Train Tracks, Orbigraphs and CAT(0) Free-by-Cyclic Groups

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September 10, 2019

Abstract

Gersten gave an example of a polynomially-growing automorphism of $F_3$ whose mapping torus $F_3 \rtimes \mathbb{Z}$ cannot act properly by semi-simple isometries on a CAT(0) metric space. By contrast, we show that if $\phi$ is a polynomially-growing automorphism belonging to one of several related groups, there exists $k > 0$ such that the mapping torus of $\phi^k$ acts properly and cocompactly on a CAT(0) metric space. This $k$ can often be bounded uniformly. Our results apply to automorphisms of a free product of $n$ copies of a finite group $A$, as well as to palindromic and symmetric automorphisms of a free group of finite rank. Of independent interest, a key tool in our proof is the construction of relative train track maps on orbigraphs, certain graphs of groups thought of as orbi-spaces.

1 Introduction

Fix a positive integer $n$. Let $A_1, \ldots, A_n$ be a collection of finite groups and $W$ their free product $A_1 \ast \cdots \ast A_n$. We reserve the notation $W_n$ for the case where each $A_i$ is $\mathbb{Z}/2\mathbb{Z}$. That is, $W_n$ is the free Coxeter group on $n$ letters (some authors prefer the term “universal”) with Coxeter presentation $(a_1, \ldots, a_n | a_i^2 = 1)$. Let $\text{Aut}(W)$ be the group of automorphisms of $W$, and let $\text{Inn}(W)$ the normal subgroup of inner automorphisms. In this paper we study the group of outer automorphisms of $W$, $\text{Out}(W) = \text{Aut}(W)/\text{Inn}(W)$, with an eye, especially in the examples, towards $\text{Out}(W_n)$.

One facet that mapping class groups, $\text{GL}_n(\mathbb{Z})$ and $\text{Out}(F_n)$ have in common is that they are “big” groups: although they are finitely presented, their elements and subgroups exhibit a varied and interesting array of dynamical behavior. The Nielsen–Thurston normal form, Jordan normal form, and relative train track representative, respectively, all attempt to organize this information to aid us in reasoning about this behavior.

One of the goals of this paper is to argue that the groups $\text{Out}(W)$ are also “big”: we show how to extend the work of Bestvina, Feighn and Handel [BH92, BFH00, FH11, FH17] on relative train track representatives for outer automorphisms $\varphi \in \text{Out}(F_n)$ to the setting of $\text{Out}(W)$.

Theorem A. Every outer automorphism $\varphi \in \text{Out}(W)$ is represented by a homotopy equivalence $f : G \to G$ of an orbigraph $G$ that is a relative train track map in the sense of [BH92] and [FH11] (see Theorem 5.2.1).

If $\varphi$ is irreducible (that is, leaves invariant no conjugacy class of an infinite free factor), then $\varphi$ may be represented by a train track map (see Theorem 4.3.3).
Our goal in proving Theorem A is to find the correct equivariant perspective for our case so that proofs of Bestvina, Feighn and Handel can be adapted without too much extra work. We hope that the availability of train track technology will spur further development in the study of \(\text{Out}(W)\), so we have given the theory a reasonably thorough exposition.

An interesting, immediate consequence of the work in Section 4 and Remark 2.2.1 is the following:

**Corollary 1.** If \(S\) is a surface with nonempty boundary, and \(\iota: S \to S\) is a hyperelliptic involution—that is, an involutive homeomorphism whose action on the homology of \(S\) is by minus the identity—then if \(f: S \to S\) is a homeomorphism that commutes with \(\iota\) up to homotopy, the outer action of \(f\) on \(\pi_1(S)\) induces an outer automorphism \(f_\sharp\) in \(\text{Out}(W_n)\), where \(n - 1\) is the rank of \(\pi_1(S)\).

If \(S\) has one boundary component and \(f\) is Pseudo-Anosov, then \(f_\sharp\) is an irreducible outer automorphism of \(W_n\).

It is worth noting that this is not the first construction of relative train track maps for automorphisms of free products. Collins and Turner in [CT94] formulated and proved a version of the Scott conjecture using an analogue of relative train track maps constructed with graphs of 2-complexes. More recently, Francaviglia and Martino in [FM15] developed the Lipschitz metric theory of Guirardel–Levitt outer space to prove the existence of relative train track representatives for outer automorphisms of free products. By working with orbigraph quotients, rather than graphs of 2-complexes or \(W\)-trees, the objects we work with are finitary, and like [FH18], the methods we develop have the advantage of being algorithmic.

A careful reading of the proof of Theorem A shows that our construction (like that of Francaviglia and Martino) holds for more automorphisms of free products than we have stated it. In particular, we do not rely on the finiteness of the groups \(A_i\), nor do we require they be freely indecomposable. The correct hypothesis is that \(W\) is the free product of the \(A_i\), and the outer automorphism \(\varphi\) permutes their conjugacy classes. We only use this observation for the proof of Corollary 5, so to avoid awkward notation, we have written the proof assuming the \(A_i\) are finite.

It is also worth noting that our paper does not the cover the entirety of the train track theory as developed in [BH92], [BPH00] and [FH11]. In particular, we have not developed the theory of attracting laminations, nor the action of a relative train track map on the “space of lines” in an orbigraph. Because of this, we have not constructed an \(\text{Out}(W)\)-analogue of improved relative train track maps, nor completely-split relative train track maps, simply because our application does not require them. We expect such a construction building on work in this paper to be possible.

As a first application of this technology, we investigate mapping tori of a class of outer automorphisms \(\varphi \in \text{Out}(W)\). Given a group \(G\) and \(\varphi \in \text{Out}(G)\) an outer automorphism represented by an automorphism \(\Phi: G \to G\), the mapping torus of \(\varphi\) (or \(\Phi\)) is the semidirect extension \(G \rtimes_{\Phi} \mathbb{Z}\). Mapping tori of free group automorphisms are an interesting class of groups that, while widely studied, remain somewhat mysterious. A full introduction to the study of mapping tori of free group automorphisms is beyond the scope of this paper. We will content ourselves to note that our result highlights a difference in behavior between \(\text{Out}(W)\) and \(\text{Out}(F_n)\).
Before stating the theorem, we give an example of our result. Consider the following automorphisms of $F_3 = \langle a, b, c \rangle$.

\[
\begin{align*}
\Phi &: \begin{cases}
a &\mapsto a \\
b &\mapsto ba \\
c &\mapsto ca^2 
\end{cases} \\
\Psi &: \begin{cases}
a &\mapsto a \\
b &\mapsto aba \\
c &\mapsto a^2ca^2 
\end{cases}
\end{align*}
\]

Gersten [Ger94] gave $\Phi$ as an example of a free group automorphism whose mapping torus cannot be a subgroup of a CAT(0) group. It is a “poison subgroup” for nonpositive-curvature. Our Corollary 4 implies that although the monodromy automorphism is visually very similar, the mapping torus of $\Psi$ acts properly and cocompactly on a CAT(0) 2-complex.

**Theorem B.** Let $A$ be a finite abelian group and $W = A * \cdots * A$ a free product of $n$ copies of $A$. Let $\varphi \in \text{Out}^0(W)$ be a polynomially-growing outer automorphism represented by $\Phi: W \to W$. Then the mapping torus $W \rtimes_{\varphi} \mathbb{Z}$ acts geometrically on a CAT(0) 2-complex $\tilde{X}_\varphi$.

If $A$ is not assumed to be abelian, or if $\varphi$ is not assumed to be in $\text{Out}^0(W)$, there exists some $k > 0$ such that the mapping torus of $\Phi^k$ acts geometrically on a CAT(0) 2-complex.

The subgroup $\text{Out}^0(W)$ is defined in section 2; it is a finite index subgroup of $\text{Out}(W)$. Key to our argument is the fact that outer automorphisms of $\varphi \in \text{Out}^0(W)$ are represented by relative train track maps whose strata are particularly well-behaved. In the analogy with $\text{Out}(F_n)$, we expect that $\varphi \in \text{Out}^0(W)$ correspond to “rotationless” outer automorphisms in the sense of [FH11].

Given a conjugacy class $[w]$ in a group $G$ with finite generating set $S$, denote by $|[w]|$ the minimal word length of an element of $G$ representing $[w]$ with respect to $S$. Recall that an outer automorphism $\varphi \in \text{Out}(G)$ is polynomially-growing if there exists a polynomial $p(x)$ such that the length $|\varphi^k([w])|$ is bounded above by $p(k)$ for all conjugacy classes $[w]$. Gersten’s example above is polynomially-growing, so general polynomially-growing automorphisms of $F_n$ are typically not geometrically well-behaved.

The group $W = A * \cdots * A$ splits as $F \rtimes \mathbb{Z}$, where $F$ is a free group of rank $(|A|-1)(n-1)$ (see Section 2). If an outer automorphism $\varphi$ belongs to $\text{Out}^0(W)$, then it preserves the conjugacy class of $F$ and thus induces an outer automorphism $\varphi|_F \in \text{Out}(F)$.

**Corollary 2.** If $\varphi \in \text{Out}(W)$ satisfies the conclusions of Theorem B, then $F \rtimes_{\varphi|_F} \mathbb{Z}$ is the fundamental group of a nonpositively-curved 2-complex $X_\varphi$.

**Corollary 3.** If $\varphi \in \text{Out}(W)$ is polynomially-growing, then $W \rtimes_{\varphi} \mathbb{Z}$ and $F \rtimes_{\varphi|_F} \mathbb{Z}$ act properly by semi-simple isometries on CAT(0) spaces. In particular none of these groups are “poison subgroups.”

An interesting special case of Corollary 2 is the following. Recall that an automorphism $\Phi: F_n \to F_n$ is said to be palindromic if there exists a free basis $x_1, \ldots, x_n$ for $F_n$ such that $\Phi$ sends the $x_i$ to palindromes—words in the $x_i$ that are the same spelled forwards and backwards. Palindromic automorphisms were first studied by Collins in [Co95].
Corollary 4. If $\Phi : F_n \to F_n$ is palindromic and polynomially-growing, there exists $0 < k < n!$ such that the mapping torus of $\Phi$ is the fundamental group of a nonpositively-curved 2-complex.

Recall that an automorphism $\Phi : F_n \to F_n$ is said to be symmetric if there exists a free basis $x_1, \ldots, x_n$ for $F_n$ such that $\Phi$ permutes the conjugacy classes of the $x_i$, and pure symmetric, if there exist words $w_1, \ldots, w_n \in F_n$ such that $\Phi(x_i) = w_i^{-1}x_iw_i$. Although not a direct corollary of Theorem B, entirely analogous arguments prove the following.

Corollary 5. If $\Phi : F_n \to F_n$ is a polynomially-growing, pure symmetric automorphism, then the mapping torus of $\Phi$ is the fundamental group of a nonpositively-curved 2-complex.

One of our original motivations in this paper was work of Samuelson [Sam06], who studied mapping tori of certain “upper triangular” (thus polynomially-growing) automorphisms of $F_n$ in order to probe how frequently such mapping tori are CAT(0) or, like Gersten’s example, are “poison subgroups.” A key step in our proof shows that if $\varphi$ is a polynomially-growing outer automorphism in one of the classes we have just described, then there exists $k < n!$ such that $\varphi^k$ has an upper-triangular representative $\Phi^k$. We view Corollary 2 as an extension of Samuelson’s work, and like Samuelson, our construction of $\check{X}_{\varphi}$ is an application of a general construction of Bridson–Haefliger [BH99].

In Section 2 we review geometric and algebraic information about $\text{Out}(W)$ and connections of $\text{Out}(W_n)$ to groups of outer automorphisms of $F_{n-1}$. In Section 3 we develop the equivariant topological perspective we use to construct relative train track maps. The constructions in the proof of Theorem A are split across Sections 4 and 5. Having read Section 3, a reader familiar with the literature on $\text{Out}(F_n)$ may feel free to skim the latter two sections, noting Theorem 5.2.1 for the full definition of a relative train track map. In Section 6 we collect observations and consequences of Theorem A. Finally, in Section 7 we turn to mapping tori of automorphisms of $W$ and the proof of Theorem B and its corollaries.

2 Background on $W$ and $\text{Out}(W)$

Basic Conventions. Throughout this paper, $n$ will be a positive integer. $A_1, \ldots, A_n$ and $A$ will denote (nontrivial) finite groups, and $W = A_1 \ast \cdots \ast A_n$ will usually denote the free product of the $A_i$. We reserve $W_n$ for the case where each $A_i = \mathbb{Z}/2\mathbb{Z}$. We will use lowercase Greek letters like $\varphi$ to denote outer automorphisms (typically of $W$, but occasionally of a free group), and the corresponding capital Greek letter to denote an automorphism that represents it.

We will typically view conjugation as a right action, writing $x^y = y^{-1}xy$. We make an exception for the acting letter of a mapping torus: if $t$ generates the cyclic factor of $G \rtimes \varphi \mathbb{Z}$, say, we will write $t^g = \Phi(g)$.

2.1 History and Geometry.

The groups $W$ sit in the intersection of several well-studied classes of groups: they are free products, graph products, and virtually free. By virtue of this, the groups $\text{Out}(W)$
have often been studied as examples of groups of outer automorphisms of groups within these larger classes.

The modern approach to $\text{Out}(F_n)$ begins with Culler and Vogtmann’s construction of Outer Space, a contractible space on which $\text{Out}(F_n)$ acts properly, as well as its spine, a simplicial equivariant deformation retract where the action is also cocompact.

Outer Space has analogs among the groups $\text{Out}(W)$. McCullough and Miller, and independently Krstić and Vogtmann constructed simplicial complexes that the $\text{Out}(W)$, or closely related groups, act on properly and cocompactly. McCullough and Miller’s construction focused on “symmetric” (that is, conjugating) automorphisms of free products, while Krstić and Vogtmann investigated subcomplexes of Outer Space corresponding to automorphisms of virtually free groups. More recently Guirardel and Levitt reinterpreted McCullough–Miller space as a space of actions on trees. Our work allows an interpretation as a space of “marked metric orbigraphs.”

These spaces have proved very useful for understanding $\text{Out}(W)$ and much about them remains to be studied. From these spaces one can derive that the $\text{Out}(W)$ are virtually of type $F$, and have virtual cohomological dimension $n - 2$. Recently, using the geometry of McCullough–Miller space, Das [Das18] showed that for $n \geq 4$, the group $\text{Out}(W)$ is thick in the sense of Behrstock–Drutu–Mosher [BDM09] and is thus not relatively hyperbolic. Since for $n \leq 3$, $\text{Out}(W)$ is hyperbolic (in fact, virtually free), this result is sharp.

Somewhat more is known about $\text{Out}(W_n)$. Mirroring results of Bridson–Vogtmann and Bridson for Culler–Vogtmann outer space respectively, Piggott [Pig12] showed that (again, for $n \geq 4$) any simplicial isometry of McCullough–Miller space for $W_n$ is induced by an element of $\text{Out}(W_n)$, and Cunningham in his thesis [Cum15] showed that McCullough–Miller space does not support an $\text{Out}(W_n)$-equivariant $\text{CAT}(0)$ metric. The groups $\text{Out}(W_n)$ are closely related to palindromic automorphisms of free groups, and to the centralizer in $\text{Out}(F_{n-1})$ of a hyperelliptic involution; see Remark 2.2.1.

2.2 Algebra

**The Subgroup $\text{Out}^0(W)$**. The Grushko decomposition theorem implies that the $A_i$ form a set of representatives for the conjugacy classes of maximal (with respect to inclusion) finite subgroups of $W$. The action of $\text{Out}(W)$ on the conjugacy classes of the $A_i$ defines a map $\pi : \text{Out}(W) \rightarrow S_n$ to the symmetric group on $n$ letters. An element $\varphi \in \text{Out}(W)$ in the kernel of this map sends each $A_i$ to a conjugate of itself. For each $i$, some automorphism $\Phi_i : W \rightarrow W$ representing $\varphi$ satisfies $\Phi_i(A_i) = A_i$, and the restriction $\Phi_i|_{A_i} : A_i \rightarrow A_i$ is well defined up to an inner automorphism of $A_i$. Thus elements $\varphi$ in the kernel of the map $\pi$ induce outer automorphisms $\varphi|_{A_i}$ in $\text{Out}(A_i)$ for each $i$, $1 \leq i \leq n$. We define the subgroup $\text{Out}^0(W)$ as

$$\text{Out}^0(W) := \{ \varphi \in \text{Out}(W) \mid \pi(\varphi) = 1 \text{ and } \varphi|_{A_i} = 1 \text{ for all } i \}.$$ 

Thus if $\Phi : W \rightarrow W$ is an automorphism representing $\varphi \in \text{Out}^0(W)$, then there exist $w_i \in W, 1 \leq i \leq n$ such that if $a_i \in A_i$, $\Phi(a_i) = w_i^{-1}a_ww_i$. For each $W$, there is a natural number $M$ such that for any outer automorphism $\varphi$, we have $\varphi^M \in \text{Out}^0(W)$. In the case of $W_n$, the number $M$ is Landau’s function $g(n)$, the maximum order of an element in $S_n$. Note that Gutierrez–Piggott–Ruane [GPR12] and Krstić–Vogtmann [KV93] define a subgroup they call $\text{Out}^0(G)$. All three subgroups are slightly different.
The Case of $A \ast \cdots \ast A$. In the case where $W = A \ast \cdots \ast A$ is a free product of $n$ copies of a single finite group $A$, for $a \in A$, let $a_i$ denote the element of $W$ corresponding to $a$ in the $i$th free factor. There is a map $\pi: W \rightarrow A$ sending each free factor isomorphically to $A$. The kernel $F$ of the map is free of rank $(n-1)(|A| - 1)$; a free basis is

$$a_1^{-1}a_2 \cdots a_n^{-1}a_n$$

for all $a \in A \setminus \{1\}$.

The map $W \rightarrow A$ splits; send $a \in A$ to $a_1$. This yields a description of $W$ as a semidirect product $F \rtimes A$. Elements of $\Out^0(W)$ preserve $F$ and induce the identity on $A$. The conjugation action of $A$ on $F$ yields a map $\theta: A \rightarrow \Out(F)$ and the restriction $\Out^0(W)|_F$ centralizes $\theta(A)$; see [Krs92, Section 2] for more details.

Remark 2.2.1. Applying the above to $W_n$, we see that $W_n \cong F_{n-1} \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts as a hyperelliptic involution $\iota$ of $F_{n-1}$—it inverts each element of a fixed free basis $x_1, \ldots, x_{n-1}$.

Note that an automorphism $\varphi$ of $F_{n-1}$ commutes with $\iota$ when inverting each letter of $\varphi(x_i)$ in place inverts $\varphi(x_i)$ as a group element. In other words, $\varphi(x_i)$ must be spelled the same forwards and backwards—a palindrome. Such a $\varphi$ is called a palindromic automorphism. Palindromic automorphisms were defined in [Col95].

The subgroup $F_{n-1} \leq W_n$ is characteristic, so restricting $\Phi$ in $\Aut(W_n)$ to its action on $F_{n-1}$, $\Phi|_{F_{n-1}}$, yields a representation $\rho: \Aut(W_n) \rightarrow \Aut(F_{n-1})$. It is not hard to check that $\rho$ is injective. In fact, by work of Krstić [Krs92, Section 2], the map $\rho$ induces an isomorphism from $\Aut(W_n)$ to the full preimage in $\Aut(F_{n-1})$ of the group Bregman–Fullarton [BF18] term the hyperelliptic outer automorphism group, $\mathrm{HOut}(F_{n-1})$, the centralizer of $[\iota]$ in $\Out(F_{n-1})$. The automorphism $\iota$ is an inner automorphism of $W_n$ but not an inner automorphism of $F_{n-1}$. Abusing notation by identifying $\Aut(W_n)$ with its image under $\rho$, we have the following diagram of short exact sequences. The map $\pi$ is the natural map $\Aut(F_{n-1}) \rightarrow \Out(F_{n-1})$.

$$
\begin{array}{cccccc}
1 & \longrightarrow & \Inn(W_n) & \longrightarrow & \Aut(W_n) & \longrightarrow & \Out(W_n) & \longrightarrow & 1 \\
& & \downarrow \pi & & \downarrow \pi & & \cong \\
1 & \longrightarrow & \langle \iota \rangle & \longrightarrow & \Out(F_{n-1}) & \longrightarrow & \Out(W_n) & \longrightarrow & 1 \\
\end{array}
$$

More generally, a hyperelliptic involution in $\Out(F_{n-1})$ is an involution whose action on the homology of $F_{n-1}$ is by minus the identity. Any two hyperelliptic involutions are conjugate in $\Out(F_{n-1})$ [BF18, Lemma 6.1] [GJ00, Proposition 2.4], so their centralizers are all isomorphic.

3 Topology of Finite Orbigraphs

Throughout the paper, $G$ will denote a graph of groups in the sense of Bass–Serre.

Cone Points, Half-Edges. We view our graphs as genuine topological objects with the obvious CW complex structure. We typically write $e$ or $E$, possibly with subscripts, for edges (1-cells) in $G$, $\tilde{e}$ or $\tilde{E}$ for edges in the universal cover $\tilde{G}$. We reserve the term vertex for those 0-cells in $G$ with trivial stabilizer, preferring the term cone point for
those 0-cells with nontrivial stabilizer. If an edge meets one or more cone points, we will sometimes call it a half edge, but will be content to call all 1-cells in $G$ or $\tilde{G}$ edges. We think of edges $E$ or $\tilde{E}$ as coming with a choice of orientation, and write $E$ or $\tilde{E}$ for $E$ or $\tilde{E}$ with its orientation reversed. We write $\mathcal{V}$ for the set of vertices in an orbigraph $G$ and $\mathcal{C}$ for the set of cone points, and $G^0$ for the 0-skeleton of $G$, that is, $G^0 = \mathcal{V} \cup \mathcal{C}$.

### 3.1 Orbifolds, Equivariant Homotopy

**Definition 3.1.1.** A $W$-orbigraph is a finite graph of groups $G$ with fundamental group $W = A_1 \ast \cdots \ast A_n$ with each $A_i$ finite such that the action of $W$ on the universal cover $\tilde{G}$ is without inversions, is properly discontinuous (and thus geometric), and has trivial edge stabilizers.

In the sequel, we will just write orbigraph, since the fundamental group will be clear from context. Just as finite graphs may be thought of as parametrizing geometric actions of the free group on locally finite trees, orbifolds parametrize geometric actions of $W$ on locally finite trees without edge stabilizers.

**Remark 3.1.2.** Not all graphs of groups with fundamental group $W_n$ are orbifolds. For instance, those corresponding to the splittings $W_n = W_n \ast_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ or $W_n = W_{n-2} \ast D_{\infty}$ (for $n \geq 2$) are not orbifolds, the former because it has a nontrivial edge stabilizer, the latter because it has an infinite vertex stabilizer. Orbifolds are a proper subset of $W$-trees.

Thus orbifolds are just finite simplicial trees, with $n$ of the 0-cells having nontrivial stabilizer equal to one of the $A_i$.

**Example 3.1.3.** The Davis complex for $W_n$ is an $n$-valent tree $\Gamma(W_n)$, which may be obtained from the Cayley graph for the Coxeter presentation of $W_n$ by identifying each pair of edges with the same initial and terminal vertices labeled $a_i$ to one edge. The quotient of the barycentric subdivision of $\Gamma(W_n)$ by the action of $W_n$ is an orbigraph with one vertex $*$, $n$ cone points each with $\mathbb{Z}/2\mathbb{Z}$ stabilizer, and $n$ half-edges, each one connecting the vertex to a cone point.

More generally, for $W = A_1 \ast \cdots \ast A_n$ we define an orbigraph $\mathcal{T}_n$ again with one vertex $*$, $n$ cone points with stabilizers the $A_i$ and $n$ half-edges, each connecting $*$ to a unique cone point. We will call this orbigraph the “thistle with $n$ prickles”, $\mathcal{T}_n$. Label the half-edge connecting $*$ to the cone point with stabilizer $A_i$ as $E_i$ and orient it pointing toward $*$. See Figure [1].

If $c$ is a cone point in $\mathcal{T}_n$ with stabilizer $A_i$ and corresponding half-edge $E_i$, there are $\#(A_i)$ lifts $\tilde{E}_i$ of $E_i$ incident to each lift $\tilde{c}$ of $c$ in the universal cover $\tilde{\mathcal{T}}_n$. For each $\tilde{c}$, some conjugate of $A_i$ in $W$ acting on $\tilde{\mathcal{T}}_n$ fixes $\tilde{c}$ and each element of the conjugate permutes the $\tilde{E}_i$ according to the multiplication in $A_i$. Fixing a lift $*\tilde{}$ in $\tilde{\mathcal{T}}_n$ of the vertex $*$ in $\mathcal{T}_n$, one checks that the orbit $W*\tilde{}$ of $*\tilde{}$ is in one-to-one correspondence with the elements of $W$.

**Example 3.1.4.** The isomorphism $W_n \cong F_{n-1} \rtimes \mathbb{Z}/2\mathbb{Z}$ from Remark [2.2.1] yields an action of $W_n$ on the standard Cayley graph $\Gamma(F_{n-1})$ for $F_{n-1}$: $a_1$ acts as the hyperelliptic involution, fixing the identity vertex and sending $w \in F_{n-1}$ to $\iota(w)$. In particular, $a_1$
permutes edges incident to the identity vertex, sending the edge labeled \(x_i\) to the edge labeled \(x_i^{-1}\). Meanwhile \(a_{i+1} \cdot 1 \leq i \leq n - 1\), reflects across the edge incident to the identity vertex labeled \(x_i^{-1}\). One checks that the (left) action of \(a_1a_{i+1}\) recovers the action of \(x_i\).

The quotient of the barycentric subdivision of \(\Gamma(F_{n-1})\) by this action of \(W\) is an orbigraph \(H_n\) with \(n\) cone points, no vertices and \(n - 1\) half-edges. We can define a similar orbigraph for general \(W\) still denoted \(H_n\). One cone point, corresponding to \((a_1)\) in our example, has degree (number of incident edges) \(n - 1\), while the others have degree \(1\). Label the half-edges \(E_1, \ldots, E_{n-1}\) and orient them so that they point towards their common incident cone point. We will call this orbigraph (and any other that differs only by which \(A_i\) stabilizes the degree \(n - 1\) cone point) the hedgehog with \(n - 1\) spikes. See Figure 1.

There is an obvious map \(\tau: T_n \rightarrow H_n\) (as in the previous example) that collapses \(E_1\) in \(T_n\) to a point and sends \(E_{i+1}\) in \(T_n\) homeomorphically to \(E_i\) in \(H_n\). The map \(\tau\) has a \(W\)-equivariant lift to the universal covers \(\tilde{T}_n \rightarrow H_n\) that collapses each lift \(\tilde{E}_1\) of \(E_1\), thus mapping each lift of \(*\) to the lift of the cone point with stabilizer \(A_1\) it is nearest to. Thinking of \(H_n\) as being tiled by copies of \(H_n\) as a fundamental domain for the action of \(W\), a map \([0, 1] \rightarrow H_n\) changes fundamental domains at the degree \(n - 1\) cone point exactly when its preimage in \(\tilde{T}_n\) crosses a pair of \(\tilde{E}_1\) edges.

![Figure 1: The thistle, \(T_{10}\), and the hedgehog, \(H_{10}\) for \(W_{10}\)](image)

Now consider the map \(\sigma: H_n \rightarrow T_n\) that expands each \(E_i\) linearly (with respect to some metric) over the edge-path \(E_1E_{i+1}\). \(\sigma\) also has a \(W\)-equivariant lift \(\tilde{\sigma}: \tilde{H}_n \rightarrow \tilde{T}_n\) that maps each edge labeled \(\tilde{E}_i\) to an edge path labeled \(\tilde{E}_1 \tilde{E}_i\) so that the map \(\tilde{\sigma}\) sends fundamental domains for \(H_n\) to fundamental domains for \(T_n\).

One checks that \(\sigma\) and \(\tau\) are homotopy inverses for each other, and that \(\tilde{\sigma}\) and \(\tilde{\tau}\) are as well. In fact, something stronger is true: \(\tilde{\sigma}\tilde{\tau}\) and \(\tilde{\tau}\tilde{\sigma}\) are \(W\)-equivariantly homotopic to the identity, say through homotopies \(f_t\) and \(g_t\), that may be chosen such that at each time \(t\), the quotients \(f_t(\tilde{T}_n)/W\) and \(g_t(\tilde{H}_n)/W\) are still orbigraphs.

**Definition 3.1.5.** We say that maps \(f, g: G \rightarrow G'\) between orbigraphs are **equivariantly homotopic through orbigraphs**, when there are \(W\)-equivariant lifts \(\tilde{f}, \tilde{g}: \tilde{G} \rightarrow \tilde{G}'\) and a
Remark 3.2.1. Suppose forests be nontrivial, we follow the convention in [BH92].

Example 3.1.6. We will use the analogous notion of contractibility (namely, equivariantly contractible through orbigraphs). Thus although the underlying graph of any orbigraph is contractible, none of them (when \( n \geq 2 \)) are equivariantly contractible through orbigraphs. Indeed, the only subgraphs of \( T_n \) that are contractible are those that contain at most one edge, while the only subgraphs of \( H_n \) that are contractible are those consisting of a single cone point.

3.2 Forests, Paths, Circuits

A subgraph of an orbigraph is nontrivial if it contains at least one edge, and a forest if it is nontrivial and each of its connected components is contractible. (In requiring that forests be nontrivial, we follow the convention in [BH92].)

Remark 3.2.1. Suppose \( G \) is a graph of groups with a map \( \gamma: [a, b] \to G \). If the action of \( \pi_1(G) \) on the universal cover \( \hat{G} \) is not free, a choice of lift of \( \gamma(a) \) to \( \hat{G} \) is not sufficient to uniquely specify a lift \( \hat{\gamma}: [a, b] \to \hat{G} \). At each lift of a vertex in \( G \) with nontrivial stabilizer, we need a choice of fundamental domain to cross into. Of course, this is already implicit in the definition (cf. [Ser03]) of the fundamental group of a graph of groups!

Definition 3.2.2. Let \( I \subset \mathbb{R} \) denote a closed interval, possibly infinite. Every proper map \( \hat{\sigma}: I \to \hat{G} \) from such an interval is homotopic rel endpoints to a path, \([\hat{\sigma}]\), which is either a linear embedding or (if \( I \) is finite) a point, in which case \([\hat{\sigma}]\) is a trivial path. The image of \([\hat{\sigma}]\) is unique, and it is really the image we care about. If \( f: \hat{G} \to G \) is a lift to the universal cover of a homotopy equivalence \( f: G \to G \), we write \( \tilde{f}(\hat{\sigma}) \) for \([\tilde{f}(\hat{\sigma})]\). Passing from \( \hat{\sigma} \) to \([\hat{\sigma}]\) is called tightening, and \([\hat{\sigma}]\) is tight.

A path \( \hat{\sigma} \subset \hat{G} \) has an obvious decomposition into a sequence of edges or segments of edges in \( \hat{G} \). For example, if the domain of \( \hat{\sigma} \) is finite, this sequence takes the form \( E'_1 E_2 \cdots E_{k-1} E'_k \), where \( E'_1 \) is a terminal segment of some edge \( E_1 \), \( E_k \) is an initial segment of some edge \( E_k \), and the other \( E_i \) are edges in \( \hat{G} \). A similar decomposition, only singly or doubly infinite occurs when the domain of \( \hat{\sigma} \) is infinite. We will usually identify \( \hat{\sigma} \) with such an edge path decomposition.

Definition 3.2.3. A path \( \sigma \) in an orbigraph \( G \) is a composition \( \pi \circ \hat{\sigma}: I \to \hat{G} \to G \), where \( \hat{\sigma} \) is a path in the universal cover \( \hat{G} \) of \( G \) and \( \pi: \hat{G} \to G \) is the covering map. Any map \( \sigma: I \to G \) with a prescribed lift \( \hat{\sigma}: I \to \hat{G} \) to the universal cover can be tightened rel endpoints to a unique path \([\sigma] = \pi([\hat{\sigma}]])\).

Think of the orbigraph \( G \) as a fundamental domain for the action of \( W = \pi_1(G) \) on \( \hat{G} \). A path \( \hat{\sigma} \) in \( \hat{G} \) changes fundamental domains at lifts of cone points. We record this information as follows: If \( \hat{E} \) and \( \hat{E}' \) share a lift of a cone point \( v \) and lie in different fundamental domains, a conjugate of some \( A_i \in W \) stabilizes \( v \), and in particular, some
conjugate of \( a \in A \) takes the fundamental domain containing \( \hat{E} \) to the fundamental domain containing \( \hat{E}' \). If \( \pi : \hat{G} \to G \) is the covering map, the information \( \pi(\hat{E})\alpha \pi(\hat{E}') \) is sufficient to recover the edge path \( \hat{E}\hat{E}' \) up to a choice of lift of the initial edge.

Thus we will often think of paths \( \sigma \) in \( G \) as their edge path decomposition: a sequence of edges and elements of the cone point stabilizers \( A \).

**Definition 3.2.4.** A circuit in an orbigraph \( G \) is a cyclically ordered sequence of edges and elements of cone point stabilizers \( A \) such that for any choice of initial point the lift to the universal cover is a path.

With our understanding of equivariant homotopy, thus a circuit is an immersion \( S^1 \to G \), and any homotopically nontrivial map \( S^1 \to G \) can be tightened to a unique circuit.

**Remark 3.2.5.** For \( W_n \)-orbigraphs, we will write \( E_iE_{i+1} \) for \( E_i g_i E_{i+1} \) when \( g_i \) is the nonidentity element of \( \mathbb{Z}/2\mathbb{Z} \), and will usually write \( \hat{E} \) for \( \hat{E}\hat{E} \) or \( \hat{E}\hat{E} \), since in most cases only one choice of orientation will be possible. We speak of such paths as taking the cone point. When we consider paths that are the images of paths under homotopy equivalences, we allow paths that begin at cone points to begin or end by taking the cone point. This alerts us that in the universal cover, the new path (or its inverse) begins by traveling in a different fundamental domain than the original path did.

**Example 3.2.6.** Write \( W_3 = \langle a, b, c \rangle \) and label the edges of \( T_3 \) \( A \), \( B \), and \( C \). Fix a lift \( \hat{*} \) of \( * \) to be the identity vertex and identify the other lifts of \( * \) with elements of \( W_n \) via the action. Consider the midpoints of the edge paths in \( \hat{T}_3 \) joining the identity vertex to \( bcbcb \) and \( bacabacab \), respectively. In each case it is the lift of a cone point with stabilizer \( \langle b \rangle \). The edge paths from these midpoints to the identity vertex correspond to the paths \( B\hat{C}B \) and \( B\hat{A}C\hat{A}B \) in \( T_3 \). Writing \( X \) and \( Y \) for the edges of the orbigraph \( \mathcal{H}_3 \) as in Example 3.1.4 The images of these paths under the map \( \tau \) are \( \hat{X}\hat{Y}\hat{X} \) and \( \hat{X}.\hat{Y}.\hat{X} \), respectively. Note that \( \hat{X}.\hat{X} \) is a path, but \( \hat{X}.\hat{Y} \) can be tightened to a point (in either orientation for each).

### 3.3 Markings, Topological Representatives

The understanding developed in Remark 3.2.1 allows us to once and for all identify \( W \) with the group of (tight) closed paths in \( \hat{T}_n \) based at \( * \), where the group operation is concatenation followed by tightening. Indeed, this agrees completely with the usual identification of \( W \) with \( \pi_1(\mathcal{T}_n,\ast) \) as a fundamental group of a graph of groups. This allows us to identify automorphisms \( \Phi : W \to W \) with the automorphisms \( f_{\#} : \pi_1(\mathcal{T}_n,\ast) \to \pi_1(\mathcal{T}_n,\ast) \) induced by (orbigraph) homotopy equivalences \( f : (\mathcal{T}_n,\ast) \to (\mathcal{T}_n,\ast) \) and vice versa.

**Example 3.3.1.** In light of Remark 2.2.1 consider the following palindromic automorphisms of \( F_2 = \langle x, y \rangle \).

\[
\begin{align*}
x &\mapsto xyxyx \\
y &\mapsto xyx
\end{align*}
\]

One checks that these correspond to the automorphisms \( \alpha \) and \( \beta \) of \( W_3 = \langle a, b, c \rangle \) defined
as

\[
\begin{align*}
\alpha: & \begin{cases} 
a \mapsto a \\
b \mapsto bacabacab \\
c \mapsto bacab 
\end{cases} \\
\beta: & \begin{cases} 
a \mapsto a \\
b \mapsto bcbc \\
c \mapsto bc 
\end{cases}
\end{align*}
\]

And thus (cf. Example 3.2.6) homotopy equivalences \( F_\alpha \) and \( F_\beta \) of \( T_3 \) given by (writing \( A, B \) and \( C \) for the edges of \( T_3 \))

\[
\begin{align*}
F_\alpha: & \begin{cases} 
A \mapsto A \\
B \mapsto B\hat{A}\hat{C}\hat{A}B \\
C \mapsto C\hat{A}B 
\end{cases} \\
F_\beta: & \begin{cases} 
A \mapsto A \\
B \mapsto B\hat{C}\hat{B} \\
C \mapsto C\hat{B} 
\end{cases}
\end{align*}
\]

**Definition 3.3.2.** More generally, a marked orbigraph is an orbigraph \( G \) along with a homotopy equivalence \( \tau: T_n \to G \). An orbigraph homotopy equivalence \( f: G \to G \) determines an outer automorphism \( f_2 \) of \( \pi_1(G, \tau(\ast)) \) via \( f_2(\sigma) := [f(\sigma)] \) and thus an outer automorphism \( \varphi \) of \( W \). We assume that \( f(\mathcal{V}) \subset \mathcal{V} \cup \mathcal{C} \) and \( f(\mathcal{C}) = \mathcal{C} \). If in addition the \( f \)-image of each edge is a path, we say that \( f: G \to G \) is a topological representative of \( \varphi \). (cf. [BH92])

A topological representative is determined by the paths making up the images of its edges, together with isomorphisms \( A_i \to f(A_i) \). Thus, for instance if \( \sigma = e_1g_1e_2 \), \( f_2(\sigma) = [f(\sigma)] \) is obtained from \( f(e_1)f(g_1)f(e_2) \) by performing multiplication in the \( A_i \) containing \( f(g_1) \) and then possibly tightening.

**Example 3.3.3.** Consider \( H_3 \) and \( \tau \) as in Examples 3.1.4 and 3.2.6 as a marked orbigraph. The automorphisms \( \alpha \) and \( \beta \) induce topological representatives \( f_\alpha, f_\beta: H_3 \to H_3 \) defined as

\[
\begin{align*}
f_\alpha: & \begin{cases} 
X \mapsto X\hat{Y}\hat{X} \\
Y \mapsto Y\hat{X} 
\end{cases} \\
f_\beta: & \begin{cases} 
X \mapsto X\hat{Y}\hat{X} \\
Y \mapsto Y\hat{X} 
\end{cases}
\end{align*}
\]

### 4 Train Tracks

The goal of this section is to prove the irreducible case of Theorem A, which is Theorem 4.3.3 below. In the Out\((F_n)\) setting, an outer automorphism is irreducible if it preserves the conjugacy class of no free factor. For Out\((W)\), the meaning of irreducible needs alteration: the definition we give below is equivalent to saying that \( \varphi \in \text{Out}(W) \) is irreducible if it preserves the conjugacy class of no infinite free factor.

#### 4.1 Transition Matrices, Irreducibility

We follow [BH92] p. 5: the transition matrix \( M \) associated to \( f: G \to G \) has entries \( a_{ij} \) defined as the number of times the \( f \)-image of the \( j \)th edge crosses the \( i \)th edge in either direction. Both \( f_\alpha \) and \( f_\beta \) have the same transition matrix,

\[
\begin{pmatrix}
3 & 2 \\
2 & 1
\end{pmatrix}
\]
Recall that an \( m \times m \) matrix \( M \) with nonnegative integer entries is \textit{irreducible} \cite{Sen81} if for each pair \( i, j \), there is some power of \( M \) with positive \( a_{ij} \) entry. Every nonnegative integral matrix describes a graph \( \Gamma_M \) with vertices \( v_1, \ldots, v_m \) and \( a_{ij} \) oriented edges from \( v_j \) to \( v_i \). Equivalently, \( M \) is irreducible if and only if for each pair \( i, j \), there is an oriented path from \( v_j \) to \( v_i \). The graph \( \Gamma_M \) for the matrix above has two vertices, \( E_1 \) and \( E_2 \), 3 directed edges from \( E_1 \) to itself, 1 from \( E_2 \) to itself, and 2 each from \( E_1 \) to \( E_2 \) and from \( E_2 \) to \( E_1 \).

A subgraph \( G_0 \) of an orbigraph is \textit{invariant}, with respect to a topological representative \( f: G \to G \) if \( f(G_0) \subseteq G_0 \). A topological representative \( f: G \to G \) is \textit{irreducible} if \( G \) does not contain any nontrivial \( f \)-invariant subgraphs. This is true if and only if its transition matrix is irreducible.

An outer automorphism \( \varphi \) is \textit{irreducible} if every topological representative \( f: G \to G \) of \( \varphi \), for which \( G \) has no valence-one vertices (recall that we do not consider cone points to be vertices) and no invariant forests, is irreducible. If \( \varphi \) is not irreducible we say that it is \textit{reducible}. Thus there must be some orbigraph \( G \) with no valence-one vertices and a topological representative \( f: G \to G \) of \( \varphi \) such that \( G \) contains a nontrivial invariant subgraph, but does not contain a nontrivial invariant forest. Such a representative is a \textit{reduction} for \( \varphi \).

We obtain a \( W \)-analogue of Bestvina–Handel’s criterion \cite{BH92} Lemma 1.2 for reducibility:

\textbf{Lemma 4.1.1.} \textit{If there is a proper, infinite free factor \( W' \) of \( W \) that is invariant up to conjugacy under the action of \( \varphi \), then \( \varphi \) is reducible.}

\textit{Proof.} Choose an automorphism \( \Phi: W \to W \) representing \( \varphi \) with \( \varphi(W') = W' \), and a free factor \( W'' \) such that \( W = W' \ast W'' \). Identify \( \pi_1(\mathcal{T}_n, \ast) \) with \( W \) such that the first \( k \) edges of \( \mathcal{T}_n \) correspond to \( W' \) and the remaining \( (n-k) \) edges correspond to \( W'' \). Then \( \Phi: W \to W \) is represented by a homotopy equivalence \( f: \mathcal{T}_n \to \mathcal{T}_n \) that has \( \mathcal{T}_k \) as a nontrivial invariant subgraph. Perhaps after composing \( \Phi \) with an inner automorphism, we may assume that \( \mathcal{T}_n \) has no invariant forests. Therefore \( \varphi \) is reducible. \hfill \Box

\textbf{Remark 4.1.2.} (cf. \cite{BH92} Remark 1.3) Suppose \( f: G \to G \) is a reduction for \( \varphi \) and that \( G_i = f^i(G_0) \), \( 0 \leq i \leq k-1 \), are distinct noncontractible components of an \( f \)-invariant subgraph. Then each \( G_i \) determines an infinite free factor \( W^i \) such that \( W^1 \ast \cdots \ast W^k \) is a free factor of \( W \) and such that \( \varphi \) cyclically permutes the conjugacy classes of the \( W^i \).

The converse of Lemma 4.1.1 is true, as it is in the \text{Out}(F_n) setting. Although the proof is analogous to \cite{BH92} Lemma 1.16, the presence of contractible subgraphs in the thistle graphs, \( \mathcal{T}_k \), requires a little care. We prove the converse at the end of this section.

Next, we begin adapting Bestvina–Handel’s method of altering an arbitrary homotopy equivalence representing \( \varphi \in \text{Out}(W) \) in order to better reason about its properties.

\subsection*{4.2 Forest Collapse}

We say that a homotopy equivalence \( f: G \to G \) is \textit{tight} if for each edge \( e \in G \), either \( f(e) \) is a (tight) edge path, or \( f(e) \) is a point in \( \mathcal{G} \cup \mathcal{V} \). A homotopy equivalence \( f: G \to G \) can be \textit{tightened} to a tight homotopy equivalence by a homotopy rel \( \mathcal{G} \cup \mathcal{V} \).
Lemma 4.2.1 ([BH92] p. 7). If \( f: G \to G \) is a tight homotopy equivalence, collapsing a maximal (with respect to inclusion) pretrivial forest in \( G \) produces a topological representative \( f': G' \to G' \). If instead \( f: G \to G \) is a topological representative of an irreducible outer automorphism \( \varphi \in \text{Out}(W) \) and \( G \) has no valence-one vertices, collapsing a maximal invariant forest yields an irreducible topological representative \( f': G' \to G' \).

A forest in an orbigraph \( G \) is pretrivial with respect to a homotopy equivalence \( f: G \to G \) if each edge in the forest is eventually mapped to a point. Maximal pretrivial forests are, in particular, invariant. We describe how to collapse invariant forests.

If \( f: G \to G \) is a tight homotopy equivalence and \( G_0 \subseteq G \) is an invariant forest, define \( G_1 = G/G_0 \) to be the quotient orbigraph obtained by collapsing each component of \( G_0 \) to a point. Such a point is a cone point in \( G_1 \) if its preimage contains a cone point. Let \( \pi: G \to G_1 \) be the quotient map, and define \( f_1 = \pi f \pi^{-1}: G_1 \to G_1 \). Since \( G_0 \) was \( f \)-invariant, this is well-defined. If \( e \subseteq G \) is an edge not in \( G_0 \), then the edge path for \( f_1(e) \) is obtained from \( f(e) \) by deleting all occurrences of edges in \( G_0 \). Since \( f \) was tight, if \( e \sigma \bar{e} \) is a subpath of the \( f \)-image of some edge \( e' \) not in \( G_0 \), where \( \sigma \) is a nontrivial path in \( G_0 \), then \( \sigma \) must be of the form \( \sigma' a \sigma' \) for some path \( \sigma' \subseteq G_0 \) and \( a \) an element of a cone point stabilizer. In \( f_1(e') \), \( e \sigma \bar{e} \) is replaced by \( ea \bar{ea} \). This implies that \( f_1: G_1 \to G_1 \) is tight. The transition matrix for \( f_1: G_1 \to G_1 \) is obtained from the transition matrix for \( f: G \to G \) by deleting the rows and columns associated to the edges of \( G_0 \).

4.3 Turns, Train Track Maps

A turn in \( G \) is an pair of edges of \( G \) originating at a common vertex or cone point. If originating at a cone point, as in Remark 3.2.1, we require an element of the cone point stabilizer to make an unambiguous choice of lift to the universal cover \( \tilde{G} \) up to the action of \( W \). Thus in the language of Example 3.3.3, \( (\bar{X},\bar{Y}) \) and \( (\bar{X},\bar{Y}) \) are distinct turns in \( \mathcal{H}_3 \). If each \( A_t \) is abelian, we may take our turns to be unordered. A turn is nondegenerate if it is defined by distinct oriented edges, or if the stabilizer element is nontrivial, and is degenerate otherwise.

As in [BH92], we write \( Df \) for the self-map on the set of oriented edges of \( G \) induced by a topological representative \( f: G \to G \) that sends each oriented edge to the first oriented edge in its \( f \)-image. We write \( Tf \) for the corresponding map on the set of turns; \( Tf \) acts as \( f \) on the stabilizer element. For example, if \( f(E_1) = g_1E_1, f(E_2) = g_2E_2 \), then \( Tf \) sends the turn \((E_1,g,E_2)\) to \((E_1,g_1g_2,E_2)\); hence our requirement of an ordering. In Example 3.3.3, both \( f_\alpha \) and \( f_\beta \) define the same map \( Df \), namely \( X,Y \mapsto X, \bar{X} \mapsto \bar{X}, \bar{Y} \mapsto \bar{Y} \).

A turn is illegal if some iterate of \( Tf \) maps it to a degenerate turn, and is legal otherwise. In Example 3.3.3, \( (X,Y) \) is the only illegal turn. Given a path \( \sigma = E_1' g_1 E_2 g_2 \ldots E_k' \) in \( G \), as in Remark 3.2.1, we say \( \sigma \) crosses or contains the turns \( \{E_i, g_i, E_{i+1}\} \). A path is legal if it contains only legal turns.

Definition 4.3.1. A topological representative \( f: G \to G \) for an irreducible outer automorphism \( \varphi \in \text{Out}(W) \) is a train track map if for each edge \( e \) in \( G \), \( f(e) \) is a legal path. (Compare [BH92] p. 8.)

Thus \( f_\alpha \) is a train track map but \( f_\beta \) is not, since \((X,.,Y)\) is legal while \((X,Y)\) is not. The main tool Bestvina–Handel use to construct train tracks is the following:
Theorem 4.3.2 (Perron–Frobenius \cite{Sen81}). Suppose $M$ is an irreducible, nonnegative integral matrix. There is a unique positive eigenvector $\vec{w}$ of norm one for $M$, and its associated eigenvalue satisfies $\lambda \geq 1$. If $\lambda = 1$, $M$ is a transitive permutation matrix. Moreover if $\vec{v}$ is a positive vector and $\mu > 0$ satisfies $(M\vec{v})_i \leq \mu v_i$ for each $i$ and $(M\vec{v})_j < \mu v_j$ for some $j$, then $\lambda < \mu$.

The idea is to reduce the eigenvalue to a minimum through a series of moves until a train track is reached.

Theorem 4.3.3. Every irreducible outer automorphism $\varphi$ of $W_n$ is topologically represented by an irreducible train track map on an orbigraph. Any irreducible topological representative $f : G \to G$ whose transition matrix has minimal Perron–Frobenius eigenvalue $\lambda$ is a train track map. If $\lambda = 1$, then $f : G \to G$ is a finite-order homeomorphism. cf. \cite[Theorem 1.7]{BH92}

Lemma 4.3.4 (Thurston \cite{BH92} Remark 1.8). If $\varphi \in \text{Out}(W)$ is an irreducible outer automorphism, any train track map representing $\varphi$ has the same Perron–Frobenius eigenvalue $\lambda$. In particular, $\lambda$ is equal to the exponential growth rate of $\varphi$.

Remark 4.3.5. An irreducible outer automorphism $\varphi \in \text{Out}(W_n)$ may have a different Perron–Frobenius eigenvalue than $\varphi^{-1}$. For example, let $\{X, Y, Z\}$ be the edges of $\mathcal{H}_4$, let $f : \mathcal{H}_4 \to \mathcal{H}_4$ be defined by $X \mapsto Y$, $Y \mapsto Z$ and $Z \mapsto XY$. (the image of $Z$ begins by taking a cone point) and let $g : \mathcal{H}_4 \to \mathcal{H}_4$ be defined by $X \mapsto Z.X$, $Y \mapsto X$ and $Z \mapsto Y$. Then $f$ and $g$ are irreducible train track maps that are homotopy inverses of each other, but the transition matrix for $f$ has Perron–Frobenius eigenvalue the “golden ratio” $\frac{1+\sqrt{5}}{2}$, while the transition matrix for $g$ has Perron–Frobenius eigenvalue the real root of the polynomial $x^3 - 2x^2 - 1$, which is greater than 2. (cf. \cite[Remark 1.8]{BH92})

4.4 Elementary Moves

With the understanding developed in this and the previous section, we now adapt the tools Bestvina–Handel use to prove Theorem 4.3.3, namely subdivision, folding, valence-one homotopy, and valence-two homotopy. The main difference in our setting is that cone points require more care than vertices. As a convenience to the reader, we will briefly recall the definition of these moves, note the areas where care is needed in the presence of cone points, and the effect each move has on the Perron–Frobenius eigenvalue $\lambda$. We refer the reader to \cite[Lemmas 1.10–1.15]{BH92} for proofs. An expert reader may safely proceed to the proof.

Throughout suppose $f : G \to G$ is a topological representative on a marked orbigraph $G$. When we mention $\varphi \in \text{Out}(W)$, it will be assumed to be an irreducible outer automorphism. As we alter $G$, the marking is changed in the obvious way.

Subdivision. \cite[Lemma 1.10]{BH92} If $w$ is not a vertex nor a cone point of $G$, but $f(w)$ is, we may give $G$ a new orbigraph structure $G_1$ by declaring $w$ to be a vertex. This yields a new topological representative $f_1 : G_1 \to G_1$. The new topological representative is irreducible if $f : G \to G$ was, and the associated Perron–Frobenius eigenvalues are equal. We say $f_1 : G_1 \to G_1$ is obtained from $f : G \to G$ by subdivision.
Folding. [BH92] Lemma 1.15] Suppose some pair of edges $e_1, e_2$ in $G$ have the same $f$-image. Define a new orbigraph $G_1$ by identifying $e_1$ and $e_2$ to a single edge $e$ such that $f: G \to G$ descends to a well-defined homotopy equivalence $f_1: G_1 \to G_1$. This is an elementary fold. More generally, if $e_1'$ and $e_2'$ are maximal initial segments of $e_1$ and $e_2$ with endpoints in $f^{-1}(e \cup f')$, we first subdivide at the endpoints of $e_1'$ and $e_2'$ if they are not already vertices and then perform an elementary fold on the resulting edges. Note that because $f$ is a homotopy equivalence, $e_1' \cup e_2'$ contains at most one cone point, whose image in $G_1$ remains a cone point.

If $f: G \to G$ was an irreducible representative of $\varphi \in \text{Out}(W)$, then the topological representative $f_2: G_2 \to G_2$ obtained from $f_1: G_1 \to G_1$ by tightening, collapsing a maximal pretrivial forest and collapsing a maximal invariant forest is irreducible. The Perron–Frobenius eigenvalue $\lambda_2$ of $f_2: G_2 \to G_2$ is equal to $\lambda$ if no tightening occurs, and satisfies $\lambda_2 < \lambda$ otherwise.

Valence-One Homotopy. [BH92] Lemma 1.11] Suppose $v$ is a valence-one vertex (not a cone point) of $G$ with incident edge $e$. Let $\pi_1$ be the deformation retraction that collapses $v$ and its incident edge to a point in $G_1 = G \setminus \{v, \text{int}(e)\}$. Let $f_2: G_2 \to G_2$ be the topological representative obtained from $\pi_1 f |_{G_1}: G_1 \to G_1$ by tightening and collapsing a maximal pretrivial forest. We call this operation valence-one homotopy. Note that this operation, if we allowed it in the case that $v$ is a valence-one cone point, would change the fundamental group of $G$.

If $f: G \to G$ was an irreducible representative of $\varphi \in \text{Out}(W)$ and $f_3: G_3 \to G_3$ is the irreducible representative obtained by a sequence of valence-one homotopies followed by the collapse of a maximal invariant forest, then the associated Perron–Frobenius eigenvalue $\lambda_3$ satisfies $\lambda_3 < \lambda$.

Valence-Two Homotopy. [BH92] Lemma 1.13] Suppose $v$ is a vertex (again, not a cone point) of $G$ with valence two. Assume it is the terminal vertex of $e$ and the initial vertex of $e'$. Let $g_1: G \to G$ be a homotopy with support in $e \cup e'$ that collapses $e'$ and stretches $e$ across the path $ee'$. Notice that $v$ is not in the image of $f$ under the map $g_1 f: G \to G$. Give $G$ a new orbigraph structure by removing $v$ from the set of vertices, and then define a new topological representative $f_1: G_1 \to G_1$ by tightening $g_1 f$ on the new orbigraph and then collapsing a maximal pretrivial forest. This operation is called valence-two homotopy of $e$ across $e'$.

Let us continue to call the resulting edge $e'$. Then if $e''$ is an edge in $G \setminus \{e \cup e'\}$ that is not collapsed, then $f_1(e'')$ is obtained from $f(e'')$ by removing all occurrences of $e'$ and $e''$, tightening, and then removing all edges that are collapsed in the pretrivial forest. $f_1(e)$, likewise, is obtained from $f(e \cup e')$ by the same operations if $e$ is not collapsed.

If $v$ were a cone point, then $g_1 f$ would not satisfy $f(e') = e'$, since following $f$ with $g_1$ does not move the cone point, just the image of $f$.

Suppose $f: G \to G$ was an irreducible representative of $\varphi \in \text{Out}(W)$ with Perron–Frobenius eigenvalue $\lambda$ and a choice of associated positive eigenvector $\vec{v}$. Write $v$ for the eigenvector coefficient of $\vec{v}$ corresponding to $e$ and $v'$ for the eigenvector coefficient corresponding to $e'$. If $f_2: G_2 \to G_2$ is the irreducible representative obtained by a valence-two homotopy of $e$ across $e'$ followed by the collapse of a maximal invariant forest and $v' \leq v$, then the associated Perron–Frobenius eigenvalue $\lambda_2$ satisfies $\lambda_2 \leq \lambda$. 

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If $v' < v$, then $\lambda_2 < \lambda$. Thus we always perform valence-two homotopies by collapsing the edge with smaller eigenvector coefficient when they differ.

We are now ready to adapt Bestvina–Handel’s proof of Theorem 4.3.3 to the orbigraph setting. (Cf. [BH92, Theorem 1.7 p. 16–17])

**Proof of Theorem 4.3.3.** Let $f : G \to G$ be an irreducible topological representative of $\varphi$ such that $G$ has no valence-one or valence-two vertices. (Recall that cone points are not vertices.) For example, if a chosen representative of $\varphi$ on $\mathcal{T}_n$ is not irreducible, we may collapse a maximal invariant forest to yield an irreducible representative on $\mathcal{H}_n$. If $f$ is not a train track map, we will construct a new irreducible topological representative $f_4 : G_4 \to G_4$ with $1 \leq \lambda_4 < \lambda$.

In this case, $G$ has at most $2n - 3$ edges. One way to see this is that the underlying space of $G$ is a tree with a total of $n$ valence-one and valence-two vertices (the cone points of $G$). Cyclically ordering these $n$ vertices and attaching $n$ edges connecting each vertex to its neighbors in the order yields a graph $G'$ with fundamental group $F_n$ without any valence-one or valence-two vertices. An Euler characteristic argument shows that $G'$ has at most $3n - 3$ vertices, proving the claim.

Crucially, this allows us to argue, as in the $\text{Out}(F_n)$ setting, that there are only a finite number of possible transition matrices that our process can yield, so eventually our eigenvalue reaches a minimum, at which point we must have a train track map.

If $\lambda = 1$, $M$ is a transitive permutation matrix and our homotopy equivalence $f$ must actually be a homeomorphism and thus finite order.

So assume $\lambda > 1$ and that $f : G \to G$ is not a train track map. By assumption, there is a point $P \in G \setminus G^0$ with $f(P)$ a cone point or vertex such that $f^k(P)$ is not injective at $P$ for some $k > 1$. Choose a small neighborhood $U$ of $P$ such that the following hold.

1. $\partial U = \{s, t\} \subset f^{-\ell}(G^0)$;
2. $f^i|_U$ is one-to-one for $1 \leq i \leq k - 1$;
3. $f^k(U \setminus P)$ is two-to-one onto a subset of a single edge;
4. $P \notin f^i(U)$, $1 \leq i \leq k$.

The first step is a repeated subdivision, adding $P$ and $\{f^i(s), f^i(t) \mid 0 \leq j \leq \ell - 1\}$ to the set of vertices in reverse order, first $P$ and $f^{\ell-1}(s)$ and $f^{\ell-1}(t)$, and so on. The subdivisions yield a new irreducible topological representative $f_1 : G_1 \to G_1$.

Next, we fold. In $G_1$, $P$ has valence two. Call the incident edges $\alpha$ and $\beta$. First we fold $Df^{\ell-1}_{\alpha}$ and $Df^{\ell-1}_{\beta}$. Either nontrivial tightening occurs and $\lambda_2$, the Perron–Frobenius eigenvalue for the resulting topological representative satisfies $\lambda_2 < \lambda$, or $P$ still has valence two and we may fold $Df^{k-2}(\alpha)$ and $Df^{k-2}(\beta)$. After folding $k$ times if necessary, we have either decreased $\lambda$, or $P$ is a valence-one vertex of the resulting orbigraph.

We remove valence-one vertices (not cone points) via homotopies. This yields a new irreducible topological representative $f_3 : G_3 \to G_3$ with $\lambda_3 < \lambda$. The only concern is that this process may have created valence-two vertices (not cone points), and our assumption requires only cone points to have valence one or two. So we remove the valence-two vertices via homotopies, yielding an irreducible topological representative $f_4 : G_4 \to G_4$, with $\lambda_4 < \lambda$. 

\[\square\]
We conclude by stating and proving a converse to Lemma 4.1.1. Since every outer automorphism \( \varphi \in \text{Out}(W) \) permutes the conjugacy classes of the finite free factors of \( W \), for irreducibility to be not vacuous, we consider infinite free factors (or more generally, free factors that are not conjugate to vertex groups). The proof is analogous to [BH92, Lemma 1.16].

**Lemma 4.4.1.** If there are proper, infinite free factors \( W^1, \ldots, W^k \) of \( W \) such that \( W^1 * \cdots * W^k \) is a free factor of \( W \) and \( \varphi \in \text{Out}(W) \) cyclically permutes the conjugacy classes of the \( W^i \), then \( \varphi \) is reducible.

**Proof.** Let \( n_i \) be the Kurosh rank of the free factors \( W^i \) and \( n_{k+1} \) the Kurosh rank of a complementary free factor \( W^{k+1} \) so that \( W = W^1 * \cdots * W^k * W^{k+1} \). For \( 1 \leq i \leq k+1 \), let \( T^i \) be the thistle with \( n_i \) prickles (if \( n_{k+1} = 0, T^{k+1} \) is a vertex) and vertex \( v_i \). For each \( i \) with \( 1 \leq i \leq k \), we choose automorphisms \( \Phi_i : W \to W \) representing \( \varphi \) such that \( \Phi(W^i) = W^i + 1 \), with indices taken mod \( k \). We let \( f_i : T^i \to T^{i+1} \) be the corresponding homotopy equivalence fixing the vertex of \( T_i \) (again, for \( 1 \leq i \leq k \), with indices mod \( k \)). Define \( G \) to be the union of the \( T^i, 1 \leq i \leq k+1 \), together with, for \( 1 \leq i \leq k \), an edge \( E_i \) connecting \( v_i \) to \( v_{i+1} \) (in that orientation).

Collapsing the \( E_i \) to a point yields a homotopy equivalence \( G \to T_n \). Identifying the image of \( \pi_1(T^i, v_i) \) with \( W^i \) will serve as a marking. We will use \( \Phi_1 \) to create a topological representative \( f : G \to G \) for \( \varphi \). Define \( f(T^i) = f_i(T^i) \) for \( 1 \leq i \leq k \). By assumption, for \( 1 \leq i \leq k \), there exist \( c_i \in W \) such that \( \Phi_1(x) = c_i \Phi_1(x) \). Choose \( \gamma_i \) a closed path based at \( v_{k+1} \) representing \( c_i \) (so \( \gamma \) is the trivial path) and define \( f(E_i) = \gamma_i E_{i+1} \), with indices taken mod \( k \). Finally, define \( f(T^{k+1}) \) by \( \Phi_1 \) and the marking on \( G \).

This is a reduction for \( \varphi \) unless \( G \) has an invariant forest. Since thistles have contractible subgraphs, there are a few possibilities. If there is a family of edges \( e_1, \ldots, e_k \) with \( e_i \in T^i \) and \( f(e_i) = e_{i+1} \) with indices mod \( k \), we may collapse each of these edges. Likewise if some edge in \( T^{k+1} \) is sent to itself, we may collapse it. If each \( c_i = 1 \in W \), then the \( E_i \) also form an invariant forest if they are contractible (i.e. the subgraph spanned by them contains at most one cone point.) After all these forest collapsings, the only worry is that \( n_{k+1} = 0 \) and the \( E_i \) would be collapsed, leaving \( G \) as the only \( f \)-invariant subgraph. In this case, choose \( A \) an edge of \( T^1 \) sharing an initial vertex with \( E_1 \), and change \( f \) via a homotopy with support in \( E_1 \) so that \( f(E_1) = f(A) f(A) E_2 \). Then fold the initial segment of \( E_1 \) mapping to \( f(A) \) with all of \( A \). The resulting graph is combinatorially identical to \( G \) but the markings differ. Now \( f(E_1) = f(A) E_2 \) and \( f(E_k) = A E_1 \), so the \( E_i \) no longer form an invariant forest.

5 **Relative Train Tracks**

Having constructed train tracks for irreducible automorphisms in Theorem 4.3.3, our goal in this section is to prove the general case of Theorem A by constructing relative train track maps for each \( \varphi \in \text{Out}(W) \). In [PH11], Feighn and Handel add a number of properties to relative train track maps, which were originally defined in [BH92]. We reserve the term relative train track map for topological representatives \( f : G \to G \) for outer automorphisms \( \varphi \in \text{Out}(W) \) that satisfy all of these properties. To state the definition of a relative train track map on an orbigraph (Theorem 5.2.1 below), we need more notation. We have mostly followed the conventions in [PH11 Section 2], so a reader
familiar with the literature on Out($F_n$) might need only note that we ignore finite free factors of $W$ (more generally, free factors that are conjugate to vertex groups) and safely skip ahead to Section 6.

### 5.1 Preliminaries

Throughout this subsection, $f : G \to G$ will denote a topological representative of an outer automorphism $\varphi \in \text{Out}(W)$ on an orbigraph $G$.

#### Splittings.
We refer the reader to Section 3 for details on paths and circuits. A decomposition of a path or circuit $\sigma$ into subpaths is a splitting for $f : G \to G$, denoted $\ldots \sigma_1 \cdot \sigma_2 \cdot \ldots$ with centered dots if $f_k^k(\sigma) = \ldots f_k^k(\sigma_1) f_k^k(\sigma_2) \ldots$ for all $k \geq 0$. That is, one can tighten the $f_k$-image of $\sigma$ just by tightening the images of the $\sigma_i$ and then concatenating. From the perspective of $f_k$, the $\sigma_i$ do not interact.

#### Nielsen Paths.
A path $\sigma$ is a periodic Nielsen path if $\sigma$ is nontrivial and $f_k^k(\sigma) = \sigma$ for some $k \geq 1$. The minimal such $k$ is the period of $\sigma$, and $\sigma$ is a Nielsen path if it has period 1. A periodic Nielsen path is indivisible if it cannot be written as a concatenation of nontrivial periodic Nielsen paths.

#### Filtrations.
A filtration on an orbigraph is an increasing sequence, $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$ of $f$-invariant subgraphs. That is, for each $i$, $f(G_i) \subset G_i$. The subgraphs are not required to be connected.

#### Strata.
The $r$th stratum is the subgraph $H_r = G_r \setminus G_{r-1}$. A path or circuit has height $r$ if it intersects the interior of $H_r$. Recall from Section 4 that a topological representative $f : G \to G$ induces a map $Df$ on the set of edges in $G$ and a map $Tf$ on the set of turns in $G$. If both edges of a turn $T$ are contained in a stratum $H_r$, then $T$ is a turn in $H_r$. If a path or circuit $\sigma$ has height $r$ and contains no illegal turns in $H_r$, then $\sigma$ is $r$-legal.

#### Transition Submatrices.
Relabeling the edges of $G$ and thus permuting the rows and columns of $M$ so that the edges of $H_i$ precede those of $H_{i+1}$, $M$ becomes block upper-triangular, with the $i$th block $M_i$ equal to the square submatrix of $M$ containing those rows and columns corresponding to edges in $H_i$.

A filtration is maximal when each $M_i$ is either irreducible or the zero matrix. If $M_i$ is irreducible, call $H_i$ an irreducible stratum and a zero stratum otherwise. If $H_i$ is irreducible, $M_i$ has an associated Perron–Frobenius eigenvalue $\lambda_i \geq 1$. If $\lambda_i > 1$, then $H_i$ is an exponentially-growing stratum.

Otherwise, $\lambda_i = 1$, $H_i$ is non-exponentially-growing and $M_i$ is a transitive permutation matrix. If $H_r$ is an exponentially-growing stratum and $\sigma \subset G_{r-1}$ is a nontrivial path with endpoints at vertices in $H_r \cap G_{r-1}$, then $\sigma$ is a connecting path for $H_r$.

Throughout the paper, we will assume our filtrations are maximal unless otherwise specified. For the remainder of the subsection, we will denote the filtration associated to $f : G \to G$ as $\emptyset = G_0 \subset \cdots \subset G_m = G$. Given such a filtration, the following
Lemma 5.1.1. Given a topological representative $f: G \to G$ and a filtration $\emptyset = G_0 \subset \cdots \subset G_m = G$, there is a maximal filtration that contains the $G_i$ as filtration elements.

Proof. Let $H_i$ be a stratum of the original filtration with transition matrix $M_i$. If $M_i$ is already irreducible or the zero matrix, nothing need be done. Otherwise, consider $\Gamma_{M_i}$, the associated directed graph. For each edge in $H_i$, $\Gamma_{M_i}$ has a vertex, and for each time the $f$-image of the $j$th edge crosses the $i$th edge in either direction, there is a directed edge from the $j$th vertex to the $i$th vertex. (See Section 3 or [Sen81].) A subgraph $\Gamma'$ of $\Gamma_{M_i}$ is strongly-connected if between each pair of vertices $v$ and $w$ in $\Gamma'$, there are directed edge paths from $v$ to $w$ and from $w$ to $v$. The maximal strongly-connected subgraphs of $\Gamma_{M_i}$ are its strongly-connected components. Each component $\Gamma'$ determines an irreducible stratum in the new filtration.

Not every vertex need belong to a strongly-connected component. The ones that do not are partitioned into zero strata that are as large as possible as follows. For vertices $v$ and $w$ in $\Gamma_k$ not contained in any strongly-connected component, write $v \sim w$ if there is no directed edge path from $v$ to $w$, nor from $w$ to $v$. The equivalence classes of the equivalence relation generated by $\sim$ are the zero strata.

We have our strata, now we need an order on them. Given two strata $H_j$ and $H_k$ resulting from our operations, put $H_j$ before $H_k$ if there exists $e \in H_j$ and $e' \in H_k$ such that in $\Gamma_{M_i}$ there is a directed edge path from the vertex corresponding to $e'$ to the vertex corresponding to $e$. Complete this to a total order on the strata arbitrarily.

Non-exponentially-growing Strata. If $H_r$ is non-exponentially-growing but not periodic, each edge $e$ has a subinterval which is eventually mapped back over $e$, so the subinterval contains a periodic point. After declaring all of these periodic points to be vertices, reordering, reorienting, and possibly replacing $H_r$ with two non-exponentially-growing strata, we may assume the edges $E_1, \ldots, E_k$ of $H_r$ satisfy $f(E_i) = E_{i+1}u_i$, where the indices are taken mod $k$ and $u_i$ is a path in $G_{r-1}$. We always adopt this convention.

Eigenvalues. Let $H_{r_1}, \ldots, H_{r_k}$ be the exponentially-growing strata for $f: G \to G$. We define $\text{PF}(f)$ to be the sequence of associated Perron–Frobenius eigenvalues, $\lambda_{r_1}, \ldots, \lambda_{r_k}$ in nonincreasing order. We order the set

$$\{\text{PF}(f) \mid f: G \to G \text{ is a topological representative for some } \varphi \in \text{Out}(W)\}$$

lexicographically; thus if $\text{PF}(f) = \lambda_1, \ldots, \lambda_k$ and $\text{PF}(f') = \lambda'_1, \ldots, \lambda'_\ell$, then $\text{PF}(f) < \text{PF}(f')$ if there is some $j$ with $\lambda_j < \lambda'_j$ and $\lambda_i = \lambda'_i$ for $1 \leq i < j$, or $k < \ell$ and $\lambda_i = \lambda'_i$ for $1 \leq i \leq k$.

Free Factor Systems. In Section 4 we declared an outer automorphism $\varphi \in \text{Out}(W)$ to be irreducible if it preserved the conjugacy class of no infinite free factor. In this section we will call a free factor of $W$ trivial if it is finite (or more generally, if it is conjugate to a vertex group). If $W^r$ is a free factor of $W$, let $[[W^r]]$ denote its conjugacy class. If $W^1, \ldots, W^k$ are nontrivial free factors and $W^1 * \cdots * W^k$ is a free factor of $W$, the collection $\{[[W^1]], \ldots, [[W^k]]\}$ is a free factor system.
Example 5.1.2. If \( f : G \to G \) is a topological representative and \( G_r \subset G \) is an \( f \)-invariant subgraph with noncontractible (connected) components \( C_1, \ldots, C_k \), then the conjugacy classes \([\pi_1(C_i)]\) of the fundamental groups of the \( C_i \) are well-defined. We define
\[
\mathcal{F}(G_r) := \{[\pi_1(C_1)], \ldots, [\pi_1(C_k)]\}.
\]
Our convention on contractibility implies that each \([\pi_1(C_i)]\) is infinite. We say that \( G_r \) realizes \( \mathcal{F}(G_r) \).

Out(\( W \)) (or more generally, the \( \varphi \) we are interested in) acts on the set of conjugacy classes of free factors of \( W \). If \( \varphi \in \text{Out}(W) \) is an outer automorphism and \( W' \) a free factor, \([W']\) is \( \varphi \)-invariant if \( \varphi([W']) = [W'] \). In this case, there is some automorphism \( \Phi : W \to W \) representing \( \varphi \) such that \( \Phi(W') = W' \) and \( \Phi|_{W'} \) is well-defined up to an inner automorphism of \( W' \), so it induces an outer automorphism \( \varphi|_{W'} \in \text{Out}(W') \), which we will call the restriction of \( \varphi \) to \( W' \).

There is a partial order \( \sqsubseteq \) on free factor systems: We say \([W_1]\) \( \sqsubseteq \) \([W_2]\) if \( W_1 \) is conjugate to a free factor of \( W_2 \). We say \( \mathcal{F}_1 \sqsubseteq \mathcal{F}_2 \) for free factor systems \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) if for each \([W']_i \in \mathcal{F}_1 \) there exists \([W']_j \in \mathcal{F}_2 \) such that \([W']_i \sqsubseteq [W']_j \).

Thus if we order the set of complexities of free factor systems of \( W \) lexicographically, then \( \mathcal{F}_1 \sqsubseteq \mathcal{F}_2 \) implies \( \text{cx}(\mathcal{F}_1) \leq \text{cx}(\mathcal{F}_2) \).

Dynamics on \( G \). Let \( \text{Per}(f) \) denote the set of \( f \)-periodic points in \( G \). The subset of points with period one is \( \text{Fix}(f) \). A subgraph \( C \subset G \) is wandering if \( f^k(C) \subset G \setminus C \) for all \( k \geq 1 \) and is non-wandering otherwise.

Definition 5.1.3. The core of a subgraph \( C \subset G \) is the minimal subgraph \( K \) of \( C \) such that the inclusion is a homotopy equivalence. Equivalently, it is the set of edges in \( C \) that are crossed by some circuit contained in \( C \).

Definition 5.1.4. Suppose that \( u < r \) and that the following hold.

1. \( H_u \) is irreducible.
2. \( H_r \) is exponentially-growing and all components of \( G_r \) are noncontractible.
3. For each \( i \) with \( u < i < r \), \( H_i \) is a zero stratum that is a component of \( G_{r-1} \), and each vertex of \( H_i \) has valence at least two in \( G_r \).

Then we say that \( H_i \) is enveloped by \( H_r \), and write \( H_r^* = \bigcup_{k=u+1}^r H_k \).

5.2 Relative Train Track Maps

In this section we state and prove the existence of relative train track maps for outer automorphisms \( \varphi \in \text{Out}(W) \).

Theorem 5.2.1 (cf. [FH11] Theorem 2.19). Given an outer automorphism \( \varphi \in \text{Out}(W) \), there exists a homotopy equivalence \( f : G \to G \) on an orbigraph representing \( \varphi \), together with a filtration \( \emptyset = G_0 \subset \cdots \subset G_\ell = G \) of \( f \)-invariant subgraphs satisfying the following properties:

(V) The endpoints of all indivisible periodic Nielsen paths are vertices.
If a periodic stratum $H_m \subset \text{Per}(f)$ is a forest, then there exists a filtration element $G_j$ such that $\mathcal{F}(G_j) \neq \mathcal{F}(G_\ell \cup H_m)$ for any filtration element $G_\ell$.

Each zero stratum $H_i$ is enveloped by an exponentially-growing stratum $H_r$. Each vertex of $H_i$ is contained in $H_r$ and meets only edges in $H_i \cup H_r$.

The terminal endpoint of an edge in a non-periodic, non-exponentially-growing stratum $H_i$ is a cone point, hence periodic and is contained in a filtration element $G_j$ with $j < i$ that is its own core.

The core of a filtration element is a filtration element.

For every exponentially-growing stratum $H_r$, we have

(EG-i) $Df$ maps the set of edges in $H_r$ to itself; every turn with one edge in $H_r$ and the other in $G_{r-1}$ is legal.

(EG-ii) If $\sigma \subset G_{r-1}$ is a connecting path for $H_r$, then $f_\sharp(\sigma)$ is as well. In particular, $f_\sharp(\sigma)$ is nontrivial.

Moreover, if $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d$ is a nested sequence of $\phi$-invariant free factor systems, we may assume that each free factor system is realized by some filtration element.

Any homotopy equivalence $f: G \to G$ and filtration $\emptyset = G_0 \subset \cdots \subset G_m = G$ that satisfies the premises and conclusions of this theorem is called a relative train track map.

The definition allows for homotopy equivalences that are not topological representatives. In the course of the proof we will construct topological representatives $f: G \to G$ that satisfy the conclusions of the proposition. The advantage of this definition is that it allows iteration: if $f: G \to G$ is a relative train track map, then possibly after enlarging the filtration, so is $f^k$ for $k > 1$.

Historically, relative train track maps for $\phi \in \text{Out}(F_n)$ were defined by Bestvina–Handel as topological representatives satisfying (EG-i), (EG-ii) and (EG-iii). In this section, we will call topological representatives satisfying these three properties pre-relative train track maps, PRTTs.

In much of the rest of this section we will follow the constructions in [BH92, Section 5], [FH11, Section 2] and [FH18, Section 2]. We sketch the outline of the proof:

Given an outer automorphism $\phi \in \text{Out}(W)$, we begin with a topological representative that is bounded, a term which will be defined below. We construct a bounded topological representative $f: G \to G$ that satisfies the “moreover” statement of Theorem 5.2.1. The construction is analogous to that used in the proof of Lemma 4.4.1. We use two new operations, described in Lemma 5.4.1 and Lemma 5.4.2 so that the resulting topological representative satisfies (EG-i) and (EG-ii). If (EG-iii) is not satisfied, as in [BH92] and [FH18], we modify the algorithm in the proof of Theorem 4.3.3 to reduce $PF(f)$, the set of Perron–Frobenius eigenvalues for the exponentially-growing strata of $f: G \to G$, while remaining bounded. The boundedness assumption ensures that we will hit a minimum value after a finite number of moves. We accomplish this as Theorem 5.4.4. Afterwards, we adapt arguments of Feighn–Handel in [FH11, Theorem 2.19, pp. 56–61] to satisfy the remaining properties.
Bounded Representatives. As we observed in the proof of Theorem 4.3.3 if \( G \) is a \( W \)-orbigraph without valence-one or valence-two vertices (not cone points), then \( G \) has at most \( 2n - 3 \) edges. Our assumption that \( \varphi \) was irreducible allowed us to remove valence-two vertices, but we cannot always do this in the general case. Instead, we will call a topological representative \( f: G \to G \) bounded if there are at most \( 2n - 3 \) exponentially-growing strata, and if, for each exponentially-growing stratum \( H_r \), the associated Perron–Frobenius eigenvalue \( \lambda_r \) is also the Perron–Frobenius eigenvalue of a matrix with at most \( 2n - 3 \) rows and columns (compare [BH92, p. 37]). As in the proof of Theorem 4.3.3 if \( f: G \to G \) is bounded, the set of \( \text{PF}(f') \) for \( f' \) also representing \( \varphi \) with \( \text{PF}(f') \leq \text{PF}(f) \) is finite, so operations decreasing it will eventually reach a minimum, which we will denote \( \text{PF}_{\text{min}} \).

5.3 Elementary Moves Revisited

We briefly revisit the four elementary moves defined in Section 4 with an eye to their effect on \( \text{PF}(f) \). Proofs are contained in [BH92, Lemmas 5.1–5.4] Throughout we will assume \( f: G \to G \) is a topological representative of an outer automorphism \( \varphi \in \text{Out}(W) \). We denote the associated filtration \( \emptyset = G_0 \subset \cdots \subset G_m = G \).

Subdivision. [BH92, Lemma 5.1] Suppose \( f': G' \to G' \) is obtained from \( f: G \to G \) by subdividing an edge \( e \) in \( H_r \) into two edges \( e_1 \) and \( e_2 \). If \( H_r \) is a zero stratum, then \( e_1, e_2 \) and the remaining edges in \( H_r \) determine a zero stratum. If \( H_r \) is irreducible, it is possible that at most one of \( e_1 \) or \( e_2 \) is entirely mapped into \( G_{r-1} \). If so, that edge determines a new zero stratum. In any case, \( \text{PF}(f') = \text{PF}(f) \).

Folding. [BH92, Lemma 5.3] Suppose \( f': G' \to G' \) is obtained from \( f: G \to G \) by folding a pair of edges \( e_1, e_2 \) in \( G \). If \( e_1 \) and \( e_2 \) lie in different strata, the higher one is a zero stratum, no edges are collapsed, and \( \text{PF}(f') = \text{PF}(f) \). In fact, \( \text{PF}(f') = \text{PF}(f) \) unless \( e_1 \) and \( e_2 \) lie in a single exponentially-growing stratum \( H_r \) and nontrivial tightening occurs, necessarily in \( H_r \). In this case, \( \lambda_r \) is replaced by some number of strictly smaller \( \lambda' \), so \( \text{PF}(f') < \text{PF}(f) \).

Valence-One Homotopy. [BH92, Lemma 5.2] Suppose \( f': G' \to G' \) is obtained from \( f: G \to G \) by a valence-one homotopy at a vertex \( v \). If the edge incident to \( v \) is contained in an exponentially-growing stratum \( H_r \), then as above, \( \lambda_r \) is replaced by some number of strictly smaller \( \lambda' \) and \( \text{PF}(f') < \text{PF}(f) \). Otherwise \( \text{PF}(f') = \text{PF}(f) \).

Valence-Two Homotopy. [BH92, Lemma 5.4] Suppose \( f': G' \to G' \) is obtained from \( f: G \to G \) by performing a valence two homotopy of \( E_j \) across \( E_i \), where \( E_i \) and \( E_j \) are edges in strata \( H_i \) and \( H_j \), respectively, with \( i \leq j \). If \( H_i \) is not exponentially-growing, \( \text{PF}(f') = \text{PF}(f) \). If \( i < j \) and \( H_i \) is exponentially-growing, then \( \text{PF}(f') < \text{PF}(f) \). If \( i = j \) and \( H_i \) is exponentially-growing, then \( \lambda_i \) is replaced by some number of \( \lambda' \) that satisfy \( \lambda' \leq \lambda_i \). Thus in this case it is possible that \( \text{PF}(f') > \text{PF}(f) \).

In order to preserve boundedness, we need more control of valence-two homotopy. Call an elementary move safe if performing it on a topological representative \( f: G \to G \) yields a new topological representative \( f': G' \to G' \) with \( \text{PF}(f') \leq \text{PF}(f) \).
Lemma 5.3.1 ([BH92] Lemma 5.5). If \( f : G \rightarrow G \) is a bounded topological representative and \( f' : G' \rightarrow G' \) is obtained from \( f \) by a sequence of safe moves, with \( \text{PF}(f') < \text{PF}(f) \). Then there is a bounded topological representative \( f'' : G'' \rightarrow G'' \) with \( \text{PF}(f'') < \text{PF}(f) \).

The idea of the proof is the following: first, given \( f' : G' \rightarrow G' \), we perform all valence-one homotopies and safe valence-two homotopies. Then we perform dangerous valence-two homotopies until the resulting homotopy equivalence is bounded. The resulting topological representative satisfies \( \text{PF}(f') \leq \text{PF}(f'') < \text{PF}(f) \).

5.4 (EG-i) and (EG-ii)

We recall the construction of the *invariant core subdivision* of an exponentially-growing stratum \( H_r \). Assume for the moment that a topological representative \( f : G \rightarrow G \) linearly expands edges over edge paths with respect to some metric on \( G \). If \( f(H_r) \) is not entirely contained in \( H_r \), then the set

\[
I_r := \{ x \in H_r \mid f^k(x) \in H_r \text{ for all } k > 0 \}
\]

is an \( f \)-invariant Cantor set. The *invariant core* of an edge \( e \) in \( H_r \) is the smallest closed subinterval of \( e \) containing the intersection of \( I_r \) with the interior of \( e \). The endpoints of invariant cores of edges in \( H_r \) form a finite set which \( f \) sends to itself. Declaring elements of this finite set of points to be vertices is called *invariant core subdivision*. (We have added the adjective “invariant” in an attempt to avoid potential confusion vis-à-vis multiple uses of the word “core.”)

The following lemma says that invariant core subdivision can be used to create topological representatives whose exponentially-growing strata satisfy [EG-i]

Lemma 5.4.1 ([BH92] Lemma 5.13). If \( f' : G' \rightarrow G' \) is obtained from \( f : G \rightarrow G \) by an invariant core subdivision of an exponentially-growing stratum \( H_r \), then \( \text{PF}(f') = \text{PF}(f) \), \( Df' \) maps the set of edges in \( H_r' \) to itself, so \( H_r' \) satisfies [EG-i]. If \( H_j \) is another exponentially-growing stratum for \( f : G \rightarrow G \) that satisfies [EG-i] or [EG-ii], then the resulting exponentially-growing stratum \( H_j' \) for \( f' : G' \rightarrow G' \) still satisfies these properties.

In fact, invariant core subdivision affects only edges in \( H_r \). If new vertices are created, then one or more non-exponentially-growing strata are added to the filtration below \( H_r \). Orienting the resulting edges away from the newly created vertices, these strata already satisfy our standing assumption on non-exponentially-growing strata.

The following lemma says that an application of operations already defined may be used to construct topological representatives whose exponentially-growing strata satisfy [EG-ii]

Lemma 5.4.2 ([BH92] Lemma 5.14). Let \( f : G \rightarrow G \) be a topological representative with exponentially-growing stratum \( H_r \). If \( \alpha \subset G_{r-1} \) is a path with endpoints in \( H_r \cap G_{r-1} \) such that \( f_\alpha \) is trivial, we construct a new topological representative \( f' : G' \rightarrow G' \). Let \( H_r' \) be the stratum of \( G' \) determined by \( H_r \). The intersection \( H_r' \cap G_{r-1}' \) has fewer points than \( H_r \cap G_{r-1} \).

The new topological representative \( f' : G' \rightarrow G' \) is constructed by subdividing at points in \( \alpha \cap f^{-1}(G_0) \) and repeatedly folding, following by tightening and collapsing a pretrivial forest. As such, \( \text{PF}(f') \leq \text{PF}(f) \). If \( H_j \) is an exponentially-growing stratum for \( f : G \rightarrow \)
G with \( j > r \) that satisfies (EG-i) or (EG-ii), the resulting exponentially-growing stratum \( H_j' \) still satisfies these properties. If \( H_r \) satisfies (EG-i), then so does \( H_j' \).

Moreover, by paying attention to the action of an iterate of \( f \) on the set of points in \( H_r \cap G_{r-1} \), Feighn and Handel prove the following.

**Lemma 5.4.3** ([FH18] Lemma 2.4). There is an algorithm that checks whether a topological representative \( f : G \to G \) satisfies (EG-ii). Since (EG-i) and (EG-iii) are finite properties, there is an algorithm that checks whether \( f : G \to G \) is a PRTT.

**Proof.** Since (EG-i) is a finite property, we may assume that each exponentially-growing stratum satisfies (EG-i). So suppose \( H_j \) is an exponentially-growing stratum and that \( C \) is a component of \( G_{j-1} \). If \( C \) is contractible, then there are only finitely many paths in \( C \) with endpoints at vertices, so checking (EG-ii) for connecting paths in \( C \) is a finite property. So suppose \( C \) is noncontractible, or more generally non-wandering. We claim that (EG-ii) for \( C \) is equivalent to the property that each vertex in \( H_j \cap C \) is periodic. Note that cone points are assumed to be periodic. If this latter property fails, there is some \( k > 0 \) and points \( v \) and \( w \) in \( H_j \cap C \), with \( f^k(v) = f^k(w) \). In this case, there is some path connecting \( f^{k-1}(v) \) and \( f^{k-1}(w) \) whose \( f^k \)-image is trivial (consider what a homotopy inverse for \( f \) does to \( f^k(v) \)), so (EG-ii) fails. If each vertex in \( H_j \cap C \) is periodic, each connecting path \( \alpha \) for \( H_j \) contained in \( C \) has either distinct endpoints or determines a loop in its free homotopy class that takes at least one cone point. Both of these properties are preserved by \( f_j \), so (EG-ii) holds. Finally, (EG-iii) for \( H_j \) is equivalent to checking that \( f_j(e) \) is \( j \)-legal for each edge \( e \in H_j \), so is a finite property.

We are now able to prove the existence of PRTTs.

**Theorem 5.4.4.** For every outer automorphism \( \varphi \in \text{Out}(W) \), there exists a topological representative \( f : G \to G \) on an orbigraph that satisfies (EG-i), (EG-ii) and (EG-iii). If \( \mathcal{F}_1 \sqsubseteq \cdots \sqsubseteq \mathcal{F}_d \) is a nested sequence of \( \varphi \)-invariant free factor systems, we may assume that each free factor system is realized by a filtration element.

Our proof is adapted from [BFH00] Lemma 2.6.7, [BH92] Theorem 5.12 and Lemma 5.9 and [FH18] Theorem 2.2 and Corollary 2.10.

**Proof.** The first step is to construct a bounded topological representative \( f : G \to G \) and associated filtration such that each \( \mathcal{F}_i \) is realized by a filtration element. We proceed by induction on \( d \), the length of the nested sequence of \( \varphi \)-invariant free factor systems. The case \( d = 1 \) is accomplished in Lemma 4.4.1. Otherwise, let \( \mathcal{F}_d = \{[[W^1]], \ldots, [[W^k]]\} \). For each \( j \) with \( 1 \leq j \leq d-1 \) and each \( i \) with \( 1 \leq i \leq k \), write \( \mathcal{F}_j^i \) for the set of conjugacy classes of free factors in \( \mathcal{F}_j \) that are conjugate into \( W^i \). Then for each \( i \),

\[
\mathcal{F}_1^i \sqsubseteq \cdots \sqsubseteq \mathcal{F}_{d-1}^i
\]

is a nested sequence of \( \varphi \)-invariant free factor systems. By induction, there are orbigraphs \( K^i \) and topological representatives \( f_i : K^i \to K^i \) representing the restriction \( \varphi|_{W^i} \) of \( \varphi \) to \( W^i \) together with associated (not maximal!) filtrations \( K_1^i \sqsubseteq \cdots \sqsubseteq K_{j-1}^i \) such that \( K_j^i \) realizes \( \mathcal{F}_j^i \). Inductively, we may assume that \( f_i \) fixes some vertex or cone point \( v_i \) of \( K^i \), and that \( K^i \) has no valence-one or valence-two vertices.
As in the proof of Lemma 4.4.1, take a complementary free factor $W^{k+1}$ so that $W = W^1 \ast \cdots \ast W^k \ast W^{k+1}$, and an associated thistle $T^{k+1}$. The orbigraph $G$ is constructed as follows: begin with the disjoint union of the $K^i$. Glue $T^{k+1}$ to $K^1$ by identifying the vertex of the thistle with the fixed point $v_1$. Then attach an edge connecting $v_1$ to $v_i$ for $2 \leq i \leq k$. The resulting orbigraph has no valence-one or valence-two vertices, so is bounded. Define $f: G \to G$ from the $f_i$ as in the proof of Lemma 4.4.1. For $1 \leq j \leq d-1$, let $G_j = \bigcup_{i=1}^k K^i_j$, and define $G_d = \bigcup_{i=1}^k K^i$. Then $\emptyset = G_0 \subset \cdots \subset G_d \subset G_{d+1} = G$ is an $f$-invariant filtration, and each $F_i$, $1 \leq i \leq d$ is realized by $G_i$. Apply Lemma 5.1.1 to complete this filtration to a maximal filtration. This completes the first step.

The next step is to promote our topological representative to a PRTT. Firstly, we note (cf. the proof of [BPH00, Lemma 2.6.7]) that the moves described in the previous subsection and in Section 4 all preserve the property of realizing free factors. More precisely, suppose $C_1$ and $C_2$ are disjoint noncontractible components of some filtration element $G_r$ and $f': G' \to G'$ is obtained from $f: G \to G$ by collapsing a pretrivial forest, folding, subdivision, invariant core subdivision, valence-one homotopy or (properly restricted) valence-two homotopy. If $p: G \to G'$ is the identifying homotopy equivalence, then $p(C_1)$ and $p(C_2)$ are disjoint, noncontractible subgraphs of $G'$. Thus, we may work freely without worrying about our free factor systems.

We begin with the highest exponentially-growing stratum $H_r$ of $G$. We check whether $H_r$ satisfies (EG-i) and (EG-ii) using Lemma 5.4.3. If not, apply Lemma 5.4.1 and Lemma 5.4.2 to create a new topological representative, still called $f: G \to G$, such that the resulting exponentially-growing stratum $H_r$ satisfies (EG-i) and (EG-ii). Repeat with the next highest exponentially-growing stratum until all exponentially-growing strata satisfy these properties. Check whether the resulting topological representative, which we still call $f: G \to G$, satisfies (EG-iii). If it does, we are done.

If not, then there is some edge $e$ in an exponentially-growing stratum $H_r$ such that $f_r(e)$ is not $r$-legal. (Here we need (EG-i) to be satisfied.) We apply the algorithm in the proof of Theorem 4.3.3: there is a point $P$ in $H_r$ where $f^k$ is not injective at $P$ for some $k > 1$. We subdivide and then repeatedly fold. Either we have reduced the eigenvalue or created a valence-one vertex. We remove all valence-one vertices via homotopies and perform all valence-two homotopies which do not increase $	ext{PF}(f)$. At this point we have created a new topological representative $f': G' \to G'$ with $\text{PF}(f') < \text{PF}(f)$, but $f'$ may not be bounded. Use Lemma 5.3.1 to produce a new bounded topological representative $f'': G'' \to G''$ with $\text{PF}(f'') < \text{PF}(f)$. If (EG-i) and (EG-ii) are not satisfied by $f''$, we may restore these properties without increasing $\text{PF}(f'')$. Because $\text{PF}(f)$ can only be decreased a finite number of times before reaching $\text{PF}_{\text{min}}$, eventually this process terminates, yielding a PRTT.

\[\square\]

**Corollary 5.4.5.** If $f: G \to G$ is a bounded topological representative satisfying (EG-i) and with $\text{PF}(f) = \text{PF}_{\text{min}}$, then the exponentially-growing strata of $f$ satisfy (EG-iii).

The main applications of the properties (EG-i) through (EG-iii) are the following:

**Lemma 5.4.6** ([FHH11, Lemma 2.9]). Suppose $f: G \to G$ is a PRTT and that $H_r$ is an exponentially-growing stratum.

1. Suppose $v$ is a vertex or cone point of $H_r$ that is contained in a component $C$ of
that is noncontractible or more generally satisfies $f^i(C) \subset C$ for some $i > 0$. Then $v$ is periodic and at least one edge of $H_r$ incident to $v$ is $Df$ periodic.

2. If $\sigma$ is an $r$-legal circuit or path of height $r$ with endpoints, if any, at vertices of $H_r$, then the decomposition of $\sigma$ into single edges in $H_r$ and maximal subpaths in $G_{r-1}$ is a splitting.

Lemma 5.4.7 ([FH11] Lemma 2.10). Let $H_r$ be an exponentially-growing stratum of a PRTT $f: G \to G$, and let $v$ be a vertex of $H_r$. Then there is a legal turn in $G_r$ based at $v$. In particular, no vertex of $H_r$ has valence one in $G_r$.

Proof. Let $v \in H_r$ be a vertex as in the statement. By [EG-i] and the fact that the transition submatrix for $H_r$ is irreducible, there is some point $w$ in the interior of an edge of $H_r$ that is mapped to $v$. By [EG-iii] this implies $v$ is in the interior of an $r$-legal path. So the turn based at $v$ taken by this path is legal.

5.5 The Remaining Properties

Proof of Theorem 5.2.1. To complete the proof, we adapt the proof of [FH11] Theorem 2.19 pp. 56–62.

Property (V). To prove (V) we need to collect more information about Nielsen paths.

Lemma 5.5.1 (cf. [FH11] Lemma 2.11). Suppose $f: G \to G$ is a PRTT and $H_r$ is an exponentially-growing stratum.

1. There are only finitely many indivisible periodic Nielsen paths of height $r$.

2. If $\sigma$ is an indivisible periodic Nielsen path of height $r$, then $\sigma$ contains exactly one illegal turn in $H_r$.

3. After perhaps declaring the endpoints to be vertices, we may assume the initial and terminal points of $\sigma$ are in $G^0$. Let $E$ and $E'$ be the initial and terminal edges of $\sigma$. Then $\sigma$ has period 1 if and only if $E$ and $E'$ are fixed by $Df$.

Feighn and Handel use the foregoing to show that in fact,

Lemma 5.5.2 ([FH11] Lemma 2.12). If $f: G \to G$ is a PRTT, there are only finitely many points in $G$ that are the endpoints of an indivisible periodic Nielsen path. If these points are not already in $G^0$, they lie in the interior of exponentially-growing strata.

Note that if $E$ is a periodic edge, then it is a Nielsen path, but it is not indivisible. This latter lemma implies that (V) can be accomplished by declaring these periodic points to be vertices. This process preserves [EG-i] through [EG-iii].

Sliding. The move sliding was introduced in [BFH00] Section 5.4, p. 579]. Suppose $H_i$ is a non-periodic, non-exponentially-growing stratum that satisfies our convention: the edges $E_1, \ldots, E_k$ satisfy $f(E_i) = E_{i+1}u_i$ where indices are taken mod $k$ and $u_i$ is a path in $G_{i-1}$. We will call the edge of $H_i$ we focus on $E_1$. Let $\alpha$ be a path in $G_{i-1}$ from the terminal endpoint of $E_1$ to some vertex or cone point of $G_{i-1}$. Define a new
orbigraph $G'$ by removing $E_1$ from $G$ and gluing in a new edge, $E_1'$ with initial point the same as the initial point of $E_1$ and terminal point the terminal point of $\alpha$. See Figure 2.

Define homotopy equivalences $p: G \to G'$ and $p': G' \to G$ by sending each edge other than $E_1$ and $E_1'$ to itself, and $p(E_1) = E_1', p'(E_1') = E_1\alpha$. Define $f': G' \to G'$ by tightening $pf'$: $G' \to G'$. If $G_r$ is a filtration element of $G$, define $G'_r := p(G_r)$. The $G'_r$ form the filtration for $f': G' \to G'$.

Lemma 5.5.3 (Lemma 2.17 of [FH11]). Suppose $f': G' \to G'$ is obtained from $f: G \to G$ by sliding $E_1$ along $\alpha$ as described above. Let $H_i$ be the non-exponentially-growing stratum of $G$ containing $E_1$, and let $k$ be the number of edges in $H_i$.

1. $f': G' \to G'$ is a PRTT if $f$ was.
2. $f'|_{G'_{i-1}} = f|_{G_{i-1}}$.
3. If $k = 1$, then $f'(E_1') = E_1'[\bar{\alpha}u_1f(\alpha)]$.
4. If $k \neq 1$, then $f'(E_k) = E_1'[\bar{\alpha}uk]$, $f'(E_1') = E_2[u_1f(\alpha)]$ and $f'(E_j) = E_{j+1}u_j$ for $2 \leq j \leq k-1$.
5. For each exponentially-growing stratum $H_r$, $p_\sharp$ defines a bijection between the set of indivisible periodic Nielsen paths in $G$ of height $r$ and the indivisible periodic Nielsen paths in $G'$ of height $r'$.

![Figure 2: Sliding $E_1$ along $\alpha$](image)

(NEG) Part One. We will first show that the terminal endpoint of an edge in a non-exponentially-growing stratum $H_i$ is either periodic or has valence at least three. Let $E_1, \ldots, E_k$ be the edges of $H_i$. As usual, assume $f(E_i) = E_{i+1}u_i$ where indices are taken mod $k$ and $u_i \subset G_{i-1}$ is a path in lower strata. Suppose the terminal endpoint $v_1$ of $E_1$ is not periodic and has valence two. Then $v_1$ is not a cone point. If $E$ is the other edge incident to $v_1$, then $E$ does not belong to an exponentially-growing stratum $H_r$, by Lemma 5.4.7.
We perform a valence-two homotopy of $\bar{E}_1$ over $E$. The argument above shows that if $v$ is a vertex or a cone point of an exponentially-growing stratum, $f(v) \not= v_1$, so before collapsing the pretrivial forest, [(EG-i)] through [(EG-iii)] are preserved. The pretrivial forest is inductively constructed as follows: any edge which was mapped to $E$ is added, then any edge which is mapped into the pretrivial forest is added. Again, the argument above shows that no vertex or cone point of an exponentially-growing stratum is incident to any edge in the pretrivial forest. After repeating this dichotomy finitely many times, the terminal endpoint $v_1$ of $E_1$ is either periodic or has valence at least three in $G$. If $f: G \to G$ was a PRTT and a topological representative, it remains so.

Finally, we arrange that $v_1$ is periodic: the component of $G_{i-1}$ containing $v_1$ is nonwandering (because $f^{k-1}(u_1)$ is contained in it), so contains a periodic vertex $w_1$. Choose a path $\alpha$ from $v_1$ to $w_1$ and slide $E_1$ along $\alpha$. No valence-one vertices are created, because $v_1$ was assumed to have valence at least three. Repeating this process finitely many times for each edge in non-exponentially-growing strata we establish the first part of [(NEG)] namely the following.

(NEG*) The terminal endpoint of each edge in a non-exponentially-growing stratum is periodic.

(Z) Part One. Property [(Z)] has several parts. Let $H_i$ be a zero stratum. For $H_i$ to be enveloped by $H_r$, let $H_u$ be the first irreducible stratum below $H_i$ and $H_r$ the first irreducible stratum above. We need to show that $H_r$ is exponentially-growing, that no component of $G_r$ is contractible, and another condition on $H_i$, see Definition 5.1.4. We need to show also that each vertex of $H_i$ is contained in $H_r$ and meets only edges of $H_i$ and $H_r$. Following [FH11, p. 58], we will prove almost all of these properties now; here we only show that each component of $G_r$ is non-wandering, rather than that it is non-contractible.

First we arrange that if a filtration element $G_i$ has a wandering component, then $H_i$ is a wandering component. Suppose $G_i$ has wandering components. Call their union $W$ and their complement $N$. If $N$ is $f$-invariant, so it is contained in a union of strata. If $N$ is not precisely equal to a union of strata, the difference is that $N$ contains part but not all of a zero stratum, so we may divide this zero stratum to arrange so that $N$ is a union of strata. Thus $W$ is a union of zero strata. Since $N$ is $f$-invariant, we may push all strata in $W$ higher than all strata in $N$. We define a new filtration. Strata in $N$ and higher than $G_i$ remain unchanged. The strata that make up $W$ will be the components of $W$. If $C$ and $C'$ are such components, $C'$ will be higher than $C$ if $C' \cap f^k(C) = \emptyset$ for all $k \geq 0$. We complete this to an ordering on the components of $W$, yielding the desired result.

Now we work toward showing that zero strata are enveloped by exponentially-growing strata. Suppose that $K$ is a component of the union of all zero strata in $G$, that $H_i$ is the highest stratum that contains an edge of $K$ and that $H_u$ is the highest irreducible stratum below $H_i$. We aim to show $K \cap G_u = \emptyset$ (we want $H_u$ to play the role it does in Definition 5.1.4). So assume $K \cap G_u \not= \emptyset$. By the previous paragraph, because $H_u$ is irreducible, each component of $G_u$ is nonwandering, so $K$ meets $G_u$ in a unique component $C$ of $G_u$. $K$ has at most one cone point, being a zero stratum, and this cone point, if it exists, belongs to $K \cap C$. We will show that $K$ has a vertex $v$ that has valence one in $K \cup C$. If not, because the underlying graph of $C$ is a tree, by connectivity, $K$ is
properly contained in \( C \). This is a contradiction, since \( K \) is assumed to contain an edge that is not in \( G_u \).

This valence-one vertex \( v \) is not periodic, because \( f \) eventually maps \( K \cup C \) into \( C \), so by (NEG*) \( v \) is not an endpoint of an edge in a non-exponentially growing stratum. But \( v \) is also not an endpoint of an edge in an exponentially-growing stratum \( H_i \) by part (1) of Lemma 4.6. Since \( K \) is a component of the union of all zero strata, \( v \) must have valence one in \( G \). But PRRTs are constructed without valence-one vertices, and we have not created any valence-one vertices so far. This contradiction implies that \( K \cap G_u = \emptyset \). In particular, \( K \) contains no cone points. The lowest edge in \( K \) is mapped either to another zero stratum or into \( G_u \). In any case, by connectivity, \( K \) is wandering, so we can reorganize zero strata so that \( K = H_i \). Repeating this for each component of the union of all zero strata, we have arranged that if \( H_i \) is a zero stratum and \( H_i \) is the first irreducible stratum above \( H_i \), then \( H_i \) is a component of \( G_{r-1} \), most of part (3) of Definition 5.1.4.

Let \( H_r \) be the first irreducible stratum above \( H_i \). Because \( H_r \) is irreducible, no component of \( G_r \) is wandering, so the component that contains \( H_i \) intersects \( H_r \). No vertex of \( H_i \) is periodic, so (NEG*) implies \( H_i \) is exponentially-growing. What’s more, part (1) of Lemma 5.4.6 implies that vertices of \( H_i \) only meet edges of \( H_i \) and \( H_r \), and every vertex of \( H_i \) has valence at least two in \( H_r \). This satisfies every part of Definition 5.1.4 except that we have not shown that all components of \( G_r \) are noncontractible, only that they are nonwandering.

Let \( C \) be the component of \( G_r \) containing \( H_i \). Choose \( k \geq 1 \) so that \( f^k(C) \subset C \). Note that the homotopy equivalence \( f^k|_C : C \to C \) is a PRRT, so no vertex of \( H_i \cap C \) has valence one in \( C \); that is, the leaves of the underlying graph of \( C \) are cone points in \( C \), so every edge of \( H_i \) is contained in some circuit in \( G_r \). That is, \( H_i \) is contained in the core of \( G_r \).

**Tree Replacement.** The remaining part of property \([Z]\) we will show now is that every vertex of \( H_i \) is contained in \( H_r \). We do so by Feighn and Handel’s method of tree replacement. Replace \( H_i \) with a tree \( H'_i \) whose vertex set is exactly \( H_i \cap H_r \). We may do so for every zero stratum at once, (with \textit{a priori} different exponentially-growing strata \( H_r \), of course) and call the resulting orbigraph \( G' \). Let \( X \) denote the union of all irreducible strata. There is a homotopy equivalence \( p' : G' \to G \) that is the identity on edges in \( X \) and sends each edge in a zero stratum \( H'_i \) to the unique path in \( H_i \) with the same endpoints. Choose a homotopy inverse \( p : G \to G' \) that also restricts to the identity on edges in \( X \) and maps each zero stratum \( H_i \) to the corresponding tree \( H'_i \). Define \( f' : G' \to G' \) by tightening \( pfp' : G' \to G' \). Vertices in \( X \) are \( f \)-invariant, by (EG-i) and (NEG*), and \( f' \) still satisfies (EG-i). Because \( p \) and \( p' \) send nontrivial paths with endpoints in \( X \) to nontrivial paths with endpoints in \( X \), (EG-ii) is preserved as well. Because \( PF(f) = PF(f') \), Corollary 5.4.5 implies \( f' \) is still a PRRT. The other properties are likewise preserved.

**Property (P).** Following [HTT][p. 56], we will show that if \( H_m \subset \text{Per}(f) \) is a periodic forest and \( F_1 \subset \cdots \subset F_d \) is our chosen nested sequence of \( \varphi \)-invariant free factor systems realized by the filtration, then there is some \( F_i \) that is not realized by \( H_m \cup G_f \) for any filtration element \( G_f \). Assume that this does not hold for some periodic forest \( H_m \). Then,
for all $F_i \leq i \leq d$, there is some filtration element $G_i$ such that $F(H_m \cup G_i) = F_i$.
In this case, we will collapse an invariant forest containing $H_m$, reducing the number of non-exponentially-growing strata. Iterating this process establishes the result, which implies $[P]$

Let $Y$ be the set of all edges in $G \setminus H_m$ eventually mapped into $H_m$ by some iterate of $f$. Each edge of $Y$ is thus contained in a zero stratum. We want to arrange that if $\alpha$ is a path in a zero stratum with endpoints at vertices that is not contained in $Y$, then $f_j(\alpha)$ is not contained in $Y \cup H_m$. If there is such a path $\alpha$, let $E_i$ be an edge crossed by $\alpha$ and not contained in $Y$. Perform a tree replacement as above, removing $E_i$ and adding in an edge with endpoints at the endpoints of $\alpha$. By our preliminary form of $(Z)$, if a vertex incident to $E_i$ has valence two, then it is an endpoint of $\alpha$, so this does not create valence-one vertices. The image of the new edge is contained in $Y \cup H_m$, so we add it to $Y$. Because there are only finitely many paths in zero strata with endpoints at vertices, we need only repeat this process finitely many times if necessary.

Let $G'$ be the orbigraph obtained by collapsing each component of $H_m \cup Y$ to a point, and let $p: G \to G'$ be the quotient map. Identify the edges of $G \setminus (Y \cup H_m)$ with the edges of $G'$, and define $f': G' \to G'$ on each edge $E$ of the complement as $[pf(E)]$. By construction, $f'$ is a topological representative, and $f'(E)$ is obtained from $f(E)$ by removing all incidences of edges in $Y \cup H_m$. If $p(H_r)$ is not empty, the strata $H_r$ and $p(H_r)$ are thus of the same type, and that $f'$ has one fewer non-exponentially-growing stratum and possibly fewer zero strata. The previous properties, $(\text{NEG}2)$ and our preliminary form of $(Z)$ are still satisfied.

Let $H_r$ be an exponentially-growing stratum. By Lemma 5.4.3, checking $(\text{EG-ii})$ for $p(H_r)$ is equivalent to checking that each vertex of $p(H_r) \cap C$ is periodic for each non-wandering component $C$ of $p(G_{r-1})$. Let $v'$ be such a vertex. By assumption, there is a vertex $v \in H_r$ such that $p(v) = v'$. If $v$ is periodic, we are done. If not, Lemma 5.4.3 implies that the component of $G_{r-1}$ containing $v$ is wandering, contradicting the assumption that $C$ is wandering. But then $p^{-1}(v') = \{v\}$, $v$ is also contained in a non-wandering component of $G_{r-1}$ and part (1) of Lemma 5.4.6 implies $v$ is periodic. This verifies $(\text{EG-ii})$. It is easy to see that $(\text{EG-iii})$ is still satisfied, and that $PF(f) = PF(f')$, so Corollary 5.4.5 implies $f'$ is still a PRTT.

It remains to check that our family of free factor systems is still realized. Let $F_j$ be such a free factor system. By assumption on $H_m$, there is $G_j$ (depending on $j$) such that $F(G_j \cup H_m) = F_j$. Each non-contractible component of $G_j \cup H_m$ is mapped into itself by some iterate of $f$. Since $Y$ is eventually mapped into $H_m$, some power of $f$ induces a bijection between the non-contractible components of $G_j \cup H_m \cup Y$ and those of $G_j \cup H_m$, so $p(G_j)$ realizes $F_j$ by assumption. Repeating this process decreases the number of non-exponentially-growing strata, so eventually property $[P]$ is established.

**Property (Z).** Suppose that $C$ is a non-wandering component of some filtration element. We will show that $C$ is noncontractible. The lowest stratum $H_i$ containing an edge of $C$ is either exponentially-growing or periodic. If $H_i$ is exponentially-growing and $v$ is the endpoint of some edge of $H_i$, Lemma 5.4.7 implies that $v$ is either a cone point or has valence at least two in $H_i$, showing that $C$ is noncontractible. If instead $H_i$ is periodic, we show that $[P]$ implies $H_i$ is not a forest. If it were, $[P]$ says in particular that there is a filtration element $G_j$ such that $F(G_j) \neq F(G_j \cup H_m)$. Thus $j < m$. This is only possible if $H_m$ is not disjoint from $G_j$, but it is by assumption. Therefore $H_i$
must not be a forest, and in particular \( C \) is not contractible. This proves that \([Z]\) follows from the form of \([Z]\) we have already established.

**Property (NEG).** Let \( E \) be an edge in a non-exponentially-growing stratum \( H_i \). Let \( C \) be the component of \( G_{i-1} \) containing the terminal point \( v \) of \( H_i \); it is non-wandering by our work proving \([\text{NEG}^*]\). By the argument in the previous paragraph, if \( H_i \) is the lowest stratum containing an edge of \( C \), \( H_i \) is either periodic or exponentially-growing. In both cases, the argument in the previous paragraph shows that \( H_i \) is noncontractible; in fact \( H_i \) is its own core, since the argument in either case shows that no vertex of \( H_i \) has valence one in \( H_i \). In fact, in each case, \( f(H_i) \subset H_i \), so we may rearrange the filtration so that the \( H_i \) are at the bottom of the filtration.

Choose a cone point \( w \) of \( H_i \), a path \( \tau \) in \( G_{i-1} \) from \( v \) to \( w \) and slide \( E \) along \( \tau \). If we work up through the filtration, repeating this process arranges for \([\text{NEG}]\) to be satisfied. The resulting homotopy equivalence is still a PRTT by Corollary 5.4.5, still realizes the nested sequence of invariant free factor systems and still satisfies \([Z]\). This time, sliding may have introduced vertices of valence one. But \([\text{NEG}]\) together with \([Z]\) and Lemma 5.4.7 imply that only valence-one vertices are mapped to the valence-one vertices created. We perform valence-one homotopies to remove each of these vertices. If property \([P]\) is not satisfied, restore it and repeat. Since the number of non-exponentially-growing strata decreases, this process terminates.

**Property (F).** We want to show that the core of each filtration element is a filtration element. If \( H_i \) is a zero stratum, then \( F(G_i) = F(G_{i-1}) \), so assume that \( H_i \) is irreducible, and thus \( G_i \) has no contractible components. If a vertex \( v \) has valence one in \( G_i \), then \([\text{NEG}]\) implies it is not the terminal endpoint of a non-periodic, non-exponentially-growing edge. Lemma 5.4.7 shows that \( H_k \), the stratum containing \( v \), is not exponentially-growing, and \([Z]\) implies that it is not a zero stratum. If \( H_k \) were periodic, it would be a forest, because every edge would be incident to a vertex of valence one in \( H_k \), so \([P]\) implies that some and hence every valence-one vertex of \( H_k \) is contained in some lower filtration element. This exhausts the possibilities: \( v \) must be the initial endpoint of a non-exponentially-growing edge. All edges in such a stratum have initial endpoint a valence-one vertex of \( G_i \), and no vertex of of valence at least two in \( G_i \) maps to them. Thus we may push all such non-exponentially-growing strata \( H_k \) above \( G \setminus H_k \). After repeating this process finitely many times, \( F(G_i) \) is realized by a filtration element that is its own core. Working upwards through the strata, \([F]\) is satisfied. 

6 Properties of Relative Train Tracks, \( \text{Out}^0(W) \)

In this section, we make useful observations about the behavior of cone points in relative train track maps. Throughout this section, let \( f: G \to G \) be a relative train track map representing \( \varphi \in \text{Out}(W) \), let \( F_1 \subset \cdots \subset F_d \) the chosen nested sequence of \( \varphi \)-invariant free factor systems, and let \( G = G_0 \subset \cdots \subset G_m = G \) be the associated filtration.

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6.1 Cone Points

**Lemma 6.1.1.** If $C$ is a subgraph of $G$, the core of $C$ is the convex hull (within $C$) of the cone points it contains. If a component of $C$ contains only one cone point $c$, then $c$ is a component of the core of $C$.

*Proof.* A circuit $\sigma$ in an orbifigraph looks something like a rubber band. As a closed path in the underlying graph, $\sigma$ appears to backtrack over itself. Wherever it does so there is a cone point in $G$ that is taken by $\sigma$; otherwise $\sigma$ would not be a tight path. Since the underlying graph of an orbifigraph or a connected component of one is a tree, there is a unique path between any two cone points. Such a path obviously gives rise to a circuit.

**Lemma 6.1.2.** If a subgraph $C$ of $G$ is wandering, it contains at most one cone point, which is contained in $G \setminus C$. If $f: G \to G$ is a relative train track map, then (Z) implies that zero strata in $G$ have no cone points.

*Proof.* The first statement follows from the fact that $f$ is a homotopy equivalence and cone points are periodic. The second statement follows because if $H_k$ is a zero stratum, (Z) implies that $G_{k-1} \cap H_k$ is empty. 

Thus, cone points are incident only to irreducible strata.

### 6.2 Out$^0(W)$

Recall that Out$^0(W)$ is the finite-index subgroup of Out$(W)$ such that if $W = A_1 \ast \cdots \ast A_n$, then $\varphi$ is in Out$^0(W)$ if it preserves the conjugacy class of each $A_i$ and induces the trivial element of Out$(A_i)$. (More generally, the results in this subsection hold for $\varphi$ that preserve the conjugacy class of each vertex group.) Suppose now that $f: G \to G$ is a relative train track map representing an outer automorphism $\varphi \in$ Out$^0(W)$. The following properties are immediate from Theorem 5.2.1 and the definition of Out$^0(W)$.

**Lemma 6.2.1.** If $f: G \to G$ is a relative train track map representing $\varphi \in$ Out$^0(W)$, then each cone point of $G$ is fixed, and the induced isomorphism $f(A_i) \to A_i$ is an inner automorphism.

**Lemma 6.2.2.** If $H_i$ is a non-periodic, non-exponentially-growing stratum, the terminal endpoint of an edge in $H_i$ is a cone point and hence fixed. Thus all the edges of $H_i$ share a common terminal endpoint. If any initial endpoint of an edge of $H_i$ is a cone point as well, then $H_i$ consists of a single edge whose initial and terminal endpoints are fixed.

**Lemma 6.2.3.** Let $C$ be a noncontractible component of a filtration element $G_r$, and let $H_k$ be the lowest stratum containing an edge of $C$. Then $H_k \subset C$ is its own core and is either exponentially-growing or a periodic stratum. If periodic, $H_k$ is a single edge $E_k$, and the initial and terminal endpoints of $E_k$ are cone points.

*Proof.* The fact that $H_k$ is either exponentially-growing or periodic and its own core was established in the proof of Theorem 5.2.1 Property (Z). Thus $H_k$ contains a cone point $c$. Edges incident to $c$ are mapped to paths that begin at $c$, so are contained in $C$. This establishes that $H_k \subset C$. If $H_k$ is periodic, its underlying graph is a tree whose leaves are cone points, and the edges of $H_k$ are transitively permuted. If $H_k$ contains more than one edge, this is impossible.
Corollary 6.2.4. If $G_r$ contains a cone point not contained in $G_{r-1}$, then $H_r$ is connected and irreducible.

Proof. $H_r$ is irreducible by Lemma 6.1.2. That $H_r$ is connected follows from the proof of the previous lemma.

7 CAT(0) Mapping Tori

In [Ger94], Gersten gave a short proof that $\text{Aut}(F_n)$ is not a CAT(0) group (for $n \geq 3$) by exhibiting an automorphism of $F_3$ whose mapping torus cannot act properly by semi-simple isometries on a CAT(0) metric space. In particular, such groups are not subgroups of CAT(0) groups.

Gersten’s example is polynomially-growing. The main result of this section is that by contrast, if $\varphi \in \text{Out}^0(W)$ is a polynomially-growing outer automorphism, then the mapping torus of $\varphi$ is a CAT(0) group.

Theorem 7.0.1. Let $A$ be a finite abelian group and $W = A \ast \cdots \ast A$. Let $\varphi \in \text{Out}^0(W)$ be a polynomially-growing outer automorphism represented by $\Phi: W \to W$. Then there is a CAT(0) 2-complex $X_\varphi$ on which the mapping torus $W_n \rtimes_\varphi \mathbb{Z}$ acts properly and cocompactly.

If $\varphi$ is not assumed to be in $\text{Out}^0(W)$, or if $A$ is not assumed to be abelian, a finite index subgroup of $W \rtimes_\varphi \mathbb{Z}$ is a CAT(0) group, so $W \rtimes_\varphi \mathbb{Z}$ acts properly on a CAT(0) space by semi-simple isometries.

Our proof of the stated theorem requires inner automorphisms of $A$ to act trivially, hence the restriction to $A$ abelian. This restriction can be dropped at the cost of taking high powers of the automorphism.

Our methods also apply to free-group automorphisms: recall that $W = A \ast \cdots \ast A = F \rtimes A$, where $F$ is free of rank $(n-1)(|A|-1)$. If $\Phi: W \to W$ represents $\varphi \in \text{Out}^0(W)$, we may choose $\Phi$ so that $\Phi(F) = F$, so we can restrict $\Phi: W \to W$ to $F$. The difficulty above is in arranging that $F \rtimes_{\Phi\mid_F} \mathbb{Z}$ splits as an HNN extension. The mapping torus of $\Phi\mid_F$ is a finite index subgroup of the mapping torus of $\Phi$, so the former is also a CAT(0) group.

Finally, our techniques also apply to pure symmetric automorphisms of free groups. An automorphism $\Phi: F_n \to F_n$ is pure symmetric when it sends each basis element $x_i$ to a conjugate.

Corollary 7.0.2. If $\Phi: F_n \to F_n$ is a pure symmetric automorphism, the mapping torus $F_n \rtimes \Phi\mathbb{Z}$ is the fundamental group of a nonpositively-curved 2-complex.

As a warm-up for the proof, we sketch Gersten’s argument and discuss an example of our construction arising from a palindromic automorphism of $F_3$.
7.1 Gersten’s Example

Gersten’s example concerns the following automorphism \( \Psi : F_3 \to F_3 \). We write \( F_3 = \langle a, b, c \rangle \).

\[
\Psi \begin{cases} 
  a &\mapsto a \\
  b &\mapsto ba \\
  c &\mapsto ca^2 
\end{cases}
\]

Gersten’s first observation is to rewrite this group as a double HNN extension of \( \langle a, t \rangle \cong \mathbb{Z}^2 \) with \( b \) and \( c \) as stable letters. He does this by rewriting the relators.

\[
tbt^{-1} = ba \sim b^{-1}tb = at \quad \text{and} \quad tct^{-1} = ca^2 \sim c^{-1}tc = a^2t
\]

This observation generalizes: every polynomially-growing free-by-cyclic group that can be represented by an improved relative train track (IRTT) or completely-split relative train track map (CT) admits a hierarchy, a repeated graph-of-groups decomposition with cyclic edge stabilizers and polynomially-growing free-by-cyclic groups of lower rank as vertex stabilizers, terminating with \( \mathbb{Z}^2 \). Levitt records this fact in the case of IRTTs as [Lev09, Definition 1.1]; in the case of CTs it follows from [FH11, Lemma 4.21]. This allows for arguments by induction.

We return to Gersten’s proof. Suppose, aiming for a contradiction, that \( F_3 \rtimes_{\psi} \mathbb{Z} \) acts properly by semi-simple isometries on a CAT(0) metric space \( X \). By the Flat Torus Theorem [BH99, Theorem II.7.1, p. 244], there is an isometrically embedded Euclidean plane \( Y \subset X \). This plane is preserved by \( H = \langle a, t \rangle \), which acts on \( Y \) by translation, and the quotient \( Y/H \) is a 2-torus.

Fix a point \( p \in Y \). The content of the HNN extension is that in \( F_3 \rtimes_{\psi} \mathbb{Z} \), \( t \), \( at \) and \( a^2t \) are all conjugate, so in the action of \( H \) on \( Y \), these elements have the same translation length. Thus there is a circle of radius \( d(p, t.p) \) in \( Y \) centered at \( p \) that meets the point \( t.p \), \( at.p \) and \( a^2t.p \). But on the other hand, these three points lie along an axis for the translation action of \( a \). But in Euclidean geometry, a straight line cannot meet a circle in three points. See Figure 3.

![Figure 3: Gersten’s example would force a line to intersect a circle in three points.](image)

The lesson here is that in order for an HNN extension of a CAT(0) group with cyclic associated subgroups to be a CAT(0) group, there must be a “good reason” for the generators of the associated cyclic subgroups to have the same translation length.
7.2 Bridson–Haefliger’s Construction

Bridson and Haefliger give a general construction providing a sufficient condition for an HNN extension of a CAT(0) group to be a CAT(0) group. Because the geometry of this space will be important to our arguments, we describe their construction in the setting where the associated subgroups are infinite cyclic.

**Theorem 7.2.1** ([BH99] Proposition II.11.21, p. 358). Let \( H \) be a group acting properly and cocompactly on a CAT(0) space \( X \). Given \( x \) and \( y \) infinite order elements of \( H \) whose translation length on \( X \) are equal, there is a CAT(0) space \( Y \) on which the HNN extension \( G = H *_{x^t = y} \) acts properly and cocompactly.

Informally, the construction proceeds by “blowing up” the Bass–Serre tree \( T \) for the HNN extension. For each vertex of \( T \), \( Y \) contains an isometric copy of \( X \). When two vertices share an edge, there is a strip, that is, a space \( S := \mathbb{R} \times [0,1] \) glued in, with \( \mathbb{R} \times \{0\} \) glued to one copy of \( X \) along an axis for \( x \), and \( \mathbb{R} \times \{1\} \) glued to another copy of \( X \) along an axis for \( y \). See Figure 4.

In the case where \( X \) is the universal cover of a space with fundamental group \( H \), one might imagine attaching a cylinder to \( X/H \) with one end attached along a loop representing \( x \) and the other along a loop representing \( y \) in \( \pi_1(X/H) \). If this is done carefully, the universal cover of the resulting space is \( Y \).

Formally, fix geodesic axes \( \gamma \) and \( \eta \) for the actions of \( x \) and \( y \) on \( X \), respectively. We think of \( \gamma \) and \( \eta \) as isometric embeddings of \( \mathbb{R} \) into \( X \). Let \( \alpha \) be the translation length of both \( x \) and \( y \) in \( X \). Recall that the vertices of the Bass–Serre tree \( T \) correspond to cosets of \( H \) in \( G \) and edges of \( T \) correspond to cosets of \( K = \langle \gamma \rangle \). The vertices \( gH \) and \( gtH \) are connected by the edge \( gK \) in \( T \). Let \( K \) act on \( S \) by translation by \( \alpha \) in the first factor. The CAT(0) space \( Y \) is a quotient of the disjoint union \( G \times X \cup G \times S \) by the equivalence relation generated by the following.

1. \((gh, p) \sim (g, h.p)\)
2. \((gx, t, \theta) \sim (g, x.t, \theta)\)
3. \((g, \gamma(t)) \sim (g, t, 0)\)
4. \((gt, \eta(t)) \sim (g, t, 1)\)

Where \( g \in G \), \( h \in H \), \( p \in X \), \( t \in \mathbb{R} \) and \( \theta \in [0,1] \). The group \( G \) acts on \( Y \) by multiplication in the labels, and it is easy to see that \( Y \) contains distinct, isometrically embedded copies of \( X \) for each coset \( G/H \), and likewise for copies of \( S \) indexed by the cosets \( G/K \).

7.3 A Palindromic Automorphism of \( F_3 \)

Recall that \( W_n = F_{n-1} \rtimes \mathbb{Z}/2\mathbb{Z} \). In our example, \( n = 4 \); we will write \( F_3 = \langle x, y, z \rangle \), and write \( a \) for the generator of the \( \mathbb{Z}/2\mathbb{Z} \) factor. We have \( a^{-1}xa = x^{-1} \), and similarly for \( y \) and \( z \). Automorphisms of \( F_{n-1} \) that commute with the conjugation action of \( a \) send basis elements to palindromes. Consider the following palindromic automorphism of \( F_3 \).

\[
\Phi \begin{cases} 
  x \mapsto x \\
  y \mapsto xyx \\
  z \mapsto yzy 
\end{cases} 
\]

\( F_3 \rtimes_{\Phi} \mathbb{Z} = \langle x, y, z, t \mid [x, t], (xt)^y = x^{-1}t, (yt)^2 = y^{-1}t \rangle \)
Let $b = ax$, $c = ay$, and $d = az$ be Coxeter generators for $W_4$. We can see that $\Phi$ represents an outer automorphism $\varphi \in \text{Out}^0(W_4)$.

\[
\Phi = \begin{cases} 
  a &\mapsto a \\
  b &\mapsto b \\
  c &\mapsto bacab \\
  d &\mapsto cadac
\end{cases}
\]

Our aim is to inductively apply Theorem 7.2.1 to show that $F_3 \rtimes \Phi Z$ is the fundamental group of a CAT(0) 2-complex. Along the way we will also show the complex admits a compatible action of $a$, so the resulting mapping torus, $W_4 \rtimes \Phi Z$ acts properly and cocompactly on the same space. Above we have rewritten the presentation for $F_3 \rtimes \Phi Z$ to make the hierarchy clearer. Write $G_0 = \langle x, t \rangle \cong \mathbb{Z}^2$, $G_1 = G_0 *_{(xt)^2 = x^{-1}t} y$ and $G_2 = G_1 *_{(yt)^2 = y^{-1}t} z = F_3 \rtimes \Phi Z$. Write $K_0 = \langle xt \rangle$ and $K_1 = \langle yt \rangle$, respectively.

**Step One.** The first CAT(0) space, $X_0$ for $G_0$ to act on is the Euclidean plane by translation. Letting $\bar{x}$ and $\bar{t}$ be the translation vectors for $x$ and $t$, notice that the translation lengths of $xt$ and $x^{-1}t$ are equal to the lengths of the diagonals of the parallelogram determined by $\bar{x}$ and $\bar{t}$. This implies that $xt$ and $x^{-1}t$ have the same translation length exactly when $\bar{x}$ and $\bar{t}$ are orthogonal.

In this situation, $X_0$ admits an isometric action of $a$ by reflecting across a fixed geodesic axis for $t$. Choose an axis $\gamma$ for $xt$ and $\eta := a . \gamma$ for $x^{-1}t$. With this data, we apply Theorem 7.2.1 to yield a new CAT(0) space $X_1$ on which $G_1$ acts on properly and cocompactly.

**Step Two.** We extend $a$ to an isometry of $X_1$: $a$ acts on the copy of $X_0$ corresponding to the identity coset of $G_1/G_0$ in the previous paragraph. If $h \in G_0$, $a$ takes $h . \gamma$ to $h^a . \eta$, and vice versa, so we extend our definition of $a$ so that it swaps the associated
strips $S = \mathbb{R} \times [0, 1]$ and sends $(s, \theta)$ to $(s, 1 - \theta)$. More generally, $a$ takes $gG_0 \times X_0$ to $g^aG_0 \times X_0$, takes $gK_0 \times S$ to $g^a y^{-1} K_0$ and acts as above in the $X_0$ and $S$ factors. One checks that because $a$ is an isometry of the pieces and respects the gluing, this defines an isometry of $X_1$.

**Step Three.** This done, notice that $(yt)^a = y^{-1}t$, so these elements must have the same translation length in $X_1$! Now we repeat: applying Theorem 7.2.1 one more time yields a CAT(0) space on which $G_2$ acts properly and cocompactly. In fact, an identical argument as above allows us to again extend $a$ to an isometry of $X_2$, as desired.

### 7.4 Polynomially-Growing Outer Automorphisms

Let $\varphi \in \text{Out}(W)$ be an outer automorphism. We say $\varphi$ is polynomially-growing if some (and hence every) relative train track map $f: G \to G$ representing $\varphi$ has no exponentially-growing strata. It follows from work of Levitt [Lev09] that equivalently, $\varphi$ is polynomially-growing if the word length of every conjugacy class grows at most polynomially under iteration of $\varphi$.

**Lemma 7.4.1.** If $f: G \to G$ is a relative train track map representing $\varphi \in \text{Out}(W)$ a polynomially-growing outer automorphism, then $G$ has no zero strata.

**Proof.** This is immediate from the definition and [Z].

**Lemma 7.4.2.** Let $f: G \to G$ be a relative train track map representing a polynomially-growing outer automorphism $\varphi \in \text{Out}^0(W)$. Then each stratum consists of a single edge $E$, and $f(E) = Eu$, where $u$ is a closed path in lower strata.

**Proof.** Suppose, aiming to a contradiction, that $H_i$ is the lowest stratum, necessarily non-exponentially-growing, that contains more than one edge. Call the edges $E_1, \ldots, E_m$. By [NEG], the $E_j$ share a common terminal endpoint, and because cone points are fixed by $f$, the initial endpoints of the $E_j$ are distinct and all vertices. Call the vertices $v_1, \ldots, v_m$. Because $H_i$ is the lowest such stratum, each $v_j$ has valence one in $G_i$. There must be another edge incident to $v_j$, say $E'$. If the stratum containing $E'$ is not periodic, then the terminal endpoint of $E'$ is fixed. This contradicts the fact that the underlying graph of $G$ is a tree. In fact, this argument shows that the $v_j$ lie in distinct components of $G \setminus G_i$, and that each of said components contains no cone points. Since we may assume $G$ has no valence one vertices, this contradiction proves the claim.

We prove an analogue of Levitt’s tool for inducting on the rank of polynomially-growing $\varphi \in \text{Out}^0(F_n)$, [Lev09] Definition 1.1.

**Proposition 7.4.3.** If $\varphi \in \text{Out}^0(W)$ is represented by an relative train track $f: G \to G$ whose highest strata is polynomially-growing, so it consists of a single edge $E$, one of two decompositions occur:

1. There is a nontrivial decomposition $W = W^1 \ast W^2$ such that $\varphi$ has a representative $\Phi$ with $\Phi(W^i) = W^i$ for $i = 1, 2$. We write $\Phi_i = \Phi|_{W^i}$, so $\Phi = \Phi_1 \ast \Phi_2$. This happens whenever $E$ separates $G$.  

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2. If $E$ does not separate, there is a decomposition $W = W^1 \ast A$, with $W^1$ a free product of finite groups of Kurosh rank $n - 1$, and a representative $\Phi$ such that $\Phi(W^1) = W^1$ and $\Phi(a) = u^{-1}au$ with $u \in W^1$ and $a \in A$. We write $\Phi|_{W^1} = \Phi_1$.

Proof. If $E$ separates $G$, write $G_1$ and $G_2$ for the components of the complement. By Lemma 7.4.2, $f(E) = Eu$, with $u$ a path in lower strata, so $u$ is contained within one component of the complement, say $G_1$. The initial endpoint, $v$, is fixed by $f$, so provides a lift $f_2$: $\pi_1(G, v) \to \pi_1(G, v)$. For each cone point $c$ in $G_1$, fix a path $\sigma_c$ that travels in $G_i$ from $v$ to $c$. The paths $[\sigma_c a \sigma_c]$ as $c$ varies over the cone points of $G$ and $a$ varies over a set of generators of the cone point stabilizer generate $\pi_1(G, v)$. Write $W^1 = \pi_1(G, v)$. It is clear that $f_2(W^1) = W^1$.

If $E$ does not separate $G$, then $E$ is the unique edge incident to a valence-one cone point $c$. If $E$ (as a stratum) is periodic, then by $[P]$ $E$ is not a forest, so both endpoints of $E$ are cone points and correspond to an (infinite) free factor that is fixed up to conjugacy by $\varphi$.

Suppose $E$ is not periodic, so it is oriented away from the cone point $c$ and $f(E) = Eu$, where $u$ is a closed path based at the terminal endpoint of $E$. By $[\text{NEG}]$, this endpoint $v$ is a cone point, but we do not need this fact. Let $G_1$ be the complement of $E$ (as a stratum). It is a filtration element, so $f(G_1) \subset G_1$. If we take the lift $f_2$: $\pi_1(G, v) \to \pi_1(G, v)$ based at $v$, $\pi_1(G_1, v) = W^1$ satisfies the conclusions of the proposition. Let $A$ be the finite free factor corresponding to the paths $[EaE]$, where $a$ is an element of the stabilizer of $c$. Abusing notation, we will write $a$ for this element of $\pi_1(G, v)$ and $u$ for the class $[u] \in \pi_1(G, v)$ of the path such that $f(E) = Eu$. Then $f_2(a) = [uEaEu] = u^{-1}au$, as required.

Levitt notes that if $u$ as in the second case can be written as $w \varphi(w^{-1})$, we reduce to the first case, because then $\varphi(w^{-1}aw) = u^{-1}au$ for $a \in A$. In the case where $\varphi \in \text{Out}^0(W)$ is polynomially growing, repeating this process allows us to prove the following:

**Corollary 7.4.4.** If each $A_i$ is abelian and $\varphi \in \text{Out}^0(W)$ is polynomially growing, there is some basis of $W$ and representative $\Phi: W \to W$ representing $\varphi$ with respect to which $\Phi$ is upper triangular. That is, there are finite free factors $B_1, \ldots, B_n$ with $W = B_1 \ast \cdots \ast B_n$ and words $w_k \in B_1 \ast \cdots \ast B_{k-1}$ such that if $b_k \in B_k$, then $\Phi(b_k) = w_k^{-1}b_kw_k$. In particular, by Lemma 6.1.2 and Lemma 6.2.3 we may assume $\Phi(b_1) = b_1$ and $\Phi(b_2) = b_2$, i.e. that $w_1 = w_2 = 1$. If the $A_i$ are not assumed to be abelian, then this statement holds for some power of $\Phi$.

**Remark 7.4.5.** Note that in the first case, $W \rtimes_\Phi \mathbb{Z} = (W^1 \rtimes_\Phi_1 \mathbb{Z}) *_{t_1= \epsilon_2} (W^2 \rtimes_\Phi_2 \mathbb{Z})$, where $t_1$ and $t_2$ generate the $\mathbb{Z}$ factors of each.

### 7.5 Proof of Theorem 7.0.1

We are now ready for the proof. The argument establishes that our approach to the example in Section 7.3 generalizes. The case of symmetric automorphisms of free groups is slightly different and also easier, so we begin with that proof.

**Proof of Corollary 7.4.4** Suppose $\varphi$ is a pure symmetric outer automorphism of $F_n$. Proposition 7.4.3 and Corollary 7.4.4 apply to a relative train track map $f: G \to G$ on
an orbigraph representing \( \varphi \), yielding a basis \( x_1, \ldots, x_n \) for \( F_n \) and an automorphism \( \Phi: F_n \to F_n \) that represents \( \varphi \) and is upper-triangular with respect to \( A_1, \ldots, A_n \).

\[
F_n \rtimes_\Phi \mathbb{Z} = \langle x_1, \ldots, x_n, t \mid tx_k t^{-1} = w_k^{-1} x_k w_k \rangle
\]

As above, \( w_k \in F_{k-1} = \langle x_1, \ldots, x_{k-1} \rangle \), and \( w_1 = 1 \). Note that the relation \( tx_k t^{-1} = w_k^{-1} x_k w_k \) can be rewritten as \( x_k^{-1} w_k t x_k = w_k t \), yielding a hierarchy for \( F_n \rtimes_\Phi \mathbb{Z} \) as an iterated HNN extension. The first stage is the base group \( \langle x_1, t \rangle \cong \mathbb{Z}^2 \). At each stage, the hypotheses of Theorem 7.2.1 are obviously satisfied, so iteratively applying Theorem 7.2.1 beginning with any geometric action of \( \langle x_1, t \rangle \) on the Euclidean plane proves the result.

Proof of Theorem 7.0.1 Fix \( A \) a finite group. For clarity in the proof, we suspend our convention and write \( W_n = A \ast \cdots \ast A = F \rtimes A \). We proceed by induction on \( n \), the Kurosh rank of \( W_n \). The base case is \( n = 2 \). Since \( \text{Out}^0(W_2) \) is trivial, we may consider the action on the (metric) product \( \mathcal{H}_2 \times \mathbb{R} \), where \( \mathcal{H}_2 \) is the universal cover of the hedgehog graph with one spike; it is a regular \( |A| \)-valent tree with two orbits of vertices.

So assume that for \( k < n \) and for all polynomially-growing outer automorphism \( \psi \in \text{Out}^0(W_k) \), the conclusions of the theorem hold. That is, we are allowed to replace \( \varphi \) or \( \Phi \) representing it by a power only when \( A \) is nonabelian. We prove the result holds for \( n \). Let \( \varphi \in \text{Out}^0(W_n) \) be an outer automorphism. As in Corollary 7.4.4 choose finite free factors \( A_1, \ldots, A_n \) and an automorphism \( \Phi \) representing \( \varphi \) that is upper-triangular. We assume that the mapping torus of \( \Phi|_{A_1 \ast \cdots \ast A_{n-1}} \) acts properly and cocompactly on a CAT(0) 2-complex \( X \). If \( A \) is not abelian, arrange for this by replacing \( \Phi \) by a power.

Recall our notation from Section 2 for \( a \in A \), we write \( a_i \) for the image of \( a \) in \( A_i \). We have \( W_n = F \rtimes A \), where \( F = \langle a_1^{-1} a_n \mid 2 \leq i \leq n \text{ and } a \in A \setminus \{1\} \rangle \). By Corollary 7.4.4 there is \( w_n \in A_1 \ast \cdots \ast A_{n-1} \) such that \( \varphi(a_n) = w_n^{-1} a_n w_n \) for all \( a_n \in A_n \). We may also assume \( \Phi(a_1) = a_1 \) for all \( a_1 \in A_1 \). We need that \( w_n \in F \cap W_{n-1} \). This can be arranged by composing \( \Phi \) with the inner automorphism corresponding to conjugation by some \( a \in A_1 \). This does not change the isomorphism type of the mapping torus of \( \Phi|_{A_1 \ast \cdots \ast A_{n-1}} \), and we continue to work with \( X \).

If \( A \) is not abelian, it may no longer be the case that \( \Phi(a_1) = a_1 \) for all \( a_1 \in A_1 \). Restore this property by replacing \( \Phi \) by a power.

This done, notice that \( t a_1^{-1} a_n t^{-1} = a_1^{-1} w_n^{-1} a_n w_n = (w_n^{-1}) a_1^{-1} a_n w_n \). This implies that \( a_1^{-1} a_n \), thought of as the stable letter for our HNN extension, conjugates \( (w_n t)^{a_1} \) to \( w_n t \). Since \( a_1 \) is an isometry of \( X \), as in Section 7.3 we may apply Theorem 7.2.1 for each \( a_n \in A_n \). We do this by first fixing an axis \( \gamma \) for the action of \( w_n t \) on \( X \), and then using \( a_1 \gamma \) as \( a_1 \in A_1 \) varies to work as the geodesic axes of interest. This yields a CAT(0) space \( Y \) that \( F \rtimes_{\Phi|F} \mathbb{Z} \) acts on geometrically. We check that once again, the isometric actions of \( a_1 \in A_1 \) on \( X \) also extend to isometries of \( Y \), proving the claim.

Acknowledgements

The author wishes to thank her advisor, Kim Ruane, for her patience, helpful feedback, and continual gentle suggestions to not avoid torsion; Mladen Bestvina for pointing out that our proof does not use finiteness of the \( A_i \); Santana Afton, Mark Hagen and Robert Kropholler for many helpful discussions about preliminary versions of these arguments;
and Matthew Zaremsky for asking about pure symmetric automorphisms. Daniel Keliher and Ian Runnels offered suggestions improving the exposition.

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