Complex networks based on discrete-mode lasers

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Abstract. We propose a method to combine a number of discrete-mode lasers to construct a complex network with a nontrivial topology. The topological properties of this network are only encoded in the selection of the active modes for each laser, while the optical fibre coupling between the lasers is trivial. In a very simplified model of this network, we establish the stability of the fixed point with all lasing modes switched on.

1. Introduction
Many complex systems encountered in nature are made up of a number of relatively simple dynamical units which are however connected in a non-trivial way. In particular, a different connection topology can change the properties of the resulting complex network completely, even if the dynamics of the basic entities are unchanged. While in many disciplines researchers have traditionally focused on the identification of those basic entities and their dynamics on various hierarchy levels, the study of the connection topology itself and its consequence for the dynamical behaviour of the overall system received only little attention for a long time.

This situation radically changed at the end of the 1990s, when it was realized that topological properties of naturally occurring networks can be explained by simple rules governing how the connections between the nodes are to be constructed. It was discovered that a simple random rewiring of a small fraction of links in a regular network leads to a small world network with a short characteristic path length between any two nodes [1]. At the same time the clustering coefficient, which measures the probability that two neighbors of a node are themselves directly linked remains at a high value close to the case of the regular network. A second breakthrough came with the introduction of growing networks [2], which follow the preferential attachment rule. In this case, new nodes are added to the networks by linking them with a high probability to existing nodes, which already have a high degree (i.e. a large number of links). This leads to the characteristic power law form of the degree distribution with a few hub nodes of high degree and many nodes with a small degree.

Since then, research on complex networks has dramatically increased and complex networks have been applied to many fascinating phenomena in biology, technology and society, such as the brain [3], power grids [4], transportation networks [5] or financial markets [6]. For a recent review we refer to [7].

In spite of this impressive research effort, so far no technological realization of a large scale complex network exists. The research on complex networks today relies mostly on computer simulations, and compares with data from existing complex networks, such as the brain or the Internet, with only very limited possibilities to modify the topology. In particular, it has not yet proved possible to construct a large scale network with a prescribed topology as a physical system in the laboratory. This would be highly desirable from a fundamental point of view,
since it would allow one to test many of the concepts in complex network theory experimentally, and could lead the way to highly interesting applications.

In this paper we will theoretically explore the possibility of implementing an optical complex network on the basis of the recently developed discrete-mode Fabry-Pérot lasers [8, 9]. To this end, we will first review the relevant properties of the uncoupled two-mode lasers in Sec. 2, then we will describe how a complex topology can be realised using such lasers in Sec. 3 and finally consider how the stability of a synchronized state in such a network can be assessed in Sec. 4.

2. Properties of free running two-mode lasers
The two-mode lasers, which form the basic building blocks of the proposed complex network, are based on the edge-emitting Fabry-Pérot semiconductor laser diode. Due to the broad gain peak of the active medium, an unperturbed Fabry-Pérot laser emits light at many equally spaced frequencies simultaneously. In order to limit the number of lasing modes to, for example, only two modes, we introduce a small number of perturbative features along the cavity. The positions of the features are determined via the solution of the corresponding inverse problem [8, 9]. In principle this technique allows us to arbitrarily select two lasing modes among a pool of the order of 30 or more fundamental Fabry-Pérot modes that are available.

In this section we review the dynamical properties of the resulting two-mode laser, which are relevant for the purpose of building a complex network. We concentrate on the case of a two-mode laser, where the two lasing modes are chosen to be four fundamental Fabry-Pérot modes apart, since so far this is the experimentally most studied case [9, 10].

A typical optical spectrum for this type of laser is shown in Fig. 1, with the pump current adjusted to obtain identical average power in each mode.

A mode resolved time trace of the light output power, as shown in the inset of Fig. 2 reveals

Figure 1. Typical optical spectrum of a two-mode laser
that the power in the individual modes fluctuates in time. The important feature, which makes the two-mode laser a highly interesting device for the construction of a network, is that due to the mode competition, the two modes in the laser are coupled via negative feedback, and therefore give rise to anti-correlated fluctuations [9] at low frequencies. This is demonstrated more precisely in the mode-resolved power spectra of the free running two-mode laser diode shown in Fig. 2. We observe that the low frequency components in the spectra below 2GHz are far more pronounced in the individual modes, than they are in the sum of the two modes. The reason for this is that the low frequency fluctuations of the individual modes are anti-correlated, and therefore the total output power of this laser has largely suppressed low frequency oscillations. This feature is also visible from the time traces of the output power of the individual modes and the total output power, as shown in the inset of Fig. 2.

As we will explain in more detail in the next section, we propose to build a complex network by coupling many two-mode lasers. A first step towards an understanding of the dynamics of such a coupled system, is to examine the response of only one two-mode laser under the external injection of light from a tunable laser [10]. If the frequency of the injected light is close to the frequency of one of the two modes, one observes a rich bifurcation scenario including bistability and hysteresis between single-mode and two-mode equilibria. Then the dynamical equations for the complex electric field amplitude of the two modes $E_1$ and $E_2$ and the excess carrier density...
$n$ can be written as [10]:

\begin{align}
\dot{E}_1 &= \left[ \frac{1}{2} (1 + i\alpha)(g_1(2n + 1) - 1) - i\Delta\omega \right] E_1 + K \\
\dot{E}_2 &= \frac{1}{2} (1 + i\alpha)(g_2(2n + 1) - 1)E_2 \\
T\dot{n} &= P - n - (1 + 2n) \left( g_1|E_1|^2 + g_2|E_2|^2 \right),
\end{align}

with the nonlinear modal gain function

$$g_m = g_m^{(0)} \left( 1 + \epsilon \sum_n \beta_{mn}|E_n|^2 \right)^{-1}. \quad (4)$$

Here $K$ is the injected power and $\Delta\omega$ is the angular frequency detuning between injected light and the first lasing mode. Furthermore $g_m^{(0)}$ is the linear modal gain, $T$ is the carrier lifetime, $P$ is the pump current, and $\alpha$ is the phase-amplitude coupling. The $\epsilon\beta_{mn}$ determine the cross and self saturation. All parameters, the time variable and the dynamical variables are conveniently normalized. A detailed analysis of the dynamical equations (1-3) shows a remarkable agreement between experimental data and theoretical prediction [10], which motivates our use of this theoretical framework as the basis for the proposed system of many coupled two-mode lasers.

For the purpose of the current paper the following two points will be important to remember: (i) only injected light which is close to the selected lasing modes of the two mode laser has appreciable influence on the internal dynamics; (ii) the field variables are complex variables with amplitudes and phases, however only the amplitudes couple to other dynamical variables in the system.

3. Building a complex network topology

After we have clarified the dynamics of a single two-mode laser, let us now turn to the problem of connecting them to form a complex network.

A straightforward way to connect a number of lasers in a topologically non-trivial way would be via a network of optical fibre links. Then the topology of the network is directly represented by the physical fibre connection between the individual lasers. In practice such an approach is however technically difficult and cumbersome even for a relatively small amount of lasers, and would not scale to construct a network with a large number of nodes.

We therefore propose an alternative way of representing a non-trivial topology within a system of coupled lasers. Our method makes use of the fact that it is relatively easy to select a different pair of lasing modes for each laser, by simply introducing suitable perturbative features along the cavity of each laser diode individually. The light output of each of these lasers can then be coupled into a global wavelength multiplexer which is terminated by a mirror.

An example of such a setup using four two-mode lasers is illustrated in Fig. 3(a). The multiplexer and mirror cause the light output from every laser to be fed back into every other laser, which at first glance seems to give rise to a trivial complete graph topology. However we also know from the previous section that only light which is injected at a frequency similar to one of the two lasing modes of a given laser can affect its internal dynamics. If we now assume that the individual modes of the four lasers in our example are chosen as in Fig. 3(b), then we see that the dynamically relevant coupling structure in this setup is much more complicated than a simple all-to-all coupling. For example, the top laser is not directly coupled with the bottom laser, since they do not have any selected mode with a common frequency, and therefore light from the top laser can not directly influence the internal dynamics of the bottom laser or vice versa. On the other hand, the bottom three lasers are all coupled, since they share
selected modes of a common frequency. In order to represent the structure of the dynamical connections of this setup, we could for example draw an abstract diagram as in Fig. 3(c). Here the modes of each laser are represented by coloured circles, where circles of the same colour are connected via a thick black link (representing the optical fibre connection). The two modes for each individual laser are connected via a coloured link, and therefore we can regard the coloured links as representing the internal dynamics of individual lasers. We see that the resulting network has in this representation a non-trivial topology.

In order to make contact with the well developed theory of complex networks, we will now proceed with the following two simplifications. First we assume that the speed of light is infinite (i.e. vanishing delay time for light propagation between the lasers) and secondly we replace the degrees of freedom of all modes in the individual lasers that share a common frequency by one single degree of freedom, namely the field intensity of this mode at the reflecting mirror. We stress however that this represents a substantially simplified view of the original coupled laser system, since we neglect that the individual modes are in general separate degrees of freedom which couple at different ends of the optical fibre, and the optical signal needs a finite time to travel between the two input points. This simplification allows us to represent each global mode associated with one frequency by a coloured circle in Fig. 3(d), replacing the connected circles of equal colour in Fig. 3(c). At the same time we replace the coloured lines in Fig. 3(c), which represent the internal degrees of freedom of the individual lasers, by coloured squares in Fig. 3(d). The degree of a circle is given by the number of lasers sharing the associated mode, and the degree of a box is given by the number of selected modes in the associated laser.

This type of graph with two types of nodes (squares and circles) and connections only between different types of nodes is commonly called a bipartite network. In a bipartite lattice all links are assumed to couple to the same degree of freedom, which we achieve by the assumptions discussed in the preceding paragraph. It turns out that a suitable generalisation of the concept of a bipartite network is in fact able to describe the coupled laser system even if these assumptions are relaxed. This will be discussed in a future work. For the purpose of the current paper we simply assume that the essential dynamics in the laser network can be represented via a conventional bipartite network. The advantage of this simplification is that it is possible to apply the known theory of synchronisation in a bipartite network [11], as we shall do in the next section.

4. Stability analysis of the synchronised state
The purpose of the current section is to demonstrate that, in principle, the Master Stability Function (MSF) [12, 13] technique for bipartite networks [11] can be applied to assess the stability of a suitably simplified coupled laser system. Although our presentation is motivated by the MSF technique, no prior knowledge of the MSF is required to follow this section, since we restrict ourselves to the simplest case of analysing the stability of a fixed point.

Even a single optically injected two-mode laser shows a highly nontrivial and complex bifurcation scenario experimentally, and a model of at least the complexity of Eq. (1)–(3) is required to adequately describe the involved phenomena [10]. Nevertheless, we here consider a drastically simplified model for our coupled laser system in order to be able to apply the existing techniques for complex networks.

Let us start with the following simplified model equations for the $N$ uncoupled two-mode lasers:

\[
\frac{d|E^{k}_{1}|^2}{dt} = [(2n^{k} + 1)g^0 - 1]|E^{k}_{1}|^2 \quad (5)
\]

\[
\frac{d|E^{k}_{2}|^2}{dt} = [(2n^{k} + 1)g^0 - 1]|E^{k}_{2}|^2 \quad k = 1, \ldots, N \quad (6)
\]
Figure 3. (a) Physical coupling scheme of a set of four two-mode lasers which are coupled via a wavelength multiplexer into a single waveguide. This single waveguide is terminated by a reflecting mirror (orange). (b) Selected modes for each laser diode. (c) Abstract network topology arising from the simple physical all-to-all coupling as in panel (a) and the mode spectrum of panel (b). (d) Resulting bipartite network. The square nodes represent the two-mode lasers, and the circular nodes represent light modes at given wavelengths connecting the lasers.

\[
T \frac{dn^k}{dt} = P - n^k - (2n^k + 1)g^0 \left( |E_{1k}^k|^2 + |E_{2k}^k|^2 \right).
\]  

Compared to Eq. (1)–(3) we have neglected the phases of the field variables, and only deal with the amplitudes. Furthermore we neglect the fact that the parameters in Eq. (1)–(3) will in general depend on the frequency difference between the selected lasing modes. We also put \( \epsilon = 0 \) and \( g_1^0 = g_2^0 = g^0 \) for simplicity.

In order to now introduce a coupling between the lasers, we use the assumptions discussed in the previous section and replace the degrees of freedom of all \( |E_{1j/2}^k|^2 \) pertaining to a common fundamental mode \( j \) by simply one global field amplitude \( A_j \), which we define to be the optical field amplitude at the mirror.

Without worrying at the moment about the validity of our approximations, we can write down an extremely simple form of the system of \( N \) coupled lasers,

\[
\frac{dA_j}{dt} = -A_j + g^0 A_j \sum_k K_{jk}(2n^k + 1) \quad j = 1, \ldots, M
\]
Here $M$ denotes the number of modes, and $K$ is the adjacency matrix of the system (i.e. $K_{jk} = 1$ ⇔ laser $k$ lases at fundamental mode $j$). The coupling structure of (8)–(9) is now in the form of a bipartite complex network, since the coupling only happens between the $A_j$ and $n^k$ variables.

An obvious necessary condition for a synchronous state $A_1(t) = \ldots = A_M(t)$ and $n_1(t) = \ldots = n_N(t)$ to be a solution of (8)–(9) is that the row sums $\sum_{k} K_{jk} = K_r$ and column sums $\sum_{j} K_{jk} = K_c$ need to be constant [7]. If we use only two mode lasers, we have $K_c = 2$ and the column sum is naturally fulfilled. The condition of the constant row sum is fulfilled if we consider network topologies, which have the same number of lasers lasing at each mode. Then the synchronous state follows the dynamical equations

$$
\frac{dA_s}{dt} = -A_s + g_0 A_s K_r (2n_s + 1) \quad (10)
$$

$$
T \frac{dn_s}{dt} = P - n_s - (2n_s + 1)g_0 K_c A_s. \quad (11)
$$

The structure of this set of equations is just as in a single mode laser, with however $K_r$ and $K_c$ modifying the gain coefficient. Elementary stability analysis reveals [14] that the point

$$
A_s = \frac{K_r}{K_c} (P - n_s) \quad (12)
$$

$$
2n_s = \frac{1}{g_0 K_r} - 1 \quad (13)
$$

becomes a stable fixed point of the synchronous state equations (10)–(11) for $P > n_s$. However, this does not automatically imply that the point $(A_s, n_s)$ is also stable fixed point of the full set of dynamical equations (8)–(9).

To assess the global stability of the synchronous state we perform a straightforward linear stability analysis of equations (8)–(9) around $(A_s, n_s)$ which leads to

$$
\frac{d\delta A_j}{dt} = g_0 A_s \sum_{k} K_{jk} 2 \delta n^k \quad j = 1, \ldots, M \quad (14)
$$

$$
T \frac{d\delta n^k}{dt} = -g_0 K_r (2P + 1) \delta n^k - \frac{1}{K_r} \sum_{j} K_{jk} \delta A_j, \quad k = 1, \ldots, N. \quad (15)
$$

The synchronous is stable if all eigenvalues of

$$
G = \begin{pmatrix}
0 & g_0 A_s K_r \\
-\frac{1}{K_r} K^T & -\frac{g_0 A_s K_r (2P + 1)}{T} \Id
\end{pmatrix} \quad (16)
$$

have negative real part.

While the problem of finding the eigenvalues of (16) is elementary, we choose to tackle this problem in a way which is similar to the MSF technique for bipartite networks [11], although we restrict ourselves here to the case of the stability analysis of fixed points. As a first step we need to determine the eigenvectors $(u_i, v_i)^T$ and eigenvalues $\lambda_i$ fulfilling

$$
H \begin{pmatrix}
u_i \\
v_i
\end{pmatrix} = \begin{pmatrix}
0 & K \\
K^T & 0
\end{pmatrix} \begin{pmatrix}
u_i \\
v_i
\end{pmatrix} = \lambda_i \begin{pmatrix}
u_i \\
v_i
\end{pmatrix} \quad (17)
$$
Note that the matrix $H$ only depends on the network topology, and not on the dynamical properties of the lasers. Given any specific adjacency matrix $K$ with positive entries and constant row and column sums ($\sum_k K_{jk} = K_r$ and $\sum_j K_{jk} = K_c$) the eigenvalues $\lambda_i$ will turn out to be real, since

$$H^2 = \begin{pmatrix} KK^T & 0 \\ 0 & K^T K \end{pmatrix}$$

(18)

is positive semidefinite.

For each of these eigenvectors we can then make the ansatz

$$G \left( \begin{array}{c} c_1 u_i \\ c_2 v_i \end{array} \right) = \left( -\frac{g_0 A_s \lambda_i c_1}{TK_r} - \frac{g_0 K_r (2P+1) c_2}{T} \right) v_i = \Lambda_i \left( \begin{array}{c} c_1 u_i \\ c_2 v_i \end{array} \right)$$

(19)

which via (16) simply means that

$$\left( \begin{array}{c} 0 \\ -\frac{g_0 A_s \lambda_i}{TK_r} \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) = \Lambda_i \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right)$$

(20)

Stability is obtained if all eigenvalues $\Lambda_i$ of the matrix $G$ (16) have negative real part. From our considerations above, we simply need to ensure that all eigenvalues of the matrix

$$L(\lambda) = \begin{pmatrix} 0 & \frac{g_0 A_s \lambda^2}{TK_r} \\ -\frac{1}{TK_r} \lambda & -\frac{g_0 K_r (2P+1)}{T} \end{pmatrix}$$

(21)

have negative real part, for $\lambda \in \{\lambda_1, \ldots, \lambda_{M+N}\}$. Since $\text{Tr} L(\lambda) < 0$ we only need to ensure that

$$\det L(\lambda) = g_0 A_s \lambda^2 > 0$$

(22)

which is the case for real $\lambda$. Since $\lambda$ is taken from the set of eigenvalues of the matrix $H$ and we have shown above that the eigenvalues $\lambda_i$ are real, we have therefore demonstrated the stability of the synchronous fixed point.

The result of this section is therefore that the simplified model of our coupled laser system has always a trivial stable fixed-point of all modes lasing simultaneously. We have arrived at this result by separating the stability analysis into two parts. The topological part of the problem is to consider the possible eigenvalues of the matrix $H$, which only contains information on the adjacency matrix. The dynamical part of the problem is to determine the eigenvalues of the matrix $L(\lambda)$, which only contains properties of the local system. In fact this separation of the stability problem into a topological and a dynamical part, is what is the essence of the Master Stability Function (MSF) approach [12, 13], and what we have followed in this section can be considered to be a special case of the MSF for bipartite networks, as developed in [11].

5. Summary

In summary we have theoretically investigated a means to construct a network of two mode lasers, which avoids the need for cumbersome optical fibre coupling. Instead all lasers are trivially coupled and the topology of the network is obtained by individually selecting different modes for each laser. Using a very simplified model we have demonstrated that the interacting network has a stable synchronous fixed point. We expect more interesting dynamics to emerge, as more realistic features are added to our model. Such a network may open the way for a technological realization of complex networks on the basis of optical elements.

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