Partial differential equations/Optimal control

Partial regularity for solutions to subelliptic eikonal equations

Sur la régularité partielle des solutions de l’équation eikonale sous-elliptique

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\section*{A B S T R A C T}

On a bounded domain $\Omega$ in the Euclidean space $\mathbb{R}^n$, we study the homogeneous Dirichlet problem for the eikonal equation associated with a system of smooth vector fields, which satisfies Hörmander’s bracket generating condition. We prove that the solution is smooth in the complement of a closed set of Lebesgue measure zero.

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\section*{R É S U M É}

Soit $\Omega$ un ouvert borné à bord lisse de $\mathbb{R}^n$. Nous étudions le problème de Dirichlet homogène sur $\Omega$ pour l’équation eikonale associée à un système de champs de vecteurs qui satisfait la condition de Hörmander. Nous montrons que la solution de ce problème est régulière dans le complémentaire d’un ensemble fermé de mesure de Lebesgue nulle.

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\section*{Version française abrégée}

Soit $\Omega$ un ouvert borné de $\mathbb{R}^n$, de bord $\Gamma$ donné par une surface lisse de dimension $n - 1$. On considère un système de champs de vecteurs $X_1, \ldots, X_N$ lisses – c’est-à-dire de classe $C^\infty$, soit analytiques – sur un voisinage de $\Omega$, noté $\Omega'$. Supposons que ce système satisfasse la \textit{condition de Hörmander}, i.e. $\operatorname{Lie}\{X_1, \ldots, X_N\}(x) = \mathbb{R}^n$ pour tout $x \in \Omega'$, où $\operatorname{Lie}\{X_1, \ldots, X_N\}$

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désigne la sous-algèbre de Lie engendrée par \([X_1, \ldots, X_N]\). Il est à noter que, dans ce papier, les champs de vecteurs ne sont pas supposés linéairement indépendants, et \(N < n\) non plus.

Sous cette hypothèse, le problème de Dirichlet suivant :

\[
\sum_{j=1}^{N} (X_j T)^2(x) = 1 \text{ dans } \Omega, \quad T = 0 \text{ sur } \Gamma,
\]

admet une solution de viscosité unique \(T\). De plus, \(T\) est hölderienne – mais pas plus lisse en général. La régularité de \(T\) a été étudiée par les auteurs de ce papier dans [3]. On se concentre ici sur l’analyse du support singulier de \(T\).

**Définition 0.1.** Le support singulier d’une fonction \(f : \Omega \to \mathbb{R}\) est le complémentaire dans \(\Omega\) de l’ensemble des points \(x \in \Omega\) qui ont un voisinage où la fonction \(f\) est indéfiniment différentiable – soit analytique ; on le note par \(\text{Sing supp } f\).

On peut de même définir le support \(C^{1,1}\) et le support Lipschitz de \(T\) ; on les note par \(\text{Sing supp}_{C^{1,1}} T\) et \(\text{Sing supp}_{\text{Lip}} T\), respectivement. On montre tout d’abord le résultat suivant.

**Théorème 0.2.** \(\text{Sing supp } T = \text{Sing supp}_{C^{1,1}} T\).

Une fois établi ce résultat, nous montrons que le support singulier de \(T\) est négligeable pour la mesure de Lebesgue.

**Théorème 0.3.** La mesure de Lebesgue de l’ensemble \(\text{Sing supp } T\) est nulle.

Le Théorème 0.3 est lié à la conjecture de Sard minimisante en géométrie sous-riemannienne (voir, par exemple, [11]). Une formulation de cette conjecture, adaptée au cas d’une cible donnée par une hypersurface lisse \(\Gamma\), est la suivante : l’ensemble \(S_{\text{min}}\) des points atteignables par les trajectoires singulières minimisantes est de mesure de Lebesgue nulle. Comme \(S_{\text{min}}\) coïncide avec l’ensemble sur lequel la distance sous-riemannienne n’est pas Lipschitz ([3, Théorème 3.2]), la conjecture ci-dessus peut être déduite des résultats de [9], où on montre la différentiabilité presque partout de la distance sous-riemannienne d’une cible qui a la propriété de boule interne.

Une deuxième partie de la même conjecture peut être reformulée en disant que l’ensemble de tous les points qui ont un voisinage où la distance sous-riemannienne est lisse doit être de mesure pleine de Lebesgue. Comme cet ensemble coïncide avec le complémentaire du support singulier de la distance sous-riemannienne, le Théorème 0.3 montre la conjecture pour des cibles régulières de codimension 1.

1. Introduction

Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set with boundary \(\Gamma\), given by a smooth manifold of dimension \(n – 1\). Let \(X_1, \ldots, X_N\) be a system of smooth vector fields defined on some open neighbourhood of \(\Omega\), say \(\Omega'\). Hereafter, the term smooth stands for either \(C^\infty\) or \(C^m\), the latter meaning real analytic functions. We shall assume that Hörmander’s bracket-generating condition is satisfied, i.e. \(\text{Lie}\{X_1, \ldots, X_N\}(x) = \mathbb{R}^n\), \(\forall x \in \Omega'\), where \(\text{Lie}\{X_1, \ldots, X_N\}(x)\) denotes the space of all values, at \(x\), of the vector fields of the Lie algebra generated by \(\{X_1, \ldots, X_N\}\). We point out that we need not suppose such vector fields to be linearly independent, nor that \(N < n\).

Under the above assumptions—that will be in force throughout the paper—it is well known that the boundary value problem

\[
\sum_{j=1}^{N} (X_j T)^2(x) = 1 \text{ in } \Omega, \quad T = 0 \text{ on } \Gamma,
\]

admits a unique continuous viscosity solution. Moreover, \(T\) is Hölder continuous, but fails to be more regular, in general.

In [3], we investigated the regularity of \(T\). Building on such results, in this paper we analyse the singular support of \(T\).

**Définition 1.1.** The singular support of a function \(f : \Omega \to \mathbb{R}\), \(\text{Sing supp } f\) in short, is the complement in \(\Omega\) of the set of all points \(x \in \Omega\) that have an open neighbourhood on which \(f\) is smooth.

In a similar way, one can define the \(C^{1,1}\) singular support and the Lipschitz singular support of \(T\), which are denoted by \(\text{Sing supp}_{C^{1,1}} T\) and \(\text{Sing supp}_{\text{Lip}} T\), respectively. (It is clear that \(\text{Sing supp } T\) is closed in \(\Omega\).) We first prove the following result.

**Theorem 1.2.** \(\text{Sing supp } T = \text{Sing supp}_{C^{1,1}} T\).
Moreover, we show that the singular support of $T$ is a negligible set.

**Theorem 1.3.** Sing supp $T$ has Lebesgue measure zero.

We note that Theorem 1.3 is related to the so-called minimizing Sard conjecture in sub-Riemannian geometry (see, e.g., [11, Conjecture 1, p. 158]). One of the formulations of such a conjecture, adapted to the case of a smooth target, claims that the set $S_{\text{min}}$, which consists of all points lying on a singular minimizing trajectory, should have Lebesgue measure zero. Since, by [3, Theorem 3.2], $S_{\text{min}}$ coincides with the set on which the sub-Riemannian distance fails to be Lipschitz, the above conjecture can be proved by appealing to [9], where the almost everywhere differentiability of the sub-Riemannian distance to a closed set with the inner ball property is obtained.

A further part of the same conjecture could be rephrased saying that the set of all points, on a neighbourhood of which the sub-Riemannian distance is smooth, should have full Lebesgue measure. Since such a set is nothing but the complement of the singular support of the sub-Riemannian distance, Theorem 1.3 above shows the conjecture to be true for smooth targets of codimension 1.

2. Proofs

The proof of Theorems 1.2 and 1.3 relies on the fact that the solution $T$ of (2) is the value function of a suitable time optimal control problem.

Let $x \in \bar{B}$. For any measurable function $u = (u_1, \ldots, u_N): [0, +\infty[ \to \mathbb{R}^N$ taking values in $\overline{B}_1(0)$, the unit closed ball of $\mathbb{R}^N$, we denote by $y^{x,u}$ the unique maximal solution to the Cauchy problem

$$\begin{cases}
y'(t) = \sum_{j=1}^N u_j(t) X_j(y(t)) & (t \geq 0), \\
y(0) = x. 
\end{cases}$$

(3)

The time needed to steer $x$ to $\Gamma$ along $y^{x,u}$ is given by

$$\tau_\Gamma(x,u) = \inf \{ t \geq 0 : y^{x,u}(t) \in \Gamma \}.$$ 

Given any $y \in \Omega$, the Minimum Time Problem with target $\Gamma$ is the following:

(MTP) minimize $\tau_\Gamma(x,u)$ over all controls $u : [0, +\infty[ \to \overline{B}_1(0)$.

The minimum time function is defined by

$$T(x) = \inf_{u(\cdot)} \tau_\Gamma(x,u) \quad (x \in \Omega).$$

It is well known that $T$ is the unique viscosity solution to the Dirichlet problem (2). Moreover, Hörmander’s bracket generating condition implies that (3) is small time locally controllable, so that $T$ is finite and continuous (see, for instance, [5, Proposition 1.6, Chapter IV]).

We recall that a $u(\cdot)$ is called an optimal control at a point $x \in \Omega$ if $T(x) = \tau_\Gamma(x,u)$. The corresponding solution to (3), $y^{x,u}$, is called the time-optimal trajectory at $x$ associated with $u$.

We now recall the definition of singular time-optimal trajectories. For any point $z \in \Gamma$, we denote by $\nu(z)$ the outward unit normal to $\Gamma$ at $z$.

**Definition 2.1.** We say that a time-optimal trajectory $y(\cdot) = y^{x,u}(\cdot)$ at a point $x \in \Omega$ is singular if there exists an absolutely continuous arc $p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$ such that

(a) $p_k'(t) = \sum_{j=1}^N u_j(t) \langle \partial_{q_k} X_j(y(t)), p(t) \rangle$ a.e. $(k = 1, \ldots, N),$ \hspace{2cm} (4)

(b) $\langle X_\ell(y(t)), p(t) \rangle = 0$ \hspace{1cm} $\forall t \in [0, T(x)]$ \hspace{1cm} $(k = 1, \ldots, N),$ \hspace{2cm} (5)

(c) $\exists \lambda > 0 : p(T(x)) = \lambda \nu(y(T(x))).$ \hspace{2cm} (6)

Notice that (5) and (6) imply that all the $X_j(y(T(x)))$’s are tangent to $\Gamma$, that is,

$$\text{span} \{ X_1(y(T(x))), \ldots, X_N(y(T(x))) \} \subset T_\Gamma(y(T(x))).$$

So, $y(T(x))$ is a characteristic point.

In order to connect the lack of regularity of $T$ with the presence of singular trajectories, it is useful to look at the Lipschitz singular set of $T$, i.e.
Proof. of parametrization assertion, Theorem along relations Hamilton–Jacobi Then, Lemma (S1) \( \sum_{j=1}^{N} (p_j X_j(x))^2 \) is the Hamiltonian associated with \( \{X_1, \ldots, X_N\} \). For any \( \xi \in V_0 \), denote by \( (X(\cdot, \xi), P(\cdot, \xi)) \) the solution to

\[
\begin{align*}
-\dot{X} &= \nabla_x H(X, P), \\
\dot{P} &= \nabla_x H(X, P), \\
X(0) &= \xi, \\
P(0) &= H(\xi, \nu(\xi))^{-1} \nu(\xi),
\end{align*}
\]

defined on some maximal interval \( [0, \tau] \), and by \( X_{t, \xi} \) and \( P_{t, \xi} \) the Jacobian of the maps \( X \) and \( P \) composed with a local parametrization of \( \Gamma \) (such matrix-valued functions solve a certain system of ODE’s, i.e. the linearization of (10)), Observe that \( \tau > T_0 \) for all \( \xi \) in a suitable relatively open set \( V \subset V_0 \) because \( y_0 \)-coupled with a suitable dual arc \( p_0 \)-solves (10) backward in time for \( \xi = \xi_0 \), i.e.

\[
(X(t, \xi_0), P(t, \xi_0)) = (y_0(t - T_0), p_0(t - T_0)) \quad (t \in [0, T_0]).
\]

So, proving that \( y_0(\cdot) \) contains no conjugate point amounts to showing \( \det X_{t, \xi}(t, \xi_0) \neq 0 \) for all \( t \in [0, T_0] \). If this is not the case, let \( t_0 \in [0, T_0] \) be the first time at which \( \det X_{t, \xi}(t, \xi_0) = 0 \). Then, by the classical method of characteristics, \( T \) is smooth at \( X(t, \xi_0) \) and \( \nabla T(X(t, \xi_0)) = -P(t, \xi_0) \) for all \( t \in [0, t_0] \). So,

(S1) \( x \in \text{Sing}_1 T \) if and only if \( x \) is the initial point of a singular trajectory ([3, Theorem 3.2]);
(S2) \( \text{Sing}_1 T \) is closed in \( \Omega \) ([3, Proposition 4.1]);
(S3) \( T \) is locally semiconcave in \( \Omega \setminus \text{Sing}_1 T \) ([3, Theorem 4.3]).

We recall that a function is semiconcave if it can be locally represented as the sum of a smooth function plus a concave one. Notice that property (S3) above ensures that \( \text{Sing}_1 T = \text{Sing} \supp_{\text{up}} T \).

The fact that the existence of singular time-optimal trajectories may destroy the regularity of a solution to a first-order Hamilton–Jacobi equation was observed (implicitly) by Sussmann in [12] and (explicitly) by Agrachev in [1]. The regularity that these authors considered is the subanalyticity of the point-to-point distance function associated with real-analytic distributions. The aforementioned subanalyticity results were extended to solutions to the Dirichlet problem in [13].

We recall that a vector \( p \in \mathbb{R}^n \) is a proximal subgradient of \( T \) at \( x \in \Omega \) if \( \exists c, \rho > 0 \) such that

\[
T(y) - T(x) - \langle p, y - x \rangle \geq -c|y - x|^2, \quad \forall y \in B(x, \rho) \cap \Omega.
\]

The set of all proximal subgradients of \( T \) at \( x \) is denoted by \( \partial_{p} T(x) \).

The following lemma identifies proximal subdifferentiability as a threshold for local smoothness.

Lemma 2.2. Let \( x_0 \in \Omega \) be such that

(a) \( \partial_{p} T(x_0) \) is nonempty,
(b) \( T \) is semiconcave on an open neighbourhood \( U_0 \subset \Omega \) of \( x_0 \).

Then, \( T \) is of class \( C^\infty \) on some open neighbourhood \( U \subset U_0 \) of \( x_0 \).

Proof. To begin with, we note that a) and b) force \( T \) to be differentiable at \( x_0 \). Then, standard arguments based on sensitivity relations guarantee the existence of a unique optimal trajectory, \( y_0(\cdot) \), starting from \( x_0 \), and ensure that \( T \) stays differentiable along such a trajectory, which, therefore, is not singular in view of (S1). So, by (S3), \( T \) is semiconcave on a relatively open neighbourhood, \( W_0 \), of \( \{y_0(t) : t \in [0, T_0]\} \), where we have set \( T_0 = T(x_0) \). Thus, there exists a constant \( C_1 \) such that

\[
\nabla^2 T \leq C_1 I
\]

in the sense of distributions on \( W_0 \). Moreover, by the propagation of proximal subdifferentiability (see [7, Theorem 3] or [8, Theorem 2.3]), a) implies that there exists a constant \( C_2 \geq 0 \) such that, for all \( t \in [0, T_0] \) and \( h \in \mathbb{R}^n \) sufficiently small,

\[
T(y_0(t) + h) - T(y_0(t)) - \langle \nabla T(y_0(t)), h \rangle \geq -C_2 |h|^2.
\]

The key idea of the proof is to deduce the local smoothness of \( T \) along \( y_0(\cdot) \), in particular, near \( x_0 \), from [10, Theorem 3.1]. For this, we must prove that \( \{y_0(t) : t \in [0, T_0]\} \) contains no conjugate points.\(^4\) In order to check such an assertion, we identify \( y_0 \) as a backward solution to the characteristic system as follows. Since \( \xi_0 := y_0(T_0) \) is not a characteristic boundary point, there exists an open neighbourhood \( V_0 \subset \Gamma \) of \( \xi_0 \) such that \( H(\xi, \nu(\xi)) > 0 \) for all \( \xi \in V_0 \), where

\[
H(x, p) = (\sum_{j=1}^{N} |p_j X_j(x)|^2)^{1/2}
\]

is the Hamiltonian associated with \( \{X_1, \ldots, X_N\} \). For any \( \xi \in V_0 \), denote by \( (X(\cdot, \xi), P(\cdot, \xi)) \) the solution to

\[
\begin{align*}
-\dot{X} &= \nabla_p H(X, P), \\
\dot{P} &= \nabla_x H(X, P), \\
X(0) &= \xi, \\
P(0) &= H(\xi, \nu(\xi))^{-1} \nu(\xi),
\end{align*}
\]

\(^4\) Notice that, in [10], structural assumptions—that are not satisfied in our settings—are imposed. However, such assumptions are not needed for the proof of [10, Theorem 3.1].
\[\nabla^2 T(X(t, \xi_0)) X_{t, \xi}(t, \xi_0) = -P_{t, \xi}(t, \xi_0), \quad \forall t \in [0, t_0]. \tag{11}\]

Since, by the well-known properties of solutions to linear systems (see, e.g., [6, p. 155]), \(P_{t, \xi}\) can be singular at no point at which \(\det X_{t, \xi} = 0\), from (11) it follows that
\[\lim_{t \to t_0} \left| \det (\nabla^2 T(X(t, \xi_0))) \right| = \infty. \tag{12}\]

Using the fact that for all \(t \in [0, t_0]\), the left-hand side of (9) is equal to \(\langle \nabla^2 T(X(t, \xi_0)) h, h \rangle + o(\|h\|^2)\), we deduce that \(\langle \nabla^2 T(X(t, \xi_0)) h, h \rangle \geq -C_2 \|h\|^2 + o(\|h\|^2)\). Then, we conclude that there exists \(C_2 > 0\) such that \(\langle \nabla^2 T(X(t, \xi_0)) \eta, \eta \rangle \geq -C_2\) for all \(\eta \in S^{n-1}\) and \(t \in [0, t_0]\). Finally, the last inequality, together with (8), yields that \(\nabla^2 T(X(\cdot, \xi_0))\) is bounded on \([0, t_0]\) in contrast with (12), thus completing the proof. □

**Proof of Theorem 1.2.** Let \(\Sigma_1(T) = \text{Sing}\supp\mathcal{C}^{1,1} T\) and \(\Sigma(T) = \text{Sing}\supp T\). Since \(\Sigma_1(T) \subseteq \Sigma(T)\), we just need to show that \(\Omega \setminus \Sigma_1(T) \subseteq \Omega \setminus \Sigma(T)\). As mentioned above, \(T\) is semiconcave on \(\Omega \setminus \Sigma_1(T)\). Moreover, from the very definition (7) of proximal subgradients, it follows that \(\partial_T T(x) \neq \emptyset\) for any \(x \in \Omega \setminus \Sigma_1(T)\). Then, the conclusion follows from Lemma 2.2. □

**Proof of Theorem 1.3.** We keep the notation \(\Sigma_1(T)\) of the previous proof and set \(\Sigma_{\text{Lip}}(T) = \text{Sing}\supp_{\text{Lip}} T\). We observe that, by Theorem 1.2, it suffices to show that the \(C^{1,1}\) singular support of \(T\) has null measure. For this purpose, we decompose such a support as \(\Sigma_1(T) = \Sigma_{\text{Lip}}(T) \cup (\Sigma_1(T) \setminus \Sigma_{\text{Lip}}(T))\). By [9, Corollary 3.3], we deduce that \(\Sigma_{\text{Lip}}(T)\) has measure zero. In order to prove that \(\Sigma_1(T) \setminus \Sigma_{\text{Lip}}(T)\) has measure zero we use an idea from [2]. Recall that, by [3, Theorem 4.1], \(T\) is locally semiconcave in \(\Omega \setminus \Sigma_{\text{Lip}}(T)\). Then, Alexandroff Theorem (see [4]) guarantees that \(T\) has a second order Taylor expansion at a.e. point of \(\Omega \setminus \Sigma_{\text{Lip}}(T)\). Hence, \(\partial_T T(x)\) is nonempty for a.e. \(x \in \Omega \setminus \Sigma_{\text{Lip}}(T)\). So, thanks to Lemma 2.2 we conclude that there exists a set of full measure in \(\Omega \setminus \Sigma_{\text{Lip}}(T)\) which lies in the complement of \(\Sigma_1(T) \setminus \Sigma_{\text{Lip}}(T)\). This proves that the set \(\Sigma_1(T) \setminus \Sigma_{\text{Lip}}(T)\) has null measure and completes the proof. □

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