Phase diagram of the $Z_3$ Parafermionic Chain with Chiral Interactions

Ye Zhuang, Hitesh J. Changlani, Norm M. Tubman, Taylor L. Hughes

$^1$Department of Physics and Institute of Condensed Matter Theory, University of Illinois, Urbana, Illinois 61801, USA

(Dated: March 2, 2015)

Parafermions are exotic quasiparticles with non-Abelian fractional statistics that can be realized and stabilized in 1-dimensional models that are generalizations of the Kitaev p-wave wire. We study the simplest generalization, i.e. the $Z_3$ parafermionic chain. Using a Jordan-Wigner transform we focus on the equivalent three-state chiral clock model, and study its rich phase diagram using the density matrix renormalization group technique. We perform our analyses using quantum entanglement diagnostics which allow us to determine phase boundaries, and data about the nature of the phase transitions. In particular, we study the transition between the topological and trivial phases, as well as to an intervening incommensurate phase which appears in a wide region of the phase diagram. The phase diagram is predicted to contain a Lifshitz type transition which we confirm using entanglement measures. We also attempt to locate and characterize a putative tricritical point in the phase diagram where the three above mentioned phases meet at a single point.

Introduction— There has been concerted effort to engineer systems with stable Majorana bound states, and other anyonic quasiparticles, for use in the topological quantum computing architecture [1–7]. For example, there has been recent progress in attempts to isolate Majorana bound states in quantum nanowires [5, 8–10] and in superconductor surfaces implanted with a line of magnetic impurities [11]. These quasi-1D systems effectively realize a version of the Kitaev p-wave wire model [12], and are predicted to have a gapped topological phase which supports characteristic Majorana bound states at the ends of the wire.

While the boundary modes in these heterostructure systems are non-Abelian anyons, they are unfortunately known to be insufficient for universal quantum computation. A possible remedy for this problem has been to look for more exotic non-Abelian excitations. For example, Fendley has recently suggested exploring one-dimensional $Z_N$ para-fermionic models which support topological phases with more computationally efficient non-Abelian anyon bound states [13]. Still, the $Z_N$ non-Abelian anyons are not able to perform universal quantum computation, however they can be leveraged to create a 2D phase with Fibonacci anyons, which are universal [14]. These promising features have spurred wide spread interest in these models, and has led to many analytical and numerical studies, including several experimental proposals for realizing these topological phases [15–39].

In this work, we continue along these lines of research by exploring the rich phase diagram of the $Z_3$ para-fermionic chain; though for ease of calculation we actually study the Jordan-Wigner transformed para-fermionic chain[40], including chiral interactions. The resulting model is the three state chiral clock model. This model re-surfaced in this context in Ref. 13 as a candidate for exhibiting non-Abelian bound states beyond Majorana fermions. It was shown analytically that para-fermionic boundary zero modes can exist in this model when spatial-parity and time-reversal symmetries are broken via chiral interactions [13]. This was verified numerically in Ref. 41, which confirms that chiral interactions can help to stabilize the boundary zero modes, although the zero modes themselves are more fragile than one might initially expect.

Here we are interested in studying the full phase diagram of the chiral clock model as a function of two chiral-interaction phase-parameters $(\theta, \phi)$, as well as the relative strength of the nearest neighbor coupling $(J)$ to the local Zeeman field $(f)$. Using entanglement techniques, we have been able to locate the phase boundaries that separate the topological phase from the trivial gapped phase, and a critical incommensurate phase, the latter of which has no analog in the Kitaev p-wave wire model. We have conclusively identified the region in which there is a topological phase, and have explored the nature of the quantum phase transitions in and out of the three adjoining phases. In addition, by studying oscillatory properties of the system in, or near, the incommensurate phase, we establish the approximate location of a putative tricritical point[42, 43], and further support the entanglement signatures that were recently proposed for identifying Lifshitz transitions[44].

The article is arranged as follows. We first discuss the details of the model, and the criteria used to map out its phase diagram. For our numerical simulations, the density matrix renormalization group (DMRG) [45, 46] algorithm is employed, as it gives immediate access to the entanglement entropy (EE), and therefore the central charge, at putative critical points/regions in the phase diagram [47]. Next, we discuss the general features of the phase diagram and locate regions in the topological phase (where para-fermion boundary modes may exist). We also discuss the nature of the phase transitions out of the topological phase. For part of our study we discuss our observations pertaining to a critical incommensurate phase, and the possibility of a tricritical point [42, 43] in the phase diagram at the intersection of the topological, trivial, and incommensurate phases. We also find a region of the phase diagram which exhibits the critical entanglement features of a Lifshitz transition [44]. Finally, we conclude by discussing future directions and possible relevance to experiments looking for para-fermions. We also include two appendices which discuss some subtleties of the numerical analysis.

The Model— For our study we use the 1D 3-state ($Z_3$) chiral clock model [13, 42, 48, 49]. This model is related to the para-fermionic chain through a Jordan-Wigner transformation [40], similar to the well-known, analogous case that the Kitaev p-
wave wire is related to the transverse-field Ising model via the same type of transformation. The Hamiltonian for the 3-state chiral clock model is:

\[
H_3 = -f \sum_{j=1}^{L} \tau_j^+ e^{-i\phi} - J \sum_{j=1}^{L-1} \sigma_j^+ \sigma_{j+1} e^{-i\theta} + \text{h.c.} \tag{1}
\]

following the notation in previous work [13], where \( f, J, \theta \) and \( \phi \) are scalar parameters, and \( \sigma_j \) and \( \tau_j \) are local three state spin operators on site \( i \). The spin operators have the properties \( \tau^3 = \sigma^3 = 1, \sigma \tau = \omega \tau \sigma, \) where \( \omega = e^{2\pi i/3} \). Specifically, we use the matrix representation

\[
\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

This model has a global \( Z_3 \) symmetry that can be represented with \( \chi \equiv \prod_{j=1}^{L} \tau_j \equiv e^{2\pi i j Z} \). Here \( Z \) is the generator of the symmetry, and has three different eigenvalues 0,1,2. In addition, when all of the coefficients in the Hamiltonian are real, then the Hamiltonian is invariant under time-reversal, charge-conjugation, and parity symmetries, i.e. when the system is a \( Z_3 \)-ferromagnet or \( Z_3 \)-anti-ferromagnet Hamiltonian. This can be easily seen from the following definitions of these symmetries. Charge conjugation \( C \) acts on the spin operators via \( C \sigma_j C = \sigma_j^\dagger, C \tau_j C = \tau_j^\dagger, C^2 = 1 \). As an aside, note that charge conjugation, together with the \( Z_3 \) symmetry, forms the \( S_3 \) permutation symmetry, i.e. the symmetry obeyed when the 3-state clock model is restricted to the 3-state Potts model. Time reversal \( T \) acts on the spin operators via \( T \sigma_j T = \sigma_j, T \tau_j T = \tau_j^\dagger, T^2 = 1 \), and complex conjugates any scalar coefficients. Spatial parity \( P \) acts on the spin operators via \( P \sigma_j P = \sigma_{\bar{j}}, P \tau_j P = \tau_{\bar{j}}, P^2 = 1 \). Finally, we note two things: (i) due to the symmetry of the Hamiltonian with respect to \( \phi \) and \( \theta \), we only need to consider the region of the phase diagram where \( \phi / \theta \) each range from 0 to \( \frac{\pi}{3} \), and (ii) for \( f = J \), the system is self-dual along the line \( \phi = \theta \).

There are many previously known results about this model (Eq. 1), beginning with the original proposals of Ostlund [42] and Huse [48]. For example, the corresponding two-dimensional classical Hamiltonian for \( \phi = 0 \) was studied in Ref. 42, and the the one-dimensional quantum Hamiltonian was studied in Ref. 43 for the restricted case \( \phi = \theta \). One of the most important early results is that Eq. (1) has a second order quantum phase transition at \( f = J \) when \( \theta = 0 \). At this point the model realizes the full \( S_3 \) permutation symmetry (instead of just \( Z_3 \)), and the critical point is described by the critical conformal field theory for the 3-state Potts model, which has central charge 4/5 [50].

Generically, it is known that the phase diagram is divided up into two gapped regions, one of which is identified with small values of \( f \) (compared with \( J \)), and the other with large values of \( f \). These regions are separated by continuous quantum phase transitions that we will identify and discuss further below. Using a more modern terminology, the gapped phase for small \( f \) is a symmetry broken phase of the 3-state clock model and it exactly corresponds to the “topological” phase in the Jordan-Wigner transformed para-fermionic chain. The gapped phase for large \( f \) is a disordered phase of the 3-state clock model, and maps onto the “trivial” phase of the para-fermionic chain. This gives another example of a case where the degeneracy associated to symmetry breaking is mapped to topological degeneracy via the Jordan-Wigner transformation [51, 52]. Hence, in either representation this phase has a three-fold ground-state degeneracy, which can be detected by measuring the ground state EE. On the other hand, the trivial phase is equivalent to the spin disordered phase, which does not have a generic ground-state degeneracy. The parameter \( f \) is thus an important tuning parameter for the phase diagram, and analogous to the external transverse field in the Ising model.

While we expect these general features to pervade the phase diagram, the phase space for generic \( \theta \) and \( \phi \) is largely unexplored. Additionally, it is known that the combination of the \( Z_3 \) symmetry and the chiral nature of the interactions, gives rise to interesting behavior that cannot be found in the Majorana/Ising case. For example, this model supports a so-called “incommensurate phase” which is not present in the transverse-field Ising model with chiral interactions [42].

This motivates the main objective of our article, which is to characterize the phases and the nature of the phase transitions over the entire phase space. We will show that there are two types of phase transitions that occur to destabilize the topological phase, and there is a large region of critical incommensurate phase that separates the topological from the trivial phase over a wide range of parameters. Let us now move on to a discussion of the methods we employ.

Methods—We primarily use the spatial EE in order to characterize the phase diagram. This measure has been widely used to detect topological order in 2D [53, 54], and has been applied more recently to 1D topological phases [55]. The EE can be derived by partitioning the system into two regions \( A \) and \( B \), and then calculating the reduced density matrix of region \( A \) by tracing over all the degrees of freedom in region \( B \). Mathematically, the reduced density matrix is given by \( \rho_A \equiv \text{Tr}_B \rho \), and the corresponding entanglement entropy is defined to be:

\[
S \equiv -\text{Tr}(\rho_A \ln \rho_A) \tag{2}
\]

There are two useful entanglement indicators we will employ to identify the phases and phase transitions for the chiral clock model. First, for the gapped regions of the phase diagram, it is known that for one dimensional gapped systems the entanglement entropy increases with the the block size \( l \) (the size of region \( A \)), and saturates when \( l \) reaches the correlation length [47]. Furthermore, if there is topological ground-state degeneracy we would expect an entanglement of order \( \sim \log D \) where \( D \) is the degeneracy [55]. To eliminate the most harmful finite-size effects we will take the central-cut, i.e. cutting the chain in half, to identify the nature of the gapped phases.
For critical regions of the phase diagram, it is known that the entanglement entropy will grow logarithmically with system size, and the scaling is characterized by the central charge [47]. More specifically, for critical systems with open boundary conditions, the form of the entanglement scaling law is [47]:

\[ S = \frac{c}{6} \ln \left( \frac{L}{\pi} \sin \frac{\pi l}{L} \right) + S_0 \]  

(3)

where \( l \) the length of the subsystem, \( c \) is the central charge, and \( S_0 \) contains the sub-leading corrections. Once we know the central charge we will have an important piece of information about the phase transition/critical phase, and can then appeal to previously known analytic results in restricted parts of the phase diagram to help further specify the phase diagram. Below we will see the efficacy of these two indicators for determining the phase diagram.

To arrive at our results for the phase diagram (and to obtain reasonable estimates of the phase boundaries in the thermodynamic limit), we simulated Hamiltonians using open-boundary DMRG with 100 sites, and a bond dimension \( m = 100 \). We find this to be sufficient for the phases with low entanglement entropy. For the critical phases, additional checks were performed with bond dimension \( m = 200 \). For establishing characteristics of other phases, for example, the region of critical incommensurate phase, larger lengths of 400 sites were also tested.

They share a common/direct phase boundary between them when \( \theta \) and \( \phi \) are small. For large \( \theta \) or \( \phi \), an intermediate incommensurate phase appears between the two.

We show the central-cut EE in Fig. 2(a), (b), (c) for several 2D cross-sections of the 3D phase diagram. These plots help to identify the gapped phases and the topology of the phase boundaries. To more clearly identify the nature of the critical regions/boundaries we also calculate the central charge via the scaling relation. It is interesting to see that the observed locations of the phase boundaries for cross sections \( \phi = 0 \) and \( \theta = \phi \) are broadly consistent with earlier works [42, 43], and that the topological phase itself is stable over a large part of the phase diagram [57].

We indicate several special points on these cross sections: Point A in Fig. 2(a) and Fig. 2(c) is the transition point of the three-state Potts model associated with \( c = 4/5 \) [50], and Point B and C are putative tricritical points. We indicate approximate locations of the phase boundaries with solid, dashed, or dot-dashed lines, depending on the nature of the phase transition, as indicated in the figure caption.

From the central-cut EE we see that the trivial phase is characterized by a small EE, while the topological phase has a nearly uniform EE of \( \approx \ln 3 \) indicating a three-fold degeneracy of the ground state. The change of EE is abrupt between the two phases as can clearly been seen in Fig. 2(a) and Fig. 2(c) for \( \theta \approx \pi/4 \) and \( \theta \approx \pi/6 \) respectively. We also verified that this transition is accompanied by a divergence in the second order derivative of the ground state energy (not shown).

The third phase in the phase diagram is the incommensurate phase. This is a critical phase in which the correlation functions generically behave as \( A(r) e^{i(2\pi r/3)Qr} \), where \( A \) decays algebraically and \( Q \) is irrational. The oscillatory properties of the correlation functions also manifest themselves in oscillatory behavior seen in energy gaps, which we address later. Although there is not an extremely sharp distinction between the central-cut EE for the topological and incommensurate phases, the EE scaling with system size is markedly different. The former has an EE that quickly saturates to a constant value of \( \ln 3 \) with sub-system size, while the latter has EE that diverges logarithmically with sub-system size. By fitting our data to Eq. (3), we establish that the incommensurate phase is critical and its central charge is \( c = 1 \) over the entire phase.

While constructing the detailed phase diagram cross sections, we found that while it was easy to approximate the locations of the phase boundaries, we often encountered difficulties in precisely nailing down the central charge of the corresponding critical points. As an example, we note the appearance of a few points with (apparently) high central charge, indicated by red color, on the direct topological-trivial phase boundary in Fig. 2(d). While in some cases there may be real physics associated to this behavior, we show in Appendix B that a primary source for these spurious effects is fitting to a region of the phase diagram that is just slightly off-criticality. We show that the central charge is very sensitive to the precise location of the critical point, and can easily give \( O(1) \) errors even when only slightly tuned away from criticality, and even

![Diagram](image-url)
Figure 2. Three cross-sections corresponding to (a) $\phi = 0$ (b) $\phi = \pi/3$ and (c) $\phi = \theta$ of the three dimensional phase diagram, and all for $L = 100$. Topological, trivial, and incommensurate (IC) phases are identified by the central-cut entanglement entropy (color coded). For (a) and (b) a 2D grid in increments of 0.01 was used to resolve finite features of the transitions. (c) was mapped out on a 2D grid in increments of 0.05. Point A is the transition point of the 3-state Potts model, i.e. the chiral clock model for ($\theta = \phi = 0$). Points B and C are Lifshitz points and are associated with putative tricritical behavior. The solid lines, dashed lines, and dotted-dashed lines indicate direct topological-trivial ($c = 4/5$) type, Kosterlitz-Thouless type, and Pokrovskii-Talapov type [56] transitions respectively. The thick circularly-dotted line represents an upper bound on the region where exact parafermionic zero modes can exist [13]. Panels (d), (e) and (f) show the corresponding central charges for cross sections (a), (b), (c) respectively. The IC phase is associated with central charge $c = 1$ (yellow) whereas the critical regions close to point A have $c = 4/5$ (green).

with reasonably large-size calculations.

Additionally, although most phase boundaries were easily identified, there are three regions where difficulties arise: (i) the trivial-incommensurate phase transition at $\phi = 0$ and large $\theta$ (lower-right corner of Fig. 2(d)), (ii) the topological-incommensurate phase transition at $\phi = \pi/3$ and small $\theta$ (upper-left corner of Fig. 2(e)), and (iii) the Lifshitz transition area for $f = 0.5$ and $\phi = \theta \sim \pi/6$ as seen in Fig. 2(f). Regions (i) and (ii) are related by duality, and the explanation of the numerical difficulties in these regions may have a common origin. To explain, we recall that the trivial-incommensurate phase transition at $\phi = 0$ and large $\theta$, i.e. region (i), is of the Kosterlitz-Thouless type [42]. Hence, the correlation length decays as $\exp\left(c(T - T_{KT})^{-1/2}\right)$ away from the transition point [58, 59], and this results in a long correlation length (compared to our system size $L = 100$) for this this region of the phase diagram. The duality indicates that region (ii) may also be near a Kosterlitz-Thouless phase transition point. Thus, we attribute the issues with these regions as likely artifacts due to finite size effects. We elaborate further on this in Appendix A. The remaining region (iii) requires more discussion, to which we now turn.

Lifshitz behavior—Let us now focus on the cross-section in Figs. 2(c), 2(f), which corresponds to $\phi = \theta$. Since the system is self-dual on the line $f = J$, the trivial-topological phase boundary should just be the line $f = J = 0.5$, a fact verified in our numerical calculations when $\theta = \phi$ are small. On top of the phase diagram we also plot the function $f = [2 \sin(3\phi)][1 + 2 \sin(3\phi)]^{-1}$, (in a thick circular dotted line), which represents an upper bound on the region in which exact parafermionic zero modes are expected to exist as proven in Ref. 13. The region of the phase diagram above this curve are guaranteed to not have exact parafermionic zero modes, despite still being in the topological phase with the topological ground state degeneracy. Along the critical line $f = J = 0.5$, $c = 4/5$ at the ferromagnetic point ($\phi = \theta = 0$), and $c = 1$ at the antiferromagnetic point ($\phi = \theta = \pi/3$) [50]. It is a priori unclear how the central charge transitions from $c = 4/5$ to $c = 1$, i.e. is an abrupt jump at some transition point or does it change incrementally in stages, or perhaps something else entirely? Only a few studies address this question directly: among them is the work of Howes et al. [43] who
used fermion analyses and series expansions to conjecture that a tricritical point connecting the ordered (topological), disordered (trivial), and incommensurate phases exists at exactly $\phi = \theta = \pi/6$. Our results here suggest a modified picture which we develop and present below.

To address the questions posed above, we studied the critical line $f = J$ carefully. We observed (see Fig 3(a)) that before we reach the putative tricritical (Lifshitz) point at $\phi = \theta = \pi/6$, the EE starts to show oscillatory behavior [60]. The frequency of the oscillations increases as we approach the Lifshitz point from small $\phi = \theta$, and when further increasing $\phi = \theta$ its amplitude dies out after the system clearly enters the incommensurate phase. Conventionally, a Lifshitz transition point of this nature corresponds to a continuously varying oscillation length, and in this case it is the length scale associated with the incommensurate order. Interestingly, the shapes of the EE oscillation curves match those observed recently in 1D free, and interacting, fermion systems near Lifshitz points where the Fermi surface is augmented by additional Fermi points [44]. Thus, our result adds to the evidence of Ref. 44 that these types of EE oscillations are a fingerprint of the Lifshitz-type phase transition. As an aside, we mention that the Lifshitz oscillations are only present in the EE when one uses open boundary conditions. One can easily check this by calculating the EE for free fermions as a function of next-nearest neighbor hopping[44], but with periodic boundary conditions.

To quantitatively study the nature of this critical regime, we need to calculate the variation of the central charge. However, in the presence of oscillations in the EE, we must modify Eq. (3) if we wish to extract the central charge. Empirically, the observed oscillations appear to have a similar form to those in the work Ref. 61, and we propose a phenomenological scaling form which can fit the EE with oscillations:

$$S(l)_{cor} = \frac{c}{6} \ln \left( \frac{L \sin \pi l}{\pi L} \right) + S_0 + \frac{\cos(2\pi l/\xi + \phi)}{(L/2 - |L/2 - l|)^w},$$

where the first two terms are the same as in Eq. (3), and the third term incorporates oscillations and a symmetrized damping function. The parameter $\xi$ is the oscillation length and $p$ is a phase factor. These parameters, along with the exponent $w$, are free-parameters determined by fitting. Some representative fits are shown in Figs. 4(a) and 4(b), which clearly capture the sub-leading oscillations accurately.

The results of calculating the central charge from this procedure are shown as a function of $\phi$ in Fig. 4(c). One can see that there is still an unaccounted for effect that leads to a peak in the central charge at a system-size dependent $\phi$ value. More careful inspection reveals that the peak is located at a $\phi^*$ that corresponds to an oscillation length $\xi \approx L/2$. Thus, as seen in the figure, the peak location $\phi^*$, occurs at values closer and closer to $\phi = \theta = 0$ when system size is increased, and all other parameters remain fixed. Our observations indicate that the central charge converges to $c \approx 1$ when $\phi \geq \phi^*$, and $c \approx 4/5$ for $\phi < \phi^*$. This strongly suggests that the transition from $c = 4/5$ to $c = 1$ along the line $f = J = 0.5$ is an abrupt one that occurs at $\phi = \theta = \phi^*$. From our numerical data it appears that $\phi^* \rightarrow 0$ as $L \rightarrow \infty$. Hence, our data supports a scenario where there is an immediate onset of oscillations as one tunes away from $\phi = \theta = 0$ in the thermodynamic limit.

We corroborate this by observing that oscillations are not seen in the EE if the oscillation length itself exceeds the system size $L$. For example, for $L = 200$, the oscillations are not explicitly visible for $\phi \lesssim \pi/12$, however upon increasing the system size, with all other parameters fixed, the oscillations appear over a larger region of $\phi$, as is shown in Fig. 4(a). As $\phi$ is decreased the correlation length increases, and thus we must use larger and larger systems to observe the oscillations. Thus, we believe that this is evidence that, in the thermodynamic limit, the oscillations are a feature for all $\theta = \phi$ except $\theta = \phi = 0$. An alternate scenario, which we can not rule out completely based on this numerical data, is that the incommensurate phase persists to small but non-zero values of $\theta = \phi$. Thus a conservative estimate of the location of the tricritical point is $0 \leq (\theta = \phi) < 0.25$, which is well below the previously conjectured location of $\theta = \phi = \pi/6$. We
Central Charge

Figure 4. Panels (a) and (b) show the profile of the entanglement entropy (as a function of block size) for various values of system size at $\phi = \theta = 0.25$ and $\phi = \theta = 0.35$ respectively. Panel (c) shows the central charge obtained by fitting the entanglement entropy with the corrected formula along the line $\phi = \theta$ and $f = J = 0.5$. The two dashed lines are at $c = 0.8$ and $c = 1$. The arrow indicates the trend of the peak when $L$ is increased.

We aim to shed further light on this transition through larger scale simulations in future work.

Finally, we note that matching oscillations are observed in the splitting of the lowest two energy states (Fig. 3(b)), as a function of system size. We can extract the characteristic length scale $\xi$ of the oscillations from both the EE (for a given system length), and the energy gap (as a function of system length). Our results are shown in Fig. 3(c) where a clear correlation between the two is observed for $\phi = \theta$ and $f = J = 0.5$. The two dashed lines are at $c = 0.8$ and $c = 1$. The arrow indicates the trend of the peak when $L$ is increased.

Conclusions— In summary, we have mapped out the three dimensional phase diagram of the $Z_3$ chiral clock model using the density matrix renormalization group method. Using the entanglement entropy (of the half-chain) as a diagnostic, we have been able to locate the phase boundaries of the various topological-trivial-incommensurate phase transitions. Quantitatively, we have also been able to see the variation of the central charge along the various critical surfaces that divide these phases. Another outcome of this study is the identification of the Lifshitz transition using the entanglement entropy, along with an estimate of the location of the putative tricritical point. We discussed several competing qualitative scenarios for the cross section of the phase diagram in which the tricritical point has been predicted to exist. Our data suggests that the tricritical point (along $f = J = 1/2$) is not at $\phi = \theta = \pi/6$; rather we find it to be shifted to a much smaller value in the range $0 \leq \theta = \phi < 0.25$.

Finally, our results must be viewed in a broader context as providing further confirmation of the stability of the parafermionic topological phase to chiral interactions, over a wide range of parameters. We expect a further study of this and related models to elucidate the conditions under which these phases can be practically realized.

Acknowledgement— We thank P. Fendley and G. Ortiz for discussions. HJC was supported by SciDac grant DOE FG02-12ER46875. NMT was supported by DOE DE-NA0001789. Computer time was provided by XSEDE, supported by the National Science Foundation Grant No. OCI-1053575, the Oak Ridge Leadership Computing Facility at the Oak Ridge National Laboratory, which is supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC05-00OR22725 and Taub campus cluster at UIUC/NCSA. TLH is supported by the US National Science Foundation under grant DMR 1351895-CAR.

Appendix A: Kosterlitz Thouless transition in the 3-state chiral clock model phase diagram

In the main text, we studied several 2D cross sections of the 3D ($f, J = 1 - f, \theta, \phi$) phase diagram of the chiral Potts model. The 2D cross sections corresponding to $\phi = 0$ (see Figs. 2(a) and 2(d)) and $\phi = \pi/3$ (see Figs. 2(b) and 2(e)) showed some regions whose phase boundaries could not be located. This was attributed to finite size errors, which we now address.

We first discuss the features seen in Figs. 2(a) and 2(d), i.e. the cross section for $\phi = 0$. For small $f$ and large $\theta$, the phase transition between the topological and trivial phase is indirect: it is mediated by the incommensurate phase. To establish the fact that the incommensurate region is of non-zero extent, we performed finite size analyses on both the entanglement entropy and central charge as is shown in Figs 5(a) and 5(b). This extent is found to be from $f \approx 0.07$ to $f \approx 0.15$. We find that the central charge of the trivial-incommensurate transition is consistent with that of the Kosterlitz-Thouless (KT) type [42].

Because of the duality in the Hamiltonian (Eq. 1), the phase diagram is symmetric with respect to the line $f = J = 0.5$, $\phi = \theta$. Thus, the above mentioned phase transition is dual to
Figure 5. Panel (a) shows the entanglement entropy (for the central cut), as a function of $f$ for $\theta = 0$ (b) shows the corresponding central charge calculated for various system sizes in the incommensurate-topological phase transition, for large $\phi$ and small $\theta$. That is to say, the region with the smooth change of the central charge in the lower-right corner of Fig.2(d) is dual to the (red) region in the upper-left corner of Fig.2(e). This region, being near the KT phase transition point is also plagued by finite size errors: the correlation length is long compared with the system size ($L = 100$).

To test this assertion, we studied the (apparently) large central charge that was calculated near the critical region, as shown in Fig. 6. For example, as is shown in Fig. 6(a), the point $\phi = \pi/3$, $\theta = 0$, and $f = 0.8$ appears to be critical, but for larger system sizes is shown to be gapped. We base this conclusion on the appearance of a saturation plateau in the profile of the EE scaling as a function of subsystem size. As a comparative check, we went deeper into the critical regime (i.e. $f = 0.9$). As can be seen in Fig.6(b) and as is expected, we found no such plateau in the EE.

**Appendix B: Extracting Central Charge Near Critical Points**

While constructing the cross sections 2(d), we found that it was easy to approximate the location of the critical line. However, we often encountered difficulties in precisely nailing down the central charge of corresponding critical point. To illustrate this issue, we note the appearance of a few points with (apparently) high central charge, indicated by red color, on the topological-trivial phase boundary. We will show below that this is a spurious effect of fitting to a region that is slightly off-criticality.

When performing the fit to EE data obtained from a finite size system, and for a point in parameter space that is close to (but not at) a critical point, it is often difficult to obtain a reasonable estimate of the central charge. One possible explanation is that when the system size is smaller than the correlation length, the fit to Eq. (3) may be good, but the central charge obtained from the fit may not match central charge of the nearby critical point. This is not unique to our model, and we were also able to observe this effect for free Dirac fermions.
with a tunable mass term. Eventually, when off criticality, and when the system size is larger than the correlation length, the EE saturates, revealing the gapped phase.

To provide an example of such behavior, we refer to known analytic results that the central charge should be $4/5$ at $(f = J = 0.5, \phi = \theta = 0)$, and zero for all other $f$ at $\phi = \theta = 0$. In Fig. 7, we show that at the critical point $f = J = 0.5$, the central charge is $c = 0.81 \pm 0.01$, close to the analytical result. However, when we are slightly away from this point, say $f = 0.499$, the system still appears critical with an (apparent) central charge of $c = 1.58$, much larger than the expected value of 0.80. On going slightly further away, $f = 0.495$, a plateau in the EE profile is seen consistent with our expectation of a gapped phase. Thus, the fitting procedure produces misleading results small distances away from a critical point, and makes it difficult to determine the central charge for critical points in which the position of the point is not known to extremely high accuracy.

Figure 7. The EE as a function of the subsystem size $l$ at $\phi = \theta = 0$ and several different $f$ close to or at the critical point ($f = 0.50$). From the highest curve to the lowest one, the corresponding $f$ is $0.495, 0.499, 0.500$, and $0.501$. The central charges obtained from the fitting are shown in the legend. For $f = 0.495$ and $f = 0.501$, a plateau in the EE is seen indicating a gapped phase. For $f = 0.499$, an apparent critical phase is seen which is attributed to an artifact of finite size effects.

[1] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[2] L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
[3] L. Fu and C. L. Kane, Phys. Rev. Lett. 102, 216403 (2009).
[4] R. M. Lutchyn, J. D. Sau, and S. D. Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[5] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
[6] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. Fisher, Nature Physics 7, 412 (2011).
[7] J. Alicea, Rep. Prog. Phys. 75, 076501 (2012).
[8] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
[9] L. P. Rokhinson, X. Liu, and J. K. Furdyna, Nature Physics 8, 795 (2012).
[10] A. Finck, D. Van Harlingen, P. Mohseni, K. Jung, and X. Li, Phys. Rev. Lett. 110, 126406 (2013).
[11] S. Nadji-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Science 346, 602 (2014).
[12] A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001).
[13] P. Fendley, Journal of Statistical Mechanics: Theory and Experiment 2012, P11020 (2012).
[14] R. S. K. Mong, D. J. Clarke, J. Alicea, N. H. Lindner, P. Fendley, C. Nayak, Y. Oreg, A. Stern, E. Berg, K. Shtengel, and M. P. A. Fisher, Phys. Rev. X 4, 011036 (2014).
[15] G. Ortiz, E. Cobanera, and Z. Nussinov, Nuclear Physics B 854, 780 (2012).
[16] N. H. Lindner, E. Berg, G. Refael, and A. Stern, Phys. Rev. X 2, 041002 (2012).
[17] M. Cheng, Phys. Rev. B 86, 195126 (2012).
[18] D. J. Clarke, J. Alicea, and K. Shtengel, Nature communications 4, 1348 (2013).
[19] A. Vaezi, Phys. Rev. B 87, 035132 (2013).
[20] A. Rapp, P. Schmitteckert, G. Takacs, and G. Zaran, New Journal of Physics 15, 013058 (2013).
[21] M. B. Hastings, C. Nayak, and Z. Wang, Phys. Rev. B 87, 165421 (2013).
[22] M. Burrello, B. van Heck, and E. Cobanera, Phys. Rev. B 87, 195422 (2013).
[23] R. Bondesan and T. Quella, Journal of Statistical Mechanics: Theory and Experiment 2013, P10024 (2013).
[24] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B 88, 235103 (2013).
[25] M. Barkeshli, Y. Oreg, and X.-L. Qi, arXiv preprint arXiv:1401.3750 (2014).
[26] E. Cobanera and G. Ortiz, Phys. Rev. A 89, 012328 (2014).
[27] J. C. Y. Teo and C. L. Kane, Phys. Rev. B 89, 085101 (2014).
[28] Y. Oreg, E. Sela, and A. Stern, Phys. Rev. B 89, 115402 (2014).
[29] C. P. Orth, R. P. Tiwari, T. Meng, and T. L. Schmidt, arXiv preprint arXiv:1405.4353 (2014).
[30] J. Klinovaja and D. Loss, Phys. Rev. Lett. 112, 246403 (2014).
[31] F. Zhang and C. Kane, Phys. Rev. Lett. 113, 036401 (2014).
[32] W. Li, S. Yang, H. Tu, and M. Cheng, arXiv preprint, arXiv:1407.3790 (2014).
[33] A. Tsvilik, arXiv preprint arXiv:1407.4002 (2014).
[34] J. Klinovaja and D. Loss, Phys. Rev. B 90, 045118 (2014).
[35] J. Klinovaja, A. Yacoby, and D. Loss, Phys. Rev. B 90, 155447 (2014).
[36] R. S. Mong, D. J. Clarke, J. Alicea, N. H. Lindner, and P. Fendley, Journal of Physics A: Mathematical and Theoretical 47, 452001 (2014).
[37] J. Alicea and A. Stern, arXiv e-prints (2014), arXiv:1410.0359 [cond-mat.str-el].
[38] A. Milsted, E. Cobanera, M. Burrello, and G. Ortiz, Phys. Rev. B 90, 195101 (2014).
[39] M. Dalmonte, J. Carrasquilla, L. Taddia, E. Ercolessi, and M. Rigol, ArXiv e-prints (2014), arXiv:1412.5624 [cond-mat.str-el].
[40] E. Fradkin and L. P. Kadanoff, Nuclear Physics B 170, 1 (1980).
[41] A. S. Jermyn, R. S. K. Mong, J. Alicea, and P. Fendley, Phys. Rev. B 90, 165106 (2014).
[42] S. Ostlund, Phys. Rev. B 24, 398 (1981).
[43] S. Howes, L. P. Kadanoff, and M. D. Nijs, Nuclear Physics B 215, 169 (1983).
[44] M. Rodney, H. F. Song, S.-S. Lee, K. Le Hur, and E. S. Sørensen, Phys. Rev. B 87, 115132 (2013).
[45] S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
[46] U. Schollwöck, Annals of Physics 326, 96 (2011), January 2011 Special Issue.
[47] P. Calabrese and J. Cardy, Journal of Statistical Mechanics: Theory and Experiment 2004, P06002 (2004).
[48] D. A. Huse, Phys. Rev. B 24, 5180 (1981).
[49] H.-C. Jiang, Z. Wang, and L. Balents, Nature Physics 8, 902 (2012).
[50] P. H. Ginsparg, arXiv preprint hep-th/9108028 63 (1988).
[51] H.-D. Chen and J. Hu, Phys. Rev. B 76, 193101 (2007).
[52] M. Greiter, V. Schnells, and R. Thomale, Annals of Physics 351, 1026 (2014).
[53] M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
[54] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
[55] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B 81, 064439 (2010).
[56] V. L. Pokrovsky and A. L. Talapov, Phys. Rev. Lett. 42, 65 (1979).
[57] By this we mean that the system remains gapped and the topological ground-state degeneracy is robust. We do not mean that the edge zero-modes remain exact over the entire phase range. See Ref. 41 for discussion on this distinction.
[58] J. M. Kosterlitz and D. J. Thouless, Journal of Physics C: Solid State Physics 6, 1181 (1973).
[59] J. Kosterlitz, Journal of Physics C: Solid State Physics 7, 1046 (1974).
[60] Note, in Fig. 3(a) the EE curves in the incommensurate phase are not shown because they overlap with the curve at $\phi = \theta = 0.5$.
[61] B. Swingle, J. McMinis, and N. M. Tubman, Phys. Rev. B 87, 235112 (2013).