TWO FINITE DIFFERENCE METHODS BASED ON AN H2N2 INTERPOLATION FOR TWO-DIMENSIONAL TIME FRACTIONAL MIXED DIFFUSION AND DIFFUSION-WAVE EQUATIONS

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Abstract. In this work, two fully novel finite difference schemes for two-dimensional time-fractional mixed diffusion and diffusion-wave equation (TFMD-DWEs) are presented. Firstly, a Hermite and Newton quadratic interpolation polynomial have been used for time discretization and central quotient has used in spatial direction. The H2N2 finite difference is constructed. Secondly, in order to increase computational efficiency, the sum-of-exponential is used to approximate the kernel function in the fractional-order operator. The fast H2N2 finite difference is obtained. Thirdly, the stability and convergence of two schemes are studied by energy method. When the tolerance error $\epsilon$ of fast algorithm is sufficiently small, it proves that both of difference schemes are of $3-\beta$ ($1<\beta<2$) order convergence in time and of second order convergence in space. Finally, numerical results demonstrate the theoretical convergence and effectiveness of the fast algorithm.

1. Introduction. The objective of this study is to presented a novel H2N2 finite difference scheme and corresponding fast H2N2 difference scheme for two-dimensional time-fractional mixed diffusion and diffusion-wave equation (TFMD-DWEs). The underlying idea of this work is totally different for previous work. On the one hand, the convergence order of the H2N2 difference scheme is $3-\beta$ ($1<\beta<2$) which only depends on $\beta$ in temporal direction. On the other hand, in order to enhance computational efficiency, we propose a new H2N2 fast difference scheme, meanwhile, we use the fast Fourier transforms (FFTs) to solve the system of linear algebraic equations. Furthermore, the stability and convergence of the two difference schemes are investigated in detail.

Fractional differential equations have received much attention over the past few years, because of its non-local property, they have become significant mathematical model in many fields of science and engineering, such as viscoelastic models in blood
flow [33, 4, 5], underground transport [2, 37, 21, 9, 14], options pricing models in
financial markets [29, 30, 35, 6]. Compared to the single term fractional diffusion
or diffusion-wave equation, TFMDDWEs may be more accurate and flexible for
describing some undergoing process or properties of media, successfully capturing
power-law frequency dependence [20], adequately modeling various types of visco-
elastic damping [8], properly simulating the unsteady floe of a fractional Maxwell
fluid [13, 34], and Oldroyd-B fluid [12].

There are a few research which study the exact solution of multi-term time-
fractional partial differential equations (TFPDEs). Existence, uniqueness and a
priori estimates for a class of these equations were obtained by Luchko [27] by ap-
plying the maximum principle and the Fourier method. Schneider and Wyss [31]
obtained the corresponding Green’s functions in terms of Fox functions for the frac-
tional diffusion and wave equations. Daftardar-Gejji and Bhalekar [10] considered
the solution of multi-term time fractional diffusion-wave equation under the homoge-
nous/nonhomogeneous boundary conditions and the statement followed the separa-
tion of variables. Jiang et al. [17] derived the analytical solutions for the multi-term
time-space fractional advection-diffusion equations. However, most of these studies
focus on multi-term TFPDEs with all the fractional orders being in (0,1) or (1,2),
then there are limited literatures on the (especially numerical) solutions of multi-
term TFPDEs in which the fractional orders are in both (0,1) and (1,2). Hao and
Lin [16] presented an implicit and compact finite difference scheme for multi-term
TFMDDWEs with the first order accuracy in time and fourth order accuracy in
space. Zhao et al. [41] and Wei et al. [36] respectively established two full-discrete
approximate schemes for two-term TFMDDWEs and 2D multi-term TFMDDWEs
with spatial variable coefficient by using linear triangle finite element method in
space and classic $L^1$ time-stepping method combined with Crank-Nicolson scheme
in time. The convergence order were of $\min\{2 - \alpha, 3 - \beta\}$ ($0 < \alpha < 1 < \beta < 2$) in
time and second order in space. Feng et al. [11] used mixes $L$ scheme to approx-
imate the time fractional sub-diffusion term, diffusion-wave term, and the couple
time-space derivative, respectively and the finite element method was applied in
space. He constructed numerical scheme with an accuracy of $O(\tau^{\min\{2 - \alpha, 3 - \beta\}} + h^2)$
for 2D multi-term TFMDDWEs. Liu et al. [23] and Chen et al. [7] respectively
proposed unified numerical schemes for multi-term TFMDDWEs based on weighted
and shifted Grünwald difference (WSGD) operator discretization in time and Le-
gendre spectral approximation in space. The convergence rate of both of two scheme
in $L^2$ norm were $O(\tau^2 + N_s^{-r})$, where $\tau$, $N_s$ and $r$ were the time-step size, polyno-
mial degree, and regularity in space. Especially, Liu et al. attempted to deal with
the non-smooth solution case by adding some initial correction terms.

Since the nonlocal property and weak singularity of fractional-order operators
bring large computational cost and storage of numerical methods, developing ef-
icient method becomes greatly significant. In order to overcome such difficulties,
some techniques were adopted to reduce the computational cost and storage of the
derived methods. For example, some progress has been made to reduce the memory
requirement and computational cost of the discrete convolution for approximating
the Riemann-Liouville (R-L) fractional operators [1, 22, 24, 39]. Basing on
the piecewise polynomial interpolation, the kernel function in the fractional operators
was approximated by the sum-of-exponentials (SOE) [28, 38]. And other fast (and
parallel) approximation to time-fractional derivative was based on the block triangu-
lar Toeplitz matrix or block triangular Toeplitz-like matrix [19, 26, 15]. Especially,
the SOE was used to approximate the kernel function, Jiang et. al [18] presented fast $L^1$ recursion formula to approximate the Caputo derivative ($0 < \alpha < 2$) and constructed fast $L^1$ difference scheme for fractional diffusion equation. The computational cost of this scheme was $O(MNN_{exp})$ as compared with $O(MN^2)$ for direct methods and the storage requirement was only $O(MNN_{exp})$ as compared with $O(MN)$ for direct methods in one dimension case, where $N, M, N_{exp}$ were respectively the total number of time steps, the number of grid points in space and the number of exponentials. For high dimension case, Zeng [40] et al. employed a fast Poisson solver for the system of linear algebraic equations for high-dimensional time fractional subdiffusion equations. The fast Poisson solver allowed us to handle the linear system directly with $O(M^2 \log M)$ operations in two dimension space when $M$ is the number of the spatial mesh points in both $x$ and $y$ directions in [3].

Inspired by our recent work [32], we extend and adapt its H2N2 method which was first introduced for single-term time-fractional diffusion-wave equations to numerically solve the following time fractional mixed sub-diffusion and diffusion-wave equations equation with non-homogeneous source term:

\[
\begin{cases}
\partial_0^\alpha D^\beta u(x,t) + \partial_0^\beta D^\gamma u(x,t) = \Delta u + q(x,t), & (x,t) \in \Omega \times [0,T], \\
u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in \Omega, \\
u(x,t) = 0, & x \in \partial \Omega, \quad 0 \leq t \leq T,
\end{cases}
\]

(1)

where $\Omega = \prod_{j=1}^d (l_j^{(j)}, r_j^{(j)}) \subset \mathbb{R}^d$, $\partial \Omega$ is the boundary of $\Omega$, $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \Omega$, $\Delta = \sum_{j=1}^d \partial^2_{x_j} u$, $q(x,t)$, $\phi(x)$, $\psi(x)$ are given sufficiently smooth functions satisfying $\phi(x) = 0$ when $x \in \partial \Omega$. Meanwhile, $\partial_0^\alpha D^\beta u(x,t)$ and $\partial_0^\beta D^\gamma u(x,t)$ with $0 < \alpha < 1 < \beta < 2$ denote the Caputo derivatives defined by

\[
\partial_0^\alpha D^\beta u(x,t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \partial^n u(x,s) \frac{1}{(t-s)^{\gamma-n+1}} ds, \quad n-1 < \gamma \leq n.
\]

It is different from the previous work, the main contributions are highlighted as follows:

- A novel H2N2 finite difference scheme for TFMDDEs, which can reach the convergence of $3 - \beta$ order only depending on $\beta$ in time and second order in space;
- A fully new fast H2N2 difference scheme for TFMDDEs, under the condition of tolerance error $\epsilon = 10^{-12}$ of fast algorithm, can reach the same convergence order as the H2N2 difference scheme;
- The truncation error of two approximation formulas and convergence rate of two schemes are analyzed in detail. Meanwhile, numerical examples not only verify the derived convergence theorems but also show the fast algorithm takes less CPU time than the direct algorithm with the same accuracy.

The rest of this paper is organized as follows. In Section 2, firstly, some notations are shown. Secondly, the truncation error of a H2N2 approximation formula is analyzed. Then the H2N2 difference scheme is constructed. Finally, the stability and convergence order are proved by energy method. In Section 4, a novel fast H2N2 formula is proposed for fractional derivative and corresponding truncation error is obtained. Then applying this formula, the fast H2N2 difference scheme
is presented and convergence is analyzed. Numerical experiments are reported in Section 5. In Section 6, we give a brief conclusion.

In this paper $c_1$ and $C_1$ denote generic constants that depend on the data of the problem but are independent of $N$ and can take different values in different places.

2. The discretization of the Caputo derivative.

2.1. Notations. In this section, we are going to consider the approximation of the Caputo derivative of order $\alpha \in (0,1)$ and $\beta \in (1,2)$:

$$C_0^\alpha D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds \quad \text{ and } \quad C_0^\beta D_t^\beta f(t) = c_\beta \int_0^t \frac{f''(s)}{(t-s)^{\beta-1}} ds,$$

where $c_\beta = 1/\Gamma(2-\beta)$, if $\epsilon = \alpha$ or $\beta$. Then we construct a finite difference scheme for solving the problem (1) and analyze the convergence of this scheme. The convergence order will be equal to the order of the approximation error.

Before presenting our numerical schemes, we introduce some notations. Denote $L^j = r^j - l^j, \ j = 1,2,\ldots,d$. We discretize spatial domain $\Omega$ by spatial step size $h_j = L^j/l^j$ for positive integers $l^j, \ j = 1,2,\ldots,d$ and denote $h = \max_{1 \leq j \leq d} h_j$. Let

$$x_{i_1}^j = l^j + i^j h_j, \ i^j = 0,1,\ldots,l^j, \ j = 1,2,\ldots,d.$$

The discrete grids in space are

$$\Omega_h = \{(x_{i_1}^{(1)},x_{i_2}^{(2)},\ldots,x_{i_d}^{(d)}) | i^{(k)} = 0,1,\ldots,l^{(k)}, \ k = 1,2,\ldots,d\},$$

$$\Omega_h = \Omega_h \cap \Omega, \quad \partial\Omega_h = \Omega_h \cap \partial\Omega,$$

and $M$ is the total number of grid points in space equal to $\prod_{j=1}^d (l^{(j)} + 1)$. Denote the index vector $i = (i^{(1)},i^{(2)},\ldots,i^{(d)})$ and the spatial point $x_i = (x_{i_1}^{(1)},x_{i_2}^{(2)},\ldots,x_{i_d}^{(d)})$, then we can define the index space

$$\tilde{I} = \{i | x_i \in \tilde{\Omega}_h\}, \quad I = \{i | x_i \in \Omega_h\}, \quad \partial I = \{i | x_i \in \partial\Omega_h\}.$$

Thus the grid function space is defined by

$$\mathcal{V}_h = \{u \ | \ u \text{ being a grid function on } \tilde{\Omega}_h\}, \quad \mathcal{V}_h = \{u \ | \ u \in \mathcal{V}_h; u_i = 0 \text{ when } i \in \partial I\}.$$

For $u \in \mathcal{V}_h$, introduce the following notations for $j = 1,2,\ldots,d$,

$$\delta_j u_{i_1,i_2,\ldots,i_{j-1}} = \frac{1}{h_j^2} (u_{i_1+\delta_j} - u_{i_1}), \quad \delta_j^2 u_i = \frac{1}{h_j^2} (u_{i+\delta_j} - 2u_i + u_{i-\delta_j}), \quad \Delta_h u_i = \sum_{j=1}^d \delta_j^2 u_i,$$

where $\delta_j = (0,\ldots,1,\ldots,0)$ is an index with 1 at the $j$th position and 0 at other positions.

For any $u, w \in \mathcal{V}_h$, the inner products and norms are defined by

$$(u,w) = \left(\prod_{j=1}^d h_j\right) \sum_{i \in I} u_i w_i, \quad \|u\| = \sqrt{(u,u)},$$

$$(\delta_j u,\delta_j w) = \left(\prod_{j=1}^d h_j\right) \sum_{i^{(j)}=0}^{l^{(j)}-1} \left(\prod_{\substack{l=1,l\neq j \ \text{and} \ i^{(l)}=1}}^d (\delta_j u_{i_1,i_2,\ldots,i_{j-1}}) (\delta_j w_{i_1,i_2,\ldots,i_{j-1}})\right),$$

$$|u|_{1,j}^0 = \sqrt{(\delta_j u,\delta_j u)}, \quad \|
abla_h u\| = \sqrt{\sum_{j=1}^d |u|_{1,j}^2}, \quad \|u\|_\infty = \max_{i \in \tilde{I}} |u_i|.$$
Taking the integer and denote $N$, $\tau = \frac{T}{N}$, $t_n = n\tau$, ($0 \leq n \leq N$), $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$; $\Omega_x = \{t_k | 0 \leq k \leq N\}$. For $w = \{w^n | 0 \leq n \leq N\}$ defined on $\Omega_x$, introduce the following notations

$$w^{n-\frac{1}{2}} = \frac{1}{2}(w^n + w^{n-1}), \quad \delta_t w^{n-\frac{1}{2}} = \frac{1}{\tau}(w^n - w^{n-1}), \quad 1 \leq n \leq N.$$

2.2. The H2N2 formula. An H2N2 approximation for mixed Caputo derivative

$\frac{\gamma}{\check{\alpha}}D_0^\beta f(t) + \frac{\gamma}{\check{\alpha}}D_0^\beta f(t)$, $\alpha \in (0, 1)$, $\beta \in (1, 2)$. For any function $f$ defined on the interval $[0, t_1]$, using $f(t_0)$, $f(t_1)$ and $f'(t_0)$, we denote by $H_{2,0}(t)$ its Hermite quadratic interpolation polynomial

$$H_{2,0}(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{1}{2\tau}\left[\delta_t f^{\frac{1}{2}} - f'(t_0)\right](t - t_0)^2,$$

$$H'_{2,0}(t) = f'(t_0) + \frac{2}{\tau}\left[\delta_t f^{\frac{1}{2}} - f'(t_0)\right](t - t_0), \quad (2)$$

$$H''_{2,0}(t) = \frac{2}{\tau}\left[\delta_t f^{\frac{1}{2}} - f'(t_0)\right]. \quad (3)$$

There exists $\theta_0, \xi_0 \in (t_0, t_1)$ such that

$$f(t) - H_{2,0}(t) = \frac{f'''(\theta_0)}{6}(t - t_0)^2(t - t_1) \quad \text{and} \quad \frac{2}{\tau}\left[\delta_t f^{\frac{1}{2}} - f'(t_0)\right] = f''(\xi_0).$$

For any function $f$ defined on the interval $[t_{k-1}, t_{k+1}]$ $(1 \leq k \leq N - 1)$, using $f(t_{k-1})$, $f(t_k)$ and $f(t_{k+1})$, we denote by $N_{2,k}(t)$ its Newton quadratic interpolation polynomial

$$N_{2,k}(t) = f(t_{k-1}) + \delta_t f^{k-\frac{1}{2}}(t - t_{k-\frac{1}{2}}) + \frac{1}{2}(\delta_t^2 f^k)(t - t_{k-1})(t - t_k),$$

$$N'_{2,k}(t) = \delta_t f^{k-\frac{1}{2}} + \frac{1}{2}(\delta_t^2 f^k)(2t - t_k - t_{k-1}), \quad (4)$$

$$N''_{2,k}(t) = \delta_t^2 f^k = \frac{1}{\tau}\left(\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k-\frac{1}{2}}\right). \quad (5)$$

There exist $\theta_k, \xi_k \in (t_{k-1}, t_{k+1})$ such that

$$f(t) - N_{2,k}(t) = \frac{f'''(\theta_k)}{6}(t - t_{k-1})(t - t_k)(t - t_{k+1}) \quad \text{and} \quad \delta_t^2 f^k = f''(\xi_k). \quad (6)$$

A numerical approximation formula [32] for the Caputo derivative $\frac{\gamma}{\check{\alpha}}D_0^\beta f(t)$ is given as follows

$$\frac{\gamma}{\check{\alpha}}D_0^\beta f(t_{n-\frac{1}{2}}) \approx c_\beta \left[\int_{t_0}^{t_n} \left(\frac{H''_{2,0}(t)}{2}\right)(t_{n-\frac{1}{2}} - t)^{1-\beta} dt + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N''_{2,k}(t)(t_{n-\frac{1}{2}} - t)^{1-\beta} dt\right]$$

$$= c_\beta \left[\frac{1}{\tau_0}(\delta_t f^{\frac{1}{2}} - f'(t_0)) + \sum_{k=1}^{n-1} b_k^{(n, \beta)}(\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k-\frac{1}{2}})\right]$$

$$= D_0^\beta f(t_{n-\frac{1}{2}}),$$

where $c_\beta = \psi(\beta)$, $\tau_0 = \frac{\tau}{\alpha}$, and $\psi(\beta)$ is the digamma function.
where $1 \leq k \leq n - 1$ for
\begin{align*}
v_0^{(n, \beta)} &= \frac{2}{\tau} \int_{t_0}^{t_0 + \frac{1}{2}} \frac{1}{-1} (t_n - s)^{\beta - 1} ds = \frac{2 \tau^{1-\beta}}{2 - \beta} \left[ (n - 1/2)^{2-\beta} - (n - 1)^{2-\beta} \right], \quad (8) \\
v_k^{(n, \beta)} &= \frac{1}{\tau} \int_{t_k - \frac{1}{2}}^{t_k + \frac{1}{2}} \frac{1}{-1} (t_n - s)^{\beta - 1} ds = \frac{\tau^{1-\beta}}{2 - \beta} \left[ (n - k)^{2-\beta} - (n - k - 1)^{2-\beta} \right]. \quad (9)
\end{align*}

**Theorem 2.1.** [32] Suppose $f(t) \in C^3[t_0, t_n]$. Denote $r_n = \delta D_0^\alpha f(t_n - \frac{1}{2}) - D_t^\alpha f(t_n - \frac{1}{2})$. Then, we have
\begin{equation}
|r_n| \leq \left[ \frac{C_\beta}{8} + \frac{1}{12\Gamma(3 - \beta)} + \frac{\beta - 1}{2\Gamma(4 - \beta)} \right] \max_{t_0 \leq t \leq t_n} |f''(t)| \tau^{3-\beta}.
\end{equation}

Now, we discuss the H2N2 approximation of the Caputo derivative $\delta D_0^\alpha f(t)$ on the special grid points of the form $\{t_0, t_\frac{1}{2}, t_\frac{3}{2}, \cdots, t_{n-\frac{1}{2}}, \cdots\}$. Using Eqs. (2)–(3) and Eqs. (4)–(5), it follows that
\begin{align*}
\delta D_0^\alpha f(t_n - \frac{1}{2}) &= \frac{1}{\Gamma(1 - \alpha)} \left[ \int_{t_0}^{t_0 + \frac{1}{2}} H_{2,0}(t)(t_n - t - \tau)^{-\alpha} dt + \sum_{k=1}^{n-1} \int_{t_k - \frac{1}{2}}^{t_k + \frac{1}{2}} N'_{2,k}(t)(t_n - t - \tau)^{-\alpha} dt \right] + \tilde{r}^n \\
&= c_\alpha \left( (t_n - t_0)^{1-\alpha} f'(t_0) - (t_n - t_{\frac{1}{2}} - \frac{1}{2})^{1-\alpha} \delta_t f^{\frac{1}{2}} + \int_{t_0}^{t_0 + \frac{1}{2}} H_{2,0}(t)(t_n - t - \tau)^{-\alpha} dt + \sum_{k=1}^{n-1} \left( (t_n - t_{\frac{1}{2}} - \frac{1}{2})^{1-\alpha} \delta_t f^{k+\frac{1}{2}} - (t_n - t_{k+\frac{1}{2}} - \frac{1}{2})^{1-\alpha} \delta_t f^{k+\frac{1}{2}} + \int_{t_k - \frac{1}{2}}^{t_{k+\frac{1}{2}}} N'_{2,k}(t)(t_n - t - \tau)^{-\alpha} dt \right) \right) + \tilde{r}^n \\
&= c_\alpha \left[ (A_0^{(n, \alpha)} f(t_0) + a_0^{(n, \alpha)} (\delta_t f^{\frac{1}{2}} - f'(t_0))) + \sum_{k=1}^{n-1} a_k^{(n, \alpha)} (\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k+\frac{1}{2}}) \right] + \tilde{r}^n \\
&= c_\alpha \left[ a_{n-1}^{(n, \alpha)} (\delta_t f^{n-\frac{1}{2}} + \sum_{k=1}^{n-1} (a_k^{(n, \alpha)} - a_{k-1}^{(n, \alpha)}) \delta_t f^{k-\frac{1}{2}} + (A_0^{(n, \alpha)} - a_0^{(n, \alpha)}) f'(t_0) \right] + \tilde{r}^n
\end{align*}
where the coefficients read
\begin{align*}
A_0^{(n, \alpha)} &= (t_n - t_0)^{1-\alpha} = \tau^{1-\alpha}(n - \frac{1}{2})^{1-\alpha}, \quad (11) \\
a_0^{(n, \alpha)} &= \frac{2}{\tau} \int_{t_0}^{t_0 + \frac{1}{2}} (t_n - t - \tau)^{-\alpha} dt = \frac{2 \tau^{1-\alpha}}{2 - \alpha} \left[ (n - \frac{1}{2})^{2-\alpha} - (n - 1)^{2-\alpha} \right]. \quad (12)
\end{align*}
\[ a^{(n,\alpha)}_k = \frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\alpha} dt \]
\[ = \frac{\tau^{1-\alpha}}{2-\alpha} [(n-k)^{2-\alpha} - (n-k-1)^{2-\alpha}], \quad 1 \leq k \leq n-1, \quad (13) \]

**Theorem 2.2.** Assume that \( f(t) \in C^3[t_0, t_n] \), it holds
\[ |\hat{p}_1| \leq \frac{\tau^{3-\alpha} c_\alpha}{8} \max_{t_0 \leq t \leq t_1} |f'''(t)|, \quad |\hat{p}_k| \leq \frac{\tau^{3-\alpha}}{3\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_n} |f'''(t)|, \quad 1 \leq k \leq n-1. \]

**Proof.**
(i) For the first step truncation error, we have from Eq. (10)
\[ |\hat{p}_1| \leq \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_{\frac{1}{2}}} (f'(t) - H_2^{\alpha}(t))(t_{\frac{1}{2}} - t)^{-\alpha} dt \]
\[ \leq \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_{\frac{1}{2}}} \frac{f'''(\xi)}{2} (t-t_0)^2(t_{\frac{1}{2}} - t)^{-\alpha} dt \]
\[ \leq \frac{\tau^{3-\alpha}}{8\Gamma(2-\alpha)} \max_{t_0 \leq t \leq t_1} |f'''(t)|. \quad (14) \]
Thus the first conclusion is proven.

(ii) Now we estimate \( \hat{p}_k \), \( 1 \leq k \leq n-2 \). It follows from Eq. (10), Eq. (6) and integration by parts that
\[ \hat{p}_k = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n-2} \left[ (f(t) - N_{2,k}(t)) \cdot (t_{n-\frac{1}{2}} - t)^{-\alpha} \mid_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \right. \]
\[ - \alpha \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (f(t) - N_{2,k}(t)) \cdot (t_{n-\frac{1}{2}} - t)^{-1-\alpha} dt \]
\[ = A_1 + B_1, \quad 1 \leq k \leq n-2. \]
Using Eq. (6), one has
\[ |A_1| \leq \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n-2} \left[ (f(t) - N_{2,k}(t)) \cdot (t_{n-\frac{1}{2}} - t)^{-\alpha} \right] \mid_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \]
\[ \leq \frac{\tau^3}{3\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{n-1}} |f'''(t)| \left[ |(t_{n-\frac{1}{2}} - t_{n-2})^{-\alpha} - (t_{n-\frac{1}{2}} - t_{k-\frac{1}{2}})^{-\alpha}| \right. \]
\[ + \left. \| (f(t_{n-\frac{1}{2}}) - N_{2,n-2}(t_{n-\frac{1}{2}})) - (f(t_{\frac{1}{2}}) - N_{2,1}(t_{\frac{1}{2}}))(t_{n-\frac{1}{2}} - t_{n-\frac{1}{2}})^{-\alpha} \| \right] \quad (15) \]
Since \( |(t-t_{k-1})(t-t_k)(t-t_{k+1})| \leq 2\tau^3 \) for all \( t \in (t_{k-1}, t_{k+1}) \), we have
\[ |B_1| \leq \frac{\alpha \tau^3}{3\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{n-1}} |f'''(t)| \int_{t_0}^{t_{n-\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{-\alpha-1} dt \]
\[ = \frac{\alpha \tau^3}{3\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{n-1}} |f'''(t)| \left( \frac{\tau^{-\alpha}}{\alpha} - \frac{t_{n-\frac{1}{2}}^{-\alpha}}{\alpha} \right) \quad (16) \]
Now we estimate $\hat{r}^{n-1}$. A direct calculation shows that
\[
|f'(t) - N_{2,n-1}'| \leq \frac{1}{3} \tau^2 \max_{t_{n-2} \leq t \leq t_n} |f'''(t)|.
\]
Hence
\[
|\hat{r}^{n-1}| \leq \frac{1}{3(1-\alpha)} \tau^2 \max_{t_{n-2} \leq t \leq t_n} |f'''(t)| \int_{t_{n-2}}^{t_{n-1}} \frac{1}{(t-n+1/2)^{\alpha}} \text{d}s.
\]  \hfill (17)
Using Eqs. (14)-(17), we obtained the desired result. \hfill \Box

**Lemma 2.3.** For $\{A_0^{(n,\alpha)}, a_k^{(n,\alpha)}, b_k^{(n,\beta)}\}$, $0 \leq k \leq n-1$ defined by Eqs. (11)-(13) and Eqs. (8)-(9), it holds that
\[
0 < a_{n-1}^{(n,\alpha)} < a_{n-2}^{(n,\alpha)} < \cdots < a_0^{(n,\alpha)} < A_0^{(n,\alpha)},
\]  \hfill (18)
\[
b_k^{(n,\beta)} > b_{n-2}^{(n,\beta)} > b_1^{(n,\beta)} > \cdots > b_0^{(n,\beta)} > 0.
\]  \hfill (19)

**Proof.** Using the integral mean-value theorem and the definition of $\{A_0^{(n,\alpha)}, a_k^{(n,\alpha)}, b_k^{(n,\beta)}\}$ for $k = 0, 2, \cdots, n-1$, there exist numbers $\theta_0, \xi_0 \in (t_0, t_{1/2}), \theta_k, \xi_k \in (t_{k-1/2}, t_{k+1/2})$ such that
\[
a_k^{(n,\alpha)} = (t_n - \frac{1}{2} - \theta_k)^{1-\alpha}, \quad b_k^{(n,\beta)} = (t_n - \frac{1}{2} - \xi_k)^{1-\beta}, \quad 0 \leq k \leq n-1.
\]
Using the monotonicity of the function $(t_n - \frac{1}{2} - s)^{1-\alpha}$, $(t_n - \frac{1}{2} - s)^{1-\beta}$, we obtain the inequalities (18)-(19). \hfill \Box

Denote $g_0^{(n,\alpha)} = a_{n-1}^{(n,\alpha)}$, $g_k^{(n,\alpha)} = a_n^{(n,\alpha)} - a_{n-k}^{(n,\alpha)}$, $1 \leq k \leq n-1$, then
\[
D_t^\alpha f(t_{n-2}) = \frac{1}{\Gamma(2-\alpha)} \left[ g_0^{(n,\alpha)} \delta_1 f^{n-\frac{1}{2}} + \sum_{k=1}^{n-1} g_k^{(n,\alpha)} \delta_1 f^{n-k-\frac{1}{2}} \right] + (A_0^{(n,\alpha)} - a_0^{(n,\alpha)}) f'(t_0),
\]  \hfill (20)
Next, the property of $g_k^{(n,\alpha)}$, $1 \leq k \leq n-1$ is vital for the proof of convergence.

**Lemma 2.4.** For any $0 < \alpha < 1$, the coefficients $\{g_k, 0 \leq k \leq n-1\}$ satisfy
\begin{itemize}
  \item[(i)] $g_k^{(n,\alpha)} \geq 0$, $0 \leq k \leq n-1$,
  \item[(ii)] $g_1^{(n,\alpha)} \leq g_0^{(n,\alpha)}$, $g_{k+1}^{(n,\alpha)} \leq g_k^{(n,\alpha)}$, $1 \leq k \leq n-3$, $g_{n-2}^{(n,\alpha)} \leq g_{n-1}^{(n,\alpha)}$,
  \item[(iii)] $g_2^{(n,\alpha)} - 2g_1^{(n,\alpha)} + g_0^{(n,\alpha)} \geq 0$, $g_k^{(n,\alpha)} - 2g_{k+1}^{(n,\alpha)} + g_{k+3}^{(n,\alpha)} \geq 0$, $2 \leq k \leq n-3$,
\end{itemize}
where there are $\tilde{g}_0^{(n,\alpha)} = 8g_0^{(n,\alpha)}$ and
\[
\tilde{g}_{n-1}^{(n,\alpha)} = \tilde{g}_{n-1}^{(n,\alpha)} + \frac{\tau^{1-\alpha}}{2-\alpha} \left[ n^{2-\alpha} - 2(n - \frac{1}{2})^{2-\alpha} + (n-1)^{2-\alpha} \right]
\]
\[
= \tilde{g}_{n-1}^{(n,\alpha)} + (1-\alpha) \tau^{1-\alpha} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{dz_1 dz_2}{(n-1+z_1+z_2)\alpha}.
\]
Proof. (i) It is easy to get the first conclusion from the definition of $a_k^{(n,\alpha)}$, $0 \leq k \leq n-1$.

(ii) At first, a direct calculation is that
\[
g_0^{(n,\alpha)} - g_1^{(n,\alpha)} = 7a_{n-1}^{(n,\alpha)} - a_{n-2}^{(n,\alpha)} + a_{n-1}^{(n,\alpha)} = \frac{\tau^{1-\alpha}}{2-\alpha} [9 - 2^{2-\alpha}] \geq 0.
\]

Secondly, for $1 \leq k \leq n-3$, one has
\[
g_k^{(n,\alpha)} - g_{k+1}^{(n,\alpha)} = 2a_{n-k-1}^{(n,\alpha)} - a_{n-k}^{(n,\alpha)} - a_{n-k-2}^{(n,\alpha)}
= -\frac{\tau^{1-\alpha}}{2-\alpha} [(k+2)^{2-\alpha} - 3(k+1)^{2-\alpha} + 3k^{2-\alpha} - (k-1)^{2-\alpha}]
= (1-\alpha)\alpha \tau^{1-\alpha} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dz_1 dz_2 dz_3}{(k-1 + z_1 + z_2 + z_3)^{1-\alpha}} \geq 0.
\]

Finally, for $k = n-2$, similarly, we have
\[
g_{n-2}^{(n,\alpha)} - g_{n-1}^{(n,\alpha)} = 2a_{n-3}^{(n,\alpha)} - a_{n-2}^{(n,\alpha)} - a_{n-4}^{(n,\alpha)} - \frac{\tau^{1-\alpha}}{2-\alpha} [n^{2-\alpha} - 2(n-2)^{2-\alpha} + (n-1)^{2-\alpha}]
= (1-\alpha)\alpha \tau^{1-\alpha} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dz_1 dz_2 dz_3}{(n-3 + z_1 + z_2 + z_3)^{1-\alpha}} \geq 0.
\]

(iii) Firstly, we can get following inequality by a direct calculation
\[
g_2^{(n,\alpha)} - 2g_1^{(n,\alpha)} + g_0^{(n,\alpha)} = a_{n-3}^{(n,\alpha)} - 3a_{n-2}^{(n,\alpha)} + 2a_{n-1}^{(n,\alpha)} + 8a_{n-1}^{(n,\alpha)}
= \frac{\tau^{1-\alpha}}{2-\alpha} [3^{2-\alpha} - 2^{2-\alpha} - 3(2^{2-\alpha} - 1) + 10] \geq 0.
\]

Then, for $2 \leq k \leq n-3$, we have
\[
g_k^{(n,\alpha)} - 2g_{k-1}^{(n,\alpha)} + g_{k-2}^{(n,\alpha)} = a_{n-k-2}^{(n,\alpha)} - 3a_{n-k-1}^{(n,\alpha)} + 3a_{n-k}^{(n,\alpha)} - a_{n-k+1}^{(n,\alpha)}
= \frac{\tau^{1-\alpha}}{2-\alpha} [(k+2)^{2-\alpha} - 4(k+1)^{2-\alpha} + 6k^{2-\alpha} - 4(k-1)^{2-\alpha}]
= \rho_\alpha \tau^{1-\alpha} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dz_1 dz_2 dz_3 dz_4}{(k-2 + z_1 + z_2 + z_3 + z_4)^{\alpha+2}} \geq 0,
\]

where $\rho_\alpha = (1-\alpha)\alpha \alpha + 1)$. At last, one has
\[
g_{n-3}^{(n,\alpha)} - 2g_{n-2}^{(n,\alpha)} + g_{n-1}^{(n,\alpha)} = a_0^{(n,\alpha)} - 3a_1^{(n,\alpha)} + 3a_2^{(n,\alpha)} - a_3^{(n,\alpha)} + \frac{\tau^{1-\alpha}}{2-\alpha} [n^{2-\alpha}]
- 2(n-1)^{2-\alpha} + (n-1)^{2-\alpha}]
= \frac{\tau^{1-\alpha}}{2-\alpha} [n^{2-\alpha} - 4(n-1)^{2-\alpha} + 6(n-2)^{2-\alpha} - 4(n-3)^{2-\alpha}]
+ (n-4)^{2-\alpha}]
= \rho_\alpha \tau^{1-\alpha} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dz_1 dz_2 dz_3 dz_4}{(n-4 + z_1 + z_2 + z_3 + z_4)^{\alpha+2}} \geq 0,
\]

which completes the proof.
3. A difference scheme.

3.1. The derivation of the difference scheme. Suppose \( u(x, t) \in C^{(\alpha, \beta)}(\bar{\Omega} \times [0, T]) \) and denote

\[
U^n_i = u(x_i, t_n), \quad i \in I, \quad 0 \leq n \leq N,
\]

\[
\varphi_i = \varphi(x_i), \quad \psi_i = \psi(x_i), \quad i \in I.
\]

Considering the first equation of Eq. (1) at the point \((x_i, t_{n-\frac{1}{2}})\), it follows

\[
\frac{\partial}{\partial t} u(x_i, t_n) + \frac{\partial}{\partial x} u(x_i, t_n) = \Delta u(x_i, t_n) + q(x_i, t_{n-\frac{1}{2}}), \quad i \in I, \quad 1 \leq n \leq N,
\]

Applying Eq. (20) and Eq. (7) for approximating the time fractional derivative and central difference quotient for approximating the spatial derivative, we can obtain

\[
c_\alpha \left[ g^{(n, \alpha)}_0 \delta_t U^n_i + \sum_{k=1}^{n-1} g^{(n, \alpha)}_k \delta_t U^n_{i-k} - \frac{1}{2} \right] + (A^{(n, \alpha)}_0 - a^{(n, \alpha)}_0) \psi_i
\]

\[
+ c_\beta \left[ \sum_{k=1}^{n-1} b^{(n, \beta)}_k \left( \delta_t U^n_{i+k} - \delta_t U^n_{i-k} \right) + b^{(n, \beta)}_0 \left( \delta_t U^n_i - \psi_i \right) \right] = \Delta_h U^n_i + q_i - R^n_i, \quad i \in I, \quad 1 \leq n \leq N,
\]

and \( q^n_i - \frac{1}{2} = q(x_i, t_{n-\frac{1}{2}}) \). There exists a positive constants \( C_1 \) such that

\[
|R^n_i| \leq C_1 \left( h^{3-\beta} + h^2 \right), \quad i \in I, \quad 1 \leq n \leq N.
\]

Noticing the initial and boundary value conditions of Eq. (1), we have

\[
U^n_i = \varphi_i, \quad i \in I,
\]

\[
U^n_i = 0, \quad i \in \partial I, \quad 0 \leq n \leq N.
\]

Omitting \( R^n_i \) in Eq. (21) and replacing the grid function \( U^n_i \) by its numerical approximation \( u^n_i \), we construct the difference scheme for solving the problem (1) as follows:

\[
c_\alpha \left[ g^{(n, \alpha)}_0 \delta_t u^n_i + \sum_{k=1}^{n-1} g^{(n, \alpha)}_k \delta_t u^n_{i-k} - \frac{1}{2} \right] + (A^{(n, \alpha)}_0 - a^{(n, \alpha)}_0) \psi_i
\]

\[
+ c_\beta \left[ \sum_{k=1}^{n-1} b^{(n, \beta)}_k \left( \delta_t u^n_{i+k} - \delta_t u^n_{i-k} \right) + b^{(n, \beta)}_0 \left( \delta_t u^n_i - \psi_i \right) \right] = \Delta_h u^n_i + q_i, \quad i \in I, \quad 1 \leq n \leq N,
\]

\[
U^0_i = \varphi_i, \quad i \in I,
\]

\[
u^n_i = 0, \quad i \in \partial I, \quad 0 \leq n \leq N.
\]

At each time level, the finite difference scheme (25)-(27) leads to a block tridiagonal linear system and the coefficient matrix is strictly diagonally dominant [32]. Therefore, it has a unique solution.
3.2. The stability and convergence. Next, the stability and convergence of finite difference scheme (25)-(27) will be analyzed. It is easy to verify that

\[(\Delta_h u, u) = -\sum_{j=1}^{d}(\delta_j u, \delta_j u) = -\|\nabla_h u\|^2.\]  

(28)

Lemma 3.1. [25] Let \(\{c_0, c_1, \cdots, c_n\}\) be a sequence of real numbers with the properties

\[c_n \geq 0, \ c_n - c_{n-1} \leq 0, \ c_{n+1} - 2c_n + c_{n-1} \geq 0.\]

Then for any positive integer \(M\), and for each vector \([V_1, V_2, \cdots, V_M]\), holds that

\[\sum_{n=1}^{M} \left( \sum_{p=0}^{n-1} c_p V_{n-p} \right) V_n \geq 0.\]

Lemma 3.2. For \(V_1, V_2, \cdots, V_N \in \tilde{V}_h\), it holds that

\[\sum_{n=1}^{m} g^{(n,\alpha)}_0 \|V^n\|^2 + \sum_{n=2}^{m} \left( \sum_{k=1}^{n-2} g^{(n,\alpha)}_k (V^{n-k}, V^n) + \tilde{g}^{(n,\alpha)}_n \right) (V^1, V^n) \geq 0, \ 1 \leq m \leq N.\]

Proof. It follows from Lemma 2.4 and Lemma 3.1 to obtain the conclusion. □

Lemma 3.3. For \(\psi, V_1, V_2, \cdots, V_N \in \tilde{V}_h\), it holds that

\[\sum_{n=1}^{m} \left( \sum_{k=1}^{n-1} b^{(n,\beta)}_k \|V^k\|^2 - \sum_{n=1}^{m} b^{(n,\beta)}_0 \|\psi\|^2 \right) \geq \frac{1}{2} \left[ \sum_{k=1}^{m} b^{(m,\beta)}_k \|V^k\|^2 - \sum_{n=1}^{m} b^{(n,\beta)}_0 \|\psi\|^2 \right]\]

for \(1 \leq m \leq N\).

Proof. We introduce \(\tilde{Q} = \sum_{n=1}^{m} [b^{(n,\beta)}_n \|V^n\|^2 - \sum_{k=1}^{n-1} (b^{(n,\beta)}_k - b^{(n,\beta)}_{k-1}) (V^k, V^n) - b^{(n,\beta)}_0 \|\psi\|^2],\) then

\[\frac{1}{\Gamma(2-\beta)} \tilde{Q} \geq \frac{1}{\Gamma(2-\beta)} \sum_{n=1}^{m} \left[ b^{(n,\beta)}_n \|V^n\|^2 - \frac{1}{2} \sum_{k=1}^{n-1} (b^{(n,\beta)}_k - b^{(n,\beta)}_{k-1}) (\|V^k\|^2 + \|V^n\|^2) \right.\]

\[- \left. \frac{b^{(n,\beta)}_0}{2} (\|\psi\|^2 + \|V^n\|^2) \right] \geq \frac{1}{2\Gamma(2-\beta)} \left[ \sum_{k=1}^{m} b^{(m,\beta)}_k \|V^k\|^2 - \sum_{n=1}^{m} b^{(n,\beta)}_0 \|\psi\|^2 \right],\]

which completes the proof. □
Theorem 3.4. Assume \{u^n_i\} \in \tilde{I}, \ 0 \leq n \leq N\} satisfy
\begin{align}
c_{\alpha}\left[9_0^{(1,\alpha)}\delta_t u^\frac{1}{2}_i + (A_0^{(1,\alpha)} - a_0^{(1,\alpha)})\psi_i\right] + c_\beta b_0^{(1,\beta)}\left(\delta_t u^\frac{1}{2}_i - \psi_i\right) &= \Delta_h u^\frac{1}{2}_i + p^i, \quad (29) \\
c_{\alpha}\left[9_0^{(n,\alpha)}\delta_t u^{n-\frac{1}{2}}_i + \sum_{k=1}^{n-1} \frac{g_k^{(n,\alpha)}\delta_t u^{n-k-\frac{1}{2}}_i + (A_0^{(n,\alpha)} - a_0^{(n,\alpha)})\psi_i} + c_\beta b_{n-1}^{(n,\beta)}\delta_t u^{n-\frac{1}{2}}_i - \sum_{k=1}^{n-1} (b_k^{(n,\beta)} - b_{k-1}^{(n,\beta)})\delta_t u^{k-\frac{1}{2}}_i + b_0^{(n,\beta)}\psi_i\right] &= \Delta_h u^{n-\frac{1}{2}}_i + p^n_i, \quad i \in I, \ 2 \leq n \leq N, \quad (30) \\
u^n_0 &= \varphi_i, \quad i \in I, \\
u^n_i &= 0, \quad i \in \partial I, \quad 0 \leq n \leq N. \quad (31) (32)
\end{align}

If \(\tau \leq \tau_0, \tau_0 = (\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{4M(2-\beta)})^\frac{1}{1-2\alpha}\), it holds that
\begin{align}
\|\nabla_h u^n\|^2 &\leq (1 + \frac{4C_2\Gamma(2-\beta)}{b_0^{(1,\beta)}})\|\nabla_h u^0\|^2 + \sum_{k=1}^{m} \left(\frac{\tau b_0^{(k,\beta)}}{\Gamma(2-\beta)}\right)^2 + 6\tau \frac{\Gamma(2-\beta)(A_0^{(n,\alpha)} - a_0^{(n,\alpha)})^2}{\Gamma^2(2-\alpha)b_0^{(n,\beta)}}\|\psi\|^2 + 6\tau \frac{\|p_k\|^2}{b_0^{(n,\beta)}} + \frac{8c_0\tau}{(b_0^{(1,\beta)})^2}\|p_1\|^2 + (8C_2\tau + \frac{8C_2\Gamma^2(2-\beta)\tau(A_0^{(1,\alpha)} - a_0^{(1,\alpha)})^2}{\Gamma^2(2-\alpha)(b_0^{(1,\beta)})^2})\|\psi\|^2.
\end{align}

Proof. (I) Case \(n = 1\). Taking an inner product (29) with \(\delta_t u^\frac{1}{2}\) and using Eq. (28), we have
\begin{align}
c_{\alpha}\left[9_0^{(1,\alpha)}\|\delta_t u^\frac{1}{2}\|^2 + (A_0^{(1,\alpha)} - a_0^{(1,\alpha)})\psi_\alpha + c_\beta \left[b_0^{(1,\beta)}\|\delta_t u^\frac{1}{2}\|^2 - b_0^{(1,\beta)}(\psi, \delta_t u^\frac{1}{2})\right]
\right] = (\Delta_h u^\frac{1}{2}, \delta_t u^\frac{1}{2}) + (p^1, \delta_t u^\frac{1}{2}),
\end{align}
which follows
\begin{align}
c_{\alpha}(9_0^{(1,\alpha)} - b_0^{(1,\alpha)})\|\delta_t u^\frac{1}{2}\|^2 + c_\beta \left[b_0^{(1,\beta)}\|\delta_t u^\frac{1}{2}\|^2 - b_0^{(1,\beta)}(\psi, \delta_t u^\frac{1}{2})\right]
\right] = (\Delta_h u^\frac{1}{2}, \delta_t u^\frac{1}{2}) + (p^1, \delta_t u^\frac{1}{2}) - \frac{(A_0^{(1,\alpha)} - a_0^{(1,\alpha)})}{\Gamma(2-\alpha)}(\psi, \delta_t u^\frac{1}{2}). \quad (33)
\end{align}

(II) Case \(n \geq 2\). Taking an inner product (30) with \(\delta_t u^{n-\frac{1}{2}}\) and using Eq. (28), we can get
\begin{align}
c_{\alpha}\left[9_0^{(n,\alpha)}\|\delta_t u^{n-\frac{1}{2}}\|^2 + \sum_{k=1}^{n-1} \frac{g_k^{(n,\alpha)}(\delta_t u^{n-k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}})} + c_\beta b_{n-1}^{(n,\beta)}\|\delta_t u^{n-\frac{1}{2}}\|^2
\right]
\right] - \sum_{k=1}^{n-1} (b_k^{(n,\beta)} - b_{k-1}^{(n,\beta)})(\delta_t u^{k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) - b_0^{(n,\beta)}(\psi, \delta_t u^{n-\frac{1}{2}})
\right] = (\Delta_h u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) + (p^n, \delta_t u^{n-\frac{1}{2}}) - \frac{(A_0^{(n,\alpha)} - a_0^{(n,\alpha)})}{\Gamma(2-\alpha)}(\psi, \delta_t u^{n-\frac{1}{2}}), \quad 2 \leq n \leq N.
\end{align}
Adding the first term of Eq. (33) with the first term of Eq. (34) and using Lemma 3.2, we have following estimation

\[
\sum_{n=0}^{m} \left[ \frac{\tau^{\alpha}}{\Gamma(2 - \alpha)} \right] \sum_{k=1}^{n-1} g_k^{(n, \alpha)} (\delta_t u^{n-k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) + c_\alpha \sum_{n=2}^{m} \frac{\tau^{\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{n-2} g_k^{(n, \alpha)} (\delta_t u^{n-k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}})
\]

Adding the first term of Eq. (33) with the first term of Eq. (34) and using Lemma 3.2, we have following estimation

\[
\sum_{n=0}^{m} \frac{\tau^{\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{n-1} g_k^{(n, \alpha)} (\delta_t u^{n-k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) + \frac{\tau^{\alpha}}{\Gamma(2 - \alpha)} \sum_{n=2}^{m} \frac{\tau^{\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{n-2} g_k^{(n, \alpha)} (\delta_t u^{n-k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}})
\]

Using the Lemma 3.3 and adding the second term of Eq. (34) with the second term of Eq. (33), it follows that

\[
c_\beta \sum_{n=1}^{m} b_k^{(n, \beta)} ||\delta_t u^{n-\frac{1}{2}}||^2 - \frac{1}{2} \sum_{k=1}^{m} b_k^{(n, \beta)} (\delta_t u^{k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) - b_0^{(n, \beta)} (\psi, \delta_t u^{n-\frac{1}{2}})
\]

Then, it is easy to know that

\[
\sum_{n=1}^{m} (\Delta_h u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) = - \frac{1}{2} \left( ||\nabla_h u^m||^2 - ||\nabla_h u^0||^2 \right). \quad (37)
\]
Using Eqs. (35)-(36), we have

\[
\frac{c_β}{6} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 - \frac{c_β}{2} \sum_{n=1}^{m} b_0^{(n, β)} \|\psi\|^2 + \frac{c_β}{6} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 -
\]

\[
\frac{11 t^{1-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=1}^{m} \|\delta_t u^{n-\frac{1}{2}}\|^2 + \frac{c_β}{2} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 - \frac{\tau^{1-\alpha}}{8\Gamma(1-\alpha)} \sum_{k=2}^{m} \|\delta_t u^{\frac{1}{2}}\|^2
\]

\[
\geq \frac{c_β}{6} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 - \frac{c_β}{2} \sum_{n=1}^{m} b_0^{(n, β)} \|\psi\|^2 + \left[ \frac{c_β T^{1-\beta}}{6} - \frac{T}{8\Gamma(1-\alpha)\tau^\alpha} \right] \|\delta_t u^{\frac{1}{2}}\|^2
\]

\[
\geq \frac{c_β}{6} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 - \frac{c_β}{2} \sum_{n=1}^{m} b_0^{(n, β)} \|\psi\|^2 + \left[ \frac{c_β T^{1-\beta}}{6} - \frac{T}{8\Gamma(1-\alpha)\tau^\alpha} \right] \|\delta_t u^{\frac{1}{2}}\|^2.
\]

(38)

Using Eqs. (33), (34), (37) and (38) and setting \(C_2 = \max_{0 < \alpha < 1, 1 < \beta < 2} \left| \frac{T^{1-\beta}}{6\Gamma(2-\beta)} - \frac{T}{8\Gamma(1-\alpha)\tau^\alpha} \right| \), it follows that

\[
\frac{1}{6\Gamma(2-\beta)} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 + \frac{1}{2T} \left( \|\nabla_h u^m\|^2 - \|\nabla_h u^0\|^2 \right)
\]

\[
\leq \frac{1}{2T(2-\beta)} \sum_{n=1}^{m} b_0^{(n, β)} \|\psi\|^2 + \sum_{n=1}^{m} \left( p_n, \delta_t u^{n-\frac{1}{2}} \right) - \sum_{n=1}^{m} \left( A_0^{(n, α)} - a_0^{(n, α)} \right) \left( \psi, \delta_t u^{n-\frac{1}{2}} \right) + C_2 \|\delta_t u^{\frac{1}{2}}\|^2
\]

\[
\leq \frac{1}{2T(2-\beta)} \sum_{n=1}^{m} b_0^{(n, β)} \|\psi\|^2 + \frac{1}{12T(2-\beta)} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 + 3 \sum_{k=1}^{m} \left( A_0^{(2-\beta)} - a_0^{(n, α)} \right)^2 \|\psi\|^2
\]

\[
+ \frac{1}{12T(2-β)} \sum_{k=1}^{m} b_{k-1}^{(m, β)} \|\delta_t u^{k-\frac{1}{2}}\|^2 + 3 \sum_{k=1}^{m} \left( A_0^{(2-\beta)} - a_0^{(n, α)} \right)^2 \|\psi\|^2
\]

\[
+ \frac{2C_2^2 T^{2-\beta}}{\tau b_0^{(1, β)}} \|\nabla_h u^0\|^2 + 4C_2 \|\psi\|^2 + \frac{4C_2^2 T^{2-\beta}}{(b_0^{(1, β)})^2} \|p^1\|^2
\]

\[
+ \frac{4C_2^2 T^{2-\beta}(A_0^{(1, α)} - a_0^{(1, α)})^2}{\tau^2(b_0^{(1, β)})^2} \|\psi\|^2.
\]

(39)

The last inequality of (39), from Eq. (33), one can similarly use

\[
\frac{b_0^{(1, β)}}{\Gamma(2-\beta)} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{1}{2T} \|\nabla_h u^0\|^2 \leq \frac{1}{2T} \|\nabla_h u^0\|^2 + \frac{b_0^{(1, β)}}{4\Gamma(2-\beta)} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{b_0^{(1, β)}}{\Gamma(2-\beta)} \|\psi\|^2
\]

\[
+ \frac{b_0^{(1, β)}}{4\Gamma(2-\beta)} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{b_0^{(1, β)}}{b_0^{(1, β)}} \|p^1\|^2 +
\]

\[
\frac{b_0^{(1, β)}}{4\Gamma(2-\beta)} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{b_0^{(1, β)}}{b_0^{(1, β)}} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{b_0^{(1, β)}}{\Gamma(2-\beta)} \|\psi\|^2,
\]

\[
\frac{b_0^{(1, β)}}{\Gamma(2-\beta)} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{b_0^{(1, β)}}{b_0^{(1, β)}} \|\delta_t u^{\frac{1}{2}}\|^2 + \frac{b_0^{(1, β)}}{\Gamma(2-\beta)} \|\psi\|^2.
\]
it follows from that
\[
\|\delta tu^\frac{1}{2}\|^2 \leq \frac{2\Gamma(2-\beta)}{\tau_0^{1(1,\beta)}} \|\nabla h_0u^0\|^2 + 4\|\psi\|^2 + \frac{4\Gamma^2(2-\beta)}{(b_0^{1(1,\beta)})^2} \|p^1\|^2 \\
+ \frac{4\Gamma^2(2-\beta)(A_0^{1(1,\alpha)} - a_0^{1(1,\alpha)})^2}{\Gamma^2(2-\alpha)(b_0^{1(1,\beta)})^2} \|\psi\|^2.
\]

Thus, we rewrite Eq. (39) as
\[
\|\nabla h^n u^n\|^2 - \|\nabla h^0 u^0\|^2 \leq \frac{\tau}{\Gamma(2-\beta)} \sum_{k=1}^m b_k^{(k,\beta)} \|\psi\|^2 + 6\Gamma(2-\beta)\tau \sum_{k=1}^m \|p^k\|^2 + 6\Gamma(2-\beta)\tau \sum_{k=1}^m \frac{(A_0^{(n,\alpha)} - a_0^{(n,\alpha)})^2}{\Gamma^2(2-\alpha)b_k^{(k,\beta)\}} \|\psi\|^2 + \frac{4C_2\Gamma^2(2-\beta)}{(b_0^{1(1,\beta)})^2} \|\nabla h_0u^0\|^2 \\
+ 8C_2\tau\|\psi\|^2 + \frac{8C_2\Gamma^2(2-\beta)\tau}{(b_0^{1(1,\beta)})^2} \|p^1\|^2 + \frac{8C_2\Gamma^2(2-\beta)\tau(A_0^{1(1,\alpha)} - a_0^{1(1,\alpha)})^2}{\Gamma^2(2-\alpha)(b_0^{1(1,\beta)})^2} \|\psi\|^2,
\]

which completes the proof. \(\square\)

**Theorem 3.5.** Assume \(\{U^n_i\}\) and \(\{u^n_i\}\) are solutions of the problem (1) and difference scheme (25)-(27), respectively. Denote
\[
\tilde{u}^n_i = U^n_i - u^n_i, \quad i \in I, \quad 0 \leq n \leq N.
\]

Then there exists a positive constant \(C_3\) such that
\[
\|\nabla h\tilde{u}^n\| \leq C_3(\tau^{3-\beta} + h^2), \quad 0 \leq n \leq N.
\]

**Proof.** Subtracting Eqs. (25)-(27) from Eqs. (21), (23)-(24), we obtain the following error equations
\[
c_\alpha \left[ g_0^{(n,\alpha)} \delta_t \tilde{u}^{n-\frac{1}{2}} + \sum_{k=1}^{n-1} g_k^{(n,\alpha)} \delta_t \tilde{u}^{n-k-\frac{1}{2}} \right] + \sum_{k=1}^{n-1} b_k^{(n,\beta)} \left( \delta_t \tilde{u}^{n-k+\frac{1}{2}} - \delta_t \tilde{u}^{n-k-\frac{1}{2}} \right) \\
+ b_0^{(n,\beta)} \delta_t \tilde{u}^{\frac{1}{2}} = \Delta h \tilde{u}^{n-\frac{1}{2}} + R^n_i, \quad i \in I, \quad 1 \leq n \leq N,
\]
\[
\tilde{u}^0_i = 0, \quad i \in I,
\]
\[
\tilde{u}^0_i = 0, \quad i \in \partial I, \quad 0 \leq n \leq N.
\]

Applying Eq. (22) and Theorem 3.4, it yields
\[
\|\nabla h\tilde{u}^n\| \leq 6\Gamma(2-\beta)C_1^2 \tau \sum_{k=1}^m \frac{1}{b_k^{(n,\beta)}} \left( \tau^{3-\beta} + h^2 \right)^2 + 8\Gamma^2(2-\beta)C_1C_2\tau(\tau^{3-\beta} + h^2)^2 \\
\leq C_2^2(\tau^{3-\beta} + h^2)^2, \quad 0 \leq n \leq N.
\]

Here, the inequality has been used
\[
\frac{1}{b_k^{(n,\beta)}} \leq \frac{1}{b_0^{(n,\beta)}} \leq t^{\beta-1}n^{-\frac{1}{2}},
\]

which completes the proof. \(\square\)
4. A fast difference scheme. In order to improve the computational efficiency, we apply Jiang’s fast evaluation [18] of the L1 scheme to accelerate the proposed scheme for solving our problem (1). In their work, an efficient sum-of-exponentials approximation for the kernel $t^{-\alpha}$ is proposed to reduce the computation cost and memory for the evaluation of the Caputo fractional derivative. We present an H2N2 fast approximation formula for mixed fractional derivatives $\frac{\partial}{\partial t}^\alpha D_0^\beta f(t) + \frac{\partial}{\partial t}^\alpha D_0^\beta f(t)$. Then applying this formula, we presented a corresponding fast difference scheme.

4.1. Fast time discretization.

Lemma 4.1. ([18]) For the given $\alpha \in (0, 1)$ and tolerance error $\epsilon$, cut-off time restriction $\delta$ and final time $T$, there are one positive integer $N_{exp}$, positive points $\{s_l \mid l = 1, 2, \cdots, N_{exp}\}$ and corresponding positive weights $\{w_l \mid l = 1, 2, \cdots, N_{exp}\}$ such that

$$\left| t^{-\alpha} - \sum_{l=1}^{N_{exp}} w_l e^{-s_l t} \right| \leq \epsilon, \quad \forall t \in [\delta, T],$$

where

$$N_{exp} = O \left( \left( \log \frac{1}{\epsilon} \right) \left( \log \log \frac{1}{\epsilon} + \log \frac{T}{\delta} \right) + \left( \log \frac{1}{\delta} \right) \left( \log \log \frac{1}{\epsilon} + \log \frac{1}{\delta} \right) \right).$$

Now we will derive a fast algorithm for approximating the mixed Caputo fractional derivatives $\frac{\partial}{\partial t}^\alpha D_0^\beta f(t_{n-\frac{1}{2}})$, $\frac{\partial}{\partial t}^\alpha D_0^\beta f(t_{n-\frac{1}{2}})$, respectively. Let $\delta = \frac{T}{2}$, $\delta f^{-\frac{1}{2}} = f'(t_0)$. From Eq. (2), Eq. (4) and Lemma 4.1, we have

$$\frac{\partial}{\partial t}^\alpha D_0^\beta f(t_{n-\frac{1}{2}}) \approx \frac{1}{\Gamma(1 - \alpha)} \left\{ \int_{t_0}^{t_{\frac{1}{2}}} H_{2,0}^{\prime}(t) \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{l_1} (t_{n-\frac{1}{2}} - t)} dt + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_{2,k}(t) \right\}$$

$$= \frac{1}{\Gamma(1 - \alpha)} \left\{ \int_{t_0}^{t_{\frac{1}{2}}} (f'(t_0) + H_{2,0}^{\prime}(t) - t_0) \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{l_1} (t_{n-\frac{1}{2}} - t)} dt + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \left[ \delta f^{k-\frac{1}{2}} + \frac{1}{2} (\delta f^{k})^2(2t - t_k - t) \right] \right\}$$

$$= \frac{1}{\Gamma(1 - \alpha)} \left\{ \sum_{l_1=1}^{N_{1,exp}} w_{l_1} F_{l_1} + c_\alpha r^{1-\alpha} \delta f^{-\frac{3}{2}} + c_\alpha a_{n-1}(\alpha) (\delta f^{-\frac{3}{2}} - \delta f^{-\frac{1}{2}}) \right\}$$

$$= \frac{\partial}{\partial t}^\alpha D_0^\beta f(t_{n-\frac{1}{2}}), \quad 1 \leq n \leq N$$
and
\[
\begin{align*}
C_0D^\beta_t f(t_{n-\frac{1}{2}}) &= \frac{1}{\Gamma(2-\beta)} \left[ \int_0^{t_{n-\frac{1}{2}}} f''(t) \left( t_n - \frac{1}{2} - t \right)^{\beta-1} dt + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} f''(t) \left( t_n - \frac{1}{2} - t \right)^{\beta-1} dt \right] \\
&\approx \frac{1}{\Gamma(2-\beta)} \left[ \int_{t_0}^{t_{n-\frac{1}{2}}} H'_2(t) \sum_{l=1}^{N_{2,\exp}} w_{l2} e^{-sl_2(t_n - \frac{1}{2} - t)} dt \\
&\quad + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N'_{2,k}(t) \sum_{l=1}^{N_{2,\exp}} w_{l2} e^{-sl_2(t_n - \frac{1}{2} - t)} dt \\
&\quad + \int_{t_{n-\frac{1}{2}}}^{t_n} N''_{2,n-1}(t)(t_n - \frac{1}{2} - t)^{1-\beta} dt \right] \\
&= \frac{1}{\Gamma(2-\beta)} \sum_{l=1}^{N_{2,\exp}} w_{l2} F^n_{l2} + \frac{\eta(n,\beta)}{\Gamma(2-\beta)} \left( \delta_t f^{n-\frac{1}{2}} - \delta_t f^{n-\frac{3}{2}} \right) \\
&= C_0D^\beta_t f(t_{n-\frac{1}{2}}), \quad 1 \leq n \leq N,
\end{align*}
\]
where \( \delta t f^{-\frac{1}{2}} = f'(t_0) \), \( F_1^1 = 0 \), \( F_2^1 = 0 \), \( B_{1,t_1}^2 = \int_{t_0}^{t_1} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds \),

\[
B_{1,t_1}^{n} = \int_{t_{n_{-\frac{1}{2}}}}^{t_{n_{-\frac{1}{2}}}} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds, \quad n \geq 3, \quad 1 \leq l_1 \leq N_{1,exp},
\]

\[
B_{2,t_1}^{n} = \frac{2}{\tau} \int_{t_0}^{t_{\frac{1}{2}}} (s-t_0) e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds,
\]

\[
B_{3,t_1}^{n} = \frac{1}{2\tau} \int_{t_{n_{-\frac{1}{2}}}^{2}} (2s-t_{n_{-2}}-t_{n_{-3}}) e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds, \quad n \geq 3, \quad 1 \leq l_1 \leq N_{1,exp},
\]

\[
B_{3,t_2}^{n} = \frac{2}{\tau} \int_{t_0}^{t_{\frac{1}{2}}} e^{-s_{12}(t_{n_{-\frac{1}{2}}}-s)} \, ds, \quad B_{3,t_2}^{n} = \frac{1}{\tau} \int_{t_{n_{-\frac{1}{2}}}^{2}} e^{-s_{12}(t_{n_{-\frac{1}{2}}}-s)} \, ds
\]

with \( n \geq 3, \quad 1 \leq l_2 \leq N_{2,exp} \).

Besides, one can see that

\[
F D_{t}^{\alpha} f(t_{n_{-\frac{1}{2}}}) = -\frac{1}{\Gamma(1-\alpha)} \sum_{l_1=1}^{N_{1,exp}} w_{l_1} f'(t_0) s_{l_1}^{-\frac{1}{2}} + \frac{1}{\Gamma(1-\alpha)} \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds
\]

\[
\sum_{l_1=1}^{N_{1,exp}} w_{l_1} \frac{e^{-s_{11}}}{} \frac{1}{s_{l_1}^{s_{11}}} \delta t f^{n_{-\frac{1}{2}}} + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \delta t f^{n_{-\frac{1}{2}}}
\]

\[
\frac{H_{t_0}^{n_{-\frac{1}{2}}}}{\Gamma(1-\alpha)} \int_{t_0}^{t_{\frac{1}{2}}} \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-l)} \, dt
\]

\[
\frac{N_{n_{-\frac{1}{2}}}^{n_{-\frac{1}{2}}}}{\Gamma(2-\alpha)} \int_{t_{n_{-\frac{1}{2}}}^{2}} (t_{n_{-\frac{1}{2}}}-s_{11})^{-1-\alpha} \, ds_{11}
\]

\[
= \frac{A_{0}^{(n_{-\frac{1}{2}})}}{\Gamma(1-\alpha)} f'(t_0) + \frac{\delta_{0}^{(n_{-\frac{1}{2}})}}{\Gamma(1-\alpha)} (\delta t f^{n_{-\frac{1}{2}}} - f'(t_0)) + \frac{2}{\tau} \int_{t_{0}}^{t_{\frac{1}{2}}} \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds_{11}
\]

\[
\sum_{l_1=1}^{N_{1,exp}} w_{l_1} \frac{e^{-s_{11}}}{} \frac{1}{s_{l_1}^{s_{11}}} \delta t f^{n_{-\frac{1}{2}}}
\]

\[
\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \delta t f^{n_{-\frac{1}{2}}} + \frac{\delta_{0}^{(n_{-\frac{1}{2}})}}{\Gamma(2-\alpha)} (\delta t f^{n_{-\frac{1}{2}}} - \delta t f^{n_{-\frac{1}{2}}})
\]

where \( \tilde{A}_{0}^{(n_{-\frac{1}{2}})} = -\sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{11} t_{n_{-\frac{1}{2}}}} \),

\[
\tilde{A}_{k}^{(n_{-\frac{1}{2}})} = \begin{cases} 
-\frac{2}{\tau} \int_{t_{0}}^{t_{\frac{1}{2}}} \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds_{11}, \quad k = 0, \\
-\frac{1}{\tau} \int_{t_{k_{-\frac{1}{2}}}^{2}} \sum_{l_1=1}^{N_{1,exp}} w_{l_1} e^{-s_{11}(t_{n_{-\frac{1}{2}}}-s)} \, ds_{11}, \quad k = 1, \ldots, n-2, \\
\delta e_{n_{-\frac{1}{2}}}, \quad k = n-1.
\end{cases}
\]
On the other hand,

\[ F D_t^\beta f(t_{n-\frac{1}{2}}) = \frac{2}{\Gamma(2-\beta)} \int_{t_0}^{t_{k-\frac{1}{2}}} \left[ \frac{\delta_t f^\frac{\alpha}{2} - f'(t_0)}{\tau} \right] N_{2,\exp} \sum_{t_2=1}^{N_{2,\exp}} w_{t_2} e^{-s_{t_2}(t_{n-\frac{1}{2}} - t)} dt + \frac{1}{\Gamma(2-\beta)} \sum_{k=2}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k-\frac{1}{2}}} \left[ \frac{\delta_t f^\frac{\alpha}{2} - \delta_t f^\frac{\alpha}{2}}{\tau} \right] N_{2,\exp} \sum_{t_2=1}^{N_{2,\exp}} w_{t_2} e^{-s_{t_2}(t_{n-\frac{1}{2}} - t)} dt + \frac{c^{(n,\beta)}_{n-1}}{\Gamma(2-\beta)} \left[ \delta_t f^n - \delta_t f^n \right] \]

where

\[ c^{(n,\beta)}_k = \begin{cases} \frac{2}{\tau} \int_{t_0}^{t_{k-\frac{1}{2}}} \sum_{t_2=1}^{N_{2,\exp}} w_{t_2} e^{-s_{t_2}(t_{n-\frac{1}{2}} - t)} dt, & k = 0, \\
\frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k-\frac{1}{2}}} \sum_{t_2=1}^{N_{2,\exp}} w_{t_2} e^{-s_{t_2}(t_{n-\frac{1}{2}} - t)} dt, & k = 1, \ldots, n-2, \\
\gamma^{(n,\beta)}_{n-1}, & k = n-1. \end{cases} \tag{41} \]

By the definition of \( a^{(n,\alpha)}_k \) and \( \tilde{a}^{(n,\alpha)}_k \), it easy to know that

\[ a^{(n,\alpha)}_{n-1} - \tilde{a}^{(n,\alpha)}_{n-1} = 0; \quad \left| a^{(n,\alpha)}_k - \tilde{a}^{(n,\alpha)}_k \right| \leq \epsilon, \quad 0 \leq k \leq n-2. \tag{42} \]

**Lemma 4.2.** The coefficient \( \{\tilde{a}^{(n,\alpha)}_k \mid n = 1, 2, \ldots, N; k = 0, 1, \ldots, n-1\} \) defined in Eq. (40) satisfies

\[ \tilde{a}^{(n,\alpha)}_0 > \tilde{a}^{(n,\alpha)}_1 > \cdots > \tilde{a}^{(n,\alpha)}_{n-2}. \tag{43} \]

If \( \epsilon < \frac{2-2^{-\alpha}}{2-\alpha} \sqrt{\tau}^{1-\alpha} \), then \( \tilde{a}^{(n,\alpha)}_n < a^{(n,\alpha)}_n \).

**Proof.** It is clear that all \( \tilde{a}^{(n,\alpha)}_k \) is negative. Using the mean value theorem, we write

\[ \tilde{a}^{(n,\alpha)}_k = -\sum_{l_1=1}^{N_{1,\exp}} \frac{w_{l_1}}{s_{l_1}} e^{-s_{l_1}(t_{n-\frac{1}{2}} - \xi_k)}, 1 \leq k \leq n-2, \xi_0 \in (t_0, t_{\frac{1}{2}}), \xi_k \in (t_{k-\frac{1}{2}}, t_{k-\frac{1}{2}}), \]

which leads to inequalities (43). Furthermore, we have

\[ \tilde{a}^{(n,\alpha)}_{n-1} - \tilde{a}^{(n,\alpha)}_{n-2} = a^{(n,\alpha)}_{n-1} - a^{(n,\alpha)}_{n-2} = (a^{(n,\alpha)}_{n-1} - a^{(n,\alpha)}_{n-2}) + (a^{(n,\alpha)}_{n-2} - a^{(n,\alpha)}_{n-2}) \geq (a^{(n,\alpha)}_{n-1} - a^{(n,\alpha)}_{n-2}) = \epsilon \geq (2-2^{-\alpha}) \frac{\sqrt{\tau}^{1-\alpha} - \epsilon > 0.} \]

This completes the the proof of this Lemma. \( \square \)

By the definition of \( b^{(n,\beta)}_k \) and \( c^{(n,\beta)}_k \), it easy to know that

\[ b^{(n,\beta)}_{n-1} - c^{(n,\beta)}_{n-1} = 0; \quad \left| b^{(n,\beta)}_k - c^{(n,\beta)}_k \right| \leq \epsilon, \quad 0 \leq k \leq n-2. \tag{44} \]
Lemma 4.3. The coefficient \(\{c_k^{(n,\beta)}| n=1,2,\cdots,N, \ k=0,1,\cdots,n-1\}\) defined in Eq. (41) satisfies
\[
0 < c_0^{(n,\beta)} < c_1^{(n,\beta)} < \cdots < c_{n-2}^{(n,\beta)} .
\]
If \(\epsilon < \frac{2-2^2-\beta}{2-\beta} \epsilon^1-\beta\), then \(c_{n-2}^{(n,\beta)} < c_{n-1}^{(n,\beta)}\).

Proof. The proof of this lemma is very similar to [32, Lemma 4.2], we omit the details.

\[
\text{Theorem 4.4. Suppose the function } f \in C^3[0,T], \ \alpha \in (0,1), \ \text{it holds that}
\]
\[
\left| \frac{C}{0}D_t^\alpha f(t_{n-\frac{1}{2}}) - F D_t^\alpha f(t_{n-\frac{1}{2}}) \right| \leq c_0 \tau^{3-\alpha} + \frac{\epsilon}{\Gamma(1-\alpha)} t_{n-\frac{1}{2}} \left\{ \max_{t_0 \leq t \leq t_n} |f'(t)| + \max_{t_0 \leq t \leq t_n} |f''(t)| \right\} , \ n=1,2,\cdots,N .
\] (45)

Proof. Since
\[
\frac{C}{0}D_t^\alpha f(t_{n-\frac{1}{2}}) - F D_t^\alpha f(t_{n-\frac{1}{2}}) = \left[ \frac{C}{0}D_t^\alpha f(t_{n-\frac{1}{2}}) - D_t^\alpha f(t_{n-\frac{1}{2}}) \right] + \left[ D_t^\alpha f(t_{n-\frac{1}{2}}) - F D_t^\alpha f(t_{n-\frac{1}{2}}) \right]
\equiv D_n + E_n .
\] (46)

Using the Theorem 2.2, there exists a constant \(c_0\) such that \(|D_n| \leq c_0 \tau^{3-\alpha}\) and
\[
E_n = \frac{1}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t_{n-1/2}} H_{2,0}(t) \left[ (t_{n-1/2} - t)^{-\alpha} - \sum_{l=1}^{n-1} w_l e^{-\lambda_l (t_{n-1/2} - t)} \right] dt 
+ \sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} N_{2,k}(t) \left[ (t_{n-1/2} - t)^{-\alpha} - \sum_{l=1}^{n-1} w_l e^{-\lambda_l (t_{n-1/2} - t)} \right] dt \right\} ,
\]
which can be estimated as
\[
|E_n| \leq \frac{\epsilon}{\Gamma(1-\alpha)} \left[ \int_{t_0}^{t_{1/2}} |H_{2,0}'(t)| dt + \sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} |N_{2,k}'(t)| dt \right] 
\leq \frac{\epsilon}{\Gamma(1-\alpha)} \left[ \int_{t_0}^{t_{1/2}} |f'(t)| + H_{2,0}'(t_0)(t-t_0) dt + \sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} \delta_k f_{k-\frac{1}{2}}^{k-\frac{1}{2}} \right] + \frac{1}{2} N_{2,k}(2t - t_k - t_{k-1}) dt \right] 
\leq \frac{\epsilon}{\Gamma(1-\alpha)} t_{n-\frac{1}{2}} \left\{ \max_{t_0 \leq t \leq t_n} |f'(t)| + \max_{t_0 \leq t \leq t_n} |f''(t)| \right\} .
\] (47)

Substituting (47) into (46) yields the representation (45).

\[
\text{Theorem 4.5. [32] Suppose the function } f \in C^4[0,T], \ \beta \in (1,2), \ \text{we have}
\]
\[
\left| \frac{C}{0}D_t^\beta f(t_{n-\frac{1}{2}}) - F D_t^\beta f(t_{n-\frac{1}{2}}) \right| \leq c_1 \tau^{3-\beta} + \frac{\epsilon}{\Gamma(2-\beta)} t_{n-\frac{1}{2}} \max_{t_0 \leq t \leq t_n} |f''(t)|
\]
for \(n=1,2,\cdots,N\).
Fast approximation formula of $\frac{C^\alpha}{\alpha}D^\alpha_t f(t) + \frac{C^\beta}{\beta}D^\beta_t f(t)$: Applying Lemma 4.1, Theorem 4.4 and Theorem 4.5, we can get

$$
F D^\alpha_t f^{n-\frac{1}{2}} + F D^\beta_t f^{n-\frac{1}{2}} = \frac{1}{\Gamma(1-\alpha)} \sum_{l_1=1}^{N_1,\text{exp}} w_{l_1} F_{l_1}^n + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \delta_t f^{n-\frac{3}{2}}
$$

$$
+ \frac{a^{(n,\alpha)}_{n-1}}{\Gamma(2-\alpha)} (\delta_t f^{n-\frac{1}{2}} - \delta_t f^{n-\frac{3}{2}}) + \frac{1}{\Gamma(2-\beta)} \sum_{l_2=1}^{N_2,\text{exp}} w_{l_2} F_{l_2}^n
$$

$$
+ \frac{b^{(n,\beta)}_{n-1}}{\Gamma(2-\beta)} (\delta_t f^{n-\frac{1}{2}} - \delta_t f^{n-\frac{3}{2}}) + O (\tau^{3-\beta + \epsilon}),
$$

$$
n \geq 1, \ 1 \leq l_1 \leq N_{1,\text{exp}}, \ 1 \leq l_2 \leq N_{2,\text{exp}}, \quad (48)
$$

$$
F_{l_1}^n = e^{-s_1 \tau} F_{l_1}^{n-1} + B_{l_1}^n (\delta_t f^{n-\frac{1}{2}} - \delta_t f^{n-\frac{3}{2}}), \quad (49)
$$

$$
F_{l_2}^n = e^{-s_2 \tau} F_{l_2}^{n-1} + B_{l_2}^n (\delta_t f^{n-\frac{1}{2}} - \delta_t f^{n-\frac{3}{2}}), \quad (50)
$$

$$
F_{l_1}^1 = 0, \ F_{l_2}^1 = 0, \ \delta_t f^{n-\frac{3}{2}} = f'(t_0). \quad (51)
$$

4.2. The fast difference scheme. Next, we present a fast difference scheme for the problem (1). Define the grid function

$$
U_i^n = u(x_i, t_n), \ i \in I, \ 0 \leq n \leq N.
$$

Considering the first equation of Eq. (1) at the point $(x_i, t_{n-\frac{1}{2}})$, we have

$$
\frac{C^\alpha}{\alpha}D^\alpha_t u(x_i, t_{n-\frac{1}{2}}) + \frac{C^\beta}{\beta}D^\beta_t u(x_i, t_{n-\frac{1}{2}}) = \Delta u(x_i, t_{n-\frac{1}{2}}) + q(x_i, t_{n-\frac{1}{2}}), \ i \in I, \ 1 \leq n \leq N.
$$

Applying Eqs. (48)-(51) in temporal direction, we get

$$
\left[ \frac{g_0^{(1,\alpha)}}{\Gamma(2-\alpha)} + \frac{b_0^{(1,\beta)}}{\Gamma(2-\beta)} \right] \delta_t U_i^\frac{3}{2} + \left[ \frac{A_0^{(1,\alpha)} - a_0^{(1,\alpha)}}{\Gamma(2-\alpha)} - \frac{b_0^{(1,\beta)}}{\Gamma(2-\beta)} \right] \psi_i
$$

$$
= \Delta_h U_i^{n-\frac{1}{2}} + q_i^{n-\frac{1}{2}} + R_i^1, \quad i \in I, \quad (52)
$$

$$
\frac{1}{\Gamma(1-\alpha)} \sum_{l_1=1}^{N_1,\text{exp}} w_{l_1} F_{l_1,i}^n + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \delta_t U_i^{n-\frac{1}{2}} + \frac{a^{(n,\alpha)}_{n-1}}{\Gamma(2-\alpha)} (\delta_t U_i^{n-\frac{1}{2}} - \delta_t U_i^{n-\frac{3}{2}})
$$

$$
+ \frac{1}{\Gamma(2-\beta)} \sum_{l_2=1}^{N_2,\text{exp}} w_{l_2} F_{l_2,i}^n + \frac{b^{(n,\beta)}_{n-1}}{\Gamma(2-\beta)} (\delta_t U_i^{n-\frac{1}{2}} - \delta_t U_i^{n-\frac{3}{2}}) = \Delta_h U_i^{n-\frac{1}{2}}
$$

$$
+ g_i^{n-\frac{1}{2}} + R_i^n, \quad 1 \leq l_1 \leq N_{1,\text{exp}}, \ 1 \leq l_2 \leq N_{2,\text{exp}}, \ i \in I, \ 1 \leq n \leq N, \quad (53)
$$

$$
F_{l_1,i}^n = e^{-s_1 \tau} F_{l_1,i}^{n-1} + B_{l_1}^n (\delta_t U_i^{n-\frac{1}{2}} - \delta_t U_i^{n-\frac{3}{2}}), \quad 1 \leq l_1 \leq N_{1,\text{exp}}, \ i \in I, \ 2 \leq n \leq N,
$$

$$
F_{l_2,i}^n = e^{-s_2 \tau} F_{l_2,i}^{n-1} + B_{l_2}^n (\delta_t U_i^{n-\frac{1}{2}} - \delta_t U_i^{n-\frac{3}{2}}), \quad 1 \leq l_2 \leq N_{2,\text{exp}}, \ i \in I, \ 2 \leq n \leq N,
$$

$$
\delta_t U_i^{n-\frac{1}{2}} = \psi_i, \ F_{l_1,i}^1 = 0, \ F_{l_2,i}^1 = 0, \ 1 \leq l_1 \leq N_{1,\text{exp}}, \ 1 \leq l_2 \leq N_{2,\text{exp}}, \ i \in I,
$$
where there exists a constant $C_3$ such that

$$|\tilde{R}^i_n| \leq C_3 (\tau^{3-\beta} + h^2), \quad |\tilde{R}^n_i| \leq C_3 (\tau^{3-\beta} + h^2 + \epsilon), \quad i \in I, \ 1 \leq n \leq N.$$ 

Omitting the small term $\tilde{R}^n_i$, $n \geq 1$ in Eqs. (52)-(53), replacing the grid function $U^n_i, F^n_{1,i}, F^n_{2,i}$ by its numerical approximation $\tilde{u}^n_i, \tilde{f}^n_{1,i}, \tilde{f}^n_{2,i}$ and noticing the initial-boundary value conditions (23)-(24). We construct the fast difference scheme for the problem (1) as follows

$$\left[ \frac{g_0^{(1, \alpha)}}{\Gamma(2-\alpha)} + \frac{b_0^{(1, \beta)}}{\Gamma(2-\beta)} \right] \delta_t \tilde{u}^n_i + \left[ \frac{A_0^{(1, \alpha)} - a_0^{(1, \alpha)}}{\Gamma(2-\alpha)} - \frac{b_0^{(1, \beta)}}{\Gamma(2-\beta)} \right] \psi_i = \Delta h \tilde{u}^n_i$$

$$+ q_i^n, \quad i \in I,$$

$$\frac{1}{\Gamma(1-\alpha)} \sum_{l=1}^{N_1, \text{exp}} w_{l,i} f^n_{1,i} + \tau^{1-\alpha} \delta_t \tilde{u}^n_i \tilde{u}^{n-\frac{1}{2}} + \frac{a_{n-1}^{(n, \alpha)}}{\Gamma(2-\alpha)} (\delta_t \tilde{u}^n_i \tilde{u}^{n-\frac{1}{2}} - \delta_t \tilde{u}^{n-\frac{1}{2}})$$

$$+ \frac{1}{\Gamma(2-\beta)} \sum_{l=1}^{N_2, \text{exp}} w_{l,i} f^n_{2,i} + \frac{b_{n-1}^{(n, \beta)}}{\Gamma(2-\beta)} (\delta_t \tilde{u}^n_i \tilde{u}^{n-\frac{1}{2}} - \delta_t \tilde{u}^{n-\frac{1}{2}}) = \Delta h \tilde{u}^n_i + q^n_i,$$

$$1 \leq l \leq N_1, 1 \leq l \leq N_2, \quad i \in I, \ 1 \leq n \leq N,$$

$$f^n_{1,i} = e^{-s_1 \tau} f_{1,i}^{n-1} + B_{1,i} \delta_t \tilde{u}^n_i \tilde{u}^{n-\frac{1}{2}} + B_{2,i} \delta_t \tilde{u}^{n-\frac{1}{2}} - \delta_t \tilde{u}^{n-\frac{1}{2}},$$

$$1 \leq l \leq N_1, \quad i \in I, \ 2 \leq n \leq N,$$

$$f^n_{2,i} = e^{-s_2 \tau} f_{2,i}^{n-1} + B_3 \delta_t \tilde{u}^n_i \tilde{u}^{n-\frac{1}{2}} - \delta_t \tilde{u}^{n-\frac{1}{2}},$$

$$1 \leq l \leq N_2, \quad i \in I, \ 2 \leq n \leq N,$$

$$f^1_{1,i} = 0, \quad f^1_{2,i} = 0, \quad 1 \leq l \leq N_1, \quad 1 \leq l \leq N_2, \quad i \in I,$$

$$\delta_t \tilde{u}^{n-\frac{1}{2}} = \psi_i, \quad u^0_i = \phi_i, \quad i \in I,$$

$$u^n_i = 0, \quad i \in \partial I, \ 0 \leq n \leq N.$$ 

\textbf{Theorem 4.6.} Suppose that $\{U^n_i\}$ and $\{\tilde{u}^n_i\}$ are solutions of the problem (1) and fast difference scheme (54)-(60), respectively. Denote

$$\tilde{u}^n_i = U^n_i - \tilde{u}^n_i, \quad i \in \bar{I}, \ 0 \leq n \leq N.$$ 

Then, there exists a positive constant $C_4$ such that

$$\|\nabla h \tilde{u}^n\| \leq C_4 (\tau^{3-\beta} + h^2 + \epsilon), \ 1 \leq n \leq N.$$ 

The proof of this theorem is fairly similar with Theorem 3.5.

5. Numerical examples. In this section, we will report two numerical experiments. First, we present the accuracy of the difference scheme (DS) (25)-(27) and fast difference scheme (FS) (54)-(60) for one-dimensional problem. The second example discuss for two-dimensional problem, we use the FFTs to fast solve the system of linear algebraic equations. It not only illustrates the convergence order of DS (25)-(27) and FS (54)-(60) but also demonstrates FS can greatly reduce the storage cost and CPU time (measured in seconds).

To examine the accuracy and efficiency of the difference schemes, we compute the $L^2$ norm error of the numerical solution

$$E(h, \tau) = \max_{0 \leq k \leq N} \|U^k - u^k\|.$$
and the convergence orders

\[ R_\tau = \log_2 \left( \frac{E(h, 2\tau)}{E(h, \tau)} \right), \quad R_h = \log_2 \left( \frac{E(2h, \tau)}{E(h, \tau)} \right). \]

**Example 5.1.** In the model (1), take \( T = 1, \ [0, L] = [0, 1], \ \sigma = 2 + \beta. \) and the source term

\[
q(x, t) = \sin(\pi x) \left[ \frac{\Gamma(1 + \sigma)}{\Gamma(1 + \sigma - \alpha)} t^{\sigma - \alpha} + \frac{3\Gamma(3)}{\Gamma(3 - \alpha)} t^{2-\alpha} + \frac{\Gamma(1 + \sigma)}{\Gamma(1 + \sigma - \beta)} t^{\sigma - \beta} \right.
\]

\[
+ \frac{3\Gamma(3)}{\Gamma(3 - \beta)} t^{2-\beta} + \pi^2(t^{\sigma} + 3t^2 + 1) \left. \right],
\]

\[
\psi(x, t) = \sin(\pi x)(\sigma t^{\sigma-1} + 6t).
\]

The problem has an exact solution reads \( u(x, t) = \sin(\pi x)(t^{\sigma} + 3t^2 + 1). \)

**Table 1.** Comparison of the temporal convergence order and elapsed CPU time of both DS (25)-(27) and FS (54)-(60) for Example 5.1 with different \((\alpha, \beta)\), \( M = \lceil N^{3/2} \rceil \) and \( \epsilon = 10^{-12}. \)

| \( \beta \) | \( N \) | \( \alpha = 0.3 \) | \( E(M, N)(\text{DS}) \) | \( E(M, N)(\text{FA}) \) | \( R_\tau (\text{DS}) \) | \( R_\tau (\text{FA}) \) | \( \text{CPU(DA)} \) | \( \text{CPU(FA)} \) |
|---|---|---|---|---|---|---|---|---|
| 1.3 | 1280 | 1.18e-5 | 1.10e-5 | – | – | 5.84 | 3.70 |
| | 2560 | 3.62e-6 | 3.43e-6 | 1.70 | 1.68 | 18.22 | 6.13 |
| | 5120 | 1.11e-6 | 1.06e-6 | 1.70 | 1.69 | 76.42 | 22.10 |
| | 10240 | 3.42e-7 | 3.32e-7 | 1.70 | 1.67 | 319.5 | 76.91 |
| 1.5 | 1280 | 5.11e-5 | 4.95e-5 | – | – | 3.75 | 2.88 |
| | 2560 | 1.80e-5 | 1.75e-5 | 1.51 | 1.50 | 12.83 | 7.14 |
| | 5120 | 6.34e-6 | 6.19e-6 | 1.51 | 1.50 | 44.41 | 14.91 |
| | 10240 | 2.24e-6 | 2.18e-6 | 1.50 | 1.50 | 160.5 | 32.34 |
| 1.7 | 1280 | 2.36e-4 | 2.26e-4 | – | – | 4.93 | 1.94 |
| | 2560 | 9.52e-5 | 9.10e-5 | 1.31 | 1.31 | 12.44 | 3.47 |
| | 5120 | 3.87e-5 | 3.71e-5 | 1.30 | 1.30 | 34.97 | 11.20 |
| | 10240 | 1.57e-5 | 1.50e-5 | 1.30 | 1.30 | 112.4 | 25.5 |

| \( \beta \) | \( N \) | \( \alpha = 0.9 \) | \( E(M, N)(\text{DS}) \) | \( E(M, N)(\text{FA}) \) | \( R_\tau (\text{DS}) \) | \( R_\tau (\text{FA}) \) | \( \text{CPU(DA)} \) | \( \text{CPU(FA)} \) |
|---|---|---|---|---|---|---|---|---|
| 1.3 | 1280 | 1.13e-5 | 1.06e-5 | – | – | 5.84 | 4.09 |
| | 2560 | 3.48e-6 | 3.29e-6 | 1.70 | 1.68 | 19.45 | 8.90 |
| | 5120 | 1.07e-6 | 1.02e-6 | 1.70 | 1.69 | 77.95 | 26.44 |
| | 10240 | 3.29e-7 | 3.17e-7 | 1.70 | 1.67 | 301.6 | 59.34 |
| 1.5 | 1280 | 4.91e-5 | 4.79e-5 | – | – | 4.22 | 2.09 |
| | 2560 | 1.73e-5 | 1.69e-5 | 1.51 | 1.50 | 13.23 | 10.00 |
| | 5120 | 6.09e-6 | 5.99e-6 | 1.51 | 1.50 | 42.39 | 13.30 |
| | 10240 | 2.15e-6 | 2.12e-6 | 1.50 | 1.50 | 159.8 | 30.58 |
| 1.7 | 1280 | 2.21e-4 | 2.16e-4 | – | – | 3.56 | 2.06 |
| | 2560 | 8.88e-5 | 8.70e-5 | 1.31 | 1.31 | 9.20 | 3.45 |
| | 5120 | 3.62e-5 | 3.54e-5 | 1.30 | 1.30 | 30.73 | 12.98 |
| | 10240 | 1.46e-5 | 1.44e-5 | 1.30 | 1.30 | 116.9 | 25.63 |

Our first concern is to test the numerical accuracy of our proposed algorithm in time. Taking \( M = \lceil N^{3/2} \rceil \), Table 1 contains the \( L_2 \) norm errors, convergence orders and CPU time of (25)-(27) and (54)-(60) for different parameters \((\alpha, \beta)\). It shows that for \( \epsilon = 10^{-12} \) the convergence orders of two schemes can be as high as \( \tau^{3-\beta} \), which are consistent with Theorem 3.5 and Theorem 4.6. In particular, if \( N \)}
Table 2. Comparison of the spatial convergence order and elapsed CPU time for implementing the DS (25)-(27) and FS (54)-(60) with different \((\alpha, \beta)\), \(N = 30000\), \(\epsilon = 10^{-12}\) (Example 5.1).

| \(\beta\) | \(M\) | \(\alpha = 0.3\) | \(E(M, N)\) | \(R_M\) | \(CPU(DA)\) | \(CPU(FA)\) | \(\alpha = 0.9\) | \(E(M, N)\) | \(R_M\) | \(CPU(DA)\) | \(CPU(FA)\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.3 | 16 | 8.35e-3 | 512 | 7.09 | 8.01e-3 | 680.7 | 3.84 |
| | 32 | 2.09e-3 | 2.00 | 528 | 4.89 | 2.00e-3 | 2.00 | 717.3 | 4.64 |
| | 64 | 5.22e-4 | 2.00 | 553 | 4.97 | 5.00e-4 | 2.00 | 788.5 | 5.42 |
| 128 | 1.30e-4 | 2.00 | 668 | 6.78 | 1.25e-4 | 2.00 | 898.0 | 7.14 |
| 1.5 | 16 | 8.13e-3 | 407.6 | 2.47 | 7.82e-3 | 420.2 | 4.08 |
| | 32 | 2.03e-3 | 2.00 | 419.9 | 3.05 | 1.96e-3 | 2.00 | 412.4 | 4.64 |
| | 64 | 5.07e-4 | 2.00 | 448.1 | 3.58 | 4.89e-4 | 2.00 | 424.8 | 5.69 |
| 128 | 1.27e-4 | 2.00 | 471.7 | 5.00 | 1.22e-4 | 2.00 | 456.4 | 7.31 |
| 1.7 | 16 | 8.97e-3 | 567.4 | 4.00 | 7.72e-3 | 565.1 | 3.91 |
| | 32 | 2.91e-3 | 2.00 | 556.8 | 5.02 | 1.93e-3 | 2.00 | 556.2 | 4.61 |
| | 64 | 5.045e-4 | 2.00 | 543.1 | 4.84 | 1.96e-3 | 2.00 | 555.2 | 4.89 |
| 128 | 1.267e-4 | 1.99 | 709.9 | 7.11 | 1.21e-4 | 1.99 | 822.5 | 7.34 |

As the second set of tests, the spatial convergence rate of the presented algorithms is measured by the example. The step sizes in time are kept sufficiently small and fixed \(\tau = \frac{1}{2000}\). Taking varying step sizes in space, Table 2 shows that the second-order convergence of two schemes in space. Owing to taking a large \(N\), we can observe from Table 2, the fast scheme spends less CPU time (in seconds) than the direct algorithm with the same accuracy.

In the following, we will present an example for solving the two-dimensional time fractional mixed diffusion and diffusion-wave equation with smooth solution by applying both DS (25)-(27) and FS (54)-(60).

We extend the fast Poisson solver [3] to solve the difference scheme (54) efficiently and write the matrix representation of Eqs. (25), (54)-(55) as by applying both DS (25)-(27) and FS (54)-(60).

\[
\begin{align*}
\frac{u^n_{i+1} - u^n_i}{\tau} & + \mu_1^n S_{M1-1} u^{n-\frac{1}{2}} + \mu_2^n S_{M2-1} = G^n, \\
\end{align*}
\]

where \(d^n = \frac{\alpha_{n-1}^{(n, \alpha)} + \beta_{n-1}^{(n, \beta)}}{\Gamma(2-\alpha)} \), \(\mu_1^n = \frac{\tau}{2d^n \Gamma(2-\alpha)} \), \(\mu_2^n = \frac{\tau}{2d^n \Gamma(2-\beta)} \), \((u^{n-\frac{1}{2}}) = u_i^{n-\frac{1}{2}}, i \in I\). The matrix \(S_M \in \mathbb{R}^{M \times M}\) is defined by

\[
S_M = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
& & & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & \cdots & -1 & 2 & \\
\end{pmatrix}_{M \times M}
\]

and the matrix \(G^n \in \mathbb{R}^{(M-1) \times (M-1)}\) in Eq. (61) are respectively given from DS and FS by \(GD^n, GF^n\)

\[
GD^n = u_i^{n-1} + \frac{\tau}{2d^n} \sum_{k=1}^{n-1} \left( \frac{a_k^{(n, \alpha)} - a_{k-1}^{(n, \alpha)}}{\Gamma(2-\alpha)} + \frac{b_k^{(n, \beta)} - b_{k-1}^{(n, \beta)}}{\Gamma(2-\beta)} \right) \delta_i u_i^{k-\frac{1}{2}}
\]

\[
- \frac{\tau}{2d^n} \left( \frac{a_0^{(n, \alpha)} - a_0^{(n, \alpha)}}{\Gamma(2-\alpha)} - \frac{b_0^{(n, \beta)}}{\Gamma(2-\beta)} \right) \psi_i + \frac{\tau}{2(a_{n-1}^{(n, \alpha)} + b_{n-1}^{(n, \beta)})} u_i^{n-\frac{1}{2}}, n \geq 1,
\]
\[
GF^i_n = u^{n-1} + \frac{\tau}{2} \delta u^{n-\frac{1}{2}} - \frac{\tau^{2-\alpha}}{2\Gamma(2-\alpha)d^n} \delta u^{n-\frac{1}{2}} - \frac{\tau}{2\Gamma(1-\alpha)d^n} \sum_{i=1}^{N_1} w_i f^n_{i,i}
\]

where \(1 \leq l_1 \leq N_1, 1 \leq l_2 \leq N_2, i \in I, n \geq 2\). The matrix equation (61) can be quickly solved by some FFTs.

Now we can employ a fast Poisson solver from [3, 40] for the sequence of discretized linear systems. Let \(\lambda_i^{(d)}\) and \(q_i^{(d)}\) be the \(i\)-th eigenvalue and the associated eigenvector of \(S_{M_d}, M_d (d = 1, 2, \cdots)\) is a positive integer. Then we have

\[
S_{M_d-1} Q^{(d)} = Q^{(d)} \Lambda^{(d)}, \quad (Q^{(d)})^T Q^{(d)} = \frac{1}{2h_d} E_{M_d-1},
\]

where \(Q^{(d)} = [q_1^{(d)}, q_2^{(d)}, \cdots, q_{M_d-1}^{(d)}], \Lambda^{(d)} = \text{diag}(\lambda_1^{(d)}, \lambda_2^{(d)}, \cdots, \lambda_{M_d-1}^{(d)}),\) and \(E_{M_d}\) is an identity matrix of order \(M_d\). Also, we have explicit representation of \(\lambda_i^{(d)}\) and \(q_i^{(d)}\):

\[
\lambda_i^{(d)} = 4 \sin^2 \left( \frac{i \pi h_d}{2} \right), \quad h_d = 1/M_d, \quad i = 1, 2, \cdots, M_d - 1,
\]

\[
q_i^{(d)} = (\sin(i \pi h_d), \sin(2i \pi h_d), \cdots, \sin((M_d-1)i \pi h_d))^T.
\]

Define \(W^n \in \mathbb{R}^{(M_1-1) \times (M_2-1)}, 1 \leq n \leq N\) such that

\[
u^{n-\frac{1}{2}} = Q^{(1)} W^n Q^{(2)} = Q^n, 1 \leq n \leq N.
\]

Substituting Eq. (62) into Eq. (61) and multiplying this equality by \((Q^{(1)})^T\) on the left and \((Q^{(2)})^T\) on the right, we can get

\[
W^n + \mu^n_1 \Lambda^{(1)} W^n + \mu^n_2 W^n \Lambda^{(2)} = 4h_1 h_2 Q^{(1)} C^n Q^{(2)} = H^n.
\]

The linear system (63) can be solved simply by

\[
W^n_i = \frac{H^n_i}{1 + \mu^n_1 \lambda^{(1)}_i + \mu^n_2 \lambda^{(2)}_i}, \quad i \in I, 1 \leq n \leq N.
\]

Once \(W^n, 1 \leq n \leq N\) is obtained, we can obtained the solution \(u^n_i (i \in \bar{I})\) from Eq. (62) with \(u^n_i = 2u^{n-\frac{1}{2}} - u^{n-1}, (i \in \bar{I})\).

Note that \(H\) in the right hand side of Eq. (63) can be computed by using the FFTs in \((M_1 M_2 \log(M_1) + M_1 M_2 \log(M_2))N\) operations and \(u^n\) can be computed similarly. Thus, the computational cost of DS (25)–(27) is \(O(M_1 M_2 (\log M_1 + \log M_2)N + M_1 M_2 N^2)\), but that of FS (54)–(60) is \(O(M_1 M_2 (\log M_1 + \log M_2)N + M_1 M_2 N N_{\exp})\). Generally speaking, owing to \(N_{\exp} < 100\), FS has the great advantage of reducing the computational and storage cost. The method can be similarly extended to three dimension problems.

Example 5.2. Consider the following time-fractional mixed diffusion and diffusion-wave equation

\[
\begin{cases}
C_0^\alpha D_t^\alpha u(x, y, t) + C_0^2 D_t^2 u(x, y, t) = \Delta u(x, y, t) + s(x, y, t), \quad (x, y, t) \in \Omega \times [0, 1], \\
u(x, y, 0) = 100 \sin(\pi x) \sin(\pi y), \quad (x, y) \in \Omega, \\
u_t(x, y, 0) = 0, \quad (x, y) \in \Omega \cup \partial \Omega, \\
u(x, y, t) = 0, \quad (x, y, t) \in \partial \Omega \times [0, 1],
\end{cases}
\]
where $\beta \in (1, 2)$ and $\Omega = (0, 1)^2$ with its boundary $\partial \Omega$. The exact solution of the above equation is $u(x, y, t) = 100(t^{2+\beta} + 3t^2 + 1) \sin(\pi x) \sin(\pi y)$ and the right hand function is

$$s(x, y, t) = 100\left[\frac{3 + \beta}{\Gamma(3 + \beta - \alpha)} t^{2+\beta-\alpha} + \frac{3\Gamma(3)}{\Gamma(3 - \alpha)} t^{2-\alpha} + \frac{\Gamma(3 + \beta)}{\Gamma(3)} t^2 \sin(\pi x) \sin(\pi y) + 200\pi^2(t^{2+\beta} + 3t^2 + 1) \sin(\pi x) \sin(\pi y)\right].$$

Table 3. Comparison of the temporal convergence order and elapsed CPU time of DS (25)-(27) and FS (54)-(60) for Example 5.2 with different $(\alpha, \beta)$, $M = \lceil N^{\frac{2+\beta-\alpha}{2}} \rceil$, $\epsilon = 10^{-12}$.

| $\beta$ | $N$ | $\alpha = 0.3$ | $\alpha = 0.9$ |
| --- | --- | --- | --- |
| $M$ | $E(M, N)$ | $R_M$ | CPU(DA) | CPU(FA) | $E(M, N)$ | $R_M$ | CPU(DA) | CPU(FA) |
| 1.3 | 512 | 200 | 4.141e-3 | 1.67 | 221.3 | 1.284e-2 | 1.67 | 205.9 | 134.6 |
| | | | | | | | | | |
| 1.5 | 1024 | 362 | 1.282e-3 | 1.69 | 1658 | 1.248e-3 | 1.69 | 1627 | 1305 |
| | | | | | | | | | |
| 1.7 | 2048 | 652 | 3.998e-4 | 1.68 | 17970 | 3.894e-4 | 1.68 | 18860 | 8096 |

| $\beta$ | $N$ | $\alpha = 0.3$ | $\alpha = 0.9$ |
| --- | --- | --- | --- |
| $M$ | $E(M, N)$ | $R_M$ | CPU(DA) | CPU(FA) | $E(M, N)$ | $R_M$ | CPU(DA) | CPU(FA) |
| 1.3 | 256 | 111 | 1.306e-2 | 1.16 | 23.94 | 1.298e-2 | 1.16 | 22.72 | 59.27 |
| | | | | | | | | | |
| 1.5 | 1024 | 362 | 1.282e-3 | 1.69 | 1658 | 1.248e-3 | 1.69 | 1627 | 1305 |
| | | | | | | | | | |
| 1.7 | 2048 | 652 | 3.998e-4 | 1.68 | 17970 | 3.894e-4 | 1.68 | 18860 | 8096 |

Table 4. Comparison of the spatial convergence order and elapsed CPU time of the DS (25)-(27) and FS (54)-(60) for Example 5.2 with different $(\alpha, \beta)$, $N = 20000$, $\epsilon = 10^{-12}$.

| $\beta$ | $M$ | $\alpha = 0.3$ | $\alpha = 0.9$ |
| --- | --- | --- | --- |
| $N$ | $E(M, N)$ | $R_M$ | CPU(DA) | CPU(FA) | $E(M, N)$ | $R_M$ | CPU(DA) | CPU(FA) |
| 8 | 2.744e+0 | 546.8 | 20.41 | 2.654e+0 | 495.1 | 22.78 |
| 1.3 | 16 | 6.831e-1 | 2.01 | 591.7 | 19.48 | 6.656e-1 | 2.01 | 596.5 | 22.00 |
| | 32 | 1.766e-1 | 2.00 | 1055 | 33.09 | 1.602e-1 | 2.00 | 1061 | 35.67 |
| | 64 | 4.264e-2 | 2.00 | 207.0 | 224.54 | 4.155e-2 | 2.00 | 2099 | 211.3 |
| 8 | 2.686e+0 | 444.3 | 19.80 | 2.619e+0 | 441.7 | 19.89 |
| 1.5 | 16 | 6.688e-1 | 2.01 | 522.7 | 19.92 | 6.316e-1 | 2.01 | 526.1 | 21.67 |
| | 32 | 1.670e-1 | 2.00 | 716.8 | 33.13 | 1.578e-1 | 2.00 | 726.5 | 34.25 |
| | 64 | 4.175e-2 | 2.00 | 2039 | 208.4 | 4.071e-2 | 2.00 | 2084 | 224.2 |
| 8 | 2.558e+0 | 496.5 | 21.30 | 2.537e+0 | 490.6 | 20.52 |
| 1.7 | 16 | 6.358e-1 | 2.01 | 653.1 | 22.45 | 6.316e-1 | 2.01 | 655.5 | 21.67 |
| | 32 | 1.588e-1 | 2.00 | 1124 | 36.06 | 1.578e-1 | 2.00 | 875.2 | 36.80 |
| | 64 | 3.972e-2 | 2.00 | 3042 | 229.2 | 3.946e-2 | 2.00 | 2092 | 233.5 |

Let $h_1 = h_2 = h$, $h = \lceil \tau(3-\beta)/2 \rceil$, $\tau = 1/20000$. Then the direct scheme (25)-(27) and the fast scheme (54)-(60) proposed in the present work will be used to solve this example successively. Firstly, the numerical accuracy and CPU time of two difference schemes in time will be tested. Taking $\epsilon = 10^{-12}$ in FS, the numerical errors, convergence orders and CPU time are recorded in Table 3. From Table 3, the convergence orders are of $3-\beta$ order in time, which is consistent with our obtained Theorem 3.5 and Theorem 4.6. Furthermore, one can see that the computational cost of DS is nearly the same as the one of FS because of the small $N$.

Now, we aim to examine the numerical accuracy and computational cost of the presented difference schemes in space. To the end, the computational results are collected into Table 4 with varying step sizes in space and fixed $\tau = 1/20000$, $\epsilon = 10^{-12}$. 
This table not only shows convergence orders of two method are good agreement with theoretical results but also illustrates the computational cost of DS is largely higher than that of FS with the large $N$. In short, we can conclude that the algorithm presented in this work is greatly efficient.

6. Conclusion. In this paper, we proposed a new H2N2 formula to approximate the fractional derivative $\frac{C}{0} D_\alpha^t f(t) + \frac{C}{0} D_\beta^t f(t)$, $\alpha \in (0, 1)$, $\beta \in (1, 2)$. The truncation error is strictly analyzed in Theorem 2.1. Based on the proposed numerical formula in time and the centered difference approximation for the spatial discretization, we have presented direct difference scheme and fast difference scheme for the mixed diffusion and time-fractional wave equation and proved they are of second-order convergence in space and can reach the optimal convergence order $3 - \beta$ in time. To increase the efficiency of the scheme, we develop fast algorithm from Jiang et al. [18] and FFTs to get the numerical solution. Numerical examples are not only illustrated the numerical result is consist with theoretical conclusion, but also shown that when $N$ is large, the fast scheme can significantly decrease the computational cost without losing any accuracy. In the future work, the $L^2-1_\delta$ formula which can reach the second order accuracy in time will be considered in this problem.

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