A PROOF OF THE IRREVERSIBILITY
OF RENORMALIZATION GROUP FLOWS IN FOUR DIMENSIONS

Stefano Forte\(^{(a,c)}\) and José I. Latorre\(^{(a,b)}\)

\(^{(a)}\)Departament d’Estructura i Constituents de la Matèria,
Universitat de Barcelona
and
\(^{(b)}\)I.F.A.E.,
Diagonal 647, E-08028 Barcelona, Spain

\(^{(c)}\)INFN, Sezione di Torino,
Via P. Giuria 1, I-10125 Torino, Italy

Abstract

We present a proof of the irreversibility of renormalization group flows, i.e.
the \(c\)-theorem for unitary, renormalizable theories in four (or generally even)
dimensions. Using Ward identities for scale transformations and spectral rep-
resentation arguments, we show that the \(c\)-function based on the trace of the
energy-momentum tensor (originally suggested by Cardy) decreases monotonically
along renormalization group trajectories. At fixed points this \(c\)-function
is stationary and coincides with the coefficient of the Euler density in the trace
anomaly, while away from fixed points its decrease is due to the decoupling of
positive–norm massive modes.

Submitted to: *Nuclear Physics B*

April 1998
1. Zamolodchikov’s two dimensional c–theorem

Fundamental theorems that impose constraints on the long-distance realization of a quantum field theory are rare. An instance of such a powerful, non-perturbative tool is provided by the t’Hooft anomaly matching conditions [1]. If an ultraviolet (UV) theory has an axial anomaly, as a consequence of the Adler-Bardeen non-renomalization theorem [2] the anomaly coefficient must be matched by that of its infrared (IR) effective realization. This exact result is useful to check the mutual consistency of short vs. long distance realizations of a given theory. Yet, the same non-renormalization theorem which provides the basis of anomaly matching trivializes the relation between different scales and thus the way the IR limit is attained. This motivates the search for other theorems which may add further and more general constraints to the IR realization of quantum field theories.

The mechanism controlling the modification of a physical theory through a change of scale is the renormalization group (RG). Following Wilson [3], we may set up an exact RG equation which describes the change of the effective hamiltonian of a theory as we flow towards the IR. The derivation of this equation proceeds in two steps: an integration of modes (Kadanoff block-spin transformation) followed by a rescaling. Since integrating out modes seems an irreversible operation, one is naturally lead to ask whether the RG flow itself is irreversible. This is equivalent to asking whether there is a fundamental obstruction to recover microscopic physics from macroscopic physics, or, more generally, whether there is a net information loss along RG trajectories. Indeed, realistic quantum field theories seem intuitively irreversible; on the other hand, however, it is known that some theories may display limit cycles, and thus their RG flow is manifestly not irreversible. The question is, thus, to see under which conditions irreversibility may follow.

Efforts to answer this question started with the work of ref. [4], where the perturbative expansion of a theory of $N$ scalars with interaction $g_{abcd} \phi^a \phi^b \phi^c \phi^d$ ($a,b,c,d = 1 \ldots N$) is analyzed up to three loops. The beta functions of the theory turn out to be gradients of a scalar function of the couplings, thus yielding irreversible flows at this order of perturbation theory.

A fundamental theorem was later proven by Zamolodchikov [5] in the context of two dimensional field theories. This result is based on the construction of a function of the couplings $c(g^i)$ from two-point correlators of energy-momentum tensors, which is then proven to decrease monotonically along RG trajectories for unitary theories. More precisely, defining at the operator level a theory as

$$S = g^i \int_x O_i(x), \quad (1.1)$$

one can construct a function $c(g^i)$ whose RG flow is

$$- \beta^i \frac{\partial}{\partial g^i} c(g^j) = - \beta^i \beta^j G_{ij} \leq 0, \quad (1.2)$$

where $G_{ij}$ (known as the Zamolodchikov metric) is given by

$$G_{ij} \equiv \langle O_i(x)O_j(0) \rangle|_{x=\mu^{-1}} \geq 0 \quad \mu = \text{reference scale}, \quad (1.3)$$
and is positive definite due to reflection positivity. Because at fixed points the function \( c(g^i) \) reduces to the central charge of the conformal field theory which defines it, this remarkable result is known as the \( c \)-theorem. It can furthermore be proven that the beta functions are gradients of \( c(g^i) \) to the first few orders of conformal perturbation theory, namely, in a perturbative expansion around any (gaussian or non-gaussian) fixed point.

The result of ref. [5] shows that the sought–for necessary condition for irreversibility is unitarity. The proof of the theorem makes also use of renormalizability and Poincaré invariance via the absence of anomalous dimensions for the energy momentum tensor and its conservation. The construction of a natural positive definite metric \( G_{ij} \) in the space of couplings introduces the concept of distance between theories, each defined at a given scale. The difference of the \( c \)-charges between two fixed points is reparametrization invariant. RG equations are simply equations of motion in the space of theories.

The essence of the \( c \)-theorem in two dimensions can thus be summarized as follows. For every unitary, renormalizable, Poincaré invariant quantum field theory there exists a universal \( c \)-function which decreases along RG flows, while it is only stationary at (conformal) fixed points, where it reduces to the central charge. This sets an arrow on RG flows, and implies their irreversibility. It follows that a theory can be the IR realization of a given UV theory only if their central charges satisfy the inequality \( c_{IR} < c_{UV} \). It is important to notice that the identification of the \( c \)-function with the central charge at conformal points makes its determination possible and thus the theorem useful for realistic applications.

One may thus conjecture that irreversibility may also hold in four and higher dimensions. Evidence supporting this conjecture is provided by perturbative computations [3], by the non-trivial leading order of the derivative expansion of the exact RG equation [4], and by a large number of explicit non-trivial RG flows [8]. However, the generalization of the \( c \)-theorem itself to higher dimensional field theories has proven to be problematic [10-12]. It is the purpose of this paper to present a proof of irreversibility of RG group flows in four and higher dimensions. This proof will include and synthesize ideas which originate from various lines of research. First and more importantly, we will make use of a four-dimensional generalization of the \( c \)-function proposed by Cardy [10]. Furthermore, our construction will make use of the physical insight provided by the alternative proof of the two-dimensional \( c \)-theorem of ref. [12], based on the spectral analysis of two-point energy-momentum tensor correlators. Combining this with the use of scale Ward identities will allow us to prove that Cardy’s \( c \)-function decreases along RG flows, thus establishing the \( c \)-theorem in four dimensions, and generally even dimensions.

2. Spectral proof of Zamolodchikov’s theorem

The meaning of the two–dimensional \( c \)-theorem can be understood in a somewhat more general way by means of techniques first introduced in ref. [12], which we now briefly
review. The correlator of two energy-momentum tensors can be written using a spectral representation as

\[
\langle T_{\alpha\beta}(x)T_{\mu\nu}(0) \rangle = \frac{\pi}{3} \int d\lambda \, \rho(\lambda, \mu) \left( \partial_\alpha \partial_\beta - g_{\alpha\beta} \Box \right) \left( \partial_\mu \partial_\nu - g_{\mu\nu} \Box \right) G(x, \lambda) ,
\]

(2.1)

where \( \lambda \) is the spectral parameter (with dimensions of mass), \( \rho(\lambda, \mu) \) is the spectral function, which depends on \( \lambda \) and on the subtraction point \( \mu \), and \( G(x, \lambda) \) is the free scalar propagator of a particle with mass \( \lambda \).

At a fixed point, the spectral function reduces to a delta function

\[
\rho(\lambda, \mu) \bigg|_{\text{cft}} = c \, \delta(\lambda) ,
\]

(2.2)

where the coefficient \( c \), being dimensionless, cannot depend on \( \mu \) on dimensional grounds, and is thus necessarily a constant. This reflects the fact that all physical intermediate states in (2.1) are massless. At an UV fixed point we thus get that

\[
c_{UV} = \int d\lambda \, \rho(\lambda, \mu) .
\]

(2.3)

By considering \( \rho(\lambda, \mu) \) in the vicinity of an UV fixed point it can also be shown explicitly that \( c \) in eq. (2.3) coincides with the central charge of the conformal field theory. Now, the integral of the spectral density cannot depend on \( \mu \), again on dimensional grounds, so that eq. (2.3) actually holds for all finite \( \mu \) (i.e., away from the IR limit \( \mu \to \infty \)).

On the other hand, in the IR limit only massless modes survive, so that the spectral function can in general be written as

\[
\rho(\lambda, \mu) = c_{IR} \, \delta(\lambda) + \rho_{\text{smooth}}(\lambda, \mu) ,
\]

(2.4)

where the contribution of all massive modes is contained in \( \rho_{\text{smooth}}(\lambda, \mu) \). It thus follows that

\[
c_{UV} = c_{IR} + \int d\lambda \, \rho_{\text{smooth}}(\lambda, \mu) ,
\]

(2.5)

where the second term on the r.h.s. is necessarily \( \mu \)-independent. Notice that this means that the IR limit is not uniform, i.e., it does not commute with the integration over \( \lambda \), so that the IR limit of \( c \) does not coincide with the constant value \( c_{UV} \). Finally, unitarity guarantees that \( \rho_{\text{smooth}} \) is positive, so

\[
c_{UV} - c_{IR} \geq 0 .
\]

(2.6)

Several examples illustrating the above construction can be found in ref. [13].

This alternative proof of the \( c \)-theorem provides us with a number of physical insights. First, we see that unitarity is necessary for irreversibility because it guarantees positivity of the spectral representation. Furthermore, we understand that the decrease of the \( c \)-function is due to the decoupling of massive modes in the spectrum which appear as
intermediate states in the spectral decomposition. Finally, we see that the central charge provides an effective counting of massless degrees of freedom, and must thus be an additive function: RG flows are then such that the effective counting of massless modes in the IR is lower than in the UV. The physical picture is thus the following. At the UV fixed point, all modes are massless and appear as a delta term in $\rho$. When the scale parameter $\mu$ starts to move, only some modes remain massless and form $c_{IR}$, the rest acquire a mass and start to decouple. In the IR theory only the $c_{IR}$ contribution remains.

The original proof of ref. [5] is based on the construction of a particular combination of correlators, which is then shown to decrease along RG flows. It is now clear, however, that the $c$–theorem can be formulated at the level of the density of states. One may then construct many combinations of correlators to prove the theorem, but all of them are related to a unique spectral density.

A technical but important point concerning this proof is related to subtractions in the two-point correlator (2.1). The variation $\Delta c = c_{UV} - c_{IR}$ is given by the integral of $\rho_{\text{smooth}}$, which is free of subtraction ambiguities, as it picks up the non-local contribution of intermediate massive modes. In other words, $\Delta c$ can be computed from the imaginary part of $\langle TT \rangle$, which is free of contact terms and, thus, scheme independent. However, the correlator itself is only defined up to contact terms, which can be freely changed by choice of renormalization scheme. This freedom is restricted by the fact that the energy-momentum tensor obeys an algebra which fixes the relative subtractions in one– and two–point functions. This point will be of great relevance in the proof of the $c$–theorem in four dimensions.

A first important obstacle in trying to generalize the spectral arguments to higher dimensions is due to the fact that the proof of ref. [5] makes essential use of a peculiarity of two dimensions, namely the fact that there is no spin. It follows that correlators of two energy-momentum tensors are saturated by spinless intermediate states, i.e., the response to scale and shear transformations are controlled by one single spin structure. In four (and higher) dimensions, we find two separate structures in the spectral decomposition: one corresponds to spin–zero and the other to spin–two intermediate states. A number of branching avenues thus opens as one tries to define a candidate for a $c$–function [11,12,14]. One may in particular conjecture [6] that the natural generalization of the theorem should lie in the spin–zero part of the spectral decomposition of the $\langle TT \rangle$ correlator, on the grounds that this is related to the trace of the energy-momentum tensor and thus to the response of the system upon dilatations. However, dimensional analysis shows that in $d$ dimensions

$$\rho^{\text{spin}=0}(\lambda, t)_{\text{eft}} = c \lambda^{d-2} \delta(\lambda),$$

that is, the would-be analogue of $c$ in eq.(2.2) is now inaccessible to physical observables as it multiplies a vanishing distribution for $d > 2$.

3. Cardy’s proposal for a $c$–charge

A different avenue towards the derivation of a higher–dimensional generalization of the $c$–theorem is based on reconsidering the role of the central charge in conformal field
theories which underlies the two–dimensional proof. Recall that the central charge of a
conformal field theory appears as the coefficient of the identity operator in the operator
product expansion (OPE) of two energy-momentum tensors

$$T(z)T(0) \sim \frac{c}{2z^4} I + \ldots .$$

and, consequently, in the numerator of the $\langle T(z)T(0) \rangle$ correlator. Now, $c$ can also be
understood \[16\] as the coefficient of the trace anomaly of a conformal field theory in
curved space

$$\langle T^\alpha_\alpha \rangle_{cft} \equiv \langle \theta \rangle_{cft} = -\frac{c}{12} R .$$

Moreover, the integrated trace anomaly turns out to be proportional to the Euler number,
$\chi(M)$, of the background manifold $M$

$$\int_M \sqrt{g} \langle \theta \rangle_{cft} = -\frac{c}{12} \int_M \sqrt{g} R = -c \frac{\pi}{3} \chi(M) .$$

The central charge can thus also be viewed as the coefficient of a contribution to the trace
anomaly which admits a topological interpretation.

This different understanding of the central charge suggests a new generalization to
higher dimensions \[10\]: At fixed points, the $c$–function should reduce to a coefficient hidden
in the trace anomaly. In four dimensions, we explicitly have \[17,18\]

$$\langle \theta(x) \rangle_{cft} = \frac{1}{2880} (-3c_F F(x) + c_G G(x) + c_R \Box R(x))$$

where

$$F = C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3} R^2$$
$$G = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 .$$

The coefficients $c_F$ and $c_G$ seem a priori equally good candidates for the $c$–functions, while
$c_R$ isn’t, since it multiplies a divergence operator and it is thus scheme dependent. The
coefficient $c_F$, however, which can be shown \[12\] to coincide with the positive coefficient
of the spin–two structure in the correlators of two energy-momentum tensors, is found not
to decrease along RG flows in specific examples, and is ruled out.

The only remaining possibility is thus $c_G$. This can be singled out by taking an integral
over e.g. a sphere,

$$\langle \int_{S^n} \sqrt{g} \theta \rangle_{cft} = \frac{\pi^2}{90} c_G \chi(S^n) ,$$

where $\chi(M) = \int_{S^n} \sqrt{g} G$ is the Euler characteristic of the manifold. The $c_R$ contribution
then vanishes because it is a total derivative, while the $c_F$ contribution vanishes because
it is proportional to the square of the Weyl tensor and the sphere is conformally flat. It is thus natural to propose to consider the integrated trace anomaly on a sphere

\[ c = \int_{S^n} \sqrt{g} \langle \theta \rangle \]  

(3.7)
as a candidate for the c–function in higher (even) dimensions. Notice, however, that away from conformal points a variety of different choices for the c–function is possible, all of which reduce to the same c–charge eq. (3.6) at conformal points: namely, any combination of the charge of eq. (3.6) and terms proportional to the beta–functions of the theory. In particular, the choice eq. (3.7) itself, away from conformal points, does not reduce to the charge of eq. (3.6), since the trace of the energy momentum tensor then receives extra contributions (proportional to the beta functions) on top of it, due to the breaking (canonical and anomalous) of dilatation symmetry.

The candidate c–function eq. (3.7) can be determined in the first few orders of conformal perturbation theory and shown explicitly to decrease along RG flows [10]. However, in order to establish a general theorem one must determine the variation of eq. (3.7) along generic RG trajectories. Naively, this is related to the correlator of two traces of the energy momentum tensor, but this contains subtractions which render the proof of positivity non-trivial.

The c–function eq. (3.7) has a number of desirable properties. It is based on the energy-momentum tensor, which exists for any theory and couples to all degrees of freedom. It is related to the integrated trace of this operator, which generates scale transformations, and thus implement RG flows. Finally, it is additive, as required for a function that should count effective degrees of freedom.

On the other hand, the need to resort to a curved background may seem unnatural. However, the coefficient \( c_G \) is also unambiguously determined as a coefficient in three point functions of the energy momentum tensor. In general, in \( d = 2n \) dimensions, anomalies show up in \( n+1 \)–point functions, as well as in the one–point trace anomaly in curved space, and the choice of the latter form is a matter of technical convenience. The background metric plays here a similar role to that of an external current when discussing anomalies in fermionic loops.

The conjecture that \( c \) defined in eq. (3.7) may be a candidate for the c–function has been tested in various ways. First, this c–function can be verified to be positive in all known conformal field theories. This fact can be related to a modified weak positivity theorem in quantum field theory [19]. Furthermore, it can be explicitly checked that the coefficient of the trace anomaly obeys the desired inequality eq. (2.6) in an impressively large number of non-trivial cases where the UV and IR realizations are known exactly [8,9]. Finally, it can be shown that no combination of \( c_F \) and \( c_G \) can satisfy the inequality except \( c_G \) itself [9].

A different proposal for a c–function away from conformal field theory, consisting of a different combination of the charge eq. (3.6) and a term proportional to \( \beta^i \), has also been considered [11] (see also ref. [21]). Of course, this candidate c–function also passes all
4. Proof of irreversibility

We will now present a proof of irreversibility of RG trajectories for the trace of the energy-momentum tensor in four dimensions. Our proof will actually remain valid for any even number of dimensions. We will consider the integrated trace of the energy-momentum tensor on curved space. For simplicity, we take a $d$-dimensional maximally symmetric space, where the energy-momentum tensor satisfies

$$\langle T^{\mu\nu}(x) \rangle = \langle \theta \rangle \frac{g^{\mu\nu}(x)}{d},$$

(4.1)

where $\langle \theta \rangle$ is an $x$–independent constant, so that integration of the trace over space amounts to a volume factor which can be absorbed in the normalization. We further take the curvature to be the negative constant $R = -a^2 d(d - 1)$, where $a$ is a constant parameter with the dimensions of mass. None of these choices entails loss of generality, even though in the generic non-compact case, where the volume is infinite, the space integration has to be defined more carefully.

We then define a dimensionless $c$-function depending on the subtraction point $\mu$ and the renormalized couplings $g^i$ as

$$c \left( \frac{\mu}{a}, g^i \right) \equiv a^{-d} V \frac{\langle \theta \rangle}{A_d},$$

(4.2)

where $V$ is the volume of the $(d - 1)$–dimensional sphere and $A_d$ is a normalization factor which we fix as

$$V = \frac{2\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right)} , \quad A_d = B_d (d(d - 1))^\frac{d}{2} V,$$

(4.3)

$B_d$ being the Bernoulli numbers. With this choice the anomaly for a conformal theory of a massless scalar is

$$\langle \theta \rangle_{cft} = B_d (-R)^{\frac{d}{2}} = \frac{A_d a^d}{V},$$

(4.4)

thus,

$$c = 1 \quad \text{for a massless scalar} .$$

(4.5)

Furthermore, $c = 11$ for free massless fermions and $c = 62$ for vectors [17,18].

The operator $\theta$ can be further decomposed as

$$\theta = \theta_{an} + \theta_{dyn} ,$$

(4.6)
where \( \theta_{\text{dyn}} \), which we call the dynamical trace anomaly (also called internal anomaly), carries all the dependence on the beta-functions of the theory

\[
\theta_{\text{dyn}} = \beta^i \mathcal{O}_i .
\]  

(4.7)

Note that masses are treated as couplings and therefore contributions to the trace of the energy-momentum tensor which are present at the classical level are included in \( \theta_{\text{dyn}} \). At fixed points, \( \theta_{\text{dyn}} \) vanishes and \( \theta \) reduces to the background anomaly, which is proportional to the identity operator:

\[
\langle \theta \rangle_{\text{cft}} = \langle \theta_{\text{an}} \rangle_{\text{cft}} = cB_d(-R)^{\frac{d}{2}} .
\]  

(4.8)

Because of the \( x \)-independence of \( \langle \theta \rangle \), the function \( c(\mu/a, g^i) \) only depends on the subtraction point \( \mu \) explicitly through the dimensionless ratio \( \mu/a \) and, implicitly, through the renormalized couplings \( g^i \). Furthermore, since the energy–momentum tensor has vanishing anomalous dimension (being a conserved current), the \( c \)-function obeys the homogeneous RG equation

\[
\mu \frac{d}{d\mu} c \left( \frac{\mu}{a}, g^i \right) = \left( \mu \frac{\partial}{\partial \mu} + \beta^i(g) \frac{\partial}{\partial g^i} \right) c \left( \frac{\mu}{a}, g^i \right) = 0 .
\]  

(4.9)

Note that \( g^i \) stands for all the couplings in the theory, including those to the background, as well as masses and any other dimensionful parameters. The couplings are all rendered dimensionless by dividing out the appropriate power of \( \mu \) (as it is customary in the Wilsonian renormalization group approach). The explicit \( \mu \) dependence can be traded for the dependence on the only dimensionful physical parameter \( a \), leading to

\[
\frac{d}{da} c \left( \frac{\mu}{a}, g^i \right) = \beta^j(g) \frac{\partial}{\partial g^j} c \left( \frac{\mu}{a}, g^i \right) .
\]  

(4.10)

Now, in a symmetric space, a rescaling of \( a \) is equivalent to a general scale transformation:

\[
\delta_s \equiv a \frac{d}{da} = -2 \int d^d x \ g^{\alpha \beta}(x) \frac{\delta}{\delta g^{\alpha \beta}(x)} .
\]  

(4.11)

However, the response of a generic Green function for a composite operator \( \mathcal{O}(y) \) to scale transformations is fixed by the scale Ward identity \([21-23]\):

\[
-2g^{\alpha \beta}(x) \frac{\delta}{\delta g^{\alpha \beta}(x)} \langle \mathcal{O}(y) \rangle = -\nabla_\mu \langle j_D^{\mu}(x) \mathcal{O}(y) \rangle + \delta^{(d)}(x-y) \langle \delta_s \mathcal{O}(y) \rangle + \langle \theta(x) \mathcal{O}(y) \rangle ,
\]  

(4.12)

where \( j_D^{\mu}(x) \) is the dilatation current, whose charge generates scale transformations. The second term on the r.h.s. of eq. (4.12) gives the canonical scale transformation of the operator \( \delta_s \mathcal{O}(y) = \gamma_\mathcal{O} \mathcal{O}(y) \), where \( \gamma_\mathcal{O} \) is the dimension (engineering plus anomalous) of

\[1\] The same result can be of course obtained by first solving eq. (4.9) in terms of running couplings \( \bar{g}^i \left( \frac{\Lambda}{a} \right) \), where \( \Lambda \) is a RG invariant dynamically generated scale, and then differentiating with respect to \( a \).
the operator $O(y)$, while the third term is the anomalous divergence of the dilatation current, which satisfies the anomaly equation

$$\nabla_\mu j^\mu_D(x) = \theta(x)$$ (4.13)

at the operator level. All the operators in eqs. (4.12)-(4.13) are renormalized composite operators [22]; the expectation values correspond to $T^*$-ordered proper Green functions, i.e. only receive contributions from connected diagrams.

The scale dependence of $c(\frac{\mu}{a}, g^i)$ can be obtained by integrating the anomalous scale Ward identity eq. (4.12) over all space, and specializing to the case in which $O(x) = \theta(x)$. Upon integration, the contribution proportional to the divergence of the current provides a surface term and thus vanishes exponentially unless there is a massless particle (dilaton) in the spectrum which couples to $\nabla_\mu j^\mu_D$ and $\theta$. The anomalous dimension of $\theta$ vanishes and its engineering dimension coincides with the space dimension, so $\gamma_\theta = d$. Furthermore, the anomalous contribution $\theta_{an}$ to the energy-momentum tensor eq. (4.6), being proportional to the identity operator, can only contribute to the one–point Green function, but not to $n$–point proper Green functions of $\theta$. It follows that the trace Ward identity for $\theta$ reads

$$a \frac{d}{da} \langle \theta \rangle = \frac{1}{V} \int d^d x \sqrt{g(x)} \langle \theta_{dyn}(x) \theta_{dyn}(0) \rangle_s + d \langle \theta \rangle,$$ (4.14)

where the subscript on the two–point function indicates that the surface (contact) contribution due to the current divergence term when dilatons are present has been included as a subtraction in the definition of the correlator. Note however that, quite in general, no dilaton contribution is expected, in which case the correlators $\langle \theta \theta \rangle_s$ simply coincides with the $T^*$ ordered correlator which appears in the Ward identity eq. (4.12).

The scale transformation of $c(\frac{\mu}{a}, g^i)$ is thus fully determined as

$$a \frac{d}{da} c = \frac{a^{-d}}{A_d} \int d^d x \sqrt{g(x)} \langle \theta_{dyn}(x) \theta_{dyn}(0) \rangle_s .$$ (4.15)

The contact term due to the canonical transformation of $\theta$ upon dilatations is exactly canceled by an equal and opposite contact term due to the dimensionful factor $a^{-d}$ which relates $\theta$ to the dimensionless $c$-function. Furthermore,

$$\langle \theta(x) \theta(0) \rangle_{non\ local} \equiv \langle \theta(x)_{dyn} \theta(0)_{dyn} \rangle_s = \beta^i \beta^j \langle O_i(x) O_j(0) \rangle_s \geq 0,$$ (4.16)

---

2 It may be instructive to compare the scale Ward identity eq. (4.12) with the well–known chiral Ward identities which describe the way chiral symmetry is realized, for instance in QCD or in the sigma model. In the chiral case, the dilatation current $j^\mu_D$ is replaced by the axial current $j^\mu_5$. Since chiral symmetry is an internal symmetry, the l.h.s. of eq. (4.12), which corresponds to the transformation of the space-time coordinates, is then simply zero. In the absence of an axial anomaly (for instance in the non-singlet sector) the last term on the r.h.s. of eq. (4.12) also vanishes. The two remaining terms are then either separately zero, or else must cancel. The latter case, in which there exists a chirally non-invariant operator $O$ with a non-vanishing vacuum expectation value, and a massless mode that couples to the divergence of the axial current, corresponds to the Goldstone realization of the chiral symmetry.

9
where the last inequality can be derived from positivity of the spectral representation along the lines of the argument presented in sect. 2, and will be discussed in the next section. We conclude that

$$-\beta^i \partial_i c = -\frac{a^{-d}}{A_d} \int d^d x \sqrt{g(x)} \langle \theta(x)_{\text{dyn}} \theta(0)_{\text{dyn}} \rangle_s \leq 0 ,$$

(4.17)

thus establishing the desired result, i.e. the monotonic decrease of $c(\frac{L}{a}, g^i)$ along RG flows.

Note that at fixed points $c$ coincides, up to a normalization, with the coefficient of the Euler density in the trace anomaly and its derivative is consistently zero. Away from fixed points a positive definite, invertible, scheme-dependent metric in the space of couplings can be defined as

$$G_{ij} = \frac{1}{V} \int d^d x \sqrt{g(x)} \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle_s .$$

(4.18)

It is then easy to show, along the lines of the two-dimensional argument of ref. [5], that to the first few orders of conformal perturbation theory

$$\beta^i = G^{ij} \partial_j c ,$$

i.e. the beta functions are gradients of the $c$–function.

The physical interpretation of eq. (4.17) is simple. The RG flow of $c(\frac{L}{a}, g^i)$ is related by the Ward identity to scale non-invariance and hence to the presence of massive states in the Hilbert space. The monotonic decrease of the $c$–function when moving towards the IR is controlled by the decoupling of these massive modes from the correlator eq. (4.16). At fixed points, no such modes are present and the $c$–charge is accordingly stationary. The $c$–function thus provides an effective counting of massless degrees of freedom in the theory. By deriving the positivity of the correlator eq. (4.16) from that of the spectral representation (see sect. 5 below) the monotonic character of the decoupling is directly derived from the unitarity of the spectrum.

We shall now discuss in somewhat greater detail some technical issues which we have set aside in our proof of the theorem, specifically in relation to the choice of the background metric and to the precise definition of subtractions in the definition of Green functions.

**Background metric**

As explained in sect. 3, the background metric is used as a device to deal with the coefficient of the Euler density in the trace anomaly, which is also present in correlators of three point functions in flat space [24]. For the sake of definiteness, we chose to present our argument in the hyperbolic space $H_d$. This space is maximally symmetric and provides the natural Euclidean continuation of anti-de-Sitter space. Unlike what happens on the sphere, massive propagators decay at infinity, as we shall see in the next section. Nevertheless, the proof of the Ward identity is easily generalized to a form which makes no explicit reference to the choice of background metric.
Specifically, the $c$–function can be generally defined as

$$c_M = \int_M d^d x \sqrt{g(x)}\langle\theta(x)\rangle,$$  \hspace{1cm} (4.20)

where $\theta$ is defined as the trace of the energy momentum tensor, $\theta(x) \equiv T^\mu_\mu(x)$. In general, scale transformations are implemented by (4.11) and it then follows that

$$\delta_s c_M = \frac{-2}{V} \int d^d x \frac{\delta}{\delta g^{\alpha\beta}(x)} \int d^d y \sqrt{g(y)}\langle\theta(y)\rangle,$$  \hspace{1cm} (4.21)

The contact term in the Ward identity eq. (4.14) is then removed by the contact term due to the variation of $\sqrt{g}$ on the r.h.s. of eq. (4.21), thus implying

$$\delta_s c_M = \frac{1}{V} \int d^d x \sqrt{g(x)} \int d^d y \sqrt{g(y)} \langle \theta_{dyn}(x)\theta_{dyn}(0) \rangle_s,$$  \hspace{1cm} (4.22)

which is the general form of the Ward identity satisfied by the $c$–function.

Even though the Ward identity which expresses the scale variation of the $c$–function is easily generalized to generic space, the proof of positivity of the correlator on the r.h.s. of eq. (4.17) in sect. 5 will make crucial use of the assumption that space is maximally symmetric: on generic spaces, positivity of the correlator (4.22) is not guaranteed. In fact, the $c$–charge eq. (3.7) on a generic space will also pick up contributions proportional to the $c_F$ and $c_R$ terms in eq. (3.3), but it has been shown by explicit computation [9] that no combination of the $c_F$ and $c_G$ coefficients decreases in general along RG flows. Here, we are only interested in the validity of the $c$–theorem in the flat space limit. We will therefore not address the issue of classifying the properties of the background space which are necessary and sufficient for the $c$–theorem to hold, and we will limit ourselves to establishing our result on $H_d$.

Subtractions and scale Ward identities

All the arguments in the proof of irreversibility are done at the level of renormalized correlators. This means that first, we must define precisely the subtraction scheme used to define the one–point function $\langle\theta\rangle$, and furthermore, we must fix the contact terms which in general appear in the definition of the two–point function $\langle\theta(x)\theta(y)\rangle$.

The guiding principle in this issue is provided by the Ward identities. Specifically, the scale Ward identity eq. (4.12) is derived upon the assumption that the energy-momentum tensor is equal to the variation of the renormalized action upon diffeomorphism, which in turns means that

$$\langle \sqrt{g}T^{\alpha\beta}(x) \rangle = -2V \frac{\delta \ln Z}{\delta g^{\alpha\beta}(x)},$$  \hspace{1cm} (4.23)

(where $V$ is the volume of the $d−1$–dimensional sphere eq. (4.3)), so that the $\theta$ one–point function is defined as

$$\sqrt{g(x)}\langle\theta(x)\rangle = -2V g_{\alpha\beta}(x) \frac{\delta}{\delta g_{\alpha\beta}(x)} \ln Z.$$  \hspace{1cm} (4.24)
Furthermore, the Ward identity eq. (4.12) only holds for the specific choice of contact terms in the definition of correlators of composite operators such that equations of motion are satisfied at the operator level by the renormalized operators [22]. For the correlator of two traces of energy-momentum tensors this means that

$$\langle \theta(x) \theta(y) \rangle = 4V^2 \frac{\delta}{\delta g_{\mu\nu}(x)} g^{\alpha\beta}(y) \frac{\delta}{\delta g_{\alpha\beta}(y)} \ln Z + 2V \langle \frac{\delta}{\delta g_{\mu\nu}(x)} \theta(y) \rangle. \quad (4.25)$$

With this definition, the correlator reduces to a purely dynamical contribution and thus vanishes at conformal points.

Different definitions of the correlators are of course possible: for instance, in ref. [12] the correlator is constructed to satisfy covariant conservation

$$\nabla_\mu \langle T^{\mu\nu} T^{\alpha\beta} \rangle_{CC} = 0. \quad (4.26)$$

This fixes the contact terms as

$$\langle \sqrt{g} T^{\mu\nu}(x) \sqrt{g} T^{\alpha\beta}(y) \rangle_{CC} = 4V^2 \frac{\delta}{\delta g_{\mu\nu}(x)} \frac{\delta}{\delta g_{\alpha\beta}(y)} \ln Z + \frac{V}{d} \langle \theta \rangle \delta^d(x - y) \sqrt{g} \left( g^{\mu\nu} g^{\alpha\beta} - g^{\nu(\alpha} g^{\mu)} \right), \quad (4.27)$$

which then implies

$$-2g_{\alpha\beta}(x) \frac{\delta}{\delta g_{\alpha\beta}(x)} \langle \theta(y) \rangle = \frac{1}{V} \sqrt{g(x)} \langle \theta(x) \theta(y) \rangle_{CC}. \quad (4.28)$$

instead of the Ward identity eq. (4.12), i.e., with this definition, the contact term which appears on the r.h.s. of the Ward identity is included in the definition of the correlator. It thus follows that

$$\langle \theta(x) \theta(y) \rangle_{CC} = \langle \theta(x) \theta(y) \rangle_s + d V \langle \theta \rangle \frac{\delta^d(x - y)}{\sqrt{g}}. \quad (4.29)$$

Of course, the cancellation of contact terms leading to eq. (4.17), which follows from the fact that $c(a, g^i)$ is a dimensionless object, will always take place regardless of the choice of scheme; however in different schemes the Ward identities will look different.

5. Spectral analysis and positivity of correlators

Unitarity implies that all the states in the Hilbert space of a theory have positive norm. The spectral representation of a two–point correlator is based on saturating the correlator with a resolution of the identity made with all the physical states in the Hilbert space of the theory. In this way, unitarity translates into positivity of correlators. We will
now use this general framework to prove the positivity of the two–point function of $\theta_{\text{dyn}}$ eq. (1.17) which is needed to complete our proof.

The states in the Hilbert space of a theory provide representations of the symmetry group of space-time. In our case, the background geometry is the hyperboloid $H_d$, which is understood as a Wick rotation of $\text{AdS}_d$. The corresponding isometry group is then $SO(d - 1, 2)$. A discussion of the basic elements of this group and the construction of spectral representations in $H_d$ can be found in refs. [25,26] and [12] whose notations and conventions we will follow.

The spectral representation for the renormalized two–point correlator of the trace of the energy-momentum tensor reads in general

$$
\langle \theta(x)\theta(0) \rangle = A_d a^{d-2} \int_{\lambda_{\text{min}}}^{\infty} d\lambda \, \rho \left( \lambda; \frac{\mu}{a}, g^i \right) (\Delta + da^2)^2 G_{\lambda}(z),
$$

(5.1)

where $\lambda$ is the spectral parameter, $\rho(\lambda; \frac{\mu}{a}, g^i)$ is the positive spectral function which depends on the spectral parameter, on the dimensionless ratio $\mu/a$ and on the renormalized couplings $g^i$, and $G_{\lambda}$ is the propagator of a scalar particle carrying the highest weight representation of $SO(d - 1, 2)$. Notice that eq. (5.1) will hold for any choice of contact terms in the definition of the correlator.

The explicit form of the highest–weight propagator is

$$
G_{\lambda} = \frac{1}{2\pi^{\frac{d}{2}}} \left( -\frac{a^2}{2} \right)^{\frac{d-2}{2}} (z^2 - 1)^{\frac{2-d}{2}} Q_{\frac{d-2}{2}}^\mu(z),
$$

(5.2)

where $Q_{\mu}^\nu(z)$ are associated Legendre functions and $z = \cosh ar$. This propagator decays at infinity as $z^{-\lambda}$ and satisfies by definition

$$
(\Delta + a^2 c_2(\lambda)) G_{\lambda} = \frac{\delta^d(x)}{\sqrt{g(x)}},
$$

(5.3)

where $c_2(\lambda) = \lambda(\lambda - d + 1)$ is the second Casimir operator of the group. The unitarity bound for scalar representations is $\lambda \geq (d - 3)/2$. The value of $c_s(\lambda)$ is related to the mass of the intermediate state by

$$
m^2 - \frac{d(d-2)}{4} a^2 = a^2 \lambda(\lambda - d + 1).
$$

(5.4)

The second term in the l.h.s. of eq. (5.4) is due to the coupling of the fields to the curvature required to preserve conformal invariance of the massless theory at the classical level; a massless scalar corresponds thus to the representation $\lambda = \frac{d}{2}$.

Using the definition of the propagator eq. (5.3) the spectral representation eq. (5.1) can be explicitly rewritten as

$$
\langle \theta(x)\theta(0) \rangle = A_d a^{d-2} \int_{\lambda_{\text{min}}}^{\infty} d\lambda \, \rho \left( \lambda; \frac{\mu}{a}, g^i \right) (\Delta + da^2)^2 \frac{1}{\Delta + a^2 \lambda(\lambda - d + 1)} \frac{\delta^d(x)}{\sqrt{g(x)}}.
$$

(5.5)
From this expression it is apparent that, within the unitary region $\lambda \geq (d - 3)/2$, the spectral representation becomes local in space if and only if $\lambda = d$. For this value, the correlator reduces to a pure contact term:

$$\rho \left( \lambda; \frac{\mu}{a}, g^i \right) \propto \delta(\lambda - d) \Rightarrow \langle \theta(0) \theta(x) \rangle \propto a^{d-2} \left( \Delta + da^2 \right) \frac{\delta^d(x)}{\sqrt{g(x)}} . \quad (5.6)$$

This case corresponds to that of a conformal field theory. This is thus also the lowest value of the spectral parameter $\lambda$, i.e. $\lambda_{\text{min}} = d$, as required demanding that the correlator of two $c$–functions be well–behaved at spatial infinity \cite{12}.

It is important to notice a fundamental difference between the spectral representation in flat space (relevant for the two–dimensional construction of sect. 2) and curved space. In flat space, the coefficient of the delta contribution is constant on dimensional grounds. As a consequence, the IR limit is non–uniform, and the IR and UV values of the central charge are related by an additive scale–independent constant. In curved space, an extra scale is available, so that the general form of the local contribution to the spectral function is

$$\rho \left( \lambda; \frac{\mu}{a}, g^i \right) = \rho \left( \frac{\mu}{a}, g^i \right) \delta(\lambda - d) . \quad (5.7)$$

At fixed points the whole spectral function reduces to

$$\rho \left( \frac{\mu}{a}, g^i \right) \rightarrow \rho_{\text{cft}} \Rightarrow \rho(\lambda, t) \rightarrow \rho_{\text{cft}} \delta(\lambda - d) . \quad (5.8)$$

However the value $\rho_{\text{cft}}$ is attained through smooth variation of the function $\rho \left( \lambda; \frac{\mu}{a}, g^i \right)$ along the flow; therefore, we cannot simply derive the irreversibility of the flow from a comparison of the UV and IR limits, as in sect. 2: a study of the derivative of the $c$–function along the flow, such as that of sect. 4, is required to prove the $c$–theorem.

Now, in order to prove positivity of the correlator which gives the evolution of our $c$–function, eq. (4.17), we must specify the precise definition of the correlator. The definition used in the Ward identity is characterized by the fact that only $\theta_{\text{dyn}}$, proportional to the beta–functions, contributes to it. Furthermore, the surface (contact) contribution coming from the current divergence term upon integration of the Ward identity eq. (4.12) is also included in the correlator. However, consider surface contributions to the unintegrated Ward identity eq. (4.12) with $O = c \left( \frac{\mu}{a}, g^i \right)$. The l.h.s. of the equation is manifestly vanishing, since surface contributions to $c \left( \frac{\mu}{a}, g^i \right)$ single out its constant IR limit. Therefore, on the r.h.s. the surface contribution from the current divergence must exactly cancel the current correlator. But at infinity massive contributions decouple, so the latter also reduces to the contribution of intermediate massless states to the spectral decomposition of eq. (5.5). In other words, the contribution of intermediate massless states to the $\theta$–correlator exactly matches the surface contribution from the divergence of the dilatation current, consistently with the anomalous conservation law eq. (4.13). The physical origin of the ensuing cancellation is simple: the anomalous scale dependence of the $c$–function is driven by the scale–dependence of the vacuum, controlled by the trace anomaly through the $\langle \theta \theta \rangle_s$ correlator, and massless intermediate states decouple from it.
The correlator appearing in the Ward identity (4.14) is thus free of contact terms:

$$\langle \theta^{\text{dyn}}(x)\theta^{\text{dyn}}(0) \rangle_s = A_d a^{d-2} \int_{\lambda=d+\epsilon}^{\infty} \text{d}\lambda \, \rho \left( \frac{\lambda \mu}{a}, g^i \right) \left( \Delta + a^2 \right)^2 G_{\lambda}(z) , \quad (5.9)$$

where $\epsilon$ denotes that the contact term has been removed. All intermediate states are massive and yield non-local contributions to the correlator. At fixed points the whole correlator vanishes since massless modes do not contribute to it. Then, upon integration we find

$$\int \text{d}^d x \sqrt{g(x)} \langle \theta^{\text{dyn}}(x)\theta^{\text{dyn}}(0) \rangle = \frac{A_d d^2}{2^d - 2} a^d \int_{\lambda=d+\epsilon}^{\infty} \text{d}\lambda \, \frac{\rho \left( \lambda \frac{\mu}{a}, g^i \right)}{\lambda(\lambda - d + 1)} \geq 0 , \quad (5.10)$$

which is a scheme–independent statement and establishes the result we set out to prove: unitarity imposes the positivity for the correlator that controls the sign of the change of $c(\frac{\mu}{a}, g^i)$. Notice that the explicit form of the spectral representation eq. (5.4), the cancellation of long-distance contribution responsible for the absence of contact terms in the correlator eq. (5.9), and the positivity of the spectral function $\rho$ are all derived upon the assumption that space is maximally symmetric.

6. An explicit example

In order to illustrate our proof of the $c$–theorem we study now a simple explicit example, namely, the case of a free massive scalar field in the hyperbolic background $H_4$. We will thus be able to check the scale Ward identity and the manifest positivity of the $c$–function in a simple setting.

The relevant field theory is discussed in refs. [27,12,28] to which we refer for details. The bare trace of the energy-momentum tensor is

$$\theta(x) = V m^2 \phi^2(x) , \quad (6.1)$$

with

$$\frac{m^2}{a^2} = (\lambda - 2)(\lambda - 1) = \sigma^2 - \frac{1}{4} . \quad (6.2)$$

We can verify explicitly our $c$–theorem eq. (4.17) by computing the $c$–function, demonstrating that its derivative is indeed given by a correlator free of contact terms, and checking the positivity of this correlator.

The $c$–function is given by eq. (4.2) in terms of the trace anomaly which, for the theory at hand, has been determined in ref. [27] (see also ref. [28]):

$$a^{-4}(\theta) = 30 \left( \frac{m}{a} \right)^4 \left( \psi \left( \sigma + \frac{1}{2} \right) - \log \frac{\nu}{a} \right) - \frac{3}{4} \left( \frac{m}{a} \right)^4 - \frac{1}{6} \left( \frac{m}{a} \right)^2 + \frac{1}{30} \right) , \quad (6.3)$$
where $\psi(z)$ is the polygamma function and $\nu$ is a renormalization scale. In order to fix the renormalization scheme, we require that the $m \to \infty$ limit (IR limit) be a trivial fixed point. This corresponds to the fact that in this limit there are no propagating modes. This means that we require $(\langle \theta(x)\theta(0) \rangle)_{\text{IR}} \equiv \lim_{m \to \infty} \langle \theta(x)\theta(0) \rangle = 0$, which fixes uniquely $\log \frac{m}{a} = \frac{3}{4}$. The $c$-function is thus completely determined as

$$c\left(\frac{m}{a}\right) = 30 \left(\frac{m}{a}\right)^4 \left(\psi\left(\sigma + \frac{1}{2}\right) - \log \frac{m}{a}\right) - \frac{1}{6} \left(\frac{m}{a}\right)^2 + \frac{1}{30}.$$  \hfill (6.4)

In order to verify the theorem we must now compute the correlator of two energy-momentum tensors, which, recalling that the scalar propagator is $G_\lambda$, is simply found to be

$$\langle \theta(x)\theta(0) \rangle_{\text{pert}} = 2V^2 m^4 (G_\lambda(z))^2.$$  \hfill (6.5)

This expression manifestly needs renormalization since the product of two propagators which appears on the r.h.s. is not a distribution. Furthermore, the perturbative definition of the correlator implicit in eq. (6.5) does not necessarily coincide to that used in the Ward identity (4.12), so we will also need to perform an extra subtraction in order to be able to compare to eq. (4.17).

The UV divergence of the correlator is clearly seen when considering the integrated correlator,

$$\int d^4x \sqrt{g(x)} \langle \theta(x)\theta(0) \rangle_{\text{pert}} = \frac{V^3}{8\pi^4} m^4 \int_1^\infty dz \left(Q_{\lambda-2}^1(z)\right)^2,$$  \hfill (6.6)

which displays a log singularity at $z = 1$. A simple way to renormalize it consists of using differential renormalization [29], whereby the integrand on the r.h.s. of eq. (6.6) is rewritten by pulling out a derivative, which is then understood to act on its left side in the sense of distributions. This softens the singularity and renders it integrable while leaving the integrand unchanged away from the singular point. In our case, this amounts to writing

$$\left(Q_{\lambda-2}^1(z)\right)^2 = -\frac{d}{dz} \left(\sqrt{z^2 - 1} Q_{\lambda-2}(z)Q_{\lambda-2}^1(z)\right) - Q_{\lambda-2}(z) \frac{d}{dz} \left(\sqrt{z^2 - 1} Q_{\lambda-2}^1(z)\right),$$  \hfill (6.7)

where the total derivative is understood to act on its left. We can then perform the $z$-integral and get a finite result by using

$$\int_1^\infty dz \left(Q_{\lambda-2}(z)\right)^2 = \frac{\psi'(\lambda - \frac{1}{2})}{\lambda - \frac{3}{2}}.$$  \hfill (6.8)

This result is still only defined up to a choice of renormalization scheme, i.e. a polynomial in $m$, which we may again fix by requiring the IR limit to be trivial, i.e. by subtracting from the correlator its $m \to \infty$ limit. This prescription in principle still allows the subtraction of terms which are subleading in the IR limit, i.e. inverse powers of $m$. These must however all vanish because they would diverge in the UV limit. Note also that no
massless infrared modes are present in this example and thus no subtraction of the surface term related to these modes (as discussed below eq. (4.14)) is required. We thus get

$$\int d^4x \sqrt{g(x)} \langle \theta(x)\theta(0) \rangle_{\text{pert}} = -\frac{\pi^2}{2} a^4 \left( \left( \frac{m}{a} \right)^6 \frac{\psi'\left(\sigma + \frac{1}{2}\right)}{\sigma} - \left( \frac{m}{a} \right)^4 + \frac{1}{3} \left( \frac{m}{a} \right)^2 - \frac{2}{15} \right).$$

(6.9)

We can now determine the subtraction required to define the correlator

$$\int d^4x \sqrt{g(x)} \langle \theta(x)\theta(0) \rangle = \int d^4x \sqrt{g(x)} \langle \theta(x)\theta(0) \rangle_{\text{pert}} + \kappa c \left( \frac{m}{a} \right)$$

(6.10)

by imposing that the correlator vanishes at the conformal point, i.e., specifically, that it vanishes as $m \to 0$. This fixes $\kappa = -4$, thus showing that the perturbative definition of the correlator coincides with the covariantly conserved one eq. (4.27). With this choice we see explicitly that the correlator is equal to the derivative of the $c$–function, thus verifying the $c$–theorem eq. (4.17):

$$a \frac{\partial}{\partial a} c \left( \frac{m}{a} \right) = 30 \left[ - \left( \frac{m}{a} \right)^6 \frac{\psi'\left(\sigma + \frac{1}{2}\right)}{\sigma} - 4 \left( \frac{m}{a} \right)^4 \psi\left(\sigma + \frac{1}{2}\right) 
+ 4 \left( \frac{m}{a} \right)^4 \log \frac{m}{a} + \left( \frac{m}{a} \right)^4 + \frac{1}{3} \left( \frac{m}{a} \right)^2 \right].$$

(6.11)

We can now check explicitly that the $c$–function and its derivative satisfy the desired properties. Note in particular that according to eq. (6.11) the derivative of the $c$–function is positive and proportional to $m^2$, that is, to the square of the beta–function. Plots of the $c$–function and its derivative are displayed in Fig. 1. We can finally get the Zamolodchikov metric directly from eq. (6.11), by dividing it by $m^2$: the metric is positive definite and invertible away from the fixed points.

7. Anomaly constraints

We can summarize the results we have obtained in the following way. Irreversibility of RG flows in four–dimensional (or, more generally, even-dimensional) renormalizable quantum field theories follows from unitarity. The integrated trace anomaly plays the role of a $c$–function which decreases along RG trajectories. This $c$–function is stationary at fixed points, where it gives a $c$–charge that coincides with the coefficient of the Euler density in the trace anomaly. Irreversibility is established using the scale Ward identities, and affords a very natural physical interpretation: the decrease of the $c$–function along the RG flow is due to the decoupling of intermediate massive states from the Hilbert space. At fixed points the $c$–charge is additive and counts effective degrees of freedom.

The $c$–theorem provides a new instrument to relate short– vs. long–distance realizations of quantum field theories. The set of constraints emerging from the combination of
‘t Hooft anomaly matching and the irreversibility of RG flows can be dubbed anomaly constraints. Axial and trace anomalies are characterized at fixed points by coefficients (charges) that multiply topological terms. The behavior of these charges under RG transformations is not the same: axial anomalies are protected by the Adler-Bardeen non-renormalization theorem and thus scale-independent, whereas the trace anomaly scales in an irreversible way.

The power of anomaly matching relies in that it requires exact matching of the charge at all scales, while its drawback comes from the need of the corresponding axial symmetry to be present in the theory in the first place. On the other hand, the decrease of the \( c \)-charge along RG flows is of more general nature since it exists for any theory. The constraint coming from its irreversibility, however, is an inequality rather than an exact equality.

The way this inequality is satisfied seems to be related to the way symmetries are realized at various scales. Indeed, with the definition of the \( c \)-function which we studied, the \( c \)-charge of free fermions and vectors are larger than the one associated to scalars. This suggests that, very crudely speaking, scalars should be “preferred” in IR realizations. This agrees with the empirical observation that in many cases in which the decrease of the \( c \)-function can be verified by explicit computation, this turns out to be

Figure 1: The function \( c(m/a) \) and its logarithmic derivative \( c'(m/a) \equiv -a \frac{dc}{da}(m/a) \).
due to the fact that the IR theory displays spontaneous chiral symmetry breaking and thus contains Nambu-Goldstone scalar bosons. One is thus lead to speculate that the fact that chiral symmetry is realized in the Nambu-Goldstone mode at low energy and in the Wigner-Weyl mode at high energy in realistic theories such as QCD may be due to a deep interplay between scale and chiral symmetry.

Acknowledgments We thank D. Z. Freedman, H. Osborn and R. Tarrach for a critical reading of the manuscript. S.F. is supported by an IBERDROLA visiting professorship at the University of Barcelona; J.I.L. acknowledges financial support from CICYT (contract AEN95-0590), CIRIT (contract 1996GR00066) and NATO (contract CRG 910890).
References

[1] G. ’t Hooft, *Recent developments in gauge theories*, Eds. G. ’t Hooft et al., Plenum Press, New York, 1980.

[2] S. L. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517.

[3] K. G. Wilson and J. Kogut, Phys. Rep. 12C (1974) 75.

[4] D. J. Wallace and R. K. P. Zia, Ann. of Phys. 92 (1975) 142.

[5] A. B. Zamolodchikov, JETP Lett. 43 (1986) 730.

[6] A. Cappelli, J. I. Latorre and X. Vilasís-Cardona, Nucl. Phys. B376 (1992) 510; hep-th/9109041.

[7] G. Zumbach, Nucl.Phys. B413 (1994) 754; Phys. Lett. A190 (1994) 225.; J. Generowicz, C. Harvey-Fros, T. R. Morris, Phys. Lett. B407 (1997) 27; hep-th/9705088.

[8] F. Bastianelli, Phys. Lett. B369 (1996) 249-254; hep-th/9511065.

[9] D. Anselmi, J. Erlich, D. Z. Freedman, A. Johansen; hep-th/9711035.

[10] J. L. Cardy, Phys. Lett. B215 (1988) 749.

[11] H. Osborn, Phys. Lett. B222 (1989) 97;

[12] A. Cappelli, D. Friedan and J.I. Latorre, Nucl. Phys. B352 (1991) 616.

[13] D. Z. Freedman, J. I. Latorre and X. Vilasís-Cardona, Mod. Phys. Lett. A6 (1991) 531.

[14] A. Cappelli, D. Friedan and J. I. Latorre, unpublished work.

[15] G. M. Shore, Phys. Lett. B253 (1991) 380; B256 (1991) 407.

[16] A. Polyakov, Phys. Lett. B103 (1981) 207.

[17] M. J. Duff, Nucl. Phys. B125 (1977) 334.

[18] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge Univ. Press (1982).

[19] J. I. Latorre and H. Osborn, Nucl. Phys. B511 (1998) 737; hep-th/9803196.

[20] H. Osborn and D. Z. Freedman, hep-th/9804101.

[21] C. Callan, S. Coleman and R. Jackiw, Ann. of Phys. 59 (1970) 42;

[22] J. C. Collins, *Renormalization*, (Cambridge, Cambridge, U.K., 1984).

[23] J. C. Collins, A. Duncan and S. D. Joglekar, Phys. Rev. D16 (1977) 438; N. K. Nielsen, Nucl. Phys. B120 (1977) 212;

R. Tarrach, Nucl. Phys. B196 (1982) 45.
[24] H. Osborn and S. Petkou Ann. Phys. 231 (1994) 311; hep-th/9307010.
[25] D. W. Düsedau and D. Z. Freedman, Phys. Rev. D33 (1986) 395.
[26] N. Ja. Vilenkin, Special functions and the theory of group representations, American Math. Soc. 1968.
[27] R. Camporesi, Phys. Rev. D43 (1991) 3958;
    R. Camporesi and A. Higuchi, Phys. Rev. D45 (1992) 3591.
[28] C. J. C. Burges et al., Ann. Phys. 167 (1986) 285.
[29] D. Z. Freedman, K. Johnson and J.I. Latorre, Nucl. Phys. B371 (1992) 353.