On universal solution to reflection equation*

J. Donin†, P. P. Kulish‡, and A. I. Mudrov†

October 26, 2018

† Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel.
‡ St.-Petersburg Department of Steklov Mathematical Institute, Fontanka 27, 191011 St.-Petersburg, Russia

Abstract

For a given quasitriangular Hopf algebra $\mathcal{H}$ we study relations between the braided group $\tilde{\mathcal{H}}^*$ and Drinfeld’s twist. We show that the braided bialgebra structure of $\tilde{\mathcal{H}}^*$ is naturally described by means of twisted tensor powers of $\mathcal{H}$ and their module algebras. We introduce universal solution to the reflection equation (RE) and deduce a fusion prescription for RE-matrices.

Key words: reflection equation, twist, fusion procedure.
AMS classification codes: 17B37, 16W30.

1 Introduction

There are two important algebras in the quantum group theory, the Faddeev-Reshetikhin-Takhtajan (FRT) and reflection equation (RE) algebras. As was shown in [DM], they are related by a twist of the underlying quasitriangular Hopf algebra $\mathcal{H}$ squared. This twist transforms any bimodule over $\mathcal{H}$ to an $\mathcal{H}^{\hat{\otimes}2}$-module, so there is an analog of the dual algebra $\mathcal{H}^*$, an $\mathcal{H}^{\hat{\otimes}2}$-module algebra $\tilde{\mathcal{H}}^*$ . This algebra turns out to be isomorphic to the Hopf algebra in the quasitensor category of $\mathcal{H}$-modules, [M], thus equipped with an additional

---

*This research is partially supported by the Israel Academy of Sciences grant no. 8007/99-01 and by the RFBR grant no. 02-01-00085.
structure of braided coalgebra. In the present paper we develop further the approach of studying this structure from the twist point of view. The algebra \( \tilde{\mathcal{H}}^* \) is a module over the twisted tensor square \( \mathcal{H}^{\otimes 2} \). This twist can be extended to all tensor powers of \( \mathcal{H} \) giving \( \mathcal{H}^{\otimes n} \), \( n = 0, 1, \ldots \), where the zero power is set to be the ring of scalars with the natural Hopf algebra structure. It appears natural to consider modules over all Hopf algebras \( \mathcal{H}^{\otimes n} \) simultaneously. They form a monoidal category with respect to twisted tensor product, and the iterated comultiplications \( \Delta^n \colon \mathcal{H} \to \mathcal{H}^{\otimes n} \) induce a functor from that category to the quasitensor category of \( \mathcal{H} \)-modules. The iterated braided coproducts \( \Delta^n_{\tilde{\mathcal{H}}^*} \) on \( \tilde{\mathcal{H}}^* \) take their values in module algebras over \( \mathcal{H}^{\otimes 2n} \), \( n = 2, 3, \ldots \)

We introduce universal K-matrix, \( K \in \mathcal{H} \otimes \tilde{\mathcal{H}}^* \), satisfying the characteristic equation
\[
(\Delta \otimes \text{id})(K) = R^{-1} K_1 R K_2.
\]
This equation implies the "abstract" reflection equation
\[
R_{21} K_2 R K_1 = K_1 R_{21} K_2
\]
in \( \mathcal{H} \otimes \mathcal{H} \otimes \tilde{\mathcal{H}}^* \) involving the universal R-matrix. We prove that \( K \) is equal to the canonical element of \( \mathcal{H} \otimes \tilde{\mathcal{H}}^* \) with respect to the Hopf pairing between \( \mathcal{H} \) and \( \mathcal{H}^* \), which coincides with \( \tilde{\mathcal{H}}^* \) as a linear space. The reason for considering this object is the same as for the universal R-matrix: it gives a solution to the matrix RE in every representation of \( \mathcal{H} \).

A fusion procedure for R-matrices was the first contact of the Yang-Baxter equation with algebraic structures [KRS]. Later it was related with properties of the universal R-matrix in the theory of quantum groups [Dr1]. Although an analogous fusion procedure was considered for matrix solutions to the reflection equation [MeN, KSk1], to our knowledge no universal element was proposed in any algebraic approach to the RE (see, e.g., [DeN] and references therein). Using the characteristic equation on the universal element \( K \), we formulate a version of fusion procedure for RE matrices, suggesting an algorithm of tensoring RE matrices.

The paper is organized as follows. Section 2 contains a summary on Hopf algebras, universal R-matrix, and twist. There we introduce RE matrices and prove an auxiliary proposition about their restriction to submodules. In Section 3 we consider a special type of twist of tensor product Hopf algebras and their modules when the twisting cocycle satisfies the bicharacter identities. We show that the braided tensor product of module algebras is a twisted tensor product. In Section 4 we extend this construction to higher tensor powers of
Hopf algebras. In Section 5 we focus our study on the RE dual $\tilde{H}^*$ to a quasitriangular Hopf algebra $H$. There, we introduce the universal RE matrix $K$ and deduce the characteristic equation for it. Using the bicharacter twist and the universal K-matrix, we study the braided bialgebra structure $\tilde{H}^*$ in Section 6. Section 7 is devoted to the central result of the paper, a fusion procedure for RE matrices. Appendix contains the proof of Theorem 17.

2 Preliminaries

2.1. Let $k$ be a commutative algebra over a field of zero characteristic. Let $H$ be a quasitriangular Hopf algebra over $k$, with the coproduct $\Delta$, counit $\varepsilon$, antipode $\gamma$, and the universal R-matrix $\mathcal{R} \in H \otimes H$. By definition, it satisfies following Drinfeld’s equations, $[\text{Dr1}]:$

\[(\Delta \otimes \text{id})(\mathcal{R}) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = R_{13}R_{12},\]

and, for any $x \in H$,

\[\mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}.\]  

We adopt the usual convention of marking tensor components with subscripts indicating the supporting tensor factors. The subscript $op$ will stand for the opposite multiplication while the superscript $op$ for the opposite coproduct. The standard symbolic notation with suppressed summation is used for the coproduct, $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

By $\mathbb{N}$ we assume a set of non-negative integers, i.e., including zero. By $\Delta^n$, $n \in \mathbb{N}$, we denote the n-fold coproduct, $\Delta^n : H \to H^{\otimes n}$, setting

\[\Delta^0 = \varepsilon, \quad \Delta^1 = \text{id}, \quad \Delta^2 = \Delta, \quad \Delta^3 = (\Delta \otimes \text{id}) \circ \Delta, \quad \ldots\]

Here and further on we view the ring of scalars $k$ as equipped with the structure of Hopf algebra over $k$. It is then convenient to put $H^{\otimes n} = k$ for $n = 0$.

The antipode $\gamma$ is treated as a Hopf algebra isomorphism between $H_{op}$ and $H^{op}$. The universal R-matrix defines two homomorphisms from the dual coopposite Hopf algebra $H^{*op}$ to $H$:

\[\mathcal{R}^\pm(a) = \langle a, \mathcal{R}_1^\pm \rangle \mathcal{R}_2^\pm, \quad a \in H^{*op}, \quad \text{where} \quad \mathcal{R}^+ = \mathcal{R} \quad \text{and} \quad \mathcal{R}^- = \mathcal{R}_{21}^{-1}.\]

1One may think of $k$ as either $\mathbb{C}$ or $\mathbb{C}[[h]]$. In the latter case all algebras are assumed complete in the $h$-adic topology.

2 For a guide in quasitriangular Hopf algebras, the reader is referred to original Drinfeld’s report, $[\text{Dr1}]$, and to one of the textbooks, e.g. $[\text{ChPr}]$ or $[\text{Mj}]$.  

3
2.2. Twist of a Hopf algebra $\mathcal{H}$ by a cocycle $\mathcal{F}$ is a Hopf algebra $\tilde{\mathcal{H}}$ whose comultiplication is obtained from $\Delta$ by a similarity transformation, $[Dr2]$, 

$$\tilde{\Delta}(x) = \mathcal{F}^{-1}\Delta(x)\mathcal{F}, \quad x \in \tilde{\mathcal{H}}.$$ 

To preserve coassociativity, it is sufficient for the element $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ to satisfy the cocycle equation

$$(\Delta \otimes \text{id})(\mathcal{F})_{12} = (\text{id} \otimes \Delta)(\mathcal{F})_{23} \tag{5}$$

and the normalization condition

$$(\varepsilon \otimes \text{id})(\mathcal{F}) = (\text{id} \otimes \varepsilon)(\mathcal{F}) = 1 \otimes 1. \tag{6}$$

The multiplication and counit in $\tilde{\mathcal{H}}$ remain the same as in $\mathcal{H}$. The antipode changes by a similarity transformation, $\tilde{\gamma}(x) = u^{-1}\gamma(x)u$ with $u = \mathcal{F}(\mathcal{F}_2) \in \mathcal{H}$, and the universal R-matrix is

$$\tilde{\mathcal{R}} = \mathcal{F}^{-1}_{21}\mathcal{R}\mathcal{F}. \tag{7}$$

Twist establishes an equivalence relation among Hopf algebras, so we call $\tilde{\mathcal{H}}$ and $\mathcal{H}$ twist-equivalent. We use notation $\tilde{\mathcal{H}} \sim \mathcal{H}$; then $\mathcal{H} \mathcal{F}^{-1} \sim \tilde{\mathcal{H}}$.

Recall that a left $\mathcal{H}$-module algebra $\mathcal{A}$ is an associative algebra over $k$ endowed with the left action $\triangleright$ of $\mathcal{H}$ such that the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is $\mathcal{H}$-equivariant. Given two $\mathcal{H}$-module algebras $\mathcal{A}$ and $\mathcal{B}$, there exists their braided tensor product, $[M]$. This is an $\mathcal{H}$-module algebra coinciding with $\mathcal{A} \otimes \mathcal{B}$ as a linear space and endowed with multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1(\mathcal{R}_2 \triangleright a_2) \otimes (\mathcal{R}_1 \triangleright b_1) b_2, \tag{8}$$

for $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$, $i = 1, 2$.

An $\mathcal{H}$-module algebra $\mathcal{A}$ becomes a left $\tilde{\mathcal{H}}$-module algebra $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}} \mathcal{F} \sim \mathcal{A}$, when equipped with the multiplication

$$a \circ b = (\mathcal{F}_1 \triangleright a)(\mathcal{F}_2 \triangleright b), \quad a, b \in \mathcal{A}. \tag{9}$$

The action of $\tilde{\mathcal{H}}$ on $\tilde{\mathcal{A}}$ is that of $\mathcal{H}$ on $\mathcal{A}$, having in mind $\tilde{\mathcal{H}} \simeq \mathcal{H}$ as associative algebras.

2.3 (Reflection equation). Let $(\rho, V)$ be a representation of $\mathcal{H}$ on a module $V$, and $\mathcal{R}$ the image of the universal R-matrix, $\mathcal{R} = (\rho \otimes \rho)(\mathcal{R}) \in \text{End}^{\otimes 2}(V)$. Let $\mathcal{A}$ be an associative
algebra. An element \( K \in \text{End}(V) \otimes A \) is said to be a solution to (constant) RE or an (constant) RE matrix in representation \( \rho \) with coefficients in \( A \) if

\[
R_{21}K_1RK_2 = K_2R_{21}K_1R.
\]

This equation is supported in \( \text{End}^{\otimes 2}(V) \otimes A \), and the subscripts label the components belonging to the different tensor factors \( \text{End}(V) \). Let us prove the following elementary proposition.

**Proposition 1.** Let \( K \in \text{End}(V) \otimes A \) be an RE matrix and suppose the \( H \)-module \( V \) is semisimple. Let \( \rho_0 \) be a representation of \( H \) on the submodule \( V_0 \subset V \) and \( V_0 \xrightarrow{\pi} V \xrightarrow{\iota} V_0 \) the intertwiners. Then, the matrix \( K_0 = \pi K \iota \in \text{End}(V_0) \otimes A \) is a solution to the RE in the representation \( \rho_0 \).

**Proof.** Multiply equation (10) by \( \pi \otimes \pi \) from the left and by \( \iota \otimes \iota \) from the right and use the intertwining formulas \( \pi \rho(x) = \rho_0(x) \pi \), \( \rho(x) \iota = \iota \rho_0(x) \) valid for every \( x \in H \). The result will be the RE on the matrix \( K_0 \) with the R-matrix \( (\rho_0 \otimes \rho_0)(R) \).

\[
\square
\]

3 Twisted tensor product of Hopf algebras

3.1. In this section we recall a specific case of twist as applied to tensor products of Hopf algebras, [RS]. This construction is of particular importance for our consideration. Denote by \( H' \) the tensor product \( H^{(1)} \otimes H^{(2)} \) of two Hopf algebras. It is equipped with the standard tensor product multiplication and comultiplication. An element \( F \in H^{(2)} \otimes H^{(1)} \) may be viewed as that from \( H' \otimes H' \) via the embedding

\[
F \in (1 \otimes H^{(2)}) \otimes (H^{(1)} \otimes 1) \subset H' \otimes H'.
\]

If \( F \) satisfies the bicharacter identities

\[
(\Delta^{(2)} \otimes \text{id})(F) = F_{13}F_{23} \in H^{(2)} \otimes H^{(2)} \otimes H^{(1)}, \tag{11}
\]

\[
(\text{id} \otimes \Delta^{(1)})(F) = F_{13}F_{12} \in H^{(2)} \otimes H^{(1)} \otimes H^{(1)},
\]

then it fulfills cocycle condition (3) in \( H'^{\otimes 2} \) and condition (3).

**Definition 1.** Twisted tensor product \( H^{(1)} \tilde{\otimes} H^{(2)} \) of two Hopf algebras \( H^{(i)} \), \( i = 1, 2 \), is the twist of \( H^{(1)} \otimes H^{(2)} \) with a bicharacter cocycle \( F \) satisfying (11).
Proposition 2. The maps

$$id \otimes \varepsilon^{(2)}: \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \to \mathcal{H}^{(1)}, \quad \varepsilon^{(1)} \otimes id: \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \to \mathcal{H}^{(2)}$$

are Hopf algebra homomorphisms.

Proof. This is a corollary of the equalities $$(\varepsilon^{(2)} \otimes id)(\mathcal{F}) = 1 \otimes 1 = (id \otimes \varepsilon^{(1)})(\mathcal{F})$$ which follow from (11).

3.2. A particular case of twisted tensor product is the twisted tensor square $\mathcal{H} \otimes \mathcal{H}$ of a quasitriangular Hopf algebra $\mathcal{H}$. The universal $R$-matrix satisfies bicharacter conditions (11) by virtue of (1). If one takes the universal $R$-matrix of $\mathcal{H} \otimes \mathcal{H}$ equal to $R_{13}^{-1}R_{24}^+$, then formula (7) gives the universal $R$-matrix of $\mathcal{H} \otimes \mathcal{H}$:

$$R_{14}^{-1}R_{13}R_{24}^+R_{23} = R_{41}^{-1}R_{31}R_{24}R_{23} \in (\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H}). \quad (12)$$

Proposition 3. The Hopf algebra $\mathcal{H} \otimes \mathcal{H}$ is twist-equivalent to $\mathcal{H}^{\text{op}} \otimes \mathcal{H}^{\text{op}}$ via the cocycle $\mathcal{F} = \mathcal{R}_{13}R_{23} \in (\mathcal{H}^{\text{op}} \otimes \mathcal{H}) \otimes (\mathcal{H}^{\text{op}} \otimes \mathcal{H})$.

Proof. This twist is a composition of the two, $\mathcal{H} \otimes \mathcal{H} \overset{\mathcal{R}_{13}}{\sim} \mathcal{H}^{\text{op}} \otimes \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{H} \overset{\mathcal{R}_{23}}{\sim} \mathcal{H} \otimes \mathcal{H}$.

3.3 (Twisted tensor product of module algebras). Let $\mathcal{H}^{(i)}$, $i = 1, 2$, be Hopf algebras and $\mathcal{A}^{(i)}$ their left module algebras. Consider the twisted tensor product $\mathcal{H}' = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ by a bicharacter $\mathcal{F}$. There is a left $\mathcal{H}'$-module algebra, $\mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$, which is built on $\mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ as an $\mathcal{H}'$-module and contains $\mathcal{A}^{(1)} \otimes 1$ and $1 \otimes \mathcal{A}^{(2)}$ as invariant subalgebras. The multiplication is given by

$$(a_1^{(1)} \otimes a_2^{(1)})(a_1^{(2)} \otimes a_2^{(2)}) = a_1^{(1)}(\mathcal{F}_2 \triangleright a_2^{(1)}) \otimes (\mathcal{F}_1 \triangleright a_1^{(2)}) a_2^{(2)}, \quad (13)$$

where $a_j^{(i)} \in \mathcal{A}^{(i)}$. This multiplication satisfies the permutation relation

$$(1 \otimes a^{(2)})(a^{(1)} \otimes 1) = (\mathcal{F}_2 \triangleright a^{(1)} \otimes 1)(1 \otimes \mathcal{F}_1 \triangleright a^{(2)}), \quad a^{(i)} \in \mathcal{A}^{(i)}. \quad (14)$$

Remark 4. A particular case of this construction, when $\mathcal{H}^{(1)} = \mathcal{H}^{(2)} = \mathcal{H}$ and $\mathcal{F} = \mathcal{R}$ (cf. Subsection 3.2), gives the braided tensor product (8) of $\mathcal{H}$-module algebras, which acquires an $\mathcal{H}$-module algebra structure because of the Hopf algebra homomorphism $\Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. 6
4 Twisted tensor powers $\mathcal{H}^{\otimes n}$

4.1. The purpose of this section is to extend the collection of $\mathcal{H}$-modules by modules over certain Hopf algebras related to $\mathcal{H}$. The reason for that is the observation that the braided tensor product of two $\mathcal{H}$-module algebras, is a particular case of the twisted tensor product of modules (cf. Remark 4) and thus a module over $\mathcal{H}^{\otimes \mathcal{H}}$.

4.2. Consider the set $\mathcal{S}(\mathcal{H}) = \bigcup_{\mathcal{H}'} \text{Hom}(\mathcal{H}, \mathcal{H}')$ of Hopf algebra homomorphisms, where $\mathcal{H}'$ runs over Hopf algebras. We treat the ring $k$ of scalars as a Hopf algebra over $k$, so $\varepsilon \in \mathcal{S}(\mathcal{H})$.

Let $\phi^{(i)} \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$, be two homomorphisms from $\mathcal{H}$ to Hopf algebras $\mathcal{H}^{(i)}$. Put the bicharacter $F \in \mathcal{H}^{(2)} \otimes \mathcal{H}^{(1)}$ to be the image of the universal R-matrix, $F = (\phi^{(2)} \otimes \phi^{(1)})(R)$.

**Proposition 5.** The map

$$\phi^{(1)} \tilde{\otimes} \phi^{(2)} = (\phi^{(1)} \otimes \phi^{(2)}) \circ \Delta: \mathcal{H} \to \mathcal{H}^{(1)} \tilde{\otimes} \mathcal{H}^{(2)}$$

is a Hopf algebra homomorphism.

**Proof.** This follows from the readily verified fact that the map $\Delta: \mathcal{H} \to \mathcal{H}^{\otimes \mathcal{H}}$ is a Hopf algebra embedding. \qed

**Proposition 6.** The operation $\tilde{\otimes}$ makes the set $\mathcal{S}(\mathcal{H})$ a monoid with $\varepsilon$ being the neutral element. The operation $\tilde{\otimes}$ is natural: if $f^{(i)}: \mathcal{H}^{(i)} \to \mathcal{H}_1^{(i)}$ are Hopf algebra homomorphisms, then

$$(f^{(1)} \circ \phi^{(1)}) \tilde{\otimes} (f^{(2)} \circ \phi^{(2)}) = (f^{(1)} \otimes f^{(2)}) \circ (\phi^{(1)} \tilde{\otimes} \phi^{(2)}).$$

**Proof.** It follows immediately from coassociativity of $\Delta$ that the operation $\tilde{\otimes}$ is associative. It is obviously natural because so is the construction of twisted tensor product. The counit map is the identity due to the arguments used in the proof of Proposition 2. \qed

We will use the notation $\mathcal{H}^{(1)} \tilde{\otimes} \mathcal{H}^{(2)}$ for $\mathcal{H}^{(1)} \tilde{\otimes} \mathcal{H}^{(2)}$ when the specific form of homomorphisms $\phi^{(i)}$ is clear from the context. In particular, we consider the set $\{\text{id}^{\otimes n} | n \in \mathbb{N}\}$, where $\text{id}^0 = \varepsilon$, and the corresponding twisted tensor powers $\{\mathcal{H}^{\otimes n} | n \in \mathbb{N}\}$. The projections $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes k}$ obtained by applying $\varepsilon$ to some tensor factors $n - k$ times are Hopf algebra homomorphisms. Also, for every $k \in \mathbb{N}$, the coproduct $\Delta^k$ applied to any component of $\mathcal{H}^{\otimes n}$ induces a Hopf algebra homomorphism from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}^{\otimes (n+k)}$.
4.3. Let \( \mathcal{A}^{(i)} \) be two left \( \mathcal{H} \)-module algebras. By \( \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)} \) we denote the \( \mathcal{H}^{\otimes 2} \)-module algebra \( \mathcal{A}^{(1)} \otimes \tilde{\mathcal{A}}^{(2)} \), see Subsection 3.3. Suppose \( \mathcal{A}^{(i)} \) are module algebras over Hopf algebras \( \mathcal{H}^{(i)} \) and the \( \mathcal{H} \)-module structures on \( \mathcal{A}^{(i)} \) are induced by homomorphisms \( \phi^{(i)} : \mathcal{H} \rightarrow \mathcal{H}^{(i)} \). Obviously, the algebra \( \mathcal{A}^{(1)} \otimes \tilde{\mathcal{A}}^{(2)} \) is isomorphic to \( \mathcal{A}^{(1)} \otimes F \mathcal{A}^{(2)} \), with \( F = (\phi^{(2)} \otimes \phi^{(1)})(\mathcal{R}) \), and the \( \mathcal{H}^{\otimes 2} \)-module structure on it is induced by the homomorphism \( \phi^{(1)} \otimes \phi^{(2)} : \mathcal{H}^{\otimes 2} \rightarrow \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \) of Hopf algebras.

Example 1. The operation \( \otimes \) is associative and allows to introduce twisted tensor powers of an \( \mathcal{H} \)-module algebra \( \mathcal{A} \). By induction, we can define the \( n \)-th tensor power \( \mathcal{A}^{\otimes n} \), setting \( \mathcal{A}^{\otimes n} = \mathcal{A}^{\otimes (n-1)} \otimes \mathcal{A} \).

Example 2. Let \( \mathcal{H}' \) be another Hopf algebra such that there exists a Hopf algebra homomorphism \( \phi : \mathcal{H} \rightarrow \mathcal{H}' \). Suppose that the algebra \( \mathcal{A} \) from the previous example is also an \( \mathcal{H}' \)-module algebra and the \( \mathcal{H} \)-module structure on \( \mathcal{A} \) is induced by \( \phi \). Then, \( \mathcal{A}^{\otimes n} \) is a module over \( \mathcal{H}'^{\otimes n} \), \( \mathcal{H}^{\otimes n} \), and \( \mathcal{H} \) (but not over \( \mathcal{H}' \) since, in general, there is no natural map from \( \mathcal{H}' \) to \( \mathcal{H}'^{\otimes n} \), which is twisted over \( \mathcal{H} \)).

5 RE dual \( \tilde{\mathcal{H}}^* \)

5.1 (RE twist). Consider a quasitriangular Hopf algebra \( \mathcal{H}' \) with the universal R-matrix \( \mathcal{R}' \) and a left \( \mathcal{H}' \)-module algebra \( \mathcal{A} \).

Definition 2. \( \mathcal{A} \) is called quasi-commutative if for any \( a, b \in \mathcal{A} \)

\[
(\mathcal{R}'_{2} \triangleright b)(\mathcal{R}'_{1} \triangleright a) = ab. \tag{16}
\]

The following statement is elementary.

Proposition 7. Let \( \mathcal{H}' \) be a quasitriangular Hopf algebra and \( \tilde{\mathcal{H}}' \cong \mathcal{H}' \). If \( \mathcal{A} \) is a quasi-commutative \( \mathcal{H}' \)-module algebra, then the twisted algebra \( \tilde{\mathcal{A}} \), \( \tilde{\mathcal{A}} \cong \mathcal{A} \), is a quasi-commutative \( \mathcal{H}' \)-module algebra.

Proof. Clear. \( \square \)

Corollary 8. Let \( \mathcal{A}^{(i)} \) be quasi-commutative module algebras over Hopf algebras \( \mathcal{H}^{(i)} \), where \( i = 1, 2 \). Let \( \mathcal{F} \in \mathcal{H}^{(2)} \otimes \mathcal{H}^{(1)} \) be a bicharacter and \( \mathcal{H}' \) the twisted tensor product \( \mathcal{H}^{(1)} \tilde{\otimes} \mathcal{H}^{(2)} \). Then, the twisted tensor product \( \mathcal{A}^{(1)} \tilde{\otimes} \mathcal{A}^{(2)} \) is a quasi-commutative \( \mathcal{H}' \)-module algebra.
**Proof.** The ordinary tensor product $A^{(1)} \otimes A^{(2)}$ is a quasi-commutative $H^{(1)} \otimes H^{(2)}$-module algebra. Now apply Proposition 7.

**Example 3.** The Hopf dual $H^*$ is an $H$-bimodule with respect to the left and right coregular actions. We reserve for them the notation $\triangleright$ and $\triangleleft$, respectively:

$$x \triangleright a = a_{(1)} \langle a_{(2)}, x \rangle, \quad a \triangleleft x = \langle a_{(1)}, x \rangle a_{(2)},$$

where $x \in H$, $a \in H^*$, and $a_{(1)} \otimes a_{(2)} = \Delta_H(a)$. $H^*$ can be viewed as a left $H' = H^{op} \otimes H$-module algebra with the universal $R$-matrix taken to be $R' = R_{13}^{-1} R_{24}$. It is quasi-commutative, with condition (16) turning to the equation

$$(a \triangleleft R_1)(b \triangleleft R_2) = (R_2 \triangleright b)(R_1 \triangleright a)$$

for any $a, b \in H^*$. This equation is fulfilled due to (2).

**Example 4.** Along the line of Proposition 3, consider the twist from the Hopf algebra $H^{op} \otimes H$ to $^S H \otimes H$ by the cocycle $R_{13} R_{23}$. Denote by $\tilde{H}^*$ the quasi-commutative $H \otimes H$-module algebra that is twist-equivalent to $H^*$. The equation

$$(R_1 \triangleright a \triangleleft R_2)(b \triangleleft R_1 \triangleleft R_2') = (R_2 \triangleright b)(R_1 \triangleright a \triangleleft R_2')$$

taking place for any $a$ and $b$ from $\tilde{H}^*$ is a consequence of (18). Here, the primes distinguish different copies of $R$.

**5.2.** The left action of $H^{op} \otimes H$ on $H^*$ is expressed through the left and right coregular actions:

$$(x \otimes y) \triangleright a = y \triangleright a \triangleleft \gamma(x), \quad x \otimes y \in H^{op} \otimes H, \quad a \in H^*.$$ (20)

Let $m$ be the multiplication in $H^*$ and $\tilde{m}$ the multiplication in $\tilde{H}^*$. They are related by the formulas

$$\tilde{m}(a \otimes b) = m(R_1 \triangleright a \triangleleft R_{1'} \otimes b \triangleleft \gamma(R_2) \triangleleft R_{2'})$$ (21)

$$m(a \otimes b) = \tilde{m}(R_1 \triangleright a \triangleleft \gamma(R_{1'}) \otimes b \triangleleft R_{2'} \triangleleft R_2).$$ (22)

These expressions prove that the associative algebra $\tilde{H}^*$ is exactly the Hopf algebra in the quasitensor category of $H$-modules, [M].
5.3 (Universal K-matrix). Let \( \{e_i\} \) be a base in \( \mathcal{H} \) and \( \{e^i\} \) its dual in \( \mathcal{H}^* \) with respect to the canonical Hopf pairing. Let us consider the element \( T = \sum_i e_i \otimes e^i \in \mathcal{H} \otimes \mathcal{H}^* \). In case \( \mathcal{H} \) is infinite dimensional this element can be defined by introducing an appropriate topology. Using the identities
\[
\sum_i e_i \otimes e^i \triangleright x = \sum_i x e_i \otimes e^i, \quad \sum_i e_i \otimes x \triangleright e^i = \sum_i e_i x \otimes e^i, \tag{23}
\]
which are true for any \( x \in \mathcal{H} \), we rewrite relations (18) in a compact form that involves the universal T-matrix \( T \):
\[
RT_1 T_2 = T_2 T_1 R. \tag{24}
\]
Now we consider the same base \( \{e^i\} \) as that in \( \tilde{\mathcal{H}}^* \). Then, the canonical element \( K = \sum_i e_i \otimes e^i \in \mathcal{H} \otimes \tilde{\mathcal{H}}^* \) may be thought of as a universal K-matrix. Indeed, applying identities (23) to (19) we rewrite it in the form of the RE,
\[
R_{21} K_1 R K_2 = K_2 R_{21} K_1 R, \tag{25}
\]
in \( \mathcal{H} \otimes \mathcal{H} \otimes \tilde{\mathcal{H}}^* \).

**Remark 9.** In applications, elements of the dual base \( \{e^i\} \) are usually expressed as polynomials in generators of the algebra \( \mathcal{H}^* \). Although belonging to the same linear space, \( e^i \) are expressed differently through generators of the deformed algebra \( \tilde{\mathcal{H}}^* \). While fixing the universal K-matrix as an element of \( \mathcal{H} \otimes \tilde{\mathcal{H}}^* \), the proposed formula does not suggest an easy way of computing it.

5.4. Let us consider an element \( K^A \in \mathcal{H} \otimes A \) of the form \( K^A = \sum_i e_i \otimes a^i \), were \( \{e_i\} \) is a base in \( \mathcal{H} \). It can be viewed as a linear map \( f : \tilde{\mathcal{H}}^* \to A, f : u \mapsto \sum_i \langle u, e_i \rangle a^i \).

**Proposition 10.** The map \( f : \tilde{\mathcal{H}}^* \to A \) is an algebra homomorphism if and only if
\[
(\Delta \otimes \text{id})(K^A) = R^{-1} K_1^A R K_2^A \in \mathcal{H} \otimes \mathcal{H} \otimes A. \tag{26}
\]

**Proof.** Let us prove the statement for \( K^A = K \) first. This case corresponds to \( A = \tilde{\mathcal{H}}^* \) and \( f = \text{id} \). The universal T-matrix \( T \in \mathcal{H} \otimes \mathcal{H}^* \) obeys the identity \( (\Delta \otimes \text{id})(T) = T_1 T_2 \). Now applying formula (22) to the right-hand side of this equality we come to equation (26) for \( K^A = K \). The general case follows from here since the element \( K^A \) is equal to \( (\text{id} \otimes f)(K) \). \( \square \)
5.5 (Characters of $\mathcal{H}^*$). Let $\chi$ be a character (an algebra homomorphism to $k$) of the RE dual $\mathcal{H}^*$. We can think of it as an element from $\mathcal{H}$, then $\chi = (\text{id} \otimes \chi)(\mathcal{K})$; so $\rho(\chi)$ gives a numeric solution to the RE in any representation $\rho$ of $\mathcal{H}$. Using formula (26) we find the necessary and sufficient condition for an element $\chi \in \mathcal{H}$ to be a character of $\mathcal{H}^*$:

$$\Delta \chi = R^{-1} \chi_1 R \chi_2. \quad (27)$$

It is easy to see that this equation is preserved by the similarity transformation $\chi \rightarrow \eta \chi \eta^{-1}$, where $\eta$ is a group-like element or, in other words, a character of the algebra $\mathcal{H}^*$ (the tensor square of a group-like element commutes with $R$).

The unit of $\mathcal{H}$ satisfies equation (27), so the counit $\varepsilon_{\mathcal{H}^*}$ of $\mathcal{H}^*$ is a character of $\mathcal{H}^*$.

5.6. The comultiplication map $\Delta: \mathcal{H} \rightarrow \mathcal{H}^{\otimes 2}$ is a Hopf algebra embedding. The composition

$$(\mathcal{R}^+ \otimes \mathcal{R}^-) \circ \Delta_{\mathcal{H}^*}: \mathcal{H}^{\text{op}} \rightarrow \mathcal{H}^{\otimes 2}, \quad (28)$$

is a Hopf algebra homomorphism too. These maps extend to a Hopf algebra homomorphism from the double $\text{D}(\mathcal{H})$ to $\mathcal{H}^{\hat{\otimes} 2}$, $\text{[RS]}$. It follows that any $\mathcal{H}^{\hat{\otimes} 2}$-module is that over $\mathcal{H}$, $\mathcal{H}^{\text{op}}$, and $\text{D}(\mathcal{H})$. This is true for the algebra $\mathcal{H}^*$ in particular.

The element $Q = R_{21} R \in \mathcal{H} \otimes \mathcal{H}$ defines a map

$$Q(a) = \langle a, Q_1 \rangle Q_2, \quad a \in \mathcal{H}^*, \quad (29)$$

from $\mathcal{H}^*$ to $\mathcal{H}$. This map is an $\mathcal{H}$-equivariant algebra homomorphism, where $\mathcal{H}$ is considered as the adjoint module over itself. $\text{[M]}$. The existence of this homomorphism follows from the fact that the algebra $\mathcal{H}^*$ is a left module algebra over $\mathcal{H}^{\text{op}}$. The dual version of this statement claims that $\mathcal{H}^*$ is a right comodule algebra over $\mathcal{H}_{\text{op}}$. Then the map $Q$ has the form $(\varepsilon_{\mathcal{H}^*} \otimes \gamma) \circ \delta$, where $\delta$ is the coaction $\mathcal{H}^* \rightarrow \mathcal{H}^* \otimes \mathcal{H}_{\text{op}}$, $\text{[DM]}$:

$$\delta(a) = a_{(2)} \otimes \langle a_{(3)}, \mathcal{R}_1^{-1} \gamma(a_{(1)}), \mathcal{R}_2 \mathcal{R}_2^{-1} \mathcal{R}_1 \rangle, \quad a \in \mathcal{H}^*. \quad (30)$$

Here the element $a$ is considered as belonging to the linear space $\mathcal{H}^* \simeq \tilde{\mathcal{H}}^*$ and $a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ is the symbolic decomposition of the three-fold coproduct $\Delta_3^{\mathcal{H}^*}(a)$. The counit $\varepsilon_{\mathcal{H}^*}$ is a character of $\mathcal{H}^*$, cf. Subsection $\text{[ESS]}$, and this implies that $Q$ is an algebra map. Clearly any character of the associative algebra $\mathcal{H}^*$ defines in this way a homomorphism from $\mathcal{H}^*$ to $\mathcal{H}$.

The element $Q$ is the image of the universal K-matrix in $\mathcal{H}^{\otimes 2}$, $Q = (\text{id} \otimes Q)(\mathcal{K})$, so it satisfies the abstract reflection equation in $\mathcal{H}^{\hat{\otimes} 3}$, $\text{[S]}$.

$\text{3}$$\text{It appeared in [FRT] as a map from the FRT algebra, which is an extension of $\mathcal{H}^*$, to $\mathcal{H}$.}$
6 Braided comultiplication and twist

6.1. Let $\iota_k : \mathcal{H}^* \to \mathcal{H}^* \otimes^n$ be the embedding sending $\mathcal{H}^*$ to the $k$-th tensor factor. It is an algebra homomorphism. Consider the elements $\mathcal{K}^{(k)} = (\text{id} \otimes \iota_k)(\mathcal{K})$.

Proposition 11. The multiplication in $\mathcal{H}^* \otimes^n$ satisfies by the following commutation relations

\begin{align*}
\mathcal{R}_{21} \mathcal{K}^{(k)}_1 \mathcal{R}_{12} \mathcal{K}^{(k)}_2 &= \mathcal{K}^{(k)}_2 \mathcal{R}_{21} \mathcal{K}^{(k)}_1 \mathcal{R}_{12}, \quad k = 1, \ldots, n, \\
\mathcal{R}_{12}^{-1} \mathcal{K}^{(k)}_1 \mathcal{R}_{12} \mathcal{K}^{(l)}_2 &= \mathcal{K}^{(l)}_2 \mathcal{R}_{12}^{-1} \mathcal{K}^{(k)}_1 \mathcal{R}_{12}, \quad l, k = 1, \ldots, n, \quad l < k.
\end{align*}

(31)

Proof. Denote by $\mathcal{H}^{\otimes 2}_k$ the algebra $\mathcal{H}^{\otimes 2}_k$ embedded in the $k$-th place in $(\mathcal{H} \otimes \mathcal{H}) \otimes^n$ and similarly $\mathcal{H}^{* \otimes n}_k = \iota_k(\mathcal{H}^*) \subset \mathcal{H}^* \otimes^n$. The first line in (31) is the RE for $\mathcal{K}^{(k)}_k$, which holds since $\mathcal{H}^{* \otimes n}_k$ is a subalgebra in $\mathcal{H}^* \otimes^n$. The action of $\mathcal{H}^{\otimes 2}_k$ restricted to $\mathcal{H}^{* \otimes n}_k$ is induced by that of $\mathcal{H}^{\otimes 2}_k$ via the Hopf algebra projection $\mathcal{H}^{\otimes 2n} \to \mathcal{H}^{\otimes 2}_k$. Then the cross-relations between $\mathcal{H}^{* \otimes n}_k$ and $\mathcal{H}^{* \otimes n}_l$ are verified straightforwardly.

Remark 12. Relations (31) reflect the fact that $\mathcal{H}^* \otimes^n$ is a quasi-commutative $\mathcal{H}^{\otimes 2n}$-module algebra, cf. Corollary 8.

6.2. In this subsection, we give an interpretation of the braided bialgebra structure on $\mathcal{H}^*$ from the Drinfeld’s twist point of view. Let us denote by $\Delta \mathcal{H}^*$ the comultiplication $\Delta : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^*$ considered as a $k$-linear map from $\mathcal{H}^{* \otimes n}$.

Theorem 13 (Mj). The map $\Delta \mathcal{H}^*$ from $\mathcal{H}^*$ to the braided tensor product $\mathcal{H}^* \otimes \mathcal{H}^*$ is an algebra homomorphism.

The proof of this theorem that is given in [Mj] is quite complicated. Here we give another proof using Hopf algebra twist and the universal K-matrix. The key observation is that the braided tensor product of $\mathcal{H}^{\otimes k}$ and $\mathcal{H}^{\otimes m}$-module algebras, which is defined by (8), is in fact a particular case of twisted tensor product from (13), as pointed out in Remark 4, and hence a module algebra over $\mathcal{H}^{\otimes (k+m)}$. Thus we reformulate Theorem 13 in the following way.

Theorem 14. For any $n \in \mathbb{N}$ the $n$-fold coproduct $\Delta^n : \mathcal{H}^* \to \mathcal{H}^* \otimes^n$ considered as a $k$-linear map from $\mathcal{H}^*$ to the twisted tensor product $\mathcal{H}^* \otimes^n$ is an algebra homomorphism.

Proof. Put $\mathcal{K}^{(1 \ldots m)} = \mathcal{K}^{(1)} \ldots \mathcal{K}^{(m)} \in \mathcal{H} \otimes \mathcal{H}^* \otimes^m$, $m \in \mathbb{N}$. A straightforward induction using formula (13) shows that $(\text{id} \otimes \Delta^n)(\mathcal{K}) = \mathcal{K}^{(1 \ldots m)}$ for all $m \in \mathbb{N}$. We shall prove the theorem
by induction on \( n \). The case \( n = 0 \) holds because \( \varepsilon \HH^* \) is a character of \( \HH^* \), cf. Section 5.3 and the case \( n = 1 \) is trivial. Let us assume that the statement is true for \( n \geq 1 \). From the bottom line of equation (31), we find

\[
\mathcal{R}^{-1} \mathcal{K}_1^{(n+1)} \mathcal{R} \mathcal{K}_2^{(1...n)} = \mathcal{K}_2^{(1...n)} \mathcal{R}^{-1} \mathcal{K}_1^{(n+1)} \mathcal{R}.
\]

Put, in notation of Subsection 5.4, \( A = \HH^* \otimes (n+1) \) and \( a^i = \Delta_{\HH^*}^{n+1}(c^i) \in A \). Then, the element

\[
\mathcal{K}^A = \mathcal{K}^{(1...n+1)} = \mathcal{K}^{(1...n)} \mathcal{K}^{(n+1)}
\]

fulfills condition (26), as follows from the induction assumption and equation (32).

6.3. In this subsection we give, following [KS], a topological interpretation of the algebra \( \HH^* \otimes n \). It is known that a representation \( (\rho, V) \) of \( \HH \) induces a representation of the braid group \( B_n \) of \( n \) strands in \( \mathbb{R}^3 \). Let \( \{\sigma_i\}_{i=1}^{n-1} \) be the set of generators of \( B_n \) satisfying relations

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \ldots, n-2 \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |j-i| > 1.
\]

The representation of \( B_n \) on \( V \otimes n \) is defined via the correspondence \( \sigma_i \to S_{i,i+1} \), where \( S = PR \) is expressed through the image of the universal R-matrix \( R = (\rho \otimes \rho)(\mathcal{R}) \) and the flip operator \( P \) on \( V \otimes 2 \). As usually, the subscripts specialize embedding \( \text{End}(V) \subset \text{End}^\otimes (n)(V) \).

Let \( A_m \) denote a set of \( m \) parallel lines in \( \mathbb{R}^3 \). The braid group \( B_n^m \) of \( n \) strands in \( \mathbb{R}^3 \setminus A_m \) is generated by \( \{\sigma_i\}_{i=1}^{n-1} \subset B_n \subset B_n^m \) and the elements \( \{\tau_k\}_{k=1}^{m} \) subject to relations

\[
\sigma_{n-1} \tau_k \sigma_{n-1} \tau_k = \tau_k \sigma_{n-1} \tau_k \sigma_{n-1}, \quad k = 1, \ldots, m,
\]

\[
\sigma_{n-1} \tau_k \sigma_{n-1} \tau_l = \tau_l \sigma_{n-1} \tau_k \sigma_{n-1}, \quad k, l = 1, \ldots, m, \quad k > l.
\]

Together with \( \rho \), a representation \( \theta \) of the algebra \( \HH^* \otimes n \) on a module \( W \) defines a representation of \( B_n^m \) on the module \( V \otimes n \otimes W \). It extends the representation of the subgroup \( B_n \subset B_n^m \) in \( V \otimes n \) via the natural embedding \( V \otimes n \subset V \otimes n \otimes W \). Further, assigned to every \( \tau_k \) is the operator \( (\rho \otimes \theta) \circ (id \otimes \iota_k)(\mathcal{K}) \), where the embedding \( \iota_k \) is defined in Subsection 5.1.

7 Fusion rules for RE matrices

7.1. In this section we consider the following problem. Let \( (\rho^{(1)}, V^{(1)}) \) and \( (\rho^{(2)}, V^{(2)}) \) be two representations of \( \HH \). Let \( \mathcal{A} \) be an associative algebra and \( K^{(i)}, i = 1, 2 \), two constant RE matrices with coefficients in \( \mathcal{A} \) in the representations \( \rho^{(1)} \) and \( \rho^{(2)} \). How to build an RE matrix in the tensor product representation \( \rho^{(1)} \otimes \rho^{(2)} \) or in its sub-representation?
7.2. Let $\rho^{(i)}$ be representations of $\mathcal{H}$ on modules $V^{(i)}$, where $i$ runs over a set of indices, $i \in I$. Denote by $R^{(i,j)} = (\rho^{(i)} \otimes \rho^{(j)})(\mathcal{R})$ the image of the universal R-matrix in $\text{End}(V^{(i)}) \otimes \text{End}(V^{(j)})$, $i, j \in I$. These matrices satisfy the Yang-Baxter equation

$$R^{(i,j)}_{12} R^{(i,k)}_{13} R^{(j,k)}_{23} = R^{(j,k)}_{23} R^{(i,k)}_{13} R^{(i,j)}_{12}, \quad i, j, k \in I.$$  

Let $K^{(i)} \in \text{End}(V^{(i)}) \otimes A$ be RE matrices in representations $\rho^{(i)}$, $i \in I$, and consider the elements

$$K^{(i,j)} = (R^{(i,j)})^{-1} K^{(i)}_1 R^{(i,j)} K^{(j)}_2 \in \text{End}(V^{(i)} \otimes V^{(j)}) \otimes A, \quad i, j \in I. \quad (35)$$

**Proposition 15.** Suppose the RE matrices have the form $K^{(i)} = (\rho^{(i)} \otimes f)(\mathcal{K})$, $i \in I$, where $f$ is an algebra homomorphism $\mathcal{H}^* \rightarrow A$. Then, the elements $K^{(i,j)}$ defined by (35) are RE matrices in the tensor product representation $\rho^{(i)} \otimes \rho^{(j)}$. They can be restricted to any sub-representation.

**Proof.** Apply the algebra homomorphism $(\rho^{(i)} \otimes \rho^{(j)} \otimes f) \circ (\Delta \otimes \text{id})$ to the universal K-matrix and use Proposition 10. \qed

7.3. Proposition 13 allows to build tensor product of solutions to the RE of special type, namely when they are images of the same universal K-matrix (cf. concluding remarks below) via the same homomorphism $f$. Since a priori there are no criterias for this to be true, one has to substitute this condition by certain compatibility requirements.

**Definition 3.** An RE data is a set $\{(\rho^{(i)}, V^{(i)}, K^{(i)})\}_{i \in I}$ of triples, where $\rho^{(i)}$ are finite dimensional representations of $\mathcal{H}$ on $V^{(i)}$ and the matrices $K^{(i)} \in \text{End}(V^{(i)}) \otimes A$ satisfy the equations

$$R^{(j,i)}_{21} K^{(i)}_1 R^{(i,j)} K^{(j)}_2 = K^{(j)}_2 R^{(j,i)}_{21} K^{(i)}_1 R^{(i,j)} \quad (36)$$

in $\text{End}(V^{(i)} \otimes V^{(j)}) \otimes A$, $i, j \in I$.

Clearly restriction of the set I to any its subset gives an RE data. In particular, each triple $(\rho^{(i)}, V^{(i)}, K^{(i)})$ is an RE data for $I = \{i\}$; condition (36) reduces to the RE equation on $K^{(i)}$. Given an RE triple $(\rho, V, K)$ and a set I one can form an RE data putting $(\rho^{(i)}, V^{(i)}, K^{(i)}) = (\rho, V, K)$ for each $i \in I$. Also, for any homomorphism $f: \mathcal{H}^* \rightarrow A$ the matrices $K^{(i)} = (\rho^{(i)} \otimes f)(\mathcal{K})$ satisfy condition (36).

\[\text{Even in the case } A = \mathbb{C} \text{ and the fundamental vector representation of } \mathcal{U}_q(gl(n)) \text{ there is a variety of solutions to the same RE, } \mathcal{K} \mathrm{SS} \mathrm{S}.\]
Proposition 16. Consider a triple $(\rho, V, K)$ from an RE data $\mathcal{K}$. Suppose the module $V$ is semisimple. Let $(\rho_0, V_0)$ be a sub-representation of $(\rho, V)$ and $K_0$ the restriction of $K$ to $V_0$ from Proposition 1. Then, the set $\mathcal{K} \cup \{(\rho_0, V_0, K_0)\}$ is an RE data.

Proof. The apparent modification of the proof of Proposition 1. \hfill \Box

The following theorem generalizes Proposition 15.

Theorem 17 (Fusion procedure). Let $\{(\rho^{(i)}, V^{(i)}, K^{(i)})\}_{i \in I}$ be an RE data. Then, the union

$$\big\{(\rho^{(i)}, V^{(i)}, K^{(i)})\}_{i \in I} \cup \big\{(\rho^{(i)} \otimes \rho^{(j)}, V^{(i)} \otimes V^{(j)}, K^{(i,j)})\}_{(i,j) \in I \times I},$$

where $K^{(i,j)}$ is defined in (35), is an RE data, too.

Proof. See Appendix. \hfill \Box

Remark 18. It is easy to prove "associativity" of the fusion: $K^{\{i,j,k\}} = K^{\{i,\{j,k\}\}}$.

Let us apply Proposition 16 and Theorem 17 to the quantum group $U_q(\mathfrak{g})$, where \( \mathfrak{g} \) is a semisimple Lie algebra. Its finite dimensional modules are semisimple, and each irreducible representation can be realized as a submodule in the tensor power of a fundamental representation, \cite{ChP3}. Starting from an RE matrix in any fundamental representation, one can build an RE matrix in every irreducible representation by tensoring and projecting.

8 Concluding remarks

Suppose a quasitriangular Hopf algebra $\mathcal{H}$ is a subbialgebra in a bialgebra $\mathcal{B}$ which is quasitriangular in the sense that condition (3) is fulfilled for any $x \in \mathcal{B}$. One can take as $\mathcal{B}$ the dual to the FRT bialgebra associated with a finite dimensional representation of $\mathcal{H}$. All the constructions of this paper can be literally carried over from the dual Hopf algebra $\mathcal{H}^*$ to the bialgebra $\mathcal{B}^*$, which is a quasi-commutative bimodule algebra over $\mathcal{H}$. One can twist $\mathcal{B}^*$ along the line of Proposition 3 to obtain an $\mathcal{H}$-$\mathcal{S}$-$\mathcal{H}$-module algebra, $\tilde{\mathcal{B}}^*$. There is a braided bialgebra structure on $\tilde{\mathcal{B}}^*$ given by the coproduct of $\mathcal{B}^*$, which is considered as a map from $\mathcal{B}^*$ to $\mathcal{B}^* \otimes \mathcal{B}^*$. The algebra $\mathcal{B}^*$ is a comodule over $\mathcal{H}^{\text{cop}}$ and the counit $\epsilon_{\mathcal{B}^*}$ is its character. So there is an $\mathcal{H}$-equivariant homomorphism $Q: \tilde{\mathcal{B}}^* \to \mathcal{H}$ defined by formula (29) where $a \in \tilde{\mathcal{B}}^*$. Likewise for $\mathcal{B} = \mathcal{H}$, one can introduce the universal K-matrix $\mathcal{K} \in \mathcal{B} \otimes \tilde{\mathcal{B}}^*$, which is the
canonical element of the bialgebra pairing between $B$ and $B^*$. It will satisfy equations (23) and (24) in $B \otimes B \otimes \tilde{B}^*$. Any representation of $B$ gives a solution to the corresponding RE matrix, and the formula (13) provides a fusion procedure for such RE matrices.

There is an algebra map $\tilde{B}^* \rightarrow \tilde{H}^*$ coming from the bialgebra homomorphism $B^* \rightarrow H^*$; the latter is dual to the embedding $H \rightarrow B$. Therefore $\tilde{B}^*$ gives, in general, more solutions to the RE than $\tilde{H}^*$ does. If, for instance, $H$ is a factorizable Hopf algebra, $[RS]$, the map $Q: \tilde{H}^* \rightarrow H$ is an algebra isomorphism. So the characters of $\tilde{H}^*$ are exactly those of $H$. Consider, e.g., the quantum group $U_q(gl(n))$. It has only one-parameter family of one-dimensional representations assigning a scalar to the central generator. On the other hand, there are much more solutions to the RE in the fundamental vector representation of $U_q(gl(n))$ with the same R-matrix, $[KSS, M]$. They are characters of $\tilde{B}^*$, which is in this case the matrix RE algebra associated with the given representation of $U_q(gl(n))$.

9 Appendix: proof of Theorem 17

9.1. First we claim that for every $i, j, k \in I$ the following identities hold true:

\[
R^{(k,j)}_{32} R^{(k,i)}_{31} \left\{ \left( R^{(i,j)}_{12} \right)^{-1} K_1^{(i)} R^{(i,j)}_{12} K_2^{(j)} \right\} R^{(i,k)}_{13} R^{(j,k)}_{23} \{ K_3^{(k)} \} = \\
= \{ K_3^{(k)} \} R^{(k,j)}_{32} R^{(k,i)}_{31} \left\{ \left( R^{(i,j)}_{12} \right)^{-1} K_1^{(i)} R^{(i,j)}_{12} K_2^{(j)} \right\} R^{(i,k)}_{13} R^{(j,k)}_{23} , \tag{38}
\]

\[
R^{(j,i)}_{21} R^{(k,i)}_{31} K_1^{(i)} R^{(i,j)}_{13} R^{(j,k)}_{23} \left\{ \left( R^{(j,k)}_{23} \right)^{-1} K_2^{(j)} R^{(j,k)}_{23} K_3^{(k)} \right\} = \\
= \left\{ \left( R^{(j,k)}_{23} \right)^{-1} K_2^{(j)} R^{(j,k)}_{23} K_3^{(k)} \right\} R^{(j,i)}_{21} R^{(k,i)}_{31} K_1^{(i)} R^{(i,j)}_{13} . \tag{39}
\]

Let us prove equation (38). Pulling $R^{(k,j)}_{32}$ from left to right in the l.h.s. of (38) and using the Yang-Baxter equation twice we obtain

\[
\left( R^{(i,j)}_{12} \right)^{-1} R^{(k,i)}_{31} K_1^{(i)} R^{(i,k)}_{13} R^{(i,j)}_{32} K_2^{(j)} R^{(j,k)}_{23} K_3^{(k)} .
\]

Now pulling $K_3^{(k)}$ from right to left and employing (30) twice, we get

\[
K_3^{(k)} \left( R^{(i,j)}_{12} \right)^{-1} R^{(k,i)}_{31} K_1^{(i)} R^{(i,k)}_{13} R^{(i,j)}_{32} K_2^{(j)} R^{(j,k)}_{23} .
\]

This time we pull $R^{(k,j)}_{32}$ from right to to left and apply the Yang-Baxter equation twice to obtain the r.h.s. of (38).

Now we turn to equation (39). Pulling $\left( R^{(j,k)}_{23} \right)^{-1}$ to the left in the l.h.s. and using the Yang-Baxter equation twice we come to

\[
\left\{ \left( R^{(j,k)}_{23} \right)^{-1} \right\} R^{(k,i)}_{31} R^{(j,i)}_{21} K_1^{(i)} R^{(i,j)}_{12} R^{(i,k)}_{13} \left\{ K_2^{(j)} R^{(j,k)}_{23} K_3^{(k)} \right\} .
\]
This is equal
\[ \left\{ \left( R^{(j,k)}_{23} \right)^{-1} K_2^{(j)} \right\} R^{(k,j)}_{31} R^{(j,i)}_{21} \left\{ K_1^{(i)} \right\} R^{(i,k)}_{12} R^{(i,j)}_{13} \left\{ R^{(j,k)}_{23} K_3^{(k)} \right\}, \]
due to (33). Pulling \( R^{(j,k)}_{23} \) to the left and apply the Yang-Baxter equation twice we obtain
\[ \left\{ \left( R^{(j,k)}_{23} \right)^{-1} K_2^{(j)} R^{(j,k)}_{23} \right\} R^{(j,i)}_{21} R^{(k,i)}_{31} \left\{ K_1^{(i)} \right\} R^{(i,k)}_{13} R^{(i,j)}_{12} \left\{ K_3^{(k)} \right\}. \]

Now we push \( K_3^{(k)} \) to the left, employ (33), and come to the r.h.s. of (33).

9.2. Now we prove that the matrix \( K^{(i,j)}, i, j \in I \), obeys the reflection equation in the representation \( \rho^{(i)} \otimes \rho^{(j)} \). The image of the universal R-matrix in the representation \( \rho^{(i)} \otimes \rho^{(j)} \) is equal to
\[ R^{(i,j)}_{14} R^{(i,i)}_{13} R^{(j,j)}_{24} R^{(j,i)}_{23} \in \{ \text{End}(V^{(i)}) \otimes \text{End}(V^{(j)}) \} \otimes \{ \text{End}(V^{(i)}) \otimes \text{End}(V^{(j)}) \}. \]
The left-hand side of the RE on \( K^{(i,j)} \) reads
\[ R^{(i,j)}_{32} R^{(i,i)}_{31} R^{(j,j)}_{42} R^{(j,i)}_{41} \left\{ \left( R^{(i,j)}_{12} \right)^{-1} K_1^{(i)} R^{(i,j)}_{12} K_2^{(j)} \right\} \times \]
\[ \times R^{(i,j)}_{14} R^{(i,i)}_{13} R^{(j,j)}_{24} R^{(j,i)}_{23} \left\{ \left( R^{(i,j)}_{34} \right)^{-1} K_3^{(i)} R^{(j,j)}_{34} K_4^{(j)} \right\}, \]
while the r.h.s. of the RE is obtained from this by placing the last factor concluded in \( \} \) to the leftmost position. Pulling \( \left( R^{(i,j)}_{34} \right)^{-1} \) to the left and using the Yang-Baxter equation four times, we obtain
\[ \left\{ \left( R^{(i,j)}_{34} \right)^{-1} \right\} R^{(j,i)}_{42} R^{(i,i)}_{41} R^{(j,j)}_{32} R^{(j,i)}_{31} \left\{ \left( R^{(i,j)}_{12} \right)^{-1} K_1^{(i)} R^{(i,j)}_{12} K_2^{(j)} \right\} \times \]
\[ \times R^{(i,i)}_{13} R^{(j,j)}_{23} R^{(j,j)}_{24} R^{(j,i)}_{23} \left\{ \left( R^{(i,j)}_{34} \right)^{-1} K_3^{(i)} R^{(j,j)}_{34} K_4^{(j)} \right\}. \]

From this, using (38) for \( k = i \), we come to
\[ \left\{ \left( R^{(i,j)}_{34} \right)^{-1} K_3^{(i)} \right\} R^{(j,i)}_{42} R^{(i,i)}_{41} R^{(j,j)}_{32} R^{(j,i)}_{31} \left\{ \left( R^{(i,j)}_{12} \right)^{-1} K_1^{(i)} R^{(i,j)}_{12} K_2^{(j)} \right\} \times \]
\[ \times R^{(i,i)}_{13} R^{(j,j)}_{23} R^{(j,j)}_{24} R^{(j,i)}_{23} \left\{ \left( R^{(i,j)}_{34} \right)^{-1} K_4^{(i)} \right\}. \]

Now we push \( R^{(i,j)}_{34} \) to the left and use the Yang-Baxter equation four times. The resulting expression is
\[ \left\{ \left( R^{(i,j)}_{34} \right)^{-1} K_3^{(i)} R^{(i,j)}_{34} \right\} R^{(j,i)}_{32} R^{(i,i)}_{31} R^{(j,j)}_{42} R^{(j,i)}_{41} \left\{ \left( R^{(i,j)}_{12} \right)^{-1} K_1^{(i)} R^{(j,j)}_{12} K_2^{(j)} \right\} \times \]
\[ \times R^{(i,j)}_{14} R^{(j,i)}_{24} R^{(i,i)}_{13} R^{(j,j)}_{23} \left\{ K_4^{(j)} \right\}. \]

To accomplish the proof, one should push \( K_4^{(j)} \) to the left employing the identity (38) where \( k \) is equal to \( j \) and 3 is replaced by 4.
9.3. It remains to check that for any choice of $i, j, k \in I$, the matrices $K^{(i,j)}$, $K^{(k)}$ are compatible in the sense of (38). This statement splits into several assertions. First of all, the matrices $K^{(i,j)}$ and $K^{(k)}$ must solve the RE in their own representations. In what concerns $K^{(k)}$, this holds by assumption; as to the matrix $K^{(i,j)}$, this has been proven in the previous subsection. We must verify the cross-relations between $K^{(i,j)}$ and $K^{(k)}$. It is not difficult to see that they are encoded in the system of equations (38–39).

References

[ChPr] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[Dr1] V. G. Drinfeld: *Quantum Groups*, in Proc. Int. Congress of Mathematicians, Berkeley, 1986, ed. A.V. Gleason, AMS, Providence (1987) 798.

[Dr2] V. G. Drinfeld: *Quasi-Hopf algebras*, Leningrad Math. J. 1 (1990) 1419–1457.

[DM] J. Donin and A. Mudrov: *Reflection Equation, Twist, and Equivariant Quantization* Isr. J. Math., in press [math.QA/0204295].

[DeN] G. W. Delius and R. I. Nepomechie: *Solutions of the boundary Yang-Baxter equation for arbitrary spin*, J. Phys. A: Math. Gen. 35 (2002) L341-L348.

[FRT] L. Faddeev, N. Reshetikhin, and L. Takhtajan: *Quantization of Lie groups and Lie algebras*, Leningrad Math. J., 1 (1990) 193-226.

[KRS] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin: *Yang-Baxter equation and representation theory*, Lett. Math. Phys. 5 (1981) 393-403.

[KSk] P. P. Kulish, E. K. Sklyanin: *Algebraic structure related to the reflection equation*, J. Phys. A, 25 (1992) 5963–5975.

[KS] P. P. Kulish, R. Sasaki: *Covariance properties of reflection equation algebras*, Prog. Theor. Phys. 89 #3 (1993) 741–761.

[KSS] P. P. Kulish, R. Sasaki, and C. Schweibert: *Constant solutions of reflection equations and quantum groups*, J. Math. Phys., 34 # 1 (1993) 286–304.

[M] A. I. Mudrov: *Characters of the $\mathcal{U}_q(sl(n))$-reflection equation algebra*, Lett. Math. Phys, 60 (2002) 283–291.
[MeN] L. Mezincescu and R. I. Nepomechie: *Fusion procedure for open chains* J. Phys. A: Math. Gen. 25 (1992) 2533–2543.

[Mj] S. Majid: *Foundations of quantum group theory*, Cambridge University Press, 1995.

[RS] N. Yu. Reshetikhin, M. A. Semenov-Tian-Shansky: *Quantum R-matrices and factorization problem*, J. Geom. Phys. 5 (1988), 533–550.

[S] M. A. Semenov-Tian-Shansky: *Poisson Lie Groups, Quantum Duality Principle, and the Quantum Double*, Contemp. Math. 175 (1994) 219–248.