Shot noise in semiclassical chaotic cavities

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We construct a trajectory-based semiclassical theory of shot noise in clean chaotic cavities. In the universal regime of vanishing Ehrenfest time $\tau_E$, we reproduce the random matrix theory result, and show that the Fano factor is exponentially suppressed as $\tau_E$ increases. We demonstrate how our theory preserves the unitarity of the scattering matrix even in the regime of finite $\tau_E$. We discuss the range of validity of our semiclassical approach and point out subtleties relevant to the recent semiclassical treatment of shot noise in the universal regime by Braun et al. cond-mat/0511292.

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Introduction. Quantum transport through chaotic ballistic cavities is often well described by Random Matrix Theory (RMT) [1]. Despite its many successes, or should we say, because of these successes, one might wonder what is the origin of this RMT universality, and under what conditions do system specificities modify the RMT of transport. System specific contributions to transport originate from the underlying classical dynamics, which suggests that one employs semiclassical methods based on classical trajectories [2]. Indeed, the semiclassical program toward a microscopic foundation for the RMT of transport, including explicit bounds for its regime of applicability, is currently on its way to being completed successfully [3, 4, 5, 6, 7, 8, 9, 10].

Here we contribute to this program by deriving the zero-frequency shot noise power $S$ for quantum chaotic systems. The interest in shot-noise, the intrinsically quantum part of the fluctuations of a non-equilibrium electronic current, is that it often contains information on the system that cannot be obtained through conductance measurements. For instance, shot-noise experiments have determined the charge and statistics of the charge carriers in superconducting heterostructures and in the fractional quantum hall effect [11]. In this paper, we consider an open ballistic quantum dot [11] carrying a large number of conducting channels and accordingly neglect electron-electron interactions. We reproduce the RMT result, and show how shot noise deviates from RMT predictions in the semiclassical limit. We calculate the Fano factor $F = S/S_p$, given by the ratio of $S$ to the Poissonian noise $S_p = 2e\langle J \rangle$ that would be generated by a current flow of uncorrelated electrons. According to the scattering theory of transport one has $F = \text{Tr}[t^\dagger t(1-t^\dagger t)]/\text{Tr}[t^\dagger t]$ [11]. If one makes the RMT assumption that the transmission matrix $t$ is the $N_L \times N_R$ off-diagonal block of a $(N_L + N_R) \times (N_L + N_R)$ random unitary scattering matrix, one gets $F = N_L N_R/(N_L + N_R)^2$ [11, 12], in term of the number of quantum channels $N_L$ and $N_R$ carried by the contacts to the left and right leads. Ref. [2] carried out the first semiclassical calculation of $F$ for the specific case of quantum graphs. The difficulty is to calculate $\text{Tr}[t^\dagger t^i t^j t^k] = \sum_{i,j,q} |t_{j,i}|^2 |t_{q,i}|^2 + \sum_{i \neq j, p \neq q} t_{j,i}^* t_{j,p} t_{p,q}^* t_{q,i}$. Ref. [6] employed a diagonal approximation to calculate the first two terms and identified the dominant four-trajectory contributions to the third one. Quantum graphs fundamentally differ from continuum models which we treat here. In our semiclassical derivation we find the dominant contributions to $F$ from the path pairings shown in Fig. 1. These pairings are similar to those considered in Ref. [3] for quantum graphs. However, unlike quantum graphs, chaotic systems have continuous families of scattering trajectories with similar actions, which means in particular that we cannot make a diagonal approximation to evaluate the contributions $D_2$ and $D_3$ shown in Fig. 1. This important point was not addressed in Ref. [3].

Exploring the range of validity of RMT for chaotic systems, we find $F$ to be exponentially reduced [12].

$$F = N_L N_R (N_L + N_R)^{-2} \exp[-\tau_E^2/\tau_D],$$

(1)
for systems with left (right) lead width, $W_L (W_R)$, such that the width of leads $W_{LR} \gtrsim h_{\text{eff}}^{1/2} L$. These systems witness the emergence of the new Ehrenfest time scale $	au_{E}^\gamma = \lambda^{-1} \ln[\hbar_{\text{eff}}^{1/2} (\tau_1 / \tau_D)^2]$, which generally induces significant deviations from the RMT of transport. Here, $h_{\text{eff}} = \hbar / (p_F L)$, $L$ is the linear system size, $p_F$ the Fermi momentum of the particle with mass $m$, $\tau_1$ the time of flight, $\tau_D$ the dwell time through the system, and $\lambda$ the Lyapunov exponent of the chaotic classical dynamics.

Our semiclassical calculation correctly captures both the universal regime with $\tau_{E}^\gamma / \tau_D \ll 1$ and the deep semiclassical regime where $\tau_{E}^\gamma$ becomes comparable to or exceeds $\tau_D$. We reproduce Eq. (1) and explicitly show that the exponential suppression of $F$ is due to paths shorter than $\tau_{E}^\gamma$ which become noiseless. We demonstrate the unitarity of the theory by calculating both $F = \langle t | t \rangle / \langle t | t \rangle$ and $F = \langle t | t \rangle / \langle t | t \rangle$. We finally comment on the current limitations of the trajectory-based semiclassical approach.

We consider a two-dimensional chaotic quantum dot ideally connected to two external leads. We require that the size of the openings to the leads is much smaller than the perimeter of the system but is still semiclassically large, $1 \ll N_L, N_R \ll L/\lambda f$. This ensures that the chaotic dynamics inside the dot has enough time to develop. The system’s transport properties are given by its scattering matrix $S$, with an $N_L \times N_R$ transmission block $t$, and an $N_L \times N_R$ reflection block $r$. To calculate the Fano factor, one needs to calculate the conductance $g = \langle t | t \rangle$, as well as $\langle t | t \rangle$. Semiclassically, the transmission matrix reads [12],

$$t_{ji} = -(2\pi \hbar)^{-1/2} \int_L dy_0 \int_R dy \sum_\gamma (dp_y / dp_0)_\gamma^{1/2} \langle j | y \rangle \langle y_0 | i \rangle \exp[i S_{ji} / \hbar + i \pi \mu_{ji} / 2],$$

where $| i \rangle$ is the transverse wavefunction of the $i$th lead mode. This expression sums over all paths $\gamma$ (with classical action $S_{ji}$ and Maslov index $\mu_{ji}$) starting at $y_0$ on a cross-section of the injection (L) lead and ending by $y$ on the exit (R) lead. We approximate $\sum_\gamma (y' | n) \langle n | y \rangle \approx \delta(y' - y)$ to write $\langle t | t \rangle$ as a sum over four paths, $\gamma 1$ from $y_0 1$ to $y_1$, $\gamma 2$ from $y_0 3$ to $y_3$, and $\gamma 4$ from $y_0 1$ to $y_3$.

$$\langle t | t \rangle = \frac{1}{(2\pi \hbar)^2} \int_L dy_0 \int_R dy_0 dy_3 \int_R dy_1 dy_3 \times \sum_{\gamma 1, \ldots, \gamma 4} A_{\gamma 1} A_{\gamma 4} A_{\gamma 2} A_{\gamma 3} \exp[i \delta S / \hbar].$$

Here, $A_\gamma = [dp_y / dp_0]^{1/2}$ and $\delta S = S_{\gamma 1} - S_{\gamma 2} + S_{\gamma 3} - S_{\gamma 4}$ (we absorbed all Maslov indices into the actions $S_{\gamma i}$). We are interested in quantities averaged over variations in the energy or the system shape. For most contributions, $\delta S / \hbar$ oscillates wildly with these variations. The dominant contributions that survive averaging are those for which the fluctuations of $\delta S / \hbar$ are minimal. They are shown in Fig. 1. Their paths are in pairs almost everywhere except in the vicinity of encounters. Going through an encounter, one of the four paths cross each other, while the other two avoid the crossing. They remain in pairs, though the pairing switches, e.g. from $(\gamma 1; \gamma 4)$ and $(\gamma 2; \gamma 3)$ to $(\gamma 1; \gamma 2)$ and $(\gamma 3; \gamma 4)$ in Fig. 1. Paths are always close enough to their partner that their stability is the same. Thus, for all pairings in Fig. 1

$$\sum_{\gamma 1, \ldots, \gamma 4} A_{\gamma 1} A_{\gamma 4} A_{\gamma 2} A_{\gamma 3} \rightarrow A_{\gamma 2}^2 A_{\gamma 3}^2.$$

We define $P(Y, Y_0; t) = \delta y_0 \delta \theta dt$ as the product of the momentum along the injection lead, $p_F \cos \theta_0$, and the classical probability to go from an initial position and angle $Y_0 = (y_0, \theta_0)$ to within $(\delta y, \delta \theta)$ of $Y$ in a time within $\delta t$. Then the sum over all paths $\gamma$ from $y_0$ to $y$ is

$$\sum_{\gamma} A_\gamma^2 [\cdots]_\gamma = \int_0^\infty dt \int \delta \theta \int \delta \theta P(Y, Y_0; t) [\cdots]_Y.$$ 

For an individual system, $P$ has $\delta$-functions for all classical trajectories. However averaging over an ensemble of systems or over energy gives a smooth function

$$\langle P(Y, Y_0; t) \rangle = \frac{p_F \cos \theta_0 \cos \theta}{2(W_L + W_R) \tau_D} \exp[-t / \tau_D].$$

Using Eqs. (1) and (3) to calculate the conductance within the diagonal approximation directly leads to the Drude conductance $\langle \langle t | t \rangle \rangle \approx g_D = N_L N_R / (N_L + N_R)$. This level of approximation for $\langle \langle t | t \rangle \rangle$ is sufficient to obtain $F$ to leading order in $N_{LR}$. We now use Eqs. (3), (4) and (5) to analyze the contributions in Fig. 1.

There are two things that can happen to two pairs of paths as they leave an encounter. The first is uncorrelated escape. The pairs of paths escape when the perpendicular distance between them is larger than $W_{LR}$, which requires a minimal time $T_W(\epsilon) / 2 = \lambda^{-1} \ln[\epsilon^{-1} W / L]$ between encounter and escape. The two pairs of paths then escape in an uncorrelated manner, typically at completely different times, with completely different momenta (and possibly through different leads). The second is correlated escape. Pairs of paths escape when the distance between them is less than $W_{LR}$, then the two pairs of paths escape together, at the same time through the same lead.

**Contributions to the Fano factor.** Taking into account the two escape scenarios just described, we write $\langle \langle t | t \rangle \rangle = D_1 + D_2 + D_3 + D_4$. Each of these four contributions, sketched in Fig. 2, can be written as

$$D_i = \frac{1}{(2\pi \hbar)^2} \int_L dy_0 \int_R dy_0 \int dy_3 \int dy_3 \int dt_3 \int dt_3 \times \langle \langle P(Y, Y_0; t) P(Y, Y_3; t_3) \rangle \rangle \exp[i \delta S_{D_i} / \hbar],$$

where subscripts 1, 3 make the connection to Fig. 1. When evaluating Eq. (7) the joint exit probability for two crossing paths has to be computed.
paths in the same region of phase-space (shaded areas in Fig. 1) have highly correlated escape probabilities. Here the action difference is \( \delta S_{D_2} = E_F e^2/\lambda \), where \( \epsilon \) is the crossing angle shown in Fig. 1. We write

\[
P(Y_1, Y_{03}; t_1) = \int dR_3 \tilde{P}(Y_1, R_3; t_1 - t_1') P(R_3, Y_{03}; t_1'),
\]

where \( \tilde{P} \) is the probability for the classical path to exist (not multiplied by the injection momentum), and \( R_3 \) is a point in the system’s phase-space \( (r_1, \phi_1) \) visited at time \( t_1' \), with \( \phi_1 \) giving the direction of the momentum. We choose \( R_1 \) and \( R_3 \) as the points at which the paths cross, so \( R_3 = (r_1, \phi_1 \pm \epsilon) \) and \( dR_3 = v_F^2 \sin \epsilon dt_1' dt_3' d\epsilon \). Thus

\[
D_1 = 2(2\pi)^{-2} \int_0^\infty dY_{03} dY_03 \times \int_0^\pi \text{Re} \left[ e^{i\delta S_{D_1}/\hbar} \right] \langle I(Y_{01}, Y_{03}; \epsilon) \rangle.
\]

where \( I(Y_{01}, Y_{03}; \epsilon) \) is related to the probability that \( \gamma_3 \) crosses \( \gamma_1 \) at angle \( \pm \epsilon \). Its average is independent of \( Y_{01,03} \), so \( \langle I(Y_{01}, Y_{03}; \epsilon) \rangle = \langle I(\epsilon) \rangle \). For \( D_1 \), injections/escapes are more than \( T_W(\epsilon)/2 \) from the crossing.

\[
\langle I(\epsilon) \rangle = 2v_F^2 \sin \epsilon \int_R dY_1 dY_3 \int dR_1 \times \int_T dt_1 \int_{-T/2}^{t_1 - T/2} dt_1' \int_T^{T + t_1 - T/2} dt_3 \times \langle \tilde{P}(Y_1, R_1; t_1 - t_1') P(R_1, Y_{01}; t_1') \rangle \times \tilde{P}(Y_3, R_3; t_3 - t_1') P(R_3, Y_{03}; t_3'),
\]

where \( T \) is shorthand for \( T_W(\epsilon) \). We next note that within \( T_W(\epsilon)/2 \) of the crossing, paths \( \gamma_1 \) and \( \gamma_3 \) are so close to each other that their joint escape probability is the same as for a single path (this was absent from Ref. [3] and was first noted in Ref. [17]). Elsewhere \( \gamma_1, \gamma_3 \) escape independently through either lead, hence

\[
\langle I(\epsilon) \rangle = \frac{p_F^2 T_D N_R^2 \cos \theta_{01} \cos \theta_{03} \sin \epsilon}{(N_L + N_R)^3} e^{-T_W(\epsilon)/\tau_D},
\]

where we used \( N_R = (\pi \hbar)^{-1} p_F W_R \), and assumed that the probability that \( \gamma_3 \) is at \( R_3 \) at time \( t_1' \) in a system of area \( A \) is \( (2\pi \hbar)^{-1} = m^2 [2\pi \hbar \tau_D(N_L + N_R)^{-1}] \). Then the \( Y_{01,03} \)-integral in Eq. (8) gives \( 2W_1^2 \), while the \( \epsilon \)-integral is dominated by \( \epsilon \ll 1 \) and yields a factor of \( -\pi \hbar (2E_F \tau_D)^{-1} e^{-\epsilon^2/\tau_D} [1 + O(1/\lambda)] \). Thus

\[
D_1 = -N_R^2 N_R^2 (N_L + N_R)^{-3} \exp[-T_W(\epsilon)/\tau_D].
\]

The contribution \( D_2 \) is shown in Fig. 1b, with Fig. 2 showing the paths in detail in the L lead. This contribution can be evaluated in a way similar to \( D_2 \), the difference being that the paths escape before time \( T_W(\epsilon) \), i.e. before becoming a distance \( W \) apart. The paths are always correlated, so the escape probability for the two paths equals that for one. Moreover, both paths will automatically escape through the same lead, hence

\[
\int_R dY_1 dY_3 \int_0^{T_W} dt_1 dt_3 \langle P(Y_1, Y_{01}; t_1) P(Y_3, Y_{03}; t_3) \rangle = \frac{N_R p_F^2 \cos \theta_{01} \cos \theta_{03}}{N_L + N_R} (1 - e^{-T_W(\epsilon)/\tau_D}).
\]
Performing the same analysis as for $D_2$ we find that

$$D_4 = N_L N_R (N_L + N_R)^{-1} (1 - \exp[-\tau_E^2/\tau_0^2]).$$  \hfill (18)

The Fano factor is given by $F = 1 - g_0^{-1} (D_1 + D_2 + D_3 + D_4)$. Our results of Eqs. (15) and (16) show that $D_3 + D_2 + D_1 = g_0$. One hence gets $F = -D_1/g_0$. From Eq. (11), one finally obtains our main result, Eq. (1). The splitting of phase-space discussed in Refs. [6, 7, 14] for $\tau_E^2 > \tau_0^2$ naturally emerges here. For paths shorter than $\tau_E^2$, only $D_4$ is non-zero. This cancels these path’s $\text{Tr}[t^\dagger t]$-contribution, making them noiseless.

**Preservation of Unitarity.** The unitarity of the scattering matrix ensures that $t^\dagger t + r^\dagger r = 1$ and hence the Fano factor can be written as $F = g_0^{-1} \langle \text{Tr}[t^\dagger tr^\dagger r] \rangle$. We calculate this expression to explicitly show that our method preserves unitarity. We first note that there is no contribution $D_3$ nor $D_4$ to $\text{Tr}[t^\dagger tr^\dagger r]$. We are left with the calculation of two contributions, $D_1'$ and $D_2'$, obtained from $D_1$ and $D_2$ shown in Fig. 4a,b with $y_{01}$, $y_{03}$ and $y_{13}$ on the left lead and $y_1$ on the right lead. The calculation proceeds as for $D_1$ and $D_2$, with one factor of $N_L/(N_L + N_R)$ replaced by $N_L/(N_L + N_R)$ in both contributions. The sum of these two contributions is $D_1' + D_2' = e^{-\tau_E^2/\tau_0^2} N_L^2 N_R^2 (N_L + N_R)^{-3}$, the Fano factor is then $F = (D_1' + D_2')/g_0$, which reproduces Eq. (1).

**Off-diagonal nature of all contributions.** In our analysis we allow for the fact that open chaotic systems have continuous families of paths with highly correlated actions coupling to multiple lead modes. For example paths $\gamma_1$ and $\gamma_2$ in Fig. 2 have an action difference given in Eq. (12), which does not fluctuate under energy or sample averaging. The stationary phase integral for $D_{2,3,4}$ over such paths is dominated by paths $\gamma_1$ and $\gamma_3$ with $p_{0\perp} \sim m \lambda \eta_{0\perp}$. Since $\eta_{0\perp}$ is integrated over the width $W$ of the lead, $p_{0\perp}$ varies over a range of order $mAW$, these contributions are clearly not diagonal in the lead mode basis. Upon completion of this manuscript, we became aware of Ref. 8, which presents a semiclassical calculation of $F$ for $\tau_E^2 = 0$. While their method is superficially similar to ours, they make a diagonal assumption to get the contributions that we call $D_{2,3,4}$. Our analysis shows that this is unjustifiable. Such an assumption would moreover violate unitarity for finite $\tau_E^2$.

**Regime of applicability of these semiclassics.** We appear to be the first to report that all trajectory-based semiclassical methods used so far in the theory of transport (including in the present article) are only applicable in the regime $W \geq \hbar^{1/2}_\text{eff} L$. Dominant off-diagonal contributions such as those discussed above have encounters of a typical size $\sim \hbar^{1/2}_\text{eff} L$. When $W < \hbar^{1/2}_\text{eff} L$, the two non-crossing paths (i.e. $\gamma_2$ and $\gamma_4$ in Fig. 1) at an encounter are a distance apart greater than $W$. The probability that one of the four paths escapes while the other three paths remain in the system is of order $\tau_E/\tau_0$, where $\tau_E \sim \hbar^{-1} \ln(\hbar L/W^2)$ is the time over which this path is a distance of order $W$ from any of the other paths. The current methods fail once this is taken into account, suggesting that diffraction effects may become important. We believe that the regime $\hbar^{1/2}_\text{eff} < (W/L) \leq \hbar^{1/2}_\text{eff}$ is well described by RMT, and thus suspect this diffraction may be the microscopic source of RMT universality in this regime. Clearly this regime merits further study.

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