ON THE NON-ABELIAN MONOPOLES ON THE BACKGROUND OF SPACES WITH CONSTANT CURVATURE

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Procedure of constructing the BPS solutions in SO(3) model on the background of 4D-space-time with the spatial part as a model of constant curvature: Euclid, Riemann, Lobachevsky, is reexamined. It is shown that among possible solutions $W^k_\alpha(x)$ there exist just three ones which in a one-to-one correspondence can be associated with respective geometries, the known non-singular BPS-solution in the flat Minkowski space can be understood as a result of somewhat artificial combining the Minkowski space model with a possibility naturally linked up with the Lobachevsky geometry. A special solution $W^k_{(\text{triv})\alpha}(x)$ in three spaces is described, which can be understood as result of embedding the Abelian monopole potential into the non-Abelian model.

The problem of Dirac fermion doublet in the external BPS-monopole potential in these curved spaces is examined on the base of generally covariant tetrad formalism by Tetrode-Weyl-Fock-Ivanenko. In the frame of spherical coordinates, and (Schrödinger’s) tetrad basis, and special unitary basis in isotopic space, an analog of Schwinger’s one in Abelian case, there arises a Schrödinger’s structure for extended operator $\mathbf{J} = \mathbf{l} + \mathbf{S} + \mathbf{T}$. Correspondingly, instead of monopole harmonics, the language of conventional Wigner $D$-functions is used, radial equations are founds in all three models, and solved in the case of trivial $W^k_{(\text{triv})\alpha}(x)$ in Lobachevsky and Riemann models. In the particular case $W^k_{(\text{triv})\alpha}(x)$, the doublet-monopole Hamiltonian is invariant under additional one-parametric group. This symmetry results in a freedom in choosing a discrete operator $\hat{N}_A$ entering the complete set of quantum variables.

Keywords: Monopole, constant curvature spaces, tetrad formalism, Pauli – Schrödinger’s basis, isotopic multiplet, monopole harmonics, Wigner’s functions
1 Introduction

While there not exists at present definitive succeeded experiments concerning monopoles, it is nevertheless true that there exists a veritable jungle of literature on the monopole theories. Moreover, properties of more general monopoles, associated with large gauge groups now thought to be relevant in physics. As evidenced even by a cursory examination of some popular surveys, the whole monopole area covers and touches quite a variety of fundamental problems. The most outstanding of them are: the electric charge quantization, \( P \)-violation in purely electromagnetic processes, scattering on the Dirac string, spin from monopole and spin from isospin, bound states in fermion-monopole system and violation of the Hermiticity property, fermion-number breaking in the presence of a magnetic monopole and monopole catalysis of baryon decay.

The tremendous volume of publications on monopole topics (and there is no hint that its raise will stop) attests the interest which they enjoy among theoretical physicists, but the same token, clearly indicates the unsettled and problematical nature of those objects: the puzzle of monopole seems to be one of the still yet unsolved problems of particle physics.

Many physicists have contributed to investigation of the monopole-based theories. The wide scope of the field and the prodigious number of investigators associated with various of its developments make it all but hopeless to list even the principal contributors. The list of references given in the end is not complete and the paper does not pretend to be a survey in this matter, most of references may be useful to the readers who wish some supplementary material or are interested in more technical developments beyond the scope of the present treatment.

In general, there are several ways of approaching the monopole problems. As known, together with geometrically topological way of exploration into them, another approach to studying such configurations is possible; namely, that concerns any physical manifestations of monopoles when they are considered as external potentials. Moreover, from the physical standpoint, this latter method can thought of as a more visualized one in comparison with less obvious and direct topological language; in the present treatment the accent is made just on this aspect.

The basic frame of the present investigation is the study of a Dirac particle isotopic doublet in the external monopole potentials on the background of curved spaces, these are 4D-spaces with 3-spatial geometry of constant curvature: Euclid \( E_3 \), Riemann \( S_3 \), and Lobachevsky \( H_3 \).

For convenience of the readers, some remarks about the approach used
in the work are to be given.

The technical and geometrical novelty is that, in the paper, the tetrad generally relativistic method of Tetrode – Weyl – Fock – Ivanenko for describing a spinor particle will be exploited (the first publications were for describing a spinor particle will be exploited are [1, 2, 3, 4, 5, 6, 7, 8, 9]). Choosing this method is not an accidental. It is matter that, the use of a special spherical tetrad in the theory of a spin $1/2$ particle had led Schrödinger [14, 18, 19] to a basis of remarkable features. This Schrödinger’s basis had been used with great efficiency by Pauli in his investigation [20] on the problem of allowed spherically symmetric wave functions in quantum mechanics; also see Möglich [17] cited in Pauli’s paper. In particular, the following explicit expression for (spin $1/2$ particle’s) angular momentum operator had been found

$$J_1 = l_1 + i\sigma^{12}\cos \phi \frac{\cos \theta}{\sin \theta}, \quad J_2 = l_2 + i\sigma^{12}\sin \phi \frac{\cos \theta}{\sin \theta}, \quad J_3 = l_3;$$

such a structure for $J_i$ typifies this frame in bispinor space. This Schrödinger’s basis had been used with great efficiency by Pauli in his investigation [20] on the problem of allowed spherically symmetric wave functions in quantum mechanics. For our purposes, just several simple rules extracted from the much more comprehensive Pauli’s analysis will be quite sufficient (those are almost mnemonic working regulations). They can be explained on the base of $S = 1/2$ particle case. indeed, using Weyl’s representation of Dirac matrices where $\sigma^{12} = \frac{1}{2} (\sigma_3 \oplus \sigma_3)$ and taking into account the explicit form for $\vec{J}_2, J_3$ according to (1), it is readily verified that the most general bispinor functions with fixed quantum numbers $j, m$ are to be

$$\Phi_{jm}(t, r, \theta, \phi) = \begin{vmatrix}
    f_1(t, r) & D_{j-m,-1/2}^j
    f_2(t, r) & D_{j-m,+1/2}^j
    f_3(t, r) & D_{j-m,-1/2}^j
    f_4(t, r) & D_{j-m,+1/2}^j
\end{vmatrix};$$

where $D_{mm'}(\phi, \theta, 0)$ designates the Wigner’s $D$-functions [12, 13] (the notation and subsequently required formulas according to [68] are adopted). One should take notice of the low right indices $-1/2$ and $+1/2$ of $D$-functions in (2), which correlate with the explicit diagonal structure of the matrix $\sigma^{12} = \frac{1}{2} (\sigma_3 \oplus \sigma_3)$. The Pauli criterion allows only half integer values for $j$.

So, one may remember several very simple facts of $D$-functions theory and then produce, almost automatically, proper wave functions. There
may exist a generalized analog of such a representation for $J_i$-operators, that might be successfully used whenever in a linear problem there exists a spherical symmetry.

In particular, the case of electron in the external Abelian monopole field, together with the problem of selecting the allowed wave functions as well as the Dirac charge quantization condition [10], completely come under that Shrödinger-Pauli method. In particular, components of the generalized conserved momentum can be expressed as follows

$$
\begin{align*}
J_1^e g &= l_1 + (i\sigma^{12} - e g) \frac{\cos \phi}{\sin \theta}, \\
J_2^e g &= l_2 + (i\sigma^{12} - e g) \frac{\sin \phi}{\sin \theta}, \\
J_3^e g &= l_3,
\end{align*}
$$

where $e$ and $g$ are an electric and magnetic charges respectively. In accordance with the above rules, the corresponding electron-monopole wave functions can be constructed like in the purely electron pattern (2) but with a single change $D^j_{-m, \pm 1/2} \implies D^j_{-m, e g \pm 1/2}$. The Pauli criterion produces two results: first, $|e g| = 0, 1/2, 1, 3/2, \ldots$ (what is called the Dirac charge quantization condition; second, the quantum number $j$ may take the values $|e g| - 1/2, |e g| + 1/2, |e g| + 3/2, \ldots$ that selects the proper spinor particle-monopole functions.

So, it seems rather a natural step to use some generalized Schrödinger’s basis at analyzing the problem of particles in the Abelian and non-Abelian monopole fields.

There exists additional reasons justifying the interest to just the aforementioned approach: the Shrödinger’s tetrad basis and Wigner’s $D$-functions are deeply connected with what is called the formalism of spin-weight harmonics: Goldberg – Macfarlane – Newman – Rohrlich – Sudarshan [49], developed in the frame of the Newman-Penrose method of light (or isotropic) tetrad by Newman and Penrose [40]; see also Frolov [96], Alexeev and Khlebnikov [116], Penrose and Rindler [174]. On relationships between spinor monopole harmonics of Wu and Yang [93, 98] and spin-weight see in: Dray [182, 186], Gal’tsov – Ershov [201], also see Krolikowski – Rzewuski – Turski [89, 187, 189]. Also see [33, Lochak [34], Halbwachs – Hillion – Vigier [35, 36, 37, 38], Pandres [44]. The present work follows the notation used in [202].

There is still more reason for special attention just to the Schrödinger’s basis on the background of non-Abelian monopole matter. As will be seen
subsequently, that basis can be associated with the unitary isotopic gauge in the non-Abelian monopole problem.

The main guideline of the present paper is as follows.

In Sections 2 – 5 (PART I), we examine constructing the BPS solutions in $SO(3)$ model on the background of 4D-space-time with the spatial part as a model of constant curvature: Euclid, Riemann, Lobachevsky. It is shown that among possible solutions $W^k_\alpha(x)$ (constructed in conformally flat coordinates) there exist just three ones which in a one-to-one correspondence can be associated with respective geometries, the known non-singular BPS-solution in the flat Minkowski space can be understood as a result of somewhat artificial combining the Minkowski space model with a possibility naturally linked up with the Lobachevsky geometry. Besides, a special solution $W^k_{(triv)\alpha}(x)$ in three spaces is described, which can be understood as result of embedding the Abelian monopole potential into the non-Abelian model (first, such a specific non-Abelian solution was found out in [78]).

In Sections 6 – 10 (PART II), we look into the problem of particle in monopole background. Firstly we we consider main points of spin 1/2 quantum particle in the presence of the Abelian external field.

In Section 11 – 15 (PART III) we consider a doublet of Dirac particles in non-Abelian monopole potentials. Because the above mentioned special solution $W^k_{(triv)\alpha}(x)$ in three spaces can be understood as result of embedding the Abelian monopole potential into the non-Abelian model we assume that such a trivial potential is presented in the well-known monopole solutions by t’Hooft and Polyakov [69] [74] [76] we establish explicitly that constituent structure. The use of the spherical coordinates and special gauge transformation enables us to introduce heuristically useful concepts of three gauges: Cartesian, Dirac and Schwinger’s; both later are unitary ones in isotopic space. The use of Schwinger’s isotopic gauge enables us to reduce the non-Abelian doublet-monopole problem to the above Schrödinger’s type. The Pauli criterion allows here all positive integer values for $j : j = 0, 1, 2, 3, \ldots$

As known, an important case in theoretical investigation is the electron-monopole system at the minimal value of the quantum number $j$; so, the case $j = 0$ should be considered especially carefully, and we do this. In the chosen frame, it is the independence on $\theta, \phi$-variables that sets the wave functions of minimal $j$ apart from all other particle multiplet states. Correspondingly, the relevant angular term in the wave equation will be effectively eliminated.

The systems of radial equations found by separation of variables (4 and
8 equations in the cases of $j = 0$ and $j > 0$, respectively) are simplified by searching a suitable operator that can be diagonalized simultaneously with $\vec{J}^2, J_3$. The usual space reflection ($P$-inversion) operator for a bispinor doublet field has to be followed by a certain discrete transformation in the isotopic space, so that a required quantity could be constructed. The problem of discrete symmetry in presence of monopole has been studied intensively in the literature, but previous results are not general as much as possible.

As a result we find out that there are two different possibilities depending on what type of external monopole potential is taken. So, in case of the non-trivial potential, the composite reflection operator with required properties is

$$\hat{N}^S = \hat{\pi} \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}, \quad \hat{\pi} = +\sigma_1$$

(4)

here, the quantities $\hat{\pi}$ and $\hat{P}_{\text{bisp.}}$ represent fixed matrices acting in the isotopic and bispinor space, respectively, and changing simultaneously with any variations of relevant bases. A totally different situation occurs in case of the simplest monopole potential. Now, a possible additional operator, suitable for separating the variables, depends on an arbitrary real numerical parameter $A$ (for some detail in case of complex-valued $A$ see in [261]):

$$\hat{N}^A = \hat{\pi}_A \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}, \quad \hat{\pi}_A = e^{iA\sigma_3} \sigma_1 .$$

(5)

The same quantity $A$ appears also in expressions for the corresponding eigenfunction (the eigenvalues $N_A = \delta(-1)^{j+1}; \delta = \pm 1)$:

$$\Psi^{A}_{\epsilon jm\delta}(x) = T_{+1/2} \otimes F(x) + \delta e^{iA} T_{-1/2} \otimes G(x) .$$

(6)

Further the fermion doublet just in this simplest monopole field. In the first place, we have constructed a remaining operator from a supposedly complete set: $\{ \hat{H}, \vec{J}^2, J_3, \hat{N}_A, \hat{K} = ? \}$. That $\hat{K}$ is determined as a natural extension of the well-known (Abelian) Dirac operator to the non-Abelian case. Correspondingly, the set of radial equations is eventually reduced to a set of two ones; which can be solved in hyper geometrical functions in all three spaces. The spectrum of energy in the space $S_3$ is discrete.

On simple comparing the non-Abelian doublet functions with the Abelian ones, we arrive at an explicit factorization of the doublet functions by Abelian ones and isotopic basis vectors. The relevant decompositions have
been found for the composite states with all values of $j$, including the minimal one $j_{\text{min}} = 0$ too.

We are especially interested in the question: where does the above ambiguity come from? It is quite easily understandable that this possibility is closely connected with the fact of decoupling of two isotopic components in the wave equation. The situation can be formulated in terms of an additional hidden symmetry: there are two operators, $t_3$ and $\hat{N}_A$, commuting with the Hamiltonian but not commuting with each other. In formal mathematical terms, the origin of the above freedom in discrete symmetry operations lies in the existence of an additional (one parametric) operation $U(A)$ that leaves the doublet-monopole Hamiltonian invariant. Just this operation $U(A)$ changes $\hat{N}_{A=0}$ into $\hat{N}_A$. Different values for $A$ lead to the same whole functional space; each fixed $A$ governs only the basis states $\Psi^A(x)$ of it, and the symmetry operation acts transitively on those states: $\Psi^A(x) = U(A)\Psi^{A'=0}(x)$.

Additionally, we draw an analogy between this isotopic symmetry and more familiar chiral symmetry transformation ($\gamma^5$ symmetry in massless Dirac field theory [125]). The role of the Abelian $\gamma^5$-matrix is taken by the isotopic $\sigma_3$-matrix: its form in the Schwinger’s isotopic gauge is $U^S(A) = \exp(A/2) \exp(-i A \vec{n}_{\theta,\phi})$.

Also some technical details touching the discrete operation $\hat{N}_A$ are given; in particular, the form of that transformation in the Cartesian isotopic gauge is calculated:

$$U_C(A) = e^{+iA/2} \exp(-i \frac{A}{2} \vec{n}_{\theta,\phi}) ;$$

(7)
correspondingly, the discrete operator looks

$$\hat{N}^C_A = (-i) \exp(-i A \vec{n}_{\theta,\phi}) \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P} .$$

(8)
The explicit coordinate dependence in Cartesian gauge results from the non-commutation $\sigma_3$ with a gauge transformation involved into transition from Shwinger’s to Cartesian isotopic basis. In the analogous Abelian situation, the form of the chiral transformation remains the same because $\gamma^5$ and the relevant gauge matrix (that belongs to the bispinor local representation of the group $SL(2,C)$) are commutative with each other.

It may be stressed that these symmetry operations occur only in the case of special monopole potential; instead, for the ’t Hooft-Polyakov potential as well as for the free isotopic doublet case no such additional symmetry occur.
2 BPS-monopole, radial equations

In the literature, a $SU(2)$-monopole problem in the limit of Bogomolny – Prasad – Sommerfield for Minkowski flat and curved space-time backgrounds has attracted great interest.

Polyakov [69], t’Hooft [73], Julia and Zee [76], Prasad and Sommerfield [77], Bais and Russel [78], Wang [79], Nieuwenhuizen et al [83], Benguria et al [101], Witten [93], Ray [111], Goddard and Olive [113], Cervero and Jacobs [117], Boutaleb et al [118], Actor [122], Harnad et al [123], Maison [137], Clement [138], Gu [139], Schigolev [147], [166], Kamata [148], Henneaux [149], Hitchin [150], Kasuya [151], Hitchin [165], Melnikov and Shigolev [167], Comtet et al [170], Deser [171], Atiyah [172], Chakrabarti [183], Nash [191], Gibbons and Manton [190], Chakrabarti [193], Atiyah and Hitchin [198], Garland and Murray [211], Pajput and Rashmi [212], Ershov and Gal’tsov [214], Yaffe [215], Yang [216], Bartnik [218], Austin and Braam 1990-Austin-Braam, Pedersen and Tod [223], Ortiz [227], Balakrishna and Wali [228], Breitenlohner er al [233], Hitchin et al [235], Volkov [240], Jarvis and Norbury [243], Kraan and van Baal [246], Kimyeong Lee and Changhai Lu [252], Houghton et al [263], Volkov and Gal’tsov [264], Norbury et al [267], [274], [275], Meng [276], Landweber [282], Gibbons and Warnick [283], Weinberg and Yi [285], Harland [286].

In a space-time with a metrics tensor $g_{\alpha\beta}(x)$ let us consider the Yang-Mills - Higgs system. Lagrangian of that system is given by

$$L = \frac{1}{2} g^{\alpha\beta}(x) D_\alpha \Phi^a D_\beta \Phi^a - \frac{1}{4} g^{\alpha\rho}(x) g^{\beta\sigma}(x) F_{\alpha\beta}^a F_{\rho\sigma}^a - \frac{\lambda}{4}(\Phi^2 - V^2)^2.$$ 

Three scalar fields $\Phi^a(x)$ are supposed to be real; correspondingly, the Lagrangian is invariant under local $SO(3,R)$ group transformations. The operator $D_\alpha$ is

$$D_\alpha \Phi^a = \partial_\alpha \Phi^a + e \epsilon_{abc} W_\alpha^b \Phi^c.$$ 

The $W_\alpha^b$ stands for the Yang-Mills isotriplet. Antisymmetric generally co-variant strength tensor is given by

$$F_{\alpha\beta}^a = \partial_\alpha W_\beta^a - \partial_\beta W_\alpha^a + e \epsilon_{abc} W_\alpha^b W_\beta^c.$$
In accordance with the variational principle one can derive equations

\[
\frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} D^\alpha \Phi^a + e \epsilon_{abc} W^b_\alpha D^\alpha \Phi^c = -\lambda (\Phi^2 - V^2) \Phi^a ,
\]

\[
\frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} F^\alpha_{\beta} + e \epsilon_{abc} W^b_\alpha F^\alpha_{\beta} = -e \epsilon_{abc} \Phi^b D^\beta \Phi^c .
\] (9)

In the following, all analysis will be done for three (curved) space models: Euclid’s – $E_3$, Riemann’s – $S_3$, and Lobachevsky’s – $H_3$; conformally flat coordinates will be used (we employ dimensionless variables $x^\alpha / \rho \Rightarrow x^\alpha$, where $\rho$ is a curvature radius):

\[
dS^2 = (dx^0)^2 - \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\Sigma^2} .
\] (10)

To $E_3$-model there corresponds $\Sigma = 1$, to

$S_3 - \Sigma = 1 + r^2/4 , \quad H_3 - \Sigma = 1 - r^2/4 , \quad r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

Starting with the the known dyon substitution

\[
\Phi^a(x) = x^a \Phi(r) , \quad W^a_0(x) = x^a f(r) , \quad W^a_i(x) = \epsilon_{iab} x^b K(r) ,
\] (11)

after simple calculation, we get the radial equations for $\Phi, f, K$ – below only the situation in absence of self-interactions between components of scalar triplet will be examined (the Bogomolny-Prasad-Sommerfield limit)

\[
\Phi'' + \frac{4}{r} \Phi' - 2e\Phi (2 + er^2 K) K - \frac{\Sigma'}{\Sigma} (\Phi' + \frac{\Phi}{r}) = 0 ,
\]

\[
f'' + \frac{4}{r} f' - 2ef (2 + er^2 K) K - \frac{\Sigma'}{\Sigma} (f' + \frac{f}{r}) = 0 ,
\]

\[
K'' + \frac{4K'}{r} + e \frac{(f^2 - \Phi^2) (1 + er^2 K)}{\Sigma^2} - eK^2 (3 + er^2 K) + \frac{\Sigma'}{\Sigma} (K' + \frac{2K}{r}) = 0 .
\] (12)

## 3 Solutions in flat space

Now let us turn to eqs. (12) specified for the flat Minkowski space (in next Sections we will extend the solving procedure to $H_3$ and $S_3$ models). As
Σ = 1 eqs. (12) take the form

\[ \Phi'' + \frac{4}{r} \Phi' - 2e\Phi(2 + er^2 K) K = 0, \]

\[ f'' + \frac{4}{r} f' - 2ef(2 + er^2 K) K = 0, \]

\[ K'' + \frac{4K'}{r} + e(f^2 - \Phi^2)(1 + er^2 K) - eK^2(3 + er^2 K) = 0. \] (13)

It is known that the dyon system (11) can be solved on the base of solution for a purely monopole system:

\[ \Phi^a(x) = x^0 \Phi(r), \ W^a_0(x) = 0, \ W^a_i(x) = \epsilon_{iab} x^b K(r), \] (14)

when radial equations are

\[ \Phi'' + \frac{4}{r} \Phi' - 2e\Phi(2 + er^2 K) K = 0, \]

\[ K'' + \frac{4K'}{r} - e\Phi^2(1 + er^2 K) - eK^2(3 + er^2 K) = 0. \] (15)

Indeed, turning to eqs. (13) and setting \( f = c\Phi \), where \( c \) is a constant, one comes to

\[ f = c\Phi, \ \frac{d^2}{dr^2}\Phi + \frac{4}{r} \frac{d}{dr} \Phi - 2e\Phi(2 + er^2 K) K = 0, \]

\[ \frac{1}{1 - c^2} \left( \frac{d^2}{dr^2}K + \frac{4}{r} \frac{d}{dr}K \right) - e\Phi^2(1 + er^2 K) - \frac{eK^2}{1 - c^2}(3 + er^2 K) = 0. \]

From these, having introduced a new radial variable and a new function \( \tilde{K} \):

\[ r \rightarrow (1 - c^2)^{1/4} r = \tilde{r}, \ \frac{K(r)}{\sqrt{1 - c^2}} = \tilde{K}((1 - c^2)^{1/4} r), \]

one obtains a system of the above type (15). Therefore the dyon functions have been reduced to monopole ones:

\[ \Phi(r) = \tilde{\Phi}((1 - c^2)^{1/4} r), \ f(r) = c\Phi(r), \]

\[ K(r) = \sqrt{1 - c^2} \tilde{K}((1 - c^2)^{1/4} r). \] (16)

Bearing this in mind, we will examine only the purely monopole equations (15). For further work instead of \( \Phi(r) \) and \( K(r) \) in (15) it is convenient to use new functions \( f_1 \) and \( f_2 \):

\[ 1 + e r^2 K = r f_1(r), \quad 1 + e r^2 \Phi = r f_2(r); \] (17)
correspondingly eqs. (15) transform into
\[
2 \left( f_2' + f_1^2 \right) + (f_2'' - 2 f_1^2 f_2) = 0,
2 \left( f_1' + f_1 f_2 \right) + r \left( f_1'' - f_1 f_2^2 - f_1^3 \right) = 0.
\] (18)

One can solve these equations by satisfying four equations
\[
f_2' + f_1^2 = 0, \quad f_2'' - 2 f_1^2 f_2 = 0,
f_1' + f_1 f_2 = 0, \quad f_1'' - f_1 f_2^2 - f_1^3 = 0.
\] (19)

Second and fourth equations are consequences of the first and third, so we have only two independent ones
\[
f_1' = -f_1 f_2, \quad f_2' = -f_1^2, \quad \text{or} \quad f_2 = -\frac{f_1'}{f_1}, \quad \left( \frac{f_1'}{f_1} \right)' = f_1^2; \quad (20)
\]
the task reduces to a single differential equation
\[
\left( \frac{f_1'}{f_1} \right)' = f_1^2.
\] (21)

From whence one gets
\[
(ln f_1)'' = f_1^2, \quad \frac{d}{dr} \left[ (ln f_1)' \right]^2 = \frac{d}{dr} f_1^2.
\]

From this it follows
\[
\int \frac{d f_1}{f_1 \sqrt{c + f_1^2}} = \pm (r + \text{const}).
\]

Depending on the sign of the constant $c$ we have three types of solutions:
\[
c = 0, \quad f_1 = \pm \frac{A}{Ar + B}, \quad f_2 = \frac{A}{Ar + B};
c < 0, \quad f_1 = \pm \frac{A}{\text{sh} \ (Ar + B)}, \quad f_2 = \frac{A}{\tanh \ (Ar + B)};
c > 0, \quad f_1 = \pm \frac{A}{\sin \ (Ar + B)}, \quad f_2 = \frac{A}{\tan \ (Ar + B)};
\] (22)

where $A$ and $B$ are arbitrary constants. Turning back to (17), we get
\[
K(r) = \frac{1}{e r^2} \left( r f_1 - 1 \right), \quad \Phi(r) = \frac{1}{e r^2} \left( r f_2 - 1 \right); \quad (23)
\]
in usual unites, \( A \) is measured in (meter\(^{-1} \)), and \( B \) is dimensionless. Thus, we arrive at six different solutions:

\[
\begin{align*}
K_1^\pm &= \frac{1}{e \, r^2} \left[ \pm \frac{A}{Ar + B} - 1 \right], \\
\Phi_1(r) &= \frac{1}{e \, r^2} \left[ \frac{A}{Ar + B} - 1 \right], \\
K_2^\pm &= \frac{1}{e \, r^2} \left[ \pm \frac{A}{\sinh (Ar + B)} - 1 \right], \\
\Phi_2(r) &= \frac{1}{e \, r^2} \left[ \frac{A}{\tanh (Ar + B)} - 1 \right], \\
K_3^\pm &= \frac{1}{e \, r^2} \left[ \pm \frac{Ar}{\sin (Ar + B)} - 1 \right], \\
\Phi_3(r) &= \frac{1}{e \, r^2} \left[ \frac{Ar}{\tan (Ar + B)} - 1 \right].
\end{align*}
\]

(24)

Here it should be noted that in going from (20) to (21) we have missed one simple solution (which is to be interpreted as Abelian Dirac’s nonopole being placed into background of the non-Abelian theory)

\[
\begin{align*}
f_1(r) &= 0, \quad f_2(r) = C, \quad \text{or} \\
K &= -\frac{1}{e \, r^2}, \quad \Phi(r) = \frac{1}{e \, r^2} (C \, r - 1).
\end{align*}
\]

(25)

It should be noted that if \( f_1 = er^2K + 1 = 0 \), the initial equations (15) become just one linear and other nonlinear equations:

\[
\begin{align*}
\Phi'' + \frac{4}{r} \Phi' + 2\Phi &= 0, \\
K'' + \frac{4K'}{r} - 2eK^2 &= 0.
\end{align*}
\]

The nonlinear one is satisfied by the function \( K = -1/er^2 \); whereas a general solution \( \Phi(r) \) is a linear combination

\[
\Phi = \frac{c_1}{r} + \frac{c_2}{r^2}.
\]

(26)

4 Some technical details for curved models

In curved models \( H_3 \) and \( S_3 \), analogously to the flat space \( E_3 \), there exists possibility to construct dyon functions in terms of purely monopole’s ones (all details are omitted). By this reason, further we will examine only the purely monopole case:

\[
\begin{align*}
\Phi'' + \frac{4}{r} \Phi' - 2e\Phi (2 + er^2K) K - \frac{\Sigma'}{\Sigma} (\Phi' + \Phi \frac{r}{K}) &= 0, \\
K'' + \frac{4K'}{r} - e\Phi^2 \frac{(1 + er^2K)}{\Sigma^2} - eK^2 (3 + er^2K) + \frac{\Sigma'}{\Sigma} (K' + 2 \frac{K}{r}) &= 0.
\end{align*}
\]

(27)
Instead of $K(r)$ and $\Phi(r)$ let us introduce $A(r)$ and $B(r)$:

$$1 + e^2 r^2 K = A(r), \quad er^2 \Phi = B(r), \quad (28)$$

then eqs. (27) transform into

$$B'' - \frac{2B A^2}{r^2} + \frac{\Sigma'}{\Sigma} \left( \frac{B}{r} - B' \right) = 0, \quad (29)$$

$$A'' - \frac{A B^2}{r^2 \Sigma^2} + \frac{A(1 - A^2)}{r^2} + \frac{\Sigma'}{\Sigma} A' = 0. \quad (30)$$

For $A(r)$ and $B(r)$ let us use substitutions

$$A = c f_1(R), \quad B = a f_2(R) + b; \quad (31)$$

where $a(r), b(r), c(r), R(r)$ stand for some yet unknown functions of $r$, whereas $f_1(R)$ and $f_2(R)$ are assumed to obey two relationships (see (20))

$$\frac{d}{dR} f_1 = -f_1 f_2, \quad \frac{d}{dR} f_2 = -f_1^2,$$

so that $f_1$, $f_2$ coincide with those listed in (20). Initial functions look as follows:

$$K(r) = \frac{1}{er^2} \left[ c(r) f_1(R) - 1 \right], \quad \Phi(r) = \frac{1}{er^2} \left[ a(r) f_2(R) + b(r) \right]; \quad (32)$$

limiting transition to the case of the flat space should be

$$c(r) \Longrightarrow r, \quad a(r) \Longrightarrow r, \quad b(r) \Longrightarrow -1, \quad R(r) \Longrightarrow r. \quad (33)$$

Substituting (31) into (29) we arrive at

$$a'' f_2 - \left(2a' R' + a R''\right) f_1^2$$

$$+ 2a \left(R'\right)^2 f_1^2 f_2 + b'' - \frac{2}{r^2} \left(a f_2 + b\right) c^2 f_1^2$$

$$+ \frac{\Sigma'}{\Sigma} \left[ \frac{a f_2 + b}{r} - \left(a f_2 - a R' f_1^2 + b' \right) \right] = 0. \quad (34)$$

Setting factors at 1, $f_2$, $f_1^2 f_2$ equal to zero, we get four equations:

$$1: \quad b'' + \frac{\Sigma'}{\Sigma} \left(\frac{b}{r} - b'\right) = 0,$$

$$f_2: \quad a'' + \frac{\Sigma'}{\Sigma} \left(\frac{a}{r} - a'\right) = 0,$$

$$f_1^2: \quad -2a' R' - a R'' - \frac{2bc^2}{r^2} + \frac{\Sigma'}{\Sigma} a R' = 0,$$

$$f_1^2 f_2: \quad 2a \left(R'\right)^2 - \frac{2ac^2}{r^2} = 0. \quad (35)$$
Analogously, substituting (80) into (30), we get
\[ \begin{align*}
    c'' f_1 - (2c' R' + c R'') f_1 f_2 + c (R')^2 f_1^3 + c (R')^2 f_1 f_2^2 \\
    - \frac{c}{r^2} f_1 (a^2 f_2^2 + 2ab f_2 + b^2) \frac{1}{\Sigma} + \frac{cf_1}{r^2} (1 - c^2 f_1^2) \\
    + \Sigma' \sum (c' f_1 - c R' f_1 f_2) = 0 , \quad (36)
\end{align*} \]
from where it follows four equations:

\[ \begin{align*}
    f_1 : & \quad c'' - \frac{cb^2}{r^2 \Sigma^2} + \frac{c}{r^2} \sum c' = 0 , \\
    f_1 f_2 : & \quad -2c' R' - c R'' - \frac{2abc}{r^2 \Sigma^2} - \frac{\Sigma'}{\Sigma} c R' = 0 , \\
    f_1^3 : & \quad c (R')^2 - \frac{c^3}{r^2} = 0 , \\
    f_1 f_2^2 : & \quad c(R')^2 - \frac{ca^2}{r^2 \Sigma^2} = 0 . \quad (37)
\end{align*} \]

Collecting eqs. (35) and (86) together, we get the system

\[ \begin{align*}
    (R')^2 &= \frac{c^2}{r^2} , \quad (R')^2 = \frac{a^2}{r^2 \Sigma^2} , \quad (38) \\
    a'' + \frac{\Sigma'}{\Sigma} \left( \frac{a}{r} - a' \right) &= 0 , \quad b'' + \frac{\Sigma'}{\Sigma} \left( \frac{b}{r} - b' \right) = 0 , \quad (39) \\
    -2a' R' - a R'' - \frac{2bc^2}{r^2} + \frac{\Sigma'}{\Sigma} a R' &= 0 , \quad (40) \\
    -2c' R' - c R'' - \frac{2abc}{r^2 \Sigma^2} - \frac{\Sigma'}{\Sigma} c R' &= 0 . \quad (41)
\end{align*} \]

It is readily seen that in the system, eq. (40) can be derived from others. Indeed, let us multiply eq. (41) by \( c \), then
\[ -2cc' R' - c^2 R'' - \frac{2abc^2}{r^2 \Sigma^2} - \frac{\Sigma'}{\Sigma} c^2 R' = 0 ; \]
in turn, from eqs. (38) it follows
\[ c^2 = \frac{a^2}{\Sigma^2} , \quad \Rightarrow \quad cc' = \frac{aa'}{\Sigma^2} - \frac{a^2}{\Sigma^2} \frac{\Sigma'}{\Sigma} . \]

Therefore, previous relation can be transformed to the form
\[ -2R' a \frac{a}{\Sigma^2} \left( a' - a \frac{\Sigma'}{\Sigma} \right) - \frac{a^2}{\Sigma^2} R'' - \frac{2abc^2}{r^2 \Sigma^2} - \frac{\Sigma'}{\Sigma} \frac{a^2}{\Sigma^2} R' = 0 . \]
From the latter it follows

\[-2R' \left( a' - a \frac{\Sigma'}{\Sigma} \right) - a R'' - \frac{2bc^2}{r^2} - \frac{\Sigma' a}{\Sigma} R' = 0.\]

which coincides with (40). Therefore, independent equations are

\[(R')^2 = \frac{c^2}{r^2}, \quad c^2 = \frac{a^2}{\Sigma^2}, \quad (42)\]

\[a'' + \frac{\Sigma'}{\Sigma} \left( \frac{a}{r} - a' \right) = 0, \quad b'' + \frac{\Sigma'}{\Sigma} \left( \frac{b}{r} - b' \right) = 0, \quad (43)\]

\[-2c' R' - c R'' - \frac{2abc}{r^2 \Sigma^2} - \frac{\Sigma'}{\Sigma} c R' = 0. \quad (44)\]

Eq. (44) can be simplified. Indeed, let us multiply it by \(cR'\):

\[-(c^2)'(R')^2 - \frac{1}{2} \frac{c^2}{r^2} [(R')^2]' - \frac{2abc}{r^2 \Sigma^2} R' - \frac{\Sigma'}{\Sigma} c^2 (R')^2 = 0,\]

and allow for expressions for \(c^2\) and \((R')^2\) according to (42):

\[-\frac{a^2}{r^2 \Sigma^2} \frac{d}{dr} \left( \frac{a^2}{\Sigma^2} \right) - \frac{1}{2} \frac{a^2}{\Sigma^2} \frac{d}{dr} \frac{a^2}{r^2 \Sigma^2} - \frac{2ab}{r^2 \Sigma^2} a^2 - \frac{\Sigma'}{\Sigma} a^2 - \frac{a^2}{r^2 \Sigma^2} \frac{\Sigma'}{\Sigma} a^2 = 0.\]

After simple calculation, we get to

\[b = \frac{1}{2R'} \left( -3a' + 2 \frac{\Sigma'}{\Sigma} a + \frac{a}{r} \right).\]

Further, bearing in mind the identity

\[R' = \delta \frac{a}{r \Sigma}, \quad \delta = \pm 1,\]

we arrive at

\[2ab = \delta r \Sigma \left( -3a' + 2 \frac{\Sigma'}{\Sigma} a + \frac{a}{r} \right);\]

Thus, the radial system will take the form

\[a'' + \frac{\Sigma'}{\Sigma} \left( \frac{a}{r} - a' \right) = 0, \quad b'' + \frac{\Sigma'}{\Sigma} \left( \frac{b}{r} - b' \right) = 0, \quad (45)\]

\[c = \epsilon \frac{a}{\Sigma}, \quad R' = \delta \frac{a}{r \Sigma}, \quad (46)\]

\[2ab = \delta r \Sigma \left( -3a' + 2 \frac{\Sigma'}{\Sigma} a + \frac{a}{r} \right); \quad (47)\]
here \( \delta^2 = 1, \epsilon^2 = 1 \), and these two parameters are independent. The quantity \( \epsilon \) may be excluded by inserting it into \( a(r) \), solution of the linear differential equation. So, we have more simple system:

\[
a'' + \frac{\Sigma'}{\Sigma} \left( \frac{a}{r} - a' \right) = 0, \quad b'' + \frac{\Sigma'}{\Sigma} \left( \frac{b}{r} - b' \right) = 0, \quad (48)
\]

\[
c = \frac{a}{\Sigma}, \quad R' = \delta \frac{a}{r \Sigma}, \quad (49)
\]

\[
2ab = \delta r \Sigma \left( -3a' + 2 \frac{\Sigma'}{\Sigma} a + \frac{a}{r} \right). \quad (50)
\]

The way to solve the task is to be as follows: first, one can find general expressions for \( a(r), b(r) \); then determine \( c(r) \) and \( R(r) \) from (49); and finally one should substitute \( a(r) \) and \( b(r) \) into eq. (50).

5 Radial solutions in the Riemann and Lobachevsky models

First, let us examine the case of Riemann space model. Equations for \( a(r) \) and \( b(r) \) are the same \( (a, b = g) \):

\[
\frac{d^2}{dr^2} g - \frac{2r}{4 \rho^2 (1 + r^2/4 \rho^2)} \frac{d}{dr} g + \frac{2}{4 \rho^2 (1 + r^2/4 \rho^2)} g = 0; \quad (51)
\]

in this section we will use usual unites for \( r \). General solutions are

\[
a = a_1 r + a_2 (1 - r^2/4 \rho^2), \quad b = b_1 r + b_2 (1 - r^2/4 \rho^2); \quad (52)
\]

\( a_1, a_2, b_1, b_2 \) are constants. Correspondingly, from (49) for \( c(r) \) and \( R(r) \) we have

\[
c(r) = a_1 \frac{r}{1 + r^2/4 \rho^2} + a_2 \frac{1 - r^2/4 \rho^2}{1 + r^2/4 \rho^2}, \quad (53)
\]

\[
R(r) = \delta \left( a_1 2 \rho \arctan \frac{r}{2 \rho} + a_2 \ln \frac{r/\rho}{1 + r^2/4 \rho^2} \right) + C. \quad (54)
\]

Substituting \( a(r), b(r) \) into eq. (50), we get the system of algebraic relations:

\[
2a_2 b_2 = \delta a_2, \quad a_1 b_2 + a_2 b_1 = -\delta a_1, \quad (55)
\]

\[
a_1 b_1 - \frac{1}{2} b_2 a_2 \frac{1}{\rho^2} = \delta \frac{5}{4} a_2 \frac{1}{\rho^2}.
\]
First, let \( a_2 \neq 0 \). From first relation in (55) it follows \( b_2 = \delta / 2 \), and two remaining ones take on the form
\[
b_1 a_2 = -\frac{3}{2} \delta a_1 , \quad a_1 b_1 = +\frac{3}{2} \delta a_2 \frac{1}{\rho^2} ,
\]
from where it follows
\[
\frac{a_2}{a_1} = -\frac{a_1}{a_2} \frac{\rho^2}{\rho^2} \implies \left( \frac{a_2}{a_1} \right)^2 = -\rho^2 ,
\]
and therefore , \( a_2 = \pm i \rho a_1 \); at this for \( b_1 \) we have complex values:
\[
b_1 = \pm \frac{3}{2} i \delta \rho .
\]
Thus, we arrive at the complex-valued solution:
\[
a = a_1 r \pm i a_1 \rho \left( 1 - \frac{r^2}{4 \rho^2} \right) , \quad b = \pm \frac{3i \delta}{2} \frac{r}{\rho} + \frac{\delta}{2} \left( 1 - \frac{r^2}{4 \rho^2} \right) .
\]
(56)
Correspondingly, \( c(r) \) and \( R(r) \) are
\[
c(r) = a_1 \frac{r}{1 + r^2/4\rho^2} \pm i a_1 \rho \frac{1 - r^2/4\rho^2}{1 + r^2/4\rho^2} ,
\]
\[
R(r) = \delta \left( a_1 2\rho \arctan \frac{r}{2\rho} \pm i a_1 \rho \ln \frac{r/\rho}{1 + r^2/4\rho^2} \right) + C .
\]
(57)
In the limit \( \rho \to \infty \), they behave
\[
a = \pm i a_1 \rho , \quad b = +\frac{\delta}{2} , \quad c(r) = \pm i a_1 \rho , \quad R(r) = \delta \left[ a_1 r \pm i a_1 \rho \ln \frac{r}{\rho} \right] + C .
\]
(58)
This solution is complex-valued and it has no physical meaning in the limit of the flat space, in the following this solution will not be considered.

Now, let \( a_2 = 0 \), then eqs. (55) give \( a_1 b_2 = -\delta a_1 , \quad a_1 b_1 = 0 \), from where we arrive at two solutions:

I \quad \begin{align*}
a_2 &= 0 , & a_1 &\neq 0 , & b_1 &= 0 , & b_2 &= -\delta ; \\
a(r) &= a_1 r , & b(r) &= -\delta \left( 1 - \frac{r^2}{4\rho^2} \right) , \\
c(r) &= \frac{a_1 r}{1 + r^2/4\rho^2} , & R(r) &= \delta a_1 \left( 2\rho \arctan \frac{r}{2\rho} \right) + C .
\end{align*}
(59)

II \quad \begin{align*}
a_2 &= 0 , & a_1 &= 0 : & b &= b_1 r + b_2 \left( 1 - \frac{r^2}{4\rho^2} \right) , \\
a(r) &= 0 , & c(r) &= 0 , & R(r) &= C .
\end{align*}
(60)
In order to have a needed behavior in the limit of the flat space, one must consider only the following solutions:

\[
I \quad a(r) = a_1 r, \quad b(r) = -(1 - \frac{r^2}{4 \rho^2}), \\
c(r) = \frac{a_1 r}{1 + r^2/4 \rho^2}, \quad R(r) = a_1 (2 \rho \arctan \frac{r}{2 \rho}) + C. \tag{61}
\]

\[
\begin{align*}
\text{II} & \quad b = b_1 r + b_2 (1 - \frac{r^2}{4 \rho^2}), \\
a(r) = 0, \quad c(r) = 0, \quad R(r) = C. \tag{62}
\end{align*}
\]

Respective expressions for \( K(r) \) and \( \Phi(r) \) look as

\[
I \quad K(r) = \frac{1}{er^2} \left[ \frac{a_1 r}{1 + r^2/4 \rho^2} f_1[a_1 (2 \rho \arctan \frac{r}{2 \rho}) + C] - 1 \right], \\
\Phi(r) = \frac{1}{er^2} \left[ a_1 r f_2[a_1 (2 \rho \arctan \frac{r}{2 \rho}) + C] - (1 - \frac{r^2}{4 \rho^2}) \right]; \tag{63}
\]

\[
\begin{align*}
\text{II} & \quad K(r) = -\frac{1}{er^2}, \quad \Phi(r) = \frac{1}{er^2} \left[ b_1 r + b_2 (1 - \frac{r^2}{4 \rho^2}) \right]; \tag{64}
\end{align*}
\]

It is readily verified that solution of the type \( I \) (depending on \( f_1, f_2 \) there are three different possibilities) has a good behavior in flat space limit. Indeed, in the limit \( \rho \to \infty \) we get

\[
K(r) = \frac{1}{er^2} \left[ a_1 r f_1(a_1 r + C) - 1 \right], \\
\Phi(r) = \frac{1}{er^2} \left[ a_1 r f_2(a_1 r + C) \right] - 1. \tag{65}
\]

From this, choosing for instance \( f_1 \) and \( f_2 \) according to (see (22))

\[
f_1(x) = \pm \frac{\alpha}{\sin(\alpha x + \beta)}, \quad f_2(x) = \frac{\alpha}{\tan(\alpha x + \beta)},
\]

we get

\[
K(r) = \frac{1}{er^2} \left( \frac{\pm \alpha a_1 r}{\sin(\alpha(a_1 r + C) + \beta)} - 1 \right), \\
\Phi(r) = \frac{1}{er^2} \left( \frac{\alpha a_1 r}{\tan(\alpha(a_1 r + C) + \beta)} - 1 \right); \tag{66}
\]

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from where, with the notation $\alpha a_1 = A$, $\alpha C + \beta = B$, we arrive at

$$K(r) = \frac{1}{er^2} \left( \frac{\pm Ar}{\sin(Ar + B)} - 1 \right), \quad \Phi(r) = \frac{1}{er^2} \left( \frac{Ar}{\tan(Ar + B)} - 1 \right);$$

which coincides with $K_3$ and $\Phi_3$, according to (23). In the same manner can be considered two other cases from (22).

Now, let us show that solution of the type II will give a trivial monopole solution in the flat space limit. Indeed, in this limit, eqs. (108) look as

$$K(r) = \frac{1}{er^2}, \quad \Phi(r) = \frac{1}{er^2} (b_1 r + b_2),$$

which coincides with (26). It is readily verified that for such a trivial solution, Yang-Mills equations become just two independent differential equations (linear and nonlinear). Indeed, let $er^2K(r) + 1 = 0$, then equations become

$$\Phi'' + \frac{4}{r} \Phi' + 2\Phi \frac{1}{r^2} - \frac{\Sigma'}{\Sigma} (\Phi' + \Phi) = 0,$$

$$K'' + \frac{4K'}{r} - 2\varepsilon K^2 + \frac{\Sigma'}{\Sigma} (K' + \frac{2K}{r}) = 0.$$

Evidently, the nonlinear equation is satisfied by $K(r) = -1/er^2$. In turn equation for $\Phi(r)$

$$\Phi'' + \frac{4}{r} \Phi' + 2\Phi \frac{1}{r^2} - \frac{r/2\rho^2}{1 + r^2/4\rho^2} (\Phi' + \Phi) = 0,$$

has two independent solutions

$$\Phi_1 = \frac{1}{r}, \quad \Phi_2 = \frac{1 - r^2/4\rho^2}{r^2}, \quad (66)$$

which are in accordance with (108).

The case of Lobachevsky space is treated in the similar manner. The results are

$$K(r) = \frac{1}{er^2} \left[ \frac{a_1 r}{1 - r^2/4\rho^2} f_1 \left[ a_1 (2\rho \arctanh \frac{r}{2\rho}) + C \right] - 1 \right],$$

$$\Phi(r) = \frac{1}{er^2} \left[ a_1 r f_2 \left[ a_1 (2\rho \arctanh \frac{r}{2\rho}) + C \right] - (1 + \frac{r^2}{4\rho^2}) \right]; \quad (67)$$
\[ K(r) = -\frac{1}{er^2}, \quad \Phi(r) = \frac{1}{er^2} \left[ b_1 r + b_2 (1 + \frac{r^2}{4\rho^2}) \right]. \quad (68) \]

Solution of the type \(I\) is analogue of the known monopole solution in flat space \(23\). Let us show that solution of the type \(II\) will give a trivial monopole solution in the flat space limit. Indeed, its limit at \(\rho \to \infty\) looks as follows

\[ K(r) = -\frac{1}{er^2}, \quad \Phi(r) = \frac{1}{er^2} \left[ b_1 r + b_2 \right], \quad (69) \]

which coincide with \(25\) - \(26\). It is readily verified that for such a trivial solution, Yang-Mills equations become just two independent differential equations (linear and nonlinear). Indeed, let \(er^2 K(r) + 1 = 0\), then equations become

\[ \Phi'' + \frac{4}{r} \Phi' + 2\Phi \frac{1}{r^2} - \frac{\Sigma'}{\Sigma} (\Phi' + \frac{\Phi}{r}) = 0, \]
\[ K'' + \frac{4K'}{r} - 2eK^2 + \frac{\Sigma'}{\Sigma} (K' + \frac{2K}{r}) = 0; \]

Nonlinear equation is satisfied by \(K(r) = -1/er^2\). Linear equation

\[ \Phi'' + \frac{4}{r} \Phi' + 2\Phi \frac{1}{r^2} + \frac{r/2\rho^2}{1 - r^2/4\rho^2} (\Phi' + \frac{\Phi}{r}) = 0, \]

has two independent solutions

\[ \Phi_1 = \frac{1}{r}, \quad \Phi_1 = \frac{1 + r^2/4\rho^2}{r^2}. \quad (70) \]

PART II

6 The Pauli criterion

Let the \(J_i^\lambda\) denote

\[ J_1 = l_1 + \lambda \frac{\cos \phi}{\sin \theta}, \quad J_2 = l_2 + \lambda \frac{\sin \phi}{\sin \theta}, \quad J_3 = l_3, \quad (71) \]

where \(l_i\) stand for the components of orbital momentum operator \(67\):

\[ l_1 = i (\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi), \]
\[ l_2 = i (-\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi), \quad l_3 = -i \partial_\phi. \]
At arbitrary $\lambda$, as readily verified, those $J_i$ satisfy the commutation rules of the Lie algebra $SU(2)$. As known, all irreducible representations of such an abstract algebra are determined by a set of weights

$$ j = 0, 1/2, 1, 3/2, \ldots ; \dim j = 2j + 1. $$

Given the explicit expressions of $J_a$ above, we will find functions $\Phi_{\lambda jm}(\theta, \phi)$ on which the representation of weight $j$ is realized. In agreement with the general approach [67], those solutions are to be established by the following relations

$$ J_+ \Phi_{\lambda jj} = 0, \quad \Phi_{\lambda jm} = \sqrt{(j + m)!/(j - m)! (2j)!} J^{j-m}_{\lambda} \Phi_{\lambda jj}, $$

$$ J_\pm = J_1 \pm iJ_2 = e^{\pm i\phi} (\pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} + \frac{\lambda}{\sin \theta}). \quad (72) $$

From the equations $J_+ \Phi_{\lambda jj} = 0$ and $J_3 \Phi_{\lambda jj} = j \Phi_{\lambda jj}$ it follows that

$$ \Phi_{\lambda jj} = N_{\lambda jj} e^{ij\phi} \sin^j \theta (1 + \cos \theta)^{j+\lambda/2} / (1 - \cos \theta)^{j-\lambda/2}, $$

$$ N_{\lambda jj} = \frac{1}{\sqrt{2\pi}} \frac{1}{2^j} \sqrt{\frac{(2j + 1)}{(j + m + 1)(j - m + 1)}}. $$

Further, employing (72) we produce the functions $\Phi_{\lambda jm}$

$$ \Phi_{\lambda jm} = N_{\lambda jm} e^{im\phi} \frac{1}{\sin^m \theta} (1 - \cos \theta)^{j-\lambda/2} / (1 + \cos \theta)^{j+\lambda/2} \times (\frac{d}{d \cos \theta})^{j-m} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right], \quad (73) $$

where

$$ N_{\lambda jm} = \frac{1}{\sqrt{2\pi}} \frac{1}{2^j} \sqrt{\frac{(2j + 1)(j + m)!}{2(j - m)! \Gamma(j + \lambda + 1) \Gamma(j - \lambda + 1)}}. $$

The Pauli criterion tells us that the $(2j + 1)$ functions $\Phi_{\lambda jm}(\theta, \phi)$, so constructed are guaranteed to be a basis for a finite-dimension representation, providing that the function $\Phi_{\lambda j,-j}(\theta, \phi)$ found by this procedure obeys the identity

$$ J_- \Phi_{\lambda j,-j} = 0. \quad (74) $$
After substituting the function $\Phi_{\lambda}^{\lambda}(\theta, \phi)$ to the (163), the latter reads

$$J_{-} \Phi_{\lambda}^{\lambda} = N_{\lambda}^{\lambda} e^{-i(j+1)\phi} (\sin \theta)^{j+1} \left( 1 - \cos \theta \right)^{\lambda/2} \times$$

$$\times \left( \frac{d}{d \cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0 , \quad (75)$$

which in turn gives the following restriction on $j$ and $\lambda$

$$\left( \frac{d}{d \cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0 . \quad (76)$$

But the relation (76) can be satisfied only if the factor $P(\theta)$ subjected to the operation of taking derivative $(d/d \cos \theta)^{2j+1}$ is a polynomial of degree $2j$ in $\cos \theta$. So, we have (as a result of the Pauli criterion)

1. the $\lambda$ is allowed to take values $+1/2, -1/2, +1, -1, \ldots$

Besides, as the latter condition is satisfied, $P(\theta)$ takes different forms depending on the $(j - \lambda)$-correlation:

$$P(\theta) = (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} = P^{2j}(\cos \theta), \quad if \quad j = |\lambda|, |\lambda| + 1, \ldots$$

or

$$P(\theta) = \frac{P^{2j+1}(\cos \theta)}{\sin \theta}, \quad if \quad j = |\lambda| + 1/2, |\lambda| + 3/2, \ldots$$

so the second necessary condition resulting from the Pauli criterion is

2. given $\lambda$ according to 1, the number $j$ is allowed to take values

$$j = |\lambda|, |\lambda| + 1, \ldots$$

Hereafter, these two conditions: 1 and 2 will be referred as the first and the second Pauli consequences respectively. It should be noted that the angular variable $\phi$ is not affected (charged) by this Pauli condition; in other words, it is effectively eliminated out of this criterion, but a variable that worked above is the $\theta$. Significantly, in the contrast to this, the well-known procedure of deriving the Dirac quantization condition from investigating continuity properties of quantum mechanical wave functions, such a working variable is the $\phi$. 

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If the first and second Pauli consequences fail, then we face rather unpleasant mathematical and physical problems.

As a simple illustration, we may indicate the familiar case when $\lambda = 0$; if in those circumstances, the second Pauli condition had failed, then we would have the integer and half-integer values of the orbital angular momentum number $l = 0, 1/2, 1, 3/2, \ldots$

As regards the Dirac electron with the components of the total angular momentum in the form

\[ J_1 = l_1 + \frac{\cos \phi}{\sin \theta} \Sigma_3, \quad J_2 = l_2 + \frac{\sin \phi}{\sin \theta} \Sigma_3, \quad J_3 = l_3 \]

we are to employ the Pauli criterion in the constituent form ($\lambda$ changes into $\Sigma_3$):

\[
\Sigma_3 = \begin{pmatrix}
  +1/2 & 0 & 0 & 0 \\
  0 & -1/2 & 0 & 0 \\
  0 & 0 & +1/2 & 0 \\
  0 & 0 & 0 & -1/2
\end{pmatrix}.
\]

In this case, we obtain the allowable set $j = 1/2, 3/2, \ldots$

Significantly that the functions $\Phi_{j,m}^\lambda(\theta, \phi)$ constructed above relate directly to the well-known Wigner $D$-functions (bellow we will use the notation according to [68]):

\[ \Phi_{j,m}^\lambda(\theta, \phi) = (-1)^{j-m} D_{j-m,\lambda}^j(\phi, \theta, 0). \] \hspace{1cm} (77)

Because of the detailed development of $D$-function theory, relation (2.5) will be of vital importance in the following.

Closing this paragraph, we draw attention to that the Pauli criterion

\[ J_\perp \Phi_{j,-j}(t, r, \theta, \phi) = 0 \]

(here $\Phi_{j,-j}(\theta, \phi)$ denotes a spherically symmetrical wave function) affords a condition that is invariant relative to possible gauge transformations. The function $\Phi_{j,m}(t, r, \theta, \phi)$ may be subjected to any $U(1)$ transformation, but if all the components $J_i$ vary in a corresponding way too, then the Pauli condition provides the same result on $J$-quantization. In contrast to this, the common requirement to be a single-valued function of spatial points is often applied to producing a criterion on selection of allowable wave functions in quantum mechanics; but that is not invariant under gauge transformations.

\[ \text{1} \text{Reader is referred to the Pauli article [20] for more detail about those peculiarities. However, all these peculiarities may be ignored and then there arise new possibilities – see Hunter et al [258]-[259] and references therein.} \]
7 Electron in a spherically symmetric geometric background and Wigner D-functions

Below we review briefly some relevant facts about the tetrad formalism. In the presence of an external gravitational field, the ordinary Dirac equation

\[ (i\gamma^\alpha \partial_\alpha - m)\Psi(x) = 0 \]

is generalized into \[202, 247]\n
\[ i\gamma^\alpha(x)(\partial_\alpha + \Gamma_\alpha(x)) - m \] \[\Psi(x) = 0 , \tag{78}\]

where \(e^\alpha_{(a)}(x)\) is a tetrad:

\[ e^\alpha_{(a)}(x) = \gamma^\alpha e^\alpha_{(a)}(x), \quad e^\alpha_{(a)}(x)e^\beta_{(b)}(x)\eta^{ab} = g^{\alpha\beta}(x) ; \]

\(\Gamma_\alpha(x)\) is the bispinor connection:

\[ \Gamma_\alpha(x) = \frac{1}{2}\sigma^{ab} e^\alpha_{(a)} \nabla_\alpha(e^\alpha_{(b)\beta}) ; \]

\(\nabla_\alpha\) is the covariant derivative symbol. In the spinor basis \[125\]

\[ \sigma^a = (I, +\sigma^k) , \quad \bar{\sigma}^a = (I, -\sigma^k) , \quad \gamma^a = \begin{vmatrix} \sigma^a & \bar{\sigma}^a \\ \bar{\sigma}^a & \sigma^a \end{vmatrix} \]

\[ \psi(x) = \begin{vmatrix} \xi(x) \\ \eta(x) \end{vmatrix} , \quad \xi(x) = \begin{vmatrix} \xi^1 \\ \xi^2 \end{vmatrix} , \quad \eta(x) = \begin{vmatrix} \eta_1 \\ \eta_2 \end{vmatrix} , \]

we have two equations

\[ i\sigma^a(x) [ \partial_\alpha + \Sigma_\alpha(x) ] \xi(x) = m \eta(x) , \]

\[ i\bar{\sigma}^a(x) [ \partial_\alpha + \bar{\Sigma}_\alpha(x) ] \eta(x) = m \xi(x) ; \tag{79}\]

the symbols \(\sigma^a(x), \bar{\sigma}^a(x), \Sigma_\alpha(x), \bar{\Sigma}_\alpha(x)\) denote respectively

\[ \sigma^a(x) = \sigma^a e^\alpha_{(a)}(x) , \quad \Sigma_\alpha(x) = \frac{1}{2}\Sigma^{ab} e^\beta_{(a)} \nabla_\alpha(e^\beta_{(b)\beta}) , \quad \Sigma^{ab} = \frac{1}{4}(\sigma^a\sigma^b - \sigma^b\sigma^a) , \]

\[ \bar{\sigma}^a(x) = \bar{\sigma}^a e^\alpha_{(a)}(x) , \quad \bar{\Sigma}_\alpha(x) = \frac{1}{2}\bar{\Sigma}^{ab} e^\beta_{(a)} \nabla_\alpha(e^\beta_{(b)\beta}) , \quad \bar{\Sigma}^{ab} = \frac{1}{4}(\sigma^a\sigma^b - \sigma^b\sigma^a) . \]

Setting \(m\) equal to zero, we obtain the Weyl equations for neutrino \(\eta(x)\) and anti-neutrino \(\xi(x)\), or Dirac’s equation for a massless particle.

The form of equations \[78\] – \[79\] implies quite definite their symmetry properties. It is common, considering the Dirac equation in the same
space-time, to use some different tetrads $e^\beta_a(x)$ and $e^\beta_{(b)}(x)$, so that we have the equation (3.1) and analogous one with a new tetrad mark. In other words, together with (3.1) there exists an equation on $\Psi^\prime(x)$ where the quantities $\gamma^\prime\alpha(x)$ and $\Gamma^\prime_{\alpha}(x)$, in comparison with $\gamma^\alpha(x)$ and $\Gamma_{\alpha}(x)$, are based on another tetrad $e^\beta_{(b)}(x)$ related to $e^\beta_a(x)$ through some local Lorentz matrix $e^\beta_{(b)}(x) = L_{b}^a(x) e^\beta_a(x)$. It may be shown that these two Dirac equations on functions $\Psi(x)$ and $\Psi^\prime(x)$ are related to each other by a quite definite bispinor transformation

$$\xi^\prime(x) = B(k(x)) \xi(x), \quad \eta^\prime(x) = B^+(\bar{k}(x)) \eta(x).$$

Here, $B(k(x)) = \sigma^a k_a(x)$ is a local matrix from the $SL(2.C)$ group; 4-vector $k_a$ is the well-known parameter on this group (for instance, see Wightman [287], Macfarlane [288], Fedorov [121], Red’kov [289]). The matrix $L_{b}^a(x)$ can be expressed as a function of arguments $k_a(x)$ and $k^\ast_a(x)$:

$$L_{b}^a(k,k^\ast) = \delta^c_b [ -\delta^a_c k^\ast_n k^a_n + k_c k^\ast_a + k^a_c k^\ast + i \epsilon^a_{c m n} k_n k_m^\ast ]$$

where $\delta^c_b$ is a special Kronecker’s symbol

$$\delta^c_b = \begin{cases} 0, & \text{if } c \neq b, \\ +1, & \text{if } c = b = 0, \\ -1, & \text{if } c = b = 1,2,3. \end{cases}$$

It is normal practice that some different tetrads are used at examining the Dirac equation on the background of a given Riemanniann space-time. If there is a need for analysis of the correlation between solutions in such distinct tetrads, then it is important to know how to calculate the corresponding gauge transformations over the spinor wave functions.

First, the need for taking into account such gauge transformations was especially emphasized by Fock V.I. [7]. The first who were interested in explicit expressions for such spinor matrices, were Schrödinger [14, 18, 19] and Pauli [20]. Thus, Schrödinger found the matrix relating spinor wave functions in Cartesian and spherical tetrads:

$$x^\alpha = (x^0, x^1, x^2, x^3), \quad e^\alpha_a(x) = \delta^\alpha_a,$$

$$dS^2 = [(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2],$$

and

$$x^\prime = (t, r, \theta, \phi), \quad dS^2 = [dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)],$$

$$e^\alpha_{(0)} = (1, 0, 0, 0), \quad e^\alpha_{(1)} = (0, 0, 1/r, 0),$$

$$e^\alpha_{(2)} = (0, 0, 0, \frac{1}{r \sin \theta}), \quad e^\alpha_{(3)} = (0, 1, 0, 0);$$

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the relevant spinor matrix is
\[ B = \pm \begin{vmatrix} \cos \theta/2 e^{i\phi/2} & \sin \theta/2 e^{-i\phi/2} \\ -\sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{-i\phi/2} \end{vmatrix} . \tag{82} \]

This basis of spherical tetrad will play a substantial role in our work.

Now, let us reexamine the problem of free electron in the external spherically symmetric gravitational field, but centering upon some facts which will be of great importance at extending that method to an electron-monopole system.

In particular, we consider briefly a question of separating the angular variables in the Dirac equation on the background of a spherically symmetric Riemannian space-time. As a starting point we take a flat space-time model, so that an original equation (78) being specified for the spherical tetrad takes on the form
\[ \left[ i \gamma^0 \partial_t + i (\gamma^3 \partial_r + \frac{\gamma^1 \sigma_{31} + \gamma^2 \sigma_{32}}{r}) + \frac{1}{r} \Sigma_{\theta\phi} - m \right] \Psi(x) = 0 , \tag{83} \]
where
\[ \Sigma_{\theta\phi} = i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + i \sigma_{12}}{\sin \theta} . \tag{84} \]

We specialize the electronic wave function through substitution
\[ \Psi_{\epsilon jm}(x) = e^{-i \epsilon t} \frac{f_1(r) D_{-1/2}}{r} \begin{vmatrix} f_1(r) & D_{-1/2} \\ f_2(r) & D_{+1/2} \\ f_3(r) & D_{-1/2} \\ f_4(r) & D_{+1/2} \end{vmatrix} ; \tag{85} \]
Wigner functions are designated by \( D^j_{-m,\sigma}(\phi, \theta, 0) \equiv D_\sigma \). Using recursive formulas [68]
\[
\begin{align*}
\partial_\theta D_{+1/2} &= a D_{-1/2} - b D_{-3/2}, \\
\partial_\theta D_{-1/2} &= b D_{-3/2} - a D_{+1/2}, \\
-\frac{m - 1/2 \cos \theta}{\sin \theta} D_{+1/2} &= -a D_{-1/2} - b D_{+3/2}, \\
-\frac{m + 1/2 \cos \theta}{\sin \theta} D_{-1/2} &= -b D_{-3/2} - a D_{+1/2},
\end{align*}
\]
where \( a = (j + 1)/2, \nu = (j + 1/2)/2 \), we find

\[
\Sigma_{\theta,\phi} \Psi_{\epsilon jm}(x) = i \nu \frac{e^{-i \epsilon t}}{r} \begin{vmatrix}
-f_4(r) & D_{-1/2} \\
+f_3(r) & D_{+1/2} \\
+f_2(r) & D_{-1/2} \\
-f_1(r) & D_{+1/2}
\end{vmatrix}.
\]  
(86)

Further one gets the following set of radial equations

\[
\begin{align*}
\epsilon f_3 - i \frac{d}{dr} f_3 - i \nu \frac{f_4 - m f_1}{r} &= 0, \\
\epsilon f_4 + i \frac{d}{dr} f_4 + i \nu \frac{f_3 - m f_2}{r} &= 0, \\
\epsilon f_1 + i \frac{d}{dr} f_1 + i \nu \frac{f_2 - m f_3}{r} &= 0, \\
\epsilon f_2 - i \frac{d}{dr} f_2 - i \nu \frac{f_1 - m f_4}{r} &= 0.
\end{align*}
\]  
(87)

The usual \( P \)-reflection symmetry operator in the Cartesian tetrad basis is \( \hat{\Pi}_C = i \gamma^0 \otimes \hat{P} \) (see in [67]), or in a more detailed form

\[
\hat{\Pi}_C = \begin{vmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{vmatrix} \otimes \hat{P}, \quad \hat{P}(\theta, \phi) = (\pi - \theta, \phi + \pi)
\]

being subjected to translation into the spherical tetrad basis (see (82))

\[
\hat{\Pi}_{sph.} = S(\theta, \phi) \hat{\Pi}_C S^{-1}(\theta, \phi)
\]
gives us the result

\[
\hat{\Pi}_{sph.} = \begin{vmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{vmatrix} \otimes \hat{P} = \Pi_{sph.} \otimes \hat{P}.
\]  
(88)

With the help of identity [68]

\[
\hat{P} D_{m,\sigma}^j(\phi, \theta, 0) = (-1)^j D_{-m,-\sigma}^j(\phi, \theta, 0),
\]

from the equation on proper values \( \hat{\Pi}_{sph.} \Psi_{jm} = \Pi \Psi_{jm} \) we get

\[
\Pi = \delta (-1)^{j+1}, \quad \delta = \pm 1, \quad f_4 = \delta f_1, \quad f_3 = \delta f_2
\]  
(89)

so that \( \Psi_{\epsilon jm\delta}(x) \) looks

\[
\Psi(x)_{\epsilon jm\delta} = \frac{e^{-i \epsilon t}}{r} \begin{vmatrix}
f_1(r) & D_{-1/2} \\
f_2(r) & D_{+1/2} \\
\delta f_2(r) & D_{-1/2} \\
\delta f_1(r) & D_{+1/2}
\end{vmatrix}.
\]  
(90)
Noting (89), we readily simplify the system (87); it is reduced to a (no imaginary $i$) form:

\[
\begin{align*}
\left(\frac{d}{dr} + \frac{\nu}{r}\right) f + (\epsilon + \delta m) g &= 0, \\
\left(\frac{d}{dr} - \frac{\nu}{r}\right) g - (\epsilon - \delta m) f &= 0,
\end{align*}
\]

(91)

where instead of $f_1$ and $f_2$ we have employed their linear combinations

\[
\begin{align*}
f &= \frac{f_1 + f_2}{\sqrt{2}}, \\
g &= \frac{f_1 - f_2}{i\sqrt{2}}; \\
f_1 &= \frac{f + ig}{\sqrt{2}}, \\
f_2 &= \frac{f - ig}{\sqrt{2}}.
\end{align*}
\]

(92)

It should be noticed that the above simplification ($\Psi_{\epsilon jm} \rightarrow \Psi_{\epsilon jm\delta}$) can also be obtained through the diagonalization of the operator $\hat{K}$ – in Cartesian tetrad basis it is given in [125]; usually it is called the Johnson – Lippmann operator [29]; though the following (spherical tetrad-based) form had been presented yet in Pauli’s paper [20]; also see [279, 280]:

\[
\hat{K} = -\gamma^0\gamma^3 \Sigma_{\theta,\phi}.
\]

Actually, from $\hat{K} \Psi_{\epsilon jm} = K \Psi_{\epsilon jm}$ we produce

\[
K = -\delta (j + 1/2), \quad \delta = \pm 1, \quad f_4 = \delta f_1, \quad f_3 = \delta f_2,
\]

which coincides with (89).

Everything established above for the flat space-time model can be readily generalized into an arbitrary curved space-time with a spherical metric $g_{\alpha\beta}(x)$

\[
dS^2 = e^\nu dt^2 - e^\mu dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

and its naturally corresponding diagonal tetrad $e^\alpha_{(a)}(x)$

\[
\begin{align*}
e^\beta_{(0)} &= (e^{-\nu/2}, 0, 0, 0), & e^\beta_{(3)} &= (0, e^{-\mu/2}, 0, 0), \\
e^\beta_{(1)} &= (0, 0, 1/r, 0), & e^\beta_{(2)} &= (0, 0, 0, 1/r \sin\theta).
\end{align*}
\]

The Dirac equation can be specified for an arbitrary diagonal tetrad as follows

\[
\left[ i \gamma^a (e^\beta_{(a)} \partial_\beta + \frac{1}{2} e^\beta_{(a);\beta} ) - m \right] \Psi(x) = 0,
\]

\[
e^\beta_{(a);\beta} = \frac{1}{\sqrt{-\det g}} \frac{\partial}{\partial x^\beta} \sqrt{-\det g} \ e^\beta_{(a)}.
\]
So, for the function $\Phi(x)$ defined by

$$\Psi(t, r, \theta, \phi) = \exp\left(-\frac{1}{4}(\nu + \mu)\right) \frac{1}{r} \Phi(t, r, \theta, \phi)$$

we produce the equation

$$\left[ \gamma^0 e^{-\nu/2} \partial_t + \gamma^3 e^{-\mu/2} \partial_r + \frac{1}{r} \Sigma_{\theta, \phi} - m \right] \Phi(t, r, \theta, \phi) = 0 . \quad (93)$$

On comparing (93) with (83), it follows immediately that all the calculations carried out above for the flat space-time case are still valid only with some evident modifications. Thus,

$$\Phi_{jm\delta}(x) = \begin{vmatrix} f_1(r, t) D_{-1/2}(\theta, \phi, 0) \\ f_2(r, t) D_{+1/2}(\theta, \phi, 0) \\ \delta f_2(r, t) D_{-1/2}(\theta, \phi, 0) \\ \delta f_1(r, t) D_{+1/2}(\theta, \phi, 0) \end{vmatrix} \quad (94)$$

and instead of (91) now we find

$$(e^{-\mu/2} \frac{\partial}{\partial r} + \frac{\nu}{r}) f + (ie^{-\nu/2} \frac{\partial}{\partial t} + \delta m) g = 0 ,$$

$$(e^{-\mu/2} \frac{d}{dr} - \frac{\nu}{r}) g - (ie^{-\nu/2} \frac{\partial}{\partial t} - \delta m) f = 0 . \quad (95)$$

## 8 About electron functions in the monopole field

In the literature, a particle-monopole system has attracted a lot of attention being in a sense a 'classical' problem:

Dirac [10], Tamm [11], Groönbloom [15], Jordan [16], Fierz [21], Banderet [22], Harish-Chandra [23], Wilson [25], Eldridge [26], Saha [27], Johnson – Lippmann [30], Case [28], Ramsey [32], Eliezer and Roy [39], Goldhaber [43], Schwinger [46, 46], Dulock and McIntosh [47, 51, 52, 61], Peres [50, 57], Zwanziger [54, 55], Harst [56], Lipkin – Weisberger – Peshkin [58], collection of paper edited by Bolotovskiy and Usachev [59], Zwanziger [62], Barut [63, 64], Magne [70], Schwinger [73], Strachew and Tomilchik [75], Boulnwe et al [87], Schwinger et al [88], Goldhaber [90], Wu and Yang [93, 98], Tomilchik et al [97], Callias [99], Kazama and Yang [100, 101], Frenkel and Hrasko [102], Petry [103], Margolin and Tomilchik [104], Kazama et al [105], Goldhaber [106].
In particular, the various properties of occurring so-called monopole harmonics were investigated in detail. Here, we are going to look into this problem in the context of generalized Pauli-Schrödinger formalism reviewed in Sections 2-3: this technique provides us with an ideal tool to solve many of monopole-triggered problems.

For our further purpose it will be convenient to use the Abelian monopole potential in Schwinger’s form [73]:

\[
A^\alpha(x) = (A^0, A^i) = \left[0, g \frac{(\vec{r} \times \vec{n}) (\vec{r} \cdot \vec{n})}{r (r^2 - (\vec{r} \cdot \vec{n})^2)} \right]
\] (96)

after translating the 4-vector potential \(A^\alpha\) to the spherical coordinates and specifying \(\vec{n} = (0, 0, +1)\), we get

\[
A_0 = 0, \quad A_r = 0, \quad A_\theta = 0, \quad A_\phi = g \cos \theta.
\] (97)

Correspondingly, the Dirac equation in this electromagnetic potential takes the form

\[
\left[ i \gamma^0 \partial_t + i \gamma^3 (\partial_r + \frac{1}{r}) + \frac{1}{r} \Sigma_{\theta, \phi}^k \right] \Psi(x) = 0,
\] (98)

where

\[
\Sigma_{\theta, \phi}^k = i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + (i \sigma_{12} - k) \cos \theta}{\sin \theta},
\] (99)
and \( k \equiv eg/hc \). As readily verified, the wave operator in (98) commutes with the following three ones

\[
J_1^k = l_1 + (i\sigma^{12} - k) \frac{\cos \phi}{\sin \theta}, \\
J_2^k = l_2 + (i\sigma^{12} - k) \frac{\sin \phi}{\sin \theta}, \\
J_3^k = l_3
\]

which in turn obey the \( SU(2) \) Lie algebra. Clearly, this monopole situation come entirely under the Schrödinger-Pauli approach, so that our further work will be a matter of quite elementary calculations.

Corresponding to diagonalization of the \( J_2^k \) and \( J_3^k \), the function \( \Psi \) is to be taken as

\[
\Psi^k_{\epsilon jm}(t, r, \theta, \phi) = \frac{e^{-i\epsilon t} r}{r} \begin{vmatrix}
 f_1 D_{k-1/2} \\
 f_2 D_{k+1/2} \\
 f_3 D_{k-1/2} \\
 f_4 D_{k+1/2}
\end{vmatrix};
\]

(101)

\( D_\sigma \equiv D^j_{-m, \sigma}(\phi, \theta, 0) \). Further, noting recursive relations [68]

\[
\partial_\theta D_{k+1/2} = (+a D_{k-1/2} - b D_{k+3/2}), \\
\partial_\theta D_{k-1/2} = (+c D_{k-3/2} - a D_{k+1/2}), \\
\sin^{-1} \theta \left[ -m - (k + 1/2) \cos \theta \right] D_{k+1/2} = (-a D_{k-1/2} - b D_{k+3/2}), \\
\sin^{-1} \theta \left[ -m - (k - 1/2) \cos \theta \right] D_{k-1/2} = (-c D_{k-3/2} - a D_{k+1/2}), \\
a = \frac{1}{2} \sqrt{(j + 1/2)^2 - k^2}, \\
b = \frac{1}{2} \sqrt{(j - k - 1/2)(j + k + 3/2)}, \\
c = \frac{1}{2} \sqrt{(j + k - 1/2)(j - k + 3/2)}
\]

we find how the \( \Sigma^k_{\theta, \phi} \) acts on \( \Psi \):

\[
\Sigma^k_{\theta, \phi} \Psi^k_{\epsilon jm} = i \sqrt{(j + 1/2)^2 - k^2} \frac{e^{-i\epsilon t} r}{r} \begin{vmatrix}
 -f_4 D_{k-1/2} \\
 +f_3 D_{k+1/2} \\
 +f_2 D_{k-1/2} \\
 -f_1 D_{k+1/2}
\end{vmatrix},
\]

(102)
hereafter the factor $\sqrt{(j + 1/2)^2 - k^2}$ will be denoted by $\nu$. For the $f_i(r)$ we establish four equations

$$
\begin{align*}
\epsilon f_3 - i \frac{d}{dr} f_3 - i \frac{\nu}{r} f_4 - m f_1 &= 0 , \\
\epsilon f_4 + i \frac{d}{dr} f_4 + i \frac{\nu}{r} f_3 - m f_2 &= 0 , \\
\epsilon f_1 + i \frac{d}{dr} f_1 + i \frac{\nu}{r} f_2 - m f_3 &= 0 , \\
\epsilon f_2 - i \frac{d}{dr} f_2 - i \frac{\nu}{r} f_1 - m f_4 &= 0 .
\end{align*}
$$

(103)

As evidenced by analogy with Sec. 3 and also on direct calculation, yet other operator can be diagonalized together with $\{i\partial_t, \vec{J}_k^2, J_k^3\}$: namely, a generalized Dirac operator

$$
\hat{K}^k = - i \gamma^0 \gamma^3 \Sigma^k_{\phi,\theta} .
$$

(104)

From the equation $\hat{K}^k \Psi_{ejm} = K \Psi_{ejm}$ we can produce two possible values for this $K$ and the corresponding limitations on $f_i(r)$:

$$
K = -\delta \sqrt{(j + 1/2)^2 - k^2} : \quad f_4 = \delta f_1 , \quad f_3 = \delta f_2
$$

(105)

the system (103) is reduced to

$$
\begin{align*}
\left( \frac{d}{dr} + \frac{\nu}{r} \right) f + (\epsilon + \delta m) g &= 0 , \\
\left( \frac{d}{dr} - \frac{\nu}{r} \right) g - (\epsilon - \delta m) f &= 0 .
\end{align*}
$$

(106)

On direct comparing (106) with analogous system in Sec. 3, we can conclude that these two systems are formally similar apart from the difference between $\nu = j + 1/2$ and $\nu = \sqrt{(j + 1/2)^2 - k^2}$.

Now let us pass over to quantization of $k = eg/hc$ and $j$. As a direct result from the first Pauli condition we derive

$$
\frac{eg}{hc} = \pm 1/2, \pm 1, \pm 3/2, \ldots
$$

(107)

which coincides with the Dirac’s quantization, and from the second Pauli condition it follows immediately that

$$
\begin{align*}
\begin{array}{c}
j = |k| - 1/2, |k| + 1/2, |k| + 3/2, \ldots
\end{array}
\end{align*}
$$

(108)
The case of minimal allowable value \( j_{\text{min}} = |k| - 1/2 \) must be separated out and looked in a special way. For example, let \( k = +1/2 \), then to the minimal value \( j = 0 \) there corresponds the wave function in terms of solely \((t, r)\)-dependent quantities

\[
\Psi_{k=+1/2}^{(j=0)}(x) = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix}
 f_1(r) \\
 0 \\
 f_2(r) \\
 0
\end{pmatrix}.
\]  

(109)

At \( k = -1/2 \), in an analogous way, we have

\[
\Psi_{k=-1/2}^{(j=0)}(x) = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix}
 0 \\
 f_2(r) \\
 0 \\
 f_4(r)
\end{pmatrix}.
\]  

(110)

Thus, if \( k = \pm 1/2 \), then to the minimal allowed values \( j_{\text{min}} \) there correspond the function substitutions which do not depend at all on the angular variables \((\theta, \phi)\); at this point there exists some formal analogy between these electron-monopole states and \( S \)-states (with \( l = 0 \)) for a boson field of spin zero: \( \Phi_{l=0} = \Phi(r,t) \). However, it would be unwise to attach too much significance to this formal coincidence because that \((\theta, \phi)\)-independence of \((e-g)\)-states is not the fact invariant under tetrad gauge transformations. In contrast, the relation below (let \( k = +1/2 \))

\[
\Sigma_{\theta, \phi}^{+1/2} \Psi_{k=+1/2}^{(j=0)}(x) = \gamma^2 \cot \theta \left(i\sigma^{12} - 1/2\right) \Psi_{k=+1/2}^{(j=0)} \equiv 0
\]  

(111)

is invariant under any gauge transformations. The identity (4.10a) holds because all the zeros in the \( \Psi_{k=+1/2}^{(j=0)} \) are adjusted to the non-zeros in \((i\sigma^{12} - 1/2)\); the non-vanishing constituents in \( \Psi_{k=+1/2}^{(j=0)} \) are canceled out by zeros in \((i\sigma^{12} - 1/2)\). Correspondingly, the matter equation (98) takes on the form

\[
\left[i \gamma^0 \partial_t + i \gamma^3 (\partial_r + \frac{1}{r}) - m\right] \Psi_{(j=0)} = 0.
\]

(112)

It is readily verified that both (4.9a) and (110) representations are directly extended to \((e-g)\)-states with \( j = j_{\text{min}} \) at all the other \( k = \pm 1, \pm 3/2, \ldots \)
Indeed,

\[
k = +1, +3/2, +2, \ldots : \quad \Psi_{j_{\min}}^{k>0}(x) = \frac{e^{-ikt}}{r} \begin{vmatrix}
f_1(r) & D_{k-1/2} \\
f_3(r) & D_{k-1/2}
\end{vmatrix};
\]

\[
k = -1, -3/2, -2, \ldots : \quad \Psi_{j_{\min}}^{k<0}(x) = \frac{e^{-ikt}}{r} \begin{vmatrix}
f_2(r) & D_{k+1/2} \\
f_4(r) & D_{k+1/2}
\end{vmatrix}
\]

(113)

and, as can be shown, the relation \( \Sigma_{\theta,\phi} \Psi_{j_{\min}}^{k>0} = 0 \) still holds. For instance, let us consider in more detail the case of positive \( k \). Using the recursive relations

\[
\partial_\theta D_{k-1/2} = \frac{1}{2}\sqrt{2k-1}D_{k-3/2},
\]

\[
\sin^{-1}\theta[-m - (k-1/2)\cos\theta]D_{k-1/2} = -\frac{1}{2}\sqrt{2k-1}D_{k-3/2},
\]

we get

\[
i\gamma_1 \partial_\theta \begin{vmatrix}
f_1(r) & D_{k-1/2} \\
f_3(r) & D_{k-1/2}
\end{vmatrix} = i\frac{\sqrt{2k-1}}{2} \begin{vmatrix}
f_1(r) & D_{k-3/2} \\
f_3(r) & D_{k-3/2}
\end{vmatrix} + f_1(r) D_{k-3/2};
\]

\[
\gamma^2 \frac{i\partial_\phi + (i\sigma^{12} - k)\cos\theta}{\sin\theta} \begin{vmatrix}
f_1(r) & D_{k-1/2} \\
f_3(r) & D_{k-1/2}
\end{vmatrix} = i\frac{\sqrt{2k-1}}{2} \begin{vmatrix}
f_1(r) & D_{k-3/2} \\
f_3(r) & D_{k-3/2}
\end{vmatrix} + f_1(r) D_{k-3/2};
\]

in a sequence, the identity \( \Sigma_{\theta,\phi} \Psi_{j_{\min}}^{k>0} = 0 \) is proved. The case of negative \( k \) can be considered in the same way. Thus, at every \( k \), the \( j_{\min} \)-state's equation has the same unique form

\[
\left[ i\gamma^0 \partial_t + i\gamma^3(\partial_r + \frac{1}{r}) - mc/\hbar \right] \Psi_{j_{\min}} = 0
\]

(114)

which leads to the same unique radial system:

\[
k = +1/2, +1, \ldots
\]

\[
\epsilon f_3 - i \frac{d}{dr} f_3 - mf_1 = 0, \quad \epsilon f_1 + i \frac{d}{dr} f_1 - mf_3 = 0;
\]

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\[ k = -1/2, -1, \ldots \]

\[ \epsilon f_4 + i \frac{d}{dr} f_4 - m f_2 = 0, \quad \epsilon f_2 - i \frac{d}{dr} f_2 - m f_4 = 0. \quad (115) \]

These equations are equivalent respectively to

\[
\begin{align*}
&k = +1/2, +1, \ldots \\
&\left( \frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_1 = 0, \quad f_3 = \frac{1}{m} (\epsilon + i \frac{d}{dr}) f_1 ; \\
&k = -1/2, -1, \ldots \\
&\left( \frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_4 = 0, \quad f_2 = \frac{1}{m} (\epsilon + i \frac{d}{dr}) f_4 \\
\end{align*}
\]

which both end up with the function

\[ f(t, r) = e^{\pm \sqrt{m^2 - \epsilon^2} r} , \]

one of these at \( \epsilon < m \) looks as

\[ f(t, r) = e^{-\sqrt{m^2 - \epsilon^2} r} . \quad (117) \]

The function given by (117) which seems to be appropriate to describe a bound state in the electron-monopole system. It should be emphasized that today the \( j_{\text{min}} \) bound state problem remains still yet a question to understand. In particular, the important question is of finding a physical and mathematical criterion on selecting values for \( \epsilon \): whether \( \epsilon < m \), or \( \epsilon = m \), or \( \epsilon > m \); and what value of \( \epsilon \) is to be chosen after specifying an interval above.

Now let us proceed with studying the properties which stem from the \( \theta, \phi \)-dependence of the wave functions. In particular, we restrict ourselves to the \( P \)-parity problem in the presence of the monopole. This problem was investigated in the literature both in Abelian and non-Abelian cases: Frampton et al [208], Tolkachev et al [200, 201, 215, 225, 229] Ryzov and Savinkov et al [221, 222, 204, 205, 206, 207, 210, 226], Red’kov [202, 203, 247, 248, 249, 250, 251, 253, 256], so our first step is to particularize some relevant facts in accordance with the formalism and notation used in the present treatment.

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As evidenced by straightforward computation, the well-known purely geometrical bispinor $P$-reflection operator does not commute with the Hamiltonian $\hat{H}$ under consideration. The same conclusion is also arrived at by attempt to solve directly the proper value equation

$$\hat{\Pi}_{sph.} \Psi_{e jm} = \Pi \Psi_{e jm}$$

which leads to

$$(-1)^{j+1} \begin{vmatrix} f_4 D_{-k-1/2} & f_1 D_{-k-1/2} \\ f_3 D_{-k+1/2} & f_2 D_{-k+1/2} \\ f_2 D_{-k-1/2} & f_3 D_{-k+1/2} \\ f_1 D_{-k+1/2} & f_4 D_{-k+1/2} \end{vmatrix} = P \begin{vmatrix} f_4 D_{-k-1/2} & f_1 D_{-k-1/2} \\ f_3 D_{-k+1/2} & f_2 D_{-k+1/2} \\ f_2 D_{-k-1/2} & f_3 D_{-k+1/2} \\ f_1 D_{-k+1/2} & f_4 D_{-k+1/2} \end{vmatrix}$$

the latter matrix relation is satisfied only by the trivial substitution $f_i = 0$ for all $i$. The relation above indicates how a required discrete transformation can be constructed (further we will denote it as $\hat{N}_{sph.}$)

$$\hat{N}_{sph.} = \hat{\pi} \otimes \hat{\Pi}_{sph.}, \quad \hat{\Pi}_{sph.} = \Pi_{sph.} \otimes \hat{P}$$

(118)

where $\hat{\pi}$ is a special discrete operator changing $k$ into $-k$:

$$\hat{\pi} F(k) = F(-k).$$

From the equation

$$\hat{N}_{sph.} \Psi_{e jm} = N \Psi_{e jm}$$

it follows

$$N = \mu (-1)^{j+1}, \quad \mu = \pm 1, \quad f_4 = \mu f_1, \quad f_3 = \mu f_2.$$ (119)

These relations are compatible with the above radial system – eqs. (103) transform into

$$\left( \frac{d}{dr} + \frac{\nu}{r} \right) f + (\epsilon + \mu m) g = 0,$$

$$\left( \frac{d}{dr} - \frac{\nu}{r} \right) g - (\epsilon - \mu m) f = 0.$$ (120)

$f(r)$ and $g(r)$ are already used combinations from $f_1(r)$ and $f_2(r)$ – see (92).

Here some additional remarks must be done. Everything just said about diagonalizing the $\hat{N}_{sph.}$ is applied only to the cases when $j > j_{\text{min}}$. As regards the lower value of $j$, the situation turns out to be very specific and
unexpected. Actually, let \( j = 0 \) then from equation \( \hat{N}_{sph.} \Psi^{(j=0)} = N \Psi^{(j=0)} \), considering the cases \( k = +1/2 \) and \(-1/2\), we get respectively

\[
\begin{vmatrix}
0 & f_1 \\
-f_3 & 0 \\
0 & -f_2 \\
-f_4 & 0
\end{vmatrix} = N
\begin{vmatrix}
f_2 \\
0 \\
0 \\
f_4
\end{vmatrix}.
\]

Evidently they both have no solutions, excluding trivially null ones (and therefore being of no interest). Moreover, as may be easily seen, in both cases a function \( \Phi(x) \), defined by \( \hat{N}_{sph.} \Psi^{(j=0)} = \Phi(x) \), lies outside a fixed totality of states that are only valid as possible quantum states of the system under consideration. At greater values of this \( k \), we come to analogous relations: the equation \( \hat{N}_{sph.} \Psi_{j_{\text{min.}}} = N \Psi_{j_{\text{min.}}} \) leads to (at positive \( k \) and negative \( k \) respectively)

\[
(-1)^{j+1} \begin{vmatrix}
0 & f_3D_{k+1/2} \\
f_1D_{k+1/2} & 0 \\
f_4D_{k-1/2} & 0 \\
f_2D_{k-1/2} & 0
\end{vmatrix} = N
\begin{vmatrix}
f_2D_{k-1/2} \\
0 \\
0 \\
f_4D_{k+1/2}
\end{vmatrix};
\]

and the above arguments may be repeated again.

In turn, as regards the operator \( \hat{K}^k \), for the \( j_{\text{min.}} \) states we get \( \hat{K}^k \Psi_{j_{\text{min.}}} = 0 \); that is, this state represents the proper function of the \( \hat{K} \) with the null proper value. So, application of this \( \hat{K} \) instead of the \( \hat{N} \) has an advantage of avoiding the paradoxical and puzzling situation when \( \hat{N}_{sph.} \Psi^{(j_{\text{min.}})} \notin \{ \Psi \} \). In a sense, this second alternative (the use of \( \hat{K}^k \) instead of \( \hat{N} \) at separating the variables and constructing the complete set of mutually commuting operators) gives us a possibility not to attach great significance to the monopole discrete operator \( \hat{N} \) but to focus our attention solely on the continual operator \( \hat{K}^k \).

9 Discrete symmetry in external monopole field and selection rules

It is known that the quantum mechanics, when dealing with some specific operator \( \hat{A} \), implies its self-conjugacy property: \(< \Psi | \hat{A} \Phi > = < \hat{A} \Psi | \Phi >\).
Φ > . For example, the usual bispinor P-reflection presents evidently a self-conjugate one, since one has
\[
< \Psi(\vec{r}) | \gamma^0 \hat{P} \Phi(\vec{r}) > = \int \tilde{\Psi}^*(\vec{r}) \Phi(-\vec{r}) \, dV,
\]
\[
< \gamma^0 \hat{P} \Psi(\vec{r}) | \Phi(\vec{r}) > = \int \tilde{\Phi}^*(-\vec{r}) \Phi(\vec{r}) \, dV.
\] (121)

The Ψ with over symbol \(\sim\) denotes a transposed column-function, that is, a row-function; and the asterisk \(*\) designates the operation of complex conjugation.

In the presence of the external monopole field, the whole situation is completely different from the above. Indeed
\[
< \psi^{+eg}(\vec{r}) | \hat{N} \Phi^{+eg}(\vec{r}) > = \int (\tilde{\Psi}^{+eg}(\vec{r}))^* \Phi^{-eg}(-\vec{r}) \, dV ,
\]
\[
< \hat{N} \Psi^{+eg}(\vec{r}) | \Phi^{+eg}(\vec{r}) > = \int (\tilde{\Phi}^{-eg}(\vec{r}))^* \Phi^{+eg}(-\vec{r}) \, dV
\] (122)
it is evident that right-handed sides of these two equalities vary in sign at \(eg\) parameter; thereby it follows that the discrete operator \(\hat{N}\) does not possess a self-adjoint one.

In this connection, one must take notice of the manner in which the \(eg\) parameter enters the radial system for \(f_1, \ldots, f_4\): it occurs through \(\nu = \sqrt{(j + 1/2)^2 - \kappa^2}\). The latter leads to independence on \(\kappa\)'s sign. Therefore, the two distinct systems with the characteristics \(+eg\) and \(-eg\) respectively have their radial systems exactly identical:
\[
F^{+eg}_{s=1/2}(f_1, \ldots, f_4) = F^{-eg}_{s=1/2}(f_1, \ldots, f_4).
\] (123)

As an illustration to manifestations of the non-self-adjointness property of the \(N\)-operator, let us consider a question concerning \(P\)-parity selection rules in presence of the monopole. Here Though there exists a seemingly appropriate operator
\[
\hat{N} = \hat{\pi} \otimes \Pi_{sph.} \otimes \hat{P}, \quad \hat{\pi} \Psi^{+eg}_{\epsilon j m \mu}(\vec{r}) = \Psi^{-eg}_{\epsilon j m \mu}(\vec{r}) ,
\]
\[
\hat{N} \Psi^{eg}_{\epsilon j m \mu}(x) = \mu (-1)^{j+1} \Psi^{eg}_{\epsilon j m \mu}(x)
\] (124)
but this does not allow us to obtain any \(N\)-parity selection rules. Let us consider this question in more detail. A matrix element for some physical observable \(\hat{G}^0(x)\) is to be
\[
\int \tilde{\Psi}^{eg}_{\epsilon j m \mu}(\vec{r}) \hat{G}^0(\vec{r}) \Psi^{eg}_{\epsilon j' m' \mu'}(\vec{r}) \, dV \equiv \int r^2 \, dr \int \tilde{f}(\vec{r}) \, d\Omega.
\] (125)
First we examine the case $eg = 0$, in order to compare it with the situation at $eg \neq 0$. Let us relate $f(-\vec{r})$ with $f(\vec{r})$. Considering the equality (and the same with $j^m',\delta'$)

$$\Psi^0_{\epsilon jm\delta}(-\vec{r}) = \Pi_{sph.} \delta (-1)^{j+1} \Psi^0_{\epsilon jm\delta}(\vec{r})$$

we get

$$f^0(-\vec{r}) = \delta \delta' (-1)^{j+j'+1}\Psi^0_{\epsilon jm\delta}(\vec{r}) \left[ \Pi_{sph.} \hat{G}^0(-\vec{r}) \Pi_{sph.} \right] \Psi^0_{\epsilon jm',\delta'}(\vec{r}).$$

If $\hat{G}^0(\vec{r})$ obeys the equation

$$\Pi_{sph.} \hat{G}^0(-\vec{r}) \Pi_{sph.} = \omega^0 \hat{G}^0(\vec{r})$$

here $\omega^0$ defined to be $+1$ or $-1$ relates to the scalar and pseudo scalar, respectively, then $f(\vec{r})$ can be brought to

$$f^0(-\vec{r}) = \omega \delta \delta' (-1)^{j+j'+1} f^0(\vec{r}).$$

The latter generates the well known $P$-parity selection rules:

$$\int \Psi^0_{\epsilon jm\mu}(r) \hat{G}^0(r) \Psi^0_{\epsilon jm',\mu'}(r) dV =$$

$$= \left[ 1 + \omega \delta \delta' (-1)^{j+j'+1} \right] \int r^2 dr \int_{1/2} f^0(\vec{r}) d\Omega$$

where the $\theta, \phi$-integration is performed on a half-sphere.

The situation at $eg \neq 0$ is completely different since here any equality in the form $(5.4a)$ does not exist at all. In other words, because of the absence any correlation between $f^e g(\vec{r})$ and $f^e g(-\vec{r})$, there is no selection rules on discrete quantum number $N$. In accordance with this, for instance, an expectation value for the usual operator of space coordinate $\vec{x}$ need not equal zero and one follows this (see in Tolkachev et al [200, 209, 215, 225, 229], Ryzhov and Savinkov et al [221, 222, 204, 205, 206, 207, 210, 226]).

In the same time, from the above it follows that there exist quite definite correlations between $\Psi^{\pm eg}(-\vec{r})$ and $\Psi^{\mp eg}(\vec{r})$:

$$\Psi^{\pm eg}(-\vec{r}) = \Pi_{sph.} \Psi^{\mp eg}(\vec{r}),$$

$$f^{\pm eg}(-\vec{r}) = \omega \delta \delta' (-1)^{j+j'+1} f^{\mp eg}(\vec{r}).$$

Those latter provide certain indications that in a non-Abelian (monopole-contained) model no problems with discrete $P$-inversion-like symmetry might occur.
The above study has shown that the general outlook on this matter which prescribes to consider a magnetic charge as pseudo-scalar under $P$-reflection (just that interpretation is implied by the use of additional $\pi$-transformation changing $g$ into $-g$ and accompanying the ordinary $P$-reflection) is not effective one as we touch relevant selection rules.

10 Some technical facts on the Abelian monopole system

Now let us consider relationship between $D$-functions used above and the spinor monopole harmonics. To this end one ought to perform two translations: from the spherical tetrad and Weyl’s spinor frame in bispinor space into the Cartesian tetrad and the so-called Pauli’s (bispinor) frame. In the first place, it is convenient to accomplish those translations for a free electronic function; so as, in the second place, to follow this pattern further in the monopole case.

So, subjecting the free electronic function (spherical solution from Sec. 3) to the local gauge transformation associated with the tetrad change $\epsilon_{sph.} \rightarrow \epsilon_{Cart.}$:

$$
\Psi_{Cart.} = \begin{vmatrix}
U^{-1} & 0 \\
0 & U^{-1}
\end{vmatrix}
\Psi_{sph.},
U^{-1} = \begin{vmatrix}
\cos \theta/2e^{-i\phi/2} & -\sin \theta/2e^{-i\phi/2} \\
\sin \theta/2e^{i\phi/2} & \cos \theta/2e^{i\phi/2}
\end{vmatrix}
$$

and further, taking the bispinor frame from the Weyl’s one to the Pauli’s:

$$
\Psi_{Cart.}^P = \begin{vmatrix}
\phi \\
\xi
\end{vmatrix},
\Psi_{Cart.} = \begin{vmatrix}
\xi \\
\eta
\end{vmatrix},
\varphi = \frac{\xi + \eta}{\sqrt{2}},
\chi = \frac{\xi - \eta}{\sqrt{2}}
$$

we get to

$$
\varphi = \frac{f_1 + f_3}{\sqrt{2}} \begin{vmatrix}
\cos \theta/2e^{-i\phi/2} \\
\sin \theta/2e^{i\phi/2}
\end{vmatrix},
D_{-1/2} + \frac{f_2 + f_4}{\sqrt{2}} \begin{vmatrix}
\cos \theta/2e^{i\phi/2} \\
\sin \theta/2e^{-i\phi/2}
\end{vmatrix},
\varphi = \frac{f_1 - f_3}{\sqrt{2}} \begin{vmatrix}
\cos \theta/2e^{-i\phi/2} \\
\sin \theta/2e^{i\phi/2}
\end{vmatrix},
D_{-1/2} + \frac{f_2 - f_4}{\sqrt{2}} \begin{vmatrix}
\cos \theta/2e^{i\phi/2} \\
\sin \theta/2e^{-i\phi/2}
\end{vmatrix}
$$

Introducing special notation, $\chi_{+1/2}$ and $\chi_{-1/2}$, for columns of matrix $U^{-1}(\theta, \phi)$, sometime they are termed as helicity spinors:

$$
\chi_{+1/2} = \begin{vmatrix}
\cos \theta/2e^{-i\phi/2} \\
\sin \theta/2e^{i\phi/2}
\end{vmatrix},
\chi_{-1/2} = \begin{vmatrix}
-sin \theta/2e^{-i\phi/2} \\
\cos \theta/2e^{i\phi/2}
\end{vmatrix}
$$
previous formulas can be rewritten in the form

\[
\varphi = \frac{f_1 + f_3}{\sqrt{2}} \chi_{+1/2} D_{-1/2} + \frac{f_2 + f_4}{\sqrt{2}} \chi_{-1/2} D_{+1/2}, \\
\chi = \frac{f_1 - f_3}{\sqrt{2}} \chi_{+1/2} D_{-1/2} + \frac{f_2 - f_4}{\sqrt{2}} \chi_{-1/2} D_{+1/2}.
\]  
(130)

Further, for the above solutions with fixed proper values of the operator \( \hat{\Pi}_{sph} \) – see (89):

\[
\Pi = (-1)^{j+1}, \\
\Psi_{\text{Cart.}} = \frac{e^{-i\epsilon t}}{r\sqrt{2}} \begin{vmatrix} (f_1 + f_2) (\chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2}) \\ (f_1 - f_2) (\chi_{+1/2} D_{-1/2} - \chi_{-1/2} D_{+1/2}) \end{vmatrix}, 
\]  
(131)

\[
\Pi = (-1)^{j}, \\
\Psi_{\text{Cart.}} = \frac{e^{-i\epsilon t}}{r\sqrt{2}} \begin{vmatrix} (f_1 - f_2) (\chi_{+1/2} D_{-1/2} - \chi_{-1/2} D_{+1/2}) \\ (f_1 + f_2) (\chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2}) \end{vmatrix}.
\]  
(132)

Now, using the known extensions for spherical spinors \( \Omega_{jm}^{j+1/2}(\theta, \phi) \) in terms of \( \chi_{\pm 1/2} \) and \( D \)-functions we arrive at the common representation of the spinor spherical solutions

\[
\Pi = (-1)^{j+1}, \quad \Psi_{\text{Cart.}} = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} +f(r) \Omega_{jm}^{j+1/2}(\theta, \phi) \\ -i g(r) \Omega_{jm}^{j-1/2}(\theta, \phi) \end{vmatrix}; \\
\Pi = (-1)^{j}, \quad \Psi_{\text{Cart.}} = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} -i g(r) \Omega_{jm}^{j-1/2}(\theta, \phi) \\ f(r) \Omega_{jm}^{j+1/2}(\theta, \phi) \end{vmatrix}.
\]  
(134)

The monopole situation can be considered in the same way. As a result, we produce the following representation of the monopole-electron functions
in terms of ‘new’ angular harmonics

\[ \begin{align*}
N = (-1)^{j+1} : & \quad \Psi_{\text{Cart.}} = \frac{e^{-i\xi t}}{r} \left( \begin{array}{c} +f(r) \xi_{jmk}^{(1)}(\theta, \phi) \\ -ig(r) \xi_{jmk}^{(2)}(\theta, \phi) \end{array} \right) ; \\
N = (-1)^j : & \quad \Psi_{\text{Cart.}} = \frac{e^{-i\xi t}}{r} \left( \begin{array}{c} -ig(r) \xi_{jmk}^{(1)}(\theta, \phi) \\ +f(r) \xi_{jmk}^{(2)}(\theta, \phi) \end{array} \right). 
\end{align*} \] (135)

Here, the two column functions \( \xi_{jmk}^{(1)}(\theta, \phi) \) and \( \xi_{jmk}^{(2)}(\theta, \phi) \) denote special combinations of \( \chi_{\pm 1/2}(\theta, \phi) \) and \( D_{\pm m, eg/\hbar c \pm 1/2}(\phi, \theta, 0) \):

\[ \begin{align*}
\xi_{jmk}^{(1)} &= \chi_{-1/2} D_{k+1/2} + \chi_{+1/2} D_{k-1/2} , \\
\xi_{jmk}^{(2)} &= \chi_{-1/2} D_{k+1/2} - \chi_{+1/2} D_{k-1/2} ; 
\end{align*} \] (136)

compare them with analogous extensions (6.3a) for \( \Omega_{jm}^{\pm 1/2}(\theta, \phi) \).

These 2-component functions \( \xi_{jmk}^{(1)}(\theta, \phi) \) and \( \xi_{jmk}^{(2)}(\theta, \phi) \) just provide what is called spinor monopole harmonics. It should be useful to write out the detailed explicit form of these generalized harmonics. Given the known expressions for \( \chi \)- and \( D \)-functions, the formulas (136) yield the following

\[ \begin{align*}
\xi_{jmk}^{(1,2)} &= e^{im\phi} \left( \begin{array}{c} -\sin \theta/2 e^{-i\phi/2} \\ \cos \theta/2 e^{+i\phi/2} \end{array} \right) d^j_{m,k+1/2}(\cos \theta) \\
&\pm e^{im\phi} \left( \begin{array}{c} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{+i\phi/2} \end{array} \right) d^j_{m,k-1/2}(\cos \theta) ; 
\end{align*} \] (137)

here, the signs + (plus) and – (minus) refer to \( \xi^{(1)} \) and \( \xi^{(2)} \) respectively. When \( k = 0 \) from (6.6) it follow relations for a pure fermion case in absence of monopole potential.

Above, at translating the electron-monopole functions into the Cartesian tetrad and Pauli’s spin frame, we had overlooked the case of minimal \( j \). Returning to it, on straightforward calculation we find (for \( k < 0 \) and \( k > 0 \), respectively)

positive \( \kappa \):

\[ \begin{align*}
\Psi_{j_{\text{min.}}}^{\text{Cart.}} &= \frac{e^{-i\xi t}}{\sqrt{2r}} \left( \begin{array}{c} (f_1 + f_3) \chi_{+1/2} \\ (f_1 - f_3) \chi_{+1/2} \end{array} \right) D^{k-1/2}_{m,k-1/2}(\theta, \phi, 0) ; 
\end{align*} \] (138)
negative $\kappa$:

$$\Psi_{\text{Cart.}}^{j_{\text{min.}}} = \frac{e^{-i\epsilon t}}{\sqrt{2r}} \left( (f_2 + f_4) \chi_{-1/2} \right) \left( f_2 - f_4 \right) \chi_{-1/2} D^{(k_{j_{\text{min.}}})_{-m,k_{j_{\text{min.}}}}^{1/2}} (\theta, \phi, 0) .$$  \tag{139}

Concluding note: one can equally work whether in terms of monopole harmonics $\xi^{(1,2)}(\theta, \phi)$ or directly in terms of $D$-functions, but the latter alternative has an advantage over the former because of the straightforward access to the ‘unlimited’ $D$-function apparatus; instead of proving and producing just disguised old results.

Now we pass on to another subject and take up demonstrating how the major facts obtained so far are extended to a curved background geometry (of spherical symmetry). All above, the flat space monopole potential $A_\phi = g \cos \theta$ preserves its simple form at changing the flat space model into a curved one of spherical symmetry

$$A_\phi = g \cos \theta \implies F_{\theta\phi} = -F_{\phi\theta} = -g \sin \theta$$

and the general covariant Maxwell equation in such a curved space yields

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} F^\alpha\beta = 0 \implies$$

$$\frac{\partial}{\partial \theta} e^{\nu+\mu} r^2 \sin \theta \frac{g \sin \theta}{r^4 \sin^2 \theta} \equiv 0 , \quad (\theta \neq 0, \pi) .$$

So, the monopole potential (for a curved background geometry) is given again as $A_\phi = g \cos \theta$. In a sequence, the problem of electron in external monopole field (in a curved background) remains, in a whole, unchanged.

There are only some new features brought about by curvature, but they do not affect the $(\theta, \phi)$-aspects of the problem. For instance, consider the case of $j_{\text{min.}}$ at $k > 0$ (the case $j_{\text{min.}}$, $k < 0$ can be considered in the same way):

$$\kappa = +1, +3/2, +2, \dotsc , \quad \Psi_{j_{\text{min.}}}^{k>0} (x) = \frac{1}{r} \left| \begin{array}{c} f_1 (r, t) D_{k-1/2} \vspace{1mm} \\ f_3 (r, t) D_{k-1/2} \end{array} \right| ; \tag{140}$$

from that it follows

$$ie^{-\nu/2} \partial_t f_1 + ie^{-\mu/2} \partial_r f_1 - mf_3 = 0 ,$$

$$ie^{-\nu/2} \partial_t f_3 - ie^{-\mu/2} \partial_r f_3 - mf_1 = 0 , \tag{141}$$
and further
\[
f_3 = \frac{i}{m} \left( e^{-\nu/2} \partial_t + e^{-\mu/2} \partial_r \right) f_1(r,t)
\] 

\[
\left( e^{-\nu/2} \partial_t - e^{-\mu/2} \partial_r \right) (e^{-\nu/2} \partial_t + e^{-\mu/2} \partial_r) + m^2 \right) f_1 = 0. \tag{142}
\]

Finally, let us consider the question of gauge choice for description of the monopole potential. From general considerations we can conclude that, for the problems considered above, it was not basically essential whether to use the Schwinger’s form of the monopole potential or to use any other form. Every possible choice could bring about some technical incidental variation in a corresponding description, but this will not affect the applicability of \(D\)-function apparatus to the procedure of separating out the variables \(\theta, \phi\) in the electron-monopole system.

For example, in the Dirac gauge the monopole potential is given by
\[
(A_\alpha)^D = \begin{bmatrix} 0, & g \vec{n} \times \vec{r} \end{bmatrix}
\]
which after translating to spherical coordinates becomes
\[
A_\alpha^D = (A_t = 0, A_r = 0, A_\theta = 0, A_\phi = g(\cos \theta - 1)) . \tag{144}
\]

On comparing \(A_\phi^D\) with \(A_\phi^S\), it follows immediately that we can relate these electron-monopole pictures by means of a simple gauge transformation:
\[
S(\phi) = e^{ik\phi}, \quad \Psi^D(x) = S(\phi) \Psi^S,
A_\alpha^D(x) = A_\alpha^S(x) - \frac{\hbar c}{e} S \frac{\partial}{\partial x_\alpha} S^{-1}. \tag{145}
\]

Simultaneously translating the operators \(\hat{J}^k, \hat{K}, \hat{N}\) from S.- to D.-gauge
\[
\hat{J}_j^D = S \hat{J}_j^S S^{-1}, \quad \hat{K}^D = S \hat{K}^S S^{-1}, \quad \hat{N}^D = S \hat{N}^S S^{-1}
\]
we produce
\[
\hat{J}_1^D = l_1 + \frac{\cos \phi}{\sin \theta} (i\sigma^{12} - k(1 - \cos \theta)) ,
\hat{J}_2^D = l_2 + \frac{\sin \phi}{\sin \theta} (i\sigma^{12} - k(1 - \cos \theta)) , \quad \hat{J}_3^D = l_3 - k , \\
\hat{K}^D = -i \gamma^0 \gamma^3 ( i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + k + (i\sigma^{12} - k) \cos \theta}{\sin \theta} ) , \\
\hat{N}^D = e^{ik(2\phi + \pi)} \hat{N}^S . \tag{146}
\]
Thus, the explicit forms of the operators vary from one representation to another, but their proper values remain unchanged; any alterations in operators and corresponding modifications in wave functions cancel out each other completely. That is, as it certainly might be expected, the complete set of proper values provides such a description that is invariant, by its implications, under any possible \( U(1) \) gauge transformations.

Now, let us consider else one variation in \( U(1) \) gauge, namely, from Schwinger’s gauge \[73\] to the Wu-Yang’s \[71\]-\[93\]. In the Wu-Yang (hereafter, designated as (W-Y)-gauge, the monopole potential is characterized by two different respective expressions in two complementary spatial regions

\[
0 \leq \theta < (\pi/2 + \epsilon) \implies A^{(N)}_{\phi} = g(\cos \theta - 1), \\
(\pi/2 - \epsilon) < \theta \leq \pi \implies A_{\phi}(S) = g(\cos \theta + 1),
\]

and the transition from the \( S. \)-basis too \( W - Y \)'s can be obtained by

\[
\Psi^{S}(x) \implies \Psi^{W-Y}(x) = \begin{cases} 
\Psi^{(N)}(x) = S^{(N)}(\phi) \Psi^{S}(x), & S^{(N)}(\phi) = e^{+ik\phi}, \\
\Psi^{(S)}(x) = S^{(S)}(\phi) \Psi^{S}(x), & S^{(S)}(\phi) = e^{-ik\phi}.
\end{cases}
\]

Correspondingly, for the operators \( \hat{J}^{k}, \hat{K}, \hat{N} \) we get two different forms in \( N \)- and \( S \)-regions, respectively:

\[
\hat{J}_{1}^{\pm} = l_{1} + \frac{\cos \phi}{\sin \theta} \left( i\sigma^{12} - k(1 \pm \cos \theta) \right), \\
\hat{J}_{2}^{\pm} = l_{2} + \frac{\sin \phi}{\sin \theta} \left( i\sigma^{12} - k(1 \pm \cos \theta) \right), \\
\hat{J}_{3}^{D} = l_{3} \pm k, \\
\hat{K}^{\pm} = -i \gamma^{0} \gamma^{3} \left( i \gamma^{1} \partial_{\theta} + \gamma^{2} \frac{i\partial_{\phi} \mp k + (i\sigma^{12} - k) \cos \theta}{\sin \theta} \right), \\
\hat{N}^{\pm} = \exp(\mp ik(2\phi + \pi)) \hat{N}^{S}.
\]

where the over sign (+ or −) relates to \( S \)-region, and the lower one (− or +, respectively) to \( N \)-region.

It should be noted that only the Schwinger’s \( U(1) \) gauge, in virtue of the relation \( \hat{J}_{3} = -i \partial_{\phi} \), represents analogue of the Schrödinger’s (tetrad) basis discussed in Sec.2, whereas the Dirac and Wu-Yang gauges are not. The explicit form of the third component of a total conserved momentum

\[
J_{3} = -i \partial_{\phi} \equiv J_{3}^{Schr}.
\]
can be regarded as a determining characteristic, which specifies this basis (and its possible generalizations). The situations in \( S, \ D, \) and \( W - Y \) gauges are characterized by

\[
J_3^S = l_3; \quad J_3^D = l_3 - k; \quad J_3^{(N)} = l_3 - k, \quad J_3^{(s)} = l_3 + k. \quad (150)
\]

**PART III**

11 The Dirac and Schwinger gauges in isotopic space

Together with topological way of studying monopole configurations, another approach to monopoles is possible: namely, which is based on manifestations of monopoles playing the role of external potentials. Moreover, from the physical standpoint the latter method can be thought of as a more visualizing one in comparison with less obvious topological language. So, the basic frame of the further investigation is analysis of particles in the external monopole potentials; see also Swank et al [80], Jackiw and Rebbi [84, 85], Hasenfratz and Hooft [92], Callias [99], Goddard and Olive [113], Jackiw and Manton [127], Jackiw [128], Proxhvatilov and Franke [86], Rossi 1982-Rossi, Blaer et al [141], Tang Ju-Fei [142], Callan [143, 144], Henneaux [149], Farhi and D’Hoker [152], Marciano and Muzinich [154, 155], Din and [156], Bhakuni et al [157], Tolkachev [173], Barut et al [229], Red’kov [251, 253, 254, 255], Volkov and Gal’tsov [264], Mezincescu [266], Tokarevskaya et al [271, 278], Milton 2006-Milton, Weinberg and Yi [285].

It is well-known that the usual Abelian monopole potential generates a certain non-Abelian potential being a solution of the Yang-Mills (Y-M) equations. First, such a specific non-Abelian solution was found out in [19]. The procedure itself of that embedding the Abelian monopole 4-vector \( A_\mu(x) \) in the non-Abelian scheme: \( A_\mu(x) \to W_\mu^{(a)}(x) \equiv (0, 0, A_\phi^{(3)} = A_\mu(x)) \) ensures automatically that \( W_\mu^{(a)}(x) \) will satisfy the free Y-M equations. Thus, it may be readily verified that the vector \( A_\mu(x) = (0, 0, 0, A_\phi = g \cos \theta) \) obeys the Maxwell general covariant equations in every space-time with the spherical
symmetry:
\[ dS^2 = e^{2\nu} dt^2 - e^{2\mu} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \]
\[ \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} F^{\alpha \beta} = 0, \quad A_\phi = g \cos \theta, \quad F_{\theta \phi} = -g \sin \theta ; \quad (151) \]

Here we get essentially a single equation. One the same potential \( A_\phi = g \cos \theta \) describes the Abelian monopole in arbitrary spherically-symmetric space-time.

In turn, the non-Abelian strength tensor \( F_{\mu \nu}^a(x) \) associated with the \( A_{\mu}^a \) above has a very simple isotopic structure: \( F_{\theta \phi}^{(3)} = -g \sin \theta \) and all other \( F_{\nu \mu}^{(a)} \) are equal to zero. So, this substitution \( F_{\nu \mu}^{(a)} = (0, 0, 0) \) leads the Y-M equations to the single equation of the Abelian case. Thus, this monopole potential may be interpreted as a trivially non-Abelian solution of Y-M equations. Supposing that such a sub-potential is presented in the well-known monopole solutions of t’Hooft-Polyakov, we will establish explicitly that constituent structure.

The well-known form of the monopole solution \( (11) \) may be taken as a starting point. The field \( W_{\alpha}^{(a)}(x) \) represents a covariant vector with the usual transformation law, and our first step is a change of variables in 3-space, so let us replace \( x_i \) by the spherical coordinates \( (r, \theta, \phi) \). Thus, the given potentials \( (W_{\alpha}^{(a)}) \) convert into \( (W_t^{(a)}, W_r^{(a)}, W_{\theta}^{(a)}, W_{\phi}^{(a)}) \).

Our second step will be a special gauge transformation in the isotopic space. A required gauge matrix can be determined by the condition
\[ (0_{ab} \Phi^b(x)) = (0, 0, r \Phi(r)). \]

This equation has a set of solutions since the isotopic rotation by every angle about the third axis \( (0, 0, 1) \) will not change the finishing vector \( (0, 0, r \Phi(r)) \). We shall seek to fix such an ambiguity by deciding in favor of the simplest transformation matrix. It will be convenient to utilize the known group \( SO(3,R) \) parametrization through the Gibbs 3-vector: see Fedorov [121]:
\[ O(\mathbf{c}) = I + 2 \frac{[\mathbf{c} \times + (\mathbf{c} \times)^2]}{1 + c^2}, \quad (\mathbf{c} \times)_{ij} = -\epsilon_{ijk} c_k . \]

The simplest rotation above is characterized by
\[ D = O(\mathbf{c}) \mathbf{B}, \quad \mathbf{c} = |DB|/(B + D)B, \]
\[ \mathbf{B} = r \Phi \vec{n}_{\theta, \phi}, \quad D = r \Phi (0, 0, 1), \]
\[ \mathbf{c} = \frac{\sin \theta}{1 + \cos \theta} (\sin \phi, -\cos \phi, 0). \quad (152) \]
Together with varying scalar field $\Phi^a(x)$, the vector triplet $W_\alpha^{(a)}(x)$ is to be transformed from one isotopic gauge to another under the law [...] 

$$W_\alpha^{(a)} = O_{ab}(c(x)) \ W_\alpha^{(b)} + \frac{1}{e} \ \Delta_{ab}(c(x)) \ \frac{\partial e_\theta}{\partial x^\alpha},$$

$$\Delta(c) = -2 \frac{1 + c^\alpha}{1 + c^2}. \quad (153)$$

For definiteness let specify calculation the spherical space, in which we need hyper spherical coordinates $y^a = (x^0, \chi, \theta, \phi)$:

$$x^i = 2 \tan \frac{\chi}{2} \ n_i, \quad n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

With the use of tensor law $W^a_\alpha(y) = (\partial x^\beta / \partial y^a) \ W^b_\beta(x)$, starting from (11), we obtain a hyper spherical representation for the dyon substitution (note that $r = 2 \tan \frac{\chi}{2}$):

$$\Phi^b = \Phi(r) \begin{vmatrix} n_1 \\ n_2 \\ n_3 \end{vmatrix}, \quad W^b_0 = f(r) \begin{vmatrix} n_1 \\ n_2 \\ n_3 \end{vmatrix}, \quad W^b_\chi = 0, \quad W^b_\phi = 0,$$

$$W^b_\theta = K(r)r^2 \begin{vmatrix} -\sin \phi \\ + \cos \phi \\ 0 \end{vmatrix}, \quad W^b_\phi = K(r)r^2 \begin{vmatrix} -\sin \theta \cos \theta \cos \phi \\ -\sin \theta \cos \theta \sin \phi \\ 0 \end{vmatrix}, \quad (154)$$

from which with the use of (153) we arrive at

$$\Phi'^b = \Phi(r) \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}, \quad W'^b_0 = f(r) \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}, \quad W'^b_\chi = 0, \quad W'^b_\phi = 0,$$

$$W'^b_\theta = (r^2K(r) + 1/e) \begin{vmatrix} -\sin \phi \\ + \cos \phi \\ 0 \end{vmatrix},$$

$$W'^b_\phi = \begin{vmatrix} -(r^2K(r) + 1/e) \sin \theta \cos \phi \\ -(r^2K(r) + 1/e) \sin \theta \sin \phi \\ \frac{1}{e} (\cos \theta - 1) \end{vmatrix}. \quad (155)$$

The factor $(r^2K + 1/e)$ vanishes when $K = -1/er^2$. In other words, only the delicate fitting of single proportional coefficient results in the actual formal simplification of the non-Abelian monopole potential (155).
There exists a close link between $W_{\phi}^{(a)}$ from \[155\] and the Dirac’s expression for the Abelian monopole potential (supposing $\vec{n} = (0, 0, 1)$)

$$A^D. = g \frac{[nr]}{(r + n n) r}, \text{ or } A^D._{\phi} = g (\cos \theta - 1),$$

so that $W_{\phi}^{triv. (a)}$ from (155) and the Dirac’s expression for the Abelian monopole potential (supposing $\vec{n} = (0, 0, 1)$)

$$A^D. = g \frac{nr}{(r + n n) r}, \text{ or } A^D._{\phi} = g (\cos \theta - 1),$$

or $A^D. = g \frac{nr}{(r + n n) r}$, or $A^D._{\phi} = g \cos \theta$.

It is possible to draw an analogy between Abelian and non-Abelian models. Thus, we may introduce the Schwinger’s non-Abelian basis in the isotopic space:

$$(\Phi^D.(a), W^D.(a)) \implies (\Phi^S.(a), W^S.(a)),$$

with $c' = (0, 0, -\tan \phi/2)$. Now an explicit form of the $\theta$ and $\phi$ components of the monopole potential is given by

$$W^S.(a)_{\theta} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ r^2 K + 1/e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad W^S.(a)_{\phi} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

where the symbol $S.$ stands for the Schwinger’s gauge. Both $D.$- and $S.$- gauges are unitary ones in the isotopic space since the corresponding scalar fields $\Phi^D.(a)(x)$ and $\Phi^S.(a)(x)$ are $x_3$-directional, but one of them (Schwinger’s) seems simpler than another (Dirac’s). To the above-mentioned special monopole field \[25\] corresponds to the $K(r) = -1/e r^2$, so that the relations from (2) turn out to be very simple and related to the Abelian potential embedded into the non-Abelian scheme.

Let us determine the matrix $O(c'') = O(c')O(c)$ relating the Cartesian gauge of isotopic space with Schwinger’s

$\vec{c}'' = (+ \tan \theta/2 \tan \phi/2, - \tan \theta/2, - \tan \phi/2)$,

$$O(c'') = \begin{vmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & - \sin \theta \\ - \sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{vmatrix}.$$
This matrix is also well-known in other context as a matrix linking Cartesian and spherical tetrads in the flat space-time.

12 Dirac particles isotopic multiplet, separation of the variables in Schrödinger’s tetrad basis

In this Section we enter on analyzing the isotopic doublet of Dirac fermions in the external t’Hooft-Polyakov monopole field. We are going to reexamine this problem, using the general relativity tetrad formalism. Instead of the so-called monopole harmonics, the more conventional formalism of the Wigner’s D-functions is used.

We will specify the case of spherical space $S_3$, transition to Euclidean or Lobachewski models is achieved by a simple formal change (see below). In spherical coordinates the metric and corresponding tetrad are

$$dS^2 = dt^2 - d\chi^2 - \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) ,$$
$$e^0_0 = (1, 0, 0, 0), \quad e^0_1 = (0, 0, \sin^{-1} \chi, 0),$$
$$e^0_2 = (0, 0, 0, \sin^{-1} \chi \sin^{-1} \theta), \quad e^0_3 = (0, 1, 0, 0),$$

and the Schwinger unitary gauge of the monopole potentials, the Dirac equation for an isotopic doublet

$$\left[ \gamma^\alpha (x) \left( i\partial_\alpha + \Gamma_\alpha(x) + e t^a W^{(a)}_{\alpha} \right) - (m + \kappa \Phi^{(a)} t^a) \right] \Psi(x) = 0 .$$

takes the form (note that $r = 2 \tan(\chi/2)$)

$$\left[ \gamma^0 \left( i \partial_t + e r F(r) t^3 \right) + i\gamma^3 \left( \partial_\chi + \frac{1}{\tan \chi} \right) + \frac{1}{\sin \chi} \Sigma^S_{\theta, \phi} + \frac{e r^2 K + 1}{\sin \chi} \left( \gamma^1 \otimes I^2 - \gamma^2 \otimes I^1 \right) - (m + \kappa r \Phi(r) t^3) \right] \Psi^S = 0 ,$$

$$\Sigma^S_{\theta, \phi} = i \gamma^1 \partial_\theta + \gamma^2 \frac{id\phi + (i\sigma^{12} + t^3) \cos \theta}{\sin \theta}, \quad t^j = (1/2) \sigma^j .$$

A characteristic feature of such a correlated choice of frames in both these spaces is the explicit form of the total angular momentum operator (the sum of orbital, spin, and isotopic ones)

$$J^S_1 = l_1 + \frac{(i\sigma^{12} + t^3) \cos \phi}{\sin \theta},$$
$$J^S_2 = l_2 + \frac{(i\sigma^{12} + t^3) \sin \phi}{\sin \theta}, \quad J^S_3 = l_3 ;$$

50
so that the present case entirely comes under the situation considered by Pauli in [...]. The Pauli criterion allows here the following values for $j : j = 0, 1, 2, 3, \ldots$ The $\theta, \phi$-dependence of the doublet wave function $\Psi_{jm}$ is to be built up in terms of the Wigner $D$-functions: $D^j_{-m,\sigma}(\phi, \theta, 0)$, where the lower right index $\sigma$ takes the values from $(-1, 0, +1)$, which correlates with the explicit diagonal structure of the matrix $(i\sigma^1 + \tau^3)$:

$$
\Psi_{ejm}(x) = \frac{e^{-ict}}{\sin \chi} \left[ T_{+1/2} \otimes F(\chi, \theta, \phi) + T_{-1/2} \otimes G(\chi, \theta, \phi) \right] ; \quad (161)
$$

here the fixed symbols $j$ and $(-m)$ in $D^j_{-m,\sigma}(\phi, \theta, 0)$ are omitted and

$$
F = \begin{bmatrix}
  f_1(\chi)D_{-1} \\
  f_2(\chi)D_0 \\
  f_3(\chi)D_{-1} \\
  f_4(\chi)D_0
\end{bmatrix}, \quad G = \begin{bmatrix}
  g_1(\chi)D_0 \\
  g_2(\chi)D_{+1} \\
  g_3(\chi)D_0 \\
  g_4(\chi)D_{+1}
\end{bmatrix}, \quad T_{+1/2} = \begin{bmatrix}
  1 \\
  0
\end{bmatrix}, \quad T_{-1/2} = \begin{bmatrix}
  0 \\
  1
\end{bmatrix} ;
$$

throughout the paper the factor $e^{-ict}/\sin \chi$ will be omitted.

Another essential feature of the given frame in the

\[(Lorentz) \otimes (isotopic)-space\]

is the appearance of the very simple expression (proportional to $e^{r^2K + 1}$) for the term that mixes up together two distinct components of the isotopic doublet (see eq. (159)).

An important case in the electron-monopole problem is the minimal value of quantum number $j$. The allowed values for $j$ are $0, 1, 2, \ldots$; the case of $j = 0$ needs a careful separate consideration. When $j = 0$, the symbols $D^0_{0,\pm1}$ are meaningless, and the wave function $\Psi_{e0}(x)$ is to be constructed as

$$
\Psi_{e0} = T_{+1/2} \otimes \begin{bmatrix}
  0 \\
  f_2(\chi) \\
  0
\end{bmatrix} + T_{-1/2} \otimes \begin{bmatrix}
  g_1(\chi) \\
  0 \\
  g_3(\chi)
\end{bmatrix} . \quad (162)
$$

Using the required recursive relations for Wigner functions
\[ (\nu = \sqrt{j(j+1)}, \omega = \sqrt{(j-1)(j+2)}, j \neq 0), \]

\[ \partial_\theta D_{-1} = \frac{1}{2}(\omega D_{-2} - \nu D_0), \quad \frac{m - \cos \theta}{\sin \theta} D_{-1} = \frac{1}{2}(\omega D_{-2} + \nu D_0), \]

\[ \partial_\theta D_0 = \frac{1}{2}(\nu D_{-1} - \nu D_{+1}), \quad \frac{m}{\sin \theta} D_0 = \frac{1}{2}(\nu D_{-1} + \nu D_0), \]

\[ \partial_\theta D_{+1} = \frac{1}{2}(\nu D_0 - \omega D_{+2}), \quad \frac{m + \cos \theta}{\sin \theta} D_{+1} = \frac{1}{2}(\nu D_0 + \omega D_{+2}), \]

we find

\[ \Sigma_{\theta, \phi}^S \Psi_{jm}^S = \nu \begin{bmatrix} T_{+1/2} \otimes \left[ \begin{array}{c} -if_4 D_{-1} \\ +if_3 D_0 \\ +if_2 D_{-1} \\ -if_1 D_0 \end{array} \right] + T_{-1/2} \otimes \left[ \begin{array}{c} -ig_4 D_0 \\ +ig_3 D_{+1} \\ +ig_2 D_0 \\ -ig_1 D_{+1} \end{array} \right] \end{bmatrix}. \quad (164) \]

Further, let us write down the expression for the term mixing up the isotopic components

\[ \frac{er^2 K + 1}{\sin \chi} \left( \gamma^1 \otimes t^2 - \gamma^2 \otimes t^1 \right) \Psi_{jm} = \frac{er^2 K + 1}{2\sin \chi} \]

\[ \times \begin{bmatrix} T_{+1/2} \otimes \left[ \begin{array}{c} 0 \\ +ig_3 D_0 \\ 0 \\ -ig_1 D_0 \end{array} \right] + T_{-1/2} \otimes \left[ \begin{array}{c} 0 \\ +if_2 D_0 \\ 0 \\ 0 \end{array} \right] \end{bmatrix}. \quad (165) \]

After a simple calculation one finds the system of radial equations (for short-
ness we set $W \equiv (e^2 K + 1)/2$, $\tilde{F} \equiv e F/2$, $\tilde{\Phi} \equiv \kappa r \Phi/2$)

\[
(-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_3 - i \frac{\nu}{\sin \chi} f_4 - (m + \tilde{\Phi}) f_1 = 0,
\]

\[
(+i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_4 + i \frac{\nu}{\sin \chi} f_3 + i \frac{W}{\sin \chi} g_3 - (m + \tilde{\Phi}) f_2 = 0,
\]

\[
(-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_2 - i \frac{\nu}{\sin \chi} f_1 + i \frac{W}{\sin \chi} g_2 - (m + \tilde{\Phi}) f_3 = 0,
\]

\[
(+i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_1 + i \frac{\nu}{\sin \chi} f_2 - (m + \tilde{\Phi}) f_1 = 0,
\]

\[
(-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_3 - i \frac{\nu}{\sin \chi} g_4 - i \frac{W}{\sin \chi} f_4 - (m + \tilde{\Phi}) f_1 = 0,
\]

\[
(+i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_1 + i \frac{\nu}{\sin \chi} g_3 - (m + \tilde{\Phi}) g_1 = 0,
\]

\[
(-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_4 - i \frac{\nu}{\sin \chi} g_2 - (m + \tilde{\Phi}) g_3 = 0,
\]

\[
(+i \frac{d}{d\chi} + \epsilon - \tilde{F}) g_3 + i \frac{\nu}{\sin \chi} f_2 - (m - \tilde{\Phi}) g_4 = 0.
\] (166)

When $j$ takes on value 0 (then $\Sigma_{\theta, \phi} \Psi_{\epsilon 0} \equiv 0$), the radial system is simpler:

\[
(+i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_4 + i \frac{W}{\sin \chi} g_3 - (m + \tilde{\Phi}) f_2 = 0,
\]

\[
(-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_2 - i \frac{W}{\sin \chi} g_1 - (m + \tilde{\Phi}) f_4 = 0,
\]

\[
(-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_3 - i \frac{W}{\sin \chi} f_4 - (m - \tilde{\Phi}) g_3 = 0,
\]

\[
(+i \frac{d}{d\chi} + \epsilon + \tilde{F}) g_1 + i \frac{W}{\sin \chi} g_2 - (m + \tilde{\Phi}) g_4 = 0.
\] (167)

Both systems (166) and (167) are sufficiently complicated. To proceed further with a situation like that, it is normal practice to have searched a suitable operator which could be diagonalized additionally. It is known that the usual $P$-inversion operator for a bispinor field cannot be completely appropriate for this purpose and a required quantity is to be constructed as a combination of bispinor $P$-inversion operator and a certain discrete transformation in the isotopic space. Indeed, considering that the usual $P$-inversion operator for a bispinor field (in the basis of Cartesian tetrad, it is $P_{\text{Cart.}} \otimes \tilde{P} = i \gamma^0 \otimes \tilde{P}$, where $\tilde{P}$ causes the usual $P$-reflection of space
coordinates) is determined in the given (spherical) basis as
\[ \hat{P}_{\text{sph. bisp.}} \otimes \hat{P} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \hat{P} = -\left( \gamma^5 \gamma^1 \right) \otimes \hat{P} \]
and it acts upon the wave function \( \Psi_{jm}(x) \) as follows
\[ (\hat{P}_{\text{sph. bisp.}} \otimes \hat{P}) \Psi_{jm}(x) = (-1)^{j+1} \times [ T_{+1/2} \otimes \begin{vmatrix} f_4 D_0 \\ f_3 D_{+1} \\ f_2 D_0 \\ f_1 D_{+1} \end{vmatrix} + T_{-1/2} \otimes \begin{vmatrix} g_4 D_{-1} \\ g_3 D_0 \\ g_2 D_{-1} \\ g_1 D_0 \end{vmatrix} ] . \tag{168} \]

The latter points the way towards the search for a required discrete operator: it would have the structure
\[ \hat{N}_{\text{sph.}}^S \equiv \hat{\pi}^S \otimes \hat{P}_{\text{sph. bisp.}} \otimes \hat{P} , \quad \hat{\pi}^S = a\sigma^1 + b\sigma^2 , \tag{169} \]
so that \( \hat{\pi}^S \cdot T_{\pm 1/2} = (a \pm ib) \cdot T_{\mp 1/2} \). The total multiplier at the quantity \( \hat{\pi}^S \) is not material for separating the variables, below one sets \( (\hat{\pi}^S)^2 = (a^2 + b^2) = +1 \). In the following we restrict ourselves to real valued \( a \) and \( b \) and use notation:
\[ a + ib = e^{iA} . \]

From the equation \( \hat{N}_{\text{sph.}}^S \cdot \Psi_{jm} = N_A \Psi_{jm} \) one finds two proper values \( N_A \) and corresponding limitation on the functions \( f_i(r) \) and \( g_i(r) \):
\[ N_A = \delta (-1)^{j+1} , \quad \delta = \pm 1 , \]
\[ g_1 = \delta e^{iA} f_4 , \quad g_2 = \delta e^{iA} f_3 , \]
\[ g_3 = \delta e^{iA} f_2 , \quad g_4 = \delta e^{iA} f_1 . \tag{170} \]
Taking into account the relations (170), one produces the equations

\[ (-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_3 - \frac{\nu}{\sin \chi} f_4 - (m + \tilde{\Phi}) f_1 = 0 , \]

\[ (+i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_4 + \frac{\nu}{\sin \chi} f_3 + i \frac{W}{\sin \chi} - \delta e^{iA} f_2 - (m + \tilde{\Phi}) f_2 = 0 , \]

\[ (+i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_1 + \frac{\nu}{\sin \chi} f_2 - (m + \tilde{\Phi}) f_3 = 0 , \]

\[ (-i \frac{d}{d\chi} + \epsilon + \tilde{F}) f_2 - \frac{\nu}{\sin \chi} f_1 - i \frac{W}{\sin \chi} \delta e^{iA} f_2 - (m + \tilde{\Phi}) f_4 = 0 , \]

\[ (+i \frac{d}{d\chi} + \epsilon - \tilde{F}) f_1 + \frac{\nu}{\sin \chi} f_2 - (m - \tilde{\Phi}) f_3 = 0 , \]

\[ (+i \frac{d}{d\chi} + \epsilon - \tilde{F}) f_4 + \frac{\nu}{\sin \chi} f_3 + i \frac{W}{\sin \chi} e^{-iA} \delta f_2 - (m - \tilde{\Phi}) f_2 = 0 , \]

\[ (-i \frac{d}{d\chi} + \epsilon - \tilde{F}) f_3 - \frac{\nu}{\sin \chi} f_4 - (m - \tilde{\Phi}) f_1 = 0 . \]  

(171)

It is evident at once that the system (171) would be compatible with itself provided that \( \tilde{F}(\chi) = 0 \) and \( \tilde{\Phi}(\chi) = 0 \). In other words, the above-mentioned operator \( \hat{N}^S \) can be diagonalized on the functions \( \Psi_{\epsilon jm}(x) \) if and only if \( W^t(a) = 0 \) and \( \kappa = 0 \); below we suppose that these requirements will be satisfied. Moreover, given this limitation satisfied, it is necessary to draw distinction between two cases depending on expression for \( W(r) \).

If \( W(\chi) = 0 \), the difference between \( e^{iA} \) and \( e^{-iA} \) in the equations (171) is not essential in simplifying these equations (because the relevant terms just vanish). Thus, for the first case, the system (171) converts into

\[ W(\chi) = 0 , \]

\[ (-i \frac{d}{d\chi} + \epsilon) f_3 - \frac{\nu}{\sin \chi} f_4 - m f_1 = 0 , \]

\[ (+i \frac{d}{d\chi} + \epsilon) f_4 + \frac{\nu}{\sin \chi} f_3 - m f_2 = 0 , \]

\[ (+i \frac{d}{d\chi} + \epsilon) f_1 + \frac{\nu}{\sin \chi} f_2 - m f_3 = 0 , \]

\[ (-i \frac{d}{d\chi} + \epsilon) f_2 - \frac{\nu}{\sin \chi} f_1 - m f_4 = 0 . \]  

(172)
There exists sharply distinct situation at \( W \neq 0 \). Here, the equations are consistent with each other only if \( e^{iA} = e^{-iA} \); therefore \( e^{iA} = a + ib = \pm 1 \) (for definiteness, let this parameter \( a \) be equal +1). The corresponding set of radial equations, obtained from (171), is

\[
W(\chi) \neq 0 ,
\]

\[
\begin{align*}
(-i \frac{d}{d\chi} + \epsilon)f_3 - \frac{\nu}{\sin \chi} f_4 - mf_1 &= 0 , \\
(+i \frac{d}{d\chi} + \epsilon)f_4 + \frac{\nu}{\sin \chi} f_3 + i \frac{W(\chi)}{\sin \chi} \delta f_2 - mf_2 &= 0 , \\
(+i \frac{d}{d\chi} + \epsilon)f_1 + \frac{\nu}{\sin \chi} f_2 - mf_3 &= 0 , \\
(-i \frac{d}{d\chi} + \epsilon)f_2 - \frac{\nu}{\sin \chi} f_1 - i \frac{W(\chi)}{\sin \chi} \delta f_4 - mf_4 &= 0 .
\end{align*}
\] (173)

The case \( j = 0 \) can be considered in the same way. Here the \( N_A \)-symmetry produces

\[
N_A = -\delta , \quad \delta = \pm 1 : \quad g_1 = \delta e^{iA} f_4 , \quad g_3 = \delta e^{iA} f_2 . \quad (174)
\]

Further, the quantities \( \tilde{F} \) and \( \tilde{\Phi} \) are to be equated to zero; again there are two possibilities depending on \( W \):

\[
\begin{align*}
\underline{j = 0} , W(\chi) = 0 : & \quad (i \frac{d}{d\chi} + \epsilon)f_4 - mf_2 = 0 , \\
& \quad (-i \frac{d}{d\chi} + \epsilon)f_2 - mf_4 = 0 ; \quad (175)
\end{align*}
\]

\[
\begin{align*}
\underline{j = 0} , W(\chi) \neq 0 : & \quad (i \frac{d}{d\chi} + \epsilon)f_4 - (m - i \frac{\delta W(\chi)}{\sin \chi}) f_2 = 0 , \\
& \quad (-i \frac{d}{d\chi} + \epsilon)f_2 - (m + i \frac{\delta W(\chi)}{\sin \chi}) f_4 = 0 . \quad (176)
\end{align*}
\]

The explicit forms of the wave functions \( \Psi_{ejm\delta}(x) \) and \( \Psi_{eo\delta}(x) \) are as follows:

the case \( W(\chi) \neq 0 , \quad j > 0 \),

\[
\Psi_{ejm}(x) = T_{+1/2} \otimes \begin{vmatrix} f_1 & D_{-1} \\ f_2 & D_0 \\ f_3 & D_{-1} \\ f_4 & D_0 \end{vmatrix} + \delta T_{-1/2} \otimes \begin{vmatrix} f_4 & D_0 \\ f_3 & D_{+1} \\ f_2 & D_0 \\ f_1 & D_{+1} \end{vmatrix} ; \quad (177)
\]
the case \( W(\chi) \neq 0, \ j = 0 \),

\[
\Psi_{e0} = T_{+1/2} \otimes \begin{pmatrix}
0 & f_2(r) \\
0 & f_4(r)
\end{pmatrix} + \delta T_{-1/2} \otimes \begin{pmatrix}
0 & f_4(r) \\
0 & f_2(r)
\end{pmatrix};
\]

(178)

when \( W = 0 \), the term \( \delta T_{-1/2} \) is to be changed to \( \delta e^{iA} T_{-1/2} \).

In the end of this Section let us specify explicit form of \( W(\chi)/\sin \chi \), in this point we consider all three model, \( S_3, H_3, E_3 \):

\begin{align*}
\text{in } S_3 - \text{space, } & \quad \frac{W(\chi)}{\sin \chi} = \frac{er^2K + 1}{2\sin \chi} = \frac{1}{2}af_1(a\chi + b), \ \chi \in [0, \pi]; \\
\text{in } H_3 - \text{space, } & \quad \frac{W(\chi)}{\sh \chi} = \frac{er^2K + 1}{2\sh \chi} = \frac{1}{2}af_1(a\chi + b), \ \chi \in [0, +\infty); \\
\text{in } E_3 - \text{space, } & \quad \frac{W(r)}{r} = \frac{er^2K + 1}{2r} = \frac{1}{2}af_1(ar + b), \ r \in [0, +\infty). \\
\end{align*}

(179)

According to see (22) we have three different possibilities to choose \( f_1 \):

\[
f_1 = \pm \frac{A}{\sin (Ar + B)}, \pm \frac{A}{\sh (Ar + B)}, \pm \frac{A}{Ar + B}.
\]

One may feel that among the above monopole solutions in models \( E_3, H_3, S_3 \) there exist three ones which can be naturally associated with respective geometries. The situation can be illustrated by the schema

\[
\begin{array}{ccc}
E_3 & H_3 & S_3 \\
(ar + b) & * & - \\
\sh (ar + b) & - & * \\
\sin (ar + b) & - & *
\end{array}
\]

It should be noted that the known non-singular BPS-solution in the flat Minkowski space can be understood as a result of somewhat artificial combining the Minkowski space background with a possibility naturally linked up with the Lobachevsky geometry.
13 Analysis of the case of singular monopole field

Now, some added aspects of the simplest monopole are examined more closely. The system of radial equations, specified for this potential, is basically simpler than in general case, so that the whole problem including the radial functions can be carried out to its complete conclusion. Actually, the equation (172) admits of some further simplifications owing to diagonalizing the operator \( \hat{K}_{\theta,\phi} = -i\gamma_0^0\Sigma_{\theta,\phi} \). From the equation \( \hat{K}_{\theta,\phi}\Psi_{jm} = \lambda\Psi_{jm} \), it follows that \( \lambda = -\mu \sqrt{j(j+1)} \), \( \mu = \pm 1 \) and

\[
f_4 = \mu f_1, \quad f_3 = \mu f_2, \quad g_4 = \mu g_1, \quad g_3 = \mu g_2. \tag{180}
\]

Correspondingly, the system (172) yields

\[
(+i\frac{d}{d\chi} + \epsilon)f_1 + i\frac{\nu}{\sin\chi}f_2 - \mu m f_2 = 0, \tag{181}
\]

\[
(-i\frac{d}{d\chi} + \epsilon)f_2 - i\frac{\nu}{\sin\chi}f_1 - \mu m f_1 = 0.
\]

The wave function with quantum numbers \((\epsilon, j, m, \delta, \mu)\) has the form

\[
\Psi_A^{\epsilon jm\delta\mu}(x) = T_{+1/2} \otimes \begin{vmatrix} f_1 & D_{-1} \\ f_2 & D_0 \\ \mu & f_2 D_{-1} \\ \mu & f_1 D_0 \end{vmatrix} + e^{iA}\mu\delta T_{-1/2} \otimes \begin{vmatrix} f_1 & D_0 \\ f_2 & D_{+1} \\ \mu & f_2 D_0 \\ \mu & f_1 D_{+1} \end{vmatrix}. \tag{182}
\]

Let us relate the non-Abelian functions (29) and (178) with the wave functions satisfying the Dirac equation in the Abelian monopole potential. Those latter were investigated by many authors in the case of flat space; below we will use the notation according to \([...]\).

At \( j > j_{\text{min}} \) these Abelian functions are described as follows (the factor \( e^{-i\epsilon t}/\sin\chi \) is omitted)

\[
\Phi^{(eg)}_{jm\mu} = \begin{vmatrix} f_1(\chi) & D^j_{-m,eg - 1/2} \\ f_2(\chi) & D^j_{-m,eg + 1/2} \\ \mu f_2(\chi) & D^j_{m,eg - 1/2} \\ \mu f_1(\chi) & D^j_{m,eg + 1/2} \end{vmatrix}. \tag{183}
\]
For the minimal values \( j = j_{\text{min.}} = \left| eg \right| -1/2 \), they are

\[
\Phi_{\epsilon_0}^{(eg)} = \begin{vmatrix}
  f_1(\chi) & D_{-m,eg-1/2}^j \\
  0 & 0 \\
  f_3(\chi) & D_{-m,eg+1/2}^j \\
\end{vmatrix} ;
\]

\[
(184)
\]

\[
\Phi_{\epsilon_0}^{(eg)} = \begin{vmatrix}
  f_2(\chi) & D_{-m,eg+1/2}^j \\
  0 & 0 \\
  f_4(\chi) & D_{-m,eg+1/2}^j \\
\end{vmatrix} .
\]

\[
(185)
\]

On comparing the formulas (29) and (178) with these Abelian fermion-monopole functions, the following expansions can be easily found (respectively, for \( j > 0 \) and \( j = 0 \) cases):

\[
\Psi^{A\delta\mu}_{\epsilon_{jm}}(x) = T_{+1/2} \otimes \Phi_{\epsilon_{jm\mu}}^{eg=-1/2}(x) + \mu \delta e^{iA} T_{-1/2} \otimes \Phi_{\epsilon_{jm\mu}}^{eg=+1/2}(x) ,
\]

\[
(186)
\]

\[
\Psi_{\epsilon_{0\mu}}^{A}(x) = T_{+1/2} \otimes \Phi_{\epsilon_{0\mu}}^{eg=-1/2}(x) + \delta e^{iA} T_{-1/2} \otimes \Phi_{\epsilon_{0\mu}}^{eg=+1/2}(x) .
\]

In connection with the formulas (186), one additional remark should be given. Though, as evidenced by (3.4a,b), definite close relationships between the non-Abelian doublet wave functions and Abelian fermion-monopole functions can be explicitly discerned, in reality, the non-Abelian situation is intrinsically non-monopole-like (non-singular one). Indeed, in the non-Abelian case, the totality of possible transformations (upon the relevant wave functions) which bear the gauge status are very different from ones that there are in the purely Abelian theory. In a consequence of this, the non-Abelian fermion doublet wave functions can be readily transformed, by carrying out some special gauge transformations in Lorentzian and isotopic spaces together, into the form where they are single-valued functions of spatial points. In the Abelian monopole situation, the analogous particle-monopole functions can by no means be translated to any single-valued ones.

14 Free parameter and \( N_A \)-parity selection rules

Now we proceed with analyzing the totality of the discrete operators \( \hat{N}_A \), which all are suitable for separation of variables. What is the meaning of the parameter \( A \)? In other words, how can this \( A \) manifest itself and why
does such an unexpected ambiguity exist? We remember that the A fixes up one of the complete set of operators \{ \text{id}, \mathbf{J}^2, J_3, \hat{N}_A, \hat{K} \}, and correspondingly this A also labels all basic wave functions. It is obvious, that this parameter A can manifest itself in matrix elements of physical quantities.

As a simple example let us consider a new form of the above-mentioned selection rules depending on the A-parameter. Now, the matrix element examined is

$$\int \tilde{\Psi}^A_{\epsilon J M \delta \mu} (x) \hat{G}(x) \Psi^A_{\epsilon J' M' \delta' \mu'} (x) \, dV \equiv \int r^2 dr \int f^A (\bar{x}) \, d\Omega$$

then

$$f^A (\bar{x}) = \delta \delta' (1)^{J+J'} \tilde{\Psi}^A_{\epsilon J M \delta \mu} (x) \times \left[ (a^* \sigma^1 + b^* \sigma^2) \otimes \hat{P}_{\text{bisp.}} \hat{G} (\bar{x}) (a \sigma^1 + b \sigma^2) \otimes \hat{P}_{\text{bisp.}} \right] \Psi^A_{\epsilon J' M' \delta' \mu'} (\bar{x}) .$$

(187)

If this \( \hat{G} \) obeys the condition

$$\left[ (a^* \sigma^1 + b^* \sigma^2) \otimes \hat{P}_{\text{bisp.}} \right] \hat{G} (\bar{x}) \left[ (a \sigma^2 + b \sigma^1) \otimes \hat{P}_{\text{bisp.}} \right] = \Omega^A \hat{G} (\bar{x}) \quad (188)$$

which is equivalent to

$$\begin{pmatrix} e^{i(A-A^*)} \hat{g}_{22} (\bar{x}) & e^{-i(A+A^*)} \hat{g}_{21} (\bar{x}) \\ e^{i(A+A^*)} \hat{g}_{12} (\bar{x}) & e^{-i(A-A^*)} \hat{g}_{11} (\bar{x}) \end{pmatrix} \otimes \left[ \hat{P}_{\text{bisp.}} \hat{G}^0 (\bar{x}) \hat{P}_{\text{bisp.}} \right] = \Omega^A \begin{pmatrix} \hat{g}_{11} (\bar{x}) & \hat{g}_{12} (\bar{x}) \\ \hat{g}_{21} (\bar{x}) & \hat{g}_{22} (\bar{x}) \end{pmatrix} \otimes \hat{G} (\bar{x}) \quad (189)$$

where \( \Omega^A = +1 \) or \(-1\), then the relationship (187) comes to

$$f^A (\bar{x}) = \Omega^A \delta \delta' (1)^{J+J'} \hat{G} (\bar{x}).$$

(190)

Taking into account (190), we bring the matrix element’s integral above to the form

$$\int \tilde{\Psi}^A_{\epsilon J M \delta \mu} (x) \hat{G}(x) \Phi^A_{\epsilon J' M' \delta' \mu'} (x) \, dV \equiv \int r^2 dr \int f^A (\bar{x}) \, d\Omega$$

$$= \left[ 1 + \Omega^A \delta \delta' (1)^{J+J'} \right] \int_{V_{1/2}} f^A (\bar{x}) \, dV \quad (191)$$
where the integration in the right-hand side is done on the half-space. This expansion provides the following selection rules:

\[ ME \equiv 0 \quad \iff \quad \left[ 1 + \Omega^A \delta \delta' \left( -1 \right)^{J+J'} \right] = 0 \, . \] (192)

\( \Omega^A \) involves its own particular limitations on composite scalars or pseudoscalars because it implies definite configuration of their isotopic parts, obtained by delicate fitting all the quantities \( \hat{g}_{ij} \). Therefore, each of those \( A \) will generate its own distinctive selection rules.

15 Parameter \( A \) and additional isotopic symmetry

Where does this \( A \)-ambiguity come from and what is the meaning of this parameter \( A \)? To proceed further with this problem, one is to realize that the all different values for \( A \) lead to the same whole functional space; each fixed value for \( A \) governs only the basis states \( \Psi^A_{\epsilon JM\delta\mu}(x) \) associated with \( A \). Connection between any two sets of functions \( \{ \Psi(x)^A \} \) and \( \{ \Psi(x)^{A=0} \} \) is characterized by

\[ \Psi^{A, S.}_{\epsilon JM\delta\mu} = U^S_{\epsilon JM\delta\mu}(A) \Psi^{A'=0, S.}_{\epsilon JM\delta\mu}(x) \, , \quad U^S_{\epsilon JM\delta\mu}(A) = \begin{pmatrix} 1 & 0 \\ 0 & e^{iA} \end{pmatrix} \otimes I \, . \] (193)

It is readily verified that the operator \( \hat{N}^S_{\epsilon JM\delta\mu}(x) \) (depending on \( A \)) can be obtained from the operator \( \hat{N}^S_{\epsilon JM\delta\mu}(x) \) as follows

\[ \hat{N}^S_{\epsilon JM\delta\mu} = U^S_{\epsilon JM\delta\mu}(A) \hat{N}^S_{\epsilon JM\delta\mu}(A) \, . \] (194)

The matrix \( U^S_{\epsilon JM\delta\mu}(A) \) is so simple only in the Schwinger basis; after translating that into Cartesian one

\[ \Psi^{A, C.}_{\epsilon JM\delta\mu}(x) = U^C_{\epsilon JM\delta\mu}(A) \Psi^{A'=0, C.}_{\epsilon JM\delta\mu}(x) \, , \]

it becomes

\[ U^C_{\epsilon JM\delta\mu}(A) = \begin{pmatrix} (e^{iA} \sin^2 \theta/2 + \cos^2 \theta/2) & \frac{1}{2} \left( 1 - e^{iA} \right) \sin \theta e^{-i\phi} \\ \frac{1}{2} \left( 1 - e^{iA} \right) \sin \theta e^{i\phi} & (\sin^2 \theta/2 + e^{iA} \cos^2 \theta/2) \end{pmatrix} \otimes I \, . \] (195)

The transformation \( U^C_{\epsilon JM\delta\mu}(A) \) can be brought to the form

\[ U^C_{\epsilon JM\delta\mu}(A) = \frac{1 + e^{iA}}{2} + \frac{1 - e^{iA}}{2} \sigma \bar{n}_{\theta,\phi} \, . \]
Separating out the factor $e^{iA/2}$ in the right-hand side of this formula, we can rewrite the $U^C$: in the form

$$U^C_A = e^{iA/2} \exp(-i \frac{A}{2} \vec{n}_{\theta,\phi})$$  \hspace{1cm} (196)$$

where the second factor lies in the (local) spinor representation of the rotation group $SO(3,R)$. This matrix provides a very special transformation upon the isotopic fermion doublet and can be thought of as an analogue of the Abelian chiral symmetry transformation. This symmetry leads to the $A$-ambiguity (6.5) and permits to choose an arbitrary reflection operator from the totality $\{\hat{N}_A\}$.

Let us add some generalities. As well known, when analyzing any Lie group problems (or their algebra’s) there indeed exists a concept of equivalent representations: $U M_k U^{-1} = M'_k$ and $M_k \sim M'_k$. In this context, the two sets of operators $\{J_i^S, \hat{N}_A^S\}$ and $\{J_i^S, \hat{N}_A^S\}$ provide basically just the same representation of the $O(3,R)$-algebra

$$\{J_i^S, \hat{N}_A^S\} = U_{S,A}(A) \{J_i^S, \hat{N}_A^S\} U^{-1}_{S,A}(A).$$  \hspace{1cm} (197)$$

The totally different situation occurs in the context of the use of those two operator sets as physical observables concerning the system with the fixed Hamiltonian

$$\{\hat{J}_i^S, \hat{N}_A^S\}^{\hat{H}} \text{ and } \{\hat{J}_i^S, \hat{N}_A^S\}^{\hat{H}}.$$

Actually, in this case the two operator sets represent different observables at the same physical system: both of them are followed by the same Hamiltonian $\hat{H}$ and also lead to the same functional space, changing only its basis vectors $\{\Psi_{J_i M_k}(x)^A\}$. Moreover, in the quantum mechanics it seems always possible to relate two arbitrary complete sets of operators by some unitary transformation:

$$\{\hat{X}_\mu, \mu = 1, \ldots\}^{\hat{H}} \implies \{\hat{Y}_\mu, \mu = 1, \ldots\}^{\hat{H}}, \{\Phi_{x_1 \ldots x_s}\} \implies \{\Phi_{y_1 \ldots y_s}\}.$$  \hspace{1cm} (198)$$

But arbitrary transformations $U$ cannot generate, through converting

$$U \{\hat{X}_\mu\} U^{-1} = \hat{Y}_\mu,$$

a new complete set of variables; instead, only some Hamiltonian symmetry’s operations are suitable for this: $U \hat{H} U^{-1} = H$.

In this connection, we may recall a more familiar situation for Dirac massless field. The wave equation for this system has the form

$$i\sigma^\alpha(x) (\partial_\alpha + \Sigma_\alpha) \xi(x) = 0 \quad , \quad i\sigma^\alpha(x) (\partial_\alpha + \Sigma_\alpha) \eta(x) = 0.$$  \hspace{1cm} (199)$$
If the function $\Phi(x) = (\xi(x), \eta(x))$ is subjected to the transformation

$$
\begin{bmatrix}
\xi'(x) \\
\eta'(x)
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & z & I
\end{bmatrix}
\begin{bmatrix}
\xi(x) \\
\eta(x)
\end{bmatrix},
$$

(200)

where $z$ is an arbitrary complex number, then the new function $\Phi'(x) = (\xi'(x), \eta'(x))$ satisfies again the equation in the form (199). This manifests the Dirac massless field’s symmetry with respect to the transformation

$$
\hat{H}' = U \hat{H} U^{-1} = \hat{H}, \quad \Phi'(x) = U \Phi(x).
$$

(201)

The existence of the symmetry raises the question as to whether this symmetry affects determination of complete set of diagonalized operators and constructing spherical wave solutions. These solutions, conformed to diagonalizing the usual bispinor $P$-inversion operator are as in [2]. In the same time, other spherical solutions, together with corresponding diagonalized discrete operator, can be produced:

$$
\Phi_{\epsilon jm\mu} = \frac{e^{-i\epsilon t}}{r} \begin{bmatrix}
f_1 D^{j}_{-m-1/2} \\
f_2 D^{j}_{-m+1/2} \\
z \mu f_2 D^{j}_{-m-1/2} \\
z \mu f_1 D^{j}_{-m+1/2}
\end{bmatrix},
U \left( \hat{P}_{\text{sph.}} \otimes \hat{P} \right) U^{-1} =
$$

$$
= \left[ \frac{1}{2} (z + \frac{1}{z}) (-\gamma^5 \gamma^1) + \frac{1}{2} (z - \frac{1}{z}) (-\gamma^1) \right] \otimes \hat{P}.
$$

(202)

Introducing another complex variable $A$ instead of the parameter $z : z = (\cos A + i \sin A) = e^{iA}$; so that the operator from (6.4b) is rewritten in the form

$$
(\cos A + i \sin A \gamma^5) (-\gamma^5 \gamma^1) \otimes \hat{P} \equiv e^{+iA\gamma^5} \hat{P}_{\text{sph.}} \otimes \hat{P} \equiv e^{+iA/2} \exp (+i\gamma^5 A/2) \Phi(x).
$$

(203)

[204] may be expressed as follows

$$
\Phi'(x) = e^{+iA/2} \exp (+i\gamma^5 A/2) \Phi(x).
$$

(204)

Those are Abelian analogues of

$$
\tilde{N}_A^C = (i) \exp \left[ -i A \bar{\sigma} \bar{n}_{\theta,\phi} \right] \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P},
$$

(205)

$$
U^C. = e^{iA/2} \exp \left[ -i \frac{A}{2} \bar{\sigma} \bar{n}_{\theta,\phi} \right].
$$

(206)

This symmetry leads to the $A$-ambiguity (6.5) and permits to choose an arbitrary reflection operator from the totality $\{\tilde{N}_A\}$. 

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