Vector-weighted Mechanisms for Utility Maximization under Differential Privacy

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Abstract. We present practical aspects of implementing a pseudo posterior synthesizer for microdata dissemination under a new re-weighting strategy for utility maximization. Our re-weighting strategy applies to any vector-weighting approach under which a vector of observation-indexed weight are used to downweight likelihood contributions for high disclosure risk records. We demonstrate our method on two different vector-weighted schemes that target high-risk records by exponentiating each of their likelihood contributions with a record-indexed weight, \( \alpha_i \in [0, 1] \) for record \( i \in (1, \ldots, n) \). The first vector-weighted synthesizing mechanism computes the maximum (Lipschitz) bound, \( \Delta_{x_i} \), of each log-likelihood contribution over the space of parameter values, and sets the by-record weight \( \alpha_i \propto 1/\Delta_{x_i} \). The second vector-weighted synthesizer is based on constructing an identification disclosure risk probability, \( IR_i \), of record \( i \), and setting the by-record weight \( \alpha_i \propto 1/IR_i \). We compute the overall Lipschitz bound, \( \Delta_{\alpha, x} \), for the database \( x \), under each vector-weighted scheme where a local \( \epsilon_x = 2\Delta_{\alpha, x} \). Our new method for constructing record-indexed downweighting maximizes the data utility under any privacy budget for the vector-weighted synthesizers by adjusting the by-record weights, \( (\alpha_i)_{i=1}^n \), such that their individual Lipschitz bounds, \( \Delta_{\alpha, x_i} \), approach the bound for the entire database, \( \Delta_{\alpha, x} \). Our method asymptotically (as sample size grows) achieves an \( (\epsilon = 2\Delta_{\alpha}, \delta) \)– probabilistic differential privacy (pDP) guarantee, globally, over the space of databases, \( x \in X \) where \( \delta \downarrow 0 \). We illustrate our methods using simulated count data with and without over-dispersion-induced skewness and compare the results to a scalar-weighted synthesizer under the Exponential Mechanism (EM). We demonstrate our pDP result in a simulation study. We further demonstrate our methods on a sample of the Survey of Doctorate Recipients.

Keywords. Data privacy protection, Microdata dissemination, Pseudo posterior, Pseudo posterior mechanism, Synthetic data, Utility maximization, Vector-weighted

1 Introduction

A commonly-used data privacy approach generates synthetic data from statistical models estimated on closely-held, private data for proposed release by statistical agencies [Rubin, 1993; Little, 1993]. We denote such a release mechanism used to generate synthetic data, \( \xi(\theta \mid x) \), where \( x \) are the protected data and \( \theta \) are the parameters used to generate the
synthetic data. In particular, we consider the case where $\xi(\theta | x)$ is a posterior distribution of a Bayesian hierarchical probability model.

The Exponential Mechanism (EM) by McSherry and Talwar [2007] is a popular approach to generating draws of parameters, $\theta$, and associated synthetic data which is differentially private (Definition 2).

**Definition 1.** The Exponential Mechanism releases values of $\theta$ from a distribution proportional to

$$
\exp \left( \frac{\epsilon u(x, \theta)}{2\Delta_u} \right) \xi(\theta),
$$

where $u(x, \theta)$ is a utility function and $\xi(\theta)$ is a base or prior distribution.

Let $\Delta_u = \sup_{x \in X^n} \sup_{x, y : \delta(x, y) = 1} \sup_{\theta \in \Theta} |u(x, \theta) - u(y, \theta)|$ denote the sensitivity of $u(x, \theta)$, defined globally over $x = (x_1, \ldots, x_n) \in X^n$, the $\sigma$-algebra of datasets, $x$, governed by product measure, $P_n$; $\delta(x, y) = \# \{ i : x_i \neq y_i \}$ is the Hamming distance between $x, y \in X^n$. Then each draw of $\theta$ from the Exponential Mechanism satisfies $\epsilon$-DP, where $\epsilon$ is the privacy budget supplied by the publishing statistical agency.

The EM requires the availability of the sensitivity $\Delta_u$ for a chosen utility function $u(x, \theta)$. [Wasserman and Zhou 2010] and [Snoke and Slavkovic 2018] construct utility functions that are naturally bounded over all $x \in X^n$; however, they are not generally applicable to any population model and in the latter case are very difficult to implement in a computationally tractable manner since the EM distribution must be sampled by an inefficient random-walk Metropolis-Hastings scheme.

For a Bayesian model utilizing the data log-likelihood as the utility function of the EM, [Savitksy et al. 2020] demonstrate the EM mechanism becomes the model posterior distribution, which provides a straightforward mechanism from which to draw samples. Dimitrakakis et al. [2017] define a model-based sensitivity, $\sup_{x, y \in X^n, \delta(x, y) = 1} \sup_{\theta \in \Theta} |f_\theta(x) - f_\theta(y)| \leq \Delta$, that is constructed as a Lipschitz bound. They demonstrate a connection between the Lipschitz bound, $\Delta$ and $\epsilon \leq \Delta \Delta_u$ for each draw of parameters, $\theta$, where $f_\theta(x)$ is the model log-likelihood. The guarantee applies to all databases $x$, in the space of databases of size $n, X^n$. Other works studying posterior sampling for DP include [Wang et al. 2015], [Minami et al. 2016], [Zhang et al. 2016], and [Geumlek et al. 2017].

However, computing a finite $\Delta < \infty$ in practice, as acknowledged by Dimitrakakis et al. [2017], is difficult-to-impossible for an unbounded parameter space (e.g. a normal distribution) under simple models, which requires truncation of the parameter space to achieve a finite $\Delta$ and the truncation only works for some models to achieve a finite $\Delta$. Moreover, parameter truncation becomes intractable for practical models that utilize a multidimensional parameter space.

To guarantee the achievement of a finite $\Delta < \infty$ for any synthesizing model over an unbounded parameter space, [Savitksy et al. 2020] propose the pseudo posterior mechanism that uses a log-pseudo likelihood with a vector of weights $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ where each $\alpha_i$ exponentiates the likelihood contribution, $p(x_i | \theta)$, for each record $i \in \{1, \ldots, n\}$. Each weight, $\alpha_i \in [0, 1]$ is set to be inversely proportional to a measure of disclosure risk for record, $i$, such that the model used to generate synthetic data will be less influenced by relatively high-risk records.

The pseudo posterior mechanism of [Savitksy et al. 2020] is formulated as

$$
\xi^{\alpha(x)}(\theta | x) \propto \prod_{i=1}^{n} p(x_i | \theta)^{\alpha_i} \times \xi(\theta),
$$

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where the $\alpha_i \in [0, 1]$ serve to downweight the likelihood contributions with each $\alpha_i \propto 1/\sup_{B \in \beta_0}|f_\theta(x_i)|$ such that highly risky records are more strongly downweighted. The differential downweighting of each record intends to better preserve utility by focusing the downweighting on high-risk records. High-risk records tend to be those located in the tails of the distribution where the log-likelihood, $|f_\theta(x)|$, is highest, which allows the preservation of the high mass portions of the data distribution in the generated synthetic data. The method sets $\alpha_i = 0$ for any record with a non-finite log-likelihood, which ensures a finite $\Delta_\alpha = \sup_{x,y \in \mathcal{X}_n, \delta(x,y) = 1} \sup_{B \in \beta_0} |\alpha(x) \times f_\theta(x) - \alpha(y) \times f_\theta(y)| < \infty$. We see that $\Delta_\alpha \leq \Delta$ since $\alpha_i \leq 1$.

**Definition 1.** (Differential Privacy) The $\alpha$-weighted pseudo synthesizer, $\xi^{\alpha}(\theta | x)$, is a privacy mechanism defined in Equation 2 which satisfies $\epsilon$–DP if the following inequality holds.

$$
\sup_{x,y \in \mathcal{X}_n, \delta(x,y) = 1} \sup_{B \in \beta_0} \frac{\xi^{\alpha}(x)(B | x)}{\xi^{\alpha}(y)(B | y)} \leq e^\epsilon,
$$

which ensures the downweighting on high-risk records. High-risk records tend to be those located in the tails of the distribution where the log-likelihood, $|f_\theta(x)|$, is highest, which allows the preservation of the high mass portions of the data distribution in the generated synthetic data. The method sets $\alpha_i = 0$ for any record with a non-finite log-likelihood, which ensures a finite $\Delta_\alpha = \sup_{x,y \in \mathcal{X}_n, \delta(x,y) = 1} \sup_{B \in \beta_0} |\alpha(x) \times f_\theta(x) - \alpha(y) \times f_\theta(y)| < \infty$. We see that $\Delta_\alpha \leq \Delta$ since $\alpha_i \leq 1$.

**Definition 2.** (Differential Privacy) The $\alpha$–weighted pseudo synthesizer, $\xi^{\alpha}(\theta | x)$, is a privacy mechanism defined in Equation 2 which satisfies $\epsilon$–DP if the following inequality holds.

$$
\sup_{x,y \in \mathcal{X}_n, \delta(x,y) = 1} \sup_{B \in \beta_0} \frac{\xi^{\alpha}(x)(B | x)}{\xi^{\alpha}(y)(B | y)} \leq e^\epsilon,
$$

where $\xi^{\alpha}(x)(B | x) = \int_{\theta \in B} \xi^{\alpha}(x)(\theta | x)d\theta$.

**Definition 3.** (Probabilistic Differential Privacy) Let $\epsilon > 0$ and $0 < \delta < 1$. We say that our pseudo posterior mechanism is $(\epsilon, \delta)$–probabilistically differentially private (pDP) if $\forall x \in \mathcal{X}_n$,

$$
\Pr(x \in \text{Disc}(x, \epsilon)) \leq \delta,
$$

where the probability is taken over $x \in \mathcal{X}_n$ and $\text{Disc}(x, \epsilon)$ denotes the disclosure set,

$$
\{x \in \mathcal{X}_n : \sup_{B \in \beta_0} \log \left( \frac{\xi^{\alpha}(x)(B | x)}{\xi^{\alpha}(y)(B | y)} \right) > \epsilon, \forall y : \delta(x,y) = 1 \}.
$$

This set comprises a subspace of values $x \in \mathcal{X}_n$ where the log-pseudo posterior ratio exceeds an $\epsilon$–DP guarantee for any $y : \delta(x,y) = 1$. Less formally, $\delta \in [0, 1]$ denotes the probability that there is any database, $x'$ within $\mathcal{X}_n$ whose $\epsilon_{x'}$ exceeds the global $\epsilon$.

**Savitsky et al. [2020]** provide the following pDP guarantee.

**Theorem 4.** (Contraction of $\Delta_{\alpha,x}$ onto $\Delta_{\alpha}$).

$$
\delta \downarrow 0
$$

in $P_{\beta_0}$–probability as $n \uparrow \infty$, where $x_1, \ldots, x_n \overset{\text{ind}}{\sim} P_{\beta_0}$.
This theorem guarantees the probability that there is some database, \( x' \in X \) where \( \varepsilon_{x'} > \varepsilon \) limits to 0 for \( n \) sufficiently large. The implication is that the local \( \Delta_{\alpha,x} \) estimated on database, \( x \), contracts onto the global \( \Delta_{\alpha} \) over all databases in \( X \) or, equivalently, that the local \( \varepsilon_{x} \), contracts onto the global \( \varepsilon \).

The important condition, stated in Savitsky et al. [2020], for the contraction requires the downweighting (e.g., set of \( i \in \{1, \ldots, n\} : \alpha_i \leq 1 \)) to become progressively more sparse as the sample size, \( n \), increases. This condition limits the increase in the number of records downweighted to grow more slowly than the total number records. Since the downweighted records concentrate in the low probability mass tail region of the distribution, this condition is readily met. This contraction theorem summarizes Theorem 2 in Savitsky et al. [2020] and the proof is therein for independent (but not identically distributed models, \( \Theta \)). Intuitively, as the sample size increases, the space \( \Theta \) collapses onto a point, \( \theta \in \Theta^* \) that is guaranteed to be the minimum (Kullback-Liebler) divergence between the true generating, \( \theta_0 \) and the subspace, \( \Theta_\alpha \subset \Theta \) due to the distortion induced by \( \alpha \). This contraction onto \( \theta_\star \) reduces the space of distributions to a single distribution for \( n \) sufficiently large, which causes \( \Delta_{\alpha,x} \) on the observed database, \( x \), to become arbitrarily close to the global, \( \Delta_{\alpha} \). So we may use the local Lipschitz as the global Lipschitz for \( n \) sufficiently large. We implement a Monte Carlo simulation study in Section 5 we demonstrate the contraction of the local Lipschitz onto the global Lipschitz.

Savitsky et al. [2020] focus on the theoretical foundation of the pseudo posterior mechanism, and readers might be left with the question of how to effectively set the \( \alpha \) weights. While Savitsky et al. [2020] set each \( \alpha_i \in [0, 1] \) to be inversely proportional to the maximum (over \( \theta \)) log-likelihood for the record, there are possibly many ways to measure the disclosure risk for each record other than through the absolute value of the log-likelihood. We will illustrate one alternative method for measuring risk and setting \( \alpha \) to construct the vector-weighted synthesizer in the sequel.

In this work, we propose a new re-weighting strategy that uses the computation of maximum value of the absolute value of log-pseudo likelihood value over the parameters sampled from the pseudo posterior synthesizer, \( \Delta_{\alpha,x,i} \), for each data record after computing the weights, \( \alpha_i \), and re-estimating the synthesizer under the \( \alpha \)-weighted pseudo posterior model. The privacy guarantee is driven by the maximum over the data records of the \( \Delta_{\alpha,x,i} \). So any record, \( i' \) with a \( \Delta_{\alpha,x,i'} < \max_{i \in \{1, \ldots, n\}} \Delta_{\alpha,x,i} = \Delta_{\alpha,x} \) is overly downweighted since it does not determine the privacy protection for the overall database. We scale up or increase the weight values, \( \alpha_{i'} \), for these overly downweighted data records in a re-weighting step under the same privacy guarantee as achieved with the original weights, regardless of how disclosure risks are measured. The re-weighting strategy improves the utility of the vector-weighted synthesizer while maintaining an equivalent privacy budget. In summary, the re-weighting step targets records whose Lipschitz values for the pseudo posterior synthesizer are relatively low compared to the Lipschitz bound for the overall database. The re-weighting strategy increases the weights for those records, resulting in re-weighted, by-record Lipschitz bounds approaching the Lipschitz bound for the overall database.

The remainder of the article is organized as follows. Section 2 describes the key steps and the implementation algorithms of two vector-weighted synthesizers under a pDP guarantee. We present two simulation studies in Section 3, then proceed to introduce the new re-weighting strategy in Section 4 with an algorithm and a simulation study. Section 5 implements a Monte Carlo simulation study where we generate a collection of local databases on which we estimate our Lipschitz-weighted synthesizer and measure the local Lipschitz for each. The results demonstrate contraction of these local Lipschitz values onto a global
Lipschitz as an illustration of our pDP guarantee. We apply our methods to the highly skewed salary variable from a sample of the Survey of Doctorate Recipients in Section 6. Section 7 ends this article with a few concluding remarks.

2 Vector-weighted Synthesizer Algorithms

We proceed to construct two vector-weighted (pseudo posterior) synthesizers, both under a pDP guarantee, for microdata dissemination. The first synthesizer, labeled as Lipschitz-weighted (LW), sets each by-record weight, \( \alpha_i \in [0, 1] \) such that \( \alpha_i \propto \frac{1}{\Delta_{\alpha,x_i}} \), where \( \Delta_{\alpha,x_i} \) represents the maximum value of log-likelihood of record \( i \) over the space of parameter values for \( i \in (1, \ldots, n) \). This maximum bound is referred to as the Lipschitz bound for \( x_i \).

The second synthesizer, labeled as count-weighted (CW), sets each by-record weight, \( \alpha_i \in [0, 1] \), such that \( \alpha_i \propto \frac{1}{IR_i} \), where \( IR_i \) denotes the disclosure risk probability (\( \in [0, 1] \)) of record \( i \). The \( IR_i \) is a measure of a record’s isolation from other records and is constructed by counting the number of records whose values are outside a radius around the true value for the target record divided by the total number of records [Hu et al., forthcoming]. A record whose true value is not well-covered by the values of other records is relatively more isolated and, therefore, at higher disclosure risk. The radius is a measure of closeness that is tuned by the owner of the closely-held data.

In both synthesizers, the record-indexed vector weights \( \alpha = (\alpha_1 \in [0, 1], \ldots, \alpha_n \in [0, 1]) \) are used to exponentiate the likelihood contributions where the weights are designed to target high-risk records by downweighting their likelihood contributions. These two measures of risk - LW and CW - are related in that the notion of record isolation underlies both. The value of the response variable for an isolated target record is near to or within a close radius to the values of many other records. As earlier mentioned, such isolated records generally appear in the tails of the distribution where there is little distribution mass. As a result, downweighting the likelihood contribution of isolated records tends to preserve the high mass regions of the real data distribution in the resulting synthetic data generated for release.

We specify the method for formulation of vector weights \( \alpha = (\alpha_1, \ldots, \alpha_n) \) for the LW and CW synthesizers in Section 2.1 and Section 2.2, with Algorithm 1 and Algorithm 2, respectively.

Each algorithm starts by computing the weights, \( \alpha = (\alpha_1, \ldots, \alpha_n) \), which are then used to construct the pseudo likelihood and estimate the pseudo posterior. Next, we draw parameters from the estimated pseudo posterior distribution and compute the overall Lipschitz bound, \( \Delta_{\alpha,x} \) for database, \( x \). The resulting DP guarantee is \( \epsilon = 2\Delta_{\alpha,x} \) and is “local” to the database, \( x \), and the \( \epsilon \) is indirectly controlled through the weights, \( \alpha \). Synthetic data are then generated using the drawn \( \theta_s \) from the \( \alpha \)-weighted pseudo posterior distribution from step 5 in each algorithm of the corresponding data generating model.

As earlier discussed, the local Lipschitz bound, \( \Delta_{\alpha,x} \), contracts onto the “global” Lipschitz bound, \( \Delta_{\alpha} \), over all databases, \( x \in X^n \) of size \( n \), as \( n \) increases such that \( \epsilon \) contracts onto \( \epsilon \) at a relatively modest sample size.

2.1 Generating synthetic data under the LW vector-weighted synthesizer

We specify the algorithm for generating synthetic data under the LW \( \alpha \)-weighted pseudo posterior distribution that produces synthetic data for database, \( x \).
To implement the LW vector-weighted synthesizer, we first fit an unweighted synthesizer and obtain the absolute value of the log-pseudo likelihood for each data base record $i$ and each Markov chain Monte Carlo (MCMC) draw $s$ of $\theta$ from the unweighted posterior distribution. A Lipschitz bound for each record is computed by taking the maximum of the log-likelihoods over the $S$ draws of $\theta$. We formulate by-record weights, $\alpha = (\alpha_1, \ldots, \alpha_n)$, to be inversely proportional to the by-record Lipschitz bounds. See step 1 to step 4 in Algorithm 1.

Algorithm 1: Steps to implement the LW vector-weighted synthesizer

1. Let $[f_{\theta, i}]$ denote the absolute value of the log-likelihood computed from the unweighted synthesizer for database record, $i \in (1, \ldots, n)$ and MCMC draw, $s \in (1, \ldots, S)$ of $\theta$ from its unweighted posterior distribution;
2. Compute the $S \times n$ matrix of by-record absolute value of log-likelihoods, $L = \{[f_{\theta, i}]\}_{i=1,\ldots,n, s=1,\ldots, S}$;
3. Compute the maximum over each $S \times 1$ column of $L$ to produce the $n \times 1$ (database record-indexed) vector, $f = (f_1, \ldots, f_n)$. We use a linear transformation of each $f_i$ to $\tilde{f}_i \in [0, 1]$ where values of $\tilde{f}_i$ closer to 1 indicates relatively higher identification disclosure risk: $\tilde{f}_i = \frac{f_i - \min f_j}{\max f_j - \min f_j}$;
4. Formulate by-record weights, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i = c \times (1 - \tilde{f}_i) + g$, where $c$ and $g$ denote a scaling and a shift parameters, respectively, of the $\alpha_i$ used to tune the risk-utility trade-off for setting $\epsilon_x = 2\Delta_{\alpha, x}$;
5. Use $\alpha = (\alpha_1, \ldots, \alpha_n)$ to construct the $\alpha$–weighted pseudo posterior distribution, $\xi(\theta | x) \propto \prod_{i=1}^n p(x_i | \theta)^{\alpha_i} \times \xi(\theta)$. Draw $(\theta_1)_{s=1,\ldots,S}$ from the $\alpha$–weighted pseudo posterior distribution, where $S$ denotes the number of draws of $\theta$ from the $\alpha$–weighted pseudo posterior distribution;
6. Compute the $S \times n$ matrix of log-pseudo likelihood values, $L^\alpha = \{[f_{\theta, i}^\alpha]\}_{i=1,\ldots,n, s=1,\ldots, S}$;
7. Compute $\Delta_{\alpha, X} = \max_{s,i} [f_{\theta, i}^\alpha]$, that defines the local DP guarantee, $\epsilon_x = 2\Delta_{\alpha, x}$, for database $x$.

Algorithm 1 is implemented on the observed database, $x \in \mathcal{X}^n$, under which we compute the local (specific-to-database $x$) Lipschitz bound, $\Delta_{\alpha, x}$, to achieve a local privacy privacy guarantee, $\epsilon_x = 2\Delta_{\alpha, x}$. The asymptotic contraction result of Savitsky et al. [2020] proves that the local Lipschitz, $\Delta_{\alpha, x}$, contracts onto a global Lipschitz, $\Delta_\alpha$, to achieve a global $(\epsilon = 2\Delta_\alpha, \delta)$–pDP guarantee, where $\delta \in [0, 1]$ denotes the probability that there exists some database, $x' \in X$ whose $\epsilon_{x'} > \epsilon$.

We emphasize that LW indirectly achieves the pDP guarantee, $(\epsilon = 2\Delta_\alpha, \delta)$, through the computation of the likelihood weights, $\alpha$. Sample R scripts implementing Algorithm 1 are available in the Appendix.

The LW algorithm constructs weights intended to directly minimize the overall Lipschitz bound for the synthetic data by downweighting the likelihood contribution for each record inversely proportional to how large is the absolute value of its log-likelihood. We recall that the Lipschitz is the maximum over the parameter space and records of this absolute value of the log-likelihood quantity, so our LW weighting scheme will be efficient at targeting those high risk records that most effect the privacy guarantee to produce a relatively moderate distortion of the closely held, real data distribution expressed in the publicly-released synthetic data.
2.2 Generating synthetic data under the CW vector-weighted synthesizer

Next, we present the algorithm of [Hu et al. [forthcoming]] for generating synthetic data under the CW $\alpha$-weighted synthesizer. The weights $\alpha$ are estimated as probabilities of identification disclosure, and each $\alpha_i \in [0, 1]$, based on the assumption that a putative intruder guesses randomly from a collection of records whose values are near to or within some set radius of the record being identified.

To compute a weight for each record, $i \in (1, \ldots, n)$, we first calculate its estimated probability of identification disclosure. We assume that an intruder knows the data value of the record she seeks and that she will randomly choose among records that are close to that value. More formally, we cast a ball, $B(y_i, r)$, around the true value of $y_i$ for record $i$ with a radius $r$. The radius, $r$, is a policy hyperparameter set by the agency who owns the closely-held data. We count the number of records whose values fall outside of the radius around the target, and take the ratio of this count over the total number of records, a proportion that we label the risk probability of identification. A target record where the values for most other records lie outside the radius are viewed as isolated because the target record value is sparsely covered by the values of other records, and therefore at a higher risk of identification disclosure. We then formulate by-record weights, $\alpha = (\alpha_1, \ldots, \alpha_n)$, that are inversely proportional to the by-record risk probabilities. See step 1 to step 4 in Algorithm 2.

Algorithm 2: Steps to implement the CW vector-weighted synthesizer.

1. Let $M_i$ denote the set of records in the original data, and $|M_i|$ denote the number of records in the set;
2. Cast a ball, $B(y_i, r)$ with a radius $r$ around the true value of record $i$, and count the number of records falling outside the radius $\sum_{j \in M_i} I(y_j \notin B(y_i, r))$;
3. Compute the record-level risk probability, $IR_i$ as $IR_i = \sum_{j \in M_i} I(y_j \notin B(y_i, r)) / |M_i|$, such that $IR_i \in [0, 1]$;
4. Formulate by-record weights, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i = c \times (1 - IR_i) + g$, where $c$ and $g$ denote a scaling and a shift parameters, respectively, of the $\alpha_i$ used to tune the risk-utility trade-off;
5. Use $\alpha = (\alpha_1, \ldots, \alpha_n)$ to construct the pseudo likelihood from which the pseudo posterior is estimated. Draw $(\theta_s)_{s=1,\ldots,S}$ from the $\alpha$-weighted pseudo posterior distribution;
6. Compute the $S \times n$ matrix of log-pseudo likelihood values, $L^\alpha = \{|f_{\theta_s,i}^\alpha|\}_{s=1,\ldots,S,i=1,\ldots,n}$;
7. Compute $\Delta_{\alpha, x} = \max_{s,i} |f_{\theta_s,i}^\alpha|$, that defines the local DP guarantee for database $x$.

Even though the weights under CW are computed based on assumptions about the intruder behavior, we are yet able to compute its $\epsilon_x = 2\Delta_{\alpha, x}$ and invoke the asymptotic pDP guarantee of [Savitsky et al. [2020]] since any weighting scheme with $\alpha \in [0, 1]$ produces this formal privacy guarantee.

3 Comparison of Performances on Simulated Data

We demonstrate the properties of both vector-weighted synthesizers using two sets of simulated data of size 1000: i) simulated data from Poisson($\mu = 100$), which is nearly symmet-
ric to slight skew due to the large value of $\mu$; and ii) simulated data from a mixture of two negative binomial, $\text{NB}(\mu = 100, \phi = 5)$ and $\text{NB}(\mu = 100, \phi = 20)$, where $\phi$ denotes an over-dispersion parameter under which the variance is allowed to be larger than the mean (with mixture weights of $\pi = 0.2$ and $(1 - \pi) = 0.8$), which produces data with a highly skewed distribution.

For all results, we label the vector-weighted synthesizer in Section 2.1 as LW, and that in Section 2.2 as CW. For comparison, we include a scalar-weighted synthesizer with a scalar weight for every record set as $\alpha_i = \Delta_{\alpha,x}/\Delta_x$, where $\Delta_x$ is the local Lipschitz bound for the unweighted synthesizer. This scalar-weighted pseudo posterior synthesizer has been shown equivalent to the EM under a log-likelihood utility and produces an ($\epsilon_x = 2\Delta_{\alpha,x}$) [Savitsky et al., 2020]. This allows us to set the $\epsilon_x$ for the scalar-weighted synthesizer (SW) to that for the LW and the CW vector-weighted synthesizers, so that we may compare their relative utility performances at equivalent level of privacy protection. Finally, we include the unweighted synthesizer, labeled as “Unweighted”, which is a Poisson synthesizer for the Poisson-simulated data, and a negative binomial synthesizer for the mixture of negative binomial-simulated data. All model estimations are performed in Stan [Stan Development Team, 2016]. The Stan script for a weighted Poisson synthesizer is available in the Appendix.

3.1 Less skewed simulated data

Figure 1a plots distributions of the record-level Lipschitz bound $\Delta_{\alpha,x,i}$. The overall Lipschitz bound, $\Delta_{\alpha,x}$, is the maximum Lipschitz bound across all records; e.g., the maximum value on the y-axis of each violin plot on the y-axis for each synthesizer, weighted or unweighted. All three weighted synthesizers have equivalent $\Delta_{\alpha,x}$’s, by design, to allow comparisons of the weighting distributions and their utility performances. All three overall Lipschitz bounds are substantially lower than that of Unweighted, indicating that the weights, $\alpha$, may be used to control the ($\epsilon = 2\Delta_{\alpha}$)–pDP asymptotic privacy guarantee. Figure 1b plots the associated distribution or record-level weights $(\alpha_i)_{i=1}^n \in [0, 1]$.

Figure 1: Violin Plots for the Poisson: Lipschitz Bounds and Weights

When the data distribution is less skewed, we see in Figure 1a that more of the records for LW express relatively high values for $\Delta_{\alpha,x,i}$, concentrated around the overall Lipschitz, $\Delta_{\alpha,x}$. This suggests that LW generally avoids overly downweighting in a fashion that would produce lower by record $\Delta_{\alpha,x,i}$ values. The overall privacy guarantee is governed solely by $\Delta_{\alpha,x}$, the maximum value of the by-record Lipschitz bounds such that having records with even lower bounds does not improve the privacy guarantee.
Moreover, the distribution of the \((\alpha_i)_{i=1}^n\) for LW, shown in Figure 1b, is skewed towards values closer to 1, which indicates that we expect relatively good utility performance in the resulting synthetic data in terms of preserving the real data distribution \([Hu et al., forthcoming]\). Figure 1a and Figure 1b together indicate that the LW is relatively efficient with large values of weights across the data records and concentrated Lipschitz bounds. A relatively more efficient weighting scheme will produce synthetic data that better preserves the properties of the real, closely-held data distribution than a less efficient weighting scheme.

By contrast, CW shows relatively more records where \(\Delta_{\alpha_i,x_i}\) and \(\alpha_i\) are low relative to those of LW, which means that CW is overly downweighting more records and is less targeted than LW. This suggests CW as a less efficient weighting scheme that requires more downweighting to achieve the same privacy guarantee. We will see the negative impact of this overly downweighting feature of CW on data utility in the sequel.

SW utilizes a scalar weight and shows most records have Lipschitz bounds much lower than the overall Lipschitz bound that governs the privacy guarantee. Since all records are equally downweighted, this synthesizer has the effect of reducing the effective sample size, which would be expected to increase uncertainty estimation model parameters, such as \(\mu\) in the Poisson and negative binomial models.

The synthetic data is deemed to achieve a higher utility if its distribution is close to the real, closely-held data distribution. We may compare statistics estimated from both the closely-held and synthetic data distributions to assess the relative similarities of the two distributions. For utility evaluation, we include violin plots of the 15th and 90th quantile estimates of the generated synthetic data under each synthesizer. For comparison, we include that from the data, labeled as “Data”, in Figure 2a. We can see that between the two vector-weighted synthesizers, CW performs relatively worse with notably biased quantile estimates. This is because CW, compared to LW, more heavily downweights the high mass region of the data distribution (Figure 1a and Figure 1b), which translates into its reduction of utility in Figure 2a. These suggest that LW is a more efficient weighting scheme than the probability-based weighting scheme of CW. An additional utility plot of mean and median estimates with a similar outcome is available in the Appendix.

While the scalar-weighted synthesizer SW produces synthetic data that reasonably well preserves the quantiles of the real data distribution in Figure 2a when we look at Figure 2b of the posterior density of the mean parameter \(\mu\) of the real data, the credibility interval for SW is much wider than the two vector-weighted schemes, indicating a substantial loss of information and inferential power. Going back to Figure 1b, we see that SW uses a scalar weight around 0.375 for every record (to achieve an equivalent \(\Delta_{\alpha,x}\) as LW and CW), which
is substantially lower than most of the weights of LW and CW. This suggests that SW has the effect of reducing the sample size with a scalar-weight. Figure 2b also shows the utility distortion induced by CW due to overly downweighting relatively low-risk records.

### 3.2 Highly skewed simulated data

When the closely-held data distribution is highly skewed, the comparisons of the distributions of by-record Lipschitz bounds among the synthesizer alternatives, as well as the associated distributions of the by-record weights show similar patterns as those for the Poisson simulation; see Figure 3a and Figure 3b.

![Violin Plots for Negative Binomial Mixture](image)

Figure 3: Violin Plots for Negative Binomial Mixture

However, utility plots in Figure 3c to Figure 3e show that for the two vector-weighted synthesizers: i) LW does not achieve as high a utility performance in the skewed data as compared to when the data are less skewed—it overestimates the 15th quantile and underestimates the 90th quantile, indicating a worse control of the tails; however, due to the fact that the mass of the distribution is not much downweighted, the mean and median
estimates are relatively robust; ii) CW’s utility performance also worsens, and is in some situations, unacceptable—the quantile estimates are more extreme than LW, indicating an even worse control of the tails; moreover, its mean and median are overestimated, indicating a worse control of the distribution mass. As in Section 3.1, the scalar-weighted SW achieves correctly-centered estimates but with too much estimation uncertainty.

Figure 3f compares data distribution plots across the true, closely-held data and the four synthesizers. Similar to the previous discussion, the vector-weighted synthesizers, LW and CW, concentrate downweighting to the tails, resulting in synthetic data with shorter tails on both sides. CW downweights the tails more severely for the same privacy protection.

We also see competing effects of SW that distort the data distribution in Figure 3f. On the one hand, SW is based on the riskiest records in the tails and downweights those likelihood contributions, which will pull in or truncate the tails. On the other hand, since the weight is the same for all records, the main mode of the data distribution is flattened more than the vector-weighted synthesizers of LW and SW, which then reverses some of the truncation in the tails. This also explains the relatively good performance on the quantiles in Figure 3c.

One way to improve utility is to increase the weights [Hu et al., forthcoming, Savitsky et al., 2020]. At the same time, we want to keep an equivalent level for the overall Lipschitz bound to maintain the $(\epsilon_x = 2\Delta_{\alpha,x})$. We now turn to a re-weighting strategy to maximize utility of any vector-weighted synthesizer under equivalent $(\epsilon_x = 2\Delta_{\alpha,x})$.

\section{Re-weighting strategy to maximize utility for any privacy budget}

\subsection{Motivation and proposed method}

To motivate our re-weighting strategy, we revisit Figure 3a and focus on the two vector-weighted synthesizers, LW and CW. We know that the $(\epsilon_x = 2\Delta_{\alpha,x})$ privacy guarantee is controlled by the maximum Lipschitz bound $\Delta_{\alpha,x}$. As long as the maximum of the by-record Lipschitz bounds stays unchanged, we can increase the by-record Lipschitz bounds for other records below the maximum value to be closer to the overall maximum value. Increased by-record Lipschitz bounds are associated with increased weights, which will result in improved utility [Hu et al., forthcoming, Savitsky et al., 2020].

Figure 4 shows scatter plots of by-record Lipschitz bound $\Delta_{\alpha,x}$ (on the y-axis) against by-record weight, $\alpha$, (on the x-axis) for LW and CW. In each case, the red dashed line indicates the maximum Lipschitz bound $\Delta_{\alpha,x}$. Only a small number of records express low Lipschitz bounds for the LW synthesizer and the majority of records express relatively high.
Lipschitz bounds due to high weight values. By contrast, CW produces many records with low Lipschitz bounds, and many of them also have low weights. The relative concentration of higher weights and Lipschitz bounds for LW than for CW serve as a further justification of why LW is a more efficient vector-weighted scheme than CW. In particular, LW has done a relatively good job of targeting high-risk records and down-weighting them, while CW is overly downweighting lower-risk records and is less targeted than LW (to achieve the same privacy guarantee).

Moreover, Figure 4 reveals that both vector-weighted synthesizers could improve their weighting efficiency to achieve a given maximum Lipschitz bound $\Delta_{\alpha,x}$. We can increase the weights so that the by-record Lipschitz bounds $\Delta_{\alpha,x,i}$'s become closer to the red dashed line. In this way, we can improve the utility performance of LW and CW while maintaining an equivalent overall $\Delta_{\alpha,x}$. The utility will improve because less downweighting of records produces less distortion of the real data distribution in the generated synthetic data. In the limit, the best efficiency that may be achieved by a vector-weighted scheme is one where the plot of by-record weights on the x-axis and the by-record Lipschitz bounds on the y-axis is horizontal; that is, there is no relationship between the weights and the Lipschitz bounds.

Our re-weighting strategy constructs re-weighted weights $\alpha^w = (\alpha^w_1, \cdots, \alpha^w_n)$ by:

$$\alpha^w_i = k \times \alpha_i \times \frac{\Delta_{\alpha,x}}{\Delta_{\alpha,x,i}}$$

where $\Delta_{\alpha,x}$ is the maximum Lipschitz bound, and $\Delta_{\alpha,x,i}$ is the Lipschitz bound for record $i$. $\alpha_i$ is the weight used in the pseudo posterior synthesizers before the re-weighting step, and a constant, $k < 1$, is used to ensure that the final maximum Lipschitz bound remains equivalent before and after this re-weighting step. Both $\Delta_{\alpha,x}$ and $\Delta_{\alpha,x,i}$ are computed from the $\alpha$-weighted pseudo posterior before re-weighting. The implementation for the new re-weighting step is outlined in Algorithm 3 and should be inserted between step 4 and step 5 in Algorithm 1 and Algorithm 2 for LW and CW, respectively.

**Algorithm 3: Re-weighting step to obtain $\alpha^w$**

1. Use the calculated $\alpha = (\alpha_1, \cdots, \alpha_n)$ from the unweighted synthesizer. Use the overall Lipschitz bound, $\Delta_{\alpha,x}$, and the by record Lipschitz bounds, $\{\Delta_{\alpha,x,i}, i = 1, \cdots, n\}$, computed from the $\alpha$-weighted pseudo posterior synthesizer. Construct a constant $k < 1$ to compute $\alpha^w$, where each $\alpha^w_i = k \times \alpha_i \times \frac{\Delta_{\alpha,x}}{\Delta_{\alpha,x,i}} \in [0, 1]$;

2. Run step 5 to 7 in Algorithm 1 / Algorithm 2 again, to re-estimate the synthesizers under an $\alpha^w$-weighted pseudo posterior to obtain $\Delta_{\alpha^w,x}$ to make sure that $\Delta_{\alpha,x} \approx \Delta_{\alpha^w,x}$. If not, try another $k < 1$ and repeat.

### 4.2 Application to highly skewed data

We demonstrate our re-weighting strategy under the highly skewed negative binomial mixture data, where in Section 3.2 we have seen unsatisfactory utility outcomes of LW and CW. Using $k = 0.95$ produces an equivalent overall Lipschitz bound. The re-weighted synthetic data results are labeled “LW_final” and “CW_final”, for LW and CW respectively.
Figure 5: Lipschitz Bounds vs Weights, Before and After Re-weighting

Figure 5a and Figure 5b show the before vs after re-weighting scatter plots of Lipschitz bounds and weights. As is evident in Figure 5a, the curve showing the Lipschitz-weight relationship becomes nearly horizontal at the maximum Lipschitz bound $\Delta_{\alpha, x}$ as we move from LW to LW_final, which indicates maximum efficiency. The curve in Figure 5b becomes much less vertical from CW to CW_final, indicating much improved efficiency.

Figure 6: Before vs After Re-weighting for the Negative Binomial Mixture: Lipschitz Bounds and Weights

Figure 6a confirms that our re-weighting strategy increases the weighting efficiency of LW and CW in that the by-record Lipschitz bounds are increased while maintaining an equivalent maximum Lipschitz bound. At the record level, Figure 6b illustrates that every record has received a higher weight from LW to LW_final, and from CW to CW_final. We receive further confirmation of the improved efficiency of implementing the re-weighting step from the weight plots in Figure 6c and Figure 6d that show weights increase after...
re-weighting.

Turning to the utility performances, Figure 7a and Figure 7b show huge improvement in utility of CW after re-weighting for all estimates of the generated synthetic data. The deterioration in the preservation of the real data distribution tails due to overly downweighting records in the tails before re-weighting, as we have seen in Section 3.2, is greatly mitigated by the re-weighting strategy. We observe that all of the extreme quantiles, the mean, and the median estimates are much more accurate. The improvement of utility of LW is less impressive because the relative improvement in the efficiency of the weighting scheme is relatively smaller. However, we can certainly see improvement in estimating the mean, median, and 90th quantile. For example, compared to a 95% confidence interval of median in the data [96.0, 101.0], the CW_final achieved [96.1, 100.1] improved from CW’s [98.7, 102.4], and LW_final achieved [96.0 100.0] improved from LW’s [95.6, 99.7]. We include a table of comparisons of all estimands in the Appendix for brevity.

Moreover, utility of the mean parameter $\mu$ estimation in Figure 7c improves after re-weighting for LW and CW, with a bigger improvement for CW. When we turn to violin plot for the data distribution, on the one hand, as compared to the synthesizer distributions, on the other hand, that is displayed in Figure 7d, we see that the re-weighting strategy improves the tails in LW_final and CW_final, compared to LW and CW, respectively. This reduced down-weighting of the distribution tails is a feature of the re-weighting strategy for any vector-weighted scheme applied to highly skewed data.

![Figure 7: Before vs After Re-weighting for the Negative Binomial Mixture: Utility](image)

5 Moving from Local-to-Global Privacy Guarantee

We proceed to implement a Monte Carlo simulation study under each of our less skewed Poisson and more skewed mixture of negative binomials data generating models to walk
from the local privacy guarantee for a specific database, $x$, to a global (pDP) guarantee over the space of databases, $\forall x \in X^n$. We generate $R = 100$ local databases under each generating model, estimate the unweighted and LW weighted synthesizers on each database, $r \in \{1, \ldots, (R = 100)\}$, and compute a local Lipschitz bound, $\Delta_{\alpha,x_r}$, on each $x_r$ under each synthesizer. We plot the distributions of the $(\Delta_{\alpha,x_r})_{r=1}^{R=100}$ for each synthesizer and conclude that we have achieved a global pDP result if this distribution contracts around a global $\Delta_{\alpha}$. We summarize our Monte Carlo simulation procedure, below:

1. For $r = 1, \ldots, (R = 200)$:
   - Generate $x_r \sim \text{Pois}(\mu)$ or $x_r \sim \pi_1 \text{NB}(\mu_1 = 100, \phi_1 = 5) + \pi_2 \text{NB}(\mu_2 = 100, \phi_2 = 5)$, each of size $n = 1000$.
   - Compute the local Lipschitz bound, $\Delta_{\alpha,x_r}$, for the unweighted and $\alpha$–re-weighted synthesizers.
   - Construct the distribution of $\Delta_{\alpha,x_r}$ and note the maximum of the distribution and difference between the maximum and minimum values of the distribution of the local Lipschitz bounds.

2. Assess contraction of the $\max_r \Delta_{\alpha,x_r}$ to a single (global) value and whether the minimum and maximum values collapse together.

Figure 8 presents a violin plot of the local $(\Delta_{\alpha,x_r})_{r=1}^{R=100}$ for the $R = 100$ Monte Carlo iterations of the less skewed data generating model for the Unweighted (left) and LW-weighed (right) synthesizers. We readily observe that the LW synthesizer contracts onto the global value, $\epsilon = 2\Delta_{\alpha} = 7$. That this contraction is consistent with a relaxed, pDP guarantee comes from the small distribution mass above $\Delta_{\alpha} = 3.5$, though we see the probability that the Lipschitz bound any local database exceeds 3.5 is nearly 0 at $n = 1000$.

Figure 9 presents the associated distributions of a set of estimands over the Monte Carlo iterations for each synthesizer as compared to the confidential data. There is an expected loss of utility, though inference is reasonably well-preserved.

Figure 8: Violin plot of Lipschitz bounds over the Monte Carlo iterations under the Poisson generating model.
The following set of figures repeat the earlier set, but here under the highly skewed data generation process from a negative binomial mixture. The conclusions are the same that we observe substantial contraction around the global Lipschitz (where here $\Delta_{\alpha} = 6$).

Figure 10: Violin plot of Lipschitz bounds over the Monte Carlo iterations under the mixtures of negative binomials generating model.

We have illustrated the theory of Savitsky et al. [2020] that guarantees a pDP result by demonstrating a contraction of a collection of Lipschitz bounds for local databases onto a
global Lipschitz bound under both of our low and highly skewed data generating scenarios, with our LW synthesizer and re-weighting strategy. The key conclusion is that for $n$ sufficiently large that a local Lipschitz bound estimated on a specific database, $x$, becomes arbitrarily close to the global Lipschitz bound.

6 Application to The Survey of Doctorate Recipients

The Survey of Doctorate Recipients (SDR) provides demographic, education, and career history information from individuals with a U.S. research doctoral degree in a science, engineering, or health (SEH) field. The SDR is sponsored by the National Center for Science and Engineering Statistics and by the National Institutes of Health. In this section, we demonstrate our re-weighting strategy to a sample of 1000 observations of the SDR, focused on the highly skewed salary variable. The sample comes from the 2017 Survey of Doctorate Recipients public use file [https://ncsesdata.nsf.gov/datadownload/]. The highly skewed salary variable has a mean of $107,609, a median of$95,000, a range of [$0, $509,000], and a standard deviation of $69,718. We use a negative binomial unweighted synthesizer for this highly skewed variable salary.

6.1 Before re-weighting

Before re-weighting, the results of LW, CW and SW on the real skewed data sample tell a similar story as on simulated skewed data in Section 3.2. The results are included in the Appendix for brevity and we summarize the findings here: i) LW has the highest utility among the three, though its relatively heavy downweighting of records in the tails of the real data distribution under highly skewed data results in reduced utility compared to that of less skewed data; ii) CW’s utility performance on the real skewed data is worse than that on the simulated skewed data—it has assigned low weights to many more records, resulting in low weighting efficiency and therefore low utility; iii) SW achieves relatively correctly-centered estimates but with too much estimation uncertainty.

Moreover, as evident in Figure 12, LW’s weighting efficiency is close to optimal, therefore the re-weighting strategy might not improve much. However, CW’s weighting efficiency is expected to see huge improvement after the re-weighting strategy, which in turn, will improve its utility performance.

Figure 12: Lipschitz Bounds vs Weights, LW (left) and CW (right)
6.2 After re-weighting

We apply the re-weighting strategy to LW and CW to maximize their utility performances. We set $k = 0.95$ to maintain an equivalent overall Lipschitz bound. The results are labeled as “LW_final” and “CW_final” respectively.

![Figure 13: Lipschitz Bounds vs Weights, Before and After Re-weighting](image)

Figure 13a shows that the re-weighting strategy has pushed the Lipschitz-to-weight association to almost horizontal at the maximum Lipschitz bound, $\Delta_{\alpha,\gamma}$, for LW, indicating maximum efficiency. Figure 13b shows that the re-weighting strategy has also produced a Lipschitz-to-weight association that is less vertical for CW. Therefore, we expect to see minor utility improvement of LW_final, and a huge utility improvement of CW_final.

![Figure 14: Before vs After Re-weighting for Salary: Lipschitz Bounds and Weights](image)

Examining the distributions for the weights under both of LW and CW in Figure 14 shows how much weighting efficiency LW_final and CW_final have gained after the re-weighting.
strategy. Focusing on the walk between CW to CW_final in Figure 14d, the distribution of weights is highly diffuse with large mass assigned to low values before re-weighting. After re-weighting, by contrast, many more records receive higher weights. Even though the by-record weights ($\alpha$) have increased after re-weighting, Figure 14a shows that the re-weighting strategy has maintained an equivalent maximum Lipschitz bound $\Delta_{\alpha,x}$.

Finally, the utility results in Figure 15 demonstrate the utility maximizing ability of our proposed re-weighting strategy on the real data sample. Whether it is the preservation of statistics of the closely-held data distribution, shown in Figure 15a and Figure 15b, or the relative accuracy of parameter estimates in Figure 15c, or similarity of the synthetic data density to that of the closely-held data in Figure 15d, CW_final has produced much higher utility than CW across the board, a result that we expect to see given its improved weighting efficiency previously discussed. We make particular mention that the comparisons of the closely-held and synthetic data distributions in Figure 15d show that re-weighting reduces or mitigates the shrinking of the tail of the closely-held data in the resulting synthetic data. Overall, there is a minor utility improvement from LW to LW_final, another result we expect to see given its minor improvement of weighting efficiency. Nevertheless, our proposed re-weighting strategy maximizes the utility of any vector-weighted scheme, while maintaining an equivalent maximum Lipschitz bound $\Delta_{\alpha,x}$.

![Figure 15: Before vs After Re-weighting for Salary: Utility](image)

**7 Concluding Remarks**

In this article, we enumerate a practical approach to implement a vector-weighted pseudo posterior synthesizer that uses a vector of record-indexed weights, $(\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$, that exponentiate data likelihood contributions to surgically downweight high-risk records. We demonstrate that the LW vector-weighted scheme provides better utility for equivalent
privacy protection than does the CW vector-weighted synthesizer. When the data distribution is highly skewed, more records express high risk, especially in the right tail. LW performs worse than under less skewed data in reproducing the tail, though it keeps the mass of the data distribution relatively unaffected such that statistical inference on the released, privacy-protected synthetic data well preserves inference from the closely-held data.

We introduce a new re-weighting strategy that improves utility of any vector-weighted scheme in the difficult case of a highly-skewed data distribution, while maintaining an equivalent privacy budget. Applied to both LW and CW, this strategy improves their weighting efficiency by increasing by-record weights that produce Lipschitz bounds below the maximum bound. Improved weighting efficiency substantially mitigates the tendency for vector-weighted schemes to overly downweight the tails, especially for CW.

Lastly, we use a Monte Carlo simulation study to demonstrate that our a local DP privacy guarantee, estimated on the observed database, contracts onto the global DP privacy guarantee at a relatively low sample size, $n = 1000$, such that we may take the local result to be nearly global. Our result illustrates the theoretical pDP guarantee provided by Savitsky et al. [2020] under which the probability that any database exceeds the global $\epsilon$–DP guarantee quickly limits to 0.

We have demonstrated that our vector-weighted pseudo posterior synthesizers are both easy to implement with little change to the posterior sampler for the closely-held data while providing a global pDP guarantee.

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Appendix

1. R scripts of Algorithm 1 in Section 2.1

Computing weights $\alpha$:

The `stan_estimate_unweighted` below is the Stan output of the unweighted synthesizer.

```r
## step 1
log_lik <- stan_estimate_unweighted$log_lik # S x N matrix
N <- ncol(log_lik)
S <- nrow(log_lik)
log_ratio <- matrix(0,S,N)
log_ratio_theta <- matrix(0,S,1)
pos <- rep(TRUE,N)

## step 2
for( s in 1:S ){
  log_like_xs <- sum(log_lik[s,]) # full data
  for(i in 1:N){
    pos_i <- pos
    pos_i[i] <- FALSE
    log_like_xsi <- sum(log_lik[s,pos_i])
    log_ratio[s,i] <- abs(log_like_xs - log_like_xsi)
  }
}
log_ratio_theta <- rowMaxs(log_ratio, value = TRUE)
L <- quantile(log_ratio_theta,thresh)

## step 3
log_ratio_data <- colMaxs(logthresh_ratio,value=TRUE)
risks <- f_linres( log_ratio_data )

## step 4
weights <- c * (1 - risks) + g
weights[weights <= 0] <- 0
weights[weights >= 1] <- 1

Compute Lipschitz bound, $\Delta_{\alpha,x}$:

The `stan_estimate_weighted` below is the Stan output of the weighted synthesizer. Step 5 is done by Stan estimation.

```r
## step 6
log_lik <- stan_estimate_weighted$log_lik # S x N matrix
N <- ncol(log_lik)
S <- nrow(log_lik)
log_ratio <- matrix(0,S,N)
log_ratio_theta <- matrix(0,S,1)
```
pos <- rep(TRUE, N)
for( s in 1:S ){
    log_like_xs <- sum(log_lik[s,])  ## full data
    for(i in 1:N){
        pos_i <- pos
        pos_i[i] <- FALSE
        log_like_xsi <- sum(log_lik[s, pos_i])
        log_ratio[s,i] <- abs(log_like_xs - log_like_xsi)
    }
}

## step 7
log_ratio_theta <- rowMaxs(log_ratio, value = TRUE)
L <- quantile(log_ratio_theta, thresh)

2. Stan script for a weighted Poisson synthesizer

functions{
    real wt_pois_lpmf(int[] y, vector mu, vector weights, int n){
        real check_term;
        check_term = 0.0;
        for( i in 1:n )
            { check_term += weights[i] * poisson_log_lpmf(y[i] | mu[i]); }
        return check_term;
    }
    real wt_poisi_lpmf(int y_i, real mu_i, real weights_i){
        real check_term;
        check_term = weights_i * poisson_log_lpmf(y_i | mu_i); return check_term;}}

data {
    int<lower=1> n;
    int<lower=1> K;
    int<lower=0> y[n];
    vector<lower=0>[n] weights;
    matrix[n, K] X; }

transformed data{
    vector<lower=0>[K] zeros_beta;
    zeros_beta = rep_vector(0, K);}

parameters{
    vector[K] beta;
    vector<lower=0>[K] sigma_beta;
    cholesky_factor_corr[K] L_beta; }

transformed parameters{
    vector[n] mu;
    mu = X * beta;}

model{
    L_beta ~ lkj_corr_cholesky(6);
    sigma_beta ~ student_t(3,0,1);
    beta ~ multi_normal_cholesky(zeros_beta,
          diag_pre_multiply(sigma_beta,L_beta));
    target += wt_pois_lpmf(y | mu, weights, n);
}

generated quantities{
    vector[n] log_lik;
    for (i in 1:n) {
      log_lik[i] = wt_pois_lpmf(y[i] | mu[i], weights[i]);
    }
}

3. Additional utility plots of Poisson in Section 3.1

Figure 16 provides 3 sets of violin plots of several utility measures.

![Violin Plots of Utility for Poisson](image)

(a) Mean and Median  
(b) Synthetic Data

Figure 16: Violin Plots of Utility for Poisson

4. Utility comparison before and after re-weighting in Section 4.2

Table 1 presents utility comparison of LW and CW before and after re-weighting.

|                | Data     | LW       | LW_final | CW       | CW_final  |
|----------------|----------|----------|----------|----------|-----------|
| 15th quantile  | [70.0, 75.0] | [70.0, 74.3] | [70.6, 74.8] | [77.2, 80.8] | [71.6, 75.8] |
| 90th quantile  | [132.0, 140.0] | [132.1, 139.0] | [133.1, 140.3] | [128.5, 134.0] | [131.0, 137.7] |
| mean           | [98.8, 102.3] | [98.0, 101.4] | [98.7, 102.0] | [100.6, 103.3] | [98.6, 101.8] |
| median         | [96.0, 101.0] | [95.6, 99.7]  | [96.0, 100.0] | [98.7, 102.4]  | [96.1, 100.1] |

Table 1: Utility comparison before and after re-weighting for the negative binomial mixture: 95% confidence interval

5. Plots of LW, CW, and SW before re-weighting in Section 6.1

Figure 17 provides results of LW, CW, and SW before re-weighting.
Figure 17: Violin Plots for Salary