IRREDUCIBILITY OF SEVERI VARIETIES ON K3 SURFACES

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Abstract. Let \((Y, L)\) be a general primitively polarized K3 surface of genus \(g\). For every \(0 \leq \delta \leq g\) we consider the Severi variety parametrizing integral curves in \(|L|\) with exactly \(\delta\) nodes as singularities. We prove that its closure in \(|L|\) is connected as soon as \(\delta \leq g - 1\). If \(\delta \leq g - 4\), we obtain the stronger result that the Severi variety is irreducible, as predicted by a well-known conjecture. The results are obtained by degeneration to Halphen surfaces.

1. Introduction.

Let \(L\) be a polarization on a smooth irreducible projective surface \(S\) defined over the field of complex numbers, and denote by \(g\) the arithmetic genus of all curves in \(|L|\). For any fixed integer \(0 \leq \delta \leq g\) the Severi variety of \(\delta\)-nodal curves in \(|L|\) is the locally closed subscheme of \(|L|\) defined as

\[ V_{\delta}(S, L) := \{C \in |L| \text{ s.t. } C \text{ is integral with exactly } \delta \text{ nodes as singularities} \}; \]

the same definition applies to singular surfaces \(S\) with the further requirement that the curves \(C\) lie in the smooth locus of \(S\). These varieties are named after Severi, who introduced them in the case \(S = \mathbb{P}^2\) [Se], where he proved that they are nonempty and smooth of the expected dimension, namely, \(\dim |L| - \delta\). Severi also claimed that they are irreducible, but a rigorous proof of this fact was accomplished only some sixty years later by Harris [Ha]. Since then, Severi varieties were thoroughly investigated for many types of surfaces, in particular as regards their nonemptiness, their local geometry and their irreducibility; the last issue became known as the Severi problem. Nonemptiness has been established in many cases, as for instance K3 surfaces [MM, Ch1], abelian surfaces [KLM, KL], Enriques surfaces [CDGK]. As concerns their local geometry, Severi varieties behave well on rational surfaces and surfaces of Kodaira dimension 0, while on surfaces of general type wild unexpected phenomena occur, as highlighted in [CS, CC].

On the other hand, very little is known about the global geometry of Severi varieties even for surfaces of non-maximal Kodaira dimension. In particular, the Severi problem proves very challenging and has been solved in very few cases: for Hirzebruch surfaces by Tyomkin [Tyo], for Del Pezzo surfaces in the case of rational curves (that is, for maximal \(\delta\)) by Testa [Tes], while
partial results for blow-ups of the projective plane are due to Greuel-Lossen-Shustin [GLS]. In recent times, many papers focused on the case of toric surfaces [Bo, LT], and Zahariuc [Za] worked out the Severi problem for a general abelian surface with a polarization of any primitive type.

A vast literature is devoted to the case of $K_3$ surfaces, motivated by the following well-known folklore conjecture.

**Conjecture 1.1.** Let $(Y, L)$ be a general polarized $K_3$ surface of genus $g \geq 2$. Then, for any fixed $0 \leq \delta \leq g - 1$, the Severi variety $V_{\delta}(Y, L)$ is irreducible.

We recall that $\dim |L| = g$. The constraint $\delta \leq g - 1$ is necessary because it is well-known that the linear system $|L|$ contains finitely many rational curves: since this number is computed by the Yau-Zaslow formula [Be] and is different from 1, the Severi variety $V_{\delta}(Y, L)$ is definitely reducible for $\delta = g$.

Quite surprisingly, the above conjecture has remained open until now, despite numerous attempts. We will prove it for primitive linear systems as soon as $\delta \leq g - 4$ and $g \geq 5$.

**Theorem 1.2.** Let $(Y, L)$ be a general primitively polarized $K_3$ surface of genus $g \geq 2$. Then the following hold:

1. for every $0 \leq \delta \leq g - 1$, the closure of the Severi variety $V_{\delta}(Y, L) \subset |L|$ is connected;
2. if $g \geq 5$ and $0 \leq \delta \leq g - 4$, the Severi variety $V_{\delta}(Y, L)$ is irreducible.

Previous results in the literature due to Keilen [Kei], Kemeny [Kem], Ciliberto-Dedieu [CD2], Dedieu [De2] only concerned cases where $\delta$ is small with respect to the arithmetic genus $g$ (roughly bounded by $g/4$) and it was clear that they cannot be further improved with similar proof techniques. A weaker form of the conjecture concerning the so-called universal Severi variety $V_{g, \delta}$ was considered more approachable. Let $\mathcal{F}_g$ be the irreducible 19-dimensional moduli stack of genus $g$ primitively polarized $K_3$ surfaces. The stack $\mathcal{V}_{g, \delta}$ is smooth of pure dimension $19 + g - \delta$ and admits a morphism $\phi_{g, \delta} : V_{g, \delta} \to \mathcal{F}_g$ to a suitable open substack $\mathcal{F}_g^o$ of $\mathcal{F}_g$ whose fiber over a general point $(Y, L) \in \mathcal{F}_g$ equals the Severi variety $V_{\delta}(Y, L)$.

**Conjecture 1.3.** For every $0 \leq \delta \leq g$, the universal Severi variety $V_{g, \delta}$ is irreducible.

This prediction makes perfect sense even for $\delta = g$, when it becomes a question on the monodromy of the finite morphism $\phi_{g,g}$. It is related to the non-existence of self-rational maps of degree $> 1$ on a general $K_3$ surface in $\mathcal{F}_g$, which was predicted by Dedieu in [De1] and achieved by Chen in [Ch4]. Conjecture 1.3 was proved by Ciliberto-Dedieu [CD] for $2 \leq g \leq 11$ and $g \neq 10$, which is exactly the range where a general genus $g$ curve lies on a $K_3$ surface. We remark that, since the morphism $\phi_{g, \delta}$ is known to be smooth and dominant on all components of $\mathcal{V}_{g, \delta}$ for every $\delta$ [FKPS], Conjecture 1.1 implies Conjecture 1.3 for every $0 \leq \delta \leq g - 1$. In particular, the following result comes straightforward from Theorem 1.2.
Corollary 1.4. For every \( g \geq 5 \) and every \( 0 \leq \delta \leq g - 4 \) the universal Severi variety \( V_{g, \delta} \) is irreducible.

The assumption \( \delta \leq g - 4 \) in Theorem 1.2 and in Corollary 1.4 is due to proof technique, and is only used in the proof of Theorem 5.2. However, there is no evidence for the existence of counterexamples to Conjecture 1.1 in the remaining cases \( g - 3 \leq \delta \leq g - 1 \).

1.1. Strategy and organization of the paper. Theorem 1.2 is proved by degeneration to a so-called Halphen surface \( S_g \subset \mathbb{P}^g \), which has an elliptic singularity and is limit of primitively embedded K3 surfaces of genus \( g \). These surfaces, introduced in \([CD]\), appeared in the characterization of hyperplane sections of K3 surfaces accomplished by Arbarello-Bruno-Sernesi \([ABS]\), and were first exploited in \([ABFS]\) and then in \([AB, FT]\). We recall their construction. Let \( S \) be the blow-up of \( \mathbb{P}^g \) at 9 general points and denote by \( |L_g| \) the \( g \)-dimensional linear system on \( S \) parametrizing the strict transforms of plane curves of degree \( 3g \) having multiplicity \( g \) at the first 8 points that we have blown up and multiplicity \( g - 1 \) at the last one; these are called Du Val curves of genus \( g \) after Du Val, who first considered them \([Du]\).

The surface \( S_g \) is realized as the closure in \( \mathbb{P}^g \) of the rational map \( S \dashrightarrow \mathbb{P}^g \) defined by \( |L_g| \). In particular, Severi varieties of nodal hyperplane curves on \( S_g \) are linked to Severi varieties \( V_{g}(S, L_g) \) on \( S \). A major advantage is that the surface \( S \) possesses polarizations \( L_g \) for every genus \( g \geq 2 \) and is thus the right environment where to perform some sort of induction. In Section 2 after recalling the main features of Halphen surfaces, we show that the results known for Severi varieties on a general K3 surface of genus \( g \) still hold true for \( V_{g}(S, L_g) \). In particular, Chen’s proof of the density of Severi varieties in any equigeneric locus on a general polarized K3 surface \([Ch2, Ch3]\) works with essentially no change in the context of Halphen surfaces. Moreover, the irreducibility of \( V_{g}(S, L_g) \) is easily obtained when \( \delta \) is small with respect to \( g \); this is the basis for our induction.

In Section 3 we show that for any fixed integer \( k \geq 1 \) the linear system \(|L_g|\) sits as a linear space of codimension \( k \) inside of \(|L_{g+k}| \). However, the subspaces \(|L_{g+k-j}| \subset |L_{g+k}| \) for \( j \geq 2 \) have excess intersection with the Severi varieties \( V_{g}(S, L_{g+k}) \), as follows from the following equality:

\[
|L_{g+k-1}| \cap \overline{V_{\delta}(S, L_{g+k})} = \bigcup_{h=0}^{\delta} V_{g-h}(S, L_{g+k-h-1}).
\]

To circumvent this problem, we perform a sequence of blow-ups with smooth centers \( |L_{g+k}| \rightarrow |L_{g+k}|. \) Denoting by \( \overline{V_{g}(S, L_{g+k})} \) the strict transform of \( V_{g}(S, L_{g+k}) \), we prove in Proposition 3.3 that \( V_{g}(S, L_g) \) is isomorphic to the intersection of \( V_{g}(S, L_{g+k}) \) with the strict transform of \( |L_{g+k-1}| \) and the exceptional divisors. We then construct a surjective map

\[\psi : V_{g}(S, L_{g+k}) \rightarrow \mathbb{P}^k\]
where $\mathbb{P}^k$ is obtained from $\mathbb{P}^k$ again by a sequence of blow-ups. Lemma 3.4 and Theorem 3.5 prove that $V_\delta(S, L_g)$ is isomorphic a fiber of $\psi$ and that $\psi$ admits a section. By a standard argument using Stein factorization and Zariski’s Main Theorem, we conclude that $V_\delta(S, L_g)$ is connected and thus the same holds true for $\overline{V_\delta(S, L_g)}$.

In order to deduce connectedness for a general $K3$ surface, we consider a stable type II degeneration $Y_0 := S \cup J S'$ constructed by appropriately gluing two surfaces $S, S'$, which are both a blow-up of $\mathbb{P}^2$ at 9 general points as above, have isomorphic anticanonical divisor $J$ and satisfy $N_{J/S} \simeq N_{J/S'}^\vee$. However the limit on $Y_0$ of a relative genus $g$ polarization on a family of $K3$ surfaces degenerating to it is not unique (for instance, one of such limits contracts $S'$ and maps $S$ to $\overline{S}_g \subset \mathbb{P}^g$). "Good limits" on $Y_0$ of moduli spaces of stable maps and Severi varieties on general fibers of the family are obtained by applying the theory of expanded degenerations and moduli stacks of stable maps $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)$ to such expansions introduced by Jun Li in [Li1, Li2]. The theory of good degenerations of relative Hilbert schemes developed in [LM] is used to define an expanded linear system $|L_g|^{\exp}$. We recall that an expanded degeneration of $Y_0$ is a semistable model

$$S \cup J R \cup J \ldots \cup J R \cup J S'$$

obtained from $Y_0$ inserting a chain of ruled surfaces $R := \mathbb{P}(O_J \oplus N_{J/S})$ over $J$. Points of $|L_g|^{\exp}$ parametrize curves that live in some expansion of $Y_0$ and have no components in its singular locus. Analogously, the moduli stack $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)$ parametrizes stable maps to some expansion of $Y_0$ mapping no component of the domain curve to the singular locus of the target expansion. We let $\overline{V_\delta(Y_0^{\exp}, L)}$ be the image in $|L_g|^{\exp}$ of the semi-normalization of the substack of $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)$ parametrizing smoothable maps. We exploit Li’s decomposition (also used in [MPT]) of $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)$ as a non-disjoint union

$$\mathcal{M}_{g-\delta}(Y_0^{\exp}, L) = \bigcup_{g_1 + g_2 = g} \mathcal{M}_{h_1}(S/J, L_{g_1}) \times \mathcal{M}_{h_2}(S'/J, L'_{g_2}),$$

where $\mathcal{M}_{h_1}(S/J, L_{g_1})$ stands for the moduli stack of stable relative maps to expanded degenerations $S \cup J R \cup J \ldots \cup J R$ of $(S, J)$ with multiplicity 1 along the relative divisor $J$. A similar decomposition

$$\overline{V_\delta(Y_0^{\exp}, L)} = \bigcup_{g_1 + g_2 = g} \overline{V_{h_1}(S/J, L_{g_1}) \times V_{h_2}(S'/J, L'_{g_2})}$$

is obtained in Lemma 4.1. To prove that $\overline{V_\delta(Y_0^{\exp}, L)}$ is connected, in Proposition 4.2 one reduces to showing connectedness of the stack $V_\delta(S/J, L_g)$ as soon as $\delta \leq g - 1$. This is done by showing that the map $\psi$ mentioned above, along with its section, lift to the the stack $\mathcal{M}_{g+k-\delta}(S/J, L_{g+k})$ (cf. Lemma 3.4 and Theorem 3.5). The advantage of using stable maps is that
one also obtains connectedness of the relative normalization $\overline{V_\delta(S/J, L_g)}^0$ of $\overline{V_\delta(S/J, L_g)}$ along $\overline{V_{\delta+1}(S/J, L_g)}$.

Part (1) of Theorem 1.2 is the content of Theorem 4.3; a key point is that any two components of the degeneration $\overline{V_\delta(Y^{\exp}_0, L)}$ can be connected through a sequence of components whose pairwise intersection is generically reduced.

Section 5 is then devoted to the proof of part (2). First of all, we show that, if $(S, L) \in F_g$ is general and $0 \leq \delta \leq g-1$, two irreducible components of the Severi variety $\overline{V_\delta(S, L)}$ intersect in codimension 1, if they meet at all (cf. Propositions 5.1, 2.7 and Lemma 2.8). This is obtained by realizing $\overline{V_\delta(S, L)} \subset |L|$ as the image under a generically finite map of a degeneracy locus in $S^{[\delta]} \times |L|$ and using the fact that degeneracy loci of the expected dimension are Cohen-Macaulay. Knowing that $\overline{V_\delta(S, L)}$ is connected, in order to prove its irreducibility it is thus enough to show that the codimension 1 components of its singular locus cannot contain the intersection of two irreducible components. This holds true for $\overline{V_{\delta+1}(S, L)} \subset \text{Sing}\overline{V_\delta(S, L)}$ by the connectedness of the relative normalization of $\overline{V_\delta(S, L)}$ along $\overline{V_{\delta+1}(S, L)}$.

Let $W$ be any codimension 1 component of $\text{Sing}\overline{V_\delta(S, L)}$ not contained in $\overline{V_{\delta+1}(S, L)}$. By deformation theory (cf. [CD2] for similar arguments), we show that a general point of $W$ parametrizes either a curve whose singularities consist of (possibly non-transverse) smooth linear branches except at most for one cusp, or a curve whose normalization is hyperelliptic of genus $g - \delta$. In the former case, it turns out that $\overline{V_\delta(S, L)}$ is unibranched along $W$. The latter case can be excluded as soon as $W$ has dimension $\geq 3$, or equivalently, $\delta \leq g - 4$, because curves in $|L|$ with hyperelliptic normalization of any fixed geometric genus $\geq 2$ are known to move in dimension 2 (cf. [KLM], Rmk. 5.6); this is the only part of the proof where the assumption $\delta \leq g - 4$ is used.

1.2. Preliminaries on Severi varieties on K3 surfaces. We will here collect known properties of Severi varieties on K3 surfaces that are relevant for this paper and will be generalized to Halphen surfaces in Section 2. Standard deformation theory yields the following result (cf., e.g., [DS, §3–4]):

**Proposition 1.5.** Let $(Y, L)$ be a polarized K3 surface of genus $g$. For any fixed integer $0 \leq \delta \leq g$ the Severi variety $V_\delta(Y, L)$, if nonempty, is smooth of dimension $g - \delta$.

Indeed, for any $C \in V_\delta(Y, L)$ the projective tangent space to $V_\delta(Y, L)$ at $C$ coincides with $\mathbb{P}(H^0(Y, L \otimes I_N))$, where $N$ is the scheme of nodes of $C$. Furthermore, the nodes of any such curve $C$ can be smoothed independently. Therefore, the nonemptiness of $V_\delta(Y, L)$ for every $\delta$ reduces to the existence in the linear system $|L|$ of a nodal rational curve. This was achieved by Mori-Mukai for a general primitively polarized K3 surface, and was then generalized by Chen to non primitive polarizations.
Theorem 1.6 ([MM, Ch1]). Let \((Y, L)\) be a general K3 surface of genus \(g\).
For any fixed integer \(0 \leq \delta \leq g\), the Severi variety \(V_{\delta}(Y, L)\) is nonempty.

For primitive polarizations, Chen obtained the following much stronger result:

Theorem 1.7 ([Ch2]). Let \((Y, L)\) be a general primitively polarized K3 surface of genus \(g\). Then, all rational curves in the linear system \(|L|\) are nodal.

The above result is deeply linked to the natural question whether every curve in \(|L|\) can be deformed to a nodal curve having the same geometric genus. A positive answer is again due to Chen and, defining the equigeneric locus

\[ V^h(Y, L) := \{ C \in |L| \text{ s.t. } C \text{ is integral of geometric genus } h \} \]

for every \(0 \leq h \leq g\), it can be phrased in the following way.

Theorem 1.8 ([Ch3]). Let \((Y, L)\) be a general primitively polarized K3 surface of genus \(g\). Then, for every \(0 \leq \delta \leq g\), the Severi variety \(V_{\delta}(Y, L)\) and the equigeneric locus \(V^{g-\delta}(Y, L)\) have the same closure in \(|L|\).

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2. Halphen surfaces and their Severi varieties

Let \(S\) be the blow-up of \(\mathbb{P}^2\) at nine general points \(p_1, \ldots, p_9\) and denote by \(E_1, \ldots, E_9\) the exceptional curves of this blow-up. As the points \(p_i\) are general, there exists a unique plane cubic passing through the \(p_i\)'s, whose strict transform on \(S\) we denote by \(J\). Hence, \(J\) is the only anticanonical divisor on \(S\) and satisfies

\[ J \sim -K_S \sim 3\ell - E_1 - \cdots - E_9, \]

where \(\ell\) is the strict transform of a line in \(\mathbb{P}^2\). For any fixed \(g \geq 1\), let \(C\) be the strict transform on \(S\) of a so-called Du Val curve of genus \(g\), that is, a plane curve of degree \(3g\) having points of multiplicity \(g\) at \(p_1, \ldots, p_8\) and a point of multiplicity \(g - 1\) at \(p_9\):

\[ C \sim 3g\ell - gE_1 - \cdots - gE_8 - (g - 1)E_9. \]

Defining \(L_g := O_S(C) \in \text{Pic}(S)\), the linear system \(|L_g|\) has dimension \(g\) and its general element is a smooth irreducible curve of genus \(g\). Since \(C \cdot J = 1\), every irreducible curve \(C \in |L_g|\) intersects \(J\) at the same point, that is
denote by $p_{10}(g)$. It turns out that $p_{10}(g)$ is the only base point of $|L_g|$ (cf. \cite{ABFS} Lem. 2.4) and is uniquely determined by the condition

$$g p_1 + \ldots + g p_8 + (g - 1) p_9 + p_{10}(g) \in |O_J(3g\ell)|.$$  

We will sometimes use the notation $L_0 := E_9$ and $p_{10}(0) = p_9$.

Let $\sigma : \tilde{S} \rightarrow S$ be the blow-up of $S$ at $p_{10}(g)$. We still denote by $E_1, \ldots, E_9$ the inverse images under $\sigma$ of the exceptional curves on $S$ and by $E_{10}$ the exceptional divisor of $\sigma$. Let $\tilde{J}$ be the strict transform of $J$ and $\tilde{C}$ be the strict transform of a curve $C \in |L_g|$. The following relations hold on $\tilde{S}$:

$$-K_{\tilde{S}} \sim \tilde{J} \sim 3\ell - E_1 - \cdots - E_{10},$$

(2.1)  

$$\tilde{C} \sim 3g\ell - gE_1 - \cdots - gE_8 - (g - 1)E_9 - E_{10},$$

$$\tilde{C} \cdot \tilde{J} = 0.$$  

The line bundle $\tilde{L}_g := \mathcal{O}_{\tilde{S}}(\tilde{C})$ is base-point-free \cite{ABFS} Lem. 2.4 and thus defines a morphism from $\tilde{S}$ to a surface $\overline{S}_g \subset \mathbb{P}^g$ having trivial dualizing sheaf, canonical hyperplane sections and a single elliptic singularity $o$ resulting from the contraction of $\tilde{J}$. As in \cite{AB}, we call such a surface $\overline{S}_g \subset \mathbb{P}^g$ a polarized Halphen surface of genus $g$. A general hyperplane section of $\overline{S}_g$ is a smooth irreducible curve of genus \cite{ABFS} Lem. 2.4, while a general hyperplane section of $\overline{S}_g$ passing through $o$ has a cusp at $o$. The following result is due to Arbarello-Bruno-Sernesi:

**Proposition 2.1** (\cite{ABS}, Cor. 10.5). If the points $p_1, \ldots, p_9$ are general, the surface $\overline{S}_g$ is the limit of smooth K3 surfaces in $\mathbb{P}^g$.

Halphen surfaces $\overline{S}_g$ as above share some common behaviour with K3 surfaces of Picard rank 1. This depends on the following property, firstly exploited by Arbarello-Bruno-Farkas-Saccà \cite{ABFS}.

**Lemma 2.2.** If the points $p_1, \ldots, p_9$ are general, for any fixed integer $g \geq 1$ the only possible decompositions of $L_g$ into two effective line bundles are of the form

$$L_g \simeq \mathcal{O}_{S}(kJ) \otimes L_{g-k}$$

for some $0 \leq k \leq g - 1$.

**Proof.** By choosing $p_1, \ldots, p_9$ general, we may assume that $S$ contains no $(-2)$-curves and that $h^0(S, \mathcal{O}_S(kJ)) = 1$ for every $k \geq 1$; in other words, $p_1, \ldots, p_9$ are chosen $k$-general in the sense of \cite{AB} Def. 2.2] for every $k \geq 1$ (cf. also \cite{CD}). Let $L_g \simeq N \otimes M$ be a decomposition into two effective line bundles $N, M \in \text{Pic}(S)$. Since $c_1(L_g) \cdot J = 1$ and $(J)^2 = 0$, possibly by exchanging $N$ and $M$ we obtain $J \cdot c_1(N) = 0$ and $J \cdot c_1(M) = 1$. The statement thus follows by a theorem of Nagata \cite{ABFS} Prop. 2.3], ensuring that under the genericity assumption the only effective divisors having vanishing intersection with $J$ are the nonnegative multiples of $J$. \hfill \Box
The above result was used by Arbarello-Bruno-Farkas-Saccà in order to prove the following analogue of Lazarsfeld’s Theorem.

**Theorem 2.3** ([ABFS], Thm. 4.4). *If the points \( p_1, \ldots, p_9 \) are general, then a general curve \( C \in |L_g| \) satisfies Petri’s Theorem and all irreducible nodal curves in \( |L_g| \) satisfy the Brill-Noether Theorem.*

We now investigate Severi varieties \( V_\delta(S, L_g) \) and equigeneric loci \( V^h(S, L_g) \) on \( S \). We recall that the normalization \( \tilde{V}^h(S, L_g) \) of \( V^h(S, L_g) \) admits a universal family \( C \rightarrow \tilde{V}^h(S, L_g) \) together with a simultaneous resolution of singularities \( \tilde{C} \rightarrow C \) (cf. [Tei] I, Thm. 1.3.2] and also [DS, Thm. 1.5]). This implies the existence of an étale cover \( W \rightarrow \tilde{V}^h(S, L_g) \) along with a generically injective morphism \( w : W \rightarrow M_h(S, L_g) \) to the coarse moduli space of genus \( h \) stable maps in \( |L_g| \). The image of \( w \) consists of the irreducible components of \( M_h(S, L_g) \) parametrizing stable maps which are smoothable, that is, can be deformed to a map from a nonsingular curve, birational to its image (cf. [Va]). We denote by \( M_h(S, L_g)^{\text{sm}} \) the closure in \( M_h(S, L_g) \) of the image of \( w \).

Vice versa, by [Kol] I.6 the semi-normalization \( \tilde{M}_h(S, L_g)^{\text{sm}} \) of \( M_h(S, L_g)^{\text{sm}} \) admits a morphism

\[
\mu : \tilde{M}_h(S, L_g)^{\text{sm}} \to \tilde{V}^h(S, L_g) \subset |L_g|,
\]

that maps a stable map \( f : C \to S \) to its image \( f(C) \).

**Proposition 2.4.** *The following hold true:*

(i) *For every \( 0 \leq \delta \leq g \) the Severi variety \( V_\delta(S, L_g) \) is nonempty and smooth of dimension \( g - \delta \).*

(ii) *For every \( 0 \leq h \leq g \) the equigeneric locus \( V^h(S, L_g) \) and \( M_h(S, L_g)^{\text{sm}} \) have pure dimension \( h \).*

(iii) *For every \( 0 \leq h \leq g \) a general point \( C \) in any irreducible component of \( V^h(S, L_g) \) is immersed; equivalently, a general map \( f \) in any irreducible component of \( M_h(S, L_g)^{\text{sm}} \) is unramified. In particular, both \( V^h(S, L_g) \) and \( M_h(S, L_g)^{\text{sm}} \) are generically reduced.*

**Proof.** We recall that the expected dimension \( V_\delta(S, L_g) \) is \( g-\delta \). The nonemptiness statement in (i) follows from [GLS] Thm. B]. By standard deformation theory, the projective tangent space to \( V_\delta(S, L_g) \) at a point \( C \) is isomorphic to \( \mathbb{P}(H^0(S, L_g \otimes I_N)) \), where \( N \) is the scheme of nodes of \( C \). Hence, \( V_\delta(S, L_g) \) is smooth at \( C \) of dimension \( g - \delta \) if and only if \( h^0(L_g \otimes I_N) = g + 1 - \delta \), or equivalently, \( h^1(L_g \otimes I_N) = 0 \). This vanishing can be easily deduced by the short exact sequence

\[
0 \to \mathcal{O}_S \to L_g \otimes I_N \to \omega_C(p_{10}(g)) \otimes I_N \to 0,
\]

using the isomorphism \( \omega_C(p_{10}(g)) \otimes I_N \simeq \nu_*\omega_{\tilde{C}}(p) \), where \( \nu : \tilde{C} \to C \) is the normalization map and \( p = \nu^{-1}(p_{10}(g)) \).

\[\text{A curve is called immersed if the differential of its normalization map is everywhere injective.}\]
As concerns part (ii), let $C$ be a general point in any irreducible component $V$ of $V^h(S, L_g)$ and let $f : \tilde{C} \to S$ be the stable map defined as the composition of the normalization map $\nu : \tilde{C} \to C$ with the inclusion $C \subset S$. The discussion above the statement of this proposition yields that $\dim_{\nu(C)}V = \dim_[f]M_h(S, L_g)$ and, by standard deformation theory, the latter is bounded below by $\chi(N_f)$, where $N_f$ is the normal sheaf to $f$ defined by the short exact sequence

$$0 \to T_{\tilde{C}} \to f^*T_S \to N_f \to 0.$$  

It is then easy to check that $\chi(N_f) = \chi(\omega_{\tilde{C}}(p)) = h$ and thus $\dim V \geq h$.

In order to prove equality, we apply a result by Arbarello and Cornalba [AC] p. 26 as in [DS] proof of Thm. 2.8 getting

$$\dim V = \dim_{\nu(C)}V \leq h^0(\tilde{C}, \Omega f) = h^0(\omega_{\tilde{C}}(p - R)) \leq h,$$

where $\Omega f$ denotes the quotient of $N_f$ by its torsion subsheaf, which coincides with the zero divisor $R \subset \tilde{C}$ of the differential of $f$. Since $\omega_{\tilde{C}}(p)$ is globally generated off $p$ and $p_{10}(g) = f(p)$ is a smooth point of $C$, we conclude that $\dim V = h$ (thus getting (ii)) and $R = 0$. Hence, $C$ is immersed and this yields (iii) because $T_{[f]}M_h(S, L_g) = h^0(N_f) = h$ and $\mu$ is an isomorphism locally around $[f]$.

It is natural to ask whether the closure in $|L_g|$ of the Severi variety $\mathcal{V}_0(S, L_g)$ coincides with that of the equigeneric locus $\mathcal{V}^{g-\delta}(S, L_g)$, as it happens on a general $K3$ surface. The following result generalizes Theorem [18] to our setting.

**Proposition 2.5.** If the points $p_1, \ldots, p_q$ are general, then for every $g \geq 1$ and $0 \leq \delta \leq g$ one has the equality

$$\mathcal{V}_0(S, L_g) = \mathcal{V}^{g-\delta}(S, L_g)$$

in the linear system $|L_g|$.

**Proof.** We follow Chen’s proof of the analogous result for a general genus $g$ polarized $K3$ surface [Ch3 Cor. 1.2]. Let $V$ be any irreducible component of the equigeneric locus $\mathcal{V}^h(S, L_g)$ with $0 \leq h \leq g$. In order to prove that a general point of $V$ parametrizes a nodal curve, it is enough to show that $V$ contains a component of $\mathcal{V}^{h-1}(S, L_g)$ as soon as $h \geq 1$, and that all rational curves in $|L_g|$ are nodal. Both the statements were proved for a general genus $g$ polarized $K3$ surface by Chen (in [Ch3 Thm. 1.1] and [Ch2 Thm. 1.1], respectively), by specialization to a so-called Bryan-Leung $K3$ surface, that is, a $K3$ surface $X_0$ admitting an elliptic fibration $\pi : X_0 \to \mathbb{P}^1$ with a section $s$ and 24 nodal singular fibers. If $f$ is a fiber, the line bundle $L_0 := \mathcal{O}_{X_0}(s + gf)$ is a genus $g$ polarization on $X_0$ and every element in $|L_0|$ is completely reducible, that is, it is union of $s$ and $g$ fibers of $\pi$.

We now exhibit a limit of our surfaces $S$ that appeared in [ABFS §4.1] and is very similar to a Bryan-Leung $K3$ surface. By specializing the points
$p_1, \ldots, p_9 \in \mathbb{P}^2$ to the base locus of a general pencil of plane cubics, the surface $S$ specializes to a rational elliptic surface $q : S_0 \to \mathbb{P}^1$; the fibers of $q$ are the anticanonical divisors of $S_0$ and thus $q$ admits precisely 12 nodal singular fibers. It is easy to verify that on $S_0$ the exceptional divisor $E_9$ becomes a section of $q$ and every element in the linear system $|L_g|$ is the union of $E_9$ with $g$ fibers of $q$. Chen’s proof of [Ch3] Thm. 1.1 works in our setting with no change, yielding that on $S_0$ (and thus on a general $S$) every component of $V^h(S, L_g)$ with $h \geq 1$ contains a component of $V^{h-1}(S, L_g)$. Also the proofs in [Ch2] still work if, instead of a family of $K3$ surfaces whose central fiber is a Bryan-Leung $K3$ surface, one considers a family of surfaces $S \to \Delta$ whose general fibers are general $S$ and whose central fiber is $S_0$. The only difference that is worth remarking concerns [Ch2] Prop. 2.1, whose proof becomes even simpler in our case because every vector of the space $H^1(T_{S_0})$ parametrizing first order deformation of $S_0$ can be realized as the Kodaira-Spencer class of a projective family $S$. \[\square\]

The following result is a generalization of [Kem] [CD2], ensuring irreducibility of Severi varieties in $|L_g|$ when $\delta$ is small with respect to $g$.

**Proposition 2.6.** If $\delta \leq \frac{1}{6}g - \frac{1}{12}$, then $V_\delta(S, L_g)$ is irreducible.

**Proof.** Let $U_\delta \subset S^{[d]}$ be the open subset parametrizing 0-dimensional subschemes consisting of $\delta$ distinct points none of which lies on $J$. The nodes of any curve $C \in V_\delta(S, L_g)$ define a point in $U_\delta$ because they all lie outside of $J$ as $C \cdot J = 1$. As in [Kem] App. A, proof of Thm. A.0.6, the Severi variety is an open subset of a projective bundle over $U_\delta$ as soon as $H^1(L_g \otimes I_z^2) = 0$ for all $z \in U_\delta$. We will show that $H^1(L_g \otimes I_w) = 0$ for every $w \in S^{[3\delta]}$ whose support is disjoint from $J$. By contradiction, if this is not the case, up to replacing $w$ with a subscheme of length $d \leq 3\delta$, we may assume that $h^1(L_g \otimes I_w) = 1$ (use [BS] Lem. 1.2) and $h^1(L_g \otimes I_w) = 0$ for every proper subscheme $w'$ of $w$ (that is, $w$ is $L_g$-stable in the sense of Tyurin [Tyu] Def. 1.2]). By [Tyu] Lem. 1.2 there exists a rank 2 vector bundle $E$ fitting into an extension

\begin{equation}
0 \to \mathcal{O}_S \to E \to L_{g-1} \otimes I_w \to 0,
\end{equation}

where we have used that $L_g \otimes K_S \simeq L_{g-1}$. Since $c_1(E) = c_1(L_{g-1})$ and $c_2(E) = d \leq 3\delta$, the Riemann-Roch formula yields

\[\chi(E \otimes E^\vee) = c_1(E)^2 - 4c_2(E) + 4\chi(\mathcal{O}_S) = 2g - 3 - 4d + 4 \geq 2g - 12\delta + 1 \geq 2,\]

and thus either $h^0(E \otimes E^\vee) \geq 2$ or $h^2(E \otimes E^\vee) = h^0(E \otimes E^\vee \otimes K_S) \geq 1$. In both cases, $E$ is not $\mu_{L_{g-1}}$-stable and thus sits in a destabilizing short exact sequence

\begin{equation}
0 \to N \to E \to M \otimes I_\xi \to 0,
\end{equation}

where $\xi \subset S$ is a 0-dimensional subscheme and $N, M \in \text{Pic}(S)$ satisfy

\[\mu_{L_{g-1}}(N) \geq \mu_{L_{g-1}}(E) = \frac{2g - 3}{2} \geq \mu_{L_{g-1}}(M).\]
In particular, one gets $h^0(N^\vee) = 0$. We will use short exact sequence (2.4) and Lemma 2.2 to reach a contradiction. As in [Kn] Lem. 3.6, by tensoring (2.4) with $N^\vee$ and taking global sections, one obtains $h^0(M \otimes I_w) > 0$ and thus $M$ possesses a global section vanishing along a divisor that contains $w$; in particular, $M$ is effective and $M \not\cong \mathcal{O}(kJ)$ for any $k \geq 0$. If (2.4) splits, the same holds true for $N$ by inverting the roles of $N$ and $M$. Since $c_1(L_{g-1}) = c_1(N) + c_1(M)$, this would contradict Lemma 2.2, and we conclude that (2.4) does not split.

In the case where $h^0(E \otimes E^\vee) \geq 2$, by standard computations (cf., e.g., [AR] Lem. 3.4) one concludes that $h^0(N \otimes M^\vee) > 0$ and thus $N$ is effective, too. By Lemma 2.2 we get that $N \cong \mathcal{O}(kJ)$ and $M \cong L_{g-k}$ for some $0 \leq k \leq g - 1$ and this contradicts the inequalities on the slopes.

In order to arrive at the same conclusion in the case where $h^0(E \otimes E^\vee \otimes K_S) \geq 1$, we tensor (2.3) with $K_S$ and then apply $\text{Hom}(E, -)$ in order to get

$$h^0(E \otimes E^\vee \otimes K_S) \leq \dim \text{Hom}(E, N \otimes K_S) + \dim \text{Hom}(E, M \otimes K_S \otimes I_\xi).$$

By applying $\text{Hom}(-, M \otimes K_S \otimes I_\xi)$ to (2.3) and using the fact that $\text{Hom}(N, M \otimes K_S \otimes I_\xi) = 0$ as $\mu_{L_{g-1}}(N) > \mu_{L_{g-1}}(M \otimes K_S)$, one obtains that $\text{Hom}(E, M \otimes K_S \otimes I_\xi) = 0$. Analogously, applying $\text{Hom}(-, N \otimes K_S)$ to (2.3), one shows that $1 \leq \text{Hom}(E, N \otimes K_S) \cong H^0(M^\vee \otimes N \otimes K_S)$; hence, $N$ is effective yielding the same contradiction as above.

The following result controls the intersection of two irreducible components of $\overline{V}_\delta(S, L_g)$.

**Proposition 2.7.** Fix $g \geq 2$ and $0 \leq \delta \leq g - 1$. Let $V$ and $W$ be two intersecting components of $\overline{V}_\delta(S, L_g)$. Then every irreducible component of $V \cap W$ not contained in $|L_{g-1}|$ has pure codimension 1 and is generically reduced.

**Proof.** Set $U := |L_g| \setminus |L_{g-1}| \subset |L_g|$ and consider the incidence variety

$$I := \{(C, z) \in U \times S^{[\delta]} \text{ s.t. } C \subset |L_g \otimes I_\delta| \} \subset |L_g| \times S^{[\delta]}.$$

We will express $I$ as the degeneracy locus of a map of vector bundles on $|L_g| \times S^{[\delta]}$. Let $p : S \times S^{[\delta]} \rightarrow S$ and $q : S \times S^{[\delta]} \rightarrow S^{[\delta]}$ be the projections, and denote by $\Delta \subset S \times S^{[\delta]}$ the universal subscheme. Let $E := q_* (p^* L_g)$ denote the vector bundle of rank $g + 1$ on $S^{[\delta]}$ whose fiber over any point $z \in S^{[\delta]}$ equals $H^0(S, L_g)$. Let $F := q_* (p^* L_g|_{2\Delta})$ be the vector bundle of rank $3\delta$ on $S^{[\delta]}$ whose fiber over a point $z \in S^{[\delta]}$ is the vector space $H^0(S, L_g|_{2z})$, where $2z$ denotes the 0-dimensional subscheme of $S$ defined by the ideal $I_\delta^2$.

There is a natural map

$$\phi : E \longrightarrow F$$

of vector bundles on $S^{[\delta]}$. Note that $|L_g| \times S^{[\delta]}$ is isomorphic to the projective bundle

$$\pi : \mathbb{P}(E) \longrightarrow S^{[\delta]}.$$
Denoting by $\mathcal{U} \subset \pi^*E$ the universal subbundle, we consider the degeneracy locus $D(\bar{\phi})$ of the map

$$\bar{\phi} : \mathcal{U} \to \pi^*F,$$

of vector bundles on $\mathbb{P}(E) \simeq |L_g| \times S[0]$ obtained by composing $\pi^*\phi$ with the inclusion of $\mathcal{U}$ in $\pi^* E$. By construction, the incidence variety $I$ is contained in $D(\bar{\phi})$ and, if $(C, z) \in D(\bar{\phi}) \setminus I$, then $C \in |L_{g-1}|$. It can be easily checked that the expected dimension of $D(\bar{\phi})$ equals $g - \delta$. In order to show that $D(\bar{\phi})$ has the expected dimension along $I$, we consider the projection $t : D(\bar{\phi}) \to |L_g|$. If $(C, z) \in D(\bar{\phi})$, the curve $C$ is singular along $z$. Hence, if $(C, z) \in I$ and $C$ is reduced, then the $\delta$-invariant of $C$ is $\geq \delta$: this implies that $t(I) \subset \overline{V^{g-\delta}(S, L_g)} = \overline{V_\delta(S, L_g)}$, where the equality follows from Proposition 2.5. On the other hand, if $(C, z) \in D(\bar{\phi}) \setminus I$, then $C \in |L_{g-1}| \subset |L_g|$. We conclude that $I$ consists of the irreducible components of $D(\bar{\phi})$ whose image is not entirely contained in $|L_{g-1}|$. In particular, a general curve in $t(I)$ is reduced. The following Lemma 2.8 yields that the locus in $I \setminus (t^{-1}[L_{g-1}] \cap I)$ where the fibers of $t_I := t|_I$ are not finite has dimension $\leq g - \delta - 2$, and thus

$$g - \delta = \dim \overline{V_\delta(S, L_g)} \geq \dim I,$$

and $I$ consists of irreducible components of $D(\bar{\phi})$ that dominate $\overline{V_\delta(S, L_g)}$ and have the expected dimension. In particular, $I$ is locally Cohen-Macaulay (cf. [ACGH], II, Prop. 4.1) outside of its intersection with $t^{-1}[L_{g-1}]$. Hence, every irreducible component $I'$ of the intersection of two components of $I$ has codimension 1 by Hartshorne’s Connectedness Theorem (cf. [E], Thm. 18.12) and is generically reduced, unless possibly when $I'$ is contained in $t^{-1}[L_{g-1}]$. Let $Z$ be a component of the intersection of two irreducible components of $\overline{V_\delta(S, L_g)}$ such that $Z$ is not contained in $|L_{g-1}|$. Since any component $I'$ of $t^{-1}(Z)$ has codimension 1 in $I$ and is generically reduced, a general fiber of the restriction of $t$ to $I'$ is finite by the following Lemma 2.8 and thus $Z$ has codimension 1 in $\overline{V_\delta(S, L_g)}$. If a general curve in $Z$ has $\delta$-invariant precisely $\delta$, then the restriction of $t$ to $t^{-1}(Z)$ is birational and we may conclude that $Z$ is generically reduced. If instead a general curve in $Z$ has $\delta$-invariant $> \delta$, by dimensional reasons $Z$ is a component of $\overline{V_{\delta+1}(S, L_g)} = \overline{V^{g-\delta-1}(S, L_g)}$. We recall that $\overline{V_{\delta}(S, L_g)} = \overline{V^{g-\delta}(S, L_g)}$ is singular at the points of $\overline{V^{g-\delta-1}(S, L_g)}$ (cf. [DII]) as, in a neighborhood of a general $C \in V^{g-\delta-1}(S, L_g)$, the locus $\overline{V^{g-\delta}(S, L_g)}$ is the union of at most $\delta + 1$ sheets corresponding to the partial normalizations of $C$ of arithmetic genus $g - \delta$. In particular, $\overline{V^{g-\delta}(S, L_g)}$ is generically reduced along $\overline{V^{g-\delta-1}(S, L_g)}$ as soon as $\overline{V^{g-\delta-1}(S, L_g)}$ is generically reduced and this holds true in our case by Proposition 2.4.

Lemma 2.8. Let $I \subset |L_g| \times S[0]$ be the incidence variety defined in (2.5) and let $t_I : I \to |L_g|$ be the first projection. Then, the locus in $I \setminus t_I^{-1}[L_{g-1}]$ where the fibers of $t_I$ are not finite has dimension $\leq g - \delta - 2$. 

□
Proof. For any \( k \geq 1 \), let \( Z_{\delta,k} \subset \overline{V_\delta(S,L_g)} \) be the locus of irreducible curves \( C \in \overline{V_\delta(S,L_g)} \) such that, denoting by \( \nu : \tilde{C} \to C \) the normalization of \( C \) and by \( A_C := \text{Hom}_{O_C}(\nu_*O_C,O_C) \) its adjoint ideal, the subscheme \( E_C \subset C \) defined by \( A_C \) contains a \( k \)-dimensional family of subschemes of length-\( \delta \); in particular, if \( \dim t^{-1}_I(t_I(C)) \geq k \) then \( C \in Z_{\delta,k} \). We will show that \( Z_{\delta,k} \subset \overline{V_{\delta+k+2}(S,L_g)} \) for all \( 0 \leq \delta \leq g - 1 \) and \( k \geq 1 \) and thus

\[
\dim t^{-1}_I(Z_{\delta,k}) = \dim Z_{\delta,k} + k \leq \dim \overline{V_{\delta+k+2}(S,L_g)} + k = g - \delta - 2,
\]

that yields our statement.

We proceed by induction on \( k \). The case \( k = 1 \) amounts to showing that, if \( C \in Z_{\delta,1} \), then the \( \delta \)-invariant \( \delta(C) \) of \( C \) (i.e., the length of \( E_C \)) is \( \geq \delta + 3 \); this holds true because any subscheme \( \xi_t \) of length \( \delta \) contained in \( E_C \) corresponds to a partial normalization \( \nu_t : \tilde{C}_t \to C \) with \( p_a(\tilde{C}_t) = p_a(C) - \delta \). If \( \delta(C) = \delta \), then necessarily \( \nu_t = \nu \), and thus \( \xi_t = E_C \) is unique. Analogously, if \( \delta(C) = \delta + 1 \), then any such \( \tilde{C}_t \) is obtained from \( \tilde{C} \) by creating either one node or one cusp at the finitely many points of \( \tilde{C} \) mapping to the singular locus of \( C \); hence, the partial normalizations \( \nu_t \) (or, equivalently, the subschemes \( \xi_t \)) are finitely many in this case. The remaining case \( \delta(C) = \delta + 2 \) is treated in the same way using the fact that the only singularity having \( \delta \)-invariant equal to 2 are tacnodes, ramphoid cusps and triple points of embedded dimension 3.

We now assume that the inclusion \( Z_{\delta,h} \subset \overline{V_{\delta+h+2}(S,L_g)} \) holds for any \( 0 \leq \delta \leq g - 1 \) and \( 1 \leq h \leq k - 1 \), and prove it for \( h = k \geq 2 \). Fix a general \( C \in Z_{\delta,k} \); that is, \( C \) has a \( k \)-dimensional family of length-\( \delta \) subschemes contained in \( E_C \). We will prove that \( C \) possesses a \((k-1)\)-dimensional family of subschemes of length \( \delta + 1 \); this is enough to conclude because it implies that \( C \in Z_{\delta+1,k-1} \subset \overline{V_{\delta+k+2}(S,L_g)} \), where the inclusion follows from the induction assumption. In order to pass from subschemes of length \( \delta \) to subschemes of length \( \delta + 1 \), we consider the nested Hilbert scheme \( S^{[\delta,\delta+1]} \) parametrizing pairs \((\xi,\xi') \in S^{[\delta]} \times S^{[\delta+1]}\) such that \( \xi \subset \xi' \). This is endowed with two natural morphisms

\[
S^{[\delta]} \times S \xrightarrow{\phi} S^{[\delta,\delta+1]} \xleftarrow{\psi} S^{[\delta+1]} \times S,
\]

mapping a pair \((\xi,\xi')\) to \((\xi, x)\) and \((\xi', x)\), respectively, with \( x \in S \) being the point where \( \xi \) and \( \xi' \) differ. As explained in [Lc, p.12], the dimensions of the fibers of \( \phi \) and \( \psi \) are related as follows: given \((\xi,\xi') \in S^{[\delta,\delta+1]}\), if \( \phi^{-1}(\phi(\xi,\xi')) \cong \mathbb{P}^{i-1} \), then \( \psi^{-1}(\psi(\xi,\xi')) \cong \mathbb{P}^{i'-2} \) for some integer \( i' \) satisfying \( |i - i'| \leq 1 \). Let us consider the \( k \)-dimensional family

\[
B := \{ \xi \in S^{[\delta]} \mid \xi \subset E_C \subset C \},
\]
the subscheme
\[ W := \{(\xi, \xi') \in S^{[\delta, \delta+1]} \mid \xi \subset \xi' \subset E_C \} \subset \phi^{-1}(B \times \text{Supp}(E_C)), \]
and set \( B' := \psi(W) \subset \psi(\phi^{-1}(B \times \text{Supp}(E_C))) \). We want to show that \( B' \) has dimension \( \geq k - 1 \). Let \( (\xi, \xi') \in \phi^{-1}(B \times \text{Supp}(E_C)) \) be general, and denote by \( i - 1 \) the dimension of \( \phi^{-1}(\phi(\xi, \xi')) \) and by \( i' - 2 \) the dimension of \( \psi^{-1}(\psi(\xi, \xi')) \). Since \( i - i' \geq 1 \), we obtain
\[ \dim \psi(\phi^{-1}(B \times \text{Supp}(E_C))) = \dim B + i - 1 - (i' - 2) \geq k. \]
In order to conclude that \( \dim B' \geq k - 1 \), it is thus enough to show that the fiber at \( (\xi, x) \) of the restriction \( \phi|_W \) has codimension at most 1 in the fiber \( \phi^{-1}(\xi, x) \). We need to recall that \( \phi^{-1}(\xi, x) \simeq \mathbb{P}((I_{E}/m_xI_{E})^\vee) \) where \( m_x \) is the maximal ideal of the point \( x \); indeed, any pair \( (\xi, \xi') \in \phi^{-1}(\xi, x) \) corresponds to a short exact sequence
\[ 0 \rightarrow I_{\xi'} \rightarrow I_{\xi} \xrightarrow{\alpha} \mathcal{O}_x \rightarrow 0, \]
and thus to a linear map \( \alpha_x : I_{\xi}/m_xI_{\xi} \rightarrow \mathbb{C} \). Since \( \xi \subset E_C \), we have an inclusion \( i : I_{E}/S \rightarrow I_{\xi} \) and a linear map \( i_x : I_{E}/S/m_xI_{E}/S \rightarrow I_{\xi}/m_xI_{\xi} \); the subscheme \( \xi' \) is contained in \( E_C \) precisely when the composition \( \alpha_x \circ i_x \) is zero. In order to show that this is a codimension 1 condition on \( \mathbb{P}((I_{E}/m_xI_{E})^\vee) \) we prove that \( I_{E}/S/m_xI_{E}/S \) is 1-dimensional, or equivalently, \( E_C \) is contained at most one subscheme of \( S \) of length \( \delta(C) + 1 \). Consider the standard short exact sequence of ideals
\[ 0 \rightarrow \mathcal{O}_S(-C) \xrightarrow{j} I_{E}/S \rightarrow A_C \rightarrow 0, \]
where \( j \) is the multiplication by the section of \( L_g \) defining \( C \). Since \( C \) is singular along \( E_C \) and \( x \in \text{Supp}(E_C) \), then the image of \( j \) is contained in \( m_xI_{E}/S \) and we have an isomorphism \( I_{E}/S/m_xI_{E}/S \simeq A_C/m_xA_C \). It is thus enough to verify that \( A_C/m_xA_C \simeq \mathbb{C} \), or equivalently, \( E_C \) is contained at most one subscheme of \( C \) of length \( \delta(C) + 1 \). This holds true because any such subscheme corresponds to a rank 1 torsion free sheaf on \( C \) of the form \( \nu_\ast \mathcal{O}_C(y) \) for one of the finitely many points \( y \) mapping to \( x \).

3. Connectedness on Halphen surfaces

From now on, we will always assume that \( S \) is obtained by blowing-up 9 general points so that the equality \( \mathcal{V}_\delta(S, L_g) = \mathcal{V}_{g-\delta}(S, L_g) \) holds by Proposition 2.5 for every \( g \geq 1 \) and \( 0 \leq \delta \leq g \).

For every \( g \geq 0 \) and \( k \geq 1 \), we consider the natural injection
\[ i_{g,k} : |L_g| \hookrightarrow |L_{g+k}| \]
mapping a curve \( C \in |L_g| \) to the divisor \( C + kJ \in |L_{g+k}| \).

Lemma 3.1. The image \( i_{g,k}(|L_g|) \subset |L_{g+k}| \) coincides with the codimension \( k \) linear subspace \( |L_{g+k} \otimes I_x^k| \), where \( x \in J \) is a general point. In particular,
identifying $|L_p|$ with its image in $|L_q|$ under the map $i_{p,q} : p 	o q$ for every $q \geq 2$ and $0 \leq p \leq q$, we have the following chain of inclusions in $|L_q|$:

\[(3.1)\]

\[
\{ E_q \} = |L_0| \subset |L_1| \subset \cdots \subset |L_{q-1}| \subset |L_q|
\]

**Proof.** The inclusion $i_{g,k}(|L_g|) \subset |L_{g+k} \otimes I_x^k|$ is obvious. In order to prove equality, it is enough to show that $h^0(L_{g+k} \otimes I_x^k) = g + 1$. We proceed by induction on $k$. The case $k = 1$ is trivial and the induction step follows from the short exact sequences

\[
0 \to L_{g+k-1} \otimes I_x^{k-1} \to L_{g+k} \otimes I_x^k \to L_{g+k} \otimes I_x^k|_J \to 0,
\]

along with the isomorphism $L_{g+k} \otimes I_x^k|_J \simeq \mathcal{O}_J(p_{10}(g+k) - kx)$. \qed

### 3.1. Connectedness of the closure of Severi varieties.

**Proposition 3.2.** For every $g \geq 2$ and $0 \leq \delta \leq g - 1$, the following equality holds in $|L_g|$:

\[(3.2)\]

\[
|L_{g-1}| \cap V_\delta(S, L_g) = \bigcup_{h=0}^\delta V_{\delta-h}(S, L_{g-h-1}).
\]

**Proof.** First of all, we verify the inclusion $\supset$ in (3.2) by showing that, if $0 \leq h \leq \delta$ and $C$ is general in any irreducible component of $V_{\delta-h}(S, L_{g-h-1})$, the curve $X = C + (h + 1)J \in |L_g|$ can be deformed to an irreducible curve in $|L_g|$ of geometric genus $g - \delta$ (which thus lies in $V_\delta(S, L_g)$ by Proposition 2.5). The curve $X$ is the image of a stable map $f : \tilde{C} \cup_p \tilde{J} \to S$ of genus $g - \delta$, where $\tilde{J}$ is a smooth elliptic degree $h+1$ cover of $J$, the curve $\tilde{C}$ is the normalization of $C$ and has genus $g - \delta - 1$, and the gluing point $p$ is mapped to $C \cap J = \{ p_{10}(g - h - 1) \}$. We denote by $f_C := f|_{\tilde{C}} : \tilde{C} \to C \subset S$ and by $f_J := f|_J : \tilde{J} \to J \subset S$ the restrictions of $f$ to $\tilde{C}$ and $\tilde{J}$, respectively. As $f_J$ is étale and $C$ is nodal, both $f_J$ and $f_C$ are unramified and the same holds true for the map $f$ since $C$ and $J$ intersect transversally at $p_{10}(g - h - 1)$. The normal sheaf $N_f$ sits in the following short exact sequence:

\[
0 \to N_f(-p)|_J \to N_f \to N_f|_{\tilde{C}} \to 0.
\]

By [GHS] Lem. 2.5] we have isomorphisms

\[
N_f|_{\tilde{C}} \simeq N_{f_C}(p) \simeq \omega_{\tilde{C}}(2p),
\]

and analogously

\[
N_f(-p)|_J \simeq N_{f_J} \simeq f_J^*\mathcal{O}_J(J).
\]

Since the line bundle $f_J^*\mathcal{O}_J(J)$ is non-trivial of degree 0, we obtain that $h^0(N_{f_J}) = h^0(\omega_{\tilde{C}}(2p)) = g - \delta$ and $h^1(N_{f_J}) = 0$, and thus $f$ defines a smooth point of a $(g - \delta)$-dimensional component of $M_{g-\delta}(S, L_g)$. However, $f_C$ is an unramified stable map of genus $g-1-\delta$ and thus $\dim(f_C^*) M_{g-1-\delta}(S, L_{g-h-1}) = g - 1 - \delta$. Analogously, the map $f_J$ is rigid in $M_1(S, (h + 1)J)$. Hence, a general deformation of $f$ parametrizes a stable map from an integral (and smooth by dimensional arguments) curve of genus $g - \delta$. This proves that
$f$ is smoothable and thus the existence of the morphism \([3.12]\) yields that $X \in V_\delta(S, L_g)$.

It remains to verify the inclusion $\subset$ in \([3.2]\). Any irreducible component $V$ of $|L_{g-1}| \cap V_\delta(S, L_g)$ satisfies $\dim V = g - 1 - \delta$ because no component of $V_\delta(S, L_g)$ is contained in $|L_{g-1}|$ by Proposition \([2.4]\) and $|L_{g-1}|$ is a hyperplane in $|L_g|$. Assume now that a general element $X$ of $V$ parametrizes a curve in $|L_{g-h-1}| \setminus |L_{g-h-2}|$ for some $h \geq 0$, that is, $X = C + (h+1)J$ with $C$ irreducible. We need to show that $h \leq \delta$ and $C \in V_{\delta-h}(S, L_{g-h-1})$; by dimensional reasons and Proposition \([2.5]\) it is enough to check that $C$ has geometric genus $\leq g-1-\delta$. This trivially follows because $X = C+ (h+1)J \in \overline{V_\delta(S, L_g)} = V^{g-\delta}(S, L_g)$ and $J$ is an elliptic curve.

$$\square$$

We fix $k \gg 0$ so that $\overline{V_\delta(S, L_{g+k})}$ is irreducible by Proposition \([2.6]\). Inside $|L_{g+k}|$ we consider the chain of inclusions provided by Lemma \([3.1]\)

$$\{E_0\} = |L_0| \subset |L_1| \subset \cdots \subset |L_{g+k-1}| \subset |L_{g+k}|$$

and choose hyperplanes $H_1, \ldots, H_{g+k} \subset |L_{g+k}|$ such that for each $1 \leq i \leq g+k$ the following equality holds:

$$|L_{g+k-i}| = \left( \bigcap_{j=1}^i H_j \right) \cap |L_{g+k}|.$$

By Proposition \([3.2]\) $\overline{V_\delta(S, L_g)}$ does not coincide with the intersection of $\overline{V_\delta(S, L_{g+k})}$ with a linear subspace. In order to circumvent this problem we perform a sequence of blow-ups. We start by blowing up $|L_{g+k}|$ along $|L_{g-\delta}|$ and denote by $E_{g-\delta}$ the exceptional divisor. We then blow up the strict transform of $|L_{g-\delta+1}|$ and denote by $E_{g-\delta+1}$ the exceptional divisor, and so on until we finally blow up the strict transform of $|L_{g+k-2}|$ and get the last exceptional divisors $E_{g+k-2}$. Let

$$\pi_{k,\delta}: \overline{E_{g+k}} \rightarrow |L_{g+k}|$$

be the composition of these $k + \delta - 1$ blow-ups and still write $E_j$ for the strict transforms of the exceptional divisors in $|L_{g+k}|$.

We stress that for all $g-\delta \leq j \leq g+k-2$ the fiber of the restriction $\pi_{k,\delta}|E_j : E_j \rightarrow |L_j|$ over any $X \in |L_j|$ is the projective space $\mathbb{P}(N_{|L_j|}|/|L_{g+k}|, X) = \mathbb{P}^{g+k-j-1}$ blown-up at the point $\mathbb{P}(N_{|L_j+1|}|/|L_{g+k}|, X) = \mathbb{P}^0$ (so that the exceptional divisor is $\mathbb{P}(N_{|L_j+1|}|/|L_{g+k}|, X) = \mathbb{P}^{g+k-j-2}$) and then at the strict transform of the line $\mathbb{P}(N_{|L_j+1|}|/|L_{g+k}|, X) = \mathbb{P}^1$ (with exceptional divisor given by a $\mathbb{P}(N_{|L_j+2|}|/|L_{g+k}|, X) = \mathbb{P}^{g+k-j-3}$-bundle) and so on, until finally at the strict transform of $\mathbb{P}(N_{|L_j+1|}|/|L_{g+k}|, X)$ (the exceptional divisor over it being a $\mathbb{P}(N_{|L_j+2|}|/|L_{g+k}|, X) = \mathbb{P}^1$-bundle).
For every $1 \leq i \leq k$, we consider the line bundle

$$\mathcal{S}_l := \pi^*_{k, \tilde{\delta}} \mathcal{O}_{|L_{g+k}|}(1) - \sum_{j=1}^{\delta + k - 1} E_{g+k-1-j},$$

whose general section is the strict transform of a general hyperplane in $|L_{g+k}|$ containing $|L_{g+k-1-i}|$. Denoting by $\tilde{H}_i$ the strict transform of $H_i$ in $|L_{g+k}|$, we get that

$$\tilde{H}_1 \in |\mathcal{S}_1| \simeq \mathbb{P}^1,$$

$$\tilde{H}_1 + E_{g+k-i} \in |\mathcal{S}_i| \simeq \mathbb{P}^i, \text{ for } 2 \leq i \leq k.$$

**Proposition 3.3.** Fix $g \geq 2$ and $0 \leq \delta \leq g - 1$, and for every $k \geq 1$ let $\pi_{k, \delta} : |L_{g+k}| \to |L_{g+k}|$ be the composition of $k + \delta - 1$ blow-ups described above. Denoting by $V_\delta(S, L_{g+k})$ the strict transform of $V_\delta(S, L_{g+k})$ in $|L_{g+k}|$, the following isomorphism holds in $|L_{g+k}|$:

$$V_\delta(S, L_{g}) \simeq V_\delta(S, L_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{k} \tilde{H}_i + E_{g+k-i} \right) =$$

$$= V_\delta(S, L_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{k} E_{g+k-i} \right).$$

(3.3)

**Proof.** We will prove, by induction on $1 \leq l \leq k$, that

$$V_\delta(S, L_{g+k-l}) \simeq V_\delta(S, L_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l} \tilde{H}_i + E_{g+k-i} \right) =$$

$$= V_\delta(S, L_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l} E_{g+k-i} \right).$$

(3.4)

The case $l = 1$ amounts to show that:

$$V_\delta(S, L_{g+k}) \cap \tilde{H}_1 \simeq V_\delta(S, L_{g+k-1}).$$

Proposition 3.2 yields the following equality in $|L_{g+k}|$:

$$V_\delta(S, L_{g+k}) \cap H_1 = \bigcup_{h=0}^{\delta} V_\delta(S, L_{g+k-1-h}).$$

(3.5)

In particular, for every $1 \leq h \leq \delta$ the intersection $V_\delta(S, L_{g+k}) \cap |L_{g+k-1-h}|$ has codimension 1 in $V_\delta(S, L_{g+k})$. For a general $X \in V_\delta(S, L_{g+k-1-h})$ one has the identification

$$N_{V_\delta(S, L_{g+k-1-h})/V_\delta(S, L_{g+k}), X} = N_{V_\delta(S, L_{g+k}) \cap H_1 / V_\delta(S, L_{g+k}), X}.$$
and thus $V_\delta(S, \tilde{L}_{g+k})$ intersects the fiber over $X$ of the exceptional divisor $E_{g+k-1-h} \to \lfloor L_{g+k-1-h} \rfloor$ at the point
\[ \xi_X = \mathbb{P}(N_{V_\delta(S, \tilde{L}_{g+k}) \cap H_1} / V_\delta(S, \tilde{L}_{g+k}), X) \in \mathbb{P}(N_{\lfloor L_{g+k-1} \rfloor / L_{g+k}}).
\]

On the other hand, the fiber of $\tilde{H}_1 \cap E_{g+k-1-h} \to \lfloor L_{g+k-1-h} \rfloor$ over $X$ equals $\mathbb{P}(N_{\lfloor L_{g+k-1-h} \rfloor / H_1, X}) = \mathbb{P}^{h-1}$ blown-up at the point $\mathbb{P}(N_{\lfloor L_{g+k-1-h} \rfloor / \lfloor L_{g+k-1-h} \rfloor, X})$ and then, one after the other, at all the strict transforms of the subspaces $\mathbb{P}(N_{\lfloor L_{g+k-1-h} \rfloor / \lfloor L_{g+k-1-h} \rfloor, X})$ for $0 \leq j \leq h-2$ (so that the corresponding exceptional divisors is a $\mathbb{P}(N_{\lfloor L_{g+k-h+j} \rfloor / H_1, X}) = \mathbb{P}^{h-j-2}$-bundle). For a general choice of $X$, there is a deformation of $X$ inside $V_\delta(S, \tilde{L}_{g+k})$ pointing out of $V_\delta(S, \tilde{L}_{g+k}) \cap H_1$. Hence, $\xi_X$ is not contained in $\tilde{H}_1 \cap E_{g+k-1-h}$ and (3.5) follows.

Standard properties of blow-ups (cf. [EH, Prop. IV-21]) yield $\tilde{H}_1 \simeq \lfloor L_{g+k-1} \rfloor$ and, under this isomorphism, the exceptional divisors $E_{g+k-3}, \ldots, E_{g-\delta}$ restricts to the $\delta + k - 2$ exceptional divisors of $\pi_{k-1,\delta} : \lfloor L_{g+k-1} \rfloor \to \lfloor L_{g+k-1} \rfloor$. Furthermore, $(\tilde{H}_2 + E_{g+k-2}) \cap \tilde{H}_1 = E_{g+k-2} \cap \tilde{H}_1$ (where the equality follows from the fact that $\tilde{H}_1 \cap \tilde{H}_2 = \emptyset$ by construction) is isomorphic to the strict transform of $\lfloor L_{g+k-2} \rfloor$ under $\pi_{k-1,\delta}$ and is thus isomorphic to $\lfloor L_{g+k-2} \rfloor$. In this way, by induction on $2 \leq l \leq k$, one obtains the isomorphism
\[ \lfloor L_{g+k-l} \rfloor \simeq \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l+1} \tilde{H}_i + E_{g+k-i} \right) = \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l} E_{g+k-i} \right)
\]
and shows that, if $l \leq k - 1$, the divisor $\tilde{H}_{l+1} + E_{g+k-l-1}$ (or equivalently, $\tilde{H}_{l+1} + E_{g+k-l-1}$) restricts to the strict transform of $\lfloor L_{g+k-l-1} \rfloor$ under $\pi_{k-1,\delta} : \lfloor L_{g+k-l-1} \rfloor \to \lfloor L_{g+k-l-1} \rfloor$. Assuming now that (3.4) holds for $l$, we obtain it for $l + 1$ because
\begin{align*}
V_\delta(S, \tilde{L}_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l+1} \tilde{H}_i + E_{g+k-i} \right) & = \left( V_\delta(S, \tilde{L}_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l} \tilde{H}_i + E_{g+k-i} \right) \right) \cap \left( \tilde{H}_{l+1} + E_{g+k-l-1} \right) \\
& = \left( V_\delta(S, \tilde{L}_{g+k}) \cap \tilde{H}_1 \cap \left( \bigcap_{i=2}^{l} E_{g+k-i} \right) \right) \cap E_{g+k-l-1} \\
& \simeq V_\delta(S, \tilde{L}_{g+k-l}) \cap |L_{g+k-l-1}| \simeq V_\delta(S, \tilde{L}_{g+k-l-1}).
\end{align*}

We now consider the morphism
\[ \Psi = (\Psi_1, \ldots, \Psi_k) : \lfloor L_{g+k} \rfloor \longrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^k,
\]
whose $i$th component $\Psi_i$ is defined by the linear system $|O_1|$; in particular, $\Psi_i$ is a (non-minimal) resolution of the projection $\pi_i : |L_{g+k}| \to \mathbb{P}^i$ of $|L_{g+k}|$ from $|L_{g+k-1-i}|$.

For $2 \leq j \leq k$ consider the projective subspace $W_j = \pi_k([L_{g+k-j}] \subset \mathbb{P}^k$ of codimension $j$, and blow-up $\mathbb{P}^k$ first at the point $W_k$, then at the strict transform of the line $W_{k-1}$ and so on, until finally at the strict transform of $W_2$. We denote by $\mathbb{P}^k \to \mathbb{P}^k$ the composition of these $k-1$ blow-ups and recall that $\mathbb{P}^k$ is naturally a subvariety of the product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^k$. We denote by

$$\psi = (\psi_1, \ldots, \psi_k) : V_\delta(S, L_{g+k}) \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^k$$

the restriction of $\Psi$ to $V_\delta(S, L_{g+k})$ and prove the following result.

**Lemma 3.4.** The images of both $\Psi$ and $\psi$ coincide with $\tilde{\mathbb{P}}^k$. Furthermore, $V_\delta(S, L_g)$ is isomorphic to a fiber of $\psi$.

**Proof.** We first show, by induction on $h$, that the image of

$$(\Psi_1, \ldots, \Psi_h) : [L_{g+k}] \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^h$$

c有意义 with subvariety $\tilde{\mathbb{P}}^h$ obtained by blowing-up $\mathbb{P}^h$ first at the point $Q_h := \pi_h([L_{g+k-h}] \subset \mathbb{P}^h$, then at the strict transform of the line $\pi_h([L_{g+k-h+1}]$ and so on, until finally, at the strict transform of the codimension 2 subspace $\pi_h([L_{g+k-2}] \subset \mathbb{P}^h$. The case $h = 1$ is trivial and the statement for $h - 1$ yields that the image of $(\Psi_1, \ldots, \Psi_h)$ is contained in $\tilde{\mathbb{P}}^{h-1} \times \mathbb{P}^h$. Denoting by $q : \tilde{\mathbb{P}}^h \to \mathbb{P}^h$ the blow-up map, the universal property of blow-ups yields a factorization $\Psi_h = \tilde{\Psi}_h \circ q$ for some morphism

$$\tilde{\Psi}_h : [L_{g+k}] \to \tilde{\mathbb{P}}^h.$$ 

By resolving the projection $\mathbb{P}^h \to \mathbb{P}^{h-1}$ from the point $Q_h$, we obtain a morphism $\text{Bl}_{Q_h} \mathbb{P}^h \to \mathbb{P}^{h-1}$ and an isomorphism $\tilde{\mathbb{P}}^h \cong \mathbb{P}^{h-1} \times \mathbb{P}^{h-1}$. In particular, by considering the projection $p : \tilde{\mathbb{P}}^h \to \mathbb{P}^{h-1} \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^{h-1}$, we get a factorization $(\Psi_1, \ldots, \Psi_{h-1}) = \tilde{\Psi}_h \circ p$. We have thus proved that

$$(\Psi_1, \ldots, \Psi_h) = \tilde{\Psi}_h \circ (p, q)$$

and, since the map $(p, q) : \tilde{\mathbb{P}}^h \to \mathbb{P}^{h-1} \times \mathbb{P}^h \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^h$ is the natural inclusion, this implies that the image of $(\Psi_1, \ldots, \Psi_h)$ is $\tilde{\mathbb{P}}^h$.

We now denote by $e_0, \ldots, e_{k-2}$ the exceptional divisors of $q : \tilde{\mathbb{P}}^k \to \mathbb{P}^k$, by numbering them so that $q(e_j)$ has dimension $j$. By construction, we have $E_{g+j} = \Psi^*(e_j)$ for every $0 \leq j \leq k - 2$. Furthermore, on $\mathbb{P}^k$ there exist divisors $D_k \in |q^*O_{\mathbb{P}^k}(1)|$ and $D_i \in |q^*O_{\mathbb{P}^k}(1) - \sum_{j=i-1}^{k-2} e_j|$ for $1 \leq i \leq k - 1$
such that $\widetilde{H}_i = \Psi^*\widetilde{D}_i$ for $1 \leq i \leq k$. It then follows that the intersection

$$\widetilde{H}_1 \cap \left( \bigcap_{i=2}^{k} \widetilde{H}_i + E_{g+k-i} \right)$$

is the inverse image under $\Psi$ of $\widetilde{D}_1 \cap \left( \bigcap_{i=2}^{k} \widetilde{D}_i + e_{k-i} \right)$ and this consists of the point $\xi \in e_0 \cap e_1 \cap \ldots \cap e_{k-2}$ determined by $\mathbb{P}(N_{D_1/\mathbb{P}^k, Q_1})$, where $D_1 = q(\widetilde{D}_1)$. Proposition \[\text{3.3}\] then implies that $\psi^{-1}(\xi) \simeq V_{\delta}(S, L_g)$; since this has dimension $g - \delta$, the restriction $\psi$ of $\widetilde{H}_1$ onto $\mathbb{P}^k$.

\[\text{Theorem 3.5.} \quad \text{For every } g \geq 1 \text{ and every } 0 \leq \delta \leq g - 1 \text{ the closure of the Severi variety } \widetilde{V}_{\delta}(S, L_g) \subset |L_g| \text{ is connected.} \]

\[\text{Proof.} \quad \text{We fix } k >> 0 \text{ and consider the intersection } |L_g| \cap \widetilde{V}_{\delta}(S, L_{g+k}) \text{ inside } |L_{g+k}|. \text{ Since } \delta \text{ is small with respect to } g + k, \text{ then } \widetilde{V}_{\delta}(S, L_{g+k}) \text{ is irreducible by Proposition } \[\text{2.6}\]. \text{ Since } \widetilde{V}_{\delta}(S, L_g) = \pi_{k,\delta}(\widetilde{V}_{\delta}(S, L_g)), \text{ it is enough to prove the connectedness of } \widetilde{V}_{\delta}(S, L_g), \text{ which is a fiber of } \psi : \widetilde{V}_{\delta}(S, L_g) \to \mathbb{P}^k \text{ by Lemma } \[\text{3.3}\]. \text{ By construction, the fiber } \pi_{k,\delta}^{-1}(X) \text{ of } \pi_{k,\delta} \text{ over any } X \in |L_{g-1}| \setminus |L_{g-2}| \text{ is a } \mathbb{P}^k = \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) \text{ blown-up at the point } \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) \text{ (the exceptional divisor being identified with } \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) \text{ and then at the strict transform of the line } \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) \text{ with exceptional divisor given by } \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) = \mathbb{P}^{k-2}-\text{bundle}) \text{ and so on until, finally, at the strict transform of } \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) \text{ the exceptional divisor over it being a } \mathbb{P}(N_{L_{g-1}/L_{g+k}}, X) = \mathbb{P}^1\text{-bundle). Hence, } \Psi \text{ maps } \pi_{k,\delta}^{-1}(X) \text{ isomorphically onto } \mathbb{P}^k, \text{ that is, } \pi_{k,\delta}^{-1}(X) \text{ defines a section of } \Psi : |L_{g+k}| \to \mathbb{P}^k. \text{ In order to ensure that this restricts to a section of } \psi, \text{ we need to choose } X \text{ so that } X = (k + 1)J + C \in |L_{g-1}| \subset |L_{g+k}| \text{ for some irreducible } C \in |L_{g-1}| \text{ with precisely } \delta \text{ nodes, that is, } C \in \mathbb{V}_{\delta}(S, L_{g-1}). \text{ We claim that with this choice of } X \text{ the fiber } \pi_{k,\delta}^{-1}(X) \text{ is contained in } \mathbb{V}_{\delta}(S, L_{g+k}) \text{ and thus defines a section of } \psi. \text{ Assuming this, we may apply, e.g., [Kol] Lem. 3] and conclude that a general fiber of } \psi \text{ is connected. Since } \mathbb{P}^k \text{ is smooth, by a classical argument using the Stein factorization of } \psi \text{ (cf., e.g., [Fl] proof of Thm. 2.1)] we obtain connectedness of any fiber of } \psi \text{ and thus in particular of } \mathbb{V}_{\delta}(S, L_g). \text{ It remains to prove the claim. Proposition } \[\text{3.2}\] \text{ yields } \mathbb{V}_{\delta}(S, L_{g-1}) \subset \mathbb{V}_{\delta+k}(S, L_{g+k}) \subset \mathbb{V}_{\delta}(S, L_{g+k}), \text{ and we recall that } \mathbb{V}_{\delta}(S, L_{g+k}) \text{ is singular at the points of } \mathbb{V}_{\delta+k}(S, L_{g+k}) \text{ (cf. [DH]). More precisely, in a neighborhood of a general } Y \in \mathbb{V}_{\delta+k}(S, L_{g+k}) \text{ the locus } \mathbb{V}_{\delta}(S, L_{g+k}) \text{ is the union of } \binom{\delta+k}{\delta} \text{ sheets corresponding to the possibilities of choosing } \delta \text{ nodes among the } \delta + k \text{ nodes of } Y. \text{ Let } V \text{ be the sheet of }
We thus have the following commutative diagram:

\[
\begin{array}{ccc}
\gamma & \rightarrow & \delta \\
\downarrow & & \downarrow \\
\frac{\delta}{\gamma} & \rightarrow & \frac{\delta}{\gamma}
\end{array}
\]

Since \( V_{\delta}(S, L_{g+k}) \) corresponds to the choice of the \( \delta \) nodes of \( X \) lying on \( C \), and denote by \( N \) the scheme of nodes of \( C \). In order to show that \( \pi_{k,\delta}^{-1}(X) \) is contained in \( V_{\delta}(S, L_{g+k}) \), we need to check that

\[
N_{V\cap|L_{g-1+h}|/V,X} \simeq N_{|L_{g-1+h}|/|L_{g+k}|,X}, \quad \text{for every } 0 \leq h \leq k - 1.
\]

Since \( V_{\delta}(S, L_{g-1}) \) is smooth at \( C \), then \( H^1(S, L_{g-1} \otimes I_N) = 0 \) and one obtains \( H^1(S, L_{g-1+h} \otimes I_N) = 0 \) for every \( 1 \leq h \leq k - 1 \) by considering the following short exact sequence:

\[
0 \rightarrow L_{g-1} \otimes I_N \rightarrow L_{g-1+h} \otimes I_N \rightarrow L_{g-1+h}|_{h,J} \rightarrow 0.
\]

We thus have the following commutative diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & H^0(S, L_{g-1+h} \otimes I_N) & \rightarrow H^0(S, L_{g+k} \otimes I_N) & \rightarrow H^0(L_{g+k}|_{(k+1-h),J}) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^0(S, L_{g-1+h}) & \rightarrow H^0(S, L_{g+k}) & \rightarrow H^0(L_{g+k}|_{(k+1-h),J}) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(O_N) & \rightarrow & H^0(O_N) & \rightarrow & H^0(O_N) & \rightarrow 0 \\
\end{array}
\]

The middle horizontal short exact provides an identification of the normal space \( N_{|L_{g-1+h}|/|L_{g+k}|,X} \) with \( H^0(L_{g+k}|_{(k+1-h),J}) \simeq C^{k+1-h} \). Since the sheet \( V \) corresponds to the choice of the \( \delta \) nodes of \( X \) lying on \( C \), the projective tangent space to \( V \) at \( X \) is isomorphic to \( \mathbb{P}(H^0(S, L_{g+k} \otimes I_N)) \), and analogously the projective tangent space to \( V \cap |L_{g-1+h}| \) at \( X \) is isomorphic to \( \mathbb{P}(H^0(S, L_{g-1+h} \otimes I_N)) \). Hence, the upper horizontal short exact sequence identifies the normal space \( N_{V\cap|L_{g-1+h}|/V,X} \) with \( H^0(L_{g+k}|_{(k+1-h),J}) \) and this implies the desired isomorphisms \((3.7)\).

### 3.2. Connectedness result for expanded degenerations

We consider the moduli stack \( \mathcal{M}_{g-\delta}(S/J, L_g) \) of stable relative maps to expanded degenerations of \((S,J)\) with multiplicity 1 along the relative divisor \( J \), introduced by Jun Li [Li1, Li2]. An expanded degeneration of \( S \) along \( J \) is a semistable model of \( S \)

\[
S[n]_0 := S \cup_J R \cup_J \cdots \cup_J R, \quad R := \mathbb{P}(O_J \oplus N_{J/S})
\]

which is union of \( S \) with a length-\( n \) tree of ruled surfaces \( R \) as above for some \( n \geq 0 \). More precisely, denoting by \( J_0 \) and \( J_\infty \) the two distinguished sections on \( R \) such that \( N_{J_0/R} \simeq N_{J/S} \) and \( N_{J_\infty/R} \simeq N_{J/S} \), the above expansion
$S[n]_0$ is obtained by gluing the first copy of $R$ with $S$ along $J_0$, while two adjacent copies of $R$ are glued identifying the $J_\infty$ on the left surface with the $J_0$ on the right one. The relative divisor $J$ is the section $J_\infty$ on the latter copy of $R$.

Since $J \cdot L_g = 1$ on $S$, stable maps on expansions of $S$ can be easily described. In particular, a stable relative map of genus $g - \delta$ to the expansion $S[n]_0$ is a map $f : C \to S[n]_0$ from a connected prestable curve of genus $g - \delta$ such that no component of $C$ is mapped entirely to the singular locus of $S[n]_0$ or to the divisor $J_\infty$ on the last copy of $R$, the inverse image of every component of the singular locus of $S[n]_0$ is a node of $C$ connecting two irreducible components of $C$ that are mapped to two adjacent components of $S[n]_0$, and the automorphism group of $f$ is finite. Furthermore, in order for $f$ to define a point of $\mathcal{M}_{g-\delta}(S/J, L_g)$, we require that the image of the curve $f(C) \subset S[n]_0$ under the projection $S[n]_0 \to S$ lies in $|L_g|$. Any such map can be thus decomposed as

$$f = f_0 \cup \cdots \cup f_n : C = C_0 \cup C_1 \cup \cdots \cup C_n \to S[n]_0,$$

where every $C_i$ is an irreducible curve, two adjacent $C_i$ share a node of $C$, and $f_i(C_i)$ is contained in the $i$-th copy of $R$ if $i \geq 1$, while $f_0(C_0) \subset S$. Denoting by $h_i \geq 0$ the arithmetic genus of $C_i$ and by $g_i \geq h_i$ the arithmetic genus of $f(C_i)$, the integers $h_i$ and $g_i$ satisfy $\sum_{i=0}^n g_i = g$ and $\sum_{i=0}^n h_i = g - \delta$.

Furthermore, the curve $f_0(C_0) \in |L_{g_0}|$, while for $i \geq 1$ the numerical class of $f_i(C_i)$ is $g_i J_0 + f_i$ on $R$ (where $f$ is the class of a fiber of $R \to J$), and its linear equivalence class is determined by the gluing condition as follows. Setting $g = (g_0, \ldots, g_n)$ and $x_i(g) := p_{10}(g_0 + g_i)$, we have $f(C_i) \in |N_1(g)|$ with $N_1(g) := g_1 J_0 + f_{x_i}(g)$. As in [FT, §2], one verifies that $N_1(g)$ has two base points, namely, $p_{10}(g_0) \in J_0$ and $x_1(g) \in J_\infty$. Analogously, setting $x_i(g) := p_{10}(g_0 + \cdots + g_i)$ for every $1 \leq i \leq n$, we have $f(C_i) \in |N_i(g)|$ where the line bundle $N_i(g) := g_i J_0 + f_{x_i}(g)$ has base points $x_{i-1}(g) \in J_0$ and $x_i(g) \in J_\infty$. In particular, the evaluation map

$$\mathcal{M}_{g-\delta}(S/J, L_g) \to J$$

at the relative divisor always takes the value $p_{10}(g)$. The stability condition implies that a stable map to $S[n]_0$ has no component mapping to a fiber of a ruled surface $R$. In particular, we have $g_i > 0$ for $i > 0$ and, since the linear systems $|N_i(g)|$ contains no rational curve when $g_i > 0$, also $h_i > 0$.

The multiplicative group $\mathbb{C}^*$ acts fiberwise on $R$ preserving the sections $J_0$ and $J_\infty$; this induces an action of $(\mathbb{C}^*)^n$ on $S[n]_0$ for every $n \geq 1$. Two stable maps with target $S[n]_0$ are considered equivalent if they differ by the action of an element of $(\mathbb{C}^*)^n$ on $S[n]_0$. Summing up, thanks to the decomposition (3.8), a point $[f] \in \mathcal{M}_{g-\delta}(S/J, L_g)$ representing a stable map with target $S[n]_0$ defines points of the following moduli stacks

$$[f_0] \in \mathcal{M}_{h_0}(S, L_{g_0}) \ , \ [f_i] \in \mathcal{M}_{h_i}(R, N_i(g))/\mathbb{C}^* \text{ for } i \geq 1.$$
We will later use the following result concerning equigeneric loci in the linear systems $|N_i(g)|$ on $R$.

**Proposition 3.6.** Let $R$ and $N_i(g) := g_i J_0 + f_{x_i}(g) \in \text{Pic}(R)$ be as above. Then for every integer $1 \leq h_i \leq g_i$ the following hold:

(i) Both the equigeneric locus $V^{h_i}(R, N_i(g)) \subset |N_i(g)|$ and the moduli stack of smoothable stable maps $\mathcal{M}_{h_i}(R, N_i(g))^{sm}$ with image in the linear system $|N_i(g)|$ have pure dimension $h_i$ and are generically reduced.

(ii) Let $V$ and $W$ be two intersecting components of $\overline{V^{h_i}(R, N_i(g))}$ and let $Z$ be an irreducible component of $V \cap W$ whose general point parametrizes a curve containing neither $J_0$ nor $J_\infty$; then $Z$ has pure codimension 1 and is generically reduced.

**Proof.** Let $\eta \in \text{Pic}^0(J)$ be the line bundle such that $R = \mathbb{P}(O_J \oplus \eta)$ and denote by $\phi : R \to J$ the natural projection; we have $J_\infty \equiv J_0 - \phi^*(\eta)$. We recall from [FT] that curves in $|N_i(g)|$ have arithmetic genus $g_i$ and $\dim |N_i(g)| = g_i$. According to our notation, the moduli stack $\mathcal{M}_{h_i}(R, N_i(g))$ parametrizes maps $f$ such that $f(C)$ lies in the linear system $|N_i(g)|$; this is a closed substack of the moduli stack $\mathcal{M}_{h_i}(R, g_i J_0 + f)$ where only the numerical class of $f(C)$ is fixed. Let $\mathcal{M}_{h_i}(R, N_i(g))^{sm}$ and $\mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$ be the closed substacks parametrizing smoothable maps. The deformations of a map $[f] \in \mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$ are governed by the normal sheaf $N_f$. As in the proof of Proposition [2.4](ii)-(iii), one shows that a general $[f]$ in any irreducible component of $\mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$ is unramified and thus $\mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$ is generically reduced and has pure dimension $h_i + 1$. Being a fiber of the evaluation map $\mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm} \to J_0$, the stack $\mathcal{M}_{h_i}(R, N_i(g))^{sm}$ is generically reduced and of pure dimension $h_i$. The same holds true for the equigeneric locus $\overline{V^{h_i}(R, N_i(g))}$ thanks to the existence of a birational map $\tilde{\mu} : \tilde{\mathcal{M}}_{h_i}(R, g_i J_0 + f)^{sm} \to \overline{V^{h_i}(R, N_i(g))}$ from the semi-normalization $\tilde{\mathcal{M}}_{h_i}(R, g_i J_0 + f)^{sm}$ of $\mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$. This proves (i).

To obtain (ii) one proceeds exactly as in the proofs of Proposition [2.4](ii)-(iii) and Lemma [3.2] where the density of the Severi variety in the equigeneric locus (whose validity on $R$ is unknown) was never used. The proofs work the same way because all curves in the linear system $|N_i(g)|$ are irreducible except for those lying in the two hyperplanes of $|N_i(g)|$ defined by the linear subsystem $J_0 + |(g_i - 1) J_0 + f_{x_i}(g)|$ and $J_\infty + |(g_i - 1) J_0 + f_{y_i}(g)|$, where we have set $y_i(g) := p_{10}(g_0 + \cdots + g_i - 1)$. \[\square\]

Coming back to $\mathcal{M}_{g-\delta}(S/J, L_g)$, this is a proper and separated Deligne-Mumford stack by [Li] Thm. 3.10. We briefly recall why it is DM. By Li’s construction, there is a scheme

$$S[n] \subset S \times \mathbb{A}^n$$
combining all possible expansions \(S[k]_0\) for \(0 \leq k \leq n\); in particular, a general fiber of the projection \(S[n] \to \mathbb{A}^n\) is isomorphic to \(S\), the central fiber over \(0 \in \mathbb{A}^n\) is the \(n\)-th expansion \(S[n]_0\), while the fibers over any coordinate \((n-k)\)-dimensional plane in \(\mathbb{A}^n\) are isomorphic to \(S[k]_0\). The natural action of \((\mathbb{C}^*)^n\) on \(\mathbb{A}^n\) lifts to an action on \(S[n]\) so that its restriction to \(S[n]_0\) is trivial on \(S\), while the \(i\)-th copy of \(\mathbb{C}^*\) acts on the \(i\)-th copy of \(R\) fiberwise so that \(J_0\) and \(J_\infty\) are fixed. Let us consider the projection

\[
\beta_n : S[n] \to S
\]

and denote by \(\mathcal{M}_{g-\delta}(S[n], L_g)\) the DM stack of ordinary stable maps \(f : C \to S[n]\) such that the image of \(\beta_n(f(C)) \in [L_g]\). Let \(\mathcal{M}_{g-\delta}(S[n], L_g)L_i^n\) be the closed DM substack parametrizing stable (in J. Li’s sense recalled above) maps to some expanded degeneration \(S[k]_0\) with \(k \leq n\); this admits a \((\mathbb{C}^*)^n\) action that is induced by the one on \(S[n]\) and has finite stabilizers. The fact that \(\mathcal{M}_{g-\delta}(S/J, L_g)\) is DM thus follows from the existence of a surjective étale map

\[
\mathcal{M}_{g-\delta}(S[n], L_g)L_i^n/(\mathbb{C}^*)^n \to \mathcal{M}_{g-\delta}(S/J, L_g).
\]

We denote by \(\mathcal{M}_{g-\delta}(S/J, L_g)^{st}\) the locus in \(\mathcal{M}_{g-\delta}(S/J, L_g)\) parametrizing stable maps which are smoothable. In particular, a general point in any component of \(\mathcal{M}_{g-\delta}(S/J, L_g)^{st}\) parametrizes a stable map with irreducible domain and target precisely \(S\).

Now we fix \(k \gg 0\) as in the proof of Theorem 3.5 so that \(\overline{V_\delta(S, L_{g+k})}\) is irreducible. In particular, the moduli stack \(\mathcal{M}_{g+k-\delta}(S/J, L_{g+k})^{st}\) and its semi-normalization \(\tilde{\mathcal{M}}_{g+k-\delta}(S/J, L_{g+k})^{st}\) are irreducible, too. We consider the natural map

\[
\alpha : \tilde{\mathcal{M}}_{g+k-\delta}(S/J, L_{g+k})^{st} \to |L_{g+k}|
\]

that sends the class of a stable map \(f : C \to S[n]_0\) for \(0 \leq n \leq g + k\) to the image of \(f(C)\) under the map \(S[n]_0 \to S\).

**Lemma 3.7.** The map \(\alpha\) factors through a map

\[
(3.9) \quad \tilde{\alpha} : \tilde{\mathcal{M}}_{g+k-\delta}(S/J, L_{g+k})^{st} \to \overline{|L_{g+k}|},
\]

so that the image of \(\tilde{\alpha}\) is the strict transform \(V_\delta(S, L_{g+k})\) of \(\overline{V_\delta(S, L_{g+k})}\) in the blow-up \(\overline{|L_{g+k}|}\) of \(|L_{g+k}|\).

**Proof.** We exploit the family \(S[1] \to \mathbb{A}^1\), whose general fibers \(S_t\) are isomorphic to \(S\) and whose central fiber is \(S[1]_0 = S \cup R\). Denote by \(\mathcal{L}\) the equivalence class (up to twisting by multiples of a component of the central fiber) of line bundles on \(S[1]\) restricting to \(L_{g+k}\) on a general fiber \(S_t\) and to a line bundle of the form \((L_{g_0}, (g + k - g_0)J_0 + f_{p_0(g+k)})\) with \(0 \leq g_0 \leq g + k - 1\) on \(S[1]_0\); since all pairs \((L_{g_0}, (g + k - g_0)J_0 + f_{p_0(g+k)})\) are limits of the same family of line bundles on \(S_t\), they are considered equivalent and we denote by \(L := [(L_{g_0}, (g + k - g_0)J_0 + f_{p_0(g+k)})]\) their class. Let

\[
\mathcal{M}_{g+k-\delta}(S[1]\exp, \mathcal{L}) \to \mathbb{A}^1
\]
be the moduli stack of connected stable maps to expanded degenerations of \( \chi \) constructed in \[Li1, Li2\]. Over a point \( t \in \mathbb{A}^1 \setminus \{0\} \), the fiber of \( \chi_0 \) is simply the moduli stack \( \mathcal{M}_{g,k-\delta}(S,L_{g+k}) \) of ordinary stable maps on \( S \), while the fiber over \( 0 \) is the stack \( \mathcal{M}_g(\mathcal{S}[1]^\exp_0,L) \) parametrizing stable maps (in the sense of Jun Li) to some expanded degeneration of \( \mathcal{S}[1]_0 \), or equivalently, to some expanded degenerations \( \mathcal{S}[n]_0 \) of \( S \) with \( n \geq 1 \). By \[Li1, \S3\], \( \mathcal{M}_g-\delta(\mathcal{S}[1]^\exp_0,L) \) admits the following decomposition:

\[
\mathcal{M}_g-\delta(\mathcal{S}[1]^\exp_0,L) = \bigcup_{(g_0,h_0) \in I} \mathcal{M}_{h_0}(S/J,L_{g_0}) \times \mathcal{M}_{g+k-\delta-h_0}(R/J_0,(g+k-g_0)J_0+f_{p_10(g+k)}),
\]

where \( I \) is the set of indices \( I := \{(g_0,h_0) \in \mathbb{Z}^2 | 0 \leq g_0 < g+k, \max\{0,g_0-\delta\} \leq h_0 \leq \min\{g_0,g+k-\delta-1\}\} \), and \( \mathcal{M}_{h_0}(S/J,L_{g_0}) \) and \( \mathcal{M}_{g+k-\delta-h_0}(R/J_0,(g+k-g_0)J_0+f_{p_10(g+k)}) \) are stacks of stable relative maps to expanded degenerations of \( (S,J) \) and \( (R,J_0) \), respectively. Each factor in the decomposition (3.10) appears with multiplicity 1 by \[Li1, \Prop. 4.13\] and, by \[Li2, \S3.1\], defines a Cartier divisor in \( \mathcal{M}_{g+k-\delta}(\mathcal{S}[1]^\exp,L) \) that we denote by \( \mathcal{D}_{g_0,h_0} \).

Recalling that points of \( \mathcal{M}_{g+k-\delta}(S/J,L_{g+k}) \) parametrize maps to expanded degenerations \( \mathcal{S}[n]_0 \) of \( S \) with \( 0 \leq n \leq g+k \), there is a natural map

\[
\gamma : \mathcal{M}_{g+k-\delta}(S/J,L_{g+k}) \to \mathcal{M}_{g+k-\delta}(\mathcal{S}[1]^\exp,L)
\]

induced by the projection \( S[g+k] \to \mathcal{S}[1] \). Pulling back \( \mathcal{D}_{g_0,h_0} \) via \( \gamma \), we obtain a Cartier divisor on \( \mathcal{M}_{g+k-\delta}(S/J,L_{g+k}) \) that we denote by \( \mathcal{D}_{g_0,h_0} \). In particular, for any fixed \( 0 \leq g' < g+k \) the union

\[
\bigcup_{0 \leq g_0 \leq g', \max\{0,g_0-\delta\} \leq h_0 \leq \min\{g_0,g+k-\delta-1\}} \mathcal{D}_{g_0,h_0}
\]

is a Cartier divisor on \( \mathcal{M}_{g+k-\delta}(S/J,L_{g+k}) \). On the other hand, \( \alpha^{-1}(\mathcal{L}_{g'}) \) coincides with the pullback under the semi-normalization map

\[
\nu : \mathcal{M}_{g+k-\delta}(S/J,L_{g+k})^{sm} \to \mathcal{M}_{g+k-\delta}(S/J,L_{g+k})^{sm}
\]

(which is a universal homeomorphism) of the restriction of (3.11) to the smoothable locus; this implies that \( \alpha^{-1}(\mathcal{L}_{g'}) \) is a Cartier divisor on \( \mathcal{M}_{g+k-\delta}(S/J,L_{g+k})^{sm} \).

The case \( g' = g-\delta \) yields, by the universal property of blow-ups \[St, 70.17\], a factorization of \( \alpha \) through the blow-up of \( |L_{g+k}| \) along \( |L_{g-\delta}| \):

\[
\tilde{\mathcal{M}}_{g+k-\delta}(S/J,L_{g+k})^{sm} \xrightarrow{\alpha_1} B|_{|L_{g-\delta}|} |L_{g+k}| \xrightarrow{b_1} |L_{g+k}|.
\]

Denoting by \( |L_{g-\delta+1}| \) the strict transform of \( |L_{g-\delta+1}| \) in \( B|_{|L_{g-\delta}|} |L_{g+k}| \), we have

\[
\alpha^{-1}(|L_{g-\delta+1}|) = \alpha_1^{-1}(|L_{g-\delta+1}| + E_{g-\delta}) = \alpha_1^{-1}(|L_{g-\delta+1}| + \alpha^{-1}(|L_{g-\delta}|),
\]
and thus conclude that $\alpha_1^{-1}(|L_{g-\delta+1}|)$ is a Cartier divisor as it is difference of two Cartier divisors. Therefore, $\alpha_1$ factors through the blow-up of $Bl_{|L_{g-\delta}|L_{g+k}}$ along $|L_{g-\delta+1}|$. By the same argument, after $g + \delta - 1$ steps one obtains a factorization of $\alpha$ through a map

$$\tilde{\alpha} : \tilde{\mathcal{M}}_{g+k-\delta}(S/J, L_{g+k})^{sm} \to |L_{g+k}|.$$ 

Since $\tilde{\mathcal{M}}_{g+k-\delta}(S/J, L_{g+k})^{sm}$ is irreducible and its image under $\alpha$ is $\widetilde{V_\delta(S, L_{g+k})}$, the image of $\tilde{\alpha}$ is precisely the strict transform $\widetilde{V_\delta(S, L_{g+k})}$.

**Theorem 3.8.** For every $g \geq 1$ and every $0 \leq \delta \leq g - 1$ there exists a closed substack $\mathcal{M}_{g-\delta}(S/J, L_g)^{\dagger}$ of $\mathcal{M}_{g-\delta}(S/J, L_g)$ that is connected and contains $\mathcal{M}_{g-\delta}(S/J, L_g)^{sm}$. The image in $S$ of a stable map parametrized by a point $[f] \in \mathcal{M}_{g-\delta}(S/J, L_g)^{\dagger}$ always lies in the closure of the Severi variety $\widetilde{V_\delta(S, L_g)}$. Moreover, for every $[f] \in \mathcal{M}_{g-\delta}(S/J, L_g)^{\dagger}$ and its image is a nodal curve, then $[f] \in \mathcal{M}_{g-\delta}(S/J, L_g)^{sm}$.

**Proof.** We use the same notation as in the proof of Lemma 3.7. We compose $\tilde{\alpha}$ with the morphism $\psi : \widetilde{V_\delta(S, L_{g+k})} \to \widetilde{\mathbb{P}^k}$ introduced in (3.6). As proved in Lemma 3.4 and Theorem 3.5, $\widetilde{V_\delta(S, L_g)}$ is isomorphic to the intersection

$$V_\delta(S, L_{g+k}) \cap \widetilde{H_1} \cap \left( \bigcap_{i=2}^{k} \widetilde{H_1} + E_{g+k-i} \right) \simeq V_\delta(S, L_{g+k}) \cap \widetilde{H_1} \cap \left( \bigcap_{i=2}^{k} E_{g+k-i} \right)$$

which is a fiber of $\psi$ and is connected because $\psi$ admits a section and $\widetilde{\mathbb{P}^k}$ is smooth.

We will now prove the existence of an injective morphism

$$r : \tilde{\alpha}^{-1} \left( \widetilde{H_1} \cap \left( \bigcap_{i=2}^{k} E_{g+k-i} \right) \right) \to \mathcal{M}_{g-\delta}(S/J, L_g)$$

whose image contains $\mathcal{M}_{g-\delta}(S/J, L_g)^{sm}$ and will be denoted by $\mathcal{M}_{g-\delta}(S/J, L_g)^{\dagger}$. To do that, we denote by $\mathcal{D}_{h_0, g_0}^{sm}$ the pullback under the map $\nu$ in (3.12) of the restriction of $\mathcal{D}_{h_0, g_0}$ to the smoothable locus, and observe that

$$\tilde{\alpha}^{-1}(E_{g-\delta}) = \alpha^{-1}|L_{g-\delta}| = \bigcup_{0 \leq g_0 \leq g-\delta, \max\{0, g_0-\delta\} \leq h_0 \leq \min\{g_0, g+k-\delta-1\}} \mathcal{D}_{h_0, g_0}^{sm},$$

and then

$$\tilde{\alpha}^{-1}(E_{g-\delta+1}) = \alpha^{-1}|L_{g-\delta+1}| - \tilde{\alpha}^{-1}(E_{g-\delta}) = \bigcup_{\max\{0, g-2\delta+1\} \leq h_0 \leq \min\{g-\delta+1, g+k-\delta-1\}} \mathcal{D}_{h_0, g-\delta+1}^{sm},$$
Analogously, for every $1 \leq i \leq k + \delta - 2$ one has

$$\tilde{\alpha}^{-1}(E_{g-\delta+i}) = \alpha^{-1}|L_{g-\delta+i}| - \sum_{j=g-\delta}^{g-\delta+i-1} \tilde{\alpha}^{-1}(E_j) = \max \{0, g-2\delta+i\} \leq h_0 \leq \min \{g-\delta+i, g+k-\delta-1\}.$$

Hence, a general point of any irreducible components of $\tilde{\alpha}^{-1}(E_{g-\delta+i})$ represents a stable morphism $f = f_0 \cup f_1 : C_0 \cup C_1 \to S[1]_0$ such that $[f_0] \in M_{h_0}(S, L_{g-\delta+i})$ and $[f_1] \in M_{h_0}(R, (k+\delta-i)J_0 + f_{p10(g+k)})$ for some $\max \{0, g-2\delta+i\} \leq h_0 \leq \min \{g-\delta+i, g+k-\delta-1\}$. Finally, one computes that

$$\tilde{\alpha}^{-1}(\tilde{H}_1) = \alpha^{-1}|L_{g+k-1}| - \sum_{j=g-\delta}^{g+k-2} \tilde{\alpha}^{-1}(E_j) \simeq D_{g+k-\delta-1, g+k-1},$$

so that a general point of any irreducible components of $\tilde{\alpha}^{-1}(\tilde{H}_1)$ represents a stable map

$$f = f_0 \cup f_1 : C_0 \cup J \to S[1]_0$$

with $[f_0] \in M_{g+k-\delta-1}(S, L_{g+k-1})$ and $[f_1] \in M_{1}(R, J_0 + f_{p10(g+k)})$. A general point in the intersection $\tilde{\alpha}^{-1}(\tilde{H}_1 \cap E_{g+k-3})$ then represents a map $f$ with a chain of at least 2 copies of $J$, that is, $f$ can be written as

$$f = f_0 \cup f_1 \cup f_2 : C_0 \cup J \cup J \to S \cup R \cup R = S[2]_0$$

with $f(C_0) \in |L_{g+k-2}|$, $f(C_1) \in |J_0 + f_{p10(g+k-1)}|$ and $f(C_2) \in |J_0 + f_{p10(g+k)}|$. By further intersecting with $\tilde{\alpha}^{-1}(E_{g+k-3})$, we select maps with a chain of at least 3 copies of $J$, and so on.

In conclusion, every point of $M_{g+k-\delta}(S/J, L_{g+k})^{sm}$ lying in the intersection $\tilde{\alpha}^{-1}(\tilde{H}_1 \cap \bigcap_{i=2}^{k} E_{g+k-1})$ parametrizes a stable map $f$ with a chain of $k$ copies of $J$, that is, $f$ can be written as

$$f = f_0 \cup f_1 \cup \ldots \cup f_k : C_0 \cup J \cup \ldots \cup J \to S[m]_0 \cup R \cup \ldots \cup R = S[m+k]_0$$

for some $0 \leq m \leq g$ so that $[f_0] \in M_{g-\delta}(S/J, L_g)$ and $[f_i] \in M_{1}(R, J_0 + f_{p10(g+i)})/\mathbb{C}^*$ for $1 \leq i \leq k$. In particular, for $1 \leq i \leq k$ the class of $f_i$ is uniquely determined and this proves the existence of an injective morphism $r$ as in [3.3]. Its image, denoted by $M_{g-\delta}(S/J, L_g)^l$, is a closed substack of $M_{g-\delta}(S/J, L_g)$. Let $f_0 : C_0 \to S$ define a general point of $M_{g-\delta}(S/L_g)^{sm}$ and take $[f_i] \in M_{1}(R, J_0 + f_{p10(g+i)})/\mathbb{C}^*$ for $1 \leq i \leq k$; the map $f = f_0 \cup f_1 \cup \ldots \cup f_k$ is unramified and thus, by dimensional reasons using Propositions 2.4 and 3.6 one may check that $[f] \in M_{g+k-\delta}(S/J, L_{g+k})^{sm}$. This proves that $M_{g-\delta}(S/J, L_g)^l$ contains $M_{g-\delta}(S/J, L_g)^{sm}$.

By construction the stack $M_{g-\delta}(S/J, L_g)^l$ is homeomorphic to a fiber of $\psi \circ \tilde{\alpha}$ and thus, in order to prove that it is connected, it only remains to
show that the section of $\psi$ constructed in the proof of Theorem 3.5 lifts to a section of $\psi \circ \tilde{\alpha}$.

Take $X = (k + 1)J + C \in |L_{g-1}| \subset |L_{g+k}|$ with $C \in V_\delta(S, L_{g-1})$ as in the proof of Theorem 3.5. By dimensional reasons using Propositions 2.4 and 3.6 all stable maps $f = f_0 \cup f_1$ with $f_0 : \tilde{C} \to C \subset S$ the normalization of $C$ and $[f_1] \in \mathcal{M}_{k+1}(R/J_0, (k + 1)J_0 + f_{p_{l_0}(g)})^{sm}/\mathbb{C}^*$ are smoothable and thus contained in the fiber $\alpha^{-1}(X)$. Setting $\mathcal{M} := \mathcal{M}_{k+1}(R/J_0, (k + 1)J_0 + f_{p_{l_0}(g)})^{sm}$, this provides an inclusion

$$\mathcal{M}/\mathbb{C}^* \subset \tilde{\alpha}^{-1}(\pi_{k,\delta}^{-1}(X)) = \alpha^{-1}(X).$$

The open substack $\mathcal{M}^0 \subset \mathcal{M}$ parametrizing maps with irreducible domain is isomorphic to the open subset $U \subset |(k + 1)J_0 + f_{p_{l_0}(g)}| \cong \mathbb{P}^{k+1}$ parametrizing irreducible curves. It is trivial to check that $U$ is the complement of two hyperplanes and the restriction to $\mathcal{M}^0/\mathbb{C}^* \cong U/\mathbb{C}^*$ of

$$\tilde{\alpha}|_{\mathcal{M}/\mathbb{C}^*} : \mathcal{M}/\mathbb{C}^* \to \pi_{k,\delta}^{-1}(X) \cong \mathbb{P}^k$$

is an isomorphism onto its image. In particular, $\tilde{\alpha}|_{\mathcal{M}/\mathbb{C}^*}$ is birational and thus an isomorphism by Zariski’s Main Theorem. Since $\pi_{k,\delta}^{-1}(X)$ is a section of $\psi$, we get that $\mathcal{M}/\mathbb{C}^*$ is a section of $\psi \circ \tilde{\alpha}$ and we can conclude that all fibers of $\psi \circ \tilde{\alpha}$ are connected by considering the Stein factorization of $\psi \circ \tilde{\alpha}$ and again applying Zariski’s Main Theorem.

As concerns the last part of the statement, let

$$f = f_0 \cup f_1 \cup \ldots \cup f_n : C_0 \cup C_1 \cup \ldots \cup C_n \to S[n]_0$$

define a point of $\mathcal{M}_{g-\delta}(S/J, L_g)^\dagger \setminus \mathcal{M}_{g-\delta}(S/J, L_g)^{sm}$. Again by dimensional reasons using Propositions 2.4 and 3.6 for some $i$ the map $f_i$ contracts a component $C'_i$ of $C_i$. Since $\mathcal{M}_{g-\delta}(S/J, L_g)^\dagger \subset \mathcal{M}_{g+k-\delta}(S/J, L_{g+k})^{sm}$, the image curve $f(C_i)$ has a singularity at the point $f_i(C'_i)$ that is worse than a node by, e.g., [Va].

We will now define a moduli stack $\mathcal{V}_\delta(S/J, L_g)$ of stable relative $\delta$-nodal curves to expanded degenerations of $(S, J)$ with multiplicity 1 along the relative divisor $J$. We apply Li and Wu’s construction [LW] (cf. also [Li3] for a nice survey) of stacks of relative ideal sheaves with fixed Hilbert polynomial to obtain a moduli stack $|L_g|^{exp}$, whose closed points parametrize equivalence classes of curves $X = X_0 \cup X_1 \cup \ldots \cup X_n$ with finite automorphism group living in some expanded degeneration $S[n]_0$ of $S$, such that $X$ has no component contained in the singular locus of $S[n]_0$ or in the divisor $J_\infty$ on the $n$-th copy of $R$, and $X_0 \in |L_{g_0}|$ while $X_i \in |N_i(g)|$ for $1 \leq i \leq n$. Two such curves $X$ and $X'$ define the same point of $|L_g|^{exp}$ if they live in the same $S[n]_0$ and lie in the same orbit under the natural action of $(\mathbb{C}^*)^n$. The stack $|L_g|^{exp}$ is Deligne-Mumford, proper, separated and of finite type.
normalization of the Severi variety $V_\delta(S,J,L_g)$ to be the closure in $|L_g|^{\exp}$ of the Severi variety $V_\delta(S,J,L_g)$. Denoting by $\hat{M}_{g-\delta}(S/J,L_g)^{sm}$ the semi-normalization of $M_{g-\delta}(S/J,L_g)^{sm}$, the stack $\tilde{V}_\delta(S/J,L_g)$ can be alternatively described as the image of the natural map

$$\tilde{\mu} : \hat{M}_{g-\delta}(S/J,L_g)^{sm} \to |L_g|^{\exp}$$

sending a stable map to its image. The points of $\tilde{V}_\delta(S/J,L_g)$ thus parameterize curves $X = X_0 \cup X_1 \cup \ldots \cup X_n$ in $|L_g|^{\exp}$ whose normalization outside of the nodes $n_i := X_i \cap X_{i+1}$ is a nodal connected curve of arithmetic genus $\leq g - \delta$.

Let $M_{g-\delta}(S/J,L_g)^{\dagger}$ be as in the statement of Theorem 3.8 and denote by $\hat{M}_{g-\delta}(S/J,L_g)^{\dagger}$ its semi-normalization, which also admits a map

$$\mu^\dagger : \hat{M}_{g-\delta}(S/J,L_g)^{\dagger} \to |L_g|^{\exp}$$

sending a stable map to its image. We may compose $\mu^\dagger$ with the natural map $|L_g|^{\exp} \to |L_g|$. By Theorem 3.8, the image of this composition is $\tilde{V}_\delta(S,L_g)$ and this implies that the image of $\mu^\dagger$ is again $\tilde{V}_\delta(S,J,L_g)$.

**Proposition 3.9.** Fix $g \geq 2$ and $0 \leq \delta \leq g - 1$. Then the following hold.

(i) The stack $\tilde{V}_\delta(S/J,L_g)$ is connected and the same holds true for the relative normalization $\tilde{V}_\delta(S/J,L_g)^\dagger$ of $\tilde{V}_\delta(S/J,L_g)$ along $\tilde{V}_{\delta+1}(S/J,L_g)$.

(ii) Let $V$ and $W$ be two intersecting irreducible components of $\tilde{V}_\delta(S/J,L_g)^{\dagger}$ and let $Z$ be a component of their intersection; then $Z$ is generically reduced.

**Proof.** The stack $\tilde{V}_\delta(S/J,L_g)$ coincides with the image of $\mu^\dagger$ and is thus connected. We consider the divisor $\tilde{V}_{\delta+1}(S/J,L_g) \subset \tilde{V}_\delta(S/J,L_g)$. Since a general point $[X] \in \tilde{V}_{\delta+1}(S/J,L_g)$ parametrizes a nodal irreducible curve $X$ with $\delta + 1$ nodes, then $(\mu^\dagger)^{-1}([X]) \subset \hat{M}_{g-\delta}(S/J,L_g)^{sm}$ by the last statement in Theorem 3.8. Hence, $(\mu^\dagger)^{-1}([X])$ consists of the $\delta + 1$ partial normalizations of $X$; in particular, $M_{g-\delta}(S/J,L_g)^{\dagger}$ is generically smooth along $(\mu^\dagger)^{-1}([\tilde{V}_{\delta+1}(S/J,L_g)])$. Hence, $\mu^\dagger$ factors through a map

$$\mu^n : \hat{M}_{g-\delta}(S/J,L_g)^{\dagger} \to \tilde{V}_\delta(S/J,L_g)^{\dagger}$$

and point (i) directly follows from Theorem 3.8.

Let now $Z$ be as in point (ii). Since normal singularities are unibranch (cf., e.g., [Ko2]), then $Z$ is not contained in the inverse image of $\tilde{V}_{\delta+1}(S/J,L_g)$ under the normalization map $\tilde{V}_\delta(S/J,L_g)^{\dagger} \to \tilde{V}_\delta(S/J,L_g)$.

We first assume that a general point $\zeta$ of $Z$ parametrizes an irreducible curve, so that locally around $\zeta$ the morphism

$$\tilde{V}_\delta(S/J,L_g) \to \tilde{V}_\delta(S,L_g)$$

(derived as restriction of $|L_g|^{\exp} \to |L_g|$) is an isomorphism; the result thus follows from Proposition 2.1.
We now treat the case where a general point $\zeta$ of $Z$ parametrizes a curve $X = X_0 \cup \ldots \cup X_n \subset S[n]_0$ for some $n \geq 1$. More precisely, there exist $h = (h_0, \ldots, h_n) \in \mathbb{Z}^{n+1}$ and $g = (g_0, \ldots, g_n) \in \mathbb{Z}^{n+1}$ with $\sum_{i=0}^n h_i = g - \delta$, $\sum_{i=0}^n g_i = g$, $0 \leq h_0 \leq g_0$ and $1 \leq h_i \leq g_i$ for $i > 0$, such that $Z$ is contained in the substack $\mathcal{V}(h, g)$ of $\mathcal{V}(S/J, L_g)$ parametrizing curves $X = X_0 \cup X_1 \cup \ldots \cup X_n \subset S \cup R \cup \ldots \cup R = S[n]_0$

such that $X_i$ has arithmetic genus $g_i$ and geometric genus $h_i$. The substack $\mathcal{V}(h, g)$ lies in the intersection $\mathcal{Y}$ of $\mathcal{V}(S/J, L_g)$ with $n$ Cartier divisors and can be identified with the open substack $U \subset \mathcal{V}^{h_0}(S, L_{g_0}) \times \left[\mathcal{V}^{h_1}(R, N_1(g))/\mathbb{C}^*\right] \times \cdots \times \left[\mathcal{V}^{h_n}(R, N_n(g))/\mathbb{C}^*\right]$ parametrizing curves with no components in the singular locus of $S[n]_0$. Propositions 2.4 and 3.6 then imply that $\mathcal{V}(h, g)$ has codimension $n$ in $\mathcal{V}(S/J, L_g)$ and is generically reduced, thus proving that $Z$ is generically reduced if $\text{codim} \mathcal{Z} = n$. If instead $\text{codim} \mathcal{Z} > n$, then $\mathcal{V}' := \mathcal{V} \cap \mathcal{V}(h, g)$ and $\mathcal{W}' := \mathcal{W} \cap \mathcal{V}(h, g)$ are unions of components of $\mathcal{V}(h, g)$ and $Z$ is a component of the intersection $\mathcal{V}' \cap \mathcal{W}'$. Hence, $\mathcal{V}'$ and $\mathcal{W}'$ can be identified with open substacks of two products $\mathcal{V}_0' \times [\mathcal{V}_1'/\mathbb{C}^*] \times \cdots \times [\mathcal{V}_n'/\mathbb{C}^*]$ and $\mathcal{W}_0' \times [\mathcal{W}_1'/\mathbb{C}^*] \times \cdots \times [\mathcal{W}_n'/\mathbb{C}^*]$, where $\mathcal{V}_0', \mathcal{W}_0'$ are components of $\mathcal{V}^{h_0}(S, L_{g_0})$, while $\mathcal{V}_i', \mathcal{W}_i'$ are components of $\mathcal{V}^{h_i}(R, N_i(g))$ for $i \geq 1$. Hence, $Z$ can be identified with an open subset of an irreducible component of $(\mathcal{V}_0' \cap \mathcal{W}_0') \times [\mathcal{V}_1' \cap \mathcal{W}_1'/\mathbb{C}^*] \times \cdots \times [\mathcal{V}_n' \cap \mathcal{W}_n'/\mathbb{C}^*]$, which is generically reduced again by Propositions 2.4 and 3.6 (using the fact that a categorical quotient of a generically reduced object is still generically reduced by the universal property).

4. Connectedness on a general $K3$ surface

In this section we will show how Theorems 3.5 and 3.8 imply analogous results on a general polarized $K3$ surface.

Let $S$ and $S'$ be two surfaces both obtained as blow-ups of $\mathbb{P}^2$ at two 9-uples of general points, $p_1, \ldots, p_9$ and $p'_1, \ldots, p'_9$, respectively. We assume that the anticanonical divisors on $S$ and $S'$ are represented by the same elliptic curve $J$ and that the relation $N_{J/S} \simeq N_{J/S}'$ holds. We glue $S$ and $S'$ along $J$ so that $p'_9 = p_{10}(g)$. Since $N_{J/S} \simeq O_J(p_{10}(h) - p_{10}(h - 1))$ for every $h \geq 0$, and the same holds for $S'$, our assumptions yield $p_{10}(h) = p_{10}(g - h)'$ for every $0 \leq h \leq g$. In particular, all the pairs $(L_h, L'_{g-h}) \in \text{Pic}(S) \times \text{Pic}(S')$ define equivalent polarizations on $Y_0 := S \cup J S'$, as they differ only by the twist for a multiple of $(N_{J/S}, N_{J/S}')$; we set $L := [(L_h, L'_{g-h})]$. The surface $Y_0$ is a stable $K3$ surface of type II and thus occurs as the central fiber of a flat map $\chi : \mathcal{Y} \rightarrow \mathbb{D}$.
over a disc whose general fiber $Y_t$ is a smooth K3 surface of genus $g$ (cf. [Fr, Prop. 2.5, Thm. 5.10]). Furthermore, for every $0 \leq h \leq g$ the family $\mathcal{Y}$ comes equipped with a relative line bundle $\mathcal{L}(h)$ restricting to the genus $g$ polarization $L_t$ on a general fiber $Y_t$ and to the polarization $(L_h, L'_{g-h})$ on $Y_0$. Since the line bundles $\mathcal{L}(h)$ only differ by a twist for a multiple of some component of the central fiber, they are all equivalent and we call $\mathcal{L}$ their class. We remark that this degeneration in the particular case where $S = S'$ and the points $p_1, \ldots, p_9 \in \mathbb{P}^2$ are the base locus of a general pencil of plane cubics is the one used in [MPT] §4.

We denote by

$$\chi_{\delta} : \mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L}) \to \mathbb{D}$$

the moduli stack of connected stable maps to expanded degenerations of $\chi$ constructed in [Li1, Li2]. Over a point $t \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$, the fiber of $\chi_{\delta}$ is simply the moduli stack $\mathcal{M}_{g-\delta}(Y_t, L_t)$ of ordinary stable maps on $Y_t$, while the fiber over 0 is the stack $\mathcal{M}_{g-\delta}(Y_{0}^{\exp}, L)$ parametrizing stable maps (in the sense of J. Li) to some expanded target degeneration of $Y_0$ of the form:

$$Y_0[n]_0 := S \cup_f R \cup_f \ldots \cup_f R \cup_f S'$$

for some $n \geq 0$. As already used in [MPT], a stable map to such an expanded degeneration can be split in a non-unique way into relative stable maps to $(S, J)$ and $(S', J)$. In particular, $\mathcal{M}_{g-\delta}(Y_{0}^{\exp}, L)$ can be written as a non-disjoint union

$$(4.1) \quad \mathcal{M}_{g-\delta}(Y_{0}^{\exp}, L) = \bigcup_{h_1 + h_2 = g-\delta \atop h_1, h_2 \leq g} \mathcal{M}_{h_1}(S/J, L_{g_1}) \times \mathcal{M}_{h_2}(S'/J, L'_{g_2}),$$

where each factor in the above decomposition can be realized as a Cartier divisor on $\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})$. Let

$$\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})^{sm} \to \mathbb{D}$$

be the substack of $\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})$ whose fiber over $t \in \mathbb{D}^*$ is the substack $\mathcal{M}_{g-\delta}(Y_t, L_t)^{sm}$ of $\mathcal{M}_{g-\delta}(Y_t, L_t)$ parametrizing smoothable stable maps, and set $\mathcal{M}_{g-\delta}(Y_{0}^{\exp}, L)^{sm} := \mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})^{sm} \times_\mathbb{D} 0$.

Similarly, we denote by

$$e_{\delta} : |\mathcal{L}|^{\exp} \to \mathbb{D}$$

the good degeneration of the relative linear system $|\mathcal{L}|^* \to \mathbb{D}^*$, which is a particular case of good degenerations of relative Hilbert schemes introduced and studied in [LV]. The space $|\mathcal{L}|^{\exp}$ is a Deligne-Mumford stack, proper over $\mathbb{D}$ and of finite type. A fiber of $e_{\delta}$ over a general $t \in \mathbb{D}$ is isomorphic to the linear system $|L_t|$ on $Y_t$. Points of the central fiber, that we denote by $|L|^{\exp}$, parametrize equivalence classes of curves $X = X_0 \cup X_1 \cup \ldots \cup X_9 \cup X'$ in some expanded target degenerations $Y_0[n]_0$ of $Y_0$ with no components in the singular locus of $Y_0[n]_0$. Since Severi varieties may be defined functorially, for any fixed $0 \leq \delta \leq g$ there is a $\chi$-relative Severi variety $s_{\delta} : V_\delta(\mathcal{Y}, \mathcal{L})^* \to \mathbb{D}^*$ such that the fiber over $t \in \mathbb{D}^*$ is the Severi
variety $V_{\delta}(Y_1,L_1)$. We denote by $\overline{V_{\delta}(Y^{\exp},L)}$ the closure of $V_{\delta}(Y,L)^{\ast}$ in $|\mathcal{L}|^{\exp}$ and by

$$\overline{\pi_{\delta}} : \overline{V_{\delta}(Y^{\exp},L)} \to \mathbb{D}$$

the natural morphism. The stack $\overline{V_{\delta}(Y^{\exp},L)}$ can be alternatively realized as the image of the natural map

$$\mathcal{M}_{g-\delta}(Y^{\exp},L)^{sm} \to |\mathcal{L}|^{\exp}.$$

The analogue of (4.1) for the central fiber $V_{\delta}(Y_0^{\exp},L) := \overline{V_{\delta}(Y^{\exp},L)} \times_{\mathbb{D}} 0$ of $\overline{\pi_{\delta}}$ is then stated in the following result.

**Lemma 4.1.** The stack $\overline{V_{\delta}(Y_0^{\exp},L)}$ decomposes in the following non-disjoint union:

$$V_{\delta}(Y_0^{\exp},L) = \bigcup_{g_1+g_2=g \atop \delta_1+\delta_2=\delta} \overline{V_{\delta_1}(S/J,L_{g_1})} \times \overline{V_{\delta_2}(S'/J,L'_{g_2})}. \tag{4.2}$$

Each factor in the decomposition (4.2) appears with multiplicity 1 and defines a Cartier divisor in $\overline{V_{\delta}(Y^{\exp},L)}$.

**Proof.** As concerns the inclusion $\subset$, we recall that a general point in any irreducible component of $\overline{V_{\delta_1}(S/J,L_{g_1})}$ (respectively, $\overline{V_{\delta_2}(S'/J,L'_{g_2})}$) parametrizes an irreducible curve $C \in V_{\delta_1}(S,L_{g_1})$ (respectively, $C' \in V_{\delta_2}(S',L'_{g_2})$). By gluing $C$ and $C'$ along their intersection point $p_{10}(g_1) = p_{10}(g_2)'$, with $J$, one obtains a curve $X = C \cup C' \subset Y_0 = S \cup S'$ with $\delta_1 + \delta_2$ nodes outside of $X$; there is no obstruction to deforming such an $X$ outside of the central fiber of $\overline{\pi_{\delta}}$ and this proves $\subset$.

We now prove the opposite inclusion $\supset$. Let $V$ be a component of the central fiber $\overline{V_{\delta}(Y_0^{\exp},L)}$ of $\overline{\pi_{\delta}}$. Then $V$ has dimension $g - \delta$ by upper semi-continuity along with the fact that $\mathbb{D}$ is 1-dimensional. A general point of $V$ parametrizes a curve $X \subset Y_0[s]_n$ for some $n \geq 0$ such that $X \in |\mathcal{L}|^{\exp}$ and the normalization of $X$ outside of the singular locus of $Y_0[s]_n$ has arithmetic genus $h \leq g - \delta$. Propositions 2.4 and 3.6 then yield $n = 0$ and $h = g - \delta$, so that $X = C \cup C'$ with $C \in \overline{V^{g_1-h_1}(S,L_{g_1})} = \overline{V_{\delta_1}(S,L_{g_1})}$ and $C' \in \overline{V^{g_2-h_2}(S',L'_{g_2})} = \overline{V_{\delta_2}(S',L'_{g_2})}$ for some integers $0 \leq \delta_1 \leq g_1$, $0 \leq \delta_2 \leq g_2$ such that $g_1 + g_2 = g$ and $\delta_1 + \delta_2 = \delta$. This proves $\supset$.

The last part of the statement is a consequence of the same property for the decomposition (4.1), that follows from [Li2] since $L_h \cdot J = 1$ for every $h \geq 0$. \hfill $\square$

**Proposition 4.2.** If $0 \leq \delta \leq g - 1$, every component $V$ of $\overline{V_{\delta}(Y_0^{\exp},L)}$ can be connected to $|L_0| \times \overline{V_{g-\delta}(S'/J,L'_{g})}$ through a sequence of irreducible components

$$V = V_0, V_1, \ldots, V_r \subset |L_0| \times \overline{V_{g-\delta}(S'/J,L'_{g})}$$

such that for all $0 \leq i \leq r - 1$ the following hold:

(i) the intersection $V_i \cap V_{i+1}$ has codimension 1;
In particular, the intersection \( V_i \cap V_{i+1} \) is generically reduced.

**Proof.** By Lemma [4.4] there exist integers \( 0 \leq \delta_1 \leq g_1, 0 \leq \delta_2 \leq g_2 \) such that \( g_1 + g_2 = g \) and \( \delta_1 + \delta_2 = \delta \) and \( V_0 := \mathcal{V} = \mathcal{W}_0 \times \mathcal{W}_0' \) for some irreducible components \( \mathcal{W}_0 \) of \( \overline{\mathcal{V}}_g(S/J, L_g(\delta_1)) \) and \( \mathcal{W}_0' \) of \( \overline{\mathcal{V}}_g(S'/J, L'_{g_2+1}) \).

If \( g_1 - \delta_1 \geq 1 \), then \( \mathcal{W}_0 \) contains in codimension 1 points that parametrize curves \( C = C_0 \cup C_1 \subset S[1]_0 \) with \( C_0 \in \mathcal{V}_g(S, L_{g_1-1}) \) and \( C_1 \simeq J \in \{ g_0 + f_{p_{10}}(\delta_1) \} \). For any \( C'_0 \in \mathcal{V}_g(S', L'_{g_2+1}) \) the nodal curve \( C := C_0 \cup C_1 \cup C'_0 \) also defines a point of \( \overline{\mathcal{V}}_g(S/J, L_{g_1-1}) \times \overline{\mathcal{V}}_g(S'/J, L'_{g_2+1}) \) and this proves that \( V_0 := \mathcal{W}_0 \times \mathcal{W}_0' \) can be connected to a component

\[
\mathcal{V}_1 = \mathcal{V}_1 \times \mathcal{W}_1' \subset \overline{\mathcal{V}}_g(S/J, L_{g_1-1}) \times \overline{\mathcal{V}}_g(S'/J, L'_{g_2+1})
\]

so that the intersection \( V_0 \cap V_1 \) satisfies conditions (i)-(ii) in the statement. By repeating the same argument \( g_1 - \delta_1 \) times, we find a sequence

\[
\mathcal{V} = V_0, V_1, \ldots, V_{g_1-\delta_1}
\]

of irreducible components of \( \overline{\mathcal{V}}_g(\mathbf{Y}_0^{\text{exp}}, L) \) such that \( V_i = \mathcal{W}_i \times \mathcal{W}_i' \) is an irreducible component of \( \overline{\mathcal{V}}_g(S/J, L_{g_1-1}) \times \mathcal{V}_g(S'/J, L'_{g_2+1}) \) and that conditions (i)-(ii) hold for the intersection \( V_i \cap V_{i+1} \).

We now start from

\[
\mathcal{V}_{g_1-\delta_1} = \mathcal{W}_{g_1-\delta_1} \times \mathcal{W}_{g_1-\delta_1}' \subset \overline{\mathcal{V}}_g(S/J, L_{g_1-1}) \times \overline{\mathcal{V}}_g(S'/J, L'_{g_2+1})
\]

and notice that \( \mathcal{W}_{g_1-\delta_1}' \subset \overline{\mathcal{V}}_g(S'/J, L'_{g_1-\delta_1}) \) contains in codimension 1 points that parametrize curves \( D = \tilde{J} \cup D'_0 \subset S[1]_0' \), where \( D'_0 \in \{ \delta - 1 \} \) and \( \tilde{J} \) is an irreducible elliptic curve such that \( \tilde{J} \in V^1(R_1, \delta + 1) J_0 + f_{p_{10}}(\delta+1) \). For every rational curve \( D_0 \in \mathcal{V}_g(S, L_{\delta_1}) \) the curve \( D := D_0 \cup \tilde{J} \cup D'_0 \) also defines a point of \( \overline{\mathcal{V}}_g(S/J, L_{\delta+1}) \times |L'_{g_1-\delta_1}|^{\text{exp}} \); this proves that \( \mathcal{V}_{g_1-\delta_1} \) can be connected to a component

\[
\mathcal{V}_{g_1-\delta_1+1} := \mathcal{W}_{g_1-\delta_1+1} \times \mathcal{W}_{g_1-\delta_1}' \subset \overline{\mathcal{V}}_g(S/J, L_{\delta+1}) \times \overline{\mathcal{V}}_g(S'/J, L'_{\delta+1})
\]

so that (i) and (ii) hold.

Finally, we use the fact that the component \( \mathcal{W}_{g_1-\delta_1+1} \subset \overline{\mathcal{V}}_g(S/J, L_{\delta+1}) \) contains in codimension 1 curves of the form \( E_g + \overline{\mathcal{J}} \), where \( E_g \) is the ninth exceptional divisor on \( S \) (and thus the only curve in the linear system \( |L_0| \)) and \( \overline{\mathcal{J}} \) is an irreducible curve such that \( \overline{\mathcal{J}} \in V^1(R_1, \delta + 1) J_0 + f_{p_{10}}(\delta+1) \). For any curve \( F'_0 \in |L'_{g_1-\delta_1}| \), the divisor \( E_9 + \overline{\mathcal{J}} + F'_0 \in Y_0[1]_0 \) also defines a point
of $|L_0| \times \overline{V}_{\delta}(S'/J, L'_g)$. This proves that $V_{g_1-\delta_1+1}$ is connected to a component $V_{g_1-\delta_1+2}$ of $|L_0| \times \overline{V}_{\delta}(S'/J, L'_g)$ so that (i)-(ii) hold for $V_{g_1-\delta_1+1} \cap V_{g_1-\delta_1+2}$.

It only remains to prove that conditions (i)-(iii) imply that the intersection $V_i \cap V_{i+1}$ is generically reduced. Conditions (i)-(ii) together with Propositions 2.4 and 3.6 yield that every component of $V_i \cap V_{i+1}$ is birational to an open substack of

$$\overline{V}_{\delta_0}(S, L_{g_0}) \times \left[ \overline{V}^h(1, R_1 g_1 + t_0 (g_0 + g_1))/\mathbb{C}^* \right] \times \overline{V}_{\delta_0}(S', L'_{g_0})$$

for some integers $0 \leq \delta_0 \leq g_0$, $0 \leq \delta'_0 \leq g'_0$, $1 \leq h_1 \leq g_1$ such that $g_0 + g_1 + g'_0 = g$ and $\delta_0 + g_1 - h_1 + \delta'_0 = \delta$. The generic reducedness of $V_i \cap V_{i+1}$ thus follows from Propositions 2.4 and 3.6.

**Theorem 4.3.** Let $(Y, L)$ be a general primitively polarized $K3$ surface of genus $g \geq 2$ and fix $0 \leq \delta \leq g - 1$. Then the closure of the Severi variety $\overline{V}_{\delta}(Y, L) \subset |L|$ is connected and the same holds true for the relative normalization $\overline{V}_{\delta}(Y, L)^n$ of $\overline{V}_{\delta}(Y, L)$ along $\overline{V}_{\delta+1}(Y, L)$.

**Proof.** Let $\overline{s}_{\delta} : \overline{V}_{\delta}(Y_{0}, L) \to \mathbb{D}$ be the good degeneration of the relative Severi variety to the family $\chi : Y \to \mathbb{D}$ at the beginning of this section. We denote by $s^n_{\delta} : \overline{V}_{\delta}(Y_{0}, L)^n \to \mathbb{D}$ the relative normalization of $\overline{V}_{\delta}(Y_{0}, L)^n$ along $\overline{V}_{\delta+1}(Y_{0}, L)^n$. In particular, a general fiber of $s^n_{\delta}$ is the normalization of $\overline{V}_{\delta}(Y_i, L_i)$ along $\overline{V}_{\delta+1}(Y_i, L_i)$, while the central fiber is the normalization of $\overline{V}_{\delta}(Y_{0}, L)$ along $\overline{V}_{\delta+1}(Y_{0}, L)$. We need to show that a general fiber of $s^n_{\delta}$ is connected.

First of all, we note that the central fiber $\overline{V}_{0}(Y_{0}, L)$ of $\overline{s}_{\delta}$ is generically reduced. Indeed, this follows directly from Lemma 4.1 and Proposition 2.4(iii). Obviously, the same holds true for the central fiber of $s^n_{\delta}$, thus implying that two components of a general fiber of $s^n_{\delta}$ remain distinct also on the central fiber.

By Proposition 4.2, every component of the central fiber $\overline{V}_{\delta}(Y_{0}, L)$ of $\overline{s}_{\delta}$ can be connected to $|L_0| \times \overline{V}_{g-\delta}(S'/J, L'_g)$ through a sequence of irreducible components $\mathcal{V} = V_0, V_1, \ldots, V_r \subset |L_0| \times \overline{V}_{g-\delta}(S'/J, L'_g)$ such that the intersections $V_i \cap V_{i+1}$ are generically reduced and not contained in $\overline{V}_{\delta+1}(Y_{0}, L)$. In particular, the latter property also implies that every component $\mathcal{V}^n$ of the relative normalization $\overline{V}_{\delta}(Y_{0}, L)^n$ of $s^n_{\delta}$ can be connected to $|L_0| \times \overline{V}_{g-\delta}(S'/J, L'_g)$ through a sequence of irreducible components $\mathcal{V}^n = V^n_0, V^n_1, \ldots, V^n_r \subset |L_0| \times \overline{V}_{g-\delta}(S'/J, L'_g)^n$ such that the intersections $V^n_i \cap V^n_{i+1}$ are generically reduced. Since $|L_0|$ is a point, Proposition 3.9 implies that $|L_0| \times \overline{V}_{g-\delta}(S'/J, L'_g)$ is connected and that the intersection of two components of $|L_0| \times \overline{V}_{g-\delta}(S'/J, L'_g)$ is generically reduced.

We use this in order to conclude that a general fiber of $s^n_{\delta}$ is connected, too. If these were not the case, there would exist two irreducible components $Z_1$ and $Z_2$ of $\overline{V}_{\delta}(Y_{0}, L)$ (whose restriction to the central fiber of $s^n_{\delta}$ we
denote by \( Z_1 \) and \( Z_2 \), respectively) such that \( Z_1 \cap Z_2 = Z_1 \cap Z_2 \); in particular, a general point of \( Z_1 \cap Z_2 \) should be a nonreduced point of the central fiber by, e.g., [St, 76.36.8]. However, we may assume that this is not the case because the above discussion yields that any two components of the central fiber can be connected through a sequence of components whose intersection is generically reduced.

\[ \square \]

5. IRREDUCIBILITY ON A GENERAL K3 SURFACE

In this section, we will focus on the Severi problem for general polarized K3 surfaces. As we have already proved that Severi varieties of positive dimension are connected, the irreducibility problem can be approached by investigating how two irreducible components may intersect.

**Proposition 5.1.** Let \((Y, L)\) be a general primitively polarized K3 surface of genus \( g \geq 2 \) and fix \( 0 \leq \delta \leq g - 1 \). The intersection of two irreducible components of \( V_\delta(Y, L) \), if nonempty, has pure codimension 1.

**Proof.** Since \((Y, L)\) is general, we may assume that all curves in \(|L|\) are integral and that \( V_\delta(Y, L) = V_{g-\delta}(Y, L) \). The proof proceeds as the one of Proposition 2.7 and is actually easier because all curves in \(|L|\) are integral. In this case \( I = D(\tilde{\phi}) \) and the same proof as that of Lemma 2.8 implies that the locus in \( I \) where the fibers of the projection \( t : I \rightarrow |L| \) have positive dimension has dimension \( \leq g - \delta - 2 \).

\[ \square \]

**Theorem 5.2.** Let \((Y, L)\) be a general primitively polarized K3 surface of genus \( g \geq 4 \) and fix \( 0 \leq \delta \leq g - 4 \). Then, the Severi variety \( V_\delta(Y, L) \) is irreducible.

**Proof.** Since \((Y, L)\) is general, we may assume that all curves in \(|L|\) are integral. By Theorem 1.3, \( V_\delta(Y, L) \) and its relative normalization \( \overline{V_\delta(Y, L)}^\alpha \) along \( V_{\delta+1}(Y, L) \) are connected. If reducible, \( V_\delta(Y, L) \) contains two irreducible components \( V, W \) whose intersection \( V \cap W \) is nonempty and thus of codimension 1 by Proposition 5.1. Since \( V_{\delta}(Y, L)^{\alpha} \) is still connected, we may further assume that \( V \cap W \) is not contained in \( V_{\delta+1}(Y, L) \). It is therefore enough to show that no codimension 1 component of the singular locus of \( V_\delta(Y, L) \) may contain such an intersection.

Let \( Z \) be a component of \( \text{Sing} \overline{V_\delta(Y, L)} \) such that \( Z \) is not contained in \( V_{\delta+1}(Y, L) \) and \( Z \) has codimension 1. Let \( C \in Z \) be a general point and denote by \( f \) the composition of the normalization map \( \nu : \tilde{C} \rightarrow C \) with the inclusion of \( C \) in \( Y \). Since \( Z \) has dimension \( g - \delta - 1 \), Theorem 1.8 and Proposition 1.5 imply that \( \tilde{C} \) is a smooth irreducible curve of genus either \( g - \delta \) or \( g - \delta - 1 \), and the latter case does not occur because \( Z \) is not contained in \( V_{\delta+1}(Y, L) \).

Hence, \( \tilde{C} \) has genus \( g - \delta \) and, by generality, we can assume that all points in a dense open subset of \( Z \) parametrize curves with the same singularities.
as $C$. We may thus apply $[AC$, p. 26] as in the proof of Proposition 2.4 to obtain
\begin{equation}
g - \delta - 1 = \dim Z \leq h^0(\overline{N}_f),
\end{equation}
where $\overline{N}_f \simeq \omega_{\tilde{C}}(-R)$ with $R$ being the ramification divisor of $f$. Inequality (5.1) then yields $\deg R \leq 2$.

If $\deg R = 0$, then $f$ is unramified and $N_f = \overline{N}_f = \omega_{\tilde{C}}$, If $\deg R = 1$, then $C$ has only one ordinary cusp and, denoting by $Q$ the point of $\tilde{C}$ mapping to it, we have $N_f = \omega_{\tilde{C}}(-Q) \oplus \mathcal{O}_Q$. In both cases one has $h^0(N_f) = g - \delta$ and thus $f$ defines a smooth point of the moduli space of genus $g - \delta$ stable maps $M_{g-\delta}(Y, L)$, and more precisely of the locus $M_{g-\delta}(Y, L)^{sm}$ parametrizing smoothable stable maps. Let $\mu$ be the morphism from the semi-normalization of $M_{g-\delta}(Y, L)^{sm}$ to $\overline{V}_{g-\delta}(Y, L) = \overline{V}_g(Y, L)$. Locally around $f$ the morphism $\mu$ is injective; indeed, the inverse image under $\mu$ of an irreducible curve of geometric genus exactly $g - \delta$ is the only point defined by the composition of its normalization map with its inclusion in $S$. Therefore, the smoothness of $M_{g-\delta}(Y, L)^{sm}$ at $f$ yields that $\overline{V}_\delta(S, L)$, if singular at $C$, has a unibranched singularity there. In particular, $W$ cannot lie in the intersection of two irreducible components of $\overline{V}_\delta(S, L)$.

We now treat the remaining case $\deg R = 2$, where (5.1) implies that $\tilde{C}$ is hyperelliptic and $R$ is a divisor in the $g^1_2$. By [KLM] Rmk. 5.6, curves in $|L|$ with hyperelliptic normalization of any fixed geometric genus $\geq 2$ move in dimension 2 and hence $g - \delta - 1 = \dim W \leq 2$ yielding a contradiction as soon as $\delta \leq g - 4$. \hfill $\Box$

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IRREDUCIBILITY OF SEVERI VARIETIES ON K3 SURFACES

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