TOROIDAL AUTOMORPHIC FORMS FOR SOME FUNCTION FIELDS

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Abstract. Zagier introduced toroidal automorphic forms to study the zeros of zeta functions: an automorphic form on $GL_2$ is toroidal if all its right translates integrate to zero over all nonsplit tori in $GL_2$, and an Eisenstein series is toroidal if its weight is a zero of the zeta function of the corresponding field. We compute the space of such forms for the global function fields of class number one and genus $g \leq 1$, and with a rational place. The space has dimension $g$ and is spanned by the expected Eisenstein series. We deduce an "automorphic" proof for the Riemann hypothesis for the zeta function of those curves.

1. Introduction

Let $X$ denote a smooth projective curve over a finite field $F_q$ with $q$ elements, $A$ the adeles over its function field $F := F_q(X)$, $B$ its standard (upper-triangular) Borel subgroup, $K = G(\mathcal{O}_A)$ the standard maximal compact subgroup of $G_A$, with $\mathcal{O}_A$ the maximal compact subring of $A$, and $Z$ the center of $G$. Let $\mathcal{A}$ denote the space of unramified automorphic forms $f : G_F \backslash G_A / KZ_A \rightarrow \mathbb{C}$. We use the following notations for matrices:

\[ \text{diag}(a, b) = (a \ 0 \\ 0 \ b) \quad \text{and} \quad [ \begin{array}{cc} a & b \\ 0 & 1 \end{array} ] = (a \ b) \]

There is a bijection between quadratic separable field extensions $E/F$ and conjugacy classes of maximal non-split tori in $G_F$ via

\[ E^\times = \text{Aut}_E(E) \subset \text{Aut}_F(E) \cong G_F. \]

If $T$ is a non-split torus in $G$ with $T_F \cong E^\times$, define the space of toroidal automorphic forms for $F$ with respect to $T$ (or $E$) to be

\[ T_F(E) = \{ f \in \mathcal{A} \mid \forall g \in G_A, \quad \int_{T_FZ_A \backslash T_A} f(tg) \, dt = 0 \}. \]

The integral makes sense since $T_FZ_A \backslash T_A$ is compact, and the space only depends on $E$, viz., the conjugacy class of $T$. The space of toroidal automorphic forms for $F$ is

\[ T_F = \bigcap_E T_F(E), \]

where the intersection is over all quadratic separable $E/F$. The interest in these spaces lies in the following version of a formula of Hecke ([5], Werke p. 201); see Zagier, [15] pp. 298–299 for this formulation, in which the result essentially follows from Tate’s thesis:

**Proposition 1.1.** Let $\zeta_E$ denote the zeta function of the field $E$. Let $\varphi : \mathbb{A}^2 \rightarrow \mathbb{C}$ be a Schwartz-Bruhat function. Set

\[ f(g, s) = |\det g|_F^{s} \int_{\mathbb{A}_F^\times} \varphi((0, a)g)|a|^{2s} \, d^\times a. \]
An Eisenstein series $E(s)$

$$E(s)(g) := \sum_{\gamma \in B_F \backslash G_F} f(\gamma g, s) \quad (\text{Re}(s) > 1)$$

satisfies

$$\int_{T_F Z_A \backslash T_A} E(s)(tg) \, dt = c(\varphi, g, s) | \det g|^s \zeta_E(s)$$

for some holomorphic function $c(\varphi, g, s)$. For every $g$ and $s$, there exists a function $\varphi$ such that $c(\varphi, g, s) \neq 0$. In particular, $E(s) \in T_F(E) \iff \zeta_E(s) = 0$. \hfill \Box

**Remark 1.2.** Toroidal integrals of parabolic forms are ubiquitous in the work of Waldspurger ([13], for recent applications, see Clozel and Ullmo [1] and Lysenko [10]). Wielonsky and Lachaud studied analogues of toroidal integrals of parabolic forms.

In particular, $E(s) \in T_F(E) \iff \zeta_E(s) = 0$.

**Lemma 1.3.** The spaces $T_F(E)$ (for each $E$ with corresponding torus $T$) and $T_F$ are invariant under the Hecke algebra $\mathcal{H}$, and

$$T_F(E) \subseteq \{ f \in \mathcal{A} \mid \forall \Phi \in \mathcal{H}, \int_{T_F Z_A \backslash T_A} \Phi(f)(t) \, dt = 0 \} \quad \Box$$

Now assume $F$ has class number one and there exists a place $\infty$ of degree one for $F$; let $t$ denote a local uniformizer at $\infty$. Strong approximation implies that we have a bijection

$$G_F \backslash G_A K Z_\infty \sim \Gamma \backslash G_\infty K Z_\infty,$$

where $\Gamma = G(A)$ with $A$ the ring of functions in $F$ holomorphic outside $\infty$, and a subscript $\infty$ refers to the $\infty$-component. We define a graph $\mathcal{G}$ with vertices $V \mathcal{G} = G_\infty / K Z_\infty$. If $\sim$ denotes equivalence of matrices modulo $K Z_\infty$, then we call vertices in $V \mathcal{G}$ given by classes represented by matrices $g_1$ and $g_2$ adjacent, if $g_1^{-1} g_2 \sim [t, b]$ or $[t^{-1}, 0]$ for some $b \in \mathcal{O}_\infty / t$. Then $\mathcal{G}$ is a tree that only depends on $q$ (the so-called Bruhat-Tits tree of $\text{PGL}(2, F_\infty)$, cf. [11], Ch. II).

The Hecke operator $\Phi_\infty \in \mathcal{H}$ given by the characteristic function of $K[t, 0] K$ maps a vertex of $\mathcal{G}$ to its neighbouring vertices. The action of $\Phi_\infty$ on the quotient graph $\Gamma \backslash \mathcal{G}$ can be computed from the orders of the $\Gamma$-stabilizers of vertices and edges in $\mathcal{G}$. When drawing a picture of $\Gamma \backslash \mathcal{G}$, we agree to label a vertex along the edge towards an adjacent vertex by the corresponding weight of a Hecke operator, as in the next example.

**Example 1.4.** In Figure 1 one sees the graph $\Gamma \backslash \mathcal{G}$ for the function field of $X = \mathbb{P}^1$, with the well-known vertices representing $\{ c_i = [\pi^{-i}, 0] \}_{i \geq 0}$ and the weights of $\Phi_\infty$, meaning

$$\Phi_\infty(f)(c_n) = qf(c_{n-1}) + f(c_{n+1}) \quad \text{and} \quad \Phi_\infty(f)(c_0) = (q + 1)f(c_1).$$

![Figure 1. The graph $\Gamma \backslash \mathcal{G}$ for $X = \mathbb{P}^1$](image-url)
2. The rational function field

First, assume \( X = \mathbb{P}^1 \) over \( \mathbb{F}_q \), so \( F \) is a rational function field. Set \( E = \mathbb{F}_{q^2} F \) the quadratic constant extension of \( F \).

**Theorem 2.1.** \( T_F = T_F(E) = \{0\} \).

**Proof.** Let \( T \) be a torus with \( T_F = E^\times \), that has a basis over \( F \) contained in the constant extension \( \mathbb{F}_{q^2} \).

The integral defining \( f \in T_F(E) \) in equation (1) for the element \( g = 1 \in G_A \) becomes

\[
\int_{T_F/T_A \cap T_A} f(t) \, dt = \kappa \cdot \int_{T_F/Z_A \cap T_A} f(t) \, dt = \kappa \cdot \int_{E^\times A_E^0 \cap A_E^0/\mathcal{O}_E} f(t) \, dt = \kappa \cdot f(c_0),
\]

with \( \kappa = \mu(T_A \cap K) \neq 0 \). Indeed, by our choice of “constant” basis, we have \( T_A \cap K \cong O^\times \). For the final equality, note that the integration domain \( E^\times A_E^0 \cap A_E^0/\mathcal{O}_E \) is isomorphic to the quotient of the class group of \( E \) by that of \( F \), and that both of these groups are trivial, so map to the identity matrix \( c_0 \in \Gamma \cap J \).

Hence we first of all find \( f(c_0) = 0 \). For \( \Phi = \Phi_k \), this equation transforms into \( (\Phi^k f)(c_0) = 0 \) (cf. 2), and with 3 this leads to a system of equations for \( f(c_i) \) (\( i = 1, 2, \ldots \)) that can easily be shown inductively only to have the zero solution \( f = 0 \). \( \square \)

3. Three elliptic curves

Now assume that \( F \) is not rational, has class number one, a rational point \( \infty \) and genus \( \leq 1 \). In this paper, we focus on such fields \( F \), since it turns out that the space \( T_F \) can be understood elaborating only existing structure results about the graph \( \Gamma \setminus J \).

The Hasse-Weil theorem implies that there are only three possibilities for \( F \), which we conveniently number as follows: \( \{F_q\}_{q=2}^4 \) with \( F_q \) the function field of the projective curve \( X_q/\mathbb{F}_q \) \((q = 2, 3, 4)\) are the respective elliptic curves

\[
y^2 + y = x^3 + x + 1, \quad y^2 = x^3 - x - 1 \quad \text{and} \quad y^2 + y = x^3 + \alpha
\]

with \( \mathbb{F}_4 = \mathbb{F}_2(\alpha) \). Let \( F_q^{(2)} = \mathbb{F}_{q^2} F_q \) denote the quadratic constant extension of \( F_q \).

\[
\begin{array}{c}
\bullet & 1 & \bullet \\
\bullet & 1 & \bullet \\
\bullet & 1 & \bullet \\
\end{array}
\]

\[
\begin{array}{c}
t_1 & 1 & t_q \\
q + 1 & 1 & q + 1 \\
q + 1 & 1 & q + 1 \\
\end{array}
\]

\[
\begin{array}{c}
c_0 & 1 & c_1 \\
q - 1 & 1 & q \\
z_0 & 1 & z_1 \\
\end{array}
\]

\[
\begin{array}{c}
c_2 & 1 & \cdots \\
q & 1 & q \\
\end{array}
\]

**Figure 2.** The graph \( \Gamma \setminus J \) for \( F_q \) \((q = 2, 3, 4)\)

The graph \( \Gamma \setminus J \) for \( F_q \) \((q = 2, 3, 4)\) with the \( \Phi_\infty \)-weights is displayed in Figure 2 cf. Serre [11, 2.4.4 and Ex. 3b)+3c] on page 117 and/or Takahashi [12] for these facts.

Further useful facts: One easily calculates that \( X_q(\mathbb{F}_{q^2}) \) is cyclic of order \( 2q + 1 \); let \( Q \) denote any generator. We will use lateron that the vertices \( t_i \) correspond to classes of rank-two vector bundles on \( X_q(\mathbb{F}_q) \) that are pushed down from line bundles on \( X_q(\mathbb{F}_{q^2}) \) given by multiples \( Q, 2Q, \ldots, qQ \) of \( Q \), cf. Serre, loc. cit. For a representation in terms of matrices, one may refer to [12]: if \( iQ = (\ell, \ast) \in X_q(\mathbb{F}_{q^2}) \), then \( t_i = [\ell^2, t^{-1} + \ell t] \).
We denote a function $f$ on $\Gamma \backslash \mathcal{S}$ by a vector
\[
f = [f(t_1), \ldots, f(t_q) \mid f(z_0), f(z_1) \mid f(c_0), f(c_1), f(c_2), \ldots].
\]

**Proposition 3.1.** A function $f \in T_{F_q}(F_q^{(2)})$ ($q = 2, 3, 4$) belongs to the $\Phi_\infty$-stable linear space $\mathcal{S}$ of functions
\[
\mathcal{S} := \{ [T_1, \ldots, T_q \mid Z_0, Z_1 \mid C_0, C_1, C_2, \ldots] \}
\]
with $C_0 = -2(T_1 + \cdots + T_q)$ and for $k \geq 0$,
\[
C_k = \begin{cases}
\lambda_k Z_0 + \mu_k (T_1 + \cdots + T_q) & \text{if } k \text{ even} \\
\nu_k Z_1 & \text{if } k \text{ odd}
\end{cases}
\]
for some constants $\lambda_k, \mu_k, \nu_k$. In particular,
\[
\dim T_{F_q}(F_q^{(2)}) \leq \dim \mathcal{S} = q + 2,
\]
and $\dim T_{F_q}$ is finite.

**Proof.** We choose arbitrary values $T_j$ at $t_j$ ($j = 1, \ldots, q$) and $Z_j$ at $z_j$ ($j = 1, 2$), and set $T = T_1 + \cdots + T_q$. We have
\[
\int_{T \mathcal{A} \backslash T\mathcal{A}} f(t) \, dt = C_0 + 2\tau.
\]
Indeed, by the same reasoning as in the proof of Theorem 2.1, the integration area maps to the image of
\[
\text{Pic}(X_q(F_{q^2}))/\text{Pic}(X_q(F_q)) = X_q(F_{q^2}) / X_q(F_q) = X_q(F_{q^2})
\]
(the final equality since $X_q$ is assumed to have class number one) in $\Gamma \backslash \mathcal{S}$, and these are exactly the vertices $c_0$ and $t_j$ (the latter with multiplicity two, since $\pm Q \in E(F_{q^2})$ map to the same vertex). The integral is zero exactly if $C_0 = -2\tau$. Applying the Hecke operator $\Phi_\infty$ to this equation (cf. (2)) gives $C_1 = -2Z_1$, then applying $\Phi_\infty$ again gives $C_2 = -(q + 1)Z_0$. The rest follows by induction. If we apply $\Phi_\infty$ to the equations (5) for $k \geq 2$, we find by induction for $k$ even that
\[
C_{k+1} = \lambda_k C_1 + (\lambda_k q + \mu_k q(q + 1) - q\nu_{k-1})Z_0
\]
and for $k$ odd that
\[
C_{k+1} = (\nu_k - q\lambda_{k-1})Z_0 + (\nu_k - q\mu_{k-1})\tau.
\]

**Lemma 3.2.** The space $\mathcal{S}$ from (3) has a basis of $q + 2$ $\Phi_\infty$-eigenforms, of which exactly $q - 1$ are cusp forms with eigenvalue zero and support in the set of vertices $\{t_j\}$, and three are non-cuspidal forms with respective eigenvalues $0, q, -q$.

**Proof.** With $\tau = T_1 + \cdots + T_q$, the function
\[
f = [T_1, \ldots, T_q \mid Z_0, Z_1 \mid -2\tau, C_1, C_2, \ldots]
\]
is a $\Phi_\infty$-eigenform with eigenvalue $\lambda$ if and only if
\[
\lambda T_j = (q + 1)Z_1; \ \lambda Z_1 = \tau + Z_0; \ \lambda Z_0 = qZ_1 + C_1; \ \lambda(-2\tau) = (q + 1)C_1; \ \text{etc.}
\]
We consider two cases:

(a) if $\lambda = 0$, we find $q$ forms
\[
f_k = [0, \ldots, 0, 1, 0, \ldots, 0 \mid 0, -1 \mid -q, \ldots]
\]
with $T_j = 1 \iff j = k$. 

Corollary 3.3. The Riemann hypothesis is true for $\zeta_{F_q}$ ($q = 2, 3, 4$).

**Proof.** From Lemma 3.2, we deduce that the only possible $\Phi_\infty$-eigenvalue of a toroidal Eisenstein series is $\pm q$ or $0$, but on the other hand, from Lemma 3.4, we know this eigenvalue is $q^s + q^{1-s}$ where $\zeta_{F_q}(s) = 0$. We deduce easily that $s$ has real part $1/2$. \qed

Remark 3.4. One may verify that this proves the Riemann Hypothesis for the fields $F_q$ without actually computing $\zeta_{F_q}$: it only uses the expression for the zeta function by a Tate integral. Using a sledgehammer to crack a nut, one may equally deduce from Theorem 2.1 that $\zeta_{F_2}$ doesn’t have any zeros. At least the above corollary shows how enough knowledge about the space of toroidal automorphic forms does allow one to deduce a Riemann Hypothesis, in line with a hope expressed by Zagier 15.

**Theorem 3.5.** For $q = 2, 3, 4$, $T_{F_q}$ is one-dimensional, spanned by the Eisenstein series of weight $s$ equal to a zero of the zeta function $\zeta_{F_q}$ of $F_q$.

**Remark 3.6.** Note that the functional equation for $E(s)$ implies that $E(s)$ and $E(1-s)$ are linearly dependent, so it doesn’t matter which zero of $\zeta_{F_q}$ is taken.

**Proof.** By Lemma 3.2, $T_{F_q}$ is a $\Phi_\infty$-stable subspace of the finite dimensional space $\mathcal{S}$, and $\Phi_\infty$ is diagonalizable on $\mathcal{S}$. By linear algebra, the restriction of $\Phi_\infty$ is also diagonalizable on $T_{F_q}$ with a subset of the given eigenvalues, hence $T_{F_q}$ is a subspace of the space of automorphic forms for the corresponding eigenvalues of $\Phi_\infty$. By [8], Theorem 7.1, it can therefore be split into a direct sum of a space of Eisenstein series $\mathcal{E}$, a space of residues of Eisenstein series $\mathcal{R}$, and a space of cusp forms $\mathcal{C}$ (note that in the slightly different notations of [8], “residues of Eisenstein series” are called “cusp series”, too). We treat these spaces separately.

$\mathcal{E}$: By Proposition 3.1, $T_{F_q}(F_q(2))$ contains exactly two Eisenstein series, one corresponding to a zero $s_0$ of $\zeta_{F_q}$, and one corresponding to a zero $s_1$ of $L_q(s) := \zeta_{F_q(2)}(s)/\zeta_{F_q}(s)$.

Now consider the torus $\tilde{T}$ corresponding to the quadratic extension $E_q = F_q(z)/F_q$ of genus two defined by $x = z(z + 1)$. Set

$$\tilde{L}_q(s) := \zeta_{E_q}(s)/\zeta_{F_q}(s)$$

and $T = q^{-s}$. One computes immediately that $L_q = qT^2 + qT + 1$ but

$$\tilde{L}_2 = 2T^2 + 1, \quad \tilde{L}_3 = 3T^2 + T + 1 \quad \text{and} \quad \tilde{L}_4 = 4T^2 + 1.$$

Since $L_q$ and $\tilde{L}_q$ have no common zero, the $\tilde{T}$-integral of the Eisenstein series of weight $s_1$ is non-zero, and hence it doesn’t belong to $T_{F_q}$. Hence $\mathcal{E}$ is as expected.

$\mathcal{R}$: Elements in $\mathcal{R}$ have $\Phi_\infty$-eigenvalues $\neq 0, \pm q$, so cannot even occur in $\mathcal{S}$: since the class number of $F_q$ is one, $\mathcal{R}$ is spanned by the two forms

$$r_\pm := [1, \ldots, 1 \mid \pm 1, 1 \mid 1, \pm 1, 1, \pm 1, \ldots]$$

with $r(c_i) = (\pm 1)^i$, and this is a $\Phi_\infty$-eigenform with eigenvalue $\pm(q + 1)$. (In general, the space is spanned by elements of the form $\chi \circ \det$ with $\chi$ a class group character, cf. [8], p. 174.)
\( \mathcal{C} : \) By multiplicity one, \( \mathcal{C} \) has a basis of simultaneous \( \mathcal{H} \)-eigenforms. From Lemma 3.2, we know that potential cusp forms in \( \mathbb{T}_{\mathcal{F}} \) have support in the set of vertices \( \{ t_i \} \). To prove that \( \mathcal{C} = \{ 0 \} \), the following therefore suffices:

**Proposition 3.7.** The only cusp form which is a simultaneous eigenform for the Hecke algebra \( \mathcal{H} \) and has support in \( \{ t_i \} \) is \( f = 0 \).

**Proof.** Let \( f \) denote such a form. Fix a vertex \( t \in \{ t_i \} \). It corresponds to a point \( P = (t, \ast) \) on \( X_q(\mathbb{F}_q) \), which is a place of degree two of \( \mathbb{F}_q(X_q) \). Let \( \Phi_P \) denote the corresponding Hecke operator. We claim that

**Lemma 3.8.** \( \Phi_P(c_0) = (q + 1)c_2 + q(q - 1)t \).

Given this claim, we finish the proof as follows: we assume that \( f \) is a \( \Phi_P \)-eigenform with eigenvalue \( \lambda_P \). Then

\[
0 = \lambda_P f(c_0) = \Phi_P f(c_0) = q(q - 1)f(t) + (q + 1)f(c_2) = q(q - 1)f(t)
\]

since \( f(c_2) = 0 \), hence \( f(t) = 0 \) for all \( t \).

**Proof of Lemma 3.8.** As in [3], 3.7, the Hecke operator \( \Phi_P \) maps the identity matrix (= the vertex \( c_0 \)) to the set of vertices corresponding to the matrices \( m_\infty := \text{diag}(\pi, 1) \) and \( m_b := \begin{pmatrix} b & \pi \\ 0 & \pi \end{pmatrix} \), where \( \pi = x - \ell \) is a local uniformizer at \( P \) and \( b \) runs through the residue field at \( P \), which is

\[
\mathbb{F}_q[X]/(x - \ell) = \mathbb{F}_q[y]/F(\ell, y) \cong \mathbb{F}_q.
\]

if \( F(x, y) = 0 \) is the defining equation for \( X_q \). Hence we can represent every such \( b \) as \( b = b_0 + b_1y \). We now reduce these matrices to a standard form \( F \setminus \mathcal{T} \) from [12], §2. By right multiplication with \( [1, -b_0] \), we are reduced to considering only \( b = b_1y \).

If \( b_1 = 0 \), then the matrix is \( m_b = \text{diag}(1, \pi) \sim \text{diag}(\pi^{-1}, 1) \). Recall that \( t = x/y \) is a uniformizer at \( \infty \), so \( x - \ell = t^{-2} \cdot A \) for some \( A \in \mathbb{F}_q[\ell] \). Hence right multiplication by \( \text{diag}(A^{-1}, 1) \) gives that this matrix reduces to \( c_2 \). The same is true for \( m_\infty \).

On the other hand, if \( b_1 \neq 0 \), multiplication on the left by \( \text{diag}(1, b_1) \) and on the right by \( \text{diag}(1, b_1^{-1}) \) reduces us to considering \( m_y \). By multiplication on the right with

\[
\text{diag}((x - \ell)^{-1} \cdot A, (x - \ell)^{-1}),
\]

we get \( m_y \sim [t^2, y/(x - \ell)] \). Now note that

\[
\frac{y}{x - \ell} = \frac{y}{x} \cdot \left( 1 + \frac{\ell}{x} + \left( \frac{\ell}{x} \right)^2 + \ldots \right) = t^{-1} + \ell t + \beta(t)t^2
\]

for some \( \beta \in \mathbb{F}_q[\ell] \). Hence right multiplication with \( [1, -\beta] \) gives \( m_y \sim [t^2, t^{-1} + \ell t] \), and this is exactly the vertex \( t \).

**Remark 3.9.** Using different methods, more akin the geometrical Langlands programme, the second author ([9]) has generalized the above results as follows. For a general function field \( F \) of genus \( g \) and class number \( h \), one may show that \( \mathbb{T}_F \) is finite dimensional. Its Eisenstein part is of dimension at least \( h(g - 1) + 1 \). Residues of Eisenstein series are never toroidal. For general elliptic function fields, there are no toroidal cusp forms. For a general function field, the analogue of a result of Waldspurger ([13], Prop. 7) implies that the cusp forms in \( \mathbb{T}_F \) are exactly those having vanishing central \( L \)-value.
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