Theory of phase-locking in generalized hybrid Josephson junction arrays

M. Basler*, W. Krech† and K. Yu. Platov‡

Friedrich-Schiller-Universität Jena,
Institut für Festkörperphysik,
Max-Wien-Platz 1, D-07743 Jena, Germany
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Abstract

A recently proposed scheme for the analytical treatment of the dynamics of two-dimensional hybrid Josephson junction arrays is extended to a class of generalized hybrid arrays with "horizontal" shunts involving a capacitive as well as an inductive component. This class of arrays is of special interest, because the internal cell coupling has been shown numerically to favor in-phase synchronization for certain parameter values. As a result, we derive limits on the circuit design parameters for realizing this state. In addition, we obtain formulas for the flux-dependent frequency including flux-induced switching processes between the in-phase and anti-phase oscillation regime. The treatment covers unloaded arrays as well as arrays shunted via an external load.

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*pmb@rz.uni-jena.de
†owk@rz.uni-jena.de
‡okp@rz.uni-jena.de
I. INTRODUCTION

Two-dimensional Josephson junction arrays are considered as strong candidates for tunable microwave oscillators. Since the pioneering works by Benz and Burroughs\textsuperscript{1,2} there were some attempts to fabricate arrays of this type\textsuperscript{3–5} as well as to understand them theoretically\textsuperscript{6–9}. (For two recent reviews on 2D Josephson junction arrays, see Lachenmann\textsuperscript{10} and Boo\textsuperscript{11}.) While radiation output of two-dimensional arrays should be much larger than that of one-dimensional arrays (for quadratic arrays in the matched case typically $\sim N^2$, with $N=$ number of rows, compared to being $\sim N$, with $N=$ number of junctions, for one-dimensional arrays\textsuperscript{6,12}) observations point more to the opposite direction. While in one-dimensional arrays there were observed up to $160\mu W$\textsuperscript{13,11} the output power reported in two-dimensional arrays is several orders of magnitude smaller with a maximum of around $400nW$\textsuperscript{14,15}.

Potentially, there can be several reasons responsible for this discrepancy. Besides low critical currents/normal resistances, mismatch to the external load, or parameter tolerances some more basic problems might be responsible for this. Indeed, some recent theoretical investigations show, that the radiating in-phase mode is neutrally stable in a unshunted array without external flux\textsuperscript{6} and, even worse, that it is unstable even for a small flux entering the cell\textsuperscript{16}. As a result, the natural state of at least the simple model circuit studied in\textsuperscript{16} is a non-radiating one with both cells oscillating against each other. The situation can be improved by adding an appropriate external shunt synchronizing in-phase via its long-range interaction, but generally there remains a tendency that pairs of cells lock anti-phase and drop out of the radiating mode.

A recently proposed layout\textsuperscript{17} removes this difficulty by introducing an additional capacitive shunt in the "horizontal" branches thus turning the internal coupling to favor the in-phase state. It is the aim of the present investigation, to give this idea an analytical foundation and derive some rigorous results, notably on the parameter boundaries separating in-phase from anti-phase oscillations. In addition, we study the interplay with an
external load leading to a rather complex picture of possible stability regions as a result of the competition of external and internal synchronization.

We start with an exposition of the problem including the basic equations in Sec. II. In Sec. III and IV these equations are solved by an analytical approximation, combining ideas of the strong coupling method appropriate for small-inductance Josephson junction cells \cite{15} with the standard weak-coupling procedure of slowly-varying phase \cite{16,17,18} for the treatment of inter-cell coupling. While Sec. III contains lowest-order results corresponding to vanishing cell inductance, Sec. IV includes the effects caused by a small, but non-vanishing inductance being essential for understanding the inter-cell coupling. Sec. V contains several results including a comparison with numerical simulations. The interplay with an external load is treated in Sec. VI, and Sec. VII contains several more general conclusions relevant for the layout of two-dimensional Josephson junction arrays.

II. THE MODEL AND THE BASIC EQUATIONS

For making the problem accessible to an analytical treatment, we have to make several propositions. Fig. 1 shows the circuit under consideration. To make the physical mechanisms more transparent, the external shunt $Z_S$ is removed in the beginning, and will only be included in Sec. VI. Despite its simplicity this model has all the main features present in larger arrays, too: It is truly two-dimensional with a possible flux entering the cells. Notice, that unlike to conventional hybrid arrays \cite{5,8}, the horizontal branch contains a more general shunt consisting not only of the usual inductive connection, but of a parallel capacitance and resistance. Numerical results obtained before indicate that an intrinsic shunt of this type can favor in-phase locking even in an externally unshunted array. Here, we will confirm and extend this result by developing an analytical formalism which should be applicable to larger two-dimensional arrays with the same general structure as well.

Some more restrictions have to be put on the array: (i) Josephson junctions are described by the RSJ model \cite{12}. (ii) All junctions are considered to be identical. (iii) Junctions are
overdamped with a McCumber parameter set to zero. (iv) Self-inductance is taken into account while mutual inductance is neglected. (v) The normalized ring inductance between the two loops

\[ l = 2\pi I_C L / \Phi_0 \]  

is supposed to be small \((l \ll 1)\). From the beginning, one has to understand that the inductance of the horizontal connection acts in a two-fold way. At first, it contributes to both ring inductances thus determining the SQUID coupling within each loop. At second, it is part of the shunt common to both loops and as such it influences the inter-cell coupling. With Eq. (1) we request the SQUID coupling to be strong, which is a necessary prerequisite for our approximation scheme to work. On the other hand, we will not fix the ratio between the inductive and the capacitive horizontal impedances from the beginning.

In the following, we will exploit some more normalized quantities as follows,

\[ s = \frac{2e}{\hbar} I_C R_N t, \]  

\[ \phi = 2\pi \Phi / \Phi_0, \]  

\[ c = \frac{2e}{\hbar} I_C R_N^2 C, \]  

\[ r = R / R_N, \]  

\[ i = I / I_C, \]  

with \(I_C\) the junction critical current, \(R_N\) the normal resistance of one of the (identical) junctions, \(\Phi\) the external flux per cell, and the last normalization being valid for all currents entering the calculation. Adopting these normalizations, the circuit can be described by the RSJ equations for the Josephson phases,

\[ \dot{\phi}_{ij} + \sin \phi_{ij} = i_{ij}, \quad \{i, j\} = \{1, 2\} \]  

in conjunction with the two flux quantization conditions

\[ \phi_{12} - \phi_{11} - \varphi \mp li_l = 0 \]
(minus sign refers to \(i = 1\)) and Kirchhoff’s current laws

\[
i_0 = \frac{1}{2}(i_{11} + i_{12}),
\]

\[
\bar{i} = i_{11} - i_{21} = i_{22} - i_{12},
\]

\[
i_l = \bar{i} + i_{rc}.
\]

These have to be supplemented by Kirchhoff’s voltage law

\[
\dot{i}_l + \frac{r}{l}i_{rc} + \frac{1}{lc}i_{rc} = 0.
\]

We would like to point out that while the inductive branch carrying current \(i_l\) is part of both superconducting loops thus contributing to the flux quantization conditions (8), the branch \(i_{rc}\) enters only via the ordinary Kirchhoff’s law (12). As a result, it is impossible to simply substitute the three elements \(l, c\) and \(r\) by a single impedance \(Z\) from the beginning.

Before, it has proven useful in the treatment of strongly coupled SQUID cells\(^8\) to combine the Josephson phases within each cell via

\[
\Sigma_k = \frac{1}{2}(\phi_{k2} + \phi_{k1}),
\]

\[
\Delta_k = \frac{1}{2}(\phi_{k2} - \phi_{k1}).
\]

In addition, we introduce the circular currents

\[
i_k^c = (i_{k2} - i_{k1})/2.
\]

With the help of Eqs. (13) – (15) we finally obtain the system

\[
\dot{\Sigma}_k + \sin \Sigma_k \cos \Delta_k = i_0,
\]

\[
\dot{\Delta}_k + \sin \Delta_k \cos \Sigma_k = i_k^c,
\]

\[
\Delta_1 + \Delta_2 - \varphi = 0,
\]

\[
\Delta_1 - \Delta_2 = li_l = l(i_2^c - i_1^c + i_{rc}),
\]

\[
\dot{i}_{rc} + \frac{r}{l}i_{rc} + \frac{1}{lc}i_{rc} = (\dot{i}_1^c - \dot{i}_2^c)
\]

which our analytical approximation scheme is based on. As there are seven equations for the seven variables \(\Sigma_k, \Delta_k, i_k^c, i_{rc}\), this is a well-posed problem.
III. ANALYTICAL APPROXIMATION SCHEME AND LOWEST ORDER RESULTS

Our strategy for solving system (16) will be based on a perturbative treatment valid for small $l$ (for the basic idea compare our earlier paper $^{18}$). Thus, we start solving Eqs. (16) for $l = 0$, and only later include corrections $\sim l$ exploiting the lowest order results obtained before. This procedure is favored by the fact that $l$ enters equation (16d) only. We start by evaluating (16d) in conjunction with (16c). The solutions for $\Delta_k$ can be used to evaluate $\Sigma_k$ from (16a). Next, we find the $i^\circ_k$ (not the $\Delta_k$, which are already known in this order!) from (16b). Finally, with the ring currents $i^\circ_k$ on the right hand side of (16e) known we can evaluate the current $i_{rc}$ by solving the corresponding differential equation. All other quantities, like $i_l$ or $\Sigma_k$, are secondary and can be derived from the seven variables mentioned so far. Afterwards, we insert the lowest order result on the right hand side of Eq. (16d) and start a second cycle in the same sequence.

The procedure described above gives the following lowest-order results. First, the Josephson phase differences in both loops are found to be identical,

$$\Delta_{k,0} = \varphi/2. \quad (17)$$

In the following, comma-delimited indices refer to the order of approximation. From (16a), the Josephson phases are found to coincide with the corresponding solutions for an autonomous junction, $^{12}$

$$\Sigma_{k,0} = \frac{\pi}{2} + 2 \arctan\left( \frac{\zeta_0}{i_0 + \cos(\varphi/2)} \tan \left( \frac{\zeta_0 s - \delta_k}{2} \right) \right), \quad (18)$$

with the important modification that the frequency $\zeta_0$ becomes flux-dependent according to

$$\zeta_0 = \sqrt{i_0^2 - \cos^2(\varphi/2)}. \quad (19)$$

Next, the circular currents can be evaluated from (16b). Note, that this equation originating from the original Josephson equations does not lead to a differential equation, because the constant Josephson phase differences $\Delta_k$ are already known. The result is
$$i_k^o = \sin(\varphi/2) \cos \Sigma_{k,0}. \quad (20)$$

It is a trivial task to evaluate Eq. (20) using (18); in the further calculation we will only need the lowest harmonics of the circular currents,

$$i_{k,0}^o = -2 \frac{\zeta_0}{i_0 + \zeta_0} \sin(\varphi/2) \sin(\zeta_0 s - \delta_k). \quad (21)$$

The corresponding difference of the ring currents,

$$\tilde{i}_0 = i_{2,0}^o - i_{1,0}^o = 4 \frac{\zeta_0}{i_0 + \zeta_0} \sin(\varphi/2) \sin \left(\frac{\delta_1 - \delta_2}{2}\right) \cos \left(\zeta_0 s - \frac{\delta_1 + \delta_2}{2}\right), \quad (22)$$

enters the horizontal connection thus acting as a driving force for the oscillatory circuit according to Eq. (16e). This equation can be solved with standard methods. The stationary oscillating solution reads

$$i_{r,0} = -\frac{4l\zeta_0^2}{|\mathcal{Z}(\zeta_0)||i_0 + \zeta_0|} \sin(\varphi/2) \sin \left(\frac{\delta_1 - \delta_2}{2}\right) \sin \left(\zeta_0 s - \frac{\delta_1 + \delta_2}{2} - \psi(\zeta_0)\right). \quad (23)$$

Here, we introduced the series circuit impedance \(\mathcal{Z}\) with

$$|\mathcal{Z}(\zeta_0)| = \sqrt{r^2 + \left(\frac{1}{\zeta_0} - l\zeta_0\right)^2} \quad (24)$$

and the phase angle \(\psi\),

$$\cos \psi(\zeta_0) = \frac{r}{|\mathcal{Z}(\zeta_0)|}, \quad \sin \psi(\zeta_0) = \frac{l\zeta_0 - \frac{1}{\zeta_0}}{|\mathcal{Z}(\zeta_0)|}. \quad (25)$$

For later purposes we need \(i_t = \tilde{i} + i_{r,c}\) rather than \(i_{r,c}\), because it is just \(i_t\) which potentially may split the oscillation phases between cell 1 and cell 2 via Eq. (16d). Combining (23) with (22) after some algebra we obtain

$$i_{t,0} = -\frac{4\zeta_0 |\mathcal{Z}(\zeta_0)|}{i_0 + \zeta_0 |\mathcal{Z}(\zeta_0)|} \sin(\varphi/2) \sin \left(\frac{\delta_1 - \delta_2}{2}\right) \cos \left(\zeta_0 s - \frac{\delta_1 + \delta_2}{2} - \chi(\zeta_0)\right), \quad (26)$$

where we introduced the \(r/c\) impedance \(|z|\) with

$$|z(\zeta_0)| = \sqrt{r^2 + \frac{1}{(c\zeta_0)^2}} \quad (27)$$
\[
\sin \chi(\zeta_0) = \frac{r l c \zeta_0^2}{|Z(\zeta_0)| \sqrt{1 + (r c \zeta_0)^2}}, \quad \cos \chi(\zeta_0) = \frac{r^2 c \zeta_0 + \left( \frac{1}{c \zeta_0} - l \zeta_0 \right)}{|Z(\zeta_0)| \sqrt{1 + (r c \zeta_0)^2}}.
\]  
(28) 

(In principle, one could evaluate \(i_t\) directly from an equation similar to (16e) of course, and we checked that the result is the same. The procedure described here has the advantage of additionally providing an expression for the current flowing through the capacitive line.) 

To summarize, we observe the following lowest-order results: All four junctions oscillate with the same flux-dependent frequency \(\zeta_0 = \sqrt{i_0^2 - \cos^2(\varphi/2)}\). Because of (17), the junctions within each cell are exactly in phase, while the relative phase between cell 1 and cell 2 (given by \(\delta_1\) and \(\delta_2\), respectively) is undetermined, up to now. If both cells are in phase, there is no current through the horizontal line, because of the \(\sin[(\delta_1 - \delta_2)/2]\) present in (22). On the other hand, the horizontal current reaches its maximum if both cells oscillate anti-phase with \(\delta_1 - \delta_2 = \pi\).

### IV. INDUCTANCE EFFECTS

Now we are ready to include inductance effects. Again starting with (16d) and (16d), we insert the lowest-order result (26) on the right hand side of (16d). This leads to 

\[
\Delta_{1/2} = \Delta_{1/2,0} + l \Delta_{1/2,1} \\
= \frac{\varphi}{2} \pm 2l \frac{\zeta_0}{i_0 + \zeta_0 |Z|} \sin(\varphi/2) \sin \frac{\delta_2 - \delta_1}{2} \cos \left( \zeta_0 s - \frac{\delta_1 + \delta_2}{2} - \chi \right). 
\]  
(29) 

Note, that the first index in (29) refers to cell 1 and cell 2, respectively, while the second one indicates the order of evaluation; the + sign refers to \(\Delta_1\). This has to be inserted into (16d), 

\[
\dot{\Sigma}_k + \cos(\Delta_{k,0} + l \Delta_{k,1}) \sin \Sigma_k = i_0. 
\]  
(30) 

For evaluating these equations the cosine on the left hand side is expanded according to 

\[
\cos(\Delta_{k,0} + l \Delta_{k,1}) \approx \cos \Delta_{k,0} - l \Delta_{k,1} \sin \Delta_{k,0}. 
\]  
(31)
After transferring the correction term $\sim l$ to the right hand side of (30) one makes the crucial observation, that it acts in a similar way as, for example, an external shunt synchronizing the cells\textsuperscript{19}.

The resulting equations are evaluated with the conventional phase-slip method (see, for instance\textsuperscript{19–21}). According to this procedure which has proven useful in the treatment of linear arrays before, the up to now constant phases $\delta_1$ and $\delta_2$ are considered as time dependent,

$$\delta_k = \delta_k(s),$$

(32)

with the subsidiary condition that this time-dependence is only an adiabatic one,

$$\dot{\delta} \ll \zeta_0.$$

(33)

Physically, this means that the phases are required to be nearly constant during one Josephson oscillation.

With these assumptions, the same ansatz\textsuperscript{18} with $\delta_k(s)$ and $\zeta$ instead of $\zeta_0$ leads to the sum voltages $\dot{\Sigma}_k$,

$$\dot{\Sigma}_k = \frac{\zeta_0(\zeta - \dot{\delta}_k)}{i_0 + \cos(\varphi/2) \cos(\zeta s - \delta_k)}.$$ 

(34)

Writing $\zeta$ instead of $\zeta_0$ we have allowed for a possible (small) deviation of the actual oscillation frequency from $\zeta_0$. Inserting (34) into (30) leads to the reduced equations

$$\zeta_0(\zeta - \zeta_0 - \dot{\delta}_k) = l \sin(\varphi/2) \Delta_{k,1} (\cos(\varphi/2) + i_0 \cos(\zeta s - \delta_k)).$$

(35)

After averaging over one time period and applying some algebra we arrive at the following system of equations (for details see, for instance\textsuperscript{14–21})

$$\zeta_0(\zeta - \zeta_0 - <\dot{\delta}_1>) = li_0 \frac{\zeta_0}{i_0 + \zeta_0} \frac{|z|}{|Z|} \sin(\varphi/2) \sin \frac{<\delta_2> - <\delta_1>}{2} \sin(\varphi/2) \times \cos \left( \frac{<\delta_2> - <\delta_1>}{2} + \chi \right),$$

(36a)

$$\zeta_0(\zeta - \zeta_0 - <\dot{\delta}_2>) = -li_0 \frac{\zeta_0}{i_0 + \zeta_0} \frac{|z|}{|Z|} \sin(\varphi/2) \sin \frac{<\delta_2> - <\delta_1>}{2} \sin(\varphi/2) \times \cos \left( \frac{<\delta_2> - <\delta_1>}{2} + \chi \right),$$

(36b)
where $<>$ denotes the time average over one Josephson oscillation. The difference of (36a) and (36b) gives an evolution equation for the phase difference $<\delta>$,

$$<\delta> = \frac{i_0 l_i}{i_0 + \zeta_0 |Z|} |z| \sin^2(\phi/2) \cos \chi \sin <\delta> .$$

(37)

Eq. (37) is the basic equation determining the possible phase differences between the oscillations of both cells as well as the corresponding regions of stability.

V. PHASE LOCKING, STABILITY AND OSCILLATION FREQUENCY

We will not go into the question of general solutions of (37) but concentrate on phase-locking, being characterized by a time-independent phase-shift between cell 1 and cell 2,

$$<\delta_{lock}> = 0 .$$

(38)

Within the range $0 \leq \delta < 2\pi$ there are obviously only two possibilities for Eq. (38) to be valid,

$$<\delta_{lock}> = 0 \quad \text{and} \quad <\delta_{lock}> = \pi ,$$

(39)

the first one describing in-phase oscillations and the second one anti-phase oscillations of the cells.

The crucial question of the range of stability of these two solutions can be answered on the basis of the evolution equation (37), too. The ansatz

$$<\delta> = <\delta_{lock}> + ae^{\lambda t}$$

(|$a| \ll |1|)$ leads to the Lyapunov coefficient

$$\lambda = \frac{i_0 l_i}{i_0 + \zeta_0 |Z|} |z| \sin^2(\phi/2) \cos \chi \cos <\delta_{lock}> .$$

(41)

One recovers, that the stability is solely determined by the $\cos \chi$; all the remaining factors, except $\delta_{lock}$, are positive definite. In detail, the
in-phase solution \(< \delta^{\text{lock}} > = 0\) is stable for \(\cos \chi < 0\),
\[(42)\]
while the
anti-phase solution \(< \delta^{\text{lock}} > = \pi\) is stable for \(\cos \chi > 0\).
\[(43)\]

Before further evaluating this condition we will consider the oscillation frequency which can be derived from (36a) (or (36b)). With
\[
< \delta_1 > = < \delta_2 > = \text{const} = 0
\]
\[(44)\]
one easily recovers
\[
\zeta_{\text{in}} = \zeta_0 = \sqrt{i_0^2 - \cos^2(\varphi/2)}.
\]
\[(45)\]
Evaluating the anti-phase frequency with
\[
< \delta_1 > - < \delta_2 > = \pi
\]
\[(46)\]
needs a bit more algebra. The result is
\[
\zeta_{\text{anti}} = \zeta_0 \left( 1 - \frac{i_0 l^2 r \sin^2(\varphi/2)}{|Z|^2 (i_0 + \zeta_0)} \right).
\]
\[(47)\]
Thus, if both cells oscillate in-phase their frequency is identical to the autonomous oscillation frequency. On the other hand, if the cells oscillate anti-phase the frequency will be lower than \(\zeta_0\). The physical reason for this behavior can be understood by comparing with other (even linearly) oscillating systems: If the bindings (in our case realized by the horizontal impedance) are not loaded, the oscillation frequency remains the same as for uncoupled oscillators; if the bindings are loaded (i.e., in case of an ac current flowing through the horizontal line) the system oscillates with a different frequency.

Unfortunately, one has to respect a certain limit of validity of Eq. (47). Using the method of slowly varying phase we have adopted the supposition mentioned before that the frequency must not deviate too much from \(\zeta_0\),
\[
\zeta \approx \zeta_0.
\]
\[(48)\]
Thus, the correction in Eq. (47) is required to be small compared to the frequency itself. A rough estimate valid for \( i_0 > 1.15 \) leads to the condition

\[ l^2 \ll r. \]  

\((49)\)

Our experience shows that usually a factor of 2...3 is sufficient for this condition to be fulfilled.

Now we return to the question of anti-phase ↔ in-phase transitions described by (42) and (43), resp. Considering the numerator of \( \cos \chi \) one observes that the boundary separating in-phase and anti-phase oscillations of the cells is given by

\[ \left( \frac{1}{c\zeta} - l\zeta \right) + r^2 c\zeta = 0 \]  

\((50)\)

with the cells oscillating anti-phase if the left hand side is positive and in-phase if it is negative. In other words, the transition between both regimes lies in the vicinity of the resonance curve of the \( l-c-r \) connection with deviations becoming important for small \( l \). Fig. 2 shows the boundary between the two regimes for a frequency \( \zeta = 1.11 \) in comparison to numerical results.

To summarize, the in-phase regime is favored for not too large \( r \) as long as the inductive impedance dominates over the capacitive one, while for the capacitive impedance dominating the cells oscillate anti-phase. There is a simple physical explanation for this: Anti-phase oscillations are caused by the flux coupling via the joint inductive line carrying current \( i_l \). For a sufficiently large capacitive shunt, the current prefers the capacitive way which does not produce any such flux.

In conventional hybrid arrays horizontal lines are purely inductive. Formally, this limit can be observed letting \( c \to 0 \). In this case the capacitive impedance goes to infinity while the correction \( \sim r^2 c \) tends to zero. Then, there is no possibility for the current to be shunted, and the cells remain in the anti-phase regime. The more general question, for which parameter values \( l, c, \) and \( r \) there are no transitions can be answered on the basis of Eq. (50). This equation does only have real solutions for \( \zeta \) if
For all smaller $l$, the current in the inductive line is strong enough to keep the cells oscillating anti-phase.

Considering the circuit parameters $i_0, l$ etc. as constant and leaving the external flux $\varphi$ as the only free parameter one can observe flux-induced transitions between both regimes. The difference between the frequencies $\zeta^{\text{in}}$ and $\zeta^{\text{anti}}$ leads to a hysteresis, which has been observed in numerical simulations before\[17]. In more detail, in-phase $\rightarrow$ anti-phase transitions are observed at

$$\varphi^{\text{ia}} = 2 \arccos \left[ \pm \sqrt{i_0^2 - \zeta^{\text{tr}}^2} \right], \tag{52}$$

where we introduced the transition frequency

$$\zeta^{\text{tr}} = \frac{1}{\sqrt{lc - r^2c^2}} \tag{53}$$

as can be easily deduced from (50). The transition from the anti-phase to the in-phase regime needs a bit more algebra. It can be determined from the requirement, that the anti-phase frequency (47) be equal to the transition frequency (53). Unfortunately, the resulting equation can not be solved in closed form. However, as a first approximation, one can equate the in$\rightarrow$anti transition frequency (53) with (47) and evaluate for $\varphi$, substituting $\zeta \rightarrow \zeta^{\text{tr}}$ on the right hand side of (47),

$$\varphi^{\text{ai}} = 2 \arccos \left( \pm \sqrt{\frac{i_0^2 - (\zeta^{\text{tr}})^2 - \frac{2i_0l}{cr(i_0 + \zeta^{\text{tr}})}}{\frac{2i_0l}{cr(i_0 + \zeta^{\text{tr}})}}} \right). \tag{54}$$

It can be deduced, that $\varphi^{\text{ai}}$ is always larger than $\varphi^{\text{ia}}$. A better result for $\varphi^{\text{ai}}$ is obtained by graphically finding the transition frequency on the curve at $\zeta = \zeta^{\text{tr}}$.

Thus, if there are any transitions between both regimes at all, for small values of the external flux the cells oscillate with the lower anti-phase frequency switching to in-phase oscillations at $\varphi^{\text{ai}}$. Because of $\varphi^{\text{ai}} > \varphi^{\text{ia}}$ (for $0 < \varphi \leq \pi$) switching back to the anti-phase
state occurs at a lower flux, leading to the hysteresis mentioned above. Fig. 3 shows a plot of frequency against flux in comparison with the outcome of a numerical simulation. The frequencies are in excellent agreement, and even the transition points, which depend rather sensibly on the parameters, are located within the same region.

This last result concerning hysteresis has to be taken with some care. It was obtained by combining the anti-phase frequency formula (47) with Eq. (50) and evaluating for $\varphi$. However, (50) as originating from (37) is already a first order result, thus inserting (17) might not be fully justified while second order terms in (37) are neglected. Nonetheless, it gives a plausible explanation for the mechanism causing the hysteresis observed in numerical simulations.

VI. LONG-RANGE SYNCHRONIZATION VIA AN EXTERNAL LOAD

It has been well-known for a long time that synchronization in a one-dimensional array can be achieved and controlled by shunting the array via an external load\cite{11,22}. In a similar manner one may hope to be able to control row locking in two-dimensional arrays, too. For studying this mechanism within our model we now add the external load already indicated in Fig. 1. As a result, we have to supplement the basic equations (16). At first, we add the mesh rule for the load current $i_L$,

$$\sum_{k=1,2} \tilde{\Sigma}_k - l_L \dot{i}_L - r_L \dot{i}_L - \frac{1}{c_L} \dot{i}_L = 0.$$  \hspace{1cm} (55)

Here, $r_L$, $l_L$, and $c_L$ are the load impedances normalized in the same manner as (1), (4), and (5). In addition, the load current couples back to the junctions, thus supplementing Eq. (16a),

$$\tilde{\Sigma}_k + \sin \Sigma_k \cos \Delta_k = i_0 - \frac{1}{2} \dot{i}_L.$$  \hspace{1cm} (56)

As has been observed in the study of similar one-dimensional synchronization problems before, the reciprocal impedance $1/|Z_L| \ll 1$ provides another perturbation parameter for a
sufficiently large load; thus we evaluate the system perturbatively, neglecting terms \( \sim l/|Z_L| \).

To lowest order with respect to \(|Z_L|\) the load current vanishes, and we end up with the results described in Sec. III. Based on the lowest order Josephson oscillations (18) and the corresponding voltages \( \dot{\Sigma}_{k,0} \) we obtain the first order (with respect to \( 1/|Z_L| \)) load current \( i_{L,0} \),

\[
i_{L,0} = \frac{4 \cos(\varphi/2)}{|Z_L|} \frac{\zeta_0}{i_0 + \zeta_0} \cos \left( \frac{\delta_1 - \delta_2}{2} \right) \sin \left( \zeta_0 s - \frac{\delta_1 + \delta_2}{2} - \psi_L \right)
\]  

(57)

with

\[
|Z_L(\zeta_0)| = \sqrt{(r_L + 1)^2 + \left( \frac{1}{c_L \zeta_0} - l_L \zeta_0 \right)^2},
\]

(58a)

\[
\sin \psi_L(\zeta_0) = \frac{r_L + 1}{|Z_L(\zeta_0)|},
\]

(58b)

\[
\cos \psi_L(\zeta_0) = \frac{c_L \zeta_0 - l_L \zeta_0}{|Z_L(\zeta_0)|}.
\]

(58c)

Its structure is obviously quite similar to that of the horizontal current (22). However, one should note two differences: (i) While the load current is maximal for \( \varphi = 0 \), the horizontal current reaches its maximum for \( \varphi = \pi/2 \). (ii) The horizontal current vanishes if both cells oscillate in-phase, while the load current vanishes for both cells oscillating anti-phase.

The load current (57) provides the additional contribution to (56) and, as a result, the phase slip equations (36) get an additional term, too. After performing the time-averages we get

\[
\zeta_0(\zeta - \zeta - < \delta_1 >) = -\frac{i_0 l}{2 i_0 + \zeta_0 |Z|} \sin^2(\varphi/2) (\sin \psi + \sin(< \delta_1 > - < \delta_2 > - \chi))
\]

\[+ \frac{1}{2 i_0 + \zeta_0 |Z_L|} \cos^2(\varphi/2) (\sin \psi_L - \sin(< \delta_1 > - < \delta_2 > - \psi_L)) \]

(59a)

\[
\zeta_0(\zeta - \zeta - < \delta_2 >) = \frac{i_0 l}{2 i_0 + \zeta_0 |Z|} \sin^2(\varphi/2) (\sin \psi + \sin(< \delta_1 > - < \delta_2 > - \chi))
\]

\[+ \frac{1}{2 i_0 + \zeta_0 |Z_L|} \cos^2(\varphi/2) (\sin \psi_L + \sin(< \delta_1 > - < \delta_2 > - \psi_L)) \]

(59b)

By subtracting (59a) and (59b), we finally get the evolution equation for the averaged oscillation phase difference,
\[
< \delta > = \frac{1}{i_0 + \zeta_0} \left( \frac{1}{|Z_L|} \cos^2(\varphi/2) \cos \psi_L + i_0 |z| \sin^2(\varphi/2) \cos \chi \right) \sin < \delta >. \tag{60}
\]

Despite the relatively complicated interplay between cell interaction via the horizontal line and long range coupling via the external load there remain only the same two phase locking solutions as before,

\[
< \delta^{\text{lock}} > = 0 \quad \text{and} \quad < \delta^{\text{lock}} >= \pi, \tag{61}
\]

the stability of which is determined by the Lyapunov coefficient

\[
\lambda = \frac{1}{i_0 + \zeta_0} \left( \frac{1}{|Z_L|} \cos^2(\varphi/2) \cos \psi_L + i_0 |z| \sin^2(\varphi/2) \cos \chi \right) \cos < \delta^{\text{lock}} >. \tag{62}
\]

In-phase oscillations of the cells are stable if the term in parenthesis is lower than zero while anti-phase oscillation are stable if it is greater than zero. Thus, the desired stability for the in-phase mode is reached for

\[
\frac{1}{|Z_L|} \cos^2(\varphi/2) \cos \psi_L + i_0 |z| \sin^2(\varphi/2) \cos \chi < 0. \tag{63}
\]

Eq. (63) shows a rather complex parameter dependence, relating the seven parameters \( r, l, c, r_L, l_L, c_L, \) and \( \varphi \). Its physical meaning is best discovered considering several limiting cases.

(i) For a sufficiently large external load,

\[
\frac{1}{|Z_L|} \ll i_0 |z| |Z|, \tag{64}
\]

the relative phase of the cells is determined by the internal coupling alone. This has to be compared to the case of two externally loaded separate cells\textsuperscript{23}. In this case – as for linear arrays – the relative phase depends on the character of the external load only: While for inductively dominated loads the cells are locked in-phase, they are locked anti-phase for capacitively dominated loads, independently of the magnitude of the external load.

(ii) The contributions from the external load and from the internal shunt show a different flux dependence. For sufficiently small values of external flux the last term can be neglected,
and the locking regime is controlled by the load only. On the other hand, for flux values of around half a flux quantum the first term becomes negligible, and the internal horizontal line determines the phase difference of the cells.

(iii) For \( l \to 0 \), the second term can be neglected, and the result agrees with that obtained for two separate cells before\(^2\), as it should be. In this limit the cells internally decouple, while the external coupling remains in force.

(iv) The usual hybrid arrays without the internal \( R-C \)-line are contained as a limiting case. For \( r \to \infty \), the in-phase-condition, Eq. (63), reduces to

\[
\frac{1}{|Z_L|} \cos \psi_L \cos(\varphi^2/2) + i_0 l \sin^2(\varphi/2) < 0.
\]

(65)

It states, that for sufficiently large inductances,

\[
l > l^* = -\frac{\cos \psi_L}{i_0 \tan^2(\varphi/2)|Z_L|},
\]

(66)

ordinary pure inductive hybrid arrays may switch to the anti-phase state even for inductive external loads.

The indicated transition was indeed observed in a numerical simulation (boxes in Fig. 4). Having in mind that Eq. (63) is the result of several approximations, concerning the external shunt as well as the internal inductive coupling, the agreement is remarkably good.

The influence of changing parameters can be nicely illustrated by performing a second simulation with exactly the same parameter set, but distributing ring inductance \( l \) regularly around the loops. The result denoted by the dots in Fig. 4 clearly deviates from that obtained for inductance concentrated on the horizontal line considered before. This can be taken as a strong indication that the coupling is not provided by the loop inductances but by the inductance on the line common to both cells.

**VII. CONCLUSIONS**

Although our work is devoted to the study of a simple model circuit several results are expected to be valid for larger arrays, too. At first, the short range coupling between
neighboring cells leads to an anti-phase synchronization in conventional Josephson junction hybrid arrays. This may be one reason for explaining the very small radiation output in 2D Josephson junction arrays obtained so far. At second, we show a way to improve the situation by adding a capacitive shunt parallel to the horizontal lines. In this way, the flux-generating current potentially responsible for the anti-phase coupling is redirected through the capacitive line which is not part of a flux quantization condition.

Combining Fig. 2 with some already known facts on synchronization in strongly coupled SQUID cells\[18\] the following design criteria for generalized hybrid 2D Josephson junction arrays can be derived. (i) For synchronizing horizontal lines in-phase the ring inductances have to be kept small ($l \ll 1$). (ii) In-phase synchronization between neighboring cells in vertical direction is observed for $l > 1/c\zeta^2 + r^2c$. Based on this, we will derive some estimates for reasonable $c$ and $l$. For a given $l$, the boundary between in- and anti-phase oscillations is given by Eq. (51). Fig. 2 shows already that the additional term $\sim r$ restricts the possible $l$ by setting a lower bound. This bound is obtained from

$$\frac{dl}{dc} = 0$$

as

$$r = \frac{1}{c\zeta}, \text{ resp. } l = 2r/\zeta.$$  \hspace{1cm} (68)

Thus, for obtaining in-phase oscillations the condition

$$r < \frac{\zeta l}{2}$$

has to be respected. Because of (68) this means

$$c > \frac{2}{l\zeta}. \hspace{1cm} (70)$$

Obviously, the requirement to have a small $l$ for horizontal in-phase synchronization leads to the demand to have a sufficiently high capacitance $c > 2/l\zeta$ as well as a small resistance $r < l\zeta/2$. A reasonable compromise might for instance be
Of cause, all these estimates should be considered as very rough, and on the other hand, one has to check carefully how large these quantities on chip actually are.

On the other hand, we would like to point out that these suggestions are based on an analytical approximation scheme and are founded on solid formulae. Of cause, it still has to be shown rigorously that they can be transferred to larger arrays as well. Some preliminary results from numerical simulations indeed indicate this. We hope, that the general procedure described here can be transferred to larger arrays of the type considered here as well, and some work is on the way to actually extend it to a ladder configuration.

If the arrays are externally loaded, which is usually done via an inductive load, the parameters have to be chosen in such a manner to respect Eq. \((71)\). The best way for obtaining in-phase synchronization is to make both contributions to the Lyapunov-coefficient lower than zero separately, which is possible because the parameters of the external load can be chosen independently of those from the internal shunt. In general, one should select values such, that (i) the external load is dominated by its inductive contribution, (ii) the internal horizontal shunts are dominated by the inductive impedance, too. Because of the frequency-dependence of the characters of the shunts, one has to make sure, that these conditions are met for all values of external flux.

Of cause, the circuit studied here has several features requiring a more detailed investigation, either analytically or numerically. Usually one exploits shunted tunnel junctions for building arrays, thus one may ask for the influence of non-vanishing McCumber parameters. On the other hand, the influence of parameter splitting needs to be investigated, and in addition, in real arrays, noise comes into play. While this last aspect is to be expected to play only a minor role within the small inductance loops, it will be sure have some influence on the coupling between the cells.
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FIGURES

FIG. 1. The generalized Josephson junction hybrid array model circuit under investigation.

FIG. 2. The boundary between in-phase and anti-phase oscillations. Solid line: analytical approximation. Crosses: numerical simulation. Parameters: $i_0 = 1.5, r = 0.1, \varphi = 1.0$.

FIG. 3. Frequency against flux with a transition from anti-phase to in-phase oscillations. Parameters: $i_0 = 1.5, r = 0.1, l = 0.2, c = 4.0$. (a) Analytical approximation. (b) Numerical simulation.

FIG. 4. Transition between in-phase and anti-phase state caused by the internal inductive coupling present in a hybrid array. Parameters: $i_0 = 1.5, r_L = 1.0, l_L = 1.0, c_L = 2.0$. Solid line: analytical approximation, boxes: numerical simulation, open dots: numerical simulation with inductance regularly distributed around loops.
REFERENCES

1 S. P. Benz and C. J. Burroughs, Supercond. Sci. Technol. 4, 561 (1991).

2 S. P. Benz and C. J. Burroughs, Appl. Phys. Lett. 58, 2162 (1991).

3 P. A. A. Booi et al., IEEE Trans. Appl. Supercond. 3, 2493 (1993).

4 L. L. Sohn et al., Phys. Rev. B 47, 975 (1993).

5 H. Shea, M. Itzler, and M. Tinkham, Phys. Rev. B 51, 12690 (1995).

6 K. Wiesenfeld, S. Benz, and P. Booi, J. Appl. Phys. 76, 3835 (1994).

7 G. Filatrella and K. Wiesenfeld, J. Appl. Phys. 78, 1878 (1995).

8 R. L. Kautz, IEEE Trans. Appl. Supercond. 5, 2702 (1995).

9 M. Darula, P. Seidel, F. Busse, and S. Benacka, J. Appl. Phys. 74, 2674 (1993).

10 S. Lachenmann, Ph.D. thesis, Eberhard-Karls-Universität Tübingen, 1995.

11 P. A. Booi, Ph.D. thesis, Twente University, 1995.

12 K. K. Likharev, Dynamics of Josephson junctions and circuits (Gordon and Breach, Philadelphia, 1991).

13 J. E. Lukens, in Superconducting Devices, edited by S. T. Ruggiero and D. A. Rudman (Academic Press, New York, 1990), pp. 135–167.

14 J. A. Stern, H. G. LeDuc, and J. Zmudzinas, Trans. Appl. Supercond. 3, 2485 (1993).

15 M. Octavio, C. B. Whan, and C. J. Lobb, Appl. Phys. Lett. 60, 766 (1992).

16 M. Basler, W. Krech, and K. Platov, subm. to Journ. Appl. Phys. (1996).

17 W. Krech and K. Platov, subm. to Europhys. Lett. (1996).

18 M. Basler, W. Krech, and K. Y. Platov, Phys. Rev. B 52, 7504 (1995).
19 A. K. Jain, K. K. Likharev, J. E. Lukens, and J. E. Sauvageau, Phys. Rep. 109, 310 (1984).

20 W. Krech, Ann. Phys. (Leipzig) 39, 117 (1982).

21 W. Krech, Ann. Phys. (Leipzig) 39, 349 (1982).

22 W. Krech, Ann. Phys (Leipzig) 39, 50 (1982).

23 M. Basler, W. Krech, and K. Y. Platov, in Macroscopic Quantum Phenomena and Coherence in Superconducting Networks, edited by C. Giovanella and M. Tinkham (World Scientific, Singapore, New Jersey, London, Hong Kong, 1995), pp. 225–234.
