Risk-aware Multi-armed Bandits Using Conditional Value-at-Risk

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Abstract

Traditional multi-armed bandit problems are geared towards finding the arm with the highest expected value – an objective that is risk-neutral. In several practical applications, e.g., finance, a risk-sensitive objective is to control the worst-case losses and Conditional Value-at-Risk (CVaR) is a popular risk measure for modelling the aforementioned objective. We consider the CVaR optimization problem in a best-arm identification framework under a fixed budget. First, we derive a novel two-sided concentration bound for a well-known CVaR estimator using empirical distribution function, assuming that the underlying distribution is unbounded, but either sub-Gaussian or light-tailed. This bound may be of independent interest. Second, we adapt the well-known successive rejects algorithm to incorporate a CVaR-based criterion and derive an upper-bound on the probability of incorrect identification of our proposed algorithm.

1 Introduction

In the stochastic multi-armed bandits literature, the expected value of an arm’s distribution is considered as a good metric for measuring the quality of that arm. However, there are applications such as portfolio optimization in finance, wherein the quality of a portfolio is measured by risk sensitive metrics such as
Value-at-Risk (VaR) and Conditional-Value-at-Risk (CVaR), rather than the standard expected value. Consider a random variable which models the losses incurred by a portfolio. VaR at level $\alpha \in (0, 1)$ conveys the maximum loss incurred by the portfolio with a confidence of $\alpha$. In other words, the portfolio incurs a loss greater than VaR at level $\alpha$ with probability $1 - \alpha$. In turn, CVaR at level $\alpha \in (0, 1)$ captures the expected loss incurred by the portfolio, given that the losses exceed VaR at level $\alpha$.

In this paper, we consider the stochastic bandit set-up with a risk-sensitive metric for measuring the quality of an arm. In particular, we treat the CVaR as the metric for the quality of an arm, and study the best arm identification problem with fixed budget. The reason for considering the CVaR as the metric is due to its advantage of being a coherent risk measure [Artzner et al., 1999]. A risk measure is said to be coherent, if it is monotonic, translation invariant, sub-additive, and positive homogeneous. Note that VaR is not a coherent risk measure, and hence, unlike CVaR, does not lend itself to stochastic programming techniques well.

From a practical standpoint, the best arm identification approach is quite appropriate in the context of CVaR optimization. Consider an application such as portfolio optimization in the domain of finance. The goal here would be to find an investment strategy such that the losses incurred in the worst-case scenario (e.g., stock market crash) are minimized. Such a problem can be modeled well with a CVaR objective. However, estimating CVaR in a real-life market scenario is challenging, while one could model the various processes in the financial application considered, and build a simulator. The problem of finding the CVaR-optimal investment strategy can be addressed by using the simulator – an approach that falls under the realm of simulation optimization [Fu, 2015].

We consider a $K$-armed stochastic bandit setting, and study the problem of finding the arm with the lowest CVaR value (at a fixed level $\alpha \in (0, 1)$) in a fixed budget setting. Our specific contributions under this paradigm are as follows: First, we derive novel concentration bounds for a well-known CVaR estimation scheme, under two different assumptions on the underlying distributions. Specifically, we assume that the arm distributions are either sub-Gaussian [Wainwright], or light-tailed [Nair, 2012, Section 2.2]. To the best of our knowledge, there does not exist any two-sided CVaR concentration bound for the case of unbounded r.v.s (although a one-sided result has been proposed, using a different proof technique, in [Kolla et al., 2018]). Second, we propose an algorithm for the best CVaR arm identification that is inspired by successive-rejects [Audibert et al., 2010]. We establish an upper bound on the probability of incorrect identification by our algorithm at the end of the given budget. Note that the CVaR concentration bound we derive forms a crucial ingredient for the bandit algorithm as well as its analysis.

While bandit learning has a long history, dating back to Thompson [1933], risk-based criteria have been considered only recently. Sani et al. [2012] consider mean-variance optimization in a regret minimization framework. In the best arm identification setting, VaR-based criteria has been studied by David et al. [2018] and David and Shimkin [2016]. Unlike VaR, CVaR is a coherent risk measure [Artzner et al., 1999], and hence, preferred over VaR in financial applications. Moreover, in comparison to CVaR, obtaining a concentration result for VaR is relatively easier, and does not require assumptions on the tail of the distribution – see Propositions 1 and 2 in [Kolla et al., 2018]. CVaR-based criteria has been explored in a bandit context by Galichet et al. [2013], albeit with an assumption of bounded arms’ distributions. Under the latter assumption, a popular CVaR estimate has been shown to exponentially concentrate around the true CVaR – see [Brown, 2007, Wang and Gao, 2010], while there does not exist CVaR concentration bounds for the case when the underlying distribution is unbounded. Our CVaR concentration bound addresses this gap, for the case of sub-Gaussian and light-tailed distributions.

The rest of this paper is organized as follows: Section 2 presents the preliminaries. Section 3 introduces the estimators for VaR, CVaR and presents the concentration bounds for sub-Gaussian and light-tailed dis-
tributions. Section 4 provides bandit algorithms and their analyses for the problem of the best CVaR arm identification with fixed budget under $K$-armed stochastic bandits. Section 6 presents the results from simulation experiments on illustrative bandit problem settings. Section 7 concludes the paper.

2 Preliminaries

Given a r.v. $X$ with cumulative distribution function (CDF) $F(\cdot)$, the VaR $v_\alpha(X)$ and CVaR $c_\alpha(X)$ at level $\alpha \in (0, 1)$ are defined as follows:

\[
v_\alpha(X) = \inf \{ \xi : P[X \leq \xi] \geq \alpha \}, \tag{1}
\]

\[
c_\alpha(X) = v_\alpha(X) + \frac{1}{1 - \alpha} E[X - v_\alpha(X)]^+, \tag{2}
\]

where we have used the notation $[X]^+ = \max(0, X)$. Typical values of $\alpha$ chosen in practice are 0.95 and 0.99. We make the following assumption for the purpose of CVaR estimation as well as for the concentration bounds derived later.

(C1) The r.v. $X$ is continuous with strictly increasing CDF.

Under (C1), $v_\alpha(X)$ is a solution to $P[X \leq \xi] = \alpha$, i.e., $v_\alpha(X) = F^{-1}(\alpha)$. Further, if $X$ has a positive density at $v_\alpha(X)$, then $c_\alpha(X) = E[X|X \geq v_\alpha(X)]$ [cf. Sun and Hong, 2010].

3 CVaR estimation and concentration

In this section, we define empirical CVaR, and subsequently present a concentration result for CVaR.

3.1 VaR and CVaR estimation

Let $\{X_i\}_{i=1}^n$ be $n$ i.i.d. samples drawn from the distribution of $X$. Let $\{X_{[i]}\}_{i=1}^n$ be the order statistics of $\{X_i\}_{i=1}^n$, i.e., $X_{[1]} \geq X_{[2]} \cdots \geq X_{[n]}$. Let $\hat{F}_n(\cdot)$ be the empirical distribution function calculated using $\{X_i\}_{i=1}^n$, defined as

\[
\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n I\{X_i \leq x\}, \quad \forall x \in \mathbb{R}.
\]

Notice that CVaR is a conditional expectation, where the conditioning event requires VaR. Thus, CVaR estimation requires VaR to be estimated as well. Let $\hat{v}_{n,\alpha}$ and $\hat{c}_{n,\alpha}$ denote the estimates of VaR and CVaR at level $\alpha$ using the $n$ samples above. These quantities are defined as follows:

\[
\hat{v}_{n,\alpha} = X_{\lfloor n(1-\alpha) \rfloor}, \tag{3}
\]

\[
\hat{c}_{n,\alpha} = \frac{1}{n(1-\alpha)} \sum_{i=1}^n X_i I\{X_i \geq \hat{v}_{n,\alpha}\}. \tag{4}
\]

\footnote{For notational brevity, we omit $X$ from the notations $v_\alpha(X)$ and $c_\alpha(X)$ whenever the underlying r.v. can be understood from the context.}


3.2 Concentration bounds

In the case of distributions with bounded support, a concentration result for the above CVaR estimator exists in the literature [Gao et al., 2010]. However, there are no two-sided non-asymptotic CVaR concentration results for the case of distributions with unbounded support. For the case of unbounded distributions, deriving a CVaR concentration result becomes considerably easier when the form of distributions are known, i.e., when the closed-form expressions of VaR and CVaR can be derived. To illustrate, consider the case of a Gaussian r.v. \( X \) with mean \( \mu \) and variance \( \sigma^2 \). Let \( Q(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\xi}^{\infty} \exp(-x^2/2) \, dx \). Notice that \( Q(-x) = 1 - Q(x) \) and also that \( F_X(\xi) = Q\left(\frac{\xi - \mu}{\sigma}\right) \). Hence, \( v_\alpha(X) \) is the solution to \( Q\left(\frac{\mu - \xi}{\sigma}\right) = \alpha \), which implies that

\[
v_\alpha(X) = \mu - \sigma Q^{-1}(\alpha).
\]

The CVaR \( c_\alpha(X) \) for Gaussian \( X \) can be shown, using Acerbi’s formula [Chatterjee, 2014, pp. 329], to be equal to \( \mu \left( \frac{\alpha}{1-\alpha} \right) + \sigma c_\alpha(Z) \), where \( Z \) is the standard Gaussian random variable i.e., \( Z \sim \mathcal{N}(0,1) \) (See Appendix A for a proof).

It is clear from the above argument that estimates of \( \mu \) and \( \sigma \) are sufficient to estimate \( c_\alpha(X) \) for the Gaussian case. Sample mean \( \hat{\mu}_n \) and sample variance \( \hat{\sigma}_n^2 \) (computed using \( n \) samples from the distribution of \( X \)) would serve this purpose and we obtain \( \hat{c}_n = \hat{\mu} \left( \frac{\alpha}{1-\alpha} \right) + \hat{\sigma} c_\alpha(Z) \) as a proxy for \( c_\alpha(X) \). Given standard concentration bounds for these quantities through Hoeffding and Bernstein’s inequalities, it is straightforward to establish that \( \hat{c}_{n,\alpha} \) concentrates exponentially around \( c_\alpha(X) \).

We therefore focus on distributions that do not have closed-form expressions for VaR and CVaR. In such a setting, the CVaR has to be estimated directly from the available samples. However, for establishing concentration bounds for the CVaR, which involves conditioning on a tail event, it is necessary to assume that the distribution is not heavy-tailed. In fact, even for the case of estimating the expected value of a r.v., exponential concentration bounds are available under an assumption that restricts the tail to be light (cf. Chapter 2 of Boucheron et al. [2013]). In this paper, we consider two classes of distributions: (i) Sub-Gaussian distributions, i.e., the class of distributions whose tail decays at least as fast as tail of a Gaussian r.v., and (ii) Light-tailed distributions, i.e., the class of distributions whose tail decays exponentially fast. Note that the sub-Gaussian distributions are light-tailed — however, we obtain sharper concentration bounds for the sub-Gaussian case.

We now define these classes of distributions more formally.

**Definition 1.** Let \( \sigma > 0 \). A centered r.v. \( X \) is said to be \( \sigma \)-sub-Gaussian if \( \mathbb{E}[\exp(\lambda X)] \leq \exp\left(\frac{\lambda^2 \sigma^2}{2}\right) \) for all \( \lambda \in \mathbb{R} \). Equivalently,

\[
\mathbb{P}[|X| \geq t] \leq 8 \exp\left(-\frac{t^2}{4\sigma^2}\right), \forall t \geq 0.
\]

**Definition 2.** A r.v. \( X \) is said to be light-tailed if there exists a \( c_0 > 0 \) such that \( \mathbb{E}[\exp(\lambda X)] < \infty \) for all \( |\lambda| < c_0 \). Equivalently, there exists constants \( \eta_1, \eta_2 > 0 \) such that

\[
\mathbb{P}[|X| \geq t] \leq \eta_1 \exp(-\eta_2 t), \quad \forall t > 0.
\]

The main result below provides a novel two-sided CVaR concentration bound for the case of sub-Gaussian as well as light-tailed r.v.s.
Theorem 1 (Empirical CVaR concentration). Let \( \{X_i\}_{i=1}^n \) be a sequence of i.i.d. r.v.s. Assume (C1). Let \( \hat{c}_{n,\alpha} \) be the CVaR estimate given in (4) formed using the above set of samples. Fix \( \epsilon > 0 \).

(i) Suppose that \( X_i, i = 1, \ldots, n \) are 1-sub-Gaussian. Let \( g(\tau) = \frac{8(2+\tau)}{(1-\alpha)} \exp \left( -\tau^2/2 \right) \). Let \( n_0 \) be the smallest integer such that \( g(n^{1/4}\sqrt{\epsilon}) < \epsilon/2 \). Then, \( \forall n > n_0 \), we have

\[
P \left[ |\hat{c}_{n,\alpha} - c_{\alpha}| > \epsilon \right] \leq G \exp \left[ -\frac{\sqrt{n} (1-\alpha)^2 \min \{\epsilon, \epsilon^2 \} \min \{1, \eta_f, 1/v_{\alpha}^2 \} }{32} \right],
\]

where \( \eta_f \) depends on the value of the density \( f(\cdot) \) in a neighborhood of \( v_{\alpha} \) and \( G = 4 \max \{G_1, G_2, 16\} \), \( G_1 = \exp \left[ \frac{\epsilon (1-\alpha)}{2v_{\alpha}} \right] \), \( G_2 = \exp \left[ 2\sqrt{\epsilon} (1-\alpha) \right] \).

(ii) Suppose that \( X_i, i = 1, \ldots, n \) are light-tailed with constants \( \eta_1 > 0 \) and \( \eta_2 = 1 \). Let \( \tilde{n}_0 \) be the smallest integer such that \( g(n^{1/3}\epsilon^{2/3}) < \epsilon/2 \), with \( g(\tau) = \frac{\eta_1 (1+\tau)}{1-\alpha} \exp (-\tau) \). Then, \( \forall n > \tilde{n}_0 \), we have

\[
P \left[ |\hat{c}_{n,\alpha} - c_{\alpha}| > \epsilon \right] \leq \tilde{G} \exp \left[ -\frac{n^{1/3} (1-\alpha)^2 \min \{\epsilon^{2/3}, \epsilon^2 \} \min \{1, \eta_f, 1/v_{\alpha}^2 \} }{32} \right],
\]

where \( \tilde{G} = 5 \max \{G_1, G_2, 2, 4 \max \{1, \eta_1 \} \} \), \( G_1 \) and \( G_2 \) are as defined in part (i).

Proof. Refer Section 5.1.

Remark 1. In comparison to the one-sided bound, for the sub-Gaussian case, obtained by [Kolla et al., 2018], the bound in the theorem above has the same dependence on the accuracy \( \epsilon \), while exhibiting a weaker dependence on \( n \). The bound in the paper mentioned above featured \( n \) inside the exponential term, while we have \( \sqrt{n} \), and we believe this is an artifact of the proof technique. On the positive side, since our bound is two-sided, it opens avenues for a bandit application, while a one-sided bound is insufficient for this purpose.

Remark 2. In comparison to the bound for sub-Gaussian r.v.s, the light-tailed bound above has a weaker dependence on the number of samples \( n \) as well as accuracy \( \epsilon \), and this is not surprising, considering the fact that sub-Gaussian tails decay faster than light-tailed ones.

4 Best CVaR arm identification

4.1 Learning model and CVaR-SR algorithm

We consider a \( K \)-armed stochastic bandit problem, with arms’ distributions \( \mathcal{P}_1, \ldots, \mathcal{P}_K \). In this setting, a bandit algorithm interacts with the environment over a given budget of \( n \) rounds. In each round \( t = 1, \ldots, n \), the algorithm pulls an arm \( I_t \in \{1, \ldots, K\} \) and observes a sample cost from the distribution \( \mathcal{P}_{I_t} \). At the end of the budget \( n \) rounds, the bandit algorithm recommends an arm \( J_n \) and is judged based on the probability of incorrect identification, i.e., \( \mathbb{P} [ J_n \neq i^* ] \), where \( i^* \) denotes the best arm. Earlier works use the expected value to define the best arm, while we use CVaR.

Let \( c_{\alpha}^i \) and \( v_{\alpha}^i \) denote the CVaR and VaR of the arm \( i \) at level \( \alpha \). Let \( c^* = \min_{i=1,\ldots,K} c_{\alpha}^i \), and \( i^* \) be the arm that achieves this minimum. The goal is to devise an algorithm for which \( \mathbb{P} [ J_n \neq i^* ] \) is small after \( n \) rounds of sampling. Let arm-[\( i \)] denotes the \( i \)-th lowest CVaR valued arm. Let \( \Delta_i = c_{\alpha}^i - c_{\alpha}^{i^*} \) denote the gap between the CVaR values of arm-\( i \) and the optimal arm. Let \( \Delta_L \) and \( \Delta_U \) be lower and upper bounds on the gaps \( \Delta_i \)’s i.e., \( \Delta_L \leq \Delta_i \leq \Delta_U \) for all \( 1 \leq i \leq K \).
Algorithm 1 CVaR-SR algorithm

**Input**: Budget $n'$, $\Delta_L$ and $\Delta_U$.

**Initialization**: Play each arm $m_0$ times.

Set $A_1 = \{1, \ldots, K\}$, $\log K = \frac{1}{2} + \sum_{i=2}^{K} \frac{1}{i}, n_0 = 0, n_k = \left\lceil \frac{1}{\log K} \frac{n-K}{K+1-k} \right\rceil$ for $k \in \{1, \ldots, K - 1\}$.

for $k = 1, 2, \ldots, K - 1$ do

- Play each arm in $A_k$ for $(n_k - n_{k-1})$ times.
- Compute the CVaR estimate $\hat{\gamma}^i_{\alpha,n_k}$ for each arm $i \in A_k$ using (4).
- Set $A_{k+1} = A_k \setminus \arg\max_{i \in A_k} \hat{\gamma}^i_{\alpha,n_k}$, i.e., remove the arm with the highest empirical CVaR, with ties broken arbitrarily.

end for

**Output**: Return the arm $A_K$ that is the last remaining arm after $K - 1$ phases.

Algorithm 1 presents the pseudo code of our CVaR-SR algorithm, designed to find the CVaR-optimal arm under a fixed budget. The algorithm is a variation of the regular successive rejects (SR) algorithm [Audibert et al. 2010], with the following differences: (i) Unlike regular SR, each arm is pulled $m_0$ times, before the elimination phases begin, and this is to ensure that the concentration bounds, derived in Theorem 1 are applicable; and (ii) Regular SR uses sample mean to estimate the expected value of each arm, while CVaR-SR used empirical CVaR, as defined in (4), to estimate CVaR for each arm. The elimination logic, i.e., having $K - 1$ phases, and removing the worst arm (according to sample estimates of CVaR) at the end of each phase, is borrowed from regular SR.

4.2 Error bounds

In the following result, we analyze the performance of CVaR-SR algorithm for the case of sub-Gaussian as well as light-tailed arms’ distributions.

**Theorem 2 (Probability of incorrect identification).** Consider a $K$-armed stochastic bandit, where the arms’ distributions satisfy (C1).

(i) Let $m_0$ be the smallest integer such that the following happens:

$$2 + m_0^{1/4} \left[\sqrt{0.5\Delta_L + 1}\right] \exp\left(-\frac{\sqrt{m_0} \left[\sqrt{0.5\Delta_U + 1}\right]^2}{4}\right) < \frac{\Delta_L}{4},$$

where $\Delta_L$ (resp. $\Delta_U$) is a lower (resp. upper) bound on the gaps. Suppose that the arms’ distributions are 1-sub-Gaussian. For a given budget $n' = Km_0 + n$, the arm $J_{n'}$ returned by the CVaR-SR algorithm satisfies:

$$\mathbb{P}[J_{n'} \neq i^*] \leq \frac{K(K - 1)G_{\max}}{2} \times \exp\left(-\sqrt{\frac{(n-K)}{\log KH_1(1-\alpha)^2}} \beta_{\min}\right),$$

where $G_{\max}$ is the maximum gap among all arms.
where $H_1 = \max_{i \in \{1, 2, \ldots, K\}} \frac{i}{\min(\alpha, \Delta^2)}$, $G_{\text{max}}$ and $\beta_{\text{min}}$ are constants independent of $n$. 

(ii) Let $m_0$ be the smallest integer such that the following happens:

\[
1 + m_0^{1/3} \left[ \frac{(0.5\Delta_L)^{2/3} + 1}{1 - \alpha} \right] \exp \left( -m_0^{1/3} \left[ \left( \frac{\Delta_U}{2} \right)^{2/3} + 1 \right] \right) < \frac{\Delta_L}{4}.
\]

Suppose that the arms’ distributions are light-tailed with constants $\eta_1 > 0$ and $\eta_2 = 1$. For a given budget $n' = Km_0 + n$, the arm $J_{n'}$ returned by the CVaR-SR algorithm satisfies:

\[
\mathbb{P} [J_{n'} \neq i^*] \leq \frac{K(K - 1)\tilde{G}_{\text{max}}}{2} \times \exp \left( -\left( \frac{n - K}{\log KH^2} \right)^{1/3} (1 - \alpha)^2 \beta_{\text{min}} \right),
\]

where $H_2 = \max_{i \in \{1, 2, \ldots, K\}} \frac{i}{\min(\alpha, \Delta^2)}$, $\beta_{\text{min}}$ and $\tilde{G}_{\text{max}}$ are constants independent of $n$.

Proof. Refer Section 5.2.

From the results above, it is apparent that to get a good bound on the probability of incorrect identification, the parameter $m_0$ that governs the number of initial pulls of each arm, has to be set using $\Delta_L$ and $\Delta_U$. This is a drawback, and it would be interesting to get rid of the initialization phase, and commence elimination phases, as in regular SR algorithm. On a related note, several bandit algorithms either assume bounded rewards, with known bounds (e.g., UCB in Auer et al. [2002], MARAB algorithm for CVaR optimization Galichet [2015]), or consider unbounded, albeit sub-Gaussian arms’ distributions, with knowledge of the parameter $\sigma$.

5 Proofs

5.1 Proof of Theorem 1

Empirical CVaR, as defined in (4), involves empirical VaR, and it is natural to expect that empirical CVaR concentration would require empirical VaR to concentrate as well. VaR concentration bounds have been derived recently in Kolla et al. [2018], and we recall their result below.

Lemma 3 (VaR concentration). Suppose that (C1) holds. For any $\epsilon > 0$, we have

\[
\mathbb{P} [|\hat{v}_{n,\alpha} - v_{\alpha}| \geq \epsilon] \leq 2 \exp \left( -2n\epsilon c^2 \right),
\]

where $c$ is a constant that depends on the value of the density $f$ of the r.v. $X$ in a neighbourhood of $v_{\alpha}(X)$.

Proof. See Kolla et al. [2018].

Proof. (Theorem 1)

(i): sub-Gaussian case

Let $n_\epsilon$ be a positive scalar, that we specify later.

\[
\mathbb{P} [|\hat{c}_{n,\alpha} - c_\alpha| > \epsilon] = \mathbb{P} \left[ \frac{1}{n(1 - \alpha)} \sum_{i=1}^{n} X_i \left\{ X_i \geq \hat{v}_{n,\alpha} \right\} - c_\alpha > \epsilon \right] \leq I_1 + I_2,
\]
We now bound the

\[ I_1 = P \left[ \left| \sum_{i=1}^{n} X_i \mathbb{I} \{ X_i \geq \hat{v}_{n,\alpha} \} - c_n \right| > \epsilon, \max \{ X_1, \ldots, X_n \} \leq n_\epsilon \right] \]

\[ I_2 = P \left[ \max \{ X_1, \ldots, X_n \} > n_\epsilon \right]. \]

We now proceed to derive an upper bound on \( I_1 \) here.

\[ I_1 = P \left[ \left| \sum_{i=1}^{n} X_i \mathbb{I} \{ X_i \geq \hat{v}_{n,\alpha} \} - \int_{v_\alpha}^{\infty} \frac{xf_X(x)}{1-\alpha} \, dx \right| > \epsilon, \max \{ X_1, \ldots, X_n \} \leq n_\epsilon \right]. \]

Note that,

\[ \frac{1}{1-\alpha} \int_{n_\epsilon}^{\infty} x f_X(x) \, dx = \frac{1}{1-\alpha} \int_{x=n_\epsilon}^{\infty} \int_{y=0}^{x} \, dy f_X(x) \, dx \]

\[ = \frac{1}{1-\alpha} \left[ \int_{y=0}^{n_\epsilon} \int_{x=n_\epsilon}^{\infty} f_X(x) \, dx \, dy + \int_{y=n_\epsilon}^{\infty} \int_{x=y}^{\infty} f_X(x) \, dx \, dy \right] \]

\[ = \frac{1}{1-\alpha} \left[ \int_{y=0}^{n_\epsilon} \mathbb{P} \{ X \geq n_\epsilon \} \, dy + \int_{y=n_\epsilon}^{\infty} \mathbb{P} \{ X \geq y \} \, dy \right] \]

\[ = \frac{1}{1-\alpha} \left[ n_\epsilon \mathbb{P} \{ X \geq n_\epsilon \} + \int_{y=n_\epsilon}^{\infty} \mathbb{P} \{ X \geq y \} \, dy \right] \]

We now bound the \( \int_{y=n_\epsilon}^{\infty} \mathbb{P} \{ X \geq y \} \, dy \) as follows.

\[ \int_{y=n_\epsilon}^{\infty} \mathbb{P} \{ X \geq y \} \, dy \overset{(a)}{\leq} \int_{y=n_\epsilon}^{\infty} 8 \exp \left( -y^2/4 \right) \, dy \]

\[ \leq \int_{y=n_\epsilon}^{\infty} 8y \exp \left( -y^2/4 \right) \, dy \]

\[ \leq \int_{y=n_\epsilon^2/4}^{\infty} 16 \exp \left( -t \right) \, dt \]

\[ = 16 \exp \left( -n_\epsilon^2/4 \right), \]

where (a) follows by sub-Gaussianity. Hence, we obtain

\[ \frac{1}{1-\alpha} \int_{n_\epsilon}^{\infty} x f_X(x) \, dx \leq \frac{8 (2 + n_\epsilon)}{1-\alpha} \exp \left( -n_\epsilon^2/4 \right). \]

Assuming that \( n_\epsilon \) increases with \( n \), we have that \( \left\{ \frac{8 (2 + n_\epsilon)}{1-\alpha} \exp \left( -n_\epsilon^2/4 \right) \right\}_{n=1}^{\infty} \) is a non-increasing sequence and goes to zero as \( n \to \infty \). Hence, there exists an \( n_0 \in \mathbb{N} \) such that, all the terms beyond \( n_0 \) in the aforementioned sequence are at most \( \epsilon/2 \). Hence we obtain, for all \( n > n_0 \),

\[ I_1 \leq P \left[ \left| \sum_{i=1}^{n} X_i \mathbb{I} \{ X_i \geq \hat{v}_{n,\alpha} \} - \int_{v_\alpha}^{n_\epsilon} \frac{xf_X(x)}{1-\alpha} \, dx \right| > \frac{\epsilon}{2}, \max \{ X_1, \ldots, X_n \} \leq n_\epsilon \right] \]

\[ = P \left[ \left| \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \left[ \mathbb{I} \{ X_i \geq \hat{v}_{n,\alpha} \} - \mathbb{I} \{ X_i \geq v_\alpha \} + \mathbb{I} \{ X_i \geq v_\alpha \} \right] - \int_{v_\alpha}^{n_\epsilon} \frac{xf_X(x)}{1-\alpha} \, dx \right| > \epsilon/2, \right] \]
\[
\begin{align*}
\max \{X_1, \ldots, X_n\} & \leq n_{\varepsilon} \\
= P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \mathbb{I} \{X_i \geq v_\alpha\} \mathbb{I} \{X_i \leq n_{\varepsilon}\} - \int_{v_\alpha}^{n_{\varepsilon}} \frac{xf_X(x)}{1-\alpha} dx \right. \\
& \quad \left. + \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \left[ \mathbb{I} \{X_i \geq \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i \geq v_\alpha\} \right] > \varepsilon/2, \max \{X_1, \ldots, X_n\} \leq n_{\varepsilon} \right]
\end{align*}
\]
\[
\leq P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \mathbb{I} \{X_i \geq v_\alpha\} \mathbb{I} \{X_i \leq n_{\varepsilon}\} - \int_{v_\alpha}^{n_{\varepsilon}} \frac{xf_X(x)}{1-\alpha} dx \right. \\
& \quad \left. + \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \left[ \mathbb{I} \{X_i \geq \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i \geq v_\alpha\} \right] > \varepsilon/2 \right]
\leq I_{11} + I_{12},
\]

where
\[
I_{11} = P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \mathbb{I} \{X_i \geq v_\alpha\} \mathbb{I} \{X_i \leq n_{\varepsilon}\} - \frac{1}{1-\alpha} \int_{v_\alpha}^{n_{\varepsilon}} xf_X(x) dx \right] > \varepsilon/4 \right]
\]
\[
I_{12} = P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} X_i \left[ \mathbb{I} \{X_i \geq \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i \geq v_\alpha\} \right] > \varepsilon/4 \right].
\]

Note that, in the \( I_{11} \) above, the r.v.s \( Z_i = \frac{1}{1-\alpha} X_i \mathbb{I} \{X_i \geq v_\alpha\} \mathbb{I} \{X_i \leq n_{\varepsilon}\} \) are i.i.d. and bounded, and its expectation equals to \( \frac{1}{1-\alpha} \int_{v_\alpha}^{n_{\varepsilon}} xf_X(x) dx \). Hence, we can apply Hoeffding’s inequality to the \( I_{11} \) above. Hence, we obtain that
\[
I_{11} \leq 2 \exp \left( -\frac{n(1-\alpha)^2 \varepsilon^2}{8n_{\varepsilon}^2} \right).
\]

We now bound \( I_{12} \) as follows:
\[
I_{12} = P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} (X_i - v_\alpha + v_\alpha) \left[ \mathbb{I} \{X_i \geq \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i \geq v_\alpha\} \right] > \varepsilon/4 \right]
\leq I_{121} + I_{122},
\]

where
\[
I_{121} = P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} (X_i - v_\alpha) \left[ \mathbb{I} \{X_i \geq \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i \geq v_\alpha\} \right] > \varepsilon/4 \right],
\]
\[
I_{122} = P \left[ \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} v_\alpha \left[ \mathbb{I} \{X_i \geq \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i \geq v_\alpha\} \right] > \varepsilon/4 \right].
\]

The term \( I_{122} \) can be bounded as follows:
\[
I_{122} = P \left[ \frac{v_\alpha}{1-\alpha} \left( \hat{F}_n(v_\alpha) - \hat{F}_n(\hat{v}_{n,\alpha}) \right) + \frac{1}{n} \sum_{i=1}^{n} \left[ \mathbb{I} \{X_i = \hat{v}_{n,\alpha}\} - \mathbb{I} \{X_i = v_\alpha\} \right] > \varepsilon \right]
\]
\[
\overset{(a)}{=} P \left[ \left| \hat{F}_n(\hat{v}_{n,\alpha}) - \hat{F}_n(v_\alpha) \right| > \frac{\varepsilon(1-\alpha)}{8|v_\alpha|} \right] \quad \text{w.p. 1}
\]
where \((a)\) is due to the fact that \(\frac{1}{n}\sum_{i=1}^{n} I\{X_i = \hat{v}_{n,\alpha}\}\) and \(\frac{1}{n}\sum_{i=1}^{n} I\{X_i = v_{\alpha}\}\) take value zero w.p. 1, since \(X_i\) is continuous for each \(i\). Note that \(|\hat{F}_n(\hat{v}_{n,\alpha}) - F(v_{\alpha})| \leq 1/n\). Hence, we obtain

\[
I_{122} \leq \mathbb{P}\left[ |F(v_{\alpha}) - \hat{F}_n(v_{\alpha})| > \frac{\epsilon(1 - \alpha)}{8|v_{\alpha}|} - \frac{1}{n} \right] \\
\leq \exp\left( -2n \left( \frac{\epsilon(1 - \alpha)}{8v_{\alpha}} - \frac{1}{n} \right)^2 \right),
\]

where the last inequality is due to an application of the DKW inequality. A straightforward calculation yields

\[
I_{122} \leq G_1 \exp\left( -\frac{n\epsilon^2(1 - \alpha)^2}{32v_{\alpha}^2} \right),
\]

where \(G_1 = \exp\left( \frac{\epsilon(1 - \alpha)}{2v_{\alpha}} \right)\).

We now turn to bound \(I_{121}\). Notice that

\[
\left| \sum_{i=1}^{n} \frac{(X_i - v_{\alpha})}{n(1 - \alpha)} \right| = \left| \left\{ X_i \geq \hat{v}_{n,\alpha}\right\} - \left\{ X_i \geq v_{\alpha}\right\} \right| \leq \frac{1}{1 - \alpha} |v_{\alpha} - \hat{v}_{n,\alpha}| \hat{F}_n(v_{\alpha}) - \hat{F}_n(\hat{v}_{n,\alpha}) |.
\]

Using the inequality above, we have

\[
I_{121} \leq \mathbb{P}\left[ \frac{|v_{\alpha} - \hat{v}_{n,\alpha}|}{1 - \alpha} |\hat{F}_n(v_{\alpha}) - \hat{F}_n(\hat{v}_{n,\alpha})| > \epsilon/8 \right] \\
= \mathbb{P}\left[ \frac{|v_{\alpha} - \hat{v}_{n,\alpha}|}{1 - \alpha} |\hat{F}_n(v_{\alpha}) - \hat{F}_n(\hat{v}_{n,\alpha})| > \frac{\epsilon}{8} \right] \left| v_{\alpha} - \hat{v}_{n,\alpha} \right| \leq \frac{\sqrt{\epsilon}}{8} \\
+ \mathbb{P}\left[ \frac{|v_{\alpha} - \hat{v}_{n,\alpha}|}{1 - \alpha} |\hat{F}_n(v_{\alpha}) - \hat{F}_n(\hat{v}_{n,\alpha})| > \frac{\epsilon}{8} \right] \left| v_{\alpha} - \hat{v}_{n,\alpha} \right| > \frac{\sqrt{\epsilon}}{8} \\
\leq \mathbb{P}\left[ \hat{F}_n(v_{\alpha}) - \hat{F}_n(\hat{v}_{n,\alpha}) | > \sqrt{\epsilon}(1 - \alpha) \right] + \mathbb{P}\left[ |v_{\alpha} - \hat{v}_{n,\alpha}| > \frac{\sqrt{\epsilon}}{8} \right] \\
\leq \mathbb{P}\left[ \hat{F}_n(v_{\alpha}) - F(v_{\alpha}) | + |\hat{F}_n(\hat{v}_{n,\alpha}) - F(v_{\alpha})| > \sqrt{\epsilon}(1 - \alpha) \right] + \mathbb{P}\left[ |v_{\alpha} - \hat{v}_{n,\alpha}| > \frac{\sqrt{\epsilon}}{8} \right] \\
\leq \mathbb{P}\left[ |\hat{F}_n(v_{\alpha}) - F(v_{\alpha})| > \sqrt{\epsilon}(1 - \alpha) - \frac{1}{n} \right] + \mathbb{P}\left[ |v_{\alpha} - \hat{v}_{n,\alpha}| > \frac{\sqrt{\epsilon}}{8} \right] \\
\leq \exp\left( -2n \left( \sqrt{\epsilon}(1 - \alpha) - \frac{1}{n} \right)^2 \right) + 2 \exp\left( -2n\delta_{\epsilon_1}^2 \right),
\]

where \((a)\) is due to \(|\hat{F}_n(\hat{v}_{n,\alpha}) - F(v_{\alpha})| \leq \frac{1}{n}\), and \((b)\) is due to an application of the DKW inequality and Lemma 3. In the above, \(\delta_{\epsilon_1} = \min\{F\left( v_{\alpha} + \frac{\sqrt{\epsilon}}{8} \right) - F(v_{\alpha}), F(v_{\alpha}) - F\left( v_{\alpha} - \frac{\sqrt{\epsilon}}{8} \right) \}\). It is easy to see that

\[
\exp\left( -2n \left( \sqrt{\epsilon}(1 - \alpha) - \frac{1}{n} \right)^2 \right) \leq G_2 \exp\left( -n\epsilon(1 - \alpha)^2 \right),
\]

(13)
where \( G_2 = \exp(2\sqrt{\epsilon}(1 - \alpha)) \). Since the density exists for the r.v. \( X \), we have
\[
F(v_\alpha + \eta_1) - F(v_\alpha - \eta_2) = f(\bar{v})(\eta_1 + \eta_2), \]
for some \( \bar{v} \in [v_\alpha - \eta_2, v_\alpha + \eta_1] \). Using the identity above for the expression inside \( \delta_{\epsilon_1} \), we obtain
\[
\delta_{\epsilon_1} = \min(f(\bar{v}_1), f(\bar{v}_2)) / \sqrt{\epsilon}, \]
for some \( \bar{v}_1 \in [v_\alpha, v_\alpha + \sqrt{\epsilon}/8] \) and \( \bar{v}_2 \in [v_\alpha - \sqrt{\epsilon}/8, v_\alpha] \). Thus, we have
\[
2 \exp(-2n\delta_{\epsilon_1}^2) \leq 2 \exp(-n\eta_f(1 - \alpha)^2), \tag{14} \]
where \( \eta_f \) depends on the value of the density in a small neighbourhood of \( v_\alpha \). Using (8), (7), (8), (11) (12), (13) and (14) we obtain
\[
I_1 \leq 2 \exp(-n(1 - \alpha)^2/8n\epsilon^2) + G_1 \exp(-n\epsilon^2(1 - \alpha)^2/32v_\alpha^2) + G_2 \exp(-n\epsilon(1 - \alpha)^2) + 2 \exp(-n\eta_f(1 - \alpha)^2) \tag{15} \]
We now upper bound \( I_2 \) as follows. Recall that the sequence of r.v.s \( \{X_i\}_i \) are i.i.d. 1-sub-Gaussian.
\[
I_2 = \mathbb{P}[\max\{X_1, \ldots, X_n\} > n\epsilon] < 8n \exp\left[-\frac{n\epsilon^2}{4}\right], \tag{16} \]
where we have used a union bound together the fact that \( \mathbb{P}(X \geq \xi) \leq 8 \exp(-\xi^2/4) \) for a 1-sub-Gaussian \( X \) (see Theorem 2.1 of [Wainwright]).

From (15) and (16), it is easy to observe that the terms \( 2 \exp(-n(1 - \alpha)^2/8n\epsilon^2) \) and \( \eta_f \) increase and decrease respectively, with \( n\epsilon \). Equalizing these two terms leads to \( n\epsilon \) of the order of \( n^{1/4}/\sqrt{\epsilon} \) and hence we choose \( n\epsilon = n^{1/4} / \sqrt{\epsilon} + 1 \). From (7), for this choice of \( n\epsilon \), we get that \( I_{11} \leq 2 \exp(-\sqrt{n(1 - \alpha)^2}\epsilon / 32) \) when \( \epsilon \geq 1 \) and \( I_{11} \leq 2 \exp(-\sqrt{n(1 - \alpha)^2}\epsilon^2 / 32) \) when \( \epsilon \leq 1 \). Hence, we obtain that \( I_{11} \leq 2 \exp(-\sqrt{n(1 - \alpha)^2}\epsilon^2 / 32) \).

Then, the overall bound turns out as follows: \( \forall n > n_0 \),
\[
\mathbb{P}[|\hat{c}_{n,\alpha} - c_{\alpha}| > \epsilon] \leq 2 \exp\left(-\frac{\sqrt{n}(1 - \alpha)^2\min\{\epsilon, \epsilon^2\}}{32}\right) + G_1 \exp\left(-\frac{n\epsilon^2(1 - \alpha)^2}{32v_\alpha^2}\right) + G_2 \exp(-n\epsilon(1 - \alpha)^2) + 2 \exp(-n\eta_f(1 - \alpha)^2) + 8 \exp\left(-\frac{n\epsilon}{4}\right) \tag{17} \]
\[
\leq 10 \exp\left(-\frac{\sqrt{n}(1 - \alpha)^2\min\{\epsilon, \epsilon^2\}}{32}\right) + 3G_3 \exp\left(-\frac{\sqrt{n}(1 - \alpha)^2\min\{\epsilon, \epsilon^2\}\min\{1, \eta_f, 1/v_\alpha^2\}}{32}\right) \tag{17} \]
\[
\leq G \exp\left(-\frac{\sqrt{n}(1 - \alpha)^2\min\{\epsilon, \epsilon^2\}\min\{1, \eta_f, 1/v_\alpha^2\}}{32}\right) \tag{17} \]
where \( G_3 = 4 \max\{G_1, G_2, 2\} \) and \( G = \max\{G_1, G_2, 16\} \), which establishes the result. The claim follows.

(ii): Light-tailed case

The proof passage leading up to (15) in the proof of Theorem 1 holds for the case of light-tailed distributions as well. However, the bounding of the term \( I_2 \), and hence, the choice of \( n\epsilon \) change, and we provide the relevant details below.
\[
I_2 = \mathbb{P}[\max\{X_1, \ldots, X_n\} > n\epsilon] \leq 2n\eta_1 \exp(-n\epsilon). \]
The inequality above follows by using the definition of light-tailed distributions. As in the proof for sub-Gaussian case, we equalize the competing terms involving \( n_\epsilon \), i.e., \( 2 \exp \left( -\frac{n(1-\alpha)^2 \epsilon^2}{sn^2} \right) \) and \( 2n\eta_1 \exp \left( -n_\epsilon \right) \), leading to \( n_\epsilon = n^{1/3} (\epsilon^2/3 + 1) \). Similar to the calculation of \( n_0 \) in the sub-Gaussian case, here, \( n_0' \) is an integer beyond which \( \frac{1}{1-\alpha} \int_{x=n_\epsilon}^\infty x f_X(x) dx \) is at most \( \epsilon/2 \). It is easy to see that, \( n_0' \) is the first instant \( n \in \{ 1, 2, \cdots \} \) such that the following sequence \( \{ \frac{n_0'}{1-\alpha} (1 + n_\epsilon) \exp \left( -n_\epsilon \right) \}_{n=1}^\infty \) falls below \( \epsilon/2 \). For the above choice of \( n_\epsilon \), the overall bound can be inferred using arguments similar to those used in the sub-Gaussian case (see the passage following (17)).

\[ \square \]

### 5.2 Proof of Theorem 2

**Proof.** (i): sub-Gaussian case

Note that, if the CVaR-SR algorithm has eliminated the optimal arm in phase \( i \) then it implies that at least one of the last \( i \) worst arms i.e., one of the arms in \( \{ [K], [K-1], \cdots, [K-i+1] \} \) must not have been eliminated in phase \( i \). Hence, we get that

\[
P \left[ J_{n'} \neq i^* \right] \leq \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} P \left[ c^i_{n_k+m_0,\alpha} \geq -c^i_{n_k+m_0,\alpha} \right] = \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} P \left[ c^i_{n_k+m_0,\alpha} - c^i_{\alpha} + c^i_{n_k+m_0,\alpha} + c^i_{\alpha} - c^i_{\alpha} \geq c^i_{\alpha} - c^i_{\alpha} \right]
\]

\[
\leq \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} \left( P \left[ c^i_{n_k+m_0,\alpha} - c^i_{\alpha} \geq \Delta^i \right] + P \left[ c^i_{n_k+m_0,\alpha} - c^i_{\alpha} \leq -\Delta^i \right] \right)
\leq \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} P \left[ c^i_{n_k+m_0,\alpha} - c^i_{\alpha} \geq \Delta^i \right] + \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} P \left[ c^i_{n_k+m_0,\alpha} - c^i_{\alpha} \leq -\Delta^i \right]
\]

(18)

We now bound the above terms individually as follows.

\[
\sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} P \left[ c^i_{n_k+m_0,\alpha} - c^i_{\alpha} \geq \Delta^i \right] \leq \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} P \left[ \left| c^i_{n_k+m_0,\alpha} - c^i_{\alpha} \right| \geq \Delta^i \right]
\]

\[
\leq \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} G_i \times \exp \left( -\sqrt{n_k+m_0(1-\alpha)^2} \min \{ \Delta^i, \Delta^2 \} \beta_i \right)
\]

\[
\leq \sum_{k=1}^{K-1} G_{\max} \times \exp \left( -\sqrt{n_k+m_0(1-\alpha)^2} \times \min \{ \Delta_{[K+1-k]}, \Delta^2_{[K+1-k]} \} \beta_{\min} \right)
\]

(19)

where \( \beta_i = \frac{\min \{ 1, c^i, (1/c^i) \}^2 \sqrt{n_k+m_0(1-\alpha)^2}}{128} \), \( \beta_{\min} = \min_i \beta_i \), \( G_{\max} = \max_i G_i \), and \( (a) \) is due to the following:

For \( 1 \leq i \leq K \), let \( m_i \) be the first instant \( m \geq 1 \) such that the sequence \( \left\{ \frac{2^{i-1}1^{m_i/16} \sqrt{\log \Delta_i}}{1-\alpha} \exp \left( -\frac{\sqrt{m_i} \Delta_i}{8} \right) \right\}_{m=1}^\infty \) falls below \( \frac{\Delta^i}{2} \). It is easy to see that, the quantity \( m_0 \) defined the theorem statement is larger than \( m_i \) for \( 1 \leq i \leq K \), and hence we invoke the Theorem 1. Further, note that

\[
\sqrt{n_k+m_0} \min \{ \Delta_{[K+1-k]}, \Delta^2_{[K+1-k]} \} \geq \sqrt{n_k} \min \{ \Delta_{[K+1-k]}, \Delta^2_{[K+1-k]} \}
\]
\[ \sqrt{\frac{\min\{\Delta_{[K+1-k]}^2, \Delta_{[K+1-k]}^2\}}{\log K(K+1-k)}} \geq \sqrt{\frac{n-K}{\log KH_1}} \]

where \( H_1 \) is as defined in the theorem statement. By substituting (20) in (19) we get that

\[ \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} \mathbb{P}\left[ \hat{c}_{n_k+m_0}^i - c_i^* \geq \frac{\Delta_i}{2} \right] \leq \sum_{k=1}^{K-1} kG_{\text{max}} \exp\left(-\sqrt{\frac{(n-K)(1-\alpha)^2 \beta_{\min}}{\log KH_1}}\right). \] (21)

Similarly, we can show that

\[ \sum_{k=1}^{K-1} \sum_{i=K+1-k}^{K} \mathbb{P}\left[ \hat{c}_{n_k+m_0}^i - c_i^* \geq \frac{\Delta_i}{2} \right] \leq \sum_{k=1}^{K-1} kG_{\text{max}} \exp\left(-\sqrt{\frac{(n-K)(1-\alpha)^2 \beta_{\min}}{\log KH_1}}\right). \] (22)

By substituting (21) and (22) in (18), the main claim follows.

(ii): Light-tailed case

Follows along the lines of the proof above by applying the CVaR concentration result for the light-tailed distributions (part (ii) in Theorem 1) at respective places. The quantity \( \tilde{G}_{\text{max}} = \max_i \tilde{G}_i \), where \( \tilde{G}_i \) is a quantity corresponds to the arm \( i \) which is defined in part (ii) in Theorem 1. This completes the proof.

6 Numerical experiments

In this section, we present the simulation results of our algorithms tested on a \( K \)-armed bandit problem with Gaussian arm distributions. All the simulation have been carried out using MATLAB and averaged over 100 sample paths. We consider 10-armed bandit problem with Gaussian arm distributions whose details are given in the Table 1.

| \( \mu \) | 0.2694 | 0.6254 | 0.0552 | 0.2796 | 0.0912 | 0.6961 | 0.7485 | 0.9683 | 0.8151 | 0.6658 |
| \( \sigma^2 \) | 0.5363 | 0.3390 | 0.2668 | 0.2032 | 0.2960 | 0.1425 | 0.9266 | 0.9903 | 0.6404 | 0.7894 |
| \( c_{0.95} \) | 1.3754 | 1.324 | 0.6056 | 0.6987 | 0.7015 | 0.9899 | 2.6594 | 3.0111 | 2.1358 | 2.2947 |
| \( c_{0.99} \) | 1.6994 | 1.5288 | 0.7656 | 0.8212 | 0.8809 | 1.0759 | 3.2190 | 3.6074 | 2.5213 | 2.7691 |

The CVaR values mentioned in Table 1 are calculated empirically by using a large number of i.i.d. samples. It is clear that the arm-3 is the best arm. We run our CVaR-SR algorithm for various budgets such as \( 10^3 \), \( 10^4 \), \( 5 \times 10^4 \), \( 10^5 \), \( 5 \times 10^5 \), over 100 sample paths, and report the results in Figure 1.

As we expect, the probability of correct identification increases with the increase of budget available to the algorithm. Further, the probability of correct identification decreases as the parameter level \( \alpha \) is increased, and this is because at higher values of \( \alpha \), the tail event, that CVaR is based upon, occurs with smaller probabilities.
7 Conclusions and future directions

We considered the best CVaR arm identification problem in $K$-armed stochastic bandits under the fixed budget setting. We derived two-sided concentration bounds for a well-known CVaR estimator, assuming that the underlying distribution is either sub-Gaussian or light-tailed. Using the CVaR concentration bound, we proposed a non-trivial adaptation of the successive rejects algorithm to the setting where the goal is to find an arm with the lowest CVaR, and established error bounds for the proposed algorithm.

An interesting line of future work is to derive CVaR concentration bounds for sub-Gaussian and light-tailed distributions that do not require a lower bound on the number of samples, to be applicable. Such a bound would get rid of the initialization phase, where each arm is pulled a certain number of times, in our proposed bandit algorithm.

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A Calculation of CVaR for Gaussian r.v.s

If (C1) holds and X has a positive density at $v_\alpha(X)$, then $c_\alpha(X)$ admit the following equivalent form [cf. Kisiala, 2015]:

$$c_\alpha(X) = \frac{1}{1 - \alpha} \int_{1 - \alpha}^{1} v_\beta(X) d\beta.$$  \hspace{1cm} (23)
Using the formula above, CVaR of a Gaussian r.v. with mean $\mu$ and variance $\sigma^2$ can be calculated as follows:

$$c_\alpha(X) = \frac{1}{1 - \alpha} \int_{1-\alpha}^{1} v_\beta(X) d\beta$$

$$= \frac{1}{1 - \alpha} \int_{1-\alpha}^{1} (\mu - \sigma Q^{-1}(\beta)) d\beta$$

$$= \mu \left( \frac{\alpha}{1 - \alpha} \right) + \sigma c_\alpha(Z),$$

For the last equality above, we used the fact that $-Q^{-1}(\beta)$ is the VaR of $Z$. 