CHARACTERIZATION OF CMO VIA COMPACTNESS OF THE
COMMUTATORS OF BILINEAR FRACTIONAL INTEGRAL
OPERATORS

DINGHUAI WANG, JIANG ZOU∗ AND WENYI CHEN

ABSTRACT. Let $I_\alpha$ be the bilinear fractional integral operator, $B_\alpha$ be a more singular family of bilinear fractional integral operators and $\vec{b} = (b, b)$. Bényi et al. in [1] showed that if $b \in \text{CMO}$, the BMO-closure of $C_\infty^\infty(\mathbb{R}^n)$, the commutator $[b, B_\alpha]_i(i = 1, 2)$ is a separately compact operator. In this paper, it is proved that $b \in \text{CMO}$ is necessary for $[b, B_\alpha]_i(i = 1, 2)$ is a compact operator. Also, the authors characterize the compactness of the iterated commutator $[\Pi \vec{b}, I_\alpha]$ of bilinear fractional integral operator. More precisely, the commutator $[\Pi \vec{b}, I_\alpha]$ is a compact operator if and only if $b \in \text{CMO}$.

1. Introduction

A locally integrable function $f$ is said to belong to BMO space if there exists a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq C,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$ and the minimal constant $C$ is defined by $\|f\|_*$. There are a number of classical results that demonstrate BMO functions are the right collections to do harmonic analysis on the boundedness of commutators. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator

$$[b, T](f) = bT(f) - T(bf)$$

is bounded on some $L^p$, $1 < p < \infty$, if and only if $b \in \text{BMO}$, where $T$ be the classical Calderón-Zygmund operator. In 1978, Uchiyama [21] refined the boundedness results on the commutator to compactness. This is achieved by requiring the commutator with symbol to be in CMO, which is the closure in BMO of the space of $C^\infty$ functions with compact...

2010 Mathematics Subject Classification. Primary 42B20, 47B07; Secondary: 42B25, 47G99.

Key words and phrases. Bilinear Fractional Integral Operators; Characterization; Compactness; Iterated Commutator

The research was supported by National Natural Science Foundation of China (Grant No.11661075 and No.11261055).

* Corresponding author, Email: zhoujiangshuxue@126.com.
support. In recent years, the compactness of commutators has been extensively studied already, as Chen, Ding and Wang [5], [6] and Wang [22]. The interest in the compactness of $[b, T]$ in complex analysis is from the connection between the commutators and the Hankel-type operators. In fact, the authors of [13] and [14] have applied commutator theory to give a compactness characterization of Hankel operators on holomorphic Hardy spaces $H^2(D)$, where $D$ is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$. It is perhaps for this important reason that the compactness of $[b, T]$ attracted ones attention among researchers in PDEs.

In the multilinear setting, the boundedness results for commutators with symbols in BMO started to receive attention only a few years ago, see [15], [17], [18] or [20]. Compactness results in the multilinear setting have just began to be studied. Bényi and Torres [3], Bényi et al. [1] and [2] showed that symbols in CMO again produce compact commutators. Specially, Bényi et al. in [1] showed that if $b \in \text{CMO}$, the commutator $[b, B_\alpha]_i(i = 1, 2)$ is a separately compact operator. More precisely, it is obtained that if $b \in \text{CMO}$ and $g \in L^{p_2}$ is fixed, $[b, B_\alpha]_i(\cdot, g)(i = 1, 2)$ is a compact operator from $L^{p_1}$ to $L^q$. Unfortunately, it is unknown that if $b \in \text{CMO}$, are the commutators $[b, B_\alpha]_i, i = 1, 2$, jointly compact? We intend to study this question in future work, however, in this paper, we first give the necessary condition for commutators $[b, B_\alpha]_i$ are jointly compact.

Another subject of this paper is to consider the characterization of compactness of the iterated commutator of $I_\alpha$. In 2015, Chaffee and Torres [4] characterized the compactness of the linear commutators of bilinear fractional integral operators acting on product of Lebesgue spaces. In this paper, the characterization of compactness of the iterated commutators will be considered.

To state the main result of this paper, we first recall some necessary notions and notation.

It is well known that the fractional integral $I_\alpha$ of order $\alpha (0 < \alpha < n)$ plays an important role in harmonic analysis, PDE and potential theory (see [19]). Recall that $I_\alpha$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

For the bilinear case, the bilinear fractional integral operator $I_\alpha$, $0 < \alpha < 2n$, is defined by

$$I_\alpha (f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(y_1) f_2(y_2)}{(|x - y_1| + |x - y_2|)^{2n-\alpha}} dy_1 dy_2.$$
In this paper, we will consider the following equivalent operator

\[ I_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(y_1) f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1 dy_2. \]

Its iterated commutators with \( \vec{b} = (b_1, b_2) \) is given by

\[ I_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b_1(x) - b_1(y_1))(b_2(x) - b_2(y_2)) f_1(y_1) f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1 dy_2. \]

We will now examine a more singular family of bilinear fractional integral operators,

\[ B_\alpha(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x - y)g(x + y)}{|y|^{n-\alpha}} dy. \]

This operator was first introduced by Grafakos in \([9]\), and later studied by Grafakos and Kalton \([10]\) and Kenig and Stein \([12]\). The commutators \([b, B_\alpha]_i\) of \(B_\alpha\) with \(b\) can be written as

\[ [b, B_\alpha]_1(f, g)(x) = bB_\alpha(f, g) - B_\alpha(bf, g) \]

\[ = \int_{\mathbb{R}^n} b(x) - b(y) f(y) g(2x - y) dy, \]

the definition of \([b, B_\alpha]_2\) be the similar as \([b, B_\alpha]_1\). In what follows, we need only consider one of these two commutators.

For \(1 < p \leq q < \infty\), recall that the Muckenhoupt class of weights consists of all nonnegative, locally integrable functions \(\omega\) such that

\[ \sup_Q \left( \frac{1}{|Q|} \int_Q \omega^q(x) dx \right) \left( \frac{1}{|Q|'} \int_Q \omega(x)^{-p'} dx \right)^{q/p'} < \infty. \]

We also recall the definition of the multiple or vector weights used in the bilinear setting. For \(1 < p_1, p_2 < \infty\), \(P = (p_1, p_2)\), \(0 < \alpha < 2n\), \(\frac{2}{n} < \frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2}\), and \(q\) such that \(\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}\), a vector weight \(\omega = (\omega_1, \omega_2)\) belongs to \(A_{P, q}\) if

\[ \sup_Q \left( \frac{1}{|Q|} \int_Q \mu_\omega(x) dx \right) \left( \frac{1}{|Q|'} \int_Q \omega_1(x)^{-p_1'} dx \right)^{q/p_1'} \left( \frac{1}{|Q|'} \int_Q \omega_2(x)^{-p_2'} dx \right)^{q/p_2'} < \infty. \]

where the notation \(\mu_\omega = \omega_1^{q_1} \omega_2^{q_2}\). It was shown by Moen in \([16]\) that if \(\omega \in A_{P, q}\) then \(\omega_1^{-p_1'} \in A_{2p_1'}\) and \(\mu_\omega \in A_{2q}\). In addition, the weights in \(A_{P, q}\) are precisely those for which

\[ I_\alpha : L^{p_1}(\omega_1^{p_1}) \times L^{p_2}(\omega_2^{p_2}) \rightarrow L^q(\mu_\omega) \]

is bounded.
Now we return to our main results.

**Theorem 1.1.** Let \( 1 < p, p_1, p_2, q < \infty, 0 < \alpha < n \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} \). If \( [b, B_\alpha] \) is a compact operator from \( L^{p_1} \times L^{p_2} \) to \( L^q \), then \( b \in \text{CMO} \).

**Theorem 1.2.** Let \( 1 < p, p_1, p_2, q < \infty, 0 < \alpha < 2n \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \frac{\alpha}{n} < \frac{1}{p_1} + \frac{1}{p_2} \), and \( \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} \). For the local integral function \( b \) and \( \vec{b} = (b, b) \), the following are equivalent,

(A1) \( b \in \text{CMO} \).

(A2) \( [\Pi \vec{b}, I_\alpha] : L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \rightarrow L^q(\omega_1, \omega_2) \) is a compact operator for all \( \omega = (\omega_1, \omega_2) \) such that \( \omega_1^{p_1}, \omega_2^{p_2} \in A_p \).

(A3) \( [\Pi \vec{b}, I_\alpha] : L^{p_1} \times L^{p_2} \rightarrow L^q \) is a compact operator.

### 2. Main Lemmas

As mentioned in the introduction, CMO is the closure in BMO of the space of \( C^\infty \) functions with compact support. In [21], it was shown that CMO can be characterized in the following way.

**Lemma 2.1.** ([21]) Let \( b \in \text{BMO} \). Then \( b \) is in \( \text{CMO} \) if and only if

\[
(2.1) \quad \lim_{a \to 0} \sup_{|Q| = a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0;
\]

\[
(2.2) \quad \lim_{a \to \infty} \sup_{|Q| = a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0;
\]

\[
(2.3) \quad \lim_{|y| \to 0} \frac{1}{|Q|} \int_Q |b(x + y) - b_Q| dx = 0, \text{ for each } Q.
\]

To prove Theorem 1.1 and Theorem 1.2, we need the following results.

**Lemma 2.2.** Support that \( b \in \text{BMO} \) with \( \|b\|_* = 1 \). If for some \( 0 < \epsilon < 1 \) and a cube \( Q \) with its center at \( x_Q \) and \( r_Q \), \( b \) is not a constant on cube \( Q \) and satisfies

\[
(2.4) \quad \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy > \epsilon,
\]

then for the functions \( g(y) = \frac{|Q| \chi_{(2Q)(y)}}{|y - x_Q|^{1/p_2 + n}} \) and \( f \) is defined by

\[
(2.5) \quad f(y) = |Q|^{-1/p_1} (\text{sgn}(b(y) - b_Q)) \chi_Q(y),
\]
There exists constants $\gamma_1, \gamma_2, \gamma_3$ satisfying $\gamma_2 > \gamma_1 > 2$ and $\gamma_3 > 0$, such that

\begin{equation}
\int_{|x-x_Q| < \gamma_2 r_Q} |(b, B\alpha)1(f, g)(x)|^q dx \geq \gamma_3^q,
\end{equation}

\begin{equation}
\int_{|x-x_Q| > \gamma_2 r_Q} |(b, B\alpha)1(f, g)(x)|^q dx \leq \gamma_3^q \frac{\gamma_2^q}{4^q}.
\end{equation}

Moreover, there exists a constant $0 < \beta << \gamma_2$ depending only on $p_1, p_2, n$ such that for all measurable subsets $E \subset \{ x : \gamma_1 r_Q < |x-x_Q| < \gamma_2 r_Q \}$ satisfying $\frac{|E|}{|Q|} < \beta^n$, we have

\begin{equation}
\int_E |(b, B\alpha)1(f, g)(x)|^q dx \leq \gamma_3^q \frac{\gamma_2^q}{4^q}.
\end{equation}

Proof. It is easy to check that $f$ satisfies

$$
supp f \subset Q,
$$

$$
f(y)(b(y) - b_Q) = |Q|^{-1/p_1} |b(y) - b_Q| \chi_Q(y) \geq 0,
$$

$$
|f(y)| \leq |Q|^{-1/p_1},
$$

$$
\|f\|_{L^{p_1}} \leq 1,
$$

$$
\int (b(y) - b_Q) f(y) dy = |Q|^{-1/p_1} \int |b(y) - b_Q| dy,
$$

and $g$ satisfies that $\|g\|_{L^{p_2}} = C$ and for $x \in (2nQ)^c, y \in Q$, we get

$$
|2x - y - y| = 2|x - y| \geq 2|x - x_Q| - 2|y - x_Q| \geq 2nr_Q - \sqrt{n}r_Q = \sqrt{n}r_Q,
$$

which implies that $2x - y \in (Q(y, 2\sqrt{n}r_Q))^c \subset (2Q)^c$ and $g(2x - y) \approx |Q| \cdot |x - x_Q|^{-n/p_2 - n}.$

We first establish the following several technical estimates. For a cube $Q$ with center $x_Q$ and satisfying $2.4$ for some $\epsilon > 0$ and $x \in (2nQ)^c$, the following point-wise estimates hold:

\begin{equation}
|B\alpha((b - b_Q)f, g)(x)| \lesssim |Q|^{\frac{1}{p_1} + 1} |x - x_Q|^{-2n - n/p_2 + \alpha},
\end{equation}

\begin{equation}
|B\alpha((b - b_Q)f, g)(x)| \gtrsim |Q|^{\frac{1}{p_1} + 1} |x - x_Q|^{-2n - n/p_2 + \alpha},
\end{equation}

\begin{equation}
|B\alpha(f, g)(x)| \lesssim |Q|^{\frac{1}{p_1} + 1} |x - x_Q|^{-2n - n/p_2 + \alpha},
\end{equation}

where $f, g$ as above and the constants involved are independent of $b, f, g$ and $\epsilon$.

To prove (2.9), from the fact that $\|b\|_{\ast} = 1$ and $x \in (2nQ)^c$, we have

$$
|B\alpha((b - b_Q)f, g)(x)| = \left| \int_Q \frac{(b(y) - b_Q)f(y)g(2x - y)}{|x - y|^{n-\alpha}} dy \right|
$$
Finally using that

\[
|Q|^{\frac{1}{p}} |x - x_Q|^{-2n - n/p_2 + \alpha} \int_Q (b(y) - b_Q) f(y) dy
\]

\[
\lesssim |Q|^{\frac{1}{p_1} + 1} |x - x_Q|^{-2n - n/p_2 + \alpha}.
\]

For (2.10), by \(x \in (2nQ)^c\) and \(y \in Q\), we have \(|x - y| \approx |x - x_Q|\). Using that \((b(y) - b_Q) f(y) \geq 0\), we can compute

\[
|B_\alpha((b - b_Q) f, g)(x)| = \left| \int_Q \frac{(b(y) - b_Q) f(y) g(2x - y)}{|x - y|^{n-\alpha}} dy \right|
\]

\[
\gtrsim |Q| |x - x_Q|^{-2n - n/p_2 + \alpha} \int_Q (b(y) - b_Q) f(y) dy
\]

\[
= |Q|^{1/p_1'} |x - x_Q|^{-2n - n/p_2 + \alpha} \int_Q |b(y) - b_Q| |y|^{n-\alpha} dy
\]

\[
\gtrsim \epsilon |Q|^{\frac{1}{p_1} + 1} |x - x_Q|^{-2n - n/p_2 + \alpha}.
\]

Finally using that \(|f(y)| \leq |Q|^{-1/p_1}\) we obtain (2.20) as follows.

\[
|B_\alpha(f, g)(x)| = \left| \int_Q \frac{f(y) g(2x - y)}{|x - y|^{n-\alpha}} dy \right| \lesssim |Q|^{\frac{1}{p_1} + 1} |x - x_Q|^{-2n - n/p_2 + \alpha}.
\]

Now, we give the proofs of (2.6)-(2.7). Note that for \(b \in \text{BMO}\), we have

\[
\left( \int_{2^s r_Q < |x - x_Q| < 2^{s+1} d_j} |b(x) - b_Q|^q dx \right)^{1/q} \lesssim s 2^{sn/q} |Q|^{1/q}.
\]

Taking \(\nu > 16\), by (2.9) we obtain

\[
\left( \int_{|x - x_Q| > \nu r_Q} |(b(x) - b_Q) B_\alpha(f, g)(x)|^q dx \right)^{\frac{1}{q}}
\]

\[
\leq C |Q|^{\frac{1}{p_1} + 1} \sum_{s = \log_2 \nu} \left( \int_{2^s r_Q < |x - x_Q| < 2^{s+1} r_Q} \frac{|b(x) - b_Q|^q}{|x - x_Q|^{q(2n - \alpha + n/p_2)}} dx \right)^{\frac{1}{q}}
\]

\[
\leq C |Q|^{\frac{1}{p_1} + 1} \sum_{s = \log_2 \nu} 2^{-s(2n - \alpha + n/p_2)} |Q|^{-\frac{2 + \alpha}{n} - \frac{1}{p_2}} \left( \int_{2^s r_Q < |x - x_Q| < 2^{s+1} d_j} |b(x) - b_Q|^q dx \right)^{\frac{1}{q}}
\]

\[
\leq C \sum_{s = \log_2 \nu} 2^{-s(2n - \alpha + \frac{\alpha}{p_2} - \frac{n}{q} - \frac{1}{2})}
\]

\[
\leq C \sum_{s = \log_2 \nu} 2^{-s(2n - \alpha + \frac{n}{p_2} - \frac{n}{q} - \frac{1}{2})}
\]
≤ Cν^{-(2n-α+n/q_p - \frac{s}{q} - \frac{1}{q})},

where we have used that \(2n - \alpha + \frac{n}{p_q} - \frac{n}{q} - \frac{1}{2} > 0\) and \(s \leq 2^{s/2}\) for \(4 \leq |\log_2 \nu| \leq s\).

For \(\mu > \nu\), using (2.10) and the estimates above, we get

\[
\int_{\nu r_Q < |x-x_Q| < \mu r_Q} |[b, B_\alpha](f, g)(x)|^q dx
\]

\[
≥ C \left( |B_\alpha((b-b_Q)f, g)(x)|^q dx \right)^{1/q}
\]

\[
-C \left( \int_{\nu r_Q < |x-x_Q| < \mu r_Q} |(b(x) - b_Q)B_\alpha(f, g)(x)|^q dx \right)^{1/q}
\]

\[
≥ C\epsilon Q^{\frac{1}{q_1} + 1} \left( \int_{\nu r_Q < |x-x_Q| < \mu r_Q} |x-x_Q|^{q(2n-n/p_2+\alpha)} dx \right)^{1/q} - C\nu^{-(2n-\alpha+n/q_p - \frac{s}{q} - \frac{1}{q})}
\]

\[
≥ C\epsilon \left( \nu^{2q+nq/n} - \mu^{2q+nq/n} \right)^{\frac{1}{q}} - C\nu^{-(2n-\alpha+n/q_p - \frac{s}{q} - \frac{1}{q})}.
\]

Once again, the constants appearing above are independent of \(Q\). It is easy to see that we can select \(\gamma_1, \gamma_2\) in place of \(\nu, \mu\) with \(\gamma_2 >> \gamma_1\), then (2.13) and (2.14) are verified for some \(\gamma_3 > 0\).

We now verified (2.8). Let \(E \subset \{\gamma_1 r_Q < |x-x_Q| < \gamma_2 r_Q\}\) be an arbitrary measurable set. It follows from Minkowski inequality that

\[
\left( \int_E |[b, B_\alpha](f, g)(x)|^q dx \right)^{\frac{1}{q}}
\]

\[
≤ |Q|^{\frac{1}{q_1} + 1} \left( \int_E |x-x_Q|^{-q(2n+n/p_2-\alpha)} dx \right)^{\frac{1}{q}} + |Q|^{\frac{1}{q_1} + 1} \left( \int_E \frac{|b(x) - b_Q|^q}{|x-x_Q|^{q(2n-n/p_2)}} dx \right)^{\frac{1}{q}}
\]

\[
≤ \left( \frac{|E|^{1/q}}{|Q|^{1/q}} + \left( \frac{1}{|Q|} \int_E |b(x) - b_Q|^q dx \right)^{\frac{1}{q}} \right)
\]

It is proved in [6] p.269 that

\[
\frac{1}{|Q|} \int_E |b(x) - b_Q|^q dx \lesssim \frac{|E|}{|Q|} \left( 1 + \log \left( \frac{|Q|}{|E|} \right) \right)^{[\alpha]+1}.
\]

Taking \(0 < \beta < \min \{C^{1/n}, \gamma_2\}\) and sufficiently small, then (2.8) holds. \hfill \Box

**Lemma 2.3.** Support that \(b \in BMO\) with \(\|b\|_* = 1\). If for some \(0 < \epsilon < 1\) and a cube \(Q\) with its center at \(x_Q\) and \(r_Q\), \(b\) is not a constant on cube \(Q\) and satisfies

\[
\frac{1}{|Q|} \int_Q |b(y) - b_Q| dy > \epsilon^{1/2},
\]

...
then for the function $f_i (i = 1, 2)$ defined by
\begin{equation}
(2.12) \quad f_i (y_i) = |Q|^{-1/p_i} (\text{sgn}(b(y_i) - b_Q) - c_0) \chi_Q(y_i),
\end{equation}
where $c_0 = |Q|^{-1} \int_Q \text{sgn}(b(y) - b_Q) \, dy$, and $i = 1, 2$. There exists constants $\gamma_1, \gamma_2, \gamma_3$ satisfying $\gamma_2 > \gamma_1 > 2$ and $\gamma_3 > 0$, such that
\begin{equation}
(2.13) \quad \int_{|x - x_Q| < \gamma_2 r_Q} |\Pi b, I_\alpha| (f_1, f_2)(x)|^q \, dx \geq \gamma_3^q,
\end{equation}
\begin{equation}
(2.14) \quad \int_{|x - x_Q| > \gamma_2 r_Q} |\Pi b, I_\alpha| (f_1, f_2)(x)|^q \, dx \leq \frac{\gamma_3^q}{4^q}.
\end{equation}
Moreover, there exists a constant $0 < \beta << \gamma_2$ depending only on $p_1, p_2, n$ such that for all measurable subsets $E \subset \{ x : \gamma_1 r_Q < |x - x_Q| < \gamma_2 r_Q \}$ satisfying $\frac{|E|}{|Q|} < \beta^n$, we have
\begin{equation}
(2.15) \quad \int_E |\Pi b, I_\alpha| (f_1, f_2)(x)|^q \, dx \leq \frac{\gamma_3^q}{4^q}.
\end{equation}

Proof. Since $\int_Q (b(y) - b_Q) \, dy = 0$, it is easy to check that $f_i$ satisfies
\[
\text{supp} f_i \subset Q,
\]
\[
f_i (y_i) (b(y) - b_Q) \geq 0,
\]
\[
\int f_i (y_i) \, dy_i = 0,
\]
\[
|f_i (y_i)| \leq 2 |Q|^{-1/p_i},
\]
\[
\|f_i\|_{L^{p_i}} \leq 2,
\]
\[
\int (b(y) - b_Q) f_i \, dy = |Q|^{-1/p_i} \int_Q |b(y_i) - b_Q| \, dy.
\]
For a cube $Q$ with center $x_Q$ and $x \in (2\sqrt{n})^c$, the following point-wise estimates hold:
\begin{equation}
(2.16) \quad |I_\alpha((b - b_Q)f_1, (b - b_Q)f_2)(x)| \lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2}} |x - x_Q|^{-2n + \alpha},
\end{equation}
\begin{equation}
(2.17) \quad |I_\alpha((b - b_Q)f_1, (b - b_Q)f_2)(x)| \gtrsim \epsilon |Q|^{\frac{1}{p_1} + \frac{1}{p_2}} |x - x_Q|^{-2n + \alpha},
\end{equation}
\begin{equation}
(2.18) \quad |I_\alpha((b - b_Q)f_1, f_2)(x)| \lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{n}} |x - x_Q|^{-2n + \alpha - 1},
\end{equation}
\begin{equation}
(2.19) \quad |I_\alpha(f_1, (b - b_Q)f_2)(x)| \lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{n}} |x - x_Q|^{-2n + \alpha - 1},
\end{equation}
\begin{equation}
(2.20) \quad |I_\alpha(f_1, f_2)(x)| \lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{n}} |x - x_Q|^{-2n + \alpha - 1},
\end{equation}
where $f_i$ as above and the constants involved are independent of $b, f_i$ and $\epsilon$. 
To prove (2.16), from the fact that $\|b\|_* = 1$ and $x \in (2\sqrt{Q})^c$, we have

$$|I_\alpha((b - b_Q)f_1, (b - b_Q)f_2)(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y_1) - b_Q)(b(y_2) - b_Q)f_1(y_1)f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1dy_2 \right|$$

$$\lesssim |Q|^{-\frac{1}{p_1} - \frac{1}{p_2}} |x - x_Q|^{-2n+\alpha} \prod_{i=1}^2 \int_Q (b(y_i) - b_Q)f_i(y_i)dy_i$$

$$\lesssim |Q|^{-\frac{1}{p_1} - \frac{1}{p_2}} |x - x_Q|^{-2n+\alpha} \prod_{i=1}^2 \int_Q |b(y_i) - b_Q|dy_i$$

$$\lesssim |Q|^{-\frac{1}{p_1} - \frac{1}{p_2}} |x - x_Q|^{-2n+\alpha}.$$

For (2.17), using that $(b(y_i) - b_Q)f_i(y_i) \geq 0$, we can compute

$$|I_\alpha((b - b_Q)f_1, (b - b_Q)f_2)(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y_1) - b_Q)(b(y_2) - b_Q)f_1(y_1)f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1dy_2 \right|$$

$$\gtrsim |x - x_Q|^{-2n+\alpha} \prod_{i=1}^2 \int_Q (b(y_i) - b_Q)f_i(y_i)dy_i$$

$$= |x - x_Q|^{-2n+\alpha} \prod_{i=1}^2 |Q|^{-\frac{1}{p_i}} \int_Q |b(y_i) - b_Q|dy_i$$

$$\gtrsim \epsilon |Q|^{-\frac{1}{p_1} + \frac{1}{p_2}} |x - x_Q|^{-2n+\alpha}.$$

For (2.18), by the fact $|f_2(y_2)| \leq 2|Q|^{-1/p_2}$ and $\int_Q f_2(y_2)dy_2 = 0$, we can also estimate

$$|I_\alpha((b - b_Q)f_1, f_2)(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y_1) - b_Q)f_1(y_1)f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1dy_2 \right|$$

$$= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y_1) - b_Q)f_1(y_1)f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1dy_2 \right.$$}

$$- \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y_1) - b_Q)f_1(y_1)f_2(y_2)}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} dy_1dy_2 \right|$$

$$\lesssim |Q|^{-\frac{1}{p_1} - \frac{1}{p_2}} \int_Q \int_Q \frac{|y_2 - y_2|}{(|x - y_1| + |x - y_2|)^{2n-\alpha+1}} dy_1dy_2$$
\[
\lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{\alpha} + \frac{1}{n}} |x - x_Q|^{-2n + \alpha - 1} \int_Q |b(y_1) - b_Q| dy_1
\]

\[
\lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{\alpha} + \frac{1}{n}} |x - x_Q|^{-2n + \alpha - 1}.
\]

It is easy to see that \(|I_a((b - b_Q)f_1, f_2)(x)| = |I_a(f_1, (b - b_Q)f_2)(x)|\), then (2.19) holds.

Finally using that \(f_1\) has mean zero we obtain (2.20) as follows.

\[
|I_a(f_1, f_2)(x)|
= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(y_1)f_2(y_2)}{|x - y_1|^2} \frac{f_1(y_1)f_2(y_2)}{|x - y_2|^2} dy_1 dy_2 \right|
\lesssim |Q|^{-\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{\alpha} + \frac{1}{n}} \int_Q \int_Q \frac{|y_1 - y_2| |f_1(y_1)||f_2(y_2)|}{(|x - y_1| + |x - y_2|)^{2n + \alpha + 1}} dy_1 dy_2
\lesssim |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{\alpha} + \frac{1}{n}} |x - x_Q|^{-2n + \alpha - 1}.
\]

Now, we give the proofs of (2.13)-(2.15). Taking \(\nu > 16\), by (2.18) we obtain

\[
\left( \int_{|x-x_Q| > \nu r_Q} |(b(x) - b_Q)I_a((b - b_Q)f_1, f_2)(x)| \right)^\frac{1}{q}
\leq C |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{\alpha} + \frac{1}{n}} \sum_{s = \log_2 \nu} \left( \int_{2^s r_Q < |x - x_Q| < 2^{s+1} r_Q} \frac{|b(x) - b_Q|^q}{|x - x_Q|^{q(2n + \alpha + 1)} dx} \right)^\frac{1}{q}
\leq C |Q|^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{\alpha} + \frac{1}{n}} \sum_{s = \log_2 \nu} 2^{-s(2n + \alpha + 1)} |Q|^{-2s + \frac{1}{n}} \left( \int_{2^s r_Q < |x - x_Q| < 2^{s+1} d_j} |b(x) - b_Q|^q dx \right)^\frac{1}{q}
\leq C \sum_{s = \log_2 \nu} s^{-s(2n + \alpha - \frac{\alpha}{q} + \frac{1}{q})}
\leq C \sum_{s = \log_2 \nu} 2^{-s(2n - \frac{\alpha}{q} + \frac{1}{q})}
\leq C \nu^{-(2n - \alpha + \frac{1}{q})},
\]

where we have used that \(s \leq 2s/2\) for \(4 \leq \log_2 \nu \leq s\).

Similarly, we also have

\[
\left( \int_{|x-x_Q| > \nu r_Q} |(b(x) - b_Q)^2 I_a(f_1, f_2)(x)| \right)^\frac{1}{q} \leq C \nu^{-(2n - \alpha + \frac{1}{q})},
\]

\[
\left( \int_{|x-x_Q| > \nu r_Q} |(b(x) - b_Q)^2 I_a(f_1, f_2)(x)| \right)^\frac{1}{q} \leq C \nu^{-(2n - \alpha + \frac{1}{q})}.
\]
Then for \( \mu > \nu \), using (2.16), (2.17) and the estimates above, we get

\[
\int_{\nu r_Q < |x-x_Q| < \mu r_Q} |[\Pi \tilde{b}, I_\alpha](f_1, f_2)(x)|^q dx \\
\geq C \left( I_\alpha((b-b_Q)f_1, (b-b_Q)f_2)(x) \right)^{1/q} \\
- C \left( \int_{\nu r_Q < |x-x_Q| < \mu r_Q} |(b(x)-b_Q)I_\alpha((b-b_Q)f_1, f_2)(x)|^q dx \right)^{1/q} \\
- C \left( \int_{\nu r_Q < |x-x_Q| < \mu r_Q} |(b(x)-b_Q)I_\alpha(f_1, (b-b_Q)f_2)(x)|^q dx \right)^{1/q} \\
- C \left( \int_{\nu r_Q < |x-x_Q| < \mu r_Q} |(b(x)-b_Q)^2 I_\alpha(f_1, f_2)(x)|^q dx \right)^{1/q} \\
\geq C \epsilon |Q|^{\frac{1}{r_1} + \frac{1}{r_2}} \left( \int_{\nu r_Q < |x-x_Q| < \mu r_Q} |x-x_Q|^{q(2n+\alpha)} dx \right)^{1/q} - C \nu^{-2n+\alpha+n_q \frac{1}{q}} \\
\geq C \epsilon \left( \nu^{-2n+\alpha+n_q} - \mu^{-2n+\alpha+n_q} \right)^{\frac{1}{q}} - C \nu^{-2n+\alpha+n_q \frac{1}{q}}.
\]

We can select \( \gamma_1, \gamma_2 \) in place of \( \nu, \mu \) with \( \gamma_2 >> \gamma_1 \), then (2.13) and (2.14) are verified for some \( \gamma_3 > 0 \).

We now verified (2.15). Let \( E \subset \{ \gamma_1 r_Q < |x-x_Q| < \gamma_2 r_Q \} \) be an arbitrary measurable set. It follows from Minkowski inequality that

\[
\left( \int_E |[\Pi \tilde{b}, I_\alpha](f_1, f_2)(x)|^q dx \right)^{\frac{1}{q}} \\
\leq |Q|^{\frac{1}{r_1} + \frac{1}{r_2}} \left( \int_E |x-x_Q|^{-q(2n-\alpha)} dx \right)^{\frac{1}{q}} + |Q|^{\frac{1}{r_1} + \frac{1}{r_2}} \left( \int_E |b(x)-b_Q|^q dx \right)^{\frac{1}{q}} \\
+ |Q|^{\frac{1}{r_1} + \frac{1}{r_2}} \left( \int_E |x-x_Q|^{q(2n-\alpha+1)} dx \right)^{\frac{1}{q}} \\
\leq \left( \frac{|E|}{|Q|^{\frac{1}{r_1}}} \right)^{\frac{1}{q}} + \left( \frac{1}{|Q|} \int_E |b(x)-b_Q|^q dx \right)^{\frac{1}{q}} + \left( \frac{1}{|Q|} \int_E |b(x)-b_Q|^{2q} dx \right)^{\frac{1}{q}}
\]

The same estimate as Lemma 2.7 and taking \( 0 < \beta < \min\{\tilde{C}^{1/n}, \gamma_2\} \), we can obtain the desired result.

\[\square\]

In the proof of the boundedness of the iterated commutators, the following two important properties of the weights we will be using.
Lemma 2.4. \([4]\) Let \(1 < p_1, p_2 < \infty\), \(P = (p_1, p_2)\), \(0 < \alpha < 2n\), \(\frac{\alpha}{n} < \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}\) and \(q\) such that \(\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}\). Suppose that \(\omega_1^p, \omega_2^p \in A_p\). Then,

(i) \(\omega = (\omega_1, \omega_2) \in A_{P,q}\);

(ii) \(\mu_\omega = \omega_1^q \omega_2^q \in A_p \subset A_q\).

In the proof of Theorem 1.2, we need the following weighted version of the Frechét-Kolmogorov-Riesz theorem. We refer to works by Hanche-Olsen and Holden \([11]\) and Clop and Cruz \([7]\).

Lemma 2.5. Let \(1 < q < \infty\) and \(\omega \in A_q\). Suppose that the subset \(\mathcal{F} \subset L^q(\omega)\) satisfies the following conditions:

(i) norm boundedness uniformly

\[
(2.21) \quad \sup_{f \in \mathcal{F}} \|f\|_{L^q(\omega)} < \infty;
\]

(ii) translation continuity uniformly

\[
(2.22) \quad \lim_{f \to 0} \|f(\cdot + y) - f(\cdot)\|_{L^q(\omega)} = 0 \text{ uniformly in } f \in \mathcal{F};
\]

(iii) control uniformly away from the origin

\[
(2.23) \quad \lim_{A \to \infty} \int_{|x| > A} |f(x)|^q \omega(x) dx = 0 \text{ uniformly in } f \in \mathcal{F};
\]

then \(\mathcal{F}\) is pre-compact in \(L^q(\omega)\).

Another reduction in the proof of Theorem 1.2 will be made by slightly modifying the bilinear fractional integral operator. This technique comes from Krantz and Li \([14]\) (see also \([11, 4]\)). More precisely, for any \(\delta > 0\) small enough, the kernel \(K^\delta(x, y_1, y_2)\) in \(\mathbb{R}^{3n}\) such that for \(\max\{|x - y_1|, |x - y_2|\} > 2\delta\),

\[
K^\delta(x, y_1, y_2) = \frac{1}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}};
\]

for \(\max\{|x - y_1|, |x - y_2|\} < \delta\),

\[
K^\delta(x, y_1, y_2) = 0;
\]

and for all multi-indexes with \(|\gamma| \leq 1\),

\[
\partial^\gamma K^\delta(x, y_1, y_2) \lesssim \frac{1}{(|x - y_1| + |x - y_2|)^{2n-\alpha + |\gamma|}}.
\]
The operators \( I^\delta_\alpha \) are defined by
\[
I^\delta_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^\delta(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.
\]

**Lemma 2.6.** If \( b \in C^\infty_c \) and \( \omega \in A_{p, q} \), then
\[
\lim_{\delta \to 0} \| [\Pi b, f^\delta_\alpha] - [\Pi b, I_\alpha] \|_{L^p(\omega_1^{p_1} \times L^p(\omega_2^{p_2}) \to L^q(\mu))} = 0.
\]

**Proof.** The proof of this result is very similar to that of Lemma 2.1 in [1] and we omit the details. \( \square \)

### 3. Proofs of Theorem 1.1 and Theorem 1.2

**Proof of Theorem 1.1.** We first show that if \([b, B_0]\) is bounded from \(L^{p_1} \times L^{p_2}\) to \(L^p\), then \(b \in \text{BMO}\). For \( z_0 \in \mathbb{R}^n \setminus \{0\}\), let \( \delta = \frac{|z_0|}{2^m}\) and \( Q_0(z_0, \delta)\) denote the open cube centered at \( z_0 \) with side length \( 2\delta \). Then \(|x|^{n-\alpha}\) has an absolutely convergent Fourier series
\[
|x|^{n-\alpha} = \sum a_m e^{i m \cdot x}
\]
with \( \sum |a_m| < \infty \), where the exact form of the vectors \( v_m \) is unrelated. Taking \( z_1 = \frac{z_0}{\delta}\) we have the expansion
\[
|x|^{n-\alpha} = \delta^{-n+\alpha} |\delta x|^{n-\alpha} = \delta^{-n+\alpha} \sum a_m e^{i m \cdot \delta x} \text{ for } |x - z_1| < \sqrt{n}.
\]

Given cubes \( Q = Q(x_0, r) \) and \( Q' = Q(x_0 - rz_1, r) \), if \( x \in Q \) and \( y \in Q' \), then
\[
\left| \frac{x - y}{r} - \frac{z_0}{\delta} \right| \leq \left| \frac{x - x_0}{r} \right| + \left| \frac{y - (x_0 - \frac{rz_0}{r})}{r} \right| < \sqrt{n},
\]
and
\[
|2x - y - (x_0 - rz_1)| \leq |x - x_0| + |x - y - rz_1| \leq \sqrt{n} \frac{r}{2} + r \left| \frac{x - y}{r} - \frac{z_0}{\delta} \right| \leq 3\sqrt{n} r,
\]
which implies that \( 2x - y \in \tilde{Q} = 3\sqrt{n} Q' \).

Let \( s(x) = \text{sgn}(\int_{Q'} (b(x) - b(y)) dy) \). Then
\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q'}| dx
\]
\[
= \frac{1}{|Q|} \int_Q \left( \int_{Q'} (b(x) - b(y)) dy \right) dx
\]
\[
= \frac{1}{|Q|^2} \int_Q \int_{Q'} s(x) (b(x) - b(y)) dy dx
\]
\[
= \frac{1}{|Q|^2} \int_Q \int_{Q'} \frac{r^{n-\alpha} s(x) (b(x) - b(y))}{|x - y|^{n-\alpha}} \frac{|x - y|^{n-\alpha}}{r} dy dx
\]
It follows that
\[ \frac{1}{|Q|} \int_Q \int_{Q'} \frac{s(x)(b(x) - b(y))}{|x - y|^{n-\alpha}} \sum a_m e^{iv_m \frac{x-y}{r}} dy dx \]
\[ = \frac{1}{|Q|} \int_Q \int_{Q'} \frac{s(x)(b(x) - b(y))}{|x - y|^{n-\alpha}} \sum a_m e^{iv_m \frac{x-y}{r}} e^{-iv_m \frac{x-y}{r}} dy dx \]
\[ = \frac{1}{|Q|} \int_Q \int_{Q'} \frac{s(x)(b(x) - b(y))}{|x - y|^{n-\alpha}} \sum a_m e^{iv_m \frac{x-y}{r}} e^{-iv_m \frac{x-y}{r}} \chi_Q(x) \chi_{Q'}(y) \chi_{Q}(2x - y) dy dx \]

Setting
\[ f_m(y) = e^{-iv_m \frac{y}{r}} \chi_{Q'}(y), \]
\[ g_m(z) = e^{-iv_m \frac{z}{r}} \chi_{Q}(z), \]
\[ h_m(x) = e^{iv_m \frac{x}{r}} s(x) \chi_Q(x), \]
we have
\[ \frac{1}{|Q|} \int_Q |b(x) - b_{Q'}| dx = \frac{1}{|Q|} \sum a_m \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f_m(y)g_m(2x - y)}{|x - y|^{n-\alpha}} h_m(x) dy dx \]
\[ = \frac{1}{|Q|} \sum a_m \int_{\mathbb{R}^n} [b, B_\alpha]_1(f_m, g_m)(x) h_m(x) dx. \]

It follows that
\[ \frac{1}{|Q|} \int_Q |b(x) - b_{Q'}| dx \leq \frac{1}{|Q|} \int_Q |b(x) - b_{Q'}| dx \]
\[ \leq \frac{1}{|Q|^{1/q}} \sum |a_m| \|[b, B_\alpha]_1(f_m, g_m)\|_{L^q} \]
\[ \leq \frac{\|f_m\|_{L^{p_1}} \|g_m\|_{L^{p_2}}}{|Q|^{1/p}} \|[b, B_\alpha]_1\|_{L^{p_1} \times L^{p_2} \to L^q} \]
\[ \leq \|[b, B_\alpha]_1\|_{L^{p_1} \times L^{p_2} \to L^q}, \]
which yields \( b \in \text{BMO} \) and \( \|b\|_* \lesssim C \|[b, B_\alpha]_1\|_{L^{p_1} \times L^{p_2} \to L^q}. \)

To prove \( b \) be an element of CMO, we will adapt some arguments from [5], see also [4], which in turn are based on the original work in [21]. The approach is the following: if one of the conditions Eqs. (2.1)-(2.3) in Lemma 2.1 failed, we will show that there exist sequences of functions, \( \{f_j\}_j \) uniformly bounded on \( L^{p_1} \) and \( \{g_j\}_j \) uniformly bounded on \( L^{p_2} \), such that \( [b, B_\alpha]_1(f_j, g_j) \) has no convergent subsequence, which contradicts the assumption that \( [b, B_\alpha]_1 \) is compact. It gives us that if \( [b, B_\alpha]_1 \) is compact, \( b \) must satisfy all three conditions; that is \( b \in \text{CMO} \).

By Lemma 2.2 it is sufficient to once again repeat the steps performed in [5] (or [4], [6]) to obtain the desired result. Hence, the proof of Theorem 1.1 is completed. \( \square \)
Proof of Theorem \([A1] \Rightarrow (A2)\): Note that \((2.21)\) is immediate since for \(b \in C_c^{\infty}\), 
\([b, I_\alpha^\delta]\) is bounded from \(L^{p_1}(\omega_1^{p_1}) \times L^{p_2}(\omega_2^{p_2})\) to \(L^q(\mu_\omega)\), because \(\omega \in A_{P_q}\) by Lemma 2.6.

To show that \((2.22)\) holds, we can use a similar method as in [4, p.491], the proof of this results is very similar to that of linear commutator case, we omit the detail.

Now we give the estimate for \((2.23)\). Denote

\[ \mathcal{F} = \{ [\Pi \vec{b}, I_\alpha^\delta](f_1, f_2) : f_i \in L^{p_i}(\omega_i^{p_i}), \| f_i \|_{L^{p_i}(\omega_i)} \leq 1, i = 1, 2 \}. \]

Then \(\mathcal{F}\) is uniformly bounded, because \([\Pi \vec{b}, I_\alpha^\delta]\) is a bounded operator from \(L^{p_1}(\omega_1^{p_1}) \times L^{p_2}(\omega_2^{p_2})\) to \(L^q(\mu_\omega)\), because \(\omega \in A_{P_q}\) by Lemma 2.1. To prove the uniform equicontinuity of \(\mathcal{F}\), we must see that

\[ \lim_{t \to 0} \| [\Pi \vec{b}, I_\alpha^\delta](f_1, f_2)(\cdot + t) - [\Pi \vec{b}, I_\alpha^\delta](f_1, f_2)(\cdot) \|_{L^q(\mu_\omega)} = 0. \]

To do this, we write

\[
[\Pi \vec{b}, I_\alpha^\delta](f_1, f_2)(x) - [\Pi \vec{b}, I_\alpha^\delta](f_1, f_2)(x + t) \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y_1)) (b(x) - b(y_2)) K^\delta(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x + t) - b(y_1)) (b(x + t) - b(y_2)) K^\delta(x + t, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
= (b(x) - b(x + t))^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^\delta(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
+ (b(x) - b(x + t)) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x + t) - b(y_2)) K^\delta(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
+ (b(x) - b(x + t)) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x + t) - b(y_1)) K^\delta(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x + t) - b(y_1)) (b(x + t) - b(y_2)) \\
\times (K^\delta(x, y_1, y_2) - K^\delta(x + t, y_1, y_2)) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
= I_1(x, t) + I_2(x, t) + I_3(x, t) + I_4(x, t). 
\]

For \(I_1(x, t)\), we simply have

\[ |I_1(x, t)| \lesssim |t|^2 \| \nabla b \|_{\infty} I_\alpha(|f_1|, |f_2|)(x), \]

which implies that

\[ \| I_1(\cdot, t) \|_{L^q(\omega)} \lesssim |t|^2. \]

Similarly, we also have that for \(j = 2, 3\)

\[ |I_j(x, t)| \lesssim |t| \| \nabla b \|_{\infty} \| b \|_{\infty} I_\alpha(|f_1|, |f_2|)(x), \]

\[ \| I_j(\cdot, t) \|_{L^q(\omega)} \lesssim |t|^2. \]
and 
\[ \|I_j(\cdot, t)\|_{L^q(\omega)} \lesssim |t|. \]

We now give the estimate for \( I_4(x, t) \). We may assume that \(|t| \in (0, \delta/4)\). Thus, if \( \max\{|x - y_1|, |x - y_2|\} \leq \delta/2 \) we have
\[ K^\delta(x + t, y_1, y_2) - K^\delta(x, y_1, y_2) = 0 \]
and \( \max\{|x - y_1|, |x - y_2|\} > \delta/2 \) we have
\[ \max\{|x - y_1|, |x - y_2|\} > 2t. \]

This gives us that
\[
|I_4(x, t)| \lesssim |t| \|b\|_\infty^2 \int \int_{\max\{|x-y_1|,|x-y_2|\} > \delta/2} \left( \frac{|f_1(y_1) f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n-\alpha+1}} \right) dy_1 dy_2
\]
\[
\lesssim |t| \|b\|_\infty^2 \sum_{j \geq 0} \int \int_{2^{-j-1} \delta < \max\{|x-y_1|,|x-y_2|\} < 2^j \delta} \left( \frac{|f_1(y_1) f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n-\alpha+1}} \right) dy_1 dy_2
\]
\[
\lesssim |t| \|b\|_\infty^2 \sum_{j \geq 0} \int \int_{2^{-j-1} \delta < \max\{|x-y_1|,|x-y_2|\} < 2^j \delta} \left( \frac{|f_1(y_1) f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n-\alpha+1}} \right) \times \frac{1}{\max\{|x-y_1|,|x-y_2|\}} dy_1 dy_2
\]
\[
\lesssim \frac{|t| \|b\|_\infty^2}{\delta} \int \int_{|x-y_1|,|x-y_2| < 1} |I_\alpha(|f_1|,|f_2|)(x)|,
\]
which gives
\[ \|I_4(\cdot, t)\|_{L^q(\mu_\omega)} \leq |t|. \]

Combining the estimates above and let \(|t| < 1\), we have
\[ \|\Pi_b I_4^\delta(f_1, f_2)(\cdot + t) - \Pi_b I_4^\delta(f_1, f_2)(\cdot)\|_{L^q(\mu_\omega)} \lesssim |t|. \]

Thus, if \( b \in \text{CMO} \), \( \Pi_b I_\alpha \) is a compact operator from \( L^{p_1}(\omega_1^{\gamma_1}) \times L^{p_2}(\omega_2^{\gamma_2}) \) to \( L^q(\mu_\omega) \).

Obviously (A2) \( \Rightarrow \) (A3). So it remains to show that (A3) \( \Rightarrow \) (A1). By Lemma 2.3 and the same argument as Theorem 1.1 we need only to prove \( b \in \text{BMO} \).

(A3) \( \Rightarrow \) (A1): Let \( z_0 \in \mathbb{R}^n \) such that \(|(z_0, z_0)| > 2\sqrt{n}\) and let \( \delta \) small enough such that \( \delta < 1 \). Take \( B = B((z_0, z_0), \delta \sqrt{2n}) \subset \mathbb{R}^{2n} \) be the ball for which we can express \((|y_1|^2 + |y_2|^2)^{n-\alpha/2}\) as an absolutely convergent Fourier series of the form
\[ (|y_1|^2 + |y_2|^2)^{n-\alpha/2} = \sum_j a_j e^{iv_j \cdot (y_1, y_2)}, \quad (y_1, y_2) \in B, \]
where \( \sum_j |a_j| < \infty \) and we do not care about the vectors \( v_j \in \mathbb{R}^{2n} \), but we will at times express them as \( v_j = (v_j^1, v_j^2) \in \mathbb{R}^n \times \mathbb{R}^n \).
Set $z_1 = \delta^{-1}z_0$ and note that
\[
(\|y_1 - z_1\|^2 + |y_2 - z_1|^2)^{1/2} < \sqrt{2n} \Rightarrow (\|\delta y_1 - z_0\|^2 + |\delta y_2 - z_0|^2)^{1/2} < \delta \sqrt{2n}.
\]
Then for any $(y_1, y_2)$ satisfying the inequality on the left, we have
\[
(|y_1|^2 + |y_2|^2)^{n-\alpha/2} = \delta^{-2n+\alpha}(|\delta y_1|^2 + |\delta y_2|^2)^{n-\alpha/2} = \delta^{-2n+\alpha} \sum_j a_j e^{i \delta v_j \cdot (y_1, y_2)}.
\]
Let $Q = Q(x_0, r)$ be any arbitrary cube in $\mathbb{R}^n$. Set $\tilde{z} = x_0 + rz_1$ and take $Q' = Q(\tilde{z}, r) \subset \mathbb{R}^n$. So for any $x \in Q$ and $y_1, y_2 \in Q'$, we have
\[
\left| \frac{x - y_1}{r} - z_1 \right| \leq \left| \frac{x - x_0}{r} \right| + \left| \frac{y_1 - \tilde{z}}{r} \right| \leq \sqrt{n}, \quad \left| \frac{x - y_2}{r} - z_1 \right| \leq \left| \frac{x - x_0}{r} \right| + \left| \frac{y_2 - \tilde{z}}{r} \right| \leq \sqrt{n},
\]
which implies that
\[
\left( \left| \frac{x - y_1}{r} - z_1 \right|^2 + \left| \frac{x - y_2}{r} - z_1 \right|^2 \right)^{1/2} \leq \sqrt{2n}.
\]
Let $s(x) = \text{sgn}(\int_{Q'} (b(x) - b(y))dy)$. We have the following estimate,
\[
\left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|dx \right)^2 \lesssim \left( \frac{1}{|Q|} \int_Q |b(x) - b_{Q'}|dx \right)^2
\]
\[
\lesssim \frac{1}{|Q|} \int_Q |b(x) - b_Q|^2dx \lesssim \frac{1}{|Q|} \int_Q s(x)^2(b(x) - b_{Q'})^2dx
\]
\[
\lesssim \frac{1}{|Q||Q'|^2} \int_Q \int_{Q'} \int_{Q'} s(x)^2(b(x) - b(y_1))(b(x) - b(y_2))dy_1dy_2dx
\]
\[
\lesssim \frac{r^{2n-\alpha} \delta^{-2n+\alpha}}{|Q|^3} \int_Q \int_{Q'} \int_{Q'} \frac{s(x)^2(b(x) - b(y_1))(b(x) - b(y_2))}{(|x - y_1|^2 + |x - y_2|^2)^{n-\alpha/2}} \sum_j a_j e^{i \delta v_j \cdot (x-y_1,x-y_2)}dy_1dy_2dx.
\]
Setting
\[
g_j(y_1) = e^{-i Q_y^1 \cdot y_1} \chi_{Q'}(y_1),
\]
\[
h_j(y_2) = e^{-i Q_y^2 \cdot y_2} \chi_{Q'}(y_2),
\]
\[
m_j(x) = e^{i Q_y \cdot (x,x)} \chi_{Q}(x)s(x)^2.
\]
We have
\[
\left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|dx \right)^2
\]
\[
\lesssim \frac{r^{2n-\alpha} \delta^{-2n+\alpha}}{|Q|^3} \sum_j |a_j| \int_{\mathbb{R}^n} |[\Pi_b, I_\alpha](g_j, h_j)(x)m_j(x)|dx
\]
\[
\lesssim r^{-n-\alpha} \delta^{-2n+\alpha} \sum_j |a_j| ||[\Pi_b, I_\alpha]||_{L^p \times L^p \to L^q} ||g_j||_{L^p} ||h_j||_{L^p} ||m_j||_{L^q}
\]
\[ \lesssim \delta^{-2n+\alpha} \| \left[ \Pi \vec{b}, I_\alpha \right] \|_{L^{p_1} \times L^{p_2} \to L^{q}} \sum_j |a_j|. \]

The desired result follows from here. \(\square\)

**Acknowledgments** We would like to thank the anonymous referee for his/her comments.

**References**

[1] Béyi, Á., Damián, W., Moen, K., Torres, R.H.: Compactness properties of commutators of bilinear fractional integrals. *Math. Z.*, (2015), doi: 10.1007/s00209-015-1437-4.

[2] Béyi, Á., Damián, W., Moen, K., Torres, R.H.: Compact bilinear operators: the weighted case. *Mich. Math. J.*, 4, 39-51 (2015)

[3] Béyi, Á., Torres, R.H.: Compact bilinear operators and commutators. *Proc. Amer. Math. Soc.*, 141(10), 3609-3621 (2013)

[4] Chaffee, L., Torres, R.H.: Characterization of Compactness of the Commutators of Bilinear Fractional Integral Operators. *Potential Anal.* 43, 481-494 (2015)

[5] Chen, Y., Ding, Y., Wang, X.: Compactness of commutators of Riesz potential on Morrey spaces. *Potential Anal.* 30, 301-313(2009)

[6] Chen, Y., Ding, Y., Wang, X.: Compactness of commutators for Singular integrals on Morrey spaces. *Canad. J. Math.* 64, 257-281(2012)

[7] Clop, A., Cruz, V.: Weighted estimates for Beltrami equations. *Ann. Acad. Sci. Fenn. Math.* 38, 91-113 (2013)

[8] Coifman, R., Rochberg, R., Weiss,G.: Factorization theorems for Hardy spaces in several variables. *Ann. of Math.*, 103, 611-635 (1976)

[9] Grafakos, L.: On multilinear fractional integrals. *Studia Math.* 102, 49-56 (1992)

[10] L. Grafakos, L., Kalton, N.: Some remarks on multilinear maps and interpolation, *Math. Ann.* 319, 151-180 (2001)

[11] Hanche-Olsen, H., Holden, H.: The Kolmogorov-Riesz compactness theorem. *Expo. Math.* 28, 385-394 (2010)

[12] Kenig, C., Stein, E.: Multilinear estimates and fractional integration, *Math. Res. Lett.* 6 1-15 (1999)

[13] Krantz, S., Li, S.-Y.: Boundedness and compactness of integral operators on spaces of homogeneous type and applications, I. *J. Math. Anal. Appl.* 258, 629-641 (2001)

[14] Krantz, S., Li, S.-Y.: Boundedness and compactness of integral operators on spaces of homogeneous type and applications, II. *J. Math. Anal. Appl.* 258, 642-657 (2001)

[15] Lerner, A.K., Ombrosi, S., Pérez, C., Torres, R.H., Trujillo-González, R.: New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. *Adv. Math.*, 220, 1222-1264 (2009)

[16] Moen, K.: Weighted inequalities for multilinear fractional integral operators. *Collect Math.* 60, 213-238 (2009)

[17] Pérez, C.: Endpoint estimates for commutators of singular integral operators. *J. Funct. Anal.*, 128, 163-185 (1995)
[18] Pérez, C., Torres, R.H.: Sharp maximal function estimates for multilinear singular integrals. *Contemp. Math.*, **320**, 323-331 (2003)

[19] Stein, E.M.: Singular Integral and Differentiability Properties of Functions. Princeton University Press, Princeton (1971)

[20] Tang, L.: Weighted estimates for vector-valued commutators of multilinear operators. *Proc. Roy. Soc. Edinburgh Sect. A*, **138**, 897-922 (2008)

[21] Uchiyama, A.: On the compactness of operators of Hankel type. *Tohoku Math. J.*, **30**, 163-171 (1978)

[22] Wang, S.: The compactness of the commutator of fractional integral operator (in Chinese). *Chin. Ann. Math* **8**(A), 475-482 (1987)

**College of Mathematics and System Sciences**  
Xinjiang University  
Urumqi 830046  
Republic of China  
*E-mail address:* Wangdh1990@126.com; zhoujiangshuxue@126.com

**School of Mathematics and Statistics**  
Wuhan University  
Wuhan 430072  
Republic of China  
*E-mail address:* wychencn@hotmail.com