Spectroscopy for a few atoms harmonically-trapped in one dimension

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Symmetries for any* trap and any† interaction

\[
\hat{H} = \sum_{i=1}^{N} \left( -\frac{1}{2m} \frac{\partial^2}{\partial x_i^2} + V_{\text{trap}} (|x_i|) \right) + \sum_{\langle i, j \rangle} V_{\text{int}} (|x_i - x_j|)
\]

\[S_N \times O(1) \times O(1)\]

Permutation symmetry

Relative parity

COM parity

*one-dimensional, symmetric, spin-independent
†one-dimensional, Galilean-invariant, spin-independent
Symmetries for harmonic trap and contact interaction

\[
\frac{\hat{H}}{\hbar \omega} = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + x_i^2 \right) + g \sum_{\langle i, j \rangle} \delta(x_i - x_j)
\]

\[S_N \times O(1) \times U(1)\]

Permutation symmetry
Relative parity
COM harmonic oscillator

Additional symmetries:
- Contact symmetry \(U(N)\) when \(g = 0\)
- Spectrum generating symmetry \(SO(2,1)\) when \(g = 0\) or \(g \to \infty\)
Outline

• Motivation
• Symmetries of Configuration Space
• Spectroscopic Classification of Spatial States
• Secret Motivation
Motivation, Part I

\[
\frac{\hat{H}}{\hbar \omega} = \frac{1}{2} \sum_{i=1}^{N} \left(- \frac{\partial^2}{\partial x_i^2} + x_i^2 \right) + g \sum_{\langle i,j \rangle} \delta(x_i - x_j)
\]

• Mathematical physics
  – Symmetry and integrability
• Universality in few-body physics
  – Recent experiments
• Power of symmetry
  – Calculation and control: exact diagonalization, adiabatic evolution, quenches
• Limits of symmetry
  – Emergent complexity with increasing DOF
Harmonic Trap and Separation of Variables

\[ \hat{H} = \frac{\hbar \omega}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + x_i^2 \right) + \sum_{\langle i,j \rangle} V_{\text{int}} \left( |x_i - x_j| \right) \]

\[
\begin{pmatrix}
    r_1 \\
    r_2 \\
    \vdots \\
    R
\end{pmatrix}
= J_N
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_N
\end{pmatrix}
= \begin{pmatrix}
    \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_2 \\
    \frac{1}{\sqrt{6}} x_1 + \frac{1}{\sqrt{6}} x_2 - \sqrt{\frac{2}{3}} x_3 \\
    \vdots \\
    \frac{1}{\sqrt{N}} x_1 + \frac{1}{\sqrt{N}} x_2 + \cdots + \frac{1}{\sqrt{N}} x_N
\end{pmatrix}

\rho = \sqrt{r_1^2 + \cdots + r_{N-1}^2}

\[ \hat{H} = \frac{\hbar \omega}{2} \left( -\frac{\partial^2}{\partial R^2} + R^2 \right) + \frac{\hbar \omega}{2} \sum_{i=1}^{N-1} \left( -\frac{\partial^2}{\partial r_i^2} + r_i^2 \right) + \sum_{\langle i,j \rangle} V_{\text{int}} \left( |\tilde{d}_{ij} \cdot \vec{r}| \right) \]

\[ \left( \tilde{d}_{ij} \right)_k = (J_N)_{ki} - (J_N)_{kj} \]
Symmetries for harmonic trap
and any† interaction

\[ \hat{H} = \frac{1}{2} \left( -\frac{\partial^2}{\partial R^2} + R^2 \right) + \sum_{i=1}^{N-1} \left( -\frac{\partial^2}{\partial r_i^2} + r_i^2 \right) + \sum_{\langle i,j \rangle} V_{\text{int}} \left( |\vec{d}_{ij} \cdot \vec{r}| \right) \]

\[ S_N \times O(1) \times U(1) \]
Two-Particle Configuration Space

\[ m_1 = m_2 \]

\[ R = 0 \]

\[ r_1 = 0 \]

\[ x_2 \]

\[ x_1 \]

Symmetry group: \( S_2 \times O(1) \)

Transformations:

\[ \pi, (12) \to \sigma_{r1} \]

\[ \Pi \to C_2 \]

\[ \Pi(12) \to \sigma_R \]

\[ 2m_1 = m_2 \]

\[ R = 0 \]

\[ r_1 = 0 \]

\[ x_2 \]

\[ x_1 \]

Symmetry group: \( O(1) \)

Transformations:

\[ \Pi \to C_2 \]
Three-Particle Configuration Space

Symmetry group:

\[ S_3 \times O(1) \times O(1) \cong D_{6h} \]

\[ S_3 \times O(1) \cong C_{6v} \]
Three-Particle Configuration Space

Symmetry group:

\[ S_3 \times O(1) \cong C_{6v} \]

Relative inversion:

\( \pi \rightarrow C_2 \)

Reflections:

\( (12), (13), (23) \)

\( \pi(12), \pi(13), \pi(23) \)

Rotations:

\( (123), (132) \rightarrow 2C_3 \)

\( \pi(123), \pi(132) \rightarrow 2C_6 \)
Four-Particle Configuration Space

Symmetry group:

\( S_4 \times O(1) \sqsubset O_h \)

- \((12)\ldots \rightarrow 6\sigma_d\)
- \((123)\ldots \rightarrow 8C_3\)
- \((12)(34)\ldots \rightarrow 3C_2\)
- \((1234)\ldots \rightarrow 6S_4\)
- \(\pi \rightarrow i\)
- \(\pi (12)\ldots \rightarrow 6C_2'\)
- \(\pi (123)\ldots \rightarrow 8S_6\)
- \(\pi (12)(34)\ldots \rightarrow 3\sigma_h\)
- \(\pi (1234)\ldots \rightarrow 6C_4\)

Body diagonals are projections of particle axes, 3+1 clusters

Edge diagonals, 1+2+1 clusters

Face diagonals, 2+2 clusters
Five-Particle Configuration Space

\[ S_5 \times O(1) \]

It is known, Khaleeshi.
Quantum numbers

$S_N \times O(1) \times U(1)$

$U(1), S_N^{O(1)} \overset{\text{COM excitation number}}{\longrightarrow} \begin{array}{c} n_R \left[ p \right] ^\pi \tau; i \end{array}_g \overset{\text{Symmetric group irrep}}{\longrightarrow}$
Symmetric Group Irreps

[2]

[2]

[2]

[3]

[21]

[1^3]

[4]

[31]

[2^2]

[21^2]

[1^4]
Quantum numbers

\[ S_N \times O(1) \times U(1) \]

\[ U(1), S_N^O(1) \]

\[ \hat{H} \left| n_R[p]^{\pi \tau; i} \right\rangle_g = E_{n_R[p]^{\pi \tau}} \left| n_R[p]^{\pi \tau; i} \right\rangle_g \]

Each energy is distinct except at non-interacting and infinite repulsion limit.

\[ E_{n_R[p]^{\pi \tau}} = \hbar \omega \left( n_R + X_{[p]^{\pi \tau}} + \frac{N}{2} \right) \]
Symmetries for contact interaction

\[ \hat{H} = \frac{1}{2} \left( -\frac{\partial^2}{\partial R^2} + R^2 \right) + \frac{1}{2} \sum_{i=1}^{N-1} \left( -\frac{\partial^2}{\partial r_i^2} + r_i^2 \right) + g \sum_{\langle i, j \rangle} \delta(\vec{d}_{ij} \cdot \vec{r}) \]

Contact symmetry

Spectrum generating symmetry

For any interaction strength

\[ S_N \times O(1) \times U(1) \]

For zero interaction strength

\[ U(N) \]

For infinite interaction strength

\[ S_N \times O(1) \times U(1) \]

...and something else for \( N > 3 \... \]

\[ SO(2,1) \]

\[ X = 2n_\rho + \lambda \]
Three Particles: Harmonic Trap, Not Interacting

\[ X_{n_\rho \lambda} = 2n_\rho + \lambda \]

\[ d(X) = X + 1 \]

\[ \epsilon(\lambda > 0) = 2 \]

\[ \pi = (-1)^\lambda \]

\[ n_\rho = 0; \lambda = 0 \]

\[ n_\rho = 0; \lambda = 1 \]

\[ n_\rho = 0; \lambda = 2 \]

\[ n_\rho = 0; \lambda = 3 \]
Three Particles: Harmonic Trap, Unitary Limit

\[ X_{n_{\lambda}} = 2n_{\lambda} + \lambda \quad \text{d}(X) = 3! \quad \lambda \in 3, 6, 9, 12, \ldots \quad \pi = (-1)^{\lambda} \]
Three Particle:
Adiabatic Mapping

\[ \lambda = 3 + 6k \rightarrow [3]^+ \oplus [21]^+ \oplus [21]^- \oplus [1^3]^- \]

\[ \lambda = 6 + 6k \rightarrow [3]^- \oplus [21]^- \oplus [21]^+ \oplus [1^3]^+ \]
$[3]^+ \quad X_{\text{max}} = 30 \quad 40 \text{ states}$
Four Particles:
Harmonic Trap, Not Interacting

\[ X_{n_\rho \lambda \tau} = 2n_\rho + \lambda \quad d(X) = \frac{(X+1)(X+2)}{2!} \quad \varepsilon(\lambda) = 2\lambda + 1 \quad \pi = (-1)^\lambda \]
Lowest Energy Antisymmetric State

\[ |n_R[p]^{\pi \tau; i} \rangle_g \rightarrow |0[1^4]^+ 0 \rangle_g \]

Particle basis, Slater determinant:

\[ |0[1^4]^+ 0 \rangle_g = \frac{1}{\sqrt{24}} \sum_{p \in S_4} \text{sign}(p) |n_{p_1} n_{p_2} n_{p_3} n_{p_4} \rangle \]

Jacobi hypercylindrical:

\[ |0[1^4]^+ 0 \rangle_g = \sum_{\mu=-6}^{6} c_\mu |n_R n_\rho 6 \mu \rangle \]
Four Particles: Harmonic Trap, Non-interacting Including Spin

$$\mathcal{H} = S \otimes K$$

| number of components | component pattern | number of components | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) |
|----------------------|------------------|--------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1                   | \((4)_B\)        | 1                  | 1              | 0              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 2              |                |               |               |               |               |
| 2                   | \((31)_B\)       | 2                  | 1              | 1              | 1              | 1              | 1              | 2              | 2              | 3              | 3              | 3              | 4              | 4              | 4              | 4              | 5              |
| 2                   | \((22)_B\)       | 3                  | 1              | 1              | 1              | 2              | 3              | 3              | 4              | 4              | 4              | 4              | 5              | 5              | 5              | 6              | 6              | 7              |
| 3                   | \((2111)_B\)     | 4                  | 1              | 1              | 2              | 3              | 3              | 4              | 4              | 4              | 4              | 5              | 5              | 5              | 6              | 6              | 7              | 7              |
| 1                   | \((4)_F\)        | 5                  | 1              | 0              | 0              | 0              | 0              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              |
| 2                   | \((31)_F\)       | 6                  | 1              | 0              | 0              | 1              | 1              | 2              | 2              | 3              | 3              | 3              | 3              | 3              | 3              | 4              |               |               |
| 2                   | \((22)_F\)       | 7                  | 1              | 0              | 0              | 1              | 1              | 2              | 3              | 3              | 3              | 3              | 3              | 3              | 4              |               |               |
| 3                   | \((2111)_F\)     | 8                  | 1              | 0              | 0              | 1              | 1              | 2              | 3              | 3              | 4              | 4              | 4              | 4              | 5              | 5              | 5              | 6              |
| 4                   | \((11111)\)      | 9                  | 1              | 3              | 5              | 7              | 9              | 11             | 13             | 15             | 17             | 19             | 21             | 23             | 25             |               |               |               |
|                     |                  | 1                  | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              |
|                     |                  | 2                  | 1              | 2              | 3              | 4              | 5              | 6              | 7              | 8              | 9              | 10             | 11             | 12             | 13             | 14             | 15             | 16             |
|                     |                  | 3                  | 1              | 2              | 3              | 4              | 5              | 6              | 7              | 8              | 9              | 10             | 11             | 12             | 13             | 14             | 15             | 16             |
|                     |                  | 4                  | 1              | 2              | 3              | 4              | 5              | 6              | 7              | 8              | 9              | 10             | 11             | 12             | 13             | 14             | 15             | 16             |
|                     |                  | 5                  | 1              | 2              | 3              | 4              | 5              | 6              | 7              | 8              | 9              | 10             | 11             | 12             | 13             | 14             | 15             | 16             |

\( s = 0 \) Total: 1

\( 2D^0 \) Total: 1
Four Particles: Harmonic Trap, Unitary Limit

\[ X_{n_\rho, \lambda} = 2n_\rho + \lambda \quad d(X) = 4! \lambda \in 6,8,9,10,12,13,\ldots \quad \pi = (-1)^\lambda \]

\[ \lambda = 6 \]

\[ [1^4]^+ \quad [4]^+ \]
\[ \lambda = \text{even} \rightarrow [4]^+ \oplus [31]^+ \oplus 2[2^2]^+ \oplus [21^2]^+ \oplus [1^4]^+ \oplus 2[31]^- \oplus 2[21^2]^- \]

\[ \lambda = \text{odd} \rightarrow 2[31]^+ \oplus 2[21^2]^+ \oplus [4]^- \oplus [31]^- \oplus 2[2^2]^- \oplus [21^2]^- \oplus [1^4]^- \]

MYSTERY SYMMETRY
Caveat emptor
Irrep Reduction for Five Particles

Non-interacting

| λ | 5 | 4 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | 1 | 0 |
| 4 | 1 | 0 |
| 5 | 1 | 0 |
| 6 | 1 | 0 |

| no. of comps. | component pattern | \([5]\) | \([41]\) | \([3^2]\) | \([3]^2\) | \([2^21]\) | \([21^3]\) | \([1^4]\) |
|---------------|-------------------|------|------|------|------|------|------|------|
| 1             | \((5)_B\)         | 1    | 0    | 0    | 0    | 0    | 0    | 0    |
| 2             | \((41)_B\)        | 1    | 1    | 0    | 0    | 0    | 0    | 0    |
| 2             | \((32)_B\)        | 1    | 1    | 1    | 0    | 0    | 0    | 0    |
| 2             | \((311)_B\)       | 1    | 2    | 1    | 1    | 0    | 0    | 0    |
| 3             | \((221)_B\)       | 1    | 2    | 2    | 1    | 1    | 0    | 0    |
| 4             | \((2111)_B\)      | 1    | 3    | 3    | 3    | 2    | 1    | 0    |
| 1             | \((5)_F\)         | 0    | 0    | 0    | 0    | 0    | 0    | 1    |
| 2             | \((41)_F\)        | 0    | 0    | 0    | 0    | 0    | 1    | 1    |
| 2             | \((32)_F\)        | 0    | 0    | 0    | 0    | 1    | 1    | 1    |
| 2             | \((311)_F\)       | 0    | 0    | 0    | 1    | 2    | 1    | 1    |
| 3             | \((221)_F\)       | 0    | 0    | 1    | 1    | 2    | 2    | 1    |
| 4             | \((2111)_F\)      | 0    | 1    | 2    | 3    | 3    | 3    | 1    |
| 5             | \((11111)\)       | 1    | 4    | 5    | 6    | 5    | 4    | 1    |

| odd λ | 3 | 2 |
|-------|---|---|
| 0     | 2 | 2 |
| 2     | 4 | 2 |
| 4     | 2 | 2 |
| 6     | 2 | 2 |

Unitary limit

Working on more efficient state construction method
Secret Motivation

Integrability, separability and entanglement

- Abstractly: Particles and Tailored Observables
- Directly: Few body systems as a resource for quantum information processing
THE FUTURE

• Identify mystery symmetry
• Basis transformation coefficients, matrix elements
• More particles
  – Five is different
  – Heterogeneous particle mixtures
• More dimensions
  – Symmetry is less constraining
• More traps, more interactions
  – Lose spectrum generating group and $U(N)$ symmetry

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