Forcing Parameters in Fully Connected Cubic Networks

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Abstract: Domination in graphs has been extensively studied and adopted in many real life applications. The monitoring electrical power system is a variant of a domination problem called power domination problem. Another variant is the zero forcing problem. Determining minimum cardinality of a power dominating set and zero forcing set in a graph are the power domination problem and zero forcing problem, respectively. Both problems are \(NP\)-complete. In this paper, we compute the power domination number and the zero forcing number for fully connected cubic networks.

Keywords: dominating set; power dominating set; zero forcing set; electrical power network; fully connected cubic networks

MSC: 05C69

1. Introduction

Monitoring electrical power systems by placing as few Phase Measurement Units (PMUs) at selected regions in the system is modeled as a graph theoretic problem. The cost of such a synchronized devise is very high, and hence it is required to fetch the smallest set of devices while maintaining the ability to supervise the entire system. In 2002, Hayens et al. [1] considered this problem as the power domination problem in graphs, which is a variation of the domination problem. An electric power network is designed by a graph where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes. In 2012, Paul Dorbec et al. [2] presented the idea of a \(k\)-power domination problem, which is a generalization of power domination problem in graphs.

A graph \(G = (V, E)\) is defined as a nonempty set of vertices \(V = V(G)\) together with a set of edges \(E = E(G)\) joining certain pairs of vertices. Vertices \(u\) and \(v\) are said to be adjacent if \(u\) and \(v\) are the end vertices of an edge in \(G\). For \(u \in V\), the set of all vertices adjacent to \(u\) are said to be in the neighbors of \(u\) and is denoted by \(N(u)\). Then, the closed neighborhood of \(u\) is defined as \(N[u] = N(u) \cup \{u\}\).

A subset \(S \subseteq V\) in a graph is said to be a domination set \(S\) if every vertex in \(V\) is either in \(S\) or is adjacent to some vertices in \(S\) [1]. In 2002, Haynes et al. introduced power domination by formulating propagation rules in terms of vertices and edges in a graph.

Let \(G\) be a graph with a vertex set \(V\). Let \(K \subseteq V\). Vertices \(M^i(K)\) monitored by \(K\) at level \(i\), \(i \geq 0\), inductively are as follows:

1. \(M^0(K) = N[K]\).
2. \(M^{i+1}(K) = M^i(K) \cup \{v \in V : \exists u \in M^i(K), N(u) \cap (V(G) \setminus M^i(K)) = \{v\}\} \).
At some stage, if \( K \) monitors the entire vertex set \( V \), we say that \( K \) is a power dominating set of \( G \). The minimum cardinality of power dominating set of \( G \) is called the power domination number of \( G \) and is denoted by \( \gamma_p(G) \) [3].

The zero forcing process can be treated as a coloring process on the vertices of the graph. If vertex \( x \) is colored red and exactly one neighbor \( y \) of \( x \) is green, then change the color of \( y \) to red, and we say that \( x \) forces \( y \). A zero forcing set for \( G \) is a subset of vertices \( H \) such that if initially the vertices in \( H \) are colored red and the remaining vertices are colored green, then repeated application of the above process can color all vertices of \( G \) red. The cardinality of a minimum zero forcing set of \( G \) is represented by \( \zeta(G) \) [3]. It is customary to address ‘red vertices’ as monitored vertices and the ‘green vertices’ as unmonitored vertices.

The power domination problem is \( NP \)-complete [1]. Bounds on the power domination number for any graph \( G \) were obtained in [1]. The power domination problem studied for trees [1] and grids [4], split graphs [5]. Barrera et al. [6] studied power domination in Petersen graphs. Cockayne et al. [7] presented new concepts of domination in graphs. Chen et al. [8] investigated networks with complete connection. Dorbec et al. [9] defined power domination in product graphs. Ho et al. [10] given Hamiltonian connectivity in fully connected cubic networks. Kosari et al. [11,12] studied double roman domination and total domination in graphs. Xu et al. [13] introduced power domination in block graphs. Zhao et al. [14] presented new results of power domination in graph theory.

2. Fully Connected Cubic Networks (FCCNs)

FCCNs are networks that provide excellent expandability. Optical and electronic technologies can be utilized in FCCN to build a new hybrid computer architecture.

Let \( a^n = aa\ldots a(n \text{ times}) \). FCCN of level \( r, r \geq 1 \) denoted by \( FCCN_r \), is defined recursively as follows [15]:

1. \( FCCN_1 \) has the set of integers modulo 8 as the node set and all ordered pairs \( x, y \), \( x < y \), \( x, y \in \{0, 1, \ldots, 7\} \) as the edge set.
2. When \( r \geq 2 \), \( FCCN_r \) is built from eight node-disjoint copies of \( FCCN_{r-1} \) by adding 28 edges. Specifically, if, for \( 0 \leq k \leq 7 \), we let \( kFCCN_{r-1} \) denote a copy of \( FCCN_{r-1} \) with each node being prefixed with \( k \), then \( FCCN_r \) is defined by:
   \[
   V(FCCN_r) = \bigcup_{k=0}^{7} V(kFCCN_{r-1}),
   \]
   \[
   E(FCCN_r) = \bigcup_{k=0}^{7} E(kFCCN_{r-1}) \cup \{(pq^{r-1}, qp^{r-1})|0 \leq p < q \leq 7\}
   \]
   For \( 0 \leq k \leq 7 \), \( kFCCN_{r-1} \) is named an \((r-1)\)-level sub-FCCN of \( FCCN_r \), or simply a sub-FCCN of \( FCCN_r \), if there is no uncertainty.
3. Given an \( FCCN_r \), \( r \geq 2 \), a boundary node is a node of the form \( kp \). An ICE is an edge of the form \((pq^{n-1}, qp^{n-1})\). Each ICV of \( FCCN \) is of degree 4 except the boundary nodes of degree 3. Obviously, \( kFCCN_{r-1} \) has seven ICV and one boundary node for \( 0 \leq k \leq 7 \) and \( r \geq 2 \) [15] (see Figure 1).

**Note:** In what follows, we denote \( FCCN_r \) by \( H_r \), \( r \geq 1 \).

**Remark 1.** \( H_r \) has \( 8^r \) nodes, eight of which degree 3 and the rest \( 8^r - 8 \) nodes of degree 4. \( H_r \) has \( 2^{3r+1} - 4 \) edges. Its diameter is \( 2^{(r+1)} - 1 \). For \( 2 \leq i \leq r \), each level \( i \) contains \( 8^{i-1} \) node disjoint copies of \( H_i \). \( 1 \leq j \leq i \). This implies that there are \( 8^{r-2} \) node disjoint copies of \( H_2 \) in \( H_r \).
3. Main Results

In this section, we compute PDN and ZFN for \( H_r, \ r \geq 2 \).

3.1. Power Domination in FCCNs

We obtain lower bounds for the PDN of \( H_r, \ r \geq 2 \), and prove that the bounds are sharp.

**Lemma 1.** The power domination number of \( H_2 \) is at least 4. In other words \( \gamma_p(H_2) \geq 4 \).

**Proof.** \( H_2 \) contains eight node disjoint copies of \( H_1 \), say \( 0H_1, 1H_1, \ldots, 7H_1 \). Let \( S \) be a PDS of \( G \). We claim that \( |S| \geq 4 \). Suppose \( S \subseteq 0H_1 \), any node in \( iH_1, \ 1 \leq i \leq 7 \) is neighbor to at most one node of \( 0H_1 \). This implies that every node in \( iH_1, \ 1 \leq i \leq 7 \) is neighbor to at least three unmonitored nodes, a contradiction. Therefore, \( S \not\subseteq 0H_1 \) (see Figure 2a). Suppose \( S \subseteq 0H_1 \cup 1H_1 \). Then, \( S \) can monitor at most two nodes of \( iH_1, \ 2 \leq i \leq 7 \), each of which is neighbor to at least two unmonitored nodes, a contradiction. Therefore, \( S \not\subseteq 0H_1 \cup 1H_1 \) (see Figure 2b).

Suppose \( S \subseteq 0H_1 \cup 1H_1 \cup 2H_1 \). Then, \( S \) can monitor at most three nodes of each \( iH_1, \ 3 \leq i \leq 7 \). No three nodes in \( iH_1 \) induce an independent set. Hence, these three nodes induce an edge and an isolated node. Then, each end node of the edge is neighbor to two unmonitored nodes, and the independent node is neighbor to three unmonitored nodes, a contradiction (see Figure 2c). On the other hand, if three nodes induce a path say, \( uvw \), then \( u \) and \( w \) are neighbor to two unmonitored nodes each and \( v \) is neighbor to exactly one unmonitored node, which in turn is neighbor to at least two unmonitored nodes, a contradiction (see Figure 2d). Therefore, \( S \not\subseteq 0H_1 \cup 1H_1 \cup 2H_1 \). Hence, nodes in \( S \) are in at least 4 copies of \( H_1 \). Therefore, \( |S| \geq 4. \)
Lemma 2. Let S be a PDS of H3 and H be a subgraph of H3 isomorphic to H2. Then, |V(H) ∩ S| ≥ 4.

Proof. H3 is composed of eight copies of H2, denoted by 0H, 1H, . . . , 7H. Further, each iH, 1 ≤ i ≤ 7 contains 8 cubes as subgraphs each denoted by Q3. Consider 0H. Let 0H = \bigcup_{k=0}^{7} kQ^3. The worst case arises when all the 7 intercubic nodes 0ii, 1 ≤ i ≤ 7, of 0H are monitored by nodes of S not in 0H. Suppose |V(0H) ∩ S| = 3. Let V(0H) ∩ S = \{a, β, γ\}. Without loss of generality, let a, β, γ be in 0Q^3. Suppose a = 00i, β = 00j and γ = 00k, i ≠ j ≠ k, 1 ≤ i, j, k < 7. Let (a, a'), (β, β') and (γ, γ') be the intercubic edges in 0H with a' = 0ii, β' = 0jj and γ' = 0kk in iQ^3, jQ^3 and kQ^3, respectively, 1 ≤ i, j, k < 7. Even if 0ii, 0ii are neighbors, each of 0ii and 0ii is neighbor to two distinct unmonitored nodes in iQ^3, a contradiction (see Figure 3a).

Suppose a, β are in 0Q^3 and γ is in 1Q^3. Assume that a = 00i, β = 00j and γ = 01k, i ≠ j ≠ k, 2 ≤ i, j, k ≤ 7. Let (a, a'), (β, β') and (γ, γ') be the intercubic edges in 0H with a' = 0ii, β' = 0jj and γ' = 0kk in iQ^3, jQ^3 and kQ^3, respectively, 2 ≤ i, j, k ≤ 7. Even if 0jj, 0jj and 0jk induce a path, then 0jj and 0jk are neighbors to two

Figure 2. Circled vertices indicate in (a–d) be a PDS of H2.
unmonitored nodes each and 0/j0 is neighbor to exactly one unmonitored node, which in turn is adjacent to at least two unmonitored nodes, a contradiction (see Figure 3b).

Suppose $\alpha \in 0Q^3$, $\beta \in 1Q^3$ and $\gamma \in 2Q^3$. Let $\alpha = 00i$, $\beta = 01j$ and $\gamma = 02k$, $i \neq j \neq k$, $3 \leq i, j, k \leq 7$. Let $(\alpha, \alpha')$, $(\beta, \beta')$ and $(\gamma, \gamma')$ be the intercubic edges in $0H$ with $\alpha' = 0i0$, $\beta' = 0j1$ and $\gamma' = 0k2$ in $iQ^3$, $jQ^3$ and $kQ^3$, respectively, $3 \leq i, j, k \leq 7$. Even if 0k0, 0kk and 0ik induce a path, the nodes 0ii and 0ik are neighbor to two unmonitored nodes each and 0k0 is neighbor to exactly one unmonitored node, which in turn is neighbor to at least two unmonitored nodes, a contradiction. Therefore, $|V(H) \cap S| \geq 4$ (see Figure 3c). □

![Figure 3. Circled vertices indicates in (a–c) be a PDS of 0H2 of H3.](image)

**Lemma 3.** The power domination number of $H_r$, $r \geq 3$ is at least $2^{3r-4}$. In other words, $\gamma_p(H_r) \geq 2^{3r-4}$. 


Theorem 1. The power domination number of $H_r$, $r \geq 2$, is $2^{3r-4}$. In other words $\gamma_p(H_r) = 2^{3r-4}$. 

Proof. We prove the result by induction on $r$. We consider the case when $r = 3$. $H_3$ contains eight node disjoint copies of $H_2$, say $0H_2, 1H_2, \ldots, 7H_2$. Let $S$ be a PDS of $H_r$. We claim that $|S| \geq 32$. By Lemma 1 and Lemma 2, there are at least 4 nodes in each copy of $H_2$ to monitor $H_3$. Hence, $|S| \geq 4 \times 8$. Therefore, $\gamma_p(H_3) \geq 32$.

Assume the result is true for $r = k$, $r \geq 3$. That is, $\gamma_p(H_k) \geq 2^{3k-4}$. Consider the case when $r = k + 1$. Let $S$ be a PDS $H_{k+1}$. We have to prove that $\gamma_p(H_{k+1}) \geq 2^{3k-1}$.

Suppose not, let $|S| < 2^{3k-1}$. In $H_{k+1}$, there are $8^{k-1}$ node disjoint copies of $H_2$. With the deletion of one node from $S$ in a copy of $H$, there is at least one node say, $u \in H_1$ monitored by intercubic edges. With this monitored node $u$, we claim that 3 nodes in $H_2$, one each in 3 copies of $H_i$ say $1H_1$, $2H_1$, $3H_1$, are not sufficient to monitor all nodes in any of $H_i$, $i = 1, 2, 3$. In the worst case, suppose all the nodes in the copy of $H_1$ containing $u$ are already monitored, then, the saturated node in $1H_1$ has two unmonitored nodes neighbor to it, a contradiction. Thus, $|S| \geq 2^{3k-1}$. Therefore, $\gamma_p(H_k) \geq 2^{3k-4}$. $\square$

Radix-lexicographic ordering:

Name the nodes of $H_r$, $r \geq 1$ as follows:

(i) Name the nodes of $H_1$ as 1 digit radix $Z_7$ number, say 0, 1, $\ldots$, 7 by lexicographic order.

(ii) Name the nodes of $H_2$, as 2 digit radix $Z_7$ number, 0$H_1$, 1$H_1$, $\ldots$, 7$H_1$.

(iii) Inductively, name the nodes of $H_r$, as $r$ digit radix $Z_7$ number, $kH_{r-1}$, $0 \leq k \leq 7$ (see Figure 1).

The following is the Algorithm 1.

Algorithm 1 PD Algorithm

Input: $H_r$, $r \geq 2$, with radix-lexicographic ordering.

Algorithm: (i) Select $S_2 = \{01, 13, 20, 32\}$ in $H_2$ and let $S_3 = \bigcup_{k=0}^{7} kS_2$ in $H_3$.

(ii) Inductively select $S_r = \bigcup_{k=0}^{7} kS_{r-1}$ in $H_r$.

Output: $\gamma_p(H_r) \leq 2^{3r-4}$, $r \geq 2$.

Proof of Correctness. Let $S_r$ be a PDS of $H_r$. Consider $S_r = \bigcup_{k=0}^{7} k^{r-2}S'$ where $S' = \{01, 13, 20, 32, 00, 10, 03, 05, 02, 21, 22, 24, 12, 17, 11, 31, 33, 30, 36, 32\}$, $k^{r-2} = kk \ldots k(r - 2)$ times. Then, node in $M^0(S_r)$ say, $\bigcup_{k=0}^{7} k^{r-2}s''$ where $s'' = \{00, 02, 03, 10, 11, 12, 21, 22, 23, 30, 31, 33\}$ is neighbor to exactly one unmonitored node say, $\bigcup_{k=0}^{7} k^{r-2}s'''$ where $s''' = \{04, 06, 07, 14, 15, 16, 25, 26, 27, 34, 35, 37\}$. Now $M^1(S_r) = M^0(S_r) \cup \bigcup_{k=0}^{7} k^{r-2}s'''$. Then, for every node $v \in M^1(S_r)$, $|N[v] \setminus M^1(S_r)| = 1$. Now $M^2(S_r) = M^1(S_r) \cup \bigcup_{k=0}^{7} k^{r-2}ij$, $0 \leq i \leq 3$, $4 \leq j \leq 7$. Similarly, $M^3(S_r) = M^2(S_r) \cup \bigcup_{k=0}^{7} k^{r-2}ij$, $4 \leq i, j \leq 7$. Thus, $M^3(S_r) = V(H_r)$. Inductively, we arrive at $\gamma_p(H_r) \leq 2^{3r-4}$. $\square$

PD Algorithm (Algorithm 1) together with Lemma 3 imply the following theorem.
3.2. Zero Forcing in FCCNs

The PD process on a graph $G$ is choosing a set $S \subseteq V(G)$ and applying the ZF process to the closed neighborhood $N[S]$ of $S$. The set $S$ is a PDS of $G$ if and only if $N[S]$ is a ZFS for $G$.

The following theorem was proved in 2015 by Ferrero et al. [3], which shows the relationship between ZFS and PDS.

**Theorem 2 ([3])**. Let $G$ be a graph with no isolated vertices, and let $S = \{u_1, u_2, \ldots, u_t\}$ be a PDS for $G$. Then $\xi(G) \leq \sum_{i=1}^{t} \deg(u_i)$.

**Theorem 3 ([3])**. Let $G$ be a graph. Then, $\left[\frac{\xi(G)}{\gamma_p(G)}\right] \leq \gamma_p(G)$ and this bound is tight.

In what follows, we obtain a sharp lower bound for the zero forcing number of FCCNs.

**Lemma 4.** The zero forcing number of $H_2$ is at least 16. In other words, $\xi(H_2) \geq 16$.

**Proof.** $H_2$ contains eight node disjoint copies of $H_1$, say $0H_1, 1H_1, \ldots, 7H_1$. Let $S$ be a ZFS of $G$. We claim that $|S| \geq 16$. Suppose $S \subseteq 0H_1 \cup 1H_1$. Then, $S$ can monitor at most two nodes of $iH_1$, $2 \leq i \leq 7$, each of which is neighbor to at least 2 unmonitored nodes, a contradiction. Therefore, $S \not\subseteq 0H_1 \cup 1H_1$ (see Figure 2b).

Suppose $S \subseteq 0H_1 \cup 1H_1 \cup 2H_1$. Then, $S$ can monitor at most three nodes of each $iH_1$, $3 \leq i \leq 7$. No three nodes in $iH_1$ induce an independent set. Hence, these three nodes induce an edge and an isolated node. Each end node of the edge is neighbor to 2 unmonitored nodes and the independent vertex is neighbor to three unmonitored nodes, a contradiction (see Figure 2c). On the other hand, if three nodes induce a path, say $uvw$, then $u$ and $w$ are neighbor to 2 unmonitored nodes each and $v$ is neighbor to exactly one node, which in turn is neighbor to at least 2 unmonitored nodes, a contradiction (see Figure 2d). Therefore, $S \not\subseteq 0H_1 \cup 1H_1 \cup 2H_1$.

Suppose $S \subseteq 0H_1 \cup A \cup B$, where $A = \{11, 13, 15, 17\}$ and $B = \{31, 32, 33, 36\}$. $A$ induces four cycles and $B$ induces two independent edges. Even if all nodes of $0H_1$ and $3H_1$ are monitored, $S$ can color at most two independent nodes in $2H_1$, $4H_1$, $5H_1$, $6H_1$, $7H_1$ as red, and each node labeled as $10, 12, 16$ in $1H_1$ is neighbor to 2 unmonitored nodes, a contradiction. Therefore, $S \not\subseteq 0H_1 \cup A \cup B$ (see Figure 4a).

Suppose $S \subseteq A \cup B \cup C \cup D$, where $A = \{00, 01, 02, 03\}$, $B = \{10, 11, 12, 13\}$, $C = \{20, 21, 23, 24\}$, $D = \{34, 35, 36, 37\}$. $A, B, C, D$ induces a four cycle. Then, $S$ can induce a path say, $uvw$ in $iH_1$, $i = 0, 1, 2, 4$. Then, the end nodes of a path say, $u$ and $w$ are neighbor to 2 unmonitored nodes each and $v$ is neighbor to exactly one unmonitored node which in turn is neighbor to at least 2 unmonitored nodes, a contradiction. This implies that, $S \not\subseteq A \cup B \cup C \cup D$. Therefore, $|S| \geq 16$ (see Figure 4b). Hence, nodes in $S$ are in consecutive 4 cycle of at least 4 consecutive copies of $H_1$. \[\]

**Lemma 5.** Let $S$ be a zero forcing set of $H_3$. Let $H$ be a subgraph of $H_3$ isomorphic to $H_2$. Then, $|V(H) \cap S| \geq 16$.

**Proof.** $H_3$ is composed of eight copies of $H_2$, denoted by $0H, 1H, \ldots, 7H$. Taking into account the symmetric nature of $H_3$, without loss of generality consider $0H$. Let $0H = \bigcup_{k=0}^{7} kQ^3$. The worst case arises when all the 7 intercubic nodes $0ii$, $1 \leq i \leq 7$, of $0H$ are monitored nodes of $S$ not in $0H$. Suppose $|V(0H) \cap S| = 15$.

Let $V(0H) \cap S = A \cup B \cup C \cup D \cup E \cup F \cup G \cup I$, where $A = \{000, 001\}$, $B = \{002, 006\}$, $C = \{003, 007\}$, $D = \{010, 011\}$, $E = \{013, 017\}$, $F = \{021, 023\}$, $G = \{031, 033\}$, $H = \{032\}$, each $A, B, C, D, E, F, G, I$ induces an edge and an independent set. Even if all nodes of $0H_1$, $1H_1$ and $3H_1$ are monitored and $S$ induces a path, say $uvw$ in $iH_1$, $4 \leq i \leq 7$ or $2H_1$, then $u$ and $w$ are neighbor to 2 unmonitored nodes each and $v$ is neighbor to
when \( r \) ends in a path say, \(uvw \) end nodes of a path say, \( S \), and \( |S| \geq 16 \) (see Figure 5a).

Let \( V(0H) \cap S = A \cup B \cup C \cup D \cup E \), where \( A = \{000,001,002\} \), \( B = \{010,012,016\} \), \( C = \{011,013,017\} \), \( D = \{020,022,026\} \), \( E = \{021,023,027\} \), each \( A, B, C, D, E \) induce a path say, \( vw \). Even if all nodes of \( 1H_1 \) and \( 2H_1 \) are monitored and suppose \( S \) induces an independent set in \( iH_1 \), \( 3 \leq i \leq 7 \). Then, the independent node is neighbor to 3 unmonitored nodes, which in turn is neighbor to at least 2 unmonitored nodes, a contradiction. Therefore, \( |S| \geq 16 \) (see Figure 5b).

Let \( V(0H) \cap S = A \cup B \cup C \cup D \), where \( A = \{000,001,002,003\} \), \( B = \{010,011,012,013\} \), \( C = \{020,021,022,023\} \), \( D = \{030,032,033\} \). Even if, all nodes of \( 0H_1 \) and \( 1H_1 \) are monitored and \( S \) induces an edge in \( iH_1 \), \( 2 \leq i \leq 7 \). Then, the node labeled as 23 in \( 2H_1 \in S \), and end nodes of a path say, \( uvw \) in \( 3H_1 \in S \) and each end node of an edge in \( iH_1 \), \( 4 \leq i \leq 7 \) is neighbor to at least 2 unmonitored nodes, a contradiction. Therefore, \( |S| \geq 16 \) (see Figure 5c).

![Figure 4.](image)

Figure 4. Circled vertices indicates in (a,b) be a zero forcing sets of \( H_2 \).

**Lemma 6.** The zero forcing number of \( H_r \), \( r \geq 3 \) is at least \( 2^{3r-2} \). In other words, \( \zeta(H_r) \geq 2^{3r-2} \).

**Proof.** We prove the result by induction on \( r \). We consider the case when \( r = 3 \). \( H_3 \) contains eight node disjoint copies of \( H_2 \), say \( 0H_2, 1H_2, \ldots, 7H_2 \). Let \( S \) be a ZFS of \( H_r \). We claim that \( |S| \geq 128 \). By Lemmas 4 and 5, there are at least 16 nodes in each copy of \( H_2 \) to monitor all nodes of \( H_3 \). Hence, \( |S| \geq 16 \times 8 \). Therefore, \( \zeta(H_3) \geq 128 \).

Assume the result is true for \( r = k \), \( r \geq 3 \). That is, \( \zeta(H_k) \geq 2^{3k-2} \). Consider the case when \( r = k + 1 \). Let \( S \) be a ZFS of \( H_{k+1} \). We have to prove that \( \zeta(H_{k+1}) \geq 2^{3k+1} \). Suppose not, let \( |S| < 2^{3k+1} \). In \( H_{k+1} \), there are \( 8^{k-1} \) node disjoint copies of \( H_2 \). With the deletion of one node from \( S \) in a copy of \( H_2 \), there is at least one node, say \( u \) in \( H_1 \) monitored by intercubic edges. With this monitored node \( u \), we claim that 15 nodes in \( H_2 \), four each in 3 copies of \( H_1 \), say \( 1H_1, 2H_1, 3H_1 \), and 3 in \( 4H_1 \) are not sufficient to monitor all nodes in any of \( iH_1 \), \( i = 1, 2, 3, 4 \). In the worst case, suppose all the nodes in the copy of \( H_1 \) containing \( u \) are all already monitored, then the monitored node in \( 1H_1 \) has 2 unmonitored nodes adjacent to it, a contradiction. Thus, \( |S| \geq 2^{3k+1} \). Therefore, \( \zeta(H_k) \geq 2^{3k-2} \). □

The following is the Algorithm 2.
Algorithm 2 ZF Algorithm

Input: \( H_r, r \geq 2 \), with radix-lexicographic ordering.

Algorithm: (i) Select \( S_2 = \{i0,i1,i2,i3\}, 0 \leq i \leq 3 \) in \( H_2 \) and let \( S_3 = \bigcup_{k=0}^{7}kS_2 \) in \( H_3 \). See Figure 5d.

(ii) Inductively select \( S_r = \bigcup_{k=0}^{7}kS_{r-1} \) in \( H_r \).

Output: \( \zeta(G) \leq 2^{3r-2} \).

Figure 5. (a–c) \( H_3 \) of 0H2 (d) Zero forcing sets of \( H_2 \).

Proof of Correctness. Let \( S_r \) be a ZF of \( H_r \). Let \( S_r = \bigcup_{k=0}^{7}k^{-2}S' \), where \( S' = \{i0,i1,i2,i3\}, 0 \leq i \leq 3, k^{-2} = kk...k(r - 2) \) times. Every node \( v \in S_r \) is neighbor to exactly one unmonitored node which in turn forces exactly one unmonitored node say, \( N(S_r) = \bigcup_{k=0}^{7}k^{-2}S'' \) where \( S'' = \{i4,i5,i6,i7\}, 0 \leq i \leq 3 \). Then, every node \( v \in N(S_r) \) is neighbor...
to exactly one unmonitored node, which in turn forces exactly one unmonitored node 
say, \( A = \bigcup_{k=0}^{7} k^{r-2}S'' \) where \( S'' = \{i0, i1, i2, i3\}, \ 4 \leq i \leq 7 \). In the next step, each node 
in \( A \) is neighbor to exactly one green node, which in turn forces exactly one green node, 
say \( B = \bigcup_{k=0}^{7} k^{r-2}S''' \) where \( S''' = \{i4, i5, i6, i7\}, \ 4 \leq i \leq 7 \). Now, \( S_r = \bigcup_{k=0}^{7} k^{r-2}S' \), where 
\( S' = \{i0, i1, i2, i3\}, \ 0 \leq i \leq 3, k^{r-2} = kk \ldots (r-2) \) times is a ZFS of \( H_r, r \geq 2 \). This 
implies that \( \zeta(H_r) \leq 2^{3r-2} \). \[\square\]

ZF Algorithm (Algorithm 2) together with Lemma 6 imply the following theorem.

**Theorem 4.** The zero forcing number of \( H_r, r \geq 2 \), is \( 2^{3r-2} \). In other words, \( \zeta(H_r) = 2^{3r-2} \).

4. Conclusions

In this paper, we have obtained the PDN and ZFN for the fully connected cubic 
networks \( H_r, r \geq 2 \), identifying classes of graphs for which \( \left\lceil \frac{\zeta(G)}{\delta(G)} \right\rceil = \gamma_p(G) \) is an open 
problem. Another interesting line of research would be to determine the zero forcing 
number of networks such as hypercubes and circulant networks.

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**Abbreviations**

The following abbreviations are used in this manuscript:

| Abbreviation | Description                      |
|--------------|----------------------------------|
| PD           | Power Domination                 |
| PDS          | Power Dominating Set             |
| PDN          | Power Domination Number          |
| ZF           | Zero Forcing                     |
| ZFS          | Zero Forcing Set                 |
| ZFN          | Zero Forcing Number              |
| FCCN         | Fully Connected Cubic Networks   |
| ICV          | Intercubic Vertices              |
| ICE          | Intercubic Edges                 |

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