Free gravitational field in the metric gravity as a Hamilton system

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Abstract

The closed system of Hamilton equations is derived for all components of the free gravitational field and corresponding momenta in the metric General Relativity. The Hamilton-Jacobi equation for the free gravitational field is also derived and discussed. In general, all methods and procedures based on the Hamilton and Hamilton-Jacobi approaches are very effective in actual applications to many problems known in the metric GR. In particular, these methods are more effective than traditional methods based on the direct solution of the Einstein equation.

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I. INTRODUCTION

In this communication we derive the both Hamiltonian and Hamiltonian-Jacobi equations of the free gravitational field in the metric General Relativity (or GR, for short). In our earlier studies [1] and [2] we have developed the two non-contradictory Hamilton approaches for the metric GR. Also, in [1] and [2] we derived the explicit formulas for the total $H_t$ and canonical $H_C$ Hamiltonians in the both approaches. Furthermore, it was shown in [2] that these two Hamiltonian-based formulations of metric GR are related to each other by some canonical transformation of all dynamical variables, i.e., the corresponding ‘momenta’ and ‘coordinates’. Such canonical transformations are allowed transformations in any Hamiltonian-based approach and they are often used to simplify analysis of various Hamiltonian systems, including dynamical systems combined from different fields and particles. Theory developed below is substantially based on the procedure proposed by Dirac [3] - [5] and results obtained in our earlier studies [1] and [2] with the use of this Dirac’s procedure. A number of fundamental facts known for the general Hamilton-Jacobi theory [6] - [8] are extensively used below. Since the volume of this contribution is limited, below we restrict ourselves to the derivation of basic equations only.

First of all, we need to introduce a few principal notations. Everywhere in this study the notations $g_{\alpha\beta}$ stand for the covariant components of the metric tensor which are dimensionless functions. Analogous notations $\pi^{\alpha\beta}$ designate the corresponding contravariant components of momenta conjugate to the $g_{\alpha\beta}$ components (for more detail, see [1] and [2]). The determinant of the metric $g_{\alpha\beta}$ tensor is denoted by its traditional notation $-g(>0)$. The Latin alphabet is used for spatial components, while the index 0 means the temporal component (or time-component). Everywhere below the notation $d$ designates the dimension of the space-time manifold ($d \geq 3$). This means that an arbitrary Greek index $\alpha$ varies between 0 and $d$, while an arbitrary Latin index varies between 1 and $d$. The tensors such as $B^{\ell(\alpha\beta\gamma\mu\nu\lambda)}$, $I_{mnpq}$, etc, have been defined in earlier papers [1], [2] and [5]. In this study the definitions of all these tensors are exactly the same as in [1] and [2] and here we do not want to repeat it. The short notations $g_{\alpha\beta,k}$ and $g_{\gamma\rho,0}$ are used below for the spatial and temporal derivatives of the corresponding components of the metric tensor. Any ‘tensor’ expression containing a pair of the same (or repeated) indexes, where one index is covariant and another is contravariant, means summation over this ‘dummy’ index. This convention
is very convenient and drastically simplifies the explicit formulas derived in the metric GR.

Now, we can introduce the Lagrangian of the metric general relativity. The original Einstein-Hilbert Lagrangian (see, e.g., [9]), which was used in the first papers on the metric GR, cannot be applied for the purposes of this study, since it contains the second-order derivative(s). However, the same Einstein-Hilbert Lagrangian can be transformed into the $\Gamma - \Gamma$ Lagrangian which is reduced to the following general form

$$L = \frac{1}{4\sqrt{-g}} B^{\alpha\beta\gamma\mu\nu\rho} \left( \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right) \left( \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right) = \frac{1}{4\sqrt{-g}} B^{\alpha\beta\gamma\mu\nu\rho} g_{\alpha\beta,\gamma} g_{\mu\nu,\rho}$$ (1)

$B^{\alpha\beta\gamma\mu\nu\rho} = g^{\alpha\beta} g^{\gamma\rho} g_{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} + 2g^{\alpha\rho} g^{\beta\nu} g^{\gamma\mu} - 2g^{\alpha\beta} g^{\gamma\mu} g^{\nu\rho}$ is a homogeneous cubic function of the contravariant components of the metric tensor $g^{\alpha\beta}$. In the right-hand side of the formula, Eq.[1], we have used the short notation $g_{\alpha\beta,\gamma}$ to designate the partial derivatives $\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$. Note that the $\Gamma - \Gamma$ Lagrangian $L$, Eq.[1], contains the partial temporal derivatives $g_{0\sigma,0}$ of the first-order only, and it is used below to derive the actual Hamiltonian of metric GR. In fact, to derive the closed formula for the Hamiltonian of metric GR we need a slightly different form of the $\Gamma - \Gamma$ Lagrangian which explicitly contains all temporal derivatives (or time-derivatives) is (see, e.g., [1])

$$L = \frac{1}{4\sqrt{-g}} B^{\alpha\beta\gamma\mu\nu\rho} g_{\alpha\beta,0} g_{\mu\nu,0} + \frac{1}{2\sqrt{-g}} B^{(\alpha\beta,0|\mu\nu\rho)} g_{\alpha\beta,0} g_{\mu\nu,k} + \frac{1}{4\sqrt{-g}} B^{\alpha\beta\gamma\mu\nu\rho\lambda} g_{\alpha\beta,k} g_{\mu\nu,l}$$ (2)

where the notation $B^{(\alpha\beta,\gamma|\mu\nu\rho)}$ means a tensor symmetrized in respect to the permutation of the two groups of indexes, i.e.,

$$B^{(\alpha\beta,\gamma|\mu\nu\rho)} = \frac{1}{2} \left( B^{\alpha\beta\gamma\mu\nu\rho} + B^{\mu\nu\rho\alpha\beta\gamma} \right)$$ (3)

II. HAMILTONIAN(S) OF THE METRIC GENERAL RELATIVITY

The knowledge of the Lagrangian $L$, represented above in the form, Eq.[2], allows us to derive the closed formula for the Hamiltonian of the metric GR. The total Hamiltonian $H_t$ of the free gravitational field in metric GR derived in [1] is written in our notations in the form

$$H_t = \pi^{\alpha\beta} g_{\alpha\beta,0} - L = H_C + g_{0\sigma,0} \phi^{0\sigma}$$ (4)

where $L$ is the $\Gamma - \Gamma$ Lagrangian of the metric GR, $\phi^{0\sigma}$ are the primary constraints $\phi^{0\sigma} = \pi^{0\sigma} - \frac{1}{2\sqrt{-g}} B^{((0\sigma),0|\mu\nu\rho)} g_{\mu\nu,k}$, while $g_{0\sigma,0}$ are the corresponding $\sigma-$velocities' and $H_C$ is the
canonical Hamiltonian of the metric GR

\[
H_C = \frac{1}{\sqrt{-g}g^{00}}I_{mn}^{pq} \pi_{m}^{n} \pi_{p}^{q} - \frac{1}{g^{00}}I_{mn}^{pq} \pi_{m}^{n} B^{(pq)0|\mu\nu k} g_{\mu\nu,k} \\
+ \frac{1}{4} \sqrt{-g} \left[ \frac{1}{g^{00}}I_{mn}^{pq} B^{((mn)0|\mu\nu k)} B^{(pq)0|\alpha\beta l} - B^{\mu\nu k|\alpha\beta l} \right] g_{\mu\nu,k} g_{\alpha\beta,l}
\]

which does not contain any primary constraint \( \phi^{0\sigma} \). The primary constraints arise during transition from the \( \Gamma - \Gamma \) Lagrangian \( \mathcal{L} \) to the Hamiltonians \( H_t \) and \( H_C \). The \( \Gamma - \Gamma \) Lagrangian \( \mathcal{L} \) is a linear (not quadratic!) function of all \( d \) momenta \( \pi^{0\sigma} = \frac{\delta \mathcal{L}}{\delta g^{0\sigma,0}} \) each of which include at least one temporal index \([1]\). The Poisson brackets (see, Eqs. (6) and (7) below) between all primary constraints equal zero identically, i.e., \([\phi^{0\sigma}, \phi^{0\tau}] = 0\). This allows us to predict that the Poisson brackets between the primary constraints \( \phi^{0\gamma} \) and Hamiltonian \( H_C \) cannot be proportional to any of the primary constraints. Therefore, these Poisson brackets \([\phi^{0\sigma}, H_C] \) lead one directly to the secondary constraints \( \chi^{0\sigma} \), i.e., \([\phi^{0\sigma}, H_C] = \chi^{0\sigma} \). The explicit formulas for all \( d \) secondary constraints are given by Eq.(13) in \([1]\) and they are very cumbersome. Note that all found \( d + d = 2d \) constraints \( \phi^{0\gamma} \) and \( \chi^{0\sigma} \), where \( \gamma = 0,1,\ldots,d \), are the first-class constraints \([4]\). The Poisson brackets between canonical Hamiltonian \( H_C \) and secondary constraints \( \chi^{0\sigma} \) are expressed as ‘quasi-linear’ \([10]\) combinations of the same secondary constraains \( \chi^{0\sigma} \). This indicates that the Hamilton procedure developed for the metric GR in \([1]\) and \([2]\) does not lead to any tertiary, or other constraints of higher order(s). Briefly, this proves that the found system of the canonical Hamiltonian \( H_C \) and all primary and secondary constraints is closed in terms of some simplectic geometry which is determined by the Poisson brackets (see discussion below). In other words, we proved the complete closure of the Dirac procedure \([1]\) for the metric GR with the canonical \( H_C \) Hamiltonian, Eq.(5).

Note that the formulas for the Hamiltonians \( H_t, H_C \) presented above and explicit expressions for all secondary constraints \([1]\) allow one to derive (with the use of Castellani procedure \([11]\)) the correct generators of gauge transformations, which directly lead to the diffeomorphism invariance, i.e., to the invariance which is well known for the free gravitational field(s) in the metric GR (see, e.g., \([9]\), more details and discussion can be found in \([1]\)). The reconstruction of the diffeomorphism invariance is a relatively simple problem for the Lagrangian-based approaches which are often used in the metric GR (see, e.g., \([13]\)). In contrast with this, for any Hamiltonian-based approach the complete solution of similar problem requires a substantial work. On the other hand, analytical and explicit derivation
of the diffeomorphism invariance becomes a very good test for the total $H_t$ and canonical $H_C$ Hamiltonians and for all primary $\phi^{0\sigma}$ and secondary $\chi^{0\sigma}$ constraints derived in any ‘new’ Hamiltonian approach. To this moment only two Hamiltonian-based approaches considered in [1] and [2] (see also [3]) were able to reproduce the complete diffeomorphism invariance. On the other hand, there are many ‘advanced’ Hamiltonian-based procedures known in the metric GR which cannot pass this fundamental test, in principle. In particular, the long list of such ‘unlucky’ procedures includes the Hamiltonian(s) derived and used in ADM-gravity (also known as the ‘American gravity’, or ‘geometro-dynamics’). Analysis of the fundamental reasons of such a failure can be found in [14].

Now, we need to define the Poisson brackets between two arbitrary (analytical) functions of the $\pi^{\alpha\beta}$ and $g_{\rho\gamma}$ variables. In general, the properly defined Poisson brackets play a central role in the construction of an arbitrary Hamilton (dynamical) system. Note, that analytical calculation of the Poisson brackets between two arbitrary functions of dynamical variables in Hamilton GR is always reduced to the calculation of the fundamental Poisson brackets which are defined as follows

$$\left[ g_{\alpha\beta}, \pi^{\mu\nu} \right] = -\left[ \pi^{\mu\nu}, g_{\alpha\beta} \right] = g_{\alpha\beta} \pi^{\mu\nu} - \pi^{\mu\nu} g_{\alpha\beta} = \frac{1}{2} \left( \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha} \delta^{\mu}_{\beta} \right) = \Delta^{\mu\nu}_{\alpha\beta},$$

where the symbol $\delta^{\mu}_{\beta}$ is the Kronecker delta, while the notation $\Delta^{\mu\nu}_{\alpha\beta}$ stands for the gravitational (or tensor) delta-function. All other fundamental Poisson brackets between dynamical variables of the metric GR equal zero identically, i.e., $\left[ g_{\alpha\beta}, g_{\mu\nu} \right] = 0$ and $\left[ \pi^{\alpha\beta}, \pi^{\mu\nu} \right] = 0$. This indicates clearly that the dynamical variables $g_{\alpha\beta}$ and $\pi^{\mu\nu}$ (see, [1], [2]) in the metric GR have been defined correctly, and now we can apply these variables and fundamental Poisson brackets to analyze and solve various gravitational problems. For instance, based on Eq.(6) we can define the gravitational Poisson bracket of the two arbitrary functions $U$ and $V$ each of which depends upon the dynamical variables $g_{\alpha\beta}$ and $\pi^{\mu\nu}$. This definition is

$$\left[ U, V \right] = \sum_{(\alpha\beta)} \left( \frac{\partial U}{\partial g_{\alpha\beta}} \frac{\partial V}{\partial \pi^{\alpha\beta}} - \frac{\partial U}{\partial \pi^{\alpha\beta}} \frac{\partial V}{\partial g_{\alpha\beta}} \right).$$

It is easy to check that these Poisson brackets obey all general rules and conditions known for the Poisson brackets (see, e.g., §21 in [12]). For instance, for the Poisson bracket between a regular function $F(g_{\alpha\beta})$, which depends upon the components of metric tensor $g_{\alpha\beta}$ only, and $\gamma\rho$-component of the momentum tensor $(\pi^{\gamma\rho})$ one finds

$$\left[ F(g_{\alpha\beta}), \pi^{\gamma\rho} \right] = \frac{\partial F}{\partial g_{\alpha\beta}} \Delta^{\gamma\rho}_{\alpha\beta} = \frac{\partial F}{\partial g_{\gamma\rho}},$$
This formula can easily be obtained by representing the function \( F(g_{\alpha\beta}) \) by its Maclaurin (or Taylor) series.

**III. HAMILTON EQUATIONS OF THE FREE GRAVITATIONAL FIELD**

By using the formulas, Eqs. (4) and (5), for the total and canonical Hamiltonians, i.e., for \( H_t \) and \( H_C \), respectively, we obtain the following Hamilton equations (or system of Hamilton equations) which describe time-evolution of all dynamical variables in the metric GR, i.e., time-evolution of each component of the \( g_{\alpha\beta} \) and \( \pi^{\gamma\rho} \) tensors. These equations are

\[
\frac{dg_{\alpha\beta}}{dx_0} = [g_{\alpha\beta}, H_t] \quad \text{and} \quad \frac{d\pi^{\gamma\rho}}{dx_0} = [\pi^{\gamma\rho}, H_t] \tag{9}
\]

where the notation \( x_0 \) denotes the temporal variable. In particular, for the spatial components \( g_{ij} \) of the metric tensor one finds the following equations

\[
\frac{dg_{ij}}{dx_0} = [g_{ij}, H_c] = \frac{2}{\sqrt{-g}g^{00}}I((ij)pq)\pi^{pq} - \frac{1}{g^{00}}I_{(ij)pq}B^{(pq)0\mu\nu}g_{\mu\nu,k} \tag{10}
\]

where the notations \( I_{(ij)pq} \) and \( I((ij)pq) \) stand for the following ‘symmetrized’ values

\[
I_{(ij)pq} = \frac{1}{2}(I_{ijpq} + I_{jipi}) \quad \text{and} \quad I((ij)pq) = \frac{1}{2}(I_{(ij)pq} + I_{(ij)qp}) \tag{11}
\]

Analogously, for the \( g_{0\sigma} \) components of the metric tensor one finds the following equations of time-evolution

\[
\frac{dg_{0\sigma}}{dx_0} = [g_{0\sigma}, H_t] = g_{0\sigma,0} \tag{12}
\]

since all \( g_{0\sigma} \) components commute with the canonical Hamiltonian \( H_c \), Eq. (5). This result could be expected, since the equation, Eq. (12), is, in fact, a definition of the \( \sigma \)-velocities, or the \( g_{0\sigma,0} \)-values, where \( \sigma = 0, 1, \ldots, d \).

The Hamilton equations for the tensor components of momenta \( \pi^{\alpha\beta} \), Eq. (9), are substantially more complicated. They are derived by calculating the Poisson brackets between \( H_t \) and \( \pi^{\gamma\rho} \). This leads us to the following general formula

\[
\frac{d\pi^{\gamma\rho}}{dx_0} = -[H_t, \pi^{\gamma\rho}] = -\left[ \frac{\partial}{\partial g_{\gamma\rho}} \left( \frac{I_{mn\rho}}{\sqrt{-g}g^{00}} \right) \right] \pi^{mn} \pi^{pq} + \left[ \frac{\partial}{\partial g_{\gamma\rho}} \left( \frac{I_{mn\rho}}{g^{00}} \right) \right] \pi^{mn} B^{(pq)0\mu\nu} g_{\mu\nu,k} + \frac{1}{g^{00}} I_{mn\rho\pi} \left[ \frac{\partial}{\partial g_{\gamma\rho}} B^{(pq)0\mu\nu} \right] g_{\mu\nu,k} + \frac{1}{2} \Delta_{\omega\rho} \left[ \frac{\partial}{\partial g_{0\sigma}} \left( \sqrt{-g}g^{B((0\sigma)0\mu\nu)} \right) \right] g_{\mu\nu,k} \tag{13}
\]

\[-\frac{1}{4} \frac{\partial}{\partial g_{\gamma\rho}} \left[ \sqrt{-g}g^{00} I_{mn\rho\pi} B^{((mn)0\mu\nu)} B^{(pq)0\alpha\beta} - \sqrt{-g}B^{\mu\nu\alpha\beta\lambda} \right] g_{\mu\nu,k} g_{\alpha\beta,\lambda} \]
The arising Hamilton equations for the dynamical variables of metric GR, i.e., for the components of the $g_{\alpha\beta}$ and $\pi^{\gamma\rho}$ tensors, are significantly more complicated than analogous equations for the electro-magnetic fields known in Maxwell electrodynamics. Nevertheless, the Hamilton equations (see Eqs. (10), (12) and (13) above), which govern time-evolution of the metric gravitational field(s), have been derived in this study, and now one can apply them to solve various problems known for the free gravitational fields in metric GR. Note that these equations, i.e., Eq. (10), Eq. (12) and Eq. (13), form a conservative (or autonomous) system of differential equations. Theory of such equations and systems of such equations is a well developed area of modern Mathematics (see, e.g., [16], [17] and references therein). For instance, we can say that every phase trajectory of this system of differential equations belongs to one of three following types: (a) a smooth curve without self-intersections, (b) a closed smooth curve (cycle), and (c) a point. There are many other facts known from the general theory of conservative systems of differential equations, and all of them can be used to determine the explicit form of the $g_{\alpha\beta}(x_0)$ and $\pi^{\gamma\rho}(x_0)$ dependencies. Furthermore, the arising (Hamilton) system of differential equations (see, Eqs. (10), (12) and Eq. (13) above) conserves the total phase volume, i.e., the free gravitational field is a Liouville’s dynamical system. In general, for any conservative (Hamilton) system of differential equations one finds a number of useful theorems and each of these theorems can now be used to solve (or simplify) the actual problems existing in the modern metric GR.

IV. GENERAL METHODOLOGY OF THE HAMILTON APPROACHES

Let us discuss the general methodology of the Hamilton method which explains main advantages of this approach in applications to the metric General Relativity. Formally, the Hamilton mechanics can be defined as a geometry in the phase space (see, e.g., [6] and [8]). In other words, by determining some dynamical (or mechanical) systems as a Hamilton system we reduce the original (e.g., mechanical) problem to some geometrical problem which can be re-formulated in very clear and transparent form. However, the main achievement of the Hamiltonian approach is an introduction of the double geometric structure, or double metric(s) in the phase space. One of these metrics is the fundamental metric (e.g., Euclidean, pseudo-Euclidean, Riemannian, etc) based on the scalar products between two arbitrary vectors in multi-dimensional space. The second (or additional) ‘symplectic’
metric is uniformly defined by the Hamiltonian system (see below) considered in the same space. A possibility to apply analytical transformations in the both metrics instantaneously provides a number of great advantages for obtaining analytical and numerical solutions of the Hamilton equations of motions. Such a situation can be compared, e.g., with the Lagrange method(s) (or methods based on the use of Lagrangians), where only one (usual) metrics in the configuration space can be used.

Below, we restrict ourselves to the analysis of the Hamilton approach developed for the metric GR. As is shown above in the case of \( d \)-dimensional space-time our Hamiltonian-based approach developed above for the metric General Relativity (or GR) leads to the actual Hamilton theory with \( d \)-primary and \( d \)-secondary constraints, i.e., the total number of constraints in the equations of the free gravitational field (in metric GR) equals \( 2d \). The total dimension of the corresponding phase (tensor) space, which is often called the cotangent space \([g_{\alpha\beta}, \pi^{\gamma\rho}]\)-space, is \( N_t = d(d + 1) \), which is always even. As is well known from Classical Mechanics (see, e.g., \([8]\) and references therein), an arbitrary Hamiltonian system is defined by an even-dimensional manifold (or phase space), a symplectic structure determined on this manifold (which is uniformly related with the relative Poincaré integral invariant) and a function on it (the Hamilton function, or Hamiltonian, for short).

In other words, the Hamilton system in the metric GR is defined by the \( N_t \)-dimensional cotangent \([g_{\alpha\beta}, \pi^{\gamma\rho}]\)-space, the Hamiltonian \( H_t \) and the corresponding simplectic structure is determined by the following differential \( \omega^2 \) and \( \omega^1 \) forms

\[
\omega^2 = d\pi^{\alpha\beta} \wedge dg_{\alpha\beta}, \quad \omega^1 = \pi^{\alpha\beta} dg_{\alpha\beta} \tag{14}
\]

The one-dimensional integral, which contain the differential form \( \omega^1 \), is called the relative integral invariant of the metric GR. The relation between these two forms is simple and transparent: \( \omega^2 = d\omega^1 \).

In a large number of textbooks on Classical Mechanics the simplectic structure on the even-dimensional phase space is determined by the corresponding Poisson brackets between all (basic) dynamical variables, i.e., between ‘coordinates and momenta’. In other words, the Hamiltonian systems is defined by: (1) the even-dimensional manifold (or phase space), (2) the scalar function \( H \) (or Hamiltonian) which is determined on this manifold and has continuous derivatives of all orders, and (3) the simplectic structure on this manifold (or phase space) which is determined by the corresponding (canonical) Poisson brackets. In the
case of metric GR we have the properly defined Hamiltonian \( H_t \), Poisson brackets between all covariant components of the metric tensor \( g_{\alpha \beta} \) and contravariant components of the momenta \( \pi^{\alpha \beta} \). This Hamilton system is considered in the \( N_t \)-dimensional (or \( d(d+1) \)-dimensional) cotangent \( [g_{\alpha \beta}, \pi^{\gamma \rho}] \)-space which is even-dimensional space. Note that each of the differential \( \omega^2 \) and \( \omega^1 \) forms uniformly determines the correct and complete set of Poisson brackets (and vice-versa) which transforms the \( N_t \)-dimensional cotangent \( [g_{\alpha \beta}, \pi^{\gamma \rho}] \)-space into a simplectic even-dimensional manifold.

The definitions of the Hamilton dynamical systems presented and discuss above are absolutely correct only for non-constraint dynamical systems. For dynamical systems with actual system of the first-class constraints such a definition is not complete. Indeed, for such systems one finds another Hamilton system which has the Hamiltonian \( H_C \) (or the canonical Hamiltonian, see above). In the case of metric GR such a system is defined in the \( N_C = N_t - 2d = d(d-1) \)-dimensional cotangent \( [g_{kl}, \pi^{mn}] \)-space. The corresponding simplectic structure in this space (or manifold) is defined by the following differential \( \omega_C^2 \) and \( \omega_C^1 \) forms

\[
\omega_C^2 = d\pi^{mn} \wedge dg_{mn} \quad , \quad \omega_C^1 = \pi^{kl} dg_{kl}
\]

where the two differential forms are also related to each other by the equation \( \omega_C^2 = d\omega_C^1 \). Each of these two differential forms determines the correct and complete set of Poisson brackets in the even-dimensional \( R^{NC} \) space. Thus, in the case of constrained dynamical system we have two different Hamilton systems defined in the \( N_t \) and \( N_C \)-dimensional simplectic spaces. Since these two simplectic spaces have different (but even) dimensions, which equal \( d(d+1) \) and \( d(d-1) \), respectively, they cannot be isomorphic to each other, in principle. For dynamical systems with constraints these two dynamical Hamilton systems must be considered independently and simultaneously.

Another fundamental fact which should be emphasized here is formulated in the form: the both Hamilton systems, which are generated by the differential \( \omega^2 \) and \( \omega_C^2 \), respectively, forms can be correct, if (and only if) the both \( R^{N_t} \) and \( R^{NC} \) spaces are even-dimensional. In other words, the correct Hamilton approach cannot be develop in those cases, when the dimension of the \( R^{NC} \) space is odd. This means that for any Hamilton (dynamical) system the total number of actual first-class constraints must be even. This statement is true for the Maxwell Electrodynamics, where one finds one primary and one secondary constraints
(see, e.g., [4], [21]). In this case the corresponding dimensions are: $N_t = d(d + 1) = 6$ and $N_C = d(d - 1) = 2$. For the $d$-dimensional metric General Relativity, where we have $d$-primary and $d$-secondary constraints (or $2d$ first-class constraints total, see, e.g., [1] and [2]). Therefore, we have $N_t = d(d + 1)$ and $N_C = d(d - 1)$. In our regular four-dimensional space-time for the metric GR one finds four primary and four-secondary constraints (i.e., we have eight essential constraints). In this case, the $R^{N_t}$-space has dimension twenty, while and corresponding dimension of the $R^{N_C}$-space equals twelve. The systems of equations arising in the both $R^{20}$- and $R^{12}$-spaces are the Hamilton systems.

V. JACOBI EQUATION FOR THE FREE GRAVITATIONAL FIELD

It was shown in [2] that the two different Hamiltonian approaches, developed in [1] and [2] (see also [5]) are related by a canonical transformation of dynamical variables. Therefore, it is possible to discuss the generating functions of these canonical transformations and apply a number of powerful tools known in the general Hamilton-Jacobi theory. In particular, this allows us to derive the Hamilton-Jacobi equation of the free gravitational field. Based on this equation we can develop an alternative method which can be used to solve (both analytically and numerically) some important problems which are currently known in the metric GR. Note that in many books and textbooks all methods based on the Hamilton-Jacobi equation are recognized as the most effective, transparent and powerful tool for theoretical analysis of various Hamilton systems and applications to such systems (see, e.g., [8] and [15]).

The closed system of Hamilton equations for the metric GR derived above allows one to obtain the Jacobi equation. This can be achieved in a number of different ways. Probably, the simplest approach is to investigate the closed system of Hamilton equations under some different angle which is directly related to the general theory of partial differential equations (see, e.g., [18], [19]). First, we note that each of the Hamilton equations is the first-order differential equation upon the corresponding time-derivatives $\frac{dg_{\alpha\beta}}{dx^0}$ and $\frac{d\pi^{\rho\sigma}}{dx^0}$. This means that these Hamilton equations can be considered as a system of characteristic equations for some non-linear equation. As is was shown in a large number of books (see, e.g., [18], [19] and [20]) this partial differential equation is the Jacobi equation (also called the Hamilton-Jacobi equation). For the non-constrained dynamical systems the Hamilton-Jacobi equation takes
the form

$$\frac{\partial S}{\partial t} + H(t, y_1, \ldots, y_n; p_1, \ldots, p_n) = \frac{\partial S}{\partial t} + H(t, y_1, \ldots, y_n; \frac{\partial S}{\partial y_1}, \ldots, \frac{\partial S}{\partial y_n}) = 0 \quad \text{(16)}$$

where $H$ is the Hamiltonian of this dynamical system, $t$ is the time (or temporal coordinate), while $y_1, \ldots, y_n$ are the dynamical coordinates of the system and $p_1 = \frac{\partial S}{\partial y_1}, \ldots, p_n = \frac{\partial S}{\partial y_n}$ are the corresponding momenta. The function $S(t, y_1, \ldots, y_n)$ is called the Jacobi function.

Below, we shall follow the general theory described in [19]. This theory can be applied successfully, if the two following conditions are obeyed: (a) the function $H(x, p)$ (i.e., the Hamiltonian), has all derivatives of the first and second order which are the continuous functions everywhere in the $R^{2n}$-space; and (b) all derivatives $\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}$ do not equal zero at the same time. Note that the arising equation, Eq.(16), is the non-linear (quadratic) equation upon the $\frac{\partial S}{\partial y_k}$ derivatives, where $k = 1, \ldots, n$. However, this equation is a linear equation upon the time-derivative (or temporal-derivative) of the Jacobi function.

For dynamical systems with first-class constraints the actual situation is more complicated. However, all arising problems can be solved, and finally, we arrive to the following Jacobi (or Hamilton-Jacobi) equation for the free gravitational field

$$-\left(\frac{\partial S}{\partial x_0}\right) = \frac{1}{\sqrt{-g}} \frac{1}{g^{00}} I_{mnpq} \frac{\partial S}{\partial g_{mn}} \left( \frac{\partial S}{\partial g_{pq}} \right) - \frac{1}{g^{00}} I_{mnpq} \frac{\partial S}{\partial g_{mn}} B^{(pq0)(\mu\nu k)} g_{\mu\nu,k}$$

$$+ \frac{1}{4} \sqrt{-g} \left[ \frac{1}{g^{00}} I_{mnpq} B^{((mn)(0)0)(\mu\nu k)} B^{(pq0)(\alpha\beta l)} - B^{\mu\nu k\alpha\beta l} \right] g_{\mu\nu,k} g_{\alpha\beta,l}$$

$$+ g_{0\sigma,0} \left[ \left( \frac{\partial S}{\partial g_{0\sigma}} \right) - \frac{1}{2} \sqrt{-g} B^{((0\sigma)(0)0)(\mu\nu k)} g_{\mu\nu,k} \right]$$

where $S(x_0, \{g_{\alpha\beta}\})$ is the Jacobi function of the gravitation field(s), while $\frac{\partial S}{\partial x_0}$ and $\frac{\partial S}{\partial g_{mn}}$ are its temporal and space-space derivatives, respectively.

The explicit derivation of the Jacobi equation, Eq.(17), for the free gravitational field is a great advantage of this study. Indeed, analytical solution of the Jacobi equation is a well developed procedure which provides many advantages in comparison to the direct solution of the Hamilton equations. Moreover, the Jacobi equation, Eq.(17), opens a new avenue for derivation of a number of true and/or adiabatic invariants of the free gravitational field. As was mentioned above in analytical mechanics (see, e.g., [8]) the methods based on the Hamilton-Jacobi equation are often called and considered as the most effective procedures which can be used to solve the equations of motion known for an arbitrary, in principle, Hamilton system. In particular, the Hamilton-Jacobi methods are very effective for the
dynamical systems with Hamiltonians which contains only lower powers \( n \) of all essential momenta, i.e. \( n \leq 2 \). This is the case for the free gravitational field in the metric gravity this Hamiltonian is a quadratic functions of momenta \( \pi^{mn} \), (see, Eq.(5)). Furthermore, based on the Hamilton-Jacobi equation one can determine a number of ‘integrals of motion’, derive the closed algebras of such integrals and investigate the properties of such algebras and commutation relations between these integrals. Another interesting reason to deal with the Hamilton-Jacobi equation is a possibility to apply a wide class of analytical transformations which reduce this equation to relatively simple forms. For instance, by introducing the new ‘absolute’ time \( t \), which is simply related with the temporal coordinate \( x_0 \) used above: 
\[
dx_0 = \sqrt{-g}g^{00}dt,
\]
\begin{equation}
- \frac{\partial S}{\partial t} = I_{mnop}(\partial S/\partial g_{mn})(\partial S/\partial g_{pq}) - \sqrt{-g}I_{mnop}(\partial S/\partial g_{mn})B^{(pq0)\mu\nu\kappa}g_{\mu\nu,k}
\end{equation}
\begin{equation}
+ \frac{1}{4}(-g)[I_{mnop}B^{(mn0)\mu\nu\kappa}B^{(pq0)\alpha\beta\lambda} - g^{00}B^{\mu\nu\alpha\beta\kappa\lambda}g_{\mu\nu,k}g_{\alpha\beta,l}]
\end{equation}
where we chose the gauge in which all \( \sigma \)-velocities equal zero. This form of the Jacobi equation can be useful to simplify and solve a number of problems. It is clear that now the Jacobi function of temporal coordinate \( x_0 \) (or \( t \)) and \( n_C = \frac{N_C}{2} = \frac{d(d-1)}{2} \) real functions each of which is the covariant space-space (covariant) component \( g_{mn} \) of the metric tensor. To describe the stationary situations, when the gravitational field does not change with time, we can introduce the ‘short’ Jacobi function \( S_0(\{g_{mn}\}) \) which does not depend upon temporal coordinate. The relation between the complete \( (S) \) and short \( (S_0) \) Jacobi functions is simple: 
\[
S(\{g_{\alpha\beta}\}) = S_0(\{g_{mn}\}) - Et\ (\text{here the notations from Eq.(18) are applied}),
\]
where \( E \) is the numerical (or scalar) parameter which is called the total energy of the system. Now, from Eq.(18) one finds the following expression for the total energy \( E \) of the free gravitational field
\begin{equation}
E = I_{mnop}(\partial S/\partial g_{mn})(\partial S/\partial g_{pq}) - \sqrt{-g}I_{mnop}(\partial S/\partial g_{mn})B^{(pq0)\mu\nu\kappa}g_{\mu\nu,k}
\end{equation}
\begin{equation}
+ \frac{1}{4}(-g)[I_{mnop}B^{(mn0)\mu\nu\kappa}B^{(pq0)\alpha\beta\lambda} - g^{00}B^{\mu\nu\alpha\beta\kappa\lambda}g_{\mu\nu,k}g_{\alpha\beta,l}]
\end{equation}
where \( I_{mnop} \) is the following space-like tensor
\begin{equation}
I_{mnop} = \frac{1}{d-2}g_{mn}g_{pq} - g_{mp}g_{nq} = \frac{1}{d-2}\left[g_{mn}g_{pq} - (d-1)g_{mp}g_{nq}\right]
\end{equation}
This tensor is a positively defined, i.e., all its eigenvalues are positive. Furthermore, the
\( I_{mnpq} \) tensor is inverse to the tensor \( E^{pqkl} \) introduced by Dirac in 1950’s (for more detail, see, e.g., [1]).

VI. CONCLUSION

In conclusion, we want to note that since the end of 1930’s it was widely assumed that it is impossible to define any non-trivial symplectic structure in the \( R^{d(d+1)} \)-space based on the Lagrangians similar to the \( \Gamma - \Gamma \) Lagrangian used in the metric GR (see, e.g., [1], [2]). However, in 1950 Dirac found [3] an elegant way which allows one to restore such a symplectic structure in the tensor \( R^{d(d+1)} \)-space, which is the regular, working space for the \( d \)-dimensional metric GR. The following creation of the Hamilton approach for the metric gravity was just a ‘matter of technique’. In particular, by using the appropriate re-definition of the canonical momenta, which are closely related to the primary ‘constraints’, Dirac developed [3] the actual Hamilton approach for the metric GR. In general, it can be shown that all ‘correct’ Hamiltonians arising in the metric GR are the quadratic functions of the space-like components of momenta \( \pi^{mn} \). Based on the results of a number of recent studies we can conclude that this approach works very well in applications to the metric GR. For instance, nobody could detect any internal contradiction in this approach for more than 60 years, after this approach was created by Dirac in 1958. In this study we make another (new and important) step in the development of Dirac’s theory of the constrained dynamical systems.

In general, the non-contradictory Hamilton approaches developed for the metric gravity can be considered as the first step to perform quantization of gravitational fields in the metric GR. In this form the problem was originally formulated by P.A.M. Dirac in 1950 [3] (see also discussion in [4]). In reality, the quantization of the gravitational fields is not an actual task for modern physics. Indeed, the overall contribution of the gravitational interactions in the total energy of an arbitrary pair of elementary particles is significantly smaller (\( \approx 10^{38} \) times smaller), than the corresponding electromagnetic interaction(s). However, the modern QED allows one to describe the leading electromagnetic interaction(s) only to the maximal accuracy \( \approx 1 \cdot 10^{-15} \) (for the total energy). Current experimental accuracy does not allow one to detect any contribution from the gravitational interactions between elementary particles, which are at least in \( \approx 10^{23} \) times smaller the overall uncertainty arising from electromagnetic...
interactions. In respect to this fact, in this study we do not want to consider the quantization of gravitational (metric) fields.

On the other hand, it is clear that the both Hamilton and Hamilton-Jacobi equations for the free gravitational fields are much easier to solve than deal with the original Einstein equation. Indeed, by using the Hamilton-Jacobi formulation of the metric GR we can always apply a very powerful and reliable apparatus developed for Hamilton dynamical systems. Direct solutions of the Einstein equation cannot be compared with the Hamilton-Jacobi approach, since such ‘direct’ methods are usually based on some quasi-empirical choice and following matching of all components of the metric tensor. Hopefully, soon the methods based on the both Hamilton and Hamilton-Jacobi equations will be transformed (after some additional development) into the main working methods and procedures which can be applied to many problems in the metric GR.

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