Expansions of the Riemann Zeta function in the critical strip

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Introduction

We introduce the functions defined for $t \in ]0, +\infty[$ by

$$
\Psi_m(t) = \sqrt{2} \left( \frac{t^2 - 1}{t^2 + 1} \right)^m \frac{1}{\sqrt{1 + t^2}}
$$

where $m \geq 0$ is an integer. Their Mellin transform are

$$
\mathcal{M}(\Psi_m)(s) = \int_0^{+\infty} t^{s-1} \Psi_m(t) dt = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) Q_m(s)
$$

where $Q_m$ are polynomials in $\mathbb{R}[X]$ with their roots on the line $Re(s) = 1/2$. We use these functions $\Psi_m$ to get the expansion

$$
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} - 1 - \frac{1}{t} = \sum_{m \geq 0} \alpha_{2m} \Psi_{2m}(t) \quad \text{for } t \in ]0, +\infty[
$$

with

$$
\alpha_{2m} = \frac{2^{-4m}}{(2m)!} \left( \sum_{n \in \mathbb{Z}} H_{4m}(\sqrt{2\pi} n) e^{-\pi n^2} - 2 \frac{(4m)!}{(2m)!} \right)
$$

where $H_n$ are the Hermite polynomials.

In the strip $0 < Re(s) < 1$ the well-known classical result

$$
\int_0^{+\infty} t^{s-1} \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} - 1 - \frac{1}{t} \right) dt = \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s)
$$

allows us to conjecture the following expansion of Zeta for $0 < Re(s) < 1$

$$
\zeta(s) = \frac{1}{\sqrt{2\pi}} \pi^{\frac{s}{2}} \Gamma\left(\frac{1-s}{2}\right) \sum_{m \geq 0} \alpha_{2m} Q_{2m}(s)
$$

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1 Functions related to the quantum harmonic oscillator

1.1 Hermite functions

For every integer $m \geq 0$ let us consider the Hermite function

$$\Phi_m(x) = H_m(\sqrt{2\pi}x)e^{-\pi x^2}$$

where $H_m \in \mathbb{R}[X]$ are the Hermite polynomials defined by the generating function

$$e^{-t^2+2xt} = \sum_{m \geq 0} \frac{H_m(x)}{m!} t^m$$

or directly by $H_m(x) = (-1)^m e^{x^2} \partial^m e^{-x^2}$.

The Hermite functions $\Phi_m \in L^2(\mathbb{R})$ are known to form an orthogonal system of eigenfunctions of the quantum harmonic oscillator

$$2\pi(x^2 - \frac{1}{4\pi^2} \partial^2)\Phi_m = (2m + 1)\Phi_m \quad \text{with} \quad \int_\mathbb{R} (\Phi_m(x))^2 dx = \frac{1}{\sqrt{2}} 2^m m!$$

The function $\Phi_m$ has same parity as $m$. We are interested with the even functions $\Phi_{2m}$, we have (cf. [4])

$$\int_\mathbb{R} e^{-2i\pi x\xi} \Phi_{2m}(x) dx = (-1)^m \Phi_{2m}(\xi)$$

Thus for $\xi = 0$

$$\int_\mathbb{R} \Phi_{2m}(x) dx = (-1)^m \Phi_{2m}(0) = \frac{(2m)!}{m!}$$

The function $\Phi_{2m}$ is bounded (cf. [4]) by

$$B1) \quad |\Phi_{2m}(x)| \leq K 2^m \sqrt{(2m)!} \quad \text{with} \quad K = 1.086435 \quad \text{for} \quad x \in \mathbb{R}$$

The function $\Phi_{2m}$ is (cf. [5]) oscillating in the interval

$$I_m = \left[-\frac{1}{\sqrt{\pi}} \sqrt{2m+1}, \frac{1}{\sqrt{\pi}} \sqrt{2m+1}\right]$$

and exponentially decreasing when $x \notin I_m$, more precisely (cf. [4]) we have

$$B2) \quad |\Phi_{2m}(x)| \leq \frac{(2m)!}{m!} e^{2x\sqrt{2\pi m}} e^{-\pi x^2} \quad \text{for} \quad x > 0$$
In the series expansions of the following sections we use the normalized sums

\[ S_{2m} = \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} \Phi_{2m}(n) \]

**Lemma 0**

For \( m \to +\infty \) we have

\[ \frac{2^{-2m}}{m!} \sum_{|n| \geq 2\sqrt{2m}} \Phi_{2m}(n) = O(e^{-2\pi\sqrt{2m}}) \]

and

\[ \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} \Phi_{2m}(n) = O(m^{1/4}) \]

**Proof**

We have for \( m \geq 1 \)

\[ 2x\sqrt{2\pi m} - \pi x^2 \leq -\pi x \text{ for } x \geq 2\sqrt{2m} \]

thus using inequality (B2) we get

\[ \left| \frac{2^{-2m}}{m!} \Phi_{2m}(x) \right| \leq \frac{2^{-2m}(2m)!}{(m!)^2} e^{-\pi|x|} \text{ for } |x| \geq 2\sqrt{2m} \]

Thus by summation for \(|n| \geq 2\sqrt{2m}\) and with the Stirling formula we get

\[ \frac{2^{-2m}}{m!} \sum_{|n| \geq 2\sqrt{2m}} \Phi_{2m}(n) \leq \frac{2^{-2m+1}(2m)!}{(m!)^2} \frac{e^{-\pi(2\sqrt{2m})}}{1 - e^{-\pi}} = O(e^{-2\pi\sqrt{2m}}) \]

From inequality (B1) we deduce that

\[ \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} \Phi_{2m}(n) = \sum_{|n| < 2\sqrt{2m}} \frac{2^{-2m}}{m!} \Phi_{2m}(n) + \frac{2^{-2m}}{m!} \sum_{|n| \geq 2\sqrt{2m}} \Phi_{2m}(n) \]

\[ \leq K \sqrt{2m} \frac{2^{-m+1}(2m)!}{m!} + O(e^{-\pi\sqrt{2m}}) \]

thus by Stirling formula we get \( \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} \Phi_{2m}(n) = O(m^{1/4}). \)

\( \square \)
1.2 The functions $\Psi_m$

The function $x \mapsto e^{-2\pi a^2 x^2}$, $\text{Re}(a^2) > -\frac{1}{2}$, expands (cf. [6] p.71-75) as the following series of Hermite polynomials, for $x \in \mathbb{R}$ we have

$$e^{-2\pi a^2 x^2} = \frac{1}{\sqrt{1 + a^2}} \sum_{m \geq 0} \frac{(-1)^m a^{2m}}{2^{2m}(1 + a^2)^m m!} H_{2m}(\sqrt{2\pi x})$$

Multiplying by $e^{-\pi x^2}$ we get, with $t^2 = 1 + 2a^2$

$$e^{-\pi x^2 t^2} = \sum_{m \geq 0} (-1)^m \frac{1}{2^{2m}} \Phi_{2m}(x) \frac{\Psi_m(t)}{m!} \enspace \text{for } \text{Re}(t^2) > 0 \quad (1)$$

where we define for $t \in S = \{ re^{i\theta} \mid r > 0, -\pi/4 < \theta < \pi/4 \}$ the function

$$\Psi_m(t) = \sqrt{2} \left( \frac{t^2 - 1}{t^2 + 1} \right)^m \frac{1}{\sqrt{1 + t^2}}$$

**Lemma 1**

The functions $\Psi_m$ are related to the Hermite functions by

$$\frac{\Psi_m}{m!} = \left( \frac{2\sqrt{2}}{x} e^{-\pi/x^2} \right) \ast \frac{\Phi_{2m}}{(2m)!}$$

where $\ast$ is the multiplicative convolution of functions defined on $]0, +\infty[$

$$(f \ast g)(t) = \int_0^{+\infty} f\left( \frac{t}{x} \right) g(x) \frac{1}{x} dx$$

**Proof**

With the classical relation

$$\int_{-\infty}^{+\infty} e^{-ax^2} e^{bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \enspace \text{where } a > 0, \ b \in \mathbb{C}$$

we get

$$e^{\frac{t^2}{t^2+1}} \frac{t}{\sqrt{1 + t^2}} = \int_{-\infty}^{+\infty} e^{-\pi x^2 \frac{t^2}{t^2+1}} e^{-\frac{z^2}{2} + 2\sqrt{2\pi} x z} dx$$
and using the power series expansion
\[ e^{-z^2+2\sqrt{2\pi xz}} = \sum_{m \geq 0} \frac{z^m}{m!} H_m(\sqrt{2\pi x}) \]
we get by identification
\[ \frac{1}{m!} \sqrt{2} \left( \frac{t^2 - 1}{t^2 + 1} \right)^m \frac{t}{\sqrt{1 + t^2}} = 2\sqrt{2} \int_0^{+\infty} e^{-\pi x^2 t^2 + 1} \frac{1}{(2m)!} H_{2m}(\sqrt{2\pi x}) \, dx \]
This gives
\[ \frac{1}{m!} \Psi_m(t) = 2\sqrt{2} \int_0^{+\infty} e^{-\pi x^2/t^2} \frac{1}{t} \frac{1}{(2m)!} \Phi_{2m}(x) \, dx \]
and we see that this last integral is the multiplicative convolution
\[ (f \ast g)(t) = \int_0^{+\infty} f\left( \frac{t}{x} \right) g(x) \frac{1}{x} \, dx \]
with \( f(x) = 2\sqrt{2} e^{-\pi/x^2} \frac{1}{x} \) and \( g(x) = \frac{\Phi_{2m}(x)}{(2m)!} \).
\[ \square \]

2 Series expansions

Let \( z \mapsto \sqrt{z} \) the principal determination of the square root, the holomorphic function
\[ u \mapsto t = \frac{\sqrt{1 + u}}{1 - u} \]
maps the open unit disk \( D(0, 1) = \{ z \in \mathbb{C} \mid |z| < 1 \} \) onto the sector
\[ S = \{ re^{i\theta} \mid r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4} \} \]
For any function \( f \) holomorphic in \( S \) let us define the function
\[ Tf(u) = \frac{1}{\sqrt{1 - u}} f\left( \frac{\sqrt{1 + u}}{1 - u} \right) \]
which is holomorphic in the open disk \( D(0, 1) \).
For every integer \( m \geq 0 \) we verify immediately that we have
\[ T\Psi_m(u) = u^m \]
For a function $f$ defined on $S$ the expansion

$$f(t) = \sum_{m \geq 0} a_m \frac{\Psi_m(t)}{m!}$$

follows the Taylor expansion of $Tf$

$$Tf(u) = \sum_{m \geq 0} \frac{a_m}{m!} u^m$$

**Remark.** Note that $\Psi_m(t) = (-1)^m \frac{1}{t} \Psi_m(\frac{1}{t})$ for all $t \in S$. For a function $f$ on $S$ the relation

$$f(t) = \frac{1}{t} f\left(\frac{1}{t}\right)$$

is equivalent to the parity of the function $Tf$

$$Tf(u) = Tf(-u)$$

in this case the expansion of $f$ is of the form

$$f(t) = \sum_{m \geq 0} a_{2m} \frac{\Psi_{2m}(t)}{(2m)!}$$

**Example**

For the function $f = \frac{1}{1+t}, t \in S$, we have

$$Tf(u) = \frac{\sqrt{1+u} - \sqrt{1-u}}{2u}$$

This gives for $t \in S$

$$\frac{1}{1+t} = \frac{1}{2} \sum_{m \geq 0} \frac{(4m)!}{2^{4m}(2m+1)!} \frac{\Psi_{2m}(t)}{(2m)!}$$

(2)
2.1 Expansion of the theta function

The theta function defined for $t \in S$ by

$$G(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2}$$

is holomorphic in $S$ and we have for $u \in D(0,1)$

$$TG(u) = \frac{1}{\sqrt{1-u}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 \frac{1-u}{1+u}}$$

Let

$$TG(u) = \sum_{m \geq 0} g_m \frac{1}{m!} u^m$$

be the power series expansion of the holomorphic function $TG$ in the open disk $D(0,1)$. The Jacobi identity (cf. [3])

$$\frac{1}{t} G\left(\frac{1}{t}\right) = G(t)$$

gives the parity of $TG$ and we get

$$TG(u) = \sum_{n \geq 0} g_{2m} \frac{1}{(2m)!} u^{2m}.$$ 

Thus we have for $t \in S$

$$G(t) = \sum_{m \geq 0} g_{2m} \frac{\Psi_{2m}(t)}{(2m)!}$$

**Lemma 2**

We have for $t \in S = \{re^{i\theta} | r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$

$$G(t) = \sum_{m \geq 0} S_{4m} \Psi_{2m}(t) \quad \text{where} \quad S_{4m} = \frac{2^{-4m}}{(2m)!} \sum_{n \in \mathbb{Z}} \Phi_{4m}(n)$$

**Proof**

Take the relation (1) with $x = n \in \mathbb{Z}$, by summation we get

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} = \sum_{n \in \mathbb{Z}} \sum_{m \geq 0} \frac{(-1)^m}{m!} 2^{-2m} \Phi_{2m}(n) \Psi_m(t)$$

$$= \sum_{m \geq 0} \frac{(-1)^m}{m!} 2^{-2m} \Psi_m(t) \sum_{n \in \mathbb{Z}} \Phi_{2m}(n)$$

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To justify the interchange of summations \( \sum_{n \in \mathbb{Z}} \sum_{m \geq 0} = \sum_{m \geq 0} \sum_{n \in \mathbb{Z}} \) we observe that
\[
\left| \frac{t^2 - 1}{t^2 + 1} \right| < 1 \text{ for } t \in S
\]
and by Lemma 0 we have \( \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} |\Phi_{2m}(n)| = O(m^{1/4}) \) thus for \( t \in S \)
\[
\sum_{m \geq 0} \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} |\Phi_{2m}(n)||\Psi_m(t)| < +\infty
\]
This gives
\[
G(t) = \sum_{m \geq 0} (-1)^m S_{2m} \Psi_m(t)
\]
Since we have seen that the function \( TG \) is even we deduce that in this last sum, only the constants \( S_{4m} \) are non zero. 
\[\square\]

**Remark**

We have also (cf. Appendix) for the constants \( S_{4m} \) another expression
\[
S_{4m} = \frac{(\pi/2)^{2m}}{(2m)! S_0 \sqrt{2}} \sum_{(k,l) \in \mathbb{Z}^2} (-1)^k e^{-\pi(4k^2+l^2)} (k + il)^{4m}
\]

**Theorem**

For \( t \in S = \{ re^{i\theta} | r > 0, -\pi/4 < \theta < \pi/4 \} \) we have
\[
G(t) - 1 - \frac{1}{t} = \sum_{m \geq 0} \alpha_{2m} \Psi_{2m}(t) \text{ with } \alpha_{2m} = S_{4m} - \frac{2^{-4m+1}(4m)!}{(2m)!(2m)!}
\]

**Proof**

Using Lemma 2, to get the expansion of \( G(t) - 1 - \frac{1}{t} \) in terms of \( \Psi_m(t) \) it is now sufficient to expand \( 1 + \frac{1}{t} \). For \( f(t) = 1 + \frac{1}{t} \) one has
\[
Tf(u) = \frac{1}{\sqrt{1+u}} + \frac{1}{\sqrt{1-u}}
\]
and we obtain for \( t \in S \)
\[
1 + \frac{1}{t} = 2 \sum_{m \geq 0} \frac{(4m)!}{2^{4m}(2m)! (2m)!} \Psi_{2m}(t)
\]
\[\square\]
Remark
We see that
\[ \alpha_{2m} = \frac{2^{-4m}}{(2m)!} \left( \sum_{n \neq 0} \Phi_{4m}(n) - [\Phi_{4m}(0) + \int_{\mathbb{R}} \Phi_{4m}(x) dx] \right) \]

This is easily explained if we look at the general Müntz formula (cf. [7]):
let \( F \) be an even continuously differentiable function such that \( F \) and \( F' \) are \( O(x^{-a}), (a > 1) \) when \( x \to \infty \), then for \( 0 < \text{Re}(s) < 1 \) we have
\[ 2 \zeta(s) \mathcal{M}F(s) = \mathcal{M} \left( G(t) - [F(0) + \int_{\mathbb{R}} F(xt) dx] \right)(s) \]

with \( G(t) = \sum_{n \in \mathbb{Z}} F(nt) \), this is our case with \( F(x) = e^{-\pi x^2} \).
If there exist functions a sequence of functions \( \varphi_m \) and \( \psi_m \) such that we have an expansion
\[ F(xt) = \sum_{m \geq 0} \varphi_m(x) \psi_m(t) \]
then, at least formally, we get
\[ G(t) - [F(0) + \int_{\mathbb{R}} F(xt) dx] = \sum_{m \geq 0} \left( \sum_{n \in \mathbb{Z}} \varphi_m(n) - [\varphi_m(0) + \int_{\mathbb{R}} \varphi_m(x) dx] \right) \psi_m(t) \]
in our case \( \varphi_m = \frac{2^{-4m}}{(2m)!} \Phi_{4m} \) and \( \psi_m(t) = \Psi_{2m}(t) \).

3 Mellin transforms

3.1 The polynomials \( Q_m \)
For \( \text{Re}(s) > 0 \), the Mellin transforms of the Hermite functions \( \Phi_{2m} \) are
\[ \int_{0}^{+\infty} \frac{\Phi_{2m}(x)}{(2m)!} x^{s-1} dx = \frac{1}{2} \pi^{-s/2} \Gamma(s/2) \frac{Q_m(s)}{m!} \]
where \( Q_m \) are polynomials in \( \mathbb{R}[X] \). This is simply a consequence of the relation
\[ \int_{0}^{+\infty} e^{-\pi x^2} x^{s+2k-1} dx = \frac{1}{2} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \pi^{-k} \frac{s}{2} \Gamma(\frac{s}{2} + 1) \ldots (\frac{s}{2} + k - 1) \]
We get \( Q_0(s) = 1, Q_1(s) = 2s - 1, Q_2(s) = \frac{4}{3}s^2 - \frac{4}{3}s + 1, \ldots \).

More generally we have

\[
Q_m(s) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{(m-k)! (2k)!} s(s+2)...(s+2(k-1))
\]

and (cf. [4]) an expression of \( Q_m \) in terms of the hypergeometric function

\[ Q_m(s) = \left(-1\right)^m \, _2F_1\left(-m, s/2; 1/2; 2\right) \]

The roots of \( Q_m \) are on the line \( \text{Re}(s) = 1/2 \) (cf. [1], [2]). This can be proved (cf. [1]) by observing that the orthogonality relation of the Hermite functions \( \Phi_{2m} \) implies the orthogonality of the family of polynomials

\[ t \mapsto Q_m\left(\frac{1}{2} + it\right) \]

with respect to the Borel measure \( |\Gamma(\frac{1}{4} + i\frac{t}{2})|^2 dt \) on \( \mathbb{R} \).

More explicitly, using the Parseval’s formula for Mellin transform

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} (\mathcal{M}(f)\mathcal{M}(g))\left(\frac{1}{2} + it\right) dt = \int_0^{+\infty} (f\overline{g})(x) dx
\]

we get

\[
\frac{1}{4\pi \sqrt{\pi}} \int_\mathbb{R} |\Gamma(\frac{1}{4} + i\frac{t}{2})|^2 \left(\frac{Q_{m_1}}{m_1!} \frac{Q_{m_2}}{m_2!}\right)\left(\frac{1}{2} + it\right) dt = \int_\mathbb{R} \left(\frac{\Phi_{2m_1}}{(2m_1)!} \frac{\Phi_{2m_2}}{(2m_2)!}\right)(x) dx
\]

### 3.2 Mellin transform of \( \Psi_m \)

**Lemma 3**

For \( 0 < \text{Re}(s) < 1 \) we have

\[
\int_0^{+\infty} t^{s-1} \Psi_m(t) dt = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) Q_m(s)
\]

**Proof**

By Mellin transform of the relation of Lemma 1, we get

\[
\int_0^{+\infty} t^{s-1} \frac{1}{m!} \Psi_m(t) dt = \left( \int_0^{+\infty} 2\sqrt{2} e^{-\pi/4} u^{s-1} du \right) \left( \int_0^{+\infty} \frac{1}{(2m)!} \Phi_{2m}(x)x^{s-1} dx \right)
\]
that is
\[
\int_0^{+\infty} t^{s-1} \frac{1}{m!} \Psi_m(t) dt = \sqrt{2\pi} \frac{(-1)^{s}}{2} \int_0^{+\infty} \frac{1}{(2m)!} \Phi_{2m}(x)x^{s-1} dx
\]

\[\square\]

**Remark**

Using
\[
\frac{1}{t} \Psi_m\left(\frac{1}{t}\right) = (-1)^m \Psi_m(t)
\]

we get with the change of variable \( t \mapsto \frac{1}{t} \)

\[
\int_0^{+\infty} t^{s-1} \frac{1}{m!} \Psi_m(t) dt = (-1)^m \int_0^{+\infty} t^{-s} \frac{1}{m!} \Psi_m(t) dt
\]

for \( 0 < Re(s) < 1 \).

By the preceding lemma this gives
\[
Q_m(1-s) = (-1)^m Q_m(s)
\]

As a consequence of this relation we see that for \( s = \frac{1}{2} + it \) the polynomials \( t \mapsto Q_{2m}(\frac{1}{2} + it) \) are in \( \mathbb{R}[X] \).

### 3.3 Expansion of Mellin transforms in terms of the polynomials \( Q_m \)

If we have for a function \( f \) holomorphic in \( S \) an expansion

\[
f(t) = \sum_{m \geq 0} a_{2m} \frac{\Psi_{2m}(t)}{(2m)!}
\]

and if we can evaluate the Mellin transform of \( f \) for \( 0 < Re(s) < 1 \) by integration of the terms of the series:

\[
\int_0^{+\infty} \left( \sum_{m \geq 0} a_{2m} \frac{\Psi_{2m}(t)}{(2m)!} \right) t^{s-1} dt = \sum_{m \geq 0} \frac{a_{2m}}{(2m)!} \int_0^{+\infty} t^{-s} \Psi_{2m}(t) dt
\]

then we get for \( 0 < Re(s) < 1 \)

\[
\int_0^{+\infty} f(t)t^{s-1} dt = \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{s}{2}\right)}{2} \Gamma\left(\frac{1-s}{2}\right) \sum_{m \geq 0} \frac{a_{2m}}{(2m)!} Q_{2m}(s)
\]
A simple condition to justify this calculation is

\[ \sum_{m \geq 0} \frac{|a_{2m}|}{(2m)!} < +\infty \]

Since in this case we have for \(0 < \text{Re}(s) < 1\)

\[
\int_0^{+\infty} \sum_{m \geq 0} |t^{s-1} a_{2m} \Psi_{2m}(t) \Psi_{2m}(t) (2m)!| dt \leq \sum_{m \geq 0} \frac{|a_{2m}|}{(2m)!} \int_0^{+\infty} t^{\text{Re}(s)-1} \frac{\sqrt{2}}{\sqrt{1 + t^2}} dt < +\infty
\]

**Example**

We have by relation (2)

\[
\frac{1}{1 + t} = \frac{1}{2} \sum_{m \geq 0} a_{2m} \Psi_{2m}(t) (2m)! \quad \text{with} \quad a_{2m} = \frac{(4m)!}{2^{4m}(2m + 1)!}
\]

By Stirling formula we have \(a_{2m} = O(m^{-\frac{3}{2}})\) thus \(\sum_{m \geq 0} \frac{|a_{2m}|}{(2m)!} < +\infty\).

And for \(0 < \text{Re}(s) < 1\) we get

\[
\frac{\pi}{\sin(\pi s)} = \frac{1}{2\sqrt{2\pi}} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{1 - s}{2} \right) \sum_{m \geq 0} \frac{(4m)!}{(2m)!(2m + 1)!} 2^{-4m} Q_{2m}(s)
\]

### 3.4 A conjecture for an expansion of Zeta in the critical strip

For \(0 < \text{Re}(s) < 1\) it is known (cf. [3]) that the Mellin transform of the function

\[ t \mapsto G(t) - 1 - \frac{1}{t} \]

is

\[
\int_0^{+\infty} t^{s-1} (G(t) - 1 - \frac{1}{t}) dt = \Gamma \left( \frac{s}{2} \right) \pi^{-s/2} \zeta(s)
\]

We have seen in 2.1 that

\[
G(t) - 1 - \frac{1}{t} = \sum_{m \geq 0} \alpha_{2m} \Psi_{2m}(t) \quad \text{with} \quad \alpha_{2m} = S_{4m} - \frac{2^{-4m+1}(4m)!}{(2m)!(2m)!}
\]

If we proceed by integration of the terms of the preceding series we get

\[
\Gamma \left( \frac{s}{2} \right) \pi^{-s/2} \zeta(s) = \frac{1}{\sqrt{2\pi}} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{1 - s}{2} \right) \sum_{m \geq 0} \alpha_{2m} Q_{2m}(s)
\]
Unfortunately it seems that in this case \( \sum_{m \geq 0} |\alpha_{2m}| = +\infty \) and the justification of the preceding section does not work.

**Conjecture**

For \( 0 < Re(s) < 1 \) the evaluation of the Mellin transform of \( G(t) - 1 - \frac{1}{t} \) by integration of the terms of the preceding series is valid and we get

\[
\zeta(s) = \frac{1}{\sqrt{2\pi}} \frac{\pi^{\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)} \sum_{m \geq 0} \alpha_{2m} Q_{2m}(s)
\]

with \( \alpha_{2m} = S_{4m} - \frac{2^{-4m+1}(4m)!}{(2m)!(2m)!} \)

As we have seen the polynomials

\[ Q_{2m}(s) = 2F_1(-2m, s/2; 1/2; 2) \]

are related to Mellin transforms of the Hermite functions \( \Phi_{4m} \) and they have their roots on the line \( Re(s) = 1/2 \).

**Remarks**

1) For the Riemann-Hardy function (cf. [3]) defined for \( t \in \mathbb{R} \) by

\[
Z(t) = \pi^{-\frac{it}{2}} \frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)} \zeta\left(\frac{1}{2} + it\right)
\]

the preceding conjecture gives

\[
Z(t) = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 0} \alpha_{2m} f_{2m}(t)
\]

where the functions

\[
f_{2m}(t) = \pi^\frac{1}{4} |\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)| Q_{2m}\left(\frac{1}{2} + it\right)
\]

are orthogonal in \( L^2(\mathbb{R}) \).
2) Other expressions of $\zeta(s)$ in the critical strip are obtained by the use of the Müntz formula (cf. [7]): for a continuously differentiable function $F$ on $[0, +\infty[$ such that $F$ and $F'$ are $O(x^{-a})$, ($a > 1$) when $x \to \infty$, we have for $0 < \text{Re}(s) < 1$

$$\zeta(s) \mathcal{M}F(s) = \mathcal{M}\left(\sum_{n \geq 1} F(nt) - \frac{1}{t} \int_0^{+\infty} F(x)dx\right)(s)$$

We now show that, with our preceding method, we can obtain a simple expansion of Zeta in the critical strip by applying this formula to the function $F(x) = e^{-2\pi x}$.

For $0 < \text{Re}(s) < 1$ we have

$$(2\pi)^{-s}\Gamma(s)\zeta(s) = \mathcal{M}\left(\sum_{n \geq 1} e^{-2\pi n}t - \frac{1}{2\pi t}\right) = \int_0^{+\infty} f(t)t^{s-1}dt$$

where

$$f(t) = \frac{1}{e^{2\pi t} - 1} - \frac{1}{2\pi t}$$

It is possible to get an expansion of $f(t)$ using the Laguerre functions

$$\varphi_m(x) = e^{-x}L_m(2x)$$

defined by the generating function

$$\frac{1}{1-u}e^{-x\frac{1+u}{1-u}} = \sum_{m \geq 0} e^{-x}L_m(2x)u^m.$$ 

These functions are orthogonal in $L^2([0, +\infty[)$ and if $J_0$ is the Bessel function of order 0 then (cf.[6])

$$\int_0^{+\infty} J_0(2\sqrt{\xi x})\varphi_m(x)dx = (-1)^m\varphi_m(\xi)$$

For $t > 0$ we set

$$\psi_m(t) = \left(\frac{t - 1}{t + 1}\right)^m \frac{2}{1 + t}$$

Using the generating function of the $\varphi_m$ we get

$$e^{-2\pi xt} = \sum_{m \geq 0} \varphi_m(2\pi x)\psi_m(t)$$  (3)
Summing (3) for \( x = n \geq 1 \) we have formally for \( \Re(t) > 0 \)

\[
\frac{1}{e^{2\pi t} - 1} = \sum_{m \geq 0} s_m \psi_m(t) \quad \text{with} \quad s_m = \sum_{n \geq 1} \varphi_m(2\pi n)
\]

Since \( \frac{1}{2\pi} = \frac{1}{2\pi} \sum_{m \geq 0} (-1)^m \psi_m(t) \) we get for \( \Re(t) > 0 \)

\[
f(t) = \sum_{m \geq 0} \left( s_m - (-1)^m \frac{1}{2\pi} \right) \psi_m(t)
\]

(4)

The functions \( \psi_m \) and \( \varphi_m \) are related by the multiplicative convolution

\[
\psi_m = 2(-1)^m (e^{-\frac{1}{x}})^* \varphi_m
\]

(5)

The Mellin transform of \( \varphi_m \) is (cf. [4]) for \( \Re(s) > 0 \)

\[
\int_0^{+\infty} \varphi_m(x) x^{s-1} dx = \Gamma(s) q_m(s)
\]

where \( q_m \) is the polynomial \( q_m(s) = 2 F_1(-m, s; 1; 2) \).

By the orthogonality relation of the \( \varphi_m \) we deduce that the polynomials \( t \mapsto q_m(\frac{1}{2} + it) \) are orthogonal with respect to the Borel measure \( \lvert \Gamma(\frac{1}{2} + it) \rvert^2 dt \)
on \([0, +\infty[. Thus \( q_m \) has his roots on the line \( \Re(s) = \frac{1}{2} \).

By (5) for \( 0 < \Re(s) < 1 \) we have the Mellin transform of \( \psi_m \)

\[
\mathcal{M}(\psi_m)(s) = 2 \Gamma(s) \Gamma(1 - s)(-1)^m q_m(s)
\]

By Mellin transform of (4) we get formally

\[
\zeta(s) = 2(2\pi)^s \Gamma(1 - s) \sum_{m \geq 0} \left( (-1)^m s_m - \frac{1}{2\pi} \right) q_m(s)
\]

A more simple expansion can be obtained using the Mellin transform of the function \( g(t) = \frac{1}{t} f(\frac{1}{t}) \).

Using the Poisson formula we have

\[
g(t) = \frac{1}{t} \sum_{n \geq 1} e^{-2\pi \frac{n}{t}} - \frac{1}{2\pi} = \frac{1}{\pi} \sum_{n \geq 1} \frac{1}{1 + n^2 t^2} - \frac{1}{2t}
\]
By Müntz formula we get

\[ M(g)(s) = M\left(\frac{1}{\pi} \sum_{n \geq 1} \frac{1}{1 + n^2 t^2} - \frac{1}{2t}\right)(s) = \zeta(s) \frac{1}{2\pi} \Gamma\left(\frac{s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) \]

We now apply our preceding method, to the function \( F(x) = \frac{1}{\pi} \frac{1}{1+x^2} \).

We verify immediately that

\[ \frac{1}{\pi} \frac{1}{1 + x^2 t^2} = \frac{1}{2\pi} \sum_{m \geq 0} (-1)^m \psi_m(x^2) \psi_m(t^2) \quad (6) \]

We have for \( t > 0 \)

\[ \frac{1}{\pi} \sum_{n \geq 1} \frac{1}{1 + n^2 t^2} = \frac{1}{2\pi} \sum_{m \geq 0} (-1)^m \sigma_m \psi_m(t^2) \text{ where } \sigma_m = \sum_{n \geq 1} \psi_m(n^2) \]

With \( u = \frac{t^2 - 1}{t^2 + 1} \) we have \( \frac{1}{t} = \frac{2}{1+t^2} (1 - u^2)^{-1/2} \), thus we get

\[ \frac{1}{t} = \sum_{m \geq 0} c_m \psi_m(t) \text{ with } c_{2n} = \frac{(2n)!}{2^{2n} (n!)^2} \text{ and } c_{2n+1} = 0 \]

Finally we have the expansion

\[ g(t) = \frac{1}{2\pi} \sum_{m \geq 0} (-1)^m (\sigma_m - \pi c_m) \psi_m(t^2) \quad (7) \]

By Mellin transform of (7) we get formally

\[ \zeta(s) = \sum_{m \geq 0} \left( \sigma_m - \pi c_m \right) q_m\left(\frac{s}{2}\right) \]

Note that, unlike the preceding expansions related to Hermite and Laguerre functions, in this expansion the sequence

\[ m \mapsto \sigma_m - \pi c_m = \sum_{n \geq 1} \left( \frac{n^2 - 1}{n^2 + 1} \right)^m \frac{2}{1 + n^2} - \pi c_m \]

has a very regular oscillation with amplitude near \( \sqrt{\pi} \frac{1}{\sqrt{2} \sqrt{m}} \), but the polynomials \( s \mapsto q_m\left(\frac{s}{2}\right) \) have their roots on the line \( \Re(s) = 1 \).


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4 Appendix. Another expression for the constants $S_{4m}$

Using Poisson summation formula we deduce that for $\varphi \in \mathcal{S}(\mathbb{R})$

$$
\sum_{(k,l) \in \mathbb{Z}^2} \int_{\mathbb{R}} e^{-2i\pi xk} e^{-\pi(x-l)^2} \varphi(x) dx = \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} e^{-\pi l^2} \varphi(k) = S_0 \sum_{k \in \mathbb{Z}} \varphi(k)
$$

Taking $u \in \mathbb{C}$ and $\varphi(x) = e^{-2\pi xu} e^{-\pi x^2}$, we have

$$
\int_{\mathbb{R}} e^{-2i\pi xk} e^{-\pi(x-l)^2} \varphi(x) dx = \frac{1}{\sqrt{2}} (-1)^{kl} e^{-\frac{\pi}{2} l^2} e^{i\pi u(k+il)} e^{\pi u^2 / 2}
$$

This gives for $u \in \mathbb{C}$ the relation

$$
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 + 2\pi nu - \pi u^2 / 2} = \frac{1}{S_0 \sqrt{2}} \sum_{(k,l) \in \mathbb{Z}^2} (-1)^{kl} e^{-\frac{\pi}{2} l^2} e^{i\pi u(k+il)}
$$ (8)
Let us now define for every integer $m \geq 0$
\[
T_m = \sum_{(k,l) \in \mathbb{Z}^2} (-1)^{kl} e^{-\frac{\pi}{2}((k^2+l^2)(k+il)^m}
\]

We have clearly $T_{2m+1} = 0$ since
\[
(-1)^{l} e^{-\frac{\pi}{2}((-l)^2)} (-k-il)^{2m+1} = -[(-1)^{kl} e^{-\frac{\pi}{2}(k^2+l^2)} (k+il)^{2m+1}]
\]
thus $T_{4m+1} = T_{4m+3} = 0$ and also $T_{4m+2} = 0$ because
\[
(-1)^{l} e^{-\frac{\pi}{2}(l^2)} (-k+il)^{4m+2} = -[(-1)^{kl} e^{-\frac{\pi}{2}(k^2+l^2)} (l+ik)^{4m+2}]
\]

Thus only the constants $T_{4m}$ are non zero, and by derivation with respect to $u$ of the holomorphic function defined by the right side of (6) we have
\[
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 + 2\pi nu - \pi u^2/2} = \frac{1}{S_0 \sqrt{2}} \sum_{m \geq 0} \pi^{4m} T_{4m} u^{4m} \frac{u^m}{m!}
\]

Now using the generating function of Hermite polynomials we have
\[
e^{-\pi n^2 + 2\pi nu - \pi u^2/2} = \sum_{m \geq 0} \left(\frac{\pi}{2}\right)^m \Phi_m(n) \frac{u^m}{m!}
\]

By summation with $n \in \mathbb{Z}$ of this relation we deduce that for $|u| < 1$
\[
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 + 2\pi nu - \pi u^2/2} = \sum_{m \geq 0} \left(\frac{\pi}{2}\right)^m \left(\sum_{n \in \mathbb{Z}} \Phi_m(n)\right) \frac{u^m}{m!}
\]
(the interchange of $\sum_{n \in \mathbb{Z}}$ and $\sum_{m \geq 0}$ is easily justified using Lemma 0).

Thus we have for $|u| < 1$
\[
\sum_{m \geq 0} \left(\frac{\pi}{2}\right)^m \left(\sum_{n \in \mathbb{Z}} \Phi_m(n)\right) \frac{u^m}{m!} = \frac{1}{S_0 \sqrt{2}} \sum_{m \geq 0} \pi^{4m} T_{4m} \frac{u^{4m}}{(4m)!}
\]

and by identification we get
\[
\sum_{n \in \mathbb{Z}} \Phi_{4m+2}(n) = 0 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \Phi_{4m}(n) = \frac{(2\pi)^{2m}}{S_0 \sqrt{2}} T_{4m}
\]