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SOME EXAMPLES OF NON-SMOOTHABLE GORENSTEIN
FANO TORIC THREEFOLDS

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ABSTRACT. We present a combinatorial criterion on reflexive polytopes of di-
mension 3 which gives a local-to-global obstruction for the smoothability of
the corresponding Fano toric threefolds. As a result, we show an example of a
singular Gorenstein Fano toric threefold which has compound Du Val, hence
smoothable, singularities but is not smoothable.

1. INTRODUCTION

In this note we consider a specific feature of the deformation theory of Fano
toric threefolds with Gorenstein singularities. Such varieties are in one-to-one cor-
respondence with the 4319 reflexive polytopes of dimension 3, which were classified
by Kreuzer and Skarke [6].

Fix such a polytope $P$ and denote by $X_P$ the corresponding Fano toric variety,
i.e. the toric variety associated to the spanning fan of $P$. The singularities of $X_P$
are detected by the shape of the facets of $P$. Here we will ignore the problem
of understanding which singularities are smoothable. Instead, we will present a
local-to-global obstruction to the smoothability of $X_P$. In other words, we will
show examples where there exists an open non-affine subscheme $Y \hookrightarrow X_P$
such that $Y$ is singular, $Y$ has smoothable singularities, and $Y$ is not smoothable (and
consequently $X_P$ is not smoothable). These examples are constructed by means of
the following combinatorial criterion — the relevant definitions are given in §3.

Theorem 1.1. Let $P$ be a reflexive polytope of dimension 3 and let $X_P$ be the Fano
toric threefold associated to the spanning fan of $P$. If, for some integer $n \geq 1$, the
polytope $P$ has “two adjacent almost-flat $A_n$-triangles” as facets, then $X_P$ is not
smoothable.

A particular polytope, which satisfies the hypothesis of Theorem 1.1 allows us
to prove the following result.

Theorem 1.2. There exists a singular Fano toric threefold $X$ such that the singular
locus of $X$ is isomorphic to $\mathbb{P}^1$, $X$ has only $cA_1$-singularities, and every infinitesimal
deformation of $X$ is trivial. In particular, $X$ is not smoothable.

This refutes a conjecture made by Prokhorov [10, Conjecture 1.9], according to
which all Fano threefolds with only compound Du Val singularities are smoothable.
This conjecture was motivated by Namikawa’s result [8] on the smoothability of
Fano threefolds with Gorenstein terminal singularities.

Idea of the proof of Theorem 1.1. Fix an integer $n \geq 1$. An $A_n$-triangle (see
Definition 3.1) corresponds, via toric geometry, to the $cA_n$ threfold singularity
$\text{Spec } \mathbb{C}[x, y, z, w]/(xy - z^{n+1})$. 

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If a reflexive polytope $P$ of dimension 3 has two adjacent $A_n$-triangles as facets, then there is an open non-affine toric subscheme $Y$ of $X_P$ such that the singular locus of $Y$ is isomorphic to $\mathbb{P}^1$ and the singularities are transverse $A_n$. Here $A_n$ denotes the affine toric surface $\text{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1})$. More precisely, $Y$ is an $A_n$-bundle over $\mathbb{P}^1$ (see Definition 2.1), i.e. there exists a map $\pi: Y \to \mathbb{P}^1$ such that, Zariski locally on the target, it is the trivial projection with fibre $A_n$. The map $\pi$ may be globally non-trivial, depending on the relative position of the two adjacent $A_n$-triangles. It is possible to express the sheaf $\pi^*\text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)$, which is a vector bundle on $\mathbb{P}^1$ of rank $n$, in terms of the combinatorics of the two triangles. In particular, we get to know when this sheaf is the direct sum of negative line bundles on $\mathbb{P}^1$. This gives a combinatorial condition for $\text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)$ not to have global sections; the condition is expressed by insisting that the two triangles almost lie on the same plane, i.e. they are “almost-flat” (see Definition 3.2). If this happens, then every infinitesimal deformation of $Y$ is locally trivial and, thus, $X_P$ is not smoothable.

**Relation to Mirror Symmetry for Fano varieties.** In the context of Mirror Symmetry for Fano varieties [1,3], Akhtar–Coates–Galkin–Kasprzyk [2] introduced the notion of “mutation”. Starting from some combinatorial datum, a mutation transforms a Fano polytope (i.e. the lattice polytope associated to a Fano toric variety) into another Fano polytope. Varying the combinatorial datum gives different mutations of the same Fano polytope.

In the setting of Theorem 1.1, if a 3-dimensional reflexive polytope $P$ has two adjacent $A_n$-triangle facets ($n \geq 1$), then these are almost-flat if and only if the polytope $P$ does not admit a special kind of mutation, which we will not specify here. Therefore, Theorem 1.1 says that, in some cases, a Gorenstein Fano toric threefold is not smoothable if the corresponding polytope does not admit a special kind of mutation. This agrees with Ilten’s observation [5] that mutations of Fano polytopes induce deformations of the corresponding Fano toric varieties.

**Higher dimensions.** The methods of this paper could be easily adapted to study obstructions to deformations of toric $A_n$-bundles on smooth toric varieties of any dimension. This would give a local-to-global obstruction to the smoothability of toric varieties of dimension $d \geq 4$ which contain, as an open toric subscheme, a toric $A_n$-bundle over a smooth toric variety of dimension $d - 2$.

**Notation and conventions.** We work over $\mathbb{C}$, but everything will hold over a field of characteristic zero or over a perfect field of large characteristic. If $N$ is a lattice, its dual is denoted by $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and the symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $M$ and $N$.

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2. $A_n$-Bundles and Their Deformations

For any integer $n \geq 1$, let $A_n$ denote the toric surface singularity associated to the cone spanned by $(0,1)$ and $(n+1,1)$ inside the lattice $\mathbb{Z}^2$, i.e. the affine hypersurface

$$A_n = \text{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1}).$$

The conormal sequence of the closed embedding $A_n \hookrightarrow \mathbb{A}^3$ produces a free resolution of $\Omega^1_{A_n}$:

$$0 \rightarrow I/I^2 = \mathcal{O}_{A_n} \rightarrow \mathcal{O}_{A_n}^{\oplus 3} \rightarrow \Omega^1_{A_n} \rightarrow 0$$

(1)

where $I$ is the ideal of $A_n$ in $\mathbb{A}^3$. This allows us to compute

$$\text{Ext}^1_{A_n}(\mathcal{O}^1_{A_n}, \mathcal{O}_{A_n}) = \text{coker} \left( \mathcal{O}_{A_n}^{\oplus 3} \rightarrow \mathcal{O}_{A_n} \right) = \mathcal{O}_{A_n}/(y, x, z^n) = \mathcal{O}_{D_n}$$

where $D_n \simeq \text{Spec} \mathbb{C}[z]/(z^n)$ is the closed subscheme of $A_n$ defined by the ideal generated by $y, x$ and $z^n$. Notice that $D_n$ is the singular locus of $A_n$ equipped with the schematic structure given by the second Fitting ideal of $\Omega^1_{A_n}$.

We want to define the notion of an $A_n$-bundle and globalise this computation of the Ext group. Informally, an $A_n$-bundle is a morphism $Y \to S$ which, Zariski-locally, is the projection $A_n \times S \to S$. More precisely we have to insist that an $A_n$-bundle is a closed subscheme in a split vector bundle over $S$ of rank 3.

**Definition 2.1.** An $A_n$-bundle over a $\mathbb{C}$-scheme $S$ is a morphism of schemes $\pi_Y : Y \to S$ such that there exist three line bundles $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z \in \text{Pic}(S)$, a closed embedding of $S$-schemes

$$\iota : Y \hookrightarrow E = \text{Spec} S \text{Sym}^\bullet_{\mathcal{O}_S}(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee$$

of $Y$ into the total space of $\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z$, and an affine open cover $\{S_i\}_i$ of $S$ satisfying the following condition: for each $i$, there are trivializations $\mathcal{L}_x|_{S_i} \simeq \mathcal{O}_{S_i}$, $\mathcal{L}_y|_{S_i} \simeq \mathcal{O}_{S_i}$, $\mathcal{L}_z|_{S_i} \simeq \mathcal{O}_{S_i}$ and a commutative diagram of $S_i$-schemes

$$
\begin{align*}
\pi_Y^{-1}(S_i) & \xrightarrow{\cong} \text{Spec} \mathcal{O}_{S_i}(S_i)[x_i, y_i, z_i]/(x_i y_i - z_i^{n+1}) \\
& \xrightarrow{\iota_{S_i}} \text{Spec} \mathcal{O}_{S_i}(S_i)[x_i, y_i, z_i] = \mathbb{A}^3_{S_i}
\end{align*}
$$

where $\pi_E$ denotes the projection $E \to S$, the coordinates $x_i \in \Gamma(S_i, \mathcal{L}_x^\vee)$, $y_i \in \Gamma(S_i, \mathcal{L}_y^\vee)$ and $z_i \in \Gamma(S_i, \mathcal{L}_z^\vee)$ are the local sections corresponding to the trivializations above, the horizontal arrows are isomorphisms, the left vertical arrow is the restriction of the closed embedding $\iota : Y \hookrightarrow E$, and the right vertical arrow is the base change of the standard embedding $A_n \hookrightarrow \mathbb{A}^3$ to $S_i$.

**Remark 2.2.** A posteriori one can see that $\mathcal{L}_x \otimes \mathcal{L}_y \simeq \mathcal{L}_z^{\otimes (n+1)}$. This follows from the following easy fact in commutative algebra: let $A$ be a ring and $f \in A$ be an invertible element; if the ideal of $A[x, y, z]$ generated by $xy - z^{n+1}$ coincides with the ideal generated by $xy - fz^{n+1}$, then $f = 1$.  


Lemma 2.3. Let $S$ be a scheme with a line bundle $\mathcal{L} \in \text{Pic}(S)$. Let $D$ be the $k$-th order thickening of the zero section of the total space of $\mathcal{L}$, i.e. the closed subscheme of $\text{Spec}_S \text{Sym}_S \mathcal{L}^\vee$ locally defined by the equation $x^{k+1} = 0$ where $x$ is a nowhere vanishing local section of $\mathcal{L}^\vee$. Let $\pi: D \to S$ be the projection. Then

$$\pi_* \mathcal{O}_D = \bigoplus_{i=0}^k (\mathcal{L}^\vee)^{\otimes i}.$$ 

Proof. Let $\{S_i\}_i$ be an affine open cover of $S$ which trivializes $\mathcal{L}$. Let $x_i \in \Gamma(S_i, \mathcal{L}^\vee)$ be a local coordinate. Then we have the isomorphism of $S_i$-schemes

$$\pi^{-1}(S_i) \simeq \text{Spec} \mathcal{O}_S(S_i)[x_i]/(x_i^{k+1}).$$

Therefore $\pi_* \mathcal{O}_D|_{S_i}$ is the free $\mathcal{O}_{S_i}$-module with basis $\{1, x_i, \ldots, x_i^k\}$, which is a local frame of $\mathcal{O}_S \oplus \mathcal{L}^\vee \oplus \cdots \oplus (\mathcal{L}^\vee)^{\otimes k}$.

Another way to see this is to notice that $D = \text{Spec}_S (\text{Sym}_S \mathcal{L}^\vee)/\mathcal{I}$, and consequently $\pi_* \mathcal{O}_D = (\text{Sym}_S \mathcal{L}^\vee)/\mathcal{I}$, where $\mathcal{I} \subseteq \text{Sym}_S \mathcal{L}^\vee$ is the ideal made up of elements of degree greater than $k$. 

□

Proposition 2.4. Let $S$ be a $\mathbb{C}$-scheme and $\pi_Y: Y \to S$ be an $A_n$-bundle, with $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z \in \text{Pic}(S)$ as in Definition 2.1. Then there is an isomorphism of $\mathcal{O}_S$-modules

$$(\pi_Y)_* \left( \mathcal{E}xt^1_Y (\Omega^1_Y/\mathcal{O}_Y) \right) \simeq \bigoplus_{2 \leq j \leq n+1} \mathcal{L}_z^{\otimes j}.$$ 

Proof. Assume we are in the setting of Definition 2.1, with projections $\pi_Y: Y \to S$ and $\pi_E: E \to S$, closed embedding $\iota: Y \to E$, and a trivialising affine open cover $\{S_i\}_i$ of $S$ with local sections $x_i, y_i, z_i$.

We consider the conormal sequence of $Y \hookrightarrow E \to S$:

$$\mathcal{I}_Y/E/\mathcal{I}_E^2 \to \Omega^1_{E/S}|Y \to \Omega^1_{Y/S} \to 0,$$

where $\mathcal{I}_Y/E$ is the ideal sheaf of the closed embedding $\iota: Y \to E$. We restrict this sequence to $S_i$ and we get the conormal sequence of $Y_i = \pi_Y^{-1}(S_i)$ $\iota_{S_i}^*: E_i = \pi_E^{-1}(S_i) \to S_i$:

$$(2) \mathcal{I}_{Y_i/E_i}/\mathcal{I}_{E_i}^2 \to \Omega^1_{E_i/S_i}|Y_i \to \Omega^1_{Y_i/S_i} \to 0;$$

this is the base change to $S_i$ of (1), the conormal sequence of $A_n \hookrightarrow \mathbb{A}^3 \to \text{Spec} \mathbb{C}$. As $S_i \to \text{Spec} \mathbb{C}$ is flat, we have that (2) is left exact for all $i$. As $\{S_i\}_i$ is an open cover of $S$, we have that also (1) is left exact.

Since $\pi_E: E \to S$ is the vector bundle whose sheaf of sections is $\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z$, we have that $\Omega^1_{E/S} = \pi_E^* (\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee$. Therefore $\Omega^1_{E/S}|Y = \pi_Y^* (\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee$.

One can check that $\mathcal{I}_{Y_i/E_i}/\mathcal{I}_{E_i}^2 \simeq \pi_Y^* (\mathcal{L}_x \oplus \mathcal{L}_y)^\vee$. On the intersection $S_{ij} = S_i \cap S_j$ we have the equalities $x_i = g_{ij}^x x_j$, $y_i = g_{ij}^y y_j$, and $z_i = g_{ij}^z z_j$, where $g_{ij}^x, g_{ij}^y, g_{ij}^z \in \Gamma(S_{ij}, \mathcal{O}_S)$ are invertible functions such that $g_{ij}^x g_{ij}^y = (g_{ij}^z)^{n+1}$ (by Remark 2.2). Then the restriction of the map

$$\pi_Y^* (\mathcal{L}_x \oplus \mathcal{L}_y)^\vee$$


in \([2]\) to \(Y_{ij} = \pi_Y^{-1}(S_{ij})\) produces the following commutative diagram.

\[
\begin{array}{ccc}
O_{Y_{ij}} & \xrightarrow{g^x_{ij}} & O_{Y_{ij}}^\otimes 3 \\
g^y_{ij} & & \text{diag}(g^x_{ij}, g^y_{ij}, g^z_{ij})
\end{array}
\]

Therefore the sequence \([2]\) becomes

\[
0 \to \pi_Y^*(L_x \otimes L_y)^\vee \to \pi_Y^*(L_x \oplus L_y \oplus L_z)^\vee \to \Omega^1_{Y/S} \to 0,
\]
which gives a locally free resolution of \(\Omega^1_{Y/S}\). Hence

\[
\mathcal{E}xt^1_Y(\Omega^1_{Y/S}, O_Y) = \text{coker} (\pi_Y^*(L_x \oplus L_y \oplus L_z) \to \pi_Y^*(L_x \otimes L_y))
\]

\[
= \pi_Y^*(L_x \otimes L_y) \otimes_{O_Y} O_D
\]

\[
= \pi_Y^*(L_z)^\otimes (n+1) \otimes_{O_Y} O_D
\]

where \(D \to Y\) is the closed subscheme locally defined by \(x_i = y_i = z^n_i = 0\). Denote by \(\pi_D : D \to S\) the projection. It is clear that \(D\) is the \((n-1)\)-th order thickening of the zero section in the total space \(L_z\) over \(S\). By Lemma \([2,3]\) we have

\[
(\pi_D)_* O_D = \bigoplus_{i=0}^{n-1} (L_z^\vee)^\otimes i.
\]

Thus

\[
(\pi_Y)_* \mathcal{E}xt^1_Y(\Omega^1_{Y/S}, O_Y) = (\pi_Y)_* (\pi_Y^*(L_z)^\otimes (n+1) \otimes_{O_Y} O_D)
\]

\[
= (\pi_D)_* (\pi_D^*(L_z)^\otimes (n+1))
\]

\[
= (\pi_D)_* O_D \otimes_{O_S} L_z^\otimes (n+1)
\]

\[
= \bigoplus_{i=0}^{n-1} (L_z^\vee)^\otimes i \otimes_{O_S} L_z^\otimes (n+1)
\]

\[
= \bigoplus_{2 \leq j \leq n+1} L_z^\otimes j.
\]

This concludes the proof of Proposition \([2,4] \)

The following lemma is well known in deformation theory.

**Lemma 2.5.** Let \(Y\) be a reduced \(\mathbb{C}\)-scheme. Assume that \(Y \to \text{Spec} \mathbb{C}\) is a local complete intersection morphism and that \(H^0(Y, \mathcal{E}xt^1_Y(\Omega^1_Y, O_Y)) = 0\).

Then all infinitesimal deformations of \(Y\) are locally trivial. In particular, if \(Y\) is not smooth, then \(Y\) is not smoothable.

**Proof.** Let \((\text{Art})\) be the category of local artinian \(\mathbb{C}\)-algebras with residue field \(\mathbb{C}\). Let \(D_{efY}\) be the functor of infinitesimal deformations of \(Y\), i.e. the covariant functor from \((\text{Art})\) to the category of sets which maps each \(A \in (\text{Art})\) to the set \(D_{efY}(A)\) of isomorphism classes of deformations of \(Y\) over \(\text{Spec} A\) and acts on arrows by base
change. For every $A \in (\text{Art})$, let $Def_Y'(A)$ be the subset of $Def_Y(A)$ made up of the locally trivial deformations. This gives a subfunctor $\phi: Def_Y' \hookrightarrow Def_Y$. We refer the reader to [11, §2.4] or to [7] for details.

We want to show that the natural transformation $\phi$ is an isomorphism. It is enough to show that the injective function $\phi_A: Def_Y'(A) \hookrightarrow Def_Y(A)$ is surjective for every $A \in (\text{Art})$. This is implied by the smoothness of $\phi$ (see [7, Definition 3.9]). This is what we will prove below.

Let $\mathcal{T}_Y = \text{Hom}_Y(\Omega^1_Y, \mathcal{O}_Y)$ be the sheaf of derivations on $Y$. By [11, Theorem 2.4.1] the tangent space of $Def_Y'$ is $H^1(Y, \mathcal{T}_Y)$ and the tangent space of $Def_Y$ is $\text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)$. By [11, Proposition 2.4.6], $H^2(Y, \mathcal{T}_Y)$ is an obstruction space for $Def_Y'$. By [11, Proposition 2.4.8] or [13, Theorem 4.4], $\text{Ext}^2_Y(\Omega^1_Y, \mathcal{O}_Y)$ is an obstruction space for $Def_Y$.

The local-to-global spectral sequence for $\text{Ext}$ gives the following five term exact sequence
\[
0 \longrightarrow H^1(Y, \mathcal{T}_Y) \longrightarrow \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow H^0(Y, \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)) \longrightarrow H^2(Y, \mathcal{T}_Y) \longrightarrow \text{Ext}^2_Y(\Omega^1_Y, \mathcal{O}_Y).
\]

With the identifications above, the vanishing of $H^0(Y, \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y))$ implies that $\phi$ induces an isomorphism on tangent spaces and an injection on obstruction spaces. By [7, Remark 4.12] we get that $\phi$ is smooth. \hfill \Box

**Corollary 2.6.** Let $S$ be a smooth $\mathbb{C}$-scheme and $\pi_Y: Y \rightarrow S$ be an $A_n$-bundle, with $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z \in \text{Pic}(S)$ as in Definition 2.7. Then we have:

(i) the sheaf $\text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)$ is isomorphic to $\text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)$;

(ii) if $H^0(S, \mathcal{L}_z^{\otimes j}) = 0$ for all $2 \leq j \leq n + 1$, then all infinitesimal deformations of $Y$ are locally trivial and $Y$ is not smoothable.

**Proof.** As $Y \rightarrow S$ is a Zariski-locally trivial fibration, the sequence of Kähler differentials of $Y \rightarrow S \rightarrow \text{Spec} \mathbb{C}$ is left exact and locally split:
\[
0 \longrightarrow \pi_Y^* \Omega^1_S \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0.
\]

This implies that the dual sequence
\[
0 \longrightarrow H^0(\mathcal{L}_z^{\otimes j}) \longrightarrow H^0(S, \mathcal{L}_z^{\otimes j}) \longrightarrow 0
\]

is exact. From the long exact sequence of $\text{Ext}$ sheaves we get the following exact sequence of $\mathcal{O}_Y$-modules:
\[
0 \longrightarrow \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \text{Ext}^1_Y(\pi_Y^* \Omega^1_S, \mathcal{O}_Y).
\]

But the last sheaf is zero because $S$ is smooth over $\mathbb{C}$. This proves (i).

By Proposition 2.4 we deduce that
\[
H^0(Y, \text{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)) = \bigoplus_{2 \leq j \leq n + 1} H^0(S, \mathcal{L}_z^{\otimes j}) = 0.
\]

From Lemma 2.5 we deduce (ii). \hfill \Box

3. Toric $A_n$-bundles over $\mathbb{P}^1$

**Definition 3.1.** Fix an integer $n \geq 1$ and a 3-dimensional lattice $N$. An $A_n$-triangle in $N$ is a lattice triangle $T \subseteq N_\mathbb{R}$ such that:

(1) there are no lattice points in the relative interior of $T$;
(2) the edges of $T$ have lattice lengths $1, 1$, and $n + 1$;
(3) $T$ is contained in a plane which has height $1$ with respect to the origin, i.e. there exists a linear form $w \in \mathbb{M}(\mathbb{N}, \mathbb{Z})$ such that $T$ is contained in the affine plane $H_{w,1} := \{v \in \mathbb{N}\mathbb{R} \mid \langle w, v \rangle = 1\}$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathbb{M}$ and $\mathbb{N}$.

If $T$ is an $A_n$-triangle in the 3-dimensional lattice $\mathbb{N}$, consider the cone $\sigma \subseteq \mathbb{N}\mathbb{R}$ spanned by the vertices of $T$. Then the affine toric variety associated to the cone $\sigma$, namely $\text{Spec} \mathbb{C}[\mathbb{N}\mathbb{R} \cap M]$, is isomorphic to $\text{Spec} \mathbb{C}[x, y, z, w]/(xy - z^{n+1})$; every point with $x = y = z = 0$ is a $cA_n$ singularity.

**Definition 3.2.** Fix an integer $n \geq 1$ and a 3-dimensional lattice $\mathbb{N}$. Two adjacent $A_n$-triangles in $\mathbb{N}$ are two $A_n$-triangles $T_0$ and $T_1$ in $\mathbb{N}$ such that:

(4) $T_0 \cap T_1$ is the edge of length $n + 1$ for both $T_0$ and $T_1$;
(5) $T_0$ and $T_1$ lie in the two different half-spaces of $\mathbb{N}\mathbb{R}$ defined by the plane $\text{span}_{\mathbb{R}}(T_0 \cap T_1)$.

We say that $T_0$ and $T_1$ are almost-flat if $\langle w_1, \rho_0 \rangle = 0$, where $\rho_0$ is the vertex of the triangle $T_0$ not in the segment $T_0 \cap T_1$ and $w_1 \in \mathbb{M}$ is the linear form such that $T_1$ is contained in the plane $H_{w_1,1}$.

Notice that the condition of almost-flatness is symmetric between $T_0$ and $T_1$ because $\langle w_1, \rho_0 \rangle = \langle w_0, \rho_1 \rangle$.

**Remark 3.3.** Let $P$ be a reflexive polytope in the lattice $\mathbb{N}$ of rank $3$ and let $T_0$ and $T_1$ be two adjacent $A_n$-triangles which are facets of $P$. The convexity of $P$ implies $\langle w_1, \rho_0 \rangle \leq 0$.

Consider the dual polytope $P^* = \{u \in \mathbb{M}\mathbb{R} \mid \forall v \in P, \langle u, v \rangle \geq -1\}$.

The dual face of $T_0$ (resp. $T_1$) is the vertex $-w_0$ (resp. $-w_1$) of $P^*$. The dual face of the edge $T_0 \cap T_1$ is the edge conv $\{-w_0, -w_1\}$ of $P^*$. The segment conv $\{-w_0, -w_1\}$ has lattice length equal to $1 - \langle w_1, \rho_0 \rangle$.

**Setup 3.4.** Let $T_0$ and $T_1$ be two adjacent $A_n$-triangles in a 3-dimensional lattice $\mathbb{N}$. We denote by $\rho_u$ and $\rho_v$ the vertices of the segment $T_0 \cap T_1$. Let $\rho_0$ (resp. $\rho_1$) be the vertex of $T_0$ (resp. $T_1$) which does not lie on $T_0 \cap T_1$ (see Figure 2). Let $Y$ be the toric variety associated to the fan in $\mathbb{N}$ generated by cone $\{\rho_0, \rho_u, \rho_v\}$ and cone $\{\rho_1, \rho_u, \rho_v\}$. The projection $N \rightarrow N/(N \cap (\mathbb{R}\rho_u + \mathbb{R}\rho_v)) \simeq \mathbb{Z}$ induces a toric morphism $\pi: Y \rightarrow \mathbb{P}^1$.

**Proposition 3.5.** Let $T_0$ and $T_1$ be two adjacent $A_n$-triangles in a 3-dimensional lattice $\mathbb{N}$. Then the toric morphism $\pi: Y \rightarrow \mathbb{P}^1$, constructed in Setup 3.4, is an $A_n$-bundle and there exists an isomorphism

\[
\pi_2\mathcal{E}xt_Y^1(\Omega_Y^1, \mathcal{O}_Y) \simeq \bigoplus_{2 \leq j \leq n+1} \mathcal{O}_{\mathbb{P}^1}(-j(\langle w_1, \rho_0 \rangle + 1)).
\]
Moreover, if \( \langle w_1, \rho_0 \rangle \geq 0 \) then all infinitesimal deformations of \( Y \) are locally trivial and \( Y \) is not smoothable.

Before proving this proposition we prove the following lemma.

**Lemma 3.6.** After a \( \text{GL}_3(\mathbb{Z}) \)-transformation, in Setup \( 3.4 \) we may assume that \( N = \mathbb{Z}^3 \) and

\[
\rho_0 = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \rho_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_v = \begin{pmatrix} -n \\ n+1 \\ 0 \end{pmatrix},
\]

for some \( a, b \in \mathbb{Z} \).

**Proof.** Let \( \hat{\rho} \in N \) be the lattice point on the segment between \( \rho_u \) and \( \rho_v \) which is the closest one to \( \rho_u \). The triangle with vertices \( \rho_u, \rho_1, \hat{\rho} \) is an empty triangle at height 1, so \( \{ \rho_u, \rho_1, \hat{\rho} \} \) is a basis of \( N \). Without loss of generality we may assume that \( \rho_u = (1, 0, 0), \hat{\rho} = (0, 1, 0) \) and \( \rho_1 = (0, 0, 1) \). Since on the edge between \( \rho_u \) and \( \rho_v \) there are \( n + 2 \) lattice points, we have \( \rho_v = \rho_u + (n + 1)(\hat{\rho} - \rho_u) = (-n, n+1, 0) \).

Assume \( \rho_0 = (a, b, c) \) for some \( a, b, c \in \mathbb{Z} \). Since \( \rho_u, \rho, \rho_0 \) are the vertices of an empty triangle at height 1, they constitute a basis of \( N \). Therefore \( c = \text{det}(\rho_u | \rho | \rho_0) = \pm 1 \).

Since \( \rho_0 \) and \( \rho_1 \) have to be in the two different half-spaces in which the plane \( \mathbb{R}\rho_u + \mathbb{R}\rho_v = (0, 0, 1)^\perp \) divides \( N_{\mathbb{R}} \), we have \( c < 0 \), so \( c = -1 \). \( \square \)

**Proof of Proposition 3.5.** By Lemma 3.6 the ray map \( \mathbb{Z}^4 \rightarrow N = \mathbb{Z}^3 \) of \( Y \) is given by the matrix

\[
\begin{pmatrix}
a & 0 & 1 & -n \\
0 & 0 & n+1 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix}.
\]

One can see that the ideal of \( \mathbb{Z} \) generated by the \( 2 \times 2 \) minors is \( \mathbb{Z} \) itself and the ideal generated by the \( 3 \times 3 \) minors is \( r\mathbb{Z} \), where \( r = \gcd(n+1, b) > 0 \). Let \( p, q \in \mathbb{Z} \) be such that \( b = rp \) and \( n + 1 = rq \). The kernel of the ray map is generated by the primitive vector \( (q, q, -np - aq, -p) \). By Bézout let \( s, t \in \mathbb{Z} \) be such that \( sp + tq = 1 \). The cokernel of the transpose of the ray map is the homomorphism \( \mathbb{Z}^4 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \). 

![Diagram](image_url)
given by the matrix
\[
\begin{pmatrix}
q & q & -qa - pn & -p \\
\bar{s} & \bar{s} & -\bar{s}a + \bar{t}n & \bar{t}
\end{pmatrix},
\]
where \( \bar{\sigma} \) denotes the reduction modulo \( r \). By [4, Theorem 4.1.3], the divisor class group of \( Y \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \).

Let the group
\[ G = \left\{ \left( \lambda x^s, \lambda y^s, \lambda -qa - pn \varepsilon^{-st + tn}, \lambda^{-p} \varepsilon^t \right) \in \mathbb{C}_m^4 \middle| \lambda \in \mathbb{G}_m, \varepsilon \in \mu_r \right\} \]
act linearly on the affine space \( \mathbb{A}^4 = \text{Spec} \mathbb{C}[x_0, x_1, u, v] \). By [4, §5.1], \( Y \) is the geometric quotient of \( \mathbb{A}^4 \setminus V(x_0, x_1) = \text{Spec} \mathbb{C}[x_0^s, x_1, u, v] \cup \text{Spec} \mathbb{C}[x_0, x_1^t, u, v] \) with respect to this action. The variables \( x_0, x_1, u, v \) can be identified with the Cox coordinates of \( Y \) associated to the rays \( \rho_0, \rho_1, \rho_u, \rho_v \), respectively. The toric morphism \( \pi: Y \to \mathbb{P}^1 \) is defined by
\[
[x_0 : x_1 : u : v] \mapsto [x_0 : x_1],
\]
where \([x_0 : x_1 : u : v]\) denotes the point of \( Y \) corresponding to the \( G \)-orbit of the point \((x_0, x_1, u, v) \) in \( \mathbb{A}^4 \).

We consider the following integers
\[
d_x = b - (n + 1)(a + b),
\]
\[
d_y = -b,
\]
\[
d_z = -a - b.
\]

We consider the line bundles \( \mathcal{L}_x = \mathcal{O}_{\mathbb{P}^1}(d_x), \mathcal{L}_y = \mathcal{O}_{\mathbb{P}^1}(d_y), \mathcal{L}_z = \mathcal{O}_{\mathbb{P}^1}(d_z) \) and the sheaf \( \mathcal{E} = \mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z \) on \( \mathbb{P}^1 \). Let \( \pi_E: E \to \mathbb{P}^1 \) be the total space of \( \mathcal{E} \) over \( \mathbb{P}^1 \). Then \( E \) is the geometric quotient of \( \text{Spec} \mathbb{C}[x_0, x_1, x, y, z] \setminus V(x_0, x_1) \) with respect to the linear action of \( \mathbb{G}_m \) with weights \((1, 1, d_x, d_y, d_z)\). The variables \( x_0, x_1, x, y, z \) can be identified with the Cox coordinates of the toric variety \( E \). We denote by \([x_0 : x_1 : x : y : z]\) the point of \( E \) corresponding to the \( \mathbb{G}_m \)-orbit of \((x_0, x_1, x, y, z) \) in \( \mathbb{A}^5 \).

It is easy to check that the map \( \iota: Y \to E \) given by
\[
[x_0 : x_1 : u : v] \mapsto [x_0 : x_1 : u^{n+1} : v^{n+1} : uv]
\]
is a closed embedding, locally defined by \( xy - z^{n+1} = 0 \). So \( \pi: Y \to \mathbb{P}^1 \) is an \( A_n \)-bundle and we are in the situation of Definition 2.1.

The triangle \( T_1 \) is contained in the plane \( H_{w_1,1} \), where \( w_1 = (1, 1, 1) \). Therefore \( \langle w_1, \rho_0 \rangle = a + b - 1 = -d_z - 1 \). By Proposition 2.4 and Corollary 2.6 we have the isomorphism \( 4 \).

The inequality \( \langle w_1, \rho_0 \rangle \geq 0 \) implies that \( \mathcal{L}_z \) is a negative line bundle on \( \mathbb{P}^1 \) and, by Corollary 2.6, that all infinitesimal deformations of \( Y \) are locally trivial. \( \square \)

**Proof of Theorem 1.1** It is an immediate consequence of Proposition 3.5. \( \square \)

**Remark 3.7.** There are 273 reflexive polytopes of dimension 3 which satisfy the condition of Theorem 1.1; the complete list is given in [9, Remark 4.15]. Therefore, there are at least 273 non-smoothable Gorenstein Fano toric threefolds.
Proof of Theorem 1.2. In the lattice $N = \mathbb{Z}^3$ consider the reflexive polytope $P$ that is the convex hull of the following vectors:

$$
\rho_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \rho_u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \rho_v = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
$$

Let $\Sigma$ be the spanning fan of $P$. The maximal cones of $\Sigma$ are:

- $\text{cone} \{ \rho_0, \rho_u, \rho_v \}$
- $\text{cone} \{ \rho_0, \rho_u, \xi \}$
- $\text{cone} \{ \rho_0, \rho_v, \xi \}$
- $\text{cone} \{ \rho_1, \rho_u, \xi \}$
- $\text{cone} \{ \rho_1, \rho_v, \xi \}$

The singular cones of $\Sigma$ are the ones in the first row and cone $\{ \rho_u, \rho_v \}$. The corresponding facets of $P$ are two adjacent $A_1$-triangles. We have $w_1 = (-1,1,0)$ and $\langle w_1, \rho_0 \rangle = 0$, so the two $A_1$-triangles are almost flat.

Let $X$ be the Fano toric threefold associated to the fan $\Sigma$. The singular locus of $X$ is the curve $C$, which is the closure of the torus-orbit corresponding to cone $\{ \rho_u, \rho_v \}$. The curve $C$ is isomorphic to $\mathbb{P}^1$ and the singularities of $X$ along $C$ are transverse $A_1$.

By Proposition 3.3 the sheaf $\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)$ is the line bundle $\mathcal{O}_C(-2)$ on $C$. Therefore $H^0(X, \mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)) = 0$.

Let $j: U \hookrightarrow X$ be the inclusion of the smooth locus of $X$. Notice that the sheaf of derivations $\mathcal{T}_X = \mathcal{H}om_X(\Omega^1_X, \mathcal{O}_X)$ is isomorphic to $j_*\Omega^1_U \otimes \mathcal{O}_X(-K_X)$, because these two sheaves are both reflexive and coincide on the open subset $U$ whose complement has codimension $2$. As $-K_X$ is ample, by Bott–Steenbrink–Danilov vanishing [4, Theorem 9.3.1] we have $H^1(X, \mathcal{T}_X) = 0$. This argument comes from the proof of [12, Theorem 5.1].

From the five term exact sequence for Ext, which is rewritten in the proof of Lemma 2.5 we deduce that $\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X) = 0$. This implies that all infinitesimal deformations of $X$ are trivial. In particular, $X$ is not smoothable. 

\[\square\]

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