The generalized MIC-Kepler system
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Abstract
This paper deals with the dynamical system that generalizes the MIC-Kepler system. It is shown that the Schrödinger equation for this generalized MIC-Kepler system can be separated in spherical and parabolic coordinates. The spectral problem in spherical and parabolic coordinates is solved.

1 Introduction
The system described by the Hamiltonian
\[
\hat{H} = \frac{1}{2}(-i\nabla - s\mathbf{A})^2 + \frac{s^2}{2r^2} - \frac{1}{r} + \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)},
\]
where \(c_1\) and \(c_2\) nonnegative constants, later on will be called the generalized MIC-Kepler system. (We use the system of units for which \(\hbar = m = e = c = 1\).)

The MIC-Kepler integrable system was constructed by Zwanziger \[1\] and rediscovered by McIntosh and Cisneros \[2\]. This system is described by the Hamiltonian
\[
\hat{H}_0 = \frac{1}{2}(-i\nabla - s\mathbf{A})^2 + \frac{s^2}{2r^2} - \frac{1}{r},
\]
where
\[
\mathbf{A} = \frac{1}{r(r-z)}(y,-x,0), \quad \text{and} \quad \text{rot}\mathbf{A} = \frac{r}{r^3}.
\]

Its distinctive peculiarity is the Coulomb hidden symmetry given by the following constants of motion \[3\]
\[
\hat{I} = \frac{1}{2} \left[ (-i\nabla - s\mathbf{A}) \times \hat{\mathbf{J}} - \hat{\mathbf{J}} \times (-i\nabla - s\mathbf{A}) \right] + \frac{r}{r}, \quad \hat{\mathbf{J}} = r \times (-i\nabla - s\mathbf{A}) - s \frac{r}{r}.
\]
Here, the operator \(\hat{\mathbf{J}}\) defines the angular momentum of the system, while operator \(\hat{I}\) is the analog of the Runge-Lenz vector. These constants of motion, together with the Hamiltonian, form the quadratic symmetry algebra of the Coulomb problem. For fixed negative energy values the motion integrals make up algebra \(\text{so}(4)\), whereas for positive energy values - \(\text{so}(3.1)\). Due to the hidden symmetry the MIC-Kepler problem is factorized.
not only in the spherical but parabolic coordinates as well. Hence, the MIC-Kepler system is a natural generalization of the Coulomb problem in the presence of Dirac’s monopole. In both cases the monopole number \( s \) satisfies the Dirac’s rule of charge quantization \( s = 0, \pm 1/2, \pm 1, \ldots \).

The MIC-Kepler system could be constructed by the reduction of the four-dimensional isotropic oscillator by the use of the so-called Kustaanheimo-Stiefel transformation both on classical and quantum mechanical levels [3]. In the similar way, reducing the two- and eight-dimensional isotropic oscillator, one can obtain the two- and five-dimensional [5] analogs of MIC-Kepler system. An infinitely thin solenoid providing the system by the spin \( 1/2 \), plays the role of monopole in two-dimensional case, whereas in the five-dimensional case this role is performed by the \( SU(2) \) Yang monopole [6], endowing the system by the isospin. All the above-mentioned systems have Coulomb symmetries and are solved in spherical and parabolic coordinates both in discrete and continuous parts of energy spectra [4]. There are generalizations of MIC-Kepler systems on three-dimensional sphere [8] and hyperboloid [9] as well. The MIC-Kepler system has been worked out from different points of view in Refs. [10, 11, 12, 13, 14].

For integer values \( s \) the MIC-Kepler system describes the relative motion of the two Dirac’s dyons (charged magnetic monopoles), where vector \( \mathbf{r} \) determines the position of the second dyon with respect to the first one [11]. For half-integer \( s \) the presence of the solenoid magnetic field, endowing the system with the spin \( 1/2 \), is presupposed (see, e.g. [4]).

The Hamiltonian (1.1) for \( s = 0 \) and \( c_i \neq 0 \) \((i = 1, 2)\) reduces to the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \Delta - \frac{1}{r} + \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)},
\]

(1.4)
of the generalized Kepler-Coulomb system [15].

The potential

\[
V = -\frac{\alpha}{r} + \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)};
\]

(1.5)
is one of the Smorodinsky-Winternitz type potentials [16]. The Smorodinsky-Winternitz type potentials were revived and investigated in the 1990 by Evans [17]. In the case where \( c_1 = c_2 \), the potential (1.5) reduces to the Hartmann potential that has been used for describing axially symmetric systems like ring-shaped molecules [18] and investigated from different points of view in Refs. [19]-[31]. In particular, the (quantum mechanical) discrete spectrum for the for the generalized Kepler-Coulomb system (1.4) is well known [24, 27, 29], even for the so-called \((q,p)\)-analogue of this system [29]. Furthermore, a path integral treatment of the potential (1.5) has been given in Refs. [24, 27]. Recently, the dynamical symmetry of the generalized Kepler-Coulomb system has been studied in Refs. [29, 30, 31], the classical motion of a particle moving in the potential (1.5) has been considered in [30], and the coefficients connecting the parabolic and spherical bases have been identified in [31] as Clebsch-Gordan coefficients of the pseudo-unitary group \( SU(1,1) \).

The purpose of the present paper is to further study the bound states of the generalized MIC-Kepler system in spherical and parabolic coordinates.
2 Spherical Basis

The Schrödinger equation with Hamiltonian (1.1) in spherical coordinates \((r, \theta, \varphi)\) may be solved by seeking a wavefunction \(\psi\) of the form

\[
\psi(r, \theta, \varphi) = R(r)Z(\theta, \varphi).
\]

(2.1)

This amounts to finding the eigenfunctions of the set \(\{\hat{H}, \hat{J}_z, \hat{M}\}\) of commuting operators, where the constant of motion \(\hat{M}\) reads

\[
\hat{M} = \hat{J}_z^2 + \frac{2c_1}{1 + \cos \theta} + \frac{2c_2}{1 - \cos \theta}.
\]

(2.2)

Here \(\hat{J}_z^2\) is the square of the angular momentum, \(\hat{J}_z = s - i \partial/\partial \varphi\) its \(z\)-component and \(\hat{J}_z \psi = m \psi\).

After substitution the expression (2.1) the variables in the Schrödinger equation are separated and we arrive at the following system of coupled differential equations:

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Z}{\partial \theta} \right) + \frac{1}{4 \cos^2 \frac{\theta}{2}} \left( \frac{\partial^2}{\partial \varphi^2} - 4c_1 \right) Z +
\]

\[
+ \frac{1}{4 \sin^2 \frac{\theta}{2}} \left[ \left( \frac{\partial}{\partial \varphi} + 2is \right)^2 - 4c_2 \right] Z = -\mathcal{A} Z,
\]

(2.3)

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{A}{r^2} R + 2 \left( E + \frac{1}{r} \right) R = 0,
\]

(2.4)

where \(\mathcal{A}\) is a separation constant in spherical coordinates.

The solution of (2.3) is easily found to be

\[
Z_{jm}(\theta, \varphi; \delta_1, \delta_2) = N_{jm}(\delta_1, \delta_2) \left( \cos \frac{\theta}{2} \right)^{m_1} \left( \sin \frac{\theta}{2} \right)^{m_2} P_{j-m_+}^{(m_2, m_1)}(\cos \theta)e^{i(m-s)\varphi},
\]

(2.5)

where \(m_1 = |m - s| + \delta_1 = \sqrt{(m-s)^2 + 4c_1}, m_2 = |m + s| + \delta_2 = \sqrt{(m+s)^2 + 4c_2}, m_+ = (|m + s| + |m - s|)/2\) and \(P_n^{(a,b)}\) denotes a Jacobi polynomial. The quantum numbers \(m\) and \(j\) run through values: \(m = -j, -j+1, \ldots, j-1, j\) and

\[
j = \frac{|m + s| + |m - s|}{2}, \frac{|m + s| + |m - s|}{2} + 1, \ldots.
\]

The quantum numbers \(j, m\) characterize the total momentum of the system and its projection on the axis \(z\). For the (half)integer \(s, j, m\) are (half)integers.

Furthermore, the separation constant \(\mathcal{A}\) is quantized as

\[
\mathcal{A} = \left( j + \frac{\delta_1 + \delta_2}{2} \right) \left( j + \frac{\delta_1 + \delta_2}{2} + 1 \right).
\]

(2.6)
The normalization constant $N_{jm}(\delta_1, \delta_2)$ in (2.5) is given (up to a phase factor) by

$$N_{jm}(\delta_1, \delta_2) = \frac{1}{2^m} \sqrt{\frac{(2j + \delta_1 + \delta_2 + 1)(j - m_+)! \Gamma(j + m_+ + \delta_1 + \delta_2 + 1)}{2^{m_+ + \delta_2 + 2} \pi \Gamma(j - m_- + \delta_1 + 1) \Gamma(j + m_- + \delta_2 + 1)}}. \quad (2.7)$$

where $m_- = (|m + s| - |m - s|)/2$. The angular wavefunctions $Z_{jm}^{(s)}$ [see Eq. (2.5)] are convenient to call the ring-shaped monopole harmonics by analogy with the term "monopole harmonics" studied by Tamm [32]. These ring-shaped monopole harmonics generalize the harmonics studied by Hartmann [18] in the case $s = 0, \delta_1 = \delta_2$. Due to the connecting formula [33]

$$\left(\lambda + \frac{1}{2}\right) C_n^\lambda(x) = (2\lambda)_n P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x) \quad (2.8)$$

between the Jacobi polynomial $P_n^{(\alpha, \beta)}$ and the Gegenbauer polynomial $C_n^\lambda$, the case $s = 0, \delta_1 = \delta_2 = \delta$ yields

$$Z_{jm}^{(0)}(\theta, \varphi; \delta, \delta) = 2^{|m|+\delta} \frac{\Gamma(|m| + \delta + 1/2)}{\sqrt{\pi}} \frac{(2j + 2\delta + 1)(j - |m|)!}{4\pi^2 \Gamma(j + |m| + 2\delta + 1)}$$

$$ \times (\sin \theta)^{|m|+\delta} C_{j-|m|}^{|m|+\delta+\frac{1}{2}}(\cos \theta) e^{i m \varphi}, \quad (2.9)$$

the result already obtained in Ref. [15]. [In (2.8) $(a)_n$ stands for a Pochhammer symbol.] The case $\delta = 0$ (i.e. $c_1 = c_2 = 0$) can be treated by using the connecting formula

$$P_j^{|m|}(x) = \frac{(-2)^{|m|}}{\sqrt{\pi}} \frac{\Gamma(|m| + 1/2)}{\Gamma(|m| + 1)} \left(1 - x^2\right)^{|m|/2} C_j^{|m|}(x) \quad (2.10)$$

between the Gegenbauer polynomial $C_n^\lambda$ and the associated Legendre function [33]. In fact for the $\delta = 0$, Eq. (2.9) can be reduced to

$$Z_{jm}^{(0)}(\theta, \varphi; 0, 0) = \sqrt{\frac{(2j + 1)(j - |m|)!}{4\pi(j + |m|)!}} P_j^{|m|}(\cos \theta) e^{i m \varphi}, \quad (2.11)$$

an expression (up to a phase factor) that coincides with the usual (surface) spherical harmonics $Y_{jm}(\theta, \varphi)$.

Let us go now to radial equation (2.4). The introduction of (2.6) into the (2.4) leads to

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{1}{r^2} \left( j + \frac{\delta_1 + \delta_2}{2} \right) \left( j + \frac{\delta_1 + \delta_2}{2} + 1 \right) R + 2 \left( E + \frac{1}{r} \right) R = 0. \quad (2.12)$$

which is reminiscent of the radial equation for the hydrogen atom except that the orbital quantum number $l$ is replaced here by $j + (\delta_1 + \delta_2)/2$. The solution of (2.12) for the discrete spectrum is

$$R_{nj}^{(s)}(r) = C_n j(\delta_1, \delta_2)(2s)j^{j_1 + j_2} e^{-s} F \left( -n + j + 1; 2j + \delta_1 + \delta_2 + 2; 2s \right), \quad (2.13)$$
where \( n = |s| + 1, |s| + 2, \ldots \). In (2.13), the normalization factor \( C_{nj}(\delta_1, \delta_2) \) reads

\[
C_{nj}(\delta_1, \delta_2) = \frac{2\varepsilon^2}{\Gamma(2j + \delta_1 + \delta_2 + 1)} \sqrt{\Gamma(n + j + \delta_1 + \delta_2 + 1)} (n - j - 1)!
\]

and the parameter \( \varepsilon \) is defined by

\[
\varepsilon = \sqrt{-2E} = \frac{1}{n + \frac{\delta_1 + \delta_2}{2}}.
\]

The eigenvalues \( E \) are then given by

\[
E \equiv E_n^{(s)} = -\frac{1}{2 \left(n + \frac{\delta_1 + \delta_2}{2}\right)^2}.
\]

In the limiting case \( \delta_1 = \delta_2 = 0 \), we recover the familiar results for charge-dyon bound system [1].

## 3 Parabolic Basis

Let us consider the generalized MIC-Kepler system in the parabolic coordinates. In the parabolic coordinates \( \xi, \eta \in [0, \infty), \varphi \in [0, 2\pi) \), defined by the formulae

\[
x = \sqrt{\xi \eta} \cos \varphi, \quad y = \sqrt{\xi \eta} \sin \varphi, \quad z = \frac{1}{2}(\xi - \eta),
\]

the differential elements of length and volume read

\[
dl^2 = \frac{\xi + \eta}{4} \left( \frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta} \right) + \xi \eta d\varphi^2, \quad dV = \frac{1}{4}(\xi + \eta)d\xi d\eta d\varphi,
\]

while the Laplace operator looks like

\[
\Delta = \frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \varphi^2}.
\]

The substitution

\[
\psi(\xi, \eta, \varphi) = \Phi_1(\xi)\Phi_2(\eta) e^{i(m-s)\varphi} \sqrt{2\pi}.
\]

separates the variables in the Schrödinger equation and we arrive at the following system of equations

\[
\frac{d}{d\xi} \left( \xi \frac{d\Phi_1}{d\xi} \right) + \left[ \frac{E}{2} \xi - \frac{m_1^2}{4\xi} + \frac{1}{2} \beta + \frac{1}{2} \right] \Phi_1 = 0,
\]

\[
\frac{d}{d\eta} \left( \eta \frac{d\Phi_2}{d\eta} \right) + \left[ \frac{E}{2} \eta - \frac{m_2^2}{4\eta} - \frac{1}{2} \beta + \frac{1}{2} \right] \Phi_2 = 0,
\]

where $\beta$ is the separation constant.

These equations are analogous with the equations of the hydrogen atom in the parabolic coordinates [34]. Thus, we get

$$
\psi_{n_1n_2m}(\xi, \eta, \varphi; \delta_1, \delta_2) = \sqrt{2}\varepsilon^2 \Phi_{n_1m_1}(\xi)\Phi_{n_2m_2}(\eta) \frac{e^{i(m-s)\varphi}}{\sqrt{2\pi}}, \quad (3.7)
$$

where

$$
\Phi_{n_1m_1}(x) = \frac{1}{\Gamma(m_i + 1)} \sqrt{\frac{\Gamma(n_i + m_i + 1)}{(n_i)!}} e^{-\frac{\varepsilon x^2}{2}} F(-n_i; m_i + 1; \varepsilon x). \quad (3.8)
$$

Here $n_1$ and $n_2$ are nonnegative integers

$$
n_1 = -\frac{|m-s| + \delta_1 + 1}{2} + \frac{\beta + 1}{2\varepsilon}, \quad n_2 = -\frac{|m+s| + \delta_2 + 1}{2} - \frac{\beta - 1}{2\varepsilon}. \quad (3.9)
$$

From the last relations, taking into account (2.16), we get that the parabolic quantum numbers $n_1$ and $n_2$ are connected with the principal quantum number $n$ as follows

$$
n = n_1 + n_2 + \frac{|m-s| + |m+s|}{2} + 1. \quad (3.10)
$$

Excluding the energy $E$ from Eqs. (3.5) and (3.6), we obtain the additional integral of motion

$$
\hat{X} = \frac{2}{\xi + \eta} \left[ \xi \frac{\partial}{\partial \eta} \left( \eta \frac{\partial}{\partial \eta} \right) - \eta \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) \right] + \frac{\xi - \eta}{2\xi\eta} \frac{\partial^2}{\partial \varphi^2} + \frac{is}{\xi\eta(\xi + \eta)} \frac{\partial}{\partial \varphi} - \frac{s^2(\xi - \eta)}{2\xi\eta} + \frac{2c_1\eta}{\xi(\xi + \eta)} - \frac{2c_2\xi}{\eta(\xi + \eta)} + \frac{\xi - \eta}{\xi + \eta} \quad (3.11)
$$

with the eigenvalues

$$
\beta = \varepsilon \left( n_1 - n_2 + \frac{|m-s| - |m+s| + \delta_1 - \delta_2}{2} \right) \quad (3.12)
$$

and eigenfunctions $\psi_{n_1n_2m}^{(s)}(\xi, \eta, \varphi; \delta_1, \delta_2)$.

In Cartesian coordinates, the operator $\hat{X}$ can be rewritten as

$$
\hat{X} = z \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - x \frac{\partial^2}{\partial x \partial z} - y \frac{\partial^2}{\partial y \partial z} + \frac{is}{r(r-z)} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial z} - \frac{s^2}{r(r-z)} + \frac{c_1}{r(r+z)} - \frac{c_2}{r(r-z)} + \frac{z}{r}, \quad (3.13)
$$

so that it immediately follows that $\hat{X}$ is connected to the $z$-component $\hat{I}_z$ of the analog of the Runge-Lenz vector (1.3) via

$$
\hat{X} = \hat{I}_z + c_1 \frac{r-z}{r(r+z)} - c_2 \frac{r+z}{r(r-z)}, \quad (3.14)
$$

6
and coincides with \( \hat{I}_z \) when \( c_1 = c_2 = 0 \).

Thus we have solved the spectral problem in spherical

\[
\hat{H}\psi = E\psi, \quad \hat{M}\psi = \left(j + \frac{\delta_1 + \delta_2}{2}\right) \left(j + \frac{\delta_1 + \delta_2}{2} + 1\right) \psi, \quad \hat{J}_z\psi = m\psi \tag{3.15}
\]

and in parabolic coordinates

\[
\hat{H}\psi = E\psi, \quad \hat{X}\psi = \beta\psi, \quad \hat{J}_z\psi = m\psi, \tag{3.16}
\]

where \( \hat{H}, \hat{J}_z, \hat{M} \) and \( \hat{X} \) are defined by the expressions \((1.1), (1.3), (2.2)\) and \((3.14)\).

It is mentioned that all the formulae obtained for \( s = 0 \) yield the corresponding formulae for the generalized Kepler-Coulomb system \([15]\).

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