Stationary Electromagnetic Fields of a Slowly Rotating Magnetized Neutron Star in General Relativity

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Abstract

Following the general formalism presented by Rezzolla, Ahmedov and Miller\textsuperscript{1}, we here derive analytic solutions of the electromagnetic fields equations in the internal and external background spacetime of a slowly rotating highly conducting magnetized neutron star. The star is assumed to be isolated and in vacuum, with a dipolar magnetic field not aligned with the axis of rotation. Our results indicate that the electromagnetic fields of a slowly rotating neutron star are modified by general relativistic effects arising from both the monopolar and the dipolar parts of the gravitational field. The results presented here differ from the ones discussed by Rezzolla, Ahmedov and Miller\textsuperscript{1} mainly in that we here consider the interior magnetic field to be dipolar with the same radial dependence as the external one. While this assumption might not be a realistic one, it should be seen as the application of our formalism to a case often discussed in the literature.

1 INTRODUCTION

The rich phenomenology observed from pulsars has long motivated the study of the electromagnetic fields interior and exterior to a rotating neutron star. Furthermore, since neutron stars are among the most relativistic stellar objects, the study of electromagnetic fields in strongly curved spacetimes has been the subject of
past and recent interest. Indeed, the general relativistic modifications to the solutions of the Maxwell equations in the curved spacetime of a relativistic star were studied since the 1960s. The initial work of Ginzburg and Ozernoy\cite{2}, Anderson and Cohen\cite{3} and of Petterson\cite{4} on the stationary electromagnetic fields in a Schwarzschild external background geometry have revealed that, in general, the spacetime curvature amplifies the magnetic fields outside compact magnetic stars. More recent work by Geppert, Page and Zannias\cite{5} has reconsidered this problem also in the context of non-stationary magnetic fields.

The general relativistic effects induced by rotation of the star were first investigated by Muslimov and Tsygan\cite{6} who considered the case of a slowly rotating star surrounded by vacuum and with a prescribed dipole magnetic field aligned with the rotation axis. Muslimov and Harding\cite{7} then extended this to treat a non-vacuum exterior. More recently, Rezzolla, Ahmedov and Miller\cite{1} have considered stationary and non-stationary solutions of the Maxwell equations in the internal and external background spacetime of a slowly rotating magnetized relativistic star surrounded by vacuum and with a dipole magnetic field not aligned with the axis of rotation. In an independent study, Konno and Kojima\cite{8} have also considered a similar scenario but have limited their investigation to the stationary solution of an aligned dipole.

In this paper we make use of the general expressions formulated by Rezzolla et al.\cite{1} to study the case, recently discussed in the literature\cite{8}, in which the magnetic field interior to the star is dipolar and has the same radial dependence as the magnetic field outside it. While this is strictly speaking incorrect since the interior magnetic field cannot be determined independently of the structure of the rotating star, the error made with this assumption is small and is compensated by the possibility of dealing with a simple but instructive case.

The paper is organized as follows: in Section 2 we write the general relativistic equations for the electromagnetic fields in the background metric of a slowly rotating star in the form they assume in the orthonormal tetrads of a family of zero angular momentum observers. In Section 3 we find the stationary general relativistic solutions to the Maxwell equations in the spacetime internal and external to the rotating star. In Section 4 we summarize our conclusions.

Throughout, we use a space-like signature \((- , + , + , + )\) and geometrical units where \(c = G = 1\) (However, for those expressions with an astrophysical application we make exceptions by writing the speed of light explicitly.). Greek indices run from 0 to 3 and Latin indices from 1 to 3; semi-colons denote covariant derivatives, while commas denote partial derivatives.

\section{The Maxwell Equations in a Slowly Rotating Spacetime}

Hereafter we will assume that the contribution of the electromagnetic energy density to the total mass density is negligibly small, even for very highly magnetized stars. Under this approximation we can solve the general relativistic Maxwell equations on a given fixed curved background, rather than the coupled Einstein-Maxwell equations. In particular, we will consider the background metric as being that of a stationary, axially symmetric star approximated to first order in the angular velocity \(\Omega\). Using a Boyer-Lindquist coordinate system \((t, r, \theta, \phi)\) this metric takes the form\cite{9,10,11}:

\[
 ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 - 2\omega(r)r^2 \sin^2 \theta dtd\phi + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2 ,
\] (1)

where the differential rotation \(\omega(r)\) is caused by frame-dragging and can be interpreted as the angular velocity of a free falling (inertial) frame. The radial dependence of \(\omega\) in the region of spacetime internal to
the star has to be found as the solution of the differential equation
\[ \frac{1}{r^3} \frac{d}{dr} \left( r^4 \frac{d\tilde{\omega}}{dr} \right) + 4 \frac{dj}{dr} \tilde{\omega} = 0, \tag{2} \]
where we have defined
\[ \tilde{j} \equiv e^{-(\Phi+\Lambda)} \tag{3} \]
and where
\[ \tilde{\omega} \equiv \Omega - \omega \tag{4} \]
is the angular velocity of the fluid as measured from the local free falling (inertial) frame. In the vacuum region of spacetime external to the star, on the other hand, \( \omega(r) \) is given by the simple algebraic expression
\[ \omega(r) \equiv \frac{d\phi}{dt} = -\frac{2J}{r^3}, \tag{5} \]
where \( J = I(M, R)\Omega \) is the total angular moment of metric source as measured from infinity and \( I(M, R) \) its momentum of inertia (see [13] for a discussion of \( I \) and its numerical calculation). Outside the star, the metric (1) is completely known and explicit expressions for the other metric functions are given by
\[ e^{2\Phi(r)} \equiv \left( 1 - \frac{2M}{r} \right) = e^{-2\Lambda(r)}, \quad r > R, \tag{6} \]
where \( M \) and \( R \) are the mass and radius of the star as measured from infinity.

Note that the use of a vacuum Schwarzschild metric in place of (1) should not be considered satisfactory since the Schwarzschild metric is intrinsically inadequate for describing physical systems such as pulsars in which the coupling of the magnetic field and rotation plays a crucial role in the production of the observable electric field.

We start our analysis by recalling the general relativistic form of the Maxwell equations
\[ 3!F_{[\alpha\beta,\gamma]} = 2 \left( F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} \right) = 0, \tag{7} \]
\[ F^{\alpha\beta} \equiv 4\pi J^\alpha, \tag{8} \]
where the four-current \( J^\alpha \) can be decomposed into a conduction current density \( j^\alpha \) and a proper charge current density \( \rho_c w^\alpha \)
\[ J^\alpha = \rho_c w^\alpha + j^\alpha, \quad j^\alpha w_\alpha \equiv 0, \tag{9} \]
with \( w_\alpha \) being the components of the conductor’s four-velocity. If the conduction current is carried by electrons with electrical conductivity \( \sigma \), Ohm’s law can then be written in its special relativistic form
\[ j_\alpha = \sigma F_{\alpha\beta} w^\beta, \tag{10} \]
(A general relativistic expression can be found in [15].)

The electromagnetic field tensor \( F_{\alpha\beta} \) can be expressed through the electric and magnetic four-vector fields \( E^\alpha, B^\alpha \) measured by an observer with four-velocity \( u^\alpha \)
\[ F_{\alpha\beta} \equiv 2u_{[\alpha} E_{\beta]} + \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta, \tag{11} \]
\(^1\)This is a reasonable assumption if the neutron star has a temperature such that the atomic nuclei are frozen into a lattice and the electrons form a completely relativistic, and degenerate gas.
where \( T_{\alpha\beta} \equiv \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) \) and \( \eta_{\alpha\beta\gamma\delta} \) is the pseudo-tensorial expression for the Levi-Civita symbol \( \epsilon_{\alpha\beta\gamma\delta} \) [11]

\[
\eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta}, \quad \eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta},
\]

with \( g \equiv \det[g_{\alpha\beta}] = -e^{2(\Phi+\Lambda)}r^4 \sin^2 \theta \) for the metric [11].

It will be useful to consider the class of “zero angular momentum observers” or ZAMOs [11]. These are observers that are locally stationary (i.e. at fixed values of \( r \) and \( \theta \)) but who are “dragged” into differential rotation with respect to a reference frame fixed with respect to distant observers. At first order in \( \Omega \) they have four-velocity components given by

\[
(u^\alpha)_{\text{ZAMO}} \equiv e^{-\Phi(r)} \left( 1, 0, 0, \omega \right); \quad (u_\alpha)_{\text{ZAMO}} \equiv e^{\Phi(r)} \left( -1, 0, 0, 0 \right). \tag{13}
\]

We can now rewrite the Maxwell equations in the ZAMO reference frame by projecting the electromagnetic vector fields onto a locally orthonormal tetrad and indicating their components with “hatted” indices: i.e.

\[
\sin \theta \left( r^2 B^\phi \right)_{,r} + e^\Lambda r \left( \sin \theta B^\phi \right)_{,\theta} + e^\Lambda r B^\phi_{,\phi} = 0, \tag{14}
\]

\[
(r \sin \theta) \frac{\partial B^\phi}{\partial t} = e^\Phi \left[ E_{,\phi}^\theta - \left( \sin \theta E^\phi_{,\phi} \right)_{,\theta} - (\omega r \sin \theta) B^\phi_{,\phi} \right], \tag{15}
\]

\[
(e^\Lambda r \sin \theta) \frac{\partial B^\phi}{\partial t} = -e^{\Phi+\Lambda} E_{,\phi}^\theta + \sin \theta \left( re^\Phi E^\phi_{,r} \right)_{,r} - (\omega e^\Lambda r \sin \theta) B^\phi_{,\phi}, \tag{16}
\]

\[
(e^\Lambda r) \frac{\partial B^\phi}{\partial t} = - \left( re^\Phi E^\phi_{,r} \right)_{,r} + e^{\Phi+\Lambda} E^\phi_{,r} + \sin \theta \left( \omega r^2 B^\phi \right)_{,r} + \omega e^\Lambda r \left( \sin \theta B^\phi \right)_{,\theta}, \tag{17}
\]

and

\[
\sin \theta \left( r^2 E^\phi \right)_{,r} + e^\Lambda r \left( \sin \theta E^\phi \right)_{,\theta} + e^\Lambda r E^\phi_{,\phi} = 4\pi e^\Lambda r^2 \sin \theta J^i, \tag{18}
\]

\[
e^\Phi \left[ \left( \sin \theta B^\phi \right)_{,\theta} - B^\phi_{,\phi} \right] - (\omega r \sin \theta) E^\phi_{,\phi} = \left( r \sin \theta \right) \frac{\partial E^\phi_{,r}}{\partial t} + 4\pi e^\Phi r \sin \theta J^i, \tag{19}
\]

\[
e^{\Phi+\Lambda} B^\phi_{,r} - \sin \theta \left( r e^\Phi B^\phi \right)_{,r} - (\omega e^\Lambda r \sin \theta) E^\phi_{,\phi} = \left( e^\Lambda r \sin \theta \right) \frac{\partial E^\phi_{,r}}{\partial t} + 4\pi e^{\Phi+\Lambda} r \sin \theta J^\phi. \tag{20}
\]

\[
\left( e^\Phi r B^\phi \right)_{,r} - e^{\Phi+\Lambda} B^\phi_{,r} + \sin \theta \left( \omega r^2 E^\phi \right)_{,r} + \omega e^\Lambda r \left( \sin \theta E^\phi \right)_{,\theta} = \left( e^\Lambda r \right) \frac{\partial E^\phi_{,r}}{\partial t} + 4\pi e^{\Phi+\Lambda} r J^\phi + 4\pi e^\Lambda \omega r^2 \sin \theta J^i. \tag{21}
\]

Note that, apart from the general relativistic corrections produced by the static part of gravitational field and proportional to the metric functions \( \Lambda \) and \( \Phi \), the Maxwell equations [14]–[21] contain new general relativistic terms due to the coupling between the frame-dragging effect and the electromagnetic fields. This coupling can then act as a source of electric and magnetic fields.

Taking our conductor to be the star with four-velocity components

\[
w^\alpha \equiv e^{-\Phi(r)} \left( 1, 0, 0, \Omega \right), \quad w_\alpha \equiv e^{\Phi(r)} \left( -1, 0, 0, \frac{\omega r^2 \sin^2 \theta}{e^{2\Phi(r)}} \right), \tag{22}
\]
we can use Ohm’s law (10) to derive the following explicit components of $J^\alpha$ in the ZAMO frame

\[ J^i = \rho_e + \sigma \frac{\bar{\omega} r \sin \theta}{e^\Phi} E^\phi, \]
(23)

\[ J^r = \sigma \left( E^r - \frac{\bar{\omega} r \sin \theta}{e^\Phi} B^\theta \right), \]
(24)

\[ J^\theta = \sigma \left( E^\theta + \frac{\bar{\omega} r \sin \theta}{e^\Phi} B^r \right), \]
(25)

\[ J^\phi = \sigma E^\phi + \frac{\bar{\omega} r \sin \theta}{e^\Phi} \rho_e. \]
(26)

To simplify the problem, hereafter we will adopt the following assumptions. Firstly, for simplicity we consider the case when there is no matter outside the star so that the conductivity $\sigma = 0$ for $r > R$ and $\sigma \neq 0$ only in a shell with $R_{IN} \leq r \leq R$ (e.g. the neutron star crust). Finally, we require $\sigma$ to be uniform within this shell (Note that this might be incorrect in the outermost layers of the neutron star but is a rather good approximation in the crust as a whole.).

3 STATIONARY SOLUTIONS

In this Section we will look for stationary solutions of the Maxwell equations, i.e. for solutions in which the magnetic moment of the magnetic star does not vary in time. Note that this does not mean that the electromagnetic fields are independent of time. As a result of the misalignment between the magnetic dipole $\mu$ and the angular velocity vector $\Omega$, in fact, both the magnetic and the electric fields will have a periodic time dependence produced by the precession of $\mu$ around $\Omega$.

Our strategy in searching for the solution is based on extending the Deutsch solution \[20\] (i.e. the solution to the Maxwell equations for a misaligned rotating sphere in a Minkowski spacetime) to a curved spacetime. In particular, we will distinguish between an exterior vacuum solution to the Maxwell equations (for which fully analytic solutions can be found) and an interior non-vacuum solution. These two solutions will then be matched at the surface of the star.

A considerable simplification in the problem comes from the fact that, at first order in $\Omega$, the solutions for the electromagnetic fields do not acquire general relativistic corrections in their angular parts, and the corresponding Deutsch solution can be used directly. We will therefore search for a magnetic vector field with components

\[ B^r (r, \theta, \phi, \chi, t) = F(r) \left[ \cos \chi \cos \theta + \sin \chi \sin \theta \cos \lambda(t) \right], \]
(27)

\[ B^\theta (r, \theta, \phi, \chi, t) = G(r) \left[ \cos \chi \sin \theta - \sin \chi \cos \theta \cos \lambda(t) \right], \]
(28)

\[ B^\phi (r, \theta, \phi, \chi, t) = H(r) \sin \chi \sin \lambda(t). \]
(29)

Imposing now that $J^r = J^\theta = J^\phi = 0$ (the interior of the star is a perfect conductor and the exterior of the star is a vacuum), the radial part of the Maxwell equations \[14,19-21\] will reduce to three equations for
the radial eigenfunctions $F(r), G(r),$ and $H(r)$

$$\left(r^2 F\right)_r + 2e^\Lambda r G = 0 , \quad (30)$$

$$\left(re^\Phi H\right)_r + e^{\Phi+\Lambda} F = 0 , \quad (31)$$

$$H - G = 0 . \quad (32)$$

Equations (30)–(32) show that, in the case of infinite conductivity and as far as the magnetic field is concerned, the use of a slow rotation metric provides no additional information with respect to a non-rotating static metric\[1\].

### 3.1 Interior Solution

It is important to notice that the system of equations (30)–(32) combines information about the structure and physics of the star (through the metric functions $\Phi$ and $\Lambda$) with information about the microphysics of the magnetic field (through the radial eigenfunctions $F$ and $G$). As a result, and as mentioned in the Introduction, the general relativistic solution for the interior electromagnetic fields cannot be given independently of a self-consistent solution of the Einstein equations for the structure of the star. In practice, to calculate a generic solution to (30)–(32), it is necessary to start with a (realistic) equation of state and obtain a full solution of the relativistic star. Once the latter is known, the system of equations (30)–(32) can be solved for a magnetic field which is consistent with the star’s structure and corresponds to a magnetic configuration of some astrophysical interest. In the case of a star with constant density, for example, the exact analytical solution of the inner magnetic field can be found if a “stiff matter” equation of state is used for the stellar matter\[1\].

Given a dipolar exterior magnetic field, a simple model for the interior magnetic field comes from assuming that the latter is continuous through the surface of the star and down to some inner radius $r = R_{IN}$. The use of an inner radius removes the problem of suitable boundary conditions for $r \to 0$, and reflects the basic ignorance of the properties of magnetic fields in the interior regions of neutron stars. In this case, the radial eigenfunctions $F_{IN}$ and $G_{IN}$ are obtained through the numerical solution of a system of coupled ordinary differential equations. An even simpler but also cruder model, adopted by Konno and Kojima\[8\] for an aligned dipole, assumes that the radial eigenfunctions of the interior magnetic field have the same functional form as the corresponding eigenfunctions in the exterior of the star

$$F_{IN} = C_1 F_{EXT} \equiv C_1 F$$

and

$$G_{IN} = C_1 G_{EXT} \equiv C_1 G , \quad (33)$$

where $C_1$ is a constant to be determined through the imposition of boundary conditions. This is the model which we will also consider hereafter as a simple application of the generic equations derived by Rezzolla, Ahmedov and Miller\[1\].

The form of the internal electric field is straightforward to derive in the absence of conduction currents. In this case, in fact, Ohm’s laws (23) and (24) yield the simple expressions

$$E^\varphi = \frac{\bar{\omega} r \sin \theta}{c e^\Phi} \B^\varphi = \frac{e^{-(\Phi+\Lambda)} r \sin \theta}{c} \bar{\omega} (\cos \chi \sin \theta - \sin \chi \cos \theta \cos \lambda) C_1 G , \quad (34)$$

$$E^\theta = -\frac{\bar{\omega} r \sin \theta}{c e^\Phi} B^\theta = -\frac{e^{-\Phi} r \sin \theta}{c} \bar{\omega} (\cos \chi \cos \theta + \sin \chi \sin \theta \cos \lambda) C_1 F , \quad (35)$$

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\[ E^\phi = 0 , \]  

where we have taken into account that \( \rho_e = O(\omega) \) and that the contribution proportional to \( \bar{\omega} \rho_e \) is therefore of higher order. Note that, apart from the red-shift correction proportional to \( e^{-\Phi} \), equations (34)–(36) are the same as those in flat spacetime with \( \Omega \) being replaced by the effective fluid velocity measured by a free falling observer \( \bar{\omega} \).

### 3.2 Exterior Solution

The exterior solution for the magnetic field is simplified by the knowledge of explicit analytic expressions for the metric functions \( \Phi \) and \( \Lambda \). In particular, after defining \( N \equiv e^\Phi = e^{-\Lambda} = (1 - 2M/r)^{1/2} \), the system (30)–(32) can be written as a single, second-order ordinary differential equation for the unknown function

\[ \frac{d}{dr} \left( \frac{1 - 2M}{r} \frac{d}{dr} (r^2 F) \right) - 2F = 0. \]

Let now \( x \equiv 1 - r/M \), then equation (37) becomes

\[ \frac{d}{dx} \left\{ \left( \frac{1 + x}{1 - x} \right) \frac{d}{dx} \left( (1 - x)^2 F \right) \right\} + 2F = 0, \]

which has an exact solution expressed through the Legendre functions of the second kind \( Q_\ell \). In particular, the radial eigenfunctions \( F(r) \), \( G(r) \), and \( H(r) \), are found to be

\[ F(r) = -\frac{3}{4M^3} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \mu , \]

\[ G(r) = \frac{3N}{4M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] \mu , \]

\[ H(r) = G(r) . \]

Using expressions (39)–(41) we can now determine the value of the matching constant \( C_1 \) by requiring that the radial magnetic field is continuous across the surface of the star, i.e. that \( [B^r(r = R)]_{IN} = [B^r(r = R)]_{EXT} \). As a result, we obtain

\[ C_1 = 1 . \]

Collecting all of the expressions for the radial eigenfunctions, we obtain the following expressions for the components of the stationary magnetic field

\[ B^r = -\frac{3}{4M^3} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] (\cos \chi \cos \theta + \sin \chi \sin \theta \cos \lambda) \mu , \]

\[ B^\theta = \frac{3N}{4M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] (\cos \chi \sin \theta - \sin \chi \cos \theta \cos \lambda) \mu , \]

\[ B^\phi = \frac{3N}{4M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] (\sin \chi \sin \lambda) \mu . \]
The search for the form of the electric field is much more involved than for the magnetic field. Using as a reference the Deutsch solution, we look for the simplest solutions of the vacuum Maxwell equations in the form

\[
E^r = (f_1 + f_3) \left[ \cos \chi (3 \cos^2 \theta - 1) + 3 \sin \chi \cos \lambda \sin \theta \cos \theta \right],
\]

\[
E^\theta = \frac{1}{2} \left( f_2 + f_4 \right) \left[ \cos \chi \sin 2\theta - (1 - 2 \sin^2 \theta) \sin \chi \cos \lambda \right] + (g_1 + g_2) \sin \chi \cos \lambda,
\]

\[
E^\phi = \left[ \frac{1}{2} (f_1 + f_4) - (g_1 + g_2) \right] \sin \chi \cos \theta \sin \lambda,
\]

where the unknown eigenfunctions \( f_1 - f_4 \) and \( g_1 - g_2 \) can be found as solutions to the vacuum Maxwell equations and have radial dependence only. Skipping here the details of the derivation of the expressions for the radial eigenfunctions \( f_i \) (whose explicit form is presented in the Appendix) we here report only the final form of the components of the stationary vacuum electric field external to the misaligned magnetized relativistic star

\[
E^r = \frac{1}{4M^6r^3c} \left\{ - \left( \frac{5\omega r^3}{8M^4 C_3 + \frac{M \Omega}{9R^2 C_2}} \right) \left[ 6M^4 r^3 (2r - 3M) \ln N^2 + 4M^5 r (6r^2 - 3Mr + M^2) \right] \\
+ \frac{3\omega M^3 r^4}{2} \ln N^2 + 3M^4 \omega r^4 \right\} \cos \chi (3 \cos^2 \theta - 1) + 3 \sin \chi \cos \lambda \sin \theta \cos \theta \lambda \mu,
\]

\[
E^\theta = -\frac{3}{2M^6r^4Nc} \left\{ - \left( \frac{5\omega r^3}{8M^4 C_3 + \frac{M \Omega}{9R^2 C_2}} \right) \left[ 3M^4 r^3 (r^2 - 3Mr + 2M^2) \ln N^2 + 2M^5 r^2 (3r^2 - 6Mr + M^2) \right] + \frac{\omega r^3 M^2}{2} \right\} \\
\times \left[ 2 \cos \chi \sin \theta \cos \theta - (\cos^2 \theta - \sin^2 \theta) \sin \chi \cos \lambda \right] \mu
\]

\[
E^\phi = -\left\{ \frac{3}{2M^6r^4Nc} \left\{ - \left( \frac{5\omega r^3}{8M^4 C_3 + \frac{M \Omega}{9R^2 C_2}} \right) \left[ 3M^4 r^3 (r^2 - 3Mr + 2M^2) \ln N^2 + 2M^5 r^2 (3r^2 - 6Mr + M^2) \right] + \frac{\omega r^3 M^2}{2} \right\} \\
- \frac{3\omega r}{8M^3cN} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \right\} \sin \chi \cos \theta \sin \lambda \lambda \mu,
\]

where \( C_2 \) and \( C_3 \) are arbitrary constants to be determined through the imposition of boundary conditions.

Expressions [49]–[51] confirm that the general relativistic dragging of reference frames introduces a new contribution to the form of the electric field which does not have a flat spacetime analogue. This effect is \( O(\omega) \) and therefore is present already in our first-order slow rotation approximation. This is in contrast with what happens for the magnetic fields, where higher order approximations of the form of the metric are necessary for frame dragging corrections to appear.
The values of $C_2$ and $C_3$ can be found after imposing the continuity of the tangential electric field across the stellar surface. Using (34)–(36) as solutions for the internal electric field and imposing that 

$$[E^\theta (r = R)]_{IN} = [E^\theta (r = R)]_{EX}$$

as well as 

$$[E^\phi (r = R)]_{IN} = [E^\phi (r = R)]_{EX},$$

yields

$$C_2 = \frac{9R^5}{8M^3 (3R^2 - 6MR + M^2)} \left[ \ln N^2_R + 2M/R \right] (1 + M/R) \ln N^2_R,$$

$$C_3 = -\frac{2M^3}{30MR^2 - 60MR^2 + 10M^3 + (15R^3 - 45MR^2 + 30M^2R) \ln N^2_R},$$

with $N^2_R \equiv N^2(r = R) = 1 - 2M/R$.

Note that in the limit of an aligned dipole ($\chi = 0$), equations (49)–(50) reduce, after some algebraic manipulations, to the vacuum solutions found by Konno and Kojima.

$$E^r = \frac{1}{4M^6 r c} \left\{ C \left[ 6M^4 r^3 (2r - 3M) \ln N^2 + 4M^3 r (6r^2 - 3Mr + M^2) \right] + \frac{3\omega M^3 r^4}{2} \ln N^2 + 3M^2 \omega r^3 \right\} (3\cos^2 \theta - 1)\mu,$$

$$E^\theta = -\frac{3}{M^6 r^4 c} \left\{ C \left[ 3M^4 r^3 (r^2 - 3Mr + 2M^2) \ln N^2 + 2M^5 r^2 (3r^2 - 6Mr + M^2) \right] + \omega r^3 M^2 \right\} \sin \theta \cos \theta \mu,$$

where the constant $C$ is related to the integration constant $c_2$ given by formula (2.11) of Konno and Kojima’s paper by

$$C = -\left( \frac{5\omega r^3}{8M^4 C_3} + \frac{M\Omega}{9R^2 C_2} \right) = \frac{c_2}{\mu} - \frac{\omega r^3}{M^4}.$$  

4 CONCLUSION

In a recent related work, we have presented analytic general relativistic expressions for the stationary electromagnetic fields in the crust and vacuum region of a slowly-rotating magnetized neutron star. There, the star was considered isolated and in vacuum, but no assumption was made about the orientation of the dipolar magnetic field with respect to the rotation axis. As an application of the general formalism developed, we then found an exact solution for the interior magnetic field for a star with uniform density and a stiff matter equation of state.

In this paper we have instead considered the case of an internal magnetic field with a radial dependence which is the same as that of the external dipolar magnetic field. Strictly speaking this cannot be entirely correct since the interior magnetic field cannot be determined independently of the structure of the rotating star. However, this remains an interesting application of our equations to a case which has a Newtonian analogue and has recently been discussed in the literature.

The solutions presented in this paper provide a lowest order analytic form for the electromagnetic fields in the spacetime of a slowly rotating misaligned dipole and agree, in the case of an aligned dipole, with the
results of Konno and Kojima. While our solutions have been found with some simplifying assumptions, they also allow the major features of a realistic solution to be seen and could therefore be used in a variety of astrophysical situations.

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APPENDIX

For completeness, we here report the explicit forms of the functions \( f_1 - f_4 \), and \( g_1, g_2 \) used in expressions (56)–(58) for the components of the external electric field

\[
    f_1 = \frac{\Omega}{6cR^2} C_2 \left[ \left( 3 - \frac{2r}{M} \right) \ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4 \right] \mu ,
\]

\[
    f_2 = -\frac{\Omega}{cR^2} C_2 N \left[ \left( 1 - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2N^2} \right] \mu ,
\]

\[
    f_3 = \frac{15\omega r^3}{16M^5c} C_3 \left[ \left( 3 - \frac{2r}{M} \right) \ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4 \right] + \frac{2M^2}{5r^2} \ln N^2 + \frac{4M^3}{5r^3} \right] \mu ,
\]

\[
    f_4 = -\frac{45\omega r^3}{8M^3cN} \left[ C_3 \left[ \left( 1 - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2N^2} \right] + \frac{4M^4}{15r^4N^2} \right] \mu ,
\]

\[
    g_1 = \frac{3\Omega r}{8M^3cN} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \mu ,
\]

\[
    g_2 = -\frac{\omega}{\Omega} g_2 = -\frac{3\omega r}{8M^3cN} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \mu ,
\]

where \( g_1 + g_2 = (\bar{\omega}/\Omega)g_1 \).

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