COUNTABLE TIGHTNESS AND $\mathcal{G}$-BASES ON FREE TOPOLOGICAL GROUPS

FUCAI LIN, ALEX RAVSKY, AND JING ZHANG

Abstract. Given a Tychonoff space $X$, let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group over $X$ in the sense of Markov [30]. For every $n \in \mathbb{N}$, by $F_n(X)$ we denote the subspace of $F(X)$ that consists of all words of reduced length at most $n$ with respect to the free basis $N$. The subspace $A_n(X)$ is defined similarly. We always use $G(X)$ to denote $F(X)$ or $A(X)$, and $G_n(X)$ to $F_n(X)$ or $A_n(X)$ for each $n \in \mathbb{N}$. Therefore, any statement about $G(X)$ applies to $F(X)$ and $A(X)$, and about $G_n(X)$ applies to $F_n(X)$ and $A_n(X)$.

One of the techniques of studying the topological structure of free topological groups is to clarify the relations of subspaces $X$, $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$, where $n \in \mathbb{N}$. It is well known that only when the space $X$ is discrete, $F(X)$ and $A(X)$ can be first-countable. Therefore, the space $F(X)$ is first-countable if and only if $X$ is discrete [20]. Similarly, the groups $F(X)$ and $A(X)$ are locally compact if and only if the space $X$ is discrete [11]. More generally, P. Nickolas and M. Tkachenko proved that if one of the groups $F(X)$ or $A(X)$ is almost metrizable, then the space $X$ is discrete [31]. Further, K. Yamada gave a characterization for a metrizable space $X$ such that some the spaces $F_n(X)$ and $A_n(X)$ are first-countable [40].

Recently, Z. Li et al. in [24] proved that for each stratifiable $k$-space, the group $F(X)$ is of countable tightness if and only if the space $X$ is separable or discrete. In Section 3, we refine this result by giving a characterization of a space $X$ such that the countable tightness of the space $F(X)$ implies the countable tightness of the group $F(X)$. Furthermore, since each space with the countable fan-tightness or the strong Pytkeev property is of countable tightness, we also discuss the topological properties of the countable fan-tightness and the strong Pytkeev property of free topological group $F(X)$ or some $F_n(X)$.

Ferrando et al. in [12] introduced the concept of $\mathcal{G}$-base in the frame of locally convex spaces. Now the concept of $\mathcal{G}$-base plays an important role in the study of function spaces, see [7] [13] [15] [17] [18] [19] [20]. From [16], we know that the strong Pytkeev property for general topological groups is closely related to the notion of a $\mathcal{G}$-base. For instance, each topological group which is a $k$-space with a $\mathcal{G}$-base has the strong Pytkeev property. In Section 4, we shall

2000 Mathematics Subject Classification. Primary 54H11, 22A05; Secondary 54E20; 54E35; 54D50; 54D55.

Key words and phrases. Free topological group; free Abelian topological group; countable tightness; countable fan-tightness; $\mathcal{G}$-base; strong Pytkeev property; universally uniform $\mathcal{G}$-base.

The first author is supported by the NSFC (Nos. 11571158, 11201414, 11471153), the Natural Science Foundation of Fujian Province (Nos. 2016J05014, 2016J01671, 2016J01672) of China and the project of Abroad Fund of Minnan Normal University. This paper was partially written when the first author was visiting the School of Computer and Mathematical Sciences at Auckland University of Technology from March to September 2015, and he wishes to thank the hospitality of his host.
continuously discuss the properties of free topological groups with a $\mathcal{G}$-base, which are motivated by the following interesting Questions [11] and [12].

**Question 1.1.** [15] Question 4.17 | Let $X$ be a submetrizable $k_\omega$-space. Does the group $F(X)$ have a $\mathcal{G}$-base?

**Question 1.2.** [15] Question 4.15 | For which $X$ the groups $A(X)$ and $F(X)$ have a $\mathcal{G}$-base?

Recently, S.S. Gabriyelyan and J. Kąkol in the paper [17] and A.G. Leiderman, V.G. Pestov, A.H. Tomita in the paper [26] have given an answer to Questions 1.1 and 1.2 respectively.

### 2. Notations and Terminology

In this section, we introduce the necessary notations and terminology. Throughout this paper, all topological spaces are assumed to be Tychonoff, unless otherwise is explicitly stated. For undefined notations and terminology, refer to [3], [11] and [21]. First of all, let $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of positive integers and rational numbers, respectively.

Let $X$ be a topological space and $A \subseteq X$. The **closure** of a subspace $A$ of $X$ is denoted by $\overline{A}$. The subspace $A$ is called **bounded** if every continuous real-valued function $f$ defined on the subspace $A$ is bounded. If the closure of every bounded set in $X$ is compact, then the space $X$ is called $\mu$-**complete**. The space $X$ is called a $c_f$-**space** if every compact subset of it is finite. The space $X$ is called a $k$-**space** provided that a subset $C \subseteq X$ is closed in $X$ if and only if $C \cap K$ is closed in $K$ for each compact subset $K$ of the space $X$. In particular, the space $X$ is called a $k_{\omega}$-**space** if there exists a family of countably many compact subsets $\{K_n : n \in \mathbb{N}\}$ of $X$ such that each subset $F$ of the space $X$ is closed in $X$ if and only if $F \cap K_n$ is closed in $K_n$ for each $n \in \mathbb{N}$. A subset $A$ of the space $X$ is **sequentially open** if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ converging to a point of $A$ is eventually in $A$. The space $X$ is called **sequential** if every sequentially open subset of $X$ is open. A space $X$ is of **countable tightness** if whenever $A \subset X$ and $x \in \overline{A}$, there exists a countable set $B \subset A$ such that $x \in \overline{B}$. A space $X$ is of **countable fan-tightness** [1] if for any countable family $\{A_n : n \in \mathbb{N}\}$ of subsets of $X$ satisfying $x \in \bigcap_{n \in \mathbb{N}} F_n$, it is possible to select a finite set $K_n \subseteq A_n$ for each $n \in \mathbb{N}$, in such a way that $x \in \bigcup_{n \in \mathbb{N}} K_n$. A sequence $\{x_n\}$, convergent to a point $x$, is called non-trivial, provided all points $x_n$ and $x$ are mutually distinct.

Let $\mathcal{P}$ be a family of subsets of a space $X$. The family $\mathcal{P}$ is called a **network** at a point $x \in X$ if for each open neighborhood $U$ of $x$ in $X$ there exists an element $P \in \mathcal{P}$ such that $x \in P \subset U$. The family $\mathcal{P}$ is called a $cs$-**network** [24] at a point $x \in X$ if whenever a sequence $\{x_n : n \in \mathbb{N}\}$ converges to the point $x$ and $U$ is an arbitrary open neighborhood of the point $x$ in $X$ there exist a number $m \in \mathbb{N}$ and an element $P \in \mathcal{P}$ such that

$$\{x\} \cup \{x_n : n \geq m\} \subseteq P \subseteq U.$$ 

The space $X$ is called $csf$-**countable** if $X$ has a countable $cs$-network at each point $x \in X$. We call the family $\mathcal{P}$ a $cs^*$-**network** at a point $x \in X$ [29] if whenever a sequence $\{x_n : n \in \mathbb{N}\}$ converges to the point $x$ and $U$ is an arbitrary open neighborhood of the point $x$ in $X$, there are an element $P \in \mathcal{P}$ and a subsequence $\{x_{n_i} : i \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ such that $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subseteq P \subseteq U$. Furthermore, the family $\mathcal{P}$ is called a $k$-**network** [32] if whenever $K$ is a compact subset of $X$ and $U \subset X$ is an arbitrary open neighborhood of the point $x$ in $X$, there is a finite subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that $K \subseteq \bigcup \mathcal{P}' \subseteq U$. Recall that a regular space $X$ is $k_0$ if $X$ has a countable $k$-network. The family $\mathcal{P}$ is called a Pytkeev network [35] at a point $x \in X$ if $\mathcal{P}$ is a network at $x$ and for every open set $U$ in $X$ and a set $A$ accumulating at $x$ there exists $P \in \mathcal{P}$ such that $P \subset U$ and $P \cap A$ is infinite; the family $\mathcal{P}$ is a Pytkeev network in $X$ if $\mathcal{P}$ is a Pytkeev network at each point $x \in X$. The space $X$ is said to have the **strong Pytkeev property** [37] if at each point of $X$ there is a countable Pytkeev network. The space $X$ is called $\mathcal{P}_0$ if $X$ is regular and has a countable Pytkeev network.

The following implications follow directly from definitions. However, none of them can be reversed. By [19] Proposition 4.1, we see that a space is first-countable if and only if it has the strong Pytkeev property and is of countable fan-tightness.
Theorem 3.1. A topological space $X$ is a stratifiable space if $X$ is $T_1$ and, to each open $U$ in $X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of $X$ such that

(a) $\bigcap_{n=1}^{\infty} U_n \subset U$;
(b) $\bigcup_{n=1}^{\infty} U_n = U$;
(c) $U_n \subset V_n$ whenever $U \subset V$.

Note: Clearly, each metrizable space is stratifiable.

We consider the product $\mathbb{N}^\omega$ with the natural partial order, i.e., $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ and $\beta = (\beta_i)_{i \in \mathbb{N}}$. A topological space $(X, \tau)$ has a small base if there exist a subset $M$ of $\mathbb{N}^\omega$ and a family of open subsets $\mathcal{U} = \{U_\alpha : \alpha \in M\}$ in $X$ such that $\mathcal{U}$ is a base for $X$ and $U_\alpha \subset U_\beta$ for all $\alpha, \beta \in M$ with $\alpha \leq \beta$. In particular, we say that $(X, \tau)$ has a $\mathfrak{G}$-base if $M = \mathbb{N}^\omega$. If a space has a $\mathfrak{G}$-base, then it has a countable $cs^*$-character, see Proposition 3.5.

Given a group $G$, the letter $e_G$ denotes the neutral element of $G$. If no confusion occurs, we simply use $e$ instead of $e_G$ to denote the neutral element of $G$.

Let $X$ be a non-empty Tychonoff space. Throughout this paper, $X^{-1} := \{x^{-1} : x \in X\}$ and $-X := \{-x : x \in X\}$, which are copies of $X$. Let $e$ be the neutral element of $F(X)$ (i.e., the empty word) and $0$ be that of $A(X)$. For every $n \in \mathbb{N}$ and an element $(x_1, x_2, \ldots, x_n)$ of $(X \bigoplus X^{-1} \bigoplus \{e\})^n$ we call a word $g = x_1 x_2 \cdots x_n$ a form. In the Abelian case, a word $x_1 + x_2 + \cdots + x_n$ is also called a form for $(x_1, x_2, \ldots, x_n) \in (X \bigoplus X^{-1} \bigoplus \{e\})^n$. This word $g$ is called reduced if it does not contains $e$ or any pair of consecutive symbols of the form $xx^{-1}$ or $x^{-1}x$. It follows that if the word $g$ is reduced and non-empty, then it is different from the neutral element $e$ of $F(X)$. In particular, each element $g \in F(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for each $i = 1, \ldots, n - 1$, and the support of $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ is defined as supp($g$) := $\{x_1, \ldots, x_n\}$. Given a subset $K$ of $F(X)$, we put supp($K$) = $\bigcup_{g \in K}$ supp($g$). Similar assertions (with the obvious changes for commutativity) are valid for $A(X)$. For every $n \in \mathbb{N}$, let $i_n : (X \bigoplus X^{-1} \bigoplus \{e\})^n \to F_n(X)$ be the natural mapping defined by $i_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$ for each $(x_1, x_2, \ldots, x_n) \in (X \bigoplus X^{-1} \bigoplus \{e\})^n$. We also use the same symbol in the Abelian case, that is, it means the natural mapping from $(X \bigoplus X^{-1} \bigoplus \{e\})^n$ onto $A_n(X)$. Clearly, each $i_n$ is a continuous mapping.

3. The characterization of countable tightness in free topological groups

In this section, we mainly discuss the countable tightness and countable fan-tightness of free topological groups. First, we give a characterization of a stratifiable $k$-space $X$ such that $F_\delta(X)$ has countable tightness (or, equivalently, $F(X)$ has countable tightness). Then we show that a space $X$ must belong to some special class of spaces if $F_\delta(X)$ is of countable fan-tightness.

The following theorem generalizes a result in [27].

Theorem 3.1. Let $X$ be a stratifiable $k$-space. Then the following are equivalent:

1. $F_\delta(X)$ is of countable tightness;
2. $F(X)$ is of countable tightness;
3. The space $X$ is separable or discrete.
Proof. Since the equivalence (2) ⇔ (3) was proved in [27], it suffices to show that (1) ⇒ (3).

Assume that $X$ is neither separable nor discrete. Since each stratifiable space has $G_{δ}$-diagonal, by Śniadecki Theorem [25], each compact subspace of $X$ is metrizable. Thus $X$ is sequential. Since the space $X$ is non-discrete, it contains a non-isolated point $x ∈ X$. This means that the set \{x\} is not sequentially open, that is, there exists a non-trivial convergent sequence. Hence take an arbitrary non-trivial convergent sequence $C := \{x_{n} : n ∈ N\} ⊂ X$ with a limit point $x$. Moreover, assume that the space $X$ contains no uncountable closed discrete subset. This means that the extend of the space $X$ is countable. But, by [25], each stratifiable space is a $σ$-space, and a $σ$-space of countable extent is cosmic (that is, has countable network), and, therefore, separable. Obtained contradiction shows that there exists an uncountable discrete closed subset $D := \{d_{α} : α ∈ ω_{1}\}$ of $X$. Without loss of generality, we may assume that $C ∩ D = ∅$.

For each $α ∈ ω_{1}$ let $f_{α} : ω_{1} → ω$ be a function such that $f_{α} |_{α} : α → ω$ is a bijection. For any distinct $α, β ∈ ω_{1}$, put

$$E_{α, β} := \{d_{β}x_{m}x_{m}^{-1}d_{β}^{-1}d_{α}x_{f_{α}(β)}x_{m}^{-1}d_{α}^{-1} : m ≤ f_{α}(β)\}.$$  

Then put

$$E := \bigcup_{α, β ∈ ω_{1}, α ≠ β} E_{α, β}.$$  

Clearly, $e ∉ E$, and $e ∈ E$ by the proof of [11] Proposition 2.2. In order to obtain a contradiction, it suffices to show that each countable infinite subset $B ⊂ E$ is closed in $F(X)$. Let $Y := \text{supp}(B)$. The set $Y$ contains the point $x$, so $Y ⊂ C ∪ D ∪ \{x\}$, which implies that $F(Y)$ is a $k$-space by [4] Theorem 3.7. Since $X$ is a stratifiable space and $Y$ is closed in $X$, it follows from [34] that the subgroup $F(Y, X)$ of $F(X)$ generated by $Y$ is naturally topologically isomorphic to $F(Y)$. Then $F(Y)$ is a closed $k$-subspace in $F(C ⊔ D)$. Furthermore, we claim that for each compact subset $K$ of $F_{k}(Y)$, the set $K ∩ B$ is finite. Assume on the contrary that there is a compact subset $K$ in $F_{k}(Y)$ such that $K ∩ B$ is infinite. Clearly, the set $K ∩ B$ is a bounded subset in $F_{k}(Y)$, hence the subspace $\text{supp}(K ∩ B)$ is bounded in $Y$ by [4] Theorem 1.5. Since the space $Y$ is paracompact, $\text{supp}(K ∩ B)$ is compact in $Y$. However, the set $\text{supp}(K ∩ B)$ contains infinite many elements $d_{β}^{-1}s$ since $B$ is infinite, which is a contradiction with the compactness of the subspace $\text{supp}(K ∩ B)$. Therefore, the subset $B$ is closed in $F_{k}(Y)$, that is, the subset $B$ is closed in $F(X)$. Hence $F_{k}(X)$ is not of countable tightness since $e ∈ E$, which is a contradiction.

Obviously, we have the following corollary.

**Corollary 3.2.** Let $X$ be a stratifiable $k$-space. If $F_{k}(X)$ has the strong Pytkeev property, then $X$ is separable or discrete.

**Remark 3.3.** Let $X := D ⊔ K$, where $D$ is an uncountable discrete space and $K$ is an infinite compact metric space. Then $F_{4}(X)$ is first-countable by Theorem 4.5 in [40], hence it is of countable tightness. However, the space $X$ is not separable and discrete. We do not know whether $F_{4}(X)$ is of countable tightness. Therefore, we have the following Question 3.4.

**Question 3.4.** Let $X := C ⊔ D$, where $C$ is a non-trivial convergent sequence with its limit point and $D$ is an uncountable discrete space. For each $n ∈ \{4, 5, 6, 7\}$, does $F_{n}(X)$ have the countable tightness?

Furthermore, we also do not know the answer to the following question.

**Question 3.5.** Let $X$ be a space. If $F_{2}(X)$ is of countable tightness, does $F_{3}(X)$ have the countable tightness?

It is natural to ask whether Theorem 3.1 holds in the class of free Abelian topological groups. Next we shall give a partial answer to this question.

**Theorem 3.6.** Let $X$ be a stratifiable $k$-space. If $A_{4}(X)$ is of countable tightness, then the set of all non-isolated points of $X$ is a separable subspace in $X$.  

Theorem 3.7. The proof is omitted in this paper. We only consider space $X$ tightness in free topological groups. First, we shall give a characterization of a space $X$ to Theorem 3.1, we shall find that the situation changes dramatically for the countable fan-tightness. Let $G$ be a discrete. Clearly, we assume the converse. Suppose that $G$ is not a discrete. Then the set of all non-isolated points of $G$ is closed and discrete in $X$. Since $X$ is stratifiable and $C(\omega_1)$ is closed in $X$, it follows from [34] that the subgroup $A(C(\omega_1), X)$ of $A(X)$ generated by $C(\omega_1)$ is naturally topologically isomorphic to the free Abelian topological group $A(C(\omega_1))$. Then $A_4(C(\omega_1))$ is topologically isomorphic to $A(C(\omega_1), X) \cap A_4(X)$, which implies that the tightness of $A_4(C(\omega_1))$ is countable. However, it follows from [39, Theorem 3.2] and [23] that the tightness of $A_4(C(\omega_1))$ is uncountable, which is a contradiction.

However, the converse of Theorem 3.7 does not hold, see [39, Theorem 3.2]. Moreover, the proof of the following result is similar to [11, Proposition 2.2], and thus, the proof is omitted in this paper.

Theorem 3.7. For a stratifiable space $X$, if $F_s(X)$ is a $k$-space, then $X$ is separable or discrete.

Next, we shall discuss the countable fan-tightness in free topological groups. In contrast to Theorem 3.7, we shall find that the situation changes dramatically for the countable fan-tightness in free topological groups. First, we shall give a characterization of a space $X$ such that $G(X)$ has the countable fan-tightness.

Theorem 3.8. Let $X$ be a space. Then $G(X)$ has the countable fan-tightness if and only if the space $X$ is discrete.

Proof. We only consider $F(X)$, as the proof of the case of $A(X)$ is quite similar. Since the sufficiency is obvious, we shall prove only the necessity. In order to obtain a contradiction, assume the converse. Suppose that $F(X)$ has a countable fan-tightness but the space $X$ is not discrete. Clearly, $e \in \bigcap_{n \in \mathbb{N}} \left( \bigcup_{i \geq n} (F_i(X)) \setminus F_{i-1}(X) \right)$. However, if for each natural $n$ we take an arbitrary finite subset $D_n \subset \bigcup_{i \geq n} (F_i(X)) \setminus F_{i-1}(X)$, then an intersection $(\bigcup_{n \in \mathbb{N}} D_n) \cap F_m(X)$ will be finite for each natural $m$. Therefore $\bigcup_{n \in \mathbb{N}} D_n$ is closed and discrete in $F(X)$ by [3, Corollary 7.4.3]. Thus $e \not\in \bigcap_{n \in \mathbb{N}} D_n$, a contradiction.

It turns out that the countable fan-tightness of $F_4(X)$ imposes strong restrictions on the space $X$. Recall that a subspace $Y$ of a space $X$ is said to be $P$-embedded in $X$ if each continuous pseudometric on $Y$ admits a continuous extensions over $X$.

Theorem 3.9. Let $X$ be a space. If $F_4(X)$ has the countable fan-tightness, then $X$ is either pseudocompact or a $c$-space.

Proof. Suppose $X$ is not a $c$-space. Then there exists an infinite compact subset $C$ in $X$. Next we shall show that $X$ is pseudocompact.

Assume the converse. Then there exists a discrete family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of the space $X$. It can be easily verified that the family $\{U_n : n \in \mathbb{N}\}$ is also discrete, hence $\bigcup_{n \in \mathbb{N}} U_n$ is closed in $X$. Since the set $C$ is compact, it can intersect at most finitely many $U_n$'s. Thus without loss of generality, we may assume that $C \cap \bigcup_{n \in \mathbb{N}} U_n = \emptyset$. Then the family $\{C\} \cup \{U_n : n \in \mathbb{N}\}$ is discrete in $X$.

Since $C$ is an infinite compact set, it contains a non-isolated point $x$. For each $n \in \mathbb{N}$, pick $y_n \in U_n$, and put $C_n := y_n^{-1}x^{-1}Cy_n$. Let

$$Y := C \cup \{y_n : n \in \mathbb{N}\}, \text{ and } Z := \bigcup_{n \in \mathbb{N}} C_n.$$

Obviously, the set $Y$ is closed, $\sigma$-compact and non-discrete in $X$. Moreover, $Y$ is $P$-embedded in $X$ by [30]. By [34], the subgroup $F(Y, X)$ of $F(X)$ generated by $Y$ is naturally topologically
C

Question 3.15. Let $P$ be a topological group over $X$. Problem 3.14. [5] Let $F$ has the strong Pytkeev property in Theorem 3.13. from [17, Theorem 1.7] that $F$ is an $k_0$-space with the strong Pytkeev property. By Theorem 3.12, the group $F$ is isomorphic to $F \mathbb{Z}$ which shows that $F$ is an $k_0$-space. Therefore, it is interesting to discuss the strong Pytkeev property on free topological groups.

Finally, we shall discuss the strong Pytkeev property in free topological groups. It is well known that if a space has the strong Pytkeev property then it is of countable tightness and is $\text{csf}$-countable. In [6], the authors showed that a space is first-countable if and only if it has the strong Pytkeev property and countable fan-tightness. Therefore, it is interesting to discuss the strong Pytkeev property on free topological groups. First, we shall give a theorem which has just been proved in [28].

Recall that a space $X$ is said to be Lašnev if it is the closed image of some metric space.

Theorem 3.12. [28] Let $X$ be a non-discrete Lašnev space. Then $F_0(X)$ is of $\text{csf}$-countable if and only if $F(X)$ is an $\aleph_0$-space.

Theorem 3.13. Let $X$ be a non-discrete Lašnev space. Then $F(X)$ has the strong Pytkeev property if and only if $F(X)$ is a $\aleph_0$-space.

Proof. Obviously, it suffices to show the necessity. Suppose that $F(X)$ has the strong Pytkeev property. By Theorem 3.12, the group $F_0(X)$ is an $\aleph_0$-space, hence $F(X)$ is separable. It follows from [17] Theorem 1.7 that $F(X)$ is a $\aleph_0$-space.

We do not know whether an $\aleph_0$-space with the the strong Pytkeev property is a $\aleph_0$-space. If this answer is positive, then we can replace “$F(X)$ has the strong Pytkeev property” by “$F_0(X)$ has the strong Pytkeev property” in Theorem 3.13.

In [5], T. Banakh posed the following problem.

Problem 3.14. [5] Let $X$ be a (sequential) $\aleph_0$-space. Is the free topological group over $X$ a $\aleph_0$-space?

But we even do not know an answer to the following question.

Question 3.15. Let $X$ be the rational number subspace $\mathbb{Q}$ with the usual topology. Is the free topological group over $X$ a $\aleph_0$-space?
However, we have the following result.

**Theorem 3.16.** Let $X$ be a $\mathfrak{P}_0$-space. Then $F(X)$ is the union of countably many $\mathfrak{P}_0$-subspaces.

*Proof.* For each $n \in \mathbb{N}$, it follows from [5, Corollary 3.12] that $(X \bigoplus X^{-1} \bigoplus \{e\})^n$ is a $\mathfrak{P}_0$-space. For each $n \in \mathbb{N}$, since the mapping

$$i_n|_{i_n^{-1}(F_n(X) \setminus F_{n-1}(X))}: i_n^{-1}(F_n(X) \setminus F_{n-1}(X)) \to F_n(X) \setminus F_{n-1}(X)$$

is a homeomorphism, it follows from [5, Corollary 3.12] that $i_n^{-1}(F_n(X) \setminus F_{n-1}(X))$ is a $\mathfrak{P}_0$-subspace in $(X \bigoplus X^{-1} \bigoplus \{e\})^n$, hence $F_n(X) \setminus F_{n-1}(X)$ is a $\mathfrak{P}_0$-subspace in $F(X)$. Since $F(X) = \bigcup_{n \in \mathbb{N}} (F_n(X) \setminus F_{n-1}(X))$, $F(X)$ is the union of disjoint countably many $\mathfrak{P}_0$-subspaces. □

For closing this section, we discuss the strong Pytkeev property in topological spaces. It is well known that in the class of regular countably compact spaces the property of countable tightness is equivalent to the countable fan-tightness [2, Corollary 2]. Moreover, each compact sequential non-first-countable space is of countable tightness, hence it does not have the strong Pytkeev property. Hence in the class of regular countably compact spaces the property of countable tightness is not equivalent to the strong Pytkeev property. Moreover, it is well known that there exists a countably compact non-metrizable space with a point-countable $k$-network. It is natural to ask whether in the class of regular countably compact spaces the existence of a point-countable $k$-network implies the existence of a point-countable Pytkeev network. The answer is also negative, see Example 3.17.

**Example 3.17.** There exists an infinite countably compact space $X$ which satisfies the following conditions:

1. The space $X$ contains no infinite compact subset;
2. The space $X$ has a point-countable $k$-network;
3. The space $X$ does not have the strong Pytkeev property.

*Proof.* By [14], there exists an infinite, countably compact subspace $X$ of $\beta\mathbb{N}$, the Stone-Cech compactification of the of the space of natural numbers endowed with discrete topology, such that $\mathbb{N} \subset X$ and every compact subset of the space $X$ is finite. Therefore, the space $X$ has a point-countable $k$-network. However, the space $X$ does not have the strong Pytkeev property, see [5]. □

From the opposite side, recently, Z. Cai and S. Lin has proved that each sequentially compact space with a point-countable $k$-network is metrizable [5]. Remark, that the proof of [5, Proposition 1.4], implies that the space $X$ has a point-countable $k$-network provided it has a point-countable Pytkeev network. So the next theorem is a counterpart of this result.

**Theorem 3.18.** Let $X$ be a Hausdorff countably compact space with a point-countable Pytkeev network. Then the space $X$ is a metrizable compact space.

*Proof.* Let $\mathcal{N}$ be a point-countable Pytkeev network on the space $X$. We first show the following claim.

**Claim:** For each countably compact subset $K$ of $X$ and arbitrary open subset $U$ with $K \subset U$, there exists a finite subfamily $\mathcal{F}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Suppose not. For each $x \in K$, let $\{N \in \mathcal{N} : x \in N\} = \{N_n(x) : n \in \omega\}$. Inductively choose $x_n \in K$ such that $x_n \notin N_j(x_i)$ for $i, j < n$. Put $A := \{x_n : n \in \omega\}$. Since $K$ is countably compact and $A \subset K$, the set $A$ has a cluster point $x^*$ in $K$. By the definition of Pytkeev network, it follows that there exists some $N \in \mathcal{N}$ such that $N$ contains infinitely many $x_n$’s. Therefore, we have $N = N_j(x_i)$ for some $i$ and $j$, and there exists $n > i, j$ such that $x_n \in N_j(x_i)$, contradicting the way the $x_n$’s were chosen.

By Claim and [22, Theorem 4.1], the space $X$ is metrizable (and thus compact). □
4. Free topological groups with a $\mathfrak{G}$-base

In this section, we shall discuss the properties of free topological groups with a $\mathfrak{G}$-base, which are motivated by Questions 1.1 and 1.2. First, we recall a lemma. Then we shall give a characterization of free topological groups which are $k$-spaces having a $\mathfrak{G}$-base. Let $TG_\mathfrak{G}$ be the class of all topological groups having a $\mathfrak{G}$-base.

Lemma 4.1. Let $G \in TG_\mathfrak{G}$. Then the following are equivalent:

1. The group $G$ is a $k$-space;
2. The group $G$ is a sequential space;
3. The group $G$ is metrizable or contains a submetrizable open $k_\omega$-subgroup.

By Lemma 4.1, we know that the $k$-property and sequentiality are equivalent in the class of all topological groups having a $\mathfrak{G}$-base. In [15], the authors also said that “It would be interesting to know whether the $k$-property and sequentiality are equivalent for the class of all topological groups having countable $cs^*$-character”. Indeed, the answer is negative, see the following example.

Example 4.2. There exists a topological group $G$ such that it is a $k$-space with a countable $cs^*$-character. However, $G$ is not sequential.

Proof. Let $X$ be the Stone-Čech compactification $\beta D$ of any infinite discrete space $D$. Let $G := F(X) or G := A(X)$. Obviously, the group $G$ is a $k$-space by [3, Theorem 7.4.1]. It follows from a result of [25] that $G$ is of countable $cs^*$-character. However, it is well known that $\beta D$ is not a sequential space. Since $\beta D$ is closed in $G$, the free topological group $G$ is not sequential.

However, the topological group $G$ in the proof of Example 4.2 does not have the strong Pytkeev property by Example 4.1, hence it is natural to pose the following question.

Question 4.3. Let a topological group $G$ be a $k$-space. If $G$ has the strong Pytkeev property, is it sequential?

In [17], the authors gave an affirmative answer to Question 1.1. The following theorem complements it.

Theorem 4.4. Let $X$ be a space. Then $F(X)$ is a $k$-space with a $\mathfrak{G}$-base if and only if either $X$ is discrete or $X$ is a submetrizable $k_\omega$-space.

Proof. The sufficiency was proved in [17]. It suffices to show the necessity.

Let $F(X)$ be a $k$-space with a $\mathfrak{G}$-base. Then it follows from Lemma 4.1 that $F(X)$ is metrizable or contains a submetrizable open $k_\omega$-subgroup. If $F(X)$ is metrizable, then it is well known that $X$ is discrete. Hence we may assume that $F(X)$ is non-metrizable, and then $F(X)$ contains a submetrizable open $k_\omega$-subgroup. Then $F(X) = \bigoplus_{\alpha \in \Gamma} G_\alpha$, where each $G_\alpha$ is a submetrizable open $k_\omega$-subset in $F(X)$. Since each $G_\alpha$ has a countable $k$-network, it follows that $F(X)$ has a $\sigma$-compact finite $k$-network. Moreover, it is obvious that $F(X)$ is locally Lindelöf. It is well known that a locally Lindelöf topological group is paracompact [3, Problem 3.2.A], then $X$ is paracompact since $X$ is closed in $F(X)$. Thus $X$ is a paracompact space with a $\sigma$-compact finite $k$-network. Since $F(X)$ is a non-metrizable $k$-space, it follows from [27, Theorem 4.14] that $X$ has a countable $k$-network, hence $X$ is Lindelöf and submetrizable. Therefore, it is easy to see that $X$ is a submetrizable $k_\omega$-space.

Remark 4.5. However, there exists a space $X$ such that $A(X)$ is a $k$-space with a $\mathfrak{G}$-base and $X$ is not a Lindelöf space. Indeed, let $X := C \bigoplus D$, where $C$ is a non-trivial convergent sequence with its limit point and $D$ is an uncountable discrete space $D$. Then $A(X) \cong A(C) \times A(D)$, thus $A(X)$ is a $k$-space. Since both $A(X)$ and $A(D)$ have $\mathfrak{G}$-bases, it follows from [15] that $A(S) \times A(D)$ has a $\mathfrak{G}$-base. However, it is obvious that $X$ is not Lindelöf, hence it is not a $k_\omega$-space.
By Remark 4.3, we see that we can not replace “$F(X)$” by “$A(X)$” in Theorem 4.4. However, we have the following theorem when we add some additional assumption on the space $X.$

**Theorem 4.6.** Let $X$ be a separable space. Then $A(X)$ is a $k$-space with a $\mathfrak{G}$-base if and only if $X$ is either countable discrete or a submetrizable $k_\omega$-space.

**Proof.** We adapt the proof of Theorem 4.4 for the group $A(X)$ instead of $F(X).$ It suffices to show that $X$ is a submetrizable $k_\omega$-space if $A(X)$ is a non-metrizable $k$-space with a $\mathfrak{G}$-base. Since $X$ is separable, $A(X)$ is separable. Similarly to the proof of Theorem 4.4 we see that the index set $\Gamma$ in Theorem 4.4 is countable. Therefore, $A(X)$ has a countable $k$-network, and then $X$ is a submetrizable $k_\omega$-space. \hfill $\Box$

Next we consider the topological properties of $X$ such that the free topological group over $X$ has a $\mathfrak{G}$-base.

By the proof of [15, Theorem 3.12], we can easily obtain the following proposition.

**Proposition 4.7.** If a space $X$ has a $\mathfrak{G}$-base at point $x \in X$, then $X$ is of countable $cs^*$-character at $x$.

Therefore, we have the following proposition.

**Proposition 4.8.** Let $X$ be a space. If each $G_n(X)$ has a $\mathfrak{G}$-base at point $e$, then $G(X)$ is $csf$-countable.

**Proof.** It suffices to note that for each compact subset $K$ in $G(X)$ there exists an $n \in \mathbb{N}$ such that $K \subset G_n(X)$, see [3, Corollary 7.4.4]. \hfill $\Box$

An answer to the following question is still unknown for us.

**Question 4.9.** Let $X$ be a space. If each $G_n(X)$ has a $\mathfrak{G}$-base at $e$, does $G(X)$ have a $\mathfrak{G}$-base?

**Theorem 4.10.** Let $X$ be a collectionwise normal space containing a non-trivial convergent sequence. If $F(X)$ has a $\mathfrak{G}$-base, then $X$ is $\aleph_1$-compact.

**Proof.** Suppose that $X$ is not $\aleph_1$-compact. Hence there exists an uncountable closed discrete subset $D$ in $X.$ Moreover, by the assumption, there exists a non-trivial convergent sequence $S$ with its limit point in $X.$ Without loss of generality, we may assume that $S \cap D = \emptyset.$ Let $Y = S \cup D.$ Since $X$ is collectionwise normal, the subspace $Y$ is a retract of $X,$ and then $Y$ is $P$-embedded in $X$ [3, Exercises 7.7.a]. By [34], the subgroup $F(Y,X)$ of $F(X)$ generated by $Y$ is naturally topologically isomorphic to $F(Y).$ However, $F(Y)$ is not of $csf$-countable by a result in [28], thus $F(Y)$ is not $cs^*$-countable. Then $F(X)$ is not $cs^*$-countable. However, since $F(X)$ has a $\mathfrak{G}$-base, it follows from Proposition 4.7 that $F(X)$ is $cs^*$-countable, which is a contradiction. \hfill $\Box$

**Corollary 4.11.** Let $X$ be a stratifiable $k$-space. If $F(X)$ has a $\mathfrak{G}$-base, then $X$ is either discrete or separable.

**Proof.** Assume that $X$ is not discrete. Since a stratifiable $k$-space is paracompact and sequential, $X$ is $\aleph_1$-compact by Theorem 4.10. By [25], each stratifiable space is a $\sigma$-space, and each $\aleph_1$-compact $\sigma$-space is cosmic, and therefore, separable. \hfill $\Box$

Recently, A.G. Leiderman, V.G. Pestov and A.H. Tomita in [26] showed the following two results:

**Theorem 4.12.** [26] The free Abelian topological group $A(X)$ on a uniform space $X$ has a $\mathfrak{G}$-base if and only if $X$ has a $\mathfrak{G}$-base.

**Corollary 4.13.** [26] Let $X$ be a metrizable space and the set of all non-isolated points of $X$ is a $\sigma$-compact subset of $X.$ Then $A(X)$ has a $\mathfrak{G}$-base.
For a metrizable space $X$, it follows from \[4\] that $A(X)$ is a $k$-space if and only if $X$ is a locally compact space and the set of all non-isolated points of $X$ is separable. From Corollary \[4.13\] it is easy to see that there exists a space $Y$ which is not a $k$-space such that $A(Y)$ has a $\mathfrak{G}$-base.

However, the situation changes much for (non-Abelian) free topological groups. Let $X = C \bigoplus D$, where $C$ is a non-trivial convergent sequence with the limit point and $D$ is a closed discrete space of cardinality $\aleph_1$. From \[25\], $F_1(X)$ is not csf-countable, hence $F_1(X)$ does not have a $\mathfrak{G}$-base. In particular, we see that $F(X)$ does not have a $\mathfrak{G}$-base. However, we have the following Theorem \[4.15\].

By a similar proof of \[15\] Proposition 2.7, we can obtain the following proposition.

**Proposition 4.14.** Suppose that, for each $n \in \mathbb{N}$, $X_n$ is a space with a $\mathfrak{G}$-base. Then the countable product $\prod_{n \in \mathbb{N}} X_n$ has a $\mathfrak{G}$-base.

Given a uniformizable space $X$ there is a finest uniformity on $X$ compatible with the topology of $X$ called the fine uniformity or universal uniformity. A Tychonoff space $X$ is said to have a uniform $\mathfrak{G}$-base if there exists a uniform structure $\mathcal{U}$ on $X$, which induces the topology of $X$, such that $\mathcal{U}$ has a $\mathfrak{G}$-base. In particular, if $\mathcal{U}$ is the universal uniformity on $X$ with a uniform $\mathfrak{G}$-base, then we say that $X$ has an universally uniform $\mathfrak{G}$-base.

**Theorem 4.15.** Let $X$ have an universally uniform $\mathfrak{G}$-base. Then $F_2(X)$ has a $\mathfrak{G}$-base at each point.

**Proof.** Since $X$ has an universally uniform $\mathfrak{G}$-base, it is easy to see $X$ has a local $\mathfrak{G}$-base at each point. By Proposition \[4.14\] we see that $(X \bigoplus X^{-1} \bigoplus \{e\})^2$ has a local $\mathfrak{G}$-base at each point. Then $F_2(X)$ has a local $\mathfrak{G}$-base at each point $x \in X \cup X^{-1}$ since $X \cup X^{-1}$ is open and closed in $F_2(X)$. It is well known that $F_2(X) \setminus F_1(X)$ is homeomorphic to a subspace of $(X \bigoplus X^{-1} \bigoplus \{e\})^2$, and then $F_2(X)$ has a local $\mathfrak{G}$-base at each point $x \in F_2(X) \setminus F_1(X)$ since $F_2(X) \setminus F_1(X)$ is open in $F_2(X)$. It suffices to show that $F_2(X)$ has a $\mathfrak{G}$-base at $e$.

Suppose that $\mathcal{U}$ is the universally uniformity on $X$. Then one can take a basis $\mathcal{B} = \{U_\alpha : \alpha \in \mathbb{N}^2\}$ for $\mathcal{U}$ such that for any $\alpha$ and $\beta$ in $\mathbb{N}^2$ with $\alpha \leq \beta$, we have $U_\beta \subset U_\alpha$.

For each $\alpha \in \mathbb{N}^2$, let $W_\alpha = \{x^\alpha y^{-\varepsilon} : (x, y) \in U_\alpha, \varepsilon = \pm 1\}$. Then the family $\{W_\alpha : \alpha \in \mathbb{N}^2\}$ is a base at $e$ in $F_2(X)$ by \[40\]. Obviously, $\{W_\alpha : \alpha \in \mathbb{N}^2\}$ satisfies that for any $\alpha$ and $\beta$ in $\mathbb{N}^2$ with $\alpha \leq \beta$, $W_\beta \subset W_\alpha$. Therefore, $\{W_\alpha : \alpha \in \mathbb{N}^2\}$ is a local $\mathfrak{G}$-base at $e$. \hfill $\Box$

However, the following question is still unknown for us.

**Question 4.16.** Let $X$ be a space. If $F_2(X)$ has a $\mathfrak{G}$-base, does $F_3(X)$ have a $\mathfrak{G}$-base?

**Acknowledgements.** The authors wish to thank professors Salvador Hernández and Boaz Tsaban for telling us some information of the paper \[9\]. Moreover, the authors wish to thank professor Chuan Liu for reading parts of this paper and making comments. Finally, we hope to thank professor Shou Lin for finding a gap in our proof of Theorem 3.18 in the previous version and giving some key for us to supplement the proof.

**REFERENCES**

[1] A.V. Arhangel’skiĭ, Hurewicz spaces, analytic sets and fan-tightness of spaces of functions, Soviet Math. Dokl., 33(2)(1986), 396–399.
[2] A.V. Arhangel’skiĭ, A. Bella, Countable fan-tightness versus countable tightness, Comment. Math. Univ. Carolinae, 37(3)(1996), 567–578.
[3] A.V. Arhangel’skiĭ, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Sci., Paris, 2008.
[4] A.V. Arhangel’skiĭ, O.G. Okunev, V.G. Pestov, Free topological groups over metrizable spaces, Topology Appl., 33(1989), 63–76.
[5] T. Banakh, $\mathfrak{P}_\omega$-spaces, Topology Appl., 195(2015), 151–173.
[6] T. Banakh, The strong Pytkeev property in topological spaces, http://arXiv:1412.4268v1.
[7] T. Banakh, $\omega^n$-bases in topological and uniform spaces, http://arxiv:1607.07978v1.
[8] Z. Cai, S. Lin, Sequentially compact spaces with a point-countable k-network, Topology Appl., 193(2015), 162–166.
COUNTABLE TIGHTNESS AND $\mathfrak{B}$-BASES ON FREE TOPOLOGICAL GROUPS

[9] C. Chis, M. Vincenta Ferrer, Salvador Hernández, Boaz Tsaban, *The character of topological groups, via bounded systems, Pontryagin-van Kampen duality and pcf theory*, J. Algebra, 420(2014), 86–119.

[10] R.M. Dudley, *Continuity of homomorphisms*, Duke Math. J., 28(1961), 587–594.

[11] R. Engelking, *General Topology* (revised and completed edition), Heldermann Verlag, Berlin, 1989.

[12] J.C. Ferrando, J. Kąkol, M. López Pellicer, S.A. Saxon, *Tightness and distinguished Fréchet spaces*, J. Math. Anal. Appl., 324(2006), 862–881.

[13] P. Fletcher, W.F. Lindgren, *Quasi-uniform spaces*, Marcel Dekker, New York, 1982.

[14] Z. Frolík, *Generalizations of the $G$-property of complete metric spaces*, Czech. Math. J., 10(1960), 359–379.

[15] S.S. Gabriyelyan, J. Kąkol, A. Leiderman, *On topological groups with a small base and metrizability*, Fund. Math., 229(2015), 129–158.

[16] S.S. Gabriyelyan, J. Kąkol, A. Leiderman, *The strong Pytkeev property for topological groups and topological vector spaces*, Monatsh Math., 175(2014), 519–542.

[17] S.S. Gabriyelyan, J. Kąkol, On topological spaces and topological groups with certain local countable networks, Topology Appl., 190(2015), 59–73.

[18] S.S. Gabriyelyan, J. Kąkol, A. Kubzdela and M. Lopez Pellicer, On topological properties of Fréchet locally convex spaces with the weak topology, Topology Appl., 192(2015), 123–137.

[19] S.S. Gabriyelyan, J. Kąkol, On $\mathfrak{B}$-spaces and related concepts, Topology Appl., 191(2015), 178–198.

[20] M.I. Graev, *Free topological groups*, In: Topology and Topological Algebra, Translations Series 1, vol. 8 (1962), pp. 305–364. American Mathematical Society. Russian original in: Izvestiya Akad. Nauk SSSR Ser. Mat., 12(1948), 279–323.

[21] G. Gruenhage, *Generalized metric spaces*, In: K. Kunen, J. E. Vaughan(Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984, 423–501.

[22] G. Gruenhage, E.A Michael, Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math., 113(1984), 303–332.

[23] G. Gruenhage, Y. Tanaka, *Products of k-spaces and spaces of countable tightness*, Trans. Amer. Math. Soc., 273(1982), 299–308.

[24] J.A. Guthrie, *A characterization of No-spaces*, General Topology Appl., 1(1971), 105–110.

[25] Y. Kanatani, N. Sasaki, J. Nagata, *New characterizations of some generalized metric spaces*, Math Japonica, 30(1985), 805–820.

[26] A.G. Leiderman, V.G. Pestov, A.H. Tomita, *On topological groups admitting a base at identity indexed with $\omega^{\omega}$*, http://arxiv.org/1511.07062v1.

[27] Z. Li, F. Lin, C. Liu, *Networks on free topological groups*, Topology Appl., 180(2015), 186–198.

[28] F. Lin, C. Liu, J. Cao, *Weak Countability Axioms in Free Topological Groups*, submitted.

[29] S. Lin, Y. Tanaka, *Point-countable k-networks, closed maps, and related results*, Topology Appl., 59(1994), 79–86.

[30] A.A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat., 9(1945), 3–64 (in Russian); Amer. Math. Soc. Transl., 8(1962), 195–272.

[31] P. Nickolas, M. Tkachenko, *Local compactness in free topological groups*, Bull. Austral. Math. Soc., 68(2)(2003), 243–265.

[32] P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc., 29(1971), 183–189.

[33] E.G. Pytkeev, *Maximally decomposable spaces*, Trudy Mat. Inst. Steklov., 154(1983), 209–213.

[34] O.V. Sipacheva, *Free topological groups of spaces and their subspaces*, Topology Appl., 101(2000), 181–212.

[35] V. Śnieders, *Continuous images of Souslin and Borel sets; metrization theorems*, Dokl. Acad. Nauk USSR, 50(1945), 77–79.

[36] M. G. Tkachenko, *On a spectral decomposition of free topological groups*, Usp. Mat. Nauk, 39(2)(1984), 191–192.

[37] Boaz Tsaban, L. Zdomskyy, *On the Pytkeev property in spaces of continuous functions (II)*, Houston J. Math., 35(2009), 563–571.

[38] K. Yamada, *Characterizations of a metrizable space such that every $A_n(X)$ is a k-space*, Topology Appl., 49(1993), 74–94.

[39] K. Yamada, *Tightness of free Abelian topological groups and of finite product of sequential fans*, Topology Proc., 22(1997), 363–381.

[40] K. Yamada, *Metrizable subspaces of free topological groups on metrizable spaces*, Topology Proc., 23(1998), 379–400.

[41] K. Yamada, *The natural mappings $i_n$ and $k$-subspaces of free topological groups on metrizable spaces*, Topology Appl., 146-147(2005), 239–251.
(Fucai Lin): School of mathematics and statistics, Minnan Normal University, Zhangzhou 363000, P. R. China
E-mail address: linfucai2008@aliyun.com; linfucai@mnnu.edu.cn

(Alex Ravsky): Pidstrygach Institute for Applied Problems of Mechanics and Mathematics of NASU, Naukova 3b, Lviv, 79060, Ukraine
E-mail address: oravsky@mail.ru

(Jing Zhang): School of mathematics and statistics, Minnan Normal University, Zhangzhou 363000, P. R. China
E-mail address: zhangjing86@126.com