GENUS TWO CURVES ON ABELIAN SURFACES

ANDREAS LEOPOLD KNUTSEN AND MARGHERITA LELLI-CHIESA

ABSTRACT. This paper deals with singularities of genus 2 curves on a general $(d_1, d_2)$-polarized abelian surface $(S, L)$. In analogy with Chen’s results concerning rational curves on K3 surfaces [Ch1, Ch2], it is natural to ask whether all such curves are nodal. We prove that this holds true if and only if $d_2$ is not divisible by 4. In the cases where $d_2$ is a multiple of 4, we exhibit genus 2 curves in $|L|$ that have a triple, 4-tuple or 6-tuple point. We show that these are the only possible types of unnodal singularities of a genus 2 curve in $|L|$. Furthermore, with no assumption on $d_1$ and $d_2$, we prove the existence of at least one nodal curve in $|L|$. As a corollary, we obtain nonemptiness of all Severi varieties on general abelian surfaces and hence generalize [KLM, Thm. 1.1] to nonprimitive polarizations.

1. INTRODUCTION

The minimal geometric genus of any curve lying on a general abelian surface is 2 and there are finitely many curves of such genus in a fixed linear system. The role of genus two curves on abelian surfaces is thus analogous to that of rational curves on K3 surfaces, but until now it has not been investigated as extensively. Their enumeration is now well understood. Their count in the primitive case was carried out by Göttsche [Go], Debarre [De] and Lange-Sernesi [LS1], and used in [De] in order to compute the Euler characteristic of generalized Kummer varieties. Only recently, Bryan, Oberdieck, Pandharipande and Yin [BOPY] handled the nonprimitive case, thus obtaining a formula parallel to the full Yau-Zaslow conjecture for rational curves on K3 surfaces (cf. [KMPS]).

Singularities of rational curves on K3 surfaces have received plenty of attention. Mumford [MM, Appendix] first proved the existence of a nodal rational curve in the primitive linear system $|L|$ on a general genus $g$ polarized K3 surface $(S, L)$; as a byproduct, he obtained nonemptiness of the Severi variety $|L|_\delta$ parametrizing $\delta$-nodal curves in $|L|$ for any integer $0 \leq \delta \leq g$. His results were then generalized by Chen [Ch1, Ch2] to nonprimitive linear systems. In the primitive case, Chen managed to deal with all rational curves in $|L|$ showing that they are all nodal; the analogue for nonprimitive linear systems is still an open problem.

Singularities of genus 2 curves on abelian surfaces are not as well understood, even though they are necessarily ordinary (cf. [LS2, Prop. 2.2]). The natural question whether any genus 2 curve on a general $(d_1, d_2)$-polarized abelian surface is nodal [LC, Pb. 2.7] has negative answer if one does not make any assumption on $d_1$ and $d_2$. Indeed, multiplication by 2 on a principally polarized abelian surface $(A, L)$ identifies the six Weierstrass points of its theta divisor, whose image is thus a genus 2 curve with a 6-tuple point lying in (a translate) of the linear system $|L^{\otimes 4}|$ (cf. Example 2). Since this is a polarization of type $(4, 4)$, all genus 2 curves may still be expected to be nodal in primitive linear systems (or even in linear systems not divisible by 4, cf. [LC, Conj. 2.10]). Our main result is that this expectation does not hold in its full generality and detects all the cases where it fails, thus completely answering the question.
Theorem 1.1. Let \((S, L)\) be a general abelian surface with a polarization of type \((d_1, d_2)\). Then any genus 2 curve in the linear system \(|L|\) is nodal if and only if 4 does not divide \(d_2\).

When \(d_2\) is a multiple of 4, we exhibit genus 2 curves in \(|L|\) that have an unnodal singularity and, more precisely, a triple, a 4-tuple or a 6-tuple point (cf. Examples 1 and 2). We also show that these are the only types of unnodal singularities that a genus 2 curve on a general abelian surface may acquire (cf. Remark 1). To our knowledge, the best bound on the order of such a singularity in the literature was \(\frac{1}{2} (1 + \sqrt{8d_1 d_2} - 7)\) by Lange-Sernesi, cf. [LS2 Prop. 2.2]. The existence of unnodal genus 2 curves in all primitive linear systems of type \((1, 4k)\) is quite striking and highlights a major difference with the \(K3\) case.

When 4 divides \(d_2\), the above theorem does not exclude the existence in \(|L|\) of some nodal genus 2 curves. This is indeed proved in the following:

Theorem 1.2. Let \((S, L)\) be a general \((d_1, d_2)\)-polarized abelian surface. Then the linear system \(|L|\) contains a nodal curve of genus 2.

Given a nodal genus 2 curve as above, standard deformation theory enables one to smooth any of its nodes independently remaining inside the linear system \(|L|\). As a consequence, Theorem 1.2 yields nonemptiness of all Severi varieties on general abelian surfaces:

Corollary 1.3. Let \((S, L)\) be a general \((d_1, d_2)\)-polarized abelian surface. Then, for any \(0 \leq \delta \leq d_1 d_2 - 1\) the Severi variety \(|L|_\delta\) is nonempty and smooth of dimension equal to \(d_1 d_2 - 1 - \delta\).

This generalizes [KLM Thm. 1.1] to nonprimitive linear systems. Note that, since \(S\) has trivial canonical bundle, the regularity of \(|L|_\delta\) stated in Corollary 1.3 follows for free from its nonemptiness by the proofs of Propositions 1.1 and 1.2 in [LS2]. We mention that the irreducible components of the Severi varieties on a general primitively polarized abelian surface have been determined very recently by Zahariuc in [Za].

We now spend some words on the proofs of Theorems 1.1 and 1.2. In contrast to the methods proposed in [Ch1 Ch2 KLM], we need neither to degenerate \(S\) to a singular surface nor to specialize it to an abelian surface with large Neron-Severi group. Instead, we exploit the universal property of Jacobians in order to translate the if part of Theorem 1.1 and Theorem 1.2 into the following statement concerning Brill-Noether theory on a general curve of genus 2:

Theorem 1.4. Let \(|C| \in \mathcal{M}_2\) be a general genus 2 curve and fix any integer \(d \geq 4\). If \(C\) admits a \(g^4\) totally ramified at three points \(P_1, P_2, P_3\), then \(d\) is even and \(P_1, P_2, P_3\) are Weierstrass points.

We refer to Section 2 for the details of this reduction, that we mention here only briefly. The key fact is that any genus 2 curve \(C\) on a \((d_1, d_2)\)-polarized abelian surface \(S\) with normalization \(\bar{C}\) arises as image of a composition

\[
C \xrightarrow{u} J(C) \xrightarrow{\lambda} S,
\]

where \(u\) is the Abel-Jacobi map and \(\lambda\) is an isogeny. Three points \(P_1, P_2, P_3 \in C = u(C)\) identified by \(\lambda\) necessarily differ by elements in its kernel. Since the order of any such element is divisible by \(d_2\), the three divisors \(d_2 P_1, d_2 P_2, d_2 P_3 \in C_d\) are linearly equivalent and thus define (for \(d_2 \geq 4\)) a \(g^2\) on \(C\) totally ramified at three points. Theorem 1.4 excludes the existence of such a linear series for \(C\) general and odd values of \(d_2\), thus implying our main results in these cases. If instead \(d_2\) is even, a \(g^2\) totally ramified at three points does exist: as soon as \(P_1, P_2, P_3\) are Weierstrass points of \(C\), the divisors
2P_1, 2P_2, 2P_3 are linearly equivalent and thus the same holds true for d_2P_1, d_2P_2, d_2P_3. Conversely, by Theorem 1.4, any g^2 d_2 with three points of total ramification on C is of this type. This characterization is used in Section 2 to prove the if part of Theorem 1.1 and Theorem 1.2 for d_2 \equiv 2 \mod 4, and to provide examples of genus 2 curves with a triple, 4-tuple or 6-tuple point (cf. Examples 1 and 2) when d_2 \equiv 0 \mod 4 implying the only if part of Theorem 1.1. These examples are based on the construction of suitable isogenies \lambda as in (1) or, equivalently by taking their kernels, suitable isotropic (with respect to the commutator pairing) subgroups of the group J(C)[d_1d_2] of d_1d_2-torsion points of J(C).

Section 3 is devoted to the proof of Theorem 1.4. This is done by a double degeneration. First, we degenerate C to the transversal union of two elliptic curves meeting at a point and reduce Theorem 1.4 into a statement of Brill-Noether theory with ramification on a general elliptic curve (cf. Proposition 3.1). This reduction seriously involves the theory of limit linear series on curves of compact type, for which we refer to the original papers by Eisenbud and Harris [EH1, EH2, EH3]. Proposition 3.1 is then proved by means of a second degeneration to a a cycle of rational curves. Since this curve is not of compact type, limit linear series cannot be applied and we invoke results by Esteves [E5] and to provide examples of genus 2 curves with a triple, 4-tuple or 6-tuple point (cf. Examples 1 and 2) when d_2 \equiv 0 \mod 4 implying the only if part of Theorem 1.1.

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2. **Polarized isogenies and proof of the main theorems**

In this section we review some known facts concerning polarized isogenies and genus 2 curves on complex abelian surfaces and reduce the proof of Theorems 1.1 and 1.2 to a statement concerning Brill-Noether theory with prescribed ramification on a general curve of genus 2.

2.1. **Polarized isogenies and genus 2 curves.** Let S be an abelian surface defined over \mathbb{C} and consider a genus 2 curve \overline{C} \subset S such that the line bundle \mathcal{L} := \mathcal{O}_S(\overline{C}) is a polarization of type (d_1, d_2). The normalization map \nu : C \to \overline{C} \subset S then factors as

\[ C \xrightarrow{u} J(C) \xrightarrow{\lambda} S, \]

where u is the Abel-Jacobi map (that is an embedding only defined up to translation) and \lambda is an isogeny. We set \Lambda := J(C).

By the Push-Pull formula, the above isogeny \lambda has degree d_1d_2 and thus \lambda^* \mathcal{L} \simeq L_1^{\otimes d_1}L_2^{\otimes d_2}, where L_1 is a principal polarization on A. We write A = V/\Lambda, where V is a 2-dimensional \mathbb{C}-vector space and \Lambda is a rank 4 lattice. Chosen a symplectic basis \lambda_1, \lambda_2, \mu_1, \mu_2 of \Lambda, we denote by \epsilon_1 := \lambda_1/(d_1d_2), \epsilon_2 := \lambda_2/(d_1d_2), \mu_1 := \mu_1/(d_1d_2), \mu_2 := \mu_2/(d_1d_2) the standard generators of the group A[d_1d_2] of d_1d_2-torsion points of A. By definition, the commutator pairing on A[d_1d_2] is the nondegenerate multiplicative alternating form

\[ e_{d_1d_2} : A[d_1d_2] \times A[d_1d_2] \to \mathbb{C}^* \]

that takes value 1 on all pairs of standard generators with the only two following exceptions:

\[ e_{d_1d_2}(\epsilon_1, \mu_1) = e_{d_1d_2}(\epsilon_2, \mu_2) = e^{\frac{2\pi i}{d_1d_2}}. \]
For a fixed principally polarized abelian surface $A$ with a fixed theta divisor $\Theta$, by [Mu, §23] there is a bijection between the following two sets:

(*) polarized isogenies $\lambda: A \to S$ onto abelian surfaces $S$ such that $\lambda(\Theta) \in |L|$ for some polarization $L$ on $S$ of type $(d_1, d_2)$;

(**) isotropic subgroups $G$ of $A[d_1d_2]$ of cardinality $d_1d_2$ such that $G^\perp /G \simeq \mathbb{Z}^{\oplus 2}_{d_1} \oplus \mathbb{Z}^{\oplus 2}_{d_2}$.

Indeed, the kernel $G$ of any isogeny $\lambda$ in (*) is a subgroup of $A[d_1d_2]$ in (**); furthermore, $G^\perp /G$ is isomorphic to the kernel $K(L)$ of the isogeny defined by $L$:

$$\phi_L : S \to \hat{S}, \quad \phi_L(x) = t_x^*L \otimes L^\vee,$$

where $t_x$ denotes the translation by $x$ on $S$. Vice versa, given a subgroup $G$ in (**), the quotient map $\lambda : A \to A/G$ is an isogeny as in (*).

Given $\lambda$ as in (*), let

$$\hat{\lambda} : \hat{S} \to \hat{A}$$

be its dual isogeny and denote by $\hat{L}$ the dual polarization of $L$. Again by [Mu, §23] the kernel $\hat{G}$ of $\hat{\lambda}$ is a maximal isotropic subgroup of $K(\hat{L}) \simeq \mathbb{Z}^{\oplus 2}_{d_1} \oplus \mathbb{Z}^{\oplus 2}_{d_2}$. On the other hand, $\hat{G}$ is the character group of $G$ (cf. [BL, Prop. 2.4.3]) and thus $\hat{G} \simeq G$. In particular, the order of any element of $G$ divides $d_2$.

2.2. Reduction of Theorem 1.1 to Theorem 1.4 for odd values of $d_0$. Since isogenies are étale, all singularities of the image $\lambda(\Theta)$ of a theta divisor under an isogeny $\lambda$ as in (*) are ordinary (cf. [LS2, Prop. 2.2]). The only pathology that might prevent $\lambda(\Theta)$ from being nodal is thus the existence of three points $x, y, z \in \Theta$ such that $\lambda(x) = \lambda(y) = \lambda(z)$. Since the order of any element in the kernel of $\lambda$ divides $d_2$, such a triple of points $x, y, z$ would be identified by the multiplication map

$$m_{d_2} : A \to A;$$

equivalently, if $A = J(C)$, the three divisors $d_2x, d_2y, d_2z$ on $C$ would be linearly equivalent. For $d_2 \geq 4$ this implies the existence of a $g^2_{d_2}$ on $C$ totally ramified at $x, y, z$. In the cases $d_2 = 2$ and $d_2 = 3$ the same conclusion holds up to replacing $d_2$ with a multiple of it. Theorem 1.1 for odd values of $d_2$ then follows from Theorem 1.4.

2.3. Theorem 1.4 for even values of $d_2$. Theorem 1.4 also implies that the image of a theta divisor under an isogeny $\lambda$ as in (*) for even values of $d_2$ may have non-nodal singularities only at the image of its Weierstrass points. In order to analyze this possibility, we denote by $\epsilon_1 := \lambda_1/2$, $\epsilon_2 := \lambda_2/2$, $f_1 := \mu_1/2$, $f_2 := \mu_2/2$ the standard generators of the group $A[2]$ of 2-torsion points of $A$. As the Abel-Jacobi map $u : C \to J(C) = A$ is defined up to translation, we may assume its image to coincide with a symmetric theta divisor $\Theta$ so that (the image under $u$ of) the six Weierstrass points of $C$ are exactly the 2-torsion points of $A$ contained in $\Theta$, namely, without loss of generality by [BL, Ex. 10.2.7], the points:

$$\epsilon_1, f_1, \epsilon_1 + f_1, \epsilon_2, f_2, \epsilon_2 + f_2.$$

Theorem 1.4 for $d_2 \equiv 2 \mod 4$ then follows from the following Lemma.

Lemma 2.1. Let $G$ be an isotropic subgroup of $A[d_1d_2]$ such that $|G| = d_1d_2$, $G^\perp /G \simeq \mathbb{Z}^{\oplus 2}_{d_1} \oplus \mathbb{Z}^{\oplus 2}_{d_2}$ and at least three among the six 2-torsion points in (2) lie in the same $G$-orbit of $A[d_1d_2]$. Then, one necessarily has $d_2 \equiv 0 \mod 4$. 

Proof. Up to exchanging the $e_i$'s with the $f_j$'s and up to relabelling the indices $i$, we may assume that one of the three points in the same $G$-orbit of $A[d_1d_2]$ is $e_1$. It follows that $G$ contains at least two elements $g_1, g_2$ in the following set:

$$\{e_1 + f_1, f_1, e_2 + e_1, f_2 + e_1, e_2 + f_2 + e_1\}.$$ (3)

One easily verifies that $g_1 := \frac{d_1d_2}{e_1}g_1'$ and $g_2 := \frac{d_1d_2}{e_2}g_2'$ where $g_1', g_2' \in A[d_1d_2]$ satisfy

$$e_{d_1d_2}(g_1', g_2') = e^{\pm \frac{2\pi i}{d_1d_2}}.$$ (4)

In particular, one has $e_{d_1d_2}(g_1, g_2) = e^{\pm \frac{2\pi i}{d_1d_2}}$ and obtains

$$d_1d_2 \equiv 0 \mod 4.$$ (5)

since $G$ is isotropic.

In order to exclude the case $d_2 \equiv 2 \mod 4$ (that would also imply $d_1 \equiv 2 \mod 4$ by (5)), we proceed by contradiction. By (4) along with the fact that $(g_1', g_2') \cong \mathbb{Z}_{d_1d_2} \oplus \mathbb{Z}_{d_1d_2}$, there exists an automorphism $\varphi$ of $A[d_1d_2]$ preserving the alternating form $e_{d_1d_2}$ such that $\varphi(g_1') = e'_1$ and $\varphi(g_2') = e'_2$. In particular, we may assume $g_1 = e_1$ and $g_2 = f_1$. We consider the group

$$K := \langle e_1, f_1 \rangle,$$

and its orthogonal $K^\perp = \langle 2e_1', 2f_1', e_2, f_2 \rangle$. It is trivial to check that

$$K^\perp/K \cong \mathbb{Z}_{d_1d_2}^{\oplus 2} \oplus \mathbb{Z}_{d_1d_2}^{\oplus 2} \cong \mathbb{Z}_{d_1d_2}^{\oplus 4} \oplus \mathbb{Z}_{d_1d_2}^{\oplus 4}$$

with $d_1d_2$ odd, where the second isomorphism follows from the assumption $d_2 \equiv 2 \mod 4$. The inclusions $K < G < G^\perp < K^\perp$ imply that

$$K^\perp/K > G^\perp/K$$

and $G^\perp/G \cong (G^\perp/K)/(K/K)$: in particular $G^\perp/G$ is a quotient of a subgroup of $K^\perp/K$. However, our assumption yields

$$G^\perp/G \cong \mathbb{Z}_{d_1d_2}^{\oplus 2} \oplus \mathbb{Z}_{d_1d_2}^{\oplus 2} \cong \mathbb{Z}_2^{\oplus 4} \oplus \mathbb{Z}_2^{\oplus 4} \oplus \mathbb{Z}_4^{\oplus 2} / 2$$

with $d_1d_2$ odd. As a consequence, $\mathbb{Z}_2^{\oplus 4}$ is a quotient of a subgroup of $K^\perp/K$ and thus of $\mathbb{Z}_4^{\oplus 2}$ by (6). This is a contradiction because the only quotient of a subgroup of $\mathbb{Z}_2^{\oplus 2}$ having cardinality 16 is $\mathbb{Z}_4^{\oplus 2}$ itself.\]

By the following example, as soon as $d_2 \equiv 0 \mod 4$, there do exist isotropic subgroups $G$ of $A[d_1d_2]$ as in Lemma 2.1. As a consequence, a general polarized abelian surface of type $(d_1, d_2)$ contains an unnodal genus 2 curve and the only if part of Theorem 1.1 follows.

**Example 1.** We fix positive integers $d_1, d_2, a, b$ such that $d_1|d_2$ and the relation $ab = d_1^2d_2$ holds. We consider the following subgroup of $A[d_1d_2]$

$$G := \langle a\epsilon_1', b\epsilon_2', d_2\epsilon_2' \rangle.$$ One has $|G| = d_1d_2$ and

$$G^\perp = \langle d_1d_2 \epsilon_1', d_1d_2 \epsilon_2', d_1d_2 \epsilon_2' \rangle,$$

and hence $G^\perp/G \cong \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2}$. In particular, the group $G$ corresponds to a polarized isogeny from the principally polarized abelian surface $A$ to a $(d_1, d_2)$-polarized abelian surface $(S, L)$. If both $a$ and $b$ divide $d_1d_2/2$ (and thus $ab = d_1^2d_2$ divide $(d_1d_2)^2/4$, or equivalently, $d_2 \equiv 0 \mod 4$), then $G$ contains the 2-torsion points $e_1, f_1, e_1 + f_1$ and we
find a genus 2 curve in \(|L|\) with a singularity that is (at least) a triple point. Note that for any values of \(d_1, d_2\) such that \(d_1|d_2\) and \(d_2 \equiv 0 \mod 4\), the integers \(a = d_1d_2/2\) and \(b = 2d_1\) satisfy all the above conditions. If moreover \(d_1\) is even, then \(G\) contains also the point \(e_2\) and the image of the theta divisor acquires a 4-tuple point.

To our knowledge the only example of an unnodal genus 2 curve on an abelian surface from the published literature until now was the image of the theta divisor under the multiplication by two on a principally polarized abelian surface. This example has played interesting roles in various works concerning curve singularities (cf. [DS Ex. 4.14]), Seshadri constants (cf. [St Pf. of Prop. 2], [Ba Rmk. 6.3], [KSS Ex. 4.2]) and construction of surfaces of general type (cf. [PP]). We recall and generalize this example:

**Example 2.** Assume we have an isogeny \(\lambda\) as in (*) identifying all the six Weierstrass \(\lambda\)-points of \(\Theta\). The group \(A[2]\) of 2-torsion points of \(A\) is necessarily contained in the kernel of such a \(\lambda\), that hence factors through the multiplication by 2

\[
m_2 : A \to A.
\]

In fact, the image \(m_2(\Theta)\) has only one singularity at the image of the six Weierstrass points, that is thus a 6-tuple point. Furthermore, \(m_2(\Theta) \in |L_1^{\otimes 4}|\) where \(L_1\) is a principal polarization on \(A\). As soon as \(d_1 \equiv 0 \mod 4\), one constructs a genus 2 curve with a sextuple point on a general \((d_1, d_2)\)-polarized abelian surface \((S, L)\) by composing \(m_2\) with an isogeny \(\lambda' : A \to S\) such that \(\lambda'(\Theta) \in |L'|\) where \(L'\) is a polarization on \(S\) satisfying \(L'^{\otimes 4} \cong L\).

**Remark 1.** Theorem [1.4] along with the fact that any smooth genus 6 curve has exactly 6 Weierstrass points yields that 6 is the maximal order of any singularity of a genus 2 curve on a general abelian surface. Examples [1] and [2] exhibit genus 2 curves with a triple, a 4-tuple or a 6-tuple point. It is natural to ask whether one can construct a genus 2 curve with a 5-tuple point. Such a curve would correspond to an isotropic subgroup \(G\) containing exactly 4 points in the set \(\{\}\). However, one easily verifies that any subset of \(\{\}\) having cardinality 4 generates the whole \(A[2]\). As a consequence, if one requires \(G\) to contain four points in \(\{\}\), then \(G\) contains the whole \(A[2]\) and one falls in Example 2 thus obtaining a 6-tuple and not a 5-tuple point.

**Remark 2.** While looking for a proof of Theorem [1.1] we realized that the proof of [DL Proposition 3.1] contains a gap since it somehow assumes that an isogeny between two principally polarized abelian surfaces \(\lambda : A \to B\) never identifies three or more points on the theta divisor of \(A\). In [DL] the abelian varieties are defined over an algebraically closed field \(k\) of arbitrary characteristic and the kernel of \(\lambda\) is a maximal isotropic subgroup of \(A[p]\) for some prime integer \(p \neq \text{char } k\). Theorem [1.4] repairs the mentioned gap for \(k = \mathbb{C}\).

2.4. Reduction of Theorem 1.2 to Theorem 1.4. We conclude this section by proving the following lemma, to which Theorem 1.2 reduces thanks to Theorem 1.4.

**Lemma 2.2.** For any positive integers \(d_1, d_2\) with \(d_1|d_2\), there exists an isotropic subgroup \(G\) of \(A[d_1d_2]\) as in (***) such that any \(G\)-orbit of \(A[d_1d_2]\) contains at most two points in \(\{\}\).

**Proof.** The group

\[
G := \langle d_1e_1', d_2e_2' \rangle
\]

is clearly isotropic and the only set of points contained in \(\{\}\) that may possibly lie in the same \(G\)-orbit are \(\{f_1, e_1 + f_1\}\) for even values of \(d_2\), and \(\{f_2, e_2 + f_2\}\) if \(d_1\) is even.
One easily checks that

$$G^\perp := \langle e_1', d_2 f_1', e_2', d_1 f_2' \rangle$$

so that $G^\perp / G \simeq \mathbb{Z}_{d_1}^{g^2} \oplus \mathbb{Z}_{d_2}^{g^2}$.

\[ \square \]

3. Proof of Theorem 1.4

We proceed by degeneration to a curve $C_0$ having two irreducible smooth elliptic components $E_1$ and $E_2$ meeting at a point $P$.

Let $\pi : C \to B$ be a flat family of curves over a local one-dimensional base $B$ (that is, $B = \text{Spec } \mathcal{O}$ for some discrete valuation ring $\mathcal{O}$) with special fiber $C_0$ and generic fiber $C_0$ being a smooth irreducible curve of genus $2$; also assume that the total space $C$ is smooth. A relative linear series of type $g_3^3$ on $C$ is a pair $l = (A, \mathcal{V})$ such that $A$ is a line bundle on $C$ flat over $B$ and $\mathcal{V}$ is a rank-$3$ subbundle of $\pi_*A$. We assume the existence of a linear series $l_b = (A_b, V_b)$ of type $g_3^3$ on the generic fiber $C_0$ of $\pi$ totally ramified at three points. Possibly after finitely many sequences of base changes and blow-ups at the nodes of the special fiber, we obtain a family $\pi' : C' \to B'$ such that:

(i) the generic fiber of $\pi'$ is again $C_0$;
(ii) the special fiber $C_0'$ of $\pi'$ is obtained from $C_0$ by inserting a chain of $h \geq 0$ rational curves at the node $P$;
(iii) $l_b$ is the restriction of a relative linear series $l = (A, \mathcal{V})$ on $C'$;
(iv) there are three sections $\sigma_1, \sigma_2, \sigma_3$ of $\pi'$ such that $l_b$ is totally ramified at the points $\sigma_1(b), \sigma_2(b), \sigma_3(b)$;
(v) the points $x_1 := \sigma_1(0), x_2 := \sigma_2(0), x_3 := \sigma_3(0)$ lie in the smooth locus of $C_0'$ (but are allowed to coincide).

We label the rational components inserted at $P$ with $\gamma_1, \ldots , \gamma_h$ and set $\gamma_0 := E_1$, $\gamma_{h+1} := E_2$ and $P_i := \gamma_{i-1} \cap \gamma_i$ for $1 \leq i \leq h + 1$. The restriction of $l$ to $C_0'$ is a (crude) limit linear series $[\mathcal{E}2]$, whose aspect on $\gamma_i$ (cf. [\mathcal{E}2] Def. p. 348) is denoted by $l_i = (A_i, V_i)$. If $P_j \in \gamma_i$, let $\alpha_j^i(P_j) = (\alpha_0^i(P_j), \alpha_1^i(P_j), \alpha_2^i(P_j))$ denote the ramification sequence of $l_i$ at $P_j$. We recall the following compatibility conditions $[\mathcal{E}2]$ p. 346:

\[ \alpha_{j-1}^i(P_j) + \alpha_{2-j}^i(P_i) \geq d - 2 \text{ for } 1 \leq i \leq h + 1 \text{ and } 0 \leq j \leq 2. \]

Furthermore, any two points $Q, Q'$ on the same component $\gamma_i$ satisfy:

\[ \alpha_j^i(Q) + \alpha_{2-j}^i(Q') \leq d - 2 \text{ for } 0 \leq i \leq h + 1 \text{ and } 0 \leq j \leq 2. \]

Since $E_2 = \gamma_{h+1}$ is elliptic, then $\alpha_1^{h+1}(P_{h+1}) \leq d - 3$ and thus $\alpha_2^{h+1}(P_{h+1}) \geq 1$ by (7).

Inequality (8) then yields $\alpha_1^h(P_h) \leq d - 3$. By the same argument, we obtain that

\[ \alpha_{i-1}^h(P_i) \geq 1 \text{ for } 1 \leq i \leq h + 1. \]

Analogously, using the fact that $E_1 = \gamma_0$ is elliptic, one proves that

\[ \alpha_i^1(P_i) \geq 1 \text{ for } 1 \leq i \leq h + 1. \]

In particular, $\gamma_0 = E_1$ has at least a cusp at $P_1$ and $\gamma_{h+1} = E_2$ has at least a cusp at $P_{h+1}$.

If $x_1$ lies on the component $\gamma_i$, then $\alpha_2^i(x_1) = d - 2$ and thus (8) yields $\alpha_0^i(P_i) = 0$ as soon as $i \neq 0$ and $\alpha_0^i(P_{i+1}) = 0$ for $i \neq h + 1$. By (7), we get that both $\alpha_{i-1}^i(P_i) \geq d - 2$ if $i \neq 0$ and $\alpha_2^{i+1}(P_{i+1}) \geq d - 2$ if $i \neq h + 1$. Inductively, we obtain

\[ \alpha_2^j(P_j) \geq d - 2 \text{ for } i + 1 \leq j \leq h + 1 \text{ (if } i \neq h + 1). \]

\[ \alpha_2^j(P_j) \geq d - 2 \text{ for } 0 \leq j \leq i - 1 \text{ (if } i \neq 0). \]
We get the same conclusion if \( x_2 \in \gamma_i \) or \( x_3 \in \gamma_i \). In particular, \( l_0 \) has total ramification at \( P_1 \) as soon as at least one among the points \( x_1, x_2, x_3 \) does not lie on \( \gamma_0 = E_1 \). Analogously, if at least one among \( x_1, x_2, x_3 \) lies outside of \( \gamma_{h+1} = E_2 \) we obtain that \( P_{h+1} \) is a total ramification point for \( l_{h+1} \).

By abuse of notation, we set \( \alpha^i(x_1) \) to be the 0-sequence if \( x_1 \) does not lie on \( \gamma_i \), and the same for \( x_2 \) and \( x_3 \). In the case where \( x_1, x_2, x_3 \) are distinct the additivity of the Brill-Noether number (cf. [EH2 Lem. 3.6]) then yields:

\[
-4 = \rho(2, 2, d, (0, \ldots, 0, d - 2), (0, \ldots, 0, d - 2), (0, \ldots, 0, d - 2)) \\
\geq \rho(1, 2, d, \alpha^0(P_1), \alpha^0(x_1), \alpha^0(x_2), \alpha^0(x_3)) \\
+ \sum_{i=1}^{h} \rho(0, 2, d, \alpha^i(P_{i+1}), \alpha^i(x_1), \alpha^i(x_2), \alpha^i(x_3)) \\
+ \rho(1, 2, d, \alpha^{h+1}(P_{h+1}), \alpha^{h+1}(x_1), \alpha^{h+1}(x_2), \alpha^{h+1}(x_3)).
\]

If \( x_2 = x_1 \) and \( x_3 \neq x_1 \), the above inequality still holds up to deleting all the \( \alpha^i(x_2) \).

The cases where \( x_3 \) coincides with \( x_1 \) and/or \( x_2 \) can be treated similarly. We recall that:

- the adjusted Brill-Noether number of any linear series on \( \mathbb{P}^1 \) with respect to any collection of points is nonnegative (cf. [EH3 Thm. 1.1]);
- the adjusted Brill-Noether number of any linear series on an elliptic curve with respect to any point is nonnegative (cf. [EH3 Thm. 1.1]);
- the adjusted Brill-Noether number of any \( g_3^2 \) on an elliptic curve with respect to any two points is \( \geq -2 \) (cf. [E Prop. 4.1]).

Concerning the position of the points \( x_1, x_2, x_3 \), we can thus conclude (up to relabelling them) that either

(a) \( x_1 \) lies on \( E_1 \), \( x_2 \) lies on \( E_2 \) and \( x_3 \) lies on \( \gamma_i \) for some \( 1 \leq i \leq h \), or
(b) \( x_1 \) and \( x_2 \) are distinct and lie on the same elliptic component.

In case (a), one has \( \alpha^i(x_3) \geq (0, 0, d - 2) \) and inequalities (2), (10), (11), (12) imply both \( \alpha^i(P_{i+1}) \geq (0, 1, d - 2) \) and \( \alpha^i(P_{i+1}) \geq (0, 1, d - 2) \); this contradicts the Plücker Formula [EH1 Prop. 1.1] according to which the total ramification of any \( g_3^2 \) on \( \mathbb{P}^1 \) equals \((r + 1)d - r(r + 1)\).

Thus we necessarily fall in case (b). Without loss of generality, we assume that \( x_1, x_2 \in E_1 = \gamma_0 \). If \( x_3 = x_1 \) or \( x_3 = x_2 \), the ramification weight of \( l_0 \) at \( x_3 \) is \( \geq 2(d - 2) \) since it equals the sum of the weights of the ramification points of \( C_h \) tending to \( x_3 \) (cf., e.g., [HM p. 263]). On the other hand, \( x_3 \) cannot be a base point and thus \( \alpha^0(x_3) = (0, d - 2, d - 2) \) and this is a contradiction because \( E_1 \) is elliptic. We conclude that \( l_0 \) is totally ramified at three distinct points, namely, \( x_1, x_2, x_3 \) if \( x_3 \in E_1 \) and \( x_1, x_2, P_1 \) if \( x_3 \notin E_1 \); in both cases, \( l_0 \) also has a cusp at \( P_1 \). The next proposition then yields that \( d \) is even, the point \( x_3 \notin E_1 \) and the relation \( 2x_1 \sim 2x_2 \sim 2P_1 \) holds on \( E_1 \).

Let \( \pi_0' : J_{\pi'} \to B' \) be the relative generalized jacobian of the family \( \pi' \), whose generic fiber is the jacobian \( J(C_h) \) and whose special fiber is the generalized jacobian \( J(X_0') \) parametrizing isomorphism classes of line bundles having degree 0 on every irreducible component of \( X_0' \). Hence, one has \( J(X_0') \simeq \text{Pic}^0(E_1) \times \text{Pic}^0(E_2) \) and \( \pi_0' \) is a family of smooth principally polarized abelian surfaces. The relative degree-0 line bundle \( \mathcal{O}_{C_0}(\sigma_2 - \sigma_1) \) defines a torsion section of \( \pi_0' \) (since \( d\sigma_1(b) \sim d\sigma_2(b) \) by (iv)) intersecting the special fiber \( J(X_0') \) in the 2-torsion point \((\mathcal{O}_{E_1}(x_2 - x_1), \mathcal{O}_{E_2})\). By [M] Pf. of Prop. VII.3.2 and Cor. VII.3.3 (that works for families of abelian varieties of arbitrary dimension), the group of torsion sections of \( \pi_0' \) injects in the torsion subgroup of
any fiber of $\pi'_0$ and thus we conclude that $O_C'(\sigma_2 - \sigma_1)$ is 2-torsion. In particular, on the generic fiber $C_b$ of $\pi'$ the divisors $2\sigma_1(b)$ and $2\sigma_2(b)$ are linearly equivalent, that is, $\sigma_1(b)$ and $\sigma_2(b)$ are Weierstrass points.

We claim that $\sigma_3(b)$ is a Weierstrass point, as well. Let $\iota_b$ be the hyperelliptic involution on $C_b$ and set $\sigma_4(b) := \iota_b(\sigma_3(b))$. By contradiction, we assume $\sigma_4(b) \neq \sigma_3(b)$. As $d$ is even, then $d\sigma_3(b) \sim d\sigma_1(b) \sim \frac{d}{2}(\sigma_3(b) + \sigma_4(b))$ and thus $d\sigma_3(b) \sim d\sigma_3(b) \sim d\sigma_1(b)$. As a consequence, the linear series $l'_b := (O_{C_b}(d\sigma_1(b)), (d\sigma_1(b), d\sigma_3(b), d\sigma_4(b)))$ is a $g^d_4$ on $C_b$ totally ramified at $\sigma_1(b), \sigma_3(b), \sigma_4(b)$. The first part of the proof applied to $l'_b$, thus yields that at least two points among $\sigma_1(b), \sigma_3(b), \sigma_4(b)$ are Weierstrass points of $C_b$ and thus a contradiction.

**Proposition 3.1.** Fix an integer $d \geq 3$. If a general elliptic curve possesses a $g^d_4$ totally ramified at three points $P_1, P_2, P_3$ and with a cusp, then $d$ is even, the cusp coincides with one among $P_1, P_2, P_3$ and the relation $2P_1 \sim 2P_2 \sim 2P_3$ holds.

**Proof.** We proceed by degeneration to a rational curve $C_0$ with a node. Let $\pi : C \to B$ be a smoothing of $C_0$ over a local 1-dimensional base and, by contradiction, assume that the generic fiber $C_b$ of $\pi$ possesses a linear series $l_b = (L_b, V_b)$ of type $g^d_4$ totally ramified at three points and with a cusp. As in the proof of Theorem 1.3 after various base-changes and blow-ups we obtain another family $\pi' : C' \to B'$ such that

(a) the generic fiber of $\pi'$ is equal to that of $\pi$;
(b) the special fiber $C'_0$ of $\pi'$ is obtained from $C_0$ by inserting a chain of rational curves at its node;
(c) there exist a relative linear series $l = (L, V)$ on $C'$ and four sections $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ of $\pi'$ such that $l_b$ is totally ramified at the points $\sigma_1(b), \sigma_2(b), \sigma_3(b)$ and has a cuspidal fiber at $\sigma_0(b)$;
(d) the points $x_i := \sigma_i(0)$ lie in the smooth locus of $C'_0$.

Since in our statement the cusp might coincide with one of the points of total ramification, the sections $\sigma_1, \sigma_2, \sigma_3$ can be assumed to be distinct while $\sigma_0$ might coincide with one of them. As identity section of $\pi'$ (that is, the section defining the neutral element for the group law in each fiber of $\pi'$) we choose one among $\sigma_1, \sigma_2, \sigma_3$. As a consequence, the other two sections are $d$-torsion sections of $\pi'$.

By [Mi, Pf. of Prop. VII.3.2 and Cor. VII.3.3], the group of torsion sections of $\pi'$ injects in the torsion subgroup of any fiber of $\pi'$ and hence in particular the points $x_1, x_2, x_3$ on the special fiber $C'_0$ have to be distinct.

By [Es, Thm. 1 and Prop. 2] for every irreducible component $\gamma_i$ of $C'_0$ there is a unique relative linear series $l_i = (L_i, V_i)$ on $C'_0$ such that

(i) the line bundle $L_i$ is a twist of $L$ by suitable multiples of the Cartier divisors defined by the irreducible components of $C'_0$ and $V_i := V_i \cap H^0(C', L_i) \subset H^0(C_b, L_b)$;
(ii) the restriction maps $\pi_{ii} : V_i \to H^0(\gamma_i, L_i|_{\gamma_i})$ are all injective;
(iii) for $j \neq i$ the restriction maps $\pi_{ij} : V_i \to H^0(\gamma_j, L_i|_{\gamma_j})$ are all non-zero.

Let $L_i = (L_i|_{\gamma_i}, V_i)$ be the restriction of $L_i$ to $\gamma_i$, that is, $V_i$ is the image of $V_i$ under the restriction map $\pi_{ii}$. By [Es, Thm. 7] a smooth point of $C'_0$ is a point of total ramification for $l$ if and only if it is a point of total ramification for some $l_i$, in which case $L_i|_{\gamma_i}$ has degree $d$ and $L_i|_{\gamma_i} \simeq O_{\gamma_i}$ for $j \neq i$. We number the irreducible components of $C'_0$ and possibly relabel the sections $\sigma_i$ so that $x_0$ lies on $\gamma_0$, $x_1$ on $\gamma_k$, $x_2$ on $\gamma_{k+l}$ and $x_3$ on $\gamma_{k+l+m}$ for some integers $k, l, m \geq 0$. Let $N := k + l + m + n$ for some $n \geq 0$ be the number of components of $C'_0$; this notation yields $\gamma_N = \gamma_0$. We set $P_i := \gamma_{i-1} \cap \gamma_i$ for $1 \leq i \leq N$. For $1 \leq h \leq N - 1$, let $a_h \geq 0, b_h \geq 0$ and $c_h \geq 0$ be the unique integers.
such that

\begin{align}
\mathcal{L}_0 & \simeq \mathcal{L}_k \left( \sum_{h=1}^{N-1} a_h \gamma_h \right), \\
\mathcal{L}_0 & \simeq \mathcal{L}_{k+l} \left( \sum_{h=1}^{N-1} b_h \gamma_h \right), \\
\mathcal{L}_0 & \simeq \mathcal{L}_{k+l+m} \left( \sum_{h=1}^{N-1} c_h \gamma_h \right).
\end{align}

Assume that $k \neq 0$ so that $\mathcal{L}_k|_{\gamma_0} \simeq \mathcal{O}_{\gamma_0}$. Since the divisor at the right hand side of (13) is $\gamma_0$-free (that is, its support does not contain $\gamma_0$), then the map

$$H^0(\gamma_0, \mathcal{L}_k|_{\gamma_0}) = H^0(\gamma_0, \mathcal{O}_{\gamma_0}) \to H^0(\gamma_0, \mathcal{L}_0|_{\gamma_0})$$

is an embedding and its image is generated by the divisor $D_k := a_1 P_1 + a_{N-1} P_N$. Since $V_0 := V_b \cap H^0(\mathcal{O}', \mathcal{L}_0)$ and $V_k := V_b \cap H^0(\mathcal{O}', \mathcal{L}_k)$, the isomorphism (13) also provides the left vertical morphism in the following commutative diagram

\[
\begin{array}{ccc}
V_0 & \xrightarrow{\pi_{00}} & H^0(\gamma_0, \mathcal{L}_0|_{\gamma_0}) \\
\downarrow & & \downarrow \\
V_k & \xrightarrow{\pi_{k0}} & H^0(\gamma_0, \mathcal{O}_{\gamma_0}),
\end{array}
\]

which yields $D_k \in V_0 = \text{Im}\pi_{00}$. Analogously, one shows that the divisors $D_{k+l} := b_1 P_1 + b_{N-1} P_N$ and $D_{k+l+m} := c_1 P_1 + c_{N-1} P_N$ lie in $V_0$ as soon as $k + l \not\equiv \{0, N\}$ and $k + l + m \not\equiv \{0, N\}$, respectively.

We are going to show that at most one integer among $\{k, l, m\}$ is nonzero. Assume this is not the case and, without loss of generality, let $k > 0$ and $l > 0$. We claim that this implies that the two divisors $D_k$ and $D_{k+l}$ are non-trivial and distinct. Indeed, if $a_1 = b_1$ and $a_{N-1} = b_{N-1}$ then we have

\begin{align}
\mathcal{L}_k & \simeq \mathcal{L}_{k+l} \left( \sum_{h=2}^{N-1} (b_h - a_k) \gamma_h \right) \\
& \simeq \mathcal{L}_{k+l} \left( (a_k - b_k)\gamma_0 + (a_k - b_k)\gamma_1 + (a_k - b_k)\gamma_{N-1} + \sum_{h \neq k, h = 2}^{N-2} (b_h - a_h - b_k + a_k) \gamma_h \right),
\end{align}

where we have used that the divisor $\sum_{h=0}^{N-1} \gamma_h$ is trivial (cf., e.g., [Es, (0.1)]). On the other hand, the unique integers $a_h$ such that

\begin{align}
\mathcal{L}_k & \simeq \mathcal{L}_{k+l} \left( \sum_{h \neq k, h = 0}^{N-1} a_h \gamma_h \right)
\end{align}

can be computed by letting the degrees of the restrictions to the $\gamma_i$ of both sides of (17) match. This translates into solving a system of $N$ linear equations in $N - 1$ unknowns,
namely, for \( l \neq 1 \):
\[
\begin{align*}
\alpha_{k-1} + \alpha_{k+1} &= d \\
-2\alpha_{k+1} + \alpha_{k+2} &= 0 \\
\alpha_{i-1} - 2\alpha_i + \alpha_{i+1} &= 0 \quad \text{for } i \notin \{k-1, k, k+1, k+l\} \\
-2\alpha_{k-1} + \alpha_{k-2} &= 0 \\
d + \alpha_{l+k-1} - 2\alpha_{l+k} + \alpha_{l+k+1} &= 0,
\end{align*}
\]
where we have used that any \( \gamma_i \) is a \((-2)\)-curve. The unique solution of the above system is:
\[
\alpha_i = \begin{cases} 
\frac{m+n+k}{l+m+n+k} d & \text{for } i = k+1, \\
(i-k)\alpha_{k+1} & \text{for } k+2 \leq i \leq k+l, \\
(i-k)\alpha_{k+1} + (k+l-i)d & \text{for } k+l+1 \leq i \leq N-1, \\
(N+i-k)\alpha_{k+1} + (k+l-N-i)d & \text{for } 0 \leq i \leq k.
\end{cases}
\]
In the case \( l = 1 \) the system is slightly different but its solution is the same. In any case one has \( \alpha_0 = \alpha_1 = \alpha_{N-1} \) as in (16) if and only if \( \alpha_{k+1} = d \), which is impossible as we have assumed \( l > 0 \).

Therefore, as soon as \( k > 0 \) and \( l > 0 \), the linear system \( l_0 \) defines a \( g_{d_0}^2 \) on \( \gamma_0 \) with \( d_0 = \deg L_0|_{\gamma_0} \) having two divisors \( D_k \) and \( D_{k+l} \) only supported at \( P_l \) and \( P_N \). They generate a \( g_{d_0}^1 \) contained in \( l_0 \) that has no ramification off \( P_1 \) and \( P_N \). This is a contradiction since the point \( x_0 \) is a cusp for \( l_0 \) and hence any \( g_{d_0}^1 \) contained in \( l_0 \) should be ramified at it.

To summarize, until now we have proved that the points \( x_0, x_1, x_2, x_3 \) lie on at most two components of \( C'_{0, \gamma} \) without loss of generality, \( \gamma_0 \) (containing the cusp \( x_0 \)) and \( \gamma_k \) for some \( k \geq 0 \).

The points of total ramification \( x_1, x_2, x_3 \) cannot all lie on \( \gamma_0 \) because otherwise the ramification weights of \( l_0 \) at the points \( x_0, x_1, x_2, x_3 \) would sum to \( 3(d - 2) + 2 \), thus contradicting Plücker’s Formula. We conclude that the linear series \( l_k \) is a \( g_{\gamma_k}^0 \) on \( \gamma_k \) without base points. This is obvious if \( \gamma_k \) contains at least two among the points \( x_i \). If instead \( x_i \in \gamma_k \) for at most one index \( i \), the global generation of \( l_k \) follows from the fact that it contains both a divisor only supported at a point \( x_i \) and a divisor only supported at \( P_k \) and \( P_{k+1} \).

Note that \( l_k = (L_k, V_k) \) defines a relative linear series also on the (possibly singular) family \( \pi_k : C_k \to B' \) obtained contracting all central components of \( \pi' \) apart from \( \gamma_k \); the image of the section \( \sigma_0 \) in \( C_k \) intersects the special fiber of \( \pi_k \) at the node \( P_k = P_{k+1} \).

Since a singularity that is limit of plane cusps can never be immersed (that is, such that the differential at it of the normalization map is injective), \( l_k \) defines a morphism from \( \gamma_k \) onto a plane rational curve with a generalized cusp at the image of \( F_k \) and \( F_{k+1} \). Hence, one excludes that \( x_1, x_2, x_3 \) all lie on \( \gamma_k \) since this would again contradict Plücker’s Formula. In particular, \( l_0 \) is a \( g_{\gamma_0}^2 \) on \( \gamma_0 \) with no base points as it contains both a divisor only supported at \( P_1 \) and \( P_N \) and a divisor only supported at one of the \( x_i \).

We now exclude that two among \( x_1, x_2, x_3 \) lie on \( \gamma_k \). Indeed, if by contradiction both \( x_1 \) and \( x_2 \) lied on \( \gamma_k \), then they would define a \( g_{d_0}^1 \) contained in \( l_k \) totally ramified at \( x_1 \) and \( x_2 \) and with no ramification elsewhere. This would contradict the existence of a generalized cusp at the image of \( P_k \) and \( P_{k+1} \) since a cusp for \( l_k \) is necessarily a ramification point for any \( g_{d_0}^1 \) contained in it.

Therefore, we can assume that \( x_0, x_1, x_2 \in \gamma_0 \) and \( x_3 \in \gamma_k \). The cusp \( x_0 \) has to coincide with either \( x_1 \) or \( x_2 \) because the \( g_{d_0}^1 \) contained in \( l_0 \) generated by \( dx_1 \) and \( dx_2 \).
has no ramification off $x_1$ and $x_2$. Without loss of generality, we assume $x_0 = x_2$ and choose $\sigma_1$ as identity section of $\pi'$. The next lemma then implies that $d$ is even, $x_1 = 1$ and $x_2 = -1$. Since the group of torsion sections of $\pi'$ injects in the torsion subgroup of the special fiber $C'_0$, we conclude that

$$\sigma_2 \text{ is a } 2\text{-torsion section of } \pi'. $$

We now claim that:

$$\sigma_0 \text{ coincides with } \sigma_2. $$

Since two torsion sections do not intersect and $x_0 = x_2$, it is enough to show that $\sigma_0$ is a torsion section. On the generic fiber $C_b$ of $\pi'$ we set $y_i := \sigma_i(b)$ for $0 \leq i \leq 3$. Since $y_0$ is a cusp of the $g^4_d$ defined by $\langle dy_1, dy_2, dy_3 \rangle$, then $y_0$ is a ramification point of the $g^4_d$ totally ramified at $y_1$ and $y_2$. The relation $2y_1 \sim 2y_2$ implied by (18) forces the cover $f$ defined by this $g^4_d$ to factor as

$$C_b \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^1,$$

where $g$ is the double cover corresponding to the $g^1_2$ defined by $\langle 2y_1, 2y_2 \rangle$, and $h$ is the $d/2$ to 1 cover of $\mathbb{P}^1$ totally ramified at $g(y_1)$ and $g(y_2)$ and with no ramification elsewhere. As a consequence, the ramification points of $f$ (and in particular $y_0$) are exactly those of $g$ and thus define 2-torsion points of $C_b$. In particular, $\sigma_0$ is a 2-torsion section and (19) follows.

We now show that:

$$\sigma_3 \text{ is a } 2\text{-torsion section}. $$

By contradiction, assume this is not the case. As above, let $y_i := \sigma_i(b)$ for $1 \leq i \leq 4$ on the generic fiber $C_b$ of $\pi'$. We denote by $\iota_b$ the involution on $C_b$ defined by $\langle 2y_1, 2y_2 \rangle$, and set $y_1 := \iota_b(y_3)$; we are assuming $y_1 \neq y_3$. By (19), the point $y_2$ is a cusp of the $g^2_2$ defined by $\langle dy_1, dy_2, dy_3 \rangle$ and thus a ramification point of the $g^4_d$ totally ramified at $y_1$ and $y_3$, or equivalently, $2y_2 + D_2 \in \langle dy_1, dy_3 \rangle$ for some divisor $D_2$ of degree $d - 2$. Note that $D_2$ cannot be $\iota_b$-invariant because the $\iota_b$-invariant subspace $\langle dy_1, dy_3 \rangle^b$ is spanned by $dy_1$. Since $2y_2 + \iota_b^* D_2 \in \langle dy_1, dy_4 \rangle$, then $y_2$ is a ramification point for $\langle dy_1, dy_4 \rangle$ and hence a cusp for the $g^4_d$ defined by $\langle dy_1, dy_3, dy_4 \rangle$. In other words, the linear series $l^* := (\mathcal{O}_{C_b}(dy_1), \langle dy_1, dy_3, dy_4 \rangle)$ is a $g^2_2$ on $C_b$ totally ramified at $y_1, y_3, y_4$ and with a cusp at $y_2$. By the first part of the proof (cf. (19)), the cusp $y_2$ has to coincide with one among $y_1, y_3, y_4$; however, $y_1, y_2, y_3$ are distinct by assumption and hence $y_2 = y_4 = \iota_b(y_3)$ contradicting the assumption $y_3 \neq y_4$. 

\begin{lemma}
Assume the existence of three integers $d, a, b > 0$ with $a + b = d$ and four distinct points $x, y, P, Q$ on $\mathbb{P}^1$ such that the divisors $dx, dy, aP + bQ$ define a $g^2_2$ on $\mathbb{P}^1$ that has a cusp at $y$ and identifies $P$ and $Q$. Then $a = b = d/2$ and $d$ is even; furthermore, up to the automorphisms of $\mathbb{P}^1$, one has $P = 0, Q = \infty, x = 1$ and $y = -1$.
\end{lemma}

\begin{proof}
Let us assume that $\langle dx, dy, aP + bQ \rangle$ is a $g^2_2$ on $\mathbb{P}^1$ that identifies $P$ and $Q$ and has a cusp at $y$. We use that

(a) the $g^1_1$ generated by $dx$ and $aP + bQ$ has a unique ramification point different from $x, P, Q$ and this has to coincide with $y$;
(b) $P$ and $Q$ are identified by the unique $g^1_1$ on $\mathbb{P}^1$ totally ramified at $x$ and $y$.
\end{proof}
Acting with the automorphism group of $\mathbb{P}^1$ we can assume $x = 1$, $P = 0$ and $Q = \infty$ and use (a) in order to compute $y$. The pencil $\langle dx, aP + bQ \rangle$ corresponds to the following meromorphic function on $\mathbb{P}^1$:

$$F(z) = \frac{k(z - 1)^d}{z^a}, \text{ with } k \in \mathbb{C};$$

it is then trivial to show that $y = -\frac{a}{b}$.

On the other hand, again up to an automorphism of $\mathbb{P}^1$, we can assume $x = 0$, $y = \infty$, $P = 1$ and use (b) in order to compute $Q$. Since the only degree $d$ self-map of $\mathbb{P}^1$ totally ramified at $0$ and $\infty$ is given by the function

$$G(z) = k z^d, \text{ with } k \in \mathbb{C},$$

the point $Q$ is a $d^{th}$ root of unity that we call $\xi$.

Alltogether, there exists an automorphism $\varphi$ of $\mathbb{P}^1$ such that $\varphi(0) = 1$, $\varphi(1) = 0$, $\varphi(\infty) = \xi$ and $\varphi\left(-\frac{a}{b}\right) = \infty$. Such a $\varphi$ can be written in the form

$$\varphi(z) = \frac{az + \beta}{\gamma z + \delta}, \quad \left(\begin{array}{cc} a & \beta \\ \gamma & \delta \end{array}\right) \in PGL(2, \mathbb{C}).$$

This easily yields $\xi = -\frac{a}{b}$. In particular, $\xi$ is a rational root of unity; the only possibility is thus $\xi = -1$ and our thesis follows.

□

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A. L. KNUTSEN AND M. LELLI-CHIESA

ANDREAS LEOPOLD KNUTSEN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BERGEN, POSTBOKS 7800, 5020 BERGEN, NORWAY

E-mail address: andreas.knutsen@math.uib.no

MARGHERITA LELLI-CHIESA, DISIM, UNIVERSITY OF L’AQUILA, VIA VETOIO, LOC. COPPITO I, 67100 L’AQUILA, ITALY

E-mail address: margherita.lellichiesa@univaq.it