Mixing for the primitive equations under bounded non-degenerate noise

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Abstract

We study the stochastic 3D primitive equations of the atmospheric mechanics. We consider them under a bounded and non-degenerate noise, which is statistically periodic in time with period 1. In such a case we prove that the associated integer-time Markov chain is exponentially mixing, which means that there exists a unique stationary measure to which the laws of all trajectories of this Markov chain converge exponentially fast.

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1 Introduction

In the space $\mathbb{R}^3$, provided with a system of coordinates $\zeta := (x, y, z)$, we consider the system of primitive equations of atmospheric mechanics (see Section 2.3 in [17]). This set of equations is of crucial importance for meteorology as it outlays the evolution of the cinematic fields of the atmosphere. Normally it describes the two-dimensional horizontal velocity field coupled with a temperature one. Here though, the latter will be omitted for simplicity. Our results remain true for the complete system, with a very similar but more cumbersome proof.

Thus cinematic field $u = (u_1, u_2, u_3)$ abides by the following system:

$$\frac{\partial u_k}{\partial t} - \Delta u_k + \sum_{j=1}^{3} \partial_j (u_j u_k) + \partial_k p = f_k, \quad k = 1, 2, \quad (1)$$

$$\text{div } u = \sum_{j=1}^{3} \partial_j u = 0, \quad (2)$$

where the pressure $p$ does not depend on $z$, $p = p(x, y, t)$. We supplement the equations with:

for $j = 1$ or $2$,
$$u_j(x + L, y, z, t) = u_j(x, y, z, t), \quad u_j(x, y, z + L, t) = u_j(x, y, z, t),$$
$$u_j(x, y, z + h, t) = u_j(x, y, z, t), \quad u_j(x, y, -z, t) = u_j(x, y, z, t).$$

Thus $u_1$ and $u_2$ are periodic in $x$, $y$ and even in $z$. Moreover,
$$u_3(x + L, y, z, t) = u_3(x, y, z, t), \quad u_3(x, y, z + L, t) = u_3(x, y, z, t),$$
$$u_3(x, y, z + h, t) = u_3(x, y, z, t), \quad u_3(x, y, -z, t) = -u_3(x, y, z, t),$$

i.e $u_3$ is periodic in $x$, $y$ and $z$ and odd in $z$. The goal of this work is to study some stochastic aspects of our system, in the case of a random, bounded, right-hand term, which is statistically periodic in time.

1.1 The reduced equation

We first simplify (1) noting that $u_3$ can be deduced from the other two components of the velocity field using (2). Indeed, since $u_3$ is odd, periodic, then
$$u_3(x, y, kh) = 0, \quad k \in \mathbb{Z}.$$

Thus, we may write that
$$u_3(\zeta, t) = - \int_{-h}^{t} \text{div}_2(u_1, u_2)(x, y, \alpha, t) d\alpha,$$

where $\text{div}_2(u) := \partial_x u_1 + \partial_y u_2$. So, denoting $v := (u_1, u_2)$ we get the equation for $v$
\[
\frac{dv}{dt} - \Delta v + (v \cdot \nabla_2)v - \left( \int_{-h}^{z} \text{div}_2 v(\cdot, \cdot, \xi, \cdot) d\xi \right) \frac{\partial v}{\partial z} + \nabla_2 p = f, \tag{3}
\]
with the following boundary conditions:

\begin{align}
&v(x + L, y, z, t) = v(x, y + L, z, t) = v(x, y, z + h, t) = v(x, y, z, t), \tag{4} \\
v(x, y, -z, t) = v(x, y, z, t). \tag{5}
\end{align}

Note that since \(v(x, y, z, t)\) satisfies (4), (5), we may consider it as an even in \(z\), non-autonomous vector field on the torus,

\[v : \mathbb{O} \times \mathbb{R}^+ \to \mathbb{R}^2, \quad \mathbb{O} = (\mathbb{R}/L\mathbb{Z})^2 \times (\mathbb{R}/h\mathbb{Z}).\]

Denote \(T^2_L = (\mathbb{R}/L\mathbb{Z})^2\) and \(T_h = \mathbb{R}/h\mathbb{Z}\) then \(\mathbb{O} = T^2_L \times T_h\). In addition, \(v\) meets an incompressibility condition derived from (2). Indeed by integrating the latter in \(z\) and adding the boundary conditions, setting

\(\bar{v} := \int_{-h}^{0} v(\cdot, \cdot, \xi, \cdot) d\xi\),

one gets

\[\text{div}_2 \bar{v} = 0. \tag{6}\]

We now add some initial conditions

\[v(\cdot, 0) = v_0, \tag{7}\]

with \(v_0\) displaying, in addition to (4), (5) and (6), the zero-meanvalue property:

\[\int_{\mathbb{O}} v_0 d\zeta = 0. \tag{8}\]

As we furthermore require that \(f\) follows

\[\left( \int_{\mathbb{O}} f d\zeta \right)(t) = 0, \quad \forall t \geq 0, \tag{9}\]

we get that the solution \(v\) of (3), also satisfies

\[\left( \int_{\mathbb{O}} v d\zeta \right)(t) = 0, \quad \forall t \geq 0. \tag{10}\]

Let us now briefly discuss some major milestones in the study of this problem: the groundwork was laid in [15] in a much more general and complicated physical setting, the domain being a manifold and the density of the fluid not being constant. In addition to defining the natural spaces for the solutions the authors introduce a variational setting of the problem and solve the problem, albeit in a weak sense. To reach a better regularity for solutions the main obstacle to overcome is the non-linear term and in particular its \(z\) component. A first
way to circumvent this difficulty, introduced in \[6\], is to write a solution of the primitive equations as a deviation from a solution of the associated Stokes system. Then the boundedness of our solution in the space $H^1$ can be derived using the regularity of the solution of the Stokes system. This approach allows one to prove, among other things, a local existence of a solution of the primitive equations where the time of existence depends on the norm of the solution of the Stokes system. Supplementing this idea with a small depth hypothesis the authors of \[8\] are able to prove the existence of a global solution. A major breakthrough was then achieved in \[1\] where a careful study of the $L^6$ a priori estimates implies the boundedness of the derivatives of a solution and thus proves the well-posedness of the problem. Further regularity results for the solutions were obtained first for the second order Sobolev norm in \[9\] and then for higher Sobolev norms in \[16\]. The notation and analytic considerations of our article follow that of \[1\] and \[16\]; the setting of the latter is close to ours.

Now let us set up the exact setting in which the equations will be considered and state the well-posedness of the corresponding deterministic problem.

First let us re-write (3) more compactly: defining

$$ b(u, v) = (u \cdot \nabla^2) v - \left( \int_{-h}^{z} \operatorname{div} u(\cdot, \cdot, \xi, \cdot) d\xi \right) \frac{\partial v}{\partial z}, $$

we may write the former as

$$ \frac{\partial v}{\partial t} - \Delta v + b(v, v) + \nabla^2 p = f. \tag{12} $$

For any $j \in \mathbb{Z}$ we denote by $H^j(\Omega)$ the Sobolev space of degree $j$ on $\Omega$ and

$$ H^j := \{ v \in (H^j(\Omega))^2 : \int_{\Omega} v d\zeta = 0, v \text{ is even in } z \}, $$

and provide it with the homogenous scalar product

$$ \langle u, v \rangle_j = \langle (\nabla)^j u, (\nabla)^j v \rangle. $$

Next for $j \in \mathbb{N} \cup \{0\}$ we set

$$ V^j := \{ w \in H^j : w \text{ satisfies } (11) \}. $$

We further note:

$$ H^0 \oplus \mathbb{R} \mathbb{1} := \{ v(t) + c, v \in H^0, c \in \mathbb{R} \}. $$

Moreover, we abbreviate $H^0$ by $H$ and $V^0$ by $V$.

For $j \in \mathbb{N} \cup 0$, let us define:

- $E_j([t_1, t_2]) := L^2([t_1, t_2], V^j)$
- $U_j([t_1, t_2]) := \{ u \in E_{j+1}([t_1, t_2]), \frac{\partial u}{\partial t} \in E_{j-1}([t_1, t_2]) \}$. Notice that $U_j([t_1, t_2]) \hookrightarrow C([t_1, t_2], V^j)$, see Theorem 3.1 in the first chapter of \[14\].

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\[ E_j(T) := E_j([0, T]) \quad \text{and} \quad U_j(T) := U_j([0, T]) \], and denote 
\[ ||f||_k := \sup_{t} ||f(t)||_V. \]

- lastly, for a Banach space \( E \) we denote by \( B_E(a, R) \) the open ball in \( E \) centred in \( a \) with the radius \( R \) and \( \overline{B}_E(a, R) \) the closed ball; we shall omit \( a \) when it be zero.

This done we may state, grounding ourselves on the results of [16], the existence and unicity of solutions of the primitive equations, which we define in the following way:

**Definition 1.** Let \( T > 0, \ m \geq 1 \) and \( v_0 \in V^m \). Then \( v \) is a strong solution of \( (3), (7) \) if \( v \in U_m(T) \) and \( \exists p \in L^2(\mathbb{R}^+, H^m(T^2_L)) \) such that \( (v, p) \) verifies \( (3) \) and \( (7) \).

Moreover, if \( v(\zeta, t), t \geq 0 \) is such that \( v(\zeta, t)|_{0 \leq t \leq T} \) is a solution of \( (3), (7) \) for every \( T > 0 \), then \( v \) is called a solution over \( \mathbb{R}^+ \).

To establish the well-posedness of the problem \( (3), (7) \) we refer to [16] (see also [2] for a similar result) Namely, Theorems 2.1 and 3.1 from [16] imply:

**Theorem 1.** Let \( m \geq 1, v_0 \in V^m \) and \( f \in L^\infty(\mathbb{R}^+, V^{m-1}) \). Then problem \( (3), (7) \) has a unique solution \( v \in C(\mathbb{R}^+, V^m) \cap L^2_{loc}(\mathbb{R}^+, V^{m+1}) \). Moreover, for any \( T > 0 \), the norm of \( v \) in \( C(\mathbb{R}^+, V^m) \cap L^2_{loc}(\mathbb{R}^+, V^{m+1}) \) depends only on \( T \), \( ||v_0||^m \) and \( ||f||_{m} \).

The deterministic problem thus settled we may now turn to the stochastic one.

Given the observation-induced uncertainties in the empirical parameters of the problem described by the primitive equations, it is natural to consider the its probabilistic side and study \( (3) \) and \( (7) \) with a stochastic force \( f \), and maybe a random initial data \( v_0 \). An important point in this setting is to see whether or not the system defined by \( (3) \) is mixing. It means that for any starting point \( v_0 \in V^m \), a solution \( v(t) \) of the problem \( (3), (7) \) weakly converges to a certain measure \( \mu \) in the space \( V^m \). That is, for every bounded and continuous functional \( g \) on \( V^m \) we have

\[ \mathbb{E}[f(v(t))] \to \int_{\Omega} f(u)\mu(du), \quad \text{as} \ t \to \infty. \]  

One important case, which was the subject of many studies in the past years, is having the white noise in time as a random force. First in [4], was established the trajectory-wise existence and uniqueness of a solution (strong in space and probabilities) when the force is white in time, multiplicative noise. Next in [7] the authors show the well-posedness and existence of trajectory-wise attractors for the solutions of the problem with a white, additive noise. The next step was a proof of the existence of an invariant measures for strong solutions of the problem under a white multiplicative noise, which was undertaken in [5]; however in such a case the unicity of the said measure is not proven.
However, for equations (3), (7) with white in time random force $f$, it still is not proven that the second moment of high Sobolev norms of solutions remains bounded, in the sense that

$$E[||v(t)||_m^2] < \infty, \quad \forall t > 0,$$

for some $m \geq 2$.

Without such a result it seems impossible to prove the uniqueness of a stationary measure and mixing.

To avoid this difficulty we study the primitive equations stirred by bounded random forces, statistically periodic in time given by non-degenerate random Haar series which we call red forces or red noise (see section 6). Thus by Theorem 1 the problem (3), (7) is well posed.

The main result of this work is the following proposition:

**Theorem A.** The stochastic process generated in $V^m$ by (3) stirred by red-noise $f$ as above, evaluated at integer moments of time define a Markov chain which is exponentially mixing in spaces $V^m$, for $m \geq 2$.

Note that for the related 2D Navier-Stokes system, the mixing is established for various classes of random forces, bounded and unbounded, see in [11] and see [10, 12]. For the 3D Navier-Stokes system, on the toroidal layer $T^2_L \times (0, \epsilon)$ (with suitable boundary conditions), in [3], the mixing is proved for bounded kick-forces, if $\epsilon$ is small in terms of the force. Finally, in [2], the primitive equations (3), (7) are studied for the case when $f$ is a bounded kick-force, and a sketch of the proof of mixing is given. The case of the kick-forces, treated in [2], is significantly easier than the class of coloured noises which we consider, and is less adequate for applications.

The first part of this work is devoted to stating some deterministic results about our system. In the second part we study its stochastic side, introducing the red noise before stating a theorem from [12] which enables us to prove that our system is exponentially mixing.

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**2 Preliminary results**

Now let us introduce some useful results for the calculations, lying ahead.
Formula 1. For any \( v \in V^1 \) and \( p \in H^1(T_L^2) \)

\[
\int_{\Omega} v \cdot \nabla_2 p = \int_{\Omega} v(x, y, z) \cdot \nabla_2 p(x, y) dx dy dz = \int_{\mathbb{T}_L^2} \tilde{v}(x, y) \cdot \nabla_2 p(x, y) dx dy = \int_{\mathbb{T}_L^2} \text{div}_2(\tilde{v})(x, y) p(x, y) dx dy = 0.
\]

Moreover,

**Lemma 1.** We have the following orthogonal decomposition:

\[
H = V \oplus \nabla_2 H^1(T_L^2),
\]

where \( \nabla_2 H^1(T_L^2) := \{ \nabla_2 p, p \in H^1(T_L^2) \} \).

**Proof.** Clearly \( \nabla_2 H^1(T_L^2) \) belongs to \( H \). Now let us consider the complex base of \( H^1(T_L^2) \) made of vectors \( e^{2i\pi/L(mx+ny)} \), for \( m \) and \( n \) in \( \mathbb{Z} \). Then \( \nabla_2 H^1(T_L^2) \) is spanned by

\[
\Upsilon_{m,n} := \left( \frac{m}{n} \right) e^{(2i\pi/L)(mx+ny)}, \quad \forall m, n \in \mathbb{Z}, \quad |m| + |n| \neq 0.
\]

In view of Formula 1, the spaces \( V \) and \( \nabla_2 H^1(T_L^2) \) are orthogonal. So it remains to show that each vector, orthogonal to \( \nabla_2 H^1(T_L^2) \), belongs to \( V \). The space \( H \) admits the orthogonal complex basis \( \{ \Lambda_{m,n}^{k,\pm} : m, n \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\} \} \), where

\[
\Lambda_{m,n}^{k,+} = \left( \frac{m}{n} \right) e^{(2i\pi/L)(mx+ny)} \cos(kz),
\]

\[
\Lambda_{m,n}^{k,-} = \left( \frac{-n}{m} \right) e^{(2i\pi/L)(mx+ny)} \cos(kz).
\]

Decomposing any \( v \in H \), such that \( v \perp \nabla_2 H^1(T_L^2) \) in this basis as

\[
v = \sum_{m,n,k} \left( v_{m,n}^{k,+} \Lambda_{m,n}^{k,+} + v_{m,n}^{k,-} \Lambda_{m,n}^{k,-} \right),
\]

we see that

\[
0 = \langle v, \Upsilon_{m,n} \rangle = hL(m^2 + n^2) v_{m,n}^{0,+} \delta_{k,0}.
\]

Thus

\[
v = \sum_{m,n,k \neq 0} v_{m,n}^{k,+} \Lambda_{m,n}^{k,+} + \sum_{m,n} v_{m,n}^{0,-} \Lambda_{m,n}^{0,-}.
\]

The function \( v \) satisfies (5) and (10) since it belongs to \( H \) and it satisfies (6) since the functions \( \Lambda_{m,n}^{k,-} \) and \( \Lambda_{m,n}^{0,\pm} \) obviously do. Therefore \( v \in V \).

We note that the basis \( \{ \Lambda_{m,n}^{k,\pm} \} \) is an orthogonal basis of space any \( H^1 \) and we see that the projection \( \mathcal{P} \) has the form

\[
\mathcal{P} : \sum_{m,n,k} \left( v_{m,n}^{k,+} \Lambda_{m,n}^{k,+} + v_{m,n}^{k,-} \Lambda_{m,n}^{k,-} \right) \rightarrow \sum_{m,n,k \neq 0} v_{m,n}^{k,+} \Lambda_{m,n}^{k,+} + \sum_{m,n} v_{m,n}^{0,-} \Lambda_{m,n}^{0,-}.
\]
Accordingly, the collection of functions \( \{ \Lambda_{m,n}^{\pm} \mid m, n \in \mathbb{Z}, k \in \mathbb{N} \} \cup \{ \Lambda_0^{m,n} \mid m, n \in \mathbb{Z} \} \) is an orthogonal basis of each space \( V_j \). We denote by \( (e_n)_{n \in \mathbb{N}} \) the orthonormal basis of \( V \), obtained by the normalization of \( \{ \Lambda_{m,n}^{\pm} \mid m, n \in \mathbb{Z}, k \in \mathbb{N} \} \cup \{ \Lambda_0^{m,n} \mid m, n \in \mathbb{Z} \} \) with respect to the \( L^2 \)-norm. The functions \( \Lambda_{m,n}^{\pm} \) are eigenvectors for the operator \( \Delta \). Indeed

\[
-\Delta \Lambda_{m,n}^{\pm} = \left( \frac{(2\pi/L)^2}{2}(m^2 + n^2) + h^2k^2 \right) \Lambda_{m,n}^{\pm}.
\]

From this fact making use of the form (14) of the operator \( \mathcal{P} \), we immediately get:

**Lemma 2.** For any \( r \in \mathbb{N} \) the operator \( \mathcal{P} : H^r \oplus \mathbb{R} \mapsto V^r \) is an orthogonal projection.

Thanks to the eigenvalues of \( -\Delta \) associated with the orthogonal basis \( \Lambda_{m,n}^{\pm} \) being all positive we get the Poincaré inequality.

**Lemma 3.** There exists a \( C > 0 \) such that for any \( v \in V_1 \),

\[
\|v\|_{L^2} \leq C\|\nabla v\|_{L^2}.
\]

Now, considering the non-linear term of equation (3) we get,

**Lemma 4.** For any \( w \in V, v \in (H^2(\Omega))^2 \)

\[
\langle b(w, v), v \rangle = 0.
\]

**Proof.** Indeed, \( \langle (w \cdot \nabla_2)v, v \rangle = -\frac{1}{2}\langle \text{div}_2(w)v, v \rangle \) and

\[
\langle \left( \int_{-h}^{z} \text{div}_2 w(\cdot, \cdot, \xi) d\xi \right) \partial_z v, v \rangle = -\frac{1}{2}(\text{div}_2(w)v, v).
\]

The first formula is obtained through integration by part (over the two horizontal variables only):

\[
\int_{T_L^z} ((w \cdot \nabla_2)v)v = -\int_{T_L^z} ((w \cdot \nabla_2)v)v - \int_{T_L^z} \text{div}_2(w)v^2.
\]

As for the second it is yielded by the integration by parts over \( z \) compounded with \( \int_{-h}^{z} \text{div}_2(w) \big|_{z=0} = 0 \). Hence the result.

Moreover, we have the following result

**Lemma 5.** Setting \( u = (u_1, u_2), v = (v_1, v_2) \in (L^2(\Omega))^2 \), we may write \( \nabla b(u, v) \) as a combination of terms.

\[
D^k u_j D^{2-k} v_j', \quad \left( D^k \int_{-h}^{z} \text{div}_2 u_j(\cdot, \cdot, \xi) d\xi \right) D^{1-k} \frac{\partial v_j'}{\partial \theta}, \quad k = 0, 1, \quad j, j' = 1, 2,
\]

where \( D^l \) is any monomial term of degree \( l \) made from the spatial derivatives \( (\partial_x, \partial_y, \partial_z) \). Likewise, \( \Delta b(u, v) \) may be out-laid as a sum of terms

\[
D^k u_j D^{3-k} v_j', \quad \left( D^k \int_{-h}^{z} \text{div}_2 u_j(\cdot, \cdot, \xi) d\xi \right) D^{2-k} \frac{\partial v_j'}{\partial z}, \quad k = 0, 1, 2, \quad j, j' = 1, 2.
\]
Lemma 6. We have the following estimates:

\[ \|b(u, v)\|_{L^2} \leq C \min(\|u\|_{V^1}, \|u\|_{V^1}, \|v\|_{V^1}, \|v\|_{V^1}), \]

\[ \|\nabla b(u, v)\|_{L^2} \leq C \min(\|u\|_{V^1} \|v\|_{V^2}, \|u\|_{V^1} \|v\|_{V^2}). \]

Proof. For the first one, it is enough to write

\[ \| (u \cdot \nabla_2) v \|_{L^2} \leq \| u \|_{L^\infty} \| \nabla_2 v \|_{L^2} \leq \| u \|_{V^2} \| v \|_{V^1}, \]

or

\[ \| u \|_{L^6} \| \nabla_2 v \|_{L^3} \leq \| u \|_{V^1} \| v \|_{V^2}, \]

Thus

\[ \left\| \int_{-h}^z \nabla_2 u \frac{\partial v}{\partial z} \right\|_{L^2} \leq \| \nabla_2 u \|_{L^\infty} \left\| \frac{\partial v}{\partial z} \right\|_{L^2} \leq \| u \|_{V^1} \| v \|_{V^1}. \]

or

\[ \| \nabla_2 u \|_{L^2} \left\| \frac{\partial v}{\partial z} \right\|_{L^\infty} \leq \| u \|_{V^1} \| v \|_{V^1}. \]

For the second, by virtue of lemma 5 we only need to prove the following bounds

\[ \| D^k u D^{2-k} v \|_{L^2} \leq C \| u \|_{L^\infty} \| D^2 v \|_{L^2} \leq C \| u \|_{V^2} \| v \|_{V^2} \quad k = 0, \]

\[ C \| D u \|_{L^6} \| D v \|_{L^3} \leq C \| u \|_{V^2} \| v \|_{V^2} \quad k = 1. \]

\[ \left\| D^k \int_{-h}^z \nabla_2 u D^{1-k} \frac{\partial v}{\partial z} \right\|_{L^2} \leq C \| \nabla_2 u \|_{L^\infty} \left\| D \frac{\partial v}{\partial z} \right\|_{L^2} \leq C \| u \|_{V^1} \| v \|_{V^1} \quad k = 0, \]

or

\[ C \| \nabla_2 u \|_{L^6} \left\| D \frac{\partial v}{\partial z} \right\|_{L^3} \leq C \| u \|_{V^2} \| v \|_{V^1} \quad k = 0, \]

\[ C \| D \nabla_2 u \|_{L^6} \left\| \frac{\partial v}{\partial z} \right\|_{L^3} \leq C \| u \|_{V^2} \| v \|_{V^2} \quad k = 1, \]

or

\[ C \| D \nabla_2 u \|_{L^2} \left\| \frac{\partial v}{\partial z} \right\|_{L^\infty} \leq C \| u \|_{V^2} \| v \|_{V^1} \quad k = 1. \]

\[ \square \]

Lemmas 1 and 6 allow us to rewrite (12) in a new form, seemingly weaker but actually equally strong. Let \( u, v \in C([0, T], V^2) \cap E_3 \), then thanks to Lemma 6, \( b(u, v) \in L^2 \left( [0, T], L^2(\mathbb{O}) \right)^2 \). Moreover, \( u \) and \( v \) being even in \( z \), so is \( b(u, v) \). Thus \( b(u, v) \in H \oplus \mathbb{R} \) and thanks to Lemma 2 applied to \( r = 0 \), we define

\[ B(u, v) := \nabla b(u, v), \quad u, v \in C([0, T], V^2) \cap E_3. \]

Then for \( m \geq 2 \), \( f \in E_{m-1} \) and \( v_0 \in V^m \), (39), (7) is equivalent to:

\[ \frac{\partial v}{\partial t} = \Delta v + B(v, v) = f. \quad (16) \]

supplemented with (7). Indeed we may consider \( v \) a solution of (39) and apply \( \Psi \) to (39). Then we use Formula 1 to prove that \( \nabla_2 p = 0 \) and (14) to show that for \( v \in V^r \), \( \nabla \Delta v = \Delta v \). We thus get (15). Now let \( v \in U_m(T) \) be a solution of (16), we reconstruct \( p \) thanks to the relation

\[ \nabla_2 p = b(v, v) - B(v, v), \]
and then, \((v, p)\) is a solution of (3). Moreover using Lemma 2 we have the following corollary to Lemma 6

**Corollary 1.** We have the following estimates:

\[
\|B(u, v)\|_{L^2} \leq \|b(u, v)\|_{L^2} \leq C \min(\|u\|_{V^1}, \|v\|_{V^1}, \|u\|_{V^2}, \|v\|_{V^2}),
\]

\[
\|B(u, v)\|_{V^1} \leq \|\nabla b(u, v)\|_{L^2} \leq C \min(\|u\|_{V^2}, \|v\|_{V^2}, \|u\|_{V^3}, \|v\|_{V^3}).
\]

Besides

**Lemma 7.** The bilinear component \(B\) abides by the estimates

\[
\|B(u, v)\|_{V^m} \leq C \|u\|_{V^{m+1}} \|v\|_{V^{m+1}}, \quad m \geq 2
\]

**Proof.** For \(m \geq 2\), the space \(H^m\) is an algebra; thus, firstly

\[
\|(u \cdot \nabla^2 v)\|_{H^m} \leq C \|u\|_{V^m} \|Dv\|_{H^m} \leq C \|u\|_{V^m} \|v\|_{V^m+1}
\]

secondly

\[
\left\| \int_{-h}^t \nabla^2 u \frac{\partial v}{\partial z} \right\|_{H^m} \leq C \|Du\|_{H^m} \|Dv\|_{V^m} \leq C \|u\|_{V^{m+1}} \|v\|_{V^{m+1}}.
\]

Thus by virtue of (16), an immediate corollary to Theorem 1 is the following:

**Corollary 2.** For \(m \geq 2\), under the assumptions of Theorem 1 we further state that the solution \(v\) belongs to the space \(U_m(T)\) for all \(T > 0\).

### 2.1 Absorbing set for the system

First here we shall state the existence of an absorbing set for our system as written in (3). This result, one proves by using the a priori estimates, which through Gronwall’s lemma yield long term boundedness of the solutions of (3) (see 2 for more details for the case of a constant force):

**Theorem 2.** For an integer \(m \geq 1\), suppose that there is a constant \(C^*\) such that

\[
\|f(t)\|_{V^{m-1}} \leq C^*, \quad \forall t \geq 0
\]

then there exists \(R = R(C^*)\) such that for all \(M > 0\), there exist \(T = T(M, C^*)\) and \(K = K(M, C^*) \geq 2R\) for which a solution of (3) with initial value \(\|v_0\|_{V^m} \leq M\) verifies:

\[
\begin{aligned}
\|v(t)\|_{V^m} &\leq K, \quad t \geq 0, \\
\|v(t)\|_{V^m} &\leq R, \quad t \geq T
\end{aligned}
\]

Without loss of generality in the following we assume that \(T \geq 1\) and \(R \geq 1\). Now let us consider the regularity of the resolving operator for the problem (3), (7).
3 Analyticity of the resolving operator

Let us define the following sets and operators:

- For $T \leq \infty$, \( \mathcal{E}_j(T) := \bar{B}_{L^\infty([0,T],V_j)} \) for \( j \in N \cup \{0\} \), where \( C^* \) is the same as in Theorem 2.
- For $0 < T < \infty$ and some \( \epsilon > 0 \) which we shall fix later, we set \( \hat{E}_j(T, \epsilon) := E_j(T) + B_{E_j(T)}(\epsilon) \subset E_j(T) \).
- \( \mathcal{L} \) is the operator describing equation (16), (7):
  \[ \mathcal{L} : v \mapsto (\mathcal{L}^1 v := v(0), \mathcal{L}^2 v := \frac{\partial v}{\partial t} - \Delta v + B(v,v)) \]
- \( \text{Sol} : (u_0, \eta) \mapsto u \) where \( u \) is a solution of (16), (7), that is \( \text{Sol} \) is the inverse of \( \mathcal{L} \),
- \( d\mathcal{L} \) the linearisation of \( \mathcal{L} \):
  \[ d\mathcal{L}(v) : w \mapsto (d\mathcal{L}^1 w := w(0), d\mathcal{L}^2 w := \frac{\partial w}{\partial t} - \Delta w + B(v,w) + B(w,v)) \]

Now, by virtue of Theorem 1 for all \( m \geq 1, M > 0, 0 \leq T' \leq T \) and \( (v'_0, f') \in B_{V^m}(M) \times \mathcal{E}_{m-1}(T') \), \( \text{Sol}(v'_0, f')(t) \in V^m \) is defined for \( t \in [0,T'] \). In this setting, we have the following result

**Theorem 3.** For any \( T > 0, M > 0 \) and an integer \( m \geq 2 \), there exists \( \rho(M,T, m) > 0 \) such that for any \( T' \in [\min(1,T/2),T] \), the mapping

\[ \text{Sol} : B_{V^m}(M) \times \mathcal{E}_{m-1}(T', \rho) \mapsto U_m(T'), \tag{17} \]

is well defined and analytical. Besides it is Lipschitz with a Lipschitz constant \( L \) which depends only on \( M, T \) and \( m \).

3.1 Properties of operator \( \mathcal{L} \)

First let us establish the set on which operator \( \mathcal{L} \) is well-defined. To this end we begin by stating a corollary to Lemmas 1 and 7.

**Corollary 3.** We have

\[ ||B(u,v)||_{E_1} \leq C ||u||_{U_2} ||v||_{U_2}, \]

and for \( m \geq 2 \),

\[ ||B(u,v)||_{E_m} \leq C \min( ||u||_{U_{m+1}}, ||v||_{U_{m+1}}, ||v||_{U_{m+1}}, ||u||_{U_m}). \]
Proof. The first inequality immediately follows from Corollary 1
\[
\int_0^T ||B(u,v)||_{V^1}^2 \leq \int_0^T C||u||_{V^2}^2||v||_{V^3}^2 \leq C||u||_{C([0,T],V^2)}^2||v||_{E^3}^2 \leq C||u||_{L^2}^2||v||_{L^2}^2.
\]
To get the second one, we proceed likewise, this time starting from Lemma 7.

From this corollary we derive the following

Lemma 8. The operator
\[
\mathcal{L} : U_m \to V^m \times E_{m-1}, \quad m \geq 2.
\]
is analytical.

Proof. We analyze \(\mathcal{L}\) term-wise. By virtue of Corollary 3, we have that
\[
\int_0^T ||B(u,u)||_{V^m-1}^2 \leq ||u||_{U^m}^4.
\]
Thus \(B\) is an homogeneous operator of rank 2, continuous and thus analytic.

Another term of \(\mathcal{L}^2\), the operator
\[
u \mapsto \frac{du}{dt} - \Delta u, \quad U_m \to E_{m-1},
\]
obviously is linear continuous. Finally, the mapping \(\mathcal{L}^1\) is linear and continuous since \(U_m \subset C([0,T],V^m)\). Therefore, \(\mathcal{L}\) is analytical.

Then let us state an injection property for \(\mathcal{L}\).

Lemma 9. Let \(m \geq 2\), \(T > 0\) and \(u_1, u_2 \in U_m(T)\) such that \(\mathcal{L}(u_1) = \mathcal{L}(u_2)\). Then \(u_1(t) = u_2(t)\) for any \(t \in [0,T]\).

Proof. If \(\mathcal{L}(u_1) = \mathcal{L}(u_2)\), then \(\tilde{u} = u_1 - u_2\) is a solution of
\[
\frac{d\tilde{u}}{dt} - \Delta \tilde{u} + B(u_1, \tilde{u}) + B(\tilde{u}, u_2) = 0, \quad \tilde{u}_0 = 0.
\]

Thus applying to the equation the scalar product with \(\tilde{u}\), we get, thanks to Lemma 4 and Corollary 1
\[
\frac{d||\tilde{u}||_{L^2}^2}{dt} + ||\nabla \tilde{u}||_{L^2}^2 \leq C||\tilde{u}||_{V^1}||u_2||_{V^3}||\tilde{u}||_{L^2}.
\]
Therefore, by virtue of Young’s inequality, \(\frac{d||\tilde{u}||_{L^2}^2}{dt} \leq C||u_2||_{V^3}^2||\tilde{u}||_{L^2}^2\). Whence
\[
||\tilde{u}(t)||_{L^2}^2 \leq ||\tilde{u}_0||_{L^2}^2 e^K,
\]
so that \(\tilde{u}(t) = 0\) for any \(t \in [0,T]\).
Now let us examine the non-degeneracy of \( \mathcal{L} \).

**Lemma 10.** For any \( T > 0, u \in U_2(T) \) and \( r = 0, 1, 2 \), the mapping \( d\mathcal{L}(u) \) is an isomorphism between \( U_r(T) \) and \( V^r \times E_{r-1}(T) \), whose inverse has a norm depending solely on \( ||u||_{U_r} \).

**Proof.** To prove the existence of a solution to the linearised equation, i.e an inverse to \( d\mathcal{L} \), and the boundedness of this inverse, let us start with preliminary estimates on the linearised equation. In what follows, for \( r = 0, 1 \) we shall content ourselves with stating the a priori estimates, proving the full result for \( r = 2 \) alone.

For the following considerations, we take any \((v_0, f) \in V^r \times E_{r-1}\) and denote \( v = \text{Sol}(v_0, f) \), i.e. \( v \) is a solution of

\[
\frac{dv}{dt} - \Delta v + B(u, v) + B(v, u) = f, \quad (18)
\]
\[
v(0) = v_0. \quad (19)
\]

**A priori estimates for the case** \( r = 0 \): applying to (18) the scalar product with \( v \) in \( L^2 \) and using Lemma 4 and Corollary 1 and Cauchy-Schwartz inequality, we get

\[
\frac{1}{2} \frac{d||v||^2}{dt} + ||\nabla v||_{L^2}^2 \leq ||f||_{V^{-1}} ||v||_{V^1} + ||B(v, u)||_{L^2} ||v||_{L^2}.
\]

Henceforth we shall replace all constants by polyvalents \( C \) and \( C' \). Therefore

\[
\frac{d||v||^2}{dt} + C||\nabla v||_{L^2}^2 \leq C'(||f||_{V^{-1}} ||v||_{V^1} + ||u||_{V^3} ||v||_{V^1} ||v||_{L^2}).
\]

Thus, applying once more Young’s inequality to the last right-hand term, we get

\[
\frac{d||v||^2}{dt} + C||\nabla v||_{L^2}^2 \leq C'(||f||_{V^{-1}}^2 + ||u||_{V^3}^2 ||v||_{L^2}^2).
\]

Therefore, for any \( 0 \leq t \leq T \),

\[
||v||_{L^2}^2(t) \leq (||v(0)||_{L^2} + C'||f||_{E_{r-1}}^2) e^{C't} ||u||_{V^3}^2, \]

and thus

\[
||v||_{L^2}^2(t) \leq (||v(0)||_{L^2} + C'||f||_{E_{r-1}}^2) \chi(||u||_{U_r}), \quad \text{where } \chi(||u||_{U_r}) := e^{C't} ||u||_{U_r}^2. \quad (20)
\]

Moreover

\[
\int_0^t ||\nabla v||_{L^2}^2 \leq \frac{1}{C} \left( ||v(0)||_{L^2} + C' \int_0^t ||f||_{E_{r-1}}^2(s) + ||u||_{V^3}^2(s) ||v||_{L^2}^2(s) \right) ds
\]
\[
\leq \frac{1}{C} \left( ||v(0)||_{L^2} + C'||f||_{E_{r-1}}^2 + \int_0^t ||u||_{V^3}^2(s) ||v||_{E_r}^2(s) ds \right) ,
\]

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thus by virtue of (20), we get

$$\int_0^t ||\nabla v||^2_{L^2} \leq \left(||v(0)||_{L^2} + C'||f||^2_{E_{-1}}\right) \chi'(||u||_{U_2})$$

with \(\chi'(||u||_{U_2}) := \frac{1}{C'} \left(1 + ||u||^2_{E^2} e^{C'||u||_{U_2}}\right)\). (21)

A priori estimates for the case \(r = 1\): applying to (18) the scalar product with \(-\Delta v\), Cauchy-Schwarz and Young’s inequality to the first right-hand term as well as Corollary \(1\) and using once more Young’s inequality on the second right-hand term, we get

$$\frac{d||\nabla v||^2_{L^2}}{dt} + C||\Delta v||^2_{L^2} \leq C'||f||^2_{E_0} + ||v(0)||^2_{V_1}) \chi(||u||_{U_2}).$$

Therefore in a similar as for \(r = 1\),

$$||\nabla v||^2_{L^2}(t) \leq (C'||f||^2_{E_0} + ||v(0)||^2_{V_1}) \chi(||u||_{U_2}).$$

and

$$\int_0^t ||\Delta v||^2_{L^2} \leq (||v(0)||^2_{V_1} + C'||f||^2_{E_0}) \chi'(||u||_{U_2}).$$

Complete proof for the case \(r = 2\): now let’s apply to (18) the scalar product with \(-\Delta^2 v\), yielding

$$\frac{1}{2} \frac{d||\Delta v||^2_{L^2}}{dt} + ||\nabla \Delta v||^2_{L^2} \leq ||\nabla f||^2_{V_2} + ||B(u, v) + B(v, u)||^2_{V_2}.$$

Thus by reusing Corollary \(1\) Cauchy-Schwarz and Young’s inequalities, we get

$$\frac{d||\Delta v||^2_{L^2}}{dt} + C||\nabla \Delta v||^2_{L^2} \leq C'||f||^2_{V_1} + ||u||^2_{V_2} ||v||^2_{V_2}.$$ (24)

Therefore, proceeding as we did for \(r = 0\), we get

$$||\Delta v||^2_{L^2}(t) \leq (C'||f||^2_{V_1} + ||v(0)||^2_{V_2}) \chi(||u||_{U_2}).$$

and

$$\int_0^t ||\nabla \Delta v||^2_{L^2} \leq (||v(0)||^2_{V_2} + C'||f||^2_{V_1}) \chi'(||u||_{U_2}).$$ (26)

Thanks to all those estimates we may state (through Galerkin’s approximations) the existence of a solution for the problem (18), (19) and that it belongs to the space \(C([0, T], V^2) \cap E_3(T)\). Since the equation is linear, the estimates also implies the uniqueness of a solution. By virtue of Corollary \(3\) we have

$$B(u, v) + B(v, u) \in E_1(T).$$

Now, since \(\Delta v \in L^2([0, T], V^1)\), then in view of (18), we have \(\frac{\partial u}{\partial t} \in L^2([0, T], V^1)\). Therefore \(v \in U_2(T)\). Thus the existence of an inverse operator is proven. Furthermore, we get from this argument that the norm of this operator is bounded by a value depending only on \(||u||_{U_2}\). Thus our assertion is proved. \(\square\)
Naturally, a better smoothness of the curve \( u(t) \) implies that the mapping \( d\mathcal{L}(u) \) operates in smoother spaces:

**Lemma 11.** For \( m \geq 3 \) and \( u \in U_{m-1} \), the operator \( d\mathcal{L}(u) \) is an isomorphism between \( U_m \) and \( V^m \times E_{m-1} \). Its inverse has a norm depending solely on \( ||u||_{U_{m-1}} \).

**Proof.** We start by improving the regularity of the solution \( v \) constructed in Lemma 10.

Applying to (18) the scalar product with \((-\Delta)^m v\), we get the following estimate

\[
\frac{1}{2} \frac{d||\nabla^m v||^2_{L^2}}{dt} + ||\nabla^{m+1} v||^2_{L^2} \leq ||B(u, v) + B(v, u)\Delta^m v|| + ||f, \Delta^m v||.
\]

Thus,

\[
\frac{d||\nabla^m v||^2_{L^2}}{dt} + C||\nabla^{m+1} v||^2_{L^2} \leq ||B(u, v) + B(v, u)||_{V^{m-1}}||\nabla^{m+1} v||_{L^2}
\]

\[+ ||f||_{V^{m-1}}||\nabla^{m+1} v||_{L^2}.
\]

Therefore, by virtue of Lemma 7, we get

\[
\frac{d||\nabla^m v||^2_{L^2}}{dt} + C||\nabla^{m+1} v||^2_{L^2} \leq C'||u||_{V^m}||\nabla^{m+1} v||_{L^2}^2 + ||f||_{V^{m-1}}||\nabla^{m+1} v||_{L^2}^2,
\]

and applying Young’s inequality to the last right-hand term,

\[
\frac{d||\nabla^m v||^2_{L^2}}{dt} + C||\nabla^{m+1} v||^2_{L^2} \leq C'||u||_{V^m}||\nabla^{m+1} v||_{L^2}^2 + C''||f||_{V^{m-1}}^2.
\]

Now, Gronwall’s lemma allows us to conclude that for \( u \in U_{m-1} \), the solution \( v \in C([0, T], V^m) \cap L^2([0, T], V^{m+1}) \) exists and that the norm of the inversion depends on \( ||u||_{L^2([0, T], V^m)} \), thus on \( ||u||_{U_{m-1}} \) alone. Finally Corollary 8 allows us to conclude that \( v \in U_m(T) \).

We now may prove the existence and analyticity of operator \( \text{Sol} \):

**Proof of Theorem 8** The assertion follows from the real-analytic part of Theorem 8 where \( f = \mathcal{L} \) and \( E = U_m(T') \), \( F = V_m \times E_{m-1}(T') \), \( Z = B_{V^m}(M) \times E_{m-1}(T') \). Indeed, by Lemma 8 the mapping \( \mathcal{L} \) is analytic and by Lemma 9 it is injective. By the last assertion of Theorem 11 for any \((u_0, f) \in Z\) a solution \( \mathcal{L}^{-1}(u_0, f) =: v \) exists, and \( ||u||_{U^m} \leq R(T', C^*, m) \). Let us choose \( U = B_{U^m}(R + 1) \).

Then a) in Theorem 8 holds with \( r = 1 \). Estimate b) follows from Corollary 8 (which is valid both for complex and real analytic functions \( u(t, x) \)), and estimate c) follows from the same Corollary. Finally d) is the assertion of Lemmas 10 11. Now application of Theorem 8 implies the existence and analyticity of the mapping (17). Its Lipschitz constant is bounded by

\[
\sup_{||u_0||_{V^m} \leq M, f \in E_{m-1}(T', \rho)} ||d\text{Sol}(u_0, f)||_{V^m \times E_{m-1} \to U_m} \leq L(M, T, m).
\]
4 Study of the generated discrete-time process

Now let us take for $f$ in (3) a function, defined piecewise in time:

$$f := \sum_{k=0}^{\infty} 1_{[k,k+1]}(t)\eta_k(t-k),$$

and such that

$$f \in \mathcal{E}_m(\infty), \text{ for some } m \geq 2$$

By Theorem 1 and Corollary 2 the primitive equations (3),(7) have a unique solution $v \in U_{m}(T)$ for all $T > 0$ for any $v_0 \in V^m$. We calculate the values of the solution $v(t)$ at integer moments of time,

$$v_k = v(k), \quad v_k \in V^m,$$

and define its transition operator as

$$S : (v_n, \eta_n) \mapsto v_{n+1}, \quad S : V_m \times \mathcal{E}_{m-1}(1) \mapsto V_m.$$  \hspace{1cm} (27)

Let us now study the discrete time process obtained by iterating the applications of this operator.

4.1 Invariant subset for $S$ and its analyticity

Let us fix $m \geq 2$. Using the notation of Theorem 2 we choose $M = 2R \geq 2$ and find the corresponding $T(M) \geq 1$ and $K = K(M) \geq 2R$. Let us apply the last assertion of Theorem 3. We may set some $\epsilon(M, 2T)$ such that for $||u_0||_{V^m} \leq 2R$ and any $f' \in \hat{E}_{m-1}(2T, \epsilon(M, 2T))$, so that

$$||f - f'||_{E_{m-1}(T')} \leq \epsilon(M, 2T), \quad f \in \mathcal{E}_{m-1}(2T),$$

we have that

$$||\text{Sol}(u_0, f')(t) - \text{Sol}(u_0, f)(t)||_{V^m} \leq \epsilon(M, 2T)L, \quad 1 \leq t \leq 2T.$$  \hspace{1cm} (28)

Let us fix $\epsilon(M, 2T)$ so small $\epsilon L \leq 1$. Then by virtue of Theorem 2 for any $f' \in \hat{E}_{m-1}(2T, \epsilon(M, 2T))$ we have

$$\begin{aligned}
||\text{Sol}(u_0, f')(t)||_{V^m} &\leq K + 1, \quad 1 \leq t \leq T, \\
||\text{Sol}(u_0, f')(t)||_{V^m} &\leq R + 1, \quad T \leq t \leq 2T.
\end{aligned}$$  \hspace{1cm} (29)

From now on we set $\epsilon = \epsilon(R, M, 2T)$, $\hat{E}_{m-1}(2T) = \hat{E}_{m-1}(2T, \epsilon)$ and we define

$$O_{m} := \{ v, \exists 0 \leq t \leq 2T, v_0 \in B_{V^m}(2R), \eta \in \hat{E}_{m-1}(2T), v = \text{Sol}(v_0, \eta)(t) \}.$$

Due to (28)

$$B_{V^m}(2R) \subset O_{m} \subset B_{V^m}(K+1).$$  \hspace{1cm} (29)

Moreover we have the following result:
Lemma 12. The set \( \mathcal{O}_m \) is open in \( V^m \)

Proof. The set

\[
W_m := \{ v \in U_m(2T), v = \text{Sol}(v_0, \eta), v_0 \in B_{V^m}(2R), \eta \in \hat{E}_{m-1}(2T) \}
\]

is open in \( U_m(2T) \) as the inverse image by \( \mathcal{L} \) of \( B_{V^m}(2R) \times \hat{E}_{m-1}(2T) \) which is an open subset of \( V^m \times E_m(2T) \). We now study the following mappings:

\[
\mathcal{L}_\tau : U_m(2T) \to V^m, \quad v \mapsto v(\tau).
\]

Due to Theorem 3.2 of the first chapter in [14] these mappings are continuous and surjective. Therefore, by virtue of the open mapping theorem applied to the Banach spaces \( U_m(2T) \), the sets \( \mathcal{L}_\tau W_m \subset V^m \) are open for any \( \tau \in [0,2T] \) and \( \mathcal{O}_m = \bigcup_{\tau \in [0,2T]} \mathcal{L}_\tau W_m \) is open as well.

Moreover

Theorem 4. The set \( \mathcal{O}_m \) is invariant for \( S \) when the external force belongs to \( \mathcal{E}_{m-1}(1) \). In other words,

\[
S(\mathcal{O}_m \times \mathcal{E}_{m-1}(1)) \subset \mathcal{O}_m.
\]

Moreover, the mapping \( S \) is analytical from \( \mathcal{O}_m \times \hat{E}_{m-1}(1) \) into \( V^m \).

Proof. The analyticity of \( S \) follows from that of the mapping \( \text{Sol} : B_{V^m}(2R) \times \hat{E}_{m-1}(1) \to U_m(1) \) (cf Theorem 3).

Now let us deal with the stability of \( \mathcal{O}_m \). We consider any \( \mathcal{O} \in \mathcal{O}_m \). Then there exist \( t_0 \geq 0, l_0 \in B_{V^m}(R) \) and \( \eta_0 \in \hat{E}_{m-1}(t_0) \) such that

\[
\mathcal{O} = \text{Sol}(l_0, \eta_0)(t_0).
\]

Let \( z_0 \in \mathcal{E}_{m-1}(t_0) \) be an approximation of \( \eta_0 \) such that

\[
\|z_0 - \eta_0\|_{\mathcal{E}_{m-1}(1)} < \epsilon.
\]

Let us now consider any \( \mathcal{E} \in \mathcal{E}_{m-1}(1) \) and undertake a dichotomy depending on the definition of \( \mathcal{O} \). If \( t_0 \geq T \) then by virtue of (28), \( \|u(t)\|_{V^m} \leq R+1 \leq 2R = M \), thus \( S(\mathcal{O}, \mathcal{E}) \in \mathcal{O}_m \) by definition of \( \mathcal{O}_m \).

If, now, \( 0 \leq t_0 \leq T \), we shall write \( S(\mathcal{O}, \mathcal{E}) \) under the exact form required in the definition of \( \mathcal{O}_m \); to this end let us define

\[
z_2(t) = \begin{cases} 
z_0(t), & 0 \leq t \leq t_0, \\
z_1(t - t_0), & t_0 \leq t < t_0 + 1, \\
z_1(1) & t \geq t_0 + 1, \end{cases}
\]

and

\[
\eta_2(t) = \begin{cases} 
\eta_0(t), & 0 \leq t \leq t_0, \\
z_1(t - t_0), & t_0 \leq t < t_0 + 1, \\
z_1(1) & t \geq t_0 + 1. \end{cases}
\]
Then $\eta_2 \in \hat{E}_{m-1}(2T)$ is the control applied to the trajectory starting from $l_0$, going through $l$, defined up to $t_0 + 1$, and extended up to $2T$ by $z_1(1)$.

Clearly

\[ \|\eta_2 - z_2\|_{E_{m-1}(2T)} = \|\eta_0 - z_0\|_{E_{m-1}(2T)} \leq \epsilon. \]

Now since,

\[ S(l, z_1) = Sol(l_0, \eta_2)(t_0 + 1), \quad l_0 \in B_{V^m}(2R), \eta_2 \in \hat{E}_{m-1}(t_0 + 1), \]

we get $S(l, z_1) \in O_m$ and the proposition is proved.

A consequence of this is, that for $m \geq 2$, we may reduce the study of the long-time behaviour of trajectories of the system (27) to the behaviour of the trajectories on the invariant sets $O_m$.

5 Regularization property of the mapping $S$

By virtue of Theorem 3, for $m \geq 2$, the operator $S$ is analytical from $V^m \times \hat{E}_{m-1}(1)$ with values in $V^m$. Thanks to the following proposition we may raise the regularity of the image set by increasing the smoothness of $f$ in (18).

Proposition 1. For all $m \geq 2$, operator $S$ is analytical from $B_{V^m}(2R) \times \hat{E}_{m-1}(1)$ to $V^{m+1}$.

To prove this proposition, we first state the following result

Lemma 13. Let $j \geq 2$ be fixed and $u \in U_j$, $h \in E_j$ and $v_0 \in V^{j+1}$ be given. Then, the solution $v$ of

\[ dL(u)v = (v_0, h) \]

satisfies $v \in U_{j+1}$ and analytically depends on $u \in U_j$, $v_0 \in V^{j+1}$ and $h \in E_j$.

Proof. By virtue of Lemmas 10 and 11, $v \in U_{j+1}$ exists and is unique for all $u \in U_j$, $h \in E_j$ and $v_0 \in V^{j+1}$. To prove its analyticity as a function of those parameters, we apply the implicit function theorem. In our case, we set $f(u, v) := dL(u)v$.

Thanks to Lemma 8, $f$ is analytical as a mapping

\[ U_j \times U_{j+1} \mapsto V^{j+1} \times E_j. \]

Moreover, for any $(u, v)$, its differential $d_u f(u, v) = dL(u) : U_j \times U_{j+1} \mapsto V^{j+1} \times E_j$ is invertible between those spaces as shown by Lemmas 10 and 11. Thus, by virtue of the implicit mapping theorem, for all $\bar{u} \in U_j$ and $(\bar{v}_0, \bar{h}) \in V^{j+1} \times E_j$, there exists a

\[ \phi : U_j \times V^{j+1} \times E_j \supset A \to U_{j+1}, \]

defined and analytic on a neighbourhood $A$ of $(\bar{u}, \bar{v}_0, \bar{h})$ such that

\[ f(u, \phi(u, v_0, h)) = (h, v_0), \quad \forall (u, v_0, h) \in A. \]

Therefore $v = \phi(u, v_0, h)$ is an analytical function everywhere on $U_j \times V^{j+1} \times E_j$. 

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Proof of Proposition 1. To prove this we notice that as $S(0, 0) = 0$, then

$$S(u_0, \eta) = \int_0^1 \frac{d}{d\gamma} S(\gamma u_0, \gamma \eta) d\gamma = \int_0^1 D_u S(\gamma u_0, \gamma \eta) u_0 d\gamma + \int_0^1 D_\eta S(\gamma u_0, \gamma \eta) \eta d\gamma,$$

and set $u_\gamma := S(\gamma u_0, \gamma \eta)$. Since

$$\gamma \in (0, 1),$$

is an analytical mapping, then $u_\gamma$ analytically depends on $(\gamma u_0, \gamma \eta)$ for $(u_0, \eta) \in B_{V_0}(2R) \times \bar{E}_m(1)$ and $0 \leq \gamma \leq 1$ (we use that, $\bar{E}_m(1)$ is star-shaped).

Now let us consider $\int_0^1 D_\eta S(\gamma u_0, \gamma \eta) \eta d\gamma$ and notice that $D_\eta S(\gamma u_0, \gamma \eta) \eta$ is the value at time $t = 1$ of a solution $v_\gamma$ of

$$\frac{dv_\gamma}{dt} - \Delta v_\gamma + B(u_\gamma, v_\gamma) + B(v_\gamma, u_\gamma) = \eta, \quad v_\gamma(0) = 0.$$

In other words, it satisfies the equation $dL(u_\gamma) v_\gamma = (\eta, 0)$. As $\eta \in E_m(1)$ and $u_\gamma \in U_m(1)$ we get, applying Lemma 13 that $v_\gamma \in U_{m+1}(1)$ analytically depends on $u_\gamma$ and $\eta$, and therefore on $u_0 \in B_{V_0}(2R)$ and $\eta \in \bar{E}_m(1)$.

Thus $D_\eta S(\gamma u_0, \gamma \eta) \eta$ is analytic with values in $V^{m+1}$ and by integration the mapping $\int_0^1 D_\eta S(\gamma u_0, \gamma \eta) \eta d\gamma$ is analytic from $B_{V_0}(2R) \times \bar{E}_m(1)$ to $V^{m+1}$.

To study the integral $\int_0^1 D_u S(\gamma u_0, \gamma \eta) u_0 d\gamma$ we note that $D_u S(\gamma u_0, \gamma \eta) u_0$ is the value at $t = 1$ of the solution $\tilde{v}_\gamma$ of

$$\frac{d\tilde{v}_\gamma}{dt} - \Delta \tilde{v}_\gamma + B(u_\gamma, \tilde{v}_\gamma) + B(\tilde{v}_\gamma, u_\gamma) = 0, \quad \tilde{v}_\gamma(0) = u_0.$$

In other words, $\tilde{v}_\gamma$ satisfies $dL(u_\gamma) \tilde{v}_\gamma = (0, u_0)$. As $u_0 \in V^m$, applying Lemma 14, with $j = m-1$, we get that $\tilde{v}_\gamma \in U_{m}(1)$ is an analytical function of $u_0 \in V^m$ and $u_\gamma \in U_{m}(1)$. Now considering $\hat{v}_\gamma = t\tilde{v}_\gamma$ we see $\hat{v}_\gamma$ satisfies

$$\frac{d\hat{v}_\gamma}{dt} - \Delta \hat{v}_\gamma + B(u_\gamma, \hat{v}_\gamma) + B(\hat{v}_\gamma, u_\gamma) = \hat{v} d\gamma, \quad \hat{v}_\gamma(0) = 0.$$

Therefore,

$$dL(u_\gamma, \hat{v}_\gamma) = (0, \hat{v}_\gamma).$$

Now, since $(0, \hat{v}_\gamma) \in V^{m+1} \times E_{m}(1)$, then by virtue of Lemma 14 we have that $\hat{v}_\gamma \in U_{m+1}$ is an analytical function of $(u_\gamma, \hat{v}_\gamma)$. Besides, as $\hat{v}_\gamma(1) = \tilde{v}_\gamma(1)$, then

$$\int_0^1 D_u S(\gamma u_0, \gamma \eta) u_0 d\gamma$$

is analytical from $B_{V_0}(2R) \times \bar{E}_m(1)$ to $V^{m+1}$.

\[\square\]

5.1 Dissipativity of the system

First let us prove the following result:

Lemma 14. Let $m \geq 2$, then there exists a $C_m$ such that for all $u_0 \in C_m$

$$\|S(u_0, 0)\|_{V_m} \leq C_m \|u_0\|_{L^2}. \quad \text{(31)}$$

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Proof. We have

\[ S(u_0, 0) = \int_0^1 D_u S(\gamma u_0, 0)u_0 d\gamma, \]

because \( S(0, 0) = 0 \). Therefore it suffices to show that \( ||D_u S(\gamma u_0, 0)u_0||_{V^m} \leq C||u_0||_{L^2} \), for \( 0 \leq \gamma \leq 1 \). Thus as \( u_0 \in O_m \), using (29) and applying Theorem 1 we get that there is a constant \( A(K) \) such that

\[ ||S_{\gamma u_0}(\gamma u_0, 0)u_0||_{V^m} \leq A(K), \quad \forall t \geq 0. \]

Moreover \( D_u S(\gamma u_0, 0)u_0 \) is the value at time \( t = 1 \) of a solution \( \tilde{v} \) of

\[ dL(S(\gamma u_0, 0))\tilde{v} = (u_0, 0). \]

Thus, by virtue of Lemma 10 with \( r = 0 \),

\[ ||\tilde{v}(t)||_{V^r} \leq C(K)||u_0||_{L^2}. \]

Then, reusing the notation \( \tilde{v}(t) = t\tilde{v}(t) \) we see that \( \tilde{v}(t) \) is a solution of

\[ dL(S(\gamma u_0, 0))\tilde{v} = (0, \tilde{v}). \]

Whence, using Lemma 10 for \( r = 1 \) we get

\[ ||\tilde{v}(t)||_{V^1} \leq C'(K)||u_0||_{L^2}. \]

More particularly, for a given \( t_1 < 1 \)

\[ ||\tilde{v}(t_1)||_{V^1} \leq C'(K)||u_0||_{L^2}. \]

We shall repeat this argument between \( t_1 \) and 1. Then \( \tilde{v}(t) \) for \( t \in [t_1, 1] \) is a solution of an equation similar to

\[ dL(S(\gamma u_0, 0))\tilde{v} = (0, \tilde{v}(t_1)). \]

Arguing as above step we have that

\[ ||D_u S(\gamma u_0, 0)u_0||_{V^2} \leq C(K)||u_0||_{L^2}. \]

Therefore,

\[ ||S(u_0, 0)||_{V^2} \leq C_2(K)||u_0||_{L^2}, \]

and more generally

\[ ||Sol(u_0, 0)(t)||_{V^2} \leq C_2(K, t)||u_0||_{L^2}, \]

for any \( 0 < t < 1 \). Now, for any \( m > 2 \) iterating the process on \( m \) time intervals \([t_i, t_i + 1]_{0 \leq i \leq m+1} \) such that \( t_0 = 0 \) and \( t_{m+1} = 1 \), we get the assertion of the lemma.

From this, we obtain the following
Proposition 2. For \( m \geq 2 \) consider a norm
\[
|| \cdot ||_{V^m} := || \cdot ||_{L^2} + \delta || \cdot ||_{V^m},
\]
equivalent to \( || \cdot ||_{V^m} \). Then, there exist \( \delta = \delta_m > 0 \), and \( 0 < \gamma < 1 \) such that the system (27) abides by the following dissipativity rule:
\[
||S(u_0, 0)||_{V^m} \leq \gamma ||u_0||_{L^2}, \quad \forall u_0 \in \mathcal{O}_m.
\]
Thus,
\[
||S(u_0, 0)||_{V^m} \leq \gamma ||u_0||_{V^m}, \quad \gamma < 1, \quad \forall u_0 \in \mathcal{O}_m.
\]

Proof. First let us state a simple contraction result for our system: applying to (1) the scalar product with \( v \) and thanks to Lemma 4 and to Formula 1 we get the following estimate
\[
\frac{d||v||_{L^2}^2}{dt} + ||\nabla v||_{L^2}^2 \leq 0, \quad \text{for } v = \text{Sol}(u_0, 0).
\]
Thus by virtue of Poincaré’s inequality and following Gronwall’s lemma we get that there is a \( \kappa > 0 \) such that
\[
||S(u_0, 0)||_{L^2} \leq e^{-\kappa} ||u_0||_{L^2}.
\]
Then by virtue of lemma 14 we get that
\[
||S(u_0, 0)||_{V^m} = e^{-\kappa} ||u_0||_{V^m} + \delta C_m ||u_0||_{L^2} \leq (e^{-\kappa} + \delta C_m) ||u_0||_{L^2} \leq \gamma ||u_0||_{L^2}, \quad \gamma < 1
\]
for \( \delta = \frac{\gamma - e^{-\kappa}}{C_m} \) where \( \gamma \in (e^{-\kappa}, 1) \).

5.2 Non degeneracy of operator \( S \)

The aim of this section is to prove the density of the range of the operator \( D_{\eta}S(u_0, \eta) \), as an application between \( E_m \) and \( V^m \), for any \( m \geq 2 \) and any \( (u_0, \eta) \in \mathcal{O}_m \times \hat{E}_{m-1} \). For simplicity’s sake, in what follows we shall only establish the result in the case \( m = 2 \). To this end we will first study operators
\[
S^t_{12} : V^2 \rightarrow V^2, \quad v \mapsto v', \quad 0 \leq t_1 \leq t_2,
\]
where \( v' \) is the value at \( t_2 \) of a solution \( v(t) \) of equation \( d\mathcal{L}^2(u)v = 0 \), \( u = \text{Sol}(u_0, \eta) \in U_2(1) \), with initial value \( v(t_1) = v \).

Now, for any \( 0 \leq t_1 < t_2 \leq 1 \), let us determine the dual operator of \( S^t_{12} \) in \( (V^2, || \cdot ||_{V^o}) \). Let us start with the following result

Lemma 15. For all \( u, v, w \in U_2(1) \), we have the following equality
\[
\langle B(u, v) + B(v, u), w \rangle = \langle v, \mathfrak{B}_u(w) \rangle,
\]
where
\[
\mathfrak{B}_u(w) := \mathcal{P} \left( -B(u, w) + (d_2 u^*) w + \int_{-h}^z \nabla_2 \left( w \frac{\partial u}{\partial z} \right)(x, y, \xi) d\xi \right).
\]
and

\[(d_2 u^*) w^i = \sum_{l=1}^{2} w_l \partial_i u_l \text{ for } i = 1, 2.\]

**Proof.** First let us notice that thanks to Lemma 4, we have \(\langle B(u, v + w), v + w \rangle = 0\), so that

\[\langle B(u, v), w \rangle = -\langle v, B(u, w) \rangle, \quad \forall u, v, w \in V^2.\]

Now let us examine \(\langle B(v, u), w \rangle\). Concerning its first term, we have that

\[\langle (v \cdot \nabla_2) u, w \rangle = \int_\Omega \sum_{i=1}^{2} v_l \left( \sum_{l=1}^{2} w_l \partial_i u_l \right) = \langle v, (d_2 u^*) w \rangle.\]

Then we study the second term of \(\langle B(v, u), w \rangle\). We have

\[\langle \left( \int_{-h}^{z} \text{div}_2 v d\xi \right) \frac{\partial u}{\partial z}, w \rangle = \int_{\Omega} dx dy \left[ \int_{-h}^{z} \text{div}_2 v d\xi \right] \frac{\partial u}{\partial z} \cdot w dz.\]

Writing \(\frac{\partial u}{\partial z} \cdot w = \frac{\partial u}{\partial z} \left( \int_{-h}^{z} \frac{\partial u}{\partial z} \cdot w d\xi \right)\) we get

\[\langle \left( \int_{-h}^{z} \text{div}_2 v d\xi \right) \frac{\partial u}{\partial z}, w \rangle = -\int_{\Omega} dx dy \left[ \int_{-h}^{z} \text{div}_2 v \left( \int_{-h}^{z} \frac{\partial u}{\partial z} \cdot w d\xi \right) dz \right],\]

because \(\int_{-h}^{z} \text{div}_2 v d\xi = 0\). Thus

\[\langle \left( \int_{-h}^{z} \text{div}_2 v d\xi \right) \frac{\partial u}{\partial z}, w \rangle = \int_{\Omega} v(\zeta) \nabla_2 \left( \int_{-h}^{z} \frac{\partial u}{\partial z} \cdot w d\xi \right) (\zeta) d\zeta.\]

Now let us study the equation, generated by the dual operator \(\mathbb{B}_u\).

**Lemma 16.** If \(u \in U_2(1)\), and \(0 \leq t_1 \leq t_2 \leq 1\) then for \(r = 0, 1, 2\) and \(w'' \in V^r\), the problem

\[
\frac{dw}{dt} + \Delta w - \mathbb{B}_u (t)(w) = 0, \quad t_1 \leq t \leq t_2, \quad w(t_2) = w'',
\]

has a unique solution \(w \in U_r([t_1, t_2])\).

**Proof.** In order for \(w\) to satisfy (33) it is necessary and sufficient that \(\mathcal{\hat{w}} : t \mapsto w(t_2 + t_1 - t)\) verify equation

\[
\frac{d\mathcal{\hat{w}}}{dt} - \Delta \mathcal{\hat{w}} + \mathbb{B}_u (\mathcal{\hat{w}}) = 0
\]

with \(\mathcal{\hat{w}}(t_1) = w''\) (and \(\mathcal{\hat{u}}(t) = u(t_2 + t_1 - t)\)). The analysis of this equation being similar to the one of (18), we straightforwardly get that \(w\) is in an unique way determined by \(w''\) and that \(w \in U_r([t_1, t_2])\). □

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We then set
\[ S_{t_2}^{t_1} : V^2 \to V^2, \quad w'' \mapsto w', \quad 0 \leq t_1 \leq t_2 \leq 1, \]
where \( w' = w(t_1) \) and \( w \) is a solution of \( \mathfrak{R} \). Now let us recall that \( S_{t_1}^{t_2} v = v(t_2) \) where \( v(t) \), defined for \( t_1 \leq t \leq t_2 \) solves equations \( dL^2(u)v = 0 \) and \( v(t_1) = v_0 \).

For this \( v(t) \) and the solution \( w(t) \) of \( \mathfrak{R} \), used to define the operator \( S_{t_1}^{t_2} \), we have:
\[ \langle v, \frac{dw}{dt} + \Delta w - B(u)w \rangle = 0. \]

Integrating by parts, we get
\[ \langle \Delta v - B(u,v) - B(v,u), w \rangle + \langle v, \frac{dw}{dt} \rangle = 0, \]
which in turn becomes
\[ \langle \frac{dv}{dt}, w \rangle + \langle v, \frac{dw}{dt} \rangle = 0. \]

Whence,
\[ \frac{d}{dt} \langle v, w \rangle = 0. \]

We then get that for all \( 0 \leq t_1 < t_2 \leq 1, \)
\[ \langle S_{t_1}^{t_2} v', w'' \rangle = \langle v', S_{t_1}^{t_2} w'' \rangle, \quad \forall v', w'' \in V^2, \quad (34) \]
i.e., \( S_{t_2}^{t_1} \) is the dual operator to \( S_{t_1}^{t_2} \) in \((V^2, || \cdot ||_{V^o})\).

**Proposition 3.** For \( m \geq 2 \), the range of \( D_\eta S(u_0, \eta) : E_m \to V^m \) is dense in \( V^m \).

**Proof.** We will prove the result only for the case \( m = 2 \). To do this, it is enough to check that the orthogonal complement to the range of \( D_\eta S(u_0, \eta) : E_2(1) \to V^2 \) in \( V^2 \) is \( \{0\} \). To this end, we show for any \( w \in V^2 \) that
\[ \langle D_\eta S(u_0, \eta) h, w \rangle_2, \forall h \in E_2(1) \implies w = 0, \]
in other words we show that \( \ker(S^*) = \{0\} \), where \( S^* : V^2 \to E_2(1) \) is the dual operator to \( D_\eta S(u_0, \eta) \). Let us denote \( Sol(u_0, \eta)(t) = u(t), 0 \leq t \leq 1 \) and consider the corresponding operators \( S_{t_1}^{t_2} \) and \( S_{t_2}^{t_1} \). As \( (D_\eta S(u_0, \eta) h \rangle \) is the value at time \( t = 1 \) of the solution \( v \) of
\[ dL(Sol(u_0, \eta))v = (0, h), \]
then, applying Duhamel’s formula, we get that
\[
\langle D_\eta S(u_0, \eta) h, w \rangle_2 = \int_0^1 \langle S_1^t(h(t)), w \rangle_2 dt = \int_0^1 \langle S_1^t(h(t)), \Delta^2 w \rangle_{V^o} dt = \int_0^1 \langle h(t), S_1^t \Delta^2 w \rangle_{V^o} dt = \int_0^1 \langle h(t), \Delta^{-2} S_1^t \Delta^2 w \rangle_2, \quad \forall w \in V^4.
\]
Thus for \( w \in V^4 \), we may define \( S^*(w) \) as:

\[
S^*(w)(t) = \Delta^{-2} \tilde{S}_1^t \Delta^2 w, \quad \forall w \in V^4.
\]  \hspace{1cm} (35)

This done, let us characterize the kernel of \( S^* \) in \( V^2 \). We now consider any \( w \in \ker(S^*) \subset V^2 \) and approximate it in \( V^2 \) by a sequence \((w_n) \in (V^4)^{\mathbb{N}\cup\{0\}}\), such that \(||w_n||_{V^2} \leq 2||w||_{V^2}\) for all \( n \geq 0 \). Next, for each \( n \) let us build a curve \([0,1] \ni t \mapsto \xi^n_t \) defined by the relation, 

\[
\tilde{S}_1^t \Delta^2 w_n = \Delta \xi^n_t,
\]

then \( \xi^n = \Delta S^*(w_n) \). Since \( \Delta^2 w_n \in V^0 \), then by virtue of Lemma \[10\] with \( r = 0 \), we have that \( \xi^n \in U_2(1) \). Let us now prove that the set \((\xi^n)_{n \in \mathbb{N} \cup \{0\}}\) is bounded in \( U_0(1) \). The curve \( \eta^n := \tilde{S}_1^t \Delta^2 w_n \) satisfies

\[
\frac{d\eta^n}{dt} + \Delta \eta^n - B_u(\eta^n) = 0, \quad 0 \leq t \leq 1, \quad \eta^n = \Delta^2 w_n \in V^0.
\]

Thus \( \xi^n = \Delta^{-1} \eta^n \) abides by the relation

\[
\frac{\partial \xi^n}{\partial t} + \Delta \xi^n - \Delta^{-1}(B_u(\Delta \xi^n)) = 0, \quad \xi^n = \Delta w_n.
\]

To study this equation, we revert the time-flow, that is we apply the transformation \( t \to 1-t \), keeping the name \( \xi^n_t \) for the transformed variable. Then, applying to the obtained equation the scalar product with \( \xi^n_t \), we get

\[
\frac{\partial ||\xi^n_t||_{L^2}^2}{\partial t} + ||\nabla \xi^n_t||_{L^2}^2 \leq C||\Delta^{-1} B_u(\Delta \xi^n), \xi^n_t||.
\]

Since, by virtue of Lemma \[7\]

\[
||\Delta^{-1} B_u(\Delta \xi^n), \xi^n_t|| = ||B_u(\Delta \xi^n), \Delta^{-1} \xi^n_t||
\]

\[
= ||B(u, \Delta^{-1} \xi^n_t) + B(\Delta^{-1} \xi^n_t, u), \Delta \xi^n_t)|| \leq C||B(u, \Delta^{-1} \xi^n_t) + B(\Delta^{-1} \xi^n_t, u)||_{V^2}||\xi^n_t||_{L^2}
\]

\[
\leq C||u||_{V^3}||\nabla \xi^n_t||_{L^2}||\xi^n_t||_{L^2},
\]

then

\[
\frac{\partial ||\xi^n_t||_{L^2}^2}{\partial t} + ||\nabla \xi^n_t||_{L^2}^2 \leq C||u||_{V^3}||\nabla \xi^n_t||_{L^2}||\xi^n_t||_{L^2}.
\]

Since \( ||u||_{E_3} \leq ||u||_{V^2} \), then from here

\[
||\xi^n_t||_{U_0} \leq C(||u||_{V^2}, ||w_n||_{V^2}), \quad \forall n \geq 0.
\]

We then get that \((\xi^n)_n\) is compact in the weak topology of the space \( U_0(1) \). Let \((\xi^n)_n\) be a subsequence of \((\xi^n)_n\) weakly converging to some \((\xi)\) in \( U_0(1) \). Since \( \xi^n = \Delta S^* w_n \), then,

\[
\xi \leftarrow \xi^n = \Delta S^* w_n \rightarrow \Delta S^* w = 0, \quad \text{in } E_0(1).
\]

Thus \( \xi = 0 \) i.e \( \xi^n \rightharpoonup 0 \) in \( U_0(1) \). From here we have \( \tilde{\xi}^n_t \rightharpoonup 0 \) in \( V^0 \). But \( \xi^n_t = \Delta w_n \rightarrow \Delta w \) in \( V^0 \). Thus \( \Delta w = 0 \) and \( w = 0 \), whence

\[
\ker(S^*) = \{0\}.
\]
Now that the deterministic side is settled let us introduce the associated stochastic problem. Henceforth we will consider the \( \eta_n \) in (27) and the associated \( f \) as random variables whose exact form we shall now specify.

### 6 Red noise: definition and property

We consider the following Haar wavelets of argument \( t \in [0, 1] \):

\[
h_0(t) := \mathbb{1}_{[0,1]}(t),
\]

and for \( j \geq 0 \) and \( 0 \leq k < 2^j \)

\[
h_{j,k} := \begin{cases} 0, & t \leq \frac{k}{2^j}, \\ 1, & \frac{k}{2^j} \leq t < \frac{k+1/2}{2^j}, \\ -1, & \frac{k+1/2}{2^j} \leq t \leq \frac{k+1}{2^j}, \\ 0, & t \geq \frac{k+1}{2^j}. \end{cases}
\]

Classically the \( \{(2^j/2^j h_{j,k}) \}_{j \geq 0, 0 \leq k < 2^j} \) make up an orthonormal basis of \( L^2([0,1], \mathbb{R}) \); for the proof one may see [13]. Consequently

**Lemma 17.** The set \( \{2^j/2^j h_{j,k} \lambda_i^{-m/2} e_i \}_{j \geq 0, 0 \leq k < 2^j, i \geq 1} \) is an orthonormal basis of \( L^2([0,1], V^m) \).

Define a random process \( \tilde{\eta}(t) \in V \) for \( 0 \leq t \leq 1 \) as follows:

\[
\tilde{\eta}(t) := \sum_{i=0}^{\infty} b_i \tilde{\eta}^i(t) \lambda_i^{-m/2} e_i, \quad 0 \leq t \leq 1,
\]

\[
\tilde{\eta}^i(t) := \left( \xi^i_0(t) h_0(t) + \sum_{j=0}^{\infty} c^i_j \sum_{k=0}^{2^j-1} \xi^i_{j,k} h_{j,k}(t) \right).
\]

Here the real random variables \( \xi^i_0 \) and \( \xi^i_{j,k} \) are independent, identically distributed (i.i.d) such that \( |\xi^i_0| \leq 1 \) and \( |\xi^i_{j,k}| \leq 1 \) for any \( \omega \). In addition, we assume that the coefficients verify:

\[
\sum_{i=1}^{\infty} b_i^2 \left( 1 + \left( \sum_{j=0}^{\infty} c^i_j \right)^2 \right) < \infty.
\]

Now let us consider a sequence \( \eta_k = \sum_{i=0}^{\infty} b_i \tilde{\eta}^i_k(t) \lambda_i^{-m/2} e_i \) of independent copies of the process \( \tilde{\eta} \) and set

\[
f(t) := \sum_{k=0}^{\infty} \mathbb{1}_{[k,k+1]}(t) \eta_k(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[k,k+1]}(t) \sum_{i=1}^{\infty} b_i \tilde{\eta}^i_k \lambda_i^{-m/2} e_i. \tag{36}
\]

Now let us prove that \( f \) verifies \( f^\omega(t) \in V^m \) and study its time and space regularities.
Lemma 18. Random variable $\tilde{\eta}$ is bounded in $L^\infty([0, 1], V^m)$, for all $\omega$:

$$\exists C^* > 0, \quad ||\tilde{\eta}(t)||_{V^m}^2 < C^*, \quad \forall t \in [0, 1], \quad \forall \omega.$$ 

Proof. Let us define

$$\Sigma^i_j(t) := \sum_{k=0}^{2^j-1} \xi^i_j k h_{j,k}(t).$$

Since the supports of the functions $h_{j,k}$ for $0 \leq k \leq 2^j - 1$ are disjointed, and, as $|h_{j,k}| \leq 1$ for all $t \in [0, 1]$ and $|\xi^i_j k| \leq 1$ for all $\omega$, we have that

$$|\Sigma^i_j(t)| \leq 1 \quad \forall t, \quad \forall \omega.$$ 

Therefore

$$||\tilde{\eta}(t)||_{V^m}^2 = \sum_{i=0}^{\infty} |b_i|^2 \left( |\Sigma^i_0(t)| + \sum_{j=0}^{\infty} |c^i_j| |\Sigma^i_j(t)| \right)^2 \leq \sum_{i=0}^{\infty} |b_i|^2 \left( |\Sigma^i_0(t)|^2 + \sum_{j=0}^{\infty} |c^i_j| |\Sigma^i_j(t)| \right) \leq \sum_{i=0}^{\infty} |b_i|^2 (1 + \sum_{j=0}^{\infty} c^i_j)^2 < \infty,$$

for all $t \in [0, 1]$ and all $\omega$. \qed

Now we shall study the properties of the stochastic system (27) where $f$ is given by (36). Namely in the following section we will strengthen the notion of mixing which we first exposed in (13), replacing it with the exponential mixing, and we present a general result on mixing in stochastic systems which we shall apply to (27) in the subsequent sections.

7 An abstract theorem on mixing

To show that a stochastic process is exponentially mixing, we rely on Theorem 1.3 in [12] which we discuss in the following context. Let $H$ and $E$ be separable Hilbert spaces and $V$ a Banach space with a compact injection into $H$. Let $(\eta_k)_{k \geq 0}$ be a sequence of independent identically distributed random variables with values in $E$. Let $S : H \times E \to H$ be a continuous operator. Let us fix $u_0 \in H$ and consider the random dynamic system in $H$:

$$u_{k+1} = S(u_k, \eta_k), \quad k \geq 0. \quad (37)$$

It defines a Markov chain in the space $H$. Denote by $(u_k(u_0), k \geq 0)$ a trajectory of this system, equal to $u_0$ when $k = 0$. To proceed we lay out some additional notation. Namely, for a complete separable metric space, $U$ we denote by $C_b(U)$ the space of continuous bounded functions $g$ on $U$ equipped with the sup-norm $||g||_{L^\infty}$. For a function $g \in C_b(U)$, denote by $Lip(g) \leq \infty$ its Lipschitz constant and set:

$$||g||_L := ||g||_{L^\infty} + Lip(g).$$

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Moreover, we name $\mathcal{P}(U)$ the space of probability Borel measures on $U$. We equip it with the weak convergence of measures and with the dual Lipschitz distance

$$\|\mu - \nu\|_L^* = \sup_{g \in C_b(U), \|g\|_L \leq 1} \langle g, \mu - \nu \rangle \leq 2,$$

where $\langle g, \mu \rangle$ stands for the integral of $g$ against $\mu$. It is known that the weak convergence of measures is equivalent to the convergence in the dual-Lipschitz distance (see [11]). Finally for a random variable $\xi \in U$ we denote by $D\xi$ its law $D\xi \in \mathcal{P}(U)$. We recall that a measure $\mu \in \mathcal{P}(H)$ is stationary for the system (37) if any trajectory $(u_k(u_0))_k$ where $u_0 \in H$ is a random variable such that $D(u_0) = \mu$, satisfies $D(u_k) = \mu, \forall k$. In [10, 12], the following abstract theorem is proved:

**Theorem 5.** Under the following hypotheses

- (A1) **Regularity:** $\bar{S} : H \times E \to V$ is twice continuously differentiable and its derivatives up to the second order are bounded on bounded sets.
- (H1) **Decomposability and non-degeneracy** there exists an orthonormal basis $(\sigma_j)_{j \geq 1}$ of $E$, such that the random variables $\eta_k$ can be written in the following way

$$\eta_k := \sum_{j=1}^{\infty} b_j \xi_{j,k} \sigma_j, \quad b_j \neq 0, \quad \sum b_j^2 < \infty.$$

Here the $\xi_{j,k}$ are random i.i.d variables almost surely smaller than 1 with laws of the form $\rho(r) dr$, where $\rho$ is a Lipschitz function with a non-zero value at zero. We name $l := D(\eta_k)$ and denote by $K$ the support of $l$.
- (H2) **Dissipativity:** for any $\tau > 0$, any $u \in H$ and any $\eta \in K$ we have

$$a) ||\bar{S}(u, \eta)||_H \leq \gamma ||u||_H + \beta, \quad b) ||\bar{S}(u, 0)||_H \leq \gamma ||u||_H \quad \gamma < 1, \beta \geq 0,$$

- (H3) **Non-degeneracy:** for any $u \in H$ and $\eta_0 \in K$ the application $D_{\eta} \bar{S}(u, \eta_0) : E \to V$ has a dense range in $H$.

Then the system (37) is exponentially mixing: there is a unique stationary measure $\mu$ which is supported by $B_H(R^*)$ for some $R^* > 0$, such that for any $R > 0$, for all $u_0 \in B_H(R)$, we have

$$||D(u_k(u_0)) - \mu||_L^* \leq C \kappa^k, \quad (38)$$

with constants $0 < \kappa(R) < 1$, $C = C(R) > 0$.

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1 In [10] a more general result is proved with stronger restrictions on the system.
8 Reformulation of the hypotheses

We seek to apply to the system (27) a variation of Theorem 5 where the hypotheses are somewhat relaxed. To this end we analyse the demonstration of the theorem: in [12] the authors work in two steps. First, they state the existence of an open bounded subset $\mathcal{O}$ of the phase space that is both invariant and absorbant for the process. In their case, $\mathcal{O} = B_H(R)$ for a specific $R$. Using hypothesis (A1), they define some $\mathcal{R}(\mathcal{O}, \mathcal{K}) > 0$ such that

$$\bar{S}: \mathcal{O} \times \mathcal{K} \mapsto B_V(\mathcal{R}(\mathcal{O}, \mathcal{K})),$$

and name $X$, the completion of $\mathcal{O} \cap B_V(\mathcal{R}(\mathcal{O}, \mathcal{K}))$ in $H$. Then they consider the following restrained system:

$$\bar{S}: X \times \mathcal{K} \mapsto X,$$

where $\bar{S}$ extends to a $C^2$ smooth mapping

$$\tilde{S}: H \times E \mapsto V.$$  \hspace{1cm} (39)

In this context we formulate the following reduced hypotheses on $\bar{S}$:

- (A1'): $\tilde{S}$ verifies (A1)
- (H1)
- (H2'): (H2) b) holds
- (H3'): (H3) holds for $(u, \eta_0) \in \mathcal{O} \times \mathcal{K}$.

Then, in their demonstration of Theorem 1.3 (which is the second step of the proof of mixing for the system), they prove the following:

\textbf{Theorem 6.} From theorem 1.3 in [12]. Suppose that our system verifies (A1'), (H1'), (H2) and (H3'). Then the system (27) has a unique stationary measure $\mu \in \mathcal{P}(\mathcal{O})$ and for any $u, u' \in \mathcal{H}$, we have

$$||D(u_k(u)) - D(u_k(u'))||_L^2 \leq C||u - u'||_{H^\kappa},$$ \hspace{1cm} (40)

for some $0 < \kappa < 1, C > 0$ that only depend on $\mathcal{O}$.

Moreover we may relax the constraint on $\mathcal{O}$ which instead of being an open ball of $H$ can merely be a bounded open domain of it. Furthermore, there is no need to demand the existence of an extension $\bar{S}$ as in (39). It is enough that $S$ be defined, twice differentiable and bounded in $C^2$-norm on $X_\epsilon \times K_\epsilon$, where $X_\epsilon$ and $K_\epsilon$ are the $\epsilon$-neighbourhoods of $X$ and $K$ respectively in $H$ and $E$.

This clarified, we will verify hypotheses (A1'), (H1'), (H2) and (H3') for our system. In our case, $\bar{S} = \tilde{S}, H = V^m, V = V^{m+1}, E = E_m(1)$ and $\mathcal{O} = \mathcal{O}_m$ for some $m \geq 2$. By virtue of Theorem 1 and of Theorem 4, this last set is indeed invariant by $S$ and absorbing for it.
9 Verification of the hypotheses and conclusion

We will verify the hypotheses \((A1')-(H3)\) with the norm \(\| \cdot \|_{V^m} \) replace by the equivalent norm \(\| \cdot \|_V^{m}\). Then \((H2')\) holds by Proposition 2 and hypothesis \((A1')\) stands true by virtue of Theorem 4, the validity of \((H1)\) is obvious considering the structure of our noise, as for hypothesis \((H3')\) Proposition 3 allows us to verify it here.

Thus we get

**Theorem 7.** Assume that the force \(f\) in \((3)\) is the red noise \((36)\), and consider the corresponding system \((37)\) in a space \(V^m\) with \(m \geq 2\). Then this system has a unique stationary measure \(\mu\). It is supported by the set \(\Omega_m\) and for any random variable \(u_0 \in \Omega_m\), we have

\[
\|D(u_k(u_0)) - \mu\|_{L(V^m)}^* \leq C_m \kappa_m^k.
\]

where \(C_m \geq 0\) and \(0 < \kappa_m < 1\).

Moreover, by virtue of Theorem 2, the set \(\Omega_m\) is absorbing for our system. Therefore we have the following corollary to the previous theorem:

**Corollary 4.** Let us still suppose that the noise \(f\) be of the form \((36)\), with \(m \geq 2\). Then, for any \(R' > 0\), and for any random variable \(u_0 \in B_{V^m}(R')\), there exists \(C_m(R') > 0\) and \(\kappa(R') < 1\) such that for any \(k \geq 0\),

\[
\|D(u_k(u_0)) - \mu\|_{L(V^m)}^* \leq C_m(R') \kappa_m^k.
\]

Since the convergence in the dual-Lipschitz distance is equivalent to the weak convergence of measures, for any random variable \(u_0\) as in the assumptions of this corollary, we have

\[
\mathbb{E}[g(u_k(u_0))] \xrightarrow{k \to \infty} \int_{V^m} g(u) d\mu(u), \quad \forall g \in C_b(V^m),
\]

cf \((13)\).

**Proof of Corollary 4.** We set \(k_0(R)\) as the smallest integer such that \(k_0(R) \geq T_0 = T(R,C^*)\) with \(T\) defined in Theorem 2. When \(k \geq k_0(R)\), the system is in \(\mathcal{O}_m\), thus we may apply Theorem 4 hence the result for \(k \geq k_0\) with \(C_m(R') := C_m \kappa^{-k_0}\). Moreover, as \(\|\mu_k(u_0) - \mu\|_{L(V^m)}^* \leq 2\), then increasing \(C_m(R')\) if needed we achieve that \((41)\) also holds for any \(k \leq k_0(R)\).

10 Appendix

**Theorem 8** (Uniform local inverse theorem). Let \(E\) and \(F\) be two complex Banach spaces and \(U\) be an open bounded subset of \(E\). Let \(r > 0\) and

\[
f : U \mapsto F.
\]
is an analytical mapping which is injective. Let \( Z \subset F \) be such that \( Z \subset f(U) \) and a): \( G_r := f^{-1}(Z) + B_F(r) \subset U \).

Moreover, we assume that:

• b): \( \|df(x)\|_{E \rightarrow F} \leq K_1, \ \forall x \in G_r, \)

• c): \( \|d^2 f(x)\|_{E \times E \rightarrow F} \leq K_2, \ \forall x \in G_r \) and

• d): \( \forall x \in f^{-1}(Z), \ \|(df(x))^{-1}\|_{F \rightarrow E} \leq K_3. \)

Then \( \exists \rho = \rho(K_1, K_2, K_3, r) \) and \( \exists L = L(K_1, K_2, K_3, r) \) such that on the set \( Z_\rho = Z + B_F(\rho) \) the inverse mapping \( f^{-1}: Z_\rho \rightarrow U \) is well defined and analytic. Moreover \( \|df^{-1}(z)\|_{F \rightarrow E} \leq L, \forall z \in Z_\rho. \) The assertion remains true if \( E \) and \( F \) are real Hilbert spaces, the map \( f \) is real-analytic and \( G_r \) is an \( r \)-neighbourhood of \( f^{-1}(Z) \) in the complexification of the Hilbert space \( E. \)

Proof. Here we shall only prove that \( f^{-1} \) may be built globally on \( Z_\rho \) when its components on the \( B_F(z, \rho) \) for all \( z \in Z \) are each well defined. To do so, we only need to remark that \( f \) being injective, if two balls \( B_F(z, \rho) \) have a non-void intersection, then the locally built \( f^{-1} \) must coincide on said intersection, by virtue of \( f \) being injective. Thus \( f \) is unequivocally defined on the reunion \( Z_\rho \) of all \( B_F(z, \rho) \) with \( z \in Z. \)

\( \square \)

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