The rational cohomology Hopf algebra of a generic Kac-Moody group

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Abstract

In this paper we determine the rational homotopy type of the classifying space of a generic Kac-Moody group by computing its rational cohomology ring. As an application we determine the rational homology Hopf algebra of the generic Kac-Moody group.

Key words: Kac-Moody group, Classifying space, Hopf algebra.
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1 Introduction

Let $A = (a_{ij})$ be an $n \times n$ Cartan matrix. By Kac\cite{Kac2} and Moody\cite{Moody1}, it is well known that there is a Kac-Moody Lie algebra $g(A)$ associated to $A$. In \cite{Kac3, Kac4, Kac5} Kac and Peterson constructed the corresponding Kac-Moody group $G(A)$. In this paper for convenience we consider the derived Lie algebra $g'(A)$ and the associated simply connected group $G'(A)$. But we still use the symbols $g(A)$ and $G(A)$.

Cartan matrices are divided into three types, i.e. finite type, affine type and indefinite type. A Cartan matrix $A$ is indecomposable if $A$ can’t be written as a direct sum of two Cartan matrices $A_1$ and $A_2$. $A$ is symmetrizable if there exists an invertible diagonal matrix $D$ and a symmetric matrix $B$ such that $A = DB$, see Kac\cite{Kac3} for details. A Cartan matrix $A$ is generic if $a_{ij}a_{ji} \geq 4$ for all $i, j$. $A$ is generic if and only if all its principal sub-matrices of rank 2 are not of finite type. All these properties for Cartan matrices can be used for the associated Kac-Moody Lie algebras and Kac-Moody groups. For example a generic Kac-Moody group is indecomposable.

In \cite{Kac6}, the authors determined the rational homotopy type of the indefinite Kac-Moody group $G(A)$. Since $G(A)$ has a multiplication, it is a Hopf space. It is important to determine the rational Hopf homotopy type of $G(A)$. This is equivalent to determine the rational cohomology Hopf algebra $H^*(G(A))$ or the dual rational homology Hopf algebra $H_*(G(A))$. It is further equivalent to determine the rational cohomology algebra $H^*(BG(A))$ of the classifying space $BG(A)$.

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On the rational homotopy group $\pi_*(G(A))$, the Samelson product $[\ , ] : \pi_p(G(A)) \times \pi_q(G(A)) \to \pi_{p+q}(G(A))$ is defined as

$$[\alpha, \beta](s \wedge t) = \alpha(s)\beta(t)\alpha(s)^{-1}\beta(t)^{-1}, s \in S^p, t \in S^q.$$ 

$(\pi_*(G(A)), [\ , ])$ is a rational graded Lie algebra.

Let $\chi : \pi_*(G(A)) \to H_*(G(A))$ be the Hurewicz morphism of graded Lie algebras. By Milnor and Moore\cite{10}, the induced morphism $\tilde{\chi} : U(\pi_*(G(A))) \to H_*(G(A))$ is an isomorphism of Hopf algebras, where $U(\pi_*(G(A)))$ is the universal enveloping algebra of $\pi_*(G(A))$. And $H_*(G(A))$ is primitively generated by $\pi_*(G(A))$. So to determine the Hopf algebra structure on $H_*(G(A))$, it is enough to compute the graded Lie algebra $\pi_*(G(A))$. By combining rational homotopy theory(see \cite{12}) with the cohomology ring $H^*(G(A))(see \cite{15})$, one knows that for a generic Cartan matrix $A$, $\pi_{odd}(G(A)) \cong \mathbb{Q}$ or $\{0\}$ depending on whether $A$ is symmetrizable or not. And the decomposition $\pi_*(G(A)) = \pi_{even}(G(A)) \oplus \pi_{odd}(G(A))$ is a decomposition of Lie algebras. $H_{even}(G(A))(i.e. the rational Chow ring of G(A))$ is isomorphic to the universal enveloping algebra $U(\pi_{even}(G(A)))$. By \cite{6} the Poincaré series of $H_{even}(G(A))$ is

$$C_A(q) = P_{F(A)}(q)(1 - q^2)^n(1 - q^4)^{-\epsilon(A)}.$$ 

Here $P_{F(A)}(q)$ is the Poincaré series of the flag manifolds $F(A)$ associate to Kac-Moody group $G(A)$ and $\epsilon(A) = \dim \pi_{odd}(G(A))$. By \cite{15}, $P_{F(A)}(q) = \frac{1 + q^2}{1 - (n-1)q^2}$. Hence we can compute the Poincaré series of $H_{even}(G(A))$.

For a non-symmetrizable Cartan matrix $A$, we write the Poincaré series $C_A(q) = \frac{1}{1 - (n-1)q^2}(1 - q^4)^{-\epsilon(A)}$ as

$$\frac{1}{1 - a_4q^4 - a_4q^6 - \cdots - a_{2i}q^{2i} - \cdots},$$

and for a symmetrizable Cartan matrix $A$, we write $C_A(q) = \frac{1}{1 - (n-1)q^2}(1 - q^4)^{-\epsilon(A)}$ as

$$\frac{1}{1 - b_4q^4 - b_6q^6 - \cdots - b_{2i}q^{2i} - \cdots},$$

where for $i \geq 2$, $a_{2i} = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (i - 1 - 2k)(n+i-2k-3)$, $b_{2i} = (i-1)(n+i-3)$ are natural numbers depending on $n$.

These two Poincaré series are the same as the Poincaré series of the tensor Hopf algebra with $a_{2i}$ and $b_{2i}$ generators of degree $2i$ for each $i \geq 2$. In \cite{16} we gave the following conjecture.

**Conjecture:** For a non-symmetrizable(or symmetrizable) generic Kac-Moody group $G(A)$, the graded Lie algebra $\pi_{even}(G(A))$ is a free Lie algebra with $a_{2i}$ (or $b_{2i}$) generators of degree $2i$ for each $i \geq 2$.

Since the universal enveloping algebra of an even graded free Lie algebra is a tensor algebra, if the conjecture is true, then $H_{even}(G(A))$ is a tensor Hopf algebra with $a_{2i}$ (or $b_{2i}$) generators of degree $2i$ for each $i \geq 2$.

In this paper we compute the rational cohomology ring $H^*(BG(A))$ at first. This determines the rational homotopy type of $BG(A)$. Then we compute the graded Lie algebra $\pi_*(BG(A))$ with
Whitehead product. Since the graded Lie algebra $\pi_*(G(A))$ with Samelson product is determined by $\pi_*(BG(A))$ with Whitehead product, we determine the graded Lie algebra $\pi_*(G(A))$ and prove the conjecture at last.

The main results in this paper are the following two theorems.

**Theorem 1:** If $A = (a_{ij})_{n \times n}$ is a non-symmetrizable generic Cartan matrix, $n \geq 3$, then the Poincaré series of $BG(A)$ is $P_n(q) = q \left[ \frac{(n-1)q^2 - 1}{(1-q^2)^{n-1}(1-q^4)} + 1 \right] + 1$.

**Theorem 2:** If $A = (a_{ij})_{n \times n}$ is a symmetrizable generic Cartan matrix, $n \geq 2$, then the Poincaré series of $BG(A)$ is $Q_n(q) = \frac{1}{1-q^4} \left[ \frac{(n-1)q^2 - 1}{(1-q^2)^{n-1}} + 1 \right]$.

The contents of this paper are as follows: in section 2 we give some preparing lemmas, in section 3 we prove Theorem 1 and 2, in section 4 we give some results derived from these theorems, including the above conjecture.

## 2 Some preparing lemmas

In the following all the Cartan matrices are assumed to be generic. All the homology and cohomology are of rational coefficients.

Let $S$ be the set of integers $1, 2, \cdots, n$, and $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the simple root system of $G(A)$. For each $I \subset S$, the matrix $A_I = (a_{ij})_{i,j \in I}$ is also a Cartan matrix. Corresponding to $I \subset S$, there is a parabolic subgroup $G_I(A)$ of $G(A)$ whose simple root system is $\Pi_I = \{\alpha_i | i \in I\}$. All the proper subsets of $S$ form a category $C$ with object $I \subset S$ and morphism $I \subset J$. By constructing classifying spaces we have a functor $F : C \rightarrow Top$ which sends $I$ to $BG_I(A)$ and $I \subset J$ to map $BG_I(A) \rightarrow BG_J(A)$.

Since we only consider the homotopy type of the Kac-Moody group we replace the group $G(A)(or G_I(A))$ by its unitary form and use the same symbol.

We need the following lemmas to prove the main theorems.

**Lemma 2.1.** For a Kac-Moody group $G(A)$ and $I \subset S$, the subgroup $G_I(A)$ is isomorphic to $G(A_I) \times T^{n-|I|}$, the semi-direct product of $G(A_I)$ and $T^{n-|I|}$. As a result there is an isomorphism $H^*(BG_I(A)) \cong H^*(BG(A_I)) \times H^*(BT^{n-|I|})$.

By this lemma, the Poincaré series of $BG_I(A)$ is obtained from the Poincaré series of $BG(A_I)$ by multiplying a factor $\frac{1}{(1-q^2)^{n-|I|} \cdot \cdots \cdot (1-q^4)}$.

By Kichiloo[7],[8], for a Cartan matrix of infinite type, the homotopy colimit of the functor $F$ gives the homotopy type of $BG(A)$. For any $I \subset C$, let $C_I$ be the full subcategory of $C$ whose objects are proper subsets of $I$. If $|I| \geq 2$, then $G_I(A)$ is of infinite type. By using the result of Kichiloo to $BG_I(A)$, we get $H^*(BG_I(A)) \simeq \text{colimit } F|_{C_I}$. As a consequence we have

**Lemma 2.2.** Let $C'$ be the full subcategory of $C$ which contains only objects $\emptyset, \{1\}, \{2\}, \cdots, \{n\}$, then for a generic Kac-Moody group $G(A)$,

$$BG(A) \simeq \text{colimit } F|_{C'}$$

$$\simeq BG_1(A) \cup_{BT} BG_2(A) \cup_{BT} \cdots \cup_{BT} BG_n(A)$$

$$\simeq BG_{\{1, 2, \cdots, n-1\}}(A) \cup_{BT} BG_n(A).$$
The action of Weyl group \( W(A) \) (or \( W_I(A) \)) of \( G(A) \) (or \( G_I(A) \)) on the maximal torus \( T \) induces the action of \( W(A) \) (or \( W_I(A) \)) on \( H^*(BT) \).

**Lemma 2.3.** For a generic Kac-Moody group \( G(A) \), the image of the homomorphism \( Bi_i^* : H^*(BG_I(A)) \to H^*(BT) \) induced by the inclusion \( i_I : T \subset G(A) \) is \( H^*(BT)^W(A) \), i.e. the \( W_I(A) \) invariants. In particularly the image of the homomorphism \( H^*(BG(A)) \to H^*(BT) \) is \( H^*(BT)^W(A) \).

This lemma is the generalization of a result of Borel[1] for compact Lie groups. It can be proved in the inductive procedure of the proofs for the main theorems. see [17] for details.

**Lemma 2.4.** If \( A \) is a non-symmetrizable generic Cartan matrix, then there exist \( i, j, k \in S, i < j < k \) such that \( A_{\{i,j,k\}} \) is non-symmetrizable.

**Proof:** Suppose this lemma is not true for Cartan matrix \( A \). Then for any \( i < j < k \), \( A_{\{i,j,k\}} \) is symmetrizable. Hence \( a_{ij}a_{jk}a_{ki} = a_{ik}a_{kj}a_{ji} \). We set \( d_i = \frac{a_{ii}}{a_{11}} \). For \( 1 < i < j \), we have \( a_{1i}a_{ij}a_{1j} = a_{1j}a_{ji}a_{i1} \), hence \( \frac{a_{ij}}{a_{ji}} = \frac{d_j}{d_i} \). But this means that \( A \) is symmetrizable, a contradiction.

**Lemma 2.5.** Let \( A \) be a generic Cartan matrix. If \( A \) is symmetrizable, then \( H^*(BT)^W(A) \cong \mathbb{Q}[\psi] \), where \( \psi \) is the Killing form; If \( A \) is non-symmetrizable, then \( H^*(BT)^W(A) \cong \mathbb{Q} \).

This result was proved in Zhao-Jin[14]. In fact it is valid for an arbitrary indefinite and indecomposable Cartan matrix.

**Lemma 2.6.** Let \( X = X_1 \cup X_0 X_2 \) be the push-out of the diagram \( X_1 \xleftarrow{j_1} X_0 \xrightarrow{j_2} X_2 \). The homomorphism \( j : H^*(X_1) \oplus H^*(X_2) \to H^*(X_0) \) is given by \( j(u,v) = j_1^*(u) - j_2^*(v) \). If \( X_1, X_2 \) are deformation retracts of some open subspaces of \( X \), then there exists a short exact sequence

\[
0 \to \Sigma \text{coker} j \to H^*(X) \to \ker j \to 0.
\]

**Proof:** We have the Mayer-Vietoris exact sequence

\[
\cdots \to H^{*,-1}(X_1) \oplus H^{*,-1}(X_2) \xrightarrow{j} H^{*-1}(X_0) \xrightarrow{\delta} H^*(X) \xrightarrow{j} H^*(X_1) \oplus H^*(X_2) \xrightarrow{j} H^*(X_0) \to \cdots
\]

From this sequence we get the short exact sequence \( 0 \to \text{im} \delta \to H^*(X) \to \text{im} j \to 0 \). By the exactness of this sequence, we have \( \text{im} \delta \cong \ker j \) and \( \text{im} \delta \cong H^{*,-1}(X_0) / \ker \delta \cong H^{*,-1}(X_0) / \text{im} j \cong \text{coker} j \). This proves the lemma.

**Lemma 2.7.** Let \( A \) be a generic \( 2 \times 2 \) Cartan matrix, then \( H^*(BG(A)) \cong \mathbb{Q}[\psi] \), where \( \psi \) corresponds to Killing form which has degree 4. The Poincaré series of \( BG(A) \) is \( \frac{1}{1 - q^4} \).

**Proof:** By Lemma 2.6, we have the short exact sequence \( 0 \to \Sigma \text{coker} j \to H^*(BG(A)) \to \ker j \to 0 \) with \( j : H^*(BG_{\{1\}}(A)) \oplus H^*(BG_{\{2\}}(A)) \to H^*(BT) \).

The Poincaré series of \( \text{coker} j \) is

\[
\frac{1}{(1 - q^2)^2} - \frac{2}{(1 - q^2)(1 - q^4)} + \frac{1}{1 - q^4} = 0.
\]

Since \( \text{ker} j \) is isomorphic to \( H^*(BT)^W(A) = \mathbb{Q}[\psi] \), its Poincaré series is \( \frac{1}{1 - q^4} \). Hence \( H^*(BG(A)) \cong \ker j \cong \mathbb{Q}[\psi] \).

This lemma is the special case of \( n = 2 \) for Theorem 2.
3 The proofs of the main theorems

In this section we denote $BG_I(A)$ by $X_I$ for simplicity. For $I = \{i_1, i_2, \cdots, i_k\}$, we always denote $X_I$ by $X_{i_1i_2\cdots i_k}$. So we have $X_0 = BT$ and $X_{12\cdots k} \simeq X_{12\cdots k-1} \cup_{BT} X_k$.

**Proof of Theorem 1:** For $n = 3$, $BG(A)$ is homotopic equivalent to $X_{12} \cup_{BT} X_3$. By Lemma 2.6 we have the short exact sequence $0 \to \Sigma \ker j \to H^*(BG(A)) \to \ker j \to 0$. The homomorphism $j : H^*(X_{12}) \oplus H^*(X_3) \to H^*(BT)$ is given by $j(u, v) = Bi_{i_2}^*(u) - Bi_3^*(v)$, where $Bi_{i_2}^*$ and $Bi_3^*$ are induced by the homomorphisms $i_{12} : T \to G_{12}(A)$ and $i_3 : T \to G_3(A)$. By Lemma 2.3, we observe that $\ker j$ is the sub-ring of Weyl group invariants. Since $A$ is non-symmetrizable, by Lemma 2.5, $\ker j \cong \mathbb{Q}$. We have $\text{im } j = \text{im}(Bi_{i_2}^*)+\text{im}(Bi_3^*)$ and $\text{im}(Bi_{i_2}^*)\cap\text{im}(Bi_3^*) = \ker j \cong \mathbb{Q}$. By Lemma 2.1, the Poincaré series of $\text{im}(Bi_{i_2}^*)$ and $\text{im}(Bi_3^*)$ are $\frac{1}{(1-q^2)(1-q^4)}$ and $\frac{1}{(1-q^2)^2(1-q^4)}$ respectively. Combining these results, the Poincaré series of $H^*(BG(A))$ is $q\left[\frac{2q^2-1}{(1-q^2)^2(1-q^4)}+1\right]$. Hence for $n = 3$ the Poincaré series of $H^*(BG(A))$ is $q[\frac{2q^2-1}{(1-q^2)^2(1-q^4)}+1] + 1$.

For $n \geq 4$, we prove this theorem by induction on $n$. We assume the theorem is true for $n - 1$. Since $A$ is non-symmetrizable, by Lemma 2.4, without loss of generality we can assume $A_{123}$ is non-symmetrizable. Then $A' = A_{12\cdots n-1}$ is also non-symmetrizable. By Lemma 2.1, $H^*(X_{12\cdots n-1}) \cong H^*(BG(A')) \otimes H^*(BS^1)$. By the induction assumption, the Poincaré series of $BG(A')$ is $P_{n-1}(q) = q[\frac{(n-2)q^2-1}{(1-q^2)^{n-2}(1-q^4)}+1]$. This means that the reduced cohomology $H^*(BG(A'))$ concentrates in odd dimensions. Since $X \cong X_{12\cdots n-1} \cup_{BT} X_n$, we use Lemma 2.6 to compute $H^*(BG(A))$. By the decomposition $H^*(X_{12\cdots n-1}) \cong H^*(BG(A')) \otimes H^*(BS^1) \otimes \mathbb{Q} \otimes H^*(BS^1)$, we have $\ker j \cong H^*(BG(A')) \otimes H^*(BS^1) \otimes \mathbb{Q}$. $\text{im } j = \text{im}Bi_{12\cdots n-1}^* + \text{im}Bi_n^*$. And the intersection of $\text{im}Bi_{12\cdots n-1}^*$ and $\text{im}Bi_n^*$ is the sub-ring of Weyl group invariants. It is isomorphic to $\mathbb{Q}$. The Poincaré series of $\ker j$ is $\frac{1}{(1-q^2)^n} - \frac{1}{(1-q^2)^{n-1}(1-q^4)} - \frac{1}{1-q^4} + 1$. Therefore the Poincaré series of $BG(A)$ is $q\left[\frac{(n-1)q^2-1}{(1-q^2)^{n-1}(1-q^4)}+1\right] + 1$. The Poincaré series of $BG(A)$ is $q\left[\frac{(n-1)q^2-1}{(1-q^2)^{n-1}(1-q^4)}+1\right] + 1$. Therefore the Poincaré series of $BG(A)$ is $q\left[\frac{(n-1)q^2-1}{(1-q^2)^{n-1}(1-q^4)}+1\right] + 1$.

**Proof of Theorem 2:** The proof of this theorem is similar to that of the Theorem 1. The difference is that for the symmetrizable case, the invariants of Weyl group is generated by the Killing form which is in degree 4.

We use induction on $n$. If $n = 2$, by Lemma 2.7, the theorem is true. We assume this theorem is true for $n - 1$. For an $n \times n$ symmetrizable generic Cartan matrix $A$, $A' = A_{12\cdots n-1}$ is a symmetrizable generic Cartan matrix. By induction assumption, the Poincaré series of $BG(A')$ is $Q_{n-1} = \frac{1}{1-q^4}(q\frac{(n-2)q^2-1}{(1-q^2)^{n-1}+1}+1)$. Since $BG(A)$ is homotopy equivalent to $X_{12\cdots n-1} \cup_{BT} X_n$. By a similar Mayer-Vietoris sequence computation, we get that the Poincaré series of $BG(A)$ is $Q_n = \frac{q}{(1-q^2)^n} - \frac{q}{(1-q^2)^{n-1}(1-q^4)} - \frac{q}{(1-q^2)(1-q^4)} + \frac{1}{(1-q^4)(1-q^4)}(Q_{n-1})$.
Hence we get a Vietoris sequence. In fact all the cohomology groups appear in the sequence are free nilpotent, there exists integer \( k > 0 \). This proves the lemma.

In the proof of Theorem 2, we need the \( H^*(BG(A)) \)-module structure on the Mayer-Vietoris sequences. In fact all the cohomology groups appear in the sequence are free \( \mathbb{Q}[\psi] \)-modules.

4 Some results derived from the main theorems

In this section we need some general results in algebraic topology. For details see Whitehead[13].

From the expressions of the Poincaré series of \( BG(A) \) in Theorem 1, we can see that for the non-symmetrizable case, the even dimensional cohomology group is \( H^0(BG(A)) \). Hence the cup product on \( H^*(BG(A)) \) is trivial and we have

**Corollary 4.1.** For a generic \( n \times n \) non-symmetrizable Cartan matrix \( A \), the rational homotopy type of \( BG(A) \) is \( \bigvee_{i=1}^{\infty} S^{2i+1} \) with \( P_n(q) = 1 + \alpha_2 q^5 + \alpha_3 q^7 + \cdots + \alpha_i q^{2i+1} + \cdots \).

**Lemma 4.1.** For all \( i \geq 2 \), \( \alpha_i = a_{2i} \).

**Proof:** By definition we have

\[
\alpha_2 q^4 + \alpha_3 q^6 + \cdots = \frac{P_n(q) - 1}{q} = \frac{(n-1)q^2 - 1}{(1-q^2)(1-q^4)} + 1 = 1 - \frac{1}{C_A(q)} = a_4 q^4 + a_6 q^6 + \cdots
\]

This proves the lemma.

By Milnor-Hilton theorem (see[13]) and the homotopy equivalence \( G(A) \simeq \Omega BG(A) \), we get

**Corollary 4.2.** The homotopy Lie algebra \( \pi_*(G(A)) \) with Samelson product is the free graded Lie algebra generated by \( \Sigma^{-1} \tilde{H}_*(BG(A)) \).

The Hopf algebra \( H_*(G(A)) \) is isomorphic to the tensor algebra \( T(\Sigma^{-1} \tilde{H}_*(BG(A))) \).

From the expressions of the Poincaré series in Theorem 2, we can see that for the symmetrizable case as \( \mathbb{Q}[\psi] \)-module the only even dimensional generator of \( H^*(BG(A)) \) is \( 1 \in H^0(BG(A)) \). The cup product on \( H^*(BG(A)) \) can be determined by the following lemma.

**Lemma 4.2.** Let \( R = R_0 \oplus R_1 \) be a \( Z_2 \) graded ring and \( R_1 \) be a free \( R_0 \) module, if all the elements in \( R_1 \) are nilpotent, then \( R_1 R_1 = 0 \).

**Proof:** Let \( b_1, b_2 \neq 0 \) be two elements in \( R_1 \) and \( b_1 b_2 = a \in R_0 \). We show \( a = 0 \). Since \( b_2 \) is nilpotent, there exists integer \( k > 0 \) such that \( b_2^{k+1} \neq 0 \) but \( b_2^k = 0 \). Then we have \( a b_2^{k+1} = b_1 b_2^k = 0 \). Hence we get \( a = 0 \) from the fact that \( R_1 \) is a free \( R_0 \) module.

Set \( R_0 = H^{even}(BG(A)) \) and \( R_1 = H^{odd}(BG(A)) \), by this lemma, we get

**Corollary 4.3.** For a generic \( n \times n \) symmetrizable Cartan matrix \( A \), the rational homotopy type of \( BG(A) \) is \( BS^3 \times \bigvee_{i=2}^{\infty} S^{2i+1} \) with \( (1 - q^4)Q_n = 1 + \beta_2 q^5 + \beta_3 q^7 + \cdots + \beta_i q^{2i+1} + \cdots \).

Similarly we have
Lemma 4.3. For all $i \geq 2$, $\beta_i = b_{2i}$.

Corollary 4.4. The homotopy Lie algebra $\pi_*(G(A))$ with Samelson product is the direct sum of $\pi_*(S^3)$ and the free graded Lie algebra generated by $\Sigma^{-1}\tilde{H}_*(BG(A))$, where $\tilde{H}_*(BG(A)) \cong \bigvee_{i=2}^{\infty} \bigvee_{j=1}^{\beta_i} S^{2i+1}$.

The Hopf algebra $H_*(G(A))$ is isomorphic to the $\mathbb{Q}$-algebra $H_*(S^3) \times T(\Sigma^{-1}\tilde{H}_*(BG(A)))$.

Now we have the following result.

Proposition 4.1. If $A_1, A_2$ are two generic Cartan matrices of size $n_1$ and $n_2$, then $G(A_1)$ and $G(A_2)$ are rational homotopy equivalent Hopf spaces if and only if $n_1 = n_2$ and $\epsilon(A_1) = \epsilon(A_2)$.

References

[1] A. Borel, Sur la cohomologie des espaces fibres principaux et des espaces homogenes de groupes de Lie compacts, Ann.math. 57(1953), 115-207.

[2] V. Kac, Simple irreducible graded Lie algebras of finite growth. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 32 1968 1323-1367.

[3] V. Kac, Infinite Dimensional Lie Algebras, Cambridge University Press, 1982

[4] V. Kac, D. Peterson, Regular functions on certain infinite-dimensional groups. Arithmetic and geometry, Vol. II, 141-166, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.

[5] V. Kac, D. Peterson, Defining relations of certain infinite-dimensional groups. The mathematical heritage of Elie Cartan (Lyon, 1984). Astérisque 1985, Numéro Hors Série, 165-208.

[6] V. Kac, Constructing groups associated to infinite-dimensional Lie algebras. Infinite dimensional groups with applications (Berkeley, Calif., 1984), 167-216, Math. Sci. Res. Inst. Publ., 4, Springer, New York, 1985.

[7] N. R. Kitchloo, Topology of Kac-Moody groups, Ph D thesis of California Institute of Technology, 1998.

[8] Broto and N. Kitchloo, Classifying spaces of Kac-Moody group, Mathematische Zeitschrift, No. 240, 621-649, 2002.

[9] S. Kumar, Kac-Moody groups, their flag varieties and representation theory. (English summary) Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002.

[10] J. W. Milnor; J. C. Moore, On the structure of Hopf algebras. Ann. of Math. (2) 81 1965 211-264.

[11] R. V. Moody, A new class of Lie algebras. J. Algebra 10 1968 211-230.

[12] D. Sullivan, Infinitesimal computations in topology. Inst. Hautes études Sci. Publ. Math. No. 47 (1977), 269-331 (1978).
[13] G. W. Whitehead, Elements of homotopy theory, Springer-Verlag, 1978.

[14] Zhao Xu-an, Jin Chunhua, Polynomial invariants of Weyl groups for Kac-Moody groups. Pacific J. Math. 269 (2014), no. 2, 491-509.

[15] Zhao Xu-an, Jin Chunhua, Poincaré series and rational cohomology rings of Kac-Moody groups and their flag manifolds, [arXiv:1301.2647].

[16] Zhao Xu-an, Jin Chunhua, The ranks of homotopy groups of Kac-Moody groups, [arXiv:1501.04380].

[17] Zhao Xu-an, Gao Hongzhu, Ruan Yangyang, The cohomology rings of classifying spaces of Kac-Moody groups, preprint.