Natural star-products on symplectic manifolds and related quantum mechanical operators

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In this paper is considered a problem of defining natural star-products on symplectic manifolds, admissible for quantization of classical Hamiltonian systems. First, a construction of a star-product on a cotangent bundle to an Euclidean configuration space is given with the use of a sequence of pair-wise commuting vector fields. The connection with a covariant representation of such a star-product is also presented. Then, an extension of the construction to symplectic manifolds over flat and non-flat pseudo-Riemannian configuration spaces is discussed. Finally, a coordinate free construction of related quantum mechanical operators from Hilbert space over respective configuration space is presented.

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I. INTRODUCTION

The formalism of quantization of systems described by configuration spaces in the form of Euclidean spaces is well established and confirmed by experiments. The next step should be theory of quantization of systems defined on curved spaces, e.g. systems with constraints or systems coupled with classical gravitational fields. This task however constitutes some problems as, because of the lack of experiments, it is difficult to find a proper generalization of the quantization formalism. The only thing one can do is to work on the mathematical level and try to find some distinguished quantization schemes with interesting properties from the vast number of possibilities.

This paper aims in a discussion of this problem from a point of view of deformation quantization theory. In this approach to quantum mechanics the quantization is basically given by introducing a star-product on a phase space. Thus in this paper we will deal first with a problem of defining natural star-products on symplectic manifolds (phase spaces), and second with their appropriate operator representation in a Hilbert space over configuration space.

In the work of Bayen et al. [1, 2] there was presented a construction of a star-product on a symplectic manifold endowed with a flat symplectic linear connection. Later Fedosov [3] presented a construction of an admissible star-product for a general symplectic connection. The resulting star-products were given in a covariant form independent on the coordinate system. This results, although elegant, are difficult to use in computations. In this paper first we discuss an alternative way of introducing a star-product. It is base on a definition of a star-product with the use of a sequence of pair-wise commuting vector fields defined on a symplectic manifold. In this way equations for star-products are of simpler form and can be easier used in computations. Moreover, we discuss the connection of the vector field representation of the star-product to the covariant form of the star-product (Section II).

An important property of the star-product is an equivalence with the Moyal product. This allows introduction of the operator approach to quantum mechanics [4, 5]. It is known how to pass to the operator representation of quantum mechanics in the case of Euclidean configuration spaces. In a general case we can use the fact that for any classical and quantum canonical coordinate system the star-product is equivalent with the Moyal product. This property allows to construct the operator representation of quantum mechanics from the knowledge of this construction in Euclidean case. In Section III we construct the equivalence for the star-product written in a covariant form on a flat symplectic manifold.

In Section IV we discuss how to introduce star-products on a general symplectic manifold in a natural manner. We also present an example of such products, which construction involves symplectic linear connection on a symplectic manifold.

Section V is devoted to a problem of associating to star-algebras certain algebras of operators defined on particular Hilbert spaces. Usually, in the literature, one can find this connection for a Moyal star-product written in Cartesian
coordinates. The general case seems not to be considered yet. We describe a connection between star-algebras and respective operator algebras for a very general family of star-products considered in the paper. In particular we describe a procedure of associating, in a coordinate independent way, to every phase space function an operator defined on a Hilbert space of square integrable functions defined on a configuration space. We also give examples of operators linear, quadratic and cubic in momenta written in an invariant form and derived for a very general star-product defined on a symplectic manifold over a curved pseudo-Riemannian space.

In Section VI are made some remarks about quantization of classical Hamiltonian systems. We also discuss a problem, using the results presented in the paper, of choosing a physically admissible quantizations for Hamiltonian systems from phase spaces considered in the paper.

II. THE CASE OF A SYMPLECTIC MANIFOLD $T^*E^N$

Let us consider an $N$-dimensional Euclidean space $E^N$. The cotangent bundle $T^*E^N$ to this space is an $2N$-dimensional manifold naturally endowed with a symplectic structure $\omega$. Let us choose some Euclidean coordinate system $(x^1,\ldots,x^N)$ on $E^N$. We can extend this coordinate system to a canonical (Darboux) coordinate system $(x^1,\ldots,x^N,p_1,\ldots,p_N)$ on $T^*E^N$, which we will call an Euclidean coordinate system on the symplectic manifold $T^*E^N$. In this coordinates the symplectic form $\omega$ takes the form $dp_i \wedge dx^i$. Also the Poisson tensor $P = \omega^{-1}$ related to the symplectic form $\omega$ can be written in the form

$$P = \partial_{x^i} \wedge \partial_{p_i}. \quad (\text{II.1})$$

Equation (II.1) shows that the Poisson tensor $P$ can be decomposed into a wedge product of pair-wise commuting vector fields. However, such decomposition is not unique. There are different sets of commuting vector fields $X_1,\ldots,X_N,Y_1,\ldots,Y_N$ such that

$$P = \sum_{i=1}^N X_i \wedge Y_i. \quad (\text{II.2})$$

In what follows we will define a family of star-products on the symplectic manifold $T^*E^N$. Let $X_i,Y_i$ be a sequence of pair-wise commuting global vector fields from the decomposition (II.2) of the Poisson tensor $P$. Define a star-product by the formula

$$f \ast g = f \exp \left( \frac{1}{2}i\hbar \sum_{i} x_i y_i - \frac{1}{2}i\hbar \sum_{i} y_i x_i \right) g. \quad (\text{II.3})$$

From the commutativity of vector fields $X_i,Y_i$ follows the associativity of the star-product. As was pointed out earlier the sequence $X_i,Y_i$ is not uniquely specified by the Poisson tensor, thus we can define the whole family of star-products related to the same Poisson tensor.

For a given sequence of vector fields $X_i,Y_i$ from the decomposition (II.2) of the Poisson tensor $P$ there exists a global coordinate system $(x,p)$ in which $X_i,Y_i$ are coordinate vector fields, i.e. $X_i = \partial_x x_i$, $Y_i = \partial_p p_i$. Such coordinate system is of course a Darboux coordinate system associated with the symplectic form $\omega$. In this coordinates the star-product (II.3) takes the form of a product

$$f \ast g = f \exp \left( \frac{1}{2}i\hbar \partial_x x_i \partial_p p_i - \frac{1}{2}i\hbar \partial_x p_i \partial_p x_i \right) g, \quad (\text{II.4})$$

which is called a Moyal product [1, 2, 6]. The coordinate system $(x,p)$ we will call the natural coordinate system of the star-product.

The structure of the symplectic manifold $T^*E^N$ distinguishes one product from the presented family of star-products, namely the one for which the natural coordinate system is the Euclidean coordinate system. Such star-product is indeed uniquely defined since coordinate vector fields of Euclidean coordinate systems are related to each other by linear symplectic transformations and such transformations do not change the star-product (II.3). This distinguished star-product will be called a canonical star-product on $T^*E^N$.

In what follows let us write the canonical star-product on $T^*E^N$ in a different form. To do this let us first write it in a Darboux coordinate system induced from an arbitrary curvilinear coordinates on $E^N$. Let $\phi: (x^1,\ldots,x^N) \mapsto (x^1,\ldots,x^N)$ be a change of coordinates from arbitrary curvilinear coordinates $(x^1,\ldots,x^N)$ to Euclidean coordinates.
(x', p'). The transformation φ on EN induces a canonical transformation (x', p') \mapsto T(x', p') = (x, p) on the symplectic manifold T*EN:

\[
x^i = \phi^i(x'),
p_i = [(\phi'(x'))^{-1}]^i_j p'_j,
\]

where \([(\phi'(x'))^{-1}]^i_j\) denotes an inverse matrix to the Jacobian matrix \([\phi'(x')]^i_j\) = \(\frac{\partial \phi^i}{\partial x^j}(x')\) of \(\phi\). The transformation \(T\) is called a point transformation.

The canonical star-product in Euclidean coordinates takes the form of a Moyal product (II.4). The Moyal product (II.4) under the point transformation \(T\) transforms to the following star-product:

\[
f \ast_{(x', p')} g = f \exp \left(\frac{i}{2} \hbar \sum_{j} \left(D_{x'^i} - \frac{1}{2} i \hbar \sum j \right) \frac{D_{p'_j}}{D_{x'^i}} \right) g,
\]

where

\[
D_{x'^i} = [(\phi'(x'))^{-1}]^i_j \left( \partial_{x'^i} + \Gamma^i_{jk}(x') p'_j \partial_{p'_k} \right),
\]

\[
D_{p'_j} = \left[\phi'(x')\right]^j_i \partial_{p'_j}
\]
is a transformation of Euclidean coordinate vector fields \(\partial_{x'^i}, \partial_{p'_j}\) to a new coordinate chart, and \(\Gamma^i_{jk}(x') = \left[\phi'(x')\right]^{-1} \left[\phi''(x')\right]_{jk} \left[\phi''(x')\right]^i_l \frac{\partial^2 \phi^i}{\partial x^l \partial x^k}(x')\) is the Hessian of \(\phi\). Note that the symbols \(\Gamma^i_{jk}(x')\) are the Christoffel symbols for the \((x'^1, \ldots, x'^N)\) coordinates, associated to the standard linear connection \(\nabla\) on the configuration space EN. Formula (II.5) can be written in the form

\[
f \ast (x', p') g = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} (-1)^m \left(\frac{i \hbar}{2}\right)^{n+m} (D_{j_1 \ldots j_n} f)(D_{j_1 \ldots j_m} g),
\]

where operators \(D_{i_1 \ldots i_n}\) are given recursively by

\[
D_{i_1 \ldots i_{m+1}} = D_{i_{m+1}}(D_{i_1 \ldots i_m} f) - \Gamma_{i_{m+1} i_1 i_2} D_{i_2 \ldots i_m} f - \cdots - \Gamma_{i_{m+1} i_n i_{n+1}} D_{i_{n+1} \ldots i_1} f,
\]

\[
D_{i_1 \ldots i_{m+1}} = D_{j_1 \ldots j_m} + \sum_{j=m+1}^{n} \Gamma_{i_{m+1} j_{m+1}} D_{j_{m+1} \ldots j_m} f,
\]

\[
D_{i} f = \partial_{x'^i} f + \Gamma_{i j}^k p'_k \partial_{p'_j} f,
\]

\[
D_{j} f = \partial_{p'_j} f,
\]

where \(\{D_i, D_j\}\) is a so called adopted frame on T*EN [7]. Note that the upper indices in the operator \(D_{i_1 \ldots i_n}\) commute with the lower indices, i.e. it does not matter if, when calculating \(D_{i_1 \ldots i_n} f\), we first use formula (II.7a) and then (II.7b) or vice versa.

Equation (II.6) can be written in the form

\[
f \ast (x', p') g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i \hbar}{2}\right)^k \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} \left(\tilde{\nabla} \cdots \tilde{\nabla} f\right)_{i_1 \ldots i_n j_1 \ldots j_{k-n}} \left(\tilde{\nabla} \cdots \tilde{\nabla} g\right)_{\bar{i}_1 \ldots \bar{i}_{n} \bar{j}_1 \ldots \bar{j}_{k-n}},
\]

where \(\bar{i} = N + i\) and \(\tilde{\nabla}\) is a linear connection on the symplectic manifold T*EN, which components in the frame \(\{D_i, D_j\}\) are equal

\[
\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^i_{ik}
\]
with the remaining components equal zero. Equation (II.8) can be written in the form
\[
f * (x', p') \ g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{n=0}^{k} \left(\begin{array}{c} k \\ n \end{array}\right) A^\mu_1 \nu_1 \ldots A^\mu_n \nu_n B^{\mu_n+1} \nu_{n+1} \ldots B^\mu_k \nu_k \nabla f \mu_1 \ldots \mu_k \nabla g \nu_1 \ldots \nu_k, \tag{II.9}
\]

where

\[
A = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}.
\]

Equation (II.9) takes the form

\[
f * (x', p') \ g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k (A + B)^\mu_1 \nu_1 \ldots (A + B)^\mu_k \nu_k \nabla f \mu_1 \ldots \mu_k \nabla g \nu_1 \ldots \nu_k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \omega^\mu_1 \nu_1 \ldots \omega^\mu_k \nu_k \nabla f \mu_1 \ldots \mu_k \nabla g \nu_1 \ldots \nu_k, \tag{II.10}
\]

where

\[
\omega = A + B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

Since \(D_i \wedge \delta^j = \partial_{x^i} \wedge \partial_{p^j}\), \(\omega^{\mu \nu}\) are components of the Poisson tensor in the Darboux frame \(\{\partial_{x^i}, \partial_{p^j}\}\) as well as in the adopted frame \(\{D_i, D^j\}\).

The Christoffel symbols of the linear connection \(\tilde{\nabla}\) in the Darboux coordinate frame take the form

\[
\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^k_{ij}, \quad \tilde{\Gamma}^i_{jk} = p_i (\Gamma^r_{jk} \Gamma^l_{ri} + \Gamma^r_{ik} \Gamma^l_{rj} - \Gamma^l_{ij,k}), \tag{II.11}
\]

with the remaining components equal zero. It is straightforward to check that \(\tilde{\nabla}\) is symplectic, i.e. \(\tilde{\nabla} \omega = 0\). Moreover, from flatness of the configuration space \(E^N\) follows that \(\tilde{\nabla}\) is flat and torsionless.

Thus we wrote the canonical star-product on \(T^*E^N\) in a covariant form (II.10), where \(\tilde{\nabla}\) is a connection induced from a standard Levi-Civita connection on \(E^N\). Other star-products on \(E^N\) also can be written in a covariant form (II.10). As a linear connection \(\tilde{\nabla}\) one has to take a connection which components in a natural coordinate system vanish. However, such connection is not related to a standard Levi-Civita connection on \(E^N\).

Equation (II.11) defines a lift of the Levi-Civita connection on \(E^N\) to a symplectic connection on \(T^*Q\). It is possible to define a lift of the Levi-Civita connection \(\Gamma^i_{jk}\) on a general Riemannian manifold \(Q\) to a symplectic and torsionless connection \(\tilde{\Gamma}^i_{jk}\) on the cotangent bundle \(T^*Q\). The resulting connection in the Darboux coordinate frame is given by the formulas

\[
\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^k_{ij}, \quad \tilde{\Gamma}^i_{jk} = p_i (\Gamma^r_{jk} \Gamma^l_{ri} + \Gamma^r_{ik} \Gamma^l_{rj} - \Gamma^l_{ij,k} - \frac{1}{3} R^l_{ij,k} + \frac{1}{3} R^l_{ijk}), \tag{II.12}
\]

with the remaining components equal zero. In the adopted frame \(\{D_i, D^j\}\) the connection \(\tilde{\Gamma}^i_{jk}\) takes the form

\[
\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^i_{jk} = -\Gamma^k_{ij}, \quad \tilde{\Gamma}^i_{jk} = \frac{1}{3} p_i (R^l_{ij,k} + R^l_{ijk}), \tag{II.13}
\]

with the remaining components equal zero. As we will see later on a symplectic manifold endowed with a symplectic torsionless connection it is possible to distinguish a star-product. In the majority of physically interesting cases as the symplectic manifold is taken the cotangent bundle to a configuration space being a Riemannian manifold. In such case there exists a distinguished connection and thus a star-product which can be used to introduce quantization. More about lifts of connections can be found in [7, 8].

**III. THE CASE OF A SYMPLECTIC MANIFOLD \(T^*Q\) WITH A FLAT BASE MANIFOLD \(Q\)**

The star-product (II.3) can be defined on more general symplectic manifolds. Let \(Q\) be an \(N\)-dimensional flat pseudo-Riemannian manifold with a property that every two points of \(Q\) can be connected by exactly one geodesic.
On such manifold there exists a global Riemann normal coordinate system \((x^1, \ldots, x^N)\). Every such coordinate system is parametrized by a point \(x \in \mathcal{Q}\) and a basis \(e_1, \ldots, e_N\) in \(T_x \mathcal{Q}\). Using the flatness of the manifold \(\mathcal{Q}\) one can check that Riemann normal coordinate systems transform according to the rule

\[
x'^i = A^i_j x^j + x'_0,
\]

(III.1)

where \(x'_0\) are the coordinates of the origin of the second coordinate system from the perspective of the first coordinate system, and \(A^i_j\) is a matrix transforming the basis \(e_1, \ldots, e_N\) of the first coordinate system to a parallel transported basis \(e'_1, \ldots, e'_N\) of the second coordinate system.

The Riemann normal coordinate system \((x^1, \ldots, x^N)\) induces a global canonical coordinate system \((x^1, \ldots, x^N, p_1, \ldots, p_N)\) on a symplectic manifold \(T^* \mathcal{Q}\). We will call this coordinate system a Riemann normal coordinate system on \(T^* \mathcal{Q}\). The canonical Poisson tensor \(\mathcal{P}\) on \(T^* \mathcal{Q}\) using the Riemann normal coordinates can be globally written in the form (II.1).

Using the coordinate vector fields of the Riemann normal coordinate system on \(T^* \mathcal{Q}\) we can introduce a star-product on the symplectic manifold \(T^* \mathcal{Q}\) by the formula (II.4). The Riemann normal coordinate system is then a natural coordinate system for this star-product. Such star-product is independent on the choice of the Riemann normal coordinate system, and \(A\) or equivalently, by distinguishing a flat torsionless symplectic linear connection \(\tilde{\nabla}\) related to each other by linear symplectic transformations and such transformations do not change the star-product.

Thus on the symplectic manifold \(\mathcal{Q}\) there is a distinguished star-product from the family of star-products (II.3) given by the decompositions (II.2) of the Poisson tensor. We will call this product a canonical star-product on \(T^* \mathcal{Q}\).

For Riemann normal coordinates the Christoffel symbols \(\Gamma^i_{jk}\) of the Levi-Civita connection \(\nabla\) on \(\mathcal{Q}\) vanish. Thus also vanish the Christoffel symbols \(\tilde{\Gamma}^i_{jk}\) of the lift (II.12) of the connection \(\nabla\) to a connection \(\nabla\) on \(T^* \mathcal{Q}\). This shows that the canonical star-product on \(T^* \mathcal{Q}\) can be written in a covariant form

\[
f \star g = \sum_{k=0}^{\infty} \frac{1}{k!} \left( i\hbar \frac{\partial}{\partial x} \right)^k \omega^\mu_1 \nu_1 \cdots \omega^\mu_k \nu_k \left( \nabla \cdots \nabla \right) f_{\mu_1 \cdots \mu_k} \left( \nabla \cdots \nabla \right) g_{\nu_1 \cdots \nu_k},
\]

(III.2)

since for Riemann normal coordinates both products coincide. The flatness of the linear connection \(\nabla\) guaranties that the star-product (III.2) is associative.

**Remark III.1.** The star-product (III.3) is also a valid star-product on more general symplectic manifolds. Let us consider a symplectic manifold \(M\) whose Poisson tensor can be written in the form (II.2). In addition, let us assume that the first de Rham cohomology class \(H^1(M)\) vanishes. This will guarantee the existence of global natural coordinate systems associated to the star-products (II.3). On such symplectic manifold \(M\) the product (III.3) is a valid star-product, which can also be written in a covariant form (III.2) with an appropriate linear connection \(\nabla\). However, in this case there is no distinguished star-product from the family of products (II.3). To distinguish a star-product we have to distinguish a sequence of commuting vector fields \(X_i, Y_j\) from the decomposition (II.2) of the Poisson tensor, or equivalently, by distinguishing a flat torsionless symplectic linear connection \(\nabla\) on \(M\).

An important property of the star-product (III.2) used in quantum mechanics (see Section V) is the fact that for a given classical and quantum canonical coordinate system \((x, p)\) the star-product (III.2) is equivalent with a Moyal product associated to the coordinates \((x, p)\) (for details and a definition of a quantum canonical coordinate system see [4, 5])

\[
f \star_M^{(x,p)} g = f \exp \left( \frac{1}{2} i\hbar \hat{\partial}_x \hat{\partial}_{p_i} - \frac{1}{2} i\hbar \hat{\partial}_{p_i} \hat{\partial}_x \right) g.
\]

(III.3)

In other words there exists a formal series of operators

\[S = \text{id} + \sum_{k=1}^{\infty} S_k\]

such that

\[S(f \star_M^{(x,p)} g) = S f \star^{(x,p)} S g.\]

A procedure of a systematic construction of such morphisms can be found in [9]. Using the results presented in this paper we can construct the morphism \(S\) order by order in \(\hbar\). Let us derive the form of \(S\) to the second order in \(\hbar\). It
happens that only terms with even powers in \( \hbar \) are non-zero, thus we only have to calculate \( S_2 \). To find the form of \( S_2 \) we have to solve the following system of equations

\[
[S_2, z^\alpha] = -\frac{1}{4} A^\alpha_k, \quad (\text{III.4a})
\]

\[
[S_2, \partial^\alpha] = -\frac{1}{4} A^\alpha_k, \quad (\text{III.4b})
\]

where

\[
A^\alpha_k f = \frac{1}{k!} \omega^{\mu_1 \nu_1} \cdots \omega^{\mu_k \nu_k} \left( \tilde{\nabla}_{\mu_1} \cdots \tilde{\nabla}_{\mu_k} z^\alpha \right)_{\nu_1 \cdots \nu_k}, \quad (\text{III.5})
\]

and \( z^i = x^i, \ z^{i+N} = p_i \) for \( i = 1, \ldots, N \), \( \partial^\alpha = \omega^{\alpha \beta} \partial_\beta \).

In what follows we will show that the solution to (III.4) is of the form

\[
S_2 = \frac{1}{24} \tilde{\Gamma}^\alpha_{\beta \gamma} \partial^\alpha \partial^\beta \partial^\gamma + \frac{1}{16} \tilde{\Gamma}^\mu_{\alpha \beta} \tilde{\Gamma}^\nu_{\mu \beta} \partial^\alpha \partial^\beta,
\]

where \( \tilde{\Gamma}^\alpha_{\beta \gamma} = \omega_{\alpha \delta} \tilde{\Gamma}^\delta_{\beta \gamma} \) (see Appendix for the proof). Note that the condition that \( \tilde{\nabla} \) has vanishing torsion can be restated as

\[
\tilde{\Gamma}^\alpha_{\beta \gamma} = \tilde{\Gamma}^\alpha_{\gamma \beta}, \quad (\text{III.7})
\]

and the condition that \( \tilde{\nabla} \) is symplectic (\( \omega_{\mu \nu ; \alpha} = 0, \omega^{\mu \nu :\alpha} = 0 \)) in Darboux coordinates can be restated as

\[
\omega^{\delta \beta} \tilde{\Gamma}^\alpha_{\beta \gamma} = \omega^{\alpha \beta} \tilde{\Gamma}^\delta_{\beta \gamma}, \quad (\text{III.8a})
\]

\[
\omega^{\delta \alpha} \tilde{\Gamma}^\alpha_{\beta \gamma} = \omega^{\beta \alpha} \tilde{\Gamma}^\delta_{\beta \gamma}. \quad (\text{III.8b})
\]

From conditions (III.7) and (III.8b) we get that \( \tilde{\nabla} \) is symplectic and torsionless iff \( \tilde{\Gamma}^\alpha_{\beta \gamma} \) is symmetric with respect to indices \( \alpha, \beta, \gamma \) [8].

**IV. THE CASE OF A SYMPLECTIC MANIFOLD \( T^*Q \) WITH A NON-FLAT BASE MANIFOLD \( Q \)**

In this section we will describe a procedure of introducing star-products on a symplectic manifold \( M = T^*Q \) over a pseudo-Riemannian manifold \( (Q, g) \) with a Levi-Civita connection induced by a non-flat metric tensor \( g \), where \( Q \) is not necessarily flat and for which does not necessarily exist a global Riemann normal coordinate system. In such case it is not possible to introduce a star-product by the formula (II.3), and even if there would exist global Riemann normal coordinate systems on \( Q \) they would not be related by the formula (III.1), and because of this different Riemann normal coordinate systems would define different star-products of the form (II.3).

Henceforth, in such general case we will use a connection \( \tilde{\nabla} \) on \( T^*Q \), induced from a Levi-Civita connection \( \nabla \) on \( Q \), to define a star-product. However, a star-product in the form (III.2) for a curved linear connection \( \tilde{\nabla} \) is not a proper star-product (it is not associative). Thus we have to change the star-product (III.2) in such a way that for a curved linear connection \( \tilde{\nabla} \) it would remain associative. Moreover, we would like it to be equivalent with the Moyal product for every classical and quantum canonical coordinate system.

As a special case we can consider a symplectic manifold \( T^*E^N \) with a non-flat symplectic connection (II.12), (II.13) induced by a non-flat connection defined on \( E^N \) (possibly by some non-flat metric). Although in this case there is a global coordinate chart, the star-product of the form (III.2) is not admissible as well.

The general way of defining on a symplectic manifold \( M \) a star-product equivalent with the Moyal product is as follows. As in the general case there is no single global coordinate chart, in order to define a product, which will be equivalent with the Moyal product, it is necessary to do this locally for every classical and quantum canonical coordinate chart. Let us take an atlas of classical and quantum canonical coordinate charts \( (x_\alpha, p_\alpha) \) defined on open subsets \( U_\alpha \) of the symplectic manifold \( M \). Moreover, let us take some family of linear automorphisms \( S_\alpha \) of \( C^\infty(U_\alpha) \) with the property: two morphisms \( S_\alpha \) and \( S_\beta \) when acted on the Moyal products \( \star^{(x_\alpha, p_\alpha)}_M \) and \( \star^{(x_\beta, p_\beta)}_M \) give star-products, which on the intersection \( U_\alpha \cap U_\beta \) are related to each other by the change of variables \( (x_\alpha, p_\alpha) \to (x_\beta, p_\beta) \).

Every such automorphism \( S_\alpha \) can be used to define a star-product on \( C^\infty(U_\alpha) \) by acting on the Moyal product \( \star^{(x_\alpha, p_\alpha)}_M \). All these star-products are consistent on the intersections \( U_\alpha \cap U_\beta \) and hence glue together to give a global star-product.
on $C^\infty(M)$. The question whether such family of automorphisms $S_\alpha$ always exists is nontrivial. Moreover, in the case when such family exists it is not specified uniquely.

In what follows we will show a way of defining a natural star-product on a symplectic manifold $M = T^*\mathcal{Q}$ endowed with a non-flat symplectic torsionless linear connection $\nabla$ induced by a Levi-Civita connection $\nabla$ on $\mathcal{Q}$. We will present the construction to the third order in $\hbar$. Let us take the admissible morphisms $S (S_\alpha)$ in the similar form as for the flat case (see formula (III.6))

$$S = \text{id} + \hbar^2 \left( -\frac{1}{24} \tilde{\Gamma}_{\alpha\beta\gamma} \partial^\alpha \partial^\beta \partial^\gamma + \frac{1}{16} (\tilde{\Gamma}_{\alpha\mu} \tilde{\Gamma}_{\mu\beta} + a \tilde{R}_{\alpha\beta}) \partial^\alpha \partial^\beta \right) + o(\hbar^4), \quad (IV.1)$$

where $a$ is some real parameter and $\tilde{R}_{\alpha\beta}$ is the Ricci curvature tensor. Then we will receive the one-parameter family of star-products in the form

$$f \ast_a g = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \omega_{\mu_1\nu_1} \cdots \omega_{\mu_k\nu_k} \left( (\nabla \cdots \nabla f)_{\mu_1 \cdots \mu_k} (\nabla \cdots \nabla g)_{\nu_1 \cdots \nu_k} + B_{\mu_1 \cdots \mu_k \nu_1 \cdots \nu_k} (f, g) \right), \quad (IV.2)$$

where $B_{\mu_1 \cdots \mu_k \nu_1 \cdots \nu_k}$ are bilinear operators given by

$$B_0 (f, g) = 0, \quad (IV.3a)$$
$$B_{\mu_1 \nu_1} (f, g) = 0, \quad (IV.3b)$$
$$B_{\mu_1 \mu_2 \nu_1 \nu_2} (f, g) = -a \tilde{R}_{\mu_1 \mu_2} (\nabla_{\nu_1} f)(\nabla_{\nu_2} g), \quad (IV.3c)$$
$$B_{\mu_1 \mu_2 \nu_1 \nu_2} (f, g) = -a \tilde{R}_{\mu_1 \mu_2} (\nabla_{\nu_1} f)(\nabla_{\nu_2} g) = -\tilde{R}_{\nu_1 \nu_2 \alpha \beta} \omega^{\alpha\beta} (\nabla \cdots \nabla f)_{\mu_1 \mu_2 \mu_3} (\nabla_{\beta} g) - \tilde{R}_{\mu_1 \mu_2 \mu_3 \alpha \omega} (\nabla_{\mu_3} f)(\nabla \cdots \nabla g)_{\nu_1 \nu_2 \nu_3}$$
$$- \frac{3}{2} a \tilde{R}_{\mu_1 \mu_2 \mu_3} (\nabla_{\nu_1} f)(\nabla_{\nu_2} g)_{\nu_1 \nu_2} + \frac{3}{2} a \tilde{R}_{\mu_1 \mu_2 \mu_3} (\nabla_{\nu_1} f)(\nabla_{\nu_2} g)_{\nu_1 \nu_2}$$
$$+ 3 a \tilde{R}_{\mu_1 \mu_2 \mu_3} (\nabla_{\nu_1} f)(\nabla_{\nu_2} g)_{\nu_1 \nu_2} + \tilde{R}_{\mu_1 \mu_2 \mu_3 \alpha \beta} \omega^{\alpha\beta} \omega^{\gamma\delta} (\nabla_{\beta} f)(\nabla_{\gamma} g), \quad (IV.3d)$$

and $\tilde{R}_{\alpha\beta\gamma\delta} = \omega^{\alpha\beta} \tilde{R}_{\gamma\delta}^{\gamma\delta}$ is the curvature tensor. Analogical considerations as in the previous section prove that the star-products (IV.2) with the four first operators $B_{\mu_1 \cdots \mu_k \nu_1 \cdots \nu_k}$ given by (IV.3) are equivalent with the Moyal product, up to third order in $\hbar$. Clearly for the flat linear connection $\nabla$ the products (IV.2) reduce to (III.2).

In a special case $a = 0$ the star-product (IV.2) reduces to

$$f \ast g = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \omega_{\mu_1\nu_1} \cdots \omega_{\mu_k\nu_k} (D_{\mu_1 \cdots \mu_k} f)(D_{\nu_1 \cdots \nu_k} g), \quad (IV.4)$$

where $D_{\mu_1 \cdots \mu_k}$ are linear operators mapping functions to $k$-times covariant tensor fields given by

$$D_0 f = f, \quad (IV.5a)$$
$$D_{\mu_1} f = \tilde{\nabla}_{\mu_1} f, \quad (IV.5b)$$
$$D_{\mu_1 \mu_2} f = (\tilde{\nabla} \tilde{\nabla} f)_{\mu_1 \mu_2}, \quad (IV.5c)$$
$$D_{\mu_1 \mu_2 \mu_3} f = (\tilde{\nabla} \tilde{\nabla} \tilde{\nabla} f)_{\mu_1 \mu_2 \mu_3} - \tilde{R}_{\mu_1 \mu_2 \mu_3 \alpha} \omega^{\alpha\beta} \tilde{\nabla}_{\beta} f. \quad (IV.5d)$$

A simple calculation, with the help of the Ricci identity

$$\tilde{R}_{\alpha\beta\gamma\delta} + \tilde{R}_{\alpha\gamma\delta\beta} + \tilde{R}_{\alpha\delta\beta\gamma} = 0,$$

shows that operators (IV.5) are symmetric with respect to indices $\mu_1, \mu_2, \ldots$. It is remarkable that the star-product (IV.4) up to at least third order in $\hbar$ is a Fedosov star-product associated with the Weyl curvature form $\Omega = \omega$ [3]. Whether the Fedosov star-product for any order in $\hbar$ is of the form (IV.4) with operators $D_{\mu_1 \cdots \mu_k}$ independent on $\hbar$ is an open question and would be an interesting problem to investigate. It should be noted that for $a \neq 0$ the star-product (IV.2) is not a Fedosov star-product.

From the presented construction it is clear that when the configuration space $\mathcal{Q}$ is curved there is no single natural star-product on $T^*\mathcal{Q}$ but the whole family of natural star-products. In the considered case (see formula (IV.1)) the natural star-products are parametrized by a real number $a$. Also the Fedosov construction of star-products has freedom in taking different Weyl curvature forms $\Omega$. 
Let us generalize the formula (IV.6) in the following way

\[ S = \text{id} + \frac{\hbar^2}{4!} \left( 3 \left( \Gamma^i_{jk}(x) \Gamma^j_{ik}(x) + aR_{jk}(x) \right) \partial_p \partial_{p_k} + 3\Gamma^i_{jk}(x) \partial_x \partial_j \partial_p \partial_k \right. \]
\[ + \left. \left( 2\Gamma^i_{nl}(x) \Gamma^j_{nk}(x) - \partial_x \Gamma^i_{jk}(x) \right) \partial_j \partial_p \partial_l \partial_p \partial_m + o(\hbar^4) \right). \]  

(IV.6)

Let us generalize the formula (IV.1) in the following way

\[ S = \text{id} + \frac{\hbar^2}{4!} \left( 3 \left( \Gamma^i_{jk}(x) \Gamma^j_{ik}(x) + aR_{jk}(x) \right) \partial_p \partial_{p_k} + 3\Gamma^i_{jk}(x) \partial_x \partial_j \partial_p \partial_k \right. \]
\[ + \left. \left( 2\Gamma^i_{nl}(x) \Gamma^j_{nk}(x) - \partial_x \Gamma^i_{jk}(x) \right) \partial_j \partial_p \partial_l \partial_p \partial_m - 3b \partial_p \partial_x (\partial_x + \Gamma^i_{jd} \partial_d \partial_p \partial_m) + o(\hbar^4) \right). \]  

(IV.7)

where \( b \) is some real parameter. The star-product induced by the above morphism \( S \) for \( a = 1 \) and \( b = 1 \) leads to what was called in a paper [10] a “minimal” quantization. Moreover, the same quantization was used in [11–13] in order to investigate the quantum integrability and quantum separability of classical Stäckel systems.

V. QUANTUM MECHANICAL OPERATORS

To star-algebras \((C^\infty(M), \ast)\) are associated algebras of operators defined on certain Hilbert spaces. In [4, 5] was presented a construction of such algebras of operators for a given classical and quantum canonical coordinate system. In this section we will use the results from [4, 5] to derive a construction of such algebras of operators in a coordinate independent way.

We will be considering a symplectic manifold \( M = T^*Q \) over a pseudo-Riemannian manifold \((Q, g)\), and a family of star-products on \( M \) considered in Section IV. Let us introduce the notion of an almost global coordinate system. The coordinate system \( \phi: Q \supset U \rightarrow V \subset \mathbb{R}^N \) is called an almost global coordinate system on \( Q \) if \( Q \setminus U \) is of measure zero with respect to a measure given by the metric volume form \( \omega_g \). Similarly we define an almost global coordinate system on \( T^*Q \) where as a measure on \( T^*Q \) we take a measure induced by a Liouville form

\[ \Omega = \frac{1}{N!} \omega \wedge \cdots \wedge \omega. \]

A Darboux coordinate system induced from an almost global coordinate system on \( Q \) is the almost global coordinate system on \( T^*Q \). In what follows we will consider only spaces \( Q \) which admit an almost global coordinate system.

Let us consider a Hilbert space \( L^2(M, \Omega) \) of functions defined on the symplectic manifold \( M = T^*Q \), square integrable with respect to the Liouville form \( \Omega \). Let us also consider a Hilbert space \( L^2(Q, \omega_g) \) of functions defined on \( Q \) and square integrable with respect to the metric volume form \( \omega_g \). To every \( A \in C^\infty(M) \) we can associate an operator \( \hat{A} \), defined on the Hilbert space \( L^2(M, \Omega) \), by the formula

\[ \hat{A}\Psi = A \ast \Psi, \]

for every smooth \( \Psi \in L^2(M, \Omega) \). To function \( A \) we can also associate an operator defined on the Hilbert space \( L^2(Q, \omega_g) \). To construct such operator first let us consider an almost global coordinate system on \( Q, \phi: Q \supset U \rightarrow V \subset \mathbb{R}^N \), and related to it an almost global classical and quantum canonical coordinate system on \( M, \tilde{\phi}: M \supset U \rightarrow \mathcal{V} \subset \mathbb{R}^{2N} \). The coordinate system \( \phi \) defines a natural isomorphism \( F_\phi: L^2(Q, \omega_g) \rightarrow L^2(V, \mu) \) between the Hilbert space \( L^2(Q, \omega_g) \) and a Hilbert space \( L^2(V, \mu) \), where \( d\mu(x) = |\text{det}([g_{ij}(x)])|^{1/2} \, dx \):

\[ F_\phi\psi = \psi|_U \circ \phi^{-1}. \]

Similarly, the coordinate system \( \tilde{\phi} \) defines a natural isomorphism \( \tilde{F}_\phi: L^2(M, \Omega) \rightarrow L^2(\mathcal{V}) \) between the Hilbert space \( L^2(M, \Omega) \) and a Hilbert space \( L^2(\mathcal{V}) \) of functions defined on \( \mathcal{V} \) and square integrable with respect to the Lebesgue measure:

\[ \tilde{F}_\phi\Psi = \Psi|_U \circ \tilde{\phi}^{-1}. \]
According to [5] the Hilbert space $L^2(\mathcal{V})$ can be written as the following tensor product

$$L^2(\mathcal{V}) = (L^2(V, \mu))^* \otimes_S L^2(V, \mu) = S((L^2(V, \mu))^* \otimes_M L^2(V, \mu)),$$

where $S$ is the morphism (IV.7) intertwining the $*(x,p)$-product with the Moyal product $*_{M}$, $(L^2(V, \mu))^*$ is the dual space to $L^2(V, \mu)$, and $\otimes_M$ is a Wigner-Moyal transform [14]. Using the isomorphisms $F_\phi$ and $\tilde{F}_\phi$ we can write $L^2(M, \Omega)$ as the following tensor product

$$L^2(M, \Omega) = (L^2(Q, \omega_y))^* \otimes L^2(Q, \omega_y),$$

where

$$\phi^* \otimes \psi = (F^{-1}_\phi)^* \phi^* \otimes_S F_\phi \psi, \quad \phi, \psi \in L^2(Q, \omega_y).$$

Note that the above definition of the tensor product $\otimes$ is independent of the choice of the coordinate system $\phi$. Moreover, to an operator $A_{*(x,p)}$, where $A \in C^\infty(V)$, we can associate an $S$-ordered operator $A_S(q, p)$ by the formula [5]

$$A_{*(x,p)} = \hat{1} \otimes_S A_S(q, p),$$

where

$$A_S(q, p) = (S^{-1}A)W(q, p),$$

$S$ relates star-product $*(x,p)$ with Moyal product $*_{M}$ and $W$ means the Weyl (symmetric) ordering of operators $q^i$, $p_j$, which are canonical operators of position and momentum associated to the Levi-Civita connection $\nabla$ in the coordinate system $\phi$:

$$\hat{q}^i = x^i,$$

$$\hat{p}_j = -i\hbar \left( \partial_{x^j} + \frac{1}{2} \Gamma^k_{jk}(x) \right).$$

Again, using the isomorphisms $F_\phi$ and $\tilde{F}_\phi$, we can see that to every operator $A_{*}$, where $A \in C^\infty(M)$, we can associate an operator $\hat{A}$, defined on the Hilbert space $L^2(Q, \omega_y)$, by the formula

$$A_* = \hat{1} \otimes_A \hat{A}.$$

The operator $\hat{A}$ has the property that for any almost global coordinate system on $Q$ it takes the form of an $S$-ordered operator $A_S(q, p)$.

In what follows let us give examples of operators, defined on the Hilbert space $L^2(Q, \omega_y)$ and written in an invariant form, associated to functions (observables) linear, quadratic and cubic in momenta. The derivation of the formulas presented below is analogous as in [5]. The connection $\nabla$ is fixed by $g$ and an appropriate $*$ (quantization) is chosen by fixing a particular $S$ (IV.7). Let $H$ be a function on $M$ which in some Darboux coordinate system $(x, p)$ takes the form

$$H(x, p) = K^i(x)p_i,$$

where $K^i(x)$ are components of some vector field $K$ on $Q$. To the function $H$ corresponds the following hermitian operator $\hat{H}$ in $L^2(Q, \omega_y)$

$$\hat{H} = -\frac{i\hbar}{2} \left( K^i \nabla_i + \nabla_i K^i \right).$$

Similarly, let now $H$ be a function on $M$ which in $(x, p)$ coordinates takes the form

$$H(x, p) = K^{ij}(x)p_ip_j,$$

where $K^{ij}(x)$ are components of some symmetric second order tensor field $K$ on $Q$. To the function $H$ corresponds the hermitian operator

$$\hat{H} = -\hbar^2 \left( \nabla_i K^{ij} \nabla_j + \frac{1}{4}(1-b)K^{ij} :ij - \frac{1}{4}(1-a)K^{ij} R_{ij} \right),$$

where $R_{ij}$ is the Riemann curvature tensor.
where \( i \) denotes the covariant derivative in the direction of the vector field \( \partial_x \). Finally, let \( H \) be a function on \( M \) which in \((x,p)\) coordinates takes the form

\[
H(x,p) = K^{ijk}(x)p_ip_jp_k,
\]

where \( K^{ijk}(x) \) are components of some symmetric third order tensor field \( K \) on \( \mathcal{Q} \). To the function \( H \) corresponds the respective hermitian operator

\[
\hat{H} = \frac{1}{2}\hbar^2 \left( \nabla_i K^{ijk}\nabla_j \nabla_k + \nabla_i \nabla_j K^{ijk} \nabla_k + \frac{1}{4}(1-b)\nabla_k K^{ijk};ij + \frac{1}{4}(1-b)K^{ijk};ij \nabla_k - \frac{3}{4}(1-a)\nabla_i K^{ijk} R_{jk} - \frac{3}{4}(1-a)K^{ijk} R_{jk} \nabla_i \right).
\]

Observe that for flat connections we deal with a one parameter \((b)\) family of admissible quantizations. By admissible we understand these quantizations which coincide for a class of ‘natural’ Hamiltonians

\[
H(x,p) = \frac{1}{2}g^{ij}(x)p_ip_j + V(x).
\] (V.1)

Notice also that the well known Weyl quantization, written in a coordinate free form, is the one with \( b = 0 \). The case \( b = 1 \) represents a so called flat minimal quantization. Then, for non-flat connections, we introduced a two parameter \((a,b)\) family of quantum observables which particular representatives the reader can find in the literature \([10, 15–17]\). The quantizations \((a,b) = (a,0)\) represent non-flat generalizations of Weyl quantization, while the case \((a,b) = (1,1)\) is mentioned previously non-flat minimal quantization.

**VI. FINAL REMARKS**

In this paper we investigated a problem of defining natural star-products on symplectic manifolds \( M = T^*\mathcal{Q} \). We also associated to considered star-algebras operator algebras defined on certain Hilbert spaces. All this is the main ingredient of a quantization procedure of classical Hamiltonian systems \([5]\). Thus the first step in quantizing a classical Hamiltonian system is to \( \hbar \)-deform a classical Poisson algebra \( C^\infty(M) \) to a quantum Poisson algebra \( (C^\infty(M;\hbar),\star) \). The presented construction of the star-products depended on the linear connection \( \nabla \) on \( \mathcal{Q} \). Thus the quantization is partly fixed by fixing a linear connection on \( \mathcal{Q} \). However, this does not fix the quantization entirely as was seen in Section IV where we introduced \((a,b)\)-parameter family of star-products for a given linear connection \( \nabla \).

Moreover, we have a freedom in choosing quantum observables. Thus the second step in quantizing a classical Hamiltonian system is to \( \hbar \)-deform classical observables \( A_C \in C^\infty(M) \), in particular Hamiltonian functions \( H \), to quantum observables \( A_Q \in C^\infty(M;\hbar) \). However, the choice of a star-product, for a given linear connection \( \nabla \), and the choice of quantum observables is somewhat connected. If we choose two star-products \( \star \) and \( \star' \), such that there exists a morphism \( S \) intertwining these two star-products, and if we choose two algebras of quantum observables in a way that they will also be related by the morphism \( S \), then such two quantizations will be equivalent.

As an example let us consider a star-product (IV.2) written in some local coordinate system. Instead of using this extremely complex product and quantum observables equal to the classical ones: \( A_Q = A_C \), it is reasonable to use the Moyal star-product (III.3) in these coordinates and take quantum observables \( A_Q \) as an \( \hbar \) deformation of the classical ones

\[
A_Q = S^{-1}A_C = A_C - \frac{\hbar^2}{4!} \left( 3\left( \Gamma_{ij}^l(x)\Gamma^{lk}_j(x) + aR_{jkl}(x) \right) \partial_{p_i} \partial_{p_j} + 3\Gamma^{ij}_l(x)\partial_{p_i} \partial_{p_l} \partial_{p_j} \right),
\]

where the morphism \( S \) (IV.6) relates the star-product (IV.2) with the Moyal one (III.3).

Hence, an explicit choice of quantization of a classical Hamiltonian system is fixed by a choice of a linear connection \( \nabla \) on \( \mathcal{Q} \), and a star-product related to \( \nabla \) (or just the morphism \( S \) relating this star-product with the Moyal one). The choice of the linear connection \( \nabla \) on the configuration space \( \mathcal{Q} \) is dictated by the classical system being quantized. For example, if a Hamiltonian of the system is in the natural form (V.1) then the only natural choice is the Levi-Civita connection. However, if the Hamiltonian of a system is of the form

\[
H(x,p) = \frac{1}{2}K^{ij}p_ip_j + V(x) = \frac{1}{2}K^{ij}_rg^{ij}p_ip_j + V(x),
\]
where $K$ is some symmetric non-degenerate tensor, then we have two different natural choices of a connection. One is again the Levi-Civita connection induced by $g$ and the second one is the connection induced by a new metric $\tilde{g} = K$.

Thus, there is a freedom in choosing a quantization of a given Hamiltonian system. Only in limited cases we can verify through experiment which quantization scheme realizes in nature.

It should be noted that we considered quantization of systems over a phase space $T^*\mathcal{Q}$. For this special case of a phase space it was possible to introduce an operator representation of the quantum system in a Hilbert space $L^2(\mathcal{Q},\omega_g)$. The quantization procedure described in the paper can be generalized to systems defined over a phase space $M$ being a general symplectic manifold, provided that we fix on $M$ a symplectic torsionless linear connection. In this case, however, it is difficult to introduce an operator representation in a Hilbert space being an analog of $L^2(\mathcal{Q},\omega_g)$.

APPENDIX

Let us check if $S_2$ in the form (III.6) satisfies the system of equations (III.4). From (III.5) and (III.8a) we get that

\[
A^2 = \frac{1}{2}\omega^\mu_1\nu_1\omega^\mu_2\nu_2\Gamma^\alpha_{\mu_1\mu_2}(\partial_{\nu_1}\partial_{\nu_2} - \tilde{\Gamma}^\beta_{\nu_1\nu_2}\partial_{\beta}) = -\frac{1}{2}\tilde{\Gamma}^\alpha_{\mu_1\mu_2}\partial^\mu_1\partial^\mu_2 = -\frac{1}{2}\omega^\mu_1\alpha\Gamma^\nu_1_{\mu_1\mu_2}\tilde{\Gamma}^\mu_2_{\nu_1\nu_2}\partial^\nu_2.
\]

On the other hand

\[
[S_2, z^\alpha] = -\frac{1}{24}\omega^\beta\alpha\Gamma^\gamma_{\delta\beta\gamma}\partial^\beta\partial^\gamma - \frac{1}{24}\omega^\beta\alpha\Gamma^\gamma_{\delta\beta\gamma}\partial^\beta\partial^\gamma - \frac{1}{24}\omega^\gamma\alpha\Gamma^\mu_{\delta\gamma\mu}\partial^\beta - \frac{1}{16}\omega^\gamma\alpha\Gamma^\nu_{\mu\gamma}\partial^\beta + \frac{1}{16}\omega^\alpha\beta\Gamma^\nu_{\mu\beta}\partial^\gamma,
\]

which proves (III.4a). From (III.5) we can calculate that

\[
A^3 = \frac{1}{6}\omega^\mu_1\nu_1\omega^\mu_2\nu_2\omega^3z^\alpha(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\alpha)_{\mu_1\mu_2\mu_3}(\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3} - \tilde{\Gamma}^\beta_{\nu_1\nu_2}\partial_{\nu_3}\partial_{\beta} - \tilde{\Gamma}^\beta_{\nu_1\nu_3}\partial_{\nu_2}\partial_{\beta} - \tilde{\Gamma}^\beta_{\nu_2\nu_3}\partial_{\nu_1}\partial_{\beta} + (\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\beta)_{\nu_1\nu_2\nu_3}\partial_{\beta}).
\]

The above equation can be rewritten in a different form. To do this first let us prove that

\[
\omega^\mu_1\nu_1(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\alpha)_{\mu_1\mu_2\mu_3} = \omega^\mu_1\nu_1\tilde{A}^3_{\mu_2\mu_3\mu_1} + \omega^\mu_1\nu_1\tilde{R}^3_{\mu_2\mu_3\mu_1}, \quad \text{(A.1a)}
\]

\[
\omega^\mu_2\nu_2(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\alpha)_{\mu_1\mu_2\mu_3} = \omega^\mu_2\nu_2\tilde{A}^3_{\mu_2\mu_3\mu_1} + \omega^\mu_2\nu_2\tilde{R}^3_{\mu_2\mu_3\mu_1}. \quad \text{(A.1b)}
\]

Indeed, with the help of (III.8) we can calculate that

\[
\omega^\mu_1\nu_1(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\alpha)_{\mu_1\mu_2\mu_3} = \omega^\mu_1\nu_1(-\tilde{\Gamma}^\beta_{\mu_2\mu_1}\delta^\alpha_{\beta\mu_3} + \tilde{\Gamma}^\beta_{\mu_3\mu_2}\delta^\alpha_{\beta\mu_1} + \tilde{\Gamma}^\beta_{\mu_1\mu_3}\delta^\alpha_{\beta\mu_2})
\]

\[
= \omega^\mu_1\alpha\tilde{R}^\beta_{\mu_2\mu_1\mu_3} - \tilde{\Gamma}^\beta_{\mu_2\mu_1\mu_3} + \tilde{\Gamma}^\beta_{\mu_3\mu_2\mu_1} + \tilde{\Gamma}^\beta_{\mu_1\mu_3\mu_2},
\]

and that

\[
\omega^\mu_1\beta\tilde{\Gamma}^\alpha_{\mu_1\mu_3\beta\mu_2} = \omega^\mu_1\beta\tilde{\Gamma}^\alpha_{\mu_1\mu_3\beta\mu_2} = \omega^\mu_1\beta\omega^{\delta\gamma}\tilde{\Gamma}^\alpha_{\delta\gamma\mu_1\mu_3\beta\mu_2} = \omega^\mu_1\beta\omega^{\delta\gamma}\tilde{\Gamma}^\alpha_{\delta\gamma\mu_1\mu_3\beta\mu_2} = \omega^\mu_1\beta\omega^{\delta\gamma}\tilde{\Gamma}^\alpha_{\delta\gamma\mu_1\mu_3\beta\mu_2}
\]

\[
= \delta^\alpha_\gamma\omega^{\delta\gamma}\tilde{\Gamma}^\alpha_{\mu_1\mu_3\beta\mu_2} = \omega^\alpha\beta\omega^{\delta\gamma}\tilde{\Gamma}^\alpha_{\mu_1\mu_3\beta\mu_2},
\]

from which follows (A.1a). (A.1b) can be proved analogically. Hence using (III.8a), (A.1) and the condition

\[
\omega^{\mu_1\nu_1}\ldots\omega^{\mu_k\nu_k}(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\alpha)_{\mu_1\ldots\mu_k}(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}z^\beta)_{\nu_1\ldots\nu_k} = 0, \quad k = 3, 5, \ldots
\]

following from the quantum canonical of the coordinate system $(z^1, \ldots, z^{2N})$ we get

\[
A^3_{\alpha} = \frac{1}{6}\omega^\mu_1\nu_1\tilde{R}^\nu_1_{\mu_2\mu_3\mu_1} + \tilde{R}^\nu_1_{\mu_2\mu_3\mu_1}\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3} = -\frac{1}{2}\omega^\mu_1\alpha\tilde{R}^\nu_1_{\mu_2\mu_3\mu_1} + \frac{1}{3}\tilde{R}^\nu_1_{\mu_2\mu_3\mu_1} + \frac{2}{3}\tilde{R}^\nu_1_{\mu_2\mu_3\mu_1}\tilde{\Gamma}^\mu_2_{\nu_1\nu_2}\partial^\nu_2.
\]

On the other hand

\[
[S_2, \partial^\alpha] = -\frac{1}{24}\omega^{\delta\gamma}\tilde{\Gamma}^\lambda_{\beta\gamma\delta}\partial^\lambda\partial^\beta\partial^\gamma - \frac{1}{8}\omega^{\delta\gamma}\tilde{\Gamma}^\mu_{\beta\lambda\gamma}\partial^\mu\partial^\beta\partial^\gamma
\]

which shows that $S_2$ in the form (III.6) will satisfy (III.4b) since from flatness assumption $\tilde{R}^\alpha_{\gamma\delta} = 0$. 

11
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