On Duality in Supersymmetric Yang-Mills Theory

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We discuss non-abelian $SU(N_c)$ gauge theory coupled to an adjoint chiral superfield $X$, and a number of fundamental chiral superfields $Q^i$. Using duality, we show that turning on a superpotential $W(X) = \text{Tr} \sum_{l=1}^{k} g_l X^{l+1}$ leads to non-trivial long distance dynamics, a large number of multicritical IR fixed points and vacua, connected to each other by varying the coefficients $g_l$. 
1. Introduction.

The recent progress in understanding the role of holomorphy [1-10] and duality [11-13] in four dimensional supersymmetric field theories can be used in many cases to study strongly coupled theories. In this note we discuss a class of theories where strong coupling effects lead to a rich pattern of fixed points exhibiting new duality symmetries.

We study supersymmetric Yang-Mills theory with gauge group $SU(N_c)$, a chiral matter superfield $X$ in the adjoint representation of the gauge group and $N_f$ fundamental multiplets $Q_i$ accompanied by $N_f$ anti–fundamental multiplets $\tilde{Q}_i$, $i = 1, \cdots, N_f$. This theory is asymptotically free for $N_f < 2N_c$. The model without a superpotential for the matter fields is governed by a non-trivial infrared fixed point, and is in a non abelian Coulomb phase (for $N_f > 0$). This interesting model has so far resisted all attempts at a detailed understanding. We will instead discuss the model with a superpotential:\footnote{We consider $k < N_c$ for simplicity.}

$$W = g_k \text{Tr } X^{k+1}. \quad (1.1)$$

Adding the superpotential (1.1) has the following consequences:

a) If the operator $\text{Tr } X^{k+1}$ is relevant at the infrared fixed point of the theory with $W = 0$, adding (1.1) drives the system to a new fixed point. For $k > 2$, $\text{Tr } X^{k+1}$ is irrelevant near the UV fixed point. We will argue below that there is a range (which depends on $k$) of $N_f$ for which $\text{Tr } X^{k+1}$ is relevant in the IR. For given $N_c > k > 2$ there is a critical number of flavors $N_0(k, N_c)$, $2N_c/k < N_0(k, N_c) < 2N_c$ such that if $N_f < N_0(k, N_c)$ the operator $X^{k+1}$ is relevant in the IR limit of the $W = 0$ theory. Conversely, for given $N_f < 2N_c$ all operators $\text{Tr } X^{k+1}$ with $k \leq k_0(N_f, N_c)$ are relevant ($k_0 > 2$) in the infrared.

b) Whether or not adding (1.1) leads to a new fixed point, one can not ignore this superpotential. It has the effect of lifting the flat directions of the theory with no superpotential corresponding to giving $X$ an expectation value. In addition, the superpotential (1.1) leads to a truncation of the chiral ring. The equations of motion for $X$ set:

$$X^k - \frac{1}{N} (\text{Tr } X^k) 1 = \text{D term} \quad (1.2)$$

Hence the chiral operators involving $X$ in the presence of the superpotential (1.1) are $\text{Tr } X^l$, $l = 2, \cdots, k$, and operators involving the matrix $X^l$ with $l < k$. There are two kinds of gauge invariant operators that will be of interest below. Meson operators

$$M_j^l = \tilde{Q}_i X^{j-1} Q^i; \quad j = 1, 2, \cdots, k \quad (1.3)$$
and baryon operators that are defined as follows. Introduce “dressed quarks:"

\[ Q(l) = X^{l-1}Q; \quad l = 1, \ldots, k. \]  

(1.4)

Then construct baryon–like operators

\[ B^{(n_1, n_2, \ldots, n_k)} = Q^{n_1}_{(1)} \cdots Q^{n_k}_{(k)}; \quad \sum_{l=1}^{k} n_l = N_c \]  

(1.5)

where the color indices are contracted with an \( \epsilon \) tensor. The total number of baryon operators of the form (1.5) is:

\[ \sum_{\{n_l\}} \left( \begin{array}{c} N_f \\ n_1 \\ \vdots \\ n_k \end{array} \right) = \left( \begin{array}{c} kN_f \\ N_c \end{array} \right) \]  

(1.6)

c) The theory with no superpotential has two independent \( R \) symmetries. Adding (1.1) leaves just one of the two unbroken, with \( X \) carrying \( R \) charge \( 2/(k+1) \). When (1.1) is relevant, the IR scaling dimensions of \( X, Q^i \) are determined by their \( R \) charges.

For \( k = 1 \) (1.1) is a mass term for \( X \); the adjoint superfield decouples in the IR and one is led back to supersymmetric QCD [1-4]. The case \( k = 2 \) was discussed in [13].

2. Stability.

Before turning to duality, we prove, using an idea that will be useful later, that the theory described by (1.1) has a stable vacuum iff

\[ N_f \geq \frac{N_c}{k}. \]  

(2.1)

Consider a deformation of the superpotential of the theory (1.1) to include lower order terms:

\[ W(X) = \text{Tr} \sum_{l=1}^{k} g_l X^{l+1} + \lambda \text{Tr} X \]  

(2.2)

We have introduced a Lagrange multiplier \( \lambda \) to enforce the tracelessness condition \( \text{Tr} X = 0 \). Eq. (2.2) describes a soft perturbation of (1.1), which does not change the large field behavior of \( W \), hence if there is no stable vacuum for small \( g_l \), the theory with \( g_l = \delta_{l,k} g_k \)

\(^2\) Except at very strong coupling where, as we will see, one can use a dual picture to study the IR scaling.

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has no vacuum either (and vice versa). Thus, consider the theory with small $g_l$ (2.2). The system has multiple vacua with $Q = \tilde{Q} = 0$ and non vanishing expectation values of the eigenvalues of $X$. Vacua of the theory are found by setting the potential for the eigenvalues $x_i$ of $X$ to zero, $W'(x_i) = 0$. $W'(x)$ is a polynomial of degree $k$, hence there are $k$ solutions, which are generically distinct. Ground states are labeled by sequences of integers $i_1 \leq i_2 \leq \cdots \leq i_k$, where $i_l$ is the number of eigenvalues of the matrix $X$ residing in the $l$’th minimum of the potential. Clearly,

$$\sum_{l=1}^{k} i_l = N_c. \quad (2.3)$$

$\lambda$ is then determined by requiring that the sum of the eigenvalues (which depend on $\lambda$) vanishes.

In each vacuum $X$ has a quadratic superpotential, i.e. it is massive and can be integrated out. The gauge group is broken by the $X$ expectation value:

$$SU(N_c) \to SU(i_1) \times SU(i_2) \times \cdots \times SU(i_k) \times U(1)^{k-1} \quad (2.4)$$

Some of the $i_l$ may vanish, in which case (2.4) is modified in an obvious way. Each of the $SU(i_l)$ factors describes a supersymmetric QCD model. It is well known [1], [4] that SQCD has no stable vacuum when the number of flavors is smaller than the number of colors. Here, this implies that the system has a stable vacuum iff

$$i_l \leq N_f; \quad \forall \ 1 \leq l \leq k. \quad (2.5)$$

Eq. (2.3) then implies that a stable vacuum exists iff (2.1) is satisfied. Finally, taking $g_l \to 0 \ (l < k)$ in (2.2) we conclude that the same is true for the theory (1.1). It would be interesting to analyze the superpotential that destabilizes the theory when $N_f < N_c/k$ directly, generalizing the discussion of the case $k = 1$ [1].

By fine tuning the coefficients $g_l$ one can arrange for some of the roots of $W'$ to coincide. In that case $X$ does not decouple in the different vacua but rather is governed by a superpotential of the form (1.1) with a lower value of $k$ equal to the order of a particular root of $W'$. The stability analysis can be repeated for this case too, with the same conclusions (2.1).
3. Duality.

The anomaly free global symmetry of the $SU(N_c)$ gauge theory described above is

$$SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$$  \hspace{1cm} (3.1)

with the matter fields transforming as:

$$Q \quad (N_f, 1, 1, 1 - \frac{2}{k+1} \frac{N_c}{N_f})$$

$$\bar{Q} \quad (1, N_f, -1, 1 - \frac{2}{k+1} \frac{N_c}{N_f})$$ \hspace{1cm} (3.2)

$$X \quad (1, 1, 0, \frac{2}{k+1}).$$

Following [11], [13] it is natural to propose a dual theory with gauge group $SU(kN_f - N_c)$ and the following matter content: $N_f$ flavors of (dual) quarks $q_i, \bar{q}_i$, an adjoint field $Y$, and gauge singlets $M_j$ representing (1.3), $j = 1, \cdots, k$, with the transformation properties under the global symmetry (3.1):

$$q \quad (N_f, 1, \frac{N_c}{kN_f - N_c}, 1 - \frac{2}{k+1} \frac{kN_f - N_c}{N_f})$$

$$\bar{q} \quad (1, N_f, -1, \frac{N_c}{kN_f - N_c}, 1 - \frac{2}{k+1} \frac{kN_f - N_c}{N_f})$$ \hspace{1cm} (3.3)

$$Y \quad (1, 1, 0, \frac{2}{k+1})$$

$$M_j \quad (N_f, N_f, 0, 2 - \frac{4}{k+1} \frac{N_c}{N_f} + \frac{2}{k+1} (j-1))$$

Note again that $j \leq k$ since $X^l$ is not an independent chiral operator for $l \geq k$ (1.2). The superpotential in the dual, “magnetic”, theory is taken to be: \hspace{1cm} (3.4)

$$W_{mag} = \text{Tr} \ Y^{k+1} + \sum_{j=1}^{k} M_j \bar{q} Y^{k-j} q.$$  \hspace{1cm} (3.4)

For simplicity we set the coefficients in (3.4) to one. These coefficients are calculable and relevant for a more detailed understanding of duality. One can check using (3.3) that $W_{mag}$ preserves the $R$ symmetry $U(1)_R$. The case $k = 1$ corresponds to the duality of [11], since $X, Y$ are then massive and can be integrated out. For $k = 2$ we recover the case described\hspace{1cm} (3.4)

$^3$ Again, we assume $k < kN_f - N_c$. 

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Below we shall see that theories with different $k$’s are connected via the flows (2.2), so in a sense (3.3) generalizes the previous results.

The ‘t Hooft anomalies of the dual theories match, and are given by:

$$SU(N_f)^3 \quad N_c d^{(3)}(N_f)$$

$$SU(N_f)^2 U(1)_R \quad -\frac{2}{k+1} \frac{N_c^2}{N_f} d^{(2)}(N_f)$$

$$SU(N_f)^2 U(1)_B \quad N_c d^{(2)}(N_f)$$

$$U(1)_R \quad -\frac{2}{k+1} (N_c^2 + 1)$$

$$U(1)_R^3 \quad \left( \frac{2}{k+1} - 1 \right)^3 + 1 \left( N_c^2 - 1 \right) - \frac{16}{(k+1)^3} \frac{N_c^4}{N_f^2}$$

$$U(1)_B^2 U(1)_R \quad -\frac{4}{k+1} N_c^2.$$

The discussion of operator matching proceeds again along the lines of [1], [13]: the mesons $\tilde{Q}^i X_{j+1} Q^i$, $j = 1, \cdots, k$ (1.3) are as is by now standard explicitly introduced in the dual theory as additional gauge singlet fields $(M_j)_i^\rho$, (3.3). Tr $X^j$, $j = 2, \cdots, k$ are mapped to Tr $Y^j$. The mapping of the baryons (1.5) between the two dual theories is:

$$B^{(n_1, n_2, \cdots, n_k)}_{\text{el}} \leftrightarrow B^{(m_1, m_2, \cdots, m_k)}_{\text{mag}}; \quad m_l = N_f - n_{k+1-l}; \quad l = 1, 2, \cdots, k \quad (3.6)$$

A non-trivial check of duality is the statement that the charge assignments (3.2), (3.3) necessary for ‘t Hooft anomaly matching are also compatible with the map (3.6).

4. Deformations.

There are many interesting deformations of the theories (1.1), (3.4) that provide further checks on the duality of the previous section. We will only discuss two here. The first involves giving a mass to one of the original, “electric”, quarks. Thus we add a term to the electric superpotential (1.1):

$$W_{\text{el}} = g_k \text{Tr} X^{k+1} + m \tilde{Q}_{N_f} Q^{N_f} \quad (4.1)$$

This gives a mass to $Q^{N_f}$, $\tilde{Q}_{N_f}$ and reduces the number of flavors in the IR by one unit keeping $N_c$ fixed: $(N_c, N_f) \to (N_c, N_f - 1)$. Since the theory with $(N_c, N_f - 1)$ is dual to one with $(kN_f - N_c - k, N_f - 1)$, we expect that in the dual “magnetic” theory (3.4),...
(4.1) reduces the number of colors by \( k \) units while reducing \( N_f \) by one. The magnetic superpotential is in this case:

\[
W_{\text{mag}} = g_k \text{Tr} \ Y^{k+1} + \sum_{j=1}^{k} M_j \tilde{q} Y^{k-j} q + m(M_1)^{N_f}_{N_f} \tag{4.2}
\]

Integrating out the massive fields we find that the vacuum satisfies:

\[
q_{N_f} Y^{l-1} \tilde{q}^{N_f} = -\delta_{l,k} m; \quad l = 1, \ldots, k \tag{4.3}
\]

which together with some additional conditions fixes the expectation values:

\[
\begin{align*}
\tilde{q}^{N_f}_\alpha &= \delta_{\alpha,1}; \\
q_{N_f}^{\alpha} &= \delta^{\alpha,k}; \\
Y^{\alpha}_{\beta} &= \begin{cases} \\
\delta^{\alpha}_{\beta+1} & \beta = 1, \ldots, k - 1 \\
0 & \text{otherwise}
\end{cases}
\tag{4.4}
\end{align*}
\]

The Higgs mechanism reduces the number of colors by \( k \) units, and takes \( N_f \rightarrow N_f - 1 \). It is not difficult to extend the discussion to perturbations of the form \( m_j \tilde{Q}_{N_f} X^{j-1} Q^{N_f} \). These reduce the number of colors in the dual, magnetic, theory by \( k + 1 - j \) and give rise (in general) to a superpotential for the quarks coming from the reduction of the adjoint field \( Y \).

The second deformation we will discuss involves perturbations of the superpotential (1.1) given by (2.2). Consider first, for simplicity, the case \( k = 2 \), where in the electric theory (1.1):

\[
W(X) = \text{Tr} \left( X^3 + \frac{m}{2} X^2 + \lambda X \right). \tag{4.5}
\]

Here \( X \) is a general Hermitean matrix, and \( \lambda \) is the Lagrange multiplier introduced in (2.2) to enforce the condition \( \text{Tr} \ X = 0 \). Vacuum solutions are diagonal matrices \( X \) with eigenvalues \( x_i \) satisfying a quadratic equation, \( 3x^2 + mx + \lambda = 0 \). There are two solutions \( x^\pm \) corresponding to the two minima of the bosonic potential \( V = |W'(X)|^2 \). There are generically \( N_c + 1 \) possible vacua labeled by \( r = 0, 1, \ldots, N_c \), the number of eigenvalues \( x_i \) which equal \( x^+ \) the other \( N_c - r \) having the value \( x^- \); vacua related by the \( \mathbb{Z}_2 \) operation \( r \rightarrow N_c - r \) are identical. The Lagrange multiplier \( \lambda \) is determined by setting \( \text{Tr} \ X = 0 \), i.e. \( rx^+(\lambda) + (N_c-r)x^-(\lambda) = 0 \). The gauge group is broken to:

\[
SU(N_c) \rightarrow SU(r) \times SU(N_c - r) \times U(1). \tag{4.6}
\]
For $r = 0$, $N_c$ $SU(N_c)$ remains unbroken. As discussed in section 2, in each vacuum the theory reduces to SQCD ($X$ is massive) and if, without loss of generality, we take $N_f > N_c$, all $N_c + 1$ vacua are stable.

Consider now the dual, magnetic, theory. A similar analysis seems to suggest that there are $2N_f - N_c + 1 (> N_c + 1)$ vacua with:

$$SU(2N_f - N_c) \rightarrow SU(l) \times SU(2N_f - N_c - l) \times U(1).$$

(4.7)

However, using the results of [1] we know that the $l$’th vacuum is stable iff $l \leq N_f$ and $2N_f - N_c - l \leq N_f$. Thus, $l = N_f - N_c, \ldots, N_f$ and there are again $N_c + 1$ vacua, as required by duality. The precise map between the vacua (4.6) and (4.7) is $l = N_f - r$, and the equivalence between the two is the duality of [11]. In particular, certain linear combinations of the two gauge singlets $M_1, M_2$ (3.3) (which were denoted by $M, N$ in [13]) become the meson fields in the two vacua needed for the duality of [11].

For $r = 0$, $N_c$ in (4.6) the $SU(N_c)$ gauge group remains unbroken. It is interesting that the duality described above takes the trivial electric vacuum $\langle X \rangle = 0$ to a magnetic vacuum with $\langle Y \rangle \neq 0$. In the appropriate vacuum of the magnetic theory, the gauge group is broken at tree level to:

$$SU(2N_f - N_c) \rightarrow SU(N_f) \times SU(N_f - N_c) \times U(1).$$

(4.8)

Denoting the dual quarks of the $SU(N_f)$ sector in (4.8) by $q, \tilde{q}$ and the singlet meson combination that couples to $q, \tilde{q}$ by $M$, we have in the $SU(N_f)$ theory the standard [11] superpotential $W = M\tilde{q}q$. The $M$ equation of motion sets $\tilde{q}q$ to zero. But from [3] we know that in this theory which has the same number of colors and flavors, there is a constraint on the quantum moduli space, $\det \tilde{q}q - B\tilde{B} = \Lambda^{2N_f}$, relating the mesons $\tilde{q}q$ and baryons, $B, \tilde{B}$. Since $\tilde{q}q = 0$, we have $B\tilde{B} = -\Lambda^{2N_f}$. The expectation value of $B$ breaks the $U(1)$ symmetry in (4.8) and again reduces the prediction of the duality of [13] to that of [11].

For $k > 2$ there is a much richer set of deformations of the superpotential (2.2) connecting the different dualities. Since the analysis is conceptually similar to the $k = 2$ case described above, we only sketch the structure here.

For generic $W$ of degree $k + 1$, there are many vacua found by solving the polynomial equation $W'(X) = 0$. As described above, ground states are labeled by the number of eigenvalues $i_l$ residing in the $l$’th minimum of the bosonic potential ($l = 1, \cdots k$). The
gauge group is broken as in (2.4). In the dual theory the situation is similar with $j_l$ eigenvalues in the $l'$th minimum, $\sum_l j_l = kN_f - N_c$, and:

$$SU(kN_f - N_c) \to SU(j_1) \times SU(j_2) \times \cdots \times SU(j_k) \times U(1)^{k-1}. \quad (4.9)$$

The different vacua of the two dual theories are mapped to each other by the duality map, $j_l = N_f - i_l$.

By fine tuning the coefficients $g_l$ in (2.2) one can make two or more roots of $W'$ coincide. This leads in general to:

$$W'(x) = \prod_i (x - a_i)^{n_i}; \quad \sum n_i = k. \quad (4.10)$$

The theory near $X = a_i$ has $W \sim (x - a_i)^{n_i+1}$. If $r_i$ eigenvalues of $\langle X \rangle$ are equal to $a_i$, the gauge group is broken to:

$$SU(N_c) \to \prod_i SU(r_i) \times U(1)^{k-1}; \quad \sum r_i = N_c. \quad (4.11)$$

In the magnetic theory, in the corresponding vacua the gauge group is broken as follows:

$$SU(kN_f - N_c) \to \prod_i SU(\tilde{r}_i) \times U(1)^{k-1}; \quad \sum \tilde{r}_i = kN_f - N_c. \quad (4.12)$$

It is easy to show using the fact that stable vacua of a theory with superpotential $W = X^{k+1}$ exist only for $N_f \geq N_c/k$ (see section 2), that there is a one to one correspondence of the vacua (4.11), (4.12) with $\tilde{r}_i = n_iN_f - r_i$. We see that the duality transformation (3.2), (3.3) with a certain $k$ gives rise after perturbing the superpotential as in (2.2) to products of theories dual under the same duality with smaller values of $k$. The perturbations (2.2) therefore connect the different dualities. The consistency of the resulting picture is further evidence for the duality of the previous section.

5. Comments.

1) At the self dual points of the duality transformations discussed above, $N_f = 2N_c/k$ many new operators with R charge two appear. Examples include $M_jM_{k+1-j}$, $j = 0, \cdots, \lfloor \frac{k+1}{2} \rfloor$ ($M_j$ are defined in (1.3); $M_0 \equiv 1$). It is likely [14] that these operators are actually exactly marginal in the IR conformal field theory and lead to manifolds of fixed points. The appearance of new marginal operators at self dual points seems to be a very
general phenomenon in four dimensional duality, and is reminiscent of similar phenomena in two dimensional theories, where at self dual points one usually encounters enhanced symmetries and new moduli.

The duality described in the previous sections should act in this case on the manifold of fixed points described by the superpotential:

\[ W = g_k \text{Tr} \ X^{k+1} + \frac{1}{2} \sum_{j=1}^{k} \lambda_j M_j M_{k+1-j} \]  

\( \lambda_j (= \lambda_{k+1-j}) \) are coordinates on the moduli space of IR fixed points. Duality presumably interchanges large and small \( \lambda_j \), and electric and magnetic variables. One can generalize the discussion of \[14\] to study some aspects of this duality, such as the appearance of the singlet mesons in the magnetic theory (3.3). Rewrite (5.1) as \((h_j = h_{k+1-j})\):

\[ W = g_k \text{Tr} \ X^{k+1} - \sum_{j=1}^{k} \left( \frac{h_j^2}{2\lambda_j} N_j N_{k+1-j} + h_j N_j M_{k+1-j} \right) \]  

in terms of auxiliary fields \((N_j)\), which become dynamical at large distances. In the limit \( \lambda_j \to 0 \) we see from (5.1), (5.2) that the theory approaches the electric theory described above (1.1). As \( \lambda_j \to \infty \) the mass of \( N_j \) goes to zero, and the theory approaches the magnetic theory (3.3), (3.4), with \( N_j \) playing the role of the gauge singlet mesons in (3.3). For generic \( \lambda_j \) the singlet mesons \( N_j \) are massive, the global symmetry (3.1) \( SU(N_f) \times SU(N_f) \) is broken to \( SU(N_f) \), and the anomaly matching does not require elementary “meson” chiral superfields. The full symmetry (3.1) is restored at \( \lambda_j = 0, \infty \).

One can also study the exactly marginal deformation induced by the operator corresponding to \( j = 0 \) in (5.1):

\[ W_{fl} = g_k \text{Tr} \ X^{k+1} + h_k \tilde{Q}_i X^k Q^i. \]  

For \( k = 1, \ g_k = 0 \) (5.3) describes the line of fixed points of finite \( \mathcal{N} = 2 \) supersymmetric theories with \( N_f = 2 N_c \) [8]. For \( k > 1, \ g_k = 0 \) one can think of (5.3) as describing the finite \( \mathcal{N} = 2 \) model with all but \( 2N_c/k \) flavors given masses and integrated out [14]. Thus, it is possible that the structure described in this paper is related to \( \mathcal{N} = 2 \) duality [8].

2) Our results suggest the following picture for the theory with \( W = 0 \). As \( N_f \) decreases from \( 2N_c \) the IR dimension of \( X \) becomes smaller, so that \( X^{k+1} \) become relevant for any \( k < N_c \), when \( N_f < N_0(k) \). In this regime, adding the superpotential (1.1) takes
the theory to the fixed point we have discussed here, which is distinct from the $W = 0$ one. Duality suggests that $N_0(k) > 2N_c/k$, so that at the self dual point $N_f = 2N_c/k$, Tr $X^{k+1}$ is strongly relevant. For $N_f > N_0(k)$ the $W = 0$ fixed point is unique, ignoring deformations and restrictions of the chiral ring. Duality implies a mirror image of this picture for $N_f < 2N_c/k$. Calculating $N_0(k)$ is tantamount to calculating the scaling dimension of $X$ in the $W = 0$ theory and would be an important clue to the structure of the theory.

3) For large $k$ the scaling dimension of $X$ at the fixed point governed by (1.1) is small and the dimensions of certain gauge invariant operators like Tr $X^2$ are not governed by their $R$ charges, due to unitarity bounds [15]. Consider the theory with no superpotential, $W = 0$. As discussed above, when $N_f$ decreases, the dimension of $X$ decreases. At a certain $N_f = N_2$, the dimension of Tr $X^2$ descends to one. The operator becomes a free field [13], and decouples from the dynamics. This remains the case for $N_f < N_2$ and clearly also when we turn on the superpotential (1.1). At some lower value, $N_f = N_3$ the same happens to Tr $X^3$, etc. By duality, these operators “recouple” at some lower values of $N_f < 2N_c/k$, which are more conveniently studied in the dual, magnetic theory. The role of the decoupled, free fields deserves further investigation. They should play an important role in a dual description of the theory with no superpotential, perhaps appearing as elementary fields in such a description.

Note added: Some related issues are discussed in a recent preprint [16].

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