0.1 LAWS
SH550

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Abstract. We give a framework for dealing with 0-1 laws (for first order logic) such that expanding by further random structure tend to give us another case of the framework. From another perspective we deal with 0-1 laws when the number of solutions of first order formulas with parameters behave dichotomically.

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§0 Introduction

§1 The context: probability and model theory

[We define the context: random models $\mathcal{M}_n$ for which we may ask if the 0-1 law holds. Having in mind cases the random structure there are relations which were drawn with constant probability as well as one decaying with $n$. We give sufficient condition for the satisfaction of the 0-1 laws, semi-nice].

§2 More accurate measure and drawing again

[We restrict ourselves further by fixing, when $A <_{pr} B$, the number of copies of $B$ over a copy of $A$ in $\mathcal{M}_n$, up to some factor. The clearest case is when this approximate value is $\sim ||\mathcal{M}_n||^{\alpha(A,B)}$ (the polynomial case). We also give the framework for redrawing].

§3 Regaining the context for $\mathcal{R}^+$: reduction

[We deal with $\mathcal{R}^+$; i.e. having a 0-1 context $\mathcal{R}$ we expand $\mathcal{M}_n$ randomly to $\mathcal{M}_n^+$. Assume in random enough $\mathcal{M}_n^+$, all relevant cases behave close enough to the expected value, we can see what should be $\leq_\alpha$, $\leq^+_{\alpha}$, $\leq^+_{pr}$, so we define such relations called for the time being $\leq_\alpha^+$, $\leq^+_{ij}$, $\leq^+_{tr}$, $\leq^+_{pr}$ respectively and investigate them, proving they behave similarly enough to what we hope to prove for $\mathcal{R}^+$. Then we restrict ourselves to the polynomial case and phrase a sufficient condition for succeeding: $\leq_\alpha^+ = \leq^+_{ij}$, etc. and for preservation of semi-niceness. We point out two extreme cases: when the probabilities are essentially constant and when the probabilities are of the form $||\mathcal{M}_n||^{\alpha}, \alpha \in (0,1)$. The restriction to the polynomial case is for simplicity.]

§4 Clarifying the probability problem

[We deal more explicitly with what is needed for showing that the 0-1 content we get by redrawing is again nice enough. We restrict ourselves to what is needed.]

§5 The probability argument

[We replace the content of §4 by a slightly more general one and then prove the required inequality.]

§6 Free amalgamation

[We axiomatize the “edgeless disjoint amalgamation” used in earlier works to “free amalgamation”.]

§7 Variant of niceness

[We consider some variants of semi-nice (and semi-good) and their relations. In earlier version §7, §8 we done inside §1, §2.]
§0 Introduction

Here we continue Shelah Spencer [ShSp 304], Baldwin Shelah [BlSh 528], and (in the model theory) Shelah [Sh 467] (see earlier [GKLT], [Fa76], Lynch [Ly85], see the survey [Sp]); we are trying to get results of reasonable generality. In particular, we want: if our random model $\mathcal{M}^+$ expands a given $\mathcal{M}$, where the class of $\mathcal{M}$'s is in our context, then also the class of $\mathcal{M}^+$'s is in our context. We shall be most detailed on the “nice polynomial” case (see definitions in the text).

Let us turn to trying to explain the results. $\mathcal{R}$ is a 0-1-context (see Definition 1.1) give for each $n$, a probability distribution for the $n$-th random model $\mathcal{M}_n$. The paper is self contained and describes a reasonable result of drawing (finite) models, where a finite sequence $\bar{a}$ has a closure, $c^k(\bar{a}, \mathcal{M}_n)$ in the model $\mathcal{M}_n$, presumably random enough relative to $k + \ell g(\bar{a})$, and the number of elements of $c^m(\bar{a}, \mathcal{M})$ has an a priori bound; i.e. depending on $\ell g(\bar{a})$ and $m$ only. And as far as first order formulas are concerned, this is all there is (see §1). However, here we allow relations with constant probability; this implies that the limit theory are not necessarily stable (as proved in [BlSh 528] on the theories from [ShSp 304] and more), but they are simple ([Sh:93]). Model theoretically our approach allows non-symmetric relations $R(x)$: i.e. $\neg(R(\ldots,x_i,\ldots)_{i=1,n} = R(\ldots,x_{\sigma(i)},\ldots)_{i=1,n})$ for permutation $\sigma$ of $\{1,\ldots,n\}$.

An extreme case is when for some $m = m_{\ell g(\bar{a})}, k > m \Rightarrow c^k(\bar{a}, M) = c^m(\bar{a}, M)$ (when $M$ is random enough relative to $\ell g(\bar{a}) + k$). Still a very nice case is when for every $m$ and $k$ for some $\ell$, for $M$ random enough and $\bar{a} \in kM$ we have: if $A_0 = \operatorname{Rang}(\bar{a})$ and $A_{i+1} = c^m(A_i, M)$ then $A_\ell = A_{\ell+1}$ can be proved for $\ell = \ell(k, \ell g(\bar{a}))$. But we may have a successor function to begin with and then this is impossible.

Note that if there are $\sim |.|\mathcal{M}_n|$ uniformly definable sets with $\sim \sqrt{\log |.|\mathcal{M}_n}$ elements and we draw a two-place relation with the probability of each pair being e.g. $1/2$ we get example of all two-place relations on such definable small sets (up to isomorphism), not what we want. So it is natural to ask that definable sets $\phi(x, \bar{a})$ with no apriori bound, are quite large, say of size in $[h^d_{\phi,\bar{a}}(\mathcal{M}_n), h^u_{\phi,\bar{a}}(\mathcal{M}_n)]$ (with $d$ for down, $u$ for up. The size of $h^d_{\phi,\bar{a}}$ is discussed below. Actually instead of dealing with such $\phi$’s, we deal with $\phi$’s with $A \prec B$; for each copy of $A \in \mathcal{M}_n$ we look at the number of copies of $B$ above it. In addition, concerning the drawing we can weaken the natural demand of independency to: if $A \subseteq \mathcal{M}_n$ (|A| small compared to $n$) the probability of $\mathcal{M}_n^+ \upharpoonright A = A^+$ is in $[p^d_{A/\mathcal{M}_n}[\mathcal{M}_n], p^u_{A/\mathcal{M}_n}[\mathcal{M}_n]]$ even knowing $\mathcal{M}_n^+ \upharpoonright B$ for every $B \subseteq \mathcal{M}_n, B \not\subseteq A$ (i.e. the conditional probability). Again the probabilities should be such that the phenomena described in the first sentence of this paragraph do occur.

We can ask more: the weak independence described in the last paragraph is even knowing all other instances of relations. We also can ask for true independence.

We may look at more strict cases:

Case 1 - Polynomial Case. $\frac{1}{n}$ $h^d_{A/\mathcal{M}_n}[\mathcal{M}_n]$ and $h^u_{A/\mathcal{M}_n}[\mathcal{M}_n]$ are $\mathcal{M}_n^\|^{\alpha(A/\mathcal{M}_n)}$ (for $A \in \mathcal{K}_\infty$) and $p^d_{A/\mathcal{M}_n}[\mathcal{M}_n]$ and $p^u_{A/\mathcal{M}_n}[\mathcal{M}_n]$ are $\mathcal{M}_n^\|^{\beta(A/\mathcal{M}_n)}$ where $A \in \mathcal{K}_\infty$.

Note: here $n^\alpha, \alpha \in \mathbb{R}$, is considered polynomial, $2\sqrt{\log n}$ is not, so not only integer powers are considered polynomial.
In this case the main danger in drawing is that some \( \alpha(A) \)'s and \( \beta(A) \)'s are linearly dependent (looking at \( \mathbb{R} \) as a vector space over \( \mathbb{Q} \)) similar to [ShSp304]; i.e. for having a \( 0-1 \) law (or convergence) we need the so-called irrationality condition.

Note that here the resulting framework also falls under Case 1. The reader can concentrate on Case 1, and we usually do it here.

**Case 2.** The \( h \)'s are near 1.

Here \( h^{A/\|M\|}_{\|\mathcal{M}\|} \) are constants \( \in (0,1) \mathbb{R} \) or at least are in

\[
\left( \frac{1}{h(M)}, 1 - \frac{1}{h(M)} \right)
\]

where \( h \) goes to infinity more slowly than any \( n^\varepsilon \) (where \( \varepsilon \in \mathbb{R}^{>0} \)).

Here we may allow the \( p^A_{\|M\|} \), \( p^u_{\|M\|} \) more freedom - just to be large enough (and/or small enough than 1) compared to \( \log(\|M\|) \). Here the irrationality condition includes requiring that the \( p^A_{\|M\|} \), \( p^u_{\|M\|} \) are close enough so that we get definite answers.

**Case 3.** The \( p \)'s are near 1.

Here \( p^A_{\|M\|} \), \( p^u_{\|M\|} \) are constant \( \in (0,1) \mathbb{R} \) or at least are in

\[
\left( \frac{1}{h(M)}, 1 - \frac{1}{h(M)} \right)
\]

where \( h \) goes to infinity more slowly than any \( n^\varepsilon \) (where \( \varepsilon \in \mathbb{R}^{>0} \)).

Here we may allow the \( h^{A/\|M\|}_{\|\mathcal{M}\|} \), \( h^u_{\|M\|} \) more freedom - just the fraction of \( M_n \) they are far enough from 0 and 1 compared to \( \log(\|M\|) \). The irrationality condition will include requirements that the \( h^{A/\|M\|}_{\|\mathcal{M}\|} \), \( h^{u}_{\|M\|} \) are close enough to get definite answers.

To carry the probability argument (and get that the resulting class \( \mathcal{R}^+ \) is also in our framework), we have to deal with the following problem.

For \( \mathcal{M}_n \) large enough it is natural to define a tree \( T \) (i.e. a set \( T \) and a partial order \( \leq_T \) such that \( x \in T \Rightarrow \{ y : y <_T x \} \) is linearly ordered by \( <_T \) and have \( \text{lev}(x) \in \mathbb{N} \) elements). Let \( T_\ell = \{ x \in T : \text{lev}(x) = \ell \} \), assume \( T \) has \( m \) levels, and for \( \ell < m \), and \( \eta \in T_\ell \), the number \( k_\eta = |\text{Suc}_T(\eta)| \in \{ |k^{3,4}_\ell|, |k^{2,4}_\ell| \} \). Suppose further we have \( A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_m \) (see Definition 1.3(2), \( A_{\ell} \) is a “small model” in the original vocabulary, increasing with \( \ell \)) and we have \( \{ f_\eta : \eta \in T \} \) such that \( f_\eta \) embeds \( A_{\eta(\eta)} \) into \( \mathcal{M}_n \), and \( \eta <_T \nu \in T \Rightarrow f_\eta \subseteq f_\nu \). Assume further \( A^+_{\ell} \) is an expansion of \( A_{\ell} \) to the larger vocabulary, increasing with \( \ell \) so formally \( A^+_{\ell} \leq A^+_{\ell+1} \). Lastly assume that for each \( c \in \mathcal{M}_n \) the set \( \{ \nu \in \text{Suc}(\eta) : c \in \text{Rang}(f_\eta) \backslash \text{Rang}(f_\nu) \} \) has an apriori bound (i.e. not depending on \( n \)). Under the condition that \( f_{\text{Min}(T)} \) embeds \( A^+_0 \) into \( \mathcal{M}_n^+ \), what is the number of \( \eta \in T_m \) for which \( f_\eta \) embeds \( A^+_m \) into \( \mathcal{M}_n^{+2?} \) (at least approximately).

We can easily give bounds to the expected value, and, working harder, for variance. But we want to show that the probability of “large” deviation from the expected value are \( < \|\mathcal{M}_n\|^{-\alpha} \) for every \( \alpha \in \mathbb{R}^{+} \). This is enough to really ignore those cases (not just say they occur very rarely but show that, for \( \mathcal{M}_n \) random enough they do not occur at all). This is easy if \( m = 1 \), and so we have, essentially, many independent events.

Note the following obstruction: the independence is violated as possibly \( \eta_1 \neq \eta_2, \nu_1 \in \text{Suc}(\eta_1), \nu_2 \in \text{Suc}(\eta_2) \) and \( \text{Rang}(f_{\eta_1}) \backslash \text{Rang}(f_{\eta_2}) \backslash \text{Rang}(f_{\nu_2}) \backslash \text{Rang}(f_{\eta_2}) \).
are not disjoint. Particularly disturbing case is when for some \( x \neq y \) from \( A_{\ell g(\nu_1)},A_{\ell g(\nu_2)} \) respectively we have \( f_{\nu_1}(x) = f_{\nu_2}(y) \). However, this chaotic obstruction can be overcome by shrinking somewhat the tree, so we can get:

\[
\bigoplus f_{\eta_1}(a_1) = f_{\eta_2}(a_2) \Rightarrow a_1 = a_2.
\]

Still we have two extreme cases which are quite different

\( \bigoplus_1 \) for each \( a_1 \) there is \( \ell \) such that \( a_1 \in A_{\ell+1} \) and for every \( a_2 \):

\[
f_{\eta_1}(a_1) = f_{\eta_2}(a_2) \Leftrightarrow (a_1 = a_2 \in A_{\ell+1} \& \eta_1 | (\ell + 1) = \eta_2 | (\ell + 1))
\]

\( \bigoplus_2 \)

\[
T = \bigcup_{\ell \leq m < \ell} m_i \text{ and: if } a \in A_{\ell+1}\setminus A_\ell \& \eta_1,\eta_2 \in T_{\ell+1} \text{ then: }
\eta_1(\ell) = \eta_2(\ell) \Leftrightarrow f_{\eta_1}(a) = f_{\eta_2}(a).
\]

What we need are good upper bounds on deviation uniformly under the circumstances (including, in particular, those two cases).

We deal with probability only as required: anything with probability \(< ||\mathcal{M}_n||^c \) (i.e. for every \( c \in \mathbb{R}^+ \), for every random enough \( \mathcal{M}_n \) can be discarded; also we need to eliminate the extreme cases (= large deviation) but do not care about the exact distribution in the middle.

Having explained the probability side problems, let us turn to the model theoretic ones. First, we want to include cases like the successor function so that, possibly, the exact distribution in the middle.

An interesting phenomena is the dichotomy: either the limit theory is very simple, analyzable or is complicated (and related to \( \text{Sh} \)). Lately Tyszkieiczew proves related theorem for monadic theories of for the classes of groups; I think the right starting point should be the parallel infinite problem with no probability, what can be the (monadic) infinitary theory of models of first order \( T \) monadically expanded, see Baldwin Shelah [BSh 156], survey [Baldwin, handbook of model theoretic logics]. I do not know enough to conjecture the dividing line, but can note that all the “complicated” limit theories \( T_\omega \) are “complex” which we define as: there is a formula \( \varphi_t(x,y,z,t) \), \( \varphi_z(x,y,z,t) \) such that \( T_\omega \models “ \text{for some } t, \varphi_+(x,y,z,t), \psi_+(x,y,z,t) \text{ define a model which satisfies the axioms of PA including induction but adding “there is a last element, there are } \geq n \text{ elements” (can add induction scheme for formulas of quantifier depth } \leq n. \) See more in Baldwin [B][nearby model complete and 0-1]. See Baldwin Shelah [Sh 639].

We can deal with the example of directed graphs, edge probability \( n^{-\alpha} \) both directions has same probability.
Baldwin has asked me to explicate here how the theory of \(((n), S, R)\) where \(S\) is the successor relation, \(R\) a random graph with edge probability being \(n^{-\alpha}\), i.e. how the fact that every element has an immediate successor, is reflected in the present treatment (that is, got into the axiomatization we get; this is unlike [BlSh 528]).

We can consider three cases: with successor being modulo \(n\), with usual successor and (e.g.) with successor being \((i, i+1)\) for \(i\) not divisible by the square root of \(n\) (rounded). The limit models are: in the first case it has copies of \(\mathbb{Z}\), in the second case one copy of \(\omega\), one copy of \(\omega^*\) and copies of \(\mathbb{Z}\), and in the third case many copies of \(\omega\), many copies of \(\omega^*\) and many copies of \(\mathbb{Z}\). Now in the semi-nice case (see Definition 1.9) we should look at the set of semi-(\(k, r\))-good quadruples, now the pairs of \((A, A^+)\) which appear in such quadruples (so \(A \subseteq A^+\)) gives us the distinctions.

For \(x \in A\) let

\[
\ell^+(x, A) = \min\{\ell : \text{there are no } x_0, \ldots, x_\ell \in A \text{ such that } x = x_0 \text{ and } A \models S(x_i, x_{i+1}) \text{ for } i < \ell\}
\]

\[
\ell^-(x, A) = \min\{\ell : \text{there are no } x_0, \ldots, x_\ell \in A^+ \text{ such that } x = x_\ell \text{ and } A \models S(x_i, x_{i+1}) \text{ for } i < \ell\}
\]

Let \(m(k, r)\) be large enough.

Now in the first case, in any such pair for every \(x \in A\) we have \(\ell^+(x, A^+) \geq m(k, r)\) and \(\ell^-(x, A^+) \geq m(k, r)\), in the third case for no \(x \in A\), \(\ell^+(x, A^+) < m(k, r)\) & \(\ell^-(x, A^+) < m(k, r)\) and in the second case, there may be at most one \(x \in A\) with \(\ell^+(x, A) < m(k, r)\) and at most one \(x \in A\) with \(\ell^-(x, A^+) < m(k, r)\) (but they are not the same).

In the strict polynomial (or even less) case we can also deal with properties suggested by Lynch [Ly90]. He asks for the results in Shelah Spencer [ShSp 304] for more accurate numerical (asymptotic) results, particularly in the case the probability is zero he proved

\((*)\) for every first order sentence \(\psi\) such that \(\Pr(\mathcal{M}_n \models \psi)\) converge to zero one of the following occurs:

\( (\alpha) \) \(\Pr(\mathcal{M}_n \models \psi) = c||\mathcal{M}_n||^{-\beta} + o(n^{-\beta-\varepsilon})\) for some \(c, \beta, \varepsilon \in \mathbb{R}^+\),

\( (\beta) \) \(\Pr(\mathcal{M}_n \models \psi) = O(||\mathcal{M}_n||^{\varepsilon})\) for every \(\varepsilon \in \mathbb{R}^+\).

Confirming his conjecture in [Sh 551] we prove

\( (\beta)^+ \) \(\Pr(\mathcal{M}_n \models \psi) = O(e^{-||\mathcal{M}_n||^{\varepsilon}})\) for every \(\varepsilon \in \mathbb{R}^+\).

We shall explicate this elsewhere.

The starting point of this research was a question of Lynch communicated to me by Spencer in Fall ’91 on whether we can do [ShSp 304] starting with a successor
function; but I thought the real problem was to have a general framework and I
lectured on it in Rutgers Fall '95; see [BSh 528].

I thank Shmuel Lefschas and John Baldwin for comments and corrections. Earlier
we have used \([\cup]\) and version of niceness from the beginning of the paper.

0.1 Notation. \(\mathbb{R}\) set of reals, \(\mathbb{R}^0 > = \{\alpha \in \mathbb{R} : \alpha > 0\}\), \(\mathbb{R}^0 = \mathbb{R}^0 \cup \{0\}\), \(\mu_n\) the nth probability (= distribution). Here \(n\) is the index for \(\mathcal{A}_n\), which is always used for
the model chosen \(\mu_n\) - randomly (we do not assume \(\mathcal{A}_n\) necessarily has exactly \(n\) elements). \(\mathbb{N}\) is the set of natural numbers. We use \(k, f, m, n, i, j, r, s\) for natural
numbers and we use \(\alpha, \beta, \gamma\) for reals, \(\varepsilon, \zeta\) for positive reals.

Let \(A, B, C, D\) denote finite models \(M, N\) models and \(f, g\) denote embeddings.
Let \(h\) denote a function with range \(\subseteq \mathbb{R}\).

0.2 Notation. 1) We use \(\tau\) for vocabularies, \(\tau\) consisting of predicates only (for
simplicity), \(n(R)\) the number of places of \(R\) (= the arity of \(R\)).
2) In general treatment we can demand that each \(R \in \tau\) will be interpreted as
irreflexive relation; i.e. \(\bar{a} \in R^M \Rightarrow \bar{a}\) without repetition; (by this demand we do
not lose any generality as we can add suitable predicates). We call such \(\tau\) irreflexive,
but we do not require symmetry (so directed graphs are allowed).

We use \(A, B, C, D\) for models which are finite, if not explicitly said otherwise, and
\(M, N\) for models; we notionally do not strictly distinguish between a model and its
universe. Those are \(\tau\)-models and \(A^+, \ldots, N^+\) are \(\tau^+\)-models if not explicitly said
otherwise. We use \(a, b, c, d, e\) for elements of models, bar signifies a finite sequence.
3) We call \(\tau\) locally finite if for every \(n\) the set \(\{R \in \tau : n(R) = n\}\) is finite. Note:
the number of \(\tau\)-model with the finite universe \(A\), is finite when \(\tau\) is finite or locally
finite irreflexive.
4) Let \(f : A \rightarrow B\) mean both are models with the same vocabulary, and \(f\) is an
embedding, i.e. \(f\) is one to one and for any predicate \(R\) (in \(\tau\), the vocabulary
of \(A\) and \(B\)) which is \(k\)-place we have: \(a_1, \ldots, a_k \in A \Rightarrow [(a_1, \ldots, a_k) \in R^A \leftrightarrow
\langle f(a_1), \ldots, f(a_k) \rangle \in R^B]\). Let \(id_A\) be the identity map on \(A\). Let \(A \subseteq B\) mean
\(id_A : A \rightarrow B\) and we say: \(A\) is a submodel of \(B\).
5) If \(A, B\) are submodels of \(C\) then \(A \cup B\) means \(C \uparrow (A \cup B)\).
6) We say \(A, C\) are freely amalgamated over \(B\) in \(M\) if \(B \subseteq A \subseteq M, B \subseteq C \subseteq M, A\cap
C = B\) and: if \(R\) is a predicate of \(M\), for no \(\bar{a} \in R\), do we have \(Rang(\bar{a}) \notin B \cup A, \)
\(Rang(\bar{a}) \notin B \cup C\) and \(Rang(\bar{a}) \subseteq B \cup A \cup C\); we also say \(A, C\) are free over \(B\) inside
\(M\). (But \(\cup\)-free means according to the definition of \(\cup\), but this generalization is
done only in \(\S6, \S7\).

* * *

0.3 Notation. 1) If \(f\) is an embedding of \(A\) into \(M\) and \(A \subseteq B\), we say
\(\bar{g} = \langle g_i : i = 1, k \rangle\) are disjoint extensions of \(f\) (for \((A, B)\)) if:

(a) \(g_i\) is an extension of \(f\) to an embedding of \(B\) into \(M\)
(b) \(1 \leq i < j \leq k \Rightarrow Rang(g_i) \cap Rang(g_j) = Rang(f)\).
We say $\bar{g}$ is a disjoint $k$-sequence of extensions of $f$ if the above holds; we also say: $\bar{g}$ is a sequence disjoint over $A$, $\bar{g}$ of length $k$.

2) $\text{ex}(f, A, B, M)$ where $A \subseteq B$, $f$ an embedding of $A$ into $M$ is the set of extensions $g$ of $f$ to embedding of $B$ into $M$.

$\text{nu}(f, A, B, M)$ is the number of elements in $\text{ex}(f, A, B, M)$.

3) Let $\mathbb{N}$ and also $\omega$ denote the set of natural numbers.
§1 The Context: Probability and Model Theory

We start by defining a 0-1 context (in Definition 1.1), defining the derived \( A \leq_i B \) (\( B \) algebraic over \( A \)), \( A \leq_s B \) (dual), \( c^k(A, M) \), (in Definition 1.3, 1.4) and point out the basic properties (in 1.6). We define “\( \mathfrak{R} \) is weakly nice” and state its main property, (see 1.7, 1.8). Then we define our main version of nice, (Definition 1.9, semi-nice) and investigate some variant (1.10, 1.11) and define the 0-1 laws and variants (1.13, 1.14, 1.15). We prove that \( \mathfrak{R} \) is semi-nice implies elimination of quantifiers and phrase what this gives for 0-1 laws (in 1.16, see Definition 1.13, 1.14).

1.1 Definition. 1) \( \mathfrak{R} \) is a \( 0-1 \) context if it consists of \( \tau, \mathcal{K}, \leq \) and
\[ \mathcal{K} = \langle \mathcal{K}_n : n \in \mathbb{N} \rangle \]
satisfying (a)-(c) below where:
(a) \( \tau \) a vocabulary consisting of predicates only (for simplicity, \( \tau \) irreflexive, see 0.2), \( \tau \) finite or at least locally finite.
(b) \( \mathcal{K} \) a family of finite \( \tau \)-models, closed under isomorphisms and submodels; we denote members by \( A, B, C, D \) (sometimes \( M, N \)) and for notational simplicity the empty model belongs to \( \mathcal{K} \).
(c) \( \mathcal{K}_n \subseteq \mathcal{K} \). \( \mu_n \) is a probability measure on \( \mathcal{K}_n \). \( \mathcal{M}_n \) varies on \( \mathcal{K}_n \); for notational simplicity assume \( n_1 \neq n_2 \Rightarrow \mathcal{K}_{n_1} \cap \mathcal{K}_{n_2} = \emptyset \) and \( M \in \mathcal{K}_n \Rightarrow \| M \| > 1 \); also we sometime “forget” the possibility \( M_{n_1} \in \mathcal{K}_{n_1} \) \& \( M_{n_2} \in \mathcal{K}_{n_2} \) \& \( n_1 \neq n_2 \) \& \( \| M_{n_1} \| = \| M_{n_2} \| \) but no confusion should arise.

1.2 Definition. Let: “every random enough \( \mathcal{M}_n \) satisfies \( \psi \)” mean
\[ 1 = \liminf_n \text{Prob}_{\mu_n}(\mathcal{M}_n \models \psi) \]. Similarly “almost surely \( \mathcal{M}_n \models \psi \)” and “a.s. \( \mathcal{M}_n \)
satisfies \( \psi \)”.

1.3 Definition. For \( \mathfrak{R} \) as in 1.1(1) we define:
1) \( \mathcal{K}_\infty = \{ A \in \mathcal{K} : 0 < \limsup_n \text{Prob}_{\mu_n}(A \text{ is embeddable into } \mathcal{M}_n) \} \).
2) We define some two place relations on \( \mathcal{K} \) (mostly on \( \mathcal{K}_\infty \)):
   (a) \( A \leq B \) if \( A \subseteq B \) (being submodels) and \( B \in \mathcal{K}_\infty \) (hence \( A \in \mathcal{K}_\infty \), see 1.6(1))
   (b) \( A \leq_i B \) if \( A \leq B \in \mathcal{K}_\infty \) and for some \( m \in \mathbb{N} \) we have \( 1 = \lim_n \text{Prob}_{\mu_n}(\text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n, \text{there are at most } m \text{ extensions of } f \text{ to an embedding of } B \text{ into } \mathcal{M}_n) \)
      \[ \text{[the intention is: } B \text{ is algebraic over } A] \]
   (c) \( A \leq_s B \) if \( A \leq B \in \mathcal{K}_\infty \) and for no \( B' \) do we have \( A \leq_s B' \leq B \)
      \[ \text{[the intention is: “strongly” in some sense } A \text{ is very closed inside } B] \]
   (d) \( A \leq_{pr} B \) if \( A, B \in \mathcal{K}_\infty \) and \( A <_{s} B \) and for no \( C \) do we have \( A <_{s} C <_{s} B \)
   (e) \( \bar{A} \) a decomposition of \( A <_{s} B \) if \( \bar{A} = \langle A_\ell : \ell \leq k \rangle \) and
      \[ A = A_0 <_{pr} A_1 < \cdots <_{pr} A_k = B \]
      \[ \text{[the intention of pr is “primitive”, cannot be decomposed]} \]
   (f) \( A \leq_{a} B \) means \( A \in \mathcal{K}_\infty, B \in \mathcal{K}_\infty, A \subseteq B \) and \( A = B \) or for some \( m \in \mathbb{N} \) we have \( 1 = \lim_n \text{Prob}_{\mu_n}(\text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n, \text{there is no sequence } \langle g_\ell : \ell < m \rangle \text{ of embeddings of } B \text{ into } \mathcal{M}_n, \text{pairwise disjoint over } A) \)
[the intention is: B is algebraic in a weak sense over A; more accurately A is not strongly a submodel of A].

1.4 Definition. 1) $T_\infty = \{ \varphi : \varphi \text{ a first order sentence such that} 1 = \lim_n \text{Prob}^\mathcal{M}_n (\mathcal{M}_n \models \varphi) \}.$

2) $\mathcal{K}^{\lim} = \{ M : M \text{ a model of } T_\infty \}.$

3) If $A \subseteq M \in \mathcal{K}$ and $k \in \mathbb{N}$ let

\[ \text{ct}^k(A, M) = \bigcup \{ B : B \cap A \leq_1 B \subseteq M \text{ and } |B| \leq k \}. \]

4) For $A \subseteq M \in \mathcal{K}, A \in \mathcal{K}_\infty$ and $k, m \in \mathbb{N}$ we define $\text{ct}^{k, m}(A, M)$ by induction on $m$ as follows $\text{ct}^{k, 0}(A, M) = A$ and $\text{ct}^{k, m+1}(A, M) = \text{ct}^k(\text{ct}^{k, m}(A, M), M).$ Also let $\text{ct}^{k, \infty}(A, M) = \bigcup_m \text{ct}^{k, m}(A, M)$ and $\text{ct}^\infty(A, M) = \bigcup_k \text{ct}^k(A, M).$

5) For any $(A, B, D)$ satisfying $A \leq D \in \mathcal{K}_\infty, B \leq D$ and $k \in \mathbb{N}$ and embedding $f : A \rightarrow M$ let

\[ \text{ex}^k(f, A, B, D, M) = \left\{ g : g \text{ is an embedding of } D \text{ into } M \text{ extending } f \right\} \]

such that $\text{ct}^k(g(B), M) = g(\text{ct}^k(B, D))$ with

\[ \text{nu}^k(f, A, B, D, M) = |\text{ex}^k(f, A, B, D, M)| \]

(see Definition 1.3(2)).

1.5 Remark. We have chosen the present definition of $\text{ct}^k$ so as to have more cases where iterating the operation lead shortly to a fix point and to be compatible with [Sh 467] where the possibility $\text{ct}^k(A, M) \notin \mathcal{K}_\infty$ exists.

1.6 Claim. 1) $\mathcal{K}_\infty \subseteq \mathcal{K}$ is closed under isomorphisms and submodels.

[why? reread Definition 1.3(1)].

2) $\subseteq_1$ is a partial order on $\mathcal{K}_\infty$; if $A \leq B \leq C, A \leq_1 C$ then $B \leq_1 C.$ Also $A \leq_1 B$ iff $B \in \mathcal{K}_\infty, A \leq B$ and for some $m \in \mathbb{N},$ for no $D \in \mathcal{K}_\infty, A \subseteq D,$ is there a sequence of $m$ (distinct) embeddings of $B$ into $D$ over $A.$

[why? reread Definition 1.3(2)(b)].

3) If $A_1 \leq_1 A_2 \leq C \in \mathcal{K}_\infty$ and $B \leq C \in \mathcal{K}_\infty$ and $A_1 \subseteq B$ then $B \leq_1 B \cup A_2$

[why? see Definition 1.3(2)(b)].

4) If $A \leq C$ are in $\mathcal{K}_\infty$ then for one and only one $B \in K_\infty$ we have $A \leq_1 B \leq_s C$

[why? let $B \leq C$ be maximal such that $A \leq B,$ it exists as $A$ satisfies the demand and $C$ is finite, now $B \leq_s C$ by 1.6(2) + Definition 1.3(2)(c). Hence at least one $B$ exists, so suppose $A \leq B \leq_1 C$ for $\ell = 1, 2$ and $B_1 \neq B_2$ so without loss of generality $B_2 \setminus B_1 \neq \emptyset.$ Now by 1.6(3), $B_1 \leq_1 B_1 \cup B_2$ hence $B_1 <_s B_1 \cup B_2 \leq C,$ but this contradicts $B_1 \leq_s C$ (see Definition 1.3(2)(c)).]

5) If $A <_s B$ then there is a decomposition $\bar{A}$ of $A <_s B$

[why? see Definition 1.3(2)(e) and Definition 1.3(2)(d), remembering $B$ is finite].

6) If $A <_s B$ and $C \leq B$ then $C \cap A \leq_s C;$ (note also $A \leq_s C \wedge A \leq B \leq C \Rightarrow A \leq_s B$)
[why? otherwise for some \( C' \) we have \( C \cap A <_i C' \leq C \), then by \( 1.6(3) \) we have \( A <_i A \cup C' \), contradiction to \( A \leq_s B \). The second phrase holds by Definition 1.3(2)(c)].

7) The relations \( \leq_i, \leq_s, \leq_{pr}, \leq_n \) are preserved by isomorphisms.

[why? read Definition 1.3(2)].

8) If \( A <_{pr} B \) then for every \( b \in B \setminus A \) we have \( (A \cup \{b\}) \leq_i B \); also \( A < C \leq B \Rightarrow C \leq_i B \)

[why? if not, then by \( 1.6(4) \) for some \( C, (A \cup \{a\}) \leq_i C <_{pr} B \), but \( A <_{pr} B \Rightarrow A < C \leq (by Definition 1.3(2)(b) and 1.6(10) respectively) so \( A < C < B \) contradicting \( A <_{pr} B \). The second phrase is proved similarly.]

9) \( A \leq_i B \) if and only if \( A \leq_i C \leq B \) and for every \( A' \) we have \( A < A' < B \Rightarrow (A' <_{pr} B) \)

[why? trivially \( A < B \Rightarrow A <_{pr} B \); now the implication \( \Rightarrow \) by \( 1.6(2) \) second phrase + Definition 1.3(2)(b),(f); the implication \( \Leftarrow \) by the \( \Delta \)-system lemma and the definitions].

10) \( \leq_s \) is a partial order on \( \mathcal{X}_\infty \)

[why? read definition \( A \leq_s A \), as \( A \leq_s B \Rightarrow A \leq B \) and clearly \( A \leq_s B \leq A \Rightarrow A = B \), so the problem is transitivity. So assume \( A \leq_s B \leq C \) but \( \neg(A \leq_s C) \) and we shall derive a contradiction. As \( \neg(A \leq_s C) \) by Definition 1.3(2)(c) there is \( B_1 \) such that \( A < B_1 \leq C \). If \( B_1 \subseteq B \) we get contradiction to \( A \leq_s B \) by Definition 1.3(2)(c). If \( B_1 \not\subseteq B \), then by \( 1.6(3) \) we get \( B \leq (B_1 \cup B) \), but as \( B_1 \subseteq B \) we have \( B < (B_1 \cup B) \) but clearly \( (B_1 \cup B) \leq C \) we get contradiction to \( B \leq_s C \) by Definition 1.3(2)(c), so in any case we have gotten the desired contradiction].

11) \( A <_{pr} B \) if and only if for every \( m, 0 < \limsup \text{Prob}_\mu \) (for some embedding \( f \) of \( A \) into \( \mathcal{M}_n \), there are \( m \) disjoint extensions \( g : B \rightarrow \mathcal{M}_n \) of \( f \)). If \( A <_{pr} B \) then the inverse statement holds.

[why? read definition \( 1.3(2)(b),(c) \), see details in the proof of 1.8(1)].

12) If \( A <_{pr} B \leq D, A \leq C \leq D \), then \( C <_{pr} B \cup C \) or \( C \leq_i B \cup C \).

[why? by \( 1.6(4) \) for some \( C_1 \) we have \( \leq_i C_1 \leq_s B \cup C \). If \( C_1 \cap B \neq \emptyset \), then \( A \leq C_1 \cap B \leq B \) hence by \( 1.6(8) \) we have \( C_1 \cap B \leq B \) so by \( 1.6(3) \) we have \( C_1 \leq_i B \cup C_1 \) and, of course, \( B \cup C_1 = B \cup C \), so \( C \leq_i C_1 \leq_i C \cup B \) by \( 1.6(2) \) we have \( C \leq_i C \cup B \), one of the possible conclusions. So assume \( B \cap C_1 = A \) hence \( C = C_1 \), so \( C \leq_s B \cup C \), now if \( C = B \cup C \) clearly \( C \leq_i B \cup C \). Hence we assume \( C \neq B \cup C \) if \( C \geq_{pr} B \cup C \) we get one of the possible conclusions as above. So assume \( \neg(C \leq_{pr} B \cup C) \), necessarily for some \( C_2, C <_{pr} C_2 \leq C \cup B \).

By \( 1.6(6) \) we have \( C \cap B <_s C_2 \cap B <_s B \) so as \( C \cap B = C_1 \cap B = A \) clearly \( A <_s C_2 \cap B < B \) hence we get a contradiction finishing the proof.]

13) For every \( \ell, k \in \mathbb{N} \) there is \( m(k, \ell) \in \mathbb{N} \) such that:

\[
\text{if } A \leq B \in \mathcal{X}_\infty \text{ and } |A| \leq \ell \text{ then } cl^k(A, B) \text{ has } \leq m(k, \ell) \text{ elements.}
\]

[why? read the definitions noting that (even if \( \tau \) is only locally finite) the number of pairs \( (C_1, C_2), C_1 \leq A, C_1 \leq C_2, |C_2| \leq k \) up to isomorphism over \( C_1 \) has a bound depending only on \( C_1 \)].

14) \( a \) \( cl^k(A, M) \subseteq M \) and \( cl^k(A, M) \in \mathcal{X}_\infty \Rightarrow A \leq_i cl^k(A, M) \)

\( b \) \( A \subseteq B \subseteq M \Rightarrow cl^k(A, M) \subseteq cl^k(B, M) \)
(c) if $c^k(A, M) \subseteq N \subseteq M$ then $c^k(A, N) = c^k(A, M)$

(d) if $A \subseteq N \subseteq M$ then $c^k(A, N) \subseteq c^k(A, M)$

(e) for $k \leq \ell$ we have $c^k(A, M) \subseteq c^\ell(A, M)$

(f) for every $A \in \mathcal{X}_\infty$ and $k$ for every random enough $M_n$ and embedding $f : A \rightarrow M_n$ we have 
$M_n \upharpoonright c^k(f(A), M_n) \in \mathcal{X}_\infty$

(g) for every $k, m$ for some $\ell$ for every $A \subseteq M \in \mathcal{X}_\infty$ we have $c^m(c^k(A, M), M) \subseteq c^\ell(A, M)$.

[Why? Just check.]

15) $T_\infty$ is a consistent (first order) theory which has infinite models if
$0 < \limsup \text{Prob}_\mu_n(\|M_n\| > k)$ for every $k$.

Remark. Note that not necessarily in 1.6(11), we have “iff”. Why? e.g. if 
$\tau = \{P_1, P_2\}$ with $P_1, P_2$ unary predicates, $B = \{b_1, b_2\} \in K_\infty, A = \emptyset, P^B_1 = \{b_1\}$, and for every $M \in \mathcal{X}, |P^M_1| = 0 \lor |P^M_2| = 0$ and $M \in K_n$ & $n$ even $\Rightarrow |P^M_1| \geq n$ and $M \in K_n$ & $n$ odd $\Rightarrow |P^M_2| \geq n$.

1.7 Definition. For $\mathcal{R}$ as in Definition 1.1, we say $\mathcal{R}$ is weakly nice if we have:

$(\ast)_1$ for every $A <_{pr} B$ (from $\mathcal{X}_\infty$) and $m \in \mathbb{N}$ we have

$1 = \lim_n \text{Prob}_\mu_n \left( \text{for every embedding } f \text{ of } A \text{ into } M_n \text{ there are } m \right.$

disjoint extensions of $f$ to embedding of $B$ into $M_n$).

1.8 Claim. Assume $\mathcal{R}$ is weakly nice.

1) If $A < B \in \mathcal{X}_\infty$, then the following are equivalent:

(a) $A <_s B$

(b) for every $m < \omega$

$1 = \lim_n \text{Prob}_\mu_n \left( \text{for every embedding } f \text{ of } A \text{ into } M_n \text{ there are} \right.$

embeddings $g_\ell : B \rightarrow M_n$ extending $f$ for

$\ell < m \text{ such that } (g_\ell : \ell < m) \text{ is disjoint over } A \right)$.

(For $(b) \Rightarrow (a), \mathcal{R}$ weakly nice is not needed.)

Proof. 1) The direction $(a) \Rightarrow (b)$ holds as $\mathcal{R}$ is weakly nice, more elaborately, we prove $(b)$ assuming $(a)$ by induction on $m$ where $(A_\ell : \ell < m)$ is a decomposition of $(A, B)$ (which exists by 1.6(5)): for $m = 0$ this is trivial and for $m + 1$ by straight combinatorics. Next we prove $\neg(a) \Rightarrow \neg(b)$ even without using “$\mathcal{R}$ weakly nice”. So
assume \( b \) & \( \neg (a) \) and we shall get a contradiction. As \( \neg (a) \), by 1.6(4) for some \( A_1 \) we have \( A <_n A_1 \leq B \), hence by Definition 1.3(2)(b) for some \( m^* \in \mathbb{N} \) we have:

\[
(*) \quad 1 = \lim_n \text{Prob}_{\mathcal{M}_n} \left( \text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n \text{ there are at most } m^* \text{ extensions of } f \text{ to an embedding of } A_1 \text{ into } \mathcal{M}_n \right).
\]

As Clause (b) holds, the limit there is 1 also for the \( m^* \) we have just chosen. The contradiction is immediate. \( \square \)

1.9 Definition. 1) We say \((A^+, A, B, D)\) is a semi-(\( k, r \))-good quadruple if:

\[
(*)_{A^+, A, B, D} \quad A \leq A^+ \in \mathcal{K}_\infty \text{ and } A \leq D, B \leq D \in \mathcal{K}_\infty \text{ and for every random enough } \mathcal{M}_n \text{ we have:} \]

\[
(**) \text{ for every embedding } f : A^+ \rightarrow \mathcal{M}_n \text{ satisfying } c^{\ell r}(f(A), \mathcal{M}_n) \subseteq f(A^+) \text{ there is an extension } g \text{ of } f \upharpoonright A, \text{ embedding } D \text{ into } \mathcal{M}_n \text{ such that } c^{\ell k}(g(B), \mathcal{M}_n) = g(c^{\ell k}(B, D)).
\]

If \( r = k \) we may write \( k \) instead of \((k, r)\).

2) We say that \( \mathcal{K} \) is semi-nice if it is weakly nice and for every \( A \in \mathcal{K}_\infty \) and \( k \) for some \( \ell, m, r \) we have:

\[
(*) \text{ for every random enough } \mathcal{M}_n, \text{ and embedding } f : A \rightarrow \mathcal{M}_n \text{ and } b \in \mathcal{M}_n \text{ we can find } A_0, A^+, B, D \text{ such that:} \]

\[
(\alpha) \quad f(A) \leq A_0 \leq A^+ \leq c^m(f(A), \mathcal{M}_n)
\]

\[
(\beta) \quad f(A) \cup \{b\} \subseteq B \subseteq D \subseteq \mathcal{M}_n
\]

\[
(\gamma) \quad |D| \leq \ell
\]

\[
(\delta) \quad (A^+, A_0, B, D) \text{ is semi-(} k, r \text{-)good}
\]

\[
(\varepsilon) \quad c^{\ell r}(A_0, \mathcal{M}_n) \subseteq A^+
\]

\[
(\zeta) \quad c^{\ell k}(B, \mathcal{M}_n) \subseteq D.
\]

1.10 Claim. 1) Assume

\[
(a) \quad (A^+, A, B, D) \text{ is semi-(} k, r \text{-)good}
\]

\[
(b) \quad A \leq A^+_1 \leq A^+
\]

\[
(c) \quad B_1 \leq B, A \leq D_1 \leq D, B_1 \leq D_1
\]

\[
(d) \quad c^m(A, A^+) \subseteq A^+_1 \text{ (follows from (b))}
\]

\[
(e) \quad c^{\ell k}(B_1, D) \subseteq D_1.
\]
Then \((A_1^+, A, B_1, D_1)\) is semi-\((k, r)\)-good.

2) If \((a)\) of part (1) and \(k' \leq k, r' \geq r\) and \(A^+ \subseteq A^* \in \mathcal{K}_\infty\) satisfies \(c^{r'}(A, A^*) \subseteq A^+\), then \((A^*, A, B, D)\) is semi-\((k', r')\)-good. (We can combine parts (1) and (2)).

3) If \((a)\) of part (1) and \(r, r_1, m\) satisfies the statement \((*)\) below and \(A_1 \leq A \leq A_1^+ \in \mathcal{K}_\infty\), \(A \subseteq c^{r}(A_1, A_1^+)\) and \(c^{r'}(A, A_1^+) \subseteq A^+\), then \((A_1^+, A_1, B, D)\) is semi-\((k, r_1)\)-good where

\[
(*) = \left( \right)_{m, r}^{r_1} \text{ if } A' \subseteq B' \subseteq C' \in \mathcal{K}_\infty, \ c^{rk}(A', A''') \subseteq B' \Rightarrow c^{rk}(A', C') = c^{rk}(A', B')
\]

so in our case (see assumption \((e)\))

\[
c^{rk}(g(B_1), \mathcal{M}_n) = c^{rk}(g(B_1), g(D)).
\]

As \(g\) embeds \(D\) into \(\mathcal{M}_n\), \(c^{rk}(g(B_1), g(D)) = g(c^{rk}(B_1, D))\), and by \((*)\) above and assumption \((e)\) we have \(c^{rk}(B_1, D) = c^{rk}(B_1, D_1)\). So together with earlier equality

\[
c^{rk}(g(B_1), \mathcal{M}_n) = g(c^{rk}(B_1, D_1))
\]

as required, that is \(g \upharpoonright D_1\) is as required.

2) Easier (by 1.6(14), clause (e)).

3) Let \(\mathcal{M}_n\) be random enough and \(f : A_1^+ \rightarrow \mathcal{M}_n\) be such that \(c^{rk}(f(A_1), \mathcal{M}_n) \subseteq f(A_1^+)\). We know \(A \subseteq c^{rk}(A_1, A_1^+)\). Now by the assumption on \(r_1, m, r\) for every \(A', A_1^+ \leq A' \in \mathcal{K}_\infty\) we have \(c^{rk}(A_1, A') \subseteq c^{rk}(A, A')\) hence \(c^{rk}(f(A), \mathcal{M}_n) \subseteq c^{rk}(f(A_1), \mathcal{M}_n) \subseteq f(A_1^+)\). So we can apply the property “\((A^+, A, B, D)\) is semi-\((k, r)\)-nice”.

\(\square\)_1.10

1.11 Claim. In the Definition of semi-nice, \(1.9(2)\), we can equivalently omit \(\ell, m\) (just \(r\) suffice) and replace \((*)\) by

\[
(*)' \text{ for every random enough } \mathcal{M}_n \text{ and } f : A \rightarrow \mathcal{M}_n \text{ and } b \in \mathcal{M}_n \text{ letting } B = A \cup \{b\} \text{ we have:}
\]

\[
(c^{rk}(f(A), \mathcal{M}_n), f(A), B, f(A) \cup c^{rk}(B, \mathcal{M}_n)) \text{ is semi-}\((k, r)\)-good.
\]

Proof. Original definition implies new definition

Let \(\ell, m, r\) be as guaranteed by the original definition. Without loss of generality \(m \geq r\). For the new definition we choose \(r_1 > r, m\) such that \(A' \leq B' \in \mathcal{K}_\infty\Rightarrow c^{rk}(c^{rk}(A', B'), B') \subseteq c^{rk}(A', B')\), they exist by 1.6(14)(e),(g). Let us check \((*)'\) of the new definition, so \(\mathcal{M}_n\) is random enough and \(f : A \rightarrow \mathcal{M}_n\). So by
the old definition there are \(A_0, A^+, B, D\) satisfying \((\alpha) - (\zeta)\) of \((\ast)\). In particular \((A^+, A_0, B, D)\) is semi-\((k, r)\)-good. As \(A^+ \leq c\ell^m(f(A), \mathcal{M}_n)\) and \(f(A) \leq A_0\) by 1.10(3) also the quadruple \((c\ell^r(f(A), \mathcal{M}_n), f(A), B, D)\) is semi-\((k, r_1)\)-good. (Remember \(r_1 \geq m, r\).) Let \(B_1 = f(A) \cup \{b\}, D_1 = f(A) \cup c\ell^k(B_1, \mathcal{M}_n)\). Now apply 1.10(1) and get that \((c\ell^r(f(A), \mathcal{M}_n), f(A), f(A) \cup \{b\}, f(A) \cup c\ell^k(f(A) \cup \{b\}, \mathcal{M}_n))\) is semi-\((k, r_1)\)-good as required.

**New definition implies old definition**

Immediately, letting \(m = r\) in \((\ast)\) of 1.9(2) let \(A_0 = f(A), A^+ = c\ell^r(f(A), \mathcal{M}_n), B = f(A) \cup \{b\}\) and \(D = c\ell^k(f(A), \mathcal{M}_n)\). What about \(\ell\)? It exists by 1.6(13).

**1.12 Conclusion.** The definition of semi-nice is equivalent to:

for every \(k\) and \(\ell\) for some \(r\) we have

\[(\ast)^n\] If \(A \in \mathcal{X}_\infty, |A| \leq \ell\) and \(\mathcal{M}_n\) is random enough and \(f : A \to \mathcal{M}_n\) and \(b \in \mathcal{M}_n\) then \((c\ell^r(f(A), \mathcal{M}_n), f(A), f(A) \cup \{b\}, c\ell^k(f(A) \cup \{b\}, \mathcal{M}_n))\) is semi-\((k, r)\)-good.

**Proof.** \(\text{old} \Rightarrow \text{new}\)

Let \(\{A_i : i < i^*\} \subseteq \mathcal{X}_\infty\) list the \(A \in \mathcal{X}_\infty\) with \(\leq \ell\) elements up to isomorphism. For each \(A_i\) there is \(r_i \in \mathbb{N}\) as guaranteed in 1.11. Let \(r = \text{Max}\{r_i : i < i^*\}\).

So let \(A \in \mathcal{X}_\infty, |A| \leq \ell\) be given so for some \(i, A \cong A_i\); if \(\mathcal{M}_n\) is random enough and \(f : A \to \mathcal{M}_n\) and \(b \in \mathcal{M}_n\) and \(B = f(A) \cup \{b\}\), then \((c\ell^r(f(A), \mathcal{M}_n), f(A), B, c\ell^k(B, \mathcal{M}_n))\) is semi-\((k, r_i)\)-good.

(Why? By the choice of \(r_i\).) Now by 1.10(2) as \(r_i \leq r\) we know that \((c\ell^r(f(A), \mathcal{M}_n), f(A), B, f(A) \cup c\ell^k(B, \mathcal{M}_n))\) is semi-\((k, r)\)-good, as required because \(f(A) \leq c\ell^k(B, \mathcal{M}_n)\).

**New \(\Rightarrow\) old**

Easier (and not used).

**1.13 Definition.** Let \(\mathcal{R}\) be a 0-1 context.

1) \(\mathcal{R}\) is complete \(\overline{\text{iff}}\) for every \(A \in \mathcal{X}\), the sequence

\[\langle \text{Prob}_{\mu_n}(A \text{ is embeddable into } \mathcal{M}_n) : n \in \mathbb{N} \rangle\]

converges to zero or converges to one.

2) \(\mathcal{R}\) is weakly complete \(\overline{\text{iff}}\) the sequence above converges.

3) \(\mathcal{R}\) is very weakly complete \(\overline{\text{iff}}\) for every \(A \in \mathcal{X}\), the sequence

\[\langle \text{Prob}_{\mu_n}(A \text{ embeddable into } \mathcal{M}_{n+1}) - \text{Prob}_{\mu_n}(A \text{ embeddable into } \mathcal{M}_n) : n < \omega \rangle\]

converges to zero.

So if \(h(n) = n + 1\), we get very weakly complete (similarly in 1.14(4)).

4) \(\mathcal{R}\) is \(h\)-very weakly complete \(\overline{\text{iff}}\) for every \(A \in \mathcal{X}\), the sequence

\[\langle \text{Lim}_n \text{ Max}_{n_1, n_2 \in \omega}[\text{Prob}_{\mu_{\alpha_n}}(A \text{ embeddable into } \mathcal{M}_{n_2}) - \text{Prob}_{\mu_{\alpha_n}}(A \text{ embeddable into } \mathcal{M}_{n_1})] : n < \omega \rangle\]

converges to zero.
1.14 Definition. Let $\mathcal{R}$ be a 0-1 context.
1) $\mathcal{R}$ satisfies the 0-1 law for the logic $\mathcal{L}$ if for every sentence $\varphi \in \mathcal{L}(\tau)$ (i.e. the logic $\mathcal{L}$ with vocabulary $\tau$) the sequence
\[
\langle \text{Prob}_{\mu_n}(\mathcal{M}_n \models \varphi) : n \in \mathbb{N} \rangle
\]
converges to zero or converges to one.
2) $\mathcal{R}$ satisfies the weak 0-1 law or convergence law for the logic $\mathcal{L}$ if for every sentence $\varphi \in \mathcal{L}(\tau)$, the sequence
\[
\langle \text{Prob}_{\mu_n}(\mathcal{M}_n \models \varphi) : n \in \mathbb{N} \rangle
\]
converges.
3) $\mathcal{R}$ satisfies the very weak 0-1 law for $\mathcal{L}$ if for every sentence $\varphi \in \mathcal{L}(\tau)$ the sequence
\[
\langle \text{Prob}_{\mu_{n+1}}(\mathcal{M}_{n+1} \models \varphi) - \text{Prob}_{\mu_n}(\mathcal{M}_n \models \varphi) : n \in \mathbb{N} \rangle
\]
converges to zero.
4) $\mathcal{R}$ satisfies the $h$-very weak 0-1 law for $\mathcal{L}$ if for every sentence $\varphi \in \mathcal{L}(\tau)$, the sequence
\[
\langle \max\{n_1, n_2\} \leq n \leq n+h(n) \mid \text{Prob}_{\mu_{n_1}}(\mathcal{M}_{n_1} \models \varphi) - \text{Prob}_{\mu_{n_2}}(\mathcal{M}_{n_2} \models \varphi) : n \in \mathbb{N} \rangle
\]
converge to zero.
5) If the logic $\mathcal{L}$ is first order logic, we may omit it.

1.15 Fact
1) If $\mathcal{R}$ is complete, then it is weakly complete which implies it is very weakly complete.
2) If $h_1, h_2$ are functions from $\mathbb{N}$ to $\mathbb{N}$ and $(\forall n)(h_1(n) \leq h_2(n))$ and $\mathcal{R}$ is $h_2$-very weakly complete, then $\mathcal{R}$ is $h_1$-very weakly complete.
3) Similarly for 0-1 laws.

1.16 Lemma. 1) Assume $\mathcal{R}$ is semi-nice. Modulo the theory $T_{\infty}$, every formula of the form $\psi(x_0, \ldots, x_{m-1})$ is equivalent to a Boolean combination of formulas of the form $(\exists x_m, \ldots, x_{k-1}) \varphi(x_0, \ldots, x_{m-1}, x_m, \ldots, x_{k-1})$, where for some $A \leq_i B \in \mathcal{H}_\infty$ we have $A = \{a_\ell : \ell < m\}$, $B = \{a_\ell : \ell < k\}$ (so $m \leq k$) and
\[
\varphi = \bigwedge\left\{R(\ldots, x_\ell, \ldots, \ell < k) : B \models R(\ldots, a_\ell, \ldots) \ell < k, R \text{ an atomic or negation of atomic formula}\right\}.
\]

1A) Another way of saying it, is: there is $k$ computable from $\psi$ such that: for every random enough $\mathcal{M}_n$ and $a_0, \ldots, a_{m-1} \in M_n$, the truth value of
\( \mathcal{M}_n \models \psi(a_0, \ldots, a_{m-1}) \) is computable from
\( (\mathcal{M}_n \models \text{cl}^k(a_0, \ldots, a_{m-1}), \mathcal{M}_n), a_0, \ldots, a_{m-1})/ \equiv. \)

2) If \( \mathcal{R} \) is semi-nice and weakly complete, then the weak 0-1 holds (i.e. convergence see Definition 1.14(2)).

3) If \( \mathcal{R} \) is semi-nice and complete (see Definition 1.13) then \( T_{\infty} \) is a complete theory;
and \( \mathcal{R} \) satisfies the 0-1 law for first order sentences (see 1.14(1)).

4) If \( T_{\infty} \) is a complete theory, then \( \mathcal{R} \) is complete.

5) The parallel of 2), 3) holds for h-very weak.

Proof. 1) By (1A).

1A) We prove it by induction on the quantifier depth of \( \psi \). For \( \psi \) atomic, or a conjunction or a disjunction or a negation this should be clear. So assume
\( \psi(x_0, \ldots, x_{s-1}) = (\exists x_s)\varphi(x_0, \ldots, x_s) \), by the induction hypothesis there is a function \( F_\varphi \) and number \( k_\varphi \) such that:

\((*)_{\varphi,F_\varphi} \text{ for every random enough } \mathcal{M}_n \text{ for every } a_0, \ldots, a_s \in \mathcal{M}_n \text{ we have: the truth value of } \mathcal{M}_n \models \varphi(a_0, \ldots, a_s) \text{ is } \]
\( F_\varphi((\mathcal{M}_n \models \text{cl}^{k_\varphi}(\{a_0, \ldots, a_s\}, \mathcal{M}_n), a_0, \ldots, a_s)/ \equiv) \).

By Definition 1.9(2) (of semi-nice) for any \( A \in \mathcal{K}_\infty \) and \( k \) there are \( \ell = \ell(A, k) \) and \( m = m(A, k) \) and \( r = r(A, k) \) as there. Let \( m^* = \max\{m(A, k_\varphi) : A \in \mathcal{K}_\infty \} \) and \( \ell^* = \max\{\ell(A, k_\varphi) : A \in \mathcal{K}_\infty \} \) and \( |A| \leq s + 1 \) and see below
\((*)_5(ii) \) and \((*)_6(ii) \) and let \( r^* = \max\{r(A, k_\varphi) : A \in \mathcal{K}_\infty \} \) and \( |A| \leq s + 1 \).

Now for \( \mathcal{M}_n \) random enough, for any \( a_0, \ldots, a_{s-1} \in \mathcal{M}_n \), we shall prove a sequence of conditions one implying the next (usually also the inverse), then close the circle thus proving they are all equivalent:

\( (*)_1 \mathcal{M}_n \models \psi(a_0, \ldots, a_{s-1}) \)
\( (*)_2 \mathcal{M}_n \models (\exists x_s)\varphi(a_0, \ldots, a_{s-1}, x_s). \)

[Clearly \( (*)_1 \iff (*)_2 \)]

\( (*)_3 \) for some \( b \in \mathcal{M}_n \) we have \( \mathcal{M}_n \models \varphi(a_0, \ldots, a_{s-1}, b). \)

[Clearly \( (*)_2 \iff (*)_3 \)]

\( (*)_4(i) \) for some \( b \in \text{cl}^{m^*}(\{a_0, \ldots, a_{s-1}\}, \mathcal{M}_n) \) we have
\( \mathcal{M}_n \models \varphi(a_0, \ldots, a_{s-1}, b) \)
or
\( (ii) \) for some \( b \in \mathcal{M}_n \setminus \text{cl}^{m^*}(\{a_0, \ldots, a_{s-1}\}, \mathcal{M}_n) \) we have \( \mathcal{M}_n \models \varphi(a_0, \ldots, a_{s-1}, b). \)

[Clearly \( (*)_3 \iff (*)_4 \)]

\( (*)_5(i) \) letting \( N = \mathcal{M}_n \models \text{cl}^k(\{a_0, \ldots, a_{s-1}\}, \mathcal{M}_n) \) the following holds:
for some \( b \in \text{cl}^{m^*}(\{a_0, \ldots, a_{s-1}\}, N) = \text{cl}^{m^*}(\{a_0, \ldots, a_{s-1}\}, \mathcal{M}_n) \) we have
\( \text{truth} = F_\varphi((N \models \text{cl}^{k_\varphi}(\{a_0, \ldots, a_{s-1}, b\}, N), a_0, \ldots, a_{s-1}, b)/ \equiv). \)
Clearly $(*)_4(i) \leftrightarrow (*)_5(i)$ by the choice of $k$ as: $A \subseteq N \subseteq M \& c\ell^t(A, M) \subseteq N \Rightarrow c\ell^t(A, N) = c\ell^t(A, M)$, by $1.6(13)(c)$ and the induction hypothesis.

$(*)_5(ii)$ letting $N = c\ell^t(\{a_0, \ldots, a_{s-1}\}, \mathcal{M}_n)$ and $A = \{a_0, \ldots, a_{s-1}\}$ we have: for some $A_0, A^+$ we have $A \leq A_0 \leq A^+ \subseteq c\ell^m(A,k_\varphi)(\{a_0, \ldots, a_{s-1}\}, N)$, and $c\ell^r(A_k, \mathcal{M}_n) \subseteq A^+$ and there are $B^+, b$ such that $c\ell^k(\mathcal{A}_0 \cup \{b\}, \mathcal{M}_n) \subseteq B^+ \subseteq \mathcal{M}_n, |B^+| \leq \ell^t$ and $(A^+, A_0, A \cup \{b\}, B^+)$ is semi-$(k_\varphi, r(A, k_\varphi))$-good and $M \models \varphi(a_0, \ldots, a_{s-1}, b)$.

Clearly by the induction hypothesis $(*)_5(ii) \Rightarrow (*)_6(ii)$.

Lastly $(*)_6(ii) \Rightarrow (*)_3$ by Definition $1.9(1)$ + the induction hypothesis thus we have equivalence. So $(*_1) \leftrightarrow [(*)_5(i) \lor (*)_6(ii)]$, but the two later ones depend just on $\mathcal{M}_n \upharpoonright c\ell^t(\{a_0, \ldots, a_{s-1}\}, \mathcal{M}_n), a_0, \ldots, a_{s-1}) / \cong$, thus we have finished.

2) By $1.16(1)$ it is enough to prove that the sequence

$$\langle \text{Prob}_{\mu_n}(A \text{ is embeddable into } \mathcal{M}_n) : n < \omega \rangle$$

converge. This holds by weak completeness.

3),4),5) Left to the reader.

\[1.16\]


1.17 Remark. Note: If $\mathcal{R}$ is complete, then $T_\infty$ has a unique (up to isomorphisms) countable model $M$ such that for some $\langle A_n : n \in \mathbb{N} \rangle$ we have: $M = \bigcup_{n \in \mathbb{N}} A_n, A_n <_s A_{n+1} \in \mathcal{X}_\infty$ and every $A \in \mathcal{X}_\infty$ can be embedded into some $A_n$, and if $n \in \mathbb{N}, A \leq A_n, A \leq_\mathcal{R} B$, then for some $m$ there is an embedding $f$ of $B$ into $A_n$ such that $f \upharpoonright A = \text{id}_A$ and $f(B) \leq_\mathcal{R} A_m$ (see Baldwin, Shelah [BlSh 528], not used).

1.18 Claim. 1) A sufficient condition for “$\mathcal{R}$ is weakly nice” is

$$(*) \text{ for every } A < B, \text{ if } \neg(A \leq_\mathcal{R} B) \text{ then for some } k < \omega \text{ we have }$$

$$1 = \lim_n \text{Prob}_{\mu_n}(\text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n \text{ there are no embedding } g_\ell : B \to \mathcal{M}_n \text{ extending } f \text{ for } \ell < k, \text{ disjoint over } f).$$

Proof. Easy.
§2 More accurate measure and drawing again

We define when a pair of functions \( \bar{h} = (h^d, h^u) \) giving, up to a factor, the values \( nu(f, A, B, M) \) for \( M \) random enough, \( A <_{\text{pr}} B \) and \( f : A \to M \). We also define when \( \hat{h} \) obeys \( h \) (i.e. \( h \) bounds the error factor). From \( h_{A,B} \) for \( A <_{\text{pr}} B \) we define \( h_{A,B} \) also for the case \( A <_{s} B \) and then define a good case when the functions are polynomial in \( \|\mathcal{M}_n\| \) (see Definition 2.1, 2.3).

We then see how large is the factor error for the derived cases and deduce some natural properties (in 2.4).

Then we deal with the polynomial case.

Lastly, (2.10-2.15) we introduce our framework for adding random relations to random \( \mathcal{M}_n \). Reading, you may assume “every \( A \in K_\infty \) is embeddable into every random enough \( \mathcal{M}_n \)”.

2.1 Definition. 1) We say the 0–1 context \( \mathcal{R} \) obeys \( \bar{h} = (h^d, h^u) \) with error \( h^e \) where \( d \) is for down, \( u \) is for up and \( e \) is for error if:

(a) for \( A <_{\text{pr}} B \) we have \( h^d_{A,B} \) and \( h^u_{A,B} \) and \( h^e \) are functions from \( \bigcup_{n<\omega} \mathcal{K}_n \) to \( \mathbb{R}^\geq 0 \)

(b) for some \( \varepsilon \in \mathbb{R}^>0 \) for every random enough \( \mathcal{M}_n \) we have
\[
(h^e[\mathcal{M}_n])^\varepsilon \leq h^d_{A,B}[\mathcal{M}_n] \leq h^u_{A,B}[\mathcal{M}_n] \text{ and } h^e[\mathcal{M}_n] \geq 1
\]

(c) for every \( \varepsilon \in \mathbb{R}^>0 \) we have
\[
1 = \lim_n \text{Prob}_{\mu_n}( \text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n,
\text{ we have } h^d_{A,B}[\mathcal{M}_n] \cdot (h^e[\mathcal{M}_n])^{-\varepsilon} \leq nu(f, A, B, \mathcal{M}_n) \leq h^u_{A,B}[\mathcal{M}_n] \cdot (h^e[\mathcal{M}_n])^\varepsilon
\)
(see 0.3(2)).

1A) If \( h^e \) is identically 1 we may omit it. If \( h^u = h^d \), then we may write \( h^u \) instead of \( h \). If \( h^e[\mathcal{M}_n] = \|\mathcal{M}_n\| \) we may say “simply”.

2) We say \( \bar{h} \) is uniform if \( h^x_{A,B}[M] \) (for \( x \in \{u, d\} \)) depends on \( \|M\| \) (and \( x, A, B \)) only but not on \( M \) and then write \( h^x_{A,B}([\mathcal{M}_n]) = h^x_{A,B}([1\mathcal{M}_n]) \). Similarly for \( h^e \), \( h \) used above and below. We say \( h \) goes to infinity if for every \( m \) for every random enough \( \mathcal{M}_n, h[\mathcal{M}_n] > m \).

3) We say that \( \bar{h} \) is bounded (or bounded +) by \( h \) (for \( \bar{h} \) as above) if:

(a) \( h \) is a function from \( \bigcup_{n\in\mathbb{N}} \mathcal{K}_n \) to \( \mathbb{R}^+ \) (remember that the \( \mathcal{K}_n \)’s are pairwise disjoint)

(b) for every random enough \( \mathcal{M}_n \) we have \( h[\mathcal{M}_n] \geq 1 \)

(c) for every \( \varepsilon > 0 \) and \( A <_{\text{pr}} B \) for every random enough \( \mathcal{M}_n \) we have
\[
1 \leq h^u_{A,B}[\mathcal{M}_n]/h^d_{A,B}[\mathcal{M}_n] \leq (h[\mathcal{M}_n])^\varepsilon
\]

(d) for every \( A <_{\text{pr}} B \) and \( m \in \mathbb{N}\setminus\{0\} \) for some \( \varepsilon > 0 \) for every random enough \( \mathcal{M}_n \) we have
\[ h_{A,B}^d[\mathcal{M}_n] > (h[\mathcal{M}_n])^\varepsilon \times m \]

(e) for every \( A \in \mathcal{X}_\infty \) for every \( \varepsilon > 0 \) for every random enough \( \mathcal{M}_n \) we have

\[
\text{Prob}_{\mu_n}(A \text{ is embeddable into } \mathcal{M}_n) \geq 1/(h[\mathcal{M}_n])^\varepsilon.
\]

3A) In the context of (3), let “\( \mathcal{M}_n \) random enough” mean that for every \( \varepsilon \), the probability of failure is \( \leq 1/(h[\mathcal{M}_n])^\varepsilon \). We say \( h \) is standard if for each \( m \), for every random enough \( \mathcal{M}_n \), \( h[\mathcal{M}_n] > m \).

3B) We\(^3\) say \( \bar{h} \) is bounded by \( h \) (for \( \bar{h} \)) as above if: clauses (a), (b) as above, but in clauses (c),(e) we replace “every \( \varepsilon > 0 \)” by “for some \( m = m(A, B) \in \mathbb{N} \)” in clause (d) we replace “some \( \varepsilon > 0 \)” by “every \( m \in \mathbb{N} \)”, i.e.

(a) \( h \) is a function from \( \bigcup_{n \in \mathbb{N}} \mathcal{X}_n \) to \( \mathbb{R}^+ \) (remember that the \( \mathcal{X}_n \)’s are pairwise disjoint)

(b) for every random enough \( \mathcal{M}_n \) we have \( h[\mathcal{M}_n] \geq 1 \)

(c) for every \( A <_{pr} B \), for some \( m = m(A, B) \in \mathbb{N} \setminus \{0\} \) for every random enough \( \mathcal{M}_n \) we have

\[ 1 \leq h_{A,B}^d[\mathcal{M}_n]/h_{A,B}^d[\mathcal{M}_n] \leq (h[\mathcal{M}_n])^m \]

(d) for every \( A <_{pr} B \) and \( m \in \mathbb{N} \) for some \( m(A, B) \in \mathbb{N} \setminus \{0\} \) for every random enough \( \mathcal{M}_n \) we have

\[ h_{A,B}^d[\mathcal{M}_n] > (h[\mathcal{M}_n])^{m(A, B)}, m \]

(e) for every \( A \in \mathcal{X}_\infty \) for some \( m(A) \in \mathbb{N} \setminus \{0\} \) for every random enough \( \mathcal{M}_n \) we have

\[
\text{Prob}_{\mu_n}(A \text{ is embeddable into } \mathcal{M}_n) \geq 1/(h[\mathcal{M}_n])^m
\]

(part (3B) is an alternative to 2.1(3)).

4) Assume \( \mathfrak{K} \) obeys \( \bar{h} \). For \( A <_{s} B \) and \( M \in \bigcup_{n} \mathcal{X}_n \) we let

\[ h_{A,B}^{+n}[M] =: \max \left\{ \prod_{\ell < h} h_{A_{\ell},A_{\ell+1}}^n[M] : \bar{A} = \{A_{\ell} : \ell \leq k\} \text{ is a decomposition of } (A, B) \right\} \]

\(^2\)this, of course, will not suffice for 0-1 law

\(^3\)this is an alternative to part (3), this does not matter really so we shall use the one of part (3), the same applies to other cases

\(^4\)this, of course, will not suffice for 0-1 law, and though more natural, we shall not follow it here
\[ h_{A,B}^u[M] = \min \left\{ \prod_{\ell<h} h_{A\ell,A\ell+1}^u[M] : \bar{A} = \langle A_\ell : \ell \leq k \rangle \right\} \]

is a decomposition of \((A, B)\)

\[ h_{A,B}^d[M] = \min \left\{ \prod_{\ell<h} h_{A\ell,A\ell+1}^d[M] : \bar{A} = \langle A_\ell : \ell \leq k \rangle \right\} \]

is a decomposition of \((A, B)\)

\[ h_{A,B}^{+d}[M] = \max \left\{ \prod_{\ell<h} h_{A\ell,A\ell+1}^{d^+}[M] : \bar{A} = \langle A_\ell : \ell \leq k \rangle \right\} \]

is a decomposition of \((A, B)\)



Let \( h_{A,B}^u[M] = h_{A,B}^{-u}[M] \) and \( h_{A,B}^d[M] = h_{A,B}^{+d}(M) \).

2.2 Discussion. For the semi-nice case, we may consider it natural to have the functions \( h \) below be \( h_{X^+,A,B,D}^+ \) giving information on \( \nu^k(f, A, B, D, M) = |\text{ex}^k(f, A, B, D, M)| \) where \( \text{ex}^k(f, A, B, D, M) = \{ g \mid B : g \) embeds \( D \) into \( M \), it extends \( f \) (which embeds \( A \) into \( M \)) and \( c\ell^k(g(B), M) \subseteq g(D) \} \) and we restrict ourselves to the case that there is an embedding \( f^+ \) of \( A^+ \) into \( M \) extending \( f \) such that \( c\ell^r(f(A), M) \subseteq f(A^+) \). So we may write \( h_{X^+,A,B,D}^{+k,r} \) and \( \text{ex}^{k,r}(f, A, A^+, B, D, M) \). Note that the variables here of \( \text{ex}, \nu \) are different than in the usual case.

2.3 Definition. 1) We say \( \mathcal{R} \) obeys the polynomial \( \tilde{h} \) over (or modulo) \( h \) if \( \tilde{h} = \langle h^u, h^d \rangle \) and \( h^u = h^d \), \( h \) are functions from \( \bigcup_n \mathcal{X}_n \) to \( \mathbb{R}^{\geq 0} \) and \( h^u, h \) are uniform (see Definition 2.1(2)) and for every \( A <_{pr} B \) a real \( \alpha(A, B) \in \mathbb{R}^{>0} \) is well defined and we have:

(a) \( h : \bigcup_n \mathcal{X}_n \rightarrow \mathbb{R}^+ \)

(b) for every \( m \), for random enough \( \mathcal{M}_n \) we have \( h[\mathcal{M}_n] > 1 \) and \( ||\mathcal{M}_n|| \geq m \)

(c) for every \( \varepsilon > 0 \) for random enough \( \mathcal{M}_n \) we have \( h[\mathcal{M}_n] < ||\mathcal{M}||^\varepsilon \)

(d) if \( A <_{pr} B \) and \( m \in \mathbb{N} \), then for every \( \mathcal{M}_n \) random enough\(^5\)

\[ h_{A,B}^d[\mathcal{M}_n] = h_{A,B}^u[\mathcal{M}_n] = ||\mathcal{M}_n||^{\alpha(A,B)} ; \text{ and if } f \text{ embeds } A \text{ into } \mathcal{M}_n \text{ then } h[\mathcal{M}_n]^{-m} h_{A,B}^d[\mathcal{M}_n] \leq \nu(f, A, B, \mathcal{M}_n) \leq h[\mathcal{M}_n]^{m} h_{A,B}^u[\mathcal{M}_n] \]

(e) for some \( \varepsilon > 0 \) for every \( \mathcal{M}_n \) random enough \( ||\mathcal{M}_n|| > m \times (h[\mathcal{M}_n])^\varepsilon \)

(f) if \( A \in \mathcal{X}_\infty \) then for each \( k \) for some \( m \)

\( \text{Prob}_{\mu_n}(A \text{ is embeddable into } \mathcal{M}_n \text{ assuming } ||A|| = k) \geq 1/h[\mathcal{M}_n]^{m} \).

\(^5\) note this is not as in 2.1(3)(c)
2) We say \( \bar{h} \) is strictly polynomial if

\[
\text{(a) if } A <_{pr} B \text{ then for some } c = c(A, B) \in \mathbb{R}^+ \text{ for some } \varepsilon > 0, \text{ for every random enough } \mathcal{M}_n \text{ and every } f : A \to \mathcal{M}_n \text{ we have }
\]

\[
c(A, B)\|\mathcal{M}_n\|^{\alpha(A, B)}(1 - \|\mathcal{M}_n\|^{-\varepsilon}) \leq h^{d}_{A, B}(\mathcal{M}_n) \leq h^{u}_{A, B}(\mathcal{M}_n) \leq c(A, B)\|\mathcal{M}_n\|^{\alpha(A, B)}(1 + \|\mathcal{M}_n\|^{-\varepsilon}).
\]

3) We say \( \bar{h} \) is a polynomial if \( \bar{h} \) is polynomial over some \( h \). We say \( \bar{R} \) is polynomial over \( h \) (strictly polynomial), if this holds for some \( \bar{h} \).

2.4 Fact. 1) Assume \( \bar{R} \) obeys \( \bar{h} \) with error \( h^\varepsilon \). If \( A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_k \) and \( \varepsilon > 0 \), then every random enough \( \mathcal{M}_n \) satisfies:

\[
(*) \text{ for every embedding } f \text{ of } A_0 \text{ into } \mathcal{M}_n,
\]

\[
\prod_{\ell < k} h^{d}_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \times (h^\varepsilon[\mathcal{M}_n])^{-\varepsilon} \leq nu(f, A_0, A_1, \cdots, \mathcal{M}_n)
\]

\[
\leq \prod_{\ell < k} h^{u}_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \times (h^\varepsilon[\mathcal{M}_n])^\varepsilon.
\]

2) Assume \( \bar{R} \) obeys \( \bar{h} \) with error \( h^\varepsilon \), and for clause (\( \delta \)) assume also that \( \bar{h} \) is bounded by \( h \) and \( h \geq h^\varepsilon \).

(\( \alpha \)) if \( A \leq_{s} B \), then for every random enough \( \mathcal{M}_n \) we have\(^6\)

\[
h^{-d}_{A, B}[\mathcal{M}_n] \leq h^{+d}_{A, B}[\mathcal{M}_n] = h^{d}_{A, B}[\mathcal{M}_n] \leq h^{u}_{A, B}[\mathcal{M}_n] = h^{-u}_{A, B}[\mathcal{M}_n] \leq h^{+u}_{A, B}[\mathcal{M}_n]
\]

(\( \beta \)) if \( A <_{s} B \) and \( \varepsilon \in \mathbb{R}^+ \), then for every random enough \( \mathcal{M}_n \) and embedding \( f : A \to \mathcal{M}_n \) we have:

\[
h^{d}_{A, B}[\mathcal{M}_n] \times (h^\varepsilon[\mathcal{M}_n])^{-\varepsilon} \leq nu(f, A, B, \mathcal{M}_n) \leq h^{u}_{A, B}[\mathcal{M}_n] \times (h^\varepsilon[\mathcal{M}_n])^\varepsilon
\]

(\( \gamma \)) if \( A <_{pr} B \), then

\[
h^{+d}_{A, B}[M] = h^{-d}_{A, B}[M] = h^{d}_{A, B}[M] = h^{u}_{A, B}[M]
\]

(\( \delta \)) if \( A <_{s} B \) then for every \( \varepsilon > 0 \), for every random enough \( \mathcal{M}_n \) we have:

\[
(h[\mathcal{M}_n])^{-\varepsilon} \leq h^{u}_{A, B}[\mathcal{M}_n]/h^{d}_{A, B}[\mathcal{M}_n] \leq (h[\mathcal{M}_n])^\varepsilon., \text{ moreover}
\]

\[
(h[\mathcal{M}_n])^{-\varepsilon} \leq h^{u}_{A, B}[\mathcal{M}_n]/h^{-u}_{A, B}[\mathcal{M}_n] \leq (h[M])^\varepsilon
\]

(\( \varepsilon \)) if \( A_0 <_{s} A_1 <_{s} A_2 \) then for any random enough \( \mathcal{M}_n \):

\[
h^{d}_{A_0, A_1}[\mathcal{M}_n] \times h^{d}_{A_1, A_2}[\mathcal{M}_n] \leq h^{d}_{A_0, A_2}[\mathcal{M}_n] \leq h^{u}_{A_0, A_1}[\mathcal{M}_n] \times h^{u}_{A_1, A_2}[\mathcal{M}_n]
\]

\(^6\)if \( A \) is embeddable into \( \mathcal{M}_n \) of course as otherwise \( h^{+u}_{A, B}[\mathcal{M}_n] \) is not actually well defined, we tend to “forget” to state this.
Proof. 1) Easy by induction on $k$.
2) Clause (α):
The first and last inequality holds as $\text{Min}(X) \leq \text{Max}(X)$ for $X \subseteq \mathbb{R}$ finite non-empty (by 1.6(5)) as in this case. The equalities hold by Definition 2.3(1). The middle inequality holds by clause (β) below.

Clause (β):
By (∗) of 2.4(1).

Clause (γ):
As $A <_{pr} B$ implies $(A, B)$ has a unique decomposition.

Clause (δ):
Let $\bar{A} = \langle A_\ell : \ell \leq k \rangle$ be a decomposition of $(A, B)$ and $\varepsilon \in \mathbb{R}^{>0}$, hence for every random enough $\mathcal{M}_n$ for every embedding $f$ of $A$ into $M$

$$
\prod_{\ell < k} h^d_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \times (h[\mathcal{M}_n])^{-\varepsilon} \leq h^u_{A,B}[\mathcal{M}_n] \times (h[\mathcal{M}_n])^{-\varepsilon} \\
\leq nu(f, A, B, \mathcal{M}_n) \leq h^u_{A,B}[\mathcal{M}_n] \times (h[\mathcal{M}_n])^{\varepsilon} \\
\leq \prod_{\ell < k} h^u_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \times (h[\mathcal{M}_n])^{\varepsilon}.
$$

This gives the first inequality part of the inequalities. Let $\varepsilon > 0$ be given.

Now for each $\ell < k$ for every random enough $\mathcal{M}_n$ we have

$$
1 \leq h^u_{A_\ell, A_{\ell+1}}[\mathcal{M}_n]/h^d_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \leq (h[\mathcal{M}_n])^{\varepsilon/k}.
$$

Hence for every random enough $\mathcal{M}_n$ we have

$$
h^u_{A,B}[\mathcal{M}_n]/h^d_{A,B}[\mathcal{M}_n] \leq \left( \prod_{\ell < k} h^u_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \right) / \left( \prod_{\ell < k} h^d_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \right) \\
= \prod_{\ell < k} \left( h^u_{A_\ell, A_{\ell+1}}[\mathcal{M}_n]/h^d_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \right) \\
\leq \prod_{\ell < k} (h[\mathcal{M}_n])^{\varepsilon/k} = (h[\mathcal{M}_n])^{\varepsilon}.
$$

For the second phrase of clause (δ) (the moreover) note that for every random enough $\mathcal{M}_n$ for every $f : A \to \mathcal{M}_n$ we have: for some decomposition $\bar{A}$ of $(A, B)$

$$
1 \leq h^u_{A,B}[\mathcal{M}_n]/nu(f, A, B, \mathcal{M}_n) = \prod_{\ell < k} h^u_{A_\ell, A_{\ell+1}}[\mathcal{M}_n]/nu(f(A, B, \mathcal{M}_n)) \\
\leq \prod_{\ell < k} h^u_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] / \prod_{\ell < k} h^d_{A_\ell, A_{\ell+1}}[\mathcal{M}_n] \\
\leq ((h[\mathcal{M}_n])^{\varepsilon/k})^k = (h[\mathcal{M}_n])^{\varepsilon}
$$

and for possibly other decomposition $\bar{A}$ of $(A, B)$
1 \leq nu(f, A, B, \mathcal{M}_n)/h_{A,B}^{-d}[\mathcal{M}_n] = nu(f(A, B, \mathcal{M}_n))/\prod_{\ell<k} h_{A,\ell+1}^{d}[\mathcal{M}_n] \\
\leq \prod_{\ell<k} h_{A,\ell+1}^{n}[\mathcal{M}_n]/\prod_{\ell<k} h_{A,\ell+1}^{d}[\mathcal{M}_n] \\
\leq (h[\mathcal{M}_n]^{-\varepsilon/k})^k = h[\mathcal{M}_n]^{-\varepsilon}.

Together we get the desired inequality (well, for $2\varepsilon$).

Clause $(\varepsilon)$:
Easy (using 2.1(4)).

\[Q.E.D.\] _2.4

2.5 Claim. Assume the 0-1 law context $\mathfrak{R}$ obeys $\bar{h} = (h^d, h^u)$ with error $h^\varepsilon$.

1) A sufficient condition for "$\mathfrak{R}$ is weakly nice" is

\((\ast)_1\) if $A <_{pr} B$ and $m^* \in \mathbb{N}$ and $\varepsilon > \mathbb{R}^{>0}$ small enough, then for every random enough $\mathcal{M}_n$ we have $\bar{h}_{A,B}[\mathcal{M}_n] \times (h^\varepsilon[\mathcal{M}_n])^{-\varepsilon} > m^*$.

2) If $\bar{h}$ is bounded by $h$ or $\bar{h}$ is polynomial over $h$, then $(\ast)_1$ from above holds hence $\mathfrak{R}$ is weakly nice.

Proof. 1) Assume $A <_{pr} B$ and we show that $(\ast)_1$ of Definition 1.7 holds in this case.

If $|B \setminus A| = 1$, as $A <_{pr} B$, for every $m$, for every random enough $\mathcal{M}_n$, for any $f : A \to \mathcal{M}_n$, by $(\ast)_1$ of 2.5(1) there are distinct $g_1, \ldots, g_m$ which are embedding of $B$ into $\mathcal{M}_n$ extending $f$, now they are necessarily pairwise disjoint over $A$ so the demand in 1.3(1) holds.

So assume $|B \setminus A| > 1$ and $m^*$ be given. For each $b \in B \setminus A$ by 1.6(8) we know that $(A \cup \{b\}) \leq_i B$ but $|A \cup \{b\}| \leq |A| + 1 < |B|$ so $A \cup \{b\} <_i B$. Hence by Definition 1.3(2)(b) for some $n_b, m_b \in \mathbb{N}$ for every $n \geq n_b$ we have $1 - \varepsilon/(|B|+1) < \text{Prob}_{\mu_n}(\mathcal{E}_{n}^b)$ where $\mathcal{E}_{n}^b$ is the event:

for every embedding $f$ of $(A \cup \{b\})$ into $\mathcal{M}_n$, there are at most

$m_b$ extensions of $f$ to an embedding of $B$ into $\mathcal{M}_n$.

Let $m^{**} = |B \setminus A| \times |B \setminus A| \times (\max_{b \in B \setminus A} m_b) \times m^*$.

Also by $(\ast)_1$ above and clause (c) of 2.1(1) for some $n^{*} \in \mathbb{N}$ for every $n \geq n^*$ the event $\mathcal{E}_{n}^b \cap \mathcal{E}_{n}^{**}$ has probability $\geq 1 - \varepsilon/(|B|+1)$ where $\mathcal{E}_{n}^{**} = [h_{A,B}[\mathcal{M}_n] \leq nu(f, A, B, \mathcal{M}_n)]$ for every embedding $f : A \to \mathcal{M}_n$ and $\mathcal{E}_{n}^b = [h_{A,B}[\mathcal{M}_n] > (m^{**} + 1)]$.

Let $n^{\diamond} = \text{Max}\{n_b : b \in B \setminus A \cup \{n^{*}\}\}$.

Now suppose $n \geq n^{\diamond}$, then with probability $\geq 1 - \varepsilon$ all the events $\mathcal{E}_{n}^b$ for $b \in B \setminus A$ and $\mathcal{E}_{n}^{**} = [h_{A,B}[\mathcal{M}_n] > m^{**}]$ and $\mathcal{E}_{n}^{**}$ occurs for $\mathcal{M}_n$. It suffices to show that then $(\ast)_1$ of 1.7 occurs. So let $f$ be an embedding of $A$ into $\mathcal{M}_n$, so as both $\mathcal{E}_{n}^b$ and $\mathcal{E}_{n}^{**}$ occur necessarily there are distinct extension $g_1, \ldots, g_{m^{**}}$ of $f$ embedding $B$ into $\mathcal{M}_n$. For $i \in \{1, \ldots, m^{**}\}$ let $u_i = \{j : j \in \{1, \ldots, m^{**}\}$ and $\text{Rang}(g_j) \cap \text{Rang}(g_i) \neq \emptyset$.
Rang(f)}, and for \( b \in B \setminus A \) and \( c \in \mathcal{M}_n \setminus \text{Rang}(f) \) let \( v_{b,c} = \{ i : i \in \{1, \ldots, m^*\} \) and \( g_i(b) = c \). Now clearly \( |v_{b,c}| \leq m_b \) as the event \( \delta^*_b \) occurs and

\[
u_i = \bigcup_{b \in B \setminus A} \bigcup_{c \in \text{Rang}(g_i) \setminus \text{Rang}(f)} v_{b,c}
\]

hence

\[
|u_i| \leq |B \setminus A| \times |B \setminus A| \times \max_{b \in B \setminus A} m_b \leq m^{**}/m^*.
\]

So easily we can find \( w \subseteq \{1, \ldots, m^{**}\} \) such that \( |w| = m^* \) and \( i \in w \) & \( j \in w \) & \( i \neq j \Rightarrow j \notin u_i \). So \( \{ g_i : i \in w \} \) is as required.

2) Check (see in particular 2.1(3)(b),(d)). \( \Box_{2.5} \)

2.6 Claim. 1) Assume \( h \) obeys \( \tilde{h} \) with error \( h^\varepsilon \) and: \( A \leq_s B \leq D \) and \( A \leq C \leq_s D \) and \( D = B \cup C \). For every \( \varepsilon \in \mathbb{R}^>0 \), for every random enough \( \mathcal{M}_n \), if \( C \) is embeddable into \( \mathcal{M}_n \), then

\[
h_{A,B}^u[A_n] \geq h_{C,D}^d[A_n] \times (h^\varepsilon[A_n])^{-\varepsilon}.
\]

2) If in addition, \( \tilde{h} \) is bounded by \( h \) and \( h \geq h^\varepsilon \), then for every \( \varepsilon > 0 \), for every random enough \( \mathcal{M}_n \) and \( x \in \{u,d\} \)

\[
h_{A,B}^x[A_n] \geq h_{C,D}^e[D_n] \times (h[A_n])^{\varepsilon}
\]

3) If \( A_0 \leq A_1 \leq \ldots \leq A_k, A_0 \leq S A_k \) and \( \varepsilon \in \mathbb{R}^>0 \), then for every \( \mathcal{M}_n \) random enough into which \( A_{k-1} \) is embeddable

\[
h_{A_0,A_k}^d[A_n] \leq \Pi \{h_{A_i,A_{i+1}}^u[A_n] : A_i \leq_s A_{i+1}\} \times (h^\varepsilon[A_n])^{\varepsilon}.
\]

2.7 Definition. 1) We say that \((\bar{h}, \tilde{h}, h^\varepsilon)\) is semi-nice if

\( (a) \) \( \bar{h} \) is a 0-1 context

\( (b) \) \( \bar{h} \) obeys \( \tilde{h} \) with error \( h^\varepsilon \)

\( (c) \) for every \( A \in \mathcal{X}_\infty \) and \( k \) for some \( r \) we have:

\[
(\ast) \text{ for every random enough } \mathcal{M}_n, \text{ and embedding } f : A \rightarrow \mathcal{M}_n \text{ and } b \in \mathcal{M}_n, (cl^r(f(A), \mathcal{M}_n), f(A), f(A) \cup \{b\}, cl^k(f(A) \cup \{b\}, \mathcal{M}_n)) \text{ is semi}^* - (k, r)-\text{good for } (\bar{h}, \tilde{h}, h^\varepsilon), \text{ see below.}
\]

\( (d) \) Condition \((\ast)_1\) of 2.5(1) holds.

2) \((A^*, A, B, D)\) is semi\(^*\) - \((k, r)\)-good for \((\bar{h}, \tilde{h}, h^\varepsilon)\) if\(^7\) for some \( A_0 \) we have:

\( (a) \) \( A \leq A_1 \leq A^* \in \mathcal{X}_\infty, A_0 \leq_s D \in \mathcal{X}_\infty, B \leq D \in K_\infty \) and

\( (\beta) \) for every \( \varepsilon > 0 \), for every random enough \( \mathcal{M}_n \), for every embedding \( f^* \) of \( A^* \) into \( \mathcal{M}_n \) satisfying \( cl^r(f^*(A), \mathcal{M}_n) \subseteq f^*(A^*) \), we have, letting \( f = f^* \mid A_1 \)

\( \text{note that because of } A \leq_s D, \text{ this does not copy the definition in §1 even in “nice” cases} \)
the inequality 
\[ (h^e[\mathcal{M}_n])^{-1} \times h^d_{A,D}[\mathcal{M}_n] \leq nu^k(f, f(A_0), B, D, \mathcal{M}_n) \leq h^e_{A,D}[\mathcal{M}_n] \times (h^e[\mathcal{M}_n])^e \]
(on nu^k see below).

2A) We say \((A^*, A, B, D)\) is semi**-nice-(\(k, r\))-good for \((\mathfrak{R}, \bar{h}, h)\) if: \(A_1 = A^*\) in part (2).
3) If \(A \leq_s D, B \leq D, k \in \mathbb{N}\) and \(f : A \rightarrow \mathcal{M}\) we let \(nu^k(f, A, B, D, \mathcal{M}) = |ex^k(f, A, B, D, \mathcal{M})|\) where

\[ ex^k(f, A, B, D, \mathcal{M}) = \{ g : g \text{ is an embedding of } D \text{ into } M \]

extending \(f\) and satisfying \(cl^k(g(B), M) = g(cl^k(B, D))\).

4) We say that \((\mathfrak{R}, \bar{h}, h)\) is polynomially semi-nice if a), b), c), d) of part (1) holds and

\((e) \ \bar{h}\) is polynomial over \(h^e\).

We can list some obvious implications.

2.8 Claim. 1) Assume \((\mathfrak{R}, \bar{h}, h^e)\) is semi-nice and \(h^e\) goes to infinity and \((A_1, A, B, D)\) is semi** - (\(k, r\))-good for \((\mathfrak{R}, \bar{h}, h)\). Then we can find \(B', D', g^*\) such that

\(a) \ A \leq A_1 \leq_s D'\) and \(B' \leq D' \in \mathcal{K}_\infty \)
\(b) \ g^*\) is an embedding of \(D\) into \(D'\)
\(c) \ B' = g^*(B), D' = A_1 \cup g^*(D)\) and \(cl^k(g^*(B), D') = g^*(cl^k(B, D))\)
\(d) \) for every random enough \(\mathcal{M}_n\) and \(f : A_1 \rightarrow \mathcal{M}_n\) satisfying \(cl^r (f(A_1), \mathcal{M}_n) \subseteq f(A_1)\) there is \(g' : D' \rightarrow \mathcal{M}_n\) extending \(f\) such that \(cl^k (g'(B'), \mathcal{M}_n) = g'(cl^k(B', D'))\), that is \((A_1, A, B', D')\) is semi** - (\(k, r\))-nice
\(e) \) if \((\mathfrak{R}, \bar{h}, h^e)\) is polynomially semi-nice then \(\alpha(A, D) = \alpha(A_1, D')\).

2) Assume \(\mathfrak{R}\) obeys \(\bar{h}\) with error \(h^e, h^e\) going to infinity. If \((A, A_0, B, D)\) is semi** - (\(k, r\))-good for \((\mathfrak{R}, \bar{h}, h)\) then it is semi-(\(k, r\))-good (see Definition 1.9(1)).
3) If \((\mathfrak{R}, \bar{h}, h^e)\) is semi-nice, then \(\mathfrak{R}\) is semi-nice.

Proof. 1) Straight by counting.
2) By part (1).
3) By part (2).

\(\square_{2.8}\)

2.9 Claim. 1) Assume \(\mathfrak{R}\) is semi-nice. Then for every \(A \in \mathcal{K}_\infty\) and \(k\) and \(\ell\), for some \(r\) we have:

\((*)\) for every random enough \(\mathcal{M}_n\), for every \(f : A \rightarrow \mathcal{M}_n\) and \(B \leq \mathcal{M}_n, |B| \leq \ell,\) we have
(cℓ∗(f(A),M), f(A), B, cℓk(f(A) ∪ B, M)) is semi-(k, r)-good.

2) Similarly for semi*-nice for (R, h, h).

Proof. 1) We prove it by induction on ℓ. Now for ℓ = 0 this is trivial by 1.11, so let us prove it for ℓ + 1 (assuming we have proved it for ℓ). So let A ∈ H∞ be given. Let r(1) be such that (exists by 1.12 applied to k′ ≥ k and ℓ′ = ℓ + |A|)

(*)1 if M is random enough, A′ ∈ H∞, |A′| ≤ |A| + ℓ, A′ ≤ M and b ∈ M, then (cℓ∗(1)(A′, M), A′, A′ ∪ {b}, cℓk(A′ ∪ {b}, M)) is semi-(k, r(1))-good.

Similarly by the induction hypothesis, for some r(2)

(*)2 if M is random enough, A′ ∈ H∞, |A′| ≤ |A|, B′ ≤ M, |B′| ≤ ℓ then (cℓ∗(2)(f(A′), M), f(A′) ∪ B′, cℓ∗(1)(A ∪ B′, M)) is semi-(r(1), r(2))-good.

We shall show that r(2) is as required. So let M be random enough and f : A → M and A, B ⊆ M, |B| = ℓ + 1. Let B = B0 ∪ {b}, |B0| ≤ ℓ. So by (*)1, the quadruple (cℓ∗(1)(f(A) ∪ B, M), f(A) ∪ B, B0 ∪ {b}, cℓk(A ∪ B0 ∪ {b}, M)) is semi-(k, r(1))-good.

Similarly by (*)2 the quadruple

(cℓ∗(2)(f(A), M), f(A), B0, cℓ∗(1)(f(A) ∪ B0, M))

is semi-(r(1), r(2))-good.

By “transitivity” of the property easily the quadruple

(cℓ∗(2)(f(A), M), f(A), B, cℓk(A ∪ B, M))

is (k, r(2))-good.

2) Similar to the proof of part (1), using Definition 2.7 instead of 1.12.

□ 2.9

* * *

We now turn to “redrawing”.

2.10 Definition. Assume R, R+ are 0-1 contexts.

1) We say R+ expands R if:

(a) τ+ a vocabulary extending τ, (hence consisting of predicates only, τ+ locally finite, of course)

(b) H+ is the family of τ+-models satisfying M+ ⌞ τ ∈ H and

Hn+ = {M+ ∈ H+: M+ ⌞ τ ∈ Hn}

(c) we let τ[ℓ] = τ ∪ {R : R ∈ τ+ has ℓ places} and τ<ℓ = ∪m<ℓ τ[m] ∪ τ

(d) for M ∈ Hn we have

μn(M) = ∑{μn(M+) : M+ ∈ Hn+ and M+ ⌞ τ = M}.
For simplicity, $\tau, \tau^+$ are irreflexive and $M \in K_n \Rightarrow \mu_n(M) = \mu_n(\{M\}) > 0.$

2) For $M_n \in K_n$, we define $\mu^+_n$, a distribution on $K^+_n = \{M^+ \in K^+_n : M^+ \upharpoonright \tau = M_n\}$ by $(\mu^+_n)(M^+_n) = \mu^+_n(M^+_n)/\mu_n(M_n)$, we write $\mu^+_n$ when $n$ is clear from context (this is even formally clear when the $K^+_n$'s are pairwise disjoint).

We will be mostly interested in the case $M^+_n$ is drawn as in Definition 2.12(2) below, but first define less general cases.

2.11 Definition. 1) We define when 0-1 context $\mathcal{R}^+$ is independently derived from $\mathcal{K}$ by the function $p$ (everything related to $\mathcal{R}^+$ has superscript $+$, below $\mathcal{X}^+$), $\mathcal{X}^+$ are as in 2.1(1), and for $x \in \{a, i, s, pr\}, \leq x$ is defined by 1.3(2)).

The crux of the matter is defining $\mu^+_n$; it suffices to define each $\mu^+_n$. We can think of it as choosing a $\mu^+_n$-random model $M^+_n$ by expanding $M_n$, defining $M^+_n \upharpoonright \tau_{<\ell}$ by induction on $\ell$ by flipping coins: for $\ell = 0$, $\mu^+_n \upharpoonright \tau_{<0}$ is chosen $\mu_n$-randomly from $K^+_n$ (i.e. is $M_n$). By induction on $\ell$, for each set $A \in \mathcal{K}^+_n$ (i.e. $A \subseteq \mathcal{K}_n, |A| = \ell$): we choose $A^+_\ell = (\mu^+_n \upharpoonright \tau_{<\ell}) \upharpoonright A$, each possibility $A^+_\ell$ has probability $p_{A^+_\ell}$, $\mu^+_n \upharpoonright \tau_{<\ell}$ = $p_{A^+_\ell}[M^+_n \upharpoonright \tau_{<\ell}]$ depending on $M^+_n \upharpoonright \tau_{<\ell}$ and $(\mu^+_n \upharpoonright A) \upharpoonright \tau_{<\ell}$ (not just on the isomorphism type), note that the second one, $(\mu^+_n \upharpoonright A) \upharpoonright \tau_{<\ell}$ is determined by $A^+_\ell$ as $A^+_\ell \upharpoonright \tau_{<\ell}$. Lastly, the drawings above (in stage $\ell$) are done independently for distinct $A$ (for each $\mu^+_n \upharpoonright \tau_{<\ell}$).

2) We say $\mathcal{R}^+$ is derived uniformly and independently if in addition $p_{A^+_\ell}[M^+_n \upharpoonright \tau_{<\ell}]$ depends on $M_n, A^+_\ell / \cong$ only and is derived very uniformly if it depends on $A^+/\cong$ and $\|M_n\|$ (and $n$) only.

2.12 Definition. 1) Suppose the 0-1 context $\mathcal{R}^+$ is independently derived from $\mathcal{K}$ by the function $p$ (see Definition 2.11). We say $p$ has uniform bounds $\bar{p}$ if:

(a) $\bar{p} = (p^d, p^n)$

(b) for $A^+ \in \mathcal{X}^+$ and $\ell = |A^+| \in \mathbb{N}, p_{A^+ \upharpoonright \tau_{<\ell}}^d, p_{A^+ \upharpoonright \tau_{<\ell}}^n$ are functions from $\mathcal{K}^+$ to $[0, 1]_{\mathbb{R}}$ (depending only on $A^+ \upharpoonright \tau_{<\ell}$ up to isomorphism) such that for every random enough $M_n$:

(*) for every embedding $f$ of $A = A^+ \upharpoonright \tau$ into $\mathcal{M}^+_n \upharpoonright \tau$, letting $B^+ = f''(A^+)$ and $A^+ = f(A^+)$ we have $p_{A^+ \upharpoonright \tau_{<\ell}}^d[M_n] \leq p_{B^+ \upharpoonright \tau_{<\ell}}^d[M_n \upharpoonright \tau_{<\ell}] \leq p_{A^+ \upharpoonright \tau_{<\ell}}^d[M_n]$ and $p_{A^+ \upharpoonright \tau_{<\ell}}^n[M_n]$ depends on $(A^+ \upharpoonright \tau_{<\ell}) / \cong$ only, (and, of course, $M_n$ and $n$); independently for the relevant distinct $A^+$'s.

So we can write $p_{A^+ \upharpoonright \tau_{<\ell}}^d[\mathcal{M}_n]$ for $p_{A^+ \upharpoonright \tau_{<\ell}}^d[\mathcal{M}_n]$. (Note that essentially $A^+ = A^+ \upharpoonright \tau_{<\ell}$ as $|A^+| = \ell$ and the relation are assumed to be irreflexive, so we can waive the $\upharpoonright \tau_{<\ell}$ abusing notation.

2) Suppose the 0-1 context $\mathcal{R}^+$ is an expansion of $\mathcal{K}$. We say that the drawing of $\mathcal{M}^+_n$ (or for $\mathcal{R}^+$ obeys (the pair of functions) $\bar{p} = (p^d, p^n)$ with error $h^e$ over $\mathcal{K}$ if for every $\varepsilon \in \mathbb{R}^{>0}$ and $A^+ \in \mathcal{X}^+$, letting $\ell = |A^+|, A = A^+ \upharpoonright \tau$ and given $M_n \in \mathcal{K}_n$, and an embedding $f : A \rightarrow \mathcal{M}_n$, assuming $\mathcal{M}^+_n \upharpoonright B$ was already drawn for every
B ∈ [ℳn]≤ℓ such that B ≠ f(A), and f is an embedding of A+ ↾ τ≤ℓ to ℳn+ then the probability (by µ+ℳn, modulo the assumptions above) that f embeds A+ ↾ τ≤ℓ into ℳn+ is at least pB+τ≤ℓ[ℳn] × (h[ℳn])−ε and at most pB+τ≤ℓ[ℳn] × (h[ℳn])ε. (so we assume always pB+τ≤ℓ[ℳn] ≤ pB+τ≤ℓ[ℳn] × (h[ℳn])−ε, at least for random enough ℳn.)

3) If pA+τ≤ℓ[ℳn] depends only on ||ℳn|| and n, we may write pA+τ≤ℓ[ℳn], n; if ||ℳn|| determines n we may omit the latter (and when the intention is clear from context also in other cases).

4) For A+ ≤ B+ ∈ ℋ∞ and x ∈ {d,u} we let:

$$p^x_{A+,B+}[ℳ_n] =: \Pi\{p^x_C[ℳ_n] : C ≤ B^+ and C ∉ A^+\}.$$

5) We omit hε if hε = 1, we say simply if hε[ℳn] = ||ℳn||.

6) Let h1, h2, hε be functions from ∪n<ω ℋn to R>0, R≥1, respectively. We say h1 ∼ h2 if for every random enough ℳn, (hε[ℳn])−ε ≤ h1[ℳn]/h2[ℳn] ≤ (hε[ℳn])ε.

2.13 Remark. 1) “Obeys” (Definition 2.12(2)) means we have independence but only approximately, so we shall be able to give later other distributions in which the drawing are independent and which give lower and upper bounds to the situation for ℋ.

2) Among those variants we use mainly Definition 2.12(2) and even more the polynomial case.

2.14 Definition. In Definition 2.12 we say that p is polynomial over h if:

(a) h is a function from ∪n<ω ℋn to N converging to infinity

(b) for every ε ∈ R>0 for every ζ ∈ R>0 for ℳn random enough ζ > h[ℳn]/||ℳn||

(c) for every A+ = B+ ↾ τ≤ℓ, ℓ = |B|, B+ ∈ ℋ+, for some β(A+) ∈ R we have:

(∗) there are constants c_A+,c_A+ such that:

$$p^d_{A+}(M) = (c^d_{A+})||M||^{β(A+)} / h[M]$$

$$p^u_{A+}(M) = (c^u_{A+})||M||^{β(A+)} / h[M].$$

so this is not necessarily the very uniform case.

Remark. Of course, we can replace “constant” by any slow enough function.
2.15 Claim. 1) Definition 2.11(1) is a particular case of Definition 2.12(2). Also Definition 2.11(2), is a particular case of Definition 2.12(1), (with $p^u = p^d$), and all of them are particular cases of Definition 2.10(1). Also, if we have 2.11(1) + 2.12(2), then we have 2.12(1).

2) In 2.10(1) necessarily

(a) $\mathcal{X}_\infty = \{ A^+ \upharpoonright \tau : A^+ \in \mathcal{X}_\infty^+ \}$
(b) if $A^+ \subseteq B^+ \in \mathcal{X}_\infty^+$ and $(A^+ \upharpoonright \tau) \leq_i (B^+ \upharpoonright \tau)$ then $A^+ \leq_i B^+$
(c) if $A^+ \subseteq B^+ \in \mathcal{X}_\infty^- \quad \text{and} \quad A^+ \leq_s B^+$ then $(A^+ \upharpoonright \tau) \leq_s (B^+ \upharpoonright \tau)$.

Proof. Check.
§3 Regaining the context for \( \mathcal{R}^+ \): reduction

We write down the expected values for \( \nu(f, A^+, B^+, M_n^+) \) (see Definition 0.3(2) and 3.3 below) then we define \( \preccurlyeq_b^+ \) as what will be a variant of \( \preccurlyeq_n^+ \) if things are close enough to the expected value and derive \( \preccurlyeq_i^+ \) (the parallel to \( \preccurlyeq_i^+ \)), \( \preccurlyeq_i^+ \) (the parallel to \( \preccurlyeq_i^+ \)) (all done in Definition 3.4). We phrase a natural sufficient condition to (the probability condition part for) \( A^+ \preccurlyeq_b^+ B^+ \) (in ?) and show that when it is equivalent to a natural strengthening then \( A^+ \preccurlyeq_b^+ B \Rightarrow \) scite\{4.5\} undefined

\( A^+ \preccurlyeq_b^+ B \) and moreover \( \preccurlyeq_j^+ = \preccurlyeq_i^+ = \preccurlyeq_t^+ \) (and some obvious fact, all this in 3.7).

We then concentrate on the “polynomial” case, ending with sufficient conditions for \( K \) being semi-nice (3.12 - 6.13).

3.1 Context. \( \mathcal{R}, \bar{h} \) as in 2.1 and \( \mathcal{R}^+, \bar{p} \) as in 2.12(2) and we use 2.12(4).

3.2 Definition. Assume \( f: A \to \mathcal{M}_n \) (an embedding),

\[ A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_k \text{ and } A = A_0, B = A_k \text{ and } \bar{A} = \langle A_\ell : \ell = 0, \ldots, k \rangle. \]

Let for \( i \leq k \):

\[ T^{[\ell]} = T^{[\ell]}(f, A, B, \bar{A}, \mathcal{M}_n) =: \{ g : g \text{ an embedding of } A_\ell \text{ into } \mathcal{M}_n \text{ extending } f \} \]

\[ T(f, A, B, \bar{A}, \mathcal{M}_n) = \bigcup_{\ell \leq k} T^{[\ell]}(f, A, B, \bar{A}, \mathcal{M}_n). \]

in fact, we can omit \( A, B \) as they are determined by \( \bar{A} \).

3.3 Claim. Assume

\( (\alpha) \) \( \mathcal{R} \) obeys \( \bar{h} \) with error \( h^\tau_1 \)

\( (\beta) \) the drawing of \( \mathcal{M}_n^+ \) obeys \( \bar{p} \) with error \( h^\tau_2 \) (see 2.12(2))

\( (\gamma) \) \( \bar{A} = \langle A_\ell : \ell \leq k \rangle \) is a decomposition of \( A <_s B \) (for \( \mathcal{R} \))

\( (\delta) \) \( \varepsilon \in \mathbb{R} > 0 \) and \( \mathcal{M}_n \) is random enough

\( (\varepsilon) \) \( f: A \to \mathcal{M}_n \subseteq \mathcal{X}_n \) an embedding, and \( A^+ \upharpoonright \tau = A, B^+ \upharpoonright \tau = B, A^+ \leq B^+ \) and \( A^+_\ell = : B^+_\ell \upharpoonright A_\ell \)

\( (\zeta) \) \( \varepsilon \in \mathbb{R} > 0 \).

Then the expected value of \( \nu(f, A^+, B^+, \mathcal{M}_n^+) \) under the distribution \( \mu^+_{\mathcal{M}_n} \) (see 1.4(1)) and the assumption “\( f \) will be an embedding of \( A^+ \) into \( \mathcal{M}_n^+ \) and \( \mathcal{M}_n \) random enough” is:

\[ \text{at least } \prod_{\ell < k} \left( p^d_{A_\ell^+, A_{\ell+1}^+, [\mathcal{M}_n]} \times h^d_{A_{\ell}, A_{\ell+1}} [\mathcal{M}_n] \right) \times (h[\mathcal{M}_n])^{-\varepsilon} \]
and at most \( \prod_{\ell<k} (p_{A^+_{\ell},A^+_{\ell+1}}^n(\mathcal{M}_n) \times h_{A^+_{\ell+1}}^n(\mathcal{M}_n)) \times (h[\mathcal{M}_n])^\circ. \)

Proof. Straight. \(\square\)

So by Claim 3.3, if the upper and lower bounds are close enough, we can show that other decompositions of \((A_0, A_k)\) give similar results.

In the following, for the interesting case (here), \(\leq^+_J, \leq^+_I, \leq^+_q\) will be proved to be equal to \(\leq^*_k, \leq^*_s, \leq^*_q\) respectively (but \(\leq^*_b\) will not be \(\leq^*_k\)).

3.4 Definition. 1) \(\mathcal{K}_\infty^\oplus = \{ A^+ : A^+ \in \mathcal{K}^+ \text{ and } A^+ \upharpoonright \tau \in \mathcal{X}_\infty \} \) and, of course, \(K_\infty^\oplus = \{ A^+ : A^+ \in \mathcal{K}^+ \text{ and } 0 < \limsup_n \text{Prob}_{\mu_n^+}(A^+ \text{ embeddable into } \mathcal{M}_n^+) \} \).

2) For \(A^+, B^+ \in \mathcal{X}_\infty^\oplus\) let \(A^+ \leq^+_b B^+\) hold if \((A^+ \upharpoonright \tau) \leq_s (B^+ \upharpoonright \tau)\) or \((A^+ \upharpoonright \tau) \leq_s (B^+ \upharpoonright \tau)\) and

\[ \otimes^0_{A^+, B^+} \text{ for some } k \in \mathbb{N}, \text{ for every } \zeta > 0 \text{ we have} \]

\[ 1 = \text{Lim}_n \text{Prob}_{\mu_n}(\text{we have } \zeta \text{ is larger than the probability that for some embedding} \]

\[ f \text{ of } (A^+ \upharpoonright \tau) \text{ into } \mathcal{M}_n, \text{ the number of extensions } g \text{ of} \]

\[ f \text{ to embedding of } B^+ \text{ into } \mathcal{M}_n^+ \text{ is } \geq k, \text{ by the distribution } \mu_n^+[\mathcal{M}_n^+] \]

\[ \text{under the assumption that } f \text{ embeds } A^+ \text{ into } \mathcal{M}_n^+. \]

3) \(A^+ \leq^+_J B^+\) if for every \(A^+_k\), we have \(A^+ \leq^+_J A^+_k \leq^+_q B \Rightarrow A^+_k <^+_b B.\)

4) For \(A^+, B^+ \in \mathcal{X}_\infty^\oplus\) let \(A^+ \leq^+_I B^+\) if \(A^+ \leq^+_I B^+\) and for no \(C^+ \in K_\infty^\oplus\) do we have \(A^+ \leq^+_J C^+ \leq^+_I B^+.\)

5) For \(A^+, B^+ \in \mathcal{X}_\infty^\oplus\) let \(A^+ \leq^+_q B^+\) if \(A^+ \leq^+_q B^+\) but for no \(C^+ \in K_\infty^\oplus\) do we have \(A^+ \leq^+_J C^+ \leq^+_q B^+.\)

6) We say \(A^+\) is a \(\leq^+_q\) decomposition of \(A^+ \leq^+_q B^+\) if \(\tilde{A}^+ = (A^+_\ell : \ell \leq k), A^+_\ell \leq^+_q A^+_{\ell+1}, A^+_0 \leq^+_J A^+\).

7) \(\mathcal{X}_\infty^\oplus = \{ A^+ : A^+ \in \mathcal{X}^+ \text{ and for some } \tilde{A} = (A^+_\ell : \ell \leq k) \}

\[ \text{and } \tilde{A}^+ = (A^+_\ell : \ell \leq k) \text{ we have:} \]

\[ A^+_\ell \leq^+_q A^+_\ell \text{ at } A^+_\ell \leq^+_q A^+_\ell+1, 0 \leq A^+_0 \text{ and for some } \varepsilon \in \mathbb{R}^{>0} \text{ we have} \]

\[ 0 < \text{Lim sup}_n \text{Prob}_{\mu_n}(\varepsilon < \prod_{\ell<k} (p_{A^+_{\ell},A^+_{\ell+1}}(\mathcal{M}_n) \times h_{A^+_{\ell+1}}(\mathcal{M}_n)) \times (h[\mathcal{M}_n])^\circ. \}

3.5 Claim. Assume \(A^+ \subseteq B^+ \subseteq D^+, A^+ \subseteq C^+ \subseteq D^+\) are in \(K_\infty^+\) and \(D^+ = B^+ \cup C^+.\)

1) If \(A^+ \leq^+_b B^+\), then \(C^+ \leq^+_b D^+.\)

2) If \(A^+ \leq^+_J B^+\), then \(C^+ \leq^+_J D^+.\)

Proof. 1) Reflect.

2) Follows from part (1). \(\square\)
3.6 Claim. 1) $\mathcal{K}_+^\infty \subseteq \mathcal{K} \subseteq \mathcal{K}_-^\infty \subseteq \mathcal{K}^+$ are closed under submodels and isomorphisms.

2) If $A^1 <^+_j A^1 <^+_j C^+ \subseteq \mathcal{K}_-^\infty$ and $B^+ \leq^+_\ell C^+$ and $A^1 \subseteq B^+$ then $B^+ \leq^+_j B^+ \cup A^1_2$. If $A^+ \leq^+_\ell B^+ \leq^+_\ell C^+$ then $B^+ \leq^+_j C^+$ and $B^+ \leq^+_j C^+$

3) On $\mathcal{K}_-^\infty$ the relation $\leq^+_b$ is a partial order and also the relation $\leq^+_j$ is a partial order.

4) If $A^+ \leq^+_\ell C^+$ are in $\mathcal{K}_+^\infty$ then for one and only one $B^+ \in \mathcal{K}_+^\infty$ we have $A^+ \leq^+_\ell B^+ \leq^+_\ell C^+$.

5) Let $B^+ \leq^+_\ell C^+$ be maximal such that $A^+ \leq^+_j B^+$, it exists as $C^+$ is finite and $A^+ \leq^+_j A^+$ (because $A \leq^+_j A$ where $A = A^+ \mid \tau$), now $B^+ \leq^+_\ell C^+$ by 3.6(3) + Definition 3.4(4). Hence at least one $B^+$ exists, so suppose $A^+ \leq^+_j B^+_1 \leq^+_\ell C^+$ for $\ell = 1, 2$ and $B^+_1 \neq B^+_2$ so without loss of generality $B^+_2 \setminus B^+_1 \neq \emptyset$, by 3.6(2), $B^+_1 \leq^+_j B^+_1 \cup B^+_2$ hence $B^+_1 <^+_j B^+_1 \cup B^+_2 \leq^+_\ell C^+$, but this contradicts $B^+_1 \leq^+_\ell C^+$ (see Definition 3.4(3)).

6) If $A^+ <^+_\ell B^+$ then there is a $<_{qr}$-decomposition $\tilde{A}^+$ of $A^+ <^+_\ell B^+$ [see Definition 3.4(5)], remembering $B$ is finite.

7) The relations $\leq^+_b$, $\leq^+_j$, $\leq^+_i$, $<_{qr}$ are preserved by isomorphisms.

8) If $A^+ <^+_\ell B^+ \Rightarrow A^+ \leq^+_i C^+$ and $C^+ \cap A^+ <^+_\ell C^+$, then by 3.6(2) we have $A^+_+ \leq^+_i A^+ \cup C^+$, contradiction to $A^+ \leq^+_i B^+$.

9) $T_+^\infty$ is a consistent (first order) theory.

3.7 Claim. Assume that $\mathfrak{R}^+$ obeys $\mathfrak{p}$ (over $\mathfrak{R}$), $\mathfrak{R}$ obeys $\mathfrak{h}$ and

\[ \star \quad \text{if } A^+, B^+ \in \mathcal{K}_-^\infty, (A^+ \mid \tau) <^s (B^+ \mid \tau), \tilde{A} = (A_\ell : \ell \leq n), \text{ a decomposition} \]

so $A^+ \mid \tau = A_0 \leq^s \ldots \leq^s A_n = B, A^+_\ell = B^+ \mid A_\ell$, then $\otimes^0_{A^+, B^+}$ of 3.4(2) and $\otimes^1_{A^+, B^+, \tilde{A}}$ and $\otimes^2_{A^+, B^+, \tilde{A}}$ below are equivalent where

\[ \otimes^1_{A^+, B^+, \tilde{A}} \quad \text{for some } \varepsilon > 0 \text{ we have} \]
\[1 = \text{Lim}_n \text{Prob}_{\mu_n}\left(\varepsilon > \prod_{\ell<k} (p^{u}_{A^+_{\ell}, A^+_{\ell+1}} M_n) \times h^u_{A^+_{\ell}, A^+_{\ell+1}} [M_n] \right).\]

\[\otimes^{2}_{A^+, B^+, \bar{A}} \text{ for some } \varepsilon \in \mathbb{R}^{>0} \text{ we have} \]

\[1 = \text{Lim}_n \text{Prob}_{\mu_n}\left(\| M_n \|^\varepsilon \prod_{\ell<k} (p^{u}_{A^+_{\ell}, A^+_{\ell+1}} M_n) \times h^u_{A^+_{\ell}, A^+_{\ell+1}} [M_n] \right).\]

1) If \( A^+ <_b B^+ (\text{in } \mathcal{K}^n) \) then for some \( m \in \mathbb{N} \) we have:

\[1 = \text{Lim}_n \text{Prob}_{\mu_n}\left( \text{there is no } f : A^+ \rightarrow \mathcal{M}_n^+ \text{ and } g_\ell : B^+ \rightarrow \mathcal{M}_n^+ \text{ for } \ell < m \right. \]

\[\text{such that } (g_\ell : \ell < m) \text{ is a disjoint sequence of extensions of } f.\]

2) If \( A^+ <_j B^+ (\text{in } \mathcal{K}^n) \) then for some \( k \in \mathbb{N} \) we have:

\[1 = \text{Lim}_n \text{Prob}_{\mu_n}\left( \text{there is no } f : A^+ \rightarrow \mathcal{M}_n^+ \text{ and } g_\ell : B^+ \rightarrow \mathcal{M}_n^+ \text{ for } \ell < k \right. \]

\[\text{such that } (g_\ell : \ell < k) \text{ is a sequence of distinct extensions of } f.\]

3) If \( A^+ \leq_b B^+ \in \mathcal{K}^n, \text{ then } A^+ \leq^b B^+. \)

4) If \( A^+ \leq^j B^+ \in \mathcal{K}^n, \text{ then } A^+ \leq^j B^+. \)

5) If \( (A^+ \upharpoonright \tau) \leq_a (B^+ \upharpoonright \tau) \text{ and } A^+ \leq^+ B^+ \in \mathcal{K}^n, \text{ then } A^+ \leq^b B^+. \)

6) If \( (A^+ \upharpoonright \tau) \leq_1 (B^+ \upharpoonright \tau) \text{ and } A^+ \leq B^+ \in \mathcal{K}^n, \text{ then } A^+ \leq^j B^+. \)

3.8 Remark. 1) Are there cases we may be interested in which are not covered by this claim? If \( \mathcal{A} \) obeys \( h \), and

\[h^u_{A,B}(n) \sim n^{1/(\log n)^{1/2}} \sim h^d_{A,B}(n).\]

2) We may rephrase the assumption in 3.7 to cover those cases.

3) Note: if all is polynomial, then \( \otimes^{2}_{A^+, B^+, \bar{A}} \) is equivalent to \( \otimes^{1}_{A^+, B^+, \bar{A}} \).

4) We can use \( h \) as in 2.1(3),2.1(3A); see latter.

Proof. 1) Let \( \varepsilon > 0 \) be as in \( \otimes^{2}_{A^+, B^+, \bar{A}} \), and let \( m \in \mathbb{N} \) be such that \( \varepsilon m > |A^+| \) so \( \zeta = : \varepsilon k - |A^+| \in \mathbb{R}^{>0} \). Let \( (A^+_{\ell} : \ell \leq k), (A^+_{\ell} : \ell \leq k), \) be as in \( ? \) and let \( A = A^+ \upharpoonright \tau; \)

\[\text{scite\{4.5\} undefined}\]

considering our aim, without loss of generality \( A_0 = A \) (see Definition 3.4(1)).

\[\text{of course, we can think of cases that there are few copies of } A^+_0, \text{ then } \| M_n \| \text{ can be replaced}
\]

by such upper bounds; this has no influence in the polynomial case.
So for every $M \in \mathcal{X}$, $\|M\|^{-\varepsilon m} \times |\{f : f \text{ an embedding of } A \text{ into } M\}|$ is
$\leq \|M\|^{-\varepsilon m - |A| - \varepsilon}$, hence it suffices to prove:

(*) if $\mathcal{M}_n$ is random enough and $f$ is an embedding of $A$ into $\mathcal{M}_n$ then

$$\|\mathcal{M}_n\|^{-\varepsilon m} \geq \text{Prob}_{\mathcal{M}_n}[\text{there are a sequence of } k \text{ disjoint extensions of } f \text{ to an embedding of } B^+ \text{ into } \mathcal{M}^+_n, \text{ under the assumption } f \text{ is an embedding of } A \text{ into } \mathcal{M}^+_n].$$

Now if $\mathcal{M}_n$ is random enough then by 7.7(1) we know

$$\nu(f, A, B, \mathcal{M}_n) \leq \prod_{\ell < k} h_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n], \text{ i.e. } \text{ex}(f, A, B, \mathcal{M}_n) \text{ has } \leq \prod_{\ell < k} h_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n] \text{ members.}$$

Let $F = F^m = F^m[\mathcal{M}_n] = \{\bar{f} : \bar{f} = (f_\ell : \ell < m) \text{ and } f_\ell \in \text{ex}(f, A, B, \mathcal{M}_n) \text{ and } \bar{f} \text{ is disjoint over } A\}$. So $|F_m| \leq |\text{ex}(f, A, B, \mathcal{M}_n)|^m \leq \left(\prod_{\ell < k} h_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n]\right)^m$.

Hence, if we draw $\mathcal{M}_n^+$ (by the distribution $\mu_n[\mathcal{M}_n]$) under the assumption "$f$ is an embedding of $A$ into $\mathcal{M}^+_n$", then the expected value of $|\{\bar{f} \in F^m[\mathcal{M}_n] : \ell < m, f_\ell \text{ is an embedding of } B^+ \text{ into } \mathcal{M}^+_n\}|$ is

$$\leq \sum_{f \in F} \text{Prob}_{\mathcal{M}_n^+}[\text{for } \ell < m, f_\ell \text{ is an embedding of } B^+ \text{ into } \mathcal{M}^+_n | f \text{ is an embedding of } A^+ \text{ into } \mathcal{M}_n]$$

$$\leq \sum_{f \in F} \left(\prod_{\ell < k} p_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n]\right)^m$$

$$= |F| \times \left(\prod_{\ell < k} p_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n]\right)^m$$

$$\leq \left(\prod_{\ell < k} h_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n]\right)^m \times \left(\prod_{\ell < k} p_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n]\right)^m$$

$$= \left(\prod_{\ell < k} \left(p_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n] \times h_{A^+_{\ell}, A^+_{\ell+1}}^u[\mathcal{M}_n]\right)\right)^m$$

but if $\mathcal{M}_n$ is random enough $\otimes^2 A^+, B^+, \mathcal{A}$ (i.e. by the assumption of 3.7 and $\otimes^0 A^+, B^+$ which holds as we are assuming $A^+ \leq^+ B^+$) this last number is

$$\leq \left(\|\mathcal{M}_n\|^{-\varepsilon}\right)^m = \|\mathcal{M}_n\|^{-\varepsilon m}.$$ 

As said above, this suffices.

2) By the $\triangle$-system lemma and 3.7(1) (and the definitions).
3) Follows by 3.7(1).
4) Follows from 3.7(2).
5) Read the definitions. □
3.9 Claim. 1) A sufficient condition for \( \leq^+ = \leq^+ \mid \mathcal{K}_\infty \) is:

\((*)\) like \((*)\) of 3.7 but we add another equivalent condition

\[ \otimes^{1,d}_{A^+,B^+,A^+} \quad \text{for some } \varepsilon > 0 \text{ we have} \]

\[ 1 = \lim_n \text{Prob}_n \left( \varepsilon > \prod_{k \leq k} \left( p^d_{A^+,A^+} \cdot A^+ \times h^d_{A^+,A^+} \cdot A^+ \right) \right). \]

2) Note that \( \otimes^2_{A^+,B^+,A^+} \Rightarrow \otimes^1_{A^+,B^+,A^+} \Rightarrow \otimes^{1,u}_{A^+,B^+,A^+} \) and

\[ \otimes^2_{A^+,B^+,A^+} \Rightarrow \otimes^1_{A^+,B^+,A^+} \Rightarrow \otimes^{1,u}_{A^+,B^+,A^+}. \]

Proof. By 3.7(4), it suffices to prove, assuming \( A^+ \leq^+ B^+ \) & \( \neg(A^+ \leq^+ B^+) \), that \( \neg(A^+ \leq^+ B^+) \). As \( \neg(A^+ \leq^+ B^+) \) necessarily there is \( A^+_1 \) such that \( A^+ \leq^+ A^+_1 \leq^+ B^+ \), \( \neg(A^+_1 \leq^+ B^+) \).

The rest is easy, too. \( \square_{3.9} \)

3.10 Definition. 1) Assume

\((*)\) \( \aleph \) obeys \( \tilde{h} \) with error \( h^e, \mathcal{R}^+ \) is drawing obeys \( \tilde{p} \) with error \( h^f_1 \).

We say that \( (A^+,A^+_0,B^+,D^+) \) is a pretender to semi* - \((k,r)\)-good for \( (\mathcal{R}^+,\tilde{h},\tilde{h}^e,\tilde{p},h^f_1) \) if: for some witness \( A^+_1 \),

\( a) \ A^+_0 \leq^+_j A^+_1 \leq^+_j A^+ \leq^+_j D^+, B^+ \leq^+ D^+ \\
 b) (A^+_j \mid \tau, A^+_0 \mid \tau, B^+ \mid \tau, D^+ \mid \tau) \) is semi* - \((k,r)\)-good for \( (\mathcal{R},\tilde{h},h) \) witnessed by \( A^+_1 \mid \tau \)

\( c) \) if \( D^+ \leq^+ D^+_1 \leq^+_i D^+_1 \) and \( D^+ \mid \tau \leq^+_i D^+_1 \mid \tau \) and \( C^+_1 \leq^+_i C^+_1 \) and \( |C^+_1| \leq k \) and \( h^+_A \leq^+_i h^+_A \leq^+_i h^+_A \leq^+_i h^+_A \leq^+_i h^+_A \leq^+_i h^+_A \), then \( C^+_2 \leq^+_i D^+ \).

2) We in part one write semi* if \( A^+_1 = A^+ \).

3.11 Claim. 1) Assume \( \aleph \) obeys \( \tilde{h} \) which is bounded by \( h, K^+ \) drawn by \( p \) which obeys \( \tilde{p} \) which is bounded by \( h \). Sufficient conditions for \((*)_1 + (*)_2 + (*)_3\) are \( \otimes_1 + \otimes_2 \) where

\( (*)_1 \) \( (\mathcal{K}^+_1, \leq^+_j \mid \mathcal{K}^+_0, \leq^+_j \mid \mathcal{K}_\infty, \leq^+_j \mid \mathcal{K}_\infty) = (\mathcal{K}^+_1, \leq^+_j, \leq^+_j, \leq^+_j) \)

\( (*)_2 \) \( \mathcal{R}^+ \) is weakly nice

\( (*)_3 \) \( \mathcal{R}^+ \) obeys \( h^+ \) with error \( h^e \)

and

\[ \otimes_1 \tilde{h}^+ = (h^+_A, h^+_D), \text{ if } A^+ \leq^+_i D^+. \]

(\( \alpha \)) \( h^+_A \leq^+_i B^+ \)

\[ \otimes_1 \tilde{h}^+ = (\mathcal{M}_n^+) = \]
\[ +h^n_{A^+, B^+}[\mathcal{M}_n] = h^n_{A^+, B^+}[\mathcal{M}_n^+ | \tau] \times p^n_{A^+, B^+}[\mathcal{M}_n^+ | \tau] \]

(\beta) \[ +h^d_{A^+, B^+}[\mathcal{M}_n^+] = +h^d_{A^+, B^+}[\mathcal{M}_n] = h^d_{A^+, B^+}[\mathcal{M}_n^+ | \tau] \times p^d_{A^+, B^+}[\mathcal{M}_n^+ | \tau] \]

(\gamma) \[ h^n[\mathcal{M}_n^+] \geq h_1[\mathcal{M}_n^+ | \tau] \times h_2[\mathcal{M}_n^+ | \tau] \text { goes to infinity} \]

\[ \otimes_2 \text{ for } A^+ < q_\sigma B^+ \text{ and } \varepsilon \in \mathbb{R}^{>0}, \text{ for random enough } \mathcal{M}_n \text{ we have if } f \text{ embeds } A^+ \upharpoonright \tau \text{ into } \mathcal{M}_n, \text{ under the assumption that } f \text{ embeds } A^+ \text{ into } \mathcal{M}_n^+, \text{ the probability that } nu(f, A^+, B^+, \mathcal{M}_n^+) \text{ is not in the interval } (+h^d_{A^+, B^+}[\mathcal{M}_n] \times (h^c[\mathcal{M}_n])^{-\varepsilon}, +h^d_{A^+, B^+}[\mathcal{M}_n^+] \times (h^c([\mathcal{M}_n])^\varepsilon)) \text{ is } < 1/||\mathcal{M}_n||^{1/\varepsilon} \]

\[ \otimes_3 \text{ for } A^+ < q_\sigma B^+, \varepsilon \in \mathbb{R}^{>0} \text{ and } m \in \mathbb{N} \text{ for random enough } \mathcal{M}_n \text{ and } f : A^+ \upharpoonright \tau \to \mathcal{M}_n \text{ we have } \]

\[ +h^d_{A^+, B^+}[\mathcal{M}_n]/(h^e_3[\mathcal{M}_n])^\varepsilon \geq m. \]

2) In part (1) if in addition \( \otimes_2^+ \) then in addition \((*)_4 + (*)_5\)

\[ (*)_4 \text{ if } (A^+, A_0^+, B^+, D^+) \text{ is a pretender to being semi}^*- (k, r)-\text{good for } (\mathcal{K}^+, \tilde{h}, h^{e_0}_0, h^{e_1}_0), \text{ then it is semi}^*- (k, r)-\text{good for } (\mathcal{K}^+, \tilde{h}^+, h^+) \]

\[ (*)_5 \mathcal{K}^+ \text{ is semi-nice, moreover even } (\mathcal{K}^+, \tilde{h}^+, h^+) \text{ is semi-nice, } \mathcal{M}_n^+ \text{ random enough and } \]

\[ \otimes_2^+ \text{ if } (A^+, A_0^+, B^+, D^+) \text{ is a pretender to semi}^* - (k, r)-\text{good, then it is } f : A^+ \to \mathcal{M}_n^+ \text{ and } |C_1^+| < f(C_1^+) \leq \mathcal{M}_n^+ \text{ & } C_1^+ \leq f(A) \text{ & } |C_2^+| \leq r \Rightarrow C_2^+ \subseteq f(A^+), \text{ then the inequalities in } \otimes_2 \text{ holds for } nu^e(f, A^+, B^+, D^+). \]

**Proof.** Straight.

\[ * \quad * \quad * \]

Now we deal with the polynomial case.

### 3.12 Definition. Assume

\((*)_{ap}(a) \tilde{\mathcal{K}} \text{ is a 0-1 law context (b) } \tilde{\mathcal{K}} \text{ obeys } \tilde{h} = (h^d, h^u) \text{ (see Definition 2.1)} \]

\((c) \tilde{h} \text{ is polynomial over } h_1 \text{ with the function } \alpha(-, -) \text{ (see Definition 2.3)} \]

\((d) \text{ the 0-1 law context } \tilde{\mathcal{K}}^+ \text{ is an expansion of } \tilde{\mathcal{K}} \text{ obeying the pairs of functions } \tilde{\rho} = (p^d, p^u) \text{ (see Definition 2.12(2))} \]

\((e) \tilde{\rho} \text{ is polynomial over } h_2 \text{ (see Definition 2.14) by the function } \beta(-). \]

Here let \( A^+ \leq_\sigma B^+ \text{ mean } A^+ \subseteq B^+ \in \mathcal{K}^\| \).

\((f) \text{ for } \varepsilon > 0, \text{ for random enough } \mathcal{M}_n, h_1[\mathcal{M}_n], h_2[\mathcal{M}_n] \leq ||\mathcal{M}_n||^\varepsilon. \)
1) For $A^+ \leq^+ B^+$ satisfying
\[(A^+ | \tau) <_s (B^+ | \tau)\] we define
\[\beta(A^+, B^+) =: \alpha(A^+ | \tau, B^+ | \tau) + \sum \{ \beta(C^+) : C^+ \subseteq B^+, C^+ \not\subseteq A^+ \}.
\]

2) For $A^+ \leq^+ B^+$ let $A^+ \leq^+_b B^+$ mean:
for some $A_1^+$ we have:
\[(i)\ A^+ \leq^+ A_1^+ \leq^+ B^+
(ii) (A^+ \upharpoonright \tau) \leq^\ast (A_1^+ \upharpoonright \tau) \leq (B^+ \upharpoonright \tau)
(iii) A^+ \not= A_2^+ or for some $A_2^+$ we have $A_1^+ <^+ A_2^+ \leq^+ B^+$ and
$\beta(A_1^+, A_2^+) < 0$.

3) For $A^+ \leq^+ B$ let $A^+ \leq^+_t B^+$ mean:
for every $A_1^+$ we have $A^+ \leq^+ A_1^+ <^+ B^+ \Rightarrow A_1^+ \leq^+_b B^+$.

4) For $A^+ \leq^+ B$ let $A^+ \leq^+_t B^+$ mean:
for no $A_1^+$ do we have $A^+ <^+_t A_1^+ \leq^+ B$.

5) For $A^+ \leq^+ B$ let $A^+ \leq^+_p B^+$ mean:
$A^+ \leq^+ B^+$ but for no $A_1^+$ do we have $A^+ <^+_p A_1^+ <^+_p B^+$.

6) $\mathcal{X}^+_p = \left\{ A^+ \in \mathcal{X}^+ : \text{letting } (A^+ \upharpoonright \tau | \emptyset) \leq^\ast A_1 <^\ast A^+ \upharpoonright \tau \text{ and } A_1^+ =: A^+ \upharpoonright A_1,
\text{we have: } |A_1^+ \subseteq A_1^+ \Rightarrow \beta(A_1^+) = 0| \text{ and } \beta(A_1^+, A^+) \geq 0 \text{ and } 0 < \liminf_n \text{Prob}_{\mu_n} \left( \text{we can embed } A_1^+ \text{ into } \mathcal{M}_n \right) \right\}.$

3.13 Discussion: Note that there can be $A^+$ such that the sequence
$\langle \text{Prob}_{\mu_n}(A^+ \text{ embeddable into } \mathcal{M}_n^+) : n \in \mathbb{N} \rangle$ does not converge, but essentially this
occurs only when $((A^+ | \emptyset | \tau) \leq^\ast (A^+ | \tau)).$ More exactly if $A^+ \upharpoonright \emptyset \upharpoonright \tau \leq^\ast (A_1^+ \upharpoonright \tau) \leq (A^+ \upharpoonright \tau)$ assuming the answer for $(A_1^+ \text{ embeddable into } \mathcal{M}_n)$, we almost
surely know if $A^+$ can be embedded into $\mathcal{M}_n^+$.

We first note
3.14 Fact. Assume $(*)_{\text{ap}}$ of 3.12 and
\[\circ_{1} \text{ the irrationality assumption for } (\bar{h}, \bar{h}, \bar{R}, \bar{p}) \text{ which means: if } A^+ <^+_p B^+,
(A^+ \upharpoonright \tau) < (B^+ \upharpoonright \tau) \text{ then } \beta(A^+, B^+) \neq 0.
\]
1) If $A^+ \subseteq B^+ \subseteq C^+$ and $(A^+ \upharpoonright \tau) < (B^+ \upharpoonright \tau) < (C^+ \upharpoonright \tau)$ then
$\beta(A^+, B^+) + \beta(B^+, C^+) = \beta(A^+, C^+)$, (of course, $\circ_{1}$ is not used).
2) If $A^+ \leq^+ B^+ \in \mathcal{X}^+_n$ and $(A^+ \upharpoonright \tau) < (B^+ \upharpoonright \tau)$ and $\beta(A^+, B^+) < 0$ then
for some $m$ for every random enough $\mathcal{M}_n^+$ and any embedding $f : A^+ \rightarrow \mathcal{M}_n^+$, there
are no $m$ disjoint extensions $g : B^+ \rightarrow \mathcal{M}_n^+$ of $f$.

2A) The four conditions in $(*)$ of 3.9 (see also $(*)$ of 3.7) are equivalent (for any
$A^+, B^+, A$ as there).
3) $\leq^+_t = \leq^+_t$ for $x \in \{b, j, t, qr\}$ and $\mathcal{X}^+_n = \mathcal{X}^+_n = \mathcal{X}^+_p$. 

4) Assume $A^+ \subseteq B^+ \subseteq D^+, A^+ \subseteq C^+ \subseteq D^+, D^+ = B^+ \cup C^+, B^+ \cap C^+ = A^+, A^+ \leq p$ and $(C^+ \mid \tau) \leq_s (D^+ \mid \tau)$ then $\beta(A^+, B^+) \geq \beta(C^+, D^+)$. \\

**Proof.** 1) \\

$$\beta(A^+, C^+) = \alpha(A^+, C^+) + \sum \{ \beta(D^+) : D^+ \subseteq C^+ \text{ and } D^+ \notin A^+ \}$$

$$= \alpha(A^+, B^+) + \alpha(B^+, C^+) + \sum \{ \beta(D^+) : D^+ \subseteq C^+ \text{ and } D^+ \notin A^+ \}$$

$$= \alpha(A^+, B^+) + \alpha(B^+, C^+) + \sum \{ \beta(D^+) : D^+ \subseteq B^+, D^+ \notin A^+ \}$$

$$= \beta(A^+, B^+) + \beta(B^+, C^+).$$

2) For $\mathcal{M}_n$ which is random enough and $f : (A^+ \mid \tau) \to \mathcal{M}_n$, the set $Y = ex(f, A, B, \mathcal{M}_n)$ has $\leq h^u_{A,B}(\mathcal{M}_n)$ members (see Definition 2.1). Hence the set $Y_m = \{ g : g = \{ g_\ell \mid \ell < m \} \text{ is a sequence of members of } Y \text{ disjoint over } f \}$ has $\leq (h^u_{A,B}(\mathcal{M}_n))^m$ members which is $\leq (h_1[\mathcal{M}_n])^t \cdot ||\mathcal{M}_n||^{t\beta(A^+, B^+)}$. Now under the assumption $f : A^+ \to \mathcal{M}_n$, for each $g \in Y_m$, the probability that $g_\ell$ embeds $B^+$ into $\mathcal{M}_n^+$ for appropriate $t_2 \in \mathbb{N}$ is:

$$\leq (h_2[\mathcal{M}_n]^{t_2} \cdot ||\mathcal{M}_n||^{t\beta(C^+, D^+) \subseteq B^+, C^+ \notin A^+})^m.$$

So the expected value is

$$\leq h_1[\mathcal{M}_n]^{t_1} \times h_2[\mathcal{M}_n]^{mt_2} \times (||\mathcal{M}_n||^{t\beta(A^+, B^+)})^m$$

$$= h_1[\mathcal{M}_n]^{t_1} \times h_2[\mathcal{M}_n]^{mt_2} \times ||\mathcal{M}_n||^{t\beta(A^+, B^+)}.$$

So as $\beta(A^+, B^+) < 0$, and the assumptions on $h_1, h_2$ we have: for $m$ large enough, this probability is $\leq ||\mathcal{M}_n||^{t\beta(A^+, B^+)}$ so the conclusion follows.

2A) Straight. \\
3) Assume $A^+ \leq B^+$ and we shall prove that $A^+ \leq^p B^+ \iff A^+ \leq^p B^+$. \\
Let $A^+_1$ be such that $A^+ \leq^p A^+_1 \leq B^+$ and $(A^+ \mid \tau) \leq_s (A^+_1 \mid \tau) \leq_s (B^+ \mid \tau)$. \\
Now if $A^+ \neq A^+_1$ then both $A^+ \leq^p B^+, A^+ \leq^p B^+$ hold and we are done so assume $A^+ = A^+_1$. Now compare condition $\otimes^0_{A^+, B^+}$ of 3.4(2) and $\beta(A^+, B^+) < 0$, (as in the proof of part (2)) they are the same. So $\leq^p \leq^+$, and the other equalities follow. \\
4) Let $\mathcal{M}_n$ be random enough but such that some $f_1$ embeds $C^+ \mid \tau$ into it (note $C^+ \in \mathcal{K}^{\leq p}$ and see Definition 3.12(7)). Assume $f_1$ embeds $C^+ \mid \tau$ into $\mathcal{M}_n^+$ and let $f_0 = f \mid A^+$; now the expected value of the number of $g_0 : B^+ \to \mathcal{M}_n^+$ is $\approx ||\mathcal{M}_n||^{\beta(C^+, D^+)}$, so the desired inequality follows. □
3.15 Claim. 1) Assume \((*)_{ap}\) of Definition 3.12.
A sufficient condition for \((*)_1 + (*)_2 + (*)_3\) below is \(\otimes_1 + \otimes_2 + \otimes_3\) below where:

\((*)_1\) \( (\mathcal{X}_{\infty}^+, \leq_{j}^p \upharpoonright \mathcal{X}_{\infty}^+, \leq_{i}^p \upharpoonright \mathcal{X}_{\infty}^+, \leq_{q}^p \upharpoonright \mathcal{X}_{\infty}^+) = (\mathcal{X}_{\infty}^+, \leq_{i}^+, \leq_{i}^+, \leq_{p}^+)\)

\((*)_2\) \( \mathbb{R}^e \) is weakly nice

\((*)_3\) \( \mathbb{R}^e \) obeys a pair \(\tilde{h}\) which is polynomial over some \(h^e\)

with the function \((A^+, B^+) \rightarrow \beta(A^+, B^+)\) when \(A^+ <^+_B B^+\).

\(\otimes_1\) for some function \(h_3 : \mathcal{X} \rightarrow \mathbb{N}\) satisfying \(e \in \mathbb{R}^+ \Rightarrow \) for random enough

\(\mathcal{M} \ni h_3(\mathcal{M}_n) / ||\mathcal{M}_n|| = 0, (h_3 \text{ somewhat above})\)

\(h_1[\mathcal{M}_n] \times h_2(\mathcal{M}_n)\)

\(\otimes_2\) the irrationality assumption: for \((\mathbb{R}, \tilde{h}, \mathbb{R}^e, \tilde{p})\):

if \(A^+ <^+_q B^+, (A^+ \upharpoonright \tau) <_s (B^+ \upharpoonright \tau) \Rightarrow \) then \(\beta(A^+, B^+) \neq 0\)

\(\otimes_3\) if \(A^+ <^+_q B^+\), then for every \(\mathcal{M}^+\) random enough, for every \(f_0 : A^+ \rightarrow \mathcal{M}^+\) we have:

\[||\mathcal{M}|| \beta(A^+, B^+) / h_3(\mathcal{M}_n) \leq \|\mathcal{M}_n\| \beta(A^+, B^+) \times h_3(\mathcal{M}_n)\]

2) In part (1) we add \((*)_4, (*)_5\) if we assume also \(\otimes^+_3\)

\((*)_4\) if \((A^+, A_0^+, B^+, D^+)\) is a pretender to semi-\((k, r)\)-good, then it is semi-\((k, r)\)-good

\((*)_5\) \((\mathbb{R}^e, \tilde{h}^+, h^+)\) is semi-\((k, r)\)-good

where

\(\otimes^+_3\) if \((A^+, A_0^+, B^+, D^+)\) is a pretender to semi-\((k, r)\)-good, then for every random enough \(\mathcal{M}_n\) and \(f : A^+ \rightarrow \mathcal{M}_n\) letting \(f_0 = f \upharpoonright A_0^+\) we have

\[||\mathcal{M}_n|| \beta(A^+, D^+) / h_3(\mathcal{M}_n) \leq n\|f_0, A_0^+, B^+, D^+, \mathcal{M}_n\| \leq ||\mathcal{M}|| \beta(A^+, D^+) \times h_3(\mathcal{M}_n)\]

Proof. Straightforward.

3.16 Conclusion. Assume \((*)_{ap}\) of 3.12 and \(\otimes_1\) of 3.14 (the irrationality) and for simplicity let \(h^e(M) = ||M||\).

Then

(a) the conditions of 3.15(1), 3.15(2) hold hence their conclusions

(b) for \(A^+ \leq B^+ \in \mathcal{X}_{\infty}^\oplus\), we have

(i) \(A^+ \leq B^+ \in \mathcal{X}_{\infty}^\oplus\) and

(ii) \(A^+ \leq B^+ \Rightarrow A^+ \upharpoonright \tau \leq A\upharpoonright B\upharpoonright \tau\)

(iii) \(A^+ \leq B^+ \Rightarrow A^+ \leq B^+ \Leftarrow A^+ \upharpoonright \tau \leq A\upharpoonright B\upharpoonright \tau\)

(iv) \(A^+ \leq B^+ \Rightarrow A^+ \leq B^+ \Leftarrow A^+ \upharpoonright \tau \leq B\upharpoonright \tau\)

(v) \(A^+ \leq B^+ \Rightarrow A^+ \leq B^+ \Leftarrow A^+ \upharpoonright \tau \leq B\upharpoonright \tau\)
In more details,

Proof. All is reduced to the case of the binomial distribution by §4, §5.

In more details,

Stage A: Clauses (b)(i)-(v) and (d), the first iff

The first by 3.14, the second (i.e. clause (d)) left to the reader.

Stage B: Clauses (c) + (e)

Let \( \varepsilon > 0 \) and let \( \mathcal{M}_n \) be random enough. We need to consider all \( f \in F = \{ f : f \text{ an embedding of } A^+ \mid \tau \text{ into } \mathcal{M}_n \} \), for each of them the appropriate inequality should hold. So it is enough if the probability of failure is \( < \|\mathcal{M}_n\|^{1/\varepsilon}/|F| \) so \( \|\mathcal{M}_n\|^{-|A^+|^{1-1/\varepsilon}} \) suffice. Success means that under the hypothesis \( f : A^+ \rightarrow \mathcal{M}_n^+ \), we should consider the candidates \( g \in G = \{ g : g \text{ an embedding of } B^+ \mid \tau \text{ into } \mathcal{M}_n \} \), how many of them will be in \( G^+ = \exp(f, A^+, B^+, \mathcal{M}_n) \). Well the events \( g \in G^+ \) are not independent (in \( \mu_{\mathcal{M}_n^+} \), of course).

Note: We should prove that the probability of deviating from the expected number of extensions of \( f : A^+ \rightarrow \mathcal{M}_n^+ \), is much smaller than the number of \( f \)'s which is \( \leq \|\mathcal{M}_n\|^{4/4} \). By 4.11 we can restrict ourselves to the separated case (see Definition 4.10(3)). By 4.7 we can ignore the case that in \( G[\mathcal{M}_n^+] \) (for our \( f, A, B \) or \( f, A, A_0, B_0, B, k \) as in 4.1(2)) every connectivity component has \( < c \) element for some \( c \) depending on \( A, B \) only.

Now our problem is a particular case of the context in 5.1 and by the previous sentence we can deal separately giving an interval into which the number of components of isomorphism type \( t \), for the \( t \) with \( < c \) elements (see 5.3). Actually each one is another instance of 5.1 only separated is replaced by weakly separated. So clearly it is enough to deal with \( L_{t^*}[\mathcal{M}_n^+] \) (\( t^* \) the isomorphism type of a singleton). With well known estimates, 5.8(2) gives “very low probability” for the value \( L_{t^*}[\mathcal{M}_n^+] \) being too small. The dual estimate is given by 5.10 (remember in our case having instances of the new relations has low probability so we can use 5.8(2) to show that \( \tau \) is there small even for \( \alpha < 1 \) very near to 1).

For clause (e) just note that the number of extensions violating the desired conclusion is much smaller (the definition of pretender is just made for this).
Stage C: Rest.
Straight by now.

□ \(3.16\)
§4 Clarifying the probability problem

We are still in the context 3.1. Our aim is to clarify the probability problem to which we reduce our aim in 3.15 (in the polynomial case).

So our aim is to get good enough upper bounds and lower bounds to the $h_{A^+, B^+}(\mathcal{M}_n)$.

4.1 Hypothesis. 1) $\mathcal{M}_n \in \mathcal{K}_n, A^+ \leq q_{pr} B^+, \overline{A} = (A_\ell : \ell \leq k), A = A^+ \upharpoonright \tau,$

$B = B^+ \upharpoonright \tau, A = A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_k = B, A^*_\ell = A^+ \upharpoonright A_\ell$

and an embedding $f^* : A \to \mathcal{M}_n$.

We try to approximate $\mu(f^*, A^+, B^+, \mathcal{M}_n^+)$ under the condition “$f^*$ embed $A^+$ into $\mathcal{M}_n^+$”.

2) (a variant) Assume further $(A^+, A_0^+, B_0^+, B^+)$ is a pretender to being semi-$(k, r)$-good $B_1^* = B \upharpoonright B_1, f_1^* : A \to \mathcal{M}_n$. We try to approximate $\mu^k(f^*, A^+, B_1^+, B^+, \mathcal{M}_n^+)$ under the assumption $cf^k(f(A_0^+), \mathcal{M}_n^+) \subseteq A^+$.

Notation: Lastly let $x \in \{d, u\}$.

We shall speak mainly for 4.1(1), and then indicate the changes for 4.1(2).

4.2 Notation. $T_\ell = T_\ell[f^*, \mathcal{M}_n] = \text{ex}(f^*, A, A_\ell, \mathcal{M}_n),$

$T_\ell[f^*, \mathcal{M}_n^+] = T_\ell \cap \text{ex}(f^*, A^+, A^*_\ell, \mathcal{M}_n^+),$

$T = \bigcup_{\ell \leq k} T_\ell$ and $T[f^*, \mathcal{M}_n^+] = \bigcup_{\ell \leq k} T_\ell[f^*, \mathcal{M}_n^+]$. For $g \in T_\ell$, let $\text{lev}(g) = \ell$.

For $g_1, g_2 \in T$, let $g_1 \oplus g_2$ be $g_1 \upharpoonright A_\ell$ where $\ell \leq k$ is maximal such that $g_1 \upharpoonright A_\ell = g_2 \upharpoonright A_\ell$. Let

$m_\ell^d := h_{A_\ell, A_\ell+1}^d[\mathcal{M}_n]$,

$m_\ell^u := h_{A_\ell, A_\ell+1}^u[\mathcal{M}_n]$,

$p_\ell^d := p_{A_\ell^+, A_\ell+1}^d[\mathcal{M}_n]$,

$p_\ell^u := p_{A_\ell^+, A_\ell+1}^u[\mathcal{M}_n]$.

Lastly let

$\mathcal{K}_{\mathcal{M}_n, f^*} := \{\mathcal{M}_n^+ \in \mathcal{K}_n^+, \mathcal{M}_n^+ \text{ expand } \mathcal{M}_n \text{ and } f^* \text{ is an embedding of } A^+ \text{ into } \mathcal{M}_n^+\}$.

Let $\mu_n^+[f^*, \mathcal{M}_n]$ be the distribution that $\mu_1^+[\mathcal{M}_n]$ induce on $\mathcal{K}_{\mathcal{M}_n, f^*}$.

4.3 Hypothesis. $\mathcal{M}_n$ is random enough, so that: for $\ell < k, f \in T_\ell$ the number of $g, f \subseteq g \in T_{\ell+1}$, is in the interval $[m_\ell^d, m_\ell^u]$ where $m_\ell^d = h_{A_\ell, A_{\ell+1}}^d[\mathcal{M}_n]$.

4.4 Observation. For $\mathcal{M}_n^+$ random enough and $f^* : A^+ \to \mathcal{M}_n^+$.
(a) each \( f \in T_\ell[\mathcal{M}_n] \) with \( \ell < k \), has at least \( m_\ell^d \) immediate successors and at most \( m_\ell^u \) immediate successors

(b) if \( f \in T_\ell[\mathcal{M}_n], x \in \mathcal{M}_n \) then
|\{g \in T_{\ell+1} : f \subseteq g, x \in \text{Rang } g \setminus \text{Rang } f\}| \leq c_\ell^x
(where \( c_\ell^x \) is a constant depending on \((A_\ell, A_{\ell+1})\) only)

(c) the set of immediate successors of \( f \) can be represented as \( \bigcup_{i<\ell} T^{\ell,f,i} \).

4.5 Observation. : So when \( \mathcal{M}_n \) is random enough if \( f \in T_\ell \) embed \( A_\ell^+ \) into \( \mathcal{M}_n^+ \), then the number of \( f' \in T_{\ell+1} \) extending \( f \) which embed \( A_{\ell+1}^+ \) into \( \mathcal{M}_n^+ \) is in the expected case in the interval \([p_\ell^{m_\ell^d}, p_\ell^{m_\ell^u}]\) except for the case \( \ell = k - 1 \) (then this interval is \( \subseteq [0, 1) \), so the number of \( f' \) has a bound, but its expected value is < 1).

Of course, for \( \ell < k - 1 \) we expect that for various \( f \)'s the number will deviate (from the expected value), but for \( \mathcal{M}_n^+ \) random enough none will deviate too much.

4.6 Definition. For \( \mathcal{M}_n^+ \in \mathcal{X}_{\mathcal{M}_n, f^*} \), we define a graph

\[
G[\mathcal{M}_n^+] = G[f^*, \mathcal{M}_n^+] = G[f^*, \mathcal{M}_n^+, T].
\]

Its nodes are \( G[\mathcal{M}_n^+] = \{ f \in T_k : f \text{ embed } B^+ \text{ into } \mathcal{M}_n^+ \} \).

Its set of edges is

\[
R[\mathcal{M}_n^+] = \left\{ (g_1, g_2) : g_1 \in G[\mathcal{M}_n^+], g_2 \in G[\mathcal{M}_n^+],
\right. \]

and \( \text{Rang}(g_1) \cap \text{Rang}(g_2) \neq \text{Rang}(g_1 \cap g_2) \} \).

If \( \epsilon = \{f_1, \ldots, f_m\} \) is a component, its domain is \( \bigcup_{\ell=1}^m \text{Rang } f_\ell \setminus \text{Rang } f^* \).

4.7 Claim. Assume that the tuple \((\bar{h}, \bar{h}, \bar{p})\) satisfies \((\ast)\) of 3.12 and the irrationality inequality, i.e. \( \otimes_1 \) of 3.15.

For any \( c \in \mathbb{R}_+^+ \), for some \( m^\otimes(c) \in \mathbb{N} \), if \( \mathcal{M}_n^+ \) is random enough, then every component of \( G[\mathcal{M}_n^+] \) has \( \leq m^\otimes(c) \) members, with the probability of failure \( \leq \|\mathcal{M}_n^+\|^{-\epsilon} \).

Proof. Choose \( \epsilon > 0 \) such that \( \epsilon < 1 \) and

\((\ast)_1\) if \( A_0^+ \leq C^+ < A_k^+ \), \( C^+ \nmid \tau < A_k \) then \( \beta(C^+, A_k^+) \leq -\epsilon \)

\((\ast)_2\) \( p(A_0^+, A_k^+) \leq -\epsilon \)
(as $A^+_k <_q r A^+_k$ each $\beta(C^+, A^+_k) < 0$ by the irrationality condition and $p(A^+_0, A^+_k) < 0$ as otherwise $A^+_0 <_l c A^+_0 <_l c A_2 <_l \ldots <_l \ldots$; so $\varepsilon$ has just to be below finitely many reals which are $> 0$).

Next we choose $m^0$ such that

\[(*)_3 \ \alpha(\varepsilon(\emptyset, A_0), A_k) - m^0 \times \varepsilon < -e\]

(clearly possible). Next choose $m^1$ such that

\[(*)_4 \ \text{if } B \in K \text{ and } f_\ell : A_k \to B \text{ for } \ell < m^0 - 1 \text{ are embeddings } f_\ell \upharpoonright A_0 = f_0 \upharpoonright A_0 \text{ and } C = c^{\ell[A_k]}(\bigcup_\ell f_\ell(A_k), B), \text{ then } |C||A_k \setminus A_0| < m^1\]

(actually, somewhat less is needed).

So

\[(*)_5 \ \text{if } \{f_\ell : \ell < m^1\} \subseteq G[A_n^+] \text{ is connected then reordering we have:} \ \text{for every } \ell \in (0, m^0) \text{ one of the following occurs:}\]

\[(a) \ \text{Rang}(f_\ell \upharpoonright (A_k \setminus A_0)) \cap \bigcup_{m < \ell} \text{Rang}(f_m) \neq \emptyset \text{ but } \text{Rang}(f_\ell) \not\subset c^{\ell[A_k]}(\bigcup_{m < \ell} \text{Rang}(f_m), A_n)\]

\[(b) \ \text{Rang}(f_\ell \upharpoonright (A_k \setminus A_0)) \cap \bigcup_{m < \ell} \text{Rang}(f_m) = \emptyset \text{ but } \text{Rang}(f_\ell) \cap c^{\ell[A_k]}(\bigcup_{m < \ell} \text{Rang}(f_m), A_n) \neq \emptyset.\]

Why? Suppose this holds for $\ell \in (0, m')$, with $m'$ maximal and assume that $m' < m^0$ and we shall derive a contradiction; note that for $m' = 0$ this holds trivially. So if we can find $\ell \in (m', m^1)$ satisfying (a) or (b) we get contradiction to “$m'$ maximal” so there is no such $\ell$.

Let $S = \{\ell < m' : \text{Rang}(f_\ell \upharpoonright (A_k \setminus A_0)) \cap \bigcup_{m < m'} \text{Rang}(f_m) \neq \emptyset\}$.

Note that

\[(a) \ \ell \leq m' \Rightarrow \ell \in S \Rightarrow \text{Rang}(f_\ell \upharpoonright (A_k \setminus A_0)) \subseteq \bigcup_{m < m'} \text{Rang}(f_m) \Rightarrow \text{Rang}(f_\ell \upharpoonright (A_k \setminus A_0)) \subseteq c^{\ell[A_k]}(\bigcup_{m < m'} \text{Rang}(f_m), A_n)\]

\[(b) \ \ell \in S \& \ell > m' \Rightarrow \ell \text{ fails clause } (a) \Rightarrow \text{Rang}(f_\ell) \subseteq c^{\ell[A_k]}(\bigcup_{m \leq m'} \text{Rang}(f_m), A_n).\]

By \((*)_4\) we have $|S| < m^1$, so $S \neq \{\ell : \ell < m\}$. Also $0 < m'$ so $0 \in S$, hence $S \neq \emptyset$ & $\emptyset \neq S \neq \{\ell : \ell < m\}$. So by the connectivity of $\{f_\ell : \ell < m\}$, i.e. of the graph $G[A_n^+]$, for some $\ell_1 \in S, \ell_2 < m^1, \ell_2 \notin S$ we have $\text{Rang}(f_{\ell_1} \upharpoonright (A_k \setminus A_0)) \cap \text{Rang}(f_{\ell_2} \upharpoonright (A_k \setminus A_0)) \neq \emptyset$. Now by \((a) + (b)\) above $\text{Rang}(f_{\ell_1}) \subseteq c^{\ell[A_k]}(\bigcup_{m \leq m'} \text{Rang}(f_m), A_n)$

hence $\text{Rang}(f_{\ell_2} \upharpoonright (A_k \setminus A_0))$ has an element in this set, but as $\ell_2 \notin S$ it has no
element in $\bigcup_{m\leq m'} \text{Rang}(f_m)$, so $\ell_2$ satisfies clause (b) above. But this contradicts the maximality of $m'$. So we are done.]

Now it is enough to fix the isomorphism type of $(\mathcal{M}_n \mid \bigcup_{\ell \leq m^0} \text{Rang}(f_\ell), f_\ell(d))_{\ell < m^0, d \in A_k}$, call it $t$ (as their number is fixed not depending on $n$). Let for $g: A_0 \rightarrow \mathcal{M}_n$

$$F_t(\mathcal{M}_n, g) = \left\{ (f_\ell: \ell \leq m^0) : f_\ell \text{ embed } A_k \text{ into } \mathcal{M}_n, f_\ell \text{ extends } g \text{ and } t \text{ is the isomorphism type of } \mathcal{M}_n \mid \bigcup_{\ell \leq m^0} \text{Rang}(f_\ell), f_\ell(d)_{\ell < m^0, d \in A_k} \right\}.$$  

(Note: the “$\leq m^0$” rather than “$< m^0$” is intentional).

Let $F_t[\mathcal{M}_n, g] = \{ f \in F[\mathcal{M}_n, g] : \text{for } \ell \leq m^0, f_\ell \text{ embeds } A_k^+ \text{ into } \mathcal{M}_n^+ \}$. We will show that for each $t$, for random enough $\mathcal{M}_n$, the expected value of $|F_t[\mathcal{M}_n^+, g]|$ under the assumption that $g$ embeds $A_0^+$ into $\mathcal{M}_n$ is $\leq \|\mathcal{M}_n\|^{m^0} \times \|\mathcal{M}_n\|^\alpha(\text{c}(\emptyset, A_0), A_k) < \|\mathcal{M}_n\|^{-\varepsilon}$ this clearly suffices.

The rest is straight, still we first note

4.8 Observation: 1) Assume $A_0 \leq A_1 \leq B_1, A_0 \leq B_0 \leq B_1, A_1 \cup B_0 = B_1$. Then $\alpha(\text{c}(A_0, A_1), A_1) \geq \alpha(\text{c}(B_0, B_1), B_1)$.

Proof. 1) Let $\varepsilon > 0$ and let $\mathcal{M}_n$ be random enough. We can find $g: B_1 \rightarrow \mathcal{M}_n$ and so let

$$F_1 = \{ f : f \text{ embed } A_1 \text{ into } \mathcal{M}_n, f \upharpoonright A_0 \subseteq g \}$$

$$F_2 = \{ f : f \text{ embed } A_1 \text{ into } \mathcal{M}_n, f \upharpoonright \text{cl}(A_0, A_1) \subseteq g \}$$

$$F_3 = \{ f : f \text{ embed } B_1 \text{ into } \mathcal{M}_n, f \upharpoonright B_0 \subseteq g \}$$

$$F_4 = \{ f : f \text{ embed } B_1 \text{ into } \mathcal{M}_n, f \upharpoonright \text{cl}(B_0, B_1) \subseteq g \}.$$  

Clearly

(a) $|F_2| \leq \|\mathcal{M}_n\|^\alpha(\text{c}(A_0, A_1), A_1) + \varepsilon$

(b) $|F_4| \geq \|\mathcal{M}_n\|^\alpha(\text{c}(B_0, B_1), B_1) - \varepsilon$

(c) $|F_3| \leq |F_1|$

(d) $|F_1| \leq c|F_2|$ where $c > 0$ is a real depending on $A_0, A_1$ only

(e) $|F_4| \leq |F_3|$. 

Together (if \( \| M_n \|_c > c \)) we get \( (\alpha(\ell(A_0, A_1), A_1) + \varepsilon \geq \alpha(\ell(B_0, B_1), B_1)) - 2\varepsilon \)
but \( \varepsilon \) was any positive real so we are done. \( \square_{4.8} \)

Continuation of the proof of 4.7:

\[
B_\ell^+ = \bigcup_{m \leq \ell} \text{Rang}(f_m) \quad \text{and} \quad B_\ell = B_\ell \mid \tau, \quad \text{so} \quad B_0 = A = A_0.
\]

Let \( D_\ell = \ell(B_\ell, B_{\ell+1}) \) and \( C_\ell^+ = \{ a \in B^+ : f_\ell(a) \in D_\ell^+ \} \) and \( C_\ell = C_\ell^+ \mid \tau. \)

\((*)\) \( \alpha(D_\ell \mid B_{\ell+1}) \leq \alpha(C_\ell, B) \)
[why? apply observation 4.8 with \( A_0, A_1, B_0, B_1 \) there standing for \( \text{Rang}(f_{\ell+1}) \cap D_\ell, (\text{Rang} f_{\ell+1}), D_\ell, B_{\ell+1} \) (note: \( \text{Rang}(f_\ell) \cap D_\ell \subseteq D_\ell_{\ell+1} \)). \( \text{Rang}(f_\ell) \subseteq B_{\ell+1} \) and \( B_{\ell+1} = D_\ell \cup (\text{Rang} f_{\ell+1}), B_{\ell+1} \subseteq A_\ell \)). Together we get \((*)\).] Now

\[
\begin{align*}
\beta(A^+, B_{\ell+1}^+) &= \alpha(A, B_{\ell+1}) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq A^+ \} \\
&= \alpha(A, D_\ell) + \alpha(D_\ell, B_{\ell+1}) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq A^+ \} \\
&\leq \alpha(A, B_\ell) + \alpha(D_\ell, B_{\ell+1}) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq A^+ \} \\
&= \alpha(A, B_\ell) + \sum \{ \beta(C) : C \subseteq B_\ell^+, C \not\subseteq A \} \\
&+ \alpha(D_\ell, B_{\ell+1}) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq B_\ell^+ \} \\
&\leq \beta(A^+, B_{\ell+1}^+) + \alpha(C_\ell, B) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq B_\ell^+ \} \\
&\quad [\text{by \((*)\) above}] \\
&\leq \beta(A^+, B_{\ell+1}^+) + \beta(C_\ell, B^+) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq B_\ell^+ \} \\
&\quad \quad \text{but} \quad C^+ \subseteq f_\ell(C_\ell^+) \lor C^+ \not\subseteq f_\ell(B).
\end{align*}
\]

**Case 1:** \( C_\ell^+ \not\subseteq B^+ \).

Hence \( \beta(A^+, B_{\ell+1}^+) \leq \beta(A^+, B_{\ell+1}^+) + \beta(C_\ell^+, B^+) \leq \beta(A^+, B_{\ell+1}^+) - \varepsilon \)
(why? first inequality as the third summand above is a sum of reals \( \leq 0 \), the second
inequality by definition of \( <_{\ell+1}^+ \) and choice of \( \varepsilon \)).

**Case 2:** \( C_\ell^+ = B^+ \).

So \( \beta(C_\ell^+, B^+) = 0 \), so we get

\[
\beta(A^+, B_{\ell+1}^+) \leq \beta(A, B_\ell) + \sum \{ \beta(C^+) : C^+ \subseteq B_{\ell+1}^+, C^+ \not\subseteq B_\ell^+ \} \quad \text{but} \quad C^+ \subseteq f_\ell(C_\ell^+) \lor C^+ \subseteq f_\ell(B^+) \leq \beta(A^+, B_{\ell+1}^+) - \varepsilon. \quad \square_{4.7}
\]
The following definition points to the fact that there may be quite a different situation in spite of our treating them together as they are similar enough for our aims.

4.9 Definition.  1) We say $T$ is simple of the first kind if:
$g_1, g_2 \in T_i \Rightarrow \text{Rang}(g_1) \cap \text{Rang}(g_2) = \text{Rang}(g_1 \cap g_2)$.
2) We say $T$ is simple of the second kind if:
for every $g_1, g_2 \in T_\ell$, we have

$$\{g_1 \upharpoonright (A_{\ell+1} \setminus A_\ell) : g_1 \leq g_1' \in T_{\ell+1}\} = \{g_2 \upharpoonright (A_{\ell+1} \setminus A_\ell) : g_2 \leq g_2' \in T_{\ell+1}\}.$$  

3) $T$ is separated when:

$$\{g_1, g_2\} \subseteq T, \{y_1, y_2\} \subseteq A_k, g_1(y_1) = g_2(y_2) \Rightarrow y_1 = y_2.$$  

4) $T$ is locally disjoint if for $f \in T_\ell$, $f \subseteq g_1 \in T_{\ell+1}$, $f \subseteq g_2 \in T_{\ell+1}$, $g_1 \neq g_2$ we have $\text{Rang}(g_1) \cap \text{Rang}(g_2) = \text{Rang}(f)$.

4.10 Remark. Note the separated case assumption helps as it gives monotonicity in the probability. The following indicates that we can assume $T$ is separated.

4.11 Claim. Each problem (i.e. from 4.1) we can split $T_k[f^*, \mathcal{M}_n]$ to $\leq (\log_2 ||\mathcal{M}_n||)^{|B^*|+1}$ sets, each of them separable. (So if our estimates can absorb the inaccuracy involved we have reduced our problem to a separable one).

Proof. Just choose for each $c \in \mathcal{M}_n$ a sequence $\rho_c$ of zeroes and ones of length $\leq \|\mathcal{M}_n\|$ such that $c \neq d \Rightarrow \rho_c \neq \rho_d$. For each $g \in T_k[f^*, \mathcal{M}_n]$, let

$$w_g = \{\min\{i : \rho_c(i) \neq \rho_d(i)\} : c \neq d \in \text{Rang} g \setminus \text{Rang} f^*\},$$

and let

$$v_g = \{\{c, \rho_{g_c}(i)\} : c \in B^* \setminus A^*, i \in w_g\}.$$  

Define an equivalence relation $\equiv$ on $T_k[f^*, \mathcal{M}_n]$ as follows: $g_1 \equiv g_2$ iff $w_{g_1} = w_{g_2}$ & $v_{g_1} = v_{g_2}$. Now $\equiv$ gives a division as required. $\square$.

4.12 Discussion. Now 4.11 is sufficient for our aims, but we can get better: division to constant number, and preserving the order of magnitude of the splitting of the tree.

4.13 Claim. Consider the probability space of all $c$ where $c$ is a function from $A^* = \bigcup \{\text{Rang}(f) : f \in T_k\} \setminus \text{Rang}(f^*)$ to $A_k \setminus A_0$; all of the same probability. Let the space measure be called $\mu_{\text{sep}}$. Let $T_{[c]} = \{\eta : c \circ f_\eta \text{ is the identity on } A_{\ell g(\eta)}\}$ and $T^{[c]}_\ell = T^{[c]} \cap T_\ell$.

Then

(1) The probability that all the splitting in $T^{[c]}$ are nearly the expected value (meaning if the expected value is $v$, error is $\leq v^{\frac{1}{2} + \varepsilon}$) is very near to 1 assuming the largeness condition. (e.g. in the polynomial case)
(2) Let \( e^* = (|A_k\backslash A_0^*|)|A_k^* \backslash A_0^*|^{-|A_k^* \backslash A_0^*|} \in (0, 1)_\mathbb{R} \) and \( a_{dn} \) (or \( a_{up} \)) be the natural lower (or upper bound) of the expected value of \(|T[\mathcal{M}_n^+]|\). Then for any \( \varepsilon > 0 \) we have \( \mu_{sep} \)-almost surely the chosen \( c \) satisfies

\[
\text{Prob}_{\mu_n} \left( |T_k \cap T[\mathcal{M}_n^+]| \leq a_{dn} - a_{dn}^{1/2 + \varepsilon} \right) \leq \\
\text{Prob}_{\mu_n} \left( |T_k^c \cap T[\mathcal{M}_n^+]| \leq e^* a_{dn} - a_{dn}^{1/2 + \varepsilon} \right) + \|\mathcal{M}_n\|^{-\varepsilon}.
\]

(3) \[ \text{Prob}_{\mu} \left( |T_k \cap T[\mathcal{M}_n^+]| \geq a_{up} + a_{up}^{1/2 + \varepsilon} \right) \leq \\
\text{Prob} \left( T_k^c \cap T[\mathcal{M}_n^+] \leq e^* a_{up} + a_{up}^{1/2 + \varepsilon} \right) + \|\mathcal{M}_n\|^{-\varepsilon}.
\]

**Proof.** If we first draw \( \mathcal{M}_n^+ \) then ignoring an event with probability \( \|\mathcal{M}_n\|^{-\varepsilon} \), the components of \( T[\mathcal{M}_n^+] \) are all of size \( \leq m^\otimes(e)(\in \mathbb{N}, \text{ from 4.7}) \). So the number we get after drawing a \( \mu_{sep} \)-random \( c \), behave by a multinomial distribution, so almost surely for \( c \), we get the expected number with small error. By commutativity of probability, this implies the conclusion. \( \square_{4.13} \)
§5 The probability arguments

We relax our framework, forget about the tree (from §4), and just have a family $F$ of one-to-one functions from $[m]$ to $[n]$ (thinking $n, |F|$ are much larger than $m$), $F$ separative for simplicity (i.e. $f_1(\ell) = f_2(\ell') \Rightarrow \ell_1 = \ell_2$), and $A^*$ a $\tau^+$-model with set of elements $[m]$ with vocabulary $\tau^+$. Now we draw some relations on $[n]$ to get $\mathcal{M}_n^*$ independently and want to know enough on the number $\mathcal{L}$ of $f \in F$ such that all appropriate relations were chosen. The easiest case is when $f_1 \neq f_2 \in F \Rightarrow \text{Rang}(f_1) \cap \text{Rang}(f_2) = \emptyset$, then we get a binomial distribution. Still we are interested in the case that for every “successful” $f \in F$, the number of successful $f' \in F \setminus \{f\}$ not disjoint to $f$ is small; i.e. the expected number is $\ll 1$. So we define components of the set of successful $f$, and look what is their number. We first show that for $L$ which is larger than the expected value the probability of having “the number successes is $L$” to decrease with $L$ as in the binomial distributions. Then we get a slightly worse statement on what occurs for $L$ smaller than the expected value; the “error” term comes from the number of $f \in F$ such that in $f([m])$ there is no relation but if we change $\mathcal{M}_n^*$ such that $f$ is successful, it is not a singleton. But by the first argument (or direct checking in the cases from §4) this is small. Of course, we have larger components, but for each isomorphic type the problem of the distribution of their number is like the original one except being only weakly separative. Clearly this framework is wide enough to include what is needed in §4.

Note: Clearly the higher components contribute little but we do not elaborate as there is no need: we may restrict ourselves to finitely many components.

5.1 Definition. 1) We say $\bar{y} = (m, \bar{p}, n, F^\bar{y}) = (m, \bar{p}, n, F) = (m^\bar{y}, p^\bar{y}, n^\bar{y}, F^\bar{y})$ is a system (or $m$-system or $(m, n)$-system) if:

a) $F$ is a family of one-to-one functions with domain $[m] = \{1, \ldots, m\}$ into the set $[n] = \{1, \ldots, n\}$

b) $\mathcal{P} \subseteq \{u : u \subseteq [m], u \neq \emptyset\}$, $e$ an equivalent relation on $\mathcal{P}$,

c) $\bar{p} = (p_u/e : u \in \mathcal{P}/e)$, where $p_u/e$ is a probability and let $p_u = p_{u/e}$.

Let $R_u = \{f(u) : f \in F\}, R_{u/e} = \bigcup\{R_u' : u' \in u/e\}$

(we can look at them as symmetric relations)

d) we choose for each $u \in \mathcal{P}, R_{u/e} \subseteq R_{u/e}$ by drawing for each $v \in R_{u/e}$ a decision for $v \in R_{u/e}$, independently (for distinct $(u, v)$) with probability $p_{u/e}$. The distribution (on $\mathcal{N}^*$ (see below)) is called $\mu_n^*$.

2) We call $\mathcal{M}_n^* = ([n], \ldots, R_{u/e}, R_{u/e}, \ldots)_{u \in \mathcal{P}}$ a $\mu_n^*$-random model; we may omit $\bar{y}$. Let $\mathcal{K}_n^* = \mathcal{K}_n^*[\bar{y}] = K_\bar{y}$ be the set of all possible $\mathcal{M}_n^*$. Note that $\mathcal{P}, e$ can be defined from $\bar{p}$, so we write $\mathcal{P} = \mathcal{P}^\bar{y}, e = e^\bar{y}$ or $\mathcal{P}^n, e^n$. Note that from $M_n \in \mathcal{K}_n^*$ we can reconstruct $m^\bar{y}, n^\bar{y}$ and $F^\bar{y}$ (though not $\bar{p}^\bar{y}$) and from $F^\bar{y}$ we can reconstruct $m^\bar{y}, n^\bar{y}$ if $[m] = \bigcup\{u : u \in \mathcal{P}\}$.

* * *
It is natural to demand $F$ is separative (see Definition 5.2(1) below), as we can reduce the general case to this one (though increasing the “error” term see 4.11, 4.13). But why do we consider “weakly separative” (see Definition 5.2(2) below)? The main arguments here gives reasonable estimates if we have estimated the number of occurrences of non-trivial components, so we need to estimate them. In order to bound the “error” gotten by making the estimation of the number of occurrences of a component, we weaken the definition to include this (we could somewhat further weaken “weakly separative”, but do not, just as it gives no reasonable gain now).

5.2 Definition. 1) We say $F = F^n$ (or $\bar{y} = (m, \bar{p}^\alpha, n, F^n)$ is separative if:

   \[
   \{f_1, f_2\} \subseteq F_n \& \{\ell_1, \ell_2\} \subseteq [m] \& f_1(\ell_1) = f_2(\ell_2) \Rightarrow \ell_1 = \ell_2.
   \]

2) We call $\bar{y}$ semi-separative if:

   there is an equivalence relation $e^*$ on $[m]$ such that

   \[\begin{align*}
   (i) & \quad u \in \mathcal{P} \Rightarrow (\forall \ell, k)(\ell \in u \& k \in u \& \ell \neq k \rightarrow \neg \ell e^* k) \text{ and} \\
   (ii) & \quad (\forall u_1, u_2 \in \mathcal{P})((\forall k)[u_1 \cap (k/e^*) \neq \emptyset \leftrightarrow u_2(k \cap (k/e^*)) \rightarrow u_1 \mathcal{E} u_2]);
   \end{align*}\]

   i.e. $e$ refines $e^{**} := \{(u_1, u_2) : \{u_1, u_2\} \subseteq \mathcal{P} \text{ and } (\forall k)[u_1 \cap (k/e^*) \neq \emptyset \leftrightarrow u_2(k \cap (k/e^*) \neq \emptyset)\}

   (iii) there is an equivalence relation $e'$ on $[n]$ such that:

   if $\{f_1, f_2\} \subseteq F, \{m_1, m_2\} \subseteq [m], \text{ then } m_1 e^* m_2 \Rightarrow f_1(m_1) e' f_2(m_2)$

   (iv) if $\{f_1, f_2\} \subseteq F^n, \{u_1, u_2\} \subseteq \mathcal{P}$ and $f_1(u_1) = f_2(u_2)$ then

   $u_1 \mathcal{E} u_2 \& f_1 \upharpoonright u_1 = f_2 \upharpoonright u_2$.

3) We say $\bar{y}$ is weakly separative if (i), (ii), (iii) of part (2) holds.

4) For $X \subseteq \mathcal{P}$ let

   \[q_X = \prod_{u \in X} p_u \times \prod_{u \in \mathcal{P} \setminus X} (1 - p_u).\]

Remark. Note: we are thinking of the cases the $p_u$’s are small. If some are essentially constant, we treat them separately.

5.3 Definition. 1) For a system $\bar{y} = (m, \bar{p}, n, F)$ and $M_n^* \in \mathcal{X}_\bar{y}^*$, we define a graph $G[M_n^*] = G_{\bar{y}}[M_n^*] = (F[M_n^*], E[M_n^*])$.

   Its set of nodes is

   \[F[M_n^*] = \{f \in F : f(u) \in R_{u/e}^{M_n^*} \text{ for every } u \in \mathcal{P}\}\]

   \[E[M_n^*] = E[F] \upharpoonright F[M_n^*]\]

   where $E[F] = \{(f_1, f_2) : f_1 \in F, f_2 \in F, f_1 \neq f_2 \text{ and } \text{Rang}(f_1) \cap \text{Rang}(f_2) \neq \emptyset\}$. If $F_{\mathcal{E}} = \bigcup_{f \in \mathcal{E}} \text{Rang}(f)$. For a component $\mathcal{E}$ of $G[M_n^*]$ let $V[\mathcal{E}] = \bigcup_{f \in \mathcal{E}} \text{Rang}(f)$. 
2) We say that two components \( C_1, C_2 \) of \( G[M^*_n] \) are isomorphic if there is an embedding \( f : C_1 \to C_2 \) such that for any \( a_1, a_2 \in [m] \) and \( f_1, f_2 \in C_1 \) we have:

\[
(f(f_1))(a_1) = (f(f_2))(a_2) \iff f_1(a_1) = f_2(a_2)
\]

2A) We say that \( f \) is an embedding of a possible component \( C_1 \) into a possible component \( C_2 \) (possible means it is a component in some \( M^*_n \)) if: \( f \) is a one to one function from \( C_1 \) into \( C_2 \) satisfying the demand in (2). The isomorphism type \( C / \cong \) of a component \( C \), is naturally defined.

3) Let \( T_k = T_k^p = \{ C / \cong : \) for some \( \bar{y} \) an \( m \)-system and some \( M^*_n \in \mathcal{X}^*_p, C \subseteq F^\bar{y} \) is connected component in the graph \( G[M^*_n] \) and 

\[
|C| = k \}
\]

4) Assume \( \bar{y} \) is semi-separative, \( f_\ell : [m] \to [m^*] \) for \( \ell = 1, \ldots, k \). Let \( T = T^m = \bigcup_k T_k^{p^*} \). Let \( t^* \) be the isomorphic type of singleton. Normally \( m \) is constant so we may omit it.

5) Let \( L_4[M^*_n] \) be \( \mathcal{L}_4[M^*_n] \) where \( \mathcal{L}_4[M^*_n] = \{ C : \) a component of \( G[M^*_n] \) such that the isomorphic type of \( C \) is \( t \} \)

6) \( \bar{L} = \bar{L}[M^*_n] = (L_4[M^*_n] : t \in \bigcup_k T_k) \)

7) \( \mathcal{X}^*_p[F, \bar{L}] = \{ M_n \in \mathcal{X}^*_p[F] : \bar{L}[M^*_n] = \bar{L} \} \)

8) \( F_X[M^*_n] = \{ f \in F : \) \( u \in \mathcal{P} \Rightarrow [f(u) \in \mathcal{R} \equiv u \in X] \) \( \} \) for \( X \subseteq \mathcal{P} \)

9) \( F^*_{\bar{y}}[M^*_n] = F^\bar{y}[M^*_n] \) where for \( X \subseteq \mathcal{P} \) we let:

\[
F^*_{\bar{y}}[M^*_n] = \left\{ f \in F : f \in F_X[M^*_n] \text{ and for no } f' \in F \backslash \{ f \} \text{ do we have:} \right. \]

\[
u \in \mathcal{P} \& f'(u) \not\in f([m]) \Rightarrow f'(u) \in \mathcal{R}_{u/e}[M^*_n] \left\} \right.
\]

10) \( F_{\bar{y}}[M^*_n] = F^\bar{y}[M^*_n] \) where for \( X \subseteq \mathcal{P} \) we let \( F^\bar{y}[M^*_n] = : \{ f \in F : f \in \mathcal{L}_{t^*}[M^*_n] \} \)

5.4 Claim. 1) Separative implies semi-separative which implies weakly separative.

2) If \( \bar{y} \) is weakly separative, \( f_\ell : [m] \to [m^*] \) for \( \ell = 1, \ldots, k \) \{ \( f_\ell : \ell \in [k] \} \) is weakly separative, \( \bigcup_{\ell \in [k]} \text{Rang}(f_\ell) = [m^*] \) and

\[
F^* = \{ g : g : [m^*] \to [n] \text{ is one to one, } \ell \in [k] \Rightarrow g \circ f_\ell \in F^\bar{y} \}, p_{F^*} = p_u,
\]

\( \mathcal{P}^* = \{ f_\ell(u) : u \in \mathcal{P} \} \) then \( \bar{y}^* = (m^*, \bar{y}^*, n, F^*) \) is weakly separative.

3) Similarly with semi-separative instead weakly separative.

Proof. Straight.

Remark. The following claim says that above the expected value, the probability goes down fast enough.
5.5 Claim. Assume \( F \) is weakly separative and \( \bar{L}^\ell = \langle L^\ell_t : t \in T \rangle \), for \( \ell = 1, 2 \) and

\[
L^\ell_t = \begin{cases} 
L^\ell_t + 1 & \text{if } t = t^* \text{ (is singleton)} \\
L^\ell_t & \text{otherwise.}
\end{cases}
\]

Then \( (q_0, q_\varnothing) \) are defined in 5.2(4):

\[
\text{Prob}_{\mu_*^n}(\bar{L}[\mathcal{M}_n^*] = \bar{L}^1) \geq q_0 \left( \frac{L^2_\varnothing}{q_\varnothing |F|} \right) \text{Prob}_{\mu_*^n}(\bar{L}[\mathcal{M}_n^*] = \bar{L}^2).
\]

5.6 Remark. 1) On \( F_\varnothing \) see Definition 5.3(7).
2) Note: \( \text{Exp}(F_\varnothing[\mathcal{M}_n^*]) = q_\varnothing |F| \) and \( |F_\varnothing[\mathcal{M}_n^*]| = \sum_{t \in T} |t|L_t[\mathcal{M}_n^*] \) where \( |t| \) is the number of \( f \in C \) for any \( C \) of isomorphism type \( t \).

Proof. Let us consider for \( X \subseteq \mathcal{P} \),

\[
W_X = \begin{cases} 
(M^1, M^2) : M^1 \in \mathcal{K}_n^*[F, \bar{L}^1], M^2 \in \mathcal{K}_n^*[F, \bar{L}^2], \\
\text{for } t \in T \setminus \{t^*\}, \mathcal{L}_t[\mathcal{M}^1] = \mathcal{L}_t[\mathcal{M}^2] \text{ and} \\
\mathcal{L}_t^+[\mathcal{M}^1] \subseteq \mathcal{L}_t^+[\mathcal{M}^2] \text{ and if } \{f\} = \mathcal{L}_t^+[\mathcal{M}^2] \setminus \mathcal{L}_t^+[\mathcal{M}^1] \\
\text{then } f(u) \in \mathcal{R}^M_{u/e} \leftrightarrow u \in X \text{ for each } u \in \mathcal{P} \\
\text{and: if } f' \in F, u \in \mathcal{P}^+, f'(u) \notin \text{Rang}(f) \\
\text{then } u \in \mathcal{R}^M_{u/e} \leftrightarrow u \in \mathcal{R}^M_{u/e} 
\end{cases}
\]

\[
W = \bigcup_{X \subseteq \mathcal{P}} W_X.
\]

Clearly

\((*)_1\) for every \( X \subseteq \mathcal{P} \) and \( M^2 \in \mathcal{K}_n^*[F, \bar{L}^2] \) the set 
\{ \( M^1 : (M^1, M^2) \in W_X \) \} has exactly \( L^2_{\varnothing} \) members 

(here we use “\( F \) is weakly separative”) and

\((*)_2\) for every \( M^1 \in \mathcal{K}_n^*[F, \bar{L}^1] \), the number of \( \{M^2 : (M^1, M^2) \in W_X \} \) has at most \( |F| - \sum_{t \in T} |t|L^1_t \leq |F| \) members

(note: this is a quite crude bound as it does not take into account that not only each \( f \in C \subseteq L^1_t \) is disqualified as the possible member of \( L^2_{\varnothing} \setminus L^1_t \), but also each \( f' \in F \) such that \( \exists u \in \mathcal{P} \) \((f(u) = f'(u)) \); but at present the effect does not disturb us).

\((*)_3\) for every \( (M^1, M^2) \in W_X \) and \( X \subseteq \mathcal{P} \), we have (see Definition 5.2(4))

\[
\text{Prob}_{\mu_*^n}(\mathcal{M}_n^* = M^1)/q_X = \text{Prob}_{\mu_*^n}(\mathcal{M}_n^* = M^2)/q_\varnothing.
\]
Now

\[ |F| \text{ Prob}_{\mu_n}(\vec{L}[\mathcal{M}_n^*] = \vec{L}^1)/q_0 \]

[by the definition of \( \mathcal{X}_n[F, \vec{L}^1] \)]

\[ = \sum_{M^1 \in \mathcal{X}_n[F, \vec{L}^1]} |F| \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^1)/q_0 \]

[by (\#)_2]

\[ \geq \sum_{M^1 \in \mathcal{X}_n[F, \vec{L}^1]} \sum_{M^2 \text{ satisfies } (M^1, M^2) \in W_\emptyset} \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^1)/q_0 \]

[by interchanging sums]

\[ = \sum_{M^2 \in \mathcal{X}_n[F, \vec{L}^2]} \sum_{M^1 \text{ satisfies } (M^1, M^2) \in W_\emptyset} \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^1)/q_0 \]

[by (\#)_3]

\[ = \sum_{M^2 \in \mathcal{X}_n[F, \vec{L}^2]} L^2_{\emptyset} \times \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^2)/q_{\emptyset} \]

[by the definition of \( \mathcal{X}_n[F, \vec{L}^2] \)]

\[ = L^2_{\emptyset} \times \text{ Prob}(\vec{L}_4 \cdot [\mathcal{M}_n^*] = \vec{L}_2)/q_{\emptyset}. \]

Now the conclusion follows. \( \Box_{5.5} \)

5.7 Claim. Assume \( F \) is weakly separative. If \( L^1 + 1 = L^2 \), then

\[ \text{ Prob}_{\mu_n}(L_4 \cdot [\mathcal{M}_n^*] = L^1) \geq q_0 \left( \frac{L^2}{q_{\emptyset} \times |F|} \right) \times \text{ Prob}_{\mu_n}(L_4 \cdot [\mathcal{M}_n^*] = L^2). \]

Proof. By 5.5, dividing the event to cases. \( \Box_{5.7} \)
5.8 Conclusion. Assume $F$ is weakly separative.

1) If $L^* \geq 2(q_{\mathcal{P}}|F|/q_0)$ then $\text{Prob}_{\mu_n}(L_4^* \cdot [\mathcal{M}_n] > L^*) \leq \text{Prob}_{\mu_n}(L_4^* \cdot [\mathcal{M}_n] = L^*)$.

2) If $L^* \geq |q_{\mathcal{P}}|F|/q_0$ then

$$\text{Prob}_{\mu_n}(L_4^* \cdot [\mathcal{M}_n^*] = L^*) \leq \prod_{L = |q_{\mathcal{P}}|F|/q_0}^{|L^*| - 1} \frac{L}{|q_{\mathcal{P}}|F|/q_0}.$$  

Proof. Iterate 5.5, i.e. by induction on $L^*$.

5.9 Claim. Let $F$ be weakly separative. Assume $\bar{L}^\ell$ for $\ell = 1, 2$ are as in 5.5 and (see Definition 5.3(8))

$$\otimes \alpha \in (0,1)_{\mathbb{R}} \text{ and } \zeta \in (0,1)_{\mathbb{R}} \text{ and }$$

$$\zeta \geq \text{Prob}_{\mu_n}(\bar{L}^1 \cdot [\mathcal{M}_n^*] = \bar{L}^1 \text{ and } |F^* \cdot [\mathcal{M}_n^*]| < \alpha |F|)$$

(On $F^* \cdot [\mathcal{M}_n^*]$ see Definition 5.3(8)). Then

$$\text{Prob}_{\mu_n}(\bar{L}^1 \cdot [\mathcal{M}_n^*] = \bar{L}^1) \leq q_0 \frac{L_2^2}{q_{\mathcal{P}}\alpha |F|} \text{ Prob}_{\mu_n}(\bar{L}^2 \cdot [\mathcal{M}_n^*] = \bar{L}^2) + \zeta.$$  

Remark. In the cases we have in mind, for $n$ going to infinity, $\alpha$ goes to 1, $\zeta$ goes to 0, very fast indeed.

Proof. Start as in the proof of 5.5 getting $(\ast)_1, (\ast)_2, (\ast)_3$ but then:
\(\alpha |F| \text{ Prob}_{\mu_n}(\bar{L}, \mathcal{M}_n^* = \bar{L})/q_0\)

[by \(\bigotimes\) in the assumption and by the Definition of \(\mathcal{K}_n[F, \bar{L}]\)]

\[\leq \alpha |F| \times \zeta + \sum_{M^1 \in \mathcal{K}_n[F, \bar{L}]} \alpha |F| \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^1)/q_0\]

[by (*)_2]

\[\leq \alpha |F| \times \zeta + \sum_{M^1 \in \mathcal{K}_n[F, \bar{L}]} \sum_{M^2 \text{ satisfies } (M^1, M^2) \in W_\beta} \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^1)/q_0\]

[by interchanging sums]

\[= \alpha |F| \times \zeta + \sum_{M^2 \in \mathcal{K}_n[F, \bar{L}^2]} \sum_{M^1 \text{ satisfies } (M^1, M^2) \in W_\beta} \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^1)/q_0\]

[by (*)\_3]

\[= \alpha |F| \times \zeta + \sum_{M^2 \in \mathcal{K}_n[F, \bar{L}^2]} \sum_{M^1 \text{ satisfies } (M^1, M^2) \in W_\beta} \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^2)/q_\varnothing\]

[by (*)\_4]

\[= \alpha |F| \times \zeta + \sum_{M^2 \in \mathcal{K}_n[F, \bar{L}^2]} L_{\xi^*} \times \text{ Prob}_{\mu_n}(\mathcal{M}_n^* = M^2)/q_\varnothing\]

[by the definition of \(\mathcal{K}_n[F, L^2]\)]

\[= \alpha |F| \times \zeta + L_{\xi^*} \times \text{ Prob}(\bar{L}_\xi \cdot (\mathcal{M}_n) = \bar{L}^2)/q_\varnothing.\]

Dividing by \(\alpha |F|/q_\varnothing\) we get the desired conclusion. \(\square_{5.9}\)

* * *

5.10 Claim. Assume \(F\) is weakly separative. Assume \(L^2 = L^1 + 1\) are given and

(*) \(\alpha \in (0, 1)_R\) and \(\zeta \in (0, 1)_R\) and

\[\zeta \geq \text{ Prob}_{\mu_*}(L_{\xi^*} \cdot [\mathcal{M}_n^*] = L^1 \text{ and } |F_*[\mathcal{M}_n^*]| < \alpha |F|).\]
Then

\[ \text{Prob}_{\nu_n^*} \left( L_{t^*} [\mathcal{M}_n^*] = L_1 \right) \leq \frac{\zeta^*}{t^*} + q_0 \frac{L_1^2}{q \alpha |F|} \cdot \text{Prob}_{\nu_n^*} \left( L_{t^*} [\mathcal{M}_n^*] = L_2 \right). \]

**Proof.** By 5.9, dividing the even to case, noting that:

\[ \text{Prob}_{\nu_n^*} \left( L_{t^*} [\mathcal{M}_n^*] = L_1 \text{ and } |F_1 [\mathcal{M}_n^*]| < \alpha |F| \right) \]

\[ = \sum \{ \text{Prob}_{\nu_n^*} \left( L [\mathcal{M}_n^*] = L \text{ and } |F_1 [\mathcal{M}_n^*]| < \alpha |F| \right) : L \}
\]

\[ = (L_t : t, L_{t^*} = L_1). \]

$\square$ 5.10
§ 6 Free Amalgamation

We like to axiomatize “free amalgamation” in its connection to 0-1 laws (in previous cases the “edgeless disjoint amalgamation” serves).

* * *

We first define a context having “free amalgamation”. The idea is that it is not necessarily a “disjoint amalgamation with no additional relations” as we may allow say a two-place relation with probability $\frac{1}{2}$, so cases of this relation has no influence on the amalgamation being free.

6.1 Definition. 1) We say $(\mathcal{K}, \sqcup)$ is a 0-1 context [or weakly 0-1 context] (no contradiction to 1.1(1)) if $\mathcal{K}$ satisfies (a),(b),(c) as in 1.1(1) and $(\mathcal{X}_{\infty}, <, s)$ are defined as in 1.3(1),1.3(2)(c) above and):

(d) $\sqcup$ is a four-place relation on $\mathcal{X}_{\infty}$ written as $B \cup_{A}^{D} C$ or $\sqcup (A, B, C, D)$. This relation is preserved under isomorphism and we say: $B, C$ are $\sqcup$-freely amalgamated over $A$ inside $D$ (and omit $D$ if clear from the context)

(e) $B \cup_{A}^{D} C$ implies $A \leq s, B \subseteq D, A \subseteq C \subseteq D, B \cap C = A$.

(f) (base increasing): if $B \cup_{A}^{D} C$ and $A \subseteq C_{1} \subseteq C$ then $B \cup_{C_{1}}^{D} C$.

(g) monotonicity: $A \subseteq B \leq s, B \subseteq C \subseteq C$ and $B \cup_{A}^{D} C$ implies $B' \cup_{A}^{D} C'$.

(h) monotonicity: if $B \subseteq D \subseteq D, C \subseteq D \subseteq C$ then $B \cup_{A}^{D} C \iff B \cup_{A}^{D} C$ but in the “weakly” version for $\iff$ we add the assumption $\Box_{D', D}$ where we let $\Box_{A_{1}, A_{2}}$ be defined as

$\Box_{A_{1}, A_{2}} \quad A_{1} \subseteq A_{2} \in K_{\infty}$ and if $\emptyset \leq s, A' \subseteq s, A_{2}$ then$^{9} A' \subseteq A_{1}$

(i) existence: if $A \leq s, B$ and $A \subseteq C$ then for some $D$ and $f$ we have: $C \subseteq D, f$ is an embedding of $B$ into $D$ over $A$ and $\sqcup (A, f(B), C, D)$ (but in the “weakly” version this is omitted).

(j) Right Transitivity: if $B_{0} \sqcup_{A_{0}}^{B_{2}} A_{1}$ and $B_{1} \sqcup_{A_{1}}^{B_{2}} A_{2}$ and $B_{0} \subseteq B_{1}$ then $B_{0} \sqcup_{A_{0}}^{B_{2}} A_{2}$

$^{9}$of course, if $A \in \mathcal{X}_{\infty} \Rightarrow \emptyset \leq s, A$ this condition holds trivially. We expect that in “reasonable” cases such assumption can be removed (in clauses (g) and (h))
(k) Left Transitivity: If \( A_1 \cup B_0 \) and \( A_2 \cup B_1 \) and \( B_0 \subseteq B_1 \) then
\[
A_2 \cup B_0
\]

(l) \((\mathfrak{R}, \cup)\) has symmetry which means

\[(*) \text{ if } B \cup C \text{ and } A \leq C \text{ then } C \cup B\]

(m) Smoothness: \( B \cup C \) implies \( C \subseteq B \cup C \) (of course, \((d) + (f)\) implies this).

2) We say \((\mathfrak{R}, \cup)\) has the strong finite basis property when

(n) for every \( \ell \) for some \( m \)
\[(*) \text{ if } B \subseteq D, C \subseteq D, |B| \leq \ell \text{ then for some } A \subseteq C, |A| \leq m \text{ and }\]
\[
B \cup A \cup C
\]
or at least

(n) for every \( \ell, r \) for some \( m \)
\[(*) \text{ if } |B| \leq \ell, B \subseteq D, A_i \subseteq D \text{ for } i \leq m \text{ such that } A_i \subseteq A_{i+1}, \text{ then for some } i \text{ and } C \subseteq A_i \text{ we have:}\]
\[
B \cap A_{i+1} \subseteq C \subseteq A_i \text{ and } B \cup C \cup \ell r(C, A_{i+1})
\]

3) We say \( \cup \) (or \((\mathfrak{R}, \cup)\)) has the uniqueness if the \( \cup \)-free amalgamation is unique, that is: suppose that for \( \ell = 1, 2 \) \( B \cup \ell \cup C \) and \( D \cup \ell \cup C \) and \( f \) is an isomorphism from \( B_1 \) onto \( B_2 \) and \( g \) is an isomorphism from \( C_1 \) onto \( C_2 \) and \( A_1 = A_2, f \upharpoonright A_1 = g \upharpoonright A_2 \), then \( f \cup g \) is an isomorphism from \( D_1 \) onto \( D_2 \).

4) We say \( \cup \) (or \((\mathfrak{R}, \cup)\)) has dual transitivity when:

\[(*) \ A_1 \cup C_0, A_2 \cup C_1 \text{ then } A_2 \cup C_2 \text{ but in the weak case assume } \exists_{C_1, C_2}.
\]

6.2 Fact

1) If \( \mathfrak{R} \) is a 0-1 context (see 1.1, with \( \leq_s \) as in Definition 1.3(2)(c)) and \( \cup = \{(A, B, C, D) : A \leq_s B \subseteq D, A \subseteq C \subseteq D, B \cap C = A \text{ and the quadruple is freely}\)
amalgamated in the sense of 0.2(6) (no new instances of the relations)) then \((\mathcal{R}, \mathcal{U})\) is a 0-1 context except possibly \((f)(\text{base increasing}), (i)(\text{existence}), (m)(\text{smoothness}); \text{with uniqueness (see 6.1(3)).}

2) Assume \((\mathcal{R}, \mathcal{U})\) is a 0-1 context

\[(a) \text{ if } B \bigcup_A D, C, D = B \cup C, B' \le B, A \le A' \le C, \]
\[r = k + |A|, cf^A(A, C) \subseteq A^+ \text{ then } cf^A(B', D) \subseteq B \cup A^+ \]
\[(b) \text{ if } B \bigcup_A C, A \le_i D', D' \le B \cup C \text{ then } A \le_i D' \cap C. \]

**Proof.** 1) Straight.

**clauses (a)-(e): By the definitions.**

**clause (g):** Check the definitions, and by 1.6(6) we have \(A \le_s B'\).

**clauses (h),(i):** Check.

**clause (k):** The point is that \(A_0 \le_s A_1 \le_s A_2 \Rightarrow A_0 \le_s A_2\) by 1.6(10).

**clause (l):** Read on the meaning of \(\mathcal{U}\).

**uniqueness:** Reflect on the meaning of \(\mathcal{U}\).

2) **clause (a):** By monotonicity of \(\mathcal{U}\) (see Definition 6.1(1)(g) we may assume that \(B = B'\). Assume toward contradiction that the conclusion fails. So there are \(C' \le D, |C'| \le k, d \in C \setminus B \cup A^+ \subseteq C \setminus A^+\) and \(C' \cap B <_i C', \text{ hence (see 1.6(3)), } B <_i B \cup C', \text{ and let } C_0 \text{ be such that } A \le_i C_0 \le_s C_1 \text{ (exists by 1.6(4)). By clauses (f) + (g) of Definition 6.1(1) we have } B \cup C_0 \bigcup_A D, C, D \subseteq B \cup C_0 \text{ and } B \cup C_0 \le_s B \cup C_1. \text{ So as } C' \cap B <_i C' \subseteq B \cup C_1 \text{ (as } C_1 = A \cup (C' \cap C) \text{ and } D = B \cup C) \text{ by 1.6(3) we necessarily have } C' \subseteq B \cup C_0, \text{ so as } d \in C \setminus B \text{ necessarily } d \in C_0, \text{ but } |C_0| \le |A| + |C' \cap C \setminus A| \le |A| + k = r. \text{ Remember } A \le_i C_0, \text{ so } C_0 \subseteq cf^A(A, C) \subseteq A^+, \text{ hence } d \in A^+ \text{ a contradiction.}

**clause (b):** If the conclusion fails, then for some \(C_1, A \le_i C_1 <_s D' \cap C. \) By monotonicity (= clause (g) of Definition 6.1(1)) we have \(B \bigcup_A D' \cap C, \) so by smoothness (= clause (m) of Definition 6.1(1) we have \(D' \cap C \le_s (B \cup (D' \cap C) \text{ but } \le_s \text{ is transitive (by 1.6(10)) so } C_1 <_s B \cup (D' \cap C) \text{ but } C_1 \subseteq (B \cup C) \cap D' \subseteq B \cup (D' \cap C) \text{ (as } C_1 \subseteq (D' \cap C)) \text{ so } C_1 <_s (B \cup C) \cap D' = D' \) (as \(D' \subseteq B \cup C\)) but this contradicts \(A \le_i D', A \subseteq C_1. \) \(\square_{6.2}\)
6.3 **Definition.** Let \((\mathcal{R}, \|\|)\) be a 0-1 context (or weakly 0-1 context). If \(A \leq B_{\ell} \leq D\) for \(\ell < m\), we say \(\{B_{\ell} : 0 \leq \ell < m\}\) is free over \(A\) inside \(D\) (or the \(B_{\ell}\)'s are free over \(A\) inside \(D\)) if for every \(\ell, A \leq_s B_{\ell}\) and \(B_{\ell} \bigcup \bigcup_{k < \ell} B_k\).

Justification is:

6.4 **Claim.** The order (in Definition 6.3) is immaterial.

**Proof.** If \(\ell + 1 < m\), then (letting \(B_{\ell}^+ = \bigcup_{j < \ell} B_j \cup A\), \(B_{\ell}^+ \cup B_{\ell+1} \bigcup B_{\ell+1}^+\) (by clause (f)

of Definition 6.1) hence \(B_{\ell+1}^+ \bigcup B_{\ell+1}^+\) (by clause (1), symmetry, of Definition 6.1) that is \(B_{\ell}^+ \cup B_{\ell}^+ \bigcup B_{\ell+1}\). We have \(B_{\ell} \bigcup B_{\ell}^+\) so \(B_{\ell} \bigcup B_{\ell}^+\) (by clause (g) of of Definition 6.1(1)), and similarly \(B_{\ell}^+ \cup B_{\ell}^+ \bigcup B_{\ell+1}\) hence we get \(B_{\ell} \bigcup B_{\ell}^+\).

Now concerning 1.8 we can say more

6.5 **Claim.** Assume \((\mathcal{R}, \|\|)\) is a 0-1 context which is weakly nice.

1) If \(A < B \in \mathcal{X}_{\text{nifty}}\) then the following are equivalent

   (a) \(A <_s B\)

   (b) \(1 = \text{Lim}_n \text{Prob}_{\mu_n}(\text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n \text{ there are embeddings } g_\ell : B \rightarrow \mathcal{M}_n \text{ extending } f \text{ for } \ell < m \text{ such that } \langle g_\ell : \ell < m \rangle \text{ is disjoint over } A)\)

   (c) for every \(m < \omega\), for every \(k < \omega\)

   \(1 = \text{Lim}_n \text{Prob}_{\mu_n}(\text{for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n \text{ there are } \text{embd} \text{ of } f \text{ extending } f, \text{ for } \ell < m \text{ such that } \langle g_\ell : \ell < m \rangle \text{ is disjoint over } A \text{ and for } \ell < m \text{ we have } c\ell^k(f(A), \mathcal{M}_n) \text{ and the } g_\ell(B)\)'s are } \bigcup\text{-free over } g_\ell(A) \text{ inside } \mathcal{M}_n).

2) If \(A_0 \leq A_1 \in \mathcal{X}_\infty\) and \(A_0 \leq_s B\), then there are \(C \in \mathcal{X}_\infty\) and \(f\) such that \(A_1 \leq C, f\) is an embedding of \(B\) into \(C\) over \(A\), say \(B_1 = f(B)\) and \(B_1 \cap A_1 = A_0\) and \(A_1 \leq_s B_1 \cup A_1\).

**Proof.** 1) By 1.8 clearly (a) \(\Leftrightarrow\) (b).

1) Trivially (c) \(\rightarrow\) (b). Let us prove (a) \(\rightarrow\) (c).
Let \( m, k \in \mathbb{N} \). We now choose by induction on \( \ell \leq m, D_m \) and if \( \ell < m \) also \( g_\ell \) such that: \( D_0 = A, D_\ell \subseteq D_{\ell+1}, g_\ell^0 \) is an embedding of \( B \) over \( A \) into \( D_{\ell+1} \) \( \bigcup_{A}^D \), the induction step is done by clause (i)(existence) of Definition 6.1(1). By clause (h) of Definition 6.1(1) without loss of generality \( D_{\ell+1} = D_\ell \cup g_\ell(B) \). By clause (m)(smoothness) of Definition 6.1(1) we know that \( D_\ell \leq_s D_{\ell+1} \) and by clause (e) we know that \( g_\ell^0(B) \cap D_\ell = A \). So \( A \leq_s D_m \) so by “\( \mathfrak{R} \) weakly nice”, if \( \mathfrak{M}_n \) is random enough and \( f : A \to \mathfrak{M}_n \) an embedding then we can find an embedding \( g : D_m \to \mathfrak{M}_n \) extending \( f \). We let \( g_\ell = g \circ g_\ell^0 \) for \( \ell < m \).

Now for \( \ell_1 < \ell_2 < m, g_{\ell_1}(B) \cap g_{\ell_2}(B) = g(g_{\ell_1}(B) \cap g_{\ell_2}(B)) \subseteq g(g_{\ell_1}(B) \cap D_{\ell_2}) = g(g_{\ell_1}(B)) = g_{\ell_1}(B) \), so the disjointness demand holds. The freeness holds by the construction, and 6.3(1).

2) Without loss of generality, (not embedded yet in one model) \( B \cap A_1 = A_0 \). Let \( r^* \) be the number of structures \( C \in \mathcal{K}_\infty \) with set of elements \( B \cup A_1 \) such that \( B \subseteq C \) & \( A_1 \subseteq C \). Now let \( n_0(\varepsilon) \) be such that as \( A_1 \in \mathcal{K}_\infty \) clearly for some \( \varepsilon^* > 0 \)

\[ (*)_2 \text{ for arbitrarily large natural number } n: \]
\[ \varepsilon^* \leq \text{Prob}_{\mu_n} \left( \text{there is an embedding } f \text{ of } A_1 \text{ into } \mathfrak{M}_n \right). \]

Let (the function) \( n_1(\varepsilon) \) be such that:

\[ (*)_3 \text{ for every } \varepsilon \in \mathbb{R}^+, \text{ if } n \geq n_1(\varepsilon) \text{ then:} \]
\[ 1 - \varepsilon/4 \leq \text{Prob}_{\mu_n} \left( \text{if } C \text{ is embeddable into } \mathfrak{M}_n \text{ and } C \text{ has at most } |B| + |A_1| \text{ elements then } C \in \mathcal{K}_\infty \right). \]

As \( A_0 \leq_s B \) by part (1) for some function \( n_2(\varepsilon, m) \) we have:

\[ (*)_4 \text{ for every } \varepsilon \in \mathbb{R}^+ \text{ and } m \in \mathbb{N}, \text{ for every } m \geq n_2(\varepsilon, m) \text{ we have:} \]
\[ 1 - \varepsilon/4 \leq \text{Prob}_{\mu_n} \left( \text{every embedding } f \text{ of } A_0 \text{ into } \mathfrak{M}_n \text{ has } m \text{ extensions } g \text{ to embeddings of } B \text{ into } \mathfrak{M}_n, \text{ pairwise disjoint over } A \right). \]

Now by 1.6(11)

\[ (*)_5 \text{ for } C \in \mathcal{K} \text{ with the set of elements of } C \text{ being } B \cup A_1, \text{ such that } (A_1 \subseteq C) \& \neg(A_1 \leq_s C) \text{ there is } \ell_3 = \ell_3(C) \text{ such that for each } \varepsilon \in \mathbb{R}^+ \text{ for some } n = n_3(C, \varepsilon) \text{ we have: for every } n \geq n_3(C, \varepsilon), \]
\[ 1 - \varepsilon/4 \leq \text{Prob}_{\mu_n} \left( \text{there is no sequence } \langle g_\ell : \ell < \ell_3 \rangle \text{ of embedding of } C \text{ into } \mathfrak{M}_n \text{ pairwise disjoint over } A_1 \right). \]
Let $\varepsilon \in \mathbb{R}^+$ be given.

Let $\ell^* = \max\{\ell_3(C) : C \in \mathcal{K}\}$ has set of elements $B \cup A_1$ and extends $B$ and $A_1$ but $-(A_1 \leq_s C)$.

Let $m^* = r^* + |A_1 \setminus A_0| + 1$.

Let $n^*(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon, m^*), n_3(\varepsilon) : C \in \mathcal{K}\}$ has universe $B \cup A_1$ and extends $B$ and $A_1$ but $-(A_1 \leq_s C)$.

Let $\varepsilon < \varepsilon^*$ and let $n \in Y = \{n :$ the statement $(\ast)_2$ holds for $n\}$.

For each $M \in \mathcal{K}_n$ choose if possible an embedding $f^M$ of $A_1$ into $M$.

Now if $f^M$ is defined, choose if possible a sequence $(g^M_m : m < m^*)$ of embeddings of $B$ into $M$ extending $f^M \upharpoonright A_0$ and pairwise disjoint over $A$. As $(g^M_m(\mathcal{B}\setminus A_0) : m < m^*)$ is a sequence of pairwise disjoint subsets of $M$, the set $u = \{m < m^* : g^M_m(\mathcal{B}\setminus A_0) \text{ is disjoint to } f^M(A_1 \setminus A_0), \text{ equivalently } g^M_m(B) \cap f^M(A_1) = f^M(A_0)\}$ has at least $m^* - |A_0|\setminus A_0$ members which is $r^* \times \ell^* + 1$.

For $m \in u$ let $C_m$ be the model with universe $B \cup A_1$ such that $g^M_m \cup f^M$ is an isomorphism from $C_m$ into $M \upharpoonright (g^M_m(B) \cup f^M(A_1))$ (note: this function is one to one as $n \in u$ and by the definition of $u$). By the choice of $r^*$ for some model $C$ with set of elements $B \cup A_1$ we have $|\{m \in u : C = C_m\}| \geq |u|/r^*$. But $|u| \geq m^* - |A_1 \setminus A_0| = r^* \times \ell^* + 1$ hence $u' = \{m \in u_m : C = C_m\}$ has $> \ell^*$ members.

Now with probability $\geq \varepsilon^* - \varepsilon > 0$, $\mathcal{M}_n$ satisfies demands $(\ast)_2 - (\ast)_5$ above hence $f^\mathcal{M}_n$ is well defined (by $(\ast)_2$) and $(g^\mathcal{M}_n : m < m^*)$ is well defined (by $(\ast)_4$ and the choice of $n(\varepsilon)$). By $(\ast)_3$, we know $C \in \mathcal{K}_\infty$, obviously $A_1 \leq C, B \leq C$ (as each $C_m$ satisfies this). Lastly by $(\ast)_5$ the sequence $(g^\mathcal{M}_n : m < m^*)$ witnesses $A_1 \leq_s C$, so we have finished. \(\square_{1.8}

6.6 Comment

Our real interest is in instances of $\mathfrak{R}$ but $\mathfrak{U}$ is central in the following way

(a) it is a way to express assumptions on $\mathfrak{R}$ helping to analyze the limit behaviour (for having a 0-1 law is a reasonable criterion)

(b) assuming the random enough $\mathcal{M}_n$ satisfies $(\ast)$ below we can define $\mathfrak{U}$ and prove $(\mathfrak{R}, \mathfrak{U})$ is a 0-1 context where

$(\ast)$ for quantifiers free $\varphi(\vec{x}, \vec{y})$, the numbers $|\varphi(\mathcal{M}_n, \vec{b})|$ for $\vec{b} \in \ell^g(\vec{y})(\mathcal{M}_n)$ behave regularly enough; where

$$\varphi(\mathcal{M}_n, \vec{b}) = \{\vec{a} \in \ell^g(\vec{x})(\mathcal{M}_n) : \mathcal{M}_n \models \varphi(\vec{a}, \vec{b})\}.$$
6.7 Claim. 1) If $\mathcal{X}_n = \{([n])\}$ (so $\mu_n$ trivial) and $B \bigcup_D^A C$ means $B \cap C = A$ & $B \cup C \subseteq D$ then $(\mathcal{R}, \bigcup)$ is explicitly nice 0-1 context. Also $A \leq_s B$ iff $A \subseteq B$ and $\mathcal{X}_\infty$ is the family of finite models.

2) Let $\mathcal{X}_n = \{(n), S, c_f, c_\ell \}$ with $c_f = 1, c_\ell = n$ and $S$ the successor relation (so $\mu_n$-trivial). Then $A <_s B$ means $A \subseteq B$ & $(\forall x \in A)(\forall y \in B \setminus A)[\neg xS^B y \& \neg yS^B x]$ and $A \in \mathcal{X}_\infty$ iff $A$ is isomorphic to $([n], S, c_f, c_\ell) \upharpoonright X$ for some $X \subseteq [n]$.

3) Let in 2), $B \bigcup_D^A C$ means $A \leq_s B \leq D \in \mathcal{X}_\infty, A \leq C \leq D, B \cap C = A$ and $(\forall x \in C)(\forall y \in B \setminus A)[\neg xS^D y \& \neg yS^D x]$. Then $(\mathcal{R}, \bigcup)$ is 0-1 context, $\mathcal{R}$ is explicitly almost nice.

Proof. Straight, e.g. Why 6.1(f)?

3) Let us check clause (f) of Definition 6.1(1), (base increasing) so we have $B \bigcup_D^A C$ and $A \leq C_2 \leq C$ and we have to prove $B \cup C_1 \bigcup_{C_1}^D C$. Now $C_1 \subseteq B \cup C_1$ trivially, and for $C_1 \leq_s B \cup C_1$ see the definition of $\bigcup_{C_1}^D B \cup C_1 \subseteq D$ as $B \subseteq D, C_1 \subseteq C \subseteq D$, and $C_1 \subseteq C$ holds by assumption as $C \subseteq D$ as $B \bigcup_D^A C$. Lastly, assume $x \in C_1, y \in B \cup C \setminus C_1$ then $x \in C, y \in B \setminus C$ so by the definition of $B \bigcup_D^A C$ we have $\neg xS^D y \& \neg yS^D x$, as required. $\Box_{6.7}$

* * *

We connect the free amalgamation with §2, i.e. context $\mathcal{R}$ which obeys $\bar{h}$ which is bound by $h_1$, this gives a natural definition for free amalgamation.

Continuing Definition 2.1 let

6.8 Definition. 1) For a 0-1 context $\mathcal{R}$ obeying $\bar{h}$ and function $h$ from $\bigcup_n \mathcal{X}_n$ to $\mathbb{R}^+$ we define a four place relation $\bigcup_h (A, B, C, D)$ if

(a) $A \leq_s B \leq D, A \leq C \leq D, A = B \cap C$ and,
(b) letting $D_1 = D \setminus B \cup C$ we have $C \leq_s D_1$ and
(c) for every $\varepsilon > 0$ for every random enough $\mathcal{M}_n$ into which $D$ is embeddable we have\footnote{Inequality is the other direction follows from the definitions as proved in clause (ζ) of 2.4(2)}
\[ h^u_{C,D_1}[\mathcal{M}_n] \times h[\mathcal{M}_n] \geq h^u_{A,B}[\mathcal{M}_n]. \]

note that (b) follows by (a), (c)).

We define \( \bigcup = \bigcup_h = \bigcup[\tilde{h}, \tilde{h}] \) similarly: \( \bigcup[A, B, C, D] \) iff

(a) \( A \leq_s B \leq D, A \leq C \leq D, A = B \cap D \) and

(b) letting \( D_1 = D \upharpoonright (B \cup C) \) we have \( C \leq_s D_1 \)

(c) for some \( m = m(A, B, C, D) \in \mathbb{N} \setminus \{0\} \), for every random enough \( \mathcal{M}_n \) we have

\[ h^u_{C,D_1}[\mathcal{M}_n] \times h[\mathcal{M}_n]^m \geq h^u_{A,B}[\mathcal{M}_n]. \]

2) Let \( \bigcup[\tilde{h}] = \bigcup[\tilde{h}, h'] \) for \( h'(M) = \|M\| \) (see 7.8(3) below).

Continuing 2.4 we note

6.9 Claim. In 2.4(2) we can add if \( \bigcup^D \bigcup\bigcup^A \) (so \( A \leq_s B \leq D, A \leq C \leq D \)) and \( D = B \cup C \) then for every \( \varepsilon > 0 \) for every random enough \( \mathcal{M}_n \) we have

\[ h[\mathcal{M}_n] \geq h^u_{A,B}[\mathcal{M}_n]/h^u_{C,D}[\mathcal{M}_n] \geq 1/(h[\mathcal{M}_n])^\varepsilon \] (when \( C \) is embeddable into \( \mathcal{M}_n \) of course).

6.10 Claim. Assume \( \mathcal{R} \) obeys \( \tilde{h} \) which is bounded by \( h \) and \( \bigcup = \bigcup_h \).

1) \((\mathcal{R}, \bigcup)\) is a weak 0-1 context (see Definition 6.1(1) second version).

2) If in addition \((*)\) below holds, then \((\mathcal{R}, \bigcup)\) is a 0-1 context

\((*)\) if \( A <_pr B \leq D, A < C <_pr D, B \cup C = D \) and \( B,C \) are not \( \bigcup \)-free by amalgamation over \( A \) inside \( D \), then for some \( \varepsilon > 0 \) for every random enough \( \mathcal{M}_n \), we have \( h^d_{A,B}[\mathcal{M}_n]/h^u_{C,D}[\mathcal{M}_n] \geq (h[\mathcal{M}_n])^\varepsilon\).

3) If condition \((*)\) from part (2) holds then also

\((*)^+\) if \( A \leq_s B \leq D, A \leq C \leq_s D, D = B \cup D \) and \( B,C \) are not \( \bigcup \)-free by amalgamation over \( A \) inside \( D \) then for some \( \varepsilon > 0 \) for every random enough \( \mathcal{M}_n \) we have

\[ h^d_{A,B}[\mathcal{M}_n]/h^u_{C,D}[\mathcal{M}_n] \geq (h[\mathcal{M}_n])^\varepsilon; \]
Remark. 1) Note that (\*) of 6.10(2) just say a dichotomy; i.e. the condition (c) in Definition 2.1(5), either holds for every random enough $\mathcal{M}_n$, or fails for every random enough $\mathcal{M}_n$.

2) Note that (\*) of 6.10 exclude the case of having a successor relation.

Proof. The order is that first we prove 6.10(3) (and inside it a restricted version of transitivity (clause (j) of Definition 1.1(2))), and only then we prove 6.10(1)+(2) by going over all the clauses of Definition 1.1(2), so in some clauses there is a difference between 6.10(1) and 6.10(2), and then in the second case we may use (\*) of 6.10(2) (and so 6.10(3)).

3) Let $\bar{A} = \langle A_i : i \leq k \rangle$ be a decomposition of $A < s B$, so $A_i \prec_{pr} A_{i+1}$ for $i < k$. Let $C_i = A_i \cup C$, hence for every $i < k, C_i \leq_C C_{i+1}$ or $C_i \prec_{pr} C_{i+1}$.

For each $i < k$ let $\varepsilon(i) \in \mathbb{R}_{\geq 0}$ be such that:

Case 1: If $c_i \leq c_{i+1}$ then $\varepsilon(i) > 0$ and $h^n_{A_i,A_{i+1}}[\mathcal{M}_n] \geq (h[\mathcal{M}_n])^{\varepsilon(i)}$ for every $\mathcal{M}_n$ random enough.

Case 2: If $\neg(c_i \leq c_{i+1})$ hence $c_i \prec_{pr} c_{i+1}$ but $\neg\neg\langle A_i \cup \bigcup_{A_i} C_i \rangle$ then $\varepsilon(i) > 0$ and $h^n_{A_i,A_{i+1}}[\mathcal{M}_n] / h^{d}_{C_i,C_{i+1}}[\mathcal{M}_n] \geq (h[\mathcal{M}_n])^{\varepsilon(i)}$.

Case 3: If neither Case 1 nor Case 2, then $\varepsilon(i) = 0$. Note that by 2.6(2) for $\mathcal{M}_n$ random enough $h^n_{A_i,A_{i+1}}[\mathcal{M}_n] \geq h^{d}_{C_i,C_{i+1}}[\mathcal{M}_n]$ (if $D$ embeddable into $\mathcal{M}_n$). Let $w_2 = \{ i < k : \text{case } \ell \text{ occurs} \}$.

First assume $\sum_{i<k} \varepsilon(i) > 0$, and let $\zeta \in \mathbb{R}^+$ be $< \sum_{i<k} \varepsilon(i)/2$.

So let $\mathcal{M}_n$ be random enough such that $C$ is embeddable into $\mathcal{M}_n$ (hence $D$ is embeddable into $\mathcal{M}_n$ hence also the $C_i$’s are).

(a) $h^n_{A_i,A_{i+1}}[\mathcal{M}_n] \geq h^d_{C_i,C_{i+1}}[\mathcal{M}_n]$ when $C_i \prec_{pr} C_{i+1}

[Why? See 2.6(2).]

(b) $1 \leq \prod_{i<k} h^n_{A_i,A_{i+1}}[\mathcal{M}_n] / h^n_{A,A,B}[M] \leq (h[\mathcal{M}_n])^\varepsilon/k$.

[Why? The first inequality by the definition in 2.1(4), as $h^n_{A,A,B}(M) = h^n_{A,A,B}(M)$. Second inequality by the second inequality in 2.4(2)(\delta).]

So

$$h^d_{C,D}[\mathcal{M}] \leq \prod_{i \in w_2 \cup w_3} h^d_{C_i,C_{i+1}}[\mathcal{M}_n] \leq ?$$

[trivially]
\[(\prod_{i \in \mathcal{W}_1} 1) \times (\prod_{i \in \mathcal{W}_2} h^{d}_{C_i, C_{i+1}}[\mathcal{M}_n]) \times (\prod_{i \in \mathcal{W}_3} h^{d}_{C_i, C_{i+1}}[\mathcal{M}_n]) \leq ?
\]

(by the statements in the cases)

\[(\prod_{i \in \mathcal{W}_1} h^{n}_{A_i, A_{i+1}}[\mathcal{M}_n]/(h[\mathcal{M}_n])^{\varepsilon(i)}) \times (\prod_{i \in \mathcal{W}_2} h^{u}_{A_i, A_{i+1}}[\mathcal{M}_n]/h[\mathcal{M}_n]^{\varepsilon(i)}) \times (\prod_{i \in \mathcal{W}_3} h^{u}_{A_i, A_{i+1}}[\mathcal{M}_n])
\]

\[= (\prod_{i \leq k} h^{n}_{A_i, A_{i+1}}[\mathcal{M}_n]) \times (h[\mathcal{M}_n])^{-\sum \varepsilon(i)} \times (h[\mathcal{M}_n])^{\varepsilon} \times (h[\mathcal{M}_n])^{-\sum \varepsilon(i)}
\]

[trivially]

So \(\sum_{i<k} \varepsilon(i) - \zeta > \mathbb{R}^+\) can serve as the desired \(\varepsilon\).

Second, assume \(\sum_{i<k} \varepsilon(i) = 0\) then, as \(C_k = D\), we have

\[(c) \ 1 \leq \prod_{i < k} h^{u}_{C_i, C_{i+1}}[M]/h^{u}_{C, D}[M] \leq (h[M])^\varepsilon
\]

and by Definition 2.1(5)

\[(d)\] for every \(\varepsilon \in \mathbb{R}^+\) for every random enough \(\mathcal{M}_n\) into which \(C_{i+1}\) is embeddable

\[(h[\mathcal{M}_n])^{-\varepsilon} 1 \leq h^{u}_{A_i, A_{i+1}}[\mathcal{M}_n]/h^{u}_{C, D}[\mathcal{M}_n] \leq (h[\mathcal{M}])^{\varepsilon}.
\]

Using (b)+(c)+(d) we can get “\(B, C\) are \(\mathcal{M}\)-free over \(A\) inside \(D\)” so an assumption of \((*)^+\) fail.

**Proof of (1)+(2)**

The proof is split according to the clause in Definition 6.1(1); i.e. to \((d), \ldots, (m)\).

Clause \((d)\):

Trivial.

Clause \((e)\):

Trivial.

Clause \((f)\): Monotonicity in \(C\)
The point to notice is that “$D$” is embeddable into $\mathcal{M}_n$” does not imply “$D$ is embeddable into $\mathcal{M}_n$” but read 2.1(3), 2.1(3A).

* * *

Clause (h): (Reading Definition 2.1(5)), but here there is a difference between 6.10(1) and 6.10(2). We have to restrict ourselves to models into which $D$ can be embeddable. But there may be many into which $D'$ is embeddable but $D$ is not.

As $D \in \mathcal{X}_\infty$, for some $\zeta \in \mathbb{R}^+$ for arbitrarily large $n$, the probability that $\mathcal{M}_n$ satisfies the desired inequalities is $\geq \zeta$, so for 6.10(2), the assumption $(\ast)^+$ there guarantees they hold for every random enough $\mathcal{M}_n$.

For the weakly version in Definition 1.6(1) clause (h) we have restricted ourselves to the case $\mathbb{E}_{D',D}$ (see Definition 1.6(1)).

**Symmetry:** So assume $B \bigcup A \subseteq C <_s C$, and we should prove $C \bigcup A \subseteq D$. Without loss of generality $A \neq B$, $A \neq C$. So we have $A \subseteq B, B \cap C = A, C >_s B$.

We should prove $B <_s B \cup C$, and the inequality. Let $m^* \in \mathbb{N}$, let $\epsilon \in \mathbb{R}^+$ be small enough.

So let $\mathcal{M}_n$ be random enough and $f_0$ an embedding of $A$ into $\mathcal{M}_n$. So

$$F_1 = \{g_1 : f_1 \text{ is an embedding of } C \text{ into } \mathcal{M}_n \text{ extending } f_0\}.$$ 

Clearly $h \geq h_{A,C}^d[\mathcal{M}_n]$ and for each $f_1 \in F_1$,

$$F_{f_1}^2 = \{f_2 : f_2 \text{ is an embedding of } C \cup B \text{ into } \mathcal{M}_n \text{ extending } f_1\}$$

has $\geq h_{C,B \cup C}^d[\mathcal{M}_n]$ members.

So $F_2 = \bigcup\{F_{f_1}^2 : f_1 \in F_1\}$ has $\geq h_{A,C}^d[\mathcal{M}_n] \times h_{C,B \cup C}^d[\mathcal{M}_n]$ member. Consider the mapping $G_2 : F_2 \to F_1 = \{f_1 : f_1 \text{ is an embedding of } B \text{ into } \mathcal{M}_n \text{ extending } f_0\}$.

So $G_2$ is a mapping from $F_2$ into $F_1$. Also $F_1$ has at most $h_{A,B}^n[\mathcal{M}_n]$ which is $\geq h_{C,D,B \cup C}^u[\mathcal{M}_n]/(h[\mathcal{M}_n])^\epsilon$ as $\bigcup\{f_2 \in F^2 : f_2 \upharpoonright B = f_1\}$ averaging on all $f^1 \in F^1$ is $\geq h_{A,C}^d[\mathcal{M}_n]/(h[\mathcal{M}_n])^\epsilon$ which is $> m^*$ (as $\mathcal{M}_n$ is random enough). So for some $f^1 \in F^1$ the actual number is $> m^*$, hence $B <_s B \cup C$.

Similarly we get the inequality.

Clause (g):

I.e. we assume $A \subseteq B' \subseteq_s B, A \subseteq C' \subseteq C$ and $D \bigcup A \bigcup C$ (and we should prove $B' \bigcup C'$). Now $B \bigcup A \bigcup C$ means that $A \subseteq B \subseteq D, A \subseteq C \subseteq D, B \cap C = A$ (this is clause (a) of 2.14(5)) and letting $D_1 = B \cup C$, also $C \subseteq D_1$ (this is clause (b) of 2.14(5)) and similarly $C' \subseteq B \cup C'$ and for every $\epsilon_1 > 0$ for every random enough $\mathcal{M}_n$ we have $1 \leq h_{A,B}^n[\mathcal{M}_n]/h_{C,D_1}^u[\mathcal{M}_n] < (h[\mathcal{M}_n])^{\epsilon_1}$ (this is clause (c) of
Definition 2.10(5)). Looking at the desired conclusion (in particular the definition of $|\bigcup|$) we can restrict ourselves to $\mathcal{M}_n$ into which $D$ is embeddable; hence $C’, C$ are embeddable.
Without loss of generality $B’ = B$ or $C’ = C$.

In the first case, we note that for random enough $\mathcal{M}_n$ into which $D$ is embeddable, we have that by clause 2.6(1) above:

$$h_{A,B}^u[\mathcal{M}_n] \geq h_{C,B \cup C'}^u[\mathcal{M}_n] / h(\mathcal{M}_n) \geq h_{C,D_1}^u[\mathcal{M}_n] / h(\mathcal{M}_n)^{2\varepsilon}$$

and by the definition of $\bigcup$ also $h_{A,B}^u(\mathcal{M}_n) \leq h_{C,D_1}^u(\mathcal{M}_n)(h(\mathcal{M}_n))^{\varepsilon}$, together we have the desired inequality.

In the second case ($C’ = C$) we have $A < s B’ < s B$, and similar inequalities using clause $(\varepsilon)$ of 2.4(2) give the result.

Clause (i):

Note that the number of members of $\mathcal{K}_\infty$ with $|A| + |C| - |A|$ elements and, for simplicity, set of elements $\subseteq \{1, \ldots, |B| + |C| - |A|\}$, is finite.

So let $D_j, f_j$ for $j < j^*$ be such that

(a) $C \leq D_j \in \mathcal{K}_\infty$
(b) $f_j$ is an embedding of $B$ into $D_j$ over $C$
(c) $D_j = C \cup f_j(B)$
(d) for $j_1 \neq j_2$, $(f_{j_2} \circ f_{j_1}^{-1} \cup \text{id}_C) : D_{j_1} \rightarrow D_{j_2}$ is not an isomorphism
(e) under (a)-(d), $j^*$ is maximal.

So as said above, $j^* \in \mathbb{N}$. Now let $M \in \mathcal{K}$ be any model to which $C$ is embeddable, say by $g_M : C \rightarrow M$ such that there are distinct embeddings $f_i$ of $B$ into $M$ for $i < h_{A,B}^u(M)$ extending $g_M \upharpoonright A$ (remembering $A \leq C$); i.e. $M$ is random enough.

So for each $i < h_{A,B}^u(M)$ for some unique $j_i < j^*$ we have $(f_i \circ f_{j_i}^{-1} \cup g_M)$ is an isomorphism from $D_{j_i}$ onto $M \upharpoonright (\text{Rang}(f_i) \cup \text{Rang}(g_M))$. So for some $j = j_M < j^*$ we have

$$w = \{ i < h_{A,B}^d(M) : (f_i \circ f_{j_i}^{-1} \cup g_M) \text{ embed } D_j \text{ into } M \text{ equivalently } j_i = j_M \}$$

has $\geq h_{A,B}^d[M] / j^*$ members. So for some $j_1$ not for every random enough $\mathcal{M}_n$ do we have $j_{\mathcal{M}_n} \neq j$. Hence by 6.10(3) we are done.

Clause (j):

Restricting ourselves to models into which $B_2$ can be embeddable, by (g) + (h) we can deal with $B’_\ell (\ell \leq 2)$, $B’_\ell = A_\ell \cup B_0$, and for this look at the proof of clause (h) above.

What about the models into which $B_2$ is not embeddable? Look at the proof of clause (h) above.
Clause (k):
Like the proof of (j).

Clause (f):
Straight by computing $h_{A,D_1}(\mathcal{A}_n)$ (approximately) in two ways
where $D_1 = B \cup C$.

Clause (m): smoothness
Follows from (d) and (f). $\square_{2.4}$

We deal in 7.8 - 7.11 with the polynomial case; in the general case we deal in 7.7.

Concerning 3.12 we add

**6.11 Definition.** We define a four place relation

$$\mathcal{A} = \bigcup[p] : \mathcal{K}(A^+, B^+, C^+, D^+) \text{ iff }$$

$$A^+ \leq B^+ \leq + D^+, A^+ \leq + C^+ \leq D^+, B^+ \cap C^+ = A^+ \text{ and }$$

$$\beta(C^+, D^+) = \beta(A^+, B^+).$$

We normally write $\mathcal{A}$ when no confusion arises and may write $B^+ \bigcup C^+$.

Concerning 3.14 we add

**6.12 Fact.** Using $<_{x/p}^+$ for $x = b, j, t, qr$ instead $<_{x/p}^+$ for $x = a, i, s, pr$ respectively

$(\mathcal{R}^+, \mathcal{A})$ satisfies Definition 6.1 and satisfies $(\ast)$ of 6.10(2). Moreover, it satisfies

$(\ast)$ of 7.8(1) if $\bar{h}$ is polynomial and $\otimes_3 + \otimes_4$ of 7.9, in general.

**6.13 Claim.** Assume $(\ast)_{ap}$ of 3.12 and $\otimes_1 + \otimes_2$ of 3.15.

1) A sufficient condition for “$(\mathcal{R}^+, \mathcal{A})$ is nice” is

$$\otimes_3 (\mathcal{R}, \mathcal{A}) \text{ is nice.}$$

2) A sufficient condition for “$(\mathcal{R}^+, \mathcal{A})$ obeys $\bar{h}$ very nicely” (see Definition 7.12)

$$\otimes_4 (\mathcal{R}, \mathcal{A}) \text{ obeys some $\bar{h}$ very nicely}$$

3) A sufficient condition for “$\mathcal{R}^+$ is polynomially nice” is

$$\otimes_3 (\mathcal{R}, \mathcal{A}) \text{ is polynomially nice.}$$

**Proof.** 1) We use 7.9(4). Now $(\ast)_4$ there holds by 3.14(6) or use 7(2).

$\rightarrow$ scite{3.14} undefined

2) Straight.

3) Define the conditions for non-polynomial cases.
\[ \leq \left( \prod_{p^+ \in \mathcal{P}^+} \left( C^+ \subseteq B^+, C^+ \not\subseteq A^+ \right) \right)^m \]
\[ \leq (h_2(\mathcal{M}_n))^{t_2} \| \mathcal{M}_n \| \sum \{ \beta(C^+): C^+ \subseteq B^+, C^+ \not\subseteq A^+ \}. \]

So the expected value of the number of such $\bar{g}$’s is

\[
(h_1(\mathcal{M}_n), h_2(\mathcal{M}_n))^{t_2 + t_2} \cdot (\| M_n \|^\alpha(A,B))^m \cdot (\| \mathcal{M}_n \| \sum \{ \beta(C^+): C^+ \subseteq B^+, C^+ \subseteq A^+ \})^n
\]
\[ = (h_1(\mathcal{M}_n))^{t_1 + t_2} \| \mathcal{M}_n \|^{m\beta(A,B)}. \]

So the expected number of such $\bar{g}$ for some $f$ is

\[
\leq \{ f : f \text{ embeds } A \text{ into } \mathcal{M}_n \} \times (h_1(\mathcal{M}_n) \times h_2(\mathcal{M}_n))^{t_1 + t_2} \times \| \mathcal{M}_n \|^{m\beta(A,B)}
\]
\[ < \| (h_1(\mathcal{M}_n) \times h_2(\mathcal{M}_n))^{t_1 + t_2} \times \| \mathcal{M}_n \|^{m\beta(A,B)+|A|}. \]

This converges to zero if $m \times \beta(A, B) > |A|$ which holds for $m$ large enough.
Below we define some properties of \((\mathfrak{A}, \bigcup)\), variants of nice, we will use semi-nice to prove elimination of quantifiers (hence 0-1 laws), it is essentially the weakest among those discussed below. But it will be natural in various contexts to verify stronger ones.

The reader may e.g. ignore Definition 7.1(1),(2),(4),(5),(7)-(10), 7.3(7),(8) and the version of 7.3(5),(6) with “explicit”; in [Sh 637] we present only one variant.

7.1 Definition. 1) We say \((\mathfrak{A}, \bigcup)\) is explicitly\(^\text{11}\) nice if we have:

\[
(*)_2 \quad \text{for every } A <_{\text{pr}} B \text{ (in } \mathcal{K}_\infty) \text{ and } k \in \mathbb{N} \text{ for some } r = r(A, B, k) \in \mathbb{N} \text{ for every } C, D \text{ such that } B \bigcup A \subseteq C \text{ and } D = B \cup C \text{ we have:}
\]

\[
1 = \lim_n \text{Prob}_{\mu_n} \left( \text{for every embedding } f \text{ of } C \text{ into } \mathcal{M}_n \text{ satisfying}
\right.
\]

\[
\text{cl}^r(f(A), \mathcal{M}_n) \subseteq f(C) \text{ there is } g : D \rightarrow \mathcal{M}_n
\]

\[
\text{extending } f \text{ such that } cl^k(g(B), \mathcal{M}_n) \subseteq g(D).
\]

If \(r(A, B, k) = k + |A|\) or \(k\) we say \((\mathfrak{A}, \bigcup)\) is explicitly\(^+\) nice or explicitly\(^++\) nice respectively.

[Note: to deal with e.g. successor functions we need slightly more.]

2) We say \((A, A_0, B, D)\) is an almost \((k, r)\)-good (quadruple) if:

\[
(*)_{A, A_0, B, D}^k \quad A_0 \leq A \leq D \in \mathcal{K}_\infty \text{ and } B \leq D \text{ and for every random enough } \mathcal{M}_n
\]

\[
\text{we have:}
\]

\[
(**) \quad \text{every embedding } f : A \rightarrow \mathcal{M}_n \text{ satisfying}
\]

\[
\text{cl}^r(f(A_0), \mathcal{M}_n) \subseteq f(A) \text{ has an extension } g : D \rightarrow \mathcal{M}_n \text{ satisfying}
\]

\[
\text{cl}^k(g(B), \mathcal{M}_n) = g(\text{cl}^k(B, D)).
\]

If \(r = k\) we may write \(k\) instead of \((k, r)\).

3) We say \((A^+, A, B, D)\) is a semi \((k, r)\)-good quadruple if:

\[
(*)_{A^+, A, B, D}^k \quad A \leq A^+ \in \mathcal{K}_\infty \text{ and } A \leq D, B \leq D \in \mathcal{K}_\infty
\]

\[
\text{and for every random enough } \mathcal{M}_n \text{ we have:}
\]

\[
(**) \quad \text{for every embedding } f : A^+ \rightarrow \mathcal{M}_n \text{ satisfying}
\]

\[
\text{cl}^r(f(A), \mathcal{M}_n) \subseteq f(A^+) \text{ there is an extension } g \text{ of } f \upharpoonright A, \text{ embedding}
\]

\[
D \text{ into } \mathcal{M}_n \text{ such that}
\]

\[
\text{cl}^k(g(B), \mathcal{M}_n) = g(\text{cl}^k(B, D)).
\]

\[^{11}\text{the “nice” appears in 7.1(8) below}\]
If \( r = k \) we may write \( k \) instead of \( (k, r) \).

4) We say \((A, B, D)\) is semi \((k, r)\)-good if: \( A \leq A^+ \in \mathcal{K}_\infty \Rightarrow (A^+, A, B, D)\) is semi-\((k, r)\)-good.

5) We say \( \mathcal{K} \) is almost nice if it is weakly nice and, for every \( A \in \mathcal{K}_\infty \) and \( k \) for some \( \ell, m, r \) we have:

\[ (*) \text{ for every random enough } M_n, \text{ for every } f : A \to \mathcal{M}_n \text{ we have:} \]

\[ (**) \text{ for every } b \in \mathcal{M}_n \text{ we can find } A_0, A^+, B^+ \text{ such that:} \]

\[ \begin{align*}
\alpha & \quad f(A) \subseteq A_0 \subseteq A^+ \subseteq \mathcal{M}(f(A), \mathcal{M}_n), \\
\beta & \quad A^+ \cup \{b\} \subseteq B^+ \subseteq \mathcal{M}_n \\
\gamma & \quad |B^+| \leq \ell \\
\delta & \quad (A^+, A_0, A_0 \cup \{b\}, B^+) \text{ is almost } (k, r)\text{-good} \\
\epsilon & \quad \mathcal{M}^k(A_0, \mathcal{M}_n) \subseteq A^+ \\
\zeta & \quad \mathcal{M}^k(f(A) \cup \{b\}, \mathcal{M}_n) \subseteq B^+. 
\end{align*} \]

6) We say that \( \mathcal{K} \) is semi-nice if it is weakly nice and for every \( A \in \mathcal{K}_\infty \) and \( k \) for some \( \ell, m, r \) we have:

\[ (*) \text{ for every random enough } M_n, \text{ and embedding } f : A \to \mathcal{M}_n \text{ and } b \in \mathcal{M}_n \text{ we can find } A_0, A^+, B, D \text{ such that:} \]

\[ \begin{align*}
\alpha & \quad f(A) \leq A_0 \leq A^+ \leq \mathcal{M}(f(A), \mathcal{M}_n), \\
\beta & \quad f(A) \cup \{b\} \subseteq B \subseteq D \subseteq \mathcal{M}_n \\
\gamma & \quad |D| \leq \ell \\
\delta & \quad (A^+, A_0, B, D) \text{ is semi } (k, r)\text{-good} \\
\epsilon & \quad \mathcal{M}^k(A_0, \mathcal{M}_n) \subseteq A^+ \\
\zeta & \quad \mathcal{M}^k(B, \mathcal{M}_n) \subseteq D. 
\end{align*} \]

7) We say the pair \((A, B)\) is \((k, r)\)-good when \( A \leq_s B \) and: if \( B \leq D, A \leq_s D \) then \((A, B, D)\) is semi-\((k, r)\)-good. We say \((A, B)\) is \(*\)-good if it is \((k, k)\)-good; good if it is \((k, k)\)-good for every \( k \); and \(*\)-good, if for every \( k \) for some \( r \) it is \((k, r)\)-good.

8) We say \( \mathcal{K} \) is nice if \( A \leq_s B \) implies \((A, B)\) is \(*\)-good.

9) We say \( \mathcal{K} \) is explicitly almost nice when it is weakly nice and for every \( k, \ell \) for some \( r, m \), for some random enough \( \mathcal{M}_n \), (i.e. \( 0 < \limsup \text{Prob}_{\mu_n} \)), if \( A_0 \leq A \leq_s D, A_0 \leq B \leq D \subseteq \mathcal{M}_n \) and \( |D| \leq \ell \) are such that \( \mathcal{M}(B, \mathcal{M}_n) = \mathcal{M}(B, D) \) and \( \mathcal{M}^k(0, \mathcal{M}_n) \subseteq A \Rightarrow (A, A_0, B, D) \) is almost \((k, r)\)-good.

10) We say \( \mathcal{K} \) is explicitly semi-nice when it is weakly nice and for every \( \ell \) and \( k \) for some \( r \) for some random enough \( \mathcal{M}_n \) we have:

\[ (*) \text{ if } A \leq_s D, B \leq D \subseteq \mathcal{M}_n, |D| \leq \ell, A^+ = \mathcal{M}(A, \mathcal{M}_n) \text{ and} \]

\[ c\mathcal{M}^k(B, \mathcal{M}_n) \subseteq D \text{ then } (A^+, A, B, D) \text{ is semi } (k, r)\text{-good}. \]

7.2 Remark. 1) We may consider other candidates to 7.1(7), 7.1(8).

2) We may consider in the Definition of semi-good (or explicitly) semi-good/nice to split \( \ell \) to two: in assumption and in conclusion.

3) Also there to demand \( b \notin \mathcal{M}(A, \mathcal{M}_n) \).
7.3 Fact

1) In Definition 7.1(1) (of explicitly nice) we can replace $A <_{pr} B$ by $A <_s B$ and/or replace $cf^k(g(B), \mathcal{M}_n) \subseteq g(D)$ by $cf^k(g(B), \mathcal{M}_n) = g(cf^k(B, D))$. If $(\mathcal{R}, \mathcal{J})$ is explicitly $+ +$ nice then it is explicitly $+$ nice which implies it is explicitly nice.

2) If $(A, A_0, B, D)$ is almost $(k, r)$-good, then $(A, A_0, B, D)$ is semi $(k, r)$-good.

3) $(\mathcal{R}, \mathcal{J})$ being explicitly nice implies $\mathcal{R}$ is weakly nice.

4) If $(A, A_0, B, D)$ is semi $(k, r)$-good and $A \cup cf^k(B, D) \subseteq D'$ then $(A, A_0, B, D')$ is semi $(k, r)$-good.

5) If the definition of semi-nice we can demand $B = f(A) \cup \{b\}$.

Proof. 1) For the first phrase, clearly the new version implies the old as $A <_{pr} B \Rightarrow A <_s B$. So assume the old version and let $A <_s B$, so by 1.6 we can find $n$ and $A_0 <_{pr} A_1 <_{pr} \ldots <_{pr} A_n = A, A_n = B$. By 6.1,

$$A_{\ell+1} \bigcup_{A_{\ell}} C \cup A_{\ell}.$$ Define $k(\ell)$ for $\ell \subseteq n$ by downward induction on $\ell$. For $\ell = 0$ let $k(0) = k$, for $\ell$ let it be the $r(r_{\ell+1}, A_{\ell}, A_{\ell+1})$ guaranteed by 7.1(1). Now for random enough $\mathcal{M}_n$ and embedding $f : C \to \mathcal{M}_n$ such that $cf^k(f(A), \mathcal{M}_n) \subseteq f(C)$, we choose by induction on $\ell \leq n$ an embedding $f_\ell : C \cup A_\ell \to \mathcal{M}_n$ increase with $\ell$ such that $cf(f_\ell(C \cup A_\ell)) \subseteq C \cup A_{\ell+1}$. For $\ell = 0$ this is given, for $\ell + 1$ use the choice of $r_{\ell}$. For the second phrase, clearly $cf^k(g(B), \mathcal{M}_n) = g(cf^k(B, D))$ implies $cf^k(A, N_2) \leq N_1 \leq N_2 \Rightarrow cf^k(A, N_1) = cf^k(A, N_2)$.

In the second sentence, first implication holds as $cf^n(A', B')$ increase with $m$; the second implication holds by the definition.

2) Read the definitions.

3) $(\mathcal{R}, \mathcal{J})$ explicitly nice $\Rightarrow \mathcal{R}$ is weakly nice.

Let $A <_{pr} B$ and $m \in \mathbb{N}$ and $\varepsilon > 0$ be given. Let $r$ be as guaranteed by 7.1(1) and let $m^*$ be such that $A \leq A' \Rightarrow (cf^r(A, A')) \leq m^*$ (exists by 1.6(13)). Let $\{(C_i, D_i) : i < i^*\}$ list with no repetitions (up to isomorphism over $B$) of the pairs $(C, D)$ such that the quadruple $(A, B, C, D)$ is as in $(*)_2$ of Definition 7.1(1) with $|D| \leq |B| \times m + m^*$, and let $n^* \in \mathbb{N}$ be large enough such that for every $n \geq n^*$ the probability of the event $\mathcal{E}_n^* = \{\text{for every embedding } f : C_i \to \mathcal{M}_n \text{ such that } cf^r((A), \mathcal{M}_n) \subseteq f(C_i) \text{ there is an embedding } g : D_i \to \mathcal{M}_n \text{ extending } f'' \text{ is } \geq 1 - \varepsilon/i^*\}$. So for $n \geq n^*$ the probability that all the events $\mathcal{E}_n^0, \ldots, \mathcal{E}_{i^*-1}^*$ occur is $\geq 1 - \varepsilon$, and it suffices to prove that for such $\mathcal{M}_n$ for every embedding $f : A \to \mathcal{M}_n$, there are $m$ disjoint extensions to $g : B \to \mathcal{M}_n$. Choose by induction on $\ell$ an embedding $g_\ell : B \to \mathcal{M}_n$ extending $f$ with $\text{Rang}(g_\ell) \setminus \text{Rang}(f)$ disjoint to $\bigcup_{i < \ell} \text{Rang}(g_i)$. If we succeed to get $g_0, \ldots, g_{m-1}$, we are done, so assume we are stuck for some $\ell < m$. By Definition 6.1(2)(i) (existence for $\mathcal{J}$) we can find $D \in \mathcal{K}_\infty$ such that $C = \mathcal{M}_n \upharpoonright \left( \bigcup_{j < \ell} \text{Rang}(g_j) \cup cf^r(f(A), \mathcal{M}_n) \right) \subseteq D$ and there is an embedding $f^+ : B \to D$ extending $f$ such that $f^+(B) \setminus f(A)$ is disjoint to $C$ and
\[ D = C \cup f^+(B) \] and \[ f^+(B) \bigcup_{f(A)} D \] \[ C. \] Now there is \( i < i^* \) such that \((D, C) \cong (D_i, C_i)\) more exactly there is an isomorphism \( h \) from \( D \) onto \( D_i \) such that \( h(C) = C_i \) and \( f^+ = h \mid B \) and apply \( ^*\mathcal{E}_n \) occurs to \( \mathcal{M}_n \) to get contradiction.

4) Read definitions.

7.4 Claim. 1) Assume \((\mathcal{R}, \cup)\) is an explicitly nice 0-1 context and \( r(A, B, k) \) is as in (\(*\)_2 of Definition 7.1(1).

(a) if \((A \cup B) \bigcup_{A} D, r = r(A, A \cup B, k) \) (of Definition 7.1(1)(\(*\)_2), then \((C, A, B, D)\)

is almost \((k, r)\)-good

(b) if \( A \leq A^+ \in \mathcal{K}_\infty \) and \( A \leq_s D \) and \( B \leq D \in \mathcal{K}_\infty \) and \( k \leq r, r(A, D, k) \leq r \) where \( r(A, D, k) \) is as guaranteed by 7.3(1) then \((A^+, A, B, D)\)

is semi \((k, r)\)-good

(c) if \( A \leq_s D, B \leq D \in \mathcal{K}_\infty \) then \((A, B, D)\)

is semi \((k, r)\)-good.

2) In Definition 7.1(5), in \((\ast\ast)\) we can allow any \( b \in \mathcal{M}_n\).

3) In Definition 7.1(6), in \((\ast)\) we can restrict ourselves to \( b \in \mathcal{M}_n \setminus c^{\ell m}(f(A), \mathcal{M}_n)\) and/or replace in clause \((\gamma)\) the demand \( |D| \leq \ell \) by \( |A^+ \cup D| \leq \ell|\).

Proof. 1)

(a) Reread Definition 7.1(1) particularly \((\ast)_2\) and Definition 7.1(2).

(b) By clause (i) of Definition 6.1(1) (= existence) without loss of generality

\[ D \bigcup_{A} A^+ \] for some \( D^+ \in \mathcal{K}_\infty \). By clause (a) we know \((A^+, A, D, D^+)\)

is almost \((k, r)\)-good. By 7.3(2) we get the desired conclusion.

(c) Left to the reader.

2) Follows by part (2) (and the Definition 7.1(4)).

3) Left to the reader.

7.5 Claim. 1) If \((\mathcal{R}, \cup)\) is explicitly nice, \underline{then} \( \mathcal{R} \) is explicitly semi-nice, and \( \mathcal{R} \) is semi-nice.

2) If \((\mathcal{R}, \cup)\) is explicitly nice and \underline{then} \( \mathcal{R} \) is explicitly almost nice and, if in addition, it has the strong finite basis property it is almost nice.

3) If \( \mathcal{R} \) is explicitly semi-nice, \underline{then} \( \mathcal{R} \) is semi-nice.

4) If \( \mathcal{R} \) is explicitly almost nice and \((\mathcal{R}, \cup)\) has the strong finite basis property \underline{then} \( \mathcal{R} \) is almost nice.

5) If \( \mathcal{R} \) is almost nice \underline{then} \( \mathcal{R} \) is semi-nice.

Proof. 1) So assume that \((\mathcal{R}, \cup)\) is explicitly nice, so by 7.3(1), even if just

\[ A <_s B, k \in \mathbb{N} \] then for some \( r = r(A, B, k) \geq k \) we have \((\ast)_2\) of 7.1(1). Let us
prove that $\mathfrak{R}$ is explicitly semi-nice; i.e. Definition 7.1(10), so let $\ell$ and $k$ be given, and we should provide $r$ as required there. Let $r = \text{Max}\{r(A,B,k) : A <_s D \in \mathcal{K}_\infty, |D| \leq \ell\}$.

We should verify (5) of Definition 7.1(10), so let $\mathcal{M}_n$ be random enough, $A <_s D, B \leq D \subseteq \mathcal{M}_n, |D| \leq \ell, A^+ = cf^n(A, \mathcal{M}_n)$ and assume that $cf^k(B_\mathcal{M}_n, \mathcal{M}_n) \subseteq D$. We should prove that $(A^+, A, B, D)$ is semi-$(k, r)$-good. For this it suffices to verify the assumptions of 7.4(1)(b) but they are obvious. To finish the proof note that by 7.5(3) below $\mathfrak{R}$ is explicitly semi-nice (see Definition 7.1(10)) implies that $\mathfrak{R}$ is semi-nice.

2) Similar to part (1) using this time 7.4(1)(a) above and 7.5(4) below.

3) So let $A \in \mathcal{K}_\infty$ and $k \in \mathbb{N}$ be given and we have to find $\ell, m, r$ as required in Definition 7.1(6).

Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be such that: for any $i, j$ if $\mathcal{M}_n$ is random enough and $A' \subseteq \mathcal{M}_n, |A'| \leq j$ then $cf^n(A', \mathcal{M}_n)$ has at most $f(i,j)$ elements.

Let $\ell^* = f(k, |A| + 1)$, now let $r^*$ be the $r$ guaranteed to exist in Definition 7.1(10) for $k$ and $\ell$. Now define by induction on $i \leq \ell^* + 1$ a number $m_i$ as follows: $m_0 = |A|, m_{i+1} = f(r, \ell^* \times (m_i)^{\ell^*})$ and lastly let $m = m_{\ell^* + 1}$.

So we have chosen $\ell, m, r$ and have to show that they are as required in Definition 7.1(6).

So let $\mathcal{M}_n$ be random enough and $f : A \rightarrow \mathcal{M}_n$ and $b \in \mathcal{M}_n$. We define by induction on $i \leq \ell^* + 1$ a set $A_i$ increasing with $i$ as follows:

$$A_0 = f(A), A_{i+1} = A_i \cup \bigcup \{cf^n(A', \mathcal{M}_n) : A' \subseteq A_i \text{ and } |A'| \leq \ell\}.$$ 

Clearly we can prove by induction on $i$ that $A_0 \subseteq_1 A_i$ and $|A_i| \leq m_i$ hence $A_i \subseteq cf^{m_i}(f(A), \mathcal{M}_n)$.

As $cf^k(A_0 \cup \{b\}, \mathcal{M}_n) \setminus A_0$ has $\leq \ell^*$ members necessarily for some $i < \ell + 1$ we have: $cf^k(A_0 \cup \{b\}, \mathcal{M}_n)$ is disjoint to $A_{i+1} \setminus A_i$, and choose the maximal such $i$. So $cf^k(A_0 \cup \{b\}, \mathcal{M}_n) \setminus A_i$ has at most $\ell - |A| - i$ members, so by the definition of $A_{i+1}$ there is no $A'$, such that $A_i \cap cf^k(A_0 \cup \{b\}, \mathcal{M}_n) \leq A' \leq cf^k(A \cup \{b\}, \mathcal{M}_n)$ hence letting $A^* = cf^k(A_0 \cup \{b\}, \mathcal{M}_n) \setminus A_i$ we have $A^* \leq _s cf^k(A_0 \cup \{b\}, \mathcal{M}_n)$.

Let $D = : cf^k(A_0 \cup \{b\}, \mathcal{M}_n) \setminus A_i$ and $B = A_0 \cup \{b\} \text{ and } A^* = cf^n(A^*, \mathcal{M}_n)$ hence $A^* \subseteq A_{i+1}$. Now we use Definition 1.6(10) (i.e. the choice of $r$) with $(A^+, A^*, B, D), k, \ell, r$ here standing for $A^+, A, B, D, k, \ell, r$ there, clearly the assumption of 1.6(10)(*) holds (i.e. $A^* \leq A^* \leq \mathcal{M}_n, A^* \leq D \leq \mathcal{M}_n, B \leq D, |D| \leq \ell, f(A) = A_0 \subseteq A^* \subseteq A^+ = cf^n(A^*, \mathcal{M}_n) \subseteq A_{m_{i+1}} = cf^{m_{i+1}}(f(A), \mathcal{M}_n)$ and $r, m^*$ as required for $k, \ell$). Hence we get $(A^+, A^*, B, D)$ is semi-$(k, r)$-nice.

Now let us check requirements (α) – (ζ) in (5) of Definition 7.1(6). Now clauses (α), (β), (γ) holds by the suitable choices, and clause (δ) holds by a previous sentence and (ζ) as $D = cf^k(B, \mathcal{M}_n)$.

4) Similar to the proof of part (3).

So assume $\mathfrak{R}$ is explicitly almost nice and $(\mathfrak{R}, \mathfrak{U})$ has the strong finite basis property and we should prove that $\mathfrak{R}$ is almost nice. So we have to check Definition 7.1(5), clearly $\mathfrak{R}$ is weakly nice, so we are given $A \in \mathcal{K}_\infty$ and $k \in \mathbb{N}$ and we should find $\ell, m, r$ satisfying (5) from Definition 7.1(5).

Choose $r^*, m^*$ as in 7.1(9) for with $(k, |A| + 1)$ here standing for $(k, \ell)$ there. Let $i(5)$ be such that:
Clearly we should find $\ell,m,r$ and we should find $A$ as in Definition 7.1(5). Now let $A$ we should find $\ast\ast$ clauses in Definition 7.1(5)(7.1(6). Obviously, $K_5$ So assume that $K$ is weakly nice and for every $A$ have to check clause $(\gamma)$ of 7.1(5), now $(\alpha)$ of (**) of 7.1(5). Let $B_1 = cl^m(A \cup \{b\},\mathcal{M}_n).$

We define by induction on $i$, a set $A_i$ as follows:

$$A_i^0 = f(A), A_i^+ = cl^\ell (A_i^+,\mathcal{M}_n).$$

Clearly $|A_i| \leq m_i$. As $(\mathfrak{R},\mathcal{U})$ has the finite basis property (and the choice of above)

$$A_i^{r+1} \cup B_{i} \cup A_i^{r+1}.$$ 

we can find $i$ such that $A_i^r \cup B_{i} \cup A_i^{r+1}$. 

Now choose $A_0 = A_i^r, A^+ = A_{i+1}^r, B^+ = B_1 \cup A_{i+1}^r$ and let us check clause $(\alpha)-(\varepsilon)$ of (**) of 7.1(5), now $(\alpha),(\beta)$ hold by the choices $A_0, A^+, B^+$ and clause $(\varepsilon)$ holds by the choice of $A_{i+1}^r = A^+$, also clause $(\gamma)$ holds as $|B^+| \leq |cl_m(A \cup \{b\},\mathcal{M}_n)| + |A_{i+1}^r| \leq f(m,|A|+1) + m_{i+1}^r \leq f(m,|A|)+m$. 

5) So assume that $\mathfrak{R}$ is almost nice (i.e. Definition 7.1(5)) and let us check Definition 7.1(6). Obviously, $\mathfrak{R}$ is weakly nice. So let $A \in \mathcal{X}$ and $k \in \mathbb{N}$ be given and we should find $\ell,m,r$ such that $(*)$ of Definition 7.1(6) holds. We just choose them as in Definition 7.1(5). Now let $\mathcal{M}_n$ be random enough and $f : A \rightarrow \mathcal{M}_n$ and we should find $A_0, A^+, B, D$ as in $(*)$ of Definition 7.1(6). Let $A_0, A^+, B^+$ be as required in (**) of Definition 7.1(5). Let us choose $B = A_0 \cup \{b\}, D = B^+$, and we have to check clause $(\alpha)-(\delta)$ of Definition 7.1(6). Now they hold by the respective clauses in Definition 7.1(5)(**), but for $(\delta)$ we have to use 7.3(2).

$\square_{7.5}$

7.6 Remark. Let “$\mathfrak{R}$ is explicitly-semi-nice” means

$\otimes$ $\mathfrak{R}$ is weakly nice and for every $\ell,k$ there is $m$ such that for every $\ell_1$ there is $r$ such that:

$(*)$ if $\mathcal{M}_n$ is random enough, $A \leq D, B \leq D \leq \mathcal{M}_n, |A \cup B| \leq \ell_0, |D| \leq \ell_1$ and $A^+ = cl^\ell(A,\mathcal{M}_n), cl^m(B,\mathcal{M}_n) \leq D$ then $(A^+, A, B, D)$ is semi $(\ell,k)$-good.

Now explicitly-semi-nice $\Rightarrow$ explicitly-semi-nice $\Rightarrow$ semi-nice.

7.7 Claim. Assume that $\mathfrak{R}$ is a 0-1 context obeying $\bar{h} = (h^d, h^u)$ which is bounded by $h^*$, and $\mathcal{U} = \bigcup \bar{h}, h^*$ and assume $(*)$ of 6.10(2).
1) \( \mathcal{R} \) is weakly nice.
2) A sufficient condition for “\((\mathcal{R}, \emptyset)\) is explicitly+ nice” is

\[
\bigotimes_2 \text{if } A \prec B \prec D \text{ and } A \leq \varnothing \text{ and } A \leq D, \text{ then for some } \varepsilon \in \mathbb{R}^+ \text{ for every random enough } \mathcal{M}_n, \text{ we have } (h[\mathcal{M}_n])^{-\varepsilon} > h_{A,D}[\mathcal{M}_n]/h_{A,B}[\mathcal{M}_n].
\]

3) A sufficient condition for \((A^+, A, B, D)\) being semi \((k, r)\)-good is

\[
\bigotimes_3 \text{ if } A \leq A^+, B \leq D \in \mathcal{X}_\infty \text{ and } A \leq \varnothing \text{ and } \varnothing \geq k + |A| \text{ and:}
\]

\[
\bigvee_{A,B,D} \text{ if } D \leq D' = D \cup C, |C| \leq k, D' \neq D \cup \mathcal{C}(A, D') \text{ and } C \cap B \prec_i C, A \leq A' \leq s D' \text{ then for every } \varepsilon \in \mathbb{R}^+ \text{ for every random enough } \mathcal{M}_n, \text{ we have } \varepsilon > h_{A,D}[\mathcal{M}_n]/h_{A,D'}[\mathcal{M}_n].
\]

4) A sufficient condition for \( \mathcal{R} \) being explicitly semi nice (see Definition 7.1(10) hence semi nice, by 7.5(3)) is [Saharon copied]

\[
\bigotimes_4 \text{ if } B \leq D \in \mathcal{X}_\infty, A \leq \varnothing \text{ and } \varnothing \geq k \text{ and } r \geq |A| + k \text{ then }
\]

\[
\bigvee_{A,B,D} \text{ if } D \leq D' = D \cup C, |C| \leq k, D' \neq D \cup \mathcal{C}(A, D') \text{ and } C \cap B \prec_i C, A \leq A' \leq s D' \text{ then for every } \varepsilon \in \mathbb{R}^+ \text{ for every random enough } \mathcal{M}_n, \text{ we have } \varepsilon > h_{A,D}[\mathcal{M}_n]/h_{A,D'}[\mathcal{M}_n].
\]

5) A sufficient condition for \( \mathcal{R} \) being explicitly semi-nice (hence semi-nice by 7.5(3)) is

\[
\bigotimes_5 \text{ if } k, \ell \in \mathbb{N}, \mathcal{M}_n \text{ is random enough, } B \leq D \leq \mathcal{M}_n \text{ and } |D| \leq \ell \text{ and } \mathcal{C}^k(B, \mathcal{M}_n) \subseteq D \text{ and } A \prec D \text{ and } r = k + |A| \text{ then }
\]

\[
\bigvee_{A,B,D} \text{ if } D \leq D' = D \cup C, |C| \leq k, D' \neq D \cup \mathcal{C}(A, D') \text{ and } C \cap B \prec_i C, A \leq A' \leq s D' \text{ then for every } \varepsilon \in \mathbb{R}^+ \text{ for every random enough } \mathcal{M}_n, \text{ we have } \varepsilon > h_{A,D}[\mathcal{M}_n]/h_{A,D'}[\mathcal{M}_n].
\]

6) A sufficient condition for \( \mathcal{R} \) being explicitly- semi-nice (see 7.6) is

\[
\bigotimes_6 \text{ for any } \ell, k \in \mathbb{N} \text{ there are } m, r \in \mathbb{N} \text{ such that: if } \mathcal{M}_n \text{ is random enough, } A \leq B \leq \mathcal{M}_n, |B| \leq \ell, D = \mathcal{C}^k(B, \mathcal{M}_n) \text{ and } D \leq D', D \cup C, |C| \leq k, B \cup C \prec_i C, A \leq A' \leq s D' \text{ then } \bigvee_{A,B,D} \text{ holds.}
\]

Remark. 1) When \( \mathcal{C}^k(A, M) \) has no bound by \( k, |A| \), we may consider cases \( \mathcal{C}^{k+1}(A, M) \), is repeated closure under \( \mathcal{C}^k \) where \( \ell \) is the length of the iteration. What about the bound on the size of \( \mathcal{C}^{k+1}(A, M) \)? Consider \( \log[||\mathcal{M}_n||] \).

2) We may consider giving to \( A \prec_i B \) possibly a negative \( \alpha(A, B) \) so measure how few such cases arise, this may help make additive formulas meaningful.

Proof. 1) By 2.5(2).

2) In Definition 7.1(1), the “\((\mathcal{R}, \emptyset)\) is a 0-1 context” holds by 6.10(2) as we are
assuming $(*)_2$ of 6.10(2). Let $r(A, B, k) = |A| + k$. So assume $B \square_A^D C$ and $r = |A| + k$ and $k$ be given and without loss of generality $D = B \cup C$. Let $\mathcal{M}_n$ be random enough and $f : C \to \mathcal{M}_n$. Now we know the “order of magnitude” of

$$G = \{ g : g \text{ is an embedding of } D \text{ into } \mathcal{M}_n \text{ extending } f \}.$$ 

Also by the definition of $\sqcup$ (in 2.1(5))

$$G_1 = \{ g \in G : \neg (g(D) \sqcup_{\ell<k} c\ell^*(f(C), \mathcal{M}_n)) \}$$

has smaller order of magnitude and also, by the assumption $\otimes_2$

$$G_2 = \{ g \in G : c\ell^*(g(D), \mathcal{M}_n) \subseteq g(D) \cup c\ell^*(f(C), \mathcal{M}_n) \}.$$ 

Now every $g \in G \setminus G_1 \setminus G_2$ is as required, by 6.7(1) Shmuel?

3) Similar proof.

4) So let $\ell, k$ be given, we have to choose $r$ as required in Definition 7.1(10). Let $r = |A| + k$ and let $\mathcal{M}_n$ be random enough, so we have just to check $(*)$ from Definition 7.1(10). So assume $\mathcal{M}_n$ is random enough, $A <_s D, B \leq D \leq \mathcal{M}_n, |D| \leq \ell, c\ell^*(B, \mathcal{M}_n) \subseteq D$ and $A^+ = c\ell^*(A, \mathcal{M}_n)$, and it suffices to prove that the quadruple $(A^+, A, B, D)$ is semi $(k, r)$-good. For this we use the criterion from part (3) which holds (by assumptions above and) the assumption $\otimes_4$.

5), 6) Shmuel - details.

**7.8 Claim.** Assume that $\mathcal{R}$ obeys $\bar{h}$ and $\tilde{h}$ is polynomial over $h$.

1) If $A <_s B$ then for some $\alpha(A, B) \in \mathbb{R}^+$ and $k = k(A, B)$ (see Definition 2.3) every random enough $\mathcal{M}_n$ satisfies:

$$\text{(*) for every embedding } f \text{ of } A \text{ into } \mathcal{M}_n,$$

$$\|\mathcal{M}_n\|^{\alpha(A, B)} / h[\mathcal{M}_n]^k \leq n(u(f, A, B, \mathcal{M}_n) \leq \|\mathcal{M}_n\|^{\alpha(A, B)} \bar{h}[\mathcal{M}_n]^k).$$

2) In fact if $A = A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_k = B$ then we can let $k(A, B) = \sum_{i<k} k(A_i, A_{i+1})$ and $\alpha(A, B) = \sum_{\ell<k} \alpha(A_\ell, A_{\ell+1})$.

So the sum $\alpha(A, B)$ does not depend on the choice of the decomposition, i.e. of $\langle A_\ell : \ell \leq k \rangle$ but only on $(A, B)$ (and $\mathcal{R}$).

3) Some $h^+$ bounds $h$ (see Definition 2.1(3)), we can choose $h^+(n) = n$, but even can demand that it satisfies: for every $\varepsilon > 0$ for every random enough $\mathcal{M}_n$ we have $h^+(\mathcal{M}_n) < \|\mathcal{M}_n\|^\varepsilon$.

4) If $A <_s B <_s C$, then (see 7.8(2) above):

$$\alpha(A, C) = \alpha(A, B) + \alpha(B, C).$$

5) If $A <_s B \leq_s D, A <_s D$, then $\alpha(A, B) \geq \alpha(A, D)$.

6) If $A <_s B \leq D = B \cup C$ and $A \leq C <_s B$ then
\[ \alpha(A, B) \geq \alpha(C, D). \]

**Proof.** Straightforward.

Saharon: about almost nice?

**7.9 Claim.** Assume that \( \mathcal{R} \) is a 0-1 context obeying \( \hat{h} \) and \( \bar{h} \) is polynomial over \( h \), so \( \bar{h} \) is bound by some \( h^* \) (see 2.5) and let \( \mathcal{U} = \langle \hat{h}, h^* \rangle \).

1) \( \mathcal{R} \) is weakly nice.

2) \( \begin{align*}
D & \cup C \\
\mathcal{A} & \quad \text{iff} \quad \begin{cases}
A \leq B \leq D, A \leq C \leq D, B \cap C = A \quad \text{and} \quad \alpha(A, B) = \alpha(C, C \cup B).
\end{cases}
\end{align*} \)

3) If the condition \( \otimes_3 \) below holds then condition \((*)_2\) of 6.10(2) holds hence \( (\mathcal{R}, \mathcal{U}) \) is a 0-1 context, where

\[ \otimes_3 \quad \text{if} \quad A \leq B \leq D, A \leq C \leq D, A <_s B \quad \text{and} \quad \neg(C <_s D) \quad \text{or} \quad \neg(B \cup C) \quad \text{and} \quad D = B \cup C \quad \text{then} \quad \alpha(A, B) > \alpha(C, D). \]

4) A sufficient condition for “(\( \mathcal{R}, \mathcal{U} \)) is explicitly\(^+\) nice” is

\[ \otimes_4 \quad \text{if} \quad A <_s B <_i D \quad \text{and} \quad A \leq D, \quad \text{then} \quad \alpha(A, B) > \alpha(A, D). \]

5) A sufficient\(^{12}\) condition for “\( \mathcal{R} \) is semi-nice” is

\[ \otimes_5 \quad \text{for every} \quad i \in \mathbb{N}, \text{for some} \quad r \in \mathbb{N} \quad \text{we have:} \]

\begin{align*}
&\text{for every random enough} \quad \mathcal{M}_n, \text{and every} \quad A \leq B \leq \mathcal{M}_n, |B| \leq \ell, \text{letting} \\
&D = c\ell(B, \mathcal{M}_n), \quad A \leq A_0 \leq D \quad \text{we have} \\
\otimes_{k, \ell}^{A_0, B, D} &\quad \text{if} \quad D < D_1 = D \cup C, |C| \leq k, C \cap B <_i C, \quad A_0 \leq A_1 <_s D, \quad A_1 \text{ embeddable into} \quad \mathcal{M}_n \quad \text{over} \quad A \\
&\text{then} \quad \alpha(A_0, D) > \alpha(A_1, D_1). \end{align*}

6) A sufficient condition for \( \otimes_3 \) of 7.9(3) above, is

\[ \otimes_6 \quad \text{if} \quad A <_{pr} B \leq D, A \leq C <_{pr} D = B \cup C \quad \text{and} \]

\[ \neg(B \cup C), \quad \text{then} \quad \alpha(A, B) > \alpha(C, D). \]

**Proof.**

1) Should be clear.

2) Should be clear.

3) Read \((*)_2\) of 6.10(2).

4) Read the definition.

5) Easy.

6) Easy.
7.10 Definition. \( R \) is polynomially nice if it obeys \( \tilde{h} \) which is polynomial over \( h \) and satisfies \( \otimes_3 + \otimes_4 \) of 7.9.

7.11 Definition. We say \( R \) is polynomially semi-nice if it obeys \( \tilde{h} \) which is polynomial over \( h \) and satisfies \( \otimes_3, \otimes_5 \) of 7.9.

7.12 Definition. 1) We say \( R \), a 0-1 law context obeys \( \tilde{h} \) very nicely it obeys \( \tilde{h} \) and satisfies \((*)_2\) of 6.10(2) and \((*)_2\) of 7.7(2) (hence it is explicitly nice), with the \( \bigcup \) of Definition 2.1(5), of course.

2) We say \( R \), a 0-1 law context obeys a polynomial \( \tilde{h} \) very nicely if it satisfies \( \otimes_3 \) of 7.9(3) and \( \otimes_4 \) of 7.9(4).

3) By 2.5(2), clearly \( R \) is weakly nice. As for \((*)\) of 6.10(2), reread the definitions.

7.13 Definition. We say \(( \tilde{R}, c, (\bigcup) \) is polynomially almost nice if:

\((*)_4\) for every \( k, \ell \) for some \( m, t, s \) for every \( A, B \leq D, A \leq_s D \) we have:

for every random enough \( \mathcal{M}_n \), if

\[ f : D \to \mathcal{M}_n, A \leq_s D, B \leq D, c \ell^s(f(B), \mathcal{M}_n) \subseteq f(D), \]

\[ D \cap \ell^s(f(A), \mathcal{M}_n) \equiv f(A) \] then \( (A, B, D) \) is \( k \)-good.
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