ON THE FRECHET DIFFERENTIABILITY IN OPTIMAL CONTROL OF COEFFICIENTS IN PARABOLIC FREE BOUNDARY PROBLEMS

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ABSTRACT. We consider the inverse Stefan type free boundary problem, where the coefficients, boundary heat flux, and density of the sources are missing and must be found along with the temperature and the free boundary. We pursue an optimal control framework where boundary heat flux, density of sources, and free boundary are components of the control vector. The optimality criteria consists of the minimization of the $L^2$-norm declinations of the temperature measurements at the final moment, phase transition temperature, and final position of the free boundary. We prove the Frechet differentiability in Besov-Hölder spaces, and derive the formula for the Frechet differential under minimal regularity assumptions on the data. The result implies a necessary condition for optimal control and opens the way to the application of projective gradient methods in Besov-Hölder spaces for the numerical solution of the inverse Stefan problem.

1. Inverse Stefan Problem (ISP). The general one-phase Stefan problem is to find the temperature $u$ and boundary $s$ satisfying

\begin{align*}
Lu &\equiv (au_x)_x + bu_x + cu - u_t = f, \quad \text{in } \Omega \\
u(x,0) &\equiv \phi(x), \quad 0 \leq x \leq s(0) = s_0 \\
a(0,t)u_x(0,t) &\equiv g(t), \quad 0 \leq t \leq T \\
a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s(t) &\equiv \chi(s(t),t), \quad 0 \leq t \leq T \\
u(s(t),t) &\equiv \mu(t), \quad 0 \leq t \leq T,
\end{align*}

where

\begin{align*}
\Omega &\equiv \{(x,t) : 0 < x < s(t), \quad 0 < t \leq T\}
\end{align*}

where $a, b, c, f, \phi, g, \gamma, \chi, \mu$ are given functions. Assume now that $a, b, c, f, a(0,t)$ are not known, where $a$ is the coefficient of diffusion, $b$ is the coefficient of advection, $c$ is the coefficient of reaction, $f$ is the density of heat sources, and $g$ is the heat flux at $x = 0$. In order to find $a, b, c, f, a(0,t)$ and $g$ along with $u$ and $s$, we must have

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additional information. Assume that this additional information comes in the form of a measurement of the temperature as well as the position of the free boundary at the final moment $T$.

$u(x, T) = w(x), \ 0 \leq x \leq s(T) = s_\ast.$  \hspace{1cm} (7)

Under these conditions, we are required to solve an inverse Stefan problem (ISP): find a tuple

$\{u, a, b, c, f, g, s\}$

that satisfy conditions (1)–(7).

Motivation for this type of inverse problem arose, in particular, from the modeling of bioengineering problems on the laser ablation of biological tissues through a Stefan problem (1)–(7), where $s(t)$ is the ablation depth at the moment $t$. The unknown parameters of the model such as $a, b, c, f, g$ are very difficult to measure through experiments. Lab experiments pursued on laser ablation of biological tissues allow the measure of final temperature distribution and final ablation depth; the ISP must be solved for the identification of some, or all, of the unknown parameters $a, b, c, f, g$. Our approach allows us to regularize an error contained in a final moment temperature measurement $w(x)$ and final moment ablation depth $s_\ast$.

Another advantage of this approach is that, in fact, the condition (5) can be treated as a measurement of the temperature on the ablation front, allowing us to regularize an error contained in temperature measurement $\mu (t)$ on the ablation front. Still another important motivation arises from the optimal control of the laser ablation process. A typical control problem arises when unknown control parameters, such as the intensity of the laser source $f$, heat flux $g$ on the known boundary, and any of the coefficients $a, b, c$, must be chosen with the purpose of achieving a desired ablation depth and temperature distribution at the end of the time interval.

ISP is not well posed in the sense of Hadamard: the solution may not exist; if it exists, it may not be unique, and in general it does not exhibit continuous dependence on the data. The goal of this paper is to prove the Frechet differentiability and necessary condition for optimality in the optimal control problem introduced recently as a variational formulation of the inverse Stefan problem (ISP) in [1, 2].

The inverse Stefan problem first appeared in [12]; the problem discussed was the determination of a heat flux on the fixed boundary for which the solution of the Stefan problem has a desired free boundary. The variational approach for solving this ill-posed inverse Stefan problem was developed in [8, 9, 10]. In [36], the problem of finding the optimal value for the external temperature in order to achieve a given measurement of temperature at the final moment was considered, and existence was proven. In [38], the Frechet differentiability and convergence of difference schemes was proven for the same problem, and Tikhonov regularization was suggested.

Later development of the inverse Stefan problem proceeded along two lines: inverse Stefan problems with given phase boundaries in [5, 10, 11, 13, 14, 15, 17, 18, 32], or inverse problems with unknown phase boundaries in [4, 16, 17, 19, 20, 22, 21, 23, 25, 26, 27, 29, 30, 31, 35]. We refer to the monograph [17] for a complete list of references for both types of inverse Stefan problem, both for linear and quasilinear parabolic equations.

The established variational methods in earlier works fail in general to address two issues:
• The solution of ISP does not depend continuously on the phase transition temperature. A small perturbation of the phase transition temperature may imply significant change of the solution to the ISP.

• In the existing formulation, at each step of the iterative method a Stefan problem must be solved which incurs a high computational cost.

A new method developed in [1, 2] addresses both issues with a new variational formulation. Existence of the optimal control and the convergence of the sequence of discrete optimal control problems to the continuous optimal control problem was proved in [1, 2]. In [3], the Frechet differentiability of the variational formulation as well as existence were established under minimal conditions on the data, using precise estimates in Besov spaces. The previous work focused on the identification of the boundary $s$, the heat flux $g$, and the density of sources $f$; our goal in this work is to extend the Frechet differentiability framework used in [3] to the problem with unknown coefficients.

The structure of the remainder of the paper is as follows: in Section 2 we define all the functional spaces. Section 3 formulates optimal control problem. Section 4 describes the main results: Theorem 4.4 states the Frechet differentiability result and presents the formula for the Frechet differential; in Corollary 1 we present the necessary condition for the optimal control in the form of the variational inequality. Section 5 gives a heuristic derivation of the Frechet gradient which suggests the form of the adjoint problem and the Frechet differential. Section 6 describes important preliminary results. In Section 6.1 we recall the existence, uniqueness and energy estimates in Besov spaces for the Neumann problem to the second order linear parabolic PDEs. In Section 6.2 we formulate optimal trace embedding results for the Besov spaces. In Section 6.3 we give three important technical lemmas. Lemmas 6.4 and 6.5 are on the estimation of the Neumann problem, and adjoint PDE problem in respective Besov space norm. In Lemma 6.6 we prove an estimate on the increment of the state vector with respect to the control vector in a Besov space norm. By applying Lemmas 6.4–6.6 in Section 7 we complete the proof of the main results. Finally, conclusions are presented in Section 8.

2. Notation. We will use the notation

$$1_I(x) = \begin{cases} 1, & x \in I \\ 0, & x \notin I \end{cases}$$

for the indicator function of the set $I$, and $\lfloor r \rfloor$ for the integer part of the real number $r$.

We will require the notions of Sobolev-Slobodeckij or Besov spaces [6, 25, 24, 28, 33, 34]. In this section, assume $U$ is a domain in $\mathbb{R}$ and denote by

$$Q_T = (0,1) \times (0,T].$$

• For $\ell \in \mathbb{Z}_+$, $W^{\ell}_p(U)$ is the Banach space of measurable functions with finite norm

$$\|u\|_{W^{\ell}_p(U)} := \left( \int_U \sum_{k=0}^\ell \left| \frac{d^k u}{dx^k} \right|^p \, dx \right)^{1/p}$$

• For $\ell \notin \mathbb{Z}_+$, $B^{\ell}_p(U)$ is the Banach space of measurable functions with finite norm

$$\|u\|_{B^{\ell}_p(U)} := \|u\|_{W^{\lfloor \ell \rfloor}_p(U)} + [u]_{B^{\ell}_p(U)}$$
The minimization of the functional

\[ \beta \]

Define \( \beta \)

Fix any \( \beta \)

If \( \ell \in \mathbb{Z}_+ \), the seminorm \( [u]_{B^\ell_p(U)} \) is given by

\[
[u]_{B^\ell_p(U)} := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\partial^{\ell-1} u(x) - \partial^{\ell-1} u(y)|}{|x-y|^{1+p(\ell-\ell')}} dx dy
\]

By [7, §18, thm. 9], it follows that for \( p = 2 \) and \( \ell \in \mathbb{Z}_+ \), the \( B^\ell_p(U) \) norm is equivalent to the \( W^\ell_p(U) \) norm (i.e. the two spaces coincide.)

- Let \( 1 \leq p < \infty \), \( 0 < \ell_1, \ell_2 \). The Besov space \( B^{\ell_1, \ell_2}_{p,x,t}(Q_T) \) is defined as the closure of the set of smooth functions under the norm

\[
\|u\|_{B^{\ell_1, \ell_2}_{p,x,t}(Q_T)} = \left( \int_0^T \|u(x,t)\|_{B^{\ell_1}_{p,[0,1]}}^p dt \right)^{1/p} + \left( \int_0^1 \|u(x,t)\|_{B^{\ell_2}_{p,[0,T]} dx}^p \right)^{1/p}
\]

When \( p = 2 \), if either \( \ell_1 \) or \( \ell_2 \) is an integer, the Besov seminorm may be replaced with the corresponding Sobolev seminorm due to equivalence of the norms and the corresponding Sobolev-Besov space may be denoted by \( W^{\ell_1, \ell_2}_2 \).

- The Hölder space \( C^{\alpha, \alpha/2}_{x,t}(Q_T) \) is the set of continuous functions with \( [\alpha] \) \( x \)-derivatives and \( [\alpha/2] \) \( t \)-derivatives, and for which the highest order \( x \)- and \( t \)-derivatives satisfy Hölder conditions of order \( \alpha - [\alpha] \) and \( \alpha/2 - [\alpha/2] \), respectively.

### 3. Optimal control problem.

Fix any \( \alpha, \alpha^* \) such that \( 0 < \alpha < \alpha^* \). Consider the minimization of the functional

\[ J(v) = \beta_0 \int_0^{s(T)} |u(x,T;v) - w(x)|^2 dx + \beta_1 \int_0^T |u(s(t),t;v) - \mu(t)|^2 dt \\
+ \beta_2 |s(T) - s_4|^2
\]  (8)

on the control set

\[ V_R = \{ v = (a,b,c,f,g,s) \in H : \|v\|_H \leq R; \ s(0) = s_0, \ g(0) = a(0,0)g'(0), \ a_0 \leq a(x,t), \ \chi(s_0,0) = \phi'(s_0)a(s_0,0) + \gamma(s_0,0)s'(0), \ 0 < \delta \leq s(t) \}, \]  (9)

where \( \beta_0, \beta_1, \beta_2 \geq 0 \) and \( a_0, \delta, R > 0 \) are given.

\[ H := C^{3/2+2\alpha^*,3/4+\alpha^*}_{x,t}(D) \times C^{1/2+2\alpha^*,1/4+\alpha^*}_{x,t}(D) \times C^{1/2+2\alpha^*,1/4+\alpha^*}_{x,t}(D) \\
\times B^{11/4+\alpha}_{2,x,t}(D) \times B^{1/2+\alpha}_{2}[0,T] \times B^2_{2}[0,T]
\]

\[ \|v\|_H := \max \left( \|u\|_{C^{3/2+2\alpha^*,3/4+\alpha^*}_{x,t}(D)}, \|b\|_{C^{1/2+2\alpha^*,1/4+\alpha^*}_{x,t}(D)}, \|c\|_{C^{1/2+2\alpha^*,1/4+\alpha^*}_{x,t}(D)}, \|s\|_{B^2_{2}[0,T]}, \|g\|_{B^{1/2+\alpha}_{2}[0,T]}, \|f\|_{B^{11/4+\alpha}_{2,x,t}(D)} \right)
\]

Define

\[ D := \{(x,t) : 0 \leq x \leq \ell, \ 0 \leq t \leq T \}, \]
where \( \ell = \ell(R) > 0 \) is chosen such that for any control \( v \in V_R \), its component \( s \) satisfies \( s(t) \leq \ell \). This follows from Morrey inequality \([25, 6]\):

\[
\|s\|_{C^{1,\frac{1}{2}}[0,T]} \leq C\|s\|_{B^{1,1}_2[0,T]}, \tag{10}
\]

For a given control vector \( v \in V_R \), the state vector \( u(x,t;v) \) is a solution to the Neumann problem \((1)–(4)\). The formulated optimal control problem \((8)–(9)\) will be called Problem \( \mathcal{I} \).

Since the data appearing in the Neumann problem \((1)–(4)\) are in general non-smooth, the solutions may not exist in the classical sense. The notion of solution must be understood in a weak sense, i.e. for a fixed control vector \( v \in V_R \), \( u \in W^{2,1}_2(\Omega) \) is called a solution of the Neumann problem \((1)–(4)\) if it satisfies the equation \((1)\) and conditions \((2)–(4)\) pointwise almost everywhere.

**Remark 1.** Note that Problem \( \mathcal{I} \) imposes additional requirements on ISP. The requirements

\[
\chi(s_0,0) = \phi'(s_0)a(s_0,0) + \gamma(s_0,0)s'(0),
\]

\[
g(0) = a(0,0)\phi'(0), \quad 0 < a_0 < a(x,t), \quad \text{and} \quad 0 < \delta \leq s(t) \]

are necessary to arrange well-posedness of the Neumann problem \((1)–(4)\) in respective Besov space which is essential for the proof of the Frechet differentiability, derivation of the Frechet gradient and necessary condition for the optimality. The constant bound \( R \) provides for weak compactness of the control set \( V_R \) in respective Hilbert space and is necessary for the existence result. If ISP has a solution in the class of described data, then the constant \( R \) is chosen large enough to guarantee that the solution of the ISP is contained in \( V_R \). Otherwise, in practical applications Problem \( \mathcal{I} \) is numerically solved for the increasing sequence of bounds \( R \).

### 4. Main results

Let \( \alpha, \alpha^* \) be fixed as in \((9)\). The main results are established under the assumptions

\[
w \in B^1_2[0,\ell], \quad \phi \in B^{3/2+2\alpha}_2[0,s_0], \quad \chi, \gamma \in B^{3/4+3/2+2\alpha^*,2+3/4+3/2+2\alpha^*}_2(\Omega), \quad \mu \in B^1_2[0,T] \tag{11}
\]

Given a control vector \( v \in V_R \), under the conditions \((11)\) there exists a unique pointwise a.e. solution \( u \in W^{2,1}_2(\Omega) \) of the Neumann problem \((1)–(4)\)(\([25, 33]\)).

**Definition 4.1.** Let \( V \) be a convex and closed subset of the Banach space \( H \). We say that the functional \( \mathcal{J} : V \rightarrow \mathbb{R} \) is differentiable in the sense of Frechet at the point \( v \in V \) if there exists an element \( \mathcal{J}'(v) \in H' \) of the dual space such that

\[
\mathcal{J}(v+h) - \mathcal{J}(v) = \langle \mathcal{J}'(v), h \rangle_H + o(h,v), \tag{12}
\]

where \( v + h \in V \cap \{ u : |u| < \gamma \} \) for some \( \gamma > 0 \); \( \langle \cdot, \cdot \rangle_H \) is a pairing between \( H \) and its dual \( H' \), and

\[
o(h,v) \sim |h| \quad \text{as} \quad |h| \to 0.
\]

The expression \( d\mathcal{J}(v) = \langle \mathcal{J}'(v), \cdot \rangle_H \) is called a Frechet differential of \( \mathcal{J} \) at \( v \in V \), and the element \( \mathcal{J}'(v) \in H' \) is called Frechet derivative or gradient of \( \mathcal{J} \) at \( v \in V \).

Note that if Frechet gradient \( \mathcal{J}'(v) \) exists at \( v \in V \), then the Frechet differential \( d\mathcal{J}(v) \) is uniquely defined on a convex cone

\[
\mathcal{H}_v = \{ w \in H : w = \lambda(u-v), \lambda \in [0,+) \cup V \}.
\]
Indeed, let $J'_1(v)$ and $J'_2(v)$ be two Fréchet gradients of $J$ at $v \in V$. Choose $e \in H_c, |e| = 1$. For some $\delta > 0$ we have

$$u = v + te \in V, \quad for \ 0 \leq t \leq \delta.$$ 

From (12) it follows that

$$(J'_1(v) - J'_2(v), te)_H = o(t), \quad as \ t \downarrow 0.$$ 

Dividing by $t$ and passing to limit as $t \downarrow 0$ we have

$$(J'_1(v) - J'_2(v), e)_H = 0,$$

which proves the assertion.

**Definition 4.2.** For given $v$ and $u = u(x, t; v), \psi \in W^{2,1}_2(\Omega)$ is a solution to the adjoint problem if

$$L^* \psi \equiv (a\psi_x)_x - (b\psi)_x + \psi c + \psi_i = 0 \ in \ \Omega \quad (13)$$

$$\psi(x, T) = 2\beta_0(u(x, T) - w(x)), \ 0 < x < s(T) \quad (14)$$

$$[a\psi_x - b\psi]_{x=0} = 0, \ 0 < t < T \quad (15)$$

$$[a\psi_x + (b + s')\psi]_{x=s(t)} = 2\beta_1(u - \mu)_{x=s(t)}, \ 0 < t < T \quad (16)$$

**Theorem 4.3 (Existence of an Optimal Control).** Problem $I$ has a solution. That is,

$$V_* = \left\{ v \in V_R : J(v) = J_* = \inf_{v \in V_R} J(v) \right\} \neq \emptyset.$$

**Theorem 4.4.** The functional $J(v)$ is differentiable in the sense of Fréchet, and the Fréchet differential is

$$\langle J'(v), \Delta v \rangle_H = \int_0^T \left[ 2\beta_1 (u - \mu) u_x + \psi (\chi_x - \gamma s') - (au)_x \right]_{x=s(t)} \Delta s(t) dt$$

$$+ \left[ \beta_0 |u(w(T) - w(s(T))|^2 + 2\beta_2 (s(T) - s_a) \right] \Delta s(T) - \int_0^T [\Delta au_x \psi]_{x=s(t)} dt$$

$$+ \int_{\Omega} \left[ (\Delta au)_x + \Delta bu_x + \Delta cu - \Delta f \right] \psi dx dt - \int_0^T [\psi \gamma]_{x=s(t)} \Delta s(t) dt$$

$$- \int_0^T \psi(0, t) \Delta g(t) dt - \int_0^T [\Delta au_x \psi]_{x=0} dt \quad (17)$$

where $J'(v) \in H'$ is the Fréchet derivative, $\psi$ is a solution to the adjoint problem in the sense of definition 4.2, and $\Delta v = (\Delta a, \Delta b, \Delta c, \Delta f, \Delta g, \Delta s)$ is a variation of the control vector $v \in V_R$ such that $v + \Delta v \in V_R$.

**Corollary 1 (Optimality Condition).** If $v$ is an optimal control, then the following variational inequality is satisfied:

$$\langle J'(v), v - v \rangle_H \geq 0 \quad (18)$$

for arbitrary $v \in V_R$.

Note that $V_R$ is a closed, bounded, and convex subset of $H$, so the left hand side of the optimality condition (18) is uniquely defined for Frechet gradient $J'(v)$ defined in the sense of definition 4.1.
5. **Heuristic derivation of Frechet gradient.** To give a first indication of the form of the gradient, we apply the heuristic method of Lagrange-type multipliers; the rigorous proof follows in Section 7. Consider the functional

\[
\mathcal{L}(a, b, c, f, g, s, u, \psi) = \mathcal{J}(v) + \int_{\Omega} \psi [(au_x)_x + bu_x + cu - u_t - f] \, dx \, dt.
\]

Define \( \Delta v = (\Delta a, \Delta b, \Delta c, \Delta f, \Delta g, \Delta s) \), \( \bar{v} = v + \Delta v = (\bar{a}, \bar{b}, \bar{c}, \bar{f}, \bar{g}, \bar{s}) \) such that \( \bar{v} \in V_R \). Let \( \bar{\pi}(x, t) = \pi(x, t, \bar{v}) \). We will also denote by \( \bar{s}(t) = s(t) + \theta(t) \Delta s(t) \) where \( 0 \leq \theta(t) \leq 1 \) standing for all functions arising from application of mean value theorem in the region between \( s(t) \) and \( \bar{s}(t) \). Define

\[
\hat{s} = \min(s, \bar{s}), \quad 0 \leq t \leq T, \quad \hat{\Omega} = \{(x, t) : 0 < x < \hat{s}(t), 0 < t \leq T\}
\]

\[
\Delta u(x, t) = \bar{\pi}(x, t) - u(x, t) \text{ in } \hat{\Omega}
\]

In what follows, all terms of higher than linear order with respect to \( \Delta v \) will be absorbed into the expression \( R \). Partition the time domain as \( [0, T] = T_1 \cup T_2 \) where

\[
T_1 = \{ t \in [0, T] : \Delta s(t) < 0 \}, \quad T_2 = [0, T] \setminus T_1 = \{ t \in [0, T] : \Delta s(t) \geq 0 \}
\]

Consider the increment

\[
\Delta \mathcal{L} = \Delta \mathcal{J} + \Delta \mathcal{I}, \quad \Delta \mathcal{J} = \mathcal{J}(\bar{v}) - \mathcal{J}(v), \Delta \mathcal{I} = I_1 + I_2,
\]

\[
I_1 = \int_0^T \int_0^{\hat{s}(t)} \left[ (\Delta a \bar{u})_x + \Delta b \bar{u}_x + \Delta c \bar{u} - \Delta f \right] \psi \, dx \, dt
\]

\[
I_2 = \int_{\Omega} \psi \left[ (a \Delta u)_x + b \Delta u_x + c \Delta u - \Delta u_t \right] \, dx \, dt
\]

Transforming \( \Delta \mathcal{J} \) as in [3], we derive

\[
\Delta \mathcal{J} = \beta_0 \int_0^{\hat{s}(T)} \left[ 2(u - w) \Delta u \right]_{t=T} \, dx + \beta_0 |u(s(T), T) - w(s(T))|^2 \Delta s(T)
\]

\[
+ \beta_1 \int_0^T 2(u(s(t), t) - \mu(t)) u_x(s(t), t) \Delta s(t) \, dt
\]

\[
+ \beta_1 \int_0^T 2(u(s(t), t) - \mu(t)) \Delta u(\hat{s}(t), t) \, dt
\]

\[
+ 2 \beta_2 (s(T) - s_*) \Delta s(T) + R.
\]  \hspace{1cm} (19)

Each term in \( I_1 \) is transformed in a similar way; for example,

\[
\int_{\hat{\Omega}} (\Delta a \bar{u})_x \, dx \, dt = \int_{T_1} \int_0^{\hat{s}(t)} (\Delta a \bar{u})_x \psi \, dx \, dt + \int_{T_2} \int_0^{\hat{s}(t)} (\Delta a \bar{u})_x \psi \, dx \, dt
\]

\[
= \int_{T_1} \int_0^{\hat{s}(t)} \left[ (\Delta a u)_x + (\Delta a \Delta u)_x \right] \psi \, dx \, dt
\]

\[
+ \int_{T_2} \int_0^{\hat{s}(t)} \left[ (\Delta a u)_x + (\Delta a \Delta u)_x \right] \psi \, dx \, dt
\]

hence

\[
\int_{\hat{\Omega}} \Delta a \bar{u}_x \, dx \, dt = \int_{\hat{\Omega}} (\Delta a u)_x \psi \, dx \, dt + R.
\]
Treating the other terms similarly, we derive
\[ I_1 = \int_\Omega \left[ (\Delta u_{x})_x + \Delta b u_x + \Delta c u - \Delta f \right] \psi \, dx \, dt + R, \tag{20} \]
Transform \( I_2 \) using integration by parts to derive
\[
I_2 = \int_\Omega \left[ (\psi b)_x + \psi c + \psi_t \right] \Delta u \, dx \, dt + \int_{T_1} \left[ \alpha \psi \Delta u_x \right]_{x=\bar{T}(t)} \, dt
+ \int_0^T \left[ -a \psi_x + (b + s') \psi \right]_{x=s(t)} \Delta u(\bar{s}(t), t) \, dt
+ \int_{T_2} \left[ \alpha \psi \Delta u_x \right]_{x=s(t)} \, dt - \int_0^T \psi(0, t) \Delta g(t) \, dt + \int_0^T \left[ (\alpha \psi_x - b \psi) \Delta u \right]_{x=0} \, dt
- \int_0^{\bar{T}(T)} \psi(x, T) \Delta u(x, T) \, dx - \int_0^T \psi(0, t) u_x(0, t) \Delta a(0, t) \, dt + R. \tag{21} \]
Using the boundary condition (4) , it follows that for \( t \in T_1 \),
\[
a(\bar{\sigma}(t), t) \Delta u_x(\bar{\sigma}(t), t) = \chi_x(\bar{s}(t), t) \Delta s(t) - \gamma_x(\bar{s}(t), t) \Delta s(t) \bar{s}'(t) - \gamma(s(t), t) \Delta s'(t)
- (au)_x |_{x=\bar{s}(t)} \Delta s(t) - \Delta a(\bar{\sigma}(t), t) \Delta u_x(\bar{\sigma}(t), t) - \left[ \Delta a(\bar{u})_x \right]_{x=\bar{s}(t)}
- \Delta a(s(t), t) u_x(s(t), t). \tag{22} \]
Hence
\[
\int_{T_1} \left[ \psi \Delta u_x \right]_{x=\bar{T}(t)} \, dt = \int_{T_1} \left[ \psi \left( \chi_x \Delta s - \gamma_x \Delta s' \bar{s}' - \gamma \Delta s' - (au)_x \Delta s \right) \right]_{x=s(t)} \, dt
- \int_{T_1} \left[ \psi \Delta a u_x \right]_{x=s(t)} \, dt + R. \tag{23} \]
Similarly, boundary condition (4) implies that for \( t \in T_2 \),
\[
a(s(t), t) \Delta u_x(s(t), t) = \chi_x(\bar{s}(t), t) \Delta s(t) - \gamma_x(\bar{s}(t), t) \Delta s(t) \bar{s}'(t) - \gamma(s(t), t) \Delta s'(t)
- (au)_x |_{x=s(t)} \Delta s(t) - \Delta a(s(t), t) u_x(s(t), t) - \left[ \Delta a(\bar{u})_x \right]_{x=s(t)}
+ a_x(s(t), t) (\bar{u}(\bar{s}(t), t) - u_x(s(t), t)) \Delta s(t) - \left[ a(\bar{u})_x \right]_{x=s(t)} \Delta s(t)
- \Delta a(s(t), t) \bar{u}_x(\bar{s}(t), t) \Delta s(t) - a(s(t), t) \left[ \bar{u}_x(x,t) \right]_{x=s(t)} \Delta s(t)
- \left[ a(x,t) \right]_{x=s(t)} \Delta s(t). \tag{24} \]
So it follows that
\[
\int_{T_2} \left[ \psi \Delta u_x \right]_{x=s(t)} \, dt = \int_{T_2} \left[ \psi \left( \chi_x \Delta s - \gamma_x s' \Delta s - \gamma \Delta s' - (au)_x \Delta s \right) \right]_{x=s(t)} \, dt
- \int_{T_2} \left[ \psi \Delta a u_x \right]_{x=s(t)} \, dt + R. \tag{25} \]
Combining (21), (23), (25), it follows that
\[
I_2 = \int_\Omega \left[ (\psi b)_x + \psi c + \psi_t \right] \Delta u \, dx \, dt
+ \int_0^T \left[ \psi \left( \chi_x \Delta s - \gamma_x s' \Delta s - \gamma \Delta s' - (au)_x \Delta s - \Delta a u_x \right) \right]_{x=s(t)} \, dt
Combining (19), (20), (26) it follows that

$$
\Delta \mathcal{L} = \int_0^{\tau(u)} \left[ 2\beta_0(u - w) - \psi(x, T) \right] \Delta u(x, T) \, dx \\
+ \int_0^\tau \left[ \psi \Delta u \right]_{x=s(t)} \Delta u(x, T) \, dx \\
- \int_0^\tau \psi(0, t) \Delta g(t) \, dt + \int_0^\tau \left[ (av_x - b\psi) \right]_{x=0} \Delta u \, dt \\
- \int_0^\tau \psi(x, T) \Delta u(x, T) \, dx - \int_0^\tau \psi(0, t) u_x(0, t) \Delta a(0, t) \, dt + R 
$$

(26)

Due to arbitrariness of \( \Delta u \), its coefficients in the 1st, 4th, 8th, and 10th terms in (27) must be zero, and hence \( \psi \) is a solution of (13)–(16) and the Frechet differential \( \Delta \mathcal{F} \) is as in (17) if \( R = o(\Delta u) \).

6. Preliminary results.

6.1. \( B_{p,x,t}^{2\ell,\ell}(Q_T) \)-solutions. Consider the problem

$$
u_{xx} + bu_x + cu - u_t = f \text{ in } Q_T
$$

(28)

$$
a(0, t) u_x(0, t) = \chi_1(t), \quad 0 \leq t \leq T
$$

(29)

$$
a(1, t) u_x(1, t) = \chi_2(t), \quad 0 \leq t \leq T
$$

(30)

$$
u(x, 0) = \phi(x), \quad 0 \leq x \leq 1
$$

(31)

Let \( \ell \geq 1 \) be fixed, \( p > 1 \). The following key result is due to Solonnikov [33]

**Lemma 6.1.** [33, §7, thm. 17] Suppose that

$$
a, b, c \in C^{2\ell^* - 2, \ell^* - 1}_{x,t}(Q_T), \text{ arbitrary } \ell^* > \ell
$$

$$
f \in B_{p,x,t}^{2\ell - 2, \ell - 1}(Q_T), \quad \phi \in B_p^{2\ell - 2} [0, 1], \quad \chi_1, \chi_2 \in B_p^{\ell - 2 - \frac{1}{2p}} [0, T]
$$

and the consistency condition of order \( k \) holds; that is,

$$
\frac{\partial^j}{\partial x^j} (u_x)(0, 0) = \frac{d^j}{dt^j} \chi_1(0), \quad \frac{\partial^j}{\partial x^j} (u_x)(1, 0) = \frac{d^j}{dt^j} \chi_2(0), \quad j = 0, \ldots, k
$$
Then the solution $u$ of (28)–(31) satisfies the energy estimate
\[
\|u\|_{B^{2\ell,t}_{p,x,t}(\Omega)} \leq C \left[ \|f\|_{B^{2\ell-1,t-1}_{p,x,t}(\Omega)} + \|\phi\|_{B^{2\ell-2/p}_{p}(0,1)} + \|\chi\|_{B^{\ell-rac{1}{2}-\frac{1}{2p}}_{p}(0,T)} \right]
\]
when $\ell, \ell - \frac{3}{2p} \not\in \mathbb{Z}_+$, and when $\ell \in \mathbb{Z}_+$,
\[
\|u\|_{W^{2\ell,t}_{p,x,t}(\Omega)} \leq C \left[ \|f\|_{W^{2\ell-1,t-1}_{p,x,t}(\Omega)} + \|\phi\|_{B^{2\ell-2/p}_{p}(0,1)} + \|\chi\|_{B^{\ell-rac{1}{2}-\frac{1}{2p}}_{p}(0,T)} \right]
\]
In particular, energy estimates (32), (33) imply the existence and uniqueness of the solution in respective spaces $B^{2\ell,t}_{p,x,t}(\Omega)$ or $W^{2\ell,t}_{p,x,t}(\Omega)$.

6.2. Traces and embeddings of Besov functions. For functions $u \in W^{2,1}_{2,1}(\Omega)$, the applicability of the boundary conditions are justified by the following trace and regularity results.

Lemma 6.2. [25, lem. II.3.3] If $u \in W^{2,1}_{2,1}(\Omega)$, then $u$ has a Hölder continuous representative in $\Omega$; in particular, $u \in C^{1/2,1/4}_{\text{loc}}(\overline{\Omega})$. Moreover [25, lem. II.3.4], the following bounded embeddings of traces hold:

\[
u(s(t),t), \ u(0,t) \in B^{3/4}_{2}[0,T], \ u_{x}(s(t),t), \ u_{x}(0,t) \in B^{1/4}_{2}[0,T] \]

and for any fixed $0 \leq \bar{t} \leq T$,

\[u(\cdot, \bar{t}) \in W^{1}_{2}[0,s(\bar{t})]\]

Lemma 6.3. [33, §4, thm. 9] For a function $u \in B^{2\ell,t}_{2,x,t}(\Omega)$, the following bounded embeddings of traces hold: for any fixed $0 \leq t \leq T$,

\[u(\cdot, t) \in B^{2\ell-1}_{2}[0,1] \text{ when } \ell > 1/2 \]

For any fixed $0 \leq x \leq 1$,

\[
\begin{align*}
u(x,\cdot) &\in B^{\ell-1/4}_{2}[0,T] \text{ when } \ell > 1/4 \\
u_{x}(x,\cdot) &\in B^{\ell-3/4}_{2}[0,T] \text{ when } \ell > 3/4 \\
u_{xx}(x,\cdot) &\in B^{\ell-5/4}_{2}[0,T] \text{ when } \ell > 5/4 
\end{align*}
\]

6.3. Consequences of energy estimates and embeddings. For a given control vector $v = (a, b, c, f, g, s) \in V_{R}$ transform the domain $\Omega$ to the cylindrical domain $Q_{T}$ by the change of variables $y = x/s(t)$. Let $d = d(x,t), (x,t) \in \Omega$ stand for any of $u, a, b, c, f, \gamma, \chi$, define the function $\tilde{d}$ by

\[d(x,t) = \tilde{d}(xs(t),t), \ \text{and} \ \tilde{\phi}(x) = \phi(xs(0)) \]

The transformed function $\tilde{u}$ is a pointwise a.e. solution of the Neumann problem

\[\tilde{L}\tilde{u} := \frac{1}{s}(\tilde{a}\tilde{u}_{y})_{y} + \frac{1}{s}(\tilde{b} + \gamma s(t))\tilde{u}_{y} + \tilde{c}\tilde{u} - \tilde{u}_{t} = \tilde{f}, \text{ in } Q_{T} \]

(34)
\[
\bar{u}(x,0) = \phi(x), \ 0 \leq x \leq 1
\]
\[
\bar{a}(0,t)\bar{u}_y(0, t) = g(t) s(t), \ 0 \leq t \leq T
\]
\[
\bar{a}(1,t)\bar{u}_y(1, t) = \bar{\chi}(1,t) s(t) - \bar{\gamma}(1,t) s'(t)s(t), \ 0 \leq t \leq T
\]  

**Lemma 6.4.** For fixed \( v \in V_R \), there exists a unique solution \( u \in W^{2,1}_2(\Omega) \) of the Neumann problem \((1)-(4)\) for which the transformed function \( \bar{u} \in B^{2} _{2,2}(Q_T) \) with \( \ell = 5/4 + \alpha \) solves \((34)-(37)\) and satisfies the following energy estimate

\[
\| \bar{u} \|_{B^{2,2,2+2\alpha,5/4+\alpha}_2(Q_T)} \leq C \left( \| f \|_{B^{1/4+\alpha}_2(D)} + \| \phi \|_{B^{3/2+2\alpha}_2[0,\infty)} + \| g \|_{B^{1/\alpha}[0,T]} + \| \chi \|_{B^{1/2+2\alpha,\frac{3}{4}+\alpha}_2[0,T]} + \| \gamma \|_{B^{3/2+2\alpha,3/4+\alpha}_2(D)} \right)
\]

where \( \alpha^* > \alpha \) is arbitrary.

**Lemma 6.5.** For fixed \( v \in V_R \), given the corresponding state vector \( u = u(x,t;v) \), there exists a unique solution \( \psi \in W^{2,1}_2(\Omega) \) of the adjoint problem \((13)-(16)\) and the following energy estimate is valid

\[
\| \psi \|_{W^{2,1}_2(\Omega)} \leq C \left( \| f \|_{L^2(\Omega)} + \| \phi \|_{W^2_2[0,\infty)} + \| g \|_{B^{1/2,\frac{1}{2}+\alpha}_2[0,T]} + \| \chi \|_{B^{1/2+2\alpha,\frac{3}{4}+\alpha}_2[0,T]} + \| \gamma \|_{B^{1/2+2\alpha,\frac{3}{4}+\alpha}_2[0,T]} \right)
\]

**Lemma 6.6.** If \( a, b, c, f, g, s \) satisfy \((9)\), and \( \chi, \gamma \) satisfy \((11)\), then

\[
\| \Delta \bar{u} \|_{B^{5/2+2\alpha,5/4+\alpha}_2(Q_T)} \to 0 \quad \text{as} \quad \Delta v \to 0 \quad \text{in} \quad H.
\]

**Proof.** A straightforward calculation shows that \( \Delta \bar{u} \) solves

\[
\frac{\bar{\alpha}}{s^2} \Delta \bar{u}_{yy} + \frac{1}{s} \left( \frac{\bar{\alpha}_y}{s} + \bar{\alpha} + \bar{b}_y + \frac{1}{s} \bar{y} \bar{s} \right) \Delta \bar{u}_y + \bar{\chi} \Delta \bar{u} - \Delta \bar{u}_t = \Delta \bar{f} + \left( \frac{\bar{a}}{s^2} - \frac{\bar{\alpha}}{s^2} \right) \bar{u}_{yy} + \left( \frac{\bar{a}_y}{s^2} - \frac{\bar{\alpha}_y}{s^2} + \bar{b}_y - \frac{1}{s} \bar{b} + \frac{\bar{s}_y}{s} - \frac{\bar{s}'}{s} \right) \bar{u}_y - \Delta \bar{c}_y
\]

in \( Q_T \),

\[
\Delta \bar{u}(x,0) = 0, \ 0 \leq x \leq 1
\]
\[
\bar{u}_y(0, t) = \bar{s}(t) g(t) + s(t) \Delta g(t) + s(t) \Delta g(t) + \Delta s(t) \Delta g(t), \ 0 \leq t \leq T
\]

and

\[
\bar{u}_y(1, t) = - \Delta \bar{a} \bar{u}_y(1, t) \Delta s(t) + \bar{\gamma} \bar{u}_y(1, t) + \left( \bar{\chi}(1, t) - \bar{\gamma}(1, t) s'(t) \right) \Delta s(t)
\]

\[
+ \left( \Delta \bar{\gamma}(1, t) - \bar{\gamma}(1, t) \Delta s'(t) - \Delta \bar{s}(1, t) s'(t) \right) s(t), \ 0 \leq t \leq T
\]
By the energy estimate (32) for functions in $B_{2, x, t}^{5/2+2\alpha, 5/4+\alpha}$, it follows from (41)–(44), and Minkowski’s inequality that

$$\|\Delta \tilde{u}\|_{B_{2, x, t}^{5/2+2\alpha, 5/4+\alpha}(Q_T)} \leq C \left[ \sum_{i=1}^{6} \|\Gamma_i\|_{B_{2, x, t}^{5/2+2\alpha, 1/4+\alpha}(Q_T)} \right]$$

$$+ \|\tau(0, t)\Delta \tilde{u}_y(0, t)\|_{B_{2}^{1/2+\alpha}[0, T]} + \|\tau(1, t)\Delta \tilde{u}_y(1, t)\|_{B_{2}^{1/2+\alpha}[0, T]}$$

(45)

where

$$\Gamma_1 = \Delta \tilde{f}, \quad \Gamma_2 = \left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy}, \quad \Gamma_3 = \left( \frac{\tilde{a}_y - \tilde{\alpha}_y}{s^2} \right) \tilde{u}_y$$

$$\Gamma_4 = \left( \frac{\tilde{b}}{s} - \frac{\tilde{\beta}}{s} \right) \tilde{u}_y, \quad \Gamma_5 = \left( \frac{s'}{s} - \frac{s''}{s} \right) \tilde{u}_y, \quad \Gamma_6 = \Delta \tilde{c}\tilde{u}$$

We will first show

$$\|\Gamma_2\|_{B_{2, x, t}^{1/2+2\alpha, 1/4+\alpha}(Q_T)} = \left\| \left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy} \right\|_{B_{2, x, t}^{1/2+2\alpha, 1/4+\alpha}(Q_T)} \rightarrow 0$$

as $\Delta v \rightarrow 0$ in $H$. Note that

$$\left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy} = \left( s^2 \tilde{a} - s^2 \tilde{\alpha} \right) \frac{\tilde{u}_{yy}}{s^2 s^2} = \left( s^2 - s^2 \right) \frac{\tilde{a} \tilde{u}_{yy}}{s^2 s^2} - \left( \frac{\tilde{\alpha} - \tilde{a}}{s^2} \right) \tilde{u}_{yy}$$

$$= (s^2 - s^2) \frac{\tilde{a} \tilde{u}_{yy}}{s^2 s^2} - (\tilde{a}(x\tilde{s}(t), t) - a(x\tilde{s}(t), t) + a(x\tilde{s}(t), t) - a(xs(t), t)) \tilde{u}_{yy}$$

$$= (s^2 - s^2) \frac{\tilde{a} \tilde{u}_{yy}}{s^2 s^2} - (\Delta a(x\tilde{s}(t), t) + a(x\tilde{s}(t), t) - a(xs(t), t)) \tilde{u}_{yy}$$

(46)

By definition,

$$\left\| \left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy} \right\|_{B_{2, x, t}^{1/2+2\alpha, 1/4+\alpha}(Q_T)} = I_1 + I_2 + I_3$$

where

$$I_1 = \left\| \left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy} \right\|_{L_2(Q_T)}$$

$$I_2 = \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy} \left| \frac{x}{y} \right|^2 dx dy dt \right)^{1/2}$$

$$I_3 = \left( \int_0^T \int_0^T \int_0^1 \left( \frac{\tilde{a}}{s^2} - \frac{\tilde{\alpha}}{s^2} \right) \tilde{u}_{yy} \left| \frac{t}{\tau} \right|^2 dt d\tau \right)^{1/2}$$

Estimate $I_1$ using (46) and Minkowski’s inequality

$$I_1 \leq \frac{1}{\delta^2} \left[ \frac{2\ell}{\delta^2} \|\Delta s\|_{W_{2}^{1/2+2\alpha, 0}[0, T]} + \|\Delta a\|_{C(Q_T)} \right.$$

$$\left. + \|a\|_{C^{1/2+2\alpha, 0}(Q_T)} \|\Delta s'\|_{W_{2}^{1/2+2\alpha}[0, T]} \right]$$

(47)
Estimate (47) implies that $I_1 \to 0$ as $\Delta v \to 0$. By using (46) and the Minkowski’s inequality we estimate $I_2$ as follows:

$$I_2 \leq I_{21} + I_{22} + I_{23},$$

$$I_{21} = \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{(s^2 - s^2)}{\alpha_{yy} x^2} \right) \frac{\hat{a}(x, t)}{\alpha} \left| \frac{x}{y} \right|^2 \int dx \, dy \, dt \right)^\frac{1}{2}$$

$$I_{22} = \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{\Delta a(x, t)}{\alpha} \frac{\hat{u}_{yy} x^2}{|y|^2} \right) \frac{\hat{a}(x, t)}{\alpha} \left| \frac{x}{y} \right|^2 \int dx \, dy \, dt \right)^\frac{1}{2}$$

$$I_{23} = \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{(a(x, t) - a(x, t))}{\alpha} \frac{\hat{u}_{yy} x^2}{|y|^2} \right) \frac{\hat{a}(x, t)}{\alpha} \left| \frac{x}{y} \right|^2 \int dx \, dy \, dt \right)^\frac{1}{2}$$

$I_{21}$ is easily estimated using Minkowski’s inequality as

$$I_{21} \leq \frac{2^2}{\alpha \delta} \|\Delta s'\|_{W^1_2[0, T]} \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{\hat{a}(x, t) - \hat{a}(y, t)}{\alpha} \left| \frac{x}{y} \right|^2 \int dx \, dy \, dt \right)^\frac{1}{2}$$

$$+ \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{\hat{a}(y, t)}{\alpha} \left| \frac{x}{y} \right|^2 \int dx \, dy \, dt \right) \right)^\frac{1}{2}$$

By condition (9), it follows that

$$I_{21} \leq \frac{4^2}{\alpha \delta^2} \|\Delta s'\|_{W^1_2[0, T]} \|a\|_{C^{1, 1/2 + 2\alpha, 0} (D)} \|\hat{u}\|_{B^{5/2 + 2\alpha, 5/4 + \alpha}(Q_T)} \tag{48}$$

$I_{22}$ is also estimated using Minkowski’s inequality as

$$I_{22} \leq \frac{1}{\delta^2} \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{\Delta a(x, t)}{\alpha} \frac{\hat{u}_{yy} x^2}{|y|^2} \right) \int dx \, dy \, dt \right)^\frac{1}{2}$$

$$+ \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{\Delta a(y, t)}{\alpha} \frac{\hat{u}_{yy} x^2}{|y|^2} \right) \int dx \, dy \, dt \right)^\frac{1}{2}$$

Using the boundedness of $\Delta a$ and membership of $\Delta a$ in $C^{1, 1/2 + 2\alpha, 0}$, it follows that

$$I_{22} \leq \frac{C^2 \delta^{1/2 + 2\alpha}}{\delta^2} \|\Delta a\|_{C^{1, 1/2 + 2\alpha, 0} (D)} + \|\Delta a\|_{C(D)} \left\| \hat{u} \right\|_{B^{5/2 + 2\alpha, 5/4 + \alpha}(Q_T)} \tag{49}$$

Estimation of $I_{23}$ is completed as in [3, eq. 45]. Therefore, from (48), (49) it follows that $I_2 \to 0$ as $\Delta v \to 0$. Consider the last term $I_3$; by (46) and Minkowski’s inequality,

$$I_3 \leq I_{31} + I_{32} + I_{33}$$

where

$$I_{31} = \left( \int_0^T \int_0^1 \int_0^1 \left( \frac{(s^2 - s^2)}{\alpha_{yy} x^2} \right) \frac{\hat{a}(x, t)}{\alpha} \left| \frac{x}{y} \right|^2 \int dx \, dy \, dt \right)^\frac{1}{2}$$
\[ I_{32} = \left( \int_0^T \int_0^T \int_0^1 \frac{\Delta a(x \xi(t), t) \frac{\partial u}{\partial x}(t)}{|t - \tau|^{1+2(1/4+\alpha)}} \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

\[ I_{33} = \left( \int_0^T \int_0^T \int_0^1 \frac{\Delta a(x \xi(t), t) - \Delta a(x \xi(\tau), \tau)}{|t - \tau|^{1+2(1/4+\alpha)}} \frac{\partial u}{\partial x}(t) \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

The estimation of \( I_{31} \) and \( I_{33} \) coincide with the estimation of the same term in the proof of Lemma 6.6 in [3]; \( I_{32} \) will be estimated in a similar way as well. Write

\[ I_{32} \leq \frac{1}{\delta^2} (I^1_{32} + I^2_{32} + I^3_{32} + I^4_{32}) \]

where

\[ I^1_{32} = \left( \int_0^T \int_0^T \int_0^1 \frac{\Delta a(x \xi(t), t) - \Delta a(x \xi(t), t)}{|t - \tau|^{1+2(1/4+\alpha)}} \frac{\partial u}{\partial x}(t) \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

\[ I^2_{32} = \left( \int_0^T \int_0^T \int_0^1 \frac{\Delta a(x \xi(\tau), \tau) - \Delta a(x \xi(\tau), \tau)}{|t - \tau|^{1+2(1/4+\alpha)}} \frac{\partial u}{\partial x}(t) \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

\[ I^3_{32} = \left( \int_0^T \int_0^T \int_0^1 \frac{\Delta a(x \xi(\tau), \tau) - \Delta a(x \xi(t), t)}{|t - \tau|^{1+2(1/4+\alpha)}} \frac{\partial u}{\partial x}(t) \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

\[ I^4_{32} = \left( \int_0^T \int_0^T \int_0^1 \frac{\Delta a(x \xi(\tau), \tau) - \Delta a(x \xi(t), t)}{|t - \tau|^{1+2(1/4+\alpha)}} \frac{\partial u}{\partial x}(t) \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

Estimate in \( I^1_{32} \) using condition (9), mean value theorem, and Morrey’s inequality, it follows that

\[ I^1_{32} \leq C\ell^2 \|\Delta a\|_{C^{1/2+\alpha}(D)} \|\xi\|_{W^{1/2+\alpha}_t[0,T]} \|\overline{u}\|_{L_2(Q_T)} \]

(50)

Similarly,

\[ I^2_{32} \leq C\ell^2 \|\Delta a\|_{C^{1/2+\alpha}(D)} \|\overline{u}\|_{L_2(Q_T)} \]

(51)

Estimate in \( I^3_{32} \) using condition (9) to derive

\[ I^3_{32} \leq \|\Delta a\|_{C(D)} \left( \int_0^T \int_0^T \int_0^1 \frac{|\dot{s}(t) - \dot{s}(\tau)|^2 |\ddot{s}(\tau) + \ddot{s}(t)|^2 |\overline{u}|_{\partial x}(t)}{|t - \tau|^{1+2(1/4+\alpha)}} \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

Applying mean value theorem and Morrey’s inequality, it follows that

\[ I^3_{32} \leq 2\ell \|\Delta a\|_{C(D)} \|\xi\|_{W^{1/2}_x[0,T]T^{1/4-\alpha}} \|\overline{u}\|_{L_2(Q_T)} \]

(52)

The last term, \( I^4_{32} \), requires more careful treatment. Estimating as for the other terms, we derive

\[ I^4_{32} = \ell^2 \|\Delta a\|_{C(D)} \left( \int_0^T \int_0^T \int_0^1 \frac{|\overline{u}|_{\partial x}(t) - \overline{u}|_{\partial x}(\tau)|^2}{|t - \tau|^{1+2(1/4+\alpha)}} \ d\tau \ d\tau \ dt \right)^{\frac{1}{2}} \]

For arbitrary \( u \in B^{5/2+2\alpha,5/4+\alpha}_2(Q_T) \), the highest order space derivative does not have the required time regularity. However, the fact that \( \overline{u} \) is a pointwise a.e.
solution of (34) does imply the required regularity through the transformation of this term. Indeed, it follows that

\[
I_{32} \leq \ell^2 \|\Delta a\|_{C(D)} \left( \left( \int_0^T \int_0^T \int_0^1 \left| \frac{a_{x}+s\hat{b}+ys's\hat{u}_y}{a} \right|^2 |t-\tau|^{1+2(1/4+\alpha)} \, dx \, d\tau \, dt \right)^{\frac{1}{2}} + \right.
\]

\[
+ \left( \int_0^T \int_0^T \int_0^1 \frac{|s|^2(t)\tilde{c}u}{a|\tau|^4} \, dx \, d\tau \, dt \right)^{\frac{1}{2}} + \left. \left( \int_0^T \int_0^T \int_0^1 \frac{|s|^2f|t|^4}{a|\tau|^4} \, dx \, d\tau \, dt \right)^{\frac{1}{2}} + \right)
\]

\[
\left( \int_0^T \int_0^T \int_0^1 \frac{\tilde{u}_x(x,t)}{a(x,t)} - \frac{\tilde{u}_x(x,\tau)}{a(x,\tau)} |t-\tau|^{1+2(1/4+\alpha)} \, dx \, d\tau \, dt \right)^{\frac{1}{2}} \right) \tag{53}
\]

Each term is now handled in a relatively easy way using elementary estimates and Sobolev embedding as in [3, lem. 6.6, pp. 19–20]. Having (50)–(53), it follows that

\[
I_3 \to 0 \text{ as } \Delta v \to 0, \text{ and hence } \|\Gamma_2\| \to 0 \text{ as } \Delta v \to 0. \]

We would next like to show

\[
\|\Gamma_6\|_{B^{1/2+2\alpha,1/4+\alpha}_{2+x,t}(Q_T)} = \|\Delta \tilde{c}u\|_{B^{1/2+2\alpha,1/4+\alpha}_{2+x,t}(Q_T)} \to 0 \text{ as } \Delta v \to 0 \text{ in } H
\]

By definition,

\[
\|\Delta \tilde{c}u\|_{B^{1/2+2\alpha,1/4+\alpha}_{2+x,t}(Q_T)} = I_1 + I_2 + I_3
\]

where

\[
I_1 = \|\Delta \tilde{c}u\|_{L^2(Q_T)}
\]

\[
I_2 = \left( \int_0^T \int_0^T \int_0^1 \frac{|\Delta \tilde{c}(x,t)\tilde{u}(x,t) - \Delta \tilde{c}(y,t)\tilde{u}(y,t)|^2}{|x-y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}}
\]

\[
I_3 = \left( \int_0^T \int_0^T \int_0^1 \frac{|\Delta \tilde{c}(x,t)\tilde{u}(x,t) - \Delta \tilde{c}(x,\tau)\tilde{u}(x,\tau)|^2}{|t-\tau|^{1+2(1/4+\alpha)}} \, dx \, d\tau \, dt \right)^{\frac{1}{2}}
\]

Estimate \( I_1 \) as

\[
I_1 = \left( \int_0^T \int_0^1 |\Delta \tilde{c}(x,t)\tilde{u}(x,t)|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \|\tilde{u}\|_{C(Q_T)} (I_{11} + I_{12})
\]

where

\[
I_{11} = \left( \int_0^T \int_0^1 |\tilde{c}(x\tilde{s}(t), t) - \tilde{c}(x_s(t), t)|^2 \, dx \, dt \right)^{\frac{1}{2}},
\]

\[
I_{12} = \left( \int_0^T \int_0^1 |\Delta \tilde{c}(x_s(t), t)|^2 \, dx \, dt \right)^{\frac{1}{2}}
\]
where

\[ I_{11} \leq \|\tilde{c}\|_{C^{1/2+\alpha, 0}(Q_T)} \left( \int_0^T \int_0^1 |\Delta s(t)|^{1+4\alpha^*} \, dx \, dt \right)^{\frac{1}{2}} \]

\[ \leq T^{\frac{1}{2}} \|\tilde{c}\|_{C^{1/2+2\alpha, 0}(Q_T)} \|\Delta s\|_{W_2^{1/2+2\alpha}} \]  

and \( I_{12} \) is bounded by

\[ I_{12} \leq \left( \int_0^T \int_0^1 |\Delta c(xs(t), t)|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\delta}} \|\Delta c\|_{L^2(Q_T)} \]  

By (54)–(55), it follows that \( I_2 \to 0 \) as \( \Delta v \to 0 \).

By Minkowski’s inequality,

\[ I_2 \leq I_{2,1} + I_{2,2} \]

where

\[ I_{2,1} = \left( \int_0^T \int_0^1 \int_0^1 \frac{|\Delta \tilde{c}(x, t)|^2 |\tilde{u}(x, t) - \tilde{u}(y, t)|^2}{|x - y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

\[ I_{2,2} = \left( \int_0^T \int_0^1 \int_0^1 \frac{|\Delta \tilde{c}(x, t) - \Delta \tilde{c}(y, t)|^2 |\tilde{u}(y, t)|^2}{|x - y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

Estimate \( I_{2,1} \) using boundedness of \( \Delta c \) as

\[ I_{2,1} \leq \|\Delta \tilde{c}\|_{C(Q_T)} \left( \int_0^T \int_0^1 \int_0^1 \frac{|\tilde{u}(x, t) - \tilde{u}(y, t)|^2}{|x - y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

\[ \leq \|\Delta \tilde{c}\|_{C(Q_T)} \|\tilde{u}\|_{B^{1/2+2\alpha, 0}(Q_T)} \]  

\[ (56) \]

\( I_{2,2} \) is estimated similarly, using boundedness of \( \tilde{u} \):

\[ I_{2,2} \leq \|\tilde{u}\|_{C(Q_T)} \left( \int_0^T \int_0^1 \int_0^1 \frac{|\Delta \tilde{c}(x, t) - \Delta \tilde{c}(y, t)|^2}{|x - y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

\[ \leq \|\tilde{u}\|_{C(Q_T)} (N_1 + N_2) \]

where

\[ N_1 = \left( \int_0^T \int_0^1 \int_0^1 \frac{|\Delta c(xs(t), t) - \Delta c(y\tilde{s}(t), t)|^2}{|x - y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

\[ N_2 = \left( \int_0^T \int_0^1 \int_0^1 \frac{|c', (\cdot) t|^{x\pi(t)} - c', (\cdot) t| y\pi(t)}{|x - y|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

Estimate \( N_1 \) as

\[ N_1 \leq \left( \int_0^T \int_0^1 \int_0^1 \frac{|\Delta c(u, t) - \Delta c(v, t)|^2}{|\tilde{s}(t)|^{4\alpha}|u - v|^{1+2(1/2+2\alpha)}} \, dx \, dy \, dt \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{\delta^{2\alpha}} \|\Delta c\|_{B^{1/2+2\alpha, 0}(D)} \]  

\[ (57) \]
We show $N_2$ goes to zero by applying Lebesgue’s Dominated Convergence Theorem. Since the integrand converges pointwise to zero, we must show it is bounded by an integrable function. Estimate the integrand as follows:

$$
\left| \frac{c(x\bar{s}(t), t) - c(xs(t), t) - (c(y\bar{s}(t), t) - c(y,s(t), t))}{|x - y|^{1+2(1/2+2\alpha)}} \right|^2
\leq 2 \left| \frac{c(x\bar{s}(t), t) - c(y\bar{s}(t), t)}{|x - y|^{1+2(1/2+2\alpha)}} \right|^2 + \left| \frac{c(xs(t), t) - c(y,s(t), t)}{|x - y|^{1+2(1/2+2\alpha)}} \right|^2
\leq 2 \left\| c \right\|_{C^{1/2+2\alpha}(D)} \left| \frac{s(t)}{|x - y|^{1+4\alpha}} \right| \left| \frac{1+4\alpha}{|x - y|^{1+2(1/2+2\alpha)}} \right|
\leq 2 \left\| c \right\|_{C^{1/2+2\alpha}(D)} \left( \frac{1+2(1/2+2\alpha)}{|x - y|^{1-4(\alpha^* - \alpha)}} \right)
$$

which is integrable since $1 - 4(\alpha^* - \alpha) < 1$. Therefore, Lebesgue’s Dominated Convergence theorem implies that $N_2 \to 0$ as $\Delta \nu \to 0$. Having (56)–(58), it follows that $I_2 \to 0$ as $\Delta \nu \to 0$.

Considering $I_3$, applying Minkowski’s inequality again we derive

$$I_3 \leq I_{3,1} + I_{3,2},$$

where

$$I_{3,1} = \left( \int_0^1 \int_0^T \int_0^T \left| \frac{\Delta c(x, \tau)}{|x - t|^{1+2(1/4+\alpha)}} \right|^2 d\tau dt dx \right)^{\frac{1}{2}}$$

$$I_{3,2} = \left( \int_0^1 \int_0^T \int_0^T \left| \Delta c(x, \tau) - \Delta \tilde{c}(x, t) \right|^2 \left| \frac{\tilde{u}(x, t)}{|x - t|^{1+2(1/4+\alpha)}} \right|^2 d\tau dt dx \right)^{\frac{1}{2}}$$

$I_{3,1}$ is estimated in a similar way to $I_{2,1}$:

$$I_{3,1} \leq \left\| \Delta c \right\|_{C(Q_T)} \left( \int_0^1 \int_0^T \int_0^T \left| \frac{\tilde{u}(x, \tau) - \tilde{u}(x, t)}{|x - t|^{1+2(1/4+\alpha)}} \right|^2 d\tau dt dx \right)^{\frac{1}{2}}$$

$$\leq \left\| \Delta c \right\|_{C(Q_T)} \left\| \tilde{u} \right\|_{B^{0,1/4+\alpha}(Q_T)}$$

(59)

Lastly, $I_{3,2}$ is estimated through the boundedness of $\tilde{u}$:

$$I_{3,2} \leq \left\| \tilde{u} \right\|_{C(Q_T)} \left( \int_0^1 \int_0^T \int_0^T \left| \frac{\Delta c(x, \tau) - \Delta \tilde{c}(x, t)}{|x - t|^{1+2(1/4+\alpha)}} \right|^2 d\tau dt dx \right)^{\frac{1}{2}}$$

$$\leq \left\| \tilde{u} \right\|_{C(Q_T)} (N_3 + N_4 + N_5)$$

where

$$N_3 = \left( \int_0^1 \int_0^T \int_0^T \left| \frac{\Delta c(x, \tau) - \Delta c(x, \tilde{s}(\tau), \tau)}{|x - t|^{1+2(1/4+\alpha)}} \right|^2 d\tau dt dx \right)^{\frac{1}{2}}$$

$$N_4 = \left( \int_0^1 \int_0^T \int_0^T \left| \frac{\Delta c(x, \tilde{s}(\tau), \tau) - \Delta c(x, \tilde{s}(t), \tau)}{|x - t|^{1+2(1/4+\alpha)}} \right|^2 d\tau dt dx \right)^{\frac{1}{2}}$$
\[ N_5 = \left( \int_0^1 \int_0^T \int_0^T \left| \frac{\tilde{F} - \tilde{F}^e}{\tau - t} \right|^2 \, d\tau \, dt \, dx \right)^\frac{1}{2} \]

By condition (9), it follows that

\[ N_3 \leq \| \Delta c \|_{C^{1/2+\alpha, 0}(D)} \left( \int_0^T \int_0^T \int_0^1 \frac{|x(s(\tau) - \tilde{s}(t))|^{1+2\alpha}}{|\tau - t|^{1+2(1/4+\alpha)}} \, dx \, d\tau \, dt \right)^\frac{1}{2} \]

\[ N_3 \leq C \| \Delta c \|_{C^{1/2+\alpha, 0}(D)} \| s' \|_{W^{1,1}_2[0,T]} \] \hspace{1cm} (60)

Considering \( N_4 \), by letting \( u = x\tilde{s}(t) \), derive

\[ N_4 \leq \frac{1}{\delta^{1/2}} \left( \int_0^1 \int_0^T \int_0^T \frac{|\Delta c(u, \tau) - \Delta c(u, t)|^2}{|\tau - t|^{1+2(1/4+\alpha)}} \, d\tau \, dt \, dx \right)^\frac{1}{2} \]

\[ \leq \frac{1}{\delta^{1/2}} \| \Delta c \|_{B^{0,1/4+\alpha}(Q_T)} \] \hspace{1cm} (61)

We show \( N_5 \) goes to zero by applying Lebesgue’s Dominated Convergence Theorem. Since the integrand converges pointwise to zero, we must show it is bounded by an integrable function. Estimate the integrand as

\[ \frac{|c(x\tilde{s}(\tau), \tau) - c(x\tilde{s}(t), t) - (c(xs(\tau), \tau) - c(xs(t), t))|^2}{|\tau - t|^{1+2(1/4+\alpha)}} \leq 4(N_6 + N_7 + N_8 + N_9) \]

where

\[ N_6 = \frac{|c(x\tilde{s}(\tau), \tau) - c(x\tilde{s}(t), \tau)|^2}{|\tau - t|^{1+2(1/4+\alpha)}}, \quad N_7 = \frac{|c(x\tilde{s}(t), \tau) - c(x\tilde{s}(t), t)|^2}{|\tau - t|^{1+2(1/4+\alpha)}}, \]

\[ N_8 = \frac{|c(xs(\tau), \tau) - c(xs(t), \tau)|^2}{|\tau - t|^{1+2(1/4+\alpha)}}, \quad N_9 = \frac{|c(xs(t), \tau) - c(xs(t), t)|^2}{|\tau - t|^{1+2(1/4+\alpha)}}. \]

Estimate \( N_6 \) as

\[ N_6 \leq \| c \|_{C^{1/2+2\alpha^*, 0}(Q_T)} \left( \frac{|x(s(\tau) - \tilde{s}(t))|^{1+2\alpha^*}}{|\tau - t|^{1+2(1/4+\alpha)}} \right)^2 \]

\[ \leq \| s' \|_{C(0,T)} \| c \|_{C^{1/2+2\alpha^*, 0}(Q_T)} \frac{1}{|\tau - t|^{1/2-2(\alpha^* - \alpha)}} \] \hspace{1cm} (62)

Which is integrable since \( 1/2 - 2(\alpha^* - \alpha) < 1 \). Similarly, estimate \( N_7 \) as

\[ N_7 \leq \| c \|_{C^{0,1/4+\alpha^*, 0}(Q_T)} \frac{1}{|\tau - t|^{1-2(\alpha^* - \alpha)}} \] \hspace{1cm} (63)

Which is integrable since \( 1-2(\alpha^* - \alpha) < 1 \). \( N_8 \) and \( N_9 \) are estimated in a similar way to \( N_6 \) and \( N_7 \), respectively. Hence, Lebesgue’s Dominated Convergence theorem implies \( J_{32} \to 0 \) as \( \Delta v \to 0 \). Having (59)–(63), it follows that \( J_3 \to 0 \) as \( \Delta v \to 0 \), and hence \( \| \Gamma_6 \| \to 0 \) as \( \Delta v \to 0 \). \[\square\]
7. Proofs of the main results.

Proof of Theorem 4.3. Let \( \{v_n = (a_n, b_n, c_n, f_n, g_n, s_n)\} \subseteq V_R \) be a minimizing sequence, i.e.
\[
J(v_n) \to J_*
\]
Since the components \((a_n, b_n, c_n)\) are members of a bounded subset of
\[
\tilde{H}_* := C^{3/2+2\alpha^*,3/4+\alpha^*}(D) \times C^{1/2+2\alpha^*,1/4+\alpha^*}(D) \times C^{1/2+2\alpha^*,1/4+\alpha^*}(D),
\]
from Arzela-Ascoli theorem [24, thm. 1.5.10], it follows that there is a subsequence \((a_{n_k}, b_{n_k}, c_{n_k})\) which converges to an element \((a_*, b_*, c_*)\) in
\[
\tilde{H} := C^{3/2+2\alpha,3/4+\alpha}(D) \times C^{1/2+2\alpha,1/4+\alpha}(D) \times C^{1/2+2\alpha,1/4+\alpha}(D).
\]
Assume the whole sequence \((a_n, b_n, c_n)\) converges in \(\tilde{H}\). Similarly, \((f_n, g_n, s_n)\) are members of a bounded and closed subset of an Hilbert space \(B^{1,1/4+\alpha}(D) \times B^{1/2+\alpha}(0, T) \times B^{2}_2(0, T)\), the sequence is weakly precompact (see e.g. [37, ch. V, § 2, p.126]); that is, there exists a subsequence \((f_{n_k}, g_{n_k}, s_{n_k})\) which converges weakly. Assume the whole sequence converges weakly to \((f_*, g_*, s_*)\). Note that in fact, from the condition (9) it follows that \((a_{n_k}, b_{n_k}, c_{n_k})\) is uniformly bounded by (38); that is, there exists \(C > 0\) such that
\[
\|\tilde{u}_n\|_{W^{2,1}(Q_T)} \leq C.
\]
It follows that there exists an element \(\tilde{w} \in W^{2,1}(Q_T)\) such that for some subsequence \(n_k\),
\[
\tilde{u}_{n_k} \to \tilde{w} \text{ weakly in } W^{2,1}(Q_T)
\]
Multiply (1) written for \(\tilde{u}_{n_k}\) by an arbitrary test function \(\Phi \in L_2(Q_T)\) to derive
\[
0 = \int\int_{Q_T} \left[\tilde{L}_{n_k} \tilde{u}_{n_k} - \tilde{f}_{n_k}\right] \Phi \, dx \, dt = I_1 + I_2
\]
where
\[
I_1 := \int\int_{Q_T} \left[\tilde{L}_{n_k} \tilde{u}_{n_k} - \tilde{L}_{n_k}\tilde{u}_{n_k}\right] \Phi \, dx \, dt
\]
\[
I_2 := \int\int_{Q_T} \left[\tilde{L}_{n_k} \tilde{u}_{n_k} - \tilde{f}_{n_k}\right] \Phi \, dx \, dt
\]
Uniform convergence of \((a_{n_k}, b_{n_k}, c_{n_k})\) and boundedness of \(\tilde{u}_{n_k}\) and \(\Phi \) in \(L_2(Q_T)\) imply that \(I_1 \to 0\) as \(n_k \to \infty\): weak convergence of \(\tilde{u}_{n_k}\) to \(\tilde{w}\) implies that \(\tilde{w}\) is a \(W^{2,1}_2(Q_T)\) weak solution of (34)–(37). Since all weak limit points of \(\{\tilde{u}_n\}\) are \(W^{2,1}_2(Q_T)\) weak solutions of the same equation, by uniqueness of the weak solution, it follows that \(\tilde{w} = \tilde{u}_*\) and the whole sequence \(\tilde{u}_n \to \tilde{u}_*\) weakly in \(W^{2,1}_2(Q_T)\). Sobolev trace theorem [6] then implies that
\[
\tilde{u}_n(y, T) \to \tilde{u}_*(y, T) \text{ in } C[0,1]
\]
and hence
\[ u_n(y s_n(T), T) \to u_*(y s_*(T), T) \text{ in } C[0, 1] \]
Together with the convergence of \( s_n(t) \to s_*(t) \) uniformly on \([0, T]\), it follows that
\[ \int_0^{s_n(T)} |u_n(x, T) - w(x)|^2 \, dx - \int_0^{s_n(T)} |u_*(x, T) - w(x)|^2 \, dx \to 0 \]
The other two terms in \( \mathcal{J}(v) \) are handled similarly, so it follows that
\[ \lim_{n \to \infty} \mathcal{J}(v_n) = \mathcal{J}(v_*) = \mathcal{J}_* \]
Lemma is proved.

**Proof of Theorem 4.4.** Consider the increment
\[ \Delta \mathcal{J} := \mathcal{J}(\tilde{v}) - \mathcal{J}(v) = J_1 + J_2' + J_3', \tag{64} \]
where
\[ J_1 := \beta_0 \int_0^{\hat{s}(T)} [2(u - w) \Delta u]_{t=T} \, dx + \beta_0 |u(s(T), T) - w(s(T))|^2 \Delta s(T) \]
\[ R_1 + \cdots + R_4, \tag{66} \]
\[ J_2' := \beta_1 \int_{T_1}^{\hat{s}(T)} \left| \mathfrak{P}(\bar{s}(t), t) - \bar{s}(t) \right|^2 - \| u(s(t), t) - \bar{s}(t) \|^2 \, dt, \]
\[ J_3' := \beta_2 |\bar{s}(T) - s_*|^2 - |s(T) - s_*|^2, \tag{65} \]
We have
\[ J_1 = \beta_0 \int_0^{\hat{s}(T)} [2(u - w) \Delta u]_{t=T} \, dx + \beta_0 |u(s(T), T) - w(s(T))|^2 \Delta s(T) \]
\[ + R_1 + \cdots + R_4, \]
\[ R_1 := \beta_0 \int_0^{\hat{s}(T)} \left| \Delta u(x, T) \right|^2 \, dx, \quad R_2 := 1_{T_1}(T) \beta_0 \Delta s(T) |u(x, T) - w(x)|^2 \big|_{x=s(T)}, \]
\[ R_3 := 1_{T_2}(T) \beta_0 \left( |\mathfrak{P}(\bar{s}(T), T) - w(\bar{s}(T))|^2 - |\mathfrak{P}(s(T), T) - w(s(T))|^2 \right) \Delta s(T), \]
\[ R_4 := 1_{T_2}(T) \beta_0 \left( |\mathfrak{P}(s(T), T) - w(s(T))|^2 - |u(s(T), T) - w(s(T))|^2 \right) \Delta s(T). \]
On applying mean value theorem to \( u(\cdot, t) \) we have
\[ \bar{u}(\tilde{s}(t), t) - u(s(t), t) = \begin{cases} u_x(\tilde{s}(t), t) \Delta s(t) + \Delta u(\tilde{s}(t), t), & \text{if } t \in T_1, \\ \Delta u(s(t), t) + \mathfrak{P}_x(\tilde{s}(t), t) \Delta s(t), & \text{if } t \in T_2 \end{cases} \]
where \( \tilde{s}(t) = s(t) + \theta(t) \Delta s(t), 0 \leq \theta(t) \leq 1 \). It follows that
\[ J_2' = \beta_1 \int_{T_1}^{\hat{s}(T)} 2(u(s(t), t) - \mu(t)) u_x(s(t), t) \Delta s(t) \, dt \]
\[ + \beta_1 \int_{T_1}^{\hat{s}(T)} 2(u(s(t), t) - \mu(t)) \Delta u(\tilde{s}(t), t) \, dt + R_5 + R_6, \tag{67} \]
\[ R_5 := \int_{T_1}^{\hat{s}(T)} \beta_1 |u_x(\tilde{s}(t), t) \Delta s(t) + \Delta u(\tilde{s}(t), t)|^2 \, dt, \]
\[ R_6 := \int_{T_1}^{\hat{s}(T)} 2\beta_1 (u - \mu)_{x=s(t)} [u_x]_{x=s(t)} \Delta s \, dt, \]
and

\[ J''_2 = \beta_1 \int_{T_2} 2(u(s(t), t) - \mu(t)) \Delta u(s(t), t) dt \]
\[ + \beta_1 \int_{T_2} 2(u(s(t), t) - \mu(t))u_x(s(t), t) \Delta s(t) dt + R_7 + R_8 + R_9, \]  

(68)

\[ R_7 := \int_{T_2} \beta_1 |\Delta u(s(t), t) + \pi_x(s(t), t)|^2 dt, \]
\[ R_8 := \int_{T_2} 2\beta_1 (u - \mu)_{x=s(t)} |\pi_x|_{x=s(t)} \Delta s dt, \]
\[ R_9 := \int_{T_2} 2\beta_1 (u(s(t), t) - \mu(t)) \Delta u_x(s(t), t) \Delta s dt. \]

Hence from (65), (66), (67), (68), it follows that

\[ \Delta J = \beta_0 \int_0^{s(T)} [2(u-w)\Delta u]_{t=T} dx + \beta_0 |u(s(T), T) - w(s(T))|^2 \Delta s(T) \]
\[ + \beta_1 \int_0^T 2(u(s(t), t) - \mu(t))u_x(s(t), t) \Delta s(t) dt \]
\[ + \beta_1 \int_0^T 2(u(s(t), t) - \mu(t)) \Delta u(s(t), t) dt \]
\[ + 2\beta_2 (s(T) - s_* \Delta s(T) + \sum_{i=1}^{10} R_i, \quad R_{10} := \beta_2 |\Delta s(T)|^2. \]  

(69)

Define

\[ \Delta L u \equiv (\Delta au_x)_x + \Delta bu_x + \Delta cu, \quad \bar{L} u = (\Delta L + L) u. \]  

(70)

Let \( \psi \) be a solution of the adjoint problem, and consider

\[ 0 = \int_{\Omega} (\bar{L} u - f) \psi dx dt - \int_{\Omega} [L u - f] \psi dx dt - \int_{\Omega} L^* \psi \Delta u dx dt \]
\[ = I_1 + I_2, \]
\[ I_1 = \int_{\Omega} [\Delta L \psi - \Delta f] \psi dx dt, \quad I_2 = \int_{\Omega} L \Delta u \psi dx dt - \int_{\Omega} L^* \psi \Delta u dx dt \]

Each term in \( I_1 \) is transformed in a similar way; for example,

\[ \int_{\Omega} (\Delta a \bar{u}_x)_x \psi = \int_{T_1} \int_0^{s(t)} (\Delta a \bar{u}_x)_x \psi dx dt + \int_{T_2} \int_0^{\pi(t)} (\Delta a \bar{u}_x)_x \psi dx dt \]
\[ = \int_{T_1} \int_0^{\pi(t)} [(\Delta a u_x)_x + (\Delta a \Delta u_x)_x] \psi dx dt \]
\[ + \int_{T_2} \int_0^{s(t)} [(\Delta a u_x)_x + (\Delta a \Delta u_x)_x] \psi dx dt \]

hence

\[ \int_{\Omega} (\Delta a \bar{u}_x)_x = \int_{\Omega} (\Delta a u_x)_x \psi dx dt + R_{11} + R_{12}, \]
\[ R_{11} = \int_{\Omega} (\Delta a \Delta u_x)_x \psi dx dt, \quad R_{12} = \int_{T_1} \int_0^{\pi(t)} (\Delta a u_x)_x \psi dx dt \]
Transforming the other terms similarly, we derive

\[ I_1 = \int_\Omega [(\Delta u_x)_x + \Delta b u_x + \Delta c u - \Delta f] \psi \, dx \, dt + \sum_{i=13}^{17} R_i, \]  

\[ R_{13} = \int_\Omega \Delta b \Delta u_x \psi \, dx \, dt, \quad R_{14} = \int_{T_1} J_{s(t)} \Delta b u_x \psi \, dx \, dt, \]

\[ R_{15} = \int_\Omega \Delta c \Delta u \psi \, dx \, dt, \quad R_{16} = \int_{T_1} J_{s(t)} \Delta c u \psi \, dx \, dt, \]

\[ R_{17} = \int_{T_1} \int_{s(t)} \Delta f \psi \, dx \, dt \]

Transform \( I_2 \) using integration by parts to derive

\[ I_2 = \int_{T_1} [\psi \Delta u_x]_{x=\pi(t)} \, dx \, dt + \int_0^T [-\psi u_x + (b + s') \psi]_{x=s(t)} \Delta u(\tilde{s}(t), t) \, dt \]

\[ - \int_\Omega \psi \Delta f \, dx \, dt + \int_{T_2} [\psi \Delta u_x]_{x=s(t)} \, dx \, dt - \int_0^T \psi(0, t) \Delta g(t) \, dt \]

\[ + \int_0^T [(\psi u_x - b \psi) \Delta u]_{x=0} \, dt - \int_0^T \tilde{s}(T) \psi(x, T) \Delta u(x, T) \, dx \]

\[ - \int_0^T \psi(0, t) \Delta a(0, t) \Delta u_x(0, t) \, dt + R_{18} + R_{19} + R_{20} \]

where

\[ R_{18} := \int_{T_1} \int_{s(t)} \psi \Delta f \, dx \, dt, \quad R_{19} := \int_{T_1} \left( -\psi u_x + b \psi \right)_{x=s(t)} \Delta u(\tilde{s}(t), t) \, dt, \]

\[ R_{20} := -\int_{T_1} \psi(0, t) \Delta a(0, t) \Delta u_x(0, t) \, dt \]

From (22), it follows that

\[ \int_{T_1} [\psi \Delta u_x]_{x=\pi(t)} \, dx \, dt = \int_{T_1} \left[ \psi \left( \chi_x \Delta s - \gamma_x \Delta ss' - \gamma s' - (au_x)x \Delta s \right) \right]_{x=s(t)} \, dx \, dt \]

\[ - \int_{T_1} [\psi \Delta a u_x]_{x=s(t)} \, dx \, dt + \sum_{i=21}^{28} R_i, \]

\[ R_{21} := \int_{T_1} [\psi]_{x=s(t)} \Delta u_x(\tilde{s}(t), t) \, dt, \]

\[ R_{22} := \int_{T_1} \psi \left|_{x=s(t)} \chi_x - \gamma_x \tilde{s}'(t) - (au_x)x \right|_{x=s(t)} \Delta s \, dt, \]

\[ R_{24} := -\int_{T_1} \psi \left|_{x=s(t)} \Delta s \Delta s' \right| \, dt, \quad R_{26} := -\int_{T_1} \psi(s(t), t) \left[ \Delta a u_x \right]_{x=s(t)} \, dt, \]

\[ R_{27} := \int_{T_1} \psi(s(t), t) \Delta a \Delta u_x \left|_{x=\pi(t)} \right. \, dt, \]

Similarly, from (24) it follows that

\[ \int_{T_2} [\psi \Delta u_x]_{x=\pi(t)} \, dx \, dt = \int_{T_2} \left[ \psi \left( \chi_x \Delta s - \gamma_x s' \Delta s - \gamma \Delta s' - (au_x)x \Delta s \right) \right]_{x=s(t)} \, dx \, dt \]
Combining (72), (73), (74), it follows that
\[- \int_{T_2}^T \left[ \psi \Delta a u_x \right]_{x=s(t)} dt + \sum_{i=28}^{37} R_i, \tag{74}\]

\[R_{28} := \int_{T_2}^T \psi(s(t), t) \left[ \chi_x - \gamma_x s'(t) \right]_{x=s(t)} \Delta s dt,\]
\[R_{29} := - \int_{T_2}^T \psi \gamma_x \left[ x = s(t) \right] \Delta s \Delta s' dt,\]
\[R_{30} := - \int_{T_2}^T \psi(s(t), t) \left[ (a \pi(x))_x \right]_{x=s(t)} \Delta s dt,\]
\[R_{31} := - \int_{T_2}^T \psi(s(t), t) \left[ (a u_x)_x \right]_{x=s(t)} \Delta s dt,\]
\[R_{32} := - \int_{T_2}^T \psi(s(t), t) \left[ \Delta a u_x \right]_{x=s(t)} dt,\]
\[R_{33} := \int_{T_2}^T \psi(s(t), t) u_x(s(t), t) \left( \bar{u}_x(\bar{s}(t), t) - u_x(s(t), t) \right) \Delta s dt,\]
\[R_{34} := - \int_{T_2}^T \psi(s(t), t) \bar{u}_x \left[ \bar{a}_x \right]_{x=s(t)} \Delta s dt,\]
\[R_{35} := - \int_{T_2}^T \psi(s(t), t) \bar{u}_x \left[ a \right]_{x=s(t)} \Delta s dt,\]
\[R_{36} := - \int_{T_2}^T \psi(s(t), t) \bar{u}_x \left[ \bar{a} \right]_{x=s(t)} \Delta s dt,\]
\[R_{37} := - \int_{T_2}^T \psi(s(t), t) \pi_x(\bar{s}(t), t) \left[ \bar{a} \right]_{x=s(t)} \Delta s dt.\]

Combining (72), (73), (74), it follows that
\[I_2 = \int_0^T \left[ \psi \left( \chi_x \Delta s - \gamma_x s' \Delta s - \gamma \Delta s' \right) \right]_{x=s(t)} dt \]
\[+ \int_0^T \left[ -a \psi_x + (b + s') \psi \right]_{x=s(t)} \Delta u(\bar{s}(t), t) dt - \int_0^T \psi \Delta f dx dt \]
\[- \int_0^T \psi(0, t) \Delta g(t) dt + \int_0^T \left[ (a \psi_x - b \psi) \Delta u \right]_{x=0} dt \]
\[- \int_0^T \psi(x, T) \Delta u(x, T) dx + \sum_{i=19}^{37} R_i \tag{75}\]

Adding (69), (71), (75) it follows that
\[\Delta J = \langle J'(v), \Delta v \rangle + \sum_{i=1}^{37} R_i \tag{76}\]

Where the main linear part of the increment is as in (17). It remains to show that \(\sum_{i=1}^{37} R_i = o(\Delta v)\). All of the remainder terms \(R_1-R_{10}\), and \(R_{17}, R_{19}-R_{28}\) are common with the remainder terms derived in [3]; in particular, \(R_{24}\) of this paper is shown to satisfy \(R_{24} = o(\Delta v)\). Therefore, we demonstrate that fact here for the new terms \(R_{11}\) and \(R_{12}\), as the other new terms are estimated similarly. By
condition (9) and CBS inequality,
\[
R_{11} \leq \|a\|_{C_{1,0}^1(D)} \int_{\Omega} \|\Delta u_x + |\Delta u_{xx}| \| \psi \| \, dx \, dt
\]
\[
\leq \|a\|_{C_{1,0}^1(D)} \|\Delta u\|_{W_2^0(\Omega)} \|\psi\|_{L_2(\Omega)}
\]
Uniform boundedness of \(\psi\) in \(L_2(\Omega)\) follows from Lemma 6.5, hence it follows that \(R_{11} \to 0\) from (40). Similarly,
\[
R_{12} \leq \|\Delta a\|_{C_{1,0}^1(D)} \|\psi\|_{C(\Omega)} \int_{T_1} \int_{s(t)}^{\pi(t)} |u_x| + |u_{xx}| \, dx \, dt
\]
By CBS and Morrey’s inequalities, it follows that
\[
R_{12} \leq \|\Delta a\|_{C_{1,0}^1(D)} \|\psi\|_{C(\Omega)} \|u\|_{W_2^0(\Omega)} \|\Delta s\|_{W_2^0([0,T])}^{1/2}
\]
By Sobolev embedding theorem and Lemma 6.5, it follows that \(R_{12} = o(\Delta v)\). All of the other terms are proven in a similar way.

8. Conclusions. The new variational formulation of the inverse Stefan problem introduced in [1, 2], and the Frechet differentiability result established within a Besov spaces framework in [3], are extended to the ISP with unknown coefficients. Existence of an optimal control and Frechet differentiability is proved under minimal regularity assumptions on the data.

The result implies a necessary condition for the optimality in the form of variational inequality, and opens a way for the implementation of an effective numerical method for identification of the unknown coefficients based on the projective gradient method in Besov-Hölder space framework. The main idea of the new variational formulation is optimal control setting, where the free boundary is the component of the control vector. This allows for the development of an iterative gradient type numerical method of low computational cost. It also creates a framework for the regularization of the error existing in the information on the phase transition temperature.

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