On Properties of a Regular Simplex Inscribed into a Ball

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Abstract—Let $B$ be a Euclidean ball in $\mathbb{R}^n$ and let $C(B)$ be a space of continuous functions $f : B \to \mathbb{R}$ with the uniform norm $\|f\|_{C(B)} := \max_{x \in B} |f(x)|$. By $\Pi_1(\mathbb{R}^n)$ we mean a set of polynomials of degree $\leq 1$, i.e., a set of linear functions on $\mathbb{R}^n$. The interpolation projector $P : C(B) \to \Pi_1(\mathbb{R}^n)$ with the nodes $x^{(j)} \in B$ is defined by the equalities $Pf(x^{(j)}) = f(x^{(j)})$, $j = 1, \ldots, n+1$. The norm of $P$ as an operator from $C(B)$ to $C(B)$ can be calculated by the formula $\|P\|_B = \max_{x \in B} \sum_{j=1}^{n+1} |\lambda_j(x)|$. Here $\lambda_j$ are the basic Lagrange polynomials corresponding to the $n$-dimensional nondegenerate simplex $S$ with vertices $x^{(j)}$. Let $P'$ be a projector having the nodes in the vertices of a regular simplex inscribed into the ball. We describe the points $y \in B$ with the property $\|P'\|_B = \sum |\lambda_j(y)|$. Also we formulate a geometric conjecture which implies that $\|P'\|_B$ is equal to the minimal norm of an interpolation projector with nodes in $B$. We prove that this conjecture holds true at least for $n = 1, 2, 3, 4$.

Keywords: regular simplex, ball, linear interpolation, projector, norm

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INTRODUCTION

Let $\Omega$ be a convex body in $\mathbb{R}^n$. Denote by $C(\Omega)$ a space of continuous functions $f : \Omega \to \mathbb{R}$ with the uniform norm

$\|f\|_{C(\Omega)} := \max_{x \in \Omega} |f(x)|$.

By $\Pi_1(\mathbb{R}^n)$ we mean a set of polynomials in $n$ variables of degree $\leq 1$, i.e., a set of linear functions on $\mathbb{R}^n$. For $x^{(0)} \in \mathbb{R}^n$, $R > 0$, by $B(x^{(0)}; R)$ we denote the $n$-dimensional Euclidean ball given by the inequality $\|x - x^{(0)}\| \leq R$. Here

$\|x\| := \sqrt{(x, x)} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$.

By definition, $B_n := B(0; 1)$. Further $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$.

Let $S$ be a nondegenerate simplex in $\mathbb{R}^n$ with vertices $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$, $1 \leq j \leq n+1$. Consider the following vertex matrix of this simplex:

$S := \begin{pmatrix}
    x_1^{(1)} & \cdots & x_n^{(1)} & 1 \\
    x_1^{(2)} & \cdots & x_n^{(2)} & 1 \\
    \vdots & \vdots & \vdots & \vdots \\
    x_1^{(n+1)} & \cdots & x_n^{(n+1)} & 1
\end{pmatrix}$. 

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Let $S^{-1} = (l_{ij})$. Linear polynomials $\lambda_j(x) = l_{ij}x_i + \ldots + l_{jn}x_n + l_{n+1,j}$ have a property $\lambda_j(x^{(k)}) = \delta_j^k$. We call $\lambda_j$ the basic Lagrange polynomials corresponding to $S$. For an arbitrary $x \in \mathbb{R}^n$,

$$x = \sum_{j=1}^{n+1} \lambda_j(x)x^{(j)}, \quad \sum_{j=1}^{n+1} \lambda_j(x) = 1.$$ 

These equalities mean that $\lambda_j(x)$ are the barycentric coordinates of $x$. For details, see [1, §1.1].

An interpolation projector $P : C(\Omega) \rightarrow \Pi_1(\mathbb{R}^n)$ corresponds to a simplex $S \subset \Omega$ if the nodes of $P$ coincide with the vertices of $S$. This projector is defined by the equalities $Pf(x^{(j)}) = f(x^{(j)})$. The following analogue of Lagrange interpolation formula holds:

$$Pf(x) = \sum_{j=1}^{n+1} f(x^{(j)}) \lambda_j(x).$$

(1)

Denote by $\|P\|_2$ the norm of $P$ as an operator from $C(\Omega)$ in $C(\Omega)$. From (1), it follows that

$$\|P\|_2 = \max_{x \in \Omega} \sum_{j=1}^{n+1} |\lambda_j(x)|.$$ 

Since $\lambda_j(x)$ are the barycentric coordinates of a point $x$ we have also

$$\|P\|_2 = \max \left\{ \sum_{j=1}^{n+1} |\beta_j| : \sum_{j=1}^{n+1} \beta_jx^{(j)} \in \Omega, \quad \sum_{j=1}^{n+1} \beta_j = 1 \right\}.$$ 

(2)

Consider the case $\Omega = B := B(x^{(0)}; R)$. It is proved in [2] that

$$\|P\|_B = \max_{\ell = -1, 1} \left[ R \left( \sum_{j=1}^{n} \left( \sum_{j=1}^{n+1} f_j \right)^2 \right)^{1/2} + \sum_{j=1}^{n+1} f_j \lambda_j(x^{(0)}) \right].$$

(3)

If $S$ is a regular simplex inscribed into the ball, then $\|P\|_B$ depends neither on the center $x^{(0)}$ nor on the radius $R$ of the ball nor on the choice of such a simplex. In this case (see [2, Theorem 2])

$$\|P\|_B = \max\{\psi(a), \psi(a + 1)\},$$

(4)

where $a = \left\lfloor \frac{n + 1}{2} \right\rfloor - \frac{\sqrt{n + 1}}{2}$,

$$\psi(t) := \frac{2\sqrt{n}}{n + 1} t^{(n + 1 - t)} 1^{1/2} + \left| 1 - \frac{2t}{n + 1} \right|, \quad 0 \leq t \leq n + 1.$$ 

(5)

Various geometric estimates concerning polynomial interpolation are given in [1]. In particular, this book contains the results corresponding to the linear interpolation on the unit cube $Q_n = [0,1]^n$. Later some estimates for concrete $n$ were improved (e. g., see [3, 4]). Interpolation by linear functions on a Euclidean ball in $\mathbb{R}^n$ and related questions were considered in [2, 5, 6].

The present paper supplements the results obtained in [2] for a regular simplex inscribed into a ball. In Section 1, we find the maximal points of the function $\lambda(x) := \sum \lambda_j(x)|$. In any such a point $y \in B$

$$\lambda(y) = \sum_{j=1}^{n+1} |\lambda_j(y)| = \|P\|_B = \max\{\psi(a), \psi(a + 1)\}.$$

The points belong to the boundary sphere of $B$. The number of these points is equal to $N = \left( \begin{array}{c} n + 1 \\ k \end{array} \right)$, where $k$ coincides with that number $a$ or $a + 1$ on which $\psi(t)$ takes a bigger value. Evidently, $N$ is the number of $(k - 1)$-dimensional faces of an $n$-dimensional nondegenerate simplex. In Section 2, we discuss the questions related to the projector's norm invariance under an affine transform. In Section 3, we formulate
some geometric conjecture. The validity of this conjecture implies that the projector corresponding to a regular inscribed simplex has the minimal norm. The conjecture holds true at least for $n = 1, 2, 3, 4$.

## 1. THE MAXIMUM POINTS OF THE FUNCTION $\lambda(x)$ FOR A REGULAR INSCRIBED SIMPLEX

By definition, put

$$k = k(n) := \begin{cases} a + 1, & \text{if } \psi(a + 1) \geq \psi(a), \\ a, & \text{if } \psi(a + 1) < \psi(a). \end{cases}$$

The detailed analysis and first values $n$ and $k$ are given in [2]. For these data and also for the numbers $N = \binom{n+1}{k}$, see Table 1.

For $n = 1, 2, 3$, we have $k = 1$. If $n > 3$, then $\sqrt{n+1} > 2$, hence,

$$a + 1 \left[ \frac{n+1}{2} - \frac{\sqrt{n+1}}{2} \right] + 1 \leq \frac{n+1}{2} - \frac{\sqrt{n+1}}{2} + 1 < \frac{n+1}{2}.$$

Thus, for $n > 3$ holds $k < \frac{n+1}{2}$. Since $k$ is an integer, for all $n \geq 2$ we have $k \leq \frac{n}{2}$.

Suppose $S$ is a regular simplex inscribed into a ball $B$, $\lambda_j$ are the basic Lagrange polynomials of this simplex, and $P : C(B) \to \Pi_1(\mathbb{R}^n)$ is the corresponding interpolation projector.

**Theorem 1.** Consider an arbitrary $(k - 1)$-dimensional face $G$ of the simplex $S$. Let $H$ be the $(n - k)$-dimensional face of $S$ which contains the vertices not belonging to $G$. Denote by $g$ and $h$ the centers of gravity of $G$ and $H$. Assume that $y$ is a point there the straight line $(gh)$ inersects the boundary sphere in direction from $g$ to $h$. Then
\[ \lambda(y) = \sum_{j=1}^{n+1} |\lambda_j(y)| = \|P\|_B. \]  

(6)

**Proof.** It is sufficient to consider some given ball \( B \subset \mathbb{R}^n \), some regular simplex \( S \) inscribed into \( B \), and also \( G = \text{conv} \{x^{(0)}, \ldots, x^{(k)}\} \).

If \( n = 1 \), then \( \psi(t) = \sqrt{2(t-1)} + |t-1|, \ a = 0, \ \psi(a) = \psi(a+1) = 1, \ k = a+1 = 1 \). Let us take \( x^{(1)} = 0, \ x^{(2)} = 1, \ i.e., \ S = B = [0,1] \). In this case, \( g = 0, \ h = 1, \ \lambda_1(x) = -x + 1, \ \lambda_2(x) = x, \ \lambda(x) = 1 \). Since \( \|P\|_B = 1 \), for \( y = h = 1 \) Eq. (6) holds true. Note that in this trivial case a set of maximum points of the function \( \lambda(x) \) coincides with the ball \( B \).

Now, let \( n \geq 2 \). First, note that the center of gravity \( c \) of the simplex \( S \) belongs to the segment \([g,h]\). Indeed, the equalities

\[ c = \frac{1}{n+1} \sum_{j=1}^{n+1} x^{(j)}, \ g = \frac{1}{k} \sum_{j=1}^{k} x^{(j)}, \ h = \frac{1}{n+1-k} \sum_{j=k+1}^{n+1} x^{(j)} \]  

(7)

mean that \((n+1)c = kg + (n+1-k)h\),

\[ c = \frac{k}{n+1} g + \frac{n+1-k}{n+1} h. \]  

(8)

For proving (6), it is sufficient to indicate a linear polynomial \( p \) which takes in the nodes values \( \pm 1 \) and such that \( p(y) = \|P\|_B \). The equalities \( p(x^{(j)}) = \pm 1 \) imply

\[ p(y) = \sum_{j=1}^{n+1} p(x^{(j)}) \lambda_j(y) \leq \lambda(y) = \sum_{j=1}^{n+1} |\lambda_j(y)| \leq \max_{x \in B} \sum_{j=1}^{n+1} |\lambda_j(x)| = \|P\|_B. \]

If \( p(y) = \|P\|_B \), then all the values in this chain coincide; therefore, \( \lambda(y) = \|P\|_B \).

Let us show that the above property is fulfilled for the polynomial \( p \in \Pi_1(\mathbb{R}^n) \) with the values

\[ p(x^{(0)}) = \ldots = p(x^{(k)}) = -1, \ p(x^{(k+1)}) = \ldots = p(x^{(n+1)}) = 1. \]  

(9)

Since \( p \) is a linear function, from (7) and (9) it follows that \( p(g) = -1, \ p(h) = 1 \). Making use of (8), we get

\[ p(c) = \frac{k}{n+1} p(g) + \frac{n+1-k}{n+1} p(h) = \frac{n+1-2k}{n+1}. \]

The center of gravity of a regular inscribed simplex coincides with the center of the ball. The increment of \( p \) is proportional to the distance between the points, hence

\[ \frac{\|g-h\|}{p(g) - p(h)} = \frac{R}{p(y) - p(c)}, \]

where \( R \) is the radius of the ball. Utilizing the found values, we obtain

\[ p(y) = \frac{n+1-2k}{n+1} + \frac{2R}{\|g-h\|}. \]  

(10)

The value of the latter fraction does not depend on choice of a ball and a regular inscribed simplex. Let us calculate this value for the concrete \( S \) and \( B \).

Namely, as \( S \) we take the regular simplex with vertices

\[ x^{(1)} = e_1, \ldots, x^{(n)} = e_n, \ x^{(n+1)} = \left( \frac{1-\sqrt{n+1}}{n}, \ldots, \frac{1-\sqrt{n+1}}{n} \right). \]

The length of any edge of \( S \) is equal to \( \sqrt{2} \). This simplex is inscribed into the ball \( B = B(x^{(0)}; R) \), where

\[ x^{(0)} = \left( \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right), \ R = \frac{n}{\sqrt{n+1}}. \]
In accordance to (7), the coordinates of $g$ and $h$ are
\[ g_1 = \ldots = g_k = \frac{1}{k}, \quad g_{k+1} = \ldots = g_n = 0, \]
\[ h_1 = \ldots = h_k = \frac{1}{n+1-k}, \quad h_{k+1} = \ldots = h_{n+1} = \frac{n+1-\sqrt{n+1}}{n}. \]
From this,
\[ \|g - h\|^2 = \left(\frac{1}{k} - \frac{1}{n(n+1-k)}\right)^2 k + \left(\frac{n+1-\sqrt{n+1}}{n(n+1-k)}\right)^2 (n-k). \]
The simple calculation yeilds
\[ \|g - h\|^2 = \frac{n+1}{k(n+1-k)}. \]
Thus, in this case
\[ \frac{2R}{\|g - h\|} = 2\sqrt{n} \frac{(k(n+1-k))^{\frac{1}{2}}}{n+1} (k(n+1-k))^{\frac{1}{2}}. \]
Continuing (10), we can write
\[ p(y) = \frac{n+1-2k}{n+1} + \frac{2R}{\|g - h\|} = 1 - \frac{2k}{n+1} + \frac{2\sqrt{n}}{n+1} (k(n+1-k))^{\frac{1}{2}}. \]
If $1 \leq k \leq \frac{n+1}{2}$, then the last expression coincides with $\psi(k)$. The noted inequality is true. Moreover, $k$ coincides with that number $a$ or $a+1$ on which $\psi(t)$ takes a bigger value. Therefore, $p(y) = \max\{\psi(a), \psi(a+1)\} = \|P\|_B$.

The proof is complete.

**Theorem 2.** In the denotations of the previous theorem, $[g, h]$ is the segment of maximal length in $S$ parallel to vector $gh$.

**Proof.** In [7], the author obtained the calculation formulae for length and endpoints of the maximal segment in $S$ of a given direction. One can apply these formulas for some simplex and take into account the similarity arguments. But it is much simpler to use the following characterization of the maximal segment proved in [7] (see there Lemmas 1 and 2). A segment in a simplex parallel to a given vector has maximal length iff every $(n-1)$-dimensional face of the simplex contains at least one endpoint of this segment.

Let the notation $x = \{\beta_1, \ldots, \beta_{n+1}\}$ means that a point $x$ has barycentric coordinates $\beta_1, \ldots, \beta_{n+1}$ with respect to $S$. By $G_j$ denote the $(n-1)$-dimensional face of the simplex not containing the $j$th vertex. For points of $G_j$, all barycentric coordinates are nonnegative and $\beta_j = 0$. We have
\[ g = \frac{1}{k} \sum_{j=1}^{k} x^{(j)} = \left\{\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0\right\}, \]
\[ h = \frac{1}{n+1-k} \sum_{j=k+1}^{n+1} x^{(j)} = \left\{0, \ldots, 0, \frac{1}{n+1-k}, \ldots, \frac{1}{n+1-k}\right\}. \]
The number of nonzero barycentric coordinates in these equalities is equal to $k$ and $n+1-k$ respectively. Clearly, $g \in G_{k+1}, \ldots, G_{n+1}$, $h \in G_k, \ldots, G_1$. So, every $(n-1)$-dimensional face of $S$ contains an endpoint of the segment $[g, h]$. Consequently, this segment has maximal length of all the segments of given direction in $S$. Note that this argument is suitable for any simplex and $k = 1, \ldots, n$. 

2. THE PROJECTOR’S NORM INVARIANCE UNDER AN AFFINE TRANSFORM

In 1948, F. John [8] proved that any convex body in \( \mathbb{R}^n \) contains a unique ellipsoid of maximum volume. Also he gave characterization of those convex bodies for which a maximal ellipsoid is the unit Euclidean ball \( B_n \) (in details see, e. g., [9, 10]). John’s theorem implies the analogous statement which characterizes a unique minimum volume ellipsoid containing a given convex body.

We shall consider a minimum volume ellipsoid containing a given nondegenerate simplex. For brevity, such ellipsoid will be called a minimal ellipsoid. Obviously, a minimal ellipsoid of a simplex is circumscribed around this simplex. The center of the ellipsoid coincides with the center of gravity of the simplex. A minimal ellipsoid of a simplex is a Euclidean ball if this simplex is regular. This is equivalent to the well-known fact that the volume of a simplex contained in a ball is maximal iff this simplex is regular and inscribed into the ball (see, e. g., [11–13]).

By definition, put \( \chi_n := \text{vol}(B_n) \). Denote by \( \sigma_n \) the volume of a regular simplex inscribed into the unit ball \( B_n \). Suppose \( S \) is an arbitrary \( n \)-dimensional simplex and \( E \) is the minimal ellipsoid of \( S \). If a nondegenerate affine transform maps \( S \) into a regular simplex inscribed into \( B_n \), then the image of \( E \) under this transform coincides with \( B_n \). Hence,

\[
\frac{\text{vol}(E)}{\text{vol}(S)} = \frac{\chi_n}{\sigma_n}.
\]

It is known that

\[
\chi_n = \frac{n!}{\Gamma\left(n^2 + 1\right)}, \quad \sigma_n = \frac{1}{n} \sqrt{n + 1} \left(n + \frac{1}{n}\right)^{\frac{n}{2}},
\]

\[
\chi_{2m} = \frac{\pi^m}{m!}, \quad \chi_{2m+1} = \frac{2^{m+1} \pi^m}{(2m+1)!!} \frac{2(m!)^m}{(2m+1)!}.
\]

(see, e.g., [1, 14, 15]). Therefore,

\[
\text{vol}(E) = K_n \text{vol}(S), \quad K_n := \frac{\chi_n}{\sigma_n} \frac{n!(\pi n)^{\frac{n}{2}}}{\Gamma\left(n^2 + 1\right) (n + 1)^{\frac{n+1}{2}}}.
\]

In addition,

\[
K_{2m} = \frac{(2m!(2\pi m))^{\frac{m}{2}}}{m!(2m+1)^{\frac{m+1}{2}}}, \quad K_{2m+1} = 2^{m+1} \left(2 - \frac{1}{m+1}\right)^{\frac{m+1}{2} \pi^m} m!.
\]

The value \( K_n \) is included in the lower bound of the norm of a projector with nodes in \( B_n \). Let \( \chi_n(t) \) be the standardized Legendre polynomial of degree \( n \):

\[
\chi_n(t) := \frac{1}{2^n n!} \left[ (t^2 - 1)^n \right]^{\frac{n}{2}}.
\]

There exists a constant \( C > 0 \) not depending on \( n \) such that for any interpolation projector \( P : C(B_n) \to \Pi_1(\mathbb{R}^n) \)

\[
\|P\|_{B_n} \geq \chi_n^{-1}(K_n) > C \sqrt{n}.
\]

Inequalities (11) were obtained by the author in [6]. The right-hand estimate holds true, if we take, e. g.,

\[
C \frac{\sqrt{3\pi}}{\sqrt{12 e^2}} = 0.2135\ldots
\]

Assume \( S \) and \( S' \) are nondegenerate simplices in \( \mathbb{R}^n \) with vertices \( x^{(j)}, \ldots, x^{(n+1)} \) and \( y^{(1)}, \ldots, y^{(n+1)} \) respectively. Let \( S \) be the vertex matrix of \( S \). Denote by \( Y \) the \( n(n+1) \)-matrix whose \( j \)th column contains the coordinates of \( y^{(j)} \). Let \( \lambda_1, \ldots, \lambda_{n+1} \) be the basic Lagrange polynomials of \( S \).
Lemma 1. There exists a unique affine transform $F$ of space $\mathbb{R}^n$ which maps $S$ into $S'$ and such that $y^{(j)} = F(x^{(j)})$. The equality $y = F(x)$ is equivalent to any relation

$$
\begin{bmatrix}
  y_1 \\
  \vdots \\
  y_n \\
\end{bmatrix} = Y(S^{-1})^T \begin{bmatrix}
  x_1 \\
  \vdots \\
  x_n \\
  1 \\
\end{bmatrix},
$$

(12)

$$
y = \sum_{j=1}^{n+1} \lambda_j(x)y^{(j)}. 
$$

Proof. Each nondegenerate affine transform of $\mathbb{R}^n$ has the form $F(x) = A(x) + b$, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a nondegenerate linear operator. Let $A = (a_{ij})$ be the matrix of the operator $A$ in the canonical basis. In coordinate form, the equality $y = A(x) + b$ is equivalent to the relation

$$
\begin{bmatrix}
  y_1 \\
  \vdots \\
  y_n \\
\end{bmatrix} = \begin{bmatrix}
  a_{11} \ldots a_{1n} b_1 \\
  \vdots \\
  a_{n1} \ldots a_{nn} b_n \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  \vdots \\
  x_n \\
  1 \\
\end{bmatrix}.
$$

Define $M$ as the $n(n+1)$-matrix standing on right-hand side. The conditions $y^{(j)} = F(x^{(j)})$ are equivalent to the equality $Y = MS^T$. Consequently,

$$
M = Y(S^T)^{-1} = Y(S^{-1})^T.
$$

This means that an affine transform satisfying the conditions of the theorem is unique and has the form (12).

Since $\lambda_j \in \Pi_1(\mathbb{R}^n)$ and $\lambda_j(x^{(k)}) = \delta^{k,j}$, the Eq. (3) also gives an affine transform $y = F(x)$ such that $F(x^{(k)}) = y^{(k)}$. From the uniqueness of $F$, it follows that (13) is equivalent to (12). This equivalence can be proved also directly. Let us rewrite (13) in the coordinate form using the coefficients of the polynomials $\lambda_j$:

$$
y = \sum_{j=1}^{n+1} \lambda_j(x)y^{(j)} = \sum_{j=1}^{n+1} \left( \sum_{k=1}^{n} l_{kj}x_k + l_{n+1,j} \right)y^{(j)},
$$

$$
y_j = \sum_{j=1}^{n+1} \left( \sum_{k=1}^{n} l_{kj}x_k + l_{n+1,j} \right)y^{(j)} = \sum_{k=1}^{n} \left( \sum_{j=1}^{n+1} l_{kj}y_j^{(j)} \right)x_k + \sum_{j=1}^{n+1} y_j^{(j)}l_{n+1,j}.
$$

Thus, (13) means that

$$
y_j = \sum_{k=1}^{n} a_{ik}x_k + b_i, \quad a_{ik} = \sum_{j=1}^{n+1} y_j^{(j)}l_{kj}, \quad b_i = \sum_{j=1}^{n+1} y_j^{(j)}l_{n+1,j}.
$$

Since $S^{-1} = (l_{ij})$, these equalities are equivalent to (12).

We note that the norm of an interpolation projector is invariant under a nondegenerate affine transform.

Theorem 3. Suppose $\Omega$ is a convex body in $\mathbb{R}^n$ containing a nondegenerate simplex $S$, $\Omega'$ and $S'$ are their images under a nondegenerate affine transform, $P : C(\Omega) \to \Pi_1(\mathbb{R}^n)$ and $P' : C(\Omega') \to \Pi_1(\mathbb{R}^n)$ are interpolation projectors with nodes in vertices of $S$ and $S'$ respectively. Then $\|P\|_{\Omega} = \|P\|_{\Omega'}$.

Proof. Let $x_1, \ldots, x_{n+1}$ be the vertices of simplex $S$. We will assume that the vertices of simplex $S'$ are numerated so that $y^{(j)} = F(x^{(j)})$. Under this condition, the set of barycentric coordinates of an arbitrary point $x \in \mathbb{R}^n$ with respect to $S$ coincides with the set of barycentric coordinates of the point $y = F(x)$ with respect to $S'$. This follows from the equalities

$$
x = \sum_{j=1}^{n+1} \lambda_j(x)x^{(j)}, \quad y = \sum_{j=1}^{n+1} \lambda_j(x)y^{(j)}.
$$
The second equality coincides with relation (13) of Lemma 1. In accordance with formula (2), we have

\[ \| P \|_{\Omega} = \max \left\{ \sum_{j=1}^{n+1} \beta_j \left| \sum_{j=1}^{n+1} \beta_j \lambda^{(j)} \in \Omega, \sum_{j=1}^{n+1} \beta_j = 1 \right| \right\} = \max \left\{ \sum_{j=1}^{n+1} \beta_j \left| \sum_{j=1}^{n+1} \beta_j \lambda^{(j)} \in \Omega', \sum_{j=1}^{n+1} \beta_j = 1 \right| \right\} = \| P' \|_{\Omega'} . \]

**Corollary 1.** Suppose \( S \) is a nondegenerate simplex with minimal ellipsoid \( E \) and \( S' \) is an arbitrary regular simplex inscribed into \( B_n \). If \( P : C(E) \to \Pi_n(\mathbb{R}^n) \) and \( P' : C(B_n) \to \Pi_n(\mathbb{R}^n) \) are the projectors having nodes in the vertices of \( S \) and \( S' \) respectively, then \( \| P \|_E = \| P' \|_{B_n} \).

**Proof.** Consider the nondegenerate affine transform which maps the simplex \( S \) into the regular simplex \( S' \). This transform maps the ellipsoid \( E \) into the ball \( B_n \). It remains to apply Theorem 3 in the case when \( \Omega \) is the minimal ellipsoid of \( S' \).

Let us supplement Corollary 1 with the following remark. Denote here by \( \lambda_j \) the basic Lagrange polynomials of a simplex \( S \). The maximum points of the function \( \lambda(x) = \sum \lambda_j(x) \) lying in the minimal ellipsoid \( E \) have the same geometric description that is formulated in Theorem 1. In the condition of this theorem, we must replace the regular simplex by an arbitrary one, and the circumscribed ball by the minimal ellipsoid of the simplex. At the specified points of the boundary of the ellipsoid, \( \lambda(x) \) takes maximal value equal to \( \| P \|_E \). This result can be established according to the scheme above.

**Corollary 2.** There exists a universal constant \( C > 0 \) such that for every ellipsoid \( E \subset \mathbb{R}^n \) and every interpolation projector having the nodes in \( E \) we have \( \| P \|_E \geq \chi^{-1}(K_n) > C\sqrt{n} \).

This follows immediately from (11) and Corollary 1.

### 3. ON SOME EXTREMAL PROPERTY OF A REGULAR SIMPLEX INSCRIBED INTO A BALL

Consider a nondegenerate simplex \( S \subset \mathbb{R}^n \). Let \( E \) be the minimal ellipsoid containing \( S \). Fix a natural number \( m \leq \frac{n}{2} \). To each set of \( m \) vertices of \( S \) assign the point \( y \in E \) defined as follows. Let \( g \) be the center of gravity of the \((m-1)\)-dimensional face of \( S \) containing the selected vertices, and let \( h \) be the center of gravity of the \((n-m)\)-dimensional face containing the remaining \( n + 1 - m \) vertices. Then \( y \) is the intersection point of the straight line \((gh)\) with the boundary of \( E \) in direction from \( g \) to \( h \).

Now we formulate the following conjecture.

\( (H1) \) For a given \( m \leq \frac{n}{2} \) and any nondegenerate simplex \( S \subset B_n \), there exists a set of \( m \) vertices of \( S \) such that \( y \in B_n \).

A stronger version of the hypothesis asserts that the specified property holds for any \( m \leq \frac{n}{2} \) \((H2)\). For our purposes, it is sufficient that \((H1)\) was true for \( m = k(n) \). The number \( k = k(n) \) is defined in Section 1.

**Theorem 4.** For \( m = 1 \) the conjecture \((H1)\) holds true.

**Proof.** Suppose \( S \) is a simplex with vertices \( x^{(j)} \in B_n \) and the center of gravity \( c \). The center of the minimal ellipsoid containing \( S \) also lies in \( c \). Hence, in the case \( m = 1 \) the points \( y \) has the form \( y^{(j)} = 2c - x^{(j)}, \ j = 1, \ldots, n + 1 \). We need to show that there exists a vertex \( x \) of the simplex such that \( \| 2c - x \| \leq 1 \). Since \( S \) is nondegenerate, for some vertex \( x \) we have \( (c, x - c) \geq 0 \). This means that

\[ \| 2c - x \|^2 = (2c - x, 2c - x) = 4(c, c - x) + \| x \|^2 \leq \| x \|^2 \leq 1, \]

i. e., the vertex \( x \) is suitable. The theorem is proved.
Denote by $\theta_n(B_n)$ the minimal norm of an interpolation projector $P : C(B_n) \to \Pi_1(\mathbb{R}^n)$ with the nodes in $B_n$. By $P'$ denote a projector whose nodes coincide with the vertices of a regular simplex $S'$ inscribed into $B_n$.

**Theorem 5.** Suppose (H1) is true for $m = k(n)$. Then $\theta_n(B_n) = \|P\|_{B_n}$.

**Proof.** Consider an arbitrary projector $P$ with the nodes $x^{(j)} \in B_n$. Let $S$ be the simplex with these vertices and let $\lambda_j$ be the basic Lagrange polynomials corresponding to $S$. Denote by $E$ the minimal ellipsoid of the simplex. Since $S \subset B_n$, for some set consisting of $k(k(n))$ vertices of the simplex the corresponding point $y$ is contained in the ball. Let us fix $y$ and write the following relations:

$$\|P\|_{B_n} = \|P\|_E = \max_{x \in E} \sum_{j=1}^{n+1} |\lambda_j(x)| = \sum_{j=1}^{n+1} |\lambda_j(y)| \leq \max_{x \in E} \sum_{j=1}^{n+1} |\lambda_j(x)| = \|P\|_{B_n}.$$  

We made use of the formula for the projector norm, Theorem 1, Corollary 1, and also the remark after this corollary. The inequality in the above chain follows from the condition $y \in B_n$. Note that if $y$ lies inside the ball, then this equality becomes strict.

Therefore, for any projector with nodes in $B_n$ we have $\|P\|_{B_n} \leq \|P\|_{B_n}$. This implies that $\theta_n(B_n) = \|P\|_{B_n}$. The proof is complete.

**Corollary 3.** If $1 \leq n \leq 4$, then $\theta_n(B_n) = \|P\|_{B_n}$.

**Proof.** In the case $n = 1$, the proposition is equivalent to the fact that the norm of an interpolation projector $P : C([-1, 1]) \to \Pi_1(\mathbb{R})$ becomes minimal for the projector having the nodes at the endpoints of the segment $[-1,1]$. If $2 \leq n \leq 4$, then $k(n) = 1$, and the required result follows immediately from Theorems 4 and 5.

Corollary 3 was proved in [2] by another method suitable only for dimensions $n$ with the property $k(n) = 1$. However, starting from $n = 5$, we have $k(n) > 1$ (see [2]). Nevertheless, the equality $\theta_n(B_n) = \|P\|_{B_n}$ still can be obtained on the way directed by Theorem 5.

In the propositions of this section, the unit ball $B_n$ may be replaced by an arbitrary Euclidean ball $B$; this leads to the equivalent results.

**CONFLICT OF INTEREST**

The author declares that he has no conflicts of interest.

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