Almost-Commutative Geometry, massive Neutrinos and the Orientability Axiom in $K\bar{O}$-Dimension 6

Christoph A. Stephan $^{1,2}$

Abstract

In recent publications Alain Connes [1] and John Barrett [2] proposed to change the $K\bar{O}$-dimension of the internal space of the standard model in its noncommutative representation [3] from zero to six. This apparently minor modification allowed to resolve the fermion doubling problem [4], and the introduction of Majorana mass terms for the right-handed neutrino. The price which had to be paid was that at least the orientability axiom of noncommutative geometry [5,6] may not be obeyed by the underlying geometry.

In this publication we review three internal geometries, all three failing to meet the orientability axiom of noncommutative geometry. They will serve as examples to illustrate the nature of this lack of orientability. We will present an extension of the minimal standard model found in [7] by a right-handed neutrino, where only the sub-representation associated to this neutrino is not orientable.

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$^1$ Unité Mixed de Recherche (UMR) 6207 du CNRS et des Universités Aix-Marseille 1 et 2
Sud Toulon-Var, Laboratoire affilié à la FRUMAM (FR 2291)

$^2$ stephan@cpt.univ-mrs.fr
1 Introduction

During the last two decades Alain Connes developed noncommutative geometry [5], which allows to unify two of the basic theories of modern physics: general relativity and the standard model of particle physics as classical field theories [3]. In the noncommutative framework the Higgs boson, which had previously to be put in by hand, and many of the ad hoc features of the standard model appear in a natural way.

The basic geometric notion in the noncommutative framework is a spectral triple, a set consisting of an algebra, a Hilbert space, a Dirac operator, a chirality operator and a real structure, which corresponds the the charge conjugation from particle physics [5]. For a pedagogical introduction see [8].

For the unification of the standard model and general relativity one finds that the spectral triple is the tensor product of an internal geometry for the standard model and a continuous geometry for space time. In this notion of geometry two different notions of dimension appear. On the one hand the metric or spectral dimension which is given by the behaviour of the eigenvalues of the Dirac operator. For space time the relevant Dirac operator is just the ordinary Dirac operator on curved space time, so one finds the metric dimension to be four. The internal Dirac operator consists of the Fermionic mass matrix, which has a finite number of eigenvalues, thus the internal metric dimension is zero.

A second dimension which is assigned to each spectral triple is the $KO$-dimension, an algebraic dimension with has its roots in K-theory. Until recently [9] it was suspected that the metric dimension and the $KO$-dimension should be equal. Especially the internal space was taken to be of $KO$-dimension zero. But John Barrett [2] and Alain Connes [1] propose to depart from this paradigm, although with different initial motivations, and to take the internal space of the standard model to be of $KO$-dimension six.

This change of $KO$-dimension has several physical advantages. It resolves the fermion doubling [4] problem by allowing to project out the unphysical degrees of freedom resting in the internal space. Furthermore Majorana masses could be introduced, if the orientability axiom [5,6] is weakened, in the sense that it does not apply to the right-handed leptonic singlets.

In this article we will present three internal or matrix geometries. They all exhibit the general features of the model with three summands in the matrix algebra presented in [1, 2]. All three have right-handed Fermions which do explicitly violate the axiom of orientability.

The first example is a model of electro-weak type with two summands in the matrix algebra. It has its origin in the early days of noncommutative geometry [10, 11] and it also appeared in the classification of the spectral triples corresponding to internal spaces in $KO$-dimension zero [12–15] and $KO$-dimension six [7]. It will serve as a toy model to illustrate the general features of the lack of orientability.

The second example will be the standard model in its version with three summands summands in the matrix algebra [16]. Here the right-handed leptonic sector is not orientable. Furthermore in $KO$-dimension six [1, 2] the model admits a Majorana mass for the right-handed neutrinos and Lepto-Quark mass terms, connecting particles to antiparticles. The Lepto-Quark terms are unwanted and have to be put to zero by hand [2], or by imposing extra conditions on the Dirac operator [1].
As a third example we will present an extension of the minimal standard model with four summands in the matrix algebra which was found in \([7, 15]\). This minimal model, containing at least one massless neutrino, is extended by a right-handed neutrino with a Majorana mass term. It should be noted that the minimal standard model obeys all the axioms of noncommutative geometry. It becomes only necessary to violate the axiom of orientability when right-handed neutrinos are introduced in all three generations. The extension of the minimal standard model is based on its counterpart in \(KO\)-dimension zero presented in \([17]\). A further advantage of this model is that no Lepto-Quark terms are allowed.

We will end this publication with a few speculations on the relationship of the physical properties of the right-handed neutrino with Majorana mass term and the axiom of orientability. Furthermore we will give some remarks on the possibility of models beyond the particle content of the standard model, especially the \(AC\)-model presented in \([18]\).

### 2 Basic Definitions

In this section we will give the necessary basic definitions for finite noncommutative geometries with \(K0\)-dimension six \([1, 2]\). We restrict ourselves to real, finite spectral triples \((\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \chi)\). The algebra \(\mathcal{A}\) is a finite sum of matrix algebras \(\mathcal{A} = \oplus_{i=1}^{N} M_{n_i}(\mathbb{K}_i)\) with \(\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}\) where \(\mathbb{H}\) denotes the quaternions. A faithful representation \(\rho\) of \(\mathcal{A}\) is given on the finite dimensional Hilbert space \(\mathcal{H}\). The Dirac operator \(\mathcal{D}\) is a selfadjoint operator on \(\mathcal{H}\) and plays the role of the fermionic mass matrix. \(J\) is an antunitary involution, \(J^2 = 1\), and is interpreted as the charge conjugation operator of particle physics. The chirality \(\chi\) is a unitary involution, \(\chi^2 = 1\), whose eigenstates with eigenvalue \(+1\) are interpreted as right-handed particle states and left-handed antiparticle states, whereas the eigenstates with eigenvalue \(-1\) represent the left-handed particle states and right-handed antiparticle states. These operators are required to fulfill Connes’ axioms for spectral triples:

- \([J, \mathcal{D}] = \{J, \chi\} = 0, \mathcal{D}\chi = -\chi\mathcal{D}\),
- \([\chi, \rho(a)] = [\rho(a), J\rho(a')J^{-1}] = [[\mathcal{D}, \rho(a)], J\rho(a')J^{-1}] = 0, \forall a, a' \in \mathcal{A}\).

Note the change of the commutator \([J, \chi] = 0\) from \(KO\)-dimension zero to the \(KO\)-dimension six.

- The intersection form \(\cap_{ij} := \text{tr}(\chi \rho(p_i)J\rho(p_j)J^{-1})\) is non-degenerate, \(\det \cap \neq 0\). The \(p_i\) are minimal rank projections in \(\mathcal{A}\). This condition is called Poincaré duality. Demanding the Poincaré duality to hold requires an even number of summands in the matrix algebra \([2, 7]\).

- The chirality can be written as a finite sum \(\chi = \sum_i \rho(a_i)J\rho(a_i^\prime)J^{-1}\), which is a 0-dim Hochschild cycle. This condition is called orientability.

As mentioned in the introduction it will be necessary to weaken the last axiom since right-handed neutrinos with Majorana masses cannot fulfill the orientability axiom.
3 The Models

In this section we will treat three matrix geometries which will not in general obey all the axioms of finite spectral triples stated above. The first model which we present is an electro-weak model based on the geometry with two summands in the matrix algebra [10, 11]. It is the purpose of this model to clarify why the orientability axiom does not hold if a Majorana mass term is desired.

The second model is the standard model with three summands in the matrix algebra [16] in its version with \( KO \)-dimension six [1, 2]. For this model the Poincaré duality has to be slightly modified. Instead of being valid for the spectral triple in its whole, the leptonic sector and the quark sector are treated separately and for each sector the Poincaré duality holds [19].

The last model will be based on the standard model in its formulation with four summands in the matrix algebra. This model appeared for the first time in the four summand case with \( KO \)-dimension zero [15] and again in the classification for the case of \( KO \)-dimension six [7]. We will enlarge the particle content following [17], by introducing a right-handed neutrino and additionally furnishing it with a Majorana mass.

3.1 The Electro-Weak Model

We will start with the electro-weak model with two summands in the matrix algebra, [10, 11], which also appeared in the classification [12]. The geometry is furnished by a matrix algebra with two summands \( \mathcal{A}_{EW} = \mathbb{C} \oplus M_2(\mathbb{C}) \ni (a, b) \). It has been shown that the physical models based on this geometry suffer from Yang-Mills-anomalies [15], but nonetheless it will serve us as a toy model to exemplify some basic properties connected the orientability axiom in \( KO \)-dimension six.

We take the left- (right-) handed sub-representation to be

\[
\rho_L(a, b) = b, \quad \rho_R(a, b) = \bar{a}, \quad \rho_L^c(a, b) = a_{12}, \quad \rho_R^c(a, b, c) = a,
\]

where \( 1_{2} \) denotes the 2-dim. unity matrix. The complete representation is given by

\[
\rho(a, b) = \rho_L(a, b) \oplus \rho_R(a, b) \oplus \rho_L^c(a, b) \oplus \rho_R^c(a, b)
\]

The left- and right-handed particle and antiparticle subspaces

\[
\mathcal{H}^{PL} = \left( \begin{array}{c} e \\ \nu \end{array} \right)^L, \quad \mathcal{H}^{PR} = \nu^R, \quad \mathcal{H}^{AL} = \left( \begin{array}{c} e^c \\ \nu^c \end{array} \right)^L, \quad \mathcal{H}^{AR} = \nu^{cR}.
\]

build the complete Hilbert space given by \( \mathcal{H} = \mathcal{H}^{PL} \oplus \mathcal{H}^{PR} \oplus \mathcal{H}^{AL} \oplus \mathcal{H}^{AR} \). Note that this notation can lead to confusion because \( \nu^{cR} \) is the charge conjugate of \( \nu^R \). Since chirality and charge conjugation anti-commute \( \nu^{cR} \) is actually left-handed, confirm [17]. It follows from the axioms [1, 2, 7] that the Dirac operator has the general form

\[
\mathcal{D} = \begin{pmatrix} \Delta & H \\ H^* & \Delta \end{pmatrix},
\]

\[4\]
with $\Delta = \Delta^*$ being a complex matrix, connecting left-handed to right-handed particles and vice versa, and $H$ being a complex symmetric matrix connecting particles to antiparticles. In the basis defined by the Hilbert space, $\Delta$ can thus be written

$$\Delta = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \tag{3.5}$$

with $M$ a complex $1 \times 2$ matrix. To be consistent with the first order axiom $[[D, \rho(a)], J\rho(a')J^{-1}] = 0$, for all $a, a' \in \mathcal{A}_{EW}$ we find that the matrix connecting particles and antiparticles has to take the form

$$H = \begin{pmatrix} 0 & 0 \\ 0 & M_{\nu\bar{\nu}} \end{pmatrix}. \tag{3.6}$$

$M_{\nu\bar{\nu}}$ is a complex number, the Majorana mass of the neutrino. Note that a corresponding Majorana mass for a right-handed electron would not be possible. The electro-weak model with a right-handed electron at the place of the right-handed neutrino has as its algebra representation

$$\tilde{\rho}_L(a,b) = b, \quad \tilde{\rho}_R(a,b) = a, \quad \tilde{\rho}_L^c(a,b) = \bar{a}1_2, \quad \tilde{\rho}_R^c(a,b,c) = \bar{a}. \tag{3.7}$$

Here the first order axiom demands $H$ to be identically zero.

Let us return to the electro-weak model with one right-handed neutrino. We can write down the real structure

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \text{complex conjugation} \tag{3.8}$$

which is nothing but the charge conjugation operator from particle physics. With the chirality

$$\chi = \text{diag}(-1_2, 1, 1_2, -1) \tag{3.9}$$

and the fact that the algebra $\mathcal{A}_{EW}$ has only two projectors $p_1 = (1,0)$ and $p_2 = (0,1_2)$ it is easy to check that the Poincaré duality holds. The same is true for all other axioms safe the orientability.

For the orientability we need to be able to write $\chi$ as a 0-dim. Hochschild cycle, i.e.

$$\chi = \sum_i \rho(a_i) J \rho(a_i') J^{-1}. \tag{3.10}$$

But, taking the representation (3.2) and the real structure (3.8) we find

$$\sum_i \begin{pmatrix} b_i & 0 & 0 & 0 \\ 0 & a_i & 0 & 0 \\ 0 & 0 & \bar{a}_i & 0 \\ 0 & 0 & 0 & \bar{a}_i \end{pmatrix} \begin{pmatrix} a_i'1_2 & 0 & 0 & 0 \\ 0 & a_i' & 0 & 0 \\ 0 & 0 & b_i' & 0 \\ 0 & 0 & 0 & \bar{a}_i' \end{pmatrix} = \sum_i \begin{pmatrix} b_i a_i' & 0 & 0 & 0 \\ 0 & a_i a_i' & 0 & 0 \\ 0 & 0 & \bar{a}_i b_i' & 0 \\ 0 & 0 & 0 & \bar{a}_i \bar{a}_i' \end{pmatrix} \neq \begin{pmatrix} -1_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \chi \tag{3.11}$$
since the matrix algebra is taken to be an algebra over the real numbers. It follows that
the orientability axiom does not hold. This is a general feature of representations where
the particle sub-representation for a given particle multiplet consists of elements stem-
mring from the same sub-matrix algebra as the sub-representation from the corresponding
antiparticle multiplet. Here the case for the right-handed neutrino and its antiparticle is
treated but the same is true if one takes the model with a right-handed electron (3.7).
For a general treatment we refer to [7]

3.2 The Three Summands Model

We will present the standard model with three summands in the matrix algebra [16]
and one generation of Fermions. As in the electro-weak case the $KO$-dimension will be
taken to be six [1, 2]. The second and third generation of Fermions can be introduced by
simply copying the algebraic set-up of the first generation. This also allows to introduce
CKM-mixing matrices for the quarks and the leptons [1, 8].

The internal algebra is chosen to be $A_3 = C \oplus H \oplus M_3(C) \ni (a, b, c)$ and the represen-
tation of the algebra, ordered into left and right, particle and antiparticle part is given
by:

$$\rho_L(a, b, c) = \begin{pmatrix} b \otimes 1_3 & 0 \\ 0 & b \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} a_{13} & 0 & 0 & 0 \\ 0 & \bar{a}_{13} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix},$$

$$\rho^c_L(a, b, c) = \begin{pmatrix} 1_2 \otimes c & 0 & 0 \\ 0 & \bar{a}_{12} \end{pmatrix}, \quad \rho^c_R(a, b, c) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix},$$

where $1_3$ is the unit matrix and the complex conjugates were chosen in order to reproduce
the standard model. The complete representation is defined as in (3.2). The Hilbert space
is copied from particle physics with the usual Fermion multiplets. One has for the left-
and right-handed particle subspaces

$$\mathcal{H}^{PL} = \begin{pmatrix} d_L \\ u_L \\ \nu_e \\ e_L \end{pmatrix}, \quad \mathcal{H}^{PR} = \begin{pmatrix} d_R \\ u_R \\ \nu_e \\ e_R \end{pmatrix},$$

and for the left- and right-handed antiparticle subspaces

$$\mathcal{H}^{AL} = \begin{pmatrix} d^c_L \\ u^c_L \\ \nu^c_e \\ e^c_L \end{pmatrix}, \quad \mathcal{H}^{AR} = \begin{pmatrix} d^{cR} \\ u^{cR} \\ \nu^{cR} \\ e^{cR} \end{pmatrix}.$$  

(3.13)

The complete Hilbert space is then given by $\mathcal{H} = \mathcal{H}^{PL} \oplus \mathcal{H}^{PR} \oplus \mathcal{H}^{AL} \oplus \mathcal{H}^{AR}$.  

6
The particle part \( \Delta \) of the Dirac operator with the general form (3.5) is

\[
\Delta = \begin{pmatrix}
0 & 0 & M_d \otimes 1_3 & M_u \otimes 1_3 & 0 & 0 \\
0 & 0 & 0 & 0 & M_e & M_{\nu e} \\
M_d^* \otimes 1_3 & 0 & 0 & 0 & 0 & 0 \\
M_u^* \otimes 1_3 & 0 & 0 & 0 & 0 & 0 \\
0 & M_{\nu e}^* & 0 & 0 & 0 & 0 \\
0 & M_e^* & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

where the four mass matrices \( M_d, M_u, M_{\nu e} \) and \( M_e \) are 2 \( \times \) 1 complex matrices which may be chosen conveniently.

For the Majorana part \( H \) of the Dirac operator the most general form is

\[
H = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{u\bar{\nu}} & M_{u\bar{e}} & 0 \\
0 & 0 & M_{u\bar{\nu}}^T & M_{u\bar{e}} & 0 & 0 \\
0 & 0 & M_{u\bar{\nu}}^T & M_{u\bar{e}} & 0 & 0 \\
\end{pmatrix},
\]

where the Lepto-Quark matrices \( M_{u\bar{\nu}} \) and \( M_{u\bar{\nu}} \) are complex 3 \( \times \) 1 matrices and the Majorana mass term \( M_{u\nu} \) of the right-handed neutrino is a complex number. The matrix \( H \) is symmetric as desired. It is possible to eliminate the unwanted mass terms mixing leptons and quarks by demanding the Dirac operator to commute with \((a, a_1, 0) \in A_3, [1]\).

From the electro-weak example we learn, that the geometry based on the algebra \( A_3 \) does not obey the axiom of orientability in the right-handed leptonic sector \([19]\). It seems especially worrisome that the right-handed electron does not furnish a sub-representation which obeys the requirements of the orientability.

We will propose a model in which the particles of the minimal standard model meet the requirements of the orientability axiom \([7]\) and only the right-handed neutrino does not. By minimal standard model we mean the standard model with only a left-handed neutrino in the first generation. We will see that this model also allows for a Majorana mass term.

### 3.3 The Four Summands Model

As a basis for the first generation of the standard model with a right-handed neutrino and its Majorana mass we will take the minimal standard model based on the Krajewski diagram depicted in figure 1.
which is the diagram 12 in [7] with the first two summands exchanged. Here the quark sector is encoded in the double arrow and the lepton sector in the single arrow. For a detailed guide for translation of Krajewski diagrams to spectral triples we refer to [12,20] and [7]. The minimal standard model with four summands in the matrix algebra fulfils all the axioms of non-commutative geometry, including the orientability axiom. One should note the similarity of this model to the Connes-Lott model [21].

Following [17], the Hilbert space and consequently the representation and the Dirac operator will be enlarged. We give the corresponding Krajewski diagram in figure 2, from which the representation and the Dirac operator can be read off. Note that in figure 2 the particles and the antiparticles are depicted.

The particle mass terms are represented by the horizontal arrows, the antiparticle mass terms by the vertical arrows and the Majorana mass term, which connects the right-handed neutrino with its charge conjugate, by a dashed diagonal arrow. One should perhaps also note the aesthetic appeal of this Krajewski diagram.

The only axioms that will be violated in this model is the orientability axiom, i.e. all the other axioms of noncommutative geometry still hold. Furthermore the Majorana
mass term is now permitted by the first order axiom. Note, that Lepto-Quark-terms are prohibited by the same axiom.

The internal algebra is chosen to be \( \mathcal{A}_4 = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \ni (a, b, c, d) \) and the representation of the algebra, ordered into left and right, particle and antiparticle part is given by:

\[
\begin{align*}
\rho_L(a, b, c, d) &= \begin{pmatrix} b \otimes 1_3 & 0 \\ 0 & b \end{pmatrix}, \quad \rho_R(a, b, c, d) = \begin{pmatrix} a_{13} & 0 & 0 & 0 \\ 0 & \bar{a}_{13} & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix}, \\
\rho_L^c(a, b, c, d) &= \begin{pmatrix} 1_2 \otimes c & 0 \\ 0 & d_{12} \end{pmatrix}, \quad \rho_R^c(a, b, c, d) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix},
\end{align*}
\tag{3.17}
\]

where the complex conjugates where chosen in order to reproduce the standard model. The complete representation is then defined as in the case of the three summand model, as the direct sum of the particle representations and the complex conjugates of the antiparticle representations. It acts on the Hilbert space \( \mathcal{H} = \mathcal{H}^{PL} \oplus \mathcal{H}^{PR} \oplus \mathcal{H}^{AL} \oplus \mathcal{H}^{AR} \) where the subspaces are copied from the three summand model \( (3.13), (3.14) \). Note furthermore that the particle and antiparticle sub-representations of the right-handed electron contain elements from different summands of the matrix algebra \( \mathcal{A}_4 \):

\[
\rho_{e,R} = \bar{a}, \quad \rho_{e,R}^c = d.
\tag{3.18}
\]

Therefore this sub-representation obeys the orientability axiom. This should be contrasted with the sub-representations of the right-handed neutrino,

\[
\rho_{\nu,R} = \bar{d}, \quad \rho_{\nu,R}^c = d,
\tag{3.19}
\]

which violates the orientability axiom.

The particle part \( \Delta \) of the Dirac operator coincides with the three summand model \( (3.15) \). But the sub-matrix \( H \) differs considerably since the representation \( (3.17) \) prohibits mass terms mixing leptons and quarks. It allows a Majorana mass term for the right-handed neutrino,

\[
H = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{\nu \bar{\nu}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\tag{3.20}
\]

It is of course straight forward to enlarge this model to three generations of fermions. This allows to introduce a CKM-matrix for the quark mixing a and a leptonic mixing matrix. Note that this Majorana mass term of the right-handed neutrino is gauge invariant and therefore obeys the first order axiom. Furthermore it does not participate in the
fluctuation of the Dirac operator and enters in a different way into the spectral action than the other Dirac mass terms [1].

One should furthermore point out, that minimal model allows to introduce massive, right-handed neutrinos in the second and third Fermion generation, obeying all axioms of non-commutative geometry. But in the minimal case at least one neutrino has to remain purely left-handed and massless. This model is not in conflict with current experiments. It would be ruled out if the masses of the neutrinos could be measured directly, and would all be found to be non-zero, or if the neutrinoless double beta-decay would be observed.

4 Speculations

How has one to interpret this violation of the orientability axiom which necessarily appears, at least for right-handed neutrinos with Majorana masses? Since the right-handed neutrino is completely sterile with respect to the gauge-group, it does not possess electro-weak charges or colour charge, the chirality is the only way to discriminate between particle and antiparticle. In the case of KO-dimension zero this rôle was played by the $S^0$-real structure. But from the physical point of view the right-handed neutrino may be considered as its proper antiparticle if a Majorana mass term exists. So there is no difference between the particle and its antiparticle and one could expect that the orientability axiom does not apply to this scenario. The corresponding physical process to decide over the Majorana nature of the right-handed electron-neutrino would be the neutrinoless double beta-decay.

Furthermore the Majorana mass $M_{\nu\bar{\nu}}$ is usually taken to be of the order of the Planck mass. Therefore the right-handed neutrino, $\nu^R_e$ would be extremely heavy, and therefore would not be present in the low-energy. We do expect the noncommutative geometry of the standard model to be a low-energy approximation which should be replaced by a different geometry at high energies close to the Planck mass. This high-energy geometry may be able to include the right-handed neutrino into the complete noncommutative geometric frame-work.

As a remark we would like to point out that the extension of the standard model by new Fermions, baptised AC-Fermions [18], is compatible with the approach of KO-dimension six. These particles originate from the combination of the standard model and the electro-strong model presented in [7, 15]. It allows to enlarge the particle content by two oppositely electro-magnetically charged Fermions which may be suitable candidates for dark matter [22, 23]. These particles are as well consistent with current high-energy experiments [24]. It is remarkable that this extension is among the few possible physically sensible almost-commutative geometries found in [7].

5 Conclusion

Inspired by the recent articles of Alain Connes [1] and John Barrett [2] we presented an illustration of the problems this type of models has concerning the orientability axiom of noncommutative geometry. It turned out that especially right-handed neutrinos with
a Majorana mass term, a vital ingredient for the See-Saw-mechanism, do not obey this axiom.

We furthermore extended the minimal standard model with four summands in the matrix algebra by a right-handed neutrino. This model suffers of course from the same short-commings concerning the orientability. But we think one can argue that this effect may be considered as a low energy effect, since the problematic parts of the representation correspond to extremely heavy right-handed neutrinos, with a Majorana mass of the order of the Planck mass. Furthermore the lack of orientability of the right-handed neutrino may be linked to the physical point of view that this particle is its own antiparticle.

We are now in the very interesting situation that experimental physics may give us deep insights into the internal geometry of space time. If the neutrinoless double beta-decay would be experimentally confirmed, i.e. the electron-neutrino would possess a Majorana mass, one would have to consider one of the above spectral geometries as internal space. This would exclude the case of $KO$-dimension zero, where Majorana masses may not exist and would require to take geometries into account, which do not respect the orientability axiom.

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