On Sobolev extension domains in $\mathbb{R}^n$

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Abstract
We describe a class of Sobolev $W^{k,p}_p$-extension domains $\Omega \subset \mathbb{R}^n$ determined by a certain inner subhyperbolic metric in $\Omega$. This enables us to characterize finitely connected Sobolev $W^{1,p}_p$-extension domains in $\mathbb{R}^2$ for each $p > 2$.

1. Introduction.
Let $\Omega$ be a domain in $\mathbb{R}^n$. This paper is devoted to the problem of extendability of functions from the Sobolev space $W^{k,p}_k(\Omega)$ to functions from $W^{k,p}_k(\mathbb{R}^n)$. We recall that, given $k \in \mathbb{N}$ and $p \in [1, \infty]$, the Sobolev space $W^{k,p}_k(\Omega)$, see e.g. Maz’ja [M], consists of all functions $f \in L^1_{\text{loc}}(\Omega)$ whose distributional partial derivatives on $\Omega$ of all orders up to $k$ belong to $L^p(\Omega)$. $W^{k,p}_k(\Omega)$ is normed by
$$\|f\|_{W^{k,p}_k(\Omega)} := \sum \{\|D^\alpha f\|_{L^p(\Omega)} : |\alpha| \leq k\}.$$ A domain $\Omega$ in $\mathbb{R}^n$ is said to be a Sobolev $W^{k,p}_k$-extension domain if the following isomorphism $W^{k,p}_k(\Omega) = W^{k,p}_k(\mathbb{R}^n)|_\Omega$ holds. In other words, $\Omega$ is a Sobolev extension domain (for the space $W^{k,p}_k(\mathbb{R}^n)$) if every Sobolev function $f \in W^{k,p}_k(\Omega)$ can be extended to a Sobolev $W^{k,p}_k$-function $F$ defined on all of $\mathbb{R}^n$. For instance, Lipschitz domains (Calderón [C2], $1 < p < \infty$, Stein [St], $p = 1, \infty$) in $\mathbb{R}^n$ are $W^{k,p}_k$-extension domains for every $p \in [1, \infty]$ and every $k \in \mathbb{N}$. Jones [Jn] introduced a wider class of $(\varepsilon, \delta)$-domains and proved that every $(\varepsilon, \delta)$-domain is a Sobolev $W^{k,p}_k$-extension domain in $\mathbb{R}^n$ for every $k \geq 1$ and every $p \geq 1$. Burago and Maz’ya [BM], [M], Ch. 6, described extension domains for the space $BV(\mathbb{R}^n)$ of functions whose distributional derivatives of the first order are finite Radon measures.

Our main result is the following

**Theorem 1.1** Let $n < p < \infty$ and let $\Omega$ be a domain in $\mathbb{R}^n$. Suppose that there exist constants $C, \theta > 0$ such that the following condition is satisfied: for every $x, y \in \Omega$ such that $\|x - y\| \leq \theta$, there exists a rectifiable curve $\gamma \subset \Omega$ joining $x$ to $y$ such that
$$\int_\gamma \text{dist}(z, \partial \Omega) \frac{1-n}{p-1} \, ds(z) \leq C \|x - y\|^{\frac{p-n}{p-1}}. \tag{1.1}$$

Here $\partial \Omega$ denotes the boundary of $\Omega$ and $ds$ denotes arc length measure.

Then $\Omega$ is a Sobolev $W^{k,p}_k$-extension domain for every $k \geq 1$ and every $q > p^*$ where $p^* \in (n, p)$ is a constant depending only on $n, p$ and $C$.

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For $k = 1$ and $q > p$ this result has been proved by Koskela [K].

Observe that this theorem is also known for the case $p = \infty$ (with $p^* = q = \infty$). In that case every domain $\Omega$ satisfying inequality (1.1) is quasi-Euclidean, i.e., its inner metric is (locally) equivalent to the Euclidean distance. This case was studied by Whitney [W3] who proved that every quasi-Euclidean domain is a $W^k_\infty$-extension domain for every $k \geq 1$.

Our next result, Theorem 1.2, relates to description of Sobolev extension domains in $\mathbb{R}^2$. The first result in this direction was obtained by Gol’dstein, Latfullin and Vodop’janov [GLV, GV1, GV2] who proved that a finitely connected bounded planar domain $\Omega$ is a Sobolev $W^1_2$-extension domain if and only if $\Omega$ is an $(\varepsilon, \delta)-$domain in $\mathbb{R}^2$ for some $\varepsilon, \delta > 0$. Maz’ja [M, MP] gave an example of a simply connected domain $\Omega \subset \mathbb{R}^2$ such that $\Omega$ is a $W^1_\infty$-extension domain for every $p > 2$. However $\Omega$ is not an $(\varepsilon, \delta)-$domain for any $\varepsilon$ and $\delta$.

Let us briefly indicate the main ideas of our approach for the case $k = 1$, i.e., for the Sobolev space $W^1_p(\mathbb{R}^n)$. Recall that, when $p > n$, it follows from the Sobolev embedding theorem that every function $f \in W^1_p(\Omega), p > n$, can be redefined, if necessary, on a subset of $\Omega$ of Lebesgue measure zero so that it satisfies a local Hölder condition of order $\alpha := 1 - \frac{n}{p}$ on $\Omega$: i.e., for every ball $B \subset \Omega$

$$\left| f(x) - f(y) \right| \leq C(n, p) \| f \|_{W^1_p(\Omega)} \| x - y \|^{1 - \frac{n}{p}}, \quad x, y \in B. \quad (1.2)$$

We will identify each element of $W^1_p(\Omega)$ with its unique continuous representative. Thus we will be able to restrict our attention to the case of continuous Sobolev functions.

Following Buckley and Stanoyevitch [BSt3], given $\alpha \in [0, 1]$ and a rectifiable curve $\gamma \subset \Omega$, we define the subhyperbolic length of $\gamma$ by

$$\text{len}_{\alpha, \Omega}(\gamma) := \int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z).$$
Then we let \( d_{\alpha,\Omega} \) denote the corresponding subhyperbolic metric on \( \Omega \) given, for each \( x, y \in \Omega \), by
\[
d_{\alpha,\Omega}(x, y) := \inf_{\gamma} \text{len}_{\alpha,\Omega}(\gamma)
\]
where the infimum is taken over all rectifiable curves \( \gamma \subset \Omega \) joining \( x \) to \( y \).

The metric \( d_{\alpha,\Omega} \) was introduced and studied by Gehring and Martio in [GM]. See also [AHHL, L, BKos] for various further results using this metric. Note also that \( \text{len}_{0,\Omega} \) and \( d_{0,\Omega} \) are the well-known quasihyperbolic length and quasihyperbolic distance, and \( d_{1,\Omega} \) is the inner (or geodesic) metric on \( \Omega \).

The subhyperbolic metric \( d_{\alpha,\Omega} \) with \( \alpha = (p-n)/(p-1) \) arises naturally in the study of Sobolev \( W^1_p(\Omega) \)-functions for \( p > n \). In particular, Buckley and Stanoyevitch [BST2] proved that the local Hölder condition (1.2) is equivalent to the following Hölder-type condition
\[
|f(x) - f(y)| \leq C(n,p)\|f\|_{W^1_p(\Omega)} \{ d_{\alpha,\Omega}(x, y)^{1-\frac{1}{p}} + \|x - y\|^{1-\frac{n}{p}} \}, \quad x, y \in \Omega,
\]
with \( \alpha = (p-n)/(p-1) \).

In turn, since any extension \( F \in W^1_p(\mathbb{R}^n) \) of \( f \) satisfies the global Hölder condition
\[
|F(x) - F(y)| \leq C(n,p)\|F\|_{W^1_p(\mathbb{R}^n)} \|x - y\|^{1-\frac{n}{p}}, \quad x, y \in \mathbb{R}^n,
\]
we have
\[
|f(x) - f(y)| \leq C(n,p)\|F\|_{W^1_p(\mathbb{R}^n)} \|x - y\|^{1-\frac{n}{p}}, \quad x, y \in \Omega.
\]

Of course the conditions (1.4) and (1.5) with \( \|f\|_{W^1_p(\Omega)} \) and \( \|F\|_{W^1_p(\mathbb{R}^n)} \) replaced by unspecified constants are not equivalent to membership of \( f \) in \( W^1_p(\Omega) \) or in \( W^1_p(\mathbb{R}^n)|_\Omega \) respectively. However the preceding remarks suggest that a reasonable property which might perhaps be necessary or perhaps sufficient for a domain \( \Omega \) to be a Sobolev extension domain could be this: Whenever a function \( f : \Omega \to \mathbb{R} \) satisfies
\[
|f(x) - f(y)| \leq d_{\alpha,\Omega}(x, y)^{1-\frac{1}{p}} + \|x - y\|^{1-\frac{n}{p}}
\]
for all \( x, y \in \Omega \) and \( \alpha = (p-n)/(p-1) \) then it also satisfies
\[
|f(x) - f(y)| \leq C(n,p)\|x - y\|^{1-\frac{n}{p}}
\]
for all \( x, y \in \Omega \) and for some constant \( C(n,p) \) depending only on \( n \) and \( p \).

One would like to have a simpler condition on \( \Omega \) which would be sufficient to imply the above “reasonable property”. It is clear that the following property, which has already been considered and studied by other authors, namely
\[
d_{\alpha,\Omega}(x, y)^{1-\frac{1}{p}} \leq C\|x - y\|^{1-\frac{n}{p}} \quad \text{for all } x, y \in \Omega \text{ and } \alpha = (p-n)/(p-1)
\]
or, equivalently, \( d_{\alpha,\Omega}(x, y) \leq C\|x - y\|^\alpha \) for all \( x, y \in \Omega \), is such a condition.

These considerations lead us to work with a certain class of domains, essentially those which were introduced in [GM]. In our context here, it seems convenient to use terminology different from that of [GM] and other papers.
Definition 1.3 For each \( \alpha \in (0, 1] \), the domain \( \Omega \subset \mathbb{R}^n \) is said to be \( \alpha \)-subhyperbolic if there exist constants \( C_{\alpha, \Omega} > 0 \) and \( \theta_{\alpha, \Omega} > 0 \) such that

\[
d_{\alpha, \Omega}(x, y) \leq C_{\alpha, \Omega} \|x - y\|^\alpha
\]

for every \( x, y \in \Omega \) satisfying \( \|x - y\| \leq \theta_{\alpha, \Omega} \).

We denote the class of \( \alpha \)-subhyperbolic domains in \( \mathbb{R}^n \) by \( U_\alpha(\mathbb{R}^n) \).

In [GM] and also in [L] these domains are called “Lip\( \alpha \)-extension domains”. (This name is derived from the fact that \( \Omega \in U_\alpha(\mathbb{R}^n) \) iff all functions which are locally Lipschitz of order \( \alpha \) on \( \Omega \) are Lipschitz of order \( \alpha \) on \( \Omega \).) These domains have also been studied in [BST2, BST, BST3] where they are called “\( \alpha \)-cigar domains”, and in [BKos] where they are termed “local weak \( \alpha \)-cigar domains”.

Now Theorem 1.1 can be reformulated as follows: For each \( p > n \) and for each \( \frac{p-n}{p-1} \)-subhyperbolic domain \( \Omega \in \mathbb{R}^n \), there exists a constant \( p^* \in (n, p) \) depending only on \( n \), \( p \) and \( \Omega \), such that \( \Omega \) is a Sobolev \( W^q \)-extension domain for every \( q \geq p^* \).

In turn, Theorem 1.2 admits the following reformulation: For each \( p > 2 \), a finitely connected bounded domain \( \Omega \subset \mathbb{R}^2 \) is a Sobolev \( W^1_p \)-extension domain if and only if \( \Omega \) is a \( \frac{p-2}{p-1} \)-subhyperbolic domain.

The family \( \{U_\alpha(\mathbb{R}^n) : \alpha \in (0, 1]\} \) is an “increasing family”, i.e.,

\[
U_{\alpha'}(\mathbb{R}^n) \subset U_{\alpha''}(\mathbb{R}^n)
\]

whenever \( 0 < \alpha' < \alpha'' \leq 1 \),

see, e.g. [BKos]. Lappalainen [L] proved that

\[
U_\alpha(\mathbb{R}^n) \subset \bigcap_{\alpha < \tau \leq 1} U_\tau(\mathbb{R}^n)
\]

for every \( \alpha \in (0, 1] \).

This last result motivates our discussion presented in Section 2, which is devoted to the following question: Does the equality

\[
U_\alpha(\mathbb{R}^n) = \bigcup_{0 < \tau < \alpha} U_\tau(\mathbb{R}^n) \tag{1.6}
\]

hold? In other words, do \( \alpha \)-subhyperbolic domains have the following “self-improving” property that whenever \( \Omega \) is an \( \alpha \)-subhyperbolic domain in \( \mathbb{R}^n \) for some \( \alpha \in (0, 1] \), it is also \( \tau \)-subhyperbolic for some positive \( \tau \) which is strictly less than \( \alpha \) ? (Of course, \( \tau \) can depend on \( \Omega \).)

We do not know the answer to this question in general. We do know that the answer is affirmative for an arbitrary finitely connected bounded domain \( \Omega \in U_\alpha(\mathbb{R}^2) \), \( \alpha \in (0, 1] \), as it follows from Theorem 1.1 and Theorem 1.2. We also know that for a certain subfamily of \( U_\alpha(\mathbb{R}^n) \), the so-called strongly \( \alpha \)-subhyperbolic domains (Definition 2.4) the answer to the above question is affirmative. (See Proposition 2.6.) It should be pointed out that we have no examples of subhyperbolic domains which are not strongly subhyperbolic.

We are able to show that the following weaker version of the self-improving property (1.6) holds for an arbitrary subhyperbolic domain in \( \mathbb{R}^n \).
Theorem 1.4 Let $\alpha \in (0, 1)$ and let $\Omega$ be an $\alpha$-subhyperbolic domain in $\mathbb{R}^n$. There exist a constant $\alpha^*$, $0 < \alpha^* < \alpha$, and constants $\theta, C > 0$ such that the following is true:

For every $\varepsilon > 0$ and every $x, y \in \Omega$, $\|x - y\| \leq \theta$, there exist a rectifiable curve $\Gamma \subset \Omega$ joining $x$ to $y$ and a subset $\tilde{\Gamma} \subset \Gamma$ consisting of a finite number of arcs such that the following conditions are satisfied:

(i). For every $\tau \in [\alpha^*, \alpha]$

$$\int_{\tilde{\Gamma}} \text{dist}(z, \partial \Omega)^{\tau - 1} \, ds(z) \leq C \|x - y\|^\tau. \quad (1.7)$$

In addition, for every ball $B$ centered in $\tilde{\Gamma}$ of radius at most $\|x - y\|$, 

$$\text{diam } B \leq C \text{ length}(B \cap \tilde{\Gamma}). \quad (1.8)$$

(ii). We have $\text{length}(\Gamma) \leq C \|x - y\|$ and 

$$\text{length}(\Gamma \setminus \tilde{\Gamma}) < \varepsilon. \quad (1.9)$$

Moreover,

$$\int_{\Gamma \setminus \tilde{\Gamma}} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq C \|x - y\|^\alpha. \quad (1.10)$$

The constants $\alpha^*, \theta$ and $C$ depend only on $n$, $\alpha$, and the constants $C_{\alpha, \Omega}$ and $\theta_{\alpha, \Omega}$ introduced in Definition 1.3.

The proof of this result, presented in Section 2, is based on the reverse Hölder inequality for $m$-dyadic $A_1$-weights. (See Melas [Mel].)

Theorem 1.4 is an important ingredient in the proof of the extension Theorem 1.1. It enables us to prove the following version of the Sobolev-Poincaré inequality for subhyperbolic domains (for $p > n$ and $k \geq 1$): Let $\Omega$ be an $\alpha$-subhyperbolic domain in $\mathbb{R}^n$ with $\alpha = (p - n)/(p - 1)$. Given $f \in C^{k-1}(\Omega)$ and $x \in \Omega$ we let $T^k_x(f)$ denote the Taylor polynomial of $f$ at $x$ of degree at most $k - 1$. We prove that there exists $p^* \in (n, p)$ and constants $\theta, \lambda, C > 0$ such that for every function $f \in C^{k-1}(\Omega) \cap W^{k}_{p^*}(\Omega)$ and every $x, y \in \Omega$, $\|x - y\| \leq \theta$, the following inequality

$$|f(y) - T^k_x(f)(y)| \leq C \|x - y\|^{k - \frac{n}{p^*}} \left(\int_{B \cap \Omega} \|\nabla^k f\|^{p^*} \, dx\right)^{\frac{1}{p^*}}$$

holds. Here $B = B(x, \lambda \|x - y\|)$ is the ball centered at $x$ of radius $r = \lambda \|x - y\|$. This inequality is a particular case of Theorem 3.1 which we prove in Section 3.

In Section 4 we prove a corollary of this result related to the sharp maximal function

$$f^*_{k, \Omega}(x) := \sup_{r > 0} r^{-k} \inf_{P \in \mathcal{F}_{k-1}} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} |f - P| \, dx, \quad x \in \Omega.$$
Here $\mathcal{P}_{k-1}$ is the space of polynomials of degree at most $k - 1$ defined on $\mathbb{R}^n$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$. We show that for every $f \in W^k_p(\Omega)$ and every $x \in \Omega$ the following inequality
\[
f^k_{x,\Omega}(x) \leq C \left\{ (\mathcal{M}[\|\nabla f\|^\wedge p^\gamma](x))^{\frac{1}{p^\gamma}} + \mathcal{M}[f^\wedge](x) \right\}
\] (1.11)
holds. Here $\mathcal{M}$ denotes the Hardy-Littlewood maximal function and the symbol $g^\wedge$ stands for the extension by zero of a function from $\Omega$ to all of $\mathbb{R}^n$.

The sharp maximal function is a useful tool in the study of Sobolev functions. In [C] Calderón proved that, for $p > 1$, a function $f$ is in $W^k_p(\mathbb{R}^n)$ if and only if $f$ and $f^k_{x,\mathbb{R}^n}$ are both in $L_p(\mathbb{R}^n)$. In [S4] this description has been generalized to the case of the so-called regular subsets of $\mathbb{R}^n$, i.e., the sets $S$ such that $|B \cap S| \sim |B|$ for all balls $B$ centered in $S$ of radius at most 1. We proved in [S4] that if $S$ is regular and $f \in L_p(S), p > 1$, then $f$ can be extended to a function $F \in W^k_p(\mathbb{R}^n)$ if and only if its sharp maximal function $f^k_{x,S} \in L_p(S)$. (For the case $k = 1$ see also [S3, HKT, HKT1].) Observe that every Sobolev $W^1_p$-extension domain is a regular subset of $\mathbb{R}^n$, see Hajlasz, Koskela and Tuominen [HKT]. In [S7] we present a description of the trace space $W^1_p(\mathbb{R}^n)|_S, p > n$, for an arbitrary set $S \subset \mathbb{R}^n$ via an $L_\infty$-version of the sharp maximal function).

Every subhyperbolic domain is a regular set, as shown in Lemma 2.3. So, in order to prove, for some given $q > p^*$, that a function $f \in W^k_p(\Omega)$ extends to a Sobolev $W^k_q$-function on $\mathbb{R}^n$, it suffices to show that $f^k_{x,\Omega} \in L_q(\Omega)$. We do this by applying the Hardy-Littlewood maximal theorem to inequality (1.11). This gives us the inequality $\|f^k_{x,\Omega}\|_{L_q(S)} \leq C\|f\|_{W^k_q(\Omega)}$ which completes the proof of Theorem 1.1.

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2. Subhyperbolic domains: intrinsic metrics and self-improvement.

Throughout the paper $C, C_1, C_2, \ldots$ will be generic positive constants which depend only on parameters determining sets (say, $n, \alpha$, the constants $C_{\alpha,\Omega}$ or $\theta_{\alpha,\Omega}$, etc.) or function spaces $(p, q, \ldots)$, etc. These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(n, p)$.

The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ will be denoted by $|A|$. Given subsets $A, B \subset \mathbb{R}^n$, we put $\text{diam } A := \sup \{ \| a - a' \| : a, a' \in A \}$ and
\[
dist(A, B) := \inf \{ \| a - b \| : a \in A, b \in B \}.
\]
For $x \in \mathbb{R}^n$ we also set $\text{dist}(x, A) := \text{dist}(\{ x \}, A)$.

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$, and let $u = \gamma(t_1), v = \gamma(t_2)$ where $a \leq t_1 < t_2 \leq b$. By $\gamma_{uv}$ we denote the arc of $\gamma$ joining $u$ to $v$.

We will be needed the following auxiliary lemma.
Lemma 2.1 (i). Let \( x, y \in \Omega \) and let
\[
\max(\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)) \leq 2\|x - y\|. \tag{2.1}
\]
Let \( \gamma \) be a rectifiable curve joining \( x \) to \( y \) in \( \Omega \). Assume that for some \( \alpha \in (0, 1) \) and \( C > 0 \) the following inequality
\[
\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq C\|x - y\|^\alpha \tag{2.2}
\]
holds. Then
\[
\text{length}(\gamma) \leq 2e^C\|x - y\|.
\]

(ii). Let \( x, y \in \Omega \) and let
\[
\max(\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)) > 2\|x - y\|. \tag{2.3}
\]
Then the line segment \([x, y] \subset \Omega\) and for every \( \beta \in (0, 1) \) we have
\[
\int_{[x,y]} \text{dist}(z, \partial \Omega)^{\beta - 1} \, ds(z) \leq \|x - y\|^\beta. \tag{2.4}
\]

Proof. (i). Let us parameterize \( \gamma \) by arclength; thus we identify \( \gamma \) with a function \( \gamma : [0, \ell] \to \Omega \) satisfying \( \gamma(0) = x, \gamma(\ell) = y \). Now (2.2) is equivalent to
\[
\int_{0}^{\ell} \text{dist}(\gamma(t), \partial \Omega)^{\alpha - 1} \, dt \leq C\|x - y\|^\alpha.
\]
Since \( \text{dist}(\cdot, \partial \Omega) \) is a Lipschitz function on \( \mathbb{R}^n \),
\[
\text{dist}(u, \partial \Omega) \leq \text{dist}(v, \partial \Omega) + \|u - v\|, \quad u, v \in \Omega, \tag{2.5}
\]
so that for every \( t \in (0, \ell] \)
\[
\text{dist}(\gamma(t), \partial \Omega) \leq \text{dist}(x, \partial \Omega) + \|x - \gamma(t)\|.
\]
Since \( \gamma \) is parameterized by arclength,
\[
\|x - \gamma(t)\| \leq \text{length}(\gamma_{x\gamma(t)}) = t,
\]
so that
\[
\text{dist}(\gamma(t), \partial \Omega) \leq \text{dist}(x, \partial \Omega) + t, \quad t \in [0, \ell].
\]
This inequality and (2.2) imply
\[
C\|x - y\|^\alpha \geq \int_{0}^{\ell} \text{dist}(\gamma(t), \partial \Omega)^{\alpha - 1} \, dt \geq \int_{0}^{\ell} (\text{dist}(x, \partial \Omega) + t)^{\alpha - 1} \, dt
\]
\[
= \alpha^{-1}((\text{dist}(x, \partial \Omega) + \ell)^{\alpha} - \text{dist}(x, \partial \Omega)^{\alpha})
\]
\[
\geq \alpha^{-1}(\ell^\alpha - \text{dist}(x, \partial \Omega)^{\alpha}).
\]
But $2\|x - y\| > \text{dist}(x, \partial \Omega)$ so that

$$C\|x - y\|^\alpha \geq \alpha^{-1}(\ell^\alpha - (2\|x - y\|)^\alpha).$$

Hence

$$\ell \leq (\alpha C + 2^\alpha)^{\frac{1}{\alpha}} \|x - y\| \leq 2eC \|x - y\|$$

proving (i).

(ii). Clearly, (2.3) implies $[x, y] \subset \Omega$. Prove (2.4).

We may assume that $\text{dist}(x, \partial \Omega) > 2\|x - y\|$. Also note that $\|x - z\| \leq \|x - y\|$ for every $z \in [x, y]$. These inequalities and (2.5) imply the following:

$$\frac{1}{2} \text{dist}(x, \partial \Omega) \leq \text{dist}(x, \partial \Omega) - \|x - y\| \leq \text{dist}(x, \partial \Omega) - \|x - z\| \leq \text{dist}(z, \partial \Omega).$$

Hence,

$$\int_{[x,y]} \text{dist}(z, \partial \Omega)^{\beta - 1} ds(z) \leq \int_{[x,y]} 2^{1-\beta} \text{dist}(x, \partial \Omega)^{\beta - 1} ds(z) \leq 2^{1-\beta} \|x - y\| \text{dist}(x, \partial \Omega)^{\beta - 1} \leq 2^{1-\beta} \|x - y\| (2\|x - y\|)^{\beta - 1} = \|x - y\|^\beta$$

proving the lemma.  \(\square\)

**Lemma 2.2** Let $x, y \in \Omega$ and let $\gamma \subset \Omega$ be a rectifiable curve joining $x$ to $y$. Suppose that for some $\alpha \in (0, 1)$ and $C \geq 1$ the following inequality

$$\int_\gamma \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \leq C \text{length}^\alpha(\gamma) \quad (2.6)$$

holds. Then

(i). There exists a point $\bar{z} \in \gamma$ such that

$$\text{length}(\gamma) \leq C^{\frac{1}{1-\alpha}} \text{dist}(\bar{z}, \partial \Omega)$$

(ii). We have

$$\frac{1}{\text{length}(\gamma)} \int_\gamma \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \leq 2C \inf_{z \in \gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1}.$$  

Proof. (i). Put $\ell = \text{length}(\gamma)$. Let $\bar{z}$ be a point in $\gamma$ such that

$$\max\{\text{dist}(z, \partial \Omega) : z \in \gamma\} = \text{dist}(\bar{z}, \partial \Omega).$$

Then

$$\int_\gamma \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \geq \int_\gamma \text{dist}(\bar{z}, \partial \Omega)^{\alpha - 1} ds(z) = \ell \text{dist}(\bar{z}, \partial \Omega)^{\alpha - 1}.$$
so that, by (2.6),
\[ \ell \, \text{dist}(z, \partial \Omega)^{\alpha-1} \leq C \ell^\alpha. \]

Hence
\[ \ell \leq C^{\frac{1}{1-\alpha}} \, \text{dist}(z, \partial \Omega) \]
proving (i).

(ii). Put \( w(z) := \text{dist}(z, \partial \Omega) \). Then, by (2.6),
\[
\frac{1}{\ell} \int_\gamma w(z)^{\alpha-1} \, ds(z) \leq \ell^{-1} (C \ell^\alpha) = C \ell^{\alpha-1}.
\] (2.7)

For every \( z_1, z_2 \in \gamma \) we have
\[ |w(z_1) - w(z_2)| = |\text{dist}(z_1, \partial \Omega) - \text{dist}(z_2, \partial \Omega)| \leq \|z_1 - z_2\| \leq \ell \]
so that
\[
\max_{z \in \gamma} w(z) \leq \min_{z \in \gamma} w(z) + \ell.
\] (2.8)

Let us consider two cases. First suppose that \( \max_{z \in \gamma} w(z) \leq 2\ell \). Since \( \alpha \in (0, 1) \), we obtain
\[ \ell^{\alpha-1} \leq 2^{1-\alpha} \min_{z \in \gamma} w(z)^{\alpha-1} \]
so that, by (2.7),
\[
\frac{1}{\ell} \int_\gamma w(z)^{\alpha-1} \, ds(z) \leq 2^{1-\alpha} C \min_{z \in \gamma} w(z)^{\alpha-1} \leq 2C \min_{z \in \gamma} w(z)^{\alpha-1}.
\]

Now assume that \( 2\ell < \max_{z \in \gamma} w(z) \). Then, by (2.8),
\[ \max_{z \in \gamma} w(z) \leq \min_{z \in \gamma} w(z) + \frac{1}{2} \max_{z \in \gamma} w(z) \]
so that \( \max_{z \in \gamma} w(z) \leq 2 \min_{z \in \gamma} w(z) \). Hence
\[ \max_{z \in \gamma} w(z)^{\alpha-1} \leq 2^{1-\alpha} \min_{z \in \gamma} w(z)^{\alpha-1} \leq 2 \min_{z \in \gamma} w(z)^{\alpha-1}. \]

Finally, we have
\[
\frac{1}{\ell} \int_\gamma w(z)^{\alpha-1} \, ds(z) \leq \max_{z \in \gamma} w(z)^{\alpha-1} \leq 2 \min_{z \in \gamma} w(z)^{\alpha-1}.
\]

The lemma is proved.

This lemma implies the following important property of subhyperbolic domains.

**Lemma 2.3** Let \( \alpha \in (0, 1) \) and let \( \Omega \) be an \( \alpha \)-subhyperbolic domain.

There exist constant \( \delta > 0 \) and \( \sigma \in (0, 1] \) depending only on \( n, \alpha, C_{\alpha, \Omega} \) and \( \theta_{\alpha, \Omega} \) such that every ball \( B \) centered in \( \Omega \) of diameter at most \( \delta \) contains a ball \( B' \subset \Omega \) of diameter at least \( \sigma \, \text{diam} \ B \).
Remark 2.5

Given a rectifiable curve \( \gamma \) if there exist constants \( C, \theta > 0 \) such that every \( x, y \in \Omega, \| x - y \| \leq \theta \), can be joined by a rectifiable curve \( \gamma \subset \Omega \) satisfying the following condition: for every \( u, v \in \gamma \)

\[
\int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha-1} \, ds(z) \leq C \| u - v \|^\alpha. \tag{2.9}
\]

Remark 2.5 Given \( x, y \in \Omega \) a rectifiable curve \( \gamma \subset \Omega \) joining \( x \) to \( y \) is said to be \( d_{\alpha, \Omega} \)-geodesic if

\[
d_{\alpha, \Omega}(x, y) = \text{len}_{\alpha, \Omega}(\gamma) := \int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} \, ds(z).
\]

(See definition (1.3).)
Clearly, if \( \Omega \) is \( \alpha \)-subhyperbolic and for every \( x, y \in \Omega \) there exists \( d_{\alpha,\Omega} \)-geodesic, then \( \Omega \) is strongly \( \alpha \)-subhyperbolic. In fact, in this case every arc of \( d_{\alpha,\Omega} \)-geodesic curve is \( d_{\alpha,\Omega} \)-geodesic as well so that inequality (2.9) holds.

However, for every \( \alpha \in (0, 1] \) there exists a domain \( \Omega \in \mathbb{R}^n \) and \( x, y \in \Omega \) such that \( d_{\alpha,\Omega} \)-geodesic for \( x, y \) does not exist. This is trivial for \( \alpha = 1 \), i.e., for quasi-Euclidean domains. For the case \( \alpha \in (0, 1) \) see [BSt2].

Let us slightly generalize this example. Fix \( C \geq 1 \). We say that a rectifiable curve \( \gamma \subset \Omega \) joining \( x \) to \( y \) is \((C, d_{\alpha,\Omega})\)-geodesic if for every \( u, v \in \gamma \) the following inequality

\[
\text{len}_{\alpha,\Omega}(\gamma_{uv}) \leq C d_{\alpha,\Omega}(u, v).
\]

holds. Clearly, a rectifiable curve \( \gamma \) is \((1, d_{\alpha,\Omega})\)-geodesic iff it is \( d_{\alpha,\Omega} \)-geodesic. Moreover, if \( \Omega \in U_\alpha(\mathbb{R}^n) \) and for every \( x, y \in \Omega \), \( \| x - y \| \leq \theta \), there exists \((C, d_{\alpha,\Omega})\)-geodesic joining \( x \) to \( y \) in \( \Omega \), then \( \Omega \) is strongly \( \alpha \)-subhyperbolic.

This observation motivates the following question: Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( \alpha \in (0, 1] \). Does there exist a constant \( C = C_\Omega > 1 \) such that every two points \( x, y \in \Omega \) can be joined by a \((C, d_{\alpha,\Omega})\)-geodesic curve? Even for the quasi-Euclidean domains, i.e., for \( \alpha = 1 \), we do not know the answer to this question.

**Proposition 2.6** Let \( \alpha \in (0, 1) \) and let \( \Omega \) be a strongly \( \alpha \)-subhyperbolic domain in \( \mathbb{R}^n \). Then \( \Omega \) is \( \tau \)-subhyperbolic for some \( \tau \in (0, \alpha) \).

**Proof.** Since \( \Omega \) is strongly \( \alpha \)-subhyperbolic, there exist constants \( \theta > 0 \) and \( C \geq 1 \) such that every \( x, y \in \Omega \), \( \| x - y \| \leq \theta \), can be joined by a rectifiable curve \( \gamma \subset \Omega \) satisfying the following condition: for every \( u, v \in \gamma \)

\[
\int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \leq C \| u - v \|^\alpha.
\]

In particular,

\[
\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \leq C \| x - y \|^\alpha. \quad (2.10)
\]

Let \( \ell := \text{length}(\gamma) \). We parameterize \( \gamma \) by arclength: thus \( \gamma : [0, \ell] \to \Omega \), \( \gamma(0) = x \), \( \gamma(\ell) = y \). Let \( u = \gamma(t_1) \), \( v = \gamma(t_2) \) where \( 0 < t_1 < t_2 \leq \ell \) (recall that by \( \gamma_{uv} \) we denote the arc of \( \gamma \) joining \( u \) to \( v \)).

Applying part (ii) of Lemma 2.2 to the arc \( \gamma_{uv} \), we obtain

\[
\frac{1}{\text{length}(\gamma_{uv})} \int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \leq 2C \inf_{z \in \gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha - 1},
\]

or, in the parametric form,

\[
\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \text{dist}(\gamma(t), \partial \Omega)^{\alpha - 1} dt \leq 2C \inf_{t \in [t_1, t_2]} \text{dist}(\gamma(t), \partial \Omega)^{\alpha - 1}. \quad (2.11)
\]
Put 
\[ w(t) := \text{dist}(\gamma(t), \partial \Omega). \]

By (2.11), the function \( w = w(t) \) has the following property: for every subinterval \( I \subset [0, \ell] \)
\[
\frac{1}{|I|} \int_I w(t)^{\alpha - 1} \, dt \leq 2C \inf_I w(t)^{\alpha - 1}.
\]

Thus the function \( h := w^{\alpha - 1} \) is a Muckenhoupt’s \( A_1 \) weight on \( [0, \ell] \), see, e.g., [GR], so that \( h \) satisfies the reverse Hölder inequality on \( [0, \ell] \) (see Mackenhoupt [Mac], Gehring [G], Coiffman and Fefferman [CF]): There exist constants \( \tilde{q} > 1 \) and \( C_1 \geq 1 \) (depending only on \( C \)) such that
\[
\left( \frac{1}{\ell} \int_0^\ell h^{\tilde{q}}(t) \, dt \right)^{1/\tilde{q}} \leq C_1 \frac{1}{\ell} \int_0^\ell h(t) \, dt.
\]

By (2.10),
\[
\int_0^\ell w^{\alpha - 1}(t) \, dt \leq C \| x - y \|^\alpha
\]
so that
\[
\left( \frac{1}{\ell} \int_0^\ell w^{(\alpha - 1)\tilde{q}}(t) \, dt \right)^{1/\tilde{q}} \leq C_1 \frac{1}{\ell} \int_0^\ell w^{\alpha - 1}(t) \, dt \leq C_1 C_2 \ell \| x - y \|^\alpha.
\]

We put \( q := \min\{\tilde{q}, \frac{1-\alpha/2}{1-\alpha}\} \) and \( \tau = q(\alpha - 1) + 1 \). Clearly, \( 1 < q \leq \tilde{q} \) and \( 0 < \tau < \alpha \).

Hence,
\[
\left( \frac{1}{\ell} \int_0^\ell w^{\tau - 1}(t) \, dt \right)^{1/q} = \left( \frac{1}{\ell} \int_0^\ell w^{q(\alpha - 1)}(t) \, dt \right)^{1/q} \leq \left( \frac{1}{\ell} \int_0^\ell w^{\tilde{q}(\alpha - 1)}(t) \, dt \right)^{1/\tilde{q}} \leq C_2 \frac{1}{\ell} \| x - y \|^\alpha
\]
where \( C_2 := C_1 C \). Finally, we obtain
\[
\int_\gamma \text{dist}(z, \partial \Omega)^{\tau - 1} \, ds(z) = \int_0^\ell w^{\tau - 1}(t) \, dt \leq \frac{C_2^q}{\ell^{q-1}} \| x - y \|^{aq} \leq C_2^q \| x - y \|^{aq - q + 1} = C_2^q \| x - y \|^{\alpha q - q + \tau}
\]
proving that \( \Omega \) is a \( \tau \)-subhyperbolic domain. \( \square \)

**Proof of Theorem 1.4.** Let \( \varepsilon > 0 \) and let \( \Omega \in U_\alpha(\mathbb{R}^n) \). We will assume that the constant \( C_{\alpha, \Omega} \geq 1 \). Put
\[
\theta := \frac{1}{2} e^{-2C_{\alpha, \Omega}} \theta_{\alpha, \Omega}
\]
and fix $x, y \in \Omega$ such that $\|x - y\| \leq \theta$.

By part (ii) of Lemma 2.1, if inequality (2.3) is satisfied, then the statement of Theorem 1.4 is true with $\Gamma = \tilde{\Gamma} = [x, y]$ and any $\alpha^* \in (0, \alpha)$.

Now suppose that $x, y$ satisfy inequality (2.1), i.e.,

$$\max(\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)) \leq 2\|x - y\|.$$ 

**Lemma 2.7** Let $\Omega \in U_\alpha(\mathbb{R}^n)$ and let $x, y \in \Omega$, $\|x - y\| \leq \theta$. Let $0 < \delta \leq d_{\alpha, \Omega}(x, y)$ and let $\gamma \subset \Omega$ be a rectifiable curve joining $x$ to $y$ such that

$$\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) = \int_{\gamma_{xu}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) + \int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) + \int_{\gamma_{vy}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \geq d_{\alpha, \Omega}(x, u) + (d_{\alpha, \Omega}(u, v) + \delta) + d_{\alpha, \Omega}(v, y)$$

so that, by the triangle inequality for the metric $d_{\alpha, \Omega}$,

$$\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \geq d_{\alpha, \Omega}(x, y) + \delta.$$ 

Proof. First prove that

$$\int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) < d_{\alpha, \Omega}(u, v) + \delta.$$ 

(2.14)

In fact, assume that

$$d_{\alpha, \Omega}(u, v) + \delta \leq \int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z).$$

Then

$$\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) = \int_{\gamma_{xu}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) + \int_{\gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) + \int_{\gamma_{vy}} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z)$$

so that, by the triangle inequality for the metric $d_{\alpha, \Omega}$,
which contradicts inequality (2.12).

Since $0 < \delta \leq d_{a,\Omega}(x, y)$, by (2.12),

$$
\int_\gamma \dist(z, \partial \Omega)^{\alpha-1} \, ds(z) < 2d_{a,\Omega}(x, y).
$$

Since $\theta \leq \theta_{a,\Omega}$ and $\Omega \in U_a(R^n)$, we have $d_{a,\Omega}(x, y) \leq C_{a,\Omega}\|x - y\|^\alpha$ so that

$$
\int_\gamma \dist(z, \partial \Omega)^{\alpha-1} \, ds(z) < 2C_{a,\Omega}\|x - y\|^\alpha.
$$

By Lemma 2.1, part (i),

$$
\length(\gamma) \leq 2e^{2C_{a,\Omega}\|x - y\|}
$$

proving (i). Hence,

$$
\length(\gamma) \leq 2e^{2C_{a,\Omega}\theta} = 2e^{2C_{a,\Omega}(\frac{1}{2} e^{-2C_{a,\Omega}})\theta_{a,\Omega}} = \theta_{a,\Omega}
$$

so that for every $u, v \in \gamma$ we have $\|u - v\| \leq \length(\gamma) \leq \theta_{a,\Omega}$. Since $\Omega \in U_a(R^n)$, this implies

$$
d_{a,\Omega}(u, v) \leq C_{a,\Omega}\|u - v\|^\alpha \leq C_{a,\Omega} \length^\alpha(\gamma_{uv}).
$$

Combining this inequality with (2.14) and (2.13), we obtain

$$
\int_\gamma \dist(z, \partial \Omega)^{\alpha-1} \, ds(z) \leq C_{a,\Omega} \length^\alpha(\gamma_{uv}) + \delta
$$

$$
\quad \leq C_{a,\Omega} \length^\alpha(\gamma_{uv}) + \length^\alpha(\gamma_{uv})
$$

proving (ii) and the lemma. \hfill \Box

Put

$$
m := \left\lfloor 2(2C_{a,\Omega})^{\frac{1}{1-\alpha}} \right\rfloor + 1. \quad (2.15)
$$

Let $k$ be a positive integer such that

$$
2e^{2C_{a,\Omega}\|x - y\|(1 - 1/m)^k} \leq \varepsilon. \quad (2.16)
$$

Finally, we put

$$
\delta := \min\{d_{a,\Omega}(x, y), m^{-\alpha k}\|x - y\|^\alpha\}.
$$

Thus $0 < \delta \leq d_{a,\Omega}(x, y)$ and

$$
\|x - y\| m^{-k} \geq \delta^{\frac{1}{\alpha}}. \quad (2.17)
$$

Let $\Gamma \subset \Omega$ be a rectifiable curve joining $x$ to $y$ such that

$$
\int_\Gamma \dist(z, \partial \Omega)^{\alpha-1} \, ds(z) < d_{a,\Omega}(x, y) + \delta.
$$
Then, by Lemma 2.7, 
\[ \int_{\Gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq 2C_{\alpha, \Omega} \|x - y\|^\alpha \]  
(2.18)
and
\[ \text{length}(\Gamma) \leq 2e^{2C_{\alpha, \Omega}} \|x - y\|. \]  
(2.19)

Moreover, for every \( u, v \in \Gamma \) such that
\[ \text{length}(\Gamma_{uv}) \geq \delta_1^\alpha \]  
(2.20)
the following inequality
\[ \int_{\Gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq 2C_{\alpha, \Omega} \text{length}^\alpha(\Gamma_{uv}) \]  
(2.21)
holds.

Inequality (2.21) and part (ii) of Lemma 2.2 imply:
\[ \frac{1}{\text{length}(\Gamma_{uv})} \int_{\Gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq 4C_{\alpha, \Omega} \inf_{z \in \Gamma_{uv}} \text{dist}(z, \partial \Omega)^{\alpha - 1}. \]  
(2.22)

Put
\[ L := \text{length}(\Gamma). \]

Since \( \|x - y\| \leq L \), by (2.17) and (2.20), for every \( u, v \in \Gamma \) such that
\[ \text{length}(\Gamma_{uv}) \geq L m^{-k} \]  
(2.23)
inequality (2.21) is satisfied.

By \( I_m \) we denote the family of all \( m \)-adic closed subintervals of the interval \( I_0 := [0, L] \).

Recall that this family of intervals can be obtained by the standard iterative procedure: we start with the entire interval \([0, L]\) and, at each level of the construction, we split every interval of the given level into \( m \) equally sized closed subintervals.

Let
\[ I_{0,m} := \{0, L\}, \quad I_{1,m} := \{[[Li/m, L(i + 1)/m]] : i = 0, 1, \ldots, m - 1\} \text{ etc.} \]

Clearly, \( |I| = L m^{-j} \) for every interval \( I \) of the \( j \)-th level. Put
\[ S_{k,m} := \bigcup_{j=0}^{k} I_{j,m} = \{\text{all } m\text{-adic intervals of the level at most } k\}. \]

Let us parameterize \( \Gamma \) by arclength; thus we identify \( \gamma \) with a function \( \Gamma : [0, L] \to \Omega \) satisfying \( \Gamma(0) = x, \Gamma(L) = y \). Finally, put
\[ g(t) := \text{dist}(\Gamma(t), \partial \Omega)^{\alpha - 1}, \quad t \in [0, L]. \]
Then, by (2.23) and (2.22), the following is true: For each \( m \)-adic interval \( I \in S_{k,m} \) we have

\[
\frac{1}{|I|} \int_I g(t) \, dt \leq C_g \inf_{t \in I} g(t) \tag{2.24}
\]

with \( C_g := 4C_{\alpha, \Omega} \).

Following Melas [Mel] we say that \( g \) is a Muckenhoupt \( A_1 \)-weight on \([0, L]\) with respect to the family \( S := S_{k,m} \) of all \( m \)-adic intervals of the level at most \( k \). We let \( \mathcal{M}_S \) denote the corresponding maximal operator for the family \( S \):

\[
\mathcal{M}_S g(t) := \sup \left\{ \frac{1}{|I|} \int_I |g(t)| \, dt : I \ni t, I \in S \right\} \tag{2.25}
\]

Thus (2.24) is equivalent to the inequality

\[
\mathcal{M}_S g(t) \leq C_g g(t), \quad t \in [0, L].
\]

Put

\[
q^\# := \frac{\log m}{\log (m - (m - 1)/C_g)}
\]

and \( q^* := (1 + q^\#)/2 \). Clearly, \( 1 \leq C_g < \infty \) so that \( q^\#, q^* > 1 \).

We will be needed the following corollary of a general result proved in [Mel].

**Theorem 2.8** For any \( A_1 \)-weight \( g \) (with respect to \( S \)) and any \( q \), \( 1 \leq q \leq q^* \), the following inequality

\[
\left( \frac{1}{L} \int_0^L (\mathcal{M}_S g)^q \, dt \right)^{\frac{1}{q}} \leq \widetilde{C} \left( \frac{1}{L} \int_0^L g \, dt \right)
\]

holds. Here \( \widetilde{C} \) is a constant depending only on \( m \) and \( C_g \).

**Remark 2.9** Actually the theorem is true for \( 1 \leq q < q^\# \) but with \( \widetilde{C} \) depending on \( m, C_g \) and \( q \), see [Mel].

**Corollary 2.10** For any \( A_1 \)-weight \( g \) (with respect to \( S \)), any family \( \mathcal{A} \) of non-overlapping \( m \)-adic intervals of the level at most \( k \) and any \( q \), \( 1 \leq q \leq q^* \), we have

\[
\left( \frac{1}{L} \sum_{I \in \mathcal{A}} \left( \frac{1}{|I|} \int_I g \, dt \right)^q |I| \right)^{\frac{1}{q}} \leq \widetilde{C} \left( \frac{1}{L} \int_0^L g \, dt \right). \tag{2.27}
\]
Proof. In fact, by definition (2.25), for every $I \in A$ and every $t \in I$

$$\frac{1}{|I|} \int_I g \, ds \leq \mathcal{M}_S g(t)$$

so that

$$\left( \frac{1}{|I|} \int_I g \, dt \right)^q |I| \leq \int_I (\mathcal{M}_S g)^q \, dt.$$ 

Therefore the left-hand side of (2.27) does not exceed

$$\left( \frac{1}{L} \sum_{I \in A} \int_I (\mathcal{M}_S g)^q \, dt \right)^\frac{1}{q} \leq \left( \frac{1}{L} \int_0^L (\mathcal{M}_S g)^q \, dt \right)^\frac{1}{q}$$

which together with (2.26) implies the required inequality (2.27). \qed

We turn to construction of a family $A \subset S_{k,m}$ of non-overlapping $m$-adic intervals of the level at most $k$ such that for each $I \in A$

$$\sup_I g \leq C \inf_I g$$

and

$$| [0, L] \setminus \{ \cup I : I \in A \} | < \varepsilon.$$ 

Here $C$ is a constant depending only on $n, \alpha$, and $C_{\alpha, \Omega}$.

Let $I = [t_1, t_2] \in S_{k-1,m}$ be an $m$-adic interval of the level at most $k - 1$ and let $u := \Gamma(t_1), v := \Gamma(t_2)$. By (2.21) and part (i) of Lemma 2.2, there exists $t_I \in I$ such the point $z_I = \Gamma(t_I) \in \Gamma_{uv}$ satisfies the following inequality:

$$\text{length}(\Gamma_{uv}) \leq C' \text{dist}(z_I, \partial \Omega) \quad (2.28)$$

with $C' := (2C_{\alpha, \Omega})^{1-\alpha}$.

Let us split the interval $I$ into $m$ equal subintervals $I^{(1)}, ..., I^{(m)}$. Then $t_I \in I^{(j)}$ for some $j \in \{1, ..., m\}$. By $\Gamma^{(j)} := \Gamma|_{I^{(j)}}$ we denote the arc corresponding to the interval $I^{(j)}$. Thus $I^{(j)} \in S_{k,m}$ is an $m$-adic interval of the level at most $k$ and

$$\text{length}(\Gamma^{(j)}) = |I^{(j)}| = |I|/m = \text{length}(\Gamma_{uv})/m.$$

Since $\text{dist}(\cdot, \partial \Omega)$ is a Lipschitz function, for every $t \in I^{(j)}$ we have

$$| \text{dist}(z_I, \partial \Omega) - \text{dist}(z(t), \partial \Omega) | \leq \|z_I - z(t)\| \leq \text{length}(\Gamma^{(j)}) = \text{length}(\Gamma_{uv})/m.$$ 

Combining this inequality with (2.28) we obtain

$$| \text{dist}(z_I, \partial \Omega) - \text{dist}(z(t), \partial \Omega) | \leq \frac{C'}{m} \text{dist}(z_I, \partial \Omega).$$
But $m := [2C'] + 1$, see (2.15), so that

$$|\text{dist}(z_I, \partial \Omega) - \text{dist}(z(t), \partial \Omega)| \leq \frac{1}{2} \text{dist}(z_I, \partial \Omega).$$

Hence

$$\frac{1}{2} \text{dist}(z_I, \partial \Omega) \leq \text{dist}(z(t), \partial \Omega) \leq \frac{3}{2} \text{dist}(z_I, \partial \Omega), \quad t \in I^{(j)}.$$

We let $\tilde{I}$ denote the interval $I^{(j)}$. Thus we have proved that for each $I \in S_{k-1,m}$ there exists a subinterval $\tilde{I} \in S_{k,m}$, $\tilde{I} \subset I$, such that

$$\max_{t \in \tilde{I}} \text{dist}(z(t), \partial \Omega) \leq 3 \min_{t \in \tilde{I}} \text{dist}(z(t), \partial \Omega).$$

Since $g(t) := \text{dist}(z(t), \partial \Omega)^{\alpha - 1}$, we obtain

$$\max_{t \in \tilde{I}} g(t) \leq 3^{1-\alpha} \min_{t \in \tilde{I}} g(t).$$

Now we construct the family $\mathcal{A} \subset S_{k,m}$ as follows. At the first stage for the interval $I_0 := [0, L]$ we determine an $m$-adic interval $\tilde{I}_0 \in \mathcal{I}_{1,m}$ of the first level and put $\mathcal{A}_1 := \{ \tilde{I}_0 \}$ and $U_1 := \tilde{I}_0$.

Let us consider the set $[0, L] \setminus U_1$ which consists of $m - 1$ $m$-adic intervals of the first level. We let $\mathcal{B}_1$ denote the family of these intervals. For every $I \in \mathcal{B}_1$ we construct the interval $\tilde{I} \in \mathcal{I}_{2,m}$ and put

$$\mathcal{A}_2 := \{ \tilde{I} \in \mathcal{I}_{2,m} : I \in \mathcal{B}_1 \}.$$

By $U_2$ we denote the set

$$U_2 := U_1 \cup \{ I : I \in \mathcal{A}_2 \}.$$

Now the set $[0, L] \setminus U_2$ consists of $(m - 1)^2$ $m$-adic intervals of the second level. We denote the family of these intervals by $\mathcal{B}_2$ and finish the second stage of the procedure.

After the $k$-th stages of this procedure we obtain the families $\mathcal{A}_j \subset \mathcal{I}_{j,m}$, $j = 1, 2, ..., k$, of $m$-adic intervals. We put

$$\mathcal{A} = \cup \{ \mathcal{A}_j : j = 1, ..., k \}.$$

Thus $\mathcal{A} \subset S_{k,m}$ is a family of $m$-adic intervals of the level at most $k$. We know that for every interval $I \in \mathcal{A}$ the following inequality

$$\max_{t \in I} g(t) \leq 3^{1-\alpha} \min_{t \in I} g(t)$$

holds. We also know that the set

$$U = U_k := \cup \{ I : I \in \mathcal{A} \}$$

has the following property: the set

$$E := [0, L] \setminus U = [0, L] \setminus \cup \{ I : I \in \mathcal{A} \}$$
consists of \((m - 1)^k\) \(m\)-adic intervals of the \(k\)-th level. Since \(|I| = m^{-k} L\) for each \(I \in \mathcal{I}_{k,m}\), we obtain

\[ |E| = \frac{(m - 1)^k}{m^k} L. \]

But, by (2.19),

\[ L = \text{length}(\Gamma) \leq C'' \|x - y\| \]

where \(C'' := 2e^{2C_{\alpha,\Omega}}\). Hence,

\[ |E| \leq C'' \|x - y\|(1 - 1/m)^k. \]

Combining this inequality with (2.16), we obtain the required estimate

\[ |E| = |[0, L] \setminus U| \leq \varepsilon. \quad (2.30) \]

Now for the family \(\mathcal{A}\) constructed above let us estimate from below the quantity

\[ T := \left( \frac{1}{L} \sum_{I \in \mathcal{A}} \left( \frac{1}{|I|} \int_I g\, dt \right)^q |I| \right)^\frac{1}{q} \]

from the left-hand side of inequality (2.27). By (2.29), for each \(I \in \mathcal{A}\) we have

\[ \int_I g^q\, dt \leq |I| \max_{t \in I} g^q(t) \leq \beta qg(1-\alpha)|I| (\min_{t \in I} g(t))^q \leq \beta qg(1-\alpha)|I| \left( \frac{1}{|I|} \int_I g\, dt \right)^q \]

so that

\[ T^q \geq \beta qg(1-\alpha) \frac{1}{L} \sum_{I \in \mathcal{A}} \int_I g^q\, dt = \beta qg(1-\alpha) \frac{1}{L} \int_U g^q\, dt. \]

(Recall that \(U = \bigcup\{I : I \in \mathcal{A}\}\).)

By Corollary 2.10,

\[ T \leq \tilde{C} \left( \frac{1}{L} \int_0^L g\, dt \right) \]

so that

\[ \frac{1}{L} \int_U g^q\, dt \leq \beta qg(1-\alpha) T^q \leq C_1 \left( \frac{1}{L} \int_0^L g\, dt \right)^q. \]

with \(C_1 := \beta qg(1-\alpha)\tilde{C}\). On the other hand, by inequality (2.18),

\[ \int_0^L g\, dt = \int_{\Gamma} \dist(z, \partial\Omega)^{\alpha-1}\, ds(z) \leq 2C_{\alpha,\Omega}\|x - y\|^\alpha. \]
Hence

\[ \frac{1}{L} \int_U g^q \, dt \leq C_1 \left( \frac{1}{L} \int_0^L g \, dt \right)^q \leq C_2 \|x - y\|^{q^\alpha} / L^q \]

with \( C_2 := (2C_{\alpha, \alpha})^q C_1 \).

Recall that this inequality holds for all \( q \in [1, q^*] \), see Corollary 2.10. We put \( \bar{q} := \min\{q^*, \frac{1 - \alpha/2}{1 - \alpha}\} \) and \( \alpha^* := 1 - \bar{q}(1 - \alpha) \). Since \( q^* > 1 \) and \( 0 < \alpha < 1 \), we have \( 1 < \bar{q} \leq q \) and \( 0 < \alpha^* < \alpha \).

Let \( \tau \in [\alpha^*, \alpha] \) and let \( q := \frac{1 - \tau}{1 - \alpha} \). Then \( q \in [1, q^*] \) so that

\[ \int_U \text{dist}(\Gamma(t), \partial \Omega) \tau^{-1} \, dt = \int_U g^{\frac{1 - \tau}{1 - \alpha}}(t) \, dt = \int_U g^q(t) \, dt \leq \int_0^L g^q(t) \, dt \leq C_2 \|x - y\|^{q^\alpha} / L^q \]
Since $\Gamma(I) \subset B$, we have $|I \cap U| \leq \text{length}(B \cap \tilde{\Gamma})$ so that
\[
diam B = 2r \leq 2\ell = |I| \leq 4m |I \cap U| \leq 4m \text{length}(B \cap \tilde{\Gamma})
\]
proving (1.8).

Theorem 1.4 is completely proved. \(\square\)

3. Sobolev functions on subhyperbolic domains.

Let us fix some additional notation. In what follows, the terminology “cube” will mean a closed cube in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes. We let $Q(x, r)$ denote the cube in $\mathbb{R}^n$ centered at $x$ with side length $2r$. Given $\lambda > 0$ and a cube $Q$, we let $\lambda Q$ denote the dilation of $Q$ with respect to its center by a factor of $\lambda$. (Thus $\lambda Q(x, r) = Q(x, \lambda r)$.)

It will be convenient for us to measure distances in $\mathbb{R}^n$ in the uniform norm $\|x\| := \max\{|x_i| : i = 1, ..., n\}$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

Thus every cube $Q = Q(x, r) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$ is a “ball” in $\| \cdot \|$-norm of “radius” $r$ centered at $x$.

By $\chi_A$ we denote the characteristic function of $A$; we put $\chi_A \equiv 0$ whenever $A = \emptyset$.

Let $A = \{Q\}$ be a family of cubes in $\mathbb{R}^n$. By $M(A)$ we denote its covering multiplicity, i.e., the minimal positive integer $M$ such that every point $x \in \mathbb{R}^n$ is covered by at most $M$ cubes from $A$. Thus
\[
M(A) := \sup_{x \in \mathbb{R}^n} \sum_{Q \in A} \chi_Q(x).
\]

Recall that $\mathcal{P}_k$ denotes the space of polynomials of degree at most $k$ defined on $\mathbb{R}^n$. Also recall that, given a $k$-times differentiable function $F$ and a point $x \in \mathbb{R}^n$, we let $T^k_x(F)$ denote the Taylor polynomial of $F$ at $x$ of degree at most $k$:
\[
T^k_x(F)(y) := \sum_{|\beta| \leq k} \frac{1}{\beta!} (D^\beta F)(x)(y - x)^\beta, \quad y \in \mathbb{R}^n.
\]

Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $k \in \mathbb{N}$ and $p \in [1, \infty]$. We let $L^k_p(\Omega)$ denote the (homogeneous) Sobolev space of all functions $f \in L^1_{1, \text{loc}}(\Omega)$ whose distributional partial derivatives on $\Omega$ of order $k$ belong to $L^p(\Omega)$. $L^k_p(\Omega)$ is normed by
\[
\|f\|_{L^k_p(\Omega)} := \left( \int_{\Omega} \|\nabla^k f\|^p \, dx \right)^{\frac{1}{p}}
\]
where $\nabla^k f$ denotes the vector with components $D^\beta f$, $|\beta| = k$, and
\[
\|\nabla^k f\|(x) := \left( \sum_{|\beta| = k} |D^\beta f(x)|^2 \right)^{\frac{1}{2}}, \quad x \in \Omega.
\]
By the Sobolev imbedding theorem, see e.g., [M], p. 60, every $f \in L^k_p(\Omega), p > n$, can be redefined, if necessary, in a set of Lebesgue measure zero so that it belongs to the space $C^{k-1}(\Omega)$. Moreover, for every cube $Q \subset \Omega$, every $x,y \in Q$ and every multiindex $\beta, |\beta| \leq k-1$, the following inequality

$$|D^\beta T^{k-1}_x(f)(x) - D^\beta T^{k-1}_y(f)(x)| \leq C(n,p)\|x - y\|^{k-|\beta|-\frac{n}{p}} \left(\int_Q \|\nabla f\|^{p} \, dx\right)^{\frac{1}{p}} \tag{3.1}$$

holds. In particular, the partial derivatives of order $k-1$ satisfy a (local) Hölder condition of order $\alpha := 1 - \frac{2}{p}$:

$$|D^\beta f(x) - D^\beta f(y)| \leq C(n,p)\|f\|_{L^k_p(\Omega)}\|x - y\|^{1-\frac{n}{p}}, \quad |\beta| = k-1,$$

provided $Q$ is a cube in $\Omega$ and $x, y \in Q$.

Thus, for $p > n$, we can identify each element $f \in L^k_p(\Omega)$ with its unique $C^{k-1}$-representative on $\Omega$. This will allow us to restrict our attention to the case of Sobolev $C^{k-1}$-functions.

The main result of this section is the following

**Theorem 3.1** Let $n < p < \infty$, $\alpha = (p-n)/(p-1)$, and let $\Omega$ be an $\alpha$-subhyperbolic domain in $\mathbb{R}^n$. There exists a constant $p^*, n < p^* < p$, and constants $\lambda, \theta, C > 0$ depending only on $n,p,k,C_{\alpha,\Omega}$ and $\theta_{\alpha,\Omega}$, such that the following is true: Let $f \in L^k_p(\Omega), x,y \in \Omega, \|x - y\| \leq \theta$, and let $Q_{x,y} := Q(x, \|x - y\|)$. Then for every multiindex $\beta, |\beta| \leq k-1$, the following inequality

$$|D^\beta T^{k-1}_x(f)(x) - D^\beta T^{k-1}_y(f)(x)| \leq C\|x - y\|^{k-|\beta|-\frac{n}{p^*}} \left(\int_{(\lambda Q_{x,y}) \cap \Omega} \|\nabla f\|^{p^*} \, dx\right)^{\frac{1}{p^*}} \tag{3.2}$$

holds.

**Proof.** We will be needed the following

**Lemma 3.2** Let $x,y \in \mathbb{R}^n$ and let $\gamma \subset \Omega$ be a continuous curve joining $x$ to $y$. There is a finite family of cubes $Q = \{Q_0, \ldots, Q_m\}$ such that:

(i). $Q_0 \ni x, Q_m \ni y, Q_i \neq Q_j, i \neq j, 0 \leq i, j \leq m$, and

$$Q_i \cap Q_{i+1} = \emptyset, \quad i = 0, \ldots, m-1.$$

(ii). For every cube $Q = Q(z,r) \in Q$ we have $z \in \gamma$ and $r = \frac{1}{8}\text{dist}(z, \partial \Omega)$.

(iii). For each $Q \in Q$ the cube $2Q \subset \Omega$. Moreover, the covering multiplicity of the family of cubes $2Q := \{2Q : Q \in Q\}$ is bounded by a constant $C = C(n)$.

**Proof.** For every $z \in \Gamma$ we let $Q^{(z)}$ denote the cube

$$Q^{(z)} := Q(z, \frac{1}{8}\text{dist}(z, \partial \Omega)).$$
We put \( A := \{ Q(z) : z \in \Gamma \} \). By the Besicovitch covering theorem, see e.g. [G], there exists a finite subcollection \( B \subset A \) such that \( B \) still covers \( \Gamma \) but no point which lies in more than \( C(n) \) of the cubes of \( B \). (Thus the covering multiplicity \( M(B) \leq C(n) \).)

Given \( Q', Q'' \in B \) we write \( Q' \sim Q'' \) if there exists a family \( \{ K_0, ..., K_\ell \} \subset B \) such that \( K_0 = Q', K_\ell = Q'', K_i \neq K_j, i \neq j, 0 \leq i, j \leq \ell \), and \( K_i \cap K_{i+1} \neq \emptyset \) for every \( i = 0, ..., \ell - 1 \). Fix a cube \( \bar{Q} \in B \) such that \( x \in \bar{Q} \), and put

\[
B' := \{ Q \in B : Q \sim \bar{Q} \} \tag{3.3}
\]

and \( E := \cup \{ Q : Q \in B' \} \).

Prove that \( y \in E \). Assume that this is not true, i.e., \( y \notin E \). Since \( \bar{Q} \ni x \), we have \( x \in E \) so that \( E \cap \gamma \neq \emptyset \). Put \( F := \cup \{ Q : Q \in B \setminus B' \} \). Observe that every cube \( Q \in B \) such that \( Q \cap E \neq \emptyset \) belongs to \( B' \) so that \( E \cap F = \emptyset \).

On the other hand, since \( \gamma \subset E \cup F \) and \( y \notin E \), we have \( y \in F \) proving that \( F \cap \gamma \neq \emptyset \). Thus the sets \( E \cap \gamma \) and \( F \cap \gamma \) is a partition of the continuous curve \( \gamma \) into two closed disjoint sets; a contradiction. We have proved that \( y \in Q^* \) for some \( Q^* \in B' \) so that, by definition (3.3), there exists a family of cubes \( Q = \{ Q_0, ..., Q_m \} \) satisfying conditions (i) and (ii) of the lemma.

Prove (iii). Let \( Q = Q(z, r) \in Q \). Then \( z \in \gamma \) and \( r = \frac{1}{8} \text{dist}(z, \partial \Omega) \) so that for every \( u \in 2Q = Q(z, 2r) \) we have

\[
\text{dist}(z, \partial \Omega) \leq \text{dist}(u, \partial \Omega) + \| u - z \| \leq \text{dist}(u, \partial \Omega) + 2r \leq \text{dist}(u, \partial \Omega) + \frac{1}{4} \text{dist}(z, \partial \Omega).
\]

Hence,

\[
0 < \frac{3}{4} \text{dist}(z, \partial \Omega) \leq \text{dist}(u, \partial \Omega)
\]

proving that \( 2Q \subset \Omega \).

It remains to prove that the covering multiplicity \( M(2Q) \leq C(n) \). We know that \( M(Q) \leq M(B) \leq C(n) \). Fix a cube \( Q = Q(z, r) \in Q \). Let \( Q_i = Q(z_i, r_i) \in Q \) be an arbitrary cube such that

\[
(2Q) \cap (2Q_i) \neq \emptyset. \tag{3.4}
\]

Then \( \| z - z_i \| \leq 2r + 2r_i \) so that

\[
r = \frac{1}{8} \text{dist}(z, \partial \Omega) \leq \frac{1}{8} \text{dist}(z_i, \partial \Omega) + \frac{1}{8} \| z - z_i \| \leq r_i + \frac{1}{8}(2r + 2r_i) = \frac{1}{4}r + \frac{5}{8}r_i.
\]

Hence \( r \leq \frac{5}{2}r_i \). In the same way we prove that \( r_i \leq \frac{5}{2}r \).

Since \( Q \) has the covering multiplicity at most \( C(n) \), this collection of cubes can be partitioned into at most \( N(n) \) families of pairwise disjoint cubes, see e.g. [BrK]. Therefore, without loss of generality, we may assume that \( Q \) itself is a collection of pairwise disjoint cubes.

Since \( \frac{1}{2}r \leq r_i \leq 2r \), we have \( 2^{-n}|Q| \leq |Q_i| \leq 2^n|Q| \). Also, by (3.4) and the inequality \( r_i \leq 2r \), we have \( Q_i \subset 7Q \). Thus the number of cubes \( Q_i \), satisfying (3.4) is bounded by \( |7Q|/(2^{-n}|Q|) = 2^n7^n \). The lemma is proved. \( \square \)

Let \( x, y \in \Omega, \| x - y \| \leq \theta \), where \( \theta \) is the constant from Theorem 1.4. By this theorem there exist constants \( \alpha^* = \alpha^*(n, p), 0 < \alpha^* < \alpha \), and \( C = C(n, p) > 0 \) such that for every
\( \varepsilon > 0 \) there exists a rectifiable curve \( \Gamma \subset \Omega \) and a finite family of arcs \( \tilde{\Gamma} \subset \Gamma \) satisfying conditions (i),(ii) of the theorem.

Observe that, by inequality (1.7) (with \( \tau = \alpha \)) and by (1.10),

\[
\int_{\Gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq C \|x - y\|^\alpha. \tag{3.5}
\]

Also, by part (ii) of Theorem 1.4,

\[
\text{length}(\Gamma) \leq \tilde{C} \|x - y\| \tag{3.6}
\]

with \( \tilde{C} = 2e^{2C_{\alpha, \Omega}}, \) see (2.19).

By Lemma 3.2, there exists a collection of cubes \( Q = \{Q_0, \ldots, Q_m\} \)
satisfying conditions (i)-(iii) of the lemma. In particular, by (i), \( Q_0 \ni x, Q_m \ni y, Q_i \not\ni Q_j, i \neq j, \) and

\[
Q_i \cap Q_{i+1} \neq \emptyset, \quad i = 0, \ldots, m - 1.
\]

Let \( Q_i = Q(z_i, r_i), i = 0, \ldots, m, \) (recall that by (ii) \( z_i \in \Gamma \) and \( r_i = \frac{1}{8} \text{dist}(z_i, \partial \Omega) \)). Let \( a_i \in Q_{i-1} \cap Q_i, \) \( i = 1, \ldots, m. \) We also put \( a_0 := x, a_{m+1} := y. \)

We may assume that for every \( Q \in Q \) either \( x \notin Q \) or \( y \notin Q. \) In fact, otherwise \( x, y \in Q. \) But by condition (iii) of Lemma 3.2, \( 2Q \subset \Omega. \) Then the cube \( Q(x, \|x - y\|) \subset 2Q \subset \Omega \)
as well. It remains to apply inequality (3.1) to \( x \) and \( y \) with \( p = p^*, n < p^* < p, \) and (3.2) follows.

Thus we assume that for every cube \( Q_i = Q(z_i, r_i) \in Q \)

either \( x \notin Q_i \) or \( y \notin Q_i. \)

Since \( x, y, z_i \in \Gamma, \) we have \( \partial Q_i \cap \Gamma \neq \emptyset \) so that there exists a point \( a_i \in \partial Q_i \cap \Gamma. \) Hence, by (3.6),

\[
r_i = \|z_i - a_i\| \leq \text{length}(\Gamma \cap Q_i) \tag{3.7}
\]

so that, by (3.6),

\[
r_i \leq \text{length}(\Gamma) \leq \tilde{C} \|x - y\|, \quad i = 0, \ldots, m. \tag{3.8}
\]

Now we have

\[
A := |D^\beta T_x^{k-1}(f)(x) - D^\beta T_y^{k-1}(f)(x)| = |D^\beta T_{a_0}^{k-1}(f)(a_0) - D^\beta T_{a_{m+1}}^{k-1}(f)(a_0)| \\
\leq \sum_{i=0}^{m} |D^\beta T_{a_i}^{k-1}(f)(a_0) - D^\beta T_{a_{i+1}}^{k-1}(f)(a_0)|.
\]

Put

\[
P_i(z) := T_{a_i}^{k-1}(f)(z) - T_{a_{i+1}}^{k-1}(f)(z), \quad i = 0, \ldots, m.
\]
The polynomial \( P_i \in \mathcal{P}_{k−1} \) so that for every multiindex \( \beta, |\beta| \leq k − 1 \), we have
\[
D^\beta P_i(z) = \sum_{|\eta| \leq k−1−|\beta|} \frac{1}{\eta!} D^{\eta+\beta} P_i(a_i) (z - a_i)^\eta, \quad z \in \mathbb{R}^n.
\]

Hence
\[
|D^\beta P_i(a_0)| \leq C \sum_{|\eta| \leq k−1−|\beta|} |D^{\eta+\beta} P_i(a_i)| \|a_0 - a_i\|^{ |\eta|}.
\]

We put
\[
\mathcal{Q}_1 := \{ Q \in \mathcal{Q} : Q \cap \tilde{\Gamma} \neq \emptyset \}, \quad I_1 := \{ i \in \{0,...,m\} : Q_i \in \mathcal{Q}_1 \},
\]
and
\[
\mathcal{Q}_2 := \mathcal{Q} \setminus \mathcal{Q}_1, \quad I_2 := \{ i \in \{0,...,m\} : Q_i \in \mathcal{Q}_2 \}.
\]

Then
\[
A \leq \sum_{i=0}^m |D^\beta P_i(a_0)| \leq C \sum_{i=0}^m \sum_{|\eta| \leq k−1−|\beta|} |D^{\eta+\beta} P_i(a_i)| \|a_0 - a_i\|^{ |\eta|} = C \sum_{|\eta| \leq k−1−|\beta|} \left( \sum_{i=0}^m |D^{\eta+\beta} P_i(a_i)| \|a_0 - a_i\|^{ |\eta|} \right).
\]

Let
\[
A'_\eta := \sum_{i \in I_1} |D^{\eta+\beta} P_i(a_i)| \|a_0 - a_i\|^{ |\eta|}
\]
and
\[
A''_\eta := \sum_{i \in I_2} |D^{\eta+\beta} P_i(a_i)| \|a_0 - a_i\|^{ |\eta|}.
\]

We have proved that
\[
A \leq C \sum_{|\eta| \leq k−1−|\beta|} (A'_\eta + A''_\eta).
\] (3.11)

Let \( |\eta| \leq k−1−|\beta| \). Our next aim is to show that
\[
A'_\eta \leq C \|x - y\|^{k−|\beta|−\frac{n−\alpha^*}{p^*}} \left( \int_{(\lambda Q_{xy}) \cap \Omega} \|\nabla^k f\|^p \, dx \right)^{\frac{1}{p^*}}
\] (3.12)
where
\[
p^* := \frac{n−\alpha^*}{1−\alpha^*}
\]
and \( \lambda := 2\tilde{C} \). (Recall that \( Q_{xy} := Q(x, \|x - y\|) \).) Also we will prove that

\[
A''_\eta \leq C \|x - y\|^{k-|\beta|-\frac{n}{p}} \left( \int_E \|\nabla^k f\|^p dx \right)^{\frac{1}{p}}
\]  

(3.13)

where is \( E \) is a subset of \( \Omega \) of the Lebesgue measure \( |E| \leq C\varepsilon^n \).

Since

\[
0 < \alpha^* = \frac{p^* - n}{p^* - 1} < \alpha = \frac{p - n}{p - 1} < 1,
\]

we have \( n < p^* < p \). Since \( a_i, a_{i+1} \in Q_i = Q(z_i, r_i), i \in I_1 \), by inequality (3.1) (with \( p = p^* \)),

\[
|D^{n+\beta} P_i(a_i)| = |D^{n+\beta} T_{a_i}^{k-1}(f)(a_i) - D^{n+\beta} T_{a_{i+1}}^{k-1}(f)(a_i)|
\]

\[
\leq C \|a_i - a_{i+1}\|^{k-|\eta|-|\beta|} \left( \int_{Q_i} \|\nabla^k f\|^p dx \right)^{\frac{1}{p^*}}
\]

so that

\[
|D^{n+\beta} P_i(a_i)| \leq C r_i^{k-|\eta|-|\beta|-\frac{n}{p}} \left( \int_{Q_i} \|\nabla^k f\|^p dx \right)^{\frac{1}{p}}, \quad i \in I_1.
\]  

(3.14)

In a similar way we prove that

\[
|D^{n+\beta} P_i(a_i)| \leq C r_i^{k-|\eta|-|\beta|-\frac{n}{p}} \left( \int_{Q_i} \|\nabla^k f\|^p dx \right)^{\frac{1}{p}}, \quad i \in I_2.
\]

Prove that for each \( i \in \{0, ..., m\} \) we have

\[
\|a_0 - a_i\| \leq 2\tilde{C}\|x - y\|.
\]  

(3.15)

In fact,

\[
\|a_0 - a_i\| = \|x - a_i\| \leq \|x - z_i\| + \|z_i - a_i\| \leq \|x - z_i\| + r_i.
\]

Since \( x, z_i \in \Gamma \), by (3.6), \( \|x - z_i\| \leq \text{length}(\Gamma) \leq \tilde{C}\|x - y\| \). Also, by (3.8), \( r_i \leq \tilde{C}\|x - y\| \), proving (3.15).

Hence, by (3.14),

\[
A''_\eta \leq \left(2\tilde{C}\right)^{|\eta|} \sum_{i \in I_1} |D^{n+\beta} P_i(a_i)| \|x - y\|^{|\eta|} \leq C \sum_{i \in I_1} r_i^{k-|\eta|-|\beta|-\frac{n}{p}} \|x - y\|^{|\eta|} \left( \int_{Q_i} \|\nabla^k f\|^p dx \right)^{\frac{1}{p^*}}.
\]

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Since \( k - |\eta| - |\beta| \geq 1 \) and \( r_i \leq \tilde{C} \| x - y \| \), we have
\[
r_i^{k-|\eta|}|\eta| - r_i^{1-|\beta|} \| x - y \| \leq C r_i^{1-|\beta|} \| x - y \|^{k-|\eta|} \| x - y \| \leq C r_i^{1-|\beta|} \| x - y \|^{k-|\beta|}.
\]
Hence
\[
A'_\eta \leq C \| x - y \|^{k-|\beta|} - \sum_{i \in I_1} r_i^{1-\frac{m}{p^*}} \left( \int_{\tilde{Q}_i} \| \nabla^k f \| \ dx \right) \left( \sum_{i \in I_1} \int_{\tilde{Q}_i} \| \nabla^k f \| \ dx \right)^{\frac{1}{p^*}}.
\]
By the Hölder inequality,
\[
A'_\eta \leq C \| x - y \|^{k-|\beta|} - \left( \sum_{i \in I_1} r_i^{1-\frac{m}{p^*}} \right)^{1-\frac{1}{p^*}} \left( \sum_{i \in I_1} \int_{\tilde{Q}_i} \| \nabla^k f \| \ dx \right)^{\frac{1}{p^*}}
\]
so that
\[
A'_\eta \leq C \| x - y \|^{k-|\beta|} - \left( \sum_{i \in I_1} r_i^{\alpha^*} \right)^{1-\frac{1}{p}} \left( M(Q_1) \int_{U_1} \| \nabla^k f \| \ dx \right)^{\frac{1}{p}}
\]
where
\[
U_1 := \bigcup \{ Q : Q \in Q_1 \}.
\]
Recall that \( M(Q_1) \) stands for the covering multiplicity of the collection \( Q_1 \). Since \( M(Q_1) \leq M(Q) \leq C(n) \), we obtain
\[
A'_\eta \leq C \| x - y \|^{k-|\beta|} - \left( \sum_{i \in I_1} r_i^{\alpha^*} \right)^{1-\frac{1}{p}} \left( \int_{U_1} \| \nabla^k f \| \ dx \right)^{\frac{1}{p}} \quad (3.16)
\]
In a similar way we prove that
\[
A''_\eta \leq C \| x - y \|^{k-|\beta|} - \left( \sum_{i \in I_2} r_i^{\alpha^*} \right)^{1-\frac{1}{p}} \left( \int_{U_2} \| \nabla^k f \| \ dx \right)^{\frac{1}{p}} \quad (3.17)
\]
where
\[
U_2 := \bigcup \{ Q : Q \in Q_2 \}.
\]
Let us prove that
\[
\sum_{i \in I_1} r_i^{\alpha^*} \leq C \| x - y \|^{\alpha^*} \quad (3.18)
\]
\[ \sum_{i \in I_2} r_i^\alpha \leq C \| x - y \|^\alpha. \] (3.19)

We begin with the proof of inequality (3.18). Let \( i \in I_1 \) and let
\[ Q_i = Q(z_i, r_i), \quad r_i = \frac{1}{8} \\text{dist}(z_i, \partial \Omega). \]
Recall that \( Q_i \cap \tilde{\Gamma} \neq \emptyset \) so that there exist a point \( b_i \in Q_i \cap \tilde{\Gamma} \). By (1.8),
\[ r_i \leq C \text{ length}(\tilde{\Gamma} \cap Q(b_i, r_i)). \]
But \( Q(b_i, r_i) \subset 2Q_i = Q(z_i, 2r_i) \) so that
\[ r_i \leq C \text{ length}(2Q_i \cap \tilde{\Gamma}). \]
Hence,
\[ r_i^{\alpha^*} \leq C r_i^{\alpha^*-1} \text{ length}(2Q_i \cap \tilde{\Gamma}). \]
Let \( z \in 2Q_i \cap \tilde{\Gamma} \). Then
\[ | \text{dist}(z, \partial \Omega) - \text{dist}(z_i, \partial \Omega) | \leq \| z - z_i \| \leq 2r_i. \]
Since \( \text{dist}(z_i, \partial \Omega) = 8r_i \), we have
\[ \text{dist}(z, \partial \Omega) \leq \text{dist}(z_i, \partial \Omega) + 2r_i = 10r_i. \]
Hence,
\[ r_i^{\alpha^*-1} \leq C \text{ dist}(z, \partial \Omega)^{\alpha^*-1}, \quad z \in 2Q_i \cap \tilde{\Gamma}, \]
so that
\[ r_i^{\alpha^*} \leq C \int_{2Q_i \cap \tilde{\Gamma}} \text{dist}(z, \partial \Omega)^{\alpha^*-1} \text{ds}(z). \]
We put \( 2Q_1 := \{ 2Q : Q \in Q_1 \} \). We have
\[ \sum_{i \in I_1} r_i^{\alpha^*} \leq C \sum_{i \in I_1} \int_{2Q_i \cap \tilde{\Gamma}} \text{dist}(z, \partial \Omega)^{\alpha^*-1} \text{ds}(z) \]
so that
\[ \sum_{i \in I_1} r_i^{\alpha^*} \leq C M(2Q_1) \int_{\tilde{\Gamma}} \text{dist}(z, \partial \Omega)^{\alpha^*-1} \text{ds}(z). \]
By part (iii) of Lemma 3.2, the covering multiplicity
\[ M(2Q_1) \leq M(2Q) \leq C(n). \]
Hence,

\[ \sum_{i \in I_1} r_i^{\alpha^*} \leq C \int_{\Gamma} \text{dist}(z, \partial \Omega)^{\alpha^* - 1} \, ds(z). \]

In a similar way we prove that

\[ \sum_{i \in I_2} r_i^{\alpha} \leq C \int_{\Gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z). \]

Combining these inequalities with (1.7) (where we put \( \tau = \alpha^* \)) and (3.5), we obtain the required inequalities (3.18) and (3.19).

Hence, by (3.16),

\[
A'_\eta \leq C \|x - y\|^{k - |\beta| - \frac{1}{p^*}} \left( \int_{U_1} \|\nabla^k f\|^{p^*} \, dx \right)^{\frac{1}{p^*}}
\]

and, by (3.17),

\[
A''_\eta \leq C \|x - y\|^{k - |\beta| - \frac{n}{p}} \left( \int_{U_2} \|\nabla^k f\|^{p} \, dx \right)^{\frac{1}{p}}.
\] (3.20)

Recall that for each \( Q = Q(z, r) \in Q \) its center, the point \( z \), belongs to \( \Gamma \). Moreover, by (3.8), \( r \leq \tilde{C} \|x - y\| \), and, by (3.6), \( \text{length}(\Gamma) \leq \tilde{C} \|x - y\| \). Hence,

\[ Q \subset Q(x, 2\tilde{C}\|x - y\|) = \lambda Q_{xy}, \quad Q \in Q, \]

with \( \lambda := 2\tilde{C} \). Now we have

\[ U_1 := \cup \{Q: Q \in Q_1\} \subset (\lambda Q_{xy}) \cap \Omega \]

so that

\[
A'_\eta \leq C \|x - y\|^{k - |\beta| - \frac{n}{p}} \left( \int_{(\lambda Q_{xy}) \cap \Omega} \|\nabla^k f\|^{p^*} \, dx \right)^{\frac{1}{p^*}}
\]

proving (3.12).

Let us put \( E := U_2 \) and prove that \( |E| \leq C \varepsilon^n \). We have

\[ |E| = | \cup \{Q: Q \in Q_2\}| \leq \sum_{Q \in Q_2} |Q| = 2^n \sum_{i \in I_2} r_i^n \leq 2^n \left( \sum_{i \in I_2} r_i \right)^n. \] (3.21)
By (3.9) and (3.10),

\[ Q_i \cap \tilde{\Gamma} = \emptyset \text{ for every } Q_i \in \mathcal{Q}_2, \]

so that

\[ Q_i \cap \Gamma = Q_i \cap (\Gamma \setminus \tilde{\Gamma}). \]

But, by (3.7), \( r_i \leq \text{length}(Q_i \cap \Gamma) \) so that \( r_i \leq \text{length}(Q_i \cap (\Gamma \setminus \tilde{\Gamma})) \). Hence

\[
\sum_{I \in I_2} r_i \leq \sum_{I \in I_2} \text{length}(Q_i \cap (\Gamma \setminus \tilde{\Gamma})) \leq M(Q_2) \text{length}(\Gamma \setminus \tilde{\Gamma}).
\]

Since \( M(Q_2) \leq M(\mathcal{Q}) \leq C(n) \), we obtain

\[
\sum_{I \in I_2} r_i \leq C \text{length}(\Gamma \setminus \tilde{\Gamma})
\]

so that, by (1.9),

\[
\sum_{I \in I_2} r_i \leq C \varepsilon.
\]

Combining this inequality with (3.21), we obtain the required inequality \(|E| \leq C \varepsilon^n \). This inequality and (3.20) imply (3.13).

Now, by (3.11), (3.12) and (3.13),

\[
A := |D^\beta T_x^{k-1}(f)(x) - D^\beta T_y^{k-1}(f)(x)| \leq C \sum_{|\eta| \leq k-1-|\beta|} (A'_n + A''_n)
\]

\[
\leq C \sum_{|\eta| \leq k-1-|\beta|} \|x - y\|^{k-|\beta| - \frac{\beta}{p}} \left( \int_{(\lambda Q_{xy}) \cap \Omega} \|\nabla^k f\|^{p^*} dx \right)^{\frac{1}{p^*}}
\]

\[
+ C \sum_{|\eta| \leq k-1-|\beta|} \|x - y\|^{k-|\beta| - \frac{\beta}{p}} \left( \int_{E} \|\nabla^k f\|^{p} dx \right)^{\frac{1}{p}}.
\]

We obtain

\[
A \leq C \|x - y\|^{k-|\beta| - \frac{\beta}{p^*}} \left( \int_{(\lambda Q_{xy}) \cap \Omega} \|\nabla^k f\|^{p^*} dx \right)^{\frac{1}{p^*}}
\]

\[
+ C \|x - y\|^{k-|\beta| - \frac{\beta}{p}} \left( \int_{E} \|\nabla^k f\|^{p} dx \right)^{\frac{1}{p}}.
\]

But \( f \in L_p^k(\Omega) \) so that

\[
\int_{\Omega} \|\nabla^k f\|^{p} dx = \|f\|_{L_p^k(\Omega)}^p < \infty.
\]
Hence,
\[ \int_E \| \nabla^k f \|^p \, dx \to 0 \quad \text{as} \quad |E| = C \varepsilon^n \to 0, \]
proving the theorem. \( \square \)

4. Extension of Sobolev functions defined on subhyperbolic domains.

Given a cube \( Q \subset \mathbb{R}^n \) and a function \( f \in L_q(Q), 0 < q \leq \infty \), we let \( \mathcal{E}_k(f; Q)_{L_q} \) denote the normalized local best approximation of \( f \) on \( Q \) in \( L_q \)-norm by polynomials of degree at most \( k - 1 \), see Brudnyi [Br1]. More explicitly, we define
\[
\mathcal{E}_k(f; Q)_{L_q} := |Q|^{-\frac{1}{q}} \inf_{P \in \mathcal{P}_{k-1}} \| f - P \|_{L_q(Q)} = \inf_{P \in \mathcal{P}_{k-1}} \left( \frac{1}{|Q|} \int_Q |f - P|^q \, dx \right)^{\frac{1}{q}}.
\]

In the literature \( \mathcal{E}_k(f; Q)_{L_q} \) is also sometimes called the local oscillation of \( f \), see e.g. Triebel [T2].

Given a locally integrable function \( f \) on \( \mathbb{R}^n \), we define its sharp maximal function \( f^\sharp \) by letting
\[
f^\sharp_k(x) := \sup_{r > 0} r^{-k} \mathcal{E}_k(f; Q(x, r))_{L_1}.
\]

Recall that a function \( f \in W^k_p(\mathbb{R}^n), 1 < p \leq \infty \), if and only if \( f \) and \( f^\sharp \) are both in \( L_p(\mathbb{R}^n) \), see Calderón [C]. Moreover, up to constants depending only on \( n, k \) and \( p \) the following equivalence,
\[
\| f \|_{W^k_p(\mathbb{R}^n)} \sim \| f \|_{L_p(\mathbb{R}^n)} + \| f^\sharp \|_{L_p(\mathbb{R}^n)},
\]
holds.

This characterization motivates the following definition. Let \( S \) be a measurable subset of \( \mathbb{R}^n \). Given a function \( f \in L_{q,\text{loc}}(S) \), and a cube \( Q \) whose center is in \( S \), we let \( \mathcal{E}_k(f; Q)_{L_q(S)} \) denote the normalized best approximation of \( f \) on \( Q \) in \( L_q(S) \)-norm:
\[
\mathcal{E}_k(f; Q)_{L_q(S)} := |Q|^{-\frac{1}{q}} \inf_{P \in \mathcal{P}_{k}} \| f - P \|_{L_q(Q \cap S)} = \inf_{P \in \mathcal{P}_{k-1}} \left( \frac{1}{|Q|} \int_{Q \cap S} |f - P|^q \, dx \right)^{\frac{1}{q}}. \tag{4.2}
\]

By \( f^\sharp_{k,S} \), we denote the sharp maximal function of \( f \) on \( S \),
\[
f^\sharp_{k,S}(x) := \sup_{r > 0} r^{-k} \mathcal{E}_k(f; Q(x, r))_{L_1(S)}, \quad x \in S.
\]
(Thus, \( f^\sharp_k = f^\sharp_{k,\mathbb{R}^n} \).

Let \( \Omega \subset \mathbb{R}^n \) be a subhyperbolic domain. The following two corollaries of Theorem 3.1 present estimates of the local best approximations and the sharp maximal function of a function \( f \in W^k_p(\Omega) \) via the local \( L_p \)-norms and the maximal function of \( \nabla^k f \).
Corollary 4.1 Let $n < p < \infty$, $\alpha = (p - n)/(p - 1)$, and let $\Omega$ be an $\alpha$-subhyperbolic domain in $\mathbb{R}^n$. There exists a constant $p^* \in (n, p)$ and constants $\theta, \lambda, C > 0$ depending only on $n, p, k, C_{\alpha, \Omega}$ and $\theta_{\alpha, \Omega}$, such that the following is true: Let $f \in L_{k}^{p}(\Omega)$. Then for every cube $Q = Q(x, r)$ with $x \in \Omega$ and $0 < r \leq \theta$ the following inequality holds.

Proof. Let $p^*$ and $\theta$ be the constant from Theorem 3.1. Let $y \in Q(x, r)$ so that $\|y - x\| \leq r \leq \theta$. Applying Theorem 3.1 to the points $y, x$ (with $\beta = 0$), we obtain

$$|f(y) - T_{x}^{k-1}(f)(y)| \leq C\|x - y\|^{k - \frac{n}{p^*}} \left( \int_{(\lambda Q) \cap \Omega} \|\nabla^{k}f\|^{p^*} \, dx \right)^{\frac{1}{p^*}}.$$

Recall that $Q_{xy} := Q(x, \|x - y\|)$.

Since $n < p^*$, we have $k - \frac{n}{p^*} > 0$ so that

$$|f(y) - T_{x}^{k-1}(f)(y)| \leq Cr^{k - \frac{n}{p^*}} \left( \int_{(\lambda Q) \cap \Omega} \|\nabla^{k}f\|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq Cr^{k} \left( \frac{1}{|\lambda Q|} \int_{(\lambda Q) \cap \Omega} \|\nabla^{k}f\|^{p^*} \, dx \right)^{\frac{1}{p^*}}.$$

Hence,

$$E_{k}(f; Q)_{L^\infty(\Omega)} := \inf_{P \in \mathcal{P}_{k-1}} \sup_{y \in Q \cap \Omega} |f(y) - P(y)| \leq \sup_{y \in Q \cap \Omega} |f(y) - T_{x}^{k-1}(f)(y)| \leq Cr^{k} \left( \frac{1}{|\lambda Q|} \int_{(\lambda Q) \cap \Omega} \|\nabla^{k}f\|^{p^*} \, dx \right)^{\frac{1}{p^*}},$$

proving the corollary. \qed

Given a function $g$ defined on $\Omega$ we let $g^{\lambda}$ denote its extension by zero to all of $\mathbb{R}^n$. Thus $g^{\lambda}(x) := g(x), x \in \Omega$, and $g^{\lambda}(x) := 0, x \notin \Omega$.

As usual, given a function $u \in L_{1,loc}(\mathbb{R}^n)$ by $\mathcal{M}f$ we denote the Hardy-Littlewood maximal function

$$\mathcal{M}[f](x) := \sup_{t > 0} \frac{1}{|Q(x, t)|} \int_{Q(x, t)} |f(y)| dy.$$
Corollary 4.2 Let \( n < p < \infty \), \( \alpha = (p-n)/(p-1) \), and let \( \Omega \) be an \( \alpha \)-subhyperbolic domain in \( \mathbb{R}^n \). There exists a constant \( p^* \), \( n < p^* < p \), such that for every function \( f \in L^k_p(\Omega) \) and every \( x \in \Omega \) the following inequality

\[
f^k_{p^*}(x) \leq C \left\{ (\mathcal{M}[\|\nabla^k f\|^{p^*}](x))^{\frac{1}{p^*}} + \mathcal{M}[f^\lambda](x) \right\}
\]

holds. The constants \( p^* \) and \( C \) depend only on \( n, p, k, C_{\alpha,\Omega} \) and \( \theta_{a,\Omega} \).

**Proof.** Let \( p^*, \lambda \) and \( \theta \) be the constants from Corollary 4.1. By this corollary,

\[
\sup_{0 < r \leq \theta} r^{-k}\mathcal{E}_k(f; Q(x, r))_{L^1(\Omega)} \leq \sup_{0 < r \leq \theta} r^{-k}\mathcal{E}_k(f; Q(x, r))_{L^\infty(\Omega)}
\]

\[
\leq C \sup_{0 < r \leq \theta} \left( \frac{1}{|Q(x, \lambda r)|} \int_{Q(x, \lambda r) \cap \Omega} \|\nabla^k f\|^{p^*} \, dx \right)^{\frac{1}{p^*}}
\]

\[
\leq C \left\{ (\mathcal{M}[\|\nabla^k f\|^{p^*}](x))^{\frac{1}{p^*}} \right\}.
\]

On the other hand, by (4.2),

\[
\sup_{r > \theta} r^{-k}\mathcal{E}_k(f; Q(x, r))_{L^1(\Omega)} \leq \theta^{-k} \sup_{r > \theta} \mathcal{E}_k(f; Q(x, r))_{L^1(\Omega)}
\]

\[
\leq \theta^{-k} \sup_{r > \theta} \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r) \cap \Omega} |f| \, dx \right)
\]

\[
\leq \theta^{-k} \mathcal{M}(f^\lambda)(x)
\]

proving the lemma. \( \square \)

In [S4] we show that the restrictions of Sobolev functions to regular subsets of \( \mathbb{R}^n \) can be described in a way similar to the Calderón's criterion (4.1), i.e., via \( L^p \)-norms of a function and its sharp maximal function on a set. We recall that a measurable set \( S \subset \mathbb{R}^n \) is said to be regular if there are constants \( \sigma_S \geq 1 \) and \( \delta_S > 0 \) such that, for every cube \( Q \) with center in \( S \) and with diameter \( \text{diam} \, Q \leq \delta_S \),

\[
|Q| \leq \sigma_S |Q \cap S|.
\]

**Theorem 4.3 ([S4])** Let \( S \) be a regular subset of \( \mathbb{R}^n \). Then a function \( f \in L_p(S) \), \( 1 < p \leq \infty \), can be extended to a function \( F \in W^k_p(\mathbb{R}^n) \) if and only if its sharp maximal function \( f^k_{p,S} \in L_p(S) \). In addition,

\[
\|f\|_{W^k_p(\mathbb{R}^n)} \sim \|f\|_{L_p(S)} + \|f^k_{p,S}\|_{L_p(S)}
\]

with constants of equivalence depending only on \( n, k, p, \sigma_S \) and \( \delta_S \).

**Proof of Theorem 1.1.** By inequality (1.1), \( \Omega \) is an \( \alpha \)-subhyperbolic domain with \( \alpha = \frac{p-n}{p-1} \), so that, by Lemma 2.3, \( \Omega \) is a regular subset of \( \mathbb{R}^n \).
Let \( p^* \in (n, p) \) be the constant from Corollary 4.2. Let \( q > p^* \) and let \( f \in W^k_q(\Omega) \). We have to prove that \( f \) can be extended to a function \( F \in W^k_q(\mathbb{R}^n) \). Since \( \Omega \) is regular, by Theorem 4.3 it suffices to show that the sharp maximal function \( f^k_{\Omega} \) belongs to \( L_q(\Omega) \).

By Corollary 4.2,
\[
f^k_{\Omega}(x) \leq C \left\{ (\mathcal{M}[(\|\nabla k f\|^{\lambda})^p^*](x))^{\frac{1}{p^*}} + \mathcal{M}[f^\lambda](x) \right\}, \quad x \in \Omega,
\]
so that
\[
\|f^k_{\Omega}\|_{L_q(\Omega)} \leq C \left\{ \|\mathcal{M}[(\|\nabla k f\|^{\lambda})^p^*]\|_{L_q(\Omega)} + \|\mathcal{M}[f^\lambda]\|_{L_q(\Omega)} \right\}
\]

By the Hardy-Littlewood maximal theorem
\[
\|\mathcal{M}[f^\lambda]\|_{L_q(\mathbb{R}^n)} \leq C \|f^\lambda\|_{L_q(\mathbb{R}^n)} = C \|f\|_{L_q(\Omega)}.
\]
(Recall that \( f^\lambda \) denotes the extension of \( f \) by zero to all of \( \mathbb{R}^n \).)

Applying the Hardy-Littlewood maximal theorem to the function \( g := (\|\nabla k f\|^{\lambda})^{p^*} \) in the space \( L_s(\mathbb{R}^n) \) with \( s := q/p^* > 1 \), we obtain
\[
A := \|\mathcal{M}[(\|\nabla k f\|^{\lambda})^{p^*}]\|_{L_q(\mathbb{R}^n)} = \left( \|\mathcal{M}[g]\|_{L_s(\mathbb{R}^n)} \right)^{\frac{1}{p^*}} \leq C \left( \|g\|_{L_s(\mathbb{R}^n)} \right)^{\frac{1}{p^*}}
\]
\[
= C \left\{ \left( \int_{\mathbb{R}^n} \left(\|\nabla k f\|^{\lambda}\right)^{p^*} \right)^{\frac{1}{p^*}} \right\}^{\frac{p^*}{q}}
\]
so that
\[
A \leq C \left( \int_{\mathbb{R}^n} \left(\|\nabla k f\|^{\lambda}\right)^q \right)^{\frac{1}{q}} = C \left( \int_{\Omega} \|\nabla k f\|^{q} \right)^{\frac{1}{q}} = C \|f\|_{L^q_k(\Omega)}.
\]

Hence
\[
\|f^k_{\Omega}\|_{L_q(\Omega)} \leq C(\|f\|_{L^q_k(\Omega)} + \|f\|_{L_q(\Omega)}) \leq C\|f\|_{W^k_q(\mathbb{R}^n)}.
\]

Theorem 1.1 is completely proved.

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