Coherence measures induced by norm functions

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Coherence measures induced by norm functions are studied. It is shown that a coherence measure cannot be induced by a unitary similarity invariant norm. As a consequence, one can deduce the known fact that the Schatten p-norms do not induce coherence measures. For \((p, q) \in \{1, \infty\} \times \{1, \infty\}\), the \(\ell_{p,q}\)-norm of a matrix with columns \(A_1, \ldots, A_n\) is defined as the \(\ell_p\)-norm of the vector \((\ell_p(A_1), \ldots, \ell_p(A_n))\). It is shown that the \(\ell_{p,q}\)-norm induces a coherence measure if and only if \(q = 1\) and \(p \in \{1, 2\}\). This result gives rise to a new class of coherence measures, and explains why the \(\ell_p\)-norm, corresponding to the \(\ell_{p,q}\)-norm, induces a coherence measure if and only if \(p = 1\).

INTRODUCTION

Quantum coherence arising from quantum superposition plays a central role in quantum mechanics and so becomes an important physical resource in quantum information and quantum computation [1]. It also plays an important role in a wide variety of research fields, such as quantum biology [2–7], nanoscale physics [8, 9], and quantum metrology [10, 11].

It is well known that quantum mechanical systems differ significantly from classical systems mainly because of coherence, i.e., the superposition of physical states, that are reflected by the off-diagonal entries if quantum states are represented as density matrices. Note that coherence is basis dependent. The reference basis with respect to which coherence is measured depends on the physical problem. So, to study coherence measures of quantum states acting on an \(n\)-dimensional Hilbert space, we always fix a basis, and let \(I_n\) be the set of diagonal density matrices corresponding to the set of incoherent states. Density matrices not in \(I_n\) are coherent states.

Denote by \(M_{m,n}\) the linear space of \(m \times n\) complex matrices, and let \(M_k\) be the set of \(k \times k\) complex matrices. A quantum operation transforming quantum states in \(M_m\) to quantum states in \(M_m\) is a trace preserving completely positive map \(\mathcal{E} : M_m \rightarrow M_m\) admitting the following operator sum representation:

\[
\mathcal{E}(A) = \sum_{j=1}^r K_j A K_j^\dagger \quad \text{for all } A \in M_m,
\]

where \(r\) is a positive integer and \(K_1, \ldots, K_r \in M_{m,n}\) satisfy \(\sum_{j=1}^r K_j^\dagger K_j = I_n\). The operators \(K_1, \ldots, K_r \in M_{m,n}\) are called the Kraus operators corresponding to \(\mathcal{E}\); one may see [1] for the general background.

Denote by \(D_n\) the set of density matrices and the set of diagonal matrices in \(M_n\), respectively. If \(K_j \rho K_j^\dagger\) is a diagonal matrix for every \(\rho \in I_n\) whenever \(j \in \{1, \ldots, r\}\), we say that \(\{K_1, \ldots, K_r\}\) is a set of incoherent Kraus operators for the incoherent operation \(\mathcal{E}\).

In [12], the authors presented four defining conditions for a real-valued function \(C\) (defined on density matrices of any size) to be a coherence measure.

(B1) If \(\rho \in D_n\), then \(C(\rho) \geq 0\); the equality \(C(\rho) = 0\) holds if and only if \(\rho \in \mathcal{I}_n\).

(B2) If \(\Lambda : M_m \rightarrow M_m\) is an incoherent operation and \(\rho \in D_n\), then \(C(\rho) \geq C(\Lambda(\rho))\).

(B3) Suppose \(\rho \in D_n\) and \(\Lambda : M_m \rightarrow M_m\) is an incoherent operation with incoherent Kraus operators \(K_1, \ldots, K_r\). If \(p_j = \text{tr}(K_j \rho K_j^\dagger)\) and \(\rho_j = \frac{1}{p_j} K_j \rho K_j^\dagger\) for \(j = 1, \ldots, r\), then \(C(\rho) \geq \sum_{j=1}^r p_j C(\rho_j)\).

(B4) For any \(\{p_1, \ldots, p_r\} \subseteq D_n\) and any probability distribution \(\{p_1, \ldots, p_r\}, \sum_{j=1}^r p_j C(\rho_j) \geq C(\sum_{j=1}^r p_j \rho_j)\).

Note that condition (B3) combined with (B4) automatically imply (B2). In general, a real-valued function \(C\) is called a coherence measure if it satisfies the above four conditions; if only the conditions (B1), (B2), and (B3) are satisfied, \(C\) is usually called a coherence monotone. Researchers have considered different kinds of coherence measures; see [12–17, 19–23, 26–31].

In this paper we consider coherence measures induced by norms. Suppose \(\nu\) is a norm defined on matrices of arbitrary sizes. Given \(\rho \in D_n\), define

\[
C_\nu(\rho) = \min\{\nu(\rho - \sigma) : \sigma \in \mathcal{I}_n\}. \tag{1}
\]

Clearly, \(C_\nu\) automatically satisfies (B1). Condition (B4) is also satisfied automatically by the following argument. Suppose \(\rho_1, \ldots, \rho_r \in D_n\) and let \(\{\rho_1, \ldots, \rho_r\}\) be a probability distribution. Let \(\sigma_1^*, \ldots, \sigma_r^* \in \mathcal{I}_n\) be such that \(C(\rho_j) = \nu(\rho_j - \sigma_j^*)\), for \(j = 1, \ldots, r\), and \(\bar{\sigma} = \sum_{j=1}^r p_j \sigma_j^*\). Then

\[
\sum_{j=1}^r p_j C_\nu(\rho_j) = \sum_{j=1}^r p_j \nu(\rho_j - \sigma_j^*) \\
\geq \nu(\sum_{j=1}^r p_j \rho_j - \bar{\sigma}) \geq C_\nu(\sum_{j=1}^r p_j \rho_j).
\]

Condition (B2) holds if the norm \(\nu\) is contractive under any incoherent map \(\Lambda\), i.e., \(\nu(\rho_1 - \rho_2) \geq \nu(\Lambda(\rho_1) - \Lambda(\rho_2))\), because in this case, \(C_\nu(\rho) = \nu(\rho - \sigma^*)\) for some \(\sigma^* \in \mathcal{I}_n\), and
then
\[ \nu(\rho - \sigma^*) \geq \nu(\Lambda(\rho) - \Lambda(\sigma^*)) \geq C_\nu(\Lambda(\rho)) \]
because \( \Lambda(\sigma^*) \in I_n \). However, condition (B3) is more difficult to determine.

In [12], it was shown that if \( \nu \) is the \( \ell_1 \)-norm then \( C_\nu(\rho) = \sum_{i \neq j} |\rho_{ij}| \) is a coherence measure. In [27], the authors showed that \( C_\nu \) is not a coherence measure if \( \nu \) is the \( \ell_p \)-norm \( \nu(A) = \left( \sum |a_{ij}|^p \right)^{1/p} \) for \( p > 1 \). In the same paper, the authors also showed that \( C_\nu \) is not a coherence measure if \( \nu \) is the Schatten-\( p \)-norm \( \nu(A) = (\text{tr} |A|^p)^{1/p} \) with \( p > 1 \), where \( |A| \) is the positive semi-definite matrix \( A \) such that \( |A|^2 = A^*A \). When \( p = 1 \), the Schatten \( p \)-norm reduces to the trace norm \( \|A\| = \text{tr}|A| \), which is used frequently in quantum information science research. At one point, researchers believed that \( C_\nu \) is a coherence measure if \( \nu \) is the trace norm [18, 27]. However, this is not the case, as shown in [17] by using the following alternative framework for coherence measures.

(C1) If \( \rho \in \mathcal{D}_n \), then \( C(\rho) \geq 0 \); the equality \( C(\rho) = 0 \) holds if and only if \( \rho \in I_n \).

(C2) If \( \Lambda : M_n \to M_m \) and \( \rho \in \mathcal{D}_n \), then \( C(\rho) \geq C(\Lambda(\rho)) \).

(C3) Let \( p_1 \in \mathcal{D}_{n_1}, p_2 \in \mathcal{D}_{n_2}, p_1 + p_2 \geq 0 \) satisfy \( p_1 + p_2 = 1 \). Then \( C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(p_1) + p_2 C(p_2) \).

Clearly, (B1)-(B2) and (C1)-(C2) are the same. The advantage of this alternative framework is it changes the two inequalities in (B3)-(B4) into a concrete equality (C3). In addition to disproving that the trace norm does not induce a coherence measure, it was shown in [17] that one can modify the trace norm to define a coherence measure using the definition
\[ C(\rho) = \min \{ \|\rho - \sigma\|_1 : t \geq 0, \sigma \in I_n \} \quad \text{for} \ \rho \in \mathcal{D}_n. \]

However, it was shown in [32] that this coherence measure has some limitations.

In this paper, we obtain two general results on coherence measures induced by norm functions.

In Section 2, we show that \( C_\nu \), given in (1), cannot be a coherence measure because \( \nu \) is a unitarily similarity invariant (USI) norm, i.e., \( \nu(A) = \nu(U^*AU) \) for any \( A \in M_n \) and unitary \( U \in M_n \). However, the Schatten-\( p \)-norm is a USI norm for \( p \in [1, \infty] \). From this result, we can deduce that a Schatten-\( p \)-norm does not induce a coherence measure for any \( p \in [1, \infty] \). Some auxiliary results of independent interest are also obtained and used in our later discussion.

In Section 3, we consider the \( \ell_{q,p} \)-norm on \( M_n \), defined by
\[ \ell_{q,p}(A) = \ell_q(\ell_p(A_1), \ldots, \ell_p(A_n)) \quad \text{for} \quad p, q \in [1, \infty] \times [1, \infty], \]
where \( A_1, \ldots, A_n \) are the columns of a matrix \( A \in M_n \),
\[ \ell_\infty(x) = \max \{|x_j| : 1 \leq j \leq n\}, \]
and
\[ \ell_r(x) = \sum_{j=1}^n |x_j|^r/\|x\| \quad \text{for} \quad r \in [1, \infty), \]
for \( x = (x_1, \ldots, x_n) \). One may see [33] for some general background for the \( \ell_{q,p} \)-norm. Let
\[ C_{q,p}(\rho) = \min \{ \ell_{q,p}(\rho - \sigma) : \sigma \in I_n \}. \]
We will show that \( C_{q,p} \) is a coherence measure if and only if \( q = 1 \) and \( p \in [1,2) \). Note that the \( \ell_{p,p} \)-norm reduces to the \( \ell_p \)-norm. As a consequence of our result, we see that the \( \ell_p \)-norm induces a coherence measure if and only if \( p = 1 \).

COHERENCE MEASURES AND UNITARY SIMILARITY INVARIANT NORMS

Recall that a norm on \( M_n \) is unitarily similarity invariant (USI) if \( \|U^*AU\| = \|A\| \) for any \( A \in M_n \) and unitary \( U \in M_n \). Clearly, if \( A \) is Hermitian, then \( \|A\| \) depends only on the eigenvalues of \( A \). It is known that if \( \| \cdot \| \) is a USI norm on the space of \( n \times n \) Hermitian matrices, then there is a compact set \( S_n \subset \mathbb{R}^{1 \times n} \), depending on the norm \( \| \cdot \| \), such that the following conditions hold.

(N1) If \( (c_1, \ldots, c_n) \in S_n \) then \( \pm (c_1, \ldots, c_n)P \in S_n \) for any permutation matrix \( P \in M_n \).

(N2) For every Hermitian matrix \( A \) with eigenvalues \( \lambda_1(A) \geq \cdots \geq \lambda_n(A) \), there is a vector \( (c_1, \ldots, c_n) \in S_n \) with \( c_1 \geq \cdots \geq c_n \) such that
\[ \|A\| = \sum_{j=1}^n c_j \lambda_j(A) \]
\[ = \max \left\{ \sum_{j=1}^n d_j \lambda_j(A) : (d_1, \ldots, d_n) \in S_n \right\}. \]

One may see [35, Theorem 3.3] and also [34, Theorem 4.1] for the above result.

A norm on \( M_n \) is unitarily invariant if \( \|U^*AV\| = \|A\| \) for any \( A \in M_n \) and unitary \( U, V \in M_n \). For example, for \( p \geq 1 \), the Schatten-\( p \)-norm defined by \( \|A\|_p = (\text{tr} |A|^p)^{1/p} \) is a unitarily invariant norm. It is easy to see that a unitarily invariant norm is always a unitarily similarity invariant norm.

But there are unitarily similarity invariant norms that are not unitarily invariant; an example is the numerical radius defined by
\[ w(A) = \max \{ |\text{tr} Ap| : \rho \in \mathcal{D}_n \} \quad \text{for} \quad A \in M_n. \]

It is interesting to note that the set \( S_n \) satisfying (N1)-(N2) for the numerical radius can be \( \{ \pm (1,0,0,\ldots)P : P \in M_n \} \) is a permutation matrix \}; the set \( S_n \) satisfying (N1)-(N2) for the Schatten-\( p \)-norm can be \( \{ (d_1, \ldots, d_n) \in \mathbb{R}^{1 \times n} : \sum_{j=1}^n |d_j|^q = 1 \} \) where \( q = (1 - 1/p)^{-1} \) if \( p > 1 \), and \( \{ (d_1, \ldots, d_n) \in \mathbb{R}^{1 \times n} : \max_j |d_j| = 1 \} \) if \( p = 1 \).

As mentioned in the introduction, if \( \nu \) is the Schatten-\( p \)-norm with \( p \geq 1 \), then \( C_\nu \) is not a coherence measure. We will prove the following general result.
Theorem 1. There is no coherence measure $C_\nu$ induced by a USI norm $\nu$.

To prove this theorem we begin by establishing two auxiliary propositions which are of independent interest and will be used in Section 3.

Proposition 2. Suppose $\| \cdot \|$ is a norm on $M_n$. Define $C : \mathcal{D}_n \to [0, \infty)$ by

$$C(\rho) = \min\{\|\rho - \sigma\| : \sigma \in \mathcal{I}_n\}.$$  

Then (B1) and (B4) hold. If (B3) holds, then so does (B2).

(a) Assume (B2) holds. If $P \in \mathcal{D}_n$ is a permutation or a diagonal unitary matrix, then $C(\rho) = C(P^\dagger \rho P)$.

(b) If $\| \cdot \|$ is an absolute norm, i.e., $\|(a_{ij})\| = \|(a_{ij})\|$, then $C(\rho) = \|\rho - \rho_{\text{diag}}\|$.

Proof. As shown in Section 1, basic norm properties imply conditions (B1) and (B4). If (B3) holds, then (B3) and (B4) imply that (B2) holds.

(a) If (B2) holds, then we may consider the incoherent operation $A' = P^\dagger AP$ for a permutation or a diagonal unitary matrix $P$. Then $C(\rho) \geq C(P^\dagger \rho P)$. Now, consider $\tilde{\rho} = P^\dagger \rho P$ and $A' = PAP^\dagger$. Then

$$C(P^\dagger \rho P) = C(\tilde{\rho}) \geq C(P^\dagger \rho P) = C(\rho).$$

(b) Suppose $\| \cdot \|$ is an absolute norm. Let $\sigma^* \in \mathcal{I}_n$ satisfy $\|\rho - \sigma^*\| = \min\{\|\rho - \sigma\| : \sigma \in \mathcal{I}_n\}$. Replace the diagonal entries of $\rho - \sigma^*$ by their negative to obtain $\tilde{\tau}$. Then

$$\|\rho - \sigma^*\| = \|\tilde{\tau}\|$$

and

$$\|\rho - \sigma^*\| \leq \|\rho - \rho_{\text{diag}}\| = \frac{1}{2}\|\rho - \sigma^*\| + \|\tilde{\tau}\| \leq \frac{1}{2}(\|\rho - \sigma^*\| + \|\tilde{\tau}\|) = \|\rho - \sigma^*\|. \quad \blacksquare$$

Proposition 3. Suppose $\| \cdot \|$ is a norm on $M_n$ such that $\|P^\dagger AP\| = \|A\|$ for any trace zero Hermitian matrix $A \in M_n$ and permutation matrix $P \in M_n$. Let

$$C(\rho) = \min\{\|\rho - \sigma\| : \sigma \in \mathcal{I}_n\}.$$  

Suppose $R_k = E_{12} + \cdots + E_{1k} \in M_k$ is the basic circulant. If $\rho = \rho_1 \oplus \cdots \oplus \rho_k \in \mathcal{D}_n$ satisfies $R_1^\dagger \rho R_1 = \rho$ with $R = R_{s_1} \oplus \cdots \oplus R_{s_k}$, then

$$C(\rho) = \|(\rho_1 - s_1 I_{n_1}) \oplus \cdots \oplus (\rho_k - s_k I_{n_k})\|$$

for some $s_1, \ldots, s_k \in \mathbb{R}$. If $\rho_p = \rho_q$, we may assume that $s_p = s_q$. In particular, if $\rho_1 = \cdots = \rho_k$, then $C(\rho) = \|\rho - \rho_{\text{diag}}\|$.

Proof. Suppose $\| \cdot \|$ and $\rho = \rho_1 \oplus \cdots \oplus \rho_k \in \mathcal{D}_n$ satisfy the hypotheses of the proposition. Let $\sigma^* \in \mathcal{I}_n$ satisfy $\|\rho - \sigma^*\| = \min\{\|\rho - \sigma\| : \sigma \in \mathcal{I}_n\}$. Since $R_1^\dagger \rho R_1 = \rho$, we see that every $\rho_j$ has constant diagonal entries. Let $N = \text{lcm}(n_1, \ldots, n_k)$, the least common multiple of $n_1, \ldots, n_k$. If $\tilde{\sigma} = \frac{N}{k} \sum_{j=1}^n (R_1^\dagger \sigma^* R_1)^j$, then

$$\|\rho - \sigma^*\| \leq \|\rho - \tilde{\sigma}\| = \frac{1}{N} \sum_{j=1}^N \|R_1^\dagger (\rho - \sigma^*) R_1^j\| \leq \frac{1}{N} \sum_{j=1}^N \|R_1^\dagger (\rho - \sigma^*) R_1^j\| = \|\rho - \sigma^*\|.$$  

Thus we may replace $\sigma^*$ by $\tilde{\sigma}$, which has the form $s_1 I_{n_1} \oplus \cdots \oplus s_k I_{n_k}$.

Now assume that $\sigma^*$ has the form $s_1 I_{n_1} \oplus \cdots \oplus s_k I_{n_k}$. Suppose $\rho_1 = \rho_2$; say, without loss of generality, $\rho_1 = \rho_2$. Let $s = (s_1 + s_2)/2$, $Q = R_2 \otimes I_{n_1} \oplus I_{n_1} \otimes R_{-2n_1}$, and $\tilde{\sigma} = (\sigma^* + Q^\dagger \sigma^* Q)/2 = s I_{2n_1} \oplus s I_{2n_3} \oplus \cdots \oplus s_k I_{n_k}$. Then

$$\|\rho - \sigma^*\| \leq \|\rho - \tilde{\sigma}\| = \frac{1}{2}\|\rho - \sigma^*\| + Q^\dagger (\rho - \sigma^*) Q\| \leq \frac{1}{2}(\|\rho - \sigma^*\| + \|Q^\dagger (\rho - \sigma^*) Q\|) = \|\rho - \sigma^*\|.$$  

So, we may replace $\sigma^*$ by $\tilde{\sigma}$. \quad \blacksquare

Proof of Theorem 1. Suppose $C$ is a coherence measure induced by a USI norm $\| \cdot \|$. Then $\| \cdot \|$ satisfies the hypothesis of Proposition 3 and $C$ satisfies Proposition 2 (a). For each natural number $n$, let $S_n \subseteq \mathbb{R}^{1 \times n}$ be a compact set satisfying condition (N1) and (N2) for Hermitian matrices in $M_n$. Given $\rho \in \mathcal{D}_n$ we have

$$C(\rho) = \min\{\|\rho - \sigma\| : \sigma \in \mathcal{I}_n\}.$$  

Note the norm computation involves only trace zero matrices. We may replace every vector $c = (c_1, \ldots, c_n) \in S_n$ by $\tilde{c} = (c_1 - \gamma, \ldots, c_n - \gamma)$ so that the largest and smallest entries of $\tilde{c}$ have the form $c$ and $-c$. Now, for a trace zero Hermitian matrix $A$,

$$\sum_{j=1}^n (c_j - \gamma) \lambda_j(A) = \sum_{j=1}^n c_j \lambda_j(A) - \gamma \sum_{j=1}^n \lambda_j(A) = \sum_{j=1}^n c_j \lambda_j(A) - \gamma \text{tr} A = \sum_{j=1}^n c_j \lambda_j(A).$$

As a result, the computation of $C(\rho)$ will not be affected. So, we will assume that the largest and smallest entries of every vector $c \in S_n$ have the form $c$ and $-c$. Furthermore, we may replace $C$ by $\alpha C$ for some $\alpha > 0$ and assume that $C(J_2/2) = \|J_2 - J_2\|^2 = 1$.

Assertion 1. Let $d \in S_4$. There is a permutation matrix $P \in S_4$ such that $dP = (d, d_2, d_3, -d) \in S_4$ with $1 \geq d \geq$
Note that consequently, for any trace zero Hermitian matrix $A \in M_4$ with 
$$\lambda_1(A) \geq \lambda_2(A) \geq 0 \geq \lambda_3(A) \geq \lambda_4(A),$$
$$\|A\| = |\lambda_1(A)| + |\lambda_2(A)| + |\lambda_3(A)| + |\lambda_4(A)|.$$

Furthermore, $4/3 = C(J_3/3) = C(J_3/3 \oplus [0])$.

**Proof of Assertion 1.** Let $d \in S_4$. By (N1), there is a permutation matrix $P \in S_4$ such that $dP = (d, d_2, d_3, -d)$ in $S_4$ such that $d \geq d_2 \geq d_3 \geq -d$. Suppose $d > 1$. We may further assume that $d_2 + d_3 \leq 0$; otherwise, replace $d$ by the vector $(d, -d_3, -d_2, -d) \in S_4$. By (C3) and Proposition 3, if $\rho = J_2/2 \oplus 0_2 \in D_4$, then
$$1 = C(J_2/2) = C(\rho) = \|\rho - \sigma\|$$
for some $\sigma = \text{diag}(s, s, (1/2 - s), (1/2 - s)) \in I_4$ with $s \in [0, 1/2]$. The matrix $\rho - \sigma$ has eigenvalues $1 - s, s - 1/2, s - 1/2, s$. By (N2),
$$\|\rho - \sigma\| \geq d((1 - s) + (d_2 + d_3)(s - 1/2) - d(-s)) \geq d > 1,$$
which is a contradiction.

Next, we prove that $(1, 1, -1, -1) \in S_4$. By (C3) and Proposition 3,
$$1 = \frac{1}{2} C(J_2/2) + C(J_2/2) = C(J_2/4 \oplus J_2/4)$$
$$= \frac{1}{2} \|\rho - \sigma\|.$$

Note that $\frac{1}{2}[(J_2 - I_2) \oplus (J_2 - I_2)]$ has eigenvalues $1/4, 1/4, 1/4, -1/4, -1/4$. So, there is a vector in $S_4$ of the form $(c, c_2, c_3, -c)$ with $1 \geq c \geq c_2 \geq c_3 \geq c - 1$ such that
$$1 = \frac{1}{2} \|\rho - \sigma\| = \frac{1}{2} (2c + c_2 - c_3) \leq \frac{1}{2} (2c + |c_2| + |c_3|) \leq c \leq 1.$$

Thus, $c = 1$ and $(c, c_2, c_3, -c) = (1, 1, -1, -1)$.

Now, for every vector $d \in S_4$, there is a permutation matrix $P \in S_4$ such that $dP = (d, d_2, d_3, -d)$ with $1 \geq d \geq d_2 \geq d_3 \geq -d \geq -1$ and $(1, 1, -1, -1) \in S_4$. By (N2), for any trace zero Hermitian matrix $A \in M_4$ with $\lambda_1(A) \geq \lambda_2(A) \geq 0 \geq \lambda_3(A) \geq \lambda_4(A)$, we have
$$\|A\| = \max\{d\lambda_1(A) + d_2\lambda_2(A) + d_3\lambda_3(A) - d_4\lambda_4(A) : (d, d_2, d_3, -d) \in S_4\}$$
will be attained at the vector $(1, 1, -1, -1)$ and $\|A\| = \sum_{j=1}^4 |\lambda_j(A)|$.

Now, consider $\rho = J_3/3 \oplus [0] \in D_4$. By Proposition 3, we see that there is $\sigma = \text{diag}(s, s, s, 1 - 3s) \in I_4$ with $s \in [0, 1/3]$ such that $C(\rho) = \|\rho - \sigma\|$. Now, $\rho - \sigma$ has eigenvalues $1 - s, 1 - 3s, -s, -s$. Thus,
$$\|\rho - \sigma\| = (1 - s) + (1 - 3s) + s + s = 2 - 2s$$
which is minimized when $s = 1/3$. By (C3),
$$C(J_3/3) = C(J_3/3 \oplus [0]) = 4/3.$$

The proof of Assertion 1 is complete.

**Assertion 2.** Let $d \in S_6$. There is a permutation matrix $P \in M_6$ such that $dP = (d, d_2, \ldots, d_5, -d)$ with $1 \geq d \geq d_2 \geq \cdots \geq d_5 \geq -d \geq -1$. Moreover, $(1, 1, 1, -1, -1, 1) \in S_6$. Consequently, for any trace zero Hermitian matrix $A \in M_6$ with $\lambda_1(A) \geq \lambda_2(A) \geq \lambda_3(A) \geq 0 \geq \lambda_4(A) \geq \lambda_5(A) \geq \lambda_6(A)$,
$$\|A\| = \sum_{j=1}^6 |\lambda_j(A)|.$$

**Proof of Assertion 2.** Let $d \in S_6$. By (N1), there is a permutation matrix $P$ such that $dP$ has the form $(d, d_2, \ldots, d_5, -d) \in S_6$ with $d \geq d_2 \geq \cdots \geq d_5 \geq -d$. Suppose $d > 1$. We may assume that $d_2 + d_3 + d_4 + d_5 \leq 0$; otherwise, consider $(d, d_3, \ldots, d_2, -d) \in S_6$ instead. By Proposition 3, if $\rho = J_2/2 \oplus 0_4$, then there is $\sigma = \text{diag}(2s, 2s, 1/4 - s, 1/4 - s, 1/4 - s, 1/4 - s) \in I_6$ with $s \in [0, 1/4]$ such that $1 = C(\rho) = \|\rho - \sigma\|$. Now, $\rho - \sigma$ has eigenvalues $1 - 2s, s - 1/4, s - 1/4, s = 1/4, s = 1/4 - 2s$. By (N2),
$$\|\rho - \sigma\| \geq d((1 - 2s) + (s - 1/4)(d_2 + \cdots + d_5) + 2ds \geq d > 1,$$
which is a contradiction.

Next, we show that $(1, 1, 1, -1, -1, -1) \in S_6$. Let $\rho = (J_2 \oplus J_2 \oplus J_2)/6$. By (C3) and Proposition 3,
$$1 = \frac{1}{3} C(J_2/2) + \frac{2}{3} C(J_2/4 \oplus J_2/4) = C(\rho) = \|\rho - \rho_{\text{diag}}\|.$$}

Note that $\rho - \rho_{\text{diag}}$ has three eigenvalues equal to $1/6$ and three eigenvalues equal to $-1/6$. Thus, there is $(c, c_2, \ldots, c_5, -c) \in S_6$ such that $1 \geq c \geq c_2 \geq \cdots \geq c_5 \geq c - 1$ and
$$1 = \frac{1}{6} \|\rho - \rho_{\text{diag}}\| = \frac{1}{6} (2c + (c_2 + c_3 - c_4 - c_5)) \leq \frac{1}{6} (2c + |c_2| + |c_3| + |c_4| + |c_5|) \leq c \leq 1.$$

Thus, $c = 1$, and $c_2 = c_3 = 1 = -c_4 = -c_5$.

The proof of the last statement is similar to that in the proof of Assertion 1. The proof of Assertion 2 is complete.

To finish the proof of the theorem, consider $\rho = J_2/4 \oplus J_3/6 \oplus [0]$. Let $\sigma = J_2/2 \oplus 0_4$. Then $\rho - \sigma$ has eigenvalues $1/2, 0, 0, 0, 0, 0, 1/2$, where the three largest eigenvalues are nonnegative, and the rest are nonpositive. So, $\|\rho - \sigma\| = \sum_{j=1}^6 |\lambda_j(\rho - \sigma)| = 1$ by Assertion 2. But then by (C3) and Assertion 1,
$$1 = \|\rho - \sigma\| \geq C(\rho) = \frac{1}{5} (C(J_2/2) + C(J_3/3 \oplus [0])) = 1/2(1 + 4/3) = 7/6,$$
which is absurd. ■
COHERENCE MEASURES ASSOCIATED WITH THE $\ell_{p,q}$-NORM

Recall that, for $1 \leq p, q \leq \infty$, the $\ell_{p,q}$-norm of a matrix $A \in M_n$, with columns $A_1, \ldots, A_n$, is the $\ell_q$-norm of the vector formed by the $\ell_p$-norms of the columns of $A$; that is,

$$\ell_{p,q}(A) = \left( \sum_{j=1}^{n} \ell_p(A_j)^q \right)^{1/q}.$$ 

By Proposition 2 (b), for $\rho \in \mathcal{D}_n$, we have

$$C_{q,p}(\rho) = \min \{ \ell_{q,p}(\rho - \sigma) : \sigma \in \mathcal{I}_n \} = \ell_{q,p}(\rho - \rho_{\text{diag}}). \quad (2)$$

**Theorem 4.** The function $C_{q,p}$ in (2) is a coherence measure if and only if $q = 1$ and $p \in [1,2]$.

We first establish the necessity of Theorem 4.

**Lemma 5.** If $C_{q,p}$ is a coherence measure then $q = 1$ and $p \in [1,2]$.

**Proof.** Let $0_n$ be the $n \times n$ zero matrix and let $J_n$ be the $n \times n$ all ones matrix. Let

$$A = \frac{1}{4} (J_2 + J_2) = \frac{1}{2} \left( \frac{J_2}{2} \oplus 0_2 \right) + \frac{1}{2} \left( 0_2 \oplus \frac{J_2}{2} \right).$$

Then $C_{q,p}(A) = \frac{1}{2} 4^{1/q}$ while $C_{p,q} \left( \frac{J_2}{4}, J_2 \right) + C_{p,q} \left( J_2, \frac{J_2}{2} \right) = 2^{1/q}$; if $C_{q,p}$ were a coherence measure, then by property (C3) these two quantities must be equal, whence $q = 1$.

Now let $K_1 = (\sin \theta) I_n \oplus [0, \theta]$ for $\theta \in [0, \pi/2]$, so $A(X) = \sum_{j=1}^{n} K_j \Theta K_j^\dagger$ is an incoherent operation. Let $A = J_{n+1}$ and write $c = \cos \theta$, $s = \sin \theta$. Then $C_{1,\infty}(A) = n + 1$, $C_{1,\infty}(K_1 A K_1^\dagger) = n s^2$, $C_{1,\infty}(K_1 A K_1^\dagger) = (n + 1)c$, so

$$\sum_{j=1}^{n} C_{1,\infty}(K_j A K_j^\dagger) - C_{1,\infty}(A) = n(c - s^2) + c - 1 = (nc - 1)(1 - c)$$

is positive for $\theta \in (0, \pi/2)$ and $n$ sufficiently large. This violates property (B3), so $C_{1,\infty}$ is not a coherence measure.

For $p \neq \infty$, we have

$$C_{1,p}(A) = (n + 1)^{1/p},$$

$$C_{1,p}(K_1 A K_1^\dagger) = (\sin^2 \theta) n(n - 1)^{1/p},$$

$$C_{1,p}(K_2 A K_2^\dagger) = (\cos \theta) n^{1/p} + n(n - 1) \cos^2 \theta + \cos^p \theta \theta^{1/p}.$$ 

Let $f(n, \theta) = C_{1,p}(K_1 A K_1^\dagger) + C_{1,p}(K_2 A K_2^\dagger) - C_{1,p}(A)$. Note

$$f(n, \theta) = n^{1/p} \left( c - 1 + n(s^2(1 - 1/n)^{1/p} + (1 - 1/n)(c^p/n)^{1/p} - 1) \right) = n^{1/p}(c - 1 + g(t, \theta))$$

where $t = 1/n$ and

$$g(t, \theta) = \frac{s^2(1-t)^{1/p} + [(1-t)\theta^{2p} + tc^p]^{1/p} - 1}{t}.$$ 

By l’Hôpital’s rule,

$$\lim_{t \to 0^+} g(t, \theta) = \lim_{t \to 0^+} \frac{1}{p} \left( -s^2(1-t)^{1/p-1} + [(1-t)\theta^{2p} + tc^p]^{1/p-1}(c^p - \theta^{2p}) \right) = \frac{1}{p} \left( -s^2 + c^{2-p}(c^p - \theta^{2p}) \right) = \frac{1}{p} (c^{2-p} - 1).$$

For $p > 2$ we can make this limit arbitrarily large by making $\cos \theta$ sufficiently small. It follows that $f(n, \theta) > 0$ for $n$ sufficiently large and $\theta$ sufficiently close to $\pi/2$, violating property (B3).

The proof of sufficiency for Theorem 4 is more complicated; the key is to show a norm inequality that may be of independent interest. To this end, the following notations will be useful. Let $\mathbb{R}_+$ be the set of nonnegative real numbers. Given a set $\Omega$ we shall write $[\Omega]$ for the number of elements in $\Omega$. Let $\Omega_n = \{1, \ldots, n\}$. Given a subset $\tau \subseteq \Omega_n$, let $\tau^c = \Omega_n \setminus \tau$. Given a vector $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$ and a nonempty subset $\sigma$ of $\Omega_n$, let $v_\sigma \in \mathbb{C}^{|\sigma|}$ be the vector whose entries are $\{v_j : j \in \sigma\}$, ordered by increasing index.

We shall need two technical results first.

**Lemma 6.** Let $1 \leq p \leq 2$ and let $\Omega$ be a collection of nonempty subsets of $\Omega_n$. Suppose $\Omega$ is a cover of $\Omega_n$, and $v \in \mathbb{R}^n_+$ has no zero entries. Then

$$\left( \max_{\sigma \in \Omega} \ell_p(v_\sigma) \right)^{p-2} \ell_2(v)^2 \leq \ell_p(v)^p. \quad (3)$$

**Proof.** We shall use induction on $|\Omega|$. When $|\Omega| = 1$, we must have $\Omega = \{\Omega_n\}$ and the left-hand side of (3) becomes $\ell_p(v)^{p-2} \ell_2(v)^2 \leq \ell_p(v)^{p-2} \ell_p(v)^2 = \ell_p(v)^p$ as desired. Now suppose the assertion holds whenever $|\Omega|$ is less than $m$.

Let $\Omega$ be a cover of $\Omega_n$ with $|\Omega| = m$. Choose $\tau \in \Omega$ so that $\ell_p(v_\tau) \geq \ell_p(v_\sigma)$ for all $\sigma \in \Omega$; if $\tau = \Omega_n$ we are done, so we may assume $\tau \neq \Omega_n$ and write $K = \ell_p(v_\tau)$. Let $\Omega = \{\sigma \cap \tau^c : \sigma \in \Omega, \sigma \cap \tau^c \neq \emptyset\}$. Because $\Omega$ is a cover for $\Omega_n$, $\Omega$ is a cover for $\tau^c$ with $|\Omega| < m$. Then

$$K^{p-2} \ell_2(v)^2 = K^{p-2} \ell_2(v_\tau)^2 + K^{p-2} \ell_2(v_{\tau^c})^2 \leq K^{p-2} \ell_p(v_\tau)^2 + \left( \max_{\mu \in \Omega} \ell_p(v_\mu) \right)^{p-2} \ell_2(v_{\tau^c})^2$$

since $p \leq 2$ and for all $\sigma \in \Omega$

$$\ell_p(v_{\sigma \cap \tau^c}) \leq \ell_p(v_\sigma) \leq \ell_p(v_\tau)$$

$$= \ell_p(v_\tau)^p + \left( \max_{\mu \in \Omega} \ell_p(v_\mu) \right)^{p-2} \ell_2(v_{\tau^c})^2 \leq \ell_p(v_\tau)^p + \ell_p(v_\tau)^p$$

by the induction hypothesis

$$= \ell_p(v)^p.$$
as desired.

**Lemma 7.** Fix \( p \in [1, 2] \) and let \( n \in \mathbb{N} \). Let \( \Omega \) be a collection of nonempty subsets covering \( \{1, \ldots, n\} \) and let \( v \in \mathbb{R}^+_n \) be a nonzero vector. For each \( \sigma \in \Omega \) let \( b_\sigma \) be a nonnegative number. Suppose

\[
\sum_{\sigma \in \Omega} b_\sigma = \ell_2(v)^2 \quad \text{and} \quad \sum_{\sigma \subseteq \tau, \sigma \in \Omega} b_\sigma \leq \ell_2(v_{\tau})^2 \quad \text{for} \quad \tau \subseteq \Omega_n.
\]

Then

\[
\sum_{\sigma \in \Omega, \nu_\sigma \neq 0} \ell_p(v_\sigma)^{p-2} b_\sigma \leq \ell_p(v)^p. \tag{4}
\]

**Proof.** We prove this by induction on \( n \). When \( n = 1 \) we must have \( \Omega = \{\{1\}\} \), \( v > 0 \), and \( b_{\{1\}} = v^2 \); the assertion clearly holds.

Now suppose (4) holds whenever the length of \( v \) is less than \( n \). Consider the function \( f : \mathbb{R}^+_n \times \mathbb{R}^+_n \) defined by \( f(v; b) = 0 \) if \( v_\sigma = 0 \) for all \( \sigma \in \Omega \), and otherwise

\[
f(v; b) = \sum_{\sigma \in \Omega, \nu_\sigma \neq 0} \ell_p(v_{\sigma})^{p-2} b_\sigma.
\]

Let \( K \subset \mathbb{R}^+_n \times \mathbb{R}^+_n \) be the compact set defined by

\[
K = \{(v; b) \in \mathbb{R}^+_n \times \mathbb{R}^+_n : \ell_p(v)^p = M > 0, \sum_{\sigma \subseteq \tau, \sigma \in \Omega} b_\sigma \leq \ell_2(v_{\tau})^2 \forall \tau \subseteq \Omega_n, \text{ with equality when } \tau = \Omega_n\}.
\]

In particular, when \((v; b) \in K\) we have \( b_\sigma \leq \ell_2(v_{\sigma})^2 \), so when \( v_\sigma \) is nonzero,

\[
\ell_p(v_\sigma)^{p-2} b_\sigma \leq \ell_p(v_\sigma)^p,
\]

which approaches zero when \( v_\sigma \) approaches zero. Thus \( f \) is continuous on \( K \) and attains an absolute maximum on \( K \); it suffices to show that this maximum does not exceed \( M \).

Case 1: Suppose the maximum of \( f \) is attained on the relative boundary of \( K \). There are three possibilities.

Subcase (i): The maximum occurs at some \( v \in \mathbb{R}^+_n \) with a zero entry, say, \( v_j = 0 \). We can replace \( v \in \mathbb{R}^+_n \) with \( v_{(j)} \in \mathbb{R}^{n-1}_+ \), \( \sigma \in \Omega \) with \( \sigma \cap \{j\} = \emptyset \), and the result follows by induction.

Subcase (ii): The maximum occurs when \( b_\sigma = 0 \) for some \( \sigma \in \Omega \). We may replace \( \Omega \) by \( \Omega \setminus \{\sigma\} \) and use induction on \( |\Omega| \); note that when \( \Omega \) consists of a single element \( \sigma \),

\[
f(v; b) \leq \ell_p(v_{\sigma})^{p-2} \ell_2(v_{\sigma})^2 \leq \ell_p(v_{\sigma})^{p-2} \ell_p(v_{\sigma})^2 \leq \ell_p(v)^p.
\]

Subcase (iii): The maximum occurs when \( \sum_{\sigma \subseteq \tau, \sigma \in \Omega} b_\sigma = \ell_2(v_{\tau})^2 > 0 \) for some \( \tau \subseteq \Omega_n \). Let \( \Omega = \{\sigma \cap \tau^c : \sigma \in \Omega, \sigma \cap \tau^c \neq \emptyset\} \) and for \( \mu \in \Omega \), set

\[
\delta_\mu = \sum_{\sigma \subseteq \tau, \sigma \in \Omega} b_\sigma.
\]

Because

\[
\ell_2(v)^2 = \sum_{\sigma \in \Omega} b_\sigma = \sum_{\sigma \subseteq \tau} b_\sigma + \sum_{\sigma \in \Omega, \sigma \cap \tau^c \neq \emptyset} b_\sigma = \ell_2(v_{\tau})^2 + \sum_{\mu \in \Omega} \delta_\mu,
\]

we have \( \sum_{\mu \in \Omega} \delta_\mu = \ell_2(v_{\tau^c})^2 \). Moreover, for all \( \nu \subseteq \tau^c \) we have

\[
\sum_{\mu \subseteq \nu} \delta_\mu = \sum_{\mu \subseteq \nu} \sum_{\sigma \subseteq \tau, \sigma \in \Omega} b_\sigma = \sum_{\sigma \subseteq \tau} b_\sigma - \sum_{\sigma \subseteq \tau, \sigma \in \Omega} b_\sigma \leq \ell_2(v_{\tau^c \nu})^2 - \ell_2(v_{\tau})^2 = \ell_2(v_{\nu})^2.
\]

Then

\[
f(v, b) = \sum_{\sigma \in \Omega, \sigma \subseteq \tau} \ell_p(v_{\sigma})^{p-2} b_\sigma + \sum_{\sigma \in \Omega, \sigma \cap \tau^c \neq \emptyset} \ell_p(v_{\sigma})^{p-2} b_\sigma \leq \sum_{\sigma \in \Omega, \sigma \subseteq \tau} \ell_p(v_{\sigma})^{p-2} b_\sigma + \sum_{\sigma \in \Omega, \sigma \cap \tau^c \neq \emptyset} \ell_p(v_{\sigma})^{p-2} b_\sigma
\]

By induction (since the lengths of \( v_{\tau} \) and of \( v_{\tau^c} \) are less than \( n \)), the last two terms do not exceed \( \ell_p(v_{\tau})^p \) and \( \ell_p(v_{\tau^c})^p \) respectively and the result follows.

Case 2: Suppose the maximum of \( f \) is attained in the relative interior of \( K \). Using Lagrange multipliers we conclude that \( \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \) where

\[
g_1(v; b) = \ell_p(v)^p - M = 0
\]

and

\[
g_2(v; b) = \ell_2(v)^2 - \sum_{\sigma \in \Omega} b_\sigma = 0.
\]

From the partial derivative with respect to \( b_\sigma, \sigma \in \Omega \), we have

\[
\ell_p(v_{\sigma})^{p-2} = 0 + \lambda_2 (-1),
\]

whence \( \ell_p(v_{\sigma}) \) equals a constant \( K \) for all \( \sigma \in \Omega \). Thus the maximum of \( f \) is given by

\[
f(v; b) = \sum_{\sigma \in \Omega} K^p b_\sigma = K^p \ell_2(v)^2
\]

and the result then follows by Lemma 6.

The following result is known, e.g., see [33]. We include a proof for completeness.
Lemma 8. If \( B \) is an extreme point for the unit ball for the \( \ell_{1,p} \)-norm, then \( B \) has exactly one nonzero column.

Proof. Clearly \( B \neq 0 \). We prove the contrapositive. Suppose \( \ell_{1,p}(B) = 1 \) and \( B \) has more than one nonzero column; without loss of generality, we may suppose that the first two columns are nonzero. Let \( b_j \) be the \( j \)th column of \( B \). Let \( \epsilon = \frac{1}{2} \min\{ \ell_p(b_1), \ell_p(b_2) \} \). Then

\[
B = \frac{1}{2} \left[ \left( 1 + \frac{\epsilon}{\ell_p(b_1)} \right) b_1 \left( 1 - \frac{\epsilon}{\ell_p(b_2)} \right) b_2 b_3 \ldots b_n \right] \\
+ \frac{1}{2} \left[ \left( 1 - \frac{\epsilon}{\ell_p(b_1)} \right) b_1 \left( 1 + \frac{\epsilon}{\ell_p(b_2)} \right) b_2 b_3 \ldots b_n \right]
\]

is the average of two distinct matrices with norm 1, so \( B \) is not an extreme point. □

The next result provides the main idea for showing sufficiency in Theorem 4. The seminorm defined is in fact a norm, but that is not needed for our purposes.

Proposition 9. Let \( p \in [1, 2] \) and let \( \{ K_1, \ldots, K_m \} \) be a set of incoherent Kraus operators in \( M_{N,n} \). Define a seminorm \( \| \cdot \| \) on \( M_n \) by

\[
\| A \| = \sum_{k=1}^m \ell_{1,p}(K_k A K_k^\dagger).
\]

Then \( \| A \| \leq \ell_{1,p}(A) \) for all \( A \in M_n \).

Proof. Let \( B \) and \( B_{1,p} \) be the unit balls in \( M_n \) for \( \| \cdot \| \) and \( \ell_{1,p} \) respectively. Then \( \| A \| \leq \ell_{1,p}(A) \) for all \( p \) if only if the unit ball for the \( \ell_{1,p} \)-norm lies inside the unit ball for the \( \| \cdot \| \)-seminorm. By convexity, it suffices to show that each extreme point \( B \) of the \( \ell_{1,p} \)-ball has seminorm \( B \| \leq 1 \).

By Lemma 8, such an extreme point \( B \) has exactly one nonzero column. Let \( e_j \) be the vector whose only nonzero entry is a 1 in the \( j \)th position. We may write \( B = v e_j^\dagger \) for some \( \ell_p \)-unit vector \( v \in \mathbb{C}^n \); without loss of generality, we may assume \( j = 1 \). Thus we must show that

\[
\sum_{k=1}^m \ell_{1,p}(K_k v e_j^\dagger K_k^\dagger) \leq 1.
\]

(5)

for all \( v \in \mathbb{C}^n \) with \( \ell_p(v) = 1 \). For such a \( v \), write \( v = \sum_{j=1}^n v_j e_j \in \mathbb{C}^n \). Let

\[
F = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{bmatrix}, \quad w = F v = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix},
\]

where \( w_k = K_k v \). It is important to note that, because \( \sum_{k=1}^m K_k^\dagger K_k = I \), \( F \) is an isometry (for \( \ell_2 \)).

Since \( K_j I_n K_j^\dagger \) is diagonal, each column of \( K_k \) has at most one nonzero entry (see [25, Theorem 1]), or simply note that, if \( K_k e_j \) had nonzero entries in the \( p \)-th and \( q \)-positions, then \( K_k e_j e_q^\dagger K_k^\dagger \) would have a nonzero entry in the \((p, q)\)-position. Thus we may write \( K_k e_j = c_k e_{\sigma_k(j)} \), where \( c_k \) is a map from \( \{1, \ldots, n\} \) to \( \{1, \ldots, N\} \). Then

\[
\ell_{1,p}(K_k v e_j^\dagger K_k^\dagger) = \ell_{1,p}(w_k e_{\sigma_k(1)}^\dagger) = |c_{k1}| \ell_p(w_k),
\]

so

\[
\sum_{k=1}^m \ell_{1,p}(K_k v e_j^\dagger K_k^\dagger) = \sum_{k=1}^m |c_{k1}| \ell_p(w_k) \leq \sqrt{\sum_k |c_{k1}|^2} \sqrt{\sum_k \ell_p(w_k)^2}
\]

by the Cauchy-Schwarz inequality. Since \( F \) is an isometry, \( \sum_k |c_{k1}|^2 = \ell_2(F e_1)^2 = 1 \) because \( F \) is an isometry. For \( 1 \leq p < 2 \), however, the proof continues.

Define \( J_{k,s} = \{ j : \sigma_k(j) = s \} \). Note that

\[
w_k = K_k \left( \sum_{j=1}^n v_j e_j \right) = \sum_{j=1}^n v_j c_k e_{\sigma_k(j)}
\]

\[
= \sum_s \left( \sum_{j \in J_{k,s}} v_j c_k \right) e_s.
\]

Let \( v_{k,s} \) be the vector of length \( |J_{k,s}| \) with entries equal to \( v_j, j \in J_{k,s} \), and let \( w_{k,s} = \sum_{j \in J_{k,s}} v_j c_k \) be the \( s \)th entry of \( w_k \). Then

\[
\ell_p(w_k)^2 = \left( \sum_s |w_{k,s}|^p \right)^{2/p}
\]

\[
= \left( \sum_{s, v_{k,s} \neq 0} \ell_p(v_{k,s})^p \frac{|w_{k,s}|^p}{\ell_p(v_{k,s})^p} \right)^{2/p}
\]

\[
\leq \sum_{s, v_{k,s} \neq 0} \ell_p(v_{k,s})^p |w_{k,s}|^2 \ell_p(v_{k,s})^2
\]

since \( f(x) = x^{2/p} \) is convex for \( p \in [1, 2] \) and \( \sum_s \ell_p(v_{k,s})^p = \ell_p(v)^p = 1 \).

Thus (6) will hold if we can show that

\[
\sum_{k, s, v_{k,s} \neq 0} \ell_p(v_{k,s})^{p-2} |w_{k,s}|^2 \leq 1.
\]

(7)

Note that \( v_{k,s} \) consists of the entries of \( v \) whose indices correspond to nonzero entries in the \( s \)th row of \( K_k \), so we may regroup the sum as follows. Given a nonempty subset
\( \sigma \subseteq \{1, \ldots, n\} \), recall that \( v_{\sigma} \in \mathbb{C}^{[\sigma]} \) consists of the entries of \( v \) whose indices lie in \( \sigma \). Define
\[
    b_{\sigma} = \sum_{i} \left| (Fv)_{i} \right|^{2},
\]
where the sum is over all \( i \) such that \( \{j : F_{ij} \neq 0\} = \sigma \), and let \( \Omega \) be the collection of all nonempty \( \sigma \) for which there exists an \( i \) such that \( \{j : F_{ij} \neq 0\} = \sigma \). Then \((7)\) is equivalent to
\[
    \sum_{\sigma \in \Omega, v_{\sigma} \neq 0} \ell_{p}(v_{\sigma})^{p-2} b_{\sigma} \leq 1, \tag{8}
\]
which follows from Lemma \( 7 \) (note that the hypotheses for the lemma are satisfied because \( F \) is an isometry).

**Proof of Theorem 4.** Necessity was shown by Lemma 5. By Proposition 2, to show that \( C_{1,p} \) is a coherence measure it suffices to show that property (B3) holds. Let \( \{K_{1}, \ldots, K_{m}\} \) be a set of incoherent Kraus operators. Let \( \rho \in M_{n} \), \( p_{j} = \text{tr} K_{j}\rho K_{j}^{\dagger} \), and \( \rho_{\text{diag}} = \frac{1}{m} \sum_{j=1}^{m} K_{j}\rho K_{j}^{\dagger} \). Then
\[
    \sum_{j=1}^{m} p_{j} C_{1,p}(\rho_{j}) = \sum_{j=1}^{m} \ell_{1,p}(K_{j}\rho K_{j}^{\dagger}) \leq \sum_{j=1}^{m} \ell_{1,p}(\rho - \rho_{\text{diag}})K_{j}^{\dagger} \]
because \( K_{j}\mathcal{I} K_{j}^{\dagger} \) is diagonal for all \( j \). By Proposition 9
\[
    \sum_{j=1}^{m} \ell_{1,p}(K_{j}(\rho - \rho_{\text{diag}}))K_{j}^{\dagger} \leq \ell_{1,p}(\rho - \rho_{\text{diag}}) = C_{1,p}(\rho),
\]
so (B3) holds.

**CONCLUSION AND FURTHER RESEARCH**

In this article, we study coherence measures induced by norm functions. It is shown that no unitary similarity invariant norm induces a coherence measure; this generalizes the negative result for Schatten \( p \)-norms. On the other hand, the \( \ell_{q,p} \)-norm can induce a coherence measure, but if and only if \( q = 1 \) and \( 1 \leq p \leq 2 \). This provides a new class of potentially useful coherence measures. It would be interesting to extend our techniques to study quantum coherence in multipartite systems and related problems; [14, 19, 20, 24, 25, 36, 37].

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