Lack-of-correlation anomaly in CMB large scale polarisation maps

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Abstract. We present an assessment of the CMB large scale anomalies in polarisation using the two-point correlation function as a test case. We employ the state of the art of large scale polarisation datasets: the first based on a Planck 2018 HFI 100 and 143 GHz cross-spectrum analysis, based on SRoll2 processing, and the second from a map-based approach derived through a joint treatment of Planck 2018 LFI and WMAP-9yr. We consider the well-known $S_{1/2}$ estimator, which measures the distance of the two-point correlation function from zero at angular scales larger than $60^\circ$, and rely on realistic simulations for both datasets to assess confidence intervals. By focusing on the pure polarisation field described by either the $Q$ and $U$ Stokes parameters or by the local $E$-modes, we show that the first description is heavily influenced by the quadrupole (which is poorly constrained in both datasets) while the second one is more suited for an analysis containing higher multipoles up to $\ell \sim 10$, limit above which both datasets become markedly noise dominated. We find that both datasets exhibit a lack-of-correlation anomaly in pure polarisation, similar to the one observed in temperature, which is better constrained by the less noisy Planck HFI 100×143 data, where its significance lies at about 99.5%. We perform our analysis using realizations that are either constrained or non-constrained by the observed temperature field, and find similar results in the two cases.

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1 Introduction

The cosmic microwave background (CMB) is one of the most important cosmological observables and has greatly contributed to the success of the standard ΛCDM model. Nonetheless, anomalous features exist in the CMB large-angle anisotropy pattern which are in tension with the predictions of ΛCDM. Their statistical significance is assessed at the $2-3\sigma$ level depending on the particular estimator chosen. Several CMB anomalies exist [1]. In the following we will focus on the lack-of-correlation anomaly, which consists of a suppression in the CMB two-point correlation function at large angular scales with respect to the best-fit ΛCDM model [2–6]. This anomaly is directly connected to the so-called CMB lack-of-power anomaly, for which the lack of correlation shows up as a reduction of anisotropy power at large angular scales [7–11] and to others as well [12–15].

Two independent experiments, WMAP and Planck [16–19], agree well on these deviations, therefore limiting (but not completely excluding) the possibility of an instrumental origin. An alternative astrophysical explanation of these anomalies is the possible presence of residuals of Galactic emission. However, this explanation seems unlikely given that foreground cleaning at large angular scales is usually performed on maps and an imperfect subtraction would normally result into an increase rather than a decrease of power$^1$. A pragmatic approach is therefore to consider the CMB anomalies as correctly measured features in the CMB temperature pattern and assess their statistical significance, e.g. including correctly look-elsewhere effects [20].

If we accept the above point of view, then there are two possible explanations for these features: either we live in a rare (yet not exceedingly rare) realisation of a ΛCDM cosmology or we need a modification of ΛCDM to account for them. Discriminating between these two hypotheses can only happen based on some acceptable a posteriori probabilities to exceed. Unfortunately, the anomalies show up at large angular scales where the temperature field is

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$^1$A possible exception is the presence of chance correlations between foreground and the CMB which appears also unlikely.
already cosmic variance limited, so any additional data, while always useful for consistency tests, are not going to boost statistical significance.

Improvements can however be expected by including the CMB polarisation pattern, whose measurements are still far from reaching cosmic variance accuracy especially at large scales, where the systematic error budget is currently non negligible\(^2\). Several, anomalies oriented, analyses that include CMB polarisation have been performed on the Planck legacy data [21], using various estimators to quantify statistical significance jointly in temperature and polarization. The most adopted estimator has been proposed in [22] and only uses temperature to E-mode correlations (TE). It has been extended to incorporate polarization auto-spectra EE and BB information in [23]. The latter is employed, among others, by the Planck collaboration in their own analysis [21]. Other estimators have been proposed: see for instance [24] where a one-dimensional statistic involving TT, TE, EE angular power spectra is employed.

Incorporating polarization into a joint estimator calls for a choice. On the one hand, it is desirable to test whether the polarization observations are consistent with ΛCDM once the temperature observations are given. This can be accomplished by using constrained realizations of the joint temperature and polarization fields. On the other hand, it is also useful to test the significance of anomalies in temperature and polarization leaving both fields free to fluctuate within the ΛCDM predictions. This implies dealing with unconstrained (i.e. open) realisations. For instance, authors in [22] work under the first assumption, while the Planck collaboration [21] assumes the second.

In this paper, we analyse the consequences of either assumption. We employ two datasets: the first one is based on the cross-spectra between Planck 100 and 143 GHz channels obtained with the SRoll2 processing [25] while, the second is based on the auto-spectra obtained combining the Planck 70 GHz channel with the Ka, Q and V bands of WMAP [26]. These two datasets cannot be easily further combined because a proper combination should happen at map level (as it has been done for Planck 70 GHz and WMAP) but the Planck 100 and 143 GHz channels do not allow for this. Therefore we analyse the two datasets separately. We stress that these datasets have never been employed in the context of CMB anomalies before. This is an aspect where our analysis is entirely novel.

We focus on the lack of correlation anomaly in polarization by considering the correlation functions \(C^{QQ}\) and \(C^{UU}\), \(Q\) and \(U\) being the linear polarization Stokes parameters. We consider also the \(C^{EE}\) correlation function as proposed in [23]. The paper is organised as follows. In Section 2 we describe the datasets considered as well as our power spectra estimation procedure employed to derive correlation functions. In Section 3 we present the estimators used to assess the statistical significance of the considered anomaly. In Section 4 we set forth our main findings while in Section 5 we draw our conclusions.

2 Datasets and methodology

We consider the most constraining large-scale polarization datasets currently available, i.e. the cross-spectra between Planck 100 and 143 GHz channels [27] as presented in [25, 28] (hereafter Planck HFI 100×143) and the auto-spectra obtained combining the Planck 70 GHz channel [29] with the Ka, Q and V bands of WMAP [30] as presented in [26] (hereafter Planck LFI+WMAP). Here we briefly provide some general information useful to understand

\(^2\)The power spectrum of Planck HFI 100×143 is cosmic variance dominated between \(\ell = 3\) and \(\ell = 5\), see figure 10 of [28].
the procedure followed in preparing the former datasets. All the maps contained in the two datasets are mitigated from polarized Galactic foreground emissions (thermal dust and synchrotron) through a template fitting procedure, see e.g., [31, 32]. In the Planck LFI+WMAP dataset the auto-spectra are computed from the CMB map, built through an optimal weighting of the four foreground reduced input maps (i.e. 70 GHz, Ka, Q and V). In temperature both datasets employ the Commander Planck 2018 CMB solution smoothed through a Gaussian kernel with FWHM of 440 arcminutes and downgraded to a HEALPix $N_{\text{side}} = 16$ resolution [33]. The polarization maps are instead smoothed assuming a cosine window profile as suggested in [34, 35], and re-pixelized to the same HEALPix resolution as temperature. We select a useful sky fraction of 50% of Planck HFI $100 \times 143$ and 54% for Planck LFI+WMAP as suggested respectively in [28] and in [26].

In order to estimate the angular power spectra from the CMB maps we employ a Quadratic Maximum Likelihood (hereafter QML) method as presented in [35–37]. For a given map $x = (T, Q, U)$ the QML provides the estimated auto angular power spectra as

$$\hat{C}_\ell^X = \sum_{\ell',X'} (F^{-1})_{\ell\ell'}^{X',X} \left[ x_t^{X'} E_\ell^{X',X} x - \text{Tr}(N E_\ell^{X',X}) \right],$$

(2.1)

where $X$ and $X'$ are one of $TT$, $EE$, $BB$, $TE$, $TB$, $EB$ and $F_{\ell\ell'}^{X,X'}$ is the Fisher information matrix defined as

$$F_{\ell\ell'}^{X,X'} = \frac{1}{2} \text{Tr} \left[ C_a^{-1} \frac{\partial S}{\partial C_\ell^X} C_b^{-1} \frac{\partial S}{\partial C_\ell^{X'}} \right],$$

(2.2)

with $C \equiv S(C_\ell) + N$ being the CMB signal ($S$) plus noise ($N$) covariance matrix and $C_\ell$ a fiducial set of CMB angular power spectra. Finally, the $E$ matrix in eq. (2.1) is given by

$$E_\ell^X = \frac{1}{2} C_a^{-1} \frac{\partial S}{\partial C_\ell^X} C_b^{-1}.$$

(2.3)

Assuming uncorrelated noise between two maps $x_a$ and $x_b$, eq. (2.1) can be easily extended to cross-spectrum estimation which reads

$$\hat{C}_\ell^X = \sum_{\ell',X'} (F^{-1})_{\ell\ell'}^{X',X} x_t^{X'} E_\ell^{X',X} x_b,$$

(2.4)

having coherently defined

$$F_{\ell\ell'}^{X,X'} = \frac{1}{2} \text{Tr} \left[ C_a^{-1} \frac{\partial S}{\partial C_\ell^X} C_b^{-1} \frac{\partial S}{\partial C_\ell^{X'}} \right],$$

(2.5)

$$E_\ell^X = \frac{1}{2} C_a^{-1} \frac{\partial S}{\partial C_\ell^X} C_b^{-1},$$

(2.6)

$$C_a = S(C_\ell) + N_a,$$

$$C_b = S(C_\ell) + N_b.$$

For the two aforementioned datasets we also consider a set of 500 noise plus residual systematics simulations described in the two dedicated papers [25, 28].

\footnote{https://healpix.sourceforge.io/}
Figure 1. E-mode angular power spectrum $D_{\ell}^{EE} \equiv \ell(\ell + 1)C_{\ell}^{EE}/2\pi$ for Planck HFI 100×143 GHz (left panel) and Planck LFI+WMAP (right panel). The orange line is the data power spectrum while blue dots are the mean of power spectra extracted from 250000 constrained maps. The error bars are computed as the standard deviation of the simulations. Note that the range of values on the y-axis is the same for both panels.

3 Analysis

As discussed in the introduction we need both constrained and unconstrained simulations of the temperature and polarization fields. To build the set of constrained realizations we follow the procedure described in [22] and generate the polarized spherical harmonic coefficients, $a_{\ell m}^E$ and $a_{\ell m}^B$, as

$$a_{\ell m}^E = \frac{C_{\ell}^{TE}}{C_{\ell}^{TT}} a_{\ell m}^{Tdata} + \zeta_1 \sqrt{C_{\ell}^{EE} - \left(\frac{C_{\ell}^{TE}}{C_{\ell}^{TT}}\right)^2},$$

$$a_{\ell m}^B = \zeta_2 \sqrt{C_{\ell}^{BB}},$$

(3.1)

where $\zeta_1$ and $\zeta_2$ are random Gaussian realizations with zero mean and unit variance, $C_{\ell}^{XX}$ are the spectra corresponding to the Planck best fit $\Lambda$CDM model and $a_{\ell m}^{Tdata}$ are extracted from the observed temperature map. When building maps from spherical harmonic coefficients, we apply the same aforementioned window functions. Finally, we combine each of the 500 simulated CMB signal maps with all the 500 noise maps described in the previous section, forming a set of 250000 signal plus noise (hereafter $S+N$) realizations. In figure 1 we show the E-mode power spectra of the data maps compared with mean and standard deviations of the corresponding spectra of our $S+N$ Monte Carlo. Both Planck HFI 100×143 and Planck LFI+WMAP spectra do not show any evident outlier when compared to $S+N$ simulations.

We start computing the signal-to-noise ratio, $S/N$,

$$\frac{S}{N} = \sqrt{\sum_{\ell=2}^{\ell_{\text{max}}} \left(\frac{C_{\ell}^{EE}}{\sigma_{\ell}}\right)^2},$$

(3.2)

where $C_{\ell}^{EE}$ is a fiducial power spectrum, $\sigma_{\ell}$ is the standard deviation of the $S+N$ simulations and $\ell_{\text{max}}$ is the maximum multipole considered in the sum. In Figure 2 we show $S/N$ as a function of $\ell_{\text{max}}$ for our two datasets and for a cosmic variance limited full sky survey. As
expected, Planck HFI $100 \times 143$ has a better $S/N$ ratio with respect to Planck LFI+WMAP for almost all the $\ell_{\text{max}}$ considered. The only exception is the quadrupole, where the Planck HFI $100 \times 143$ variance is dominated by residual dipole leakage (see [25, 28] for details). Both datasets considered show a plateau above $\ell_{\text{max}} \simeq 10$ where the variance of the noise starts dominating the total variance. This justifies our choice of $\ell_{\text{max}} = 10$ as maximum multipole in the following analysis.

### 3.1 Estimators

We focus on an estimator originally suggested by the WMAP team, called $S_{1/2}$ [2]. The idea is to measure the distance between the correlation function and zero over a chosen range of angles [5]. We formally define this estimator below.

#### 3.1.1 Temperature

We start by reviewing the definition of $S_{1/2}$ in temperature, whose fluctuations are usually expanded in terms of scalar spherical harmonics:

$$\frac{\Delta T(\hat{n})}{T_0} \equiv \Theta(\hat{n}) = \sum_{\ell m} a_{\ell m}^T Y_{\ell m}(\hat{n}) ,$$

and the covariance of the coefficients $a_{\ell m}^T$ defines the anisotropy angular power spectrum,

$$\langle a_{\ell m}^T a_{\ell' m'}^{T*} \rangle = C_{\ell}^{TT} \delta_{\ell \ell'} \delta_{m m'} ,$$

standing the assumption of statistical isotropy and independence of the modes. The angular power spectrum, $C_{\ell}^{TT}$, and the two-point angular correlation function, $C(\theta)$, are related by the following expression,

$$C(\theta) \equiv \langle \Theta(\hat{n}_1) \Theta(\hat{n}_2) \rangle = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) C_{\ell}^{TT} P_{\ell}(\cos(\theta)) ,$$

\[\text{Figure 2.}\] Comparison of the integrated signal to noise ratio for the Planck HFI $100 \times 143$ (grey dots) and Planck LFI+WMAP (orange dots) dataset and for a sample of ideal simulations (green dashed line). Both datasets show a plateau above $\ell_{\text{max}} \simeq 10$, which is consequently chosen as maximum multipole in the analysis. For Planck HFI $100 \times 143$ a rise in trend is visible at $\ell_{\text{max}} > 20$, due to the corresponding increase in the E-mode signal. This effect can be safely ignored in our analysis.
where \( \hat{n}_1 \cdot \hat{n}_2 = \cos(\theta) \) and \( \mathcal{P}_\ell \) are the Legendre Polynomials. The \( S_{1/2} \) statistic in temperature, is defined as

\[
S_{1/2}^{TT} = \int_{-1}^{1/2} d(\cos \theta) |C^{TT}(\theta)|^2,
\]

and is used to quantify the lack of correlation at scales larger than 60°. Substituting (3.5) into (3.6) we can rewrite the estimator in terms of the angular power spectrum,

\[
S_{1/2}^{TT} = \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{\ell' = 2}^{\ell'_{\text{max}}} \frac{(2\ell + 1)(2\ell' + 1)}{4\pi} C^{TT}_{\ell} I_{\ell\ell'} C^{TT}_{\ell'},
\]

where the matrix \( I_{\ell\ell'} \) is defined as

\[
I_{\ell\ell'}(x) \equiv \int_{-1}^{x} \mathcal{P}_\ell(x') \mathcal{P}_{\ell'}(x') dx',
\]

and evaluated at \( x = 1/2 \), with \( x = \cos \theta \).

### 3.1.2 Polarization

Linear polarization is a spin-2 quantity and can be described by the Stokes parameters \( Q \) and \( U \) [38]. In analogy with T we can define the corresponding two-point angular correlation function as \( C^{QQ}(\theta) = \langle Q_r(\hat{n}_1)Q_r(\hat{n}_2) \rangle \) and \( C^{UU}(\theta) = \langle U_r(\hat{n}_1)U_r(\hat{n}_2) \rangle \). The Stokes parameters appearing in the correlation functions are defined with respect to a reference frame on the tangent plane with axes parallel and perpendicular to the great arch connecting \( \hat{n}_1 \) and \( \hat{n}_2 \). As in [38] we choose one point to be the north pole and the other on \( \phi = 0 \) longitude. This choice is denoted by the suffix \( r \) in the above definitions of the correlations functions. The coordinate system is hence fixed and the correlation functions depend only on the separation \( \theta \) between \( \hat{n}_1 \) and \( \hat{n}_2 \). The definition of \( S_{1/2} \) in polarization is analogous to the temperature case,

\[
S_{1/2}^{QQ,UU} = \int_{-1}^{1/2} d(\cos \theta) |C^{QQ,UU}(\theta)|^2,
\]

but it is useful again to rewrite it in terms of the angular power spectrum. The Stokes parameters can be decomposed using spin-2 spherical harmonics:

\[
Q(\hat{n}) = \sum_{\ell,m} \pm 2a_{\ell,m}^P Y_{\ell,m}(\hat{n}).
\]

A linear combination of the spin-2 spherical harmonic coefficients \( \pm 2a_{\ell,m}^P \) gives the E- and B-mode coefficients, \( a_{\ell,m}^{E/B} \), which are two scalar quantities [38]:

\[
a_{\ell,m}^E = -\frac{1}{2}[2a_{\ell,m}^P + -2a_{\ell,m}^P],
\]

\[
a_{\ell,m}^B = \frac{i}{2}[2a_{\ell,m}^P - -2a_{\ell,m}^P].
\]

The E- and B-mode power spectra, \( C^{EE/BB}_\ell \), are defined as:

\[
\langle a_{\ell,m}^E a_{\ell'm'}^{E*} \rangle = C^{EE}_\ell \delta_{\ell\ell'} \delta_{mm'},
\]

\[
\langle a_{\ell,m}^B a_{\ell'm'}^{B*} \rangle = C^{BB}_\ell \delta_{\ell\ell'} \delta_{mm'}.
\]
We can now express the correlation functions in terms of the power angular spectra \([38]\),

\[
C^{QQ}(\theta) = -\sum_{\ell} \frac{2\ell + 1}{4\pi} \left( \frac{2(\ell - 2)!}{(\ell + 2)!} \right) \left[ C^{EE}_{\ell} G^{+}_{\ell 2}(\cos(\theta)) + C^{BB}_{\ell} G^{-}_{\ell 2}(\cos(\theta)) \right],
\]

(3.13a)

\[
C^{UU}(\theta) = -\sum_{\ell} \frac{2\ell + 1}{4\pi} \left( \frac{2(\ell - 2)!}{(\ell + 2)!} \right) \left[ C^{BB}_{\ell} G^{+}_{\ell 2}(\cos(\theta)) + C^{EE}_{\ell} G^{-}_{\ell 2}(\cos(\theta)) \right],
\]

(3.13b)

where

\[
G^{+}_{\ell m}(\cos \theta) = -\left( \frac{\ell - m^2}{\sin^2 \theta} + \frac{\ell(\ell + 1)}{2} \right) \mathcal{P}^m_{\ell}(\cos \theta) + (\ell + m) \frac{\cos \theta}{\sin^2 \theta} \mathcal{P}^m_{\ell-1}(\cos \theta),
\]

(3.14a)

\[
G^{-}_{\ell m}(\cos \theta) = \frac{m}{\sin^2 \theta} ((\ell - 1) \cos(\theta) \mathcal{P}^m_{\ell}(\cos \theta) - (\ell + m) \mathcal{P}^m_{\ell-1}(\cos \theta)),
\]

(3.14b)

being the \(\mathcal{P}^m_{\ell}(\cos \theta)\) the associated Legendre polynomials. Plugging eq. (3.13) into (3.9) we obtain the following expression:

\[
S^{QQ}_{1/2} = \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{\ell'=-2}^{\ell_{\text{max}}} \frac{2\ell + 1}{8\pi} \frac{2\ell' + 1}{8\pi} \left( C^{EE}_{\ell} I^{(1)}_{\ell\ell'} C^{EE}_{\ell'} + C^{BB}_{\ell} I^{(3)}_{\ell\ell'} C^{BB}_{\ell'} + C^{EE}_{\ell} I^{(2)}_{\ell\ell'} C^{BB}_{\ell'} + C^{EE}_{\ell} I^{(4)}_{\ell\ell'} C^{BB}_{\ell'} \right),
\]

(3.15)

where we have followed the notation of [23]. For \(S^{UU}_{1/2}\) matrices \(I^{(3)}_{\ell\ell'}\) and \(I^{(1)}_{\ell\ell'}\) are swapped. More details on this calculation as well as the definition of the \(I^{(X)}_{\ell\ell'}\) matrices are given in Appendix A.

### 3.1.3 Two-point correlation functions for E- and B-modes

An alternative to Q and U is to express the polarization in terms of local E- and B-modes. These scalar quantities can be obtained from the Stokes parameters through a lowering-spin operator and their spherical harmonic coefficients are defined in eqs. (3.11a) and (3.11b). Their use has been suggested by [23] in the context of polarization correlation functions to complement the information given by Q and U, as we will show in the next section. The local correlation functions for E and B are defined as:

\[
C^{EE}(\theta) = \langle \hat{E}(\hat{n}_1) \hat{E}(\hat{n}_2) \rangle,
\]

(3.16a)

\[
C^{BB}(\theta) = \langle \hat{B}(\hat{n}_1) \hat{B}(\hat{n}_2) \rangle,
\]

(3.16b)

where the \(\hat{B}(\hat{n})\) and \(\hat{E}(\hat{n})\) functions are expanded as:

\[
\hat{E}(\hat{n}) = \sum_{\ell,m} \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} a^E_{\ell,m} Y_{\ell,m} (\hat{n}),
\]

(3.17a)

\[
\hat{B}(\hat{n}) = \sum_{\ell,m} \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} a^B_{\ell,m} Y_{\ell,m} (\hat{n}).
\]

(3.17b)
The two-point angular correlation functions can be written in terms of the angular power spectrum, in analogy with temperature:

\[ C_{EE}^\theta(\theta) = \sum_\ell \frac{2 \ell + 1}{4\pi} \frac{(\ell + 2)!}{(\ell - 2)!} \frac{(\ell' + 2)!}{(\ell' - 2)!} C_{\ell\ell'}^{EE} P_\ell(\cos \theta), \quad (3.18a) \]

\[ C_{BB}^\theta(\theta) = \sum_\ell \frac{2 \ell + 1}{4\pi} \frac{(\ell + 2)!}{(\ell - 2)!} \frac{(\ell' + 2)!}{(\ell' - 2)!} C_{\ell\ell'}^{BB} P_\ell(\cos \theta), \quad (3.18b) \]

and the expressions for the estimators \( S^{EE}_{1/2} \) and \( S^{BB}_{1/2} \) are:

\[ S_{XX}^{1/2} = \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{\ell'=2}^{\ell_{\text{max}}} 2 \ell + 1 \frac{(\ell + 2)!}{(\ell - 2)!} \frac{2 \ell' + 1}{4\pi} \frac{(\ell' + 2)!}{(\ell' - 2)!} C_{\ell\ell'}^{XX} I_{\ell\ell'} C_{\ell \ell'}^{XX}, \quad (3.19) \]

where \( X \) can be \( E \) or \( B \) and \( I_{\ell\ell'} \) is the same kernel defined in eq. (3.8) above.

4 Results

In this section we present results for the correlation functions and for the distribution of the \( S^{1/2}_{1/2} \) estimators, for the Q, U and local E-modes fields. For the sake of brevity, we only show plots for constrained simulations, while in Table 1 we report the results for both the constrained and unconstrained case. We start by discussing our results on the Q, U and local E-modes correlation functions, which are needed to better highlight the specificity of the \( S^{QQ}_{1/2} \), \( S^{UU}_{1/2} \) and \( S^{EE}_{1/2} \) estimators.

We show in Fig. 3 and 4 the QQ and UU angular correlation functions for both the Planck LFI+WMAP and Planck HFI 100×143 datasets, along with mean values and confidence intervals derived from constrained simulations, setting \( \ell_{\text{max}} = 10 \). In Fig. 5 we show instead the correlation function for local E-modes. Note that only in this latter case the different noise levels of the Planck LFI+WMAP and Planck HFI 100×143 datasets clearly show up in the plots. Such behaviour can be ascribed to the weights applied to each multipole when computing the correlation functions out of power spectra. In order to further clarify this aspect we show in Fig. 6 the geometrical weights of \( C_{\ell\ell}^{EE} \) in the definition of Q and local E-modes correlation functions, see eq. (3.14a):

\[ W_Q(\theta) = \sum_\ell \frac{2 \ell + 1}{4\pi} \frac{2(\ell - 2)!}{(\ell + 2)!} G_{\ell\ell}^+(\cos \theta), \quad (4.1a) \]

\[ W_E(\theta) = \sum_\ell \frac{2 \ell + 1}{4\pi} \frac{(\ell + 2)!}{(\ell - 2)!} P_\ell(\cos \theta) \quad (4.1b) \]

Here \( G_{\ell\ell}^+ \) is defined as in eq. (3.14a). Both quantities entering the definition of the weights (i.e. the angle \( \theta \) and the multipole \( \ell \)) are binned and the plots show the total weight inside each bin. To highlight the contribution of the quadrupole it is shown without applying any binning. Fig. 6 shows how the Q and U correlation functions are dominated by very low multipoles, in particular by the quadrupole, while the correlation function of local E-modes is more susceptible to variations at high multipoles considered here. This has a clear impact on the variance of the correlation function itself. If the Q correlation function is computed only from the quadrupole, Planck LFI+WMAP is more sensitive than Planck HFI 100×143.
Figure 3. Two point angular QQ correlation function. The orange dashed line represents data while the blue line is the mean of 250000 constrained simulations. The shaded region represents the 68% and 95% C.L.. The datasets employed are Planck HFI 100×143 on the left and Planck LFI+WMAP on the right. The analysis is performed up to ℓ = 10

Figure 4. As in Fig. 3 above but for the UU correlation function instead of QQ.

(see Fig. 7), being the latter dominated by residual dipole leakage as shown in Fig. 2. For all the higher multipoles, instead, Planck HFI 100×143 is clearly more constraining (see Fig. 8), partially, but not completely, compensating the quadrupole behaviour. Analogous results can be obtained for U Stokes field. The same multipole split for local E modes does not show the same trend, as the variance of the correlation function remains substantially unchanged if the quadrupole is excluded.

In figure 9, 10 and 11 we plot in light grey the distribution of the $S_{1/2}$ estimators for the Q and U fields respectively and for the local E modes, as defined in eq. (3.15) and (3.19). The red line represents the value of the estimators on data. In all figures the panel on the left refers to the Planck HFI 100×143 dataset and the one on the right to the Planck LFI+WMAP dataset. Specifically, we compute the integrals in eq. (3.8) and (A.4), involved in the computation of $I_{l\ell \ell'}$ matrices, in the angular range $60^\circ - 180^\circ$, coherently with previous analysis (see [2], [21]).

In Table 1 we report the value of $S_{1/2}$ estimator on data for the analysed datasets, both with and without the quadrupole contribution. We also show the percentage of simulations having a value of $S_{1/2}$ larger than data, the values reported in brackets refer to the uncon-
Figure 5. As in Fig. 3 above but for the two-point angular correlation function built from E-modes.

Figure 6. Absolute value of binned geometrical weights applied to $C_{EE}^\ell$'s entering in the definition of the Q Stokes parameter (left panel) and of the local E-modes (on the right) correlation functions.

Figure 7. Two point angular Q correlation function computed only with the quadrupole contribution. The shaded region represents the 68% and 95% C.L. The datasets employed are Planck HFI 100×143 on the left and Planck LFI+WMAP on the right.
Figure 8. Two point angular $Q$ correlation function computed starting from the octupole. The shaded region represents the 68% and 95% C.L. The datasets employed are Planck HFI $100 \times 143$ on the left and Planck LFI+WMAP on the right. The analysis is performed up to $\ell_{\text{max}} = 10$.

Figure 9. Empirical distribution for $S_{1/2}$ values from simulations (grey) and data (red). The estimator is computed on $Q$ correlation function. The maximum multipole used is $\ell_{\text{max}} = 10$. The left panel is for Planck HFI $100 \times 143$ dataset and the right one for Planck LFI+WMAP.

Figure 10. Same as in Figure 9 but for $U$. 

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Note: The text above is a natural representation of the content in the image.
Table 1. $S_{1/2}$ measured on data (second column including the quadrupole term and fourth column excluding the quadrupole term) and percentage of simulations with value of the estimator larger than the one found on data (third column including the quadrupole term and fifth column excluding the quadrupole term). The values reported in brackets refer to unconstrained simulations. The sensitivity associated to the percentage of simulations with $S_{1/2}$ higher than data is $\sim 0.2\%$.

|                  | $S_{1/2}^{data}$ | $S_{1/2} > S_{1/2}^{data}$ | $S_{1/2}^{data_{>2}}$ | $S_{1/2}^{data_{>2}} > S_{1/2}^{data_{>2}}$ [%] |
|------------------|------------------|-----------------------------|-----------------------|-----------------------------------------------|
|                  | [µK$^2$]         | [%]                         | [µK$^2$]              | [%]                                           |
| EE Planck HFI 100×143 | 1.25             | 99.5 (99.6)                 | 1.30                  | 99.5 (99.6)                                   |
| EE Planck LFI+WMAP  | 82.4             | 71.6 (72.3)                 | 82.7                  | 71.5 (72.2)                                   |
| QQ Planck HFI 100×143 | 9.57             | 19.6 (31.7)                 | 0.85                  | 48.8 (49.0)                                   |
| UU Planck HFI 100×143 | 9.0              | 34.2 (41.8)                 | 3.21                  | 49.2 (53.3)                                   |
| QQ Planck LFI+WMAP  | 5.14             | 39.4 (50.2)                 | 3.33                  | 25.5 (27.3)                                   |
| UU Planck LFI+WMAP  | 1.35             | 95.0 (95.6)                 | 2.25                  | 75.2 (77.6)                                   |

Figure 11. Same as in Figure 9 but for E-modes.

strained simulations case. As previously discussed, the local E modes estimator does not sizeably depend on the inclusion of the quadrupole, differently from what happens for $Q$ and $U$. As the Planck HFI 100×143 dataset has a higher signal to noise ratio with respect to Planck LFI+WMAP, we rely on it to make an assessment on the significance of the anomaly for E-modes. The value of the estimator on data is $S_{1/2}^{data} = 1.25$ µk for Planck HFI 100×143, with a lower tail probability of 0.5%, which suggests a low power in polarization data up to $\ell = 10$. As previously noted, the correlation functions for $Q$ and $U$ change significantly when the quadrupole is excluded from the analysis. The higher variance due to residual dipole on the quadrupole in the Planck HFI 100×143 dataset suggests to base our considerations on the Planck LFI+WMAP dataset. The trend of the latter dataset seems to indicate that the contribution of $\ell = 2$ increase the power of the low multipoles in the data with respect to simulations. This effect is highlighted by the decrease of the percentage of simulations with a value of $S_{1/2}^{data}$ higher than data when excluding the quadrupole from the analysis (compare III and V columns in table 1 for the QQ case of Planck LFI+WMAP dataset). This conclusion holds for both constrained and unconstrained simulations.

To further investigate the contribution of each multipole in determining the relative
Figure 12. Lower tail probability for the local E modes estimator computed on data w.r.t constrained simulations for both Planck HFI 100×143 (red curve) and Planck LFI+WMAP (grey curve) datasets. The region of multipoles higher than 11 is excluded from the plot as considered dominated by noise.

As a final test, we study the joint distribution of the $S_{1/2}$ estimators in temperature and polarization. In Figure 13, we show the distribution in the $(S_{1/2}^{EE}, S_{1/2}^{TT})$ plane of $N_{\text{sims}} = 500$ unconstrained Planck HFI 100×143 simulations (grey dots), compared to values computed on the actual Planck HFI 100×143 data (red dot). For each simulation the value of the estimator in temperature is normalized to the empirical mean of the $S_{1/2}^{TT}$ simulations distributions and the value of the estimator in polarization is normalized to the empirical mean of the $S_{1/2}^{EE}$ simulations distributions. The same normalization is applied to data.

Given the relatively small number of simulations available, we seek to compress the information encoded in the two-dimensional joint distribution into a single, one-dimensional estimator. To this purpose, we extend the use of the $S_{1/2}$ estimator from one to two dimensions, computing the distance of the points from zero as in formula 4.2: 
Figure 13. Joint behaviour of the $S_{1/2}$ estimator in temperature and polarization for 500 unconstrained simulations (grey dots) and data (red dot) of Planck HFI 100×143 dataset. All values of the estimator in temperature and polarization reported here are normalized to the empirical mean of the corresponding distribution. The position of data with respect to the simulations suggest a low combined power of T and E correlation functions.

Figure 14. Distribution of the $S_{EE,TT}^{1/2}$ estimator for 500 Planck HFI 100×143 simulations (grey) and data (red vertical line). The upper limit of the lower tail probability of the estimator computed on simulations with respect to data is 0.2%.

\[
S_{EE,TT}^{1/2} = \sqrt{\left( \frac{S_{TT}^{1/2}}{\langle S_{TT}^{1/2} \rangle} \right)^2 + \left( \frac{S_{EE}^{1/2}}{\langle S_{EE}^{1/2} \rangle} \right)^2}
\]  

The resulting distribution is shown in figure 14. We find that no simulations have values of $S_{EE,TT}^{1/2}$ lower than data. Given that the sensitivity associated to our Monte Carlo is $1/N_{\text{sims}} = 0.002$, we can quote this number as an upper limit to the lower tail probability associated to the $S_{EE,TT}^{1/2}$ value measured on the data.
5 Conclusions

In this paper we extend the study of the lack-of-power anomaly to the CMB polarization field, analysing the most constraining large-scale datasets currently available, which are the Planck LFI+WMAP dataset [26] and the Planck HFI 100×143 dataset [25]. Adopting a frequentist approach, we assume Planck 2018 + SRo112 cosmological fiducial model [28] and in particular a specific value for reionization optical depth $\tau = 0.0591$ which is an important choice for the angular scales probed. The value of $\tau$ used is also compatible with that obtained by [26] when using the WMAP+LFI dataset in polarization, together with the Commander 2018 solution in temperature. We employ the $S_{1/2}$ estimator [23], which is based on the two-point correlation function of $Q$ and $U$, see eq. (3.15), and of the local $E$-modes, see eq. (3.19). We compute such estimators on both data and realistic simulations, which contain signal, noise and residual systematic effects, and compare empirical distributions from simulations with data results. We employ fully polarised signal, considering both Planck CMB temperature constrained and unconstrained simulations, and limit most of the analysis to $\ell_{\text{max}} = 10$, a multipole above which both datasets are fully noise-dominated. We calculate the correlation function for $Q$ and $U$ and show that it is largely dominated by the quadrupole. This clearly impacts the results obtained, which show negligible variation when the maximum multipole included in the analysis is varied.

For both datasets considered, we do not see any anomalous behaviour, except for a mild $2\sigma$ anomaly in the case of Planck LFI+WMAP $U$ correlation function. This suggests that the power in the very low multipoles, in particular in the quadrupole, is not anomalously low in data. It is worth noting however, that both datasets include non negligible uncertainties at $\ell = 2$, mostly of systematic origin for Planck HFI 100×143 and mostly statistical for Planck LFI+WMAP, these uncertainties likely affect the constraining power of a possible polarisation anomaly.

On the other hand the estimator involving local E-modes behaves differently being more sensitive to the $\ell_{\text{max}}$ used in the analysis, and thus giving information on the integrated power of the lower multipoles. In Figure 12 we see that the lower tail probability for the two datasets follows the same descending trend and in particular the Planck HFI 100×143 seems to suggest a low power in data with respect to simulations considered.

The behaviour of the $S_{1/2}$ estimators on $Q$, $U$ and local E-modes in the case of unconstrained simulations appears to be similar to what described for the constrained case. In particular the lower tail probability for $\ell_{\text{max}} = 10$ of the Planck HFI 100×143 dataset is $0.4\%$, indicating again a low power of data with respect to simulations.

These results suggest that the large-angle CMB polarisation data behave in a similar way to temperature, exhibiting a mild low power anomaly, presumably originating not only from the quadrupole but rather than from the combined behaviour of all the multipoles $\ell \leq 10$.

This conclusion is strengthened by the analysis of the joint behaviour of the $S_{1/2}$ estimators in temperature and polarization. We employ 500 unconstrained simulations of the Planck HFI 100×143 dataset and find that the lower tail probability associated to the value of the joint estimator measured on the data is $< 0.2\%$, thus confirming the anomalous behaviour of the data with respect to the expectations in the framework of the $\Lambda$CDM model.

The present analysis has been carried out with the best datasets currently available at large angular scales, which are however limited by the still significant amount of noise in polarisation observations. This issue will be hopefully overcome by the advent of new data, such as those from LiteBIRD [39], which are expected to be cosmic variance limited at all
Figure 15. Distributions comparison between LiteBIRD like experiment (grey) and Planck HFI 100×143 dataset (magenta) simulations. The lower width and the left shift of the peak of the grey distribution are due to the lower level of noise which turns into an increase of constraining power for LiteBIRD like experiment.

scales. Through a rough estimate of the noise level from the three most sensitive LiteBIRD bands we expect an increase of five times the constraining power of this test in the case of polarization E modes, see Fig. 15. The perspectives for shedding light on this subject are thus high.

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A Appendix: calculation of $S_{1/2}^{QQ}$ and $S_{1/2}^{UU}$ in terms of power spectrum

Writing $S_{1/2}^{QQ}$ and $S_{1/2}^{UU}$ in terms of power spectrum is useful to ease computation. To obtain an analytic expression of these estimators we express the $G^\pm_\ell(\cos(\theta))$ functions in terms of the reduced Wigner matrices. We define

$$D_\ell^+ = \frac{4(\ell - 2)!}{(\ell + 2)!} G_{\ell 2}^+(\cos \theta);$$

(A.1a)

$$D_\ell^- = \frac{4(\ell - 2)!}{(\ell + 2)!} G_{\ell 2}^-(\cos \theta),$$

(A.1b)
and rewrite the $S_{1/2}^{QQ}$ statistic in terms of the power spectrum in the following way

$$S_{1/2}^{QQ} = \int_{-1}^{1/2} d(\cos \theta) [C_{1/2}^{QQ}(\theta)]^2$$

where the matrices are defined as:

$$I_{\ell^{+\ell'}}^{(1)} = \int_{-1}^{1/2} d(\cos \theta) D_{\ell}^{+} D_{\ell'}^{+}, \quad I_{\ell^{+\ell'}}^{(3)} = \int_{-1}^{1/2} d(\cos \theta) D_{\ell}^{+} D_{\ell'}^{-}.$$  

(A.4)

We write now $D_{\ell}^{\pm}$ in terms of the reduced Wigner rotation matrices, $d_{2,2}^{\pm}(\theta)$:

$$D_{\ell}^{\pm}(\cos \theta) \equiv [d_{2,2}^{\pm}(\theta) \pm d_{2,-2}^{\pm}(\theta)]$$  

(A.5)

and define

$$I_{\ell^{\pm\ell'}}^{\pm} \equiv \int_{-1}^{1/2} d\cos(\theta) d_{2,2}^{\pm}(\theta) d_{2,-2}^{\pm}(\theta),$$  

(A.6)

with $x = \cos(\theta)$. The final expression for the $I_{\ell^{\pm\ell'}}^{(X)}$ matrices is then given by

$$I_{\ell^{\pm\ell'}}^{(1)} = I_{\ell^{\pm\ell'}}^{++} + I_{\ell^{\pm\ell'}}^{+-} + I_{\ell^{\pm\ell'}}^{+\ell} + I_{\ell^{\pm\ell'}}^{-\ell};$$

(A.7)

$$I_{\ell^{\pm\ell'}}^{(2)} = I_{\ell^{\pm\ell'}}^{++} - I_{\ell^{\pm\ell'}}^{+-} + I_{\ell^{\pm\ell'}}^{+\ell} - I_{\ell^{\pm\ell'}}^{-\ell};$$

$$I_{\ell^{\pm\ell'}}^{(3)} = I_{\ell^{\pm\ell'}}^{++} + I_{\ell^{\pm\ell'}}^{-\ell} - I_{\ell^{\pm\ell'}}^{+\ell} + I_{\ell^{\pm\ell'}}^{-\ell};$$

$$I_{\ell^{\pm\ell'}}^{(4)} = I_{\ell^{\pm\ell'}}^{++} - I_{\ell^{\pm\ell'}}^{+-} - I_{\ell^{\pm\ell'}}^{+\ell} + I_{\ell^{\pm\ell'}}^{-\ell}.$$  

The matrices $I_{\ell^{\pm\ell'}}^{\pm}$ can be calculated from the relation between Wigner matrices and Clebsch-Gordan coefficients [44], as shown in Appendix A of [23].

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