Uniform endomorphisms which are isomorphic to a Bernoulli shift

By Christopher Hoffman and Daniel Rudolph

Abstract

A uniformly \( p \)-to-one endomorphism is a measure-preserving map with entropy \( \log p \) which is almost everywhere \( p \)-to-one and for which the conditional expectation of each preimage is precisely \( 1/p \). The standard example of this is a one-sided \( p \)-shift with uniform i.i.d. Bernoulli measure. We give a characterization of those uniformly finite-to-one endomorphisms conjugate to this standard example by a condition on the past tree of names which is analogous to very weakly Bernoulli or loosely Bernoulli. As a consequence we show that a large class of isometric extensions of the standard example are conjugate to it.

1. Introduction

Perhaps the most significant aspect of Ornstein’s isomorphism theory for Bernoulli shifts [5] is the very weak Bernoulli condition which Ornstein and Weiss proved characterizes isomorphism to a Bernoulli shift [6], [8]. The very weak Bernoulli (v.w.B.) condition has been exploited to show that many classes of transformations are isomorphic to Bernoulli shifts. Examples of these include ergodic toral automorphisms and geodesic flows on negatively curved space [4], [9]. The Kakutani equivalence theory of Feldman, Ornstein and Weiss has a similar condition, loosely Bernoulli (l.B.), which shows when a transformation is Kakutani equivalent to a Bernoulli shift [7]. The existence of such a criterion is the hallmark of these and parallel theories.

We consider here uniformly \( p \)-to-one endomorphisms, measure-preserving endomorphisms with entropy \( \log p \) which are a.s. \( p \)-to-one and for which the conditional expectations of the preimages are all equal to \( 1/p \). The one-sided Bernoulli shift on \( p \) symbols, each equally likely, is the standard example of a uniformly \( p \)-to-one endomorphism. This endomorphism has state space \( B = \{0, 1, \ldots, p-1\}^\mathbb{N} \). The measure \( \nu \) is defined by

\[ \nu(b \mid b_0 = a_0, b_1 = a_1, \ldots, b_m = a_m) = \frac{1}{p^{m+1}} \]
for any sequence $a_0, \ldots, a_m$ where all the $a_i \in \{0, 1, \ldots, p - 1\}$. The action on $B$ is the left shift $\sigma(b)_i = b_{i+1}$. In this paper we give a characterization of those uniformly $p$-to-one endomorphisms conjugate to the standard example via a criterion parallel to v.w.B. and l.B.

We now describe the class of uniformly two-to-one endomorphisms known as the $[T, \text{Id}]$ examples. Let $(B, \sigma, \nu)$ be the standard two-to-one endomorphism. For $Y$ any Lebesgue space and $T$ an ergodic automorphism of $Y$, $[T, \text{Id}]$ is defined by

$$[T, \text{Id}](x, y) = (\sigma(x), T^{x_0}(y)).$$

Parry [10] has shown that for $T$, an irrational rotation of the circle extremely well approximated by rationals, $[T, \text{Id}]$ is always conjugate to the standard example ($\sigma$ itself). In Section 6 we will show that $[T, \text{Id}]$ is conjugate to the standard example for all ergodic isometries $T$.

The rest of the paper is organized as follows. In Section 2 we define tree very weak Bernoulli and prove that a large natural class of factors of tree v.w.B. endomorphisms are tree v.w.B. In Section 3 we will introduce the notion of a one-sided joining of two endomorphisms. In Section 4 we introduce the concept of tree finitely determined and prove that tree very weak Bernoulli implies tree finitely determined. Section 5 contains the bulk of the proof. It will follow the same outline as the Burton-Rothstein approach to Ornstein’s theorem. First there will be a Rokhlin lemma and a strong Rokhlin lemma for endomorphisms. This is followed by the copying lemma, which is the main tool in the proof. The proof of our copying lemma is much easier than the proof of the copying lemma in Ornstein’s theorem because, perhaps surprisingly, we will not have to deal with entropy. In Section 6 we apply the tree v.w.B. characterization to show that a large class of isometric extensions of the standard endomorphism are one-sidedly conjugate to it.

The methods of this work are much more broadly applicable than to just uniformly finite-to-one endomorphisms. For any bounded finite-to-one endomorphism $(X, T, \mu)$ one can assign to each point $x$ a tree of inverse images. We will discuss this in great detail as we continue. Each point on this tree can be weighted with the expectation that the path of inverse images to that point has occurred. This assignment of a weighted tree to $x$ is a conjugacy invariant of the endomorphism. Moreover, the expectations of just the inverse images $T^{-1}(x)$ set a natural lower bound on the entropy of $T$. The uniformly finite-to-one maps are the simplest bounded finite-to-one endomorphisms in that the weighted trees do not depend on $x$ and the entropy of $T$ is at its minimum given the tree. To apply our methods to the general bounded finite-to-one case requires working relative to the sub-$\sigma$-algebra generated by the trees, with perhaps nonzero entropy relative to the trees. Both issues are standard in the theory and require no essentially new ideas. We have focused on the simplest
case here as it allows the presentation of all the new ideas without the extra baggage and also as it arises in a variety of natural situations, for example rational maps of the Riemann sphere and ergodic group endomorphisms. All of these issues will be presented separately.

Isomorphism theories for endomorphisms have been presented previously. The two cases we are aware of are del Junco [2] and Ashley, Marcus and Tuncel [1]. Del Junco considers two-sided Bernoulli shifts and is after conjugacies that are finitary, one-sided in one direction but two-sided in the other. Thus his work is significantly different from ours. It is interesting to note his use of what he calls a * joining which is strongly analogous to our one-sided joinings. The work of Ashley, Marcus and Tuncel is both more restrictive and more general than ours as they consider all finite state Markov chains. As our examples indicate, many standard endomorphisms are not directly Markov. On the other hand, the existence of a period in a Markov chain will make it nonstandard. We expect future work on relativizing our methods will provide an alternative approach to this classification of one-sided Markov chains just as relativizing Ornstein’s theorem provided a classification of two-sided Markov chains.

The authors wish to thank A. del Junco for discovering several gaps and one serious error in the previous version of this paper.

2. Tree very weak Bernoulli

In this section we will define what it means for an endomorphism to be tree very weakly Bernoulli. As with v.w.B. and l.B., our criterion concerns names on orbits. A $p$-to-1 uniform endomorphism $T$ and a point $x$ generate a $p$-ary tree of inverse images. We will describe how to define a tree name given the tree of inverse images and any partition or, more generally, any metric space-valued function on $X$. Simply stated the tree very weakly Bernoulli criterion says that for any such function and a.e. two points $x$ and $x'$ one can match these two tree names with an arbitrarily small density of errors by a map that preserves the tree structure.

Consider a $p$-ary tree with $p^n$ nodes at each index $n \geq 0$. Each node at index $n$ connects to $p$ nodes at the index $n + 1$. We assign each such set of $p$ nodes a distinct value in $\{0, \ldots, p - 1\}$. Then we label each node other than the root by the sequence of values we see moving from the root to the node. In this form we can concatenate nodes $v'$ and $v$ by concatenating their labels. Notice that when we fix a node $v$, the set of labels $vv'$ form a $p$-ary subtree rooted at node $v$. This is consistent with our convention that the root node is unlabeled.
Call this labeled tree $\mathcal{T}$ and call the tree that has all the nodes at index less than or equal to $n$, $\mathcal{T}_n$. Let $\eta$ be the set of nodes of $\mathcal{T}$ and $\eta_n$ be the set of nodes for $\mathcal{T}_n$. For $v \in \eta$ and at index $i$ (i.e. $v \in \eta_i \setminus \eta_{i-1}$) we write $|v| = i$ and where $v$ is a list of values $v_1, \ldots, v_i$ from $\{0, \ldots, p - 1\}$ which is the list of labels of the nodes along the branch from the root to $v$.

Let $\mathcal{A}$ be the collection of all bijections of the nodes of $\mathcal{T}$ that preserve the tree structure. We refer to this as the group of tree automorphisms. Let $\mathcal{A}_n$ be the bijections of the nodes of $\mathcal{T}_n$ preserving the tree structure. To give a representation to such automorphisms $A$ notice that from $A$ we obtain a permutation $\pi_v$ of $\{0, \ldots, p - 1\}$ at each node, giving the rearrangement of its $p$ predecessors. An automorphism of $\mathcal{T}_n$ will be represented by an assignment of a permutation of $\{0, \ldots, p - 1\}$ to each node of the tree except for those at index $n$.

Let $(X, T, \mu, \mathcal{F})$ be a uniformly $p$-to-one endomorphism. Then each $x \in X$ has $p$ inverse images. Select a measurable $p$ set partition $K$ of $X$ such that almost every $x$ has one preimage in each element of $K$. Label the sets of $K$ as $K_0, K_1, \ldots, K_{p-1}$. For each $i \in \{0, \ldots, p - 1\}$ and $x \in X$ define $T_i(x)$ to be the preimage of $x$ in $K_i$. We now define a set of partial inverses for $T$. For $v = (v_1, \ldots, v_i) \in \eta$ define $T_v(x) = T_{v_i}(\cdots(T_{v_1}(x)))$.

We let $R$ and $U$ be compact metric spaces. We will use $d$ generically for a metric on $R$ and $U$ and more generally for any metric space considered, and will always assume that the labeling spaces $R$ and $U$ have $d$ diameter precisely 1.

A $\mathcal{T}, R$ name $h$ is any function from $\mathcal{T}$ to $R$. We say it is tree adapted if for any $v \in \eta$ and $i, j \in \{0, \ldots, p - 1\}$ with $i \neq j$ we have $h(v_i) \neq h(v_j)$ and we say it is strongly tree adapted if $d(h(v_i), h(v_j)) = 1$. A map $f : X \to R$ generates $\mathcal{T}, R$ names by $\mathcal{T}_n(v) = f(T_v(x))$. We say that $f$ is (strongly) tree adapted if $\mathcal{T}_n(v)$ is (strongly) tree adapted for almost every $x$. Let $\mathcal{G}$ be the $\sigma$-algebra on $X$ generated by pullback of the $\sigma$-algebra on $R$. We say $f$ generates if $\forall_v T_i^{-1}(\mathcal{G}) = \mathcal{F}$. “Strongly tree adapted” depends explicitly on the choice of metric but obviously, tree adapted, does not. For any $f : X \to R$, the map $f \vee K : X \to R \times \{0, \ldots, p - 1\}$ will always be tree strongly tree adapted. More generally so long as $d(h(v_i), h(v_j))$ is bounded uniformly away from zero by some $\alpha > 0$ we can replace $d$ by an equivalent metric of diameter 1 making $h$ strongly tree adapted. Just set

$$d_{\text{new}}(x, y) = \min(1, d(x, y)/\beta)$$

for some $0 < \beta \leq \alpha$.

Any measure-preserving endomorphism $(X, T, \mu)$ and function $f : X \to R$ generate a stationary sequence of random variables $R_i = f \circ T^i, i \geq 0$. We regard such a sequence as a measure on $R^{\mathbb{N}}$ with the weak* topology. There
is a unique extension of this measure to all of $\mathbb{R}^\mathbb{Z}$, preserving stationarity. For a.e. $x \in X$ let $\{R_i(x)\}_{i<0}$ be random variables with distribution $\operatorname{Dist}(R_i, i < 0| R_i = f(T^n(x)), i \geq 0)$.

Any tree adapted $T, R$-name $h$, generates an $R$-valued sequence of random variables $\{R_i(h)\}_{i<0}$. To a cylinder set $r_{-j}, r_{-j+1}, \ldots, r_{-1}$ we assign the measure equal to $p^{-j}$ times the number of all nodes at index $j$ whose name to index $1$ is the word $r_{-j}, \ldots, r_{-1}$. For $(X, T, \mu)$ a uniform $p$-to-one endomorphism and $f : X \to R$ tree adapted and generating and $T_x$ the $T, R$-name of $x$, $\{R_i(x)\}_{i<0}$ and $\{R_i(T_x)\}_{i<0}$ are just two descriptions of the same sequence of random variables.

We now put a family of metrics on $T, R$-names (and on $T_n, R$-names). For two $T, R$-names $h$ and $h'$ we define

$$t_n(h, h') = \min_{A \in A_n} \frac{1}{n} \sum_{0 < |v| \leq n} p^{-|v|} d(h(v), h'(A(v))).$$

For the two names $h$ and $h'$ in this definition, if one follows each branch from the root and through the tree and writes down the name seen along that branch, one obtains $p^n$ different names. Giving each name a mass of $p^{-n}$ one obtains the two sequences of random variables $R_i(h)$ and $R_i(h'), -n \leq i < 0$. A matching of the trees via a tree automorphism gives a coupling of these two distributions. The weighting of nodes is such that the calculation of the $t$ distance is precisely the $d$ distance between the random variables one would calculate from this coupling. In this sense $t$ is at least as large as $d$, which would be the inf over all couplings, not just those that come from tree automorphisms.

**Definition 2.1.** Let $(X, T, \mu)$ be a uniform $p$-to-one endomorphism and $f$ a tree adapted map from $X \to R$. We say $(X, T, \mu)$ and $f$ are tree very weak Bernoulli (tree v.w.B.) if for any $\varepsilon > 0$ and all $N$ sufficiently large there is a set $G = G(\varepsilon, N)$ with $\mu(G) > 1 - \varepsilon$ such that for any $x, y \in G$,

$$t_N(T_x, T_y) < \varepsilon.$$  

It is fairly direct that tree v.w.B. endomorphisms are exact and hence always ergodic [3], [13]. Our next goal is to show that a large natural class of factors of tree v.w.B. endomorphisms is again tree v.w.B. First we define the class we are interested in.

**Definition 2.2.** We say a factor map $\phi$ from $(X, T, \mu)$ to $(Y, S, \nu)$ is tree adapted if for a.e. point $x$ the map $\phi$ restricted to $T^{-1}(x)$ is one-to-one into $S^{-1}(\phi(x))$.

It is not difficult to construct factor maps between endomorphisms that are not tree adapted. On the other hand a conjugacy is clearly tree adapted in both directions. Our constructions will never leave the class of tree adapted
factor maps. We now give a useful little technical lemma that tree adapted factors of uniform \( p \)-to-one endomorphisms are themselves uniform \( p \)-to-one endomorphisms.

**Lemma 2.3.** Suppose \((X, T, \mu)\) is a uniform \( p \)-to-one endomorphism and \((Y, S, \nu)\) is a tree adapted factor of \((X, T, \mu)\) by a map \(\phi\). Then \(\phi\) restricted to \(T^{-1}(x)\) is almost surely onto \(S^{-1}(\phi(x))\) and \((Y, S, \nu)\) is also a uniform \( p \)-to-one endomorphism.

**Proof.** Because the map \(\phi\) is tree adapted the endomorphism \(Y\) is a.e. at least \( p \)-to-one. For each \(x \in \phi^{-1}(y)\), \(x' \in T^{-1}(x)\) and \(\phi(x') = y'\) the conditional probability of \(x'\) given \(x\) is the same as the conditional probability of \(y'\) given \(x\) and \(y\) (i.e. \(1/p\)). Thus the entropy is at least \(\log p\). As the entropy of \(X\) is \(\log p\) the entropy of \(Y\) can be at most \(\log p\). Thus the conditional probability of \(y'\) given only \(y\) must be \(1/p\) and \((Y, S, \nu)\) is a uniform \( p \)-to-one endomorphism. \(\square\)

**Corollary 2.4.** For \((X, T, \mu)\) a uniform \( p \)-to-one endomorphism, \(R\) compact metric and \(f : X \to R\) tree adapted, the left shift on the sequence of random variables \(R_i = f \circ T^i\) is a uniform \( p \)-to-one endomorphism.

**Lemma 2.5.** Suppose \((X, T, \mu)\) is a uniform \( p \)-to-one endomorphism, \(f\) is tree adapted and generating, \((X, T, \mu)\) and \(f\) are tree v.w.B. and \((Y, S, \nu)\) is a factor by a tree adapted factor map \(\phi\). Then for any tree adapted \(g : Y \to R\), \((Y, S, \nu)\) and \(g\) are also tree v.w.B.

**Proof.** By Lemma 2.3, we know \((Y, S, \nu)\) is a uniform \( p \)-to-one endomorphism. Let \(G\) be the \(\sigma\)-algebra of \(f\) measurable sets; hence \(g \circ \phi\) is \(\bigvee_{i=0}^{\infty} T^{-i}(G)\) measurable. Writing this as a finite approximation, for each \(\varepsilon > 0\) we can find an \(s \in \mathbb{N}\), a \(\delta > 0\) and a subset \(G' \subseteq X\) with \(\mu(G) > 1 - \varepsilon\) so that if \(x, x' \in G'\) and

\[
d(f(T^i(x)), f(T^i(x')) < \delta \text{ for } 0 \leq i < s \text{ then }
d(g(\phi(x)), g(\phi(x'))) < \varepsilon.
\]

As \(T\) is ergodic the mean ergodic theorem tells us that

\[
\frac{1}{n} \sum_{0 < |v| \leq n} p^{-|v|} \chi_{G'}(T_v(x)) \overset{\text{a.s.}}{\to} \mu(G')
\]

in \(L^1\). (Convergence here can be shown to be pointwise but as this is not standard and we do not need it we just quote the mean convergence which follows directly from rewriting this average as an average over the forward images of \(T^{-n}(x)\).)

Suppose \(x, x'\) and \(N\) satisfy

\[
\ell_N(T_x, T_{x'}) < \frac{\delta^2}{2s},
\]
\[ \frac{1}{N} \sum_{0 < |v| \leq N} p^{-|v|} \chi_{G'}(T_v(x)) > 1 - 2\varepsilon \]

and

\[ \frac{1}{N} \sum_{0 < |v| \leq N} p^{-|v|} \chi_{G'}(T_v(x')) > 1 - 2\varepsilon. \]

Then there is an \( A \in A_n \) given by

\[ \frac{1}{N} \sum_{0 < |v| \leq N} p^{-|v|} d(f(T_v(x)), f(T_A(v)(x'))) < \frac{\delta^2}{2s}. \]

Let \( Z = \{ v | 0 < |v| \leq N, d(f(T_i \circ T_v(x)), f(T_i \circ T_A(v)(x'))) < \delta \} \) and conclude

\[ \frac{1}{N} \sum_{v \in Z} p^{-|v|} > 1 - \delta. \]

If \( v \in Z \) and both \( T_v(x) \) and \( T_A(v)(x') \) are in \( G' \) then

\[ d(g \circ \phi(T_v(x)), g \circ \phi(T_A(v)(x'))) < \varepsilon \]

and hence

\[ \frac{1}{N} \sum_{0 < |v| \leq N} p^{-|v|} d(g(S_v(\phi(x))), g(S_A(v)(\phi(x')))) < 5\varepsilon + \delta. \]

That \((Y, S, \nu)\) is tree v.w.B. now follows. \( \Box \)

3. One-sided couplings

As we are following the Burton-Rothstein approach to the isomorphism theorem we will be considering joinings of two endomorphisms \((X, T, \mu)\) and \((Y, S, \nu)\). A **coupling** of two spaces \((X, \mu)\) and \((Y, \nu)\) is a measure on \(X \times Y\) which has marginals \(\mu\) and \(\nu\). A **joining** of \((X, T, \mu)\) and \((Y, S, \mu)\) is a coupling of \((X, \mu)\) and \((Y, \nu)\) which is invariant under \(T \times S\). We will not consider all of the joinings of \((X, \mu)\) and \((Y, \nu)\) but rather a collection we call one-sided joinings. In this section we define one-sided joinings and prove some facts about them.

**Definition 3.1.** Suppose \( S \) is a subset of \( \mathbb{Z} \) and \( \{R_i\}_S \) and \( \{U_i\}_S \) are two sequences of random variables. A **one-sided coupling** is a coupling of these two sequences such that for all \( n \) and \( i > n \)

\[ \text{Dist}(\{R_j\}_{j < i} | \{R_j\}_{j \geq i}, \{U_j\}_{j < i} | \{U_j\}_{j \geq i}) = \text{Dist}(\{R_j\}_{j < i} | \{R_j\}_{j \geq i}), \]

and symmetrically

\[ \text{Dist}(\{U_j\}_{j < i} | \{R_j\}_{j \geq i}, \{U_j\}_{j < i} | \{U_j\}_{j \geq i}) = \text{Dist}(\{U_j\}_{j < i} | \{U_j\}_{j \geq i}). \]

If the sequences are stationary then the one-sided couplings that also are stationary are called **one-sided joinings**.
If along with the endomorphism \((X, T, \mu)\) there is a generating function \(f : X \to R\) we have defined \(R_i(x) = f(T^i(x))\) for any \(x\) and \(i \geq 0\). We extend this by stationarity to negative \(i\) giving a stationary sequence \(R_i, i \in \mathbb{Z}\). (Note that the map from stationary measures on \(R^\mathbb{N}\) to those on \(R^{-\mathbb{N}}\) obtained by extension to \(\mathbb{Z}\) and then restriction is a weak* homeomorphism.) If we also have an endomorphism \((Y, S, \nu)\) and a generating function \(g : Y \to R\) we define \(U_i(y) = g(S^i(y))\) and extend to indices \(i < 0\) similarly.

**Definition 3.2.** A coupling (or a joining) of \((X, T, \mu)\) and \((Y, S, \nu)\) is one-sided if there are generating functions \(f\) and \(g\) so that the coupling of \(R_i\) and \(U_i, i \geq 0\), is one-sided.

Notice that if a coupling generates a one-sided coupling for one choice of generating functions \(f\) and \(g\) it will do so for all choices.

**Lemma 3.3.** For sequences of random variables \(R_j\) and \(U_j\) and each value \(i\), those couplings for which

\[
\text{Dist}\left(\{R_j\}_{j<i} \, | \, \{R_j\}_{j \geq i}, \{U_j\}_{j \geq i}\right) = \text{Dist}\left(\{R_j\}_{j<i} \, | \, \{R_j\}_{j \geq i}\right)
\]

are a weak* closed subset of all couplings.

**Proof.** We begin with some basic reductions of the problem. Notice that this statement is simply about couplings of three measure algebras \(X_1 = \vee_{j<i} R_j, X_2 = \vee_{j \geq i} R_j\) and \(Y = \vee_{j \geq i} U_i\) where the first two are coupled by a fixed measure \(\mu\). We consider those couplings of the three where \(X_1\) and \(Y\) are coupled conditionally, independently over \(X_2\). Viewed this way we see that without loss of generality we can assume all the random variables \(R_i\) and \(U_i\) (to be explicit) are two-valued. Next notice that the condition reduces to the countable list of conditions

\[
\text{Dist}\left(\{R_j\}_{1 \leq j < i} \, | \, \{R_j\}_{j \geq i}, \{U_j\}_{j \geq i}\right) = \text{Dist}\left(\{R_j\}_{1 \leq j < i} \, | \, \{R_j\}_{j \geq i}\right).
\]

To show that each such condition is weak* closed is equivalent to proving closedness when \(X_1\) is a finite space, i.e. we have reduced to the case of showing closedness of \(\text{Dist}(P | X_2, Y) = \text{Dist}(P | X_2)\) where \(P\) is a finite partition. Now suppose \(\nu_i\) all satisfy this condition and converge weak* to \(\nu\). Upper semi-continuity of entropy tells us that \(\limsup \, H_{\nu_i}(P | X_2 \vee Y) \leq H_\nu(P | X_2, Y)\). On the other hand

\[
H_{\nu_i}(P | X_2 \vee Y) = H_{\mu}(P | X_2) = H_\nu(P | X_2) \geq H_\nu(P | X_2 \vee Y)
\]

and \(H_\nu(P | X_2 \vee Y) = H_\mu(P | X_2)\) and \(\mu\) must also satisfy the condition. \qed
Corollary 3.4. Suppose $\mu_1$ couples $U_i$ and $R_i$ one-sidedly and $\mu_2$ couples $V_i$ and $R_i$ one-sidedly. There is then a coupling $\hat{\mu}$ of all three sequences that projects to $\mu_i$ on the appropriate pair of sequences and when restricted to $U_i$ and $V_i$ is one-sided.

Proof. Suppose $\pi$ couples the three sequences $R_i$, $U_i$ and $V_i$ for $i \geq I$ with the given marginals in some way, not necessarily one-sidedly. Extend $\pi$ to index $I - 1$ by first coupling on $R_{I-1}$ relatively independently over the algebra $\bigvee_{i \geq I} R_i$. By the one-sidedness of the marginals, this maintains the marginals. Now couple on $U_{I-1}$ to this relatively independently over the algebra $\bigvee_{i \geq I-1} R_i \lor \bigvee_{i \geq I} U_i$. Couple on $V_{I-1}$ symmetrically. This preserves the marginals and

$$\text{Dist}(U_{I-1} | \{U_i\}_{i \geq I}, \{V_i\}_{i \geq I}) = \text{Dist}(U_{I-1} | \{U_i\}_{i \geq I}),$$

and symmetrically,

$$\text{Dist}(V_{I-1} | \{U_i\}_{i \geq I}, \{V_i\}_{i \geq I}) = \text{Dist}(V_{I-1} | \{V_i\}_{i \geq I});$$

i.e., for this one step the coupling is one-sided.

We continue inductively to add on variables as $I \to -\infty$ and obtain a coupling which is one-sided at all indices $i \leq I$. For each value $I \geq 0$ start with $\pi$, the relatively independent coupling of the $\mu_i$ and $\mu_2$ over the common algebra $\bigvee_{i \geq I} R_i$. Extend to the right as described above to produce a measure $\pi_I$. Let $\hat{\mu}$ be any weak* limit of the couplings $\pi_I$. Lemma 3.3 guarantees that $\hat{\mu}$ is one-sided.

Definition 3.5. Suppose $(X, T, \mu)$ and $(Y, S, \nu)$ are two endomorphisms. Then define $J^+((X, T, \mu), (Y, S, \nu))$ to be the space of all one-sided joinings of these endomorphisms. Define $J^+_e((X, T, \mu), (Y, S, \nu))$ to be the ergodic and one-sided joinings. If there is no confusion we will just write $J^+$ and $J^+_e$.

Notice that Lemma 3.3 has shown the one-sided joinings to be a closed subset of all joinings. It is not difficult to see that if $T$ and $S$ are assumed ergodic, then the one-sided joinings are convex and that the extreme points of this set are the ergodic and one-sided joinings. (One just notes that the ergodic components of a one-sided joining must themselves be one-sided as the ergodic decomposition is past measurable.) Furthermore we see that if $\hat{\mu}_1 \in J^+$ of $T$ and $\hat{\mu}_2 \in J^+$ of $T$ and $U$ then there is a stationary joining of all three which projects to these two on the appropriate pairs and is in $J^+$ of $S$ and $U$ on this pair. Just observe that Corollary 3.4 gives us a one-sided coupling and by averaging over translates and taking a weak* limit we get a joining. By extending this further, we see that if the $\hat{\mu}_i$ were in $J^+_e$, i.e. were ergodic, then the one-sided joining of $S$ and $U$ can also be chosen ergodic. Almost any ergodic component of the one-sided joining just constructed will do.
We consider two different weak* pseudometrics on processes of the form \((X, T, \mu), f\). First an endomorphism \((X, T, \mu)\) and a function \(f : X \to R\) define a measure on \(R^\mathbb{N}\). Let \(\text{dist}\) be a metric on \(C^*(R^\mathbb{N})\) which generates the weak* topology. When we refer to

\[
\text{dist}(((X, T, \mu), f), ((Y, S, \nu), g)) = \text{dist}(f, g)
\]

we mean the distance between the measures that \((X, T, \mu), f\) and \((Y, S, \nu), g\) generate on \(R^\mathbb{N}\). We mention a particular case of this \(\text{dist}\) (pseudo)metric topology to be used repeatedly. If \(\hat{\mu}\) is a joining of \((X, T, \mu)\) and \((Y, S, \nu)\) and \(f : X \to R\) and \(g : Y \to U\) then \(\hat{\mu}\) projects to a stationary measure on \((R \times U)^\mathbb{N}\). In this case we use the notation

\[
\text{dist}(\hat{\mu} \lor f, \hat{\nu} \lor g).
\]

A uniform \(p\)-to-one endomorphism \((X, T, \mu)\) and a function \(f : X \to R\) define a measure on \(R^T\) as we have associated with each point \(x \in X\) the \(T, R\) name \(T_x\). Let \(\text{tdist}\) be a metric for the weak* topology on Borel measures on \(R^T\). When we refer to

\[
\text{tdist}(((X, T, \mu), f), ((Y, S, \nu), g)) = \text{tdist}(f, g)
\]

we mean the \(\text{tdist}\) distance between the measures these processes generate on \(R^T\).

**Lemma 3.6.** The (pseudo)topologies generated by \(\text{dist}\) and \(\text{tdist}\) on uniform \(p\)-to-one endomorphisms and tree adapted functions to \(R\) are the same.

**Proof.** By Lemma 2.3 we can assume \(f\) generates. We have seen any tree adapted \(T, R\) name \(h\) generates a sequence of random variables \(R_i(h)\), i.e. a measure on \(R^{-\mathbb{N}}\). Thus any measure \(m\) on the tree adapted \(T, R\) names projects to a measure on \(R^{-\mathbb{N}}\). If this measure comes from a uniform \(p\)-to-one endomorphism then it is stationary and maps homeomorphically to a measure on \(R^\mathbb{N}\). Call this measure \(\Psi(m)\). Obviously, \(\Psi\) is a weak* continuous map which shows that \(\text{tdist}\) is at least as strong as \(\text{dist}\).

For a uniform \(p\)-to-one endomorphism \((X, T, \mu)\) and a.e. \(x\), the map \(x \to T_x\) lifted to measures is an inverse for \(\Psi\).

To see that \(\Psi^{-1}\) is continuous, suppose \(\{R_i\}\) and \(\{R_j^2\}\), \(j = 1, 2, \ldots\), are sequences of random variables (measures on \(R^\mathbb{N}\)) arising from tree-adapted functions on uniformly \(p\)-to-one endomorphisms with \(\{R_i^2\} \to \{R_i\}\) in \(\text{dist}\). This is equivalent to putting all these random variables on a common measure space \((\Omega, \hat{\mu})\) (which might as well be \((R^\mathbb{N})^\mathbb{Z} \times R^\mathbb{Z}\)) with each \(R_i^2 \to R_i\) in \(\hat{\mu}\) probability.
We write $\tilde{R} = \{R_i\}_{i \in \mathbb{Z}}$, $\tilde{R}^j = \{R^j_i\}_{i \in \mathbb{Z}}$ and $\tilde{R}^+ = \{R_i\}_{i \in \mathbb{N}}$ etc. Let $\mathcal{T}_{\tilde{R}^+}$ be the uniform $p$-adic tree of inverse images of $\tilde{R}^+$ and $\mathcal{T}_{\tilde{R}^+_j}$ that of $\tilde{R}^+_j$. All these labeled trees have the property that the labels of the $p$ predecessors of any node are precisely 1 apart in $R$. The collection of such $R$ labelings of the tree are a closed subset $C$ of $R^T$.

Let $C_n \subseteq R^{T^n}$ consist of the labelings in $C$ restricted to $T_n$. On $R^{T^n}$ use the sup metric up to tree automorphisms and on $R^n$ the sup metric.

A labeling $\eta \in C_n$ gives rise to $p^n$ distinct names in $R^n$ – the names along the $p^n$ branches. Call this set of names $N(\eta)$. The critical observation here is this: Suppose for two labelings $\eta_1$ and $\eta_2 \in C_n$, each element of $N(\eta_1)$ is within $\varepsilon < 1/2$ of some element in $N(\eta_2)$. Then the labelings $\eta_1$ and $\eta_2$ themselves must be within $\varepsilon$. Just notice that the labels along distinct branches of $\eta_1$ cannot be matched within $\varepsilon$ of the same branch of $\eta_2$ and moreover the matching must preserve the tree structure.

Both $R^n$ and $R^{T^n}$ are compact and so for each $\varepsilon > 0$ there is a closed subset $G = G(\varepsilon)$ with $\tilde{\mu}(G) > 1 - \varepsilon^2/p^n$ so that the map $\tilde{R}^+ \to \mathcal{T}_{\tilde{R}^+,n}$ is uniformly continuous on $G$. Hence there are an $N$ and $\delta$ so that if $\tilde{R}^+$ and $\tilde{R}^+_j \in G$ and $d(R_i, R_j^i) < \delta$ for $i \leq N$ then $d(\mathcal{T}_{\tilde{R}^+,n}, \mathcal{T}_{\tilde{R}^+_j,n}) < \varepsilon/2$.

As $\tilde{R}^j \to \tilde{R}$ in probability, for all $j$ large enough there will be a subset $H = H(\varepsilon, j)$ of values $\tilde{R}^j$ with $\mu(H) > 1 - 2\varepsilon/p^n$ and for each $\tilde{R}^j \in H$ there is a representative value $\tilde{R}(\tilde{R}^j)$ with $\tilde{R}^+_j(\tilde{R}^j) \in G$ so that

i) for $-n \leq i \leq N$ we have $d(R^j_i, R_i(\tilde{R}^j)) < \varepsilon/2$ and

ii) $E(\tilde{R}^j \in G)$ and $d(R_i, R_i(\tilde{R}^j))) < \delta, 1 \leq i \leq n|\tilde{R}^j| > 1 - \varepsilon$.

Let $H' \subseteq H$ consist of those $\tilde{R}^j$ for which all $p^n$ extensions of $\tilde{R}^j_i$ in $N(\mathcal{T}_{\tilde{R}^+_j}) \circ \tilde{R}^j_i$ intersect $H$ nontrivially, i.e. have good representatives. Now $\mu(H') > 1 - 2\varepsilon$ and for any $\tilde{R}^j \in H'$ all names in $N(\mathcal{T}_{\tilde{R}^+,n})$ must be within $\varepsilon$ of some branch of $\mathcal{T}_{\tilde{R}^+,n}$ and by our observation above

$$d(\mathcal{T}_{\tilde{R}^+,n}, \mathcal{T}_{\tilde{R}^+(\tilde{R}^j),n}) < \varepsilon.$$ 

Now, by ii), for $\tilde{R}^j \in H'$ we have

$$E(d(\mathcal{T}_{\tilde{R}^+,n}, \mathcal{T}_{\tilde{R}^+(\tilde{R}^j)}) < 2\varepsilon|\tilde{R}^j| > 1 - \varepsilon$$

and so

$$E(d(\mathcal{T}_{\tilde{R}^+,n}, \mathcal{T}_{\tilde{R}^+,n}) < 2\varepsilon) > 1 - 3\varepsilon.$$ 

We conclude that $\tilde{R}^j \to \tilde{R}$ in $\text{tdist}$. \(\square\)
This next lemma is important because it says that all of the joinings created in the next section are one-sided.

**Lemma 3.7.** Suppose \((X, T, \mu)\) is a uniform \(p\)-to-one endomorphism and the factor map \(\phi\) to \((Y, S, \nu)\) is tree adapted. Then the joining \(\hat{\mu}\) of \((X, T, \mu)\) and \((Y, S, \nu)\) generated by \(\phi\) is one-sided.

**Proof.** Fix \(x\) and \(y = \phi(x)\). By stationarity it is sufficient to show that the conditional probability of preimages of \(y\) is the same as the conditional probability of preimages of \(y\) given \(x\) and \(y\). As \(\phi\) is a bijection from the inverse images of \(x\) to those of \(y\) the conditional mass of any \(T^i(x)\) given \(x\) must be precisely that of \(S^i(y)\) given \(y\). This value is \(p^{-|v|}\) and hence the conditional expectation of each \(S^i(y)\) given \(x\) and \(y\) is the same as its expectation given \(y\). The other set of equalities is obvious as \(x\) determines \(y\) so that conditioning on \(x\) and \(y\) is the same as conditioning on \(x\).

**Definition 3.8.** Suppose \((X, T, \mu)\) and \((Y, S, \nu)\) are two uniform \(p\)-to-one endomorphisms and \(f\) and \(g\) are functions into the same metric space \((R, d)\) and \(n > 0\). Now,

\[
\hat{t}_n(f, g) = \ell_n(\phi(X, T, \mu), \phi(Y, S, \nu), g)) = \inf \left( \frac{1}{n} \sum_{i=1}^{n} \int d(f(T^i(x)), g(S^i(y))) \, d\hat{\mu} \right)
\]

and

\[
\bar{t}_n(f, g) = \bar{\ell}_n((X, T, \mu), (Y, S, \nu), g)) = \inf \left( \int \bar{\ell}_n(T_x, T_y) \, d\hat{\mu} \right),
\]

where the \(\inf\)'s are taken over all one-sided couplings \(\hat{\mu}\). For comparison’s sake we include the definition

\[
\bar{d}_n(f, g) = \bar{d}_n((X, T, \mu), (Y, S, \nu), g)) = \inf \left( \frac{1}{n} \sum_{i=1}^{n} \int d(f(T^i(x)), g(S^i(y))) \, d\mu \right),
\]

where the \(\inf\) is taken over all couplings \(\hat{\mu}\).

On the face of it, \(\bar{t}_n\) is a metric on random variables indexed on \(-n \leq i < 0\) and \(\hat{t}_n\) on random variables indexed on \(0 \leq i < n\). By stationarity of \(R\) these can be translated to be the same sets of random variables. The proof of the following lemma will be given later in the section.

**Lemma 3.9.** Suppose \((X, T, \mu)\) and \((Y, S, \nu)\), are uniform \(p\)-to-one endomorphisms with tree adapted functions \(f\) and \(g\) to \(R\). Then \(\bar{t}_n(f, g) = \ell_n(f, g)\).
We will not use the notation \( \hat{t}_n \) again except in the proof of Lemma 3.9, using just \( T_n \) for both notions.

**Definition 3.10.** Suppose \((X, T, \mu)\) and \((Y, S, \nu)\), are uniform \( p \)-to-one endomorphisms and \( f \) and \( g \) are functions with values in \( R \). We set

\[
\bar{t}(f, g) = \liminf_{n \to \infty} T_n(f, g).
\]

**Lemma 3.11.** Suppose \((X, T, \mu)\) and \((Y, S, \nu)\) are two uniform \( p \)-to-one endomorphisms and \( f \) and \( g \) are functions to \( R \). There is then a stationary, ergodic and one-sided joining, \( \mu \), with

\[
\bar{t}(f, g) = \int d(f(x), g(y)) \, d\mu.
\]

In particular the lim inf in the definition is a limit.

**Proof.** The simple weak* compactness argument completely analogous to that for \( \bar{t} \) works here as the set of one-sided couplings is closed and convex, and the extreme points of the stationary and one-sided couplings are the ergodic ones.

**Definition 3.12.** Suppose \((X, T, \mu)\) and \((Y, S, \nu)\) are two endomorphisms, \( f : X \to R \) is tree-adapted and \( \hat{\mu} \in J^+ \) is an ergodic one-sided joining of them. We say \( f \in \hat{\mu} \) if there is a one-sided and tree-adapted function \( \bar{f} : Y \to R \) with

\[
\int d(f(x), \bar{f}(y)) \, d\hat{\mu}(x, y) < \varepsilon.
\]

Now, \( f \in \hat{\mu} \) if \( f \in \hat{\mu} \) for all \( \varepsilon > 0 \).

Notice that if \( f \in \hat{\mu} \) one immediately obtains that for \( \bar{f} \) of the definition

\[
\bar{t}(((X, T, \mu), f), ((Y, S, \nu), \bar{f})) < \varepsilon.
\]

Also notice that if \( f \) generates and \( f \in \hat{\mu} \) then relative to \( \mu \) the endomorphism \((X, T, \mu)\) sits as a one-sided and tree-adapted factor of \((Y, S, \nu)\).

Now we show that a one-sided coupling lifts naturally to a measure on \( X \times Y \times \mathcal{A} \). This is the essential ingredient for showing that \( \hat{t} \) and \( \bar{t} \) are equal and is also necessary in the copying lemma. This lift is not unique. (In the form we now describe, the direct product of two uniform \( p \)-to-one endomorphisms has many potential lifts to a third automorphism coordinate.)
construct a joined name $\hat{h}_A(v) = (h(A^{-1}(v)), h'(v))$. Such a name will project to a measure on $R^{-N} \times U^{-N}$ that is a one-sided coupling of $R_i(h)$ and $U_i(h')$. Call it $\hat{\mu}_{(h,h',A)}$.

**Lemma 3.13.** Suppose $(X, T, \mu)$ and $(Y, S, \nu)$ are two uniform $p$-to-one endomorphisms, $x \in X$, $y \in Y$ are two points, and $f : X \to R$ and $g : Y \to U$. The one-sided couplings of the form $\hat{\mu}_{(T_x, T_y, A)}$, $A \in \mathcal{A}$, are the extreme points of the one-sided couplings of $R_i(T_x)$ and $U_i(T_y)$ and any one-sided coupling $\hat{\mu}$ of $R_i$ and $U_i$, $i < 0$, is of the form

$$\int \hat{\mu}_{(T_x, T_y, A)} dm(x,y)(A) d\hat{\mu}(x,y)$$

for some family of probability measures $m(x,y)$ on $A$.

**Proof.** We only need to show that any one-sided coupling $\overline{\mu}$ of variables of the form $R_i(h)$ and $U_i(h')$ is of the form

$$\overline{\mu} = \int \hat{\mu}_{(h,h',A)} dm(A)$$

for some measure $m$ on $A$. The proof is by induction. We first show this for a single variable. This is equivalent to showing that any self-coupling of uniform measure on $\{0, \ldots, p - 1\}$ is an average of measures supported on graphs of permutations. To see this suppose $\hat{\mu}_0$ is such a self-coupling of $\{0, \ldots, p - 1\}$. The knowing relation on $\{0, \ldots, p - 1\} \times \{0, \ldots, p - 1\}$ given by $\hat{\mu}_0(i,j) > 0$ satisfies the hypotheses of the Hall marriage lemma and hence there is a bijective subrelation; i.e., $\hat{\mu}_0 = \alpha \hat{\mu}_\pi + (1 - \alpha) \hat{\mu}_1$ with $\alpha > 0$ and $\mu_\pi$ supported on the graph of the permutation $\pi$. Repeating this for $\mu_1$ and so on, we obtain a representation of the measure as an integral of measures supported on graphs of permutations. Using the one-sidedness of $\overline{\mu}$ we complete the result inductively as we conclude that $\hat{\mu}$ is written uniquely as an integral of couplings which node by node sit on the graphs of permutations applied at each node, i.e. sit on graphs of tree automorphisms. \hfill \Box

We are ready to show that $\hat{t}$ and $\overline{t}$ agree.

**Proof of Lemma 3.9.** We begin once more noting that by translating the random variables $R_i$ and $U_i$ by $n$, both $\hat{t}_n$ and $\overline{t}_n$ are calculated as infima over couplings of variables indexed on $-n \leq i < 0$. We have already noted that a pairing of the nodes of $T_n$ by a tree automorphism $A$ when viewed on the names along the branches of the tree gives a one-sided coupling $\hat{\mu}_{(h,h',A)}$ of the distributions of names. This is enough to conclude that $\hat{t}_n \leq \overline{t}_n$. As a one-sided coupling of $R_i(T_x)$ and $U_i(T_y)$ can be written as an integral over $X \times Y \times A$ of couplings $\hat{\mu}_{(T_x, T_y, A)}$ we see the other inequality $\overline{t}_n \leq \hat{t}_n$. \hfill \Box
4. Tree finitely determined

Now that we have discussed the theory of one-sided couplings we are ready to define tree finitely determined. This will play a major role in the proof of the copying lemma.

Definition 4.1. We say \( ((X, T, \mu), f) \), where \( (X, T, \mu) \) is a uniform \( p \)-to-one endomorphism and \( f : X \to R \) is tree adapted, is tree finitely determined (tree f.d.) if for every \( \varepsilon > 0 \), there is a \( \delta \) such that for any endomorphism \( (Y, S, \mu) \) with function \( g : Y \to R \) with \( \text{tdist}(f, g) < \delta \) then \( \text{t}(f, g) < \varepsilon \).

If \( f \) is strongly tree adapted then \( \text{tdist} \) here can be replaced with \( \text{dist} \). This will enable us to work with strongly tree adapted functions. Now we need the following lemma. Later, as we will see, tree f.d. and tree v.w.b. are equivalent. We will see that all tree-adapted factors of a tree f.d. map are tree f.d. At this point we need something substantially less.

Lemma 4.2. Suppose \( ((X, T, \mu), f) \) is tree f.d. where \( f \) is a generator. Then for any bounded map \( h : X \to R \), \( ((X, T, \mu), f \lor h) \) is tree f.d.

Proof. This argument follows well established lines by approximation of \( h \) by a “finite code”. Each successive step simply requires a closer match in \( \text{tdist} \). To begin, as \( f \) is a generator, \( h \) can be approximated arbitrarily well in \( L^1(\mu) \) by maps of the form

\[
H(f(x), f(T(x)), \ldots, f(T^N(x)))
\]

where \( H \) is a continuous map from \( R^{N+1} \to R \) and \( N \) is finite. If \( ((Y, S, \nu), g \lor h') \) is sufficiently close in \( \text{tdist} \) to \( ((X, T, \mu), f \lor h) \) then \( H(g(y), g(S(y)), \ldots, g(S^N(y))) \) will of necessity also be a good \( L^1(\nu) \) approximation of \( h' \).

Now if \( ((Y, S, \nu), g \lor h') \) is close in \( \text{tdist} \) to \( ((X, T, \mu), f \lor h) \) then \( ((Y, S, \nu), g) \) is close in \( \text{tdist} \) to \( ((X, T, \mu), f) \) but is not necessarily tree-adapted. As \( f \) is tree-adapted though \( g \) must separate inverse images of most points, so some perturbation \( g' \) of \( g \) which agrees with \( g \) on most of \( Y \) will be tree-adapted. If we are close enough in \( \text{tdist} \) then we will still have \( H(g'(y), g'(S(y)), \ldots, g'(S^N(y))) \) a good approximation to \( h' \) in \( L^1(\nu) \) and \( ((Y, S, \nu), g') \) close in \( \text{tdist} \) to \( ((X, T, \mu), f) \), hence close in \( \text{t} \). Now if \( ((Y, S, \nu), g') \) and \( ((X, T, \nu), f) \) are close enough in \( \text{t} \) (notice that how close can be set after the value \( N \) and continuous map \( H \) are fixed) then \( ((Y, S, \nu), g' \lor H(g'(y), g' \circ S, \ldots, g' \circ S^N)) \) will be close in \( \text{tdist} \) to both \( ((Y, S, \nu), g \lor h') \) and to \( ((X, T, \mu), f \lor h) \) which is to say \( ((X, T, \mu), f \lor h) \) is tree f.d. \( \square \)

Corollary 4.3. Any tree f.d. process has a tree f.d. generator that is strongly tree adapted.
Proof. Choose \( h \) in Lemma 4.2 to be a map to \( \{1, 2, \ldots, p\} \) that separates inverse images. \( \square \)

**Lemma 4.4.** If \( ((X, T, \mu), f) \) is tree v.w.B. then it is tree f.d.

Proof. Suppose \( ((X, T, \mu), f) \) is tree v.w.B. Given \( \varepsilon > 0 \) choose an \( n \) so that there exists a subset \( X_0 \subseteq X \) of measure \( \geq 1 - \varepsilon \) and a fixed \( T_n, R \) name \( h_n \) so that \( \bar{t}_n(T_x, h_n) < \varepsilon \) for any \( x \in X_0 \). For each \( x \in X \) let \( A_x \) be an automorphism that realizes the minimum in the definition of \( \bar{t}_n(T_x, h_n) \). Using Lemma 3.6, choose a \( \delta > 0 \) so small that if \( (Y, S, \nu) \) is a uniform \( p \)-to-one endomorphism, \( g \) is a function to \( \mathbb{R} \), and \( \text{dist}(f, g) < \delta \) then there exists a subset \( Y_0 \subset Y \) of measure greater than or equal to \( 1 - 2\varepsilon \) such that \( \bar{t}_n(T_y, h_n) < 2\varepsilon \) for all \( y \in Y_0 \).

Consider a \( T, R \) name \( h \) constructed by tiling \( T \) with copies of \( h_n \). More precisely, for any \( v \) such that \( |v| = jn \) for some \( j \) and any \( v' \in \eta_n \) let \( h(vv') = h(v') \). For each \( x \in X \) we will inductively construct an automorphism \( A \) which will show \( T_x \) and \( h \) are close in \( \bar{t} \). The matching we use is a greedy algorithm matching \( n \) levels at a time. For each \( v \in \eta_n \) let \( A(v) = A_x(v) \). Now assume \( A \) has been defined on all \( v \in \eta_jn \). For each \( v \in \eta_jn \) let \( A(vv') = A(v)A_{T(A(v))(v')} \).

Now we calculate \( \bar{t}_{kn}(T_x, h) \). Let \( G_{kn}(x) \) be those nodes with \( |v| = jn \) for some \( 0 \leq j < k \) and \( T_n(x) \in X_0 \). Let \( M(G_{kn}(x)) \) be the sum of \( p^{-|v|} \) over all \( v \in G_{kn}(x) \). This construction leads to the calculation:

\[
\bar{t}_{kn}(T_x, h) \leq 1 - M(G_{kn}(x))/k + \varepsilon M(G_{kn}(x))/k.
\]

The fact that \( T \) is measure-preserving implies

\[
\int M(G_{kn}(x)) d\mu(x) = k\mu(X_0) \geq k(1 - \varepsilon).
\]

Hence for all but \( \sqrt{2\varepsilon} \) of the \( x \in X \),

\[
\bar{t}_{kn}(T_x, h) \leq \sqrt{2\varepsilon}.
\]

Precisely the same argument applied to \( Y \) yields that for all but \( 2\sqrt{\varepsilon} \) of the \( y \in Y \),

\[
\bar{t}_{kn}(T_y, h) \leq 2\sqrt{\varepsilon}.
\]

We conclude that

\[
\bar{T}(f, g) = \liminf_{n \to \infty} \bar{t}_n(f, g) \leq 4\sqrt{\varepsilon}
\]

which ends the proof. \( \square \)

**Corollary 4.5.** The standard endomorphism with the usual \( p \) set independent generating partition is tree v.w.B and hence tree f.d.
Proof. It is trivial that with this partition every point \( b \) has the same past tree name. Hence it is tree v.w.B. 

We postpone the converse of Lemma 4.4 as we will use the one-sided conjugacy theorem to prove it.

5. The copying lemma

In this section we prove the isomorphism theorem for uniform \( p \)-to-one endomorphisms. First we prove a Rokhlin lemma, then a strong Rokhlin lemma. Next, we prove the copying lemma. The isomorphism theorem will follow easily from the copying lemma. Notice that the strong Rokhlin lemma is proved only for finite valued \( f : X \to \mathbb{R} \); i.e., \( f \) is a partition. To emphasize this we refer to partitions \( P \) and \( Q \) instead of functions \( f \) and \( g \).

The first step is to prove a Rokhlin lemma for uniform endomorphisms. This result has appeared previously in the work of Rosenthal [11]. We present a proof here as his is perhaps too brief.

Definition 5.1. Let \((X, T, \mu)\) be a uniformly \( p \)-to-one endomorphism and \( T_v \) be some choice for the partial inverses of \( T \). A \( T^n \) Rokhlin tree is a collection of disjoint sets \( B_v \subseteq X \), \( v \in \eta_n \), with the property that \( T_v(B_\emptyset) = B_v \).

Theorem 5.2. Let \((X, T, \mu)\) be a uniformly \( p \)-to-one and ergodic endomorphism and \( T_v \) be some choice for the partial inverses of \( T \). For each \( n > 0 \) and \( \varepsilon > 0 \) there exists a \( T^n \) Rokhlin tree \( B_v \) with \( \mu(\bigcup v B_v) > 1 - \varepsilon \).

Proof. For any set \( C \) and \( n > 0 \) define

\[
B_\emptyset = \{ x \mid \min \{ i \geq 0 : T^i(x) \in C \} = 0 \mod (n + 1) \} \setminus (\bigcup_{0 \leq i \leq n} T^i(C)).
\]

Suppose \( x \in B_\emptyset \cap T_v(B_\emptyset) \) for some \( v \in \eta_n \). Then since \( \bigcup_{0 \leq i \leq n} T^{-i}(x) \cap C = \emptyset \) for any point \( x \in B_\emptyset \),

\[
\min \{ i \geq 0 : T^i(T_v(x)) \in C \} - \min \{ i \geq 0 : T^i(x) \in C \} = |v|.
\]

Both of the terms on the left-hand side cannot be equal to 0 mod \((n + 1)\) unless \(|v| = 0\). Thus \( B_\emptyset \cap T_v(B_\emptyset) = \emptyset \) for any \( 0 < |v| \leq n \) which implies \( B_v = T_v(B_\emptyset) \) forms a Rokhlin tree.

Since \( \mu(\{ x \mid (\min_{i \geq 0} T^i(x) \in C) = i \}) \) is nonincreasing we have

\[
\mu(B_\emptyset) > 1/(n + 1) - (n + 1)p^{n+1}\mu(C) \quad \text{and} \quad \mu(\bigcup v B_v) > 1 - (n + 1)^2 p^{n+1} \mu(C).
\]

This last term can be made as small as we like by choosing \( \mu(C) \) small. 

Now we prove a strong Rokhlin lemma. This says that the top level \( B_\emptyset \) can be chosen independently of any partition.
Lemma 5.3. Suppose \((X,T,\mu)\) is an ergodic uniformly \(p\)-to-one endomorphisms and \(P\) is a finite partition of \(X\). For any \(\epsilon > 0\) and \(n\) there is a \(T_n\)-Rokhlin tree \(C_v\) so that

\[ \mu(\bigcup_{v \in \eta_n} C_v) > 1 - \epsilon \]

and

\[ \text{Dist}(P) = \text{Dist}(P|C_{\emptyset}); \]

i.e., \(P\) and \(C_{\emptyset}\) are independent.

Proof. Given \(\epsilon\) choose \(m > 4(n+1)/\epsilon\). Let \(B_{\emptyset}\) be the top of a \(T_m\) Rokhlin tree with \(\bigcup_{v \in \eta_m} B_v > 1 - \epsilon/2\). For each

\[ A \in (\bigvee_{v \in \eta_m} T_v(P))|B_\emptyset \]

write \(A = \bigcup_0^\alpha A_i\) with each \(\mu(A_i) = \mu(A)/(n+1)\). Now let

\[ C_{\emptyset} = \bigcup_{A \in (\bigvee_{v \in \eta_m} T_v(P))|B_\emptyset} \bigcup_{j=0}^{m/(n+1)-2} \bigcup_{i} T^{-j(n+1)-i}(A_i). \]

Then \(C_{\emptyset}\) forms the top of a \(T_n\) Rokhlin tree with \(\bigcup_{v \in \eta_n} C_v > 1 - \epsilon\). We also have

\[ \text{Dist}(P | \bigcup_{i=0}^{m-n-1} T^{-i}(B_{\emptyset})) = \text{Dist}(P | C_{\emptyset}). \]

Thus for any element \(P_i\) of \(P\)

\[ \mu(P_i \cap C_{\emptyset}) > (1 - \epsilon)\mu(P_i)/(n+1). \]

Then we pare down \(C_{\emptyset}\) to \(C'_{\emptyset}\) so that for every element \(P_i\) of \(P\) we have precisely

\[ \mu(P_i \cap C'_{\emptyset}) = (1 - \epsilon)\mu(P_i)/(n+1) \]

and we are done. \(\square\)

Now we are ready to prove the copying lemma which is the main element in the proof of Theorem 5.5.

Lemma 5.4 (copying lemma). Suppose \((X,T,\mu)\) and \((Y,S,\nu)\) are uniform \(p\)-to-one endomorphisms with finite tree adapted functions \(f\) and \(g\) and \(\tilde{\mu} \in J^+_e\) an ergodic joining. For any \(\epsilon > 0\) there is a tree adapted function \(\tilde{f} : Y \to \mathbb{R}\) with

\[ \text{dist}(\tilde{f} \vee g, \tilde{f} \vee g) < \epsilon. \]

As a consequence, if \(R_1\) is tree f.d. then

\[ \mathcal{O}_{\tilde{e}'} = \{ \tilde{\mu} \in J^+_e | \tilde{f}' \overset{\tilde{\mu}}{\subset} Y \} \]

is an open and dense subset of \(J^+_e\).
Proof. First we show the result for finite partitions $P$ and $Q$ instead of functions $f$ and $g$. The definition of dist gives an $n$ and a $\delta$ such that if

$$\sum_{A \in \mathcal{V}^n_0(T \times S)^{-1}(P \vee Q)} |\nu(A) - \hat{\mu}(A)| < 1 - \delta,$$

then

$$\text{dist}(P, Q, \tilde{\mathcal{P}}) < \varepsilon.$$  

For any partition $P$ and any $m$ define a new partition $P^m$ so that $x$ and $x'$ are in the same element of $P^m$ if and only if $\tilde{t}_m(T_x, T_{x'}) = 0$. Choose $m > 4n/\delta$. Build $T_m$ Rokhlin trees $B_v \subset X$ and $C_v \subset Y$ so that $\mu(\cup_v B_v) = \nu(\cup_v C_v) > 1 - \delta/2$, $B_0$ is independent of $P^m$, and $C_0$ is independent of $Q^m$.

Next define a measure-preserving map $h : B_0 \to C_0$ so that the measure $h$ generates on $B_0 \times C_0$ restricted to $P^m \vee Q^m$ is the same as the measure $\hat{\mu}$ restricted to $P^m \vee Q^m$. This is possible because $B_0$ is independent of $P^m$, and $C_0$ is independent of $Q^m$.

As $\hat{\mu}$ is one-sided we can write it as

$$\hat{\mu} = \int \mu(T_x, T_y, A) \, dm(x, y) \, d\hat{\mu}(x, y).$$

Lift $\hat{\mu}$ to $X \times Y \times A$ as

$$\hat{\mu} = \int \delta_x \times \delta_y \times m(x, y) \, d\hat{\mu}(x, y).$$

Now $P^m \times Q^m \times A_m$ is a finite partition of $X \times Y \times A$. As $C_0$ is independent of $Q^m$ we can extend $\nu$ on $C_0$ to $\tilde{\nu}$ on $C_0 \times A$ to be identical in distribution on $Q^m \times A_m$ to $\tilde{\mu}$ (when normalized). We can now construct a measure-preserving map $\hat{h} : (B_0, \mu) \to (C_0 \times A, \tilde{\nu})$ so that the normalized measure supported on the graph of $\hat{h}$ restricted to $P^m \times Q^m \times A_m$ agrees in distribution with $\tilde{\mu}$.

Now we are ready to define the new partition $\tilde{\mathcal{P}}$. Write $\tilde{h}(x) = (h(x), A(x))$. For each $x \in B_0$ and every $v \in T_m$ set

$$\tilde{\mathcal{P}}(T_A(x)(v)h(x)) = P(T_v(x)).$$

On the rest of the space define $\tilde{\mathcal{P}}$ in any way such that $\tilde{\mathcal{P}}$ is tree-adapted.

Now for any set $A \in \mathcal{V}^n_0(T \times S)^{-1}(P \vee Q),$

$$\nu(A \cap (\cup_{|v| < m-n} C_v)) = \hat{\mu}(A \cap (\cup_{|v| < m-n} B_v)).$$

Since $\hat{\mu}(\cup_{|v| < m-n} B_v) = \nu(\cup_{|v| < m-n} C_v) > 1 - \delta$ this implies

$$\sum_{A \in \mathcal{V}^n_0(T \times S)^{-1}(P \vee Q)} |\nu(A) - \hat{\mu}(A)| < 1 - \delta.$$

Thus

$$\text{dist}(P \vee Q, \tilde{\mathcal{P}} \vee Q) < \varepsilon.$$  

This completes the proof of the first statement for finite partitions.
To extend the result to strongly tree-adapted functions do the following. Partition $R$ into sets of small diameter. Choose one representative point in each element of the partition. Define $F : R \to R$ to map all points in each partition element to its representative point. It follows that if the partition is fine enough then $\text{dist}(f, F \circ f)$ will be small. Do the same with $g$, i.e. construct a finite valued $G : U \to U$. Once more if the partitions are fine enough then we will have

$$\text{dist}(\hat{\mu} \vee g, F \circ f \hat{\mu} \vee G \circ g) < \frac{\varepsilon}{3}.$$ 

As $f$ and $g$ are strongly tree-adapted, if the partition elements are less than 1 in diameter $F \circ f$ and $G \circ g$ will also be strongly tree-adapted (by the discrete metric on their finite ranges). We can now apply the finite partition version proven above to $F \circ f$ and $G \circ g$ to construct $\tilde{f}$ with

$$\text{dist}(F \circ f \hat{\nu} \vee G \circ g, \tilde{f} \hat{\nu} \vee G \circ g) < \frac{\varepsilon}{3},$$

knowing $G$ comes from a fine enough partition, independent of the choice of $\tilde{f}$ that

$$\text{dist}(\tilde{f} \hat{\nu} \vee G \circ g, \tilde{f} \hat{\nu} \vee g) < \frac{\varepsilon}{3}.$$ 

For the second result notice that the previous equation implies

$$\text{dist}(f, \tilde{f}) < \varepsilon.$$ 

For any $\delta$ if $((X, T, \mu), f)$ is tree f.d. then $\varepsilon$ and $\tilde{f}$ can be chosen so that

$$\bar{t}(f, \tilde{f}) < \delta.$$ 

For any $\delta_1$ if $\delta$ and $\varepsilon'$ are small enough then by Lemma 3.4 we can extend $\nu$ to a one-sided and ergodic joining $\tilde{\mu}$ with

$$\text{dist}(\tilde{f} \hat{\nu} \vee g, f \hat{\mu} \vee g) < \delta_1 \text{ and } f \overset{\delta}{\subset} Y$$

where $O_{\varepsilon'}$ is dense. Openness follows easily from the definition. \qed

**Theorem 5.5.** A uniform $p$-to-one endomorphism $(X, T, \mu)$ is one-sidedly conjugate to the standard uniform $p$-to-one endomorphism $(B, \sigma, \nu)$ if and only if there exists a generating function $f$ so that $(X, T, \mu)$ and $f$ are tree v.w.B. (or equivalently tree f.d.).

**Proof.** Let $f$ be a strongly tree adapted function from $X$ to $R$ and $K$, the standard independent generating partition of $B$. We know $J_{\varepsilon}^+$ is a $G_\delta$ subset of $J^+$ in the weak* topology. Since $(X, T, \mu)$ and $f$ are tree v.w.B. they are also tree f.d. Thus the copying lemma tells us that the $O_{\varepsilon}$ are open and dense in $J_{\varepsilon}^+$. Intersecting over $\varepsilon = 1/n$, we see that the Baire category theorem shows that the set of $\hat{\mu}$ with $f \overset{0}{\subset} B$ is a residual subset of $J_{\varepsilon}^+$. Let $K$ be the standard
independent generating partition of $B$. As the standard example is also tree v.w.B. and tree f.d., the set of $\tilde{\mu}$ with $K^0_{\tilde{\mu}} X$ is a residual subset of $J^+_{\tilde{\mu}}$. Thus the set of $\tilde{\mu}$ with $K^0_{\tilde{\mu}} X$ and $f^0_{\tilde{\mu}} B$ is nonempty and $(X, T, \mu)$ and $(B, \sigma, \nu)$ are isomorphic.

Notice that we need only assume $f$ tree adapted here and not necessarily strongly, as we can extend $f$ to an $f \lor h$ which is still tree f.d. and now is strongly tree adapted and hence isomorphic to the standard example.

All that remains is to show tree f.d. and tree v.w.B. are equivalent. We already know that tree v.w.B. implies tree f.d. For the other implication, we have just seen that tree f.d. implies one-sidedly conjugate to the standard action which is tree v.w.B. We have also seen that tree v.w.B. descends to one-sided and tree adapted factors. Hence any uniform $p$-to-one endomorphism which is isomorphic to a tree v.w.B. endomorphism is tree v.w.B. Thus tree f.d. implies tree v.w.B.

6. Examples of tree v.w.B. skew products

We will show now that general classes of isometric extensions of standard endomorphisms are all one-sidedly Bernoulli. Among these will be the $[T, \text{Id}]$ endomorphisms, where $T$ is an irrational rotation. These were described in the first section.

Throughout this section we consider $p$ to be fixed. Remember that $(B, \sigma, \nu)$ is the standard uniform $p$-to-one endomorphism. We also fix $(Z, d)$ a compact metric space with $T$ its space of isometries. We assume $T$ acts transitively, i.e. $Z$ is a homogeneous space, and has Haar measure. We put on $T$ the uniform topology. Given a function $f : B \to T$ we construct the cocycle extension $T_f$ acting on $B \times Z$ by

$$T_f(b, z) = (\sigma(b), f(b)(z)).$$

The map $\sigma$ has a natural set of partial inverses $\sigma_v$. These extend to form a natural set of partial inverses $(T_f)_v$. Set $c_m = \sup_{|v|=m} (\text{diam}(f(\sigma_v(B))))$.

**Definition 6.1.** We say $f$ generates a summable cocycle if $\sum_m c_m < \infty$.

Notice that any $f$ depending on only finitely many coordinates $b_0, \ldots, b_m$ automatically generates a summable cocycle. Thus the $[T, \text{Id}]$ endomorphisms generate summable cocycles.

**Lemma 6.2.** Suppose $f : B \to T$ generates a summable cocycle. Then for any $\varepsilon > 0$ there exists $\delta$ such that if $d((b, z), (b', z')) < \delta$ then $d((T_f)_v(b, z), (T_f)_v(b', z')) < \varepsilon$ for all $v \in \eta$. 

Proof. Given $\varepsilon$ there exists $n_0$ so that $\sum_{n \geq n_0} c_n \leq \varepsilon/2$. Choose $\delta < \varepsilon/2$ such that $d(b, b') < \delta$ implies that $b_i = b_i'$ for all $0 \leq i < n_0$. By the definition of $c$ if $b_i = b'_i$ for all $0 \leq i < n_0$ then
\[
d((T_f)_v(b, z), (T_f)_v(b', z')) \leq d((b, z), (b', z')) + \sum_{n \geq n_0} c_n \leq \varepsilon. \tag*{$\square$}
\]

**Lemma 6.3.** Suppose $f : B \to I$ generates a summable cocycle and $T_f$ is weakly mixing. For any $\varepsilon > 0$ there is an $N > 0$ so that $(T_f)^{-N}(b, z)$ is $\varepsilon$ dense in $B \times Z$ for all $(b, z)$.

**Proof.** From [12] we know that if $T_f$ is weakly mixing then it must be v.w.B. and hence $K$. The fact that $T_f$ is a $K$-system implies that there exists an $N'$ such that for most $b$ and all $z$ we have that $(T_f)^{-N'}(b, z)$ is $\varepsilon/2$ dense in $B \times Z$. In particular this holds for one $b$. Lemma 6.2 implies that there exists $N''$ such that if $b_i = b_i'$ for all $0 \leq i < N''$ then $(T_f)^{-N'}(b', z)$ is $\varepsilon$ dense in $B \times Z$. For every $b''$ there exists $\hat{b} \in \sigma^{-N''}b''$ such that $\hat{b}_i = b_i$ for all $0 \leq i < N''$. Thus $(T_f)^{-N''}b''(b', z)$ is $\varepsilon$ dense in $B \times Z$. \tag*{$\square$}

**Theorem 6.4.** Suppose that $f$ generates a summable cocycle and that $T_f$ is weakly mixing. Then $T_f$ is tree v.w.B.

**Proof.** Given $\varepsilon$ we get a $\delta$ from Lemma 6.2 and an $N$ from Lemma 6.3 which implies that $(T_f)^{-N}(b, z)$ is $\delta$ dense in $B \times Z$ for all $(b, z) \in B \times Z$. For any $(b, z), (b', z') \in B \times Z$ we define the tree automorphism $A$ that pairs $T_{(b, z)}$ with $T_{(b', z')}$ inductively. We pair at least $N$ levels at a time. If $d((b, z), (b', z')) < \delta$ then by Lemma 6.2 we are done. Otherwise Lemma 6.3 implies that there exists a tree automorphism $A_N$ which pairs at least one preimage of $(b, z)$ with a preimage of $(b', z')$ so that the pair is within $\delta$.

Now suppose we have defined $A$ up to at least level $kN$. We will extend it to at least level $(k + 1)N$. If $A(vv')$ has been defined for all $v' \in \eta$ then we need to do nothing. If it has not and $d((T_f)_v(b, z), (T_f)_v(b', z')) < \delta$ then extend it by the identity automorphism. By this we mean, for all $v' \in \eta$, that $A(vv') = A(v)v'$. If neither of the above conditions is satisfied then we use Lemma 6.3 to tell us how to define $A(vv')$ for all $v' \in \eta_N$.

Choose $k$ so that $(1 - 1/p^N)\varepsilon^k < \varepsilon$. By the previous paragraph and Lemma 6.2 we know that for each $n > \varepsilon kN$ the fraction of preimages that is paired within $\varepsilon$ is at least $1 - \varepsilon$. Thus
\[
\tilde{t}_{kN}((b, z), (b', z')) < 3\varepsilon
\]
and $T_f$ is tree v.w.B. \tag*{$\square$}

As well as the $[T, \text{Id}]$ examples, our methods cover the following smooth endomorphisms. Replace $B$ by $x \to 2x$ on $\mathbb{R}/\mathbb{Z}$ and let $Z = \mathbb{R}/\mathbb{Z}$ as well. Set $f(x)$ to be any Hölder function to $\mathbb{R}/\mathbb{Z}$ that is not a coboundary (for example
$f(x) = \sin(2\pi x)$ giving a smooth action on the 2-torus. That $f$ is not a coboundary means $T_f$ is a weakly mixing action and that $f$ is Hölder implies that it generates a summable cocycle. Thus we conclude that such an action must be tree v.w.B. It would be interesting to know if there can be a smooth and uniformly $p$-adic action that is v.w.B. but not tree v.w.B. Our work here shows it will not be found among the isometric extensions of $x \to px \mod 1$ with summable cocycles.

University of Maryland, College Park, MD
E-mail address: djr@math.umd.edu

University of Washington, Seattle, WA
E-mail address: hoffman@math.washington.edu

References
[1] J. Ashley, B. Marcus, and S. Tuncel, The classification of one-sided Markov chains, *Ergod. Theory Dynam. Systems* 17 (1997), 269–295.
[2] A. del Junco, Bernoulli shifts of the same entropy are finitarily and unilaterally isomorphic, *Ergodic Theory Dynam. Systems* 10 (1990), 687–715.
[3] D. Heicklen, Bernoullis are standard when entropy is not an obstruction, *Israel J. Math.* 107 (1998), 141–155.
[4] Y. Katznelson, Ergodic automorphisms of $T^n$ are Bernoulli shifts, *Israel J. Math.* 10 (1971), 186–195.
[5] D. Ornstein, Bernoulli shifts with the same entropy are isomorphic, *Adv. in Math.* 4 (1970), 337–352.
[6] , Two Bernoulli shifts with infinite entropy are isomorphic, *Adv. in Math.* 5 (1970), 339–348.
[7] D. Ornstein, D. Rudolph, and B. Weiss, Equivalence of measure preserving transformations, *Mem. A. M. S.* 37 (1982), no. 262.
[8] D. Ornstein and B. Weiss, Finitely determined implies very weak Bernoulli, *Israel J. Math.* 17 (1974), 94–104.
[9] , Geodesic flow are Bernoullian, *Israel J. Math.* 14 (1973), 184–198.
[10] W. Parry, Automorphisms of the Bernoulli endomorphism and a class of skew-products, *Ergodic Theory Dynam. Systems* 16 (1996), 519–529.
[11] A. Rosenthal, Strictly ergodic models for noninvertible transformations, *Israel J. Math.* 64 (1988), 57–72.
[12] D. Rudolph, Classifying the isometric extensions of a Bernoulli shift, *J. Analyse Math.* 34 (1978), 36–60.
[13] A. M. Vershik, Decreasing sequences of measurable partitions and their applications. *Dokl. Akad. Nauk SSSR* 193 (1970), 748–751.

(Received January 20, 1999)