On Relative Length of Long Paths and Cycles in Graphs

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July 31, 2014

Abstract

Let $G$ be a graph on $n$ vertices, $p$ the order of a longest path and $\kappa$ the connectivity of $G$. In 1989, Bauer, Broersma Li and Veldman proved that if $G$ is a 2-connected graph with $d(x) + d(y) + d(z) \geq n + \kappa$ for all triples $x, y, z$ of independent vertices, then $G$ is hamiltonian. In this paper we improve this result by reducing the lower bound $n + \kappa$ to $p + \kappa$.

Key words. Hamilton cycle, dominating cycle, longest path, connectivity.

1 Introduction

Throughout this article we consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph $G$ is denoted by $V(G)$ and the set of edges by $E(G)$. A good reference for any undefined terms is [4]. For a graph $G$, we use $n$, $\delta$, $\kappa$ and $\alpha$ to denote the order (the number of vertices), the minimum degree, the connectivity and the independence number of $G$, respectively. If $\alpha \geq k$ for some integer $k$, let $\sigma_k$ be the minimum degree sum of an independent set of $k$ vertices; otherwise we let $\sigma_k = +\infty$.

Each vertex and edge in a graph can be interpreted as simple cycles of orders 1 and 2, respectively. A graph $G$ is hamiltonian if $G$ contains a Hamilton cycle, i.e. a cycle containing every vertex of $G$. A cycle $C$ of a graph $G$ is said to be dominating if $V(G \setminus C)$ is an independent set. The order of a longest path and a longest cycle in $G$ are denoted by $p$ and $c$, respectively. The difference $p - c$ is called relative length denoted by $\text{diff}(G)$. A connected graph $G$ is hamiltonian if and only if $\text{diff}(G) = 0$, that is $c = p$. It is also easy to see that if $\text{diff}(G) \leq 1$, that is $c \geq p - 1$, then any longest cycle in $G$ is a dominating cycle.

The earliest sufficient condition for a graph to be hamiltonian was developed in 1952 due to Dirac [6] in terms of order $n$ and minimum degree $\delta$.

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Theorem A [6]. Every graph with \( \delta \geq \frac{n}{2} \) is hamiltonian.

In 1960, Ore [11] improved Theorem A by replacing the minimum degree \( \delta \) with the arithmetic mean \( \frac{1}{2} \sigma_2 \) of two smallest degrees among pairwise nonadjacent vertices.

Theorem B [11]. Every graph with \( \frac{1}{2} \sigma_2 \geq \frac{n}{2} \) is hamiltonian.

The analog of Theorem A for dominating cycles was established in 1971 by Nash-Williams [9].

Theorem C [9]. If \( G \) is a 2-connected graph with \( \delta \geq \frac{n+2}{3} \) then each longest cycle in \( G \) is a dominating cycle.

In 1980, Bondy [5] proved the degree sum version of Theorem C.

Theorem D [5]. If \( G \) is a 2-connected graph with \( \frac{1}{3} \sigma_3 \geq \frac{n+2}{3} \) then each longest cycle in \( G \) is a dominating cycle.

In 1995, Enomoto, Heuvel, Kaneko and Saito [7] improved Theorem D by replacing the conclusion ”each longest cycle in \( G \) is a dominating cycle” with \( c \geq p - 1 \).

Theorem E [7]. If \( G \) is a 2-connected graph with \( \frac{1}{3} \sigma_3 \geq \frac{n+2}{3} \) then \( c \geq p - 1 \).

Using the original proof [11], Theorem B can be essentially improved by reducing the lower bound \( \frac{n}{2} \) to \( \frac{n}{2} \).

Theorem 1. If \( G \) is a connected graph with \( \frac{1}{2} \sigma_2 \geq \frac{n}{2} \) then \( c = p = n \).

Theorem E can be improved by a similar way based on a result due to Ozeki and Yamashita [12].

Theorem 2. If \( G \) is a 2-connected graph with \( \frac{1}{3} \sigma_3 \geq \frac{n+2}{3} \) then \( c \geq p - 1 \).

The minimum degree versions of Theorems 1 and 2 follow immediately.

Corollary 1. If \( G \) is a connected graph with \( \delta \geq \frac{n}{2} \) then \( c = p = n \).

Corollary 2. If \( G \) is a 2-connected graph with \( \delta \geq \frac{n+2}{3} \) then \( c \geq p - 1 \).

We propose a conjecture containing Theorems 1 and 2 as special cases when \( \lambda = 1 \) and \( \lambda = 2 \).
Conjecture 1. If $G$ is a $\lambda$-connected graph with
\[
\frac{1}{\lambda + 1} \sigma_{\lambda + 1} \geq \frac{p + 2}{\lambda + 1} + \lambda - 2
\]
then $c \geq p - \lambda + 1$.

The long cycles version of Conjecture 1 can be formulated as follows.

Conjecture 2. If $G$ is a $\lambda$-connected ($\lambda \geq 2$) graph then
\[
c \geq \min \left\{ p - \lambda + 2, \lambda \left( \frac{1}{\lambda} \sigma_{\lambda} - \lambda + 2 \right) \right\}.
\]

Conjecture 2 for $\lambda = 2$ was verified independently by Bondy [3] (1971), Bermond [2] (1976) and Linial [8] (1976).

Theorem F [2], [3], [8]. If $G$ is a 2-connected graph then either $G$ is hamiltonian or $c \geq \sigma_2$.

The minimum degree version of Theorem F was proved in 1952 by Dirac [4].

Theorem G [6]. If $G$ is a 2-connected graph then either $G$ is hamiltonian or $c \geq 2\delta$.

For $\lambda = 3$, Conjecture 2 follows immediately from the main result due to Ozeki and Yamashita [12].

Theorem H [12]. If $G$ is a 3-connected graph then either $c \geq \sigma_3 - 3$ or $c \geq p - 1$.

In 1981, the bound $n/2$ in Theorem A was reduced to $(n + \kappa)/3$ for 2-connected graphs.

Theorem I [10]. If $G$ is a 2-connected graph with $\delta \geq \frac{n + \kappa}{3}$ then $G$ is hamiltonian.

The degree sum version of Theorem I was established in 1989 due to Bauer, Broersma, Li and Veldman [1].

Theorem J [1]. If $G$ is a 2-connected graph with $\frac{1}{3} \sigma_3 \geq \frac{p + \kappa}{3}$ then $G$ is hamiltonian.

The main result of this paper can be considered as an improvement of Theorem I by reducing the bound $(n + \kappa)/3$ to $(p + \kappa)/3$.

Theorem 3. If $G$ is a 2-connected graph with $\frac{1}{3} \sigma_3 \geq \frac{p + \kappa}{3}$ then $c = p = n$. 

The minimum degree version of Theorem 3 follows immediately.

**Corollary 3.** If \( G \) is a 2-connected graph with \( \delta \geq \frac{p + \kappa}{\lambda + 1} \) then \( c = p = n \).

The following conjecture contains Theorem 3 as a special case when \( \lambda = 2 \).

**Conjecture 3.** If \( G \) is a \( \lambda \)-connected \( (\lambda \geq 2) \) graph with
\[
\frac{1}{\lambda + 1} \sigma_{\lambda + 1} \geq \frac{p + \kappa + 3}{\lambda + 1} + \lambda - 3
\]
then \( c \geq p - \lambda + 2 \).

The long cycle version of Conjecture 3 can be formulated as follows.

**Conjecture 4.** If \( G \) is a \( \lambda \)-connected \( (\lambda \geq 3) \) graph then either
\[
c \geq \lambda \left( \frac{1}{\lambda} \sigma_{\lambda} - \frac{\kappa}{\lambda} - \lambda + 3 \right)
\]
or \( c \geq p - \lambda + 3 \).

Conjecture 4 for \( \lambda = 3 \) was verified by Yamashita [13].

**Theorem K [13].** If \( G \) is a 3-connected graph then either \( c \geq \sigma_3 - \kappa \) or \( G \) is hamiltonian.

The minimum degree version of Theorem K was established by the author [10].

**Theorem L [10].** If \( G \) is a 3-connected graph then either \( c \geq 3\delta - \kappa \) or \( G \) is hamiltonian.

To prove Theorem 2, we need the following result due to Ozeki and Yamashita [12].

**Theorem M [12].** If \( G \) is a 2-connected graph then either \( c \geq p - 1 \) or \( c \geq \sigma_3 - 3 \) or \( \kappa = 2 \) and \( p \geq \sigma_3 - 1 \).

2 **Proofs**

First we introduce some additional notation.

If \( P \) is a path in a graph \( G \) then we denote by \( \vec{P} \) the path \( P \) with a given orientation, and by \( \vec{P} \) the same path with reverse orientation. If \( u, v \in V(P) \) and \( u \) precedes \( v \) on \( \vec{P} \) then \( u \vec{P} v \) denotes the consecutive vertices of \( P \) from \( u \) to \( v \). The same vertices in reverse order are given by \( v \vec{P} u \). We will consider \( u \vec{P} v \)
and $v \overrightarrow{P} u$ both as paths and as vertex sets. If $u \in V(P)$ then $u^+$ denotes the successor of $u$ on $\overrightarrow{P}$ and $u^-$ its predecessor. For $U \subseteq V(P)$, $U^+ = \{u^+ | u \in U\}$ and $U^- = \{u^- | u \in U\}$. Similar notation is used for cycles.

The proof of Theorem 1 is based on standard arguments originally proposed by Ore [11].

**Proof of Theorem 1.** Let $G$ be a connected graph with $\sigma_2 \geq p$ and let $\overrightarrow{P} = x \overrightarrow{P} y$ be a longest path in $G$ of order $p$. Clearly, $N(x) \cup N(y) \subseteq V(P)$.

**Case 1.** $xy \in E(G)$.

If $p < n$ then recalling that $G$ is connected, we can construct a path longer than $P$, a contradiction. Otherwise $p = n$, implying that $c = p = n$.

**Case 2.** $xy \notin E(G)$.

It follows that $x \notin N(x) \cup N^+(y)$. If $N(x) \cap N^+(y) = \emptyset$ then

$$p \geq |N(x)| + |N^+(y)| + |\{x\}|$$

$$= |N(x)| + |N(y)| + 1 = d(x) + d(y) + 1 \geq \sigma_2 + 1,$$

contradicting the hypothesis. Now let $N(x) \cap N^+(y) \neq \emptyset$ and $z \in N(x) \cap N^+(y)$. Then $xz \overrightarrow{P} yz \overrightarrow{P} x$ is a cycle of order $p$ and we can argue as in Case 1. ■

**Proof of Theorem 2.** Let $G$ be a 2-connected graph with $\sigma_3 \geq p + 2$. By Theorem 1, either $c \geq p - 1$ or $c \geq \sigma_3 - 3$ or $\kappa = 2$, $p \geq \sigma_3 - 1$. Recalling that $\sigma_3 \geq p + 2$ (by the hypothesis), we get either $c \geq p - 1$ or $p \geq p + 1$. Since the latter is impossible, we have $c \geq p - 1$. ■

**Proof of Theorem 3.** Let $G$ be a 2-connected graph with $\sigma_3 \geq p + \kappa$. Assume first that $\kappa \geq 3$. By Theorem 1, we can assume that $c \geq \sigma_3 - \kappa$, implying that $c \geq p$. If $c < n$ then clearly $p \geq c + 1$ (since $G$ is connected), contradicting $c \geq p$. Hence $c = p = n$, that is $G$ is hamiltonian.

Now assume that $\kappa = 2$. Since $\sigma_3 \geq p + \kappa = p + 2$, by Theorem 2, $c \geq p - 1$, implying that each longest cycle in $G$ is a dominating cycle. Let $C$ be a longest cycle in $G$.

**Case 1.** $d(x) = 2$ for some $x \in V(G\setminus C)$.

Since $C$ is a dominating cycle, we have $N(x) \subseteq V(C)$. Set $N^+_C(x) = \{y, z\}$. By the maximality of $C$, we have $xy, xz \notin E(G)$. We have also $yz \notin E(G)$, since otherwise $\overrightarrow{y \neg xzC yzC \neg y}$ is a cycle longer than $C$. Thus, $\{x, y, z\}$ is an independent set of vertices. Further, if either $N(y) \nsubseteq V(C)$ or $N(z) \nsubseteq V(C)$ then we can form a path of order at least $c + 2$, contradicting $c \geq p - 1$. Hence, $N(y) \cup N(z) \subseteq V(C)$. Put

$$A = V(y^+Cz), \quad B = V(z^+Cy).$$
If \( w \in N_A(y) \cap N_A^+(z) \) then
\[
y^{-x^{-}C \stackrel{w}{\rightarrow} y^{-}C \equiv w^{-}zC y^{-}
\]
is a cycle longer than \( C \), a contradiction. Hence \( N_A(y) \cap N_A^+(z) = \emptyset \). By a symmetric argument, \( N_B^+(y) \cap N_B(z) = \emptyset \). Then
\[
c \geq |N_A(y)| + |N_B^+(y)| + |N_A^+(z)| + |N_B(z)|
\]
= \( |N_C(y)| + |N_C(z)| = d(y) + d(z)\)
= \( d(x) + d(y) + d(z) - 2 \geq \sigma_3 - 2 \geq p.\)

**Case 2.** \( d(v) \geq 3 \) for each \( v \in V(G \setminus C).\)
Let \( S = \{v_1, v_2\} \) be a cut set of \( G \) and let \( H_1, H_2, ..., H_t \) be the components of \( G \setminus S.\)

**Case 2.1.** \( V(C) \subseteq V(H_i) \cup S \) for some \( i \in \{1, 2, ..., t\}.\)
Assume without loss of generality that \( V(C) \subseteq V(H_1) \cup S. \) Let \( u_1 \in V(H_2). \)
Since \( u_1 \not\in V(C) \), we have \( d(u_1) \geq 3. \) Then for each \( u_2 \in N(u_1) \setminus \{v_1, v_2\} \), we have \( u_1u_2 \in E(G) \) and \( u_1, u_2 \not\in V(C). \) This means that \( C \) is not a dominating cycle, a contradiction.

**Case 2.2.** \( V(C) \not\subseteq V(H_i) \cup S \ (i = 1, 2, ..., t).\)
It follows that \( V(C) \cap V(H_i) \neq \emptyset \) and \( V(C) \cap V(H_j) \neq \emptyset \) for some distinct \( i, j \in \{1, 2, ..., t\} \), say \( i = 1 \) and \( j = 2. \) Recalling also that \( |S| = 2 \), we conclude that \( V(C) \subseteq V(H_1) \cup V(H_2) \cup S \) and \( v_1, v_2 \in V(C). \) If \( t \geq 3 \) then we can argue as in Case 2.1. Hence \( t = 2. \) Clearly, \( C \) consists of two paths \( P_1 \) and \( P_2 \) with common end vertices \( v_1, v_2 \) and
\[
V(P_i) \subseteq V(H_i) \cup S \ (i = 1, 2).
\]
In other words, \( \overrightarrow{C} = v_1P_1v_2P_2v_1. \) Further, if \( V(C) = V(H_1) \cup V(H_2) \cup S \) then \( c = p = n, \) and we are done. Otherwise we can choose \( x \in V(G \setminus C). \) Since \( v_1, v_2 \in V(C) \), we have \( x \in V(H_i) \) for some \( i \in \{1, 2\}, \) say \( x \in V(H_1). \) We have \( N(x) \subseteq V(C), \) since \( C \) is a dominating cycle. Choose \( y \in N^+(x) \) such that \( |v_1P_1y| \) is as small as possible. If \( w \in N(x) \cap N^-(y) \) then
\[
v_1C y^{-}xwC ywC y y v_1
\]
is a cycle longer than \( C \), a contradiction. Hence, \( N(x) \cap N^-(y) = \emptyset, \) implying that
\[
|P_1| \geq |N(x)| + |N^-(y)| - |\{v_1\}| \geq d(x) + d(y) - 1.
\]

**Case 2.2.1.** \( V(P_2) = V(H_2) \cup S.\)
Clearly, \( |P_2| \geq |N(z)| + |\{z\}| \geq d(z) + 1 \) for each \( z \in V(H_2) \) and \( x, y, z \) is an independent set of vertices. Then
\[
c \geq |P_1| + |P_2| - |\{v_1, v_2\}|
\]

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\[
\geq (d(x) + d(y) - 1) + (d(z) + 1) - 2 \geq \sigma_3 - 2 \geq p.
\]

**Case 2.2.2.** \( V(P_2) \neq V(H_2) \cup S \).

Let \( z \in V(H_2) \setminus V(P_2) \). Since \( C \) is a dominating cycle, we have \( N(z) \subseteq V(C) \).

Then, since \( C \) is extreme, \( |P_2| \geq |N(z)| + |N^+(z)| - 1 \geq 2d(z) - 1 \). Observing also that \( \{x, y, z\} \) is an independent set of vertices, we get

\[
c \geq |P_1| + |P_2| - 2 \geq (d(x) + d(y) - 1) + (2d(z) - 1) - 2 \\
\geq (\sigma_3 - 2) + d(z) - 2 \geq \sigma_3 - 2 \geq p.
\]

\[
\]

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