Approximating the Weighted Minimum Label s-t Cut Problem

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Abstract

In the weighted (minimum) Label s-t Cut problem, we are given a (directed or undirected) graph $G = (V, E)$, a label set $L = \{\ell_1, \ell_2, \ldots, \ell_q\}$ with positive label weights $\{w_\ell\}$, a source $s \in V$ and a sink $t \in V$. Each edge $e$ of $G$ has a label $\ell(e)$ from $L$. Different edges may have the same label. The problem asks to find a minimum weight label subset $L'$ such that the removal of all edges with labels in $L'$ disconnects $s$ and $t$.

The unweighted Label s-t Cut problem (i.e., every label has a unit weight) can be approximated within $O(n^{2/3})$, where $n$ is the number of vertices of graph $G$. However, it is unknown for a long time how to approximate the weighted Label s-t Cut problem within $o(n)$. In this paper, we provide an approximation algorithm for the weighted Label s-t Cut problem with ratio $O(n^{2/3})$. The key point of the algorithm is a mechanism to interpret label weight on an edge as both its length and capacity.

1 Introduction

The (minimum) Label s-t Cut problem is a natural combinatorial optimization problem arising from several application fields, including system security [16, 25, 26], computer networks [5, 6], and control engineering [24]. The problem is defined as in Definition 1.1. Label s-t Cut has attracted much attention from researchers in computer science (see, e.g., [2, 4, 6, 9, 12, 13, 14, 15]) and researchers even in chemical engineering (see [10]). The problem finds many applications in image segmentation [19], network connectivity [29], computational biology [22], and network security [1], etc.

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**Definition 1.1.** The minimum Label $s$-$t$ Cut problem.

(Instance) We are given a (directed or undirected) graph $G = (V, E)$, a label set $L = \{\ell_1, \ell_2, \ldots, \ell_q\}$, a source $s \in V$ and a sink $t \in V$. Each edge $e$ of $G$ has a label $\ell(e)$ from $L$. Different edges may have the same label.

(Goal) The problem asks to find a minimum size label subset $L'$ such that the removal of all edges with labels in $L'$ disconnects $s$ and $t$.

In the weighted Label $s$-$t$ Cut problem, a positive label weight $w_{\ell}$ is provided further for every label $\ell \in L$. The problem asks to find a minimum weight label subset $L'$ to disconnect $s$ and $t$, where the weight $w(L')$ of label subset $L'$ is defined as $w(L') = \sum_{\ell \in L} w_{\ell}$.

We briefly introduce some origins and applications of the Label $s$-$t$ Cut problem. The detailed description can be found in [5], [16], and related references therein. Label $s$-$t$ Cut comes from an interesting scenario in system security. An intruder with some attack methods lies in his beginning state, planning to intrude a system. The intruder’s current state changes once he applies one of the attack methods. The successful state means that the intruder has intruded the system successfully. The state transition of the intruder can be modeled by an edge-labeled graph with source $s$ (representing the beginning state) and sink $t$ (representing the successful state). The system defender can disable the intruder’s attack methods by paying some costs, e.g., by strengthening the defending equipments. The system defender’s task can be depicted as finding the minimum cost of labels such that $s$ and $t$ are disconnected in the graph after removing the edges with these labels. This is precisely the Label $s$-$t$ Cut problem.

As an application scenario, let us consider computer networks, which are usually multi-layered. For simplicity, we assume that there are two layers in a network, namely, the high level logical layer and the low level physical layer. Each link in the logical layer corresponds to a path in the physical layer. Since many logical links may rely on paths that have some common physical link, a failure of some physical link may result in a failure of many seemingly unrelated logical links. This can be modeled by an edge-labeled graph. More importantly, Label $s$-$t$ Cut can be used to compute the generalized $s$-$t$ connectivity (counts in labels) in edge-labeled graphs.

**Notation.** Let $G$ be a graph in the context. We always use $n$ to denote the number of vertices of $G$, and use $m$ to denote the number of edges of $G$. Moreover, we use OPT to denote the optimal solution value of an optimization problem. Note that for the Label $s$-$t$ Cut problem, we still have a natural parameter $q$, which is the number of labels in the problem input. For simplicity, we will use $\ell$ to denote a label in $L$. Let $e$ be an edge. We
also use $\ell(e)$ to denote the label on edge $e$. Note that in the latter case $\ell$ is actually a function from $E$ to $L$. The two cases of usage of $\ell$ will be easily distinguished from the context. The same thing happens to symbol $L$. On the one hand, $L$ is the set of labels in the problem input. On the other hand, given an edge subset $E'$, we will use $L(E')$ to denote the set of labels appearing on edges in $E'$.

1.1 Related Work

It is easy to see that Label $s$-$t$ Cut is just a generalization of the classic Min $s$-$t$ Cut problem. If in Label $s$-$t$ Cut every edge has a unique label, then Label $s$-$t$ Cut degenerates to Min $s$-$t$ Cut. While it is well-known that Min $s$-$t$ Cut is polynomial-time solvable, Label $s$-$t$ Cut is NP-hard and has high approximation hardness factor $2^{(\log n)^{1-1/(\log \log n)^c}}$ for any constant $c < 1/2$ [30]. For the applications of Label $s$-$t$ Cut in system security [16] [25] [26] and computer networks [5] [6], the readers are advised to refer to the corresponding references.

The first non-trivial approximation algorithm for the weighted Label $s$-$t$ Cut problem was given by Zhang et al. [30] in 2011 with performance ratio $O(m^{1/2})$, where $m$ is the number of edges in the input graph. However, in dense graphs with $m = \Theta(n^2)$, the approximation ratio $O(m^{1/2})$ degenerates to $O(n)$. Therefore, an approximation algorithm with ratio in terms of $n$ (better than $O(n)$) is expected for weighted Label $s$-$t$ Cut.

Rohloff et al. [24] gave an $O(n^{2/3})$-approximation algorithm for unweighted Label $s$-$t$ Cut. Tang et al. [28] independently gave an $O\left(\frac{n^{2/3}}{\text{OPT}^{1/3}}\right)$-approximation algorithm for the unweighted Label $s$-$t$ Cut problem via a mixed strategy of LP-rounding and finding a min $s$-$t$ cut, where OPT denotes the value of an optimal solution to the problem. Thereafter, Zhang et al. [32] rendered the algorithm in [28] purely combinatorial, keeping the approximation ratio $O\left(\frac{n^{2/3}}{\text{OPT}^{1/3}}\right)$ unchanged. Dutta et al. [7] also gave an $O(n^{2/3})$-approximation algorithm for unweighted Label $s$-$t$ Cut.

However, all the algorithms in [24], [28], [32], and [7] only deal with the unweighted Label $s$-$t$ Cut problem. To the best of our knowledge, there is no approximation algorithm for the weighted Label $s$-$t$ Cut problem with performance ratio directly given in terms of $n$ (better than $O(n)$). Note that for an optimization problem, the complexity of the unweighted version and the weighted version could be very different. For example, the well-known $k$-MST problem in weighted graphs is NP-hard [29]. However, in unweighted graphs $k$-MST is polynomial time solvable.
1.2 Our Results

In this paper, we provide an approximation algorithm for the weighted Label s-t Cut problem with ratio $O(n^{2/3})$. This is the first approximation algorithm for weighted Label s-t Cut whose ratio is given in terms of $n$. The key point of the algorithm is a mechanism to interpret label weight on an edge as both the edge’s length and capacity.

The overall strategy is to discretize label weights $w_\ell$ with arbitrary values into integer values $\bar{w}_\ell$ in a range from one to an upper bound polynomial in $q$. Recall that $q$ is the number of labels in the problem input. Every edge $e$ in the input graph $G$ is transformed into $\bar{w}_\ell(e)$ parallel edges, resulting in a multi-graph $\tilde{G}$. Meanwhile, every label $\ell \in L$ is replaced by a group of labels $\ell^{(1)}, \ell^{(2)}, \ldots, \ell^{(\bar{w}_\ell)}$, resulting in a new label set $\tilde{L}$. In this way, we get a new Label s-t Cut instance $\tilde{I}$ on multi-graphs. Since $\bar{w}_\ell$ is polynomially bounded, instance $\tilde{I}$ can be constructed in polynomial time. Next, we develop a two-stage combinatorial algorithm for Label s-t Cut on multi-graphs to compute an approximate solution $\tilde{L}' \subseteq \tilde{L}$ to instance $\tilde{I}$. Finally, a solution $L' \subseteq L$ to the original instance is recovered from $\tilde{L}'$.

1.3 More Related Work

The Label s-t Cut problem was also studied from the parameterized perspective. Fellows et al. [9] showed that when parameterized by the number of used labels, the Label s-t Cut problem is $W[2]$-hard even in graphs whose path-width is bounded above by a small constant. On the other hand, Zhang et al. [31] showed that when parameterized by the number of used labels, Label s-t Cut is fixed-parameter tractable if the maximum length of any s-t path is bounded from above. Morawietz et al. [21] considered the parameterized complexity of Label s-t Cut with parameters related to the structure of the input graph and labels.

Jegelka et al. [15] studied a more general cut problem called Cooperative s-t Cut, which finds an s-t cut such that an objective function is minimized, where the objective function can be arbitrary submodular function defined on $2^E$. It is not difficult to see that Cooperative s-t Cut is a generalization of Label s-t Cut. Jegelka et al. [15] proved Cooperative s-t Cut cannot be approximated within $\sqrt{n \log n}$. It is interesting to note that this lower bound is based on information theory [27], instead of complexity assumptions such as P \(\neq\) NP.

We would like to introduce here the Global Label Cut problem [8, 5, 28, 12], which is closely related to Label s-t Cut. In the Global Label Cut problem,
there are no given vertices \(s\) and \(t\). The problem asks to find a minimum size (or weight) label subset \(L' \subseteq L\) such that the removal of edges whose labels are in \(L'\) makes the undirected input graph disconnected. It is easy to see that **Global Label Cut** is a natural generalization of the classic global Min Cut problem (see, e.g., [18] for Min Cut).

To the best of our knowledge, **Global Label Cut** had been proposed at least in 2006 by Faragó [8]. Zhang et al. [31] showed that in several special cases, **Global Label Cut** is polynomial-time solvable. An important progress was made by Ghaffari et al. [12], who gave a quasi-polynomial time Monte-Carlo algorithm for unweighted **Global Label Cut**, and a PTAS with high probability for weighted **Global Label Cut**. Bordini et al. [3] gave some heuristics for **Global Label Cut**. At the current time, the most intriguing open problem is whether **Global Label Cut** is in P or NP-hard.

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2 we give a simple analysis to the existing algorithm for the unweighted **Label s-t Cut** problem. In Section 3 we exhibit our high-level idea about how to approximate weighted **Label s-t Cut**. In Section 4 we show how to approximate unweighted **Label s-t Cut** on multi-graphs. Then, in Section 5 we show how to reduce the weighted **Label s-t Cut** problem to the unweighted one on multi-graphs. Finally, in Section 6 we conclude the paper.

## 2 Unweighted Label s-t Cut

It is known that unweighted **Label s-t Cut** can be approximated within \(O\left(\frac{n^{2/3}}{\text{OPT}^{1/3}}\right)\) [32]. In this section, we would like to give a simple analysis to this result.

It is known that **Label s-t Cut** on unweighted graphs can be approximated within \(O\left(\frac{n^{2/3}}{\text{OPT}^{1/3}}\right)\) [28, 32]. In this paper, we give a new, but shorter proof for this result. To the best of our knowledge, the new proof should be the shortest one known so far for the \(O\left(\frac{n^{2/3}}{\text{OPT}^{1/3}}\right)\)-approximation result.

In 2006, Rohloff et al. [24] gave an \(O(n^{2/3})\)-approximation algorithm for unweighted **Label s-t Cut**. The paper [24] deals with the sensor selection problem arising in the field of control theory and engineering. The algorithm for **Label s-t Cut** in [24] has two stages, with the first stage repeatedly finding the shortest \(s-t\) paths, and the second stage running a depth-first search.

In 2012, Tang et al. [28] independently gave an \(O\left(\frac{n^{2/3}}{\text{OPT}^{1/3}}\right)\)-approximation algorithm for the unweighted **Label s-t Cut** problem. The algorithm in [28]
is also a two-stage algorithm, with the first stage rounding a fractional solution to the following linear program (LP1), and the second stage finding a min s-t cut.

\[
\begin{align*}
\min & \quad \sum_{\ell \in L} x_{\ell} \\
\text{s.t.} & \quad \sum_{e \in P} x_{\ell(e)} \geq 1, \quad \forall s-t \text{ path } P \\
& \quad x_{\ell} \geq 0, \quad \forall \ell \in L
\end{align*}
\]  

Thereafter, Zhang et al. [32] (in 2018) rendered the algorithm in [28] purely combinatorial, keeping the approximation ratio \(O(n^{2/3}/OPT^{1/3})\) unchanged. More specifically, in the first stage of the algorithm in [32], LP-rounding is no longer needed, but a work of repeatedly finding the shortest s-t paths is placed instead.

In 2016, Dutta et al. [7] also independently gave an \(O(n^{2/3})\)-approximation algorithm for Label s-t Cut. Their algorithm is still a two-stage algorithm: LP-rounding and finding a min s-t cut. The linear program used in [7] is the following linear program (LP2), which is different to that in [28].

\[
\begin{align*}
\min & \quad \sum_{\ell \in L} x_{\ell} \\
\text{s.t.} & \quad \sum_{\ell \in L(P)} x_{\ell} \geq 1, \quad \forall s-t \text{ path } P \\
& \quad x_{\ell} \geq 0, \quad \forall \ell \in L
\end{align*}
\]

Since the separation problem to the LP in [7] is NP-hard, it is not known how to get an optimal solution to this LP in polynomial time. Dutta et al. [7] did much work to get an approximate solution to this LP. However, the LP in [28] actually is enough for the algorithm in [7].

It has mentioned that all the algorithms in [21], [28] (including [32]), and [7] are two-stage algorithms. However, the analyses of the second stages in them are different from each other. Inspired by the analysis of the second stage in [7], we give a more simpler analysis to the algorithm in [32]. To the best of our knowledge, this should be the simplest analysis to this algorithm known so far. The algorithm is shown as Algorithm 2.1 which is an approximation algorithm for the unweighted Label s-t Cut problem on (directed or undirected) simple graphs.

Algorithm 2.1 ([32]).
Input: An instance $I = (G, s, t, L)$ of Label $s$-$t$ Cut.

Output: A label subset of $L$.

1. Guess $OPT$.
2. /* Stage one */
   3. $L_1 ← \emptyset$.
   4. while the $s$-$t$ distance $\leq \frac{n^{2/3}}{OPT^{2/3}}$ do
      5. Find a shortest $s$-$t$ path $P$.
      6. Remove all the edges whose labels are in $L(P)$.
      7. $L_1 ← L_1 \cup L(P)$.
   8. endwhile /* Stage two */
9. Denote by $R$ the current remaining graph. Find a minimum size $s$-$t$ cut $E' \subseteq E(R)$ of $R$.
10. $L_2 ← L(E')$.
11. return $L_1 \cup L_2$.

**Theorem 2.1** ([11]). For a simple $n$-vertex (undirected or directed) graph $G$, if there are $k$ edge-disjoint paths between two vertices $s$ and $t$, then the average length (number of edges) of these paths is $O(n/\sqrt{k})$.

**Theorem 2.2.** Algorithm 2.1 is an $O(n^{2/3}/OPT^{1/3})$-approximation algorithm for unweighted Label $s$-$t$ Cut on (directed or undirected) simple graphs.

**Proof.** First note that the algorithm obviously gives a feasible solution in polynomial time. Every path found in stage one has length $\leq \frac{n^{2/3}}{OPT^{2/3}}$. All these paths are label-disjoint (i.e., no two paths of them contain the same label), and hence edge-disjoint. It follows that the optimal solution must select at least one label from each path, implying that the number of these path is $\leq OPT$. Therefore, we get that

$$|L_1| \leq \frac{n^{2/3}}{OPT^{1/3}}OPT.$$  \hspace{1cm} (1)

By Menger’s theorem (see, e.g., [21]), the size (number of edges) $|E'|$ of the minimum $s$-$t$ cut is equal to the number edge disjoint $s$-$t$ paths in graph $R$. By Theorem 2.1, the average length of these paths is $O(n/\sqrt{|E'|})$. Let $d_R(s, t)$ be the length of the shortest $s$-$t$ path in graph $R$. When Algorithm 2.1 enters stage two, it holds that $d_R(s, t) \geq \frac{n^{2/3}}{OPT^{1/3}}$. We thus have $\frac{n^{2/3}}{OPT^{1/3}} \leq d_R(s, t) \leq O(n/\sqrt{|E'|})$, implying that

$$|L_2| \leq |E'| = O(n^{2/3}OPT^{2/3}) = O\left(\frac{n^{2/3}}{OPT^{1/3}}\right)OPT.$$  \hspace{1cm} (2)
The theorem then follows by (1) and (2).

The guess technique in step 1 of Algorithm 2.1 is a common technique in approximation algorithms. By “guess OPT” we actually mean repeating the algorithm for each possible value of OPT in \{1, 2, \ldots, q\}, and returning the solution of minimum size ever found. The range of OPT is polynomially bounded, so the whole algorithm still runs in polynomial time.

3 High-level Idea of Approximating Weighted Label s-t Cut

Note that in the weighted Label s-t Cut problem weights are defined on labels (instead of edges). Our overall strategy dealing with weighted Label s-t Cut, as the case of unweighted Label s-t Cut, is still a two-stage algorithm. In the first stage, we repeatedly find shortest s-t paths and remove them. In the second stage, we compute an s-t cut. However, the problem we are faced is a weighted problem, and we have to consider the label weights on edges when we compute s-t path and s-t cut. This immediately brings out a troublesome problem: For s-t path, label weight on an edge has to be interpreted as edge length. However, for s-t cut, label weight has to be interpreted as edge capacity. How could we combine these two very different physical concepts together in an algorithm?

Our goal is to give an approximation algorithm for weighted Label s-t Cut whose ratio is given in terms of \(n\) (the number of vertices of the input graph), since there is already an \(O(m^{1/2})\)-approximation algorithm [30] for the problem. A straightforward approach is to use label weight \(w_{\ell(e)}\), for an edge \(e\), as both its length and capacity in Algorithm 2.1. However, the proof of Theorem 2.2 does not applies to the new algorithm. There is a result corresponding to Theorem 2.1 for edge-weighted graph by Karger and Levine [17, Theorem 7.1]. However, it also seems that this result does not help for the analysis.

Our strategy is indirect and needs some transformations. Let us examine Algorithm 2.1 again. This algorithm deals with an unweighted graph \(G\). Note that an unweighted edge can be viewed as a unit-weighted one. This means unit-weight works well for not only the first stage of computing shortest paths, but also the second stage of computing a min s-t cut. In other words, length and capacity “agree” on unit weight, although they differ very much on arbitrary weights.

Based on the above observation, our general strategy of approximating
weighted Label \(s-t\) Cut is to reduce the weighted problem to the unweighted Label \(s-t\) Cut problem on multi-graphs. Then, we design a two-stage approximation algorithm for the latter problem. In the reduction we will discretize the label weights to edge multiplicities. To guarantee that the reduction can be performed in polynomial time, we distinguish light label weights from heavy label weights. We guess a weight threshold \(W\) (which is the maximum label weight used in an optimal solution to weighted Label \(s-t\) Cut). The label weights that are at most \(W\) are called light. Otherwise, they are called heavy. Let \(e\) be an edge with light label weight \(w_{\ell(e)}\). We round the weight \(w_{\ell(e)}\) into a multiple (say \(\tilde{w}_{\ell(e)}\)) of some small quantity and represent \(e\) by \(\tilde{w}_{\ell(e)}\) multi-edges, so that the multiplicity is polynomially bounded. On the other hand, if \(e\)'s label weight is heavy, we do not do the conversion. In this way, we get a multi-graph, denoted by \(\tilde{G}\). Then we solve unweighted Label \(s-t\) Cut on multi-graph \(\tilde{G}\). The solution obtained will be converted back to a solution to weighted Label \(s-t\) Cut.

Our method solving unweighted Label \(s-t\) Cut on multi-graphs is a two-stage algorithm consisting of computing shortest \(s-t\) paths and \(s-t\) cut. However, we do not want to use the edges if their original label weights are heavy. To this aim, for an edge \(e\) in multi-graph \(\tilde{G}\), if it comes from an edge in \(G\) with light label weight, its weight is defined as one. Otherwise (i.e., it comes from an edge in \(G\) with heavy label weight), its length is defined as zero. In this way, we further turn \(\tilde{G}\) to a multi-graph with edge weights zero or one. Note that zero weight plays an important role in the algorithm. When we measure the length of an \(s-t\) path in the first stage, zero weight edges are not taken into count. More importantly, in the second stage we use a layering technique to compute an \(s-t\) cut. The layering technique guarantees that length-zero edges in \(\tilde{G}\) will not be included in the \(s-t\) cut to be found.

To make the idea clear in conception, for an edge \(e\) in multi-graph \(\tilde{G}\), if it comes from an edge in \(G\) with light label weight, its label is called admissible. Otherwise its label is called forbidden. So in the above, we actually define edge weight as one if the edge’s label is admissible, and zero if forbidden. In Section 4 we show in detail the Label \(s-t\) Cut problem on multi-graphs with forbidden labels, and its two-stage approximation algorithm.

4 Label \(s-t\) Cut on Multi-graphs with Forbidden Labels

In the Label \(s-t\) Cut problem on multi-graphs with forbidden labels, we are given a multi-graph \(G = (V, E)\), a source \(s \in V\) and a sink \(t \in V\), and two
label sets $A$ and $B$. Each edge in $E$ has one label in $A \cup B$. The problem asks to find a minimum size subset $A' \subseteq A$ such that the removal of edges with labels in $A'$ disconnects $s$ and $t$ in $G$, or report infeasibility when no solution exists. The labels in $A$ are called admissible labels, that is, these labels can be used in solutions. The labels in $B$ are called forbidden labels, which cannot be used in any solution.

If an edge’s label is an admissible label, then we call this edge an admissible edge. Likewise, if an edge’s label is a forbidden label, then we call this edge a forbidden edge.

For the Label $s$-$t$ Cut problem on multi-graphs with forbidden labels, we design a two-stage approximation algorithm, as shown in Algorithm 4.1. The algorithm first assigns weights to edges according to the types of their labels (step 1). Then, the algorithm enters its first stage (step 2 to step 9), repeatedly finding the current shortest $s$-$t$ paths and removing selected edges until the $s$-$t$ distance is relatively long. After that, the algorithm enters its second stage (step 10 to step 12), partitioning the vertices into layers and then finding an $s$-$t$ cut between two consecutive layers.

**Algorithm 4.1.**

*Input:* An instance $I = (G, s, t, A, B)$ of Label $s$-$t$ Cut on multi-graphs with forbidden labels.

*Output:* A label subset $A' \subseteq A$ such that the removal of edges with labels in $A'$ disconnects $s$ and $t$, if instance $I$ has feasible solutions (i.e., the removal of edges with admissible labels can disconnect $s$ and $t$). Returns “failure” otherwise.

1 Define weights (i.e., lengths) on edges of $G$: For every $e \in E(G)$, if $\ell(e) \in A$, then define $w(e) = 1$. Otherwise (i.e., $\ell(e) \in B$), define $w(e) = 0$. /* The length of a path $P$ is defined as the total weight of edges in $P$. The distance $\text{dist}(u, v)$ between a pair of vertices $u, v$ is defined as the length of a shortest $u$-$v$ path. If $v$ is not reachable from $u$, then define $\text{dist}(u, v) = \infty$. */

2 if $\text{dist}(s, t) = 0$ then return “failure”. /* Stage one */

3 Guess OPT. Let $\mu = \mu(G)$ be the maximum multiplicity of edges in $G$.

4 $A_1 \leftarrow \emptyset$.

5 while $\text{dist}(s, t) \leq \frac{n^{2/3} \mu^{1/3}}{\text{OPT}^{1/3}}$ do

6 Find a shortest $s$-$t$ path $P$.

7 Remove all the edges whose labels are in $L(P) \cap A$. 
8 \[ A_1 \leftarrow A_1 \cup (L(P) \cap A). \]
9 \[ \text{endwhile} \]
10 /* Stage two */
11 Denote by \( R \) the remaining graph. Let \( V_i := \{ v \in V(R) \mid \text{dist}(s, v) = i, i \neq \infty \} \)
12 for \( i = 0, 1, 2, \ldots \). This gives a partition \( \{V_0, V_1, V_2, \ldots \} \) of the vertices in \( R \). Suppose that \( t \in V_\tau \) for some integer \( \tau \geq 1 \).
13 Let \( E_i \) (0 \leq i \leq \tau - 1) be the set of edges between two consecutive layers \( V_i \) and \( V_{i+1} \). Every \( E_i \) forms an \( s\)-\( t \) cut in \( R \). Choose the minimum cardinality one among these cuts, denoted by \( E' \).
14 \[ A_2 \leftarrow L(E'). \]
15 \[ \text{return} \ A_1 \cup A_2. \]

Note that graph \( G \) varies during the execution of Algorithm 4.1. The distance \( \text{dist}(u, v) \) is always the distance between \( u \) and \( v \) in the current graph.

**Lemma 4.1.** For Algorithm 4.1, we have \( |A_1| \leq n^{2/3} (\frac{\mu}{\text{OPT}})^{1/3} \text{OPT} \).

**Proof.** Let \( h \) be the number of paths found in the first stage. Since these \( s\)-\( t \) paths do not share any admissible label, any optimal solution must select at least one admissible label from each of these paths. Therefore, we know \( \text{OPT} \geq h \). By the definition of costs on edges, the number of admissible labels on each such path is at most its length, which is in turn at most \( \frac{n^{2/3} \mu^{1/3}}{\text{OPT}^{1/3}} \). Thus, we have

\[ |A_1| \leq \frac{n^{2/3} \mu^{1/3}}{\text{OPT}^{1/3}} h \leq n^{2/3} \left( \frac{\mu}{\text{OPT}} \right)^{1/3} \text{OPT}. \]

**Lemma 4.2.** No forbidden edge is in \( E' \).

**Proof.** If \( e \) is a forbidden edge, its cost is zero. Since vertices in \( R \) are layered according to their distances to \( s \), any forbidden edge cannot lie in two different layers. By definition, \( E' \) is the set of edges between two consecutive layers. So, there is no forbidden edge in \( E' \). 

**Lemma 4.3.** For Algorithm 4.1, we have \( |E'| = O(n^{2/3} \mu^{1/3} \text{OPT}^{2/3}) \), where \( \mu \) is the maximum multiplicity of edges in \( \tilde{G} \).
Proof. When the algorithm enters stage two, we have $d(s, t) > \frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}}$. Therefore, there are at least $\frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}} + 1$ different $i$'s such that $V_i \neq \emptyset$. We omit the vertices in $G$ that are not reachable from $s$.

We claim that there exists an $i' \leq \frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}}$ such that $|V_{i'}| |V_{i'+1}| \leq 100(\frac{n \cdot OPT}{\mu})^{2/3}$. Assume for contradiction for every $0 \leq i \leq \frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}}$, we have $|V_i| |V_{i+1}| > 100(\frac{n \cdot OPT}{\mu})^{2/3}$. Then for each two consecutive layers $V_i$ and $V_{i+1}$, at least one of them has size $> 10(\frac{n \cdot OPT}{\mu})^{1/3}$. Therefore, there are at least $\frac{1}{2} \cdot \frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}}$ different $i$'s satisfying $|V_i| > 10(\frac{n \cdot OPT}{\mu})^{1/3}$, and thus there are at least $5n$ vertices in these $V_i$'s. This is obviously absurd since there are $\leq n$ vertices in $R$.

Therefore, the number of edges $|E_{i'}|$ between $V_i$ and $V_{i+1}$ is at most $|V_i| |V_{i'+1}| \mu = O(n^{2/3}OPT^{2/3} \mu^{1/3})$, considering the maximum multiplicity of edges is $\mu$. The lemma follows since $E'$ is the minimum cardinality one among all $E_i$'s.

**Theorem 4.1.** For the unweighted Label $s$-$t$ Cut problem on multi-graphs with forbidden labels, if it has feasible solutions, then Algorithm 4.1 returns an $O(\frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}})$-approximation in polynomial time. Otherwise, the algorithm returns failure in polynomial time.

Proof. First note that Algorithm 4.1 obviously runs in polynomial time.

Let $I$ be the input instance. If $I$ has no feasible solution, then there is (at least) one $s$-$t$ paths in $\tilde{G}$ whose edges are all forbidden edges. That is, there is an $s$-$t$ path of length zero. Such a path will be found at step 2 before the stage one. So, in this case Algorithm 4.1 will return failure.

Suppose that $\tilde{I}$ has feasible solutions. The labels selected by Algorithm 4.1 in stage one are all admissible labels. By Lemma 4.2 the labels selected in stage two are also all admissible labels. So, the algorithm returns a feasible solution in this case.

Let $SOL$ be the solution value of Algorithm 4.1. By Lemma 4.1 and Lemma 4.3 we get that

$$SOL = |A_1 \cup A_2| \leq \frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}} OPT + n^{2/3}OPT^{2/3} \mu^{1/3} = O \left( \frac{n^{2/3} \mu^{1/3}}{OPT^{1/3}} \right) OPT,$$

finishing the proof of the theorem.  


5 Weighted Label s-t Cut

Recall that in the weighted Label s-t Cut problem, we are given a (directed or undirected) graph $G = (V, E)$, a source $s \in V$, a sink $t \in V$, and a label set $L = \{\ell_1, \ell_2, \ldots, \ell_q\}$, where each label $\ell \in L$ has a nonnegative weight $w_\ell$. Each edge $e$ in graph $G$ has a label $\ell(e) \in L$. The goal of the problem is to find a minimum weight label subset $L' \subseteq L$ such that $s$ and $t$ are disconnected if the edges with labels in $L'$ are removed from $G$. Such a label subset $L'$ is also called a label s-t cut. Given a label subset $L'$, its weight $w(L')$ is the sum of weights of all labels in it.

We show how to approximate the weighted Label s-t Cut problem by reducing it to the (unweighted) Label s-t Cut problem on multi-graphs with forbidden labels.

5.1 The Reduction

The overall strategy of approximating weighted Label s-t Cut is to reduce the weighted problem on (simple) graphs to the unweighted problem on multi-graphs. Let $I = (G, s, t, L, w)$ be an instance of weighted Label s-t Cut. The resulting instance of unweighted Label s-t Cut will be denoted by $\tilde{\mathcal{I}} = (\tilde{G}, s, t, \tilde{A}, \tilde{B})$, where $\tilde{G}$ is a multi-graph, $\tilde{A}$ and $\tilde{B}$ are two sets of labels. For an edge $e = (u, v) \in E(G)$, there will be multiple of edges between $u$ and $v$ in $\tilde{G}$. In other words, in $\tilde{G}$ we will use the multiplicity of edges between $u$ and $v$ to reflect the weight of label $\ell(e)$.

However, the label weights of instance $\mathcal{I}$ are not polynomially bounded in general. Since the reduction have to be done in polynomial time, the crux in the reduction is to guarantee in $\tilde{G}$ the edge multiplicity between any pair of vertices is polynomially bounded, meanwhile it is yet used to reflect label weight (which may be not polynomially bounded). To realize this seemingly contradictory goal, we classify the labels in $L$ into two categories, namely, the admissible labels and the forbidden labels (which are defined in (4) and (5)). An admissible label $\ell$ in instance $\mathcal{I}$ will be translated into a group of labels in instance $\tilde{\mathcal{I}}$. The weight of $\ell$ is represented by the size of the group. On the other hand, the forbidden labels will not be used in the solution we constructed to instance $\mathcal{I}$. Therefore, we need not to translate forbidden labels.

Specifically, let $O \subseteq L$ be an optimal solution to instance $\mathcal{I}$, and $W$ be the maximum weight of labels in $O$, i.e.,

$$W = \max\{w_\ell \mid \ell \in O\}.$$
By definition, it obviously holds that

\[ W \leq \text{OPT}_w(I). \tag{3} \]

Here we use \( \text{OPT}_w(I) \) to denote the optimum of the weighted Label s-t Cut problem on instance \( I \).

By the guess technique, we may assume that we know the value \( |O| \) and \( W \), where \( |O| \) is the number of labels used in the optimal solution \( O \).

Define

\[ A = \{ \ell \in L \mid w_\ell \leq W \}, \tag{4} \]
\[ B = L \setminus A. \tag{5} \]

The labels in \( A \) are called \textit{admissible labels}, while the labels in \( B \) are called \textit{forbidden labels}.

For each admissible label \( \ell \in A \), we define

\[ \bar{w}_\ell = \left\lceil \frac{w_\ell |O|}{W} \right\rceil. \tag{6} \]

It is easy to see that \( \bar{w}_\ell \) is an integer whose value is at least one. Since \( w_\ell \leq W \) for each \( \ell \in A \), we know that \( \bar{w}_\ell \) is at most \( |O| \), which is polynomially bounded. This is a key property about \( \bar{w}_\ell \).

The guess technique is actually to try every possible values of the quantity to be guessed. For the quantity \( |O| \), we will try its each possible values from 1 to \( q \). So, the guess for \( |O| \) can be done in polynomial time and each guess of \( |O| \) is polynomially bounded. For the value \( W \), we will try each distinct label weight in \( \{ w_\ell \mid \ell \in L \} \). So, the guess for \( W \) can also be done in polynomial time.

Now we are ready to construct the instance \( \tilde{I} = (\tilde{G}, s, t, \tilde{A}, \tilde{B}) \). Initially, the label set \( \tilde{A} \) is empty, and graph \( \tilde{G} \) contains all vertices in \( V(G) \) and no edges. Then, for each admissible label \( \ell \in A \), we put \( \bar{w}_\ell \) copies of \( \ell \) into \( \tilde{A} \). Let \( \ell^{(1)}, \ell^{(2)}, \ldots, \ell^{(\bar{w}_\ell)} \) be these copies. For the sake of convenience, given an admissible label \( \ell \in A \), we define

\[ g(\ell) = \{ \ell^{(1)}, \ell^{(2)}, \ldots, \ell^{(\bar{w}_\ell)} \} \]

as the label group associated with \( \ell \). The label set \( \tilde{B} \) is just equal to \( B \). For the instance \( \tilde{I} \), the labels in \( \tilde{A} \) are admissible labels, and the labels in \( \tilde{B} \) are forbidden labels.

Let \( e = (u, v) \) be an edge in graph \( G \) with label \( \ell = \ell(e) \). If \( \ell \) is an admissible label in \( A \), we put \( \bar{w}_\ell \) edges between \( u \) and \( v \) in graph \( \tilde{G} \), with
the $i$-th copy ($1 \leq i \leq \bar{w}_\ell$) of the edge labeled with $\ell^{(i)}$. Otherwise ($\ell$ is a forbidden label in $B$), we just put $e$ itself in $\hat{G}$. The construction of instance $\hat{I}$ is finished. Note that $\hat{I}$ is just the instance $I$ in Section 4.

Here is the algorithm for weighted Label $s$-$t$ Cut.

**Algorithm 5.1.**

*Input:* An instance $I = (G, s, t, L, w)$ of weighted Label $s$-$t$ Cut.

*Output:* A label subset $L' \subseteq L$.

1. Guess $|O|$ and $W$.
2. Construct an instance $\hat{I} = (\hat{G}, s, t, \hat{A}, \hat{B})$ of Label $s$-$t$ Cut on multi-graphs as stated in Section 5.1.
3. Call Algorithm 4.1 on instance $\hat{I}$, obtaining a solution $\hat{A}' \subseteq \hat{A}$.
4. Return $L' \leftarrow \{ \ell \in A \mid g(\ell) \subseteq \hat{A}' \}$.

Algorithm 5.1 converts solution $\hat{A}'$ to a label subset $L' \subseteq L$ in its step 4 in the following way. Let $\ell^{(i)}$ be a label appeared in $\hat{A}'$. By the reduction, there is a label group $g(\ell) \subseteq \hat{A}$ created from $\ell$, to which $\ell^{(i)}$ belongs. If $g(\ell)$ appears completely in $\hat{A}'$ (i.e., $g(\ell) \subseteq \hat{A}'$), we put label $\ell$ into $L'$. Note that group $\ell$ may appear only partially in $\hat{A}'$. In this case we do not put label $\ell$ into $L'$.

### 5.2 Analysis

Let $I$ be an instance of weighted Label $s$-$t$ Cut, and $\hat{I}$ be the instance of Label $s$-$t$ Cut on multi-graphs created from $I$ by Algorithm 5.1. We now analyze the approximation ratio of Algorithm 5.1. First we show the relationship between the optimum of instance $\hat{I}$, denoted by $\text{OPT}_m(\hat{I})$, and the optimum of instance $I$, denoted by $\text{OPT}_w(I)$. For clarity, we use subscript “m” in $\text{OPT}_m(\cdot)$ to specify it is the optimum of some instance of the Label $s$-$t$ Cut problem on multi-graphs with forbidden labels. Similarly, we use subscript “w” in $\text{OPT}_w(\cdot)$ to specify it is the optimum of some instance of the weighted Label $s$-$t$ Cut problem.

**Lemma 5.1.** For the correct guess of $|O|$ and $W$, we have $\text{OPT}_m(\hat{I}) \leq \frac{|O|}{W} \text{OPT}_w(I) + |O|$.

*Proof.* By (6), we have $\bar{w}_\ell \leq \frac{w_\ell |O|}{W} + 1$, i.e., $w_\ell \geq \frac{W}{|O|}(\bar{w}_\ell - 1)$. Then it follows that

$$\text{OPT}_w(I) = \sum_{\ell \in O} w_\ell \geq \frac{W}{|O|} \sum_{\ell \in O} (\bar{w}_\ell - 1) = \frac{W}{|O|} \left( \left( \sum_{\ell \in O} \bar{w}_\ell \right) - |O| \right).$$
By replacing each label $\ell$ in $O$ with $\ell^{(1)}, \ell^{(2)}, \ldots, \ell^{(\bar{w}_\ell)}$ (note that all of them are in $\tilde{A}$, the admissible label set of $\tilde{I}$), we can get a feasible solution to instance $\tilde{I}$ with cost $\sum_{\ell \in O} \bar{w}_\ell$. This means that $\sum_{\ell \in O} \bar{w}_\ell \geq \text{OPT}_m(\tilde{I})$. So, we have

$$\text{OPT}_w(I) \geq \frac{W}{|O|} \left( \text{OPT}_m(\tilde{I}) - |O| \right),$$

obtaining the lemma.

**Lemma 5.2.** For the correct guess of $|O|$ and $W$, we have $\text{OPT}_m(\tilde{I}) \geq \frac{|O|}{W} \text{OPT}_w(I)$.

**Proof.** Since $W$ is a correct guess of the maximum weight of labels in $O$, each label in $g(\ell)$ for every $\ell \in O$ is an admissible label with respect to instance $\tilde{I}$. By replacing each label $\ell$ in $O$ with $\ell^{(1)}, \ell^{(2)}, \ldots, \ell^{(\bar{w}_\ell)}$, we can get a feasible solution to instance $\tilde{I}$. That is, instance $\tilde{I}$ has feasible solutions.

So, we may let $\tilde{O} \subseteq \tilde{A}$ be an optimal solution to instance $\tilde{I}$. From $\tilde{O}$, we can construct a feasible solution $A'$ to instance $I$, by replacing each label group $\ell^{(1)}, \ell^{(2)}, \ldots, \ell^{(\bar{w}_\ell)}$ in $\tilde{O}$ with label $\ell$. By (6), we have $\bar{w}_\ell \geq \frac{w_\ell |O|}{W}$, i.e., $w_\ell \leq \frac{W}{|O|} \bar{w}_\ell$. Then it follows that

$$\text{OPT}_w(I) \leq w(A') = \sum_{\ell \in A'} w_\ell \leq \frac{W}{|O|} \sum_{\ell \in A'} \bar{w}_\ell = \frac{W}{|O|} |\tilde{O}| = \frac{W}{|O|} \text{OPT}_m(\tilde{I}).$$

Rearranging the terms gives the lemma. □

**Lemma 5.3.** For the correct guess of $|O|$ and $W$, we have

$$w(L') = O\left( \frac{n^{2/3} \mu^{1/3}}{\text{OPT}_m(\tilde{I})^{1/3}} \right) \text{OPT}_w(I).$$

**Proof.** By (6), we have $\bar{w}_\ell \geq \frac{w_\ell |O|}{W}$, implying $w_\ell \leq \frac{W}{|O|} \bar{w}_\ell$. So, we get

$$w(L') = \sum_{\ell \in L'} w_\ell \leq \frac{W}{|O|} \sum_{\ell \in L'} \bar{w}_\ell. \quad (7)$$

By the construction of solution $L'$, we know $\sum_{\ell \in L'} \bar{w}_\ell \leq |\tilde{A}'|$. By Theorem 4.1, $|\tilde{A}'| \leq \rho(\tilde{I}) \text{OPT}(\tilde{I})$, where $\rho(\tilde{I}) = O\left( \frac{n^{2/3} \mu(\tilde{G})^{1/3}}{\text{OPT}_m(\tilde{I})^{1/3}} \right)$, $n$ is the number of vertices in $\tilde{G}$, and $\mu(\tilde{G})$ is the the maximum edge multiplicity of $\tilde{G}$. Note that the multi-graph $\tilde{G}$ has the same number of vertices as in $G$. That is, we have

$$\sum_{\ell \in L'} \bar{w}_\ell \leq \rho(\tilde{I}) \text{OPT}_m(\tilde{I}). \quad (8)$$
By (7), (8), and Lemma 5.1 we have

\[
w(L') \leq \frac{W}{|O|} \rho(\tilde{I}) \left( \frac{|O|}{W} \text{OPT}_w(I) + |O| \right) = \rho(\tilde{I}) \text{OPT}_w(I) + \rho(\tilde{I})W = O \left( \frac{n^{2/3} \mu(\tilde{G})^{1/3}}{\text{OPT}_m(I)^{1/3}} \right) \text{OPT}_w(I).
\]

\[\blacksquare\]

**Theorem 5.1.** The weighted Label s-t Cut problem can be approximated within \(O(n^{2/3})\).

**Proof.** By (6), the multiplicity \(\mu(\tilde{G})\) of multi-graph \(\tilde{G}\) is at most \(|O|\). So, by Lemma 5.3, the approximation ratio of Algorithm 5.1 is

\[
n^{2/3} \left( \frac{\mu(\tilde{G})}{\text{OPT}_m(I)} \right)^{1/3} \leq n^{2/3} \left( \frac{|O|}{\text{OPT}_m(I)} \right)^{1/3} \leq n^{2/3} \left( \frac{W}{\text{OPT}_w(I)} \right)^{1/3} \leq n^{2/3}.
\]

Finally, it is straightforward that Algorithm 5.1 runs in polynomial time. The theorem follows. \[\blacksquare\]

### 6 Concluding Remarks

In this paper, we design an approximation algorithm for the weighted Label s-t Cut problem whose ratio is \(O(n^{2/3})\), where \(n\) is the number of vertices in the input graph. An interesting open problem for Label s-t Cut is whether the exponent of the ratio \(O(n^{2/3})\) can be improved. For example, is an \(O(n^{1/2})\)-approximation for (weighted or unweighted) Label s-t Cut possible? Note that the current best approximation hardness factor for unweighted Label s-t Cut is \(2^{(\log n)^{1-c/(\log \log n)^c}}\) for any constant \(c < 1/2\) [30].

The approximation result \(O(n^{2/3})\) for weighted Label s-t Cut applies to the weighted Global Label Cut problem as well. For the Global Label Cut problem, a long-term open problem is whether it is in P or NP-hard.

17
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