Abstract. We propose two different Lagrange multiplier methods for contact problems derived from the augmented Lagrangian variational formulation. Both the obstacle problem, where a constraint on the solution is imposed in the bulk domain and the Signorini problem, where a lateral contact condition is imposed are considered. We consider both continuous and discontinuous approximation spaces for the Lagrange multiplier. In the latter case the method is unstable and a penalty on the jump of the multiplier must be applied for stability. We prove the existence and uniqueness of discrete solutions, best approximation estimates and convergence estimates that are optimal compared to the regularity of the solution.

1. Introduction. We consider the Signorini problem, find \( u \) and \( \lambda \) such that

\[
\begin{align*}
-\Delta u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \Gamma_D \\
u \leq 0, \lambda \leq 0, u\lambda &= 0 \text{ on } \Gamma_C,
\end{align*}
\]  

(1.1)

or the obstacle problem

\[
\begin{align*}
-\Delta u - \lambda &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega \\
u \leq 0, \lambda \leq 0, u\lambda &= 0 \text{ in } \Omega,
\end{align*}
\]  

(1.2)

Here \( \Omega \subset \mathbb{R}^d, d = 2, 3 \) is a bounded polyhedral (polygonal) domain and \( f \in L^2(\Omega) \). It is well known that these problems admit unique solutions \( u \in H^1(\Omega) \). This follows from the theory of Stampacchia applied to the corresponding variational inequality (see for instance [19]).

From a mechanical point of view, these equations model the deflection of a membrane in isotropic tension under the load \( f \), assuming small deformations. The membrane is either in contact with an obstacle on part of the boundary, (1.1), or in the interior of the membrane, (1.2), preventing positive displacements \( u \). In both cases the Lagrange multiplier has the interpretation of a distributed reaction force enforcing the contact condition \( u \leq 0 \).

2. Finite element discretization. Our aim in this paper is to design a consistent penalty method for contact problems that can easily be included in a standard Lagrange-multiplier method, without having to resort to the solution of variational inequalities. We consider two different choices for the multiplier spaces, either a stable choice or an unstable choice where a stabilization term is needed to ensure the stability of the formulation. In the latter case we add a penalty on the jump of the multiplier over element faces in the spirit of [11, 10].

There exists a large body of litterature treating finite element methods for contact problems [8, 22, 17, 5, 4, 6, 28, 27, 14]. Discretization of (1.1) is usually performed...
on the variational inequality or using a penalty method. The first case however leads to some nontrivial choices in the construction of the discretization spaces in order to satisfy the nonpenetration condition and associated inf-sup conditions and until recently it has proved difficult to obtain optimal error estimates [21, 16]. The latter case, on the other hand leads to the usual consistency and conditioning problems of penalty methods. Another approach proposed by Hild and Renard [20] is to use a stabilized Lagrange-multiplier in the spirit of Barbosa and Hughes [3]. As a further development one may use the reformulation of the contact condition

\[ \lambda = -\gamma^{-1}[u - \gamma\lambda]_+ \] 

(2.1)

where \([x]_+ = \max(0, x)\), introduced by Alart and Curnier [1] in an augmented Lagrangian framework. Using the close relationship between the Barbosa–Hughes method and Nitsche’s method [23] discussed by Stenberg [25], this method was then further developed in the elegant Nitsche-type formulation for the Signorini problem introduced by Chouly, Hild and Renard [13, 15]. In these works optimal error estimates for the above model problem were obtained for the first time.

Using the notation \(\langle u, v \rangle_C\) for the \(L^2\) inner product over \(C\) we have in the case of the Signorini problem (1.1) that \(C\) corresponds to \(\Gamma_C\), the boundary part where the contact conditions hold and

\[ \langle u, v \rangle_C := \int_{\Gamma_C} uv \, ds, \]

while for the obstacle problem (1.2) \(C \equiv \Omega\) and

\[ \langle u, v \rangle_C := \int_{\Omega} uv \, dx. \]

Finally, we define \(\|v\|_C := \langle v, v \rangle_C^{1/2}\). With this notation, the augmented Lagrangian multiplier seeks stationary points to the functional

\[ \mathfrak{F}(u, \lambda) := \frac{1}{2}a(u, u) + \frac{1}{2\gamma}||[u - \gamma\lambda]_+||_C^2 - \frac{\gamma}{2}||\lambda||_C^2, \] 

(2.2)

cf. Alart and Curnier [1]. Observe that formally the stationary points are given by \((u, \lambda)\) such that

\[ a(u, v) + \langle \gamma^{-1}[u - \gamma\lambda]_+, v \rangle_C = (f, v)_\Omega \]
\[ \langle \lambda + \gamma^{-1}[u - \gamma\lambda]_+, \mu \rangle_C = 0 \] 

(2.3)

for all \((v, \mu)\), or by substituting the second equation in the first

\[ a(u, v) - \langle \lambda, v \rangle_C = (f, v)_\Omega \]
\[ \langle \gamma\lambda + [u - \gamma\lambda]_+, \mu \rangle_C = 0. \] 

(2.4)

Observing now that the contact condition equally well can be written on the primal variable as \(u = -[\gamma\lambda - u]_+\) we get by adding and subtracting \(u\) in the second equation of (2.4)

\[ a(u, v) - \langle \lambda, v \rangle_C = (f, v)_\Omega \]
\[ \langle u + [\gamma\lambda - u]_+, \mu \rangle_C = 0. \] 

(2.5)
In this paper we consider two different methods, resulting from this approach. The first formulation is the straightforward discretization of (2.3) resulting in a method that gives the stationary points of the functional (2.2) over the discrete spaces. The second formulation is a discretization of (2.5) that is chosen for its closeness to the standard Lagrange multiplier method for the imposition of Dirichlet boundary conditions.

We consider discretization either with a choice of approximation spaces that results in a stable approximation, or a choice that is stable only with an added stabilizing term. Here we consider stabilization based on the interior penalty stabilized Lagrange multiplier method introduced by Burman and Hansbo [11] for solving elliptic interface problems. The appeal of this latter approach is that we may use the lowest order approximation spaces where the displacement is piecewise linear and the multiplier constant per element (or element side). When considering the Signorini problem (1.1) these spaces match the regularity of the physical problem perfectly and therefore in some sense is the most economical choice. Contact problems also present non trivial quadrature problems so that in practice it can be very difficult to integrate the terms of the formulation to a sufficient accuracy to get optimal accuracy when higher order interpolations are used. Herein we will assume that integration can be performed exactly on the interface between the contact and non-contact subdomain.

For an alternative stabilization method of Barbosa–Hughes type in the augmented Lagrangian setting, see Hansbo, Rashid, and Salomonsson [18].

We assume that \{T\}_h is a family of quasuniform meshes of \(\Omega\), such that the mesh is fitted to the zone \(C\). That is \(C\) is a subset of boundary element faces of simplices \(K\) such that \(K \cap \Gamma_C \neq \emptyset\), \(F := \partial K \cap \Gamma_C\), \(T_C := \{F\}\), \(C := \cup_{F \in T_C} C\) with \(C \subset \mathbb{R}^{d-1}\) for the Signorini problem. For the obstacle problem \(C\) is defined by \(\Omega\) and hence \(\cup_{K \in T} =: C \subset \mathbb{R}^d\) and \(T_C \equiv T\). Below we will denote the elements of \(T_C\) by \(K\) in both cases. We define \(V_h\) to be the space of \(H^1\)-conforming functions on \(T\), satisfying the homogeneous boundary condition of \(\Gamma_D\).

\[
V_h^k := \{v_h \in H^1(\Omega) : v|_{\Gamma_D} = 0; v|_K \in \mathbb{P}_k(K), \forall K \in T\},
\]

where \(\mathbb{P}_k(K)\) denotes the set of polynomials of order less than or equal to \(k\) on the simplex \(K\). Whenever the superscript is dropped we refer to the generic space of order \(k\). For the multipliers we introduce the space \(\Lambda_h\) defined as the space piecewise polynomials of order less than or equal to \(l\) defined on \(C\).

\[
\Lambda_h^l := \{\mu_h \in L^2(C) : \mu_h|_K \in \mathbb{P}_l(K), \forall K \in T_C\}.
\]

Whenever \(l = k - 1\) the superscript is dropped. We will detail the case of discontinuous multipliers, but all arguments below are valid also in case the Lagrange multiplier is approximated in the space of continuous functions, \(\Lambda_h^l \cap C^0(C), l \geq 1\), in this case no stabilization is necessary. The differences in the analysis will be outlined.

Both formulations that we consider herein take the form: Find \((u_h, \lambda_h) \in V_h \times \Lambda_h\) such that

\[
a(u_h, v_h) + b[(u_h, \lambda_h); (v_h, \mu_h)] = (f, v_h)_\Omega \quad \forall(v_h, \mu_h) \in V_h \times \Lambda_h \quad (2.6)
\]

where \((\cdot, \cdot)_\Omega\) denotes the standard \(L^2\)-inner product, \(a(u_h, v_h) := (\nabla u_h, \nabla v_h)_\Omega\) and the methods are distinguished by the definition of the form \(b[\cdot; \cdot]\) that acts only in the zone where contact may occur. The stabilization will be included in the form \(b[\cdot; \cdot]\). As already pointed out this term is necessary if the choice \(V_h \times \Lambda_h\) does not satisfy the inf-sup condition. In our framework, this is the case where the multiplier is
discontinuous over element faces. In this paper we will focus on a stabilization using a penalty on the jumps over element faces of the multiplier variable in the spirit of [11] [10].

\[
s(\lambda_h, \mu_h) := \sum_{F \in F_C} \delta \gamma \int_F h \|\lambda_h\|_{\mu_h} \, ds,
\]

(2.7)

where \( \delta > 0 \) is a parameter, \( \|x\|_F \) denotes the jump of the quantity \( x \) over the face \( F \) and \( F_C \) denotes the set of interior element faces of the elements in \( T \). The semi-norm associated with the stabilization operator will be defined as \( | \cdot |_s := s(\cdot, \cdot)^{1/2} \).

We will also below use the compact notation

\[
A_h[(u_h, \lambda_h), (v_h, \mu_h)] := a(u_h, v_h) + b[(u_h, \lambda_h); (v_h, \mu_h)]
\]

and the associated formulation, find \((u_h, \lambda_h) \in V_h \times \Lambda_h\) such that

\[
A_h[(u_h, \lambda_h), (v_h, \mu_h)] = (f, v_h)_\Omega, \quad \text{for all } (v_h, \mu_h) \in V_h \times \Lambda_h.
\]

(2.8)

We will now specify two different choices of \( b[\cdot, \cdot] \) leading to two different Lagrange-multiplier methods.

**FORMULATION 1:** In the first formulation we use the original formula for the contact condition proposed by Alart and Curnier, \( \lambda = -\gamma^{-1}[u - \gamma \lambda]_+ \)

\[
b[(u_h, \lambda_h); (v_h, \mu_h)] := \gamma^{-1}[u_h - \gamma \lambda_h]_+ \langle v_h \rangle_C
\]

\[
+ \gamma^{-1}[u_h - \gamma \lambda_h]_+ \langle \gamma \mu_h \rangle_C
\]

\[
+ \gamma \langle \lambda_h \mu_h \rangle_C + s(\lambda_h, \mu_h)
\]

(2.9)

or, writing the nonlinearity as the derivative of a quadratic form, and using the notation \( P_{\gamma \pm}(u_h, \lambda_h) := \pm(u_h - \gamma \lambda_h) \)

\[
b[(u_h, \lambda_h); (v_h, \mu_h)] := \gamma^{-1}[P_{\gamma +}(u_h, \lambda_h)]_+ \langle P_{\gamma +}(v_h, \mu_h) \rangle_C
\]

\[
- \gamma \langle \mu_h \lambda_h \rangle_C - s(\lambda_h, \mu_h),
\]

(2.10)

with \( \gamma > 0 \) a parameter to determine. In this case the finite element formulation corresponds to the approximate solutions of (2.3) in the finite element space.

**FORMULATION 2:** In the second formulation we use a reformulation of the contact condition on the displacement variable, \( u = -[\gamma \lambda - u]_+ \) to obtain the semi-linear form

\[
b[(u_h, \lambda_h); (v_h, \mu_h)] := -\langle \lambda_h, v_h \rangle_C + \langle \mu_h, u_h \rangle_C
\]

\[
+ \langle \mu_h, [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle_C + s(\lambda_h, \mu_h),
\]

(2.11)

with \( \gamma > 0 \) a parameter to determine. In this case the finite element formulation corresponds to the approximate solutions of (2.5) in the finite element space.

**2.1. Alternative formulations.** In both formulation 1 and 2 above it is possible to derive an alternative formulation of the same method using the relation

\[
[P_{\gamma -}(u_h, \lambda_h)]_+ = [P_{\gamma +}(u_h, \lambda_h)]_+ - P_{\gamma +}(u_h, \lambda_h).
\]
Considering the form (2.10) and adding and subtracting \( P_{\gamma}(u_h, \lambda_h) \) in the nonlinear term we have the alternative form (omitting the stabilization term)

\[
b[(u_h, \lambda_h); (v_h, \mu_h)] = \langle \gamma^{-1}[P_{\gamma}(u_h, \lambda_h)]+, P_{\gamma}(v_h, \mu_h) \rangle_C - \langle \gamma \mu_h, \lambda_h \rangle_C \\
= \langle \gamma^{-1}([P_{\gamma}(u_h, \lambda_h)]+ - P_{\gamma}(u_h, \lambda_h)), P_{\gamma}(v_h, \mu_h) \rangle_C \\
+ \langle \gamma^{-1}P_{\gamma}(u_h, \lambda_h), P_{\gamma}(v_h, \mu_h) \rangle_C - \langle \gamma \mu_h, \lambda_h \rangle_C \\
= - \langle \lambda_h, v_h \rangle_C + \langle \mu_h, u_h \rangle_C + \gamma^{-1} \langle \mu_h, v_h \rangle_C \\
+ \gamma^{-1} \langle [P_{\gamma}-u_h, \lambda_h]+, P_{\gamma}(v_h, \mu_h) \rangle_C. 
\] (2.12)

Similarly for formulation 2 we obtain in (2.11) omitting for simplicity the stabilization term

\[
b[(u_h, \lambda_h); (v_h, \mu_h)] = - \langle \lambda_h, v_h \rangle_C + \langle \mu_h, u_h \rangle_C \\
+ \langle \mu_h, P_{\gamma}(u_h, \lambda_h) \rangle_C + \gamma \langle \mu_h, \lambda_h \rangle_C \\
+ \gamma\langle [P_{\gamma}-u_h, \lambda_h]+, P_{\gamma}(v_h, \mu_h) \rangle_C. 
\] (2.13)

We see that this semi-linear form corresponds to a discretization of (2.4).

The methods defined by (2.11) and (2.13) or (2.10) and (2.12) respectively are equivalent, but if during the solution process the linear and nonlinear parts are separated in the nonlinear solver, one can expect the different formulations to have different behavior and give rise to different sequences of approximations in the iterative procedure.

3. Technical results. Here we will collect some useful elementary results. First recall the following inverse inequalities and trace inequalities (for a proof see, e.g., [2])

\[
\| \nabla u_h \|_K \leq C h^{-1} \| u_h \|_K, \quad \forall u_h \in V_h 
\] (3.1)

\[
\| u \|_{H^2(K)} \leq C_T (h^{-\frac{1}{2}} \| u \|_K + h^\frac{s}{2} \| \nabla u \|_K), \quad \forall u \in H^1(K) 
\] (3.2)

\[
\| \nabla u_h \|_{\partial K} \leq C_T h^{-\frac{s}{2}} \| u_h \|_K, \quad \forall u_h \in V_h 
\] (3.3)

Similar inequalities hold for functions in \( \Lambda_h \) and we will use them without making any distinction between the two cases. We let \( \pi_0 : L^2(C) \to \Lambda_h^0 \) denote the standard \( L^2 \) projection onto \( \Lambda_h^0 \) and we observe that there holds, by standard approximation properties of the projection onto constants (and a trace inequality in the case of lateral contact),

\[
\| (1 - \pi_0)v_h \|_C \leq c_0 h^s \| \nabla v_h \|_\Omega
\]

with \( s = 1 \) for the Obstacle problem where \( C \subset \Omega \) and \( s = \frac{1}{2} \) for the Signorini problem where \( C \subset \partial \Omega \). Similarly we define \( \pi_1 : L^2(C) \to \Lambda_h^1 \cap C^0(\bar{C}) \) and note that the corresponding inequality holds for \( \pi_1 \)

\[
\| (1 - \pi_1)v_h \|_C \leq c_1 h^s \| \nabla v_h \|_\Omega.
\]

We also observe for future reference that \( \| u \|_C \leq C \| u \|_{H^1(\Omega)} \) in both cases.
For the analysis below it is useful to introduce an indicator function for the contact domain $C$ defined on the space $V_h$. Let $\xi_h$ denote a finite element function such that $\xi_h \in V_h^1$ with $\xi_h(x) = 0$ for nodes in $(\bar{\Omega} \setminus C) \cup \bar{\Gamma}_D$, that is nodes outside the contact zone. For all other nodes $x_i \in K$ with $K \subset T_C$, $x_i \notin \bar{\Gamma}_D$, $\xi_h(x_i) = 1$. The following bound is well known, see for instance [12]

$$\exists c \in \mathbb{R}^+ \text{ such that } c\|\mu_h\|_C \leq \|\xi_h^2\mu_h\|_C, \quad \forall \mu_h \in \Lambda_h^l, \quad l \geq 0. \quad (3.4)$$

Stability of the method will rely on the satisfaction of the following assumption:

**Assumption 3.1.** There exists $c_D \in [0, 1)$ such that for all $\mu_h \in \Lambda_h$ there holds

$$\|(1 - \xi_h)\mu_h\|_C \leq c_D\|\mu_h\|_C.$$  

The assumption holds whenever there exists a quadrature rule on the simplex with positive weights and only interior quadrature points. This is easily shown by observing that

$$\|(1 - \xi_h)\mu_h\|_C^2 = \sum_{K \in T_C} \sum_{i \in Q_K} (1 - \xi_h(x_i))^2 \mu_h(x_i)^2 \omega_i$$

$$\leq \max_{K \in T_C} \max_{i \in Q_K} (1 - \xi_h(x_i))^2 \sum_{K \in T} \sum_{i \in Q_K} (\mu_h(x_i))^2 \omega_i$$

$$= c_D^2\|\mu_h\|_C^2,$$

where $Q_K$ is a set of integers indexing the quadrature points in $K$ and

$$c_D = \max_{K \in T_C} \max_{i \in Q_K} (1 - \xi_h(x_i))^2.$$

Since $\xi_h$ is zero only on the boundary of $C$ and no points $x_i \in Q_K$ are on the boundary we conclude that $c_D < 1$.

This is a very mild condition, on triangles it has been showed to hold at least up to integration degree 23, see [26, 29]. It follows that for the Signorini problem in three dimensions and the obstacle problem in two space dimensions the analysis holds at least up to $k = 12$. For the lowest order case where the multipliers are constant per element it is straightforward to show that $c_D \leq 1/2$ if $C \subset \mathbb{R}^2$ and $c_D \leq 1/3$ if $C \subset \mathbb{R}^3$.

**Lemma 3.1.** Let $a, b \in \mathbb{R}$; then there holds

$$([a]+ - [b]+)^2 \leq ([a]+ - [b]+)(a - b),$$

$$|[a]+ - [b]+| \leq |a - b|.$$

**Proof.** Expanding the left hand side of the expression we have

$$[a]^2_+ + [b]^2_+ - 2[a]_+[b]_+ \leq [a]+a + [b]+b - [a]+[b]+ - [a]+[b]+((-[a]+) + ([b]+)(a-b)).$$

For the proof of the second claim, this is trivially true in case both $a$ and $b$ are positive or negative. If $a$ is negative and $b$ positive then

$$|[a]+ - [b]+| = |b| \leq |b - a|$$

and similarly if $b$ is negative and $a$ positive

$$|[a]+ - [b]+| = |a| \leq |b - a|.$$
The forms \[|b|(u_1, \lambda_1); (v, \mu) - b(u_2, \lambda_2); (v, \mu)| \leq (\gamma^{-\frac{1}{2}}\|u_1 - u_2\|_{H^1(\Omega)} + \gamma^{\frac{1}{2}}\|\lambda_1 - \lambda_2\|_{C})(\gamma^{-\frac{1}{2}}\|v\|_{C} + \gamma^{\frac{1}{2}}\|\mu\|_{C}) + |\lambda_1 - \lambda_2|_{s}|\mu|_{s}.\]

Proof. Immediate by the definitions of \(b[; ;]\), the second inequality of Lemma 3.1 the Cauchy-Schwarz inequality and the assumptions on \(\|\cdot\|_{C}\). \(\Box\)

Next we define the local averaging interpolation operator \(I_{ef} : \Lambda_{h} \to \Lambda_{h} \cap C^{0}(C)\) such that for every Lagrangian node \(x_{i} \in T_{C}\)

\[I_{ef}\lambda_{h}(x_{i}) = \kappa_{i}^{-1}\sum_{K:x_{i} \in K}\lambda_{h}(x_{i}),\]

where \(\kappa_{i}\) denotes the cardinality of the set \(\{K \subset T_{C} : x_{i} \in K\}\). Observe that since \(\xi_{h} \in V_{h}^{l}\), for any \(\mu_{h} \in \Lambda_{h}\) there are functions \(R_{\mu}\) in \(V_{h}\) such that \(R_{\mu}|_{C} = I_{ef}\xi_{h}\mu_{h}\). We recall the following interpolation result between discrete spaces:

**Proposition 3.3.** For all \(\mu_{h} \in \Lambda_{h}\) there holds

\[
\|\xi_{h}\mu_{h} - I_{ef}(\xi_{h}\mu_{h})\|_{C} \leq c_{s}\|\xi_{h}\mu_{h}\|_{C, h} - 1 \leq c_{ef}\|\mu_{h}\|_{C}.
\]

Proof. For a proof of the first inequality we refer to \([9, \text{Lemma 5.3}]\). The second inequality is immediate by applying the trace inequality \([3, \text{Lemma 3.3}]\) to each term in the definition \([2.7]\) of \(s(\cdot, \cdot)\). \(\Box\)

**Lemma 3.4.** Let \(r_{h} \in \Lambda_{h} \cap C^{0}(C)\), then there exists \(R_{h} \in V_{h}\) such that \(R_{h}|_{C} = \xi_{h}r_{h}\) and \(\|R_{h}\|_{H^{1}(\Omega)} + \|R_{h}\|_{C} \leq \|h^{-s}r_{h}\|_{C}, \text{ with } s = 1/2 \text{ when } C \text{ is a subset of } \partial\Omega \text{ and } s = 1 \text{ when } C \text{ is a subset of } \Omega\).

Proof. Define \(R_{h}\) so that \(R_{h}(x) = \xi_{h}r_{h}(x)\) for all nodes \(x \in T_{C}\) and \(R_{h}(x) = 0\) for all other nodes \(x\) in the mesh. First consider the case when \(C\) is a subset of the bulk domain \(\Omega\). Then, using an inverse inequality,

\[
\|\nabla R_{h}\|_{\Omega} \leq C_{h}h^{-1}\|r_{h}\|_{\Omega} = C_{h}h^{-2}\|r_{h}\|_{\Omega} = C_{h}h^{-2}\|r_{h}\|_{C}.
\]

In the case \(C\) is a subset of the boundary of \(\Omega\) we observe that

\[
\|\nabla R_{h}\|_{\Omega} = \left(\sum_{K \subset T: \partial K \cap C \neq \emptyset} \|\nabla R_{h}\|_{K}^{2}\right)^{\frac{1}{2}} \leq \left(\sum_{K \subset T: \partial K \cap C \neq \emptyset} h^{-2}\|R_{h}\|_{K}^{2}\right)^{\frac{1}{2}}.
\]

Using that \(R_{h}\) is defined by the nodes in \(C\), combined with the shape regularity of the mesh, we may use the following inverse trace inequality \([9, \text{Lemma 3.1}]\) on every \(K : \partial K \cap C \neq \emptyset\),

\[
\|R_{h}\|_{K} \leq Ch^{\frac{1}{2}}\|R_{h}\|_{\partial K \cap C}.
\]

It follows, since \(R_{h}|_{C} = \xi_{h}r_{h}\), that

\[
\|\nabla R_{h}\|_{\Omega} \leq Ch^{-1/2}\|R_{h}\|_{C} \leq Ch^{-1/2}\|r_{h}\|_{C}.
\]

\(\Box\)
4. Existence of unique discrete solution. In the previous works on Nitsche’s method for contact problems \([13, 15]\) existence and uniqueness has been proven by using the monotonicity and semi-continuity of the operator. Here we propose a different approach where we use Brouwer’s fixed point theorem to establish existence and the monotonicity of the nonlinearity for uniqueness. To this end we introduce the finite dimensional nonlinear system corresponding to the formulation (2.6).

Let \(M := N_V + N_A\), where \(N_V\) and \(N_A\) denote the number of degrees of freedom of \(V_h\) and \(A_h\) respectively. Then define \(U, V \in \mathbb{R}^M\), where \(U = \{u_i\}_{i=1}^{N_V} \cup \{\lambda_i\}_{i=1}^{N_A}\), \(V = \{v_i\}_{i=1}^{N_V} \cup \{\mu_i\}_{i=1}^{N_A}\), where \(\{u_i\}, \{v_i\}\) and \(\{\lambda_i\}, \{\mu_i\}\) denote the vectors of unknowns associated to the basis functions of \(V_h\) and \(A_h\) respectively.

Consider the mapping \(G : \mathbb{R}^M \rightarrow \mathbb{R}^M\) defined by
\[
(G(U), V)_{\mathbb{R}^M} := A_h([u_h, \lambda_h], (v_h, \mu_h)) - (f, v_h)_{\Omega}.
\]
Existence and uniqueness of a solution to (2.6) is equivalent to showing that there exists a unique \(U \in \mathbb{R}^M\) such that \(G(U) = 0\).

We start by showing some positivity results and a priori bounds

**Lemma 4.1.** There exists \(\alpha > 0\) and an associated constant \(c_\alpha > 0\) so that with the form \(b\) defined by \([2.10]\), \(\delta > 0\) and \(\gamma = \gamma_0 h^{2s}\) with \(\gamma_0 > 0\) there holds, for all \((u_h, \lambda_h) \in V_h \times A_h\)
\[
\| \nabla u_h \|^2_{\Omega} + \gamma \| \lambda_h + \gamma^{-1} [P_{\gamma}(u_h, \lambda_h)] + \| C \| c_\alpha \gamma^{\frac{1}{2}} \lambda_h \| C \| \leq A_h([u_h, \lambda_h], (u_h - \alpha R_h, \lambda_h)), \quad (4.1)
\]
where \(R_h \in V_h\) is defined in Lemma [7.4] such that \(R_h |_{\Omega} := \gamma \xi \Omega I\).

There exists \(\alpha > 0\) and an associated constant \(c_\alpha > 0\) so that with the form \(b\) defined by \([2.11]\), \(k \geq 2\) and \(\gamma = \gamma_0 h^{2s}\) with \(\gamma_0 > 0\), \(\gamma_0\) sufficiently large, and \(\delta > 0\) there holds, for all \((u_h, \lambda_h) \in V_h \times A_h\)
\[
\| \nabla u_h \|^2_{\Omega} + \gamma^{-1} \| u_h + [P_{\gamma}(u_h, \lambda_h)]^+ \| C \| + c_\alpha \| \gamma^{\frac{1}{2}} \lambda_h \| C \| \leq A_h([u_h, \lambda_h], (u_h + \alpha R_h, \lambda_h + \gamma^{-1} \pi_0 u_h)), \quad (4.2)
\]
with \(R_h\) as before. In case \(k = 1\) \([4.2]\) holds under the additional that \(0 < \delta \leq (c_0 C^T)^2 \gamma_0^{-1}\).

Under the same conditions on the parameters as above, for both formulations there also holds, for \((u_h, \lambda_h)\) solution of (2.8),
\[
\| \nabla u_h \|_{\Omega} + \| \gamma^{\frac{1}{2}} \lambda_h \|_{C} \leq \| f \|_{\Omega}. \quad (4.3)
\]

The hidden constants are independent of \(h\).

**Remark 4.1.** For \(k \geq 2\) and continuous multiplier space the parameter \(\delta\) and the term \(\| \lambda_h \|_{C}^2\) can be dropped above.

**Proof.** First consider the claims for formulation 1. By testing in (2.8) with \(v_h = u_h\) and \(\mu_h = \lambda_h\) and observing that
\[
\left\langle \gamma^{-1} [P_{\gamma}(u_h, \lambda_h)]^+, \gamma \lambda_h \right\rangle_C + \| \gamma^{\frac{1}{2}} \lambda_h \|_{C}^2 = \| \gamma^{\frac{1}{2}} \lambda_h \|_{C}^2 + \langle \gamma^{-1} [P_{\gamma}(u_h, \lambda_h)]^+, -\gamma \lambda_h \rangle_C + 2 \left\langle \gamma^{-\frac{1}{2}} [P_{\gamma}(u_h, \lambda_h)]^+, \gamma^{\frac{1}{2}} \lambda_h \right\rangle_C,
\]
we obtain the relation
\[
\| \nabla u_h \|^2_{\Omega} + \gamma \| \gamma^{-\frac{1}{2}} [P_{\gamma}(u_h, \lambda_h)]^+ \|_{C}^2 + \| \lambda_h \|_{C}^2 = A_h([u_h, \lambda_h], (u_h, \lambda_h)).
\]
and hence by using the Cauchy-Schwarz inequality and a Poincaré inequality in the right hand side
\[
\frac{1}{2}\|\nabla u_h\|_\Omega^2 + \gamma \| \gamma^{-\frac{1}{2}}[P_{\gamma^+}(u_h, \lambda_h)]_+ + \lambda_h \|_C^2 + \|\lambda_h\|^2_s \lesssim \|f\|_\Omega^2
\] (4.4)

Using now the first equation we have testing with \( v_h = -\alpha R_h \), with \( r_h = \gamma I_{c_{e.f}}(\xi_h\lambda_h) \) and \( \mu_h = 0 \),
\[
\begin{align*}
A_h[(u_h, \lambda_h), (-R_h, 0)] &= a(u_h, -R_h) + \gamma^{-1}[P_{\gamma^+}(u_h, \lambda_h)]_+ + \gamma I_{c_{e.f}}(\xi_h\lambda_h) \bigg|_C \\
&= a(u_h, -R_h) + \gamma^{-1}[P_{\gamma^+}(u_h, \lambda_h)]_+ + \lambda_h, -\gamma I_{c_{e.f}}(\xi_h\lambda_h) \bigg|_C \\
&\quad - \langle \lambda_h, \gamma(\xi_h\lambda_h - I_{c_{e.f}}(\xi_h\lambda_h)) \rangle_C + (\lambda_h, \gamma\xi_h\lambda_h)_C. \quad (4.5)
\end{align*}
\]

For the last term in the right hand side we have by the inequality \( \|\gamma^\frac{1}{2}\lambda_h\|_C^2 \leq (\gamma\lambda_h, \xi_h\lambda_h)_C \). The second to last term of the right hand side, which is zero for continuous multiplier spaces, can be bounded using Proposition 3.3
\[
(\gamma\lambda_h, \xi_h\lambda_h - I_{c_{e.f}}(\xi_h\lambda_h))_C \leq c_\varepsilon^2 \frac{1}{4} \|\gamma^\frac{1}{2}\lambda_h\|_C^2 + c_\varepsilon^2 \varepsilon^2 \delta^{-1} |\lambda_h|^2_s. \quad (4.6)
\]

The second term is bounded using a Cauchy-Schwarz inequality and the stability of \( I_{c_{e.f}} \),
\[
\langle \gamma^{-1}[P_{\gamma^+}(u_h, \lambda_h)]_+ + \lambda_h, \gamma I_{c_{e.f}}(\xi_h\lambda_h) \rangle_C \leq \frac{1}{2}(c_{e.f}\varepsilon)^{-2}\gamma\|\gamma^{-1}[P_{\gamma^+}(u_h, \lambda_h)]_+ + \lambda_h\|_C^2 + \frac{1}{4} c_\varepsilon^2 \|\gamma^\frac{1}{2}\lambda_h\|_C^2 \quad (4.7)
\]

for the first term we use the Cauchy-Schwarz inequality followed by the stability of \( R_h \) and of \( I_{c_{e.f}} \) to obtain
\[
a(u_h, R_h) \leq C^2_h\gamma^{-2s}c_{e.f}\varepsilon^{-2}\|\nabla u_h\|_\Omega^2 + c_\varepsilon^2 \frac{1}{4} \|\gamma^\frac{1}{2}\lambda_h\|_C^2. \quad (4.8)
\]

Applying the inequalities \( (4.6)-(4.8) \) to \( (4.5) \) we have
\[
\begin{align*}
c_\varepsilon^2 \|\gamma^\frac{1}{2}\lambda_h\|_C^2 - C^2_h\gamma^{-2s}c_{e.f}\varepsilon^{-2}\|\nabla u_h\|_\Omega \\
- \frac{1}{2}(c_{e.f}\varepsilon)^{-2}\gamma\|\gamma^{-1}[P_{\gamma^+}(u_h, \lambda_h)]_+ + \lambda_h\|_C^2 \\
- c_\varepsilon^2 \varepsilon^{-2} \delta^{-1} |\lambda_h|^2_s \\
\leq A_h[(u_h, \lambda_h), (-\alpha R_h, 0)] \quad (4.9)
\end{align*}
\]

We conclude by observing that \( h^{-2s}\gamma = O(1) \) and by combining the bounds \( (3.4), (4.4) \) and \( (4.9) \) with \( \alpha \) small enough. The a priori estimate follows noting that for \( (u_h, \lambda_h) \) solution of \( (2.6) \), there holds using the Poincaré inequality and the properties of \( R_h \),
\[
A_h[(u_h, \lambda_h), (u_h - \alpha R_h, 0)] = (f, u_h - \alpha R_h) \leq C\|f\|_\Omega(\|\nabla u_h\|_\Omega + \|\gamma^\frac{1}{2}\lambda_h\|_C).
\]

To prove \( (4.2) \) we start by testing in the left hand side of \( (2.8) \) with \( v_h = u_h \) and \( \mu_h = \lambda_h + \gamma^{-1}P_{\gamma^+}(u_h, \lambda_h) + \gamma^{-1}(u_h + \pi_h u_h) \), where \( i = 0 \) if \( k = 1 \) and
\[ b[(u_h, \lambda_h), (u_h, \lambda_h + \gamma^{-1}\pi_i u_h)] = \gamma^{-1}(u_h, u_h)_C - \gamma^{-1}\|\pi_i u_h - u_h\|_C^2 \\
+ \gamma^{-1}\|P_\gamma-(u_h, \lambda_h)]+\|_C^2 \\
+ 2\langle\gamma^{-1}u_h, [P_\gamma-(u_h, \lambda_h)]+\rangle_C \\
+ \gamma^{-1}(\pi_i u_h - u_h, [P_\gamma-(u_h, \lambda_h)]+)_C \\
+ |\lambda_h|^2 + s(\lambda_h, \gamma^{-1}(\pi_i u_h - u_h)). \]

This results in

\[ \|\nabla u_h\|_\Omega^2 + \gamma^{-1}\|u_h + [P_\gamma-(u_h, \lambda_h)]+\|_C^2 + |\lambda_h|^2 - \gamma^{-1}\|\pi_i u_h - u_h\|_C^2 \\
+ \gamma^{-1}(\pi_i u_h - u_h, u_h + [P_\gamma-(u_h, \lambda_h)]+)_C + s(\lambda_h, \gamma^{-1}(\pi_i u_h - u_h)) = A_h[(u_h, \lambda_h), (\nu_h, \mu_h)]. \]

We now bound the three last terms on the left hand side. First by the properties of \( \pi_i \) we have

\[ \gamma^{-1}\|\pi_i u_h - u_h\|_C^2 \leq c_1^2 h^{2s}\gamma^{-1}\|\nabla u_h\|_\Omega^2. \tag{4.10} \]

Using a Cauchy-Schwarz inequality, the previous result and an arithmetic-geometric inequality we have

\[ \gamma^{-1}(\pi_i u_h - u_h, u_h + [P_\gamma-(u_h, \lambda_h)]+)_C \leq \frac{1}{2}c_1^2 h^{2s}\gamma^{-1}||\nabla u_h||_\Omega^2 \\
+ \frac{1}{2}\gamma^{-1}\|u_h + [P_\gamma-(u_h, \lambda_h)]+\|_C^2. \tag{4.11} \]

Finally for \( k = 1 \) we have for the last term

\[ s(\lambda_h, \gamma^{-1}(\pi_0 u_h - u_h)) \leq \frac{1}{2}|\lambda_h|^2 + \gamma^{-1}\delta C_\gamma^2||\pi_0 u_h - u_h||_C^2 \\
\leq \frac{1}{2}|\lambda_h|^2 + \frac{1}{2}\delta c_0^2 C_\gamma^2||\nabla u_h||_\Omega^2 \]

and for \( k \geq 2 \), \( s(\lambda_h, \gamma^{-1}(\pi_1 u_h - u_h)) = 0 \). Collecting the results above we obtain for \( k = 1 \)

\[ (1 - 3/2c_0^2\gamma_0^{-1} - 1/2\delta c_0^2 C_\gamma^2\gamma_0^{-1})||\nabla u_h||_\Omega^2 \\
+ \frac{1}{2}\gamma^{-1}\|u_h + [P_\gamma-(u_h, \lambda_h)]+\|_C^2 + \frac{1}{2}|\lambda_h|^2 \]

\[ \lesssim A_h[(u_h, \lambda_h), (\nu_h, \mu_h)]. \tag{4.12} \]

We see that the factor \( (1 - 3/2c_0^2\gamma_0^{-1} - 1/2\delta c_0^2 C_\gamma^2\gamma_0^{-1}) \) is positive under the assumptions on \( \gamma_0 \) and \( \delta \). The corresponding inequality for \( k \geq 2 \) is obtained by omitting the term with \( \delta \) and replacing \( c_0 \) with \( c_1 \). Observe that by using \( \nu_h = R_h \) with \( r_h = -\gamma I_c f \xi_h \lambda_h \) and \( \mu_h = 0 \) we have using similar arguments as above

\[ \gamma||\xi_h^\frac{1}{2}\lambda_h||_C^2 + (\lambda_h, \gamma(\xi_h \lambda_h - I_c f \xi_h \lambda_h))_C + a(u_h, R_h) = A_h[(u_h, \lambda_h), (R_h, 0)]. \tag{4.13} \]
Using once again (4.6) and (4.8)
\[\frac{1}{2}c_\xi \|\gamma \lambda_h\|_C - C_0^2 c_\xi \gamma^2 \lambda_h^2 \|\nabla u_h\|^2_\Omega - c_\xi^2 \gamma^2 \delta^{-1} |\lambda_h|^2 \]
\[\leq A_h[(u_h, \lambda_h), (R_h, 0)] \tag{4.14}\]

where the stabilization contribution can be dropped whenever continuous approximation is used for the multiplier space. We conclude as in the previous case by combining the bounds (4.14) and (4.12). The a priori estimate (4.3) also follows as before.

**Proposition 4.2.** The formulation (2.8) using the contact operators (2.11) or (2.10), and the same assumptions on the parameters \(\delta, \gamma\) as in Lemma 4.1, admits a unique solution.

**Proof.** By the positivity results (4.1) and (4.2) of Lemma 4.1 we have for each method that there exists a linear mapping \(B : \mathbb{R}^M \mapsto \mathbb{R}^M\) such that \(b_1|U| < |BU| \leq b_2|U|\) for some \(0 < b_1 \leq b_2\) and that for \(U\) sufficiently big
\[0 < (G(U), BU). \tag{4.15}\]

We give details regarding the construction of \(B\) only in the case of formulation 2 with \(k = 1\). The argument for \(k \geq 2\), and that for formulation 1, are similar. Let the positive constants \(c_\xi\) and \(C_\xi\) denote the smallest and the largest eigenvalues respectively of the block diagonal matrix in \(\mathbb{R}^{M \times M}\) given by
\[\begin{pmatrix}
(\nabla \varphi_i, \nabla \varphi_j)\Omega + \langle \gamma^{-1} \varphi_i, \varphi_j \rangle_C, & 1 \leq i, j \leq N \nu
\end{pmatrix}\]
where \(\varphi_i\) denotes the basis functions for the space \(V_h\) and \(\frac{1}{2}\gamma(\psi_i, \psi_j)\) where \(\psi_i\) denotes the basis functions for the space \(\Lambda_h\), \(1 \leq i, j \leq N\) such that, with \(|u_h|_{1,h}^2 := \|\nabla u_h\|^2_\Omega + \|\gamma^{-1/2} u_h\|^2_\Omega\)
\[c_\xi |U|_{1,h}^2 \leq \|u_h\|_{1,h}^2 + \frac{1}{4} \gamma \|\lambda_h\|^2_\Omega \leq C_\xi |U|_{1,h}^2.\]

Recalling the a priori bound (4.2), let \(B\) denote the transformation matrix such that the finite element function corresponding to the vector \(BU\) is the function \((u_h + \alpha R_h, \lambda_h + \gamma^{-1} \pi_0 u_h)\), with \(R_h\) defined in Lemma 4.1. First we show that for \(\alpha\) sufficiently small, there are constants \(b_1\) and \(b_2\) such that \(b_1|U|_{1,h} \leq |BU|_{1,h} \leq b_2|U|_{1,h} \). This can be seen by observing that
\[\|u_h\|^2_{1,h} + \frac{1}{4} \gamma \|\lambda_h\|^2_\Omega \leq 2\|u_h + \alpha R_h\|^2_{1,h} + \frac{1}{2} \gamma \|\lambda_h + \gamma^{-1} \pi_0 u_h\|^2_\Omega + \frac{1}{2} \|\gamma^{-1/2} \pi_0 u_h\|^2_\Omega + 2\|\alpha R_h\|^2_{1,h}\]
\[\leq 2C_\xi |BU|_{1,h}^2 + \frac{1}{2} \|u_h\|^2_{1,h} + C_\alpha \gamma \|\lambda_h\|^2_\Omega,\]
where we have used the properties of \(R_h\) from Lemma 4.2. It follows, for \(\alpha\) small enough, that
\[\frac{1}{2}c_\xi |U|_{1,h}^2 \leq (1 - \frac{1}{2})\|u_h\|^2_{1,h} + \left(\frac{1}{4} - C_\alpha \gamma\right) \|\lambda_h\|^2_\Omega \leq 2C_\xi |BU|_{1,h}^2.\]

Similarly we may prove the upper bound using that by the properties of \(R_h\) and \(\pi_0 u_h\) we have
\[c_\xi |BU|_{1,h}^2 \leq \|u_h + \alpha R_h\|^2_{1,h} + \frac{1}{4} \gamma \|\lambda_h + \gamma^{-1} \pi_0 u_h\|^2_\Omega \leq 2\|u_h\|^2_{1,h} + 2\|\alpha R_h\|^2_{1,h} + \frac{1}{4} \gamma \|\lambda_h\|^2_\Omega + \frac{1}{2} \|\gamma^{-1/2} \pi_0 u_h\|^2_\Omega \leq C(\|u_h\|^2_{1,h} + \frac{1}{4} \gamma \|\lambda_h\|^2_\Omega).\]
Assume that the positivity (4.15) holds whenever \(|U|_{\mathbb{R}^M} \geq q\). Assume now that there is no \(U\) such that \(G(U) = 0\) and define the function
\[
\phi(U) = -q/b_1 B^T G(U)/|G(U)|_{\mathbb{R}^M}.
\]
Then \(\phi : B_R \rightarrow B_R\), where \(R = q b_2 / b_1\) and, since \(G(U) \neq 0\), \(\phi\) is continuous by Lemma 3.2 and equivalence of norms on finite dimensional spaces. Hence, since \(B^T\) satisfies the same bounds as \(B\), there exists a fixed point \(X \in B_R\), with
\[
|X|_{\mathbb{R}^M} = q/b_1 |B^T G(X)|_{\mathbb{R}^M} / |G(X)|_{\mathbb{R}^M} \geq q
\]
such that
\[
X = \phi(X).
\]
It follows that
\[
0 < |X|_{\mathbb{R}^M}^2 = -q/b_1 (G(X), BX)_{\mathbb{R}^M} / |G(X)|_{\mathbb{R}^M}
\]
but by assumption \((G(X), BX) > 0\) for \(|X|_{\mathbb{R}^M} \geq q\), which leads to a contradiction.

Uniqueness is consequence of the positivity results of Lemma 4.1 and the monotonicity of Lemma 3.1. Considering first formulation 1, where the form \(b[\cdot, \cdot]\) is given by (2.10), we have
\[
\|\nabla (u_1 - u_2)\|_2^2
= -\gamma^{-1} \langle [P_{\gamma+}(u_1, \lambda_1)]_+ - [P_{\gamma+}(u_2, \lambda_2)]_+ , u_1 - u_2 + \gamma(\lambda_1 - \lambda_2) \rangle_C
- \gamma \|\lambda_1 - \lambda_2\|_C^2 - |\lambda_1 - \lambda_2|^2.
\]
It follows that, defining
\[
|||u, \lambda|||^2 := \|\nabla u\|_2^2 + |\lambda|^2,
\]
\[
|||u_1 - u_2, \lambda_1 - \lambda_2|||^2
= -\gamma \|\lambda_1 - \lambda_2\|_C^2
- \gamma^{-1} \langle [P_{\gamma+}(u_1, \lambda_1)]_+ - [P_{\gamma+}(u_2, \lambda_2)]_+ , P_{\gamma+}(u_1 - u_2, \lambda_1 - \lambda_2) \rangle_C
+ \gamma^{-1} \langle [P_{\gamma+}(u_1, \lambda_1)]_+ - [P_{\gamma+}(u_2, \lambda_2)]_+ , 2\gamma(\lambda_1 - \lambda_2) \rangle_C.
\]
Then, using the monotonicity of Lemma 3.1 we deduce
\[
|||u_1 - u_2, \lambda_1 - \lambda_2|||^2 + \|\lambda_1 - \lambda_2\|_C^2
+ \gamma^{-1} \|P_{\gamma+}(u_1, \lambda_1)]_+ - [P_{\gamma+}(u_2, \lambda_2)]_+ \|^2_C
\leq - \langle [P_{\gamma+}(u_1, \lambda_1)]_+ - [P_{\gamma+}(u_2, \lambda_2)]_+ , 2\gamma(\lambda_1 - \lambda_2) \rangle_C.
\]
Therefore
\[
|||u_1 - u_2, \lambda_1 - \lambda_2|||^2 + \gamma^{-1} \|\lambda_1 - \lambda_2\|_C^2 + \|P_{\gamma+}(u_1, \lambda_1)]_+ - [P_{\gamma+}(u_2, \lambda_2)]_+ \|^2_C = 0 \quad (4.16)
\]
and \(u_1 = u_2\). Repeating the arguments leading to (4.9) on \(\lambda_1 - \lambda_2\) and using (4.16) allows us to conclude that \(\lambda_1 = \lambda_2\).
In the case of formulation 2 we only give the details for $k \geq 2$, the case $k = 1$ is similar, but we need to handle an additional stabilization term. Assume that $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ solves (2.8) with the contact conditions defined by (2.11).

\[
\|\nabla (u_1 - u_2)\|^2 \Omega = \langle \lambda_1 - \lambda_2, u_1 - u_2 \rangle_C \\
= -\gamma \langle \lambda_1 - \lambda_2, [P_\gamma - (u_1, \lambda_1)]_+ - [P_\gamma - (u_2, \lambda_2)]_+ \rangle_C \\
- |\lambda_1 - \lambda_2|^2 s.
\]

Observing that with $\mu_h = \gamma^{-1}\pi_1(u_1 - u_2)$ we also have

\[
\gamma^{-1}\|\pi_1(u_1 - u_2)\|^2_C + \gamma^{-1}\langle \pi_1(u_1 - u_2), [P_\gamma - (u_1, \lambda_1)]_+ - [P_\gamma - (u_2, \lambda_2)]_+ \rangle_C = 0
\]

and therefore we can write

\[
\|\nabla (u_1 - u_2)\|^2 \Omega + \gamma^{-1}\|u_1 - u_2 + [P_\gamma - (u_1, \lambda_1)]_+ - [P_\gamma - (u_2, \lambda_2)]_+ \|^2_C + |\lambda_1 - \lambda_2|^2 s
\]

\[
= \gamma^{-1}\langle (1 - \pi_1)(u_1 - u_2), u_1 - u_2 + [P_\gamma - (u_1, \lambda_1)]_+ - [P_\gamma - (u_2, \lambda_2)]_+ \rangle_C.
\]

By splitting the term in the right hand side using the arithmetic-geometric inequality and using the approximation properties of $\pi_1$, we may conclude that

\[
(1 - \gamma^{-1}C_1 h^2 s^2)\|\nabla (u_1 - u_2)\|^2 \Omega \\
+ \frac{1}{2} \gamma^{-1}\|u_1 - u_2 + [P_\gamma - (u_1, \lambda_1)]_+ - [P_\gamma - (u_2, \lambda_2)]_+ \|^2_C \\
+ |\lambda_1 - \lambda_2|^2 s
\]

\[
\leq 0.
\]

As a consequence $u_1 = u_2$ when $\gamma_0$ is sufficiently large. That $\lambda_1 = \lambda_2$ is immediate from (4.13) since the first equation of (2.4) is linear.

5. Error estimates. In this section we will prove the main results of the paper which are error estimates for the two methods given by (2.9) with the two contact formulations (2.11) and (2.10). The idea of the proof is to combine the uniqueness argument with a Galerkin type perturbation analysis. Since this result is central to the present work we give full detail for both formulations.

**Theorem 5.1.** (Formulation 1) Assume that $u \in H^1(\Omega)$ and $\lambda \in L^2(C)$ is the unique stationary point of (2.2) and $(u_h, \lambda_h)$ the solution to (2.9) with $0 < \gamma = \gamma_0 h^{2s}$, where $s = 1/2$ for the Signorini problem and $s = 1$ for the Obstacle problem, $\gamma_0 \in \mathbb{R}^+$ is sufficiently small and $\delta \in \mathbb{R}^+$ sufficiently large. Then there holds for all $(v_h, \mu_h) \in V_h \times \Lambda_h$

\[
\alpha \|u - u_h\|^2_{H^1(\Omega)} + \gamma \|\lambda - \lambda_h\|^2_C + \gamma\|\lambda + \gamma^{-1}[P_\gamma + (u_h, \lambda_h)]_+\|^2_C \\
\leq \frac{1}{\alpha} \|u - v_h\|^2_{H^1(\Omega)} + \gamma\|\lambda - \mu_h\|^2_C + \gamma^{-1}\|u - v_h\|^2_C + s(\mu_h, \mu_h).
\]
Proof. Using the coercivity of $a(\cdot,\cdot)$ we may write
\[
\|u - u_h\|_{H^1(\Omega)}^2 \leq a(u - u_h, u - u_h)
\]
\[
= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)
\]
\[
\leq \frac{\alpha}{4} \|u - u_h\|_{H^1(\Omega)}^2 + \frac{1}{\alpha} \|v_h - u_h\|_{H^1(\Omega)}^2 + a(u - u_h, v_h - u_h).
\]
It follows, using Galerkin orthogonality, that
\[
a(u - u_h, v_h - u_h)
\]
\[
= \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+ , v_h - u_h \rangle_C
\]
\[
= \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+ , v_h - u_h + \gamma(\mu_h - \lambda_h) \rangle_C
\]
\[
+ \langle \gamma(\lambda_h - \lambda), (\mu_h - \lambda_h) \rangle + s(\lambda_h, \mu_h - \lambda_h).
\] (5.1)

First observe that
\[
\langle \gamma(\lambda_h - \lambda), (\mu_h - \lambda_h) \rangle = -\|\gamma^{\frac{1}{2}}(\mu_h - \lambda_h)\|_{C}^2
\]
\[
+ \|\gamma^{\frac{1}{2}}(\mu_h - \lambda)\|_{C} \|\gamma^{\frac{1}{2}}(\mu_h - \lambda)\|_{C}
\]
\[
\leq (\varepsilon_1 - 1) \|\gamma^{\frac{1}{2}}(\mu_h - \lambda_h)\|_{C}^2 + \frac{1}{4 \varepsilon_1} \|\gamma^{\frac{1}{2}}(\mu_h - \lambda)\|_{C}^2
\]
where we see that the first term can be made negative by choosing $\varepsilon_1$ small enough.

Similarly
\[
s(\lambda_h, \mu_h - \lambda_h) = -|\mu_h - \lambda_h|^2 + s(\mu_h, \mu_h - \lambda_h) \leq (\varepsilon_2 - 1) |\mu_h - \lambda_h|^2 + \frac{1}{4 \varepsilon_2} s(\mu_h, \mu_h)
\]
where once again the first term on the right hand side can be made negative by choosing $\varepsilon$ small. Considering the first term on the right hand side of equation (5.1) we may write
\[
\langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+ , v_h - u_h + \gamma(\mu_h - \lambda_h) \rangle_C
\]
\[
= \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+ , P_{\gamma+}(v_h - u, \mu_h - \lambda) \rangle_C
\]
\[
+ \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+ , P_{\gamma+}(u - u_h, \lambda - \lambda_h) \rangle_C
\]
\[
+ \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+ , 2 \gamma(\mu_h - \lambda_h) \rangle_C
\]
\[
= I + II + III
\]
The term $I$ may be bounded using the Cauchy-Schwarz inequality followed by the arithmetic geometric inequality
\[
I \leq \varepsilon_3 \|\lambda + \gamma^{-\frac{1}{2}}[P_{\gamma+}(u_h, \lambda_h)]_+\|_{C}^2 + \frac{1}{4 \varepsilon_3} \|\gamma^{-\frac{1}{2}}\gamma^{-1}[P_{\gamma+}(v_h - u, \mu_h - \lambda)\|_{C}^2.
\]

For the term $II$ we use the monotonicity property $([a+]_+ - [b+]_+)(b - a) \leq -([a+]_+ - [b+]_+)^2$ to deduce that
\[
II \leq -\|\gamma^{\frac{1}{2}}(\lambda + \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+)\|_{C}^2
\]
Finally to estimate term \( III \), let \( R_h \) be defined by Lemma 3.4 with the associated 
\( r_h := I_{c_f}(2\xi_h\gamma(\mu_h - \lambda_h)) \) and set \( \zeta_h = 1 - \xi_h \). Using the equation we may write

\[
\langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+, 2\gamma(\mu_h - \lambda_h) \rangle_C \\
\leq \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+, 2\xi_h\gamma(\mu_h - \lambda_h) \rangle_{T_{C}\cap G_h} \\
+ \langle \gamma^{-1}[P_{\gamma+}(u_h, \lambda_h)]_+ - \gamma^{-1}[P_{\gamma+}(u, \lambda)]_+, 2\xi_h\gamma(\mu_h - \lambda_h) \rangle_{G_h} \\
+ a(u - u_h, R_h) \\
= IIIa + IIIb + IIIc.
\]

We estimate \( IIIa-IIIc \) term by term. For \( IIIa \) we use the assumption 3.1

\[
IIIa \leq c_D\|\gamma^{-\frac{1}{2}}([P_{\gamma+}(u_h, \lambda_h)]_+ + \lambda)\|_C^2 + c_D\|\gamma^\frac{1}{2}(\mu_h - \lambda_h)\|_C^2.
\]

As a consequence of Lemma 3.3 we get the following bound of term \( II \)

\[
IIIb \leq \varepsilon_4\|\gamma^{-\frac{1}{2}}(\lambda + [P_{\gamma+}(u_h, \lambda_h)]_+)\|_C^2 + c_D\|\gamma^\frac{1}{2}(\mu_h - \lambda_h)\|_C^2.
\]

For the third term we observe that by the continuity of \( a \) and Lemma 3.4 we have

\[
IIIc \leq \|u - u_h\|_{H^1(\Omega)} \|R_h\|_{H^1(\Omega)} \leq C\|u - u_h\|_{H^1(\Omega)} h^{-s} \|r_h\|_C \\
\leq C\|u - u_h\|_{H^1(\Omega)} h^{-s} \gamma^\frac{1}{2}(\lambda_h - \mu_h) \|_C \\
\leq \frac{\alpha}{4}\|u - u_h\|_{H^1(\Omega)}^2 + C^2 h^{-2s}\gamma a^{-1}\|\gamma^\frac{1}{2}(\lambda_h - \mu_h)\|_C^2.
\]

Collecting the above bounds and recalling that by definition

\[
s(\lambda_h, \lambda_h) = \delta \gamma h^\frac{1}{2}\|\lambda_h\|_{T_{C}}^2,
\]

we have

\[
\frac{\alpha}{2}\|u - u_h\|_{H^1(\Omega)}^2 + (1 - \varepsilon_3 - \varepsilon_4 - c_D)\|\gamma^{-\frac{1}{2}}(\lambda + [P_{\gamma+}(u_h, \lambda_h)]_+)\|_C^2 \\
+ (1 - \varepsilon_4 - c_D - C^2\gamma_0/\alpha)\|\gamma^\frac{1}{2}(\mu_h - \lambda_h)\|_C^2 \\
+ (1 - \varepsilon_2 - c_D^2/(\delta\varepsilon_4))\|\mu_h - \lambda_h\|_C^2 \\
\leq \frac{1}{\alpha}\|u - u_h\|_{H^1(\Omega)}^2 + \frac{1}{4\varepsilon_3}\|\gamma^{-\frac{1}{2}}P_{\gamma+}(v_h - u, \lambda_h - \lambda)\|_C^2 \\
+ \frac{1}{4\varepsilon_1}\|\gamma^\frac{1}{2}(\mu_h - \lambda)\|_C^2 + \frac{1}{4\varepsilon_2}s(\mu_h, \mu_h)
\]

Observe that as usual when a continuous multiplier space is used all terms and coefficients associated to the jump operator may be omitted.

Fixing \( \varepsilon_1, \varepsilon_3, \varepsilon_4 \) and \( \gamma_0 \) sufficiently small so that

\[
\varepsilon_1 + C^2\gamma_0/\alpha = \varepsilon_3 + \varepsilon_4 = (1 - c_D)/2,
\]

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and \( \varepsilon_2 \) sufficiently small and \( \delta \) sufficiently large so that \( \varepsilon_2 + C_4^2/\delta \varepsilon_4 < 1 \), then there holds
\[
\alpha \| u - u_h \|_{H^1(\Omega)}^2 + \| \gamma^{-\frac{1}{2}}(\lambda + [P_{\gamma^+}(u_h, \lambda_h)]_+) \|_C^2
\]
\[
+ \| \gamma^{-\frac{1}{2}}(\mu_h - \lambda_h) \|_C^2 + |\mu_h - \lambda_h|^2_s
\]
\[
\lesssim \frac{1}{\alpha} \| u - v_h \|_{H^1(\Omega)}^2 + \| \gamma^{-\frac{1}{2}}P_{\gamma^+}(v_h - u, \mu_h - \lambda) \|_C^2
\]
\[
+ \| \gamma^{-\frac{1}{2}}(\mu_h - \lambda) \|_C^2 + s(\mu_h, \mu_h).
\]

The triangle inequality \( \| \gamma^{\frac{1}{2}}(\lambda - \lambda_h) \|_C^2 \leq \| \gamma^{\frac{1}{2}}(\mu_h - \lambda_h) \|_C^2 + \| \gamma^{\frac{1}{2}}(\mu_h - \lambda) \|_C^2 \) concludes the proof. \( \square \)

**Corollary 5.1.** Assume that \( u \in H^r(\Omega) \), \( 3/2 < r \leq k + 1 \) and \( \lambda \in H^{r-1-s}(\Omega) \), with \( r - 1 - s > 0 \) where \( s = 1/2 \) for the Signorini problem and \( s = 1 \) for the Obstacle problem and that \((u_h, \lambda_h)\) is the solution of (2.9) with the contact operator defined by (2.10) and under the same conditions on the parameters as in Theorem 5.1. Then there holds
\[
\alpha \| u - u_h \|_{H^1(\Omega)}^2 + \| \gamma^{\frac{1}{2}}(\lambda - \lambda_h) \|_C^2 + \| \gamma^{\frac{1}{2}}(\lambda + \gamma^{-1}[P_{\gamma^+}(u_h, \lambda_h)]_+) \|_C^2
\]
\[
\lesssim h^{r-1}(\| u \|_H^r(\Omega) + |\lambda|_{H^{r-1-s}(\Omega)}),
\]

**Proof.** Let \( v_h = i_h u \) where \( i_h \) denotes the standard nodal interpolant and let \( \mu_h = \pi_l \lambda \) where \( \pi_l \) denotes the \( L^2 \)-projection. Using standard approximation estimates and the trace inequality (3.2) we may then bound the right hand side of the estimate of Theorem 5.1,
\[
\| u - v_h \|_{H^1(\Omega)}^2 \lesssim h^{2(r-1)}|u|_{H^r(\Omega)},
\]
\[
\gamma^{-1}(u - v_h) \|_C^2 \lesssim \gamma^{-1}h^{2(r+s)}|u|_{H^r(\Omega)}^2 \lesssim h^{2(r-1)}|u|_{H^r(\Omega)}^2,
\]
\[
\gamma \| \lambda - \mu_h \|_C^2 \lesssim \lesssim \gamma h^{2(r-1-s)}|\lambda|_{H^{r-1-s}(\Omega)} \lesssim h^{2(r-1)}|\lambda|_{H^{r-1-s}(\Omega)}.
\]

Finally we have,
\[
\lesssim \gamma \| \pi_l \lambda - \lambda \|_C^2 + |\lambda - \mu_h|^2_s \lesssim h^{2(r-1-s)}|\lambda|_{H^{r-1-s}(\Omega)}
\]
and we conclude by taking square roots. \( \square \)

**Theorem 5.2.** (Formulation 2) Assume that \( u \in H^1(\Omega) \) and \( \lambda \in L^2(\Omega) \) is the unique stationary point of (2.2) and \( (u_h, \lambda_h) \) the solution to (2.8) with (2.11) and \( \gamma = \gamma_0 h^{2s} \), where \( s = 1/2 \) for the Signorini problem and \( s = 1 \) for the Obstacle problem, \( \gamma_0 \) sufficiently large and \( \delta > 0 \) then there holds for all \((v_h, \mu_h) \in V_h \times \Lambda_h\)
\[
\alpha \| u - u_h \|_{H^1(\Omega)}^2 + \| \gamma^\frac{1}{2}(\lambda - \lambda_h) \|_C^2
\]
\[
+ \| \gamma^{-\frac{1}{2}}P_{\gamma^+}(u_h, \lambda) \|_C^2 + [P_{\gamma^-}(u_h, \lambda_h)]_+ \|_C^2
\]
\[
\lesssim \frac{1}{\alpha} \| u - v_h \|_{H^1(\Omega)}^2 + \| \mu_h - \lambda \|_C^2 + |\mu_h|^2_s + \| v_h - u \|_C^2.
\]
Proof. Using the coercivity of $a(\cdot, \cdot)$ we may write

$$
\alpha \|u - u_h\|^2_{H^1(\Omega)} \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)
\leq \frac{\alpha}{4} \|u - u_h\|^2_{H^1(\Omega)} + \frac{1}{\alpha} \|v_h - u_h\|^2_{H^1(\Omega)} + a(u - u_h, v_h - u_h).
\tag{5.3}
$$

By Galerkin orthogonality we obtain the equality, and then adding and subtracting suitable quantities it follows that

$$
a(u - u_h, v_h - u_h) = \langle \lambda - \lambda_h, v_h - u_h \rangle_C
= \langle \lambda - \lambda_h, v_h - u_h \rangle_C
- \langle \mu_h - \lambda_h, u - u_h \rangle_C + s(\lambda_h, \mu_h - \lambda_h)
- \langle \mu_h - \lambda_h, [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle_C.
\tag{5.4}
$$

Then we proceed by adding and subtracting $u$ in the right slot of the first term on the right hand side, $\lambda$ in the left slot of the second term, $\mu_h$ in the left slot of the third term and finally $\lambda + \gamma^{-1}(u - u_h)$ in the left slot of the third term, leading to

$$
a(u - u_h, v_h - u_h) = \langle \lambda - \lambda_h, v_h - u \rangle_C
- \langle \mu_h - \lambda_h, [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle_C
- \gamma^{-1} \langle P_{\gamma -}(u, \lambda) - P_{\gamma -}(u_h, \lambda_h), [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle_C
- \gamma^{-1} \langle (u - u_h), [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle_C
- |\mu_h - \lambda_h|^2 + s(\mu_h, \mu_h - \lambda_h).
$$

We may then apply the monotonicity of Lemma 3.1 to obtain the bound

$$
a(u - u_h, v_h - u_h) \leq \langle \lambda - \lambda_h, v_h - u \rangle_C
- \langle \mu_h - \lambda_h, (u - u_h) + [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle_C
- \gamma^{-1} \langle [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle C
- \gamma^{-1} \langle (u - u_h), [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle C
- |\mu_h - \lambda_h|^2 + s(\mu_h, \mu_h - \lambda_h).
\tag{5.4}
$$

Summarizing (5.3) and (5.4) we have

$$
\frac{3}{4} \alpha \|u - u_h\|^2_{H^1(\Omega)} + \frac{3}{4} |\mu_h - \lambda_h|^2 + \gamma^{-1} \langle [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle C
\leq \frac{1}{\alpha} \|v_h - u_h\|^2_{H^1(\Omega)} + \langle \lambda - \lambda_h, v_h - u \rangle_C
- \langle \mu_h - \lambda_h, (u - u_h) + [P_{\gamma -}(u, \lambda)\rangle - [P_{\gamma -}(u_h, \lambda_h)]_+ \rangle C + |\mu_h|^2.
\tag{5.5}
$$
Then observe that using the second equation we may write, with \( \tilde{e} = \pi_i(u - u_h) \), with \( i = 0 \) for \( k = 0 \) and \( i = 1 \) for \( k \geq 2 \) and taking \( \mu_h = \gamma^{-1}\tilde{e} \)

\[
\gamma^{-1} \langle \tilde{e}, [P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \rangle_C + \gamma^{-1} \|\tilde{e}\|^2_C 
- \frac{1}{4} |\lambda_h|^2 - C\gamma^{-1}\delta h^2 \|\nabla(u - u_h)\|^2_\Omega
\leq \gamma^{-1} \langle \tilde{e}, [P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \rangle_C + \gamma^{-1} \|\tilde{e}\|^2_C + s(\lambda_h, \gamma^{-1}\tilde{e})
= 0
\]

where the last term vanishes for \( k \geq 2 \). It follows that

\[
\gamma^{-1} \|\tilde{e}\|^2_C - \frac{1}{4} |\lambda_h|^2 - C\gamma^{-1}\delta h^2 \|\nabla(u - u_h)\|^2_\Omega
\leq -\gamma^{-1} \langle \tilde{e}, [P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \rangle_C
\]

We recall that by the \( L^2 \)-orthogonality there holds \( \|\tilde{e}\|^2_C = \|e\|^2_C - \|e - \tilde{e}\|^2_C \) and therefore

\[
\gamma^{-\frac{1}{2}} \|e\|^2_C \leq \gamma^{-\frac{1}{2}} \|\tilde{e}\|^2_C + C\gamma^{-1}\delta h^2 \|\nabla e\|^2_\Omega
\]

and consequently using also that \( \|\tilde{e}\| \leq \|e\| \), there exists constants \( C, c \) independent of \( \gamma \) and \( h \) such that

\[
\frac{1}{2} \gamma^{-1} \|e\|^2_C - \frac{1}{2} \gamma^{-1} \|P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \|^2_C
- \frac{1}{2} |\mu_h - \lambda_h|^2 - C(\delta + 1)\gamma^{-1}\delta h^2 \|\nabla(u - u_h)\|^2_\Omega \leq \frac{1}{2} |\mu_h|^2
\]  

(5.6)

Collecting the results of equations (5.5), and (5.6) we have

\[
\left(3\alpha - C(\delta + 1)\gamma^{-1}\right) \|u - u_h\|^2_{\mathcal{H}^1(\Omega)} + \frac{1}{4} |\mu_h - \lambda_h|^2
+ \frac{1}{2} \gamma^{-1} \|e\|^2_C + \frac{1}{2} \gamma^{-1} \|P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \|^2_C
+ \gamma^{-1} \|(u - u_h), [P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \|_C
\leq \frac{1}{\alpha} \|u - v_h\|^2_{\mathcal{H}^1(\Omega)} + \langle \lambda - \lambda_h, v_h - u \rangle_C
- \langle \mu_h - \lambda, u - u_h + [P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \rangle_C + 2|\mu_h|^2.
\]

Assuming that \( C(\delta + 1)\gamma^{-1}\delta h^2 \leq \frac{1}{4}\alpha \), using that \( \frac{1}{2}a^2 + \frac{1}{2}b^2 + ab = \frac{1}{2}(a + b)^2 \) and the Cauchy-Schwarz inequality followed by the arithmetic-geometric inequality in the second to last term in the right hand side we obtain

\[
\alpha \|u - u_h\|^2_{\mathcal{H}^1(\Omega)} + |\mu_h - \lambda_h|^2
+ \gamma^{-1} \|(u - u_h), [P_{\gamma}^-(u, \lambda)]_+ - [P_{\gamma}^-(u_h, \lambda_h)]_+ \|^2_C
\leq \frac{4}{\alpha} \|u - v_h\|^2_{\mathcal{H}^1(\Omega)} + 4 \langle \lambda - \lambda_h, v_h - u \rangle_C + 4\gamma |\mu_h - \lambda|^2_C + 8|\mu_h|^2
\]  

(5.7)

Observe that the \( \delta \) in the condition may be omitted for \( k \geq 2 \).
It remains to control the Lagrange multiplier. Observe that taking \( v_h = R_h \) as defined in Lemma 3.4 with \( r_h = -\gamma I_{cf}\xi_h(u_h - \lambda_h) \) we may use Galerkin orthogonality to obtain
\[
\gamma \| \xi^2_h (\mu_h - \lambda_h) \|_{C}^2 - \gamma (\mu_h - \lambda_h, (1 - I_{cf})\xi_h(\mu_h - \lambda_h))_{C} \\
+ \gamma (\lambda - \mu_h, I_{cf}\xi_h(\mu_h - \lambda_h))_{C} + a(u - u_h, R_h) \\
= 0.
\]
Using the bound (3.4), \( c^2 \|\mu_h - \lambda_h\|^2 \leq \|\xi^2_h (\mu_h - \lambda_h)\|^2_{C} \) we have
\[
\frac{c^2}{2} \gamma (\|\lambda - \lambda_h\|_{C}^2 - \|\mu_h - \lambda_h\|_{s}^2 - \gamma \left(C + c^2 \xi^2\right) \|\mu_h\|_{C}^2 \\
+ a(u - u_h, R_h) \\
\leq 0.
\]
Recall that by Lemma 3.4 we have
\[
a(u - u_h, R_h) \leq \frac{-c^2}{4} \gamma (\|\lambda - \lambda_h\|_{C}^2 - \|\mu_h - \lambda_h\|_{s}^2 - \alpha \|u - u_h\|^2_{H^1(\Omega)} \leq \gamma \|\lambda - \mu_h\|_{C}^2.
\]
Multiplying both sides of (5.7) by 1/2 and adding it to (5.7) leads to the inequality
\[
\frac{1}{2} \alpha \|u - u_h\|^2_{H^1(\Omega)} + \frac{1}{2} \|\mu_h - \lambda_h\|_{s}^2 + c_\lambda \gamma (\|\lambda - \lambda_h\|_{C}^2 \\
+ \gamma^{-1} \|u - u_h\| + [P_{\gamma}(u, \lambda)]_{+} - [P_{\gamma}(u_h, \lambda_h)]_{+} \|_{C}^2 \\
\leq \frac{c_\lambda}{\alpha} \|u - v_h\|^2_{H^1(\Omega)} + C \gamma \|\mu_h - \lambda\|_{C}^2 + C \|\mu_h\|_{s}^2 + 4 (\lambda - \lambda_h, v_h - u)_C.
\]
where \( c_\lambda = \frac{c^2}{2\xi^2} \). Finally splitting the last term on the right hand side
\[
4 (\lambda - \lambda_h, v_h - u)_C \leq \frac{c_\lambda}{2} \gamma (\|\lambda - \lambda_h\|_{C}^2 + 2c_\lambda \gamma^{-1} \|v_h - u\|_{C}^2
\]
we conclude that
\[
\frac{1}{\alpha} \|u - u_h\|^2_{H^1(\Omega)} + \|\mu_h - \lambda_h\|_{s}^2 + c_\lambda \gamma (\|\lambda - \lambda_h\|_{C}^2 \\
+ \gamma^{-1} \|u - u_h\| + [P_{\gamma}(u, \lambda)]_{+} - [P_{\gamma}(u_h, \lambda_h)]_{+} \|_{C}^2 \\
\leq \frac{1}{\alpha} \|u - v_h\|^2_{H^1(\Omega)} + \gamma \|\mu_h - \lambda\|_{C}^2 + \|\mu_h\|_{s}^2 + 4 \gamma^{-1} \|v_h - u\|_{C}^2.
\]
\[\square\]

**Corollary 5.2.** Assume that \( u \in H^{r}(\Omega) \), \( 3/2 < r \leq k + 1 \) and \( \lambda \in H^{r-1-s}(C) \), with \( r - 1 - s > 0 \) where \( s = 1/2 \) for the Signorini problem and \( s = 1 \) for the Obstacle problem and that \( (u_h, \lambda_h) \) is the solution of (2.6) with the contact operator defined.

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Note: The document is a page from a mathematical text, discussing advanced topics in the field of partial differential equations and numerical methods, specifically focusing on the control of Lagrange multipliers in variational problems.
by (2.11) and under the same conditions on the parameters as in Theorem 5.2. Then there holds
\[
\alpha \|u - u_h\|_{H^1(\Omega)} + \gamma (\lambda - \lambda_h) + \gamma^{-1/2} \|u - u_h + [P\gamma(u, \lambda)]_+ - [P\gamma(u_h, \lambda_h)]_+\|_C \\
\lesssim h^{r-1}(\|u\|_{H^r(\Omega)} + \|\lambda\|_{H^{r-1-\varepsilon}(\Omega)}).
\]

Proof. Similar to that of Corollary 5.1.

6. Numerical examples. In the numerical examples below, we define \( h = 1/\sqrt{NNO} \), where NNO denotes the number of nodes in a uniformly refined mesh. We use the formulation (2.8) with the nonlinear term defined by (2.10). For the spaces we chose piecewise linear finite elements for the primal variable and piecewise constants for the Lagrange multipliers, constant per element for the obstacle problem, and constant per element edge on the Signorini boundary for the Signorini problem.

6.1. Smooth obstacle problem. Our smooth obstacle example, adapted from [24], is posed on the square \( \Omega = (-1, 1) \times (-1, 1) \) with \( \psi = 0 \) and
\[
f = \begin{cases} 
8r^2(1 - (r^2 - r_0^2)) & \text{if } r \leq r_0, \\
8(r^2 + (r^2 - r_0^2)) & \text{if } r > r_0,
\end{cases}
\]
where \( r = \sqrt{x^2 + y^2} \) and \( r_0 = 1/4 \), and with Dirichlet boundary conditions taken from the corresponding exact solution
\[
u = -[r^2 - r_0^2]^2_+.
\]
We choose \( \gamma = \gamma_0 h \) with \( \gamma_0 = 1/100 \) and show the convergence in the \( L^2 \) and \( H^1 \) norms in Figure 6.1. An elevation of the computed solution on one of the meshes in a sequence is given in Fig. 6.2. We note the optimal convergence of \( O(h^2) \) in \( L^2 \) (dashed line has inclination 2:1) and \( O(h) \) in \( H^1 \) (dotted line has inclination 1:1).

6.2. Nonsmooth obstacle problem. This example was proposed by Braess et al. [7]. The domain is \( \Omega = (-2, 2) \times (-2, 2) \setminus [0, 2) \times (-2, 0] \) with \( \psi = 0 \) and
\[
f(r, \varphi) = r^{2/3} \sin(2\varphi/3)(\gamma'(r)/r - \gamma''(r)) + \frac{4}{3} r^{-1/3} \gamma'(r) \sin(2\varphi/3) + \gamma_2(r)
\]
where, with \( \hat{r} = 2(r - 1/4) \),
\[
\gamma_1(r) = \begin{cases} 
1, & \hat{r} < 0 \\
-6\hat{r}^5 + 15\hat{r}^4 - 10\hat{r}^3 + 1, & 0 \leq \hat{r} < 1 \\
0, & \hat{r} \geq 1,
\end{cases}
\]
\[
\gamma_2(r) = \begin{cases} 
0, & r \leq 5/4, \\
1 & \text{elsewhere}.
\end{cases}
\]
with Dirichlet boundary conditions taken from the corresponding exact solution
\[
u(r, \varphi) = -r^{2/3} \gamma_1(r) \sin(2\varphi/3)
\]
which belongs to \( H^{5/3-\varepsilon}(\Omega) \) for arbitrary \( \varepsilon > 0 \).

For this example we plot, in Fig. 6.3, the error on consecutive refined meshes. We note the suboptimal convergence in \( L^2 \) which agrees with the regularity of the exact solution. In Fig. 6.4 we show an elevation of the approximate solution on one of the meshes used to compute convergence.
6.3. Signorini problem. The Signorini problem is posed on the unit square \((0, 1) \times (0, 1)\) with homogeneous Dirichlet boundary conditions at \(y = 1\), homogeneous Neumann boundary conditions at \(x = 0\) and \(x = 1\), and a Signorini boundary at \(y = 0\). The load is \(f = -2\pi \sin 2\pi x\) (following [5]), and we set \(\gamma_0 = 0.1\). No explicit solution is available and we instead use an overkill solution, using 66049 nodes (corresponding to \(h \approx 4 \times 10^{-3}\)) to estimate the error. In Fig. 6.5 we show the convergence in the \(L_2\) and \(H^1\) norms and again we observe optimal convergence of \(O(h^2)\) in \(L_2\) (dashed line has inclination 2:1) and \(O(h)\) in \(H^1\) (dotted line has inclination 1:1). Finally, in Fig. 6.6 we show an elevation of the computed solution.

Acknowledgments. This research was supported in part by EPSRC, UK (EP/J002313/1), the Swedish Foundation for Strategic Research (AM13-0029), the Swedish Research Council (2011-4992, 2013-4708), and the Swedish Strategic Research Program Essence.

The first author wishes to thank Dr. Franz Chouly, Prof. Patrick Hild and Prof. Yves Renard for interesting discussions on Nitsche’s method for contact problems and the augmented Lagrangian method.

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Figure 6.1. Convergence for the smooth obstacle. Dotted line has inclination 1:1, dashed line has inclination 1:2.

Figure 6.2. Elevation of the discrete solution, smooth obstacle.
Figure 6.3. Convergence for the nonsmooth obstacle. Dotted line has inclination 1:1, dashed line has inclination 1:5/3.

Figure 6.4. Elevation of the discrete solution, nonsmooth obstacle.
Figure 6.5. Convergence for the Signorini case.
Figure 6.6. Elevation of the discrete solution, Signorini case.