Matrices over local rings: stability and finite determinacy

Genrich Belitskii and Dmitry Kerner

Abstract. Consider matrices with entries in a local ring, \(Mat(m,n;R)\), suppose a group \(G\) acts on \(Mat(m,n;R)\). The matrix \(A\) is called \(G\)-stable if its orbit is open (in the given topology). For the coarse topology on \(R\) this notion gives the classical notion of stability, while for the Krull topology this notion gives finite determinacy.

In this paper we provide the general effective necessary and sufficient condition for finite determinacy of matrices. Using it we obtain detailed criteria for various particular group actions on some subspaces of \(Mat(m,n;R)\). For example, for two-sided multiplications, \(G = GL(m,R) \times GL(n,R)\), for two-sided multiplications together with the automorphisms of the ring, \(GL(m,R) \times GL(n,R) \times Aut(R)\), for congruence, for equivalence by upper triangular matrices etc.

Our results are of Mather type: for some tuples \((m,n,R,G)\) there are no finitely-determined matrices, for the others - finite determinacy is the generic property. In some cases we classify the stable matrices.

As an immediate application of finite determinacy we get various algebraizability criteria for matrices and modules and prove the Weierstrass preparation theorems for matrices.

Contents

1. Introduction 1
2. Generalities 5
3. Finite determinacy for particular groups and subspaces of \(Mat(m,n;R)\) 11
4. Stability 20
Appendix A. Applications to algebraization problems 21
References 22

1. Introduction

1.1. Setup. Let \(k\) be a field of zero characteristic. Let \((R,m)\) be a (commutative, Noetherian) local ring over \(k\). (As the simplest case, one can consider regular rings, e.g. the rational functions regular at the origin, \(k[x_1,\ldots,x_p]/(m)\) or the formal power series, \(k[[x_1,\ldots,x_p]]\). For \(k \subseteq \mathbb{C}\) (or any other normed field) one can consider converging power series, \(k\{\ldots\}\).

Let \(\dim(R) < \infty\) denote the Krull dimension of \((R,m)\). Geometrically, \(R\) is the local ring of the (algebraic/formal/analytic etc.) germ \(\text{Spec}(R)\) and (when the field \(k\) is algebraically closed) \(\dim(R)\) is the maximal among the dimensions of the irreducible components of \(\text{Spec}(R)\). Usually we assume \(\dim(R) > 0\), i.e. \(R\) is not Artinian. Let \(p\) denote the “embedding dimension” of \(\text{Spec}(R)\), i.e. the \(k\)-dimension of the vector space \(m/m^2\). (If \(R\) is regular then \(p = \dim(R)\).)

Let \(Mat(m,n;R)\) be the space of \(m \times n\) matrices with entries in \(R\). We always assume \(m \leq n\), otherwise one can transpose the matrix. Various groups act on this space, e.g.

- the left multiplications \(G_l := GL(n,R)\), the right and two-sided multiplications \(G_r := GL(m,R), G_{lr} := G_l \times G_r\);
- the ”change of coordinates”, \(Aut(R)\), and the corresponding semi-direct products, \(G_l := G_l \times Aut(R), G_{lr} := G_{lr} \times Aut(R)\). In this case we assume the ring \(R\) to be henselian.
- For square matrices, \(m = n\), one can consider e.g. congruence \((A \to UAU^T)\), for \(U \in GL(m,R)\); equivalence by upper triangular matrices etc.

In this paper the group \(G \circ Mat(m,n;R)\) is always assumed to be a pro-algebraic subgroup of \(G_{lr}\), i.e. for any projection \(Mat(m,n;R) \to Mat(m,n;R/m^{k+1})\) the image \(jet_k(G)\) is an algebraic subgroup of \(jet_k(G_{lr})\). Further, the action \(G \circ Mat(m,n;R)\) is continuous (in a given topology), the orbit \(GA \subset Mat(m,n;R)\) is an algebraic (usually infinite dimensional) subscheme over \(k\), the tangent space \(T_{(G,A)} \subset T_{Mat(m,n;R)}\) is an \(R\)-module. For more detail and examples see §2.2.

Convention. Recall that any matrix is \(G_{lr}\) equivalent to a block-diagonal, \(A \overset{G_{lr}}{\to} 1 \oplus A'\), where all the entries of \(A'\) lie in the maximal ideal \(m\), i.e. vanish at the origin. This splitting is preserved in deformation, i.e. given a family of matrices, \(A_0 \in Mat(m,n;R[[\epsilon]])\), with \(A_0\) as above, this family is \(G_{lr}\) equivalent to \(1 \oplus A'_{\epsilon}\), with \(A'_{\epsilon} \in Mat(*,*,R[[\epsilon]])\). Therefore, in this work we usually assume \(A|_0 = 0\), i.e. \(A \in Mat(m,n;m)\).

Date: May 6, 2014.
2000 Mathematics Subject Classification. Primary 58K40, 58K50 Secondary 32A19, 14B07, 15A21.
Key words and phrases. Maps of Singular Germs, Matrix Singularities, Matrix Families, Finite Determinacy, Modules over Local Rings, Algebraization, Deformation of Modules.

Many thanks to V. Goryunov and M. Leyenson.

D.K. was partially supported by the postdoctoral fellowship at the University of Toronto and by the grant FP7-People-MCA-CIG, 334347.
1.2. Stability and finite determinacy. In this work we address the following classical question. Fix some subset \( V \subseteq \text{Mat}(m, n; R) \) and some group action \( G \circ \text{Mat}(m, n; \mathfrak{m}) \). Suppose \( G \) acts on \( V \).

For a "small" deformation \( A_1 \to A_2 \in V \), are the initial and deformed matrices \( G\)-equivalent?

More precisely, for a given topology on \( V \) and a given group action: "when does the orbit \( GA \) contain an open neighborhood of \( A \in V ? " \) (Alternatively, "when is \( GA \) open in \( V ? " \)

The answer depends essentially on the notion of "smallness", i.e. the topology. If the topology \( T_1 \) (on \( \text{Mat}(m, n; R) \)) is coarser than \( T_2 \) then \( T_1 \)-stability implies \( T_2 \)-stability. Thus it is natural to study the stability for the coarsest and the finest topologies. We consider two extremal cases.

1.2.1. The strongest: (classical) stability. Consider the projection \( \text{Mat}(m, n; \mathfrak{m}) \xrightarrow{\text{jet}} \text{Mat}(m, n; \mathfrak{m}/\mathfrak{m}^2) \). Choose some (minimal) set of generators of \( \mathfrak{m}/\mathfrak{m}^2 \), identify \( \text{Mat}(m, n; \mathfrak{m}/\mathfrak{m}^2) \) with the affine space \( \mathfrak{k}^{\mathfrak{mnp}} \). Then the Zariski topology on \( \mathfrak{k}^{\mathfrak{mnp}} \) defines the topology on \( \text{Mat}(m, n; \mathfrak{m}/\mathfrak{m}^2) \). Consider the corresponding topology on \( \text{Mat}(m, n; \mathfrak{m}) \), the open sets being the preimages (under jet1) of those in \( \text{Mat}(m, n; \mathfrak{m}/\mathfrak{m}^2) \). Note that this topology is coarse, its open sets are huge, being cylinders over the large opens of \( \text{Mat}(m, n; \mathfrak{m}/\mathfrak{m}^2) \).

Call the deformation \( A \to A + B " \small " \) if \( B \) belongs to some (small enough) open neighborhood of the zero matrix \( \mathcal{O} \). A matrix \( A \) is called \( G\)-stable if \( A \xleftarrow{\mathcal{L}} A + B \) for any small \( B \).

1.2.2. The weakest: finite determinacy. The maximal ideal of \( R \) induces the natural filtration:

\[
\text{Mat}(m, n; R) \supset \text{Mat}(m, n; \mathfrak{m}) \supset \text{Mat}(m, n; \mathfrak{m}^2) \supset \cdots
\]

So, we have the Krull (or \( \mathfrak{m} \)-adic) topology, generated by all the sets of the form \( A + \text{Mat}(m, n; \mathfrak{m}) \). A matrix is called \( G - k \)-determined if \( A \xleftarrow{\mathcal{L}} A + B \) for any \( B \in \text{Mat}(m, n; \mathfrak{m}^{k+1}) \). A matrix is called finitely-\( G \)-determined if it is \( k \)-determined for some \( k \in \mathbb{N} \). (Alternatively, the orbit \( GA \) contains an open neighborhood of \( A \), in the Krull topology.

Note that the Krull topology is much finer than the coarse topology (as above), in particular the (classical) stability implies one-determinacy and is close to the zero-determinacy. (Recall that we consider only matrices vanishing at the origin.)

1.3. The main results. In §2.6 we start from the classical result on finite determinacy, reducing the problem to the study of the tangent space to the orbit \((GA, A)\). We obtain a necessary and sufficient criterion of finite determinacy in lemmas 2.8 and 2.12. This criterion is both general and very effective, for a given group \( G \) one checks the ideal of maximal minors of some associated matrix \( A_G \), alternatively one checks the left kernel of \( A_G \) on the punctured neighborhood.

The power of this criterion is shown by application to various particular groups in §3.

For example, for \( G_l, G_r, G_t \) we get the following criterion in terms of fitting ideals in §3.1 (recall that \( m \leq n \)).

**Theorem 1.1.** 1. If \( m < n \) then no \( A \in \text{Mat}(m, n; \mathfrak{m}) \) is finitely-\( G_l \)-determined.

2. \( A \in \text{Mat}(m, n; \mathfrak{m}) \) is finitely-\( G_r \)-determined iff it is finitely-\( G_l \)-determined.

3. \( A \in \text{Mat}(m, n; \mathfrak{m}) \) is finitely-\( G_t \)-determined iff \( I_m(A) \) contains a power of the maximal ideal \( \mathfrak{m} \subset R \).

The last condition can be stated also as: the quotient \( R/I_m(A) \) is a finite dimensional \( \mathfrak{k} \)-vector space, or: the module \( \text{coker}(A) \) is supported only at one point, the origin. Note that the condition "\( I_m(A) \) contains a power of the maximal ideal \( \mathfrak{m} \subset R \)" places severe restrictions on the ring. Various corollaries and examples are given in §3.1.

The case of \( G_t, G_r, G_l \) is considered in §3.4. Recall the classical degeneracy loci, defined over the algebraic closure \( \bar{R} = R \otimes \kappa \):

\[
\Sigma_r(A) := \{ pt \in \text{Spec}(\bar{R}) \mid \text{rank}(A|_{pt}) \leq r \}, \quad \{0\} := \Sigma_{-1}(A) \subseteq \Sigma_0(A) \subseteq \Sigma_1(A) \subseteq \cdots \subseteq \Sigma_m(A) = \text{Spec}(\bar{R})
\]

Algebraically, the defining ideal of \( \Sigma_r(A) \) is the fitting ideal \( I_r(A) \). As always, we assume \( m \leq n \). Recall that the expected codimension of \( \Sigma_r \), i.e. the expected height of \( I_r(A) \), is \((m-r)(n-r)\).

**Theorem 1.2.** 1. If \( \dim(\bar{R}) \leq n-m+1 \) then \( A \in \text{Mat}(m, n; \mathfrak{m}) \) is finitely-\( G_t \)-determined iff it is finitely-\( G_r \)-determined.

2. If \( \dim(\bar{R}) > n-m+1 \) then \( A \in \text{Mat}(m, n; \mathfrak{m}) \) is not finitely determined with respect to \( G_t \).

3. If for some \( 1 \leq r \leq m : (n-r)(m-r) < \dim(\bar{R}) < n(m-n-r) \) then \( A \) is not finitely determined with respect to \( G_r \). If for some \( 1 \leq r \leq m : (n-r)(m-r) < \dim(\bar{R}) < m(n-r) \) then \( A \) is not finitely determined with respect to \( G_t \).

4. Suppose \( \dim(\bar{R}) > n-m+1 \). Then \( A \in \text{Mat}(m, n; \mathfrak{m}) \) is finitely-\( G_t \)-determined iff

i. for any \( 0 \leq r < m \), the locus \( \Sigma_r(A) \) is of expected dimension and
ii. the complement \( \Sigma_r(A) \setminus \Sigma_{r-1}(A) \) is quasi-smooth at each of its points.

Here the quasi-smoothness is the substitution of the ordinary smoothness in the case when the ambient variety is singular, see definition 3.11. We emphasize that the quasi-smoothness is checked over the algebraic closure \( \kappa \).

The proof and immediate (strong) applications are given in §3.4.2.

The classical stability of \( A \in \text{Mat}(m, n; \mathfrak{m}) \) implies the stability of \( \text{jet}_1(A) \in \text{Mat}(m, n; \mathfrak{m}/\mathfrak{m}^2) \), which is essentially a question of linear algebra. In particular, to determine the (in)stability is much simpler than to (dis)prove the finite determinacy. We prove various stability criteria in §4.
1.4. Remarks and further results.

1.4.1. Our method gives absolutely explicit, ready-to-use criteria, written directly in terms of the matrix entries. This is analogous to

* "the hypersurface singularity is finitely determined iff it is isolated", as opposed to
* "the hypersurface singularity is finitely determined iff its miniversal deformation is finite dimensional".

1.4.2. Aside from particular applications, we emphasize one "theoretical" consequence of §2.6: the study of finite determinacy of matrices (for a given group action $G \circ V \subset \text{Mat}(m, n; m)$ on a given subspace of $\text{Mat}(m, n; m)$ ) is reduced to the study of finite-$G_r$-determinacy of some associated matrices. In other words, the problem for an arbitrary $(V, G)$ is "embedded" into the problem for $(\text{Mat}(m, n; R), G_r)$.

1.4.3. Base Change. It is natural to relate the order of determinacy over a given ring $R$ to that over some related ring. (For example, for $R = S/I$, with $S$-regular, to relate determinacy over $R$ and $S$.) The expected relation exists if $G$ does not involve the change of coordinates, but breaks completely otherwise. In §2.3 we discuss the behavior of determinacy under base change. In §2.4 we relate the order of determinacy over a henselian ring $R$ to that over the completion $\hat{R}$. (For example, for $k = \mathbb{C}$ and $A$ a matrix of analytic functions, the determinacy does not change under transition to the formal series.)

1.4.4. Adjustments by transformations close to identity. For any group action $G \circ \text{Mat}(m, n; R)$ and some fixed $k \in \mathbb{N}$, consider the projection $R^{\otimes k} \to R/m^k$. Then $\text{jet}_k$ maps $\text{Mat}(m, n; R)$ to $\text{Mat}(m, n; R/m^k)$ and $G$ acts on $\text{Mat}(m, n; R/m^k)$ through the corresponding group $\text{jet}_k(G)$. Consider $G^{(k)} := (\text{jet}_k)^{-1}(e) \subset G$, i.e. all the elements of $G$ that act trivially on $\text{Mat}(m, n; R/m^k)$. These are the transformations that are identities "up to $k$’th order". The finite determinacy of $A$ is established by studying the tangent space to the orbit, $T_{(G,A,A)}; \quad \S 2.6$. As the inclusion $T_{(G^{(k)},A,A)} \subset T_{(G,A,A)}$ is of finite codimension, we get: $A$ is finitely-$G$-determined iff it is finitely-$G^{(k)}$-determined.

1.4.5. Matrices of smooth functions. In this paper the ring $R$ is assumed to be Noetherian. In particular this excludes the rings of $r$-differentiable functions. However, our results are applicable to $C^\infty$ category, for $k = \mathbb{R}$. In [Belitski-Kerner] we relate the order of determinacy of $A \in \text{Mat}(m, n; C^\infty(\mathbb{R}^p, 0)/I)$ to the order of determinacy of its Taylor series, $\hat{A} \in \text{Mat}(m, n; \mathbb{R}[[x_1, \ldots, x_p]]/I)$. (Recall Mather’s result: a $C^\infty$ map is finitely-determined, for contact/left/right/left-right equivalence, iff its completion is finitely determined.)

Further, we obtain there the criterion for $w$-determinacy: when $A \in \text{Mat}(m, n; C^\infty(\mathbb{R}^p, 0)/I)$ determined by its Taylor expansion $\hat{A} \in \text{Mat}(m, n; \mathbb{R}[[x_1, \ldots, x_p]]/I)$? In other words: when the flat functions are inessential?

1.4.6. Formulation in terms of maps. It is natural to start from the biggest (reasonable) group $G$, with the biggest possible orbits. For this we ignore the matrix structure, i.e. consider $\text{Mat}(m, n; R)$ as the space of maps, $\text{Maps}(\text{Spec}(R), (k^{mn}, 0))$. The contact group, $\mathcal{K}$, acts on this space and is reasonably large. Naturally $G_{tr} \subset \mathcal{K}$ (and this inclusion is "tight", i.e. $G_{tr}$ cannot be enlarged further, §2.2). Thus the finite-$G$-determinacy of a matrix implies the finite-$\mathcal{K}$-determinacy of the corresponding map.

In the inverse direction, starting from the space $\text{Maps}(\text{Spec}(R), (k^N, 0))$, with $N = mn$, and choosing an isomorphism $k^{mn} \approx \text{Mat}(m, n, k)$, we can associate to any map the corresponding matrix. As the orbits of the group $G_{tr}$ are much smaller than those of $\mathcal{K}$, we get a much stronger notion of finite determinacy.

1.4.7. Restricting to subsets of $\text{Mat}(m, n; R)$. For square matrices one often considers the congruence, $G_{\text{congr}}: A \to UAU^T$; it preserves the (anti)symmetry. Therefore no (anti)symmetric matrix can be finitely determined in $\text{Mat}(m, m; m)$ for $G_{\text{congr}} := G_{\text{congr}} \times \text{Aut}(R)$. However, in this case it is natural to consider only deformations inside the subspaces of (anti)symmetric matrices. We treat this case in §3.5, where we prove in particular:

**Theorem 1.3.** 1. For dim$(R) \leq \left( \begin{array}{c} m \\ 2 \end{array} \right)$ there are no finitely-$G_{\text{congr}}$-determined matrices inside $\text{Mat}(m, m, m)$.

2. A matrix $A \in \text{Mat}^{sym}(m, m; m)$ is finitely-$G_{\text{congr}}$-determined iff $I_m(A)$ contains a power of the maximal ideal. In particular, if dim$(R) > 1$ then no matrix is finitely-$G_{\text{congr}}$-determined, while for dim$(R) = 1$ the generic matrix is finitely-$G_{\text{congr}}$-determined.

3. A matrix $A \in \text{Mat}^{anti-sym}(m, m; m)$ is finitely-$G_{\text{congr}}$-determined iff $I_{m-1}(A)$ contains a power of the maximal ideal.

4. $A \in \text{Mat}^{sym}(m, m, m)$ is finitely-$G_{\text{congr}}$-determined iff for any $0 \leq r < m$ such that $\Sigma_r(A)$ is of positive dimension: $\Sigma_r(A) \setminus \Sigma_{r-1}(A)$ is quasi-smooth at each of its points. Similarly for $A \in \text{Mat}^{anti-sym}(m, m, m)$.

As always, when the equivalence involves the change of variables ($G_{\text{congr}}$), the loci are considered over $\overline{k}$. For the quasi-smoothness see definition 3.11.

Sometimes one works with the upper triangular matrices, $\text{Mat}^{up}(m, m; R)$. Then the equivalence $A \sim UAV$ is by the upper triangular matrices, §3.6. Again, the general methods are directly applicable and give:
Theorem 1.4. 1. A is $G_{tr}^m$ or $G_t^m$ or $G_{tr}^m$-finitely determined, inside Mat$^{up}(m,m;m)$, iff $I_m(A)$ contains a power of $m$. (As for square matrices $I_m(A) = (\det(A))$, this condition implies in particular dim$(R) = 1$. The generic matrix over such a ring is finitely-$G$-determined for $G$ any of above.)

2. If dim$(R) > 3$ then there are no finitely-$G_{tr}^m$-determined matrices inside Mat$^{up}(m,m;m)$. 

3. If dim$(R) = 1$ then $A$ is finitely-$G_{tr}^m$-determined iff it is finitely-$G_{tr}^m$-determined.

4. If dim$(R) = 2$ then $A$ is finitely-$G_{tr}^m$-determined iff $\Sigma_{m-1} = \{\det(A) = 0\}$ is Spec$(R) \otimes \bar{k}$ is a curve-germ with isolated singularity.

5. If dim$(R) = 3$ then $A$ is finitely-$G_{tr}^m$-determined iff the locus $\Sigma_{m-1} \setminus \Sigma_{m-2}$ is of expected dimension and quasismooth.

6. In particular, for dim$(R) \leq 3$ the generic matrix is finitely-$G_{tr}^m$-determined.

For upper triangular matrices the locus $\Sigma_{m-2}$ is defined by vanishing of at least two entries on the diagonal of $A$.

More generally, for many group actions $G \odot \operatorname{Mat}(m,n;m)$, and most rings, there are no finitely determined matrices. Thus, it is natural to restrict to some subspaces/subsets of Mat$(m,n;m)$, for which one expects generic finite-$G$-determinacy. The general results of §2.6 apply also for subsets of Mat$(m,n;m)$, defined by finite sets of algebraic conditions, such that the tangent spaces to the orbits are $R$-modules. For the subsets of matrices with the given Fitting ideals we prove the (negative) criteria in §3.2.

Proposition 3.4. 1. For a given $A \in \operatorname{Mat}(m,n;m)$ consider only deformations that preserve $I_m(A)$. If dim$(R) > 2(|m| - |n| + 2)$ then $A$ cannot be finitely-$G_{tr}$-determined.

2. Similarly, for a given $A \in \operatorname{Mat}(m,n;m)$ consider only deformations that preserve all the Fitting ideals, $I_1(A), \ldots, I_m(A)$. If dim$(R) > mn$ then $A$ cannot be finitely-$G_{tr}$-determined.

1.4.8. We note that for $G \subseteq G_{tr}$, the finite determinacy criteria are often written in terms of the Fitting ideals $I_r(A)$. This is somewhat surprising as the Fitting ideals are very rough invariants of a matrix (or the corresponding module coker$(A)$). It will be very interesting to obtain some general statement of the type:

Let $M_0 \subset \operatorname{Mat}(m,n;k)$ be a vector subspace, consider the corresponding $R$ module $M := \text{jet}^{-1}_m(M_0)$. Let $G \subseteq G_{tr}$ be some "large enough" group, suppose $G$ acts on $M$. Then the finite determinacy of $A \in M$ is determined by the Fitting ideals of $A$ only.

1.4.9. Limitations of the method. A natural question is: implications to the stability of vector fields/systems of PDE. Consider a system of PDE’s: $\{\sum a_{ij}\partial_j f = 0\}$ over $(k^2,0)$. The natural equivalence is induced by the change of coordinates, $\operatorname{Aut}(k^2,0)$, and by taking linear combinations of the equations. This equivalence acts on the matrix of coefficients, $A = (a_{ij})$, by $A \to U A(f(x))(\partial_x f)$, i.e. as a subgroup of $G_{tr}$. However, the tangent space to this orbit is not an $R$-module! Indeed, the tangent space to this orbit is not an $R$-module! Therefore our methods are not applicable in this case.

1.4.10. In many cases (e.g. representation theory) one works with square matrices and the relevant equivalence is the conjugation, $A \cong G_{conj} \cup \operatorname{Aut}(R)$. Unfortunately there are no finitely-$G_{conj}$-determined matrices, for $G_{conj} = G_{conj} \cup \operatorname{Aut}(R)$, even if one restricts to the deformations inside some $R$-submodule of $\operatorname{Mat}(m,m;m)$. Indeed, for $1 \neq f \in R$ with $\text{jet}_N(f) = 1 \in R/mN+1$, the matrices $A$ and $fA$ are not $G_{conj}$-equivalent. The essential invariants for conjugation are not ideals of $R$ (or sub-loci of $\operatorname{Spec}(R)$) but the actual functions of $A$, like $\operatorname{tr}(A)$, $\det(A)$, etc.

1.5. Relations to other fields and motivation.

1.5.1. Singularity Theory. The study of stability/finite determinacy of maps, for $k = R$ or $k = \mathbb{C}$, and $R = k[x_1, \ldots, x_p]$ or $k[x_1, \ldots, x_p]$, is the usual starting point, [AGLV-1], [AGLV-2], [Looijenga], [du Plessis-Wall]. (In fact, one can take any intermediate field, $\mathbb{Q} \subseteq k \subseteq \mathbb{C}$.) The criteria are usually formulated over the algebraic closure $\bar{k} \subseteq \bar{k}$.

- For functions, $m = 1 = n$, the finite determinacy means that the singularity of $f^{-1}(0) \subset (k^2,0)$ is isolated.
- For $m = 1 \leq n$ and the regular $R$ we have maps, Maps$((k^2,0), (\bar{k}^n,0))$. The equivalence induced by $G_{tr}$ coincides with the contact equivalence, $\mathcal{K}$. A map $F \in \text{Maps}((k^2,0), (\bar{k}^n,0))$ is finitely determined iff either $F^{-1}(0) \subset (k^2,0)$ is a zero-dimensional scheme or $F^{-1}(0)$ is a complete intersection, of expected dimension $(p-n)$, with an isolated singularity. In particular, the generic map is finitely-$G_{tr}$-determined, see §2.1 for the notion of genericity.
- The case of square matrices (for $k = R$ or $k = \mathbb{C}$, $R = k[x_1, \ldots, x_p]$ and $G = G_{tr}$) was considered in [Bruce-Tari2004], and further studied in [Bruce-Goryunov-Zakalyukin2002], [Bruce03], [Goryunov-Mond05], [Goryunov-Zakalyukin03]. In particular, the generic finite determinacy was established and the simple types were classified.
- Finite determinacy is equivalent to the finite dimensionality of the miniversal deformation. In particular, the genericity of finite determinacy, for a fixed $(m,n,R,G)$, means: the stratum of matrices of Tjurina number $\infty$, $\Sigma_{G_{tr} = \infty} \subset \operatorname{Mat}(m,n;m)$, is of infinite codimension.

Remark 1.5. Regarding the stability, note that we consider local situation: $(R,m)$ is a local ring and $\operatorname{Spec}(R)$ is the (algebraic/formal/analytic etc.) germ at the origin. Thus any change of coordinates preserves the origin, i.e. for any $\phi \in \operatorname{Aut}(R)$: $\phi(m) = m$. So, e.g. the Morse critical point of a function, $f = \sum_{i=1}^p x_i^2$, is not stable in our approach.
1.5.2. Relation to Commutative Algebra. For the relevant background cf. [Eisenbud].

- Any matrix is the resolution matrix of its cokernel, $R^{\oplus m} \xrightarrow{A} R^{\oplus m} \rightarrow \text{coker}(A) \rightarrow 0$. From commutative algebra point of view, the classical stability implies the rigidity of the module, while finite determinacy means that the miniversal deformation of a module is of finite dimension.

Another relation is via embedded modules. Let $M \subset R^{\oplus m}$ be a finitely generated $R$-submodule. Then the finite determinacy of $M$ means the $G_l$ finite determinacy of the corresponding presentation matrix, while the finite determinacy up to an embedded isomorphism of $M \subset R^{\oplus m}$ corresponds to the $G_l$ finite determinacy of the presentation matrix.

- The projection $R^{jct_{k-1}} R/m^k$ induces the map of categories $\text{Mod}_R \rightarrow \text{Mod}_{R/m^k}$, defined by $M \rightarrow M/m^k M$. This map is surjective, an $R/m^k$ module $M$ is also an $R$ module, for $r \in R$ define $rM := \text{jet}_{k-1}(r)M$. But the map is usually not injective, e.g. if the presentation matrix of $M_R$ has all its entries in $m^k$ then the image is a free $R/m^k$ module. Our results give the regions of parameters (the size of presentation matrix, the dimension and embedded dimension of the ring) for which the projection map is "generically injective".

- A particularly important (and well studied) case is: $m = n$ and $\text{det}(A) \in S$ is not a zero divisor. Then $\text{coker}(A)$ is a maximally Cohen-Macaulay module over $R := S/\text{det}(A)$, [Yoshino], [Leuschke-Wiegand]. In this case there are "Maranda type results". Let $(R, m)$ be a Cohen-Macaulay ring admitting a faithfull system of parameters, $x_1, \ldots, x_{\dim(R)}$.

[Leuschke-Wiegand, §15.2] Let $\{x_i\}$ be a faithfull system of parameters for $R$, let $M, N$ be two mCM $R$-modules. If $M/\langle \{x_i^2\} \rangle M \xrightarrow{\phi} N/\langle \{x_i^2\} \rangle N$, then there exists an isomorphism $M \xrightarrow{\cong} N$ such that $\phi \otimes R/(\{x_i\}) = \phi \otimes R/(\{x_i\})$.

(The faithfull system of parameters exists e.g. for a complete Cohen-Macaulay ring with isolated singularity.)

In terms of the presentation matrices, $A_M, A_N$, this implies: if $\text{det}(A_M) = \text{det}(A_N) = 0 \in R$ and $A_M \otimes R/\langle \{x_i^2\} \rangle \cong A_N \otimes R/\langle \{x_i^2\} \rangle$, then $A_M \cong A_N$.

- The general line of research is: "which information about a module is determined by its Fitting ideals?" In particular, do the properties of stability/finite determinacy involve essentially the properties of modules or only of their Fitting ideals? As our results show, if $G$ does not involve the change of coordinates, then the finite-$G$-determinacy is a property of the zeroth Fitting ideal of a module. Another classical question is: suppose for the two modules over $R$ the (corresponding) Fitting ideals coincide. What are the additional conditions to ensure that the modules are isomorphic (i.e. their resolution matrices are $G_l$ equivalent)? Again, finite determinacy implies: we should check only the relevant Fitting ideals and to compare the modules over $R/m^N$, for some large $N$.

1.5.3. Relation to the Algebraization Problem. If a matrix $A$ is $k$-determined then, in particular, it is $G$-equivalent to a matrix whose entries are polynomials of degrees at most $k$. Therefore finite determinacy is a (significant) strengthening of the old problem of algebraization: "which objects have polynomial representatives?" Our results give immediate corollaries, in §A.1.

Moreover, using finite determinacy we give criteria for "partial algebraizability" (relative to an ideal) in §A.2. These are the natural matrix generalizations of the classical Weierstrass preparation theorem.

2. Generalities

2.1. Notations and conventions. We denote the zero matrix by $\mathbf{0}$, the identity matrix by $\mathbf{I}$.

For $0 \leq j \leq m$ and $A \in \text{Mat}(m, n; R)$ let $I_j(A) \subset R$ be the Fitting ideal, generated by all the $j \times j$ minors of $A$. (By definition $I_0(A) = R$.) Note that the chain of ideals $R = I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_m(A)$ is invariant under $G_l$, action and admits the action of $\text{Aut}(R)$.

The adjugate of a square non-degenerate matrix $A$ (or the matrix of cofactors) is uniquely defined by $AA^\vee = \text{det}(A)\mathbf{I} = A^\vee A$.

We often consider $\text{Mat}(m, n; R)$ as an infinite dimensional affine space over the base field. Explicitly, consider $R$ as a $k$-vector space, take some Hamel basis, $\{v_{x_1}\}$. Identify $\text{Mat}(m, n; k) \approx k^{\oplus mn}$. Then, as a vector space, $\text{Mat}(m, n; R) \approx \oplus_{\alpha} \{v_{x_1}k^{\oplus mn}\}$. If $R$ is complete, then one can use the jet projections, $R \rightarrow \text{jet}_k(R/m^{k+1})$. Accordingly $\text{Mat}(m, n; R)$ is the projective limit $\varprojlim \text{Mat}(m, n; R/m^{k+1})$.

For any point of this affine space, $A \in \text{Mat}(m, n; R)$, we can take the germ $(\text{Mat}(m, n; R), A)$. Further, we have the k-tangent space, $T_{(\text{Mat}(A), A)}$, which is an $R$-module and is isomorphic to $\text{Mat}(m, n; R)$.

When saying that the generic matrix satisfies some property, the genericity is taken in the sense of [Tougeron1968]: the subset of matrices that do not satisfy this property is of infinite codimension in $\text{Mat}(m, n; R)$. More precisely: the subsets of $\text{Mat}(m, n; R/m^k)$ that do not satisfy this property are Zariski closed, and their codimension tends to infinity with $k$.

2.2. Groups acting on matrices.
2.2.1. Restrictions on the groups actions. For a given group action, $G \triangleleft \text{Mat}(m,n;m)$, we consider the orbit, $GA \subseteq \text{Mat}(m,n;m)$. Use the identification of $\text{Mat}(m,n;m)$ with affine space, as above. In this paper we assume:

(3) the orbit $GA$ is an algebraic $k$-subscheme of $\text{Mat}(m,n;m)$, possibly of infinite (co)-dimension.

Consider the germ of the orbit at the matrix, $(GA, A) \subset (\text{Mat}(m,n;m), A)$. Accordingly we have the embedding of the tangent spaces, $T_{(GA,A)} \subset T_{\text{Mat}(m,n;m)}$. It is natural to identify, $T_{\text{Mat}(m,n;m)} \approx \text{Mat}(m,n;m)$. Thus $T_{\text{Mat}(m,n;m)}$ has the natural structure of $R$-module, by multiplication. In this paper we assume:

(4) the tangent space $T_{(GA,A)}$ is the $R$-submodule of $T_{\text{Mat}(m,n;m)}$.

Example 2.1. The group-actions from the introduction satisfy these assumptions. Their tangent spaces can be computed using the "infinitesimal action", $A \rightarrow (I + e\psi A)(I + e\psi)$, $e^2 = 0$.

- $G_{ir} : A \rightarrow UA^T$. Here $T_{(G_{ir}, A)} = \text{Span}_R\{uA, Av\}_{(u,v) \in \text{Mat}(m,m;k) \times \text{Mat}(n,n;k)}$. Similarly for $G_l$ and $G_r$.

- $\text{Aut}(R) : A \rightarrow A(\phi(A))$. Here $T_{(\text{Aut}(R), A)} = \text{Span}_R\{\text{Der}(A)\}_{\varphi \in \text{Der}(R,m)}$. (Here $\text{Der}(R, m)$ is the module of those derivations of $R$ that send $m$ into itself. For a regular local ring this module is generated by $\{m\partial_j\}$, where $\{\partial_j\}$ are first order partial derivatives.

- $G_{conj} : A \rightarrow UAU^{-1}$. Here $T_{(G_{conj}, A)} = \text{Span}_R\{uA - Au\}_{u \in \text{Mat}(m,m;k)}$. Similarly for $G_{cong}$.

- $G_{cong} : A \rightarrow UAU^T$. Here $T_{(G_{cong}, A)} = \text{Span}_R\{uA + Au^T\}_{u \in \text{Mat}(m,m;k)}$. Similarly for $G_{cong}$.

- Let $GL^{pp}(m, R)$ denote the group of invertible upper triangular matrices over $R$. Consider the corresponding action of $G_{pp}$. $A \rightarrow UAU$. Then $T_{(G_{pp}, A)} = \text{Span}_R\{uA, Av\}_{(u,v) \in \text{Mat}^{pp}(m,m;k) \times \text{Mat}^{pp}(n,n;k)}$.

2.2.2. The biggest group is $G_l$. When studying stability/finite determinacy, it is natural to start from the biggest possible groups, i.e. the weakest (reasonable) equivalence, even if it violates the matrix structure. In the case of maps, the contact equivalence (§1.5.1) is satisfactory in various senses. Note that we can consider $\text{Maps}((k^p, 0), (k^p, 0))$ as $\text{Mat}(1,n;m)$, (here $S$ is the local ring of $(k^p, 0)$, e.g. algebraic functions or formal power series) then the contact equivalence is induced by the action of $GL(n, S) \rtimes \text{Aut}(R)$.

Similarly, we can consider matrices over $R$ as maps from $\text{Spec}(R)$ to $\text{Mat}(m,n;k)$, i.e. consider $\text{Mat}(m,n;R)$ as $\text{Maps}(\text{Spec}(R), (k^{mn}, 0))$. Then the contact equivalence is induced by $GL(mn, R) \rtimes \text{Aut}(R)$. Therefore, it is natural to consider only those groups, $G \triangleleft \text{Mat}(m,n;R)$, that are subgroups of $GL(mn, R) \rtimes \text{Aut}(R)$. (In particular, the action is linear: $\phi(A + B) = \phi(A) + \phi(B)$.) Besides, the equivalence should, at least, distinguish between degenerate and non-degenerate matrices (more generally: matrices of distinct ranks). Groups with such properties are restricted by the following classical result.

Theorem 2.2. [Dieudonné1949] Let $T$ be an invertible map acting linearly on the vector space $\text{Mat}(m,m,k)$ (possibly violating the matrix structure). Suppose $T$ acts on the set of degenerate matrices, i.e. $\det(A) = 0$ iff $\det(T(A)) = 0$. Then either $T(A) = UA^T V$ or $T(A) = UA^TV$, for some $U, V \in GL(m,k)$.

(For the general introduction to the theory of preservers, i.e. self-maps of $\text{Mat}(m,m,k)$, that preserve some properties/structures cf. [Mohür2007].)

Therefore, in this paper $G_l$ is always a subgroup of $G_{ir}$.

2.3. Change of base ring. If the group does not involve the change of coordinates, i.e. $G \subseteq G_{ir}$, then the order of determinacy behaves well under the change of base ring.

Proposition 2.3. Fix some $G \subseteq G_{ir}$.

1. If $R = S/I$ then the order of determinacy over $S$ is bigger or equal to that over $R$.

2. Consider a homomorphism $(R, m_R) \xrightarrow{\phi} (S, m_S)$, suppose $\phi(m_R^{N_1}) \supseteq m_S^{N_2}$. If $A_R$ is $N_1$-determined then $\phi(A_R)$ is $N_2$-determined.

Proof. 1. Any relation $U(A + B)V = A$ over $S$ implies the corresponding relation over $R$, hence the statement is obvious.

2. Let $A \in \text{Mat}(m,n;R)$, let $B \in \text{Mat}(m,n;m_R^{N_1})$. By the assumption, $B = \phi(B)$, where $B \in \text{Mat}(m,n;m_S^{N_1})$. Thus $A + B \subseteq A$. As $G \subseteq G_{ir}$, one has $\phi(A) + B \subseteq \phi(A)$. ■

Thus, if $G \subseteq G_{ir}$, i.e. does not involve coordinate changes, then the stratum of not-$G$-finitely determined matrices, $\Sigma_G \subset \text{Mat}(m,n;m)$, behaves well under the base-change. Let $S \xrightarrow{\phi} R$ inducing morphism $\text{Mat}(m,n;S) \xrightarrow{\phi} \text{Mat}(m,n;R)$, which is $G$-equivariant: $\phi(gA_S) = \phi(g)A_S$. If $\text{dim}(R/\phi(S)) < \infty$ then $\phi(\Sigma_{\text{Mat}(m,n;S)}) \supseteq \Sigma_{\text{Mat}(m,n;R)}$. If $\phi$ is injective then $\phi^* (\Sigma_{\text{Mat}(m,n;R)}) \supseteq \Sigma_{\text{Mat}(m,n;S)}$.

For equivalence involving changes of coordinates the situation is more complicated, e.g. finite determinacy over $R$ neither implies nor is implied by that over $R/I$ (for some ideal $I$).

Example 2.4. The function $f = x + y^2 \in k[[x,y,z]]$ is $\text{Aut}(R)$-finitely determined (even stable!) though its image in $k[[x,y,z]]/(x)$ is not finitely determined. (Alternatively, the image of $f$ in $R/(f)$ is just zero.) In the other direction,
suppose $\text{dim}(R) = 2$, let $f \in \mathfrak{m}$ (i.e. $f$ vanishes at the origin). Then $f^2 \in R$ is not finitely-$\mathcal{K}$-determined. But for any (generic enough) ideal $J \subset R$ such that the ideal $(f, J) \subset R$ is of height two (i.e. the two germs intersect at the origin only) the element $f^2 \in R/J$ is finitely determined.

At least in one case finite determinacy descends to the quotient ring. Let $\text{Nilp}(R)$ be the ideal of all the nilpotent elements, so the ring $R/\text{Nilp}(R)$ is reduced.

**Lemma 2.5.** For any $G \subseteq \mathcal{G}_{ir}$, finite determinacy over $R$ is equivalent to that over $R_{red} := R/\text{Nilp}(R)$.

**Proof.** The direct statement is trivial. For the converse statement, choose some $q \in \mathbb{N}$ such that $\text{Nilp}(R)^q = \{0\} \subset R$ and use lemma 2.8. Then, if $I_{max}(A_G \otimes R/\text{Nilp}(R)) \supset \mathfrak{m}^N_{R_{red}}$ it follows that $I_{max}(A_G) \supset \mathfrak{m}^N_R$. So, by lemma 2.8, $A$ is finitely-$G$-determined. $\blacksquare$

2.4. Comparison of the determinacy over a henselian ring $R$ to that over the completion $\hat{R}$. The ring maps naturally to its completion, $\hat{R}$, and for many rings this map is injective. Therefore, in this section we assume: $R \to \hat{R}$.

Let $(R, \mathfrak{m})$ be a henselian local ring with base field $\mathfrak{r} = k$, let $(\hat{R}, \hat{\mathfrak{m}})$ be its completion at $\mathfrak{m}$. Take some pro-algebraic group $G \subseteq \mathcal{G}_{ir}$, defined by a finite number of polynomial equations. (For example: $\mathcal{G}_{ir}, \mathcal{G}_{congr}$ etc.) Let $M \subseteq \text{Mat}(m, n; \mathfrak{m})$ be an $R$-submodule, suppose $G$ acts on $M \subseteq \text{Mat}(m, n; \mathfrak{m})$. Then we consider the finite-$G$-determinacy problem for deformations inside $M$. Denote the $G$-order of determinacy of a matrix inside $M$ by $\text{ord}_G^{M}(A) \leq \infty$.

Consider the formal version: for the completion $\hat{M} \subseteq \text{Mat}(m, n; \hat{\mathfrak{m}})$ we have the group action $\hat{G} \subseteq \mathcal{G}_{ir} \circ \text{Mat}(m, n; \mathfrak{m})$. Hence the order of determinacy, $\text{ord}_G^{M}(A) \leq \infty$, for deformations inside $\hat{M}$. The natural map $M \to \hat{M}$ brings the corresponding order of determinacy, $\text{ord}_G^{M}(A) \leq \infty$, for deformations by matrices that are images of matrices in $M$.

**Theorem 2.6.** 1. For $A \in M$: $\text{ord}_G^{M}(A) \leq \text{ord}_G^{\hat{M}}(A) \leq \text{ord}_G^{N}(A)$.

2. If $G \subseteq \mathcal{G}_{ir}$, then for any $A \in M$: $\text{ord}_G^{M}(A) = \text{ord}_G^{\hat{M}}(A)$.

3. If for some choice of generators of $R$, all the entries of $A \in M$ are polynomials, then for $G \subseteq \mathcal{G}_{ir}$: $\text{ord}_G^{M}(A) = \text{ord}_G^{\hat{M}}(A)$.

4. Suppose $k$ is a normed field, suppose $R$ is a $k$-analytic ring, $R = k\{x_1, \ldots, x_p\}/I$. Then, for any $G \subseteq \mathcal{G}_{ir}$ and any $A \in M$: $\text{ord}_G^{M}(A) = \text{ord}_G^{\hat{M}}(A)$.

**Proof.** 1. The part $\text{ord}_G^{M}(A) \leq \text{ord}_G^{\hat{M}}(A)$ is trivial.

By the second inequality, suppose $A + B_{\geq N} \subseteq \mathcal{G} A$, for any $B_{\geq N} \in \mathfrak{m}^{N} M$. We must prove $A + C_{\geq N} \subseteq \mathcal{G} A$ for any $C_{\geq N} \in \mathfrak{m}^{N} M$. If $C_{\geq N}$ happens to be in the image of $\mathfrak{m}^{N} M$ then the equivalence follows trivially. In the general case, expand $C_{\geq N} = \sum_{i \geq N} B_i$, where each $B_i$ lies in the image of $M$, the sum is possibly infinite and $\text{jet}_{i-1}(B_i) = 0$. (One can take as $B_i$, e.g. the $i$'th homogeneous part of $B_{\geq N}$. Note that this infinite summation is well defined.)

By construction, for each fixed $j > 0$: $A + \sum_{i < j} B_i = \hat{g}_j(A + \sum_{i < j} B_i)$, with $g_j \in \mathcal{G}$. By $\S 1.4.4$, for $j$ large enough, we can (and will) choose $g_j \in G^{(k_j)} \subset \mathcal{G}$, i.e. $\text{jet}_{k_j}(g_j) = I_d$. Here $\{k_j\}$ is some sequence of natural numbers, monotonically going to infinity. Thus $\hat{A} + \sum_{i \geq j} B_i \cong \text{lim}_{l \to j}(g_j \cdots g_1) \hat{A}$ and it remains to show that the limit exists. By construction, $\text{jet}_{k_j}(g_j \cdots g_1) = \text{jet}_{k_j}(g_j + \cdots g_1)$, therefore for $l \geq j$: $\text{jet}_{k_j}(g_j \cdots g_1)$ does not depend on $l$. Thus $\text{lim}_{l \to j}(g_j \cdots g_1)$ is a well defined element of $\hat{G}$.

2. Using the first part, it is enough to prove $\text{ord}_G^{M}(A) \leq \text{ord}_G^{\hat{M}}(A)$. Suppose $\text{ord}_G^{M}(A) = N$, then for any $B_{\geq N} \in \mathfrak{m}^{N+1} M$ there exist $(U, V) \in \hat{G}$ satisfying: $A + B_{\geq N} = U \hat{A}V$. Now, over the initial ring, we should find $(U, V) \in G$ satisfying: $A + B_{\geq N} = UAV$, $(U, V) \in G$. Here $U, V$ are unknowns and their entries must satisfy a finite amount of polynomial equations. (The quadratic equations as above and the defining equations of $G \subseteq \mathcal{G}_{ir}$.) By the assumption, this system has a formal solution. Thus, by Theorem I.10 of [Artin1969] we get the existence of solutions over $R$, i.e. in $G$. Thus $\text{ord}_G^{M}(A) \leq N$.

3. If all the entries of $A$ are polynomials then, similarly to the case above, we have a finite number of polynomial equations: $A(\phi(x)) + B_{\geq N} = U\phi(x)V$, on $(U, V, \phi) \in G$. Thus, again by Artin theorem, we get a solution in $G$.

4. It is enough to prove $\text{ord}_G^{M}(A) \leq \text{ord}_G^{\hat{M}}(A)$. Note that in [Artin1969] one assumes a finite number of polynomial equations. This suffices for the case $G \subseteq \mathcal{G}_{ir}$. But, if $G \subseteq \mathcal{G}_{ir}$ involves the change of coordinates, then the equations in general are analytic. Therefore we need a separate proof.

Suppose $R = k\{x_1, \ldots, x_p\}/I$ and $\text{ord}_G^{M}(A) = N$. Identify $R$ with its embedding in $\hat{R}$. Accordingly, we write $A$ for the completion of $A$. Then for $B_{\geq N} \in \mathfrak{m}^{N+1} M$ there exist $(U, V, \phi) \in \hat{G}$ satisfying: $A(\phi(x)) + B_{\geq N} = U\hat{A}(\phi(x))V$. We should demonstrate an analytic solution to this condition.

Take some representatives of $A, B$ over $k\{x_1, \ldots, x_p\}$ and some representatives of $U, V, \phi$ over $k[[x_1, \ldots, x_p]]$. Then we get: $A(\hat{z}) + B(\hat{z}) = \hat{U}(A(\hat{z}))V + \hat{C}(\hat{z})$, where $\hat{C} \in \text{Mat}(m, n, \hat{\mathfrak{m}})$. Choose some generators $\{h_i\}$ of $I$, so $\hat{C} = \sum h_i \hat{C}_i$, where $\hat{C}_i \in \text{Mat}(m, n, k[[x_1, \ldots, x_p]])$. Thus, treating $U, V, \phi, \{C_i\}_i$, as indeterminates we get the conditions:

\[ A(\hat{z}) + B(\hat{z}) = U A(\phi(\hat{z}))V + \sum h_i C_i, \quad (U, V, \phi) \in G, \quad \phi(I) = I. \]
As \( A, B \) are matrices over \( k\{x_1, \ldots, x_p\} \), and \( \{h_i \in k\{x_1, \ldots, x_p\}\} \), these conditions are all analytic! (The condition \( \phi(I) = I \) can be written as: \( \phi(h_i) = \sum_j Q_{ij} h_j \) where \( Q_{ij} \) are again some indeterminates.)

By the assumption, there exists a formal solution to all these conditions. Therefore, by Theorem 1.2 [Artin1968], there exists an analytic solution, \( (U, V, \phi) \) over \( k\{x_1, \ldots, x_p\} \). Taking its image in \( k\{x_1, \ldots, x_p\}/I \), note that \( \phi(I) = I \), we get the needed solution. ■

2.5. Points in the neighborhood of the origin. Frequently \( R \) is the ring of "genuine" functions, i.e. for any element \( f \in R \) the germ \( \text{Spec}(R) \) has a representative that contains other closed points besides the origin and \( f \) can be actually computed at those points "off the origin". Thus, for any \( A \in \text{Mat}(m, n; R) \) we can take a small enough representative of \( \text{Spec}(R) \) and for any point of it we can evaluate the matrix, \( A|_p \), this is a numerical matrix. For example this happens for rings of rational functions or converging power series. Complete rings are not of this type, their elements, in general, cannot be computed "off the origin".

This geometric description is frequently used as the guiding tool to formulate criteria. Usually the geometric conditions are of the type a property \( \mathcal{P} \) is satisfied "generically" on some subset of \( \text{Spec}(R) \); or satisfied everywhere in the punctured neighborhood. When \( R \) is the ring of "genuine" functions this means: for a small enough representative \( U \) of \( \text{Spec}(S) \), there exists an open dense set \( V \subset U \) such that the condition \( \mathcal{P} \) is satisfied at each point of \( V \). Thus, in many places in the paper, we formulate the relevant condition both "algebraically" (in terms of \( R, A \) and some associated ideals) and "geometrically" (in terms of \( A \) computed at some points near the origin).

Finally, we remark that when speaking about the points near the origin we usually take the closure, \( \overline{p} \), so \( pt \in \text{Spec}(R \otimes \overline{k}) \).

2.6. Tangent space to the orbit, its presentation matrix and the general criterion for finite determinacy. Let \( G \subset \text{Mat}(m, n; m) \), let \( T_{(G,A)} \) be the tangent space to the orbit of \( G \) at \( A \in \text{Mat}(m, n; m) \), cf. §2.2. This tangent space is naturally embedded into \( T_{(\text{Mat}(m,n,m),A)} \approx \text{Mat}(m, n; m) \). Suppose we consider only deformations inside the subspace \( M \subset \text{Mat}(m, n; m) \), on which \( G \) acts. Suppose the germ \((M,A)\) is smooth and \( T_{(M,A)} \subseteq T_{(\text{Mat}(m,n,m),A)} \) is an \( R \)-submodule.

2.6.1. Stability vs infinitesimal stability (linearization of the problem). Suppose \( M \subset \text{Mat}(m, n; m) \) is a linear subspace. The classical criteria of stability/finite determinacy reduce the initial question to the study of vector spaces \( T_{(G,A)} \rightarrow T_{(M,A)} \).

• [AGLV-1, §III.1.1, pg.156] The matrix \( A \in M \) is \( G \)-stable iff this embedding is an isomorphism.

• [AGLV-1, §III.2.2, pg.166] The matrix is \( G \)-finitely determined iff this embedding is of finite codimension, i.e.

\[
\text{dim}_k \left( T_{(M,A)}/T_{(G,A)} \right) < \infty
\]

All the results of our work are based on these criteria.

2.6.2. The criterion in terms of the Fitting ideal. In many cases \( T_{(M,A)} \) is not free, e.g. \( T_{(\text{Mat}(m,n,m),A)} \approx \text{Mat}(m, n; m) \) is not a free \( R \)-module. However \( T_{(M,A)} \) only embeds into a free module such that the quotient is of finite \( k \)-dimension. The typical example of such an embedding is \( T_{(\text{Mat}(m,n,m),A)} \subset T_{(\text{Mat}(m,n,R),A)} \approx \text{Mat}(m, n; R) \). One does similarly when \( M \) is the set of (anti-)symmetric/upper-triangular/etc. matrices. This embedding gives the rank to \( T_{(M,A)} \), e.g.

\[
\text{rank}(T_{(\text{Mat}(m,n,m),A)}) = \text{rank}(T_{(\text{Mat}(m,n,R),A)}) = mn
\]

Let \( \mathcal{A}_G \) be a generating matrix of \( T_{(G,A)} \), i.e. \( T_{(G,A)} \) is the image of the homomorphism \( F \rightarrow T_{(M,A)} \). Here \( F \) is some free \( R \)-module. So \( \mathcal{A}_G \) is a matrix of size \( \text{rank}(T_{(M,A)}) \times \text{rank}(F) \). We assume the presentation to be minimal, i.e. \( \text{rank}(F) \) to be minimal possible. (Explicitly: no column of \( \mathcal{A}_G \) is an \( R \)-linear combination of the others.) Further, as \( T_{(G,A)} \subset T_{(M,A)} \subset \text{Mat}(m, n; m) \), all the entries of \( \mathcal{A}_G \) lie in \( m \).

Example 2.7. Let \( M = \text{Mat}(m, n; m) \) then \( \text{rank}(T_{(M,A)}) = mn \). For \( G = G_{tr} \), the matrix \( A_{G_{tr}} \), is of size \( mn \times (m^2 + n^2) \), cf. example 2.1.

Let \( M = \text{Mat}^{sym}(m, m; m) \) then \( \text{rank}(T_{(M,A)}) = \binom{m+n}{2} \). For \( G = G_{congr} \), \( \mathcal{A}_{G_{congr}} \) is of size \( \binom{m+n}{2} \times m^2 \).

Usually we have \( \text{rank}(F) \geq \text{rank}(T_{(M,A)}) \), i.e. the number of rows in the matrix \( \mathcal{A}_G \) is not bigger than the number of columns. Thus we denote \( I_{\text{max}}(\mathcal{A}_G) := I_{\text{rank}(T_{(M,A)})}(\mathcal{A}_G) \).

Theorem 2.8. 1. \( A \in M \) is finitely-\( G \)-determined iff \( I_{\text{max}}(\mathcal{A}_G) \) contains a power of the maximal ideal of \( R \).

2. If \( A \in M \) is \( G \)-stable then \( I_{\text{max}}(\mathcal{A}_G) = m^{\text{rank}(T_{(M,A)})} \subset R \). If \( R \) is a regular local ring, then the converse holds too.

Proof. Consider the cokernel, \( F \rightarrow \text{Mat}(m, n; R) \rightarrow \text{coker}(\mathcal{A}_G) \rightarrow 0 \), it is an \( R \)-module.

1. Finite determinacy means the finiteness of the codimension \( T_{(G,A)} \subset T_{(M,A)} \), which is equivalent to the finite dimensionality of the module \( \text{coker}(\mathcal{A}_G) \) (as \( k \)-vector space). And this is equivalent to: the annihilator of \( \text{coker}(\mathcal{A}_G) \) contains a power of the maximal ideal. But, by [Eisenbud, pg.498] the radicals of \( Ann(\text{coker}(\mathcal{A}_G)) \) and \( I_{\text{max}}(\mathcal{A}_G) \) coincide. Hence the first statement.

2. As \( \mathcal{A}_G \in \text{Mat}(\text{rank}(T_{(M,0)}), \text{rank}(F), m) \) we get \( I_{\text{max}}(A) \leq m^{\text{rank}(T_{(M,0)})} \). If \( A \) is stable then \( T_{(G,A)} = \text{Mat}(m, n; m) \), so \( Ann(\text{coker}(\mathcal{A}_G)) = m \). Recall, [Eisenbud, proposition 20.7], that \( I_{\text{max}}(\mathcal{A}_G) \subseteq Ann(\text{coker}(\mathcal{A}_G)) \).
and if \( \operatorname{coker}(A_G) \) can be generated by \( k \) elements then \( (\operatorname{Ann}(\operatorname{coker}(A_G)))^k \subseteq I_{\max}(A_G) \). In our case, \( \operatorname{coker}(A_G) \) is minimally generated by \( \text{rank}(T_{(M,A)}) \) elements, thus \( m^\text{rank}(T_{(M,A)}) \subseteq I_{\max}(A_G) \). Combining with the previous we get \( I_{\max}(A) = m^\text{rank}(T_{(M,A)}) \).

Vice-versa, suppose \( I_{\max}(A) = m^\text{rank}(T_{(M,A)}) \) then, as mentioned above, \( m^\text{rank}(T_{(M,A)}) \subseteq (\operatorname{Ann}(\operatorname{coker}(A_G)))^\text{rank}(T_{(M,A)}) \) while, obviously, \( \operatorname{Ann}(\operatorname{coker}(A_G)) \subseteq m \). Thus \( (\operatorname{Ann}(\operatorname{coker}(A_G)))^\text{rank}(T_{(M,A)}) = m^\text{rank}(T_{(M,A)}) \). If \( R \) is regular, then this implies \( \operatorname{Ann}(\operatorname{coker}(A_G)) = m \), i.e. \( T_{(GA,A)} = T_{(M,0)} \), which implies the stability. \( \blacksquare \)

**Remark 2.9.** In general, for non-regular ring, the condition \( I_{\max}(A) = m^mn \) does not imply stability of \( A \in \text{Mat}(m,n;m) \).

For example, given any ring \( (S,m_S) \), consider \( R = S[e]/\{e^lm_S^{m-n} \}_{j=1...mn} \), then \( m^mn_R = (m_S^mn)R \). Suppose \( A \in \text{Mat}(m,n;m_S) \) is stable, then \( I_{\max}(A) = m_S^zn \). But, of course, in general \( A \) is not stable inside \( \text{Mat}(m,n;m_R) \).

**Remark 2.10.** In many questions of geometry/singularity theory one stratifies the ambient space (the generic points, the "minimally special" points, more special points etc.) The "worst stratum" is defined as the last step of a stratification procedure. Quite often this stratum cannot be defined directly, rather it is defined inductively, raising various associated questions of complexity.

In our case, lemma 2.8 determines, for any scenario \( (R,G,M) \) the "worst" stratum \( \Sigma_{\infty} \) of not finitely-determined matrices. This worst stratum is defined directly, by the explicitly written condition: \( I_{\max}(A_G) \) does not contain any power of \( m \) (alternatively: \( I_{\max}(A_G) \) defines a subscheme of positive dimension in \( \text{Spec}(R) \)).

**Remark 2.11.** As theorem 1.1 reads, the condition "\( I_{\max}(A_G) \) contains some power of \( m \)" is equivalent to: "\( A_G \) is \( G_r \)-determined". Similarly, for regular local rings, \( I_{\max}(A_G) = m^mn \) iff \( A_G \) is \( G_r \)-stable. Therefore the lemma implies the general statement:

The stability/finiteness for an action \( G \circ M \) is reduced to those for the action \( G_r \circ \text{Mat}(\text{rank}(M), \text{rank}(F); R) \).

In other words, the problem for an arbitrary action \( G \circ M \) is "contained" in the problem for \( G_r \circ \text{Mat}(m,n;m) \).

**2.6.3. Criterion in terms of the left kernel.** For any given ring \( S \), the left kernel of a matrix \( B \in \text{Mat}(m,n;m;S) \) is defined by

\[
Ker^{(l)}(B) := \{ v \in S^\oplus m | vB = 0 \in S^\oplus n \}.
\]

By construction, it is an \( S \) submodule of the free module \( S^\oplus m \), thus it is torsion-free (or zero).

**Lemma 2.12.** Given a group action \( G \circ M \subset \text{Mat}(m,n;m) \) and \( A \in M \), let \( A_G \) be the corresponding generating matrix of \( T_{(GA,A)} \). Then \( A \) is finitely \( G \)-determined inside \( M \) iff for any radical ideal \( J \subset R \), that is not the maximal ideal, the matrix \( A_G \oplus R/J \) has no left kernel (i.e. the left kernel is trivial).

Note that instead of "any radical ideal" one can put here: "any prime ideal".

**Proof.** By the general commutative algebra, the support of the left kernel is the radical of \( I_{\max}(A_G) \). [Eisenbud, §20]. More precisely, \( A_G \oplus R/J \) has a non-trivial left kernel iff \( J \supseteq \sqrt{I_{\max}(A_G)} \). Thus, if \( I_{\max}(A_G) \) contains a power of the maximal ideal, then for any radical ideal \( J \) that is not \( m \), \( J \nsubseteq I_{\max}(A_G) \). Thus \( A_G \oplus R/J \) has no left kernel.

Suppose \( I_{\max}(A_G) \) does not contain any power of the maximal ideal, then the quotient ring \( R/I_{\max}(A_G) \) has a positive Krull dimension. Taking \( J = \sqrt{I_{\max}(A_G)} \), we get: \( I_{\max}(A_G \oplus R/J) = 0 \in R \), thus \( A_G \oplus R/J \) has the non-zero left kernel. \( \blacksquare \)

**Remark 2.13.** Suppose \( A \) is the matrix of "genuine functions", in the sense of §2.5. Then the condition, "\( I_{\max}(A) \) contains a power of the maximal ideal" is immediately translated into: for any point of the punctured neighborhood of the origin, \( pt \in \text{Spec}(R) \setminus \{0\} \), there is a function \( f \in I_{\max}(A) \) with \( f(pt) \neq 0 \). Or, alternatively, the left kernel of the numerical matrix \( A|_{pt} \) is trivial. For formal rings, we cannot compute \( A \) at a point off the origin. The statement of the lemma is just the reformulation of "\( Ker^{(l)}(A|_{pt}) = 0 \)" condition to the case of an arbitrary local ring.

Thus, to check the stability/finite determinacy, we should write down the matrix \( A_G \) for a given group \( G \).

**2.7. Example: the matrices \( A_G \), \( A_G \) and \( A_{\text{Aut}(R)} \).** Consider the (most inclusive) case \( G_r \circ \text{Mat}(m,n;m) \). The tangent space for this action was constructed in example 2.1. It is spanned by all the possible \( R \)-linear combinations of \( uA, Av, DA \), where \( D \in \text{Der}(R,m) \).

As for finite determinacy we need only the finiteness of the dimension of \( T_{(\text{Mat}(m,n,m),A)}/T_{(GA,A)} \), we can consider the ordinary module of derivations, \( \text{Der}(R) \), instead of \( \text{Der}(R,m) \). (Indeed, the quotient \( \text{Span}_R \{uA, Av, DA\} \subseteq \text{Der}(R)/\text{Span}_R \{uA, Av, DA\} \) is a finite dimensional \( k \)-vector space.)

Thus, we consider the module generated by

\[
\{ \sum_j A_{ij}E_{kj} \}_{i,k}, \{ \sum_i A_{ij}E_{ik} \}_{j,k}, \{ \partial_i A \}_{i=1\ldots p}.
\]

Here \( E_{ij} \) are elementary matrices, \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \). In the first brackets we have matrices with only one non-zero row, an arbitrary row of \( A \), in the second matrices with only one non-zero columns, an arbitrary column of \( A \).
In the third brackets, \( \partial_i \) are the generators of \( \text{Der}(R) \), as a module over \( R \). If \( R \) is a regular ring, then these are just the ordinary derivatives.

Identify \( \text{Mat}(m, n; R) \) with the space of column vectors, \( R^{\oplus mn} \). We present the matrix \( A \in \text{Mat}(mn, p + m^2 + n^2, R) \) in three blocks. The first \( mn \times p \) block corresponds to the change of variables, i.e. it generates \( T_{(A_{\text{Aut}(R)}, A)} \). The second \( mn \times n^2 \) block corresponds to \( G_J \), the third \( mn \times m^2 \) block corresponds to \( G_I \). To present the matrices in the compact form we use the notations: \( \vec{a}_{ki} := (a_{k1}, \ldots, a_{km}) \) and \( \vec{a}_{jk} := (a_{1k}, \ldots, a_{nk}) \). Then, in the basis \((\vec{a}_{11}, \vec{a}_{21}, \ldots, \vec{a}_{ml}) \) we have:

\[
A = \begin{pmatrix}
\vec{a}_{11} & \vec{a}_{12} & \cdots & \vec{a}_{1m} \\
\vec{a}_{21} & \vec{a}_{22} & \cdots & \vec{a}_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\vec{a}_{m1} & \vec{a}_{m2} & \cdots & \vec{a}_{mm}
\end{pmatrix}
\]

Here \( \vec{\partial} \) denotes the row of derivatives, \( \vec{\partial} a = (\partial_1 a, \ldots, \partial_p a) \). The over-the-matrix symbols \( \vec{E}_{ki} = (E_{k1}, \ldots, E_{km}) \) and \( \vec{E}_{jk} = (E_{1k}, \ldots, E_{nk}) \) denote the particular generators of the tangent space, as they appear in equation (8).

Sometimes we need this matrix in the basis \((\vec{a}_{j1}, \vec{a}_{j2}, \ldots, \vec{a}_{jn}) \):

\[
A = \begin{pmatrix}
\vec{a}_{j1} & 0 & \cdots & 0 \\
0 & \vec{a}_{j1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \vec{a}_{jn}
\end{pmatrix}
\]

Denote by \( A_{G_J}, A_{G_I}, A_{\text{Aut}(R)} \) the corresponding blocks of \( A \).

We restate lemma 2.12 in this particular case.

**Corollary 2.14.** \( A \) is finitely \( G \)-determined iff for any radical ideal \( J \subsetneq m \) the left kernel module is trivial:

\[
G = G_J : \quad \text{Ker}^{(l)}(A_{G_J}) = \left\{ B \in \text{Mat}(n, m, R/J) \mid BA = \Omega_{n \times n}, \ AB = \Omega_{m \times m} \right\} = \{0\}
\]

\[
G = G_J \quad \text{Ker}^{(l)}(A_{G_I}) = \left\{ B \in \text{Mat}(n, m, R/J) \mid BA = \Omega_{n \times n}, \ AB = \Omega_{m \times m}, \ \text{trace}(B\vec{\partial}A) = 0 \right\} = \{0\}
\]

Here \( \vec{0} \) is the vector of zeros, \( \vec{\partial} \) denotes the generators of all the possible derivations, \( \text{Der}(R) \). All the matrices/vectors have their entries in \( R/J \). In the "geometric" case (see \( \S 2.5 \)) this can be reformulated as:

\[
(w_1, \ldots, w_{mn}) \rightarrow \begin{pmatrix} w_1 & \cdots & w_n \\
w_{n+1} & \cdots & w_{2n} \\
\vdots & \ddots & \vdots \\
w_{n(m-1)+1} & \cdots & w_{mn} \end{pmatrix}
\]

**Proof.** This is immediate consequence of lemma 2.12. We should only check that writing the left kernel of \( A_{G_J}, A_{G_I} \) in the matrix form,
2.8. How to check the (non-)triviality of $\text{Ker}^{(l)}(A_{G_l})$. By lemma 2.12 and the subsequent remark, to check the finite determinacy (for the given $(G, M, R)$) one should check the (non-)triviality of $\text{Ker}^{(l)}(A_{G_l})$ for points $0 \neq pt \in \text{Spec}(R)$. As the example of §2.7 shows, we have some system of matrix equations. The following observations are useful.

* Usually the left kernel is equivariant with respect to $GL(m, k) \times GL(n, k)$. For example, if $A_{G_l}$ is non-zero too, i.e. there exists $0 \neq A_{G_l}$, then we can assume $A_{G_l}$ is in a suitable form one first treats the conditions on $B$ that arise from the $G \cap G_l$, part of the group. (For example, $AB = 0$ and/or $BA = 0$ and/or $AB + BA = 0$ etc.) This forces some sub-blocks of $B$ to vanish. Then one has to check the condition of the form $\text{trace}(B\tilde{A})_{|pt} = \tilde{0}$. Here $\tilde{A}$ is constructed from some blocks of $A$ (see e.g. §3.4.2) and $A_{|pt} = 0$, while $B$ is some numerical matrix. This condition is reinterpreted geometrically as follows. Write all the entries of $A$ as a column, write all the entries of $B$ as a row. Then $\tilde{A}$ gives the generating matrix $\text{Aut}(R)$, e.g. as in equation (10). And the condition $\text{trace}(B\tilde{A}) = 0$ means that (the row) $B$ is in the left kernel of $\text{Aut}(R)$. This has non-trivial solutions (for $B$) iff the rows of $\text{Aut}(R)$ are $\mathbb{k}$-linearly dependent. The later means that some associated degeneracy locus is either not quasi-smooth or not of expected dimension.

This algorithm is applied constantly in the following section.

3. Finite determinacy for particular groups and subspaces of $\text{Mat}(m, n; R)$

3.1. The case of $G_l$

Proof of theorem 1.1.

By lemma 2.8 we need to compute the ideal of maximal minors of $A$. In the absence of derivations (of $\text{Aut}(R)$), $A_{G_l}$ is a $mn \times (m^2 + n^2)$ matrix, (and $mn < m^2 + n^2$), so a minor is specified by the choice of $mn$ columns of $A_{G_l}$.

1. Note that $A_{G_l}$ is a $mn \times m^2$ matrix, thus, if $m < n$ we get $I_{mn}(A_{G_l}) = \{0\} \subset R$. In particular, (as the ring $R$ is of positive dimension, i.e. not Artinian, $I_{mn}(A_{G_l})$ cannot contain a power of $m$.

2. The direction $\Leftarrow$ is trivial, as $G_l \supset G_r$. For the direction $\Rightarrow$, suppose $A$ is not finitely $G_r$-determined, then there exists a radical ideal $J \subseteq m$ with a non-zero left kernel vector, $0 \neq v \in \text{Ker}^{(l)}(A \otimes R/J)$. As $n \geq m$, the ordinary kernel is non-zero too, i.e. there exists $0 \neq u \in \text{Ker}(A \otimes R/J)$ which is supported on $V(J)$. Namely, if $gu = 0 \in (R/J)^{\oplus n}$ then $g = 0 \in R/J$ (recall that $J$ is a radical ideal). Therefore, the matrix $B = u \otimes v \in \{u_iu_j\}_{ij} \in \text{Mat}(n, m, R/J)$ is non-zero and satisfies: $AB = 0$, $BA = 0$. Thus, $A$ is not finitely-$G_{2l}$-determined.

3. Using the form of $A$ as in equation (10), we get: $I_{\text{max}}(A_{G_l}) = (I_m(A))^n$. In particular, $I_m(A)$ contains a power of the maximal ideal iff $I_{\text{max}}(A_{G_l})$ does. Hence $A$ is $G_r$ finitely determined iff the ideal $I_m(A)$ contains a power of $m \subseteq R$.

Example 3.1. As mentioned in the introduction, to avoid trivialities we assume the entries of $A$ in $m$. Consider the trivial case: $A$ is a constant (numeric) matrix, $m \leq n$. Then $A$ is finitely $G_r$-determined iff at least one of its maximal minors is a non-zero constant, i.e. $A$ is of the full rank. (In other words, $A$ is invertible from the left, i.e. it has no left-kernel.) In this case, for $m \leq n$, $A$ is 0-determined with respect to $G_r$, even stable.

Example 3.2. Another trivial case is $\text{dim}(R) = 1$. In this case if $f \in R$ is not a zero divisor then $fR \supset m^N$ for some $N > 0$. (This follows from the existence of conductor in $R$.) So, a matrix is finitely $G_r$-determined if at least one of its maximal minors is not a zero divisor in $R$.

More generally we have:

Corollary 3.3. Suppose $\text{dim}(R) > 0$.

1. If $m = n$ then $A \in \text{Mat}(m, m, m)$ is finitely $G_{2l}$-determined iff $\text{dim}(R) = 1$ and $\det(A) \in R$ is not a zero divisor.

2. If $\text{dim}(R) > |n - m| + 1$ then no matrix in $\text{Mat}(m, n; m)$ is finitely $G_r$-determined.

3. Suppose $\text{dim}(R) \leq |n - m| + 1$. Given $A \in \text{Mat}(m, n; R)$ and $N > 0$, for any generic enough $B \in \text{Mat}(m, n; m^N)$, the matrix $A + B$ is finitely $G_r$-determined. In particular, the set of matrices that are not finitely determined is of infinite codimension in $\text{Mat}(m, n; R)$.

4. Suppose $\text{dim}(R) = 2$. If $A$ has at least two $m \times m$ blocks whose determinants are relatively prime, (i.e. if $\Delta_i = a_ih \in R$ then $h \in R$ is invertible), then $A$ is finitely $G_r$-determined.
Proof. (1) is immediate.

(2). Assume \( m \leq n \). If \( I_m \supset \mathfrak{m}^N \) for some \( N > 0 \) then the germ defined by this ideal, over \( k = \bar{k} \), \( V(I_m) \subset \text{Spec}(R) \), is supported at the origin only, i.e. the dimension of \( V(I_m) \) is zero. But the height of the ideal of maximal minors is at most \((n-m+1)\). (Or, the codimension of the corresponding germ is at most \((n-m+1)\).) So \( \text{dim}V(I_m) \geq \text{dim}(R) - (n-m+1) > 0 \), contradicting \( I_m \supset \mathfrak{m}^N \).

(3). Follows by observation that for generic enough \( B \) the codimension of \( I_m(A + B) \), or of each irreducible component of this variety, is the expected one.

(4). Let \( \Delta_1, \Delta_2 \) be two such minors then the scheme \( \{ \Delta_1 = 0 = \Delta_2 \} \) is supported at the origin only, 0 \( \in \text{Spec}(R) \). Thus the local ring contains a power of maximal ideal. \( \blacksquare \)

3.2. Finite determinacy for deformations that preserve Fitting ideals. Recall that \( G_l \), preserves all the Fitting ideals, \( \{I_j(A)\} \), in particular, in most cases there are no finitely-\( G_l \)-determined matrices. Therefore, for the action of \( G_l \) or of its subgroups, it is natural to consider deformations of \( A \in \text{Mat}(m, n; \mathfrak{m}) \) only inside the stratum

\[
\Sigma_{I_m}(A) := \{ B | I_m(B) = I_m(A) \} \subset \text{Mat}(m, n; \mathfrak{m})
\]

In terms of commutative algebra, we consider only the modules with the given support. Further, one might wish to preserve all the Fitting ideals, i.e. to consider the stratum:

\[
\Sigma_{I_m, \ldots, I_1}(A) := \{ B | I_j(B) = I_j(A), \text{ for } j = 1, \ldots, m \} \subset \text{Mat}(m, n; \mathfrak{m})
\]

Note that neither \( \Sigma_{I_m}(A) \) nor \( \Sigma_{I_m, \ldots, I_1}(A) \) are linear subspaces of \( \text{Mat}(m, n; \mathfrak{m}) \). Therefore our methods cannot establish finite determinacy for deformations inside \( \Sigma_{I_m}(A), \Sigma_{I_m, \ldots, I_1}(A) \). But in many cases it is easy to obstruct the finite determinacy.

Proposition 3.4. 1. For a given \( A \in \text{Mat}(m, n; \mathfrak{m}) \) consider only deformations inside \( \Sigma_{I_m}(A) \). If \( \text{dim}(R) > 2(|n - m| + 2) \) then \( A \) cannot be finitely-\( G_l \)-determined.

2. Similarly, for a given \( A \in \text{Mat}(m, n; \mathfrak{m}) \) consider only deformations inside \( \Sigma_{I_m, \ldots, I_1}(A) \). If \( \text{dim}(R) > mn \) then \( A \) cannot be finitely-\( G_l \)-determined.

Proof. To obstruct the finite determinacy, it is enough to prove that the embeddings of tangent spaces,

\[
T_{(G_l: A)} \subset T_{(\Sigma_{I_m}(A), A)} \subset T_{(G_l: A, A)}
\]

are of infinite codimension over \( k \).

Step 1. We compute the tangent spaces to the strata. For the deformation \( A + \epsilon B \), with \( \epsilon \) a numerical parameter, we must check the invariance of the relevant ideal to the first order in \( \epsilon \). Then, differentiating by \( \epsilon \), we get a vector in the tangent space.

Let \( C_{ij} \) denote some \( j \times j \) block of a matrix \( C \). Use the relation \( \text{det}(A + \epsilon B)_{\Box j} = \text{det}A_{\Box j} + \epsilon \text{tr}(A^\vee_{\Box j} B_{\Box j}) + \cdots \), where \( A^\vee_{\Box j} \) denotes the adjugate matrix of \( A_{\Box j} \). Then we get:

\[
T_{(\Sigma_{I_m}(A), A)} = \{ B \in \text{Mat}(m, n; \mathfrak{m}) | \forall \Box_m \subset \{1, \ldots, n\} : \text{trace}(A^\vee_{\Box_m} B_{\Box_m}) \in I_m(A) \}
\]

Similarly:

\[
T_{(\Sigma_{I_m, \ldots, I_1}(A), A)} = \{ B \in \text{Mat}(m, n; \mathfrak{m}) | \forall 1 \leq j \leq m : \forall \Box_j \subset \{1, \ldots, n\} : \text{trace}(A^\vee_{\Box_j} B_{\Box_j}) \in I_j(A) \}
\]

Step 2. Note that both strata are generically reduced, thus they are smooth at their generic points. Further, the finite determinacy is an open property. Therefore it is enough to check that the generic (smooth) points of the strata correspond to matrices that are not finitely determined.

Let a matrix \( C \in \Sigma_* \) correspond to a smooth point. Recall, \S 2.3, that if \( C \) is finitely-\( G_l \)-determined over \( R \), then it is finitely determined over \( R/J \), for any ideal. Alternatively, the restriction of \( C \) to any subgerm \( V(J) \subset \text{Spec}(R) \) remains finitely determined. Take \( J = I_{m-1}(C) \) in the first case or \( J = I_1(C) \) in the second and consider the problem over \( R/J \).

Then, in the first case, we have \( I_{m-1}(A) = \{ 0 \} \subset R/J \) hence \( T_{(\Sigma_{I_m}(A), A)} = \text{Mat}(m, n; \mathfrak{m} R/J) \). Similarly, in the second case: \( T_{(\Sigma_{I_m, \ldots, I_1}(A), A)} = \text{Mat}(m, n; \mathfrak{m} R/J) \). And we want to prove that the embedding \( T_{(G_l: C)} \subset \text{Mat}(m, n; \mathfrak{m} R/J) \) is of infinite codimension. But now we have just the ordinary problem of finite-\( G_l \)-determinacy of a matrix in \( \text{Mat}(m, n; \mathfrak{m} R/J) \). And by the assumptions on \( \text{dim}(R) \) we have: \( \text{dim}(R/J) > 0 \) in both cases, while \( I_m(C \otimes R/J) = \{ 0 \} \). So, \( C \otimes R/J \in \text{Mat}(m, n; \mathfrak{m} / J) \) cannot be finitely determined. \( \blacksquare \)
3.3. The case of $\text{Aut}(R)$. In this case the group does not use any matrix structure, we have just the classical right equivalence, $\text{Aut}(R) \cap \text{Maps}(\text{Spec}(R), (k^n, 0))$. For completeness we reprove the corresponding statements.

**Definition 3.5.** A set $\{x_1\}_{i=1,k}$ of elements of $R$ is called "a sequence of regular parameters" if:

- the quotient $R/(x_1, \ldots, x_k)$ is of Krull dimension $(\dim(R) - k)$ and
- the images of $\{x_i\}_{i=1,k}$ in $m/m^2$ are linearly independent and
- any map $\{x_i = x_i + \phi_i(x)\}_{i=1,k}$ with $\phi_i(x) \in m^N$, for some $N \geq 0$, lifts to an automorphism of the whole ring.

If $R$ is regular then the generators of $m$ (over $R$) form a maximal system of regular parameters. In general, the maximal system of regular parameters is a system of coordinates on the Noether normalization of a maximal dimensional component of $\text{Spec}(R)$.

**Proposition 3.6.** 1. If $n > 1$ then the map $A \in (\text{Spec}(R), (k^n, 0))$ is finitely-$\text{Aut}(R)$-determined iff $\dim(R) \geq n$ and the entries of $A$ form a sequence of regular parameters (of length $n$) in the sense above.

2. If $R$ is regular and $n > 1$, then in the later case $A$ is $\text{Aut}(R)$-stable.

3. If $n = 1$ then the map $A \in (\text{Spec}(R), (k^1, 0))$, i.e. an element of the ring, is finitely-$\text{Aut}(R)$-determined iff the ideal $(\partial_1 A, \ldots, \partial_p A)$ contains a power of $m$, i.e. defines a one-point-scheme on $\text{Spec}(R)$.

(If $R$ is not regular then $A^{-1}(0)$ is not necessarily reduced or has an isolated singularity, cf. example 3.10 and remark 3.22.)

**Proof.** 1. $\Rightarrow$ By lemma 2.8 we should check the ideal of maximal minors, $I_{\text{max}}(A_{\text{Aut}(R)})$, where $A_{\text{Aut}(R)}$ is a $n \times p$ matrix. In particular, if $n > p$, then $I_{\text{max}}(A_{\text{Aut}(R)}) = \{0\}$, i.e. $A$ is not finitely-$\text{Aut}(R)$-determined. This ideal is of height at most $(n - p + 1)$, thus for $n \leq p$ it defines the subgerm of $\text{Spec}(R) \otimes_k k$, which is either empty or of codimension at most $(p - n + 1)$.

The first case means that $I_{\text{max}}(A_{\text{Aut}(R)}) = R$, i.e. the determinant of one of the minors of $A_{\text{Aut}(R)}$ is an invertible element of $R$. As the elements are derivatives, we get that the entries of $A$ form a sequence of regular parameters.

The second case means that $I_{\text{max}}(A_{\text{Aut}(R)})$ defines a subgerm in $\text{Spec}(R) \otimes_k k$ of positive dimension (as $n > 1$), thus the ideal cannot contain any power of $m \subset R$.

$\Leftarrow$ If the entries of $A$ form a sequence of regular parameters then for any $B \in \text{Mat}(1, n; m^N)$, $N \gg 0$, there exists $\phi \in \text{Aut}(R)$ such that $\phi(A + B) = A$.

2. If $R$ is a regular ring then the first two conditions in the definition of regular parameters already imply that any automorphism of $\{x_i\}$ extends to an automorphism of the whole ring. Hence we get the stability.

3. This is just lemma 2.8, note that in our case $A_{\text{Aut}(R)} = (\partial_1 A, \ldots, \partial_p A)$. $\blacksquare$

3.4. The case of $G_{tr}$.

3.4.1. Conditions on Fitting ideals. If $A \in \text{Mat}(m, n; R)$ is $G_{tr}$-finitely determined then all the Fitting ideals are "simultaneously finitely determined", i.e. for any $B \geq N$ there exists $\phi \in \text{Aut}(R)$ satisfying:

\[(\phi(I_1(A + B_{\geq N})), \ldots, \phi(I_m(A + B_{\geq N}))) = (\phi(I_1(A)), \ldots, \phi(I_m(A)))\]

On this occasion we record the miniversal deformations of these ideals.

**Lemma 3.7.** The tangent space of the miniversal deformation of $I_j(A)$ is $I^{-1}_j(A)/(I_j(A), \partial I_j(A))$.

**Proof.** Consider the deformation $A + \epsilon B$, expand in powers of $\epsilon$ and use (for each square block) the relation

\[\det(A + \epsilon B) = \det(A) + \epsilon tr(A^T B) + \cdots \]

By choosing all the possible matrices $B$ we get: $T_{I_j(A)}$ is spanned by all the elements of $I^{-1}_j(A)$.

The tangent space to the orbit $\text{Aut}(R)I_j(A)$ is computed as usual. Consider an infinitesimal automorphism, whose action on the generators of $R$ is: $x_i \mapsto x_i + \epsilon \partial_i(x)$. Then $\det(A) \mapsto \det(A) + \epsilon \sum_i \partial_i det(A)$ in addition, the change $(1 + \epsilon u(x))det(A)$ does not change the ideal. Hence the statement. $\blacksquare$

For an ideal $J \subset R$ denote by $ord(J)$ the maximal number $k$ such that $J \subset m^k$.

**Corollary 3.8.** Suppose $R$ is a regular ring choose some basis $(x_1, \ldots, x_p)$ of $m$ over $R$. Suppose $ord(I_m(A)) > ord(I_{m-1}(A)) + 1$, and moreover no maximal minor of $A$ involves $x_p$. Then $A$ is not finitely-$G_{tr}$-determined.

**Proof.** We prove that in this case the miniversal deformation of $I_{m-1}(A)$ is infinite-dimensional.

Let $a \in I_{m-1}(A)$ be an element whose order coincides with that of $I_{m-1}(A)$. By lemma 3.7 the tangent plane to the miniversal deformation of $I_m(A)$ contains the (image of) the subspace generated by the elements $\{ax_j\}_{j \geq 0}$. On the other hand, by the initial assumption: if $a(\sum_j c_j x_j) \in (I_m(A), \partial I_m(A))$ then $a \in (I_m(A), \partial I_m(A))$. Which is impossible, as $ord(a) = ord(I_m(A)) - 1 = ord(\partial I_m(A))$. Thus the vector subspace (of all the possible deformations) generated by $\{ax_j\}_{j \geq 0}$ intersects trivially the subspace $< I_m(A), \partial I_m(A) >$. But then the vector space $I^{-1}_{m-1}(A)/(I_m(A), \partial I_m(A))$ is
of infinite dimension. ■

Proposition 3.9. If \( A \in \text{Mat}(m, n; R) \) is \( G_r \)-finitely determined then:
1. There exists an automorphism \( \phi \circ R \) such that \( \phi(I_1), \ldots, \phi(I_m) \) are generated by polynomials.
2. The heights of all the Fitting ideals, \( \{I_j(A)\} \), i.e. the co-dimensions of the corresponding loci \( \{V(I_j(A))\} \) are the ‘expected’ ones.
3. In particular, either \( \dim(R) > mn \) and \( I_1(A) \) is a complete intersection ideal or \( I_1(A) \) contains a power of maximal ideal.

Proof. 1. This is just a reformulation of algebraizability.
2. This is a standard statement in commutative algebra. One can consider matrices of \( \text{Mat}(m, n; R) \) as maps from \( \text{Spec}(R) \) to \( \text{Mat}(m, n; k) \). (Numerical) Matrices of rank \( \leq r \) form a closed subset of \( \text{Mat}(m, n; k) \), of codimension \( (m-r)(n-r) \). Accordingly, for \( A \in \text{Mat}(m, n; R) \), this is the expected height of ideal \( I_{r+1}(A) \). Being not of the expected height means special relations between the entries of the matrix, the relations that are violated by the generic deformation inside \( \text{Mat}(m, n; \mathbb{A}^N) \), for an arbitrary large \( N \).
3. Follows from 2. ■

Example 3.10. • If \( R \) is not a regular ring then being finitely determined does not imply that the ideal \( I_1(A) \) is radical (i.e. defines a subscheme of \( \text{Spec}A \)).
• Further, cf. remark 3.22, even if \( R \) is a complete intersection and \( A \) is finitely-\( G_r \)-determined, the Fitting ideal \( I_1(A) \) can define a scheme with multiple components.

3.4.2. The main criterion. Here we work over the algebraic closure, \( \overline{R} \).
First we define the quasi-smoothness. The substitution of smoothness of \( (X, pt) \subset (Y, pt) \) when the ambient space \( (Y, pt) \) is singular. Consider the local ring, \( \mathcal{O}_{(Y, pt)} \) and its module of derivations, \( \text{Der}(\mathcal{O}_{(Y, pt)}) \). Choose some generators, \( \text{Der}(\mathcal{O}_{(Y, pt)}) = \mathcal{O}_{(Y, pt)}(\partial_1, \ldots, \partial_p) \). If \( (Y, 0) \) is smooth then these are just ordinary partial derivatives. Let \( I_{(X, pt)} = (g_1, \ldots, g_k) \subset \mathcal{O}_{(Y, pt)} \), consider the Jacobian matrix \( J_g = (\partial_i g_j) \). Note that \( \text{rank}(J_g|_{pt}) \leq \text{codim}(\mathcal{O}_{(Y, pt)}(X, pt)). \)

Definition 3.11. A germ of positive dimension, \( (X, pt) \subset (Y, pt) \), is called quasi-smooth if \( \text{rank}(J_g|_{pt}) = \text{codim}(\mathcal{O}_{(Y, pt)}(X, pt)) \).

(We could not find the origin of this definition. If \( (Y, pt) \) is the germ of a toric variety and \( (X, pt) \) is a hypersurface, defined by a principal ideal, then we get the usual notion of quasi-smoothness.)

Example 3.12. • If \( (Y, pt) \) is smooth, then \( (X, pt) \) is quasi-smooth if it is smooth.
• Let \( (Y, 0) = (x_1, x_2) = 0 \subset (k^p, 0) \), then a hypersurface \( (X, 0) = \{f = 0\} \subset (Y, 0) \) is quasi-smooth iff jet_1(f) does not go to zero in \( k[[x_1, \ldots, x_p]]/(x_1, x_2) \).

More generally, suppose the singular locus \( \text{Sing}(Y, pt) \) is smooth, as a set. Then, \( (X, pt) \) is quasi-smooth iff it is transverse to \( \text{Sing}(Y, pt) \). Indeed, suppose \( \text{Sing}(Y, pt) \) is smooth of codimension \( k \). Rectify it so that the completion of \( \mathcal{O}_{(Y, pt)} \) is isomorphic to \( k[[x_1, \ldots, x_k]]/I \), where \( I \subset (x_1, \ldots, x_k) \). Then \( \text{Der}(\mathcal{O}_{(Y, pt)}) \subset (\partial_{k+1}, \ldots, \partial_p, m \partial_1, \ldots, m \partial_k) \). Thus the only possibly non-zero entries of the matrix \( J_g|_{pt} \) are those that come from the linear terms of \( \{g_i\} \) and the derivatives \( \{\partial_j\}_{j>k} \). Hence the statement.

Proof of theorem 1.2. 1. By theorem 1.1 if \( A \) is not finitely \( G_r \)-determined then \( I_m(A) \) contains no power of the maximal ideal. But for any \( N \), for generic enough \( B \in \text{Mat}(m, n; \mathbb{C}^{N+1}) \), the ideal \( I_m(A+B) \) has height \( m(n-m+1, \text{dim}(R)) \), i.e. defines a subscheme of \( \text{Spec}(R) \) of this codimension. (Recall that we work over \( \overline{R} \).) As \( \text{dim}(R) \leq n-m+1 \), this subscheme is a point, i.e. \( I_m(A+B) \) contains a power of maximal ideal, hence cannot be equivalent to \( I_m(A) \).
2. Follows immediately from part 3 of theorem 1.1 and observation: height of \( I_m(A) \) is \( n-m+1 \).
3. We consider only the case of \( G_r \). By corollary 2.14, the finite determinacy is equivalent to the triviality of the kernel module at each point:

\[
\{B \in \text{Mat}(m, n, \overline{R})| BA|_{pt} = 0, \text{trace}(B\partial A)|_{pt} = 0\} = \{0\}
\]

Suppose \( \text{rank}(A|_{pt}) = r \), this implies: \( \text{dim}(R) > (m-r)(n-r) \). As is explained in §2.8, we can assume \( A = (A_1 A_2 A_3 A_4) \), where \( A_1|_{pt} = 1_{r\times r} \) and \( A_2, A_3, A_4 \) vanish at \( pt \). Present \( B \) accordingly, \( (B_1 B_2 B_3 B_4) \). Then \( BA|_{pt} = 0 \) means \( B_1 = 0 \) and \( B_2 = 0 \). The only remaining condition is: \( \text{trace}(B_3\partial A_3 + B_4\partial A_4) = 0 \). Here \( A_3 \) is of size \((m-r) \times r \), \( A_4 \) is of size \((m-r) \times (n-r) \). As is explained in §2.8, if we write the entries of \( A_3, A_4 \) in one column, the trace condition means: \( \text{rank}(\partial(\_\_)) < n(m-r) \). But if \( \text{dim}(R) < n(m-r) \) then the condition is empty!

Therefore, if for some \( 1 \leq r \leq m \): \( (m-r)(n-r) < \text{dim}(R) < n(m-r) \), then for any point of \( \Sigma_r(A) \) the left kernel of \( A_{\partial_3} \) is non-trivial. Hence \( A \) is not finitely-\( G_r \)-determined.
4. As in the $G_r$-case, we should check the (non-)triviality of the kernel module

\[(21) \quad \forall \ 0 \neq pt \in \text{Spec}(R) : \quad \{B \in \text{Mat}(n, m; \mathbb{k}) \mid BA|_{pt} = 0, \ A|_{pt}B = 0, \ trace(B\partial A)|_{pt} = 0\} = \{0\}\]

Suppose $\text{rank}(A|_{pt}) = r$, we work locally near $pt$. Again, we can assume $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, where $A_1|_{pt} = I_{r \times r}$ and $A_2, A_3, A_4$ vanish at $pt$. Then $BA|_{pt} = 0$ means $B_1 = 0, B_2 = 0, B_3 = 0$. If $r = m$ then the equations force $B = 0$, thus we assume $r < m$. Finally, $\text{trace}(B\partial A)|_{pt} = trace(B_4\partial A_4)|_{pt} = 0$. Again, if we write down all the entries of $A_4$ as the column and all the entries of $B_4$ as the row, then $\text{trace}(B_4\partial A_4)|_{pt} = 0$ means that the row $B_4$ belongs to the left kernel of the $(m - r)(n - r) \times p$ matrix $\partial A_4$.

This is a linear system of $p$ equations in $(n - r)(m - r)$ variables (the entries of $B_4$). Note that $p \geq (n - r)(m - r) + 1$, otherwise the locus of rank $r$ matrices is just the origin. There is a non-zero solution for $B_4$ iff the rows of the numerical matrix $\partial A_4|_{pt}$ are linearly dependent iff the locus $\{A_4 = 0\} \subset \text{Spec}(R)$ is singular or not of expected dimension.

By direct check: $\text{jet}_1(I_{r+1}(A)) = \text{jet}_1(I_1(A_4))$. Hence, $\partial A_4|_{pt}$ coincides with the Jacobian matrix of the defining ideal of $(\Sigma_r, pt)$. Which means: the locus $\Sigma_r(A)$ is quasi-smooth at $pt \in \text{Spec}(R)$ and of expected dimension iff this happens for the locus $\{A_4 = 0\} \subset \text{Spec}(R)$. Therefore we get: $\text{Ker}^{(i)}(A_{G_r}|_{pt}) = \{0\}$ iff the germ $(\Sigma_r(A), pt)$ is quasi-smooth and of expected dimension. ■

Corollary 3.13. Suppose $R$ is regular, i.e. $\text{Spec}(R)$ is a smooth germ. Suppose the base field, $R/m = \mathbb{k}$, is algebraically closed.

1. If \( \dim(R) \leq 2(n-m+2) \), then $A$ is finitely-$G_r$-determined iff $\Sigma_{m-1}(A)$ is of expected codimension and $\text{Sing}((\Sigma_{m-1}(A)) = \Sigma_{m-2} = \{0\} \subset \text{Spec}(R)$.

2. In particular, if \( \dim(R) = n - m + 2 \), then $A$ is finitely-$G_r$-determined iff $I_{m}(A)$ defines an isolated curve singularity (necessarily reduced).

3. If $A$ is finitely-$G_r$-determined then $I_{1}(A)$ defines a subgerm of $\text{Spec}(R)$ with an isolated singularity.

4. $A \in \text{Mat}(m, n; R)$ is finitely-$G_r$-determined iff for each point $0 \neq pt \in \text{Spec}(R)$ the corresponding matrix $A_4$ (defined in the proof of the theorem) is $\text{Aut}(R)$-stable.

Proof. 1. If \( \dim(R) \leq 2(n-m+2) \) then the expected dimension of $\Sigma_r(A)$ for $r < m - 1$ is zero. Thus among the conditions of theorem 1.2 only the case $r = m - 1$ is relevant.

2. Immediate. We should only add: as $\text{Spec}(R)$ is smooth, and the determinantal ideals $I_j(A)$ are of expected height, their zero loci $V(I_j(A))$ are Cohen-Macaulay.

3. Immediate from the theorem.

4. If the ring is regular then the generators of the module of derivations are just the ordinary partial derivatives, $\text{Der}(R) = R(\partial_1, \ldots, \partial_n)$. So, the linear independence of the vectors $\{\partial_i(A_4)_{i,j}|_{pt}\}_{i=1, \ldots, m-r}$ means the linear independence of the forms $\{\text{jet}_1((A_4)_{i,j})|_{pt}\}_{i=1, \ldots, m-r}$. But then, locally in $(X, pt)$, all the higher order terms of $A_4$ can be killed just by a change of coordinates (i.e. an automorphism of the local ring $\mathcal{O}(X, pt)$). And then, by proposition 3.6, $A_4$ is $\text{Aut}(R)$-stable. ■

3.5. Congruence and (anti)symmetric matrices. For square matrices consider the congruence, $A \sim^{G_{\text{congr}}} U A U^T$, and the corresponding group $G_{\text{congr}} := G_{\text{congr}} \rtimes \text{Aut}(R)$. This group acts on the subspaces of (anti)-symmetric matrices, thus we have three cases: finite determinacy of $A \in \text{Mat}(m, m; m)$, $A \in \text{Mat}^{\text{sym}}(m, m; m)$ and $A \in \text{Mat}^{\text{anti-sym}}(m, m; m)$.

3.5.1. The matrices $A_{\text{congr}}, A_{\text{sym}}^{\text{congr}}, A_{\text{anti-sym}}^{\text{congr}}$ and their left kernels. For $A \in \text{Mat}(m, m; m)$ the tangent spaces $T(A_{\text{congr}} A, A)$ and $T(A_{\text{sym}}^{\text{congr}} A, A)$ were written in example 2.1. The generating matrix can be written as the sum of the
two matrices, in the basis \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m)\):

\[
A_{G_{\text{congr}}} = \begin{pmatrix}
A & 0 & 0 & \cdots \\
0 & A & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & \cdots & \cdots & 0 & A
\end{pmatrix}
\]

\[
\begin{pmatrix}
\tilde{a}_{j1} & 0 & \cdots & 0 \\
\tilde{a}_{j1} & \tilde{a}_{j1} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\tilde{a}_{j1} & \cdots & \cdots & 0 & \tilde{a}_{j1}
\end{pmatrix}
\]

The matrix of \(A_{G_{\text{congr}}} \) is obtained from \(A_{G_{\text{congr}}} \) by attaching the additional column:

\[
\begin{pmatrix}
\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m, \tilde{a}_m, \ldots, \tilde{a}_m, \ldots
\end{pmatrix}
\]

(22)

(23) \( \ker(A_{G_{\text{congr}}}) = \{B \mid BA^T + B^TA = 0\} \).

(24) \( \ker(A_{G_{\text{congr}}}) = \{B \mid BA^T + B^TA = 0, \text{trace}(B\tilde{a}A) = 0\} \).

For \(A \in \text{Mat}^{\text{sym}}(m, m, R)\) the matrix \(A_{G_{\text{congr}}} \) is obtained from the matrix of equation (22) by removing the rows corresponding to \(a_{ij}\) with \(i > j\). Thus we get the matrix of size \(m+1\) \times \(m^2\):

\[
\begin{pmatrix}
\tilde{a}_{11} & 0 & 0 \\
\tilde{a}_{12} & \tilde{a}_{1i} & 0 \\
\vdots & \vdots & \vdots \\
\tilde{a}_{mi} & 0 & 0 \\
2\tilde{a}_{i1} & 0 & \cdots & \cdots \\
\tilde{a}_{i1} & \tilde{a}_{1i} & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{a}_{mi} & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

For \(A \in \text{Mat}^{\text{anti-sym}}(m, m, R)\) the matrix \(A_{G_{\text{congr}}} \) is obtained from the matrix of equation (22) by removing the rows corresponding to \(a_{ij}\) with \(i < j\). Thus we get the matrix of size \(m^2/2\) \times \(m^2\):

\[
\begin{pmatrix}
\tilde{a}_{12} & a_{22} & a_{m2} & a_{11} & a_{12} & a_{1m} & 0 & \cdots & 0 \\
a_{13} & a_{23} & a_{m3} & 0 & \cdots & 0 & a_{11} & a_{1m} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1m} & a_{2m} & a_{mm} & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
\tilde{a}_{12} & a_{22} & a_{m2} & a_{11} & a_{12} & a_{1m} & 0 & \cdots & 0 \\
a_{13} & a_{23} & a_{m3} & 0 & \cdots & 0 & a_{11} & a_{1m} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1m} & a_{2m} & a_{mm} & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots
\end{pmatrix}
\]

The description of the left kernel is obtained directly either from these matrices or from corollary 3.14, assuming \(A, B\) symmetric (or anti-symmetric):
Corollary 3.15. 1. Ker\((l)(A_{G_{congr}}^{\text{sym}}) = \{B \in \text{Mat}^{\text{sym}}(m, m; k) \mid BA = 0\}\).

2. Ker\((l)(A_{G_{congr}}^{\text{anti-sym}}) = \{B \in \text{Mat}^{\text{anti-sym}}(m, m; k) \mid BA = 0\}\).

Proof. 1. The matrix \(A_{G_{congr}}^{\text{sym}}\) is of size \((m+1) \times m^2\), therefore any of its maximal minors corresponds to a choice of \((m+1) \choose 2\) columns. Note that \(A_{G_{congr}}^{\text{sym}}\) splits naturally into \((m+1) \times m\) blocks of columns. Choose some (arbitrary) ordering of these blocks, \(\{A^{(j)}\}_{j=1,...,m}\). One of the ways to choose a set of columns is by the distribution, \((m+1) \choose 2 = m + (m - 1) + \ldots + 1\), taking \(j\) columns from \(j\)th block of columns. To compute the determinant of such a minor, start from the block \(A^{(m)}\), its contribution is \(\text{det}(A^{(m)})\). Note that the blocks \(A^{(m)}\) and \(A^{(m-1)}\) have precisely one common row (which is "taken" when computing \(\text{det}(A^{(m)})\)). Thus the contribution of \(A^{(m-1)}\) is the determinant of the remaining \((m-1) \times (m-1)\) block. Continue in this way up to \(A^{(1)}\), multiplying all the contributions we get an element of \(I_m(A)I_{m-1}(A)\ldots I_1(A)\). By going over all the distributions of columns (for the given ordering \(\{A^{(j)}\}_{j=1,...,m}\)) and then over all the orderings of \(\{A^{(j)}\}_{j=1,...,m}\), we can realize all the elements of \(I_m(A)I_{m-1}(A)\ldots I_1(A)\). Thus \(I_{\text{max}}(A_{G_{congr}}^{\text{sym}}) \supseteq I_m(A)I_{m-1}(A)\ldots I_1(A)\).

Note that \(I_m(A) \supset \text{Max}(A_{G_{congr}}^{\text{sym}})\) is a statement of linear-algebra, it does not depend on the ring or the particular choice of \(A\). Thus we may assume that \(R = k[[A_{ij}]]\), i.e. \(A\) is a matrix of indeterminates that generate the complete regular ring. In particular, we can (and will) assume that the ideal \(I_m(A)\) is radical and that \(I_{m-1}(A) \otimes R/I_m(A) \neq \{0\}\). We check that \(A_{G_{congr}}^{\text{sym}} \otimes \text{R}/I_m(A)\) has a non-trivial left kernel, this will imply \(I_{\text{max}}(A_{G_{congr}}^{\text{sym}}) \subset I_m(A)\). By corollary 3.15 the left kernel in this case consists of symmetric matrices satisfying \(BA = 0 \in \text{Mat}(m, m, R/I_m(A))\). The natural candidate is the adjugate matrix, \(A^V\) is symmetric, \(A^V\) is symmetric too. Further, by the assumption of genericity: \(A^V \otimes R/I_m(A) \neq \{0\}\), as the entries of this matrix span \(I_m(A) \otimes R/I_m(A) \neq \{0\}\). Therefore \(A^V\) provides the non-trivial left kernel for \(A_{G_{congr}}^{\text{sym}} \otimes \text{R}/I_m(A)\). Hence the needed inclusion.

Finally, \(I_m(A) \neq I_{\text{max}}(A_{G_{congr}}^{\text{sym}})\) as all the entries of \(A_{G_{congr}}^{\text{sym}}\) lie in \(m\) and for \(m > 1\) the matrix \(A_{G_{congr}}^{\text{sym}}\) is of bigger size than \(A\).

2. The proof of \(I_{\text{max}}(A_{G_{congr}}^{\text{anti-sym}}) \supseteq I_{m-1}(A)I_{m-2}(A)\ldots I_1(A)\) goes as for part 1.

So, we only need to prove \(I_{m-1}(A) \supset \text{Max}(A_{G_{congr}}^{\text{anti-sym}})\). The idea of the proof is the same as in part (1). But, unlike the symmetric case, we have technical difficulties: for \(m\)-even the ideal \(I_m(A)\) is radical (and \(\sqrt{I_m(A)} \supset I_{m-1}(A)\)), while for \(m\)-odd: \(I_m(A) = \{0\}\). For simplicity we give the "geometric" proof, using the points of \(\text{Spec}(R) \setminus \{0\}\).

It is enough to demonstrate the non-trivial left kernel of \(A_{G_{congr}}^{\text{anti-sym}}\)pt for the points satisfying: \(I_{m-1}(A)_{\text{pt}} = \{0\}\). By corollary 3.15 the left kernel in this case consists of anti-symmetric matrices satisfying \(BA_{\text{pt}} = 0 \in \text{Mat}(m, m, k)\). To simplify the construction we bring \(A_{\text{pt}}\) (by congruence) to its canonical form: the only non-vanishing entries lie in \(2 \times 2\) blocks on the main diagonal and these blocks are \(\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}\). If \(m\) is odd then in this form the last row and column of \(A\) are zero. Thus the left kernel is provided by \(B\) whose only non-zero entry is \(B_{m,m} = 1\). If \(m\) is even, then by the assumption \((I_{m-1}(A)_{\text{pt}} = \{0\})\) at least one such block is zero, say \(\lambda_i = 0\). Thus the natural candidate for \(B\) consists of zeros except for the \(2 \times 2\) block corresponding to the \(\lambda_i\) block of \(A_{\text{pt}}\). In both cases we get the nontrivial left kernel of \(A_{G_{congr}}^{\text{anti-sym}}|\text{pt}\). This proves that \(I_{\text{max}}(A_{G_{congr}}^{\text{anti-sym}}|\text{pt}) = \{0\}\). Or algebraically: \(I_{\text{max}}(A_{G_{congr}}^{\text{sym}}) \otimes R/I_m(A) = \{0\}\). Hence the statement.

3.5.2. Criteria of finite determinacy for \(G_{congr}\). As in the case of ordinary matrices we have the degeneracy loci, \(\Sigma_r(A) \subset \text{Spec}(R)\). Now the expected codimension in the symmetric case is \((m-r+1) \choose 2\).

Proof. of theorem 1.3.

1. One of the possible proofs is to show that the left kernel (of corollary 3.14) is non-trivial. Fix some generic point in \(\text{Spec}(R)\). Note that the matrix \(BA^T + B^T A\) is symmetric, therefore the space \(\{B \mid BA^T + B^T A = 0\} \subset \text{Mat}(m, m; k)\) is of dimension at least \(m^2 - \frac{(m+1)}{2} = \frac{(m-1)}{2}\). Inside this space the condition \(\text{trace}(B\overline{\partial} A) = 0\) is of codimension at most \(\dim(R)\). Hence, if \(\dim(R) < \frac{(m-1)}{2}\) then \(\text{Ker}(l)(A_{G_{congr}}^{\text{sym}}) \neq \{0\}\).

2. and 3. Follow immediately from corollary 3.16 and lemma 2.8.

4. The proof is the same as in the \(G_k\) case. The conditions on the left kernel of \(A_{G_{congr}}^{\text{anti-sym}}\) are taken from the corollary above.

A symmetric matrix \(A\) is finitely \(G_{congr}\)-determined (inside \(\text{Mat}^{\text{sym}}(m, m, R)\)) iff for any \(0 \neq p \in \text{Spec}(\overline{R})\):

\[
\text{Ker}(l)(A_{G_{congr}}^{\text{sym}}|\text{pt}) = \left\{ B \in \text{Mat}^{\text{sym}}(m, m; k) \mid BA|_{\text{pt}} = 0 \in k_{m \times m}, \text{trace}(B\overline{\partial} A|_{\text{pt}}) = 0 \right\} = \{0\}.
\]
Similarly, an anti-symmetric matrix \( A \) is finitely \( \mathcal{G}_{\text{congr}} \)-determined (in \( \text{Mat}^{\text{anti-sym}}(m,m,R) \)) iff
\[
\text{Ker}^l(\mathcal{A}_{\text{anti-sym}}^\text{congr})_{pt} = \left\{ B \in \text{Mat}^{\text{anti-sym}}(m,m;\mathbb{k}) \mid BA\vert_{pt} = \mathbb{O}, \text{trace}(B\tilde{A}\vert_{pt}) = 0 \right\} = \{0\}.
\]

Now, as in the proof of theorem 1.2, we check the points \( pt \in \Sigma_r \setminus \Sigma_{r-1} \). Then \( A \sim \mathbb{1} \oplus \tilde{A}(m-r)_{(m-r) \times (m-r)} \), where \( \tilde{A} \) is a (anti-)symmetric numerical matrix. Further, the conditions \( \text{trace}(B\tilde{A}) = \text{trace}(B\tilde{B}A) = 0 \) are rewritten using the (strictly-)upper triangular parts of \( A \) and \( B \). They mean: the linear forms of the (strictly-)upper triangular parts of \( j\text{e}t_1(\tilde{A}) \) are generic.

**Corollary 3.17.** Suppose the ring \( R \) is regular and the base field, \( R/\mathfrak{m} = \mathbb{k} \), is algebraically closed. \( A \in \text{Mat}^{\text{sym}}(m,m,R) \) is finitely-\( \mathcal{G}_{\text{congr}} \)-determined iff for any point \( 0 \neq pt \in \Sigma_r \setminus \Sigma_{r-1} \) the matrix \( \tilde{A} \) defined in the proof is \( \text{Aut}(R) \)-stable inside the space of symmetric matrices. Similarly for anti-symmetric matrices.

**Remark 3.18.** The sets of (anti)symmetric matrices are the sub-spaces of \( \text{Mat}(m,m;R) \) invariant under \( \mathbb{Z}_2 \) action. More generally, we can consider some \( \mathbb{Z}_k \) actions on \( \text{Mat}(m,m;R) \). Then one has \( \mathbb{Z}_k \)-invariant subspaces and accordingly \( \mathbb{Z}_k \)-invariant \( R \)-submodules of \( \text{Mat}(m,m;R) \). Finite determinacy of matrices under deformations inside these submodules is treated in the same way as for the (anti-)symmetric case.

### 3.6. Upper triangular matrices.

#### 3.6.1. The matrices \( \mathcal{A}_{G^{up}} \), \( \mathcal{A}_{G^{up}}^{\text{left}} \) and the left kernels. Let \( \text{Mat}^{\text{up}}(m,m,R) \subset \text{Mat}(m,m,R) \) be the \( R \)-submodule of upper triangular matrices, \( A_{ij} = 0 \) for \( i > j \). Let \( G_{r}^{\text{up}} \subset G_r \) be the group of transformations \( A \to UAV \) with \( U,V \)-upper triangular and invertible.

Identify \( \text{Mat}^{\text{up}}(m,m,R) \) with \( R^{\binom{m+1}{2}} \) by choosing the basis \( (a_{11},a_{12},a_{22},a_{13},a_{33},\ldots,a_{m},\ldots,a_{mm}) \). In this basis the generating matrix of \( T(G_r^{\text{up}}A,A) \) is obtained from \( \mathcal{A}_{G_r} \) (§2.7) by erasing the columns corresponding to \( E_{ij} \) with \( i > j \) and erasing the rows corresponding to \( a_{ij} \) with \( i > j \). This matrix is of size \( \binom{m+1}{2} \times \binom{m+1}{2} \):

\[
(28) \quad \mathcal{A}_{G_{r}^{\text{up}}} = \begin{pmatrix}
\begin{array}{cccccc}
11 & 0 & 0 & \cdots & 0 & 0 \\
0 & 11 & 12 & \cdots & 0 & 0 \\
0 & 0 & 12 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 11 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & a_{mm}
\end{array}
\end{pmatrix}
\]

To write down the generating matrix of \( T(G_r^{\text{up}}A,A) \) in a convenient form, we identify \( \text{Mat}^{\text{up}}(m,m,R) \) with \( R^{\binom{m+1}{2}} \) by choosing the basis \( (a_{11},\ldots,a_{1m},a_{22},\ldots,a_{2m},\ldots) \). In this basis the generating matrix of \( T(G_r^{\text{up}}A,A) \) is obtained from \( \mathcal{A}_{G_r} \) (§2.7) by erasing the columns/rows corresponding to \( \{E_{ij}\} \) or \( \{a_{ij}\} \) with \( i > j \).

\[
(29) \quad \mathcal{A}_{G_{r}^{\text{up}}} = \begin{pmatrix}
\begin{array}{cccccc}
a_{11} & 0 & 0 & \cdots & 0 & 0 \\
a_{12} & a_{22} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1m} & a_{2m} & \cdots & a_{mm} & \cdots & \cdots \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
a_{22} & 0 & 0 & \cdots & 0 & 0 \\
a_{23} & a_{33} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{2m} & a_{3m} & \cdots & a_{mm} & \cdots & \cdots \\
\end{pmatrix}
\]

The matrix \( \mathcal{A}_{G_{r}^{\text{up}}} \) is obtained by juxtaposition of these matrices (after a series of row permutations on one of them, to bring to the same basis). To obtain the matrix \( \mathcal{A}_{G_{r}^{\text{up}}} \) one adjoins the columns (\( \tilde{\partial}a_{11},\ldots,\tilde{\partial}a_{1m},\tilde{\partial}a_{22},\ldots,\tilde{\partial}a_{2m},\ldots \)), where (\( \tilde{\partial} \)) are all the generators of \( \text{Der}(R) \).

Using these presentations (or in any other way) we get:

**Corollary 3.19.** 1. \( \text{Ker}^l(\mathcal{A}_{G_{r}^{\text{up}}}^{\text{left}}) = \{B \in \text{Mat}^{\text{up}}(m,m,R/J) \mid \text{l.t.}(B^T A) = 0 = \text{l.t.}(AB^T)\} \).
2. \( \text{Ker}^l(\mathcal{A}_{G_{r}^{\text{up}}}^{\text{left}}) = \{B \in \text{Mat}^{\text{up}}(m,m,R/J) \mid \text{l.t.}(B^T A) = 0 = \text{l.t.}(AB^T), \text{trace}(B\tilde{A}) = 0\} \)

(Here \( \text{l.t.} \) is the lower triangular part of the matrix, the condition \( \text{l.t.}(..) = 0 \) means that all the entries on or below the diagonal are zero.)
Corollary 3.20. Let \((R, m)\) be a local Noetherian ring of Krull dimension at least one, \(A \in \text{Mat}^{up}(m, m; m)\), \(m \geq 2\). Then \(I_{max}(A_G^{up}) = \prod_{i=1}^{m}(a_{ii})^{m+1-i}\) and \(I_{max}(A_G^{up}) = \prod_{i=1}^{m}(a_{ii})^{i}\).

This follows immediately from the presentation of \(A_G^{up}\) in equations (29) and (28).

Proof of theorem 1.4.

1. The statements for \(G_l^{up}, G_r^{up}\) follow immediately from corollary 3.20 and lemma 2.8. For \(G_r^{up}\) the direction \(\Leftarrow\) is immediate as \(G_r^{up} \supset G_l^{up}\).

To check the direction \(\Rightarrow\) for \(G_r^{up}\) we first prove that \(\text{dim}(R) = 1\). For this we check (non)triviality of the left kernel from corollary 3.19. Take some point \(0 \neq pt \in \text{Spec}(R)\). If \(|A|_pt\) is of full rank, then the conditions \(l.t.(B^T A) = 0 = l.t.(A B^T)\) force \(B = 0\). Therefore we should check the points in \(\text{Spec}(R)\) at which \(A\) is degenerate. Note that if \(A\) is finitely-determined (with respect to any group) then the height of ideal \(I_r(A)\) is the expected one. In particular, if \(\text{dim}(R) > 1\) then there exists a point in \(\text{Spec}(R) \otimes \mathbb{k} \setminus \{0\}\) at which e.g. \(A_{nm} = 0\). By direct check, at such a point, the matrix 

\[
B = E_{mm} (i.e. the only non-zero entry is \(B_{mm}\)) satisfies: \(l.t.(B^T A) = 0 = l.t.(A B^T)\). Hence, \(\text{Ker}^{(1)}(A_G^{up})|_{pt} \neq \{0\}\), hence \(A\) is not finitely-\(G_l^{up}\)-determined. Therefore \(\text{dim}(R) = 1\).

Now, if \(\text{dim}(R) = 1\), and \(A\) is finitely-\(G_r^{up}\)-determined then \(I_m(A)\) is of expected height, i.e. \(\{\text{det}(A) = 0\}\) defines a one-point-scheme, i.e. \(I_m(A)\) contains a power of the maximal ideal.

2. If \(\text{dim}(R) > 1\) then in \(\text{Spec}(R) \setminus \{0\}\) there are points at which \(A_{mm} = 0 = A_{m-1,m} = A_{m-1,m-1}\). At these points \(\text{Ker}^{(1)}(A_G^{up}) \neq \{0\}\), e.g. the matrix \(B = E_{m-1,m}\) satisfies \(l.t.(B^T A) = 0 = l.t.(A B^T)\) and \(\text{trace}(B \partial A) = 0\).

3. The direction \(\Leftarrow\) follows by inclusion \(G_l^{up} \supset G_r^{up}\). The direction \(\Rightarrow\) follows because finite-determinacy w.r.t any group \(G\) implies that all the fitting ideals are of expected height.

4. \(\Rightarrow\): As the Fitting ideals are of expected height, we get: \(\Sigma_{m-2} \subset \text{Spec}(R \otimes \mathbb{k})\) is the origin, while \(\Sigma_{m-1}\) is a curve.

By an arbitrarily small deformation this curve can be made generically reduced (i.e. smooth at the generic point). Hence the condition that \(\Sigma_{m-1}\) is a curve-germ with isolated singularity.

\(\Leftarrow\): Again, we check the left kernel. As mentioned above, it is enough to check the points where precisely one entry on the diagonal of \(A\) vanishes. Let this entry be \(a_{ii}\). Then \(\Sigma_{m-1}\) is locally defined by \(\{a_{ii} = 0\}\). We claim that the equations \(l.t.(B^T A_{ii}) = 0 = l.t.(A_{ii} B^T)\) forces \(B\) to have zero entries, except possibly for \(B_{ii}\). Indeed, the \(m\)th row of \(l.t.(A_{ii} B^T)\) is \((a_{mm} B_{1m}, \ldots, a_{mm} B_{mm})\). Thus \(B_{j,0} = 0\) for \(j = 1, \ldots, m\). In the same way we get: \(B_{j,k} = 0\) for \(j = 1, \ldots, m\). Similarly, the 1st column of \(l.t.(B^T A)\) is \((B_{11}, \ldots, B_{1m})\). Thus \(B_{1,j} = 0\) for \(j = 1, \ldots, m\). In the same way we get: \(B_{j,k} = 0\) for \(j = 1, \ldots, m\).

Thus the only possibly non-zero entry is \(B_{ii}\).

Now the equations \(\text{trace}(B \partial A) = 0\) mean: \(B_{ii} \partial A_{ii} |_{pt} = 0\). Which precisely means: either \(B_{ii} = 0\) or the locus \(\{a_{ii} = 0\}\) is not quasi-smooth at the point. By the assumption, \(\{a_{ii} = 0\}\) is a curve-germ with isolated singularity. Thus \(B_{ii} = 0\), i.e. \(\text{Ker}^{(1)}(A_G^{up})|_{pt} = \{0\}\). Hence the finite determinacy.

5. The points of \(\text{Spec}(R)\) where at most one diagonal entry vanishes were checked above. This forces the quasi-smoothness of \(\Sigma_{m-1} \setminus \Sigma_{m-2}\). The locus \(\Sigma_{m-3}\) is supported at the origin only. Thus we should check only the points where at least two diagonal entries of \(A\) vanish. Suppose \(A_{ii} = 0 = A_{jj}, i < j\) at some \(0 \neq pt \in \text{Spec}(R)\). (The other entries on the diagonal do not vanish and \(A_{ij} |_{pt} \neq 0\), by dimensional reasons.)

Then \(l.t.(A B^T) = 0 = l.t.(B^T A)\) forces: \(B_{kl} = 0\) for \(k > l \) or \(k < i \) or \(l > j\). So the only possibly non-zero entries of \(B\) form the triangle. Restrict to this triangle submatrix, and the corresponding submatrix of \(A\). Then one gets in addition: \(B_{ij} = 0\) for \(i < k \leq j\) and \(B_{ij} = 0\) for \(k < j\). In this way we get: the only possibly non-zero entries of \(B\) are: \(B_{ii}\) and \(B_{jj}\).

At this step the equations \(l.t.(A B^T) = 0 = l.t.(B^T A)\) are satisfied, while \(\text{trace}(B \partial A) = 0\) gives: \(B_{ii} \partial a_{ii} + B_{jj} \partial a_{jj} = 0\). Which precisely means the quasi-smoothness of the locus \(\{a_{ii} = 0 = a_{jj}\}\).

6. This is now immediate.

3.7. Applications to the singularities of maps. Suppose \(m = 1\), then \(\text{Mat}(1, n, m)\) can be considered as Maps\((\text{Spec}(R)), (k^n, 0)\) . Note that in this case the \(G_l^{k}\) and \(G_r^{k}\) equivalences coincide. Moreover, they coincide with the classical contact equivalence \(K\). Therefore in this case we consider the \(1 \times n\) matrices as maps.

Corollary 3.21. Consider the map \(\text{Spec}(R) \rightarrow (k^n, 0)\), where \(A = (a_1, \ldots, a_n)\).

1. \(A\) is finitely-\(G_r^{k}\)-determined iff \(A^{-1}(0) \subset \text{Spec}(R)\) is a (fat) point. In particular, if \(n < \text{dim}(R)\) then no such map is finitely-\(G_r^{k}\)-determined.

2. If \(\text{dim}(R) \leq n\) then \(A\) is finitely-\(K\)-determined iff it is finitely-\(G_r^{k}\)-determined. For \(\text{dim}(R) \leq n\) the generic matrix is finitely determined.

3. \(A\) is finitely-\(K\)-determined iff the ideal generated by \(\{a_i\}\) and by all the maximal minors of the Jacobian matrix, \(\{\partial_j a_i\}_{j=1, \ldots, p}\) contains \(m^N\), for some \(N\). In particular, the generic matrix is finitely-\(K\)-determined. If \(A\) is finitely-\(K\)-determined and \(p \geq n\) then \(A^{-1}(0)\) is a complete intersection (of codimension \(n\)).
Proof. 1. and 2. This is just the $G_{tr}$-criterion from theorem 1.1.
3. Note that in this case $A \in \text{Mat}(n, p + n^2, R)$ is:

$$
\left( \partial_i a_j \right)_{\text{Jacobian matrix}} = \begin{pmatrix}
\ddot{a} & \dddot{0} & \cdots & \dddot{0} \\
\dddot{0} & \ddot{a} & \dddot{0} & \cdots & \dddot{0} \\
\dddot{0} & \dddot{0} & \ddot{a} & \dddot{0} & \cdots \\
\dddot{0} & \dddot{0} & \dddot{0} & \ddot{a} & \dddot{0} \\
\dddot{a} & \dddot{0} & \dddot{0} & \dddot{0} & \dddot{0}
\end{pmatrix},
$$

here $\ddot{a} = (a_1, \ldots, a_n)$. So, if $p < n$ then $\sqrt{I_{\text{max}}(A)} = \sqrt{I_1(A)}$, while if $p \geq n$ then $\sqrt{I_{\text{max}}(A)} = \sqrt{\langle \text{Jac}, I_1(A) \rangle}$. So, the first assertion is just the statement of lemma 2.8 for the case $\text{Mat}(1, n, R)$.

Suppose $A^{-1}(0)$ is not of expected dimension, then $\{a_i\}$ are algebraically dependent, i.e. $\sum b_i a_i = 0 \in R$, for some $\{b_i\}$ in $R$. Then, by differentiation we get: $\sum_i b_i \partial a_i = 0 \in R/\langle \{a_i\} \rangle$. And not all $b_i$ are zero in $R/\langle \{a_i\} \rangle$, as $\{a_i\}$ is not a regular sequence. Thus the Jacobian matrix $\{\partial_i a_i\}_{i=1,\ldots,n}$ cannot be of the full rank over $R/\langle \{a_i\} \rangle$, i.e. its ideal of maximal minors is included into $I_1(A)$. Thus, the ideal of maximal minors of $A$ cannot contain any power of $m$, as the support of $A^{-1}(0)$ is of positive dimension. □

Remark 3.22. If $R$ is a regular ring, then from statement (3) of the last lemma we get the classical criterion [Wall-1981]: $A$ is finitely-$K$-determined iff $A^{-1}(0)$ is an isolated complete intersection. For non-regular rings the locus $A^{-1}(0)$ can have non-isolated singularity but $A$ can still be finitely determined. For example, let $R = k[[x_1, \ldots, x_n]]/(x_1^2)$, then the module of derivations is generated by $(x_1 \partial_1, \partial_2, \ldots, \partial_n)$. Consider $A \in \text{Mat}(1, 1; m)$ whose only entry is $x_1 + x_n$. Note that $A = (x_1, 0, 0, \ldots, 1, x_1 + x_n)$, thus $I_{\text{max}}(A) = R$, hence finite-$K$-determinacy. (In fact, $A$ is even $\text{Aut}(R)$-finitely determined.) But the zero locus is $\{x_1 + x_n = 0\} \approx \text{Spec}(k[x_1, x_2, \ldots, x_{n-1}]/(x_1^2))$, i.e. is a multiple hyperplane.

Remark 3.23. Using this approach we can check the determinacy of maps with respect to some subgroups of $K$. Consider $\phi \in \text{Maps}(\text{Spec}(R), (k^N, 0))$, suppose $N = mn$, with $m \leq n$. Associate to this map the matrix $A^{(m,n)}_{\phi} \in \text{Mat}(m, n; R)$. The group $G^{(m,n)}_{\text{tr}} := GL(m; R) \times GL(n; R) \circ \text{Mat}(m, n; R)$ is obviously a subgroup of $K \circ \text{Maps}(\text{Spec}(R), (k^N, 0))$, and for $m > 1$ the orbits of $G^{(m,n)}_{\text{tr}}$ are much smaller than those of $K$. Thus, the finite $G^{(m,n)}_{\text{tr}}$-determinacy is much stronger property than the finite $K$-determinacy.

Further, there is the partial ordering among $\{G^{(m,n)}_{\text{tr}}\}_{m,n=1}^{\infty}$: if $m = am'$ and $n = n'/a$, with $m \leq n$ and $a \geq 1$, then $G^{(m,n)}_{\text{tr}} \supset G^{(m',n')}_{\text{tr}}$. And their orbits embed naturally. So, the "smallest" possible subgroup is $G^{(m,n)}_{\text{tr}}$ with the smallest $n$, for the given $N$.

4. Stability

If $A$ is $G$-stable inside some linear subspace $V \subset \text{Mat}(m, n; R)$, then $\text{jet}_1(A)$ is stable inside $\text{jet}_1(V) \subset \text{Mat}(m, n; \mathbb{m}^2)$. The latter is a question of linear algebra, in particular the ring $R$ participates only through the dimension of $\mathbb{m}/\mathbb{m}^2$. Vice-versa: if $R$ is a regular ring, then the stability of $\text{jet}_1(A)$ implies that of $A$ inside $V$. Thus we study only the stability of $\text{jet}_1(A)$.

Recall that for $G_1 \subset G_2$, the $G_1$ stability implies those of $G_2$. Therefore we consider the main cases: $G_{tr}$, $G_r$, $G_t$ and $G_{tr}$. Choose some elements $x_1, \ldots, x_p \in \mathbb{m}$ that generate $\mathbb{m}$ over $R$. Accordingly, we expand $A = \sum_i x_i A^{(i)} + \cdots$, where $\{A^{(i)} \in \text{Mat}(m, n; k)\}$ and the dots mean the higher order terms. So, we have the trilinear form, $\{A^{(i)}\}$ and we study its stability with respect to the subgroups of $GL(m; k) \times GL(n, k) \times GL(p, k)$. In particular, for the fixed triple $(m, n, p)$ the stability implies that in the space of trilinear forms there is a Zariski open dense orbit of the given group. Such prehomogeneous vector spaces are well studied, [Sato-Kimura], for completeness we give the proof in down-to-earth terms.

If one of $m, n, p$ is 1, then we get to the well studied question of the stability of numerical matrices. Further, note that the question is symmetric under the permutation of $(m, n, p)$, thus we consider the particular case: $1 < m \leq n \leq p$. Associate to $A = \sum_i x_i A^{(i)}$, the numerical matrix:

$$
B = (A^{(1)}_{1,1}, A^{(2)}_{2,2}, \ldots, A^{(m,n)}_{m,n}) \in \text{Mat}(p, mn, k)
$$

Theorem 4.1. Let $1 < m \leq n \leq p$, suppose the ring $R$ is regular.
0. If $A$ is $G_{tr}$ stable then $B$ is of maximal rank, $\text{rank}(B) = \min(mn, p)$.
1. If $mn \leq p$ then $\text{jet}_1(A)$ is $G_{tr}$ stable iff it is $\text{Aut}(R)$-stable iff $\text{rank}(B) = mn$.
2. In particular, for $mn \leq p$ the generic $A$ is $\text{Aut}(R)$-stable.
3. If $mn > p$ then no matrix in $\text{Mat}(m, n; \mathbb{m})$ is stable with respect to any of $G_{tr}$, $G_t$, $G_r$. If $mn > m^2 + n^2 + p^2 - 2$ then no matrix in $\text{Mat}(m, n; \mathbb{m})$ is stable with respect to $G_{tr}$.

Proof. Note that $G_{tr}$ acts on $B$ as a subgroup of $GL(p) \times GL(mn)$.
0. Any deformation of $A$ corresponds to a deformation of $B$, in particular, the later matrix can be always deformed to a matrix of maximal rank. And $G_{tr}$ preserves the rank of $B$. 

1. and 2. If $mn \leq p$ then $G_{lr}$ stability means $rank(\mathcal{B}) = mn$. But in that case, by $Aut(R)$ transformations that act by $GL(p)$ on $B$, one can bring $B$ to the canonical form: 

\[
\begin{pmatrix}
1 \\
0 \\
\end{pmatrix}
\] . So, we get $Aut(R)$ stability.

3. The proof goes by comparison of dimensions. For the space of trilinear forms: \(dim((k^m)^* \otimes (k^n)^* \times (k^p)^*) = mnp\). For the groups: \(dim(G_{lr}) = m^2 + n^2, dim(\mathcal{G}_{lr}) = p^2 + m^2 + n^2\). The action of $G_{lr}$ has a one-parameter group of scalings, so the dimensions of its group-orbits are smaller at least by one. Similarly, the action of $\mathcal{G}_{lr}$ has a two-parameter group of scaling.

By direct check: \(mpn - (m^2 + n^2 - 1) = m(n - m) + n(p - 1) - n + 1 > 0\), i.e. there are moduli of $G_{lr}$ equivalence. So in this case there are no $G_{lr}$ stable matrices. Similarly, if \(mpn - (m^2 + n^2 + p^2 - 2) > 0\) then there are moduli of $\mathcal{G}_{lr}$-equivalence.

For the group $G_r$, apply the $GL(p, k) \times GL(n, k)$ transformations to the matrix $B$ to bring it to the form: \((I, \tilde{B})\), where $\tilde{B} \in Mat(p, mn - n, k)$. If $p = n$, then in this form the remaining freedom is the conjugation $\tilde{B} \rightarrow U\tilde{B}U^{-1}$, where $U^{-1}$ on the right belongs to $GL(n, k)$ and acts simultaneously on the blocks of $n$ columns. The matrix $\tilde{B}$ of size $n \times (m - 1)n$ obviously contains moduli under this conjugation (in fact each block of $n$-columns has moduli). If $p > n$, then in the remaining freedom, $\tilde{B} \rightarrow U\tilde{B}U^{-1}$, also $U$ on the left acts on the blocks of $n$-rows. Again, the matrix $\tilde{B}$ contains moduli.

For $G_t$ one considers the matrix $(A_{r,1}, A_{r,2}, \ldots, A_{r,n}) \in Mat(p, mn, k)$ and applies the same reasoning. ■

Remark 4.2. By permuting $m, n, p$ we get similar statements. For example, suppose $1 < m \leq p \leq n$. If $mp \leq n$ then a matrix is $G_{lr}$ stable iff it is $G_r$ stable. And the generic matrix is stable. While, if $mp > n$ then there are no stable matrices with respect to $G_{lr}, G_t, G_r$.

Appendix A. Applications to algebraization problems

Here we apply the finite determinacy criteria to various algebraization problems.

Let $S$ be the localization of an affine ring, i.e. \(S = (k[x_1, \ldots, x_p]/I)(x_1, \ldots, x_p)\). Let $(R, m)$ be a local Noetherian ring that contains $S$ and satisfies: \(\forall N\) the natural map $S/(x_1, \ldots, x_p)^N \rightarrow R/\mathfrak{m}^N$ is an isomorphism. The typical examples are henselizations/completions.

Definition A.1. $A \in Mat(m, n; R)$ is called $(G, S)$ algebraizable if $A$ is $G$ equivalent to a matrix with entries in $S$.

The case $G = G_{lr}, G_{t}$ and $R = \hat{S}$ have been intensively studied (in commutative algebra and algebraic geometry) as the algebraization of modules.

A.1. Algebraization of modules. Consider a (finitely generated) module $M_R$.

$\ast$ $G_{lr}$ : When does there exists a (finitely generated module) $N_S$ satisfying $M_R = R \otimes_S N_S$? Geometrically, given a fixed affine scheme $X$, consider its algebraic/analytic/formal germ at a point, $(X, 0)$. Which modules over $(X, 0)$ come as the stalks of some sheaves over $X$?

$\ast$ $G_{t}$ : When can $S$ be re-mapped, $S \overset{\phi}{\rightarrow} R$, such that $M_R$ comes as the restriction, $M_R = R \otimes_S N_S$? Geometrically, given some analytic/formal germ $(X, 0)$, which modules over $(X, 0)$ can be realized as the stalks of sheaves on some affine scheme, whose germ is $(X, 0)$?

The first version is the weakening of finite-$G_{lr}$-determinacy, the second is the weakening of finite-$G_{t}$-determinacy.

Therefore our results imply, in particular, various extensions and strengthening of the classical algebraization criteria for modules over local rings ([Elkik73]. For many additional results and bibliography cf. [Christensen-Sather-Wagstaff]). For example, an immediate consequence of our results from §3.1 and §3.4:

Corollary A.2. Let $M$ be an $R$-module, with the minimal presentation $R^\oplus n \xrightarrow{\Delta} R^\oplus m \rightarrow M \rightarrow 0$. Suppose $k = \overline{k}$.

1. If the support of $M$ is the origin, then $A$ is $G_{lr}$ equivalent to a matrix of polynomials.

2. If the singular locus of the support of $M$ is the origin, then $A$ is $G_t$ equivalent to a matrix of polynomials.

For example, the second condition is satisfied if the support of $M$ is a curve, possibly non-reduced, but with isolated singularity.

We mention briefly some recent developments.

• [FSWW08]:

  Proposition 3.3: Let $(R, m)$ be a local ring, $dim(R) = 1$, whose completion $\hat{R}$ is a domain. Every finitely generated module over $\hat{R}$ is the extension of a module over $R$.

  Theorem 3.4: Let $S \overset{\phi}{\rightarrow} R$ be a flat local homomorphism, assume $R$ is separable over $S$. Then every finitely generated module over $R$ is a direct summand of an extension of a module from $S$.

  Corollary 3.5: In particular, for henselization, $R \rightarrow R^h$, every finitely generated module over $R^h$ is a direct summand of an extension of a module from $R$.

  Example 3.6 of their paper shows that this property fails for completions.

• Much of research has been done for maximally Cohen-Macaulay modules (mCM). In [Keller-Murfet-Van den Bergh2008]
they prove: if \((R, \mathfrak{m})\) is a Gorenstein local ring, whose completion, \((\hat{R}, \mathfrak{m})\), has isolated singularities, then every mCM module over \(\hat{R}\) is a direct summand in the completion of a module over \(R\).

We remark that nothing of that type can hold for finite-\(G_t\)-determinacy, in view of our theorem 1.1 (for the direct sum the maximal Fitting ideal of the matrix is the product, thus the direct sum of modules is finitely-\(G_t\)-determined iff each of them is).

A.2. Algebraization relative to an ideal. For many cases of \((G, R)\) there are no finitely determined matrices in \(\text{Mat}(m, n; R)\), e.g. for \(G = G_t\) and \(\dim(R) > 1\). It is natural to consider the following weakening of the algebraization, cf. [Kucharz].

Let \(R\) be a local ring over a field, with some fixed choice of generators \(\{x_i\}\). Let \(J \subset R\) be a (proper) ideal. Suppose a subset of generators is fixed: \(x_1, \ldots, x_k\). Denote by \(S(J, x_1, \ldots, x_k)\) the affine ring generated by 1, \(J, x_1, \ldots, x_k\). The elements of \(S(J, x_1, \ldots, x_k)\) are polynomials in \(x_1, \ldots, x_k\) with coefficients in \(k[J]\).

Definition A.3. A matrix \(A \in \text{Mat}(m, n; R)\) is called \((G, S, J)\)-algebraizable (or algebraizable relative to \(J\)) if \(A\) is \(G\)-equivalent to a matrix with entries \(S(J, x_1, \ldots, x_k)\).

Example A.4. Let \(S = k[x_1, \ldots, x_p]_{(x_1, \ldots, x_p)}\), \(R = \hat{S} = k[[x_1, \ldots, x_p]]\) and \(m = 1 = n\). Then \(G_t, G_r, G_{tr}\) act by multiplication by invertible elements of \(R\). Take \(J = (x_2, \ldots, x_p)\) then the classical Weierstrass preparation theorem reads:

\[
\text{Any } f \in R \text{ is equivalent to a polynomial in } x_1, \text{ with coefficients in } k[J].
\]

In our terms: any \(f \in (R, J, S)\)-algebraizable.

The results of §3 give immediate Weierstrass-preparation-theorem for matrices:

Theorem A.5. 1. Let \(A \in \text{Mat}(m, n; R)\), let \(J \subset R\) be of height at least \(\dim R - (n - m + 1)\). Suppose the ideals \(J, I_{m}(A)\) are relatively prime. Then \(A\) is \((G_r, J)\)-algebraizable.

2. In the assumptions above, if \(A\) is symmetric then it is \((G_{onr}, J)\)-algebraizable (i.e. staying symmetric). If \(A\) is upper triangular then it is \((G_{rup}, J)\) algebraizable (i.e. staying upper-triangular).

Example A.6. Let \(R = k[x_1, \ldots, x_p]\). Every matrix \(A \in \text{Mat}(m, n; R)\) is \(G_r\) equivalent to a matrix of polynomials in \(n - m + 1\) variables, the coefficients of these polynomials being the formal power series in the remaining \(p - (n - m + 1)\) variables.

References

[AGLV-1] V.I.Arnol’d, V.V.Goryunov, O.V.Lyashko, V.A.Vasili’ev, Singularity theory. I. Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences | Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993. Springer-Verlag, Berlin, 1998. iv+245 pp. ISBN: 3-540-63711-7

[AGLV-2] V.I.Arnol’d, V.V.Vasili’ev, V.V.Goryunov, O.V.Lyashko, Singularities. II. Classification and applications. (Russian) With the collaboration of B. Z. Shapiro, I.ogi Nauki i Tekhniki, Current problems in mathematics. Fundamental directions, Vol. 39, 5–256, Akad. Nauk SSSR, 1989.

[Artin1968] M.Artin, On the solutions of analytic equations. Invent. Math. 5 1968 277–291

[Artin1969] M.Artin, Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. No. 36 1969 23–58

[Belitski-Kerner] Genrich Belitskii, Dmitry Kerner, Artin-type approximation theorem for smooth functions, Simple symmetric matrix singularities and the subgroups of Weyl groups \(W_{\mathfrak{s}}\), Algebraization relative to an ideal. Mosc. Math. J. 3 (2003), no. 2, 335–360, 741–744

[Belitski-Kerner] Genrich Belitskii, Dmitry Kerner, Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. No. 36 1969 23–58

[Birkhoff-1913] G.Birkhoff, Sur la classification des fibrés holomorphes sur la sphère de Riemann. Arch. Math. 1, (1949). 282–287

[Birkhoff] G.Birkhoff, A theorem on matrices of analytic functions. Math. Ann. 79 (1913), no. 3, 461

[Christensen-Sather-Wagstaff] L.W.Christensen, S.Sather-Wagstaff, Ascent of module structures, vanishing of Ext, and extended modules. J. Algebra 322 (2009), no. 9, 3026–3046

[Dedekind] R.Dedekind, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785 pp.

[EGA-IV-4] P.Ellkig, R.Ellkig, Solutions d’équations à coefficients dans un anneau hensélien. Ann. Sci. École Norm. Sup. (4) 6 (1973), 553–603 (1974)

[Eisenbud2003] D.Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785 pp.

[Gantmacher-book] F.R.Gantmacher, Sur une généralisation du groupe orthogonal à quatre variables. Arch. Math. 1, (1949), 282–287

[Goryunov-Zakalyukin03] V.V.Goryunov, V.M.Zakalyukin, Simple symmetric matrix singularities and the subgroups of Weyl groups \(W_{\mathfrak{s}}\), Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 507–530, 743–744

[Grothendieck-1957] A.Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann. Amer. J. Math. 79 (1957), 121–138

[Keller-Murfet-Van den Bergh2008] B.Keller, D.Murfet, M.Van den Bergh On two examples by Iyama and Yoshino, Compos. Math. 147 (2011), no. 2, 591–612.

[Kucharz] W.Kucharz, Power series and smooth functions equivalent to a polynomial. Proc. Amer. Math. Soc. 98 (1986), no. 3, 527533.
[Sato-Kimura] M.Sato, T.Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*. Nagoya Math. J. 65 (1977), 1–155.

[Leuschke-Wiegand] G.J.Leuschke, R.Wiegand, *Cohen-Macaulay representations*. Mathematical Surveys and Monographs, 181. American Mathematical Society, Providence, RI, 2012. xviii+367 pp.

[Looijenga] E. Looijenga, *Isolated Singular Points on Complete Intersections*. London Math. Soc. LNS 77, CUP, 1984.

[Mather1970] J.N.Mather, *Stability of $C^\infty$ mappings. VI: The nice dimensions*. Proceedings of Liverpool Singularities-Symposium, I (1969/70), pp. 207–253. Lecture Notes in Math., Vol. 192, Springer, Berlin, 1971.

[Mather1973] J.N.Mather, *Generic projections*. Ann. of Math. (2) 98 (1973), 226–245.

[Molnár2007] L.Molnár, *Selected preserver problems on algebraic structures of linear operators and on function spaces*. Lecture Notes in Mathematics, 1895. Springer-Verlag, Berlin, 2007.

[du Plessis-Wall] A.du Plessis, C.T.C.Wall, *The geometry of topological stability*. London Mathematical Society Monographs. New Series, 9. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[Rudin] W.Rudin, *Real and complex analysis*. Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp.

[Tougeron1968] J.C.Tougeron, *Idéaux de fonctions différentiables. I*. Ann. Inst. Fourier (Grenoble) 18 1968 fasc. 1, 177–240.

[Yoshino] Y.Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*. London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990. viii+177 pp.

[Wall-1981] C.T.C.Wall, *Finite determinacy of smooth map-germs*. Bull. London Math. Soc. 13 (1981), no. 6, 481–539.

Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be’er Sheva 84105, Israel.

E-mail address: genrich@math.bgu.ac.il

E-mail address: dmitry.kerner@gmail.com