Berry phase in a composite system

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The Berry phase in a composite system with only one subsystem being driven has been studied in this Letter. We choose two spin-½ systems with spin-spin couplings as the composite system, one of the subsystems is driven by a time-dependent magnetic field. We show how the Berry phases depend on the coupling between the two subsystems, and what is the relation between these Berry phases of the whole system and those of the subsystems.

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Geometric phases in quantum theory attracted great interest since Berry [1] shown that the state of a quantum system acquires a purely geometric feature in addition to the usual dynamical phase when it is varied slowly and eventually brought back to its initial form. The Berry phase of the composite system and those of the subsystems. As you will see, the Berry phase of the composite system is just a sum over those of the subsystems. This result is completely general, although we use spin half as an example to demonstrate the feature of the Berry phase. The Hamiltonian describing a system consisting of two interacting spin-½ particles in the presence of an external magnetic field takes the form,

\[ H = \frac{1}{2} \alpha \sigma_1 \cdot \vec{B}(t) + J(\sigma_1^+ \sigma_2^- + h.c.), \]

where \( \sigma_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z) \), \( \sigma_j^z \) are the pauli operators for subsystem \( j = 1, 2 \) and \( J = (1/2)(\sigma_j^+ + i\sigma_j^-) \). We will choose \( \vec{B}(t) = B_0 \hat{n}(t) \) with the unit vector \( \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) and have assumed that only the subsystem 1 is driven by external fields. The classical field \( \vec{B}(t) \) acts as an external control parameter, as its direction and magnitude can be experimentally changed. \( J \) stands for the constant of coupling between the two spin-½. This coupling is not a typical spin-spin coupling, but rather a toy model describing a double spin flip; nevertheless, the presentation in this Letter can be generalized to the system of nuclear magnetic resonance (NMR) in which quantum computation is implemented by geometric means [12], furthermore the observation of geometric phase for such a system is feasible by the current technology [13].

In a space spanned by \{\( |eg\rangle, |ee\rangle, |gg\rangle, |ge\rangle \} \} and in units of \( \frac{1}{2} \alpha B_0 \), the Hamiltonian Eq.(1) can be written as

\[ H = \begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{-i\phi} & 0 \\
0 & \cos \theta & g & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & g & -\cos \theta & 0 \\
0 & \sin \theta e^{i\phi} & 0 & -\cos \theta
\end{pmatrix}, \]

with \( g = \frac{2J}{\alpha B_0} \) a rescaled coupling constant. Keeping \( \theta \) constant and changing \( \phi \) slowly from 0 to \( \phi(T) = 2\pi \) the Berry phase generated after the system undergoing an adiabatic and cyclic evolution starting with an initial state \( |\Psi_j(t = 0)\rangle \) may be calculated as follows:

\[ \gamma_j = i \int_0^T dt \langle \overline{\Psi}_j | \dot{\Psi}_j \rangle, \]

where \( \overline{\Psi}_j \) is the complex conjugate of \( \Psi_j \).
Hamiltonian Eq.(1) can be calculated by solving

$$|\Psi_j\rangle = \frac{1}{\sqrt{M_j}}[a_j(\phi, \theta, g)|eg\rangle + b_j(\phi, \theta, g)|ce\rangle + c_j(\phi, \theta, g)|gg\rangle + d_j(\phi, \theta, g)|ge\rangle],$$

with

$$a_j(\phi, \theta, g) = \sin \theta e^{-i\phi}, c_j(\phi, \theta, g) = E_j - \cos \theta,$$
$$d_j(\phi, \theta, g) = \frac{g(\cos \theta - E_j)}{\sin \theta - (\cos \theta - E_j)\cos \theta + E_j} e^{i\phi},$$
$$b_j(\phi, \theta, g) = -\frac{\cos \theta + E_j}{\sin \theta} e^{-i\phi} d_j(\phi, \theta, g),$$
$$M_j = |a_j|^2 + |b_j|^2 + |c_j|^2 + |d_j|^2,$$

and $E_j$ that denote the instantaneous eigenvalues of the Hamiltonian Eq.(1) can be calculated by solving

$$(\cos^2 \theta - E_j^2)^2 + (2\sin^2 \theta + g^2)(\cos^2 \theta - E_j^2) + \sin^4 \theta = 0.$$

In the simplest case, where the coupling constant $g = 0$, the eigenvalues $E_{\pm} = \pm 1$, the corresponding eigenstates follow from Eq.10 that $a_+ = \sin \theta e^{-i\phi}$, $c_+ = 1 - \cos \theta$, $b_+ = d_+ = 0$ and $a_- = \sin \theta e^{i\phi}$, $c_- = -1 - \cos \theta$, $b_- = d_- = 0$. These give rise to the well known Berry phase $\gamma_+ = \pi(1 + \cos \theta)$ and $\gamma_- = \pi(1 - \cos \theta)$. This result is easy to understand, the subsystem 2 that evolves freely has no effects on any behaviors of the subsystem 1 as long as the whole system is initially prepared in a separable state. Hence, the Berry phase of the composite system is exactly that of the subsystem 1, while the subsystem 2 acquires no geometric phase. For a noncyclic and nonadiabatical process, the author [10, 17] draw out the same results for geometric phases. We will now turn to study the effect of the coupling between subsystems 1 and 2 on the Berry phase of the whole system, first of all we write down the four eigenvalues of the Hamiltonian as

$$E_1 = \sqrt{1 + \frac{g^2}{2} + \frac{g}{2} \sqrt{g^2 + 4 \sin^2 \theta}} = -E_2,$$
$$E_3 = \sqrt{1 + \frac{g^2}{2} - \frac{g}{2} \sqrt{g^2 + 4 \sin^2 \theta}} = -E_4.$$

Substituting these eigenvalues into Eqs.11 and 12, we can get respective Berry phases. The dependance of the Berry phase on the coupling constant as well as on the azimuthal angle $\theta$ was illustrated in Figures from 1 to 5. Figures from 1 to 4 are for the Berry phases with varying coupling constant $g$ and azimuthal angle $\theta$, whereas figure 5 shows the dependance of the Berry phase on the coupling constant with a specific azimuthal angle $\theta = \frac{\pi}{4}$. The common feature of these figures is that with the rescaled coupling constant $g \to \infty$, all Berry phases $\gamma_1 \to 0$ ( All phases are defined modulo $2\pi$ throughout this paper).
Berry phase. The contour plots presented in Fig.3 and 4 show the same symmetry indeed.

Figure 5 shows the results of Berry phases corresponding to the four instantaneous eigenstates Eq.(4) with a specific azimuthal angle $\theta = \theta_0 = \frac{\pi}{4}$. With $g \to 0$ the Berry phases approach two values (in units of $\pi$) of $\gamma_{\pm} = (1 \pm \cos \theta_0) \approx 1 \pm 0.707$ as expected, whereas they approach zero with $g \to \infty$.

Now we are in a position to study the Berry phase of the subsystem, and to show what is the relation between these phases. Generally speaking a state of a subsystem is no longer a pure one, so we have to adopt the definition of geometric phase for a mixed state $|\Psi\rangle$, that is $\phi_g = \arg \text{Tr}(\rho_0 U(t))$ with $\rho_0$ denoting the initial density matrix and $U(t)$ the transport operator which should fulfill the parallel transport evolution condition. This definition is available when the system from which we want to get geometric phases undergoes an unitary evolution. For the subsystems with non-zero couplings, however, the evolution of each subsystem is not unitary in general. So, here we borrow the idea presented in [27] to define the Berry phase for a mixed state. A non-unitary evolution of a quantal state may be conveniently modelled by attaching an ancilla to the system, in our case the ancilla can always be taken to be the other spin-$\frac{1}{2}$ system.

The geometric phase corresponding to this non-unitary evolution is then defined as the geometric phase of the whole system (system+ ancilla) that evolves unitarily. For an adiabatic cyclic evolution, this gives rise to a definition of Berry phase for a mixed state $\rho(t) = \sum_j p_j |E_j(t)\rangle\langle E_j(t)|$,

$$\gamma = \sum_j p_j \gamma_j,$$

where $\gamma_j = i \int_0^T dt \langle E_j(t)| \dot{E}_j(t) \rangle$. The Berry phase Eq.(7) for a mixed state is just an average of the individual Berry phases, weighted by their eigenvalues $p_j$. To be sure, what we have is consistent with known results, we check that this expression reduces to the standard Berry phase $\gamma = i \int_0^T dt \langle \psi(t)| \dot{\psi}(t) \rangle$ for a pure state $\rho(t) = |\psi(t)\rangle\langle \psi(t)|$.

In our case, we have four density matrices of mixed state for each subsystem, they correspond to the four instantaneous eigenstates of the Hamiltonian, respectively. For example, $\rho_1(t) = \text{Tr}_2 |\Psi_j(t)\rangle\langle \Psi_j(t)|$ represents the $j$-th density matrix for subsystem 1 among the four density matrices, where $\text{Tr}_2$ denotes a trace over subsystem 2. The Berry phase corresponding to this state $\rho_1(t)$ is then given by Eq.(7). Actually, the definition Eq.(7) can be derived by the idea of the so-called purifications as follows. We may construct a pure state

$$|\Psi(t)\rangle = \sum_j \sqrt{p_j} |E_j(t)\rangle_1 \otimes |j\rangle_a$$

for subsystem 1 + ancilla (for subsystem 2+ ancilla, in the same manner) such that

$$\text{Tr}_a |\Psi(t)\rangle\langle \Psi(t)| = \sum_j p_j |E_j(t)\rangle\langle E_j(t)| = \rho_1(t),$$
where $\text{Tr}_a$ denotes a trace over the ancilla and $|E_j(t)\rangle$ represent instantaneous eigenstates of $\rho_1(t)$. Since the states of the ancilla remain unchanged during the evolution, the Berry phase of the subsystem 1 is then the Berry phase of the compound (subsystem+ancilla), this yields the definition Eq. (7). The Berry Phase of the composite system and those of the two subsystems are illustrated in Fig. 6, a sum of the subsystem’s Berry phase is also shown. There is an evidence that the Berry phase of the composite system can be decomposed into a sum of the subsystem’s Berry phases, it reveals the relation between geometric phases of entangled bipartite systems and those of their subsystems. We can prove this point indeed by expanding the instantaneous eigenstate of the composite system via Schmidt decomposition

$$|\Psi\rangle = \sum_i \sqrt{p_i} |e_i(t)\rangle_1 \otimes |E_i(t)\rangle_2,$$

where $|\Psi\rangle$ denotes one of the instantaneous eigenstates Eq. (4). This expansion yields the reduced density operator $\rho_1(t) = \sum_i p_i |e_i(t)\rangle \langle e_i(t)|$ and $\rho_2(t) = \sum_i p_i |E_i(t)\rangle \langle E_i(t)|$ for subsystems 1 and 2, respectively. By the definition Eq. (3), the Berry phase corresponding to $|\Psi\rangle$ follows,

$$\gamma = i \int_0^T \sum_j p_j \langle e_j(t)| \dot{e}_j(t) | dt + i \int_0^T \sum_j p_j \langle E_j(t)| \dot{E}_j(t) | dt,$$

i.e., the Berry phase of the composite system adds up to be that of the composite system. This additivity holds mathematically when the Schmidt decomposition is available with time-independent coefficients $p_i$. Physically, time-independent coefficients $p_i$ indicate no population transfer among the eigenstates of the reduced density matrix of subsystem (this is what we called adiabaticity for the subsystem in this paper). Here the result that the Berry phase of the subsystem adds up to be that for the composite system remains unchanged for all two-subsystem compound when both the compound and the subsystems undergo a cyclic adiabatic evolution.

The observation of this prediction with NMR experiment is within the touch of current technology [10]. For instance, we can use Carbon-13 labelled chloroform in $d_6$ acetone as the sample, in which the single $^{13}C$ nucleus and the $^1H$ nucleus play the role of the two spin $\frac{1}{2}$. The constant of spin-spin coupling $J\sigma_i \sigma_j$ in this case is $J \approx (2\pi)214.5$Hz, and we may control the rescaled coupling constant $g$ by changing the magnitude of the external magnetic field. We would like to address that the interaction between the two spin-$\frac{1}{2}$ in our model is not a typical spin-spin coupling as that in NMR, but rather a toy model describing a double spin flip. So, we have to make a mapping when we employ the presentation in NMR system and when all subsystem are driven by the classical field. Finally, we want to discuss the problem of adiabaticity. Our study is based on an adiabatic cyclic evolution of the composite system. For any subsystem, however, the conditions of adiabaticity are not fulfilled in general, in this sense this is not a Berry phase but a geometric phase for a subsystem when the composite system itself is subject to an adiabatic evolution. But this is not the case in the Letter, it is easily to check that the eigenvalues of the reduced density matrix $\rho_1(t) = \text{Tr}_2(|\Psi_j(t)\rangle \langle \Psi_j(t)|)$ (for any $j$) are independent of time, this indicate the population on the eigenstate of $\rho_1(t)$ remain unchanged [18] while the composite system follows an adiabatic evolution.

To sum up, we have theoretically investigated the Berry phase of a composite system and that of their subsystems. The Berry phase for a mixed state to our best knowledge is a concept new project. The relation between those phases is also presented and discussed, these results provide us a new way to control the Berry phase, we thus might find some applications in quantum computation. We are investigating possible applications of this effects and its connections to other quantum effects in different systems.

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