New Computational Upper Bounds for Ramsey Numbers $R(3, k)$

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Abstract

Using computational techniques we derive six new upper bounds on the classical two-color Ramsey numbers: $R(3, 10) \leq 42$, $R(3, 11) \leq 50$, $R(3, 13) \leq 68$, $R(3, 14) \leq 77$, $R(3, 15) \leq 87$, and $R(3, 16) \leq 98$. All of them are improvements by one over the previously best published bounds.

Let $e(3, k, n)$ denote the minimum number of edges in any triangle-free graph on $n$ vertices without independent sets of order $k$. The new upper bounds on $R(3, k)$ are obtained by completing the computation of the exact values of $e(3, k, n)$ for all $n$ with $k \leq 9$ and for all $n \leq 33$ for $k = 10$, and by establishing new lower bounds on $e(3, k, n)$ for most of the open cases for $10 \leq k \leq 15$. The enumeration of all graphs witnessing the values of $e(3, k, n)$ is completed for all cases with $k \leq 9$.

We prove that the known critical graph for $R(3, 9)$ on 35 vertices is unique up to isomorphism. For the case of $R(3, 10)$, first we establish that $R(3, 10) = 43$ if and only if $e(3, 10, 42) = 189$, or equivalently, that if $R(3, 10) = 43$ then every critical graph is regular of degree 9. Then, using computations, we disprove the existence of the latter, and thus show that $R(3, 10) \leq 42$.

Keywords: Ramsey number; upper bound; computation
1 Definitions and Preliminaries

In this paper all graphs are simple and undirected. Let $G$ be such a graph. The vertex set of $G$ is denoted by $V(G)$, the edge set of $G$ by $E(G)$, and the number of edges in $G$ by $e(G)$. The set of neighbors of $v$ in $G$ will be written as $N_v(G)$ (or just $N(v)$ if $G$ is fixed). The independence number of $G$, denoted $\alpha(G)$, is the order of the largest independent set in $G$, $\deg_G(v)$ is the degree of vertex $v \in V(G)$, and $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree of vertices in $G$, respectively. For graphs $G$ and $H$, $G \cong H$ means that they are isomorphic.

For positive integers $k$ and $l$, the Ramsey number $R(k, l)$ is the smallest integer $n$ such that if we arbitrarily color the edges of the complete graph $K_n$ with $2$ colors, then it contains a monochromatic $K_k$ in the first color or a monochromatic $K_l$ in the second color. If the edges in the first color are interpreted as a graph $G$ and those in the second color as its complement $\overline{G}$, then $R(k, l)$ can be defined equivalently as the smallest $n$ such that every graph on $n$ vertices contains $K_k$ or has independence $\alpha(G) \geq l$. A regularly updated dynamic survey by the second author [17] lists the values and the best known bounds on various types of Ramsey numbers.

Any $K_k$-free graph $G$ on $n$ vertices with $\alpha(G) < l$ and $e(G) = e$ will be called a $(k, l; n, e)$-graph, and by $\mathcal{R}(k, l; n, e)$ we will denote the set of all $(k, l; n, e)$-graphs. We will often omit the parameter $e$, or both $e$ and $n$, or give some range to either of these parameters, when referring to special $(k, l; n, e)$-graphs or sets $\mathcal{R}(k, l; n, e)$. For example, a $(k, l)$-graph is a $(k, l; n, e)$-graph for some $n$ and $e$, and the set $\mathcal{R}(3, 9; 35, \leq 139)$ consists of all 35-vertex triangle-free graphs with $\alpha(G) \leq 8$ and at most 139 edges (later we will prove that this set is empty). Any $(k, l; R(k, l) - 1)$-graph will be called critical for $(k, l)$.

Let $e(k, l, n)$ denote the minimum number of edges in any $(k, l; n)$-graph (or $\infty$ if no such graph exists). The sum of the degrees of all neighbors of $v$ in $G$ will be denoted by $Z_G(v)$ (or $Z(v)$ if $G$ is fixed), i.e.

$$Z(v) = Z_G(v) = \sum_{\{u,v\} \in E(G)} \deg_G(u). \quad (1)$$

In the remainder of this paper we will study only triangle-free graphs. Note that for any $G \in \mathcal{R}(3, k)$ we have $\Delta(G) < k$, since all neighborhoods of vertices in $G$ are independent sets.

Let $G$ be a $(3, k; n, e)$-graph. For any vertex $v \in V(G)$, we will denote by $G_v$ the graph induced in $G$ by the set $V(G) \setminus (N_G(v) \cup \{v\})$. If $d = \deg_G(v)$, then clearly $G_v$ is a $(3, k - 1; n - d - 1, e(G) - Z_G(v))$-graph. Note that this implies that

$$\gamma(v) = \gamma(v, k, G) = e - Z_G(v) - e(3, k - 1, n - d - 1) \geq 0, \quad (2)$$

where $\gamma(v)$ is the so called deficiency of vertex $v$ [8]. Finally, the deficiency of the graph $G$ is defined as

$$\gamma(G) = \sum_{v \in V(G)} \gamma(v, k, G) \geq 0. \quad (3)$$
The condition that $\gamma(G) \geq 0$ will be often sufficient to derive good lower bounds on $e(k,l,n)$, though a stronger condition that all summands $\gamma(v,k,G)$ of (3) are non-negative sometimes implies even better bounds. It is easy to compute $\gamma(G)$ just from the degree sequence of $G$. If a $(3,k;n,e)$-graph $G$ has $n_i$ vertices of degree $i$, then

$$\gamma(G) = ne - \sum_i n_i(i^2 + e(3,k-1,n-i-1)) \geq 0,$$

where $n = \sum_{i=0}^{k-1} n_i$ and $2e = \sum_{i=0}^{k-1} in_i$.

2 Summary of Prior and New Results

In 1995, Kim [12] obtained a breakthrough result by establishing the exact asymptotics of $R(3,k)$ using probabilistic arguments. Recently, the fascinating story of developments and results related to the infinite aspects of $R(3,k)$ was written by Spencer [21].

Theorem 1 ([12]) $R(3,k) = \Theta(n^2 / \log n)$.

Theorem [1] gives the exact asymptotics of $R(3,k)$, while computing the values for concrete cases remains an open problem for all $k \geq 10$. Still, the progress obtained in the last 50 years in this area is remarkable. Known exact values of $R(3,k)$ for $k \leq 9$, and the best lower and upper bounds for higher $k$, are listed in [17] together with all the references. We note that much of this progress was obtained with the use of knowledge about $e(3,k,n)$. This direction is also the main focus of our paper: we compute new exact values of $e(3,k,n)$ in several cases and give improved lower bounds for many other, which in turn permits us to prove new upper bounds on $R(3,k)$ for $k = 10, 11, 13, 14, 15$ and 16. Likely, more new upper bounds could be obtained for some $17 \leq k \leq 21$, but we did not perform these computations.

General formulas for $e(3,k,n)$ are known for all $n \leq 13k/4 - 1$ and for $n = 13k/4$ when $k = 0 \mod 4$.

Theorem 2 ([18, 20]) For all $n,k \geq 1$, for which $e(3,k+1,n)$ is finite,

$$e(3,k+1,n) = \begin{cases} 0 & \text{if } n \leq k, \\ n - k & \text{if } k < n \leq 2k, \\ 3n - 5k & \text{if } 2k < n \leq 5k/2, \\ 5n - 10k & \text{if } 5k/2 < n \leq 3k, \\ 6n - 13k & \text{if } 3k < n \leq 13k/4 - 1. \end{cases}$$

Furthermore, $e(3,k+1,n) = 6n - 13k$ for $k = 4t$ and $n = 13t$, and the inequality $e(3,k+1,n) \geq 6n - 13k$ holds for all $n$ and $k$. All the critical graphs have been characterized whenever the equality in the theorem holds for $n \leq 3k$.  

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Theorem 2 is a cumulative summary of various contributions \[8, 10, 18, 19, 20\]. It captures many of the small cases, as presented in Table 3 in Section 4. For example, Theorem 2 gives the exact values of \(e(3, 9, n)\) for all \(n \leq 26\), of \(e(3, 10, n)\) for \(n \leq 28\), and of \(e(3, 13, n)\) for all \(n \leq 39\).

The inequality \(e(3, k + 1, n) \geq (40n - 91k)/6\), which is better than \(e(3, k + 1, n) \geq 6n - 13k\) for larger parameters, and a number of other improvements and characterizations of graphs realizing specific number of edges, was credited in 2001 by Lesser \[13\] to an unpublished manuscript by Backelin \[1\]. As of 2012, the manuscript by Backelin already exceeds 500 pages and it contains numerous additional related results \[1, 2\], but it still needs more work before it can be published. Therefore, in the remainder of this paper we will not rely on the results included therein, however in several places we will cite the bounds obtained there for reference. In summary, the behavior of \(e(3, k + 1, n)\) is clear for \(n \leq 13k/4 - 1\), it seems regular but very difficult to deal with for \(n\) slightly larger than \(13k/4\), and it becomes hopelessly hard for even larger \(n\). In this work we apply computational techniques to establish lower bounds for \(e(3, k, n)\) for larger \(n\), for \(k \leq 15\). Immediately, our results imply better upper bounds on \(R(3, k)\) in several cases, but we hope that they also may contribute to further progress in understanding the general behavior of \(e(3, k, n)\).

Full enumeration of the sets \(R(3, \leq 6)\) was established in \[18, 16\]. The knowledge of the exact values of \(e(3, 7, n)\) was completed in \[18\], those of \(e(3, 8, \leq 26)\) in \[19\], and the last missing value for \(\alpha(G) < 8\), namely \(e(3, 8, 27) = 85\), was obtained in \[4\]. The thesis by Lesser \[13\] contains many lower bounds on \(e(3, k, n)\) better than those in \[19\]. We match or improve them in all cases for \(k \leq 10\). For \(k \geq 11\) and \(n\) slightly exceeding \(13k/4 - 1\), the bounds by Lesser (in part credited also to \[1\]) are better than ours in several cases, however we obtain significantly better ones for larger \(n\).

The general method we use is first to compute, if feasible, the exact value of \(e(3, k, n)\) for concrete \(k\) and \(n\), or to derive a lower bound using a combination of (2), (3), and (4), and computations. Better lower bounds on \(e(3, k - 1, m)\) for \(m = n - d - 1\) and various \(d\), lead in general to better lower bounds on \(e(3, k, n)\). If we manage to show that \(e(3, k, n) = \infty\), i.e. no \((3, k; n)\)-graph exists, then we obtain an upper bound \(R(3, k) \leq n\). An additional specialized algorithm was needed to establish \(R(3, 10) \leq 42\).

Section \[3\] describes extension algorithms which we used to exhaustively construct all \((3, k; n, e)\)-graphs for a number of cases of \((n, e)\), for \(k \leq 10\). These results are described in detail in the sequel. This leads to many new lower bounds on \(e(3, k, n)\) and full enumerations of \((3, k; n)\)-graphs with the number of edges equal to or little larger than \(e(3, k, n)\), which are presented in Section \[4\] (and Appendix 1). These results are then used in Section \[5\] to prove that there exists a unique critical 35-vertex graph for the Ramsey number \(R(3, 9)\). It is known that \[5\] \(40 \leq R(3, 10) \leq 43\) \[19\]. We establish that \(R(3, 10) = 43\) if and only if \(e(3, 10, 42) = 189\), or equivalently, that if \(R(3, 10) = 43\) then every critical graph in this case is regular of degree 9. Then, in Section 6, using computations we prove that the latter do not exist, and thus obtain \(R(3, 10) \leq 42\). Finally, in Section \[7\] we describe the second stage of our computations, which imply many new
lower bounds on $e(3, \geq 11, n)$. This stage uses only degree sequence analysis of potential $(3, k; n, e)$-graphs, which have to satisfy (4). This in turn leads to the new upper bounds on the classical two-color Ramsey numbers marked in bold in Table 1 which presents the values and best bounds on the Ramsey numbers $R(3, k)$ for $k \leq 16$. All the improvements in this work are better by one over the results listed in the latest 2011 revision #13 of the survey [17]. The bound $R(3, 16) \leq 98$ was also obtained by Backelin in 2004, though it was not published [11][12]. The lower bound $R(3, 11) \geq 47$ was recently obtained by Exoo [6]. The references for all other bounds and values, and the previous upper bounds, are listed in [17].

| $k$ | $R(3, k)$ | $k$ | $R(3, k)$ |
|-----|----------|-----|----------|
| 3   | 6        | 10  | 40–42    |
| 4   | 9        | 11  | 47–50    |
| 5   | 14       | 12  | 52–59    |
| 6   | 18       | 13  | 59–68    |
| 7   | 23       | 14  | 66–77    |
| 8   | 28       | 15  | 73–87    |
| 9   | 36       | 16  | 79–98    |

Table 1: Ramsey numbers $R(3, k)$, for $k \leq 16$.

3 Algorithms

Maximum Triangle-Free Method

One method to determine $e(3, k, n)$ is by first generating all maximal triangle-free $(3, k; n)$-graphs. A maximal triangle-free graph (in short, an mtf graph) is a triangle-free graph such that the insertion of any new edge forms a triangle. It is easy to see that there exists a $(3, k; n)$-graph if and only if there is an mtf $(3, k; n)$-graph. In [4], an algorithm is described that can generate all mtf $(3, k; n)$-graphs efficiently. Using this algorithm, it is much easier to generate all mtf $(3, k; n)$-graphs instead of all $(3, k; n)$-graphs, because the number of the former is in most cases much smaller. For example, there are 477142 $(3, 8; 27)$-graphs, but only 21798 mtf graphs with the same parameters. By recursively removing edges in all possible ways from these mtf $(3, k; n)$-graphs and testing if the resulting graphs $G$ still satisfy $\alpha(G) < k$, the complete set $\mathcal{R}(3, k; n)$ can be obtained.

We applied this method to generate the sets $\mathcal{R}(3, 7; 21), \mathcal{R}(3, 7; 22), \mathcal{R}(3, 8; 26, \leq 77)$ and $\mathcal{R}(3, 8; 27)$ (see Appendix 1 for detailed results). All $(3, 7; 22)$- and $(3, 7; n, e(3, k, n))$-graphs were already known [18], other enumerations are new. This mtf method is infeasible for generating $(3, \geq 9; n)$-graphs for $n$ which were needed in this work. Nevertheless, we used it for verifying the correctness of our other enumerations, and the results agreed in all cases in which more than one method was used (see Appendix 2).
Minimum Degree Extension Method

In their 1992 paper establishing \( R(3, 8) = 28 \), McKay and Zhang \[16\] proved that the set \( R(3, 8; 28) \) is empty by generating several sets \( R(3, k; n, e) \) with additional restrictions on the minimum degree \( \delta(G) \). Suppose that one wants to generate all \((3, k; n, e)\)-graphs. If \( G \) is such a graph and one considers its minimum degree vertex \( v \), then we can reconstruct \( G \) given all possible graphs \( G_v \). McKay and Zhang described such dependencies, designed an algorithm to reconstruct \( G \), and completed the proof of \( R(3, 8) = 28 \) using this algorithm.

We implemented and used this method by McKay and Zhang \[16\], and in all cases where more than one algorithm was used it agreed with the other results. However, using this method it was not feasible to generate most classes of graphs with higher parameters needed for our project. For example, we could not generate all \((3, 9; 28, \leq 69)\)-graphs with this method, as the graphs with \( \delta(G) = 4 \) are obtained from \((3, 8; 23, \leq 53)\)-graphs, but there are already 10691100 \((3, 8; 23, \leq 52)\)-graphs (Table 13 in Appendix 1).

Neighborhood Gluuing Extension Method

Our general extension algorithm for an input \((3, k; m)\)-graph \( H \) produces all \((3, k+1; n, e)\)-graphs \( G \), often with some specific restrictions on \( n \) and \( e \), such that for some vertex \( v \in V(G) \) graph \( H \) is isomorphic to \( G_v \). We used the following strategy to determine if the parameters of input graphs to our extender were such that the output was guaranteed to contain all \((3, k+1; n, \leq e)\)-graphs.

Let \( m_i = n - i - 1 \), where \( i \) ranges over possible degrees in any graph \( G \) we look for, \( \delta(G) \leq i \leq \Delta(G) \). In the broadest case we have \( \delta(G) = \max\{n - R(3, k), 0\} \) and \( \Delta(G) = k \), but we also identified a number of special cases where this range was more restricted. Let \( t_i \) be an integer such that we have extended all \((3, k; m_i, < e(3, k, m_i) + t_i)\)-graphs as potential \( G_v \)'s of \( G \). Now, if we use \( e(3, k, m_i) + t_i \) instead of \( e(3, k, m_i) \) in (4) for all relevant values of \( i \), and (4) has no solutions for \((3, k+1; n, \leq e)\)-graphs, then we can conclude that all such graphs were already generated. We illustrate this process by an example.

Example. Table 2 lists specific parameters of the general process when used to obtain all \((3, 8; 25, \leq 65)\)-graphs. Every vertex \( v \) in any \((3, 8; 25, \leq 65)\)-graph has degree \( i \), for some \( 2 \leq i \leq 7 \). The corresponding graph \( G_v \) is of type \((3, 7; m_i, e(G_v))\). The values of \( e(3, 7, m) \) are included in Table 3 of Section 4 and let \( t_i \)'s be as in Table 2. If we use the values \( e(3, 7, m_i) + t_i \) instead of \( e(3, 7, m_i) \) in (4), then there are no solutions for degree sequences of \((3, 8; 25, \leq 65)\)-graphs. Thus, if we run the extender for all possible graphs \( G_v \) with the number of edges listed in the last column of Table 2 then we will obtain all \((3, 8; 25, e)\)-graphs for \( e \leq 65 \).

The set of increments \( t_i \) accomplishing this goal is not unique, there are others which work. We just tried to minimize the amount of required computations in a greedy way. Note that the largest increments \( t_i \) to \( e(3, 7, m_i) \) occur for \( i \)'s which are close to the average degree of \( G \).
Table 2: Obtaining all $(3, 8; 25, \leq 65)$-graphs.

| $i = \deg_G(v)$ | $m_i = |V(G_v)|$ | $e(3, 7, m_i)$ | $t_i$ | $e(G_v) = e - Z(v)$ |
|------------------|-----------------|----------------|-------|----------------------|
| 2                | 22              | 60             | 1     | 60                   |
| 3                | 21              | 51             | 1     | 51                   |
| 4                | 20              | 44             | 2     | 44, 45               |
| 5                | 19              | 37             | 3     | 37, 38, 39           |
| 6                | 18              | 30             | 2     | 30, 31               |
| 7                | 17              | 25             | 1     | 25                   |

Implementation

In this section we present some details about the extension algorithms implementations for the minimum degree and neighborhood gluing method. Implementation of the algorithm to generate maximal triangle-free Ramsey graphs is described in [4].

Given a $(3, k; n, f)$-graph $G'$ as input and an expansion degree $d$, a desired maximum number of edges $e$, and the minimum degree $d_m$ as parameters, our program constructs all $(3, k + 1; n + d + 1, \leq e)$-graphs $G$ with $\delta(G) \geq d_m$ for which there is a vertex $v \in V(G)$ such that $\deg(v) = d$ and $G_v \cong G'$. More specifically, the program adds to $G'$ a vertex $v$ with neighbors $u_1, \ldots, u_d$ and connects them to independent sets of $G'$ in all possible ways, so that the resulting graph is a $(3, k + 1; n + d + 1, \leq e)$-graph with $\delta(G) \geq d_m$. Note that the neighbors of $v$ have to be connected to independent sets of $G'$, otherwise the expanded graph would contain triangles, and, clearly, $\Delta(G) \leq k$.

The extension program first determines all independent sets of $G'$ of orders $t$ that are possible, namely $d_m - 1 \leq t \leq k - 1$. The program then recursively assigns the $d$ neighbors of $v$ to the eligible independent sets of $G'$, adds the edges joining $u_i$’s to their associated independent sets, and tests if the resulting $G$ is a valid $(3, k + 1; n + d + 1, \leq e)$-graph. If it is, then we output it. This general process is greatly accelerated by the techniques described in the following.

We bound the recursion if a given partial assignment cannot lead to any $(3, k + 1; n + d + 1, \leq e)$-graphs. Suppose that $i$ independent sets $S_1, \ldots, S_i$ have already been assigned. If $V(G') \setminus (S_1 \cup \ldots \cup S_i)$ induces an independent set $I$ of order $k + 1 - i$, then this assignment cannot lead to any output since $I \cup \{u_1, \ldots, u_i\}$ would form an independent set of order $k + 1$ in $G$. We could test this property for all subsets of $S_i$’s, but we found it to be most efficient to do it only for all pairs. Namely, if $S_1, \ldots, S_i$ is already assigned and we consider the next independent set $S$, we test if for all $j$, $1 \leq j \leq i$, $V(G') \setminus (S_j \cup S)$ does not induce any independent set of order $k - 1$. The list of independent sets which can still be assigned is dynamically updated.

For the efficiency of the algorithm it is vital that testing for independence in $V(G') \setminus (S_1 \cup \ldots \cup S_i)$ is fast, and hence we precompute the independence numbers of all induced subgraphs of $G'$. This precomputation also needs to be done very efficiently. We represent a set of vertices $S \subset V(G')$ by a bitvector. The array `indep_number[S]` of $2^n$ elements
stores the independence number of the graph induced by $S$ in $G'$. It is very important that \texttt{indep\_number[]} fits into the memory. On the computers on which we performed the expansions this was still feasible up to $n = 31$. We investigated various approaches to precompute $\texttt{indep\_number}[S]$, and Algorithm 1 below was by far the most efficient one. If the superset $S'$ of $S$ already has $\texttt{indep\_number}[S'] \geq j$, then we can break the recursion of making the supersets. Usually one can break very quickly. For small extension degrees $d \leq 3$, it is more efficient not to precompute these independence numbers, but instead to compute them as needed.

**Algorithm 1** Precomputing independence number

```plaintext
for $i = 0$ upto $2^n - 1$ do
  set $\texttt{indep\_number}[i] = 0$
end for

for $j = k - 1$ downto $k + 1 - d$ do
  for all independent sets $S$ of order $j$ in $G'$ do
    Recursively make all supersets $S'$ of $S$, and
    if $\texttt{indep\_number}[S'] = 0$ then set $\texttt{indep\_number}[S'] = j$
    else break making supersets of $S$
  end for
end for
```

If a neighbor $u_i$ of $v$ has been assigned to an independent set $S$, we also update the degrees of the vertices in $G'$. If $u_i$ is being connected to $S$, the degree of every vertex of $S$ increases by one. If the degree of a vertex $w$ of $G'$ becomes $k$, then other neighbors of $v$ cannot be assigned to independent sets which contain $w$. We call such vertices which are no longer eligible \textit{forbidden vertices}, and all of them are stored in a dynamically updated bitvector. We also dynamically update the list of independent sets which can still be assigned to $u_i$’s. Independent sets which contain forbidden vertices are removed from the list of eligible independent sets. We perform bitvector operations whenever suitable. If no eligible independent sets are left, we can bound the recursion. Note that we cannot break the recursion when the number of eligible independent sets is smaller than the number of neighbors of $v$ that still have to be considered, since they can be assigned to the same independent set. If $i$ neighbors of $v$ are already assigned and the forbidden vertices form an independent of set order $k + 1 - (d - i)$, then the recursion can also be bounded, though this criterion in general is weak.

We assign the neighbors $u_i$ of $v$ to independent sets in ascending order, i.e. if $u_i$ is assigned to $S_i$, then $|S_i| \leq |S_{i+1}|$ for all $1 \leq i < d$. Doing this rather than in descending order allows us to eliminate many candidate independent sets early in the recursion. If $|S_i|$ is small, then it is very likely that $V(G') \setminus S_i$ induces a large independent set. Hence, it is also very likely that $S_i$ cannot be assigned to a new $u_i$ or that assigning $S_i$ eliminates many eligible independent sets.

Assigning sets in ascending order also gives us an easy lower bound for the number of edges in any potential output graph which can be obtained from the current graph and
assignment. If the sets \( S_1, \ldots, S_i \) have already been assigned to neighbors of \( v \) and the current minimal order of eligible independent sets is \( t \), then any expanded graph will have at least \( f = e(G') + d + |S_1| + \ldots + |S_i| + t(d - i) \) edges. If \( f > e \), then we can bound the recursion as well.

The pseudocode of the recursive extension is listed below as Algorithm 2. It is assumed that \texttt{indep\_number[]} (see Algorithm \texttt{1}) and the list of eligible independent sets are already computed. The parameters for \texttt{Construct()} are the order of the sets which are currently being assigned and the number of neighbors of \( v \) which were already assigned to independent sets. The recursion is bounded if any of the bounding criteria described above can be applied.

\begin{algorithm}
\caption{Construct(current\_order, num\_assigned)}
\begin{algorithmic}
\IF {num\_assigned = d}
\STATE expand graph \( G' \) to \( G \)
\IF {\( G \) is a \((3, k + 1; n + d + 1, \leq e)\)-graph}
\STATE output \( G \)
\ENDIF
\ELSE
\FOR {every eligible set \( S \) of order current\_order}
\STATE assign \( S \) to \texttt{u\_num\_assigned+1}
\STATE update the set of eligible independent sets
\STATE Construct(current\_order, num\_assigned + 1)
\ENDFOR
\IF {current\_order < k - 1}
\STATE Construct(current\_order + 1, num\_assigned)
\ENDIF
\ENDIF
\end{algorithmic}
\end{algorithm}

Our extension program does not perform any isomorphism rejection. We canonically label the output graphs with \texttt{nauty} \cite{Brenner1993,Brenner2012} and remove the isomorphic copies. This is not a bottleneck as there are usually only a few \((3, k + 1; n + d + 1, \leq e)\)-graphs which are constructed by our program. The results obtained by our extension algorithms are described in Sections 4 and 6. In the appendices we describe how the correctness of our implementation was tested.

\section*{Degree Sequence Feasibility}

Suppose we know the values or lower bounds on \( e(3, k, m) \) for some fixed \( k \) and we wish to know all feasible degree sequences of \((3, k + 1; n, e)\)-graphs. We construct the system of integer constraints consisting of \( n = \sum_{i=0}^{k} n_i, 2e = \sum_{i=0}^{k} m_i, \) and (4). If it has no solutions then we conclude that \( e(3, k + 1, n) > e \). Otherwise, we obtain solutions for \( n_i \)'s which include all desired degree sequences. This algorithm is similar in functionality to the package FRANK developed by Lesser \cite{Lesser2007}.
4 Progress on Computing Small $e(3, k, n)$

Table 3: Exact values of $e(3, k, n)$, for $3 \leq k \leq 16$, $3 \leq n \leq 31$.

| Vertices $n$ | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|--------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 3            | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 4            | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |
| 5            | 5  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |
| 6            | $\infty$ | 3 | 2 | 1  |    |    |    |    |    |    |    |    |    |    |
| 7            | 6  | 3  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |
| 8            | 10 | 4  | 3  | 2  | 1  |    |    |    |    |    |    |    |    |    |
| 9            | $\infty$ | 7 | 4 | 3  | 2  | 1  |    |    |    |    |    |    |    |    |
| 10           | 10 | 5  | 4  | 3  | 2  | 1  |    |    |    |    |    |    |    |    |
| 11           | 15 | 8  | 5  | 4  | 3  | 2  | 1  |    |    |    |    |    |    |    |
| 12           | 20 | 11 | 6  | 5  | 4  | 3  | 2  | 1  |    |    |    |    |    |    |
| 13           | 26 | 15 | 9  | 6  | 5  | 4  | 3  | 2  | 1  |    |    |    |    |    |
| 14           | $\infty$ | 20 | 12 | 7  | 6  | 5  | 4  | 3  | 2  | 1  |    |    |    |    |
| 15           | 25 | 15 | 10 | 7  | 6  | 5  | 4  | 3  | 2  | 1  |    |    |    |    |
| 16           |    | 32 | 20 | 13 | 8  | 7  | 6  | 5  | 4  | 3  | 2  | 1  |    |    |
| 17           |    | 40 | 25 | 16 | 11 | 8  | 7  | 6  | 5  | 4  | 3  | 2  | 1  |    |
| 18           | $\infty$ | 30 | 20 | 14 | 9  | 8  | 7  | 6  | 5  | 4  | 3  | 2  | 1  |    |
| 19           |    | 37 | 25 | 17 | 12 | 9  | 8  | 7  | 6  | 5  | 4  |    |    |    |
| 20           |    | 44 | 30 | 20 | 15 | 10 | 9  | 8  | 7  | 6  | 5  |    |    |    |
| 21           |    | 51 | 35 | 25 | 18 | 13 | 10 | 9  | 8  | 7  | 6  |    |    |    |
| 22           |    | 60 | 42 | 30 | 21 | 16 | 11 | 9  | 8  | 7  |    |    |    |    |
| 23           | $\infty$ | 49 | 35 | 25 | 19 | 14 | 11 | 10 | 9  | 8  |    |    |    |    |
| 24           |    | 56 | 40 | 30 | 22 | 17 | 12 | 11 | 10 | 9  |    |    |    |    |
| 25           |    | 65 | 46 | 35 | 25 | 20 | 15 | 12 | 11 | 10 |    |    |    |    |
| 26           |    | 73 | 52 | 40 | 30 | 23 | 18 | 13 | 12 | 11 |    |    |    |    |
| 27           |    | 85 | 61 | 45 | 35 | 26 | 21 | 16 | 13 | 12 |    |    |    |    |
| 28           | $\infty$ | 68 | 51 | 40 | 30 | 24 | 19 | 14 | 13 |    |    |    |    |    |
| 29           |    | 77 | 58 | 45 | 35 | 27 | 22 | 17 | 14 |    |    |    |    |    |
| 30           |    | 86 | 66 | 50 | 40 | 30 | 25 | 20 | 15 |    |    |    |    |    |
| 31           |    | 95 | 73 | 56 | 45 | 35 | 28 | 23 | 18 |    |    |    |    |    |

Table 3 presents the exact values of $e(3, k, n)$ for small cases, where clear regularities are well described by Theorem 2. Empty entries in the upper-right triangle of the table are 0's, while those in the lower-left triangle are equal to $\infty$. The columns correspond to fixed values of $k$. Almost all entries are given by Theorem 2. We list them for a better perspective and completeness. The entries beyond the range of Theorem 2 are marked in bold, and they were obtained as follows: $e(3, 6, 16)$ and $e(3, 6, 17)$ in [8], all cases for
The exact counts of \((3, k; n, e_3)-graphs for k = 7, 8, 9, 10\) which were obtained by the algorithms described in Section 3 are listed in Tables 12, 13, 14, 15, respectively, in Appendix 1. All \((3, 9; n, e(3, 9, n)) -graphs which were constructed by our programs can be obtained from the House of Graphs \(H\) by searching for the keywords “minimal ramsey graph” or from [9].

**Exact values of** \(e(3, 9, n)\)

The values of \(e(3, 9, n)\) are determined by Theorem 2. The values of \(e(3, 9, n)\) for \(27 \leq n \leq 34\) were obtained by computations, mostly by the gluing extender algorithm described in Section 3 and they are presented in Table 4. All of these values improve over previously reported lower bounds [19, 13]. The equality \(e(3, 9, 35) = 140\) will be established by Theorem 3 in Section 5.

| \(n\) | \(e(3, 9, n)\) | comments |
|-------|----------------|----------|
| 27    | 61             |          |
| 28    | 68             | the same as in [1] |
| 29    | 77             |          |
| 30    | 86             |          |
| 31    | 95             |          |
| 32    | 104            | not enough for \(R(3, 10) \leq 42\) |
| 33    | 118            | just enough for Theorem 4 |
| 34    | 129            | 122 required for \(R(3, 10) \leq 43\) |
| 35    | 140            | Theorem 3 |
| 36    | \(\infty\)    | hence \(R(3, 9) \leq 36\), old bound |

**Table 4:** Exact values of \(e(3, 9, n)\), for \(n \geq 27\)

**Values and lower bounds on** \(e(3, 10, n)\)

The values of \(e(3, 10, \leq 28)\) are determined by Theorem 2. The values for \(29 \leq n \leq 34\) were obtained by the gluing extender algorithm described in Section 3. The lower bounds on \(e(3, 10, \geq 35)\) are included in the second column of Table 5. They are based on solving integer constraints (3) and (4), using the exact values of \(e(3, 9, n)\) listed in Table 4, and results from the gluing extender algorithm used similarly as in the example of Section 3. Our bounds on \(e(3, 10, n)\) improve over previously reported lower bounds [19, 13] for all \(n \geq 30\).
By Theorem 4 (see Section 5) we know that any \((3, 10; 42)-\)graph must be 9-regular with 189 edges, and thus all its graphs \(G_v\) are necessarily of the type \((3, 9; 32, 108)\). There exists a very large number of the latter graphs. Their generation, extensions to possible \((3, 10; 42, 189)-\)graphs, and implied nonexistence of any \((3, 10; 42)-\)graphs will be described in Section 6.

| \(n\) | \(e(3, 10, n)\) | comments |
|------|----------------|----------|
| 29   | 58             | exact, the same as in [13] |
| 30   | 66             | exact    |
| 31   | 73             | exact    |
| 32   | 81             | exact    |
| 33   | 90             | exact    |
| 34   | 99             | exact, \((3, 10; 34, 99)\)-graph constructed by Backelin [2] |
| 35   | 107            |          |
| 36   | 117            |          |
| 37   | 128            |          |
| 38   | 139            | 146 required for \(R(3, 11) \leq 49\) |
| 39   | 151            | as required for \(R(3, 11) \leq 50\), Theorem 7 |
| 40   | 161            |          |
| 41   | 172            | 184 maximum |
| 42   | \(\infty\)    | hence \(R(3, 10) \leq 42\), new bound, Theorem 6 |
| 43   | \(\infty\)    | hence \(R(3, 10) \leq 43\), old bound |

**Table 5:** Values and lower bounds on \(e(3, 10, n)\), for \(n \geq 29\).

All lower bounds in Tables 4 and 5 required computations of our gluing extender algorithm. We did not perform any such computations in an attempt to improve the lower bounds on \(e(3, \geq 11, n)\). All results presented in Section 7 for \(k \geq 11\) depend only on the degree sequence analysis and the results for \(k \leq 10\).

## 5 Better Lower Bounds for \(e(3, 9, 35)\) and \(e(3, 10, 42)\)

Sometimes we can improve on the lower bounds on \(e(3, k, n)\) implied by (3) and (4) by a more detailed analysis of feasible degree sequences. Such improvements typically can be done in cases for which (4) gives a small number of possible degree sequences, none of which is of a regular graph, furthermore with only one heavily dominating degree. We have such a situation in the proofs of the two following theorems.

**Theorem 3** There exists a unique \((3, 9; 35)-\)graph, and \(e(3, 9, 35) = 140\).

**Proof.** Any \((3, 9; 35)\)-graph \(G\) has \(\Delta(G) \leq 8\), hence we have \(e(G) \leq 140\). Suppose \(G \in \mathcal{R}(3, 9; 35, 140 - s)\) for some \(s \geq 0\). Since \(R(3, 8) = 28\), the degrees of vertices in \(G\)
are 7 or 8, and let there be $n_7$ and $n_8$ of them, respectively. We have $n_7 + n_8 = 35$, $n_7 = 2s$. In this case there are five solutions to (4) with $0 \leq s \leq 4$. In particular, this shows that $e(3, 9, 35) \geq 136$. If $n_7 > 0$ (equivalently $s > 0$), then consider graph $H$ induced in $G$ by $n_7$ vertices of degree 7. Observe that $\delta(H) \leq s$, since $H$ is triangle-free on $2s$ vertices. Let $v$ be a vertex in $V(G)$ of degree 7 connected to at most $s$ other vertices of degree 7. Thus we have $Z_G(v) \geq 7s + 8(7 - s) = 56 - s$, and $e(G_v) \leq (140 - s) - (56 - s) = 84$. However $G_v$ is a $(3, 8; 27)$-graph which contradicts the fact that $e(3, 8, 27) = 85$.

The computations extending all $(3, 8; 26, 76)$-graphs, using the neighborhood gluing extension method described in Section 3, established that there exists a unique (up to isomorphism) 8-regular $(3, 9; 35)$-graph. We note that it is a cyclic graph on 35 vertices with circular distances $\{1, 7, 11, 16\}$, found by Kalbfleisch [11] in 1966. Clearly, any $(3, 9; 35, 140)$-graph must be 8-regular, and thus the theorem follows. $\square$

**Theorem 4** $R(3, 10) = 43$ if and only if $e(3, 10, 42) = 189$.

**Proof.** It is known that $R(3, 10) \leq 43$ [19], i.e. there are no $(3, 10; 43)$-graphs. We will prove the theorem by showing that any $(3, 10; 42)$-graph must be regular of degree 9. The essence of the reasoning is very similar to that for $e(3, 9, 35) = 140$ in the previous theorem, except that this time it is little more complicated.

Suppose $G \in \mathcal{R}(3, 10; 42, 189 - s)$ for some $s \geq 0$. The computations described in Section 3 established that $G$ cannot have the unique $(3, 9; 35)$-graph as one of its $G_v$'s. Hence, $7 \leq \deg_G(v) \leq 9$ for all vertices $v \in V(G)$. The solutions $n_i$ to (4) which contain all possible degree sequences for $G$ with this restriction are presented in Table 6.

| $n_7$ | $n_8$ | $n_9$ | $e(G)$ | $\gamma(G)$ | $s$ |
|------|------|------|--------|-------------|-----|
| 0    | 6    | 34   | 185    | 24          | 4   |
| 1    | 6    | 35   | 185    | 25          | 4   |
| 2    | 4    | 36   | 185    | 26          | 4   |
| 3    | 2    | 37   | 185    | 27          | 4   |
| 4    | 0    | 38   | 185    | 28          | 4   |
| 0    | 6    | 36   | 186    | 60          | 3   |
| 1    | 4    | 37   | 186    | 61          | 3   |
| 2    | 2    | 38   | 186    | 62          | 3   |
| 3    | 0    | 39   | 186    | 63          | 3   |
| 0    | 4    | 38   | 187    | 96          | 2   |
| 1    | 2    | 39   | 187    | 97          | 2   |
| 2    | 0    | 40   | 187    | 98          | 2   |
| 0    | 2    | 40   | 188    | 132         | 1   |
| 1    | 0    | 41   | 188    | 133         | 1   |
| 0    | 0    | 42   | 189    | 168         | 0   |

Table 6: Solutions to (4) for $(3, 10; 42, 189 - s)$-graphs.
Note that for all $0 \leq s \leq 4$ we have $0 \leq n_7 \leq s$, $n_8 + 2n_7 = 2s$, $n_9 = 42 - n_8 - n_7$, and $e(G) = 189 - s$. Since $e(3, 9, 34) = 129$, using (2) we see that $Z(G) \leq 60 - s$ for every vertex $v$ of degree 7. Similarly, since $e(3, 9, 33) = 118$, $Z(v) \leq 71 - s$ for every vertex $v$ of degree 8. If $s = 0$, then we are done, otherwise consider graph $H$ induced in $G$ by $2s - n_7$ vertices of degree 7 or 8. Observe that $\delta(H) \leq s - n_7/2$, since $H$ is triangle-free.

**Case 1:** $n_7 = 0$. Let $v$ be a vertex in $V(G)$ of degree 8 connected to at most $s$ other vertices of degree 8. This gives $Z(G(v)) \geq 8s + 9(8 - s) = 72 - s$, which is a contradiction.

**Case 2:** $n_7 = 0$. Let $v$ be a vertex in $V(G)$ of degree 7 connected to at most $s/2$ other vertices of degree 7 (in this case $|V(H)| = s$). This gives $Z_G(v) \geq 7s/2 + 9(7 - s/2) = 63 - s$, which is a contradiction.

**Case 3:** $n_7 = 1$. If $v$ is the only vertex of degree 7, then $n_8 = 2s - 2$ and we easily have $Z_G(v) \geq 8n_8 + 9(7 - n_8) = 65 - 2s > 60 - s$, which again is a contradiction.

**Case 4:** $n_7 = 2$. Both vertices of degree 7 must have $Z_G(v) \geq 7 + 8n_8 + 9(7 - n_8 - 1) = 61 - (2s - 2n_7) = 65 - 2s$, which is a contradiction.

**Case 5:** $n_7 > 2$. The only remaining degree sequence not covered by previous cases is $n_7 = 3, 4, 4, 4, 1$, respectively; the only remaining degree sequence passing (4) for a $(3, 9; 32, 108)$-graph is $n_6 = 8, n_7 = 24$.

Potentially, the complete set of $(3, 9; 32, 108)$-graphs could be obtained by performing additional extensions of degree 6 to $(3, 8; 25, 69)$-graphs or extensions of degree 7 to $(3, 8; 24, 60)$-graphs. However, there are already 12581543 $(3, 8; 25, 68)$-graphs and 3421512 $(3, 8; 24, 59)$-graphs (see Table 13 in Appendix 1), and there are many more with one additional edge. Hence, further refinement of the construction method of the $(3, 9; 32, 108)$-graphs not in $\mathcal{X}$ was needed. It is described in the following Lemma which permitted a fast computation and the completion of the task.

**Lemma 5** Every $(3, 9; 32, 108)$-graph $H \notin \mathcal{X}$ has $n_6 = 8, n_7 = 24$, and furthermore in such $H$ every vertex of degree 6 has exactly 3 neighbors of degree 7 and every vertex of degree 7 has exactly 1 neighbor of degree 6.

6 $R(3, 10) \leq 42$

Theorem 4 implies that any $(3, 10; 42)$-graph $G$ must be regular of degree 9 with 189 edges. Removing any vertex $v$ with its neighborhood from $G$ yields a $(3, 9; 32, 108)$-graph $G_v$. Hence, our first task is to obtain all $(3, 9; 32, 108)$-graphs.

We used the neighborhood extension method to generate $(3, 9; 32, 108)$-graphs $H$ with a vertex $v$ for which $H_v$ is one of the following types: $(3, 8; 27)$, $(3, 8; 26, \leq 77)$, $(3, 8; 25, \leq 68)$, $(3, 8; 24, \leq 59)$ or $(3, 8; 23, 49)$. These extensions yielded the set of 2104151 $(3, 9; 32, 108)$-graphs $\mathcal{X}$. Using notation of the example in Section 3, now with $4 \leq i \leq 8$, $m_i = 31 - i$, and $t_i = 10, 5, 4, 4, 1$, respectively, the only remaining degree sequence passing (4) for a $(3, 9; 32, 108)$-graph is $n_6 = 8, n_7 = 24$.

Potentially, the complete set of $(3, 9; 32, 108)$-graphs could be obtained by performing additional extensions of degree 6 to $(3, 8; 25, 69)$-graphs or extensions of degree 7 to $(3, 8; 24, 60)$-graphs. However, there are already 12581543 $(3, 8; 25, 68)$-graphs and 3421512 $(3, 8; 24, \leq 59)$-graphs (see Table 13 in Appendix 1), and there are many more with one additional edge. Hence, further refinement of the construction method of the $(3, 9; 32, 108)$-graphs not in $\mathcal{X}$ was needed. It is described in the following Lemma which permitted a fast computation and the completion of the task.
Proof. As stated after the definition of $\mathcal{X}$ above, (4) implies the specified degree sequence of $H \not\in \mathcal{X}$. Suppose that $H$ has a vertex $v$ of degree 6 with at least 4 neighbors of degree 7. One can easily see that $Z_H(v) \geq 40$ and thus $e(H_v) \leq 68$. All such graphs, however, were included in the set of inputs producing $\mathcal{X}$, so we have a contradiction. Similarly, suppose that $H$ has a vertex $v$ of degree 7 with no neighbors of degree 6. Then $Z_H(v) = 49$ and $e(H_v) = 59$, but all such graphs were used as inputs producing $\mathcal{X}$, hence again we have a contradiction. Now, by the pigeonhole principle, there are exactly 24 edges connecting vertices of distinct degrees, and we can easily conclude that every vertex of degree 6 must have exactly 3 neighbors of degree 7 and every vertex of degree 7 exactly 1 neighbor of degree 6. ✷

We adapted the extension algorithm from Section 3 to generate this very restricted set of $(3, 9; 32, 108)$-graphs by performing extensions of all 64233866 $(3, 8; 24, 60)$-graphs (Table 13 in Appendix 1). The result is that there are no $(3, 9; 32, 108)$-graphs not in $\mathcal{X}$.

Theorem 6 $R(3, 10) \leq 42$.

Proof. For contradiction, suppose that $G$ is a $(3, 10; 42)$-graph. By Theorem 4 it must be a 9-regular $(3, 10; 42, 189)$-graph whose all $G_v$’s are $(3, 9; 32, 108)$-graphs. By Lemma 5 and the computations described above there are exactly 2104151 such graphs. A specialized extension algorithm (a modification of the gluing extender) was run for all of them in an attempt to obtain a 9-regular $(3, 10; 42, 189)$-graph. The neighbors of $v$ have to be connected to independent sets of order 8 in $G_v$. For every pair of (possibly equal) independent sets $\{S_i, S_j\}$ of order 8, we test if they can be assigned to two neighbors of $v$ by checking if $V(G_v) \setminus (S_i \cup S_j)$ induces an independent set of order 8 in $G_v$, and if so we can bound the recursion. We used for this task a precomputed table storing the results of such tests for all pairs of independent sets of order 8. The concept of eligible candidates (Section 3) was also used, and the condition $\Delta(G) = 9$ turned out to be particularly strong in pruning the recursion. No 9-regular $(3, 10; 42, 189)$-graphs were produced, and thus $R(3, 10) \leq 42$. ✷

Theorem 6 improves over the bound $R(3, 10) \leq 43$ obtained in 1988 [19]. The correctness tests of our implementations and the computational effort required for various parts of the computations are described in Appendix 2.

Geoffrey Exoo [6] found almost 300000 $(3, 10; 39)$-graphs, we extended this set to more than $4 \cdot 10^7$ graphs, and very likely there are more of them. The known $(3, 10; 39)$-graphs have the number of edges ranging from 161 to 175, hence we have $151 \leq e(3, 9, 39) \leq 161$. We expect that the actual value is much closer, if not equal, to 161. Despite many attempts by Exoo, us, and others, no $(3, 10; 40)$-graphs were constructed. The computations needed for the upper bound in Theorem 6 were barely feasible. Consequently, we anticipate that any further improvement to either of the bounds in $40 \leq R(3, 10) \leq 42$ will be very difficult.
7 Lower Bounds for $e(3, k, n)$ and Upper Bounds for $R(3, k)$, for $k \geq 11$

We establish five further new upper bounds on the Ramsey numbers $R(3, k)$ as listed in Theorem 7. All of the new bounds improve the results listed in the 2011 revision of the survey [17] by 1. The bound $R(3, 16) \leq 98$ was also obtained by Backelin, though it was not published [1, 2]. Note that we don’t improve the upper bound on $R(3, 12)$.

**Theorem 7** The following upper bounds hold:

$R(3, 11) \leq 50$, $R(3, 13) \leq 68$, $R(3, 14) \leq 77$, $R(3, 15) \leq 87$, and $R(3, 16) \leq 98$.

**Proof.** Each of the new upper bounds $R(3, k) \leq n$ can be obtained by showing that $e(3, k, n) = \infty$. The details of the intermediate stages of computations for all $k$ are presented in the tables and comments of the remaining part of this section. For $k = 16$ no data is shown except some comments in Table 11, in particular the data in this table implies $e(3, 16, 98) = \infty$ by (4).

In the Tables 7, 8 and 9 for $k = 11, 12$ and 13, respectively, we list several cases in the comments column, where the lower bounds on $e(3, k, n)$ listed in [13] (some of them credited to [1]) are better than our results. This is the case for $n$ slightly larger than $13k/4 - 1$, mostly due to the theorems claimed in the unpublished manuscript by Backelin [1, 2]. Our lower bounds on $e(3, k, n)$, and implied upper bounds on $R(3, k)$, do not rely on these results. We have checked that assuming the results from [1, 2, 13] would not imply, using the methods of this paper, any further improvements on the upper bounds on $R(3, k)$ for $k \leq 16$, but they may for $k \geq 17$. Hence, if the results in [1, 13] are published, then using them jointly with our results may lead to better upper bounds on $R(3, k)$, at least for some $k \geq 17$. 

\[ \square \]
Lower bounds for \( e(3, 11, n) \)

The exact values of \( e(3, 11, \leq 31) \) are determined by Theorem 2. The bounds for \( n = 32, 33 \) marked with a ‘t’ are from Theorem 2. The lower bounds on \( e(3, 11, \geq 32) \) are included in the second column of Table 7. They are based on solving integer constraints (4), using known values and lower bounds on \( e(3, 10, n) \) listed in Table 5 in Section 4. They are better than those in [13] for all \( 36 \leq n \leq 50 \).

| \( n \) | \( e(3, 11, n) \geq \) | comments |
|-------|-----------------|----------|
| 32    | 62t             | 63 [13], 63 is exact [12] |
| 33    | 68t             | 69 [13], 70 is exact [12] |
| 34    | 75              | 76 [13], 77 is exact [12] |
| 35    | 83              | 84 [13], credit to [1] |
| 36    | 92              | the same as in [12] |
| 37    | 100             |          |
| 38    | 109             |          |
| 39    | 117             | unique degree sequence solution, 6-regular |
| 40    | 128             |          |
| 41    | 138             |          |
| 42    | 149             |          |
| 43    | 159             |          |
| 44    | 170             |          |
| 45    | 182             |          |
| 46    | 195             | 199 required for \( R(3, 12) \leq 58 \) |
| 47    | 209             |          |
| 48    | 222             | unique solution: \( n_9 = 36, n_{10} = 12 \), 215 required for \( R(3, 12) \leq 59 \), old bound |
| 49    | 237             | 245 maximum |
| 50    | \( \infty \)    | hence \( R(3, 11) \leq 50 \), new bound, Theorem 7 |
| 51    | \( \infty \)    | hence \( R(3, 11) \leq 51 \), old bound |

**Table 7**: Lower bounds on \( e(3, 11, n) \), for \( n \geq 32 \).

The maximum number of edges in any \((3, 11; 49)\)-graph is that of a 10-regular graph, so a proof of \( e(3, 11, 49) > 245 \) would imply \( R(3, 11) \leq 49 \). Observe that any graph \( G_v \) of any 10-regular \((3, 11; 50)\)-graph must be a \((3, 10; 39, 150)\)-graph. Thus, our improvement of the upper bound on \( R(3, 11) \) from 51 to 50 is mainly due to the new lower bound \( e(3, 10, 39) \geq 151 \) (together with not-too-much-off adjacent bounds).
Lower bounds for $e(3, 12, n)$

The exact values of $e(3, 12, \leq 34)$ are determined by Theorem 2. The bounds for $35 \leq n \leq 37$ marked with a ‘t’ are from Theorem 2. The lower bounds on $e(3, 12, \geq 35)$ are included in the second column of Table 8. They are based on solving integer constraints (4), using known values and lower bounds on $e(3, 11, n)$ given in Table 7. They are better than those in [13] for all $43 \leq n \leq 58$.

An improvement of the upper bound on $R(3, 12)$ obtained by Lesser [13] from 60 to 59 is now immediate (it formed a significant part of her thesis), but a further improvement from 59 to 58 would require an increase of the lower bound on $e(3, 12, 58)$ by 4.

| $n$ | $e(3, 12, n)$ | comments |
|-----|--------------|----------|
| 35  | 67t          | 68 [13], 68 is exact [1] [2] |
| 36  | 73t          | 74 [13], 75 is exact [1] [2] |
| 37  | 79t          | 81 [13], 82 is exact [1] [2] |
| 38  | 86           | 88 [13], 89 [1] |
| 39  | 93           | 95 [13], 96 [1] |
| 40  | 100          | 102 [13], 103 [1] |
| 41  | 109          | 111 [13] |
| 42  | 119          | 119 [13], 120 in [1] |
| 43  | 128          | the same as in [1] |
| 44  | 138          |          |
| 45  | 148          |          |
| 46  | 158          |          |
| 47  | 167          | 168, proof based on Table 7 [2] |
| 48  | 179          | 180, proof based on Table 7 [2] |
| 49  | 191          |          |
| 50  | 203          |          |
| 51  | 216          |          |
| 52  | 229          |          |
| 53  | 241          |          |
| 54  | 255          | 259 required for $R(3, 13) \leq 67$ |
| 55  | 269          | 265 required for $R(3, 13) \leq 68$, Theorem 7 |
| 56  | 283          |          |
| 57  | 299          |          |
| 58  | 316          | 319 maximum |
| 59  | $\infty$     | hence $R(3, 12) \leq 59$, old bound |

**Table 8**: Lower bounds on $e(3, 12, n)$, for $n \geq 35$. 


Lower bounds for $e(3, 13, n)$

The exact values of $e(3, 13, \leq 39)$ are determined by Theorem 2. The bound for $n = 40$ is from Theorem 2. The lower bounds on $e(3, 13, \geq 40)$ are included in the second column of Table 9. They are based on solving integer constraints (4), using lower bounds on $e(3, 12, n)$ listed in Table 8. They are better than those in [13] for all $51 \leq n \leq 68$.

| $n$ | $e(3, 13, n)$ | comments     |
|-----|---------------|--------------|
| 40  | 84t           | 86 [13], 87 is exact [2] |
| 41  | 91            | 93 [13], 94 is exact [1] |
| 42  | 97            | 100 [13], 101 [1] |
| 43  | 104           | 107 [13], 108 [1] |
| 44  | 112           | 114 [13], 115 [1] |
| 45  | 120           | 122 [13], 123 [1] |
| 46  | 128           | 130 [13], 132 [1] |
| 47  | 136           | 139 [13], 140 [1] |
| 48  | 146           | 148 [13]     |
| 49  | 157           | 158 [13]     |
| 50  | 167           | 167 [13], 168 [1] |
| 51  | 177           | 178 [1]      |
| 52  | 189           |             |
| 53  | 200           |             |
| 54  | 212           |             |
| 55  | 223           |             |
| 56  | 234           |             |
| 57  | 247           |             |
| 58  | 260           |             |
| 59  | 275           |             |
| 60  | 289           |             |
| 61  | 303           |             |
| 62  | 319           | 326 required for $R(3, 14) \leq 76$ |
| 63  | 334           |             |
| 64  | 350           | 345 required for $R(3, 14) \leq 77$, Theorem 7 |
| 65  | 365           |             |
| 66  | 381           |             |
| 67  | 398           | 402 maximum  |
| 68  | $\infty$     | hence $R(3, 13) \leq 68$, new bound |
| 69  | $\infty$     | hence $R(3, 13) \leq 69$, old bound |

Table 9: Lower bounds on $e(3, 13, n)$, for $n \geq 40$.  

19
Lower bounds for $e(3, 14, n)$

The exact values of $e(3, 14, \leq 41)$ are determined by Theorem 2. Only lower bounds on $e(3, 14, \geq 66)$ are included in the second column of Table 10 since these are relevant for our further analysis of $R(3, 15)$ and $R(3, 16)$. They are based on solving integer constraints (4), using lower bounds on $e(3, 13, n)$ listed in Table 9. They are better than those in [13] for all $66 \leq n \leq 77$.

| $n$  | $e(3, 14, n)$ | comments                                      |
|------|--------------|-----------------------------------------------|
| 66   | 321          |                                               |
| 67   | 334          | 335, proof based on Table 9 [2]               |
| 68   | 350          |                                               |
| 69   | 365          |                                               |
| 70   | 381          |                                               |
| 71   | 398          | 407 required for $R(3, 15) \leq 86$           |
| 72   | 415          | 414 required for $R(3, 15) \leq 87$, Theorem 7|
| 73   | 432          |                                               |
| 74   | 449          |                                               |
| 75   | 468          |                                               |
| 76   | 486          | 494 maximum                                   |
| 77   | $\infty$    | hence $R(3, 14) \leq 77$, new bound           |
| 78   | $\infty$    | hence $R(3, 14) \leq 78$, old bound           |

*Table 10:* Lower bounds on $e(3, 14, n)$, for $n \geq 66$. 
Lower bounds for $e(3, 15, n)$

The exact values of $e(3, 15, \leq 44)$ are determined by Theorem 2. Only lower bounds on $e(3, 15, \geq 81)$ are included in the second column of Table 11 since these are relevant for further analysis of $R(3, 16)$. They are based on solving integer constraints (4), using lower bounds on $e(3, 14, n)$ listed in Table 10. They are better than those in [13] for all $81 \leq n \leq 87$.

| $n$ | $e(3, 15, n)$ | comments |
|-----|--------------|----------|
| 81  | 497          | 498, proof based on Table 10 [2] |
| 82  | 515          | 518 required for $R(3, 16) \leq 97$ |
|     |              | 511 required for $R(3, 16) \leq 98$, Theorem 7 |
| 83  | 533          |          |
| 84  | 552          |          |
| 85  | 572          |          |
| 86  | 592          | 602 maximum |
| 87  | $\infty$    | hence $R(3, 15) \leq 87$, new bound |
| 88  | $\infty$    | hence $R(3, 15) \leq 88$, old bound |

Table 11: Lower bounds on $e(3, 15, n)$, for $n \geq 81$.

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References

[1] J. Backelin, Contributions to a Ramsey Calculus, unpublished, 2000-2012.
[2] J. Backelin, personal communication, 2012.
[3] G. Brinkmann, K. Coolsaet, J. Goedgebeur and H. Mélot, House of Graphs: A database of interesting graphs, Discrete Applied Mathematics, 161 (2013) 311–314. Available at http://hog.grinvin.org
[4] G. Brinkmann, J. Goedgebeur and J.C. Schlage-Puchta, Ramsey Numbers $R(K_3, G)$ for Graphs of Order 10, Electronic Journal of Combinatorics, 19(4) (2012).
[5] G. Exoo, On Two Classical Ramsey Numbers of the Form $R(3, n)$, *SIAM Journal of Discrete Mathematics*, 2 (1989) 488–490.

[6] G. Exoo, *personal communication*, 2012.

[7] J. Goedgebeur, House of Graphs: Minimal Ramsey Graphs, http://hog.grinvin.org/MinRamsey.

[8] J.E. Graver and J. Yackel, Some Graph Theoretic Results Associated with Ramsey’s Theorem, *Journal of Combinatorial Theory, Series B*, 4 (1968) 125–175.

[9] R.E. Greenwood and A.M. Gleason, Combinatorial Relations and Chromatic Graphs, *Canadian Journal of Mathematics*, 7 (1955) 1–7.

[10] C. Grinstead and S. Roberts, On the Ramsey Numbers $R(3, 8)$ and $R(3, 9)$, *Journal of Combinatorial Theory, Series B*, 33 (1982) 27–51.

[11] J.G. Kalbfleisch, Chromatic Graphs and Ramsey’s Theorem, *Ph.D. thesis*, University of Waterloo, January 1966.

[12] J.H. Kim, The Ramsey Number $R(3, k)$ Has Order of Magnitude $t^2 / \log t$, *Random Structures and Algorithms*, 7 (1995) 173–207.

[13] A. Lesser, Theoretical and Computational Aspects of Ramsey Theory, *Examensarbeten i Matematik*, Matematiska Institutionen, Stockholms Universitet, 3 (2001).

[14] B.D. McKay, Practical graph isomorphism, *10th Manitoba Conference on Numerical Mathematics and Computing*, Winnipeg 1980, *Congressus Numerantium*, 30 (1981) 45–87.

[15] B.D. McKay, nauty User’s Guide (Version 2.4), Technical Report TR-CS-90-02, Department of Computer Science, Australian National University, (1990). The latest version of the software is available at http://cs.anu.edu.au/~bdm/nauty.

[16] B.D. McKay and Zhang Ke Min, The Value of the Ramsey Number $R(3, 8)$, *Journal of Graph Theory*, 16 (1992) 99–105.

[17] S.P. Radziszowski, Small Ramsey Numbers, *Electronic Journal of Combinatorics*, Dynamic Survey DS1, revision #13 (2011), http://www.combinatorics.org.

[18] S.P. Radziszowski and D.L. Kreher, On $R(3, k)$ Ramsey Graphs: Theoretical and Computational Results, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 4 (1988) 37–52.

[19] S.P. Radziszowski and D.L. Kreher, Upper Bounds for Some Ramsey Numbers $R(3, k)$, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 4 (1988) 207–212.

[20] S.P. Radziszowski and D.L. Kreher, Minimum Triangle-Free Graphs, *Ars Combinatoria*, 31 (1991) 65–92.

[21] J. Spencer, Eighty Years of Ramsey $R(3, k)$ ... and Counting!, in *Ramsey Theory: Yesterday, Today and Tomorrow* (ed. A. Soifer), *Progress in Mathematics* 285, Springer-Birkhauser 2011, 27–39.
Appendix 1: Graph Counts

Tables 12–15 below contain all known exact counts of \((3, k; n, e)\)-graphs for specified \(n\), for \(k = 7, 8, 9\) and 10, respectively. All graph counts were obtained by the algorithms described in Section 3. Empty entries indicate 0. In all cases, the maximum number of edges is bounded by \(\Delta(G)n/2 \leq (k - 1)n/2\). All \((3, \leq 9; n, \leq e(3, k, n) + 1)\)-graphs which were constructed by our programs can be obtained from the House of Graphs \[3\] by searching for the keywords “minimal ramsey graph” or from \[7\].

| edges | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|-------|----|----|----|----|----|----|----|
| 20    | 2  |    |    |    |    |    |    |
| 21    |    | 15 |    |    |    |    |    |
| 22    |    | 201|    |    |    |    |    |
| 23    |    | 2965|   |    |    |    |    |
| 24    |    | 43331|  |    |    |    |    |
| 25    |    | 498927| 2 |    |    |    |    |
| 26    |    | 4054993| 30|    |    |    |    |
| 27    | ?  | 642 |    |    |    |    |    |
| 28    | ?  | 13334|   |    |    |    |    |
| 29    | ?  | 234279|  |    |    |    |    |
| 30    | ?  | 2883293| 1 |    |    |    |    |
| 31    | ?  | ?  | 15 |    |    |    |    |
| 32    | ?  | ?  | 382|    |    |    |    |
| 33    | ?  | ?  | 8652|   |    |    |    |
| 34    | ?  | ?  | 160573|  |    |    |    |
| 35    | ?  | ?  | 2216896|  |    |    |    |
| 36    | ?  | ?  | ?  |    |    |    |    |
| 37    | ?  | ?  | ?  | 11 |    |    |    |
| 38    | ?  | ?  | ?  | 417|    |    |    |
| 39    | ?  | ?  | ?  | 10447|   |    |    |
| 40    | ?  | ?  | ?  | 172534|  |    |    |
| 41    | ?  | ?  | ?  | 1990118|  |    |    |
| 42-43 | ?  | ?  | ?  | ?  |    |    |    |
| 44    | ?  | ?  | ?  | ?  | ?  | 15 |    |
| 45    | ?  | ?  | ?  | ?  | ?  | 479 |    |
| 46    | ?  | ?  | ?  | ?  | ?  | 10119|    |
| 47    | ?  | ?  | ?  | ?  | ?  | 132965|    |
| 48    | ?  | ?  | ?  | ?  | ?  | 1090842|    |
| 49-50 | ?  | ?  | ?  | ?  | ?  |    |    |
| 51    | ?  | ?  | ?  | ?  | ?  | 4  |    |
| 52    | ?  | ?  | ?  | ?  | ?  | 70 |    |
| 53    | ?  | ?  | ?  | ?  | ?  | 717|    |
| 54    | ?  | ?  | ?  | ?  | ?  | 5167|    |
| 55    | ?  | ?  | ?  | ?  | ?  | 27289|    |
| 56    | ?  | ?  | ?  | ?  | ?  | 97249|    |
| 57    | ?  | ?  | ?  | ?  | ?  | 219623|    |
| 58    | ?  | ?  | ?  | ?  | ?  | 307464|    |
| 59    | ?  | ?  | ?  | ?  | ?  | 267374|    |
| 60    | ?  | ?  | ?  | ?  | ?  | 142741| 1 |
| 61    |    |    |    | 43923| 6 |    |    |
| 62    |    |    |    | 6484 | 30 |    |    |
| 63    |    |    |    | 331  | 60 |    |    |
| 64    |    |    |    | 59  |    |    |    |
| 65    |    |    |    | 25  |    |    |    |
| 66    |    |    |    | 10  |    |    |    |

Table 12: Number of \((3, 7; n, e)\)-graphs, for \(n \geq 16\).
| $n$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|-----|----|----|----|----|----|----|----|----|----|
| 25  | 2  |    |    |    |    |    |    |    |    |
| 26  |    | 37 |    |    |    |    |    |    |    |
| 27  | 763|    |    |    |    |    |    |    |    |
| 28  | 16939|    |    |    |    |    |    |    |    |
| 29  |    |    |    |    |    |    |    |    |    |
| 30  |    | 3  |    |    |    |    |    |    |    |
| 31  |    | 60 |    |    |    |    |    |    |    |
| 32  |    | 1980|    |    |    |    |    |    |    |
| 33  |    | 58649|    |    |    |    |    |    |    |
| 34  |    | 1594047|    |    |    |    |    |    |    |
| 35  |    |    |    |    |    |    |    |    | 1 |
| 36  |    |    |    |    |    |    | 20 |    |    |
| 37  |    |    |    |    |    |    | 950|    |    |
| 38  |    |    |    |    |    |    | 35797|    |    |
| 39  |    |    |    |    |    |    | 1079565|    |    |
| 40-41|    |    |    |    |    |    |    |    |    |
| 42  |    |    |    |    |    |    | 21 |    |    |
| 43  |    |    |    |    |    |    | 1521|    |    |
| 44  |    |    |    |    |    |    | 72353|    |    |
| 45  |    |    |    |    |    |    | 2331462|    |    |
| 46-48|    |    |    |    |    |    |    |    |    |
| 49  |    |    |    |    |    |    | 102 |    |    |
| 50  |    |    |    |    |    |    | 8241 |    |    |
| 51  |    |    |    |    |    |    | 356041|    |    |
| 52  |    |    |    |    |    |    | 10326716|    |    |
| 53-55|    |    |    |    |    |    |    |    |    |
| 56  |    |    |    |    |    |    |    |    | 51 |
| 57  |    |    |    |    |    |    | 3419|    |    |
| 58  |    |    |    |    |    |    | 129347|    |    |
| 59  |    |    |    |    |    |    | 3288695|    |    |
| 60  |    |    |    |    |    |    | 64233886|    |    |
| 61-64|    |    |    |    |    |    |    |    |    |
| 65  |    |    |    |    |    |    |    |    | 396 |
| 66  |    |    |    |    |    |    | 21493|    |    |
| 67  |    |    |    |    |    |    | 613285|    |    |
| 68  |    |    |    |    |    |    | 11946369|    |    |
| 69-72|    |    |    |    |    |    |    |    |    |
| 73  |    |    |    |    |    |    |    |    | 62 |
| 74  |    |    |    |    |    |    | 1625|    |    |
| 75  |    |    |    |    |    |    | 23409|    |    |
| 76  |    |    |    |    |    |    | 216151|    |    |
| 77  |    |    |    |    |    |    | 1526296|    |    |
| 78-84|    |    |    |    |    |    |    |    |    |
| 85  |    |    |    |    |    |    | 4 |    |    |
| 86  |    |    |    |    |    |    | 92 |    |    |
| 87  |    |    |    |    |    |    | 1374|    |    |
| 88  |    |    |    |    |    |    | 11915|    |    |
| 89  |    |    |    |    |    |    | 52807|    |    |
| 90  |    |    |    |    |    |    | 122419|    |    |
| 91  |    |    |    |    |    |    | 151308|    |    |
| 92  |    |    |    |    |    |    | 99332|    |    |
| 93  |    |    |    |    |    |    | 33145|    |    |
| 94  |    |    |    |    |    |    | 4746|    |    |

**Table 13:** Number of $(3, 8; n, e)$-graphs, for $n \geq 19$. 
| edges | number of vertices $n$ |
|-------|-----------------------|
| $e$   | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| 40    | 2  |
| 41    | 32 |
| 42    | 2089|
| 43    | 115588|
| 44-45 | ? |
| 46    | ? | 1 |
| 47    | ? | 39 |
| 48    | ? | 4113|
| 49    | ? | 306415|
| 50-51 | ? | ? |
| 52    | ? | ? | 1 |
| 53    | ? | ? | 1 |
| 54    | ? | ? | 444|
| 55    | ? | ? | 58484|
| 56-60 | ? | ? |
| 61    | ? | ? | ? | 700 |
| 62    | ? | ? | ? | 95164|
| 63    | ? | ? | ? | 6498191|
| 64-67 | ? | ? | ? |
| 68    | ? | ? | ? | ? | 126 |
| 69    | ? | ? | ? | ? | 17223|
| 70    | ? | ? | ? | ? | 1202362|
| 71-76 | ? | ? | ? | ? | ? |
| 77    | ? | ? | ? | ? | ? | 1342|
| 78    | ? | ? | ? | ? | ? | 156686|
| 79-85 | ? | ? | ? | ? | ? |
| 86    | ? | ? | ? | ? | ? | 1800|
| 87    | ? | ? | ? | ? | ? | 147335|
| 88-94 | ? | ? | ? | ? | ? | ? |
| 95    | ? | ? | ? | ? | ? | ? | 560|
| 96    | ? | ? | ? | ? | ? | ? | 35154|
| 97-103| ? | ? | ? | ? | ? | ? | ? |
| 104   | ? | ? | ? | ? | ? | ? | ? | 39 |
| 105   | ? | ? | ? | ? | ? | ? | 952|
| 106   | ? | ? | ? | ? | ? | ? | 18598|
| 107   | ? | ? | ? | ? | ? | ? | 224881|
| 108   | ? | ? | ? | ? | ? | ? | 2104151|
| 109-117| ? | ? | ? | ? | ? | ? |
| 118   | ? | ? | ? | ? | 5|
| 119   | ? | ? | ? | ? | 69|
| 120   | ? | ? | ? | ? | ? | ≥ 1223|
| 121   | ? | ? | ? | ? | ≥ 13081|
| 122   | ? | ? | ? | ? | ≥ 90235|
| 123   | ? | ? | ? | ? | ≥ 401731|
| 124   | ? | ? | ? | ? | ≥ 1188400|
| 125   | ? | ? | ? | ? | ≥ 2306474|
| 126   | ? | ? | ? | ? | ≥ 3198596|
| 127   | ? | ? | ? | ? | ≥ 2915795|
| 128   | ? | ? | ? | ? | ≥ 1758241|
| 129   | ? | ? | ? | ? | ≥ 673600 | 1 |
| 130   | ? | ? | ? | ? | ≥ 153676 | 4 |
| 131   | ? | ? | ? | ? | ≥ 18502 | ≥ 15 |
| 132   | ? | ? | ? | ? | ≥ 922 | ≥ 40 |
| 133   | ? | ? | ? | ? | ≥ 43 |
| 134   | ? | ? | ? | ? | ≥ 20 |
| 135   | ? | ? | ? | ? | ≥ 7 |
| 136   | ? | ? | ? | ? | ? |
| 137-139| ? | ? | ? | ? | ? |
| 140   | ? | ? | ? | ? | ? | ? |

Table 14: Number of $(3, 9; n, e)$-graphs, for $n \geq 24$. 
| edges | number of vertices n |
|-------|---------------------|
|       | 29    | 30    | 31    | 32    | 33    | 34    |
| 58    | 5     |       |       |       |       |       |
| 59    |       | 1364  |       |       |       |       |
| 60–65 | ?     |       |       |       |       |       |
| 66    | ?     | 5084  |       |       |       |       |
| 67    | ?     | 1048442 |       |       |       |       |
| 68–72 | ?     | ?     |       |       |       |       |
| 73    | ?     | ?     | 2657  |       |       |       |
| 74    | ?     | ?     | 580667 |       |       |       |
| 75–80 | ?     | ?     | ?     |       |       |       |
| 81    | ?     | ?     | ?     | 6592  |       |       |
| 82–89 | ?     | ?     | ?     | ?     |       |       |
| 90    | ?     | ?     | ?     | ?     | 57099 |       |
| 91–98 | ?     | ?     | ?     | ?     | ?     |       |
| 99    | ?     | ?     | ?     | ?     | ?     | ≥1    |
| ≥100  | ?     | ?     | ?     | ?     | ?     | ?     |

Table 15: Number of \((3, 10; n, e)\)-graphs, for \(29 \leq n \leq 34\).

We showed that \(e(3, 10, 34) \geq 99\) (see Section 4), a \((3, 10; 34, 99)\)-graph was constructed by Backelin [2], and thus \(e(3, 10, 34) = 99\).
Appendix 2: Testing Implementations

Correctness

Since most results obtained in this paper rely on computations, it is very important that the correctness of our programs has been thoroughly verified. Below we list the main tests and agreements with results produced by more than one computation.

- For every $(3, k)$-graph which was output by our programs, we verified that it does not contain an independent set of order $k$ by using an independent program.

- For every $(3, k; n, e(3, k, n))$-graph which was generated by our programs, we verified that dropping any edge creates an independent set of order $k$.

- For various $(3, k; n, \leq e)$-graphs we added up to $f$ edges in all possible ways to obtain $(3, k; n, \leq e + f)$-graphs. For the cases where we already had the complete set of $(3, k; n, \leq e + f)$-graphs we verified that no new $(3, k; n, \leq e + f)$-graphs were obtained. We used this, amongst other cases, to verify that no new $(3, 9; 24, \leq 43)$, $(3, 9; 28, \leq 70)$, $(3, 9; 30, \leq 87)$ or $(3, 10; 30, \leq 67)$-graphs were obtained.

- For various $(3, k; n, \leq e + f)$-graphs we dropped one edge in all possible ways and verified that no new $(3, k; n, \leq e + f - 1)$-graphs were obtained. We used this technique, amongst other cases, to verify that no new $(3, 9; 24, \leq 42)$, $(3, 9; 28, \leq 69)$, $(3, 9; 33, \leq 119)$, $(3, 9; 34, \leq 130)$, $(3, 10; 30, 66)$ or $(3, 10; 32, 81)$-graphs were obtained.

- For various sets of $(3, k + 1; n, \leq e)$-graphs we took each member $G$ and constructed from it all $G_v$'s. We then verified that this did not yield any new $(3, k; n - \deg(v) - 1, \leq e - Z(v))$-graphs for the cases where we have all such graphs. We performed this test, amongst other cases, on the sets of $(3, 9; 28, \leq 70)$- and $(3, 10; 31, \leq 74)$-graphs.

- Various sets of graphs can be obtained by both the minimum degree extension method and the neighborhood gluing extension method. We performed both extension methods for various cases (e.g. to obtain the sets of $(3, 9; 24, \leq 43)$ and $(3, 9; 25, \leq 48)$-graphs). In each of these cases the results obtained by both methods were in complete agreement.

- The sets of $(3, 7; 21, \leq 55)$, $(3, 7; 22)$, $(3, 8; 26, \leq 76)$ and $(3, 8; 27, \leq 88)$-graphs were obtained by both the maximal triangle-free method and the neighborhood gluing extension method. The results were in complete agreement. As these programs are entirely independent and the output sets are large, we think that this provides strong evidence of their correctness.

- The counts of $(3, 7; 16, 20)$, $(3, 7; 17, 25)$, $(3, 7; 18, 30)$, $(3, 7; 19, 37)$, $(3, 7; 20, 44)$, $(3, 7; 21, 51)$, and $(3, 7; 22, e)$ for all $60 \leq e \leq 66$, are confirmed by [18].
• The counts of $(3, 7; 18, 31)$, $(3, 7; 19, 38)$, $(3, 7; 20, 45)$ and $(3, 7; 21, \leq 53)$-graphs are confirmed by [19].

• The counts of $(3, 8; 19, 25)$, $(3, 8; 20, 30)$, $(3, 8; 21, 35)$ and $(3, 9; 24, 40)$-graphs are confirmed by [20].

• The counts of $(3, 7; 16, 21)$, $(3, 7; 17, 26)$, $(3, 8; 22, 42)$ and $(3, 9; 25, 47)$-graphs are confirmed by [2].

Additional implementation correctness tests of specialized algorithms described in Section 6 were as follows:

• The specialized program of Section 6 was used to extend $(3, 8; 26, 76)$- to $(3, 9; 35, 140)$-graphs and it produced the unique $(3, 9; 35, 140)$-graph.

• We relaxed the conditions to generate all $(3, 9; 32, 108)$-graphs from Lemma 5 by dropping the requirement that each vertex of degree 6 has 3 neighbors of degree 7, and enforcing just one vertex of degree 7 with exactly one neighbor of degree 6. This yielded 21602 graphs. We verified that each of these graphs was indeed already included in the set $\mathcal{X}$, and that $\mathcal{X}$ does not contain any additional such graphs.

Since our results are in complete agreement with previous results and since all our consistency tests passed, we believe that this is strong evidence for the correctness of our implementations.

**Computation Time**

The implementations of extension algorithms described in Sections 3 and 6 are written in C. Most computations were performed on a cluster with Intel Xeon L5520 CPU’s at 2.27 GHz, on which a computational effort of one CPU year can be usually completed in about 8 elapsed hours. The overall computational effort of this project is estimated to be about 50 CPU years, which includes the time used by a variety of programs. The most cpu-intensive tasks are listed in the following.

The first phase of obtaining $(3, 9; 32, 108)$-graphs required about 5.5 CPU years. The bottlenecks of this phase were the computations required for extending all $(3, 8; 24, \leq 59)$-graphs (which required approximately 3.5 CPU years), and extending the $(3, 8; 25, \leq 68)$-graphs (which took more than 2 CPU years). The second phase of obtaining the special $(3, 9; 32, 108)$-graphs with $n_6 = 8$, $n_7 = 24$ as in Lemma 5 took about 5.8 CPU years. The specialized program of Section 6 extended all $(3, 9; 32, 108)$-graphs to 9-regular $(3, 10; 42, 189)$-graphs quite fast, in about only 0.25 CPU years. Performing computations to generate all $(3, 10; 39, \leq 150)$-graphs (there are none of these), which were needed for the bound $R(3, 11) \leq 50$, took about 4.8 CPU years.

The CPU time needed to complete the computations of Section 7 was negligible, however their variety caused that they were performed during the span of several weeks.