On the distance $\alpha$-spectral radius of a connected graph

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Abstract

For a connected graph $G$ and $\alpha \in [0, 1)$, the distance $\alpha$-spectral radius of $G$ is the spectral radius of the matrix $D_\alpha(G)$ defined as $D_\alpha(G) = \alpha T(G) + (1 - \alpha)D(G)$, where $T(G)$ is a diagonal matrix of vertex transmissions of $G$ and $D(G)$ is the distance matrix of $G$. We give bounds for the distance $\alpha$-spectral radius, especially for graphs that are not transmission regular, propose local graft transformations that decrease or increase the distance $\alpha$-spectral radius, and determine the graphs that minimize and maximize the distance $\alpha$-spectral radius among several families of graphs.

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1 Introduction

We consider simple and undirected graphs. Let $G$ be a connected graph of order $n$ with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, the distance between $u$ and $v$ in $G$, denoted by $d_G(u, v)$ or simply $d_{uv}$ if the graph $G$ is clear from the context, is the length of a shortest path from $u$ to $v$ in $G$. The distance matrix of $G$ is the $n \times n$ matrix $D(G) = (d_G(u, v))_{u, v \in V(G)}$. For $u \in V(G)$, the transmission of $u$ in $G$, denoted by $T_G(u)$, is defined as the sum of distances from $u$ to all other vertices of $G$, i.e., $T_G(u) = \sum_{v \in V(G)} d_G(u, v)$. The transmission matrix $T(G)$ of $G$ is the diagonal matrix of transmissions of $G$. Then $Q(G) = T(G) + D(G)$ is the distance signless Laplacian matrix of $G$, proposed recently in [1]. Arisen from a data communication problem, the spectrum of the distance matrix was studied by Graham and Pollack [12] in 1971, early related work may be found also in [10, 11], and now it has been studied extensively, see the recent survey [2] and the very recent papers [4, 5, 17, 18, 26]. The distance signless Laplacian spectrum has also received much attention, see, e.g., [1, 3, 4, 7, 15, 16, 29].

Throughout this paper we assume that $\alpha \in [0, 1)$. Motivated by the work of Nikiforov [22], we consider the convex combinations $D_\alpha(G)$ of $T(G)$ and $D(G)$, defined as

$$D_\alpha(G) = \alpha T(G) + (1 - \alpha)D(G),$$

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see [6]. Evidently, \(D_0(G) = D(G)\) and \(2D_{1/2}(G) = Q(G)\). We call the eigenvalues of \(D_\alpha(G)\) the distance \(\alpha\)-eigenvalues of \(G\). As \(D_\alpha(G)\) is a symmetric matrix, the distance \(\alpha\)-eigenvalues of \(G\) are all real, which are denoted by \(\mu_\alpha^{[1]}(G), \ldots, \mu_\alpha^{[n]}(G)\), arranged in nonincreasing order, where \(n = |V(G)|\). The largest distance \(\alpha\)-eigenvalue \(\mu_\alpha^{[1]}(G)\) of \(G\) is called the distance \(\alpha\)-spectral radius of \(G\), written as \(\mu_\alpha(G)\). Obviously, \(\mu_0^{[1]}(G), \ldots, \mu_0^{[n]}(G)\) are the distance eigenvalues of \(G\), and \(2\mu_0^{[1]}(G), \ldots, 2\mu_0^{[n]}(G)\) are the distance signless Laplacian eigenvalues of \(G\). Particularly, \(\mu_0(G)\) is just the distance spectral radius [2] and \(2\mu_{1/2}(G)\) is just the distance signless Laplacian spectral radius of \(G\) [1].

In this paper, we give sharp bounds for the distance \(\alpha\)-spectral radius, and particularly an upper bound for the distance \(\alpha\)-spectral radius of connected graphs that are not transmission regular, and propose some types of graft transformations that decrease or increase the distance \(\alpha\)-spectral radius. We also determine the unique graphs with minimum distance \(\alpha\)-spectral radius among trees and unicyclic graphs, respectively, as well as the unique graphs (trees) with maximum and second maximum distance \(\alpha\)-spectral radii, and the unique graph with maximum distance \(\alpha\)-spectral radius among connected graphs with given clique number, and among odd-cycle unicyclic graphs, respectively.

2 Preliminaries

Let \(G\) be a connected graph with \(V(G) = \{v_1, \ldots, v_n\}\). A column vector \(x = (x_{v_1}, \ldots, x_{v_n})^\top \in \mathbb{R}^n\) can be considered as a function defined on \(V(G)\) which maps vertex \(v_i\) to \(x_{v_i}\), i.e., \(x(v_i) = x_{v_i}\) for \(i = 1, \ldots, n\). Then

\[
x^\top D_\alpha(G)x = \alpha \sum_{u \in V(G)} T_G(u)x_u^2 + 2 \sum_{\{u,v\} \subseteq V(G)} (1 - \alpha)d_G(u,v)x_ux_v,
\]

or equivalently,

\[
x^\top D_\alpha(G)x = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)(\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_ux_v).
\]

Since \(D_\alpha(G)\) is a nonnegative irreducible matrix, by the Perron–Frobenius theorem, \(\mu_\alpha(G)\) is simple and there is a unique positive unit eigenvector corresponding to \(\mu_\alpha(G)\), which is called the distance \(\alpha\)-Perron vector of \(G\). If \(x\) is the distance \(\alpha\)-Perron vector of \(G\), then for each \(u \in V(G)\),

\[
\mu_\alpha(G)x_u = \sum_{v \in V(G)} d_G(u,v)(\alpha x_v + (1 - \alpha)x_u),
\]

which is called the \(\alpha\)-equation of \(G\) at \(u\). For a unit column vector \(x \in \mathbb{R}^n\) with at least one nonnegative entry, by Rayleigh’s principle, we have \(\mu_\alpha(G) \geq x^\top D_\alpha(G)x\) with equality if and only if \(x\) is the distance \(\alpha\)-Perron vector of \(G\).

As in [27], we have the following result.

**Lemma 2.1** Suppose that \(G\) is a connected graph, \(\eta\) is an automorphism of \(G\), and \(x\) is the distance \(\alpha\)-Perron vector of \(G\). Then for \(u, v \in V(G)\), \(\eta(u) = v\) implies that \(x_u = x_v\).

**Proof** Let \(P = (p_{uv})_{u,v \in V(G)}\) be the permutation matrix such that \(p_{uu} = 1\) if and only if \(\eta(u) = v\) for \(u, v \in V(G)\). We have \(D_\alpha(G) = P^\top D_\alpha(G)P\) and \(Px\) is a positive unit vector. Thus
\(\mu_{\alpha}(G) = x^\top D_{\alpha}(G)x = (Px)^\top D_{\alpha}(G)(Px),\) implying \(Px\) is also the distance \(\alpha\)-Perron vector of \(G\). Thus \(Px = x\), and the result follows. \(\square\)

Let \(G\) be a graph. For \(v \in V(G)\), let \(N_G(v)\) be the set of neighbors of \(v\) in \(G\), and \(\deg_G(v)\) be the degree of \(v\) in \(G\). Let \(G - v\) be the subgraph of \(G\) obtained by deleting \(v\) and all edges containing \(v\). For \(S \subseteq V(G)\), let \(G[S]\) be the subgraph of \(G\) induced by \(S\). For a subset \(E'\) of \(E(G)\), \(G - E'\) denotes the graph obtained from \(G\) by deleting all the edges in \(E'\), and in particular, we write \(G - xy\) instead of \(G - \{xy\}\) if \(E' = \{xy\}\). Let \(\overline{G}\) be the complement of \(G\). For a subset \(E'\) of \(E(G)\), denote \(G + E'\) the graph obtained from \(G\) by adding all edges in \(E'\), and in particular, we write \(G + xy\) instead of \(G + \{xy\}\) if \(E' = \{xy\}\).

For a nonnegative square matrix \(A\), the Perron–Frobenius theorem implies that \(A\) has an eigenvalue that is equal to the maximum modulus of all its eigenvalues; this eigenvalue is called the spectral radius of \(A\), denoted by \(\rho(A)\). Note that \(\mu_{\alpha}(G) = \rho(D_{\alpha}(G))\) for a connected graph \(G\).

Restating Corollary 2.2 in [20, p. 38], we have

**Lemma 2.2** ([20]) Suppose that \(A\) and \(B\) are square nonnegative matrices, \(A\) is irreducible, and \(A - B\) is nonnegative but nonzero. Then \(\rho(A) > \rho(B)\).

By Lemma 2.2, we have

**Lemma 2.3** Suppose that \(G\) is a connected graph with \(u, v \in V(G)\), and \(u\) and \(v\) are not adjacent. Then \(\mu_{\alpha}(G + uv) < \mu_{\alpha}(G)\).

The transmission of a connected graph \(G\), denoted by \(\sigma(G)\), is the sum of distances between all unordered pairs of vertices in \(G\). Clearly, \(\sigma(G) = \frac{1}{2} \sum_{v \in V(G)} T_G(v)\). A graph is said to be transmission regular if \(T_G(v)\) is a constant for each \(v \in V(G)\). By Rayleigh’s principle, we have

**Lemma 2.4** Suppose that \(G\) is a connected graph of order \(n\). Then \(\mu_{\alpha}(G) \geq \frac{2\sigma(G)}{n}\) with equality if and only if \(G\) is transmission regular.

For an \(n \times n\) nonnegative matrix \(A = (a_{ij})\), let \(r_i\) be the \(i\)th row sum of \(A\), i.e., \(r_i = \sum_{j=1}^{n} a_{ij}\) for \(i = 1, \ldots, n\), and let \(r_{\min}\) and \(r_{\max}\) be the minimum and maximum row sums of \(A\), respectively.

**Lemma 2.5** ([3]) Let \(A = (a_{ij})\) be an \(n \times n\) nonnegative matrix with row sums \(r_1, \ldots, r_n\). Let \(S = \{1, \ldots, n\}\), \(r_{\min} = r_p, r_{\max} = r_q\) for some \(p\) and \(q\) with \(1 \leq p, q \leq n, \ell = \max\{r_i - a_{ip} : i \in S \setminus \{p\}\}, m = \min\{r_i - a_{iq} : i \in S \setminus \{q\}\}, s = \max\{a_{ip} : i \in S \setminus \{p\}\}\) and \(t = \min\{a_{iq} : i \in S \setminus \{q\}\}\). Then

\[
\frac{a_{qq} + m + \sqrt{(m - a_{qq})^2 + 4\ell(r_{\max} - a_{qq})}}{2} \leq \rho(A)
\]

\[
\leq a_{pp} + \ell + \sqrt{(\ell - a_{pp})^2 + 4s(r_{\min} - a_{pp})} \quad \frac{2}{2}.
\]
Moreover, the first equality holds if \( r_i - a_{iq} = m \) and \( a_{iq} = t \) for all \( i \in S \setminus \{ q \} \), and the second equality holds if \( r_i - a_{ip} = \ell \) and \( a_{ip} = s \) for all \( i \in S \setminus \{ p \} \).

Let \( f_{s \times t} \) be the \( s \times t \) matrix of all 1’s, \( I_{s \times t} \) the \( s \times t \) matrix of all 0’s, and \( I_s \) the identity matrix of order \( s \).

Let \( K_n \), \( P_n \), and \( S_n \) be the complete graph, the path, and the star of order \( n \), respectively. Let \( C_n \) denote the cycle of order \( n \geq 3 \).

For a connected graph \( G \), let \( T_{\text{min}}(G) \) and \( T_{\text{max}}(G) \) be the minimum and maximum transmissions of \( G \), respectively.

### 3 Bounds for the distance \( \alpha \)-spectral radius

Let \( G \) be a connected graph of order \( n \). Note that \( D_\alpha(K_n) = \alpha(n - 1)I_n + (1 - \alpha)(I_{n \times n} - I_n) \), and thus \( \mu_\alpha(K_n) = n - 1 \). By Lemma 2.3, we have \( \mu_\alpha(G) \geq n - 1 \) with equality if and only if \( G \cong K_n \).

If \( (d_1, \ldots, d_n) \) is the nonincreasing degree sequence of a graph \( G \) of order at least 2, then \( d_1 \) (resp. \( d_2 \)) is the maximum (resp. second maximum) degree, \( d_n \) (resp. \( d_{n-1} \)) is the minimum (resp. second minimum) degree of \( G \). The diameter of \( G \) is the maximum distance between all vertex pairs of \( G \). Using techniques from [33] by considering the first two minima or maxima of the entries of the distance \( \alpha \)-Perron vector, we may prove the following lower and upper bounds: If \( G \) is a connected graph of order \( n \geq 2 \) with maximum degree \( \Delta \) and second maximum degree \( \Delta' \), then

\[
\mu_\alpha(G) \geq \frac{1}{2} \left( \alpha(4n - 4 - \Delta - \Delta') \right) + \sqrt{\alpha^2(4n - 4 - \Delta - \Delta')^2 - 4(2\alpha - 1)(2n - 2 - \Delta)(2n - 2 - \Delta')} \]

with equality if and only if \( G \) is regular with diameter at most 2. If \( G \) is a connected graph of order \( n \geq 2 \) with minimum degree \( \delta \) and second minimum degree \( \delta' \), then

\[
\mu_\alpha(G) \leq \frac{1}{2} \left( \alpha(2dn - 2 - (d - 1)(d + \delta + \delta')) \right) + \sqrt{\alpha^2(2dn - 2 - (d - 1)(d + \delta + \delta'))^2 - 4(2\alpha - 1)SS'} \]

with equality if and only if \( G \) is regular with \( d \leq 2 \), where \( d \) is the diameter of \( G \), \( S = dn - \frac{d(d-1)}{2} - 1 - \delta(d - 1) \) and \( S' = dn - \frac{d(d-1)}{2} - 1 - \delta'(d - 1) \). The proof of the above bounds may be found in the early version of this paper at arXiv:1901.10180.

Similarly, bounds for the distance \( \alpha \)-spectral radius for connected bipartite graphs may be obtained as in [33].

A connected graph \( G \) of order \( n \) is distinguished vertex deleted regular (DVDR) if there is a vertex \( v \) of degree \( n - 1 \) such that \( G - v \) is regular. By the techniques in [3], we have the following bounds. For completeness, we include a proof here.

**Theorem 3.1** Let \( G \) be a connected graph and \( u \) and \( v \) be vertices such that \( T_G(u) = T_{\text{min}}(G) \) and \( T_G(v) = T_{\text{max}}(G) \). Let \( m_1 = \max \{ T_G(w) - (1 - \alpha)d(u, w) : w \in V(G) \setminus \{ u \} \} \), \( m_2 = \min \{ T_G(w) - (1 - \alpha)d(v, w) : w \in V(G) \setminus \{ v \} \} \), and \( e(w) = \max \{ d(w, z) : z \in V(G) \} \) for
\[ w \in V(G). \] Then
\[
\frac{m_2 + \alpha T_{\text{max}}(G) + \sqrt{(m_2 - \alpha T_{\text{max}}(G))^2 + 4(1 - \alpha)^2 T_{\text{max}}(G)}}{2} \leq \mu_{\alpha}(G)
\]
\[
\leq \frac{m_1 + \alpha T_{\text{min}}(G) + \sqrt{(m_1 - \alpha T_{\text{min}}(G))^2 + 4(1 - \alpha)^2 e(u) T_{\text{min}}(G)}}{2}.
\]

The first equality holds if and only if \( G \) is a complete graph and the second equality holds if and only if \( G \) is a DVDR graph.

**Proof** Let \( M \) be the submatrix of \( D_{\alpha}(G) \) obtained by deleting the row and column corresponding to vertex \( v \). Let \( M' \) be the matrix obtained from \( M \) by reducing some nondiagonal entries of each row with row sum greater than \( m_2 \) in \( M \) such that \( M' \) is nonnegative and each row sum in \( M' \) is \( m_2 \).

Let \( D^{(1)} \) be the matrix obtained from \( D_{\alpha}(G) \) by replacing all \((w,v)\)-entries by \( 1 - \alpha \) for \( w \in V(G) \setminus \{v\} \), and replacing the submatrix \( M \) by \( M' \). Obviously, \( D_{\alpha}(G) \) and \( D^{(1)} \) are nonnegative and irreducible, and \( D_{\alpha}(G) \geq D^{(1)} \). By Lemma 2.2, we have \( \mu_{\alpha}(G) \geq \rho(D^{(1)}) \) with equality if and only if \( D_{\alpha}(G) = D^{(1)} \). By applying Lemma 2.5 to \( D^{(1)} \), we obtain the lower bound for \( \mu_{\alpha}(G) \). Suppose that this lower bound is attained. Then \( D_{\alpha}(G) = D^{(1)} \).

As all \((w,v)\)-entries are equal to \( 1 - \alpha \) for \( w \in V(G) \setminus \{v\} \), implying \( \deg_G(v) = n - 1 \). As \( T_G(v) = T_{\text{max}}(G) \), \( G \) is a complete graph. Conversely, if \( G \) is a complete graph, then it is obvious that the lower bound for \( \mu_{\alpha}(G) \) is attained.

Let \( C \) be the submatrix of \( D_{\alpha}(G) \) obtained by deleting the row and column corresponding to vertex \( u \). Let \( C' \) be the matrix obtained from \( C \) by adding positive numbers to nondiagonal entries of each row with row sum less than \( m_1 \) in \( C \) such that each row sum in \( C' \) is \( m_1 \).

Let \( D^{(2)} \) be the matrix obtained from \( D_{\alpha}(G) \) by replacing all \((w,u)\)-entries by \((1 - \alpha)e(u)\) for \( w \in V(G) \setminus \{u\} \), and replacing the submatrix \( C \) by \( C' \). Note that \( D_{\alpha}(G) \) and \( D^{(2)} \) are nonnegative and irreducible, and \( D^{(2)} \geq D_{\alpha}(G) \). By Lemma 2.2, \( \mu_{\alpha}(G) \leq \rho(D^{(2)}) \) with equality if and only if \( D_{\alpha}(G) = D^{(2)} \). By applying Lemma 2.5 to \( D^{(2)} \), we obtain the upper bound for \( \mu_{\alpha}(G) \).

Suppose that this upper bound is attained. By Lemma 2.2, \( D_{\alpha}(G) = D^{(2)} \). As all \((w,u)\)-entries are equal to \((1 - \alpha)e(u)\) for \( w \in V(G) \setminus \{u\} \), implying \( e(u) = 1 \), i.e., \( \deg_G(u) = n - 1 \). Note that \( T_G(w) = m_1 + 1 - \alpha \) for all \( w \in V(G) \setminus \{u\} \) and \( T_{\text{min}}(G) = T_G(u) = n - 1 \). If \( m_1 + 1 - \alpha = n - 1 \), then \( G \) is a complete graph, which is a DVDR graph. Otherwise, \( m_1 + 1 - \alpha > n - 1 \).

Recall from [3] that an incomplete connected graph of order \( n \) is a DVDR graph if and only if except one vertex of degree \( n - 1 \) each other vertex has the same transmission. Thus, the upper bound for \( \mu_{\alpha}(G) \) is attained if and only if \( G \) is a DVDR graph.

We mention that more bounds for \( \mu_{\alpha}(G) \) may be derived even from some known bounds for nonnegative matrices, see, e.g., [9].

Let \( G \) be a connected graph of order \( n \). Let \( \Lambda = T_{\text{max}}(G) \). As \( \mu_{\alpha}(G) \leq \Lambda \) with equality if and only if \( G \) is transmission regular. For a connected non-transmission-regular graph \( G \) of order \( n \), Liu et al. [19] showed that
\[
\mu_{\alpha}(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(n\Lambda - 2\sigma(G) + 1)n}.
\]
and

$$\mu_{1/2}(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(2(n\Lambda - 2\sigma(G)) + 1)n},$$

Note that $4\sigma(G) < n^2\Lambda$. We show new bounds as follows:

$$\mu_0(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(n\Lambda - 2\sigma(G))\frac{\sigma(G)}{n\Lambda} + n}$$

and

$$\mu_{1/2}(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(n\Lambda - 2\sigma(G))\frac{\sigma(G)}{n\Lambda} + n}.$$ 

Instead of proving the two inequalities, we prove the following somewhat general result.

**Theorem 3.2** Let $G$ be a connected non-transmission-regular graph of order $n$. Then

$$\mu_\alpha(G) < \Lambda - \frac{(1 - \alpha)n\Lambda(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)n^2\Lambda},$$

where $\Lambda = T_{\text{max}}(G)$.

**Proof** Let $x$ be the $\alpha$-Perron vector of $G$. Denote by $x_u = \max\{x_w : w \in V(G)\}$ and $x_v = \min\{x_w : w \in V(G)\}$. Since $G$ is not transmission regular, we have $x_u > x_v$, and thus

$$\mu_\alpha(G) = x^TD_\alpha(G)x$$

$$= \alpha \sum_{w \in V(G)} T_G(w)x_w^2 + 2(1 - \alpha) \sum_{\{w,z\} \subseteq V(G)} d_{wz}x_wx_z$$

$$< 2\alpha\sigma(G)x_u^2 + 2(1 - \alpha)\sigma(G)x_v^2,$$

implying that $x_u^2 > \frac{\mu_\alpha(G)}{2\sigma(G)}$. Note that

$$\Lambda - \mu_\alpha(G)$$

$$= \Lambda - \alpha \sum_{w \in V(G)} T_G(w)x_w^2 - 2(1 - \alpha) \sum_{\{w,z\} \subseteq V(G)} d_{wz}x_wx_z$$

$$= \sum_{w \in V(G)} (\Lambda - T_G(w))x_w^2 + (1 - \alpha) \sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2$$

$$\geq \sum_{w \in V(G)} (\Lambda - T_G(w))x_v^2 + (1 - \alpha) \sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2$$

$$= (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha) \sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2.$$

We need to estimate $\sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2$. Let $P = w_0w_1 \ldots w_\ell$ be a shortest path connecting $u$ and $v$, where $w_0 = u$, $w_\ell = v$, and $\ell \geq 1$. Obviously,

$$\sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2 \geq N_1 + N_2,$$
where $N_1 = \sum_{w \in V(G) \setminus V(P)} \sum_{x \in V(P)} d_{w,c}(x_w - x_c)^2$ and $N_2 = \sum_{\{w,z\} \subseteq V(P)} d_{w,c}(x_w - x_z)^2$. For $w \in V(G) \setminus V(P)$, by the Cauchy–Schwarz inequality, we have

$$d_{w,u}(x_w - x_u)^2 + d_{w,v}(x_w - x_v)^2 \geq (x_w - x_u)^2 + (x_w - x_v)^2 \geq \frac{1}{2} (x_u - x_v)^2,$$

and thus

$$N_1 \geq \sum_{w \in V(G) \setminus V(P)} (d_{w,u}(x_w - x_u)^2 + d_{w,v}(x_w - x_v)^2) \geq \sum_{w \in V(G) \setminus V(P)} \frac{1}{2} (x_u - x_v)^2 = \frac{n - \ell - 1}{2} (x_u - x_v)^2.$$

For $1 \leq i \leq \ell - 1$ and $\ell \geq 2$, by the Cauchy–Schwarz inequality, we have

$$d_{w_0,w_i}(x_{w_0} - x_{w_i})^2 + d_{w_i,w_\ell}(x_{w_i} - x_{w_\ell})^2 \geq \min\{i, \ell - i\} ((x_{w_0} - x_{w_i})^2 + (x_{w_i} - x_{w_\ell})^2) \geq \min\{i, \ell - i\} \cdot \frac{1}{2} (x_{w_0} - x_{w_i})^2 \geq \frac{1}{2} \min\{i, \ell - i\} (x_u - x_v)^2,$$

and thus

$$N_2 \geq d_{u,v}(x_u - x_v)^2 + \sum_{i=1}^{\ell-1} (d_{w_0,w_i}(x_{w_0} - x_{w_i})^2 + d_{w_i,w_\ell}(x_{w_i} - x_{w_\ell})^2) \geq \ell (x_u - x_v)^2 + \sum_{i=1}^{\ell-1} \frac{1}{2} \min\{i, \ell - i\} (x_u - x_v)^2 = \left(\ell + \frac{1}{2} \sum_{i=1}^{\ell-1} \min\{i, \ell - i\}\right) (x_u - x_v)^2 = \begin{cases} \ell^2 + 8(\ell + 1) (x_u - x_v)^2 & \text{if } \ell \text{ is even,} \\ \ell^2 + 8(\ell - 1) (x_u - x_v)^2 & \text{if } \ell \text{ is odd.} \end{cases}$$

Case 1. $u$ and $v$ are adjacent, i.e., $\ell = 1$.

In this case, we have

$$\sum_{\{w,z\} \subseteq V(G)} d_{w,c}(x_w - x_z)^2 \geq N_1 + N_2 \geq \frac{n - 1}{2} (x_u - x_v)^2 + (x_u - x_v)^2 = \frac{n}{2} (x_u - x_v)^2.$$
Thus
\[ \Lambda - \mu_\alpha(G) \geq (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha) \sum_{[w,z] \subseteq V(G)} d_wz(x_w - x_z)^2 \]
\[ \geq (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{\mu_\alpha(G)}{2}(x_u - x_v)^2. \]

Viewed as a function of \( x_v \), \( (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{\mu_\alpha(G)}{2}(x_u - x_v)^2 \) achieves its minimum value when \( x_v > \frac{\mu_\alpha(G)}{2\sigma(G)} \). Then we have
\[ \Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)n(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)n}. \]

which implies that
\[ \Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)n\Lambda(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)n^2\Lambda}. \]

Case 2. \( u \) and \( v \) are not adjacent, i.e., \( \ell \geq 2 \).

Suppose first that \( \ell \) is even. Then
\[ \sum_{[w,z] \subseteq V(G)} d_wz(x_w - x_z)^2 \geq N_1 + N_2 \]
\[ \geq \frac{n - \ell - 1}{2}(x_u - x_v)^2 + \frac{\ell^2 + 8\ell}{8}(x_u - x_v)^2 \]
\[ = \frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2. \]

Thus
\[ \Lambda - \mu_\alpha(G) \geq (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha) \sum_{[w,z] \subseteq V(G)} d_wz(x_w - x_z)^2 \]
\[ \geq (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2. \]

Viewed as a function of \( x_v \), \( (n\Lambda - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2 \) achieves its minimum value when \( x_v > \frac{\mu_\alpha(G)}{2\sigma(G)} \). As \( x_v \geq \frac{\mu_\alpha(G)}{2\sigma(G)} \), we have
\[ \Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)(n\Lambda - 2\sigma(G))((\ell^2 + 4\ell + 4n - 4)\frac{\alpha}{8})}{\alpha\Lambda(n\Lambda - 2\sigma(G)) + (1 - \alpha)(\ell^2 + 4\ell + 4n - 4)n\Lambda}. \]

i.e.,
\[ \Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)(n\Lambda - 2\sigma(G))((\ell^2 + 4\ell + 4n - 4)\Lambda)}{16\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)(\ell^2 + 4\ell + 4n - 4)n\Lambda}. \]
As a function of \( \ell \), the expression on the right-hand side in the above inequality is strictly increasing for \( \ell \geq 2 \). Thus we have

\[
\Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)(n\Lambda - 2\sigma(G))(n + 2)\Lambda}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)(n + 2)n\Lambda} \\
> \frac{(1 - \alpha)n\Lambda(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)n^2\Lambda}.
\]

Now suppose that \( \ell \) is odd. Then

\[
\sum_{[w,z] \subseteq V(G)} d_{wz}(x_u - x_v)^2 \\
\geq N_1 + N_2 \\
\geq \frac{n - \ell - 1}{2}(x_u - x_v)^2 + \frac{\ell^2 + 8\ell - 1}{8}(x_u - x_v)^2 \\
= \frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2.
\]

Thus, as early, we have

\[
\Lambda - \mu_\alpha(G) \\
\geq (n\Lambda - 2\sigma(G))\lambda_0^2 + (1 - \alpha)\frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2 \\
\geq \frac{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5)(n\Lambda - 2\sigma(G))}{8(n\Lambda - 2\sigma(G)) + (1 - \alpha)(\ell^2 + 4\ell + 4n - 5)}\lambda_0^2 \\
> \frac{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5)(n\Lambda - 2\sigma(G))}{8(n\Lambda - 2\sigma(G)) + (1 - \alpha)(\ell^2 + 4\ell + 4n - 5)}\mu_\alpha(G) \\
\geq (1 - \alpha)(\ell^2 + 4\ell + 4n - 5)(n\Lambda - 2\sigma(G)) + (1 - \alpha)(\ell^2 + 4\ell + 4n - 5)\cdot 2\sigma(G),
\]

implying

\[
\Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)(n\Lambda - 2\sigma(G))(\ell^2 + 4\ell + 4n - 5)\Lambda}{16\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)(\ell^2 + 4\ell + 4n - 5)n\Lambda}.
\]

As a function of \( \ell \), the expression on the right-hand side in the above inequality is strictly increasing for \( \ell \geq 3 \). Thus we have

\[
\Lambda - \mu_\alpha(G) > \frac{(1 - \alpha)(n\Lambda - 2\sigma(G))(4 + n)n\Lambda}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)(4 + n)n\Lambda} \\
> \frac{(1 - \alpha)n\Lambda(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1 - \alpha)n^2\Lambda}.
\]

The result follows by combining Cases 1 and 2. \( \square \)

4 Effect of graft transformations on distance \( \alpha \)-spectral radius

In this section, we study the effect of some local graft transformations on distance \( \alpha \)-spectral radius.
A path \( u_0 \cdots u_r \) (with \( r \geq 1 \)) in a graph \( G \) is called a pendant path (of length \( r \)) at \( u_0 \) if \( \deg_G(u_0) \geq 3 \), the degrees of \( u_1, \ldots, u_{r-1} \) (if any exists) are all equal to 2 in \( G \), and \( \deg_G(u_r) = 1 \). A pendant path of length 1 at \( u_0 \) is called a pendant edge at \( u_0 \).

A vertex of a graph is a pendant vertex if its degree is 1. A cut edge of a connected graph is an edge whose removal yields a disconnected graph.

If \( P \) is a pendant path of \( G \) at \( u \) with length \( r \geq 1 \), then we say \( G \) is obtained from \( H \) by attaching a pendant path \( P \) of length \( r \) at \( u \) with \( H = G[V(G) \setminus (V(P) \setminus \{u\})] \). If the pendant path of length 1 is attached to a vertex \( u \) of \( H \), then we also say that a pendant vertex is attached to \( u \).

**Theorem 4.1** Suppose that \( G \) is a connected graph, \( uv \) is a cut edge with \( \deg_G(u) \geq 2 \), and \( v \) is adjacent to a pendant vertex \( v' \). Let

\[
G_{uv} = G - \{ uw : w \in N_G(u) \setminus \{v\} \} + \{ vw : w \in N_G(u) \setminus \{v\} \}.
\]

Then \( \mu_{\alpha}(G) > \mu_{\alpha}(G_{uv}) \).

**Proof** Let \( G_1 \) and \( G_2 \) be the components of \( G - uv \) containing \( u \) and \( v \), respectively. Let \( x \) be the distance \( \alpha \)-Perron vector of \( G_{uv} \). By Lemma 2.1, \( x_u = x_v \). As we pass from \( G \) to \( G_{uv} \), the distance between a vertex in \( V(G_1) \setminus \{u\} \) and a vertex in \( V(G_2) \) is decreased by 1, the distance between a vertex \( V(G_1) \setminus \{u\} \) and \( u \) is increased by 1, and the distances between all other vertex pairs remain unchanged. Thus

\[
\mu_{\alpha}(G) - \mu_{\alpha}(G_{uv})
\geq x^T(D_{G_{uv}}(G) - D_{G_{uv}}(G_{uv}))x
= \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V(G_2)} (\alpha(x^2_w + x^2_z) + 2(1 - \alpha)x_w x_z)
- \sum_{w \in V(G_1) \setminus \{u\}} (\alpha(x^2_w + x^2_z) + 2(1 - \alpha)x_w x_z)
+ \sum_{w \in V(G_1) \setminus \{u\}} (\alpha(x^2_w + x^2_z) + 2(1 - \alpha)x_w x_z)
- \sum_{w \in V(G_1) \setminus \{u\}} (\alpha(x^2_w + x^2_z) + 2(1 - \alpha)x_w x_z)
= \sum_{w \in V(G_1) \setminus \{u\}} (\alpha(x^2_w + x^2_z) + 2(1 - \alpha)x_w x_z)
> 0,
\]

implying \( \mu_{\alpha}(G) - \mu_{\alpha}(G_{uv}) > 0 \), i.e., \( \mu_{\alpha}(G) > \mu_{\alpha}(G_{uv}) \). \( \square \)

The previous theorem has been established for \( \alpha = 0, \frac{1}{2} \) in [16, 25].

**Theorem 4.2** Suppose that \( G \) is a connected graph with \( k \) edge-disjoint nontrivial induced subgraphs \( G_1, \ldots, G_k \) such that \( V(G_i) \cap V(G_j) = \{u\} \) for \( 1 \leq i < j \leq k \) and \( \bigcup_{i=1}^{k} V(G_i) = V(G) \),
where \( k \geq 3 \). Let \( \emptyset \neq K \subseteq \{3, \ldots, k\} \) and let \( N_K = \bigcup_{i \in K} N_G(u) \). For \( v' \in V(G_1) \setminus \{u\} \) and \( v'' \in V(G_2) \setminus \{u\} \), let

\[
G' = G - \{uw : w \in N_K\} \cup \{v'w : w \in N_K\}
\]

and

\[
G'' = G - \{uw : w \in N_K\} \cup \{v''w : w \in N_K\}.
\]

Then \( \mu_\alpha(G) < \max\{\mu_\alpha(G'), \mu_\alpha(G'')\} \).

**Proof.** Let \( x \) be the distance \( \alpha \)-Perron vector of \( G \). Let \( V_K = (\bigcup_{i \in K} V(G_i)) \setminus \{u\} \). Let

\[
\Gamma = \sum_{w \in V(G_2) \setminus \{u\}} \sum_{z \in V_K} \left( \alpha(x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z \right)
- \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} \left( \alpha(x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z \right).
\]

As we pass from \( G \) to \( G' \), the distance between a vertex in \( V(G_2) \) and a vertex in \( V_K \) is increased by \( d_G(u, v') \), the distance between a vertex \( w \) in \( V(G_1) \setminus \{u\} \) and a vertex in \( V_K \) is decreased by \( d_G(w, u) - d_G(w, v') \), which is at most \( d_G(u, v') \), and the distances between all other vertex pairs are increased or remain unchanged. Thus

\[
\mu_\alpha(G') - \mu_\alpha(G)
\geq x^\top(D_\alpha(G') - D_\alpha(G))x
\geq \sum_{w \in V(G_2) \setminus \{u\}} \sum_{z \in V_K} \left( d_G(u, v')(\alpha(x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z) \right)
- \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} \left( d_G(u, v')(\alpha(x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z) \right)
= d_G(u, v') \left( \Gamma + \sum_{z \in V_K} (\alpha(x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z) \right)
> d_G(u, v') \Gamma.
\]

If \( \Gamma \geq 0 \), then \( \mu_\alpha(G') - \mu_\alpha(G) > d_G(u, v') \Gamma \geq 0 \), implying \( \mu_\alpha(G) < \mu_\alpha(G') \). Suppose that \( \Gamma < 0 \). As we pass from \( G \) to \( G'' \), the distance between a vertex in \( V(G_1) \) and a vertex in \( V_K \) is increased by \( d_G(u, v') \), the distance between a vertex \( w \) in \( V(G_2) \setminus \{u\} \) and a vertex in \( V_K \) is decreased by \( d_G(w, u) - d_G(w, v') \), which is at most \( d_G(u, v') \), and the distances between all other vertex pairs are increased or remain unchanged. Thus

\[
\mu_\alpha(G'') - \mu_\alpha(G)
\geq x^\top(D_\alpha(G'') - D_\alpha(G))x
\geq \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} \left( d_G(u, v')(\alpha(x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z) \right)
\]
Let $H$ be a nontrivial connected graph with $u \in V(H)$. If $p \geq q \geq 1$, then
\[
\mu_\alpha(H_a(p, q)) < \mu_\alpha(H_a(p + 1, q - 1)).
\]

**Proof.** Let $G = H_a(p, q)$. Let $P = uu_1 \cdots u_p$ and $Q = vv_1 \cdots v_q$ be two pendant paths of lengths $p$ and $q$, respectively, in $G$. Using the notations in Theorem 4.2 with $k = 3, G_1 = P$, $G_2 = Q, G_3 = H$, $v' = u_{p-1}$ and $v'' = v_1$, we have $G' \cong G'' \cong H_a(p + 1, q - 1)$, and thus by Theorem 4.2, we have $\mu_\alpha(H_a(p, q)) < \mu_\alpha(H_a(p + 1, q - 1)).$ \hfill $\square$

**Theorem 4.3** Suppose that $G$ is a connected graph with three edge-disjoint induced subgraphs $G_1, G_2$ and $G_3$ such that $V(G_1) \cap V(G_3) = \{u\}, V(G_2) \cap V(G_3) = \{v\}, \bigcup_{i=1}^3 V(G_i) = V(G)$, and $G_1 - u, G_2 - v$ and $G_3 - u - v$ are all nontrivial. Suppose that $uv \in E(G_3)$. For $u' \in N_{G_1}(u)$ and $v' \in N_{G_2}(v)$, let
\[
G' = H + \{u'w : w \in N_{G_1}(u)\} + \{uw : w \in N_{G_3}(v)\}
\]
and
\[
G'' = H + \{vw : w \in N_{G_3}(u)\} + \{v'w : w \in N_{G_3}(v)\},
\]
where $H = G - \{uv : w \in N_{G_3}(u)\} - \{vw : w \in N_{G_3}(v)\}$. Then $\mu_\alpha(G) < \mu_\alpha(G')$ or $\mu_\alpha(G) < \mu_\alpha(G'')$.

**Proof.** Let $x$ be the distance $\alpha$-Perron vector of $G$. Let
\[
\Gamma = \sum_{w \in V(G_2)} \sum_{z \in V(G_3) \setminus \{u, v\}} (\alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z)
\]
\[
- \sum_{w \in V(G_1)} \sum_{z \in V(G_3) \setminus \{u, v\}} (\alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z).
\]

implying $\mu_\alpha(G'') - \mu_\alpha(G) > 0$, i.e., $\mu_\alpha(G) < \mu_\alpha(G'')$. \hfill $\square$
As we pass from $G$ to $G'$, the distance between a vertex in $V(G_2)$ and a vertex in $V(G_3) \setminus \{u, v\}$ is increased by 1, the distance between a vertex in $V(G_1)$ and a vertex in $V(G_3) \setminus \{u, v\}$ may be increased, unchanged, or decreased by 1, and the distances between any other vertex pairs remain unchanged. Thus

$$
\mu_\alpha(G') - \mu_\alpha(G) \geq x^T (D_\alpha(G') - D_\alpha(G))x
\geq \sum_{v \in V(G_2)} \sum_{z \in V(G_1) \setminus \{u,v\}} (\alpha (x_v^2 + x_z^2) + 2(1 - \alpha)x_v x_z)
- \sum_{u \in V(G_1)} \sum_{v \in V(G_2) \setminus \{u,v\}} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)
= \Gamma.
$$

If $\Gamma \geq 0$, then $\mu_\alpha(G') - \mu_\alpha(G) \geq 0$, i.e., $\mu_\alpha(G) \leq \mu_\alpha(G')$. If $\mu_\alpha(G) = \mu_\alpha(G')$, then $\mu_\alpha(G') = x^T D_\alpha(G')x$, implying $x$ is the distance $\alpha$-Perron vector of $G'$. By the $\alpha$-equations of $G$ and $G'$ at $v$, we have

$$
0 = \mu_\alpha(G')x_v - \mu_\alpha(G)x_v
= \sum_{u \in V(G_1) \setminus \{u,v\}} (d_G(v, u) - d_G(v, w))(\alpha x_v + (1 - \alpha)x_u)
= \sum_{u \in V(G_1) \setminus \{u,v\}} (\alpha x_v + (1 - \alpha)x_u)
> 0,
$$
a contradiction. Thus, if $\Gamma \geq 0$, then $\mu_\alpha(G) < \mu_\alpha(G')$.

Suppose that $\Gamma < 0$. As earlier, we have

$$
\mu_\alpha(G'') - \mu_\alpha(G) \geq x^T (D_\alpha(G'') - D_\alpha(G))x
\geq \sum_{u \in V(G_1)} \sum_{v \in V(G_2) \setminus \{u,v\}} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)
- \sum_{v \in V(G_2)} \sum_{u \in V(G_1) \setminus \{u,v\}} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)
= -\Gamma
> 0,
$$
and thus $\mu_\alpha(G) < \mu_\alpha(G'')$.

A weak version of previous theorem for $\alpha = \frac{1}{2}$ has been established in [16].

For nonnegative integers $p, q$ and a graph $G$ with $u, v \in V(G)$, let $G_{\alpha, \eta}(p, q)$ be the graph $H(v, q)$ with $H = G(u, p)$. The following corollary has been known for $\alpha = 0, \frac{1}{2}$ in [15, 32].

**Corollary 4.2** Let $H$ be a connected graph of order at least 3 with $uv \in E(H)$. Suppose that $\eta(u) = v$ for some automorphism $\eta$ of $G$. For $p \geq q \geq 1$, we have $\mu_\alpha(G_{\alpha, \eta}(p, q)) < \mu_\alpha(G_{\alpha, \eta}(p + 1, q - 1))$. 

Proof Let $G = H_u,v(p,q)$. Let $P = uu_1 \cdots u_p$ and $Q = vv_1 \cdots v_q$ be two pendant paths of lengths $p$ and $q$ in $G$ at $u$ and $v$, respectively. Using the notations of Theorem 4.3 with $G_1 = P$, $G_2 = Q$, $G_3 = H$, $u' = u_1$ and $v' = v_1$, we have $G' \cong H_{u,v}(p - 1, q + 1)$ and $G'' \cong H_{u,v}(p + 1, q - 1)$, and thus by Theorem 4.3, we have $\mu_a(H_{u,v}(p,q)) < \max\{\mu_a(H_{u,v}(p - 1, q + 1)), \mu_a(H_{u,v}(p + 1, q - 1))\}$. If $p = q = p = q + 1$, respectively, then $H_{u,v}(p - 1, q + 1) \cong H_{u,v}(p + 1, q - 1) (H_{u,v}(p,q) \cong H_{u,v}(p - 1, q + 1)$, respectively) as $\eta(u) = v$ for some automorphism $\eta$ of $G$, and thus from the above inequality, we have $\mu_a(G) < \mu_a(H_{u,v}(p + 1, q - 1))$.

Suppose that $p \geq q + 2$ and $\mu_a(G) < \mu_a(H_{u,v}(p - 1, q + 1))$. If $p \equiv q \pmod{2}$, then we have

$$\begin{align*}
\mu_a(G) & \leq \mu_a(H_{u,v}(p + q + 3, p + q - 3)) \\
& < \mu_a(H_{u,v}(p + q + 1, p + q - 1)) \\
& < \mu_a(H_{u,v}(p + q + 3, p + q - 3)),
\end{align*}$$

which is impossible. If $p \equiv q \pmod{2}$, then we have

$$\begin{align*}
\mu_a(G) & \leq \mu_a(H_{u,v}(p + q + 1, p + q - 1)) \\
& < \mu_a(H_{u,v}(p + q + 1, p + q - 1)) \\
& < \mu_a(H_{u,v}(p + q + 3, p + q - 3)),
\end{align*}$$

which is also impossible. Therefore $\mu_a(H_{u,v}(p,q)) < \mu_a(H_{u,v}(p - 1, q + 1))$. $\square$

5 Graphs with small or large distance $\alpha$-spectral radius

First we determine the graphs with minimum distance $\alpha$-spectral radius among trees and unicyclic graphs.

**Theorem 5.1** Let $G$ be a tree of order $n$. Then $\mu_a(G) \geq \mu_a(S_n)$ with equality if and only if $G \cong S_n$.

**Proof** The result is trivial if $n = 1, 2, 3$. Suppose that $n \geq 4$. Let $G$ be a tree of order $n$ such that $\mu_a(G)$ is as small as possible. Let $d$ be the diameter of $G$. Evidently, $d \geq 2$. Suppose that $d \geq 3$. Let $v_0 v_1 \cdots v_d$ be a diametral path of $G$. By Theorem 4.1, $\mu_a(G_{v_1 v_2}) < \mu_a(G)$, a contradiction. Thus $d = 2$, i.e., $G \cong S_n$. $\square$

In Theorem 5.1, the case $\alpha = 0$ has been known in [24] and the case $\alpha = \frac{1}{2}$ has been known in [16, 29].

For $n \geq 3$ and $1 \leq a \leq \lfloor \frac{n^2}{2} \rfloor$, let $D_{n,a}$ be the tree obtained from vertex-disjoint $S_{a+1}$ with center $u$ and $S_{n-a-1}$ with center $v$ by adding an edge $uv$. Let $T$ be a tree of order $n$ with minimum distance $\alpha$-spectral radius, where $T \cong S_n$. Let $d$ be the diameter of $T$. Then $d \geq 3$. Suppose that $d \geq 4$. Let $v_0 v_1 \cdots v_d$ be a diametral path of $T$. Note that $T_{v_1 v_2} \not\cong S_n$. By Theorem 4.1, $\mu_a(T_{v_1 v_2}) < \mu_a(T)$, a contradiction. Thus $d = 3$, implying $T \cong D_{n,a}$ for some $a$ with $1 \leq a \leq \lfloor \frac{n^2}{2} \rfloor$. 
Let $S^*_n$ is the graph obtained from $S_n$ by adding an edge between two vertices of degree one.

**Lemma 5.1** ([29]) Let $G$ be a unicyclic graph of order $n \geq 6$. If $G \not\cong S^*_n$, then

$$\sigma(G) \geq n^2 - n - 4 > \sigma(S^*_n) = n^2 - 2n.$$  

Note that for $n = 5$, we have $\sigma(C_5) = \sigma(S^*_5)$. So, in the above lemma, the condition $n \geq 6$ is necessary.

**Theorem 5.2** Let $G$ be a unicyclic graph of order $n \geq 8$. Then $\mu_\alpha(G) \geq \mu_\alpha(S^*_n)$ with equality if and only if $G \cong S^*_n$.

**Proof** Suppose that $G \not\cong S^*_n$. We only need to show that $\mu_\alpha(G) > \mu_\alpha(S^*_n)$.

By Lemmas 2.4 and 5.1, we have

$$\mu_\alpha(G) \geq \frac{2\sigma(G)}{n} \geq \frac{2(n^2 - n - 4)}{n}.$$ 

By [20, p. 24, Theorem 1.1] or by Theorem 3.2, we have

$$\mu_\alpha(S^*_n) < T_{\max}(S^*_n) = 2n - 3.$$ 

Since $n \geq 8$, we have

$$\mu_\alpha(G) \geq \frac{2(n^2 - n - 4)}{n} \geq 2n - 3 > \mu_\alpha(S^*_n),$$

as desired. \hfill \Box

The result in Theorem 5.2 for $\alpha = 0, \frac{1}{2}$ has been known in [29, 31].

In the following, we determine the graphs with maximum distance $\alpha$-spectral radius among some classes of graphs.

For $2 \leq \Delta \leq n - 1$, let $B_{n,\Delta}$ be a tree obtained by attaching $\Delta - 1$ pendant vertices to a terminal vertex of the path $P_{n-\Delta+1}$. In particular, $B_{n,2} = P_n$ and $B_{n,n-1} = S_n$. The following theorem for $\alpha = 0, \frac{1}{2}$ was given in [16, 24] for trees.

**Theorem 5.3** Let $G$ be a connected graph of order $n$ with maximum degree $\Delta$, where $2 \leq \Delta \leq n - 1$. Then $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta})$ with equality if and only if $G \cong B_{n,\Delta}$.

**Proof** Let $G$ be a graph among connected graphs of order $n$ with maximum degree $\Delta$ such that $\mu_\alpha(G)$ is as large as possible. Then $G$ has a spanning tree $T$ with maximum degree $\Delta$.

By Lemma 2.3, $\mu_\alpha(G) \leq \mu_\alpha(T)$ with equality if and only if $G \cong T$. Thus $G$ is a tree.

The result is trivial if $n = 3, 4$ and if $\Delta = 2, n - 1$. Suppose that $3 \leq \Delta \leq n - 2$. We only need to show that $G \cong B_{n,\Delta}$.

Let $u \in V(G)$ with $\deg_G(u) = \Delta$. Suppose that there exists a vertex different from $u$ with degree at least 3. Then we may choose such a vertex $w$ of degree at least 3 such that $d_G(u, w)$ is as large as possible. Obviously, there are two pendant paths, say $P$ and $Q$, at $w$ of lengths at least 1. Let $p$ and $q$ be the lengths of $P$ and $Q$, respectively. Assume that $p \geq q$. Let
Suppose that $\mu_\alpha$ is unchanged. Thus $\mu_\alpha$ is decreased by 1, and the distances between all other vertex pairs are increased or remain unchanged.

Proof

First suppose that $G$ is a tree. If $n = 4$, then the result follows from Theorem 4.1. Suppose that $n \geq 5$. Let $\Delta$ be the maximum degree of $G$. Since $G \not\cong P_n$, we have $\Delta \geq 3$.

By Theorem 5.3, $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta})$ with equality if and only if $G \cong B_{n,\Delta}$. By Corollary 4.1, $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta}) \leq \mu_\alpha(B_{n,3}) < \mu_\alpha(P_n)$ with equalities if and only if $\Delta = 3$ and $G \cong B_{n,3}$, i.e., $G \cong B_{n,3}$.

Now suppose that $G$ is not a tree. Then $G$ contains at least one cycle. If there is a spanning tree $T$ with $T \cong P_n$, then by Lemma 2.3 and the above argument, we have $\mu_\alpha(G) < \mu_\alpha(T) \leq \mu_\alpha(B_{n,3})$. If any spanning tree of $G$ is a path, then $G$ is a cycle $C_n$. Now we only need to show that $\mu_\alpha(C_n) < \mu_\alpha(B_{n,3})$.

Let $C_n = u_1u_2 \cdots u_n$ and $T' = C_n - \{u_1u_2, u_2u_3\} + u_2u_n$. Then $T' \cong B_{n,3}$. Let $x$ be the distance $\alpha$-Perron vector of $C_n$. By Lemma 2.3, we have $x_{u_2} = \cdots = x_{u_n}$. As we pass from $C_n$ to $T'$, the distance between $u_2$ and $u_1$ is increased by 1, the distance between $u_2$ and $u_i$ with $3 \leq i \leq \lceil \frac{n+1}{2} \rceil$ is increased by $n - 2i + 3$, the distance between $u_2$ and $u_i$ with $\lceil \frac{n+1}{2} \rceil \leq i \leq n$ is decreased by 1, and the distances between all other vertex pairs are increased or remain unchanged. Thus

$$\mu_\alpha(T') - \mu_\alpha(C_n)$$

$$= x^T(D_\alpha(T') - D_\alpha(G))x$$

$$\geq \alpha(x_{u_2}^2 + x_{u_1}^2) + 2(1 - \alpha)x_{u_2}x_{u_1}$$

$$- \sum_{i=\lceil \frac{n+1}{2} \rceil + 2}^{\lceil \frac{n+1}{2} \rceil} (n - 2i + 3)(\alpha(x_{u_2}^2 + x_{u_i}^2) + 2(1 - \alpha)x_{u_2}x_{u_i})$$

$$+ \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (n - 2i + 3)(\alpha(x_{u_2}^2 + x_{u_i}^2) + 2(1 - \alpha)x_{u_2}x_{u_i})$$

$$= 2x_{u_2}^2 \left(1 - \left(n - \left\lceil \frac{n+1}{2} \right\rceil - 1\right) - \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (n - 2i + 3)\right)$$
\[2x^2_{w_1} \left(1 + \left(n - 1 - \left\lfloor \frac{n + 1}{2} \right\rfloor \right) \left(\left\lfloor \frac{n + 1}{2} \right\rfloor - 2\right)\right) \]

\[\geq 2x^2_{w_1} > 0,\]

and therefore \(\mu_a(C_n) < \mu_a(B_{n,3})\), as desired. \(\square\)

A clique of \(G\) is a subset of vertices whose induced subgraph is a complete graph, and the clique number of \(G\) is the maximum number of vertices in a clique of \(G\). For \(2 \leq \omega \leq n\). Let \(K_{i,n,\omega}\) be the graph obtained from a complete graph \(K_n\) and a path \(P_{n-i}\) by adding an edge between a vertex of \(K_n\) and a terminal vertex of \(P_{n-i}\) if \(\omega < n\) and let \(K_{i,n,\omega} = K_n\) if \(\omega = n\). In particular, \(K_{i,n,3} \cong P_n\) for \(n \geq 2\). The following result for \(\alpha = 0, \frac{1}{2}\) was given in [15, 21].

**Theorem 5.5** Let \(G\) be a connected graph of order \(n \geq 2\) with clique number \(\omega \geq 2\). Then \(\mu_a(G) \leq \mu_a(K_{i,n,\omega})\) with equality if and only if \(G \cong K_{i,n,\omega}\).

**Proof** It is trivial if \(\omega = n\) and it follows from Theorem 5.4 if \(\omega = 2\).

Suppose that \(3 \leq \omega \leq n - 1\). Let \(G\) be a graph among connected graphs of order \(n\) with clique number \(\omega\) such that \(\mu_a(G)\) is as large as possible. We only need to show that \(G \cong K_{i,n,\omega}\).

Let \(S = \{v_1, \ldots, v_\omega\}\) be a clique of \(G\). By Lemma 2.3, \(G - E(G[S])\) is a forest. Let \(T_i\) be the component of \(G - E(G[S])\) containing \(v_i\), where \(1 \leq i \leq \omega\). For \(1 \leq i \leq \omega\), by Corollary 4.1, if \(T_i\) is nontrivial, then \(T_i\) is a pendant path at \(v_i\). Note that any two distinct vertices in \(G[S]\) are adjacent. By Corollary 4.2, there is only one nontrivial \(T_i\), and thus \(G \cong K_{i,n,\omega}\). \(\square\)

Recall that \(K_{i,n,3}\) is the unique unicyclic graph of order \(n \geq 3\) with maximum distance 0-spectral radius [31], and the unique odd-cycle unicyclic graph of order \(n \geq 3\) with maximum distance \(\frac{1}{2}\)-spectral radius [15].

**Theorem 5.6** Let \(G\) be a unicyclic odd-cycle graph of order \(n \geq 3\). Then \(\mu_a(G) \leq \mu(K_{i,n,3})\) with equality if and only if \(G \cong K_{i,n,3}\).

**Proof** If \(n = 3, 4\), the result is trivial. Suppose that \(n \geq 5\). Let \(G\) be a graph with maximum distance \(\alpha\)-spectral radius among unicyclic odd-cycle graphs of order \(n\). We only need to show that \(G \cong K_{i,n,3}\).

Let \(C = v_1 \cdots v_{2k+1} v_1\) be the unique cycle of \(G\), where \(k \geq 1\). Let \(T_i\) be the component of \(G - E(C)\) containing \(v_i\) for \(1 \leq i \leq 2k + 1\). Let \(U_1 = V(T_{2k}) \cup V(T_{2k+1})\), \(U_2 = \bigcup_{1 \leq i \leq 2k-1} V(T_i)\) and \(U_3 = \bigcup_{1 \leq i \leq 2k-1} V(T_i)\). Let \(x\) be the distance \(\alpha\)-Perron vector of \(G\). Let

\[\Gamma = \sum_{u \in U_1} \sum_{v \in U_3} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)\]

\[-\sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v).\]

Suppose that \(k \geq 2\). Let \(G' = G - v_1 v_{2k+1} + v_{2k+1} v_{2k-1}\). Note that the length of \(C\) is odd. As we pass from \(G\) to \(G'\), the distance between a vertex in \(S_1\) and a vertex in \(S_3\) is increased
by at least 1, the distance between $S_2$ and $V(T_{2k+1})$ is decreased by 1, and the distance between all other vertex pairs are increased or remain unchanged. Thus

$$\mu_{\alpha}(G') - \mu_{\alpha}(G) \geq x^T(D_{\alpha}(G') - D_{\alpha}(G))x$$

$$\geq \sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$- \sum_{u \in V(T_{2k+1})} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$> \sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$- \sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v).$$

If $\Gamma \geq 0$, then $\mu_{\alpha}(G') > \mu_{\alpha}(G)$, a contradiction. Thus $\Gamma < 0$. Let $G'' = G - v_2k v_{2k-1} + v_{2k} v_1$. As we pass from $G$ to $G''$, the distance between a vertex in $S_1$ and a vertex in $U_2$ is increased by at least 1, the distance between $U_3$ and $V(T_{2k})$ is decreased by 1, and the distance between all other vertex pairs are increased or remain unchanged. As above, we have

$$\mu_{\alpha}(G'') - \mu_{\alpha}(G) \geq x^T(D_{\alpha}(G'') - D_{\alpha}(G))x$$

$$\geq \sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$- \sum_{u \in V(T_{2k})} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$> \sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$- \sum_{u \in U_1} \sum_{v \in U_2} (\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v).$$

Thus $\mu_{\alpha}(G') > \mu_{\alpha}(G)$, also a contradiction. It follows that $k = 1$, i.e., the unique cycle of $G$ is of length 3.

Obviously, $T_i$ is a tree for $1 \leq i \leq 3$. For $1 \leq i \leq 3$, by Corollary 4.1, if $T_i$ is nontrivial, then it is a path with a terminal vertex $v_i$. Then by Corollary 4.2, only one $T_i$ is nontrivial. Thus $G \cong K_{1,3}$.

Let $G$ be a unicyclic graph of order $n \geq 4$ with maximum distance $\alpha$-spectral radius. By Corollary 4.1, the maximum degree of $G$ is 3 and all vertices of degree 3 lie on the unique cycle. Let $u$ be a vertex of degree 3 and $P$ be the pendant path at $u$. Let $v$ and $w$ be the two neighbors of $u$ on the cycle, and $z$ the neighbor of $u$ on $P$. Let $G_1 = G - uw + vw$ and $G_2 = G - uw + wz$. Then $\mu_{\alpha}(G) < \max\{\mu_{\alpha}(G_1), \mu_{\alpha}(G_2)\}$ if the length of the cycle of $G$ is odd, see [4, Lemma 6.11]. Note that the argument does not work when the length of the cycle of $G$ is even. So we need other ways to determine the unicyclic graph(s) with maximum distance $\alpha$-spectral radius even for $\alpha = \frac{1}{2}$.
6 Remarks

In this paper, we study the distance $\alpha$-spectral radius of a connected graph. We consider bounds for the distance $\alpha$-spectral radius, local transformations to change the distance $\alpha$-spectral radius, and the characterizations for graphs with minimum and/or maximum distance $\alpha$-spectral radius in some classes of connected graphs.

Besides the distance $\alpha$-spectral radius, we may concern other eigenvalues of $D_\alpha(G)$ for a connected graph $G$. We give examples.

For an $n \times n$ Hermitian matrix $C$, let $\lambda_1(C), \ldots, \lambda_n(C)$ be the eigenvalues of $C$, arranged in a nonincreasing order. Let $A, B$ be $n \times n$ Hermitian matrices. Weyl's inequalities [13, p. 181] state that

$$\lambda_j(A + B) \leq \lambda_i(A) + \lambda_j(B)$$

for $1 \leq i \leq j \leq n$, and

$$\lambda_j(A + B) \geq \lambda_i(A) + \lambda_j(B)$$

for $1 \leq j \leq i \leq n$.

Using these inequalities, and as in the recent work of Atik and Panigrahi [3], we have

**Theorem 6.1** Let $G$ be a connected graph and $\lambda$ be any eigenvalue of $D_\alpha(G)$ other than the distance $\alpha$-spectral radius. Then

$$2\alpha T_{\text{min}}(G) - T_{\text{max}}(G) + (1 - \alpha)(n - 2) \leq \lambda \leq T_{\text{max}}(G) - (1 - \alpha)n.$$

**Proof** Let $D_\alpha(G) = A + B$, where $A = (\alpha T_{\text{min}}(G) - (1 - \alpha))I_n + (1 - \alpha)J_n$. Then $B$ is a non-negative symmetric matrix with maximum row sum $T_{\text{max}}(G) - \alpha T_{\text{min}}(G) - (1 - \alpha)(n - 1)$. Thus $|\lambda_n(B)| \leq \lambda_1(B) \leq T_{\text{max}}(G) - \alpha T_{\text{min}}(G) - (1 - \alpha)(n - 1)$.

For matrix $A$, we have $\lambda_1(A) = \alpha T_{\text{min}}(G) + (1 - \alpha)(n - 1)$ and $\lambda_j(A) = \alpha T_{\text{min}}(G) - 1 + \alpha$ for $j = 2, \ldots, n$. Thus, for $j = 2, \ldots, n$, we have by the above Weyl's inequalities that

$$\lambda_j(D_\alpha(G)) \leq \lambda_1(B) + \lambda_j(A)$$

$$\leq T_{\text{max}}(G) - \alpha T_{\text{min}}(G) - (1 - \alpha)(n - 1) + \alpha T_{\text{min}}(G) - 1 + \alpha$$

$$= T_{\text{max}}(G) - (1 - \alpha)n$$

and

$$\lambda_j(D_\alpha(G)) \geq \lambda_n(B) + \lambda_j(A)$$

$$\geq -T_{\text{max}}(G) + \alpha T_{\text{min}}(G) + (1 - \alpha)(n - 1) + \alpha T_{\text{min}}(G) - 1 + \alpha$$

$$= 2\alpha T_{\text{min}}(G) - T_{\text{max}}(G) + (1 - \alpha)(n - 2).$$

This completes the proof. $\square$

Let $G$ be a connected graph and $\lambda$ be any eigenvalue of $D_\alpha(G)$ other than the distance $\alpha$-spectral radius. By previous theorem, we have

$$|\lambda| \leq T_{\text{max}}(G) - (1 - \alpha)(n - 2).$$
The distance $\alpha$-energy of a connected graph $G$ of order $n$ is defined as
\[
E_\alpha(G) = \sum_{i=1}^{n} \left| \mu_i^{(\alpha)}(G) - \frac{2\sigma(G)}{n} \right|.
\]

Then $E_0(G)$ is the distance energy of $G$ \[14, 33\], while
\[
E_{1/2}(G) = \frac{1}{2} \sum_{i=1}^{n} \left| 2\mu_i^{(1/2)}(G) - \frac{2\sigma(G)}{n} \right|
\]
is half of the distance signless Laplacian energy of $G$ \[8\]. Thus, it is possible to study the distance energy and the distance signless Laplacian energy in a unified way.

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