Spectral semi-Fredholm theory on Hilbert C*-modules

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Abstract Given an \( A \)-linear, bounded, adjointable operator \( F \) on the standard module \( H_A \), we consider the operators of the form \( F - \alpha 1 \) as \( \alpha \) varies over \( Z(A) \) and this gives rise to a different kind of spectra of \( F \) in \( Z(A) \) as a generalization of ordinary spectra of \( F \) in \( C \). Using the generalized definitions of Fredholm and semi-Fredholm operators on \( H_A \) given in [3] and [2] together with these new, generalized spectra in \( Z(A) \) we obtain several results as a generalization of the results from the classical spectral semi-Fredholm theory given in [1], [6], [7], [8], [9].

We consider first \( 2 \times 2 \) operator matrix \( M^\alpha_A = \begin{bmatrix} F & C \\ D & 0 \end{bmatrix} \), acting on \( H_A \oplus H_A \) and investigate the relationship between \( M^\alpha_A \) and \( F, D \) in the context of semi-\( A \)-Fredholm properties and the generalized \( A \)-Fredholm spectra in \( Z(A) \). Moreover, we define the generalized compressions of an operator \( F \) on \( H_A \) and give a description of various (generalized) \( A \)-Fredholm spectra in \( Z(A) \) of these compressions. Finally, we prove a chain of inclusions concerning the boundaries of several kinds of the generalized spectra in \( Z(A) \) of operator \( F \).

Keywords Hilbert C*- module · semi-\( A \)-Fredholm operator · \( A \)-Fredholm spectra · perturbations of spectra · compressions

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1 Introduction

The main aim of this paper is to establish the spectral semi-Fredholm theory on Hilbert C*-modules as a generalization of certain aspects and results of the classical spectral semi-Fredholm theory on Hilbert and Banach spaces. Some aspects of the classical semi-Fredholm theory concerning the perturbation of spectra of operator matrices were investigated in \[1\], the paper which is going to one of the main references in this paper. In \[1\] Djordjevic lets X and Y be Banach spaces and the operator \( M_C : X \oplus Y \to X \oplus Y \) be given as \( 2 \times 2 \) operator matrix

\[
\begin{bmatrix}
A & C \\
0 & B
\end{bmatrix}
\]

where \( A \in B(X) \), \( B \in B(Y) \) and \( C \in B(Y, X) \). Djordjevic investigates the relationships between certain semi-Fredholm properties of \( A, B \) and certain semi-Fredholm properties of \( M_C \). Then he deduces as corollaries the description of the intersection of spectra of \( M_C \)'s, when \( C \) varies over all operators in \( B(Y, X) \) and \( A, B \) are fixed, in terms of spectra of \( A \) and \( B \). The spectra which he considers are not in general ordinary spectra, but rather different kind of Fredholm spectra such as essential spectra, left and right Fredholm spectra etc...

Fredholm theory on Hilbert C*-modules as a generalization of Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko in \[2\]. They have elaborated the notion of a Fredholm operator on the standard module \( H_A \) and proved the generalization of the Atkinson theorem. In \[2\] one goes further in this direction and defines semi-Fredholm operators on Hilbert C*-modules. One investigates then and proves several properties of these generalized semi Fredholm operators on Hilbert C*-modules as an analogue or generalization of the well-known properties of classical semi-Fredholm operators on Hilbert and Banach spaces. One introduces new classes of operators on \( H_A \) such as \( M\Phi_\pm(H_A) \), \( M\Phi_{\pm}(H_A) \), \( M\Phi^\pm(H_A) \), \( M\Phi^\pm_{\pm}(H_A) \), \( \Phi_{\pm}(H_A) \), \( \Phi_{\pm}(H_A) \) as a various generalizations of the classes \( \Phi_{\pm}(H), \Phi_{\pm}(H) \), \( \Phi_{\pm}(H) \), \( \Phi_{\pm}(H) \) where \( H \) is a Hilbert space.

The idea in chapter 3 in this paper was to use these new classes of operators on \( H_A \) and prove that an analogue or a generalized version of certain results in \[1\], such as

- \[1\] Proposition 3.1]
- \[1\] Theorem 3.1]
- \[1\] Corollary 3.1]
- \[1\] Proposition 3.2]
- \[1\] Proposition 3.3]
- \[1\] Theorem 4.1]
- \[1\] Corollary 4.1]
- \[1\] Theorem 4.2]
- \[1\] Corollary 4.2]

hold when one considers these new classes of operators and these generalized Fredholm spectra in \( Z(A) \) of operators in \( B^\infty(H_A) \).

Furthermore, in chapter 4 we consider the compressions on \( H_A \). A compression on a Banach space \( X \) in \[2\] is defined in the following way:

Let \( P(X) \) denote the set of all bounded projections \( P \in B(X) \) such that \( \text{codim} R(P) < \infty \). For \( A \in B(X) \) and \( P \in P(X) \) the compression

\( A_P : R(P) \to R(P) \) is defined by \( A_P y = P A y, y \in R(P) \), i.e. \( A_P = PA_{|R(P)} \), where \( A_{|R(P)} : R(P) \to X \) is the restriction of \( A \). Clearly, \( R(P) \) is a Banach space and \( A_P \in B(P) \).

A natural generalization of compressions on \( H_A \) would be the following:

Let \( P(H_A) = \{ P \in B(H_A) \mid P \text{ is the projection and ker}(P) \text{ is finitely generated} \} \).

For \( F \in B(H_A) \) and \( P \in P(H_A) \), the compression \( F_P \in B(\text{Im}(P)) \) is given by \( F_P = PF_{|\text{Im}(P)} \).

We consider these generalized compressions and prove in this setting generalizations of the main results of \[2\] Lemma 2.10.1], \[3\] Theorem 2.10.2].
In these results we give a description of various kinds of \( \mathcal{A} \)-Fredholm spectra in \( Z(\mathcal{A}) \) of a given operator \( F \in B^a(\mathcal{H}_A) \) in terms of the intersection, as \( P \) varies over \( P(\mathcal{H}_A) \), of certain kinds of Fredholm spectra in \( Z(\mathcal{A}) \) of the operators of the form \( PF|_{R(P)} \) in \( B^a(R(P)) \) or \( B^a(R(P)) \).

Finally, in chapter 5, we consider the boundaries in \( Z(\mathcal{A}) \) (recall that \( Z(\mathcal{A}) \) a \( \mathcal{C}^* \)-subalgebra of \( \mathcal{A} \)) of several kinds of Fredholm spectra in \( Z(\mathcal{A}) \) of given \( F \in B^a(\mathcal{H}_A) \) and prove the chain of inclusions as a generalization of [6, Theorem 2.2.2], [6, Theorem 2.7.5], [6, Theorem 2.7.6], originally given in [7].

\section{Preliminaries}

In this section we are going to introduce the notation, the definitions and some of the results in [2] that are needed in this paper as well as some auxiliary results which are going to be used later in the proofs. Throughout this paper we let \( \mathcal{A} \) be a unital \( \mathcal{C}^* \)-algebra, \( \mathcal{H}_A \) be the standard module over \( \mathcal{A} \) and we let \( B^a(\mathcal{H}_A) \) denote the set of all bounded, adjointable operators on \( \mathcal{H}_A \). Next, for the \( \mathcal{C}^* \)-algebra \( \mathcal{A} \), we let \( Z(\mathcal{A}) = \{ \alpha \in \mathcal{A} \mid \alpha \beta = \beta \alpha \text{ for all } \beta \in \mathcal{A} \} \) and for \( \alpha \in Z(\mathcal{A}) \) we let \( \alpha I \) denote the operator from \( \mathcal{H}_A \) into \( \mathcal{H}_A \) given by \( \alpha I(x) \text{ for all } x \in \mathcal{H}_A \). The operator \( \alpha I \) is obviously \( \mathcal{A} \)-linear since \( \alpha \in Z(\mathcal{A}) \) and it is adjointable with its adjoint \( \alpha^* I \).

Set
\[
M_\Phi^+ (\mathcal{H}_A) = \{ F \in B^a(\mathcal{H}_A) \mid F \text{ is upper semi-} \mathcal{A}\text{-Fredholm } \},
\]
\[
M_\Phi^- (\mathcal{H}_A) = \{ F \in B^a(\mathcal{H}_A) \mid F \text{ is lower semi-} \mathcal{A}\text{-Fredholm } \},
\]
\[
M_\Phi (\mathcal{H}_A) = \{ F \in B^a(\mathcal{H}_A) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } \mathcal{H}_A \}.
\]

\begin{remark}
Notice that if \( M, N \) are two arbitrary Hilbert modules \( \mathcal{C}^* \)-modules, the definition above could be generalized to the classes \( M_\Phi^+ (M, N) \) and \( M_\Phi^- (M, N) \).
\end{remark}
is an $\mathcal{M}\Phi$ decomposition for $F$, then the index of $F$ is defined by index $F = [N] = [N_1] - [N_2] \in K(A)$ where $[N_1]$ and $[N_2]$ denote the isomorphism classes of $N_1$ and $N_2$ respectively. By [4] Definition 2.7.9, the index is well defined and does not depend on the choice of $\mathcal{M}\Phi$ decomposition for $F$. As regards the $K$-group $K(A)$, it is worth mentioning that it is not true in general that $[M] = [N]$ implies that $M \cong N$ for two finitely generated submodules $M, N$ of $H_A$. If $K(A)$ satisfies the property that $[N] = [M]$ implies that $N \cong M$ for any two finitely generated, closed submodules $M, N$ of $H_A$, then $K(A)$ is said to satisfy "the cancellation property", see [5] Section 6.2.

Theorem 1 [2] Theorem 2.2] Let $F \in B^n(H_A)$. The following statements are equivalent:
1) $F \in \mathcal{M}\Phi_+(H_A)$
2) There exists $D \in B^n(H_A)$ such that $DF = I + K$ for some $K \in K(H_A)$

Theorem 2 [2] Theorem 2.3] Let $D \in B^n(H_A)$. Then the following statements are equivalent:
1) $D \in \mathcal{M}\Phi_-(H_A)$
2) There exist $F \in B^n(H_A), K \in K(H_A)$ such that $DF = I + K$ for some $K \in K(H_A)$

Corollary 1 Let $M, N, W$ be Hilbert $C^*$-modules over a unital $C^*$-algebra $A$. If $F \in B^n(M, N), D \in B^n(N, W)$ and $DF \in \mathcal{M}\Phi(M, W)$, then there exists a chain of decompositions

$$M = M_2^\perp \oplus M_2 \xrightarrow{F} F(M_2^\perp) \oplus R \xrightarrow{D} W_1 \oplus W_2 = W$$

w.r.t. which $F, D$ have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix}$, respectively, where $F_1, D_1$ are isomorphisms, $M_2, W_2$ are finitely generated, $F(M_2^\perp) \oplus R = N$ and in addition $M = M_2^\perp \oplus M_2 \xrightarrow{DF} W_1 \oplus W_2 = W$ is an $\mathcal{M}\Phi$-decomposition for $DF$.

Proof By the proof of [1] Theorem 2.7.6 applied to the operator $DF \in \mathcal{M}\Phi(M, W)$, there exists an $\mathcal{M}\Phi$-decomposition

$$M = M_2^\perp \oplus M_2 \xrightarrow{DF} W_1 \oplus W_2 = W$$

for $DF$. This is because the proof of [1] Theorem 2.7.6 also holds when we consider arbitrary Hilbert $C^*$-modules $M$ and $W$ over unital $C^*$-algebra $A$ and not only the standard module $H_A$. Then we can proceed as in the proof of Theorem [1] part 2) $\Rightarrow$ 1).

Corollary 2 If $D \in \mathcal{M}\Phi_-(H_A)$, then there exists an $\mathcal{M}\Phi_-$-decomposition $H_A = N_1^\perp \oplus N_1 \xrightarrow{D} M_2 \oplus N_2' = H_A$ for $D$. Similarly, if $F \in \mathcal{M}\Phi_+(H_A)$, then there exists an $\mathcal{M}\Phi_+$-decomposition $H_A = M_2^\perp \oplus N_1 \xrightarrow{F} N_2^\perp \oplus N_2 = H_A$ for $F$.

Proof Follows from the proofs of Theorem [1] and Theorem [2] part 1) $\Rightarrow$ 2).

Corollary 3 [2] Corollary 2.4] $\mathcal{M}\Phi(H_A) = \mathcal{M}\Phi_+(H_A) \cap \mathcal{M}\Phi_-(H_A)$
Definition 2 [2, Definition 5.1] Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$. We say that $F \in \tilde{\mathcal{M}}\Phi^{-} \subset (\mathcal{M}\Phi_{-} \subset (H_{\mathcal{A}}))$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_{\mathcal{A}}$$

with respect to which $F$ has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_A \end{bmatrix},$$

where $F_1$ is an isomorphism, $N_1, N_2$ are closed, finitely generated and $N_1 \preceq N_2$, that is $N_1$ is isomorphic to a closed submodule of $N_2$. We define similarly the class $\tilde{\mathcal{M}}\Phi^{+} \subset (\mathcal{M}\Phi_{+} \subset (H_{\mathcal{A}}))$, the only difference in this case is that $N_2 \preceq N_1$. Then we set

$$\mathcal{M}\Phi^{-} \subset (\mathcal{M}\Phi_{-} \subset (H_{\mathcal{A}})) = (\tilde{\mathcal{M}}\Phi^{-} \subset (\mathcal{M}\Phi_{-} \subset (H_{\mathcal{A}}))) \cup (\mathcal{M}\Phi^{+} \subset (\mathcal{M}\Phi_{+} \subset (H_{\mathcal{A}})))$$

and

$$\mathcal{M}\Phi^{+} \subset (\mathcal{M}\Phi_{+} \subset (H_{\mathcal{A}})) = (\tilde{\mathcal{M}}\Phi^{+} \subset (\mathcal{M}\Phi_{+} \subset (H_{\mathcal{A}}))) \cup (\mathcal{M}\Phi^{-} \subset (\mathcal{M}\Phi_{-} \subset (H_{\mathcal{A}}))).$$

Remark 2 The notation $\oplus$ denotes the direct sum of modules without orthogonality, as given in [4].

At the end of this section, we also define another class of operators on $H_{\mathcal{A}}$ which we are going to use later in section 5.

Definition 3 We set

$$\mathcal{M}\Phi_0(H_{\mathcal{A}}) = \{F \in \mathcal{M}\Phi(H_{\mathcal{A}}) \mid \text{index } F = 0\}.$$ 

3 Perturbations of spectra in $Z(\mathcal{A})$ of operator matrices acting on $H_{\mathcal{A}} \oplus H_{\mathcal{A}}$

It this section we will consider the operator $M^A_{C}(F, D) : H_{\mathcal{A}} \oplus H_{\mathcal{A}} \to H_{\mathcal{A}} \oplus H_{\mathcal{A}}$ given as $2 \times 2$ operator matrix

$$\begin{bmatrix} F & C \\ 0 & D \end{bmatrix},$$

where $C \in B^a(H_{\mathcal{A}})$.

To simplify notation, throughout this paper, we will only write $M^A_{C}$ instead of $M^A_{C}(F, D)$ when $F, D \in B^a(H_{\mathcal{A}})$ are given.

Let $\sigma^A_0(M^A_{C}) = \{\alpha \in Z(\mathcal{A}) \mid M^A_{C} - \alpha I \text{ is not } A\text{-Fredholm }\}$. Then we have the following proposition.

Proposition 1 For given $F, C, D \in B^a(H_{\mathcal{A}})$, one has

$$\sigma^A_0(M^A_{C}) \subset (\sigma^A_0(F) \cup \sigma^A_0(D)).$$

Proof Observe first that

$$M^A_{C} - \alpha I = \begin{bmatrix} 1 & 0 \\ 0 & D - \alpha I \end{bmatrix} \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F - \alpha I & 0 \\ 0 & 1 \end{bmatrix}.$$
Now \[ \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix} \] is clearly invertible in \( B^\alpha(H_A \oplus H_A) \) with inverse \( \begin{bmatrix} 1 & -C \\ 0 & 1 \end{bmatrix} \), so it follows that \( \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix} \) is \( A \)-Fredholm. If, in addition both \( \begin{bmatrix} F - \alpha I & 0 \\ 0 & 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 \\ 0 & D - \alpha I \end{bmatrix} \) are \( A \)-Fredholm, then \( M_C^A - \alpha I \) is \( A \)-Fredholm being a composition of \( A \)-Fredholm operators. But, if \( F - \alpha I \) is \( A \)-Fredholm, then clearly \( \begin{bmatrix} F - \alpha I & 0 \\ 0 & 1 \end{bmatrix} \) is \( A \)-Fredholm, and similarly if \( D - \alpha I \) is \( A \)-Fredholm, then \( \begin{bmatrix} 1 & 0 \\ 0 & D - \alpha I \end{bmatrix} \) is \( A \)-Fredholm. Thus, if both \( F - \alpha I \) and \( D - \alpha I \) are \( A \)-Fredholm, then \( M_C^A - \alpha I \) is \( A \)-Fredholm. The proposition follows.

This proposition just gives an inclusion. We are going to investigate in which cases the equality holds. To this end we introduce first the following theorem.

**Theorem 3** Let \( F, D \in B^\alpha(H_A) \). If \( M_C^A \in \mathcal{M} \Phi(H_A \oplus H_A) \) for some \( C \in B^\alpha(H_A) \), then \( F \in \mathcal{M} \Phi_+(H_A), D \in \mathcal{M} \Phi_-(H_A) \) and for all decompositions

\[
H_A = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_A,
\]

\[
H_A = M'_1 \oplus N'_1 \xrightarrow{D} M'_2 \oplus N'_2 = H_A
\]

w.r.t. which \( F, D \) have matrices \( \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix} \), respectively, where \( F_1, D_1 \) are isomorphisms, and \( N_1, N'_1 \) are finitely generated, there exist closed submodules \( \tilde{N}_1, \tilde{N}'_1, \tilde{N}_2, \tilde{N}'_2 \) such that \( \tilde{N}_2 \cong \tilde{N}_1, \tilde{N}'_2 \cong \tilde{N}'_1 \) and \( N_2, N'_2 \) are finitely generated and

\[
\tilde{N}_2 \oplus \tilde{N}'_2 \cong \tilde{N}'_1 \oplus \tilde{N}_1.
\]

**Proof** Again write \( M_C^A \) as \( M_C^A = D'C'F' \) where

\[
F' = \begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix}, C' = \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix}, D' = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}.
\]

Since \( M_C^A \) is \( A \)-Fredholm, if

\[
H_A \oplus H_A = M_C^A \xrightarrow{F' \oplus H_A} M_C^A \oplus N' = H_A \oplus H_A
\]

is a decomposition w.r.t. which \( M_C^A \) has the matrix \( \begin{bmatrix} (M_C^A)_1 & 0 \\ 0 & (M_C^A)_4 \end{bmatrix} \) where \( (M_C^A)_1 \) is an isomorphism and \( N, N' \) are finitely generated, then by Corollary [11] and also using that \( C' \) is invertible, one may easily deduce that there exists a chain of decompositions

\[
H_A \oplus H_A = M_C^A \xrightarrow{F' \oplus H_A} R_1 \oplus R_2 \xrightarrow{C' \oplus H_A} C'(R_1) \oplus C'(R_2) \xrightarrow{D' \oplus H_A} M_C^A \oplus N' = H_A \oplus H_A
\]

w.r.t. which \( F', C', D' \) have matrices

\[
\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix}, \begin{bmatrix} D_1 & D_2 \\ 0 & D_4 \end{bmatrix}.
\]
respectively, where $F_1', C_1', C_4', D_1'$ are isomorphisms. So $D'$ has the matrix \[
\begin{bmatrix}
D_1' & 0 \\
0 & D_4'
\end{bmatrix}
\] w.r.t. the decomposition
\[
H_A \oplus H_A = WC'(R_1) \oplus WC'(R_2) \xrightarrow{D'} M' \oplus N' = H_A \oplus H_A,
\]
where $W$ has the matrix \[
\begin{bmatrix}
1 & -D_1'^{-1}D_2' \\
0 & 1
\end{bmatrix}
\] w.r.t the decomposition

\[
C'(R_1) \oplus C'(R_2) \xrightarrow{W} C'(R_1) \oplus C'(R_2)
\]
and is therefore an isomorphism.

It follows from this that

\[F' \in \mathcal{M}\Phi_+(H_A \oplus H_A), D' \in \mathcal{M}\Phi_-(H_A \oplus H_A),\]

as $N$ and $N'$ are finitely generated submodules of $H_A \oplus H_A$. Moreover $R_2 \cong WC'(R_2)$, as $WC'$ is an isomorphism.

Since there exists an adjointable isomorphism between $H_A$ and $H_A \oplus H_A$, using Theorem 1 and Theorem 2 it is easy to deduce that $F'$ is left invertible and $D'$ is right invertible in the Calkin algebra on $B^a(H_A \oplus H_A)/K(H_A \oplus H_A)$. It follows from this that $F$ is left invertible and $D$ is right invertible in the Calkin algebra $B^a(H_A)/K(H_A)$, hence $F \in \mathcal{M}\Phi_+(H_A)$ and $D \in \mathcal{M}\Phi_-(H_A)$ again by Theorem 1 and Theorem 2 respectively. Choose arbitrary $\mathcal{M}\Phi_+$ and $\mathcal{M}\Phi_-$ decompositions for $F$ and $D$ respectively i.e.

\[
H_A = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_A,
\]
\[
H_A = M_1' \oplus N_1' \xrightarrow{D} M_2' \oplus N_2' = H_A.
\]

Then

\[
H_A \oplus H_A = (M_1 \oplus H_A) \oplus (N_1 \oplus \{0\}) \xrightarrow{\downarrow F'}
\]

and

\[
H_A \oplus H_A = (M_2 \oplus H_A) \oplus (N_2 \oplus \{0\}) \xrightarrow{\downarrow D'}
\]

are $\mathcal{M}\Phi_+$ and $\mathcal{M}\Phi_-$ decompositions for $F'$ and $D'$ respectively. Hence the decomposition

\[
H_A \oplus H_A = M \oplus N' \xrightarrow{F'} R_1 \oplus R_2 = H_A \oplus H_A
\]

and the $\mathcal{M}\Phi_+$ decomposition given above for $F'$ are two $\mathcal{M}\Phi_+$ decompositions for $F'$. Again, since there exists an adjointable isomorphism between $H_A \oplus H_A$ and $H_A$, we may apply [2, Corollary 2.18] to operator $F'$ to deduce that \[(N_2 \oplus \{0\}) \oplus P) \cong (R_2 \oplus P)\] for some finitely generated submodules $P, \tilde{P}$ of $H_A \oplus H_A$.

Similarly, since

\[
H_A \oplus H_A = WC'(R_1) \oplus WC'(R_2) \xrightarrow{D'} M' \oplus N' = H_A \oplus H_A
\]
and
\[ H_A \oplus H_A = (H_A \oplus M_1^2) \oplus ((0) \oplus N_1') \]
\[ \downarrow D' \]
\[ H_A \oplus H_A = (H_A \oplus M_2^2) \oplus ((0) \oplus N_2') \]
are two \( M\Phi \)-decompositions for \( D' \), we may by the same arguments apply [2] Corollary 2.19] to the operator \( D' \) to deduce that
\[(\{(0) \oplus N_1'\)\(\oplus P'\} \cong (WC'(R_2)\oplus \tilde{P}')\]
for some finitely generated submodules \( P', \tilde{P}' \) of \( H_A \oplus H_A \). Since \( WC' \) is an isomorphism, we get
\[(WC'(R_2)\oplus \tilde{P}') \cong (WC'(R_2)\oplus \tilde{P}) \cong (R_2 \oplus \tilde{P} \oplus \tilde{P}) \cong ((R_2 \oplus \tilde{P}) \oplus \tilde{P}).\]
Hence
\[((N_2 \oplus \{0\})\oplus \tilde{P}') \cong (((0) \oplus N_1')\oplus \tilde{P}).\]
This gives \((N_2 \oplus P \oplus \tilde{P}') \cong (N_1' \oplus P' \oplus \tilde{P})\) (Here \( \oplus \) always denotes the direct sum of modules in the sense of [3] Example 1.3.4]). Now
\[N_2 \oplus P \oplus \tilde{P}' = (N_2 \oplus \{0\} \oplus \{0\}) \oplus (0 \oplus P \oplus P'),\]
\[N_1' \oplus P' \oplus \tilde{P} = (N_1' \oplus \{0\} \oplus \{0\}) \oplus (0 \oplus P' \oplus \tilde{P})\]
and they are submodules of \( L_5(H_4) \) which is isomorphic to \( H_A \) (the notation \( L_5(H_A) \) is as in [3] Example 1.3.4]). Call the isomorphism between \( H_A \) for and \( L_5(H_A) \) for \( U \) and set
\[\tilde{N}_2 = U(N_2 \oplus \{0\} \oplus \{0\}), \tilde{N}_2 = U((0) \oplus P \oplus P'),\]
\[\tilde{N}_1 = U(N_1' \oplus \{0\} \oplus \{0\}), \tilde{N}_1 = U((0) \oplus P' \oplus \tilde{P}).\]
Since \( P, P', \tilde{P}, \tilde{P}' \) are finitely generated, the result follows.

**Remark 3** [1] Theorem 3.1, part (1) \( \Rightarrow \) (2) follows actually as a corollary from Theorem 4 in the case when \( X = Y = H \), where \( H \) is a Hilbert space. Indeed, by Theorem 3 if \( M_C \in \Phi(H \oplus H) \), then \( F \in \Phi_+(H) \) and \( D \in \Phi_-(H) \). Hence ImF and ImD are closed, \( \dim \ker F, \dim \ker D < \infty \). W.r.t. the decompositions \( H = \ker F^\perp \oplus \ker F \overset{\sim}{\to} \im F \oplus \im D = H \) and
\[H = \ker D^\perp \oplus \ker D \overset{\sim}{\to} \im D \oplus \im D^\perp = H, \ F, D\]
have matrices
\[
\begin{bmatrix}
F_1 & 0 & 0 \\
0 & F_4 & 0 \\
0 & 0 & D_1 \\
0 & 0 & D_4
\end{bmatrix},
\begin{bmatrix}
D_1 & 0 & 0 \\
0 & D_4 & 0 \\
0 & 0 & F_4 \\
0 & 0 & F_1
\end{bmatrix},
\]
respectively, where \( F_1, D_1 \) are isomorphisms.

From Theorem 3 it follows that there exist closed subspaces \( \tilde{N}_2, \tilde{N}_2, \tilde{N}_1', \tilde{N}_1' \) such that \( \tilde{N}_2 \cong \im F^\perp, \tilde{N}_1' \cong \ker D, \dim \tilde{N}_2, \dim \tilde{N}_1' < \infty \) and
\[(\tilde{N}_2 \oplus \tilde{N}_2) \cong (\tilde{N}_1' \oplus \tilde{N}_1').\]
But this just means that \( \im F^\perp \) and \( \ker D \) are isomorphic up to a finite dimensional subspace in the sense of [1] Theorem 2.1] because we consider Hilbert subspaces now.
Proposition 2 Suppose that there exists some $C \in B^\sigma(H_A)$ such that the inclusion $\sigma_c^A(M_C^2) \subset \sigma_c^A(F) \cup \sigma_c^A(D)$ is proper. Then for any

$$\alpha \in [\sigma_c^A(F) \cup \sigma_c^A(D)] \setminus \sigma_c^A(M_C^2)$$

we have

$$\alpha \in \sigma_c^A(F) \cap \sigma_c^A(D).$$

**Proof** Assume that

$$\alpha \in [\sigma_c^A(F) \setminus \sigma_c^A(D)] \setminus \sigma_c^A(M_C^4).$$

Then $(F - \alpha) \notin M\Phi(H_A)$ and $(D - \alpha) \notin M\Phi(H_A)$. Moreover, since $\alpha \notin \sigma_c^A(M_C^4)$, then $(M_C^4 - \alpha)$ is $A$-Fredholm. From Theorem 3 it follows that $(F - \alpha) \in M\Phi^+(H_A)$. Since $(F - \alpha) \in M\Phi^+(H_A)$, $(D - \alpha) \in M\Phi^+(H_A)$, we can find decompositions

$$H_A = M_1 \oplus N_1 \overset{F-\alpha}{\sim} M_2 \oplus N_2 = H_A,$$

$$H_A = M'_1 \oplus N'_1 \overset{D-\alpha}{\sim} M'_2 \oplus N'_2 = H_A$$

w.r.t. which $F - \alpha, D - \alpha$ have matrices

$$\begin{bmatrix} (F - \alpha)_1 & 0 \\ 0 & (F - \alpha)_4 \end{bmatrix}, \begin{bmatrix} (D - \alpha)_1 & 0 \\ 0 & (D - \alpha)_4 \end{bmatrix},$$

respectively, where $(F - \alpha)_1, (D - \alpha)_1$ are isomorphisms, $N_1, N'_1$ and $N_2, N'_2$ are finitely generated. By Theorem 3 there exist then closed submodules $\tilde{N}_2, \tilde{N}'_2, \tilde{N}'_1, \tilde{N}_1$ such that $\tilde{N}_2 \cong \tilde{N}'_2, N'_1 \cong \tilde{N}'_1, (\tilde{N}_2 \oplus \tilde{N}'_2) \cong (\tilde{N}'_1 \oplus \tilde{N}_1)$ and $\tilde{N}_2, \tilde{N}'_2$ are finitely generated. But then, since $N_1'$ is finitely generated (as $(D - \alpha) \in M\Phi^+(H_A)$), we get that $\tilde{N}_1'$ is finitely generated being isomorphic to $N_1'$. Hence $(\tilde{N}'_1 \oplus \tilde{N})$ is finitely generated also (as both $\tilde{N}_2$ and $\tilde{N}'_2$ are finitely generated). Thus $(\tilde{N}_2 \oplus \tilde{N})$ is finitely generated as well, so $\tilde{N}_2$ is finitely generated. Therefore $N_2$ is finitely generated, being isomorphic to $\tilde{N}_2$. Hence $F - \alpha$ is in $M\Phi^+(H_A)$. This contradicts the choice of

$$\alpha \in [\sigma_c^A(F) \setminus \sigma_c^A(D)] \setminus \sigma_c^A(M_C^4).$$

Thus

$$[\sigma_c^A(F) \setminus \sigma_c^A(D)] \setminus \sigma_c^A(M_C^4) = \emptyset.$$

Analogously we can prove

$$[\sigma_c^A(D) \setminus \sigma_c^A(F)] \setminus \sigma_c^A(M_C^4) = \emptyset.$$

The proposition follows.

Next, we define the following classes of operators on $H_A$

$$M_{S_+}(H_A) = \{ F \in B^e(H_A) \mid (F - \alpha) \in M\Phi_+(H_A) \}$$

whenever $\alpha \in Z(A)$ and $(F - \alpha) \in M\Phi_+(H_A)$,

$$M_{S_-}(H_A) = \{ F \in B^e(H_A) \mid (F - \alpha) \in M\Phi_-(H_A) \}$$

whenever $\alpha \in Z(A)$ and $(F - \alpha) \in M\Phi_-(H_A)$. 

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Lemma 2.16, we have that $N \in \mathfrak{f}$initely generated. Thus $\tilde{N}$ is finitely generated also. So $(D, -\alpha)$

Let again $\Phi : (F - \alpha) \in \mathcal{M} \Phi(H_A \oplus H_A)$. By Theorem 3, we know that $M$ be decompositions w.r.t. which $F$ is finitely generated submodules. If $F$ are decompositions w.r.t. which $F$ are isomorphisms, then $(F - \alpha)_{1, 1}$, are finitely generated submodules of $H_A$. Again, by Theorem 3 there exist closed submodules $N_2, \tilde{N}_2, \tilde{N}_1$, such that $N_2 \cong \tilde{N}_2, N_1 \cong \tilde{N}_1$, $(N_2 \oplus \tilde{N}_2) \cong (\tilde{N}_1 \oplus \tilde{N}_1)$ and $\tilde{N}_2, \tilde{N}_1$ are finitely generated submodules.

Since $(\tilde{N}_2 \oplus \tilde{N}_2) \cong (\tilde{N}_1 \oplus \tilde{N}_1)$, it follows that $\tilde{N}_1$ is finitely generated, hence $N_1$ is finitely generated also. So $(D - \alpha) \in \mathcal{M} \Phi(H_A)$. Similarly, we can show that if $D \in S_-(H_A)$, then $(F - \alpha) \in \mathcal{M} \Phi(H_A)$. In both cases $(F - \alpha) \in \mathcal{M} \Phi(H_A)$ and $(D - \alpha) \in \mathcal{M} \Phi(H_A)$, which contradicts that $\alpha \in \sigma^A(F) \cup \sigma^A(D)$.

Theorem 4 Let $F \in \mathcal{M} \Phi_+(H_A), D \in \mathcal{M} \Phi_-(H_A)$ and suppose that there exist decompositions $H_A = M_1 \oplus N_1 \xrightarrow{F} N_2 \oplus N_2 = H_A$ $H_A = N_1 \perp H_A \xrightarrow{D} M_2 \oplus N_2 = H_A$

w.r.t. which $F, D$ have matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

respectively, where $F_1, D_1$ are isomorphisms, $N_1, N_2$ are finitely generated and assume also that one of the following statements hold:

a) There exists some $J \in B^a(N_2, N_1)$ such that $N_2 \cong \text{Im} J$ and $\text{Im} J^\perp$ is finitely generated.

b) There exists some $J' \in B^a(N_1, N_2)$ such that $N_1 \cong \text{Im} J', (\text{Im} J')^\perp$ is finitely generated.

Then $M_1^A \in \mathcal{M} \Phi(H_A \oplus H_A)$ for some $C \in B^a(H_A)$. 

**Proof** By Proposition 2, it suffices to show the inclusion. Assume that

$$\alpha \in [\sigma^A(F) \cup \sigma^A(D)] \setminus \sigma^A(M_2^A).$$

Then, $(M_1^A - \alpha) \in \mathcal{M} \Phi(H_A \oplus H_A)$. By Theorem 3, we have $(F - \alpha) \in \mathcal{M} \Phi_+(H_A), (D - \alpha) \in \mathcal{M} \Phi_-(H_A)$.
Remark 4. $\text{Im}J^\perp$ in part a) denotes the orthogonal complement of $\text{Im}J$ in $N_1'$ and $\text{Im}J'^\perp$ denotes the orthogonal complement of $\text{Im}J'$ in $N_2$.

By [3, Theorem 2.3.3], if $\text{Im}J$ is closed, then $\text{Im}J$ is indeed orthogonally complementable, so since in assumption a) above $\text{Im}J$ is closed, so $N_1' = \text{Im}J \oplus \text{Im}J^\perp$. Similarly, in b) $N_2 = \text{Im}J' \oplus \text{Im}J'^\perp$.

Proof. Suppose that b) holds, and consider the operator $\tilde{J}' = J'P_{N_1}$ where $P_{N_1}$ denotes the orthogonal projection onto $N_1'$. Then $\tilde{J}'$ can be considered as a bounded adjointable operator on $H_A$ (as $N_2$ is orthogonally complementable in $(H_A)$). To simplify notation, we let $M_2 = N_2^\perp$, $M_1' = N_1'^\perp$ and we let $M_{\tilde{J}'} = M_{\tilde{J}'}$. We claim then that w.r.t. the decomposition
\[ H_A \oplus H_A = (M_1 \oplus H_A) \oplus (N_1 \oplus \{0\}) \]
\[ \downarrow M_{\tilde{J}'} \]
then $D_{\tilde{J}'}$ has the matrix
\[ \begin{bmatrix} (M_{\tilde{J}'}),_1 & (M_{\tilde{J}'}),_2 \\ (M_{\tilde{J}'}),_3 & (M_{\tilde{J}'}),_4 \end{bmatrix} \]
where $(M_{\tilde{J}'}),_1$ is an isomorphism. To see this observe first that
\[ (M_{\tilde{J}'}),_1 = \cap_2 \cap M_2 \cap (\text{Id}_M \oplus H_A) \]
\[ \begin{bmatrix} F_{\cap M_1} \tilde{J}' \\ 0 \end{bmatrix} D_{\cap M_1} \]
( as $\cap_2 D = D_{\cap M_2}$ ), where $\cap_2 \cap M_2$ denotes the projection onto $M_2$ along $M_2'$ along $\text{Im}J'^\perp$ and $\cap_2$ denotes the projection onto $M_1'$ along $N_1'$. Clearly, $(M_{\tilde{J}'}),_1$ is onto $(M_2 \oplus \text{Im}J' \oplus M_2')$. Now, if $(M_{\tilde{J}'}),_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $x \in M_2, y \in H_A$, then $D_{\cap M_1} y = 0$, so $y \in N_1'$, as $D_{\cap M_1}$ is bounded below. Also $F x + \tilde{J}' y = 0$. But, since $y \in N_1'$, then $\tilde{J}' y = J' y$, so we get $F x + J' y = 0$. Since $F x \in M_2$, $J' y = N_2$ and $M_2 \cap N_2 = \{0\}$, we get $F x = J' y = 0$. Since $F_{\cap M_1}$ and $J'$ are bounded below, we get $x = y = 0$. So $(M_{\tilde{J}'}),_1$ is injective as well, thus an isomorphism. Recall next that $N_1 \oplus \{0\}$ and $\text{Im}J'^\perp \oplus N_2'$ are finitely generated. By using the procedure of diagonalisation of $M_{\tilde{J}'}$ as done in the proof of [3, Lemma 2.7.10], we obtain that
\[ M_{\tilde{J}'} \in M(\text{Id}_A \oplus H_A) \]
Assume now that a) holds. Then there exists $\iota \in B^a(\text{Im}J, N_2)$ s.t $\iota J = \text{id}_{N_2}$.

Let $\tilde{\iota} = iP_{\text{Im}J}$ where $P_{\text{Im}J}$ denote the orthogonal projection onto $\text{Im}J$. (notice that $\text{Im}J$ is orthogonally complementable in $H_A$ since it is orthogonally complementable in $N_1'$ and $H_A = N_1' \oplus N_2')$. Thus $\tilde{\iota} \in B^a(H_A)$. Consider $M_{\tilde{\iota}} = \begin{bmatrix} F \tilde{\iota} \\ 0 \end{bmatrix}$.

We claim that w.r.t. the decomposition
\[ H_A \oplus H_A = (M_1 \oplus (M_1' \oplus \text{Im}J)) \oplus (N_1 \oplus \text{Im}J'^\perp) \]
\[ H_A \oplus H_A = (H_A \oplus M' ) \oplus (\{0\} \oplus N' ), \]

\( M \) has the matrix
\[
\begin{bmatrix}
(M)_{11} & (M)_{12} \\
(M)_{13} & (M)_{14}
\end{bmatrix},
\]
where \((M)_{11}\) is an isomorphism. To see this, observe again that
\[ (M)_{11} = (H_A \oplus M' ) / \text{ImJ}, \]
so \((M)_{11}\) is obviously onto \( H_A \oplus M' \).
Moreover, if \( (M)_{11} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) for some \( x \in M \) and \( y \in M' \oplus \text{ImJ} \), we get that \( D \cap M' y = 0 \), so \( y \in \text{ImJ} \).

**Remark 5** We know from the Corollary 2 that since
\[ F \in \Phi_+(H_A), D \in \Phi_-(H_A), \]
we can find the decompositions
\[ H_A = M_1 \oplus N_1 \rightarrow N_2^+ \oplus N_2 = H_A, \]
\[ H_A = N_1^- \oplus N_1' \rightarrow M_2' \oplus N_2' = H_A, \]
respectively, where \( F_1, D_1 \) are isomorphisms, \( N_1, N_1' \) are finitely generated. However, in this theorem we have also the additional assumptions a) and b).

**Remark 6** [1, Theorem 3.1], part (2) \( \Rightarrow \) (1) follows as a direct consequence of Theorem 3 in the case when \( X = Y = H \), where \( H \) is a Hilbert space. Indeed, if \( F \in \Phi_+(H), D \in \Phi_-(H), \ker D \) and \( \text{Im} F^{-} \) are isomorphic up to a finite dimensional subspace, then we may let
\[ M_1 = \ker F^+, N_1 = \ker F^+, N_2^+ = \text{Im} F, N_2 = \text{Im} F^+, N_1' = \ker D, \]
\[ M_2' = \text{Im} D, N_2' = \text{Im} D^+, N_1' = \ker D. \]
Since \( \ker D \) and \( \text{Im} F^+ \) are isomorphic up to a finite dimensional subspace, by [1, Definition 2.1] this means that either the condition a) or the condition b) in Theorem 3 holds. By Theorem 3 it follows then that \( \Phi \in \Phi(H \oplus H) \).
Let $\tilde{W}(F, D)$ be the set of all $\alpha \in Z(A)$ such that there exist decompositions

$$H_A = M_1 \oplus N_1 \xrightarrow{F-\alpha} M_2 \oplus N_2 = H_A,$$

$$H_A' = M_1' \oplus N_1' \xrightarrow{D-\alpha} M_2' \oplus N_2' = H_A,'$$

w.r.t. which $F - \alpha, D - \alpha$ have matrices

$$\begin{bmatrix} (F - \alpha)_{11} & 0 \\ 0 & (F - \alpha)_{44} \end{bmatrix}, \begin{bmatrix} (D - \alpha)_{11} & 0 \\ 0 & (D - \alpha)_{44} \end{bmatrix},$$

where $(F - \alpha)_{11}, (D - \alpha)_{11}$ are isomorphisms, $N_1, N_1'$ are finitely generated submodules and such that there are no closed submodules $\tilde{N}_2, \tilde{N}_2', \tilde{N}_1, \tilde{N}_1'$ with the property that $N_2 \cong \tilde{N}_2, N_1' \cong \tilde{N}_1', \tilde{N}_2, \tilde{N}_1$ are finitely generated and

$$(\tilde{N}_2 \oplus \tilde{N}_1) \cong (\tilde{N}_2' \oplus \tilde{N}_1').$$

Set $W(F, D)$ to be the set of all $\alpha \in Z(A)$ such that there are no decompositions

$$H_A = M_1 \oplus N_1 \xrightarrow{F-\alpha} N_2' \oplus N_2 = H_A,$$

$$H_A' = N_1' \oplus N_1' \xrightarrow{D-\alpha} M_2' \oplus N_2' = H_A,'$$

w.r.t. which $F - \alpha, D - \alpha$ have matrices

$$\begin{bmatrix} (F - \alpha)_{11} & 0 \\ 0 & (F - \alpha)_{44} \end{bmatrix}, \begin{bmatrix} (D - \alpha)_{11} & 0 \\ 0 & (D - \alpha)_{44} \end{bmatrix},$$

where $(F - \alpha)_{11}, (D - \alpha)_{11}$ are isomorphisms $N_2, N_2'$ are finitely generated and with the property that a) or b) in the Theorem hold. Then we have the following corollary:

**Corollary 4** For given $F \in B^a(H_A)$ and $D \in B^a(H_A)$,

$$\tilde{W}(F, D) \subseteq \bigcap_{C \in B^a(H_A)} \sigma^a(M_1^C) \subseteq W(F, D).$$

**Theorem 5** Suppose $M_1^C \in \mathcal{MF}_-(H_A \oplus H_A)$ for some $C \in B^a(H_A)$. Then $D \in \mathcal{MF}_-(-H_A)$ and in addition the following statement holds:

Either $F \in \mathcal{MF}_-(H_A)$ or there exists decompositions

$$H_A \oplus H_A = M_1 \oplus N_1 \xrightarrow{F'} M_2 \oplus N_2 = H_A \oplus H_A,$$

$$H_A \oplus H_A = M_1' \oplus N_1' \xrightarrow{D'} M_2' \oplus N_2' = H_A \oplus H_A,$$

w.r.t. which $F', D'$ have the matrices

$$\begin{bmatrix} F'_{11} & 0 \\ 0 & F'_{44} \end{bmatrix}, \begin{bmatrix} D'_{11} & 0 \\ 0 & D'_{44} \end{bmatrix},$$

where $F', D'$ are isomorphisms, $N_1'$ is finitely generated, $N_1, N_2, N_1'$ are closed, but not finitely generated, and $M_2 \cong M_1', N_2 \cong N_1'$. 


Proof If $M_C^4 \in \mathcal{M}\Phi_-(H_A \oplus H_A)$, then there exists a decomposition

$$H_A \oplus H_A = M_1 \oplus N_1 \overset{M_2}{\rightarrow} M_2 \oplus N_2 = H_A \oplus H_A$$

w.r.t. which $M_C^4$ has the matrix $\begin{pmatrix} (M_C^4)_1 & 0 \\ 0 & (M_C^4)_4 \end{pmatrix}$, where $(M_C^4)_1$ is an isomorphism and $N_2$ is finitely generated. By Corollary 2 we may assume that $M_1 = N_1^\perp$. Hence $F'_\mid_{M_1}$ is adjointable. Since $F'_\mid_{M_1}$ can be viewed as an operator in $B^\circ (M_1, (D'C')^{-1}(M_2))$, as $M_1$ is orthogonally complementable, by [14, Theorem 2.3.3], $F'(M_1)$ is orthogonally complementable in $(D'C')^{-1}(M_2)$. By the same arguments as in the proof of Theorem 1 case (2) $\Rightarrow$ 1 we deduce that there exists a chain of decompositions

$$M_1 \oplus N_1 \overset{F'}{\rightarrow} R_1 \oplus R_2 \overset{C'}{\rightarrow} C'(R_1) \oplus C'(R_2) \overset{D'}{\rightarrow} M_2 \oplus N_2$$

w.r.t. which $F', C', D'$ have matrices $\begin{pmatrix} F'_1 & 0 \\ 0 & F'_4 \end{pmatrix}$, $\begin{pmatrix} C'_1 & 0 \\ 0 & C'_4 \end{pmatrix}$, $\begin{pmatrix} D'_1 & D'_2 \\ 0 & D'_4 \end{pmatrix}$, where $F'_1, C'_1, C'_4, D'_1$ are isomorphisms. Hence $D'$ has the matrix $\begin{pmatrix} D'_1 & 0 \\ 0 & D'_4 \end{pmatrix}$, w.r.t. the decomposition

$$H_A \oplus H_A = WC'(R_1) \oplus WC'(R_2) \overset{D'}{\rightarrow} M_2 \oplus N_2 = H_A \oplus H_A,$$

where $W$ is an isomorphism. It follows that $D' \in \mathcal{M}\Phi_-(H_A \oplus H_A)$, as $N_2$ is finitely generated. Hence $D \in \mathcal{M}\Phi_-(H_A)$ (by the same arguments as in the proof of Theorem 3). Next, assume that $F \notin \mathcal{M}\Phi_-(H_A)$, then

$$F' \notin \mathcal{M}\Phi_-(H_A \oplus H_A).$$

Therefore $R_2$ can not be finitely generated (otherwise $F'$ would be in $\mathcal{M}\Phi_-(H_A \oplus H_A)$). Now, $R_1 \cong WC'(R_1), R_2 = WC'(R_2)$.

Remark 7 In case of ordinary Hilbert spaces, [14, Theorem 4.1 part 2] $\Rightarrow$ 3) follows as a corollary from Theorem 5. Indeed, suppose that $D \in B(H)$ and that $F \in B(H)$ (where $H$ is a Hilbert space). If $\ker D < \text{Im} F^\perp$, this means by [14, Remark 4.1] that $\dim \ker D < \infty$. So, if (2) in [14, Theorem 4.1] holds, that is $M_C \in \Phi_-(H \oplus H)$ for some $C \in B(H)$, then by Theorem 5 $D \in \Phi_-(H)$ and either $F \in \Phi_-(H)$ or there exist decompositions

$$H \oplus H = M_1 \oplus N_1 \overset{F'}{\rightarrow} M_2 \oplus N_2 = H \oplus H,$$

$$H \oplus H = M'_1 \oplus N'_1 \overset{D'}{\rightarrow} M'_2 \oplus N'_2 = H \oplus H,$$

which satisfy the conditions described in Theorem 5. In particular $N_2, N'_2$ are infinite dimensional whereas $N'_2$ is finite dimensional. Suppose that $F \notin \Phi_-(H)$ and that the decompositions above exist. Observe that $\ker D' = \{0\} \oplus \ker D$. Hence, if $\dim \ker D < \infty$, then $\dim \ker D' < \infty$. Since $D'_\mid_{N'_2}$ is an isomorphism, by the same arguments as in the proof of [3, Proposition 3.6.8] one can deduce that $\ker D' \subseteq N'_1$. Assume that $\dim \ker D = \dim \ker D' < \infty$ and let $N'_1$ be the orthogonal complement of $\ker D'$ in $N'_1$, that is $N'_1 = \ker D' \oplus N'_1$. Now, since $\text{Im} D'$ is closed as $D' \in \mathcal{M}\Phi_-(H \oplus H)$, then $D'_\mid_{N'_1}$ is an isomorphism. Since $\dim N'_2 = \infty$ and $\dim \ker D' < \infty$, we have $\dim N'_1 = \infty$. Hence $D'(N'_1)$ is infinite dimensional.
Let $R$. Hence $N$. The hypotheses of Theorem 6 as long (ImF) for F and D, respectively, is one particular pair of decompositions that satisfies $\Phi$ can be deduced as a corollary from Theorem 6. Indeed, if $F$ is closed and $D \in M\Phi-(H_A)$ and either $F \in M\Phi-(H_A)$ or that there exist decompositions

$$H_A = M_1 \oplus N_1 \overset{F}{\longrightarrow} N_2' \oplus N_2 = H_A,$$

$$H_A = N_1' \oplus N_1' \overset{D}{\longrightarrow} M_2' \oplus N_2' = H_A,$$

w.r.t. which $F, D$ have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where $F_1, D_1$ are isomorphisms $N_2'$, is finitely generated and that there exists some $\iota \in B^0(N_2, N_1')$ such that $\iota$ is an isomorphism onto its image in $N_1'$. Then $M_C A \in M\Phi-(H_A \oplus H_A)$ for some $C \in B^0(H_A)$.

**Proof** Since $\text{Im} M$ is closed and $\iota \in B^0(N_2, N_1')$, $\text{Im} M$ is orthogonally complementable in $N_1'$ by [[3, Theorem 2.3.3]], that is $N_1' = \text{Im} \iota \oplus N_2'$ for some closed submodule $N_2'$.

Hence $H_A = \text{Im} \iota \oplus N_2' \oplus N_2'$, that is $\text{Im} M$ is orthogonally complementable in $H_A$. Also, there exists $J \in B^0(\text{Im} M, N_2)$ such that $J \iota = \text{id}_{N_2'}$, $J \iota = \text{id}_{\text{Im} M}$. Let $P_{\text{Im} M}$ be the orthogonal projection onto $\text{Im} M$ and set $C = J P_{\text{Im} M}$. Then $C \in B^0(H_A)$. Moreover, w.r.t. the decomposition

$$H_A \oplus H_A = (M_1 \oplus (M_1' \oplus \text{Im} M)) \oplus (N_1 \oplus N_1'), M_C A \to (H_A \oplus M_2') \oplus (N_2' \oplus N_2') = H_A \oplus H_A,$$

$M_C$ has the matrix $\begin{bmatrix} (M_C^A)_{11} & (M_C^A)_{12} \\ (M_C^A)_{21} & (M_C^A)_{22} \end{bmatrix}$, where $(M_C^A)_{11}$ is an isomorphism. This follows by the same arguments as in the proof of Theorem [[3]]. Using that $N_2'$ is finitely generated and proceeding further as in the proof of the above mentioned theorem, we reach the desired conclusion.

**Remark 8** In the case of ordinary Hilbert spaces, [[3, Theorem 4.1]] part (1) $\Rightarrow$ (2) can be deduced as a corollary from Theorem [[3]]. Indeed, if $F$ is closed and $D \in \Phi-(H)$, which gives that $\text{Im} D$ is closed also, then the pair of decompositions

$$H = (\text{ker} F)^\perp \oplus \text{ker} F \overset{F}{\longrightarrow} \text{Im} F \oplus \text{Im} F^\perp = H,$$

$$H = (\text{ker} F)^\perp \oplus \text{ker} D \overset{D}{\longrightarrow} \text{Im} D \oplus \text{Im} D^\perp = H$$

for $F$ and $D$, respectively, is one particular pair of decompositions that satisfies the hypotheses of Theorem [[3]] as long $(\text{Im} F)^\perp \leq \text{ker} D$.

Let $R(F, D)$ be the set of all $\alpha \in Z(\mathcal{A})$ such that there exists no decompositions

$$H_A = M_1 \oplus N_1 \overset{F-\alpha I}{\longrightarrow} N_2' \oplus N_2 = H_A,$$

$$H_A = N_1' \oplus N_1' \overset{D-I}{\longrightarrow} M_2' \oplus N_2' = H_A$$

subspace of $N_2'$. This is a contradiction since dim $N_2'$ is finite. Thus, if $F \notin \Phi-(H)$, we must have that ker $D$ is infinite dimensional. Hence, we deduce, as a corollary, [[1, Theorem 4.1]] in case when $X = Y = H$, where $H$ is a Hilbert space. In this case, part (3b) in [[1, Theorem 4.1]] could be reduced to the following statement: Either $F \in \Phi-(H)$ or dim ker $D = \infty$.

**Theorem 6** Let $F, D \in B^0(H_A)$ and suppose that $D \in M\Phi-(H_A)$ and either $F \in M\Phi-(H_A)$ or that there exist decompositions

$$H_A = M_1 \oplus N_1 \overset{F}{\longrightarrow} N_2' \oplus N_2 = H_A,$$

$$H_A = N_1' \oplus N_1' \overset{D}{\longrightarrow} M_2' \oplus N_2' = H_A,$$
that satisfy the hypotheses of the Theorem [6] Set $R'(F, D)$ to be the set of all 
$\alpha \in Z(A)$ such that there exist no decompositions 
$$H_A \oplus H_A = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_A \oplus H_A,$$
$$H_A \oplus H_A = M'_1 \oplus N'_1 \xrightarrow{D'} M'_2 \oplus N'_2 = H_A \oplus H_A$$
that satisfy the hypotheses of the Theorem [5].

Then we have the following corollary:

**Corollary 5** Let $F, D \in \mathcal{B}^+(H_A)$. Then

$$\sigma_{re}^A(D) \cup (\sigma_{re}^A(F) \cap R'(F, D)) \subseteq \bigcap_{C \in \mathcal{B}^+(H_A)} \sigma_{re}^A(C) \subseteq \sigma_{re}^A(D) \cup (\sigma_{re}^A(F) \cap F'(R, D))$$

**Theorem 7** Let $M^A_C \in \mathcal{M}(H_A \oplus H_A)$. Then $F' \in \mathcal{M}(H_A \oplus H_A)$ and either 
$D \in \mathcal{M}(H_A)$ or there exist decompositions 
$$H_A \oplus H_A = M_1 \oplus N_1 \xrightarrow{F'} M_2 \oplus N_2 = H_A \oplus H_A,$$
$$H_A \oplus H_A = M'_1 \oplus N'_1 \xrightarrow{D'} M'_2 \oplus N'_2 = H_A \oplus H_A,$$
where $F', D'$ have matrices $[F' \, 0 \, \, 0 \, F']$, $[D' \, 0 \, \, 0 \, D']$, respectively, where $F', D'$ are 
isomorphisms, $M_2 \cong M'_1$ and $N_2 \cong N'_1$, $N_1$ is finitely generated and $N_2, N'_1$ are closed, 
but not finitely generated.

**Proof** Since $M^A_C \in \mathcal{M}(H_A \oplus H_A)$, there exists an $\mathcal{M}(H_A)$ decomposition for $M^A_C$,
$$H_A \oplus H_A = M_1 \oplus N_1 \xrightarrow{M^A_C} M_2 \oplus N_2 = H_A \oplus H_A,$$
so $N_1$ is finitely generated. By the proof of [6] Theorem 2.7.6, we may assume 
that $M_1 = N_1$. Hence $F'_{|M_1}$ is adjointable. As in the proof of Corollary [5] and 
Theorem [6] we may consider a chain of decompositions 
$$H_A \oplus H_A = M_1 \oplus N_1 \xrightarrow{F'} R_1 \oplus R_2 \xrightarrow{C'} C'(R_1) \oplus C'(R_2) \xrightarrow{D'} M'_2 \oplus M'_2 = H_A \oplus H_A$$
with $F', C', D'$ have matrices $[F' \, 0 \, \, 0 \, F']$, $[C' \, 0 \, \, 0 \, C']$, and $[D' \, D']$, respectively, where $F', C', D'$ are isomorphisms. Then we can proceed in the same 
way as in the proof of Theorem [5].

**Remark 9** In the case of Hilbert spaces, the implication (2) $\Rightarrow$ (3) in 
[7] Theorem 4.2] follows as a corollary of Theorem [6]. Indeed, for the implication 
(2) $\Rightarrow$ (3b), we may proceed as follows: Since $\text{Im}(F)^0 \cong \text{Im}(F)^\perp$ and $(\ker D)' \cong \ker D$ when one considers Hilbert spaces, then by [7] Remark 4.1, $(\text{Im}(F)^0 \prec (\ker D)'$ means simply that $\dim \text{Im}F^\perp < \infty$ whereas $\dim \ker D = \infty$. If in addition 
$D \notin \Phi(H)$, then $D' \notin \Phi(H \oplus H)$. Now, if $\dim \text{Im}(F)^\perp < \infty$, then $\dim \ker D = \infty$, 
and $F \in \Phi(H)$ as $F \in \Phi(H)$ and $\dim \text{Im}(F)^\perp < \infty$. Then $F' \in \Phi(H \oplus H)$, so by
Lemma 2.16 | $N_2$ must be finitely generated. Thus $N_1'$ must be finitely generated being isomorphic to $N_2$. By the same arguments as earlier, we have that $\ker D' \cong \ker D$ and $\ker D' \subseteq N_1'$. Since we consider Hilbert spaces now, the fact that $N_1'$ is finitely generated means actually that $N_1'$ is finite dimensional. Hence $\ker D'$ must be finite dimensional, so $\dim \ker D = \dim \ker D' < \infty$. This is in a contradiction to $\Im F \perp \ker D$. So, in the case of Hilbert spaces, if $M_C \in \Phi_+(H \oplus H)$, from Theorem 7 it follows that $F \in \Phi_+(H)$ and either $D \in \Phi_+(H)$ or $\Im F \perp$ is infinite dimensional.

Theorem 8 Let $F \in \mathcal{M}\Phi_+(H_A)$ and suppose that either $D \in \mathcal{M}\Phi_+(H_A)$ or that there exist decompositions

\[ H_A = M_1 \oplus N_1 \xrightarrow{F} N_2 \oplus N_2 = H_A, \]

\[ H_A = N_1' \oplus N_1' \xrightarrow{D} M_2 \oplus N_2 = H_A \]

w.r.t. which $F, D$ have matrices $[F_1 \ 0]
[0 \ F_4]$, $[D_1 \ 0]
[0 \ D_4]$, respectively, where $F_1, D_1$ are isomorphisms, $N_1'$ is finitely generated and in addition there exists some $\iota \in B^a(N_1', N_2)$ such that $\iota$ is an isomorphism onto its image. Then

$M_C^{\iota} \in \mathcal{M}\Phi_+(H_A \oplus H_A)$.

Proof Let $C = P_{N_1'}$ where $P_{N_1'}$ denotes the orthogonal projection onto $N_1'$, then apply similar arguments as in the proof of Theorem 11 and Theorem 8.

Remark 10 The implication (1) $\Rightarrow$ (2) in [1] Theorem 4.2 in case of Hilbert spaces could also be deduced as a corollary from [8] Indeed, if $\Im D$ is closed, then $D$ is an isomorphism from $\ker D^\perp$ onto $\Im D$. Moreover, if $F \in \Phi_+(H)$, then $F$ is also an isomorphism from $\ker F^\perp$ onto $\Im F$ and $\dim \ker F < \infty$. If in addition $\ker D \leq \Im F^\perp$, then the pair of decompositions

\[ H = \ker F^\perp \oplus \ker F \xrightarrow{F} \Im F \oplus \Im F^\perp = H, \]

\[ H = \ker D^\perp \oplus \ker D \xrightarrow{D} \Im D \oplus \Im D^\perp = H \]

is one particular pair of decompositions that satisfies the hypotheses of Theorem 8.

Let $L'(F, D)$ be the set of all $\alpha \in Z(A)$ such that there exist no decompositions

\[ H_A \oplus H_A = M_1 \oplus N_1 \xrightarrow{F - \alpha I} M_2 \oplus N_2 = H_A \oplus H_A, \]

\[ H_A \oplus H_A = N_1' \oplus N_1' \xrightarrow{D - \alpha I} M_2 \oplus N_2 = H_A \oplus H_A. \]

for $F' - \alpha I, D' - \alpha I$ respectively, which satisfy the hypotheses of Theorem 7.

Set $L(F, D)$ to be the set of all $\alpha \in Z(A)$ such that there exist no decompositions

\[ H_A = M_1 \oplus N_1 \xrightarrow{F - \alpha I} N_2 \oplus N_2 = H_A, \]

\[ H_A = N_1' \oplus N_1' \xrightarrow{D - \alpha I} M_2 \oplus N_2 = H_A. \]

for $F - \alpha I, D - \alpha I$ respectively which satisfy the hypotheses of Theorem 8.

Then we have the following corollary:
Corollary 6 Corollary: Let \( F, D \in B^a(H_A) \). Then
\[
\sigma_{\alpha}^A(F) \cup (\sigma_{\alpha}^A(D) \cap L(F, D)) \subseteq \bigcap_{C \in B^a(H_A)} \sigma_{\alpha}^A(M_C^A) \subseteq \sigma_{\alpha}^A(F) \cup (\sigma_{\alpha}^A(D) \cap L(F, D))
\]

4 Compressions

Throughout this section, given an operator \( T \in B^a(H_A) \), we let \( R(T), N(T) \) denote the image of \( T \) and the kernel of \( T \) respectively. This change in the notation is due to the fact that in this section we generalize certain results given in \([6]\), so we follow here also the notation from \([6]\). We start with the following lemma:

Lemma 1 Let \( F, P \in B^a(H_A) \) and suppose that \( P \) is a projection such that \( N(P) \) is finitely generated. Then \( F \in M \Phi(H_A) \) if and only if
\[
PF|_{N(P)} \in M \Phi(R(P)).
\]

Proof Suppose first that \( F \in M \Phi(H_A) \). Observe that since \( N(P) \) is finitely generated, \( P \in M \Phi(H_A) \) also. Hence \( PFP \in M \Phi(H_A) \) by \([4\text{, Lemma 2.17}]\).

Let
\[
H_A = M \oplus N \xrightarrow{PFP} M' \oplus N' = H_A
\]
be a decomposition w.r.t. which \( PFP \) has the matrix \[
\begin{pmatrix}
(PFP)_1 & 0 \\
0 & (PFP)_4
\end{pmatrix}
\]
where \( (PFP)_1 \) is an isomorphism, \( N, N' \) are finitely generated. By the proof of Theorem \([4\text{, part 2) } \Rightarrow 1] \) we know that \( P(M) \) is closed. Moreover, by \([3\text{, Theorem 2.7.6}] \) we may assume that \( M \) is orthogonally complementable. Hence \( P|_M \) could be viewed as an adjointable operator from \( M \) into \( R(P) \) with closed image. By \([3\text{, Theorem 2.3.3}] \) \( P(M) \) is then orthogonally complementable in \( R(P) \), that is \( P(M) \oplus \tilde{N} = R(P) \) for some closed submodule \( \tilde{N} \). With respect to the decomposition
\[
H_A = M \oplus N \xrightarrow{P} P(M) \oplus (\tilde{N} \oplus N(P)) = H_A,
\]
P has the matrix \[
\begin{pmatrix}
P_1 & P_2 \\
0 & P_4
\end{pmatrix}
\]
where \( P_1 \) is an isomorphism. Hence \( P_1 \) has the matrix \[
\begin{pmatrix}
P_1 & 0 \\
0 & P_4
\end{pmatrix}
\]
w.r.t. the decomposition
\[
H_A = U(M) \oplus U(N) \xrightarrow{P} P(M) \oplus (\tilde{N} \oplus N(P)) = H_A,
\]
where \( U \) has the matrix \[
\begin{pmatrix}
1 & -P_1^{-1}P_2 \\
0 & 1
\end{pmatrix}
\]
w.r.t. the decomposition \( M \oplus N \xrightarrow{U} M \oplus N \), so that \( U \) is an isomorphism. Since \( P \in M \Phi(H_A) \) and \( U(N) \) is finitely generated, by \([2\text{, Lemma 2.16}] \), \( \tilde{N} \oplus N(P) \) is finitely generated. Hence \( \tilde{N} \) is finitely generated. Now, \( PF|_{P(M)} \) is an isomorphism from \( P(M) \) onto \( M' \). Since \( P(M) \) is also orthogonally complementable in \( H_A \) (because \( P|_M \in B^a(M, H_A) \), as \( M \) is orthogonally complementable, \( P \) is adjointable and \( P(M) \) is closed), it follows again that \( PF|_{P(M)} \)
can be viewed as an adjointable operator from \(P(M)\) into \(R(P)\), so \(M'\) is orthogonally complementable in \(R(P)\) by Lemma 2.3.3 since \(M' = R(PF_{\mu(M)})\). Thus \(M' \oplus \tilde{N}' = R(P)\) for some closed submodule \(\tilde{N}'\). Now,

\[
H_A = M' \oplus N' = M' \oplus \tilde{N}' \oplus N(P),
\]

so it follows that \((\tilde{N}' \oplus N(P)) \cong N'\). Since \(N'\) is finitely generated, it follows that \(\tilde{N}\) is finitely generated also. With respect to the decomposition

\[
R(P) = P(M) \oplus \tilde{N} \xrightarrow{PF} M' \oplus \tilde{N}' = R(P),
\]

\(PF\) has the matrix

\[
\begin{bmatrix}
(PF)_{11} & (PF)_{12} \\
0 & (PF)_{44}
\end{bmatrix},
\]

where \((PF)_{11}\) is an isomorphism. Then

\[
PF\) has the matrix \[
\begin{bmatrix}
(PF)_{11} & 0 \\
0 & (PF)_{44}
\end{bmatrix}
\]

w.r.t. the decomposition

\[
R(P) = \hat{U}(P(M)) \oplus \hat{U}(\tilde{N}) \xrightarrow{PF} M' \oplus \tilde{N}' = R(P),
\]

where \(\hat{U}\) is an isomorphism of \(R(P)\) onto \(R(P)\). Since \(\tilde{N}, \tilde{N}'\) and thus also \(\hat{U}(\tilde{N})\) are finitely generated, it follows that \(PF_{\mu(R(P))} = M\Phi(R(P))\).

Conversely, suppose that \(PF_{\mu(R(P))} \in M\Phi(R(P))\). Let

\[
R(P) = M' \oplus N' \xrightarrow{PF} M' \oplus \tilde{N}' = R(P)
\]

be a decomposition w.r.t. which \(PF_{\mu(R(P))}\) has the matrix

\[
\begin{bmatrix}
(PF)_{11} & 0 \\
0 & (PF)_{44}
\end{bmatrix},
\]

where \(N, N'\) are finitely generated and \((PF)_{11}\), is an isomorphism. It follows that w.r.t.

\[
H_A = M' \oplus N' \oplus N(P) \xrightarrow{F} M' \oplus N' \oplus N(P) = H_A,
\]

F has the matrix

\[
\begin{bmatrix}
F_1 & F_2 \\
F_3 & F_4
\end{bmatrix},
\]

where \(F_1\) is an isomorphism as \(F_1 = (PF)_{11}\). Indeed, \(F_1 = \cap_{M'} F_{\mu(M')}\), where \(\cap_{M'}\) denotes the projection onto \(M'\) along \(N' \oplus N(P)\). But then, since \(PF\) maps \(M\) isomorphically onto \(M'\) and so \(R(P) = M' \oplus N(P)\), it follows that \(PF_{\mu(M')} = \cap_{M'} F_{\mu(M')}\). Hence \(F_1 = \cap_{M'} F_{\mu(M')} = PF_{\mu(M')}\) is an isomorphism from \(M\) onto \(M'\). Using the techniques of diagonalization from the proof of Lemma 2.7.10] and the fact that \(N \oplus N(P)\) and \(N' \oplus N(P)\) are finitely generated, one deduces that \(F \in M\Phi(H_A)\).

**Corollary 7** Let \(F, P \in B^\alpha(H_A)\) and suppose that \(P\) is a projection such that \(N(P)\) is finitely generated. Then \(\sigma^A_{\alpha}(F) = \sigma^A_{\alpha}(PF_{\mu(M)})\) where

\[
\sigma^A_{\alpha}(PF_{\mu(R(P))}) = \{ \alpha \in Z(A) \mid \text{PF - \alpha I}_{\mu(R(P))} \notin M\Phi(R(P)) \}\}

Let now \(\tilde{M}\Phi_0(H_A)\) be the set of all \(F \in B^\alpha(H_A)\) such that there exists a decomposition

\[
H_A = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_A
\]
w.r.t. which $F$ has the matrix \[
\begin{bmatrix}
F_1 & 0 \\
0 & F_4
\end{bmatrix},
\]
where $F_1$ is an isomorphism, $N_1, N_2$ are finitely generated and 
\[N_1 \oplus N_1 = N_2 \oplus N_2 = H_A,\]
for some closed submodule $N \subseteq H_A$.

Notice that this implies that $F \in \mathcal{M}(H_A)$ and $N_1 \cong N_2$, so that index $F = [N_1] - [N_2] = 0$. Hence $\mathcal{M}(H_A) \subseteq \mathcal{M}(H_A)$.

Let $P(H_A) = \{P \in B(H_A) \mid P$ is a projection and $N(P)$ is finitely generated $\}$ and let
\[
\sigma_{cW}^A(F) = \{\alpha \in Z(A) \mid (F - \alpha I) \notin \mathcal{M}(H_A)\}
\]
for $F \in B^a(H_A)$. Then we have the following theorem.

**Theorem 9** Let $F \in B^a(H_A)$. Then
\[
\sigma_{cW}^A(F) = \cap \{\sigma_{cW}^A(PF_{[P(H_A)]}) \mid P \in P(H_A)\}
\]
where
\[
\sigma_{cW}^A(PF_{[P(H_A)]}) = \{\alpha \in Z(A) \mid (PF - \alpha I)_{[P(H_A)]}$ is not invertible in $B(R(P))\}.
\]

**Proof** Let $\alpha \notin \cap \{\sigma_{cW}^A(PF_{[P(H_A)]}) \mid P \in P(H_A)\}$. Then there exists some $P \in P(H_A)$ such that $(PF - \alpha I)_{[P(H_A)]}$ is invertible in $B(R(P))$. Hence $(PF - \alpha I)_{[P(H_A)]}$ is an isomorphism from $R(P)$ onto $R(P)$, so w.r.t. the decomposition
\[
H_A = R(P) \oplus U(N(P)) \xrightarrow{F - \alpha I} R(P) \oplus U(N(P)) = H_A,
\]
$F - \alpha I$ has the matrix \[
\begin{bmatrix}
(F - \alpha I)_1 & (F - \alpha I)_2 \\
(F - \alpha I)_3 & (F - \alpha I)_4
\end{bmatrix},
\]
where
\[
(F - \alpha I)_1 = (PF - \alpha I)_{[P(H_A)]} \text{ is an isomorphism. Then, w.r.t. the decomposition}
\]
\[
H_A = U(R(P)) \oplus U(N(P)) \xrightarrow{F - \alpha I} V^{-1}(R(P)) \oplus V^{-1}(N(P)) = H_A,
\]
$F - \alpha I$ has the matrix \[
\begin{bmatrix}
(F - \alpha I)_1 & 0 \\
0 & (F - \alpha I)_4
\end{bmatrix},
\]
where
\[
U \text{ has the matrix } \begin{bmatrix} 1 & 0 & -(F - \alpha I)_1 & (F - \alpha I)_2 \\
0 & 1 & -(F - \alpha I)_3 & (F - \alpha I)_4
\end{bmatrix} \text{, w.r.t. the decomposition}
\]
\[
R(P) \oplus U(N(P)) \xrightarrow{U} R(P) \oplus U(N(P)),
\]
$V$ has the matrix \[
\begin{bmatrix}
1 & 0 \\
-(F - \alpha I)_3 & (F - \alpha I)_4
\end{bmatrix}, \text{ w.r.t. the decomposition}
\]
\[
R(P) \oplus U(N(P)) \xrightarrow{V} R(P) \oplus U(N(P)),
\]
so $U, V$ are isomorphisms and $(F - \alpha I)_1$ is an isomorphism.

Notice that $U(R(P)) = R(P), V^{-1}(N(P)) = N(P)$. Set $M_1 = R(P), N_1 = U(N(P)), M_2 = V^{-1}(R(P)), N_2 = N(P)$ and $N = R(P)$. It follows that $(F - \alpha I) \in \mathcal{M}(H_A)$, so $\alpha \notin \sigma_{cW}^A(F)$. 
Conversely, suppose that $\alpha \notin \sigma_{\text{AF}}^A(F)$. Then, by definition of $\sigma_{\text{AF}}^A(F)$ and $\mathcal{M} \Phi_0(H_A)$, there exists a decomposition

$$H_A = M_1 \oplus N_1 \xrightarrow{F-\alpha I} M_2 \oplus N_2 = H_A$$

w.r.t. which $F-\alpha I$ has the matrix $\begin{bmatrix} (F-\alpha I)_1 & 0 \\ 0 & (F-\alpha I)_4 \end{bmatrix}$, where $(F-\alpha I)_1$ is an isomorphism, $N_1, N_2$ are finitely generated and $N_1 \oplus N_1 = N_2 \oplus N_2 = H_A$ for some closed submodule $N$.

Let $\cap_{M_1}, \cap_{M_2}$ denote the projections onto $M_1$ along $N_1$ and onto $M_2$ along $N_2$ respectively. Since $F-\alpha I$ has the matrix $\begin{bmatrix} (F-\alpha I)_1 & 0 \\ 0 & (F-\alpha I)_4 \end{bmatrix}$ w.r.t. the decomposition

$$H_A = M_1 \oplus N_1 \xrightarrow{F-\alpha I} M_2 \oplus N_2 = H_A,$$

it follows that

$$\cap_{M_2}(F-\alpha I)|_N = (F-\alpha I) \cap_{M_1}|_N.$$

As $H_A = N \oplus N_1 = M_1 \oplus N_1$, it follows that $\cap_{M_1}|_N$ is an isomorphism from $N$ onto $M_1$. Using this together with the fact that $(F-\alpha I)|_{M_1}$ is an isomorphism from $M_1$ onto $M_2$, one gets that

$$\cap_{M_2}(F-\alpha I)|_N = (F-\alpha I) \cap_{M_1}|_N$$

is an isomorphism from $N$ onto $M_2$. Therefore w.r.t. the decomposition

$$H_A = N \oplus N_1 \xrightarrow{F-\alpha I} V^{-1}(M_2) \oplus N_2 = H_A,$$

where $(F-\alpha I)_1$ is an isomorphism (as $(F-\alpha I)_1 = \cap_{M_2}(F-\alpha I)|_N$). Hence $F-\alpha I$ has the matrix $\begin{bmatrix} (F-\alpha I)_1 & 0 \\ 0 & (F-\alpha I)_4 \end{bmatrix}$ w.r.t. the decomposition

$$H_A = N \oplus N_1 \xrightarrow{F-\alpha I} V^{-1}(M_2) \oplus N_2 = H_A,$$

where $V$ has the matrix

$$\begin{bmatrix} 1 & 0 \\ -((F-\alpha I)_4(F-\alpha I)_1)^{-1} & 1 \end{bmatrix},$$

w.r.t. the decomposition

$$M_2 \oplus N_2 \xrightarrow{V} N \oplus N_2,$$

so that $V$ and $(F-\alpha I)_1$ are isomorphisms. It follows that $(F-\alpha I)|_N$ is an isomorphism from $N$ onto $V^{-1}(M_2)$. Next, since

$$H_A = N \oplus N_2 = V^{-1}(M_2) \oplus N_2,$$

it follows that $P_{\cap_{M_1}(M_2)}$ is an isomorphism from $V^{-1}(M_2)$ onto $N$, where $P$ denotes the projection onto $N$ along $N_2$. Hence $P(F-\alpha I)|_N$ is an isomorphism from $N$ onto $N$, so

$$\alpha \notin \cap \{\sigma(A(P^t(P))) \mid P \in P(H_A) \text{ and } N(P) \text{ is finitely generated}\}.$$

**Lemma 2** $\mathcal{M} \Phi_0(H_A)$ is open in $B^a(H_A)$. 


Proof If \( F \in \mathcal{M}_{\Phi_0}(H_A) \), then there exists a decomposition

\[
H_A = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_A
\]

w.r.t. which \( F \) has the matrix \( \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix} \), where \( F_1 \) is an isomorphism, \( N_1, N_2 \) are finitely generated and \( H_A = N \oplus N_1 = N \oplus N_2 \) for some closed submodule \( N \). We may w.l.g. assume that \( M_1 = N \). Indeed, as we have seen in the proof of the Theorem 4.3, we have that \( PF|_N \) is invertible in \( B(N) \), where \( P \) is the projection onto \( N \) along \( N_2 \). Then, w.r.t. the decomposition \( H_A = N \oplus N_1 \oplus N_2 = H_A \), \( F \) has the matrix \( \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix} \), where \( F_1 \) is an isomorphism, \( N_1, N_2 \) are finitely generated and \( H_A = N \oplus N_1 = N \oplus N_2 \) for some closed submodule \( N \).

We may w.l.g. assume that \( M_1 = N \). Indeed, as we have seen in the proof of the Theorem 4.3, we have that \( PF|_N \) is invertible in \( B(N) \), where \( P \) is the projection onto \( N \) along \( N_2 \). Then, w.r.t. the decomposition \( H_A = N \oplus N_1 \oplus N_2 = H_A \), \( F \) has the matrix \( \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix} \), where \( F_1 \) is an isomorphism, \( N_1, N_2 \) are finitely generated and \( H_A = N \oplus N_1 \oplus N_2 = H_A \) for some closed submodule \( N \).

Now, by the proof of lemma [3, Lemma 2.7.10], there exists some \( \epsilon > 0 \) such that if \( D \in B^a(H_A) \) and \( \|D - F\| < \epsilon \), then \( D \) has the matrix \( \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix} \), w.r.t. the decomposition

\[
H_A = N \oplus \tilde{U}(N_1) \xrightarrow{D} \tilde{V}^{-1}(N) \oplus N_2 = H_A,
\]

where \( \tilde{U}, \tilde{V} \) are isomorphisms. Hence

\[
H_A = N \oplus \tilde{U}(N_1) = N \oplus N_2,
\]

so we may assume w.l.g. that \( N = M_1 \).

We let now \( \mathcal{M}_{\Phi_0}(H_A) \) be the space of all \( F \in B^a(H_A) \) such that there exists a decomposition

\[
H_A = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = H_A,
\]

w.r.t. which \( F \) has the matrix \( \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix} \), where \( F_1 \) is an isomorphism, \( N_1 \) is finitely generated and such that there exist closed submodules \( N_2', N \) where \( N_2' \subseteq N_2, N_2' \cong N_1 \), \( H_A = N \oplus N_1 = N \oplus N_2' \) and the projection onto \( N \) along \( N_2' \) is adjointable.

Then we set

\[
\sigma_{\alpha}^A(F) := \{ \alpha \in Z(A) \mid (F - \alpha I) \notin \mathcal{M}_{\Phi_0}(H_A) \}.
\]
Theorem 10  Let $F \in B^a(H_A)$. Then

$$\sigma_{e_0}^A(F) = \cap \{\sigma_A^A(PF_{\cap(P)}) \mid P \in P^a(H_A)\}$$

where

$$\sigma_A^A(PF_{\cap(P)}) = \{\alpha \in Z(A) \mid (PF - \alpha I)_{\cap(P)} \text{ is not bounded below on } R(P)\}$$

is not bounded below on $R(P)$ and $P^a(H_A) = P(H_A) \cap B^a(H_A)$.

Proof Suppose that $\alpha \notin \sigma_{e_0}^A(P)$ for some $P \in P^a(H_A), \alpha \in Z(A)$. Then the operator $(PF - \alpha I)_{\cap(P)}$ is bounded below on $R(P)$, hence its image is closed. But $R((PF - \alpha I)_{\cap(P)}) = R(PFP - \alpha P)$. Since $(PFP - \alpha P)$ can be viewed as an adjointable operator from $H_A$ into $R(P)$, from [3 Theorem 2.3.3] it follows that $R((PF - \alpha I)_{\cap(P)}) = R(PFP - \alpha P)$ is orthogonally complementable in $R(P)$. So $R(P) = M \oplus M'$, where $M = R(PF - \alpha P)$. Hence $H_A = M \oplus M' \oplus N(P)$ and $(PF - \alpha I)_{\cap(P)}$ is an isomorphism from $R(P)$ onto $M$. It follows that w.r.t. the decomposition

$$H_A = R(P) \oplus N(P) \quad \text{F - } \alpha I \text{ has the matrix } \begin{bmatrix} (F - \alpha I)_1 & (F - \alpha I)_2 \\ (F - \alpha I)_3 & (F - \alpha I)_4 \end{bmatrix}, \text{ where } (F - \alpha I)_1 \text{ is an isomorphism.} \quad \text{Hence w.r.t. the decomposition }$$

$$H_A = R(P) \oplus U(N(P)) \quad \text{F - } \alpha I \text{ has the matrix } \begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}, \text{ where } (F - \alpha I)_1, U, V \text{ are isomorphisms. Set } N = M_1 = R(P), N_1 = U(N(P)), M_2 = V^{-1}(M), N_2 = M' \oplus N(P) \text{ and } N_2' = N(P). \text{ It follows that }$$

$$H_A = N \oplus N_1 = N_2' \oplus N_2, N_1 \cong N_2' \subseteq N_2$$

and $F - \alpha I$ has the matrix $\begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$ w.r.t. the decomposition $H_A = M_1 \oplus N_1 \quad \text{F - } \alpha I \text{ has the matrix } \begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$ w.r.t. the decomposition $H_A = M_1 \oplus N_1 \quad \text{F - } \alpha I \text{ has the matrix } \begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$ w.r.t. which $F - \alpha I$ has the matrix $\begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$, where $(F - \alpha I)_1$ is an isomorphism, $N_1$ is finitely generated and there exists some closed submodules $N'_1, N'_2$ such that $N'_1 \subseteq N_2, N'_2 \cong N_1, N \oplus N_1 = N \oplus N'_2 = H_A$ and the projection onto $N'$ along $N'_2$ is adjointable. As we have seen in the proof of Theorem 9, $\cap_{M_1}(F - \alpha I)_{\cap(P)}$ is an isomorphism then, where $\cap_{M_1}$ denotes the projection onto $M_2$ along $N_2$. Therefore, w.r.t. the decomposition

$$H_A = N_2 U(N_1) \quad \text{F - } \alpha I \text{ has the matrix } \begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$$

$H_A = N \oplus U(N_1) \quad \text{F - } \alpha I \text{ has the matrix } \begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$
(F − αI) has the matrix \[
\begin{bmatrix}
(F − αI)_1 & 0 \\
0 & (F − αI)_4
\end{bmatrix},
\]
where \((F − αI)_1, U, V\) are isomorphisms. Hence \((F − αI)|_N\) maps \(N\) isomorphically onto \(V^{-1}(M)\). Since \(N'_2 \cong N_1\), it follows that \(N'_2\) is finitely generated (as \(N_1\) is so), hence, by Lemma 2.3.7, as \(N'_2\) is a closed submodule of \(N_2\), we get that \(N_2 = N'_2 \oplus N'_2\) for some closed submodule \(N'_2\) of \(N_2\). So
\[
H_A = V^{-1}(M_2) \oplus N_2 = V^{-1}(M_2) \oplus N'_2 \oplus N'_2 = N \oplus N'_2.
\]

It follows that if \(P\) is the projection onto \(N\) along \(N'_2\), then \(P|_{V^{-1}(M_2) \oplus N'_2}\) is an isomorphism from \(V^{-1}(M_2) \oplus N'_2\) onto \(N\). Hence \(P|_{V^{-1}(M_2)}\) maps \(V^{-1}(M_2)\) isomorphically onto some closed submodule of \(N\). Using this together with the fact that \((F − αI)|_N\) is an isomorphism from \(N\) onto \(V^{-1}(M_2)\), we obtain that \(P(F − αI)|_N\) is bounded below. Thus \(α \notin \sigma^a_{(PF|_{R(P)})}\).

Remark 11 In the similar way as for \(\hat{M}(H_A)\), one can show that \(\hat{M}(H_A)\) is open in \(B^a(\hat{H}_A)\).

Definition 4 We set \(\hat{M}(H_A)\) to be the set of all \(D \in B^a(\hat{H}_A)\) such that there exists a decomposition
\[
H_A = M'_1 \oplus N'_1 \xrightarrow{D} M'_2 \oplus N'_2 = H_A
\]
w.r.t. which \(D\) has the matrix \[
\begin{bmatrix}
D_1 & 0 \\
0 & D_4
\end{bmatrix},
\]
where \(D_1\) is an isomorphism, \(N'_2\) is finitely generated and such that \(H_A = M'_1 \oplus N \oplus N'_2\) for some closed submodule \(N\), where the projection onto \(M'_1 \oplus N\) along \(N'_2\) is adjointable.

Then we set
\[
\sigma^a_{ed}(D) = \{α ∈ Z(A) | (D − αI) \notin \hat{M}(H_A)\}
\]
and for \(P \in P^a(\hat{H}_A)\) we set
\[
\sigma^a_{ed}(PD|_{R(P)}) = \{α ∈ Z(A) | (PD − αI)|_{R(P)}\) is not onto R(P)\}.
\]

We have then the following theorem.

Theorem 11 Let \(D \in B^a(\hat{H}_A)\). Then
\[
σ^A_{ed}(D) = \bigcap\{σ^a_{ed}(PD|_{R(P)}) \mid P \in P^a(\hat{H}_A)\}
\]

Proof Suppose first that \(α \notin \bigcap\{σ^a_{ed}(PD|_{R(P)}) \mid P \in P^a(\hat{H}_A)\}\), then \((PD − αI)|_{R(P)}\) is onto \(R(P)\) for some \(P \in P^a(\hat{H}_A)\). Since \(P\) is adjointable and \(R(P)\) is closed, by Theorem 2.3.3 \(R(P)\) is orthogonally complementable in \(\hat{H}_A\), hence \((PD − αI)|_{R(P)}\) can be viewed as an adjointable operator from \(R(P)\) onto \(R(P)\). Then, again by Theorem 2.3.3, \((NPD − αI)|_{R(P)}\) is orthogonally complementable in \(R(P)\), that is \(R(P) = (NPD − αI)|_{R(P)} \oplus N\) for some closed
submodule $\tilde{N}$. The operator $PD - \alpha I$ is an isomorphism from $\tilde{N}$ onto $R(P)$. Hence w.r.t. the decomposition

$$H_A = \tilde{N} \oplus ((N(PD - \alpha I)|_{R(P)}) \oplus N(P)) \xrightarrow{D - \alpha I} R(P) \oplus N(P) = H_A.$$ 

$D - \alpha I$ has the matrix

$$\begin{bmatrix}
(D - \alpha I)_1 & (D - \alpha I)_2 \\
(D - \alpha I)_3 & (D - \alpha I)_4
\end{bmatrix},$$

where $(D - \alpha I)_1$ is an isomorphism.

It follows that $D - \alpha I$ has the matrix

$$\begin{bmatrix}
(D - \alpha I)_1 & 0 \\
0 & (D - \alpha I)_2
\end{bmatrix},$$

w.r.t. the decomposition

$$H_A = \tilde{N} \oplus U((N(PD - \alpha I)|_{R(P)}) \oplus N(P)) \xrightarrow{D - \alpha I} V^{-1}(R(P)) \oplus N(P) = H_A,$$

where $U, V$ and $(D - \alpha I)_1$ are isomorphisms. Set $N = (N(PD - \alpha I)|_{R(P)})$, $M_1 = \tilde{N}, M_2 = V^{-1}(R(P)), N_1' = U((N(PD - \alpha I)|_{R(P)}) \oplus N(P)), N_2 = N(P)$ and observe that $R(P) = N \oplus \tilde{N}$. Hence $(D - \alpha I) \in \hat{M} \Phi^+_A(H_A)$. Conversely, let $\alpha \notin \sigma^A_{cd}(D)$ and let

$$H_A = M_1' \oplus N_1' \xrightarrow{D - \alpha I} M_2' \oplus N_2' = H_A$$

be decomposition w.r.t. which $D - \alpha I$ has the matrix

$$\begin{bmatrix}
(D - \alpha I)_1 & 0 \\
0 & (D - \alpha I)_2
\end{bmatrix},$$

where $(D - \alpha I)_1$ is an isomorphism and such that $H_A = M_1' \oplus N_1' \oplus N_2'$ for some closed submodule $N$, where the projection onto $M_1'$ along $N_2'$ is adjointable. It follows that $P|_{M_1'}$ is an isomorphism onto $M_1' \oplus N$, where $P$ is the projection onto $M_1' \oplus N$ along $N_2'$. Hence $P(D - \alpha I)|_{M_1'}$ is an isomorphism onto $M_1' \oplus N$. Therefore $P(D - \alpha I)|_{M_1' \oplus N}$ is onto $M_1' \oplus N$. Now $R(P) = M_1' \oplus N$ and

$$P(D - \alpha I)|_{M_1' \oplus N} = P(D - \alpha I)|_{M_1' \oplus N}.$$

Similarly as for $M \Phi^+_0(H_A)$ and $M \Phi^+_A(H_A)$, one can show that $\hat{M} \Phi^+_A(H_A)$ is open. If $A = C$, that is if $H_A = H$ is an ordinary Hilbert space, then $M \Phi^+_0(H) = \Phi^+_0(H)$, $\hat{M} \Phi^+_A(H) = \Phi^+_A(H)$ and $\hat{M} \Phi^+_0(H) = \Phi^+_0(H)$. In addition, observe that $M \Phi^+_A(H_A) \subseteq M \Phi^+_A(H)$ and $\hat{M} \Phi^+_A(H_A) \subseteq \hat{M} \Phi^+_A(H_A)$.

5 The boundary of several kinds of Fredholm spectra in $Z(A)$

Recall first [2] Definition 5.1 and Definition 5.2 in Preliminaries. We give then the following definition:

**Definition 5** Let $F \in B^d(H_A)$. We set

$$\sigma^A_{cd}(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin M \Phi^+_0(H_A) \},$$

$$\sigma^A_{cd}(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin M \Phi^+_A(H_A) \},$$

$$\sigma^A_{cd}(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin M \Phi^-_A(H_A) \},$$

$$\sigma^A_{cd}(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin M \Phi^+_A(H_A) \},$$

$$\sigma^A_{cd}(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin M \Phi(H_A) \}.$$
Theorem 12 Let \( F \in B^\alpha(H_A) \). Then the following inclusions hold:

\[
\partial \sigma_{\psi}(F) \subseteq \partial \sigma_{\psi}^A(F) \subseteq \partial \sigma_{\psi}^A(F) \subseteq \partial \sigma_{\psi}(F).
\]

(We consider the boundaries in the \( C^*-\)algebra \( Z(A) \))

Proof We will show this by proving the following inclusions:

\[
\partial \sigma_{\psi}^A(F) \subseteq \sigma_{\psi}^A(F),
\]

\[
\partial \sigma_{\psi}^A(F) \subseteq (\sigma_{\psi}^A(F) \cap \sigma_{\psi}^A(F)) = \sigma_{\psi}^A(F),
\]

\[
\partial \sigma_{\psi}^A(F) \subseteq \sigma_{\psi}^A(F) \text{ and } \partial \sigma_{\psi}^A(F) \subseteq \sigma_{\psi}^A(F).
\]

Since obviously

\[
\partial \sigma_{\psi}^A(F) \subseteq (\sigma_{\psi}^A(F) \cap \sigma_{\psi}^A(F))
\]

if we prove the inclusions above, the theorem would follow. Here we use the property that if \( S, S' \subseteq Z(A) \) , \( S \subseteq S' \) and \( \partial S' \subseteq \partial S \) , then \( \partial S' \subseteq \partial \psi \). The first inclusion follows by the same arguments as in the classical case (the proof of \([6, \text{Theorem 2.2.2.3}]\) since \( \sigma_{\psi}^A(F) \setminus \sigma_{\psi}^A(F) \) is open in \( Z(A) \) by the continuity of index, which follows from \([6, \text{Lemma 2.7.10}]\). Next, if \( \alpha \in \partial \sigma_{\psi}^A(F) \), then obviously \( F - \alpha I \) is in \( \partial M \Phi(H_A) \). Using \([2, \text{Corollary 4.2}]\) we deduce that \( (F - \alpha I) \notin M \Phi_\pm(H_A) \). This is as in the proof of \([6, 2.2.2.4] \) and \([6, 2.2.2.5] \). Hence

\[
\partial \sigma_{\psi}^A(F) \subseteq (\sigma_{\psi}^A(F) \cap \sigma_{\psi}^A(F))
\]

Suppose now that \( \tilde{\alpha} \in \partial \sigma_{\psi}(F) \). If \( \tilde{\alpha} \notin \sigma_{\psi}^A(F) \), then \( (F - \tilde{\alpha} I) \notin M \Phi_\pm(H_A) \), so there exists a decomposition

\[
H_A = M_1 \oplus N_1 \xrightarrow{F - \tilde{\alpha}I} M_2 \oplus N_2 = H_A
\]

w.r.t. which \( F - \tilde{\alpha}I \) has the matrix

\[
\begin{bmatrix}
(F - \tilde{\alpha}I) & 0 \\
0 & (F - \tilde{\alpha}I)
\end{bmatrix},
\]

where \( (F - \tilde{\alpha}I) \), is an isomorphism and \( N_2 \) is finitely generated.

By the proof of \([6, \text{Lemma 2.7.10}]\) there exists some \( \epsilon > 0 \) such that if \( \tilde{\alpha} \in A \) and \( ||\tilde{\alpha} - \tilde{\alpha}'|| < \epsilon \), then \( F - \tilde{\alpha}'I \) has the matrix

\[
\begin{bmatrix}
(F - \tilde{\alpha}'I) & 0 \\
0 & (F - \tilde{\alpha}'I)
\end{bmatrix}
\]

w.r.t. the decomposition

\[
H_A = M_1 \oplus U(N_1) \xrightarrow{F - \tilde{\alpha}'I} V^{-1}(M_2) \oplus N_2 = H_A
\]

where \( (F - \tilde{\alpha}'I) \), \( U, V \) are isomorphisms, so \( (F - \tilde{\alpha}'I) \notin M \Phi_\pm(H_A) \) in this case.

But, since \( \tilde{\alpha} \in \partial \sigma_{\psi}(F) \), we may choose \( \tilde{\alpha}' \in A \) such that \( ||\tilde{\alpha} - \tilde{\alpha}'|| < \epsilon \) and in addition \( (F - \tilde{\alpha}'I) \in M \Phi_\pm(H_A) \). Thus \( (F - \tilde{\alpha}'I) \notin M \Phi_\pm(H_A) \cap M \Phi_\pm(H_A) \) and from Corollary \([6] \) we have \( M \Phi_\pm(H_A) \cap M \Phi_\pm(H_A) = \Phi_\pm(H_A) \), so
\((F - \alpha'I) \in \mathcal{M}\Phi(H_A)\). Since \(F - \alpha'I\) has the matrix \[
abla^{-1}(M_2) \oplus N_2 = H_A,
\]
the decomposition
\[
H_A = M_1 \oplus U(N_1) \xrightarrow{F - \alpha'I} V^{-1}(M_2) \oplus N_2 = H_A,
\]
where \((F - \alpha'I)_1, V\) are isomorphisms and \(N_2\) is finitely generated, by
[2] Lemma 2.17 we must have that \(U(N_1)\) is finitely generated, as
\((F - \alpha'I) \in \mathcal{M}\Phi(H_A)\). Hence \(N_1\) is finitely generated, so \((F - \alpha I) \in \mathcal{M}\Phi(H_A)\). In
particular \((F - \alpha I) \in \mathcal{M}\Phi_a(H_A)\), which contradicts the choice of \(\alpha \in \partial \sigma_{\mathcal{M}}(F)\).
Thus \(\alpha \in \sigma_{\mathcal{M}}(F)\), so \(\alpha \in \sigma_e(F) \cap \sigma_{\mathcal{M}} = \sigma_{\mathcal{M}}(F)\). Similarly, we can show that
\(\partial \sigma_{\mathcal{M}}(F) \subseteq \sigma_{\mathcal{M}}(F)\).

Next we consider the following spectra for \(F \in \mathcal{B}(H_A)\):
\[
\sigma_e^A(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin \mathcal{M}\Phi_a(H_A) \}
\]
\[
\sigma_{\mathcal{M}}^A(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin \mathcal{M}\Phi_a(H_A) \}
\]
Clearly, \(\sigma_e^A(F) \subseteq \sigma_{\mathcal{M}}^A(F) \subseteq \sigma_e(F)\). We have the following theorem.

**Theorem 13** Let \(F \in \mathcal{B}(H_A)\). Then
\[
\partial \sigma_{\mathcal{M}}^A(F) \subseteq \partial \sigma_e^A(F) \subseteq \partial \sigma_e(F)
\]
Moreover, \(\partial \sigma_{\mathcal{M}}^A(F) \subseteq \partial \sigma_e^A(F)\) if \(K(A)\) satisfies the cancellation property.

**Proof** Again it suffices to show
\[
\partial \sigma_{\mathcal{M}}^A(F) \subseteq \sigma_e^A(F), \partial \sigma_e^A(F) \subseteq \sigma_e(F) \text{ and } \partial \sigma_{\mathcal{M}}^A(F) \subseteq \sigma_{\mathcal{M}}^A(F).
\]
The first inclusion follows as in the proof of [3] Theorem 2.7.5, since
\(\partial \sigma_{\mathcal{M}}^A(F) \subseteq \sigma_{\mathcal{M}}^A(F)\) by Theorem [12] and since \(\partial \sigma_e^A(F) \subseteq \sigma_e^A(F)\).

To deduce the second inclusion, assume first that \(\alpha \in \partial \sigma_{\mathcal{M}}^A(F) \setminus \sigma_e^A(F)\). Then
\((F - \alpha I) \in \mathcal{M}\Phi_a(H_A)\) and \((F - \alpha I) \notin \mathcal{M}\Phi_a^C(H_A)\). It follows then that \(F - \alpha I\) is in
\(\mathcal{M}\Phi_a^C(H_A)\) \(\setminus \mathcal{M}\Phi(H_A)\). But, since \(\mathcal{M}\Phi_a^C(H_A) \setminus \mathcal{M}\Phi(H_A)\) is open
by [2] Theorem 4.1 and \(\mathcal{M}\Phi_a(H_A) \subseteq \mathcal{M}\Phi(H_A)\) by definition, it follows that \((F - \alpha I) \notin \partial \mathcal{M}\Phi_a^C(H_A)\). This contradicts the choice of \(\alpha \in \partial \sigma_{\mathcal{M}}^A(F)\). Hence \(\partial \sigma_{\mathcal{M}}^A(F) \subseteq \sigma_e^A(F)\).

For the last inclusion, assume that \(\alpha \in \sigma_{\mathcal{M}}^A(F)\) and that \(\alpha \notin \sigma_{\mathcal{M}}^A(F)\).
Then \((F - \alpha I) \in \mathcal{M}\Phi_a(H_A)\) and \((F - \alpha I) \notin \mathcal{M}\Phi_a^C(H_A)\). This means, by
definitions of \(\mathcal{M}\Phi_a^C(H_A)\) and \(\mathcal{M}\Phi_a(H_A)\) that \((F - \alpha I) \in \mathcal{M}\Phi_a(H_A)\) and that given any decomposition
\[
H_A = M_1 \oplus N_1 \xrightarrow{F - \alpha I} M_2 \oplus N_2 = H_A
\]
\(w.r.t.\) which \((F - \alpha I)\) has the matrix
\[
\begin{pmatrix}
(F - \alpha I)_1 & 0 \\
0 & (F - \alpha I)_4
\end{pmatrix}
\]
\(w.r.t.\) the decomposition
\[
H_A = M_1 \oplus U(N_1) \xrightarrow{F - \alpha I} V^{-1}(M_2) \oplus N_2 = H_A,
\]
where \( (F - \tilde{\alpha}'I)_1, U, V \) are isomorphisms. As \( N_1 \) is not isomorphic to a closed submodule of \( N_2 \) and \( U \) is an isomorphism from \( H_A \) onto \( H_A \), it follows that \( U(N_1) \) is not isomorphic to a closed submodule of \( N_2 \). Now, if \( (F - \tilde{\alpha}'I) \in M_{\Phi}^- (H_A) \), then we must have \( (F - \tilde{\alpha}'I) \in M_{\Phi}^- (H_A) \), as \( (F - \tilde{\alpha}'I) \in M_{\Phi} (H_A) \) and \( M_{\Phi}^- (H_A) = M_{\Phi}^- (H_A) \cap M_{\Phi} (H_A) \) by definition. By [2] Lemma 5.2, as \( K(A) \) satisfies "the cancelation" property, we must then have that \( U(N_1) \not\leq N_2 \) which is a contradiction. So \( \partial \sigma_{e\alpha}^A(F) \subseteq \sigma_{e\alpha}^A(F) \).

Similarly one can show that

\[
\partial \sigma_{e\omega}^A(F) \subseteq \partial \sigma_{e\beta}^A(F) \subseteq \partial \sigma_{e\beta}^A(F)
\]

where

\[
\sigma_{e\beta}^A(F) = \{ \alpha \in A | (F - \alpha I) \notin M_{\Phi}^- (H_A) \}
\]

and

\[
\sigma_{e\beta}^A(F) = \{ \alpha \in A | (F - \alpha I) \notin M_{\Phi}^- (H_A) \}
\]

and in addition \( \partial \sigma_{e\beta}^A(F) \subseteq \partial \sigma_{e\beta}^A(F) \) if \( K(A) \) satisfies "the cancellation property".

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References

1. Dragan S. Djordjević, Perturbations of spectra of operator matrices, J. Operator Theory 48(2002), 467-486.
2. S. Ivković, Semi-Fredholm theory on Hilbert C*-modules, Banach Journal of Mathematical Analysis, to appear (2019)
3. A. S. Mishchenko, A.T. Fomenko, The index of elliptic operators over C*-algebras, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 831–859; English transl., Math. USSR-Izv. 15 (1980) 87–112.
4. V. M. Manuilov, E. V. Troitsky, Hilbert C*-modules, In: Translations of Mathematical Monographs. 226, American Mathematical Society, Providence, RI, 2005.
5. N. E. Wegge Olsen, K-theory and C*-algebras, Oxford Univ. Press, Oxford, 1993.
6. S. Živković Zlatanović, V. Rakočević, D.S. Djordjević, Fredholm theory, University of Niš Faculty of Sciences and Mathematics, Niš, (2019).
7. D. Milčić and K. Veselić, On the boundary of essential spectra, Glasnik Mat. tom 6 (26) No 1(1971), 73788
8. A. Pokrzywa, A characterizations of the Weyl spectrum, Proc. Am. math. Soc. 92, 215-218.
9. J. Zemanek, Compressions and the Weyl-Browder spectra, Proc. Roy. Irish Acad. Sec. A 86 (1986), 57-62.