Totally Rank One Interval Exchange Transformations

YUE WU\textsuperscript{1} DONGMEI LI\textsuperscript{2} DIQUAN LI\textsuperscript{3} YUNJIAN WANG\textsuperscript{4}

Abstract

For irreducible interval exchange transformations, we study the relation between the powers of induced map and the induced maps of powers and raise a condition of equivalence between them. And skew production of Rauzy induction map is set up and verified to be ergodic regarding to a product measure. Then we prove that almost all the interval exchange transformations are totally rank one (rank one for all powers of positive integers) by interval. As a corollary, for almost all interval exchange transformations, rank one transformations are dense $G_{δ}$ in the weak closure.

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In this paper, we extend Veech’s theorem about rank one interval exchange transformations (i.e.t.) (\cite{4}, \cite{11}) to all positive-integer powers of interval exchange transformations (totally rank one). Property of rank one is a crucial conception in the field of ergodic theory and dynamic systems. Originated from stacking construction of transformations related to symbolics dynamics, rank one transformation has been studied with regarding to many topics (\cite{2}, \cite{3}, \cite{5}), including functional spectrum, invariant measures of productions, and many other topics. If the base of each Rohlin tower of the rank one transformation is an interval, we say it is rank one by interval. If all the powers of a transformation are rank one, we say it is totally rank one (definition \cite{11}). Here it is showed that measure theoretically all the interval exchange transformations are totally rank one by interval (Theorem \cite{13}).

Definition 0.1 (Totally Rank One). We say a finite measure-preserving system $(X, β, T, μ)$ is totally rank one, if for every $q ∈ \mathbb{N}$, $T^q$ is rank one.

In Section one, we introduce the fundamental concepts of i.e.t, Rauzy-Veech induction (\cite{7}, \cite{8}, \cite{9}) and Veech’s theorems (\cite{9}) about i.e.t.’s related to this paper. At the end of section one, the main theorem of this paper is stated: almost all interval exchange transformations are totally rank one by interval. In Section 2, a condition for the induced map of a power of i.e.t. to be equal to the power of a induced map of i.e.t. is given, with redar to Rauzy-Veech induction. In Section 3, we introduce a skew product, which is extended from the Rauzy map. It is showed that this skewing transformation is ergodic and conservative relative to a product measure. In the last section, Section 4, the major propositions studied in Section 2 and Section 3 are used to set up a cutting and stacking structure, related to the Rauzy-Veech induction, of a power of i.e.t. This shows that measure theoretically, all the i.e.t.’s are totally rank one by interval. A corollary is conducted that rank one transformations are dense $G_{δ}$ in the weak closure of a i.e.t, measure theoretically.

1 Schlumberger WesternGeco Geosolution, Houston, Texas, USA.
2 Corresponding Author; Harbin University of Science and Technology, School of Applied Science, Harbin, Heilongjiang, China.
3 Central South University, School of Geosciences and Info-physics, University, Changsha, Hunan, China.
4 Schlumberger SWT Technology Center, Houston, Texas, USA.
1 Introduction

Let $\Lambda_m \subset R^m$ be a positive cone, and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \Lambda_m$. Let $G_m$ be the group of $m$-permutations, and $G_m^0$ be the subset of $G_m$ which contains all the irreducible permutations on $\{1, 2, \cdots, m\}$. A permutation $\pi$ is irreducible if and only if for any $1 \leq k < m$, $\{1, 2, \cdots, k\} \neq \{\pi(1), \cdots, \pi(k)\}$, or equivalently $\sum_{j=1}^k (\pi(j) - j) > 0$, ($1 \leq k < m$). Given $\lambda \in \Lambda_m$, $\pi \in G_m^0$, the corresponding interval exchange transformation is defined by:

$$T_{\lambda, \pi}(x) = x - \beta_{\pi^{-1}}(\lambda - \beta_{i-1}(\lambda)), \quad (x \in [\beta_{i-1}(\lambda), \beta_i(\lambda)))$$

where

$$\lambda^\pi = (\lambda_{\pi^{-1}1}, \lambda_{\pi^{-1}2}, \cdots, \lambda_{\pi^{-1}m}).$$

$$\beta_i(\lambda) = \begin{cases} 0 & i = 0 \\ \sum_{j=1}^i \lambda_j & 1 \leq i \leq m. \end{cases}$$

Obviously $\beta_{\pi^{-1}}(\lambda^\pi) = \sum_{j=1}^{\pi^{-1}1} \lambda_{\pi^{-1}j}$, and the transformation $T_{\lambda, \pi}$, which is also denoted by $(\lambda, \pi)$, sends the $i$th interval to the $\pi(i)$th position.

**Rauzy-Veech induction.** For $T_{\lambda, \pi}$, the Rauzy map sends it to its induced map on $[0, |\lambda| - \min \{\lambda_m, \lambda_{\pi^{-1}m}\})$, which is the largest admissible interval of form $J = [0, L), 0 < L < |\lambda|$. Given any permutation, two actions $a$ and $b$ are:

$$a(\pi)(i) = \begin{cases} \pi(i) & i \leq \pi^{-1}m \\ \pi(i) - 1 & \pi^{-1}m + 1 < i \leq m \\ \pi(m) & i = \pi^{-1}m + 1 \end{cases}$$

and

$$b(\pi)(i) = \begin{cases} \pi(i) & \pi(i) \leq \pi(m) \\ \pi(i) + 1 & \pi(m) + 1 < \pi(i) < m \\ \pi(m) + 1 & \pi(i) = m. \end{cases}$$

The Rauzy-Veech map $Z(\lambda, \pi) : \Lambda_m \times G_m^0 \rightarrow \Lambda_m \times G_m^0$ is determined by :

$$Z(\lambda, \pi) = (A(\pi, c)^{-1} \lambda, c \pi),$$

where $c = c(\lambda, \pi)$ is defined by

$$c(\lambda, \pi) = \begin{cases} a, & \lambda_m < \lambda_{\pi^{-1}m} \\ b, & \lambda_m > \lambda_{\pi^{-1}m}. \end{cases}$$
\( Z(\lambda, \pi) \) is a.e. defined on \( \Lambda_m \times \{\pi\} \), for each \( \pi \in G_m^0 \).

The matrices \( A = A(\pi, c) \) in (1.4) are defined as the following:

\[
A(\pi, a) = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
I_{\pi^{-1}m} & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(1.6)

\[
A(\pi, b) = \begin{pmatrix}
I_{m-1} & \cdot & \cdots & \cdot & \cdot \\
0 & 1 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdots & \cdots & \cdot & \cdot \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

(1.7)

where \( I_k \) is the \( k \)-identity matrix, and \( j = \pi^{-1}m \).

And the normalized Rauzy map \( \mathcal{R} : \Delta_{m-1} \times G_m^0 \to \Delta_{m-1} \times G_m^0 \) is defined by

\[
\mathcal{R}(\lambda, \pi) = \left( \frac{A(\pi, c)^{-1}\lambda}{|A(\pi, c)^{-1}\lambda|}, c\pi \right) = \left( \frac{\pi_1^* Z(\lambda, \pi)}{|\pi_1^* Z(\lambda, \pi)|}, \pi_2^* Z(\lambda, \pi) \right),
\]

(1.8)

where \( \pi_1^* \) and \( \pi_2^* \) are the projection to the first coordinate and the second coordinate respectively. Iteratively,

\[
Z^n(\lambda, \pi) = ((A^{(n)})^{-1}\lambda, c^{(n)}\pi) = (\lambda^{(n)}, \pi^{(n)}),
\]

(1.9)

where

\[
c^{(n)} = c_n c_{n-1} \cdots c_1, \quad (c_1, \cdots, c_n \in \{a, b\}, c_i = c(Z^{i-1}(\lambda, \pi))
\]

(1.10)

and

\[
A^{(n)} = A(\pi, c_1) A(c^{(1)}(\pi), c_2) A(c^{(2)}(\pi), c_3) \cdots A(c^{(n-1)}(\pi), c_n).
\]

(1.11)

The Rauzy class \( C \subseteq G_m \) of \( \pi \) is a set of orbits for the group of maps generated by \( a \) and \( b \). On the \( \mathcal{R} \) invariant component \( \Delta_{m-1} \times C \), we have:

**Theorem 1.1** (H. Masur [6], W. A. Veech [8]). Let \( \pi \in G_m^0 \), the set of irreducible permutations. For Lebesgue almost all \( \lambda \in \Lambda_m \), normalized Lebesgue measure on \( I_\lambda \) is the unique invariant Borel probability measure for \( T_{(\lambda, \pi)} \). In particular, \( T_{(\lambda, \pi)} \) is ergodic for almost all \( \lambda \).

**Theorem 1.2** (W. A. Veech [8]). Let \( \pi \in G_m^0 \), the set of irreducible permutations. For Lebesgue almost all \( \lambda \in \Lambda_m \), normalized Lebesgue measure on \( I_\lambda \) is rigid and rank one, thus admits simple spectrum.

We extend Veech’s result to the following theorem:

**Theorem 1.3.** All the notations as above, suppose \( m > 1, m \in \mathbb{N}, \pi \in G_m^0, q \in \mathbb{N} \). For Lebesgue almost all \( \lambda \in \Lambda \), \( T^{q}_{(\lambda, \pi)} \) is rank one by interval with flat stack, thus it is also rigid.
The theorem will be seen to imply that for the corresponding corresponding interval exchange transformations, each has a residual set of rank one transformations in its weak closure, by [KIN], also in its commutant.

Next we introduce some result which would be utilized in section 5 and section 6:

**Theorem 1.4.** Let \( m > 1, \pi \in \mathcal{G}_m^0 \), \( \mathcal{R} = \mathcal{R}(\pi) \) is the Rauzy class containing \( \pi \). Then there exists on \( \Delta_{m-1} \times \mathcal{R}(\pi) \) a unique absolutely continuous invariant measure \( \mu \) up to a scalar multiple, such that \( \mathcal{P} \) (see (2.1.9)) is conservative and ergodic on \( \Delta_{m-1} \times \mathcal{R} \) relative to \( \mu \). The density of \( \mu \) on \( \Delta_{m-1} \times \{\pi\} \) is the restriction of a function which is positive, rational, and homogeneous of degree \( m \) on \( \Lambda_m \times \{\pi\} \).

Next, the induced map of \( \mathcal{P} \) on \( \Delta_{m-1} \times \{\pi\} \) will be described, and a condition for a measurable function to be essential constant will be given.

By Theorem 1.3, induced map \( \mathcal{P}_\pi \) of \( \mathcal{P} \) on \( \Delta_{m-1} \times \{\pi\} \) is well defined measure theoretically. Then for a.e. \( x = (\alpha, \pi) \in \Delta_{m-1} \times \{\pi\} \) there exists \( n(x) \in \mathbb{N} \) such that \( \mathcal{P}_\pi(x) = \mathcal{P}^{n(x)}(x) \), and there exists \( A = A(x) \) such that \( \mathcal{P}_\pi(x) = (A^{-1}A^{-1})^{(\pi)} \). What’s more, if we let \( \Delta^\alpha = (A\Lambda_m) \cap \Delta_{m-1} \), and define \( P_A : \Delta^\alpha \rightarrow \Delta_{m-1} \) by

\[
(1.12) \quad P_A(\alpha) = \frac{A^{-1}\alpha}{|A^{-1}\alpha|} \quad (\alpha \in \Delta^\alpha)
\]

then for any \( y = (\alpha, \pi) \in \Delta^\alpha \times \{\pi\} \), \( \mathcal{P}_\pi(y) = (P_A(\alpha), \pi) \).

There exists a set \( \mathcal{F} \) containing countable many elements (which are visitation matrices), such that \( \mathcal{F}_1 = \{\Delta^\alpha | A \in \mathcal{F}\} \) is a partition of \( \Delta_{m-1} \) into infinitely many atoms.

**Note.** For any \( A \in \mathcal{F} \), for Lebesgue almost all \( \lambda \in \Delta^\alpha, \mathcal{P}_\pi^i(\lambda, \pi) \) is defined for all \( i \in \mathbb{N} \).

To end this section, we recall (9.5) of [VEE3]

**Theorem 1.5.** Let \( \mathcal{H} \) be a separable Hilbert space, and \( U_A \) be unitary operators on \( \mathcal{H} \). Suppose \( f : \Delta_{m-1} \rightarrow \mathcal{H} \) is a measurable function and \( f(\mathcal{P}_\pi x) = U_A f(x) \), for a.e. \( x \in \Delta^\alpha, A \in \mathcal{F} \), then \( f \) is constant.

## 2 Powers of the Induced Map and the Induced Map of the powers

Without normalization, the \( n \)th iteration of the Rauzy map raises an \( m \)-interval exchange transformation on a subinterval \( J_n \) with vector \( \alpha = \lambda(n) \) and \( m \)-permutation \( \pi(n) \) (see [1.9]). Note that \( |J_n| = |\alpha| \). In this section we pay more attention on the first return time of \( T_{\lambda,\pi}|J_n \). The visitation matrix \( A(n) \) shows the orbit distribution of \( I_j^{(n)} \) among \( I_j^{(n)} \)’s. That is \( A_{ij}^{(n)} \) is the number of the visitation of \( T^l(I_{0j}) \) \((0 \leq l < a_j, a_j \) is the first return time of \( I_j^{(n)} \) to \( J_n \)) on \( I_1 \). Thus the first return time of any point in \( I_j^{(n)} \) is \( a_j = \sum_{i=1}^m A_{ij}^{(n)} \), that is \( a_j \) equals the \( j \)th column sum of \( A(n) \). Canonically, this yields a stack structure with \( m \)-stacks, \( S_1, S_2, \cdots, S_m \), each corresponding to the \( T \) orbits of a subinterval of \( \alpha \). The \( j \)th stack \( S_j \) has base \( I_j^{(n)} \) and height \( a_j \). \( T \) sends each level (except for the top) of \( S_j \) to the higher level, and sends the top back into \( J_n = \cup I_j^{(n)} \). Suppose the first return time of \( T^l \) to \( J_n \) is \( r_q(x), x \in J_n \), then \( T^{qr_q(x)}(x) \in J_n, T^q(x) \notin J_n(l = 1, 2, \cdots, r_q(x) - 1) \). It is easily seen that though \( T^{qr_q(x)} \in J_n \), the return time of \( T \) to \( J_n \) may be strictly less than \( qr_q(x) \).
Lemma 2.1. All notations as above, suppose \( a_j^{(k)} \) is the \( j \)th column sum of \( A^{(k)} \), and each \( a_j^{(k)} \) is odd, then there exists \( 1 \leq j_0 \leq m \) such that \( a_{j_0}^{(k+1)} \) is even.

Proof. Suppose \( Z^k(\lambda, \pi) = (\lambda^{(k)}, \pi^{(k)}) \), let \( \pi_0 = \pi^{(k)} \), then \( A^{(k+1)} = A^{(k)} A \), where \( A \) is either 1.6 or 1.7.

For the first case we see that
\[
a^{(k+1)}_{\pi_0^{-1}m+1} = a^{(k)}_{\pi_0^{-1}m} + a^{(k)}_m
\]
for the second case we see that
\[
a^{(k+1)}_{\pi_0^{-1}m} = a^{(k)}_{\pi_0^{-1}m} + a^{(k)}_m
\]
\[\square\]

Definition 2.2. \( T_{(\lambda, \pi)} \) satisfies the infinite distinct orbit condition (i.d.o.c., by Kean [14]) if \( T^{-j}(\beta_i)_{j \in \mathbb{N}}, 1 \leq i \leq m \) (the negative trajectories of discontinuities of \( T \)) are infinite disjoint sets.

Lemma 2.3. If \( T_{(\lambda, \pi)} \) satisfies the i.d.o.c, then \( T^q_{(\lambda, \pi)} \) also satisfies the i.d.o.c.

Proof. Suppose \( T^q_{(\lambda, \pi)} \) is \( T_{(\xi, \tau)} \), then the \( q(m-1) \) discontinuities are \( D_q = \{T^{-i}\beta_i, 1 \leq j < m, 0 \leq i < q\} \).

One may concern the point 0. Since 0 = \( T^q(\beta_k) \) for some \( 1 \leq k \leq m \), the \( i \)th preimage of 0 is the \( j \)th preimage of \( \beta_k \), already contained in \( D_q \). So what need to be shown is that the \( T^q \) orbit of \( T^{-i}(\beta_k) \) and that of \( T^{-i+1}\beta_j \) are disjoint. This is true since \( i_1 \neq i_2 \) and \( 0 \leq i_1, i_2 < q \).
\[\square\]

Proposition 2.4. If there exist \( j_1, j_2 \in \mathbb{N}, 1 \leq j_1, j_2 \leq m \) such that \( a_{j_1} \neq a_{j_2} \mod q \), then \( (T^q|_{J_n}) \circ (T^q|_{J_n}) \neq (T^q|_{J_n})(T^q|_{J_n}) \).

Proof. Let \( \Gamma = \{i|a_i \equiv a_{j_1} \mod q\} \) then \( \Gamma^* = \{i|a_i \neq a_{j_1} \mod q\} \) since \( a_{j_1} \neq a_{j_2} \mod q \), we know that \( \Gamma \neq \phi, \Gamma^* \neq \phi \), and \( j_1 \in \Gamma, j_2 \in \Gamma^* \). Correspondingly, let
\[
K = \{x|x \in I^{(a)}_i, i \in \Gamma\}
\]
\[
K^* = \{x|x \in I^{(a)}_i, i \in \Gamma^*\}
\]

Then since \( T^q_{n, \pi} \) satisfies i.d.o.c., \( T^q|_{J_n}(K) \cap K^* \neq \phi \), there exits \( x \in T^q|_{J_n}(K) \cap K^* \). Since \( T^q|_{J_n} \) is invertible, there exits \( y \in K \cap (T^q|_{J_n})^{-1}(K^*) \). Thus
\[
(T^q|_{J_n}) \circ (T|_{J_n}(y)) = T^{q_1+a_{j_1}}(y), \quad l_1 \in \mathbb{Z}
\]
\[
(T|_{J_n}) \circ (T^q|_{J_n}(y)) = T^{q_2+a_{j_0}}(y), \quad l_2 \in \mathbb{Z}, j_0 \in \Gamma^*
\]
Since \( a_{j_1} \neq a_{j_0} \mod q \)
\[
(T^q|_{J_n}) \circ (T|_{J_n}(y)) \neq (T|_{J_n}) \circ (T^q|_{J_n})(y)
\]
\[\square\]

Remark 2.5. Proposition 2.4 shows that if two subintervals of the induced map have different return times modulo \( q \), then the induced map of \( T \) and that of \( T^q \) on the same interval \( J_n \) do not commute.
From Lemma 2.4 and Proposition 2.4, we can see that during the consequent iterations of Rauzy map, at least one of \( J_{2k}, J_{2k+1} \) is such an subinterval such that \( T^2v, Tv \) do not commute, \( k' \in \{2k, 2k+1\} \). One may draw more general conclusion about the general case of \( T^q_{r} \) \( (q > 1, q \in \mathbb{N}) \). From these propositions, relation between \( T^q|J_n \) and \( T|J_n \) are studied. If we equalize the degree and study \( T^q|J_n \) and \( (T|J_n)^q \), it is interesting that these two transformation with the same domain and range (both are \( J_n \)) may be equal to each other. A condition for it to be true is given right after this paragraph, one will see it is crucial for proving Theorem 1.3.

Suppose \( a_i \equiv 1 \mod q, i = 1, 2, \cdots, m \); then for any \( x \in J_n \) there exists a sequence of positive integers \( q: a_{i_1}(x), a_{i_2}(x), \cdots, a_{i_q}(x) \) such that \( T^{a^*(x)} = (T|J_n)^q(x), a^*(x) = \sum_{k=1}^{q} a_{i_k}(x) \) and \( T^{a_{i_j}(x)}((T|J_n)^{j-1}(x)) = (T^q|J_n)^j(x), 1 \leq j \leq q \). That is to say that \( \sum_{k=1}^{j} a_{i_k}(x) \) is the \( j \)th return time of \( x \in J_n \) under the transformation \( T \). Since \( a_i \equiv 1 \mod q, i = 1, 2, \cdots, m, a^* \equiv 0 \mod q \). We claim that

\[
T^q|J_n = (T|J_n)^q
\]

Suppose \( (T^q|J_n)(x) = T^{a^*(x)} \), to prove 2.4, the only thing needs to be verified is that \( a'(x) \geq a^*(x) \). \((a'(x) \leq a^*(x) \) is implied by \( a^* \equiv 0 \mod q \). On the other hand, the \( j \)th return time of \( x \in J_n \) under \( T \) is \( a_j^*(x) = \sum_{k=1}^{j} a_{i_k}(x), a_j^*(x) \equiv j \neq 0 \mod q, (1 \leq j < q) \), thus \( a'(x) \geq a^*(x) \). That proves the following theorem:

**Theorem 2.6.** All notations and definitions as above, if \( a_i \equiv 1 \mod q, T^q|J_n = (T|J_n)^q \).

§2 An Ergodic Skew Product of Rauzy map

As an extension of the Rauzy map, a skewing transformation will be defined. In this section, we will show that this skew product is ergodic and conservative relative to a certain product measure. This result is the fundamental tool to prove the major result, Theorem 1.3. We locate the case to that the skewing group is a finite group, and we reach the result a little bit indirectly, i.e. we prove the ergodic property of a skew product conjugate to that we want.

Starting with \( m, q \in \mathbb{N}, m, q \geq 2 \), a finite group \( G \) is given by \( G = GL(m, \mathbb{Z}_q) \), where \( \mathbb{Z}_q \) is the ring of integers modulo \( q \). A map \( g : G^0 \times \{a, b\} \rightarrow G \) is defined by:

\[
g(\pi, c) = A(\pi, c) \mod q, (\pi \in G^0, c \in \{a, b\})
\]

The same notation \( g \) is used for a map \( g : X \rightarrow G \), where \( X = \Delta_{m-1} \times R(\pi), R(\pi) \) the Rauzy class of \( \pi \). Since they are naturally associated, it is well understood. That is \( g(x) = g(\pi(x), c(x)), c(x) = c(\lambda(x), \pi(x)). \)

The normalized Rauzy map is \( P(\lambda, \pi) = (A^{-1}-1, \lambda^{-1} \circ (\pi)), P_\pi \) is the induced map of \( P \) on \( \Delta_{m-1} \times \{\pi\}. \)

Suppose for \( n \in \mathbb{N}, P^\mu(\pi)(x) = P^\mu(x), \) where \( r = r_1 + r_2 + \cdots + r_n \), and if \( r_i = r_1 + r_2 + \cdots + r_i \), then \( P^\mu(\pi)(x) = P^\mu(\pi), g^\nu(\pi)(x) = g^{P^{-1}(\pi)}g^{P^{r-2}(\pi)} \cdots g^{P^{r-2}(\pi)}g(x) \) is associated with \( x \). One can understand \( g^{\nu}(\pi) \) as a closed path from \( \pi \) to \( \pi \). Let \( G'(\pi) = \{g^{\nu}(\pi)|x \in X(\pi), P^\nu x \in X(\pi)\} \). Next, it is shown that \( G'(\pi) \) is a semigroup of \( G \) based on the fact that \( P_\pi \) is an expansion.

**Theorem 2.7.** \( G'(\pi) \) is closed under multiplication.
Proof. Suppose \(g^{(r)}(x), g^{(s)}(y) \in G'(\pi)\), \(x, y \in X(\pi)\). Suppose \(P^s(y)\) is associated with the visitation matrix \(A\) then \(y \in \Delta^A \times \{\pi\}\). We know that \(P^s(\Delta^A \times \{\pi\}) = \Delta^A_{m-1} \times \{\pi\}\). Therefore there exists \(y_0 \in X(\pi)\) such that \(g^{(s)}(y) = g^{(s)}(y_0)\) and \(P^s(y_0) = x\). Then

\[
g^{(r)}(x)g^{(s)}(y) = g^{(r)}(P^s(y_0))g^{(s)}(y_0) = g^{(r+s-1)}(y_0) \cdots g^{(s)}(y_0)g^{(s-1)}(y_0) \cdots g(y_0)
\]

Thus

\[
g^{(r)}(x)g^{(s)}(y) \in G'(\pi)
\]

\[\square\]

\(G'(\pi)\) is thus a sub-semigroup of \(G\). Let \(G(\pi) = G'(\pi)\), then \(G(\pi)\) is a subsemigroup of the finite group \(G\). Therefore \(G(\pi)\) is a subgroup of \(G\).

If we have two permutations \(\pi_1, \pi_2\) in the same irreducible Rauzy class \(R(\pi)\), by Theorem 1.1 (Veech’s ergodic theorem), there exists \(x \in X(\pi_1)\) and \(n \geq 0\) such that \(P^n(x) \in X(\pi_2)\). Assign \(g^{(n)}(x)\) to \(g(\pi_1, \pi_2)\), of course there may be other choice, but once a fixed one is assigned, it is well defined. \(g(\pi_1, \pi_2)\) understood as a path from \(\pi_1\) to \(\pi_2\), two 'paths' may be connected in the following sense: if \(\pi_1, \pi_2, \pi_3 \in R(\pi)\), \(y \in X(\pi_2)\), \(P^s(y) \in X(\pi_3)\) and \(g^s(y) = g(\pi_2, \pi_3)\), then there exists \(x \in X(\pi_1)\) such that \(P^r(x) = y\) and \(g^r(x) = g(\pi_1, \pi_2)\), then \(g^{r+s}(x) = g^r(y)g^r(x)g(\pi_2, \pi_3)g(\pi_1, \pi_2)\). It is obvious that \(g(\pi_2, \pi_1)g(\pi_1, \pi_2) \in G'(\pi_1)\).

Now we are ready to define a skewing map \(W_0 : X \times G_0 \to X \times G_0\) where \(G_0 = G(\pi_0)\) for some fixed \(\pi_0 \in R\). That is:

\[
W_0(x, \gamma) = (P_x, h(x)\gamma)
\]

where

\[
h(x) = g^{-1}(\pi_0, c(x)\pi(x))g(x)g(\pi_0, \pi(x))
\]

To understand \(W_0\), we do some computation

\[
W_0^2 = (P^2x, h(Px)h(x)\gamma)
\]

\[\ldots\]

\[
W_0^r = (P^r_x, h(P^{r-1}x)h(P^{r-2}x) \cdots h(Px)h(x))
\]

Let

\[
h^{(r)}(x) = h(P^{r-1}x)h(P^{r-2}x) \cdots h(Px)h(x)
\]

Next define a space \(X^*\) and a transformation \(W\) on \(X^*:\)

\[
X^* = \bigcup_{\pi \in R(\pi_0)} X(\pi) \times g(\pi_0, \pi)G_0
\]

\[
W(x, g(\pi_0, \pi(x))\gamma) = (P_x, g(x)g(\pi_0, \pi(x))\gamma)
\]

The relation between \(W\) and \(W_0\) is given by \(\varphi : X^* \to X \times G_0\) defined as \(\varphi(x, g(\pi_0, \pi(x))\gamma) = (x, \gamma)\), it is easy to see that \(\varphi\) is bijective. Since it is also true that \(\varphi W = W_0\varphi\), we have \(W, W_0\) are conjugate.

Lemma 2.8. If \(\pi \in R, c \in \{a, b\}\), then \(G(c\pi) = g(\pi, c)G(\pi)g(\pi, c)^{-1}\).
Proof. Since
\[ g(\pi, c)G(\pi)g(\pi, c)^{-1} \subseteq G(c\pi) \]
\[ g(\pi, c)^{-1}G(c\pi)g(\pi, c) \subseteq G(\pi) \]
We have
\[ g(\pi, c)g(\pi, c)^{-1}G(c\pi)g(\pi, c)g(\pi, c)^{-1} \subseteq g(\pi, c)G(\pi)g(\pi, c)^{-1} \subseteq G(c\pi) \]
Thus
\[ G(c\pi) = g(\pi, c)G(\pi)g(\pi, c)^{-1} \]
\[ \square \]

Lemma 2.9. Let \( H(\pi) = \{ h^r(x) | x \in X(\pi), P^r(x) \in X(\pi) \} \), Then \( H(\pi) = G_0 \).

Proof. It is obvious that \( g^r(x) \) generates \( G(\pi) \), since \( G(\pi) \) is finite, \( \{ g^r(x) | x \in X(\pi), P^r(x) \in X(\pi) \} = G'(\pi) \).

By Lemma 2.8 we have \( G(\pi)g(\pi_0, \pi)\gamma = g(\pi_0, \pi)G_0 \). Since \( G(c\pi) \) and \( G(\pi) \) are conjugate, \( G(\pi_1) \) and \( G(\pi_2) \) are conjugate for any \( \pi_1, \pi_2 \in R(\pi_0) \).\( \square \)

Thus \( G(\pi) \) and \( G_0 \) have the same number of elements. Since \( G(\pi)g(\pi_0, \pi)\gamma \subseteq g(\pi_0, \pi)G_0, G(\pi)g(\pi_0, \pi)\gamma = g(\pi_0, \pi)G_0 \).

Because \( \varphi \) is a conjugation, \( \varphi(W^r(x, g(\pi_0, \pi)\gamma)) = W^r_0\varphi(x, g(\pi_0, \pi)\gamma) \), it is obvious that \( \#H(\pi) = \#(G(\pi)) = \#(G_0) \). Since \( H(\pi) \subseteq G_0, H(\pi) = G_0 \).

Next we show the ergodic property of the two skewing transformations.

Lemma 2.10. All notations as above, and suppose \( \omega \otimes \delta_\pi \) is the invariant measure of \( P_\pi \), \( \sigma \) is the normalized Haar measure on \( G_0 \). Then \( W_0 \) is ergodic and conservative relative to the measure \( \nu_0 = \sum_{\pi \in R} \omega \otimes \delta_\pi \otimes \sigma \).

Proof. First the parallel properties of the induced map on \( X(\pi) \times G_0 \) is verified, then those of \( W_0 \) itself will be gained.

Denote the induced map by \( W_\pi \), that is \( W_\pi(x, \gamma) = (P_\pi x, h_\pi(x)\gamma) \), where \( P_\pi(x) = P^\pi(x) \) and \( h_\pi(x) = h^r(x) \).

Suppose \( F \in L^2(\omega \otimes \delta_\pi \otimes \sigma) \) is essentially invariant under \( W_\pi \), that is \( F \circ W_\pi(z) = F(z) \) a.e. \( z \). Then define a function \( f : X(\pi) \to \mathcal{H} = L^2(G_0) \) by \( f(x)(\gamma) = F(x, \gamma) \).

Recall the partition \( F_1 \) from Section 0. Suppose \( A = A^{(n)}(\lambda, \pi), (\lambda \in \Delta) \), let \( \Delta = \Delta^A \in F_1 \). It is also true that \( \Delta \) is associated with common \( c_1, c_2, \cdots, c_n \) and \( \pi_1, \pi_2, \cdots, \pi_n \), where \( c_i = c(P^{i-1}x) \), \( \pi_i = \pi(P^{i-1}x) \).

Since
\[ h^{(n)}(x) = h(P^{n-1}x) \cdots h(x) \]
\[ = g^{-1}(\pi_0, c(P^{n-1}x)\pi(P^{n-1}x))g(P^{n-1}x)g(\pi_0, \pi(P^{n-1}x)) \]
\[ \cdots g^{-1}(\pi_0, c(x)\pi(x))g(x)g(\pi_0, \pi(x)) \]
\[ = g^{-1}(\pi_0, c_n(x)\pi_n(x))g(x)g(\pi_0, \pi_nx) \]
\[ \cdots g^{-1}(\pi_0, c(\pi_1))g(x)g(\pi_0, \pi_1) \]
$h^n(x)$ is constant on $\Delta \times \{\pi\}$. Thus a unitary operator $U_\Delta : \mathcal{H} \rightarrow \mathcal{H}$ may be defined as $U_\Delta(\psi)(x) = \psi((h^n(x))^{-1})$, $\psi \in \mathcal{H}, x \in \Delta \times \{\pi\}$. Since $\sigma$ is the Haar measure (counting measure here) on $G_0$, $U_\Delta$ is an isometry, thus a unitary operator.

Since $F \circ \mathcal{W}_\pi(z) = F(z)$, $z \in X_\pi \times G_0$, we have:

$$f(P_\pi x)(\gamma) = F(P_\pi x, \gamma) = F(W_{\pi}^{-1}(P_\pi x, \gamma))$$

$$= F(x, (h_\pi)^{-1}\gamma) = U_\Delta(f(x))$$

(2.3)

By Theorem 1.3 we know that $f$ is essentially constant. This tells us:

i) Let $F(\gamma) = F(x_1, \gamma) = F(x_1, \gamma)$ for such an $x \in X(\pi)$, it is well understood that $F(\gamma)$ is a fixed function in $\mathcal{H} = L^2(\sigma)$.

ii) From 2.3 we know that $F(h^n(x)\gamma) = F(\gamma)$

Now we apply Lemma 2.9, $H(\pi)\gamma = G_0$, thus $F(\gamma)$ is constant on $G_0$. That is to say if $F_1$ is any $\mathcal{W}_0$ invariant function in $L^2(X \times G_0)$, the restricted map of $F_1$ on $X(\pi) \times G_0$ is constant. Since $\mathcal{W}_0$ send an element in $X(\pi) \times G_0$ to an element in $X(c(\pi)) \times G_0$ and $\mathcal{R}(\pi)$ is the set $\{c_1c_2 \cdots c_k(\pi), \pi \geq 0, c_i \in \{a, b\}, 1 \leq i \leq k\}$, it is not hard to see that $F_1$ is constant on $X \times G_0$. Therefore $\mathcal{W}_0$ is ergodic. $\mathcal{W}_0$ is also conservative, since $\mathcal{P}_\pi$ is conservative for all $\pi \in \mathcal{R}(\pi), \mathcal{R}(\pi)$ and $G_0$ are both finite. 

**Theorem 2.11.** $W$ is ergodic and conservative relative to the measure $\nu = \sum_{\pi \in \mathcal{R}} \omega \otimes \delta_\pi \otimes g(\pi_0, \pi)\sigma$.

**Proof.** Since $\mathcal{W}$ and $\mathcal{W}_0$ are conjugate, Lemma 2.10 implies this result. 

§3 Totally Rank One Property for Interval Exchange Transformations

In this section, $\pi \in G_m^*$ is fixed. That is to say an interval exchange is associated with $(\lambda, \pi)$ ($\lambda \in \Delta_{m-1}$). Based on theorem 2.11, we shall show the measure theoretic generic rank one property of the powers of the $\pi$-interval exchange transformations(Theorem 1.3).

A positive matrix can be associated with the iteration of the Rauzy transformation on

$$X_\pi = \{(\lambda, \pi)|\lambda \in \Delta_{m-1}, \pi \text{ a given } m\text{-permutation}\}$$

One way to see this is that once $(\lambda, \pi)$ satisfies i.d.o.c, it is minimal, the iterations are corresponding to the induced maps of $(\lambda, \pi)$ on smaller and smaller subintervals. What’s more, Theorem 2.11 implies that, for a.e. $\lambda$, the orbit of $(\lambda, \pi)$ will visit $U \times \pi \times \{e\}$ infinitely many times, for any open subset $U \subset \Delta_{m-1}$. If $M$ is any $m \times m$ positive matrix, then $M \cdot A(\pi^*, e)$ $(\pi^* \in \mathcal{R}(\pi))$ is also a positive matrix. Therefore, there exists $c_1, c_2, \cdots, c_n$ with each $c_i \in \{a, b\}, 1 \leq i \leq n$ such that $c_n c_{n-1} \cdots c_1 = \pi, B = A^{(n)}$ is positive and $B \equiv e \mod q$. $A^{(n)}$ may be expressed by induction:

I

$$A^{(1)} = A(\pi, c_1)$$

II

$$A^{(i+1)} = A(i) A(c_i \cdots c_2 c_1(\pi), c_{i+1})$$

(2.4)

It is nice to see that for any $\eta \in BA$, $Z^n(\eta, \pi) = (B^{-1}\eta, \pi)$ passing the same sequence of permutations as $c_1\pi, c_2c_1\pi, \cdots, c_n c_{n-1} \cdots c_1 \pi$. 

9
Given $\tau > 0$, define an open set in $\Delta_{m-1} \times \{\pi\} \times \{e\}$ ($e$ be the identity in $G_0$), that is $\mathcal{J} = \{(\alpha, \pi, e)\} | \alpha \in \Delta_{m-1}, \alpha_1 > 1 - \tau$. Let $\mathcal{J}^* = \{(\frac{B_1 \alpha}{B_0}, \pi, e)|(\alpha, \pi, e) \in \mathcal{J}\}$. Then $\omega^*(\mathcal{J}^*) > 0$.

The above process will be continued in the proof of the following lemma:

**Lemma 2.12.** All notations as above, for a.e. $\lambda \in \Delta_{m-1}$ and $\epsilon > 0$ there is an positive integer $n_0$ and an interval $J \in [0, |\lambda|]$ such that:

(a) $T^{i_0}J$ are pairwise disjoint, $0 \leq i < n_0$. 
(b) $T^n$ is linear on $T^{i_0}J$, $0 \leq i < n_0$.
(c) $\sum_{i=0}^{n_0-1} |T^{i_0}J| > (1 - \epsilon) |\lambda|$
(d) $|J \cap T^{i_0}J| > (1 - \epsilon) |\lambda|$

**Proof.** We prove the corresponding results a.e. $\lambda \in \Delta_{m-1}$, equivalently.

Assume $\lambda \in \Delta_{m-1}$ satisfies: there exists $k \in \mathbb{N}$ such that $W_\lambda(\pi, \epsilon) \in \mathcal{J}^*$, based on Theorem [2.11]

In other words such $\lambda$ form a set full measure.

Since $W_\lambda(\pi, \epsilon)$ is an element in $\mathcal{J}^*$, $W_{\lambda+k}(\pi, \epsilon)$ is an element in $\mathcal{J}$. Suppose $W_\lambda(\pi, \epsilon)$ is associated with the visitation matrix $A$. Then $A \equiv \epsilon \mod q$. Suppose $W_{\lambda+k}(\pi, \epsilon) = (\tilde{\alpha}, \pi, \epsilon)$ then the associated matrix is $A^{k+1}(\lambda, \pi) = AB$, $AB$ is positive and $AB \equiv \epsilon \mod q$. Also we know that $\tilde{\alpha} = \frac{(AB)^{-1} \lambda}{(AB)^{-1} \lambda}$. Let $\alpha = (AB)^{-1} \lambda$, $J' = [0, |\lambda|]$, then since $\tilde{\alpha}_1 > (1 - \tau)$, it is true that $\alpha_1 > (1 - \tau) |\lambda|$. Suppose $(T|_{\eta})^q = T_{(\eta, \epsilon)}$ with $\eta \in \Lambda_{q(m-1)+1}$, $|\eta| = |\alpha|$, $\epsilon \in \Gamma_{q(m-1)+1}$.

By induction, it is easy to see that there exists $1 \leq k \leq q(m-1)+1$ such that $\eta_k > (1 - 2^{q-1} \tau) |\lambda|$. Let $J$ be the $k$-th interval of $\eta$. Since $AB \equiv \epsilon \mod q$, we have $(T|_{\eta})^q = (T^q|_{\eta})$ by Theorem 3.1.6. Now visiting each $I_j(\lambda)$ the same number of times, points in each suninterval of $\eta$ come back to $J'$ coincidently for the first $q$ times under $T$. Therefore, if $I^q$ is the $i$-th interval of $\eta$, we can let $a_i^{(q)}$ be the first return time to $J'$ of $(T|_{\eta})^{i-1}(I^q_i)$ under $T$. Let $\tilde{\alpha}_i = \sum_{i=1}^{q} a_i^{(q)} = a_i^{q} |\lambda|$. Thus

$$T^q|_{J'}(I_{\eta}) = (T^q)^{\tilde{\alpha}_i}(I_{\eta})$$

Let $n_0 = a_i^{q}$, then

$$A) \quad \cdots \quad (T^q)^l(J), (0 \leq l < n_0) \text{ are } \text{pairwise disjoint and}$$

$$B) \quad \cdots \quad |(T^q)^{n_0}(J) \cap J| > (1 - 2^{q-1} \tau) |\lambda|.$$
Therefore,
\[ v(AB) \leq v(B) \]

Recall that each \( a_i^{(t)} \) is one of the column summations of \( AB \). Thus
\[ a_i^{(t)} \leq v(B)a_j^{(s)}, \quad 1 \leq i, j \leq q(m - 1) + 1, \quad 1 \leq t, s \leq q \]
This implies \( a_i^{*} \leq v(B)a_j^{*}, \quad 1 \leq i, j \leq q(m - 1) + 1. \)

So the total length(measure) of the remain of the major stack \( (\bigcup_{i=0}^{n-1}(T^q)^i(J)) \) is:
\[
|\lambda| - n_0 |J| = \sum_{i=1}^{m(q-1)+1} a_i^{*}\eta_i - a_k^{*}\eta_k
\]
\[
= \sum_{i=1, i \neq k}^{m(q-1)+1} a_i^{*}\eta_i \leq v(B)a_k^{*}\frac{2q-1}{1-2q^2} |\eta|
\]
\[
\leq v(B)a_k^{*}\frac{2q-1}{1-2q^2}\eta_k \leq v(B)\frac{2q-1}{1-2q^2}
\]
That is
\[ C) \quad \bigcup_{i=0}^{n-1}(T^q)^i(J) \supset (1 - v(B)\frac{2q-1}{1-2q^2}), (|\lambda| = 1). \]
In order to make the approximation coincide to \( \varepsilon > 0 \) choose \( \tau \) small enough such that \( 2^q \tau < \varepsilon \) and \( v(B)\frac{2q-1}{1-2q^2} \tau < \varepsilon \). Combining \( A) \quad B) \quad C) \) and Theorem \[2.11\] for a fixed \( \pi \in \mathcal{G}_m^0 \) almost all \( \lambda, (\lambda, \pi) \) will visit \( J^\pi \) infinitely often under \( W \), we attain (a)(b)(c)(d) in the Lemma. \( \square \)

It is easy to see that Theorem \[1.3\] is a corollary of Lemma \[2.12\]

**Proposition 2.13.** Let \((\mathbb{X}, \mathcal{B}, \mu, T)\) be an automorphism system on a standard measure space. Suppose all powers of \( T \), such that this subset is a dense \( G_\delta \) set in \( Wcl(T) \), and all transformations in it are rank one.

**Proof.** We say two partitions \( P_1 \) and \( P_2 \) satisfying \( P_1 >_\varepsilon P_2 \) if for any atom \( p \in P_2 \), there exists \( p_1, p_2, \ldots, p_m \in P_1 \) such that \( m(p\Delta(\bigcup_{i=1}^m p_i)) < \varepsilon \).

Next we use \( P^{(n)} \) to denote the partition by dyadic sets of rank \( n \).

Let \( R(n, q, \varepsilon) = \{ S \mid \text{there exists } B \subset [0, 1), \text{ such that } P_{B,q} \geq_{\varepsilon} P^n, \text{ where } P_{B,q} = \{ B, SB, \ldots, S^{q-1}B, X- \bigcup_{i=0}^{q-1}S^iB \} \}

Then \( R(n, q, \varepsilon) \) is an open set in \( G = Aut(\mathbb{X}, \mathcal{B}, \mu) \).

Suppose \( T \) has all powers \( (T^n, n \in \mathbb{N}) \) rank one and rigid. Then \( Wcl(T) \) has uncountable many elements. \( W(n, q, \varepsilon) = Wcl(T) \cap R(n, q, \varepsilon) \) is an open set in \( Wcl(T) \).

Since all powers of \( T \) are rank one, we have
\[
T^n \in \bigcup_{q} W(n, q, \varepsilon), n \in \mathbb{N}
\]
and \( \{ T^n, n \in \mathbb{N} \} \) are dense in \( Wcl(T) \). Thus \( \bigcup_{q} W(n, q, \varepsilon) \) is a dense open set in \( Wcl(T) \). Therefore \( \cap_n \cup_q W(n, q, \varepsilon) \) is a dense \( G_\delta \) set in \( Wcl(T) \), with all elements rank one. \( \square \)
Corollary 2.14. All the notations as above, suppose $m > 1$, $m \in \mathbb{N}$, $\pi \in G^*_m$, $q \in \mathbb{N}$. For Lebesgue almost all $\lambda \in \Lambda$, it is true that there is a dense $G_\delta$ subset of rank one transformations in the weak closure of $T^{q\lambda}_q(\lambda,\pi)$.

Proof. By theorem 1.3 and proposition 2.13.

References

[1] R. V. Chacon, Weakly mixing transformations which are not strongly mixing, Proc. Amer. Math. Soc. 22 1969 559–562.

[2] S. Ferenczi, Systems of finite rank, Collaq. Math. 73 (1997), P. 35-65.

[3] E. Glasner, Ergodic theory via joinings, Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[4] M. Keane, Interval Exchange transformations, Math Z. 141(1973), 25-31.

[5] J.L. King, The commutant is the weak closure of the powers, for rank-1 transformations, Ergodic Theory Dynam. Systems 6 (1986) no. 3, 363–384.

[6] H. Masur, Interval Exchange Transformation and measured foliation, Ann. of Math., 115 169-200.

[7] G. Rauzy, Echanges d’intervalles et transformations induites, Acta Arith 34(1979) 315-328.

[8] W.A. Veech, Gauss Measures for Transformations on the Space of Interval Exchange Map, Ann of Math, Vol 115(1982), 201-242.

[9] W.A. Veech, The Metric Theory of Interval Exchange Transformation I : Generic Spectral Properties, American Journal of Mathematics, 107(6):1331-1359,1984.

[10] W.A. Veech, Interval exchange transformations, J.D. Analyse Math. 33(1978) 222-278.

[11] M. Viana, Ergodic Theory of Interval Exchange maps, Rev. Mat. Complut., 19(2006), no. 1, 7-100.

[12] Y. Wu, Applications of Rauzy Induction on the generic ergodic theory of interval exchange transformations, Doctor of Philosophy Thesis, Rice University Electronic Theses and Dissertations(2006).

[13] Y. Wu Whirly 3-interval exchange transformations, Ergodic Theory and Dynamical Systems, available on CJO2015. doi:10.1017/etds.2015.63.

[14] Y. Wu, D. Li, Y. Wang, D. Li, A family of totally rank one two-sided shift maps, Arxiv, Cornell University, [arXiv:1604.02597](http://arxiv.org/abs/1604.02597) (2016).