Oscillation and Asymptotic Behavior of Second-Order Half-Linear Noncanonical Difference Equations of Advanced Type

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Abstract. We established new single-condition criteria for the oscillation of all solutions to a second-order half-linear advanced equation of the form \[ \Delta(\varphi(\zeta)(\Delta x(\zeta))^\nu) + \rho(\zeta)x^{\nu}(\zeta + \eta) = 0; \quad \zeta \geq \zeta_0, \] under the conditions that \[ \sum_{s=\zeta_0}^{\infty} \frac{1}{\varphi(\zeta)} < \infty. \] We derive new single-condition constraints for the oscillation of all unimprovable constant solutions to equation. Even in the linear case, the significant result is new and improves all of the prior results to the best of our knowledge. The advantage of our technique is the simple proof, solely relying on the monotonicities of a positive solution sequentially improved. Examples of our conclusions are presented.

1. Introduction

We consider the following second-order half-linear advanced difference equation of the form

\[ \Delta(\varphi(\zeta)(\Delta x(\zeta))^\nu) + \rho(\zeta)x^{\nu}(\zeta + \eta) = 0; \quad \zeta \geq \zeta_0, \] \hspace{1cm} (1)

where \( \Delta \) is forward difference operator defined by \( \Delta x(\zeta) = x(\zeta + 1) - x(\zeta) \).

We suppose that the following assumptions hold:

\( (A_1) \) \( \eta \geq 1 \) is a positive integer;
\( (A_2) \) \( \{\varphi(\zeta)\}_{\zeta=\zeta_0}^{\infty} \) is a sequence of positive real numbers;
\( (A_3) \) \( \{\rho(\zeta)\}_{\zeta=\zeta_0}^{\infty} \) is a nonincreasing sequence of non-negative real numbers and \( \rho(\zeta) \neq 0 \) for infinitely many values of \( \zeta \);
\( (A_4) \) \( \nu \in \{ \frac{a}{b} : a \text{ and } b \text{ are odd positive integers} \} \);
\( (A_5) \) The equation (1) is so-called noncanonical form i.e.,

\[ \theta(\zeta) := \sum_{s=\zeta}^{\infty} \frac{1}{\varphi(\zeta)} < \infty. \] \hspace{1cm} (2)
By a solution of (1), we mean a real sequence \( \{x(\zeta)\} \) which is defined for \( \zeta \geq \zeta_0 \) and satisfies (1) for \( \zeta \geq \zeta_0 \). A solution \( \{x(\zeta)\} \) is said to be oscillatory, if the terms \( \{x(\zeta)\} \) of the solution are not eventually positive or eventually negative; otherwise it is said to be nonoscillatory. (1) itself is said to be oscillatory if all of its solutions are oscillatory.

In addition to the pure mathematical problem, there is a reasonable interest in solutions to advanced difference equations because the mathematical modelling of many problems in the real world can lead to difference equations where unknown functions are based not only on the current but also on the current on the future state. This curiosity is reinforced as a result of the differential equation discretization. As a result, numerous investigators have examined the qualitative solutions to difference equations, particularly the study of the oscillatory behavior of solutions.

Although there were several solutions to the difference equation, significant research activity concentrated on difference equations with retarded arguments, with just a few studies dealing with equations with advanced arguments. 2 12 13 14 15 16 17 19 20.

The development of growth rates based on current arguments but extending into the future uses difference equations with advanced arguments. The advanced argument persuades activity that is immediately available and beneficial for decision-making. Inattentive complexity is the economic crises and population dynamics, for instance. 1 6 18. For the general background and basic theory of difference equations, one can refer to 1 2 3 7 8.

El- Morshedy and Grace 9 obtained oscillation results for the second-order nonlinear difference equation
\[
\Delta(\phi(\zeta)f(\Delta x(\zeta-1))) + g(\zeta, x(\zeta)) = 0
\]
and established several fundamental comparison results.

Murugesan and Ammamuthu 10 established oscillatory conditions for advanced functional difference equations
\[
\Delta[\phi(\zeta)\Delta(x(\zeta) + p(\zeta)x(\zeta + \tau))] + \rho(\zeta)f(x(\zeta + \eta)) = 0; \quad \zeta \geq \zeta_0.
\]

Dinakar et al. 5 investigated the second-order half-linear advanced difference equations of the form
\[
\Delta(a(\zeta)(\Delta x(\zeta))^\nu) + \mu(\zeta)x^\delta(\eta(\zeta)) = 0
\]
and proved that equation (6) oscillates if
\[
\sum_{\zeta=\zeta_0}^{\infty} \left( \frac{1}{a(\zeta)} \sum_{s=\zeta_0}^{\zeta-1} \pi^\nu(\eta(s))\rho(s) \right)^{\frac{1}{\nu}} = \infty.
\]

Chandrasekaran et al. 4 used a new, improved method and established oscillation conditions for the second-order advanced difference equation
\[
\Delta(a(\zeta)\Delta x(\zeta)) + \mu(\zeta)x(\eta(\zeta)) = 0.
\]

In 11, we established new oscillation conditions for the second-order noncanonical difference equation
\[
\Delta(\phi(\zeta)(\Delta x(\zeta))^\gamma) + \mu(\zeta)x^\delta(\zeta + \eta) = 0,
\]

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\[
\Delta(\phi(\zeta)(\Delta x(\zeta))^\gamma) + \mu(\zeta)x^\delta(\zeta + \eta) = 0,
\]
where \( \eta \) is an integer and proved that (9) is oscillatory if
\[
\limsup_{\zeta \to \infty} \pi'(\zeta) \sum_{s=\zeta}^{\zeta-1} \rho(s) > 0
\] (10)
and
\[
\sum_{u=\zeta_1}^{\infty} \frac{1}{\phi_1'(\zeta)} \left( \sum_{s=\zeta_1}^{u-1} \rho(s) \right) = \infty.
\] (11)
holds and \( \gamma > \delta \).

Murugesan and Jayakumar [13] discussed the oscillation properties of
\[
\Delta(\phi'(\zeta)\Delta x(\zeta)) + \rho(\zeta)x(\zeta + \eta) = 0
\] (12)
where \( \eta \) is an integer.

In [12], we derived oscillatory conditions for the second-order noncanonical difference equation
of the delay and advanced type of (12).

In this study, we derive new single-condition constraints for the oscillation of all unimprovable constant solutions to (1). Even in the linear situation, this sharp conclusion is unique and, to our knowledge, improves all previous results in the literature. Moreover, in the linear case, we can express comparable results for canonical equations.

The following is the paper’s structure: We proved some auxiliary lemmas in section 2. Then, the paper’s main results are stated and established in section 3. Finally, two examples are offered in section 4 to demonstrate our results.

2. Auxiliary Lemmas

Let us define
\[
\delta_* = \liminf_{\zeta \to \infty} \frac{1}{\nu} \phi_1^\nu(\zeta + \eta) \theta^{\nu+1}(\zeta + \eta + 1) \rho(\zeta)
\] (13)
and
\[
\mu_* = \liminf_{\zeta \to \infty} \frac{x(\zeta)}{x(\zeta + \eta)}.
\] (14)

The proofs rely on the existence of positivity \( \delta_* \), which is also required for Theorems 2 and 3 to be valid. Then there is a \( \zeta_1 \geq \zeta_0 \) for every arbitrary fixed \( \delta \in (0, \delta_*) \) and \( \mu \in [1, \mu_*] \) such that
\[
\frac{1}{\nu} \rho(\zeta) \phi_1^\nu(\zeta + \eta) \theta^{\nu+1}(\zeta + \eta + 1) \geq \delta
\] (15)
and
\[
\frac{\theta(\zeta)}{\theta(\zeta + \eta)} \geq \mu, \quad \zeta \geq \zeta_1.
\] (16)

In the following section, we presume that all functional inequalities are satisfied; eventually, that is, for all \( \zeta \) large enough.

To prove our main results, we require the following lemmas.

[6, Theorem 2] Let \( \{x(\zeta)\} \) be an eventually positive solution of (1). If
\[
\sum_{u=\zeta_0}^{\infty} \frac{1}{r_1^\nu(u)} \left( \sum_{s=\zeta_0}^{u-1} \rho(s) \right)^{\frac{1}{\nu}} = \infty,
\] (17)
then \( \{x(\zeta)\} \) is eventually decreasing with \( \lim_{\zeta \to \infty} x(\zeta) = \infty \). Let \( \delta_* > 0 \). If \( \{x(\zeta)\} \) is an eventually positive solution of (1), then
(i) \( \{ x(\zeta) \} \) is eventually decreasing with \( \lim_{\zeta \to \infty} x(\zeta) = 0 \);

(ii) \( \{ \frac{x(\zeta)}{\theta(\zeta)} \} \) is eventually nondecreasing.

Proof. Assume that \( \{ x(\zeta) \} \) is an eventually positive solution of (1). Then there is a \( \zeta_1 \geq \zeta_0 \) such that \( x(\zeta) > 0 \). Now,

\[
\sum_{u=\zeta_1}^{\zeta-1} \frac{1}{r(s)} \left( \sum_{s=\zeta_1}^{u-1} \rho(s) \right) \geq \frac{\zeta-1}{\zeta_1} \sum_{u=\zeta_1}^{\zeta-1} \frac{1}{r(s)} \left( \sum_{s=\zeta_1}^{u-1} \rho(s) \right) \geq \frac{\zeta-1}{\zeta_1} \sum_{u=\zeta_1}^{\zeta-1} \frac{1}{r(s)} \left( \frac{1}{\theta(u+\eta)} - \frac{1}{\theta(n_1+\eta)} \right)^{\frac{1}{p}}.
\]

with \( \delta \) defined by (15). Since \( \theta^{-\mu}(\zeta) \to \infty \) as \( \zeta \to \infty \), for any \( l \in (0,1) \) and \( \zeta \) is a large enough, we have \( \theta^{-\mu}(\zeta + \eta) - \theta^{-\mu}(\zeta_1 + \eta) \geq l \nu \theta^{-\mu}(\zeta + \eta) \) and hence

\[
\sum_{u=\zeta_1}^{\zeta-1} \frac{1}{r(s)} \left( \sum_{s=\zeta_1}^{u-1} \rho(s) \right) \geq l \frac{\zeta-1}{\zeta_1} \sum_{u=\zeta_1}^{\zeta-1} \frac{1}{r(s)} \left( \theta(u+\eta) \right)^{\frac{1}{p}} \geq l \frac{\zeta-1}{\zeta_1} \sum_{u=\zeta_1}^{\zeta-1} \frac{1}{r(s)} \left( \theta(u+\eta) \right)^{\frac{1}{p}} \geq l \frac{\zeta-1}{\zeta_1} \ln \frac{\theta(\zeta_1)}{\theta(\zeta)} \to \infty \quad \text{as} \quad \zeta \to \infty.
\]

Then by Lemma 2 (i) holds.

(ii) Using the fact that \( \{ r^\frac{1}{p}(n) \Delta x(n) \} \) is nonincreasing, we obtain

\[
x(\zeta) \geq - \sum_{s=\zeta}^{\infty} \frac{1}{r^\frac{1}{p}(s)} \Delta x(s) \geq - r^\frac{1}{p}(\zeta) \Delta x(\zeta) \theta(\zeta),
\]

with implies that

\[
\Delta \left( \frac{x(\zeta)}{\theta(\zeta)} \right) = \frac{r^\frac{1}{p}(\zeta) \Delta x(\zeta) \theta(\zeta) + x(\zeta)}{r^\frac{1}{p}(\zeta) \theta(\zeta) \theta(\zeta + 1)} \geq 0.
\]

The proof is complete. \( \square \)

To develop the (i) - part of Lemma 2 let us define a sequence \( \{ \delta_\zeta \} \) by

\[
\delta_0 = \sqrt{\delta_s} \quad (18)
\]

\[
\delta_k = \frac{\delta_0}{\delta_0 \delta_{k-1}} - \delta_k, \quad k \in \mathbb{N}. \quad (19)
\]

Using the induction principle, we can easily show that if for any \( k \in \mathbb{N}, \delta_1 < 1, i = 0, 1, 2, \ldots k, \) then \( \delta_{k+1} \) exists and

\[
\delta_{k+1} = \xi_k \delta_k > \delta_k, \quad (20)
\]
where \( l_k \) is defined by
\[
\xi_0 = \frac{\mu_0 \delta_0}{\sqrt{1 - \delta_0}} \\
\xi_{k+1} = \mu_0 \delta_0 (\xi_k - 1) \sqrt{1 - \delta_k}, \quad k \in \mathbb{N}. \tag{21}
\]

Let \( \delta_0 > 0 \) and \( \mu_0 < \infty \). If \( \{x(\zeta)\} \) is an eventually positive solution of \([1]\), then for any \( k \in \mathbb{N}_0 \), \( \{\frac{x(\zeta)}{\theta^\nu(\zeta)}\} \) is eventually decreasing.

**Proof.** Assume that \( \{x(\zeta)\} \) is an eventually positive solution of \([1]\). Then there exists a \( \zeta_1 \geq \zeta_0 \) such that \( x(\zeta + \eta) > 0 \) for \( \zeta \geq \zeta_1 \). Summing \([1]\) from \( \zeta_1 \) to \( \zeta - \eta - 1 \), where \( \zeta \geq \zeta_1 + \eta + 1 \), we get
\[
- \phi(\zeta - \eta)(\Delta x(\zeta - \eta))^\nu = -\phi(\zeta_1)(\Delta x(\zeta_1))^\nu + \sum_{s=\zeta_1}^{\zeta-\eta-1} \rho(s)x^\nu(s + \eta). \tag{23}
\]

Since \( \{\phi(\zeta)(\Delta x(\zeta))^\nu\} \) is decreasing, we get
\[
-\phi(\zeta)(\Delta x(\zeta))^\nu \geq -\phi(\zeta_1)(\Delta x(\zeta_1))^\nu + \sum_{s=\zeta_1}^{\zeta-\eta-1} \rho(s)x^\nu(s + \eta)
\]
\[
\geq -\phi(\zeta_1)(\Delta x(\zeta_1))^\nu + x^\nu(\zeta) \sum_{s=\zeta_1}^{\zeta-\eta-1} \rho(s). \tag{24}
\]

Using \([16]\) in the above inequality, we get
\[
-\phi(\zeta)(\Delta x(\zeta))^\nu \geq -\phi(\zeta_1)(\Delta x(\zeta_1))^\nu + \delta x^\nu(\zeta) \sum_{s=\zeta_1}^{\zeta-\eta-1} \frac{\rho(s)}{\phi^{\frac{\nu}{\nu+1}}(s + \eta + 1)}
\]
\[
\geq -\phi(\zeta_1)(\Delta x(\zeta_1))^\nu + \delta x^\nu(\zeta) \left[ \frac{1}{\theta^\nu(\zeta)} - \frac{1}{\theta^\nu(\zeta_1)} \right]. \tag{25}
\]

From \((i)\)-part of Lemma \([2]\) we have that \( \lim_{\zeta \to \infty} x(\zeta) = 0 \). Hence, there is a \( \zeta_2 \geq \zeta_1 \) such that
\[
-\phi(\zeta_1)(\Delta x(\zeta_1))^\nu - \delta x^\nu(\zeta_1) > 0, \quad \zeta \geq \zeta_2.
\]

Thus,
\[
- \phi(\zeta)(\Delta x(\zeta))^\nu > \delta x^\nu(\zeta_1) \tag{26}
\]
or
\[
- \phi(\zeta)(\Delta x(\zeta))^\nu > \sqrt{\delta} x(\zeta_1) = \epsilon_0 \delta_0 x(\zeta),
\]
where \( \epsilon_0 = \frac{\sqrt{\delta}}{x_0} \) is an arbitrary constant from \((0, 1)\). Therefore,
\[
\Delta \left( \frac{x(\zeta)}{\theta^{\sqrt{\delta}}(\zeta)} \right) = \frac{\theta^{\sqrt{\delta}}(\zeta)\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta) + \sqrt{\delta} \theta^{\sqrt{\delta}-1}(\zeta)x(\zeta)}{\phi^{\frac{1}{\nu}}(\zeta)\theta^{\sqrt{\delta}}(\zeta)\theta^{\sqrt{\delta}}(\zeta + 1)}
\]
\[
= \frac{\theta^{\sqrt{\delta}-1}(\zeta)\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta) + \sqrt{\delta} x(\zeta)}{\phi^{\frac{1}{\nu}}(\zeta)\theta^{\sqrt{\delta}}(\zeta)\theta^{\sqrt{\delta}}(\zeta + 1)} \leq 0, \quad \zeta \geq \zeta_2. \tag{27}
\]
For $\zeta \geq \zeta_2 + \eta + 1$, summing the equation [1] from $\zeta_2$ to $\zeta - \eta - 1$ and using the fact that $\left\{\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right\}$ is a decreasing sequence, we obtain

$$-\phi(\zeta - \eta)(\Delta x(\zeta - \eta))^{\nu} \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + \left(\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right)^{\nu} \sum_{s=\zeta_2}^{\zeta - \eta - 1} \rho(s)\theta^{\nu} \nu \sqrt{s}(s + \eta)$$

$$= -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + \left(\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right)^{\nu} \sum_{s=\zeta_2}^{\zeta - \eta - 1} \rho(s)\left(\frac{\theta(s + \eta)}{\theta(s + 2\eta)}\right)^{\nu} \nu \sqrt{s}(s + \eta) \theta^{\nu} \nu \sqrt{s}(s + 2\eta). \quad (28)$$

Using the decreasing nature of $\{\rho(\zeta)\}$ and $\{\phi(\zeta)(\Delta x(\zeta))^{\nu}\}$ and by [16], we get

$$-\phi(\zeta)(\Delta x(\zeta))^{\nu} \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + \mu^{\nu} \nu \sqrt{s} \left(\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right)^{\nu} \sum_{s=\zeta_2}^{\zeta - \eta - 1} \rho(s)\theta^{\nu} \nu \sqrt{s}(s + 2\eta)$$

$$- \phi(\zeta)(\Delta x(\zeta))^{\nu} \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + \frac{\delta}{1 - \sqrt{s}} \mu^{\nu} \nu \sqrt{s} \left(\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right)^{\nu} \sum_{s=\zeta_2}^{\zeta - \eta - 1} \frac{\nu(1 - \nu \sqrt{s})}{\phi(s) \theta^{\nu} + \nu \nu \sqrt{s}(s + 2\eta)} \quad (29)$$

$$- \phi(\zeta)(\Delta x(\zeta))^{\nu} \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + \frac{\delta}{1 - \sqrt{s}} \mu^{\nu} \nu \sqrt{s} \left(\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right)^{\nu} \left[\frac{1}{\theta^{\nu(1 - \nu \sqrt{s})}(\zeta + \eta)} - \frac{1}{\theta^{\nu(1 - \nu \sqrt{s})}(\zeta + 2\eta)}\right]. \quad (30)$$

Now, we claim that $\lim_{\zeta \to \infty} \frac{x(\zeta)}{\theta \sqrt{s}(\zeta)} = 0$. It suffices to show that there is $\epsilon > 0$ such that $\left\{\frac{x(\zeta)}{\theta \sqrt{s}(\zeta)}\right\}$ is eventually decreasing sequence. Since $\{\theta(\zeta)\}$ tends to zero, we can find a constant.

$$\xi \in \left(\frac{\sqrt{1 - \sqrt{s}}}{\mu \sqrt{s}}, 1\right)$$

and a $\zeta_3 \geq \zeta_2$ such that

$$\frac{1}{\theta^{\nu(1 - \nu \sqrt{s})}(\zeta + \eta)} - \frac{1}{\theta^{\nu(1 - \nu \sqrt{s})}(\zeta + 2\eta)} \geq \frac{\xi^{\nu}}{\theta^{\nu(1 - \nu \sqrt{s})}(\zeta + \eta)} \quad (\zeta \geq \zeta_3).$$

Using the above inequality in [30],

$$-\phi(\zeta)(\Delta x(\zeta))^{\nu} \geq \frac{\xi^{\nu}}{1 - \sqrt{s}} \mu^{\nu} \nu \sqrt{s} \left(\frac{x(\zeta)}{\theta(\zeta)}\right)^{\nu},$$

i.e.,

$$- \phi^\frac{\nu}{2}(\zeta) \Delta x(\zeta) \geq \left(\sqrt{s} + \epsilon\right) \frac{x(\zeta)}{\theta(\zeta)}, \quad (31)$$
where
\[ \epsilon = \sqrt{\delta} \left( \frac{\xi \mu \sqrt{\delta}}{\sqrt{1 - \sqrt{\delta}}} - 1 \right) > 0. \]

Then, from (31),
\[ \Delta \left( \frac{x(\zeta)}{\theta \sqrt{\delta + \epsilon(\zeta)}} \right) \leq 0, \quad \zeta \geq \zeta_3, \]
and hence the claim holds. Therefore, for \( \zeta_4 \geq \zeta_3 \),
\[ -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu \geq \frac{\delta}{\nu} \mu \sqrt{\delta} \left( \frac{x(\zeta)}{\theta \sqrt{\delta}(\zeta)} \right)^\nu \frac{1}{\theta^{\nu-\nu} \sqrt{\delta}(\zeta + \eta)} > 0, \quad \zeta \geq \zeta_4. \]

Using the above inequality in (30),
\[ -\phi(\zeta)(\Delta x(\zeta))^\nu \geq \frac{\delta}{\nu} \mu \sqrt{\delta} \left( \frac{x(\zeta)}{\theta \sqrt{\delta}(\zeta)} \right)^\nu \frac{1}{\theta^{\nu-\nu} \sqrt{\delta}(\zeta + \eta)} \geq \frac{\sqrt{\delta}}{\sqrt{1 - \sqrt{\delta}}} \mu \sqrt{\delta} x(\zeta) = \epsilon_1 \delta_1 x(\zeta), \quad \zeta \geq \zeta_4, \]
where
\[ \epsilon_1 = \sqrt{\frac{\delta (1 - \sqrt{\delta_0})}{\delta (1 - \sqrt{\delta})} \mu \sqrt{\delta_0}} \]
is an arbitrary constant from \((0, 1)\) approaching 1 if \( \delta \to \delta_* \) and \( \mu \to \mu_* \). Hence,
\[ \Delta \left( \frac{x(\zeta)}{\theta \delta(\zeta)} \right) < 0, \quad \zeta \geq \zeta_4. \]

By induction, one can show that for \( k \in \mathbb{N}_0 \) and \( \zeta \) large enough,
\[ \Delta \left( \frac{x(\zeta)}{\theta \delta_k(\zeta)} \right) < 0, \]
where \( \epsilon_k \) given by \( \epsilon_0 = \sqrt{\frac{\delta}{\delta_*}} \)
\[ \epsilon_{k+1} = \epsilon_0 \sqrt{\frac{1 - \delta_k}{1 - \epsilon_k \delta_k \mu \delta_*}}, \quad k \in \mathbb{N}_0 \]
is an arbitrary constant from \((0, 1)\) tends to 1 if \( \delta \to \delta_* \) and \( \mu \to \mu_* \). Now, we assert that from
any \( k \in \mathbb{N} \), \( \left\{ \frac{x(\zeta)}{\theta^{k+1}(\zeta)} \right\} \) decreasing implies that \( \left\{ \frac{x(\zeta)}{\theta^k(\zeta)} \right\} \) is decreasing sequence as well. For this, we use (20) and the fact that \( \epsilon_{k+1} \) is arbitrarily closed to 1,
\[ \epsilon_{k+1} \delta_{k+1} > \delta_k. \]
Thus, for $\zeta$ large enough, 

$$-\phi^\frac{1}{\nu}(\zeta)\Delta x(\zeta)\theta(\zeta) > \epsilon_{k+1}\delta_{k+1}x(\zeta) > \delta_kx(\zeta)$$

and so for any $k \in \mathbb{N}_0$ and $\zeta$ large enough,

$$\Delta \left( \frac{x(\zeta)}{\theta^{\delta_k}(\zeta)} \right) < 0.$$

The proof is complete. \hfill \Box

3. Main Results

We present and prove our main results in this section. Let $\mu^* < \infty$. If

$$\liminf_{\zeta \to \infty} \phi^\frac{1}{\nu}(\zeta + \eta)\theta^{\nu+1}(\zeta + \eta + 1)\rho(\zeta) > \max \{ c(\omega) : \nu\omega(1 - \omega)\mu_{-\nu}\omega : 0 < \omega < 1 \},$$

then (1) is oscillatory.

**Proof.** We may suppose on the contrary that $\{x(\zeta)\}$ is an eventually positive solution of (1). Then by Lemma 2 and 2 we have $\Delta \{ \frac{x(\zeta)}{\theta(\zeta)} \} \geq 0$ and $\Delta \{ \frac{x(\zeta)}{\theta^{\delta_k}(\zeta)} \} < 0$ for any $k \in \mathbb{N}_0$ and $\zeta$ large enough. This is the case when $\delta_k < 1$ for any $k \in \mathbb{N}_0$.

Thus the sequence $\{\delta_k\}$ defined by (19) is increasing and bounded from above, and hence there exists a finite limit

$$\lim_{k \to \infty} \delta_k = \omega,$$

where $\omega$ is the smallest positive root which satisfies

$$c(\omega) = \liminf_{\zeta \to \infty} \phi^\frac{1}{\nu}(\zeta + \eta)\theta^{\nu+1}(\zeta + \eta + 1)\rho(\zeta).$$

But, by (32), the equation (34) cannot have any positive solutions.

This contradiction completes the proof. \hfill \Box

By some calculations, we have

$$\max \{ c(\omega) : 0 < \omega < 1 \} = c(max),$$

where

$$\omega_{max} = \left\{ \begin{array}{ll}
\frac{\nu}{2\nu+1}, & \text{for } \mu_s = 1 \\
\frac{\nu}{\sqrt{(\nu\phi+\nu+1)^2 - 4\nu^2\phi+\nu^2+1}} & \text{for } \mu_s \neq 1 \text{ and } \phi = \ln \mu_s,
\end{array} \right.$$

and $c(\omega)$ is defined by (32).

Let

$$\lim_{\zeta \to \infty} \frac{\theta(\zeta)}{\theta(\zeta + \eta)} = \infty.$$  

(35)

If

$$\liminf_{\zeta \to \infty} \phi^\frac{1}{\nu}(\zeta + \eta)\theta^{\nu+1}(\zeta + \eta + 1)\rho(\zeta) > 0,$$

then (1) is oscillatory.
Proof. Assume on the contrary that \( \{x(\zeta)\} \) is an eventually positive solution of (1). Then there exists a \( \zeta_1 \geq \zeta_0 \) such that \( x(\zeta) > 0 \) for \( \zeta \geq \zeta_1 \). With regard to (35), for any \( M > 0 \) there exists a sufficiently large \( \zeta \) such that

\[
\frac{\theta(\zeta)}{\theta(\zeta + \eta)} \geq \left( \frac{M}{\sqrt{\delta} \nu} \right)^{2 \eta}.
\]  

(36)

Follow the procedure as in the proof of Lemma 2, we show that \( \{x(\zeta)/\theta(\zeta)\} \) is decreasing eventually, say for \( \zeta \geq \zeta_1 \). Summing the equation (1) from \( \zeta_2 \) to \( \zeta - \eta - 1 \), where \( \zeta \geq \zeta_2 + 2\eta + 1 \), we get

\[
-\phi(\zeta - \eta)(\Delta x(\zeta - \eta))\nu = -\phi(\zeta_2)\Delta x(\zeta_2) + \sum_{s=\zeta_2}^{\zeta-\eta-1} \rho(s)x'(s+\eta).
\]

Using the decreasing nature of \( \{x(\zeta)/\theta(\zeta)\} \), \{\rho(\zeta)\}, and \{\phi(\zeta)(\Delta x(\zeta))\nu\}, we obtain

\[
-\phi(\zeta + \eta)(\Delta x(\zeta + \eta))\nu \geq -\phi(\zeta_2)(\Delta x(\zeta_2))\nu + M\nu x'(\zeta + \eta) \sum_{s=\zeta_2}^{\zeta-\eta-1} \frac{\nu}{\phi(s+2\eta)}\theta^{\nu+1}(s+2\eta+1)
\]

\[
\geq M\nu \left( \frac{x(\zeta + \eta)}{\theta(\zeta + \eta)} \right)^\nu,
\]

which gives

\[
-\phi(\zeta)(\Delta x(\zeta))\nu \geq M\nu \left( \frac{x(\zeta)}{\theta(\zeta)} \right)^\nu,
\]

or

\[
-\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta) \geq Mx(\zeta),
\]

from which we derive that \( \{x(\zeta)/\theta(\zeta)\} \) is a decreasing sequence. Since \( M \) is arbitrary, we have a contradiction with (ii)-part of Lemma 2.

The proof is complete. \( \square \)

In the linear case \( \nu = 1 \), the oscillation property from (1) may be transferred into canonical equations

\[
\Delta(\tilde{\phi}(\zeta)\Delta u(\zeta)) + \tilde{\rho}(\zeta)u(\zeta + \eta) = 0, \quad \zeta \geq \zeta_0,
\]  

(37)

where \( \{\tilde{\phi}(\zeta)\} \) is a positive real sequence and \( \{\tilde{\rho}(\zeta)\} \) is a nonnegative real sequence and \( \rho(\zeta) \neq 0 \) for infinitely many values of \( \zeta \), and

\[
R(\zeta) = \sum_{s=\zeta_0}^{\zeta-1} \frac{1}{\tilde{\phi}(s)} \to \infty \quad \text{as} \quad \zeta \to \infty.
\]

Let

\[
\delta_* := \liminf_{\zeta \to \infty} \frac{R(\zeta + \eta)}{R(\zeta)} < \infty.
\]

If

\[
\liminf_{\zeta \to \infty} \left( \tilde{\phi}(\zeta + \eta)\tilde{\rho}(\zeta)R(\zeta + 1) \frac{R^2(\zeta + \eta)}{R(\zeta + \eta + 1)} \right) > \max \{\omega(1 - \omega)\delta_*^{-\omega} : 0 < \omega < 1\},
\]

then (37) is oscillatory.
Proof. The canonical (37) equation may be verified directly as equal to a noncanonical equation with \( \nu = 1 \); 
\[
\phi(\zeta) = \tilde{\phi}(\zeta) R(\zeta) R(\zeta + 1) \\
\rho(\zeta) = \tilde{\rho}(\zeta) R(\zeta + 1) R(\zeta + \eta)
\]
and 
\[
x(\zeta) = \frac{u(\zeta)}{R(\zeta)}.
\]
Now, 
\[
\theta(\zeta) = \sum_{s=\zeta}^{\infty} \frac{1}{\phi(s)} = \sum_{s=\zeta}^{\infty} \frac{\Delta R(s)}{R(s) R(s + 1)} = \frac{1}{R(\zeta)}.
\]
The conclusion, therefore, follows immediately from Theorem 3.

Let 
\[
\lim_{\zeta \to \infty} \frac{R(\zeta + \eta)}{R(\zeta)} = \infty.
\]
If 
\[
\liminf_{\zeta \to \infty} \left( \tilde{\phi}(\zeta + \eta) \tilde{\rho}(\zeta) \frac{R(\zeta + 1) R^2(\zeta + \eta)}{R(\zeta + \eta + 1)} \right) > 0,
\]
then (37) is oscillatory.

Proof. The conclusion comes from Theorem 3 using the noncanonical equivalent representation of (37), as in the Theorem 3.

4. Examples
Let us investigate the oscillatory behavior of the second-order advanced difference equation
\[
\Delta \left( (\zeta(\zeta - 1))^{\frac{1}{3}} (\Delta x(\zeta))^{\frac{1}{3}} \right) + \mu_0 \frac{(\zeta + 1)^{\frac{1}{3}}}{\zeta} x^{\frac{1}{3}}(\zeta + 1) = 0; \quad \zeta = 2, 3, \ldots. \tag{38}
\]
Here, we have \( \phi(\zeta) = (\zeta(\zeta - 1))^{\frac{1}{3}} \), \( \rho(\zeta) = \mu_0 \frac{(\zeta + 1)^{\frac{1}{3}}}{\zeta} \), \( \nu = \frac{1}{3} \) and \( \eta = 1 \).

By simple computation, we have 
\[
\theta(\zeta) = \frac{1}{\zeta - 1} \quad \text{and} \quad \mu_* = 1.
\]
Also, 
\[
\phi_{\nu}(\zeta + \eta) \rho(\zeta) \theta^{\nu + 1}(\zeta + \eta + 1) = \mu_0.
\]
and 
\[
\max \{ c(\omega) : \nu \omega^\nu (\omega - 1) \mu_*^{-\nu \omega} : 0 < \omega < 1 \} = \frac{1}{4 \sqrt{4}}.
\]
By Theorem 3, (38) is oscillatory if 
\[
\mu_0 > \frac{1}{4 \sqrt{4}}.
\]
Let us investigate the second-order advanced difference equation
\[
\Delta \left( \frac{1}{\zeta + 1} \Delta x(\zeta) \right) + \mu_0 \frac{(\zeta + 4)}{(\zeta + 1)(\zeta + 2)^3} x(\zeta + 2) = 0; \quad \zeta = 0, 1, 2, \ldots. \tag{39}
\]
Here, we see that $\phi(\zeta) = \frac{1}{\zeta+1}$, $\rho(\zeta) = \mu_0 \frac{(\zeta+4)(\zeta+2)}{(\zeta+1)(\zeta+2)}$, and $\eta = 2$.

Clearly

$$R(\zeta) = \sum_{s=0}^{\zeta-1} (s+1) = \frac{\zeta(\zeta+1)}{2}.$$ 

and

$$\delta_s = 1.$$ 

By simple computation, we obtain

$$\phi(\zeta + \eta)\rho(\zeta + 1)R(\zeta + 1)\frac{R^2(\zeta + \eta)}{R(\zeta + \eta + 1)} = \frac{\mu_0}{4}$$ 

and

$$\max\{\omega(1 - \omega) : 0 < \omega < 1\} = \frac{1}{4}.$$ 

By Theorem 3, the equation (39) is oscillatory if $\mu_0 > 1$.

5. Conclusion

In the paper, we derived single-condition constraints for the oscillation of all solution of second-order half-linear advanced difference equations. Our technique is based on new monotonic properties of non-oscillatory solutions. The results are supported with illustrative examples.

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