurious numerical diffusion in isopycnal coordinate models substantially reduces the number of vertical layers needed to resolve vertical density gradients when compared to $z$-level models. For example, Vitousek and Fringer (2014) showed that the number of vertical layers can be reduced by almost one order of magnitude when using isopycnal coordinates to simulate internal solitary waves while resolving the leading-order nonlinear and nonhydrostatic effects. We demonstrate that the present horizontally unstructured-grid model has the same capability as the horizontally Cartesian-grid model of Vitousek and Fringer (2014) by simulating internal wave interaction with the circular island. The different cases are listed in Table 1. In case Z_NH_80, a nonhydrostatic simulation of internal solitary-like wave propagation is conducted with 80 $z$-levels vertically distributed as shown in Figure 4(c) and given by

$$h_{(k)} = \begin{cases} 
12.1 \text{ m} & 1 \leq k \leq 32 \\
4.0 \text{ m} & 33 \leq k \leq 70 \\
6.0 \text{ m} & 71 \leq k \leq 80 
\end{cases}.$$  

(66)

In case R_NH_10, the same nonhydrostatic simulation is performed but with 10 isopycnal layers vertically distributed as shown in Figure 4(d) and given
Following Equation (63), the initial undisturbed density stratification is plotted as the thick solid line in panel (b) for reference.

By

\[ h_{(k)} = \begin{cases} 
230.0 \text{ m} & 1 \leq k \leq 2 \\
13.3 \text{ m} & 3 \leq k \leq 8 \\
30.0 \text{ m} & 9 \leq k \leq 10 
\end{cases} \]  \hspace{1cm} (67)

To assess nonhydrostatic effects, case R_H_10 is identical to case R_NH_10 but is hydrostatic. Finally, case R_NH_2 with vertical layers depicted in Figure 4(b) is conducted to show that similar nonhydrostatic results can be obtained with just two isopycnal layers. The density in the layers is assumed to be given by the density in Equation (63) at mid-layer height. The total number of grid cells in three dimensions for each test case is listed in Table 1.

We simulate the evolution of internal waves past the circular island for 10 h with a time-step size of \( \Delta t = 5 \text{s} \), which is dictated by the need to accurately resolve first-mode internal gravity waves propagating with a speed of \( c_1 = 2.74 \text{ m s}^{-1} \). This is the speed of the first-mode linear internal gravity
Table 1: Hydrostatic (H) and nonhydrostatic (NH) test cases to simulate internal solitary-like waves interacting with a circular island using z-levels (Z) and isopycnal (R) coordinates with different numbers of vertical layers. The problem size is the total number of grid cells (# Cells) in three dimensions, while $T_{wall}$ is the wall-clock time per time step.

| Case Name | Hydro/Nonhydrostatic | Coordinate | # Layers | # Cells ($\times 10^6$) | $T_{wall}$ (s) |
|-----------|-----------------------|------------|----------|--------------------------|----------------|
| Z_NH_80   | Nonhydrostatic        | z          | 80       | 31.3                     | 13.85          |
| R_NH_10   | Nonhydrostatic        | $\rho$     | 10       | 3.9                      | 2.47           |
| R_H_10    | Hydrostatic           | $\rho$     | 10       | 3.9                      | 0.62           |
| R_NH_2    | Nonhydrostatic        | $\rho$     | 2        | 0.8                      | 0.82           |

wave in a depth of 600 m with the stratification given in Equation (63). This time step results in a maximum internal wave Courant number of $C_i = c_1 \Delta t / \min(d_f) = 0.12$. Free-slip boundary conditions are applied at the four solid boundaries of the computational domain and on the island. No diffusion of momentum and scalars is assumed, and the flux-limiting scheme with the van Leer limiter (Van Leer, 1977) is used for horizontal advection of scalars (with $z$-coordinates) and layer heights (with $\rho$-coordinates). Simulations are run in parallel using 24 AMD 6378 Opteron Processors (2.4 GHz), and the wall-clock times per time step are indicated in Table 1.

Evolution of an internal solitary-like wave interacting with the circular island is illustrated for all four cases at three points in time corresponding to the three columns in Figure 5. We first discuss the features of case Z_NH_80 which we consider to be the base case given that it has a similar number of vertical $z$-levels as the cases in Vitousek and Fringer (2014). At time $t = 4$ h (Figure 5a1), wave refraction is first observed as the internal solitary wave crest at the northern part of the domain encounters the shallow bathymetry of the shelf. At time $t = 7$ h (Figure 5a2), the incident internal solitary wave has propagated across the island and there is a clear pattern of internal wave reflection. Finally, at time $t = 7$ h (Figure 5a3) the internal solitary wave has propagated past the island and there is clear internal wave reflection and oblique wave-wave interaction as the diffracted wave crests interact in the lee of the island.

The results for cases Z_NH_80 (Figure 5a) and R_NH_10 (Figure 5b) are nearly identical, indicating that 10 isopycnal layers reproduce the results with 80 $z$-levels on a horizontally-unstructured grid. As indicated by the numbers in Table 1, this implies that the isopycnal-coordinate model can reproduce
results of a z-level model with nearly one order of magnitude fewer grid points and a reduction in the computational cost by a factor of nearly six. The computational cost is not proportional to the reduction in the number of layers because of the reduction in parallel efficiency associated with the nonhydrotatic pressure solver as the problem size decreases while retaining the same number of processors. While the computational cost can be further reduced by a factor of four by eliminating the nonhydrotatic pressure and retaining 10 isopycnal-coordinate layers (compare case R_NH_10 to R_H_10 in Table 1), the nonhydrotatic pressure plays an important role in these simulations. Comparison of the results in Figure 5(b) to (c) shows that a lack of nonhydrostatic dispersion in the hydrostatic simulation produces internal solitary waves that are too short. In fact, these waves are a numerical artifact arising from a balance between nonlinear steepening and numerical dispersion, as discussed by Vitousek and Fringer (2011). Because they are too short, numerical diffusion associated with the TVD scheme for the layer heights overwhelms the hydrostatic simulations and leads to waves with significantly smaller amplitude. The smaller amplitude leads to a slightly lower internal solitary wave speed owing to reduced amplitude dispersion.

Surprisingly, comparison of the solitary wave widths obtained with the 10- and 2-layer isopycnal-coordinate models in Figure 5(b) and (d) shows that the nonhydrostatic dispersion is still resolved with just two layers. This reflects ability of the model to reproduce some of the nonlinear and nonhydrotatic physics with a reduction in computational cost by a factor of 17 when compared to the 80-layer z-level model (compare cases Z_NH_80 and R_NH_2 in Table 1). However, the two-layer model overpredicts the wave speed because the finite-thickness pycnocline, which is needed to correctly predict the wave speed, is not represented by two layers. Furthermore, we should not expect to reproduce continuous stratification results with just two or even three layers given the potential for higher internal wave modes to impact the solution. We suspect the oscillatory behavior of the two-layer solution in Figure 5(d) arises because the two-layer solution is overly dispersive leading to trailing internal waves upon interacting with the shelf. These trailing waves are absent in the multilayer solutions because energy is distributed among the higher modes that propagate in phase with the leading solitary wave.
Figure 5: Evolution of internal waves interacting with a circular island for case Z_{NH.80} (a), R_{NH.10} (b), R_{H.10} (c) and R_{NH.2} (d). Snapshots of wave amplitude (in meter) at time $t = 4, 7, 10$ h are presented in the left, middle, and right panels, respectively.
5.3. Lock-exchange gravity currents

The $r$-coordinate described in Section 4.2 is tested in this section with the simulation of lock-exchange gravity currents following the parameters used in the DNS described by Härtel et al. (2000). This is a common test case for nonhydrostatic ocean models including FVCOM-NH (Lai et al., 2010), SUNTANS (Fringer et al., 2006), GETM (Klingbeil and Burchard, 2013), and Fluidity-ICOM (Hiester et al., 2011). As shown in Figure 6, gravity currents are simulated in a two-dimensional channel of length $L = 0.8$ m and height $D = 0.1$ m. A no-slip condition is applied at the bottom while a free-slip, rigid-lid condition is implemented at the surface, which allows the study of gravity currents at both no-slip and free-slip boundaries using one simulation. The initial density field is given by a vertical interface at the center of the domain with a nondimensional density difference of $\Delta \rho / \rho_0 = 0.001$. The kinematic viscosity $\nu = 10^{-6}$ m$^2$s$^{-1}$ and there is no scalar diffusivity, following Fringer et al. (2006), Koltakov and Fringer (2013), and Hiester et al. (2011). Second-order central-differencing is used for momentum advection and first-order upwinding is employed for scalar transport. Although the code has the ability to employ higher-order TVD schemes for scalar transport, we use first-order upwinding because its linear and simplified properties allow for a better demonstration and interpretation of the behavior of the $r$-coordinate. In particular, convergence of the solution is not monotonic with respect to variations in the $\alpha_V$ coefficient in the monitor function (59) when using the nonlinear TVD schemes.

The gravity currents are allowed to propagate for a simulation time of $10T$, where $T \equiv \sqrt{D/2g'} = 2.24$ s is the gravity current time scale and $g' \equiv g\Delta \rho / \rho_0 = 0.01 \text{m}\text{s}^{-2}$ is the reduced gravity. The grid consists of a horizontally Cartesian mesh with uniform horizontal grid spacing. In the vertical, an adaptive mesh based on resolving density gradients with Equation (56)
is implemented. Effects of the coefficient $\alpha_V$ in the monitor Equation (59) on the adaptive mesh and thus on the resulting gravity currents are investigated. As discussed in Section 4.2, the monitor function $M_{33}$ is limited to a maximum value of 2.0. This limits refinement of the layer heights from one step to the next by a factor of two and prevents over-refinement of the grid near the no-slip boundary. Test cases indicated in Table 2 are performed to compare the $z$- and $r$-coordinate results using different values of $\alpha_V$ and grid resolutions. The effects of $\alpha_V$ are investigated by performing $r$-coordinate simulations on a grid with $256 \times 64$ grid points with 31 values of $\alpha_V$ in the range $5.0 \times 10^{-3} \leq \alpha_V (\Delta \rho)^2 / D^2 \leq 3.5 \times 10^{-2}$. The horizontal refinement coefficient in the monitor function (62) is $\alpha_H = 2 \alpha_V$, and we find that the results are not as sensitive to this parameter as they are to $\alpha_V$. Three additional $z$-level cases with progressively refined grids are added, namely cases Z64, Z128 and Z192.

Since the lock-exchange flow propagates at a characteristic speed that exceeds the linear, first-mode internal gravity wave speed, the time-step size $\Delta t = 0.011\,\text{s}$ is used for all adaptive-grid simulations and is dictated by stability of explicit momentum advection. With this time-step size, $C_U = u_{max} \Delta t / \Delta x + W_{max} \Delta t / \Delta z = 0.18$, where $\Delta x = 0.31\,\text{cm}$ and $\Delta z = 0.16\,\text{cm}$ based on case Z64, and the characteristic velocities $u_{max} = 0.024\,\text{m}\,\text{s}^{-1}$ and $W_{max} = 0.013\,\text{m}\,\text{s}^{-1}$. Based on the speed of the first-mode, two-layer internal wave $c_1 \equiv \sqrt{g'D/4} = 0.016\,\text{m}\,\text{s}^{-1}$, $C_i \equiv c_1 \Delta t / \Delta x = 0.057$. The time-step size in the $z$-level cases is adjusted to maintain the same Courant number $C_u = u_{max} \Delta t / \Delta x$ as the horizontal mesh is refined. Because we iterative three times to adapt the grid at each time step, the adaptive grid incurs a

| Case Name | # Hori. Grid | # Vert. Grid | Coordinate | $\alpha_V (D^2/(\Delta \rho)^2)$ |
|-----------|--------------|--------------|------------|---------------------------------|
| Z64       | 256          | 64           | $z$        | -                               |
| Z128      | 256          | 128          | $z$        | -                               |
| Z192      | 768          | 192          | $z$        | -                               |
| R64_A5    | 256          | 64           | $r$        | $5 \times 10^{-3}$              |
| R64_A6    | 256          | 64           | $r$        | $6 \times 10^{-3}$              |
| ...       | 256          | 64           | $r$        | :                               |
| R64_A35   | 256          | 64           | $r$        | $35 \times 10^{-3}$             |
Table 3: Comparison of front speeds as defined by the Froude number for the lock-exchange gravity current simulations. The percent error is relative to the DNS results.

| Boundarya | Z64  | R64_A10 | R64_A20 | R64_A30 | Z192 | Staticb | DNSc |
|-----------|------|---------|---------|---------|------|---------|------|
| Free-slip | 0.6583 | 0.6596  | 0.6607  | 0.6614  | 0.6684 | 0.6571  | 0.6750 |
| Error (%) | 2.5   | 2.3     | 2.1     | 2.0     | 1.0   | 2.7     | -    |
| No-slip   | 0.5544 | 0.5564  | 0.5571  | 0.5574  | 0.5704 | 0.5547  | 0.5740 |
| Error (%) | 3.4   | 3.1     | 2.9     | 2.9     | 0.6   | 3.4     | -    |

a The free-slip results represent the speed of the leftward-propagating front at the top while the no-slip results represent the speed of the rightward-propagating front at the bottom.
b Results of Koltakov and Fringer (2013) with a static 256 × 64 grid.
c DNS results of Härtel et al. (2000).

Evolution of the lock-exchange gravity current is illustrated with the relative density anomaly $\rho/\rho_0 - 1$, and snapshots of case Z64 at times $t = 5$ and $10T$ are presented in Figure 7(a1) and (a2), respectively. As soon as the simulation starts, a mutual intrusion flow forms driving two fronts in opposite directions at the top and bottom boundaries. At $t = 5T$, a leftward-propagating free-slip head near the top and a rightward-propagating no-slip head near the bottom are easily identified in Figure 7(a1). At time $t = 10T$, Kelvin-Helmholtz billows at the density interface are clearly observed in Figure 7(a2). Corresponding snapshots of case R64_A20 are shown in Figure 7(b1) and (b2). In spite of the small difference in the Kelvin-Helmholtz billows, case R64_A20 illustrates that stronger vertical shear and density gradients and less numerical diffusion of scalars are possible with the adaptive grid which is more concentrated in regions of stronger gradients, as illustrated in Figure 7(c1) and (c2).

The present model is validated through comparison of the simulated front speeds to those reported in the literature. To quantify the front speeds, the Froude number $F_r = u_g/u_b$ is used, where $u_g$ is the gravity current speed and the buoyancy velocity $u_b \equiv \sqrt{g'D/2} = 0.022 \text{ m s}^{-1}$. The current speed $u_g$ is computed through linear regression of the front positions as a function of time.
Figure 7: Evolution of lock-exchange gravity currents for cases Z64 (a) and R64_A20 (b). Snapshots of density anomaly (in kg m$^{-3}$) relative to $\rho_0$ at time $t = 5$ and $10T$ are presented in panels (x1) and (x2), respectively. Evolution of the vertical coordinates for case R64_A20 is illustrated in (c) as solid lines and only 16 layers are shown for clarity.

of time. A selection of simulated front speeds is listed in Table 3, with errors expressed as the % difference between the indicated results and the DNS results of Härtel et al. (2000). The z-level results are nearly identical to the static grid simulation performed by Koltakov and Fringer (2013) with the same grid resolution. Further refinement of the grid in case Z192 leads to z-level results that much more closely match the DNS results of Härtel et al. (2000), with errors of just 1% and 0.6% for the free-slip and no-slip currents, respectively. As indicated by the progressive reduction in error with increasing $\alpha_V$ in Table 3, the front speed errors can also be reduced with the same grid resolution but with adaptive $r$-coordinates.
A more quantitative picture of the effects of $\alpha_V$ on the front speeds is illustrated in Figure 8, which shows the Froude numbers for all 31 values of $\alpha_V$ compared to the $z$-level results. Increasing the value of $\alpha_V$ leads to faster propagation of the top and bottom head speeds. However, as demonstrated in Figure 8, since $M_{33} \leq 2.0$, the bottom-head Froude number for the $r$-coordinate simulations is limited by the bottom-head Froude number for the $z$-coordinate case Z128 (the $r$-coordinate grid has 64 grid points in the vertical while the $z$-coordinate case Z128 has 128). This is not the case for the free-slip head at the top boundary which is more accurate than the $z$-level case Z128 and continues to exhibit improvement in the head speed at the largest value of $\alpha_V$. In contrast to the bottom, no-slip head speed which is limited in accuracy by resolution of the thin bottom boundary layer, the top, free-slip head speed is limited in accuracy by numerical diffusion of the density interface which continues to decrease with increasing $\alpha_V$.

Following Koltakov and Fringer (2013), the numerical diffusion of the density interface is quantified by the background potential energy. According to Winters et al. (1995), the total potential energy $E_p$ can be split into the available potential energy $E_a$ and background potential energy $E_b$. The background potential energy is defined as the potential energy of the fluid if it were to come to rest adiabatically. In a numerical simulation, both physical and numerical diffusion of scalars lead to mixing which in turn leads to an increase in the background potential energy. Therefore, because the present simulations lack physical scalar diffusion, the background potential energy increases solely due to numerical diffusion. We would then expect simulations with less numerical diffusion to exhibit smaller increases in the background potential energy.

In a two-dimensional discrete domain, $E_p$ and $E_b$ can be computed with

$$E_p = g \sum_{i=1}^{N_c} \sum_{k=1}^{N_k} \rho(i,k) z(i,k) \delta V(i,k), \quad \text{and} \quad E_b = g \sum_{n=1}^{N_c \times N_k} \rho(n) z(n) \delta V(n),$$

(68)

where $N_c$ is the total number of cells in a horizontal plane, $\delta V(i,k)$ is the volume of cell $i$ in layer $k$, $z(i,k)$ is the height of the cell center, $\rho(n)$ is the sorted equivalent of $\rho(i,k)$ in descending order, and $\delta V(n)$ is the corresponding volume of the cell with density $\rho(n)$. The height of the sorted density field is computed with

$$z(n+1) = z(n) + \frac{\delta V(n+1)}{A_p(n)}, \quad \text{with} \quad z(1) = z(0) + \frac{\delta V(1)}{2A_p(0)},$$

(69)
Figure 8: Effects of coefficient \( \alpha_V \) on the Froude number at the free-slip top (a) and the no-slip bottom (b), and on the background potential energy at time \( t = 10T \) (c). Results of cases using an adaptive mesh are plotted as solid lines with filled circles. Results of cases Z64, Z128 and Z192 are presented for comparison as the dash-dot, dashed, and dot lines, respectively.

where \( z(0) \) is the vertical coordinate of the domain bottom. After computing \( E_p \) and \( E_b \) with Equation (68), the available potential energy is computed with \( E_a = E_p - E_b \). In what follows we define the relative change in the background potential energy as

\[
\Delta E_b(t) = \frac{E_b(t) - E_b(0)}{E_a(0)}. \tag{70}
\]

The relative change in the background potential energy at time \( t = 10T \) with different values of \( \alpha_V \) for the \( r \)-coordinate simulations is shown in Figure 8(c). Increasing \( \alpha_V \) leads to less numerical diffusion and a smaller value of \( \Delta E_b \) owing to improved resolution of the density interface. This drop in \( \Delta E_b \) continues even for the highest value of \( \alpha_V \), indicating that the reduc-
tion in numerical diffusion is likely contributing to the continued increase of the top head speed at this value of $\alpha_V$ in Figure 8(b). The reduction in numerical diffusion with increasing $\alpha_V$ leads to improved resolution of the density interface and Kelvin-Helmholtz billows shown in Figure 7. Similarly, the vertical shear is also better resolved, as indicated by the increase in the magnitude of the vorticity toward that in case Z192, as shown in Figure 9.

6. Conclusions

In this paper, we have presented a finite-volume, generalized vertical coordinate formulation of the RANS equations that have been implemented into the existing $z$-level SUNTANS model Fringer et al. (2006). The framework enables seamless implementation of $z$-level, terrain-following, isopycnal or adaptive vertical coordinates. Use of the general vertical coordinate transformation along with the ALE method to move the grid leads to a set of transformed equations that account for the vertical grid motion with grid
fluxes in the momentum and scalar transport equations along with a layer-
height continuity equation that governs the evolution of the layer heights. The resulting equations enable use of the existing advection schemes in the original SUNTANS model and minimal modification to the nonhydrostatic pressure solver. Although the mild-slope approximation is assumed, it is possible to modify the approach to include all terms associated with the coordinate transformation. However, mild slopes should be used to avoid pressure gradient errors. The method conserves volume and scalars both locally and globally, although momentum and energy are not conserved. Stability is governed primarily by the propagation of first-mode internal gravity waves for isopycnal-coordinate simulations, although adaptive-coordinate simulations can be limited by vertical Courant number constraints if the vertical grid spacing is not appropriately controlled.

Isopycnal-coordinate test cases on Cartesian meshes show that the model behaves similarly to the nonhydrostatic, isopycnal coordinate model of Vítousek and Fringer (2014). To validate the behavior of the isopycnal-coordinate model on unstructured grids, we simulate internal solitary-like waves interacting with a circular island over bathymetry. Results indicate that the isopycnal-coordinate model reproduces the nonlinear and nonhydrostatic processes associated with internal wave refraction, diffraction, and wave-wave interactions around the island with eight times fewer vertical levels and a reduction in computational cost by a factor of nearly six. Use of just two vertical layers also reproduces the dynamics with a reduction in computational cost by a factor of 17. However, such a simulation should not be expected to reproduce dynamics associated with multiple internal wave modes that are not represented by a two-layer model.

The second test case compares simulation of the lock-exchange problem using $z$-levels and adaptive vertical coordinates. The vertical coordinate is updated with an $r$-adaptive approach through solution of a Laplace equation with monitor functions that dictate where the vertical coordinates should be concentrated. The method is similar to the adaptive method of Hofmeister et al. (2010) in which the vertical coordinates are updated with a diffusion equation with diffusion coefficients that are analogous to the coefficients in the monitor functions. However, our method updates the grid with solution of one equation rather than a series of equations accounting for vertical and horizontal diffusion and isopycnal tendency.

Simulations show that the vertical adaptivity improves resolution of density gradients and vorticity in the lock-exchange problem, such that the
vertically-adaptive approach essentially reproduces the $z$-level result with half the number of vertical layers. Increasing the coefficient $\alpha_V$ dictating vertical grid adaptivity improves prediction of the gravity current speeds at the top and bottom boundaries. However, improvement of the bottom, no-slip gravity current speed is limited to the speed of the current in the higher-resolution $z$-level case with twice the number of layers owing to the limit on vertical refinement of the adaptive grid by at most a factor of two. Prediction of the gravity current speed at the top, free-slip boundary is not limited by the refinement constraint, and its speed is even better predicted than the $z$-level case with twice the number of layers because there is no boundary layer at the free-slip boundary. Instead, prediction of the top gravity current depends more on resolution of the density gradients and Kelvin-Helmholtz billows. These are better resolved with increasing vertical adaptivity which incurs less numerical diffusion. The numerical diffusion is quantified by the change in the background potential energy which is shown to decrease monotonically with increasing $\alpha_V$.

These results indicate that the model can employ both isopycnal and adaptive vertical coordinates and reproduce $z$-level results with significant reductions in computational cost. An ideal strategy would employ a hybrid-coordinate framework combining isopycnal, terrain-following, and $z$-coordinates in one simulation along with an adaptive step that prevents grid overturning or unreasonably small layer heights and ensures grids that satisfy the mild-slope approximation. Such an approach is the subject of ongoing work seeking to apply the model to realistic, three-dimensional settings in which nonhydrostatic effects are important.

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References

Adcroft, A., Anderson, W., Balaji, V., Blanton, C., Bushuk, M., Dufour, C.O., Dunne, J.P., Griffies, S.M., Hallberg, R., Harrison, M.J., Held, I.M., Jansen, M.F., John, J.G., Krasting, J.P., Langenhorst, A.R., Legg, S., Liang, Z., McHugh, C., Radhakrishnan, A., Reichl, B.G.,
Rosati, T., Samuels, B.L., Shao, A., Stouffer, R., Winton, M., Wittenberg, A.T., Xiang, B., Zadeh, N., Zhang, R., 2019. The GFDL global ocean and sea ice model OM4.0: Model description and simulation features. Journal of Advances in Modeling Earth Systems 11, 3167–3211. doi:10.1029/2019MS001726.

Adcroft, A., Hallberg, R., 2006. On methods for solving the oceanic equations of motion in generalized vertical coordinates. Ocean Modelling 11, 224–233. doi:10.1016/j.ocemod.2004.12.007.

Adcroft, A., Hill, C., Marshall, J., 1997. Representation of topography by shaved cells in a height coordinate ocean model. Monthly Weather Review 125, 2293–2315. doi:10.1175/1520-0493(1997)125<2293:ROTBSC>2.0.CO;2.

Armfield, S., Street, R., 2000. Fractional step methods for the Navier-Stokes equations on non-staggered grids. ANZIAM Journal 42, 134–156. doi:10.1006/jcph.1994.1146.

Auclair, F., Bordois, L., Dossmann, Y., Duhaut, T., Paci, A., Ulses, C., Nguyen, C., 2018. A non-hydrostatic non-Boussinesq algorithm for free-surface ocean modelling. Ocean Modelling 132, 12–29. doi:10.1016/j.ocemod.2018.07.011.

Auclair, F., Marsaleix, P., Estournel, C., 2000. Sigma coordinate pressure gradient errors: Evaluation and reduction by an inverse method. Journal of Atmospheric and Oceanic Technology 17, 1348–1367. doi:10.1175/1520-0426(2000)017<1348:SCPGE>2.0.CO;2.

Bleck, R., 2002. An oceanic general circulation model framed in hybrid isopycnic-Cartesian coordinates. Ocean Modelling 4, 55–88. doi:10.1016/S1463-5003(01)00012-9.

Bleck, R., Rooth, C., Hu, D., Smith, L.T., 1992. Salinity-driven thermocline transients in a wind- and thermohaline-forced isopycnic coordinate model of the North Atlantic. Journal of Physical Oceanography 22, 1486–1505. doi:10.1175/1520-0485(1992)022<1486:SDTTIA>2.0.CO;2.

Blumberg, A.F., Mellor, G.L., 1987. A description of a three-dimensional coastal ocean circulation model. American Geophysical Union. chapter 4. pp. 1–16. doi:10.1029/C0004p0001.
Burchard, H., Beckers, J.M., 2004. Non-uniform adaptive vertical grids in one-dimensional numerical ocean models. Ocean Modelling 6, 51–81. doi:10.1016/S1463-5003(02)00060-4.

Burchard, H., Bolding, K., 2002. GETM-A general estuarine transport model. Scientific documentation. Technical Report. European Commission, Joint Research Centre, Institute for Environment and Sustainability.

Casulli, V., 1999a. A semi-implicit finite difference method for non-hydrostatic, free-surface flows. International Journal for Numerical Methods in Fluids 30, 425–440. doi:10.1002/(SICI)1097-0363(19990630)30:4<425::AID-FLD847>3.0.CO;2-D.

Casulli, V., 1999b. A semi-implicit numerical method for non-hydrostatic free-surface flows on unstructured grid, in: Numerical Modeling of Hydrodynamic Systems ESF Workshop, pp. 175–193.

Casulli, V., 2009. A high-resolution wetting and drying algorithm for free-surface hydrodynamics. International Journal for Numerical Methods in Fluids 60, 391–408. doi:10.1002/fld.1896.

Casulli, V., Cattani, E., 1994. Stability, accuracy and efficiency of a semi-implicit method for three-dimensional shallow water flow. Computers & Mathematics with Applications 27, 99–112. doi:10.1016/0898-1221(94)90059-0.

Casulli, V., Walters, R.A., 2000. An unstructured grid, three-dimensional model based on the shallow water equations. International Journal for Numerical Methods in Fluids 32, 331–348. doi:10.1002/(SICI)1097-0363(20000215)32:3<331::AID-FLD941>3.0.CO;2-C.

Casulli, V., Zanolli, P., 2005. High resolution methods for multidimensional advection-diffusion problems in free-surface hydrodynamics. Ocean Modelling 10, 137–151. doi:10.1016/j.ocemod.2004.06.007.

Chassignet, E.P., 2011. Isopycnic and hybrid ocean modeling in the context of GODAE. Springer Netherlands, Dordrecht. pp. 263–293. doi:10.1007/978-94-007-0332-2_11.

Chou, Y.J., Fringer, O.B., 2010. Consistent discretization for simulations of flows with moving generalized curvilinear coordinates. International
Durran, D.R., Blossey, P.N., 2012. Implicit-explicit multistep methods for fast-wave-slow-wave problems. Monthly Weather Review 140, 1307–1325. doi:10.1175/MWR-D-11-00088.1.

Engwirda, D., 2018. Generalised primal-dual grids for unstructured co-volume schemes. Journal of Computational Physics 375, 155–176. doi:10.1016/j.jcp.2018.07.025.

Fringer, O.B., Gerritsen, M., Street, R.L., 2006. An unstructured-grid, finite-volume, nonhydrostatic, parallel coastal ocean simulator. Ocean Modelling 14, 139–173. doi:10.1016/j.ocemod.2006.03.006.

Fringer, O.B., Street, R.L., 2003. The dynamics of breaking progressive interfacial waves. Journal of Fluid Mechanics 494, 319–353. doi:10.1017/S0022112003006189.

Gräwe, U., Holtermann, P., Klingbeil, K., Burchard, H., 2015. Advantages of vertically adaptive coordinates in numerical models of stratified shelf seas. Ocean Modelling 92, 56–68. doi:10.1016/j.ocemod.2015.05.008.

Griffies, S.M., Adcroft, A., Hallberg, R.W., 2020. A primer on the vertical lagrangian-remap method in ocean models based on finite volume generalized vertical coordinates. Journal of Advances in Modeling Earth Systems 12, e2019MS001954. doi:10.1029/2019MS001954.

Griffies, S.M., Böning, C., Bryan, F.O., Chassignet, E.P., Gerdes, R., Hasumi, H., Hirst, A., Treguier, A.M., Webb, D., 2000. Developments in ocean climate modelling. Ocean Modelling 2, 123–192. doi:10.1016/S1463-5003(00)00014-7.

Gross, E.S., Bonaventura, L., Rosatti, G., 2002. Consistency with continuity in conservative advection schemes for free-surface models. International Journal for Numerical Methods in Fluids 38, 307–327. doi:10.1002/fld.222.

Hallberg, R., Adcroft, A., 2009. Reconciling estimates of the free surface height in Lagrangian vertical coordinate ocean models with mode-split
Härtel, C., Meiburg, E., Necker, F., 2000. Analysis and direct numerical simulation of the flow at a gravity-current head. Part 1. Flow topology and front speed for slip and no-slip boundaries. Journal of Fluid Mechanics 418, 189–212. doi:10.1017/S0022112000001221.

Hiester, H.R., Piggott, M.D., Allison, P.A., 2011. The impact of mesh adaptivity on the gravity current front speed in a two-dimensional lock-exchange. Ocean Modelling 38, 1–21. doi:10.1016/j.ocemod.2011.01.003.

Hirt, C.W., Amsden, A.A., Cook, J.L., 1974. An arbitrary Lagrangian-Eulerian computing method for all flow speeds. Journal of Computational Physics 14, 227–253. doi:10.1016/0021-9991(74)90051-5.

Hofmeister, R., Burchard, H., Beckers, J.M., 2010. Non-uniform adaptive vertical grids for 3D numerical ocean models. Ocean Modelling 33, 70–86. doi:10.1016/j.ocemod.2009.12.003.

Klingbeil, K., Burchard, H., 2013. Implementation of a direct nonhydrostatic pressure gradient discretisation into a layered ocean model. Ocean Modelling 65, 64–77. doi:10.1016/j.ocemod.2013.02.002.

Koltakov, S., Fringer, O.B., 2013. Moving grid method for numerical simulation of stratified flows. International Journal for Numerical Methods in Fluids 71, 1524–1545. doi:10.1002/fld.3724.

Lai, Z., Chen, C., Cowles, G.W., Beardsley, R.C., 2010. A nonhydrostatic version of FVCOM: 1. Validation experiments. Journal of Geophysical Research: Oceans 115. doi:10.1029/2009JC005525.

Li, X., Jackson, C.R., Pichel, W.G., 2013. Internal solitary wave refraction at Dongsha Atoll, South China Sea. Geophysical Research Letters 40, 3128–3132. doi:10.1002/grl.50614.

Lynett, P.J., Liu, P.L.F., 2002. A two-dimensional, depth-integrated model for internal wave propagation over variable bathymetry. Wave Motion 36, 221–240. doi:10.1016/S0165-2125(01)00115-9.
Mahadevan, A., Oliger, J., Street, R., 1996. A nonhydrostatic mesoscale ocean model. Part II: Numerical implementation. Journal of Physical Oceanography 26. doi:10.1175/1520-0485(1996)026<1881:ANMOMP>2.0.CO;2.

Mandli, K.T., 2013. A numerical method for the two layer shallow water equations with dry states. Ocean Modelling 72, 80–91. doi:10.1016/j.ocemod.2013.08.001.

Marshall, J., Adcroft, A., Hill, C., Perelman, L., Heisey, C., 1997. A finite-volume, incompressible Navier Stokes model for studies of the ocean on parallel computers. Journal of Geophysical Research: Oceans 102, 5753–5766. doi:10.1029/96JC02775.

Mellor, G.L., Oey, L.Y., Ezer, T., 1998. Sigma coordinate pressure gradient errors and the seamount problem. Journal of Atmospheric and Oceanic Technology 15, 1122–1131. doi:10.1175/1520-0426(1998)015<1122:SCPGEA>2.0.CO;2.

Perot, B., 2000. Conservation properties of unstructured staggered mesh schemes. Journal of Computational Physics 159, 58–89. doi:10.1006/jcph.2000.6424.

Ringler, T., Petersen, M., Higdon, R.L., Jacobsen, D., Jones, P.W., Maltrud, M., 2013. A multi-resolution approach to global ocean modeling. Ocean Modelling 69, 211–232. doi:10.1016/j.ocemod.2013.04.010.

Shchepetkin, A.F., McWilliams, J.C., 2005. The regional oceanic modeling system (ROMS): a split-explicit, free-surface, topography-following-coordinate oceanic model. Ocean Modelling 9, 347–404. doi:10.1016/j.ocemod.2004.08.002.

Song, Y.T., 1998. A general pressure gradient formulation for ocean models. Part I: Scheme design and diagnostic analysis. Monthly Weather Review 126, 3213–3230. doi:10.1175/1520-0493(1998)126<3213:AGPGFF>2.0.CO;2.

Stelling, G.S., Van Kester, J.A.T.M., 1994. On the approximation of horizontal gradients in sigma co-ordinates for bathymetry with steep bottom slopes. International Journal for Numerical Methods in Fluids 18, 915–935. doi:10.1002/fld.1650181003.
Tang, H., Tang, T., 2003. Adaptive mesh methods for one- and two-dimensional hyperbolic conservation laws. SIAM Journal on Numerical Analysis 41, 487–515. doi:10.1137/S003614290138437X.

Van Leer, B., 1977. Towards the ultimate conservative difference scheme. IV. A new approach to numerical convection. Journal of Computational Physics 23, 276–299. doi:10.1016/0021-9991(77)90095-X.

Vitousek, S., Fringer, O.B., 2011. Physical vs. numerical dispersion in non-hydrostatic ocean modeling. Ocean Modelling 40, 72–86. doi:10.1016/j.ocemod.2011.07.002.

Vitousek, S., Fringer, O.B., 2013. Stability and consistency of nonhydrostatic free-surface models using the semi-implicit $\theta$-method. International Journal for Numerical Methods in Fluids 72, 550–582. doi:10.1002/fld.3755.

Vitousek, S., Fringer, O.B., 2014. A nonhydrostatic, isopycnal-coordinate ocean model for internal waves. Ocean Modelling 83, 118–144. doi:10.1016/j.ocemod.2014.08.008.

Willebrand, J., Barnier, B., Bönig, C., Dieterich, C., Killworth, P.D., Le Provost, C., Jia, Y., Molines, J.M., New, A.L., 2001. Circulation characteristics in three eddy-permitting models of the North Atlantic. Progress in Oceanography 48, 123–161. doi:10.1016/S0079-6611(01)00003-9.

Winters, K.B., Lombard, P.N., Riley, J.J., D’Asaro, E.A., 1995. Available potential energy and mixing in density-stratified fluids. Journal of Fluid Mechanics 289, 115–128. doi:10.1017/S002211209500125X.

Zijlema, M., Stelling, G., Smit, P., 2011. SWASH: An operational public domain code for simulating wave fields and rapidly varied flows in coastal waters. Coastal Engineering 58, 992–1012. doi:10.1016/j.coastaleng.2011.05.015.

Zijlema, M., Stelling, G.S., 2005. Further experiences with computing non-hydrostatic free-surface flows involving water waves. International Journal for Numerical Methods in Fluids 48, 169–197. doi:10.1002/fld.821.