THE SIMILARITY DEGREE OF AN OPERATOR ALGEBRA

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Abstract. Let $A$ be a unital operator algebra. Let us assume that every bounded unital homomorphism $u: A \to B(H)$ is similar to a contractive one. Let $\text{Sim}(u) = \inf \{\|S\| \|S^{-1}\|\}$ where the infimum runs over all invertible operators $S: H \to H$ such that the "conjugate" homomorphism $a \to S^{-1}u(a)S$ is contractive. Now for all $c > 1$, let $\Phi(c) = \sup \text{Sim}(u)$ where the supremum runs over all unital homomorphism $u: A \to B(H)$ with $\|u\| \leq c$. Then, there is $\alpha \geq 0$ such that for some constant $K$ we have:

\[ (*) \quad \forall c > 1 \quad \Phi(c) \leq Kc^\alpha. \]

Moreover, the smallest $\alpha$ for which this holds is an integer, denoted by $d(A)$ (called the similarity degree of $A$) and $(*)$ still holds for some $K$ when $\alpha = d(A)$. Among the applications of these results, we give new characterizations of proper uniform algebras on one hand, and of nuclear $C^*$-algebras on the other. Moreover, we obtain a characterization of amenable groups which answers (at least partially) a question on group representations going back to a 1950 paper of Dixmier.

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§0. Introduction

Consider a unital operator algebra $A$ (i.e. a subalgebra of $B(H)$, containing $I$, not assumed self-adjoint). We are interested in the following "similarity property" of $A$:

For any bounded unital homomorphism $u: A \to B(H)$, there is an invertible operator $S : H \to H$ (= a similarity) such that $x \to S^{-1}u(x)S$ is contractive.
In other words, every bounded unital homomorphism on $A$ is similar to a contractive one.

Let $\text{Sim}(u) = \inf\{\|S\|\|S^{-1}\|\}$ where the infimum runs over all invertible operators $S: H \to H$ such that the “conjugate” homomorphism $a \to S^{-1}u(a)S$ is contractive. Now for all $c > 1$, let

$$\Phi(c) = \sup \text{Sim}(u)$$

where the supremum runs over all unital homomorphism $u: A \to B(H)$ with $\|u\| \leq c$. Assume that the above similarity property holds. Then it is easy to show that $\Phi(c)$ is finite for all $c > 1$. Our first observation, simple but crucial, will be that necessarily $\Phi(c)$ has polynomial growth, i.e. there is a number $\alpha \geq 0$ and a constant $K$ such that

$$(0.1) \quad \forall c > 1 \quad \Phi(c) \leq Kc^\alpha,$$

equivalently: any bounded unital homomorphism $u: A \to B(H)$ satisfies $\text{Sim}(u) \leq K\|u\|^\alpha$. Let $d(A)$ be the infimum of the numbers $\alpha \geq 0$ for which (0.1) holds for some constant $K$.

Our second observation (which lies a bit deeper) is that $d(A)$ is an integer, i.e. we have

$$d(A) \in \{0, 1, 2, 3, \ldots\},$$

and moreover there is a constant $K$ such that (0.1) holds for $\alpha = d(A)$.

We call $d(A)$ the similarity degree of the operator algebra $A$. If the similarity property fails, then we set $d(A) = \infty$.

By a result due to Paulsen ([Pa4]), the similarity property is closely related to the notion of complete boundedness, for which we refer to [Pa1]. To describe this connection, we will consider the following property (C) of an operator algebra $A$:

(C) Every contractive unital homomorphism $u: A \to B(H)$ is completely bounded. It is easy to see that this holds for all $C^*$-algebras and for several examples of uniform algebras (such as the disc and the bidisc algebras).

Under this assumption, (see [Pa4]) a unital homomorphism $u: A \to B(H)$ is similar to a contractive one iff it is completely bounded (c.b. in short).

Let $\mathcal{K}$ be the $C^*$-algebra of compact operators on $\ell_2$, let $C_0 \subset \mathcal{K}$ be the subspace of diagonal operators and let $\mathcal{K} \otimes_{\min} A$ be the minimal (= spatial) tensor product. Under the above assumption (C) on $A$, we will show (see Theorem 4.2) that $d(A)$ is the smallest integer $d$ with the following property: there is a constant $K$ such that any $x$ in the unit ball of $\mathcal{K} \otimes_{\min} A$ can be written as a product of the form

$$x = \alpha_0D_1\alpha_1D_2\ldots D_d\alpha_d$$

with $\alpha_i \in \mathcal{K} \otimes 1$ and $D_i \in C_0 \otimes_{\min} A$ such that

$$\prod_{i=0}^{d} \|\alpha_i\| \prod_{i=1}^{d} \|D_i\| \leq K.$$

Thus, $d(A)$ appears as the minimal “length” necessary to express any element of the unit ball of $\mathcal{K} \otimes_{\min} A$ as a alternated product as above with $2d + 1$ factors (with a good control of the norms of the factors).
More generally, if $\mathcal{A}$ is merely a Banach algebra with unit, we may consider it as embedded as a dense unital subalgebra into its enveloping unital operator algebra $\tilde{\mathcal{A}}$. The morphism $\mathcal{A} \subset \tilde{\mathcal{A}}$ is characterized by the property that a unital homomorphism $v: \mathcal{A} \to B(H)$ is contractive (i.e. has norm equal to 1) iff it extends to a completely contractive homomorphism $\tilde{v}: \tilde{\mathcal{A}} \to B(H)$. In particular, $\tilde{\mathcal{A}}$ satisfies (C).

In this situation, let us assume that every bounded unital homomorphism $u: \mathcal{A} \to B(H)$ extends to a completely bounded unital homomorphism $\tilde{u}: \tilde{\mathcal{A}} \to B(H)$. We define

\begin{equation}
(0.2) \quad d(\mathcal{A}) = \inf\{\alpha \geq 1\}
\end{equation}

where the infimum runs over all $\alpha \geq 1$ such that for some $K$ we have for all bounded unital homomorphisms $u: \mathcal{A} \to B(H)$

\begin{equation}
(0.3) \quad \|\tilde{u}\|_{cb} \leq K\|u\|^{\alpha}.
\end{equation}

If there is no such $\alpha$, then we set by convention $d(\mathcal{A}) = \infty$. Then again the same observations are valid:

(i) $d(\mathcal{A}) < \infty$

(ii) $d(\mathcal{A})$ is an integer and the infimum is attained in (0.2).

An interesting example of this situation is given by group algebras (or semi-group algebras).

Let $G$ be a discrete group (resp. semi-group with unit). Let $\mathcal{A}$ be the group (resp. semi-group) algebra of $G$ i.e. $\mathcal{A} = \ell_1(G)$ equipped with convolution. In the group case, $\tilde{\mathcal{A}}$ coincides with the (full) $C^*$-algebra of $G$, denoted by $C^*(G)$. Let $g \to \delta_g$ be the natural mapping from $G$ into $\ell_1(G)$ (i.e. $\delta_g(s) = 1$ iff $s = g$). Let $u: \ell_1(G) \to B(H)$ be a linear map and let $\pi(g) = u(\delta_g)$. Clearly $u$ is a bounded unital homomorphism iff $\pi$ is a uniformly bounded group (resp. semi-group) representation. Moreover if we define

$$
|\pi| = \sup_{g \in G} \|\pi(g)\|
$$

we have obviously

$$
|\pi| = \|u\|.
$$

Conversely, any bounded representation $\pi: G \to B(H)$ extends uniquely to a bounded linear homomorphism $u: \ell_1(G) \to B(H)$.

In this setting, we will write $d(G)$ instead of $d(\mathcal{A})$.

We can show (see Theorem 3.2 and Corollary 3.4 below) that $d(G) = 1$ iff $G$ is finite and $d(G) = 2$ iff $G$ is amenable and infinite.

This result gives some information on the “similarity problem” for uniformly bounded group representations. Namely, we can prove

**Theorem 0.1.** Let $G$ be a discrete group. The following are equivalent:

(i) $G$ is amenable.
There is a constant $K$ and $\alpha < 3$ such that for any $H$ and for any uniformly bounded group representation $\pi : G \to B(H)$ there is an invertible operator $S : H \to H$ (called “a similarity”) with

$$\|S^{-1}\| \|S\| \leq K|\pi|^\alpha$$

and such that $g \to S^{-1}\pi(g)S$ is a unitary representation of $G$.

(iii) Same as (ii) with $K = 1$ and $\alpha = 2$.

Note: A uniformly bounded representation $\pi : G \to B(H)$ is called unitarizable if there is an invertible $S : H \to H$ such that $S^{-1}\pi(\cdot)S$ is a unitary representation. The implication (i) $\Rightarrow$ (iii) is a classical fact proved in 1950 by Dixmier [Di], following earlier work by Sz.-Nagy [SN] for $G = \mathbb{Z}$. At that time, there were no known example of uniformly bounded non-unitarizable representation. The first example of this phenomenon was given in 1955 by Ehrenpreis and Mautner [EM] on the group $SL_2(\mathbb{R})$ (cf. also [KS]). See Cowling’s notes [Co] for more information on the Lie group case. Later on, many constructions were given on non-commutative free groups (or on any discrete group containing a non-commutative free group as a subgroup). See for example the references in [MPSZ] and [BF2]. See also [P1, Chapter 2].

In the same paper, Dixmier asks whether amenable groups are the only groups $G$ on which every uniformly bounded representation $\pi$ is unitarizable. This remains an open question. Our result shows that if one incorporates in Dixmier’s question the fact that the similarity $S$ can be found with $\|S\| \|S^{-1}\| \leq |\pi|^2$, then the answer is affirmative.

It seems conceivable that $d(G) < \infty \Rightarrow d(G) \leq 2$ automatically, but at the time of this writing we have not been able to prove this, and we are now more inclined to believe (in analogy with Corollary 6.2) that there are examples of discrete groups $G$ with $2 < d(G) < \infty$. Note that these would be non-amenable groups not containing $F_2$, the free group on two generators. While such examples are known to exist [O1-2], they still appear difficult to understand (see for example the exposition in [Pat]).

Recently, we proved ([P5]) that when $A$ is the disc algebra we have $d(A) = \infty$, thus solving the “Halmos problem” on polynomially bounded operators. Of course, this also holds for the polydisc algebra, the ball algebra or for any uniform algebra admitting a quotient algebra (unitally) isometric to (or completely isomorphic to) the disc algebra. It is conceivable that $d(A) = \infty$ for any proper uniform algebra, however at this point we are only able to show the following (see Theorem 5.1).

**Theorem 0.2.** Let $K$ be a compact set. Let $A \subset C(K)$ be a uniform algebra (i.e. a closed unital subalgebra which separates the points of $K$). Then $A = C(K)$ iff $d(A) \leq 2$ and $A$ satisfies (C).

We now turn to $C^*$-algebras. Unfortunately, at this time we are unable to produce examples of $C^*$-algebras $A$ for which $d(A)$ takes arbitrarily large finite values (or one for which the degree is infinite). This would solve (negatively) a well known open problem, due to Kadison [Ka] (see [P1]). We conjecture that there is a $C^*$-algebra $A$ (probably the reduced $C^*$-algebra of the free group on infinitely many generators) with $d(A) = \infty$. Unfortunately we only are able to produce examples of $C^*$-algebras $A$ with $d(A)$ equal to either 1 (finite dimensional case), 2 (nuclear case), and 3 ($B(\ell_2)$).
We give a result (Theorem 6.1) which is very close to proving that, for a $C^*$-algebra, $d(A) \leq 2$ implies $A$ nuclear. Indeed, it is known (see [CE]) that $A$ is nuclear iff, for any $*$-representation $\pi: A \to B(H)$, the von Neumann algebra generated by $\pi$ is injective. What we can prove is the following (see Theorem 6.1).

**Theorem 0.3.** Let $A$ be a unital $C^*$-algebra such that $d(A) \leq 2$. Then, whenever a $*$-representation $\pi: A \to B(H)$ generates a semi-finite von Neumann algebra, that von Neumann algebra is injective.

We are convinced that the similarity degree $d(A)$ can take arbitrary integer values when $A$ runs over all possible (non self-adjoint) operator algebras, but again we have not been able to verify this yet. However, in the more general framework of “similarity settings” considered below, it is easy to exhibit examples realizing any possible integral value of the degree, see Remark 3.6.

The present investigation was considerably influenced by several sources [Pel, B1, BRS, BP2] which I would like to acknowledge here:

1) Peller’s paper [Pel] contains a discussion (partly based on some ideas of A. Davie [Da]) of the space of coefficients of representations of a $Q$-algebra (with some consequences for operator algebras). In view of the recent characterizations in [BRS] and [B1] of operator algebras which use the Haagerup tensor product, it was natural to try to transpose these ideas from [Pel] to the “new” category of operator algebras with c.b. maps as its morphisms. This is the content of section 1 below.

2) Blecher and Paulsen’s paper [BP2] contains several striking factorization theorems for elements in the maximal tensor products of various operator algebras, analogous to the factorization of polynomials into products of polynomials of degree 1. Their factorization is into infinitely (or at least unboundedly) many matricial factors (see §7). It was natural to wonder in which case the number of factors could be bounded by a fixed integer. This is what lead to the central notion of this paper: the “similarity degree” (see Theorem 4.2).

We refer the reader to the books [Pa1] and [P1] for the precise definitions of all the undefined terminology that we will use, and to [KaR] for operator algebras in general. We recall only that an “operator space” is a closed subspace $E \subset B(H)$ of the $C^*$-algebra of all bounded operators on a Hilbert space $H$. We will use freely the notion of a completely bounded (in short c.b.) map $u: E_1 \to E_2$ between two operator spaces, as defined e.g. in [Pa1]. We denote by $\|u\|_{cb}$ the corresponding norm and by $cb(E_1, E_2)$ the Banach space of all c.b. maps from $E_1$ to $E_2$.

We denote by $K$ the $C^*$-algebra of all compact operators on $\ell_2$.

We will use repeatedly the notion of “maximal” operator space introduced in [BP1], and further studied in [Pa6]. Let us recall its definition: let $E$ be any normed space. Let $I$ be the class of all maps $u: B \to B(H_u)$ with $\|u\| \leq 1$ (and say $\dim H_u \leq \text{card}(E)$). We let $J: E \to \bigoplus_{u \in I} B(H_u)$ be the isometric embedding defined by $J(x) = \bigoplus_{u \in I} u(x)$. Then, $\max(E)$ is defined as the operator space $J(E) \subset B\left(\bigoplus_{u \in I} H_u\right)$, and any operator space which is of this form (up to complete isometry) is called “maximal”.

The “maximal” operator spaces are characterized by the property that, for any linear map $u: E \to B(H)$ we have $\|u\|_{cb} = \|u\|$. The following slightly more explicit description of their operator space structure is often useful: for any $n$ and any $x$ in
Since restricted to $E$ of the similarity degree $d$ of the OA operator algebra (denoted by $\beta$) iff $\alpha$ is the smallest number $c$ for any $OA$ quotient of $E$. A similar setting is a triple $(i, E, A)$ where $E$ is an operator space, $A \subset B(H)$ a unital subalgebra and $i: E \to A$ is an injective linear map with $\|i\|_{cb} \leq 1$, such that $A$ is generated by $i(E)$.

Given such a setting, for any $c \geq 1$ we construct the enveloping unital operator algebra $\tilde{A}_c$ which contains $A$ as a dense unital subalgebra and has the property that for any unital homomorphism $u: A \to B(H)$, we have

$$\|u_i\|_{cb(E,B(H))} \leq c \iff \|u\|_{cb(\tilde{A}_c,B(H))} \leq 1.$$  

In particular, when $c = 1$, $u$ is completely contractive on $\tilde{A}_1$ iff it is completely contractive when restricted to $E$. We also introduce in §1, the universal unital operator algebra (denoted by $OA(E)$) of an arbitrary operator space $E$. The inclusion $E \to OA(E)$ can be viewed as the “maximal” setting involving $E$. The main result of §1 is Theorem 1.7 which gives an alternate description of $A_c$ as a canonical quotient of $OA(E)$.

In analogy with the group case, we prove (see Corollary 2.7) that $d$ is an integer and that the preceding property still holds for $\alpha = d$.

In §3, we apply this to uniformly bounded group representations on a discrete group $G$, we denote the degree in this case by $d(G)$, and we prove the above Theorem 0.1, which implies that $d(G) \leq 2$ iff $G$ is amenable. We actually prove a stronger version involving the space of “coefficients” of uniformly bounded (u.b. in short) representations.

This is proved by applying §2 to the following setting: $E = \ell_1(G)$ with its (usual) maximal operator space structure (which also can be defined by duality with $c_0$, cf. [ER, BP1]), and $A \subset C^*(G)$ is the image of $\ell_1(G)$ under the canonical map from $\ell_1(G)$ into $C^*(G)$.

In §4, we come to the most natural “setting”: we consider a unital operator algebra $A \subset B(H)$ and we let $A = A$ and $E = \max(A)$ in the sense of [BP1]. Then we denote by $d(A)$ the corresponding degree. We give a number of characterizations of this number.

In §5, we investigate the class of uniform algebras, i.e. $A \subset C(K)$ ($K$ compact), $A$ is unital and separates the point of $K$. In analogy with the group case, we prove that there is a constant $C$ such that any unital homomorphism $u: A \to B(H)$ satisfies $\|u\|_{cb} \leq C\|u\|^2$ iff $A = C(K)$, or equivalently (by a result of Sheinberg [Sh]) iff $A$ is amenable.
In §6, we turn to \( C^* \)-algebras and prove an analogous result (with the assumption that \( A \) has sufficiently many semi-finite representations): \( d(A) \leq 2 \) iff \( A \) is nuclear or equivalently (by results of Connes and Haagerup, see [H4]) iff \( A \) is amenable.

Finally, in §7, we give a slightly expanded version of some of Blecher and Paulsen’s results in [BP2]. We give, as an illustration, an apparently new characterization of the elements of the space \( B(G) \) formed of the coefficients of unitary representations of a discrete group \( G \), to be compared with the case of uniformly bounded representations treated in Theorem 1.12.

§1. Enveloping operator algebras. Preliminary results

It will be convenient to work in the following very general setting: we give ourselves a unital algebra \( A \) together with a linear subspace \( E \subset A \). We assume that \( E \) is given with an operator space structure. We will denote by \( i: E \rightarrow A \) the inclusion mapping. Moreover, we assume that the unital algebra generated by \( i(E) \) is the whole of \( A \).

In addition, we assume that \( A \) can be faithfully represented in \( B(H) \) for some Hilbert space \( H \) by a unital representation \( u_0: A \rightarrow B(H) \) such that \( \|u_0i\|_{cb} \leq 1 \). We will then say that the triple \( (i, E, A) \) is a “similarity setting”.

Given such a setting, we can define for any \( c \geq 1 \) the enveloping unital operator algebra \( \tilde{A}_c \) as follows:

Consider the family \( C_c \) of all unital homomorphisms \( u: A \rightarrow B(H_u) \) with \( H_u \) a Hilbert space, such that \( \|ui\|_{cb} \leq c \). Then we equip \( A \) with the norm

\[
\|a\|_c = \sup_{u \in C_c} \|u(a)\|.
\]

Note that \( \|a\|_c < \infty \) since \( u \) is a homomorphism and \( i(E) \) generates \( A \). Moreover, since \( u_0 \in C_c \), we indeed have a norm. We denote by \( \tilde{A}_c \) the completion of \( A \) for this norm. Clearly we have an isometric unital homomorphism

\[
\tilde{A}_c \subset \bigoplus_{u \in C_c} B(H_u)
\]

which allows us to consider from now on \( \tilde{A}_c \) as a unital operator algebra (and a fortiori as an operator space).

Note that whenever \( 1 \leq c \leq d \) we have \( C_c \subset C_d \) hence we have a completely contractive unital homomorphism \( i_{c,d}: \tilde{A}_d \rightarrow \tilde{A}_c \) with

\[
\|i_{c,d}: \tilde{A}_d \rightarrow \tilde{A}_c\|_{cb} \leq 1.
\]

(1.0)

Note that \( \tilde{A}_c \) is characterized by the following property:

(1.1) any unital homomorphism \( u: A \rightarrow B(H) \) such that \( \|ui\|_{cb} \leq c \) admits a unique extension \( \tilde{u}: \tilde{A}_c \rightarrow B(H) \) with \( \|\tilde{u}\|_{cb(\tilde{A}_c, B(H))} \leq 1 \).
In this general setting, we wish to study the following

**Similarity Property:** For each $u$ in $\bigcup_{c>1} C_c$, there is an invertible operator $S: H \to H$ (= a similarity) such that the homomorphism

$$u_S: a \to S^{-1} u(a) S$$

satisfies $\|u_S i\|_{cb} \leq 1$ (or equivalently is in $C_1$).

As we will see in the examples below, our setting contains a number of fundamental similarity problems: when $A$ is a group algebra (i.e. $A = \ell_1(G)$) or when $A$ is a $C^*$-algebra, or when $A$ is the disc algebra.

**Example 1.1.** Let $G$ be a discrete group. Let $A$ be the group algebra of $G$, i.e. $A = \ell_1(G)$ equipped with convolution. Let $\Gamma \subset G$ be a set of generators for $G$ and let $E = \ell_1(\Gamma)$.

In this situation, it is easy to check that $\tilde{A}_1 = C^*(G)$ the “full” $C^*$-algebra of $G$ (= the enveloping $C^*$-algebra of $\ell_1(G)$). Then the similarity property in this context means that for any group representation $u: G \to B(H)$ such that $\sup_{\gamma \in \Gamma} \|u(\gamma)\| \leq c$ there is a similarity $S: H \to H$ such that $\sup_{t \in G} \|S^{-1} u(t) S\| \leq 1$. We study this problem in section 3.

**Example 1.2.** Let $G = \mathbb{N}$ (a discrete semi-group can also be discussed), let $E = \ell_1(\mathbb{N})$ and let $A(D)$ the disc algebra with the natural contractive inclusion $i: \ell_1(\mathbb{N}) \to A(D)$. We let $A = i(\ell_1(\mathbb{N}))$. We equip $\ell_1(\mathbb{N})$ with its maximal operator space structure, so that for any map $v: \ell_1(\mathbb{N}) \to B(H)$ we have $\|v\| = \|v\|_{cb}$. Consider a unital homomorphism $u: A(D) \to B(H)$ such that $\|u i\| = \|u i\|_{cb} \leq c$, and let $T = u(z)$. Then $T$ is a power bounded operator and

$$\|u i\| = \|u i\|_{cb} = \sup_{n \geq 1} \|T^n\|.$$ 

Since there are power bounded operators which are not similar to contractions ([Fo, Le]), the similarity property does not hold in this case.

**Example 1.3.** Let $A = A(D)$ the disc algebra and let $E = A(D)$ equipped with its “maximal” operator space structure, $i$ being the identity on $A(D)$. Then consider $u: A \to B(H)$ such that $\|u i\|_{cb} \leq c$ and let $T = u(z)$. Here $u i$ is c.b. iff $T$ is “polynomially bounded”. Moreover $\|u i\|_{cb} \leq c$ holds iff we have

$$\|P(T)\| \leq c\|P\|_{\infty}$$

for any polynomial $P$.

The similarity problem in this case is a well known problem usually attributed to Halmos. The problem was solved by a counterexample in [P5]. Analogous questions can be formulated for any uniform algebra. We will return to this topic in §5.
Example 1.4. Let $\mathcal{A}$ be a $C^*$-algebra and let $E = \max(\mathcal{A})$, with $i$ again equal to the identity. Then the similarity problem reduces again to a well known open problem raised by Kadison [Ka]:
is every bounded unital homomorphism $u: \mathcal{A} \to B(H)$ similar to a $*$-representation?
We discuss the $C^*$-algebra setting in §6.

Let $E$ be an arbitrary operator space. We wish to define the “free unital operator algebra” associated to $E$. One way to define it is as follows. We consider the free unital (noncommutative) algebra $\mathcal{P}(E)$ associated to $E$ (equivalently, this is the tensor algebra over $E$). The elements of $\mathcal{P}(E)$ may be described as the vector space of formal sums

$$P = \lambda_0 1 + \sum \lambda_{i_1} e_{i_1}^1 + \sum \lambda_{i_1 i_2} e_{i_1}^2 e_{i_2}^2 + \cdots + \sum \lambda_{i_1 \cdots i_N} e_{i_1}^N \cdots e_{i_N}^N,$$

with $\lambda_0, \lambda_{i_1}, \ldots, \lambda_{i_1 \cdots i_N} \in \mathbb{C}$ and with $e_{i_1}^1, e_{i_1}^2, \ldots$ all in $E$, equipped with the “free” product operation.

Grouping terms, we may rewrite (1.2) as

$$P = P_0 + P_1 + \cdots + P_N$$

with $P_0, P_1, \ldots, P_N$ “homogeneous”, i.e.

$$P_N = \sum \lambda_{i_1 \cdots i_N} e_{i_1}^N \cdots e_{i_N}^N.$$

We will denote by $E^{(N)}$ the linear subspace of $\mathcal{P}(E)$ spanned by all elements of the form (1.3)’. When $N = 0$ we define by convention

$$E^{(0)} = \mathbb{C}1.$$ 

The space $E^{(1)}$ is just $E$ viewed as a subset of $\mathcal{P}(E)$. Then consider the family $\mathcal{J}$ of all the mappings $v: E \to B(H_v)$ with $\|v\|_{cb} \leq 1$. Let

$$v(P) = \lambda_0 I + \sum \lambda_i^1 v(e_i^1) + \cdots + \sum \lambda_{i_1 i_2 \cdots i_N} v(e_{i_1}^1) \cdots v(e_{i_N}^N)$$

and

$$\|P\| = \sup_{v \in \mathcal{J}} \|v(P)\|.$$ 

We will denote by $OA(E)$ the completion of $\mathcal{P}(E)$ for this norm. (The fact that it is a norm easily follows from (1.10) and (1.14) below.) Clearly we have $\|PQ\| \leq \|P\| \|Q\|$ for all $P, Q$ in $\mathcal{P}(E)$, hence we have a unital Banach algebra structure on $OA(E)$.

We denote by $OA_N(E)$ the closure in $OA(E)$ of all the elements of the form (1.3). Moreover, we denote by $E_N$ the closure in $OA(E)$ of the linear subspace $E^{(N)}$.

By construction, we have a natural embedding

$$OA(E) \subset \bigoplus_{v \in \mathcal{J}} B(H_v).$$
which allows us to consider from now on $OA(E)$ as a unital operator algebra (and a fortiori as an operator space) containing $E$ completely isometrically. This operator space structure can be described as follows: consider an element $G$ in $\mathcal{K} \otimes \mathcal{P}(E)$. Clearly $G$ can be written (for some $N$) as a finite sum of the following form

\begin{equation}
G = \lambda_0 \otimes 1 + \sum_i \lambda_i^1 \otimes e_i^1 + \cdots + \sum_{i_1 \cdots i_N} \lambda_{i_1 i_2 \cdots i_N}^N \otimes e_{i_1}^N \cdots e_{i_N}^N.
\end{equation}

with $\lambda_1, \ldots, \lambda_{i_1 \cdots i_N}^N \in \mathcal{K}$ and

\begin{equation}
e_i^1, \ldots, e_i^j, \ldots, e_i^1, \ldots, e_i^N \in E.
\end{equation}

For short we will also write this as

\begin{equation}
G = G_0 + \cdots + G_N, \quad G_j \in \mathcal{K} \otimes E^{(j)}
\end{equation}

Then the following formula encodes the operator space structure of $OA(E)$:

\begin{equation}
\|G\|_{\mathcal{K} \otimes_{\min} OA(E)} = \sup_{v \in J} \left\| \lambda_0 \otimes I + \sum_i \lambda_i^1 \otimes v(e_i^1) + \cdots + \sum_{i_1 \cdots i_N} \lambda_{i_1 i_2 \cdots i_N}^N \otimes v(e_{i_1}^N) \ldots v(e_{i_N}^N) \right\|_{\mathcal{K} \otimes_{\min} B(\mathcal{H}_v)}.
\end{equation}

This algebra $OA(E)$ is characterized by the following (easily verified) property:

\begin{equation}
(1.7) \text{Let $B$ be any unital operator algebra. For any $v: E \to B$ with $\|v\|_{cb} \leq 1$ there is a unique unital homomorphism $\hat{v}: OA(E) \to B$ extending $v$ such that $\|\hat{v}\|_{cb} \leq 1$.}
\end{equation}

(Note that actually the extension $\hat{v}$ is the restriction of a $C^*$-algebra representation.)

For instance, we may consider $B = OA(E)$ and $v_z: E \to OA(E)$ defined by

\[ \forall e \in E \quad v_z(e) = ze, \]

where $z \in \mathbb{C}$ with $|z| \leq 1$. Then by construction of $OA(E)$, $\|v_z\|_{cb} \leq 1$ hence there is a unique unital homomorphism $\hat{v}_z: OA(E) \to OA(E)$ extending $v_z$ and such that $\|\hat{v}_z\|_{cb} \leq 1$. We will use the notation

\[ \omega(z) = \hat{v}_z. \]

Then it is easy to check (since $\hat{v}_z$ is a homomorphism extending $v_z$) that if $P$ is as in (1.3) we have

\[ \omega(z)P = P_0 + zP_1 + \cdots + z^NP_N. \]

Similarly, if $G$ is as in (1.5) we have

\begin{equation}
(1.8) \quad (I_\mathcal{K} \otimes \omega(z))(G) = G_0 + zG_1 + \cdots + z^NG_N.
\end{equation}

It will be useful to record here the following fact.
Lemma 1.5. Each $P$ in $\mathcal{P}(E)$ can be written in a unique way as

$P = P_0 + P_1 + \cdots + P_N,$

for some integer $N$ with $P_j \in E^{(j)}$ for all $j \geq 0$. If we define

$Q_j(P) = P_j$

(i.e. $Q_j(P) = 0$ for $j > N$) then $Q_j$ extends to a complete contraction from $OA(E)$ onto $E^{(j)}$.

Proof. By the preceding formula (1.8) we have

$G_j = \int z^j [I_K \otimes \omega(z)](G)dm(z)$

where $m$ denotes normalized Haar measure on $\{z \mid ||z|| = 1\}$. By convexity this yields

$\|G_j\|_{\min} \leq \|G\|_{\min}.$

Since $G_j = (I_K \otimes Q_j)(G)$, this shows that $\|Q_j\|_{cb} \leq 1$, and it also shows the unicity of the expression (1.9).

Remark. Another description of $OA(E)$ is as follows: we consider the $C^*$-algebra $C^* < E >$ constructed in [Pe]. This is characterized by the property that for any $v: E \to B$ ($B$ any $C^*$-algebra) with $\|v\|_{cb} \leq 1$ there is a representation $\pi: C^* < E > \to B$ extending $v$. Then we can define $OA(E)$ as the unital (non-selfadjoint) operator algebra generated by the elements of $E$ in the unitization of $C^* < E >$.

We now introduce the “product map” $\pi_1$ and a whole family of deformations $\pi_z$. Consider $z \in \mathbb{C}$ with $|z| \leq 1$ and let $c = 1/|z|$. We can define a unital homomorphism

$\pi_z: OA(E) \to \tilde{A}_c$

as follows:

Let $V_z = zi: E \to \tilde{A}_c$. Then $\|V_z\|_{cb} \leq 1$. Therefore, by (1.7) (since $\tilde{A}_c$ is an operator algebra) there is a unique unital homomorphism

$\pi_z: OA(E) \to \tilde{A}_c$

extending $V_z$ and such that $\|\pi_z\|_{cb} \leq 1$.

We will need the following simple observation.

Lemma 1.6. Consider our usual similarity setting $(i, E, A)$. Assume that $E$ contains the unit element $1_{\mathcal{A}}$ of $A$. Let $0 \leq j < N$. Then for any $g$ in $K \otimes E^{(j)}$, there is $g'$ in $K \otimes E^{(N)}$ such that

$\quad (I_K \otimes \pi_1)(g - g') = 0$

$\quad \|g'\|_{\min} \leq \|1_{\mathcal{A}}\|_{E}^{N-j} \|g\|_{\min}.$
Proof. We introduce the map $V: OA(E) \to OA(E)$ which is simply the left multiplication by $1_A$, i.e. $V(x) = x \otimes 1_A$. Clearly we have

$$\forall j \geq 0 \quad V[E^{(j)}] \subset E^{(j+1)}$$

hence

$$V^{N-j}[E^{(j)}] \subset E^{(N)}.$$

Moreover for any $P$ in $\mathcal{P}(E)$ we have clearly $\pi_1(V(P)) = \pi_1(P)$ hence

$$\pi_1(V^{N-j}(P)) = \pi_1(P).$$

Similarly for any $g$ in $\mathcal{K} \otimes E^{(j)}$, let

$$g' = (I_{\mathcal{K}} \otimes V^{N-j})(g).$$

Then we clearly have (1.11) and (1.12). (Indeed, since $OA(E)$ is an operator algebra $V: x \to 1_A x$ has $cb$ norm $\leq \|1_A\|_{E}$.)

We now come to our first result.

**Theorem 1.7.** Let $c \geq 1$ and $z = 1/c$. The mapping $\pi_z$ is a completely contractive surjection from $OA(E)$ onto $\tilde{A}_c$. Moreover, it induces canonically a completely isometric isomorphism

$$\sigma_z: OA(E)/\ker(\pi_z) \to \tilde{A}_c,$$

so that, if $Q_z: OA(E) \to OA(E)/\ker(\pi_z)$ denotes the canonical surjection, we have

$$\pi_z = \sigma_z Q_z.$$

More precisely, for any $f$ in $\mathcal{K} \otimes A$ with $\|f\|_{\mathcal{K} \otimes_{\min} \tilde{A}_c} < 1$, there is $F$ in $\mathcal{K} \otimes \mathcal{P}(E)$ with $\|F\|_{\mathcal{K} \otimes_{\min} OA(E)} < 1$ such that

$$I_{\mathcal{K}} \otimes \pi_z(F) = f.$$ 

Proof. Note that $\pi_z$ is characterized as the unital homomorphism such that

$$\forall e \in E \quad \pi_z(e) = ze(e),$$

if we view $E \subset OA(E)$ and $A \subset \tilde{A}_c$. By construction we have $\|\pi_z\|_{cb} \leq 1$. On the other hand, note that $\pi_z(OA(E))$ contains $i(E)$ and is a subalgebra, hence it contains $A \subset \tilde{A}_c$ since we assume that $i(E)$ is generating. Therefore we can define a mapping

$$u_z: A \to OA(E)/\ker(\pi_z)$$

simply by setting

$$\forall a \in A \subset \tilde{A}_c \quad u_z(a) = (\sigma_z)^{-1}(a).$$

This is clearly a unital homomorphism. Moreover we have $u_z i(e) = Q_z(z^{-1}e)$ since $\sigma_z Q_z(z^{-1}e) = \pi_z(z^{-1}e) = i(e)$. Since $u_z i = z^{-1} Q_z|_E$ it follows that

$$\|u_z i\|_{cb} \leq |z|^{-1} = c.$$
Hence by the defining property of \( \tilde{A}_c \), (since \( OA(E)/\ker(\pi_z) \) is an operator algebra [BRS]) there is a unique unital homomorphism \( \tilde{u}_z: \tilde{A}_c \to OA(E)/\ker(\pi_z) \) such that \( \|\tilde{u}_z\|_{cb} \leq 1 \). Moreover \( \tilde{u}_{z|A} = u_z \) hence we have \( \|(\sigma_z)^{-1}(a)\| = \|u_z(a)\| \leq \|a\|_{\tilde{A}_c} \) for any \( a \) in \( A \). By the density of \( A \) in \( \tilde{A}_c \), it follows that \( \sigma_z \) is a surjective isometry, and \( \tilde{u}_z = (\sigma_z)^{-1} \), so that finally

\[
\|(\sigma_z)^{-1}\|_{cb} = \|\tilde{u}_z\|_{cb} \leq 1.
\]

Thus \( \sigma_z \) is a complete isometry.

For the last assertion in Theorem 1.7, we need an obvious extension of the main result in [BRS] to non-closed unital subalgebras of \( B(H) \), as follows: let \( P \) be a unital subalgebra of \( B(H) \) and let \( I \subset P \) be a 2-sided ideal which is closed in \( P \). Then the quotient space \( Z = P/I \) can be equipped with a (non-complete) operator space structure by Ruan’s theorem [R]. Consequently, its completion \( \hat{Z} \) can be equipped with a (complete this time) operator space structure.

On the other hand, \( \hat{Z} \) is clearly a unital Banach algebra, and it is easy to check that the product mapping \( \hat{Z} \otimes_h \hat{Z} \to \hat{Z} \) is a complete contraction. Hence by [BRS] there is a completely contractive unital homomorphism from \( \hat{Z} \) into \( B(H) \) for some Hilbert space \( H \).

Returning to the situation in Theorem 1.7, let \( P = P(E) \) (equipped with the operator space structure induced by \( OA(E) \)) and let \( I = P(E) \cap \ker(\pi_z) \). Let us denote \( Z_c = P/I \) in this case. Clearly, the restriction of \( \pi_z \) to \( P(E) \) induces a completely contractive unital homomorphism \( \tilde{\sigma}_z: \tilde{Z}_c \to \tilde{A}_c \), which is injective on \( Z_c \).

Since we assume that \( i(E) \) generates \( A \), we have \( \tilde{\sigma}_z(Z_c) = \pi_z(P(E)) = A \). Thus the inverse of \( \tilde{\sigma}_z|Z_c \) defines a homomorphism \( u_z: \tilde{A}_c \to \hat{Z}_c \) such that \( \|u_zi\|_{cb} \leq c \), and repeating the preceding argument we obtain that \( \tilde{\sigma}_z \) must be a complete isometry from \( \hat{Z}_c \) onto \( \tilde{A}_c \).

The next result is a simple reformulation of Paulsen’s results in [Pa4].

**Proposition 1.8.** Let \( K \geq 0 \) be a constant. Let \( E \) be an operator space, \( A \) a unital algebra and \( i: E \to A \) an injection with \( i(E) \) generating \( A \). The following properties of a unital homomorphism \( u: A \to B(H) \) are equivalent, for each fixed \( c \geq 1 \).

(i) \( u \) extends to a c.b. homomorphism

\[
\hat{u}: \hat{A}_c \to B(H) \quad \text{with} \quad \|\hat{u}\|_{cb} \leq K.
\]

(ii) There is an invertible operator \( S: H \to H \) with \( \|S\| \|S^{-1}\| \leq K \) such that the map \( u_S: A \to B(H) \) defined by \( u_S(a) = S^{-1}u(a)S \) extends completely contractively to \( \hat{A}_c \).

(iii) There is an invertible operator \( S \) with \( \|S\| \|S^{-1}\| \leq K \) such that

\[
\|u_Si\|_{cb} \leq c.
\]
Proof. The equivalence (i) ⇔ (ii) is exactly Paulsen’s result [Pa4]. Clearly (ii) ⇒ (iii) holds since by construction \( \|i\|: E \to \tilde{A}_c \|_{cb} \leq c \). Now assume (iii). By the defining property of \( \tilde{A}_c \), \( u_S: A \to B(H) \) admits an extension \( \tilde{u}_S: \tilde{A}_c \to B(H) \) with \( \|\tilde{u}_S\|_{cb} \leq 1 \). Hence we obtain (ii). \qed

Remark 1.9. Let \( E \) be any operator space. Consider the iterated Haagerup tensor product \( X = E \otimes_h E \ldots \otimes_h E \) (\( N \)-times). Let \( x \) be arbitrary in \( K \otimes E \otimes \cdots \otimes E \). Then \( x \) can be written as a finite sum

\[
x = \sum_{i_1 \ldots i_N} \lambda_{i_1i_2 \ldots i_N} x_{i_1}^1 \otimes \cdots \otimes x_{i_N}^N
\]

with \( \lambda_{i_1 \ldots i_N} \in K \) and \( x_{i_1}^1, \ldots, x_{i_N}^N \in E \). It is proved in [CES] that we have

\[
\|x\|_{K \otimes \min X} = \sup \left\{ \left\| \sum_{i_1 \ldots i_N} \lambda_{i_1i_2 \ldots i_N} \otimes \sigma^1(x_{i_1}^1)\sigma^2(x_{i_2}^2) \cdots \sigma^N(x_{i_N}^N) \right\| \right\}_{\min}
\]

where the supremum runs over all possible choices of \( H \) and of complete contractions \( \sigma^1: E \to B(H), \ldots, \sigma^N: E \to B(H) \). We claim that actually this supremum is attained when \( \sigma^1, \ldots, \sigma^N \) are all the same, more precisely we have

\[
(1.13) \quad \|x\|_{K \otimes \min X} = \sup \left\{ \left\| \sum_{i_1 \ldots i_N} \lambda_{i_1i_2 \ldots i_N} \otimes \sigma(x_{i_1}^1)\sigma(x_{i_2}^2) \cdots \sigma(x_{i_N}^N) \right\| \right\}_{\min}
\]

where the supremum runs over all possible \( H \) and all complete contractions \( \sigma: E \to B(H) \).

Indeed, this follows from a trick already used by Blecher in [B1] and which seems to originate in Varapoulos’s paper [V]. The trick consists in replacing \( \sigma^1, \ldots, \sigma^N \) by the single map \( \sigma: E \to B\left(H \oplus \cdots \oplus H\right) \) of the form

\[
\sigma(e) = \begin{pmatrix} 0 & \sigma^1(e) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \sigma^N(e) & 0 \\ \end{pmatrix}
\]

(More precisely \( \sigma(e) \) is the \( (N+1) \times (N+1) \) matrix having \( (\sigma^1(e), \ldots, \sigma^N(e)) \) above the main diagonal and zero elsewhere).

Then it is easy to check that \( \|\sigma\|_{cb} = \sup_j \|\sigma^j\|_{cb} \) and \( \forall x^1, \ldots, x^N \in E \)

\[
[\sigma(x^1) \ldots \sigma(x^N)]_{1,N+1} = \sigma^1(x^1)\sigma^2(x^2) \cdots \sigma^N(x^N).
\]

From this our claim immediately follows. Note that our claim shows for instance that in the case \( E = \max(\ell_1^n) \), the space \( E \otimes_h \cdots \otimes_h E \) (\( N \) times) can be identified completely isometrically with a subspace of \( C^*(F_n) \) (here \( F_n \) is the free group with \( n \) generators). Namely the subspace spanned by all products

\[
\{ U_{i_1}U_{i_2} \ldots U_{i_N} \mid 1 \leq i_1 \leq n, \ldots, 1 \leq i_N \leq n \},
\]

where \( U_1, \ldots, U_n \) denote the free unitary generators of \( C^*(F_n) \). Although this useful fact might have been observed by others, it does not seem to have been recorded into print.
Proposition 1.10. Let $E$ be any operator space. Consider $E$ as embedded into $OA(E)$. Fix $N \geq 1$, recall that we denote by $E_N$ the closed subspace of $OA(E)$ generated by all products of the form $x_1 \cdot x_2 \ldots x_N$ with $x_i \in E$. Then the natural "product" mapping
\[ T_N: E \otimes_h \ldots \otimes_h E \to E_N \]
which takes $x_1 \otimes \cdots \otimes x_N$ to $x_1 \cdot x_2 \ldots x_N$ is a completely isometric isomorphism.

Proof. For simplicity let us denote $X = E \otimes_h \ldots \otimes_h E$ ($N$ times). Since the algebraic tensor product $E \otimes \cdots \otimes E$ is dense in $X$ and similarly for $E_N$, it suffices to prove that for any element $G$ in $K \otimes E \otimes \cdots \otimes E$ we have
\[ \|G\|_{K \otimes_{\min} X} = \|(I_K \otimes T_N)(G)\|_{K \otimes_{\min} E_N}. \]
(1.14)
But this is immediate by (1.13) and (1.6). \qed

We now record here several consequences of Theorem 1.7 (inspired by Peller’s results for the category of $Q$-algebras in [Pel, Prop. 4.2 and 4.3]).

Corollary 1.11. Let $W: \tilde{A}_c \to B(H)$ be a linear mapping. Let $z = 1/c$. Then
\[ \|W\|_{cb(\tilde{A}_c, B(H))} = \|W \pi_z\|_{cb(OA(E), B(H))}. \]
(1.15)
Moreover, for any linear map $w: A \to B(H)$, the following assertions are equivalent:

(i) For some $c \geq 1$ and some $K \geq 0$, $w$ extends to a c.b. map $\tilde{w}: \tilde{A}_c \to B(H)$ with
\[ \|\tilde{w}\|_{cb} \leq K. \]
(ii) There are constants $c' \geq 1$ and $K' \geq 0$ such that, for any $N \geq 1$, the mapping $w_N: E \otimes_h \ldots \otimes_h E \to B(H)$ defined by $w_N(x_1 \otimes \cdots \otimes x_N) = w(x_1 \ldots x_N)$ satisfies
\[ \|w_N\|_{cb(E \otimes_h \ldots \otimes_h E, B(H))} \leq K'(c')^N. \]
(ii)' There are constants $c' \geq 1$ and $K' \geq 0$ such that, for any $N \geq 1$, there are bounded linear mappings $u_i: E \to B(H_{i+1}, H_i)$, with $\|u_i\|_{cb} \leq c'$ for all $i$, where $H_i$ are Hilbert spaces with $H_{N+1} = H$ and $H_1 = H$, such that
\[ \forall x_1, \ldots, x_N \in E \quad w(x_1 x_2 \ldots x_N) = K' u_1(x_1) u_2(x_2) \ldots u_N(x_N). \]

Proof. The first part is an obvious consequence of the first assertion in Theorem 1.7. We now prove the second part. The equivalence between (ii) and (ii)', with the same constants $K'$, $c'$, is a particular case of the well known factorization of completely bounded multilinear forms (cf. [CS1-2, PaS]). We now turn to the remaining equivalence. Assume (i). Then $\|\tilde{w} \pi_z\|_{cb} \leq K$, hence $\|\tilde{w} \pi_z\|_{cb} \leq K$, but for $x_1, \ldots, x_N$ in $E$ we have
\[ \tilde{w} \pi_z(x_1, \ldots, x_N) = c^{-N} w(x_1 \ldots x_N) = c^{-N} w_N(x_1 \otimes \cdots \otimes x_N). \]
Hence by Proposition 1.10, we have

\[(1.16) \quad \|w_N\|_{cb} = c^N \|\tilde{w}\pi_{z|E_N}\|_{cb}\]

hence

\[\|w_N\|_{cb} \leq Kc^N.\]

This proves (i) \(\Rightarrow\) (ii).

Conversely, assume (ii). Let \(c > c'\) and \(z = 1/c\) as before. Then we have by (1.16)

\[\forall N \geq 1 \quad \|\tilde{w}\pi_{z|E_N}\|_{cb} \leq K' \left(\frac{c'}{c}\right)^N.\]

By Lemma 1.5, this implies using (1.15),

\[\|\tilde{w}\|_{cb(\mathcal{A}_z, B(H))} = \|\tilde{w}\pi_{z}\|_{cb} \leq 1 + \sum_{N \geq 1} \|\tilde{w}\pi_{z|E_N}\|_{cb} \leq 1 + K' \sum_{N \geq 1} (c'/c)^N < \infty,\]

whence (i) with \(K = 1 + K' \sum_{N \geq 1} (c'/c)^N.\) \(\square\)

We now illustrate the meaning of Corollary 1.11 in the group case.

**Theorem 1.12.** Let \(G\) be a group (or merely a semi-group). Consider a function \(f: G \to B(H)\). The following assertions are equivalent.

(i) There is a uniformly bounded representation \(\pi: G \to B(H_\pi)\) and bounded operators \(\xi: H_\pi \to H\) and \(\eta: H \to H_\pi\) such that

\[\forall f \in G \quad f(t) = \xi\pi(t)\eta.\]

(ii) There are constants \(K' \geq 0\) and \(c' \geq 1\) such that, for each \(N \geq 1\), the function \(f_N: G^N \to B(H)\) defined by

\[f_N(t_1, t_2, \ldots, t_N) = f(t_1t_2 \ldots t_N)\]

defines (with the obvious identification) an element of \(cb(\ell_1(G) \otimes_h \cdots \otimes_h \ell_1(G), B(H))\) (\(N\)-fold tensor product) with norm \(\leq K'c'^N.\)

(ii)’ There are constants \(K' \geq 0\) and \(c' \geq 1\) such that, for each \(N \geq 1\), there are bounded mappings \(\varphi_i: G \to B(H_{i+1}, H_i)\), with \(\sup_{t \in G} \|\varphi_i(t)\| \leq c'\) for all \(i\), where \(H_i\) are Hilbert spaces with \(H_{N+1} = H\) and \(H_1 = H\), such that

\[\forall t_1, \ldots, t_N \in G \quad f(t_1t_2 \ldots t_N) = K'\varphi_1(t_1)\varphi_2(t_2) \ldots \varphi_N(t_N).\]

**Proof.** We merely apply Corollary 1.11 and Proposition 1.8 with \(A = E = \ell_1(G)\).

Note that, using the factorization of cb maps, it is easy to verify that (i) holds iff the mapping \(t \to f(t)\) extends linearly to a mapping \(\tilde{w}: \mathcal{A}_c \to B(H)\) with \(\|\tilde{w}\|_{cb} \leq K\).

We leave the details to the reader. \(\square\)

**Remark.** Note that if (i) holds in the preceding statement with \(|\pi| \leq c\) then we obtain (ii) with \(K = \|\xi\| \|\eta\|\) and the same number \(c\). However, if (ii) holds we only obtain (i) with a representation \(\pi\) such that \(|\pi| \leq (1 + \varepsilon)c'\) (with \(\varepsilon > 0\)) and with \(\|\xi\| \|\eta\| \leq K_\varepsilon = 1 + K \sum_{N \geq 1} (1 + \varepsilon)^{-N}\). Indeed, these are the constants appearing in the proof of Corollary 1.11. Nevertheless, we will see below (see Corollary 7.8) that, in the particular case \(c' = 1\), we can get rid of this extra factor \((1 + \varepsilon)\).
§2. Main results

Let $E, \mathcal{A}$ and $i: E \to \mathcal{A}$ be our general setting as described in the beginning of §1. We will assume that the following holds:

Every unital homomorphism $u: \mathcal{A} \to B(H)$ such that $\|ui\|_{cb} < \infty$ is similar to a homomorphism such that $\|ui\|_{cb} \leq 1$, i.e. there is an invertible operator $S: H \to H$ such that the map $e \to S^{-1}ui(e)S$ is completely contractive.

When this holds we will say that in this setting the similarity property holds. We will need to carefully keep track of the constants involved in this phenomenon.

**Lemma 2.1.** If the similarity property holds then there is, for each $c \geq 1$, a number $\Phi(i, c)$ such that, for any unital homomorphism $u: \mathcal{A} \to B(H)$ with $\|ui\|_{cb} \leq c$, there is a similarity $S: H \to H$ with $\|S\| \cdot \|S^{-1}\| \leq \Phi(i, c)$ such that $e \to S^{-1}ui(e)S$ is a completely contractive map from $E$ to $B(H)$.

**Proof.** This is elementary. Just consider the unital homomorphism $U = \bigoplus_{u \in C_c} u$ and a similarity $S$ such that $S^{-1}US$ is contractive then restrict to the invariant subspaces associated to each $u$ in $C_c$. We get the announced bound with $\Phi(c) = \|S\| \cdot \|S^{-1}\|$. □

The preceding lemma allows us to define the following parameter associated to the similarity property

\begin{equation}
\Phi(i, c) = \sup_{u \in C_c} \inf \{\|S\| \cdot \|S^{-1}\|\}
\end{equation}

where the infimum runs over all $S: H_u \to H_u$ invertible such that $\|u_Si\|_{cb} \leq 1$ where $u_S(a) = S^{-1}u(a)S$. When the supremum is infinite, we write $\Phi(i, c) = \infty$ by convention. Equivalently by Proposition 1.8 we have

\begin{equation}
\Phi(i, c) = \sup_{u \in C_c} \|u\|_{cb(\tilde{A}_1, B(H_u))},
\end{equation}

where by $\|u\|_{cb(\tilde{A}_1, B(H_u))}$ we mean that we compute the $cb$ norm of $u: \mathcal{A} \to B(H_u)$ using the operator space structure induced by $\tilde{A}_1$ on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $\tilde{A}_1$, there is no risk of confusion.

By the definition of $\tilde{A}_c$ and by (2.2), when the similarity property holds, then the natural map $\tilde{A}_c \to \tilde{A}_1$ (which is always a complete contraction by (1.0)) is a complete isomorphism and (2.2) can be rewritten as

\begin{equation}
\Phi(i, c) = \|\tilde{A}_1 \to \tilde{A}_c\|_{cb}.
\end{equation}

It will be convenient to introduce the following notation

\begin{equation}
\text{Sim}(u) = \inf \{\|S^{-1}\| \cdot \|S\|\}
\end{equation}

over all $S$ such that $\|u_Si\|_{cb} \leq 1$. By convention we set $\text{Sim}(u) = +\infty$ if there is no such $S$. 

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Lemma 2.2. Let $c > 1$. Assume $\Phi(i, c) < \infty$. Then for any $0 < \theta < 1$ we have

$$\Phi(i, c) \leq \Phi(i, c^\theta)^{1/\theta}.$$ 

Proof. Consider $u$ in $C_c$ with $\text{Sim}(u)$ finite. By definition of $\Phi(i, c)$ for any $\varepsilon > 0$ there exist unital homomorphisms $u$ such that $\|S^1\| - \|S\| \leq \text{Sim}(u) + \varepsilon$ and $\|uS_i\|_{cb} \leq 1$ where we again denote $uS_i(a) = S^1u(a)Si$. Clearly we can assume that $S$ is hermitian. We then invoke the three lines lemma. Consider $z \in \mathbb{C}$ with $0 \leq \text{Re}(z) \leq 1$ and $e \in E$.

If $\text{Re} z = 1$ we have $\|S^{-z}ui(e)S^z\| \leq \|e\|$, and if $\text{Re} z = 0$ we have $\|S^{-z}ui(e)S^z\| \leq c\|e\|$. Hence by the subharmonicity of

$$z \to \log\|S^{-z}ui(e)S^z\|$$

we have

$$\|S^{\theta-1}ui(e)S^{1-\theta}\| \leq c^\theta\|e\|.$$ 

More generally, the same reasoning exactly yields that the map $v: A \to B(H)$ defined by $v(a) = S^{\theta-1}u(a)S^{1-\theta}$ satisfies $\|v\|_{cb} \leq c^\theta$. In other words we have $v \in C_{cb}$ so that by definition of $\Phi(i, c^\theta)$ there is a similarity $T: H \to H$ such that $\|T^{-1}\| \|T\| \leq \Phi(i, c^\theta) + \varepsilon$, and $\|vT_i\|_{cb} \leq 1$. This last inequality implies that $e \to T^{-1}S^{\theta-1}ui(e)S^{1-\theta}T$ is a complete contraction. Hence

$$\text{Sim}(u) \leq \|T^{-1}\| S^{\theta-1} \|S^{1-\theta}T\| \leq \|T^{-1}\| \|T\| (\|S\| \|S^{-1}\|)^{1-\theta}$$

i.e.

$$\text{Sim}(u) \leq (\Phi(i, c^\theta) + \varepsilon)(\text{Sim}(u) + \varepsilon)^{1-\theta}.$$ 

Since this holds for all $u$ in $C_c$, we have

$$\Phi(i, c) \leq (\Phi(i, c^\theta) + \varepsilon)(\Phi(i, c) + \varepsilon)^{1-\theta}$$

now since $\varepsilon > 0$ is arbitrary and $\Phi(i, c) < \infty$, we obtain after division by $\Phi(i, c)^{1-\theta}$

$$\Phi(i, c^\theta) \leq \Phi(i, c^\theta).$$

Remark. In case $\Phi(i, c) = \infty$ (for some $c > 1$), the preceding statement remains valid if there exist unital homomorphisms $u: A \to B(H)$ with $\|ui\|_{cb} \leq c$ for which $\text{Sim}(u)$ takes arbitrarily large finite values. (By Haagerup’s result in [H1] on cyclic homomorphisms, this condition is always satisfied in the $C^*$-setting considered in §6.) Then the preceding proof shows that $\Phi(i, c') = \infty$ for all $c' > 1$.

Lemma 2.3. Let $\varphi: [1, \infty) \to \mathbb{R}_+$ be a non-decreasing function such that $\varphi(c) \leq \varphi(c^\theta)^{1/\theta}$ whenever $0 < \theta < 1$ and $c \geq 1$. Fix $\alpha > 0$. Then, if $\varphi(c) < c^\alpha$ for some $c > 1$ we have $\varphi(x) < x^\alpha$ for all $x \geq c$ and there is a constant $K$ such that $\varphi(x) \leq Kx^\alpha$ for all $x \geq 1$.

Proof. Assume $c > 1$ and $\varphi(c) = c^\beta$ with $\beta < \alpha$. Then let $t = 1/\theta$ with $1 \leq t < \infty$. Since $\varphi(x^t) \leq \varphi(x)^t$ for any $x \geq 1$ we have $\varphi(c^t) \leq \varphi(c)^t = c^{\beta t^\alpha}$. Hence we have $\varphi(x) \leq x^\beta$ whenever $x = c^t$ with $t \geq 1$. Equivalently this holds whenever $x \geq c$.

The rest is obvious by monotonicity.  

Hence we have
**Corollary 2.4.** If the similarity property holds (i.e. if \( \Phi(i, c) < \infty \) for all \( c \geq 1 \)) then there is a constant \( K \) and an exponent \( \alpha \geq 0 \) such that

\[
\forall c \geq 1 \quad \Phi(i, c) \leq K c^\alpha.
\]

We come now to our main result.

**Theorem 2.5.** Consider our usual setting \( E, \mathcal{A}, i \) and assume that the similarity property holds. More precisely we assume that for some constants \( K > 0 \) and \( \alpha > 0 \) we have \( \Phi(i, c) \leq K c^\alpha \) for all \( c \) large enough. Let \( N \) be an integer with \( \alpha < N + 1 \). Then the restriction of \( \pi_1 \) to \( OA_N(E) \) is a complete surjection of \( OA_N(E) \) onto \( \tilde{A}_1 \), i.e. there is a constant \( K' \) such that, for any \( f \) in \( \mathcal{K} \otimes_{\min} \tilde{A}_1 \) with \( \|f\|_{\min} < 1 \), there is \( \hat{f} \) in \( \mathcal{K} \otimes_{\min} OA_N(E) \) with \( \|\hat{f}\|_{\min} < K' \) such that \( (I_{\mathcal{K}} \otimes \pi_1)(\hat{f}) = f \). Furthermore, this last property implies that for some constant \( K_1 \) we have

\[
\forall c \geq 1 \quad \Phi(i, c) \leq K_1 c^N.
\]

Finally, if \( 1_{\mathcal{A}} \) belongs to \( E \), then the restriction of \( \pi_1 \) to \( E_N \) is a complete surjection of \( E_N \) onto \( \tilde{A}_1 \).

**Proof of Theorem 2.5.** Let us denote for simplicity

\[
\mathcal{X}_N = OA_N(E).
\]

We first fix \( c > 1 \) chosen large enough so that

\[
\sum_{j > N} K c^{\alpha - j} < 1/2.
\]

Note that since \( N + 1 > \alpha \), this choice is possible. By the standard iteration argument used in the proof of the open mapping theorem, it suffices to prove the following.

**Claim.** There is a constant \( K'' \) such that for any \( f \) in the open unit ball of \( \mathcal{K} \otimes_{\min} \tilde{A}_1 \), with \( f \in \mathcal{K} \otimes \mathcal{A} \), there is an element \( \hat{f} \) in \( \mathcal{K} \otimes_{\min} \mathcal{X}_N \) with \( \|\hat{f}\|_{\min} < K'' \) and such that

\[
\| (I_{\mathcal{K}} \otimes \pi_1)(\hat{f}) - f \|_{\min} < 1/2.
\]

By our assumption, we have a natural isomorphism

\[
\varphi_z: \tilde{A}_1 \to \tilde{A}_c
\]

which is the identity on \( \mathcal{A} \), with \( \|\varphi_z\|_{cb} \leq K c^\alpha \), \( z = 1/c \). Let \( f \) be as in our present claim. Then we have

\[
\| (I_{\mathcal{K}} \otimes \varphi_z)(f) \|_{\mathcal{K} \otimes_{\min} \tilde{A}_c} < K c^\alpha.
\]

Hence by Theorem 1.7 there is \( g \) in \( \mathcal{K} \otimes_{\min} OA(E) \) such that

\[
\|g\|_{\min} < K c^\alpha \quad \text{and} \quad (I_{\mathcal{K}} \otimes \pi_1)(g) = (I_{\mathcal{K}} \otimes \varphi_z)(f).
\]
Note that since \( \varphi_z \) is the identity on \( \mathcal{A} \) and \( f \in \mathcal{K} \otimes \mathcal{A} \), we may write \((I_{\mathcal{K}} \otimes \varphi_z)(f) = f\).

We can assume that, for some \( m \), \( g \) is of the form \( g = g_0 + \cdots + g_m \) with \( g_j \in \mathcal{K} \otimes E^{(j)} \).

By Lemma 1.5 we have

\[
(2.8) \quad \forall \ j \leq m \quad \|g_j\|_{\min} \leq \|g\|_{\min} < K c^\alpha.
\]

Now let \( G_j = (I_{\mathcal{K}} \otimes \pi_1)(g_j) \in \mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_1 \). Note that

\[
f = (I_{\mathcal{K}} \otimes \pi_z)(g) = \sum_{0}^{m} z^j G_j,
\]

and \( \|G_j\|_{\min} \leq \|g_j\|_{\min} < K c^\alpha \). Hence we have

\[
\left\| f - \sum_{0}^{N} z^j G_j \right\|_{\min} \leq \sum_{j>N} |z|^j \|G_j\|_{\min} \leq \sum_{j>N} K c^{\alpha - j},
\]

therefore by (2.5), we obtain

\[
\left\| f - \sum_{0}^{N} z^j G_j \right\|_{\min} < 1/2.
\]

Let \( \tilde{f} = \sum_{0}^{N} z^j g_j \), then by (2.8) we have \( \|\tilde{f}\|_{\min} \leq (N + 1) K c^\alpha = K'' \) and (2.6) holds. This proves our claim.

Thus we have proved that the “product map” \( \pi_{1|\mathcal{X}_N} : \mathcal{X}_N \to \tilde{\mathcal{A}}_1 \) is a complete surjection. Now let us show that this implies that \( \Phi(i, c) \leq K_1 e^N \) for all \( c \geq 1 \).

To do that, consider \( u : \mathcal{A} \to B(H) \) unital homomorphism with \( \|ui\|_{cb} \leq c \), let \( \tilde{u} : \tilde{\mathcal{A}}_c \to B(H) \) be the canonical extension of \( u \). Then we have

\[
(2.9) \quad \|\tilde{u}\|_{cb(\tilde{\mathcal{A}}_1, B(H))} \leq K'' \|\tilde{\pi}_{1|\mathcal{X}_N}\|_{cb(\mathcal{X}_N, B(H))}
\]

but, for any \( j \), we have, if \( z = 1/c \)

\[
c^j \tilde{u} \pi_z|_{E_j} = \tilde{u} \pi_{1|E_j}
\]

hence

\[
\|\tilde{u} \pi_{1|E_j}\|_{cb(E_j, B(H))} = \|c^j \tilde{u} \pi_z|_{E_j}\|_{cb(E_j, B(H))} \\
\leq c^j \|\tilde{u}\|_{cb(\tilde{\mathcal{A}}_c, B(H))} \|\pi_z|_{E_j}\|_{cb(E_j, \tilde{\mathcal{A}}_c)}
\]

and by (1.1) and Theorem 1.7

\[
\leq c^j.
\]

By Lemma 1.5 this implies

\[
(2.10) \quad \|\tilde{u} \pi_{1|\mathcal{X}_N}\|_{cb(\mathcal{X}_N, B(H))} \leq \sum_{j=0}^{N} \|\tilde{u} \pi_{1|E_j}\|_{cb(E_j, B(H))} \leq \sum_{j=0}^{N} c^j \leq K'_1 c^N
\]

for some constant \( K'_1 \) independent of \( c \). Hence (2.9) and (2.10) yield

\[
\|\tilde{u}\|_{cb(\tilde{\mathcal{A}}_1, B(H))} \leq K'' K'_1 c^N.
\]
By (2.2) this gives the announced estimate (2.4).

Finally, replacing \( g_0, \ldots, g_{N-1} \) by \( g'_0, \ldots, g'_{N-1} \) according to Lemma 1.6 we can obtain, in the case \( 1_A \in E \), an element \( \hat{f} = \sum_{j=0}^{N-1} z^j g'_j + z^N g_N \) in \( K \otimes E(N) \) such that

\[
\| (I_K \otimes \pi_1)(\hat{f}) - f \|_{\min} < 1/2.
\]

Moreover by (1.12) and (2.8) we have

\[
\| \hat{f} \|_{\min} \leq \sum_{j=0}^{N-1} \| g'_j \|_{\min} + \| g_N \|_{\min} \\
\leq K_2 = NKc^\alpha \| 1_A \|^N + Kc^\alpha.
\]

This justifies the last assertion. \( \square \)

By a simple modification of the preceding proof we obtain:

**Theorem 2.6.** Fix a number \( \alpha > 0 \) and let \( N \) be an integer with \( N \leq \alpha < N + 1 \). Let \( X \subset K \) be a closed subspace for which there is a projection \( P: K \to X \) with \( \|P\|_{cb} = 1 \). Assume that there is a constant \( K \) such that, for any \( f \) in \( X \otimes A \) we have

\[
\forall c \geq 1 \quad \| f \|_{X \otimes_{\min} A_c} \leq Kc^\alpha \| f \|_{X \otimes_{\min} A_1}.
\]

Then the restriction of \( I_X \otimes \pi_1 \) to \( X \otimes_{\min} OA_N(E) \) is a surjection of \( X \otimes_{\min} OA_N(E) \) onto \( X \otimes_{\min} A_1 \), i.e. there is a constant \( K' \) such that, for any \( f \) in \( X \otimes_{\min} A_1 \) with \( \| f \|_{\min} < 1 \), there is \( \hat{f} \) in \( X \otimes_{\min} OA_N(E) \) with \( \| \hat{f} \|_{\min} < K' \) such that \( (I_X \otimes \pi_1)(\hat{f}) = f \). Furthermore, this last property implies that (2.11) actually holds with \( \alpha = N \) for some (possibly different) constant \( K \). Finally, if \( 1_A \) belongs to \( E \), then the restriction of \( I_X \otimes \pi_1 \) defines a surjection from \( X \otimes_{\min} E_N \) onto \( X \otimes_{\min} A_1 \).

**Corollary 2.7.** Consider our usual setting \( (E, A, i) \) and assume that the similarity property holds. Let \( d \) be the infimum of the numbers \( \alpha > 0 \) for which there is a constant \( K \) such that \( \Phi(i, c) \leq Kc^\alpha \) for all \( c \) large enough. Then \( d \) is an integer.

Moreover, there is a constant \( K' \) such that for all \( c \geq 1 \) we have

\[
(2.12) \quad c^d \leq \Phi(i, c) \leq K'c^d.
\]

We will call \( d \) the “similarity degree” of our setting \( (E, A, i) \).

**Proof.** Let \( N \) be the integer such that \( N \leq d < N + 1 \). Then Theorem 2.5 implies \( d \leq N \), hence \( d = N \). Thus \( d \) is an integer. Fix \( \alpha < d \). Then by Lemma 2.3, we have necessarily \( \Phi(i, c) \geq c^\alpha \) for all \( c \geq 1 \). By continuity, this must hold also for \( \alpha = d \), whence the left side of (2.12). Finally, the right side of (2.12) follows from the last part of Theorem 2.5. \( \square \)

**Remark 2.8.** The case \( d = 0 \) is of course trivial, this case happens iff \( A \) is one dimensional. The case \( d = 1 \) also is trivial, although a bit more interesting. By Theorem 2.5, \( d = 1 \) happens only if the operator space \( A_1 \) is completely isomorphic to a quotient space of the direct sum of \( C \) with \( E \). For instance, in the situation of the basic Example 1.1, we have \( d = 1 \) only if \( C^*(G) \) is completely isomorphic to a quotient space of \( \ell_1(\Gamma) \), or equivalently only if \( C^*(G) \) is a max-space, in the sense of [BP1]. By [BP1, Pa5], we know that this can happen only if \( C^*(G) \) is finite dimensional, whence only if (and a posteriori iff) \( G \) is finite.
Remark 2.9. For simplicity, we will identify $E$ with $i(E)$ in this remark, so we view $E$ simply as a subset of $A$. We also view $A$ as a subset of $\tilde{A}_1$. We will moreover assume that $E$ contains the unit. Then, by Theorem 2.5, the degree $d$ (as defined in Corollary 2.7) is equal to the smallest integer $d$ with the property that the natural product map from $E \otimes_h \cdots \otimes_h E$ ($d$ times) to $\tilde{A}_1$ is a complete surjection. By the very definition of the Haagerup tensor product, this last property can be restated as follows: there is a constant $K$ such that for any $n$, any $\varepsilon > 0$ and any $a = (a_{ij})$ in $M_n(A)$ we can find matrices $x_1, \ldots, x_d$ with (say) $x_1 \in M_{q_1q_2}(E)$, $x_2 \in M_{q_2q_3}(E), \ldots, x_d \in M_{q_dq_{d+1}}(E)$ and with $q_1 = n = q_{d+1}$ so that the matricial product $x_1 \cdot x_2 \cdots x_d$ (this is a product in $M_n(A)$, recall that we view $E$ as a subset of $A$) satisfies
\[ \|a - x_1 \cdot x_2 \cdots x_d\|_{M_n(\tilde{A}_1)} < \varepsilon, \]
and finally we have
\[ \prod_{i=1}^{d} \|x_i\|_{M_{q_iq_{i+1}}(E)} \leq K \|a\|_{M_n(\tilde{A}_1)}. \]

In most of the “concrete” examples considered below the space $E$ is a “maximal” operator space in the sense of [BP1]. In that case, we may apply the decomposition described in (0.4) to any rectangular matrix $x$ in $M_{pq}(E)$ (by just adding enough zeros to make it a square matrix). Using this fact, we obtain the following.

Proposition 2.10. Consider a setting $(i, E, A)$. Assume that the operator space $E$ is a maximal operator space (in the sense of [BP1]) and that $i(E)$ contains the unit of $A$. Then the similarity degree $d$ of $(i, E, A)$ is equal to the smallest integer $d$ with the following property: there is a constant $K$ such that for all $n$ any element $x$ in $M_n(A)$ with $\|x\|_{M_n(\tilde{A}_1)} < 1$ can be written as a limit (in the norm of $M_n(\tilde{A}_1)$) of matricial products of the form (again we view $E$ as a subset of $A$)
\[ \alpha_1 D_1 \alpha_2 D_2 \cdots D_d \alpha_{d+1} \]
where $\alpha_1, \ldots, \alpha_{d+1}$ are rectangular scalar matrices, with say $\alpha_i \in M_{p_iq_i}$, $p_1 = n$, $q_{d+1} = n$ and $D_1, \ldots, D_d$ are diagonal matrices with entries in $E$, with $D_i \in M_{q_iq_i}(E)$ (with $q_i = p_{i+1}$) and finally we have
\[ \prod_{i=1}^{d+1} \|\alpha_i\| \prod_{i=1}^{d} \|D_i\| < K. \]
(\text{Note that we can assume if we wish, by adding zero entries, that $q_2 = p_3 = q_3 = \cdots = p_d = q_d = N$ for some $N$ large enough.})

§3. Groups

Let $G$ be a discrete group. In this section, we apply our results in the case
\[ E = A = \ell_1(G), \]
with $i : E \to A$ equal to the identity. We equip $E = \ell_1(G)$ with its “maximal” operator space structure, so that for a map $u : E \to B(H)$ boundedness and complete boundedness are equivalent and
\[ \|u\|_{cb} = \|u\|. \]
Observe that $\mathcal{A} = \ell_1(G)$ is a unital (Banach) algebra for the convolution product. The unit element of $\mathcal{A}$ is $\delta_e$ defined by $\delta_e(t) = 1$ if $t = e$ and 0 otherwise. We have $\|\delta_e\|_{\mathcal{E}} = 1$.

It is classical in this case that $\tilde{\mathcal{A}}_1 = C^*(G)$ the full $C^*$-algebra of $G$. Indeed, any contractive unital homomorphism $u: \ell_1(G) \to B(H)$ induces a norm one representation $\pi: G \to B(H)$ which is automatically a unitary representation. (Indeed $\|\pi(g)\| \leq 1$ and $\|\pi(g)^{-1}\| \leq 1$ implies $\pi(g)$ unitary for any $g$ in $G$.) It also is a classical fact that the dual of $\tilde{\mathcal{A}}_1 = C^*(G)$ can be identified with the space $B(G)$ of all coefficients of the unitary representations of $G$ (cf. [Ey, FTP]). The space $B(G)$ is defined as the space of all functions $\varphi: G \to \mathbb{C}$ for which there is a unitary representation $\pi: G \to B(H_\pi)$ and vectors $\xi, \eta \in H_\pi$ such that

\begin{equation}
\forall t \in G \quad \varphi(t) = \langle \pi(t)\xi, \eta \rangle.
\end{equation}

Moreover one defines

$$
\|\varphi\|_{B(G)} = \inf\{\|\xi\|\|\eta\|\}
$$

where the infimum runs over all possible representations of $\varphi$ as in (3.1). One can imitate this definition for the algebra $\tilde{\mathcal{A}}_c$:

Let us denote by $B_c(G)$ the space of all functions $\varphi: G \to \mathbb{C}$ for which there is a uniformly bounded representation $\pi: G \to B(H_\pi)$ with $|\pi| \leq c$ and vectors $\xi, \eta$ in $H_\pi$ such that (3.1) holds. We then define again

\begin{equation}
\|\varphi\|_{B_c(G)} = \inf\{\|\xi\|\|\eta\|\}
\end{equation}

where the infimum runs over all possible such decompositions of $\varphi$. If $c = 1$, we recover the unitary case so that $B_1(G)$ is identical to $B(G)$. Now consider a function $f$ in $\mathcal{A} = \ell_1(G)$. Clearly we have

\begin{equation}
\|f\|_{\tilde{\mathcal{A}}_c} = \sup \left\{ \left\| \sum_{t \in G} f(t)\varphi(t) \right\| \left| \varphi \in B_c(G) \quad \|\varphi\|_{B_c(G)} \leq 1 \right. \right\}.
\end{equation}

More precisely, we have the following well known fact.

**Proposition 3.1.** *With the natural duality appearing in (3.3) we have

$$
B_c(G) = (\tilde{\mathcal{A}}_c)^*
$$

with equal norms.*

*Proof.* By (3.3) the unit ball of $B_c(G)$ (which is convex) is weak-* dense in the unit ball of $(\tilde{\mathcal{A}}_c)^*$. Hence for any $\varphi$ in the unit ball of $(\tilde{\mathcal{A}}_c)^*$ there is a net $\varphi_i$ in the unit ball of $B_c(G)$ which tends pointwise to $\varphi$. Then, by a standard ultraproduct argument, one can check that $\varphi$ itself is in the unit ball of $B_c(G)$. □

We will also need the space of Herz-Schur multipliers on $G$ which we denote by $M_0(G)$, we refer to [DCH, BF1-2, Bo1, H3, P2] for more information. We recall
that a function \( \varphi: G \to \mathbb{C} \) is in the space \( M_0(G) \) iff there are bounded Hilbert space valued functions \( x: G \to H \) and \( y: G \to H \) such that

\[
\forall \, s, t \in G \quad \varphi(s^{-1}t) = \langle x(t), y(s) \rangle.
\]

Moreover, we denote

\[
\|\varphi\|_{M_0(G)} = \inf \{ \sup_{s \in G} \|x(s)\|_H \cdot \sup_{t \in G} \|y(t)\|_H \}
\]

where the infimum runs over all possible factorizations of \( \varphi \). For the reader’s convenience, we will now reformulate explicitly the meaning of the constants introduced in the previous section. Let \( c \geq 1 \).

Consider a bounded representation \( \pi: G \to B(H) \) with \( |\pi| \leq c \). Assume that \( \pi \) is unitarizable then we denote

\[
\text{Sim}(\pi) = \inf \{ \|S^{-1}\| \, \|S\| \}
\]

where the infimum runs over all invertible operators \( S: H \to H \) such that \( t \to S^{-1}\pi(t)S \) is a unitary representation. Then we set

\[
\Phi_G(c) = \sup \{ \text{Sim}(\pi) \}
\]

where the sup runs over all uniformly bounded representations with \( |\pi| \leq c \). Let \( i_G: E \to \mathcal{A} \) be the setting associated to the identity map of \( \ell_1(G) = E = \mathcal{A} \). By Proposition 1.8 and (2.2)', we know that

\[
(3.5') \quad \Phi_G(c) = \Phi(i_G, c) = \|\tilde{A}_1 \to \tilde{A}_c\|_{cb}.
\]

It will be convenient for our discussion to introduce also

\[
\Psi_G(c) = \sup \{ \|f\|_{B(G)} \mid f \in B_c(G) \, \|f\|_{B_c(G)} \leq 1 \}.
\]

Note that by (2.2)' the inclusion \( \tilde{A}_1 = C^*(G) \hookrightarrow \tilde{A}_c \) has norm \( \leq \Phi_G(c) \), hence we have by Proposition 3.1

\[
(3.6) \quad \Psi_G(c) \leq \Phi_G(c).
\]

Moreover, again by Proposition 3.1

\[
\Psi_G(c) = \|\tilde{A}_1 \to \tilde{A}_c\|.
\]

**Theorem 3.2.** The following properties of a discrete group \( G \) are equivalent:

(i) \( G \) is amenable.

(ii) \( \Phi_G(c) \leq c^2 \) and \( \Psi_G(c) \leq c^2 \) for all \( c > 1 \).

(iii) There is \( \alpha < 3 \) and a constant \( K \) such that for all \( c \geq 1 \)

\[
\Phi_G(c) \leq K c^\alpha.
\]

(iv) \( \Phi_G(c) < c^3 \) for some \( c > 1 \) and \( \Phi_G(c) < \infty \) for all \( c > 1 \).

(v) \( \exists \alpha < 3 \, \exists K > 0 \) such that for all \( c \geq 1 \)

\[
\Psi_G(c) \leq K c^\alpha.
\]
Proof. (i) ⇒ (ii) is Dixmier’s classical result [Di]. (ii) ⇒ (iii) ⇒ (iv) are trivial by (3.6). Moreover, (iv) ⇒ (iii) follows from Lemmas 2.2 and 2.3 and (3.5)’. In addition, (iii) ⇒ (v) follows from (3.6).
Hence it remains only to prove (v) ⇒ (i). Assume (v). By Theorem 2.6 with X = C, the restriction of π₁ to E₂ is a surjection from E₂ onto A₁ = C∗(G). Equivalently, this means that the adjoint map w₂: B(G) → (E₂)∗ is an isomorphic embedding, so that for some δ > 0 we have

(3.7) \[ \forall \varphi \in B(G) \quad \delta \| \varphi \|_{B(G)} \leq \| w₂(\varphi) \|_{E₂^*}. \]

Now assume φ finitely supported. We have

\[ \| w₂(\varphi) \|_{E₂^*} = \sup \left\{ \left\| \sum_{s,t \in G} \varphi(st)\alpha(s,t) \right\| \right\} \]

where the supremum runs over all α = \[ \sum_{s,t \in G} \alpha(s,t)\delta_s \cdot \delta_t \] in the unit ball of E₂. By Proposition 1.10, the space E₂ can be naturally identified with \[ \ell₁(G) \otimes_h \ell₁(G), \] so that for α as above

\[ \| \alpha \|_{E₂} = \left\| \sum_{s,t \in G} \alpha(s,t)\delta_s \otimes \delta_t \right\| \]

\[ \ell₁(G) \otimes_h \ell₁(G) \]

Hence we find

(3.8) \[ \| w₂(\varphi) \|_{E₂^*} = \left\| \sum_{s,t \in G} \varphi(st)e_s \otimes e_t \right\| \]

\[ (\ell₁(G) \otimes_h \ell₁(G))^* \]

where \[ e_t \in \ell₁(G)^* = \ell_{∞}(G) \] is biorthogonal to \[ \delta_t, \] i.e. \[ e_t(\delta_s) = 1 \] if \[ t = s \] and 0 otherwise. But now it is well known (cf. [DCH, or P1]) that the right side of (3.8) is equal to \[ \| \varphi \|_{M₀(G)}. \] Hence we deduce from (3.7) that for all finitely supported φ: \[ G \to C \]
\[ \delta \| \varphi \|_{B(G)} \leq \| \varphi \|_{M₀(G)}. \]

By a result due to Bożejko [Bo2], this implies that G is amenable, whence (i). □

**Corollary 3.3.** If G is not amenable, then for any \( c > 1 \) there is a representation \( π_c: G \to B(H_c) \) with \( \| π_c \| \leq c \) such that \( c^3 \leq \inf \{ \| S \| \| S^{-1} \| \} \), where the infimum runs over all the similarities \( S \) such that \( S^{-1}π_c(\cdot)S \) is unitarizable.

**Proof.** By the preceding statement, we know that \( Φ_G(c) \geq c^3 \) for all \( c > 1 \). If \( Φ_G(c) = \infty \), we clearly have the conclusion. Otherwise \( A₁ \) and \( A_c \) are isomorphic. Then we represent \( A_c \) as a subalgebra of some \( B(H) \), say \( A_c \subset B(H_c) \) and we define \( π_c \) to be the representation on \( G \) associated to the restriction to \( E = \ell₁(G) \) of the canonical morphism \( \tilde{A}_1 \to \tilde{A}_c \). By (2.2)” we have \( \| A₁ \to A_c \|_{cb} \geq c^3 \), hence by Proposition 1.8, this representation has the desired property. □

The next result recapitulates what we know from §2.
Theorem 3.4. Assume that every uniformly bounded group representation on $G$ is unitarizable.

(i) Then the function $\Phi_G(c)$ defined in (3.5) is finite for all $c \geq 1$. Moreover, let $d(G)$ be the smallest $\alpha > 0$ such that $\Phi_G(c) \in O(c^{\alpha})$ when $\alpha \to \infty$. Then $d(G)$ is an integer. We call it the similarity degree of $G$.

(ii) We have $\Phi_G(c) \leq c^{d(G)}$ for all $c \geq 1$ and $d(G)$ is the largest integer with this property.

(iii) The degree $d(G)$ is the smallest integer $N$ such that the natural “product” mapping

$$W_N: \ell_1(G) \otimes_h \cdots \otimes_h \ell_1(G) \rightarrow C^*(G)$$

which takes $\delta_{t_1} \otimes \cdots \otimes \delta_{t_N}$ to $\delta_{t_1 t_2 \cdots t_N}$ is a complete surjection onto $C^*(G)$.

Proof. The first part follows from Theorem 2.5 and especially from (2.4). By Lemma 2.3, if $\Phi_G(c) < c^\alpha$ for some $\alpha < d(G)$ then $\Phi(c) \in O(c^\alpha)$. Therefore we must have $\Phi_G(c) \leq c^\alpha$ for all $c \geq 1$ and $\alpha < d(G)$, whence the second part.

Finally, the third part follows from Theorem 2.5 and Proposition 1.10, which tell us that $\Phi_G(c) \in O(c^N)$ when $c \to \infty$ iff $W_N$ is a complete surjection. \qed

Remark 3.5. With the preceding notation, Theorem 3.2 says that $d(G) \leq 2$ iff $G$ is amenable.

Remark 3.6. We now return to the Example 1.1. Let $G$ be a discrete group. Let $A$ be the group algebra of $G$, i.e. $A = \ell_1(G)$ equipped with convolution. Let $\Gamma \subset G$ be a set of generators for $G$ and let $E = \ell_1(\Gamma)$, equipped again with its natural (=maximal) operator space structure. Here again, we have $\hat{A}_1 = C^*(G)$, but the similarity degree now depends very much on the choice of the generators. Let us denote by $d = d(\Gamma, G)$ the similarity degree for this setting, according to Corollary 2.7. Then, by Theorem 2.5, the product map

$$p: (t_1, \ldots, t_d) \in \Gamma^d \rightarrow t_1 \cdots t_d \in G$$

extends to a (complete) surjection from $\ell_1(\Gamma) \otimes_h \cdots \otimes_h \ell_1(\Gamma)$ onto $C^*(G)$. Let us denote by $[\Gamma]^d \subset G$ the image of $\Gamma^d$ under $p$. Then, the elements supported by $[\Gamma]^d$ must be dense in $C^*(G)$, and a fortiori, say, in $\ell_2(G)$. This clearly implies (denoting by $e$ the unit element of $G$) $\{e\} \cup \cup_{j \leq d} [\Gamma]^j = G$. Therefore, every element of $G$ can be written as a product of at most $d$ elements of $\Gamma$. If we introduce the usual distance on $G$ relative to (the Cayley graph of) $\Gamma$, this means that the diameter of $G$ is at most $d$. This remark allows to produce examples of similarity settings with arbitrarily large finite similarity degrees. Indeed, just consider $G = \mathbb{Z}^N$, for some integer $N$, and take for $\Gamma$ the subset formed of all elements with only one non zero coordinate. Clearly, by the preceding remarks we have $N \leq d$ in this case.

In the converse direction, we claim that $d \leq 2N$. Indeed, let $\pi: G \rightarrow B(H)$ be a representation such that $\sup_{t \in \Gamma} \|\pi(t)\| \leq c$. Then clearly $\sup_{t \in G} \|\pi(t)\| \leq c^N$. Now, since $G$ is amenable, this implies by Dixmier’s theorem that $\Sim(\pi) \leq c^{2N}$. Hence, we have shown that $N \leq d \leq 2N$.

More precisely, consider in $C^\ast(\mathbb{Z}^N) \simeq C^\ast(\mathbb{Z}) \otimes C^\ast(\mathbb{Z}) \otimes \cdots \otimes C^\ast(\mathbb{Z})$ the subspace $E = C_1 + \cdots + C_N$ with $C_i = 1 \otimes \cdots \otimes 1 \otimes C^\ast(\mathbb{Z}) \otimes 1 \otimes \cdots \otimes 1$ where $C^\ast(\mathbb{Z})$ appears at the $i$-th place. We equip $E$ again with its maximal operator space structure and
we let $\mathcal{A}$ be the algebra generated by $E$ in $C^*(\mathbb{Z}^N)$. We will show that the degree of the “similarity setting” constituted of the inclusion $E \subset \mathcal{A} \subset C^*(\mathbb{Z}^N)$ is equal to $2N$.

Let $\pi: C^*(\mathbb{Z}^N) \to B(H)$ be a unital homomorphism such that $\Vert \pi|_E \Vert \leq c$. Then clearly $\sup_{t \in \mathbb{Z}^N} \Vert \pi(t) \Vert \leq c^N$, hence by Dixmier’s Theorem $\Vert \pi \Vert_{cb} \leq c^{2N}$. On the other hand, we clearly have $\int_{C} \Lambda$ integral on $C(T^N) = C^*(\mathbb{Z}^N)$, then $\sum_{i=1}^{N} \tau(x_i) \leq 1$ and $\sum_{i=1}^{N}(x_i - \tau(x_i)) \leq 2$, which implies (using conditional expectations) $\Vert x_i - \tau(x_i) \Vert \leq 2$, hence

$$
\Vert \pi(x) \Vert = \left\Vert \sum_{i=1}^{N} 1 \otimes u(x_i) \otimes 1 \right\Vert \\
\quad \leq \left\Vert \left( \sum_{i=1}^{N} \tau(x_i) \right) 1 \otimes \cdots \otimes 1 \right\Vert + \left\Vert \sum_{i=1}^{N} 1 \otimes \cdots (u(x_i) - \tau(x_i)1) \otimes \cdots 1 \right\Vert \\
\quad \leq 1 + \sum_{i=1}^{N} \Vert u(x_i - \tau(x_i))1 \Vert \leq 1 + Nc \Vert x_i - \tau(x_i)1 \Vert \\
\quad \leq 1 + 2Nc.
$$

On the other hand, we clearly have $\Vert \pi \Vert_{cb} \geq (c^2)^N = c^{2N}$, hence this proves that the degree $d$ of this similarity setting is exactly equal to $2N$.

Remark. In the setting described in Example 1.1, let $G$ be a discrete amenable group, so that $\hat{A}_1 = C^*(G) = C^\infty(G)$. We claim that the smallest constant $K$ appearing in Proposition 2.10 (with $d = 2$) is actually equal to 1.

Indeed, consider $x$ in $M_n(C^\infty(G))$, with $\Vert x \Vert < 1$. Since $G$ is amenable, its Fourier algebra $A(G)$ has an approximate unit in its unit ball. Hence, we may assume (by density) that $x$ is of the following form $x = \sum_{t \in G} y(t) \otimes \lambda(t) \varphi(t)$ with $y(t) \in M_n$ and $y = \sum_{t \in G} y(t) \otimes \lambda(t)$ such that $\Vert y \Vert_{M_n(C^\infty(G))} < 1$ and $\varphi$ of the form $\varphi(t) = \langle \lambda(t) \xi, \eta \rangle$ with $\Vert \xi \Vert \Vert \eta \Vert < 1$ where $\xi(\cdot), \eta(\cdot), y(\cdot)$ and $\varphi(\cdot)$ are all finitely supported.

Equivalently we have

$$
(3.9) \quad x_{ij} = \sum_{s, \theta} y_{ij}(s\theta^{-1})\lambda(s)\lambda(\theta^{-1})\overline{\eta(s)}\xi(\theta).
$$

Then a simple computation shows that we can write

$$
(3.10) \quad x = A_1 D_1 A_2 D_2 A_3
$$

where $A_1, A_2, A_3$ are rectangular scalar matrices and where $D_1, D_2$ are diagonal with entries in $A$ of the form $\lambda(t)$ for some $t$ in $G$. Moreover, we have

$$
(3.11) \quad ||A_1|| ||A_2|| ||A_3|| < 1.
$$
Explicitly, we can take
\[
A_1(i, (s, k)) = \eta(s) \delta_{ik}
\]
\[
A_2((s, k), (\theta, \ell)) = y_{k\ell}(s\theta^{-1})
\]
\[
A_3((\theta, \ell), j) = \xi(\theta) \delta_{\ell j}
\]
and the diagonal matrices defined by
\[
D_1((s, k), (s, k)) = \lambda(s)
\]
\[
D_2((\theta, \ell), (\theta, \ell)) = \lambda(\theta^{-1}).
\]
Note that we can restrict the sums in (3.9) to be over the finite subsets of \(G\) where \(\xi\) and \(\eta\) are supported, so that we indeed obtain finite matrices in (3.10), and (3.11) is easy to check. Thus the decomposition (3.10) clearly implies our claim that Proposition 2.10 holds with \(d = 2\) and \(K = 1\).

§4. OPERATOR ALGEBRAS

We now come to the main application of our results. Let \(A\) be a unital operator algebra. Let
\[
E = \max(A)
\]
in the sense of [BP1] (see (0.4) above). The operator space \(E\) is equal to \(A\) as a Banach space, but its operator space structure is characterized by the property that, for any linear map \(u: E \to B(H)\), we have
\[
\|u\| = \|u\|_{cb}.
\]
Here, of course we take \(A = A\), and we let \(i: E \to A\) be the identity of \(A\). Of course, we have \(\tilde{A}_1 = A\) isometrically (but perhaps not completely so).

Then we denote by \(d(A)\) the similarity degree of this setting \((i, E, A)\).

Note that, by definition, \(d(A) \leq \alpha\) iff there is a constant \(K\) such that, for any bounded unital homomorphism \(u: A \to B(H)\), there is an invertible \(S\) for which \(a \to S^{-1}u(a)S\) is contractive and such that \(\|S\|\|S^{-1}\| \leq K\|u\|^{\alpha}\).

It is easy to check that for any closed two sided ideal \(I \subset A\), the quotient space \(A/I\) (which, by [BRS], is an operator algebra) satisfies
\[
d(A/I) \leq d(A).
\]
Moreover, if \(B\) is another unital operator algebra and if \(A \oplus B\) denotes the direct sum (equipped with the norm \(\|(x, y)\| = \max\{\|x\|, \|y\|\}\) and the obvious “block diagonal” operator algebra structure) then we have
\[
d(A \oplus B) \leq \max\{d(A), d(B)\}.
\]
Now assume that every unital contractive homomorphism \(u: A \to B(H)\) is completely bounded. Then clearly \(A \simeq \tilde{A}_1\) completely isomorphically, and there
is a constant $K$ such that $\|u\|_{cb} \leq K$ for all unital contractive homomorphisms $u: A \to B(H)$. This implies that, if we define
\[
\Phi_A(c) = \sup\{\|u\|_{cb}\}
\]
where the supremum runs over all unital homomorphisms $u: A \to B(H)$ with $\|u\| \leq c$, then in the present setting we have
\[
(4.2) \quad \forall c \geq 1 \quad \Phi(i, c) \leq \Phi_A(c) \leq K \Phi(i, c).
\]
Thus, to recapitulate, we obtain the following two statements (note that the equivalence between (a) and (b) below is due to Paulsen [Pa4]).

**Theorem 4.1.** Let $A$ be any unital operator algebra, then the following are equivalent:

(a) Every bounded unital homomorphism $u: A \to B(H)$ is similar to a completely contractive one.

(b) Every bounded unital homomorphism $u: A \to B(H)$ is completely bounded.

(c) There is an integer $d \geq 0$ such that, for some constant $K$, any unital homomorphism $u: A \to B(H)$ satisfies $\|u\|_{cb} \leq K \|u\|^d$.

*Proof.* For the equivalence between (a) and (b) (due to Paulsen [Pa4]), see the above Proposition 1.8. If (a) or (b) holds, then in the present setting we have $A \simeq \tilde{A}_1$ completely isomorphically and (4.2) holds. Thus (b) $\Rightarrow$ (c) follows from Corollary 2.4, and the converse is obvious. □

**Theorem 4.2.** For any fixed integer $d \geq 0$, the following properties of a unital operator algebra $A$ are equivalent:

(i) There is a constant $K$ such that any unital homomorphism $u: A \to B(H)$ satisfies $\|u\|_{cb} \leq K \|u\|^d$.

(ii) There is a number $\alpha$ with $d \leq \alpha \leq d + 1$ for which there exists a constant $K$ such that any unital homomorphism $u: A \to B(H)$ satisfies $\|u\|_{cb} \leq K \|u\|^\alpha$.

(iii) The natural product mapping $T_d$ from $\text{max}(A) \otimes_h \cdots \otimes_h \text{max}(A)$ ($d$ times) onto $A$ is a complete quotient map, i.e. it induces a complete isomorphism from the quotient space $\text{max}(A) \otimes_h \cdots \otimes_h \text{max}(A)/\ker(T_d)$ onto $A$.

(iv) There is a constant $K$ such that any bounded linear map $u: A \to B(H)$ satisfies $\|u\|_{cb} \leq K \|uT_d\|_{cb}$.

(v) There is a constant $K$ such that the following holds: assume that a linear map $u: A \to B(H)$ is such that there are bounded linear maps $v_i: A \to B(H)$ such that $\forall x_1, \ldots, x_d \in A$
\[
u(x_1 x_2 \ldots x_d) = v_1(x_1) v_2(x_2) \ldots v_d(x_d),
\]
then we have
\[
\|u\|_{cb} \leq K \prod_{i=1}^{d} \|v_i\|.
\]

(vi) There is a constant $K$ such that the following holds: for all $n$, any element $x$ in $M_n(A)$ with $\|x\|_{M_n(A)} < 1$ can be written, for some integer $N$, as a matricial product of the form
\[
x = \alpha_0 D_1 \alpha_2 D_2 \ldots D_d \alpha_d
\]
where $\alpha_0 \in M_{nN}$, $\alpha_1 \in M_N$, $\ldots$, $\alpha_{d-1} \in M_N$, $\alpha_d \in M_{Nn}$ are scalar matrices (i.e. $\alpha_0$ and $\alpha_d$ are rectangular of size $n \times N$ and $N \times n$, and the others are square matrices of size $N \times N$), and $D_1, \ldots, D_d$ are $N \times N$ diagonal matrices with entries in $A$, and finally we have

$$\prod_{i=0}^{d} \|\alpha_i\| \prod_{i=1}^{d} \|D_i\| < K.$$  

(vii) There is a constant $K$ such that any $x$ in the unit ball of $K \otimes_{\min} A$ can be written as a product of the form (recall that $C_0 \subset K$ denotes the subspace of diagonal operators)

$$x = \alpha_0 D_1 \alpha_1 D_2 \ldots D_d \alpha_d$$

with $\alpha_i \in K \otimes 1$ and $D_i \in C_0 \otimes_{\min} A$ such that

$$\prod_{i=0}^{d} \|\alpha_i\| \prod_{i=1}^{d} \|D_i\| \leq K.$$

Proof. (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (iii) follows from Theorem 2.5 and (iv) is merely a restatement of (iii). Similarly (iv) $\Rightarrow$ (v) is clear by the known factorization property of c.b. multilinear maps ([CS1-2, PaS]) and by (4.1). Furthermore, if (v) holds, then every bounded unital homomorphism $u : A \to B(H)$ satisfies

$$\|u\|_{cb} \leq K \|u\|^d,$$

whence (i). This proves the equivalence of (i)-(v). The equivalence between (vi) and (vii) can be checked by a standard argument left to the reader.

Finally, note that (vi) $\Rightarrow$ (v) is obvious (with the same constant $K'$). It remains to prove the converse, so assume (v), then Proposition 2.10 is of course valid in the present setting ($E = \max(A)$, and $A = A$) with degree $d$. Note that $\|x\|_A = \|x\|_{\tilde{A}_1}$ for any $x$ in $A$, hence for any $a$ in $M_n(A) = M_n(\tilde{A}_1)$, we have obviously

$$\|a\|_{M_n(\max(A))} \leq n^2 \|a\|_{M_n(\tilde{A}_1)}.$$  

This implies, by (0.4), that any $a$ with $\|a\|_{M_n(\tilde{A}_1)} < n^{-2}$can be written as $a = \alpha D \beta$ as in (0.4). Thus, by Proposition 2.10, any $x$ with $\|x\|_{M_n(A)} < 1$ can be written as a sum $x = a + y$ with $y = \alpha_1 D_1 \alpha_2 D_2 \ldots D_d \alpha_{d+1}$ having factors $\alpha_1, D_1, \alpha_2, D_2, \ldots, D_d, \alpha_{d+1}$ as in (vi), and with $a = \alpha D \beta$ as above. Now, by adding redundant factors equal to the unit, we can assume that $a$ is of the same form as $y$, say $a = \alpha_1' \delta_1 D_1 \alpha_2' D_2 \ldots \delta_d D_d \alpha_{d+1}$, and then changing $N$ to $2N$ (and $K$ to $K+1$), it is easy to rewrite the sum $x = a + y$ as a single product as in (vi). This shows that (v) implies (vi) and concludes the proof.  

Corollary 4.3. Assume (4.2). Then the similarity degree $d(A)$ (defined as $d(A) = d(i, E, A)$ with $E = \max(A)$ and $i = I_A$) is equal to the smallest integer $d$ satisfying the equivalent conditions (i)-(vi) in Theorem 4.2. Moreover, when this integer is
finite, for any $c > 1$, there is a unital homomorphism $u_c: A \to B(H)$ with $\|u_c\| \leq c$ such that

$$\|u_c\|_{cb} \geq c^{d(A)}.$$

**Proof.** The first assertion is clear. The second one follows from (2.2)', (2.12) and the obvious fact that, for any unital operator algebra $A$, in the present setting the natural inclusion of $\tilde{A}_1$ into $A$ is completely contractive. □

**Remark 4.4.** As in Remark 2.8, $d(A) = 1$ iff $\tilde{A}_1 = \max(A)$ completely isomorphically. In contrast with the group case (see Remark 2.8), there are infinite dimensional examples when this happens. Indeed, consider a closed infinite dimensional subspace $E \subset B(H)$, which is a maximal operator space, i.e. such that $E = \max(E)$. Then consider (as in [Pa5]) the subalgebra $A_E \subset B(H \oplus H)$ formed of all elements of the form $\begin{pmatrix} \lambda I & x \\ 0 & \lambda I \end{pmatrix}$ with $x \in E$ and $\lambda \in \mathbb{C}$. Clearly, $A_E \simeq \mathbb{C} \oplus E$, hence $A_E \simeq \max(A_E)$ completely isomorphically, and (4.2) holds in this case, so this gives an example with degree 1, i.e. we have $d(A_E) = 1$. For examples with degree equal to 2 and 3, see §6 below. We conjecture that the value of $d(A)$ can be any integer, but, at the time of this writing, we do not have any example with $3 < d(A) < \infty$.

**Remark 4.5.** In the present setting $(i, E, A)$ as defined in the beginning of §4, the similarity property holds iff we have:

(SP) Every bounded unital homomorphism $u: A \to B(H)$ is similar to a contractive one.

Clearly, by §2, this holds iff in this setting the degree is finite. Up to now, in this section, we have concentrated on algebras $A$ which satisfy (4.2). Nevertheless, the above property (SP) could be of interest even if the right side of (4.2) fails. Note however that, if we replace $A$ by $\tilde{A}_1$, then we return to the situation discussed in Theorem 4.2. More precisely, the setting being still the same as throughout this section, we have

$$\forall c \geq 1 \quad \Phi_{\tilde{A}_1}(c) = \Phi(i, c),$$

and $A$ satisfies (SP) iff every bounded unital homomorphism $u: \tilde{A}_1 \to B(H)$ is c.b. Note that here $A$ and $\tilde{A}_1$ are isometric, but perhaps not completely isomorphic.

**Remark 4.6.** Fix $n \geq 1$. Let $U$ be the unitary group in $M_n$ with normalized Haar measure $m$. It is not hard to show that the mapping

$$x \to \int_U x u \otimes u^* m(du)$$

is completely contractive from $M_n$ to $\max(M_n) \otimes_h \max(M_n)$.

Thus, in the case $A = M_n$, the surjection appearing in Theorem 4.2 (iii) (with $d = 2$ here) actually admits a completely contractive lifting. Consequently, when $A = \mathcal{K}$ and $d = 2$, the constant $K$ appearing in (iii),(iv) (v) or (vi) in Theorem 4.2 is actually equal to 1. Probably a more general result holds in the context of “normal virtual diagonals” in the sense of [E].
Remark. It is probably possible to develop the theory of the “similarity degree” in the category of dual operator algebras, replacing the Haagerup tensor product by the dual variant considered in [BS] and restricting attention to weak*-continuous homomorphisms, but we have not pursued this yet. (Note added may 97: this program has now been successfully carried out by C. Le Merdy.)

Remark 4.7. Recently, Kirchberg [Ki] showed that a unital C*-algebra $A$ has the similarity property (in other words $d(A) < \infty$) iff every derivation $\delta: A \to B(H)$, relative to an arbitrary $*$-representation $\pi: A \to B(H)$ (we will call such a derivation a $\pi$-derivation) is inner. Equivalently, we have $d(A) < \infty$ iff there is a constant $K$ such that any such derivation $\delta$ satisfies

\begin{equation}
||\delta||_{cb} \leq K||\delta||.
\end{equation}

More precisely, a simple adaptation of Kirchberg’s argument shows that (4.3) implies

\begin{equation}
\|f(z)\| \leq c \text{ if } \text{Re}(z) = 1 \text{ and } \|f(z)\| \leq 1 \text{ if } \text{Re}(z) = 0. \quad (4.4)
\end{equation}

Here is a brief sketch: we follow the presentation of Kirchberg’s argument in [P1, p. 129]. Let $\pi: A \to B(H)$ be a unital $*$-representation and let $S: H \to H$ be self-adjoint and invertible. Let $\pi_S(x) = S^{-1} \pi(x) S$ and let $\delta(x) = \log(S) \pi(x) - \pi(x) \log(S)$ $(x \in A)$. We assume $\|\pi_S\| = c$ and $\|\pi_S||_{cb} = ||S^{-1}|| S||$. Fix a unitary in $A$. Consider the entire function $f(z) = S^z \pi(a) S^{-z}$. We have $\|f(z)\| \leq c$ if $\text{Re}(z) = 1$ and $\|f(z)\| \leq 1$ if $\text{Re}(z) = 0$. Hence by log-subharmonicity, we have $\|f(z)\| \leq c^{\theta}$ if $\text{Re}(z) = \theta$, $0 < \theta < 1$. Since $f(0) = \pi(a)$ is unitary, we have $\|f(\theta)\| = 1 + \theta f'(0) \pi(a)^{-1} + o(\theta) \leq 1 + \theta \log(c) + o(\theta)$ when $\theta > 0$ tends to zero. Therefore the Hermitian operator $T = f'(0) \pi(a)^{-1} = \log(S) - \pi(a) \log(S) \pi(a)^{-1}$ satisfies, for any $h$ in the unit sphere of $H$

$$\langle Th, h \rangle \leq \log c.$$ 

Applying this last estimate with $a$ replaced by its inverse, we obtain for any $h$ in the unit sphere of $H$

$$-\langle Th, h \rangle \leq \log c.$$

Consequently $\|T\| \leq \log(c)$. Hence, we find $\|\delta(a)\| = \|f'(0)\| = \|T\| \leq \log(c)$, so that $\|\delta\| \leq \log(c)$, whence by (4.3) $||\delta||_{cb} \leq K||\delta|| \leq K \log(c)$. By following Kirchberg’s argument as presented in [P1, p. 130], we then conclude that $\log(\|\pi_S\|_{cb}) \leq ||\delta||_{cb} \leq K \log(c)$, hence finally $\|\pi_S\|_{cb} \leq \|\pi_S\|^K$.

Let $A$ be a unital operator algebra and let $K(A)$ be the smallest constant $K$ such that $||\delta||_{cb} \leq K||\delta||$ for any completely contractive unital homomorphism $\pi: A \to B(H)$ and any $\pi$-derivation $\delta: A \to B(H)$.

Curiously, in almost all of E. Christensen’s works in the $C^*$-case, the upper estimates which appear for $K(A)$, are all natural integers (cf. [C1-4]). On the other hand, note that if $A$ satisfies (iv)-(vi) in Theorem 4.2, then we have $K(A) \leq Kd$. This suggests various questions which we could not answer (we ask this for $C^*$-algebras only, but the questions make sense in general):

**Problem 4.8.** Is any of the best constants $K$ appearing in the conditions (i), or (iv)-(vi) (from Theorem 4.2) automatically equal to 1?

**Problem 4.9.** Is $K(A)$ always an integer when it is finite?
A uniform algebra is a closed unital subalgebra $A$ of a commutative unital $C^*$-algebra $C$, such that $A$ generates $C$ as a $C^*$-algebra. Equivalently, we can view $A$ as a unital subalgebra of the algebra $C(T)$ of all continuous complex functions on some compact set $T$, which separates the points of $T$. We say that $A$ is proper if $A \neq C(T)$. A typical example is the disc algebra $A(D)$ formed of all continuous complex valued functions on $\partial D$ which extend continuously and analytically inside $D$. Equivalently, $A(D) \subset C(\partial D)$ can be viewed as the closure in $C(\partial D)$ of the space of all polynomials.

Recently, we produced the first example of a bounded unital homomorphism on $A(D)$ which is not c.b. (cf. [P5]). It is possible that every proper uniform algebra admits such homomorphisms and has infinite degree (note that the extension to other domains of $C^n$ such as the polydisc or the ball is trivial). However, at the time of this writing, the only general result we have in this direction is the following one.

**Theorem 5.1.** Let $A$ be a uniform algebra, such that any contractive unital homomorphism from $A$ to $B(H)$ is c.b. Then $d(A) \geq 3$ if $A$ is proper. In other words, $d(A) \leq 2$ iff $A$ is a commutative $C^*$-algebra.

**Remark 5.2.** Equivalently (by [Sh]), $d(A) \leq 2$ iff $A$ is an amenable Banach algebra in the sense of e.g. [Pi2]. Compare with Remark 3.5. Note that there seem to be no known example of an amenable operator algebra which is not a $C^*$-algebra (see [CL]).

The proof is based on the following two results. To state the first one, it will be convenient to work with a slightly unconventional version of the space $H^\infty$, which we now introduce. Let $T = \partial D$. Consider $\Omega = T^I$ with $I = \{1, 2, 3, \ldots\}$. Let $(z_i)_{i \geq 1}$ denote the coordinates on $\Omega$ and let $A_n$ be the $\sigma$-algebra generated by $(z_1, \ldots, z_n)$ with $A_0$ the trivial $\sigma$-algebra. Let $m$ be the usual probability measure on $T^I$ (= normalized Haar measure). Every $m$-integrable function $f$: $\Omega \to \mathbb{C}$ defines a martingale $(f_n)_n$ by setting $f_n = E(f \mid A_n)$. A martingale $(f_n)_n$, relative to the filtration $(A_n)$, is called “Hardy” if for each $n \geq 1$ the function $f_n$ depends analytically on $z_n$ (but arbitrarily on $z_1, \ldots, z_{n-1}$). We denote by $H^\infty_m$ the subspace of $L^\infty(\Omega, m)$ formed by all $f$ which generate a Hardy martingale.

In Harmonic Analysis terms, the space $H^\infty_m$ is indeed the version of $H^\infty$ associated to the ordered group $\mathbb{Z}^I$ (formed of all the finitely supported families $n = (n_i)_{i \in I}$ with $n_i \in \mathbb{Z}$), ordered by the lexicographic order, i.e. the order defined by setting $n' < n''$ iff the last differing coordinate (="letter" with reversed alphabetical order) satisfies $n'_i < n''_i$. As explained e.g. in [Ru, Chapter 8], this group has a “linear” behaviour and the associated $H^p$ spaces on it behave like the classical (unidimensional) ones. More generally, for any Banach space $X$, we will denote by $H^p_m(X)$ ($1 \leq p \leq \infty$) the usual $H^p$-space of $X$-valued functions on the group $\Omega$ (with ordered dual $\mathbb{Z}^I$), in Bochner’s sense.

**Lemma 5.3.** Let $I$ be any set, and let $X = (\ell_1(I) \otimes_h \ell_1(I))^*$. Then there is a constant $C$ such that for any Hardy martingale $(f_n)$ in $H^\infty_m(X)$ we have

$$\sup_N \left\| \sum_{n=1}^{N} \int \bar{z}_n df_n \, dm \otimes e_n \right\|_{X^{\otimes_{\min,\max}(\ell_2)}} \leq C \|f\|_{H^\infty_m(X)}.$$
Proof. We follow [P2, §4]. First observe that it suffices to prove this when $I$ is a finite set. Indeed, if we know (5.1) for all finite sets then we can obtain it for an arbitrary set $I$ by taking the supremum of each side over all finite subsets $I \subset \mathcal{I}$.

Let us assume that $I$ is finite, so that $X = \ell_\infty(I) \otimes_h \ell_\infty(I)$.

It clearly suffices to prove that for all functions $f$ in the open unit ball of $H_\infty^m(X)$ we have

$$
\left\| \sum_0^N \int \bar{z}_n f_n dm \otimes e_n \right\|_{X \otimes_{\min \max} \ell_2} \leq C
$$

for some absolute constant $C$ independent of $N$ and $f$. Let $f \in H_\infty^m(X)$ with $\|f\|_{H_\infty^m(X)} < 1$. By invoking [P2, Theorem 4.2 and Remark 4.4], it follows that we can write for $n = 0, \ldots, N$

$$
(5.2) \quad \int \bar{z}_n f_n dm = \int \bar{z}_n f dm = A_n + B_n
$$

with $A_n, B_n \in \ell_\infty(I) \otimes_h \ell_\infty(I)$ such that

$$
(5.3) \quad \left\| \sum_0^N A_n \otimes e_n \right\|_{\ell_\infty(I, \ell_2) \otimes \ell_\infty(I)} + \left\| \sum_0^N B_n \otimes e_n \right\|_{\ell_\infty(I) \otimes \ell_\infty(I, \ell_2)} \leq C
$$

for some absolute constant $C$. Here we denoted by $\ell_\infty(I, \ell_2)$ the Banach space of all bounded $\ell_2$-valued functions on $I$ equipped with its natural norm. In addition, we denoted by $\otimes$ the projective tensor product and we made the obvious identifications of $\ell_\infty(I) \otimes \ell_\infty(I, \ell_2)$ with a subset respectively of $\ell_\infty(I, \ell_2) \otimes \ell_\infty(I)$ and $\ell_\infty(I) \otimes \ell_\infty(I, \ell_2)$. By a simple argument, one can check that

$$
\left\| \sum_0^N A_n \otimes e_n \right\|_{X \otimes_{\min \max} \ell_2} \leq \left\| \sum_0^N A_n \otimes e_n \right\|_{\ell_\infty(I, \ell_2) \otimes \ell_\infty(I)}
$$

and similarly

$$
\left\| \sum_0^N B_n \otimes e_n \right\|_{X \otimes_{\min \max} \ell_2} \leq \left\| \sum_0^N B_n \otimes e_n \right\|_{\ell_\infty(I) \otimes \ell_\infty(I, \ell_2)}.
$$

Therefore, we conclude from (5.2) and (5.3) that

$$
\left\| \sum_0^N \int \bar{z}_n f_n dm \otimes e_n \right\|_{X \otimes_{\min \max} \ell_2} \leq C.
$$

□

Remark. It is possible to complete the proof without appealing to the projective tensor product, remaining in the framework of the Haagerup tensor product, but this option would unnecessarily lengthen the argument.

In [Kis], S. Kisliakov proved the remarkable fact that, if $A \subset C(T)$ is any proper uniform algebra, there is no bounded linear projection from $C(T)$ onto $A$. In [Ga], Garling extended Kisliakov’s result. In particular, the following result is implicit in Garling’s paper, but is proved there (see the proof of Theorem 2 in [Ga]).
**Lemma 5.4.** Let $A$ be any proper uniform algebra. Then for some $\beta > 0$ there is, for each integer $n$, a Hardy martingale $f_1, \ldots, f_n$ with values in the unit ball of $A^*$ but such that

$$\left\| \int \tilde{z}_k df_k \right\|_{A^*} \geq \beta \quad \text{for} \quad k = 1, 2, \ldots, n.$$ 

**Proof of Theorem 5.1.** Let $A \subset C(T)$ be a subspace with the induced operator space structure. By a joint result due independently to Junge and to Paulsen and the author (cf. [Pa6, Theorem 4.1]) there is a constant $C'$ such that for any sequence $(\xi_n)$ in $A^*$ we have

$$\left( \sum \|\xi_n\|^2 \right)^{1/2} \leq C' \sup_N \left\| \sum_{0}^{N} \xi_n \otimes e_n \right\|_{A^* \otimes_{\min, \max}(\ell_2)}.$$ 

By Theorem 4.1, if $d(A) \leq 2$, then $A$ is a quotient (as an operator space) of $\max(A) \otimes_{\min} \max(A)$, or a fortiori of $\ell_1(I) \otimes_{\min} \ell_1(I)$ for some index set $I$. Therefore, there is a subspace $Y \subset X$ and an complete isomorphism $w: A^* \to Y$ with $\|w\|_{cb} \leq 1$. By Lemma 5.3 and by (5.4), this implies that for all Hardy martingales $f = (f_n)$ in $H_m(A^*)$ we have

$$\left( \sum \left\| \int \tilde{z}_n df_n dm \right\|^2_{A^*} \right)^{1/2} \leq C'' \|f\|_{H_m(A^*)}$$

with $C'' = CC'||w^{-1}\|_{cb}$. Finally, by Lemma 5.4, $A$ cannot be proper (since (5.5) would imply $\beta \sqrt{n} \leq C''$ for all $n$).

**Remark 5.5.** The preceding proof establishes more than claimed in Theorem 5.1. Indeed, we conclude that if $A$ is proper then the operator space $A^*$ is not completely isomorphic to any subspace of $(\ell_1(I) \otimes_{\min} \ell_1(I))^*$ for any set $I$. Stated in that form the result cannot be improved much. Indeed, it can be shown that if $A$ is an arbitrary operator space, then for a suitable set $I$, $A^*$ embeds completely isometrically into $(\ell_1(I) \otimes_{\min} \ell_1(I) \otimes_{\min} \ell_1(I))^*$. Indeed, let

$$\mathcal{X}_N(I) = \ell_1(I) \otimes_{\min} \cdots \otimes_{\min} \ell_1(I) \quad (N \text{ times}).$$

Note that for any operator space $A$, the space $\max(A)$ is completely isometric to a quotient space of $\ell_1(I)$ for some set $I$ (cf. [BP1]). Then since $\mathcal{K}$ is nuclear, we have $d(\mathcal{K}) = 2$, so that $\mathcal{K}$ is completely isometric to a quotient space of $\mathcal{X}_2(I)$ for some suitable countable set $I$ (see Remark 4.6). Therefore, $\mathcal{K} \otimes_{\min} \mathcal{K}$ is completely isometric to a quotient space of $X_2(I) \otimes_{\min} X_2(I) = X_4(I)$. Since $R$ and $C$ are completely isometric to quotients of $\mathcal{K}$, it follows that $S_1 = R \otimes_{\min} C$ is completely isometric to a quotient of $\mathcal{K} \otimes_{\min} \mathcal{K}$. Finally, since every separable operator space is completely isometric to a quotient of $S_1$ (cf. [B2, p. 24]), we conclude that every (resp. separable) operator space is completely isometric to a quotient of $X_4(I)$ for some set $I$ (resp. countable). The modification for the non-separable case is immediate.
§6. $C^*$-algebras

Let $A$ be a unital $C^*$-algebra. The “setting” used in this section is the same as in §4, i.e. $E = \max(A)$ and $A = A$. Note that in the $C^*$-case, we have $\tilde{A}_1 = A$ completely isometrically, and (4.2) becomes $\Phi_A(c) = \Phi(i, c)$ for all $c > 1$.

It is known that any nuclear $C^*$-algebra satisfies $d(A) \leq 2$ (cf. Bunce [Bu] and Christensen [C3]). In this section we study the converse. Essentially, we show that if $A$ admits sufficiently many type II representations, then indeed the converse holds. More precisely we will prove the following.

**Theorem 6.1.** Let $A$ be a $C^*$-algebra such that $d(A) \leq 2$. Then, for any representation $\pi: A \to B(H)$ such that $\pi(A)$ generates a semi-finite von Neumann subalgebra of $B(H)$, the bi-commutant $\pi(A)^{''}$ is injective.

**Corollary 6.2.** If $\dim H = \infty$, then $d(B(H)) = 3$.

**Proof.** Indeed, on one hand we know by Haagerup’s result ([H1, Prop. 1.8]) that $d(B(H)) \leq 3$. On the other hand, by Joel Anderson’s results in [A], there is a type $II_{\infty}$ representation $\pi: B(H) \to B(\mathcal{H})$ such that $\pi(B(H))^{''} \cong M \otimes B(H)$ where $M$ is a $II_1$ factor containing a non trivial ultraproduct of matrix spaces. By Wassermann’s result [W], we know that the latter is not injective, so $\pi(B(H))^{''}$ is not injective. Thus, Theorem 6.1 implies in particular that $d(B(H)) \geq 3$. (I am grateful to Simon Wassermann for kindly directing me to Anderson’s result and explaining to me its consequences.) □

**Remark.** By [C4], for any $II_1$-factor $M$ with property $\Gamma$ we have $d(M) \leq 44$. Since these cannot be nuclear ([W]), the preceding result ensures that $3 \leq d(M) \leq 44$. It would of course be interesting to reduce the interval of possible values of $d(M)$.

The proof uses the following results.

The first lemma is a simple variant of a result from [JP].

**Lemma 6.3.** Let $A$ be any $C^*$-algebra. Then for any $n$ and any $\xi_1, \ldots, \xi_n \in A^*$ we have

$$
(\sum \|\xi_i\|^2)^{1/2} \leq 4 \left\| \sum_{i=1}^n \xi_i \otimes e_i \right\|_{A^* \otimes_{\min} \max(\ell_2)}.
$$

**Proof.** (The proof combines observations made independently by M. Junge [J] and the author.) Let $u: A \to \max(\ell_2^n)$ be the map defined by $u(a) = \sum_1^n \xi_i(a) e_i$. Let $E$ be a finite dimensional operator space. We use the same notation as in [JP], i.e. we denote $d_{SK}(E) = \inf \{\|v\|_{cb}\|v^{-1}\|_{cb}\}$ where the infimum runs over all possible isomorphisms $v: E \to \tilde{E}$ between $E$ and a subspace of the $C^*$-algebra of all compact operators on $\ell_2$, which we have denoted above by $K$.

Let $a_1, \ldots, a_n$ be a finite subset of $A$ and let $E \subset A$ be their linear span. Then the mapping $u|_{E}: E \to \max(\ell_2^n)$ factors through $A$ completely boundedly with a corresponding constant $\leq \|u\|_{cb}$. Fix $\epsilon > 0$. By Lemma 6.2.11 in [P3] this implies that $u|_{E}$ can be written as a composition $u|_{E} = u_2 u_1$ with $u_1: E \to \tilde{E}$ and $u_2: \tilde{E} \to \max(\ell_2^n)$ such that $\|u_1\|_{cb} = 1$, $d_{SK}(\tilde{E}) = 1$ and $\|u_2\|_{cb} \leq \|u\|_{cb}(1 + \epsilon)$. By
the main result in [JP], this implies

\[
\left| \sum_{1}^{n} \xi_{i}(a_{i}) \right| = \left| \sum_{1}^{n} \langle u(a_{i}), e_{i} \rangle \right|
= \left| \sum_{1}^{n} \langle u_{2}u_{1}(a_{i}), e_{i} \rangle \right|
\leq 4 \ d_{SK}(\tilde{E}) \|u_{2}\|_{cb} \left( \sum \|u_{1}(a_{i})\|^{2} \right)^{1/2}
\leq 4 \left( \sum \|a_{i}\|^{2} \right)^{1/2} \|u\|_{cb}(1 + \epsilon).
\]

Hence, since \( \epsilon > 0 \) is arbitrary, and since

\[
\|u\|_{cb} = \left\| \sum_{i=1}^{n} \xi_{i} \otimes e_{i} \right\|_{A^{\ast} \otimes_{\min \max}^{\max}(\ell_{2})},
\]

taking the supremum over all possible \( n \)-tuples \((a_{i})_{i \leq n} \) in \( A \), we obtain \( 6.1 \). \( \square \)

**Lemma 6.4.** Let \((e_{i})\) be the canonical basis of the operator space \( \max(\ell_{2}) \). Let \( H \) be any Hilbert space and let \( X \) be either \( B(C, H) \) or \( B(H^{\ast}, C) \), or equivalently let \( X \) be either the column Hilbert space or the row Hilbert space. Then for all \( x_{1}, \ldots, x_{n} \) in \( X \) we have

\[
\left\| \sum_{1}^{n} x_{i} \otimes e_{i} \right\|_{X \otimes_{\min \max}^{\max}(\ell_{2})} \leq \left( \sum \|x_{i}\|^{2} \right)^{1/2}.
\]

**Proof.** Assume \( X = B(C, H) \) or \( B(H^{\ast}, C) \). We identify \( X \) with \( H \) as a vector space. Let \((\delta_{m})\) be an orthonormal basis in \( H \). Observe that for any finite sequence \( a_{m} \) in \( B(\ell_{2}) \) we have in both cases

\[
\left\| \sum_{m} \delta_{m} \otimes a_{m} \right\|_{\min} \leq \left( \sum \|a_{m}\|^{2} \right)^{1/2}.
\]

whence we have, for any \( x_{1}, \ldots, x_{n} \) in \( X \),

\[
\left\| \sum x_{i} \otimes e_{i} \right\| = \left\| \sum_{m} \delta_{m} \otimes \sum_{i} \langle x_{i}, \delta_{m} \rangle e_{i} \right\|
\leq \left( \sum_{m} \left\| \sum_{i} \langle x_{i}, \delta_{m} \rangle e_{i} \right\|^{2} \right)^{1/2}
\leq \left( \sum_{m,i} \|\langle x_{i}, \delta_{m} \rangle\|^{2} \right)^{1/2} = \left( \sum_{i} \|x_{i}\|^{2} \right)^{1/2}.
\]

\( \square \)

**Proof of Theorem 6.1.** Recall that, since \( \pi(A) \) is a quotient \( C^{\ast} \)-algebra of \( A \), we have obviously \( d(\pi(A)) \leq d(A) \). Hence it suffices to prove the statement with \( \pi(A) \) in the place of \( A \). More precisely, we assume given \( A \subset B(H) \) such that \( M = A'' \) admits a faithful semi-finite normal trace denoted by \( \tau \) and we must show that \( d(A) \leq 2 \) implies that \( M \) is injective. First we can reduce to the finite case: indeed
it suffices to show that, for any projection $p$ in $M$ and $0 < \tau(p) < \infty$, the algebra $pMp$ is injective. Then, by a result due to Connes for factors and to Haagerup [H2] in the general case, $pMp$ is injective iff there is a constant $C$ such that for any central projection $q \neq 0$ in $pMp$, for any $n$ and any $n$-tuple $u_1, \ldots, u_n$ of unitaries in $pMp$, we have

\begin{equation}
 n \leq C \left\| \sum_{1}^{n} (qu_i) \otimes (qu_i) \right\|_{\min}.
\end{equation}

Fix $p, q$ and $u_1, \ldots, u_n$ unitary in $pMp$ as above. We will show that (6.2) holds. Let $\xi_i \in A^*$ be the functional defined by

$\xi_i(a) = \tau(qu_i a)$

and let

$t = \left\| \sum_{1}^{n} (qu_i) \otimes (qu_i) \right\|_{\min}^{1/2}.$

Let $\sigma: M \to B(L^2(M, \tau))$ be the classical representation of $M$ as left multiplications on $L^2(M, \tau)$. Clearly $\left\| \sum \sigma(qu_i) \otimes \sigma(qu_i) \right\|_{\min} \leq \left\| \sum qu_i \otimes qu_i \right\|_{\min}$ hence ([P6, Cor. 2.7]) there is a decomposition

\begin{equation}
\sigma(qu_i) = a_i + b_i
\end{equation}

in $B(L^2(M, \tau))$ with

$\sum a_i^* a_i \leq t^2 \quad \text{and} \quad \sum b_i b_i^* \leq t^2.$

Consider the mapping $u: A \to \max(\ell^2_2)$ defined by

$\forall a \in A \quad u(a) = \sum_{1}^{n} \xi_i(a)e_i.$

Then (using the tracial property of $\tau$) we have (recall $qu_i = qu_i q = u_i q$)

$u(ab) = \sum_{1}^{n} \tau(bqu_i a) e_i$

$= \sum_{1}^{n} \tau(bq(u_i) qa) e_i.$

Let us denote by $j: A \to L^2(M, \tau)$ the mapping defined by $j(a) = qa$. We can write for all $a, b$ in $A$

$u(ab) = \sum_{1}^{n} \langle (a_i + b_i) j(a), j(b^*) e_i \rangle$

(here the scalar product is in $L^2(M)$) hence

$u(ab) = \varphi_1(a, b) + \varphi_2(a, b)$

where

$\varphi_1(a, b) = \sum_{1}^{n} \langle a_i j(a), j(b^*) e_i \rangle$. 

and
\[ \varphi_2(a, b) = \sum_{i=1}^{\ell_n^2} \langle b_i, j(a), j(b^*) \rangle e_i. \]
Now we assume \( \max(\ell_n^2) \subset B(\mathcal{H}) \) completely isometrically and we claim that for some Hilbert space \( \hat{H} \) there are mappings
\[ \beta_i: A \to B(\mathcal{H}, \hat{H}) \quad \text{and} \quad \alpha_i: A \to B(\hat{H}, \mathcal{H}) \]
for \( i = 1, 2 \) such that
\[ \varphi_i(a, b) = \alpha_i(a) \beta_i(b) \]
and
\[ \|\alpha_1\| \|\beta_1\| \leq \tau(q) t, \quad \|\alpha_2\| \|\beta_2\| \leq \tau(q) t. \]
Taking this for granted for the moment, let us now complete the argument. By Theorem 4.2 our assumption \( d(A) \leq 2 \) implies that there is a constant \( K \) such that the product mapping \( P: \max(A) \otimes_{h} \max(A) \to A \) satisfies for all maps \( u: A \to B(\mathcal{H}) \)
\[ \|u\|_{cb} \leq K\|uP\|_{cb}. \]
Note that if \( \hat{u}: \max(A) \times \max(A) \to B(\mathcal{H}) \) is the bilinear form associated to \( uP \), then by [CS1-2, PaS] we know that
\[ \|uP\|_{cb} = \|\hat{u}\|_{cb} \]
and since \( \hat{u} = \varphi_1 + \varphi_2 \) we obtain
\[ \|u\|_{cb} \leq K[\|\hat{u}\|_{cb}] \leq K(\|\varphi_1\|_{cb} + \|\varphi_2\|_{cb}) \]
but by the specific factorization of \( \varphi_1 \) and \( \varphi_2 \) given above we have \( \|\varphi_i\|_{cb} \leq \|\alpha_i\| \|\beta_i\| \) whence
\[ \|u\|_{cb} = K(\|\alpha_1\| \|\beta_1\| + \|\alpha_2\| \|\beta_2\|) \]
\[ \leq 2K t \tau(q). \]
Equivalently we have
\[ \left\| \sum_{i=1}^{\ell_n} \xi_i \otimes e_i \right\|_{A^* \otimes_{\min} \max(\ell_n^2)} \leq 2\tau(q) t K. \]
By (6.1) this implies
\[ \left( \sum_{i=1}^{\ell_n} \|\xi_i\|_{A^*}^2 \right)^{1/2} \leq 8\tau(q) t K. \]
But on the other hand
\[ \|\xi_i\|_{A^*} \geq |\xi_i(u_i^*)| = \tau(q u_i u_i^*) = \tau(q) \]
hence
\[ \tau(q) \sqrt{n} \leq 8\tau(q) t K \]
and finally \( n \leq 64K^2 t^2 \). Thus we obtain (6.2).
This completes the proof modulo the claim. We now turn to the latter claim. Let \( L = L^2(M, \tau) \). We denote by \( x \to r(x) \in B(\bar{L}, \mathbb{C}) \) the canonical identification. Note that \( r(x)r(y)^* \in B(\mathbb{C}, \mathbb{C}) \) can be identified with \( \langle x, y \rangle \).

With this identification, we have for all \( a, b \) in \( A \)
\[
\varphi_1(a, b) = \sum_1^n r(a_i j(a)) r(j(b^*)) e_i = \left( \sum_1^n r(a_i j(a)) \otimes e_i \right) \odot (r(j(b^*)) \otimes I).
\]

We set \( \alpha_1(a) = \sum_1^n r(a_i j(a)) \otimes e_i \) and \( \beta_1(b) = r(j(b^*)) \otimes I \). Then \( \|\beta_1(b)\| = \|j(b^*)\|_L = \|q b^*\|_{L^2(\tau)} \leq \|b\| \tau(q)^{1/2} \) and by Lemma 6.4
\[
\|\alpha_1(a)\| \leq \left( \sum_1^n \|r(a_i j(a))\|_L^2 \right)^{1/2} \leq \left( \sum_1^n \|a_i j(a)\|_L^2 \right)^{1/2} \leq \left( \sum_1^n a_i^* a_i \right)^{1/2} \|j(a)\|_L \leq t\tau(q)^{1/2} \|a\|.
\]

Hence we obtain \( \|\alpha_1\| \|\beta_1\| \leq t\tau(q) \) as announced. Similarly we define
\[
\alpha_2(a) = r(j(a)) \otimes I
\]
and
\[
\beta_2(b) = \sum_1^n r(b_i^* j(b^*)) \otimes e_i.
\]

Then clearly \( \varphi_2(a, b) = \alpha_2(a) \beta_2(b) \) and this time we have \( \|\alpha_2\| \leq \tau(q)^{1/2} \) and \( \|\beta_2\| \leq t\tau(q)^{1/2} \), whence \( \|\alpha_2\| \|\beta_2\| \leq t\tau(q) \). This completes the proof of the claim, and also of Theorem 6.1. \( \square \)

In particular, since \( C^*(G) \) or \( C^*_v(G) \) is nuclear iff \( G \) (discrete) is amenable, (cf. e.g. [La]) we recover some results from \( \S 3 \), as follows.

**Corollary 6.5.** Let \( G \) be a discrete group and let \( A \) be either \( C^*(G) \) or the reduced \( C^* \)-algebra \( C^*_v(G) \). Then \( d(A) \leq 2 \) iff \( G \) is amenable.

**Remark.** Note however, that the equivalence with (v) in Theorem 3.2 concerning the spaces of coefficients does not follow from this new approach.

**Corollary 6.6.** Let \( A \) be a \( C^* \)-algebra which generates a non-injective semi-finite von Neumann algebra. Then for any \( c > 1 \), there is a unital homomorphism \( u_c: A \to B(H) \) with \( \|u_c\| \leq c \) and \( \|u_c\|_b \geq c^3 \).

**Remark 6.7.** A unital \( C^* \)-algebra \( A \) satisfies the similarity property (i.e. \( d(A) < \infty \)) as soon as \( \Phi_A(c) < \infty \) for some \( c > 1 \). Indeed, this follows from Lemma 2.3 and the remark preceding it.

**Remark.** The following result proved in [H1] and [C3] plays an important rôle in these papers: Let \( u: A_1 \to A_2 \) be a bounded homomorphism between \( C^* \)-algebras. Then for any finite subset \( \{x_i\} \) in \( A_1 \) we have
\[
\left\| \sum u(x_i)^* u(x_i) \right\|^{1/2} \leq \|u\|^{1/2} \left\| \sum x_i^* x_i \right\|^{1/2}.
\]

The next result shows that the exponent 2 cannot be improved in this result.
Proposition 6.8. Suppose that a number \( \alpha \geq 1 \) has the following property: there is a constant \( K \) such that for any bounded homomorphism \( u: B(H) \to B(H) \) (\( \dim H = \infty \)) and for any finite subset \( x_1, \ldots, x_n \) in \( B(H) \) we have
\[
\left\| \sum u(x_i)^* u(x_i) \right\|^{1/2} \leq K \left\| u \right\|^{\alpha} \left\| \sum x_i^* x_i \right\|^{1/2}.
\]
Then necessarily \( \alpha \geq 2 \).

Proof. Our assumption can be written as follows: for any \( c \geq 1 \) and any unital homomorphism \( u \) with \( \left\| u \right\| \leq c \), we have for any \( n \) and any \( x_1, \ldots, x_n \) in \( B(H) \)
\[
\left\| (I_K \otimes u) \left( \sum e_{i1} \otimes x_i \right) \right\|_{\min} \leq K c^{\alpha} \left\| \sum e_{i1} \otimes x_i \right\|_{\min}.
\]
In other words, the subspace \( X \) spanned in \( K \) by the sequence \( (e_{i1}) \) \( (i = 1, 2, \ldots) \) satisfies the assumption (2.11) in Theorem 2.6. Assume \( \alpha < 2 \). Then, by Theorem 2.6, (2.11) actually holds for \( \alpha = 1 \) (for some \( K \)). Thus, if \( \alpha < 2 \) we may as well assume \( \alpha = 1 \). But then Haagerup’s argument in [H1] (or the proof presented in [P1, chapter 7]) will lead to \( d(B(H)) \leq 2 \), which contradicts Corollary 6.2. Thus we must have \( \alpha \geq 2 \). \( \square \)

§7. The Blecher-Paulsen factorization

In this section, we connect our description of the enveloping algebra \( \tilde{A}_1 \) with some ideas of Blecher and Paulsen in [BP2]. We take a slightly more general viewpoint than them in order to cover the situation of a group (or an algebra) generated by a subset, but the main idea is in [BP2].

We consider our usual “setting” \( (i, E, \mathcal{A}) \), where \( E \) is an operator space, \( \mathcal{A} \) a unital operator algebra (not assumed complete), and \( i: E \to \mathcal{A} \) is a completely contractive linear injection with range generating \( \mathcal{A} \). But in addition we will assume throughout this section that \( E \) is “unital”, by which we mean that \( E \) contains a norm one element \( e \) such that \( i(e) = 1_A \).

Consider again the algebra \( \tilde{A}_1 \) as defined above, with unital embeddings \( E \subset \mathcal{A} \subset \tilde{A}_1 \).

It will be convenient to consider \( E \) as “included” into \( \mathcal{A} \) and to view \( i \) as an inclusion map. The reader should be warned however that \( i \) will generally not be assumed completely isometric: in general the operator space structure on \( \mathcal{A} \) only plays a auxiliary rôle. What really matters here is the given operator space structure on \( E \) and the resulting operator algebra one on \( \tilde{A}_1 \), which appears as “generated” by \( E \).

Theorem 7.1. With the above notation, let \( n \) be a positive integer. Then the following properties of an element \( x \) in \( M_n(\mathcal{A}) \) are equivalent:

(i) \( \left\| x \right\|_{M_n(\tilde{A}_1)} < 1 \).

(ii) The matrix \( x \) can be written, for some integer \( N \) and some integer \( d \), as a matricial product of the form
\[
x = \alpha_0 D_1 \alpha_2 D_2 \ldots D_d \alpha_d
\]
where \( \alpha_0 \in M_{nN} \), \( \alpha_1 \in M_N \), \( \alpha_{d-1} \in M_N \), \( \alpha_d \in M_{Nn} \) are scalar matrices (i.e., \( \alpha_0 \) and \( \alpha_d \) are rectangular of size \( n \times N \) and \( N \times n \), and the others are square matrices of size \( N \times N \)), and \( D_1, \ldots, D_d \) are \( N \times N \) matrices with entries in \( E \), and finally we have

\[
\prod_{i=0}^{d} \| \alpha_i \| \prod_{i=1}^{d} \| D_i \|_{M_N(E)} < 1.
\]

**Proof.** The proof follows from an immediate adaptation of an argument in [BP2]. We merely sketch it. It is clear that (ii) implies (i). Conversely assume (i). This means that there is a number \( \theta < 1 \), such that for any contractive unital homomorphism \( u : A \to B(H) \), we have

\[
\| [I_{M_n} \otimes u](x) \|_{M_n(B(H))} \leq \theta.
\]

Now, for any \( n \) and for any \( x \) in \( M_n(A) \), let us denote by \( \| x \|_{(n)} \) the infimum of

\[
\prod_{i=0}^{d} \| \alpha_i \| \prod_{i=1}^{d} \| D_i \|
\]

over all possible factorizations (with arbitrary \( N \) and \( d \)) of \( x \) as in (ii) above. Then by the main result in [BRS], there is a unital homomorphism \( u : A \to B(H) \) with the help of which \( \| . \|_{(n)} \) can be identified with \( \| u(.) \|_{M_n(B(H))} \) for all \( n \). But then, clearly \( u \) is completely contractive, since we trivially have (by definition of \( \| . \|_{(N)} \))

\[
\| [I_{M_N} \otimes u](D) \|_{M_N(B(H))} = \| D \|_{(N)} \leq \| D \|_{M_N(E)}
\]

for any \( D \) in \( M_N(E) \) and any \( N \). Hence, returning to the particular \( n \) and \( x \) appearing in (i), by (7.1) we must have \( \| x \|_{(n)} \leq \theta \). Equivalently, we obtain (ii). \( \Box \)

**Remark 7.2.** Recall that when \( A \) is a \( C^* \)-algebra (resp. \( A = C^*(G) \)) and \( E = \max(A) \) (resp. \( E = \ell_1(G) \)) as in §5 (resp. §3), then \( \tilde{A}_1 = A \) completely isometrically. When the latter holds, Theorem 7.1 gives a characterization of the elements of the unit ball of \( M_n(A) \). Note also that when \( E = \max(E) \), the elements of \( M_N(E) \) admit a specific factorization (a kind of diagonalization) described above in (0.4).

As application, we have the following apparently new characterization of the coefficients of unitary representations of a group \( G \), i.e. of the elements of the space \( B(G) \), as follows. (Take \( H \) unidimensional in the next statement, then (i) below is the same as saying that the norm of \( f \) in \( B(G) \) is \( \leq K \).)

**Corollary 7.3.** Let \( G \) be any discrete group, and let \( \Gamma \subset G \) be a subset containing the unit element and generating \( G \) in the sense that every element of \( G \) can be written as a product of elements of \( \Gamma \). Let \( K \geq 0 \) be a fixed constant. The following properties of a function \( f : G \to B(H) \) are equivalent:

(i) There are a unitary representation \( \pi : G \to B(H_\pi) \) and operators \( \xi : H_\pi \to H \) and \( \eta : H \to H_\pi \) such that \( f(t) = \xi_\pi(t)\eta \) for any \( t \) in \( G \) and \( \| \xi \| \| \eta \| \leq K \).

(ii) For each \( N \geq 1 \), the function \( f_N : \Gamma^N \to B(H) \) defined by \( f_N(t_1, \ldots, t_N) = f(t_1t_2 \ldots t_N) \) extends (with the obvious identification) to an element of \( cb(\ell_1(\Gamma) \otimes_h \cdots \otimes_h \ell_1(\Gamma), B(H)) \) (where the tensor product is \( N \)-fold) with norm \( \leq K \).
(iii) Same as (ii) with $\Gamma = G$.
(iv) For any $N \geq 1$, there are bounded functions $F_i: \Gamma \rightarrow B(H_{i+1}, H_i)$, with $\sup_{\Gamma} \|F_i\| \leq 1$ for all $i$, where $H_i$ are Hilbert spaces with $H_{N+1} = H$ and $H_1 = H$, such that

$$\forall \ t_1, \ldots, t_N \in \Gamma \quad f(t_1t_2\ldots t_N) = K \ F_1(t_1)F_2(t_2)\ldots F_N(t_N).$$

Proof. By the factorization of c.b. maps (cf. [Pa1]) (i) holds iff $f$ extends linearly to a c.b. mapping $u: C^*(G) \rightarrow B(H)$ with $\|u\|_{cb} \leq K$. We consider now the setting $(i, E, A)$ defined by $E = \ell_1(\Gamma)$, $A = \ell_1(G)$ (viewed as a subalgebra of $C^*(G)$). Then, as already mentioned in Remark 7.2, for any $n$, we have an isometric identity $M_n(\tilde{A}_1) = M_n(C^*(G))$. Let us denote as in §3, by $W_N$ the natural product map from $E_N = \ell_1(\Gamma) \otimes_h \ldots \otimes_h \ell_1(\Gamma)$ (N times) into $C^*(G)$. Then, by Theorem 7.1, we have $\|u\|_{cb} \leq K$ iff

$$\sup_N \|uW_N\|_{cb(E_N, B(H))} \leq K. \quad (7.2)$$

But now, (7.2) is but a reformulation of (ii), so that (i) is equivalent to (ii). Moreover, since (i) $\Rightarrow$ (ii) is valid for any $\Gamma$, it holds when $\Gamma = G$, whence (ii) $\Rightarrow$ (iii), and the converse is obvious. Finally, the equivalence between (ii) and (iv) follows from the well known factorization theorem of c.b. multilinear maps (cf. [CS1-2, PaS]). □

§8. BANACH ALGEBRAS

The general method of this paper can be applied in other situations when studying a Banach algebra $A$ given together with a generating system, or a family of generating subalgebras. The rôle of the “degree” is then played by the minimal length of the products necessary to generate (in a suitable Banach algebraic sense) the unit ball (or some ball centered at the origin). One can also develop our approach for a general “variety of Banach algebras” (in the sense of [Dix]) instead of that of operator algebras. To illustrate briefly what we have in mind, take the variety of all Banach algebras, then our basic idea leads to:

**Theorem 8.1.** Let $A$ be a Banach algebra with unit ball $B_A$. Consider a subset $\beta \subset B_A$ and assume that the algebra it generates, denoted by $A$, is dense in $A$. For $n = 1, 2, \ldots$, let us denote by $\beta^n$ the set of all products of $n$ elements taken in $\beta$. Let $d$ be a positive integer and let $\alpha$ be any number such that $d \leq \alpha < d + 1$. Consider the following properties:

(i) $\alpha$ There is a constant $K$ such that, for any Banach algebra $B$ and for any homomorphism $u: A \rightarrow B$, if $u$ is bounded on $\beta$, $u$ is continuous and we have $\|u\| \leq K \sup_{x \in \beta} \|u(x)\|^\alpha$.

(ii) $d$ There is a constant $K'$ such that (here $aconv$ stands for the absolutely convex hull)

$$B_A \subset K'aconv(\beta \cup \beta^2 \cup \ldots \cup \beta^d).$$
There is a constant $K''$ such that, for any Banach algebra $B$ and for any continuous homomorphism $u: A \to B$, we have

$$\|u\| \leq K'' \sup_{x \in \beta} \|u(x)\|^d.$$ 

Then (i) $\Rightarrow$ (ii). Moreover, if $\beta$ contains a unit element for $A$, then conversely (ii) $\Rightarrow$ (iii).

**Proof.** (Sketch) In this proof, we will say “morphism” for homomorphism with values in a Banach algebra. We will follow the same strategy as in §1 and §2. Let $\|u\|_{\beta} = \sup\{\|u(x)\| \mid x \in \beta\}$. Let $c \geq 1$ and let $C_c$ be the set of all morphisms $u: A \to B_u$ with $\|u\|_{\beta} \leq c$. We define the Banach algebra $A_c$ as the completion of $A$ for the embedding $J: A \to \oplus_{u \in C_c} B_u$ defined by $J(x) = \oplus_{u \in C_c} u(x)$. Let $F_\beta$ be the free semi-group with free generators indexed by $\beta$. We will consider $\beta$ as a subset of $F_\beta$. Consider the space $\ell_1(F_\beta)$, viewed as a Banach algebra for convolution. Let $\delta_t$ ($t \in F_\beta$) denote the canonical basis and let $B$ be the dense subalgebra linearly generated by $\delta_t$ ($t \in F_\beta$), equipped with the induced norm. Let $c \geq 1$ and $z = 1/c$. We have a unique morphism $\pi_z: B \to A$ such that $\pi_z(\delta_t) = zt$ for all $x \in \beta$. It is easy to check that if $u: A \to B_u$ is any morphism, then $\|u\|_{\beta} \leq c$ iff $\|u\pi_z\| \leq 1$. Moreover, $A_c$ can be identified with the completion of $B/\ker(\pi_z)$, and (i) $\alpha$ means that

$$\forall x \in A \quad \|x\|_{\tilde{A}_c} \leq Kc^\alpha \|x\|_A.$$ 

The proof can then be completed by arguing as in Theorem 2.5. We leave the remaining details to the reader. \square

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**References**

[A] J. Anderson, *Extreme points in sets of linear functionals in $\mathcal{B(H)}$, J. Funct. Anal. 31* (1979), 195–217.

[B1] D. Blecher, *A completely bounded characterization of operator algebras*, Math. Ann. 303 (1995), 227–240.

[B2], *The standard dual of an operator space*, Pacific J. Math. 153 (1992), 15–30.

[BRS] D. Blecher, Z.J. Ruan and A. Sinclair, *A characterization of operator algebras*, J. Funct. Anal. 89 (1990), 188–201.

[BP1] D. Blecher and V. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. 99 (1991), 262–292.

[BP2], *Explicit construction of universal operator algebras and applications to polynomial factorization*, Proc. Amer. Math. Soc. 112 (1991), 839–850.

[BS] D. Blecher and R. Smith, *The dual of the Haagerup tensor product*, Journal London Math. Soc. 45 (1992), 126–144.

[Bou] J. Bourgain, *On the similarity problem for polynomially bounded operators on Hilbert space*, Israel J. Math. 54 (1986), 227–241.

[Bo1] M. Bożejko, *Remarks on Herz-Schur multipliers on free groups*, Mat. Ann. 258 (1981), 11–15.

[Bo2], *Positive definite bounded matrices and a characterization of amenable groups*, Proc. A.M.S. 95 (1985), 357–360.

[BF1] M. Bożejko and G. Fendler, *Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group*, Boll. Unione Mat. Ital. (6) 3-A (1984), 297–302.
[Ki] E. Kirchberg, *The derivation and the similarity problem are equivalent*, J. Operator Th. 36 (1996), 59–62.

[Kis] S. Kisliakov, *The proper uniform algebras are uncomplemented*, Lomi preprint (1988).

[KS] R.A. Kunze and E. Stein, *Uniformly bounded representations and Harmonic Analysis of the $2 \times 2$ real unimodular group*, Amer. J. Math. 82 (1960), 1–62.

[La] C. Lance, *Tensor products and nuclear $C^*$-algebras. Operator algebras and applications*, Amer. Math. Soc. Proc. Symposia Pure Math 38 part 1 (1982), 379–399.

[Le] A. Lebow, *A power bounded operator which is not polynomially bounded*, Mich. Math. J. 15 (1968), 397–399.

[MPSZ] A.M. Mantero, T. Pytlik, R. Szwarc and A. Zappa, *Equivalence of two series of spherical representations of a free group*, Ann. di Matematica pura ed applicata (IV) CLXV (1993), 23–28.

[O1] A. Yu. Ol'shanskii, *On the problem of the existence of an invariant mean on a group*, Russian Math. Surveys 35 (1980), 180–181.

[O2] , *Geometry of defining relations in groups*, Kluwer, Dordrecht, Netherlands, 1991.

[Pat] A. Paterson, *Amenability*, Amer. Math. Soc. Math. Surveys 29 (1988).

[Pa1] V. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Math., vol. 146, Longman, Wiley, New York, 1986.

[Pa2] , *Completely bounded maps on $C^*$-algebras and invariant operator ranges*, Proc. Amer. Math. Soc. 86 (1982), 91–96.

[Pa3] , *Every completely polynomially bounded operator is similar to a contraction*, J. Funct. Anal. 55 (1984), 1–17.

[Pa4] , *Completely bounded homomorphisms of operator algebras*, Proc. Amer. Math. Soc. 92 (1984), 225–228.

[Pa5] , *Representation of Function algebras, Abstract operator spaces and Banach space Geometry*, J. Funct. Anal. 109 (1992), 113–129.

[Pa6] , *The maximal operator space of a normed space*, Proc. Edinburgh Math. Soc. (to appear).

[PaS] V. Paulsen and R. Smith, *Multilinear maps and tensor norms on operator systems*, J. Funct. Anal. 73 (1987), 258–276.

[Pel] V. Peller, *Estimates of functions of power bounded operators on Hilbert space*, J. Oper. Theory 7 (1982), 341–372.

[Pes] V. Pestov, *Operator spaces and residually finite-dimensional $C^*$-algebras*, J. Funct. Anal. 123 (1994), 308–317.

[Pi1] J.P. Pier, *Amenable locally compact groups*, Wiley Interscience, New York, 1984.

[Pi2] , *Amenable Banach algebras*, Pitman, Longman, 1988.

[P1] G. Pisier, *Similarity problems and completely bounded maps*, vol. 1618, Springer Lecture Notes, 1995.

[P2] , *Multipliers and lacunary sets in non amenable groups*, Amer. J. Math. 117 (1995), 337–376.

[P3] , *An introduction to the theory of operator spaces*, Preprint (to appear).

[P4] , *A simple proof of a theorem of Kirchberg and related results on $C^*$-norms*, J. Op. Theory. 35 (1996), 317–335.

[P5] , *Un opérateur polynômalement borné sur un Hilbert qui n'est pas semblable à une contraction*, (and article to appear in Journal Amer. Math. Soc.), Comptes Rendus Acad. Sci. Paris 322 (1996), 547–550.

[P6] , *The operator Hilbert space $OH$, complex interpolation and tensor norms*, Memoirs Amer. Math. Soc. 122, 585 (1996), 1–103.

[R] Z.J. Ruan, *Subspaces of $C^*$-algebras*, J. Funct. Anal. 76 (1988 217–230).

[Ru] W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.

[Sh] M.V. Scheinberg, *A characterization of the algebra $C(\Omega)$ in terms of cohomology groups*, Uspekhi Mat. Nauk 32 (1977), 203–204; Russian

[SN] B. Sz.-Nagy, *On uniformly bounded linear transformations on Hilbert space*, Acta Sci. Math. (Szeged) 11 (1946–48), 152–157.

[V] N. Varopoulos, *A theorem on operator algebras*, Math. Scand 37 (1975), 173–182.
[W] S. Wassermann, *On tensor products of certain group C*-algebras*, J. Funct. Anal. **23** (1976), 239–254.

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