The phase diagram of an Ising model on a polymerized random surface

Thordur Jonsson
Raunvisindastofnun Haskolans, University of Iceland
Dunhaga 3, 107 Reykjavik
Iceland

John F. Wheater
Department of Physics, University of Oxford
1 Keble Road, Oxford OX1 3NP
United Kingdom

Abstract. We construct a random surface model with a string susceptibility exponent $\gamma = 1/4$ by taking an Ising model on a random surface and introducing an additional degree of freedom which amounts to allowing certain outgrowths on the surfaces. Fine tuning the Ising temperature and the weight factor for outgrowths we find a triple point where $\gamma = 1/4$. At this point magnetized and nonmagnetized gravity phases meet a branched polymer phase.
1 Introduction

A few years ago it was discovered that by allowing surfaces in matrix models to touch at points, associating a coupling constant to the touching and fine tuning this coupling the string susceptibility exponent could jump from $\gamma$ to $\gamma/(\gamma - 1)$ [1, 2, 3]. It was then shown in [4] working directly with triangulated random surfaces that this is a generic phenomenon in random surface theories. In [5] a simple random surface model was studied where the scenarios of [4] were explicitly realized and one could construct a random surface theory with $\gamma = 1/3$ from one with $\gamma = -1/2$.

Here we generalize this construction, starting with the Ising model on a random surface and introducing an additional degree of freedom with an associated coupling. We map the phase diagram of this theory and show that there are three phases: magnetized and unmagnetized gravity phases both of which have $\gamma = -1/2$ and a branched polymer phase with $\gamma = 1/2$. The exponent $\gamma$ takes the value $-1/3$ on the line where the magnetized and unmagnetized gravity phases meet and on the line separating the branched polymer phase from the gravity phase we have generically $\gamma = 1/3$. The three phases meet in a triple point where $\gamma = 1/4$.

It has been believed for some time that $\gamma > 0$ may be associated with $c > 1$ matter fields interacting with 2-dimensional quantum gravity. This is one of the reasons why the transmutation of $\gamma$ found in [1, 2, 3, 4] is interesting. In a recent paper [6] it is argued on the basis of Liouville theory calculations that the jump of $\gamma$ to a positive value is not related to $c > 1$ but rather to a different solution of a $c < 1$ theory. From the point of view of discretized models it is not clear that this is the correct interpretation. Studies of models that manifestly have $c > 1$, e.g. many Ising or Potts models on a random surface, indicate that $\gamma$ depends only on $c$ [7] and the behaviour of such models is analogous to the ones with a transmuted $\gamma > 0$.

2 Definition of the model

Let $\mathcal{T}$ denote the collection of all triangulations of the disc where the boundary is just one link. If $T \in \mathcal{T}$ we allow two triangles in $T$ to share two links so in particular this ensemble includes tadpole surfaces. The dual graphs corresponding to surfaces in $\mathcal{T}$ are all planar $\varphi^3$-diagrams with one external leg. To each triangle $i$ in $T$ we associate an Ising spin variable $\sigma_i$.

Before defining our model let us recall some facts about the Ising model on a random surface. The one-loop function of the Ising model on a random surface is
given by

\[ \mathcal{G}(\mu, \beta) = \sum_{T \in \mathcal{T}} e^{-\mu |T|} \sum_{\{\sigma\}} e^{\beta \sum_{(ij)} (\sigma_i \sigma_j - 1)/2}, \] (1)

where the sum on \{\sigma\} is over all spin configurations on the triangulation \( T \) and the sum on \((ij)\) is over all nearest neighbour pairs of triangles in \( T \). Here \(|T|\) usually denotes the number of triangles but for later convenience we shall let \(|T|\) denote the number of interior links in the triangulation \( T \). Since the number of interior links is linearly related to the number of triangles the values of critical exponents do not depend upon this change. The susceptibility of the Ising model is defined by

\[ \bar{\chi}(\mu, \beta) = -\frac{\partial}{\partial \mu} \mathcal{G}(\mu, \beta). \] (2)

This theory was solved exactly in [8, 9] and the critical exponents calculated. We shall make use of the following facts from [8, 9]: For each \( \beta \geq 0 \) there is \( \mu_{Ic}^{\beta} \) such that \( \bar{\chi}(\mu, \beta) \) is analytic in \( \mu \) for \( \mu > \mu_{Ic}^{\beta} \) and infinite for \( \mu < \mu_{Ic}^{\beta} \). As \( \mu \downarrow \mu_{Ic}^{\beta} \)

\[ \bar{\chi}(\mu, \beta) \sim (\mu - \mu_{Ic}^{\beta})^{-\gamma^{I}(\beta)} + \text{less singular terms} \] (3)

where

\[ \gamma^{I}(\beta) = \begin{cases} -1/2, & \beta \neq \beta_{c}^{I} \\ -1/3, & \beta = \beta_{c}^{I} \end{cases} \] (4)

and \( \beta_{c}^{I} = -\frac{1}{2} \log \left( \frac{1}{2\sqrt{7}} \right) \).

The model we wish to study is an Ising model on a random surface as described above with an additional degree of freedom which we shall call \textit{outgrowths}. If \( \ell \) is an interior link in a triangulation \( T \), we put an outgrowth on the triangulation at \( \ell \) by cutting it open along \( \ell \), gluing two sides of a new triangle to the boundary of the cut and attaching a surface in \( T \) to the remaining boundary link in the new triangle. We associate a non-negative weight factor \( \lambda \) to each outgrowth. The extra triangle used for gluing an outgrowth on the underlying surface \( T \) carries an Ising spin which interacts with its neighbours, see Fig. 1. The triangles and links in the outgrowth have the same degrees of freedom as the ones in the underlying surface. One can think of the outgrowths as being defined by the phase boundaries of an additional restricted Ising spin system on the surface where phase boundaries in the restricted system are only allowed to have length 1, cf. [3].

Another way to think of the model is the following: For each triangulation we consider all interior loops of length 1. To each such loop we assign a variable with two values, blue and red say. If the colour is red this loop is the boundary of an outgrowth and there is a corresponding weight factor \( \lambda \) associated with it. If the colour is blue the loop is not the boundary of an outgrowth and no multiplicative
factor is associated to it. In the partition function we then sum over all colour assignments to the loops of length 1.

Let \( G(\mu, \beta, \lambda) \) denote the one-loop function in our theory. If the value of the boundary spin is fixed to be \( \sigma \) we denote the one-loop function by \( G_\sigma(\mu, \beta, \lambda) \). In the absence of an external magnetic field, as will be the case in this paper, \( G_\sigma \) is independent of \( \sigma \) and \( G = 2G_\sigma \). We can express the one-loop function as

\[
G(\mu, \beta, \lambda) = \sum_{T \in \mathcal{T}} e^{-\mu|T|} \sum_{\{\sigma\}} \prod_{\ell \in T} \left( e^{\beta(\sigma_i \sigma_{j-1})/2} + \lambda \sum_{\sigma, \sigma'} e^{\beta(\sigma_i \sigma_j + \sigma \sigma' + \sigma \sigma' - 3)/2} e^{-2\mu G_{\sigma'}(\mu, \beta, \lambda)} \right),
\]

where \( i \) and \( j \) are the triangles next to the link \( \ell \) in the triangulation \( T \), \( \sigma \) is the spin variable associated to the intermediate triangle for an outgrowth and \( \sigma' \) is the value of the boundary spin of the outgrowth, see Fig. 1. The first term in the parenthesis on the right side of (5) corresponds to the case when there is no outgrowth on the link \( \ell \) and the second term corresponds to the presence of an arbitrary outgrowth.

Summing over the spin variables \( \sigma \) and \( \sigma' \) the one-loop function becomes

\[
G(\mu, \beta, \lambda) = \sum_{T \in \mathcal{T}} e^{-\mu|T|} \sum_{\{\sigma\}} \prod_{\ell \in T} \left( e^{\beta(\sigma_i \sigma_{j-1})/2} + 2\lambda e^{-2\mu} e^{-3\beta/2} \cosh^{3} \frac{\beta}{2} (1 + \sigma_i \sigma_j \tanh^{2} \frac{\beta}{2}) G_{\sigma'} \right),
\]

The function \( G \) can be rewritten as the one-loop function of a single Ising model on a random surface with renormalized couplings \( \bar{\mu} \) and \( \bar{\beta} \), i.e.

\[
G(\mu, \beta, \lambda) = \mathcal{G}(\bar{\mu}, \bar{\beta})
\]

provided

\[
e^{-\bar{\mu}} e^{\bar{\beta}(\sigma_i \sigma_{j-1})/2} = e^{-\mu} \left( e^{\beta(\sigma_i \sigma_{j-1})/2} + 2\lambda e^{-2\mu} e^{-3\beta/2} \cosh^{3} \frac{\beta}{2} (1 + \sigma_i \sigma_j \tanh^{2} \frac{\beta}{2}) G_{\sigma'} \right).
\]

Since \( \sigma_i \sigma_j = \pm 1 \) we have two equations for two unknowns which are readily solved and we find

\[
\bar{\mu} = \mu - \log \left( 1 + \frac{\lambda}{2} e^{-2\mu} (1 + e^{-\beta}) (1 + e^{-2\beta}) G(\mu, \beta, \lambda) \right)
\]

\[
\bar{\beta} = -\log \frac{e^{-\beta} + \lambda e^{-2\mu} e^{-\beta} (1 + e^{-\beta}) G(\mu, \beta, \lambda)}{1 + \frac{\lambda}{2} e^{-2\mu} (1 + e^{-\beta}) (1 + e^{-2\beta}) G(\mu, \beta, \lambda)}.
\]

If the couplings \( \beta \) and \( \lambda \) are fixed there is a critical value of \( \mu \) which we denote by \( \mu_c(\beta, \lambda) \) such that the one loop function \( G \) is analytic in \( \mu \) for \( \mu > \mu_c(\beta, \lambda) \) and
infinite for $\mu < \mu_c(\beta, \lambda)$. In fact one can show that the set $\mathcal{B} = \{(\mu, \beta, \log \lambda) : G(\mu, \beta, \lambda) < \infty \}$ is a convex subset of $\mathbb{R}^3$ and $G$ is a real analytic function in its interior. Note that the renormalized couplings $\bar{\mu}$ and $\bar{\beta}$ are functions of all the unrenormalized couplings $\mu$, $\beta$ and $\lambda$. Due to (7) we clearly have

$$\bar{\mu}(\mu_c(\beta, \lambda), \beta, \lambda) \geq \mu_c(\bar{\beta}(\mu_c(\beta, \lambda), \beta, \lambda)).$$

### 3 The phase diagram

Here we derive a relation between the susceptibility of our model

$$\chi(\mu, \beta, \lambda) = -\frac{\partial}{\partial \mu} G(\mu, \beta, \lambda)$$

and the susceptibility of the random surface Ising model with renormalized couplings. This will enable us to calculate the critical exponent $\gamma(\beta, \lambda)$ defined by

$$\chi(\mu, \beta, \lambda) \sim (\mu - \mu_c(\beta, \lambda))^{-\gamma(\beta, \lambda)} + \text{less singular terms}$$

as $\mu \downarrow \mu_c(\beta, \lambda)$.

First note that

$$\chi = \chi \frac{\partial \bar{\mu}}{\partial \mu} - \frac{\partial G}{\partial \beta} \frac{\partial \bar{\beta}}{\partial \mu}.$$  \hfill (14)

If we define the function $C(\bar{\mu}, \bar{\beta})$ by the equation

$$\frac{\partial G}{\partial \beta} = \bar{\chi}(\bar{\mu}, \bar{\beta}) C(\bar{\mu}, \bar{\beta}),$$

then it easy to see that $-1/2 \leq C \leq 0$. In order to simplify some of the formulas in the sequel we put

$$\Lambda = \frac{\lambda}{2} e^{-2\mu}(1 + e^{-\beta})(1 + e^{-2\beta}).$$

After a calculation, using (7), (8), (10) and (14), we find

$$\chi(\mu, \beta, \lambda) = \frac{N(\mu, \beta, \lambda) \bar{\chi}(\bar{\mu}, \bar{\beta})}{D(\mu, \beta, \lambda)},$$

where

$$N(\mu, \beta, \lambda) = 1 + \frac{2\Lambda G}{1 + \Lambda G} - \frac{2\Lambda C G \tanh \beta}{\left(1 + \Lambda G \frac{2}{1 + e^{-2\beta}}\right) (1 + \Lambda G)}$$

and

$$D(\mu, \beta, \lambda) = 1 - \frac{\Lambda \bar{\chi}}{1 + \Lambda G} + \frac{\Lambda C \bar{\chi} \tanh \beta}{\left(1 + \Lambda G \frac{2}{1 + e^{-2\beta}}\right) (1 + \Lambda G)}.$$
Note that $N$ is a positive uniformly bounded function. Let us fix $\beta$ and $\lambda$ and take $\mu$ larger than its critical value $\mu_c(\beta, \lambda)$. As we lower $\mu$, a singularity is eventually encountered in $\chi(\mu, \beta, \lambda)$ for one or both of two reasons: Either $\bar{\mu}$ reaches the critical value of the cosmological constant of the random surface Ising model, $\mu^t_c$, or $D$ becomes zero. If $\lambda$ is small enough then the denominator $D$ is positive for any $\mu$ and $\beta$ because $\overline{G}$ and $\bar{\chi}$ are bounded functions. On the other hand, the last term in (19) is negative definite so $D$ is negative if

$$\Lambda \bar{\chi} \geq 1 + \Lambda \overline{G}. \quad (20)$$

Since $\bar{\chi} \geq 2 \overline{G}$ for all values of the coupling constants, the inequality (20) holds if $\Lambda \overline{G} \geq 1$. We shall indeed prove that there is a critical line separating the region where $D = 0$ at the critical point from a region where $D > 0$ at the critical point.

It will be convenient in the remainder of this paper to regard $D, N, \mu_c$ etc. as functions of $\Lambda$ rather than $\lambda$. This amounts to a smooth change of coordinates in the coupling constant space. We claim the following: For any value of $\beta$ there is a value of $\Lambda$ which we denote by $\Lambda_c(\beta)$ such that

$$D(\mu, \beta, \Lambda) > 0 \quad (21)$$

for all $\mu \geq \mu_c(\beta, \Lambda)$ provided $\Lambda < \Lambda_c(\beta)$ and

$$D(\mu_c(\beta, \Lambda), \beta, \Lambda) = 0 \quad (22)$$

for $\Lambda \geq \Lambda_c(\beta)$. Furthermore,

$$\bar{\mu}(\mu_c(\beta, \Lambda), \beta, \Lambda) > \mu^t_c(\bar{\beta}(\mu_c(\beta, \Lambda), \beta, \Lambda)) \quad (23)$$

in the region $\Lambda > \Lambda_c(\beta)$.

In order to prove the claim we consider lines in the coupling constant space with fixed values of $\bar{\mu}$ and $\bar{\beta}$. These lines can be parametrized by $\Lambda$ and they constitute a fibration of the coupling constant space. First we observe, using

$$e^{-\bar{\beta}} = e^{-\beta} \frac{1 + \frac{2\Lambda \overline{G}}{1 + e^{-2\bar{\beta}}}}{1 + \Lambda \overline{G}}, \quad (24)$$

that on each such line $\bar{\beta}$ is an increasing smooth function of $\Lambda$, $\bar{\beta} = \bar{\beta}$ at $\Lambda = 0$ and $\beta = \bar{\beta} + \log(1 + \sqrt{1 - e^{-2\beta}})$ at $\Lambda = \infty$. In order to prove our claim it therefore suffices to show that $D$ has exactly one zero on each line where $\bar{\beta}$ and $\bar{\mu}$ are fixed. On such a line the function $C$, defined in (13), is constant and in view of (24) we can write

$$D = 1 - \frac{\Lambda \bar{\chi}}{1 + \Lambda \overline{G}} + \frac{\Lambda C \bar{\chi} e^{\beta} e^{-\beta} \tanh \beta}{(1 + \Lambda \overline{G})^2} \quad (25)$$
when $D$ is restricted to the line. We saw above that $D < 0$ if $\Lambda \overline{G} > 1$ so in order to prove our claim it suffices to show that

$$\frac{dD}{d\Lambda} < 0$$

for $\Lambda \overline{G} \leq 1$. We find

$$\frac{dD}{d\Lambda} = \frac{-\overline{\chi}}{(1 + \Lambda \overline{G})^2} + C\overline{\chi} e^\beta e^{-\beta \tanh \beta} \frac{d}{d\Lambda} \left( \frac{\Lambda}{(1 + \Lambda \overline{G})^2} \right)$$

$$+ \frac{\Lambda C \overline{\chi} e^\beta}{(1 + \Lambda \overline{G})^2} \frac{d\beta}{d\Lambda} \left( e^{-\beta \tanh \beta} \right).$$

The second term on the right hand side above is negative definite if $\Lambda \overline{G} \leq 1$. Using

$$\frac{d\beta}{d\Lambda} = \frac{\overline{G} - e^{-\beta \cosh \beta \overline{G}}}{e^{-2\beta} + (1 + \Lambda \overline{G}) e^{-\beta \sinh \beta}}$$

and the inequality

$$1 \leq e^{-\beta + \beta} \leq \frac{3}{2},$$

which follows from (24) if $\Lambda \overline{G} \leq 1$, one can now check by an explicit calculation that the sum of the two remaining terms is negative. This completes the proof of the claim.

In the region $\Lambda > \Lambda_c(\beta)$ the functions $\overline{G}$ and $\overline{\chi}$ are analytic functions of their arguments as we reach the critical point and we can calculate the critical exponent $\gamma(\beta, \Lambda)$ by the same method as in [4, 5] and find the generic branched polymer value $\gamma = 1/2$. In this case the surfaces are in the branched polymer phase, as expected for large $\Lambda$, and the entropy is dominated by outgrowths. We shall call the line $\Lambda = \Lambda_c(\beta)$ the $\Lambda$-line. It is easily seen from the implicit function theorem that $\Lambda_c$ is a smooth function of $\beta$.

Let us now consider the region $\Lambda < \Lambda_c(\beta)$. In this case the denominator $D$ vanishes nowhere and by arguments analogous to those in [4, 5] we obtain

$$\gamma(\beta, \Lambda) = \gamma^I(\beta_c(\Lambda, \beta, \Lambda)).$$

This is the gravity phase of the model which, as expected for small $\Lambda$, is characterized by few outgrowths. It is easy to check from (9) and (10) that the equations

$$\bar{\beta}(\mu_c(\beta, \Lambda), \beta, \Lambda) = \beta^I_c$$

and

$$\overline{\mu}(\mu_c(\beta, \Lambda), \beta, \Lambda) = \mu_c^I(\beta^I_c)$$

are
have a unique solution $\beta_c(\Lambda)$ in this region for any value of $\Lambda$. We shall call this line the $\beta$-line. This line separates the gravity phase of the model into two parts, a magnetized one for $\beta > \beta_c(\Lambda)$ and an unmagnetized one for $\beta < \beta_c(\Lambda)$, see Fig. 2. In both these phases $\gamma = -1/2$ but on the $\beta$-line $\gamma = -1/3$ by (1). The $\beta$-line meets the $\Lambda$-line at a unique point; this is where the magnetized and unmagnetized gravity phases meet the branched polymer phase and is therefore a triple point of the theory. On the $\Lambda$-line the numerator and denominator in (17) are both singular and conspire to realize the scenario of [4]. We can calculate the value of $\gamma$ by the same method as in [5] and find $\gamma = 1/3$ except at the triple point where $\gamma = 1/4$.

4 Discussion

We have given an explicit construction of a random surface theory with $\gamma = 1/4$ at one particular point in the phase diagram. It is interesting, as pointed out in [4], that this value of $\gamma$ has been seen in simulations of a random surface in a three dimensional hypercubic lattice with a weak self-avoidance condition [10]. We have, however, not been able to see any relation between that model and the one studied here.

These results can be extended to other spin systems. Suppose that, instead of the Ising spin we have a vector of spins $s(s.s = 1)$ at each site and we make the replacement
\[ e^{\beta(s_1.s_2-1)/2} \rightarrow 1 + \kappa s_1.s_2, \quad 0 \leq \kappa \leq 1. \] (34)
(which in the Ising case is exact up to a factor depending only on $\beta$). This leads to the equations
\[ G(\mu, \kappa, \lambda) = \overline{G}(\bar{\mu}, \bar{\kappa}) \] (35)
\[ \bar{\mu} = \mu - \log \left(1 + \lambda e^{-2\mu} G(\mu, \kappa, \lambda) \right) \] (36)
\[ \bar{\kappa} = \kappa \frac{1 + \lambda e^{-2\mu} \kappa G(\mu, \kappa, \lambda)}{1 + \lambda e^{-2\mu} G(\mu, \kappa, \lambda)} \] (37)
which have the same structure as (10), (11), (12). Random surface models with these generalized matter fields have not been explicitly solved so $\overline{G}(\bar{\mu}, \bar{\kappa})$ is not known; however we see that such models which belong to the minimal series with $\bar{\gamma} = -1/n$ with $n = 2, 3, 4, \ldots$ must also realize the scenario of [4] and have a triple point with $\gamma = 1/(n + 1)$ when outgrowths are included.

Instead of using surfaces with boundaries of length 1 and allowing tadpoles one could define a model analogous to the one constructed here using surfaces with boundaries consisting of two links and not allowing tadpoles. In this case one has
to work with two different one-loop functions, i.e. the one where the boundary spins are aligned and another one where they point in opposite directions. It turns out that in order to verify the existence of a triple point one needs rather delicate estimates on the ratio of these two one-loop functions. There is however little doubt that this model should have the same critical behaviour as the one we have studied here.

Acknowledgement. T. J. has benefitted from discussions with J. Ambjørn and B. Durhuus and would like to acknowledge hospitality at Institut Mittag-Leffler.

References

[1] S. R. Das, A. Dhar, A. M. Sengupta and S. R. Wadia, Mod. Phys. Lett. A5 (1990) 1041.

[2] L. Alvarez-Gaume, J. L. F. Barbon, and C. Crnkovic, Nucl. Phys. B 394 (1993) 383.

[3] G. P. Korchemsky, Phys. Lett. B 296 (1992) 323; Mod. Phys. Lett. A7 (1992) 3081.

[4] B. Durhuus, Nucl. Phys. B 426 (1994) 203.

[5] J. Ambjørn, B. Durhuus and T. Jonsson, Mod. Phys. Lett. A9 (1994) 1221.

[6] I. R. Klebanov, *Touching random surfaces and Liouville gravity*, PUPT-1486, hep-th/9407167.

[7] J. Ambjørn and G. Thorleifsson, Phys. Lett. B 323 (1994) 7.

[8] V. A. Kazakov, Phys. Lett. A 119 (1986) 140.

[9] D. V. Boulatov and V. A. Kazakov, Phys. Lett. B (1987) 379.

[10] B. Baumann and B. Berg, Phys. Lett. B 164 (1985) 131.
**Figure Caption.**

**Fig. 1.** The Figure illustrates the operation of placing an outgrowth on the link joining the triangles $i$ and $j$. Here $\sigma$ is the spin on the intermediate triangle and $\sigma'$ is the boundary spin of the outgrowth.

**Fig. 2.** The phase diagram of the model compactified to fill a square. For $\Lambda > \Lambda_c$ the critical behaviour is governed by the vanishing of $D$ while $\bar{\mu}$ and $\bar{\beta}$ are analytic so the $\beta$-line is not a line of phase transition in this region. The different phases are denoted by UMG (unmagnetized gravity), MG (magnetized gravity) and BP (branched polymer).
Fig. 1
