AdS/CFT correspondence on torus

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Abstract
The AdS/CFT correspondence is established for the case of AdS$_3$ space compactified on a filled rectangular torus with the CFT field on the boundary.

1 Introduction

The AdS/CFT correspondence \cite{1, 2, 3} is currently under study in various respects. One may check its validity for the case of interacting fields \cite{4, 5} (multi-particle scattering in gravitation theory, etc.) but it would be also interesting to check whether it holds in cases where the space–time geometry is more involved than the spherical one.

In \cite{6}, it was proposed, starting from a two-dimensional (compact) manifold $M$, to consider a theory on the space $M \times \mathbb{R}_+$ endowed with the AdS metric. From the topological standpoint, this, however, results in a singularity as $s \in \mathbb{R} \to +\infty$ (because $M$ is not necessarily simply connected), and one must impose an additional condition on the fields of the theory (the fast decrasing at infinity) in order to make the field configuration smooth.

In this paper, we consider the case of AdS$_3 \times S^{d+1}$ space and confine ourselves to the massless case (the case where the internal degrees of freedom w.r.t. the internal compact group $S^{d+1}$, $d = 2, 3, \ldots$, are switched off). However, this is only for the integrity of the paper, and all the main stages of our consideration can be generalized to the massive modes of AdS$_n \times S^{d+1}$ space. We also consider the case of a homogeneous compactified AdS$_3$ manifold without (topological) singularity in the interior (we just have no boundary as $s \to \infty$; note also that our calculations technically resemble the bulk calculations of \cite{3}). We show that the classical scalar field theory on the AdS$_3$ manifold gives us the appropriate quantum correlators.

2 Geometry of AdS$_3$ manifolds

The group $SL(2, \mathbb{C})$ of conformal transformations of the complex plane admits the continuation to the upper half-space $\mathbb{H}^+_3$ endowed with the constant negative curvature (AdS$_3$ space). In the Schottky uniformization picture, Riemann surfaces of higher genera can be obtained from $\mathbb{C}$ by factorizing it over a finitely generated free-acting discrete subgroup $\Gamma \subset SL(2, \mathbb{C})$. Therefore, we can continue the action of this subgroup to the whole AdS$_3$ and, after factorization, obtain a three-dimensional manifold of constant negative curvature (an AdS$_3$ manifold) whose boundary is (topologically) a two-dimensional Riemann surface \cite{3, 4}.

We consider the simplest case of a genus one AdS$_3$-manifold, which can be obtained if we identify

$$(w, \overline{w}, s) \sim (qw, \overline{qw}, s|q|),$$

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where \( q = e^{a+ib} \) is the modular parameter, \( a, b \in \mathbb{R}, \ a > 0, \ w, \overline{w} = x + iy, \ x - iy \) are the coordinates on \( \mathbb{C} \), and \( s \) is the third coordinate on AdS\(_3\).

Adopting the AdS/CFT correspondence approach, we should first regularize expressions in order to make them finite. In the AdS\(_3\) case with the (brane or black hole) singularity at infinity, this was done by setting the boundary data on the \( \varepsilon \)-plane [2] rather than at infinity (zero plane). However, in our case, we cannot take an \( \varepsilon \)-plane because it is not invariant w.r.t. transformations [2]. Instead, we can set the boundary data on the \( \varepsilon \)-cone—the the set of points

\[
\frac{z}{r} = \varepsilon, \quad r^2 = u\overline{w} + s^2.
\]

Given the boundary data on this cone, we fix the problem setting—the Laplace equation then has a unique solution (the Dirichlet problem on a compact manifold) and we do not need to introduce additional constraints (as in the case of AdS\(_3\)).

Geometrically, performing the \( \varepsilon \)-cone regularization and factorization over the group \( \Gamma \) of transformations [2], we obtain the filled torus whose boundary is associated with the two-dimensional manifold (the torus) on which the CFT dwells. The “center” of the torus is now the only closed geodesic of the length \( \log |q| \) (the image of the vertical half-line \( z = \varepsilon = 0 \)), and the AdS-invariant distance \( r \) from this geodesic to the image of the \( \varepsilon \)-cone is constant, \( \cosh r = 1/\varepsilon \).

## 3 Scalar field on AdS\(_3\) in spherical coordinates

In order to operate with the cone geometry, it is convenient to reformulate the free-field problem on AdS\(_3\) in the spherical coordinates \( r \equiv e^\tau, \theta, \varphi \) in which relations [2] read

\[
(\tau, \theta, \varphi) \sim (\tau + ma, \theta, \varphi + mb + 2\pi n), \quad m, n \in \mathbb{Z}.
\]

(3.1)

On the cone \( \sin \theta = \varepsilon \), the usual toric periodic conditions are imposed on fields depending on the two-dimensional variables \( \tau, \varphi \).

Spherical coordinates are standard,

\[
\begin{align*}
    s &= r \cos \theta & \partial_s &= \cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta \\
    y &= r \sin \theta \sin \varphi & \partial_y &= \sin \theta \sin \varphi \partial_r + \frac{1}{r} \cos \theta \sin \varphi \partial_\theta + \frac{1}{r \sin \theta} \cos \varphi \partial_\varphi \\
    x &= r \sin \theta \cos \varphi & \partial_x &= \sin \theta \cos \varphi \partial_r + \frac{1}{r} \cos \theta \cos \varphi \partial_\theta - \frac{1}{r \sin \theta} \sin \varphi \partial_\varphi
\end{align*}
\]

(3.2)

The action of the massless scalar field \( \Phi \) on AdS\(_3\) is

\[
\int \frac{dx dy ds}{s^3} \{ s^2 \partial_x \Phi \partial_x \Phi + s^2 (\partial_y \Phi \partial_y \Phi + \partial_y \Phi \partial_y \Phi) \} = \int r \tan \theta dr d\theta d\varphi \left\{ \partial_r \Phi \partial_r \Phi + \frac{1}{r^2} \partial_\theta \Phi \partial_\theta \Phi + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi \Phi \partial_\varphi \Phi \right\} = \int \tan \theta dr d\theta d\varphi \left\{ \partial_r \Phi \partial_r \Phi + \partial_\theta \Phi \partial_\theta \Phi + \frac{1}{\sin^2 \theta} \partial_\varphi \Phi \partial_\varphi \Phi \right\} \tag{3.3}
\]

It admits the variable separation:

\[
\Phi(\tau, \theta, \varphi) = \sum_{k, m \in \mathbb{Z}} \Phi_l(\tau) Y_{k, m}(\sin \theta) X_m(\varphi), \tag{3.4}
\]

where

\[
X_m(\varphi) = e^{im\varphi}, \quad \partial^2 X_m = -m^2 X_m, \tag{3.5}
\]

and for the function \( Y_{k, m}(\sin \theta) \), the eigenvalue problem arises

\[
\frac{\cos \theta}{\sin \theta} \partial_\theta \left( \frac{\sin \theta}{\cos \theta} \partial_\theta Y_{k, m}(\sin \theta) \right) - \frac{m^2}{\sin^2 \theta} Y_{k, m}(\sin \theta) = \lambda_k Y_{k, m}(\sin \theta), \quad \lambda_k \geq 0. \tag{3.6}
\]

2
We substitute \( \rho \) for \( \sin \theta \) and consider the problem on the interval \( 1 - \varepsilon \geq \rho \geq 0 \) with the regularity condition at \( \rho = 0 \),

\[
Y''_{k,m}(\rho) + \frac{1}{\rho} Y'_{k,m}(\rho) - \frac{m^2}{\rho^2(1 - \rho^2)} Y_{k,m}(\rho) = \frac{\lambda_k}{1 - \rho^2} Y_{k,m}(\rho). \tag{3.7}
\]

The equation for the radial coordinate,

\[
\Phi''_k(\tau) + \lambda_k \Phi_k(\tau) = 0, \tag{3.8}
\]

determines the values of \( \lambda_k \) consistent with the periodicity conditions

\[
\Phi(\tau + \frac{\log|q|}{\epsilon}, \theta, \varphi + \arg q) = \Phi(\tau, \theta, \varphi), \tag{3.9}
\]

where \( q \) is the modular parameter of the torus,

\[
q = e^{a+ib}, \quad a, b \in \mathbb{R}. \tag{3.10}
\]

Now Eq. (3.8) can be easily solved, \( \lambda_k = -mb/a + 2\pi k/a \), and Eq. (3.7) can be reduced to the standard hypergeometric equation whose general solution that is regular at \( m \)

\[

\Phi(\tau, \theta, \varphi) = \sum_{m,k \in \mathbb{Z}} e^{im\varphi} e^{i\left(-\frac{mb + 2\pi k}{2a}\right)} C_{k,m} \times Y_{k,m}(\sin \theta),
\]

\[
Y_{k,m}(\sin \theta) = |\sin \theta|^m \binom{m}{2} + \frac{1}{i} \frac{\sin b + 2\pi k}{2a} \cdot \frac{|m| - \frac{m - mb + 2\pi k}{2a}; |m| + 1; \sin^2 \theta}{2F_1(a, b; c; z)} \tag{3.11}
\]

where \( 2F_1(a, b; c; z) \) is the hypergeometric series,

\[
2F_1(a, b; c; z) \equiv \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \quad (a)_k = \prod_{i=0}^{k-1} (a+i),
\]

and \( C_{k,m} \) are the mode amplitudes.

Expression (3.11) is singular at \( z = 1 \) and we must find its asymptotic behavior for \( z = 1 - \varepsilon \). The standard formula

\[
2F_1(a, b; c; z) = \Gamma \left[ \frac{c}{c-a}, \frac{c-a-b}{c-b} \right] 2F_1(a, b; 1 + a + b - c; 1 - z) + \]

\[
+ \Gamma \left[ \frac{c}{a}, \frac{a+b-c}{b} \right] (1-z)^{c-a-b} 2F_1(c-a, c-b; 1+c-a-b; 1-z) \tag{3.12}
\]

works for \( c - a - b \notin \mathbb{Z} \), while for \( c - a - b \in \mathbb{Z} \) another exact relation holds (see, e.g., formula 7.3.1.31 from [3]),

\[
2F_1(a, b; c; 1) \bigg|_{c=a+b+m} = \Gamma \left[ \frac{m}{a+m}, \frac{a+b+m}{b+m} \right] \sum_{k=0}^{m-1} \frac{(a)_k(b)_k}{k!(1-m)_k} (1-z)^k -
\]

\[
- \Gamma \left[ \frac{a+b+m}{a}, \frac{b+m}{b} \right] (z-1)^m \sum_{k=0}^{\infty} \frac{(a+m)_k(b+m)_k}{k!(m+k)!} (1-z)^k \times
\]

\[
\times \left[ \log(1-z) - \Psi(k+1) - \Psi(k+m+1) + 
\Psi(a+k+m) + \Psi(b+k+m) \right]. \tag{3.13}
\]

Here \( \Psi(x) \) is the logarithmic derivative of the \( \Gamma \)-function. In the massless case, we need only the case \( m = 1 \) in (3.13) and are interested in the asymptotic behavior as \( 1-z \sim \varepsilon \to +0 \).

Coefficients \( C_{k,m} \) determine the boundary values of the field \( \Phi \). Then, for action (3.3), we obtain

\[
\int_0^a d\tau \int_0^{2\pi} d\varphi \int_0^{\sin \theta = 1-\varepsilon} \frac{d\theta}{\sin \theta} \left\{ \partial_\tau \Phi \partial_\tau \Phi + \partial_\varphi \Phi \partial_\varphi \Phi + \frac{1}{\sin^2 \theta} \partial_\varphi \Phi \partial_\varphi \Phi \right\} =
\]

\[
= \int_0^a d\tau \int_0^{2\pi} d\varphi \Phi(\tau, 1-\varepsilon, \varphi) \sin \theta \frac{\partial}{\partial \sin \theta} \Phi(\tau, \sin \theta, \varphi) \bigg|_{\sin \theta = 1-\varepsilon}. \tag{3.14}
\]
Keeping only the logarithmically divergent and finite parts as $\varepsilon \to 0$, we obtain action (3.3) in the mode expansion ($C^*_{k,m} \equiv C_{-k,-m}$):

$$\sum_{k,m \in \mathbb{Z}} |C_{k,m}|^2 \left\{ |m| - 2 \left( \frac{m^2}{4} + w^2 \right) \left[ \log \varepsilon + \Psi(|m|/2 + 1 + iw) + \Psi(|m|/2 + 1 - iw) - 2\Psi(1) \right] \right\}.$$  (3.15)

Here

$$w = -\frac{mb}{2a} + \frac{\pi k}{a}, \quad q = e^{a+ib}, \quad a, b \in \mathbb{R}. \quad (3.16)$$

4 Massive modes

Including into consideration the “internal” (compact) degrees of freedom (assuming the compact manifold to be a sphere $S^{d+1}$), we obtain an additional term in the initial action (3.3),

$$\int \frac{dx \, dy \, ds}{s^3} \left\{ s^2 \partial_x \Phi \partial_x \Phi + s^2 (\partial_y \Phi \partial_y \Phi + \partial_y \Phi \partial_y \Phi) + l(l+d)\Phi^2 \right\} =$$

$$\int \tan \theta d\tau d\vartheta d\varphi \left\{ \partial_\tau \Phi \partial_\tau \Phi + \partial_\vartheta \Phi \partial_\vartheta \Phi + \frac{1}{\sin^2 \theta} \partial_\varphi \Phi \partial_\varphi \Phi + \frac{l(l+d)}{\cos^2 \theta} \Phi^2 \right\}. \quad (4.1)$$

Separating the variables as above, we obtain that the conditions on the “torus” coordinates $\varphi$ and $\tau$ are as in (3.5) and (3.8), and only the equation for the $\theta$-component is changed,

$$\cos \theta \frac{\partial_\theta}{\sin \theta} \left( \cos \theta \frac{\partial_\theta}{\sin \theta} Y_{k,m} \right) - \frac{m^2 Y_{k,m}}{\sin^2 \theta} - \lambda_k Y_{k,m} - \frac{l(l+d)Y_{k,m}}{\cos^2 \theta} = 0. \quad (4.2)$$

Let us introduce a new quantity $\rho$,

$$4\rho(\rho - 1) = l(l+d), \quad \rho > 0. \quad (4.3)$$

The solution to (4.2) that is regular at $\theta = 0$ is

$$Y_{k,m}(\theta) = (\cos \theta)^{2-2\rho}(\sin \theta)^{|m|} \, _2F_1 \left( \frac{|m|}{2}, 1 - \rho + iw, \frac{|m|}{2}, 1 - \rho - iw; |m| + 1; \sin^2 \theta \right), \quad (4.4)$$

where $w = -\frac{mb}{2a} + \frac{\pi k}{a}$.

Important particular case is $d = 2$ (AdS$_3 \times S^3$) where $\rho = l/2 + 1$ and we must use formula (3.13) rather than (3.12) in order to find the asymptotic behavior. In this case, the nonlocality is again encoded in the $\Psi$-function terms.

5 Exact Green’s function for massless modes

Turn now to expression (3.15). The term in braces is the Fourier transform of the Green’s function of the boundary CFT field on torus. We are interested in the nonlocal contribution to Green’s functions on torus coming from this formula and disregard all local terms (which can be removed by a proper renormalizing procedure). First, note that the $m$-dependence in (3.15) can be set analytic. Using the standard formulas

$$\Psi(x) = \Psi(1-x) - \pi \cot \pi x \quad \text{and} \quad \Psi(1+x) = \frac{1}{x} + \Psi(x),$$

we can rewrite the summand in (3.15) as follows:

$$|m| - 2 \left( \frac{m^2}{4} + w^2 \right) \left[ \Psi(1 + |m|/2 + iw) + \Psi(1 + |m|/2 - iw) - 2\Psi(1) \right] =$$

$$= - \left( \frac{m^2}{4} + w^2 \right) \left[ \Psi(1 + m/2 + iw) + \Psi(1 + m/2 - iw) + \Psi(1 - m/2 + iw) + \Psi(1 - m/2 - iw) - 4\Psi(1) \right], \quad (5.1)$$
where the cot-terms cancel each other because of the symmetry \( w \rightarrow -w \).

In what follows, it is useful to represent the \( \Psi \)-function using the formula

\[
\Psi(1 + a) - \Psi(1 + b) = \sum_{j=0}^{\infty} \left( \frac{1}{1 + j + a} - \frac{1}{1 + j + b} \right).
\]

Now we aim at finding the Green’s function of the Yang–Mills field insertions on the two-dimensional torus. First, note that the term \( \left( \frac{m^2}{4} + w^2 \right) \) in front of the summand is nothing but the Laplacian action on the Riemann surface, i.e., we obtain that the correlation (Green’s) function is

\[
G(\tau, \varphi) = \frac{1}{4} (\partial^2_\varphi + \partial^2_\varphi) \sum_{m, k \in \mathbb{Z}} e^{im\varphi + i[-\frac{m^2}{4} + \frac{2\pi k}{a}]} \tau \left[ 2 \log \varepsilon + \sum_{(\pm), \pm} \Psi(1(\pm)m/2 \pm i\varepsilon) - 4\Psi(1) \right] = \frac{1}{4} (\partial^2_\varphi + \partial^2_\varphi) \sum_{m, k \in \mathbb{Z}} e^{im\varphi + i[-\frac{m^2}{4} + \frac{2\pi k}{a}]} \tau \\
\times \left[ 2 \log \varepsilon + \sum_{(\pm), \pm} \sum_{l=1}^{\infty} \left( \frac{4}{l} - \frac{1}{l(\pm)m/2 \pm i[-\frac{m^2}{4} + \frac{2\pi k}{a}]} \right) \right].
\]

The sum \( \sum_{(\pm), \pm} \) in (5.2) means that we must take a sum over four terms with all possible appearances of the “+” and “−” signs. We distinguish between two appearances of \( \pm \) signs by taking one of them in parentheses. The triple sum over \( m, k, \) and \( l \) is rather involved. Note that constant terms are irrelevant to our discussion as they produce only local contributions (like \( \partial_z \partial_{\tau} \delta^2_\Pi(z, \overline{z}) \), where \( \delta^2_\Pi(z, \overline{z}) \) is the double-periodic two-dimensional delta function).

First, we take the sum over \( k, \)

\[
\sum_{k=-\infty}^{\infty} \frac{e^{2i\pi k \tau/a}}{l(\pm)m/2 + i(m/b - \frac{2\pi k}{a})} = f(\tau/a).
\]

The function \( f(\tau/a) \) is periodic under the shift \( \tau \rightarrow \tau + 1 \) and satisfy the functional equation

\[
\pm \frac{1}{2a} \partial_{\tau/a} f(\tau/a) + \left( \frac{l(\pm)m}{2} + i\frac{mb}{2a} \right) f(\tau/a) = \delta^1_\Pi(\tau/a).
\]

The (unique) solution to (5.4) that is periodic in \( \tau \) is a saw-tooth-like exponential curve

\[
f(\tau/a) = A e^{\tau^2/a} \quad \text{for} \quad \tau/a \in (0, 1),
\]

which is to be continued periodically to the whole \( \mathbb{R} \). Equation (5.4) gives

\[
\chi = \mp 2a(l(\pm)m/2) + imb \quad \text{and} \quad A = \frac{\pm 2a}{1 - e^{\mp 2a(l(\pm)m/2) + imb}}.
\]

Therefore, the remaining nonlocal terms are combined into the sum

\[
\sum_{m \in \mathbb{Z}} e^{im\varphi - im\tau/b/a} \left[ \text{const} - \sum_{(\pm), \pm} \frac{\pm 2a e^{\mp 2(l(\pm)m/2) + imb/\tau/a}}{1 - e^{\mp 2(l(\pm)m/2) + imb}} \right], \quad 0 \leq \tau < a.
\]

Next sum to be taken is (see, e.g., [1])

\[
\sum_{m \in \mathbb{Z}} \frac{e^{nw}}{1 - q^m e^u} = \frac{\theta(u - w | q) \theta(0 | q)}{\theta(u | q) \theta(0 | q)}, \quad |q| > 1, \ 0 < \text{Re} w, \text{Re} u < \log |q|.
\]
where \( \theta(w|q) \) is the standard antisymmetric theta function \( \theta_{11}(w|q) \), which reads in our convention as follows:
\[
\theta_{11}(w|q) \equiv \sum_{n \in \mathbb{Z}} q^{-(n+1/2)^2} e^{(w+\pi i)(n+1/2)}.
\]

We exhibit the dependence on the modular parameter as it can be varied as well. In particular, we have both the \( q \)- and \( \tau \)-dependences. Substituting \( \theta_{11}(w|q) \) in (5.5) and taking all sums over \( \pm \) signs, we obtain
\[
2\text{Re} \sum_{l=1}^{\infty} \left[ \text{local} - 2a e^{2il\varphi} \frac{\theta(2ilb - \tau - i\varphi|e^{a+ib})\theta'(0|e^{a+ib})}{\theta(\tau + i\varphi|e^{a+ib})\theta(2ilb|e^{a+ib})} + 2a e^{-2il\varphi} \frac{\theta(-2ilb - \tau - i\varphi|e^{a+ib})\theta'(0|e^{a+ib})}{\theta(\tau + i\varphi|e^{a+ib})\theta(-2ilb|e^{a+ib})} \right].
\]

(5.7)

Let us introduce the complex variables
\[
z = \tau + i\varphi, \quad \overline{\tau} = \tau - i\varphi.
\]

Unfortunately, we are unaware of whether sums of type (5.7) have been presented in the literature. However, we can proceed with an important particular case of a rectangular torus. This corresponds to taking the limit \( b \to 0 \), which must be nonsingular as follows from the problem setting.

In the asymptotic regime \( b \to 0 \), the leading term, which may lead to a divergency in (5.7), vanishes because of the real part condition resulting from the \( w \to -w \) invariance, \( \lim_{b \to 0} \theta(z + 2ib|q) = \{ \theta(z|e^{a}), \ z \neq 0, \ 2ib\theta'(0|e^{a}), \ z = 0 \} \), and only nonsingular terms contribute. These terms are obtained when taking the limit \( b \to 0 \) in both arguments of the function \( \theta(z|q) \):
\[
\theta_{11}(z - 2ib|e^{a+ib})|_{b \to 0} = \sum_{n \in \mathbb{Z}} e^{-\frac{i}{2}(a+ib)((n+1/2)^2 + (z-2ib+\pi)(n+1/2)} =
\[
= \left[ 1 - 2ib\frac{\partial}{\partial z} - \frac{b}{2} \frac{\partial^2}{\partial z^2} + O(b^2) \right] \theta_{11}(z|e^{a}).
\]

(5.8)

This formula is valid everywhere except the zero point, at which we obtain
\[
\theta_{11}(-2ib|e^{a+ib})|_{b \to 0} = -2ib\theta'_{11}(0|e^{a}) - ib^2\theta''_{11}(0|e^{a}) + O(b^3),
\]
\[
\theta'_{11}(0|e^{a+ib})|_{b \to 0} = \theta'_{11}(0|e^{a}) - \frac{ib}{2} \theta''_{11}(0|e^{a}) + O(b^2).
\]

(5.9)

(5.10)

Note that \( \theta^{(2n)}(0|q) = 0 \). Therefore, we obtain
\[
\text{Re} \sum_{l=1}^{\infty} \left[ \text{local} - 4a e^{2il\varphi} \frac{\left[ 1 - 2ib\partial - \frac{i}{2} \partial^2 \right] \theta(\tau + i\varphi|e^{a})\left[ \partial - \frac{ib}{2} \partial^2 \right] \theta(0|e^{a}) + 4a \{ l \to -l \} }{1 - \frac{i}{2} \partial^2} \theta(\tau + i\varphi|e^{a})\left[ -2ib\partial - \frac{i}{2} \partial^2 \right] \theta(0|e^{a}) + 4a \{ l \to -l \} \right] =
\[
= \left[ \text{keeping only the part independent on } b \right]
\]
\[
= \text{Re} \sum_{l=1}^{\infty} \left[ \text{local} - 4a e^{2il\varphi} \frac{\partial}{\partial z} \log \theta(\tau + i\varphi|e^{a}) + 4a \{ l \to -l \} \right].
\]

(5.11)

The remaining sum in \( l \) can be easily done:
\[
\sum_{l=1}^{\infty} \left( e^{2il\varphi} - e^{-2il\varphi} \right) = i \cot \varphi,
\]

(5.12)

which yields the nonlocal contribution to the Green’s function \( G(\tau, \varphi) \),
\[
G(\tau, \varphi) = \frac{1}{4} \partial_z \partial_{\overline{z}} (-4a)\text{Re} \left[ i \cot \varphi \frac{\partial}{\partial z} \log \theta(\tau + i\varphi|e^{a}) \right].
\]

(5.13)

As we hope, there must be such formulas in the general case enjoying nice properties of modular invariance.
Exploiting the standard product formula for the function $\theta_{11}(z|q)$,

$$\theta_{11}(z|q) = ie^{z/2} - e^{-z/2} \prod_{m>0} \left( 1 - q^m e^z \right) \left( 1 - q^m e^{-z} \right),$$

we find

$$G(\tau, \varphi) = \frac{1}{4} \partial_\tau \partial_\varphi (4a) \cot \varphi \text{Im} \left\{ \frac{1}{e^z - 1} + \sum_{m=1}^{\infty} \left[ \frac{1}{q^m e^z - 1} - \frac{1}{q^m e^{-z} - 1} \right] \right\}. \quad (5.14)$$

Since

$$\frac{\cos \varphi}{\sin \varphi} \text{Im} \frac{1}{q^m e^z - 1} = \frac{-e^{z+am} \cos \varphi}{e^{2(\tau+am)} - 2e^{\tau+am} \cos \varphi + 1} = \text{Re} \frac{-e^{z+am}}{(e^{z+am} - 1)(e^{\tau+am} - 1)}, \quad (5.15)$$

disregarding local contributions, we obtain

$$G(\tau, \varphi) = a \partial_\tau \partial_\varphi \text{Re} \left\{ \frac{1}{(e^z - 1)(e^{\tau} - 1)} - \sum_{m=1}^{\infty} \left[ \frac{e^{z+am}}{(e^{z+am} - 1)(e^{\tau+am} - 1)} + \frac{e^{am-z}}{(e^{am-z} - 1)(e^{am-\tau} - 1)} \right] \right\}. \quad (5.16)$$

The differentiation w.r.t. $z$ and $\tau$ eliminates pure holomorphic and antiholomorphic parts, and the remaining parts constitute the single expression

$$G(z, \tau) = -\frac{a}{4} \sum_{m=-\infty}^{\infty} \frac{1}{\sinh^2 \frac{z+am}{2} \sinh^2 \frac{\tau+am}{2}}. \quad (5.17)$$

This is the Green's function of two Yang–Mills tensor field insertions on torus. At singularity points, it has the proper scaling behavior $G(r) \sim 1/r^4$.

### 6 Discussion

For the case of rectangular torus, we have obtained the proper Green’s function expression (5.17) for the two-point correlator of the Yang–Mills tensor insertions. This demonstrates again that the AdS/CFT correspondence holds in our case where no singularity at the AdS time infinity is assumed. A more interesting (but far more involved technically) is the problem of verifying this correspondence in actual gravitational calculations of a four-point correlation functions. But even concerning the free-field theory, there remain questions on the mass spectrum, on the generalization to nonrectangular tori, to higher dimensions, etc. Of special interest is the question whether it is possible to consider filled Riemann surfaces of higher genera. The construction works well in this case, but the $\varepsilon$-regularized surface, or the boundary of the integration domain, cannot be described in the invariant distance terms (the structure of the set of closed geodesics becomes very involved already starting from genus two); however, we hope that one can obtain a proper answer using an approximation technique. In the present calculations, we disregarded all local contributions. However, these contributions can be important when considering additional boundary terms in the initial action. It would be interesting to check whether the Hamiltonian prescription of [11] holds in this case. These questions deserve further investigation.

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