POTENTIAL THEORY ON TREES, GRAPHS AND
AHLFORS-REGULAR METRIC SPACES

NICOLA ARCOZZI, RICHARD ROCHBERG, ERIC T. SAWYER, AND BRETT D. WICK

ABSTRACT. We investigate connections between potential theories on a Ahlfors-regular metric space \( X \), on a graph \( G \) associated with \( X \), and on the tree \( T \) obtained by removing the “horizontal edges” in \( G \). Applications to the calculation of set capacity are given.

CONTENTS

1. Introduction 2
1.1. Main Results and Outline of the Contents 4
1.2. Some Examples of Ahlfors-Regular Metric Spaces 6
2. A Metric Space, a Graph, and a Tree 7
2.1. Christ’s Theorem Revisited 7
2.2. The space \( X \) as the Boundary of \( G \) 10
2.3. The Tree and its Boundary as Metric Spaces 12
2.4. The Map \( \Lambda : \partial T \rightarrow X \) 13
2.5. Moving Measures through \( \Lambda \) 14
3. The Muckenhoupt-Wheeden Inequality on Graphs 18
4. Proof of Main Result 22
5. Potential Theory on Trees 24
5.1. Tree Capacity Seen from “Inside” the Tree 25
5.2. Capacity of Special Sets 27
5.3. An Algorithm to Compute Tree Capacities 28
5.4. Trace Inequalities on Trees and Ahlfors-regular spaces 30
5.5. Defining Capacities via Carleson Measures 34
5.6. Monotonicity of the Tree Condition 36
5.7. Trace Inequalities: The Testing Condition implies the Capacitary Condition 37
Appendix I: Nonlinear Potential Theory 39
Appendix II: Nonlinear Potential Theory on Trees 41
References 42

Date: August 25, 2011.
2000 Mathematics Subject Classification. 30LXX, 31-XX, 32U20.
Key words and phrases. Ahlfors-regular metric space, potential theory, capacity.
The first author’s work partially supported by the COFIN project Analisi Armonica, funded by the Italian Minister for Research.
The second author’s work supported by the National Science Foundation under Grant No. 0700238.
The third author’s work supported by the National Science and Engineering Council of Canada.
The fourth author’s work supported by the National Science Foundation under Grant No. 1001098 and 0955432.
1. Introduction

One of the cornerstones of potential theory is the notion of set capacity, which plays there a rôle analogous to that played by Lebesgue measure (or Haar measure) in the theory of $L^p$ spaces and harmonic analysis. The notion of capacity has its origins in physics, where it measures the maximum amount of (positive, say) electric charge which can be carried by a conductor while keeping the potential generated by that charge below a fixed threshold. The notion of capacity has been extended to nonlinear potentials, to various metric space settings, to the theory of stochastic processes and more. For a long time, capacity had its peculiar rôle in the theory of conformal mappings and it is foundational material for the theory of quasi-conformal mappings. There is an extensive literature on these topics and we merely refer the reader to a tiny sample of it see for example [1], [16], [20], [27], [32].

A basic fact about set capacity is that it is subadditive, but not additive; to the point that there are sets of positive, finite capacity with different Hausdorff dimensions. Capacity, that is, is much different from (although linked with) Hausdorff measure. This fact makes computing, or just estimating, set capacity a craft on its own. The capacity of a set, in fact, measures together its geometric size and the way its pieces are distributed in the surrounding space. The bed of nails of a fakir is a well known example: cleverly arranged nails sum up to little area, but they might be distributed so as to have sufficient capacity not to break the membrane which is laid on them.

There is one context, however, where an elementary algorithm to compute capacity exists, and that is the context of trees. The capacity of subsets of the tree’s boundary, in fact, reduce to the calculation of continued fractions of generalized type. This fact seems to have been independently observed by all researchers who, for different reasons, developed some potential theory on trees [7], [26], [30]. It is interesting then to know if these simple calculations on trees can be of some help in estimating capacities of sets in, say, Euclidean space.

In this article we show that this is in fact possible in the rather general context of Ahlfors-regular metric spaces for $p$-capacities associated with suitable Bessel-type kernels. To each such space $X$ we associate a tree $T$. The capacity of a closed subset of $X$ turns out to be estimated from above and below by the capacity of a corresponding closed subset of $T$’s boundary. The result seems new even for Bessel capacity in Euclidean space $\mathbb{R}^n$, $n \geq 2$. For linear capacities and $n = 1$ it was proved by Benjamini and Peres in [10]. For $n = 1$ and $p \neq 2$ it might be a folk theorem, for which we have no explicit reference (though see [33]). The boundary $\partial T$ of the tree $T$ is a totally disconnected set with respect to a natural metric. The problem of estimating set capacities is reduced then to an analogous problem for subsets of a generalized Cantor set.

The interest of the result, we believe, goes beyond the problem of estimating set capacities per se. Many problems in potential theory have an essentially combinatorial nature, which is best seen when they are translated in the language of trees.

Our interest in such questions developed while the first three authors were working with Carleson measures for Dirichlet-type spaces of holomorphic functions. Stegenga’s Theorem [31] characterizes Carleson measures in terms of a condition involving the logarithmic capacity of subsets of the unit circle. In [5] we found a characterization of the same measures in terms of a non capacitary testing condition, formulated using the tree structure of a Whitney decomposition of the unit disc.
We asked ourselves whether there was some convincing, direct and possibly useful explanation for the equivalence of these two conditions. The question splits into two more precise questions: (i) Why is a testing condition equivalent to a capacitary condition? (see [7] for a first answer); (ii) Why does a rather poor geometric structure, such as the tree structure, contain essentially all the potential theoretic information of a more complex geometry, such as that of the unit disc? This article tries to give a first convincing answer to those questions.

Observations of a similar flavor have been made before. In [10], Benjamini and Peres proved that the recurrence/transience dichotomy for some random walks on trees can be decided in terms of logarithmic capacity in the Euclidean plane where the trees have been suitably imbedded. In [33], Verbitsky and Wheeden proved in some generality that the study of nonlinear potentials can be reduced to “dyadic” potentials. In retrospect, it might be said that the equivalence between dyadic and continuous potentials was implicit in Wolff’s proofs in [19], and this article might be seen as a long commentary on Wolff’s work.

Let us mention an application of the results and ideas discussed in this article. In [9], where a Nehari-type theorem for bilinear forms on the holomorphic Dirichlet space is proved, one of the main tools was the holomorphic approximation of the tree potential for a subset \( E \) of the unit circle in the complex plane. Of fundamental importance were the facts that (a) the tree capacity of \( E \) is comparable with its (classical) Bessel capacity and (b) that the simple nature of potential theory on trees not only helps with the estimates we needed, but also that the discrete potential can be computed almost explicitly. In [9] we used the linear, one-dimensional, Euclidean version of results that here are discussed in greater generality. An application to estimates for condenser capacities in the complex plane is discussed in [3].

There are three main tools which we use to build the bridge between trees and Ahlfors-regular spaces. The first is Christ’s dyadic decomposition of a homogeneous metric space [13] (the easy part of it). We refine it, in the more specialized Ahlfors-regular case, to make room for the singular measures which are necessary in potential theory. Christ’s construction can be thought of as a graph \( G \) representing the geometry of \( X \) at different scales. In particular, we identify \( X \) with the bi-Lipschitz image of the graph’s boundary, a result which might have independent interest. The tree \( T \) used by Christ in his study of singular integrals is a spanning tree for the graph \( G \). The second tool (useful when \( X \) is not homeomorphic to a subset of the real line) is a technical lemma showing that measures can be moved back and forth from the metric space \( X \) to the boundary of its associated tree. We must define the (non-canonical) pullback of a measure, which poses a nontrivial measurability problem. The third tool (useful to deal with the nonlinear case) is a deep inequality by Muckenhoupt and Wheeden [29] (later independently proved by Wolff [19] with a wholly different, almost combinatorial argument). The Muckenhoupt-Wheeden-Wolff inequality, in fact, moves potentials from the space \( X \) to the graph \( G \), where it is easy to see that the “vertical edges” (those which define the tree structure) carry all the relevant quantitative information. The inequality of Muckenhoupt, Wheeden and Wolff extends the well known equivalence, for subsets of the real line, of logarithmic capacity (in the plane) and \( 1/2 \)-Bessel capacity (on the real line). In the general case, this equivalence can be stated in terms of Wolff potentials.

In order to better focus on the main ideas, in this article we only consider the case of bounded, complete Ahlfors-regular spaces. The theory could be extended to the unbounded case by choosing Bessel-type potentials with exponential decay at infinity, in order to control the tail estimates. A good source for this kind of extension is [23].
We finish with a comment about the bibliography. We have made references to articles and books where the results we needed or we made reference to could be found. We did not look for the primary source of the results we have quoted and we have probably omitted some important references. This is especially true for the section concerning potential theory on trees. Many of the results which we prove in Section 5 can be found in the literature, often independently proved by several authors at different times (including the authors of the present paper), with different degrees of generality; or they might be seen as particular cases of general results in one of the several axiomatic versions of Potential Theory. We have chosen to prove ourselves what we needed, trying to keep the exposition as self-contained as possible (there are two exceptions: the proof of Christ’s Decomposition Theorem and the basic properties of Axiomatic Potential Theory, which we took from [1]). The main point of the paper, in fact, is not developing some new Potential Theory on Trees (although some of the results we present might be new, especially those relating capacity and Carleson measures), but rather shedding light on the connection between discrete and non-discrete Potential Theory.

1.1. Main Results and Outline of the Contents. Let \((X, m, \rho)\) be an Ahlfors-regular metric measure space. In this article we are interested in local properties and we frequently will assume that \(\text{diam}(X) < +\infty\). By this we mean that \((X, \rho)\) is a complete metric space, \(m \geq 0\) is a Borel measure on \(X\), and there exist constants \(0 < c_1 < c_2, Q > 0\), such that, for all \(r \geq 0\) and \(x \in X\):

\[
(1) \quad c_1 r^Q \leq m(B(x, r)) \leq c_2 r^Q.
\]

Here, \(B(x, r) = \{y \in X : \rho(x, y) < r\}\) is the metric ball of radius \(r\), centered at \(x\). Since we are working with a bounded metric space, this means that (1) is required to hold for \(0 \leq r \leq \text{diam}(X)\). This special assumption could be removed by assuming that the potential kernel \(K\) below has exponential decay, analog to that of the Bessel kernels in \(\mathbb{R}^n\).

Given \(0 < s < 1\), define the kernel \(K : X \times X \to [0, \infty]\):

\[
(2) \quad K(x, y) = [m(x, \rho(x, y)) + m(y, \rho(x, y))]^{-s} \approx m(x, \rho(x, y))^{-s}.
\]

When \(X\) is an Ahlfors-regular, bounded domain in \(\mathbb{R}^n\), the kernel \(K\) is a Riesz kernel (though since we are in a bounded domain, it would also make sense to say Riesz-Bessel):

\[
K(x, y) = \frac{1}{|x - y|^{sn}}.
\]

Given \(1 < p, p' < \infty, \frac{1}{p} + \frac{1}{p'} = 1\), and a Borel measure \(\omega \geq 0\) on \(X\), consider the \(p\)-energy of \(\omega\) associated with the kernel \(K\):

\[
\mathcal{E}_X(\omega) := \int_X [K\omega(x)]^{p'} dm(x),
\]

where \(K\omega(x) = \int_X K(x, y)d\omega(y)\). The \(p\)-capacity relative to the kernel \(K\) associates to \(E\), a compact subset of \(X\), the nonnegative number

\[
(3) \quad \text{Cap}_X(E) = \sup_{\text{supp}(\omega) \subseteq E} \left[\frac{\omega(K)^p}{\mathcal{E}_X(\omega)^{p-1}}\right].
\]
See [1], [18], [20], [21], [23] for various approaches to nonlinear capacities in measure metric spaces. The kind of potential theory which is discussed in this paper is only interesting when \( \frac{1}{p'} \leq s < 1 \). When \( 0 < s < \frac{1}{p'} \), singletons have positive capacity.

We will show that capacities in Ahlfors-regular metric spaces can be estimated by similar capacities defined on a totally disconnected metric space.

**Theorem 1** (Main Result). To \((X, m, \rho)\) we can associate a tree \( T \), a metric \( \rho_T \) on \( \partial T \) with respect to which \( \partial T \), the boundary of \( T \), is Ahlfors \( Q \)-regular, and a Lipschitz map 
\[
\Lambda : \partial T \to X
\]
in such a way that, if \( E \) is compact in \( X \), then
\[
\text{Cap}_X(E) \approx \text{Cap}_{\partial T}(\Lambda^{-1}(E)).
\]

Here, \( \text{Cap}_{\partial T} \) is the capacity on \( \partial T \) which corresponds to the same parameters used in defining the capacity on \( X \): \( Q \), the Hausdorff dimension; \( p \), the integrability exponent; \( s \), the kernel parameter. If \( m_T \) is the \( Q \)-Hausdorff measure on \( \partial T \), the kernel used to define \( \text{Cap}_{\partial T} \) is
\[
K_T(x, y) = [m_T(x, \rho_T(x, y)) + m_T(y, \rho_T(x, y))]^{-s}.
\]

In the other direction we have that, if \( F \) is compact in \( \partial T \), then
\[
\text{Cap}_X(\Lambda(F)) \approx \text{Cap}_{\partial T}(F).
\]

One of main difficulties here is that the map \( \Lambda \) is not one-to-one, therefore sets having positive capacity in \( X \) have multiple preimages, possibly far away, in \( \partial T \) (this difficulty does not appear when \( X \) is homeomorphic to a subset of the real line, since in this case the set of the points having multiple preimages is countable).

Let us mention an immediate consequence of Theorem 1 in the case of Euclidean space \( \mathbb{R}^n \). The tree is, in that case, the tree of the dyadic cubes, which has a huge group of automorphisms, which are in turn isometries of the corresponding measure-metric structure \((T, \rho_T, m_T)\). Such automorphisms are obtained by freely shuffling the cubes which are immediately below a cube. The capacity of a set is essentially invariant under any number of such shuffling operations. This invariance property is obviously much more general than invariance under isometries of the Euclidean space (the price to pay is that set capacity is invariant under automorphisms modulo multiplicative constants, which are anyway independent of the particular automorphism). Invariance under tree automorphisms is not obvious, especially in several variables, since portions of the set having positive capacity might be doubled (in the higher dimensional case) and moved apart one from the other (in all dimensions), operations which generally increase capacity.

Here is an outline of the article. In Section 2 we construct the graph \( G \) and the tree \( T \) associated with \( X \), and the surjection \( \Lambda : \partial T \to X \). Among the properties of \( \Lambda \) in the section, we show that the natural push-forward of measures \( m \mapsto \Lambda_* m \) has a (noncanonical) right inverse. This allows us to move measures back and forth from \( \partial T \) to \( X \).

In Section 3 we prove a version of the Muckenhoupt-Wheeden-Wolff inequality for potentials on graphs. The part of the inequality we need is that independently proved by T. Wolff. In Section 4 we show that measures on \( \partial T \) and \( X \), if they are correspondent under \( \Lambda_* \) or its inverse map, have comparable energy. As a consequence, we prove Theorem 1.
Section 5 presents the basic facts of potential theory on trees. First, we show that the Muckenhoupt-Wheeden-Wolff inequality establishes a correspondence between “Bessel” potential theory on $X$ (or $\partial T$) and (in the interesting case) “logarithmic” potential theory on $T \cup \partial T$. The latter is the sort of potential theory which has been independently developed by several authors over the past twenty years, an account of which is given in the rest of the section. Potential theory on trees is simpler than its counterparts on general Ahlfors-regular spaces for several reasons. The main such reason is that capacities can be computed explicitly by means of recursive formulas, which are deduced in the present article. Moreover, in the context of trees, sets’ boundaries are often trivial and scaling arguments are elementary and natural. As a consequence, the capacitary potential of a set $E$ is a simple object, whose geometry is linked to that of the set $E$ in a very transparent way.

Some of the material presented here has already appeared in the literature, while some of it is presented here for the first time. We present, in particular, a deduction of the trace inequality for the potential $K$ from a much simpler dyadic “Carleson measure inequality” on trees. Then, we give a rather direct proof that a testing condition known to be equivalent to the trace inequality implies a capacitary condition, also known to be equivalent to the trace inequality. This answers a question Maz’ya asked some of us some years ago.

For ease of the reader, we have put in Appendix I the basic results of Nonlinear Potential Theory as they are presented in [1] and in Appendix II their particularization to the tree context.

Our hope is that the simple tree model will be useful to researchers working in or using the results of potential theory. In this paper, we do not offer new applications of the equivalence between “classical” and tree capacities, except for a new proof of the trace inequalities in the Ahlfors-regular case. We hope to return to applications in other papers and that other people will find useful the tool we have here developed.

1.2. Some Examples of Ahlfors-Regular Metric Spaces. We end the introduction by mentioning a few (often not independent) examples of the spaces to which the theory developed in the present paper applies. More material, at a much deeper level, can be found, for example, in [20].

(1) Euclidean $\mathbb{R}^n$ and the $n$-dimensional sphere $\Sigma_n$ are $n$-dimensional Ahlfors-regular spaces.

(2) The both closed and open balls in Euclidean space $\mathbb{R}^n$ are $n$-regular Ahlfors spaces.

(3) The ternary Cantor set $C$ with the metric inherited from the real line is Ahlfors-regular with $Q = \frac{\log 2}{\log 3}$.

(4) If $(X, \rho, m)$ is Ahlfors $Q$-regular, the measure $m$ can be replaced by the $Q$-Hausdorff measure $H_Q^\rho$ in $(X, \rho)$ (the metric measure structure, that is, can be reduced to the metric structure alone).

(5) The “snowflake metric” $\rho(x, y) = |x - y|^{1/2}$ makes the real line into an Ahlfors 2-regular space.

(6) Carnot groups having homogeneous dimension $Q$ with their Carnot metrics are Ahlfors $Q$-regular. (See [12] for a comprehensive introduction to the topic).

(7) An $n$-dimensional, complete Riemannian manifold with nonnegative, bounded curvature is Ahlfors $n$-regular.

(8) The unit sphere in $\mathbb{C}^n$ endowed with the Koranyi metric $\rho(z, w) = |1 - z \cdot \overline{w}|^{1/2}$ is an Ahlfors-regular space having dimension $2n + 1$. 
(9) If \((X, \rho, m)\) is Ahlfors \(Q\)-regular and \(Q_\alpha\) is one of the dyadic boxes in \(X\)'s dyadic decomposition (see Christ's Theorem in the next section), then \(\overline{Q}_\alpha\), the closure of \(Q_\alpha\) in \(X\), is \(Q\)-regular. [See Theorem 9].

(10) The boundary of any homogeneous tree (except \(\mathbb{Z}\)) with respect to the Gromov distance is a regular Ahlfors space.

Notation. If \(A(P_1, \ldots, P_n)\) and \(B(P_1, \ldots, P_n)\) are two positive (or positively infinite) quantities depending on the objects \(P_1, \ldots, P_n\), we write \(A \approx B\) if there are constants \(0 < C_1 < C_2\) (independent of \(P_1, \ldots, P_n\)) such that \(C_1A \leq B \leq C_2A\). We write \(A \lesssim B\) if there is a constant \(C > 0\) such that \(A \leq CB\). We will denote by \(c\) a positive constant which might change value within the same expression or calculation.

2. A Metric Space, a Graph, and a Tree

In this section we consider the discretization of the Ahlfors-regular space \(X\) at different scales as proved by M. Christ. While we only need the easy part of Christ's Theorem, we have to be careful: having to deal with Borel measures, we can not discard sets having null measure. Christ's Theorem can be interpreted as the construction of a graph \(\Gamma\), having \(X\) as boundary. To better clarify this point we bi-Lipschitz modify the original distance \(\rho\) to \(\overline{\rho}\), a distance which extends to a length-distance on \(\overline{\Gamma} := \Gamma \cup X\). We then consider the tree structure \(T\) of the dyadic boxes: \(T\) and \(\Gamma\) have the same vertices, but \(T\) only preserves the vertical edges. The boundary \(\partial T\) of \(T\) is “larger” than \(X\), and can be thought of as a Cantor set. We construct an onto, Lipschitz map \(\Lambda: \overline{T} \to \overline{\Gamma}\), which maps boundaries to boundaries, \(\Lambda(\partial T) = X\). The main point of this section consists in proving that the push-forward of measures \(\Lambda_*\omega := \omega \circ \Lambda\) has a (noncanonical) left inverse with nice properties. The main technical difficulty consists in proving that \(\Lambda\) maps Borel sets to Borel sets (rather, a generalization of this fact). Once all this is done, we can move Borel measures back and forth through \(\Lambda\).

2.1. Christ’s Theorem Revisited. Let \((X, m, \rho)\) be an Ahlfors \(Q\)-regular metric measure space. We denote the balls by \(B(x, r) = \{y \in X: \rho(y, x) < r\}\). In this section we do not assume \(X\) to be bounded.

In particular, \((X, \rho)\) is a homogeneous space à la Coifman-Weiss: the measure \(m\) satisfies the doubling condition

\[
m(B(x, 2r)) \leq c_3m(B(x, r)).
\]

It is well known (see, e.g., Semmes’ essay in [17]) that we can take \(m\) to be the \(Q\)-dimensional Hausdorff measure for the metric space \((X, \rho)\).

We shall also assume, in the main body of the paper, that \(X\) is bounded, \(\text{diam}(X) \leq 1\), so that (1) only is required to hold when \(r \leq 1\). Under the sole hypothesis that \((X, \rho, m)\) is a homogeneous space, M. Christ [13] proved that \(X\) admits a “dyadic” decomposition. We state here the easy part of Christ’s Theorem (the difficult part is an estimate of the mass concentrated near the boundary of the dyadic blocks).

**Theorem 2** (M. Christ, [13]). There exists a collection \(\{Q^k_\alpha, \alpha \in I_k, k \in \mathbb{Z}\}\) of open subsets of \(X\) and \(\delta > 0, a_0 > 0, c_4 > 0\) such that

(i) \(m(X \setminus \cup_{\alpha \in I_k} Q^k_\alpha) = 0\) holds for all \(k \geq 0\);

(ii) if \(l \geq k\), then either \(Q^l_\beta \subseteq Q^k_\alpha\), or \(Q^l_\beta \cap Q^k_\alpha = \emptyset\);

(iii) for all \((l, \beta)\) and \(l > k\), there is a unique \(\alpha\) in \(I_k\) such that \(Q^l_\beta \subseteq Q^k_\alpha\);
(iv) \( \text{diam}(Q^k_\alpha) \leq c_4 \delta^k; \)
(v) for all \((k, \alpha), B(z^k_\alpha, \delta^k) \subseteq Q^k_\alpha\), for distinguished points \(z^k_\alpha \in Q^k_\alpha\).

If \( X \) is bounded, after multiplying the distance times a positive constant, we can assume that \( k \in \mathbb{N} \) and that \( I_0 \) contains a unique element \( o: Q^0_\alpha = X \). We take the sets \( \alpha \equiv (k, \alpha) \leftrightarrow Q^k_\alpha \) themselves as points of a new space \( T \). When we do not want to stress the level of \( \alpha \), we simply write \( Q_\alpha \) instead of \( Q^k_\alpha \).

For \( \alpha \in I_k \), we set \( k = d(\alpha) \) to be the level of \( \alpha \). To the set \( T \) we can associate a tree structure. To do this, there is an edge of the tree between \( \alpha \) and \( \beta \) if \( d(\beta) = d(\alpha) + 1 \) and \( \beta \subseteq \alpha \), possibly interchanging the roles of \( \alpha \) and \( \beta \). The tree \( T \) has a natural, edge-counting distance \( d \), which is realized by geodesics. We can then introduce partial order on the tree by saying \( \alpha \leq \beta \) in \( T \) if \( \alpha \in [\alpha, \beta] \), the geodesic joining \( o \) and \( \beta \).

We also introduce on \( T \) a graph structure. Two distinct points \( \alpha \) and \( \beta \) in \( T \) are connected by an edge of the graph \( G \) if they are already connected by an edge of \( T \), or if \( d(\alpha) = d(\beta) = k \) and there are points \( x \in \alpha \) and \( y \in \beta \) such that \( \rho(x, y) \leq \delta^k \). In this case, we write \( \alpha \sim_G \beta \). The natural edge-counting distance in \( G \) is denoted by \( d_G \). It is realized by geodesics, but, contrary to the tree case, there might be several geodesics joining two points. We identify \( T = G \) as sets of vertices, and use different names when dealing with different edge structures. (In Graph Theory it is said that \( T \) is a spanning tree for the graph \( G \)).

Since we are interested in (possibly singular) measures on \( X \), we have to refine Christ’s construction in order to have regions whose union gives us back the whole space \( X \). This is done via the following lemmas.

**Lemma 3.** For each \( k \), we have that \( \bigcup_{\alpha \in I_k} \overline{Q^k_\alpha} = X \).

**Proof.** By contradiction, suppose that \( x \) lies in \( X \setminus \bigcup_{\alpha \in I_k} \overline{Q^k_\alpha} \). Then, any ball \( B(x, \epsilon) \) intersects \( Q^k_\alpha \) for infinitely many \( \alpha \) in \( I_k \). In fact, since \( m(B(x, \epsilon)) > 0 \), by (i) in Theorem 2 (and the assumption that open sets have positive measure, by Ahlfors-regularity) there exists \( \alpha_1 \) in \( I_k \) such that \( B(x, \epsilon) \cap Q^k_{\alpha_1} \neq \emptyset \). There must be some \( 0 < \epsilon_2 < \epsilon = \epsilon_1 \) such that \( B(x, \epsilon_2) \cap Q^k_{\alpha_1} = \emptyset \), otherwise \( x \in \overline{Q^k_{\alpha_1}} \), a contradiction. As before, there must be then \( \alpha_2 \neq \alpha_1, \alpha \) such that \( B(x, \epsilon_2) \cap Q^k_{\alpha_2} \neq \emptyset \).

Iterating this procedure, we find a sequence \( Q^k_{\alpha_n} \) of distinct (hence, pairwise disjoint) dyadic regions at generation \( k \). They must be contained in the ball \( B(x, \epsilon + 2c_4 \delta^k) \), by (iv), and each of them has volume at least \( c_4 \delta^k \), by Ahlfors-regularity (1). Hence,

\[
+\infty = \sum_1^\infty m(Q^k_{\alpha_n}) = m(\bigcup_1^\infty Q^k_{\alpha_n}) \leq m(B(x, \epsilon + 2c_4 \delta^k)) < +\infty,
\]
yielding a contradiction. Thus, \( \bigcup_{\alpha \in I_k} \overline{Q^k_\alpha} = X \).  

**Lemma 3** has the following simple corollary.

**Corollary 4.** Let \( F^k = \bigcup_{\alpha \in I_k} \partial Q^k_\alpha \) and \( F = \bigcup_k F^k \). Then,
(i) \( F^k \) is closed;
(ii) \( F^k \subseteq F^{k+1} \);
(iii) \( m(F) = 0 \).

**Lemma 5.**
(i) There exists \( c_5 \) independent of \( k \geq 0 \) and of \( x \in X \) such that \( \sharp \{ \alpha: x \in \overline{Q^k_\alpha} \} \leq c_5; \)
(ii) There exists $c_6$ independent of $\alpha$ in $G$ such that $\sharp\{\beta \in G : \beta^e_G \alpha \leq c_6\}$: the graph $G$ has bounded connectivity.

**Proof.** It is clear that (i) is a consequence of (ii). In fact, all $\alpha \in I_k$ containing $x$ in their closure are $G$-related.

Fix $x \in Q^k_\alpha$: there is a constant $c$ such that, if $\rho(Q^k_\alpha, Q^k_\beta) \leq \delta^k$ (a property implied by $\beta^e_G \alpha$), then $Q^k_\beta \subseteq B(x, c\delta^k)$. Now,

$$\delta^k \cdot \sharp\{\beta \in I_k : Q^k_\beta \cap B(x, \delta^k) \neq \emptyset\} \leq c \sum_{Q^k_\beta \cap B(x, \delta^k) \neq \emptyset} m(Q^k_\beta)$$

$$= m(\bigcup_{Q^k_\beta \cap B(x, \delta^k) \neq \emptyset} Q^k_\beta) \leq m(B(x, c\delta^k)) \leq c\delta^k,$$

which implies (ii). ■

Complete Ahlfors-regular spaces are not especially exotic among metric spaces since they have some expected properties.

**Corollary 6.** $(X, \rho)$ is locally compact.

**Proof.** To see this, mimic the proof of the Heine-Borel Theorem, using the dyadic decomposition of Christ instead of the usual dyadic decomposition of $\mathbb{R}^n$, and the completeness of $X$. ■

**Corollary 7.** $(X, \rho)$ is separable.

**Proof.** It suffices to show that each $\overline{Q}_\alpha$ is separable. By Christ’s Theorem, for each integer $n > 0$, there are finitely many points $\{z^n_j : j = 1, \ldots, k_n\}$ in $\overline{Q}_\alpha$ such that any point $x$ in $\overline{Q}_\alpha$ lies at distance at most $2^{-n}$ from some $z^n_j$. The countable set $\{z^n_j : j = 1, \ldots, k_n, n \geq 1\}$ is dense in $\overline{Q}_\alpha$. ■

**Lemma 8.** If $\alpha \in I_k$ and $h \geq 0$, then

$$\overline{Q}_\alpha = \bigcup_{\beta \geq \alpha, \beta \in I_{k+h}} \overline{Q}_\beta.$$

**Proof.** Clearly, $\overline{Q}_\alpha \supseteq \bigcup_{\beta \geq \alpha, \beta \in I_{k+h}} \overline{Q}_\beta$. In the other direction, since there are finitely many $Q_\beta$’s, it suffices to show that $\bigcup_{\beta \geq \alpha, \beta \in I_{k+h}} \overline{Q}_\beta$ is dense in $Q_\alpha$. If this were not the case, there would be a metric ball $B(z, \epsilon)$ in $Q_\alpha \setminus \bigcup_{\beta \geq \alpha, \beta \in I_{k+h}} \overline{Q}_\beta$, hence, by Ahlfors-regularity,

$$m(Q_\alpha) > \sum_{\beta \geq \alpha, \beta \in I_{k+h}} m(Q_\beta),$$

which contradicts Corollary 4. ■

**Theorem 9.** Each set $X_\alpha = \overline{Q}_\alpha$ is Ahlfors $Q$-regular. Moreover, the dyadic sets in Christ’s decomposition of $X_\alpha$ may be taken to be the sets $Q_\beta$ where $\beta \geq \alpha$.

**Proof.** Assume without loss of generality that $\alpha \in I_0$ and let $x$ be a point of $\overline{Q}_\alpha$. Let $r \in (0, 1)$, $\delta^{k+1} < r \leq \delta^k$, and let $\beta \in I_k$, $\beta \geq \alpha$, be such that $x \in \overline{Q}_\beta$ (at least one such $\beta$ exists by Lemma 8). By Christ’s Theorem, there is constant $C$ such that $Q_\alpha \cap B(x, C\delta^k) \supseteq Q_\beta$, hence,

$$m(Q_\alpha \cap B(x, C\delta^k)) \geq m(Q_\beta) \geq C'\delta^Q k.$$

Ahlfors-regularity of $\overline{Q}_\alpha$ immediately follows. ■
2.2. The space $X$ as the Boundary of $G$. We now define a new distance $\overline{\rho}$ on $G \cup X$. Let $\alpha \neq \beta$ be points of $G$ such that $d(\alpha) \leq d(\beta) \leq d(\alpha) + 1$ and $\alpha \not\sim \beta$. To the edge $[\alpha, \beta]$ we associate the weight $\ell_G([\alpha, \beta]) := \delta d(\alpha)$. The length of a path $\Gamma = (\alpha_0, \ldots, \alpha_n)$ in $G$ is defined by adding the length of its edges. If needed, we set $\ell_G((\alpha)) = 0$, where $\alpha$ is the trivial path from $\alpha$ to itself. The distance $\overline{\rho}$ is defined, on points $\alpha, \beta \in G$, as

$$\overline{\rho}(\alpha, \beta) = \inf\{\ell_G(\Gamma) : \Gamma \text{ a path having endpoints } \alpha \text{ and } \beta\}. \tag{6}$$

By the triangle inequality,

$$\rho(Q_\alpha, Q_\beta) \leq c\overline{\rho}(\alpha, \beta). \tag{7}$$

Let $(\overline{G}, \overline{\rho})$ be the completion of the metric space $(G, \rho)$. Each point $\alpha \in G$ is isolated in $\overline{G}$. Let $\partial G := \overline{G} \setminus G$.

There is a bijection $\flat$ between $\partial G$ and $X$. To define the bijection $\flat$, let $a = \{\alpha_n\}$ be a Cauchy sequence in $(\overline{G}, \overline{\rho})$, which is not constant, and let $[a]$ be the equivalence class of $a$ in $\partial G$. Define

$$b([a]) = x \in X$$

if and only if $\lim_{n \to \infty} \rho(x, Q_{\alpha_n}) = 0$. We then have the following Lemma.

**Lemma 10.** $b$ is a well defined bijection of $\partial G$ onto $X$.

**Proof.** The proof will be broken up into several steps.

- To each Cauchy sequence we associate a unique $x \in X$.
  Let $d(\alpha_n)$ be the level of $\alpha_n$ in $T$. Since the sequence is not constant, it is Cauchy and $G$ is discrete, we must have $\lim_{n \to \infty} d(\alpha_n) = +\infty$. Let $z_n$ be a distinguished point in $Q_{\alpha_n}$. Let $n, m \geq 1$. Using (7),

$$\overline{\rho}(z_n, z_{n+m}) \leq \text{diam}(Q_{\alpha_n}) + \rho(Q_{\alpha_n}, Q_{\alpha_{n+m}}) + \text{diam}(Q_{\alpha_{n+m}}) \leq c(\delta d(\alpha_n) + \overline{\rho}(\alpha_n, \alpha_{n+m}) + \delta d(\alpha_{n+m})) \to 0,$$

as $n \to \infty$. Since $(X, \rho)$ is complete and $\{z_n\}$ is Cauchy, it has a well defined limit $x$ and $\lim_{n \to \infty} \rho(x, Q_{\alpha_n}) \leq \lim_{n \to \infty} \rho(x, z_n) = 0$.

Any other choice of the distinguished points, say $\zeta_n \in Q_{\alpha_n}$, clearly gives the same limit, since $\lim_{n \to \infty} \rho(z_n, \zeta_n) = 0$ (with the sequence $\alpha_n$ non constant).

- If $[a] = [b]$, $a$ and $b$ define the same point in $X$: $b[a] = b[b]$.
  Let $a = \{\alpha_n\}$ and $b = \{\beta_n\}$. Choose distinguished points $z_n \in Q_{\alpha_n}$ and $w_n \in Q_{\beta_n}$. Reasoning as above, $\lim_{n \to \infty} \rho(z_n, w_n) = 0$, hence the two sequences converge to the same point $x \in X$.

- $b : \partial G \to X$ is surjective.
  Given $x \in X$, for each $n \geq 0$ choose a $Q_{\alpha_n}^n$ such that $x \in \overline{Q_{\alpha_n}^n}$. It is easily checked that the graph distance between $(n, \alpha_n)$ and $(n+1, \alpha_{n+1})$ is at most 2 and that $b([((n, \alpha_n))]) = x$.

- $b$ is injective.
  If $x \neq y$, $\rho(x, y) \geq \delta^k$ for some $k$. Let $a = \{\alpha_n\}$ and $b = \{\beta_n\}$ be such that $x = b([a])$ and $y = b([b])$. Since $d(\alpha_n), d(\beta_n) \to \infty$, for some $n_0$ and $n \geq n_0$ we have $n \geq k$ and $\rho(Q_{\alpha_n}^n, Q_{\beta_n}^n) \geq \frac{1}{2}\delta^k$. Then, $\overline{\rho}(\alpha_n, \beta_n) \geq \epsilon\delta^k$ for large $n$ ($\epsilon$ fixed), which in turn implies $[a] \neq [b]$.
The map \( b \) in Lemma 10 identifies the metric measure space \( X \) we started with the metric boundary of a graph \( G \). From now on, we simply write \( \partial G = X \). The theorem below says that such identification carries all the important structure of \( X \).

**Theorem 11.**

(i) The metric \( \bar{\rho} \), restricted to \( X \), is bi-Lipschitz equivalent to \( \rho \):

\[
C \rho(x, y) \leq \bar{\rho}(x, y) \leq C \rho(x, y).
\]

(ii) Let \( Z_k = \{ z^k_\alpha : \alpha \in I_k \} \) be the set of the distinguished points \( z^k_\alpha \in Q^k_\alpha \) mentioned in Theorem 2; and let \( G_k = \{ \alpha \in G : d(\alpha) = k \} \). Then, \( \rho|_{Z_k} \) is bi-Lipschitz equivalent to \( \bar{\rho}|_{G_k} \):

\[
C \rho(z^k_\alpha, z^k_\beta) \leq \bar{\rho}(\alpha, \beta) \leq C \rho(z^k_\alpha, z^k_\beta).
\]

The theorem says that all Ahlfors-regular spaces (bounded and complete, so far, but we think that these further assumptions can be easily removed) are - modulo a bi-Lipschitz change of the metric - the boundaries of rather regular and concrete graphs. Or, in other terms, that all such metric spaces admit a natural extension to a metric measure space in which the distance is given by length of curves and it is realized by geodesics.

By fattening the graph’s edges, it is conceivable that the space \( X = \partial G \) is the metric boundary (modulo bi-Lipschitz change of metric) of a Riemannian manifold \( M \) which maintains many of the properties of \( X \) (\( M \), for instance, could be taken to be Ahlfors-regular, having as dimension the least integer \( n > Q \)). This simple construction seems to have gone so far unnoticed.

**Proof.** (i) Let \( x, y \) be points in \( X \). We show first that \( \bar{\rho}(x, y) \leq C \rho(x, y) \). Suppose that \( \rho(x, y) \leq \delta^k \) and let \( \alpha, \beta \in I_k \) be such that \( x \in Q_\alpha^k, y \in Q_\beta^k \). Then, there is an edge of \( G \) between \( \alpha \) and \( \beta \), hence \( \bar{\rho}(\alpha, \beta) \leq \delta^k \). Arguing like in the proof of \( b \)'s surjectivity, we can find sequences \( a = \{ \alpha_n \} \) and \( b = \{ \beta_n \} \) such that \( \alpha_0 = \alpha, \beta_0 = \beta; \beta_n, \alpha_n \in I_{k+n}; x \in Q_{\alpha_n} \) and \( y \in Q_{\beta_n} \). Two successive elements of the sequence \( a \) have graph distance at most 2, and the same holds for \( b \). A geometric series argument applied to the sequences shows then that \( \bar{\rho}(\alpha, x) \leq c \delta^k \) and \( \bar{\rho}(y, \beta) \leq c \delta^k \). Then, \( \bar{\rho}(x, y) \leq C \delta^k \), as wished.

In the other direction, assume that \( \rho(x, y) > 0 \) (otherwise there is nothing to prove), with \( x = b((\{ \alpha_n \})) \) and \( y = b((\{ \beta_n \})) \). By Lemma 10, for each \( \epsilon > 0 \) we can take \( n = n(\epsilon) \) large enough to have:

(a) \( \bar{\rho}(x, \alpha_n), \bar{\rho}(y, \beta_n) \leq \epsilon \rho(x, y) \) (because \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are Cauchy sequences tending to \( x \) and \( y \), respectively, in \( \bar{G} \));

(b) \( \rho(x, \alpha_n), \rho(y, \beta_n) \leq \epsilon \rho(x, y) \) (by definition of \( b \));

(c) \( n \geq n_0 \) to be chosen.

Let \( \gamma_0 = \alpha_n, \gamma_1, \ldots, \gamma_m = \beta_n \) be any path in \( G \) which joins \( \alpha_n \) and \( \beta_n \), and choose distinguished points \( w_j \in Q_{\gamma_j} \) (\( 0 \leq j \leq m \)). We have then

\[
\rho(x, y) \leq \rho(x, w_0) + \sum_{j=1}^{m} \rho(w_j, w_{j-1}) + \rho(w_m, y)
\leq 2\epsilon \rho(x, y) + \text{diam}(Q_{\alpha_n}) + \text{diam}(Q_{\beta_n}) + \sum_{j=1}^{m} \rho(w_j, w_{j-1}) + \rho(w_m, y) \quad \text{(by (c))}
\leq 4\epsilon \rho(x, y) + c\ell_G((\gamma_0, \ldots, \gamma_m)),
\]
by (c) and by definition of $\ell_G$. Passing to the infimum of the lengths,

$$(1 - 4\epsilon)\rho(x, y) \leq c \overline{\rho}(\alpha_n, \beta_n)$$

$$\leq c(\overline{\rho}(x, y) + \overline{\rho}(x, \alpha_n) + \overline{\rho}(y, \beta_n))$$

$$\leq c\overline{\rho}(x, y) + 2\epsilon\rho(x, y),$$

by (a). It suffices to choose $\epsilon$ small enough to get the desired inequality.

Part (ii) of the theorem is proved similarly.

A consequence of the proof is that, given $x, y \in X$, there exists a “near geodesic” in $\overline{G}$ for the distance $\overline{\rho}(x, y)$, passing through a point at level $k$, where $\rho(x, y) \approx \delta^k$. This fact, indeed, extends the well-known relations between Euclidean and hyperbolic distances, and geodesics, in the upper half space. When the space $(X, \rho)$ has a metric defined by a length, like the spaces introduced by Heinonen and Koskela in [21], then we find “near geodesics” joining $x, y \in X$ at all levels $n$ of $\overline{G}$ such that $\delta^n \leq \rho(x, y)$. When the space doesn’t have “enough” rectifiable curves, as it happens with “snowflake” metrics on the real line, the near geodesics joining $x$ and $y$ cannot be completely contained in a small strip of $\overline{G}$, they have to reach a level $n$ such that $\delta^n \gtrsim \rho(x, y)$.

2.3. The Tree and its Boundary as Metric Spaces.

2.3.1. The Boundary of a Tree. Let $T$ be a tree with root $o$ (not necessarily the tree arising from Christ’s Theorem). We assume that $T$ has bounded connectivity and that it has no childless vertices. Namely, there is $N \geq 1$ such that each vertex $\alpha$ in $T$ has $N(\alpha)$ children, with $1 \leq N(\alpha) \leq N$.

We introduce some notation which will be frequently used. If $\alpha$ is a vertex of $T$, the predecessor set of $\alpha$ is $P(\alpha) = [\alpha, \alpha]$, while the successor set of $\alpha$ is $S(\alpha) = \{\beta \in T : \alpha \in P(\beta)\}$. Given $\alpha$ and $\beta$ in $T$, $\alpha \wedge \beta = \max(P(\alpha) \cap P(\beta))$ is their confluent. The predecessor $\alpha^{-1}$ of $\alpha \in T \setminus \{o\}$ is the only element $\beta$ in $P(\alpha)$ such that $d(\alpha, \beta) = 1$.

The combinatorial boundary $\partial T$ of $T$ is the set of all half-infinite geodesics (with respect to the edge counting distance) having an endpoint at the root. For ease of notation, we think of $\partial T$ as a set of labels $\zeta$ of the geodesics $\Gamma_\zeta$. The topology for $\partial T$ is that having as basis the sets

$$\partial S(\alpha) = \{\zeta \in \partial T : \alpha \in \Gamma_\zeta\}.$$

Let now $\delta \in (0, 1)$ be fixed. Associate to each edge $(x^{-1}, x)$ of $T$ $(x^{-1}, x^{-1})$, we recall, is the parent of $x$) the weight $w(x, x^{-1}) = \delta^d(x, o)$, where $d$ is the edge-counting distance in $T$. The metric $\rho_T$ on $T$ with parameter $\delta$ is defined as the length-metric associated with the weight $w$. The geodesics for $\rho_T$ are clearly the same as the geodesics for the edge-counting metric (actually, the poverty of the tree structure is such that all length-metrics have the same geodesics; and the edge-counting metric is a particular length metric). Let $\overline{T}$ be the metric completion of $T$ with respect to $\rho_T$. The metric boundary $\partial T$ of $T$ is $\overline{T} \setminus T$, with the topology induced by the metric $\rho_T$.

Lemma 12. For each $\zeta$ in $\overline{T}$, consider the geodesic $\Gamma_\zeta$ as a sequence of points in $T$. Then,

(i) $\Gamma_\zeta$ is a Cauchy sequence for $\rho_T$;

(ii) Let $\tilde{\zeta}(\zeta)$ be the equivalence class of $\Gamma_\zeta$ in $\partial T$. The map $\zeta \mapsto \tilde{\zeta}(\zeta)$ is a homeomorphism of $\partial T$ onto $\partial T$. 
The proof of the lemma is easy and it is left to the reader. Using this Lemma, extend \( \natural \) to a map from \( T \cup \tilde{\partial}T \) to \( T \cup \partial T \) by letting \( \natural|_T = Id|_T \). Requiring the extended \( \natural \) to be a homeomorphism gives a topological structure to \( \overline{T} = T \cup \tilde{\partial}T \).

We extend the tree notation to take into account the boundary. If \( \zeta \in \partial T \), let \( P(\zeta) = \Gamma \zeta \) be the geodesic from the root to \( \zeta \). For \( \alpha, \beta \in T \), let \( \alpha \wedge \beta = \max(P(\alpha) \cap P(\beta)) \) be the confluent of \( \alpha \) and \( \beta \). Observe that the geodesic \( [\alpha, \beta] \) between \( \alpha \) and \( \beta \) in \( T \) with respect to the metric \( \rho_T \) is

\[
[\alpha, \beta] = [P(\alpha) \cup P(\beta)] \setminus P((\alpha \wedge \beta)^{-1})
\]

(thinking of geodesics as sets of vertices, rather than sequences of adjacent edges).

If \( \alpha \) is an element of \( T \), \( \partial S(\alpha) \subseteq \partial T \) is the set of the half-infinite geodesics passing through \( \alpha \), \( \partial S(\alpha) = S(\overline{\alpha}) \setminus S(\alpha) \). The metric \( \rho_T \) can be explicitly computed. If \( \zeta, \xi \in T \), then

\[
\rho_T(\zeta, \xi) = \frac{2\delta}{1 - \delta} \left[ \delta^{d(\zeta \wedge \xi)} - \frac{1}{2} \left( \delta^{d(\zeta)} + \delta^{d(\xi)} \right) \right].
\]

In some references \( \rho_T \) is called the Gromov metric on \( T \). The restriction of \( \rho_T \) to \( \partial T \) is

\[
\rho_T(\zeta, \xi) = \frac{2\delta}{1 - \delta} \delta^{d(\zeta \wedge \xi)}.
\]

The balls in \( (\partial T, \rho_T) \) are exactly the cl-open sets \( \partial S(\alpha) \) and \( T \) itself might be seen as the tree of the metric balls in \( \partial T \).

2.3.2. The tree \( T \) associated with the graph \( G \). Let now \( T \), the rooted tree, and \( \delta > 0 \) be the same as in Christ’s Theorem. The metric \( \rho_T \) just defined on \( T \) is the length-metric associated with the length \( \ell_T \), which is the restriction of the length \( \ell_G \) to the edges of \( T \). The trivial estimate \( \rho_T(\alpha, \beta) \geq \overline{\rho}(\alpha, \beta) \) cannot in general be reversed.

We can define a Borel measure \( \tilde{m} \) on \( \partial T \) by declaring that

\[
\tilde{m}(\partial S(\alpha)) := m(Q_\alpha).
\]

It is easy to see that the definition is consistent (by Christ’s Theorem).

**Proposition 13.** The space \( (\partial T, \rho_T, \tilde{m}) \) is complete and Ahlfors \( Q \)-regular.

The statement follows immediately from the fact noted above, that the metric balls in \( \partial T \) are the sets of the form \( \partial S(\alpha) \).

2.4. The Map \( \Lambda : \partial T \to X \). Let \( P_T(\xi) = (\xi_n) \) be the geodesic starting at the root \( o \) and ending at the boundary point \( \xi \in \partial T \) (we might, and sometimes will, identify \( \xi \equiv P_T(\xi) \)). Each \( \xi_n \in T \) appearing in \( P_T(\xi) \) can be identified with a dyadic box \( Q(\xi_n) \) in \( I_n \).

We define a map \( \Lambda : \partial T \to X \),

\[
\Lambda : \xi \mapsto \Lambda(x) = \bigcap_{n \geq 0} \overline{Q(\xi_n)}.
\]

By the discussion above, \( \Lambda \) is a contraction,

\[
\overline{p}(\Lambda(\zeta), \Lambda(\xi)) \leq \rho_T(\zeta, \xi).
\]

The map \( \Lambda \) is not open, however, neither is it one-to-one, in general.
Lemma 14. There is a constant $c$ independent of $x$ such that
\[ \sharp\{\zeta \in T : \Lambda(\zeta) = x\} \leq c. \]

Proof. Let $\zeta^{(1)}, \ldots, \zeta^{(n)}$ be points in $\partial T$ such that $\Lambda(\zeta^{(j)}) = x$, $\zeta^{(j)} = (\zeta_{k}^{(j)})_{k \in \mathbb{N}}$. Let $N = \max\{d(\zeta^{(j)} \cap \zeta^{(i)}) : i \neq j = 1, \ldots, n\}$. Then, $x \in \zeta^{(1)}_{N+1} \cap \cdots \cap \zeta^{(n)}_{N+1}$, and the dyadic sets $\zeta^{(j)}_{N+1}$ are all distinct. By Lemma 5 (i), $n \leq c_5$. ■

Lemma 15. $\Lambda$ maps $\partial T$ onto $X$.

Proof. Let $x$ be a point of $X$, $k \geq 0$, consider those $Q^k_{\alpha}$ such that $x \in \overline{Q^k_{\alpha}}$, and let $J_k$ be the set of such $Q^k_{\alpha}$. Each dyadic set in $J_{k+1}$ must have its parent in $J_k$. If $x \in Q^{k+1}_{\alpha}$, this is obvious; if $x \in \partial Q^{k+1}_{\alpha}$, and $Q^{k+1}_{\beta}$ is the parent of $Q^{k+1}_{\alpha}$, then $x \in \overline{Q^{k}_{\beta}}$. This shows that $J = \bigcup_k J_k$ is a full subtree of $T$ ($o \in J$ and $\alpha, \beta \in J \implies [\alpha, \beta] \subseteq J$), having elements at each level $k \geq 0$. By the Axiom of Choice, $J$ contains an infinite geodesic $\zeta$, and clearly $\Lambda(\zeta) = x$. ■

At this point, we can be more explicit about the shape of our “near geodesics”. Let $x \neq y \in X$ and let $k \geq 0$ an integer such that $\delta^{k+1} \leq \rho(x, y) \leq \delta^k$. Consider tree geodesics $\Gamma_x, \Gamma_y$ starting at level $k$ and going all the way down to $x, y$ respectively, and let $\gamma_k(x)$ and $\gamma_k(y)$ be their points at level $k$. Clearly, $\rho(\gamma_k(x), \gamma_k(y)) \leq c\delta^h$. Hence, there is a path $\Gamma'$ in $G$, having length $\ell_G(\Gamma') \lesssim \delta^k$ joining $\gamma_k(x)$ and $\gamma_k(y)$. Furthermore, there is a fixed constant $h \in \mathbb{N}$ such that the path $\Gamma'$ consists of at most $h$ edges between level $k$ and level $k - h$. The path $\Gamma_x \cup \Gamma' \cup \Gamma_y$ is clearly a near geodesic between $x$ and $y$.

2.5. Moving Measures through $\Lambda$. We can use the map $\Lambda$ to push Borel measures from $\partial T$ to $X$. If $\nu$ is a measure on $\partial T$, let
\[ \Lambda_* \nu(E) := \nu(\Lambda^{-1}(E)), \]
whenever $E$ is Borel measurable in $X$. We want to move measures in the other direction as well. Let $\omega$ be a nonnegative Borel measure on $X$ and for each Borel subset $A$ of $\partial T$ set
\[ N(x) = \sharp\{\zeta \in \partial T : \Lambda(\zeta) = x\}, \quad N_A(x) = \sharp\{\zeta \in A : \Lambda(\zeta) = x\} \]
and define
\[ \Lambda^* \omega(A) := \int_X \frac{N_A(x)}{N(x)} d\omega(x). \]

Note that, contrary to the pushforward $\Lambda_*$, the operator $\Lambda^*$ is noncanonical. The integral (9) is well defined because $x \mapsto N_A(x)$ is Borel measurable, a fact which will be proved below.

For each $x$ in $X$, let $\nu_x(A) = \frac{\sharp(\Lambda^{-1}(x) \cap A)}{\sharp(\Lambda^{-1}(x))}$ be the normalized counting measure on $\Lambda^{-1}(x)$. Then
\[ \int_{\partial T} \varphi(\zeta) d\Lambda^* \omega(\zeta) = \int_X \left\{ \int_{\Lambda^{-1}(x)} \varphi(\zeta) d\nu_x(\zeta) \right\} d\omega(x) \]
whenever $\omega$ is a Borel measure on $X$ and $\varphi$ is Borel measurable.

We need the following lemma, whose proof is rather tedious, but reveals the inner functioning of the map $\Lambda$.

Lemma 16. Each Borel set $A$ in $\partial T$ can be decomposed $A = \bigsqcup_i A_i$ as the disjoint, countable union of Borel sets $A_i$ such that the restriction $\Lambda|_{A_i}$ of $\Lambda$ to each $A_i$ is one to one. Moreover, if $A$ is a Borel set in $\partial T$, then $\Lambda(A)$ is a Borel set in $X$. 
Proof. Let $N$ be the counting function just introduced and define the stopping time

$$n(x) := \min\{n \in \mathbb{N} : x \text{ belongs to the closure of } N(x) \text{ cubes at level } n\}.$$ 

We need the sets

$$H_{k,n} = \{x : N(x) = k \text{ and } n(x) = n\},$$

and their pieces

$$H_{k,n}(\alpha_1, \ldots, \alpha_k) = H_{k,n} \cap \overline{Q}_{\alpha_1} \cap \cdots \cap \overline{Q}_{\alpha_k},$$

where $\alpha_1, \ldots, \alpha_k \in \mathcal{I}_n$ (index set for the cubes at level $n$) are distinct and their order in the labeling of the set $H_{k,n}(\alpha_1, \ldots, \alpha_k)$ does not matter.

We claim that each set $H_{k,n}(\alpha_1, \ldots, \alpha_k)$ is Borel measurable in $X$. (Some of these sets and of the sets introduced below are empty; this causes us no trouble). To see this claim, set $E_k = \{x \in X : N(x) \geq k\}$. If $\mathcal{I}_n$ is the set of cubes at level $n$, then $E_1 = X$ and

$$E_k = \bigcup_n \bigcup_{\alpha_1, \ldots, \alpha_k \text{ distinct in } \mathcal{I}_n} (\overline{Q}_{\alpha_1} \cap \cdots \cap \overline{Q}_{\alpha_k}).$$

Each $E_k$ is Borel measurable in $X$. Set now $F_k := E_k \setminus E_{k+1} = \{x \in X : N(x) = k\}$. Set further $G_{1,1} = F_1$, and

$$G_{k,n} = F_k \cap \left[ \bigcup_{\alpha_1, \ldots, \alpha_k \text{ distinct in } \mathcal{I}_n} (\overline{Q}_{\alpha_1} \cap \cdots \cap \overline{Q}_{\alpha_k}) \right].$$

Then, $H_{k,n} = G_{k,n} \setminus G_{k,n-1}$ and the claimed measurability of the sets $H_{k,n}(\alpha_1, \ldots, \alpha_k)$ follows.

Set now, for $1 \leq j \leq k$, $H^j_{k,n}(\alpha_1, \ldots, \alpha_k)$ to be the subset of $\partial T$ defined by

$$H^j_{k,n}(\alpha_1, \ldots, \alpha_k) = \Lambda^{-1}(H_{k,n}(\alpha_1, \ldots, \alpha_k)) \cap \partial S(\alpha_j).$$

We have the following properties, easy to verify:

(i) $H^j_{k,n}(\alpha_1, \ldots, \alpha_k)$ is Borel measurable in $\partial T$;

(ii) For each $1 \leq j \leq k$ and each choice of distinct $\alpha_1, \ldots, \alpha_k$ in $\mathcal{I}_n$, $\Lambda$ maps $H^j_{k,n}(\alpha_1, \ldots, \alpha_k)$ bijectively onto $H_{k,n}(\alpha_1, \ldots, \alpha_k)$;

(iii) The sets $H^j_{k,n}(\alpha_1, \ldots, \alpha_k)$ are mutually disjoint.

We use now the sets just introduced to define a set family in $\partial T$. Namely, $\mathcal{G}$ is the family of the sets having the form

$$\prod_{k, n \leq j \leq k, \alpha_1, \ldots, \alpha_k \in \mathcal{I}_n} \left[ H^j_{k,n}(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(A(k, n; j; \alpha_1, \ldots, \alpha_k)) \right],$$

where each set $A(k, n; j; \alpha_1, \ldots, \alpha_k)$ is Borel measurable in $X$. Since $\Lambda$ is continuous, $\mathcal{G}$ is contained in the Borel $\sigma$-algebra of $\partial T$. The family of sets $\mathcal{G}$ has in addition the following properties.

(i) $\mathcal{G}$ is a $\sigma$-algebra;

(ii) The image of each set in $\mathcal{G}$ is measurable in $X$;

(iii) Each basic set $\partial S(\alpha)$ belongs to $\mathcal{G}$. 
A consequence of (i),(iii) is that \( \mathcal{G} \) is the Borel \( \sigma \)-algebra in \( \partial T \) and that the image of a Borel set of \( \partial T \) under \( \Lambda \) is Borel in \( X \), namely, both \( \Lambda \) and \( \Lambda^{-1} \) map Borel sets to Borel sets. Statement (i) is easily verified:

\[
\partial T \setminus \left\{ \prod_{k,n \leq j \leq k} \left[ \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(A(k,n;j;\alpha_1, \ldots, \alpha_k)) \right] \right\} = \\
\prod_{k,n \leq j \leq k} \left[ \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(X \setminus A(k,n;j;\alpha_1, \ldots, \alpha_k)) \right]
\]

and

\[
\bigcup_{\lambda} \left\{ \prod_{k,n \leq j \leq k} \left[ \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(A_\lambda(k,n;j;\alpha_1, \ldots, \alpha_k)) \right] \right\} = \\
\prod_{k,n \leq j \leq k} \left[ \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}\left( \bigcup_{\lambda} A(k,n;j;\alpha_1, \ldots, \alpha_k) \right) \right]
\]

For statement (ii), observe that

\[
\Lambda\left( \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(A(k,n;j;\alpha_1, \ldots, \alpha_k)) \right) = \\
H_{k,n}(\alpha_1, \ldots, \alpha_k) \cap A(k,n;j;\alpha_1, \ldots, \alpha_k),
\]

which is measurable in \( X \).

We now consider (iii). For fixed \( \alpha \in T \), we want to show that \( \partial S(\alpha) \) is an element of \( \mathcal{G} \). Consider the sets

\[
A = \prod_{\beta \geq \alpha} \prod_{n=d(\beta)} \prod_{\alpha_1, \ldots, \alpha_k} \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k)
\]

and

\[
B = \prod_{\alpha \leq \gamma \leq \alpha} \prod_{n=d(\gamma)} \prod_{\alpha_1, \ldots, \alpha_k} \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(\overline{Q_\alpha}).
\]

By definition, \( A, B \in \mathcal{G} \). It is clear that \( A \subseteq \partial S(\alpha) \). We show now that \( B \subseteq \partial S(\alpha) \). Let

\[
\zeta \in \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \cap \Lambda^{-1}(\overline{Q_\alpha}),
\]

with \( n = d(\gamma) \) and let \( x = \Lambda(\zeta) \). Then \( x \) has \( k \) preimages at level \( n \) and \( \zeta \) is the only preimage in \( \partial S(\gamma) \). Suppose that \( \zeta \notin \partial S(\alpha) \). Since \( x \in \overline{Q_\alpha} \), there is some other preimage \( \zeta' \neq \zeta \) of \( x \) which lies in \( \partial S(\alpha) \). But \( \partial S(\alpha) \subseteq \partial S(\gamma) \), hence, there are two preimages of \( x \) in \( \partial S(\gamma) \), a contradiction.

We now show that \( \partial S(\alpha) \subseteq A \cup B \). Let \( \zeta \in \partial S(\alpha) \), \( \Lambda(\zeta) = x \), \( N(x) = k \) and \( n(x) = n \). Consider two cases.

(a) First suppose \( n(x) \geq d(\alpha) \). Then there are \( k \) distinct sets \( \overline{Q_{\alpha_1}}, \ldots, \overline{Q_{\alpha_k}} \) at level \( n \) such that \( x \in H_{k,n}(\alpha_1, \ldots, \alpha_k) \) and \( \zeta \in \mathcal{H}_{k,n}^j(\alpha_1, \ldots, \alpha_k) \) for some \( 1 \leq j \leq k \). It has to be \( \alpha_j \geq \alpha \), because otherwise \( \alpha \) and \( \alpha_j \) are not order related, hence \( \partial S(\alpha) \) and \( \partial S(\alpha_j) \) are disjoint and they can not both contain \( \zeta \).
(b) Suppose $n(x) < d(\alpha)$. Again, $x \in H_{k,n}(\alpha_1, \ldots, \alpha_k)$ and $\zeta \in H_{k,n}^\dagger(\alpha_1, \ldots, \alpha_k)$ for some $1 \leq j \leq k$. We want to prove that $\gamma := \alpha_j < \alpha$. If such is not the case, then $S(\alpha_j)$ and $S(\alpha)$ are not order-related, as above, hence they are disjoint.

We list the basic properties of $\Lambda^*$.

**Proposition 17.** The measure $\Lambda^*(\omega)$ is a Borel measure on $\partial T$. Moreover,

(i) $\Lambda_s(\Lambda^*(\omega)) = \omega$;

(ii) Let $N = \max\{N(x) : x \in X\}$. Then, for each Borel measurable set $A$ in $\partial T$,

$$\frac{\omega(\Lambda(A))}{N} \leq \Lambda^*(\omega)(A) \leq \omega(\Lambda(A)).$$

In terms of the supports of the measures, we have:

(iii) $\text{supp}(\Lambda^* \omega) \subseteq \Lambda^{-1}(\text{supp}(\omega))$;

(iv) $\text{supp}(\Lambda_s \nu) = \Lambda(\text{supp}(\nu))$.

(v) $\text{supp}(\omega) \subseteq \Lambda(\text{supp}(\Lambda^* \omega))$.

A consequence of (v) is that $\Lambda^{-1}(\text{supp}(\omega)) \subseteq \Lambda^{-1}(\Lambda(\text{supp}(\Lambda^* \omega)))$. We believe that more is true:

**Conjecture 18.** $\Lambda^{-1}(\text{supp}(\omega)) \subseteq \text{supp}(\Lambda^* \omega)$ (hence, by (iii), $\Lambda^{-1}(\text{supp}(\omega)) = \text{supp}(\Lambda^* \omega)$).

Unfortunately we do not have a proof for this.

**Proof of Proposition 17.** First, we show that the function $N_A$ is measurable, so that the integral defining $\Lambda^*(\omega)$ makes sense. By Lemma 16, the set $A$ can be decomposed as the disjoint, countable union of measurable subsets $A_i$ of $\partial T$, with the property that $\Lambda$ is one to one on each of them and $\Lambda(A_i)$ is measurable in $X$. Then,

$$N_A = \sum_i N_{A_i} = \sum_i \chi_{\Lambda(A_i)},$$

which is Borel measurable.

The set function $\Lambda^*(\omega)$ is additive. Let $\{A_n\}_{n=0}^\infty$ be a disjoint family of measurable sets in $\partial T$ then

$$\int_X \frac{N_{\cup A_n}(x)}{N(x)} d\omega(x) = \int_X \frac{\#\{\zeta \in \cup A_n : \Lambda(\zeta) = x\}}{N(x)} d\omega(x)$$

$$= \sum_n \int_X \frac{\#\{\zeta \in A_n : \Lambda(\zeta) = x\}}{N(x)} d\omega(x)$$

$$= \sum_n \int_X \frac{N_{A_n}(x)}{N(x)} d\omega(x)$$

Estimate (ii) follows from Lemmas 14 and 15. Given a measurable set $A$ in $\partial T$ and $x$ in $\Lambda(A)$ then $1 \leq N_A(x) \leq N(x) \leq c$. To show (i), let $E$ be a measurable subset of $X$,

$$\Lambda_s(\Lambda^*(\omega))(E) = \Lambda^*(\omega)(\Lambda^{-1}(E)) = \int_X \frac{N_{\Lambda^{-1}(E)}(x)}{N(x)} d\omega(x)$$

$$= \int_X \frac{\#\{\zeta \in \Lambda^{-1}(E) : \Lambda(\zeta) = x\}}{N(x)} d\omega(x)$$

$$= \int_X \chi_E(x) \frac{N(x)}{N(x)} d\omega(x) = \omega(E).$$
For the proof of (v) let $K = \text{supp}(\Lambda^* \omega)$. Then,
\[
0 = \Lambda^* \omega(\partial T \setminus K) = \int_X \frac{N_{\partial T \setminus K}(x)}{N(x)} d\omega(x)
\]
if and only if $\omega$-a.e. $x$ we have $0 = N_{\partial T \setminus K}(x) = \sharp(\Lambda^{-1}(x) \cap [\partial T \setminus K])$, which is equivalent to having $\Lambda^{-1}(x) \subseteq K$ for $\omega$-a.e. $x$; hence, $x \in \Lambda(K)$ for $\omega$-a.e. $x$. Since $\Lambda(K)$ is closed in $X$, $\text{supp}(\omega) \subseteq \Lambda(K) = \Lambda(\text{supp}(\Lambda^* \omega))$, giving (v).

Turning to (iii). Let $E = \text{supp}(\omega)$. If $x \in E$, $\Lambda^{-1}(x) \subseteq \Lambda^{-1}(E)$, hence $\Lambda^{-1}(x) \cap [\partial T \setminus \Lambda^{-1}(E)] = \emptyset$. Thus,
\[
\Lambda^* \omega(\partial T \setminus \Lambda^{-1}(E)) = \int_X \frac{\sharp(\Lambda^{-1}(x) \cap [\partial T \setminus \Lambda^{-1}(E)])}{\sharp(\Lambda^{-1}(x))} d\omega(x)
= \int_{X \setminus E} \frac{\sharp(\Lambda^{-1}(x) \cap [\partial T \setminus \Lambda^{-1}(E)])}{\sharp(\Lambda^{-1}(x))} d\omega(x)
= 0.
\]
Namely, $\text{supp}(\Lambda^* \omega) \subseteq \Lambda^{-1}(E) = \Lambda^{-1}(\Lambda^* \omega)$, which is (iii).

Statement (iv) is well known and we include a proof for completeness. For $F$ closed in $\partial T$,
\[
\text{supp}(\Lambda_s \omega) \subseteq F \iff 0 = \Lambda_s \omega(X \setminus F) = \omega(\Lambda^{-1}(X \setminus F)) = \omega(\partial T \setminus \Lambda^{-1}(F)) \iff \text{supp}(\omega) \subseteq \Lambda^{-1}(F) \iff \Lambda(\text{supp}(\omega)) \subseteq F.
\]

\[\blacksquare\]

Remark 19. For the measure $m$ itself, we have that $N_A(x)/N(x) = 1$, $m$-a.e. We might, and we will, identify $m$, the measure on $X$, and $\tilde{m}$, the measure on $\partial T$.

We will also need a “localized” version of $\Lambda^*$. Let $F$ be a closed subset of $\partial T$ and $\omega$ be a Borel measure on $X$. Then,
\[
\Lambda^*_F(A) := \Lambda^*(F \cap A),
\]
is the restriction of $\Lambda^*$ to $F$. Proposition 17 localizes to

Corollary 20. Let $\omega$ be a positive Borel measure on $X$ and $F$ a closed set in $\partial T$. Then,
(i) $\text{supp}(\Lambda^*_F \omega) \subseteq \Lambda^{-1}(\text{supp}(\omega)) \cap F$.
(ii) $\Lambda^*_F \omega(A) \approx \omega(\Lambda(F \cap A))$ when $A \subseteq \partial T$ is measurable.
(iii) $\Lambda_s(\Lambda^*_F \omega)(E) \approx \omega(E \cap \Lambda(F))$ when $E \subseteq X$ is measurable.

Proof. (i) follows from (iii) in Proposition 17. (ii) follows from (ii) in Proposition 17. (iii) follows form (i) and (ii) in Proposition 17. \[\blacksquare\]

3. The Muckenhoupt-Wheeden Inequality on Graphs

In this section we prove the Muckenhoupt-Wheeden inequality on graphs. In the linear case, the inequality can be proved by Fubini’s Theorem and easy geometric considerations. The main point of the inequality consists in inverting “on average” the $\ell^1 \subseteq \ell^\infty$ inclusion. We need its corollary: an inversion “on average” of the $\ell^1 \subseteq \ell^p$ inclusion, which was independently proved by T. Wolff by means of a completely different argument. (The other
half, which inverts \( \ell^p' \subseteq \ell^\infty \), was independently proved in [4]). The usefulness of the inequality is shown in the next section.

Fix \( 0 < s < 1 \) and \( q \geq 1 \). For each \( x \) in \( X \), let \( P_G(x) = \{ \alpha \in G : d_G(\alpha, \Lambda^{-1}(x)) \leq 1 \} \), this is the set of those \( \alpha \) in \( G \) having distance at most one from a tree-geodesic ending at \( x \).

Given a Borel measure \( \omega \) on \( X \), define

\[
\mathcal{I}_G \omega (x) = \sum_{\alpha \in P_G(x)} \frac{\omega(\alpha)}{m(\alpha)^s}
\]

and

\[
S_G \omega (x) = \sup_{\alpha \in P_G(x)} \frac{\omega(\alpha)}{m(\alpha)^s}.
\]

Here, \( \omega(\alpha) := \omega(Q^k_{\alpha}) \) and \( m(\alpha) := m(Q^k_{\alpha}) = m(Q^k_{\alpha}') \) are the measures of the corresponding sets. Clearly, \( S_G \omega \leq I_G \omega \), pointwise. The following surprising theorem of Muckenhoupt and Wheeden shows that, on average, the opposite inequality holds, as well.

**Theorem 21.** Given \( q \geq 1 \), there is a constant \( c_7 \) such that

\[
(10) \quad \int_X \mathcal{I}_G \omega (x)^q dm(x) \leq c_7 \int_X S_G \omega (x)^q dm(x).
\]

The inequality was proved in [29] for Riesz potentials. In [19], T. Wolff independently rediscovered, with a different proof, one-half of it. In our context, Wolff’s inequality reads:

\[
\int_X \mathcal{I}_G \omega (x)^q dm(x) = \int_X \left( \sum_{\alpha \in P_G(x)} \frac{\omega(\alpha)}{m(\alpha)^s} \right)^q dm(x) \leq C \int_X \sum_{\alpha \in P_G(x)} \frac{\omega(\alpha)^q}{m(\alpha)^{sq}} dm(x).
\]

The left hand side is related with a classical capacity on \( X \), the right hand side with a new capacity on the tree. The equivalence of the two definitions was the key to extend to the nonlinear case important features of linear potential theory [19]. The other half of (10),

\[
(11) \quad \int_X \sum_{\alpha \in P_G(x)} \frac{\omega(\alpha)^q}{m(\alpha)^{sq}} dm(x) \leq C \int_X S_G \omega (x)^q dm(x)
\]

was later rediscovered in [4], with a completely different proof. The authors were motivated by the fact that, in the linear case \( q = 2 \), the equivalence of the two quantities in (11) a priori follows from the fact that both condition [25] and [5] have been proved to characterize the Carleson measures for the Dirichlet space.

For ease of the reader, we give the proof of Theorem 21 in the present context. First, define the maximal function

\[
M^m_G \omega (x) = \sup_{\alpha \in P_G(x)} \frac{\omega(\alpha)}{m(\alpha)}.
\]

We then have the following well-known lemma whose proof is a standard argument.

**Lemma 22.** The maximal function \( M^m_G \) is bounded on \( L^\infty \) and on \( L^{1,\infty} \),

\[
m(\{M^m_G \omega > \lambda\}) \lambda \leq c \|\omega\|_1.
\]
Proof of Theorem 21. Again, we separate the proof into some easier steps.

Step 1. Fix \( k \in \mathbb{N} \). Let \( \|\omega\|_1 = \int_X d\omega \) and using Ahlfors-regularity,

\[
\mathcal{I}_G \omega(x) = \sum_{\alpha \in P_G(x), d(\alpha) < k} \frac{\omega(\alpha)}{m(\alpha)^s} + \sum_{\alpha \in P_G(x), d(\alpha) \geq k} \frac{\omega(\alpha)}{m(\alpha)^s}
\]

(12)

\[
\leq c \|\omega\|_1 \delta^{-Qk} + c M_G^m \omega(x) \delta^{Q(1-s)k}.
\]

Choose \( k \) in such a way \( \delta^Q k \approx \|\omega\|_1 / M_G^m \omega(x) \), so that the two summands are approximately equal,

\[
\|\omega\|_1 \delta^{-Qk} \approx M_G^m \omega(x) \delta^{Q(1-s)k}.
\]

Then,

(13)

\[
\mathcal{I}_G \omega(x) \leq c \|\omega\|_1^{1-s} (M_G^m \omega(x))^s.
\]

Using Lemma 22 and the estimates for \( \mathcal{I}_G \), we obtain

(14)

\[
m(\mathcal{I}_G \omega > \lambda) \leq m(\|\omega\|_1^{1-s} (M_G^m \omega(x))^s > \lambda) \leq c \|\omega\|_1 \lambda^{-\frac{1}{s}} \|\omega\|_1^{\frac{1-s}{s}} = c \left( \frac{\|\omega\|_1}{\lambda} \right)^{1/s}.
\]

Step 2. We will show that there exist \( a > 1, b > 1 \) such that, for all \( \epsilon \in (0,1) \) and all \( \lambda > 0 \),

(15)

\[
m(\mathcal{I}_G \omega > a \lambda) \leq b \epsilon m(\mathcal{I}_G \omega > \lambda) + m(S_G \omega > \epsilon \lambda).
\]

By Corollary 4, we can replace each of the three sets \( A \) whose \( m \)-measure is considered in (15) by \( A \setminus F \), considering only those points \( x \) in \( X \) such that \( \chi \Lambda^{-1}(x) = 1 \).

First, we extend the definitions of \( \mathcal{I}_G \omega \) and \( S_G \omega \) to points \( \alpha \) of \( T \equiv G \). Let \( P_G(\alpha) = \{ \beta \in G \colon d_G(\beta, [\alpha, \alpha]) \leq 1 \) and \( d(\beta) \leq d(\alpha) \} \) and set

\[
\mathcal{I}_G \omega(\alpha) := \sum_{\beta \in P_G(\alpha)} \frac{\omega(\beta)}{m(\beta)^s}
\]

and

\[
S_G \omega(\alpha) := \sup_{\beta \in P_G(\alpha)} \frac{\omega(\beta)}{m(\beta)^s}.
\]

If \( \mathcal{I}_G \omega(\alpha) > \lambda \) and \( x \notin F \), then there is \( \alpha \in P_G(x) \) such that \( \mathcal{I}_G \omega(\alpha) > \lambda \) and it is clear that, if \( \alpha > \beta \) in \( T \), then \( \mathcal{I}_G \omega(\alpha) \geq \mathcal{I}_G \omega(\beta) \). Let \( A(\lambda) \) be the set of the \( \alpha \) in \( T \) which are maximal with the property \( \mathcal{I}_G \omega(\alpha) > \lambda \).

Then, the set \( \{ \xi \in (X \setminus F) \cup G : \mathcal{I}_G \omega(\xi) > \lambda \} \) is the disjoint union of the sets \( S_1(\alpha) = S(\alpha) \cup (Q^d(\alpha) \setminus F) \), as \( \alpha \) ranges in \( A(\lambda) \).

Observe that \( \partial \overline{S}(\alpha) = Q^d(\alpha) \) is the boundary of \( S(\alpha) \) with respect to the metric \( \overline{p} \) in \( \overline{G} \).

By maximality, if \( \alpha \) is in \( A(\lambda) \), then \( \mathcal{I}_G \omega(\alpha^{-1}) \leq \lambda < \mathcal{I}_G \omega(\alpha) \). Fix \( \alpha \) in \( A(\lambda) \). We consider two cases.

First, suppose that for \( m \)-almost all \( x \) in \( Q^d(\alpha) \) we have \( S_G \omega(x) > \epsilon \lambda \). Then,

\[
m(\{ \mathcal{I}_G \omega > a \lambda \} \cap \partial \overline{S}(\alpha)) \leq m(\partial \overline{S}(\alpha)) = m(\{ S_G \omega > \epsilon \lambda \} \cap \partial \overline{S}(\alpha)),
\]

and we have nothing to prove.
Second, suppose that, on the contrary, there is \( x_0 \) in \( Q_\alpha \setminus F \) such that \( S_G \omega(x_0) \leq \epsilon \lambda \). Let

\[
\tilde{S}(\alpha) = \bigcup_{\beta, d(\beta) = d(\alpha), d_G(\alpha, \beta) \leq 1} Q_\beta,
\]

and let \( \omega_1 = \omega|_{\tilde{S}(\alpha)} \).

We write \( \partial_{\tilde{S}} \tilde{S}(\alpha) = \bigcup_{\beta} \partial_{\tilde{S}} S(\beta) \), the union being over the \( \beta \)'s used in the definition of \( \tilde{S}(\alpha) \). By Step 1,

\[
\tag{16}
m \left( \left\{ I_G \omega_1 > \frac{a\lambda}{2} \right\} \cap Q_\alpha \right) \leq \frac{c}{a^{1/s}} \left( \frac{\|\omega_1\|_1}{\lambda} \right)^{1/s}.
\]

We estimate

\[
\tag{17} \|\omega_1\|_1 = \omega(\partial_{\tilde{S}} \tilde{S}(\alpha)) \leq c S_G \omega(\alpha) m(\alpha)^s.
\]

In fact,

\[
\omega(\partial_{\tilde{S}} \tilde{S}(\alpha)) \leq \sum_{\beta, d(\beta) = d(\alpha), d_G(\alpha, \beta) \geq 1} \omega(\partial_{\tilde{S}} S(\beta)) \leq c \max_{\beta, d(\beta) = d(\alpha), d_G(\alpha, \beta) \geq 1} \omega(\partial_{\tilde{S}} S(\beta)) \leq c \left( \frac{\omega(\beta)}{m(\beta)^s} \right) m(\alpha)^s \leq c S_G \omega(\alpha) m(\alpha)^s.
\]

Inserting \( \omega_2 \) in \( \omega_1 \), and\( \omega_2 = \omega - \omega_1 \). Then,

\[
\tag{18} I_G \omega_2(\alpha^{-1}) \leq I_G \omega(\alpha^{-1}) \leq \lambda,
\]

by the maximality of \( \alpha \). If \( x \in \alpha \setminus F \), then,

\[
I_G \omega_2(x) = I_G \omega_2(\alpha^{-1}) \leq \lambda < \frac{a\lambda}{2}.
\]

(In the first equality we use the information on \( \text{supp}(\omega_2) \). Thus,

\[
\{ I_G \omega > a\lambda \} \cap (\alpha \setminus F) = \left\{ I_G \omega_1 > \frac{a\lambda}{2} \right\} \cap (\alpha \setminus F) \cup \left\{ I_G \omega_2 > \frac{a\lambda}{2} \right\} \cap (\alpha \setminus F) \]
\]

\[
\tag{19}
= \left\{ I_G \omega_1 > \frac{a\lambda}{2} \right\} \cap (\alpha \setminus F).
\]

The wished estimate follows from \( \tag{18} \) and \( \tag{19} \), proving Step 2.

**Step 3.** A “standard argument” (see [1]) shows that the estimate follows from the good-\( \lambda \) inequality in Step 2.
For our purposes, the main point of interest of Theorem 21 is the following.

**Corollary 23.**

\[
\int_X \left( \sum_{\alpha \in P_G(x)} \frac{\omega(\alpha)}{m(\alpha)^s} \right)^q dm(x) \leq c \int_X \sum_{\alpha \in P_G(x)} \left( \frac{\omega(\alpha)}{m(\alpha)^s} \right)^q dm(x).
\]

4. **Proof of Main Result**

In this section, we first write the energy \( E(\omega) \) of a measure \( \omega \) on \( X \) in a way which resembles one side of the Muckenhoupt-Wheeden inequality, the use Wolff’s half of the inequality to approximatively write the energy in a different way. The catch is that when the same procedure is applied to a measure on \( \partial T \), one gets a very similar expression. The approximating expression for the energy, in fact, does not depend on the graph geometry, but only on the tree geometry. Using the fact that Borel measures on \( \partial T \) and on \( X \) can be moved forth and back through \( \Lambda \), and observing that energies of corresponding measures are equivalent, we easily obtain a proof of the equivalence of capacities in Theorem 1.

**Lemma 24.** There are positive constants \( c \) and \( C \) independent of the Borel measure \( \omega \) and of the point \( x \) in \( X \) such that

\[
c \sum_{\alpha \in P_G(x)} m(\alpha)^{-s} \leq K(x, y) \leq C \sum_{\alpha \in P_G(x)} m(\alpha)^{-s}
\]

and

\[
c I_G \omega(x) \leq K \omega(x) \leq C I_G \omega(x).
\]

**Proof.** We begin with the estimate from below.

\[
I_G \omega(x) = \sum_{\alpha \in P_G(x)} \frac{\omega(\alpha)}{m(\alpha)^s} = \int_X \left( \sum_{\alpha \in P_G(x)} \frac{\chi_{\alpha}(y)}{m(\alpha)^s} \right) d\omega(y) = \int_X \left( \sum_{\alpha \in P_G(x) \cap P_0^G(y)} m(\alpha)^{-s} \right) d\omega(y),
\]

where \( P_0^G(y) \) is the union of the tree geodesics ending at \( y \). Let \( \alpha \) be one of the indexes over which the sum is taken. Then, \( y \in \alpha \) and there is \( \beta \sim G \alpha \) such that \( x \in \beta \). Then, \( \rho_T(x, y) \leq c \delta d(\alpha) \). Also, at each level \( k \), there are boundedly many \( \alpha \) indexing the sum. Thus,

\[
\left( \sum_{\alpha \in P_G(x) \cap P_0^G(y)} m(\alpha)^{-s} \right) \approx \sum_{\alpha: x \in \alpha, \rho_T(x, y) \leq \delta d(\alpha)} m(\alpha)^{-s} \approx m(x, \rho_T(x, y))^{-s} \approx K(x, y).
\]

This is the first estimate in the lemma; the second follows by inserting (21) in (20). ■

Given the kernel \( K \) on \( X \), we define a new kernel \( K_{\partial T} \) on \( \partial T \), with \( \tilde{m} \) instead of \( m \) and with \( \rho_T \) instead of \( \rho \).
Theorem 25. Given a measure $\omega$ in $X$, let $\Lambda^*(\omega)$ be the measure on $\partial T$ defined by (9), and let $E_{\partial T}$ the energy defined by the kernel $K_{\partial T}$. Then,

$$E_X(\omega) \approx E_{\partial T}(\Lambda^*\omega). \quad (22)$$

On the other hand, if $\nu$ is a measure on $\partial T$, then

$$E_{\partial T}(\nu) \approx E_X(\Lambda^*\nu). \quad (23)$$

Proof.

We first prove (22)

$$E_X(\omega) := \int_X K\omega^p dm \approx \int_X [I_G \omega]^p dm \text{ by Lemma 24} \approx \int_X \sum_{\alpha \in P_G(x)} \left(\frac{\omega(\alpha)}{m(\alpha)^s}\right)^p dm(x) \text{ by Theorem 21} = \sum_{\alpha \in T} \omega(\alpha)^p m(\alpha)^{-sp'} \int_X \chi(x \in G : \alpha \in P_G(x)) dm(x) \approx \sum_{\alpha \in T} \Lambda^*(\omega)(\alpha)^p \frac{m(\alpha)^{sp'-1}}{\tilde{m}(\alpha)^{sp'-1}},$$

by Proposition 17.

The last expression in the chain of equivalences does not depend on the graph structure, but only on the structure of $T$, it is then equivalent to $E_{\partial T}(\Lambda^*(\omega))$.

We now consider (23). Recall the identification $\alpha \equiv Q_\alpha$ between points in $G$ and the closure of dyadic sets in $X$. In the tree $T$, $\alpha$ is identified with a dyadic subregion $[\alpha] \subseteq \partial T$ in the same fashion, and $[\alpha] \subseteq \Lambda^{-1}(\alpha)$. The inclusion is generally proper because of the “edge effect”. Elements in the boundary of $Q_\alpha$ have preimages in dyadic subsets on $\partial T$ at the same level of $[\alpha]$, but different from $[\alpha]$.

However, it is still true that $\Lambda^{-1}(\alpha) \subset \cup_{\beta \supseteq \alpha} [\beta]$, and this, together with Ahlfors-regularity and the uniform bound on the number of such $\beta$’s, suffices to show that (using the identification of the measures $m$ on $X$ with the measure $\Lambda^*m$ on $\partial T$, see in Remark 19):

$$E_{\partial T}(\nu) \approx \sum_{\alpha} \frac{\nu([\alpha])^p}{m(\alpha)^{sp'-1}} \approx \sum_{\beta} \Lambda^*\nu(\beta)^p \frac{m(\beta)^{sp'-1}}{\tilde{m}(\beta)^{sp'-1}} \approx E_X(\Lambda^*\nu).$$

We are now ready for the proof of Theorem 1. Observe that the first assertion (4) is implied by the second, since (4) implies that

$$\text{Cap}_{\partial T}(\Lambda^{-1})(E) \approx \text{Cap}_X(\Lambda(\Lambda^{-1})(E)) = \text{Cap}_X(E).$$
**Proof of Theorem 1.** Let $F \subseteq \partial T$ be closed. We first show that

$$\text{Cap}_X(\Lambda(F)) \lesssim \text{Cap}_{\partial T}(F).$$

Let $\omega$ be a positive Borel measure on $X$, supported on $\Lambda(F)$. Then, $\text{supp}(\Lambda_F^*\omega) \subseteq F$. Also,

$$\|\Lambda_F^*\omega\|_1 = \Lambda^*\omega(F) = \int_X \frac{\#(\Lambda^{-1}(x) \cap F)}{\#(\Lambda^{-1}(x))}d\omega(x) \approx \omega(\Lambda(F)) = \|\omega\|_1.$$

In terms of the energies, we have:

$$\mathcal{E}_X(\omega) \approx \sum_{\alpha \in T} \frac{\omega(Q_\alpha)^{p'}}{m(Q_\alpha)^{p's-1}} \approx \sum_{\alpha \in T} \frac{\Lambda^*\omega(\partial S(\alpha))^{p'}}{m(Q_\alpha)^{p's-1}} \approx \sum_{\alpha \in T} \frac{\Lambda_F^*\omega(\partial S(\alpha))^{p'}}{m(Q_\alpha)^{p's-1}} \approx \mathcal{E}_{\partial T}(\Lambda_F^*\omega).$$

These three pieces of information together give:

$$\text{Cap}_X(\Lambda(F)) = \sup \left\{ \frac{\|\omega\|_1^p}{\mathcal{E}_X(\omega)^{p-1}} : \text{supp}(\omega) \subseteq \Lambda(F) \right\} \lesssim \sup \left\{ \frac{\|\Lambda_F^*\omega\|_1^p}{\mathcal{E}_{\partial T}(\Lambda_F^*\omega)^{p-1}} : \text{supp}(\Lambda_F^*\omega) \subseteq F \right\} \leq \text{Cap}_{\partial T}(F).$$

In the other direction, we show that, if $F$ is closed in $\partial T$, then

$$\text{Cap}_X(\Lambda(F)) \gtrsim \text{Cap}_{\partial T}(F).$$

Let $\nu$ be a positive Borel measure supported on $F$. Then, $\text{supp}(\Lambda_\ast\nu) = \Lambda(\text{supp}(\nu)) \subseteq \Lambda(F)$. Hence,

$$\text{Cap}_{\partial T}(F) = \sup \left\{ \frac{\|\nu\|_1^p}{\mathcal{E}_{\partial T}(\nu)^{p-1}} : \text{supp}(\nu) \subseteq F \right\} \approx \sup \left\{ \frac{\|\Lambda_\ast\nu\|_1^p}{\mathcal{E}_X(\Lambda_\ast\nu)^{p-1}} : \text{supp}(\Lambda_\ast\nu) \subseteq \Lambda(F) \right\} \leq \text{Cap}_X(\Lambda(F)).$$

\[\square\]

5. **Potential Theory on Trees**

In this section, we develop the basic potential theory on trees. Most material is covered, with different degrees of generality, in various sources ([26] and [30], for instance).

We give a self-contained exposition, only requiring the basic facts of nonlinear potential theory (Frostman’s thesis), which are anyway recalled, without proof, in Appendix I, and translated in tree terms in Appendix II. We list main features of the section.
(i) In 5.1, we identify the capacity $\text{Cap}_{\partial T}$ on $\partial T$ considered before with (the restriction to $\partial T$ of) a new tree capacity $\text{Cap}$ defined on $\overline{T}$. This is analogous to the identification, for subsets of the unit circle, of logarithmic capacity in the complex plane with the linear $1/2$-Bessel capacity on circle. In the Ahlfors-regular case, as a consequence, we have the identification of $\text{Cap}_{\mathcal{X}}$ with the new tree capacity $\text{Cap}$.

(ii) In 5.2, we estimate the capacity $\text{Cap}$ of points and sets having the form $\partial S(\alpha)$ ($\alpha \in T$). As a corollary, we have estimates for the capacity of metric balls in Ahlfors-regular spaces. It is here explained why the cases $s \in [1/p', 1)$ are the only interesting ones in the Ahlfors-regular case and why $s = 1/p'$ is especially interesting.

(iii) We prove in Section 5.3 a recursive formula to compute capacities of closed subsets of $\overline{T}$. Indeed, the formula gives two-sided estimates for capacities in Ahlfors-regular metric spaces.

(iv) We give a simple proof, in Section 5.4, of the Trace Theorem (or Carleson Measure Theorem) on trees. The measures satisfying the Trace Inequality (or Carleson imbedding, in the language of most complex analysts) will be characterized by a testing condition, in the spirit of [24] [25].

By the general machinery developed earlier, the Trace Theorem on trees implies an analogous theorem in Ahlfors-regular spaces.

(v) In 5.5, we show that the capacity of a set in the tree can be defined by means of the “Carleson measures” supported on this set.

(vi) In 5.6, we give a direct proof -using (v) and a monotonicity property- that the testing condition of the Trace Theorem is equivalent to a certain capacitary condition. An indirect proof of this fact is that the capacitary condition, as well, characterizes the Carleson measures.

5.1. Tree Capacity Seen from “Inside” the Tree. Let $T$ be a tree, having (combinatorial) boundary $\partial T$ with respect to a fixed root $o$ in $T$, and let $\overline{T} = T \cup \partial T$ be the compactification of $T$. The tree $T$ is endowed with a positive weight $\pi$, $\pi(x) > 0 \ \forall x \in T$. We consider exponents $1 < p < \infty$ and $p^{-1} + p'^{-1} = 1$.

Let $\xi$ in $\overline{T}$. The predecessor set of $\xi$ is $\{y : o \leq y \leq \xi\} \cap T$. The successor set in $\overline{T}$ of $x \in T$ is $S(x) = \{\xi \in \overline{T} : \xi \geq x\}$. The natural distance in $T$ is denoted by $d$, while the length-distance associated with the weight $\pi^{1-p'}$ will be denoted by $d_\pi$.

The Hardy operator $I$ is defined as

$$ (24) \qquad If(\xi) = \sum_{y \in P(\xi)} f(y), \ \xi \in \overline{T}. $$

Its formal adjoint is

$$ I^*\omega(x) = \int_{S(x)} d\omega(y), x \in T. $$

where $\omega$ is a Borel measure on $\overline{T}$.

We consider the kernel $g : \overline{T} \times T \to \mathbb{R}$ defined by

$$ g(\zeta, \alpha) = \chi_{P(\zeta)}(\alpha) = \chi_{S(\alpha)}(\zeta). $$

Here, we think of $(T, \pi^{1-p'})$ as a measure space, while $\overline{T}$ is given the metric structure induced by the metric $\pi_T(\zeta, \xi) = 2^{-d(\zeta \wedge \xi)}$ (the metric itself will play no rôle, we are only interested in the topology).
It is well explained in [1] (see also Appendices I and II of the present article) how to develop the potential theory associated with the kernel \( g \). The potential of a function \( f = \pi^{p'-1} \varphi : T \to \mathbb{R} \) is \( Gf = I(f \pi^{p'-1}) = I \varphi : T \to \mathbb{R} \); while the (dual) potential of a positive, Borel measure \( \omega \) on \( T \) is given by \( \hat{G} \omega = I^* \omega = \int_{S(y)} d\omega \).

The energy of the measure \( \omega \) is
\[
E(\omega) = \sum_{\alpha \in T} (I^* \omega(\alpha))^{p'} \pi(\alpha)^{1-p'}.
\]

We can finally define the capacity \( \text{Cap}(E) \) of a closed subset \( E \) of \( T \) as
\[
\text{Cap}(E) = \inf \left\{ \| \varphi \|_{L^p(\pi)} : \varphi \geq 0, I \varphi \geq 0 \text{ on } E \right\}
\]
\[
= \sup \left\{ \frac{\omega(E)^p}{E(\omega)^{p-1}} : \text{supp}(\omega) \subseteq E \right\}.
\]

(25)

The second equality is a deep result in potential theory (see the appendices), whose proof relies on a min/max principle.

In the important case when \( T \) is the tree of Christ’s dyadic boxes of an Ahlfors \( Q \)-regular metric space \((X, \rho, m)\) (which without essential loss of generality, in view of Theorem 1 and Proposition 13, might be taken to be \( \partial T \) itself), and \( 0 < s < 1 \) is fixed, we consider the weight
\[
\pi_s(\alpha) = m(Q_\alpha)^{\frac{s p'}{p'-1}} = m(\alpha)^{\frac{s p'}{p'-1}}.
\]

The case \( s p' = 1 \) correspond to unweighted potential theory on trees, which, as we shall see, leads to “logarithmic” potentials. As before, we have identified the measure \( m \) on \( X \) with the corresponding measure \( \Lambda^* m \) on \( \partial T \).

**Theorem 26.** Let \( T \) be the tree associated with the metric measure space \((X, \rho, m)\), \( 0 < s < 1 \) fixed and \( \pi = \pi_s \) be the weight just defined. Let \( \text{Cap} \) the capacity on \( T \) associated with \( \pi_s \). Then, there are constants \( C_1 < C_2 \) such that, for all compact subsets \( K \) of \( \partial T \),
\[
C_1 \text{Cap}(K) \leq \text{Cap}_X(\Lambda(K)) \leq C_2 \text{Cap}(K).
\]
and
\[
\text{Cap}_X(F) \approx \text{Cap}(\Lambda^{-1}(F))
\]
whenever \( F \) is closed in \( \partial T \).

To better contextualize the capacities in Theorem 26, consider that, when \( \pi = \pi_s \),
\[
\mathcal{E}(\omega) := \sum_{\alpha \in T} (I^* \omega(\alpha))^{p'} \pi_s(\alpha)^{1-p'}
\]
\[
= \sum_{\alpha \in T} \frac{I^* \omega(\alpha)^{p'}}{m(\alpha)^{s p'-1}}
\]
\[
= \int_{\partial T} I \left[ (I^* \omega)^{p'-1} \pi_s^{1-p'} \right](\zeta) d\omega(\zeta)
\]
\[
= \int_{\partial T} \sum_{\alpha \in T} \frac{I^* \omega^{p'}(\alpha)}{m(\alpha)^{s p'}} d\omega(\zeta).
\]

Both equalities follow from straightforward applications of Fubini’s Theorem.
The integrand on the third line,
\begin{equation}
W(\omega) = I \left[ (I^*\omega)^{p'-1}\pi^{1-p'} \right],
\end{equation}
is the Wolff potential of the measure \(\omega\); which was introduced by Wolff in \([19]\). The integral in the fourth line is the intermediate term in Theorem 21, with \(q = p'\). A deep analysis of the Wolff potential and of its applications can be found in \([14]\) and \([23]\).

Theorem 26 is actually implicit in the proof of (22), since comparable energies give comparable capacities. We have decide to state it separately in this section because the relation between Cap and \(\text{Cap}_{\partial T}\), especially in the important case \(sp' = 1\), is similar to the relation between logarithmic capacity (in the complex plane) and Bessel 1/2-capacity (on the real line) for subsets of the real line.

For the remaining part of this section, we will consider general weights \(\pi\) on a generic tree \(T\) since the proofs are not more difficult and notation is actually more transparent. We will point out, in some cases of interest, how the results specialize to the case when \(T\) is the tree of the dyadic boxes introduced by Christ.

5.2. Capacity of Special Sets. We begin with a general fact, points having positive capacity are points having finite “potential theoretic distance” from the root.

**Proposition 27.** Let \(\zeta_0\) be a point of \(\partial T\). Then,
\[
\text{Cap}\{\zeta_0\} > 0 \iff d_\pi(o, \zeta_0) < \infty.
\]
More precisely, \(\text{Cap}\{\zeta_0\} = d_\pi(0, \zeta_0)^{1-p}\) whenever \(\zeta_0 \in T\).

**Proof.** It suffices to compute the energy of a unit mass \(\delta_\zeta_0\) concentrated at \(\zeta_0\):
\[
\mathcal{E}(\delta_\zeta_0) = \sum_\alpha (I^*\delta_\zeta_0)(\alpha)^{p'(\alpha)}\pi^{1-p'(\alpha)} = \sum_\alpha \pi^{1-p'} = d_\pi(\alpha).
\]

It is obvious from the definition of capacity in \(T\) that \(\text{Cap}\{\alpha\} \geq \text{Cap}(S(\alpha))\) (if \(I\phi\) is one on \(\alpha\), it can be taken to be one on \(\partial S(\alpha)\) without increasing the norm of \(\phi\). Interesting cases are those in which the opposite inequality holds as well; for instance, the Ahlfors-regular case.

**Lemma 28.** If \(0 < s < 1\) and \(\pi = \pi_s\), then \(\text{Cap}\{\alpha\} \approx \text{Cap}(\partial S(\alpha))\).

**Proof.** We have to prove that \(\text{Cap}\{\alpha\} \lesssim \text{Cap}(\partial S(\alpha))\). A simple calculation shows that
\[
\text{Cap}\{\alpha\} = d_{\pi_s}(\alpha) \approx \begin{cases} d(\alpha)^{1-p} & \text{if } s = 1/p', \\ m(\alpha)^{p(s-1/p')} & \text{if } 1/p' < s < 1. \end{cases}
\]

We test the definition of capacity (25) on the measure \(\omega := m|_{\partial S(\alpha)}\), so that \(\omega(\partial S(\alpha)) = m(\alpha)\). We estimate the energy of \(\omega\):
\[
\mathcal{E}(\omega) = \sum_{\beta \in [0,\alpha)} \frac{m(\alpha)^{p'}}{m(\beta)^{sp'-1}} + \sum_{\gamma \in S(\alpha)} m(\gamma)^{p'+1-sp'} = I + II.
\]
For the first term we have (recognizing a geometric series, by Ahlfors-regularity, when \(1/p' < s < 1\)):

\[
I \approx \begin{cases} 
  d(\alpha)m(\alpha)p' & \text{if } s = 1/p', \\
  m(\alpha)^{1+p'-sp'} & \text{if } 1/p' < s < 1.
\end{cases}
\]

For the second term, using Ahlfors-regularity and the fact that \(p'(1-s) > 0\),

\[
II \approx \sum_{n=0}^{\infty} \left( \sum_{\gamma \in S(\alpha), \ d(\gamma,\alpha) = n} m(\gamma) \right) \delta^{Q(d(\alpha)+n)p'(1-s)} = m(\alpha) \sum_{n=0}^{\infty} \delta^{Q(d(\alpha)+n)p'(1-s)} \approx m(\alpha)^{1+p'-sp'}.
\]

Summing \(I\) and \(II\) and using the definition of capacity,

\[
\text{Cap}(\partial S(\alpha)) \geq \frac{\omega(\alpha)^p}{E(\omega)^{p-1}} \approx \begin{cases} 
  \frac{m(\alpha)^p}{|m(\alpha)^{p'd(\alpha)}|^{p-1}} & \text{if } s = 1/p', \\
  \frac{m(\alpha)^p}{|m(\alpha)^{1-sp'+p'}|^{p-1}} & \text{if } 1/p' < s < 1,
\end{cases}
\]

\[= \begin{cases} 
  d(\alpha)^{1-p} & \text{if } s = 1/p', \\
  m(\alpha)^{p(s-1/p')} & \text{if } 1/p' < s < 1.
\end{cases} = \text{Cap}([\alpha]),
\]

as wished. \(\blacksquare\)

By Theorem 1, these conclusions apply to the case of the trees coming from Ahlfors-regular spaces and \(0 < s < 1\). A simple geometric series argument, Proposition 27 and Lemma 28 imply:

**Corollary 29.** Let \((X, m, \rho)\) be a (bounded) Ahlfors \(Q\)-regular space and \(0 < s < 1\). Then,

(i) For all points \(x\) in \(X\), \(\text{Cap}_X(\{x\}) = 0\) if and only if \(\frac{1}{p'} \leq s < 1\);

(ii) More generally,

\[
\text{Cap}_X(B(x, r)) \approx \begin{cases} 
  \log \frac{1}{r} & \text{if } s = 1/p', \\
  r^{Q(s-p' - 1)} & \text{if } 1/p' \leq s < 1.
\end{cases}
\]

As earlier anticipated, the value \(s = 1/p'\) relates the capacity \(\text{Cap}_X\) with a logarithmic capacity in the tree. Ultimately, this fact is hidden in the Wolff inequality.

### 5.3. An Algorithm to Compute Tree Capacities.

For the material in this subsection, relevant references are also \([30]\) and \([10]\). First, we show that the capacity of a set can be computed as the “derivative at infinity” of a Green function. In this case, the rôle of the point at infinity is played by the root.

**Theorem 30.** Let \(E\) be closed in \(T\) and let \(\varphi\) be the corresponding extremal function. Then,

\[
\text{Cap}(E) = \varphi^{p-1}(o)\pi(o).
\]
The Green function in question is the equilibrium potential \( \Phi = I \varphi \).

**Proof.** Let \( E \) be a closed subset of \( T \) and \( \varphi = (I^* \sigma)^{p'-1} \pi_1^{1-p'} \) be, respectively, its capacitary measure and capacitary function. Then,

\[
\text{Cap}(E) = \sigma(E) = I^* \sigma(o) = \pi(o) \varphi(o)^{p-1}.
\]

Let \( d_\pi \) be the distance associated with \( \pi \): \( d_\pi(x) = \sum_{y \in P(x)} \pi_1^{1-p'}(y) \). The capacity of a subset of \( T \) can be computed by means of a recursive algorithm.

**Theorem 31.** Let \( Z \) be a set in \( T \), \( x_0 \) a point in \( T \) and let \( x_1, \ldots, x_n \) be its children. For each \( j \), let \( Z_j = Z \cap S(x_j) \). If \( \text{Cap}_{x_j} \) denotes capacity with respect to the root \( x_j \), then

\[
\text{Cap}(Z) = \frac{\sum_j \text{Cap}_{x_j}(Z_j)}{1 + d_\pi(x) \left( \sum_j \text{Cap}_{x_j}(Z_j) \right)^{p'-1}}.
\]

The theorem and its proof also hold if the set of the children is infinite. We need the obvious lemma.

**Lemma 32.** Let \( \varphi = (I^* \sigma)^{p'-1} \pi_1^{1-p'} \), where \( \sigma \in \mathcal{M}_+(T) \) is a measure. For \( x \) in \( T \), let \( C(x) \) be the set of the children of \( x \) in \( T \). Then,

\[
\pi(x) \varphi(x)^{p-1} = \sigma(x) + \sum_{x_j \in C(x)} \pi(x_j) \varphi(x_j)^{p-1}.
\]

We will apply the Lemma for \( x \) outside the support of \( \sigma \).

**Proof of Theorem 31.** Let \( \varphi \) be the extremal function for \( \text{Cap}(Z) \). We claim that

\[
(27) \quad \varphi_j = \frac{\varphi}{1 - I \varphi(x)}.
\]

restricted to \( S(x_j) \), is extremal for \( \text{Cap}_{x_j}(Z_j) \).

For \( \zeta \in Z \), \( \sum_{y=x_j}^{\zeta} \varphi_j(y) = \frac{I_\varphi(\zeta) - I_\varphi(x)}{1 - I \varphi(x)} \geq 1 \ q.e. \) \( \zeta \), hence \( \varphi_j \) is a candidate to be extremal for \( \text{Cap}_{x_j}(Z_j) \). Suppose there exists another \( \psi \) such that \( \sum_{y=x_j}^{\zeta} \psi(y) \geq 1 \ q.e. \) \( \zeta \in Z_j \) and \( \|\psi\|_{L^p(S(x_j), \sigma)} < \|\varphi_j\|_{L^p(S(x_j), \pi)} \). Reasoning as in the proof of Theorem 38 below, we can find a function \( \varphi' \) such that \( \|\varphi'\|_{L^p(\pi)} < \|\varphi\|_{L^p(\pi)} \) and \( I \varphi' \geq 1 \ q.e. \) on \( Z \), contradicting the extremality of \( \varphi \). The claim is proved.

Next, we claim that

\[
(28) \quad \sum_{y=0}^x \varphi(y) \pi(y) = d_\pi(x) \text{Cap}(Z)^{p'}.
\]

Let \( y_0 = o, \ldots, y_N = x \) be an enumeration of the points in the geodesic \([o, x]\) between \( o \) and \( x \). Since \( E \) is contained in \( S(x) \), \( \varphi \) has is supported on \( S(x) \cup [o, x] \).

By Lemma 32,

\[
\varphi(y_j)^{p-1} \pi(y_j) = \varphi(y_{j-1})^{p-1} \pi(y_{j-1})
\]

and, by iteration,

\[
\varphi(y_j) = \frac{\pi(o)^{p'-1}}{\pi(y_j)^{p'-1}}.
\]
Summing,
\[
\sum_{y=0}^{x} \pi(y) \varphi(y)^p = \sum_{y=0}^{x} \pi(y)^{1+\rho(1-p')} \pi(o)^{p(p'-1)} \varphi(o)^p
\]
\[
= d_\pi(x) \pi(o)^p \varphi(o)^p = d_\pi(x) \text{Cap}(Z)^{p'},
\]
by Lemma 30. This proves (28). Hence,
\[
\text{Cap}(Z) = \sum_{j} \sum_{S(x_j)} \varphi^p \pi + \sum_{o} \varphi^p \pi
\]
\[
= \sum_{j} \sum_{S(x_j)} \varphi_j^p \pi[1 - I\varphi(x)]^p + d_\pi(x) \text{Cap}(Z)^{p'}
\]
\[
= \sum_{j} \text{Cap}_{x_j}(Z_j)[1 - \text{Cap}(Z)^{p'-1}d_\pi(x)]^p + d_\pi(x) \text{Cap}(Z)^{p'} \text{ (by equations (27) and (28))},
\]
since \( I\varphi(x) = \sum_{y=0}^{x} \varphi(y) = \pi(o)^{p'-1} \varphi(o)d_\pi(x) \). Thus,
\[
\text{Cap}(Z) = \sum_{j} \text{Cap}_{x_j}(Z_j)[1 - \text{Cap}(Z)^{p'-1}d_\pi(x)]^p + d_\pi(x) \text{Cap}(Z)^{p'}.
\]

Rewrite the identity as
\[
\left( \sum_{j} \text{Cap}_{x_j}(Z_j) \right)^{p'-1} = \text{Cap}(Z)^{p'-1} \left[ 1 - \text{Cap}(Z)^{p'-1}d_\pi(x) \right]^{(1-p)(p'-1)}
\]
and solve with respect to \( \text{Cap}(Z) \). The recursive formula in the statement of the theorem is proved.  ■

5.4. Trace Inequalities on Trees and Ahlfors-regular spaces.

5.4.1. Trace Inequalities on Trees. To each positive measure \( \mu \) we associate

\[
[\mu] := \sup_{a \in T} \left\{ \frac{I^*[I^* \mu]^{p'-1}}{I^* \mu(a)} \right\}^{p-1}.
\]

By definition, it is homogeneous of degree \( d = 1 \) with respect to \( \mu \) and \( d = -1 \) with respect to \( \pi \). Those measures \( \mu \) having \( [\mu] < \infty \) are called Carleson measures for \((I, \pi, p)\).

The following theorem was proved, on trees, in [5] and, in a slightly more general form, in [6]. Since the proof in [6] is short, we will give it here, for convenience of the reader. (See also [8] for the linear case with special weights. The proof of the special case is the same as that of the general one).

Trace inequalities and, more generally, weighted trace inequalities have a long history. We mention the approach via testing condition of [24] and, long before, that by means of capacitary conditions by Maz’ya [28]. See also [23] for an approach relating trace inequalities and nonlinear equations.

**Theorem 33.** There is a constant \( C(p) \) which only depends on \( p \) such that

\[
\int_T I f^p d\mu \leq C(p) [\mu] \sum_{x \in T} f^p(x)\pi(x).
\]
In fact, we have the quantitative estimates

\[ [\mu] \leq \|I\|_{L^p(\pi), L^p(\mu)}^p \leq C(p)[\mu]. \]

We say that a measure is a Carleson measure for the discrete potential \( g \) (recall that \( g(\zeta, \alpha) = \chi_{\mathcal{P}(\zeta)}(\alpha) \)) if it satisfies the imbedding \( (30) \). We say that \( \mu \) satisfies the testing condition if \( [\mu] < \infty \). The theorem says that Carleson measures are exactly those satisfying the testing condition.

**Proof of the Theorem 33.** Inequality \( (30) \) means that

\[ I : \ell^p(\pi) \to L^p(\mu) \]

is bounded and, by duality, this is equivalent to the boundedness, with the same norm (which we call \( |||\mu||| \) in the course of the proof), of

\[ I^*_\mu : L^{p'}(\mu) \to \ell^{p'}(\pi^{1-p'}) \],

where, it can be easily checked,

\[ I^*_\mu g(\alpha) = I^*(gd\mu)(\alpha) = \int_{S(\alpha)} gd\mu. \]

For the duality, we use the \( \ell^2 \) inner product to have \( [\ell^p(\pi)]^* = \ell^{p'}(\pi^{1-p'}) \) and the \( L^2(\mu) \) inner product to have \( [L^p(\mu)]^* = L^{p'}(\mu) \).

The boundedness of \( I^*_\mu \) is expressed by the inequality

\[ \sum_{\alpha \in \mathcal{T}} |I^*(gd\mu)(\alpha)|^{p'} \pi^{1-p'}(\alpha) \leq |||\mu|||^{p'} \int_{\mathcal{T}} |g|^{p'} d\mu, \]

with a finite constant \( |||\mu||| \), which obviously has to be checked on positive \( g \)'s only. We make the left hand side of \( (31) \) larger. Introduce the maximal operator

\[ \mathcal{M}_\mu g(\zeta) := \sup_{\beta \in \mathcal{P}(\zeta)} \frac{I^*(gd\mu)(\alpha)}{I^*_\mu}, \]

with \( \zeta \) in \( \mathcal{T} \) and \( g \) positive and measurable on \( \mathcal{T} \).

We will show that, in fact,

\[ \sum_{\alpha \in \mathcal{T}} [\mathcal{M}_\mu g(\alpha)]^{p'} I^*(gd\mu)^{p'}(\alpha) \pi^{1-p'}(\alpha) \leq |||\mu||| \int_{\mathcal{T}} g^{p'} d\mu. \]

In fact, \( (32) \) follows from the more general inequality

\[ \int_{\mathcal{T}} [\mathcal{M}_\mu g]^{p'} d\sigma \leq C(p) \int_{\mathcal{T}} g^{p'} \mathcal{M}_\mu (d\sigma) d\mu, \]

which holds for a positive, Borel measure \( \sigma \) on \( \mathcal{T} \) with a constant \( C(p) \) which only depends on \( p \) in \( (1, \infty) \). Inequality \( (33) \) is proved in Theorem 34 below.

Let us verify that \( (33) \) implies \( (32) \). Let \( \sigma \) be the discrete measure

\[ \sigma(\alpha) = I^*(\mu(\alpha))^{p'} \pi^{1-p'}(\alpha), \text{ if } \alpha \in \mathcal{T}. \]

The left hand side of \( (33) \) is the left hand side of \( (32) \). On the right hand side, observe that

\[ \sup_{\zeta \in \mathcal{T}} \mathcal{M}_\mu(\sigma)(\zeta) = [\mu], \]
hence,
\[
\int_T g^p \mathcal{M}_\mu(d\sigma)d\mu \leq [\mu] \int_T g^p d\mu.
\]
The proof that \(\|\mu\| \leq C(p)[\mu]\) is complete.

To prove the other direction, it suffices to test inequality (31) on functions of the form \(g = \chi_{S(\alpha)}\), \(\alpha \in T\).

Observe that testing (32) on the same functions \(g = \chi_{S(\alpha)}\), one obtains that the stronger (32) and the weaker (31) hold together with comparable constants, or together fail.

**Theorem 34.** Given positive Borel measures \(\mu\) and \(\sigma\) on \(\overline{T}\), let

\[
\mathcal{M}_\mu(\sigma) := \sup_{\beta \in \mathcal{P}(\zeta)} \frac{I^*(d\sigma)}{I^*\mu}.
\]

Then, (33) holds.

**Proof.** We show that \(\mathcal{M}_\mu\) satisfies a weak \((1, 1)\) inequality. Let \(\lambda > 0\) and let \(E = \{\zeta \in T : \mathcal{M}_\mu(\sigma)(\zeta) > \lambda\}\). Then, \(E\) is disjoint union (by the monotonicity of \(\mathcal{M}_\mu g\) with respect to to the natural partial order on \(T\)) of sets \(S(\alpha_j), j = 1, \ldots, n\) in \(T\) and

\[
\int_{S(\alpha_j)} g d\mu > \lambda \mu(S(\alpha_j))
\]
(by the minimality of \(\alpha_j\) in \(S(\alpha_j)\)).

Thus, \(\mu(S(\alpha_j)) > 0\) and

\[
\sigma(E) = \sum_j \sigma(S(\alpha_j)) \leq \frac{1}{\lambda} \sum_j \mathcal{M}_\mu(\sigma)(\alpha_j) \int_{S(\alpha_j)} g d\mu \leq \frac{1}{\lambda} \int_E g \mathcal{M}_\mu(\sigma) d\mu.
\]

The \(L^\infty\) inequality is obvious. Suppose \(g \leq C \mathcal{M}_\mu(\sigma)\)-a.e.. We can assume that \(\sigma \neq 0\) (otherwise (33) holds trivially), hence, \(\mathcal{M}_\mu(\sigma)(\zeta) > 0\) for all \(\zeta\) in \(\overline{T}\). This implies that \(g \leq C\) \(\mu\)-a.e., so that \(\int_{S(\alpha)} g d\mu/\mu(S(\alpha)) \leq C\) for all \(\alpha\) in \(T\) (set it equal to 0 if \(\mu(S(\alpha))\)). Then, \(\mathcal{M}_\mu g \leq C\) everywhere, hence \(\sigma\)-a.e.. Marcinkiewicz interpolation gives now the theorem.

5.4.2. **Trace Inequalities on Ahlfors-regular spaces.** Let \(K : X \times X \to [0, \infty]\) be the kernel defined in (2), with \(s \in [1/p'(1], \) where \(X\) is a Ahlfors \(Q\)-regular space. We say that a positive Borel measure \(\mu\) on \(X\) satisfies the trace inequality for the space \((X, m, \rho)\), the exponent \(p\) and the kernel \(K\) if the inequality

\[
(34) \quad \int_X \left( \int_X K(x, y) f(y) dm(y) \right)^p d\mu \leq C(\mu) \int_X f^p dm
\]
holds for all positive, Borel \(f\).
There is an enormous amount of literature on weighted trace inequalities (see, e.g., [24], [23], [14] and the literature quoted there) and we make no claim of originality for the results we are going present below. What we are interested in is the relationship between discrete and continuous trace inequalities, and between different necessary and sufficient conditions.

Let $T$ be the tree associated with $X$ and $K_T$ be the kernel on $\partial T$ which corresponds to the same choice of $p$ and $s$.

**Theorem 35.** The measure $\mu$ satisfies the trace inequality for $K$ if and only if $\Lambda^* \mu$ satisfies the trace inequality for $K_T$.

**Proof.** For a Borel measure $\omega$ on $X$, let $K\omega(x) = \int_X K(x,y)d\omega(y)$. By duality and symmetry of the kernel $K$, $\mu$ satisfies the trace inequality for $K$ if and only if the inequality

$$\int_X K(gd\mu)^p'd\mu \leq C(\mu) \int_X g^p'd\mu$$

hold for all positive Borel functions $g$. That is,

$$C(\mu) \int_X g^p'd\mu \geq \mathcal{E}_X(gd\mu) \approx \mathcal{E}_{\partial T}(\Lambda^*(gd\mu)).$$

For $g : X \to \mathbb{R}^+$, define $\Lambda^*g := g \circ \Lambda$. Then one can easily show that

$$\Lambda^*(gd\mu) = (\Lambda^*g)d(\Lambda^*\mu).$$

To verify (37), first check the statement for simple $g$, then use Monotone Convergence Theorem. By the (37),

$$\mathcal{E}_{\partial T}(\Lambda^*(gd\mu)) = \mathcal{E}_{\partial T}((\Lambda^*g)(d\Lambda^*\mu))$$

and, since

$$\int_X g^p'd\mu \approx \int_{\partial T} \Lambda^*(g^p'd\mu) = \int_{\partial T} (\Lambda^*g)^p'd\Lambda^*\mu,$$

the wished inequality (35) is equivalent to the tree inequality

$$\mathcal{E}_{\partial T}((\Lambda^*g)(\Lambda^*\mu))) \leq C(\mu) \int_{\partial T} (\Lambda^*g)^p'd\Lambda^*\mu.$$

Set $G = \Lambda^*g$ and $M = \Lambda^*\mu$. The inequality

$$\mathcal{E}_{\partial T}(GdM) \leq C(M) \int_{\partial T} G^p'dM$$

expresses the fact that $M$ satisfies the trace inequality on $K_{\partial T}$. Hence, if $\Lambda^*\mu$ satisfies the trace inequality on $\partial T$, $\mu$ does on $X$.

In the other direction, if (35) holds, then $\mu$ satisfies the capacitary inequality $\mu(E) \leq C(\mu) \text{Cap}_X(E)$, when $E$ is a closed subset of $X$. Simply test the inequality dual to (35) on those $f$ such that $f \geq 1$ on $E$. By the equivalence of continuous and discrete capacities, this implies that $\Lambda^*\mu(F) \leq C(\mu) \text{Cap}_{\partial T}(F)$ for closed subsets of $\partial T$. It is well known (see for example [24]) that the capacitary condition implies, via duality, the testing condition $[\Lambda^*\mu] < \infty$ (see also the next subsection) and this, in turn, implies the discrete trace inequality by Theorem 33. ■

**Remark 36.** In view of the results of Section 5.1, the fact that $M$ satisfies the trace inequality for $K_{\partial T}$ is equivalent to the fact that $M$ satisfies (30). Hence, we have a dyadic characterization for the measures satisfying the trace inequalities on Ahlfors-regular spaces.
Corollary 37. A nonnegative Borel measure \( \mu \) on \( X \) satisfies the trace inequality (34) if and only if it satisfies the testing condition of Kerman-Sawyer type

\[
\int_B \left[ \int_B K(x, y) \, d\mu(y) \right]^{p'} \, dm(x) \leq C(\mu)\mu(B)
\]

uniformly over the metric balls \( B \) in \( X \). The constant \( C(\mu) \) is comparable with \([\mu]\).

**Proof.** Clearly, the testing condition (38) is implied by the dual (35) to the trace inequality (34). In the other direction, it is easy to see that the testing condition (38) implies the testing condition on trees, \([\Lambda^*\mu] < \infty\), which implies the discrete trace inequality, which implies in turn, by Theorem 35, the trace inequality in \( X \). \( \blacksquare \)

5.5. Defining Capacities via Carleson Measures.

**Theorem 38.** Let \( E \subseteq \overline{T} \) be compact. Then,

\[
\text{Cap}(E) = \sup_{\text{supp}(\mu) \subseteq E} \frac{\mu(E)}{[\mu]}.
\]

To prove the theorem, we need another characterization of capacity, which holds for the classical capacity, but which is not included in [1]. Lacking a reference, we give a proof which works in the present context.

**Proposition 39.**

\[
\text{Cap}(K) = \inf \{ \mathcal{E}(\mu) : V(\mu) \geq 1 \text{ on } K \}.
\]

**Proof.** Suppose first that \( K \subseteq T \) and that it is finite. Given a measure \( \mu \), let \( \varphi = (I^*\mu)^{p'-1} - \pi^{1-p'} \). Then, \( \mathcal{E}(\mu) = ||\varphi||_{L^p(\pi)}^p \) and \( V(\mu) = I \varphi \). Let

\[
\mathcal{C}(K) = \inf \{ \mathcal{E}(\mu) : V(\mu) \geq 1 \text{ on } K \} = \mathcal{E}(\overline{\mu}),
\]

where \( \overline{\mu} \) is the extremal measure (it exists by elementary compactness and finiteness of \( K \)), which satisfies \( V(\overline{\mu}) = 1 \) on \( K \) and it is unique.

Let \( \overline{\varphi} = (I^*\overline{\mu})^{p'-1} - \pi^{1-p'} \). Then,

\[
\text{Cap}(K) = \inf \left\{ ||\varphi||_{L^p(\pi)}^p : I \varphi \geq 1 \text{ on } K \right\} \leq \mathcal{C}(K) = \mathcal{E}(\overline{\mu}) = ||\overline{\varphi}||_{L^p(\pi)}^p.
\]

Suppose there is \( \mu_0 \) such that, with \( \varphi_0 = (I^*\mu_0)^{p'-1} - \pi^{1-p'} \),

\[
\text{Cap}(K) = ||\varphi_0||_{L^p(\pi)}^p = \mathcal{E}(\mu_0)
\]

and

\[
I \varphi_0 = V(\mu_0) \geq 1 \text{ on } K.
\]

**Proof of Theorem 38.**

We start with inequality \( \leq \). For fixed \( a \in T \) we denote by \( \overline{T}_a = \overline{S}(a) \) the subtree of \( \overline{T} \) having root \( a \) and we add a subscript \( a \) to the corresponding tree objects. Namely, \( \text{Cap}_a \) is the capacity of subsets of \( \overline{T}_a \), \( \mathcal{E}_a \) is the energy in \( \overline{T}_a \), \( \omega_a \) is the extremal measure in the definition of capacity, and so on. Let \( E_a = E \cap \overline{T}_a \). The extremal measure \( \omega_a \) and the function \( \varphi_a = (I^*\omega)^{p'-1} - \pi^{1-p'} \) satisfy

\[
\text{Cap}_a(E_a) = \omega_a(E_a) = \mathcal{E}_a(\omega_a) = ||\varphi_a||_{L^p(\overline{T}_a, \pi)}^p.
\]
We claim that \( \omega_a \) is a rescaling of the extremal measure for \( E \) in \( T \), \( \omega \), restricted to \( E_a \):
\[
\omega_a = \left. \frac{\omega|_{E_a}}{1 - I[(I^\ast \omega)^{p' - 1} \pi^{1 - p'}](a^{-1})} \right|_a = \left. \frac{\omega|_{E_a}}{1 - V(\omega)(a^{-1})} \right|_a.
\]

In fact, \( \omega_a \) minimizes \( E_a(\mu) \) over all measures \( \mu \) such that \( I_a[(I_a^\ast \mu)^{p' - 1} \pi^{1 - p'}](\xi) = V_a(\mu)(\xi) \geq 1 \) on \( E_a \) (with the possible exception of a set having null-capacity). On the other hand, we claim that \( \omega|_{E_a} \) minimizes \( E_a(\mu) \) among all measures \( \mu \) on \( E_a \) such that \( V_a(\mu)(\xi) \geq 1 - V(\omega)(a^{-1}) \) q.e. on \( E_a \).

Suppose this is not the case. Then there exists a measure \( \nu \) on \( E_a \) such that \( V_a(\nu)(\xi) \geq 1 - V(\omega)(a^{-1}) \) for q.e. \( \xi \in E_a \) and
\[
E_a(\nu) = \sum_{T_a} (I^\ast \nu)^{p'\pi^{1 - p'}}(\xi) < \sum_{T_a} (I^\ast \omega)^{p'\pi^{1 - p'}} = E_a(\omega|_{E_a}).
\]

Let \( \varphi \) be such that \( V(\omega) = I^\varphi \) in \( T \) and \( \varphi \) so that \( V_a(\nu) = I_a^\nu \) in \( T_a \). Define now a new function \( \psi \) on \( T \) by
\[
\psi(x) = \begin{cases} 
\varphi_1(x) & \text{if } x \in T_a, \\
\varphi(x) & \text{if } x \in T \setminus T_a.
\end{cases}
\]

We have
\[
I^\psi(\xi) \geq 1 \text{ q.e. on } E,
\]
hence \( \|\psi\|_{L^p(\pi)} \geq \text{Cap}(E) \). On the other hand,
\[
\|\psi\|_{L^p(\pi)} = E_a(\nu) + [E(\omega) - E_a(\omega)] < E(\omega) = \text{Cap}(E),
\]
and we have reached a contradiction.

The measure
\[
\lambda = \frac{\omega|_{E_a}}{1 - V(\omega)(a^{-1})},
\]
then, minimizes \( E_a(\mu) \) over the set of the measures \( \mu \) such that \( V_a(\mu)(\xi) \geq 1 \) for q.e. \( \xi \in E_a \), hence \( \lambda = \omega_a \). The claim is proved.

By the homogeneity of the energy,
\[
E_a(\omega|_{E_a}) = (1 - V(\omega)(a^{-1}))^{p'} E_a(\omega_a) = (1 - V(\omega)(a^{-1}))^{p'} \omega_a(E_a) = (1 - V(\omega)(a^{-1}))^{p - 1} \omega(E_a).
\]

As a consequence,
\[
\sum_{\xi \geq a} (I^\ast \omega)^{p'\pi^{1 - p'}} = \frac{E_a(\omega|_{E_a})}{\omega(E_a)} = (1 - V(\omega)(a^{-1}))^{p' - 1} \leq 1,
\]
with equality if and only if \( a = o \) (we use the default value \( V(\omega)(o^{-1}) = 0 \)).

Hence, \( [\omega] = 1 \) and
\[
\text{Cap}(E) = \omega(E) = \frac{\omega(E)}{[\omega]}.
\]

We now prove \( \geq \). By definition of \([\cdot] \), \( E(\mu) \leq [\mu]^{p - 1} \mu(E) \) for all measures \( \mu \). Then,
\[
\frac{\mu(E)}{[\mu]} \leq \frac{\mu(E)}{(E(\mu))^{p - 1}} = \frac{\mu(E)^p}{E(\mu)^{p - 1}} \leq \text{Cap}(E),
\]
as wished. ■

The proof above has an interesting consequence.

**Corollary 40.** If \( \omega \) is the extremal measure for \( \text{Cap}(E) \), with \( E \) closed in \( T \), then \([\omega] = 1\).

5.6. **Monotonicity of the Tree Condition.** First, we give a direct proof that the testing condition for the trace inequalities on trees \( (\mu < \infty) \) is monotone: if \( \nu \leq \mu \), then \([\nu] \lesssim [\mu]\).

Then, we use this fact to give a direct proof that the testing condition and the capacitary condition are equivalent. This answers a question which was posed to us by Maz’ya a few years ago (private communication).

It is known (see [24]) that, by a duality argument, the capacitary condition implies the testing condition. We concentrate, then, on the opposite implication. For a measure \( \mu \) on \( T \), let \( \sigma_\mu = (I^*\mu)^{p'} \pi^{1-p'} \).

**Theorem 41.** Let \( \mu \) be a measure on \( T \) and let \( \lambda \) be a measurable function on \( T \), \( 0 \leq \lambda \leq 1 \). If \( I^*\sigma_\mu \leq I^*\mu \) on \( T \), then \( I^*\sigma_\lambda \mu \leq p \cdot I^*(\lambda \mu) \).

**Corollary 42.** If \( \nu \leq \mu \) and \( \mu, \nu \) are measures on \( T \), then \([\nu] \leq \frac{p'}{p' - 1} [\mu] \).

**Proof.** By rescaling, it suffices to verify the hypothesis at the root. We use a simple argument based on distribution functions.

Let 
\[
M_\mu \lambda (x) = \max_{0 \leq y \leq x} \frac{I^*(\lambda \mu)(y)}{I^*\mu(y)}
\]
be the discrete maximal function we used in [6]. If necessary, we can extend the definition to \( x \in \partial T \) in the obvious way. Then,
\[
I^*\sigma_\lambda \mu (o) = \sum_{x \in T} \left[ \frac{I^*(\lambda \mu)(x)}{I^*\mu(x)} \right]^{p'} (I^*\mu(x))^{p'} \pi^{1-p'} (x)
\leq \sum_{x \in T} [M_\mu \lambda (x)]^{p'} \sigma_\mu (x)
= 2 \int_0^{I^*\sigma_\mu(o)} t^{p'-1} \sigma_\mu (\zeta \in T : M_\mu \lambda (\zeta) > t) dt.
\]

Now, \( \{ \zeta \in \overline{T} : M_\mu \lambda (\zeta) > t \} = \bigcup_j S(x_j) \) is the disjoint union of Carleson boxes in \( T \) (by the definition of the maximal function, we do not need to consider the closure of \( S(x_j) \) in \( \overline{T} \)). Then,
\[
t \sigma_\mu (\zeta \in T : M_\mu \lambda (\zeta) > t) = \sum_j t \sigma_\mu (S(x_j))
\leq \sum_j t I^* \mu (x_j)
\leq \sum_j I^* (\lambda \mu) (x_j)
\leq I^* (\lambda \mu) (o).
\]

Inserting this estimate in the previous one and integrating, we have
\[
I^*\sigma_\lambda \mu (o) \leq \frac{p'}{p' - 1} \cdot I^*\sigma_\mu (o) = p I^*\sigma_\mu (o).
\]

■
An immediate consequence of Theorem 41 is that in (39) we do need to restrict to measures supported in $E$.

**Corollary 43.** Let $E \subseteq T$ be compact. Then,

$$
\text{Cap}(E) \leq \sup_{\mu} \frac{\mu(E)}{|\mu|} \leq p^{p-1} \text{Cap}(E).
$$

5.7. **Trace Inequalities: The Testing Condition implies the Capacitary Condition.**

We now give a direct proof that the testing condition $[\mu] < \infty$, equivalent to the test inequality (30), is equivalent to a capacitary condition (see (45) below). Both conditions are known, in a fairly general context, to characterize the Trace inequality; hence they are a priori equivalent. Here, however, we give it direct proof of their equivalence. Better, we show that (30) implies (45), since the opposite (direct) implication is known (see [24]).

**Theorem 44.** Let $\mu$ be a positive, Borel measure on $T$. If $\mu$ satisfies

$$
\sup_{x \in T} \frac{I^s \left( [I^s \mu]^p \pi^{1-p} \right)(x)}{I^s \mu(x)} \leq C_1(\mu),
$$

then $\mu$ satisfies, for all closed sets $E$ in $T$,

$$
\mu(E) \leq C_2(\mu) \text{Cap}(E).
$$

Moreover, $C_2(\mu) \leq p^{p-1} C_1(\mu)$. Conversely, (45) implies (44).

**Proof.** Without loss of generality, suppose that $\mu$ satisfies (44) with $C_1(\mu) = 1$. Then, $\mu_E := \mu|_E \leq \mu$ satisfies $[\mu_E] \leq p^{p-1}$ by Theorem 41. Hence,

$$
\text{Cap}(E) = \sup_{\sup(\nu) \in E} \frac{\nu(E)}{|\nu|} \geq \frac{\mu(E)}{|\mu_E|} \geq p^{1-p} \mu(E).
$$

Conditions of testing and of capacitary type are also both known to characterize the Carleson measures for the holomorphic Dirichlet space. Let us recall definitions and results. The Dirichlet space $D$ contains those functions $f$ which are holomorphic in the unit disc $D$ of the complex plane for which the norm

$$
\|f\|_D^2 = |f(0)|^2 + \int_D |f'(z)|^2 \, dx \, dy
$$

is finite. A measure $\mu$ on $\overline{D}$ is Carleson for $D$ if the imbedding $D \hookrightarrow L^2(\mu)$ is bounded. (The problem of the boundary values is indeed of interest, when $\mu(\partial D) \neq 0$: we direct the interested reader to [6] and [11] and to the references therein for a discussion of this problem. We just mention that such problem is intimately related with the characterization itself of the Carleson measures). Carleson measures satisfy a sort of “holomorphic trace inequality”, and it is not surprising that they can be characterized by both capacitary and testing conditions. For $z = re^{i\theta}$ in $D$, let $S(z) = \{re^{i\varphi} : r \leq \rho \leq 1, \ |\theta - \varphi| \leq 2\pi(1-r)\}$ be the usual Carleson box with vertex $z$ and let $I(z) = \partial S(z) \cap \partial D \subseteq S(z)$ be the part of its boundary lying on $\partial D$. Stegenga [31] proved that the Carleson measures for $D$ are exactly those satisfying

$$
\mu(\bigcup_{j=1}^n S(z_j)) \leq C(\mu) \text{Cap}(\bigcup_{j=1}^n I(z_j)),
$$

where $C(\mu)$ is an absolute constant depending only on $\mu$. These conditions are equivalent to

$$
\sup_{x \in T} \frac{I^s \left( [I^s \mu]^p \pi^{1-p} \right)(x)}{I^s \mu(x)} \leq C_1(\mu),
$$

and the opposite (direct) implication is known (see [24]).
for all finite subsets \( \{ z_j \} \) of \( D \) such that the arcs \( I(z_j) \) are pairwise disjoint. Here the capacity is nothing other than the logarithmic capacity which corresponds to \( s = \frac{1}{2} \) and \( p = 2 \) on the unit circle \( \mathbb{T} \). On the other hand, a finite measure \( \mu \) is Carleson if and only if it satisfies -for all \( a \) in \( D \)- the testing condition

\[
(47) \quad \int_{S(a)} [\mu(S(z) \cap S(a))]^2 \frac{dxdy}{(1-|z|^2)^2} \leq C(\mu) \mu(S(a)).
\]

This condition can be written in a discrete fashion (see, e.g., [5] and [6]), in the spirit of the present paper. We remind the reader that the first testing condition for Carleson measures was found in [25] and it is different from (47) (it is what we get from (47) after applying the Muckenhoupt-Wheeden-Wolff inequality. The left hand side of (47) is analogous to the left hand side of (11), while the left hand side of the testing condition in [25] is analogous to the right hand side of (11)).

Now, Stegenga’s condition involves a measure in the interior and a capacity for a boundary set. In Theorem 44, by contrast, we have that measure and capacity either both live in the interior, or both live on the boundary. It is then interesting, we believe, to know whether the capacity of the interior set is comparable with that of its “shadow” on the boundary. The answer on trees, which might be transferred to various metric situations, is positive, under a mild assumption on the weight \( \pi \).

**Lemma 45.** Suppose that, for all \( x \) in \( \mathbb{T} \),

\[
(48) \quad \operatorname{Cap}(\partial S(x)) \gtrsim d_\pi(x)^{1-p}
\]

(i.e., that \( d_\pi(x)^{1-p} \approx \operatorname{Cap}(\{x\}) \approx \operatorname{Cap}(\partial S(x)) \)). Then,

\[
\operatorname{Cap}(S(E)) \lesssim \operatorname{Cap}(E)
\]

(i.e., \( \operatorname{Cap}(S(E)) \lesssim \operatorname{Cap}(E) \)) whenever \( E = \cup_j \partial S(x_j) \) is finite union of sets of the form \( \partial S(x_j) \).

Observe that the converse inequality, \( \operatorname{Cap}(S(E)) \geq \operatorname{Cap}(E) \), follows from trivial comparison.

**Proof.** Let \( \varphi \) be the extremal function for \( E \) and let \( A = \{ x_j : I\varphi(x^{-1}_j) \geq 1/2 \} \) and \( B = \{ x_j : I\varphi(x^{-1}_j) < 1/2 \} \). Then,

\[
\operatorname{Cap}(E) = \sum_{x \in T} \varphi(x)\pi(x)
\geq \sum_{x \in \cup_{a \in A} [0,a]} \varphi^p(x)\pi(x) + \sum_{b \in B} \sum_{x \in S(b)} \varphi^p(x)\pi(x)
\geq 2^{-p} \operatorname{Cap}(A) + \sum_{b \in B} (1 - I\varphi(b^{-1}))^p \sum_{x \in S(b)} \left( \frac{\varphi(x)}{1 - I\varphi(b^{-1})} \right)^p \pi(x).
\]

(49)

Since, when \( \zeta \) belongs to \( S(b) \) for some \( b \in B \),

\[
\sum_{x \in [b,\zeta]} \frac{\varphi(x)}{1 - I\varphi(b^{-1})} \geq 1,
\]

the function \( \varphi(x)/(1 - I\varphi(b^{-1})) \) is admissible for \( \operatorname{Cap}_b(\partial S(b)) \), hence,
\[ \sum_{x \in S(b)} \left( \frac{\varphi(x)}{1 - I\varphi(b^{-1})} \right)^p \pi(x) \geq \text{Cap}_b(\partial S(b)) \]
\[ \geq \text{Cap}(\partial S(b)) \geq \mathcal{D}(b)^{1-p}, \]
by (48). Inserting this last inequality in (49),
\[ \text{Cap}(E) \geq \left[ \text{Cap}(A) + \sum_{b \in B} d_x(b)^{1-p} \right] \]
\[ \geq \text{Cap}(A) + \sum_{b \in B} \text{Cap}(\{b\}) \]
\[ \geq \text{Cap}(A \cup B) = \text{Cap}(S(E)). \]

**Corollary 46.** Suppose that condition (48) holds for the weight \( \pi \). Then, for a measure \( \mu \) on \( T \), the testing condition (44) is equivalent to the capacitary condition
\[ \mu(S(E)) \lesssim \text{Cap}(E), \]
to be checked over all \( E \) subsets of \( \partial T \) having the form \( E = \cup_j \partial S(x_j) \) (with the set of the \( x_j \)'s being finite).

**Appendix I: Nonlinear Potential Theory**

We report the main results from [1], 2.3 and 2.5. Let \((M, \nu)\) be a measure space and \((X, \delta)\) be a separable metric space. A kernel \( g \) on \( X \times M \) is a function \( g : X \times M \to [0, +\infty] \) such that \( g(\cdot, y) \) is lower semicontinuous on \( X \) for each \( y \in M \) and \( g(x, \cdot) \) is measurable for each \( x \in X \). Fix \( 1 < p < \infty \).

Let \( m \in \mathcal{M}_+(X) \) be a positive Borel measure on \( X \) and \( f \geq 0 \) be measurable on \( M \). We define the potentials
\[ \mathcal{G} f(x) = \int_M g(x, y)f(y)d\nu(y), \ x \in X; \]
\[ \mathcal{G}^* m(y) = \int_X g(x, y)dm(x), \ y \in M. \]

The mutual energy of \( m \) and \( f \) is
\[ \mathcal{E}(m, f) = \int_X \mathcal{G} f dm = \int_M \mathcal{G}^* m f d\nu. \]

The capacity of a subset \( E \) of \( X \) is
\[ \text{Cap}(E) = \inf \left\{ \int_M f^p : f \in \Omega_E \right\}, \]
where \( \Omega_E = \{ f \in L^p_+(\nu) : \mathcal{G} f \geq 1 \text{ on } E \} \). We say that a property holds q.e. if it holds for all \( x \) in \( X \), but for a set having null capacity. In the definition (53) we can replace \( \Omega_E \) by \( \Omega^*_E \), where \( \Omega^*_E = \{ f \in L^p_+ (\nu) : \mathcal{G} f \geq 1 \text{ q.e on } E \} \).
Theorem 47. If $\text{Cap}(E) < \infty$, then there exists a unique $f^E$ in $L^p_+(\nu)$ such that $\mathcal{G}f^E \geq 1$ q.e. on $E$ and

$$\text{Cap}(E) = \int_M (f^E)^p \, d\nu.$$ 

Proposition 48.

1. $\text{Cap}(\emptyset) = 0$;
2. If $E \subseteq F$, then $\text{Cap}(E) \leq \text{Cap}(F)$;
3. If $K_i \searrow K$ are compact sets, then $\text{Cap}(K_i) \searrow \text{Cap}(K)$;
4. If $E_i \nearrow E$ are arbitrary sets, then $\text{Cap}(E_i) \nearrow \text{Cap}(E)$.

A set $E$ in $X$ is capacitable if

$$\text{Cap}(E) = \sup \{\text{Cap}(K) : K \subseteq E \text{ is compact} \}.$$ 

Theorem 49. All Suslin sets (hence, all Borel sets) are capacitable.

Theorem 50. If $E_i \nearrow E$ are arbitrary, then $\text{Cap}(E_i) \nearrow \text{Cap}(E)$. If, moreover, $\text{Cap}(E) < \infty$, then $f^{E_i} \to f^E$ in the $L^p(\nu)$ norm.

The capacity can be defined in terms of measures, as well.

Theorem 51. If $K$ is compact in $X$, then

$$\text{Cap}(K)^{1/p} = \sup \left\{ m(K) : m \in \mathcal{M}_+(K), \|\mathcal{G}m\|_{L^p(\nu)} \leq 1 \right\}.$$ 

The formula extends to all Suslin sets $E$.

Theorem 52. If $K$ is compact in $X$, then there exists $m^K$ in $\mathcal{M}_+(K)$ such that $f^K = (\mathcal{G}m^K)^{p'-1}$ and

$$\text{Cap}(K) = m^K(K) = \int_M (\mathcal{G}m^K)^{p'} \, d\nu = \int_X \mathcal{G}f^K \, dm^K.$$ 

The nonlinear potential $V^m$ is defined by

$$V^m(x) = (\mathcal{G}m)^{p'-1}(x) = \int_M g(x, y) \left( \int_X g(z, y) \, dm(z) \right)^{p'-1} \, d\nu(y).$$ 

In particular, the energy of $m$ is

$$\mathcal{E}(m) = \int_X V^m \, dm = \int_M (\mathcal{G}m)^{p'} \, d\nu.$$ 

Theorem 53. If $K$ is compact in $X$, then

$$\mathcal{G}f^K = V^{m^K} \leq 1 \text{ on } \text{supp}(m^K)$$

and

$$\text{Cap}(K) = \max \{m(K) : m \in \mathcal{M}_+(K), V^m \leq 1 \text{ on } \text{supp}(m) \}.$$ 

To finish, we extend the above under a reasonable extra assumption.

Theorem 54. Suppose that $\text{Cap}(E) < \infty$, that $M$ is itself a locally compact space and that, whenever $\varphi \in C_0(M)$, $\mathcal{G}\varphi$ is continuous and

$$\lim_{x \to \infty} \mathcal{G}\varphi(x) = 0.$$ 

There is, then, a measure $m^E$ in $\mathcal{M}_+(E)$ such that
\textbf{Proposition 55.}

1. \( f^E = (\mathcal{G}m^E)^{p'-1} \);
2. \( \mathcal{G}f^E \geq 1 \) q.e. on \( E \);
3. \( \mathcal{G}f^E \leq 1 \) q.e. on \( \text{supp}(m^E) \);
4. \( \text{Cap}(E) = m^E(E) = \int_M (\mathcal{G}m^E)^{p'} \, d\nu = \int_X \mathcal{G}f^E \, dm^E. \)

\textbf{Appendix II: Nonlinear Potential Theory on Trees}

We set \( X = T \) in Appendix I, where \( T \) is endowed with any reasonable metric and \( M = T \) with the discrete measure \( \nu = \pi^{1-p'} \). Our kernel is
\[
g(\xi, x) = \chi_{P(\xi)}(x) = \chi_{S(x)}(\xi).
\]
We shall also let \( \varphi = f\nu \), hence, \( \varphi^p\pi = f^p\nu \). With these definitions and renormalizations, we have the following properties.

\textbf{Proposition 55.}

1. \( Gf(\zeta) = I(f\nu)(\zeta) = I\varphi \);
2. \( \mathcal{G}m(y) = I^*m(y) = \int_{S(y)} dm; \)
3. \( \text{Cap}(E) = \inf \{ \sum \varphi^p\pi : \varphi \geq 0, I\varphi \geq 1 \text{ on } E \}; \)
4. \( \text{Let } E \subset T. \text{ Then, } \text{Cap}(E) = 0 \text{ if and only if there is } \varphi \in L_+^p(\pi) \text{ such that } E \subset \{ x : I\varphi(x) = +\infty \}. \text{ In particular, } E \subset \partial T; \)
5. \( \text{Let } \Omega_E \text{ be the closure of } \Omega_E \text{ in } L^p(\pi). \text{ Then, } \)
\[
\Omega(E) := \{ \varphi \in L_+^p(\pi) : I\varphi(\xi) \geq 1 \text{ for q.e. } \xi \in E \};
\]
6. \( \text{Let } E \subset T. \text{ Then, } \exists! \varphi^E \text{ such that } \varphi^E \in L_+^p(\pi), I\varphi^E \geq 1 \text{ q.e. on } E, \text{ Cap}(E) = \sum \varphi^E(x)^p\pi(x); \)
7. \( \text{The mutual energy of } m \text{ and } f \text{ is } \mathcal{E}(m, f) = \int_T I(f\nu) \, dm = \sum_T I^*m \cdot f \cdot \nu = \mathcal{E}(m, \varphi); \)
8. \( \text{Let } K \subset T \text{ be compact. Then, } \)
\[
\text{Cap}(K)^{1/p} = \sup \left\{ m(K) : \text{ supported in } K, \sum_T [I^*m]^p\pi^{1-p'} \leq 1 \right\};
\]
9. \( \text{There exists an extremal, positive measure } m^K \text{ with support in } K, \text{ such that } \)
\[
\varphi^K = [I^*m^K]^{p'-1}\pi^{1-p'}.
\]
Moreover,
\[
\text{Cap}(K) = m^K(K) = \sum_T [I^*m^K]^p\pi^{1-p'} = \int_T I[\varphi^K] \, dm^K;
\]
10. \( \text{The nonlinear potential of a measure } m \text{ is } \)
\[
V(m)(\zeta) = I(\pi^{1-p'}[I^*m]^{p'-1})(\zeta).
\]
The energy of a measure is
\[
\mathcal{E}(m) = \int_T V(m) \, dm = \sum_x (I^*m)^p\pi^{1-p'} = \int_T I[(I^*m)^{p'-1}\pi^{1-p'}] \, dm;
\]
(11) Let $K$ be compact in $T$. Then,
$$I\varphi^K = V(m^K) \leq 1 \text{ on supp}(m^K)$$
and
$$\text{Cap}(K) = \max \{m(K) : m \geq 0 \text{ and supported in } K, \text{ such that } V(m) \leq 1 \text{ on supp}(m)\};$$

(12) Let $E \subset T$ is such that Cap$(E) < \infty$. Then, there is a measure $m^E$ such that $\varphi^E$, the capacitary function, is given by $\varphi^E = (I^*m^E)^{\frac{1}{p'}-1} \pi^{1-p'}$. Moreover,
$$I\varphi^E \geq 1 \text{ q.e. on } E, \ I\varphi^E \leq 1 \text{ q.e. on supp}(m^E)$$
and
$$m^E(E) = \sum_T (I^*m^E)^{\frac{1}{p'}-1} = \int_T I\varphi^E dm^E = \text{Cap}(E);$$

(13) The properties of capacity enumerated in Proposition 48 and the Capacitability Theorem obviously hold.

As a consequence of the above and of homogeneity, we have the following equivalent definitions of capacity.

$$\text{Cap}(K) = \inf \left\{ \|\varphi\|^p_{L^p(\pi)} : I\varphi \geq 1 \text{ on } K \right\}$$
$$= \inf \left\{ \|\varphi\|^p_{L^p(\pi)} : I\varphi \geq 1 \text{ q.e. on } K \right\}$$
$$= \inf \left\{ \frac{\|\varphi\|^p_{L^p(\pi)}}{(\inf_{x \in K} I\varphi)^p} : I\varphi \geq 1 \text{ q.e. on } K \right\}$$
$$= (\sup \{m(K) : \mathcal{E}(m) \leq 1, \text{ supp}(m) \subseteq K\})^{\frac{1}{p}}$$
$$= \sup_{\text{supp}(m) \subseteq K} \frac{m(K)^p}{\mathcal{E}(m)^{p-1}}$$
$$= (\inf \{\mathcal{E}(m) : m(K) \geq 1, \text{ supp}(m) \subseteq K\})^{1-p}$$
$$= \sup \{m(K) : V(m) \leq 1 \text{ on supp}(m), \text{ supp}(m) \subseteq K\}$$
$$= \sup \frac{m(K)}{\sup_{\xi \in \text{supp}(m)} V(m)(\xi)^{p-1}}$$
$$= \left( \inf_{m: \text{supp}(m) \subseteq K} \left\{ \sup_{\xi \in \text{supp}(m)} V(m)(\xi)^{p-1} : m(K) \geq 1 \right\} \right)^{-1}$$

References

[1] D. R. Adams, L. I. Hedberg, Function Spaces and Potential Theory, Grundlehren der Mathematischen Wissenschaften 314, Springer-Verlag, Berlin, 1996, xii+366 pp., 2, 4, 5, 6, 21, 26, 34, 39
[2] H. Aikawa, M. Essén, Potential Theory–Selected topics, Lecture Notes in Mathematics 1633, Springer-Verlag, Berlin, 1996.
[3] N. Arcozzi, Capacity of Shrinking Condensers in the Plane, preprint. 3
[4] N. Arcozzi, R. Rochberg, Topics in Dyadic Dirichlet Spaces, New York J. Math. 10 (2004), 45–67. 19
[5] N. Arcozzi, R. Rochberg, E. Sawyer, Carleson Measures for Analytic Besov Spaces, Rev. Mat. Iberoamericana 18 (2002), no. 2, 443–510. 2, 19, 30, 38
[6] N. Arcozzi, R. Rochberg, E. Sawyer, The Characterization of The Carleson Measures for Analytic Besov Spaces: A Simple Proof, Complex and Harmonic Analysis, 167–177, DEStech Publ., Inc., Lancaster, PA, 2007. 30, 36, 37, 38
| Authors | Title | Journal/Book Details |
|---------|-------|---------------------|
| N. Arcozzi, R. Rochberg, E. Sawyer | Capacity, Carleson Measures, Boundary Convergence and Exceptional Sets | Trans. Amer. Math. Soc. 291 (1989), no. 2, 557–592. |
| N. Arcozzi, R. Rochberg, E. Sawyer | Carleson Measures for the Drury-Arveson Hardy Space and other Besov-Sobolev Spaces on Complex Balls | Adv. Math. 218 (2008), no. 4, 1107–1180. |
| N. Arcozzi, R. Rochberg, E. Sawyer, B. D. Wick | Bilinear forms on the Dirichlet Space | Anal. and PDE 3 (2010), no. 1, 21–47. |
| I. Benjamini, Y. Peres | Random Walks on a Tree and Capacity in the interval | Ann. Inst. H. Poincaré Probab. Statist. 28 (1992), no. 4, 557–592. |
| A. Beurling | Ensembles exceptionnels | Acta Math. 72, (1940). 1-13. |
| A. Bonfiglioli, E. Lanconelli, F. Uguzzoni | Stratified Lie groups and Potential Theory for their sub-Laplacians | Springer Monographs in Mathematics. Springer, Berlin, 2007. xxvi+800 pp.. |
| M. Christ | A T(b) theorem with Remarks on Analytic Capacity and the Cauchy Integral | Colloq. Math. 60/61 (1990), no. 2, 601–628. |
| C. Cascante, J. M. Ortega, I. E. Verbitsky | Nonlinear Potentials and Two Weight Trace Inequalities for General Dyadic and Radial Kernels | Indiana Univ. Math. J. 53 (2004), no. 3, 845–882. |
| F. Di Biase | Fatou type theorems, Maximal functions and approach regions | Progress in Mathematics, 147. |
| J. L. Doob | Classical Potential Theory and its Probabilistic Counterpart | Grundlehren der Mathematischen Wissenschaften 262, Springer-Verlag, New York, 1984. xxiv+846 pp. Birkhäuser Boston, Inc., Boston, MA, 1998. xii+152 pp.. |
| M. Gromov | Metric structures for Riemannian and non-Riemannian spaces | Based on the 1981 French original, with appendices by M. Katz, P. Pansu and S. Semmes, translated from the French by Sean Michael Bates, Progress in Mathematics 152, Birkhauser Boston, Inc., Boston, MA, 1999, xx+585 pp.. |
| P. Hajlasz, P. Koskela | Sobolev met Poincare | Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.. |
| L. I. Hedberg, T. H. Wolff | Thin Sets in Nonlinear Potential Theory | Ann. Inst. Fourier (Grenoble) 33 (1983), no. 4, 161–187. |
| J. Heinonen | Lectures on Analysis on Metric Spaces | Universitext, Springer-Verlag, New York, 2001. x+140 pp.. |
| J. Heinonen, P. Koskela | Quasiconformal Maps in Metric Spaces with Controlled Geometry | Acta Math. 181 (1998), no. 1, 1–61. |
| I. Holopainen, P. Soardi | p-Harmonic Functions on Graphs and Manifolds | Manuscripta Math. 94 (1997), no. 1, 95–110. |
| N. J. Kalton, I. E. Verbitsky | Nonlinear Equations and Weighted Norm Inequalities | Trans. Amer. Math. Soc. 351 (1999), no. 9, 3441–3497. |
| R. Kerman, E. Sawyer | The trace inequality and eigenvalue estimates for Schrödinger operators | Ann. Inst. Fourier (Grenoble) 36 (1986), no. 4, 207-228. |
| R. Kerman, E. Sawyer | Carleson Measures and Multipliers of Dirichlet-type Spaces | Trans. Amer. Math. Soc. 309 (1988), no. 1, 87–98. |
| R. Lyons, Y. Peres | Probability on Trees and Networks | preprint available at http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html |
| V. G. Maz’ya | Sobolev Spaces | translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. xix+486 pp.. |
| V. G. Maz’ya | Imbedding Theorems and their Applications | Baku Sympos. (1966) “Nakua”, Moscow, 1970, 142-159 (Russian). |
| B. Muckenhoupt, R. Wheeden | Weighted Norm Inequalities for Fractional Integrals | Trans. Amer. Math. Soc. 192 (1974), 261–274. |
| M. Paolucci, P. M. Soardi | Potential Theory on Infinite Networks | Lecture Notes in Mathematics 1590, Springer-Verlag, Berlin, 1994. viii+187 pp.. |
| D. A. Stegenga | Multipliers of the Dirichlet Space | Illinois J. Math. 24 (1980), no. 1, 113–139. |
| M. Tsuji | Potential Theory in Modern Function Theory | Maruzen Co., Ltd., Tokyo 1959, 590 pp.. |
[33] I. E. Verbitsky, R. Wheeden, *Weighted Norm Inequalities for Integral Operators*, Trans. Amer. Math. Soc. 350 (1998), no. 8, 3371–3391.

N. Arcozzi, Dipartimento di Matematica, Università di Bologna, 40127 Bologna, ITALY
E-mail address: arcozzi@dm.unibo.it

R. Rochberg, Department of Mathematics, Washington University, St. Louis, MO 63130, U.S.A
E-mail address: rr@math.wustl.edu

E. T. Sawyer, Department of Mathematics & Statistics, McMaster University; Hamilton, Ontario, L8S 4K1, CANADA
E-mail address: sawyer@mcmaster.ca

B. D. Wick, School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA USA 30332–0160
E-mail address: wick@math.gatech.edu