Quantum Regge Calculus of Einstein-Cartan theory

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We study the Quantum Regge Calculus of Einstein-Cartan theory to describe quantum dynamics of Euclidean space-time discretized as a 4-simplices complex. Tetrad field $e_\mu(x)$ and spin-connection field $\omega_\mu(x)$ are assigned to each 1-simplex. Applying the torsion-free Cartan structure equation to each 2-simplex, we discuss parallel transports and construct a diffeomorphism and local gauge-invariant Einstein-Cartan action. Invariant holonomies of tetrad and spin-connection fields along large loops are also given. Quantization is defined by a bounded partition function with the measure of $SO(4)$-group valued $\omega_\mu(x)$ fields and Dirac-matrix valued $e_\mu(x)$ fields over 4-simplices complex.

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Introduction. Since the Regge Calculus [1] was proposed for the discretization of gravity theory in 1961, many progresses have been made in the approach of Quantum Regge Calculus [2, 3] and its variant dynamical triangulations [4]. In particular, the renormalization group treatment is applied to discuss any possible scale dependence of gravity [2]. In Lagrangian formalism, gauge-theoretic formulation [5] of quantum gravity using connection variables on a flat hypercubic lattice of the space-time was inspired by the success of lattice regularization of non-Abelian gauge theories. A locally finite model for gravity has been recently proposed [6]. In this Letter, based on the scenario of Quantum Regge Calculus, we present a diffeomorphism and local gauge-invariant invariant regularization and quantization of Euclidean Einstein-Cartan (EC) theory, invariant holonomies of tetrad and spin-connection fields $\omega_\mu(x)$ along large loops in 4-simplices complex, and some calculations in 2-dimensional case.

Euclidean Einstein-Cartan gravity. The basic gravitational variables in the Einstein-Cartan gravity constitute a pair of tetrad and spin-connection fields $(e_\mu^a, \omega_\mu^{ab})$, whose Dirac-matrix values $e_\mu = e_\mu^a \gamma_a$ and $\omega_\mu = \omega_\mu^{ab} \sigma_{ab}$. The space-time metric of 4-dimensional Euclidean manifold $\mathcal{M}$ is $g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \delta_{ab}$, where $\delta^{ab} = (+, +, +, +)$. The diffeomorphism invariance under general coordinate transformations $x \to x'(x)$ is preserved by all derivatives and $d$-form fields on $\mathcal{M}$ made to be coordinate scalars with the help of tetrad fields $e_\mu^a = \partial \xi^a / \partial x^\mu$. Under the local Lorentz
coordinate transformation $\xi'{}^a(x) = [\Lambda(x)]_b^a \xi^b(x)$, the local (w.r.t. $\xi$) gauge transformations are:

$$e'_\mu(\xi) = V(\xi) e_\mu(\xi) V^\dagger(\xi), \quad (1)$$

$$\omega'_\mu(\xi) = V(\xi) \omega_\mu(\xi) V^\dagger(\xi) + V(\xi) \partial_\mu V^\dagger(\xi); \quad (2)$$

and fermion field $\psi'(\xi) = V(\xi) \psi(\xi)$, the covariant derivative $D'_\mu = V(\xi) D_\mu V^\dagger(\xi)$, $D_\mu = \partial_\mu - ig\omega_\mu(\xi)$

where $V(\xi) = \exp i[\theta^{ab}(\xi) \sigma_{ab}] \in SO(4)$, and $\theta^{ab}(\xi)$ is an arbitrary function of $\xi$. In an $SU(2)$ gauge theory, gauge field $A_a(\xi_E)$ can be viewed as a connection $\int A_a(\xi_E) d\xi^a$ on the global flat manifold. On a locally flat manifold, the spin-connection $\omega_\mu dx^\mu = \omega_a(\xi) d\xi^a$, where $\omega_a(\xi) = e_\mu^a \omega_\mu$, one can identify that the spin-connection field $\omega_\mu(x)$ or $\omega_a(\xi)$ is the gravity analog of gauge field and its local curvature is given by

$$R^{ab} = d\omega^{ab} - g\omega^{ae} \wedge \omega^b_e, \quad (3)$$

and $R'^{ab} = V(\xi) R^{ab}(\xi) V^\dagger(\xi)$ under the transformation (1,2). The diffeomorphism and local gauge-invariant EC action for gravity is given by the Palatini action $S_P$ and Host modification $S_H$

$$S_{EC}(e, \omega) = S_P(e, \omega) + S_P(e, \omega) \quad (4)$$

$$S_P(e, \omega) = \frac{1}{4\kappa} \int_M d^4x \det(e) e_{abcd} e^a \wedge e^b \wedge R^{cd}, \quad (5)$$

$$S_H(e, \omega) = \frac{1}{2\kappa\tilde{\gamma}} \int_M d^4x \det(e) e_a \wedge e_b \wedge R^{ab} \quad (6)$$

where $\kappa \equiv 8\pi G$, the Newton constant $G = 1/m_P^{2\text{Planck}}$, and $\det(e)$ is the Jacobi of mapping $x \rightarrow \xi(x)$. The complex Ashtekar connection [7] with reality condition and the real Barbero connection [8] are linked by a canonical transformation of the connection with a finite complex Immirzi parameter $\tilde{\gamma} \neq 0$[9], which is crucial for Loop Quantum Gravity [10].

Classical equations can be obtained by the invariance of the EC action (4) under the transformation (1,2),

$$\delta S_{EC} = \frac{\delta S_{EC}}{\delta e_\mu} \delta e_\mu + \frac{\delta S_{EC}}{\delta \omega_\mu} \delta \omega_\mu = 0, \quad (7)$$

where $\delta e_\mu$ and $\delta \omega_\mu$ are infinitesimal variations, which can be expressed in terms of independent Dirac matrix bases $\gamma_5$ and $\gamma_\mu$. Therefore, for an arbitrary function $\theta_{ab}$, we have $\delta S_{EC}/\delta e_\mu = 0$ and $\delta S_{EC}/\delta \omega_\mu = 0$, respectively leading to Einstein equation and Cartan’s structure equation (torsion-free)

$$de^a - \omega^{ab} \wedge e_b = 0. \quad (8)$$
of the 2-simplex. The 2-simplex area $S$ summed topology of the manifold, which gives geometric constraints on the numbers of sub-simplices which is a simplicial manifold. The way to construct a simplicial manifold depends also on the same achoron). The 4-simplex has 5 vertexes – 0-simplex (a space-time point “$\omega$” gauge” field $N$ abele of Regularized EC action. The four-dimensional Euclidean manifold $x$ around each vertex $x$ the 2-simplex $h(x)$, and other edges (dashed lines) $e_{\mu}^1(x + a_\nu)$ and $e_{\nu}(x + a_\mu)$ are parallel transports of $e_{\mu}(x)$ and $e_{\nu}^1(x)$ along $\nu$- and $\mu$-directions respectively. Each 2-simplex in the 4-simplices complex has a closed parallelogram lying in it. Group-valued gauge fields $U_{\mu}(x)$ and $U_{\nu}(x) = U_{\nu}(x + a_\mu)$ are respectively associated to edges $e_{\mu}(x)$ and $e_{\nu}^1(x)$ of the 2-simplex $h(x)$, as indicated. The fields $e_{\rho}(x + a_\mu)$ and $U_{\rho}(x + a_\mu)$ are associated to the third edge $(x + a_\mu, x + a_\nu)$ of the 2-simplex $h(x)$.

**Regularized EC action.** The four-dimensional Euclidean manifold $\mathcal{M}$ is discretized as an ensemble of $N_0$ space-time points “$x$” and $N_1$ links (edges) “$l_{\mu}(x)$” connecting two neighboring points, which is a simplicial manifold. The way to construct a simplicial manifold depends also on the assumed topology of the manifold, which gives geometric constrains on the numbers of sub-simplices $(N_0, N_1, \cdots$, see Ref. [4]). In this Letter, analogously to the simplicial manifold adopted by Regge Calculus we consider a 4-simplices complex, whose elementary building block is a 4-simplex (penta-choron). The 4-simplex has 5 vertexes – 0-simplex (a space-time point “$x$”), 5 “faces” – 3-simplex (a tetrahedron), and each 3-simplex has 4 faces – 2-simplex (a triangle), and each 2-simplex has three faces – 1-simplex (an edge or a link “$l_{\mu}(x)$”). Different configurations of 4-simplices complex correspond to variations of relative vertex-positions \{x\}, edges \{l_{\mu}(x)\} and “deficit angle” around each vertex $x$. These configurations will be described by the configurations of dynamical fields $e_{\mu}(x)$ and $\omega_{\mu}(x)$ (its group-valued $U_{\mu}(x)$) in a regularized EC-theory [11].

To illustrate how to construct a regularized EC theory describing dynamics of 4-simplices complex, we consider a 2-simplex (triangle) $h(x)$ (see Fig. 1). The fundamental tetrad field $e_{\mu}(x)$ and “gauge” field $\omega_{\mu}(x)$ are assigned to each 1-simplex (edge) of the 4-simplices complex. The values of $e_{\mu}(x)$-field characterize edge spacings $a_{\mu}(x) \equiv |l_{\mu}(x)|$, where $l_{\mu}(x) = ae_{\mu}(x)$ and the Planck length $a = (8\pi G)^{1/2}$. The fundamental area operator $S_{h\mu}^{\nu} \equiv l_{\mu}(x) \wedge l_{\nu}(x)/2$, where $\mu \neq \nu$ indicates edges of the 2-simplex. The 2-simplex area $S_{h}(x) = |S_{h\mu}^{\nu}(x)|$.

The Cartan equation (8) is actually an equation for infinitesimal parallel transports of $e_{\nu}(x)$.
fields. Applying this equation to the 2-simplex \( h(x) \), as shown in Fig. [1], we show that \( e_\nu(x) \) undergoes its parallel transport to \( e_\nu(x + a_\mu) \) along the \( \mu \) \([\nu]\)-direction for an edge spacing \( a_\mu(x) \) \([a_\nu(x)]\), following the discretized Cartan equation

\[
e_\nu^a(x + a_\mu) - e_\nu^a(x) - a_\mu \omega_\mu^{ab}(x) \wedge e_{\nu b}(x) = 0, \tag{9}
\]

and \( \mu \leftrightarrow \nu \). The parallel transports \( e_\nu^a(x + a_\mu) \) and \( e_\nu^a(x + a_\nu) \) are neither independent fields, nor assigned to any edges of the 4-simplices complex. They are related to \( e_\mu(x) \) and \( \omega_\mu(x) \) fields assigned to edges of the 2-simplex \( h(x) \) by the Cartan equation (9). Because of torsion-free, \( e_\mu(x), e_\nu(x) \) and their parallel transports \( e_\mu(x + a_\nu), e_\nu(x + a_\mu) \) form a closed parallelogram \( CP(x) \) (Fig. [1]). Otherwise this would means the curved space-time could not be approximated locally by a flat space-time [12]. We define \( \omega_\mu(x + a_\nu) \) and \( \omega_\nu(x + a_\mu) \) by using the discretized equation for curvature (3),

\[
\omega_\nu^{ab}(x + a_\mu) - \omega_\nu^{ab}(x) - a_\mu \omega_\mu^{ae}(x) \wedge \omega_{e\nu b}(x) = a_\mu R_{\mu
u}^{ab}(x), \tag{10}
\]

and \( \mu \leftrightarrow \nu \). For zero curvature case, analogously to (9), parallel transports \( \bar{\omega}_\nu^{ab}(x + a_\mu) \) \([\bar{\omega}_\mu^{ab}(x + a_\nu)]\) can be defined as

\[
\bar{\omega}_\nu^{ab}(x + a_\mu) - \omega_\nu^{ab}(x) - a_\mu \omega_\mu^{ae}(x) \wedge \omega_{e\nu b}(x) = 0, \tag{11}
\]

and \( \mu \leftrightarrow \nu \). The difference (“deficit angle”) between \( \omega_\nu^{ab}(x + a_\mu) \) and \( \bar{\omega}_\nu^{ab}(x + a_\mu) \) is the curvature \( a_\mu R_{\mu
u}^{ab}(x) \).

Instead of \( \omega_\mu(x) \) field, we assign a group-valued field \( U_\mu(x) \) to each 1-simplex of 4-simplices complex. For example, at edges \((x, \mu)\) and \((x, \nu)\) of the 2-simplex \( h(x) \) \((\mu \neq \nu\) see Fig. [1], we define \( SO(4) \) group-valued spin-connection fields,

\[
U_\mu(x) = e^{i g \omega_\mu(x)}, \quad U_\nu(x) = e^{i g \omega_\nu(x)}, \tag{12}
\]

which take value of fundamental representation of the compact group \( SO(4) \), and their local gauge transformations,

\[
U_\mu(x) \rightarrow V(x)U_\mu(x)V^\dagger(x + a_\mu), \tag{13}
\]

and \( \mu \leftrightarrow \nu \) in accordance with (2). Actually, these group-valued fields (12) can be viewed as unitary operators for finite parallel transportations. Eq. (9) can be generalized to

\[
e_\nu(x + a_\mu) = U_\mu(x)e_\nu(x)U_{\mu}^\dagger(x), \tag{14}
\]
and $\mu \leftrightarrow \nu$. While, corresponding to \(10\) for the field $\omega_\nu(x+a_\mu)$, we define

$$U_\nu(x+a_\mu) \equiv U_\mu(x)U_\nu(x)U_\mu^\dagger(x),$$  \hspace{1cm} \text{(15)}$$

$$U_\nu(x+a_\mu) \equiv e^{iga_\omega_\nu(x+a_\mu)},$$  \hspace{1cm} \text{(16)}$$

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x) \equiv U(x+a_\mu)U(x),$$  \hspace{1cm} \text{(17)}$$

and $\mu \leftrightarrow \nu$. Eq. \(17\) characterizes relative angles $\theta_{\mu\nu}(x)$ between two neighboring edges $e_\mu(x)$ and $e_\nu(x)$ (see Fig. \(1\)). In the \textit{naive continuum limit}: $ag\omega_\mu \ll 1$ (small coupling or weak-field), indicating that the wavelengths of weak and slow-varying fields $\omega_\mu(x)$ are much larger than the edge spacing $a_{\mu\nu}$, we have

$$U_{\mu\nu}(x) = \exp \left\{i g [a_\omega_\nu(x) + a_\omega_\mu(x)] + ig a^2 \partial_\mu \omega_\nu(x) - \frac{1}{2} (ga)^2 \left[\omega_\nu(x), \omega_\mu(x)\right] + O(a^3) \right\},$$  \hspace{1cm} \text{(18)}$$

where $O(a^3)$ indicates high-order powers of $ag\omega_\mu$.

Using the tetrad fields $e_\mu(x)$ to construct coordinate and Lorentz scalars so as to obtain a regularized EC action preserving the diffeomorphism and local gauge-invariance, we define the smallest holonomy along closed triangle path of 2-simplex:

$$X_h(v,U) = \text{tr} \left[ v_{\mu\nu}(x)U_\mu(x)v_{\mu\nu}(x+a_\mu)U_\nu(x+a_\mu)v_{\mu\nu}(x+a_\nu)U_\nu(x+a_\nu) \right],$$  \hspace{1cm} \text{(19)}$$

whose orientation is anti-clock-like, and $X_h^\dagger(e,U)$ is clock-like (see Fig. \(1\)). We have following two possibilities for the vertex-field $v_{\mu\nu}(x)$. The first $v_{\mu\nu}(x) = e_{\mu\nu}(x)\gamma_5$:

$$A_P(e,U) = \frac{1}{8g^2} \sum_h \left\{X_h(v,U) + \text{h.c.} \right\},$$  \hspace{1cm} \text{(20)}$$

$$e_{\mu\nu}(x) \equiv (e^a \wedge e^b)\sigma_{ab},$$  \hspace{1cm} \text{(21)}$$

where $\sum_h$ is the sum over all 2-simplices $h(x)$. In the limit: $ag\omega_\mu \ll 1$, Eq. \(20\) becomes

$$A_P(e,U_\mu) = \frac{1}{a^2} \sum_h S_h^2(x)\epsilon_{cdab} e^c \wedge e^d \wedge R^{ab} + O(a^4).$$  \hspace{1cm} \text{(22)}$$

We define a 4-d volume element $V(x) = \sum_{h(x)} S_h^2(x)$ around the vertex $x$. The interior of 4-simplex is approximately flat, leading to

$$\sum_x V(x) \Rightarrow \int d^4x \xi(x) = \int d^4x \det[e(x)],$$  \hspace{1cm} \text{(23)}$$

and Eq. \(22\) approaches to $S_P(e,\omega)$ \(5\) with an effective Newton constant $G_{\text{eff}} = gG/4$. The second $v_{\mu\nu}(x) = e_{\mu\nu}(x)$:

$$A_H(e,U_\mu) = \frac{1}{8g^2} \sum_h [X_h(v,U) + \text{h.c.}],$$  \hspace{1cm} \text{(24)}$$
where the real parameter $\gamma = i\tilde{\gamma}$. Analogously, in the limit: $a g \omega_\mu \ll 1$, Eq. (24) approaches to

$$S_H(e, \omega) = \frac{1}{2\kappa\tilde{\gamma}} \int d^4x \det[e(x)] e_a \wedge e_b \wedge R^{ab} + O(a^4).$$

Under the gauge transformation (1),

$$v_{\mu\nu}(x) \rightarrow V(x)v_{\mu\nu}(x)V^\dagger(x).$$

The diffeomorphism and local gauge-invariant regularized EC action is then given by

$$A_{EC} = A_P + A_H.$$ (27)

Considering the following diffeomorphism and local gauge-invariant holonomies along a large loop $C$ on the Euclidean manifold $\mathcal{M}$

$$X_C(v, \omega) = \mathcal{P}_C \text{Tr} \left\{ ig \oint_C v_{\mu\nu}(x) \omega^\mu(x) dx^\nu \right\},$$

where $\mathcal{P}_C$ is the path-ordering and “Tr” denotes the trace over spinor space, we attempt to regularize these holonomies on the 4-simplices complex. Suppose that an orientating closed path $C$ passes space-time points $x_1, x_2, x_3, \cdots, x_N = x_1$ and edges connecting between neighboring points in the 4-simplices complex. At each point $x_i$ two tetrad fields $e_\mu(x_i)$ and $e_{\mu'}(x_i)$ ($\mu \neq \mu'$) respectively orientating path incoming to $(i-1 \rightarrow i)$ and outgoing from $(i \rightarrow i+1)$ the point $x_i$, we have the vertex-field $v_{\mu\mu'}(x_i)$ defined by Eqs. (21,24). Link fields $U_{\mu}(x_i)$ are defined on edges lying in the loop $C$, recalling the relationship $U_{\mu}(x_i) = U_{-\mu}(x_{i+1}) = U_{\mu}^\dagger(x_{i+1})$, we can write the regularization of the holonomies (28) as follows,

$$X_C(v, U) = \mathcal{P}_C \text{Tr} \left[ v_{\mu\mu'}(x_1) U_{\mu'}(x_1) v_{\mu'\nu}(x_2) U_{\nu}(x_2) \right. \left. \cdots v_{\mu\mu'}(x_i) U_{\mu'}(x_i) v_{\mu'\nu}(x_{i+1}) \right. \left. \cdots v_{\lambda\mu}(x_{N-1}) U_{\mu}^\dagger(x_{N-1}) \right],$$

preserving diffeomorphism and local gauge-invariances. Eq. (29) is consistent with Eq. (19).

**Euclidean partition function.** The partition function $Z_{EC}$ and effective action $A_{EC}^\text{eff}$ are

$$Z_{EC} = \exp -A_{EC}^\text{eff} = \int \mathcal{D}e\mathcal{D}U \exp -A_{EC},$$

with the diffeomorphism and local gauge-invariant measure

$$\int \mathcal{D}e\mathcal{D}U \equiv \prod_{x,\mu} \int de_\mu(x) dU_\mu(x).$$

(31)
where $\prod_{x,\mu}$ indicates the product of overall edges, $dU_\mu(x)$ is the Haar measure of compact gauge group $SO(4)$ or $SU(2)$, and $de_\mu(x)$ is the measure of Dirac-matrix valued field $e_\mu(x) = \sum_a e^a_\mu(x)\gamma_a$, determined by the functional measure $de^a_\mu(x)$ of the bosonic field $e^a_\mu(x)$. It should be mentioned that the measure (31) is just a lattice form of the standard DeWitt functional measure \[13\] over the continuum degrees, with the integral of the spin-connection field $\omega_\mu(x)$ replaced by the Haar integral over the $U_\mu(x)$'s, analytical integration or numerical simulations runs overall configuration space of continuum degrees and no gauge fixing is needed. In this path-integral quantization formalism, values of the partition function (30) presents all dynamical configurations of 4-simplices complex, described by field configurations $e_\mu(x)$ and $U_\mu(x)$ in the weight $\exp -A_{EC}$. The vacuum expectational values (v.e.v.) of diffeomorphism and local gauge-invariant quantities, for instance holonomies (29), are given by

$$\langle X_C(e,U) \rangle = \frac{1}{Z_{EC}} \int DeDU \left[ X_C(e,U) \right] \exp -A_{EC}. \quad (32)$$

In the action (20,24), $X_h(v,U)$ (19) contains the quadric term of $e_\mu(x)$-field associated to each edge $(x,\mu)$, the partition function $Z_{EC}$ (30) and v.e.v. (32) are converge.

Analogously to Eq. (7), the local gauge-invariance of the partition function (30) ($\delta Z_{EC} = 0$) leads to

$$\langle \frac{\delta A_{EC}}{\delta e_\mu} \delta e_\mu + U_\mu \frac{\delta A_{EC}}{\delta U_\mu} + \text{h.c.} \rangle = 0, \quad (33)$$

which becomes “averaged” Einstein equation $\langle \delta A_{EC}/\delta e_\mu \rangle + \text{h.c.} = 0$, and

$$\langle U_\mu \frac{\delta A_{EC}}{\delta U_\mu} - U_\mu^\dagger \frac{\delta A_{EC}}{\delta U_\mu^\dagger} \rangle = 0. \quad (34)$$

Eq. (34) is “averaged” torsion-free Cartan equation (8), which actually shows the impossibility of spontaneous breaking of local gauge symmetry. This should not be surprised, since the torsion-free (8) is a necessary condition to have a local Lorentz frame, therefore a local gauge-invariance.

The local gauge-invariance of (32) ($\delta \langle X \rangle = 0$) leads to dynamical equations for holonomies (29), which can be formally written as

$$\langle \frac{\delta X}{\delta e_\mu} \delta e_\mu + X \frac{\delta A_{EC}}{\delta e_\mu} \delta e_\mu + X U_\mu \frac{\delta A_{EC}}{\delta U_\mu} + \text{h.c.} \rangle = 0, \quad (35)$$

leading to $\langle \delta X/\delta e_\mu + X \delta A_{EC}/\delta e_\mu \rangle + \text{h.c.} = 0$, and

$$\langle X \rangle + \langle X \left( U_\mu \frac{\delta A_{EC}}{\delta U_\mu} - U_\mu^\dagger \frac{\delta A_{EC}}{\delta U_\mu^\dagger} \right) \rangle = 0. \quad (36)$$

Eq. (36) has the same form as the Schwinger-Dyson equation for Wilson loops in lattice gauge theories.
The regularized EC theory [27] can be separated into left- and right-handed parts by replacing $U_\mu(x) = U_\mu^L(x) \otimes U_\mu^R(x)$, where $U_\mu^L,R(x) \in SU_{L,R}(2)$. In addition, we can generalize the link field $U_\mu(x)$ to be all irreducible representations $U_\mu^j(x)$ of the gauge group $SO(4)$. The regularized EC action [27] should be a sum over all representations $j \equiv j_{L,R} = 1/2, 3/2, \cdots$,

$$\mathcal{A}_{EC} = \sum_j \left[ \mathcal{A}_P(e_\mu, U_\mu^j) + \mathcal{A}_H(e_\mu, U_\mu^j) \right],$$

(37)

and the measure [31] should include all representations of gauge group.

Some calculations in 2-dimensional case. We consider a 2-simplices complex, i.e., random simplicial surface, whose elementary building block is a triangle $h(x)$ (see Fig. 1). In this case, local gauge transformations [13,26] can be made so that all fields $v_\mu\rho(x + a_\mu)U_\rho(x + a_\mu)v_\rho\mu(x + a_\nu) = 1$ in Eq. (19), as if we choose a particular gauge. The partition function [30] can be calculated by integrating over $e_\mu(x)$- and $U_\mu(x)$-fields, using the Cayley-Hamilton formula for a determinant [14] and the properties of invariant Haar measure: $\int dU_\mu^j(x) = 1$, $\int dU_\mu^j(x)U_\mu^j(x) = 0$ and

$$\int dU_\mu^j(x)U_\mu^{ab}(x)U_\nu^{cd}(x') = \frac{1}{d_j} \delta_{\mu\nu} \delta^{ac} \delta^{bd} \delta(x - x'),$$

(38)

where $d_j = n_{jL,n_{jR}}$ ($n_{jL,n_{jR}} = 2j_{L,R} + 1$), the dimension of irreducible representations $j = (j_L,j_R)$ of $SU_L(2) \otimes SU_R(2)$. We obtain the entropy $S = \ln Z_{EC}$

$$S = \sum \text{Tr} \left[ \gamma^s \frac{i}{2d_j g^2} + \frac{2}{2d_j g^2 \gamma} \right] = \sum_j \frac{4}{d_j g^2 \gamma a^2} S_{surf},$$

(39)

where $\sum$ is the sum over all 2-simplices, degrees of freedom of gauge group representations and Dirac spinors. The 2-dimensional surface

$$S_{surf} = \sum_h S_h(x) = N_h P_a, \quad P_a = \frac{1}{N_h} \sum_h S_h(x)$$

(40)

where $N_h$ is the total number of 2-simplices and $P_a$ averaged area of 2-simplices. The free energy $\mathcal{F} = -\frac{1}{\beta} \ln Z_{EC}$, where the inverse “temperature” $\beta = 1/g^2$, see Eqs. (20,24). Selecting fundamental representation $d_j = 4$, we obtain $S = S_{surf}/(g^2 \gamma a^2)$ and $\mathcal{F} = -S_{surf}/(\gamma a^2)$.

In the same way, we calculate the average of regularized EC action $\mathcal{A}_{EC}$ [31],

$$\langle \mathcal{A}_{EC}^j [e_\mu, U_\mu^j] \rangle \simeq \frac{1}{d_j} \left( \frac{1}{8 g^2} \right)^2 \left( 1 + \frac{4}{\gamma^2} \right) N_h,$$

(41)

in the strong coupling (field) limit $g \gg 1$ or $g \omega_\mu \sim \mathcal{O}(1)$, which implies that $\omega_\mu$ field’s wavelength is comparable to the Planck length $a$. The average [41] of regularized EC action has discrete values corresponding to the fundamental state $d_j = 4$ and excitation states $d_j = 16$. 


Using the convexity inequality $\langle e^{-A EC} \rangle \geq e^{-\langle A EC \rangle}$, we have
\[
\langle A^i_{EC}[e_\mu, U_\mu^i] \rangle \leq \ln Z^i_{EC}(2/g^2) - \ln Z^i_{EC}(1/g^2). \tag{42}
\]
Using Eqs. (39,40), we obtain
\[
\frac{1}{d_j} \left( \frac{1}{8g^2} \right)^2 \left( 1 + \frac{4}{\gamma^2} \right) N_h \leq \frac{4}{d_j g^2 \gamma a^2} S_{surf}, \tag{43}
\]
and averaged area of a 2-simplex
\[
P_a \geq \frac{\pi}{32g^2} \left( 1 + \frac{4}{\gamma^2} \right) \frac{8\pi}{m^2_{\text{Planck}}}, \tag{44}
\]
implying that the Planck length is minimal separation between two space-time points [15].

Some remarks. Although the regularized EC action (27) approaches to the EC action (4) in the "naive continuous limit" $ag \omega_\mu \ll 1$, the regularized EC theory is physically sensible, provided it has a non-trivial continuum limit. It is crucial, on the basis of non-perturbative methods and renormalization group invariance, to find: (1) the scaling invariant regimes (ultraviolet fix points) $g_c$, where phase transition takes place and physical correlation length $\xi$ is much larger than the Planck length $a$; (2) $\beta$-function $\beta(g)$ and renormalization-group invariant equation $\xi = \text{const.} \, a \exp \int g \, dg'/\beta(g')$; (3) all relevant and renormalizable operators (one-particle irreducible (1PI) functions) with effective dimension-4 in these regimes to obtain effective low-energy theories. One may add by hand the cosmological $\Lambda$-term $\frac{\Lambda}{4!} \epsilon^{\mu\nu\rho\sigma} \sum_x \text{tr}[e_\mu e_\nu e_\rho e_\sigma] + \text{h.c.}$, where $\lambda = \Lambda a^2$, into the regularized EC action (27). However, 1PI functions $A_{EC}^{\text{eff}}$ (30) effectively contain this dimensional operator, which is related to the truncated Green function $\langle A_{EC} A_{EC} \rangle$. It is then a question what is the scaling property of this operator in terms of $\xi^{-2}$, where inverse correlation length $\xi^{-1}$ gives the mass scale of low-energy excitations of the theory.

One can consider the following regularized fermion action,
\[
A_F(e_\mu, U_\mu^i, \psi) = \frac{1}{2} \sum_{x, \mu} \left[ \bar{\psi}(x) e^\mu(x) U_\mu(x) \psi(x + a_\mu) - \bar{\psi}(x + a_\mu) U_\mu^\dagger(x) e^\mu(x) \psi(x) \right], \tag{45}
\]
where fermion fields $\psi(x)$ and $\psi(x + a_\mu)$ are defined at two neighboring points (vertexes) of 4-simplices complex, fields $U_\mu(x)$ and $e_\mu(x)$ are added to preserve local gauge and diffeomorphism invariances, and $\sum_{x, \mu}$ is the sum over all edges (1-simplices) of 4-simplices complex. This bilinear fermion action (45) introduces a non-vanishing torsion field [16, 17]. We need to study whether the regularized EC action (27) with fermion action (45) can be effectively written in form of a torsion-free part and four fermion interactions, as the EC theory in continuum. In addition, the bilinear
fermion action \cite{15} has the problem of either fermion doubling or chiral (parity) gauge symmetry breaking, due to the No-Go theorem \cite{18}. Resultant four fermion interactions can possibly be resolution to this problem \cite{19,20}. Acknowledgment: Author thanks to anonymous referee for his/her comments, to H. Kleinert and J. Maldacena for discussions on invariant holonomies in gauge theories. Author is grateful to H. W. Hamber and R. M. Williams for discussions on renormalization group invariance and the properties of Dirac-matrix valued tetrad fields.

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We are not clear now how to relate configurations of fields $e_\mu(x)$ and $\omega_\mu(x)$ to topological constrained configurations of 4-simplices complex in dynamical triangulations.

We recall the “Planck lattice”, G. Preparata and S.-S. Xue, Phys. Lett. B264, (1991) 35; and Ref. [16], a discretized space-time with minimal spacing of the Planck length due to quantum gravity.