Almost Hermitian Structures and Quaternionic Geometries

Francisco Martín Cabrera and Andrew Swann

Abstract

Gray & Hervella gave a classification of almost Hermitian structures \((g, I)\) into 16 classes. We systematically study the interaction between these classes when one has an almost hyper-Hermitian structure \((g, I, J, K)\). In general dimension we find at most 167 different almost hyper-Hermitian structures. In particular, we obtain a number of relations that give hyperKähler or locally conformal hyperKähler structures, thus generalising a result of Hitchin. We also study the types of almost quaternion-Hermitian geometries that arise and tabulate the results.

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1 Introduction

In [4] Gray & Hervella gave a classification of almost Hermitian manifolds in terms of the covariant derivatives of the Kähler 2-form. This derivative has special symmetries and may naturally be decomposed into four components lying in spaces they called $W_1, \ldots, W_4$. This gives $2^4 = 16$ different classes of almost Hermitian manifolds determined by which components are non-zero. From these one may easily read off properties such as integrability of the almost complex structure or closure of the Kähler form, cf. Table 1.

An almost quaternion-Hermitian manifold locally possesses three almost complex structures defining almost Hermitian structures with respect to a common metric and satisfying the multiplicative identities of the imaginary quaternions. An analogue of the Gray-Hervella classification may be obtained for such structures by considering the covariant derivative of a certain four-form. In general dimensions, this leads to $2^6 = 64$ classes, which were recently described in detail in [7].

It is a natural question to ask how these two classifications interact. Indeed various results in this direction are already known, the most celebrated being Hitchin’s proof [5] that for a manifold to be hyperKähler it is sufficient that the three Kähler 2-forms be closed. One first observation is that the space of covariant derivatives of three arbitrary two-forms is about twice as large as the space of covariant derivatives of a quaternionic four-form. One should therefore expect to find more relations than simply those arising from the fact that the third almost complex structure is the product of the first two. In this paper, we find systematically all such relations between these covariant derivatives. With this in place it is an easy matter to read off various consequences in the style of Hitchin’s result and to obtain generalisations.

After recalling definitions and the relevant representation theory in §2 the key technical points of the paper may be found in §3: a first decomposition of the covariant derivative $\nabla \omega_I$ is given in Lemma 3.1 and this is refined in Propositions 3.2 and 3.4. Conclusions regarding almost Hermitian types are drawn in §4 whereas the consequences for the quaternionic geometry may be found in §5. The paper ends with tables summarising all the results and a short discussion of numbers of cases and possible examples.

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| $C$      | Name                     | Characteristic property                          |
|----------|--------------------------|--------------------------------------------------|
| $\{0\}$ | Kähler                   | $\nabla \omega_I = 0$                            |
| $\mathcal{W}_2$ | almost Kähler, symplectic | $d\omega_I = 0$                                 |
| $\mathcal{W}_4$ | locally conformal Kähler  | $d\omega_I = \alpha \wedge \omega_I \neq 0$     |
| $\mathcal{W}_1 + \mathcal{W}_2$ | $(1, 2)$-symplectic       | $(d\omega_I)^{1,2} = 0$                          |
| $\mathcal{W}_3 + \mathcal{W}_4$ | integrable, Hermitian    | $d\Lambda^{1,0} \subset \Lambda^{2,0} + \Lambda^{1,1}$ |
| $\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$ | semi-Kähler, co-symplectic | $d^*\omega_I = 0$                               |

Table 1: Important classes $C$ in the Gray-Hervella classification of almost Hermitian manifolds (of dimension at least 6).

2 Preliminaries

A $4n$-dimensional manifold $M$ ($n > 1$) is said to be almost quaternion-Hermitian, if $M$ is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and a rank-three subbundle $\mathcal{G}$ of the endomorphism bundle $\text{End} \, TM$, such that locally $\mathcal{G}$ has an adapted basis $I, J, K$ satisfying $I^2 = J^2 = -1$ and $K = IJ = -JI$, and $\langle AX, AY \rangle = \langle X, Y \rangle$, for all $X, Y \in T_x M$ and $A = I, J, K$. This is equivalent to saying that $M$ has a reduction of its structure group to $\text{Sp}(n) \text{Sp}(1)$. An almost quaternion-Hermitian manifold with a global adapted basis is called an almost hyper-Hermitian manifold.

There are three local Kähler-forms $\omega_A(X, Y) = \langle X, AY \rangle$, $A = I, J, K$. From these one may define a global, non-degenerate four-form $\Omega$, the fundamental form, by the local formula

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K. \quad (2.1)$$

2.1 Covariant Derivative of one Kähler Form

Let $\nabla$ denote the Levi-Civita connection. As the metric is almost Hermitian with respect to $A = I, J, K$, one obtains [4]

$$\nabla_X \omega_A(AY, AZ) = -\nabla_X \omega_A(Y, Z). \quad (2.2)$$
On the other hand the relation $I = JK$, implies that the covariant derivative of $\omega_I$ is determined by those of $\omega_J$ and $\omega_K$. This may be expressed symmetrically by [3, 7]

$$\nabla_X \omega_I(Y, KZ) + \nabla_X \omega_J(KY, IZ) + \nabla_X \omega_K(IZ, JY) = 0,$$

or directly by

$$\nabla_X \omega_I(Y, Z) = \nabla_X \omega_J(KY, Z) - \nabla_X \omega_K(Y, JZ).$$

The two other versions of this equation obtained by cyclically permuting $I, J, K$ also hold, and in this paper we will use such results without further comment.

The following conventions will be used in the sequel. If $b$ is a $(0, s)$-tensor, we write

$$A_{(i)} b(X_1, \ldots, X_i, \ldots, X_s) = -b(X_1, \ldots, AX_i, \ldots, X_s),$$

$$Ab(X_1, \ldots, X_s) = (-1)^s b(AX_1, \ldots, AX_s),$$

for $A = I, J, K$. We also consider the natural extension of $\langle \cdot, \cdot \rangle$ to $(0, s)$-tensors given by

$$\langle a, b \rangle = \sum_{i_1, \ldots, i_s=1}^{4^n} a(e_{i_1}, \ldots, e_{i_s})b(e_{i_1}, \ldots, e_{i_s}),$$

where $\{e_1, \ldots, e_{4n}\}$ an orthonormal basis for $T_x M$. The notation $\{e_1, \ldots, e_{4n}\}$ will also denote the corresponding dual basis of one-forms. In some situations we will write $g$ for the Riemannian metric $\langle \cdot, \cdot \rangle$.

Using these conventions, we may write our expression for the covariant derivative of $\omega_I$ as

(2.3) \hspace{1cm} \nabla \omega_I = -K(2)\nabla \omega_J + J(2)\nabla \omega_K.$$

2.2 Representation Theory

Our key tool for refining the expression (2.3) for $\nabla \omega_I$ is the representation theory of $Sp(n)$, $Sp(1)$ and $U(1)$. We will follow the $E$-$H$ formalism used in [11, 12, 13] to denote irreducible $Sp(n)$-$Sp(1)$-modules. Thus, $E$ is the fundamental representation of $Sp(n)$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$ via left multiplication by quaternionic matrices, and $H$ is the representation of $Sp(1)$ on $\mathbb{C}^2 \cong \mathbb{H}$ given by $q, \zeta = \zeta \overline{q}$, for $q \in Sp(1)$ and $\zeta \in H$. An $Sp(n)$-$Sp(1)$-structure
on a manifold $M$ gives rise to local bundles $E$ and $H$ associated to these representation and identifies $TM \otimes_{\mathbb{R}} \mathbb{C} \cong E \otimes_{\mathbb{C}} H$.

On $E$, there is a $Sp(n)$-invariant complex symplectic form $\omega_E$ and a Hermitian inner product given by $\langle x, y \rangle_C = \omega_E(x, \bar{y})$, for all $x, y \in E$ and being $\bar{y} = y^j$ ($y \to \bar{y}$ is a quaternionic structure map on $E = \mathbb{C}^{2n}$ considered as right complex vector space). The mapping $x \to x^\omega = \omega_E(\cdot, x)$ gives us an identification of $E$ with its dual $E^*$. If $\{u_1, \ldots, u_n, \tilde{u}_1, \ldots, \tilde{u}_n\}$ is a complex orthonormal basis for $E$, then

$$\omega_E = u_i^\omega \wedge \tilde{u}_i^\omega = u_i^\omega \tilde{u}_i^\omega - \tilde{u}_i^\omega u_i^\omega,$$

where we have used the summation convention and omitted tensor product signs. These conventions will be used throughout the paper.

The irreducible representations of $Sp(1)$ are the symmetric powers $S^kH \cong \mathbb{C}^{k+1}$. An irreducible representation of $Sp(n)$ is determined by its dominant weight $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i$ are integers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. This representation will be denoted by $V^{(\lambda_1, \ldots, \lambda_r)}$, where $r$ is the largest integer such that $\lambda_r > 0$. Familiar notation is used for some of these modules when possible. For instance, $V^{(k)} = S^kE$, the $k$th symmetric power of $E$, and $V^{(1, \ldots, 1)} = \Lambda_0^rE$, where there are $r$ ones in the exponent and $\Lambda_0^rE$ is the $Sp(n)$-invariant complement to $\omega_E \Lambda^{r-2}E$ in $\Lambda^rE$. Also $K$ will denote the module $V^{(21)}$, which arises in the decomposition

$$E \otimes \Lambda_0^2E \cong \Lambda_0^3E + K + E,$$

where $+$ denotes direct sum.

When we fix a choice of local almost complex structure $I$, we get a reduction of the structure group to $Sp(n) U(1) \subset U(2n)$. We write $L$ for the standard representation of $U(1)$ on $\mathbb{C}$, and $\Lambda^{1,0}$ for the representation of $U(2n)$ on $\mathbb{C}^{2n}$. The latter has dual representation $\Lambda^{0,1}$, and arbitrary irreducible representations lie in some tensor product $\Lambda^{p,q} = \Lambda^p \Lambda^{1,0} \otimes \Lambda^q \Lambda^{0,1}$ and will be labelled by the minimal pair $(p, q)$. In particular, $U^{3,0}$ is the irreducible module in the decomposition

$$\Lambda^{1,0} \otimes \Lambda^{2,0} = \Lambda^{3,0} + U^{3,0}.$$

The space $\Lambda_0^{p,q}$ is the orthogonal complement of $\omega_I \Lambda^{p-1,q-1}$ in $\Lambda^{p,q}$. When we need to regard these as representations over the real numbers we will use the notation $[\Lambda_0^{p,q}]$, etc. This satisfies $[\Lambda_0^{p,q}] \otimes \mathbb{C} = \Lambda_0^{p,q} + \Lambda_0^{p,q}$. Note that $TM = [\Lambda^{1,0}]$. Elements of $\Lambda^{p,q}$ are said to have type $(p, q)$; elements of $[\Lambda^{p,q}]$ for $p \neq q$ will be said to have type $\{p, q\}$. Modules and types will be labelled by the almost complex structure $I$ when it is necessary to avoid confusion.
3 Quaternionic Decompositions

Let us reconsider the covariant derivative of a single Kähler form $\omega_I$, this time from the point of view of representation theory. Equation (2.2) shows that

$$\nabla \omega_I \in [\Lambda^{1.0}] \otimes u(2n)^\perp = [\Lambda^{3.0}] + [U^{3.0}] + [\Lambda_0^{2.1}] + [\Lambda^{1.0}],$$

where $u(2n)^\perp = [\Lambda^{2.0}]$ is the orthogonal complement of $u(2n) = \Lambda^{1.1}$ in $\Lambda^2 T^* M$. This is Gray & Hervella’s decomposition [4] (cf. [2]).

In the present context, we have also the action of $Sp(n) U(1)$ which is a subgroup of $U(2n)$. To obtain descriptions of the modules $W_i$ as representations of this subgroup, note that

$$TM \otimes \mathbb{C} = EH = E(L + \mathcal{T}) = EL + E\mathcal{T} = \Lambda^{1.0} + \Lambda^{0.1}.$$ 

From this we find

$$(3.1) \quad \Lambda^2 T^* M \otimes \mathbb{C} = S^2 E + S^2 H + \Lambda_0^2 E S^2 H$$

$$= S^2 E + C \omega_I + L^2 + \mathcal{T}^2 + \Lambda_0^2 E(L^2 + \mathcal{C} + \mathcal{T}^2)$$

and hence $u(2n) \otimes \mathbb{C} = S^2 E + C \omega_I + \Lambda_0^2 E$, $\Lambda^{2.0} = (\Lambda_0^2 E + \mathbb{C})L^2$. Taking the tensor products of these representations we get

$$(3.2) \quad W_1 \otimes \mathbb{C} = (\Lambda_0^2 E + E)(L^3 + \mathcal{T}^3), \quad W_2 \otimes \mathbb{C} = (K + E)(L^3 + \mathcal{T}^3),$$

$$W_3 \otimes \mathbb{C} = (\Lambda_0^2 E + K + E)(L + \mathcal{T}), \quad W_4 \otimes \mathbb{C} = E(L + \mathcal{T}).$$

We now set about finding the corresponding components of $\nabla \omega_I$ explicitly. First note that $S^2 H \subset \Lambda^2 T^* M \otimes \mathbb{C}$ decomposes orthogonally both as $C \omega_I + L^2 + \mathcal{T}^2$ and as $C \omega_I + C \omega_J + C \omega_K$. Thus $L^2 + \mathcal{T}^2$ is the direct sum of the tensors in $S^2 H$ which are of type $(1, 1)_J$ or of type $(1, 1)_K$. Let us write $(\Lambda_0^2 E)_I$ for the module $\Lambda_0^2 E$ in (3.1); this consists of the tensors in $\Lambda_0^2 E S^2 H$ which are of type $(1, 1)_I$. Our discussion of $S^2 H$ now shows $\Lambda_0^2 E(L^2 + \mathcal{T}^2) = (\Lambda_0^2 E)_J + (\Lambda_0^2 E)_K$. Thus

$$\nabla_{X} \omega_I \in u(n)^\perp = \mathbb{R} \omega_J + \mathbb{R} \omega_K + [(\Lambda_0^2 E)_J] + [(\Lambda_0^2 E)_K],$$

with the last decomposition being orthogonal.

In order to label these components of $\nabla_{X} \omega_I$, consider the one forms $\lambda_I$, $\lambda_J$, $\lambda_K$, defined by

$$(3.3) \quad \lambda_I(X) := \frac{1}{4\pi} \langle \nabla_{X} \omega_J, \omega_K \rangle = -\frac{1}{4\pi} \langle \nabla_{X} \omega_K, \omega_J \rangle.$$
For any two-form $\beta$ of type $(1,1)$, $I_{(1)}\beta$ is a symmetric two-tensor. Now
\[ S^2 T^* M \otimes \mathbb{C} = S^2 E S^2 H + \Lambda_0^2 E + \mathbb{C} g. \]
Thus for $\beta \in (\Lambda_0^2 E)_I$, we have that $I_{(1)}\beta \in \Lambda_0^2 E$ and in particular $I_{(1)}\beta$ is of type $(1,1)$ for $I, J$ and $K$. We therefore introduce the tensors $\alpha_I, \alpha_J, \alpha_K$ in $T^* M \otimes \Lambda_0^2 E \subset T^* M \otimes S^2 T^* M$ given by
\begin{equation}
\alpha_I := -\lambda_I \otimes g + \frac{1}{2}(J(2) - J(3))\nabla \omega_K = -\lambda_I \otimes g + \frac{1}{2}(K(3) - K(2))\nabla \omega_J.
\end{equation}
Note that they satisfy $\langle \alpha_A(X, \cdot, \cdot), g \rangle = 0$ and
\[ \alpha_A = I_{(2)}I_{(3)}\alpha_A = J_{(2)}J_{(3)}\alpha_A = K_{(2)}K_{(3)}\alpha_A, \]
for $A = I, J, K$. These forms and tensors will play a significant rôle in the present paper.

From equation (2.3) we have:

**Lemma 3.1** Using the definitions (3.3) and (3.4), the covariant derivative of $\omega_I$ is given by
\[ \nabla \omega_I = \lambda_K \otimes \omega_J - \lambda_J \otimes \omega_K + J(2)\alpha_K - K(2)\alpha_J. \]

Since $\alpha_I, \alpha_J, \alpha_K \in EH \otimes \Lambda_0^2 E = (\Lambda_0^3 E + K + E)H$, we may decompose $\alpha_I$ into three components
\[ \alpha_I = \alpha_I^{(3)} + \alpha_I^{(K)} + \alpha_I^{(E)} \in \Lambda_0^3 EH + KH + EH. \]
(When $\dim M = 8$, the module $\Lambda_0^3 E$ is trivial and the corresponding component $\alpha_I^{(3)}$ is not present.)

Similar notation will be used for the $Sp(n) U(1)$-components in the decomposition of $\nabla \omega_I$. Thus, for example, $(\nabla \omega_I)_{W_1} = (\nabla \omega_I)^{(3)}_{W_1} + (\nabla \omega_I)^{(E)}_{W_1}$. Controlling the $\Lambda_0^3 E$ and $K$ components of $\nabla \omega_I$ is relatively straightforward as we shall now see.

The $W_1 + W_2$-part of $\nabla \omega_I$ consists of the components that are of type $\{3,0\}_I$. As $\nabla \omega_I$ is already of type $\{2,0\}_I$ in the last two indices, one sees that $2(\nabla \omega_I)_{W_1 + W_2} = (1 - I_{(1)}I_{(2)})\nabla \omega_I$. Therefore, using Lemma 3.1, we have
\begin{equation}
(\nabla \omega_I)_{W_1 + W_2} = \frac{1}{2}\{K(J\lambda_J - K\lambda_K) \otimes \omega_J + J(J\lambda_J - K\lambda_K) \otimes \omega_K + (J(1)K(2) + K(1)J(2))(J(1)\alpha_J - K(1)\alpha_K)\}. \end{equation}
Similarly, the $W_3 + W_4$-component of $\nabla \omega_I$ is given by $2(\nabla \omega_I)_{W_3 + W_4} = (1 + I(2)I(3)) \nabla \omega_I$. Thus, we have

$$\nabla \omega_I = \frac{1}{2} \{ -K(J\lambda_J + K\lambda_K) \otimes \omega_J + J(J\lambda_J + K\lambda_K) \otimes \omega_K + (J(1)K(2) - K(1)J(2))(J(1)\alpha_J + K(1)\alpha_K) \}. \tag{3.6}$$

Now for $W_1 + W_2$, the module $\Lambda^3_0 E$ only occurs in $W_1$. Similarly, in $W_3 + W_4$ this module lies solely in $W_3$. Arguing in a similar way for the $K$-components, we are lead to the following result.

**Proposition 3.2** Let $\alpha_I$ be as in (3.4). Then

(a) $(\nabla \omega_I)_{W_1}^{(3)}$ is uniquely determined by $J(1)\alpha_J^{(3)} - K(1)\alpha_K^{(3)}$,

(b) $(\nabla \omega_I)_{W_3}^{(3)}$ by $J(1)\alpha_J^{(3)} + K(1)\alpha_K^{(3)}$,

(c) $(\nabla \omega_I)_{W_2}^{(K)}$ by $J(1)\alpha_J^{(K)} - K(1)\alpha_K^{(K)}$ and

(d) $(\nabla \omega_I)_{W_3}^{(K)}$ by $J(1)\alpha_J^{(K)} + K(1)\alpha_K^{(K)}$.

**Proof.** The $\Lambda^3_0 E$-parts of $(\nabla \omega_I)_{W_1}$ and $(\nabla \omega_I)_{W_3}$ are given by the corresponding parts of equations (3.5) and (3.6), i.e.,

$$\nabla \omega_I_{W_1}^{(3)} = \frac{1}{2} (J(1)K(2) + K(1)J(2))(J(1)\alpha_J^{(3)} - K(1)\alpha_K^{(3)}),$$

$$\nabla \omega_I_{W_3}^{(3)} = \frac{1}{2} (J(1)K(2) - K(1)J(2))(J(1)\alpha_J^{(3)} + K(1)\alpha_K^{(3)}),$$

and analogous formulæ for the $K$-components. The Proposition now follows from the following Lemma. \hfill \Box

**Lemma 3.3** For $\varepsilon = \pm 1$, $(J(1)K(2) + \varepsilon K(1)J(2))(J(1)\alpha_J - \varepsilon K(1)\alpha_K) = 0$ if and only if $J(1)\alpha_J = \varepsilon K(1)\alpha_K$.

**Proof.** Let us just give the proof for $\varepsilon = +1$. Clearly we only need to consider the forward implication. Since $(J(1)K(2) + \varepsilon K(1)J(2))^2 = 2(1 + I(1)I(2))$, then the first equation of the lemma gives

$$J(1)\alpha_J - K(1)\alpha_K + K(1)I(2)\alpha_J - J(1)I(2)\alpha_K = 0.$$ But $J(1)\alpha_J - K(1)\alpha_K \in T^* \otimes S^2T^*$ and $K(1)I(2)\alpha_J - J(1)I(2)\alpha_K \in T^* \otimes \Lambda^2 T^*$. Hence both of these pairs must vanish and in particular $J(1)\alpha_J - K(1)\alpha_K = 0$ as required. \hfill \Box
The situation for the $E$-components of $\nabla \omega_I$ is a little more complicated. The presence of the $EH$-component in the decomposition of $\alpha_I$ means that we can define one-forms $\eta_I$ from $\alpha_I$. We set

\[(3.7) \quad \eta_I(X) = \alpha_I(e_i, e_i, X) = \langle \alpha_I(\cdot, \cdot, X), g \rangle.\]

From a one-form $\eta$ we may produce a tensor $\alpha(\eta)$ in $EH \otimes \Lambda^2_0E$ by

\[
\alpha_I(\eta) = e_i \otimes (\eta \vee e_i + I\eta \vee Je_i + J\eta \vee Je_i + K\eta \vee Ke_i) - \frac{1}{n} \eta \otimes g,
\]

where $a \vee b = \frac{1}{2}(a \otimes b + b \otimes a)$. As the map $\eta \mapsto \alpha_I(\eta)$ is equivariant for the $Sp(n)\ Sp(1)$-action, $\alpha_I(\eta)$ lies in $EH \subset EH \otimes \Lambda^2_0E$. Computing the corresponding one-form $\eta_I$ from $\alpha_I(\eta)$ via (3.7) one gets $(2n+1)(n-1)\eta/n$. Thus in general

\[(3.8) \quad \alpha^{(E)}_I = \frac{1}{(2n+1)(n-1)}\{4n e_i \otimes (\eta_I \vee e_i)^H - \eta_I \otimes g\},\]

where $4(a \vee b)^H = a \vee b + Ia \vee Ib + Ja \vee Jb + Ka \vee Kb$.

**Proposition 3.4** Let $\lambda_I$ and $\eta_I$ be as in (3.3) and (3.7). Then

(a) $(\nabla \omega_I)^{(E)}_{W_1}$ and $(\nabla \omega_I)^{(E)}_{W_2}$ are uniquely determined by independent linear combinations of $J\lambda_I - K\lambda_K$ and $J\eta_I - K\eta_K$;

(b) $(\nabla \omega_I)^{(E)}_{W_3}$ and $(\nabla \omega_I)^{(E)}_{W_4}$ are uniquely determined by independent linear combinations of $J\lambda_I + K\lambda_K$ and $J\eta_I + K\eta_K$.

**Proof.** For (a), we compute the $E$-components of $(\nabla \omega_I)_{W_1+W_2}$ in the decompositions (3.2). For simplicity, write

\[
\lambda_I^- := J\lambda_I - K\lambda_K, \quad \eta_I^- := J\eta_I - K\eta_K.
\]

The $E$-component of $(\nabla \omega_I)_{W_1+W_2}$ is obtained from (3.5), by replacing $\alpha_J$ and $\alpha_K$ by $\alpha_J^{(E)}$ and $\alpha_K^{(E)}$, respectively. Taking (3.8) into account, we have

\[
2(\nabla \omega_I)^{(E)}_{W_1+W_2} = K \left( \lambda_I^- + k_{12}\eta_I^- \right) \otimes \omega_J + J \left( \lambda_I^- + k_{12}\eta_I^- \right) \otimes \omega_K
\]
\[
- nk_{12} \left( e_i \otimes Je_i \wedge K\eta_I^- + e_i \otimes Ke_i \wedge J\eta_I^- \right),
\]

where $k_{12} = -1/(2n+1)(n-1)$.

The $E$-part of the $W_1$-component of $\nabla \omega_I$ is obtained by alternating $(\nabla \omega_I)^{(E)}_{W_1+W_2}$, i.e.,

\[
6(\nabla \omega_I)^{(E)}_{W_1} = K \left( \lambda_I^- + k_1\eta_I^- \right) \wedge \omega_J + J \left( \lambda_I^- + k_1\eta_I^- \right) \wedge \omega_K,
\]
where \( k_1 = -1/(n-1) \). The difference \( (\nabla \omega_I)^{(E)}_{W_1+W_2} - (\nabla \omega_I)^{(E)}_{W_1} \) is the \( E \)-part of the \( \mathcal{W}_2 \)-component of \( \nabla \omega_I \), i.e.,

\[
6 (\nabla \omega_I)^{(E)}_{W_2} = 2K (\lambda_I^- + k_2 \eta_I^-) \otimes \omega_J + 2J (\lambda_I^- + k_2 \eta_I^-) \otimes \omega_I \\
+ e_i \otimes J e_i \land K (\lambda_I^- + k_2 \eta_I^-) + e_i \otimes K e_i \land J (\lambda_I^- + k_2 \eta_I^-),
\]

where \( k_2 = 1/(2n+1) \). As \( k_1 \neq k_2 \), this proves part (a).

For part (b), we introduce

\[
\lambda_I^+ = J \lambda_I + K \eta_K, \quad \eta_I^+ = J \eta_I + K \eta_K.
\]

The \( \mathcal{W}_4 \)-component of \( \nabla \omega_I \) is given by [4]

\[
(\nabla \omega_I)_{\mathcal{W}_4} = \frac{1}{2(2n-1)} \left\{ e_i \otimes e_i \land d^* \omega_I + e_i \otimes I e_i \land I d^* \omega_I \right\},
\]

where \( d^* \) denotes the coderivative operator. Since from Lemma 3.1 it follows that \( I d^* \omega_I = \lambda_I^+ + \eta_I^+ \), then we have

\[
(\nabla \omega_I)_{\mathcal{W}_4} = \frac{1}{2(2n-1)} \left\{ e_i \otimes e_i \land (\lambda_I^+ + k_4 \eta_I^+) + e_i \otimes I e_i \land (\lambda_I^+ + k_4 \eta_I^+) \right\},
\]

where \( k_4 = 1 \).

The difference \( (\nabla \omega_I)^{(E)}_{W_3+W_4} - (\nabla \omega_I)^{(E)}_{W_4} \) is the \( E \)-part of the \( \mathcal{W}_3 \)-component of \( \nabla \omega_I \), i.e.,

\[
2 (\nabla \omega_I)^{(E)}_{W_3} = -K (\lambda_I^+ + k_3 \eta_I^+) \otimes \omega_J + J (\lambda_I^+ + k_3 \eta_I^+) \otimes \omega_I \\
-\frac{1}{2(2n-1)} \left( e_i \otimes e_i \land I (\lambda_I^+ + k_3 \eta_I^+) + e_i \otimes I e_i \land (\lambda_I^+ + k_3 \eta_I^+) \right),
\]

where \( k_3 = k_{12} = -1/(2n+1)(n-1) \) as before. As \( k_3 \neq k_4 \), we have part (b).

\[\square\]

4 Almost Hyper-Hermitian Structures

In the following theorem we show the way in which the class of \( \omega_K \), as an almost Hermitian structure, is conditioned by the respective classes of \( \omega_I \) and \( \omega_J \). Moreover, from the theorem one can also deduce the list of possible triples of almost Hermitian types corresponding to \( \omega_I \), \( \omega_J \) and \( \omega_K \), respectively. Such a list is contained in the tables given in §6. The theorem also provides the essential rules for determining which triples of Gray-Hervella types may occur.
Theorem 4.1 Let $M$ be an almost hyper-Hermitian manifold.

(i) If $\nabla \omega_I \in \mathcal{W}_3 + \mathcal{W}_4$ and $\nabla \omega_J \in \mathcal{C}$, then $\nabla \omega_K \in \mathcal{C}$, where $\mathcal{C}$ means any Gray-Hervella class of almost Hermitian structures.

(ii) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{C} + \mathcal{W}_3 + \mathcal{W}_4$, then $\nabla \omega_K \in \mathcal{C} + \mathcal{W}_3 + \mathcal{W}_4$, for $\mathcal{C} = \mathcal{W}_1, \mathcal{W}_2$.

(iii) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_2$, then $\nabla \omega_K \in \mathcal{W}_3 + \mathcal{W}_4$.

(iv) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_4$ and $\nabla \omega_K \in \mathcal{C} + \mathcal{W}_3 + \mathcal{W}_4$, for $\mathcal{C} = \mathcal{W}_1, \mathcal{W}_2$, then $\nabla \omega_K \in \mathcal{W}_3 + \mathcal{W}_4$.

Moreover, if $M$ is eight-dimensional, we also have

(v) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4$, then $\nabla \omega_K \in \mathcal{W}_3 + \mathcal{W}_4$.

Proof. Follows directly from Propositions 3.2 and 3.4. □

We refer the reader to Table 1 for interpretations of some of these classes. In particular, parts (i) and (ii) each contain the statement that if $I$ and $J$ are integrable then $K = IJ$ is too, as first shown by Obata [9].

An important class of almost hyper-Hermitian manifolds are those in which all three Kähler forms are parallel. These are hyperKähler manifolds and their metrics are Ricci-flat. The following result is a consequence of the previous Theorem and shows some of the possible conditions which imply that a manifold is hyperKähler.

Theorem 4.2 Let $M$ be an almost hyper-Hermitian manifold. If one of the following conditions holds, then $M$ is hyperKähler:

(i) $\nabla \omega_I, \nabla \omega_J, \nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_2$,

(ii) $\nabla \omega_I \in \mathcal{W}_1$ and $\nabla \omega_J \in \mathcal{W}_2$,

(iii) $\nabla \omega_I = 0$ and $\nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_2$,

(iv) $\nabla \omega_I \in \mathcal{C}$ and $\nabla \omega_J \in \mathcal{W}_4$, $\mathcal{C} = \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$,

(v) $\nabla \omega_I = 0$ and $\nabla \omega_J \in \mathcal{C} + \mathcal{W}_4$, $\mathcal{C} = \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$,

(vi) $\nabla \omega_I = 0$, $\nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_4$ and $\nabla \omega_K \in \mathcal{C} + \mathcal{W}_3$, $\mathcal{C} = \mathcal{W}_1 + \mathcal{W}_2, \mathcal{W}_1 + \mathcal{W}_4, \mathcal{W}_2 + \mathcal{W}_4$, or
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(vii) \( \nabla \omega_I = 0, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4 \) and \( \nabla \omega_K \in \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4 \).

Moreover, if \( M \) is eight-dimensional, each one of the following conditions also implies that \( M \) is hyperKähler:

(viii) \( \nabla \omega_I \in \mathcal{W}_1 \) and \( \nabla \omega_J \in \mathcal{W}_3 \),

(ix) \( \nabla \omega_I = 0 \) and \( \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_3 \), or

(x) \( \nabla \omega_I = 0, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 \) and \( \nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4 \). \( \square \)

Remark 4.3 Part (i) of Theorem 4.2 was already proved in [8] and is a generalisation of Hitchin’s result [5] that if \( \omega_I, \omega_J \) and \( \omega_K \) are closed, then the manifold is hyperkähler. Part (iii) includes the statement that a Kähler manifold is automatically hyperKähler as soon as one additional two-form is closed.

The \( \mathcal{W}_4 \)-part of the covariant derivative of an almost Hermitian structure is linearly determined by its Lee form [4] defined, in the present context, by \( \theta_A = 1/(2n-1) Ad^* \omega_A \), for \( A = I, J, K \). Below it will be shown that if the structures determined by \( I, J, K \) are locally conformal Kähler, then they have a common Lee form. Thus, in such a case, we can say that the manifold is locally conformal hyperKähler. In general, with \( \lambda_I \) and \( \eta_I \) as in (3.3) and (3.7), we have the following result:

Lemma 4.4 Let \( M \) be an almost hyper-Hermitian manifold. The three almost Hermitian structures have a common Lee form if and only if

\[
I \lambda_I + I \eta_I = J \lambda_J + J \eta_J = K \lambda_K + K \eta_K.
\]

Note this happens when \( (\nabla \omega_I)^{(E)}_{\mathcal{W}_1 + \mathcal{W}_2} = (\nabla \omega_J)^{(E)}_{\mathcal{W}_1 + \mathcal{W}_2} = 0 \), which is the case if \( \nabla \omega_I, \nabla \omega_J \in \mathcal{W}_3 + \mathcal{W}_4 \).\]

Proof. From Lemma 3.1 we have \( Id^* \omega_I = J \lambda_J + K \lambda_K + J \eta_J + K \eta_K \), and the result follows. \( \square \)

Combining this Lemma with Theorem 4.1 we obtain:

Theorem 4.5 Let \( M \) be an almost hyper-Hermitian manifold. If one of the following conditions holds, then \( M \) is locally conformal hyperKähler:

(i) \( \nabla \omega_I \in \mathcal{W}_4 \) and \( \nabla \omega_J \in \mathcal{W}_i + \mathcal{W}_4, \ i = 1, 2, 3 \),
(ii) \( \nabla \omega_I, \nabla \omega_J \in W_i + W_4 \) and \( \nabla \omega_K \in W_1 + W_2 + W_4, \) \( i = 1, 2, 3, \)

(iii) \( \nabla \omega_I \in W_i + W_4, \) \( \nabla \omega_J \in W_3 + W_4 \) and \( \nabla \omega_K \in W_j + W_3 + W_4, \) \( i, j = 1, 2, 3 \) and \( i \neq j, \)

(iv) \( \nabla \omega_I \in W_1 + W_4 \) and \( \nabla \omega_J \in W_2 + W_4 \) and \( \nabla \omega_K \in W_i + W_j + W_4, \) \( i, j = 1, 2, 3 \) and \( i \neq j, \) or

(v) \( \nabla \omega_I \in W_4, \) \( \nabla \omega_J \in C + W_4 \) and \( \nabla \omega_J \in D + W_4, \) \( C, D = W_1 + W_2, W_1 + W_3, W_2 + W_3 \) and \( C \neq D. \) \hfill \( \square \)

Remark 4.6 Part (i) of Theorem is a generalisation of Obata’s result that a Kähler structure with an additional integrable complex structure is hyperKähler.

5 Almost Quaternion-Hermitian Structures

Up to this point we have concentrated on the types of the almost Hermitian structures \((g, I), (g, J)\) and \((g, K)\). However, these may be regarded as coming from an adapted basis for an almost quaternion-Hermitian structure, and in dimension at least 12 such a structure has one of 64 possible types determined by the covariant derivative of the fundamental 4-form \( \Omega \). It is therefore interesting to find what consequences the three almost Hermitian types have for the quaternionic type. The 64 quaternionic classes come from the following \( Sp(n) \) \( Sp(1) \)-decomposition.

**Proposition 5.1 (Swann)*** The covariant derivative of the fundamental form \( \Omega \) of an almost quaternion-Hermitian manifold \( M \) of dimension at least 8, has the property

\[
\nabla \Omega \in T^*M \otimes \Lambda_3^2 E S^2 H = (\Lambda_3^3 E + K + E)(S^3 H + H). \hfill \square
\]

If the dimension of \( M \) is at least 12, all the modules of the sum are non-zero.

For an eight-dimensional manifold \( M \), we have \( \Lambda_3^3 E S^3 H = \Lambda_3^3 E H = \{0\} \).

If \( \nabla \Omega = 0 \), \( M \) is said to be quaternionic Kähler and the metric \( g \) is automatically Einstein (see for example [1]). If \( \nabla \Omega \in EH \), then \( M \) is locally conformal quaternionic Kähler. The case \( \nabla \Omega \in (K + E)H \) is known as QKT geometry: there is a second \( Sp(n) \) \( Sp(1) \)-connection on \( M \) with totally skew-symmetric torsion, see for example [6]. When \( \nabla \Omega \in (\Lambda_3^3 E + K + E)H \)
the underlying almost quaternionic structure is integrable, i.e., there is a
torsion-free connection preserving the bundle spanned by \( I, J \) and \( K \).

Using Lemma 3.1 the covariant derivative \( \nabla \Omega \) is given by
\[
(5.1) \quad \nabla \Omega = 2 \mathcal{S}_{I,J,K} \left( J(2)\alpha_K - K(2)\alpha_J \right) \wedge \omega_I,
\]
where \( \mathcal{S} \) denotes the cyclic sum. Note that the one-forms \( \lambda_I, \lambda_J \) and \( \lambda_K \) do
not appear in this formula. We immediately conclude that the \( \Lambda^3_0 E \) \( (S^3H + H) \), \( K(S^3H + H) \) and \( E(S^3H + H) \) of \( \nabla \Omega \) are linearly determined by the
corresponding components of the \( \alpha \)'s. To further divide these components
we use the following result.

Let \( \alpha: \bigotimes^4 T^*M \to \Lambda^4 T^*M \) be the alternation map.

**Proposition 5.2 (Cabrera [7])** The covariant derivative of \( \Omega \) splits as
\[
\nabla \Omega = (\nabla \Omega)_{S^3H} + (\nabla \Omega)_H \in (\Lambda^3_0 E + K + E)S^3H + (\Lambda^3_0 E + K + E)H,
\]
with
\[
(5.2) \quad (\nabla \Omega)_{S^3H} = \frac{1}{6}(4\nabla \Omega - \mathcal{S}_{I,J,K} (Id \otimes \alpha)I(1)I(2) \nabla \Omega),
\]
\[
(5.3) \quad (\nabla \Omega)_H = \frac{1}{6}(2\nabla \Omega + \mathcal{S}_{I,J,K} (Id \otimes \alpha)I(1)I(2) \nabla \Omega). \quad \square
\]

The classes involved in this last result are also determined by conditions
on \( \alpha_I, \alpha_J, \alpha_K \).

**Proposition 5.3** Let \( M \) an almost quaternion-Hermitian manifold and \( U \)
an open set where the adapted basis \( I, J, K \) is defined. Then

(i) \( (\nabla \Omega)_H \) is linearly determined by \( I(1)\alpha_I + J(1)\alpha_J + K(1)\alpha_K \).

(ii) \( (\nabla \Omega)_{S^3H} \) is linearly determined by \( I(1)\alpha_I - J(1)\alpha_J \) and \( J(1)\alpha_J - K(1)\alpha_K \).

**Proof.** Using (5.1) the tensors appearing in equations (5.2) and (5.3) may
be expressed as
\[
\mathcal{S}_{I,J,K} (Id \otimes \alpha)I(1)I(2) \nabla \Omega = 4 \mathcal{S}_{I,J,K} \{ J(1)K(2)(I(1)\alpha_I + K(1)\alpha_K)
- K(1)J(2)(I(1)\alpha_I + J(1)\alpha_J) \} \wedge \omega_I,
\]
\[
\nabla \Omega = 2 \mathcal{S}_{I,J,K} ( -K(1)J(2)K(1)\alpha_K + J(1)K(2)J(1)\alpha_J) \wedge \omega_I.
\]
This gives
\[ \frac{3}{2} (\nabla \Omega)_H = \mathcal{G}_{I,J,K} (J(1)K(2) - K(1)J(2)) (I(1)\alpha_I + J(1)\alpha_J + K(1)\alpha_K) \wedge \omega_I, \]
\[ \frac{3}{2} (\nabla \Omega)_{S^3H} = \mathcal{G}_{I,J,K} \{ J(1)K(2) (2J(1)\alpha_J - K(1)\alpha_K - I(1)\alpha_I) \\ - K(1)J(2) (2K(1)\alpha_K - I(1)\alpha_I - J(1)\alpha_J) \} \wedge \omega_I. \]

Both of these expressions have the form
\[ \Xi_\beta := \mathcal{G}_{I,J,K} (K(2)\beta_J - J(2)\beta_K) \wedge \omega_I, \]
with \( \beta_I \in T^*M \otimes \Lambda^2 E \subset T^*M \otimes S^2 T^*M. \)

Now, we compute
\[ \sum_{i=1}^{4n} \Xi_\beta(X, Y, Z, e_i, Ie_i) = (4n + 2)(\beta_J(X, KY, Z) - \beta_K(X, JY, Z)). \]

However, \( (2)\beta_J(X, \cdot, \cdot) \) is an element of \( \Lambda^2 T^*M \) that is of type \((1, 1)\), \( \{2, 0\} \) \( J \) and \( \{2, 0\} \) \( K \). So \( \Xi_\beta \) is uniquely determined by \( \beta_I, \beta_J \) and \( \beta_K \) and the result follows. \( \Box \)

Combining this Proposition with the remark after equation (5.1), we find that for \( V = \Lambda^3_0 E, K, E \) the \( VH \)-component of \( \nabla \Omega \) is uniquely determined by \( I(1)\alpha_I^{(V)} + J(1)\alpha_J^{(V)} + K(1)\alpha_K^{(V)} \), and the \( V S^3H \)-component is uniquely determined by \( I(1)\alpha_I^{(V)} - J(1)\alpha_J^{(V)} \) and \( J(1)\alpha_J^{(V)} - K(1)\alpha_K^{(V)} \). We can thus fully determined the quaternionic type of the manifold from information about the \( \alpha \)'s. As these are determined by the covariant derivatives \( \nabla \omega_I \), etc., we obtain the following relations between Hermitian and quaternionic types (see also the tables in §6).

**Theorem 5.4** Let \( M \) an almost quaternion-Hermitian manifold. On an open set \( U \) where an adapted basis \( I, J, K \) is defined, one has:

(i) If \( \nabla \omega_I \in \mathcal{W}_1 + \mathcal{W}_4 \) and \( \nabla \omega_J \in \mathcal{W}_2 + \mathcal{W}_4 \), or \( \nabla \omega_I, \nabla \omega_J, \nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_4 \), then \( \nabla \Omega \in E(S^3H + H) \).

(ii) If \( \nabla \omega_I, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4 \), then \( \nabla \Omega \in (\Lambda^3_0 E + E)(S^3H + H) + KH. \)
(iii) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4$, then $\nabla \Omega \in (K + E)(S^3 H + H) + \Lambda^3 E H$.

(iv) If $\nabla \omega_I \in \mathcal{W}_1 + \mathcal{W}_4$ and $\nabla \omega_J \in \mathcal{W}_1 + C + \mathcal{W}_4$, $C = \mathcal{W}_2, \mathcal{W}_3$, then $\nabla \Omega \in (\Lambda^3 E + E)(S^3 H + H)$.

(v) If $\nabla \omega_I \in \mathcal{W}_2 + \mathcal{W}_4$ and $\nabla \omega_J \in \mathcal{W}_2 + C + \mathcal{W}_4$, $C = \mathcal{W}_1, \mathcal{W}_3$, then $\nabla \Omega \in (K + E)(S^3 H + H)$.

(vi) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{W}_3 + \mathcal{W}_4$, then $\nabla \Omega \in (\Lambda^3 E + K + E)H$.

(vii) If $\nabla \omega_I \in \mathcal{W}_1$ and $\nabla \omega_J, \nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_3$, then $\nabla \Omega \in \Lambda^3 E (S^3 H + H)$.

(viii) If $\nabla \omega_I \in \mathcal{W}_2$ and $\nabla \omega_J, \nabla \omega_K \in \mathcal{W}_2 + \mathcal{W}_3$, then $\nabla \Omega \in K(S^3 H + H)$.

(ix) If $\nabla \omega_I, \nabla \omega_J, \nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_3$, then $\nabla \Omega \in \Lambda^3 E (S^3 H + H) + (K + E)H$.

(x) If $\nabla \omega_I, \nabla \omega_J, \nabla \omega_K \in \mathcal{W}_2 + \mathcal{W}_3$, then $\nabla \Omega \in K(S^3 H + H) + (\Lambda^3 E + E)H$.

(xi) If one of the following conditions holds, then $\nabla \Omega \in (\Lambda^3 E + K)(S^3 H + H)$:

(a) $\nabla \omega_I \in \mathcal{W}_1, \nabla \omega_J \in \mathcal{W}_2 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$,

(b) $\nabla \omega_I \in \mathcal{W}_2, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$, or

(c) $\nabla \omega_I \in \mathcal{W}_1 + \mathcal{W}_2, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_2 + \mathcal{W}_3$.

(xii) If $\nabla \omega_I \in \mathcal{W}_1 + \mathcal{W}_3, \nabla \omega_J \in \mathcal{W}_2 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$, then $\nabla \Omega \in (\Lambda^3 E + K)(S^3 H + H) + EH$.

Moreover, if $M$ is eight-dimensional, one also has

(xiii) If $\nabla \omega_I \in \mathcal{W}_1 + \mathcal{W}_3, \nabla \omega_J \in \mathcal{W}_2 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_2 + \mathcal{W}_4$, then $\nabla \Omega \in K(S^3 H + H) + ES^3 H$.

(xiv) If $\nabla \omega_I, \nabla \omega_J \in \mathcal{W}_1 + \mathcal{W}_3$ and $\nabla \omega_K \in \mathcal{W}_4$, then $\nabla \Omega \in ES^3 H$. □

6 Tables and Comments

Tables 2 and 3 show the full consequences of the formulæ derived in this paper.

Hermitian types are denoted by a hexadecimal number $0, \ldots, 9, A, \ldots, F$, where $W_i$ contributes $2^{i-1}$. So, for example $B = 1 + 2 + 8$ represents $W_1 + W_2 + W_4$. 
The rows of the table give the Hermitian type of $I$, the columns the type of $J$. Due to symmetry we only need to show the cases where the Hermitian type of $J$ is greater than or equal to that of $I$.

Each rectangle in the table contains up to 16 entries corresponding to the Hermitian types of $K$ that are greater than or equal to that of $J$. These are arranged with the type of $K$ increasing in each column, so the first column potentially begins with type 0, the next type 4, and then type 8 and finally type $C$.

In each position in this rectangle there is one of two types of entry. Three hexadecimal digits $abc$ indicate that the Hermitian types of $I$, $J$ and $K$ reduce to $a$, $b$ and $c$ respectively. Two bold digits $PQ$, indicate that the Hermitian types do not reduce and specify instead the quaternionic type of the manifold. $P$ corresponds to the $S^3H$-part of $\nabla\Omega$ and $Q$ to the $H$-part, with $\Lambda^3_0E$ contributing 4, $K$ 2 and $E$ 1. Thus 36 indicates type $(K+E)S^3H + (\Lambda^3_0E+K)H$. Note that the bottom right entry in each rectangle corresponds to $\omega_K$ having type $F = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4$; this is no restriction on $\omega_K$ and so this entry tells what happens when $I$ and $J$ have a specified type.

These results are for general dimension $4n \geq 12$. In dimension 8, several conclusions may be different. These are indicated by italicising the entry in Tables 2 and 3, provided the difference does not simply arise because of the absence of $\Lambda^3_0E$ in the quaternionic type. What actually happens in dimension 8 in these special cases is then given in Table 4. This table lists the $I,J,K$ type, its reduction and finally the quaternionic type. The entries different from the general case are again italicised.

One finds that in general dimension there are 167 different almost hyper-Hermitian types, whilst in dimension 8 there are only 144. In comparison, the number of potential triples of types is $\frac{1}{6}16.17.18 = 816$. Of these 816 cases, 276 are hyperKähler and 44 are locally conformal hyperKähler in general dimension. For dimension 8, one gets instead 316 and 44, respectively.

For completeness Table 5 gives the situation for dimension 4. In this case there are no $\alpha_I$ terms and the only Hermitian types are $\{0\}$, $\mathcal{W}_2$, $\mathcal{W}_4$ and $\mathcal{W}_2 + \mathcal{W}_4$. We do not specify quaternionic types in this table as these are no longer determined by $\nabla\Omega$ (the four-form $\Omega$ is a constant times the volume form, and so parallel). We see that there are only 7 distinct almost hyper-Hermitian types in this case.

It is natural to ask whether examples of each of the 167 different almost hyper-Hermitian types occur. With so many cases this is clearly a daunting task. However, one special case that is of interest is when $I$, $J$ and $K$ have
the same type. In this situation one may check that if a given component of $\nabla \omega_I$ vanishes then the same is true of $\nabla \omega_A$, where $A = aI + bJ + cK$, with $a^2 + b^2 + c^2 = 1$ constant. The table shows that the only possible Gray-Hervella types are $\{0\}$, $W_3 + C$ (with $C \subset W_1 + W_2 + W_4$), $W_4$ or $W_1 + W_2 + W_4$. With the exception of the last case, these may all be realised in dimension 12 by considering homogeneous structures, and conformal changes of such, on $(S^3)^4$, $T^3 \times M^3$, with $M^3$ a three-dimensional Lie group, either semi-simple, nilpotent or solvable. However, the given structures do not exhibit the full predicted almost quaternionic-Hermitian types, and we have not yet found examples of the last case $W_1 + W_2 + W_4$, which will be quaternionic type $E(S^3H + H)$. We therefore reserve presentation of such examples to future work.

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| IVF | F | E | D | C | B | A | 9 | 8 | 7 | 6 |
|-----|---|---|---|---|---|---|---|---|---|---|
| 0   | 33| 00| 00| 00| 00| 00| 00| 00| 00| 00|
| 77  | 0EE| 00| 00| 00| 00| 00| 00| 00| 00| 00|
|     | 0DD| 11| 0BB| 00| 00| 00| 00| 00| 00| 00|
| 1   | 55| 11C| 11C| 11C| 11C| 00| 00| 00| 00| 00|
| 77  | 0EE| 00| 00| 00| 00| 00| 00| 00| 00| 00|
|     | 1DD| 1IC| 0BB| 55| 00| 00| 00| 00| 00| 00|
| 77  | 1DD| 1IC| 0BB| 55| 00| 00| 00| 00| 00| 00|
| 2   | 00| 22C| 22C| 00| 00| 00| 00| 00| 00| 00|
| 77  | 2DD| 22C| 22C| 00| 00| 00| 00| 00| 00| 00|
| 3   | 00| 33C| 22C| 11C| 00| 00| 00| 00| 00| 00|
| 77  | 2DD| 22C| 22C| 00| 00| 00| 00| 00| 00| 00|
| 4   | 00| 444| 444| 000| 000| 000| 000| 000| 000| 000|
| 77  | 4EE| 444| 444| 000| 000| 000| 000| 000| 000| 000|
| 5   | 00| 55C| 444| 444| 444| 444| 444| 444| 444| 444|
| 77  | 5DD| 5DD| 5DD| 5DD| 5DD| 5DD| 5DD| 5DD| 5DD| 5DD|
| 6   | 00| 444| 22C| 22C| 000| 000| 000| 000| 000| 000|
| 77  | 6DD| 6DD| 6DD| 6DD| 6DD| 6DD| 6DD| 6DD| 6DD| 6DD|
| 7   | 00| 444| 444| 444| 444| 444| 444| 444| 444| 444|
| 77  | 7DD| 7DD| 7DD| 7DD| 7DD| 7DD| 7DD| 7DD| 7DD| 7DD|
| 8   | 00| 555| 888| 888| 888| 888| 888| 888| 888| 888|
| 77  | 8EE| 888| 888| 888| 888| 888| 888| 888| 888| 888|
| 9   | 00| 555| 99C| 99C| 99C| 99C| 99C| 99C| 99C| 99C|
| 77  | 9DD| 9DD| 9DD| 9DD| 9DD| 9DD| 9DD| 9DD| 9DD| 9DD|
| A   | 00| 888| 888| 888| 888| 888| 888| 888| 888| 888|
| 77  | AEE| AEE| AEE| AEE| AEE| AEE| AEE| AEE| AEE| AEE|
| B   | 00| 888| 888| 888| 888| 888| 888| 888| 888| 888|
| 77  | BDD| BDD| BDD| BDD| BDD| BDD| BDD| BDD| BDD| BDD|
| C   | 00| 888| 888| 888| 888| 888| 888| 888| 888| 888|
| 77  | CEE| CEE| CEE| CEE| CEE| CEE| CEE| CEE| CEE| CEE|
| D   | 00| 888| 888| 888| 888| 888| 888| 888| 888| 888|
| 77  | DDD| DDD| DDD| DDD| DDD| DDD| DDD| DDD| DDD| DDD|
| E   | 00| 888| 888| 888| 888| 888| 888| 888| 888| 888|
| 77  | EEE| EEE| EEE| EEE| EEE| EEE| EEE| EEE| EEE| EEE|
| F   | 00| 888| 888| 888| 888| 888| 888| 888| 888| 888|

Table 2: General dimensions, part 1
### Table 3: General dimensions, part 2

| $I^J_\{\}$ | 5       | 4       | 3       | 2       | 1       | 0       |
|------------|---------|---------|---------|---------|---------|---------|
| 0          | 000 000 | 000 000 | 000 000 | 000 000 | 000 000 | 000 000 | 000 000 |
| 1          | 000 11C | 114 000 | 114 000 | 000 000 | 000 000 | 000 000 | 000 000 |
| 2          | 000 000 | 224 000 | 224 000 | 000 000 | 000 000 | 000 000 | 000 000 |
| 3          | 000 11C | 77 000  | 77 000  | 000 000 | 000 000 | 000 000 | 000 000 |
| 4          | 000 11C | 444 000 | 444 000 | 000 000 | 000 000 | 000 000 | 000 000 |
| 5          | 000 11C | 444 000 | 444 000 | 000 000 | 000 000 | 000 000 | 000 000 |

### Table 4: Differences in dimension 8

| $I^J_\{\}$ | 5       | 4       | 3       | 2       | 1       | 0       |
|------------|---------|---------|---------|---------|---------|---------|
| 0          | 000 000 | 000 000 | 000 000 | 000 000 | 000 000 | 000 000 |
| 1          | 000 11C | 114 000 | 114 000 | 000 000 | 000 000 | 000 000 |
| 2          | 000 000 | 224 000 | 224 000 | 000 000 | 000 000 | 000 000 |
| 3          | 000 11C | 77 000  | 77 000  | 000 000 | 000 000 | 000 000 |
| 4          | 000 11C | 444 000 | 444 000 | 000 000 | 000 000 | 000 000 |
| 5          | 000 11C | 444 000 | 444 000 | 000 000 | 000 000 | 000 000 |

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Table 5: Dimension 4

| \( I^J \) | A | 8 | 2 | 0 |
|---|---|---|---|---|
| 0 | 0AA | 000 | 000 | \( \boxed{000} \) |
|   | 000 | 000 | 000 | \( \boxed{000} \) |
| 2 | 2AA | 000 | 228 | |
|   | 282 | 000 | 228 | |
| 8 | 8AA | 888 | 228 | |
|   | 888 | 888 | 228 | |
| A | AAA |  |  |  |

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(Martín Cabrera) Department of Fundamental Mathematics, University of La Laguna, 38200 La Laguna, Tenerife, Spain

*E-mail address: fmartin@ull.es*

(Swann) Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark

*E-mail address: swann@imada.sdu.dk*