Fusion rules of chiral algebras

Matthias Gaberdiel
Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Silver Street
Cambridge, CB3 9EW, U. K.

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Abstract

Recently we showed that for the case of the WZW- and the minimal models fusion can be understood as a certain ring-like tensor product of the symmetry algebra. In this paper we generalize this analysis to arbitrary chiral algebras. We define the tensor product of conformal field theory in the general case and prove that it is associative and symmetric up to equivalence. We also determine explicitly the action of the chiral algebra on this tensor product.

In the second part of the paper we demonstrate that this framework provides a powerful tool for calculating restrictions for the fusion rules of chiral algebras. We exhibit this for the case of the $W_3$-algebra and the $N=1$ and $N=2$ NS superconformal algebras.

1 Introduction

Fusion is a central concept in conformal field theory. The fusion rules of a conformal field theory describe which of the three-point-functions of the theory are non-zero and thus de-
termine the possible couplings. Equivalently, the fusion rules can be understood to describe which conformal families appear in the operator product expansion of two vertex operators.

This latter point of view suggests that one should regard fusion as some kind of tensor product. Conformal families can be interpreted as irreducible representations of the chiral algebra and multiplying vertex operators should correspond to taking the tensor product. The operator product expansion is then the decomposition of this tensor product into the irreducible components.

Generically, the chiral algebra (of which this tensor product has to be a representation) possesses a central extension and thus there is no canonical definition of a tensor product. However, inspired by a recent proposal of Richard Borcherds [3] to regard fusion in conformal field theory as the canonical tensor product of modules of a quantum ring, a generalization of rings and vertex algebras, we showed in [10] how one can define a tensor product for the case of the WZW- and the minimal models. We determined explicitly the action of the chiral algebra on this tensor product and showed that the tensor product is associative and symmetric up to equivalence. We then determined the fusion rules of these models, analyzing the decomposition of the tensor product into its irreducible components. We thereby recovered the well-known restrictions for the fusion rules.

In this paper we want to generalize this analysis to arbitrary chiral algebras. In section 2 we derive the action of a general chiral algebra on products of vertex operators, which gives rise to an action of the chiral algebra on tensor products of representations. This action can be formulated as a comultiplication which depends on two parameters, namely the insertion points of the two vertex operators. For certain values of these parameters we have two different expressions, which agree on the underlying conformal field theory. In order to maintain the whole information of conformal field theory, we have to impose this equality on the level of representations. This leads us to defining the "true" tensor product of conformal field theory as the quotient of the vector space tensor product by all relations which arise in this way.

The action of the chiral algebra on this tensor product is given by either of the two expressions. As in [10] we can prove, that the so-defined tensor product is associative and symmetric up to equivalence. There we were also able to show, that these expressions do indeed define a comultiplication. We cannot repeat this proof here, as we have not specified the commutation (or anticommutation) relations of the chiral algebra. However, we can show on general grounds that they have to satisfy this property if they come from a well-defined conformal field theory.

We have thus established a framework within which fusion is a well-defined algebraic notion. Determining the fusion rules now amounts to decomposing this tensor product into
irreducible representations. We cannot say very much about this in the general case, however, given any algebra we can use the knowledge about the null-vectors to determine restrictions for the possible fusion rules. In particular, as we have proved the associativity of the tensor product, we only have to analyze the fusion of simple representations to obtain restrictions for the general case. We use this argument to derive restrictions for the fusion rules of the $W_3$-algebra and the $N = 1$ and $N = 2$ NS superconformal algebras in section 3-5, thereby reproducing the results of [8], [15] and [13]. Our method, however, is entirely different: it is purely algebraic and does not (for the case of the superconformal algebras) rely on any superdifferential calculus.

2 Definition of the tensor product

Let us start by defining the chiral algebra of a conformal field theory. Let $S(w)$ be a holomorphic field with conformal weight $h \in \mathbb{Z}/2$, where $L_0 S(0) \Omega = h S(0) \Omega$. Following [11] we can expand $S$ in terms of modes as

$$S(w) = \sum_{l \in \mathbb{Z}+h} w^{l-h} S_l.$$  

The operator product expansion of two holomorphic fields contains only holomorphic fields. We thus define the chiral algebra $\mathcal{A}$ to be the algebra generated by all the modes of all holomorphic fields, subject to the commutation (or anticommutation) relations induced from the (singular part of the) operator product expansion. This definition contains most “symmetry algebras” of conformal field theories, but not for example the Ramond superconformal algebras. On the other hand we do not want to include the Ramond algebras in our definition, as the operator product expansion of two Ramond fields contains only Neveu-Schwarz fields and this does not allow an interpretation as a tensor product.

The operator product expansion of a holomorphic field $S$ with a vertex operator is then given by [11]

$$S(w)V(\psi, \zeta) = \sum_{l \in \mathbb{Z}+h} (w - \zeta)^{l-h} V(S_l \psi, \zeta).$$

We want to determine the action of the modes $S_l$ on products of vertex operators. To do this we have to calculate (as in [11]) the contour integral

$$\oint_C dw w^{l+h-1} S(w) V(\psi, \zeta)V(\chi, z) \Omega,$$
where $C$ is a large contour encircling the two insertion points and we assume $|\zeta| > |z|$ in order to guarantee the existence of the operator product. We could use the usual contour integration techniques to evaluate (2.3), as the residues of the integrand are determined by (2.2). However, if we did this calculation straight away, we would obtain terms, where the $S$-modes act on the vacuum — and these terms would not allow an interpretation as a comultiplication. Instead we consider scalar products of the integrand of (2.3) with any vector in the dense subset of finite energy vectors $\phi \in \mathcal{F}$ and obtain thus (by the definition of conformal field theory) a meromorphic function of $w$

$$\langle \phi, S(w) V(\psi, \zeta) V(\chi, z) \Omega \rangle,$$

(2.4)

whose singular part is given by

$$\varepsilon_{\psi} \sum_{l=-\infty}^{h-1} (w - z)^{l-h} \langle \phi, V(\psi, \zeta) V(S_{-l} \chi, z) \Omega \rangle + \sum_{m=-\infty}^{h-1} (w - \zeta)^{-l-h} \langle \phi, V(S_{-m} \psi, \zeta) V(S_{-l} \chi, z) \Omega \rangle.$$

(2.5)

Here $l$ and $m$ are in $\mathbb{Z} + h$ and we assume that the vertex operator $V(\psi, \zeta)$ and the holomorphic field $S^a(w)$ are local with respect to each other, i.e.

$$S^a(w) V(\psi, \zeta) = \varepsilon_{\psi} V(\psi, \zeta) S^a(w)$$

(2.6)

in the sense of [11, (2.5b)], where $\varepsilon_{\psi} = \pm 1$. Using the operator product expansion of the holomorphic field $S(w)$ with the vertex operator $V(\psi, z)$ we can write the regular part of the meromorphic function (2.4) as

$$\varepsilon_{\psi} \sum_{l=h}^{\infty} (w - z)^{-l-h} \langle \phi, V(\psi, \zeta) V(S_{-l} \chi, z) \Omega \rangle - \sum_{m=-\infty}^{h-1} (w - \zeta)^{-m-h} \langle \phi, V(S_{-m} \psi, \zeta) V(S_{-l} \chi, z) \Omega \rangle;$$

(2.7)

if rewritten as a power series about $w = 0$ (assuming, that (2.7) is well-defined for $w = 0$, i.e. that $|\zeta| > |z|$ is suitably chosen), it will converge for all $w$, since this function is entire. As (2.4) is just the sum of (2.5) and (2.7), we can use these expressions to evaluate the contour integral (2.3), thereby obtaining only terms, where $S_l$ either acts on $\psi$ or on $\chi$. The action is independent of the chosen vector $\phi$. We can thus interpret the formulae as giving a comultiplication of the chiral algebra, namely

$$\oint_C dw \ w^m \langle \phi, S^a(w) V(\psi, \zeta) V(\chi, z) \Omega \rangle = \sum \langle \phi, V(\Delta_{\psi, \zeta}^{(1)}(S^a_m) \psi, \zeta) V(\Delta_{\chi, z}^{(2)}(S^a_m) \chi, z) \Omega \rangle,$$

(2.8)
where we write $\Delta_{\zeta,z}(a) \in \mathcal{A} \otimes \mathcal{A}, a \in \mathcal{A}$ as

$$\Delta_{\zeta,z}(a) = \sum \Delta_{\zeta,z}^{(1)}(a) \otimes \Delta_{\zeta,z}^{(2)}(a). \quad (2.9)$$

We calculate

$$\Delta_{\zeta,z}(S_n) = \sum_{m=1-h}^{n} \binom{n+h-1}{m+h-1} \zeta^{n-m} (S_m \otimes 1)$$

$$+ \varepsilon_1 \sum_{l=1-h}^{n} \binom{n+h-1}{l+h-1} z^{n-l} (1 \otimes S_l), \quad (2.10)$$

$$\Delta_{\zeta,z}(S_{-n}) = \sum_{m=1-h}^{\infty} \binom{n+m-1}{n-h} (-1)^{m+h-1} \zeta^{-(n+m)} (S_m \otimes 1)$$

$$+ \varepsilon_1 \sum_{l=1-h}^{\infty} \binom{l-h}{n-h} (-z)^{l-n} (1 \otimes S_{-l}), \quad (2.11)$$

where in (2.10) we have $n \geq 1 - h$, in (2.11) $n \geq h$ and $\varepsilon_1$ is $\pm 1$ according to whether the left-hand vector in the tensor product satisfies (2.6) with $\varepsilon_\psi = \pm 1$. (In (2.10) $m$ and $l$ are in $\mathbb{Z} - h$, in (2.11) $m \in \mathbb{Z} - h$ and $l \in \mathbb{Z} + h$.)

As in [10] the above formulae are formally well defined for $|\zeta| > 1$ and $|z| < 1$. If we restrict ourselves to a suitable dense subset of the tensor product (containing all tensor products of finite energy vectors) the formulae are well-defined for (a suitable subset of) $|z| < 1$ and $\zeta \in \Phi \setminus \{0\}$. We can also interchange the rôles of $\zeta$ and $z$, thereby obtaining instead of (2.10)

$$\tilde{\Delta}_{\zeta,z}(S_n) = \varepsilon_2 \sum_{m=1-h}^{n} \binom{n+h-1}{m+h-1} \zeta^{n-m} (S_m \otimes 1)$$

$$+ \sum_{l=1-h}^{n} \binom{n+h-1}{l+h-1} z^{n-l} (1 \otimes S_l), \quad (2.12)$$

and instead of (2.11)

$$\tilde{\Delta}_{\zeta,z}(S_{-n}) = \varepsilon_2 \sum_{m=n}^{\infty} \binom{m-h}{n-h} (-\zeta)^{m-n} (S_{-m} \otimes 1)$$

$$+ \sum_{l=1-h}^{\infty} \binom{n+l-1}{n-h} (-1)^{l+h-1} z^{-(n+l)} (1 \otimes S_{-l}). \quad (2.13)$$
On the underlying conformal field theory the action of (2.10, 2.11) and (2.12, 2.13) must coincide, wherever both are well-defined, e.g. on finite energy vectors for $0 < |z|, |ζ| < 1$. To maintain the whole information of conformal field theory we have to impose these relations on the level of the representations. Therefore we define the “true” tensor product of conformal field theory (for fixed ($ζ, z$)) to be the vector space tensor product quotiented by all relations which arise in this way. On this (ring-like) tensor product, the action of the chiral algebra is then given by either of the two formulae.

We remark that we recover the formulae given in [10] for the affine algebra and the Virasoro algebra, if we set $h = 1, 2$, respectively. There we were able to prove that (2.10 - 2.13) define actually an algebra homomorphism which preserves the central charge. We cannot give a similar proof here, as we have not specified the commutation (or anticommutation) relations of the chiral algebra. However, recalling the derivation from conformal field theory, it is obvious, that this has to be true on the underlying conformal field theory. Namely, we can use a usual contour deformation argument to see that the action of $AB ± BA$ on the product of vertex operators equals the action of $[A, B]_{±}$, which amounts to saying that (2.10, 2.11), resp. (2.12, 2.13) actually defines a comultiplication on the true tensor product of conformal field theory.

To proceed further we have to determine the dependence of the tensor product on the two parameters $ζ$ and $z$. To this end we first translate the transformation properties of the holomorphic field into corresponding properties of the modes. Namely, following [11], we have

$$e^{uL} S(w) e^{-uL} = S(w + u) \quad (2.14)$$

and

$$λ^L S(w) λ^{-L} = λ^h S(λw), \quad (2.15)$$

where in (2.15) it is understood that a choice of the logarithm of $λ$ has been made. Expressing $S(w)$ in terms of the modes (2.1) the above two equations become

$$e^{uL} S_m e^{-uL} = \begin{cases} \sum_{l=1-h}^{m} \binom{m+h-1}{l+h-1} (-u)^{m-l} S_l & \text{if } m \geq 1 - h \\ \sum_{l=-m}^{\infty} \binom{l-h}{-m-h} u^{l+m} S_{-l} & \text{if } m \leq -h, \end{cases} \quad (2.16)$$

and

$$λ^L S_m λ^{-L} = λ^{-m} S_m. \quad (2.17)$$

Using these equations we can show (as in [10]) that

$$(e^{uL} \otimes e^{vL}) \circ Δ_{ζ+u, z+v} \circ (e^{-uL} \otimes e^{-vL}) = Δ_{ζ,z}. \quad (2.18)$$
and
\[(\lambda L_0 \otimes \lambda L_0) \Delta_{\zeta,z}(A) (\lambda^{-L_0} \otimes \lambda^{-L_0}) = \Delta_{\lambda\zeta,\lambda z} (\lambda L_0 A \lambda^{-L_0}) ,\] (2.19)
and similarly for \(\tilde{\Delta}\). (In the second formula we again assume that a choice for the logarithm of \(\lambda\) has been made.) We remark that (2.18) implies that the tensor products corresponding to different choices of \((\zeta, z)\) are equivalent. Thus, it does make sense to talk about the tensor product of two representations without reference to the insertion points, as all different choices are equivalent.

Next we want to investigate the associativity-properties of the tensor product. As in [10] we can show, that we have for suitable \(\zeta_1, \zeta_2\) and \(z\)
\[(\Delta_{\zeta_2-w, \zeta_1-w} \otimes 1) \circ \Delta_{w,z} = (1 \otimes \Delta_{\zeta_1-w, z-w}) \circ \Delta_{\zeta_2,w},\] (2.20)
and similarly for \(\tilde{\Delta}\) (see [10] for more details). These equations imply that the quotient of the triple tensor product, we have to take out in order to obtain the true tensor product of conformal field theory, is independent of the bracketing. In addition, the action of the chiral algebra is coassociative if we choose the parameters according to (2.20). As different choices for the parameters yield equivalent tensor products, the tensor product is associative up to equivalence.

We remark that using the same methods as above we could have also determined the action of the chiral algebra on products of three vertex operators. A straightforward calculation shows that the action (2.20) agrees with this action if the vertex operators are inserted at \(\zeta_2, \zeta_1\) and \(z\) and we use in (2.7) the operator product expansion of the holomorphic field with the vertex operator at \(z\). (If we take the operator product expansion with another vertex operator, we obtain an expression corresponding to (2.20) with some \(\Delta\)’s replaced by \(\tilde{\Delta}\).) Thus fusion of \(n\) representations really corresponds to taking successively \((n-1)\) tensor products.

We can similarly show that the tensor product is symmetric up to equivalence. Namely, we can calculate the \(R\)-matrix of the comultiplication, i. e. the operator \(R(\zeta, z)\) in \(A \otimes A\), which satisfies on all tensor products
\[R(\zeta, z) \circ \Delta_{\zeta,z} \circ R(\zeta, z)^{-1} = \tau \circ \Delta_{\zeta,z},\] (2.21)
where \(\tau\) is the twist-map, interchanging the two factors of the tensor product. As in [10] we find that \(R(\zeta, z)\) is given by
\[R(\zeta, z) = e^{(\zeta-z)L_{-1}} \otimes e^{(z-\zeta)L_{-1}},\] (2.22)
where we assume, that $\zeta$ and $z$ are suitably chosen to ensure convergence \[10\].

We have thus succeeded in giving a precise definition of fusion in the general case: fusion is simply the tensor product introduced above with the action of the chiral algebra induced by either of the two formulae (2.10, 2.11), resp. (2.12, 2.13). We have shown that the tensor product is associative and symmetric up to equivalence. To obtain the fusion rules we have to study the decomposition of the tensor product into the irreducible components. We cannot — at the moment — say very much about this in the general case, however, given any chiral algebra we can use the knowledge about null-vectors to obtain restrictions for the possible fusion rules.

### 3 \(W_3\)-algebra

The \(W_3\) algebra is generated by the Virasoro algebra \(\{L_n\}\), the modes of the energy momentum tensor with \(h = 2\), and the modes \(Q_m\) of a field of conformal dimension \(h = 3\), subject to the relations \[3\], \[3\]

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12} cn (n^2 - 1) \delta_{n,-m} \quad (3.1)
\]

\[
[L_m, Q_n] = (2m - n)Q_{m+n} \quad (3.2)
\]

\[
[Q_m, Q_n] = \frac{1}{48} (22 + 5c) \frac{c}{3 \cdot 5!} (m^2 - 4) (m^2 - 1) m \delta_{m,-n} + \frac{1}{3} (m - n) \Lambda_{m+n} + \frac{1}{48} (22 + 5c) \frac{c}{30} (m - n) (2m^2 - mn + 2n^2 - 8) L_{m+n}, \quad (3.3)
\]

where \(\Lambda_k\) are the modes of a field of conformal dimension \(h_\Lambda = 4\), which are explicitly given by

\[
\Lambda_n = \sum_{k=-\infty}^{\infty} :L_{n-k}L_k: + \frac{1}{5} x_n L_n, \quad (3.4)
\]

\(x_{2l} = (l+1)(1-l)\) and \(x_{2l+1} = (l+2)(1-l)\). Evaluating the general comultiplication formulae (2.10, 2.11) for the present case we obtain

\[
\Delta_{\zeta,z}(L_n) = \sum_{m=-1}^{n} \binom{n+1}{m+1} \zeta^{n-m} (L_m \otimes \mathbb{1}) + \sum_{l=-1}^{n} \binom{n+1}{l+1} z^{n-l} (\mathbb{1} \otimes L_l) \quad (3.5)
\]
\[ \Delta_{\zeta,z}(L_{-n}) = \sum_{m=-1}^{\infty} \left( \begin{array}{c} n + m - 1 \\ n - 2 \end{array} \right) (-1)^{m+1} \zeta^{-(n+m)} (L_m \otimes \mathbb{1}) \\
\quad \quad + \sum_{l=n}^{\infty} \left( \begin{array}{c} l - 2 \\ n - 2 \end{array} \right) (-z)^{l-n} (\mathbb{1} \otimes L_{-l}), \quad (3.6) \]

and

\[ \Delta_{\zeta,z}(Q_n) = \sum_{m=-2}^{n} \left( \begin{array}{c} n + 2 \\ m + 2 \end{array} \right) \zeta^{n-m} (Q_m \otimes \mathbb{1}) + \sum_{l=-2}^{n} \left( \begin{array}{c} n + 2 \\ l + 2 \end{array} \right) z^{n-l} (\mathbb{1} \otimes Q_l) \quad (3.7) \]

\[ \Delta_{\zeta,z}(Q_{-n}) = \sum_{m=-2}^{\infty} \left( \begin{array}{c} n + m - 1 \\ n - 3 \end{array} \right) (-1)^{m} \zeta^{-(n+m)} (Q_m \otimes \mathbb{1}) \\
\quad \quad + \sum_{l=n}^{\infty} \left( \begin{array}{c} l - 3 \\ n - 3 \end{array} \right) (-z)^{l-n} (\mathbb{1} \otimes Q_{-l}), \quad (3.8) \]

where in (3.3) we have \( n \geq -1 \), in (3.6) \( n \geq 2 \), in (3.7) \( n \geq -2 \) and in (3.8) \( n \geq 3 \).

The comultiplication of \( \Lambda_n \) is determined by the formulae (2.10) and (2.11), since the \( \Lambda_n \)'s are the modes of a field of conformal dimension \( h = 4 \). We can thus show, following the methods of [10] and using the power series expansions of appendix A, that these formulae do indeed define a comultiplication.

The highest weight representations are labeled by the eigenvalues of \( L_0 \) and \( Q_0 \) of the highest weight vector \( |h,q> \), i.e.

\[ L_m |h,q> = \delta_{m,0} h |h,q>, \quad Q_m |h,q> = \delta_{m,0} q |h,q>, \quad m \geq 0. \quad (3.9) \]

We parametrize the weights of a \( W \)-highest weight vector following [8], [5] as

\[ h = \frac{1}{3} (x^2 + xy + y^2 - 3a^2) \quad q = \frac{1}{27} (x - y) (2x + y) (x + 2y), \quad (3.10) \]

\[ x = p\alpha - q/\alpha \quad y = r\alpha - s/\alpha, \quad (3.11) \]

where we define \( a \) and \( \alpha \) by

\[ c = 2 - 24a^2 \quad \alpha = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + 1}. \quad (3.12) \]

It is believed [8, 11] that the doubly degenerate primary fields, i.e. those which have two independent null-vectors, satisfy \( p, q, r, s \in \mathbb{N} \). In particular the representation \( (1,1;1,1) \) is just the vacuum.
To derive restrictions on the possible fusion rules for representations of the $W_3$ algebra, we use the explicit form of the null-vectors of the highest weight vectors $\phi_{(p,r,q,s)}$ corresponding to $(p, r; q, s) = (2, 1; 1, 1), (1, 2; 1, 1), (1, 1; 2, 1), (1, 1; 1, 2)$, given by

$$
(2 h_{(p,r,q,s)} Q_{-1} - 3 q_{(p,r,q,s)} L_{-1}) \phi_{(p,r,q,s)}
$$

(3.13)

and

$$
((5 h_{(p,r,q,s)} + 1) h_{(p,r,q,s)} Q_{-2} - 12 q_{(p,r,q,s)} L_{-1}^2 + 6 q_{(p,r,q,s)} (h_{(p,r,q,s)} + 1) L_{-2}) \phi_{(p,r,q,s)}.
$$

(3.14)

Suppose that the representation $(p', r'; q', s')$ is contained in the tensor product of $(2, 1; 1, 1)$ and $(p, r; q, s)$ corresponding to $(\zeta, z) = (\zeta, 0)$. (As the tensor product is equivalent for all different choices of the parameters $(\zeta, z)$, we can restrict ourselves without loss of generality to this case.) Then the scalar product (in the following we omit the subscripts of the tensor product and the intertwiner $\pi$), cf. [10]

$$
\langle \phi'(p', r'; q', s'), (\phi_{(p,r,q,s)} \otimes \phi_{(2,1;1,1)}) \rangle
$$

(3.15)

does not vanish. As $\phi_{(p', r'; q', s')}$ is a highest weight vector, we have the two equations (we write $\phi' = \phi_{(p', r'; q', s')}$, $\phi = \phi_{(p,r,q,s)}$ and $\phi_0 = \phi_{(2,1;1,1)}$ and similarly for $h$ and $q$

$$
0 = \langle \phi', (2 h_0 \Delta (Q_{-1}) - 3 q_0 \Delta (L_{-1})) (\phi \otimes \phi_0) \rangle
$$

(3.16)

and

$$
0 = \langle \phi', ((5 h_0 + 1) h_0 \Delta (Q_{-2}) - 12 q_0 \Delta (L_{-1})^2 + 6 q_0 (h_0 + 1) \Delta (L_{-2})) (\phi \otimes \phi_0) \rangle.
$$

(3.17)

We can use these equations to obtain necessary restrictions for the fusion rules of the $W_3$-algebra, performing an analysis similar to [10] (see appendix B for details). We arrive at

$$
\phi_{(2,1;1,1)} \otimes \phi_{(p,q,r,s)} = \phi_{(p+1,q,r,s)} \oplus \phi_{(p,q-1,r,s)} \oplus \phi_{(p-1,q+1,r,s)}
$$

(3.18)

$$
\phi_{(1,2;1,1)} \otimes \phi_{(p,q,r,s)} = \phi_{(p,q+1,r,s)} \oplus \phi_{(p,q-1,r,s)} \oplus \phi_{(p+1,q-1,r,s)}
$$

(3.19)

$$
\phi_{(1,1;2,1)} \otimes \phi_{(p,q,r,s)} = \phi_{(p,q,r+1,s)} \oplus \phi_{(p,q,r,s-1)} \oplus \phi_{(p,q,r-1,s+1)}
$$

(3.20)

$$
\phi_{(1,1;1,2)} \otimes \phi_{(p,q,r,s)} = \phi_{(p,q,r,s+1)} \oplus \phi_{(p,q,r,s-1)} \oplus \phi_{(p,q,r+1,s-1)}
$$

(3.21)

1These conditions have also been independently derived in [1] and [3].
Assuming that the representation $\phi_{(1,2;1,1)}$ is indeed contained in the tensor product of $\phi_{(2,1;1,1)} \otimes \phi_{(1,1;1,1)}$ and similarly for $\phi_{(2,1;1,1)}$ etc. — we should be able to prove this using the methods of [2], [6] — we can already derive restrictions for the general fusion rules from these equations. Firstly we observe, that we might restrict ourselves to the set of representations, which are contained in tensor products of the form

$$\phi_{(2,1;1,1)}^p \otimes \phi_{(2,1;1,1)}^q \otimes \phi_{(1,1;1,2)}^r \otimes \phi_{(1,1;1,1)}^s;$$

(3.22)

where $p, q, r, s \in \mathbb{N}_0$. If $\phi_{(p_0,q_0;r_0,s_0)}$ is contained in this set of representations, then it is automatically contained in the tensor product (3.22) with $p_0 = p - 1, q_0 = q - 1$ etc. — this can be seen using an induction argument. Next we can show by induction on $p$, that for $p \geq 2$

$$\phi_{(2,1;1,1)} \otimes \phi_{(p,1;1,1)} = \phi_{(p+1,1;1,1)} \oplus \phi_{(p-2,1;1,1)}$$
$$\phi_{(1,2;1,1)} \otimes \phi_{(p,1;1,1)} = \phi_{(p,2;1,1)} \oplus \phi_{(p-1,1;1,1)}$$

(3.23)

and similarly for $(1, q; 1, 1)$, etc. This establishes that the set of all representations $\phi_{(p,q;r,s)}$ for $p, q, r, s \in \mathbb{N}$ is closed under the operation of taking tensor products. Furthermore, using the associativity and symmetry of the tensor product as in [10], we can derive restrictions for the general fusion rules, namely,

$$\phi_{(p_2,q_2;r_2,s_2)} \otimes \phi_{(p_1,q_1;r_1,s_1)} = \sum_{(p,q) \in \Lambda(p_1,q_1,p_2,q_2)} \sum_{(r,s) \in \Lambda(r_1,s_1,r_2,s_2)} \left[ \phi_{(p,q;r,s)} \right];$$

(3.24)

where $\Lambda(p_1,q_1,p_2,q_2)$ consists of all pairs $(p, q) \in \mathbb{N} \times \mathbb{N}$, which satisfy

$$(p, q) = (p_1, q_1) + \lambda_2 = (p_2, q_2) + \lambda_1$$

(3.25)

for some $\lambda_i \in \Lambda[p_i - 1, q_i - 1]$, where $\Lambda[l, m]$ denotes the weight lattice of the irreducible representation of $A_2$ corresponding to the weight $[lm]$ in the Dynkin-basis.

So far we have only analyzed the case, where $c$ is generic. For special values of $c$, corresponding to the minimal models, we have additional null-vectors, which give rise to a truncation of the fusion rules. We then obtain a finite set of representations closed under the operation of taking tensor products. We remark that the above restrictions have already been derived in [8] by means of a free field construction and in [9] via quantized Drinfeld-Sokolov reduction.
4 The $N = 1$ NS superconformal algebra

The $N = 1$ NS superconformal algebra is generated by the Virasoro algebra \( \{ L_n, G_\alpha \} \), \( n \in \mathbb{Z} \), and the modes of the superfield $G$ with $h = \frac{3}{2}$, \( \{ G_\alpha \}, \alpha \in \mathbb{Z} + \frac{1}{2} \), subject to the relations

\[
[L_n, G_\alpha] = \left( \frac{1}{2} n - \alpha \right) G_{n+\alpha} \quad (4.1)
\]

\[
\{ G_\alpha, G_\beta \} = 2 L_{\alpha+\beta} + \frac{1}{3} c \left( \alpha^2 - \frac{1}{4} \right) \delta_{\alpha,-\beta}. \quad (4.2)
\]

Evaluating (2.10, 2.11) we obtain for the Virasoro algebra (3.5, 3.6) and for the $G_\mu$’s

\[
\Delta_{\zeta,\bar{z}}(G_\alpha) = \sum_{\mu=-\frac{1}{2}}^{\alpha} \left( \frac{\alpha + \frac{1}{2}}{\mu + \frac{1}{2}} \right) \zeta^{\alpha-\mu} (G_\mu \otimes \mathbb{1}) + \varepsilon_1 \sum_{\lambda=-\frac{1}{2}}^{\alpha} \left( \frac{\alpha + \frac{1}{2}}{\lambda + \frac{1}{2}} \right) \varepsilon_{\lambda-\alpha} (\mathbb{1} \otimes G_\lambda) \quad (4.3)
\]

\[
\Delta_{\zeta,\bar{z}}(G_{-\alpha}) = \sum_{\mu=-\frac{1}{2}}^{\alpha} \left( \frac{\alpha + \mu - 1}{\alpha - \frac{3}{2}} \right) (-1)^{\mu+\frac{1}{2}} \zeta^{-\alpha+\mu} (G_\mu \otimes \mathbb{1})
\]

\[
+ \varepsilon_1 \sum_{\lambda=\alpha}^{\infty} \left( \frac{\lambda - \frac{3}{2}}{\alpha - \frac{3}{2}} \right) (-\bar{z})^{\lambda-\alpha} (\mathbb{1} \otimes G_{-\lambda}), \quad (4.4)
\]

where in (4.3) $\alpha \geq -\frac{1}{2}$ and in (4.4) $\alpha \geq \frac{3}{2}$.

We remark that $\varepsilon_{G_\alpha \psi} = -\varepsilon_\psi$ and $\varepsilon_{L_\alpha \psi} = \varepsilon_\psi$. It is then easy to check, using the power series expansions of appendix A, that the above formulae do indeed define a comultiplication.

Following [7] we consider the degenerate representations, parametrized by the eigenvalue of $L_0$

\[
h_{m,n} = h_0 + \left( \frac{1}{2} \alpha_+ m + \frac{1}{2} \alpha_- n \right)^2 , \quad (4.5)
\]

where $m, n \in \frac{1}{2} \mathbb{N}$, $m - n \in \mathbb{Z}$, and

\[
h_0 = \frac{1}{16} \left( \frac{2}{3} c - 1 \right), \quad \alpha_\pm = \frac{1}{2} \left[ \sqrt{1 - \frac{2}{3} c} \pm \sqrt{\frac{9}{2} - \frac{2}{3} c} \right]. \quad (4.6)
\]

In this parametrization the vacuum representation is just \((\frac{1}{2}, \frac{1}{2})\).

To derive restrictions for the fusion rules we shall use the explicit form of the first few singular vectors, namely

\[
\left( G_{-\frac{1}{4}} - \frac{2}{(2 h_{m,n} + 1)} G_{-\frac{1}{4}} L_{-1} \right) \phi_{m,n} \quad (4.7)
\]
where \((m, n) = \left(\frac{1}{2}, \frac{3}{2}\right)\) or \((m, n) = \left(\frac{3}{2}, \frac{1}{2}\right)\), and
\[
\left( L_{-1}^2 - \frac{4}{3} h_{1,1} L_{-2} - G_{-\frac{3}{2}} G_{-\frac{1}{2}} \right) \phi_{1,1}. \tag{4.8}
\]
Firstly, we analyze the tensor product of \(\phi_{n, \frac{3}{2}}\) with \(\phi_{m,n}\). Suppose the representation generated by the highest weight vector \(\phi'\) is contained in the tensor product and the scalar product
\[
\langle \phi', (\phi_{m,n} \otimes \phi_{n, \frac{3}{2}}) \rangle
\] does not vanish. As \(\phi'\) is a highest weight vector, we have (setting \(z = 0\) and writing \(\phi = \phi_{m,n}, \phi_0 = \phi_{n, \frac{3}{2}}\) and similarly for \(h\))
\[
0 = \left\langle \phi', \Delta (G_{\frac{1}{2}}) \left( \Delta (G_{-\frac{1}{2}}) - \frac{2}{(2h_0 + 1)} \Delta (G_{-\frac{3}{2}}) \Delta (L_{-1}) \right) (\phi \otimes \phi_0) \right\rangle
\]
\[
= \langle \phi', (L_{-1} \otimes \text{Id}) (\phi \otimes \phi_0) \rangle + 2 h \zeta^{-1} \langle \phi', (\phi \otimes \phi_0) \rangle
\]
\[
- \frac{2}{(2h_0 + 1)} \left[ \langle \phi', \Delta (G_{-\frac{1}{2}}) \Delta (L_{-1}) (G_{-\frac{3}{2}} \otimes \text{Id}) (\phi \otimes \phi_0) \rangle \right] + \epsilon_1 \zeta \langle \phi', (G_{-\frac{1}{2}} L_{-1} \otimes G_{-\frac{3}{2}}) (\phi \otimes \phi_0) \rangle + \epsilon_1 \langle \phi', (G_{\frac{1}{2}} L_{-1} \otimes G_{-\frac{1}{2}}) (\phi \otimes \phi_0) \rangle
\]
\[
+ \langle \phi', (L_{-1} \otimes G_{\frac{1}{2}} G_{-\frac{3}{2}}) (\phi \otimes \phi_0) \rangle, \tag{4.9}
\]
where we have used (4.7). Using the identities
\[
\langle \phi', (G_{-\frac{1}{2}} L_{-1} \otimes G_{-\frac{3}{2}}) (\phi \otimes \phi_0) \rangle = \langle \phi', \Delta (G_{-\frac{1}{2}}) (L_{-1} \otimes G_{-\frac{3}{2}}) (\phi \otimes \phi_0) \rangle
\]
\[
- \epsilon_1 \langle \phi', (L_{-1} \otimes G_{\frac{1}{2}}) (\phi \otimes \phi_0) \rangle
\]
\[
= - \epsilon_1 \langle \phi', (L_{-1} L_{-1}) (\phi \otimes \phi_0) \rangle \tag{4.10}
\]
and
\[
\epsilon_1 \langle \phi', (G_{\frac{1}{2}} L_{-1} \otimes G_{-\frac{3}{2}}) (\phi \otimes \phi_0) \rangle = \epsilon_1 \langle \phi', (G_{\frac{1}{2}} L_{-1} \otimes G_{-\frac{1}{2}}) (\phi \otimes \phi_0) \rangle
\]
\[
= - \langle \phi', \Delta (G_{-\frac{3}{2}}) (G_{-\frac{3}{2}} \otimes \text{Id}) (\phi \otimes \phi_0) \rangle
\]
\[
+ \langle \phi', (L_{-1} \otimes \text{Id}) (\phi \otimes \phi_0) \rangle \tag{4.11}
\]
together with (B.1, B.2) we obtain
\[
0 = \zeta^{-1} \left[ \kappa + 2 h - \frac{2}{(2h_0 + 1)} (\kappa (\kappa - 1) + \kappa (2 h_0 + 1)) \right] \langle \phi', (\phi \otimes \phi_0) \rangle
\]
\[
= \zeta^{-1} \left[ -\kappa + 2 h - \frac{2}{(2h_0 + 1)} \kappa (\kappa - 1) \right] \langle \phi', (\phi \otimes \phi_0) \rangle, \tag{4.12}
\]
13
where $\kappa = h' - h - h_0$. Again this formula implies $[.] = 0$, which is a quadratic equation in $h'$. For each of the two fields $\phi_{\frac{3}{2}, \frac{3}{2}}$ and $\phi_{\frac{1}{2}, \frac{1}{2}}$ we have thus two solutions, giving the fusion rules \[7\]

\[
\phi_{m,n} \otimes \phi_{\frac{1}{2}, \frac{1}{2}} = \phi_{m,n+1} \oplus \phi_{m,n-1}
\] (4.14)

and

\[
\phi_{m,n} \otimes \phi_{\frac{3}{2}, \frac{3}{2}} = \phi_{m+1,n} \oplus \phi_{m-1,n}.
\] (4.15)

Similarly, we can use

\[
0 = \left\langle \phi', \left( \Delta (L_{-1})^2 - \frac{4}{3} h_0 \Delta (L_{-2}) - \Delta (G_{-\frac{3}{2}}) \Delta (G_{-\frac{1}{2}}) \right) (\phi \otimes \phi_0) \right\rangle
\] (4.16)

to derive restrictions for the possible fusions involving the field $\phi_{1,1}$. With the help of the identity

\[
\left\langle \phi', \left( G_{-\frac{1}{2}} \otimes G_{-\frac{3}{2}} \right) (\phi \otimes \phi_0) \right\rangle = -\varepsilon_1 \left\langle \phi', \Delta (G_{-\frac{3}{2}}) \left( G_{-\frac{3}{2}} \otimes 1 \right) (\phi \otimes \phi_0) \right\rangle
\]

\[
+ \varepsilon_1 \zeta^{-1} \left\langle \phi', (L_{-1} \otimes 1) (\phi \otimes \phi_0) \right\rangle
\]

\[
- \varepsilon_1 \zeta^{-2} \left\langle \phi', (G_{\frac{3}{2}} G_{-\frac{3}{2}} \otimes 1) (\phi \otimes \phi_0) \right\rangle
\]

\[
= \zeta^{-2} \varepsilon_1 (\kappa - 2h) \left\langle \phi', (\phi \otimes \phi_0) \right\rangle
\] (4.17)

we obtain

\[
0 = \zeta^{-2} \left[ -\kappa^2 - \frac{4}{3} h_0 (\kappa - h) \right] \left\langle \phi', (\phi \otimes \phi_0) \right\rangle,
\] (4.18)

which implies \[7\]

\[
\phi_{m,n} \otimes \phi_{1,1} = \phi_{m-\frac{1}{2},n-\frac{1}{2}} \oplus \phi_{m+\frac{1}{2},n+\frac{1}{2}}.
\] (4.19)

Restricting ourselves to representations, which are contained in

\[
\phi_{\frac{3}{2}, \frac{3}{2}}^p \otimes \phi_{\frac{3}{2}, \frac{3}{2}}^q \otimes \phi_{1,1}^\eta \otimes \phi_{\frac{3}{2}, \frac{3}{2}}^s,
\] (4.20)

where $p, q \in \mathbb{N}_0$ and $\eta = 0, 1$, we can again show, that if a representation is contained in this set then it is contained in the tensor product \[4.20\] with the minimal values for $p$ and $q$. As in \[10\] we can thus derive restrictions for the fusion rules in the general case, using the associativity and symmetry of the tensor product. Depending on whether $(\alpha, \beta)$ and $(\alpha', \beta')$
are half-integers or integers, we obtain

\[ \varphi_{\alpha,\beta} \otimes \varphi_{\alpha',\beta'} = \begin{cases} 
\sum_{\gamma=|\alpha-\alpha'|+\frac{1}{2}} \sum_{\delta=|\beta-\beta'|+\frac{1}{2}} \varphi_{\gamma,\delta} & \text{if } \alpha, \beta, \alpha', \beta' \in \mathbb{Z} + \frac{1}{2} \\
\sum_{\gamma=|\alpha-\alpha'|+1} \sum_{\delta=|\beta-\beta'|+1} \varphi_{\gamma,\delta} & \text{if } \alpha, \beta \in \mathbb{Z}, \alpha', \beta' \in \mathbb{Z} + \frac{1}{2} \\
\sum_{\gamma=|\alpha-\alpha'|+\frac{1}{2}} \sum_{\delta=|\beta-\beta'|+\frac{1}{2}} \varphi_{\gamma,\delta} & \text{if } \alpha, \beta, \alpha', \beta' \in \mathbb{Z} 
\end{cases} \quad (4.21) \]

where in the first two lines of (4.21) \( \gamma \) and \( \delta \) attain only every other value and in the third line of (4.21) the summands corresponding to \( \gamma = |\alpha-\alpha'|+\frac{1}{2}, \delta = |\beta-\beta'|+\frac{1}{2} \) are excluded. The above are exactly the even fusion rules of [15].

In the \( W_3 \)-case we could express the scalar products of the form \( \langle \varphi', (Q_p \otimes \mathbb{1}) (\varphi \otimes \varphi_0) \rangle \) with \( p = 1, 2 \) in terms of \( \langle \varphi', (\varphi \otimes \varphi_0) \rangle \). In the present case, however, we cannot obtain such a relation, i.e., we cannot express

\[ \langle \varphi', (\mathbb{1} \otimes G_{-\frac{1}{2}}) (\varphi_{m,n} \otimes \varphi_{\frac{1}{2},\frac{1}{2}}) \rangle = -\varepsilon_1 \langle \varphi', (\mathbb{1} \otimes G_{-\frac{1}{2}}) (\varphi_{m,n} \otimes \varphi_{\frac{1}{2},\frac{1}{2}}) \rangle \quad (4.22) \]

in terms of \( \langle \varphi', (\varphi_{m,n} \otimes \varphi_{1,\frac{1}{2}}) \rangle \). Thus there exist highest weight vectors \( \varphi' \) such that (4.9) is zero, but not (4.22). In this case the scalar product of the three highest weight vectors vanishes, but the representation generated from \( \varphi' \) is nevertheless contained in the tensor product.

We remark that the \((\zeta, z)\)-dependence of the scalar product (4.22) is given by

\[ (\zeta - z)^{h' - h_0 + \frac{1}{2}}, \quad (4.23) \]

which follows from the same reasoning as in [10]. As the scalar product corresponds essentially to the three-point-function, we expect that this case corresponds to the odd fusion rules of [15]. Indeed, carrying through the analogous analysis, we can check that we obtain in this way the same restrictions as for the odd fusion rules [15].

\footnote{I thank G. Watts for pointing this out to me.}
5 The $N = 2$ NS superconformal algebra

The $N = 2$ NS superconformal algebra is generated by the Virasoro algebra \((3.1)\), the modes of an $U(1)$-current with $h = 1 \{T_n\}, n \in \mathbb{Z}$ and the modes of two superfields of conformal dimension $h = \frac{3}{2}, \{G_\alpha^\pm\}, \alpha \in \mathbb{Z} + \frac{1}{2}$, subject to the relations \([4]\)

\[
\begin{align*}
[L_m, G_\alpha^\pm] &= \left( \frac{1}{2} m - \alpha \right) G_{\alpha+m}^\pm \\
[L_m, T_n] &= -n T_{m+n} \\
[T_m, T_n] &= \tilde{c} m \delta_{m,-n} \\
[T_m, G_\alpha^+] &= \pm G_{\alpha+m}^+ \\
\{G_\alpha^+, G_\beta^-\} &= 0 \\
\{G_\alpha^+, G_\beta^-\} &= 2 L_{\alpha+\beta} + (\alpha - \beta) T_{\alpha+\beta} + \tilde{c} \left( \alpha^2 - \frac{1}{4} \right) \delta_{\alpha,-\beta},
\end{align*}
\]  

(5.1)

where $\tilde{c} = c/3$. The comultiplication is given by \((3.5, 3.6)\) for the Virasoro algebra, \((4.3, 4.4)\) for the superfield modes and by

\[
\Delta_{\zeta,z}(T_n) = \sum_{m=0}^{n} \binom{n}{m} \zeta^{n-m} (T_m \otimes \mathbb{I}) + \sum_{l=0}^{n} \binom{n}{l} z^{n-l} (\mathbb{I} \otimes T_l)
\]  

(5.2)

\[
\Delta_{\zeta,z}(T_{-n}) = \sum_{m=0}^{\infty} \binom{n + m - 1}{n - 1} (-1)^m \zeta^{-(n+m)} (T_m \otimes \mathbb{I})
\]

\[+ \sum_{l=n}^{\infty} \binom{l - 1}{n - 1} (-z)^{l-n} (\mathbb{I} \otimes T_{-l}),
\]  

(5.3)

where in \((5.2)\) $n \geq 0$ and in \((5.3)\) $n \geq 1$.

We restrict ourselves for simplicity to the doubly-degenerate representations which have two fermionic null-vectors. These representations contain in particular all unitary representations with $c < 1$ \([4]\). We can parametrize these representations by two half-odd integers $j, k \in \mathbb{Z} + \frac{1}{2}, j, k > 0$, where the representation $(j, k)$ has null-vectors at $(\Delta h, \Delta q) = (j, 1)$ and $(\Delta h, \Delta q) = (k, -1)$ — $\Delta h$ denotes the relative $L_0$—eigenvalue and $\Delta q$ the relative $T_0$—eigenvalue. The conformal weight and charge of the corresponding highest weight vector is then given by

\[
q = \frac{1}{2} (\tilde{c} - 1) (j - k) \quad h = \frac{1}{2} (\tilde{c} - 1) \left( \frac{1}{4} - j k \right).
\]  

(5.4)
We remark that the representation \((\frac{1}{2}, \frac{1}{2})\) is just the vacuum representation. Furthermore, we shall use the explicit form of the two null-vectors of the representations \((\frac{1}{2}, \frac{3}{2})\)

\[
G^+_{\frac{1}{2}} \phi_{\frac{1}{2}, \frac{3}{2}}
\]  

(5.5)

and

\[
\left( q_{\frac{1}{2}, \frac{3}{2}} + \tilde{c} - 1 \right) G^-_{\frac{1}{2}} + L_{-1} G^-_{\frac{1}{2}} + T_{-1} G^-_{\frac{1}{2}} \right) \phi_{\frac{1}{2}, \frac{3}{2}}
\]  

(5.6)

and of the representation \((\frac{3}{2}, \frac{1}{2})\)

\[
G^-_{\frac{1}{2}} \phi_{\frac{3}{2}, \frac{1}{2}}
\]  

(5.7)

and

\[
\left( \tilde{c} - 1 - q_{\frac{3}{2}, \frac{1}{2}} \right) G^+_{\frac{1}{2}} + L_{-1} G^+_{\frac{1}{2}} - T_{-1} G^+_{\frac{1}{2}} \right) \phi_{\frac{3}{2}, \frac{1}{2}}
\]  

(5.8)

To obtain restrictions for the even fusion rules let us first analyze the tensor product of \(\phi_{j,k}\) with \(\phi_{\frac{1}{2}, \frac{3}{2}}\). Suppose the irreducible representation generated by the highest weight vector \(\phi'\) is contained in the tensor product, then the scalar product

\[
\left\langle \phi', \left( \phi_{j,k} \otimes \phi_{\frac{1}{2}, \frac{3}{2}} \right) \right\rangle
\]

does not vanish. On the other hand, as \(\phi'\) is a highest weight vector, we have (setting \(z = 0\) and writing \(\phi = \phi_{j,k}, \phi_0 = \phi_{\frac{1}{2}, \frac{3}{2}}\) and similarly for \(h\) and \(q\))

\[
(2h' - q') \left\langle \phi', (\phi \otimes \phi_0) \right\rangle = \left\langle \phi', \Delta (G^+_{\frac{1}{2}}) \Delta (G^-_{\frac{3}{2}}) (\phi \otimes \phi_0) \right\rangle,
\]

(5.9)

from which we can conclude that

\[
\left\langle \phi', \left( G^-_{\frac{1}{2}, \frac{1}{2}} G^+_{\frac{1}{2}, \frac{3}{2}} \otimes I \right) (\phi \otimes \phi_0) \right\rangle = \zeta^{-1} (2(h' - h) + q - q') \left\langle \phi', (\phi \otimes \phi_0) \right\rangle.
\]

(5.10)

Similarly, we can use the other null-vector equation to obtain

\[
0 = \left\langle \phi', \Delta (G^+_{\frac{1}{2}}) \left[ \mu \Delta (G^-_{\frac{1}{2}}) + \Delta (L_{-1}) \Delta (G^-_{\frac{1}{2}}) + \Delta (T_{-1}) \Delta (G^-_{\frac{1}{2}}) \right] (\phi \otimes \phi_0) \right\rangle
\]

\[
= \left\langle \phi', \left[ \Delta (G^+_{\frac{1}{2}}) \Delta (L_{-1}) + \Delta (G^+_{\frac{1}{2}}) \Delta (T_{-1}) \right] \left( G^-_{\frac{1}{2}} \otimes I \right) (\phi \otimes \phi_0) \right\rangle
\]

\[
+ \varepsilon_1 \left\langle \phi', \Delta (G^+_{\frac{1}{2}}) \left[ L_{-1} \otimes G^-_{\frac{1}{2}} + q \zeta^{-1} (I \otimes G^-_{\frac{1}{2}}) \right] (\phi \otimes \phi_0) \right\rangle
\]

\[
+ \mu \zeta^{-1} \left\langle \phi', \Delta (G^+_{\frac{1}{2}}) \left( G^-_{\frac{1}{2}} \otimes I \right) (\phi \otimes \phi_0) \right\rangle
\]

\[
= \varepsilon_1 (\kappa + q) \left\langle \phi', \left( G^+_{\frac{1}{2}} G^-_{\frac{1}{2}} \right) (\phi \otimes \phi_0) \right\rangle + \mu \left\langle \phi', \left( G^+_{\frac{1}{2}} G^-_{\frac{1}{2}} \otimes I \right) (\phi \otimes \phi_0) \right\rangle
\]

\[
+ (\mu (2h + q) + q (2h_0 + q_0) + \kappa (2h_0 + q_0)) \zeta^{-1} \left\langle \phi', (\phi \otimes \phi_0) \right\rangle.
\]

(5.12)
where $\mu = q_0 + \tilde{c} - 1$ and $\kappa = h' - h - h_0$. Using the identity

$$0 = \left\langle \phi', \Delta \left( G_{-\frac{1}{2}}^{-} \right) \left( G_{-\frac{1}{2}}^{+} \otimes \mathbb{1} \right) (\phi \otimes \phi_0) \right\rangle$$

$$= \left\langle \phi', \left( G_{-\frac{1}{2}}^{-} G_{-\frac{1}{2}}^{+} \otimes \mathbb{1} \right) (\phi \otimes \phi_0) \right\rangle - \varepsilon_1 \left\langle \phi', \left( G_{-\frac{1}{2}}^{+} G_{-\frac{1}{2}}^{-} \right) (\phi \otimes \phi_0) \right\rangle$$

(5.13)

and (5.11) we get

$$0 = \zeta^{-1} \left[ (\kappa + q) (2 (h' + h_0 - h) + q_0 + q - q') + \mu (2 (\kappa - h' + 2h) + q') \right] \left\langle \phi', (\phi \otimes \phi_0) \right\rangle .$$

(5.14)

Thus a necessary condition for $\phi'$ to be contained in the tensor product of $\phi$ and $\phi_0$ is that $[.] = 0$. As we know in addition — from insertion of $\Delta (T_0)$ — that $q' = q + q_0$, we obtain a quadratic equation for $h'$, which has the two solutions

$$(j, k) \otimes \left( \frac{1}{2}, \frac{3}{2} \right) = (j - 1, k) \oplus (j, k + 1) .$$

(5.15)

Similarly, we can use the knowledge of the null-vectors of $\left( \frac{3}{2}, \frac{1}{2} \right)$ to obtain

$$(j, k) \otimes \left( \frac{3}{2}, \frac{1}{2} \right) = (j + 1, k) \oplus (j, k - 1) .$$

(5.16)

As in the previous section we can use the associativity and symmetry of the tensor product to deduce from these conditions restrictions for the general fusion rules. We obtain

$$(j_1, k_1) \otimes (j_2, k_2) = \sum_{j=\max(j_2 - k_1, j_1 - k_2) + \frac{1}{2}}^{j_1 + j_2 - \frac{1}{2}} \left[ (j, j - j_1 - j_2 + k_1 + k_2) \right] .$$

(5.17)

Again these restrictions reproduce the even fusion rules of [13], [14] and were first derived in [12]. As in the previous section we find additional fusion rules — the two different odd fusion rules of [13], [14] — if we require that instead of (5.9)

$$\left\langle \phi', \left( G_{-\frac{1}{2}}^{\alpha} \otimes \mathbb{1} \right) \left( \phi_{j,k} \otimes \phi_{1, \frac{3}{2}} \right) \right\rangle = -\varepsilon_1 \left\langle \phi', \left( \mathbb{1} \otimes G_{-\frac{1}{2}}^{\alpha} \right) \left( \phi_{j,k} \otimes \phi_{1, \frac{3}{2}} \right) \right\rangle$$

(5.18)

does not vanish, where either $\alpha = +$ or $\alpha = -$. 

18
6 Conclusions

We have shown in this paper that fusion in conformal field theory can be understood as a certain ring-like tensor product of representations of the chiral algebra. We have proved that this tensor product is associative and symmetric up to equivalence. We have also derived explicit formulae for the action of the chiral algebra, under which the central extension is preserved.

Having given a precise meaning to fusion, deriving the fusion rules is now a purely algebraic problem, namely to decompose the tensor product into irreducible representations. We have demonstrated how this can be done for the case of the $W_3$-algebra and the $N = 1$ and $N = 2$ superconformal algebras, thereby recovering the known restrictions for the fusion rules. Essentially the same analysis can be carried through for any chiral algebra, e. g. the $N \geq 3$ NS superconformal algebras or other $W$-algebras. We have therefore established a unifying framework within which all chiral algebras can be treated similarly and the calculation of the fusion rules is straightforward.

A Comultiplication property

To prove the comultiplication property for the three algebras we use the same methods as in [10], i. e. we use the Cauchy product formula to obtain a power series expansion for the product of two power series and compare these coefficients with the coefficients of the power series of the product. In the following we have collected the relevant power series expansions.

\[ \sum_{m=1-h}^{n} \left( \frac{n + h - 1}{m + h - 1} \right) \zeta^{n-m} = (\zeta + 1)^{n+h-1} \] (A.1)

\[ \sum_{m=1-h}^{n} \left( \frac{n + h - 1}{m + h - 1} \right) \zeta^{n-m} m = (1-h)(\zeta + 1)^{n+h-1} + \frac{d}{d\zeta} (\zeta + \varepsilon)^{n+h-1} \bigg|_{\varepsilon=1} = (\zeta + 1)^{n+h-2} ((1-h)\zeta + n) \] (A.2)

\[ \sum_{m=1-h}^{n} \left( \frac{n + h - 1}{m + h - 1} \right) \zeta^{n-m} m^2 = (\zeta + 1)^{n+h-3} \left((1-2h+h^2)\zeta^2 + (h-1+3n-2hn)\zeta + n^2 \right) \] (A.3)
\[
\sum_{m=1-h}^{n} \binom{n+h-1}{m+h-1} \zeta^{n-m} m^3 = (\zeta + 1)^{n+h-4} \left( (1 - 3h + 3h^2 - h^3) \zeta^3 \\
+ (-4 + 7h - 3h^2 + 7n - 9hn + 3h^2n) \zeta^2 \\
+ (1 - h - 4n + 3hn + 6h^2 - 3hn^2) \zeta + n^3 \right) 
\] (A.4)

\[
\sum_{m=1-h}^{\infty} \binom{n+m-1}{n-h} (-1)^{m+h-1} \zeta^{-(n+m)} = \frac{(-1)^{n-h+1}}{(n-h)!} \frac{d^n-h}{d\zeta^{n-h}} \sum_{l=1}^{\infty} (-\zeta)^{-l} \\
= (\zeta + 1)^{-(n-h+1)} 
\] (A.5)

\[
\sum_{m=1-h}^{\infty} \binom{n+m-1}{n-h} (-1)^{m+h-1} \zeta^{-(n+m)} m = \left( -\zeta \frac{d}{d\zeta} - n \right) (\zeta + 1)^{-(n-h+1)} \\
= (\zeta + 1)^{-(n-h+2)} ((1-h)\zeta - n) 
\] (A.6)

and

\[
\sum_{m=n}^{\infty} \binom{m-h}{n-h} (-\zeta)^{m-n} = \frac{(-1)^{n-h}}{(n-h)!} \frac{d^n-h}{d\zeta^{n-h}} \sum_{l=0}^{\infty} (-\zeta)^l \\
= (\zeta + 1)^{-(n-h+1)} 
\] (A.7)

\[
\sum_{m=n}^{\infty} \binom{m-h}{n-h} (-\zeta)^{m-n} m = \left( \zeta \frac{d}{d\zeta} + n \right) (\zeta + 1)^{-(n-h+1)} \\
= (\zeta + 1)^{-(n-h+2)} ((h-1)\zeta + n) . 
\] (A.8)

**B W3-calculation**

We rewrite the equations (3.16) and (3.17) using (4.9, 4.10) of [10], namely

\[
h' \langle \phi', (\phi \otimes \phi_0) \rangle = \zeta \langle \phi', (L_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle + (h + h_0) \langle \phi', (\phi \otimes \phi_0) \rangle 
\] (B.1)

and

\[
\kappa(k - 1) \langle \phi', (\phi \otimes \phi_0) \rangle = \zeta^2 \langle \phi', (L_{-1}^2 \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle . 
\] (B.2)
where \( \kappa = h' - h - h_0 \), the formulae (3.5 - 3.7) and the two null-vector equations to obtain

\[
0 = 2 h_0 \zeta \langle \phi', (Q_{-2} \otimes \mathbb{1}) (\phi' \otimes \phi_0) \rangle + 2 h_0 \langle \phi', (Q_{-1} \otimes \mathbb{1}) (\phi' \otimes \phi_0) \rangle - 3 q_0 \kappa \zeta^{-1} \langle \phi', (\phi \otimes \phi_0) \rangle
\]

and

\[
0 = (5h_0 + 1) h_0 \langle \phi', (Q_{-2} \otimes \mathbb{1}) (\phi' \otimes \phi_0) \rangle + (12 q_0 \kappa (\kappa - 1) + 6 q_0 (h_0 + 1) (\kappa - h)) \zeta^{-2} \langle \phi', (\phi \otimes \phi_0) \rangle .
\]

Furthermore, we can use

\[
q' \langle \phi', (\phi \otimes \phi_0) \rangle = \langle \phi', \Delta (Q_0) (\phi \otimes \phi_0) \rangle
\]

\[
= \zeta^2 \langle \phi', (Q_{-2} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle + 2 \zeta \langle \phi', (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle + (q + q_0) \langle \phi', (\phi \otimes \phi_0) \rangle .
\]

Eliminating from these last three equations the terms containing \( Q_{-1} \) and \( Q_{-2} \) we obtain

\[
0 = [q_0 (12 \kappa (\kappa - 1) + 6 (\kappa - h) (h_0 + 1) + 3 \kappa (5h_0 + 1) + h_0 (5h_0 + 1))]
\]

and thus get the condition, that the bracket [.] must vanish. However, as we have two unknowns, namely \( h' \) and \( q' \), we have to find another condition. To this end we observe that

\[
q' \langle \phi', (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle = \langle \phi', \Delta (Q_0) (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]

\[
= (q + q_0) \langle \phi', (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]

\[
+ \zeta^2 \langle \phi', (Q_{-2} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]

\[
+ 2 \zeta \langle \phi', (Q_{-1} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]

\[
+ \left( \frac{2}{3} h + \frac{2}{15} - \frac{1}{240} (22 + 5c) \right) \langle \phi', (L_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle .
\]

Furthermore, we have

\[
0 = \langle \phi', (2 h_0 \Delta (Q_{-1}) - 3 q_0 \Delta (L_{-1})) (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]

\[
= 2 h_0 \zeta \langle \phi', (Q_{-2} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle + 2 h_0 \langle \phi', (Q_{-1} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]

\[
- 3 q_0 \langle \phi', (L_{-1} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle
\]
and

\[ 0 = \langle \phi', (5h_0 + 1) h_0 \Delta (Q_{-2}) - 12q_0 \Delta (L_{-1})^2 \\
+ 6q_0 (h_0 + 1) \Delta (L_{-2})) (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle \\
= (5h_0 + 1) h_0 \langle \phi', (Q_{-2} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle + 12q_0 \langle \phi', (L_{-1}^2 Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle \\
+ 6q_0 (h_0 + 1) \zeta^{-1} \langle \phi', (L_{-1} Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle + 18q_0 q (h_0 + 1) \zeta^{-3} \langle \phi', (\phi \otimes \phi_0) \rangle \\
- 6q_0 (h_0 + 1) (h + 1) \zeta^{-2} \langle \phi', (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle . \quad (B.9) \]

Eliminating the terms containing products of \(Q\)-generators from (B.7 - B.9) we obtain the relation

\[ [(5h_0 + 1) (h_0 (q + q_0 - q') + 3q_0(\kappa - 1)) + 6q_0 (h_0 + 1) (\kappa - h - 2) + 12q_0 (\kappa - 1) (\kappa - 2)] \]

\[ \langle \phi', (Q_{-1} \otimes \mathbb{1}) (\phi \otimes \phi_0) \rangle \]

\[ = - [(5h_0 + 1) h_0 \kappa \left( \frac{2}{3}h + \frac{2}{15} - \frac{22 + 5c}{240} \right) + 18q_0 (h_0 + 1) \zeta^{-1} \langle \phi', (\phi \otimes \phi_0) \rangle] , \quad (B.10) \]

which together with (B.7) and (B.3) gives us the condition

\[ 72 q_0^2 \kappa^2 + \kappa \left[ 9q_0^2 (9h_0 - 3) + 48q_0 h_0 (q + q_0 - q') + 2h_0^2 (5h_0 + 1) \left( \frac{2}{3}h + \frac{2}{15} - \frac{22 + 5c}{240} \right) \right] \]

\[ + 6q_0 h_0 (q + q_0 - q') (9h_0 - 3) + 36q_0 q_0 (h_0 + 1) = 0. \quad (B.11) \]

We can eliminate \(q'\) from (B.6) and (B.11), thereby obtaining a cubic equation in \(h'\). For each of the three possible solutions of \(h'\) there is precisely one solutions for \(q'\).

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