BLOCH-OGUS THEOREM, CYCLIC HOMOLOGY AND DEFORMATION OF CHOW GROUPS

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Abstract. Using Bloch-Ogus theorem and Chern character from K-theory to cyclic homology, we answer a question of Green and Griffiths on extending Bloch formula. Moreover, we construct a map from local Hilbert functor to local cohomology. With suitable assumptions, we use this map to answer a question of Bloch on constructing a natural transformation from local Hilbert functor to cohomological Chow groups.

1. Introduction

This paper is devoted to studying infinitesimal deformation of Chow group $CH^p(X)$ of codimension $p$ algebraic cycles modulo rational equivalence, where $X$ is a smooth projective variety over a field $k$ of characteristic zero. After many years’ intensive study, the structure of $CH^p(X)$ for general $p$ still remains largely open. To understand Chow groups infinitesimally, Bloch pioneered to study formal completions of $CH^p(X)$. One fundamental tool in this approach is Bloch formula (cf. Bloch [5], Quillen [45] and Soulé [49]),

$$CH^p(X)\cong H^p(X, K^M_p(O_X)),$$

(1.1) where $K^M_p(O_X)$ is the Milnor K-theory sheaf associated to the presheaf $U \rightarrow K^M_p(O_X(U))$ with $U \subset X$ open affine. Kerz generalized the isomorphism (1.1) in [33].

Bloch formula motivates two functors on the category $Art_k$ (see Notation (2) on page 4 below)

$$\tilde{CH}^p: A \rightarrow H^p(X, K^M_p(O_{X_A})), $$

(1.2) $$\hat{CH}^p: A \rightarrow \text{kernel of } \{H^p(X, K^M_p(O_{X_A})) \overset{\text{aug}}{\rightarrow} H^p(X, K^M_p(O_X))\},$$

(1.3) where $A \in Art_k$, $X_A = X \times_{\text{Spec}(k)} \text{Spec}(A)$ and $\text{aug}$ is the map induced by augmentation $A \rightarrow k$. The group $\tilde{CH}^p(A)$, which is called cohomological Chow group, can be considered as deformation of Chow group $CH^p(X)$, and $\hat{CH}^p(A)$ is called formal completion of $CH^p(X)$, see Bloch [6] and Stienstra [50]. These functors $\tilde{CH}^p$ and $\hat{CH}^p$ are

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closely related with some major conjectures, including variational (infinitesimal) Hodge conjecture. We refer to Bloch-Esnault-Kerz [9, 10], Green-Griffiths [26], Morrow [41] and Patel-Ravindra [43] for recent progress on these conjectures.

Bloch [6] studied these two functors in the case that $k$ is a number field and asked the following important conjecture.

**Conjecture 1.1** ([6]). Let $X$ be a smooth complex projective surface with trivial geometric genus, i.e. $p_g(X) = 0$, then the Albanese map \[ CH^2_{deg \, 0}(X) \to Alb(X) \]

is an isomorphism, where $CH^2_{deg \, 0}(X)$ is the subgroup of $CH^2(X)$ consisting of zero cycles with degree zero and $Alb(X)$ is the Albanese variety.

This conjecture is closely related with the well-known example of Mumford (cf. Lewis [37, 38], Mumford [42], Roitman [46, 47], Voisin [55] et al) and had been studied intensively, for example, see Bloch-Kas-Lieberman [11], Bloch-Srinivas [13], Hu [30], Pedrini-Weibel [44] and Voisin [56, 57].

Stienstra [50] further studied these functors $\widehat{CH}^p$ and $\widehat{CH}^p$, and computed $\widehat{CH}^p(A)$ in the case that $k$ is an extension of $\mathbb{Q}$ of finite transcendence degree. He also considered the parallel situation in positive characteristic and developed Cartier-Dieudonné theory for Chow groups in [51]. An excellent summary of these results is given by Bloch [7] (Chapter 6).

Let $K^M_p(O_{X_A})$ be the relative K-group, which is defined to be the kernel of the morphism $K^M_p(O_{X_A}) \to K^M_p(O_X)$. One challenge in studying $\widehat{CH}^p(A)$, which already appeared in the case of $p = 2$, is computation of $K^M_p(O_{X_A})$. To get a feeling of this, we recall that, for $A = k[\varepsilon]$ the ring of dual numbers, van der Kallen [53] computed that $K^M_2(O_{X_{k[\varepsilon]}})$ is isomorphic to the sheaf of absolute Kähler differentials $\Omega^1_{X/k}$. Consequently, there is an isomorphism

\[(1.4) \quad \widehat{CH}^2(k[\varepsilon]) = H^2(X, \Omega^1_{X/k}),\]

and $\widehat{CH}^2(k[\varepsilon])$ is called the formal tangent space to Chow group $CH^2(X)$.

Bloch [4, 6] and Maaßen-Stienstra [40] made further computations of relative K-groups. The following is one of basic results on understanding formal completion of $CH^2(X)$.

**Theorem 1.2** (cf. Theorem 6.2 of [7]). Let $k$ be a field of characteristic zero. Let $R$ be a local $k$-algebra, and $A$ an augmented artinian $k$-algebra.
with augmentation ideal $m_A$. We write $S = R \otimes_k A$ and $I = R \otimes_k m_A$, and define

$$K_2(S, I) = \text{kernel of } \{ K_2(S) \to K_2(R) \},$$

$$\Omega^1_{S, I} = \text{kernel of } \{ \Omega^1_{S/k} \to \Omega^1_{R/k} \}.$$

The universal derivation $d : S \to \Omega^1_{S/k}$ induces $d : I \to \Omega^1_{S, I}$ and there is an isomorphism $K_2(S, I) \cong \Omega^1_{S, I}/dI$.

In the pioneering work [27], Green and Griffiths studied deformation of algebraic cycles of a smooth projective variety $X$ and investigated geometric meaning behind the formal tangent space to $CH^2(X)$ defined via the isomorphism (1.4). In particular, they computed the tangent space to zero cycles of a surface and justified that the formal tangent space to $CH^2(X)$ carried concrete geometric meaning, see Theorem 8.47 of [27]. Inspired by a list of questions asked by Green and Griffiths in [27], Dribus, Hoffman and the author used higher K-theory to extend much of their theory in [18, 62, 63, 64, 65]. Especially relevant to the present paper is the following question in section 7.2 of [27] (see also Question 1.2 in [62]).

**Question 1.3 ([27])**. Let $X$ be a smooth projective variety over a field $k$ of characteristic zero and let $Y \subset X$ be a closed subvariety of codimension $p$, is it possible to define a map from the tangent space $T_Y \text{Hilb}^p(X)$ of the Hilbert scheme at the point $Y$ to the tangent space of the cycle group $TZ^p(X)$

$$T_Y \text{Hilb}^p(X) \to TZ^p(X)?$$

For $p = \text{dim}(X)$, Green-Griffiths [27] answered this question by studying deformations of zero cycles over the ring of dual numbers. Their method was generalized by the author [62].

The ring of dual numbers is a special local artinian $k$-algebra. Green and Griffiths’ question inspires us to compare deformation of subvarieties with that of algebraic cycles (classes) over arbitrary local artinian $k$-algebras. Then we come to the following question suggested by Bloch in the introduction of [6] (page 406).

**Question 1.4 ([6])**. Let $X$ be a smooth projective variety over a field $k$ of characteristic zero and let $Y \subset X$ be a closed subvariety of codimension $p$, is there a natural transformation from local Hilbert functor $\text{Hilb}$ (recalled in Definition 3.1 below) to the functor $\tilde{CH}^p$ (see (1.2))

$$\text{Hilb} \to \tilde{CH}^p?$$

This question is closely related with the following one suggested by Green-Griffiths on page 471 of [26].
**Question 1.5** ([26]). Let $X$ be a smooth projective variety over a field $k$ of characteristic zero. For $A \in \text{Art}_k$, we write $X_A = X \times_{\text{Spec}(k)} \text{Spec}(A)$. Is it possible to extend Bloch formula (1.1) from $X$ to its infinitesimal thickening $X_A$? In other words, do we have the following identification

$$CH^p(X_A)_\mathbb{Q} = H^p(X, K^M_p(O_{X_A}))_\mathbb{Q}?$$

By modifying Balmer’s tensor triangular Chow groups [3], we answered this question when $A$ is a truncated polynomial $k[t]/(t^j)$ in [63].

Guided by Question 1.4 and Question 1.5, this paper is organized as follows. In section 2, after recalling Bloch-Ogus theorem, cyclic homology and Milnor Chow groups, we answer Question 1.5 in Theorem 2.24. In the third section, we construct a map from local Hilbert functor to local homology in (3.2). With suitable assumptions, we use this map to answer Question 1.4 in Theorem 3.14.

**Notation:**

1. For any abelian group $M$, $M_\mathbb{Q}$ denotes $M \otimes \mathbb{Q}$.

2. If not stated otherwise, $k$ is a field of characteristic zero. Let $\text{Sch}/k$ be the category of schemes of finite type over $k$ and Let $\text{Art}_k$ denote the category of localartinian $k$-algebras with residue field $k$.

3. If not stated otherwise, K-theory in this paper is Thomason-Trobaugh non-connective K-theory. For $X \in \text{Sch}/k$, let $Y \subset X$ be closed, Keller [31, 32] defined cyclic homology complexes $HC(X)$ and $HC(X \text{ on } Y)$ from localization pairs (see Example 2.7 and 2.8 of [17] for details), which agree with the definitions of Weibel [60].

Following the convention in section 2 of [17], we use cohomological notation for cyclic homology.

4. For $F$ an abelian group-valued functor, we denote by $F(O_X)$ the sheaf on a scheme $X \in \text{Sch}/k$ obtained by localizing $F$. The functor $F$ used in this paper are Milnor K-group $K^M_*(-)$, K-group $K_*(-)$, Hochschild homology $HH_*(-)$, cyclic homology $HC_*(-)$ and their eigenspaces of Adams operations $\psi^m$, denoted $K^{(l)}_*(-)$, $HH^{(l)}_*(-)$ and $HC^{(l)}_*(-)$ respectively.

### 2. Bloch-Ogus theorem and local cohomology

In this section, after recalling Bloch-Ogus theorem, cyclic homology and Milnor Chow group, we extend Bloch formula (1.1) in Theorem 2.24.
2.1. **Bloch-Ogus Theorem.** Given a smooth algebraic variety $X$ and a cohomology theory $h$ satisfying natural axioms, the classical Bloch-Ogus theorem, says that the Zariski sheafification of the Cousin complex (formed from the coniveau spectral sequence) of $h$ is a flasque resolution of the Zariski sheaf associated to the presheaf $U \mapsto h^*(U)$.

Bloch-Ogus [12] proved their theorem for étale cohomology with coefficients in roots of unity, by reducing to the “effacement theorem” which was proved by using a geometric presentation lemma. Later, Gabber [19] gave a different proof of effacement theorem for étale cohomology.

In [15], Colliot-Thélène, Hoobler and Kahn axiomatized Gabber’s proof and showed that his argument could be applied to any “Cohomology theory with support” which satisfies étale excision and a technical lemma (called “Key lemma”). The latter follows either from homotopy invariance or from projective bundle formula. In particular, Gabber’s argument works for K-theory and (negative) cyclic homology. This was used by Dribus, Hoffman and the author [18] to study the deformation of algebraic cycles. We recall it briefly.

Let $h$ be a contravariant functor from the category $\text{Sch}/k$ to spectra or chain complexes. For $X \in \text{Sch}/k$, let $Y \subset X$ be closed, we can extend $h$ to the pair $(X,Y)$.

**Definition 2.1.** For $h$ spectrum-valued, $h(X \text{ on } Y)$ is defined as the homotopy fiber of $h(X) \rightarrow h(X - Y)$. For any integer $p$, $h^p(X \text{ on } Y)$ is defined as homotopy group $\pi_{-p}(h(X \text{ on } Y))$.

For $h$ chain complex-valued, let $C_\bullet$ be the mapping cone of $h(X) \rightarrow h(X - Y)$, then $h(X \text{ on } Y)$ is defined as $C_\bullet[-1]$. For any integer $p$, $h^p(X \text{ on } Y)$ is defined as homology group $H_{-p}(h(X \text{ on } Y))$.

This gives a “cohomology theory with support” in the sense of Definition 5.1.1 of [15]. Following [15], we recall étale excision and projective bundle formula.

**Definition 2.2** (Étale excision). A functor $h$ is said to satisfy étale excision, if it is additive and if for any étale morphism $f : X' \rightarrow X$ such that $f^{-1}(Y) \rightarrow Y$ is an isomorphism with $Y \subset X$ closed, the pullback

$$f^* : h^p(X \text{ on } Y) \rightarrow h^p(X' \text{ on } f^{-1}(Y))$$

is an isomorphism for any integer $p$.

The functor $h$ is said to satisfy Zariski excision if the pullback $f^*$ is an isomorphism for any integer $p$, when $f$ runs over all open immersions.
Definition 2.3 (Projective bundle formula for projective line). The functor $h$ is said to satisfy projective bundle formula for projective line, if

$$h^p(X) \oplus h^p(X) \xrightarrow{\sim} h^p(\mathbb{P}_X^1)$$

is an isomorphism for any $X \in \text{Sch}/k$ and for any integer $p$, where $\mathbb{P}_X^1$ is the projective line over $X$.

If the functor $h$ in Definition 2.1 satisfies Zariski excision, then there exists a convergent spectral sequence, called coniveau spectral sequence (see section 1 of [15]),

$$E_1^{q,p} = \bigoplus_{x \in X^{(q)}} h^{q+p}(X \text{ on } x) \Rightarrow h^{q+p}(X),$$

where $X^{(q)}$ denotes the set of points of codimension $q$ in $X$ and

$$h^{q+p}(X \text{ on } x) = \lim_{\rightarrow} h^{q+p}(U \text{ on } \{x\} \cap U).$$

The $E_1$-terms give rise to Cousin complex of $h$

$$0 \to \bigoplus_{x \in X^{(0)}} h^p(X \text{ on } x) \to \bigoplus_{x \in X^{(1)}} h^{p+1}(X \text{ on } x) \to \cdots.$$  (2.1)

The following setting is used below.

Setting 2.4. Let $X$ be a $d$-dimensional smooth projective variety over a field $k$ of characteristic zero, with generic point $\eta$.

For $A \in \text{Art}_k$, we write $X_A = X \times_{\text{Spec}(k)} \text{Spec}(A)$. Let $F$ be a functor as in Notation (4) on page 4, we denote by $F(O_{X_A})$ the kernel of the morphism (induced by augmentation $A \to k$) $F(O_{X_A}) \to F(O_X)$.

Theorem 2.5 (Bloch-Ogus Theorem). In notation of Setting 2.4, if a functor $h$ in Definition 2.1 satisfies étale excision and projective bundle formula for projective line, then for any integer $p$, the Zariski sheafification of the Cousin complex (2.1) is a flasque resolution of the sheaf associated to the presheaf $U \to h^p(O_X(U))$.

Proof. This was originally proved by Bloch-Ogus [12] for étale cohomology and it was extended by Gabber [19]. Colliot-Thélène, Hoobler and Kahn applied Gabber’s method to prove the theorem in a general context, see Corollary 5.1.11 and Proposition 5.4.3 of [15].

□

Universal exactness was originally introduced by Grayson [25]. For arbitrary scheme $T \in \text{Sch}/k$ ($T$ might be singular), we can derive a new functor $h^T$ from the functor $h$ in Definition 2.1

$$h^T : X \to h(X \times_{\text{Spec}(k)} T).$$
If the functor $h$ satisfies étale excision and projective bundle formula for projective line, so does the new functor $h^T$. This implies that,

**Corollary 2.6** *(Corollary 6.2.4 of [15]).* In notation of Setting 2.4, if a functor $h$ in Definition 2.1 satisfies étale excision and projective bundle formula for projective line, then for any integer $p$, the Zariski sheafification of the Cousin complex (2.1) of $h^T$ is a flasque resolution of the sheaf associated to the presheaf $U \to h^p(U \times \text{Spec}(k) T)$.

**Lemma 2.7.** Both $K$-theory and cyclic homology satisfy étale excision and projective bundle formula.

*Proof.* For $K$-theory, it was proved in Theorem 7.1 (for étale excision) and in Theorem 7.3 (for projective bundle formula) of [52]. For cyclic homology, see Example 2.8 of [17] (for étale excision) and Remark 2.11 of [17] (for projective bundle formula). □

When the functor $h$ is the $K$-theory spectrum $K(X)$, the associated Cousin complex (2.1) is the Bloch-Gersten-Quillen sequence

\[
0 \to \bigoplus_{x \in X^{(0)}} K_0(O_{X,x}) \to \bigoplus_{x \in X^{(1)}} K_{-1}(O_{X,x} \text{ on } x) \to \cdots.
\]

**Corollary 2.8.** In Setting 2.4, for each integer $p \geq 0$, the Zariski sheafifications of the Bloch-Gersten-Quillen sequences (2.2) of $X$ and $X_A$ are flasque resolutions of the $K$-theory sheaves $K_p(O_X)$ and $K_p(O_{X_A})$ respectively.

*Proof.* By Quillen’s dévissage, the Bloch-Gersten-Quillen sequence (2.2) of $X$ has the form

\[
0 \to K_p(O_{X,\eta}) \to \cdots \bigoplus_{x \in X^{(p-1)}} K_1(k(x)) \to \bigoplus_{x \in X^{(p)}} K_0(k(x)) \to 0,
\]

whose Zariski sheafification is a flasque resolution of the sheaf $K_p(O_X)$. This was used by Quillen [15] to prove Bloch formula.

By Corollary 2.6 and Lemma 2.7, the Zariski sheafification of the Bloch-Gersten-Quillen sequence (2.2) of $X_A$, which has the form

\[
0 \to K_p(O_{X_A,\eta}) \to \cdots \bigoplus_{x \in X^{(p)}} K_0(O_{X_A,x} \text{ on } x) \to \cdots,
\]

is a flasque resolution of the sheaf $K_p(O_{X_A})$. □
It is worth noting that nontrivial negative K-groups may appear in the sequence (2.3).

The closed immersion $X \to X_A$ induces a map between K-theory spectra $\mathcal{K}(X_A) \to \mathcal{K}(X)$, which further induces a map between Bloch-Gersten-Quillen sequences. Since the closed immersion $X \to X_A$ has a section $X_A \to X$, there is a split commutative diagram (2.4)

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\mathcal{K}_p(O_{X_A, \eta}) & K_p(O_{X_A, \eta}) & K_p(O_{X, \eta}) \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(1)}} \mathcal{K}_{p-1}(O_{X_A, x} \text{ on } x) & \bigoplus_{x \in X^{(1)}} K_{p-1}(O_{X_A, x} \text{ on } x) & \bigoplus_{x \in X^{(1)}} K_{p-1}(O_{X, x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(d)}} \mathcal{K}_{p-d}(O_{X_A, x} \text{ on } x) & \bigoplus_{x \in X^{(d)}} K_{p-d}(O_{X_A, x} \text{ on } x) & \bigoplus_{x \in X^{(d)}} K_{p-d}(O_{X, x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0,
\end{array}
\]

where each $\mathcal{K}_{p}(O_{X_A, x} \text{ on } x)$ is the kernel of the map (induced by augmentation $A \to k$) $K_{p}(O_{X_A, x} \text{ on } x) \to K_{p}(O_{X, x} \text{ on } x)$. Let $\mathcal{K}(X_A)$ denote the homotopy fiber of $\mathcal{K}(X_A) \to \mathcal{K}(X)$, the left column of the diagram (2.4) is the Cousin complex of the spectrum $\mathcal{K}(X_A)$.

### 2.2. Cyclic homology

Hochschild and cyclic homology are defined over $\mathbb{Q}$ here. For $R$ a commutative $\mathbb{Q}$-algebra, Hochschild homology $HH_{*}(R)$ and cyclic homology $HC_{*}(R)$ carry Lambda operations $\lambda^m$ and Adams operations $\psi^m$, see section 4.5 of [39] and section 9.4.3 of [59] for details. In fact, the action of symmetric group naturally splits Hochschild complex $HH(R)$ and cyclic homology complex $HC(R)$ into sums of sub-complexes $HH^{(i)}(R)$ and $HC^{(i)}(R)$ respectively. For each integer $p \geq 1$, this decomposes $HH_{p}(R)$ and $HC_{p}(R)$ into direct sums of eigenspaces

\[
HH_{p}(R) = HH^{(1)}_{p}(R) \oplus \cdots \oplus HH^{(p)}_{p}(R),
\]

(2.5) \[ HC_{p}(R) = HC^{(1)}_{p}(R) \oplus \cdots \oplus HC^{(p)}_{p}(R). \]

For $p = 0$, $HC_{0}(R) = HC^{(0)}_{0}(R) = R$. 
Lemma 2.9 (Ex 9.4.4 and Corollary 9.8.16 of [59]). With notation as above, there are isomorphisms

$$HH_p^{(p)}(R) \cong \Omega^p_{R/Q}, \quad HC_p^{(p)}(R) = \frac{\Omega^p_{R/Q}}{d\Omega^{p-1}_{R/Q}}.$$ 

These operations $\lambda^m$ and $\psi^m$ can be extended to cyclic homology $HC_*(X)$, where $X \in \text{Sch}/k$, see Weibel [61]. Let $Y \subset X$ be closed, since cyclic homology satisfies Zariski descent, we can identify cyclic homology $HC_*(X)$ on $Y$ with hypercohomology

$$HC_*(X \text{ on } Y) = \mathbb{H}^*_{Y}(X, HC(X)),$$

where $HC(X)$ is the cyclic homology complex of $X$. This enables us to further extend $\lambda^m$ and $\psi^m$ to $HC_*(X \text{ on } Y)$.

Combining Corollary 2.6 with Lemma 2.7, one has

Lemma 2.10. In Setting 2.4, for each integer $p \geq 0$, the Zariski sheafification of the following Cousin complex of cyclic homology of $X_A$

$$0 \to HC_p(O_{X_A, \eta}) \to \bigoplus_{x \in X^{(1)}} HC_{p-1}(O_{X_A, x \text{ on } x}) \to \cdots,$$  

is a flasque resolution of the sheaf $HC_p(O_{X_A})$.

The differentials of the complex of (2.6) respect Adams operations $\psi^m$. This yields that

Lemma 2.11. In Setting 2.4, for each integer $p \geq 0$, the Zariski sheafification of the complex

$$0 \to HC_p^{(l)}(O_{X_A, \eta}) \to \bigoplus_{x \in X^{(1)}} HC_{p-1}^{(l)}(O_{X_A, x \text{ on } x}) \to \cdots,$$  

is a flasque resolution of the sheaf $HC_p^{(l)}(O_{X_A})$, where the integer $l$ satisfying that $0 \leq l \leq p$ and each $HC_*(X)_{-}^{(l)}$ is eigenspace of Adams operations $\psi^m$.

We are mainly interested in the case $l = p$ below. Let $q$ be an integer satisfying that $1 \leq q \leq d$, where $d = \dim(X)$. For $x \in X^{(q)}$, let $\overline{HC}_{p-q}^{(p)}(O_{X_A, x \text{ on } x})$ be the kernel of the map (induced by $A \to k$) $HC_{p-q}^{(p)}(O_{X_A, x \text{ on } x}) \to HC_p^{(p)}(O_{X, x \text{ on } x})$. Let $l = p$ in (2.7), Lemma 2.11 implies that

Corollary 2.12. In Setting 2.4, for each integer $p \geq 0$, the Zariski sheafification of the complex

$$0 \to \overline{HC}_p^{(p)}(O_{X_A, \eta}) \to \bigoplus_{x \in X^{(1)}} \overline{HC}_{p-1}^{(p)}(O_{X_A, x \text{ on } x}) \to \cdots,$$
is a flasque resolution of the sheaf $\overline{HC}_p^{(p)}(O_{X_A})$.

We want to compute each group $\overline{HC}_p^{(p)}(O_{X_A,x})$ of the complex (2.8). We refer to chapter IV of [28] for definitions and properties of local cohomologies of abelian sheaves.

**Lemma 2.13.** In Setting [2.4], let $q$ be an integer satisfying that $1 \leq q \leq d$, where $d = \dim(X)$. For $x \in X^{(q)}$ and for each integer $p \geq 0$, $\overline{HC}_p^{(p)}(O_{X_A,x})$ is isomorphic to local cohomology $H^q_x(\overline{HC}_p^{(p)}(O_{X_A}))$

$$\overline{HC}_{p-q}^{(p)}(O_{X_A,x} \text{ on } x) = H^q_x(\overline{HC}_p^{(p)}(O_{X_A})).$$

As recalled in the beginning of section 2.2, the cyclic homology complexes $HC(O_{X_A,x})$ and $HC(O_{X,x})$ split into direct sums of subcomplexes $HC^{(i)}(O_{X_A,x})$ and $HC^{(i)}(O_{X,x})$ respectively. We are interested in $HC^{(p)}(O_{X_A,x})$ and $HC^{(p)}(O_{X,x})$, and denote by $\overline{HC}^{(p)}(O_{X_A,x})$ the kernel of the natural map of complexes

$$HC^{(p)}(O_{X_A,x}) \rightarrow HC^{(p)}(O_{X,x}).$$

**Proof.** Since cyclic homology satisfies Zariski descent, we can identify $\overline{HC}_p^{(p)}(O_{X_A,x})$ with hypercohomology

$$\overline{HC}_p^{(p)}(O_{X_A,x} \text{ on } x) = H^0_x(-p,q)(O_{X,x},\overline{HC}_p^{(p)}(O_{X_A,x})).$$

There exists a spectral sequence

$$E^{i,j}_2 = H^i_x(O_{X,x}, H^j(\overline{HC}_p^{(p)}(O_{X_A,x}))) \Rightarrow H^{-p,q}_x(O_{X,x}, \overline{HC}_p^{(p)}(O_{X_A,x}));$$

where $i+j = -(p-q)$. We use cohomological notation for cyclic homology (see Notation (3) on page 3), so $H^j(\overline{HC}_p^{(p)}(O_{X_A,x})) = \overline{HC}_j^{(p)}(O_{X_A,x})$, the above spectral sequence can be rewritten as

$$E^{i,j}_2 = H^i_x(O_{X,x}, \overline{HC}_j^{(p)}(O_{X_A,x})) \Rightarrow H^{-p,q}_x(O_{X,x}, \overline{HC}_p^{(p)}(O_{X_A,x}));$$

where $i$ and $j$ are non-negative integers, and $i-j = -(p-q)$.

Since the Krull dimension of $O_{X,x}$ is $q$, if $i > q$, then the local cohomology $H^i_x(O_{X,x}, \overline{HC}_j^{(p)}(O_{X_A,x})) = 0$ for each $j$. This shows that the index $i$ in non-zero terms of the spectral sequence (2.9) satisfies that $0 \leq i \leq q$. It follows that $j = i + p - q \leq p$. If $j < p$, then $\overline{HC}_j^{(p)}(O_{X_A,x}) = 0$, see (2.5) on page 8. Hence, the index $j = i + p - q$ in non-zero terms of the spectral sequence (2.9) can only be $p$, which implies that $i = q.$

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1The index $p-q$ might be negative.
In conclusion, the only non-zero term in the spectral sequence (2.9) is $H^q_x(O_{X,x}, \overline{HC}^{(p)}_p(O_{X,A,x}))$. So the spectral sequence (2.9) degenerates and

$$\overline{HC}^{(p)}_{p-q}(O_{X,A,x}) = H^q_y(O_{X,x}, \overline{HC}^{(p)}_p(O_{X,A,x})).$$

□

**Definition 2.14** (cf. Def 3.2 of [10]). For $X \in \text{Sch} / k$, an abelian sheaf $F$ on $X$ is called Cohen-Macaulay, if for every scheme point $y \in X$, $H^i_y(X,F) = 0$ for $i \neq \text{codim}\{y\}$.

To see the importance of Cohen-Macaulay sheaves, we recall that, for $X \in \text{Sch} / k$ and for an abelian sheaf $F$ on $X$, the Cousin complex of $F$ constructed in Proposition 2.3 of chapter IV of [28] has the form

$$0 \rightarrow \bigoplus_{x \in X^{(0)}} H^0_x(F) \rightarrow \bigoplus_{x \in X^{(1)}} H^1_x(F) \rightarrow \cdots$$

**Lemma 2.15** (Prop 2.6 of chapter IV of [28]). The following are equivalent:

1. the abelian sheaf $F$ is Cohen-Macaulay,
2. the Zariski sheafification of the Cousin complex (2.10) of $F$ is a flasque resolution of $F$.

For $X$ a smooth projective variety over a field $k$ of characteristic zero, $\Omega^*_{X/k}$ is Cohen-Macaulay (see page 239 of [28]). Kerz [33] proved that Milnor K-theory sheaf $K^M_*(O_X)$ is Cohen-Macaulay. Bloch-Esnault-Kerz generalized these examples in Prop 3.5 of [10] and applied it to the infinitesimal study of Chow groups. This motivates us to find more examples of Cohen-Macaulay sheaves.

In Setting 2.4, by Corollary 2.12 and Lemma 2.13, the Zariski sheafification of the Cousin complex (2.10) of the sheaf $\overline{HC}^{(p)}_p(O_{X,A})$

$$0 \rightarrow \overline{HC}^{(p)}_p(O_{X,A}) \rightarrow \bigoplus_{x \in X^{(1)}} H^1_x(\overline{HC}^{(p)}_p(O_{X,A})) \rightarrow \cdots,$$

is a flasque resolution of $\overline{HC}^{(p)}_p(O_{X,A})$. It follows from Lemma 2.13 that

**Corollary 2.16.** In Setting 2.4, for each integer $p \geq 0$, the sheaf $\overline{HC}^{(p)}_p(O_{X,A})$ is Cohen-Macaulay.

**Lemma 2.17.** In Setting 2.4, for each integer $p \geq 0$, the sheaf $\overline{HH}^{(p)}_p(O_{X,A})$ is Cohen-Macaulay.

**Proof.** By Lemma 2.9, $\overline{HH}^{(p)}_p(O_{X,A}) \cong \Omega^{\text{top}}_{X_A/Q}$. We first show that $\overline{HH}^{(p)}_p(O_{X,A})$ is Cohen-Macaulay. Let $F^j$ be the image of the map
$\Omega^j_{k/Q} \otimes_k \Omega^{p-j}_{X_A/Q} \to \Omega^p_{X_A/Q}$, where $j = 0, 1, \ldots, p$. There is a filtration on $\Omega^p_{X_A/Q}$ given by

$$\Omega^p_{X_A/Q} = F^0 \supset F^1 \supset \cdots \supset F^p \supset F^{p+1} = 0,$$

whose associated graded piece is $Gr^j \Omega^p_{X_A/Q} = F^j / F^{j+1} = \Omega^j_{k/Q} \otimes_k \Omega^{p-j}_{X_A/k}$. In particular, $Gr^p \Omega^p_{X_A/Q} = F^p = \Omega^p_{k/Q} \otimes_k O_{X_A}$.

There is an isomorphism of sheaves

$$\Omega^p_{X_A/k} = \bigoplus_{j_1 + j_2 = p-j} \Omega^{j_1}_{X/k} \otimes_k \Omega^{j_2}_{A/k},$$

which can be checked locally ($X_A$ and $X$ have the same underlying space). Since each $\Omega^{j_1}_{X/k}$ is Cohen-Macaulay, so is $\Omega^j_{X_A/k}$. It follows that each $Gr^j \Omega^p_{X_A/Q}$ is Cohen-Macaulay. There is a short exact sequence

$$0 \to F^p \to F^{p-1} \to Gr^{p-1} \to 0,$$

where both $F^p = Gr^p \Omega^p_{X_A/Q} = \Omega^p_{k/Q} \otimes_k O_{X_A}$ and $Gr^{p-1} = \Omega^{p-1}_{k/Q} \otimes_k O_{X_A/k}$ are Cohen-Macaulay, so the associated long exact sequence of local cohomology implies that $F^{p-1}$ is Cohen-Macaulay. We are able to prove that each $F^j$ is Cohen-Macaulay by continuing this procedure. In particular, $F^0 = \Omega^p_{X_A/Q}$ is Cohen-Macaulay. When $A = k$, $HH^p(O_X) \cong \Omega^p_{X/Q}$ is Cohen-Macaulay.

The short exact sequence

$$0 \to \overline{HH}^p(O_{X_A}) \to HH^p(O_{X_A}) \to HH^p(O_X) \to 0$$

is split, where both $\overline{HH}^p(O_{X_A})$ and $HH^p(O_X)$ are Cohen-Macaulay, so is $HH^p(O_{X_A})$. $\square$

2.3. Milnor Chow groups. For $X \in Sch/k$, it is well known that Grothendieck group of $X$ carries Adams operations $\psi^m$, which is induced from exterior powers of vector bundles on $X$. These operations can be extended to higher algebraic K-theory. For $Y \subset X$ a closed subscheme, Soulé [49] and Levine [36] defined Adams operations $\psi^m$ on K-groups with supports $K_n(X, Y)$, where $n \geq 0$.

Since the appearance of nontrivial negative K-groups in our study, we need to extend Adams operations $\psi^m$ to negative range. According to Weibel [58] (section 8), this can be done inductively by using Bass fundamental exact sequence. We have used this method in [18] (section 8.2).
Let $I$ be a nilpotent ideal in a commutative $\mathbb{Q}$-algebra $R$. We define the relative $K$-group $K_n(R, I)$ to be the kernel of the morphism $K_n(R) \to K_n(R/I)$ and define $K_n^{(l)}(R, I)$ to be the eigenspace of $\psi^m = m^l$, where $\psi^m$ is Adams operations on $K_n(R, I)$. The relative cyclic homology $HC_{n-1}(R, I)$ and $HC_{n-1}^{(l-1)}(R, I)$ are defined similarly. Goodwillie and Cathelineau proved that these relative groups are connected by the relative Chern character.

**Theorem 2.18 ([14, 24])**. With notation as above, the relative Chern character induces an isomorphism between $K_n(R, I)_{\mathbb{Q}}$ and $HC_{n-1}(R, I)$, which respects Adams operations

$$K_n(R, I)_{\mathbb{Q}} \overset{\cong}{\to} HC_{n-1}(R, I), \quad K_n^{(l)}(R, I)_{\mathbb{Q}} \overset{\cong}{\to} HC_{n-1}^{(l-1)}(R, I).$$

This theorem is very useful to compute relative $K$-groups. Cortiñas-Haesemeyer-Weibel [16] generalized it to space level. We adopt it to Setting 2.4 and refer to appendix B of [16] for a general form.

Let $\mathcal{HC}(X_A)$ and $\mathcal{HC}(X)$ be the Eilenberg-Mac Lane spectra associated to cyclic homogy complexes $HC(X_A)$ and $HC(X)$ respectively. Let $\overline{\mathcal{HC}(X_A)}$ be the homotopy fiber of $\mathcal{HC}(X_A) \to \mathcal{HC}(X)$, we define $\overline{\mathcal{HC}}^{(l-1)}(X_A)$ as the homotopy fiber of the map $\psi^m - m^l$ on $\overline{\mathcal{HC}(X_A)}$. The spectra $\overline{K}(X_A)$ and $\overline{K}^{(l)}(X_A)$ are defined similarly.

Theorem 2.18 can be generalized in the following way.

**Theorem 2.19** (cf. Theorem B.11 of [16]). In Setting 2.4, the relative Chern character induces homotopy equivalence of spectra

$$\overline{K}(X_A) \overset{\cong}{\to} \overline{HC}(X_A)[1], \quad \overline{K}^{(l)}(X_A) \overset{\cong}{\to} \overline{HC}^{(l-1)}(X_A)[1].$$

For each integer $p \geq 1$, the homotopy equivalence of spectra

$$\overline{K}^{(l)}(X_A) \overset{\cong}{\to} \overline{HC}^{(l-1)}(X_A)[1]$$
induces a commutative diagram of Cousin complexes

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
K_p^{(l)}(O_{X_A,\eta})_Q & \xrightarrow{=} & \HC_{p-1}^{(l-1)}(O_{X_A,\eta}) \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(1)}} K_p^{(l)}(O_{X_A, x on x})_Q & \xrightarrow{=} & \bigoplus_{x \in X^{(1)}} \HC_{p-2}^{(l-1)}(O_{X_A, x on x}) \\
\downarrow & & \downarrow \\
\vdots & \dashrightarrow & \vdots \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(d)}} K_p^{(l)}(O_{X_A, x on x})_Q & \xrightarrow{=} & \bigoplus_{x \in X^{(d)}} \HC_{p-d-1}^{(l-1)}(O_{X_A, x on x}) \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

(2.11)

We explain the notations of diagram (2.11) briefly. Let \(K_p^{(l)}(O_{X_A, x on x})\) and \(K_p^{(l)}(O_{X, x on x})\) denote eigenspaces of Adams operations \(\psi^m = m^l\) respectively, \(K_p^{(l)}(O_{X_A, x on x})\) is the kernel of the morphism

\[ K_p^{(l)}(O_{X_A, x on x}) \to K_p^{(l)}(O_{X, x on x}), \]

and \(\HC_{p-1}^{(l-1)}(O_{X_A, x on x})\) is defined similarly.

Combining Corollary 2.8 Lemma 2.11 with diagrams (2.4) and (2.11), one has

**Lemma 2.20.** In Setting [2.4], there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\HC_{p-1}^{(l-1)}(O_{X_A,\eta}) & \xleftarrow{=} & K_p^{(l)}(O_{X_A,\eta})_Q \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(1)}} \HC_{p-2}^{(l-1)}(O_{X_A, x on x}) & \xleftarrow{=} & \bigoplus_{x \in X^{(1)}} K_p^{(l-1)}(O_{X_A, x on x})_Q \\
\downarrow & & \downarrow \\
\vdots & \dashrightarrow & \vdots \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(d)}} \HC_{p-d-1}^{(l-1)}(O_{X_A, x on x}) & \xleftarrow{=} & \bigoplus_{x \in X^{(d)}} K_p^{(l-d-1)}(O_{X_A, x on x})_Q \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]
where each column is a complex whose Zariski sheafification is a flasque resolution of $\overline{HC}_{p-1}^{(l-1)}(O_{X_A})$, $K_p^{(l)}(O_{X_A})_Q$ and $K_p^{(l)}(O_X)_Q$ respectively.

Combining Lemma 2.20 (let $l = p$) with Lemma 2.13, one has

**Theorem 2.21.** In Setting 2.4, for each integer $p \geq 1$, there exists the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\overline{HC}_{p-1}^{(p-1)}(O_{X_A},\eta) & \xrightarrow{\text{Ch}} & K_p^{(p)}(O_{X_A},\eta)_Q & \rightarrow & K_p^{(p)}(O_{X,\eta})_Q & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\bigoplus_{x \in X^{(1)}} H_1^k(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{\partial} & K_p^{(p)}(O_{X,A,x} \text{ on } x)_Q & \rightarrow & K_p^{(p)}(O_{X,x} \text{ on } x)_Q & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
& \cdots & \cdots & \rightarrow & \cdots & \rightarrow \\
\bigoplus_{x \in X^{(p-1)}} H^p_{p-1}(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{\partial} & K_1^{(p)}(O_{X,A,x} \text{ on } x)_Q & \rightarrow & K_1^{(p)}(O_{X,x} \text{ on } x)_Q & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
& \cdots & \cdots & \rightarrow & \cdots & \rightarrow \\
\bigoplus_{x \in X^{(p+1)}} H^{p+1}_{p-1}(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{\partial} & K_{p-1}^{(p)}(O_{X,A,x} \text{ on } x)_Q & \rightarrow & K_{p-1}^{(p)}(O_{X,x} \text{ on } x)_Q & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
& \cdots & \cdots & \rightarrow & \cdots & \rightarrow \\
\bigoplus_{x \in X^{(d)}} H^{d}_{p-1}(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{\partial} & K_{p-d}^{(p)}(O_{X,A,x} \text{ on } x)_Q & \rightarrow & K_{p-d}^{(p)}(O_{X,x} \text{ on } x)_Q & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \\
\end{array}
\]

in which the Zariski sheafification of each column is a flasque resolution of $\overline{HC}_{p-1}^{(p-1)}(O_{X_A})$, $K_p^{(p)}(O_{X_A})_Q$ and $K_p^{(p)}(O_X)_Q$ respectively. The map from the middle column to the left one, denoted Ch, is induced by relative Chern characters from $K$-theory to cyclic homology, and the map from the middle column to the right one, denoted Pr, is induced by augmentation $A \rightarrow k$.

Using tensor triangular geometry \[2\], Balmer \[3\] defined tensor triangular Chow groups of a tensor triangulated category, which were
further explored by Klein [35]. By slight modifying Balmer’s definition, we proposed Milnor K-theoretic cycles.

**Definition 2.22** (Definition 3.4 of [63]). In notation of Theorem 2.21 let $p$ further satisfy that $1 \leq p \leq d$, where $d = \dim(X)$. The $p$-th Milnor K-theoretic cycle groups of $X$ and $X_A$, denoted $Z_p^M(D_{\text{perf}}(X))$ and $Z_p^M(D_{\text{perf}}(X_A))$ respectively, are defined to be

$$Z_p^M(D_{\text{perf}}(X)) := \text{Ker}(d_{1,X}^{-p}), \quad Z_p^M(D_{\text{perf}}(X_A)) := \text{Ker}(d_{1,X_A}^{-p}).$$

The $p$-th Milnor K-theoretic Chow groups of $X$ and $X_A$, denoted by $CH_p^M(D_{\text{perf}}(X))$ and $CH_p^M(D_{\text{perf}}(X_A))$ respectively, are defined to be

$$CH_p^M(D_{\text{perf}}(X)) := \frac{\text{Ker}(d_{1,X}^{-p})}{\text{Im}(d_{1,X}^{-p-1})}, \quad CH_p^M(D_{\text{perf}}(X_A)) := \frac{\text{Ker}(d_{1,X_A}^{-p})}{\text{Im}(d_{1,X_A}^{-p-1})}.$$

The elements of $Z_p^M(D_{\text{perf}}(X_A))$ are called Milnor K-theoretic cycles. The reason why we use the kernel of $d_{1,X}^{-p}$ to define $Z_p^M(D_{\text{perf}}(X_A))$ is explained in section 2.2 of [64], where $A$ is the ring of dual numbers $k[t]/(t^2)$.

To explain that the above definitions are a honest generalization of the classical cycle group $Z^p(X)$ and Chow group $CH^p(X)$, we recall that

**Theorem 2.23** (Theorem 3.16 of [63]). For $X$ a smooth projective variety over a field $k$ of characteristic zero, there exists the following identifications

$$Z_p^M(D_{\text{perf}}(X)) = Z^p(X)_Q, \quad CH_p^M(D_{\text{perf}}(X)) = CH^p(X)_Q.$$

In fact, the right column of the diagram in Theorem 2.21 agrees with the following complex of Milnor K-theory studied by Soulé [49]

$$0 \to K_p^M(k(X))_Q \to \bigoplus_{x \in X^{(1)}} K_{p-1}^M(k(x))_Q \to \cdots \to \bigoplus_{x \in X^{(p)}} K_0^M(k(x))_Q \to 0.$$

This is the key to prove Theorem 2.23.

The Milnor K-theoretic Chow groups $CH_p^M(D_{\text{perf}}(X_A))$ agrees with cohomological Chow group $H^p(X, K_p^M(O_{X_A}))_Q$ as follows.

**Theorem 2.24.** With notation as above, there are isomorphisms

$$(2.12) \quad CH_p^M(D_{\text{perf}}(X_A)) = H^p(X, K_p^{(p)}(O_{X_A})))_Q = H^p(X, K_p^M(O_{X_A}))_Q.$$

The first isomorphism follows from Theorem 2.21 and the second one is from the isomorphism $K_p^M(O_{X_A})_Q = K_p^{(p)}(O_{X_A})_Q$. This answers Question 1.5 i.e., extends Bloch formula (1.1) from $X$ to $X_A$. When $A = k[t]/(t^2)$, the above isomorphisms (2.12) were proved in Theorem 3.17 of [63].
3. Local Hilbert function and Chow groups

In this section, we first construct a map from local Hilbert functor to local homology in (3.2). Then, with suitable assumptions, we use this map to answer Question 1.4 in Theorem 3.14.

In notation of Setting 2.4 let $Y \subset X$ be a closed irreducible subvariety of codimension $p$.

From now on, we fix the integer $p$.

**Definition 3.1.** The local Hilbert functor $\mathcal{H}ilb$ is a functor on the category $Art_k$

$$\mathcal{H}ilb : A \rightarrow \mathcal{H}ilb(A),$$

where $A \in Art_k$ and $\mathcal{H}ilb(A)$ denotes the set of infinitesimal embedded deformations of $Y$ in $X_A$.

This functor $\mathcal{H}ilb$ had been studied intensively in literature, including [29, 48, 54]. We connect it with K-theory in the following.

Let $y$ be the generic point of $Y$. We denote by $K_0(O_{X_A,y} \text{ on } y)$ the Grothendieck group of the triangulated category $D^b(O_{X_A,y} \text{ on } y)$, which is the derived category of perfect complexes of $O_{X_A,y}$-modules with homology supported on the closed point $y \in \text{Spec}(O_{X,y})$.

The closed subvariety $Y$ is generically given by a regular sequence $\{f_1, \cdots, f_p\}$ of $O_{X,y}$. For any $Y' \in \mathcal{H}ilb(A)$, $Y'$ is generically given by a regular sequence $\{f^1_A, \cdots, f^A_p\}$ of $O_{X_A,y}$. Let $L^A_\bullet$ be the Koszul complex of the regular sequence $\{f^1_A, \cdots, f^A_p\}$, we consider the complex $L^A_\bullet$ as an element of $K_0(O_{X_A,y} \text{ on } y)_\mathbb{Q}$.

Adams operations $\psi^m$ for K-theory of perfect complexes defined in [22] has the following property.

**Lemma 3.2** (Prop 4.12 of [22]). Adams operations $\psi^m$ on $L^A_\bullet$ satisfies that

$$\psi^m(L^A_\bullet) = m^p L^A_\bullet.$$

Let $K_0^{(p)}(O_{X_A,y} \text{ on } y)$ be the eigenspace of $\psi^m = m^p$. The above Lemma implies that $L^A_\bullet \in K_0^{(p)}(O_{X_A,y} \text{ on } y)_\mathbb{Q}$.

**Definition 3.3.** With notation as above, one defines a set-theoretic map

$$\alpha_A : \mathcal{H}ilb(A) \longrightarrow K_0^{(p)}(O_{X_A,y} \text{ on } y)_\mathbb{Q}$$

$$Y' \longrightarrow L^A_\bullet.$$

It is interesting to determine whether $\alpha_A(Y')$ is a Milnor K-theoretic cycle (in the sense of Definition 2.22) or not.
Question 3.4. With notation as above, is it true that
\[ \alpha_{A}(Y') \in Z^{M}_{p}(D^{Perf}(X_{A}))? \]

In other words, is it true that \(d_{1,X_{A}}^{p,-p} \circ \alpha_{A}(Y') = 0\), where \(d_{1,X_{A}}^{p,-p}\) is the differential of the diagram in Theorem 2.21?

The subtlety of this question is the appearance of negative K-group \(K_{-1}(O_{X_{A},x} \text{ on } x)\) in the diagram in Theorem 2.21 which may not vanish. We refer to [17, 33] for recent progress on Weibel’s vanishing conjecture of negative K-theory.

In the diagram in Theorem 2.21 \(K_{-1}(O_{X,x} \text{ on } x) = K_{-1}(k(x)) = 0\), this implies that \(d_{1,X_{A}}^{p,-p} \circ \text{Pr} \circ \alpha_{A}(Y') = 0\). Since the diagram in Theorem 2.21 is split, Question 3.4 is equivalent to the following one.

Question 3.5. Let \(\text{Ch} \circ \alpha_{A}\) be the composition
\[
  \mathbb{H} \text{ilb}(A) \xrightarrow{\alpha} K^{(p)}_{0}(O_{X_{A},y} \text{ on } y)_{Q} \xrightarrow{\text{Ch}} H^{p}_{y}(\mathcal{H} \text{ilb}_{r-1}^{(p-1)}(O_{X_{A}})),
\]
does the image \(\text{Ch} \circ \alpha_{A}(Y')\) lie in the kernel of \(\partial_{1,X_{A}}^{p,-p}\), where \(\text{Ch}\) and \(\partial_{1,X_{A}}^{p,-p}\) are maps of the diagram in Theorem 2.21?

It is known that \(\text{Ch} \circ \alpha_{A}(Y')\) does not always lie in the kernel of \(\partial_{1,X_{A}}^{p,-p}\), see Example 4.4 of [62]. Hence, \(\alpha_{A}(Y')\) is not a Milnor K-theoretic cycle in general.

In the rest of this section, we strengthen the situation of Setting 2.4 as follows.

Setting 3.6. In notation of Setting 2.4, we further assume that \(Y \subset X\) is a locally complete intersection. There exists a finite open affine covering \(\{U_{i}\}_{i \in I}\) of \(X\) such that \(Y \cap U_{i}\) is given by a regular sequence \(f_{1}, \ldots, f_{n}\) of \(O_{X}(U_{i})\).

Let \(G \text{Art}_{k} \subset \text{Art}_{k}\) denote the subcategory of \(\text{Art}_{k}\) whose objects are also graded \(k\)-algebras \(A = \bigoplus_{m \geq 0} A_{m}\) such that \(A_{0} = k\).

In this setting, for \(A \in G \text{Art}_{k}\) and for \(Y' \in \mathbb{H} \text{ilb}(A)\), we will prove that \(\text{Ch} \circ \alpha_{A}(Y')\) lies in the kernel of \(\partial_{1,X_{A}}^{p,-p}\), which yields that the image \(\alpha_{A}(Y')\) is a Milnor K-theoretic cycle.

Let \(U_{i} = \text{Spec}(R) \subset X\) be open affine, for \(A \in G \text{Art}_{k}\), we note that \(R \otimes_{k} A = \bigoplus_{m \geq 0} (R \otimes_{k} A_{m})\) is a graded \(k\)-algebra with \(R \otimes_{k} A_{0} = R \otimes_{k} k = R\). Since \(Q \subset k, R \otimes_{k} A\) can be also considered as a graded \(Q\)-algebra. By Goodwillie [23], the SBI sequence (defined over \(Q\)) broke into short exact sequence
\[
  0 \to \mathcal{H} \text{ilb}_{l-1}^{(l-1)}(R \otimes_{k} A) \xrightarrow{B} \mathcal{H} \text{ilb}_{l}^{(l)}(R \otimes_{k} A) \xrightarrow{1} \mathcal{H} \text{ilb}_{l}^{(l)}(R \otimes_{k} A) \to 0,
\]
where \(R \otimes_{k} A\) is considered as a graded \(Q\)-algebra and \(l\) is any positive integer, \(\mathcal{H} \text{ilb}_{l-1}^{(l-1)}(R \otimes_{k} A)\) is defined to be the kernel of \(HC_{l-1}^{(l-1)}(R \otimes_{k} A)\).
\( A \) \( \rightarrow HC_{l-1}^{(l-1)}(R), \overline{HH}_l^{(l)}(R \otimes_k A) \) and \( \overline{HC}_l^{(l)}(R \otimes_k A) \) are defined similarly. This sequence is useful in computing cyclic homology and K-theory, for example, see Geller, Reid and Weibel [20, 21]. The following short exact sequence
\[
0 \rightarrow HC_{l-1}^{(l-1)}(O_{X_A}) \overset{B}{\rightarrow} HH_l^{(l)}(O_{X_A}) \overset{1}{\rightarrow} HC_l^{(l)}(O_{X_A}) \rightarrow 0,
\]
is a sheaf version of (3.3).

For each integer \( q \) satisfying that \( 1 \leq q \leq d \), where \( d = \dim(X) \), and for \( x \in X^{(q)} \), there is a long exact sequence associated to (3.4)
\[
\cdots \rightarrow H^q_l(\overline{HC}_{l-1}^{(l-1)}(O_{X_A})) \rightarrow H^q_l(\overline{HH}_l^{(l)}(O_{X_A})) \rightarrow H^q_l(\overline{HC}_l^{(l)}(O_{X_A})) \rightarrow H^{q+1}_l(\overline{HC}_{l-1}^{(l-1)}(O_{X_A})) \rightarrow \cdots.
\]

By Corollary 2.16 and Lemma 2.17, the sheaves of sequence (3.4) are Cohen-Macaulay, so the sequence (3.5) is indeed a short exact sequence
\[
0 \rightarrow H^q_l(\overline{HC}_{l-1}^{(l-1)}(O_{X_A})) \overset{B}{\rightarrow} H^q_l(\overline{HH}_l^{(l)}(O_{X_A})) \overset{1}{\rightarrow} H^q_l(\overline{HC}_l^{(l)}(O_{X_A})) \rightarrow 0.
\]

We recall that \( y \) is the generic point of \( Y, y \in X^{(p)} \). To investigate Question 3.5, we want to describe the composition
\[
\Hilb(A) \xrightarrow{\alpha_A} K_0^{(p)}(O_{X_A}, y) \text{ on } y \Q
\]
\[
\overset{\text{Ch}}{\rightarrow} H_y^p(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) \overset{B}{\rightarrow} H_y^p(\overline{HH}_p^{(p)}(O_{X_A})),
\]
where \( B \) is the injective map in (3.6) (let \( l = p, q = p \) and \( x = y \)).

Let \( \overline{\Omega}_{X_A/Q}^{(p)} \) be the kernel of \( \Omega_{X_A/Q}^p \rightarrow \Omega_{X/Q}^p \). By Lemma 2.9, there is an isomorphism
\[
\overline{HH}_p^{(p)}(O_{X_A}) = \overline{\Omega}_{X_A/Q}^{(p)}.
\]
Then we write the above composition (3.7) as
\[
\Hilb(A) \xrightarrow{\alpha_A} K_0^{(p)}(O_{X_A}, y) \text{ on } y \Q
\]
\[
\overset{\text{Ch}}{\rightarrow} H_y^p(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) \overset{B}{\rightarrow} H_y^p(\overline{\Omega}_{X_A/Q}^{(p)}),
\]
and describe it in the following.

We first use a construction of Angéniol and Lejeune-Jalabert [1] to describe the composition \( B \circ \text{Cl}^2 \). An element of \( K_0^{(p)}(O_{X_A}, y) \Q \) is represented by a strict perfect complex \( L \).

\[
0 \rightarrow L_n \overset{M_n}{\rightarrow} L_{n-1} \overset{M_{n-1}}{\rightarrow} \cdots \overset{M_2}{\rightarrow} L_1 \overset{M_1}{\rightarrow} L_0 \rightarrow 0,
\]

\[\text{Analogous descriptions were given in [62] (Section 3) and [65] (Section 2) by using Angéniol and Lejeune-Jalabert’s method, where } A \text{ is a truncated polynomial } k[t]/(t^l).\]
where each $L_j$ is a free $O_{X,A,y}$-modules of finite rank, each $M_j$ is a matrix with entries in $O_{X,A,y}$ and the homology of $L_\bullet$ is supported on $y$.

**Definition 3.7** (page 24 in [1]). The local fundamental class attached to this perfect complex $L_\bullet$ is defined to be the following collection

$$[L_\bullet]_{\text{loc}} = \{ \frac{1}{p!} dM_j \circ dM_{j+1} \circ \cdots \circ dM_{j+p-1} \}, j = 1, 2, \cdots,$$

where $d = d_Q$ and each $dM_j$ is the matrix of differentials. In other words,

$$dM_j \in \text{Hom}(L_j, L_{j-1} \otimes \Omega^1_{O_{X,A,y}/Q}).$$

By Lemma 3.1.1 (on page 24) and Definition 3.4 (on page 29) in [1], the local fundamental class $[L_\bullet]_{\text{loc}}$ defines a cycle of the complex $\text{Hom}(L_\bullet, \Omega^p_{O_{X,A,y}/Q} \otimes L_\bullet)$ and its image (still denoted $[L_\bullet]_{\text{loc}}$) in $\mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,A,y}/Q} \otimes L_\bullet)$, which is the $p$-th cohomology of the complex $\text{Hom}(L_\bullet, \Omega^p_{O_{X,A,y}/Q} \otimes L_\bullet)$, does not depend on the choice of the basis of $L_\bullet$.

Since $L_\bullet$ is supported on $y$, by the discussion after Definition 2.3.1 on page 98-99 in [1], there exists a trace map

$$\text{Tr} : \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,A,y}/Q} \otimes L_\bullet) \to H_y^p(\Omega^p_{X,A}/Q).$$

**Definition 3.8** (Definition 2.3.2 on page 99 in [1]). The image of $[L_\bullet]_{\text{loc}}$ under the above trace map $\text{Tr}$, denoted $\mathcal{V}^p_{L_\bullet}$, is called Newton class.

Grothendieck group of a triangulated category is the monoid of isomorphism objects modulo the submonoid formed from distinguished triangles.

**Lemma 3.9** (Proposition 4.3.1 on page 113 in [1]). The Newton class $\mathcal{V}^p_{L_\bullet}$ is well-defined on the Grothendieck group $K^p_0(O_{X,A,y}, y)$ on $y$.

The morphism $\Omega^p_{X,A/Q} \to \Omega^p_{X,A/Q}$ induces a map $\varphi : H_y^p(\Omega^p_{X,A/y}) \to H_y^p(\Omega^p_{X,A/Q})$.

**Definition 3.10.** One uses Newton class $\mathcal{V}^p_{L_\bullet}$ to defines a morphism

$$\rho : K^p_0(O_{X,A,y} \text{ on } y) \to H^p_y(\Omega^p_{X,A/y}) \to H^p_y(\Omega^p_{X,A/Q}) \quad L_\bullet \quad \longrightarrow \quad \mathcal{V}^p_{L_\bullet} \quad \longrightarrow \quad \varphi(\mathcal{V}^p_{L_\bullet}).$$

The composition $B \circ \text{Ch}$ in (3.9) can be described by $\rho$, so $B \circ \text{Ch} \circ \alpha_A$ in (3.9) is given by $\rho \circ \alpha_A$. Concretely, in notation of Setting 3.6, for any $Y' \in \mathfrak{H}ilb(A)$, $Y'$ is still a locally complete intersection. In fact, $Y' \cap U_i$ is given by a regular sequence $\{f^A_1, \cdots, f^A_p\}$ of $O_{X,A}(U_i)$. 
By considering each \( f_i \) and \( f_i^A \) as elements of \( O_{X,y} \) and \( O_{X_A,y} \) respectively, one has that \( Y \) and \( Y' \) are generically given by regular sequences \( \{f_1, \cdots, f_p\} \) and \( \{f_1^A, \cdots, f_p^A\} \) respectively.

Let \( F^A \) be the Koszul resolution of \( O_{X_A,y}/(f_1^A, \cdots, f_p^A) \), which has the form

\[
0 \to F_p^A \to \cdots \to F_0^A \to 0,
\]

where each \( F_i^A = \bigwedge^i(O_{X_A,y})^\oplus p \).

By Definition 3.3, \( \alpha_A(Y') = F^A = F_0^A \in K^p(O_{X_A,y} \text{ on } y)_Q \). The image \( B \circ \text{Ch} \circ \alpha_A(Y') \) can be described via Newton class. Concretely, the following diagram

\[
\begin{align*}
F^A & \xrightarrow{\text{loc}} O_{X_A,y}/(f_1^A, \cdots, f_p^A) \\
F^A_p = \bigwedge^p(O_{X_A,y}) & \xrightarrow{[F^A]_\text{loc}} F_0^A \otimes \Omega^p_{O_{X_A,y}/Q}(\cong \Omega^p_{O_{X_A,y}/Q}),
\end{align*}
\]

where \([F^A]_\text{loc} = df_1^A \wedge \cdots \wedge df_p^A\) is the local fundamental class attached to \( F^A \), gives an element \( \beta^A \) in \( \text{Ext}^p(O_{X_A,y}/(f_1^A, \cdots, f_p^A), \Omega^p_{O_{X_A,y}/Q}) \). There is an isomorphism

\[
H^p_y(\Omega^p_{X_A/Q}) = \lim_{n \to \infty} \text{Ext}^p(O_{X_A,y}/(f_1^A, \cdots, f_p^A)_n, \Omega^p_{O_{X_A,y}/Q}),
\]

the image \([\beta^A] \) of \( \beta^A \) under the limit is the Newton class \( \mathcal{V}^p_{F^A} \in H^p_y(\Omega^p_{X_A/Q}) \).

Let \( F_p(f_1, \cdots, f_p) \) be the Koszul resolution of \( O_{X,y}/(f_1, \cdots, f_p) \), which has the form

\[
0 \to F_p \to \cdots \to F_0 \to 0,
\]

where each \( F_i \) is defined as usually.

For \([F^A]_\text{loc} = df_1^A \wedge \cdots \wedge df_p^A \in \Omega^p_{O_{X_A,y}/Q}\), we denote by \( [F^A]_\text{loc} \) the image of \([F^A]_\text{loc} \) under the morphism \( \Omega^p_{O_{X_A,y}/Q} \to \Omega^p_{O_{X_A,y}/Q} \), where \( \Omega^p_{O_{X_A,y}/Q} \) is the kernel of \( \Omega^p_{O_{X_A,y}/Q} \to \Omega^p_{O_{X_A,y}/Q} \). Concretely, \([F^A]_\text{loc} = df_1^A \wedge \cdots \wedge df_p^A - df_1 \wedge \cdots \wedge df_p \). The following diagram (denoted \( [\beta^A] \))

\[
\begin{array}{c}
\begin{align*}
F_p(f_1, \cdots, f_p) & \xrightarrow{\text{loc}} O_{X,y}/(f_1, \cdots, f_p) \\
F_p(\cong O_{X,y}) & \xrightarrow{[F^A]_\text{loc}} F_0 \otimes \Omega^p_{O_{X_A,y}/Q}(\cong \Omega^p_{O_{X_A,y}/Q}),
\end{align*}
\end{array}
\]

defines an element in \( \text{Ext}^p(O_{X,y}/(f_1, \cdots, f_p), \Omega^p_{O_{X_A,y}/Q}) \). There is an isomorphism

\[
H^p_y(\Omega^p_{X_A/Q}) = \lim_{n \to \infty} \text{Ext}^p(O_{X,y}/(f_1, \cdots, f_p)_n, \Omega^p_{O_{X_A,y}/Q}),
\]

the image \([\beta^A] \) in \( H^p_y(\Omega^p_{X_A/Q}) \) of \( [\beta^A] \) under the limit is \( B \circ \text{Ch} \circ \alpha_A(Y') \).

To summarize, one has
Lemma 3.11. In Setting 3.6, for \( Y' \in \text{Hilb}(A) \), the image of \( Y' \) under the composition \( B \circ \text{Ch} \circ \alpha_A \) in (3.9) can be described by \( [\beta^A] \):

\[
B \circ \text{Ch} \circ \alpha_A(Y') = [\beta^A].
\]

Let \( l = p \) in the sequence (3.4), the natural map \( B : \overline{HC}_{p-1}^{(p-1)}(O_{X_A}) \to \overline{HH}_p^{(p)}(O_{X_A}) \) induces a commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\overline{HC}_{p-1}^{(p-1)}(O_{X_A}, \eta) & \xrightarrow{B} & \overline{\Omega}_{X_A/Q}^p \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(1)}} H^1_x(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{B} & \bigoplus_{x \in X^{(1)}} H^1_x(\overline{\Omega}_{X_A/Q}^p) \\
\downarrow & & \downarrow \\
\ldots & \to & \ldots \\
\bigoplus_{x \in X^{(p)}} H^p_x(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{B} & \bigoplus_{x \in X^{(p)}} H^p_x(\overline{\Omega}_{X_A/Q}^p) \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{B} & \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\overline{\Omega}_{X_A/Q}^p) \\
\downarrow & & \downarrow \\
\ldots & \to & \ldots \\
\bigoplus_{x \in X^{(d)}} H^d_x(\overline{HC}_{p-1}^{(p-1)}(O_{X_A})) & \xrightarrow{B} & \bigoplus_{x \in X^{(d)}} H^d_x(\overline{\Omega}_{X_A/Q}^p) \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array}
\]

where the two columns are Cousin complexes of \( \overline{HC}_{p-1}^{(p-1)}(O_{X_A}) \) and \( \overline{HH}_p^{(p)}(O_{X_A}) \) respectively and we use (3.8) to identify \( \overline{HH}_p^{(p)}(O_{X_A}) \) with \( \overline{\Omega}_{X_A/Q}^p \).

Lemma 3.12. With notation as above, for \( [\beta^A] \in H^p_y(\overline{\Omega}_{X_A/Q}^p) \), where \( \beta^A \) is (3.10), one has

\[
\bar{\partial}_{1,X_A}^{p,-p}([\beta^A]) = 0,
\]
where $\partial_{1,X_A}^{p,-p}$ is the differential of the right column of diagram (3.11). In other words, for $Y' \in \text{Hilb}(A)$, the image $B \circ \text{Ch} \circ \alpha(Y')$ in (3.9) lies in the kernel of $\partial_{1,X_A}^{p,-p}$

$$\partial_{1,X_A}^{p,-p} \circ B \circ \text{Ch} \circ \alpha(Y') = 0.$$ 

**Proof.** In notation of Setting [3.4], by shrinking $U_i$, we assume that $O_X(U_i)$ is local. The regular sequence $\{f_1, \ldots, f_p\}$ can be extended to a system of parameter $\{f_1, \ldots, f_p, f_{p+1}, \ldots, f_d\}$ of the regular local ring $O_X(U_i)$. The prime ideals $Q_j := (f_1, \ldots, f_p, f_j)$, where $j = p+1, \ldots, d$, define generic points $z_j \in X^{(p+1)}$. In the following, to check $\partial_{1,X_A}^{p,-p} \circ B \circ \text{Ch} \circ \alpha(Y') = 0$, we consider the prime $Q_{p+1} = (f_1, \ldots, f_p, f_{p+1})$ which defines the generic point $z_{p+1}$, other cases work similarly.

Let $Q = (f_1, \ldots, f_p)$ be the prime ideal which defines the generic point $(Y_y) y \in X^{(p)}$, then $O_{X,Y} = (O_{X,z_{p+1}})_Q$. Then $\beta^A$ (cf. (3.10)) can be rewritten as

$$\begin{cases}
F_\bullet(f_1, f_2, \ldots, f_p) & \longrightarrow & (O_{X,z_{p+1}})_Q/(f_1, f_2, \ldots, f_p) \\
F_p(\cong (O_{X,z_{p+1}})Q) & \longrightarrow & F_0 \otimes \Omega_{O_{X,z_{p+1}}/Q}(\cong \Omega_{O_{X,z_{p+1}}/Q}/Q),
\end{cases}$$

where $\Omega_{O_{X,z_{p+1}}/Q}$ is the kernel of $\Omega_{O_{X,z_{p+1}}/Q}/Q \rightarrow \Omega_{O_{X,z_{p+1}}/Q}/Q$, and $F_\bullet(f_1, f_2, \ldots, f_p)$ is of the form

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0,$$

where each $F_i = \bigwedge^i((O_{X,z_{p+1}})Q)^{\oplus p}$. Since $f_{p+1} \notin Q = (f_1, \ldots, f_p)$, $f_{p+1}^{-1}$ exists in $(O_{X,z_{p+1}})_Q$, we can write $\beta^A = \frac{f_{p+1}}{f_{p+1}}[\beta^A]_{\text{loc}}$.

The image $\partial_{1,X_A}^{p,-p}(\beta^A)$ is represented by the following diagram (denoted $\gamma$)

$$\begin{cases}
F_\bullet(f_1, f_2, \ldots, f_p, f_{p+1}) & \longrightarrow & O_{X,z_{p+1}}/(f_1, f_2, \ldots, f_p, f_{p+1}) \\
F_{p+1}(\cong O_{X,z_{p+1}}) & \longrightarrow & F_0 \otimes \Omega_{O_{X,z_{p+1}}/Q}(\cong \Omega_{O_{X,z_{p+1}}/Q}/Q),
\end{cases}$$

where $\Omega_{O_{X,z_{p+1}}/Q}$ is the kernel of $\Omega_{O_{X,z_{p+1}}/Q} \rightarrow \Omega_{O_{X,z_{p+1}}/Q}$, and the complex $F_\bullet(f_1, f_2, \ldots, f_p, f_{p+1})$ is of the form

$$0 \longrightarrow \bigwedge^{p+1}(O_{X,z_{p+1}})^{\oplus p+1} \xrightarrow{M_{p+1}} \bigwedge^p(O_{X,z_{p+1}})^{\oplus p+1} \longrightarrow \ldots.$$ 

Let $\{e_1, \ldots, e_{p+1}\}$ be a basis of $(O_{X,z_{p+1}})^{\oplus p+1}$, the map $M_{p+1}$ is

$$e_1 \wedge \cdots \wedge e_{p+1} \rightarrow \sum_{j=1}^{p+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots e_{p+1},$$
where $\hat{e}_j$ means to omit $e_j$. Since $f_{p+1}$ appears in $M_{p+1}$, one has that
\[
\gamma = 0 \in Ext^{p+1}_{O_{X,z_{p+1}}}(f_1, f_2, \ldots, f_p, f_{p+1}, \mathfrak{M}_{O_{X,z_{p+1}}}/\mathbb{Q}).
\]
Hence, $\partial^{p-1}_{1,X_A}(\beta^A) = 0$.

The commutativity of diagram (3.11) yields that $B \circ \partial^{p-1}_{1,X_A} \circ \text{Ch} \circ \alpha(Y') = \partial^{p-1}_{1,X_A} \circ B \circ \text{Ch} \circ \alpha(Y') = 0$. Each B map in diagram (3.11) is injective (see the exact sequence (3.6)), so $\partial^{p-1}_{1,X_A} \circ \text{Ch} \circ \alpha(Y') = 0$. This answers Question 3.5 in Setting 3.6. Equivalently, it answers Question 3.4 in Setting 3.6.

**Theorem 3.13.** In Setting 3.6, for any $A \in GArt_k$ and for $Y' \in \text{Hilb}(A)$, $\alpha_A(Y')$ is a Milnor K-theoretic cycle.

The Milnor K-theoretic cycle $\alpha_A(Y')$ defines an element of Milnor K-theoretic Chow group (defined in Definition 2.22), which further gives an element of the cohomological Chow group $CH^p(X, K^p_{O_{X_A}})_\mathbb{Q}$ by Theorem 2.24 denoted $[\alpha_A(Y')]$. There is a set-theoretic map

\[
(3.12) \quad \text{Hilb}(A) \to \tilde{CH}^p(A),
Y' \to [\alpha_A(Y')],
\]

where $\tilde{CH}^p(A) = CH^p(X, K^p_{O_{X_A}})_\mathbb{Q}$, see (1.2) on page 1.

Let $f : C \to A$ be a morphism in the category $GArt_k$, there exists a commutative diagram of sets

\[
\begin{array}{ccc}
\text{Hilb}(C) & \xrightarrow{\alpha_C(3.1)} & K^p_0(O_{X_{C,y}} \text{ on } y)_\mathbb{Q} \\
\downarrow f_H & & \downarrow f_K \\
\text{Hilb}(A) & \xrightarrow{\alpha_A(3.1)} & K^p_0(O_{X_{A,y}} \text{ on } y)_\mathbb{Q},
\end{array}
\]

where $f_H$ and $f_K$ are induced by $f$ respectively. Since $Y$ is a locally complete intersection, this square can be straightforwardly checked. This induces a commutative diagram of sets

\[
\begin{array}{ccc}
\text{Hilb}(C) & \xrightarrow{(3.12)} & \tilde{CH}^p(C) \\
\downarrow & & \downarrow \\
\text{Hilb}(A) & \xrightarrow{(3.12)} & \tilde{CH}^p(A).
\end{array}
\]

We deduce that
Theorem 3.14. In Setting 3.6 there exists a natural transformation between functors on $GArt_k$

$$\mathbb{T} : \text{Hilb} \to \widetilde{CH}^p,$$

which is defined to be, for any $A \in GArt_k$, $\mathbb{T}(A)$ is (3.12).

This answers Bloch’s Question 1.4 in Setting 3.6.

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