Non-Markovianity assisted Steady State Entanglement

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We analyze the steady state entanglement generated in a coherently coupled dimer system subject to dephasing noise as a function of the degree of Markovianity of the evolution. By keeping fixed the noise strength while varying the underlying dynamics, we demonstrate that non-Markovianity is an essential, quantifiable resource that may support the formation of steady state entanglement whereas purely Markovian dynamics governed by Lindblad master equations lead to separable steady states. This result illustrates possible mechanisms leading to long lived entanglement in purely decohering, possibly local, environments. We present a feasible experimental demonstration of this noise assisted phenomenon using a system of trapped ions.

The generation and ultimate persistence of quantum entanglement is normally thought to be correlated with a high degree of system isolation, while the presence of a surrounding environment tends to decohere the quantum system thus driving it towards a classically correlated state. However, driven systems, generally out of equilibrium, have been shown to tend towards steady states where quantum correlations are nonvanishing [1]. A variety of entanglement preserving mechanisms have been put forward, showing how the presence of environmental noise can be instrumental in keeping the system entangled in the steady state [1–7]. In the framework of a Markovian dynamics, the presence of local pure dephasing is normally detrimental for entanglement preservation [8]. The situation though is very different when the dephasing noise is non-Markovian. We show that the presence of steady state entanglement can be unambiguously linked to an increasing degree of non-Markovianity, even when the environments are acting locally. This result is of particular interest in the light of identifying mechanisms that assist entanglement preservation in condensed matter and biomolecular systems, where non-Markovian dephasing is a dominant noise source [9–13].

The system — We consider a dimer system coherently coupled via an exchange interaction of strength \( J \), so that the system Hamiltonian reads \( H_s = \sum_{j=1}^{2} \omega_j \sigma_j^z + J (\sigma_1^- \sigma_2^+ + \sigma_1^+ \sigma_2^-) \ (h = 1) \), and subject to the action of an environment that leaves the populations unaffected but tends to randomize the phases of superposition states (the so-called pure dephasing or transverse decoherence). We will model this situation by subjecting each element \( j = 1, 2 \) of the dimer to the action of a localized mode, being the system-mode interaction governed by a Hamiltonian of the form \( H_{s-m} = g_j \sigma_j^z (a_j + a_j^+) \), where \( a_j (a_j^+) \) denote the the operators of annihilation (creation) of mode excitations. We will assume that the local modes are damped by a conventional Markovian bath so that the global time evolution of the dimer and the vibrational modes is described by a Markovian master equation with a Liouvillian part accounting for the damping of the localized modes at a certain rate \( \kappa \). Note, however, that tracing out the local mode leads to a density matrix for the dimer system that is in general no reproducible from a purely Markovian evolution for the dimer alone. We will show that the interplay between the entangling exchange interaction and the local dephasing will result in an entanglement dynamics that depends crucially on whether of not the action of the environment can indeed be described by merely subjecting the dimer to Markovian dephasing. In the limit where the coherent coupling \( g \) is smaller than the decay \( \kappa \) we find that the effective dephasing rate \( \gamma_{\text{eff}} \), which quantifies the strength of the system-environment coupling, is proportional to the ratio \( g^2 / \kappa \) (see [14] for the case of cavity QED). When \( 2g \ll \kappa \) the dephasing is exponential and the dynamics of the system can be reproduced by a Lindblad master equation with the effective decay rate \( \gamma_{\text{eff}} \). For \( 2g < \kappa \), a regime which we denote as weakly non-Markovian, the definition of an effective dephasing rate \( \gamma_{\text{eff}} \) is still reason-
able for times exceeding $\kappa^{-1}$ when the decay is well approximated by an exponential. In the limit where $g > \kappa$ significant coherent oscillations can occur and the definition of an effective dephasing rate becomes meaningless. In this work we will always be operating in the regime where $2g < \kappa$ and will analyze the different dynamics that the dimer will undergo when subject to the same effective noise strength but experiences a different underlying dynamics. This will allow us to single out the precise influence of environmental time-correlations on the persistence/absence of stationary quantum correlations in the dimer system.

A convenient parametrization— In order to link the observed dynamics to a parameter that quantifies in some form how much the environmental action departs from strict Markovianity, we will introduce an index $f$ such that the coherent spin-mode coupling $g$ and the mode damping rate $\kappa$ are given by, respectively: $g = \sqrt{f}g_0$ and $\kappa = f\kappa_0$. In this way, the noise strength $g^2/\kappa = g_0^2/\kappa_0$ is kept fixed while varying $f$ from values much larger than 1 to much smaller than 1 leads to, respectively, Markovian and non-Markovian dynamics with, crucially, the same effective local dephasing rate $\gamma_{\text{eff}}$. In other words, the strength of the dephasing noise is kept fixed, but the underlying noisy dynamics is modified, making the coherent system-oscillator coupling to dominate over losses (small $f$ domain) or viceversa (limit of large $f$). Note, however, that the decoherence rate is kept constant as the parameter $f$ is varied, the population decay (which depends on $g$ and $\kappa$) does change, as shown in Fig. 1. The monotonous change in Markovianity of the noise with the parameter $f$ is shown in Fig. 2, where an explicit measure of non-Markovianity is considered.

**Analytical Results.**— The effective Hamiltonian for the dimer-local damped modes system takes the form

$$ H = \omega_1 \sigma_1^+ \sigma_1^- + \omega_2 \sigma_2^+ \sigma_2^- + J(\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) + (\Omega_1 - i\kappa_1)a_1^\dagger a_1 + (\Omega_2 - i\kappa_2)a_2^\dagger a_2 + g_1 \sigma_1^{(z)} (a_1 + a_1^\dagger) + g_2 \sigma_2^{(z)} (a_2 + a_2^\dagger), $$

where $\omega_i$ and $\Omega_i$ ($i = 1, 2$) denote the site and the mode frequencies, respectively, and

$$ \dot{\rho} = -iH\rho + i\rho^H + 2\kappa_1 a_1 \rho a_1^\dagger + 2\kappa_2 a_2 \rho a_2^\dagger. $$

Motivated by the situations frequently encountered in biomolecular complexes where external illumination is either weak and/or doubly excited states are strongly suppressed [13], we will focus here on situations where the dimer dynamics is confined to the one excitation sector so that the entire dynamics for the two sites takes place in the subspace spanned by the states $|01\rangle, |10\rangle$. By introducing the convenient delocalized states $|u\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|d\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and the operators

$$ \sigma_{p,m} = \frac{1}{\sqrt{2}}(\sigma_1^{(z)} + \sigma_2^{(z)}), $$

noting that $\sigma_p|u\rangle = \sigma_p|d\rangle = 0$ while $\sigma_m|u\rangle = \sqrt{2}|d\rangle$ and $\sigma_m|d\rangle = \sqrt{2}|u\rangle$, in the subspace spanned by $|u\rangle$ and $|d\rangle$, the Hamiltonian part accounting for the system-mode interaction is given by

$$ H_{s-m} = \frac{g_1}{2} \sigma_m(a_p + a_p^\dagger) + \frac{g_2}{2} \sigma_m(a_m + a_m^\dagger), $$

where $g_\pm = g_1 \pm g_2$. Note that the restriction to the one excitation sectors removes the contribution from terms involving $\sigma_p$, as this operator does not couple to the states $|u\rangle$ and $|d\rangle$. In the special case where $\omega_1 = \omega_2 = \omega, \Omega_1 = \Omega_2 = \Omega, \kappa_1 = \kappa_2 = \kappa$ and $g_1 = g_2 = g$, the total Hamiltonian takes the form

$$ H = (\omega + J)|u\rangle\langle u| + (\omega - J)|d\rangle\langle d| + (\Omega - i\kappa)a_m^\dagger a_m + \sqrt{2g}|u\rangle\langle d| + |d\rangle\langle u|(a_m + a_m^\dagger), $$

and the time evolution reads,

$$ \dot{\rho} = -iH\rho + i\rho^H + 2\kappa a_m \rho a_m^\dagger. $$

An analogous derivation could be done for the case where both subsystems in the dimer couple to a global mode. In this case, $H_{s-m} = \sigma_m g(a + a^\dagger)$, and therefore the dimer would remain decoupled if $g_1 = g_2$, while different couplings would yield to a system’s dynamics analogous to that of the effective qubit ($|d\rangle, |u\rangle$) discussed above with $g = g_1 - g_2$.

Figure 1 depicts the time evolution of the dimer’s population inversion $\sigma_2^{(z)\text{dimer}} = |10\rangle\langle 10| - |01\rangle\langle 01|$ for different values of $f$. For this example the following parameters were considered: $\omega = 0$, $J = 1$ (interdimer coupling), $\Omega = 2J$, $g_0 = J$ and $\kappa_0 = 20J$. With this, $\gamma_{\text{eff}} = J/10$. We truncated each oscillator after Fock layer 2 as the probability to have a single excitation in either mode never exceeds 10% in the parameter regime that we consider here. Varying $f$ in the range from $10^{-3}$ to 100 allows us to move from a situation where $g \ll \kappa$ to the domain where $g \sim \kappa$, so that the damped mode ranges from being Markovian to imprinting an element of non-Markovianity to the dynamics, as quantitatively exemplified when evaluating the degree of non-Markovianity $D_{NM}$ introduced in [18]. This measure provides a necessary and sufficient condition for a given evolution to depart from strict Markovianity by means of evaluating whether or not the associated dynamical map $\mathcal{E}_{(t+\epsilon,t)}$ is completely positive (CP) for any $\epsilon$. For that, it has to be $\mathcal{E}_{(t+\epsilon,t)} \otimes 1 |\Phi\rangle\langle \Phi| \geq 0$, where $|\Phi\rangle$ is maximally entangled of our open system and some ancillary system [19]. Using this condition, one can define a quantitative measure to quantify how much does the evolution departs from strict Markovianity by evaluating $I = \int_0^\infty g(t)dt$, where the function $g$ quantifies how much does the norm of state $\mathcal{E}_{(t+\epsilon,t)} \otimes 1 |\Phi\rangle\langle \Phi|$ differ from 1 in the limit of
ter equation with local dephasing rates
dimer can be reproduced by simulating a Markovian mas-
where there is an excitation in one
factorized condition where there is an excitation in one
quantified by the logarithmic negativity
long term. The time evolution of the entanglement in the
steady state of the dimer system tends towards the max-
experiencing dephasing at the same rate as before, the
weakly coupled to a purely Markovian environment and
when the index \( f \) varies from the range \( f \gg 1 \)
Markovian evolution) to \( f \ll 1 \) (non-Markovian effects).
when the evolution is strictly Markovian, the steady state
become close to one e-bit for \( f \) sufficiently small.
that can be proven using the results presented in section
4 of [8], following theorem 5.2 of [17]. When \( f \) decreases
but remaining in the weakly non-Markovian regime, the
steady state approaches the singlet state, which in this
case is the lowest energy eigenstate of the coherent evolu-
Provided that we keep on the weak non-Markovian
regime (so that the condition \( \kappa_i > g_i \) is satisfied), altering
the symmetry of the problem and considering \( g_1 \neq g_2 \)
does not reduce the overlap with the singlet below 90% up
to ratios \( g_2/g_1 = 2 \). At a finite temperature, the
steady state entanglement is decreased but remains fi-
nite. For an average photon number of \( n_{th} = 0.1 \), which
responds to a temperature of about 100 mK for GHz
frequencies, we obtain that the steady state entanglement
reaches the value 0.85 for \( f = 0.01 \) (Finite \( T \) results are
not shown in the Figures).

Possible experimental realization – In this section we
outline a possible implementation of the dimer in contact
with non-Markovian environments that was discussed in
the previous section. This realization employs proven el-
ements of ion trap technology and offers, in principle, the
possibility for control of all the relevant parameters
in the Hamiltonian Eq. (2). Three basic dynamical
elements have to be generated. The electronic degrees of
freedom of trapped ions constitute the dimer and need
to be coupled to each other, the motional degrees of free-
dom play the role of the environment and need to be
affected by a dispersive interaction and laser cooling needs to be employed
to induce damping of the motional degrees of freedom
which generates a Lorentzian spectral density.

For the implementation of the direct coupling between
the two qubits we make use of a Sørensen-Mølmer gate
To this end one applies laser fields with two different frequencies so that the two-photon process coupling $|gg\rangle \leftrightarrow |ee\rangle$ and $|eg\rangle \leftrightarrow |ge\rangle$ is resonant, i.e., $\omega_1^L + \omega_2^L = 2\omega_{eg}$, while neither of the frequencies are resonant with single excitations of the ions. This can be achieved by choosing $\omega_1^L = \omega_{eg} - \delta$ and $\omega_2^L = \omega_{eg} + \delta$, where $\delta \ll \eta\Omega_R, \nu$. Here $\eta$ denotes the Lamb-Dicke parameter, $\Omega_R$ the laser Rabi frequency and $\nu$ is the frequency of the center-of-mass mode. If we prepare the two ions representing the dimer initially in a single excitation subspace we realize in this fashion the Hamiltonian $H_{\text{dimer}} = J_{\text{eff}}(\sigma_1^z \sigma_2^z + \sigma_2^z \sigma_1^z)$ with an effective exchange coupling $J_{\text{eff}} \sim (\Omega_R\eta)^2/(\nu - \delta)$ [21]. The added advantage of this scheme is the fact that it does not require the center-of-mass mode to be in its ground state thus increasing the robustness of the scheme.

The excitation preserving coupling Hamiltonian $\sigma_z(a + a^\dagger)$ between an ion and the motional degrees of freedom can be achieved in several ways. One approach subjects the ion to a far off-resonant standing wave, which creates the state dependent potential $V(z) = V_0 \cos^2(kz + \frac{\pi}{2})\sigma_z$. Here $k$ is the wave-vector of the standing wave lasers. The operator $\hat{z}$ is readily expressed in terms of phonon operators, $\hat{z} = \sum_n M_n \sqrt{\frac{1}{2m_n a_n a_n^\dagger}}$. Traveling wave fields can also be used to generate this type of interaction in a suitably transformed basis [20]. This scheme couples to all the modes of the ion crystal which may lead to correlations between the environments acting on separate parts of the dimer. We may select specific environment modes by choosing light fields that are slightly off-resonant to specific modes while being more strongly detuned from the remaining modes.

Finally our dimer model assumes that the modes coupling to the constituents of the dimer are damped. Damping of motional degrees of freedom can be achieved, of course, by applying laser cooling to an auxiliary third ion that couples to all the motional modes of the system. To be effective, this auxiliary ion should be placed at the end of the ion chain to ensure coupling to all modes in the ion string. Combining these three elements yields the dynamical Hamiltonian discussed in Eq. [2] which underlies the non-Markovianity driven steady state entanglement in a dimer system.

**Conclusions** – We have demonstrated that, when subject to system-environment interactions of the same strength, the non-Markovian character of the noise can be the crucial property that leads to steady state entanglement where purely Markovian noise would result in the complete destruction of entanglement. A possible experimental verification of this depahsing-assisted phenomenon in ion trap physics which employs only experimentally demonstrated building blocks has been discussed. We expect these studies to contribute towards the identification of the physical mechanisms that could underpin the persistence of stationary quantum correlations in very noisy environments occurring in natural conditions [9–13]. A key issue in this context would be the evaluation of this effect for realistic (possibly strongly non-Markovian), multicomponent, biological systems operating at physiological temperatures. Numerical methods able to incorporate temperature dependence and the ability to simulate exactly structured spectral densities would be required to fully address this issue and initial steps towards this development have already been presented [23].

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