ON BOREL FIXED IDEALS GENERATED IN ONE DEGREE

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Abstract. We construct a (shellable) polyhedral cell complex that supports a minimal free resolution of a Borel fixed ideal, which is minimally generated (in the Borel sense) by just one monomial in $S = \mathbb{k}[x_1, x_2, ..., x_n]$; this includes the case of powers of the homogeneous maximal ideal $(x_1, x_2, ..., x_n)$ as a special case. In our most general result we prove that for any Borel fixed ideal $I$ generated in one degree, there exists a polyhedral cell complex that supports a minimal free resolution of $I$.

1. Introduction

We study resolutions over the polynomial ring $S = \mathbb{k}[x_1, x_2, ..., x_n]$, where $\mathbb{k}$ is a field. The idea to encode the structure of the resolution of a monomial ideal in the combinatorial structure of a simplicial complex was introduced in [3] (see also [9]). The idea was generalized later in [4], where resolutions supported on a regular cell complex were introduced. The generalization continued in [2] and [14], where monomial resolutions supported on a CW-complex were introduced and studied. Recently, the necessity for CW-resolutions is justified in [19], and their sufficiency is disproved by the existence of a monomial ideal whose resolution cannot be supported on a CW-complex.

In this paper we study Borel fixed ideals generated in the same degree $d$, which we call $d$-generated. For $d$-generated Borel fixed ideals a minimal free resolution is already well known, namely the Eliahou-Kervaire resolution (see, e.g. [10] or [16]), which is also CW-cellular, as it is proved in [2] by using discrete Morse theory. Moreover, in [2], the authors give the Morse complex that supports a minimal free resolution for powers of the homogeneous maximal ideal $(x_1, ..., x_n)$ of the polynomial ring $S = \mathbb{k}[x_1, x_2, ..., x_n]$ as a worked example. More generally, the Morse complex that supports a minimal free resolution of principal Borel fixed ideals, that is, of those ideals which are minimally generated (in the Borel sense) by just one monomial, is given in [14]. However, it is not clear whether any of those Morse complexes is regular or not. Thus, a natural question is whether there exists a regular cell complex that supports a minimal resolution of a $d$-generated Borel fixed ideal. We answer the question positively in this paper, which is organized as follows:

In Section 2, we give the basic notation and preliminaries for the rest of this paper and we refer to the literature for more details.

In Section 3, we answer the above natural question by constructing inductively a shellable polyhedral cell complex that supports the minimal free resolution of a...
principal Borel fixed ideal in $S = k[x_1, x_2, ..., x_n]$; this includes the case of powers of the homogeneous maximal ideal as a special case. Our most general result is theorem 3.19, where we prove that for any $d$-generated Borel fixed ideal $I$, there exists a polyhedral cell complex that supports a minimal free resolution of $I$. It should be noted that the basis we use in the minimal free resolution is different than the one used in the Elliahou-Kervaire resolution.

Finally, in Section 4, we consider the lcm-lattice of a $d$-generated Borel fixed ideal. In particular, in proposition 4.3, we show that it is ranked. This result was proved (in greater generality) independently in [18].

2. Notation-Preliminaries

2.1. Monomial ideals. All ideals in this paper are considered to be monomial ideals. We work over the polynomial ring $S = k[x_1, x_2, ..., x_n]$ with $\text{char}(k) = 0$.

For small $n$ we may use the letters $a, b, c, d, ...$ instead of $x_1, x_2, x_3, x_4, ...$.

For a monomial $m = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$ in $S$, we define the exponent vector to be $e(m) = (a_1, a_2, ..., a_n)$ and we set $\text{max}(m)$ to be the largest index of a variable that divides $m$.

We let $G(I)$ denote the unique minimal set of monomial generators of a (monomial) ideal $I$. A (monomial) ideal $I$ is called Borel fixed, if for every $m$ in $G(I)$ and every $x_t$ that divides $m$, $m \rightarrow s := m x_t x_s$ is in $I$ for all $1 \leq s < t$. A Borel fixed ideal $I$ is called principal Borel, and it is written as $I = \langle m \rangle$, if $I$ is the smallest Borel fixed ideal such that $m$ is in $G(I)$. In this case, we also say that $I$ is generated by just one monomial $m$ in the Borel sense.

Example 2.1. Let $S = k[a, b, c]$. The ideal $(a^2, ab, b^2, ac, bc)$ is a Borel fixed ideal, which is also principal Borel, because $(a^2, ab, b^2, ac, bc) = \langle bc \rangle$

For more on monomial ideals we refer to [8], [9] and [16].

2.2. Cellular resolutions and polyhedral complexes. As in [4], let $X$ be a regular cell complex having $G(I)$, the set of minimal generators of $I$, as its set of vertices and let $\epsilon_X$ be an incidence function on $X$. It is well known that such a function exists, (see e.g. pp. 244-248 in [15]). Next we label each nonempty face $F$ of $X$ by the least common multiple $m_F$ of the monomials $m_j$ in $G(I)$, which correspond to the vertices of $F$. The degree $a_F$ of the face $F$ is defined to be the exponent vector $e(m_F)$.

Let $SF$ be the free $S$-module with one generator $F$ in degree $a_F$. The cellular complex $F_X$ is the $\mathbb{Z}^n$-graded $S$-module $\bigoplus_{\varnothing \neq F \in X} SF$ with differential

$$\partial F = \sum_{\varnothing \neq F' \in X} \epsilon_X(F, F') \frac{m_F}{m_F'} F'.$$

For each degree $b \in \mathbb{Z}^n$ let $X_{\preceq b}$ be the subcomplex of $X$ on the vertices of degree $\preceq b$. The following results are proved in [4].
Proposition 2.2. The complex $F_X$ is a free resolution of $I$ if and only if $X_{\leq b}$ is acyclic over $k$ for all degrees $b$. In this case, $F_X$ is called a cellular resolution of $I$.

Corollary 2.3. The cellular complex $F_X$ is a resolution of $I$ if and only if the cellular complex $F_{X_{\leq b}}$ is a resolution of the monomial ideal $I_{\leq b}$ for all $b \in \mathbb{Z}^n$.

Remark 2.4. A cellular resolution $F_X$ is minimal if and only if any two comparable faces $F' \subseteq F$ of the same degree coincide.

Example 2.5. Let $I \subseteq S$ be a monomial ideal with $G(I) = \{x_1^d, x_2^d, \ldots, x_n^d\}$ for a fixed positive integer $d$. Then the labelled $(n-1)$-simplex $\Delta_{n-1}(x_1^d, x_2^d, \ldots, x_n^d)$ with vertices in $G(I)$ supports a minimal free resolution of $I$.

Note that in this paper, whenever we say cellular resolutions, we mean resolutions supported on a regular cell complex. Otherwise, we talk about CW-resolutions to emphasize the difference and avoid confusion.

The above results are presented in [16] for polyhedral complexes, which is a special case of regular cell complexes. A polyhedral cell complex $X$ is a nonempty finite collection of convex polytopes (in some real vector space $\mathbb{R}^N$), called faces of $X$, satisfying two properties:

- If $P$ is a polytope in $X$ and $F$ is a face of $P$, then $F$ is in $X$.
- If $P$ and $Q$ are in $X$, then $P \cap Q$ is a face of both $P$ and $Q$.

For more on polytopal complexes we refer to [20]. Here it suffices to mention that a basic notion that we are going to use is that of regular subdivisions of a polytope ([20], p.129, or [7], p.34). Another concept is the shellability of a polytope ([20], p.233).

2.3. Results from Algebraic Topology. We assume familiarity with the basic notions of CW-complex and regular cell complex and their differences. Recall that the closures of the cells of a regular CW-complex are homeomorphic with closed balls. For example, any polyhedral cell complex is regular. So we only state the two major theorems from algebraic topology that we use. We need the cellular version of Mayer-Vietoris theorem and the Künneth theorem with field coefficients.

Theorem 2.6 (Mayer-Vietoris). Let $X$ be a CW-complex and let $Y_1$ and $Y_2$ be CW subcomplexes of $X$ such that $X = Y_1 \cup Y_2$. Then there is an exact sequence

$$\cdots \rightarrow \tilde{H}_i(Y_1 \cap Y_2; k) \rightarrow \tilde{H}_i(Y_1; k) \oplus \tilde{H}_i(Y_2; k) \rightarrow \tilde{H}_i(X; k) \rightarrow \tilde{H}_{i-1}(Y_1 \cap Y_2; k) \rightarrow \cdots .$$

Theorem 2.7 (Künneth). Let $X$ and $Y$ be two CW-complexes. Then there is a natural isomorphism

$$\bigoplus_j (H_j(X; k) \otimes_k H_{i-j}(Y; k)) \rightarrow H_i(X \times Y; k).$$

We refer to [13] or [15], for more on these.

3. Cellular Resolutions of $d$-generated Borel fixed ideals

3.1. Three basic Lemmas. Now we may proceed to our study of $d$-generated Borel fixed ideals. Let $I$ and $J$ be two monomial ideals in $S$ and assume that $X$ and $Y$ are regular cell complexes in $\mathbb{R}^N$ (for some $N$) that support a (minimal) free resolution of $I$ and $J$, respectively.

Can we say anything about the cellular resolution of $I + J$ and/or the cellular resolution of $IJ$?
The following three lemmas give some results related to this question, which will be useful in proving our main results. The assumption that the cell complexes are regular is not necessary.

**Lemma 3.1.** Let $I$ and $J$ be two monomial ideals in $S$ such that $G(I + J) = G(I) \cup G(J)$ set-theoretically. Suppose that

(i) $X$ and $Y$ are regular cell complexes in some $\mathbb{R}^N$ that support a (minimal) free resolution of $I$ and $J$, respectively, and

(ii) $X \cap Y$ is a regular cell complex that supports a (minimal) free resolution of $I \cap J$.

Then $X \cup Y$ supports a (minimal) free resolution of $I + J$.

**Proof:** First let $Z := X \cup Y$ and note that $Z$ is a regular cell complex. From our hypothesis, we have

$$\tilde{H}_i(X_{\leq b}; k) = 0, \quad \tilde{H}_i(Y_{\leq b}; k) = 0, \quad \text{and} \quad \tilde{H}_i((X \cap Y)_{\leq b}; k) = 0$$

for all $i$ and all $b \in \mathbb{Z}^n$. Furthermore, it is clear that

$$Z_{\leq b} = (X \cup Y)_{\leq b} = X_{\leq b} \cup Y_{\leq b}$$

for all $b \in \mathbb{Z}^n$. Then the Mayer-Vietoris theorem 2.6 gives us the following exact sequence

$$\tilde{H}_i(X_{\leq b}; k) \oplus \tilde{H}_i(Y_{\leq b}; k) \to \tilde{H}_i((X \cap Y)_{\leq b}; k) \to \tilde{H}_{i-1}((X \cap Y)_{\leq b}; k)$$

Consequently, $\tilde{H}_i(Z_{\leq b}; k) = 0$ and the proof is complete from proposition 2.2.

**Remark 3.2.** For any two monomial ideals $I$ and $J$, we have

$$G(I + J) \subseteq G(I) \cup G(J).$$

Our assumption that $G(I + J) = G(I) \cup G(J)$ guarantees the right labelling of the cell complex $X \cup Y$. A case where equality becomes true is when all elements of $G(I) \cup G(J)$ are of the same degree.

Note that from the labelling of $X, Y$ and $X \cap Y$ and our assumptions above, it follows that

$$G(I \cap J) = G(I) \cap G(J).$$

**Lemma 3.3.** Let $I \subseteq k[x_1, \ldots, x_k]$ and $J \subseteq k[x_{k+1}, \ldots, x_n]$ be two monomial ideals. Suppose that $X$ and $Y$ are regular cell complexes in some $\mathbb{R}^N$ of dimension $k - 1$ and $n - k - 1$, respectively, that support a (minimal) free resolution of $I$ and $J$, respectively. Then the regular cell complex $X \times Y$ supports a (minimal) free resolution for $I J$.

**Proof:** Let $Z := X \times Y$ and let $b = (b_1, b_2) \in \mathbb{Z}^n$, where $b_1 \in \mathbb{Z}^k$ and $b_2 \in \mathbb{Z}^{n-k}$. Then, it is easy to check that

$$Z_{\leq b} = (X \times Y)_{\leq b} = X_{\leq b_1} \times Y_{\leq b_2}$$

From the Künneth theorem 2.7 for CW complexes, there is an isomorphism

$$\bigoplus_j (H_j(X_{\leq b_1}; k) \otimes_k H_{i-j}(Y_{\leq b_2}; k)) \cong H_i(X_{\leq b_1} \times Y_{\leq b_2}; k) = H_i(Z_{\leq b}; k)$$

for all $i$. From our hypothesis, we have

$$\tilde{H}_i(X_{\leq b_1}; k) = 0 \quad \text{and} \quad \tilde{H}_i(Y_{\leq b_2}; k) = 0,$$
for all $i$. Therefore,
\[ H_0(Z_{\leq b}; k) = k \otimes_k k = k, \]
while
\[ H_i(Z_{\leq b}; k) = 0, \]
for $i > 0$. Now assume that $e_X \times e_Y$ and $\sigma_X \times \sigma_Y$ are two comparable faces of $X \times Y$ with the same label. That is,
\[ e_X \subset \sigma_X \quad \text{and} \quad e_Y \subset \sigma_Y \]
and
\[ label(e_X \times e_Y) = label(\sigma_X \times \sigma_Y) = (b_1, b_2) \]
where $b_1 \in \mathbb{Z}^k$ and $b_2 \in \mathbb{Z}^{n-k}$. Then,
\[ label(e_X) = label(\sigma_X) = (b_1) \]
and
\[ label(e_Y) = label(\sigma_Y) = (b_2). \]
Therefore, $e_X = \sigma_X$ and $e_Y = \sigma_Y$, $e_X \times e_Y = \sigma_X \times \sigma_Y$. The proof is complete from proposition 2.2 and remark 2.4.

**Remark 3.4.** From our conclusion in lemma 3.3, it follows that
\[ pdim(S/IJ) = dim(X \times Y) + 1 = (k - 1) + (n - k - 1) + 1 = n - 1. \]

**Lemma 3.5.** Let $I \subset k[x_1, ..., x_k]$ and $J \subset k[x_1, ..., x_n]$ be two monomial ideals such that $|G(I)| = |G(J)|$. Suppose that there exists a regular cell complex $X$ in $\mathbb{R}^N$ (for some $N$) of dimension $k - 1$ and a regular cell complex $Y$ in $\mathbb{R}^{n-k}$ of dimension $n - k$, which support a (minimal) free resolution of $I$ and $J$ respectively. Then the regular cell complex $X \times Y$ supports a (minimal) free resolution of $IJ$.

**Proof:** Let $Z := X \times Y$ and let $b = (b_1, \beta, b_2) \in \mathbb{Z}^n$, where $b_1 \in \mathbb{Z}^{k-1}$, $\beta \in \mathbb{Z}$ and $b_2 \in \mathbb{Z}^{n-k}$. Then, for $1 \leq k \leq \beta - 1$ define iteratively $Z_{\leq b}^{(k)}$ as follows
\[ Z_{\leq b}^{(k+1)} = Z_{\leq b}^{(k)} \cup \left( X_{\leq (b_1, k)} \times Y_{\leq (\beta-k, b_2)} \right), \]
where $Z_{\leq b}^{(1)} = X_{\leq (b_1, 0)} \times Y_{\leq (\beta, b_2)}$, and note that
\[ Z_{\leq b} = Z_{\leq b}^{(\beta)} = \left( X_{\leq (b_1, 0)} \times Y_{\leq (\beta, b_2)} \right) \cup \left( X_{\leq (b_1, 1)} \times Y_{\leq (\beta-1, b_2)} \right) \cup \cdots \cup \left( X_{\leq (b_1, k)} \times Y_{\leq (\beta-k, b_2)} \right) \]
Moreover,
\[ \left( X_{\leq (b_1, 0)} \times Y_{\leq (\beta, b_2)} \right) \cap \left( X_{\leq (b_1, 1)} \times Y_{\leq (\beta-1, b_2)} \right) = X_{\leq (b_1, 0)} \times Y_{\leq (\beta-1, b_2)}, \]
or more generally,
\[ Z_{\leq b}^{(k)} \cap \left( X_{\leq (b_1, k+1)} \times Y_{\leq (\beta-k-1, b_2)} \right) = X_{\leq (b_1, 0)} \times Y_{\leq (\beta-k-1, b_2)}. \]
Therefore, by combining the Mayer-Vietoris theorem 2.6 with the Künneth formula 2.7 we get
\[ H_1(Z_{\leq b}; k) = 0. \]
Now assume that $e_X \times e_Y$ and $\sigma_X \times \sigma_Y$ are two comparable faces of $X \times Y$ with the same label. That is,
\[ e_X \subset \sigma_X \quad \text{and} \quad e_Y \subset \sigma_Y \]
and
\[ \text{label}(e_X \times e_Y) = \text{label}(\sigma_X \times \sigma_Y) = (b_1, \beta, b_2) \]
where \( b_1 \in \mathbb{Z}^{k-1}, \beta \in \mathbb{Z} \) and \( b_2 \in \mathbb{Z}^{n-k} \). Then,
\[ \text{label}(e_X) = (b_1, \beta_1) \quad \text{and} \quad \text{label}(\sigma_X) = (b_1, \beta_2), \]
which implies \( \beta_1 \leq \beta_2 \) and
\[ \text{label}(e_Y) = (\beta - \beta_1, b_2) \quad \text{and} \quad \text{label}(\sigma_Y) = (\beta - \beta_2, b_2), \]
which implies \( \beta - \beta_1 \leq \beta - \beta_2 \), that is, \( \beta_2 \leq \beta_1 \). Thus we have \( \beta_1 = \beta_2 \) and then,
\[ \text{label}(e_X) = \text{label}(\sigma_X) = (b_1, \beta_1) \]
and
\[ \text{label}(e_Y) = \text{label}(\sigma_Y) = (\beta - \beta_1, b_2). \]
Therefore, \( e_X = \sigma_X \) and \( e_Y = \sigma_Y \), and so \( e_X \times e_Y = \sigma_X \times \sigma_Y \). Thus, the resolution is minimal and the proof is complete.

**Remarks 3.6.**
1) For any two monomial ideals \( I \) and \( J \), we have
\[ G(IJ) \subseteq G(I)G(J). \]
Thus, our assumption that \( |G(IJ)| = |G(I)| \cdot |G(J)| \) forces \( G(IJ) = G(I)G(J) \).
2) Let \( F_X \) be the cellular resolution of \( I \) and let \( F_Y \) be the cellular resolution of \( J \). Then define
\[ F_{X \times Y} := F_X \otimes F_Y. \]
(see e.g. pp. 280-282 in [15]).

**Example 3.7.** Let \( S = k[a, b, c] \). The resolution of \( I = (a, b) \) is of the form
\[ 0 \to S(-2) \to S^2(-1) \to (a, b) \to 0 \]
and the resolution of \( J = (b, c) \) is of the form
\[ 0 \to S(-2) \to S^2(-1) \to (b, c) \to 0 \]
Therefore, the resolution of \( IJ = (a, b)(b, c) \) is of the form
\[ 0 \to S(-4) \to S^4(-3) \to S^4(-2) \to IJ \to 0, \]
which is the tensor product of the first two resolutions.
3) As in remark 3.4, from our conclusion in lemma 3.5, it follows that
\[ \text{pdim}(S/IJ) = \text{dim}(X \times Y) + 1 \]
\[ = (k - 1) + (n - k) + 1 \]
\[ = n. \]
4) A lemma similar to lemmas 3.3 and 3.5 for monomial ideals
\( I \subset k[x_1, \ldots, x_{k-1}, x_k] \quad \text{and} \quad J \subset k[x_{k-1}, x_k, \ldots, x_n] \)
and corresponding regular cell complexes \( X \) and \( Y \) with
\[ \dim(X) = k - 1 \quad \text{and} \quad \dim(Y) = n - k + 1 \]
would fail because we would have
\[ \dim(X \times Y) + 1 = (k - 1) + (n - k + 1) + 1 \]
\[ = n + 1 \]
\[ > \text{pdim}(S/IJ). \]
3.2. Powers of the homogeneous maximal ideal. Now we may prove our first main result, which is about the powers of the homogeneous maximal ideal in \( S \).

**Theorem 3.8.** There exists a (shellable) polyhedral cell complex \( P_d(x_1, \ldots, x_n) \) that supports a minimal free resolution of \((x_1, \ldots, x_n)^d\). Moreover, \( P_d(x_1, \ldots, x_n) \) is a polyhedral subdivision of the \((n-1)\)-simplex \( \Delta_{n-1}(x_1^d, x_2^d, \ldots, x_n^d) \).

**Proof:** The proof will be by induction on \( d \). It is clear that if \( d = 1 \), then the standard \((n-1)\)-simplex denoted by \( \Delta_{n-1}(x_1, x_2, \ldots, x_n) \), supports a minimal free resolution of \((x_1, \ldots, x_n)^d\) for all \( n \geq 1 \). Thus

\[
P_1(x_1, \ldots, x_n) = \Delta_{n-1}(x_1, x_2, \ldots, x_n)
\]

for all \( n \geq 1 \). Also, \( P_1(x_{k+1}, \ldots, x_n) \) is a subcomplex of \( P_1(x_k, \ldots, x_n) \) for all \( k < n \). Next, assume that for some \( d \geq 1 \) we have constructed \( P_d(x_1, \ldots, x_n) \) for all \( n \geq 1 \) and that \( P_d(x_{k+1}, \ldots, x_n) \) is a subcomplex of \( P_d(x_k, \ldots, x_n) \) for all \( k < n \). Define the ideals

\[
I_k = (x_1, x_2, \ldots, x_k)(x_k, x_{k+1}, \ldots, x_n)^d
\]

and note that an easy (finite) induction on \( k \) gives us

\[
I_1 + \cdots + I_k = (x_1, \ldots, x_k)(x_1, x_2, \ldots, x_n)^d
\]

for all \( 1 \leq k \leq n \). Indeed, assuming that we have proved it for \( k-1 \), for some \( k > 1 \), then we have

\[
I_1 + \cdots + I_k - I_k = (x_1, \ldots, x_k)(x_k, x_{k+1}, \ldots, x_n)^d + (x_1, \ldots, x_k)(x_k, x_{k+1}, \ldots, x_n)^d
\]

\[
= (x_1, \ldots, x_k)(x_1, x_2, \ldots, x_n)^d + (x_1, \ldots, x_k)(x_k, x_{k+1}, \ldots, x_n)^d
\]

\[
= (x_1, \ldots, x_k)(x_1, x_2, \ldots, x_n)^d + x_k(x_k, x_{k+1}, \ldots, x_n)^d
\]

\[
= (x_1, \ldots, x_k)(x_1, x_2, \ldots, x_n)^d
\]

Moreover, we see that

\[
(I_1 + \cdots + I_k) \cap I_{k+1} = (x_1, \ldots, x_k)(x_1, \ldots, x_n)^d \cap (x_1, \ldots, x_k)(x_{k+1}, \ldots, x_n)^d
\]

\[
= (x_1, \ldots, x_k)(x_{k+1}, \ldots, x_n)^d.
\]

From lemmas 3.3 and 3.5, we conclude that the polyhedral cell complexes \( C_k \) and \( D_k \) \( (k = 1, 2, \ldots, n) \) defined by

\[
C_k := \Delta_{k-1}(x_1, x_2, \ldots, x_k) \times P_d(x_k, \ldots, x_n)
\]

and

\[
D_k := C_k \cap C_{k+1} = \Delta_{k-1}(x_1, x_2, \ldots, x_k) \times P_d(x_{k+1}, \ldots, x_n)
\]

support a minimal free resolution for \( I_k \) and \((I_1 + \cdots + I_k) \cap I_{k+1} \), respectively.

Thus, from this and lemma 3.1, the polyhedral cell complex \( C'_k \), which is defined recursively by

\[
C'_1 = C_1, \quad \text{and} \quad C'_{k+1} = C'_k \cup C_{k+1}
\]

for \( k \geq 1 \), supports a (minimal) free resolution for \((x_1, \ldots, x_k)(x_1, x_2, \ldots, x_n)^d\). Accordingly, set

\[
P_{d+1}(x_1, x_2, \ldots, x_n) := C'_n = C_1 \cup C_2 \cup \cdots \cup C_n.
\]

and the construction of our polyhedral cell complex is done by induction. The fact that \( P_d(x_1, \ldots, x_n) \) is a polyhedral subdivision of the \((n-1)\)-simplex \( \Delta_{n-1}(x_1^d, x_2^d, \ldots, x_n^d) \)
is clear from our construction. \( P_d(x_1, \ldots, x_n) \) is a regular subdivision of the \((n - 1)\)-simplex \( \Delta_{n-1}(x_1^d, x_2^d, \ldots, x_n^d) \) (see, e.g., [7], p.37). Since a regular subdivision of a polytope is shellable ([20], p.243), we conclude that \( P_d(x_1, \ldots, x_n) \) is shellable.

**Example 3.9.** Let \( I = (a, b, c, d)^2 \). Using the software package MACAULAY 2 [12], we see that the polyhedral cell complex that supports the minimal free resolution of \( I \) is

This can be decomposed as follows:

Another cell complex that supports a minimal free resolution of \( I \) is the following (Morse complex), which supports the Eliahou-Kervaire resolution of \( I \). In particular, note that it is not polyhedral.
Remark 3.10. From theorem 3.8 and corollary 2.3, we may get a minimal cellular resolution for all ideals of the form $I_{\leq b}$ ($b \in \mathbb{Z}^n$) (see also [17]).

3.3. Principal Borel fixed ideals. Our next goal is to prove a more general result for principal Borel fixed ideals. Note that the following theorem includes theorem 3.8 as a special case, since

$$(x_1, ..., x_n)^d = <x_n^d>.$$ 

Theorem 3.11. There exists a (shellable) polyhedral cell complex $Q(m)$ that supports a minimal free resolution of the principal Borel fixed ideal

$$I = <m> \cap \prod_{j=1}^{s} P_{d_j},$$

where $m = x_{\lambda_1}^{d_1}x_{\lambda_2}^{d_2} \cdots x_{\lambda_s}^{d_s}, I_i = (x_1, x_2, ..., x_i)$ and $1 \leq \lambda_1 < \lambda_2 < ... < \lambda_s$. Moreover, $Q(m)$ is a subcomplex of $P_d(x_1, ..., x_n)$, where $d = \text{degree}(m)$. In particular, $Q(m)$ is the union of all the convex polytopes (i.e. the faces) of the polyhedral cell complex $P_d(x_1, ..., x_n)$, with vertices in $<m>$.

Remark 3.12. If $s = 1$, then $m = x_{\lambda_1}^{d_1}$, and so $Q(m) = P_{d_1}(x_1, x_2, ..., x_{\lambda_1})$. If $\lambda_{s-1} = 1$, then $s = 2$ and $m = x_{\lambda_1}^{d_1}x_{\lambda_2}^{d_2}$, so $Q(m)$ is obtained by multiplying all the labels of the vertices of $P_{d_2}(x_1, x_2, ..., x_{\lambda_2})$ by $x_1^{d_1}$.

Before we prove the above theorem we need a lemma. Because of the above remark, we may assume that $s > 1$ and $\lambda_{s-1} > 1$.

Lemma 3.13. Let $I$ be a principal Borel fixed ideal as above. Define the ideals

$$N_k = <x_{\lambda_1}^{d_1}x_{\lambda_2}^{d_2} \cdots x_{\lambda_j}^{d_j}x_{j+1}^{d_{j+1}+\cdots+d_{s-1}}>$$

for $\lambda_j < k \leq \lambda_{j+1}$ ($j < s - 1$ and $\lambda_0 = 0$). Then for $\lambda_{s-1} < \mu \leq \lambda_s$

(a) $N_i(x_1, ..., x_{\mu})^{d_s} = N_1(x_1, ..., x_{\mu})^{d_s} + N_2(x_2, ..., x_{\mu})^{d_s} + ... + N_i(x_i, ..., x_{\mu})^{d_s}$

for $i = 1, 2, ..., \lambda_{s-1}$, and

(b) $N_j(x_1, ..., x_{\mu})^{d_s} \cap N_{j+1}(x_{j+1}, ..., x_{\mu})^{d_s} = N_j(x_{j+1}, ..., x_{\mu})^{d_s}$.
for \( j = 1, 2, \ldots, \lambda_s - 1 \),

Proof: (a) First it is clear that
\[
N_i(x_1, \ldots, x_\mu)^{d_\mu} \supseteq N_i(x_1, \ldots, x_\mu)^{d_\mu} + N_2(x_2, \ldots, x_\mu)^{d_\mu} + \ldots + N_i(x_i, \ldots, x_\mu)^{d_\mu}.
\]
Next, note that the ideal \( N_i(x_1, \ldots, x_\mu)^{d_\mu} \) is principal Borel, minimally generated (in the Borel sense) by
\[
x_{\lambda_1}^{d_\lambda_1}x_{\lambda_2}^{d_\lambda_2} \ldots x_{\lambda_j}^{d_{\lambda_j}}x_{j+1}^{d_{j+1}} + \ldots + x_{\mu}^{d_\mu},
\]
\( (\lambda_1 < i \leq \lambda_j < \mu, \text{since } j < s - 1) \), which is contained in \( N_i(x_1, \ldots, x_\mu)^{d_\mu} \). Thus, to complete the proof of this part, it suffices to show that the sum of the ideals on the right hand side of the equality to be proved is Borel fixed. Set \( J_k = N_k(x_k, \ldots, x_\mu)^{d_\mu} (k = 1, 2, \ldots, i) \). Since \( J_1 \) is Borel fixed, assume by induction that for some \( 1 \leq k < i \), the ideal \( J_1 + \ldots + J_k \) is Borel fixed and let \( n \in J_1 + \ldots + J_k + J_k+1 \). If \( n \in J_1 + \ldots + J_k \), we are done, so assume that \( n \in G(N_k+1) \setminus (J_1 + \ldots + J_k) \). Then write
\[
n = n' + n''
\]
for some \( n' \in G(N_k+1) \) and \( n'' \in G((x_{k+1}, \ldots, x_\mu)^{d_\mu}) \). Now observe that \( x_{k+1} \) must divide \( n' \) because \( n \notin J_1 + \ldots + J_k \). Next, if \( r < t \) and \( x_t \) divides \( n \), then we see that \( n_{t-r} \) is in \( J_{k+1} \). Indeed, it is easy to verify this when \( x_t \) divides \( n' \), because \( N_k+1 \) is Borel fixed, so assume that \( x_t \) does not divide \( n' \). Then \( x_t \) divides \( n'' \), so \( t \geq k + 1 \). If \( k + 1 \leq r \), then we have
\[
\frac{n_{r-t}}{x_t} = n' \frac{x_{r-t}}{x_t} \in N_{k+1}(x_{k+1}, \ldots, x_\mu)^{d_\mu} = J_{k+1},
\]
while if \( r < k + 1 \),
\[
\frac{n_{r-t}}{x_t} = n'_{r-t} \frac{x_{r-t}}{x_t} \in N_{k+1}(x_{k+1}, \ldots, x_\mu)^{d_\mu} = J_{k+1},
\]
because \( N_{k+1} \) is Borel fixed. Thus
\[
\frac{n_{r-t}}{x_t} \in J_1 + \ldots + J_k + J_{k+1}
\]
in all cases and the proof of part (a) is complete. For part (b), let \( m \in G(N_j(x_1, \ldots, x_\mu)^{d_\mu}) \) and \( n \in G(N_{j+1}(x_{j+1}, \ldots, x_\mu)^{d_\mu}) \), and write \( m = m_1m_2 \) and \( n = n_1n_2 \), where \( m_1 \in G(N_j) \), \( m_2 \in G((x_{j+1}, \ldots, x_\mu)^{d_\mu}) \), \( n_1 \in G(N_{j+1}) \) and \( n_2 \in G((x_{j+1}, \ldots, x_\mu)^{d_\mu}) \). Then, note that \( m_1n_2 \) divides \( \text{lcm}(m, n) \). This implies that \( \text{lcm}(m, n) \) is in \( N_j(x_{j+1}, \ldots, x_\mu)^{d_\mu} \), and so
\[
N_j(x_1, \ldots, x_\mu)^{d_\mu} \cap N_{j+1}(x_{j+1}, \ldots, x_\mu)^{d_\mu} \subseteq N_j(x_{j+1}, \ldots, x_\mu)^{d_\mu}.
\]
The opposite containment is obvious, so the proof of part (b) is complete.

Remark 3.14. Part (a) with \( i = \lambda_{s-1} \) and \( \mu = \lambda_s \) yields
\[
I = N_1(x_1, \ldots, x_\lambda)^{d_\lambda} + N_2(x_2, \ldots, x_\lambda)^{d_\lambda} + \ldots + N_{\lambda_{s-1}}(x_{\lambda_{s-1}}, \ldots, x_\lambda)^{d_\lambda}.
\]
Examples 3.15. 1) For the ideal \( I = < b^2 > \) in \( \mathbb{k}[a, b, c, d] \), we have \( s = 2, \lambda_1 = 2, d_1 = 1, \lambda_2 = 4 \) and \( d_2 = 2 \). Moreover, \( N_1 = < a > \), \( N_2 = < b > = (a, b) \). Therefore,
\[
I = < a > (a, b, c, d)^2 + < b > (b, c, d)^2.
\]
2) For the ideal \( I = < bcd > \) in \( \mathbb{k}[a, b, c, d] \), we have \( s = 3, \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 4 \) and \( d_1 = d_2 = d_3 = 1 \). Moreover, \( N_1 = < a^2 > \), \( N_2 = < b^2 > \) and \( N_3 = < bc > \). Therefore,
\[
I = < a^2 > (a, b, c, d) + < b^2 > (b, c, d) + < bc > (c, d).
\]
<bcd> = <a²> (a, b, c, d) + <b²> (b, c, d) + <bc> (c, d)

Proof of Theorem 3.11: By induction on s. For s = 1, we are done. Assume that s > 1 and that we have obtained $Q \left( \prod_{j=1}^{k} I_{\lambda_{j_{i}}} \right)$ for all $k < s$. By the inductive hypothesis, there is a polyhedral cell complex $Q(N_{i})$ that supports a minimal free resolution for the ideals $N_{i}$, for all $1 \leq i \leq \lambda_{s-1}$. Moreover, from our construction we see that $Q(N_{i})$ is a subcomplex of $Q(N_{i+1})$ for all $1 \leq i < \lambda_{s-1}$. Set $J_{i} = N_{k}(x_{i}, ..., x_{\lambda_{s}})$ $(i = 1, 2, ..., \lambda_{s-1} - 1)$. From lemmas 3.3, 3.5 and 3.13 it follows that the polyhedral cell complexes $C_{i}$ and $D_{i}$ $(i = 1, 2, ..., \lambda_{s})$ defined by

$$
C_{i} := Q(N_{i}) \times P_{d_{s}}(x_{i}, ..., x_{\lambda_{s}})
$$

and

$$
D_{i} := C_{i} \cap C_{i+1} = Q(N_{i}) \times P_{d_{s}}(x_{i+1}, ..., x_{\lambda_{s}}),
$$
support a minimal free resolution of $J_{i}$ and $(J_{1} + ... + J_{i}) \cap J_{i+1}$, respectively, for all $1 \leq i < \lambda_{s-1}$. Thus, lemma 3.1 implies that the polyhedral cell complex $C'_{k}$, which is defined recursively by

$$
C'_{1} = C_{1}, \quad \text{and} \quad C'_{i+1} = C'_{i} \cup C_{i+1}
$$

for $1 \leq i < \lambda_{s-1}$, supports a (minimal) free resolution of $J_{1} + J_{2} + ... + J_{i}$. Accordingly, set

$$
Q(I) := C'_{\lambda_{s-1}} = C_{1} \cup C_{2} \cup \cdots \cup C_{\lambda_{s-1}}
$$
and the construction of our polyhedral cell complex is done by induction. Also, from our construction it follows that \( Q(m) \) is a subcomplex of \( P_d(x_1, ..., x_n) \), where \( d = \text{degree}(m) \), as desired. Finally, it is easy to see as in 3.8 that \( Q(m) \) is also shellable in order to complete the proof.

3.4. \( d \)-generated Borel fixed ideals. Next we would like to generalize theorem 3.11 to the case of any Borel fixed ideal generated in one degree. Before we prove this in 3.19, we need more preliminary results. Recall that for two monomials \( m_1 \) and \( m_2 \) of the same degree, \( m_1 \triangleright_{\text{rel}} m_2 \) means that the rightmost non-zero entry of the difference \( e(m_1) - e(m_2) \) is negative.

Lemma 3.16. Let \( m_1 \) and \( m_2 \) be two monomials of the same degree \( d \), such that \( m_1 \triangleright_{\text{rel}} m_2 \), which minimally generate in the Borel sense an ideal

\[
I = \langle m_1, m_2 \rangle.
\]

Then

\[
\langle m_1 \rangle \cap \langle m_2 \rangle
\]

is a principal Borel ideal. Moreover,

\[
Q(m_1) \cap Q(m_2) = Q(m).
\]

Proof. First assume that \( m_1 = a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \) and \( m_2 = b_1^{x_1} b_2^{x_2} \cdots b_n^{x_n} \). Then define \( \mu_n = \min\{a_n, b_n\} \) and the natural numbers \( \mu_i \) for \( i = n-1, ..., 1 \) recursively, by setting

\[
\mu_i = \min\{a_i + ... + a_n, b_i + ... + b_n\} - (\mu_{i+1} + ... + \mu_n).
\]

Define the following monomial of degree \( d \)

\[
\text{MIN}(m_1, m_2) := x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}
\]

From our choice of the \( \mu_i \)'s, we have

\[
\mu_{n-i} + \mu_{n-i+1} + ... + \mu_n \leq a_{n-i} + a_{n-i+1} + ... + a_n,
\]

and

\[
\mu_{n-i} + \mu_{n-i+1} + ... + \mu_n \leq b_{n-i} + b_{n-i+1} + ... + b_n,
\]

for all \( i = 0, 1, ..., n - 1 \). Therefore,

\[
\langle \text{MIN}(m_1, m_2) \rangle \subseteq \langle m_1 \rangle \cap \langle m_2 \rangle.
\]

Now let \( m = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \) be in \( G(< m_1 >) \) and let \( n = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \) be in \( G(< m_2 >) \). We want to show that \( \text{lc}(m, n) \) is in \( \langle \text{MIN}(m_1, m_2) \rangle \), so we may assume that \( c_1 < d_1 < d < d_1 < d_2 \). Next, let \( k \) be the largest positive integer such that

\[
\max\{c_1, d_1\} + ... + \max\{c_k, d_k\} < d.
\]

Then set \( \nu_i = \max\{c_i, d_i\} \) for \( i = 1, 2, ..., k \) and \( \nu_{k+1} = d - (\nu_1 + ... + \nu_k) \). Since

\[
\nu_1 + ... + \nu_i \geq \max\{c_1 + ... + c_i, d_1 + ... + d_i\} \geq \max\{a_1 + ... + a_i, b_1 + ... + b_i\},
\]

for all \( 1 \leq i \leq k \), we see that

\[
\nu_{i+1} + ... + \nu_{k+1} \leq d - \max\{a_1 + ... + a_i, b_1 + ... + b_i\} = \min\{a_i+1 + ... + a_n, b_i+1 + ... + b_n\} = \mu_{i+1} + ... + \mu_n.
\]
Therefore, the monomial \( x_1^{\nu_1} x_2^{\nu_2} \cdots x_{k+1}^{\nu_{k+1}} \) is a minimal generator of \( \langle \text{MIN} (m_1, m_2) \rangle \) and divides \( \text{lcm}(m, n) = x_1^{\max\{c_1, d_1\}} x_2^{\max\{c_2, d_2\}} \cdots x_n^{\max\{c_n, d_n\}} \). Therefore, \( \text{lcm}(m, n) \) is in \( \langle \text{MIN} (m_1, m_2) \rangle \), and so
\[
\langle m_1 \rangle \cap \langle m_2 \rangle \subseteq \langle \text{MIN} (m_1, m_2) \rangle .
\]
Thus, the proof of our first claim is complete with \( m := \text{MIN}(m_1, m_2) \). Now note that \( Q(m_1) \cap Q(m_2) \) is the union of all the convex polytopes of the polyhedral cell complex \( P_d(x_1, ..., x_n) \) with vertices in \( \langle m_1 \rangle \cap \langle m_2 \rangle = \langle m \rangle \). Since \( Q(m) \) is the union of all the convex polytopes of the polyhedral cell complex \( P_d(x_1, ..., x_n) \) with vertices in \( \langle m \rangle \), we must have
\[
Q(m_1) \cap Q(m_2) = Q(m).
\]

**Examples 3.17.** 1) Let \( m_1 = b^5c \), \( m_2 = ab^3c^2 \) and \( m_3 = a^2c^4 \) in \( k[a, b, c] \). Then
\[
\langle m_1 \rangle \cap \langle m_2 \rangle = \langle b^5c \rangle \cap \langle ab^3c^2 \rangle = \langle ab^4c \rangle ,
\]
\[
\langle m_1 \rangle \cap \langle m_3 \rangle = \langle b^5c \rangle \cap \langle a^2c^4 \rangle = \langle a^2b^3c \rangle .
\]
Also,
\[
\langle m_1 \rangle \cap \langle m_2, m_3 \rangle = \langle ab^4c, a^2b^3c \rangle = \langle ab^4c \rangle .
\]
The following lemma is sufficient for our purposes.

**Lemma 3.18.** Let \( m_1, m_2, \ldots, m_s \) be \( s \) monomials of the same degree \( d \), such that \( m_1 \succ_{\text{rlex}} m_2 \succ_{\text{rlex}} \ldots \succ_{\text{rlex}} m_s \), which minimally generate in the Borel sense an ideal

\[
I = \langle m_1, \ldots, m_s \rangle.
\]

Then for all \( j < s \),

\[
\langle m_j \rangle \cap \langle m_{j+1}, \ldots, m_s \rangle
\]

is a Borel fixed ideal, which is minimally generated in the Borel sense by at most \( s - j \) monomials \( n_{j+1}, \ldots, n_s \) of degree \( d \), with \( n_k \succ_{\text{rlex}} m_j \) for \( k = j + 1, \ldots, s \).

**Proof.** First note that the case where \( j = s - 1 \) is essentially the previous lemma 3.16. Now for all \( j < s \), we have

\[
\langle m_j \rangle \cap \langle m_{j+1}, \ldots, m_s \rangle = (\langle m_j \rangle \cap \langle m_{j+1} \rangle) + \ldots + (\langle m_j \rangle \cap \langle m_s \rangle)
\]

\[
= \langle \text{MIN}(m_j, m_{j+1}) \rangle + \ldots + \langle \text{MIN}(m_j, m_s) \rangle
\]

\[
= \langle n_{j+1}, \ldots, n_s \rangle
\]

where

\[
n_k := \text{MIN}(m_j, m_k)
\]

for \( k = j+1, \ldots, s \). As we saw in example 3.17, some of the \( n_k \)'s might be redundant, so the above intersection is minimally generated in the Borel sense by at most \( s - j \) monomials \( n_{j+1}, \ldots, n_s \) of degree \( d \), with \( n_k \succ_{\text{rlex}} m_j \) for \( k = j + 1, \ldots, s \).

Now we are ready to prove our most general result.

**Theorem 3.19.** Let \( m_1, m_2, \ldots, m_s \) be \( s \) monomials of the same degree \( d \), such that \( m_1 \succ_{\text{rlex}} m_2 \succ_{\text{rlex}} \ldots \succ_{\text{rlex}} m_s \), which minimally generate in the Borel sense an ideal

\[
I = \langle m_1, \ldots, m_s \rangle.
\]

Then there exists a polyhedral cell complex \( Q(m_1, \ldots, m_s) \) that supports a minimal free resolution of \( I \). Moreover, \( Q(m_1, \ldots, m_s) \) is the union of all the convex polytopes of the polyhedral cell complex \( P_d(x_1, \ldots, x_n) \) with vertices in \( I = \langle m_1, \ldots, m_s \rangle \).

**Proof:** For \( s = 2 \) both of our claims were proved in Lemma 3.16. So assume that \( s > 2 \), and for all \( j < s \) set

\[
I_j = \langle m_j, \ldots, m_s \rangle
\]

Next suppose that for some \( j < s \) we have constructed a polyhedral cell complex \( Q(K) \) that supports a minimal free resolution of any Borel fixed ideal \( K \), which is minimally generated in the Borel sense by at most \( s - j \) monomials of the same degree \( d \). Assume also that \( Q(K) \) is the union of all the convex polytopes of the polyhedral cell complex \( P_d(x_1, \ldots, x_n) \) with vertices in \( I = \langle m_1, \ldots, m_s \rangle \).

From lemma 3.18, we see that

\[
\langle m_j \rangle \cap \langle m_{j+1}, \ldots, m_s \rangle = \langle n_{j+1}, \ldots, n_s \rangle
\]

is a Borel fixed ideal, which is minimally generated by \( at \ most \ s - j \) monomials \( n_{j+1}, \ldots, n_s \) of degree \( d \). Thus, so far we have constructed the polyhedral cell complex \( Q(m_j) \) in theorem 3.11, and the polyhedral cell complexes \( Q(m_{j+1}, \ldots, m_s) \) and \( Q(n_{j+1}, \ldots, n_s) \), by the inductive hypothesis. Moreover, by the inductive hypothesis, \( Q(m_j) \cap Q(m_{j+1}, \ldots, m_s) \) is the union of all the convex polytopes of the polyhedral cell complex \( P_d(x_1, \ldots, x_n) \) with vertices in \( \langle m_j \rangle \cap \langle m_{j+1}, \ldots, m_s \rangle \).
Since \( Q(n_{j+1}, \ldots, n_s) \) is the union of all the convex polytopes of the polyhedral cell complex \( P_d(x_1, \ldots, x_n) \) with vertices in \( < n_{j+1}, \ldots, n_s > \), we must have

\[
Q(m_j) \cap Q(m_{j+1}, \ldots, m_s) = Q(n_{j+1}, \ldots, n_s).
\]

Since the rest of the hypotheses of lemma 3.1 are easily checked to be satisfied, we conclude that the complex

\[
X_j := Q(m_j) \cup Q(I_{j+1}) = Q(m_j) \cup Q(m_{j+1}) \cup \ldots \cup Q(m_s)
\]

supports a minimal free resolution of the ideal \( I_j \). Thus

\[
X := X_1 = Q(m_1) \cup Q(m_2) \cup \ldots \cup Q(m_s).
\]

supports a minimal free resolution of \( I_1 = I \).

4. The LCM-Lattice

The lcm-lattice of an arbitrary monomial ideal \( I \) was introduced in [11], where the authors show how its structure relates to the Betti numbers and the maps in the minimal free resolution of \( I \). The lcm-lattice of \( I \), with \( G(I) = \{ m_1, m_2, \ldots, m_r \} \), is denoted by \( L_I \). This is the lattice with elements labeled by the least common multiple of \( m_1, m_2, \ldots, m_r \) ordered by divisibility; that is, if \( n \) and \( m \) are distinct elements of \( L_I \), then \( m \prec n \) if and only if \( m \) divides \( n \). Moreover, we include \( \hat{0} := 1 \) as the bottom element, while \( \hat{1} = \text{lcm}(m_1, m_2, \ldots, m_r) \) is the top element. We say that \( n \) covers \( m \) and we write \( m \rightarrow n \), if \( m \prec n \) and if there is no element \( k \neq n, m \) of \( L_I \) such that \( m \prec k \prec n \).

We would like to find a labelling of the edges of \( L_I \) with the following property: for all elements \( m \) and \( n \) in \( L_I \) with \( m \prec n \), there exists a unique increasing maximal chain from \( m \) to \( n \) and it is lexicographically strictly first than all other maximal chains from \( m \) to \( n \). This would prove that \( L_I \) is shellable (see [5], [6]) in a way different than [1]. Finding such a labelling is still an open problem.

Remark 4.1. The natural labelling which assigns to each edge \( m \rightarrow n \) the integer

\[
\max \left( \frac{m}{n} \right) := \max \{ i | x_i \text{ divides } \frac{m}{n} \}
\]

does not work.

Example 4.2. Let

\[
I = < ab, ac, ad^2, b^2 cd^2 >
\]

\[
= (a^2, ab, b^5, ac, b^4 c^2, b^3 c^3, b^4 d, b^3 cd, b^2 c^2 d, ad^2, b^3 d^2, b^2 cd^2)
\]
The interval $[1, ab^2cd^2]$ of $L_I$ is

Hence there is no decreasing sequence of labels from $ab^2cd^2$ to 1 (or even to $abc$).

The above example shows also that the lcm lattice of a Borel fixed ideal need not be ranked in general. However, if $I$ is generated in the same degree then we prove the following.

**Proposition 4.3.** The lcm-lattice $L_I$ of a $d$-generated Borel fixed ideal $I$ is ranked.

**Proof:** Let $I = (m_1, m_2, ..., m_r)$ be minimally generated by $m_1, m_2, ..., m_r$ in the same degree $d$ and let $m \neq \hat{1} = \text{lcm}(m_1, m_2, ..., m_r)$ be in the lcm-lattice $L_I$ of $I$. Assume that $m = \text{lcm}(m_\alpha, m_\beta, ..., m_\gamma)$, with

$e(m_\alpha) = (a_1, a_2, ..., a_n), \quad e(m_\beta) = (b_1, b_2, ..., b_n) \quad ... \quad e(m_\gamma) = (c_1, c_2, ..., c_n)$.

In order to show that the lattice is ranked it suffices to prove that $\text{deg}(n) = 1 + \text{deg}(m)$ for all $n$ that cover $m$. There exists a $m_\delta$ in $I$, with $e(m_\delta) = (d_1, d_2, ..., d_n)$ such that $n = \text{lcm}(m_\alpha, m_\beta, ..., m_\gamma, m_\delta) = \text{lcm}(m, m_\delta)$. Also, there is at least one $j$ such that $d_j > \max\{a_j, b_j, ..., c_j\}$. Without loss of generality assume that for that $j$, it is $\max\{a_j, b_j, ..., c_j\} = a_j$. If there is some $k$ with $j < k \leq n$ such that $a_k \neq 0$, then

$\ell := \text{lcm}((m_\alpha)_{k \to j}, m_\alpha, m_\beta, ..., m_\gamma)$

has degree $\text{deg}(\ell) = 1 + \text{deg}(m)$, divides $n$ and is divisible by $m$. The minimality of $n$ forces $\ell = n$ and so $\text{deg}(n) = 1 + \text{deg}(m)$. Now assume that $a_k = 0$ for all $j < k \leq n$. Then, there is an $i < j$ such that $\max\{a_i, b_i, ..., c_i\} > d_i$. [Indeed, suppose to the contrary that $d_i \geq \max\{a_i, b_i, ..., c_i\}$ for all $i < j$. Then,

$d \geq \sum_{i=1}^{j} d_i > \sum_{i=1}^{j} \max\{a_i, b_i, ..., c_i\} \geq \sum_{i=1}^{j} a_i = \sum_{i=1}^{n} a_i = d,

a contradiction.] Then

$\ell := \text{lcm}((m_\delta)_{j \to i}, m_\alpha, m_\beta, ..., m_\gamma)$

has degree $\text{deg}(n) - 1$, divides $n$ and is divisible by $m$. Hence, $\ell = m$ and so $\text{deg}(n) = 1 + \text{deg}(m)$, as desired. The proof is complete.
Remarks 4.4. 1) The above proof applies with minor modifications to the case of a strongly stable square-free ideal generated in the same degree. A monomial ideal $I$ is called strongly stable square-free if all monomials in $G(I)$ are square-free and for every $m$ in $G(I)$, if $x_t$ divides $m$ and $x_s$ does not divide $m$ ($1 \leq s < t$), then $m_{t\rightarrow s}$ is in $I$.

2) There exists a $d$-generated Borel fixed ideal $I = (m_1, m_2, \ldots, m_r)$ minimally generated by $m_1, m_2, \ldots, m_r$ and an element $m$ of $L_I$ of degree $d + 1$, such that for some $1 \leq s < t \leq n$,

(i) $x_t$ divides $m$

(ii) $x_s^{d_t}$ does not divide $m$

where $d_t$ is the largest positive integer such that $x_t^{d_t}$ divides $\text{lcm}(m_1, m_2, \ldots, m_r)$, and

(iii) $m_{t\rightarrow s}$ is not in $L_I$.

Example 4.5. Let

$I = \langle x_1x_3^2, x_2^2x_3x_4 \rangle$.

Then $d_3 = 3$ and $x_2^2x_3^2x_4 = \text{lcm}(x_2^2x_3x_4, x_2^2x_3^2)$ is in $L_I$, but $x_2^2x_3^3 = (x_2^2x_3^2x_4)_{1\rightarrow 3}$ is not in $L_I$.

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References

[1] E.Batzies, Discrete Morse theory for cellular resolutions, Dissertation Phillips-Universität Marburg (2002).

[2] E.Batzies, V.Welker, Discrete Morse theory for cellular resolutions, J. reine angew. Math. 543 (2002), 147-168.

[3] D.Bayer, I.Peeva, and B.Sturmfels, Monomial Resolutions, Math. Research Letters, 5 (1998), 31-46.

[4] D.Bayer, B.Sturmfels, Cellular resolutions of monomial modules, J. reine angew. Math. 503 (1998), 123-140.

[5] A.Björner, M.L.Wachs, On lexicographically shellable posets, Trans. Amer. Math. Soc. 277 (1983), no. 1, 323-341.

[6] A.Björner, M.L.Wachs, Shellable nonpure complexes and posets I, Trans. Amer. Math. Soc. 348 (1996), no. 4, 1299-1327.

[7] W.Bruns, Joseph Gubelane, Polytopes, rings and K-theory, incomplete preliminary incomplete version of a book project, available at http://www.mathematik.uni-osnabrueck.de/staff/phpages/brunsw/preprints.htm

[8] D.Eisenbud, Commutative algebra with a view toward algebraic geometry, Grad. Texts Math. 150, Springer Verlag, New York, 1995.

[9] D.Eisenbud, The geometry of syzygies, Grad. Texts Math. 229, Springer Verlag, New York, 2005.

[10] S.Eliahou, M.Kervaire, Minimal resolutions of some monomial ideals, J.Algebra 129 (1990) 1-25.

[11] V.Gasharov, I.Peeva and V.Welker, The lcm-lattice in monomial resolutions, Math. Res. Lett. 6 (1999), no. 5-6, 521–532.

[12] D.Grayson, M.Stillman, MACAULAY 2, a software system devoted to supporting research in algebraic geometry and commutative algebra, available at http://www.math.uiuc.edu/Macaulay2/.
[13] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, 2002.
[14] M. Jöllenbeck, V. Welker, *Minimal resolutions via algebraic discrete Morse theory*, preprint.
[15] W. S. Massey, *A Basic Course in Algebraic Topology*, Grad. Texts Math. 127, Springer Verlag, New York, 1991.
[16] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Grad. Texts Math. 227, Springer Verlag, New York, 2005.
[17] I. Peeva, M. Velasco, *Frames and degenerations of monomial resolutions*, preprint, 2005.
[18] J. Phan, *Minimal monomial ideals and linear resolutions*, preprint, 2005.
[19] M. Velasco, *Cellular resolutions and nearly Scarf ideals*, preprint, 2005.
[20] G. Ziegler, *Lectures on Polytopes*, Grad. Texts Math. 152, Springer Verlag, New York, 1995.

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