A MATHEMATICAL THEORY OF THE TOPOLOGICAL VERTEX

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Abstract. We have developed a mathematical theory of the topological vertex—a theory that was originally proposed by M. Aganagic, A. Klemm, M. Mariño, and C. Vafa on effectively computing Gromov-Witten invariants of smooth toric Calabi-Yau threefolds derived from duality between open string theory of smooth Calabi-Yau threefolds and Chern-Simons theory on three manifolds.

1. Introduction

In [1], M. Aganagic, A. Klemm, M. Mariño and C. Vafa proposed a theory on computing Gromov-Witten invariants in all genera of any smooth toric Calabi-Yau threefold derived from duality between open string theory of smooth Calabi-Yau threefolds and Chern-Simons theory on 3-manifolds. In summary their theory says that

O1. There exist certain open Gromov-Witten invariants counting holomorphic maps from bordered Riemann surfaces to $\mathbb{C}^3$ with boundary mapped to three specific Lagrangian submanifolds. The topological vertex

$$C_{\vec{\mu}}(\lambda; n)$$

is a generating function of such invariants; it depends on a triple of partitions $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$ and a triple of integers $n = (n_1, n_2, n_3)$.

O2. The Gromov-Witten invariants of any smooth toric Calabi-Yau threefold can be expressed in terms of $C_{\vec{\mu}}(\lambda; n)$ by explicit gluing algorithms.

O3. By the duality between Chern-Simons theory and Gromov-Witten theory, the topological vertex is given by

$$C_{\vec{\mu}}(\lambda; n) = q^{\frac{1}{2}(\sum_{i=1}^{3} \kappa_i n_i)} W_{\vec{\mu}}(q),$$

where $q = e^{\sqrt{-1} \lambda}$ and $W_{\vec{\mu}}(q)$ is a combinatorial expression related to the Chern-Simons link invariants of a particular link. (cf. Section 2.1.)

As was demonstrated in the work of many, for instance [45, 21, 22], this algorithm is extremely efficient in deriving the structure result of the Gromov-Witten invariants of toric Calabi-Yau threefolds.

The purpose of this paper is to provide a mathematical theory for this algorithm. To achieve this, we need to provide a mathematical definition of the open Gromov-Witten invariants referred to in O1; secondly, we need to establish the gluing algorithms O2; finally, the duality O3 must be established mathematically.

In this paper, we shall complete the first two steps as outlined. Our theory is based on relative Gromov-Witten theory [26, 18, 19, 24, 25]. Our results can be summarized as follows.

R1. We introduce the notion of formal toric Calabi-Yau (FTCY) graphs, which is a refinement and generalization of the graph associated to a toric Calabi-Yau
threefold. An FT CY graph \( \Gamma \) determines a relative FT CY threefold \( Y^{\text{red}} = (\hat{Y}, \hat{D}) \); it can be degenerated to indecomposable ones.

R2. We define formal relative Gromov-Witten invariants for relative FT CY threefolds (Theorem 4.8). These invariants include as special cases Gromov-Witten invariants of smooth toric Calabi-Yau threefolds.

R3. We show that the formal relative Gromov-Witten invariants in R2 satisfy the degeneration formula of relative Gromov-Witten invariants of projective varieties (Theorem 7.5).

R4. Any smooth relative FT CY threefold can be degenerated to indecomposable ones, whose isomorphism classes are determined by a triple of integers \( n = (n_1, n_2, n_3) \). By degeneration formula, the formal relative Gromov-Witten invariants in R2 can be expressed in terms of the generating function \( \hat{C}_\mu(\lambda; n) \) of that of an indecomposable FT CY threefold (Proposition 7.4). This degeneration formula coincides with the gluing algorithms described in O2.

R5. We evaluate \( \hat{C}_\mu(\lambda; n) \) (Proposition 6.5, Theorem 8.1):

\[
\hat{C}_\mu(\lambda; n) = q^{\frac{1}{2} \left( \sum_{i=1}^{3} \kappa_i (n_i) \right)} \hat{W}_\mu(q),
\]

where \( \hat{W}_\mu(q) \) is a combinatorial expression defined by (2.5) in Section 2.1.

In R4, we shall define \( \hat{C}_\mu(\lambda; n) \) as local relative Gromov-Witten invariants of a formal Calabi-Yau \((\hat{Z}, \hat{D})\) that is the infinitesimal neighborhood of a configuration \( C_1 \cup C_2 \cup C_3 \) of three \( \mathbb{P}^1 \)'s meeting at a point \( p_0 \) in a relative Calabi-Yau threefold \((Z, D)\); the stable maps have ramification partition \( \mu \) around the relative divisor \( D \). Since \( \hat{Z} \) is formal, we shall define the local invariants \( \hat{C}_\mu(\lambda; n) \) via localization formula. The most technical part of the paper is to show that such local invariants exist as topological invariants; namely they are rational numbers independent of equivariant parameters (Theorem 5.2, invariance of the topological vertex).

Our results R1-R5, together with a conjectural identity \( \hat{W}_\mu(q) = W_\mu(q) \) (Conjecture 8.3), will provide a complete mathematical theory of the topological vertex theory. The conjecture holds when one of the partitions, say \( \mu_3 \), is empty (Corollary 8.8); it also holds for all low degree cases we have checked.

By virtual localization, \( \hat{C}_\mu(\lambda; n) \) can be expressed in terms of Hodge integrals (Proposition 6.6), which combined with (1.2) provides us a formula of a generating function \( G_\mu^*(\lambda; w) \) of the three-partition Hodge integrals (Theorem 8.2):

\[
G_\mu^*(\lambda; w) = \sum_{|\mu|=|\mu^i|} \prod_{i=1}^{3} \frac{\chi_{\mu^i}(\mu^i)}{z_{\mu^i}} q^{|\sum_{i=1}^{3} \kappa_i (w_i) + 1}} \hat{W}_\mu(q).
\]

(See Section 2 for precise definitions involved in (1.3)). This generalizes a formula of two-partition Hodge integrals (Theorem 8.7) proved in [31]. Our evaluation of three-partition Hodge integrals, thus \( \hat{C}_\mu(\lambda; n) \), relies on the cut-and-join equations of three partition Hodge integrals, which turn out to be equivalent to the invariance of the topological vertex mentioned above. Indeed, the cut-and-join equations are so strong that they reduce the evaluation of general three-partition Hodge integrals first to the case where \( \mu^3 = \emptyset \), and eventually to the case where \( \mu^1 = (d) \) and \( \mu^2 = \mu^3 = \emptyset \).

An important class of toric Calabi-Yau threefolds consists of local toric surfaces in a Calabi-Yau threefold; they are the total space of the canonical line bundle of a projective toric surface (e.g. \( \mathcal{O}_{\mathbb{P}^2}(-3) \)). In this case, only \( \hat{C}_{\mu^1, \mu^2, \emptyset}(\lambda; n) \) (or two-partition Hodge integrals) are involved. The algorithm in this case was described in [2]; explicit formula
was given in [17] and derived in [50] by localization, using the formula of two-partition Hodge integrals.

It is worth mentioning that, assuming the existence of $C_{\vec{\mu}}(\lambda; n_1, n_2, n_3)$ (at certain fractional $n_i$) to three-partition Hodge integrals, and derived the gluing algorithms in O2 by localization [6].

Maulik, Nekrasov, Okounkov, and Pandharipande conjectured a correspondence between the Gromov-Witten and Donaldson-Thomas theories for any non-singular projective threefold [36, 37]. This correspondence can also be formulated for certain non-compact threefolds in the presence of a torus action; the correspondence for toric Calabi-Yau threefolds is equivalent to the validity of the topological vertex [36, 42]. For non-Calabi-Yau toric threefolds the building block is the equivariant vertex (see [36, 37, 43, 44]) which depends on equivariant parameters.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and previous results, and introduce some generating functions. R1 is carried out in Section 3. R2 is carried out in Section 4; the case when the relative FTCY threefold is indecomposable gives the mathematical definition of topological vertex, and the proof of its invariance (Theorem 5.2) is given in Section 5. In Section 6, we express the topological vertex in terms of three-partition Hodge integrals and double Hurwitz numbers. In Section 7, we establish R3 and R4. In Section 8, we derive the combinatorial expression in R5. Some examples of the identity $W_{\vec{\mu}}(q) = \tilde{W}_{\vec{\mu}}(q)$ are listed in Section 8.4.

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2. Definitions and Previous Results

In this section, we recall some definitions and previous results, and introduce some generating functions. We begin with the partitions and representations of symmetric groups.

2.1. Partitions and Representations of Symmetric Groups. Recall that a partition $\mu$ of a nonnegative integer $d$ is a sequence of positive integers

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0)$$

such that $d = \mu_1 + \ldots + \mu_h$.

We write $\mu \vdash d$ or $|\mu| = d$, and call $\ell(\mu) = h$ the length of the partition. For convenience, we denote by $\emptyset$ the empty partition; thus $|\emptyset| = \ell(\emptyset) = 0$. The order of $\text{Aut}(\mu)$, the group of permutations of $\mu_1, \mu_2, \ldots$ that leave $\mu$ fixed, is

$$|\text{Aut}(\mu)| = \prod_j m_j(\mu)!,$$

where $m_j(\mu) = \#\{i : \mu_i = j\}$.

The transpose of $\mu$ is a partition $\mu^t$ defined by $m_i(\mu^t) = \mu_i - \mu_{i+1}$. Note that $|\mu^t| = |\mu|$, $(\mu^t)^t = \mu$, $\ell(\mu^t) = \mu_1$.

\footnote{During the revision of this paper, Maulik, Oblomkov, Okounkov, and Pandharipande announced a proof of GW/DT correspondence for all toric threefolds [58].}
A partition \( \mu \) also corresponds to a conjugacy class in \( S_d \), where \( S_d \) is the permutation group of \( d = |\mu| \) elements in the obvious way. With this understanding, the cardinality \( z_\mu \) of the centralizer of any element in this conjugacy class is of the form,

\[
z_\mu = a_\mu |\text{Aut}(\mu)|, \quad \text{where } a_\mu = \mu_1 \cdots \mu_{\ell(\mu)}.
\]

We let \( \mathcal{P} \) denote the set of partitions; we let

\[
\mathcal{P}_+ = \mathcal{P} - \{\emptyset\}, \quad \mathcal{P}_+^2 = \mathcal{P}_+ - \{\emptyset, \emptyset\}, \quad \mathcal{P}_+^3 = \mathcal{P}_+^2 - \{\emptyset, \emptyset, \emptyset\}.
\]

Given a triple of partitions \( \vec{\mu} = (\mu^1, \mu^2, \mu^3) \in \mathcal{P}_+^3 \), we define

\[
\ell(\vec{\mu}) = \sum_{i=1}^3 \ell(\mu^i), \quad \text{Aut}(\vec{\mu}) = \prod_{i=1}^3 \text{Aut}(\mu^i).
\]

For any partition \( \nu \), we let \( \chi_\nu \) denote the irreducible character of \( S_{|\nu|} \) indexed by \( \nu \), and let \( \chi_\nu(\mu) \) be the value of \( \chi_\nu \) on the conjugacy class determined by the partition \( \mu \). Recall that the Schur functions \( s_\mu \) are related to the Newton functions \( p_i(x) = x_1^i + x_2^i + \cdots \) by

\[
s_\mu(x) = \sum_{|\nu|=|\mu|} \frac{\chi_\nu(\mu)}{\nu} p_\nu(x), \quad x = (x_1, x_2, \ldots).
\]

The Littlewood-Richardson coefficients \( c^{\mu}_{\nu \xi} \), which are nonnegative integers, and skew Schur functions are determined according to the rules

\[
s_\mu s_\nu = \sum_\eta c^{\mu}_{\nu \eta} s_\eta \quad \text{and} \quad s_\eta / s_\mu = \sum_\nu c^{\mu}_{\nu \eta} s_\nu.
\]

In order to define the combinatorial expressions \( W_{\vec{\mu}}(q) \) and \( \tilde{W}_{\vec{\mu}}(q) \) in O3 and R5 in the introduction (Section 1), we need to introduce more notation. We define \( [m] = q^{m/2} - q^{-m/2} \), and define

\[
(2.1) \quad \kappa_\mu = \sum_{i=1}^{\ell(\mu)} \mu_i (\mu_i - 2i + 1),
\]

which for transpose partitions satisfies \( \kappa_{\mu^t} = -\kappa_\mu \).

We define

\[
(2.2) \quad W_{\mu}(q) = q^{\kappa_\mu/4} \prod_{1 \leq i < j \leq \ell(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]} \prod_{i=1}^{\ell(\mu)} \prod_{v=1}^{\mu_i} \frac{1}{[v-i+\ell(\mu)]}
\]

and define

\[
(2.3) \quad W_{\mu,\nu}(q) = q^{\nu/2} W_{\mu}(q) \cdot s_\nu(\mathcal{E}_\mu(q, t)),
\]

where

\[
\mathcal{E}_\mu(q, t) = \prod_{j=1}^{\ell(\mu)} \frac{1 + q^{t_{\mu_j-j}}}{1 + q^{t_{j-1}}} \cdot \left( 1 + \sum_{n=1}^{\infty} \frac{t^n}{\prod_{i=1}^{\nu}(q^i - 1)} \right).
\]

We also denote

\[
c^{\mu_{\rho}}_{\nu_{\rho}}(q^2) = \sum_\eta c^{\mu_{\rho}}_{\eta_{\rho}}(q^2) c^{(\nu_{\rho})^t}_{\eta_{\rho}}.
\]

**Definition 2.1.** For \( \vec{\mu} = (\mu^1, \mu^2, \mu^3) \), we define

\[
(2.4) \quad W_{\vec{\mu}}(q) = q^{\kappa_{\mu^2}/2 + \kappa_{\mu^3}/2} \sum_{\rho^1, \rho^2} c^{\mu^1_{\rho^1}}_{\rho^1} c^{\mu^2_{\rho^2}}_{\rho^2} \frac{W_{\mu^1}(q) W_{\mu^2}(q) W_{\mu^3}(q)}{W_{\vec{\mu}}(q)}.
\]
Definition 2.2. For a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ we let $2\mu$ be the partition $(2\mu_1 \geq 2\mu_2 \geq \cdots)$. For $\vec{\rho} = (\rho^1, \rho^2, \rho^3)$, we define

$$\tilde{W}_\rho(q) = q^{-(c_{\mu_1} - 2c_{\mu_2} - \frac{3}{2}c_{\mu_3})/2} \sum_{\nu^1, \nu^2, \nu^3, \nu^4} c_{\nu^1}^+ c_{\nu^2}^+ c_{\nu^3}^1 c_{\nu^4}^3 q^{\nu_1 + \nu_2 + \nu_3 + \nu_4} \chi_{\nu_1}^2 \chi_{\nu_2}^2 \chi_{\nu_3}^2 \chi_{\nu_4}^2$$

(2.5)

$$\cdot q^{(2c_{\nu_1} - n_{\nu_3})/2} W_{\nu_1 + \nu_3}(q) \sum_{\mu} \frac{1}{z_{\mu}} \chi_{\eta_1}^2(\mu) \chi_{\eta_2}^2(2\mu).$$

We have the following identities (see [51]):

$$W_{\mu,\nu}(q) = W_{\emptyset,\mu,\nu}(q) = W_{\nu,\emptyset,\nu}(q) = q^{\ell(\nu)/2} W_{\mu,\nu}(q).$$

(2.6)

(2.7)

We now come to the generating function of the double Hurwitz numbers. Let $\mu^+, \mu^-$ be partitions of $d$; let $H^*_{\chi,\mu^+,\mu^-}$ be the weighted counts of Hurwitz covers of the sphere of the type $(\mu^+, \mu^-)$ by possibly disconnected Riemann surfaces of Euler characteristic $\chi$. We form the generating function

$$\Phi^*_{\mu^+,\mu^-}(\lambda) = \sum_{\chi} \lambda^{-\chi + \ell(\mu^+) + \ell(\mu^-)} \frac{H^*_{\chi,\mu^+,\mu^-}}{(-\chi + \ell(\mu^+) + \ell(\mu^-))!}.$$

By Burnside formula,

(2.8)

$$\Phi^*_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d} e^{c_{\nu}^+ \lambda/2} \chi_{\nu}^+(\mu^+) \chi_{\nu}^-(\mu^-) \frac{1}{z_{\mu^+} z_{\mu^-}}.$$

Using the orthogonality of characters

(2.9)

$$\sum_{\rho} \frac{\chi_{\mu}(\rho) \chi_{\nu}(\rho)}{z_{\rho}} = \delta_{\mu \nu},$$

it is straightforward to check that (2.8) implies the following two identities:

(2.10)

$$\Phi^*_{\mu^+,\mu^-}(\lambda_1 + \lambda_2) = \sum_{|\nu|=d} \Phi^*_{\mu^+,\mu^-}(\lambda_1) z_{\nu} \Phi^*_{\nu,\mu^-}(\lambda_2)$$

and

(2.11)

$$\Phi^*_{\mu^+,\mu^-}(0) = \frac{\delta_{\mu^+,\mu^-}}{z_{\mu^+}}.$$
and differential operators

\[ C^\pm = \sum_{j,k} (j+k) p^\pm_j p^\pm_k \frac{\partial}{\partial p^\pm_j p^\pm_k}, \quad J^\pm = \sum_{j,k} j k p^\pm_j p^\pm_k \frac{\partial^2}{\partial p^\pm_j \partial p^\pm_k}. \]

They form a cut-and-join equation for double Hurwitz numbers:

\[ \frac{\partial \Phi^\bullet}{\partial \lambda} = \frac{1}{2} (C^+ + J^+) \Phi^\bullet = \frac{1}{2} (C^- + J^-) \Phi^\bullet. \]

(2.12)

The generating function \( \Phi^\bullet(\lambda; p^+, p^-) \) is the unique solution to this system satisfying the initial value

\[ \Phi^\bullet(0; p^+, p^-) = 1 + \sum_{\mu \in P^+} \frac{p^\mu_1 p^\mu_2}{z_\mu}. \]

2.3. Three-Partition Hodge Integrals. We shall introduce three-partition Hodge integrals in this subsection.

For the three-partition Hodge integrals we need to work with Deligne-Mumford moduli stack \( \mathcal{M}_{g,n} \) of stable \( n \)-pointed nodal curves of genus \( g \). Over this moduli stack, we let \( \pi : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \) be the universal curve, let \( s_i : \mathcal{M}_{g,n} \to \mathcal{M}_{g,n+1} \) be its section that corresponds to the \( i \)-th marked points of the family, and let \( \omega_n \) be the relative dualizing sheaf. The commonly known \( \lambda \) classes and the \( \psi \)-class are defined using these morphisms.

One forms the Hodge bundle \( E = \pi_* \omega_n \); its \( j \)-th Chern class \( \lambda_j = c_j(E) \) is the \( \lambda \)-class. One then form the pull back line bundle \( L_i = s_i^* \omega_n \); its first Chern class \( \psi_i = c_1(\mathcal{L}_i) \) is the \( \psi \)-class. A Hodge integral is then an integral of the form

\[ \int_{\mathcal{M}_{g,n}} \psi_1^{j_1} \cdots \psi_3^{j_3} \lambda_1^{k_1} \cdots \lambda_g^{k_g}. \]

We now introduce three-partition Hodge integrals. Let \( w_1, w_2, w_3 \) be formal variables. In this subsection, and in Section 6, 7, 8, we use the following convention:

\[ w = (w_1, w_2, w_3), \quad w_1 + w_2 + w_3 = 0, \quad w_4 = w_1. \]

For \( \bar{\mu} = (\mu^1, \mu^2, \mu^3) \in P^3_+ \), we let

\[ d_{\mu^1}^1 = 0, \quad d_{\mu^2}^2 = \ell(\mu^1), \quad d_{\mu^3}^3 = \ell(\mu^1) + \ell(\mu^2). \]

We define the three-partition Hodge integral to be

\[ G_{g,\bar{\mu}}(w) = \frac{(\sqrt{-1})^{\ell(\bar{\mu})}}{|\text{Aut}(\bar{\mu})|} \prod_{i=1}^3 \prod_{j=1}^{\mu^1_i - 1} \frac{\Lambda^\gamma_j(w_i) w_i^{\ell(\bar{\mu}) - 1}}{(\mu^1_i - 1)! w_i^{\ell(\bar{\mu}) - 1}} \int_{\mathcal{M}_{g,\ell(\bar{\mu})}} \prod_{j=1}^3 \prod_{i=1}^{\ell(\bar{\mu})} \Lambda^\gamma_j(w_i) (w_i - \mu^1_i \psi_i \mu^2_i + j) \]

(2.14)

where \( \Lambda^\gamma_j(u) = u^\gamma - \lambda_1 u^{\gamma-1} + \cdots + (-1)^\gamma \lambda_g. \)

It is clear from the definition that \( G_{g,\bar{\mu}}(w) \) obeys the cyclic symmetry:

\[ G_{g,\mu^1,\mu^2,\mu^3}(w_1, w_2, w_3) = G_{g,\mu^2,\mu^3,\mu^1}(w_2, w_3, w_1). \]

(2.15)

Note \( \sqrt{-1}^{\ell(\bar{\mu})} G_{g,\bar{\mu}}(w) \) is a rational function in \( w_1, w_2, w_3 \) with \( \mathbb{Q} \)-coefficients, and is homogeneous of degree 0, so it suffices to work with \( w = (1, \tau, -\tau - 1) \). For such weights, we shall write

\[ G_{g,\bar{\mu}}(\tau) = G_{g,\bar{\mu}}(1, \tau, -\tau - 1). \]

Then (2.15) becomes

\[ G_{g,\mu^1,\mu^2,\mu^3}(\tau) = G_{g,\mu^2,\mu^3,\mu^1}(-1 - \frac{1}{\tau}) = G_{g,\mu^3,\mu^1,\mu^2}(\frac{-1}{\tau + 1}). \]
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We now form three-partition generating functions. We let $\lambda$ and $p^i = (p_1^i, p_2^i, \ldots)$ be formal variables; given a partition $\mu$ we define $p_\mu^i = p_1^i \cdots p_{(\mu)}^i$ (note $p_0^i = 1$); for $p^1$, $p^2$ and $p^3$, we abbreviate $p = (p^1, p^2, p^3)$ and $p_\mu = p_\mu^1 \cdots p_\mu^3$.

The generating functions are

$$G_\mu(\lambda; w) = \sum_{g=0}^\infty \lambda^{g-2+\ell(\mu)} G_{g,\mu}(w)$$

and

$$G(\lambda; p; w) = \sum_{\mu \in P_+} G_\mu(\lambda; w)p_\mu.$$

The generating functions for not necessarily connected domain curves are

$$(2.16) \quad G^\bullet(\lambda; p; w) = \exp(G(\lambda; p; w)) = 1 + \sum_{\mu \in P_+} G_\mu^\bullet(\lambda; w)p_\mu$$

and

$$(2.17) \quad G_\mu^\bullet(\lambda; w) = \sum_{\chi \in \mathbb{Z}, x \leq 2\ell(\mu)} \lambda^{-x+\ell(\mu)} G_{\chi,\mu}(w).$$

Finally, we define $G_\mu(\lambda; \tau)$, $G(\lambda; p; \tau)$, $G^\bullet(\lambda; p; \tau)$ and $G_\mu^\bullet(\lambda; \tau)$ similarly. We will relate $G_\mu^\bullet(\lambda; \tau)$ to $\mathcal{W}_\mu(q)$ in Theorem 8.2.

3. Relative Formal Toric Calabi-Yau Threefolds

In this section, we will introduce formal toric Calabi-Yau (FTCY) graphs, and construct their associated relative FTCY threefolds.

3.1. Toric Calabi-Yau Threefolds. Given a smooth toric Calabi-Yau threefold $Y$, we let $Y^1$ (resp. $Y^0$) be the union of all $1$-dimensional (resp. $0$-dimensional) $(\mathbb{C}^*)^3$ orbit closures in $Y$. We assume that

$$Y^1 \text{ is connected and } Y^0 \text{ is nonempty.}$$

Under the above condition, we will find a distinguished subtorus $T \subset (\mathbb{C}^*)^3$ and use the $T$-action to construct a planar trivalent graph $\Gamma_Y$. FTCY graphs that will be defined in Section 3.3 are generalization of the planar trivalent graphs associated to smooth toric Calabi-Yau threefolds just mentioned.

We first describe the distinguished subtorus $T$. We pick a fixed point $p \in Y^0$ and look at the $(\mathbb{C}^*)^3$ action on the tangent space $T_p Y$ and its top wedge $\Lambda^3 T_p Y$. Clearly, the later defines a homomorphism $\alpha_p : (\mathbb{C}^*)^3 \to \mathbb{C}^*$, which by the Calabi-Yau condition and the connectedness of $Y^1$ is independent of choice $p$. We define

$$T \overset{\text{def}}{=} \ker \alpha_p \cong (\mathbb{C}^*)^2.$$

We next describe the planar trivalent graph $\Gamma_Y$. We let $\Lambda_T$ be the group of irreducible characters of $T$, i.e.,

$$\Lambda_T \overset{\text{def}}{=} \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^{\oplus 2}.$$

We let $T_R \overset{\text{def}}{=} U(1)^2$ be the maximal compact subgroup of $T$; let $t_R$ and $t_R^\vee$ be its Lie algebra and its dual; let $\mu : Y \to t_R^\vee$ be the moment map of the $T_R$-action on $Y$. Because of the canonical isomorphism $t_R^\vee \overset{\text{def}}{=} \Lambda_T \otimes \mathbb{Z} \mathbb{R}$, the image of $Y^1$ under $\mu$ forms a planar trivalent graph $\Gamma$ in $\mathbb{Z}^{\oplus 2} \otimes \mathbb{R}$. Some examples are shown in Figure 1.

The graph $\Gamma_Y$ encodes the information of $Y$ in that its edges and vertices correspond to irreducible components of $Y^1$ and fixed points $Y^0$; the slope of an edge determines the $T$-action on the corresponding component of $Y^1$. 
Let \( \hat{Y} \) be the formal completion of \( Y \) along \( Y^1 \). Then \( \hat{Y} \) is a smooth formal Calabi-Yau scheme and inherits the \( T \)-action on \( Y \). The formal Calabi-Yau scheme \( \hat{Y} \) together with the \( T \)-action can be reconstructed from the graph \( \Gamma_Y \) (cf. (a) in Section 3.2 below). The construction of a relative FTCY threefold from a FTCY graph (given in Section 3.5) can be viewed as generalization of this reconstructing procedure.

### 3.2. Relative Toric Calabi-Yau Threefolds

A smooth relative toric Calabi-Yau threefold is a pair \((Y,D)\) where \( Y \) is a smooth toric threefold, \( D \) is a possibly disconnected, smooth \((\mathbb{C}^*)^3\) invariant divisor of \( Y \), such that the relative Calabi-Yau condition holds:

\[
\Lambda^3 \Omega_Y (\log D) \cong \mathcal{O}_Y.
\]

A toric Calabi-Yau threefold can be viewed as a relative Calabi-Yau threefold where the divisor \( D \) is empty.

We now describe in details three examples of relative toric Calabi-Yau threefolds and their associated graphs, as they are the building blocks of the definitions and constructions in the rest of Section 3:

(a) \( Y \) is the total space of \( \mathcal{O}_{p1}(-1+n) \oplus \mathcal{O}_{p1}(-1-n) \).

(b) \( Y \) is the total space of \( \mathcal{O}_{p1}(n) \oplus \mathcal{O}_{p1}(-1+n) \); \( D \) is its fiber over \( q_1 = [1,0] \in \mathbb{P}^1 \).

(c) \( Y \) is the total space of \( \mathcal{O}_{p1}(n) \oplus \mathcal{O}_{p1}(-n) \); \( D \) is the union of its fibers over \( q_0 = [0,1] \) and \( q_1 = [1,0] \) in \( \mathbb{P}^1 \).

In Figure 2, the edge connecting the two vertices \( v_0 \) and \( v_1 \) corresponds to the zero section \( \mathbb{P}^1 \), which is a 1-dimensional \((\mathbb{C}^*)^3\) orbit closure in \( Y \); the vertices \( v_0, v_1 \) correspond to the \((\mathbb{C}^*)^3\) fixed points \( q_0, q_1 \in \mathbb{P}^1 \), respectively.

In Case (a), \( Y \) is a toric Calabi-Yau threefold, so we may specify a subtorus \( T \) as in Section 3.1. The weights of the \( T \)-action on the fibers of \( T \mathbb{P}^1, \mathcal{O}_{p1}(-1+n), \mathcal{O}_{p1}(-1-n) \) at the \( T \)-fixed point \( q_0 \in \mathbb{P}^1 \) are given by \( w_1, w_2, w_3 \in \Lambda_T \cong \mathbb{Z}^2 \), respectively. We must have \( w_1 + w_2 + w_3 = 0 \) because \( T \) acts on \( \Lambda^3 T_{q_0} Y \) trivially. The weights of the \( T \)-action
on the fibers of $T\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1+n), \mathcal{O}_{\mathbb{P}^1}(-1-n)$ at the other fixed point $q_1 \in \mathbb{P}^1$ are given by $-w_1, w_2 + (1-n)w_1, w_3 + (1+n)w_1 = -w_2 + nw_1$, respectively. From the graph in Figure 2(a), one can read off the degrees of the two summand of $N_{\mathbb{P}^1/Y}$ and the $T$-action on $Y$: $T$ with the $T$-action can be reconstructed from the graph.

Similarly, from the graph in Figure 2(b) (resp. (c)), one can reconstruct the pair $(Y, D)$ in (b) (resp. (c)) together with the $T$-action; the weights of the $T$-action at fixed points can be read off from the graph as follows:

|   | $T\mathbb{P}^1$ | $\mathcal{O}_{\mathbb{P}^1}(n)$ | $\mathcal{O}_{\mathbb{P}^1}(-1-n)$ |   | $T\mathbb{P}^1$ | $\mathcal{O}_{\mathbb{P}^1}(n)$ | $\mathcal{O}_{\mathbb{P}^1}(-n)$ |
|---|-----------------|-------------------------------|---------------------------------|---|-----------------|-------------------------------|---------------------------------|
| $q_0$ | $w_1$ | $w_2$ | $w_3 = -w_1 - w_2$ | $q_0$ | $w_1$ | $w_2$ | $-w_2$ |
| $q_1$ | $-w_1$ | $w_2 - nw_1$ | $w_3 + (1+n)w_1 = -w_2 + nw_2$ | $q_1$ | $-w_1$ | $w_2 - nw_1$ | $-w_2 + nw_1$ |

3.3. FTCY Graphs. In this subsection, we will introduce formal toric Calabi-Yau (FTCY) graphs, which are graphs together with local embedding into $\mathbb{R}^2$ endowed with the standard orientation and integral lattice $\mathbb{Z}^{\oplus 2} \subset \mathbb{R}^2$. As will be clear later, assigning a slope to an edge depends on the orientation of the edge. For book keeping purpose, we shall associate to each edge two (oppositely) oriented edges; for an oriented edge we can talk about its initial and terminal vertices. To recover the graph, we simply identify the two physically identical but oppositely oriented edges as one (unoriented) edge. This leads to the following definition.

**Definition 3.1** (graphs). A graph $\Gamma$ consists of a set of oriented edges $E^o(\Gamma)$, a set of vertices $V(\Gamma)$, an orientation reversing map $\text{rev}: E^o(\Gamma) \to E^o(\Gamma)$, an initial vertex map $v_0: E^o(\Gamma) \to V(\Gamma)$ and a terminal vertex map $v_1: E^o(\Gamma) \to V(\Gamma)$, such that $\text{rev}$ is a fixed point free involution; that both $v_0$ and $v_1$ are surjective and $v_1 = v_0 \circ \text{rev}$. We say $\Gamma$ is weakly trivalent if $|v^{-1}(v)| \leq 3$ for $v \in V(\Gamma)$.

For simplicity, we will abbreviate $\text{rev}(e)$ to $-e$. Then the equivalence classes $E(\Gamma) = E^o(\Gamma)/\{\pm 1\}$ is the set of edges of $\Gamma$ in the ordinary sense. In case $\Gamma$ is weakly trivalent, we shall denote by $V_1(\Gamma)$, $V_2(\Gamma)$ and $V_3(\Gamma)$ the sets of univalent, bivalent, and trivalent vertices of $\Gamma$; we shall also define

$$E^f(\Gamma) = \{e \in E^o(\Gamma) \mid v_1(e) \in V_1(\Gamma) \cup V_2(\Gamma)\}$$

which consists of oriented edges whose terminal edges are not trivalent. Finally, we fix a standard basis $\{u_1, u_2\}$ of $\mathbb{Z}^{\oplus 2}$ such that the ordered basis $(u_1, u_2)$ determines the orientation on $\mathbb{R}^2$.

**Definition 3.2** (FTCY graphs). A formal toric Calabi-Yau (FTCY) graph is a weakly trivalent graph $\Gamma$ together with a position map

$$p: E^o(\Gamma) \to \mathbb{Z}^{\oplus 2} - \{0\}$$

and a framing map

$$f: E^f(\Gamma) \to \mathbb{Z}^{\oplus 2} - \{0\},$$

such that (see Figure 3)

1. $p$ is anti-symmetric: $p(-e) = -p(e)$.

2. $p$ and $f$ are balanced:
   - For a bivalent vertex $v \in V_2(\Gamma)$ with $v_1^{-1}(v) = \{e_1, e_2\}$, $p(e_1) + p(e_2) = 0$ and $f(e_1) + f(e_2) = 0$.
   - For a trivalent $v \in V_3(\Gamma)$ with $v_0^{-1}(v) = \{e_1, e_2, e_3\}$, $p(e_1) + p(e_2) + p(e_3) = 0$.

3. $p$ and $f$ are primitive:
• For a trivalent vertex \( v \in V_3(\Gamma) \) with \( v_0^{-1}(v) = \{e_1, e_2, e_3\} \), any two vectors in \( \{p(e_1), p(e_2), p(e_3)\} \) form an integral basis of \( \mathbb{Z}^3 \).
• For \( e \in E^l(\Gamma) \), \( p(e) \wedge f(e) = u_1 \wedge u_2 \).

\[
\begin{align*}
&\xrightarrow{\ f(e) \ } \ v \\
&\xleftarrow{\ p(e) \ } \\
v_0^{-1}(v) = \{e\} \\
&\xrightarrow{\ f(e_1) \ } \ e_1 \\
&\xleftarrow{\ p(e_1) \ } \\
v_1^{-1}(v) = \{e_1, e_2\} \\
&\xrightarrow{\ f(e_2) \ } \ e_2 \\
&\xleftarrow{\ p(e_2) \ } \\
v_1^{-1}(v) = \{e_1, e_2, e_3\}
\end{align*}
\]

**Figure 3**

We say \( \Gamma \) is a regular FTCY graph if it has no bivalent vertex.

The neighborhood of an oriented edge \( e \) of an FTCY graph \( \Gamma \) looks like (a), (b), or (c) in Figure 2 (if we add vectors \( \{-f(e) \mid v_1(e) \in V_1(E)\}\)). So to each edge we may associate a relative Calabi-Yau threefold \( (Y^e, D^e) \) where \( Y^e \) is the total space of the direct sum of two line bundles over \( \mathbb{P}^1 \). Since \( e \) is oriented, we may assign \( L^e \), one of the two line bundles, to \( e \) as follows: in Figure 2(a), (b), or (c), if \( p(e) = w_1 \) points to the right, so that \( v_0 \) (resp. \( v_1 \)) is the initial (resp. terminal) vertex of \( e \), then the weight of \( T \)-action on \( L_{v_0}^e \) (resp. \( L_{v_1}^e \)) is given by the upward vector at \( v_0 \) (resp. \( v_1 \)), denoted by \( l_0(e) \) (resp. \( l_1(e) \)). Then \( Y^e \) is the total space of \( L^e \oplus L^{-e} \rightarrow \mathbb{P}^1 \). This is the geometric interpretation of the following definition:

**Definition 3.3.** Let \( \Gamma \) be an FTCY graph. We define \( l_0, l_1 : E^o(\Gamma) \rightarrow \mathbb{Z}^2 \) as follows:

\[
l_0(e) = \begin{cases} 
- f(e), & v_0(e) \notin V_3(\Gamma), \\
p(e_{01}), & v_0(e) \in V_3(\Gamma).
\end{cases}
\]

\[
l_1(e) = \begin{cases} 
f(e), & v_1(e) \notin V_3(\Gamma), \\
p(e_{11}), & v_1(e) \in V_3(\Gamma).
\end{cases}
\]

Here \( e_{11} \) is the unique oriented edge such that \( v_0(e_{11}) = v_1(e) \) and \( p(e) \wedge p(e_{11}) = u_1 \wedge u_2 \).

\[
\begin{align*}
&\xrightarrow{\ l_0(e) \ } \ p(e) \\
&\xleftarrow{\ l_1(e) \ } \\
&\xrightarrow{\ l_0(e) \ } \ p(e) \\
&\xleftarrow{\ l_1(e) \ } \\
&\xrightarrow{\ l_0(e) \ } \ p(e) \\
&\xleftarrow{\ l_1(e) \ } \\
&\xrightarrow{\ l_0(e) \ } \ p(e) \\
&\xleftarrow{\ l_1(e) \ }
\end{align*}
\]

**Figure 4**

The degree of the line bundle \( L^e \) determines an integer \( n^e \):

\[
\text{deg } L^e = \begin{cases} 
n^e - 1, & v_1 \in V_3(\Gamma), \\
n^e, & v_1 \notin V_3(\Gamma).
\end{cases}
\]

This motivates the following definition.
Definition 3.4. We define $\bar{n} : E^o(\Gamma) \to \mathbb{Z}$ by

$$t_1(e) - t_0(e) = \begin{cases} (1 - \bar{n}(e))p(e), & v_1(e) \in V_3(\Gamma), \\ -\bar{n}(e)p(e), & v_1(e) \notin V_3(\Gamma). \end{cases}$$

We write $n^e$ for $\bar{n}(e)$.

Note that $n^{-e} = -n^e$.

3.4. Operations on FTCY Graphs. In this subsection, we define four operations on FTCY graphs: smoothing, degeneration, normalization, and gluing. These operations extend natural operations on toric Calabi-Yau threefolds.

The first operation is the smoothing of a bivalent vertex $v \in V_2(\Gamma)$. In doing this, we shall eliminate the vertex $v$ and combine the two edges attached to $v$.

**Definition 3.5** (smoothing). The smoothing of $\Gamma$ along a bivalent vertex $v \in V_2(\Gamma)$ is a graph $\Gamma_v$ that has vertices $V(\Gamma) - \{v\}$, oriented edges $E^o(\Gamma)/\sim$ with the equivalence $\pm e_1 \sim \mp e_2$ for $\{e_1, e_2\} = v_1^{-1}(v)$. The maps $v_0, v_1, p$ and $f$ descend to corresponding maps on $\Gamma_v$, making it a FTCY graph. (See Figure 5: $\Gamma_3$ is the smoothing of $\Gamma_2$ along $v$.)

The reverse of the above construction is called a degeneration.

**Definition 3.6** (degeneration). Let $\Gamma$ be a FTCY graph and let $e \in E^o(\Gamma)$ be an edge. We pick a lattice point $f_0 \in \mathbb{Z}^{\mathbb{R}}$ so that $p(e) \wedge f_0 = u_1 \wedge u_2$. The degeneration of $\Gamma$ at $e$ with framing $f_0$ is a graph $\Gamma_{e,f_0}$ whose edges are $E^o(\Gamma) \cup \{\pm e_1, \pm e_2\} - \{\pm e\}$ and whose vertices are $V(\Gamma) \cup \{v_0\}$; its initial vertices $\tilde{v}_0$, terminal vertices $\tilde{v}_1$, position map $\tilde{p}$ and framing map $\tilde{f}$ are identical to those of $\Gamma$ except

$$\tilde{v}_0(e_1) = v_0(e), \quad \tilde{v}_1(e_1) = \tilde{v}_1(e_2) = v_0, \quad \tilde{v}_0(e_2) = v_1(e),$$

$$\tilde{p}(e_1) = -\tilde{p}(e_2) = p(e), \quad \tilde{f}(e_1) = \tilde{f}(e_2) = f_0.$$ 

(See Figure 5: $\Gamma_2$ is the degeneration of $\Gamma_3$ at $e$ with framing $f_0$.)

The normalization is to separate a graph along a bivalent vertex and the gluing is its inverse.

**Definition 3.7** (normalization). Let $\Gamma$ be a FTCY graph and let $v \in V_2(\Gamma)$ be a bivalent vertex. The normalization of $\Gamma$ at $v$ is a graph $\Gamma^o$ whose edges are the same as that of $\Gamma$ and whose vertices are $V(\Gamma) \cup \{v_1, v_2\} - \{v\}$; its associated maps $\tilde{v}_0, \tilde{v}_1, \tilde{p}$ and $\tilde{f}$ are identical to that of $\Gamma$ except for $\{e_1, e_2\} = v_1^{-1}(v), \tilde{v}_1(e_1) = v_1$ and $\tilde{v}_1(e_2) = v_2$. (See Figure 5: $\Gamma_1$ is the normalization of $\Gamma_2$ at $v$.)

**Definition 3.8** (gluing). Let $\Gamma$ be a FTCY graph and let $v_1, v_2 \in V_1(\Gamma)$ be two univalent vertices of $\Gamma$. Let $f_i = \tilde{f}(e_i)$, where $\{e_i\} = v_i^{-1}(v_1)$. Suppose $p(e_1) = -p(e_2)$ and $f_1 = -f_2$.

We then identify $v_1$ and $v_2$ to form a single vertex, and keep the framing $f(e_1) = f_1$. The resulting graph $\Gamma^{v_1,v_2}$ is called the gluing of $\Gamma$ at $v_1$ and $v_2$. (See Figure 5: $\Gamma_2$ is the gluing of $\Gamma_1$ at $v_1$ and $v_2$.)

It is straightforward to generalize smoothing and normalization to subset $A$ of $V_2(\Gamma)$. Given $A \subset V_2(\Gamma)$, let $\Gamma_A$ denote the smoothing of $\Gamma$ along $A$, and let $\Gamma^A$ denote the normalization of $\Gamma$ along $A$. There are surjective maps

$$\pi_A : E(\Gamma) \to E(\Gamma_A), \quad \pi^A : V(\Gamma^A) \to V(\Gamma).$$
3.5. **Relative FTCY Threefolds.** In this subsection we will introduce relative formal toric Calabi-Yau (FTCY) threefolds.

Given a FTCY graph $\Gamma$, we will construct a pair $\hat{Y}_{\text{rel}} = (\hat{Y}, \hat{D})$, where $\hat{Y}$ is a threefold, possibly with normal crossing singularities, $\hat{D} \subset \hat{Y}$ is a relative divisor, so that $\hat{Y}_{\text{rel}} = (\hat{Y}, \hat{D})$ is a formal relative Calabi-Yau threefold:

$$\wedge^3 \Omega_{\hat{Y}} (\log \hat{D}) \cong \mathcal{O}_{\hat{Y}}. \quad (3.3)$$

The pair $(\hat{Y}, \hat{D})$ admits a $T$-action so that the action on $\Lambda^3 T_p \hat{Y}$ is trivial for any fixed point $p$. As a set, the scheme $\hat{Y}$ is a union of $\mathbb{P}^1$'s, each associated to an edge of $\Gamma$; two $\mathbb{P}^1$ intersect exactly when their associated edges share a common vertex; the normal bundle to each $\mathbb{P}^1$ in $\hat{Y}$ and the $T$-action on $\hat{Y}$ are dictated by the data encoded in the graph $\Gamma$.

We will also specify a $T$-invariant divisor $\hat{L} \subset \hat{D}$.

In the following construction, we will use the notation introduced in Section 3.3.

3.5.1. **Edges.** Let $e \in E^0(\Gamma)$ with $v_0$ and $v_1$ its initial and terminal vertices. We let $T$ acts on $\mathbb{P}^1$ by

$$t \cdot [X, Y] = [p(e)(t)X, Y], \quad t \in T.$$ 

Here we view $p(e)$ as an element in $\Lambda T = \text{Hom}(T, \mathbb{C}^*)$. We denote the two fixed points by $q_0 = [0, 1]$ and $q_1 = [1, 0]$. Next we let $L^e \to \mathbb{P}^1$ be the line bundle of

$$\text{deg } L^e = \begin{cases} 
  n^e - 1, & v_1 \in V_3(\Gamma), \\
  n^e, & v_1 \notin V_3(\Gamma),
\end{cases}$$

where $n^e = \tilde{n}(e)$ is defined in Definition 3.4. We let $L^{-e}$ be the line bundle on $\mathbb{P}^1$ of degree $n^{-e} - 1$ (resp. $n^{-e}$) when $v_0$ is trivalent (resp. non-trivalent). We then assign the $T$-action at $L^{q_0}_{q_0}$ be $I_0(e)$ and at $L_{q_1}$ be $I_1(e)$; assign the $T$-action at $L^{-q_1}$ be $I_0(-e)$ and at $L^{-q_0}$ be $I_1(-e)$ (see Figures 4).

Next, we let $\Sigma(e)$ be the formal completion of the total space of $L^e \oplus L^{-e}$ along its zero section. The $T$-actions on $L^e$ and on $L^{-e}$ induce a $T$-action on $\Sigma(e)$. By construction, there is a $T$-isomorphism

$$\Sigma(e) \cong \Sigma(-e) \quad (3.4)$$

that sends $[0, 1] \in \Sigma(e)$ to $[1, 0] \in \Sigma(-e)$ and sends the first summand $L^e$ in $N_{\mathbb{P}^1/\Sigma(e)}$ to the second summand $L^{-e}$ in $N_{\mathbb{P}^1/\Sigma(-e)}$.

It is clear that

$$\wedge^3 \Omega_{\Sigma(e)} \cong p^* \mathcal{O}_{\mathbb{P}^1}(c), \quad (3.5)$$

where $p: \Sigma(e) \to \mathbb{P}^1$ is the projection, and $c = \#(\{v_0, v_1\} \cap V_3(\Gamma)) - 2$. 

---

**Figure 5**

$\Gamma_1 = (\Gamma_2)^v \Gamma_2 = (\Gamma_1)^{v_1,v_2} = (\Gamma_3)_{f_0,e} \Gamma_3 = (\Gamma_2)^v$
3.5.2. **Gluing along Trivalent Vertices.** Given \( v \in V_3(\Gamma) \) with \( v_0^{-1}(v) = \{e_1, e_2, e_3\} \) so indexed so that \( p(e_1), p(e_2), p(e_3) \) is in counter-clockwise order.

To glue the formal scheme \( \Sigma(e_1), \Sigma(e_2) \) and \( \Sigma(e_3) \), we introduce a \( T \)-scheme
\[
\Sigma(v) = \Spec \mathbb{C}[x_1, x_2, x_3], \quad \text{with } T\text{-action } t \cdot x_i = p(e_i)(t)x_i \quad \forall t \in T,
\]
and gluing morphisms
\[
(3.6) \quad \psi_{e_1,v} : \Sigma(v) \longrightarrow \Sigma(e_1)
\]
according to the following rule. First, we let \( \hat{\Sigma}(e_k) \) be the formal completion of \( \Sigma(e_k) \) along \( g_0 \in \mathbb{P}^1 \subset \Sigma(e_k) \). \( \hat{\Sigma}(e_k) \) is a formal \( T \)-scheme and is \( T \)-isomorphic to the \( T \)-scheme
\[
\Spec \mathbb{C}[y_1, y_2, y_3] : \quad t \cdot y_i = p(e_{i+k})(t)y_i
\]
such that \( L_{v_0}^e, L_{v_1}^e, T_{g_0} \mathbb{P}^1 \) are mapped to \( \mathbb{C} \frac{\partial}{\partial y_1}, \mathbb{C} \frac{\partial}{\partial y_2}, \mathbb{C} \frac{\partial}{\partial y_3} \), respectively. The gluing morphism \( \psi_{e_1,v} \) is the composite of
\[
(3.7) \quad \Spec \mathbb{C}[y_1, y_2, y_3] \longrightarrow \hat{\Sigma}(e_k) \longrightarrow \Sigma(e_k)
\]
with the \( T \)-isomorphism
\[
\Sigma(v) \equiv \Spec \mathbb{C}[x_1, x_2, x_3] \longrightarrow \Spec \mathbb{C}[y_1, y_2, y_3]
\]
defined by \( y_i \mapsto x_{k+i} \).

Using the morphisms \( \psi_{e_1,v} \), we can glue \( \Sigma(e_1) \) and \( \Sigma(e_2) \) and then glue \( \Sigma(e_3) \) onto it via the cofiber products
\[
\begin{array}{ccc}
\Sigma(e_1) & \longrightarrow & \Sigma(e_1) \coprod_{\Sigma(v)} \Sigma(e_2) \\
\uparrow & & \uparrow \\
\Sigma(v) & \longrightarrow & \Sigma(e_2)
\end{array}
\quad
\begin{array}{ccc}
\Sigma(e_3) & \longrightarrow & \Sigma(e_3) \coprod_{\Sigma(v)} \Sigma(e_2) \\
\uparrow & & \uparrow \\
\Sigma(v) & \longrightarrow & \Sigma(e_2)
\end{array}
\]
\[
\Sigma(v) \longrightarrow \Sigma(e_1) \coprod_{\Sigma(v)} \Sigma(e_2) \coprod_{\Sigma(v)} \Sigma(e_3)
\]
Since the gluing map \( \psi_{e_1,v} \) are \( T \)-equivalent, the \( T \)-actions on \( \Sigma(e_k) \) descend to the glued scheme.

3.5.3. **Gluing along Bivalent Vertices.** Next we glue \( \Sigma(e_1) \) and \( \Sigma(e_2) \) in case \( \{e_1, e_2\} = v_0^{-1}(v) \) for a \( v \in V_2(\Gamma) \). Note that \( l_0(e_1) + l_0(e_2) = 0 \). Let \( \Sigma(v) \) be the formal \( T \)-scheme
\[
\Sigma(v) = \Spec \mathbb{C}[x_1, x_2], \quad t \cdot x_i = l_0(e_i)(t)x_i;
\]
we let the gluing morphism \( \psi_{e_1,v} \) of (3.6) be the composite of (3.7) with the \( T \)-morphism
\[
\Spec \mathbb{C}[x_1, x_2] \longrightarrow \Spec \mathbb{C}[y_1, y_2, y_3]
\]
defined via \( y_1 \mapsto 0, y_1 \) and \( y_2 \) map to \( x_1 \) and \( x_2 \) respectively in case \( k = 1 \) and to \( x_2 \) and \( x_1 \) respectively in case \( k = 2 \). We can glue \( \Sigma(e_1) \) and \( \Sigma(e_2) \) along \( \Sigma(v) \) via the cofiber product as before.

3.5.4. **Univalent vertices.** Lastly, we consider the case \( e_1 = v_0^{-1}(v) \) for a \( v \in V_1(\Gamma) \). Note that \( l_0(e) + l_1(-e) = 0 \). Let \( \Sigma(v) \) be the formal \( T \)-scheme
\[
\Sigma(v) = \Spec \mathbb{C}[x_1, x_2], \quad t \cdot x_1 = l_0(e_1)(t)x_1, \quad t \cdot x_2 = l_1(-e_1)(t)x_2;
\]
and define \( \psi_{e_1,v} \) in (3.6). We let \( D^v \) be the image divisor \( \psi_{e_1,v}(\Sigma(v)) \subset \Sigma(e_1) \) and consider it as part of the relative divisor of the formal Calabi-Yau scheme \( Y_{\text{rel}} \) we are constructing.

Let \( L(v) \subset \Sigma(v) \) be the divisor defined by \( x_2 = 0 \), and let \( \tilde{L}^v = \psi_{e_1,v}(L(v)) \subset D^v \).

\[\text{Here we agree that } e_{k+3} = e_k; \text{ same for } x_{i+3} = x_i \text{ later.}\]
3.5.5. Final step. Now it is standard to glue all $\Sigma(e)$ to form a scheme $\hat{Y}$. We first form the disjoint union
\[
\coprod_{e \in E^0(\Gamma)} \Sigma(e);
\]
because of (3.4), the orientation reversing map $E^0(\Gamma) \to E^0(\Gamma)$ defines a fixed point free involution
\[
\tau : \coprod_{e \in E^0(\Gamma)} \Sigma(e) \to \coprod_{e \in E^0(\Gamma)} \Sigma(e);
\]
we define $\hat{Y}$ be its quotient by $\tau$. Next, for each trivalent vertex $v$ with $v_0^{-1}(v) = \{e_1, e_2, e_3\}$, we glue $\Sigma(e_1), \Sigma(e_2), \Sigma(e_3)$ along $\Sigma(v)$; for each bivalent vertex $v$ with $v_0^{-1}(v) = \{e_1, e_2\}$, we glue $\Sigma(e_1)$ and $\Sigma(e_2)$ along $\Sigma(v)$. We denote by $\hat{Y}$ the resulting scheme after completing all the gluing associated to all trivalent and bivalent vertices. The $T$-action on $\Sigma(e)$'s descends to a $T$-action on $\hat{Y}$. Finally, for each univalent vertex $v$ with $e = v_0^{-1}(v)$, we let $D^v \subset \Sigma(e)$ be the divisor defined in Section 3.5.4. The (disjoint) union of all such $D^v$ form a divisor $\hat{D}$ that is the relative divisor of $\hat{Y}$. Since $\hat{D}$ is invariant under $T$, the pair $Y^{rel} = (\hat{Y}, \hat{D})$ is a $T$-equivariant formal scheme. Because of (3.5), we have
\[
\wedge^3 \Omega_{\hat{Y}}(\log \hat{D}) \cong \mathcal{O}_{\hat{Y}};
\]
hence $Y^{rel} = (\hat{Y}, \hat{D})$ is a formal toric Calabi-Yau scheme.

Following the construction, the scheme $\hat{Y}$ is smooth away from the images $\psi_{ev}(\Sigma(v))$ associated to bivalent vertices $v$, and has normal crossing singularities there. Therefore $\hat{Y}$ is smooth when $\Gamma$ has no bivalent vertices. The relative divisor $\hat{D}$ is the union of smooth divisor $D^v$ indexed by $v \in V_1(\Gamma)$. Within each divisor $D^v$ there is a divisor $\hat{L}^v \subset D^v$ defined as in Section 3.5.4.

For later convenience, we introduce some notation. Let $\tilde{e}$ denote the equivalence class $\{e, -e\}$ in $E(\Gamma)$, and let $C^{\tilde{e}}$ denote the projective line in $\hat{Y}$ coming from the $P^1$ in $\Sigma(e)$. For $v \in V_1(\Gamma)$, let $z^v$ denote the point in $D^v$ coming from the closed point $q_0$ in $\Sigma(e)$, where $v_0(e) = v$.

4. Definition of Formal Relative Gromov-Witten Invariants

In this section, we will define relative Gromov-Witten invariants of relative FTCY threefolds; the case when the relative FTCY threefold is indecomposable gives the mathematical definition of topological vertex.

4.1. Moduli Spaces of Relative Stable morphisms. Let $\Gamma$ be a FTCY graph and let $Y^{rel} = (\hat{Y}, \hat{D})$ be its associated scheme. Clearly, the degrees and the ramification patterns of relative stable morphisms to $Y^{rel}$ are characterized by effective classes of $\Gamma$.

Definition 4.1 (effective class). Let $\Gamma$ be a FTCY graph. An effective class of $\Gamma$ is a pair of functions $\bar{d} : E(\Gamma) \to \mathbb{Z}_{\geq 0}$ and $\bar{\mu} : V_1(\Gamma) \to \mathcal{P}$ that satisfy
\begin{enumerate}
\item $|\bar{\mu}(v)| = \bar{d}(\tilde{e})$ if $v \in V_1(\Gamma)$ and $v_1(e) = v$;
\item $\bar{d}(\tilde{e}_1) = \bar{d}(\tilde{e}_2)$ if $v \in V_2(\Gamma)$ and $v_0^{-1}(v) = \{e_1, e_2\}$.
\end{enumerate}
We write $\mu^v$ for $\bar{\mu}(v)$, $d^{\tilde{e}}$ for $\bar{d}(\tilde{e})$.

To show that an effective class does characterize a relative stable morphism, a quick review of its definition is in order. An ordinary relative morphism $u$ to $(Y, D)$ consists of
\begin{itemize}
\item a possibly disconnected nodal curve $X$
\end{itemize}
• distinct smooth points \( \{ q^v_j \mid v \in V_0(\Gamma), 1 \leq j \leq \ell(\mu^v) \} \) in \( X \) such that each connected component of \( X \) contains at least one of these points,

• a morphism \( u: X \to \hat{Y} \) so that

\[
u^{-1}(\hat{D}^v) = \sum_{j=1}^{\ell(\mu^v)} \mu^v_j q^v_j \]

for some positive integers \( \mu^v_j \);

• \( u \) is pre-deformable along the singular loci

\[
\prod_{v \in V_2(\Gamma)} \Sigma(v)
\]

of \( \hat{Y}^{rel} \), i.e., if \( v \in V_2(\Gamma) \) and \( v_0^{-1}(v) = \{ e_1, e_2 \} \), then \( u^{-1}(\Sigma(v)) \) consists of nodes of \( X \), and for each \( y \in u^{-1}(\Sigma(v)) \), \( u|_{u^{-1}(\Sigma(e_1))} \) and \( u|_{u^{-1}(\Sigma(e_2))} \) have the same contact order to \( \Sigma(v) \) at \( y \);

• \( u \) coupled with the marked points \( q^v_i \) is a stable morphism in the ordinary sense.

Unless otherwise specified, all the stable morphisms in this paper are with not necessarily connected domain.

Since

\[ H_2(\hat{Y}; \mathbb{Z}) = \bigoplus_{\bar{e} \in E(\Gamma)} \mathbb{Z}[C^{\bar{e}}], \]

the morphism \( u \) defines a map \( \bar{d}: E(\Gamma) \to \mathbb{Z} \) via

\[(4.1) \quad u_\ast([X]) = \sum_{\bar{e} \in E(\Gamma)} \bar{d}(\bar{e})[C^{\bar{e}}]. \]

The integers \( \mu^v_j \) form a partition

\[ \mu^v = (\mu^v_1, \ldots, \mu^v_{\ell(\mu^v)}) \]

and the map \( \bar{\mu}: V_1(\Gamma) \to \mathcal{P} \) is

\[ \bar{\mu}(v) = \mu^v. \]

With this definition, the requirement (1) in Definition 4.1 follows from (4.1) and (2) holds since \( u \) is pre-deformable.

To define relative stable morphisms to \( \hat{Y}^{rel} \), we need to work with the expanded schemes of \( \hat{Y}^{rel} \) introduced in [24]. In the case studied, they are the associated formal schemes of the expanded graphs of \( \Gamma \).

\textbf{Definition 4.2.} Let \( \Gamma \) be a FTCY graph. A flat chain of length \( n \) in \( \Gamma \) is a subgraph \( \Gamma' \subset \Gamma \) that has \( n \) edges \( \pm e_1, \ldots, \pm e_n \), \( n+1 \) univalent or bivalent vertices \( v_0, \ldots, v_n \) with identical framings \( \bar{f} \) (up to sign) so that

\[ v_0(e_1) = v_0; \quad v_1(e_i) = v_0(e_{i+1}) = v_i \quad i = 1, \ldots, n-1; \quad v_1(e_n) = v_n, \]

and that all \( p(e_i) \) are identical.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{A flat chain of length \( n \)}
\end{figure}
Definition 4.3. A contraction of a FTCY graph $\Gamma$ along a flat chain $\hat{\Gamma} \subset \Gamma$ is the graph after eliminating all edges and bivalent vertices of $\hat{\Gamma}$ from $\Gamma$, identifying the univalent vertices of $\hat{\Gamma}$ while keeping their framings unchanged.

Given a FTCY graph $\Gamma$ and a function
$$m : V_1(\Gamma) \cup V_2(\Gamma) \rightarrow \mathbb{Z}_{\geq 0},$$
the expanded graph $\Gamma_m$ is obtained by replacing each $v \in V_1(\Gamma) \cup V_2(\Gamma)$ by a flat chain $\hat{\Gamma}_{m,v}$ of length $m^v = m(v)$ with framings $\pm f(e)$, where $v_1(e) = v$. In particular, $\Gamma_0 = \Gamma$, where $0(v) = 0$ for all $v \in V_1(\Gamma) \cup V_2(\Gamma)$. The original graph $\Gamma$ can be recovered by contracting $\Gamma_m$ along the flat chains
$$\{\hat{\Gamma}_{m,v} \mid v \in V_1(\Gamma) \cup V_2(\Gamma)\}.$$

We now study their associated Calabi-Yau scheme. We denote by $(\hat{\Gamma}_m, \hat{D}_m)$ the associated effective class of any degeneration of $\Gamma$, and in particular, an effective class of $\Gamma$. We recover the original scheme $\hat{Y}$ by shrinking the irreducible components of $\hat{Y}_m$ associated to the flat chains that are contracted. This way we define a projection
$$\pi_m : \hat{Y}_m \rightarrow \hat{Y}.$$

We define a relative automorphism of $\hat{Y}_m$ to be an automorphism of $\hat{Y}_m$ that is also a $\hat{Y}$-morphism; an automorphism of a relative morphism $u : X \rightarrow (\hat{Y}_m, \hat{D}_m)$ is a pair of a relative automorphism $\sigma$ of $\hat{Y}_m$ and an automorphism $h$ of $X$ so that
$$u \circ h = \sigma \circ u.$$

Definition 4.4. A relative morphism to $\hat{Y}_m^{rel}$ is an ordinary relative morphism to $(\hat{Y}_m, \hat{D}_m)$ for some $m$; it is stable if its automorphism group is finite.

Note that an effective class $(\hat{d}, \hat{\mu})$ of an FTCY graph $\Gamma$ can also be viewed as an effective class of any degeneration of $\Gamma$, and in particular, an effective class of $\Gamma_m$. We fix a FTCY graph $\Gamma$, an effective class $(\hat{d}, \hat{\mu})$, and an even integer $\chi$. We then form the moduli space $\mathcal{M}^{\bullet}_{\chi, \hat{d}, \hat{\mu}}(\hat{Y}_{m}^{rel})$ of all stable relative morphisms $u$ to $\hat{Y}_{m}^{rel}$ that satisfy
- $\chi(O_X) = \chi/2$, where $X$ is the domain curve of $u$;
- the associated effective class of $u$ is $(\hat{d}, \hat{\mu})$.

Since $\hat{Y}$ is a formal Calabi-Yau threefold with possibly normal crossing singularity and smooth singular loci, the moduli space $\mathcal{M}^{\bullet}_{\chi, \hat{d}, \hat{\mu}}(\hat{Y}_{m}^{rel})$ is a formal Deligne-Mumford stack with a perfect obstruction theory [24, 25].

Lemma 4.5. The virtual dimension of $\mathcal{M}^{\bullet}_{\chi, \hat{d}, \hat{\mu}}(\hat{Y}_{m}^{rel})$ is $\sum_{v \in V_1(\Gamma)} \lambda(\mu^v)$.

Proof. The proof is straightforward and will be omitted. \qed

4.2. Equivariant Degeneration. Let $T$ act on $\mathbb{P}^1 \times \mathbb{A}^1$ by
$$t \cdot ([X_0, X_1], s) = ([p(t)X_0, X_1], s),$$
where $p \in \Lambda_T = \text{Hom}(T, \mathbb{C}^*)$. Let $\mathcal{Y}$ be the blowup of $\mathbb{P}^1 \times \mathbb{A}^1$ at $(0, 1, 0)$. The $T$-action on $\mathbb{P}^1 \times \mathbb{A}^1$ can be lifted to $\mathcal{Y}$ such that the projection $\mathcal{Y} \rightarrow \mathbb{P}^1 \times \mathbb{A}^1$ is $T$-equivariant; composition with the projection $\mathbb{P}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ gives a $T$-equivariant family of curves
$$\mathcal{Y} \rightarrow \mathbb{A}^1$$
such that $\mathcal{Y}_s \cong \mathbb{P}^1$ for $s \neq 0$ and $\mathcal{Y}_0 \cong \mathbb{P}^1 \cup \mathbb{P}^1$. 
The above construction can be generalized as follows. Let \( \Gamma \) be a FTCY graph and let
\[ V_2(\Gamma) = \{ v_1, \ldots, v_n \} \]

Then we have a \( T \)-equivariant family
\[ (\hat{\calY}, \hat{\calD}) \to \mathbb{A}^n \]
such that
\[ (\hat{\calY}, \hat{\calD})_0 = \hat{\calY}(0, \ldots, 0) \cong \hat{\calY}_{\text{rel}} \quad \text{and} \quad (\hat{\calY}, \hat{\calD})_s = (\hat{\calY}, \hat{\calD}(s_1, \ldots, s_n)) \cong \hat{\calY}_{\text{rel}}^{\Gamma}_{(v_2(v_s) = 0)}.
\]
Recall that given a subset \( A \subset V_2(\Gamma) \), \( \Gamma_A \) is the smoothing of \( \Gamma \) along \( A \) (Section 3.4). The \( T \)-action on \( \mathbb{A}^n \) is trivial, and the \( T \)-action on each fiber is consistent with the one described in Section 3.

By the construction in [25], there is a \( T \)-equivariant family
\[ (4.3) \quad \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) \to \mathbb{A}^n \]
such that \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY})_s = \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}(\hat{\calD})) \). In particular,
\[ \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY})_0 = \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}_{\text{rel}}). \]
The total space \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) \) is a formal Deligne-Mumford stack with a perfect obstruction theory \([T^1 \to T^2]\) of virtual dimension
\[ \sum_{v \in V_1(\Gamma)} \ell(\mu^v) + |V_2(\Gamma)|. \]

For each \( v \in V_2(\Gamma) \) there is line bundle \( L^v \) over \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) \) with a section \( s^v : \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) \to L^v \) such that
\[ \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY})_0 = \{ s^v = 0 \mid v \in V_2(\Gamma) \} \subset \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}). \]
The pair \((L^v, s^v)\) corresponds to \((L_0, r_0)\) in [25, Section 3].

4.3. Perfect Obstruction Theory. Let \( \Gamma \) be a FTCY graph, and let \((\vec{d}, \vec{\mu})\) be an effective class of \( \Gamma \). We briefly describe the perfect obstruction theory on \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}_{\text{rel}}) \) constructed in [25].

Let \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) \to \mathbb{A}^{V_2(\Gamma)}, [T^1 \to T^2], \) and \( \{ L^v \mid v \in V_2(\Gamma) \} \) be defined as in Section 4.2. Let \( [\hat{T}^1 \to \hat{T}^2] \) be the perfect obstruction theory on \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}_{\text{rel}}) \). Let \( u : (X, q) \to (\hat{\calY}_m, \hat{\calD}_m) \) represent a point in \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}_{\text{rel}}) \subset \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) \), where
\[ q = \{ q^v_j \mid v \in V_1(\Gamma), 1 \leq j \leq \ell(\mu^v) \}. \]
We have the following exact sequence of vector spaces at \( u \):
\[ (4.4) \quad 0 \to \hat{T}_u^1 \to \mathbb{T}_u^1 \to \bigoplus_{v \in V_2(\Gamma)} L_u^v \to \hat{T}_u^2 \to \mathbb{T}_u^2 \to 0. \]

We will describe \( \mathbb{T}_u^1, \mathbb{T}_u^2 \), and \( L_u^v \) explicitly. When \( \Gamma \) is a regular FTCY graph, i.e., \( V_2(\Gamma) = \emptyset \), the line bundles \( L^v \) do not arise, and \( \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}) = \calM_{\chi, \vec{d}, \vec{\mu}}^{\bullet}(\hat{\calY}_{\text{rel}}) \).
We first introduce some notation. Given \( m : V_1(\Gamma) \cup V_2(\Gamma) \to \mathbb{Z}_{\geq 0} \), let \( \bar{\Gamma}_{m,v}^\nu \) be the flat chain of length \( m^\nu = m(v) \) associated to \( v \in V_1(\Gamma) \cup V_2(\Gamma) \), and let
\[
V(\bar{\Gamma}_m) = \{ \bar{v}_0^v, \ldots, \bar{v}_m^v \},
\]
where \( \bar{v}_m^v \in V_1(\Gamma_m) \) if \( v \in V_1(\Gamma) \).

Let \( v \in V_1(\Gamma) \) and \( 0 \leq l \leq m^\nu - 1 \), or let \( v \in V_2(\Gamma) \) and \( 0 \leq l \leq m^\nu \). We define a line bundle \( L_l^v \) on the divisor \( \hat{D}_l^v = \Sigma(\bar{v}_l) \) in \( Y_m \) by
\[
L_l^v = N_{\hat{D}_l^v/\Sigma(e_v)} \otimes N_{\hat{D}_l^v/\Sigma(e'_v)}
\]
where \( u_l^{-1}(\bar{v}_l^v) = \{ e_v, e'_v \} \). Note that \( L_l^v \) is a trivial line bundle on \( \hat{D}_l^v \).

With the above notation, we have
\[
L_l^v = \bigotimes_{\bar{v}_l^v \in m^\nu} H^0(\hat{D}_l^v, L_l^v).
\]

The tangent space \( T_u^1 \) and the obstruction space \( T_u^2 \) to \( \mathcal{M}^{\bullet, \hat{\chi}, j}(\hat{\mathcal{Y}}) \) at the moduli point
\[
[u : (X, \mathbf{q}) \to (\hat{\mathcal{Y}}_m, \hat{\mathcal{D}}_m)]
\]
are given by the following two exact sequences:
\[
0 \to \text{Ext}^0(\Omega_X(R_{\mathbf{q}}), \mathcal{O}_X) \to H^0(\mathcal{D}^\bullet) \to T_u^1 \to
\]
\[
\to \text{Ext}^1(\Omega_X(R_{\mathbf{q}}), \mathcal{O}_X) \to H^1(\mathcal{D}^\bullet) \to T_u^2 \to 0
\]
and
\[
0 \to H^0(u^* (\Omega_{\mathcal{Y}_m}(\log \mathcal{D}_m))^\vee) \to H^0(\mathcal{D}^\bullet) \to
\]
\[
\to \bigoplus_{v \in V_1(\Gamma)} H^0_{\text{et}}(\mathbb{R}^1_{\mathbf{q}}) \bigoplus \bigoplus_{v \in V_2(\Gamma)} H^0_{\text{et}}(\mathbb{R}^1_{\mathbf{q}}) \to H^1(u^*(\Omega_{\mathcal{Y}_m}(\log \mathcal{D}_m))^\vee) \to
\]
\[
\to H^1(\mathcal{D}^\bullet) \to \bigoplus_{v \in V_1(\Gamma)} H^1_{\text{et}}(\mathbb{R}^1_{\mathbf{q}}) \bigoplus \bigoplus_{v \in V_2(\Gamma)} H^1_{\text{et}}(\mathbb{R}^1_{\mathbf{q}}) \to 0.
\]

where
\[
R_{\mathbf{q}} = \sum_{v \in V_1(\Gamma)} \sum_{j=1}^{\ell(u^v)} q_j^v,
\]

\[
H^0_{\text{et}}(\mathbb{R}^1_{\mathbf{q}}) \cong \bigoplus_{q \in u^{-1}(\hat{\mathcal{D}}_l^v)} T_q(u^{-1}(\Sigma(e_v))) \otimes T_q(u^{-1}(\Sigma(e'_v))) \cong \mathbb{C}^{\otimes n_l^v}
\]
for \( u_l^{-1}(\bar{v}_l^v) = \{ e_v, e'_v \} \),

\[
H^1_{\text{et}}(\mathbb{R}^1_{\mathbf{q}}) \cong H_0(\hat{\mathcal{D}}_l^v, L_l^v)^{\otimes n_l^v} \big/ H^0(\hat{\mathcal{D}}_l^v, L_l^v),
\]
and \( n_l^v \) is the number of nodes over \( \hat{D}_l^v \). In (4.9),
\[
H^0(\hat{D}_l^v, L_l^v) \to H^0(\hat{D}_l^v, L_l^v)^{\otimes n_l^v}
\]
is the diagonal embedding.

We refer the reader to [25] for the definitions of \( H^i(\mathcal{D}^\bullet) \) and the maps between terms in (4.3), (4.7).
4.4. Formal Relative Gromov-Witten Invariants. Usually, the relative Gromov-Witten invariants are defined as integrations of the pull back classes from the target space over the relative divisor. In the case studied, the analogue is to integrate a total degree 2 \( \ell (\mu^v) \) class from the relative divisor \( \tilde{D} \). The class we choose is the product of the "Poincaré dual" of the divisor \( \tilde{L}^v \subset \tilde{D}^v \), one for each marked point \( q_i^v \). Equivalently, we consider the moduli space

\[
\mathcal{M}_{\chi, \Gamma, \mu}^\bullet (\tilde{Y}^\text{rel}, \tilde{L}) = \left\{ (u, X, \{ q_i^v \}) \in \mathcal{M}_{\chi, \Gamma, \mu}^\bullet (\tilde{Y}^\text{rel}) \mid u(\tilde{q}_i^v) \in \tilde{L}^v \right\}.
\]

Its virtual dimension is zero. More precisely, let \( T \) be the intersection theory developed in [7] and the localization in [8, 15]. By applying the virtual localization to the moduli scheme (4.10)

\[
\tilde{M}^\text{vir} \to \tilde{M}^\text{vir} = \bigoplus_{v} (\mathcal{N}^v_{\tilde{L}/\tilde{D}^v}) u(\tilde{q}_i^v)
\]

as virtual vector spaces.

In the rest of this subsection (Section 4.4), we fix \( \chi, \Gamma, \tilde{\mu} \), and write \( \mathcal{M} \) instead of \( \mathcal{M}_{\chi, \Gamma, \mu}^\bullet (\tilde{Y}^\text{rel}, \tilde{L}) \). We now define the formal relative Gromov-Witten invariants of \( \tilde{Y}^\text{rel} \) by applying the virtual localization to the moduli scheme \( \mathcal{M} \). We use the equivariant intersection theory developed in [7] and the localization in [8, 15].

Since \( \tilde{Y}^\text{rel} \) is toric, the moduli space \( \mathcal{M} \) and its obstruction theory are \( T \)-equivariant. We consider the fixed loci \( \mathcal{M}^T \) of the \( T \)-action on \( \mathcal{M} \). Its coarse moduli space is projective. The virtual localization is an integration of the quotient equivariant Euler classes. When \( [u] \) varies in a connected component of \( \mathcal{M}^T \), the vector spaces \( T_u^1 \) and \( T_u^2 \) form two vector bundles. We denoted them by \( T^1 \) and \( T^2 \). Since the obstruction theory are \( T \)-equivariant, both \( T^i \) are \( T \)-equivariant. We let \( T^{i,f} \) and \( T^{i,m} \) be the fixed and the moving parts of \( T^i \). Since the fixed part \( T^{i,f} \) induces a perfect obstruction theory of \( \mathcal{M}^T \), it defines a virtual cycle

\[
[\mathcal{M}^T]^\text{vir} \in A_*(\mathcal{M}^T),
\]

where \( A_*(\mathcal{M}^T) \) is the Chow group with rational coefficients.

The perfect obstruction theory \( [T^{1,f} \to T^{2,f}] \) together with the trivial \( T \)-action defines a \( T \)-equivariant virtual cycle

\[
[\mathcal{M}^T]^\text{vir, T} \in A^*_T(\mathcal{M}^T).
\]

Since \( T \) acts on \( \mathcal{M}^T \) trivially, we have

\[
A^*_T(\mathcal{M}^T) \cong A_*(\mathcal{M}) \otimes \Lambda_T
\]

where

\[
\Lambda_T = \text{Hom}(T, \mathbb{C}^*) \cong A^*_T(\text{pt}) \cong \mathbb{Q}[u_1, u_2].
\]

Under the isomorphism (4.11), we have

\[
[\mathcal{M}^T]^\text{vir, T} = [\mathcal{M}]^\text{vir} \otimes 1.
\]

The moving part \( T^{i,m} \) is the virtual normal bundles of \( \mathcal{M}_{\chi, \Gamma, \mu}^\bullet (\tilde{Y}^\text{rel}, \tilde{L}) \). Let

\[
e^T(T^{i,m}) \in A^*_T(\mathcal{M}^T)
\]
be the $T$-equivariant Euler class of $T^{1,m}$, where $A^*_T(M^T)$ is the $T$-equivariant operational Chow group (see [7, Section 2.6]). For $i = 1, 2$, $e^T(T^{1,m})$ lies in the subring

$$A^*(M^T) \otimes \mathbb{Q}[u_1, u_2] \subset A^*_T(M^T)$$

and is invertible in

$$A^*(M^T) \otimes \mathbb{Q}[u_1, u_2]_m \subset A^*_T(M^T) \otimes \mathbb{Q}[u_1, u_2]_m.$$ 

where $\mathbb{Q}[u_1, u_2]_m$ is $\mathbb{Q}[u_1, u_2]$ localized at the ideal $m = (u_1, u_2)$.

For later convenience, we introduce some notation. Let $X$ be a Deligne-Mumford stack with a $T$-action, and let $X^T$ be the $T$-fixed points. Recall that

$$(4.12) \quad A^T_*(X) \otimes \mathbb{Q}[u_1, u_2]_m \cong A^*_T(X^T) \otimes \mathbb{Q}[u_1, u_2]_m \cong A_*(X^T) \otimes \mathbb{Q}[u_1, u_2]_m.$$ 

The degree of a zero cycle defines a map $\deg : A_0(X^T) \to \mathbb{Q}$. We define

$$\deg_m : A_d(X^T) \otimes \mathbb{Q}[u_1, u_2]_m \to \mathbb{Q}[u_1, u_2]_m$$

by

$$a \otimes b \mapsto \begin{cases} \deg(a)b & d = 0, \\ 0 & d \neq 0. \end{cases}$$

This gives a ring homomorphism

$$\deg_m : A^*_T(X) \otimes \mathbb{Q}[u_1, u_2]_m \cong A_*(X^T) \otimes \mathbb{Q}[u_1, u_2]_m \to \mathbb{Q}[u_1, u_2]_m.$$ 

Given $c \in A^*_T(X) \otimes \mathbb{Q}[u_1, u_2]_m$ and $\alpha \in A^*_T(X) \otimes \mathbb{Q}[u_1, u_2]_m$, define

$$\int c = \deg_m(c \cap \alpha) \in \mathbb{Q}[u_1, u_2]_m.$$ 

Following the lead of the virtual localization formula [15], we define

**Definition 4.6** (formal relative Gromov-Witten invariants).

$$F^\bullet_{\chi^*,d,\bar{\mu}}(u_1, u_2) = \frac{1}{|\text{Aut}(\bar{\mu})|} \int_{[M^T]_{\text{vir},T}} \frac{e^T(T^{1,m})}{e^T(T^{2,m})}$$

where we view $[M^T]$ as an element in $A^*_T(M^T) \otimes \mathbb{Q}[u_1, u_2]_m$.

Note that

$$\frac{e^T(T^{1,m})}{e^T(T^{2,m})} \cap [M^T]_{\text{vir},T} \in \left(A^*_T(M^T) \otimes \mathbb{Q}[u_1, u_2]_m\right)_0$$

where $\left(A^*_T(M^T) \otimes \mathbb{Q}[u_1, u_2]_m\right)_0$ is the degree zero part of the graded ring $A^*_T(M^T) \otimes \mathbb{Q}[u_1, u_2]_m$. Therefore,

$$F^\bullet_{\chi^*,d,\bar{\mu}}(u_1, u_2) \in (\mathbb{Q}[u_1, u_2]_m)_0 = \mathbb{Q}(u_1/u_1)$$

where $(\mathbb{Q}[u_1, u_2]_m)_0$ is the degree zero part of the graded ring $\mathbb{Q}[u_1, u_2]_m$.

**Remark 4.7.** For our purpose of defining $F^\bullet_{\chi^*,d,\bar{\mu}}(u_1, u_2)$, we may consider the equivariant Borel-Moore homology $H^T_M(M) = H^T_M(M)$ instead of the equivariant Chow group $A^*_T(M)$, and consider the equivariant cohomology $H^*_T(M) = H^*_T(M)$ instead of the equivariant operational Chow group $A^*_T(M)$, where $M$ is any of the moduli spaces involved in the above discussions.
If $\mathcal{M}$ were a proper Deligne-Mumford stack then its obstruction theory would define a virtual cycle
\begin{equation}
[M]^{\text{vir}} \in A_0(\mathcal{M})
\end{equation}
and
\[F_{\chi, d, \mu}^\Gamma(u_1, u_2) = \frac{1}{|\text{Aut(}\mu)|} \deg[M]^{\text{vir}} \in \mathbb{Q}\]
would be a topological invariant independent of $u_1, u_2$. However, $\mathcal{M}$ is not proper, so (4.14) does not exist. Nevertheless, we will show that

**Theorem 4.8.** The function $F_{\chi, d, \mu}^\Gamma(u_1, u_2)$ is independent of $u_1, u_2$; hence is a rational number depending only on $\Gamma, \chi, d$ and $\mu$.

In Section 6 and Section 7, we will reduce the invariance of $F_{\chi, d, \mu}^\Gamma(u_1, u_2)$ (Theorem 4.8) to the invariance for a special topological vertex (Theorem 5.2).

5. Invariance of the Topological Vertex

We begin with the notion of topological vertex and topological vertex with standard framing.

**Definition 5.1** (topological vertex and standard framing). A topological vertex is a FTCY graph that has one trivalent vertex and three univalent vertices (see Figure 10 in Section 6). We say a topological vertex has a standard framing if its three edges $e_1$, $e_2$ and $e_3$ that share $v_0$ as their initial vertices have their position and framing maps satisfying (see Figure 7)

\[f(e_1) = p(e_2), \quad f(e_2) = p(e_3) \quad \text{and} \quad f(e_3) = p(e_1).\]

\[\text{Figure 7. A Topological vertex with standard framing}\]

In this section, we shall prove

**Theorem 5.2** (invariance of the topological vertex). Theorem 4.8 holds for any topological vertex with standard framing.

We fix a topological vertex with standard framing $\Gamma$ once and for all in this section; we let $\hat{Y}^{\text{rel}} = (\hat{Y}, \hat{D})$ be its associated FTCY threefold. As before, we continue to denote $T$ the group $(\mathbb{C}^*)^2$. We abbreviate $\mathcal{M}_{\chi, d, \mu}^\Gamma(\hat{Y}^{\text{rel}}, \hat{L})$ to $\mathcal{M}_\bullet(\hat{Y})$.

Our first step to prove the invariance of

\begin{equation}
F_{\chi, d, \mu}^\Gamma(u_1, u_2) = \frac{1}{|\text{Aut(}\mu)|} \int_{[M]^{\text{vir}}_{\mathcal{M}_\bullet(\hat{Y})}} e^T(T, m) \in \mathbb{Q}(u_1/u_2)
\end{equation}
is to construct a new pair of a nonsingular projective toric threefold $W$ with a relative divisor $D \subset W$ and a subdirector $L \subset D$ so that the similarly defined moduli $\mathcal{M}_{X,\tilde{d},\tilde{\mu}}^*(W_{\text{rel}}, L)$ (abbreviated to $\mathcal{M}^*(W)$) of relative stable morphisms with boundary constraint $L$ admits a morphism

$$\Phi : \mathcal{M}^* (\hat{Y}) \left( = \mathcal{M}^*_{X,\tilde{d},\tilde{\mu}} (\hat{Y}_{\text{rel}}, \hat{L}) \right) \longrightarrow \mathcal{M}^* (W) \left( = \mathcal{M}^*_{X,\tilde{d},\tilde{\mu}} (W_{\text{rel}}, L) \right)$$

so that the induced map on the $T$-fixed loci

$$\Phi^T : \mathcal{M}^* (\hat{Y})^T \longrightarrow \mathcal{M}^* (W)^T$$

has the property that it is an open and closed embedding and the obstruction theories of $\mathcal{M}^* (W)$ along its fixed loci is identical to that of $\mathcal{M}^* (\hat{Y})$ via $\Phi^T$. Because the equality of two obstruction theories, we have the identity:

$$(5.2) \quad \int_{[\mathcal{M}^* (\hat{Y})^T]^{\text{vir}}} e^T(T^{1,m}) = \int_{[\mathcal{M}^* (W)^T]^{\text{vir}}} e^T(T^1,T^2),$$

where $\mathcal{M}^* (W)^T$ is the image of the fixed loci of $\mathcal{M}^* (\hat{Y})$. (Here by abuse of notation, we denote by $T^{1,m}$ the moving parts of the obstruction complex $[T^1 \to T^2]$ of $\mathcal{M}^* (\hat{Y})$ as well as $\mathcal{M}^* (W)$ along their fixed loci.)

To prove the invariance of the right hand side, we shall devise a local contribution of $\text{deg} [\mathcal{M}^* (W)]^{\text{vir}}$ along $\mathcal{M}^* (\hat{Y})$; this local contribution is a sum of the desired term (5.1) with some other terms; we will show that this some other terms vanish completely. This will settle the invariance of the topological vertex $\Gamma$.

5.1. The Relative Calabi-Yau Manifold $W_{\text{rel}}$ and the Morphism $\Phi$. We begin with constructing the toric variety $W_{\text{rel}}$ as promised. Looking at the graph $\Gamma$ that we chose, the obvious choice of $W$ is the toric blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along three disjoint lines

$$(5.3) \quad \ell_1 = \infty \times \mathbb{P}^1 \times 0, \quad \ell_2 = 0 \times \infty \times \mathbb{P}^1 \quad \text{and} \quad \ell_3 = \mathbb{P}^1 \times 0 \times \infty.$$

The moment polytope of $W$, which is the image of the moment map

$$\Upsilon : W \longrightarrow \mathbb{R}^3$$

of the $U(1)^3$-action on $W$, is shown in Figure 8. It is diffeomorphic to the quotient $W/U(1)^3$. Here we follow the convention that $([z_1, z_2, z_3])$ is the point $([z_1, 1], [z_2, 1], [z_3, 1])$ in $(\mathbb{P}^1)^3$. We let $D \subset W$ be the exceptional divisor and let $D_1 \subset D$ be its connected component lying over $\ell_i$. Each $D_i$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We then let $C_1, C_2$ and $C_3$ be the proper transforms of

$$\mathbb{P}^1 \times 0 \times 0, \quad 0 \times \mathbb{P}^1 \times 0 \quad \text{and} \quad 0 \times 0 \times \mathbb{P}^1,$$

and let $L_i \subset D_i$, $i = 1, 2$ and 3, be the preimage of

$$(\infty, 0, 0) \in \ell_1, \quad (0, \infty, 0) \in \ell_2 \quad \text{and} \quad (0, 0, \infty) \in \ell_3.$$

Clearly, restricting to $C_i$ the log-canonical sheaf

$$\wedge^3 \Omega_W (\log D) |_{C_i} \cong \mathcal{O}_{C_i}. \quad (5.4)$$

Hence to the curves $C_i$ the relative pair $W_{\text{rel}} = (W, D)$ is practically a relative Calabi-Yau threefold.

For later discussion, we agree that under the isomorphisms $D_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\ell_i \cong \mathbb{P}^1$, the tautological projection $D_i \rightarrow \ell_i$ is the first projection. Under this convention, the line $L_i \subset D_i$ is the line $0 \times \mathbb{P}^1$ and the intersection $p_i = C_i \cap D_i$ is the point $(0,0)$.
Figure 8. Moment Polytope of $W$.

All faces of this polytope represent the $(\mathbb{C}^*)^3$ invariant divisors of $W$. The point $p_0$ is the image of the point $(0,0,0) \in W$. The line $\overline{p_0p_1}$ is the image of the curve $C_1 \cong \mathbb{P}^1$, and the thickened line $\overline{p_iq_i}$ is the image of the curve $L_i \cong \mathbb{P}^1$. The rectangle face containing the edge $\overline{p_ip_i}$ is the image of the relative divisor $D_i \cong \mathbb{P}^1 \times \mathbb{P}^1$.

As to the torus action, we pick the obvious one on $(\mathbb{P}^1)^3$ via

$$\begin{align*}
(z_1, z_2, z_3)(t_1, t_2, t_3) &= (t_1z_1, t_2z_2, t_3z_3), \\
(t_1, t_2, t_3) &\in (\mathbb{C}^*)^3.
\end{align*}$$

It lifts to a $(\mathbb{C}^*)^3$-action on $W$ that leaves $D_i$ and $L_i$ invariant. We let $T \subset (\mathbb{C}^*)^3$ be the subgroup defined by $t_1t_2t_3 = 1$; it is isomorphic to $(\mathbb{C}^*)^2$ and is the subgroup that leaves (5.4) invariant. In the following, we shall view $W_{\rel} = (W, D)$ as a relative Calabi-Yau $T$-manifold to the curves $C_i$.

Next we will define the moduli space $\mathcal{M}_{x, \vec{d}, \vec{\mu}}^\bullet(W_{\rel}, L)$. Clearly, each $C_i$ induces a homology class $[C_i] \in H_2(W; \mathbb{Z})$. For

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{P}_+^3,$$

we let $\vec{d}$ be the homology class

$$\vec{d} = |\mu_1|[C_1] + |\mu_2|[C_2] + |\mu_3|[C_3] \in H_2(W; \mathbb{Z}).$$

The pair $(\vec{d}, \vec{\mu})$ is an effective class of $\Gamma$:

$$\vec{d}(\vec{e}_i) = |\mu_i|, \quad \vec{\mu}(v_i) = \mu_i, \quad i = 1, 2, 3.$$

We then let

$$\mathcal{M}_{x, \vec{d}, \vec{\mu}}^\bullet(W_{\rel}, L), \quad \text{abbreviate to } \mathcal{M}^\bullet(W),$$

be the moduli of relative stable morphisms

$$u : (X; R_1, R_2, R_3) \to W_{\rel} = (W, D_1, D_2, D_3)$$

having fundamental classes $\vec{d}$, having ramification patterns $\mu^i$ along $D_i$, and satisfying $u(R_i) \subset L_i$, modulo the equivalence relation introduced in [25]. It is a proper, separated DM-stack; it has a perfect obstruction theory [24, 25], and thus admits a virtual cycle.

It follows from our construction that the scheme $Y$, which is the the closure of the three one-dimensional orbits in $\hat{Y}$, can be identified with the union $C_1 \cup C_2 \cup C_3$ in $W$; the formal
scheme $\hat{Y}$ is then the formal completion of $W$ along $Y$. Further, the relative divisor $\hat{D}$ of $Y$ (resp. the subdivisor $L \subset \hat{D}$) is the preimage of the relative divisor $D \subset W$ (resp. the subdivisor $L \subset D$); the induced morphism

\begin{equation}
\phi : (\hat{Y}, \hat{D}, \hat{L}) \longrightarrow (W, D, L)
\end{equation}

is $T$-equivariant; and the two effective classes $(\hat{d}, \hat{\mu})$ are consistent under the map $\phi$. Therefore, it induces a $T$-equivariant morphism of the moduli spaces

\begin{equation}
\Phi : M^\bullet(\hat{Y}) \longrightarrow M^\bullet(W),
\end{equation}

which induces a morphism

\[ \Phi^T : M^\bullet(\hat{Y})^T \longrightarrow M^\bullet(W)^T \]

between their respective fixed loci.

**Lemma 5.3.** The morphism $\Phi^T$ is an open and closed embedding; the obstruction theories of $M^\bullet(\hat{Y})$ and $M^\bullet(W)$ are identical under $\Phi$ along the fixed loci $M^\bullet(\hat{Y})^T$ and its image in $M^\bullet(W)$.

**Proof.** This follows immediately from that $C_1$, $C_2$ and $C_3$ are the closures of three one-dimensional orbit, that $Y = C_1 \cup C_2 \cup C_3$ and that $\hat{Y}$ is the formal completion of $W$ along $Y$. \hfill \Box

Later, we shall work with the moduli of relative stable morphisms $M^\bullet_{\chi, \hat{d}, \hat{\mu}}(Y^{rel}, L)$, similarly defined as that of $M^\bullet(W)$. Again, for notational simplicity, we shall abbreviate it to $M^\bullet(Y)$.

### 5.2. Invariant Relative Stable Morphisms

Let $a_1, a_2, a_3 \in \mathbb{Z}$ with $a_1 + a_2 + a_3 = 0$ be three relatively prime integers; $\eta = (a_1, a_2, a_3)$ defines a subgroup

\[ T_\eta = \{(t^{a_1}, t^{a_2}, t^{a_3}) \mid t \in U(1)\} \subset T. \]

Our next task is to characterize those stable relative morphisms that are invariant under $T_\eta \subset T$ and are small deformations of elements in $M^\bullet(Y)$.

To investigate relative stable morphisms to $W$, we need the expanded relative pair $(W[m], D[m])$, $m = (m_1, m_2, m_3)$ (see Figure 9). Let $\Delta$ be the projective bundle $\mathbb{P}(O_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus O_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1))$ with two sections

\[ D_+ = \mathbb{P}(O_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus 0) \quad \text{and} \quad D_- = \mathbb{P}(0 \oplus O_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)); \]

we form an $m_i$-chain of $\Delta$ by gluing $m_i$ copies of $\Delta$ via identifying the $D_-$ of one $\Delta$ to the $D_+$ of the next $\Delta$ using the canonical isomorphism $pr : D_+ \to \mathbb{P}^1 \times \mathbb{P}^1$; we then attach this chain to $D_i$ by identifying the $D_+$ of the first $\Delta$ in the chain with $D_i$ and declaring the $D_-$ of the last $\Delta$ be $D[m]$. The scheme $W[m]$ is the result after attaching such three chains, of length $m_1$, $m_2$ and $m_3$ respectively, to $D_1$, $D_2$ and $D_3$ in $W$. The union

\[ D[m] = D[m]_1 \cup D[m]_2 \cup D[m]_3 \]

is the new relative divisor of $W[m]$. Note that our construction is consistent with that the normal bundle of $D_i$ in $W$ has degree $-1$ along $L_i$.

For future convenience, we denote by $\Delta[m]$ the chain of $\Delta$’s that is attached to $D_i$; we denote by $L[m] \subset D[m]$ the same line as $L_i \subset D_i$. The new scheme $W[m]$ contains $W$ as its main irreducible component; it also admits a stable contraction $W[m] \to W$. Also, $(\mathbb{C}^*)^3$ acts on $(W[m], D[m])$ since the $(\mathbb{C}^*)^3$-action on $N_{D_1/W}$ induces a $(\mathbb{C}^*)^3$-action on each $\Delta$ attached to $D_i$. Therefore $(\mathbb{C}^*)^3$ and $T$ act on $M^\bullet(W)$. Unless otherwise
These maps are mentioned, the maps \( W \to W[m] \) and \( W[m] \to W \) are these inclusion and projection; these maps are \((\mathbb{C}^*)^3\)-equivariant.

The pair \((W, D)\) contains \((Y, p)\), \(p = p_1 + p_2 + p_3\), as its subpair. Accordingly, the pair \((W[m], D[m])\) contains a subpair \((Y[m], p[m])\) whose main part \(Y[m]\) is the preimage of \(Y\) under the contraction \(W[m] \to W\). The relative divisor \(p[m]\) is the intersection \(Y[m] \cap D[m]\). It is the embedding \(Y[m] \subset W[m]\) that induces the embedding \(\mathcal{M}^*(Y) \subset \mathcal{M}^*(W)\).

Now let \(u_0\) be a relative stable morphism in \(\mathcal{M}^*(Y)\), considered as an element in \(\mathcal{M}^*(W)\); let \(u_s\) be a small deformation of \(u_0\) in \(\mathcal{M}^*(W)^{T_\eta}\) that is not entirely contained in \(\mathcal{M}^*(Y)\). Each \(u_s\) is a morphism from its domain \(X_s\) to \(W[m]\) for some triple \(m\) depending possibly on \(s\). We let \(\tilde{u}_s : X_s \to W\) be the composite of \(u_s\) with the contraction \(W[m] \to W\). Then \(\tilde{u}_s\) form a flat family of morphisms; it specializes to \(\tilde{u}_0\) as \(s\) specializes to 0. Hence as sets, \(\tilde{u}_s(X_s)\) specializes to \(\tilde{u}_0(X_0)\) as \(s\) specializes to 0. Because \(\tilde{u}_s(X_s)\) are union of algebraic curves in \(W\) and \(\tilde{u}_0(X_0)\) is contained in \(C_1 \cup C_2 \cup C_3\), for general \(s\) the intersection \(\tilde{u}_s(X_s) \cap D\) is discrete. Hence every irreducible component \(Z \subset \tilde{u}_s^{-1}(D_i)\) must be mapped to a fiber of \(\Delta[m_i]/D_i\).

Now suppose there is such a connected component \(Z\) with \(u_s(Z)\) lies in the fiber of \(\Delta[m_i]\) over \(q \in D_i\), then by the pre-deformable requirement on relative stable morphisms forces the same \(q\) in \(D[m_i]\), to lie in \(u_s(X_s)\). Because of the requirement \(u_s(Z) \cap D[m_i] \subset L[m_i]\) we impose on the \(\mathcal{M}^*(W)\), we have

\[(5.8) \quad \tilde{u}_s(X_s) \cap D_i \subset L_i.\]

Our primary interest is to those \(u\) that are \(T_\eta\)-invariant and are small deformations of elements in \(\mathcal{M}^*(Y)\). This leads to the following definition.
Definition 5.4. We let \( \Xi(\eta) \) be the union of all connected components of
\[
\{ [u, X] \in \mathcal{M}^*(W)^T_\eta \mid \hat{u}(X) \cap D \text{ is finite} \}
\]
that intersect \( \mathcal{M}^*(Y) \) but are not entirely contained in it.

Following the discussion before Definition 5.4, all \( u \) in \( \Xi(\eta) \) satisfies (5.8). In case \( a_{i+1} \neq 0 \) (we agree \( a_4 = a_1 \)), the only \( T_\eta \)-fixed points of \( L_i \) are \( p_i \) and \( q_i \); hence all \( u \) in \( \Xi(\eta) \) satisfies a strengthened version to (5.8):
\[
(5.9) \quad \hat{u}_i(X) \cap D_i \subset p_i, \quad \text{when} \quad a_{i+1} \neq 0.
\]
Here \( q_i \) is ruled out because each connected component of \( \Xi(\eta) \) intersects \( \mathcal{M}^*(Y) \).

We now characterize elements in \( \Xi(\eta) \). We comment that we shall reserve \( a_1, a_2 \) and \( a_3 \) for the three components of \( \eta \); we always assume the three \( a_i \)'s are relatively prime and that \( a_1 + a_2 + a_3 = 0 \). In this and the next two Subsections, we shall workout the case \( a_1 > 0 \) and \( a_2, a_3 < 0 \); the case \( \eta = (1, -1, 0) \) will be considered in Subsection 5.5. Now let \( [u, X] \in \Xi(\eta) \) and let \( V \subset \hat{u}(X) \) be any irreducible component. Since \( u \) is \( T_\eta \)-invariant, \( V \) is \( T_\eta \)-invariant. Hence \( V \) must be the lift of the set
\[
V = \{ (c_1 t^{a_1}, c_2 t^{a_2}, c_3 t^{a_3}) \mid t \in \mathbb{C} \cup \{ \infty \} \} \subset (\mathbb{P}^1)^3
\]
for some \((c_1, c_2, c_3)\). Thus we see that the following cases are impossible:

1. all \( c_i \) are non-zero: should this hold, then \( V \cap \ell_2 = (0, \infty, \infty) \), which violates (5.8);
2. \( c_1 = 0 \) but the other two are non-zero: should this hold, then either \( V \cap D = V \cap D_1 = q_1 \), or \( V \cap D = V \cap D_2 = p_2' \) (see Figure 9 for the location of \( p_2' \)), so (5.9) is violated;
3. \( c_2 = 0 \) but the others are non-zero: should this hold, then \( V \cap D_1 = q_1 \) since \( a_1 > |a_3| \), which violates (5.9).

This leaves us with the only two possibilities: when only one of \( c_i \) is non-zero or \( a_3 = 0 \) but the other two are non-zero. In the first case we have \( V = C_i \) for some \( i \); in the later case \( V \) is the image of the map
\[
(5.10) \quad \phi_{k,c} : \mathbb{P}^1 \to W, \quad k \in \mathbb{Z}^+, \quad c \in \mathbb{C}^*
\]
that is the lift of \( \mathbb{P}^1 \to (\mathbb{P}^1)^3 \) defined by \( \xi \mapsto (\xi^{ka_1}, c^{-ka_2} \xi^{ka_2}, 0) \). Clearly, \( \phi_{k,c} \) is \( T_\eta \)-invariant. It is easy to see that these are the only \( T_\eta \)-equivariant maps \( Z \to W \) from irreducible \( Z \) whose images are not entirely lie in \( C_1 \cup C_2 \cup C_3 \) and the divisor \( D \). This proves

Lemma 5.5. Suppose \( a_1 > 0 \) and \( a_2, a_3 < 0 \). Then any \( (u, X) \in \Xi(\eta) \) not entirely contained in \( Y \) has at least one irreducible component \( Z \subset X \) and a pair \((k, c)\) so that \( u|_{Z} \cong \phi_{k,c} \).

Here by \( u|_{Z} \cong \phi_{k,c} \) we mean that there is an isomorphism \( Z \cong \mathbb{P}^1 \) so that under this isomorphism \( u|_{Z} \cong \phi_{k,c} \).

When \( c \) specialize to 0, the map \( \phi_{k,c} \) specializes to
\[
\phi_{k,0} : \mathbb{P}^1 \sqcup \mathbb{P}^1 \to W
\]
defined as follows. We endow the first copy (of \( \mathbb{P}^1 \sqcup \mathbb{P}^1 \)) with the coordinate \( \xi_1 \) and the second copy with \( \xi_2 \); we then form the nodal curve \( \mathbb{P}^1 \sqcup \mathbb{P}^1 \) by identifying 0 of the first \( \mathbb{P}^1 \) with 0 of the second \( \mathbb{P}^1 \); we define \( \phi_{k,0} \) to be the lift of the maps
\[
\xi_1 \mapsto (\xi_1^{ka_1}, 0, 0) \quad \text{and} \quad \xi_2 \mapsto (0, \xi_2^{-ka_2}, 0).
\]
Since $\xi_1 = 0$ and $\xi_2 = 0$ are both mapped to the origin in $(\mathbb{P}^1)^3$, they glue together to form a morphism $\phi_{k,0} : \mathbb{P}^1 \sqcup \mathbb{P}^1 \to W$.

This leads to the following definition.

**Definition 5.6.** A deformable part of a $(u, X) \in \Xi(\eta)$ consists of a curve $Z \subset X$ and an isomorphism $u|_Z \cong \phi_{k,c}$ for some $(k,c)$.

Suppose $(u, X)$ has at least two deformable parts, say $(Z_1, \phi_{k_1,c_1})$ and $(Z_2, \phi_{k_2,c_2})$, then the explicit expression of $\phi_{k,c}$ ensures that $Z_1$ and $Z_2$ share no common irreducible components. Should $Z_1 \cap Z_2 \neq \emptyset$, their intersection would be a nodal point of $X$ that could only be mapped to either $D_1$ or $D_2$ of $W$ under $u$. (Note that it could not be mapped to $p_0$ since then both $c_1$ and $c_2$ would be zero, and that node would be in more than two irreducible components of $X$.) However, the case where the node is mapped to $D_1$ or $D_2$ can also be ruled out because it violates the pre-deformable requirement of relative stable morphisms [24]. Hence $Z_1$ and $Z_2$ are disjoint. This way, we can talk about the maximal collection of deformable parts of $(u, X)$; let it be

$$(Z_1, \phi_{k_1,c_1}), \ldots, (Z_l, \phi_{k_l,c_l}).$$

For convenience, we order it so that $k_i$ is increasing.

**Definition 5.7.** We define the deformation type of $(u, X) \in \Xi(\eta)$ be

$$(k_i)_l = (k_1 \leq k_2 \leq \cdots \leq k_l)$$

It defines a function on $\Xi(\eta)$, called the deformation type function.

Let $(u, X)$ be an element in $\Xi(\eta)$ of type $(k_i)_l$. Intuitively, we should be able to deform $u$ within $\Xi(\eta)$ by varying $u|_Z$, using $\phi_{k,c}$ to generate an $A^l$-family in $\Xi(\eta)$. It is our next goal to make this precise.

To proceed, we need to show how to put $\phi_{k,t}$ into a family. We first blow up $\mathbb{P}^1 \times A^l$ at $(0,0)$ to form a family of curves $\mathfrak{Y}$ over $A^l$. The complement of the exceptional divisor $\mathfrak{Y} - E = \mathbb{P}^1 \times A^l - (0,0)$ comes with an induced coordinate $(\xi, t)$. We define

$$\Phi_k : \mathfrak{Y} \to W; \quad (\xi, t) \mapsto (\xi^{ka_1}, t^{-ka_2}, \xi^{ka_2}, 0).$$

We claim that $\Phi_k$ extends to a $\Phi_k : \mathfrak{Y} \to W$. Indeed, if we pick a local coordinate near $E$, which is $(\xi, v)$ with $t = \xi$, then

$$\Phi_k : \mathfrak{Y} \to W; \quad (\xi, v) \mapsto (\xi^{ka_1}, (\xi v)^{-ka_2} \xi^{ka_2}, 0) = (\xi, v) \mapsto (\xi^{ka_1}, v^{-ka_2}, 0),$$

which extends to a regular

$$\Phi_k : \mathfrak{Y} \to W.$$

Note that for $c \in A^l$, the fiber of $(\Phi_k, \mathfrak{Y})$ over $c$ is exactly the $\phi_{k,c}$ we defined earlier. Henceforth, we will call $(\Phi_k, \mathfrak{Y})$ the standard model of the family $\phi_{k,t}$; we will use $\mathfrak{Y}_c$ to denote the fiber of $\mathfrak{Y}$ over $c \in A^l$.

To deform $u$ using the family $\Phi_k$, we need to glue $\mathfrak{Y}$ onto the domain $X$. We let $D_1$ be the proper transform of $0 \times A^l \subset \mathbb{P}^1 \times A^l$, and let $D_2 = \infty \times A^l$ in $\mathfrak{Y}$. Both $D_1$ and $D_2$ are canonically isomorphic to $A^l$ via the second projection. For $Z \subset X$, we fix an isomorphism $Z \cong \mathfrak{Y}$, so that $u|_Z \cong \phi_{k,c}$; we specify $v_1, v_2 \in Z$ so that $u(v_1) \subset D_1$; we let $X_0$ be the closure of $X - Z$ in $X$.

We now glue $\mathfrak{Y}$ onto $X_0 \times A^l$. In case both $v_1$ and $v_2$ are nodes of $X$, we glue $\mathfrak{Y}$ onto $X_0 \times A^l$ by identifying $D_1$ with $v_1 \times A^l$ and $D_2$ with $v_2 \times A^l$, using their standard isomorphisms with $A^l$; in case $v_1$ is a marked point of $X$ and $v_2$ is a node, we glue $\mathfrak{Y}$ onto $X_0 \times A^l$ by identifying $D_2$ with $v_2 \times A^l$ and declaring $D_1$ to be the new marked points,
replacing $v_1$; in case $v_1$ is a node and $v_2$ is a marked points, we repeat the same procedure with the role of $v_1$ and $v_2$ and of $\mathcal{D}_1$ and $\mathcal{D}_2$ exchanged; finally in case both $v_1$ and $v_2$ are marked points, we simply replace $Z \times \mathbb{A}^1$ in $X \times \mathbb{A}^1$ by $\mathcal{Q}$ while declaring that $\mathcal{D}_1$ and $\mathcal{D}_2$ are the two marked points replacing $v_1$ and $v_2$. We let $\mathcal{X} \to \mathbb{A}^1$ be the resulting family.

The morphisms
\[ X_0 \times \mathbb{A}^1 \xrightarrow{pr} X_0 \xrightarrow{u|_{X_0}} W[m] \quad \text{and} \quad \Phi_k : \mathcal{Q} \longrightarrow W \]
glue together to form a morphism
\[ U : \mathcal{X} \longrightarrow W[m]. \]

The pair $(U, \mathcal{X})$ is the family in $\Xi(\eta)$ that keeps $u|_{X_0}$ fixed.

More generally, we can deform $u$ inside $\Xi(\eta)$ by identifying and altering its restriction to the deformable parts of $X$ simultaneously. This way, any $u \in \Xi(\eta)$ of type $(k_i)$ generates an $\mathbb{A}^1$ family of elements in $\Xi(\eta)$.

### 5.3. Global Structure of the Loci of Invariant Relative Morphisms

In this subsection, we shall prove that any connected component of $\Xi(\eta)$ is an $\mathbb{A}^1$-bundle.

We begin with a technical Lemma

**Lemma 5.8.** Let $v \in X_{\text{node}}$ be a node in $u^{-1}(D)$. Then $v$ remains a node when $u$ deforms infinitesimally in $\Xi(\eta)$.

The key to the proof relies on the fact that the restrictions of the automorphisms induced by $T_\eta$ on one irreducible component of $X$ that contains $v$ are infinite while on the other irreducible component are finite.

Recall that there is a natural inclusion $h : T \to \text{Aut}(W[m])$ induced by the $T$-actions on $W$ and on $N_{D_i}/W$. There is a unique homomorphism $h' : T \to \text{Aut}(W[m]/W)$ such that $h'(t) \circ h(t) \in \text{Aut}(W[m])$ acts trivially on $p_i|_{m_i}$, the fiber of $\Delta[m_i]$ over $p_i \in D_i$, for all $t \in T$. Since $u$ is $T_\eta$-invariant and the image $u(u^{-1}(\Delta[m_i]))$ is entirely contained in $p_i|_{m_i}$, there are group homomorphisms $h_1 : T_\eta \to \text{Aut}(X)$ and $h_2 : T_\eta \to \text{Aut}(W[m]/W)$ so that for all $\sigma \in T_\eta$,

\[ (5.11) \quad (i) \quad h'(\sigma) \circ h(\sigma) \circ u = h_2(\sigma) \circ u \circ h_1(\sigma) \quad \text{and} \quad (ii) \quad h_2(\sigma) \text{ acts on } p_i|_{m_i} \text{ trivially}. \]

Now let $v \in u^{-1}(D_i)$ be a node of $X$ that is mapped to $D_i$ under $u$; let $V_-$ be the irreducible component of $X$ that contains $v$ that is mapped to $W$ and let $V_+$ be the other irreducible component of $X$ that contains $v$. Then $u(V_+)$ must be contained in $\Delta[m_i]$. Since $h_1(\text{id}) = \text{id}$ and that $T_\eta$ is connected, $h_1(\sigma)(V_+) \subset V_+$. Hence $h_1(\sigma)$ are automorphisms of $V_+$ that fixed $v$. We let

\[ T_{\eta}|_{V_+} \overset{\text{def}}{=} \{ h_1(\sigma) \mid \sigma \in T_\eta \} \subset \text{Aut}(V_+, v); \]

it is a group which is a homomorphism image of $T_\eta \cong U(1)$, so it is either $U(1)$ or trivial.

**Lemma 5.9.** The group $T_{\eta}|_{V_-}$ is infinite while the group $T_{\eta}|_{V_+}$ is finite. So $T_{\eta}|_{V_-} \cong U(1)$ and $T_{\eta}|_{V_+}$ is trivial.

**Proof.** The group $T_{\eta}|_{V_-}$ is infinite is obvious. Since $u(v) = p_i$ and $u(V_-) \subset W$, $u(V_-)$ is $T_\eta$ invariant but not $T_{\eta}$ fixed. Since the induced action on $u(V_-)$ is infinite, $T_{\eta}|_{V_-}$ must be infinite because $u$ is $T_{\eta}$-invariant.

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3The automorphisms $\zeta \in \text{Aut}(W[m])$ that preserve the fibers of the map $W[m] \to W$ are called relative automorphisms of $W[m]/W$; the group of all such automorphisms is denoted by $\text{Aut}(W[m]/W)$. If $m = (m_1, m_2, m_3)$ then $\text{Aut}(W[m]/W) \cong (\mathbb{C}^*)^{m_1+m_2+m_3}$. 

Note that \( u(V_+) \) must be contained in \( p_1[m_i] \) for some \( i \). By (5.11),
\[
u|_{V_+} = u \circ h_1(\sigma)|_{V_+} : V_+ \to p_1[m_i] \quad \forall \sigma \in T_0.
\]
By stability of the relative morphism \( u \), \( T_0|_{V_+} \) is finite. \( \square \)

We now prove Lemma 5.8.

**Proof of Lemma 5.8.** Suppose \( v \) can be smoothed at least of first order within \( \Xi(\eta) \). Since \( \Xi(\eta) \) is fixed by \( T_0 \), \( T_0|_{V_+} \) is infinite forces \( T_0|_{V_+} \) to be infinite as well. This violates the assumption that \( T_0|_{V_+} \) is finite. This proves the Lemma. \( \square \)

As we argued before, each \( u \in \Xi(\eta) \) contains a deformable part that is the union of some \( \phi_{k_i, c_i} \). Our next task is to show the deformable part of \( u \) remain the same within a connected component of \( \Xi(\eta) \).

We make it more precise now. We let \( (u, X) \) be any element in \( \Xi(\eta) \); let \( Y_1, \cdots, Y_l \subset X \) be all its deformable parts so that \( u|_{Y_i} \equiv \phi_{k_i, c_i} \); we let \( v_{i,1} \) and \( v_{i,2} \in Y_i \) be the marked points so that \( u(v_{i,j}) \) be all its deformable parts so that \( u(v_{i,j}) \equiv \phi_{k_i, c_i} \). Then according to the discussion in the previous Subsection, by varying \( u|_{Y_i} \) using \( \Phi_{k_i} \) we get a copy \( h^k \) in \( \Xi(\eta) \); together they provide a copy \( h^k \) in \( \Xi(\eta) \). This is one of the fiber of the fiber bundle structure on \( \Xi(\eta) \) we are about to construct.

To extend this \( h^k \subset \Xi(\eta) \) to nearby elements of \( [u] \), we need to extend all \( Y_i \) in \( X \) to a flat family of subcurves.

**Lemma 5.10. The deformation type function on \( \Xi(\eta) \) is locally constant.**

**Proof.** We pick an analytic disk \( 0 \in S \) and an analytic map \( \psi : S \to \mathcal{M}^*(W)_{\text{def}} \) so that \( \psi(0) = [u] \). The morphism \( \psi \) pulls back the tautological family on \( \Xi(\eta) \) to a family \( \mathcal{U} : \mathcal{X} \to \mathcal{Z} \) over \( S \). The central fiber \( \mathcal{X}_0 \) is \( X \) and thus contains \( Y_i \). We let \( \mathcal{N} \subset \mathcal{X} \) be the subscheme of all fibers of \( \mathcal{X}/S \). Since \( v_{i,j} \) is either a marked point or a node of \( X \), \( v_{i,j} \in \mathcal{N} \cup \mathcal{R} \). Let \( \mathcal{P}_{i,j} \) be the connected component of \( \mathcal{N} \cup \mathcal{R} \) that contains \( v_{i,j} \). We claim that \( \mathcal{P}_{i,j} \) is a section of \( \mathcal{N} \cup \mathcal{R} \to S \). First, \( \mathcal{P}_{i,j} \) is flat over \( S \) at \( v_{i,j} \). This is true in case \( v_{i,j} \) is a marked point since \( \mathcal{R} \) is flat over \( S \) by definition; in case \( v_{i,j} \) is a node it is true because of Lemma 5.8. Therefore, \( \mathcal{P}_{i,j} \) dominates over \( S \). Then because \( \mathcal{N} \cup \mathcal{R} \) is proper and unramified over \( S \), dominating over \( S \) guarantees that \( \mathcal{P}_{i,j} \) finite etale over \( S \). But then since \( S \) is a disk, \( \mathcal{P}_{i,j} \) must be isomorphic to \( S \) via the projection.

We now pick the desired family of curves \( Y_i \). In case \( \mathcal{P}_{i,j} \) is one of the section of the marked points of \( \mathcal{X}/S \), we do nothing; otherwise, we resolve the singularity of the fibers of \( \mathcal{X} \) along \( \mathcal{P}_{i,j} \). As a result, we obtain a flat family of subcurves \( Y_i \subset X \) that contains \( Y_i \) as its central fiber. We let \( \mathcal{U}_i : Y_i \to W \) be the restriction of \( \mathcal{U} \) to \( Y_i \). Because \( \mathcal{U}_i(Y_i) \subset W \subset W[\mathfrak{m}] \), \( \mathcal{U}_i(Y_i) \subset W \times S \subset W \) as well.

Since \( \mathcal{U} : \mathcal{X} \to W \) is a family of \( T_0 \)-equivariant relative stable maps, \( \mathcal{U}_i : Y_i \to W \) is also a family of \( T_0 \)-equivariant stable morphisms. Then because \( \mathcal{U}_i|_{Y_i} \) is isomorphic to \( \phi_{k_i, c_i} \), each member of \( \mathcal{U}_i \) must be an \( \phi_{k_i, c} \) for some \( c \in \mathbb{C} \). This proves that the deformation type of \( \mathcal{U}_i \) contains that of \( \mathcal{U} \) as a subset. Because this holds true with \( 0 \) and \( s \) exchanged, it shows that the deformation type function stay constant over \( S \).

Finally, because any two elements in the same connected component of \( \Xi(\eta) \) can be connected by a chain of analytic disks, the deformation type function does take same values on such component. This proves the lemma. \( \square \)

We are now ready to exhibit a fiber bundle structure of any connected component of \( \Xi(\eta) \). Let \( Q \subset \Xi(\eta) \) be any connected component. According to the previous subsection, all elements in \( Q \) are of the same deformation types, say \( (k_i)_i \). In case \( Q \) is not entirely
contained in $\mathcal{M}^{\bullet}(Y)$, then necessarily $l > 0$. To get the fiber structure, we need to take a finite cover of $Q$, which we now construct.

**Definition 5.11.** We define the groupoid $\hat{Q}$ over $Q$ as follows. For any scheme $S$ over $Q$, we let $\hat{Q}(S)$ be the collection of data $\{(U, X, W), \rho_i, Z_i, \pi_i | i = 1 \cdots l\}$ of which

1. $U: X \to W$ is an object in $Q(S)$
2. $\rho_i$ are morphisms from $S$ to $A^1_i$, $A^1_i \cong \mathbb{A}^1$
3. $Z_i$ are flat families of subcurves in $X$ over $S$ with all marked points discarded
4. $\pi_i : Z_i \to \rho_i^* Q_k$ an isomorphism over $S$

Together they satisfy

$$U|_{Z_i} \equiv \rho_i^* \Phi_k \circ \pi_i : Z_i \to W.$$ Further, an arrow from $\{(U, X, W), \rho_i, Z_i, \pi_i\}$ to $\{(U', X', W'), \rho'_i, Z'_i, \pi'_i\}$ consists of an isomorphism $h_1 : X \to X'$ and an isomorphism $h_2 : W \to W'$ relative to $W$ so that under these isomorphisms $Z_i = Z'_i$, $\rho_i = \rho'_i$, $\pi_i = \pi'_i$ (for all $i$) and $U = U'$.

Here we use $A^1_i$ to denote the target of $\rho_i$, which is $\mathbb{A}^1_i$. We are doing this to distinguish them for different $i$.

For a fixed type $(k_i)_i$, let $G_{(k_i)}$ be the group $\{\sigma \in S_l | k_{\sigma(i)} = k_i\}$.

**Proposition 5.12.** The groupoid $\hat{Q}$ is a DM-stack acted on by $G_{(k_i)}$; it is finite and étale over $Q$, and $Q/G_{(k_i)} = \hat{Q}$. The morphisms $\rho_i$ in each object in $\hat{Q}$ glue to a morphism $\hat{\rho}_i : \hat{Q} \to A^1_i$. Let $\bar{Q}_0 = (\hat{\rho}_1, \cdots, \hat{\rho}_l)^{-1}(0)$. Then there is a canonical projection $\pi : \bar{Q} \to \bar{Q}_0$ making it an $A^1$-bundle over $\bar{Q}_0$. Finally, the morphism

$$\pi_i(\hat{\rho}_1, \cdots, \hat{\rho}_l) : \hat{Q} \to \bar{Q}_0 \times A^1_i$$

is an isomorphism of DM-stacks.

**Proof.** The proof is straightforward, following the previous discussion, and will be omitted. $\square$

**5.4. The Obstruction Sheaves.** In this subsection, we will investigate the obstruction sheaf to deforming a $[u]$ in $\Xi(\eta)$ for the case $a_2, a_3 < 0$; we will follow the convention introduced in Subsection 5.3.

We let $S \to Q$ be an $T$-equivariant étale neighborhood, we let

$$U : X \to W, \quad R \subset X, \quad D \subset W \quad \text{and} \quad Z_i \subset X$$

be the tautological family of $S$ of $\Xi(\eta)$. Here $W$ is an $S$-family of $W[m]$ of possibly varying $m$, $D$ is the relative divisor of $W$, $R \subset X$ is the sections marked points and $Z_i \subset X$ is the $i$-th deformable parts of $U$.

Let $T^2$ be the obstruction sheaf over $S$ of the obstruction theory of $\Xi(\eta)$. According to [25], its $T_n$-invariant part, indicated by the subscript $(\cdot)_T_n$, fits into the long exact sequences

\begin{align*}
&\rightarrow \mathcal{E}xt^1_{/S}(\Omega_{X/S}(R), O_X)_{T_n} \underset{\beta}{\to} A^1_{T_n} \underset{\delta}{\to} \hat{T}^2_{T_n} \rightarrow 0; \quad (5.12) \\
&\rightarrow B^0_{T_n} \underset{\alpha}{\to} R^1 \pi_* (U^* \Omega_{W^1/S}(\log D)^\vee)_{T_n} \rightarrow A^1_{T_n} \rightarrow B^1_{T_n} \rightarrow 0 \quad (5.13)
\end{align*}

and

\begin{align*}
&\rightarrow \hat{T}^1_{T_n} \rightarrow \mathcal{H}_{T_n} \rightarrow \hat{T}^2_{T_n} \rightarrow \hat{T}^2_{T_n} \rightarrow 0. \quad (5.14)
\end{align*}

\[\text{Here we consider } Q \text{ as a groupoid and } Q(S) \text{ is the collection of objects over } S.\]
Within these sequences, \( B^j_i = \bigoplus_{j=1}^{3} B^j_i \); each summand \( B^j_i \) is a sheaf that associates to the smoothing of the nodes of the fibers of \( X \) that are mapped under \( \mathcal{U} \) to \( D (\subset W) \) and the singular loci of \( \Delta [m_i] \); the \( \mathcal{W} \) is the scheme \( \mathcal{W} \) with the log structure defined in [25] and \( \Omega_{\mathcal{W}} \) is the sheaf of log differentials. In our case, \( \mathcal{U}^* \Omega_{\mathcal{W}/S} (D) = \mathcal{U}^* \Omega_W (log D) \), where \( \mathcal{U} : X \to W \) is the obvious induced morphism.

Without taking the \( T_n \) invariant part, the top two exact sequences define the obstruction sheaf \( \mathcal{T}^2 \) to deforming \([u]\) in \( \mathcal{M}_{\chi, \eta, \rho}(W^{rel}) \) — the moduli of relative stable morphisms to \( W^{rel} \) without requiring \( u(R) \subset L \). Taking the invariant part and adding the last exact sequence defines the obstruction sheaf \( \mathcal{T}^2_{\eta} \) of \( \Xi(\eta) \). The sheaf \( \mathcal{H} \) is the pull back of the normal line bundle to \( L \subset \mathcal{D} \). For the \( \eta \) we are interested, \( \mathcal{H}_{\eta} = 0 \); hence the last exact sequence reduces to \( \mathcal{T}^2_{\eta} \equiv \mathcal{T}^2_{\eta} \).

In the following we shall show that the \( \ell \) families \( \mathcal{Z}_{\ell} \subset \mathcal{X} \) of deformable parts of \( (\mathcal{U}, \mathcal{X}) \) each contributes to a weight zero trivial quotient sheaf of \( \mathcal{T}^2_{\eta} \).

We begin with the sheaf

\[
\pi_{k*} (\Phi^* \Omega_W (log D))^v
\]

where \( \Phi_k : \mathbf{F} \to W \) is the family constructed before and \( \pi_k : \mathbf{F} \to \mathbb{A}^1 \) is the projection. We let \( D_{12} \text{ (resp. } D_{31}) \) be the \( T \)-invariant divisor of \( W \) that contains \( C_1 \) and \( C_2 \) (resp. \( C_1 \) and \( C_3 \)); it is also the proper transform of the product of the first and second (resp. the first and third) copies of \( \mathbb{P}^1 \) in \( (\mathbb{P}^1)^3 \). We let \( \pi_{12} : W \to D_{12}, \pi_{31} : W \to D_{31} \) and \( \pi_1 : W \to C_1 \) be the obvious projections. We claim that \( \Omega_W (log D)|_{D_{12}} \) has a direct summand \( \pi_{12*} \mathcal{N}^\vee_{C_1/D_{31}} \).

Indeed, via the projection \( \pi_{31} \) we have homomorphism

\[
\pi_{31*} \mathcal{N}^\vee_{C_1/D_{31}} |_{D_{12}} \to \Omega_W |_{D_{12}} \to \Omega_W (log D)|_{D_{12}}.
\]

Also via the projection \( \pi_{12} \) we have homomorphism

\[
\pi_{12*} \Omega_{D_{12}} (log E_{12})|_{D_{12}} \to \Omega_W (log D)|_{D_{12}}, \quad E_{12} \equiv D_{12} \cap W.
\]

Combined, we have

\[
\pi_{31*} \mathcal{N}^\vee_{C_1/D_{31}} |_{D_{12}} \oplus \pi_{12*} \Omega_{D_{12}} (log E_{12})|_{D_{12}} \to \Omega_W (log D)|_{D_{12}},
\]

which can easily be shown an isomorphism. This proves that \( \Omega_W (log D)|_{D_{12}} \) has a direct summand \( \pi_{12*} \mathcal{N}^\vee_{C_1/D_{31}} \). Consequently, \( \Phi_k^* \Omega_W (log D)^v \) has a direct summand \( \Phi_k^* (\pi_{12*} \mathcal{N}^\vee_{C_1/D_{31}}) \).

Because of our choice, the weight of \( d_{2i} \) is \( a_i \); the weight of \( T_0 \) at \( 0 \) is \( 1/k \), and the weight of \( \Phi_k^* (\pi_{12*} \mathcal{N}^\vee_{C_1/D_{31}}) \) at \( 0 \times \mathbb{A}^1 \subset \mathbf{F} \) is \(-a_3 \). Hence, the sheaf (5.15) splits to line bundles of weights

\[
-a_3 - a_1 + \frac{1}{k}, \quad -a_3 - a_1 + \frac{2}{k}, \cdots, \quad -a_3 - \frac{1}{k}.
\]

Since all \( a_i \) are integers, and \( a_3 \leq -1 \) and \( -a_3 - a_1 = a_2 \leq -1 \), within the above list there is exactly one that is zero. Hence

\[
R^1 \pi_{k*} (\Phi_k^* \Omega_W (log D)^v) \equiv \mathcal{O}_{\mathbb{A}^1}.
\]

We now let \( \rho_i : S \to \mathbb{A}^1 \) be so that \( \mathcal{U} |_{Z_i} \equiv \rho_i^* \Phi_k_i \). Since \( Z_i \subset \mathcal{X} \) is a flat family of subcurves,

\[
R^1 \pi_* (U^* \Omega_{W}^{rel}|_S (log D)^v) |_{Z_i} \equiv \mathcal{O}_{\mathbb{A}^1}.
\]
is surjective; but the last term is isomorphic to the pull back $\rho^*_i$ of (5.16); hence we obtain a quotient sheaf
\begin{equation}
\varphi_i : R^1\pi_* (U^*\Omega_{W^i/S}(\log D)^\vee)_{T^n} \rightarrow \rho^*_i \mathcal{O}_{\widehat{X}_i}.
\end{equation}

**Lemma 5.13.** The homomorphism $\varphi_i$ canonically lifts to surjective
\begin{equation}
\tilde{\varphi}_i : T^{i^2}_{T^n} \rightarrow \rho^*_i \mathcal{O}_{\widehat{X}_i}.
\end{equation}

The default proof is to follow the construction of the sheaves and the exact sequences in (5.12-5.14); once it is done, the required vanishing will follow immediately. However, to follow this strategy, we need to set up the notation as in [25] that itself requires a lot of efforts. Instead, we will utilize the decomposition of $S$ to give a more conceptual argument; bypassing some straightforward but tedious checking.

We first decompose $U$ into four subfamilies. Since $W/S$ is a family of expanded pairs of $(W, D)$, $W[0] = W \times S$ is a closed subscheme of $W$. We then let $X[0] = U^{-1}(W[0])$. Because of Lemma 5.8,
\begin{equation}
U[0] = U |_{X[0]} : X[0] \rightarrow W[0]
\end{equation}
is an $S$-family of relative stable morphisms relative to $D[0] = D \times S \subset W[0]$. Next we consider the composite
\begin{equation}
\tilde{U} : X \rightarrow W \rightarrow W[0]
\end{equation}
and the preimage $\tilde{U}^{-1}(D_i)$. Because of the same reason, either this preimage is a flat family of nodes over $S$ or is a flat family of curves over $S$. In the former case we agree $X[i] = \emptyset$ and in the later case we define
\begin{equation}
X[i] = \tilde{U}^{-1}(D_i \times S), \quad U[i] = U |_{X[i]} : X[i] \rightarrow W[i],
\end{equation}
where the last term $W[i]$ is the $S$-family of $\Delta [m_i]$’s that were attached to $W[0]$ along $D_i \times S$ to form $W$.

Since $U[0] : X[0] \rightarrow W[0]$ is a family of $T_\eta$-equivariant relative stable morphisms, and since $U[i]$ is a family of $T_\eta$-equivariant relative stable morphisms to $\Delta$ relative to $D_-$ and $D_+$, modulo an additional equivalence induced by the $C^*$ action on $\Delta$, the obstruction sheaves $T[i, 2]$ over $S$ to deforming $U[i]$ as $T_\eta$-equivariant maps fit into similar exact sequences
\begin{align}
&\rightarrow \mathcal{E}xt^1_{\mathcal{O}_{X[i]}/S}(\Omega_{X[i]/S}(\mathcal{R}[i]), \mathcal{O}_{X[i]})_{T^n} \rightarrow \mathcal{A}[i, 1]_{T^n} \rightarrow T^{i, 2}_{T^n} \rightarrow 0 \\
&\rightarrow \mathcal{B}[i, 0]_{T^n} \rightarrow R^1\pi_* (U[i]^*\Omega_{W[i]/S}(\log D)^\vee)_{T^n} \rightarrow \mathcal{A}[i, 1]_{T^n} \rightarrow \mathcal{B}[i, 1]_{T^n} \rightarrow 0.
\end{align}
Here we have already used the observation that $T^{i, 2}_{T^n} = T^{i, 2}_{T^n}$.

Now let $N_\text{sp} \subset U^{-1}(D_i \times S)$ be any section of nodes of $X$ that separates $X[0]$ and $X[i]$. By Lemma 5.9, the induced $T_\eta$-automorphisms on the connected component of $X[0]$ adjacent to $N_\text{sp}$ is infinite and on $X[i]$ is finite. Therefore the $T_\eta$-invariant parts
\begin{equation}
\mathcal{E}xt^1_{\mathcal{O}_{X}(\mathcal{R}), \mathcal{O}_{X}}_{T^n} = \bigoplus_{i=0}^3 \mathcal{E}xt^1_{\mathcal{O}_{X[i]}/S}(\Omega_{X[i]/S}(\mathcal{R}[i]), \mathcal{O}_{X[i]})_{T^n}.
\end{equation}
For the similar reason, because the tangent bundle $T_pW$ has no weight 0 non-trivial $T_\eta$-invariant subspaces,
\begin{equation}
R^1\pi_* (U^*\Omega_{W^i/S}(\log D)^\vee)_{T^n} = \bigoplus_{i=0}^3 R^1\pi_* (U[i]^*\Omega_{W[i]/S}(\log D[i])^\vee)_{T^n}.
\end{equation}
where \( D[i] \) is the relative divisor of \( \mathcal{W}[i] \). Further, if we follow the definition of the sheaves \( B^i \) and \( A^i \), we can prove that

\[
(5.22) \quad \bigoplus_{i=0}^{3} A^i_{\mathcal{T}_n} = A^j_{\mathcal{T}_n} \quad \text{and} \quad \bigoplus_{i=0}^{3} B^i_{\mathcal{T}_n} = B^j_{\mathcal{T}_n} ;
\]

that under these isomorphisms,

\[
(5.23) \quad \bigoplus_{i=0}^{3} \alpha[i] = \alpha, \quad \bigoplus_{i=0}^{3} \beta[i] = \beta \quad \text{and} \quad \bigoplus_{i=0}^{3} \delta[i] = \delta;
\]

and

\[
(5.24) \quad \bigoplus_{i=0}^{3} \mathcal{T}^i_{\mathcal{T}_n} = \mathcal{T}^2_{\mathcal{T}_n}.
\]

Finally, the exact sequences (5.12) and (5.13) become the direct sums of the exact sequences (5.19) and (5.20).

Now we come back to the weight zero quotient \( \varphi_i \) in (5.17). By its construction, \( \varphi_i \) is merely the canonical quotient homomorphism

\[
(5.25) \quad R^1\pi_* (\mathcal{U}[0]^* \Omega_{\mathcal{W}[0]/S}(\log D[0]))^! \llap{\longrightarrow} \rightarrow R^1\pi_* (\mathcal{U}[0]^* \Omega_{\mathcal{W}[0]/S}(\log D[0])^!|_{\mathcal{Z}_i})_{\mathcal{T}_n} = \rho_i^* O_{\mathcal{A}^i};
\]

under the isomorphism (5.21). Because of (5.24), to lift \( \varphi_i \) to \( \hat{\varphi}_i \), we only need to lift (5.25) to \( \mathcal{T}^0_{\mathcal{T}_n} \rightarrow \rho_i^* O_{\mathcal{A}^i} \).

For this, we need to look at the exact sequence (5.20) for \( \mathcal{X}[0] \). Since \( \mathcal{U}[0] \) is a relative stable map to \( (W, D) \) — namely no \( \Delta \)'s have been attached to \( W \) — the sheaf \( B^0[j] = 0 \). Therefore the sequence (5.20) reduces to \( \alpha[0] = \text{id} \). On the other hand, \( \mathcal{T}^0_{\mathcal{T}_n} \) is the obstruction sheaf on \( S \) to deformations of \( \mathcal{U}[0] \). Since \( \mathcal{Z}_i \) is a family of connected components of \( \mathcal{X}[0]/S \), the exact sequence (5.19) decomposes into direct sum of individual exact sequences that contains

\[
(5.26) \quad \mathcal{E}xt^1_{\mathcal{Z}_i} (\Omega_{\mathcal{Z}_i}/S (\mathcal{R}[0]), O_{\mathcal{Z}_i})_{\mathcal{T}_n} \llap{\longrightarrow} \rightarrow R^1\pi_* (\mathcal{U}[0]^* \Omega_{\mathcal{W}[1]/S}(\log D[0])^!|_{\mathcal{Z}_i})_{\mathcal{T}_n} \llap{\longrightarrow} \rightarrow \mathcal{T}^1_{\mathcal{T}_n} \llap{\longrightarrow} \rightarrow 0,
\]

as its factors.

For \( \mathcal{Z}_i \), since it is smooth, it has expected dimension zero and has actual dimension one, the obstruction sheaf \( \mathcal{T}^1_{\mathcal{T}_n} \) must be a rank one locally free sheaf on \( S \). Then because the middle term in (5.26) is \( \rho_i^* O_{\mathcal{A}^i} \), which is a rank one locally free sheaf, the arrow \( \delta[\mathcal{Z}_i] \) must be an isomorphism while \( \beta[\mathcal{Z}_i] = 0 \). Hence \( \varphi_i \) lifts to

\[
\mathcal{T}^0_{\mathcal{T}_n} \llap{\longrightarrow} \rightarrow \mathcal{T}^{[\mathcal{Z}_i]}_{\mathcal{T}_n} \llap{\longrightarrow} \rightarrow \mathcal{T}^{[\mathcal{Z}_i]}_{\mathcal{T}_n} \llap{\longrightarrow} \rightarrow \rho_i^* O_{\mathcal{A}^i},
\]

and lifts to \( \hat{\varphi}_i : \mathcal{T}^2_{\mathcal{T}_n} \rightarrow \rho^*_i O_{\mathcal{A}^i} \), thanks to (5.24).

5.5. **The Case for** \( \eta = (1, -1, 0) \). We now investigate the structures of maps \( [u] \in \Xi(\eta) \) in case \( \eta = (1, -1, 0) \). Let \((u, X)\) be any such map, let \( R \) be the marked points and let \( \tilde{u} \) be the contraction \( X \rightarrow W \). Because \( a_3 = 0 \), \( \tilde{u}(X) \) intersects \( D_1 \) at \( p_1 \); intersects \( D_3 \) at \( p_3 \) while intersects \( D_2 \) can be any point in \( L_2 \). Thus being \( T_{\eta} \)-equivariant forces \( \tilde{u}(X) \) to be a finite union of a subset of \( C_1, C_2, C_3 \) and the lifts of the sets \( \{ z_1 z_2 = c, z_3 = 0 \} \subset (\mathbb{P}^1)^3 \).

In case all irreducible components are mapped to \( \cup C_i \) under \( \tilde{u} \), \( [u] \in \mathcal{M}^*(Y) \). For those that are not in \( \mathcal{M}^*(Y) \), there bound to be some \( Y \subset X \) so that \( \hat{u}(Y) \) is the lifts of
\{z_1z_2 = c, z_3 = 0\}. Such \(u|_Y\) are realized by the morphism \(\phi_{k,c} : \mathbb{P}^1 \to W\) that are the lifts of
\[
(5.27) \quad \xi \mapsto (\xi^k, \xi^{-k}, 0) \in (\mathbb{P}^1)^3.
\]

When \(c\) specializes to 0, the map \(\phi_{k,c}\) specializes to \(\phi_{k,0} : \mathbb{P}^1 \sqcup \mathbb{P}^1 \to W\) that is the lift of \(\xi_1 \mapsto (\xi_1^k, 0, 0)\) and \(\xi_2 \mapsto (0, \xi_2^{-k}, 0)\). Indeed, there is a family \(\mathcal{Y} \to \mathbb{A}^1\) and a morphism \(\Phi_k : \mathcal{Y} \to W\) so that its fiber over \(c \in \mathbb{A}^1\) is the \(\phi_{k,c}\) defined; also this is a complete list \(T_r\)-equivariant deformations of \(\phi_{k,c}\). Since the argument is exactly the same as in the case studied, we shall not repeat it here.

Here comes the main difference between this and the case studied earlier. In the previous case, \(\text{Im} \phi_{k,c} \cap D_1 = p_i\) for both \(i = 1\) and 2; hence we can deform each \(u|_Y \cong \phi_{k,c}\) to produce an \(\mathbb{A}^1\) family in \(\Xi(\eta)\). In the case under consideration, though \(\text{Im} \phi_{k,c} \cap D_1 = p_1\), if we fix an embedding \(\mathbb{A}^1 \subset L_2\) so that 0 is \(\mathbb{A}^1\) is the \(p_2\), then \(\text{Im} \phi_{k,c} \cap D_1 = \epsilon^k \subset L_2\).

In other words, if we deform \(u|_X \cong \phi_{k,c}\), we need to move the connected component of \(X^{[2]}\) that is connected to \(Y\).

This leads to the following definition.

**Definition 5.14.** We say that a connected component \(Y \subset X^{[0]}\) is subordinated to a connected component \(E \subset X^{[2]}\) if \(Y \cap E \neq \emptyset\); we say a connected component \(E \subset X^{[2]}\) is deformable if every connected component of \(X^{[0]}\) that is subordinate to \(E\) is of the form \(\phi_{k,c}\) for some pair \((k,c)\). We say \(u\) has deformation type \((k_1)\) if it has exactly \(l\) deformable connected components \(\phi_{k_1,c_1}, \ldots, \phi_{k_l,c_l}\) in \(X^{[2]}\).

The deformation types define a function on \(\Xi(\eta)\).

**Lemma 5.15.** The deformation type function is locally constant on \(\Xi(\eta)\).

**Proof.** The proof is parallel to the case studied, and will be omitted. \(\square\)

As in the previous case, any \(u|_I \in \Xi(\eta)\) of deformation type \((k_1)\) generates an \(\mathbb{A}^1\) in \(\Xi(\eta)\) so that its origin lies in \(\mathcal{M}^*(Y)\). Let \(E_1, \ldots, E_i \subset X^{[2]}\) be the complete set of deformable parts of \(u\); let \(Y_{i,j}, j = 1, \ldots, u_i\) be the complete set of connected components in \(X^{[0]}\) that are subordinate to \(E_i\). By definition, each \(u|_{Y_{i,j}} \cong \phi_{k_{i,j},c_{i,j}}\). To deform \(u\), we shall vary \(c_{i,j}\) in each \(\phi_{k_{i,j},c_{i,j}}\) and move \(E_i\) accordingly to get a new map.

In accordance, we shall divide \(X\) into three parts. We let \(X_0\) be the union of irreducible components of \(X\) other than the \(E_i\)'s and \(Y_{i,j}\)'s. The variation of \(u\) will remain unchanged over this part of the curve. The second part is the moving part \(E_i\)'s. Recall that each \(u|_{E_i}\) is a morphism to \(\Delta[m_2]\). Suppose it maps to the fiber \(\Delta[m_2]_c\) of \(\Delta[m_2]\) over \(c \in L_2 \subset D_2\). To deform \(u\), we need to make the new map maps \(E_i\) to \(\Delta[m_2]_{c'}\). Since the total space of \(\Delta[m_2]\) over \(L_2\) is a trivial \(\mathbb{P}^1[m_2]\) bundle, there is a canonical way to do this. We let

\[
\varphi_{c,c'} : \Delta[m_2]_c \cong \Delta[m_2]_{c'}
\]

be the isomorphism of the two fibers of \(\Delta[m_2]\) over \(c\) and \(c' \in L_2\) induced by the projection \(\Delta[m_2] \to \mathbb{P}^1[m_2]\) that is induced by the product structure on \(\Delta[m_2]\) over \(L_2\). The third parts are those \(Y_{i,j}\) that are subordinate to \(E_i\).

We now deform the map \(u\) using the parameter space \(\mathbb{A}^l\). We let \(K_i\) be the least common multiple of \((k_{i,1}, \ldots, k_{i,n_i})\); we let \(c_{i,j} = K_i/k_{i,j}\). Since \(Y_{i,j}\) and \(Y_{i,j'}\) are connected to the same connected component \(E_i \subset X^{[2]}\), \(c_{i,j} = c_{i,j'}\) and we let it be \(c_i\). For \(t = (t_1, \ldots, t_i) \in \mathbb{A}^l\), we define

\[
(5.28) \quad u^t|_{X_0} = u|_{X_0}, \quad u^t|_{E_i} = \varphi_{c_i,t_i^k} \circ u|_{E_i} \quad \text{and} \quad u^t|_{Y_{i,j}} = \phi_{k_{i,j},t_i^{k_{i,j}}}.
\]
Here in case $Y_{i,j} \cong \mathbb{P}^1$, which is the case when $c_{i,j} \neq 0$, by $u^k|_{Y_{i,j}} = \phi_{k_{i,j},0}$ we mean that we will replace $Y_{i,j}$ by $\mathbb{P}^1 \sqcup \mathbb{P}^1$ with necessarily gluing if required; and vice versa.

The $\mathbb{A}^l$ family $u^k$ is a family of $T_\eta$-equivariant relative stable morphisms in $\Xi(\eta)$; the map $u^0$ associated to $0 \in \mathbb{A}^l$ lies in $\mathcal{M}^\bullet(Y)$; the induced morphism $\mathbb{A}^l \to \Xi(\eta)$ is an embedding up to a finite quotient.

By extending this to any connected component $Q$ of $\Xi(\eta)$, we obtain

**Proposition 5.16.** Let $Q$ be any connected component of $\Xi(\eta)$ that is not entirely contained in $\mathcal{M}^\bullet(Y)$. Suppose elements of $Q$ has deformation type $(k_l)_l$. Then there is a stack $\bar{Q}$, a finite quotient morphism $\bar{Q}/G_{(k_l)} \to Q$, a closed substack $Q_0 \subset \bar{Q}$, $l$ projections $\rho_l : Q \to \mathbb{A}^l$ and a projection $\pi : \bar{Q} \to Q_0$ so that

$$ \left( \pi, (\rho_1, \ldots, \rho_l) \right) : \bar{Q} \xrightarrow{\cong} \bar{Q}_0 \times \mathbb{A}^l $$

is an isomorphism. Further, given a $[u] \in Q$, the fiber $\mathbb{A}^l$ in $\bar{Q}$ that contains a lift of $[u] \in \bar{Q}$ is the $\mathbb{A}^l$ family $\{u_t \mid t \in \mathbb{A}^l\}$; its intersection with the zero section $\bar{Q}_0$ is $u^0$.

Finally, the intersection $\bar{Q} \cap \mathcal{M}^\bullet(Y)$ is the image of $\bar{Q}_0$.

**Proof.** Let $U : X \to W$ be the tautological family over $\bar{Q}$. We choose $\bar{Q}$ so that there are families of subcurves $\mathcal{E}_1, \ldots, \mathcal{E}_L \subset X$ so that for each $z \in \bar{Q}$, $\mathcal{E}_i \cap \mathcal{X}_z, \ldots, \mathcal{E}_L \cap \mathcal{X}_z$ are exactly the $l$ deformable parts of $\mathcal{X}_z$. Then the composite $\mathcal{E}_i : W \to W$ factor through $L_2 \subset W$, and the resulting morphism $\mathcal{E}_i \to L_2$ factor through $\bar{Q} \to L_2$. Because each $\mathcal{E}_i \cap \mathcal{X}_z$ has a $\phi_{k,c}$ connected to it, the image of $\bar{Q} \to L_2$ lies in $L_2 - q_2$. We then fix an isomorphism $\mathbb{A}^l \cong L_2 - q_2$ with 0 corresponding to $p_2$. This way we obtain the desired morphism

$$ \rho_1 : \bar{Q} \to \mathbb{A}^l \cong L_2 - q_2. $$

The proof of the remainder part of the Proposition is exactly the same as the case studied; we shall not repeat it here. \qed

The last step is to investigate the obstruction sheaf over $Q$, or its lift to $\bar{Q}$.

Let $R \subset X$ the divisor of marked points. By passing to an étale covering of $\bar{Q}$, we can assume that $R \to Q$ is a union of sections; in other words, we can index the marked points of $[u]$ in $\bar{Q}$ globally. We then pick an indexing so that for $i \leq l$ the $i$-th section of the marked points $R_i$ lies in $\mathcal{E}_i$. For $i = 1, \ldots, n$, where $n$ is the number of marked points, we let $U_i : \bar{Q} \to L_2$ be

$$ U_i : U|_{R_i} : R_i \cong \bar{Q} \xrightarrow{\cong} L_2 \subset W. $$

Since $L_2 \subset D_2$ is isomorphic to $L_2 \times \bar{Q} \subset D_2 \times \bar{Q}$ under the contraction $W \to W \times \bar{Q}$ and since $R_i$ lies in $\mathcal{E}_i$, for $i \leq l$ the morphism $U_i$ is exactly the $\rho_i$ under the isomorphism $\mathbb{A}^l \cong L_2 - q_2$, and $U_i^* \mathcal{N}_{\bar{E}/D}$ is canonically isomorphic to $\rho_i^* \mathcal{N}_{L_2/D_2}$. Because $D_2$ is fixed by $T_\eta|_{N_{L_2/D_2}}$ is fixed as well, and hence $\rho_i^* \mathcal{N}_{L_2/D_2}$ is a trivial line bundle on $Q$ with trivial $T_\eta$-linearization.

Because $\mathcal{H} = \bigoplus_{i=1}^n U_i^* \mathcal{N}_{\bar{E}/D}$ (see Section 5.4), $\bigoplus_{i=1}^l \rho_i^* \mathcal{N}_{L_2/D_2}$ becomes a direct summand of $\mathcal{H}$. Because it has weight zero, it induces a canonical homomorphism

$$ \bigoplus_{i=1}^l \rho_i^* \mathcal{N}_{L_2/D_2} \xrightarrow{\cong} \mathcal{T}_{\mathcal{T}_\eta}^2, $$

a weight zero subsheaf of $\mathcal{T}_{\mathcal{T}_\eta}^2$.

**Lemma 5.17.** The homomorphism $\bigoplus_{i=1}^l \rho_i^* \mathcal{N}_{L_2/D_2} \to \mathcal{T}_{\mathcal{T}_\eta}^2$ in (5.14) is injective; thus $\mathcal{T}_{\mathcal{T}_\eta}^2$ contains $\bigoplus_{i=1}^l \rho_i^* \mathcal{N}_{L_2/D_2}$ as its subsheaf. Indeed, this subsheaf is canonically a direct summand of $\mathcal{T}_{\mathcal{T}_\eta}^2$. 


Proof. First the first \( l \) marked points lie in the connected components of \( X^{[2]} \) that are connected to the domain of at least one \( φ_{l,c} \) in \( W \). Because all deformations of \( φ_{k,c} \) as \( T^n \)-invariant maps are \( φ_{k,c} \), and they intersect \( D_2 \) in \( L_2 \) only; hence for these \( i \) even if we do not impose the condition \( U(R_i) \subset L_2 \) the condition will be satisfied automatically. In short, the arrow \( \tilde{T}^1_{T_n} \to H_T \) has image lies in the summand \( \oplus_{i=1}^n H^*_iN_{L/D} \). This proves that the homomorphism \( \oplus_{i=1}^n H^*_iN_{L/D} \to \tilde{T}^1_{T_n} \) is injective.

We now show that this subsheaf is canonically a summand of the obstruction sheaf. The ordinary moduli of stable relative morphisms \( M \) that is isomorphic to \( M \) is connected to the domain of at least one \( φ \).

Obstruction theory. As shown in [3, 4, 28], the virtual cycle \( \tilde{M}^{vir}_W \) requires that the marked points be sent to the relative divisor. The moduli space \( M(W) = M^{\bullet}_{\chi,d,\rho}(W^{rel}, L) \) we worked on imposes one more restriction: the marked points be sent to \( L \subset D \). The obstruction sheaves of the two moduli spaces are related by the exact sequence (5.14) because of the exact sequences

\[
0 \to N_{L_i/D_i} \to N_{L_i/W} \to N_{D_i/W}|L_i \to 0.
\]

In our case, \( L_i \) is a \( \mathbb{P}^1 \) and the above exact sequence splits \( T \)-equivariantly. Hence the sheaf \( \tilde{T}^1_{T_n} \) splits off a factor that is the kernel of \( \tilde{T}^1_{T_n} \to H_T \). Therefore \( \oplus_{i=1}^n \rho^*_iN_{L_2/D_2} \), which is a summand of \( H_T \) and a subsheaf of \( \tilde{T}^1_{T_n} \), becomes a summand \( \tilde{T}^1_{T_n} \).

5.6. A Criterion for Constancy. We first remark that due to the nature of the discussion in the remainder of this section, we shall use analytic topology of instead of Zariski topology. In particular, unless otherwise specified, all open sets will be analytic open sets. Accordingly, we shall view DM-stack as orbifold with analytic topology.

Before we proceed to the details of the proof, a quick review of the construction of the virtual cycles of the moduli stack is in order. Let \( \mathbf{T} = (\mathbb{C}^*)^2 \) as before and let \( \mathbf{M} \) be a proper Deligne-Mumford stack acted on by \( \mathbf{T} \) and endowed with a \( \mathbf{T} \)-equivariant perfect obstruction theory. As shown in [3, 4, 28], the virtual cycle \([M]^{vir}\) is constructed by

\begin{enumerate}
  \item identifying the perfect obstruction theory of \( \mathbf{M} \);
  \item picking a vector bundle\(^5\) \( E \) on \( \mathbf{M} \) so that it surjects onto the obstruction sheaf of \( \mathbf{M} \) and
  \item constructing an associated cone \( C \subset E \) of pure dimension rank \( \mathbf{E} \).
\end{enumerate}

The virtual cycle \([M]^{vir}\) is the image of the cycle \([C] \in H_s(E, E - M)\) under the Thom isomorphism

\[
φ_E : H_s(E, E - M) \to H_{s-2r}(M), \quad r = \text{rank } E.
\]

Here as usual, we denote by \( E \) the total space of \( E \) and denote by \( \mathbf{M} \subset E \) its zero section that is isomorphic to \( \mathbf{M} \). Also, all (co)homologies are taken with \( \mathbb{Q} \) coefficient.

Following [15], we can make the above construction \( T \)-equivariant. We choose \( E \) be a \( \mathbf{T} \)-equivariant vector bundle. Then the cone \( C \) alluded before is a \( \mathbf{T} \)-invariant subcone of \( E \). Because \( C \subset E \) is \( \mathbf{T} \)-equivariant, the composite

\[
T \times C \stackrel{pr_2}{\to} C \to E
\]

defines a \( \mathbf{T} \)-equivariant \([C]^T \in H^*_T(E, E - M)\); its image under the \( \mathbf{T} \)-equivariant Thom isomorphism \( φ_E \) is the \( \mathbf{T} \)-equivariant virtual moduli cycle

\[
φ_E([C]^T) = [M]^{vir,T} \in H^*_T(M).
\]

Note that the equivariant homologies are \( H^*_T(pt) = \mathbb{Q}[u_1, u_2] \) modules. As before, we denote by \( m = (u_1, u_2) \) the maximal ideal and \( H^*_T(\cdot)_m \) the localization at \( m \).

\(^5\)It was shown in [25] the existence of a global vector bundle \( E \) can be replaced by that \( \mathbf{M} \) is dominated by a quasi-projective scheme.
Next, we apply the localization theorem to the class $[M]^\text{vir}T$. Let
\[ M^T = \bigoplus \{ M_a \mid a \in A \} \]
be the decomposition of the $T$-fixed loci into connected components; let
\[ \tau_a : H^T_s(M_a)_m \to H^T_s(M)_m \]
be induced by the inclusion. According to [15], to each $M_a$ there is a canonically defined virtual cycle $[M_a]^\text{vir}T \in H^T(M_a)$ and a virtual $T$-equivariant normal bundle $\mathcal{N}_a^\text{vir} = \mathcal{T}^{1,m} - \mathcal{T}^{2,m}$ so that, after localization,
\[ [M]^\text{vir}T \in \sum_{a \in A} \tau_a \left( \frac{[M_a]^\text{vir}T}{e^T(\mathcal{N}_a^\text{vir})} \right) \in \left( H^T_s(M)_m \right)_0. \]
Here $(\cdot)_0$ is the degree zero part of the graded ring inside the parentheses.

Now suppose we have an $r$-invariant close substack $N \subset M$; we assume that for any $a \in A$, either $M_a \cap N = \emptyset$ or $M_a \subset N$. Such $N$ divides $A$ into a part of those that are contained in $N$ and a part that are not. Summing over those inside $N$, we obtain
\[ [M]^\text{vir}T_{m,N} = \sum_{M_a \subset N} \tau_a \left( \frac{[M_a]^\text{vir}T}{e^T(\mathcal{N}_a^\text{vir})} \right) \in \left( H^T_s(M)_m \right) \otimes \mathbb{Q}[u_1,u_2]_m. \]

Now suppose $M$ has virtual dimension zero, then taking degree (defined in Section 4.4)
\[ \deg_m : H^T_s(M)_m \to \mathbb{Q}(u_1/u_2), \]
we obtain a rational function in $u_1/u_2$:
\[ \deg_{m,[M]^\text{vir}T_{m,N}} \in \mathbb{Q}(u_2/u_2). \]

To find a sufficient condition that makes the above quantity independent of $u_1/u_2$, we shall devise a way to show that this partial sum is the localization of a purely topological quantity. To achieve this, we let $T_\mathbb{R} \subset T$ be the compact maximal real subgroup and we pick a $T_\mathbb{R}$-equivariant metric on $M$; we can do this by viewing $M$ as a possibly singular orbifold. We then form a $T_\mathbb{R}$-invariant closed tubular neighborhood of $N$:
\[ \Sigma = \{ u \in M \mid \text{dist}(u,N) \leq \epsilon \} \subset M. \]

We then close the boundary
\[ \partial \Sigma = \{ u \in M \mid \text{dist}(u,N) = \epsilon \} \subset M. \]

of $\Sigma$ by picking a one-parameter subgroup $T_\nu \subset T$ and contract individual $S^1_\nu$ orbits in $\partial \Sigma$. Namely, for $S^1_\nu$ the maximal compact subgroup of $T_\nu$ and for $z \in \partial \Sigma$ we shall contract the whole orbit $S^1_\nu \cdot z \subset \partial \Sigma$ to a single point $\{ S^1_\nu \cdot z \}$. To do so, we will pick the subgroup $T_\nu \subset T$ so that $M^T = M^T_\nu$. We denote the resulting space by $\hat{M}$. Note that because $M$ is algebraic and proper, $\hat{M}$ is a compact singular orbifold. For later study, we denote $\partial \Sigma / S^1_\nu$ by $\hat{M}_\infty$.

We next define the virtual cycle of $\hat{M}$. Because $E$ is a $T_\mathbb{R}$-equivariant vector bundle, and because points in $\partial \Sigma$ have finite $S^1_\nu$-stabilizers, $E$ descends to a vector bundle $\hat{E}$ on $\hat{M}$. For the same reason, the cone $C$ descends to a cone $\hat{C} \subset \hat{E}$. Because $C$ is algebraic and because $\epsilon$ is sufficiently small, $\hat{C}$ defines an element $[\hat{C}] \in H^T_s(\hat{E})$; since all data are $T_\mathbb{R}$-equivariant, it also defines an equivariant class
\[ [\hat{C}]^T \in H^T_s(\hat{E}), \hat{E} = \hat{M}. \]
We define their respective images under the obvious Thom isomorphisms their virtual and equivariant virtual cycles:

\[ [\tilde{M}]^{\text{vir}} \in H_*(\tilde{M}) \quad \text{and} \quad [\hat{M}]^{\text{vir}, T} \in H^T_*(\hat{M}). \]

We now localize the class \([\tilde{M}]^{\text{vir}, T}\). We let

\[ \hat{M}^T = \coprod \{ \hat{M}_a \mid a \in \mathcal{A} \} \]

be the decomposition of the \(T_0\)-fixed loci into connected components. Because either \(M_a \subset N\) or \(M_a \cap N = \emptyset\), \(M^T\) is the disjoint union of \(\{ M_a \mid M_a \subset N \}\) and \((\hat{M}_\infty)^T\). We now look at the arrows between localized modules:

\[ H^T_*(\hat{E}, \hat{E} - \hat{M})_m \xrightarrow{\text{Thom}} H^T_*(\hat{M})_m \xrightarrow{\iota} \bigoplus_{a \in \mathcal{A}} H^T_*(\hat{M}, \hat{M} - \hat{M}_a)_m. \]

Here \(\iota = \oplus \iota_a\) and

\[ \iota_a : H^T_*(\hat{M})_m \xrightarrow{\iota} H^T_*(\hat{M}, \hat{M} - \hat{M}_a)_m. \]

By localization theorem, \(\iota\) is an isomorphism.

Since \(M\) has virtual dimension zero, \(\hat{M}\) has virtual dimension zero as well. Applying the degree map \(\deg_m\) from \(H^T_*(\hat{M})_m\) and \(H^T_*(\hat{M}, \hat{M} - \hat{M}_a)_m\) to \(\mathbb{Q}(u_1/u_2)\), we obtain

\[ \deg_m [\hat{M}]^{\text{vir}, T} = \sum_{a \in \mathcal{A}} \deg_m \iota_a ([\hat{M}]^{\text{vir}, T}) \in \mathbb{Q}(u_1/u_2). \]

Since \(\hat{M}\) is compact, the left hand side is independent of \(u_1/u_2\). This immediately proves a useful criterion

**Lemma 5.18.** Let the notation be as before and suppose \(\iota_a([\hat{M}]^{\text{vir}, T}) = 0\) for all \(\hat{M}_a \subset (\hat{M}_\infty)^T\). Then the sum

\[ \sum_{M_a \subset N} \deg_m \iota_a ([\hat{M}]^{\text{vir}, T}) \in \mathbb{Q}(u_1/u_2) \]

is a constant.

### 5.7. Proof of Theorem 4.8

We shall verify this criterion for \(M = M^*(W)\) and \(N = M^*(Y)\); this will complete the proof of Theorem 4.8.

As instructed by the criterion, our first step is to classify the connected components of \((\hat{M}_\infty)^T\). Let \(\tilde{z} \in \hat{M}_\infty\) be any closed point fixed by \(T_0\). By the construction of \(\hat{M}_\infty\), \(\tilde{z}\) is an \(S^1_0\)-orbit \([S^1_0 \cdot z]\) of some \(z \in \partial \Sigma\); therefore \(\tilde{z}\) is fixed by \(T_0\) if and only if the \(S^1_0\)-orbit \(S^1_0 \cdot z\) is identical to the \(T_0\)-orbit \(T_0 \cdot z\), which is possible only if \(\dim_{\mathbb{R}} \text{stab}_{T_0}(z) \geq 1\). Because \(\tilde{z} \in \partial \Sigma\), it is not in \(M^T\); hence there must be a subgroup \(T_\eta \subset T\) so that \(\tilde{z} \in \hat{M}^{T_\eta}\). Finally, because \(\epsilon\) is sufficiently small, \(z \in \Xi(\eta)\). This shows that

\[ \coprod \{ \hat{M}_a \mid \hat{M}_a \subset (\hat{M}_\infty)^T \} = \coprod_{T_\eta \subset T} \left( \Xi(\eta) \cap \partial \Sigma \right) / S^1_0. \]

We now analyze in more details the individual connected components appear in the right hand side of the above decomposition. Before we move on, we remark that we only need to consider the case \(\eta = (a_1, a_2, a_3)\), for \(a_1 > 0\) and \(a_2, a_3 < 0\), and the case \(\eta = (1, -1, 0)\). Indeed, since the symmetry of \((\mathbb{P}^1)^3\) defined by \((z_1, z_2, z_3) \mapsto (z_2, z_3, z_1)\) lifts to a symmetry of \(W\), any statement that holds true for \(\eta = (a_1, a_2, a_3)\) holds true for \(\eta' = (a_2, a_3, a_1)\). Consequently, we only need to work with those \(\eta\) so that \(|a_1| \geq |a_2|\) and \(|a_3|\). Then because \(T_\eta = T_{-\eta}\), we can assume further that \(a_1 > 0\). Hence either \(a_2\) and \(a_3 < 0\) or one of them is zero. For former is the case one; in the later case, by applying the \(S_3\) symmetry we can reduce it to the case \(\eta = (1, -1, 0)\).
We fix a \( T_\eta \subset T \) belongs to the two classes just mentioned. We let \( \mathcal{Q}_a \) be a connected component of \( \Xi(\eta) \) associated to \( M_a \). According to Proposition 5.12 and 5.16, after a finite branched covering \( \pi_a : \hat{\mathcal{Q}}_a \to \mathcal{Q}_a \), \( \hat{\mathcal{Q}}_a \) is isomorphic to \( \mathcal{Q}_{a,0} \times \mathbb{A}^l \) for some integer \( l > 0 \); the \( S^1_a \)-action on \( \hat{\mathcal{Q}}_a \) is the product of the action on \( \mathcal{Q}_{a,0} \) induced by that on \( M^\bullet(Y) \) and the action

\[
(5.29) \quad (u_1, \ldots, u_l)^\sigma = (\sigma^{w_1} u_1, \ldots, \sigma^{w_l} u_l) \in \mathbb{A}^l
\]

for some \( \mathbf{w} = (w_1, \ldots, w_l) \) where \( w_i \) are nonzero rational numbers. In case some \( w_i \) are non-integers, we let \( d \) be the least common multiple of the denominators of all \( w_i \) and replace the \( S^1 \) action by the composing it with the degree \( d \) homomorphism \( S^1 \to S^1 \). This way the new exponents are \( dw_i \), which are integrals. Thus without loss of generality, we can assume that all \( w_i \) are integers in the first place. Hence if we let \( P_a : \mathcal{Q}_a \to \mathbb{A}^l \) be the projection, which is \( (\rho_1, \cdots, \rho_l) \) by our convention, and if we endow \( \mathbb{A}^l \) with the \( S^1_a \)-action (5.29), then \( \mathcal{Q}_a \to \mathbb{A}^l \) is \( S^1_a \)-equivariant.

We now pick an \( S^1_a \)-invariant Riemannian metric on \( \mathbb{A}^l \); we let \( S^l_{\epsilon^{-1}} \subset \mathbb{A}^l \) be the \( \epsilon \)-sphere under this metric. \( P_a^{-1}(S^l_{\epsilon^{-1}}) \). Without lose of generality, we can assume that the metric on \( \mathbb{A}^l \) and on \( M \) are chosen so that \( P_a^{-1}(S^l_{\epsilon^{-1}}) \subset \mathcal{Q}_a \) is the preimage of \( \mathcal{Q}_a \cap \partial \Sigma \) in \( \mathcal{Q}_a \). Hence \( P_a \) induces an \( S^l_a \)-equivariant map

\[
(5.30) \quad P_a^{-1}(S^l_{\epsilon^{-1}}) \to S^l_{\epsilon^{-1}}
\]

and thus induces a map between their quotients

\[
\mu_a : \hat{M}_a \overset{\text{def}}{=} P_a^{-1}(S^l_{\epsilon^{-1}}) / S^1_a \to S^l_{\epsilon^{-1}} / S^1_a = \mathbb{P}^{l-1}_\mathbb{W}.\]

Here we use the subscript \( \mathbf{w} \) to indicate the weighted and the superscript \( l-1 \) to denote the dimension of weighted projective space; to be precise, we shall view the weighted projective spaces as DM-stack. Since the specific weight is irrelevant to our study, we shall not keep track of it in our study. We let

\[
\bar{\pi}_a : \hat{M}_a \to \mathcal{M}_a
\]

the projection induced by \( \hat{\mathcal{Q}}_a \to \mathcal{Q}_a \).

We next put our prior knowledge of the invariant part of the obstruction sheaf of \( \mathcal{Q}_a \) in this setting. We let \( \mathcal{T}^2_\eta \) be the obstruction sheaf on \( \mathcal{Q} \) and let \( \mathcal{T}^2_{a,T_\eta} \) be its invariant part. By Lemma 5.13 and 5.17, there is a canonical quotient sheaf homomorphism

\[
(5.31) \quad \mathcal{T}^2_{a,T_\eta} \to \bigoplus_{\eta=1}^l \rho_i^* \mathcal{O}_{\mathbb{A}^l_i},
\]

both with trivial \( T_\eta \)-actions.

A direct check shows that to each \( i \) there is a \( S^l_i \)-linearization on \( \mathcal{O}_{\mathbb{A}^l_i} \) so that the above homomorphism is \( S^l_i \)-equivariant. Because \( S^l_i \cdot S^l_j = T_{\mathbb{R}} \), the adopted \( S^l_i \)-linearization and the trivial \( S^l_i \)-linearization on \( \mathcal{O}_{\mathbb{A}^l_i} \) makes (5.31) \( T_{\mathbb{R}} \)-equivariant.

Since the obstruction sheaf \( \mathcal{T}^2 \) on \( M \) is a \( T_{\mathbb{R}} \)-equivariant quotient sheaf of \( E|_M \), pull back to \( \hat{\mathcal{Q}}_a \), denoted by \( E|_{\hat{\mathcal{Q}}_a} \), and then composing with (5.31) give us a \( T_{\mathbb{R}} \)-equivariant quotient sheaf

\[
(5.32) \quad E|_{\hat{\mathcal{Q}}_a} \to \bigoplus_{\eta=1}^l \rho_i^* \mathcal{O}_{\mathbb{A}^l_i}.
\]

Their descents to \( \hat{M}_a \) then give rise to a quotient homomorphism

\[
\hat{E}|_{\hat{M}_a} \to \mu_a^* \mathcal{V}_a.
\]
Here $V_a$ is the descent (or the $S^1$-quotient) of $\oplus_{i=1}^l \mathcal{O}_{\tilde{M}}|_{S^1 \cong \Sigma}$ — a rank $l$ vector bundle on $\mathbb{P}^{l-1}$ with trivial $T_\mathbb{R}$-action; $\tilde{E}|_{\tilde{M}_a}$ is the lift of $E|_{\tilde{M}_a}$ to $\tilde{M}_a$.

We need a key technical Lemma recently proved in [20, Lemma 2.6] concerning the cone $C \subset E$ and its restriction to $Q_a$.

**Lemma 5.19** ([20]). Let $C|_{Q_a} \subset E|_{Q_a}$ be the restriction of $C \subset E$ to $Q_a$; let $C|_{\tilde{Q}_a} \subset E|_{\tilde{Q}_a}$ be the pull back of $C|_{Q_a}$ to $\tilde{Q}_a$. Then $C|_{\tilde{Q}_a}$ lies in the kernel bundle of the homomorphism (5.32).

We are now ready to prove Theorem 4.8.

**Proof of Theorem 4.8.** According to the constancy criterion Lemma 5.18, we only need to check that for any connected component $\tilde{M}_a \subset (\tilde{M}_\infty)^T$ and for $\iota_a$ the localization homomorphism

$$\iota_a : H^T_\mathbb{R}(\tilde{M})_m \longrightarrow H^T_\mathbb{R}(\tilde{M}, \tilde{M} - \tilde{M}_a)_m,$$

we have $\iota_a[\tilde{M}]^{\text{vir}, T} = 0$.

To prove this vanishing, we shall first construct a $T_\mathbb{R}$-invariant finite covering

$$\pi : \tilde{U} \longrightarrow U$$

of $Q_a \subset U \subset M$ that extends the existing covering $\tilde{Q}_a \longrightarrow Q_a$: we shall then extend the existing $\tilde{Q}_a \longrightarrow \mathbb{A}^1$ to a $T_\mathbb{R}$-equivariant $\varphi : \tilde{U} \longrightarrow \mathbb{A}^1$ and extend (5.32) to a $T_\mathbb{R}$-equivariant

$$\Phi : \pi^* E \longrightarrow \varphi^* \mathbf{1}_{\mathbb{A}^1},$$

where $\mathbf{1}_{\mathbb{A}^1}$ is the trivial line bundle on $\mathbb{A}^1$. Once these are done, we then pick a $T_\mathbb{R}$-invariants multi-section $\xi$ of $\mathbf{1}_{\mathbb{A}^1}$ on $\mathbb{A}^1$. We shall argue that it can be chosen to be nowhere vanishing over $A^1 - 0$. We shall also pick a $T_\mathbb{R}$-invariant $C_0$-splitting

$$\Phi' : \varphi^* \mathbf{1}_{\mathbb{A}^1} \longrightarrow \pi^* E$$

of $\Phi$. Then the section $\Phi'(\xi)$ of $\pi^* E$, by Lemma 5.19, is disjoint from the pull back cone $\pi^* C$.

To define the localization $\iota_a[M]^{\text{vir}, T}$, we shall construct an open neighborhood $\tilde{U}$ of $\tilde{M}_a \subset \tilde{M}$. For this, we form $U \cap \Sigma$; we then close the $S^1$-orbit of $U \cap \partial \Sigma$ to form $\tilde{U}$. Since $U$ is a $T_\mathbb{R}$-invariant neighborhood of $Q_a$, $\tilde{U}$ is a $T_\mathbb{R}$-invariant neighborhood of $\tilde{M}_a \subset \tilde{M}$. Next we look at the restriction to $\pi^{-1}(\Sigma) \subset U$ of $\Phi'(\xi)$, $\Phi'(\xi)|_{\pi^{-1}(\Sigma)}$, and its image in $E|_{U \cap \Sigma}$ under the projection

$$\pi^* E|_{\pi^{-1}(\Sigma)} \longrightarrow E|_{U \cap \Sigma}.$$

Since $\Phi'(\xi)$ is $T_\mathbb{R}$-invariant, this restriction descends to a multi-section of $\tilde{E}|_{\tilde{U}}$. We denote the resulting multi-section by $\tilde{\xi}$. Because $\Phi'(\xi) \cap \pi^* C = 0$, $\tilde{\xi} \cap \tilde{C}|_{\tilde{U}} = 0$. Therefore,

$$0 = \iota_a[M]^{\text{vir}, T} = [\tilde{\xi} \cap \tilde{C}|_{\tilde{U}}] = E^T_\mathbb{R}(\tilde{U}, \tilde{U} - \tilde{M}_a)_m = E^T_\mathbb{R}(\tilde{M}, \tilde{M} - \tilde{M}_a)_m.$$

This will prove the vanishing $\iota_a[M]^{\text{vir}, T} = 0$.

We now provide the details. First, we remark that since $M$ is a moduli of relative stable morphisms, its coarse moduli exists and is compact. We denote the coarse moduli of $M$ by $M_-$; for consistency, we will denote the coarse moduli of other DM-stacks using the same subscript “-”. Also, for any $z \in M$, we can find a finite group $G_z$ and a $G$ space $V$ such that that stack quotient $[V/G]$ is an open neighborhood of $z \in M$ while the quotient $V/G$ is an open neighborhood of $z \in M_-$. 
1. Extending a neighborhood of $\bar{Q}_a$. We first find an open $Q_a \subset U \subset M$ and a $G_{(k_i)} \times T_R$-orbifold $\bar{U}$ containing $\bar{Q}_a$ as sub-$G_{(k_i)} \times T_R$-orbifold so that $\bar{U}/G_{(k_i)} = U$ and that the induced $T_R$-structure on $U$ coincides with the one induced by $M$.

Indeed, we first pick a $T_R$-invariants open $U \subset M$ containing $Q_a$, we can cover $U$ by $U_\alpha$, $\alpha \in \Lambda$, so that each $U_\alpha$ is a $G_\alpha$ orbifold quotient of a $G_\alpha$-space $V_\alpha$; namely $U_\alpha = [V_\alpha/G_\alpha]$. We let $p_\alpha : V_\alpha \to U_\alpha$ be the quotient map. We next construct $\bar{U}_\alpha$, a $G_{(k_i)}$-space whose quotient by $G_{(k_i)}$ will be $U_\alpha$. To do this, we first construct a $G_{(k_i)}$-space quotient $\bar{U}_0,\alpha \to U_0,\alpha = U_\alpha \cap Q_a$.

Since $M$ is the moduli of stable morphism, we have a tautological $G_\alpha$-family of stable morphisms $[U, \chi]$ over $V_0,\alpha = p_\alpha^{-1}(Q_a)$. Using this family, we can construct a $G_{(k_i)}$-quotient $\bar{V}_0,\alpha \to V_0,\alpha$ similar to our construction of $\bar{Q}$ from $Q$ in Lemma 5.12. We claim that $G_\alpha$ acts on $\bar{V}_0,\alpha$ naturally. Let $g \in G_\alpha$ and $z \in V_\alpha$; then $g$ induces an isomorphism $\bar{\chi}_z : X_z \to X_{gz}$. Because a point $w \in U_0,\alpha$ over $z$ are $[U_\alpha, \chi_\alpha]$ together with $Z_i \subset X_\alpha$ and $U|Z_i \equiv \phi_{k_i,c_i}$, the isomorphism $\bar{\chi}_{Z_i} : X_{Z_i} \to X_{gz}$ defines $\bar{\chi}(Z_i) \subset \bar{\chi}(X_z)$ and $U|\bar{\chi}(Z_i) \equiv \phi_{k_i,c_i}$. This way we see that $G_\alpha$ acts on $\bar{V}_0,\alpha$ naturally.

We now construct $G_\alpha$ pair $\bar{V}_\alpha \to V_\alpha$ extending $\bar{V}_0,\alpha \to V_0,\alpha$. This time, since $V_\alpha$ is an analytic space, by shrinking $V_\alpha$ while keep $V_0,\alpha$ unchanged if necessarily, we can extend the $G_{(k_i)}$-covering $\bar{V}_0,\alpha \to V_0,\alpha$ to a covering $\bar{V}_\alpha \to V_\alpha$. Because this construction is unique, the $G_\alpha$ action lifts to a $G_\alpha$ action on $\bar{V}_\alpha$. We define $\bar{U}_\alpha$ be the orbifold quotient $[V_\alpha/G_\alpha]$. As to the $T_R$-action, since it is compact, we can make $U_\alpha$, $T_R$-invariant; thus $\bar{V}_\alpha \to V_\alpha$ is $T_R$-equivariant. Finally, since this construction is canonical once we have the chart $V_\alpha \to U_\alpha$, the orbifold quotient $\bar{U}_\alpha$ patch together to form an $G_{(k_i)} \times T_R$-orbifold $\pi : \bar{U} \to U$. Lastly, we comment that $\bar{U}$ has a coarse moduli the coarse moduli $\pi$ is the quotient $\bar{U}/G_{(k_i)}$.

2. Extending the map $\bar{Q}_a \to \mathbb{A}^l$. We now show that the map $\rho : \bar{Q}_a \to \mathbb{A}^l$ constructed in Proposition 5.16 extends to a $T_R$-equivariant $C^0$-map $\varphi : \bar{U} \to \mathbb{A}^l$. Here the $T_R$ action on $\mathbb{A}^l$ is induced by the trivial $S_T^l$ action and the $S_T^l$ action defined in (5.29). Since $\rho$ factors through its coarse moduli $\rho_- : \bar{Q}_a \to \mathbb{A}^l$, to extend $\rho$, we only need to extend $\rho_-$ to a $T_R$-equivariant continuous $\varphi_- : \bar{U} \to \mathbb{A}^l$. This is possible because $T_R$ is compact and that $T_R$ acts on $\mathbb{A}^l$ linearly. Once we have $\varphi_-$, we define $\varphi = \rho \circ \varphi_-$ be the composite of $\bar{U} \to \bar{U}_-$ with $\varphi_-.

3. Extending the homomorphism $E|\bar{Q}_a \to \rho^*1_{\mathbb{A}^l}$. Lastly, we shall extend the sheaf homomorphism (5.32) to a $T_R$-equivariant $\Phi : \pi^*E \to \varphi^*1_{\mathbb{A}^l}$. This is possible for the same as the previous extension problem.

To complete the proof, we need to construct an invariant hermitian metric on $\pi^*E$ that will provide us a splitting $\Phi'$. This is simple using partition of unity over $U_-$. First, we find hermitian metric $h_\alpha$ on $p_\alpha^*E$ over $V_\alpha$. Since both $T_R$ and $G_\alpha$ are compact, we can assume $h_\alpha$ is $G_\alpha \times T_R$-invariant. We then use a partition of unity $\zeta_\alpha$ subordinate to the covering $U_{\alpha} \subset U_\alpha$ and define the hermitian metric on $\pi^*E$ be the sum of the pull back of $\zeta_\alpha h_\alpha$. The resulting metric $h$ satisfies the desired property.

Finally, we need to find a $T_R$-invariant multisection of $1_{\mathbb{A}^l}$ which is nowhere vanishing over $\mathbb{A}^l - 0$. Since the $T_R$ action is via the trivial $S_T^l$ and the $S_T^l$ action specified in (5.29), we only need to find an $S_T^l$-invariant multiple section. The action of $S_T^l$ on $1_{\mathbb{A}^l}$ is given by $(z_1, \ldots, z_l)^\sigma = (\sigma^{v_1} z_1, \ldots, \sigma^{v_l} z_l)$. 


for some \((v_1, \ldots, v_l)\). For \(i = 1, \ldots, l\), let

\[
(5.33) \quad s_i(u_1, \ldots, u_l) = \begin{cases} 
\frac{u_{v_i}}{w_i} & v_i/w_i > 0, \\
\frac{-u_{v_i}}{w_i} & v_i/w_i < 0, \\
1 & v_i = 0.
\end{cases}
\]

Then \((s_1, \ldots, s_l)\) is an \(S^1_\nu\)-invariant multiple section of \(1_{\mathbb{A}^l}\) which is nowhere vanishing over \(\mathbb{A}^l\). \(\square\)

6. Topological Vertex, Hodge Integrals and Double Hurwitz Numbers

Let \(\Gamma_{n; w_1, w_2}\) be the FTCY in Figure 10, where

\[
(6.1) \quad f_1 = w_2 - n_1 w_1, \quad f_2 = w_3 - n_2 w_2, \quad f_3 = w_1 - n_3 w_3, \quad w_3 = -w_1 - w_2,
\]

and \(n = (n_1, n_2, n_3) \in \mathbb{Z}^3\). Any topological vertex (defined in Definition 5.1) is of this form.

\[
\begin{array}{c}
\text{Figure 10. The graph of a topological vertex}
\end{array}
\]

In this section, we will compute

\[
(6.2) \quad F^\bullet_{X, \mu}(n; w_1, w_2) = F^\bullet_{\Gamma_{n; w_1, w_2}}(u_1, u_2)
\]

where the RHS is defined by (4.13). To simplify the notation, we will fix \(n = (n_1, n_2, n_3)\) and \((w_1, w_2)\) and write \(\Gamma\) instead of \(\Gamma_{n; w_1, w_2}\).

6.1. Torus Fixed Points and Label Notation. In this subsection, we describe the \(T\)-fixed points in \(M^\bullet_{X, \mu}(\Gamma) \defeq M^\bullet_{X, \mu}(\hat{Y}_\text{rel}, \hat{L})\), and introduce the label notation. To each label corresponds a disjoint union of connected components of

\[
M^\bullet_{X, \mu}(\Gamma)_T = M^\bullet_{X, \mu}(\Gamma)^T,
\]

or equivalently, a collection of graphs in the graph notation.

Let \(\hat{Y}_\text{rel} = (\hat{Y}, \hat{D})\) be the FTCY associated to \(\Gamma\), and let

\[
\hat{D}^i = \hat{D}^{v_i}, \quad C^i = C^{v_i}
\]

for \(i = 1, 2, 3\). Given \(u : (X, q) \to (\hat{Y}_m, \hat{D}_m)\) which represents a point in \(M^\bullet_{X, \mu}(\Gamma)^T\), let \(\tilde{u} = \pi_m \circ u : X \to \hat{Y}_\text{rel}^\Gamma\), where \(\pi_m : \hat{Y}_m \to \hat{Y}\) be the projection defined in Section 4.1. Then \(\tilde{u}(X) \subset C^1 \cup C^2 \cup C^3\). Let \(z^0\) and \(z^i\) be the two \(T\) fixed points on \(C^i\), and let

\[
V^i = \tilde{u}^{-1}(z^i)
\]
for $i = 0, 1, 2, 3$. Let $E^i$ be the closure of $\tilde{u}^{-1}(C^i \setminus \{z^0, z^i\})$ for $i = 1, 2, 3$. Then $E^i$ is a union of projective lines, and $u|_{E^i} : E^i \to C^i$ is a degree $d^i = |\mu^i|$ cover fully ramified over $z^0$ and $z^i$.

Define

$$P^i(m^i) = \pi^{-1}_m(z^i)$$

which is a point if $m^i = 0$, and is a chain of $m^i$ copies of $P^1$ if $m^i > 0$.

For $i = 1, 2, 3$, let

$$\tilde{u}^i = u|_{V^i} : V^i \to P^i(m^i), \quad \tilde{u}^i = u|_{E^i} : E^i \to C^i.$$

The degrees of $\tilde{u}^i$ restricted to connected components of $E^i$ determine a partition $\nu^i$ of $d^i$.

For $i = 0, 1, 2, 3$, let $V^i_1, \ldots, V^i_{g_j^i}$ be the connected components of $V^i$, and let $g_j^i$ be the arithmetic genus of $V^i_j$. (We define $g_j^i = 0$ if $V^i_j$ is a point.) Define

$$\chi^i = \sum_{j=1}^{k^i} (2 - 2g_j^i).$$

Then

$$-\sum_{i=0}^{3} \chi^i + 2\sum_{i=1}^{3} \ell(\nu^i) = -\chi.$$

Note that $\chi^i \leq 2 \min\{\ell(\mu^i), \ell(\nu^i)\}$ for $i = 1, 2, 3$, so

$$-\chi^i + \ell(\nu^i) + \ell(\mu^i) \geq 0$$

and the equality holds if and only if $m^i = 0$. In this case, we have $\nu^i = \mu^i$, $\chi^i = 2\ell(\mu^i)$.

We introduce moduli spaces of relative stable maps to the non-rigid ($P^1, \{0, \infty\}$) (called rubber in [41] etc.): 

$$\mathcal{M}_{\chi, \nu, \mu} \overset{\text{def}}{=} \overline{\mathcal{M}}_{\chi}(P^1, \nu, \mu) / / \mathbb{C}^*$$

where $\overline{\mathcal{M}}_{\chi}(P^1, \nu, \mu) / / \mathbb{C}^*$ is defined as in [31, Section 5].

For each $i \in \{1, 2, 3\}$, there are two cases:

Case 1: $m^i = 0$. Then $\tilde{u}^i$ is a constant map from $\ell(\mu^i)$ points to $p^i$.

Case 2: $m^i > 0$. Then $\tilde{u}^i$ represents a point in $\mathcal{M}_{\chi^i, \nu^i, \mu^i}$.

**Definition 6.1.** An admissible label of $\mathcal{M}_{\chi, \nu, \mu}(\Gamma)$ is a pair $(\bar{\chi}, \bar{\nu})$ such that

1. $\bar{\chi} = (\chi^0, \chi^1, \chi^2, \chi^3)$, where $\chi^i \in 2\mathbb{Z}$.
2. $\bar{\nu} = (\nu^1, \nu^2, \nu^3)$, where $\nu^i$ is a partition such that $|\nu^i| = |\mu^i|$.
3. $\chi^0 \leq 2\sum_{i=1}^{3} \ell(\nu^i)$.
4. $\chi^i \leq 2 \min\{\ell(\mu^i), \ell(\nu^i)\}$ for $i = 1, 2, 3$.
5. $-\sum_{i=0}^{3} \chi^i + 2\sum_{i=1}^{3} \ell(\nu^i) = -\chi$.

Let $G_{\chi, \nu, \mu}(\Gamma)$ denote the set of all admissible labels of $\mathcal{M}_{\chi, \nu, \mu}(\Gamma)$.

For a nonnegative integer $g$ and a positive integer $h$, let $\mathcal{M}_{g, h}$ be the moduli space of stable curves of genus $g$ with $h$ marked points. Although $\mathcal{M}_{g, h}$ is empty for $(g, h) = (0, 1), (0, 2)$, for simplicity of notation we will formally assume the following integrals exist:

$$\int_{\mathcal{M}_{0, 1}} \frac{1}{1 - d\psi} = \frac{1}{d^2}, \quad \int_{\mathcal{M}_{0, 2}} \frac{1}{(1 - \mu_1 \psi_1)(1 - \mu_2 \psi_2)} = \frac{1}{\mu_1 + \mu_2}.$$

This convention will give the correct final results.

For a nonnegative integer $g$ and a positive integer $h$, let $\mathcal{M}_{\chi, h}$ be the moduli of possibly disconnected stable curves $C$ with $h$ marked points such that
• If $C_1, \ldots, C_k$ are connected components of $C$, and $g_i$ is the arithmetic genus of $C_i$, then
\[ \sum_{i=1}^{k} (2 - 2g_i) = \chi. \]

• Each connected component contains at least one marked point.

The connected components of $\mathcal{M}_{g,h}$ are of the form
\[ \mathcal{M}_{g_1, h_1} \times \cdots \times \mathcal{M}_{g_k, h_k}, \]
where
\[ \sum_{i=1}^{k} (2 - 2g_i) = \chi, \quad \sum_{i=1}^{k} h_i = h. \]

The restriction of the Hodge bundle $E \to \mathcal{M}_{g,h}$ to the above connected component is the direct sum of the Hodge bundles on each factor, and
\[ \Lambda^\vee(u) = \prod_{i=1}^{k} \Lambda_i^\vee(u). \]

We define
\[ \mathcal{M}_{\vec{\chi}, \vec{\nu}} = \prod_{i=0}^{3} \mathcal{M}_{\vec{\chi}, \vec{\nu}}^i \]
where $\mathcal{M}_{\vec{\chi}, \vec{\nu}}^0 = \mathcal{M}_{\vec{\chi}, \vec{\mu}, \ell(\vec{\nu})}$, and for $i \in \{1, 2, 3\}$,
\[ \mathcal{M}_{\vec{\chi}, \vec{\nu}}^i = \begin{cases} \{\text{pt}\}, & -\chi^i + \ell(\nu^i) + \ell(\mu^i) = 0, \\ \mathcal{M}_{\vec{\chi}, \vec{\nu}, \mu}^{\ell(\nu^i)}, & -\chi^i + \ell(\nu^i) + \ell(\mu^i) > 0. \end{cases} \]

For each $(\vec{\chi}, \vec{\nu}) \in G_{\vec{\chi}, \vec{\nu}}(\Gamma)$, there is a morphism $i_{\vec{\chi}, \vec{\nu}} : \mathcal{M}_{\vec{\chi}, \vec{\nu}} \to \mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\ell(\vec{\nu})}(\Gamma)^T$, whose image $\mathcal{F}_{\vec{\chi}, \vec{\nu}}$ is a union of connected components of $\mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\ell(\vec{\nu})}(\Gamma)^T$. The morphism $i_{\vec{\chi}, \vec{\nu}}$ induces an isomorphism
\[ \mathcal{M}_{\vec{\chi}, \vec{\nu}} / \left( \prod_{i=1}^{3} A_{\vec{\chi}, \vec{\nu}}^i \right) \cong \mathcal{F}_{\vec{\chi}, \vec{\nu}} \]
where $A_{\vec{\chi}, \vec{\nu}}^i$ is the automorphism group associated to the edge $e_i$:
\[ A_{\vec{\chi}, \vec{\nu}}^i = \prod_{j=1}^{\ell(\nu^i)} \mathbb{Z}_{\nu^i_j}, \quad -\chi^i + \ell(\nu^i) + \ell(\mu^i) = 0; \]
\[ 1 \to \prod_{j=1}^{\ell(\nu^i)} \mathbb{Z}_{\nu^i_j} \to A_{\vec{\chi}, \vec{\nu}}^i \to \text{Aut}(\nu^i) \to 1, \quad -\chi^i + \ell(\nu^i) + \ell(\mu^i) > 0. \]

The fixed points set $\mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\ell(\vec{\nu})}(\Gamma)^T$ is a disjoint union of
\[ \{\mathcal{F}_{\vec{\chi}, \vec{\nu}} \mid (\vec{\chi}, \vec{\nu}) \in G_{\vec{\chi}, \vec{\nu}}^{\ell(\vec{\nu})}(\Gamma)\}. \]

**Remark 6.2.** There are two perfect obstruction theories on $\mathcal{F}_{\vec{\chi}, \vec{\nu}}$: one is the fixed part $[\mathcal{T}^{1, \vec{f}} \to \mathcal{T}^{2, \vec{f}}]$ of the restriction of the perfect obstruction theory on $\mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\ell(\vec{\nu})}(\Gamma)$; the other comes from the perfect obstruction theory on the moduli spaces $\mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\ell(\vec{\nu})}$ and $\mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\vec{\mu}}$. It is straightforward to check that they coincide.
6.2. Contribution from Each Label. We view $w_i$ and $f_i$ in Equation (6.1) as elements in

$$Zu_1 \oplus Zu_2 = \Lambda_T \cong H^2_T(pt, \mathbb{Q}).$$

Recall that $H^*_T(pt; \mathbb{Q}) = \mathbb{Q}[u_1, u_2]$. The results of localization calculations will involve rational functions of $w_i$ and $f_i$ which are elements in $\mathbb{Q}(u_1, u_2)$.

If $m^i > 0$, let $\psi^0_i, \psi^\infty_i$ denote the target $\psi$ class of $\mathcal{M}_{\vec{\chi}, \vec{\nu}}$ (see e.g. [31, Section 5] for definitions). Let $N_{\vec{\chi}, \vec{\nu}}^{\text{vir}}$ denote the virtual bundle on $\mathcal{M}_{\vec{\chi}, \vec{\nu}}$ which is the pull back of $\mathcal{T}^1, m - \mathcal{T}^2, m$ under $i_{\vec{\chi}, \vec{\nu}}$.

With the above notation and the explicit description of $[\mathcal{T}^1 \to \mathcal{T}^2]$ in Section 4.3, calculations similar to those in [30, Appenix A] show that

$$\frac{1}{e_T(N_{\vec{\chi}, \vec{\nu}}^{\text{vir}})} = \prod_{i=0}^{3} B_{v_i} \prod_{i=1}^{3} B_{e_i},$$

where

$$B_{v_0} = \prod_{i=1}^{3} \frac{\alpha_{\mu} A^\chi(w_i) w_i^{\ell(\vec{\nu})-1}}{\prod_{j=1}^{\ell(\nu)} (w_i(w_i - \nu_i^j \psi^i_j))},$$

and for $i \in \{1, 2, 3\}$,

$$B_{v_i} = \begin{cases} 1, & -\chi^i + \ell(\nu^i) + \ell(\mu^i) = 0 \\ (-1)^{\ell(\nu^i) - \chi^i / 2} a_{\nu^i} f_{i}^{-\chi^i + \ell(\nu^i) + \ell(\mu^i)} - w_i - \psi^0_i, & -\chi^i + \ell(\nu^i) + \ell(\mu^i) > 0 \end{cases}$$

$$B_{e_i} = (-1)^{\nu_i^i |n^i + \ell(\nu^i) - |\nu^i|} \prod_{j=1}^{\nu_i^i} \prod_{a=1}^{\nu_i^j-1} (w_i(w_i + \nu_i^j \psi^i_j) + aw_i) \prod_{j=1}^{\nu_i^j-1} (\nu_i^j - 1)w_i^{\nu_i^j-1}$$

The disconnected double Hurwitz numbers $H^*_{X, \nu, \mu}$ (see Section 2.2) can be related to intersection of the target $\psi$ class (see [31, Section 5] for a derivation):

$$H^*_{X, \nu, \mu} = \frac{(-\chi + \ell(\nu) + \ell(\mu))!}{|\text{Aut}(\nu) \times \text{Aut}(\mu)|} \int_{\mathcal{M}_{\nu, \mu}^{\text{vir}}} (\psi^0)^{-\chi + \ell(\nu) + \ell(\mu) - 1}$$

The three-partition Hodge integral $G^*_{X, \vec{\nu}}(w)$ defined by (2.17) in Section 2.3 can be expressed as

$$G^*_{X, \vec{\nu}}(w) = (-\sqrt{-1})^{\ell(\vec{\mu})} V_{X, \vec{\nu}}(w) \prod_{i=1}^{3} E_{\nu^i}(w_i+1, w_i)$$

where

$$V_{X, \vec{\nu}}(w) = \frac{1}{|\text{Aut}(\nu)|} \int_{\mathcal{M}_{X, \nu}^{\text{vir}}} \prod_{i=1}^{3} \frac{\Lambda^\chi(w_i) w_i^{\ell(\vec{\nu})-1}}{\prod_{j=1}^{\ell(\nu)} (w_i(w_i - \nu_i^j \psi^i_j))}$$

$$E_{\nu^i}(x, y) = \prod_{j=1}^{\nu_i^j-1} \frac{\nu_i^j \psi^i_j + ax}{(\nu_i^j - 1)w_i^{\nu_i^j-1}}$$

Set

$$I_{X, \vec{\nu}}(n; w) = \int_{[\mathcal{M}_{\vec{\chi}, \vec{\nu}}^{\text{vir}}]} \frac{1}{e_T(N_{\vec{\chi}, \vec{\nu}}^{\text{vir}})}$$
Then
\[
I_{\chi,\bar{\mu}}(n; w) = \frac{1}{\prod_{i=1}^3 |A_{\chi,\bar{\mu}}^i|} \int_{|\mathcal{M}_{\chi,\bar{\mu}}|^\text{vir}} \frac{1}{N_{\chi,\bar{\mu}}} \\
= |\text{Aut}(\bar{\mu})|(-1)^{\sum_{i=1}^3 (n_i-1)|\mu_i|} (-\sqrt{-1})^{(\ell(\bar{\mu})+\ell(\bar{\nu}))} V_{\chi,\bar{\nu}}(w) \\
\cdot \prod_{i=1}^3 E_{\nu_i}(w_i, w_{i+1}) z_{\nu_i} \left(-\sqrt{-1} \frac{f_i}{w_i}\right) - \chi^i + \ell(\nu^i) + \ell(\mu^i) \cdot H_{\chi,\nu^i,\mu^i}^\bullet \left(-\chi^i + \ell(\nu^i) + \ell(\mu^i)\right)!
\]
So
\[
I_{\chi,\bar{\mu}}(n; w) = |\text{Aut}(\bar{\mu})|(-1)^{\sum_{i=1}^3 (n_i-1)|\mu_i|} (-\sqrt{-1})^{(\ell(\bar{\mu}))} G_{\chi,\bar{\nu}}^\bullet(w) \\
\cdot \prod_{i=1}^3 z_{\nu_i} \left(-\sqrt{-1} \left(n_i - \frac{w_{i+1}}{w_i}\right)\right) - \chi^i + \ell(\nu^i) + \ell(\mu^i) \cdot H_{\chi,\nu^i,\mu^i}^\bullet \left(-\chi^i + \ell(\nu^i) + \ell(\mu^i)\right)!
\]

6.3. Sum over Labels. We have
\[
F_{\chi,\bar{\mu}}^\bullet(n; w_1, w_2) = \frac{1}{|\text{Aut}(\bar{\mu})|} \sum_{(\chi,\bar{\nu}) \in G_{\chi,\bar{\nu}}(\Gamma)} I_{\chi,\bar{\nu}}(n; w).
\]
Define generating functions
\[
F_{\mu}^\bullet(\lambda; n_1, n_2) = \sum_{\chi \in \mathbb{Z}_\chi \leq \ell(\bar{\mu})} \lambda^{-\chi + \ell(\bar{\mu})} F_{\chi,\bar{\mu}}^\bullet(n_1, n_2)
\]
\[
\tilde{F}_{\mu}^\bullet(\lambda; n_1, n_2) = (-1)^{\sum_{i=1}^3 (n_i-1)|\mu_i|} \sqrt{-1}^{(\ell(\bar{\mu}))} F_{\mu}^\bullet(n_1, n_2)
\]
Then relation (6.7) becomes
\[
\tilde{F}_{\mu}^\bullet(\lambda; n_1, n_2) = \sum_{|\nu^i| = |\mu^i|} \Phi_{\nu^i,\mu^i}^\bullet(n_1, n_2) \prod_{i=1}^3 z_{\nu_i} \Phi_{\nu^i,\mu^i}^\bullet \left(-\sqrt{-1} \left(n_i - \frac{w_{i+1}}{w_i}\right)\right)!
\]
where \(G_{\chi,\bar{\nu}}^\bullet(\lambda; w)\) is defined by (2.16) in Section 2.3, and \(\Phi_{\nu^i,\mu^i}^\bullet(\lambda)\) is the generating function of disconnected double Hurwitz numbers defined in Section 2.2.

Equations (6.10), (2.10), and (2.11) imply that
\[
\tilde{F}_{\mu}^\bullet(\lambda; n_1, n_2) = \sum_{|\nu^i| = |\mu^i|} \tilde{F}_{\mu}^\bullet(\lambda; 0, n_1, n_2) \prod_{i=1}^3 z_{\nu_i} \Phi_{\nu^i,\mu^i}^\bullet \left(-\sqrt{-1} n_i\right)
\]
By Theorem 5.2,
\[
F_{\mu}^\bullet(\lambda; 0; n_1, n_2) = \sum_{\chi} \lambda^{-\chi + \ell(\bar{\mu})} F_{\chi,\bar{\mu}}^\bullet(\lambda)(n_1, n_2)
\]
does not depend on \(n_1, n_2\). So by (6.9) and (6.11), \(F_{\mu}^\bullet(\lambda; n_1, n_2)\) and \(\tilde{F}_{\mu}^\bullet(\lambda; n_1, n_2)\) do not depend on \(n_1, n_2\). From now on, we will write
\[
F_{\mu}^\bullet(\lambda; n), \quad \tilde{F}_{\mu}^\bullet(\lambda; n)
\]
instead of \(F_{\mu}^\bullet(\lambda; n_1, n_2), \tilde{F}_{\mu}^\bullet(\lambda; n_1, n_2)\). To summarize, for each \(\bar{\mu} \in \mathcal{P}^3\) and each \(n \in \mathbb{Z}^3\), we have defined an generating function \(F_{\mu}^\bullet(\lambda; n)\) which can be expressed in terms of Hodge integrals and double Hurwitz numbers as follows.
We will reduce the invariance of $F$ (defined in Definition 4.1). In this section, we will calculate the formal relative Gromov-Witten invariants.

Proposition 6.3.

\[ \tilde{F}_\mu^*(\lambda; n) = \sum_{|\nu^t|=|\mu^t|} G_\nu^*(\lambda; w) \prod_{i=1}^{3} z_{\nu^t_i} \Phi_{\nu^t_i, \mu^t_i} \left( \sqrt{-1} \left( n_i - \frac{u_{\nu^t_i+1}}{w_{\nu^t_i}} \right) \lambda \right). \]

Proposition 6.3 and the sum formula (2.10) of double Hurwitz numbers imply:

Corollary 6.4 (framing dependence in winding basis).

\[ \tilde{F}_\mu^*(\lambda; n) = \sum_{|\nu^t|=|\mu^t|} \tilde{F}_\nu^*(\lambda; 0) \prod_{i=1}^{3} z_{\nu^t_i} \Phi_{\nu^t_i, \mu^t_i} \left( \sqrt{-1} n_i \lambda \right) \]

Note that (6.13) is valid for any three complex numbers $n_1, n_2, n_3$.

6.4. Representation Basis. The framing dependence (6.13) is particularly simple in the representation basis used in [1]. We use the notation in Section 2.1. Define $\tilde{C}_\mu(\lambda; n)$ by

\[ \tilde{C}_\mu^*(\lambda; n) = \sum_{|\nu^t|=|\mu^t|} \tilde{F}_\nu^*(\lambda; n) \prod_{i=1}^{3} \chi_{\mu^t_i}(\nu^t_i) \]

which is equivalent to

\[ \tilde{F}_\mu^*(\lambda; n) = \sum_{|\nu^t|=|\mu^t|} \tilde{C}_\nu^*(\lambda; n) \prod_{i=1}^{3} \chi_{\mu^t_i}(\nu^t_i) \]

Then (6.13) is equivalent to

Proposition 6.5 (framing dependence in representation basis).

\[ \tilde{C}_\mu^*(\lambda; n) = e^{\frac{1}{2} \sqrt{-1} \left( \sum_{i=1}^{3} \nu^t_i \cdot n_i \right) \lambda} \tilde{C}_\mu^*(\lambda; 0). \]

Define $\tilde{C}_\mu(\lambda) = \tilde{C}_\mu(\lambda; 0)$, and let $q = e^{\sqrt{-1} \lambda}$. Then (6.14), (6.12), and the Burnside formula (2.8) of double Hurwitz numbers imply

Proposition 6.6. We have

\[ \tilde{C}_\mu^*(\lambda) = q^{\frac{1}{2} \left( \sum_{i=1}^{3} \nu^t_i \cdot \frac{w_{\nu^t_i+1}}{w_{\nu^t_i}} \right)} \sum_{|\nu^t|=|\mu^t|} G_\nu^*(\lambda; w) \prod_{i=1}^{3} \chi_{\mu^t_i}(\nu^t_i). \]

7. Gluing Formulae of Formal Relative Gromov-Witten Invariants

Let $\Gamma$ be a FCTY graph (see Definition 3.2), and let $\tilde{d}, \tilde{\mu}$ be an effective class of $\Gamma$ (defined in Definition 4.1). In this section, we will calculate the formal relative Gromov-Witten invariant

\[ F_{\chi, \tilde{d}, \tilde{\mu}}^*(u_1, u_2) \in \mathbb{Q}(u_2/u_1). \]

We will reduce the invariance of $F_{\chi, \tilde{d}, \tilde{\mu}}^*$ (Theorem 4.8) to the invariance of the topological vertex at the standard framing (Theorem 5.2). We will derive gluing formulae for such invariants.

As in Definition 4.1, we will use the abbreviation

\[ d^\tilde{e} = d(\tilde{e}), \quad \tilde{e} \in E(\Gamma); \quad \mu^v = \tilde{\mu}(v), \quad v \in V_1(\Gamma). \]
7.1. Torus Fixed Points and Label Notation. In this subsection, we describe the $T$-fixed points in $\mathcal{M}_{\chi,\bar{d},\bar{\mu}}(\bar{Y}_{\text{rel}}, \bar{L})$, and introduce the label notation. This is a generalization of Section 6.1.

Given a morphism

$$u : (X, q) \rightarrow (\bar{Y}_m, \bar{D}_m)$$

which represents a point in $\mathcal{M}_{\chi,\bar{d},\bar{\mu}}(\bar{Y}_{\text{rel}}, \bar{L})^\text{rel}$, let $\tilde{u} = \pi_m \circ u : X \rightarrow \bar{Y}$, as before. Then

$$\tilde{u}(X) \subset \bigcup_{\bar{e} \in E(\Gamma)} C_{\bar{e}}.$$

where $C_{\bar{e}}$ is defined as in Section 3.5.

Let $z^v$ be the $T$ fixed point associated to $v \in V(\Gamma)$, as in Section 3.5, and let

$$V^v = \tilde{u}^{-1}(z^v).$$

Let $E_{\bar{e}}$ be the closure of $\tilde{u}^{-1}(C_{\bar{e}} \setminus \{z_{\nu_0(e)}, z_{\nu_1(e)}\})$ for $\bar{e} = \{e, -e\} \in E(\Gamma)$. Then $E_{\bar{e}}$ is a union of projective lines, and $u|_{E_{\bar{e}}} : E_{\bar{e}} \rightarrow C_{\bar{e}}$ is a degree $d_{\bar{e}}$ cover fully ramified over $v_0(e)$ and $v_1(e)$.

For $v \in V_1(\Gamma) \cup V_2(\Gamma)$, define

$$\mathbb{P}^v(m^v) = \pi_m^{-1}(z^v)$$

which is a point if $m^v = 0$, and is a chain of $m^v$ copies of $\mathbb{P}^1$ if $m^v > 0$. Let

$$\tilde{u}^v = u|_{V^v} : V^v \rightarrow \mathbb{P}^v(m^v).$$

For $\bar{e} \in E(\Gamma)$, define

$$\tilde{u}_{\bar{e}} = u|_{E_{\bar{e}}} : E_{\bar{e}} \rightarrow C_{\bar{e}}.$$

The degrees of $\tilde{u}_{\bar{e}}$ restricted to connected components of $E_{\bar{e}}$ determine a partition $\nu_{\bar{e}} = \nu_{\bar{e}}^e$ of $d_{\bar{e}}$.

For $v \in V(\Gamma)$, let $V_1^v, \ldots, V_{k^v}^v$ be the connected components of $V^v$, and let $g_j^v$ be the arithmetic genus of $V_j^v$. (We define $g_j^v = 0$ if $V_j^v$ is a point.) Define

$$\chi^v = \sum_{j=1}^{k^v} (2 - 2g_j^v).$$

Then

$$- \sum_{v \in V(\Gamma)} \chi^v + \sum_{\bar{e} \in E(\Gamma)} \ell(\nu_{\bar{e}}) = -\chi.$$

Given $v \in V_1(\Gamma)$ with $v_1^{-1}(v) = \{e\}$, we have $\chi^v \leq 2 \min \{\ell(\nu^e), \ell(\mu^e)\}$, so

(7.2) $$r^v \overset{\text{def}}{=} -\chi^v + \ell(\nu^e) + \ell(\mu^e) \geq 0$$

and the equality holds if and only if $m^v = 0$. In this case, we have $\nu^e = \mu^e$, $\chi^v = 2\ell(\mu^e)$. For each $v \in V_1(\Gamma)$, there are two cases:

Case 1: $m^v = 0$. Then $\tilde{u}^v$ is a constant map from $\ell(\mu^e)$ points to $z^v$.

Case 2: $m^v > 0$. Then $\tilde{u}^v$ represents a point in $\mathcal{M}_{\chi^e, \nu^e, \mu^e}^{\text{rel}}$.

Given $v \in V_2(\Gamma)$ with $v_1^{-1}(v) = \{e, e\}$, we have $\chi^v \leq 2 \min \{\ell(\nu^e), \ell(\nu^{e^e})\}$, so

(7.3) $$r^v \overset{\text{def}}{=} -\chi^v + \ell(\nu^e) + \ell(\nu^{e^e}) \geq 0$$

and the equality holds if and only if $m^v = 0$. In this case, we have $\nu^e = \nu^{e^e}$, $\chi^v = 2\ell(\nu^e)$. For each $v \in V_2(\Gamma)$, there are two cases:

Case 1: $m^v = 0$. Then $\tilde{u}^v$ is a constant map from $\ell(\mu^e)$ points to $z^v$. 
Case 2: \( m^v > 0 \). Then \( \hat{a}^v \) represents a point in \( \mathcal{M}_{Y, v}^{\bullet} \).

**Definition 7.1.** An admissible label of \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(\hat{Y}_{\text{rel}}, \hat{L}) \) is a pair \((\hat{\chi}, \hat{v})\) such that

1. \( \hat{\chi} : V(\Gamma) \to 2\mathbb{Z} \). Let \( \chi^v \) denote \( \hat{\chi}(v) \).
2. \( \hat{v} : E^v(\Gamma) \to \mathcal{P} \), where \( \hat{v}(e) = \hat{v}(e) \) and \(|\hat{v}(e)| = d^e \). We write \( v^e \) for \( \hat{v}(e) \).
3. For \( v \in V_1(\Gamma) \) with \( v_1^{-1}(v) = \{ e \} \), we have \( \chi^v \leq 2 \min\{ \ell(v^e), \ell(\mu^e) \} \).
4. For \( v \in V_2(\Gamma) \) with \( v_1^{-1}(v) = \{ e, e' \} \), we have \( \chi^v \leq 2 \min\{ \ell\nu^e, \ell(\nu^e) \} \).
5. For \( v \in V_3(\Gamma) \), define \( \ell(v) = \sum_{e \in v_1^{-1}(v)} \ell(v^e) \). Then \( \chi^v \leq 2\ell(v) \).
6. \( -\sum_{v \in V(\Gamma)} \chi^v + 2 \sum_{e \in E(\Gamma)} \ell(v^e) = -\chi \).

Let \( G_{X, \hat{a}, \hat{v}}(\hat{\chi}, \hat{v}) \) denote the set of all admissible labels of \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(\hat{Y}_{\text{rel}}, \hat{L}) \).

Given \((\hat{\chi}, \hat{v}) \in G_{X, \hat{a}, \hat{v}}(\hat{\chi}, \hat{v})(\Gamma)\), define \( r^v \) as in (7.2) and (7.3) for \( v \in V_1(\Gamma) \) and \( v \in V_2(\Gamma) \), respectively. We define

\[
\mathcal{M}_{\hat{\chi}, \hat{v}} = \prod_{v \in V(\Gamma)} \mathcal{M}^{\bullet}_{\hat{\chi}, \hat{v}}
\]

where

\[
\mathcal{M}^{\bullet}_{\hat{\chi}, \hat{v}} = \begin{cases} 
\{ \text{pt} \}, & v \in V_1(\Gamma) \cup V_2, v^v = 0, \\
\mathcal{M}^{\bullet}_{\hat{\chi}, \hat{v}} & v \in V_1(\Gamma), v_1^{-1}(v) = \{ e \}, v^v > 0, \\
\mathcal{M}^{\bullet}_{\hat{\chi}, \hat{v}} & v \in V_2(\Gamma), v_1^{-1}(v) = \{ e, e' \}, v^v > 0, \\
\mathcal{M}^{\bullet}_{\hat{\chi}, \hat{v}} & v \in V_3(\Gamma),
\end{cases}
\]

For each \((\hat{\chi}, \hat{v}) \in G_{X, \hat{a}, \hat{v}}(\hat{\chi}, \hat{v})(\Gamma)\), there is a morphism \( i_{\hat{\chi}, \hat{v}} : \mathcal{M}_{\hat{\chi}, \hat{v}} \to \mathcal{M}_{X, \hat{a}, \hat{v}}(\hat{Y}_{\text{rel}}) \), whose image \( \mathcal{F}_{\hat{\chi}, \hat{v}} \) is a union of connected components of \( \mathcal{M}_{X, \hat{a}, \hat{v}}(\hat{Y}_{\text{rel}}) \). The morphism \( i_{\hat{\chi}, \hat{v}} \) induces an isomorphism

\[
\mathcal{M}_{\hat{\chi}, \hat{v}} \left/ \left( \prod_{e \in E(\Gamma)} A^{\bullet}_{\hat{\chi}, \hat{v}} \right) \right. \cong \mathcal{F}_{\hat{\chi}, \hat{v}},
\]

where \( A^{\bullet}_{\hat{\chi}, \hat{v}} \) is the automorphism group associated to the edge \( \hat{e} \):

\[
A^{\bullet}_{\hat{\chi}, \hat{v}} = \prod_{j=1}^r \mathbb{Z}_{\nu^e_j}, \quad \{ v_0(e), v_1(e) \} \cap V_1(\Gamma) = \{ v \} \neq \emptyset \quad \text{and} \quad r^v = 0;
\]

\[
1 \to \prod_{j=1}^r \mathbb{Z}_{\nu^e_j} \to A^{\bullet}_{\hat{\chi}, \hat{v}} \to \text{Aut}(v^e) \to 1, \quad \text{otherwise.}
\]

The fixed points set \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(\hat{Y}_{\text{rel}}, \hat{L})^T \) is a disjoint union of

\[
\{ \mathcal{F}_{\hat{\chi}, \hat{v}} | (\hat{\chi}, \hat{v}) \in G_{X, \hat{a}, \hat{v}}(\Gamma) \}.
\]

**7.2. Perfect Obstruction Theory on Fixed Points Set.** There are two perfect obstruction theories on \( \mathcal{F}_{\hat{\chi}, \hat{v}} \): one is the fixing part \([T^1.f \to T^2.f]\) of the restriction of the perfect obstruction theory on \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(\hat{Y}_{\text{rel}}, \hat{L}) \); the other comes from the perfect obstruction theory on the moduli spaces \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(v) \) and \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(v) \). Let \( [\mathcal{M}_{\hat{\chi}, \hat{v}}]^{\text{vir}} \) denote the virtual cycle defined by \([T^1.f \to T^2.f]\). By inspecting the \( T \)-action on the perfect obstruction theory on \( \mathcal{M}_{X, \hat{a}, \hat{v}}^{\bullet}(\hat{Y}_{\text{rel}}, \hat{L}) \) (see [25] and the description in Section 4), we get

\[
[\mathcal{M}_{\hat{\chi}, \hat{v}}]^{\text{vir}} = \prod_{v \in V(\Gamma)} [\mathcal{M}^{\bullet}_{\hat{\chi}, \hat{v}}]^{\text{vir}}.
\]
where
\[
[M^\bullet_{\chi,v}]^{\text{vir}} = \begin{cases} 
\{[\text{pt}]\}, & v \in V_1(\Gamma) \cup V_2(\Gamma), \quad r^v = 0, \\
[M^\bullet_{\chi,v\nu^+}\nu^-]^{\text{vir}}, & v \in V_1(\Gamma), \quad \nu_1^{-1}(v) = \{e\}, \quad r^v > 0, \\
c_1(L) \cap [M^\bullet_{\chi,v\nu^+\nu^-}]^{\text{vir}}, & v \in V_2(\Gamma), \quad \nu_1^{-1}(v) = \{e,e'\}, \quad r^v > 0, \\
[M^\bullet_{\chi,v\nu_1}]^{\text{vir}}, & v \in V_3(\Gamma),
\end{cases}
\]
where \(L\) is a line bundle on \(M^\bullet_{\chi,v}\) coming from the restriction of the line bundle \(L^v\) on \(M^\bullet_{\chi,v}(\hat{Y})\) (see Section 4.3).

We now give a more explicit description of \(L\). Let
\[
u : (X,q) \to (\mathbb{P}^1(m), p_0, p_m)
\]
represent a point in \(M^\bullet_{\chi,v^+,\nu^-}\), where \(\mathbb{P}^1(m)\) is a chain of \(m > 0\) copies of \(\mathbb{P}^1\) with two relative divisors \(p_0\) and \(p_m\). Let \(\Delta_l\) be the \(l\)-th irreducible component of \(\mathbb{P}^1(m)\) so that \(\Delta_l \cap \Delta_{l+1} = \{p_l\}\). The complex lines
\[
L^0_u = T_{p_0} \Delta_1, \quad L^1_u = \bigotimes_{i=1}^{m-1} T_{p_i} \Delta_l \otimes T_{p_i} \Delta_{l+1}, \quad L^\infty_u = T_{p_m} \Delta_m.
\]
form line bundles \(L^0_u, L^1_u, L^\infty_u\) on \(M^\bullet_{\chi,v^+,\nu^-}\) when we vary \(u\) in \(M^\bullet_{\chi,v^+,\nu^-}\). The line bundle \(L\) is given by
\[
L = L^0 \otimes L^1 \otimes L^\infty.
\]
Note that
\[
c_1(L^0) = -\psi^0, \quad c_1(L^\infty) = -\psi^\infty,
\]
where \(\psi^0, \psi^\infty\) are target \(\psi\) classes (see e.g. [31, Section 5]).

Let \(D\) be the divisor in \(M^\bullet_{\chi,v^+,\nu^-}\) which corresponds to morphisms with target \(\mathbb{P}^1(m)\), \(m > 1\). Then \(L^1_u = O(D)\). Let
\[
J'_{\chi,v^+,\nu^-} = \{(\chi^+, \chi^-, \sigma) \mid \chi^+, \chi^- \in 2\mathbb{Z}, \sigma \in \mathcal{P}, |\sigma| = |\nu^+| = |\nu^-|,
- \chi^+ + 2\ell(\sigma) - \chi^- = -\chi^-, \quad -\chi^\pm + \ell(\nu^\pm) + \ell(\sigma) > 0\}.
\]
For each \((\chi^+, \chi^-)\) in \(J'_{\chi,v^+,\nu^-}\), there is a morphism
\[
\pi^\chi_+^-,\chi_-^-,\sigma : M^\bullet_{\chi^+,\nu^+} \times M^\bullet_{\chi^-,\nu^-} \to M^\bullet_{\chi^+,\nu^+}\nu^-
\]
with image contained in \(D\). Moreover,
\[
[M^\bullet_{\chi^+,\nu^+}\nu^-]^{\text{vir}} \cap c_1(L^1) = \sum_{(\chi^+, \chi^-, \sigma) \in J'_{\chi^+,\nu^+}\nu^-} \frac{a_\sigma}{|\text{Aut}(\sigma)|} (\pi^\chi_+^-,\chi_-^-,\sigma)_* \left([M^\bullet_{\chi^+,\nu^+}]^{\text{vir}} \times [M^\bullet_{\chi^-,\nu^-}]^{\text{vir}}\right)
\]
where \(a_\sigma\) and \(\text{Aut}(\sigma)\) are defined in Section 2.1.

7.3. Contribution from Each Label. We use the definitions in Section 2.1 and Section 3.3.

In this subsection, we view the position \(p(e)\) and the framing \(f(e)\) as elements in
\[
Z\cup Z_2 = \Lambda T \cong H_T^2(\text{pt}, \mathbb{Q}).
\]
Recall that \(H_T^2(\text{pt}; \mathbb{Q}) = \mathbb{Q}[u_1, u_2]\). The results of localization calculations will involve rational functions of \(p(e)\) and \(f(e)\).

Let \(N^\text{vir}_{\chi,\sigma}\) denote the pull back of \(T^{1,m} - T^{2,m}\) of \(\mathcal{F}_{\chi,\sigma}\) under \(i_{\chi,\sigma}\). Let \(r^v\) be defined as (7.2) and (7.3). For \(e \in E^0(\Gamma)\), let \(\bar{e} = \{e, -e\} \in E(\Gamma)\) as before.
With the above notation and the explicit description of \([T^1 \rightarrow T^2]\) in Section 4.3, calculations similar to those in [30, Appendix A] show that

\[
\frac{1}{e_T(N_{vir}^{\chi,e})} = \prod_{v \in V(\Gamma)} B_v \prod_{e \in E(\Gamma)} B_e,
\]

where

\[
B_v = \begin{cases}
1 & v \in V_1(\Gamma) \cup V_2(\Gamma), v^v = 0, \\
(1) & v \in V_1(\Gamma), v^{-1}(v) = \{e\}, v^v > 0, \\
(1) & v \in V_2(\Gamma), v^{-1}(v) = \{e, e'\}, v^v > 0, \\
\prod_{e \in \mathcal{V}_0(v)} (1) & v \in V_3(\Gamma);
\end{cases}
\]

\[
B_e = (1)^{n^v} \cdot \begin{cases}
E_{\nu}(p(e), l_0(e)) \cdot E_{\nu}(p(-e), l_0(-e)) & v_0(0) \in V_3(\Gamma), v_0(1) \in V_3(\Gamma), v_0(2) \not\in V_3(\Gamma), v_1(1) \not\in V_3(\Gamma), \\
1 & v_0(2) \not\in V_3(\Gamma), v_1(1) \not\in V_3(\Gamma).
\end{cases}
\]

Recall that \(n^v\) is defined in Definition 3.4 and \(E_{\nu}(x,y)\) is defined by (6.6).

For \(v \in V_2(\Gamma)\), we have

\[
\int_{\mathcal{M}_{\chi,e}^{\nu,v}} \frac{f(e)^{r^v}}{(-p(e) - \psi^0)(-p(e') - \psi^\infty)} = \int_{\mathcal{M}_{\chi,e}^{\nu,v}} \frac{f(e)^{r^v} c_1(L)}{(-p(e) - \psi^0)(p(e) - \psi^\infty)}
\]

\[
= \int_{\mathcal{M}_{\chi,e}^{\nu,v}} \frac{f(e)^{r^v}}{p(e) - \psi^\infty} + \int_{\mathcal{M}_{\chi,e}^{\nu,v}} \frac{f(e)^{r^v}}{-p(e) - \psi^0}
\]

\[
+ \sum_{(x^+, \chi^-, \sigma) \in J_{x^+, \chi^-, \sigma}^{v, v', v''}} \left( \frac{a_\sigma}{\text{Aut}(\sigma)} \right) \int_{\mathcal{M}_{\chi,e}^{\nu,v,v',v''}} \frac{f(e)^{r^v}}{-p(e) - \psi^0} \int_{\mathcal{M}_{\chi,e}^{\nu,v,v',v''}} \frac{f(e)^{r^v}}{p(e) - \psi^\infty}
\]

\[
= |\text{Aut}(\nu') \times \text{Aut}(\nu'')| \sum_{(x^+, \chi^-, \sigma) \in J_{x^+, \chi^-, \sigma}^{v, v', v''}} (-1)^{r^v_{\chi^+, \sigma} + \ell(\nu')} \sqrt{H_{x^+, \chi^-, \sigma}^{v, v', v''}} \frac{f(e)^{r^v}}{p(e) - \psi^\infty}
\]

where \(r^v_{\chi^+, \sigma} = -\chi^+ + \ell(\nu') + \ell(\sigma), r^-_{\chi^-, \sigma} = -\chi^- + \ell(\sigma) + \ell(\nu''), \) and

\[
J_{x^+, \chi^-, \sigma}^{v, v', v''} = J_{x^+, \chi^-, \sigma}^{v, v', v''} \cup \{(2\ell(\nu'), \chi, \nu'), (\chi, 2\ell(\nu'''), \nu''')\}.
\]

Given \(v \in V_3(\Gamma)\), we have \(v_0^{-1}(v) = \{e_1, e_2, e_3\}\), where \(p(e_1) \wedge p(e_2) = u_1 \wedge u_2\). Then \((e_1, e_2, e_3)\) is unique up to cyclic permutation. Let

\[
p^v = (\nu^v_1, \nu^v_2, \nu^v_3), \quad w^v = (p(e_1), p(e_2), p(e_3)).
\]

Then \(V_{\chi,e}^{\nu}(w^v)\) is independent of choice of cyclic ordering of \(e_1, e_2, e_3\), where \(V_{\chi,e}^{\nu}(w)\) is defined by (6.5).

Set

\[
I_{\chi,e}(u_1, u_2) = \int_{\mathcal{M}_{\chi,e}^{\nu,v}} \frac{1}{e_T(N_{vir}^{\chi,e})}.
\]
Then

\[ I_{\chi,\nu}(u_1, u_2) = \prod_{e \in E(\Gamma)} (-1)^{n_e d_{\nu}} z_{\nu} \prod_{v \in V_3(\Gamma)} V_{\chi, \nu, v}(w^v) \prod_{e \in \mathbb{Z}[\Gamma]} E_{\nu, \nu} (p(e), l_1(e)) \]

\[ \cdot \prod_{v \in V_1(\Gamma), \nu_1(e) = v} \sqrt{-1}^{(\nu_v) + \ell(\mu_v)} (-1)^{d_{\nu}} \left( \sqrt{-1} \frac{\tilde{f}(e)}{p(e)} \right)^r \frac{H_{\chi, \nu_v, \mu_v}^*}{r_{\nu_v, \mu_v}!} \]

\[ \cdot \prod_{v \in V_2(\Gamma), \nu_2(e) = v} \left( \sqrt{-1}^{(\nu_v) + \ell(\mu_v)} \left( \sqrt{-1} \frac{\tilde{f}(e)}{p(e)} \right)^r \cdot \sum_{(\chi^+, \chi^-, \sigma) \in J_{\chi_v, \nu_v, \mu_v'}} \frac{H_{\chi, \nu_v, \sigma}^*}{r_{\chi, \sigma}^+} \frac{H_{\chi^+, \nu_v, \sigma}^*}{r_{\chi^+, \sigma}^+} \right) \]

So

\[ I_{\chi,\nu}(u_1, u_2) = \left| \text{Aut}(\bar{\mu}) \right| \prod_{e \in E(\Gamma)} (-1)^{n_e d_{\nu}} \prod_{v \in V_3(\Gamma)} G_{\chi, \nu, v}(w^v) \]

\[ \cdot \prod_{v \in V_1(\Gamma), \nu_1(e) = v} \sqrt{-1}^{(\mu_v) + \ell(\nu_v)} (-1)^{d_{\nu}} \left( \sqrt{-1} \frac{\tilde{f}(e)}{p(e)} \right)^r \frac{H_{\chi, \nu_v, \mu_v}^*}{r_{\nu_v, \mu_v}!} \]

\[ \cdot \prod_{v \in V_2(\Gamma), \nu_2(e) = v} \left( \sqrt{-1}^{(\nu_v) + \ell(\mu_v)} \left( \sqrt{-1} \frac{\tilde{f}(e)}{p(e)} \right)^r \cdot \sum_{(\chi^+, \chi^-, \sigma) \in J_{\chi_v, \nu_v, \mu_v'}} \frac{H_{\chi, \nu_v, \sigma}^*}{r_{\chi, \sigma}^+} \frac{H_{\chi^+, \nu_v, \sigma}^*}{r_{\chi^+, \sigma}^+} \right) \]

7.4. **Sum over Labels.** Finally, with the notation above, the formal relative GW invariants of a general FTCY graph \( \Gamma \) are

\[ F_{\chi, \delta, \mu}^\Gamma(u_1, u_2) = \frac{1}{\left| \text{Aut}(\bar{\mu}) \right|} \sum_{(\chi, \delta, \mu) \in G_{\Gamma}^*} I_{\chi,\delta,\mu}(u_1, u_2). \]

Define a generating function

\[ F_{\delta, \mu}^\Gamma(\lambda; u_1, u_2) = \sum_{\chi \in \mathbb{Z}, \lambda \leq \ell(\mu)} \lambda^{-\chi + \ell(\mu)} F_{\chi, \delta, \mu}^\Gamma(u_1, u_2). \]

Then (7.6) becomes

\[ F_{\chi, \delta, \mu}^\Gamma(u_1, u_2) = \sum_{|\nu'| = \delta_1(e) \in E(\Gamma)} (-1)^{n_{\nu'}} z_{\nu'} \prod_{v \in V_3(\Gamma)} G_{\chi, \nu', v}(w^v) \]

\[ \cdot \prod_{v \in V_1(\Gamma), \nu_1(e) = v} (-1)^{d_{\nu}} \sqrt{-1}^{(\nu_v) + \ell(\mu_v)} \Phi_{\nu_v, \mu_v}^\Gamma \left( \sqrt{-1} f(e) \right) \frac{p(e) \lambda}{(\sqrt{-1} f(e)) \lambda} \]

\[ \cdot \prod_{v \in V_2(\Gamma), \nu_2(e) = v} \Phi_{\nu_v, \sigma} \left( \sqrt{-1} f(e) \right) \frac{p(e) \lambda}{(\sqrt{-1} f(e)) \lambda} \]
where $G^*_{\bar{\mu}}$ is defined by (2.16), $w_v$ is defined in (7.5), and $\Phi^*_{\nu,\mu}$ is defined in Section 2.2. Equations (6.12) and (6.9) imply

$$\sqrt{-1}^{\ell(\bar{\mu})} G^*_{\bar{\mu}}(\lambda; p(e_1), p(e_2), p(e_3))$$

(7.9)

$$= \sqrt{-1}^{\ell(\bar{\mu})} \sum_{|\nu'| = |\mu'|} F^*_{\bar{\nu}}(\lambda; 0) \prod_{i=1}^3 z_{\nu'_i} \Phi^*_{\nu'_i, \mu'_i} \left( \sqrt{-1}^{l_0(e_i)} \frac{1}{p(e_i)} \right)$$

$$= (-1)^{\sum_1^3 \bar{d}(e_i)} \sum_{|\nu'| = |\mu'|} F^*_{\bar{\nu}}(\lambda; 0) \prod_{i=1}^3 \sqrt{-1}^{\ell(\nu'_i) - \ell(\mu'_i)} z_{\nu'_i} \Phi^*_{\nu'_i, \mu'_i} \left( \sqrt{-1}^{l_0(e_i)} \frac{1}{p(e_i)} \right)$$

7.5. Invariance. In this subsection, we prove that formal relative Gromov-Witten invariants are rational numbers independent of $u_1, u_2$ (Theorem 4.8). We will use operations on FTCY graphs such as smoothing and normalization (defined in Section 3.4) to reduce this to the invariance of the topological vertex (Theorem 5.2).

Let $\Gamma$ be a FTCY graph, and let $\Gamma_2 = \Gamma_{V_2(\Gamma)}$, $\Gamma^2 = \Gamma_{V_2(\Gamma)}$. Then $\Gamma_2, \Gamma^2$ are regular FTCY graphs. We call $\Gamma_2$ the full smoothing of $\Gamma$, and $\Gamma^2$ the full resolution of $\Gamma$. We have surjective maps

$$\pi_2 = \pi_{V_2(\Gamma)} : E^0(\Gamma) \to E^0(\Gamma_2), \quad \pi^2 = \pi_{V_2(\Gamma)} : V(\Gamma^2) \to V(\Gamma).$$

Definition 7.2. Let $\Gamma$ be a FTCY graph, and let $\Gamma^2$ be the full resolution of $\Gamma$.

Let $(\bar{d}, \bar{\mu})$ be an effective class of $\Gamma$. A splitting type of $(\bar{d}, \bar{\mu})$ is a map $\bar{\sigma} : V_2(\Gamma) \to P$ such that $|\bar{\sigma}(v)| = \bar{d}(v)$ if $v_1(e) = v$.

Given a splitting type $\bar{\sigma}$ of an effective class $(\bar{d}, \bar{\mu})$ of $\Gamma$, let $(\bar{d}, \bar{\mu} \sqcup \bar{\sigma})$ denote the effective class of $\Gamma^2$ defined by $\bar{d} : E(\Gamma^2) = E(\Gamma) \to Z_{\geq 0}$ and

$$\bar{\mu} \sqcup \bar{\sigma}(v) = \begin{cases} \bar{\mu}(\pi^2(v)), & \pi^2(v) \in V_1(\Gamma) \\ \bar{\sigma}(\pi^2(v)), & \pi^2(v) \in V_2(\Gamma) \end{cases}$$

Let $S_{\bar{d}, \bar{\mu}}$ denote the set of all splitting types of $(\bar{d}, \bar{\mu})$.

The following is clear from the expression (7.8).

Lemma 7.3. Let $\Gamma$ be a FTCY graph, and let $(\bar{d}, \bar{\mu})$ be an effective class of $\Gamma$. Then

$$F^*_{\bar{d}, \bar{\mu}}(\lambda; u_1, u_2) = \sum_{\sigma \in S_{\bar{d}, \bar{\mu}}} z_{\sigma} F^*_{\bar{d}, \bar{\mu} \sqcup \sigma}(\lambda; u_1, u_2)$$

where $z_{\sigma} = \prod_{v \in V_2(\Gamma)} z_{\sigma(v)}$.

By Lemma 7.3, it suffices to consider regular FTCY graphs. For a regular FTCY graph $\Gamma$, (7.8) reduces to

$$F^*_{\bar{d}, \bar{\mu}}(\lambda; u_1, u_2) = \sum_{|\nu'| = |\mu'|} \prod_{e \in E(\Gamma)} (-1)^{\nu^e} z_{\nu^e} \prod_{v \in V_4(\Gamma)} \sqrt{-1}^{l_0(v)} G^*_{\nu^e, \mu^e}(\lambda; w_v)$$

(7.10)

$$\cdot \prod_{v \in V_2(\Gamma), \nu^e(\lambda) = v} (-1)^{\nu^e} \sqrt{-1}^{(\nu^e) + \ell(\nu^e)} \Phi^*_{\nu^e, \mu^e} \left( \sqrt{-1}^{l_0(e)} \frac{1}{p(e)} \right)$$

since $V_2(\Gamma) = \emptyset$. 
Let \((\tilde{d}, \tilde{\mu})\) be the effective class of a regular FT CY graph. Let \(P_{d, \mu}^\Gamma\) be the set of all maps \(\tilde{\nu} : E^0(\Gamma) \to \mathcal{P}\) such that
- \(|\tilde{\nu}(e)| = \tilde{d}^e\).
- \(\tilde{\nu}(e) = \tilde{\mu}(v)\) if \(v_0(e) = v \in V_1(\Gamma)\).

Note that we do not require \(\tilde{\nu}(e) = \tilde{\nu}(-e)\). Denote \(\tilde{\nu}(e)\) by \(\nu^e\). Given \(v \in V_0(\Gamma)\), there exist \(e_1, e_2, e_3 \in E(\Gamma)\), unique up to a cyclic permutation, such that \(v_0^{-1}(v) = \{e_1, e_2, e_3\}\) and \(p(e_1) \wedge p(e_2) = u_1 \wedge u_2\). Define

\[
(7.11) \quad \tilde{\nu}^\mu = (\nu^{e_1}, \nu^{e_2}, \nu^{e_3}), \quad z_{\nu^e} = z_{\nu^{e_1}} z_{\nu^{e_2}} z_{\nu^{e_3}}.
\]

Note that \(\tilde{F}^\mu(\lambda; 0)\) and \(z_{\nu^e}\) are invariant under cyclic permutation of \(e_1, e_2, e_3\), thus well-defined.

Using (7.9) and the sum formula (2.10) of double Hurwitz numbers, we can rewrite (7.10) as follows:

\[
(7.12) \quad F_{d, \mu}^\Gamma(\lambda; u_1, u_2) = \sum_{\tilde{\nu} \in P_{d, \mu}^\Gamma} \prod_{e \in E(\Gamma)} F_{d, \mu}^\nu(\lambda; 0) z_{\nu^e} \prod_{e \in E(\Gamma)} \sqrt{-1}^{(\nu^e) - (\nu^{-e})} (1)^{n^e \ell} \Phi_{\nu^e, \nu^{-e}}(\sqrt{-1} n^e \lambda).
\]

Note that the right-hand side of (7.12) does not depend on \(u_1, u_2\). This completes the proof of Theorem 4.8. From now on, we write \(F_{d, \mu}^\Gamma(\lambda)\) instead of \(F_{d, \mu}^\Gamma(\lambda; u_1, u_2)\). We define

\[
F_{\chi, d, \mu}^\Gamma = F_{\chi, d, \mu}^\Gamma(\lambda; u_1, u_2),
\]

to be formal relative Gromov-Witten invariants of \(\tilde{Y}_1^{\text{reg}}\).

7.6. Gluing Formulae. Let \((\tilde{d}, \tilde{\mu})\) be an effective class of a regular FT CY graph \(\Gamma\). Let

\[
T_{d, \mu} = \left\{ \tilde{\nu} : E(\Gamma) \to \mathcal{P} \middle| |\tilde{\nu}(e)| = \tilde{d}(e), \tilde{\nu}(-e) = \tilde{\nu}(e) \right\}.
\]

Note that we do not require \(\tilde{\nu}(e) = \tilde{\mu}(v)\) if \(v_0(e) = v \in V_1(E)\). We have

\[
(7.13) \quad F_{d, \mu}^\Gamma(\lambda; 0) = \frac{(-1)^{|\mu|} |\mu|}{\sqrt{-1} \hat{\Gamma}(\tilde{\mu})} \sum_{|\nu| = |\mu|} \hat{C}_{\nu}(\lambda) \prod_{i=1}^3 \chi_{\nu^e}(\mu^i) z_{\nu^i},
\]

where \(\hat{C}_{\nu}(\lambda) = \hat{C}_{\nu}(\lambda; 0)\). Applying (7.13) and the Burnside formula (2.8) of double Hurwitz numbers, we see that (7.12) is equivalent to the following.

**Proposition 7.4.** Let \(\Gamma\) be a regular FT CY graph. Then

\[
F_{d, \mu}^\Gamma(\lambda) = \sum_{\tilde{\nu} \in T_{d, \mu}} \prod_{e \in E(\Gamma)} (-1)(n^+ + 1)^{\ell} e^{-\sqrt{-1} \kappa_{\nu^e} n^+ \lambda/2} \prod_{e \in V_1(\Gamma)} \hat{C}_{\nu^e}(\lambda) \prod_{e \in V_1(\Gamma)} \frac{\chi_{\nu^e}(\mu^e)}{\sqrt{-1} (\mu^e)}.
\]

Recall that \(\kappa_{\nu^e}\) is defined by (2.1), and we have \(n^- = -n^e\), \(\kappa_{\nu^e} = -\kappa_{\nu^e}\), so

\[
\kappa_{\nu^e} n^- = \kappa_{\nu^e} (-n^e) = \kappa_{\nu^e} n^e.
\]

**Theorem 7.5** (gluing formula). Let \(\Gamma\) be a FT CY graph, and let \(\Gamma_2\) and \(\Gamma_2\) be its full smoothing and its full resolution, respectively. Let \((\tilde{d}, \tilde{\mu})\) be an effective class of \(\Gamma\) which can also be viewed as an effective class of \(\Gamma_2\). Then

\[
(7.14) \quad F_{d, \mu}^{\Gamma_2}(\lambda) = F^{\Gamma_2}_{d, \mu}(\lambda) = \sum_{\tilde{\sigma} \in S_{d, \mu}} z_{\sigma} F_{d, \mu}^{\Gamma_2}(\lambda).
\]
We abbreviate $\vec{\nu}$ where $p$

**Corollary 7.6.**

By Lemma 7.3 and Proposition 7.4, it suffices to show that if $|\mu| = |\nu| = d$, then

$$
\sum_{|\sigma| = d} \frac{\chi_\mu(\sigma)}{\sqrt{-1} \chi_\nu(\sigma)} z_\sigma = (-1)^d \delta(\nu'),
$$

which is obvious.

### 7.7. Sum over Effective Classes

Given a regular FTCY graph, let $\text{Eff}(\Gamma)$ denote the set of effective classes of $\Gamma$. Introduce formal Kähler parameters

$$
t = \{ t^e : e \in E(\Gamma) \}
$$

and winding parameters

$$
p = \{ p^v = (p^v_1, p^v_2, \ldots) : v \in V_1(\Gamma) \}
$$

We define the formal relative Gromov-Witten partition function of $\tilde{Y}_\Gamma^{\text{rel}}$ to be

$$
Z^\Gamma_{\text{rel}}(\lambda; t; p) = \sum_{(\vec{\nu}, \vec{\mu}) \in \text{Eff}(\Gamma)} F^\Gamma_{d, \vec{\mu}}(\lambda) e^{-\sum_{e \in E(\Gamma)} d(e) t^e} \prod_{v \in V_3(\Gamma)} p^v_{\mu^v}
$$

where $p^v_{\mu^v} = p^v_{\mu^v_1} \cdots p^v_{\mu^v_{\mu^v(\nu)}}$.

Let $T^\Gamma$ denote the set of pairs $(\vec{\nu}, \vec{\mu})$ such that

- $\vec{\nu} : E^\circ(\Gamma) \rightarrow \mathcal{P}$ such that $\vec{\nu}(-e) = \vec{\nu}(e)^t$.
- $\vec{\mu} : V_1(\Gamma) \rightarrow \mathcal{P}$.
- $|\nu^e| = |\mu^e|$ if $\nu_0(e) = \nu_v$.

We abbreviate $\vec{\nu}(e)$ to $\nu^e$ for $e \in E^\circ(\Gamma)$, abbreviate $\vec{\mu}(v)$ to $\mu^v$ for $v \in V_1(\Gamma)$, and define $\nu^v$ by (7.11) for $v \in V_3(\Gamma)$. The following is a direct consequence of Proposition 7.4.

**Corollary 7.6.**

$$
Z^\Gamma_{\text{rel}}(\lambda; t; p) = \sum_{(\vec{\nu}, \vec{\mu}) \in T^\Gamma} \prod_{e \in E(\Gamma)} e^{-|\nu^e| t^e (-1)^{(n_e+1)|\nu^e|} e^{-\sqrt{-1} \kappa_\nu^e n_e^e \lambda/2}}
\cdot \prod_{v \in V_3(\Gamma)} \tilde{C}_{\vec{\nu}^v}(\lambda) \prod_{v \in V_1(\Gamma), \nu_0(e) = \nu_v} \frac{\chi_{\nu^v}(\mu^v)}{\sqrt{-1} \Gamma(\mu^v)^2} z_{\mu^v}
$$

### 8. Combinatorial Expressions for the Topological Vertex

We use the notation introduced in Section 2.1. The goal of this section is to derive the following combinatorial expression for $\tilde{C}_{\vec{\mu}}(\lambda)$:

**Theorem 8.1.** Let $\vec{\mu} \in \mathcal{P}_+^3$. Then

$$
\tilde{C}_{\vec{\mu}}(\lambda) = \tilde{W}_{\vec{\mu}}(q),
$$

where $q = e^{\sqrt{-1} \lambda}$, and $\tilde{W}_{\vec{\mu}}(q)$ is defined by (2.5).

We now outline our strategy to prove Theorem 8.1. By Proposition 6.6,

$$
\tilde{C}_{\vec{\mu}}(\lambda) = \sum_{|\nu'| = |\mu'|} G^\bullet_{\vec{\nu}'}(\lambda; w),
$$

where $w$ is as in (2.13). Since the above sum is independent of $w$, we may take $w = (1, 1, -2)$ and obtain

$$
\tilde{C}_{\vec{\mu}}(\lambda) = \sum_{|\nu'| = |\mu'|} G^\bullet_{\vec{\nu}'}(\lambda; 1, 1, -2).
$$
In Section 8.1, we will show that the main result in [31] gives a combinatorial expression of $G_{\mu,\nu,\emptyset}(\lambda; w)$ (Theorem 8.7). In Section 8.2, we will relate $G_{\emptyset,\mu^1\cup\mu^2,\mu^3}(\lambda; 1, 1, -2)$ to $G_{\emptyset,\mu^1,\mu^2,\mu^3}(\lambda; 1, 1, -2)$. This gives the combinatorial expression $\tilde{W}_{\mu}(q)$ in Theorem 8.1. Moreover, (6.18) and Theorem 8.1 imply the following formula of three-partition Hodge integrals.

**Theorem 8.2** (Formula of three-partition Hodge integrals). Let $\mu \in P_3^+$. Then

$$G_{\mu}(\lambda; w) = \sum_{|\nu| = |\mu|} \chi_{\nu,\mu}(w) \frac{1}{z_{\mu}} \tilde{W}_{\nu}(q).$$

The cyclic symmetry of $\tilde{C}_{\mu}(\lambda)$ is obvious from definition. By Theorem 8.1 we have the following cyclic symmetry

$$\tilde{W}_{\mu^1,\mu^2,\mu^3}(q) = \tilde{W}_{\mu^3,\mu^1,\mu^2}(q) = \tilde{W}_{\mu^1,\mu^2,\mu^3}(q)$$

which is far from being obvious.

Finally, we conjecture that the combinatorial expression $\tilde{W}_{\mu}(q)$ coincides with $W_{\mu}(q)$ predicted in [1]:

**Conjecture 8.3.** Let $\mu \in P_3^+$. Then

$$\tilde{W}_{\mu}(q) = W_{\mu}(q),$$

where $q = e^{\sqrt{-1} \lambda}$, and $W_{\mu}(q)$ is defined by (2.4).

We have strong evidence for Conjecture 8.3. By Theorem 8.1 and Corollary 8.8, Conjecture 8.3 holds when one of the three partitions is empty. When none of the partitions is empty, A. Klemm has checked by computer that Conjecture 8.3 holds in all the cases where $|\mu^i| \leq 6, \ i = 1, 2, 3$.

We will list some of these cases in Section 8.4.

### 8.1 One-Partition and Two-Partition Hodge Integrals

We recall some notation in [30].

$$C_{\mu}(\lambda; \tau) = \sqrt{-1}^{|\mu|} G_{\mu,\emptyset,\emptyset}(\lambda; 1, \tau, -\tau - 1).$$

$$V_{\mu}(q) = q^{-\kappa_{\mu}/4} \sqrt{-1}^{\mu} W_{\mu}(q).$$

where $W_{\mu}(q) = W_{\mu,\emptyset,\emptyset}(q)$ is defined in Section 2.1. The main result of [30] is the following formula conjectured by Mariño and Vafa [33] (see [40] for another proof):

**Theorem 8.4.**

$$C_{\mu}(\lambda; \tau) = \sum_{|\nu| = |\mu|} \frac{\chi_{\nu,\mu}(q)}{z_{\mu}} q^{\kappa_{\nu}(\tau + \frac{1}{2})/2} V_{\nu}(q).$$

Theorem 8.4 can be reformulated in our notation as follows:

**Theorem 8.5** (Formula of one-partition Hodge integrals). Let $w$ be as in (2.13), and let $\mu \in P_+$. Then

$$G_{\mu,\emptyset,\emptyset}(\lambda; w) = \sum_{|\nu| = |\mu|} \frac{\chi_{\nu,\mu}(q)}{z_{\mu}} q^{\kappa_{\nu} \frac{w}{2} \nu} W_{\nu,\emptyset,\emptyset}(q).$$
Let
\begin{equation}
G^{\ast}_{\mu^{+}, \mu^{-}}(\lambda; \tau) = (-1)^{\mu^{+} - \ell(\mu^{-})} G^{\ast}_{\mu^{+}, \mu^{-}}(\lambda; 1, \tau, -1 - \tau).
\end{equation}
The main result of [31] is the following formula conjectured in [49]:

**Theorem 8.6.** Let \((\mu^{+}, \mu^{-}) \in \mathcal{P}_{\pm}^{2}\). Then
\begin{equation}
G^{\ast}_{\mu^{+}, \mu^{-}}(\lambda; \tau) = \sum_{|\nu^{+}| = |\mu^{+}|} \frac{\chi_{\nu^{+}}(\mu^{+})}{z_{\mu^{+}}} \frac{\chi_{\nu^{-}}(\mu^{-})}{z_{\mu^{-}}} q^{(\kappa_{\nu^{+} \tau} + \kappa_{\nu^{-} \tau}^{-1})/2} \mathcal{W}_{\nu^{+}, \nu^{-}}(q).
\end{equation}

We now reformulate Theorem 8.6 in the notation of this paper.
\begin{align*}
G^{\ast}_{\mu^{1}, \mu^{2}, \emptyset}(\lambda; 1, \tau, -1 - \tau) &= (-1)^{|\mu^{1}| - \ell(\mu^{2})} \sum_{|\nu^{i}| = |\mu^{i}|} \frac{\chi_{\nu^{1}}(\mu^{1})}{z_{\mu^{1}}} \frac{\chi_{\nu^{2}}(\mu^{2})}{z_{\mu^{2}}} q^{(\kappa_{\nu^{1} \tau} + \kappa_{\nu^{2} \tau}^{-1})/2} \mathcal{W}_{\nu^{1}, \nu^{2}}(q) \\
&= \sum_{|\nu^{i}| = |\mu^{i}|} \frac{\chi_{\nu^{1}}(\mu^{1})}{z_{\mu^{1}}} \frac{\chi_{\nu^{2}}(\mu^{2})}{z_{\mu^{2}}} q^{(\kappa_{\nu^{1} \tau} + \kappa_{\nu^{2} \tau}^{-1})/2} q^{\kappa_{\nu^{2}}/2} \mathcal{W}_{\nu^{1}, \nu^{2}}(q) \\
&= \sum_{|\nu^{i}| = |\mu^{i}|} \frac{\chi_{\nu^{1}}(\mu^{1})}{z_{\mu^{1}}} \frac{\chi_{\nu^{2}}(\mu^{2})}{z_{\mu^{2}}} q^{(\kappa_{\nu^{1} \tau} + \kappa_{\nu^{2} \tau}^{-1})/2} \mathcal{W}_{\nu^{1}, \nu^{2}, \emptyset}(q)
\end{align*}

Theorem 8.6 is equivalent to the following:

**Theorem 8.7** (Formula of two-partition Hodge integrals). Let \(w\) be as in (2.13) and let \((\mu^{1}, \mu^{2}) \in \mathcal{P}_{\pm}^{2}\). Then
\begin{equation}
G^{\ast}_{\mu^{1}, \mu^{2}, \emptyset}(\lambda; w) = \sum_{|\nu^{1}| = |\mu^{1}|} \sum_{|\nu^{2}| = |\mu^{2}|} \frac{\chi_{\nu^{1}}(\mu^{1})}{z_{\mu^{1}}} \frac{\chi_{\nu^{2}}(\mu^{2})}{z_{\mu^{2}}} q^{(\kappa_{\nu^{1} \tau} + \kappa_{\nu^{2} \tau}^{-1})/2} q^{\kappa_{\nu^{2}}/2} \mathcal{W}_{\nu^{1}, \nu^{2}}(\emptyset, q).
\end{equation}

Note that Theorem 8.5 corresponds to the special case where \((\mu^{1}, \mu^{2}) = (\mu, \emptyset)\). Theorem 8.7 and (6.17) imply

**Corollary 8.8.** Let \(\bar{\mu} = (\mu^{1}, \mu^{2}, \mu^{3}) \in \mathcal{P}_{\pm}^{3}\), and let \(q = e^{\sqrt{-1} \lambda}\). Then
\begin{equation}
\tilde{G}_{\bar{\mu}}(\lambda) = \mathcal{W}_{\bar{\mu}}(q)
\end{equation}
when one of \(\mu^{1}, \mu^{2}, \mu^{3}\) is empty.

### 8.2. Reduction

Recall that
\begin{equation}
G_{g, \bar{\mu}}(\tau) = G_{g, \bar{\mu}}(1, \tau, -\tau - 1).
\end{equation}

For two partitions \(\mu^{1}\) and \(\mu^{2}\), let \(\mu^{1} \cup \mu^{2}\) be the partition with
\begin{equation*}
m_{i}(\mu^{1} \cup \mu^{2}) = m_{i}(\mu^{1}) + m_{i}(\mu^{2}), \quad \forall i \geq 1.
\end{equation*}

We have

**Lemma 8.9.** Let \(\bar{\mu} = (\mu^{1}, \mu^{2}, \mu^{3}) \in \mathcal{P}_{\pm}^{3}\). Then
\begin{equation}
G_{g, \bar{\mu}}(\lambda; 1) = (-1)^{|\mu^{1}| - \ell(\mu^{2})} \sum_{|\nu^{1}| = |\mu^{1}|} \sum_{|\nu^{2}| = |\mu^{2}|} \frac{\chi_{\nu^{1}}(\mu^{1})}{z_{\mu^{1}}} \frac{\chi_{\nu^{2}}(\mu^{2})}{z_{\mu^{2}}} q^{(\kappa_{\nu^{1} \tau} + \kappa_{\nu^{2} \tau}^{-1})/2} q^{\kappa_{\nu^{2}}/2} \mathcal{W}_{\nu^{1}, \nu^{2}}(\emptyset, q)
\end{equation}

\begin{align*}
+ \delta_{g \emptyset, \mu^{3}} \sum_{m \geq 1} \delta_{\mu^{1}(m)} \delta_{\mu^{2}(2m)} (-1)^{m-1} \frac{1}{m}
\end{align*}
Proof. Let
\[ I_{g, \bar{\mu}}(w) = \int_{\mathcal{M}_{g, \ell(\bar{\mu})}} \prod_{i=1}^{3} \frac{\Delta_Y(w_i) w_i^{\ell(\bar{\mu})}}{\prod_{j=1}^{2} (w_i - \mu_j^i \psi_{d_i^j + j})} \]
and let \( I_{g, \bar{\mu}}(\tau) = I_{g, \bar{\mu}}(1, \tau, -\tau - 1) \). Then
\[ I_{0, \bar{\mu}}(\tau) = \frac{(\tau(-\tau - 1))^{\ell(\bar{\mu})}}{\tau^{2\ell(\mu^3)}(-\tau - 1)^{2\ell(\mu^3)}} \left( |\mu_1^1| + \frac{|\mu_1^2|}{\tau} + \frac{|\mu_1^3|}{-\tau - 1} \right)^{\ell(\bar{\mu}) - 3} \]
Note that \( I_{g, \bar{\mu}}(\tau) \) has a pole at \( \tau = 1 \) only if
\[ g = 0, \quad \bar{\mu} = ((m), (2m)) \text{ or } (\emptyset, (2m)), \]
where \( m > 0 \). Let
\[ E_\mu(\tau) = \prod_{j=1}^{\ell(\mu)} \frac{\mu_j^{-1} (\mu_j + a)}{(\mu_j - 1)!}. \]
Then \( E_\mu(\tau) \) is a polynomial in \( \tau \) of degree \( |\mu| - \ell(\mu) \), and
\[ E_\mu(-\tau - 1) = (-1)^{|\mu| - \ell(\mu)} E_\mu(\tau). \]
Then
\[ G_{g, \bar{\mu}}(\tau) = \left( -\sqrt{-1} \right)^{\ell(\bar{\mu})} E_\mu(\tau) E_{\mu^2}(-1 - \tau^{-1}) \left( \frac{1}{-\tau - 1} \right) \]
while
\[ G_{g, \emptyset, \mu^1 \cup \mu^2, \mu^3}(\tau) = \left( \frac{-\sqrt{-1}}{|\text{Aut}(\mu^1 \cup \mu^2) \times \text{Aut}(\mu^3)|} E_{\mu^1 \cup \mu^2}(-1 - \tau^{-1}) \right) \]
where \( G_{\emptyset, \mu^1 \cup \mu^2, \mu^3}(\tau) = E_{\mu^1}(-1 - \tau^{-1}) E_{\mu^2}(-1 - \tau^{-1}) \).
Suppose that \((g, \bar{\mu})\) is not the exceptional case listed in (8.11). Then neither is \((g, \emptyset, \mu^1 \cup \mu^2, \mu^3)\). It is immediate from the definition that
\[ I_{g, \emptyset, \mu^1 \cup \mu^2, \mu^3}(1) = I_{g, \emptyset, \mu^1 \cup \mu^2, \mu^3}(1), \]
so
\[ G_{g, \bar{\mu}}(1) = (-1)^{|\mu| - \ell(\mu^1)} \frac{|\text{Aut}(\mu^1 \cup \mu^2)|}{|\text{Aut}(\mu^1) \times \text{Aut}(\mu^2)|} G_{g, \emptyset, \mu^1 \cup \mu^2, \mu^3}(1) \]
For the exceptional case (8.11), we have
\[ G_{0, (m), \emptyset, (2m)}(\tau) = \frac{\tau}{(\tau + 1)(m - 1)!} \prod_{a=1}^{m-1} \frac{(\tau m + a)}{2m - 1} \prod_{a=m+1}^{2m-1} \frac{2m - 1 + a}{a - 1 + a} \]
while
\[ G_{0, \emptyset, (m), (2m)}(\tau) = \frac{-1}{(\tau + 1)(m - 1)!2m!} \prod_{a=1}^{m-1} \left( \frac{-\tau - 1}{\tau} m + a \right) \]
\[ \times \prod_{a=1}^{m-1} \left( \frac{2m}{-\tau - 1} + a \right) \prod_{a=m+1}^{2m-1} \left( \frac{2m}{-\tau - 1} + a \right) \]

So
\[ G_{0, \emptyset, (m), (2m)}(1) = \frac{(-1)^{m-1}}{2m}, \quad G_{0, \emptyset, (m), (2m)}(1) = \frac{-1}{2m}. \]

Combining the general case (8.13) and the exceptional case (8.14), we obtain (8.9).

Let \( p, p^+, p^\mu \) be defined as in Section 2.3, and let \( G^*(\lambda; p; \tau) \) be defined as in (2.16). We have

**Lemma 8.10.** Let
\[ p^+_i = (-1)^{i-1}p^1_i + p^2_i, \quad p^+_\mu = \prod_{j=1}^{\ell(\mu)} p^+_\mu_j. \]

Then
\[ G^*(\lambda; p^1, p^2, p^3; 1) = G^*(\lambda; 0, p^+, p^3; 1) \exp\left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p^1_m p^3_m \right). \]

**Proof.** We have
\[ G(\lambda; p; 1) = G(\lambda; p; 1, 1, -2) = \sum_{\mu, \emptyset} \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\mu)} G_{g, \emptyset, \mu^1, \mu^2, \mu^3}(1) \left( \prod_{\mu_1 \cup \mu_2 = \mu^1} z_{\mu_1} z_{\mu_2} \right) \left( -1 \right)^{|\mu^1| - \ell(\mu^1)} p^1_\mu p^2_\mu p^3_\mu + \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p^1_m p^3_m. \]

By Lemma 8.9,
\[ G(\lambda; p; 1) \]
\[ = \sum_{\mu, \emptyset} \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\mu)} G_{g, \emptyset, \mu^1, \mu^2, \mu^3}(1) \left( \prod_{\mu_1 \cup \mu_2 = \mu^1} z_{\mu_1} z_{\mu_2} \right) \left( -1 \right)^{|\mu^1| - \ell(\mu^1)} p^1_\mu p^2_\mu p^3_\mu + \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p^1_m p^3_m. \]

It is easy to see that
\[ \sum_{\mu_1 \cup \mu_2 = \mu^1} \frac{z_{\mu^1}}{z_{\mu_1} z_{\mu_2}} (-1)^{|\mu^1| - \ell(\mu^1)} p^1_\mu p^2_\mu = p^+_\mu. \]

So
\[ G(\lambda; p^1, p^2, p^3; 1) = G(\lambda; 0, p^+, p^3; 1) + \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p^1_m p^3_m. \]

which is equivalent to (8.16).
8.3. Combinatorial Expression.

Lemma 8.11. Let \( p^+ \) be defined by (8.15).

\[
G^*(\lambda; 0, p^+, p^3; 1) = \sum_{\nu^+, \mu^+, \mu^3 \in \mathcal{P}} c_{\nu^+}^{\mu^+} g_{\nu^+, \mu^+, \mu^3}(q) (-1)^{|\mu^3| - \ell(\mu^3)} \prod_{i=1}^{3} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} p_{\mu^i}.
\]

Proof. By Theorem 8.7,

\[
G^*(\lambda; 0, p^+, p^3; 1) = \sum_{\mu^+ \in \mathcal{P}} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^3}(\mu^3)}{z_{\mu^3}} g_{\nu^+, \mu^+, \mu^3}(q) (-1)^{|\mu^3| - \ell(\mu^3)} \prod_{i=1}^{3} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} p_{\mu^i}.
\]

Recall that

\[
W_{\nu^+, \nu^3}(q) = q^{\nu^+ \cdot \nu^3} W_{\nu^+, \nu^3}(q),
\]

\[
p_{\mu^+} = \sum_{\mu^1 \cup \mu^2 = \mu^+} \frac{z_{\mu^+}}{z_{\mu^1} z_{\mu^2}} (-1)^{|\mu^1| - \ell(\mu^1)} p_{\mu^1} p_{\mu^2}^2.
\]

Let \( s^i_{\mu} \) be Schur functions. Then

\[
G^*(\lambda; 0, p^+, p^3; 1) = \sum_{\mu^+ \in \mathcal{P}} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^3}(\mu^3)}{z_{\mu^3}} g_{\nu^+, \mu^+, \mu^3}(q) (-1)^{|\mu^3| - \ell(\mu^3)} \prod_{i=1}^{3} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} p_{\mu^i}.
\]

In the above we have used (8.17) and the following identity:

\[
c_{\nu^+}^{\mu^+} = \sum_{\nu^+, \mu^-} \frac{\chi_{\nu^+}(\nu^0) \chi_{\nu^-}(\mu^-)}{z_{\nu^+} z_{\nu^-}}.
\]
Remark 8.12. By the same method we also have

\[(8.20) \quad G^\bullet(\lambda; 0, p^+, p^3; 1) = \sum_{\gamma^+, \gamma', \mu' \in \mathcal{P}} \lambda^{(\gamma^+), \gamma', \mu'} q^{(-2\kappa_{\gamma^+} + \frac{\kappa_{\gamma^+}}{2})/2} \mathcal{W}_{\gamma^+, \gamma', \mu'}(q) \prod_{i=1}^{3} \frac{\chi_{\gamma^+}(\mu_i)}{z_{\mu_i}} p_{\mu_i}^i.\]

Lemma 8.13. We have

\[(8.21) \quad \exp\left( -\sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p_m^1 p_2^3 \right) = \sum_{\mu \in \mathcal{P}} (-1)^{|\mu| - \ell(\mu)} z_{\mu} p_{\mu}^1 p_{\mu}^2 p_{\mu}^3 \]

where \(2\mu\) is the partition \((2\mu_1, 2\mu_2, \ldots, 2\mu_{\ell(\mu)})\).

Proof. Let \((x_1, \ldots, x_n, \ldots)\) be formal variables such that \(p_m^i = \sum_n (x_n^i)^m\). By standard series manipulations,

\[
\exp\left( -\sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p_m^1 p_2^3 \right) = \exp\left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \left( \sum_{n_1, n_3} (x_{n_1}^1)^m (x_{n_3}^3)^{2m} \right) \right) = \prod_{n_1, n_3} \exp\left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} (p_{n_1}^1 (p_{n_3}^3)^2)^m \right) = \prod_{n_1, n_3} (1 + x_{n_1}^1 (x_{n_3}^3)^2).\]

Now recall (cf. [32], p. 65, (4.1')):

\[
\prod_{i,j} (1 + x_i y_j) = \sum_{\mu \in \mathcal{P}} (-1)^{|\mu| - \ell(\mu)} z_{\mu} p_{\mu}(x)p_{\mu}(y).\]

Hence we have

\[
\exp\left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p_m^1 p_2^3 \right) = \sum_{\mu \in \mathcal{P}} (-1)^{|\mu| - \ell(\mu)} z_{\mu} p_{\mu}(x)p_{\mu}(x^3) = \sum_{\mu \in \mathcal{P}} (-1)^{|\mu| - \ell(\mu)} z_{\mu} p_{\mu}(x)p_{2\mu}(x^3).\]

\[\Box\]
By Lemma 8.10, Lemma 8.11, and Lemma 8.13, we have

\[ G^*(\lambda; p^1, p^2, p^3; 1) = \sum_{\mu \in \mathcal{P}} C_{(\nu_1)}^{\mu_1} q^{-(2\kappa_{\nu_1} - \frac{\kappa_{\nu_2} + \kappa_{\nu_3}}{2})/2} W_{(\nu_2), \nu_3} s_{\mu_1}^{1} s_{\mu_2}^{2} s_{\mu_3}^{3} \]

By Proposition 6.6,

\[ G^*(\lambda; p^1, p^2, p^3; 1) \]

By comparing coefficients,

\[ \sum_{\mu \in \mathcal{P}} C_{(\nu_1)}^{\mu_1} q^{-(2\kappa_{\nu_1} - \frac{\kappa_{\nu_2} + \kappa_{\nu_3}}{2})/2} W_{(\nu_2), \nu_3} s_{\mu_1}^{1} s_{\mu_2}^{2} s_{\mu_3}^{3} \]

Therefore,

\[ \tilde{C}_{\tilde{\mu}}(\lambda) = \tilde{W}_{\tilde{\mu}}(q) \]

where \( \tilde{W}_{\tilde{\mu}}(q) \) is defined by (2.5). This completes the proof of Theorem 8.1.

**Remark 8.14.** By (8.20) one gets a slightly different expression.

### 8.4. Examples of Conjecture 8.3

Recall that

**Conjecture 8.3.** Let \( \tilde{\mu} \in \mathcal{P}_3^\lambda \). Then

\[ \tilde{W}_{\tilde{\mu}}(q) = W_{\tilde{\mu}}(q), \]

where \( q = e^{\sqrt{-1} \Lambda} \), and \( W_{\tilde{\mu}}(q) \) is defined by (2.4).

We have seen in Section 8 that Conjecture 8.3 holds when one of the three partitions is empty. When none of the partitions is empty, A. Klemm has checked by computer that Conjecture 8.3 holds in all the cases where

\[ |\mu^i| \leq 6, \quad i = 1, 2, 3. \]
We list some of these cases here.

\[
\hat{W}_{1,1,1}(q) = \frac{q^4 - q^3 + q^2 - q + 1}{q^{1/2}(q-1)^3}
\]

\[
\hat{W}_{1,1,2}(q) = \frac{q^6 - q^5 + q^3 - q + 1}{(q^2 - 1)(q-1)^3}
\]

\[
\hat{W}_{1,1,3}(q) = \frac{q^6 - q^5 + q^3 - q + 1}{q(q^2 - 1)(q-1)^3}
\]

\[
\hat{W}_{1,1,4}(q) = \frac{q^{3/2}(q^8 - q^7 + q^4 - q + 1)}{(q^3 - 1)(q^2 - 1)(q-1)^3}
\]

\[
\hat{W}_{1,2,1}(q) = \frac{q^{8} - 2q^7 + 3q^6 - 3q^5 + 3q^4 - 3q^3 + 3q^2 - 2q + 1}{q^{1/2}(q^3 - 1)(q-1)^4}
\]

\[
\hat{W}_{1,1,1,1}(q) = \frac{q^8 - q^7 + q^4 - q + 1}{q^{3/2}(q^3 - 1)(q^2 - 1)(q-1)^3}
\]

\[
\hat{W}_{1,2,2}(q) = \frac{q^{1/2}(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1)}{(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,1,1,2}(q) = \frac{q^8 - q^7 + 2q^6 - q^4 + q^3 - q + 1}{q^{3/2}(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,1,1,3}(q) = \frac{q^8 - q^7 + q^5 - q^4 + q^3 + q + 1}{q^{3/2}(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,1,1,4}(q) = \frac{q^4(q^{10} - q^9 + q^5 + q + 1)}{(q^4 - 1)(q^3 - 1)(q^2 - 1)(q-1)^5}
\]

\[
\hat{W}_{1,1,2,1}(q) = \frac{q(q^{10} - 2q^9 + 2q^8 - 2q^6 + 3q^5 - 2q^4 + 2q^2 - 2q + 1)}{(q^4 - 1)(q^3 - 1)(q^2 - 1)(q-1)^4}
\]

\[
\hat{W}_{1,1,2,2}(q) = \frac{q(q^8 - 2q^6 + q^5 + q^4 + q^3 - 2q^2 + 1)}{(q^3 - 1)(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,1,2,3}(q) = \frac{q^{10} - 2q^9 + 2q^8 - 2q^6 + 3q^5 - 2q^4 + 2q^2 - 2q + 1}{q(q^4 - 1)(q^2 - 1)(q-1)^4}
\]

\[
\hat{W}_{1,1,2,4}(q) = \frac{q^{10} - q^9 + q^5 - q + 1}{q^2(q^4 - 1)(q^3 - 1)(q^2 - 1)(q-1)^3}
\]

\[
\hat{W}_{1,2,3}(q) = \frac{q^2(q^{10} - q^9 + q^6 - q^4 + q^3 + q + 1)}{(q^3 - 1)(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,3,2}(q) = \frac{q^2(q^{10} - q^9 + q^7 - q^6 + q^4 - q + 1)}{(q^3 - 1)(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,2,1,1}(q) = \frac{q^{11} - 2q^{10} + 2q^9 - q^8 + 2q^6 + q^4 - q^3 - 2q^2 + 2q + 1}{(q^3 - 1)(q^2 - 1)^2(q-1)^3}
\]

\[
\hat{W}_{1,2,1,2}(q) = \frac{q^{11} - q^{10} + q^7 - q^5 + q^4 - q^3 + 2q^2 - 2q + 1}{q(q^3 - 1)(q^2 - 1)(q-1)^4}
\]

\[
\hat{W}_{1,2,1,3}(q) = \frac{q^{12} - q^{11} + q^9 + q^8 - q^6 + q^4 - q + 1}{q(q^3 - 1)(q^2 - 1)^2(q-1)^3}
\]
\[ \mathcal{W}_{(1),(1,1),(2)}(q) = \mathcal{W}_{(1),(1,1),(2)}(q) = \frac{q^{12} - q^{11} + q^8 - q^6 + q^4 + q^3 - q^2 - q + 1}{q^3(q^3 - 1)(q^2 - 1)^2(q - 1)^3} \]

\[ \mathcal{W}_{(1),(1,1),(3)}(q) = \mathcal{W}_{(1),(1,1),(3)}(q) = \frac{q^{12} - q^{11} + q^8 - q^6 + q^4 + q^3 - q^2 - q + 1}{q^3(q^3 - 1)(q^2 - 1)^2(q - 1)^3} \]

\[ \mathcal{W}_{(1),(3),(1)}(q) = \mathcal{W}_{(1),(3),(1)}(q) = \frac{q(q^{12} - q^{11} - q^{10} + q^9 + q^8 - q^6 + q^4 - q + 1)}{(q^3 - 1)(q^2 - 1)^2(q - 1)^3} \]

\[ \mathcal{W}_{(1),(1),(1),(2,1)}(q) = \mathcal{W}_{(1),(1),(2,1)}(q) = \frac{q^{11} - q^{10} + q^7 - q^5 + q^4 - q^3 + 2q^2 - 2q + 1}{q^2(q^3 - 1)(q^2 - 1)(q - 1)^4} \]

\[ \mathcal{W}_{(1),(2),(1),(1)}(q) = \mathcal{W}_{(1),(2),(1)}(q) = \frac{q^{11} - 2q^{10} + 2q^9 - q^8 + q^7 - q^6 + q^4 - q + 1}{q^3(q^2 - 1)(q^2 - 1)(q - 1)^4} \]

\[ \mathcal{W}_{(1),(1),(1),(1,1,1)}(q) = \mathcal{W}_{(1),(1),(1,1,1)}(q) = \frac{q^{10} - q^9 + q^7 - q^6 + q^4 - q + 1}{q^2(q^3 - 1)(q^2 - 1)^2(q - 1)^3} \]

\[ \mathcal{W}_{(1),(1),(1),(1,1,1)}(q) = \mathcal{W}_{(1),(1),(1,1,1)}(q) = \frac{q^{10} - q^9 + q^6 - q^4 + q^3 - q + 1}{q^2(q^3 - 1)(q^2 - 1)^2(q - 1)^3} \]

\[ \mathcal{W}_{(2),(2),(2)}(q) = \mathcal{W}_{(2),(2)}(q) = \frac{q(q^{10} - 3q^8 + 3q^7 + 2q^6 - 5q^5 + 2q^4 + 3q^3 - 3q^2 + 1)}{(q^2 - 1)^3(q - 1)^3} \]

\[ \mathcal{W}_{(2),(2),(2,1)}(q) = \mathcal{W}_{(2),(2,1)}(q) = \frac{q^{12} - q^{11} - q^{10} + 2q^9 - q^7 + q^6 - q^5 + 2q^3 - q^2 - q + 1}{q(q^2 - 1)^3(q - 1)^3} \]

\[ \mathcal{W}_{(2),(1,1),(1,1)}(q) = \mathcal{W}_{(2),(1,1)}(q) = \frac{q^{12} - q^{11} - q^{10} + 2q^9 - q^7 + q^6 - q^5 + 2q^3 - q^2 - q + 1}{q^2(q^2 - 1)^3(q - 1)^3} \]

\[ \mathcal{W}_{(1,1),(1,1),(1,1)}(q) = \mathcal{W}_{(1,1),(1,1)}(q) = \frac{q^{10} - 3q^8 + 3q^7 + 2q^6 - 5q^5 + 2q^4 + 3q^3 - 3q^2 + 1}{q^2(q^2 - 1)^3(q - 1)^3} \]

\[ \mathcal{W}_{(1),(2),(3,1)}(q) = \mathcal{W}_{(1),(2)}(q) = \frac{q^{7/2}(q^{13} - 2q^{12} + q^{11} + 2q^{10} - 3q^9 + 2q^8 - 2q^6 + 2q^5 - q + 1)}{(q^4 - 1)(q^2 - 1)^2(q - 1)^4} \]

\[ \mathcal{W}_{(1),(2),(1,1),(3)}(q) = \mathcal{W}_{(1),(2),(1,1)}(q) = \frac{(q^{19} - q^{18} - q^{17} + q^{16} + q^{15} - q^{14} + q^{13} + q^{11} - q^{10}}{(q^4 - 1)(q^2 - 1)^2(q - 1)^4} \]

\[ \mathcal{W}_{(2),(2),(2,1,1)}(q) = \mathcal{W}_{(2),(2,1,1)}(q) = \frac{(q^{22} - q^{21} - 2q^{20} + 3q^{19} + q^{18} - 3q^{17} + 3q^{15} - q^{14} - 2q^{13} + q^{12} + q^{11} + q^{10} - 2q^9 - q^8}}{(q^4 - 1)(q^2 - 1)^2(q - 1)^4} \]

\[ \mathcal{W}_{(1),(2),(2),(3,2)}(q) = \mathcal{W}_{(1),(2),(3,2)}(q) = \frac{(q^{23} - 2q^{22} + q^{21} + q^{20} - q^{19} + q^{18} - 2q^{17} + q^{16} + q^{15} + q^{13} - 3q^{12} + q^{10} + 2q^9 + q^8}}{(q^4 - 1)(q^2 - 1)^2(q - 1)^4} \]

\[ \mathcal{W}_{(1),(2),(2,1,1)}(q) = \mathcal{W}_{(1),(2,1,1)}(q) = \frac{2q^7 - 2q^6 + 2q^4 + 2q^3 - 2q^2 - q + 1 \cdot (q(q^4 - 1)(q^3 - 1)(q^3 - 1)(q - 1)^4} \]
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