Perfect state transfer of a qudit over underlying networks of group association schemes

M A Jafarizadeh\(^1\), R Sufiani\(^{1,2}\), S F Taghavi\(^1\) and E Barati\(^1\)

\(^1\) Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz 51664, Iran
\(^2\) Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran
\(^3\) Research Institute for Fundamental Sciences, Tabriz 51664, Iran
E-mail: jafarizadeh@tabrizu.ac.ir and sofiani@tabrizu.ac.ir

Received 3 November 2008
Accepted 10 March 2009
Published 6 April 2009

Abstract. As generalizations of results of Christandl et al (2004 Phys. Rev. Lett. 92 187902; 2005 Phys. Rev. A 71 032312) and Facer et al (2008 Phys. Rev. A 77 012334), Bernasconi et al (2008 0808.0510 [quant-ph]; 2008 0806.2074 [math.CO]) studied perfect state transfer (PST) between two particles in quantum networks modeled by a large class of cubelike graphs (e.g. the hypercube) which are the Cayley graphs of the elementary Abelian group \(\mathbb{Z}^2\). In Jafarizadeh and Sufiani (2008 Phys. Rev. A 77 022315) Jafarizadeh et al (2008 J. Phys. A: Math. Theor. 41 475302) respectively, PST of a qubit over distance regular spin networks and optimal state transfer (ST) of a \(d\)-level quantum state (qudit) over pseudo-distance regular networks were discussed, where the networks considered there were not, in general, related to a certain finite group. In this paper, PST of a qudit over antipodes of more general networks, called underlying networks of association schemes, is investigated. In particular, we consider the underlying networks of group association schemes in order to employ the group properties (such as irreducible characters) and use the algebraic structure of these networks (such as Bose–Mesner algebra) in order to give an explicit analytical formula for coupling constants in the Hamiltonians so that the state of a particular qudit initially encoded on one site will perfectly evolve to the opposite site without any dynamical control. It is shown that the only necessary condition in order for PST over these networks to be achieved is that the centers
Perfect state transfer of a qudit over underlying networks of group association schemes

of the corresponding groups be non-trivial. Therefore, PST over the underlying networks of the group association schemes over all the groups with non-trivial centers such as the Abelian groups, the dihedral group $D_{2n}$ with even $n$, the Clifford group $CL(n)$ and all of the $p$-groups can be achieved.

**Keywords:** spin chains, ladders and planes (theory), exact results, new applications of statistical mechanics, quantum transport in one dimension

**ArXiv ePrint:** 0901.4510

---

1. Introduction

The quantum communication between two parts of a physical unit, e.g. a qubit, is a crucial ingredient for many quantum information processing protocols [8]. There are various physical systems that can serve as quantum channels, one of them being a quantum spin system. In view of applications like the communication between registers in quantum devices, the study of the natural evolution of permanently coupled spin networks has become increasingly important. A special case of interest consists of homogeneous networks of particles coupled by constant and fixed (nearest-neighbor) interactions. An important feature of these networks is the possibility of faithfully transferring a qubit between specific particles without tuning the couplings or altering the network topology. This phenomenon is usually called perfect state transfer (PST). Quantum communication over short distances through a spin chain, in which adjacent qubits are coupled by equal strengths, has been studied in detail and an expression for the fidelity of quantum state transfer has been obtained [9,10]. Similarly, in [11], near-perfect state transfer was...
Perfect state transfer of a qudit over underlying networks of group association schemes

achieved for uniform couplings providing a spatially varying magnetic field was introduced. After the work of Bose [9], in which the potentialities of the so-called spin chains have been shown, several strategies were proposed to increase the transmission fidelity [12] and even to achieve, under appropriate conditions, perfect state transfer [1, 2], [13]–[16], [3]. Recently, Bernasconi et al [4] have studied PST between two particles in quantum networks modeled by a large class of cubelike graphs. Since quantum networks (and communication networks in general) are naturally associated with undirected graphs, there is a growing amount of literature on the relation between graph-theoretic properties and properties that allow PST [5]. In [6], the so-called distance regular graphs have been considered as spin networks (in the sense that, with each vertex of a distance regular graph, a qubit or a spin-1/2 particle was associated) and PST over them has been investigated. In [17], state transfer over spin chains of arbitrary spin has been discussed so that an arbitrary unknown qudit can be transferred through a chain with rather good fidelity by the natural dynamics of the chain. In the recent paper [7], the authors have investigated optimal state transfer (ST) of a d-level quantum state (qudit) over pseudo-distance regular networks, where it was shown that only for pseudo-distance regular networks with some certain symmetry (mirror symmetry) in the corresponding intersection numbers (consequently their QD parameters) will PST between antipodes of the networks be achieved.

In the present paper we will consider the more general graphs that are the underlying graphs of group association schemes [18] and give necessary and sufficient conditions for PST of a qudit state in quantum networks modeled by a large class of graphs with group structure. An association scheme is a combinatorial object with useful algebraic properties (see [19] for an accessible introduction). The theory of association schemes has its origin in the design of statistical experiments. This mathematical object has very useful algebraic properties which enables one to employ them in algorithmic applications such as the shifted quadratic character problem [20] (in this problem, the association scheme is the Paley scheme which corresponds to a strongly regular graph, the Paley graph). A d-class symmetric association scheme (d is called the diameter of the scheme) has d+1 symmetric relations $R_i$ which satisfy some particular conditions. Each non-diagonal relation $R_i$ can be thought of as the network $(V, R_i)$, where we will refer to it as the underlying graph of the association scheme $(V$ is the vertex set of the association scheme which is considered as the vertex set of the underlying graph). In fact, an association scheme partitions the relationships between pairs of vertices into classes, so that for an arbitrary chosen vertex (referred to as a reference vertex), one can stratify the vertices into distinct classes or stratas according to its relationships with all of the other vertices. Moreover, this stratification is independent of the choice of reference vertex. In the problem of transfer of an arbitrary qubit state which is considered in this work, we are given N spin-1/2 particles as the corresponding vertex set; then, for a given particle associated with a vertex of the underlying graph, the strength of its interaction with other $N-1$ particles is determined according to its relationship with the other vertices defined via the relations of the corresponding association scheme. In [21, 22], algebraic properties of association schemes have been employed in order to evaluate the effective resistances in finite resistor networks, where the relations of the corresponding schemes define the kinds of resistances or conductances between any two nodes of the networks. In [23], a dynamical system with d different couplings has been investigated in which the relationships between

doi:10.1088/1742-5468/2009/04/P04004

3
the dynamical elements (couplings) are given by the relations between the vertexes according to the corresponding association schemes. Indeed, according to the relations \( R_i \), the so-called adjacency matrices \( A_i \) are defined which form a commutative algebra known as Bose–Mesner (BM) algebra. One of the important preferences of association schemes is their useful algebraic structures that enable one to find the spectrum of the adjacency matrices relatively easy; then, for different physical purposes, one can define particular spin Hamiltonians which can be written in terms of the adjacency matrices of an association scheme so that the corresponding spectra can be determined easily. Group association schemes are particular schemes in which the vertices belong to a finite group and the relations are defined based on the conjugacy classes of the corresponding group. Working with these schemes is relatively easy, since almost all of the needed information about the scheme, for instance the so-called eigenvalue matrices \( P \) and \( Q \) associated with the scheme, can be obtained via the character tables of the corresponding groups. We will use the technique of the stratification of the underlying networks of group association schemes (this technique can be used even for some graphs which are not the underlying graphs of association schemes \([24,25]\), not only for the purpose of state transfer, but also in investigating the continuous time quantum walk (CTQW) over the undirected graphs \([26]–[30]\) and employ their algebraic structures in order to calculate the transition probability amplitude between an arbitrary chosen reference vertex (the identity element of the corresponding group is associated with this vertex) and the antipode vertex (any element of the center of the group which forms one element conjugacy class or one element strata, can be considered as the corresponding antipode vertex) and optimize it in order to attain PST between them (transmission fidelity attains a maximum value of 1). As we will see, the preference of this employment is that we are able to give analytical formulae for coupling strengths in particular Hamiltonians imposed to these graphs, in terms of the irreducible characters of the corresponding group, in order for PST to be achieved. In fact, we show that for such networks in which the group \( G \) has non-trivial center (groups which have non-trivial elements commuting with all of the other group elements), an initial state encoded in one vertex of the network (referred to as the reference vertex or reference site) can be transferred perfectly to the site labeled by any element of the center of the corresponding group. We give an explicit analytical formula for the coupling strengths in terms of the parameters of the corresponding group association scheme such as the diameter \( d \) of the scheme and the so-called first and second eigenvalue matrices \( P \) and \( Q \) (which are given in terms of the irreducible characters of the group). By calculating the optimal transmission fidelity \( F_{\text{opt}} \), it is shown that for all such networks the perfect transfer \( (F_{\text{opt}} = 1) \) can be achieved. We illustrate the method for perfect transfer of a qubit over the considered networks in detail and then generalize it to the PST of a qudit over the same networks.

It should be noticed that the interesting point of this work is that, for the underlying graphs of group schemes considered in this paper, the vertices of the graphs are the elements of particular finite groups, where the interactions between qubits associated with vertices are governed by the relationship between the vertices defined by the adjacency matrices \( A_i \) of the scheme; therefore, according to different Hamiltonians (different kind of association schemes and consequently different kinds of relations or interactions) imposed on a given vertex set, one can transfer a given state from a chosen vertex to different vertices. For instance, consider a system of 8 qubits, each of which has been located at

doi:10.1088/1742-5468/2009/04/P04004
Perfect state transfer of a qudit over underlying networks of group association schemes

a corner of a hypercube; by imposing the interactions between these 8 vertices according to the relations of the group scheme over $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and taking some coupling strengths between vertices equal to each other, one can obtain a sub scheme which is the same as the Hamming scheme $H(2, 3)$, and transfer the state of an arbitrary qubit initially prepared at the vertex labeled by $|1\rangle \equiv |000\rangle$ to the antipode vertex labeled by $|8\rangle \equiv |111\rangle$ perfectly. However, by imposing the other relations defined by the other group scheme with 8 vertices, for instance the scheme over the dihedral group $D_8$, one can transfer the same state initialized at the vertex labeled by $|e\rangle \equiv |000\rangle$ to the vertex labeled by $|a^2\rangle \equiv |011\rangle$ in the same graph. In other words, for a given finite set of vertices, one can associate different relationships between the vertices by choosing different group association schemes in order to transfer a given state from a chosen vertex to different vertices.

The organization of this paper is as follows. In section 2, we recall some materials about association schemes, particularly the group association schemes, their underlying networks and the corresponding stratifications. Section 3 is devoted to the perfect transfer of a qubit over the corresponding networks, where an analytical formula for a suitable set of coupling constants in particular spin Hamiltonians is given. In section 4, we generalize the method to the perfect transfer of a qudit over the same networks. Section 5 contains two examples of the underlying networks of group association schemes for which the PST is achieved. The paper ends with a brief conclusion.

2. Underlying networks of association schemes

In this section, we review some preliminary tools about the particular networks which are considered through this paper. For material not covered in this section, as well as more detailed information about association schemes and their underlying networks, please refer to [18] and [26].

Definition 1. Assume that $V$ and $E$ are vertex and edge sets of a regular resistor network, respectively (each edge has a certain conductance). Then, the relations $\{R_i\}_{0 \leq i \leq d}$ on $V \times V$ satisfying the following conditions:

(i) $\{R_i\}_{0 \leq i \leq d}$ is a partition of $V \times V$,
(ii) $R_0 = \{(\alpha, \alpha): \alpha \in V\}$,
(iii) $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = \{(\beta, \alpha): (\alpha, \beta) \in R_i\}$,
(iv) For $(\alpha, \beta) \in R_k$, the number $p_{i,j}^k = |\{\gamma \in V: (\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j\}|$ does not depend on $(\alpha, \beta)$ but only on $i, j$ and $k$,

define a symmetric association scheme of class $d$ on $V$ which is denoted by $Y = (V, \{R_i\}_{0 \leq i \leq d})$. Furthermore, if we have $p_{i,j}^k = p_{j,i}^k$ for all $i, j, k = 0, 2, \ldots, d$, then $Y$ is called commutative.

For examples of association schemes, consider a cube known as the Hamming scheme $H(3, 2)$, in which $V$ (the vertex set) is the set of 3-tuples with entries in $F_2 = \{0, 1\}$. Two vertices are connected if and only if they differ by exactly one entry (see figure 1(a)). The distance between vertices, i.e. the length of the shortest edge path connecting them,
Figure 1. (a) A vertex set \{1, 2, \ldots, 8\} with interactions according to the relations of Hamming scheme \(H(3, 2)\), where with only one non-zero coupling strengths \(J_1\), PST between the vertex (000) and the vertex (111) is achieved. (b) The same vertex set \{1, 2, \ldots, 8\} with interactions according to the relations of the group association scheme over dihedral group \(D_8\), where with only two non-zero coupling strengths \(J_1\) and \(J_3\), PST from the vertex \(e\) to the vertex \(a^2\) is achieved; the solid lines denote the interaction coupling \(J_1\) and the dashed lines denote the coupling strength \(J_3\).
will then indicate which relation they are contained in. For example, if \( x = (0, 0, 1) \), 
\( y = (0, 1, 1) \) and \( z = (1, 0, 1) \), then \( (x, y) \in R_1 \), \( (x, z) \in R_1 \) and \( (y, z) \in R_2 \).

Let \( Y = (V, \{R_i\}_{0 \leq i \leq d}) \) be a commutative symmetric association scheme of class \( d \). Then the matrices \( A_0, A_1, \ldots, A_d \) defined by

\[
(A_i)_{\alpha,\beta} = \begin{cases} 
1 & \text{if } (\alpha, \beta) \in R_i, \\
0 & \text{otherwise } (\alpha, \beta) \in V
\end{cases}
\]  

(2.1)

are adjacency matrices of \( Y \) such that

\[
A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k.
\]  

(2.2)

From (2.2), it is seen that the adjacency matrices \( A_0, A_1, \ldots, A_d \) form a basis for a commutative algebra \( A \) known as the Bose–Mesner algebra of \( Y \). This algebra has a second basis \( E_0, \ldots, E_d \) (known as primitive idempotents of \( Y \)) so that

\[
E_0 = \frac{1}{N} J, \quad E_i E_j = \delta_{ij} E_i, \quad \sum_{i=0}^{d} E_i = I,
\]  

(2.3)

where \( N := |V| \) is the number of vertices (sites) and \( J \) is the all-one matrix in \( A \). Let \( P \) and \( Q \) be the matrices relating the two bases for \( A \):

\[
A_i = \sum_{j=0}^{d} P_{ji} E_j, \quad 0 \leq j \leq d,
\]  

\[
E_i = \frac{1}{N} \sum_{j=0}^{d} Q_{ji} A_j, \quad 0 \leq j \leq d.
\]  

(2.4)

Then clearly

\[
PQ = QP = NI.
\]  

(2.5)

It also follows that

\[
A_i E_j = P_{ji} E_j,
\]  

(2.6)

which shows that the \( P_{ji} \) (resp. \( Q_{ji} \)) is the \( j \)th eigenvalue (resp. the \( j \)th dual eigenvalue) of \( A_i \) (resp. \( E_i \)) and that the columns of \( E_j \) are the corresponding eigenvectors. Thus, \( m_i = \text{rank}(E_i) \) is the multiplicity of the eigenvalue \( P_{ji} \) of \( A_i \) (provided that \( P_{ji} \neq P_{jk} \) for \( k \neq i \)). We see that \( m_0 = 1 \), \( \sum_i m_i = N \), and \( m_i = \text{trace} E_i = N(E_i)_{jj} \) (indeed, \( E_i \) only has eigenvalues 0 and 1, so \( \text{rank}(E_k) \) equals the sum of the eigenvalues).

Clearly, each non-diagonal (symmetric) relation \( R_i \) of an association scheme \( Y = (V, \{R_i\}_{0 \leq i \leq d}) \) can be thought of as the network \( (V, R_i) \) on \( V \), where we will call it the underlying network of association scheme \( Y \). In other words, the underlying network \( \Gamma = (V, R_1) \) of an association scheme is an undirected connected network, where the set \( V \) and \( R_1 \) consist of its vertices and edges, respectively. Obviously replacing \( R_1 \) with one of the other relations such as \( R_i \), for \( i \neq 0, 1 \), will also give us an underlying network \( \Gamma = (V, R_i) \) (not necessarily a connected network) with the same set of vertices but a new set of edges \( R_i \).
Perfect state transfer of a qudit over underlying networks of group association schemes

2.1. Stratification

For an underlying network \( \Gamma \), let \( W = \mathbb{C}^n \) (with \( n = |V| \)) be the vector space over \( \mathbb{C} \) consisting of column vectors whose coordinates are indexed by vertex set \( V \) of \( \Gamma \), and whose entries are in \( \mathbb{C} \). For all \( \beta \in V \), let \( |\beta\rangle \) denote the element of \( W \) with a 1 in the \( \beta \) coordinate and 0 in all other coordinates. We observe that \( \{ |\beta\rangle | \beta \in V \} \) is an orthonormal basis for \( W \), but in this basis, \( W \) is reducible and can be reduced to irreducible \( A \)-submodules \( W_i \), \( i = 0, 1, \ldots, d \) of the Bose–Mesner algebra \( A \) (by a \( A \)-submodule we mean a subspace \( W_i \) of \( W \) such that \( AW_i \subseteq W_i \)), i.e.

\[
W = W_0 \oplus W_1 \oplus \cdots \oplus W_d, \tag{2.7}
\]

where \( d \) is the diameter of the corresponding association scheme (for more details see [26]). If we define \( \Gamma_i(o) = \{ \beta \in V : (o, \beta) \in R_i \} \) for an arbitrary chosen vertex \( o \in V \) (called the reference vertex), then the vertex set \( V \) can be written as a disjoint union of \( \Gamma_i(o) \), i.e.

\[
V = \bigcup_{i=0}^{d} \Gamma_i(o). \tag{2.8}
\]

In fact, the relation (2.8) stratifies the network into a disjoint union of strata (associate classes) \( \Gamma_i(o) \). With each stratum \( \Gamma_i(o) \) one can associate a unit vector \( |\phi_i\rangle \) in \( W \) (called the unit vector of the \( i \)th stratum) defined by

\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle, \tag{2.9}
\]

where \( |\alpha\rangle \) denotes the eigenket of the \( \alpha \)th vertex at the associate class \( \Gamma_i(o) \) and \( \kappa_i = |\Gamma_i(o)| \) is called the \( i \)th valency of the network. For \( 0 \leq i \leq d \), the unit vectors \( |\phi_i\rangle \) of equation (2.9) form a basis for an irreducible submodule of \( W \) with maximal dimension denoted by \( W_0 \). Since \( \{|\phi_i\rangle\}_{i=0}^d \) becomes a complete orthonormal basis of \( W_0 \), we often write

\[
W_0 = \sum_{i=0}^{d} \oplus \mathbb{C}|\phi_i\rangle. \tag{2.10}
\]

Let \( A_i \) be the adjacency matrix of the underlying network \( \Gamma \). From the action of \( A_i \) on reference state \( |\phi_0\rangle \) (\( |\phi_0\rangle = |o\rangle \), with \( o \in V \) as reference vertex), we have

\[
A_i|\phi_0\rangle = \sum_{\beta \in \Gamma_i(o)} |\beta\rangle. \tag{2.11}
\]

Then by using (2.9) and (2.11), we obtain

\[
A_i|\phi_i\rangle = \sqrt{\kappa_i}|\phi_i\rangle. \tag{2.12}
\]
2.1.1. Group association schemes. Group association schemes are particular association schemes for which the vertex set contains elements of a finite group $G$ and the relations $R_i$ are defined by

$$R_i = \{(x, y) | yx^{-1} \in C_i \}, \quad i = 0, 1, \ldots, d,$$

(2.13)

where $C_0 = \{e\}, C_1, \ldots, C_d$ are the conjugacy classes of $G$. Then, $(G; \{R_i\}_{0 \leq i \leq d})$ becomes a commutative association scheme and it is called the group association scheme of the finite group $G$. The $i$th adjacency matrix $A_i$ is defined as

$$A_i = \bar{C}_i := \sum_{g \in C_i} g,$$

(2.14)

where $g$ is considered in the regular representation of the group. Then, we can write

$$\bar{C}_i \bar{C}_j = \sum_{k=0}^{d} p_{ij}^k \bar{C}_k,$$

(2.15)

so that the intersection numbers $p_{ij}^k, i, j, k = 0, 1, \ldots, d$ are given by [31]

$$p_{ij}^k = \frac{|C_i| |C_j|}{|G|} \sum_{\chi} \chi(\alpha_i) \chi(\alpha_j) \chi(\alpha_k),$$

(2.16)

where the sum is taken over all the irreducible characters of $G$. Therefore, the idempotents $E_0, \ldots, E_d$ of the group association scheme are the projection operators as

$$E_k = \frac{\chi_k(1)}{|G|} \sum_{\alpha \in G} \chi_k(\alpha^{-1}) \chi(\alpha) \chi(1).$$

(2.17)

Thus eigenvalues of adjacency matrices $A_k$ and idempotents $E_k$ are

$$P_{ik} = \frac{\kappa_k \chi_i(\alpha_k)}{d_i}, \quad Q_{ik} = \frac{d_k \chi_k(\alpha_i)}{d_i},$$

(2.18)

respectively, where $d_j = \chi_j(1)$ is the dimension of the irreducible character $\chi_j$ and $\kappa_k \equiv |C_k|$ is the $k$th valency of the graph.

It should be noticed that, in the cases that some of the conjugacy classes are not real and hence some of the irreducible representations are complex, we encounter directed underlying graphs or non-symmetric association schemes. In these cases, one can form a symmetric association scheme out of a given non-symmetric association scheme (see appendix A of [26]) so that the transition probability amplitude between the vertices $\phi_0 \in C_0$ and $\beta \in C_k$ at time $t$ is given by

$$\langle \beta | \phi_0(t) \rangle = \begin{cases} 
\frac{1}{|G|} \sum_i d_i e^{-i\kappa_1 \chi_i(\alpha_1) t/d_i} \chi_i(\beta) & \text{for real representations}, \\
\frac{1}{|G|} \sum_i d_i e^{i\kappa_1 (\chi_i(\alpha_1) + \chi_i(\alpha_1)) t/d_i} (\chi_i(\beta) + \chi_i(\beta)) & \text{for complex representations},
\end{cases}$$

(2.19)

for more details, see [26].

doi:10.1088/1742-5468/2009/04/P04004
In section 3, we investigate state transfer over undirected underlying graphs of group association schemes, where we will deal with particular Hamiltonians which are defined in terms of the adjacency matrices of the corresponding association scheme; in other words, we will consider the underlying graphs of group association schemes as spin networks in which the interactions between spin states associated with the vertices of the graphs are determined by the relations \( R_i \) of the association scheme defined as in equation (2.13) (or equivalently by the adjacency matrices \( A_i \) defined as in (2.14)).

As regards the above arguments, in the underlying graphs of association schemes, the interactions between qubits associated with vertices are governed by the relationship between the vertices defined by the adjacency matrices \( A_i \) of the scheme; therefore, according to different Hamiltonians (different kinds of association schemes and consequently different kinds of relations or interactions) imposed to a given vertex set, one can transfer a given state from a chosen vertex to different vertices. For instance, consider a system of 8 qubits, each of which is located at a corner of a hypercube; by imposing the interactions between these 8 vertices according to the relations of the group scheme over \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), and taking some coupling strengths between vertices equal to each other, one can obtain a subscheme which is the same as the Hamming scheme \( H(2,3) \), and transfer state of an arbitrary qubit initially prepared at the vertex labeled by \(|1\rangle \equiv |000\rangle \) to the antipode vertex labeled by \(|8\rangle \equiv |111\rangle \) perfectly (see figure 1(a)); however, as we will see in section 5, by imposing the relations defined by the group scheme over the dihedral group \( D_8 \), even by considering only two types of relations (the coupling strengths associated with other relations are taken to be zero) (see, for example, 5.1 for details), we can transfer the same state initialized at the vertex labeled by \(|e\rangle \equiv |000\rangle \) to the vertex labeled by \(|a^2\rangle \equiv |011\rangle \) in the same graph (see figure 1(b)). In other words, for a given finite set of vertices, one can associate different relationships between the vertices by choosing different group association schemes in order to transfer a given state from a chosen vertex to different vertices.

3. Perfect state transfer of a qubit over antipodes of underlying networks of association schemes

In order to set the scene, let us first recall briefly the algorithm of quantum state transfer over a general quantum network. For more details see [1,2,6,7].

A general finite quantum network is defined to be a simple, connected, finite graph \( \Gamma = (V,E) \), where \( V \) denotes the finite set of its vertices and \( E \) the set of its edges. A two-dimensional quantum system associated with such a graph is defined by attaching a 2-level (spin-1/2) particle to each vertex of the graph so that with each vertex \( i \in V \) one can associate a Hilbert space \( \mathcal{H}_i \simeq \mathbb{C}^2 \). The Hilbert space associated with \( \Gamma \) is then given by

\[
\mathcal{H} = \bigotimes_{i \in V} \mathcal{H}_i = (\mathbb{C}^2)^{\otimes N},
\]

where \( N := |V| \) denotes the total number of vertices (sites) in \( \Gamma \). Now, suppose that we impose a particular spin Hamiltonian \( H \) (which governs the interactions between the spin-1/2 particles) to the graph so that the total z component of the spin, given by \( \sigma_{\text{tot}}^z = \sum_{i \in V} \sigma_i^z \), is conserved, i.e. \( [\sigma_{\text{tot}}^z, H] = 0 \). Hence the Hilbert space \( \mathcal{H} \) decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator \( \sigma_{\text{tot}}^z \). Then,
for the purpose of quantum state transfer, it suffices to restrict our attention to the N-dimensional eigenspace of $\sigma_z^\alpha$ corresponding to a spin configuration in which all the spins except one are up and one spin is down. A basis state for this eigenspace can hence be denoted by the $|j\rangle$, where $j$ is the vertex in $\Gamma$ at which the spin is down. Thus, $\{|j\rangle \mid j \in V\}$ denotes a complete set of orthonormal basis vectors spanning the single down subspace.

The process of transmitting a quantum state from site A to site B proceeds in two steps: initialization and evolution. First, a quantum state $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A \in \mathcal{H}_A$ (with $\alpha, \beta \in \mathcal{C}$ and $|\alpha|^2 + |\beta|^2 = 1$) to be transmitted is created. The state of the entire spin system after this step is given by

$$|\psi(t = 0)\rangle = |\psi_A0\cdots0_B\rangle = \alpha|0_A0\cdots0_B\rangle + \beta|1_A0\cdots0_B\rangle = \alpha|0\rangle + \beta|A\rangle,$$

(3.2)

where $|0\rangle := |0_A0\cdots0_B\rangle$ corresponds to the configuration of all spins up (this state is a zero-energy eigenstate of the considered Hamiltonian $H$). Then, the network couplings are switched on and the whole system is allowed to evolve under $U(t) = e^{-iHt}$ for a fixed time interval, say $t_0$. The final state becomes

$$|\psi(t_0)\rangle = \alpha|0\rangle + \beta \sum_{j=1}^N f_{jA}(t_0)|j\rangle,$$

(3.3)

where $f_{jA}(t_0) := \langle j|e^{-iHt_0}|A\rangle$. Any site B is in a mixed state if $|f_{AB}(t_0)| < 1$, which also implies that the state transfer from site A to B is imperfect. In this paper, we will focus only on PST. This means that we consider the condition

$$|f_{AB}(t_0)| = 1 \quad \text{for some } 0 < t_0 < \infty,$$

(3.4)

which can be interpreted as the signature of perfect communication (or PST) between A and B in time $t_0$. The effect of the modulus in (3.4) is that the state at B, after transmission, will no longer be $|\psi\rangle$, but will be of the form

$$\alpha|0\rangle + e^{i\phi}\beta|1\rangle.$$

(3.5)

The phase factor $e^{i\phi}$ is not a problem because $\phi$ is independent of $\alpha$ and $\beta$ and will thus be a known quantity for the graph, which we can correct for with an appropriate phase gate (for more details see, for example, [1, 2, 15, 16]).

The model we will consider is an underlying network of a group association scheme over a finite group $G$ consisting of $N = |G|$ sites labeled by $\{1, 2, \ldots, N\}$ and diameter $d$. Then we stratify the network with respect to a chosen reference site, say $1 = e$ (the unit element of the group) and assume that the group $G$ has a non-trivial conjugacy class, say $C_m$, with cardinality 1 which contains the output site $N$ (i.e. $|\phi_m\rangle = |N\rangle$). At time $t = 0$, the qubit in the first (input) site of the network is prepared in the state $|\psi_m\rangle$. We wish to transfer the state to the $N$th (output) site of the network with unit efficiency after a well-defined period of time. The standard basis for an individual qubit (e.g. one can consider the state of a spin-1/2 particle) is chosen to be $\{|0\rangle = |\downarrow\rangle, |1\rangle = |\uparrow\rangle\}$, and we shall assume that initially all spins point ‘down’ along a prescribed $z$ axis, i.e. the network is in the state $|\psi\rangle = |0_A00\cdots00_B\rangle$. Then, we consider the dynamics of the system to be
Perfect state transfer of a qudit over underlying networks of group association schemes

governed by the quantum-mechanical Hamiltonian

\[ H = \sum_{l=0}^{d} J_l \sum_{(i,j) \in R_l} \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)}, \]

(3.6)

where \( \vec{\sigma} \) is a vector with familiar Pauli matrices \( \sigma_x, \sigma_y \) and \( \sigma_z \) as its components acting on the one-site Hilbert space \( \mathcal{H}_i \cong \mathbb{C}^2 \) associated with the site \( i \in V \), and \( J_l \) is the coupling strength between the reference site 1 and all of the sites belonging to the \( l \)-th stratum with respect to 1. As can be easily seen, the Hamiltonian (3.6) commutes with the total spin operator (conservation). That is, since we have \( [\sigma_{\text{tot}}, H] = 0 \), hence the total Hilbert space \( \mathcal{H} = \bigotimes_{i \in V} \mathcal{H}_i = (\mathbb{C}^2)^{\otimes N} \) decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator \( \sigma_{\text{tot}} \). Then, as was shown in [6], one can rewrite the Hamiltonian (3.6) in the single down subspace with basis vectors \( |i\rangle = |\uparrow \cdots \uparrow 0 \cdots \uparrow \rangle \), \( i = 1, 2, \ldots, N \) as follows:

\[ H = 2 \sum_{i=0}^{d} J_i A_i + \frac{N - 4}{2} \sum_{i=0}^{d} \kappa_i J_i I. \]

(3.7)

Then, by using (2.4), we have

\[ e^{-iHt} = e^{-it(N-4)/2} \sum_{i=0}^{d} e^{2it\sum_{j=0}^{d} J_j P_{kj}} E_k. \]

Now, let the \( m \)-th stratum of the network contain only one vertex. Then one can write

\[ \langle \phi_m | e^{-iHt} | \phi_0 \rangle = \sum_{k=0}^{d} e^{-i\sum_{j=0}^{d} J_j P_{kj}} \langle \phi_m | E_k | \phi_0 \rangle = \frac{1}{|G|} \sum_{k=0}^{d} e^{-i\sum_{j=0}^{d} J_j P_{kj}} Q_{mk} = \frac{1}{|G|} \sum_{k=0}^{d} e^{-i\sum_{j=0}^{d} (J_j \kappa_j / d_k) \chi_k(\alpha_k) \chi_k(\alpha_m)} d_k \chi_k(\alpha_m). \]

(3.8)

In order that the optimal state transfer be achieved, the above probability amplitude must take its maximum value. By using (A.1), we obtain

\[ \max \left| \langle \phi_m | e^{-iHt} | \phi_0 \rangle \right|^2 = \frac{1}{|G|^2} \max \left| \sum_k |Q_{mk}| e^{-i\sum_{j=0}^{d} J_j P_{k+\epsilon_k \pi} + i\epsilon_k \pi} \right|^2, \]

where \( \epsilon_k \) is 0 or 1, depending on the sign of \( Q_{mk} \) (the sign of \( \chi_k(\alpha_m) \)). The above maximum is attained if we have

\[ -t \sum_{j} J_j P_{kj} + \epsilon_k \pi = \phi + 2l \pi \rightarrow (PJ)_k = \frac{\epsilon_k \pi - \phi - 2l \pi}{t_0}, \quad l \in \mathbb{Z} \]

(3.9)

(\( \phi \) is a constant phase). Then, by using (2.5), (2.18) and (3.9), the optimal coupling constants (for which the optimal state transfer is achieved) are obtained as

\[ J_k = \frac{1}{|G|} \sum_{i=0}^{d} \frac{\epsilon_i \pi - \phi - 2l \pi}{t_0} Q_{ki} = \frac{1}{|G|} \sum_{i=0}^{d} \frac{\epsilon_i \pi - \phi - 2l \pi}{t_0} d_i \chi_i(\alpha_k), \]

(3.10)

doi:10.1088/1742-5468/2009/04/P04004
Perfect state transfer of a qudit over underlying networks of group association schemes

so that the optimal fidelity is given by

$$F_{\text{opt.}} = \max \langle \phi_m | e^{-iHt} | \phi_0 \rangle = \frac{1}{|G|} \sum_{i=0}^{d} |Q_{mi}| = \frac{1}{|G|} \sum_{i=0}^{d} d_i |\bar{\chi}_i(\alpha_m)|. \quad (3.11)$$

From the fact that $\alpha_m \in C_m$ belongs to the center of the group (and so commutes with all elements of the group), Schur’s lemma implies that the irreducible representation of $\alpha_m$ is equal to $\mu 1$ for some $\mu \in C$. Now, let $a$ be the order of $\alpha_m \in G$ ($\alpha_m^a = 1$). Then, clearly $\mu$ must be the $a$th root of unity, i.e. we have $\mu = e^{2\pi i/a}$ and so $|\mu| = 1$. Then, we will have $|\chi_k(\alpha_m)| = |\mu| d_k = d_k$ and the optimal fidelity attains

$$F_{\text{opt.}} = \frac{1}{|G|} \sum_{k=0}^{d} d_k^2 = 1, \quad (3.12)$$

where we have used the well-known identity $\sum_{k=0}^{d} d_k^2 = |G|$ from group theory. The above result indicates that all finite groups with non-trivial center can allow state transfer with optimal fidelity 1, i.e. perfect state transfer.

4. Generalization to perfect state transfer of a $D$-level quantum state

In the $D^N$-dimensional Hilbert space associated with a system of $N$, $D$-level quantum states ($\mathcal{H} \simeq (C^D)^\otimes N$), the Hamiltonian (3.6) can be generalized as

$$H = \sum_{l=0}^d J_l \sum_{(i,j) \in R_l} \vec{\lambda}_i \cdot \vec{\lambda}_j, \quad (4.1)$$

where $\vec{\lambda}_i$ is a $D^2 - 1$-dimensional vector with generators of $SU(D)$ as its components acting on the one-site Hilbert space $\mathcal{H}_i \simeq C^D$.

Let us denote a state in which the $i$th site has been excited to the level $\nu$ by $|\nu_i \rangle \equiv |0 \cdots 0 \nu \alpha 0 \cdots 0\rangle$. Then, the Hamiltonian $H$ can be diagonalized in each subspace $S^{(\nu)}$ spanned by the vectors $|\nu_i \rangle$, $i = 1, \ldots, N$, for $\nu = 1, \ldots, D - 1$ (for more details see [7]). If we call the states with only one site excited as one-particle states and the subspace spanned by these vectors comprise the one-particle sector of the full Hilbert space, then the whole one-particle subspace $S$ can be written as

$$S = S^{(1)} \oplus S^{(2)} \oplus \cdots \oplus S^{(D-1)}.$$ 

In other words, in $D^N$-dimensional Hilbert space $\mathcal{H}$, we deal with $D - 1$ one-particle subspaces (recall that each of these subspaces has dimension $N$). In the case of a system of $N$ qubits ($D = 2$), we have only one a one-particle subspace of dimension $N$.

As was shown in [6,7], the Hamiltonian (4.1) restricted to each one-particle subspace $S^{(\nu)}$, for $\nu = 1, 2, \ldots, d - 1$, can be written in terms of the adjacency matrices $A_l$ as

$$H = 2 \sum_{l=0}^d J_l A_l + \frac{(N-2)D - 4N}{2D} \sum_{l=0}^d J_l \kappa_l 1. \quad (4.2)$$

Then the same procedure can be applied in order to obtain the optimal fidelity. Equation (4.2) is similar to equation (3.7) apart from the non-important second term.
Perfect state transfer of a qudit over underlying networks of group association schemes

which is a multiple of the identity. Therefore, maximizing the probability amplitude 
\[ \langle \phi_m | e^{-iHt} | \phi_0 \rangle \] gives the same constraints on coupling constants \( J_i \) and leads to the same optimal fidelity (3.12).

5. Examples

In this section, we consider two examples of group association schemes and investigate PST over them.

5.1. The dihedral group \( D_{2n} \)

The dihedral group \( G = D_{2n} \) is generated by two generators \( a \) and \( b \) with the following relations:
\[
D_{2n} = \langle a, b : a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle.
\]
(5.1)

We consider the case of even \( n = 2m \); the case of odd \( n \) can be considered similarly. The dihedral group \( D_{2n} \) with even \( n = 2m \) has \( m+3 \) conjugacy class so that \( C_m = \{a^m\} \). For even \( n = 2m \), the \( m+3 \) conjugacy classes are given by
\[
C_0 = \{1\}, \quad C_r = \{a^r, a^{-r}\}; \quad 1 \leq r \leq m-1,
\]
\[
C_m = \{a^m\}, \quad C_{m+1} = \{a^2j, b; j = 0, \ldots, m-1\},
\]
\[
C_{m+2} = \{a^{2j+1}b; j = 0, \ldots, m-1\}.
\]

The character table is given by

| \( D_{2n} \) | \( e \) | \( a^m \) | \( a^r (1 \leq r \leq m-1) \) | \( b \) | \( ab \) |
|-----------|-----|------|--------------------|-----|-------|
| \( \chi_0 \) | 1   | 1    | 1                  | 1   | 1     |
| \( \chi_1 \) | 1   | 1    | 1                  | -1  | -1    |
| \( \chi_2 \) | 1   | (-1)^m | (-1)^r           | 1   | -1    |
| \( \chi_3 \) | 1   | (-1)^m | (-1)^r           | -1  | 1     |
| \( \psi_j (1 \leq j \leq m-1) \) | 2   | 2(-1)^j | 2 \cos(2\pi jr/n) | 0   | 0     |

Then, by using the result (3.11) we obtain
\[
F_{\text{opt}} = \frac{1}{|G|} \sum_{k=0}^{m+2} d_k |\bar{\chi}_k(\alpha_m)| = \frac{1}{4m} \{4 + 4(m-1)\} = 1.
\]

From (3.10), one can evaluate the optimal coupling constants \( J_i \). For example, for \( n = 4 \) we obtain
\[
J_0 = J_2 = J_4 = 0, \quad J_1 = \frac{\pi}{2t_0}, \quad J_3 = \frac{2\pi}{t_0},
\]
where the above values are obtained by choosing \( \phi = \pi \), \( l_0 = 3 \), \( l_1 = -1 \), \( l_2 = -l_3 = 2 \) and \( l_4 = 0 \) in equation (3.10). It is seen that, by choosing \( \phi = \pi \), all coupling constants become zero except for \( J_1 \) and \( J_3 \) (see figure 1(b)).

doi:10.1088/1742-5468/2009/04/P04004
5.2. The cubic group $T_h$

One of the point groups with high or polyhedral symmetry is the cubic group $T_h$ of order 24 with pyritohedral symmetry. This group is isomorphic to $A_4 \times C_2$, where $A_4$ is the alternating group of order 12. It is the symmetry of a cube with, on each face, a line segment dividing the face into two equal rectangles, such that the line segments of adjacent faces do not meet at the edge. The symmetries correspond to the even permutations of the body diagonals and the same combined with inversion. The group $T_h$ has 8 conjugacy classes with the following character table:

| $T_h$ | $C_0$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\chi_0$ | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| $\chi_1$ | 1     | -1    | -1    | 1     | 1     | -1    | -1    | -1    |
| $\chi_2$ | 1     | 1     | 1     | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1     |
| $\chi_3$ | 1     | 1     | 1     | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1     |
| $\chi_4$ | 1     | -1    | -1    | $\omega$ | $\omega^2$ | $-\omega$ | $-\omega^2$ | 1     |
| $\chi_5$ | 1     | 1     | -1    | $\omega^2$ | $\omega$ | $-\omega^2$ | $-\omega$ | 1     |
| $\chi_6$ | 3     | -1    | 3     | -1    | 0     | 0     | 0     | 0     |
| $\chi_7$ | 3     | -1    | -3    | 1     | 0     | 0     | 0     | 0     |

with $\omega := e^{2\pi i/3}$. By combining the classes $C_4$ and $C_5$ and denoting it by $\tilde{C}_4 = C_4 \cup C_5$, and $C_6$ with $C_7$ and denoting the new obtained class as $\tilde{C}_5$, one can obtain a symmetric association scheme with undirected underlying graph. Now, using the result (3.10) one can evaluate the optimal coupling constants $J_i$ for, $i = 0, 1, \ldots, 5$ as follows:

$$J_0 = J_1 = J_2 = J_4 = 0, \quad J_3 = J_5 = \frac{\pi}{2l_0},$$

where we have chosen $\phi = \pi/2$, $l_0 = -l_1 = 3$ and $l_2 = l_3 = l_4 = l_5 = 0$ in equation (3.10). By these choices, all coupling constants become zero except for $J_3$ and $J_5$ (see figure 2).

5.3. The Clifford group

The Clifford algebra with $n$ generator matrices $\gamma_1, \gamma_2, \ldots, \gamma_n$, obeys the following relations [32]:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I. \quad (5.2)$$

Thus, the $\gamma$’s have square 1 and anti-commute. The Clifford group denoted by $CL(n)$ has $2^{n+1}$ elements as

$$CL(n) = \{ \pm 1, \pm \gamma_i \ldots \gamma_j ; i_1 < \cdots < i_j ; j = 1, \ldots, n \},$$

where $i_r \in \{1, 2, \ldots, n\}$. We suppose $n > 2$ throughout. It is well known that [32], the center of $CL(n)$ denoted by $Z(CL(n))$, consists of $\{ \pm 1 \}$ if $n$ is even and $\{ \pm 1, \pm \gamma_1 \ldots \gamma_n \}$ if $n$ is odd. $CL(n)$ has $2^n$ one-dimensional representations, each real. In each such representation, $U(-1) = I$; any irreducible representation with dimension greater than 1 has $U(-1) = -I$.

For even $n$, the conjugacy classes are given by

$$C_0 = \{ 1 \}, \quad C_1 = \{ -1 \}, \quad C_2 = \{ \gamma_1, -\gamma_1 \}, \ldots, \quad C_j = \{ \gamma_i_1 \ldots \gamma_i_j, -\gamma_i_1 \ldots \gamma_i_j \}, \quad C_{2^n+1} = \{ \gamma_1 \ldots \gamma_n, -\gamma_1 \ldots \gamma_1 \}.$$
Perfect state transfer of a qudit over underlying networks of group association schemes

*Figure 2.* This shows the underlying graph of the symmetrized scheme obtained from the group scheme over the cubic group $T_h$, where non-zero coupling strengths $J_3 = J_5$ and $J_0 = J_1 = J_2 = J_4 = 0$, PST from $C_0$ to $C_2$ is achieved.

whereas for odd $n$, we have

$C_0 = \{1\}, \quad C_1 = \{-1\}, \quad C_2 = \{\gamma_1, -\gamma_1\}, \ldots, \quad C_j = \{\gamma_i \cdots \gamma_j, -\gamma_i \cdots \gamma_j\},$  
$C_{2n+1} = \{\gamma_1 \cdots \gamma_n\}, \quad C_{2n+2} = \{-\gamma_1 \cdots \gamma_1\}.$

In the following we consider only the case of even $n$; the case of odd $n$ can be considered similarly.

The characters of the $2^n$ one-dimensional representations are given by

$\chi_k(1) = \chi_k(-1) = 1, \quad 0 \leq k \leq 2^n - 1,$

$\chi_{2^n}(\pm \gamma \lambda) = \pm \delta_{\lambda 0} 2^{n/2} \Rightarrow \chi_{2^n}(1) = 2^{n/2}, \quad \chi_{2^n}(-1) = -2^{n/2}.$

Then, by using the result (3.11) we obtain

$F_{\text{opt.}} = \frac{1}{2n+1} \sum_{k=0}^{2^n} d_k |\bar{\chi}_k(\alpha_m)| = \frac{1}{2n+1} \{2^n + 2^{n/2}(2^{n/2})\} = 1.$

Even though we considered only two examples of PST over underlying networks of group association schemes in detail, for all underlying networks of group association schemes for which the corresponding group has non-trivial center, PST can be achieved. For example, underlying networks of group association schemes associated with Abelian finite groups and all $p$-groups (a $p$-group is a group whose order is a power of the prime number $p$; the centers corresponding to the $p$-groups are non-trivial [31]) allow PST.

doi:10.1088/1742-5468/2009/04/P04004
Perfect state transfer of a qudit over underlying networks of group association schemes

It should also be noticed that for direct product groups $G = G_1 \times G_2$, so that PST can be achieved for $G_1$ and $G_2$ at the same time $t_0$, PST will be achievable for the group $G$, since the transition amplitudes $\langle \phi_k | e^{-iHt} | \phi_0 \rangle$ corresponding to the underlying network associated with group association scheme over $G$ are products of the amplitudes corresponding to the groups $G_1$ and $G_2$ (for more details see [26]). In particular, for any finite group $G$ with non-trivial center, not only PST over the underlying network of group association schemes over $G$ is achieved, but also it is achievable for direct product group $G \times G \times \cdots \times G$. For instance, the $n$-dimensional hypercube network ($n$-cube) can be viewed as the underlying network of group association schemes over the product group $Z_2 \times Z_2 \times \cdots \times Z_2$ and PST over it can be achieved by only nearest-neighbor coupling strengths.

6. Conclusion

Perfect state transfer of a qudit over antipodes of the so-called underlying networks of group association schemes was investigated, where by using the group properties and the algebraic structure of these networks (such as Bose–Mesner algebra), an explicit analytical formula for coupling constants in the Hamiltonians (in terms of the irreducible characters of the corresponding group) was given, so that the state of a particular qudit initially encoded on one site can be perfectly evolved to the opposite site without any dynamical control.

Appendix

As regards the arguments of section 3, for the purpose of the perfect state transfer, we consider the underlying graphs of group association schemes in which the corresponding group has non-trivial center in order for the group to have non-trivial one-element conjugacy classes or stratas with size 1, i.e. the last stratum of the graph denoted by $\Gamma_m(o)$ contains only one site ($\kappa_m = |\Gamma_m(o)| = 1$). Then, by imposing the constraint that the transition probability amplitude $\langle \phi_m | e^{-iHt} | \phi_0 \rangle$ be equal to an arbitrary phase such as $e^{i\theta}$ and consequently the transition probability amplitudes $\langle \phi_i | e^{-iHt} | \phi_0 \rangle$ be zero, for all $i \neq m$, one can achieve PST from the reference site $|\phi_0\rangle = |e\rangle$ associated with the first stratum to the site of the last stratum $|\phi_m\rangle$.

Even though, as was shown in section 3, we need to evaluate only the probability amplitude $\langle \phi_m | e^{-iHt} | \phi_0 \rangle$ in order for the corresponding fidelity to be maximized and PST be attained, here we give the transition probability amplitudes between the reference site $|\phi_0\rangle = |e\rangle$ associated with the first stratum to the site of the last stratum $|\phi_m\rangle$. To do so, given a finite group $G$, let $\beta \in G$ belong to the $l$th conjugacy class of the group. In general, assume that the group $G$ has $R$ real conjugacy classes ($C_i = C_{i-1}^{-1}$, for $i = 0, 1, \ldots, R - 1$) and so it will possess $R$ real representations; now, we use the stratification and algebraic properties of the underlying graph of the group association...
scheme over the group $G$ to write
\[ \langle \beta | e^{-iHt} | \phi_0 \rangle = \frac{1}{\sqrt{|G|}} \langle \phi_0 | e^{-iHt} | \phi_0 \rangle = \frac{1}{|G|} \left\{ \sum_{k=0}^{R-1} e^{-it\Theta_{\chi_k}} d_k \chi_k(\alpha_l) \right\} + \sum_{k=R}^{(d+1-R)/2} e^{-it\Theta_{\chi_k}} d_k (\chi_k(\alpha_l) + \chi_k(\alpha_l)), \]  
where
\[ \Theta_{\chi_k} := \sum_{l=0}^{R-1} \frac{j_{k,l}}{d_k} \chi_k(\alpha_l) + \sum_{l=R}^{(d+1-R)/2} \frac{j_{k,l}}{d_k} (\chi_k(\alpha_l) + \chi_k(\alpha_l)), \]  
where we have used the fact that $\chi_k(\tilde{\alpha}_l) = \chi_k(\alpha_l)$. As is seen from (A.2), we have $\Theta_{\chi_k} = \Theta_{\tilde{\chi}_k}$.

It should also be noticed that, for groups $G$ with non-real conjugacy classes, the corresponding group association schemes are not symmetric and so the corresponding underlying graphs are not undirected. Therefore, for the purpose of state transfer, we need to symmetrize the group schemes in order to obtain undirected graphs. Now, we recall that for any non-real conjugacy class $C_i$ ($C_i \neq C_i^{-1}$), $C_i^{-1}$ is also a conjugacy class of $G$ with the same size of $C_i$, i.e. $|C_i| = |C_i^{-1}|$; then, as was shown in [26], the symmetrized classes $C_i$, for $i = 0, \ldots, R - 1$; $\tilde{C}_i \equiv C_i \cup C_i^{-1}$, for $i = R, \ldots, (d+1-R)/2$, form a symmetric association scheme which is a subscheme of the non-symmetric group association scheme over $G$. Now, by choosing the identity element of the group as the reference vertex and stratify the underlying graph with respect to it, the resulted symmetrized subscheme can be considered for the purpose of PST if its last stratum denoted by $\Gamma_m(e)$ has size 1, i.e. we must have $|C_m = C_m^{-1}| = 1$ in order for PST to be studied.

References

[1] Christandl M, Datta N, Ekert A and Landahl A J, 2004 Phys. Rev. Lett. 92 187902
[2] Christandl M, Datta N, Dorlas T C, Ekert A, Kay A and Landahl A J, 2005 Phys. Rev. A 71 032312
[3] Facer C, Twamley J and Cresser J, 2008 Phys. Rev. A 77 012334
[4] Bernasconi A, Godsil C and Severini S, Quantum networks on cubelike graphs, 2008 arXiv:0808.0510 [quant-ph]
[5] Godsil C, Periodic graphs, 2008 arXiv:0806.2074 [math.CO]
[6] Jafarizadeh M A and Sufiani R, 2008 Phys. Rev. A 77 022315
[7] Jafarizadeh M A, Sufiani R, Taghavi S F and Barati E, 2008 J. Phys. A: Math. Theor. 41 475302
[8] Bouwmeester D, Ekert A and Zeilinger A, 2000 The Physics of Quantum Information (Berlin: Springer)
[9] Bose S, 2003 Phys. Rev. Lett. 91 207901
[10] Subrahmanyan V, 2004 Phys. Rev. A 69 034304
[11] Shi T, Li Y, Song Z and Sun C, 2004 arXiv:quant-ph/0405152
[12] Osborne T J and Linden N, 2004 Phys. Rev. A 69 052315
[13] Burgarth D and Bose S, 2005 Phys. Rev. A 71 052315
[14] Burgarth D and Bose S, 2005 New J. Phys. 7 135
[15] Yung M H and Bose S, 2005 Phys. Rev. A 71 032310
[16] Yung M H, 2006 Phys. Rev. A 74 030303
[17] Bayat A and Karimipour V, 2006 arXiv:quant-ph/0612144
[18] Bailey R A, 2004 Association Schemes: Designed Experiments, Algebra and Combinatorics (Cambridge: Cambridge University Press)
[19] Godsil C, Association schemes, lecture notes, 2005 http://quoll.uwaterloo.ca/pstuff/
[20] Childs A M, 2008 arXiv:0810.0312 [quant-ph]
Perfect state transfer of a qudit over underlying networks of group association schemes

[21] Jafarizadeh M A, Sufiani R and Jafarizadeh S, 2007 J. Phys. A: Math. Theor. 40 4949
[22] Jafarizadeh M A, Sufiani R and Jafarizadeh S, 2008 J. Math. Phys. 49 073303
[23] Jafarizadeh M A, Behnia S, Faizi E and Ahadpour S, 2008 Pramana J. Phys. 70 417
[24] Obata N, Quantum Probabilistic Approach to Spectral Analysis of Star Graphs, 2004 Interdiscip. Inf. Sci. 10 41
[25] Hora A and Obata N, 2003 Fundamental Problems in Quantum Physics (Singapore: World Scientific) p 284
[26] Jafarizadeh M A and Salimi S, 2006 J. Phys. A: Math. Gen. 39 1
[27] Jafarizadeh M A and Salimi S, 2007 Ann. Phys., NY 322 1005
[28] Jafarizadeh M A and Sufiani R, 2007 Physica A 381 116
[29] Jafarizadeh M A and Sufiani R, 2007 Int. J. Quantum Inf. 5 575
[30] Jafarizadeh M A, Sufiani R, Salimi S and Jafarizadeh S, 2007 Eur. Phys. J. B 59 199
[31] Gordon James and Martin Liebeck, 1993 Representations and Characters of Groups (Cambridge: Cambridge University Press)
[32] Barry Simon, 1996 Representations of Finite and Compact Groups (Graduate Studies in Mathematics vol 10) (Providence, RI: American Mathematical Society)