Macroscopic features of quantum fluctuations in large N qubit system

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We introduce a discrete Q-function of N qubit system projected into the space of symmetric measurements as a tool for analyzing general properties of quantum systems in the macroscopic limit. For known states the projected Q-function helps to visualize the results of collective measurements, and for unknown states it can be approximately reconstructed by measuring lowest moments of the of collective variables.

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INTRODUCTION

It was recently understood that pure states can be used for a statistical description of quantum system [1, 2]. In particular, the thermal-like pure states [3] satisfactorily depict equilibrium values of a certain class of observables [4]. From this point of view, a quantum state characterization in the limit of macroscopically large number of particles (macroscopic limit) is tightly connected to the choice of the observables involved in the measurement process. Since the complexity of the microscopic description (especially experimentally, through tomographic reconstruction) of a quantum state grows very rapidly with the dimension of the system, it is not only very difficult but also unnecessary to determine individual particle states in order to capture essential properties of large quantum systems. An adequate choice of measurable coarse-grained collective variables [5] assists to reveal several relevant features of quantum systems in the macroscopic limit. Additionally, in many cases the assessment of correlation functions of the collective operators are the only accessible type of practical measurements since it is usually impossible to distinguish between the particles in a macroscopic volume [6].

Thus, in order to get information about a system as a whole we can restrict ourselves only to measurement of collective observables symmetric with respect to particle permutations (symmetric measurements).

In this paper we introduce the concept of the discrete Husimi Q-function projected into the 3 dimensional space of symmetric measurements as a tool for analysis of the following fundamental questions: How to properly describe a quantum system state in the macroscopic limit using results of measurements of appropriate collective variables? How to visualize the result of such measurements? What higher correlation functions can be described with a reasonable accuracy by measuring lowest moments of collective observables in an unknown state?

We will show that the projected Q-function is very well suited for description of N-particle systems that behave in many ways as statistical systems when N ≫ 1, since it naturally emerges from the discrete phase-space representation and tends to a smooth distribution in the limit of large N preserving essential features of the quantum state. In particular, for known states it helps to visualize the results of collective measurements, and for unknown states it is possible to approximately reconstruct the projected Q-function by measuring lowest moments of the collective variables. We exemplify our approach for the N-qubit case; generalization to highest spin is straightforward.

In Sec.II we briefly recall the concept of the discrete phase-space distribution functions and propose a discrete Q-function in the space of collective measurements. In Sec.III we analyze the macroscopic limit of this distribution and introduce the concept of localization in the measurement space. In Sec. IV we discuss the possibility of reconstructing higher order moments of collective operators from the asymptotic form of the Q-distribution. In Sec.V we discuss the results and further topics related to our approach.

DISCRETE Q-FUNCTION IN THE MEASUREMENT SPACE

We first recall that the full non-redundant description of quantum state is provided by functions in the discrete phase-space. The discrete phase-space (DPS) for an N qubit system is a two-dimensional $2^N \times 2^N$ grid where coordinates $(\alpha, \beta)$ are specified by N-dimensional strings $\alpha = (a_1, ..., a_N), \beta = (b_1, ..., b_N)$, $a_i, b_j \in \mathbb{Z}_2$. The points $(\alpha, \beta)$ label elements of a monomial operational basis [8], [9] $Z_{\alpha}X_{\beta}$ according to

$$Z_\alpha = \sigma_2^{a_1} \otimes \ldots \otimes \sigma_2^{a_N}, \quad X_\beta = \sigma_2^{b_1} \otimes \ldots \otimes \sigma_2^{b_N},$$

where $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$. The operators $Z_\alpha$ and $X_\beta$ in the computational basis $\{|\kappa\rangle = |k_1, ..., k_N\rangle, k_i \in \mathbb{Z}_2\}$ in the full Hilbert space $\mathcal{H}_{2^N} = \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_2$ act as displacements,

$$Z_\alpha|\kappa\rangle = (-1)^{\alpha \kappa}|\kappa\rangle, \quad X_\beta|\kappa\rangle = |\kappa + \beta\rangle,$$

where the multiplication and sum are mod 2 operations, $\alpha \kappa = a_1k_1 + \ldots + a_Nk_N \in \mathbb{Z}_2$, $\kappa + \beta = (b_1 + k_1, ..., b_N + k_N)$. Defined in this way DPS (isomorphic to a product of two-dimensional discrete torus $T^2 \otimes T^2 \otimes \ldots$) is endowed with a finite geometry [10] and admits a set of discrete symplectic operations required for analysis of quasidistribution functions [11].
Each point of the DPS also labels an element of a set of the discrete coherent states (DCS), constructed as \[^{13}\]
\[
|\alpha, \beta\rangle = e^{i\varphi(\alpha, \beta)} Z_s X_\beta |\xi\rangle ,
\]
where \(e^{i\varphi(\alpha, \beta)}\) is an unessential for us phase and \(|\xi\rangle\) is a fiducial state. The set of projectors on \[^{2}\]\ resolves the identity
\[
\sum_{\alpha, \beta} |\alpha, \beta\rangle \langle \alpha, \beta| = 2^N \mathbb{I}
\]
and constitute a discrete POVM. For instance, the discrete \(Q\)- and \(P\)-symbols of an operator \(\hat{f}\) can be formally defined as the following maps:
\[
Q_f (\alpha, \beta) = \langle \alpha, \beta | \hat{f} | \alpha, \beta\rangle, \quad \hat{f} = \sum_{\alpha, \beta} P_f (\alpha, \beta) (\alpha, \beta) | \alpha, \beta\rangle .
\]

Nevertheless, not every choice of the fiducial state \(|\xi\rangle\) leads to an informationally complete POVM. It follows from the expression for the mapping kernel \[^{12}\]
\[
\Delta^{(s)} (\alpha, \beta) = \frac{1}{2^{N(s+3)/2}} \sum_{\gamma, \delta} (-1)^{\alpha \delta + \beta \gamma + \gamma \delta (1-s)/2} [\langle \xi | Z_\gamma X_\delta | \xi\rangle]^{-1} Z_\gamma X_\delta ,
\]
\[
Q_f (\alpha, \beta) = \text{tr} \left[ \hat{f} \Delta^{(s=-1)} (\alpha, \beta) \right], \quad P_f (\alpha, \beta) = \text{tr} \left[ \hat{f} \Delta^{(s=1)} (\alpha, \beta) \right],
\]
that the expansion of the density matrix on the DCS projectors is not faithful (and the \(Q\)-symbol is not an invertible mapping) for fiducial states where \(\langle \xi | Z_\gamma X_\delta | \xi\rangle = 0\) for some \(\gamma, \delta\).

It was proposed \[^{14}\]\ to take the fiducial state \(|\xi\rangle\) as a product of identical qubit states:
\[
|\vartheta, \varphi\rangle_j = e^{i\varphi/2} \sin \vartheta/2 |1\rangle_j + e^{-i\varphi/2} \cos \vartheta/2 |0\rangle_j ,
\]
with \(\vartheta = \arctan \sqrt{\varphi}, \varphi = \pi/4\), i.e. \(|\xi\rangle\) is the standard \(SU(2)\) coherent state
\[
|\vartheta, \varphi\rangle_1 \otimes \ldots \otimes |\vartheta, \varphi\rangle_N \propto |\xi\rangle = \frac{\sqrt{3} - 1}{\sqrt{2}} e^{i\pi/4} ,
\]
specified by the unitary vector \(\mathbf{n}_0 = (1, 1, 1)/\sqrt{3}\). For a single qubit DCS \[^{2}\]\ form a symmetric tetrahedron inscribed in the Bloch sphere, so that \(Q\)- and \(P\)-symbols are connected by a simple relation, \(3Q(a, b) = 2P(a, b) + 1\). In what follows we will refer to DCS with this choice of the fiducial state as symmetric coherent states \[^{14}\]. Then, the set of projectors on \[^{3}\]\ forms a non-orthogonal informationally complete operational basis. The mappings \[^{3}\]\ are invertible and the average of any operator \(\hat{f}\) is computed as a convolution of the Husimi function \(Q_{\rho}(\alpha, \beta)\) with the \(P\)-symbol of \(\hat{f}\):
\[
\langle \hat{f} \rangle = \sum_{\alpha, \beta} P_f (\alpha, \beta) Q_{\rho}(\alpha, \beta),
\]
where \(\rho\) is the density matrix.

In the case of systems with continuous symmetries (\(H(1)\) for the harmonic oscillator, \(SU(2)\) for a spin-like systems or states from the symmetric representation of \(N\)-qubit system, etc.) the quasidistribution functions are very useful for state identification \[^{15}\]. Unfortunately, the representation of states in DPS is not very intuitive \[^{11}\], mainly since there is no natural order in the space of two-dimensional \(Z_2\) strings. The \(Q\)-function usually has a form of an almost random distribution of peaks, see Fig.1 where the \(Q\)-function for the bipartite state \(|\psi_T\rangle \sim \otimes i=1^3 |00\rangle_i + |10\rangle_i + |01\rangle_i - |11\rangle_i\) in the 6-qubit case is plotted. So, it results very difficult to get a good insight on the quantum state from the corresponding discrete quasidistribution function especially when the number of qubits is large, \(N >> 1\) \[^{14}\]. In addition, the complexity of the microscopic description of a quantum state grows very rapidly with the dimension of the system. On the other hand, it is not very difficult but also unnecessary to determine individual qubit states in order to describe essential properties of a quantum system in the macroscopic limit. In order to describe global properties of \(N\)-qubit systems we choose the correlation functions of the collective spin operators
\[
S_{x,y,z} = \sum_{i=0}^N a_{x,y,z}^{(i)}
\]
as a set of (symmetric) observables. In practice, average values of powers of collective operators are non-trivial in any (not only symmetric with respect to particle permutations) state except of singlets. Thus, by measuring symmetric observables \[^{9}\] we collect information from all \(SU(2)\) irreducible subspaces that appear in the decomposition of an arbitrary \(N\) qubit state and averaging over hidden degrees of freedom (degeneration of irreps entering in such decomposition).

The crucial element of our construction is the observation that the \(P\) and \(Q\) symbols \[^{3}\]\ of symmetric operators do not
depend on the phase-space coordinates but rather only on the lengths of the strings $\alpha, \beta$ and $\alpha + \beta$ where the length $0 \leq h(\kappa) \leq N$ counts the number of nonzero coefficients $k_j \in \mathbb{Z}_2$ in the string $\kappa = (k_1, k_2, ..., k_N)$ (in the mathematical literature such invariant under permutation quantitites are usually called weights of the strings $\alpha, \beta$ and the Humming distance between them), i.e.

$$P_f(\alpha, \beta) = P_f(h(\alpha), h(\beta), h(\alpha + \beta)),$$

$$Q_f(\alpha, \beta) = Q_f(h(\alpha), h(\beta), h(\alpha + \beta)).$$

For instance, one obtains

$$P_s = \frac{\sqrt{3}}{2N} \left[ N - 2h(\alpha) \right] $$

$$P_s = \frac{\sqrt{3}}{2N} \left[ N - 2h(\alpha + \beta) \right] .$$

This feature is specific to the symmetric coherent states (2)-(7) and follows from the property of (7) that (14) contains the entire information about all collective observables. Moreover, the averaging (14) over strings of given length smooths the original discrete distribution, which makes it very useful for studying qubit states in the macroscopic limit.

For symmetric states, for which the density matrix is invariant under particle permutations, the projected $\tilde{Q}$-function according to (11) and (14) has the form

$$\tilde{Q}_\rho(m, n, k) = Q_\rho(m, n, k) R_{mnk},$$

where $R_{mnk}$ is the combinatorial factor

$$R_{mnk} = \frac{N!}{\left(\frac{m+n-k}{2}\right)! \left(\frac{2N-m-n-k}{2}\right)! \left(\frac{n+m+k}{2}\right)! \left(\frac{m+n+k}{2}\right)!};$$

For instance, the $Q$-function of the fiducial state (7) has a step-like form $Q_\xi(\alpha, \beta) = 3^{-h(\alpha)+h(\beta)+h(\alpha+\beta)/2}$, while the projected $\tilde{Q}$-function (15) asymptotically tends to a Gaussian function

$$\tilde{Q}_\xi \sim \exp \left[ -N^{-1} \left( \frac{5}{2} (m^2 + n^2 + k^2) - km - nk - mn - (m + n + k)N \right) \right],$$

in the limit $N \gg 1$ as a consequence of the smoothing produced by the combinatorial factor (16).

**LARGE N ASYMPTOTIC AND THE LOCALIZATION PROBLEM**

One can find the asymptotic form of the $\tilde{Q}$-function (14) in the macroscopic limit, $N \gg 1$, for general states (not necessarily symmetric with respect to qubit permutations). Making use of an integral representation of the $\delta$-functions we rewrite (14) as

$$\tilde{Q}(\alpha, \beta) = \int \frac{d^3 \omega}{(2\pi)^3} f(\omega) \left( \omega_1^h(\alpha) \omega_2^h(\beta) \delta(\alpha+\beta) \right),$$

where $x = (x = m/N, y = k/N, z = n/N)$ are the scaled coordinates in the measurement space, and the $P$-symbols of the collective variables $S \cdot n$ are

$$P_{S \cdot n} = \frac{N \sqrt{3}}{2N} (1 - 2x \cdot n),$$

here $n$ is a unit vector. If the highest order cumulants $\kappa_r$ of collective variables in any measurement direction do not grow very rapidly, in the sense that $\kappa_2 \gg \kappa_3^2$, $\kappa_2 \gg \kappa_4^{1/2}$ etc. then applying $\omega_j = 1 + \varepsilon_j, |\varepsilon_j| \ll 1$ we can approximate (18) as a single Gaussian function

$$\tilde{Q}(x) \approx \frac{2^{N+1}}{(\pi N)^{3/2} \sqrt{\det T}} \exp \left( -N \Delta x T^{-1} \Delta x \right),$$
where $\Delta x = x - \bar{x}$; here $\bar{x} = (1 - (S) / N \sqrt{3}) / 2$ with

$$T = \frac{1}{6N} (T + \Lambda), \quad \text{det} \ T \geq 0 \quad (21)$$

where

$$\Gamma_{ij} = \langle S_i S_j + S_j S_i \rangle / 2 - \langle S_i \rangle \langle S_j \rangle, \quad i, j = x, y, z$$

is the correlation matrix and where the symmetric matrix $\Lambda$ is defined as

$$\Lambda = \begin{bmatrix}
2N & \sqrt{3} \langle S_x \rangle & \sqrt{3} \langle S_y \rangle \\
\sqrt{3} \langle S_x \rangle & 2N & \sqrt{3} \langle S_z \rangle \\
\sqrt{3} \langle S_y \rangle & \sqrt{3} \langle S_z \rangle & 2N
\end{bmatrix},$$

if $\text{det} \ T \neq 0$. For instance, for a coherent state $|\mu, \nu\rangle = Z_\mu X_\nu |\xi\rangle$ (not fully symmetric with respect to qubit permutations) we obtain, considering the covariance of the $Q$-function with respect to displacements, $Q_{\mu, \nu}(\alpha, \beta) = Q_\xi (\alpha + \mu, \beta + \nu)$ \[14\] that $\bar{x} = (h(\mu), h(\mu + \nu), h(\nu)) / 3N + (1, 1, 1) / 3$ and

$$T = \frac{1}{3} \begin{bmatrix}
4/3 & 1 - 2\bar{z} & 1 - 2\bar{y} \\
1 - 2\bar{z} & 4/3 & 1 - 2\bar{x} \\
1 - 2\bar{y} & 1 - 2\bar{x} & 4/3
\end{bmatrix}.$$ 

It is worth noting that $\tilde{Q}_{\mu, \nu}(x)$ is totally isotropic, i.e. is described by a sphere of radius $3N^{-1/2} / 2$ centered at $\bar{x} = (1, 1, 1) / 2$, $\tilde{Q}_{\mu, \nu}(x) \sim \exp (-9N \Delta x^2 / 4)$ only in the case when $h(\mu) = h(\nu) = h(\mu + \nu) = N / 2$, which corresponds to the states with zero average values of the collective spin operators $S_{x,y,z}$.

Let us point out that the probability ellipsoid $\Delta x T^{-1} \Delta x$ in (20) depicts how well we can describe the state by two lowest moments of the collective variables rather than their fluctuations in some particular directions. For instance, the principal axes of the ellipsoid (17) for the fiducial coherent state (7) are $n_1 = (1, 1, 1) / \sqrt{3}$, $n_2 = (-1, 0, 1) / \sqrt{2}$, $n_3 = (-1, 1, 0) / \sqrt{2}$ with the corresponding lengths $\lambda_1 = 2 / 3$, $\lambda_2 = 1 / 3$, $\lambda_3 = 1 / 3$. Observe that in the direction $n_1$ the collective operator $S \cdot n_1$ does not fluctuate at all, since (7) is its eigenstate. Nevertheless, in this direction the ellipsoid has maximum length, which means that the accuracy of the state description with the first and second moments along $n_1$ is less then in the directions $n_2, n_3$. In Sec. IV we detail this general assertion when discuss the higher order moments.

In Fig.2 we plot the exact $\tilde{Q}(x)$ function for the fiducial state (7) as a set of spheres in the three dimensional space of symmetric measurements $\{m, n, k\}$. The size of the spheres and their colors indicate the density of the distribution. The largest and darkest spheres represent points of highest density. The grey envelope corresponds to the analytical approximation given by (20) centered at $\bar{x} = (1, 1, 1) / 3$.

In case where one of the eigenvalues of $T$ is zero then along the direction corresponding to this eigenvalue the $Q$-function is proportional to the $\delta$-function:

$$\tilde{Q}(x) \sim \exp \left(-N \left( x \cdot u_1 \right)^2 / \lambda_1 - N \left( x \cdot u_2 \right)^2 / \lambda_2 \right) \delta (x \cdot u_3),$$

where $\lambda_{1,2}$ are non zero eigenvalues of $T$, $\lambda_3 = 0$ and $u_{1,2,3}$ are the corresponding normalized eigenvectors. An example of such state is the $SU(2)$ coherent state with $\theta = -\arctan \sqrt{2}, \varphi = \pi / 4$, for which the $\tilde{Q}(x)$ is located in the plane $z = x + y$, so that $\lambda_1 = \lambda_2 = 2 / 3, \lambda_3 = 0$.

The dispersion matrix (21) provides important information about state localization in the space of symmetric measurements. Loosely speaking, the volume of the ellipsoid $\Delta x T^{-1} \Delta x$ is finite for localized states and becomes unbounded for non-localized states in the limit $N \to \infty$. In other words, the state is localized in some direction if the width of the distribution (20), determined by the eigenvalue of the matrix $T$ in the corresponding direction, is much less than the extension of the measurement space. For localized states the outcomes of symmetric measurements depend only on a small number of parameters, which are essentially the lowest order moments of the collective variables.

The localization in the measurement space is not the same as localization in the Hilbert space. For instance, the uniform distribution $\rho = I / 2N$ is represented as a localized sphere in the measurement space, $\tilde{Q}(x) \sim \exp (-2N \Delta x^2)$ centered at $\bar{x} = (1, 1, 1) / 2$.

It follows immediately from (21) that the localized states are characterized by $Tr T \ll N$ (so that, $\lim_{N \to \infty} Tr T / N = 0$). It is easy to see that all separable states, $\rho = \bigotimes_{i=1}^N \rho^{(i)}$, are localized since $Tr T = 3N - \sum_{i=1}^N r_i^2$, where $r_i^2$ is the Bloch vector of $i$-th particle, so that $4 / 3 \leq Tr T \leq 3 / 2$. Another examples of localized states are the bi-separable states of the form

$$|\psi_0\rangle \sim \bigotimes_{i=1}^N |00\rangle_i + a |10\rangle_i + b |01\rangle_i - |11\rangle_i,$$

and graph-like states

$$|\psi_T\rangle \sim \bigotimes_{i=1}^N (|00\rangle_i + |10\rangle_i + |01\rangle_i - |11\rangle_i). \quad (23)$$

FIG. 2: $\tilde{Q}(x)$ functions for an DCS $|0, 0\rangle$, with $N=18$. 
For all aforementioned states \( \kappa_r \sim N \).

One can observe that since for the states \( \tilde{Q} \) the matrix \( T \) has the form

\[
T = \frac{1}{3} \text{diag} \left( \frac{3 + 3a^2 + 2a}{2a^2 + 2}, \frac{3 + 3a^2 + 2a}{2a^2 + 2}, 1 \right),
\]

the projected \( \tilde{Q} \)-function for singlet ( \( a = -1 \) ) is a sphere, \( \tilde{Q}(\vec{x}) \sim \exp(-3N\Delta x^2) \) centered at \( \vec{x} = (1, 1)/2 \). It will be shown below that the \( 1/\sqrt{3} \) is the minimum possible radius that a spherically symmetric distribution can acquire.

For the fully entangled W-state the \( T \)-matrix in the limit \( N \gg 1 \) has the form corresponding to the localized states:

\[
T = \frac{1}{6} \begin{bmatrix}
5 & \sqrt{3} & 0 \\
\sqrt{3} & 5 & 0 \\
0 & 0 & 2
\end{bmatrix}.
\]

Nevertheless, taking into account the explicit form of the \( Q \)-function

\[
Q_W = \mathcal{N} |\xi|^{2n} \left| N\xi + k (\xi^{-1} - \xi) - m (\xi^{-1} + \xi) \right|^2, \quad (24)
\]

where \( \xi = \sqrt{\frac{3}{2}} e^{i\pi/4} \) and \( N^{-1} = N (1 + |\xi|^2)^N \), it results that the cumulants in \( x \) and \( y \) directions behave as \( \kappa_2 \sim N^r \), so that \( \kappa_2 \sim \kappa_4 \) (while along \( z \) direction \( \kappa_2 \sim N \)) and thus do not satisfy the condition required to obtain \( \tilde{Q} \). This is because the localized \( \tilde{Q}_W \)-function has a fine structure: inside a single Gaussian envelope centered at \( \vec{x} = 1/2(1, 1, 1 - 1/\sqrt{3}) \) there are two narrow maxima located at

\[
\vec{x}_{\pm} \approx \left( \frac{1}{2} \pm \frac{\sqrt{3} - 1}{2\sqrt{N}}, \frac{1}{2}, \frac{\sqrt{3} - 1}{2\sqrt{N}}, 2 - \sqrt{3} \right)
\]

and thus, separated by \( \sim N^{-1/2} \) in the \( x - y \) plane, as it can be observed from Fig.3, where the grey envelope corresponds to the Gaussian approximation \( \tilde{Q}(\vec{x}) \). The analytical expression for the maxima positions can be obtained from \( (15) \) and \( (24) \) by using Stirling (but not Gaussian) approximation for \( R_{nmk} \).

If at least one of the eigenvalues of the \( T \)-matrix scales like \( N \), the Gaussian \( \tilde{Q}(\vec{x}) \) spreads out over the whole measurement space along the direction corresponding to that eigenvalue. This happens when fluctuations of the collective variables in some direction behave like \( \sim N^2 \). Such non-localized states can not be described in the measurement space by a single Gaussian. Consider as an example the state \( |GHZ\rangle = |000\rangle + |111\rangle \sqrt{2} \). The correlation matrix has the form \( \Gamma = \text{diag}(N, N, N^2) \), so taking into account that \( \langle S_{x,y,z} \rangle = 0 \) we obtain from \( (20) \) the following Gaussian approximation for the \( Q \)-function

\[
\tilde{Q}(\vec{x}) \sim \exp \left[ -2N \left( \Delta x^2 + \Delta y^2 + 3\Delta z^2/N \right) \right], \quad (25)
\]

where \( \vec{x} = (1, 1, 1)/2 \): the \( \tilde{Q} \)-function is delocalized along the axis \( z \) in the measurement space, which also follows from a rapidly growing behavior of cumulants, \( \kappa_2 \sim N^2r \) along the \( z \)-direction.

In Fig.4 we plot the exact form \( \tilde{Q}(\vec{x}) \) of the \( Q \)-function for the GHZ state projected into the space of symmetric measurements. Observe that along the "localized directions" \( x \) and \( y \) the \( Q \)-function is well described by the Gaussian \( \tilde{Q}(\vec{x}) \) and although does not describe well the general behavior, still provides non-trivial bounds for the distribution. Using the explicit form of the discrete \( Q \)-function for the GHZ state

\[
Q_{GHZ} = \mathcal{N} N \left| \xi^n + (-1)^m \xi^{N-n} \right|^2/2, \quad (26)
\]

one can find from \( (15) \) and \( (24) \) that the projected to the measurement space \( Q \)-function is described by a sum of two equally weighted localized Gaussians centered at \( (1, 1, 1 \pm 1/\sqrt{3})/2 \) with corresponding dispersion matrices:

\[
T_{\pm} = \begin{bmatrix}
3 & \pm \sqrt{3} & 0 \\
\pm \sqrt{3} & 3 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

Another example of non-localized states is a superposition of elements of the computational basis \( |\kappa\rangle \): \( |\psi\rangle \sim |\kappa_1 \rangle + |\kappa_2 \rangle \) with \( h(\kappa_1 + \kappa_2) \gtrsim \sqrt{N} \); this should be contrasted with the case of the single \( |\kappa\rangle \) for which the \( Q \)-function is a localized ellipsoid centered at \( (1/2, 1/2, 1/2 + (h(\kappa)/N - 1/2)^{3/2} - 1/2) \) and characterized by

\[
T = \frac{1}{2} \begin{bmatrix}
1 & (1 - 2h(\kappa)/N)/\sqrt{3} & 0 \\
(1 - 2h(\kappa)/N)/\sqrt{3} & 1 & 0 \\
0 & 0 & 2/3
\end{bmatrix}.
\]

Localization and delocalization properties are in general not invariant under local transformations generated by \( \Gamma \). For
instance, the correlation matrix for the shifted GHZ state,

\[ X_\nu |GHZ\rangle \sim |\nu_1, \nu_2, \ldots \rangle + |1 + \nu_1, 1 + \nu_2, \ldots \rangle , \]

has the form \( \Gamma = \text{diag}(N, N, (N - 2h(\nu))^2), \nu = (\nu_1, \ldots, \nu_N), \nu_j = 0, 1. \) Thus, for translations characterized by lengths \( h(\nu) \sim N/2 \pm \Delta h, \) with \( \Delta h \sim \sqrt{N} \) the states \( X_\nu |GHZ\rangle \) become localized and can be sufficiently well described by a single Gaussian envelope \( (20). \)

In Fig. 5 we plot the \( \tilde{Q} \)-function for the shifted GHZ state with \( \nu = (1, 1, 1, 1, 0, 0, 0, 0) \) in 8 qubit case. The analytical approximation corresponds to an ellipsoid described by \( T = \text{diag}(1/2, 1/2, 1/3). \)

On the other hand the states \( (22) \) and \( (23) \) remain localized under transformations \( (1). \)

Thus, analyzing the dispersion matrix \( (21) \) obtained from measured data we can assess the localization properties of unknown states: unbounded for \( N \to \infty \) scaled eigenvalues \( \lambda_j \) of \( T, \) such that \( \lim_{N \to \infty} (\lambda_j / N) = \infty, \) determine unlocalized directions in the measurement space, while in the direction of finite eigenvalues \( \lambda_j \) the distribution function of measured collective observables is delimited by a narrow Gaussian \( (20). \)

**AVERAGE VALUE EVALUATION**

The Gaussian approximation \( (20) \) is not only helpful to describe the shape of the \( \tilde{Q} \)-function in the large \( N \) limit, but can be also used to estimate the highest moments of the collective observables by integrating \( (20) \) with \( P(x) \)-symbols of the corresponding variables

\[
\langle \hat{O} \rangle = \frac{N^3}{2} \int \int \int P_\nu(x) \tilde{Q}(x) \, d^3 x , \tag{29}
\]

where \( \hat{O} \) is an arbitrary symmetric operator and \( 1/2 \) factor appears because the summation index \( k \) in \( (13) \) runs in steps of two. The approximate \( \tilde{Q} \)-function \( (20) \) leads to exact expressions for the first and second moments of collective observables.

In the case of localized states this approximation also satisfactorily describes the leading-term asymptotic of the higher order correlations. For instance, for the shifted coherent states \( Z_\mu X_\nu |\rangle \) the deviation of the approximate expressions \( (29) \) from the exact ones for the averages \( \langle S_{zc}^3 \rangle, \langle S_{zc}^4 \rangle \) of the central moments \( S_{zc} = S_j - \langle S_j \rangle \) appears in the orders \( N \) and \( N^2 \) correspondingly:

\[
\langle S_{zc}^3 \rangle_{ex} = -3^{-3/2} N c_\nu \left[ N^2 c_\nu + 6N - 4 \right] ,
\]

\[
\langle S_{zc}^3 \rangle_{app} \approx -3^{-3/2} N c_\nu \left[ N^2 c_\nu + 6N + 12 \right] ,
\]

\[
\langle S_{zc}^4 \rangle_{ex} = \frac{1}{9} N^2 \left( N^2 c_\nu^4 + [12N - 16] c_\nu^2 + 12 \right) ,
\]

\[
\langle S_{zc}^4 \rangle_{app} \approx \frac{1}{9} N^2 \left( N^2 c_\nu^4 + [12N + 48] c_\nu^2 + 12 + \frac{96}{N} \right) ,
\]

where \( c_\nu = 2h(\nu) / N - 1. \) Similar expressions are obtained for \( S_{xc} \) and \( S_{yc} \) (just changing the index \( \nu \) to \( \mu \) and \( \nu + \mu \) correspondingly). Thus, the relative deviation is \( \sim N^{-2} \) for non-zero values of the shifting coefficients \( c_j, j = \nu, \mu, \nu + \mu. \) It is interesting to note that even for the deeply non-symmetric case, when all \( c_j = 0 \) and the projected \( \tilde{Q} \)-function takes a form of a ball in the measurement space, the Gaussian approximation \( (20) \) provides good results for the higher-order moments. In the displaced GHZ state \( X_\nu |GHZ\rangle , \) with \( h(\nu) = N/2, \) described by a localized distribution, such relative deviation is \( \sim N^{-1}. \)
For non-localized states the approximation (20) does not lead to an accurate estimate of highest moments along the directions where the distribution is spread in the measurement space. For instance, the relative deviation of the forth order moments in the non-localized $x$-direction is $\sim 1$ for the $W$-state,

$$\langle S^4_{xc}\rangle_{app} \approx 27N^2 + 12N - 20,$$
$$\langle S^4_{xc}\rangle_{ex} = 15N^2 - 30N + 16,$$

while in the localized $z$-direction it is still $\sim N^{-2}$:

$$\langle S^4_{zc}\rangle_{ex} = [N - 2]^4,$$
$$\langle S^4_{zc}\rangle_{app} \approx [N - 2]^4 + 16 [N - 2]^2.$$

A similar situation occurs in the GHZ state if the approximation (25) is used:

$$\langle S^4_{xc}\rangle_{app} \approx 3N^2 + 16N, \quad \langle S^4_{zc}\rangle_{ex} = 3N^2 - 2N,$$
$$\langle S^4_{zc}\rangle_{app} \approx 3N^4 + 16N^2, \quad \langle S^4_{zc}\rangle_{ex} = N^4.$$

The two-Gaussian approximation (27) provides a correct estimate for the leading terms of highest moments even in the non-localized direction with a relative deviation $\sim N^{-2}$:

$$\langle S^4_{zc}\rangle_{ex} \approx N^4 + 16N^2.$$

As was mentioned in Sec.II, the accuracy of estimation of higher order moments from the two lowest ones for localized states can be related to the lengths of principal axes. In general, the larger is the length of the axis the less is the accuracy of estimation along this direction (for appropriately normalized operators). For instance, for the fiducial state (7) the average value $\langle (S \cdot n_j) \rangle^k = N^k$, $n_1 = (1,1,1)/\sqrt{3}$ cannot be obtained as a result of integration of the Gaussian (17) with the corresponding $P$-function for $k > 4$. As a representative additional example, let us consider an element of computational basis $|s\rangle$, for which the T-matrix has the form (28) and thus the principal axes and the corresponding eigenvalues are

$$n_1 = (1,0,1)/\sqrt{2}, \quad n_2 = (-1,0,1)/\sqrt{2}, \quad n_3 = (0,0,1),$$
$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{6} [1 - 2\gamma], \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{6} [1 - 2\gamma], \quad \lambda_3 = \frac{1}{3}.$$

where $\gamma = h(\kappa)/N$. The exact and approximate values of the forth order moments along $n_1$ and $n_3$ are then

$$\langle S^4_{xc}\rangle_{ex} = [1 - 2\gamma]^4 N^4,$$
$$\langle S^4_{xc}\rangle_{app} \approx [1 - 2\gamma]^4 N^4 + 16 [1 - 2\gamma]^2 N^2,$$

while in the localized $z$-direction the corresponding estimates from the exact ones are

$$N^2 \left( \langle S^4_{zc}\rangle_{app} - \langle S^4_{zc}\rangle_{ex} \right) = \frac{16}{\left[ 1 - 2\gamma \right]^2},$$

$$N \left( \langle (S_{xc} + S_{ye})^4 \rangle_{app} - \langle (S_{xc} + S_{ye})^4 \rangle_{ex} \right) = 18 + 4\sqrt{3} (1 - 2\gamma).$$

It follows form this that for the state $|0\rangle$ (all qubits are non-excited, $\gamma = 0$) $\lambda_1 > \lambda_3$ and the deviation (31) is larger than (30), while for $|1\rangle$ (all qubits non-excited, $\gamma = 1$) $\lambda_3 > \lambda_1$ and the situation is inverse. The case $\gamma = 1/2$ corresponding to a half excited qubits is special, since in such state $\langle S^4_{zc}\rangle_{ex} = 0$ for all $m$.

**Spherically symmetric localized distributions.** It is clear that the collective variables $\langle S \cdot n_j \rangle = N_j$, $n_j = (1,1,1)/\sqrt{3}$ can not be obtained as a result of integration of the Gaussian (17) with the corresponding $P$-function for $k > 4$. As a representative additional example, let us consider an element of computational basis $|s\rangle$, for which the T-matrix has the form (28) and thus the principal axes and the corresponding eigenvalues are

$$n_1 = (1,0,1)/\sqrt{2}, \quad n_2 = (-1,0,1)/\sqrt{2}, \quad n_3 = (0,0,1),$$

where $\gamma = h(\kappa)/N$. The exact and approximate values of the forth order moments along $n_1$ and $n_3$ are then

$$\langle S^4_{xc}\rangle_{ex} = [1 - 2\gamma]^4 N^4,$$
$$\langle S^4_{xc}\rangle_{app} \approx [1 - 2\gamma]^4 N^4 + 16 [1 - 2\gamma]^2 N^2,$$

CONCLUSIONS

In summary, discrete quasidistribution functions projected into the space of symmetric measurements provide useful insight into the macroscopic behavior of large particle systems. They can be used both for visualization purposes and for the analysis of general properties of quantum states from measured data in the asymptotic limit $N \to \infty$. In particular, it is suitable for testing the localization of $N$-qubit states and for estimation of the higher order correlations of collective observables. The shape of the projected $Q$-function characterizes the precision of the state description with the lowest moments. In particular, the first two moments of the collective variables in the localized directions contain important information about general properties of the state (e.g. allow to estimate the higher order correlations).

We have numerically tested several types of localized states and for all of them the Gaussian approximation (20) gives exact coefficient of the leading term for all higher moments. In
other words, the higher moments of normalized collective operators \( S \cdot n / N \) are completely determined by the first two moments. In this sense some of the localized states in the measurement space are similar to the 3 dimensional harmonic oscillator coherent states.

Especially interesting in this respect are spherically symmetric localized states. Due to the statistical independence of all directions the average value computed according to (29) of the commutator of any operator of the form \((S \cdot n)^k\) (and thus an arbitrary element from the enveloping algebra of collective operators) with \( S \cdot n \) over spherically symmetric states is zero to the leading order for all \( n, n' \). Thus, the operators \( S \cdot n \) can be effectively considered as classical macroscopic observables on the symmetric states.

Some of non-localized states (GHZ states, etc) can be represented in the measurement space as a superposition of Gaussians (20), which provides a transparent physical picture about sets of measurements that allow to detect such states. For instance, the GHZ state is well described by two Gaussians (27), the assessment of which requires measurements of the fluctuations of central moments \( S_jc = S_j - \langle S_j \rangle \) at \( \langle S_j \pm \rangle = (0, 0, \pm N) \) along directions determined by the principal axes: \( n_1 = (1, -1, 0)/\sqrt{2}, n_2 = (1, 1, 0)/\sqrt{2}, n_3 = (0, 0, 1) \).

Interestingly, small subspaces (of the whole \( 2^N \) dim Hilbert space) with a fixed energy do not necessarily correspond to the localized states. For instance, the uniform state in a subspace with a fixed eigenvalue \( N(N+2) \) of the Hamiltonian \( H = S_x^2 + S_y^2 + S_z^2 \) described by
\[
\rho = N^{-1} \sum_{k=-N/2}^{N/2} |k, N/2 \rangle \langle k, N/2|,
\]
where \( |k, N/2 \rangle \) is a completely symmetric under permutation \( N \) particle state with \( S_j |k, N/2 \rangle = (2k-N) |k, N/2 \rangle \) (the Dicke state), is not localized in the measurement space. The relation of our description of large qubit systems to the thermodynamical approach through a Hamiltonian evolution is an intriguing problem and will be considered elsewhere [16].

The same approach can be extended to systems with higher symmetries (qudits), in order to analyze the difference between large numbers of two and many-level systems in the macroscopic limit.

Similarly to the projected \( \bar{Q} \)-function it could be expected that a projected into the space of symmetric measurements discrete Wigner function \( W_\rho (\alpha, \beta) \) would be an interesting tool for observation of quantum interference in the macroscopic limit. Nevertheless, there is an important difference between the Wigner and \( \bar{Q} \)-functions on the geometrical level: it is usually expected that summing the Wigner functions along appropriate directions in the discrete phase-space (rays, for the standard construction) one obtains the probability to detect the systems in the states associated to such lines [10]. In order to satisfy such property one should insert a phase \( \phi(\gamma, \delta) \) into the kernel (3) and set \( s = 0 \). Such a phase satisfies the equation \( \phi^2(\gamma, \delta) = (-1)^{\gamma \delta} \) (for the standard construction [10]) and thus, is not uniquely defined [11]. On the other hand, since the Wigner mapping is self-dual, i.e.
\[
\langle \hat{f} \rangle = \sum_{\alpha, \beta} W_f (\alpha, \beta) W_\rho (\alpha, \beta),
\]
a meaningful projected Wigner function can be defined only if the symbols of collective operators \( W_f (\alpha, \beta) \) are functions of corresponding lengths, as it happens for \( Q \) and \( P \)-functions [11-10]. This requirement establishes an additional non-trivial condition of the phase \( \phi(\gamma, \delta) \). It would be interesting to find a phase that satisfies such a condition and simultaneously guarantee the self-duality and the geometrical properties of the Wigner mapping.

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APPENDIX

In this Appendix we present some explicit expressions for computation of symbols of collective operators.

The generating function for the \( Q \)-function of the moments of the collective operator in an arbitrary direction \( S \cdot n \) has the form
\[
Q(\alpha, \beta) = \langle \alpha, \beta \rangle \exp (\lambda S \cdot n) |\alpha, \beta\rangle
= (-1)^{h(\alpha)} \left[ \sinh (\lambda) \right]^N \sqrt{3} ^{N} \left[ n_x + n_y + n_z + \sqrt{3} \coth (\lambda) \right]^{h(\alpha)+h(\beta)+h(\alpha+\beta)}
\]
\[
\left[ n_x + n_y - n_z - \sqrt{3} \coth (\lambda) \right]^{-h(\alpha)-h(\beta)+h(\alpha+\beta)}
\]
\[
\left[ n_x - n_y - n_z + \sqrt{3} \coth (\lambda) \right]^{h(\alpha)+h(\beta)-h(\alpha+\beta)}
\]
\[
\left[ n_x - n_y + n_z - \sqrt{3} \coth (\lambda) \right]^{-h(\alpha)+h(\beta)-h(\alpha+\beta)} .
\]

The \( P \)-function has the same symmetry properties, but its computation is more involved and requires evaluation of the following sums:
\[
P(\alpha, \beta) = \frac{1}{2^{MN}} \sum_{\gamma, \delta} (-1)^{\alpha \delta + \beta \gamma} h(\gamma) + h(\delta) + h(\gamma + \delta)
\sum_{\mu, \nu} (-1)^{\mu \delta + \nu \gamma} Q(\mu, \nu) ,
\]
For instance, for the powers of \( S_x \) one obtains
\[
P_{S_x^2} = \frac{3}{2 N} \left( 2 x - 1 \right)^2 N^2 - \frac{N}{2^{h-1}} ,
\]
\[
P_{S1} = - \frac{3^{3/2} (2x - 1)^3 N^3}{2^N} + \frac{3^{3/2} (2x - 1) N^2}{2^{N-1}} - \frac{\sqrt{3}N (2x - 1)^3}{2^{N-2}},
\]

\[
P_{S2} = 9 \frac{(2x - 1)^4 N^4}{2^N} - 36 \frac{(2x - 1)^2 N^3}{2^N} + 12 \frac{(16x^2 - 16x + 5) N^2}{2^N} - 32 \frac{N}{2^N},
\]

where \(x = m/N\).

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