LIE ALGEBROIDS AND CARTAN’S METHOD OF EQUVALENCE

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Abstract. Élie Cartan’s general equivalence problem is recast in the language of Lie algebroids. The resulting formalism, being coordinate and model-free, allows for a full geometric interpretation of Cartan’s method of equivalence via reduction and prolongation. We show how to construct certain normal forms (Cartan algebroids) for objects of finite-type, and are able to interpret these directly as ‘infinitesimal symmetries deformed by curvature.’

Details are developed for transitive structures but rudiments of the theory include intransitive structures (intransitive symmetry deformations). Detailed illustrations include subriemannian contact structures and conformal geometry.

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1. A new setting for Cartan’s method

This paper has its origins in attempts to understand, in as invariant a language as possible, Élie Cartan’s assertion that finite-type geometric structures are ‘symmetries deformed by curvature.’ Having identified structures well-suited to this viewpoint — namely Lie algebroids equipped with a suitably compatible linear connection, called Cartan algebroids — we turn to the problem of realizing this model in practice. To do so requires us to revisit Cartan’s method of equivalence, and even to reformulate the basic ‘problem of equivalence’ that it addresses. The advantages of this reformulation are both theoretical and computational, as we shall demonstrate.

The introduction to this paper is in two parts. In this first section we recast Cartan’s equivalence problem in the language of Lie algebroids and formulate specific objectives for the paper. We describe the basic geometric objects with which the paper will be concerned: certain infinitesimal models of geometric structures, and Cartan algebroids, which amount to normal forms thereof.

In Sect. 2 we outline those elements of Cartan’s method which allow us, in the Lie algebroid setting, to associate, with any given object of finite-type, an intrinsically defined Cartan algebroid. Limitations of the method and an outline of the remainder of the paper will also be given there.

1.1. The equivalence problem. Cartan’s method for determining when two objects are equivalent is a procedure for finding the symmetries of a given object. The method applies to an astonishing variety of objects: differential equations, polynomials and variational problems; tensor structures on smooth manifolds, such as Riemannian, conformal, symplectic, complex and almost Kähler structures; differential operators and associated smooth manifold structures, such as affine and projective structures; and so on. For an introduction to the method and a survey of applications see [15, 8, 10, 2, 14].

What makes the method so general is that the secondary data — in terms of certain secondary data of universal form, this data encoding essential information about the objects, rather than in terms of the objects themselves. In Cartan’s original approach the secondary data is a collection of one-forms (a coframe), defined pointwise up to extra ‘group parameters.’ In subsequent reformulations, the secondary data is a $G$-structure (see, e.g., [17]) or an exterior differential system (see, e.g., [2]). While in practice the construction of the secondary data can seem ad hoc, it can often be given the following interpretation: it is a geometrization of the PDE’s one would write down to find the symmetries (self-equivalences) of the given object, whether or not these symmetries actually exist.

In our Lie algebroid reformulation of Cartan’s method, the secondary data — here called an infinitesimal geometric structure — can generally be understood as a geometrization of the PDE’s one writes down to find the infinitesimal symmetries of the given object. This geometrization is coordinate-free and does not depend on any choice of group, group quotient, or other fixed model. We motivate the formal definition with the following example.

1.2. The 1-symmetries of a Riemannian metric. Consider the problem of finding the infinitesimal symmetries of a Riemannian metric $\sigma$, i.e., its Killing fields, defined on a smooth $n$-dimensional manifold $M$. Call the 1-jet of a vector field $V$ on $M$, evaluated at some point $m \in M$, a 1-symmetry of $\sigma$, whenever $\sigma$ has, at
m ∈ M, vanishing Lie derivative along V. Then the collection of all 1-symmetries is a subbundle \( g \subset J^1(TM) \) and V is a Killing field if and only if its first-order prolongation \( J^1V \) is a section of the form \( J^1V \) for some V are called holonomic.

1. The Killing fields of \( \sigma \) are in one-to-one correspondence with the holonomic sections of \( g \).

Moreover, as we show later, \( \sigma \) can be recovered from \( g \) up to a constant factor, so that little is lost by restricting attention to \( g \).

The important observation to make here is that \( g \) is the tangent bundle \( TM \), and its first jet \( J^1(TM) \). In the language associated with these objects, we have:

2. The bundle \( g \subset J^1(TM) \) of 1-symmetries is the isotropy subalgebroid of \( \sigma \) under the representation determined by the adjoint representation of \( J^1(TM) \).

The terms ‘isotropy’ and ‘adjoint representation’ are natural generalizations to Lie algebroids of familiar Lie algebra notions. The adjoint representation of a Lie algebroid is described in Sect. 3. Isotropy subalgebroids are defined in Sect. 5.

A Lie algebroid over a smooth manifold \( M \) is a vector bundle on \( M \), together with a Lie bracket on its space of sections, and a vector bundle morphism called the anchor, satisfying certain conditions making Lie algebroids generalizations of both tangent bundles (\( # : TM \rightarrow TM \) the identity) and Lie algebras (\( M \) a single point). The bundle of \( k \)-jets of sections of any Lie algebroid is itself a Lie algebroid.

Lie algebroids over \( M \) also generalize the infinitesimal actions of Lie algebras on \( M \) (see Sect. 1.7 below), in which case the image of the anchor is a distribution tangent to the foliation of \( M \) by orbits; an arbitrary Lie algebroid is always tangent to some foliation, accordingly called orbits; a Lie algebroid with surjective anchor is called transitive.

1.3. Infinitesimal geometric structures and their symmetries.

Infinitesimal geometric structures, as defined below, generalize the vector bundle of 1-symmetries of a Riemannian metric:

**Definition.** Let \( t \) be any Lie algebroid over \( M \) (the tangent bundle in the simplest case). Then an infinitesimal geometric structure on \( t \) is any subalgebroid \( g \subset J^1(t) \). In practice the infinitesimal geometric structure associated with a given object — whether it be a tensor, an operator, or whatever — is constructed as the isotropy subalgebroid of an appropriate Lie algebroid representation. Further examples, both transitive and intransitive, are given in Sect. 5. Infinitesimal geometric structures are this paper’s principal objects of investigation.

The structure kernel of \( g \) is the kernel \( h \subset T^*M \otimes TM \) of the restriction of the projection of \( g \) to \( t \), this morphism being the anchor of \( g \) when \( t = TM \). This is an analogue of the structure group \( G \) of a \( G \)-structure. The structure kernel of \( g \) is a subalgebroid whenever it has constant rank.

The image of \( g \) is the image of \( a \). If \( a : g \rightarrow t \) is surjective, we call \( g \) a surjective infinitesimal geometric structure. If \( t = TM \), then ‘surjective’ is synonymous with ‘transitive.’ For example, the bundle \( g \subset J^1(TM) \) of 1-symmetries of a Riemannian metric is a surjective infinitesimal geometric structure on \( TM \).
metric is surjective. We will see that every Poisson structure has an associated infinitesimal geometric structure $g \subset J^1(T^*M)$ which is surjective but generally not transitive. A simple example of an infinitesimal geometric structure failing to be surjective or transitive is the joint isotropy subalgebroid of a Riemannian metric and a vector field $V$ with non-degenerate energy $\|V\|^2$.

Structures sometimes viewed as transitive are actually intransitive when the notion is invariantly formulated. For example, almost complex structures are generically intransitive structures.

Associated with any $G$-structure on $M$ is a corresponding infinitesimal geometric structure on $TM$, but this structure is always surjective (equivalently, transitive).

Here now, in Lie algebroid language, is an analogue of Cartan’s problem of equivalence:

**Equivalence Problem.** Given smooth manifolds $M_1$ and $M_2$ and infinitesimal geometric structures $g_1 \subset J^1(TM_1)$ and $g_2 \subset J^1(TM_2)$, does there exist a diffeomorphism $\phi: M_1 \to M_2$, with associated tangent map $T\phi: TM_1 \to TM_2$, such that the corresponding Lie algebroid isomorphism $J^1(T\phi)$ maps $g_1$ isomorphically onto $g_2$.

**Remark.** If we want to formulate a more general equivalence problem replacing $TM_1$ and $TM_2$ with general Lie algebroids $t_1$ and $t_2$ and allowing for arbitrary Lie algebroid morphisms $t_1 \to t_2$ instead of $TM_1 \to TM_2$, or we must restrict the class of infinitesimal geometric structures considered so that morphisms of ‘coordinate change type’ can be defined. However, for the restricted purposes of the present paper, further elaboration on this point will be unnecessary.

1.4. Cartan’s method. Having formulated the equivalence problem in terms of appropriate secondary data, Cartan’s method attempts to put the data into an appropriate normal form. In the original one-form approach, for example, this normal form is a coframe, on a possibly larger space, but with group parameters eliminated. See, e.g., [8] or [15].

The normalizing algorithm involves two fundamental operations, known as reduction and prolongation. If the secondary data is regarded as a system of PDE’s, then reduction amounts to identifying coordinate changes that decouple certain of the equations from the others, these latter being rendered redundant. Prolongation means increasing the number of variables by introducing derivatives of the independent variables as new independent variables; new equations are added to account for the equality of mixed partial derivatives.

When the normalizing procedure succeeds, the normal form obtained delivers certain basic invariants, and certain ‘derived’ invariants can be constructed from these. One may test for the equivalence of two objects by comparing the values taken on by the invariants. (Specifically, one compares the associated ‘classifying manifold’; see, e.g., [15, Chapter 8]). Objects for which the normalizing procedure succeeds have, at most, finite-dimensional symmetry, and so are called finite-type objects. The basic invariants of a finite-type object may be understood as obstructions to the existence of a maximal set of symmetries (self-equivalences); they are embodied in the ‘curvature’ of the associated normal form, when this normal form...
is understood as a ‘symmetry deformed by curvature.’ In common practice, this interpretation is often obscured, however.

For objects of infinite-type, the normalizing procedure fails to terminate and an altogether different criterion for equivalence must be applied. In finite-type objects can be identified by applying Cartan’s ‘involutivity’ test and are described extensively in [2, 10]. They will not be studied here.

1.5. The symmetries of infinitesimal geometric structures.

Analogous to the Killing fields of a Riemannian structure are the symmetries of an infinitesimal geometric structure \( g \subset J^1t \). These are those sections \( V \) of \( t \) whose prolongations \( J^1V \subset J^1t \) are sections of \( g \). Evidently, the symmetries of \( g \) are in one-to-one correspondence with the holonomic sections of \( g \subset J^1t \)— those sections that are prolongations of something. Symmetries are necessarily sections of the image of \( g \) and are closed under the Lie algebroid bracket.

We will not be presenting a complete solution to the equivalence problem here. Rather, our main focus is the following:

**Obstruction Problem.** *Given an infinitesimal geometric structure \( g \subset J^1t \), find the obstructions to the existence of symmetries of \( g \), in the sense defined above.*

We now turn our attention to the normal forms that we shall be constructing out of infinitesimal geometric structures. We begin by describing such a normal form explicitly in the case of Riemannian geometry, and by showing concretely how curvature invariants enter as the obstruction to symmetry.

1.6. A normal form for Riemannian geometry.

Let \( g \subset J^1(t^*) \oplus TM \) be the bundle of 1-symmetries of a Riemannian metric \( \sigma \), as described in 1.2 above. Note that the Levi-Cevita connection \( \nabla \) associated with \( \sigma \) may be regarded as a splitting of a canonical exact sequence,

\[
0 \to T^*M \otimes TM \to J^1(TM) \to TM \to 0.
\]

We have a corresponding exact sequence,

\[
0 \to \mathfrak{h} \to g \to TM,
\]

where \( \mathfrak{h} \subset T^*M \otimes TM \) denotes the \( \mathfrak{o}(n) \)-bundle of space endomorphisms, which is \( \nabla \)-invariant. By the subbundle of \( J^1(TM) \) lies inside \( g \) and we obtain a linear connection \( \nabla^{(1)} \) by...
With the help of the Bianchi identities for linear connections,

$$\text{curv } \nabla^{(1)}(U_1, U_2)(V \oplus \phi) = 0 \oplus (\nabla_V \text{curv } \nabla) \quad (1)$$

implying that $\nabla^{(1)}$ is flat if and only if $\text{curv } \nabla$ is both $\nabla$-invariant and $\mathfrak{h}$-invariant. Now $\mathfrak{h}$-invariance implies, by purely algebraic arguments, that $\text{curv } \nabla$ has only a scalar component; $\nabla$-invariance then implies constant scalar curvature. Whence from (1) one recovers the standard criterion for maximal local homogeneity of a Riemannian manifold.

1.7. Cartan algebroids: symmetries deformed by curvature.

A linear connection $\nabla$ on a Lie algebroid $\mathfrak{g}$ is a Cartan connection if it is suitably compatible with the Lie algebroid structure [1]. The pair $(\mathfrak{g}, \nabla^{(1)})$ is then a Cartan algebroid. The formal definition and basic properties are reviewed in Sect. 4.

In the Riemannian example above, the pair $(\mathfrak{g}, \nabla^{(1)})$ is a Cartan algebroid, and we saw that $\mathfrak{g} \cong \mathfrak{g}_0 \times M$, for some Lie algebra $\mathfrak{g}_0$. In fact, whenever $\mathfrak{g}_0$ is any Lie algebra, acting smoothly on a manifold $M$, the trivial bundle $\mathfrak{g}_0 \times M$ inherits the structure of a (trajectory) algebroid, and the trivial flat connection is a Cartan connection. Conversely, any Cartan algebroid with a flat Cartan connection is locally an action algebroid (Theorem 4.6, Sect. 4). It is in this sense that Cartan algebroids are infinitesimal symmetries deformed by curvature. Moreover, the orbits of the Cartan algebroid may be regarded as deformations of orbits of a symmetry.

In [1] we described how Cartan algebroids may be viewed as model-free, and possibly intransitive versions of classical Cartan geometries, and mentioned other alternative models contained in the literature. Recently, Crampin [6] has delineated the relationship between transitive Cartan algebroids and adjoint tractor bundles [4], which like Cartan algebroids are infinitesimal objects, but unlike them are based on a transitive model fixed a priori.

One consequence of choosing a model-free approach is worth repeating here. Generally, ‘curvature’ has referred to the local deviation from an underlying model — typically $\mathbb{R}^n$ or a homogeneous space $G/H$ — and curvature merely measures the local deviation from some maximally symmetric space. From this point of view, Euclidean space, hyperbolic space, and spheres, are all ‘flat’ Riemannian manifolds. If Cartan’s method determines that a space is flat in the weaker sense, then the particular flat space one has at hand is simply part of the method’s output. In model-based approaches model ‘mutation’ (or something similar) may be needed to detect all possible cases [16].

In Appendix A, we explain how flat Cartan algebroids may be regarded as infinitesimal analogues of Lie pseudogroups of transformations, and discuss the global analogues of arbitrary Cartan algebroids, Cartan groupoids, which may be regarded as ‘curved’ Lie pseudogroups.
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Theorem. Let \( g \subset J^1 t \) be an infinitesimal geometric structure and assume the projection \( \alpha: g \to t \) has constant rank. Then \( g \subset J^1 t \) if and only if it is surjective and has structure kernel \( \mathfrak{h} \), any linear connection \( \nabla \) on \( M \) such that \( \nabla \sigma = 0 \).

Generators are indispensable in explicit computations.

The following crucial observation is not difficult to establish. (For a proof see Sect. 6.)

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Now suppose \( t \) is a Lie algebroid and \( g \subset J^1 t \) an infinitesimal geometric structure and assume the

projection \( \alpha: g \to t \) has constant rank. Then we call \( \nabla \) a generator of \( g \subset J^1 t \) if \( s(t_1, t_2) \), image of \( g \). Generators are certain ‘preferred connections’ but need not be unique. For example, the Levi-Civita connection is a generator, but so is the tautological one. (For details, see Sect. 6.)

In particular, \( \text{curv} \nabla \) is then the local obstruction to maximal symmetry.

When geometric structures do not satisfy the hypotheses of Theorem 2.1, one tries to correct this with an appropriate sequence of prolongation and reduction.

2.2. Prolongation. The prolongation of an infinitesimal geometric structure \( J^1 t \) is a natural ‘lift’ of \( g \) to a subset \( g^{(1)} \subset J^1 g: J^1(J^1 t) \), and the existence of a natural inclusion

\[ g^{(1)} := J^1 g \cap J^2 g \]

It turns out that \( g^{(1)} \) is an infinitesimal geometric structure with constant rank. Most importantly, there is a one-to-one correspondence between symmetries of \( g \) and symmetries of \( g^{(1)} \), furnishing

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Proposition. A section \( W \subset t \) is a symmetry of \( g \) if and only if it is surjective and has structure kernel \( \mathfrak{h} \). Generators are indispensable in explicit computations.

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Proposition. A section \( W \subset t \) is a symmetry of \( g^{(1)} \).

We prove this proposition in Sect. 8.
2.3. Reduction. Let $g \subset J^1 t$ be an infinitesimal geometric structure of $g$ we shall mean any subalgebroid $g' \subset g$ with the same symmetries as $g$. By a reduction of $g$ we shall mean any subalgebroid $g' \subset g$ with the same symmetries as $g$; it suffices to check that symmetries of $g$ are symmetries of $g'$. In contrast to prolongation, there is no unique way to construct reductions. Notice, however, that if $g' \subset g$ is a reduction and $g'' \subset g$ merely a subalgebroid satisfying $g' \subset g'' \subset g$, then $g''$ is automatically a reduction of $g$ also. We say $g''$ is a cruder reduction than $g'$.

We now describe the most important reduction techniques: elementary reduction and $\Theta$-reduction.

2.4. Elementary reduction. Returning to Cartan's method described above, we emphasize that transitivity is not a hypothesis of Theorem 2.1 (and that Cartan algebroids can be intransitive). Rather, one requires surjectivity. If an infinitesimal geometric structure $g \subset J^1 t$ is not surjective, we may attempt to make it so by passing to the elementary reduction $g_1$ of $g$ by definition, $g_1 := g \cap J^1 t_1$, where $t_1 \subset t$ denotes the image of $g$. Assuming constant rank, they are subalgebroids. In particular, $g_1$ is an infinitesimal geometric structure. Moreover, one easily proves:

**Proposition.** If the elementary reduction $g_1$ of $g$ has constant rank then it is a reduction of $g$ in the sense above. If $g$ is surjective then $g_1 = g$ if and only if $g$ is contained in $J^1 t_1$ and is surjective as an infinitesimal geometric structure on $t_1$.

Because surjectivity is built into the definition of reductions never appears in that setting. Elementary reduction is described further in Sect.7, together with a cruder alternative called image reduction.

2.5. $\Theta$-reduction. If an infinitesimal geometric structure $g \subset J^1 t$ is already surjective but has a non-trivial structure kernel, then, with a view to applying Theorem 2.1, one can try to shrink the structure kernel by prolonging (see above). However, the prolongation $g^{(1)}$ generally fails to be surjective itself. While one might attempt to correct this by turning to the elementary reduction of $g^{(1)}$, there is an alternative that is computationally more attractive. One anticipates the lack of surjectivity of $g^{(1)}$ by first replacing $g$ by its $\Theta$-reduction. By definition, this is the image of $g^{(1)} \subset J^1 g$, i.e., the set $g^{(1)}_1 := p(g^{(1)}) \subset g$, where $p: J^1 g \to g$ is the natural projection. The point is that $\Theta$-reductions can be computed without directly computing the larger space $g^{(1)}$. In Sect.9 we prove:

**Proposition.** If $g^{(1)}_1$ and $g^{(1)}$ have constant rank, then the $\Theta$-reduction $g^{(1)}_1$ of $g$ is a reduction of $g$ in the sense of 2.3. One has $g^{(1)}_1 = g$ if and only if $g^{(1)}$ is surjective.

Classically, 'reduction' has usually meant reduction by torsion, a notion we define in 8.5 for arbitrary surjective infinitesimal geometric structures. If $t = TM$, then reduction by torsion coincides with $\Theta$-reduction. More generally, however, torsion reduction is a cruder reduction technique. In some cases $\Theta$-reduction is not only more efficient but also more computationally convenient; conformal geometry (Sect.12) is a case in point. While $\Theta$-reductions are always defined, we have restricted their detailed analysis to the case of surjective structures over transitive Lie algebroids. This analysis appears in Sect.11.
2.6. **A specific normalizing algorithm and its limitations.**

A specific algorithm for constructing a Cartan algebroid out of an infinitesimal geometric structure of finite-type. First, we define two auxiliary procedures.

By Proposition 2.4, the following procedure, which we shall call \textit{surjectify} $g$, forces $g$ to be surjective:

\begin{verbatim}
do while $g$ is not surjective
    replace $g$ with $g_1$ (elementary reduction)
end do.
\end{verbatim}

Next, we let \textit{strongly surjectify} $g$ denote the following procedure making $g$ and $g^{(1)}$ simultaneously surjective (by Propositions 2.4 and 2.5):

\begin{verbatim}
do while $g^{(1)}$ is not surjective
    surjectify $g$
    replace $g$ with $g^{(1)}_1$ ($\Theta$-reduction)
    surjectify $g$
end do.
\end{verbatim}

To describe an implementation of this procedure it evidently suffices to describe $\Theta$-reduction in the special surjective case.

One might attempt to normalize an infinitesimal geometric structure using elementary reduction and prolongation alone. In practice, however, it is generally easier to apply the following algorithm:

\begin{verbatim}
surjectify $g$
repeat until stop encountered
    if $h = 0$ apply Theorem 2.1 and stop
    strongly surjectify $g$
    if $h = 0$ apply Theorem 2.1 and stop
    replace $g$ with $g^{(1)}$ (prolongation)
end repeat.
\end{verbatim}

Notice that prolongation is delayed as long as possible in which the above algorithm can fail.

Firstly, an execution of \textit{surjectify} $g$ or \textit{strongly surjectify} $g$ could fail because $g$, at some iteration of these procedures’ do-while loops, loses the constancy of its rank. While prolongation of $g$ might resolve this kind of singularity (by recovering rank constancy), this requires a prolongation theory for 'variable rank Lie algebroids' (or Lie pseudoalgebras) which is not provided here. Similarly, $g$ might lose rank constancy at some iteration of the repeat-until loop of the main algorithm.

Even if all singularities are successfully resolved, it may happen that no stop is ever encountered in the repeat-until loop, which then becomes perpetual. However, such cases can be detected by applying (a Lie algebroid version of) Cartan’s involutivity test; they occur for objects of infinite-type which are not described here.

Another possibility for failure concerns $\Theta$-reduction. In practice, it seems rather complicated to implement without making the added assumption that the base algebroid $t$ of $g \subset J^1 t$ is transitive. One way to handle intransitivity and singularities might be to restrict, in some way, the infinitesimal geometric structure $g$ to each orbit of $t$. This restriction will sit over a transitive base (this being a Lie algebroid over the orbit) but is not simply the pullback in the category of Lie algebroids, for one wants an infinitesimal geometric structure over the base, not merely a Lie...
algebroid. Also, one needs to understand how conclusions regarding the restricted structure combine with transverse information to solve the original problem. Fortunately, a splitting theory for Lie algebroids exists [7] and this possibly reduces the transverse problem to the case of an isolated singularity (zero-dimensional orbit). None of this is explored here either.

If the Cartan algorithm above succeeds it ends in an application of Theorem 2.1, this delivering a Cartan algebroid whose parallel sections are in natural one-to-one correspondence with the symmetries of $\mathfrak{g}$. We then say that $\mathfrak{g}$ admits an associated Cartan algebroid.

2.7. Paper outline. In Sect. 3 we review basic Lie algebroid notions and establish attendant notation. In particular, we describe the generalizations of linear (Koszul) connections afforded by Lie algebroids, these connections amounting to deformations of Lie algebroid representations, which are also defined. We recall the definition of the adjoint representation of $J^1 \mathfrak{g}$ on $\mathfrak{g}$, and write down the bracket on $J^1 \mathfrak{g}$ explicitly. We introduce the notion of associated connections, which are ubiquitous throughout, and developed further in Sect. 6.

Sect. 4 summarizes features of Cartan algebroids established in [1], stating, in particular, the result that Cartan algebroids are deformations of infinitesimal symmetries (Theorem 4.6).

Sect. 5 gives many examples of infinitesimal geometric structures, arising as the isotropy subalgebroids associated with various structures in differential geometry. We also explain how to associate an infinitesimal geometric structure with a Cartan algebroid or a classical $G$-structure. From our description of affine structures, it will be clear how one may associate an infinitesimal geometric structure with an arbitrary (but suitably regular) differential operator on $M$. We demonstrate in practice how one computes the image of an infinitesimal geometric structure, without resorting to local coordinate calculations.

Associated with an infinitesimal geometric structure $\mathfrak{g} \subset J^1 t$, with structure kernel $\mathfrak{h}$ and image $t_1$, is an exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to t_1 \to 0.$$  

A generator $\nabla$ of $\mathfrak{g}$, as defined in 2.1 above, above amounts to a splitting of this sequence, determining an identification $\mathfrak{g} \cong t_1 \oplus \mathfrak{h}$, which are ubiquitous throughout, and developed further in Sect. 6.

Sect. 7 shows how to explicitly compute an elementary reduction of $\mathfrak{g}$ once a generator is known. A simple application to functions on a Riemannian three-manifold illustrates the technique.

Sections 8 and 9 are important theoretical parts of the paper, describing prolongations, torsion reductions, and $\Theta$-reductions, and the relationships between these. An immediate application is Theorem 9.4, which furnishes general conditions under which an infinitesimal geometric structure $\mathfrak{g} \subset J^1 t$ is its own associated Cartan algebroid, Riemannian geometry being a case in point. Keen to illustrate the result in this and other concrete examples, we include additional detail in the special case $t = TM$ (Theorem 9.5) that is not established until Sect. 11. The reader preferring
a more linear presentation may skip to Sect. 11 immediately after 9.4, returning to 9.5 and the remainder of the paper thereafter.

We have not attempted substantially novel applications of Cartan’s method in the present work. In particular, our applications to intransitive phenomena are fairly superficial. We hope to correct this deficiency of making comparisons with other approaches, subriemannian contact three-manifolds in Sect. 10, is to be found in [9, 14]. In addition to constructing obstructions to symmetry, we go on to construct the invariant differential operators.

In Sect. 11 we return to prolongation, explaining in detail how to ‘prolong’ a generator, and hence how to compute prolongations in practice. This analysis includes the general case \( t \neq TM \), but we must assume \( t \) is transitive to avoid complications.

A detailed section on conformal geometry, Sect. 12, illustrates the more general prolongation results.

3. Preliminary notions

For an introduction to Lie groupoids and algebroids, see [3] or [12]. All constructions in this paper are made in the \( C^\infty \) category.

Notation. We use \( \text{Alt}^k(V) \cong \Lambda^k(V^*) \) and \( \text{Sym}^k(V) \cong S^k(V^*) \) of \( \mathbb{R} \)-valued alternating and symmetric \( k \)-linear maps on a vector space \( V \). Similar notation applies to the tensor algebra of a vector bundle \( E \) over \( M \). If \( \sigma \) is a section of \( E \), then this is indicated by writing \( \sigma \in \Gamma(E) \) or \( \sigma \subset E \). Thus \( \sigma \subset \text{Alt}^2(TM) \otimes E \) means \( \sigma \) is an \( E \)-valued differential two-form on \( M \).

3.1. Lie algebroids. A Lie algebroid over \( M \) consists of a vector bundle \( g \) over \( M \), a Lie bracket \([ \cdot, \cdot ]\) on the space of sections \( \Gamma(g) \), together with a vector bundle morphism \( \# : g \rightarrow TM \), called the anchor. One requires that the Leibnitz identity,

\[
[X, fY] = f[X, Y] + df(Y)X,
\]

where \( f \) is an arbitrary smooth function. The anchor \( \# \) is compatible with the Jacobi-Lie bracket on vector fields, and...
3.2. The definition of $\mathfrak{gl}(E)$ for a vector bundle $E$

Lie algebra $\mathfrak{g}$ is a vector space $E$, together with $\mathfrak{g} \to \mathfrak{gl}(E) := \text{Hom}(E, E)$. Turning now to the generalization of $\mathfrak{gl}(E)$ relevant to Lie algebroid representations, let $E$ be a vector bundle over the same base $M$, and consider the exact sequence

$$0 \to T^*M \otimes E \hookrightarrow J^1E \to E \to 0.$$

Here $T^*M \otimes E \hookrightarrow J^1E$ is the inclusion which, as a map on sections, sends $2d\sigma \otimes \tau$ to $fJ^1\tau - J^1(f\sigma)$. Applying $\text{Hom}(\cdot, E)$ to the sequence, and identifying $\text{Hom}(T^*M \otimes E, E)$ with $TM \otimes \text{Hom}(E, E)$, we obtain a second exact sequence

$$0 \to \text{Hom}(E, E) \hookrightarrow \text{Hom}(J^1E, E) \twoheadrightarrow TM \otimes \text{Hom}(E, E) \to 0.$$

Noticing that there is natural inclusion $TM \hookrightarrow TM \otimes \text{Hom}(E, E)$ via the map $\nu \mapsto \nu \otimes \text{id}$, we define $\mathfrak{gl}(E) \subset \text{Hom}(J^1E, E)$ to be the preimage of $TM$ under the surjective arrow $\twoheadrightarrow$, and obtain a third exact sequence

$$0 \to \text{Hom}(E, E) \hookrightarrow \mathfrak{gl}(E) \twoheadrightarrow TM \otimes \text{Hom}(E, E) \to 0.$$

**Proposition.** Regard each section $D$ of $\text{Hom}(J^1E, E)$ as a differential operator $D: \Gamma(E) \to \Gamma(E)$. Then:

1. A section $D \subset \text{Hom}(J^1E, E)$ lies in $\mathfrak{gl}(E)$ if and only if there exists a vector field $V$ such that

$$D(f\sigma) = fD\sigma + df(V)$$

for all sections $\sigma$ of $E$ and functions $f$; in that case $V = \#D$.

2. The operator commutator bracket,

$$[D_1, D_2]_{\mathfrak{gl}(E)} \sigma := D_1D_2\sigma - D_2D_1\sigma,$$

makes $\mathfrak{gl}(E)$ into a Lie algebroid with anchor $\#$.

**Remark.** $\mathfrak{gl}(E)$ is in fact a realization of the Lie algebroid $\mathbb{GL}(E)$ of isomorphisms between fibres of the vector bundle $E$. So elements of $\mathfrak{gl}(E)$ have the interpretation of 'infinitesimal moving frames.' For details, see [12] (where $\mathfrak{gl}(E)$ is denoted $\mathcal{D}(E)$).

3.3. Lie algebroid representations and $\mathfrak{g}$-connections

A representation of a Lie algebroid $\mathfrak{g}$ on $E$ is a morphism $\mathfrak{g} \to \mathfrak{gl}(E)$ of Lie algebroids. When $M$ is a single point one recovers the usual representations of a Lie algebra. Deforming the representation notion we arrive at the following:

**Definition.** Let $\mathfrak{g}$ be any Lie algebroid over $M$. A $\mathfrak{g}$-connection on a vector bundle $E$ over $M$ is a vector bundle morphism $\nabla: \mathfrak{g} \to \mathfrak{gl}(E)$ that is not required to be a Lie algebroid morphism, but nevertheless required to respect the anchors, $\nabla \# = \#\nabla$.

2. An opposite sign convention is adopted in our paper [1] and elsewhere. The present convention avoids unpleasant sign changes in 3.6(2) and 3.7(4).
Suppose $X$ is a section of $\mathfrak{g}$. When the section $\nabla(X)$ be viewed as a differential operator, we instead write
\[ \nabla X, \quad \nabla_X V := \nabla(X(J_1 V)). \]
In view of the preceding characterization of the sections of $\mathfrak{g}L(E)$ as derivations, we have the Leibnitz identity
\[ \nabla_X (f\sigma) = f\nabla_X \sigma + df(\#X)\sigma; \]
Conversely:

**Proposition.** Every vector bundle morphism $\nabla: \mathfrak{g} \to \text{Hom}(J_1 E, E)$ that is Leibnitz in the above sense is a $\mathfrak{g}$-connection.

If $\nabla$ is a $\mathfrak{g}$-connection, then the formula
\[ \text{curv } \nabla (X,Y) := [\nabla(X), \nabla(Y)]_{\mathfrak{g}}, \]
defining the Lie algebroid curvature of the map $\nabla$,
\[ \text{curv } \nabla (X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \]
The $\mathfrak{g}$-connection $\nabla$ is a $\mathfrak{g}$-representation when $\text{curv } \nabla = 0$.

**Example.** If $\mathfrak{g}$ is a Lie algebroid and $E \subset \mathfrak{g}$ is a subalgebroid contained in the kernel of its anchor then a canonical representation by $\rho X Y := [X,Y]_{\mathfrak{g}}$. Important cases in point are the kernel of the anchor itself, and the structure kernel of an infinitesimal geometric structure, when these have constant rank.

3.4. **Linear connections.** Using the language of the preceding discussion, a linear connection $\nabla$ on $E$ is just a $TM$-connection on $L^1 E$, when $\nabla$ is flat. It is an elementary fact that there is a one-to-one correspondence with the splittings $s: E \to J_1 E$ given by
\[ 0 \to T^*M \otimes E \hookrightarrow J_1 E \to 0 \]
The splitting associated with a linear connection $\nabla$ is given by
\[ s\sigma = J_1 \sigma + \nabla \sigma; \]
Here $\nabla \sigma \subset T^*M \otimes E$ is defined by $(\nabla \sigma)(V) := \nabla_V \sigma.$
3.6. The adjoint representation. The generalization of Lie algebra representations to a Lie algebroid $\mathfrak{g}$ is not a self-representation of $J^1\mathfrak{g}$ on $\mathfrak{g}$. This representation is well-defined by
\[
\text{ad}^g_{J^1X} Y = [X, Y].
\]
Using the identity
\[
[J^1X, J^1Y]_{\mathfrak{g}} = J^1\mathfrak{g} \begin{bmatrix}0 & 0 \\ \mathfrak{g} & 0 \end{bmatrix}
\]
one shows that $\text{ad}^g$ is indeed a representation (a morphism of Lie algebroids).

We note that
\[
\text{ad}^g X = \phi(#X);
\]
for all sections $\phi \subset T^*M \otimes \mathfrak{g} \subset J^1\mathfrak{g}$. If $a: \mathfrak{g} \to T^*M$, then one has the identity
\[
a(\text{ad}^g_{\xi} X) = \text{ad}^h_{(J^1a)\xi}(aX);
\]

3.7. The bracket on $J^1(\cdot)$ of a Lie algebroid $\mathfrak{g}$. $J^1\mathfrak{g}$ is implicitly defined by the requirement that this representation, we now describe this bracket concretely.

Although the exact sequence
\[
0 \to T^*M \otimes \mathfrak{g} \to J^1\mathfrak{g} \to \Gamma(T^*M \otimes \mathfrak{g}) \to \Gamma(J^1\mathfrak{g})
\]
possesses no canonical splitting, the corresponding sequence
\[
0 \to \Gamma(T^*M \otimes \mathfrak{g}) \to \Gamma(J^1\mathfrak{g})
\]
is split by $J^1: \Gamma(\mathfrak{g}) \to \Gamma(J^1\mathfrak{g})$, delivering a canonical identification
\[
\Gamma(J^1\mathfrak{g}) \cong \Gamma(\mathfrak{g}) \oplus \Gamma(T^*M \otimes \mathfrak{g})
\]
Under this identification, the Lie algebra $\Gamma(J^1\mathfrak{g})$ is uniquely determined sections $X, \phi$ where $\phi$ is the anchor of $J^1\mathfrak{g}$.

In addition to having the adjoint representation of $J^1\mathfrak{g}$ on $TM$, given by the composite
\[
J^1\mathfrak{g} \xrightarrow{J^1\#} J^1(TM) \xrightarrow{\text{ad}^g} \mathfrak{g},
\]
i.e., $J^1 X \cdot V = [#X, V]$; so we can construct a natural representation of $J^1\mathfrak{g}$ on $TM$, given by the composite
\[
J^1\mathfrak{g} \xrightarrow{J^1\#} J^1(TM) \xrightarrow{\text{ad}^T M} \mathfrak{g}.
\]
To prove the proposition one uses 3.6(1) and the fact that section $s$ of $T^*M \otimes g$ are finitely generated by those of the form $df \otimes X = fJ_1X - J_1(fX)$.

3.8. Dual connections, torsion, and associated connections.

Let $g$ be a Lie algebroid and $\nabla$ a $g$-connection on itself. We define the dual of $\nabla$ to be the $g$-connection $\nabla^*$ on $g$ defined by

$$\nabla^*_X Y := \nabla_Y X + [X,Y].$$

One has ‘duality’ in the sense that $\nabla^{**} = \nabla$.

The torsion of $\nabla$ is the section, $\text{tor } \nabla$, of $\text{Alt}^2(g) \otimes g$ measuring the difference between $\nabla$ and its dual:

$$\text{tor } \nabla (X,Y) := \nabla_X Y - \nabla^*_X Y = \nabla_X Y - \nabla^*_X Y - [X,Y].$$

The torsion or curvature of $\nabla$ can be expressed in terms of the torsion and curvature of $\nabla^*$ (and, by duality, vice versa):

**Proposition.** Let $\nabla$ be a $g$-connection on $g$, and assume $\text{tor } \nabla = - \text{tor } \nabla^*$.

\[
\begin{align}
\text{tor } \nabla &= - \text{tor } \nabla^* \\
\text{curv } \nabla (X,Y)Z &= (\nabla^*_Z \text{tor } \nabla^*) (X,Y) + \text{curv } \nabla^* (Z,Y)X.
\end{align}
\]

We now introduce two important connections generalizing the dual of a linear connection on $TM$. They are examples of associated connections, defined more generally in 6.3.

Let $\nabla$ be an arbitrary linear (i.e., $TM$-) connection on a Lie algebroid $g$. The associated $g$-connection on $g$ is defined by

$$\bar{\nabla}_X Y = \nabla_{\#Y}X + [X,Y]_g;$$

The associated $g$-connection on $TM$ is defined by

$$\bar{\nabla}_X V = \# \nabla V + [\#X, V]_{TM};$$
4.1. **Action algebroids.** Let $\mathfrak{g}_0$ be a finite-dimensional Lie algebra with bracket $[\cdot, \cdot]_{\mathfrak{g}_0}$ acting smoothly on a manifold $M$. There exists a homomorphism $\rho: \mathfrak{g}_0 \to \Gamma(TM)$. We may regard $\mathfrak{g}_0$ as the Lie algebra of infinitesimal symmetries. The trivial bundle $\mathfrak{g} := \mathfrak{g}_0 \otimes TM$ is a Lie algebroid equipped with a Lie algebroid structure. This is the associated action algebroid.

The anchor of this Lie algebroid is the ‘action map’ #:

$$
\tau(\xi, m) := [\xi, \rho(\xi)(m)]_{\mathfrak{g}_0};
$$

And let $\nabla$ denote the canonical flat connection on $\mathfrak{g}$ on $\Gamma(\mathfrak{g}_0 \times M)$ is defined by

$$
[X, Y] := \nabla_\#X Y - \nabla_\#Y X;
$$

Notice that if $\bar{\nabla}$ denotes the associated $\mathfrak{g}$-connection.

4.2. **Cartan connections.** Let $\nabla$ be a linear connection on a Lie algebroid $\mathfrak{g}$ on $\Gamma(\mathfrak{g}_0 \times M)$ is defined by

$$
[X, Y] := \nabla_\#X Y - \nabla_\#Y X;
$$

Then $\nabla$ is a *Cartan connection* if the correspondence

$$
\begin{align*}
J \sigma & := J^1 \sigma + \nabla \sigma \\
0 & \to T^*M \otimes \mathfrak{g} \hookrightarrow J^1 \mathfrak{g} \to \mathfrak{g}
\end{align*}
$$

is a Lie algebroid morphism. A *Cartan algebroid* is a Lie algebroid equipped with a Cartan connection. A *morphism* of Cartan algebroids is a Lie algebroid morphism preserving the underlying Lie algebroid structure.

It follows immediately from the definition that $\nabla$ is a *Cartan connection*.

In particular, every Cartan algebroid $\mathfrak{g}$ has a canonical flat connection $\nabla$, associated with a Cartan connection $\nabla$, a *Cartan connection*.

4.3. **Cocurvature.** Associated with an arbitrary linear connection $\nabla$ on $\mathfrak{g}$, we call its *cocurvature*. This is a section of $\text{Alt}^2(\mathfrak{g}, T^*M \otimes \mathfrak{g})$ measuring the lack of 'compatibility' of $\nabla$ with respect to all smooth functions (and consequently with respect to all smooth functions) on $\mathfrak{g}$.

Notice that if $\bar{\nabla}$ denotes the associated $\mathfrak{g}$-connection.

Then

$$
\begin{align*}
\nabla & \text{ is a Cartan connection if and only if } \\
\text{corollary below.}
\end{align*}
$$
(4) For any sections $X, Y, Z \subset g$ and $V \subset TM$ one has
\[
\text{cocurv } \nabla(X, Y)\#Z = -\text{curv } \bar{\nabla}(X, Y)Z,
\]
\[
\#\text{cocurv } \nabla(X, Y)V = -\text{curv } \bar{\nabla}(\#X, \#Y)V,
\]
where $\bar{\nabla}$ denotes the associated $g$-connection on $g$ in the first formula, and on $TM$ in the second.

(5) In particular, if $g = TM$, then
\[
\text{cocurv } \nabla = -\text{curv } \bar{\nabla},
\]
where $\bar{\nabla}$ denotes the dual linear connection on $TM$.

As simple consequences of (4) we have:

**Corollary.**

(6) Suppose $g$ is transitive. Then $\nabla$ is a Cartan connection if and only if the associated $g$-connection $\bar{\nabla}$ on $g$ is flat.

(7) Suppose $g$ has an injective anchor. Then $\nabla$ is a Cartan connection if and only if the associated $g$-connection $\bar{\nabla}$ on $TM$ is flat.

Although we shall make no use of the fact here, it is worth remarking that a Cartan connection $\nabla$ on a transitive Lie algebroid $g$ is uniquely determined by the corresponding self-representation $\bar{\nabla}$; see [1, Proposition 6.1].

### 4.4. Basic examples of Cartan algebroids.

We now list some elementary examples of Cartan algebroids. Example (7) explains the choice of name ‘Cartan algebroid.’

(1) Every action algebroid $g_0 \times M$, equipped with its canonical flat connection $\nabla$, is a Cartan algebroid. Locally this is the only flat example. See 4.6 below.

(2) As we sketch in Appendix A, every Lie pseudogroup of transformations in $M$ has a flat Cartan algebroid as its infinitesimalization.

(3) According to Proposition 4.3(5) a linear connection is a Cartan connection if and only if its dual $\nabla^*$ is flat, i.e., is an infinitesimal parallelism on $M$. By duality, every Cartan connection arises as the dual of some infinitesimal parallelism. See also 5.4.

(4) If $M$ is a Lie group, then the flat linear connection on $TM$ corresponding to left (or right) trivialization is, as a Cartan connection on $TM$,
4.5. **The symmetric part of a Cartan algebroid.**

An arbitrary Cartan algebroid $\mathfrak{g}$ has a canonical subalgebroid isomorphic to an action Lie algebroid. Indeed, let $\nabla$ denote the Cartan connection and let $\mathfrak{g}_0 \subset \Gamma(\mathfrak{g})$ be the subspace of $\nabla$-parallel sections, which is finite-dimensional. Then vanishing curvature ensures that $\mathfrak{g}_0 \subset \Gamma(\mathfrak{g})$ is a Lie subalgebra, and we obtain an action of $\mathfrak{g}_0$ on $M$ given by

$$\mathfrak{g}_0 \times M \to TM, \quad (X, m) \mapsto \#X(m).$$

Equipping the action algebroid $\mathfrak{g}_0 \times M$ with its canonical flat connection, we obtain a morphism of Cartan algebroids,

$$\mathfrak{g}_0 \times M \to \mathfrak{g}, \quad (X, m) \mapsto X(m). \tag{1}$$

Assuming $M$ is connected, this morphism is injective because $\nabla$-parallel sections vanishing at a point vanish everywhere. We call the image of the monomorphism (1) the symmetric part of $\mathfrak{g}$.

4.6. **Curvature as the local obstruction to symmetry.**

A Cartan algebroid $\mathfrak{g}$ is globally flat if it is isomorphic to an action algebroid, or, equivalently, if it coincides with its symmetric part. We call $\mathfrak{g}$ flat if every point of $M$ has an open neighborhood $U$ on which the restriction $\mathfrak{g}|_U$ is globally flat. The following theorem shows that a Cartan algebroid may be viewed as an infinitesimal symmetry deformed by curvature.

**Theorem.** Let $\mathfrak{g}$ be a Cartan algebroid with Cartan connection $\nabla$, defined over a connected manifold $M$. Then $\mathfrak{g}$ is flat if and only if $\text{curv} \nabla = 0$. When $M$ is simply-connected, flatness already implies global flatness.

In the globally flat case the bracket on the Lie algebra $\mathfrak{g}_0$ of $\nabla$-parallel sections is given by

$$[\xi, \eta]_{\mathfrak{g}_0} = \text{tor} \nabla(\xi, \eta),$$

where $\nabla$ denotes the associated representation of

$$\nabla_X Y = \nabla_# Y X + [X, Y].$$

**Proof.** The necessity of vanishing curvature is immediate. To establish the assertions in the first paragraph it suffices to show that 4.5(1) is an isomorphism whenever $\text{curv} \nabla = 0$ and $M$ is simply-connected. Indeed, in that case $\nabla$ determines a trivialization of the bundle $\mathfrak{g}$ in which constant sections correspond to the $\nabla$-parallel sections of $\mathfrak{g}$ — that is, to elements of $\mathfrak{g}_0$. In particular, $\mathfrak{g}_0 \times M$ and $\mathfrak{g}$ will have the same rank, implying the monomorphism 4.5(1) is an isomorphism.

The formula (1) holds in the globally flat case because it holds for any action algebroid, as is readily established. □

**Example.** Every Lie group possesses a dual pair of flat linear connections $\nabla$, $\nabla^*$ corresponding to the left and right trivializations of the tangent bundle (see 4.4(4) above). Conversely, whenever a simply-connected manifold $M$ supports a linear connection $\nabla$ on $TM$ such that $\nabla$ and its dual $\nabla^*$ are simultaneously flat, then $\nabla$ is a flat Cartan connection on $TM$ and the theorem above delivers an isomorphism

**3** In [1] we used symmetric and locally symmetric in place of globally flat and flat, respectively.
$TM \cong \mathfrak{g}_0 \times M$, where $\mathfrak{g}_0$ is the Lie algebra of $\mathfrak{g}$.

The adjoint representation of $\mathfrak{g}$ amounts to a $\mathfrak{g}_0$-valued Mauer-Cartan form on $\mathfrak{g}$, determining a representation of $\mathfrak{g}$.

Isotropy subalgebroids, of certain jet-bundle representations. In the case of Riemannian geometry, it is the isotropy of a rank-one subbundle; sections of this bundle are closed under the bracket of $\mathfrak{g}$.

5. EXAMPLES OF INFINITESIMAL GEOMETRIC STRUCTURES

In this section we describe the infinitesimal geometric structures occurring in nature. Subriemannian contact structures and conformal structures are described separately in Sections 10 and 12. Conformal parallelisms are described in Section 9.7.

Isotropy. Most infinitesimal geometric structures occurring in nature are best understood as isotropy (or joint isotropy) subalgebroids. In the case of Riemannian geometry it is the isotropy of a rank-one subbundle; sections of this bundle are closed under the bracket of $\mathfrak{g}$.

Assume either that $\Sigma = \Sigma^0 \subset E$ is invariant or that $\Sigma = \Sigma^1 \subset E$ is a collection of all elements $x \in \mathfrak{g}$ for which $\sigma \in \Sigma \implies \rho_x \sigma \in \Sigma$ for arbitrary local sections $\sigma \in E$; here $\rho_x \sigma := \rho_x \otimes \sigma$ is the base point of $x$.

The isotropy of $\Sigma$ is a subset of $\mathfrak{g}$ intersecting fibers in subspaces whose dimensions may vary, i.e., is a ‘variable-rank subbundle’ under the bracket of $\mathfrak{g}$. When this rank is constant the isotropy is a genuine subalgebroid, called the

For the application of Theorem 4.6 to examples 4.4(5) and 4.4(7), see [1].
The structure kernel of $\mathfrak{g}$ is the isotropy $\mathfrak{h} \subset T^*M \otimes TM$ representation. So $\mathfrak{h}$ is the bundle of $\sigma$-skew-symmetric endomorphisms of tangent spaces, a Lie algebra bundle modeled on $\mathfrak{o}(n)$, $n := \dim M$. Conformally equivalent metrics give the same structure kernel.

One way to see that $\mathfrak{g}$ is surjective (i.e., transitive) is to apply the algebraic lemma B.1 in Appendix B to the morphism $X \mapsto X \cdot \sigma : J^1(TM) \rightarrow \text{Sym}^2(TM)$, whose kernel is $\mathfrak{g}$. On account of the surjectivity of the restriction $\phi \mapsto \phi \cdot \sigma : T^*M \otimes TM \rightarrow \text{Sym}^2(TM)$ of this morphism, the lemma delivers an exact sequence
\[
0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{g} \rightarrow TM \rightarrow 0.
\]
Thus $\mathfrak{g} \subset J^1(TM)$ is surjective and has constant rank (making it a subalgebroid and thus an infinitesimal geometric structure).

The lemma just applied is very useful in determining the image and structure kernel of infinitesimal geometric structures defined by isotropy. Less trivial applications of Lemma B.1 are made in 5.3 and 5.5 below. Minimal comment will accompany subsequent applications.

The symmetries of $\mathfrak{g}$ (in the sense of 1.3) are the vector fields along which $\sigma$ has vanishing Lie derivative, i.e., its Killing fields. A linear connection $\nabla$ on $TM$ such that $\sigma$ is $\nabla$-parallel, where $\nabla$ denotes the dual of $\nabla$. The Levi-Cevita connection is thus the unique torsion-free generator of $\mathfrak{g}$.

From $\mathfrak{g}$ one can recover the metric $\sigma$ up to a positive constant (not merely its conformal class). In the simply-connected case, slightly more is true:

**Proposition.** Let $\mathfrak{h} \subset T^*M \otimes TM$ denote the arbitrary conformal structure. Then on simply-connected open subsets of $M$, every surjective infinitesimal geometric structure $\mathfrak{g} \subset J^1(TM)$ having structure kernel $\mathfrak{h}$ is the isotropy subalgebroid of some Riemannian structure $\sigma$ in the given conformal class. This structure is uniquely determined up to a constant.

**Proof.** Suppose $\mathfrak{g} \subset J^1(TM)$ has structure kernel $\mathfrak{h}$ and surjective isotropy subalgebroid $\mathfrak{g} \subset J^1(TM)$ is the isotropy subalgebroid of some Riemannian structure $\sigma$ in the given conformal class. This structure is uniquely determined up to a constant. Changing the sign of $\sigma$ if necessary, we obtain the sought after metric.

The application of Cartan’s method to Riemannian structures is given in 9.6. In analogy with the Riemannian case, the isotropy subset $\mathfrak{g} \subset J^1(TM)$ of an arbitrary tensor on $M$ is an infinitesimal geometric structure, whenever this isotropy has constant rank. Moreover, in many cases, this structure encodes all useful information (i.e., some analogue of the preceding proposition applies.) For structures defined by more than one tensor one considers the joint isotropy, defined by intersecting the individual isotropies. For example, for almost Kähler structures, one considers the joint isotropy of the symplectic and almost complex structures. Here is another example:

5.3. Vector fields on a Riemannian manifold. Let $V$ be a nowhere vanishing vector field on a Riemannian manifold $M$ with metric $\sigma$. The vector fields on $M$ that are simultaneously infinitesimal isometries of $\sigma \subset \text{Sym}^2(TM)$ and $V \subset TM$ are the symmetries of the joint isotropy of $\sigma$ and $V$, with respect to representations of $J^1(TM)$ on $\text{Sym}^2(TM)$ and $TM$ respectively. Denoting the isotropy of $\sigma$ alone
by \( g \subset J^1(TM) \) as above, \( g \) acts on \( TM \) by restriction. The isotropy is the isotropy \( g_V \subset g \) of \( V \).

The structure kernel of \( g_V \) is the \( o(n - 1) \)-bundle \( \mathcal{E}(TM \to M) \) of space endomorphisms infinitesimally fixing \( V \) (mod \( \mathcal{E}(\text{dim } TM) \)).

**Proposition.** The image of \( g_V \) is the distribution of \( \frac{1}{2} \| V \|^2 \).

In particular, \( g \) has constant rank (is an infinitesimal geometric structure) only if \( V \) has constant length or \( \frac{1}{2} \| V \|^2 \) is a free vector bundle (transitive) in the former case only.

**Proof.** Applying Lemma B.1 to the morphism \( X \) \( \rightarrow \) \( TM \) to the kernel, we deduce that \( D \) is the kernel of the following diagram commute:

\[
\begin{array}{ccc}
g & \xrightarrow{\#} & TM \\
x \mapsto x \cdot V & \downarrow & e^{-} \\
TM & \longrightarrow & TM/V
\end{array}
\]

Let \( \nabla \) be any generator of \( g \) (e.g., the Levi-Civita connection). \( g \) the corresponding splitting of \( \mathcal{E}(2) \) above. Then \( \nabla_U V \mod V^\perp \), where \( \nabla \) is the dual connection.

For a trivial line bundle \( \mathbb{R} \times M \), using \( V \), we have \( \Theta(U) = d\left( \frac{1}{2} \| V \|^2 \right)(U) \). Here we have used \( \nabla \sigma = 0 \), which is trivial.

**5.4. Parallelism.** The simplest non-trivial example of an infinitesimal geometric structure kernel. In other words, \( g \) is a subalgebraically onto \( TM \) by the anchor \( \# : J^1(TM) \rightarrow TM \) a unique generator \( \nabla \) that is a Cartan connection in \( \mathcal{E}(2) \), the dual connection \( \nabla \) is flat, i.e., an integrable. Conversely all infinitesimal parallelisms arise in this way.

When \( M \) is simply-connected the Lie algebra \( g \) integrates to a Lie groupoid morphism \( M \times M \rightarrow M \).
let \( N \subset \text{Alt}^2(TM) \otimes TM \) denote the Nijenhuis torsion of \( J \):

\[
N(U,V) = \frac{1}{4} \left( [JU,JV] - [U,V] - J[JU,JV] - J[JU,V] \right).
\]

Then:

**Proposition.** The structure kernel of \( \mathfrak{g} \) is \( T^*M \otimes TM \) and the image of \( \mathfrak{g} \) is the kernel of the morphism

\[
\Theta : TM \to (T^*M \otimes TM)/[T^*M \otimes TM, J],
\]

\[
\Theta(U) = -4N(JU, \cdot, \cdot) \mod [T^*M \otimes TM, J].
\]

In particular, \( \mathfrak{g} \) is transitive if and only if the section \( N(U, \cdot, \cdot) \subset T^*M \otimes TM \) lies in \( [T^*M \otimes TM, J] \) for all vector fields \( U \).

**Proof.** First, note that

\[
\frac{1}{2} \left[ \text{ad}_{J^1(U)} J, J \right] V = -J[JU, JV] - [JU, V] + [U, JV] - [JU, JV] = N(U, V).
\]

Next observe that \( \mathfrak{g} \) is the kernel of the morphism

\[
\theta : J^1(TM) \to T^*M \otimes TM,
\]

\[
\theta(X) = \text{ad}_X J,
\]

i.e., \( \theta(X)U = \text{ad}_X (JU) - J(U) \).

Applying Lemma [B.1] to this morphism, we obtain a morphism \( \Theta \) with the requisite kernel, and satisfying

\[
\Theta(U) = \text{ad}_{J^1(U)} J \mod [T^*M \otimes TM, J],
\]

\[
= \text{ad}_{J^1(U)} J - \frac{1}{2} \left[ \text{ad}_{J^1(U)} J, J \right]
\]

By (1), we have

\[
\left( \text{ad}_{J^1(U)} J - \frac{1}{2} \left[ \text{ad}_{J^1(U)} J, J \right] \right) V = [U, JV] - [JU, V] + J[JU, V] - J[JU, V] = N(U, V).
\]

**5.6. Poisson structures.** Although not of finite-type, Poisson structures furnish us with another interesting example of an infinitesimal geometric structure. Now generally the isotropy \( \mathfrak{g} \subset J^1(TM) \) of a Poisson tensor on \( M \) fails to have constant rank, and so fails to be an infinitesimal geometric structure on \( TM \). However, one can define an infinitesimal geometric structure on the cotangent bundle \( T^*M \), which the Poisson tensor makes into a Lie algebroid (see below). Although not transitive, this structure is surjective.

Let \( \omega \) be a symplectic structure on \( M \) and let \( # : T^*M \to TM \) denote the inverse of \( v \mapsto \omega(v, \cdot) \). Since \( # \) is an isomorphism, there is a unique bracket on \( \Gamma(T^*M) \) making \( T^*M \) into a Lie algebroid with anchor \( # \). This bracket is given by

\[
\left[ \alpha, \beta \right]_{T^*M} = L_{#\alpha} \beta - L_{#\beta} \alpha + d(\Pi(\alpha, \beta)), \quad \alpha, \beta \in \Gamma(T^*M),
\]

where \( L \) denotes Lie derivative and \( \Pi \) is the Poisson tensor. This tensor is defined by \( \Pi(\alpha, \beta) = \omega(\#, \alpha, #\beta) \) and so satisfies

\[
\langle \alpha, #\beta \rangle = \Pi(\alpha, \beta), \quad \alpha, \beta \in \Gamma(T^*M).
\]
More generally, (1) defines a Lie algebroid structure on the manifold $(M, \Pi)$, with anchor $\#_{\alpha}$ defined by (2). The symmetries of $\Pi$ are the orbits of the Lie algebroid $T^*M$.

An infinitesimal isometry of a Poisson manifold $(M, \Pi)$ is a vector field $\xi$ on $M$ such that $\mathcal{L}_\xi \Pi = 0$. Poisson manifolds have many infinitesimal isometries. In particular, every closed 1-form $\alpha \in \Omega^1(M)$ determines a Lie algebroid structure on $T^*M$. An infinitesimal isometry $\xi$ of a Poisson manifold $(M, \Pi)$ is tangent to the symplectic leaves known as a Hamiltonian vector field, or a Hamiltonian vector field if $\alpha$ is exact.

It is not too difficult to establish the following proposition.

**Proposition.** Let $\mathfrak{g} \subset J^1(T^*M)$ denote the kernel of the map $J^1(T^*M) \to \text{Alt}^2(TM)$ whose corresponding map $\mathfrak{g} \to \text{Ker} \ Sym^2(TM)$ is surjective.

Then $\mathfrak{g}$ is a surjective infinitesimal geometric structure on $T^*M$, with kernel $\text{Sym}^2(TM)$, whose symmetries are the closed 1-forms.

A linear connection $\nabla$ on $T^*M$ is a generator of a Lie algebroid if and only if $\text{curv} \nabla (V, \#\alpha)\beta - \text{curv} \nabla (V, \#\beta)\alpha = 0$ for all sections $\alpha, \beta \subset T^*M; V \subset TM$.

If $M$ is the dual of a Lie algebra, equipped with its Lie-Poisson structure (see, e.g., [13, §10.1]), then the canonical flat linear connection on $T^*M$ is torsion free. Such a linear connection on $T^*M$ if and only if

$$\text{curv} \nabla (V, \#\alpha)\beta - \text{curv} \nabla (V, \#\beta)\alpha = 0$$

for all sections $\alpha, \beta \subset T^*M; V \subset TM$.

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**5.7. Subgeometries of an Abelian Lie group.**

An Abelian Lie group, $V$ its Lie algebra, and $M \subset E$ a codimension-one submanifold whose symmetry is a V-valued one-form on $M$ obtained by restricting a V-valued one-form on $E$. Then $d\omega = 0$ and $\dim V = \dim M + 1$.

Let $\omega(TM)$ denote the tangent bundle of $M$, so that $N := (V \times M)/\omega(TM)$ is a model of the ...
is given by
\[ \Theta(U) = \mathcal{L}_U \omega \mod \omega(TM) = \frac{1}{2} (\nabla \omega)_{\text{sym}}(U, \cdot) \mod \omega(TM) \]
\[ (\mathcal{L}_{U_1} \omega)(U_2) = \omega(\nabla U_2 U_1) + \frac{1}{2} d\omega(U_1, U_2) + \cdots \]

5.8. **Affine structures.** Any suitably non-degenerate infinitesimal operator on \( M \), defines an infinitesimal geometric structure \( J^1(J^k(TM)) \). As a simple example, which will suffice to illustrate the general principle, we consider an affine structure on \( M \), i.e., \( J^1(TM) \) acts on \( M \), in which case \( k = 1 \). The relevant non-isotropy of the torsion of \( \nabla \) should have constant rank.

View an affine structure \( \nabla \) as a section of \( J^1(TM) \)
\[ \nabla(J^1W, V) := \nabla_V W; \quad W \in \mathfrak{T}M. \]

In order to associate a natural isotropy subalgebra with \( \nabla \), we consider an affine structure on \( M \), i.e., \( \nabla \) acts on \( M \) via adjoint action, and on \( TM \) \( \nabla \) acts on \( J^1(TM) \) via adjoint action, and on \( TM \)
\[ \nabla(J^1W, V) := \nabla_V W; \quad \nabla(J^1W, V) := \nabla_V W; \quad \nabla(J^1W, V) := \nabla_V W; \]

Secondly, \( J^2(TM) \) may be identified with a subcanonical embedding \( J^2(TM) \hookrightarrow J^1(J^1(TM)) \) on which \( \nabla \) acts, and the prolongations sends \( J^2V \) to \( J^1(J^1V) \). Combining the two actions of \( J^2(TM) \) on \( J^1(TM)^* \otimes T^*M \otimes TM \).

**Proposition.** Let \( \mathfrak{g} \subset J^2(TM) \) denote the isotropy and \( \mathfrak{t} \subset J^1(TM) \) the isotropy of \( \text{tor} \nabla \subset \text{Alt}^2(TM) \).

1. The symmetries of \( \mathfrak{g} \) are the prolonged infinitesimal isometries \( \mathfrak{t} \).
2. The image of \( \mathfrak{g} \subset J^1(J^1(TM)) \) is \( \mathfrak{t} \) and \( \mathfrak{g} \) has rank \( 2 \)

In particular, (2) implies that \( \mathfrak{g} \subset J^2(TM) \) has an infinitesimal geometric structure on \( J^1(TM) \).
a condition that is second-order in $U$. Unravelling the infinitesimal isometries defined above, we may write this condition as

$$J^1(J^1(U)) \cdot \nabla = 0.$$  

It easily follows that $J^1U$ is a symmetry of $\mathfrak{g}$ whenever $\nabla$ is a symmetry of $\nabla$.

Suppose, conversely, that $X \subset J^1(TM)$ is a section of $\mathfrak{g}$. This means:

(4) $$J^1 X \subset J^2(TM)$$

and $$J^1 X \cdot \nabla = 0.$$  

It is well known that (4) is equivalent to $X \subset J^1(S^1 \otimes M)$ (see §8.1). So $X = J^1U$, where $U$ is an infinitesimal isometry, reads $J^1(J^1 U) \cdot \nabla = 0$. This completes the proof of (1).

Let $\xi$ be any section of $J^2(TM)$. It is easy to see that $J^1(TM)^* \otimes T^*M \otimes TM$ is tensorial, i.e., drops to some section $U$ of $T^*M \otimes TM$. Noting that $\mathfrak{g}$ is then the kernel of the map

$$\xi \mapsto (\xi \cdot \nabla)^\vee$$

$$J^2(TM) \to T^*M \otimes T^*M,$$

whose domain $J^2(TM)$ fits into an exact sequence

$$0 \to \text{Sym}^2(TM) \otimes TM \to J^2(TM) \to 0,$$

one shows, by applying Lemma $\text{B.1}$ that $\mathfrak{g}$ fits into the sequence

$$0 \to 0 \to \mathfrak{g} \overset{b}{\to} \mathfrak{t} \to 0.$$  

Here $b$ is the restriction of the canonical projection of $\mathfrak{g}$ into $\mathfrak{t}$, which establishes (2).

5.9. **Projective structures.** Recall that two linear connections $\nabla, \nabla'$ are projectively equivalent if their geodesics coincide as unparameterized curves. Equivalently, their difference $\nabla - \nabla'$, which may be viewed as a section of $T^*M \otimes TM \otimes TM$, should take its values in the subbundle $\Sigma_0 := (\text{Alt}^2(TM) \otimes TM)$ of $T^*M \otimes T^*M \otimes TM$.
It is not hard to see that $\mathfrak{g}$ has $j_{S}(T^{*}M) \cong T^{*}M$ so that

$$\mathfrak{g} = \mathfrak{g}_{\nabla} \oplus j_{S}(T^{*}M)$$

where $\mathfrak{g}_{\nabla} \subset J^{2}(TM)$ denotes the isotropy of $\nabla$; connection $\nabla^{(1)}$ on $J^{1}(TM)$ in Corollary 5.8 is a generator of $\mathfrak{g}$ as well. An explicit formula appears in 11.5.

5.10. $G$-structures. Let $G$ be a subgroup of $GL(n, \mathbb{R})$, where $n$ is the dimension of $M$. A $G$-structure on $M$ is a $G$-reduction $P$ of the bundle of (absolute) frames on $M$; see, e.g., [11]. In particular, $P$ is a principal $G$-bundle, so that $\mathfrak{g} = T P / G$ is a transitive Lie algebroid over $M$, and the associated vector bundles of $P$ are representations of $\mathfrak{g}$; see, e.g., [12]. As $P$ is a frame bundle, $TM$ will be such a representation (see below). That is, we have a Lie algebroid morphism $\mathfrak{g} \to \mathfrak{gl}(TM) \cong J^{1}(TM)$.

This turns out to be injective, identifying $\mathfrak{g}$ with the direct sum of its image and its structure kernel. The representation of $\mathfrak{g}$ on $TM$ may be described as follows. Identify sections $X$ of $\mathfrak{g} := TP / G$ with $G$-invariant vector fields on $P$, and use the tautological 1-form on $P$ to identify sections $V$ of $TM$ with $G$-invariant $\mathbb{R}^{n}$-valued functions on $P$. Then $X \cdot V := L_{X}V$, where $L$ denotes Lie derivative on $P$.

5.11. Cartan algebroids as infinitesimal geometric structures. We have seen that all surjective infinitesimal geometric structures with trivial kernel define Cartan algebroids (Theorem 2.1). Conversely, if $t$ is a Cartan algebroid with Cartan connection $\nabla$, then $\mathfrak{g} := s_{\nabla}(t) \subset J^{1}(t)$ defines an infinitesimal geometric structure generated by $\nabla$ with trivial structure kernel. Here $s_{\nabla}: t \to J^{1}(t)$ is the splitting of

$$0 \to T^{*}M \otimes t \to J^{1}(t) \to 0$$

determined by $\nabla$.

6. Generators, associated operators, and Bianchi identities

Picking a generator for an infinitesimal geometric structure $\mathfrak{g}$ allows us to identify $\mathfrak{g}$ with the direct sum $t_{1} \oplus h$ of its image and its structure kernel. Generators are also the appropriate connections for which to develop all the usual formalisms of differential geometry: covariant differentiation, covariant exterior differentiation, Bianchi identities, etc. (It will be natural, however, to use the more encompassing descriptor 'associated' in place of 'covariant.') By virtue of 5.11, we obtain formalism for Cartan algebroids as a special case. The present section, rather formal in nature, can be scanned on a first reading. In 6.1 we address the existence and uniqueness of generators and prove the theorem in 2.1, where generators were defined. In 6.2 we see how information about $\mathfrak{g}$ is encoded in $t_{1}$, $h$, and $\nabla$. Basic algebraic invariants of an infinitesimal geometric structure $\mathfrak{g} \subset J^{1}(t)$ are the vector bundles occurring as representations of $\mathfrak{g}$. Associated with these representations, and a choice of generator $\nabla$, are the associated connections and associated differential operators, described in 6.3. The latter generalize the divergence, gradient, etc. of Riemannian geometry when $\nabla$ is the Levi-Civita connection. (In Sect. 10 we describe these objects for subriemannian...
contact three-manifolds.) In principle, any invariant can be expressed in terms of associated differential operators on separative infinitesimal geometric structures. In [6.4] we define derivative, and in [6.5] analogues of the classical Bianchi identities.

6.1. Basic properties of generators. Let \( g \subset \mathfrak{t} \) be an arbitrary subalgebroid. We define a \( \mathfrak{t} \)-connection \( \nabla \) on \( \mathfrak{t} \) in the obvious way: a generator \( \nabla \) structure kernel \( \mathfrak{h} \), image \( \mathfrak{t} \) be an arbitrary Lie algebroid, and the induced Lie algebroid \( \mathfrak{t} \). Let \( \mathfrak{h} \) is paracompact, it possesses a splitting of \( \mathfrak{t} \). To prove (3), let \( \mathfrak{t} \to \mathfrak{g} \) be an infinitesimal geometric structure, with structure kernel \( \mathfrak{h} \), and image \( \mathfrak{t} \) has constant rank if and only if \( \mathfrak{h} \subset T^*M \otimes \mathfrak{t} \) (or equivalently, are subalgebroids).

**Proposition.** If \( \mathfrak{g} \to \mathfrak{t} \) has constant rank then:

1. \( \mathfrak{g} \) admits a generator \( \nabla \).
2. \( \nabla \) is unique if and only if \( \mathfrak{g} \) is surjective and \( \mathfrak{t} \) is a symmetry.
3. Every \( \nabla \)-parallel section of \( \mathfrak{t} \) is a symmetry.

**Proof of proposition and Theorem 2.1** The construction

\[
0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{t} \to 0
\]

is an exact sequence of vector bundles. Assuming a splitting \( s: \mathfrak{t} \to \mathfrak{g} \) which can be extended to a

\[
0 \to T^*M \otimes \mathfrak{t} \to J^1 \mathfrak{t} \to 0
\]

To prove (1), let \( \nabla \) be the corresponding linear connection.

Conclusion (2) follows readily from the correspondence \( \mathfrak{t} \) and splitting of (4). To prove (3), let \( s: \mathfrak{t} \to J^1 \mathfrak{t} \) be a generator \( \nabla \), i.e., \( s \mathfrak{V} = J^1 \mathfrak{V} + \nabla \mathfrak{V} \). Then if \( \mathfrak{V} \) is \( \nabla \)-parallel then \( s \mathfrak{V} \) lies in \( \mathfrak{g} \), by the definition of generators.

Assume \( \nabla \) is a generator and \( \mathfrak{h} = 0 \). Suppose \( J^1 \mathfrak{V} = s \mathfrak{V} - \nabla \mathfrak{V} \) is a section of \( \mathfrak{g} \). Then \( s \mathfrak{V} \subset \mathfrak{g} \) by \( \mathfrak{h} = 0 \). So \( \nabla \mathfrak{V} \subset (T^*M \otimes \mathfrak{t}) \cap \mathfrak{g} = \mathfrak{h} = 0 \). Symmetry together with (3), establishes Theorem 2.1.

In the remainder of this section it is tacitly assumed that all infinitesimal geometric structures have constant rank in the sense above.

6.2. Reconstructing geometric structures
Proposition. A linear connection $\nabla$ on a Lie algebroid $\mathfrak{g} \subset J^1\mathfrak{t}$ with structure kernel $\mathfrak{h}$ is $\nabla$-invariant if and only if:

1. $\mathfrak{h} \subset T^*\mathfrak{m} \otimes \mathfrak{t}$ is $\nabla$-invariant, i.e., $\nabla_V \phi \in \mathfrak{h}$ for all sections $\phi$ of $\mathfrak{h}$ and $V$ of $\mathfrak{t}$.
2. cocurv $\nabla (V_1, V_2) \subset \mathfrak{h}$ for all sections $V_1, V_2 \subset \mathfrak{t}$.

If $\mathfrak{g} \subset J^1\mathfrak{t}$ is such an infinitesimal geometric structure with structure kernel $\mathfrak{h}$, then the Lie algebroid $\mathfrak{g} \cong \mathfrak{t}_1 \oplus \mathfrak{h}$ is given by

\[
\begin{align*}
\#(V \oplus \phi) &= \#V \\
[V_1 \oplus \phi_1, V_2 \oplus \phi_2] &= [V_1, V_2]_{\mathfrak{t}_1} \oplus ([\phi_1, \phi_2]_\mathfrak{h} + \nabla_{V_1} \phi_2 - \nabla_{V_2} \phi_1)
\end{align*}
\]

We recall that cocurvature was defined in 4.3.

6.3. Associated connections and differential operators.

An infinitesimal geometric structure with structure kernel $\mathfrak{h}$ is generated by a Lie algebra automorphism $\rho: \mathfrak{g} \to \mathfrak{gl}(E)$, we have an associated $\mathfrak{t}_1$-connection $\nabla$ on $E$ if $\mathfrak{g}$ is surjective. By definition, this is the Lie algebroid morphism $s_\nabla: \mathfrak{t} \to J^1\mathfrak{t}$ which acts on its structure $\mathfrak{t}_1$-connection via the composite $\rho$.

Examples.

1. Taking $\mathfrak{g} := J^1\mathfrak{t}$ and $\rho = \text{ad}^\mathfrak{t}$, we obtain

\[
\nabla_U V = \text{ad}^\mathfrak{t}_{s_\nabla U} V = \text{ad}^\mathfrak{t}_{\nabla_U} V
\]

i.e., $\nabla_U V = \nabla_{\#_U} V + [U, V]_{\mathfrak{t}}$.

This is the associated $\mathfrak{t}$-connection on $\mathfrak{t}$ defined in 1.6.1.

2. Let $\mathfrak{g} := J^1\mathfrak{t}$ act on $TM$ via the composite $J^1\mathfrak{t} \xrightarrow{J^1\#} J^1(TM) \xrightarrow{\text{ad}^{TM}} TM$.

Then we similarly compute

\[
\nabla_U W = \#_W U + [\#_U, W]_{TM};
\]

This is the associated $\mathfrak{t}$-connection on $TM$ defined in 1.6.2.

3. An arbitrary infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ acts on $TM$ via bracket: $\mathfrak{g}X := [X, Y]_{\mathfrak{g}}$ (for all sections $X, Y$ of $\mathfrak{t}$) and $\rho$ is the Lie algebra automorphism defined by $\mathfrak{g}X := [X, Y]_{\mathfrak{g}}$.

(Continued on the next page.)
The associated derivative of a \( g \)-tensor \( \sigma \in \Gamma(E) \), where \( \nabla \) is the associated \( t_1 \)-connection on \( E \). And we mean that \( t_1 \subset t \) is invariant under the adjoint, for example, if \( g \) is surjective. (Image reduction of \( g \)-representation, implying \( \nabla \sigma \) is another \( g \)-tensor closed under associated derivative. In particular, we obtain higher order differential operators.

Additionally supposing that all \( g \)-representations come into \( g \)-representations coming from some collection, we have

\[
t^*_1 \otimes E_i \cong E_{n_{i1}} \oplus E_{n_{i2}} \oplus E_{n_{i3}} \oplus \cdots \quad \text{(finite)}
\]

for some \( n_{ij} \in I \), and obtain a corresponding decomposition \( \nabla|\Gamma(E_i) = \partial_{i1} \oplus \partial_{i2} \oplus \partial_{i3} \). We call the differential operators \( \partial_{ij} : \Gamma(E_i) \to \Gamma(E_i) \) differential operators; all differential operators which come out of associated connections \( \nabla \) are combinations thereof.

If there is a canonical way in which to choose the \( t \)-differential operators become invariant differential operators of infinitesimal geometric structure \( g \). Significant cases are:

5. The case where \( t \) is a Cartan algebroid discussion are just \( t \)-representations because \( g \cong t \).

6. The case where the generator \( \nabla \) of \( g \) is unique, reducing the situation to case (5) above.

7. The case where torsion \( \text{tor} \ nabla \) has a natural \( t \)-structure.

For invariant differential operators associated with special manifolds, see Sect. [10].

6.4. The associated exterior derivative. Let \( g \)-algebraic geometric structure with structure kernel \( h \). Then a \( \text{degree} \ k \) is a section \( \theta \subset \text{Alt}^k(t_1) \otimes E \), where \( t_1 \) is a \( g \)-representation. (We use \( t_1 \), rather than \( t \), to emphasize the derivative \( d_{\nabla} \theta \subset \text{Alt}^k(t_1) \otimes E \) of \( \theta \) is defined by

\[
d_{\nabla} \theta (U) = \nabla_{\theta} \theta = \text{for } k = 0.
\]
(2) For any $\mathfrak{g}$-type differential form $\theta$, we have
\[d_\nabla^2 \theta = \Omega \wedge \theta.\]

Here the wedge implies a contraction $\phi \otimes \sigma \mapsto \phi \cdot \sigma : h \otimes E \to E$, defined by the representation of $h$ on $E$.

Proof of (2). The general case can easily be reduced to the $k = 0$ case that we prove now. Letting $s : t \to J^1 t$ denote the splitting of $6.1(4)$ determined by $\nabla$, we compute, for arbitrary $U_1, U_2 \subset t_1$,
\[d_\nabla^2 \theta (U_1, U_2) = \nabla_{\nabla U_1} U_2 \theta - \nabla_{\nabla U_2} U_1 \theta - \nabla [U_1, U_2] \cdot \theta = sU_1 \cdot (sU_2 \cdot \theta) - sU_2 \cdot (sU_1 \cdot \theta) - \text{cogr} \nabla (U_1, U_2) \cdot \theta = 0 + \Omega (U_1, U_2) \cdot \theta.

6.5. Bianchi identities. Generalizing the classical situation, the Bianchi identities below exhibit certain algebraic and differential dependencies between $T$ and $\Omega$, rooted in the equality of mixed partial derivatives.

First, since $T = d_\nabla \nabla_i$, where $i \subset t_1^* \otimes t$ denotes the inclusion $t_1 \subset t$, We deduce from (2) above,
\[d_\nabla^3 \theta = \Omega \wedge \theta + \nabla \Omega \wedge \theta + d_\nabla \Omega \wedge d_\nabla \theta = 0.

Applying part (2) again, we conclude that
\[d_\nabla \Omega \wedge \theta = 0.

A little manipulation allows us to write (1) and (2) in the form
\[
\begin{align*}
(3) \quad & (\nabla U_3 T)(U_1, U_2) + T(T(U_1, U_2), U_3) + \Omega(U_1, U_2) \cdot U_3 + 1-2-3-cyclic \ terms = 0 \quad \text{(Bianchi I)}, \\
(4) \quad & (\nabla U_3 \Omega)(U_1, U_2) + \Omega(T(U_1, U_2), U_3) + 1-2-3-cyclic \ terms = 0 \quad \text{(Bianchi II)}.
\end{align*}
\]

Example. If $\nabla$ is the Cartan Connection on some Lie algebroid $t$ and $g \subset J^1 t$ the corresponding infinitesimal geometric structure (see 5.11) then $T = \text{tor} \nabla$ and $\Omega = 0$. Bianchi I becomes
\[
\begin{align*}
& (\nabla U_3 \text{tor} \nabla)(U_1, U_2) + \text{tor} \nabla (\text{tor} \nabla (U_1, U_2), U_3) + 1-2-3-cyclic \ terms = 0.
\end{align*}
\]
7. Elementary reduction and image reduction

In this section we study elementary reduction, as well as a cruder alternative we call image reduction. These techniques are useful when an infinitesimal geometric structure fails to be surjective, and in particular to intransitive infinitesimal geometric structures on $TM$. A simple application to smooth functions on a Riemannian three-manifold is included.

7.1. Image reduction. Let $g \subset J^1t$ be an infinitesimal geometric structure with structure kernel $h \subset T^*M \otimes t$ and image $t_1 \subset t$. Assume $h$ (or equivalently $t_1$) has constant rank. Then the image reduction of $g$ is simply the isotropy $g_{t_1} \subset J^1g$ of $t_1 \subset t$, under the adjoint representation of $g \subset J^1t$ on $t$. It is not hard to show that image reduction is cruder than elementary reduction, as defined in 2.4, and described further below. Nevertheless, it is usually easier to apply image reduction and this may simplify the subsequent application of elementary reduction.

7.2. Elementary reduction. With $g \subset J^1t$, $h$, and $t_1$ as above, let $g_1$ denote the elementary reduction of $g$ (see 2.4). The structure kernel of $g_1$ is $h_1 := h \cap (T^*M \otimes t_1)$. One can compute the image $t_2 \subset t_1$ of $g_1$ if one knows a generator $\nabla$ of $g$:

**Proposition.** There is vector bundle morphism $t_1 \xrightarrow{b} (T^*M \otimes t)/(T^*M \otimes t_1)$ whose corresponding map of section spaces is $U \mapsto \nabla U \mod (T^*M \otimes t_1)$.

The morphism $b$ is independent of the choice of generator $\nabla$.

**Proof.** Begin by observing that the one-jet $J^1U(m)$ lies in $h$. So we define a morphism $J^1t_1 \xrightarrow{B} (T^*M \otimes t)$ which on sections is the map $J^1U \mapsto \nabla U$ mod $t$. The proposition now follows from an application of Lemma B.1 to the morphism $B$; one uses the fact that the sequence

$$0 \to T^*M \otimes t_1 \to T^*M \otimes t \to T^*M \otimes (t_1/T^*M) \to 0$$

is exact. \hfill \Box

**Remark.** A significant simplification occurs if $h_1 = h$, i.e., if $h \subset T^*M \otimes t_1$ lies entirely within $T^*M \otimes t_1$. Then we may view $b$ as the map $t_1 \xrightarrow{b} T^*M \otimes (t/t_1)$ $b(U) = \nabla_U \mod t_1$. In particular, the proposition will imply that $g_1 = g$ if and only if $t_1$ is $\nabla$-invariant. In that case $\nabla$ drops to a linear connection on $t_1$ which generates $g_1$, as a surjective infinitesimal geometric structure on $t_1$. 

7.3. Functions on a Riemannian three-manifold.

We consider the (infinitesimal) symmetries of a smooth function \( f \) on a Riemannian three-manifold \( M \), with metric \( \sigma \). By symmetries, we mean the Killing fields of \( \sigma \) preserving \( f \). In the terminology of 1.3, these are the symmetries of the joint isotropy

\[
(J^1(TM))_{\sigma,f} \subset (T^1(T^1M))_{\sigma,f}
\]
of \( \sigma \) and \( f \), under the relevant representations determined by the adjoint representation of \( J^1(TM) \) on \( TM \). Any such symmetry must also preserve \( df \), and so we have an immediate reduction,

\[
(J^1(TM))_{\sigma,f,df} \subset (T^1(T^1M))_{\sigma,f}.
\]

Let \( E := \frac{1}{2} \| \text{grad} \ f \|^2 \) denote the ‘energy’ of \( f \). Let \( df \) and \( dE \) be everywhere linearly independent. It follows that the connected components of the joint level-sets of \( f \) and \( E \) constitute a rank-one foliation on \( M \).

We denote by \( T \) the unit vector field tangent to this foliation, directed so as to make \( \{ T, \text{grad} \ f, \text{grad} \ E \} \) positively oriented.

Define \( J \subset T^*M \otimes TM \) by \( JU := n \times U \), where \( n := \text{grad} \ f / \| \text{grad} \ f \| \) (so that \( J \) restricts to a complex structure on level sets of \( f \)). Then the reduction in (1) has a rank-one structure kernel spanned by \( J \). Using Lemma B.1, it is not hard to see that its image is \( \langle T \rangle = \ker df \cap \ker dE \).

We observe that \( g \) has trivial structure kernel, \( h = 0 \), and image \( t_1 := \langle T \rangle \). To see this one applies Lemma B.1 to the morphism,

\[
J^1(TM)_{\sigma,f,df} \to TM/\langle T \rangle \quad X \mapsto \text{ad}_X T \mod \langle T \rangle,
\]

which has \( g \) as kernel.

As \( g \) itself is evidently stable under image-reduction, by Proposition 6.1 \( g \) has a generator \( \nabla \). Now, if \( g = (J^1(TM))_{\sigma,f,df,\langle T \rangle} \), we must have

\[
\nabla_T \sigma = 0, \quad \nabla_T \text{grad} \ f = 0 \quad \text{and}
\]

where \( \nabla UV := \nabla_V U + [U, V] \). From these identities

\[
\nabla T = 0,
\]

and \( \nabla_T \text{grad} \ f = 0 \) we have

\[
\nabla_T \sigma = 0.
\]
non-trivial component, \( \text{curv} \nabla (JT, \text{grad} f) \), which makes sense in general, but suppose for the moment that \( g \) is transitive. We begin, however, with a useful characterization of the subbundle \( J^2 g \subset J^1 (J^1 g) \) that is completely general. This section concludes with the reformulation associated with torsion.

8.1. Prolongation. Let \( t \) be an arbitrary vector and \( g^\ast \subset J^1 t \) a natural inclusion of vector bundles \( J^2 t \hookrightarrow J^1 (J^1 W)(m); W \subset J^1 t \). As a basic fact one has the following:

**Lemma.** For any section \( X \subset J^1 t, X \) is holonomic.

**Proof.** See Appendix [B.4]

Now the definition of prolongation, \( g^{(1)} := J^1 g \cap J^1 t \), makes sense in general, but suppose for the moment that \( g \subset J^1 t \) is an infinitesimal geometric structure subalgebroid, implying \( g^{(1)} \) is an infinitesimal geometric structure. A consequence of definitions is that \( J^1 W \) is a section of \( g^{(1)} \). In fact, it is a consequence of the lemma above that arise in this way:

**Proposition.** If \( g \subset J^1 t \) is an infinitesimal geometric structure, \( W \subset t \) is a symmetry of \( g \) if and only if \( J^1 W \subset g^{(1)} \).

Since \( J^1 : \Gamma (t) \to \Gamma (J^1 t) \) is injective, this establishes the equivalence between the symmetries of \( g \) and those of \( g^{(1)} \).
Here is a vector bundle morphism well defined by
\[ D_V(fX) = fD_VX + df \]
for arbitrary smooth functions \( f \) on \( M \).

The above construction, holding for arbitrary \( t \) in the case that \( t \) is replaced by \( J^1t \). This delivers an operator \( D \) which will also be denoted \( D \). In the formulas above the vector bundle morphism is replaced by the natural projection \( p: J^1(J^1t) \rightarrow J^1t \).

**Proposition** (Characterization of \( J^2t \subset J^1(J^1t) \) vector bundle \( t \), one has \( J^2t = \ker \omega_2 \), where
\[ \omega_2: J^2_t \rightarrow \text{Alt}^2(TM) \]
is a vector bundle morphism well defined by
\[ (\omega_2\xi)(V_1,V_2) := D_{V_1}D_{V_2}\xi - D_{V_2}D_{V_1}\xi \]
Here \( J^2_t \subset J^1(J^1t) \) is the kernel of the vector bundle morphism
\[ \omega_1: J^1(J^1t) \rightarrow T^*M \]
well defined by
\[ (\omega_1\xi)V := D_V(p\xi) - df(p\xi) \]
In this proposition some \( D \)'s are operators \( \Gamma(J^1t) \rightarrow \Gamma(T^*M \otimes J^1t) \). All ambiguity is mitigated by the context.

Since the proposition above is just a general fact about vector bundles, its proof is relegated to Appendix B.4

### 8.3. Torsion

We now return to the case that \( g \) is a structure on a Lie algebroid \( t \). Applying the general characterization of \( g^{(1)} \) as an isotropy subalgebroid.

Regard the restriction \( a: g \rightarrow t \) of \( J^1t \rightarrow t \) as this is the *tautological one-form*. The adjoint represents a representation of \( g \) on \( t \). So the exterior derivative is a \( g \)-form, of degree two. This is the *torsion* of the form
\[ da(X_1,X_2) = \text{ad}^t_{X_1}(aX_2) - \text{ad}^t_{X_2}(aX_1) \]
**Remark.** If \( \mathfrak{g} \) is intransitive, then \((J^1 \mathfrak{g})_{a,da}\) generally be the prolongation of \( \mathfrak{g} \): every section of \( \mathfrak{h} \subset T^*\mathfrak{f} \) is in the image of the anchor \# : \( \mathfrak{g} \to TM \) turns out that is not a symmetry of \( \mathfrak{g}^{(1)} \).

The proposition is an easy corollary of Proposition 1.2.

**Lemma.** Let \( \mathfrak{g} \subset J^1 \mathfrak{t} \) be a (possible intransitive) Lie algebroid on \( \mathfrak{t} \) and let \( D \) denote the deviation operator discussed in 8.2. Then for an arbitrary section \( \xi \subset J^1 \mathfrak{g} \), one has

\[
\begin{align*}
(1) \quad (\xi \cdot a)X &= D_\#X(p\xi) - aD_\#X\xi, \\
(2) \quad (\xi \cdot da)(X_1, X_2) &= d(\xi \cdot a)(X_1, X_2) + D_\#X_1D_\#X_2\xi.
\end{align*}
\]

Here \( X, X_1, X_2 \subset \mathfrak{g} \) are arbitrary sections.

**Proof of lemma.** Begin by observing that

\[
(\xi \cdot a)(X) = \text{ad}^t_{p\xi}(aX) - \text{ad}^t_{(J^1 a)\xi}(aX),
\]

Since \( a : \mathfrak{g} \to \mathfrak{t} \) is a Lie algebroid morphism, the identity

\[
\text{ad}^t_{(J^1 a)\xi}(aX),
\]

and so

\[
(\xi \cdot a)(X) = \text{ad}^t_{p\xi - (J^1 a)\xi}(aX).
\]

Note here that \( J^1 a : J^1 \mathfrak{g} \to J^1 \mathfrak{t} \) is the morphism.

Because \( p\xi - (J^1 a)\xi \) is a section of the kernel of \( J^1 \mathfrak{g} \) of \( T^*\mathfrak{g} \otimes \mathfrak{t} \) and, applying \[3.62\], obtain

\[
(3) \quad (\xi \cdot a)(X) = (p\xi - (J^1 a)\xi).
\]

On the other hand, since \( \xi = J^1(p\xi) + D\xi \), we have implying

\[
p\xi - (J^1 a)\xi = D(p\xi) - (J^1 a)D\xi
\]

\[
\implies (p\xi - (J^1 a)\xi)V = D_V(p\xi) - aV.
\]

Combining this with \[3\] gives \[1\].

It is not too difficult to show that \( \xi \cdot da = d(\xi \cdot a) \).

Therefore

\[
J^1(p\xi) \cdot da = d(J^1(p\xi) \cdot a) = d(\xi \cdot a).
\]
8.4. Normalizing torsion and the upper coboundary morphism. Identify \( g \) with \( t \oplus h \) by choosing a generator \( \nabla \) of \( g \). Then we obtain a corresponding identification,

\[
\text{Alt}^2(g) \otimes t \cong \left( \text{Alt}^2(t) \otimes t \right) \oplus \left( t^* \otimes h^* \right) \oplus \left( t^* \otimes h^* \right) \oplus \left( \text{Alt}^2(h) \otimes t \right)
\]

and a corresponding splitting of the torsion

\[
da = \text{tor } \bar{\nabla} \oplus \text{ev} \oplus 0.
\]

Here \( \bar{\nabla} \) denotes the associated \( t \)-connection on \( t \), and \( \text{ev} \) is just evaluation, \( \text{ev}(V \otimes \phi) := \phi(V) \). Notice that \( \text{tor } \bar{\nabla} \) is the only component of \( da \) depending on the choice of generator. Given two generators \( \nabla_1 \) and \( \nabla_2 \), their difference \( \nabla_2 - \nabla_1 \) may be viewed as a section of \( t^* \otimes h \) and one readily computes

\[
\text{tor } \bar{\nabla}_2 = \text{tor } \bar{\nabla}_1 + \Delta(\nabla_2 - \nabla_1)
\]

(1)

where \( \Delta \) denotes the upper coboundary morphism,

\[
t^* \otimes h \hookrightarrow t^* \otimes T^*M \otimes t \xrightarrow{\text{id} \otimes \#^* \otimes \text{id}} t^* \otimes h
\]

Here \( \#^* : T^*M \rightarrow t^* \) is the dual of the anchor \( \# : t \rightarrow TM \). As an elementary consequence of (1) above, we obtain

**Proposition.** If \( C \subset \text{Alt}^2(t) \otimes t \) is a complement for the image of \( \Delta \), then there exists a generator \( \nabla \) such that \( \text{tor } \bar{\nabla} \subset C \). If \( \Delta \) is injective, then this generator is unique.

Note that there is no need to require that \( C \) be \( g \)-invariant.

8.5. Intrinsic torsion and torsion reduction. In this section, we define the torsion bundle,

\[
H(g) := \frac{\text{Alt}^2(t) \otimes t}{\text{im} \Delta}
\]

and call the image \( \tau \) of \( \text{tor } \bar{\nabla} \), under the map \( \Gamma(\text{Alt}^2(t) \otimes t) \rightarrow H(g) \), the intrinsic torsion of the choice of generator, i.e., is an invariant of \( \nabla \). This is a \( g \)-representation whenever it is a bona fide vector bundle.
Assumption. In this section \( g \subset J^1 t \) is a surjective infinitesimal geometric structure over a transitive Lie algebroid \( t \). In particular, \( g \) has a structure kernel \( h \) of constant rank and \( g \) admits generators (Proposition 6.1(1)).

Our chief objective is a characterization of the \( \Theta \)-reduction \( g(1) \) that does not require an explicit knowledge of \( g(1) \).

9.1. The lower coboundary morphism. As in torsion reduction, a ‘coboundary morphism’ plays a central role in \( \Theta \)-reduction. However, unless \( t = TM \), the upper coboundary morphism \( \Delta \), defined in 8.4, is not the appropriate one. Rather, we need the lower coboundary morphism \( \delta \), defined as the composite

\[
T^*M \otimes h \hookrightarrow T^*M \otimes T^*M \otimes t \xrightarrow{A \otimes} \text{Alt}^2 (TM) \otimes t
\]

where \( A(\alpha \otimes \beta) := \alpha \wedge \beta \). This morphism is also a morphism of \( g \)-representations.

As we assume \( t \) is transitive, we may, by dualizing the anchor map \( \#: t \to TM \), regard \( T^*M \) as a subbundle of \( t^* \), and obtain natural inclusions

\[
T^*M \otimes h \hookrightarrow t^* \otimes h \quad \text{and} \quad \text{Alt}^2 (TM) \otimes t \hookrightarrow \text{Alt}^2 (t) \otimes t
\]

With this understanding, we may regard \( \delta : T^*M \hookrightarrow \text{Alt}^2 (TM) \otimes t \) as the restriction of the upper coboundary morphism \( \Delta \) defined in 8.4.

The analogue of the torsion bundle \( H(g) \) defines a bundle

\[
h(g) := \frac{\text{Alt}^2 (TM)}{\text{im} \delta}
\]

Whenever \( h(g) \) is a genuine vector bundle (has constant rank), there is evidently a natural morphism \( \psi : h(g) \to \text{Alt}^2 (t) \otimes t \) making the following diagram commute:

\[
\begin{array}{ccc}
\text{Alt}^2 (TM) \otimes t & \xrightarrow{/ \text{im} \delta} & \text{Alt}^2 (t) \otimes t \\
\text{inclusion} & & \xrightarrow{/ \text{im} \Delta}
\end{array}
\]
where \( p : J^1 g \to g \) is the projection. This follows e.g., [8.3][3]. One establishes (1) by applying Lemma B.1.

Now \( g^{(1)} \) is the kernel of the morphism \( \xi \mapsto \xi \cdot da \). It follows from [8.3][2] and transitivity that:

(2) For any \( \xi \in (J^1 g)_a \), the element \( \xi \cdot da \in \text{Alt}^2 \) an element \( (\xi \cdot da) \wedge \in \text{Alt}^2(TM) \otimes t \).

This means we may regard \( g^{(1)} \) as the kernel of a morphism:

\[
(J^1 g)_a \overset{\theta}{\longrightarrow} \text{Alt}^2(TM) \otimes t
\]

\( \xi \mapsto (\xi \cdot da) \wedge \). According to (1), the domain of \( \theta \) fits into an exact sequence:

\[
0 \to T^*M \otimes h \overset{}{\hookrightarrow} (J^1 g)_a
\]

Applying Lemma B.1 to the morphism \( \theta \), we obtain:

\[
0 \to \ker \delta \hookrightarrow g^{(1)} \overset{a^{(1)}}{\longrightarrow} \ker \Theta
\]

where \( \Theta \) is the unique morphism making the following diagram commute:

\[
\begin{array}{ccc}
(J^1 g)_a & \longrightarrow & \\
\downarrow \theta & & \\
\text{Alt}^2(TM) \otimes t & \overset{\text{im} \delta}{\longrightarrow} &
\end{array}
\]

Summarizing:

**Proposition.** If \( g \subset J^1 t \) is surjective and \( t \) is an arbitrary morphism \( \Theta : g \to h(g) \), constructed above, such that:

\[
0 \to \ker \delta \hookrightarrow g^{(1)} \overset{a^{(1)}}{\longrightarrow} g
\]

is exact. In particular, the structure kernel of \( g^{(1)} \) is an exact sequence.

**Remark.** By the proposition the structure kernel \( h \subset T^*M \otimes g \) and is consequently commutative.
Corollary. Suppose that the torsion bundle $H^1(\mathfrak{g}_\tau)$ of $\mathfrak{g}$ is well-defined. Then $\mathfrak{g}_1(\tau)$ coboundary morphism $\delta$ has constant rank, and hence $\mathfrak{g}_\tau$ is a reduction of $\mathfrak{g}$ in the sense of 2.3. Thus, torsion reduction coincide.

Here the rank hypotheses and Proposition 9.2 ensure that Proposition 2.5 applies. However, the result presumably holds with a constant rank hypothesis on $\mathfrak{g}_\tau$ alone.

9.4. Structures both surjective and $\Theta$-reduced

Theorem. Let $\mathfrak{g} \subset J^1t$ be a surjective infinitesimal geometric structure on a transitive Lie algebroid $\mathfrak{g}$. Assume that $\mathfrak{g}$ is $\Theta$-reduced (equivalently, that the map $\delta$ defined above vanishes). Assume that the associated lower coboundary morphism $J_\mathfrak{g}$ is injective. Then $\mathfrak{g}$ has an associated Cartan algebroid, namely $\mathfrak{g}$ itself, equipped with a canonical Cartan connection $\nabla^{(1)}$. The $\nabla^{(1)}$-parallel sections of $\mathfrak{g}$ coincide with the prolonged symmetries of $\mathfrak{g}$.

Proof. Proposition 2.5 implies the prolongation $\mathfrak{g}^{(1)}$ has trivial structure kernel, because we supposed $\mathfrak{g}^{(1)}$ is injective.

Applying Theorem 2.1 to the infinitesimal geometric structure $\nabla^{(1)}$ on $\mathfrak{g}$ whose parallel sections are the symmetries of $\mathfrak{g}$.

These are nothing but the prolonged symmetries of $\mathfrak{g}$.

In Proposition 11.1 we characterize $\nabla^{(1)}$ as the Cartan connection $\nabla^{(1)}$ on $\mathfrak{g}$ whose curvature $\text{curv} \nabla^{(1)} \subset \text{Alt}^2(TM) \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. The formula expressing $\nabla^{(1)}$ in terms of a generator of $\mathfrak{g}$.

9.5. The special case $t = TM$. When $t = TM$, the lower and upper coboundaries are the same thing, as are the upper and lower Cartan connections. We now rewrite the above theorem accordingly, $\nabla^{(1)}$ the Cartan connection that we establish later in 11.4.

Here $\nabla$ will denote the dual of $\nabla$, i.e., $\nabla_U V = J^1(TM)$ reductive if $\Delta$ has constant rank and if the $\Theta$-reduction and $\tau$-reduction coincide. We call the generator $\nabla$ normal. Proposition 8.4 guarantees the existence of normal generators when $\mathfrak{g}$ has constant rank, so that the prolongation $\mathfrak{g}_1(\tau)$ has constant rank, and both $\Theta$-reduction and $\tau$-reduction coincide.
(2) Identifying \( g \) with \( TM \oplus \mathfrak{h} \) using the generator \( e \),

\[
\nabla^{(1)}_U (V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\nabla_U \phi + \mathbf{h})
\]

(3) If \( g \) is reductive and \( \nabla \) is normal, or if \( \tau = 0 \) for any normal generator \( \nabla \).

When one of the conditions in (3) holds, obstruction theory in particular is simple to describe, as is the symmetry Lie algebra. If \( \tau \) is simply-connected, then equality holds if and only if \( \Delta \) is an isomorphism.

\[
\Delta: g - \text{parallel} \rightarrow \text{torsion-reduced}.
\]

This is trivially reductive. Applying Theorem 9.5, we obtain the Cartan connection \( \nabla \).

\[
\nabla = \nabla \circ \nabla = \text{curv } \bar{\nabla} = 0,
\]

\[
cur \nabla^{(1)}(U_1, U_2) (V \oplus \phi) = 0 \oplus \left( - (\nabla_V \text{curv } \bar{\nabla}) \right) \oplus \left( \text{tor } \nabla^{(1)} V_1 \oplus \phi_1, V_2 \oplus \phi_2 \right).
\]

Here \( \nabla^{(1)} \) denotes the representation of \( g \) on its Cartan connection \( \nabla^{(1)} \) on \( g \). Applying Theorem 4.6

Corollary. Let \( g \subset J^1(TM) \) be an infinitesimal geometric structure satisfying the hypotheses of the above theorem, and assume \( \nabla \) is normal, or that \( \tau = 0 \) and \( \nabla \) is torsion-free. If \( \mathfrak{g}_0 \) be the Lie algebra of all symmetries of \( \nabla \), then equality holds if and only if \( \mathfrak{g}_0 \) is naturally isomorphic (arbitrary) with Lie bracket given by

\[
[V_1 \oplus \phi_1, V_2 \oplus \phi_2] = (\text{tor } \nabla(V_1, V_2) + \phi_1(V_2) - \phi_2(V_1))
\]

9.6. The symmetries of Riemannian structure

The upper coboundary morphism for \( g \) is a map

\[
T^*M \otimes \mathfrak{h} \xrightarrow{\Delta} \text{Alt}^2(TM)
\]

where \( \mathfrak{h} \subset T^*M \otimes TM \) is the \( \mathfrak{o}(n) \)-bundle of skew-symmetric endomorphisms. This morphism is well-known to be...
According to Corollary 9.5, we are in the maximally symmetric case when $\nabla$ is both $\mathfrak{h}$-invariant and $\nabla$-parallel. According to a well-known representation-theoretic analysis of the curvature module, this happens if and only if

$$\text{curv} \nabla(V_1, V_2) = s \left( \sigma(V_1) \otimes V_2 - \sigma(V_2) \otimes V_1 \right)$$

for some constant $s \in \mathbb{R}$ (the scalar curvature). The Lie algebra described in the corollary is then isomorphic to the Lie algebra of infinitesimal isometries of Euclidean space, hyperbolic space, or the sphere, according to whether $s = 0$, $s < 0$, or $s > 0$.

**9.7. The symmetries of a conformal parallelism.**

Let $\omega: TM \to V$ be global parallelism ($V$ a vector space with the dimension of $M$) and let $\langle \omega \rangle \subset T^*M \otimes V$ be the line bundle spanned by $\omega$. A conformal parallelism is an equivalence class of absolute parallelisms, where $\omega, \omega': TM \to V$ are considered equivalent if $\omega' = f \omega$ for some positive function $f$. The infinitesimal isometries of the conformal parallelism having $\omega$ as representative coincide with the symmetries of the isotropy $g \subset J^1(TM)$ of $\langle \omega \rangle \subset T^*M \otimes V$.

A straightforward application of Lemma B.1 shows that $g$ is transitive, with rank-one structure kernel $\langle \text{id} \rangle \subset T^*M \otimes TM$.

The upper boundary morphism, given by

$$\Delta: T^*M \to \text{Alt}^2(TM)$$

is evidently injective ($\dim M \geq 2$). We leave it to the reader to verify that $g$ has vanishing intrinsic torsion $\tau$ precisely when

$$d\omega = \alpha \wedge \omega,$$

for some one-form $\alpha$. While $\alpha$ depends on the choice of representative $\omega$, the two-form $d\alpha$ does not.

Assuming $\tau = 0$, $g$ has a unique torsion-free generator, i.e., a torsion-free parallelism $\nabla$ (by definition of $\tau$). Moreover, it is not hard to show that

$$\nabla_U \omega = \alpha(U)\omega,$$

and accordingly that

$$\text{curv} \nabla = d\alpha \otimes \text{id}.$$
Here we shall understand $\mathcal{H}$ to be transversally oriented, so as to make the specification of the subriemannian structure to include a specific orientation. This amounts to the choice of a non-vanishing one-form $\theta$ annihilating $\mathcal{H}$. The contact hypothesis means that $\mathcal{H}$ is equipped with a symplectic structure.

The infinitesimal isometries of the subriemannian metric $\sigma$ generate the symplectic group $Sp(2,\mathbb{R})$. The infinitesimal isometries of the infinitesimal geometric structure $J^1(TM)$ is the isotropy of $\mathcal{H}$ and $J^1(TM)_{\mathcal{H},\sigma} \subset J^1(TM)_{\mathcal{H},\sigma}$

10.1. Preliminary reduction. The symplectic group $Sp(2,\mathbb{R})$ acts effectively on $\mathcal{H}$, and the isotropy $J^1(TM)_{\mathcal{H},\sigma}$ of $\mathcal{H}$ has kernel $\langle \mathbf{n} \rangle$, image $\mathcal{H}$, and the restriction of $\mathcal{H}$ to $\mathcal{H}$ relating the area form $dA$ to the subriemannian metric $\sigma$.

$\mathbf{g} := J^1(TM)_{\mathcal{H},\sigma, d\theta} \subset J^1(TM)_{\mathcal{H},\sigma}$

The reduction of $\mathcal{H}$ is a reduction of $J^1(TM)_{\mathcal{H},\sigma}$. (This reduction is in fact the first torsion-free reduction of $J^1(TM)_{\mathcal{H},\sigma}$.)

The subriemannian metric $\sigma$ has a canonical extension to a bona fide Riemannian metric, defined as follows: Let $\mathbf{n}$ be the Reeb vector field associated with the contact form $\theta$. That is, $d\theta(\mathbf{n}, \cdot) = 0$, $\theta(\mathbf{n}) := 0$. The easy proof of the following

Proposition. The reduction $\mathbf{g} \subset J^1(TM)$ above is a surjective infinitesimal geometric structure kernel $\mathbf{h} \subset T^*M \otimes TM$ is the globally trivial $\mathbf{g}$.

10.2. The complex structure on $\mathcal{H}$. Let $\times$ be determined by the extended metric $\sigma$ and define $J \subset TM$. $J$ has kernel $\langle \mathbf{n} \rangle$, image $\mathcal{H}$, and the restriction of $\mathcal{H}$ on $\mathcal{H}$ relating the area form $dA$ to the subriemannian metric $\sigma$.

Proposition. $\mathbf{g} \subset J^1(TM)$ is a surjective infinitesimal geometric structure kernel $\mathbf{h} \subset T^*M \otimes TM$ is the globally trivial $\mathbf{g}$. 

$\mathbf{g}$
Proposition.

(1) \( g \subset J^1(TM) \) is reductive, in the sense of [9.5], and hence defines a natural \( \mathfrak{g} \)-representation and a complex line-bundle.

(2) For any generator \( \nabla \) of \( g \) and all vector fields \( \nabla V \in \mathcal{H} \), allowing us to view \( \nabla \mathbf{n} \) as a \( \mathfrak{g} \)-invariant complement for its image, and applying Proposition 8.4. This analysis is not hard but a little tedious, and is relegated to Appendix B.3. For the interested reader, we include there a formula for the extended metric \( \nabla \mathbf{n} \subset (\mathcal{H}^* \otimes \mathcal{H})_\text{sym} \) and

\[ \text{Here } \nabla \sigma|\mathcal{H} \subset \mathcal{H}^* \otimes \text{Sym}^2(\mathcal{H}) \text{ denotes the restriction.} \]

With \( \nabla \) so fixed, we have:

(4) The torsion \( \text{tor } \nabla = - \text{tor } \nabla \) is given by the formula

\[ \text{tor } \nabla(U_1 + a_1 \mathbf{n}, U_2 + a_2 \mathbf{n}) = (a_1 \nabla_{U_2} \mathbf{n} - a_2 \nabla_{U_1} \mathbf{n}). \]

Here \( U_1, U_2 \in \mathcal{H} \), \( a_1, a_2 \in \mathbb{R} \).

(5) There exists a natural isomorphism of \( \mathfrak{g} \)-representations

\[ H(\mathfrak{g}) \cong \text{Alt}^2(TM) \oplus (\mathcal{H}^* \otimes \text{Sym}^2(\mathcal{H})) \]

with respect to which the intrinsic torsion of \( \mathfrak{g} \)

\[ \tau = d\theta \oplus \nabla \mathbf{n}. \]

(6) The intrinsic torsion component \( \nabla \mathbf{n} \) can be a transverse derivative of the subriemannian metric \( \sigma \):

\[ \sigma(\nabla_{U_1} \mathbf{n}, U_2) = (\nabla \mathbf{n} \sigma)(U_1, U_2); \]

The proposition is established by analyzing the detail, identifying a natural \( \mathfrak{g} \)-invariant complement of \( \mathfrak{g} \)-equivariance. For the interested reader, we write down Bianchi identities for the normalized generator \( \nabla \) in terms of the Levi-Cevita connection associated with the extended metric \( \nabla \mathbf{n} \).

10.4. Bianchi Identities and low weight differential operators. For now, we merely list those operators of 'weight' two or less.

There exists a unique and normal generator

\[ \nabla \mathbf{n} \subset (\mathcal{H}^* \otimes \mathcal{H})_\text{sym} \]
We are now ready to define two invariant operators \( \partial_- : \Gamma(H_2) \to \Gamma(H) \) according to

\[
(\partial_+ U)V = \frac{1}{2} \left( \tilde{\nabla}_V U + JU V \right),
\]

\[
(\nabla_{U_1} q)U_2 - (\nabla_{U_2} q)U_1 = dA(U_1, U_2) J\sigma(U_3).
\]

Associated with the normalized generator \( \nabla \) of quantities \( T := \text{tor} \, \nabla \) and \( \Omega := \text{cocur} \, \nabla = -\text{cur} \, \nabla \), we have \( 6.5(3) \) and \( 6.5(4) \). Of course these are also invariants of the submanifold contact structure. According to \( 10.3(4) \), \( T \) depends only on the invariant \( \nabla n \). As it turns out, one component of \( \Omega \) is a new invariant function. Recalling that \( \Omega(U_1, U_2) \subset H \) for all \( U_1, U_2 \subset TM \) (Proposition 6.2(3)) and that \( \Omega(U_1, U_2) \subset \mathfrak{h} \), there is a real-valued function \( \kappa \) well defined by

\[
(1) \quad \Omega(U_1, U_2)U_3 = -\kappa dA(U_1, U_2) JU_3;
\]

**Proposition** (Bianchi identities).

(2) \( \text{trace}(\nabla n) = 0 \), i.e., \( \nabla n \subset H_2 \).

(3) \( \partial_\kappa \kappa = -\frac{1}{2} \text{curl}_{\mathcal{H}}(\partial_-(\nabla n)) \).

(4) The cocurvature of \( \nabla \) is given by

\[
\Omega(U_1 + a_1 n, U_2 + a_2 n)(U_3 + a_3 n) =
\left( \text{trace}(\nabla n) \right) - \kappa dA(U_1, U_2) + \frac{1}{2} \sigma(\partial_-(\nabla n), a_1 n, a_2 n).
\]

**Proof.** Proposition \( 10.3(4) \) states that

\[
T(U_1 + a_1 n, U_2 + a_2 n) = (a_1 \nabla U_2 n - a_2 \nabla U_1 n).
\]

A little multilinear algebra determines that \( \Omega \) has

\[
\Omega(U_1 + a_1 n, U_2 + a_2 n)(U_3 + a_3 n) = -\left( \kappa dA(U_1, U_2) + \frac{1}{2} \sigma(\partial_-(\nabla n), a_1 n, a_2 n) \right)
\]

for some section \( \omega \subset \mathcal{H}^* \) and some \( \kappa \) as above. \( 6.5(4) \) are equations in bundle-valued three-forms that vanish if and only if \( \lambda(U_1, U_2, n) = 0 \) for all sections \( U_1, U_2, n \). In fact to the Bianchi identities gives...
Suppose that $\nabla n = 0$. Then $n$ is automatically an infinitesimal isometry of the subriemannian contact structure of $\Sigma$. Note that if the rank-one foliation generated by $\partial$ on a surface $\Sigma$, then the invariant function $\kappa$ drops to $\kappa = 0$. In any case, Theorem [9.5] applies, because of [10.3(1)]. Using the formula for $\Omega = \nabla n - \text{curv} \nabla$ above, one applies this theorem and its conclusion.

**Proposition** (Compare with [9]). Suppose $\nabla n$ is an associated Cartan algebroid, namely $g$ itself. If $U$ and $g_0$ are the Lie algebra of all infinitesimal isometries of $U$, then $\dim g_0 \leq \text{rank } g = 4$. If $U$ holds if and only if the function $\kappa$ defined by $\kappa = 0$, $\kappa < 0$, or $\kappa > 0$.

10.6. **Invariant differential operators.** The normalized generator $\kappa$ is automatically a symmetry of the Lie algebra $\mathfrak{g}$ of infinitesimal isometries $\mathfrak{h}$. When $\mathfrak{g}_0 \cong \mathfrak{b} \times \mathbb{R}$ (direct product) where $\mathfrak{b}$ is the Lie algebra of Killing fields of the Euclidean plane, hyperbolic plane, or sphere, according to whether $\kappa = 0$, $\kappa < 0$, or $\kappa > 0$.

Define $\mathcal{H}_0 := \mathbb{C} \times M$, $\mathcal{H}_1 := \mathcal{H}$, and define $\mathcal{H}_2$ where we define

$$
\mathcal{H}_k := \text{Sym}^{k-1}(\mathcal{H}) \otimes_{\mathbb{C}} \mathcal{H}
$$

where $\mathcal{H}$ is $\mathcal{H}$ with the complex structure $-J$. For any irreducible representation and a complex line-bundle, the two structures being to

$$
ad_J q = k i q; \quad q \in \mathcal{H}_k
$$

Every $\mathcal{H}_k$ is irreducible as a (real) $g$-representation of the irreducible trivial representation $\mathbb{R} \times M$.

Recall that for each section $q \in E$ of an irreducible $\mathcal{H}$-representation, the objective is to derive the decomposition of $\nabla q$ of the decomposition of $T^*M \otimes E$ into irreducibles.
for $k = 1$. In the latter case we are using the Hermitian pairing $\langle U, V \rangle = \sigma(U, V) - idA(U, V)$.

A compatible pair of splitting morphisms $\mathcal{H}_{k+1}$ are defined as follows:

$$ (\pi_+ Q)(U, V_1, \ldots, V_{k-1}) = \frac{1}{2} \langle Q(U, V_1, \ldots, V_{k-1}) \rangle $$

for all $k \geq 1$, and

$$ (sq)(U_1, U_2, V_1, \ldots, V_{k-2}) = \frac{i}{2} \langle U_1, U_2, V_1, \ldots, V_{k-2} \rangle $$

for $k \geq 2$, while

$$ (sq)U = \frac{1}{2} qU, $$

for $k = 1$ ($q$ a $\mathbb{C}$-valued function).

Let $q$ be a section of $\mathcal{H}_k$. Then we have a restricted propagator $\widetilde{\partial}q$ splitting, we have $\mathcal{H}^* \otimes \mathcal{H}_k \cong \mathcal{H}_{k-1} \oplus \mathcal{H}_{k+1}$ and an operator $\partial_+ q := \pi_-(\widetilde{\partial}q|\mathcal{H})$. That is, $\partial_+ q \subset \mathcal{H}_{k+1}$ and

$$ (\partial_+ q)(U, V_1, \ldots, V_{k-1}) = \frac{1}{2} \langle (\nabla U)q(V_1, \ldots, V_{k-1}) \rangle $$

for any $k \geq 1$,

$$ (\nabla U_1)q(U_2, V_1, \ldots, V_{k-2}) - (\nabla U_2)q(U_1, V_1, \ldots, V_{k-1}) = 0 $$

for $k \geq 2$, and

$$ (\nabla U_1)q(U_2) - (\nabla U_2)q(U_1) = dA $$

for $k = 1$. This last formula simply means, for $q \subset \mathcal{H}_k$,

$$ \partial_- q = \text{curl}_{\mathcal{H}}(q) - i dA. $$

Finally, for any section $q \subset \mathcal{H}_k$ and any $k \geq 1$, we have $\mathcal{H}^* \otimes \mathcal{H}_k \cong \mathcal{H}_{k-1}$ and $TM = \mathcal{H} \otimes \langle \mathfrak{n} \rangle$, we now obtain:

**Proposition.** For any $k \geq 1$, we have a natural isomorphism $\mathcal{H}^* \otimes \mathcal{H}_k \cong \mathcal{H}_{k-1}$.
We assume throughout that \( t \) is a \textit{transitive} Lie algebroid. For basic implications, see Sect. 9 under ‘Assumption.’ We continue to denote the structure kernel of \( g \subset J^1t \) on \( t \); in symbols, if
\[
aD\#_V X + [aX, V]_t = \text{ad}^t_X V;
\]
Here \( a: g \to t \) is the restricted projection \( J^1g \to t \) — not immediately useful in computations but is a natural intermediate result. It stands between the rather abstract Proposition 9.2 and the computationally useful Theorem 11.2 given later.

**Proposition.** Let \( D \) be any natural connection on \( g \) and consider the morphism,
\[
\hat{\Theta}: g \to \text{Alt}^2(TM) \otimes g^* \otimes h,
\]
where \( a \) is the projection \( g \to t \). Then:

1. The morphism \( \Theta: g \to h(g) \), defined in 9.2,
\[
\Theta(X)(U_1, U_2) := a(\text{curv}_D(U_1, U_2))_X;
\]

Moreover, if \( \ker \delta \) and \( \ker \Theta \) have constant rank, then \( g \) is an infinitesimal geometric structure, by Proposition 9.2, the:

2. If \( \Theta = 0 \), then all generators of \( g^{(1)} \) are natural;
3. A natural connection \( D \) on \( g \) generates \( g^{(1)} \)
\[
\text{curv}_D(U_1, U_2)X \in h \quad \text{for all} \quad X \in \ker \Theta.
\]

**Corollary.** If \( g \subset J^1t \) is a \( \Theta \)-reduced infinitesimal geometric structure, by Proposition 9.2, then:

A linear connection \( D \) on \( t \) generates \( g^{(1)} \) if and only if \( \text{curv}_D(U_1, U_2)X \in h \)

**Corollary.** The Cartan connection \( \nabla^{(1)} \) in Theorem 9.4 is the unique natural connection on \( g \) such that \( \text{curv}_{\nabla^{(1)}} \subset \text{Alt}^2(TM) \otimes g^* \otimes h \).

The proof follows a similar strategy as Proposition 9.2.
Proof. Let \( \mathcal{D} \) be the deviation operator described via:
\[
[aX, V]_t = \text{ad}^t_{J_1(aX)} V = \text{ad}^t_X V - \mathcal{D}^t [sX, V]
\]
Taking care not to confuse \( D \)'s with \( \mathcal{D} \)'s, we also have:
\[
aD^t[V]X = a(sX - J^1 X)(\#V) = a(D(sX - D^t)\#V)
\]
Here \( s \) is the splitting in (5). Combining this with:
\[
aD^t[V]X + [aX, V]_t - \text{ad}^t_X V = aD^t[V]X - \text{ad}^t_X V
\]
The claim in (4) now follows from (8.3(1)) and transitivity. In (6) is natural and, with the help of (4) and (9.2(2)), that all possibilities are derived as a consequence of (8.3(2)) and transitivity. One checks that the connection \( 's \) is always covered, establishing (6).

Proof of proposition. By (4), \( D \) generates \((J^1 g)_a \) surjective ([9.2(1)]), let \( s: g \to (J^1 g)_a \) denote the
\[
0 \to T^*M \otimes h \hookrightarrow (J^1 g)_a
\]
determined by the generator \( D \). By the commutativeness of \( \Theta \), we have:
\[
\Theta(X) = \theta(sX) \text{ mod } \text{im } \delta = (sX - D^t)\#V
\]
Invoking (5), we prove (1).

If \( \Theta = 0 \), then \( g^{(1)} \) is surjective, and so \( s(g) \subset \text{im } \delta \).
Here \( s: g \to g^{(1)} \) is the corresponding splitting of \( \Theta \). Being the case we have, in particular, \( s(g) \subset \text{im } \delta \) for \( (J^1 g)_a \) also. By (4), \( D \) is natural. This proves (2).

By (4), a natural connection \( D \) generates \( g^{(1)} \).
Here
\[ d\nabla \phi (U_1, U_2) := \nabla U_1 (\phi(U_2)) - \nabla U_2 (\phi(U_1)) \]
and
\[ (\nabla^h \phi) U := \nabla^h U \phi. \]

**Theorem** (Prolonging a generator of \( g \subset J^1 t \)).

Let \( g \) be a geometric structure on a transitive Lie algebroid \( J^1 t \).

Use \( \nabla \) to identify \( g \) with \( t \oplus \mathfrak{h} \). Then:

1. The composite morphism,
   \[ g \cong t \oplus \mathfrak{h} \xrightarrow{\tilde{\Theta}} \text{Alt}^2(TM) \otimes t \]
   coincides with the morphism \( \Theta: g \to h(g) \) defined in Proposition 9.2.

2. \( \ker \Theta \subset t \oplus \mathfrak{h} \) is precisely the set of all \( V \oplus \phi \) such that
   \[ \delta(\epsilon) = \tilde{\Theta}(V \oplus \phi) \]
   admits a solution \( \epsilon \in T^*M \otimes \mathfrak{h} \).

3. \( \ker \Theta \subset t \oplus \mathfrak{h} \) is precisely the set of all \( V \oplus \phi \) such that
   \[ \delta(\epsilon) = \tilde{\Theta}(V \oplus \phi) \]
   admits a solution \( \epsilon \in T^*M \otimes \mathfrak{h} \).

4. Assuming \( \ker \delta \) and \( \ker \Theta \) have constant rank on \( \mathfrak{h} \), an infinitesimal geometric structure, by Proposition 11.1, \( g \cong t \oplus \mathfrak{h} \) generating \( g^{(1)} \) is given by
   \[ \nabla_U^{(1)} (V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\mathfrak{h} \otimes \phi) \]
   where \( \epsilon: t \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h} \) is any of the vectors \( \epsilon := \epsilon(V \oplus \phi) \) solves the generator equation defined in \( \ker \Theta \). If \( g^{(1)} \) is surjective (i.e., \( \Theta = 0 \)) then it has this form.

**Proof.** Let \( D \) denote the general form of a natural connection with \( \epsilon: t \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h} \) completely arbitrary. From Proposition 11.1 then one computes
\[ \hat{\Theta}(V \oplus \phi)(U_1, U_2) = a \left( \text{curv} D (U_1, U_2)(V \oplus \phi) \right) \]
where \( \hat{\Theta} \) is the morphism defined by (1). Conclusion (4) follows from Proposition 11.1. Conclusion (3) is just another way of stating the definition of \( \tilde{\Theta} \) in 11.1(3) and 11.4. If \( \Theta = 0 \) then every generator is natural (Proposition 11.1).
Note that $\text{cocurv} \nabla(aX, \cdot)$ is a section of $T^*M \otimes t$.

**Proof.** Since $t$ is assumed to be transitive, there exists a section $\phi$ such that $\nabla^h\#U = \tilde{\nabla}U$ for all $U \in t$. Here $\tilde{\nabla}$ denotes the covariant derivative of $\mathfrak{h}$ discussed in [6.3](3). After a little manipulation, we get:

$$ \left( d\nabla \phi - \delta(\nabla^h \phi) \right) (\#U_1, \#U_2) = (\phi \cdot \text{tor} \nabla) \#U_1 \cdot \#U_2. $$

Note that $T^*M \otimes t$ (of which $\phi$ is a section) acts on $J^1 t$ (which acts on $t$ via adjoint action).

Replace $g$ in Proposition [3.8](3) with $t$ and replace $h$ by $\nabla^h$.

Then part (2) of that proposition delivers the formula:

$$ \text{curv} \nabla (\#U_1, \#U_2) V = (\tilde{\nabla} V \text{ tor} \tilde{\nabla})(U_1, U_2) + \text{curv} \tilde{\nabla} (V, U_2) U_1. $$

Applying Proposition [4.3](4), we may rewrite this as:

$$ \text{curv} \nabla (\#U_1, \#U_2) V = (\tilde{\nabla} V \text{ tor} \tilde{\nabla} + \Delta(\phi))(U_1, U_2). $$

Substituting (1) and (2) into the definition [11.2](2) of $\tilde{\Theta}(V \oplus \phi)(\#U_1, \#U_2) = (\tilde{\nabla} V \text{ tor} \tilde{\nabla} + \phi \cdot \text{tor} \tilde{\nabla}) U_1 \cdot U_2$, we obtain:

Under the identification $g \cong t \oplus \mathfrak{h}$ determined by $\psi$, the stated formula.

**Proof of Theorem [9.3](9).** By Theorem [11.2](2) and (2), we have:

$$ \psi(\Theta(X)) = i(\tilde{\Theta}(X)) \mod \psi(X) $$

where $i: \text{Alt}^2(TM) \otimes t \to \text{Alt}^2(t) \otimes t$ denotes the inclusion (being injective). The proposition above then gives:

$$ \psi(\Theta(X)) = X \cdot \text{tor} \tilde{\nabla} \mod \psi(X). $$

11.4. **The special case $g \subset J^1(TM)$**. We now consider the special case $t = TM$. As an application, we complete the proof of an unproven assertion of preceding sections.
(2) \( \ker \Theta \subset \mathfrak{g} \cong TM \oplus \mathfrak{h} \) is precisely the set of solutions of the generator equation,

\[
\Delta(\epsilon) = \tilde{\Theta}(V \oplus \phi)
\]

admits a solution \( \epsilon \in T^*M \otimes \mathfrak{h} \).

(3) Assume \( \ker \Delta \) and \( \ker \Theta \) have constant rank, \( \Delta \) is torsion-free linear connection (by Proposition [11.2]). Then \( \Theta(X) = X \cdot \tau \) is torsion. Also, a linear connection \( \nabla^{(1)} \) on \( \mathfrak{g} \) is given by

\[
\nabla^{(1)}_U(V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\nabla_U \phi + \epsilon(U))
\]

where \( \epsilon: TM \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h} \) is any of the values of \( \epsilon(V \oplus \phi) \) solves the generator equation defined in \( \ker \Theta \). If \( \mathfrak{g}^{(1)} \) is surjective (i.e., \( \Theta = 0 \)) then \( \nabla^{(1)} \) is torsion-free.

**Note.** The \( \epsilon \)'s solving the generator equation above, in Theorem [11.2] are different.

**Proof.** In [11.2] above take \( \tau = TM \) and let \( \nabla^\mathfrak{h} \) be the reduction with the generator \( \nabla \) (given by [6.2][1] with \( \tau = \mathfrak{h} \)) gives

\[
\tilde{\Theta}(V \oplus \phi) = \tilde{\Theta}(V \oplus \phi) + \Delta(\epsilon)
\]

Noting that \( \text{co} \text{curv} \nabla = - \text{curv} \nabla \) and \( \delta = \Delta(\epsilon) \) (both of which naturally arise in the intrinsic geometry) the stated results as a special case of Theorem [11.2] as above.

**Proof of Theorem [9.5].** The hypothesis that \( \mathfrak{g} \) be reductive and \( \nabla \) be a normal connection on \( \mathfrak{g} \) in Theorem [9.5] follows. The Cartan connection on \( \mathfrak{g} \) in Theorem [9.5]; conclusion (3) above implies that it has the form.

Suppose \( \mathfrak{g} \) is reductive and let \( \nabla \) be a normal connection on \( \mathfrak{g} \) in Theorem [9.5]. Some \( \mathfrak{g} \)-invariant complement \( C \subset \text{Alt}^2(TM) \otimes TM \).

\[
\nabla_{V} \text{tor} \nabla + \phi \cdot \text{tor} \nabla = (V \cdot \tau + \phi \cdot \tau)
\]

is a section of \( C \). However, this last also lies in
Now let $\mathfrak{g} \subset J^2(TM)$ instead denote the isotropy algebra in Proposition 5.8. Then it is not too difficult to check that, in this regard, a helpful formula, readily derived, is
\[\nabla^{(1)}(J^1V)(U_1, U_2) = 0 \oplus ((J^2V)/(J^1V))\]
for any section $V \subset TM$.

The generator $\nabla^{(1)}$ is necessarily the Cartan connection in Proposition 5.8. Its curvature is given by
\[\text{curv } \nabla^{(1)}(U_1, U_2)(V \oplus \phi) = 0 \oplus \left( - (\nabla_V curv) \right)\]
In particular, curv $\nabla^{(1)}$ vanishes if and only if curv $\nabla$ is invariant. But as $\text{id}_{TM}$ is a section of $T^*M \otimes TS^0M$, curv $\nabla = 0$. In that case we obtain
\[\text{tor } \nabla^{(1)}(V_1 \oplus \phi_1, V_2 \oplus \phi_2) = (\phi_1(V_2) - \phi_2(V_1))\]
where $\nabla^{(1)}$ denotes the representation of $J^1(TM)$ on $J^1(TM)$. Applying Theorem 4.6, we recover the following classical result:

**Proposition.** Let $\nabla$ be a torsion-free linear connection on a smooth manifold $M$ with $\text{rank } J^1(TM) = n(n+1)$, $n = \dim M$. If $\mathcal{U}$ is simply-connected and $\nabla$ is conformally parallel, with respect to the conformal metric $\mathfrak{g}$, if and only if $\text{curv } \nabla = 0$, in which case $\mathfrak{g}$ is naturally isomorphic to $\text{Sym}^e T^*m M \oplus (T^*_m M \otimes T^*_m M)$, $m \in \mathcal{U}$.

12. Application: Conformal Structures

In this section we turn to the application of Cartan's method to conformal structures. Our results are summarized in Theorems 12.1, 12.2.

12.1. The Lie Algebroid Setting. Let $\mathfrak{s}$ be a Lie algebra of smooth vector fields on a smooth $n$-dimensional manifold $M$, with $n := \dim M \geq 3$. Let $\mathfrak{g}$ be viewed as the one-dimensional subbundle of $\text{Sym}^e T^*M$.

Let $\mathfrak{g}_\mathfrak{s} \subset J^1(TM)$ denote the isotropy of $\mathfrak{g}$ of $\mathfrak{s}$.

Let $\mathfrak{g} \subset J^1(TM)$ denote the isotropy of $\langle \mathfrak{s} \rangle \subset J^1(TM)$.

This means that the 1-jet of a vector field $V$ at $p \in M$ is a section of the vector bundle $J^1(TM) \to M$, $V \mapsto J^1_p V$. Linearizing about a point $p \in M$, we get a linear representation $\mathfrak{g}_p \subset J^1(TM)$.

In this section we will apply Cartan's method to conformal structures on a manifold $M$ with $\text{dim } M \geq 3$. The Lie algebroid setting.
by skew(\(\phi\)) := (\(\phi - \phi^t\)). These morphisms and

Because the Levi-Cevita connection \(\nabla\) associated with \(\sigma\) generates \(g_\sigma\), it also generates \(g \supset g_\sigma\).

12.2. Classical ingredients. From well-known representation-theoretic arguments we know that the curvature of the Levi-Cevita connection \(\nabla\) takes values in a proper

\[ T^*M \otimes T^*M \xrightarrow{\text{coricci}} \text{Alt}^2(TM) \otimes \mathfrak{h}_\sigma, \]

\[ \text{coricci}(\Phi)(V_1, V_2) := \text{skew}(\Phi V_1 \otimes V_2 - \Phi V_2 \otimes V_1). \]

\(E_{\text{Weyl}} \subset \text{Alt}^2(TM) \otimes \mathfrak{h}_\sigma\) is the intersection of the kernels of the so-called Bianchi and Ricci morphisms; see, e.g., [16, p. 230]. Whence,

\[ \text{curv } \nabla = W + \text{coricci}(R) \]

for uniquely determined sections \(W \subset E_{\text{Weyl}}\) and \(\text{coricci}(R)\) of \(\text{Alt}^2(TM) \otimes \mathfrak{h}_\sigma\) and in particular we may speak of the isotropy \(g_W \subset g\) of \(W\).

Also of significance will be the Cotton-York tensor \(d_\nabla R\) of \(R \subset \text{Sym}^2(TM) \subset T^*M \otimes T^*M\) on \(M\):

\[ d_\nabla R(U_1, U_2) := \nabla_{U_1}(R(U_2)) - \nabla_{U_2}(R(U_1)). \]

Alternatively, by torsion-freeness, \(d_\nabla R\) is the image

\[ T^*M \otimes \text{Sym}^2(TM) \hookrightarrow T^*M \otimes T^*M \otimes T^*M \]

\[ \alpha \otimes \beta \otimes \gamma \mapsto \alpha \wedge \beta \otimes \gamma. \]

Bianchi’s second identity [6.5][4] for the generator between the Cotton-York tensor \(d_\nabla R\), and the derivative \(d_\nabla d_\nabla R\), is known that \(W = 0\) implies the vanishing of \(d_\nabla R\), where \(E_{\text{Weyl}} = 0\) and the values of \(d_\nabla R\) are restricted.
12.4. **The $W = 0$ case.** Our second theorem lists results that are essentially classical:

**Theorem.** Suppose $W = 0$. Then $\mathfrak{g}$ has an associated Cartan algebroid, namely its prolongation $\mathfrak{g}^{(1)} \subset J^2(TM)$, which is surjective. Denoting the Cartan connection on $\mathfrak{g}^{(1)}$ by $\nabla^{(2)}$, we have:

1. The $\nabla^{(2)}$-parallel sections of $\mathfrak{g}^{(1)} \subset J^2(TM)$ are precisely the twice-prolonged conformal Killing fields.
2. Each metric $\sigma$ in the conformal class determines natural isomorphisms $\mathfrak{g}^{(1)} \cong \mathfrak{g} \oplus T^*M$, and an associated explicit formula for $\nabla^{(2)}$ (see 12.11(1) and 12.9(1) below).
3. If $n \geq 4$, then $\nabla^{(2)}$ is automatically flat. If $n < 4$, then $\nabla^{(2)}$ is flat if and only if $d\nabla R = 0$. In particular, the Lie algebra $\mathfrak{g}$ of all conformal Killing fields over any simply-connected open set $U \subset M$ satisfies
   \[
   \dim \mathfrak{g}_0 \leq \text{rank } \mathfrak{g}^{(1)} = \frac{1}{2}(n + 1)(n + 2),
   \]
   with equality holding if and only if $n \geq 4$ or $d\nabla R = 0$.

12.5. **Outline of the application of Cartan’s method.** Before describing partial results for the general case $W \neq 0$, we sketch the arguments leading to the results above.

Although $\mathfrak{g} \subset J^1(TM)$ is surjective, we have not apply. In 12.7 we show that $\mathfrak{g}$ is already $\Theta$-reduced. The coboundary morphism is not injective and Theorem 9.4 is not applicable. We turn then, in 12.8 and 12.9, to the prolongation $\mathfrak{g}^{(1)} \subset J^2(TM)$, which is surjective (because $\mathfrak{g}$ is $\Theta$-reduced) but has non-trivial structure kernel.

We show in 12.10 that $\mathfrak{g}^{(1)}$ is already $\Theta$-reduced and that the coboundary morphism associated with $\mathfrak{g}^{(1)}$ is injective and $\nabla^{(2)}$ is flat, making it an associated Cartan algebroid.

12.6. **The $W \neq 0$ case and intransitivity.** If $\mathfrak{g}$ is not $\Theta$-reduced. According to Proposition 12.10 below, the structure kernel of $\mathfrak{g}^{(1)}$ is a reduction of $\mathfrak{g}$. The prolongation $\mathfrak{g}^{(1)}$ is therefore not surjective. Rather than continue to apply Cartan’s method, we remark that $\mathfrak{g}$ itself is necessarily a reduction of $\mathfrak{g}^{(1)}$ (assuming rank-constancy), and applying the algorithm to $\mathfrak{g}$ is an easier prospect.

Suppose that $W$ vanishes nowhere. Then the structure kernel of $\mathfrak{g}$ is contained within that of $\mathfrak{g}^{(1)}$ (because $\text{id} \cdot W$ is trivial). Since the upper (=lower) coboundary morphism for $\mathfrak{g}$ is injective, the same is true for $\mathfrak{g}^{(1)}$; the prolongation of $\mathfrak{g}$ will have trivial structure kernel. Singularities notwithstanding, $\mathfrak{g}$ will therefore have an associated Cartan algebroid that is some reduction of $\mathfrak{g}^{(1)}$, and in particular will be a subalgebroid of $J^1(TM)$ (in contrast to the $W = 0$ case). A detailed argument is omitted here.

However we proceed, the following result implies that any associated Cartan algebroid will be intransitive in general: Call $W$ strongly degenerate if there exists
a section $\phi \subset \mathfrak{h}$ such that $\nabla_V W = \phi \cdot W$, i.e., such that
\[
(\nabla_V W)(U_1, U_2)U_3 = \phi W(U_1, U_2)U_3 - W(\phi U_1, U_2)U_3 - W(U_1, \phi U_2)U_3
\]
for all vector fields $V, U_1, U_2, U_3$. We shall see in the remainder of this section that this definition is independent the metric within the conformal class used to fix a Levi-Cevita connection $\nabla$.

**Theorem.** The isotropy $g_W \subset J^1(TM)$ is surjective if and only if $W$ is strongly degenerate.

The remainder of this section is devoted to proofs of the three preceding theorems.

12.7. **The torsion reduction of $g$.** Since $g \subset h$ as $h$-based torsion, its torsion reduction is obtained as torsion reduction. To compute it, we turn to the upper (=lower) coboundary morphism for $g$,
\[
T^*M \otimes \mathfrak{h} \xrightarrow{\Delta} \text{Alt}^2(TM)
\]
Its restriction to $T^*M \otimes \mathfrak{h}_\sigma$ is nothing but the upper coboundary $\sigma = \Theta$. Since the latter is an isomorphism (see [9.6]), the following holds:
\[
\Delta(g) = 0,
\]
implying $g$ is already torsion-reduced. Theorem 9.5 does not apply, however, because $\Delta$ has non-trivial kernel. Indeed,
\[
\text{rank}(\ker \Delta) = \text{rank}(\ker \Theta) > 0,
\]
\[
(1)
\]
12.8. **The first prolongation $g^{(1)}$.** Since $g$ is already torsion-reduced (i.e., reduced) the prolongation $g^{(1)}$ is surjective (Proposition 9.2). Its structure kernel $\mathfrak{h}^{(1)}$ is $\ker \delta = \ker \Delta$. Define an inclusion
\[
T^*M \xrightarrow{i} \text{Sym}^2(TM) \otimes T^*M
\]
where $i(\alpha) := j_S(\alpha) - \sigma \otimes \sigma$. The map $j_S : T^*M \rightarrow \text{Sym}^2(TM) \otimes T^*M$ is the canonical map,
\[
\text{Sym}^2(TM)(V_1, V_2) = \alpha(V_1)V_2 + \alpha(V_2)V_1 - \alpha(V_1)V_2
\]
Then $i$ is a monomorphism of $g$-representations (1)
This formula may also be written

(2) \( \text{curv} \nabla^{(1)}(U_1, U_2)X = -(X \cdot \text{curv} \nabla)(U_1, U_2) \subset h \sigma; \)

\( X \subset g \). 

12.10. The \( \Theta \)-reduction of \( g^{(1)} \). Since \( g^{(1)} \subset J^1 \)
the \( \Theta \)-reduction of \( g^{(1)} \) is the kernel of a morphism \( g^{(1)} \rightarrow \Theta^{(1)} \), to distinguish it from the corresponding
\[ \Theta^{(1)} \]
The definition of \( h(g^{(1)}) \) depends on the low coboundary morphism \( \delta^{(1)} \)
which we denote by

\[
T^*M \otimes h^{(1)} \xrightarrow{\delta^{(1)}} \text{Alt}^2(TM)
\]

Identifying \( h^{(1)} \) with \( T^*M \) as described above, one shows that \( \delta^{(1)} \) is the map \( \alpha \otimes \beta \mapsto \text{coRicci}(\alpha \otimes \beta) + (\alpha \wedge \beta) \otimes \text{id}TM \).

Note that the first term on the right belongs to \( \text{Alt}^2(TM) \otimes h \sigma \) and the second to \( \text{Alt}^2(TM) \otimes \langle \text{id}TM \rangle \). In particular, the image of \( \delta^{(1)} \) lies entirely within \( \text{Alt}^2(TM) \otimes h \sigma \).

Since \( \text{coRicci} \) is injective \((n \geq 3)\) we have ker \( \delta^{(1)} = 0 \). The second prolongation \( g^{(2)} := (g^{(1)})^{(1)} \) of \( g \) has trivial structure kernel. In particular, \( h(g^{(1)}) := (\text{Alt}^2(TM) \otimes g)/\text{im} \delta^{(1)} \) has constant rank.

Next, we observe that the composite morphism

\[
E_{\text{Weyl}} \hookrightarrow \text{Alt}^2(TM) \otimes h \sigma \hookrightarrow \text{Alt}^2(TM)/\text{im} \delta^{(1)}
\]
is injective. This follows from the description of \( E_{\text{Weyl}} \cap E_{\text{Ricci}} = 0 \),
where \( E_{\text{Ricci}} = \text{coRicci}(S^2(TM)) \).

Identifying \( E_{\text{Weyl}} \) with the corresponding \( g \)-subrepresentation

**Proposition.** The following diagram commutes.

\[
\begin{array}{ccc}
g^{(1)} & \xrightarrow{\Theta^{(1)}} & h(g^{(1)}) \\
\text{projection} & & \text{inclusion} \\
g & \xrightarrow{\Theta} & E_{\text{Weyl}}
\end{array}
\]
algebra acts on $E_{\text{Weyl}}(m)$ and
\[ \mathfrak{h} \cdot W = \bigcup_{m \in M} \{ \phi \cdot W(m) \mid \phi \} \]

Evidently, $W$ is strongly degenerate if and only if $\Delta \in \mathfrak{g}$.

\textbf{Proof of proposition.} We will apply part $\mathbf{(2)}$ of Theorem 11.2, with the roles of $\mathfrak{g}, \mathfrak{t}, \mathfrak{h}, \delta, \Theta, \tilde{\Theta}, \mathfrak{g}^{(1)}$ in the theorem being played by $\mathfrak{g}, \mathfrak{t}, \mathfrak{h}, \delta, \nabla, \tilde{\Theta}, \mathfrak{g}^{(1)}$

Our first task is to choose a connection $\nabla^{\mathfrak{h}^{(1)}}$ on $\nabla$ which we claim are $\nabla$-invariant. The $\nabla$-invariance follows from Proposition \ref{6.2}(2). So the second inclusion $\nabla$-generates $\mathfrak{g}$ and because

\[ \nabla: T^*M \otimes \mathfrak{h} \to \text{Alt}^2(TM) \otimes \mathfrak{h} \]

is $\mathfrak{g}$-equivariant, it follows that $\nabla$ is $\nabla$-equivariant. $\nabla$ is torsion free, meaning $\nabla$-invariance is the same as $\nabla$-invariance. $\nabla^{\mathfrak{h}^{(1)}}$ is $\nabla$-equivariant for arbitrary sections $X \subset \mathfrak{g}$ and $\phi \subset \mathfrak{h}^{(1)}$

In the present context \ref{11.2}(1) reads
\[ \tilde{\Theta}^{(1)}(X \oplus \phi) := \text{curv} \nabla^{(1)}(\cdot, \cdot)X - \phi \]

From \ref{12.9}(2) and (1) above one obtains
\[ \tilde{\Theta}^{(1)}(X \oplus \phi) = -X \cdot \text{curv} \nabla = -X \cdot \delta^{(1)}(\mathfrak{h}^{(1)}) \]

for arbitrary sections $X \subset \mathfrak{g}$ and $\phi \subset \mathfrak{h}^{(1)}$. So the composite
\[ \mathfrak{g}^{(1)} \cong \mathfrak{g} \oplus \mathfrak{h}^{(1)} \oplus \tilde{\Theta}^{(1)} \cdot \text{Alt}^2(TM) \otimes \mathfrak{h}^{(1)} \]
We have used (2) above. Referring to the description of a solution is given by $\epsilon = -X \cdot R$. Using $\nabla^{(1)}$ to identify $\mathfrak{g}^{(1)} \cong T^*M$ implicit above, we find

$$\nabla^{(2)}_U (X \oplus \alpha) = \left( \nabla^{(1)}_U X + \text{skew}(\alpha \otimes U) + \alpha(U) \right)$$

for arbitrary sections $X \subset \mathfrak{g}$ and $\alpha \subset T^*M$.

We claim

$$\text{curv} \nabla^{(2)}(U_1, U_2)(X \oplus \alpha) = (X \cdot R)([U_1, U_2])$$

where $d_{\nabla}R$ is the Cotton-York tensor, defined in 12.2. In particular, whenever $W = 0$, the tensor $d_{\nabla}R$ is a conformal invariant which vanishes if and only if the Cartan algebroid is flat, i.e., if and only if the Cartan algebroid is flat (Theorem 4.6). This completes the proof of Theorem 12.4.

**Proof of (2).** Since $W = 0$ we have $\text{curv} \nabla = \text{coRicci}$, and one computes

$$\text{curv} \nabla^{(2)}(U_1, U_2)(X \oplus \alpha) = -((\nabla^{(1)}_U X) \cdot \alpha(U_2)) + ((\nabla^{(1)}_U X) \cdot R(U_2)) - (X \cdot R)([U_1, U_2])$$

Equation (2) now follows from the readily verified identities

$$(\nabla^{(1)}_U X) \cdot V = \nabla_U (X \cdot V) - X \cdot (\nabla_U V) + \nabla_U X \cdot V$$

$$(\nabla^{(1)}_U X) \cdot \alpha = \nabla_U (X \cdot \alpha) - X \cdot (\nabla_U \alpha) + \nabla_U X \cdot \alpha$$

One also makes use of the fact that $\text{tor} \nabla = 0$.

**Appendix A. Cartan groupoids and Lie pseudogroups**

We now explain how flat Cartan algebroids may be viewed as deformations of Lie pseudogroups; and conversely, how Lie pseudogroups integrate flat Cartan algebroids. As a byproduct of this discussion, we are led to define *groupoid etalifications.* These are the global versions of Cartan groupoids and Lie pseudogroups; and conversely, how Lie pseudogroups integrate flat Cartan algebroids. We shall understand all constructions to be made in the smooth category.

**A.1. Lie pseudogroups via pseudoactions.** Let
pseudotransformations of the pair groupoid $M \times M$ in $M$ taking possibly multiple values.

A \emph{pseudoaction} of $G$ on $M$ is any foliation $\mathcal{F}$

1. The leaves of $\mathcal{F}$ are pseudotransformations.
2. $\mathcal{F}$ is multiplicatively closed.

To define what is meant in (2) let $\hat{\mathcal{F}}$ denote the collection of those subsets of $G$ that are simultaneously an open subset of some leaf of $\mathcal{F}$, and a local bisection. Let $\hat{G}$ denote the collection of all local bisections of $G$, this being a groupoid over the power set of $M$. Then condition (2) is the requirement that $\hat{\mathcal{F}} \subset \hat{G}$ be a subgroupoid.

Given a pseudoaction $\mathcal{F}$ of $G$ on $M$, each element of $\hat{\mathcal{F}}$ defines a local diffeomorphism in $M$ and, by (2), the collection of all such local diffeomorphisms constitutes a pseudogroup of transformations in $M$. For example, if $G = G_0 \times M$, then the canonical horizontal foliation $\mathcal{F}$ furnishes us with the usual pseudogroup of transformations associated with the prescribed action of the Lie group $G_0$.

\textbf{A.2. The flat Cartan algebroid associated with a Lie pseudogroup.}

Let $G$ be a Lie pseudogroup of transformations in $M$. Then $G$ is generated by the pseudoaction $\mathcal{F}$ of some Lie groupoid $G$ over $M$. For each point $g \in G$ lies in some bisection $b \in \hat{\mathcal{F}}$ of the same one-jet at $g$. Thus $\mathcal{F}$ defines a map $D_{\mathcal{F}} : G \to \mathcal{J}^1 G$ into the Lie groupoid of all one-jets of bisections of $G$. This map, which is a right inverse for the natural projection $\mathcal{J}^1 G \to G$, is a groupoid morphism between $G$ and $\hat{G}$.

An arbitrary groupoid morphism $D : G \to \mathcal{J}^1 G$ is what we call a Cartan connection $\nabla$ viewed as certain `multiplicatively closed' distribution on $G$. If $D$ is Frobenius integrable precisely when it comes from a pseudoaction $\mathcal{F}$ as above, in which case $D$ is simply the tangent distribution. A (possibly non-integrable) Cartan connection is a Cartan groupoid. Thus Cartan groupoids are deformed Lie pseudogroups.

Differentiating a Cartan connection $D : G \to \mathcal{J}^1 G$, we obtain a splitting $\mathcal{g} \to \mathcal{J}^1 \mathcal{g}$ for the exact sequence of Lie algebroids

\begin{equation}
0 \to T^*M \otimes \mathcal{g} \to \mathcal{J}^1 \mathcal{g} \to \mathcal{g} \to 0.
\end{equation}

This splitting will be a morphism of $G$-principal

foliation $\mathcal{F}$ is a pseudoaction generating a Lie pseudogroup $\mathcal{G}$ of transformations in $M$.

For each locally defined $\nabla$-parallel section $X$ in $\mathcal{F}$, it integrates to a one-parameter family of local transformations belonging to $\mathcal{G}$. Conversely each transformation in the pseudogroup $\mathcal{G}$ — or at least each transformation ‘close’ to the identity — arises as the time-one map associated with such a vector field. In this sense $\mathcal{G}$ integrates the flat Cartan algebroid.

APPENDIX B. MISCELLANY

B.1. On morphisms whose domains sit in a short exact sequence.

In the category of vector spaces, or of vector bundles over $M$, let $\theta: B \rightarrow B_1$ be an arbitrary morphism, $B_0$ its kernel, and suppose $B$ occurs in some exact sequence, as shown below:

$$\begin{array}{cccc}
B_0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow\theta & & \downarrow & & \downarrow & & \\
& & 0 & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & & & B_1 & & & & & \\
\end{array}$$

The proof of the following is a straightforward diagram chase.

**Lemma.** Let $A_0$ and $A_1$ denote, respectively, the kernel and image of the composite morphism $A \hookrightarrow B \xrightarrow{\theta} B_1$; and define $C_1 := B_1/A_1$.

$$\begin{array}{cccc}
0 & \longrightarrow & A_1 & \hookrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
& & & & & \downarrow & & & & \\
& & & & & B & \longrightarrow & C & \longrightarrow & 0 \\
& & & & & B_0 & \longrightarrow & C_0 & \longrightarrow & 0 \\
\end{array}$$

is also exact. Then:

(1) There exists a unique morphism $C \xrightarrow{\Theta} C_1$ such that:

$$\begin{array}{cccc}
B & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow\theta & & \downarrow\Theta & & \downarrow \Theta \\
B_0 & \longrightarrow & C_0 & \longrightarrow & 0 \\
\end{array}$$
B.2. Gluing Lie algebroid ‘point’ invariants

Let $\mathfrak{g}$ be a transitive Lie algebroid over $M$ and $\mathfrak{h} \subset \mathfrak{g}$ to be a $\mathfrak{g}$-representation. Each fiber $\mathfrak{h}(m)$ of $\mathfrak{h}$ is a Lie algebra $E(m)$. The following lemma furnishes conditions under which $\mathfrak{h}(m)$-invariant elements $\sigma(m) \in E(m)$, for each $m \in M$, are $\mathfrak{g}$-invariant sections $\sigma \in E$. For applications, see Section 5.2.

**Lemma** (Extension Lemma). Suppose that $M$ is simply-connected and that the $\mathfrak{g}$-flatness and the simple-connectivity of $M$ are $\mathfrak{g}$-invariant. For any non-vanishing $\mathfrak{g}$-invariant section $\sigma$ of $T\mathfrak{g}$, we have $\sigma = \sigma(1)$. From (2) it follows that $\mathfrak{g}$-flatness and the simple-connectivity of $M$ are $\mathfrak{g}$-invariant. For applications, see Section 5.2.

**Proof.** Noting that $Y \subset \mathfrak{h}$ implies $[X,Y]_g \subset \mathfrak{h}$, the tensor $Y \cdot (X \cdot \sigma) = X \cdot (Y \cdot \sigma) - [X,Y]_g \cdot \sigma$ shows that the rank-$r$ subbundle $E^\mathfrak{h} \subset E$ is $\mathfrak{g}$-invariant. Consequently, the representation $\mathfrak{g} \rightarrow \mathfrak{gl}(E^\mathfrak{h})$ factors through the anchor, delivering a $\mathfrak{g}$-representation $TM \rightarrow \mathfrak{gl}(E^\mathfrak{h})$, i.e., a flat linear connection $\nabla$ on $E^\mathfrak{h}$ to be any non-vanishing $D$-parallel section of $E^\mathfrak{h}$. This establishes $10.3(2)$.

**B.3. Proof of Proposition 10.3.** Let $\nabla$ be any flat linear connection and $d\theta$ be all $\mathfrak{g}$-invariant, they are all $\nabla$-invariant. From (2) it follows that $\nabla n = \theta(\nabla(V, n)) = 0$. From (2) it follows that $\nabla n = 0$. From (2) it follows that $\nabla n = 0$. Therefore, compute,

$$\theta(\nabla n) = \theta(\nabla n, V) = d\theta(n, V)$$

So $\nabla n$ is $\mathcal{H}$-valued, for any $V \subset TM$. This established.

Now $\text{Alt}^2(\mathcal{H})$ is rank-one and spanned by $dA$ and $\text{tor} \nabla$ to $\mathcal{H}$ (a section of $\text{Alt}^2(\mathcal{H})$) is of the form $\text{tor} \nabla = \theta(\nabla n)$. Therefore, compute,

$$\theta(\nabla n) = \theta(\nabla n, V) = d\theta(n, V)$$

So $\nabla n$ is $\mathcal{H}$-valued, for any $V \subset TM$. This established.
Therefore, if $\nabla$ is a generator satisfying $\nabla\sigma|\mathcal{H}$, it becomes a consequence of (4) above.

If $\nabla n \subset (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}}$ then (2) implies that (to follow from (1)) (take $U := n$).

We return to supposing that $\nabla$ is an arbitrary generator. The remaining claims of the proposition we require a detailed analysis of the upper coboundary morphism,

$$T^*M \otimes \mathfrak{h} \xrightarrow{\Delta} \text{Alt}^2(TM)$$

By Proposition 10.2, we have $\mathfrak{h} \cong (\mathbb{R} \times M)$, so with the help of the $\mathfrak{g}$-invariant splitting $TM = \mathcal{H} \oplus \langle n \rangle$ and $\mathcal{H}$ identifies a natural isomorphism of $\mathfrak{g}$-representations, we get

$$\text{Alt}^2(TM) \otimes TM \xrightarrow{\phi} \text{Alt}^2(TM) \oplus (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}}$$

where $(\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}} \subset \mathcal{H}^* \otimes \mathcal{H}$ denotes the $\mathfrak{g}$-summands. We write $\phi = \phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \phi_4$ and describe it at the end. Knowing the $\phi_j$, one readily establishes:

**Lemma.** Under the identifications above, we have

$$\text{tor } \nabla = d\theta \oplus (\nabla n)_{\text{sym}} \oplus \phi$$

where $f \subset (\mathbb{R} \times M)$ is the function on $M$ described.

(7) The torsion $\text{tor } \nabla \subset \text{Alt}^2(TM) \otimes TM$ is given by

$$\text{tor } \nabla = d\theta \oplus (\nabla n)_{\text{sym}} \oplus \phi$$

(8) The upper coboundary morphism $\Delta$ takes the form

$$T^*M \xrightarrow{\Delta} \text{Alt}^2(TM) \oplus (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}} \oplus \phi$$

$$\alpha \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \phi$$

In particular, $\Delta$ is injective, and its image has complement

$$C := \text{Alt}^2(TM) \oplus (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}}$$

which is $\mathfrak{g}$-invariant because the splitting (6) is $\mathfrak{g}$-invariant.

Also, we obtain $\mathfrak{g}$-invariant isomorphisms,

$$H(\mathfrak{g}) \cong C \cong \text{Alt}^2(TM) \oplus (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}}$$

This proves the first part of 10.3(5).

Now (7) shows that $\text{tor } \nabla \subset C$ if and only if $\nabla n \subset (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}}$ and only if $\nabla \sigma|\mathcal{H} = 0$ (by (7)) and $\nabla n \subset (\mathcal{H}^* \otimes \mathcal{H})_{\text{sym}}$.
The definitions of \( \phi_1, \phi_2, \phi_3, \phi_4 \). The morphism
\[
\text{Alt}^2(TM) \otimes TM \to \text{Alt}^2(TM) \otimes TM
\]
where the first arrow is the identity on \( \text{Alt}^2(TM) \otimes TM \). The morphism \( \phi_2 \) is the orthogonal projection \( TM \to \langle n \rangle \). The morphism \( \phi_3 \) is the restriction \( \text{Alt}^2(TM) \to \text{Alt}^2(M) \otimes TM \to \mathcal{H}^* \), where the first arrow is the identity on \( \text{Alt}^2(TM) \otimes TM \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \).

\[
\text{Alt}^2(TM) \otimes TM \to \text{Alt}^2(M) \otimes TM \to \mathcal{H}
\]
where the first arrow is contraction \( \rho \mapsto \rho(n, \cdot) \). The morphism \( \phi_2 \) is the orthogonal projection \( TM \to \mathcal{H} \). The morphism \( \phi_3 \) is the restriction \( \text{Alt}^2(TM) \to \text{Alt}^2(M) \otimes TM \to \mathcal{H}^* \), where the first arrow is the identity on \( \text{Alt}^2(TM) \otimes TM \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \).

\[
\text{Alt}^2(TM) \otimes TM \to \text{Alt}^2(M) \otimes TM \to \mathcal{H}
\]
where the first arrow is the restriction \( \text{Alt}^2(TM) \otimes TM \to \mathcal{H} \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \). The morphism \( \phi_4 \) is the orthogonal projection \( TM \to \mathcal{H} \).

\[
\text{Alt}^2(TM) \otimes TM \to \text{Alt}^2(M) \otimes TM \to \mathcal{H}
\]
where the first arrow is restriction tensored with orthogonal projection.

Relationship with the Levi-Cevita connection. So, it is not difficult to express the generator \( \nabla \) of the Levi-Civita connection \( \nabla^{L-C} \) associated with \( \sigma \):

\[
\nabla_U V = \nabla^{L-C}_U V - \epsilon(U)\nabla^{L-C}_U n,
\]
where \( \epsilon \subset T^*M \otimes (T^*M \otimes TM)_{\text{alt}} \) is defined by

\[
\epsilon(n) = \frac{1}{2}(\nabla^{L-C}_U n)_{\text{alt}},
\]
\[
\epsilon(U) = (\theta \otimes \nabla^{L-C}_U n)_{\text{alt}} \quad \text{for } U, \quad \text{or } \epsilon(U) V = (J\nabla^{L-C}_U n) \times V \quad \text{for } U.
\]

Here \( \times \) denotes cross product and \( (T^*M \otimes TM)_{\text{alt}} \) is a \( g \)-subrepresentation of skew-symmetric elements.

B.4. On \( J^2t \) as a subbundle of \( J^1(J^1t) \). Here we apply Lemma 8.1 which describe properties of the second jet bundle vector bundle \( t \).

Evidently the formula for \( \omega_{12} \) in Proposition 8.2 defines, at the very least, a \( \text{Alt}^2M \otimes TM \to TM \to TM \), that is, \( \omega_{12} \).
Next, applying Lemma B.1 to the morphism $\omega_2$ we obtain an exact sequence

$$0 \to \mathrm{Sym}^2(TM) \otimes t \to \ker \omega_2 \to J^1 t \to 0.$$  

Since $J^2 t$ itself occurs in a natural exact sequence

$$0 \to \mathrm{Sym}^2(TM) \otimes t \to J^2 t,$$

the bundles $\ker \omega_2$ and $J^2 t$ have the same rank.

Recalling that $X \subset J^1 t$ is holonomic if and only if $D_X = 0$, it is not hard to see that $X$ is holonomic if and only if $J^1 X \subset \ker \omega_2$. Lemma 8.1 is then a corollary of Proposition 8.2.

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