The unitary cover of a finite group and the exponent of the Schur multiplier.

Nicola Sambonet*

In memory of David Chillag.

Abstract

For a finite group we introduce a particular central extension, the unitary cover, having minimal exponent among those satisfying the projective lifting property. We obtain new bounds for the exponent of the Schur multiplier relating to subnormal series, and we discover new families for which the bound is the exponent of the group. Finally, we show that unitary covers are controlled by the Zel’manov solution of the restricted Burnside problem for 2-generator groups.

1 Introduction

The Schur multiplier of a finite group $G$ is the second cohomology group with complex coefficients, denoted by $M(G) = H^2(G, \mathbb{C}^*)$. It was introduced in the beginning of the twentieth century by I. Schur, aimed at the study of projective representations. To determine $M(G)$ explicitly is often a difficult task. Therefore, it is of interest to provide bounds for numerical qualities of $M(G)$ as the order, the rank, and - our subject - the exponent.

In 1904 Schur already showed that $[\exp M(G)]^2$ divides the order of the group, and this bound is tight as $M(C_n \times C_n) = C_n$. Note that $C_n \times C_n$ is an example of group satisfying

$$\exp M(G) \mid \exp G,$$

property which has been proven for many classes of groups.

*Department of Mathematics, Technion, Haifa, 32000 Israel, sambonet@tx.technion.ac.il.
1.1 Groups such that $\exp M(G)$ divides $\exp G$

Firstly, (1) holds for every abelian group $G$. Indeed, consider the cyclic decomposition ordered by recursive division:

$$G = \bigoplus_{i=1}^{n} C_{d_i} \ , \ d_i \mid d_{i+1} .$$

(2)

By Schur it is known (cf. [12, p. 317]) that

$$M(G) = \bigoplus_{i=1}^{n} C_{d_i^{n-i}} .$$

(3)

Consequently, $\exp M(G) = d_{n-1}$ which in turn divides $\exp G = d_n$. A second important example of groups enjoying (1) are the finite simple groups, whose multipliers are known and listed in the Atlas [4].

A standard argument (cf. [3, Th. 10.3]) proposes to focus on $p$-groups. Indeed, the $p$-component of $M(G)$ is embedded in the multiplier of a $p$-Sylow via the restriction map. Therefore, if $\Pi(G)$ denotes the set of prime divisors of $G$, and $S_p$ denotes a $p$-Sylow of $G$ for $p \in \Pi(G)$, then

$$\exp M(G) \mid \prod_{p \in \Pi(G)} \exp M(S_p) .$$

(4)

Clearly, since

$$\exp G = \prod_{p \in \Pi(G)} \exp S_p ,$$

if (1) holds for every $p$-Sylow of $G$, then it does also for $G$.

Then, a fundamental feature of $p$-groups is the nilpotency class. Recently, P. Moravec completed a result of M. R. Jones [11, Rem. 2.8] proving (1) for groups of class at most 3, and extended this result to groups of class 4 in the odd-order case [17, Th. 12, Th. 13]. Moravec discovered many other families enjoying (1): metabelian groups of prime exponent [15, Pr. 2.12], 3-Engel groups, 4-Engel groups in case the order is coprime with 2 and 5 [16, Cor. 5.5, Cor. 4.2], $p$-groups of class lower than $p - 1$ [17, Pr. 11], and $p$-groups of maximal class [18, Th. 1.4]. Without pretending to complete a list, we mention that (1) holds for extraspecial and abelian-by-cyclic groups, as follows from an argument of R.J. Higgs [9, Pr. 2.3, Pr. 2.4].

We have not cited 2-groups of class 4. Actually, the general validity of (1) was disproved by such a group long time before all the reported examples. A. J. Bayes, J. Kautsky and J. W. Wamsley introduced a group of order $2^{68}$ and exponent 4 whose multiplier has exponent 8 [1]. Lately, Moravec described another counterexample of order $2^{11}$ and class 6 [15, Ex. 2.9]. Nevertheless, these essentially
are the only counterexamples we know: both were obtained by computer technique and satisfy \( \exp M(G) = 8 \) but \( \exp G = 4 \).

The scenario in case \( p > 2 \) is indeed not clear yet. For instance, groups of exponent 3 satisfy (1) as they have nilpotency class at most 3 (cf. [16, p. 6]). This family, as well as the metabelian groups of prime exponent which we already encountered, strengthens the idea that (1) should at least hold for groups of prime exponent.

A. Lubotzky and A. Mann proved (1) for powerful \( p \)-groups [13, Th. 2.4]. By definition, a \( p \)-group \( G \) for \( p > 2 \) is powerful if the derived subgroup \( G' \) is contained in the agemo subgroup

\[ \mathcal{U}(G) = \langle g^p \mid g \in G \rangle \]

generated by the \( p \)-powers. Abelian \( p \)-groups are powerful, and if a \( p \)-group is powerful, then its quotients of exponent \( p \) are necessarily abelian. For this reason, powerful \( p \)-group and groups of exponent \( p \) can be considered as two extremes dealing with \( p \)-groups for \( p > 2 \).

### 1.2 Bounds for \( \exp M(G) \)

Beside the result on powerful \( p \)-groups, Lubotzky and Mann provided a bound for \( \exp M(G) \) involving the exponent and the rank of \( G \) [13, Pr. 2.6,Pr. 4.2.6], this has been recently refined by J. González Sánchez and A. P. Nicolas [6, Th. 2].

Meanwhile, Moravec proved the existence of a bound only in terms of the exponent of the group [15, Pr. 2.4]. This bound relies on the Zel’manov solution of the restricted Burnside problem, with the idea that the problem of finding bounds for the exponent of the multiplier can be reduced in some extent to 2-generator groups, we will give an alternative proof of this fact.

Since the use of the Zel’manov solution gives a bound which is apparently far from being efficient, Moravec also stated a more practical bound:

\[ \exp M(G) \mid (\exp G)^{2(d-1)} \]  \hspace{1cm} (5)

where \( d \) is the derived length assumed to be greater than 1 [15, Th. 2.13]. The bound analogue to (5) with the nilpotency class \( c \) in place of \( d \) was previously discovered by Jones [11, Cor. 2.7], then modified as

\[ \exp M(G) \mid (\exp G)^{c/2} \]  \hspace{1cm} (6)

by G. Ellis [5, Th. B1]. As Moravec illustrated, (5) improves (6) for \( c \geq 11 \) via the formula

\[ d \leq \lfloor \log_2 c \rfloor + 1 \]  \hspace{1cm} (7)

relating nilpotency class and derived length (cf. [19, 5.1.11]).
2 Results

To give evidence for the content of this paper, we present some advancement concerning the problems exposed in the introduction. We improve (5) and consequently (6), also including the case of abelian groups for which (1) holds.

**Theorem A.** Let \( G \) be a \( p \)-group of derived length \( d \). Then

\[
\exp M(G) \mid \begin{cases} 
2d^{-1} \cdot (\exp G)^d & p = 2 \\
(\exp G)^d & p > 2 
\end{cases}
\]

*Comparison with (5):* in case \( p > 2 \), the bounds coincide for \( d = 2 \) and the improvement occurs for \( d > 2 \); in case \( p = 2 \), it is non-efficient for \( d = 2 \), the bounds coincide for \( d = 3 \) and \( \exp G = 4 \), and the improvement occurs in all the other cases.

*Comparison with (6) via (7):* in case \( p > 2 \), the bounds coincide for \( c = 4, 5, 6, 7 \) and \( c = 8, 9 \), and the improvement occurs for \( c = 7 \) or \( c \geq 9 \); in case \( p = 2 \) and \( \exp G = 4 \), the bounds coincide for \( c = 7 \) and the improvement occurs for \( c \geq 11 \); in case \( p = 2 \) and \( \exp G > 4 \), the bounds coincide for \( c = 7 \), and the improvement occurs for \( c \geq 9 \).

The difference between the odd and the even case in Theorem A can be explained with the concept of unitary cover (§4.2), based on the theory of central extensions and projective representations (cf. [10, §9], and §3 hereby).

**Theorem B.** There exists a canonical element \( \Gamma_u(G) \), the *unitary cover* of \( G \), which has minimal exponent in the set of central extensions of \( G \) satisfying the projective lifting property. The map \( \Gamma_u \), associating to a group its unitary cover, satisfies for any normal subgroup \( N \) of \( G \) the following properties:

i) \( \exp M(G) \mid \exp \Gamma_u(N) \cdot \exp M(G/N) \)

ii) \( \exp \Gamma_u(G) \mid \exp \Gamma_u(N) \cdot \exp \Gamma_u(G/N) \)

iii) \( \Gamma_u(G/N) \) is a homomorphic image of \( \Gamma_u(G) \).

Moreover, if \( G = N \rtimes H \), then

iv) \( \exp M(G) \mid \lcm\{\exp \Gamma_u(N), \exp M(H)\} \)

v) \( \exp \Gamma_u(G) \mid \lcm\{\exp \Gamma_u(N) \cdot \exp H, \exp \Gamma_u(H)\} \).

By minimality, one can eventually replace the unitary cover with any central extension satisfying the projective lifting property, for instance with any Schur cover. The word “canonical” refers to the fact that \( \Gamma_u(G) \) is uniquely defined, whereas two Schur covers need not to be isomorphic.
We determine the exponent of the unitary cover for abelian \( p \)-groups, and in case \( p > 2 \) we extend this result for powerful \( p \)-groups (introduced in §1.1). Readily, we describe other families of groups for which (1) holds by Theorem B.

**Lemma C.** The following holds.

i) If \( G \) is a powerful \( p \)-group for \( p > 2 \), then \( \exp \Gamma_u(G) = \exp G \).

ii) If \( G \) is an abelian 2-group of exponent \( n \). Then

\[
\exp \Gamma_u(G) = 2^\sigma \cdot \exp G
\]

for \( \sigma = \begin{cases} 1 & \text{if } G \text{ has a subgroup isomorphic with } C_n \times C_n \\ 0 & \text{otherwise.} \end{cases} \)

**Corollary D.** Let \( G \) be a \( p \)-group, and \( N \) a normal subgroup of \( G \). Assume one of the following:

\[
\begin{cases} p = 2 & N \text{ is abelian with no subgroups isomorphic with } C_{2^k} \times C_{2^k}, \\ p > 2 & N \text{ is a powerful } p \text{-group.} \end{cases}
\]

And assume one of the following:

\[
\begin{cases} M(G/N) = 1 \\ G = N \rtimes H \text{ where } H \text{ satisfies (1)}. \end{cases}
\]

Then \( G \) satisfies property (1).

At least for groups of odd order, the previous result generalizes the case of abelian-by-cyclic groups to powerful-by-trivial multiplier groups, and it reveals a closure property under semidirect products with powerful kernels.

Our next result concerns regular \( p \)-groups, which constitute one of the most important family of \( p \)-groups and were introduced by P. Hall in 1934 [7, §4]. A \( p \)-group \( G \) is regular if for every \( x, y \in G \) there exist \( c \in \langle x, y \rangle' \) such that \( (xy)^p = x^p y^p c^p \). Abelian \( p \)-groups are regular, regular 2-groups are abelian, and regular \( p \)-groups share important properties with abelian groups for any \( p \).

Many families of groups for which (1) has been proven consist of regular \( p \)-groups, at first abelian \( p \)-groups and \( p \)-groups of class lower than \( p \) (cf. [2, p. 98] and §1.1). On the other hand, if \( P = G/\mathcal{U}(G) \) belongs to some of such classes, then \( G \) is regular and it also satisfies (1). We refer to the first Hall criterion claiming that if \( |P/\mathcal{U}(P)| < p^q \), then \( G \) is regular and it is said absolutely regular.

**Proposition E.** If \( G \) is a regular \( p \)-group and \( \exp M(G/\mathcal{U}(G)) \) divides \( p \), then \( G \) satisfies (1). Moreover, (1) holds for groups of exponent \( p \) iff it holds for regular \( p \)-groups. In particular, absolutely regular \( p \)-groups enjoy this property, and in general regular 3-groups.
We shall now prove the bound concerning the derived length, since we obtain the result in its stronger versions, first involving any subnormal series, then involving abelian 2-groups and powerful \( p \)-groups for \( p > 2 \).

**Proof of Theorem**[A] By iteration of Theorem[B] if a group \( G \) admits a subnormal series

\[
G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1, \ G_i \unlhd G_{i-1}, \ Q_i = G_{i-1}/G_i, \quad (8)
\]

then

\[
\exp \Gamma_u(G) \mid \prod_{j=1}^{r} \exp \Gamma_u(Q_i).
\]

By Lemma[C] we have respectively:

I. Let \( G \) be a \( p \)-group for \( p > 2 \). Assume \( G \) admits a subnormal series (8) where \( Q_i \) are powerful \( p \)-groups. Then

\[
\exp M(G) \mid \prod_{j=2}^{r} \exp Q_j \cdot \exp M(Q_1).
\]

II. Let \( G \) be a 2-group. Assume \( G \) admits a subnormal series (8) where \( Q_i \) are abelian. Then

\[
\exp M(G) \mid 2^{|I|} \cdot \prod_{j=2}^{r} \exp Q_j \cdot \exp M(Q_1).
\]

where \( I \subseteq \{2, \ldots, r\} \) is such that \( k \in I \) iff \( Q_k \) has a subgroup isomorphic with \( C_{e_k} \times C_{e_k} \) for \( e_k = \exp Q_k \).

These bounds prove Theorem[A] considering the derived series, so that the factor \( Q_i \) are abelian. In case \( p > 2 \) apply I, as abelian \( p \)-groups are powerful. In case \( p = 2 \) apply II and substitute \( |I| \) with \( d - 1 \). Notice that \( \exp Q_k \) divides \( \exp G \) for every \( k \), as well as \( \exp M(Q_1) \) divides \( \exp G \) since \( Q_1 = G/G' \) is abelian. \( \square \)

We expose our alternative proof that the study of the exponent of the multiplier can be restricted in some extent to 2-generator groups.

**Proposition F.** Let \( \mathcal{S}(G) \) denote the set of 2-generator subgroups of \( G \), then

\[
\exp \Gamma_u(G) \mid \mathrm{lcm}_{S \in \mathcal{S}(G)} \exp \Gamma_u(S).
\]

For a fixed positive integer \( n \), let \( \mathcal{S}(n) \) denote the set of isomorphism classes of 2-generator groups whose exponent divides \( n \). Substituting \( \mathcal{S}(G) \) with \( \mathcal{S}(\exp G) \) in the bound of Proposition[F] we obtain a bound depending only on the exponent of the group (cf. [15, Pr. 2.4]).
We assume that \( \exp G = p^k \), by the Zel’manov solution of the restricted Burnside problem \([21], [22]\), there exists a finite group

\[
\mathfrak{B}_{p^k} = \text{RBP}(2, p^k)
\]
such that every element in \( \mathfrak{S}(p^k) \) is a homomorphic image of \( \mathfrak{B}_{p^k} \). Therefore, by Theorem \([\mathbb{B}]\) we have that \( \exp \Gamma_u(S) \) divides \( \exp \Gamma_u(\mathfrak{B}_{p^k}) \) for every \( S \) in \( \mathfrak{S}(p^k) \), and we can also add some information to this result.

**Proposition G.** If \( G \) is a group of exponent \( p^k \), then

\[
\exp \Gamma_u(G) \mid \exp \Gamma_u(\mathfrak{B}_{p^k} ) .
\]

Moreover,

\[
\exp \Gamma_u(\mathfrak{B}_{p^k}) = p^k \cdot \exp \text{M}(\mathfrak{B}_{p^k})
\]
and

\[
p^k \mid \exp \text{M}(\mathfrak{B}_{p^k} ).
\]

Given an account on the theory of central extensions (§3.1) and projective representations (§3.2), we discuss a generalization of the Schur’s construction which proves that covering groups always exist (§4.1), then we introduce the unitary cocycles which define the unitary cover (§4.2), finally we prove the encountered results (§4.3).

## 3 Background

Let \( G \) be a group, and \( A \) an abelian group. A 2-cocycle is a map \( \alpha : G \times G \to A \) satisfying

\[
\alpha(x, y) \cdot \alpha(xy, z) = \alpha(x, yz) \cdot \alpha(y, z) .
\]

A 2-coboundary is a cocycle obtained from a map \( \zeta : G \to A \) as

\[
\delta \zeta(x, y) = \zeta(x) \cdot \zeta(y) \cdot \zeta(xy)^{-1} .
\]

The sets of cocycles and coboundaries are denoted with \( Z^2(G, A) \) and \( B^2(G, A) \) respectively, they constitute abelian groups under pointwise multiplication. The quotient \( H^2(G, A) = Z^2(G, A)/B^2(G, A) \) is the *second cohomology group*. In the particular case \( A = \mathbb{C}^\times \), we obtain the Schur multiplier \( \text{M}(G) = H^2(G, \mathbb{C}^\times) \), and we briefly denote \( Z^2(G) = Z^2(G, \mathbb{C}^\times) \) and \( B^2(G) = B^2(G, \mathbb{C}^\times) \).

These definitions play a fundamental role in the theory of central extensions, and in the theory of projective representations. We will give an account hereby, recommending the reading of \([10\ pp.181-185]\). Accordingly with this reference we adopt the right notation \( x^y = y^{-1}xy \) and \([x, y] = x^{-1}y^{-1}xy\).
3.1 Central extensions and Schur covers

A central extension of a group $G$ is a group $\Gamma$ having a central subgroup $A \leq Z(\Gamma)$ such that $\Gamma/A$ is isomorphic with $G$. It is usually written as

$$\omega : 1 \to A \to \Gamma \overset{\pi}{\to} G \to 1$$

where $A = \ker \pi$, and $\omega$ will be now defined. Let $\phi : G \to \Gamma$ be a section, that is $\pi(\phi(g)) = g$ for every $g \in G$. By definition, every $\gamma \in \Gamma$ can be uniquely written as $\gamma = a \cdot \phi(g)$ for some $a \in A$ and some $g \in G$. Then, $\omega : G \times G \to A$ is associated with $\phi$ by the relation

$$\phi(g) \cdot \phi(h) = \omega(g, h) \cdot \phi(gh),$$

and in turn $\omega \in Z^2(G, A)$.

Consider now a different section $\phi' : G \to \Gamma$, clearly $\phi'(g) = \zeta(g) \cdot \phi(g)$ for some $\zeta : G \to A$. From the analogue relation defining $\omega'$, it follows that $\omega' = \omega \cdot \delta \zeta$. Multiplication by a coboundary correspond to a change of section.

We may also mention that the trivial cocycle $G \times G \to \{1\} \leq A$ corresponds to the trivial extension $\Gamma = G \times A$.

We briefly show how the Schur multiplier parametrizes the central extensions. Denote by $\tilde{A} = \text{Hom}(A, \mathbb{C}^\times)$ the group of the irreducible characters of $A$. Then there exists $\eta : \tilde{A} \to M(G)$ called the standard map, defined as

$$\eta : \tilde{A} \to M(G), \; \lambda \mapsto \eta(\lambda) = [\lambda \circ \omega], \; \lambda \circ \omega(x, y) = \lambda(\omega(x, y)).$$

By the discussion above $\eta$ is well-defined, and it is easy to see that $\eta$ is a homomorphism such that

$$\ker \eta = (A \cap \Gamma')^\perp, \; (A \cap \Gamma')^\perp = \{\chi \in \tilde{A} \mid A \cap \Gamma' \leq \ker \pi\}.$$ 

The standard map also leads to the definition of Schur covers: a central extension is a Schur cover of $G$ if the standard map is an isomorphism. An equivalent definition is the following: a Schur cover of $G$ is a central extension such that the kernel is isomorphic with the Schur multiplier and it is contained in the derived subgroup,

$$1 \to M \to \Gamma \to G \to 1, \; M \simeq M(G), \; M \leq Z(\Gamma_G) \cap \Gamma'_G.$$ 

If we make the weaker assumption that the standard map is onto, then $\Gamma$ has the projective lifting property. This is equivalent to the following property,

$$1 \to A \to \Gamma \to G \to 1, \; A \cap \Gamma' \simeq M(G), \; A \leq Z(\Gamma_G).$$

If $\Gamma$ has the projective lifting property, then $\exp M(G)$ has to divide $\exp \Gamma$. Therefore, it has interest to find a minimal bound for the exponent of an extensions with the projective lifting property. We remark that this lower bound has not to be realized by a Schur cover, as shown by the following example.
Example 1. Consider the semidirect product of two cyclic groups of order $p^2$ defined by
$$G = \langle x, y \mid x^{p^2} = y^{p^2} = 1, \ y^x = y^{p+1} \rangle.$$ 
It can be seen, for instance using Gap [23], that $\exp M(G) = p$ and that the group
$$\Gamma_1 = \langle \bar{x}, \bar{y} \mid \bar{x}^{p^2} = \bar{y}^{p^2} = 1, \ \bar{y}^x = \bar{y}^{p+1} \rangle$$ 
is the only Schur cover of $G$, and it has exponent $p^3$. Nevertheless, the group
$$\Gamma_2 = \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^{p^2} = \bar{y}^{p^2} = \bar{z}^{p^2} = 1, \ \bar{y}^x = \bar{y}^{p+1} \cdot \bar{z}, [\bar{z}, \bar{x}] = [\bar{z}, \bar{y}] = 1 \rangle$$
has the projective lifting property for $G$, and satisfies $\exp \Gamma_2 = p^2$.

Among the central extensions those with the projective lifting property have fundamental importance, as they permit to transfer results on ordinary representations to projective representations and vice-versa. This was depicted by Schur who also proved by a constructive method that Schur covers always exist (§4.1).

Concerning the problem of bounding the exponent of the Schur multiplier, we can focus on the order of the single elements. Therefore, we introduce a local variation: for any $\mu \in M(G)$ we will say that a central extension
$$1 \to A_\mu \to \Gamma \to G \to 1$$
is a $\mu$-cover if $A_\mu$ is cyclic and the standard map $\eta_\mu$ maps $\bar{A}_\mu$ onto $\langle \mu \rangle$. This definition is not usually stated, as for any $\mu \in M(G)$ it is possible to obtain a $\mu$-cover as a quotient of any extension with the projective lifting property.

Proposition 2. Let $\Gamma$ be an extension of $G$ with the projective lifting property, and $\mu \in M(G)$. Then a $\mu$-cover $\Gamma_\mu$ can be obtained as a quotient of $\Gamma G$. In particular, $\exp \Gamma_\mu$ divides $\exp \Gamma G$.

Proof. Since the standard map $\eta_G : \bar{A} \to M(G)$ is assumed to be onto, there exists a preimage $\lambda \in \bar{A}$ of $\mu$ under $\eta_G$, that is $\eta_G(\lambda) = \mu$. We claim that the $\mu$-cover is $\Gamma_\mu = \Gamma_G/\ker \lambda$, whose exponent divides $\exp \Gamma_G$. Set $A_\mu = A/\ker \lambda$, then $\lambda$ can be identified with a faithful irreducible character $\lambda_\mu$ of the cyclic group $A_\mu$, and the standard map $\eta_\mu : \bar{A}_\mu \to \langle \mu \rangle$ is onto. □

We write down some complementary formulas for further reference. Let $\Gamma$ be any central extension. For any section $\phi$, an element $\gamma$ of $\Gamma$ is uniquely written as $\gamma = a \phi(g)$ for some $a \in A$ and some $g \in G$. Hence, $o(\gamma)$ divides $\text{lcm}\{o(a), o(\phi(g))\}$ and since $o(\phi(g)) = o(g) \cdot o(\phi(g)^{o(g)})$ it holds
$$\exp \Gamma = \text{lcm}\{\exp A, \max_{g \in G} o(g) \cdot o(\phi(g)^{o(g)})\}.$$  (9)
Moreover, for \( g \in G \) it is not difficult to see that
\[
\phi(g)^{\gamma(g)} = \prod_{j=0}^{\omega(g)-1} \omega(g, g^j) .
\]

Finally, concerning conjugation in \( \Gamma \), by comparison of \( \phi(x) \cdot \phi(y) \) and \( \phi(y) \cdot \phi(x') \) it follows that
\[
\phi(x)^{\phi(y)} = \omega(x, y) \omega(y, x')^{-1} \cdot \phi(x')
\]
holds for any \( x, y \in G \).

**Proposition 3.** Let \( \Gamma \) be a central extension of a \( d \)-generator group \( G \). Then there exists a \( d \)-generator subgroup \( X \) of \( \Gamma \), which is a central extension of \( G \) such that \( X' = \Gamma' \). In particular, if \( \Gamma \) has the projective lifting property, then also \( X \) does, and if \( \Gamma \) is a Schur cover, then \( X = \Gamma \).

**Proof.** Let \( G = \langle x_1, \ldots, x_d \rangle \) and \( \phi : G \to \Gamma \) be any section. We claim that the desired subgroup is \( X = \langle \phi(x_1), \ldots, \phi(x_d) \rangle \). For \( g \in G \) fix a writing \( g = x_{i_1}^{e_1} \cdots x_{i_l}^{e_l} \), then \( \phi(g) = b \cdot \phi(x_{i_1})^{e_1} \cdots \phi(x_{i_l})^{e_l} = b \cdot \xi \), for \( b \in A \) and \( \xi \in X \). Any \( \gamma \in \Gamma \) is uniquely written as \( a \cdot \phi(g) \) for some \( a \in A \) and \( g \in G \). Therefore, since \( A \leq Z(\Gamma) \), then \( \Gamma' = \langle [\gamma_1, \gamma_2] \mid \gamma_i \in \Gamma \rangle = \langle [\xi_1, \xi_2] \mid \xi_i \in X \rangle = X' \). \( \square \)

### 3.2 Projective representations and twisted group algebras

In analogy to the group algebra \( \mathbb{C}[G] \) for ordinary representations, for projective representations it is defined the **twisted group algebra**, which in turn relies on the cocycles. For \( \alpha \in Z^2(G), \mathbb{C}^\alpha[G] \) is the \( \mathbb{C} \)-algebra with basis \( \tilde{G} = \{ \tilde{g} \mid g \in G \} \) identified with the group, and product \( \tilde{x} \cdot \tilde{y} = \alpha(x, y) \cdot \xi \) obeying to the group product unless a twisting coefficient.

The cocycle condition is the associative law \((\tilde{x} \cdot \tilde{y}) \cdot \tilde{z} = \tilde{x} \cdot (\tilde{y} \cdot \tilde{z})\), whereas multiplication by a coboundary represents a locally-linear change of group-basis \( \tilde{g} = \tilde{\zeta}(g) \cdot \tilde{g} \). As common we consider normalized cocycles, that is \( \alpha(1, 1) = 1 \). The meaning of this assumption is that \( \tilde{1} \) is the identity of the twisted group algebra. Hence, for normalized coboundaries \( \delta \tilde{\zeta} \) it can be assumed \( \tilde{\zeta}(1) = 1 \).

For a subgroup \( H \) of \( G \), the **restriction map** is defined by
\[
\text{res} : M(G) \to M(H) , \ [\alpha] \mapsto [\alpha_H] , \ \alpha_H(h_1, h_2) = \alpha(h_1, h_2) ,
\]
and there is a natural identification \( \mathbb{C}^{\alpha_H}[H] \leq \mathbb{C}^{\alpha}[G] \). Then, for a normal subgroup \( N \) of \( G \) the **inflation map** is defined by
\[
\text{inf} : M(G/N) \to M(G) , \ [\beta] \mapsto [\beta^*] , \ \beta^*(g_1, g_2) = \beta(g_1N, g_2N) .
\]
Clearly, the image of the inflation map from $M(G/N)$ is contained in the kernel of the restriction to $M(N)$, a description of these subgroups can be done in terms of the idempotents of $\mathbb{C}^{\alpha}[N]$. We recall that the twisted group algebra $\mathbb{C}^{\alpha}[G]$ is semi-simple: it admits a decomposition in irreducible subspaces, each one defined by an idempotent.

It can be seen that $[\alpha_H] = 1$ iff $\mathbb{C}^{\alpha_H}[H]$ admits a 1-dimensional idempotent. Moreover, for a normal subgroup $N$ of $G$, it was proven by R. J. Higgs [8, Pr. 1.5] that $[\alpha] = [\beta^*]$ for some $\beta \in Z^2(G/N)$ iff $\mathbb{C}^{\alpha}[N]$ admits a 1-dimensional idempotent which is invariant under conjugation in $\mathbb{C}^{\alpha}[G]$. In analogy to (11), for any $x, y \in G$ comparing $\bar{x} \cdot \bar{y}$ and $\bar{y} \cdot \bar{x}$ we have the relation

$$\bar{x} \cdot \bar{y} = \alpha(x, y) \cdot \alpha(y, x)^{-1} \cdot \bar{x}$$

which describes conjugation in $\mathbb{C}^{\alpha}[G]$.

**Proposition 4.** Let $N \leq G$ and $\alpha \in Z^2(G)$. If $\alpha_N = 1$ and $\alpha(n, g) = \alpha(g, n^g)$ for every $n \in N$ and $g \in G$, then $[\alpha]$ is inflated from $G/N$. If in addition $G = N \rtimes H$ and $[\alpha_H] = 1$, then it holds $[\alpha] = 1$.

**Proof.** Since $\alpha_N = 1$, then $\mathbb{C}^{\alpha}[N]$ admits the principal idempotent

$$\varepsilon_N = \frac{1}{|N|} \sum_{n \in N} \bar{n} ,$$

which is invariant in $\mathbb{C}^{\alpha}[G]$ by (12). In case $G = N \rtimes H$, since we assume $[\alpha_H] = 1$, then $\mathbb{C}^{\alpha_H}[H]$ admits a central 1-dimensional idempotent $\upsilon_H$. As in the general case, $\mathbb{C}^{\alpha}\alpha[N]$ admits the principal idempotent $\varepsilon_N$. By (12) $\varepsilon_N$ and $\upsilon_H$ commutes, so that $\varepsilon_N \cdot \upsilon_H$ is a 1-dimensional idempotent of $\mathbb{C}^{\alpha}[G]$, and $[\alpha] = 1$.  

Also for the powers there is a formula analogue to (10), as for any $g \in G$ it holds

$$\bar{g}^{\alpha(g)} = \prod_{j=0}^{\alpha(g)-1} \alpha(g, g^j) .$$  

Cocycles whose group-basis satisfy the identity $\bar{g}^{\alpha(g)} = 1$ for any $g \in G$ will play the main role in the next section.

## 4 Method

### 4.1 Schur’s construction

We abstract the fundamental tool for our main results. We give a generalization of the construction which proves Schur’s theorem on the existence of a covering group, this will lead to the definition of the unitary covers (§4.2).
**Definition 5.** Let $H$ be a finite subgroup of $Z^2(G)$. We define a central extension

$$1 \to \tilde{H} \to \hat{H} \rtimes G \to G \to 1 \quad , \quad \tilde{H} \leq Z(\hat{H} \rtimes G) .$$

The underlying set of $\hat{H} \rtimes G$ is $G \times H$, and multiplication is given by the rule

$$(g, \chi) \cdot (h, \psi) = (gh, \omega(g, h) \cdot \chi \psi) ,$$

where $\omega(g, h) \in \tilde{H}$ is defined by $\omega(g, h)(\alpha) = \alpha(g, h)$ for $\alpha \in H$.

The proof of Schur's theorem is done in this terms: since $B^2(G)$ is a divisible subgroup of finite index in $Z^2(G)$, then it has a complement $Z(G) = B^2(G) \oplus J$, and $\tilde{J} \rtimes G$ is a Schur cover of $G$ (cf. [10, Th. 11.17]).

This construction is natural respect to the standard map in the following sense. For a cyclic decomposition $H = \langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_k \rangle$, the dual group admits the decomposition

$$\tilde{H} = \langle \tilde{\alpha}_1 \rangle \oplus \ldots \oplus \langle \tilde{\alpha}_k \rangle , \quad \tilde{\alpha}_i(\alpha_j) = \begin{cases} e^{2\pi i/\sigma(\alpha_j)} & \text{if } j = i \\ 1 & \text{otherwise} \end{cases} , \quad \sigma = \sqrt{-1} ,$$

and there is a canonical identification of $H$ with the double dual

$$H^\sim \equiv H , \quad \alpha_i^\sim \equiv \alpha_i ,$$

under this identification the standard map relative to $\tilde{H} \rtimes G$ is the projection from $Z^2(G)$ to $M(G)$

$$\eta : H^\sim \equiv H \to M(G) , \quad \eta(\alpha) = [\alpha] .$$

Referring to §3.1, we immediately have the following lemma.

**Lemma 6.** The extension $\tilde{H} \rtimes G$ has the projective lifting property iff every cocycle in $Z^2(G)$ is cohomologous with a cocycle of $H$, and it is a Schur cover iff in addition $H \cap B^2(G) = 1$.

For a pair of subgroups $K \leq H \leq Z^2(G, A)$, there is a natural isomorphism

$$\tilde{K} \to \tilde{H}/K^\perp , \quad K^\perp = \{ \chi \in \tilde{H} \mid K \leq \ker \chi \} ,$$

defined choosing coherent cyclic decompositions

$$\left\{ \begin{array}{l} K = \langle \beta_1 \rangle \oplus \ldots \oplus \langle \beta_i \rangle \\ H = \langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_i \rangle \oplus \ldots \oplus \langle \alpha_k \rangle \end{array} \right. , \quad \beta_i \in \langle \alpha_i \rangle ,$$

then setting $\tilde{\beta}_i \mapsto \tilde{\alpha}_i K^\perp$, this induces an isomorphism

$$\tilde{K} \rtimes G \cong (\tilde{H} \rtimes G)/K^\perp .$$

A case of particular interest is when $K$ is obtained via the inflation map.
Lemma 7. Let $N$ be a normal subgroup of $G$, $H$ be a finite subgroup of $Z^2(G)$, and $L$ be a finite subgroup of $Z^2(G/N)$ such that $H \cap Z^2(G/N) = H \cap L^*$. Denote

$$K = H \cap L^* , \quad \tilde{N} = \langle (n, 1_H) | n \in N \rangle ,$$

then there is a surjection

$$\tilde{L} \approx (G/N) \to (\tilde{H} \approx G)/K^\perp \tilde{N} ,$$

which is an isomorphism in case $L^* \leq H$.

Proof. If $(1, \chi) \in \tilde{N} \cap \tilde{H}$, then $\chi = \prod_{j=1}^k \omega(n_{1,j}, n_{2,j})$ for some $n_{i,j} \in N$, so that $\chi \in K^\perp$. Consequently,

$$\tilde{N} \cap \tilde{H} \leq K^\perp .$$

Also, $(n, 1_H)^{(\chi, \omega)} = (n^\ast, \omega(n, g)\omega(g, n^\ast)^{-1})$ and $\omega(n, g)\omega(g, n^\ast)^{-1} \in K^\perp$. Therefore,

$$K^\perp \tilde{N} \leq \tilde{H} \approx G .$$

Therefore, we obtain the central extension

$$1 \to \tilde{H}/K^\perp \to (\tilde{H} \approx G)/K^\perp \tilde{N} \to G/N \to 1$$

and we show that $(\tilde{H} \approx G)/K^\perp \tilde{N}$ is a homomorphic image of $\tilde{L} \approx (G/N)$. Write

$$\begin{align*}
L &= \langle \gamma_1 \rangle \oplus \ldots \oplus \langle \gamma_i \rangle \oplus \ldots \oplus \langle \gamma_m \rangle \\
K &= \langle \beta_1 \rangle \oplus \ldots \oplus \langle \beta_i \rangle , \quad \beta_i \in \langle \gamma_i^* \rangle \\
H &= \langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_i \rangle \oplus \ldots \oplus \langle \alpha_k \rangle , \quad \beta_i \in \langle \alpha_i \rangle .
\end{align*}$$

If $\beta_i = (\gamma_i^m)^n$, then there is an isomorphism

$$\tilde{L}/\tilde{K}^\perp \cong \tilde{K} , \quad \tilde{\gamma}_i \tilde{K}^\perp \mapsto \tilde{\beta}_i , \quad \tilde{K} = \langle \gamma_i^{m_1} \rangle \oplus \ldots \oplus \langle \gamma_i^{m_l} \rangle ,$$

which can be composed with the canonical isomorphism $\tilde{K} \approx \tilde{H}/K^\perp$ to give

$$\tilde{L} \to \tilde{L}/\tilde{K}^\perp \approx \tilde{H}/K^\perp , \quad \tilde{\gamma}_i \mapsto \tilde{\alpha}_i \tilde{K}^\perp .$$

Then, observe that $(gn, 1_H) = (g, 1_H) \cdot (n, \omega(g, n)^{-1})$ and that $(n, \omega(g, n)^{-1}) \in K^\perp \tilde{N}$. The map

$$(gN, \tilde{\gamma}^k_1 \ldots \tilde{\gamma}^k_m) \mapsto (g, \tilde{\alpha}_1^k \ldots \tilde{\alpha}_l^k) \tilde{K}^\perp \tilde{N}$$

is well defined, and it is the desired homomorphism. In case $L^* \leq H$, then $K = L^*$ and $\tilde{L} = \tilde{K}$. Thus, the map described is one to one.

\[ \square \]
4.2 Unitary cocycles and unitary covers

We introduce a subgroup of $Z^2(G)$, whose definition is done accordingly to (13), and we mimic the Schur’s construction (§4.1) introducing the unitary cover.

**Definition 8.** A cocycle $\alpha \in Z^2(G)$ is said to be unitary if

$$\prod_{j=0}^{o(g)-1} \alpha(g, g^j) = 1$$

for every $g \in G$. The set of unitary cocycles constitutes a group denoted by $Z_u(G)$, the unitary cover of $G$ is the extension $\Gamma_u(G) = Z_u(G) \rtimes G$.

Unitary cocycles and unitary covers are the core of our main results. We begin proving that every cocycle is cohomologous with an unitary, so that cohomology can be done with unitary cocycles exclusively. We describe the unitary coboundaries and provide a relation which refers to conjugation as shown by (12).

At once we will show a clear benefit of these facts, as we ready give the first statement of Theorem B in its explicit formulation.

**Lemma 9.** Let $\alpha \in Z^2(G)$. Then:

i) There exists $\beta \in Z_u(G)$ such that $[\alpha] = [\beta]$. Therefore,

$$M(G) \cong Z_u(G)/B_u(G)$$

where

$$B_u(G) = B^2(G) \cap Z_u(G)$$

is the group of unitary coboundaries.

ii) If $\delta \zeta \in B_u(G)$ for $\zeta : G \to \mathbb{C}^\times$, then $\zeta(g)^{o(g)} = 1$ for any $g \in G$.

iii) $Z_u(G)$ and $B_u(G)$ are finite, and $\exp B_u(G)$ divides $\exp G$.

iv) If $\beta \in Z_u(G)$, then $\beta(x, g)^{o(x)} = \beta(g, x^g)^{o(x)}$ for every $x, g \in G$.

**Proof.** i) Let $\alpha$ be any cocycle, define $\xi(g)$ to be any $o(g)$-root of $\prod_{j=0}^{o(g)-1} \alpha(g, g^j)$, set $\beta = \alpha \cdot \delta \xi^{-1}$, then $\beta$ is the unitary cocycle cohomologous with $\alpha$. ii) Apply the definition of unitary cocycle to $\delta \zeta$. iii) follows from ii. iv) in $\mathbb{C}^\beta[G]$ it holds $(\overline{x^g})^{o(x)} = (\beta(x, g)\beta(g, x^g)^{-1} \cdot \overline{x^g})^{o(x)} = [\beta(x, g)\beta(g, x^g)^{-1}]^{o(x)} \cdot (\overline{x^g})^{o(x)}$ by (12). Since $\beta \in Z_u(G)$, then $(\overline{x^g})^{o(x)} = (\overline{x})^{o(x)} = 1$, and since $\sigma(x^g) = \sigma(x)$, then $(\overline{x^g})^{o(x)} = 1$. Therefore, $[\beta(n, g)\beta(g, n^g)^{-1}]^{o(x)} = 1$ proving the assertion. □
Lemma 10. The extension $\Gamma_u(G)$ has the projective lifting property for $G$, and it satisfies

$$\exp \Gamma_u(G) = \operatorname{lcm}(\exp Z_u(G), \exp G).$$

Moreover, if $\Gamma$ is a central extension of $G$ having the projective lifting property, then $\exp \Gamma_u(G)$ divides $\exp \Gamma$.

**Proof.** That $\Gamma_u(G)$ has the projective lifting property it follows by Lemma 9 and Lemma 6. To find the exponent we use (9) with the section $\phi(g) = (g, 1_\lambda)$, where $A = Z_u(G)$, by definition

$$\phi(g)^{\alpha(g)} = (g^{o(g)}_1, \prod_{j=1}^{\alpha(g)-1} \omega(g, g^j)) = 1$$

and the assertion is proven. We now prove minimality dividing the proof in two steps: first we prove a local-version, then we use this to complete the proof.

**Local-version.** Let $\mu \in M(G)$, and $\Gamma_\mu$ be a $\mu$-cover. If $\beta \in Z_u(G)$ is such that $\mu = [\beta]$, then $o(\beta)$ divides $\exp \Gamma_\mu$.

**Step I.** It is enough to prove that there exists one cocycle $\beta$ with the required assertion, since two such cocycles differ by an unitary coboundary, and by Lemma 9 unitary coboundaries have order dividing $\exp G$ thus dividing $\exp \Gamma_\mu$. Since $\Gamma_\mu$ is a $\mu$-cover, the standard map $\eta_\mu : \tilde{A}_\mu \to \langle \mu \rangle$ is onto. Therefore, there exists $\lambda \in \tilde{A}_\mu$ such that $\eta_\mu(\lambda) = \mu$, where $\eta_\mu(\lambda) = [\lambda \circ \omega_\mu]$. Reading the proof of Lemma 9 together with (10), an unitary cocycle $\beta$ cohomologous to $\lambda \circ \omega_\mu$ is found defining $\xi : G \to \mathbb{C}^\times$ to be for $g \in G$ any $o(g)$-root of $\lambda(\phi(g)^{o(g)})$, then setting $\beta = (\lambda \circ \omega_\mu) \cdot \delta \xi^{-1}$. We show the assertion by use of (9). Since $o(\omega_\mu)$ divides $\exp A_\mu$ and $\lambda$ is a homomorphism, then the order of $\lambda \circ \omega_\mu$ divides $\exp A_\mu$, and since $\lambda$ is faithful, then $o(\lambda(\phi(g)^{o(g)})) = o(\phi(g)^{o(g)})$. Therefore,

$$o(\xi(g)) = o(g) \cdot o(\lambda(\phi(g)^{o(g)})) = o(g) \cdot o(\phi(g)^{o(g)}),$$

and $o(\delta \xi)$ divides $\max_{g \in G} o(g) \cdot o(\phi(g)^{o(g)})$. Thus, $o(\beta)$ divides $\operatorname{lcm}[o(\lambda \circ \omega_\mu), o(\delta \xi)]$ which divides $\operatorname{lcm}(\exp A_\mu, \max_{g \in G} o(g) \cdot o(\phi(g)^{o(g)}))$ that by (9) is $\exp \Gamma_\mu$.

**Step II.** For $\mu \in M(G)$, by Lemma 9 there exists $\beta_\mu \in Z_u(G)$ such that $[\beta_\mu] = \mu$. By Proposition 2 there exists a $\mu$-cover $\Gamma_\mu$ obtained as a quotient from $\Gamma_G$, then by the local-version $o(\beta_\mu)$ divides $\exp \Gamma_\mu$ and consequently $\exp \Gamma_G$. In particular, defining $J = \{ \beta_\mu \mid \mu \in M(G) \}$, then $J$ is a subgroup of $Z_u(G)$ of exponent dividing $\exp \Gamma_G$. Clearly $Z_u(G) = JB_u(G)$, so that $\exp Z_u(G) = \operatorname{lcm}(\exp J, \exp B_u(G))$ which divides $\operatorname{lcm}(\exp \Gamma_G, \exp B_u(G))$. The proof is complete by Lemma 9 since $\exp B_u(G)$ divides $\exp \Gamma_G$.  \qed
4.3 Proof of the main theorems

Proof of Theorem B. The first statement is part of Lemma 10. The proof of i) and iv) is explicitly written while proving ii) and v), nevertheless, we shall give an independent proof only based on commonly known results.

i) We prove that

\[ \exp M(G) \mid \lcm[\exp Z_u(N), \exp N] \cdot \exp M(G/N), \]

then we apply Lemma 10. For any co-class \([\alpha] \in M(G)\), we can assume by Lemma 9 that \(\alpha \in Z_u(G)\). For \(r = \lcm[\exp Z_u(N), \exp N]\), we show that \(\alpha'\) satisfies the first two conditions of Proposition 4, proving that \([\alpha]'\) is inflated from \(M(G/N)\) so that it becomes trivial when risen to the exp \(M(G/N)\)-power. Since \(\exp Z_u(N)\) divides \(r\) clearly \((\alpha_N)' = 1\), and since \(\exp N\) divides \(r\) by Lemma 9 the proof is complete.

iv) We assume \(G = N \rtimes H\), and we prove that

\[ \exp M(G) \mid \lcm[\exp Z_u(N), \exp N, \exp M(H)], \]

then we apply Lemma 10. By a result of K. Tahara [20, Th. 2], \(M(G)\) is isomorphic with the direct sum of \(M(H)\) and the kernel of the restriction from \(M(G)\) to \(M(H)\). Denote by \(r\) the lcm, since \(\exp M(H)\) divides \(r\) we can consider only co-classes \([\alpha] \in M(G)\) whose restriction to \(H\) is trivial. By Lemma 9 we can also assume that \(\alpha \in Z_u(G)\). The third condition of Proposition 4 is immediate, and since \(\lcm[\exp Z_u(N), \exp N]\) divides \(r\) the first two conditions follow the general case.

ii) We prove that

\[ \exp Z_u(G) \mid \lcm[\exp Z_u(N), \exp N] \cdot \lcm[\exp Z_u(G/N), \exp G/N], \]

then we apply Lemma 10. Fix \(\alpha \in Z_u(G)\) and a transversal \(T\) for \(N\) in \(G\). Define \(\xi(g) = \alpha(t, n)\) for \(g = tn\) where \(t \in T\) and \(n \in N\). Let \(r = \lcm[\exp Z_u(N), \exp N]\), define \(\beta = \alpha'\) and \(\tilde{\beta} = \beta \cdot \delta\theta\) for \(\theta = \xi'\). We show that \(\theta(g)^{o(gN)} = 1\) for every \(g \in G\), and that \(\tilde{\beta}\) is inflated from \(Z_u(G/N)\), as \(o(\tilde{\beta})\) divides \(\lcm[o(\beta), o(\delta\theta)]\). This completes the proof. For \(g_1, g_2 \in G\), let \(g_i = t_in_i\) where \(t_i \in T\) and \(n_i \in N\), and let \(t_1t_2 = t_{1,2}n_{1,2}\) where \(t_{1,2} \in T\) and \(n_{1,2} \in N\). In the twisted group algebra \(\mathcal{O}^\theta[G]\) consider

\[ \theta(g)^{o(gN)} \cdot g^{o(gN)} = (\theta(g) \cdot g)^{o(gN)} = (I \cdot \bar{n})^{o(gN)} = \bar{p}(x^{(N)}) \cdot \bar{n} \bar{p}(x^{(N)})^{-1} \cdots \bar{n} \bar{p}(x^{(N)})^{-1}. \]

For any \(x \in G\) since \(\bar{x}^{o(x)} = (x^{o(xN)})^{o(x^{o(xN)})} = 1\), then

\[ \prod_{j=1}^{o(xN)-1} \alpha(x, x^j)^{o(x^{o(xN)})} = 1, \quad x \in G, \]

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in particular \( \bar{g}^{\sigma(gN)} = g^{\sigma(gN)} \) and \( \bar{r}^{\sigma(gN)} = r^{\sigma(gN)} \) as \( tN = gN \). Since \( \exp N \) divides \( r \), by Lemma 9 then \( \bar{n}^j = n^j \) for every \( j \), and since \( \exp Z_u(N) \) divides \( r \), then \( \beta_N = 1 \). Therefore,

\[
\bar{r}^{\sigma(gN)} \cdot \bar{n}^{\sigma(gN)-1} \cdots \bar{n} = \bar{r}^{\sigma(gN)} \cdot n^{\sigma(gN)-1} \cdots n = g^{\sigma(gN)}
\]

proving that \( \vartheta(g)^{\sigma(gN)} = 1 \). We now prove that \( \beta = \gamma^* \) for some \( \gamma \in Z_u(G/N) \).

Notice that \( \bar{\xi} = \bar{\xi} \) for any \( t \in T \) and any \( n \in N \), so that

\[
\alpha(g_1, g_2) \cdot \delta\xi(g_1, g_2) = \alpha(t_1, t_2) \cdot \delta\xi(t_1, t_2) \cdot [\alpha(n_1, t_2) \cdot \alpha(t_2, n_2^{h_1})^{-1}] \cdot \alpha(n_1, n_1^{h_1} n_2) \cdot \alpha(n_1, n_2) \cdot \alpha(n_2).
\]

Hence, define \( \gamma \in Z^2(G/N) \) by setting \( \gamma(g_1 N, g_2 N) = \bar{\beta}(g_1, g_2) \), since

\[
\prod_{j=1}^{\sigma(gN)-1} \bar{\beta}(g, g^j) = \prod_{j=1}^{\sigma(gN)-1} \alpha(g, g^j)^\ast \cdot \delta\vartheta(g, g^j) = \prod_{j=1}^{\sigma(gN)-1} \delta\vartheta(g, g^j) = \vartheta(g)^{\sigma(gN)} = 1,
\]

then it holds \( \gamma \in Z_u(G/N) \).

v) Assume \( G = N \rtimes H \), and choose the transversal \( T = H \) so that

\[
\alpha(g_1, g_2) \cdot \delta\xi(g_1, g_2) = \alpha(h_1, h_2) \cdot [\alpha(n_1, h_2) \cdot \alpha(h_2, n_1^{h_1})^{-1}] \cdot \alpha(n_1^{h_1} n_2) \cdot \alpha(n_2).
\]

The proof that \( o(\xi(g)) \) divides \( \exp \Gamma_u(N) \cdot o(gN) \) follows the general case, and this gives the bound

\[
\exp Z_u(G) \mid \text{lcm} \{\exp Z_u(H), \exp \Gamma_u(N) \cdot \exp H\}
\]

where \( \Gamma_u \) can replace \( Z_u \) by Lemma 10 with no loss.

iii) Since \( Z_u(G/N)^* \leq Z_u(G) \), then

\[
\Gamma_u(G/N) \simeq \Gamma_u(G)/(Z_u(G/N)^*)^{\perp} N
\]

by Lemma 7.

\[ \square \]

**Proof of Lemma C.** We begin finding a cover of minimal exponent for an abelian \( p \)-group \( A \). Write the cyclic decomposition (2), then

\[
\Gamma = \langle x_1, \ldots, x_m \mid o(x_i) = p^d, \ [x_i, x_j] = 1 \rangle
\]

is a cover for \( A \) (cf. [12] p. 325), which satisfies \( \exp \Gamma = \exp A \) for \( p > 2 \), and \( \exp \Gamma = 2^r \cdot \exp A \) for \( p = 2 \). It can be seen that \( \exp Z_u(A) = \exp \Gamma \), and by Lemma 10 it follows that \( \exp \Gamma_u(A) = \exp \Gamma \). The case \( p = 2 \) is proved, while for \( p > 2 \) we use inductively this result on abelian \( p \)-groups. By definition, \( G' \leq U(G) \) so that \( G/\overline{U}(G) \) is elementary abelian, and we can assume that \( \exp G > p \). Since \( \exp \Gamma_u(G) = \exp \Gamma_u(U(G)) \cdot \exp \Gamma_u(G/\overline{U}(G)) \), by Theorem B the result follows by induction as \( \overline{U}(G) \) is powerful and \( \exp \overline{U}(G) = \exp G/p \) [13 Cor. 1.5, Pr. 1.7].

\[ \square \]
Proof of Proposition E. We can assume that \( \exp G \geq p > 2 \). It is known that \( \mathcal{U}(G) \) is powerful [13, p. 497], by Theorem [13] and Lemma [13] then \( \exp M(G) \) divides \( \exp \mathcal{U}(G) \cdot \exp M(G/\mathcal{U}(G)) \). Moreover, every element of \( \mathcal{U}(G) \) is a \( p \)-power [2, Th. 7.2], so that \( \exp \mathcal{U}(G) = \exp G/p \). We remind that \( \exp G/\mathcal{U}(G) = p \), and groups of exponent \( p \) are regular. Assuming that \( \exp M(G/\mathcal{U}(G)) \) divides \( p \) the proof is completed. This is the case for \( p = 3 \), as well for absolutely regular \( p \)-groups as \( \text{cl}(G/\mathcal{U}(G)) < p \). □

Proof of Proposition F. Since there exists a \( R \in \mathfrak{Z}(G) \) such that \( \exp R = \exp G \), by Lemma [10] it is enough to prove that there exists \( S \in \mathfrak{Z}(G) \) such that \( \exp Z_u(S) \) divides \( \exp Z_\mu(G) \). Choose any element \( \mu \in M(G) \) satisfying \( o(\mu) = \exp M(G) \). By Lemma [9] there exists \( \alpha \in Z_u(G) \) such that \( \mu = [\alpha] \). Let \( x, y \in G \) such that \( o(\alpha) = o(\alpha(x, y)) \), and set \( S = \langle x, y \rangle \). Since \( \alpha S \in Z_u(S) \) and \( o(\alpha S) = o(\alpha) \), then \( o(\mu) \) divides \( o(\alpha) \) and the proof is complete. □

Proof of Lemma G. For any \( p \)-group \( G \) and any integer \( m \), the \( m \)-agemo subgroup is defined as

\[
\mathcal{U}^m(G) = \langle g^{p^m} \mid g \in G \rangle.
\]

Let \( \Gamma \) be any central extension of \( \mathcal{B}_p \), by Proposition [3] there exists a 2-generated subgroup \( X \) of \( \Gamma \) which is a central extension of \( \mathcal{B}_p \) such that \( \Gamma' = X' \). Since \( \mathcal{B}_p \) is the maximal 2-generated group of exponent \( p^k \), it follows that \( X/\mathcal{U}^k(X) \simeq \mathcal{B}_p \). Therefore, \( A \cap \Gamma' = A \cap X' \leq \mathcal{U}^k(X) \). The assertion

\[
\exp \Gamma_u(\mathcal{B}_p) = p^k \cdot \exp M(\mathcal{B}_p)
\]

follows, since \( \Gamma_u(\mathcal{B}_p) \) has the projective lifting property.

We now prove that \( p^k \) divides \( \exp M(\mathcal{B}_p) \). For any integer \( l \), the sequence

\[
1 \to \mathcal{U}^k(\mathcal{B}_{p^k+l}) \to \mathcal{B}_{p^k+l} \to \mathcal{B}_p \to 1
\]

give rise to the central extension

\[
1 \to \mathcal{U}^k(\mathcal{B}_{p^k+l})/[\mathcal{U}^k(\mathcal{B}_{p^k+l}), \mathcal{B}_{p^k+l}] \to \mathcal{B}_{p^k+l}/[\mathcal{U}^k(\mathcal{B}_{p^k+l}), \mathcal{B}_{p^k+l}] \to \mathcal{B}_p \to 1.
\]

There is an embedding (§3.1)

\[
\mathcal{U}^k(\mathcal{B}_{p^k+l}) \cap \mathcal{B}^l_{p^k+l}/[\mathcal{U}^k(\mathcal{B}_{p^k+l}), \mathcal{B}_{p^k+l}] \rightarrow M(\mathcal{B}_p),
\]

which for large enough \( l \) becomes an isomorphism, and the exponent of this group is \( p^{m_0} \) for the minimal \( m_0 \) such that

\[
\mathcal{U}^{m_0}(\mathcal{U}^k(\mathcal{B}_{p^k+l}) \cap \mathcal{B}^l_{p^k+l}) \leq [\mathcal{U}^k(\mathcal{B}_{p^k+l}), \mathcal{B}_{p^k+l}].
\]
Any 2-generator group $G$ of exponent $p^{k+1}$ is a homomorphic image of $\mathfrak{B}_{p,k}$, then
$$\mathfrak{U}^{m_0}(\mathfrak{U}^k(G) \cap G') \leq [\mathfrak{U}^k(G), G],$$
therefore, lower bounds for $\exp M(\mathfrak{B}_{p,k})$ derives from two generators groups. Consider the covering group of $C_n \times C_n$ defined by
$$G = \langle x, y \mid x^{p^k} = y^{p^{2k}} = 1, y^x = y^{p^k+1} \rangle,$$
then $G' = \mathfrak{U}^k(G) = Z(G) = \langle y^k \rangle$, so that
$$\mathfrak{U}^{m_0}(\mathfrak{U}^k(G) \cap G') = \mathfrak{U}^{m_0}(\langle y^k \rangle), \ [\mathfrak{U}^k(G), G] = 1,$$
so that $m_0 \geq k$ completing the proof. □

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