Mating quadratic maps with the modular group
III:
The modular Mandelbrot set
Shaun Bullett  Luna Lomonaco
May 2, 2023

Abstract
We prove that there exists a homeomorphism $\chi$ between the connectedness locus $\mathcal{M}_\Gamma$ for the family $\mathcal{F}_a$ of ($2 : 2$) holomorphic correspondences introduced by Bullett and Penrose, and the parabolic Mandelbrot set $\mathcal{M}_1$. The homeomorphism $\chi$ is dynamical ($\mathcal{F}_a$ is a mating between $PSL(2, \mathbb{Z})$ and $P_{\chi(a)}$), it is conformal on the interior of $\mathcal{M}_\Gamma$, and it extends to a homeomorphism between suitably defined neighbourhoods in the respective one parameter moduli spaces.

Following the recent proof by Petersen and Roesch that $\mathcal{M}_1$ is homeomorphic to the classical Mandelbrot set $\mathcal{M}$, we deduce that $\mathcal{M}_\Gamma$ is homeomorphic to $\mathcal{M}$.

MSC2020: 37F05, 37F10, 37F44, 37F46.

1 Introduction
Parallels between the dynamical behaviour of iterated rational maps and Kleinian groups have been evident since the work of Fatou and Julia over a century ago: see Sullivan’s celebrated dictionary between the two areas in [S]. In 1994 the first examples of iterated holomorphic correspondences on the Riemann sphere behaving as matings between a rational map and a Kleinian group were exhibited by the first author together with Christopher Penrose [BP]. The rational map was a quadratic polynomial $Q_c : z \rightarrow z^2 + c$ and the Kleinian group was the modular group $\Gamma = PSL(2, \mathbb{Z})$, equipped with the generators $\alpha(z) = z + 1$ and $\beta(z) = z/(z + 1)$.

In [BP] a ($2 : 2$) holomorphic correspondence $\mathcal{F} : z \rightarrow w$ is termed a mating between $Q_c$ and $PSL(2, \mathbb{Z})$ if there exists a completely invariant open simply-connected subset $\Omega$ of the sphere such that:

1. there is a conformal bijection $\phi : \Omega \rightarrow \mathbb{H}$ conjugating $\mathcal{F}|_\Omega$ to $\alpha|_\mathbb{H}$ and $\beta|_\mathbb{H}$ (where $\mathbb{H}$ denotes the upper half-plane), and
2. \( \hat{\mathbb{C}} \setminus \Omega = \Lambda_- \cup \Lambda_+ \), where \( \Lambda_- \cap \Lambda_+ \) is a single point and there exist homomorphisms \( \varphi_{\pm} : \Lambda_{\pm} \rightarrow \Gamma_c \) conjugating \( \mathcal{F}|_{\Lambda_-} \) to \( Q_c|_{\Gamma_c} \) and \( \mathcal{F}|_{\Lambda_+} \) to \( Q_c^{-1}|_{\Gamma_c} \) respectively.

The matings exhibited in [BP] were in the family of \((2 : 2)\) holomorphic correspondences \( \mathcal{F}_a : z \rightarrow w \) defined by the polynomial relation obtained from the following equation by multiplying through by denominators:

\[
\left( \frac{aw - 1}{w - 1} \right)^2 + \left( \frac{aw - 1}{w - 1} \right) \left( \frac{az + 1}{z + 1} \right) + \left( \frac{az + 1}{z + 1} \right)^2 = 3 \tag{1.1}
\]

See Section 2.1 below for an explanation of why a mating between a quadratic rational map and \( PSL(2, \mathbb{Z}) \) has to be in this family, up to conformal conjugacy. The limit set \( \Lambda(\mathcal{F}_a) = \Lambda_- (\mathcal{F}_a) \cup \Lambda_+ (\mathcal{F}_a) \) of the correspondence \( \mathcal{F}_a \) is defined for all \( a \) in the Klein combination locus \( \mathcal{K} \) (see Section 2 or [BL1] for the definition of \( \mathcal{K} \)). The connectedness locus \( \mathcal{C}_a \) for the family \( \mathcal{F}_a \) is the set of values of the parameter \( a \in \mathcal{K} \) for which the limit set \( \Lambda(\mathcal{F}_a) \) is connected. Denoting by \( \mathbb{D}(4, 3) \) the closed disc with centre \( a = 4 \) and radius \( 3 \), the intersection \( \mathcal{M}_1 := \mathcal{C}_a \cap \mathbb{D}(4, 3) \) is called the modular Mandelbrot set. In [BP] it was conjectured that for every \( a \in \mathcal{M}_1 \) the correspondence \( \mathcal{F}_a \) is a mating between \( PSL(2, \mathbb{Z}) \) and some \( Q_c \) with \( c \in \mathcal{M} \), the classical Mandelbrot set, and that \( \mathcal{M}_1 \) is homeomorphic to \( \mathcal{M} \). A computer plot of \( \mathcal{M}_1 \) is displayed in Figure 1.

Considerable progress was made on the first conjecture using ‘pinching’ techniques (see [BH]), but it was only fully resolved (in [BL1]) with the application of the theory of ‘parabolic-like mappings’ [L1], developed by the second author of the present paper. Her key observation was that the family \( \mathcal{F}_a \) presents a persistent parabolic fixed point \( P_e \) of multiplier 1 at the intersection of \( \Lambda_- \) and \( \Lambda_+ \), which indicates that a family of maps with a persistent parabolic fixed point may be a better model than quadratic polynomials, that family being \( \text{Per}_0(1) \), that is to say maps which are conjugate to some \( P_A(z) = z + 1/z + A, A \in \mathbb{C} \). These maps are the parabolic analogues of quadratic polynomials (for which there is a persistent super-attracting fixed point). The parabolic Mandelbrot set \( \mathcal{M}_1 \) (Figure 1 centre) is the connectedness locus of this family: the set of \( B = 1 - A^2 \in \mathbb{C} \) such that the complement \( K_A \) of the parabolic basin of attraction of infinity is connected. The parabolic Mandelbrot set \( \mathcal{M}_1 \) was recently proved to be homeomorphic to \( \mathcal{M} \) ([PR]).

In the present paper we resolve the second conjecture proposed in [BP]:

**Main Theorem.** The modular Mandelbrot set \( \mathcal{M}_1 \) is homeomorphic to the parabolic Mandelbrot set \( \mathcal{M}_1 \), via a homeomorphism \( \chi \) which has the properties:

(i) For every \( a \in \mathcal{M}_1 \), the correspondence \( \mathcal{F}_a \) is a mating between the rational map \( P_{\chi(a)} \) and the modular group \( PSL(2, \mathbb{Z}) \);

(ii) \( \chi \) is a conformal homeomorphism between the interior \( \mathcal{M}_1 \) of \( \mathcal{M}_1 \) and the interior \( \mathcal{M}_1 \) of \( \mathcal{M}_1 \);

(iii) when restricted to the complement of a closed neighbourhood (in the induced topology) \( N \subset \mathcal{M}_1 \) of the root point \( a = 7 \) of \( \mathcal{M}_1 \), the homeomorphism \( \chi \) extends to a homeomorphism between an open set containing \( \mathcal{M}_1 \setminus N \) in the \( a \)-plane and an open set containing \( \mathcal{M}_1 \setminus \chi(N) \) in \( \text{Per}_0(1) \).
Figure 1: On the left, the modular Mandelbrot set $\mathcal{M}_{\Gamma}$, which we prove in this paper to be homeomorphic to the parabolic Mandelbrot set $\mathcal{M}_1$ (in the centre). On the right is the classical Mandelbrot set $\mathcal{M}$. Dynamical space plots of $F_a$, for $a$ at the centres of three components of $\mathcal{M}_{\Gamma}$, are displayed in Figure 4 while dynamical plots of the elements of $\text{Per}_1(1)$ at the centres of the corresponding components of $\mathcal{M}_1$ are displayed in Figure 5. The homeomorphism $\chi : \mathcal{M}_{\Gamma} \rightarrow \mathcal{M}_1$ sends centres to centres.

Observe that (iii) tells us that the limbs and sublimbs of $\mathcal{M}_{\Gamma}$ are laid out in the $a$-plane in combinatorially the same arrangement as the corresponding limbs and sublimbs of $\mathcal{M}_1$. Another consequence of the Main Theorem is:

**Corollary.** The modular Mandelbrot set $\mathcal{M}_{\Gamma}$ is the whole of the connectedness locus $\mathcal{C}_{\Gamma}$.

Thus the conjectures of [BP] have all been answered positively, once they have been translated into what is now clearly the right setting by substituting ‘parabolic quadratic rational maps’ for ‘quadratic polynomials’, and ‘$\mathcal{M}_1$’ for ‘$\mathcal{M}$’. Moreover, following the recent proof by Petersen and Roesch [PR] of Milnor’s conjecture that $\mathcal{M}_1$ is homeomorphic to the classical Mandelbrot set $\mathcal{M}$, our Main Theorem proves that $\mathcal{M}_{\Gamma}$ is homeomorphic to $\mathcal{M}$, as originally conjectured in [BP].

There is as yet no complete combinatorial description of the classical Mandelbrot set $\mathcal{M}$, as such a description has to await the resolution of MLC, the celebrated conjecture that $\mathcal{M}$ is locally connected. The fact that $\mathcal{M}$ and $\mathcal{M}_{\Gamma}$ have now been shown to be homeomorphic offers the possibility of exploring $\mathcal{M}$ through another model, one which for example has a different set of Yoccoz inequalities, yet to be fully worked out, governing the sizes of its limbs and sub-limbs (see [BL2]).

**The anti-holomorphic case.** In independent work, Lee, Lyubich, Makarov and Mukherjee investigated anti-holomorphic matings in a recent article [LLMM]. They established the existence of a homeomorphism between a combinatorial model of the connectedness locus of a certain family of anti-holomorphic maps (matings, in the sense of [LLMM]), and a combinatorial model of the Tricorn, the anti-holomorphic analogue of the Mandelbrot Set. Their proof introduces an alternative ‘straightening’ strategy, different from the approach in [LL1] and the present paper.
Strategy of proof and layout of paper. In Sections 2.1 and 2.2 we summarise the background facts needed concerning the family $F_a$ and the family $Per_1(1)$. In Section 3 we develop a surgery construction which converts $F_a$ into a quadratic rational map in the family $P_A$, without the intermediate parabolic-like stage of [BL1], and moreover does so uniformly with the parameter. The basic idea is to replace the complement of the backwards limit set $\Lambda_-(F_a)$ in $\hat{C}$ by a copy of the basin $A_0(\infty)$ of the parabolic fixed point $\infty$ of a rational map of the form $P_A$, so that $\Lambda_-(F_a)$ becomes the filled Julia set of such a $P_A$. As a model for the parabolic basin we take the Blaschke product $h(z) = \frac{z^2 + 1/3}{z^2/3 + 1}$ on $\hat{C} \setminus \mathbb{D}$.

We borrow a stratagem from [L1] to overcome the difficulty of gluing a smooth set outside a set with cusps, and so realising the replacement: for every $F_a$ we construct forward invariant arcs $\gamma_a$ emanating from the parabolic fixed point, which move holomorphically with $a$, and which in a neighbourhood of $\Lambda_a$, separate the expanding dynamics of $F_a$ from the parabolic dynamics. We choose $a_0 \in M_{1\Gamma}$ and glue the dynamics of $h$ outside a set $O_{a_0}$ containing the limit set $\Lambda_a$, having $\gamma_{a_0}$ on the boundary, and to obtain uniformity with respect to the parameter we ensure the boundaries of $O_a$ move holomorphically with respect to $a$. To start the construction of $O_a$, we first choose a (pinched) neighbourhood of $M_{1\Gamma} \setminus \{7\}$ in parameter space in which to work. This is a lune $L_\theta$, the open set bounded by two arcs of circles intersecting with angle $2\theta$ between them at the points $a = 1$ and $a = 7$ (the root point of $M_{1\Gamma}$). The existence of such a neighbourhood for some value of $\theta$ in the half-open interval $[\pi/3, \pi/2)$ was proved in [BL2] as an application of a new Yoccoz inequality derived there. In the Appendix to the present paper we show that for every $a \in L_\theta$ there exists a dynamical space lune $V_a$ which moves holomorphically with $a$ and contains $\Lambda(F_a)$. To guarantee the holomorphic motion of the arcs $\gamma_a$ we restrict our attention to a doubly truncated lune $K \subset L_\theta$ (Section 3.2.1). The truncation at the end $a = 7$ of the lune $L_\theta$ is by the removal of an arbitrarily small disc neighbourhood $N$ of the root point, but rather than overload the notation $K$ with subscripts or superscripts we invite the reader to keep in mind that our holomorphic motions parametrised by $K$, and hence our subsequent surgery construction for $F_a$, $a \in K$ and definition of $\chi : K \to Per_1(1)$, are all a priori dependent on $N$.

In Section 3.2 we define the central set $O_a$ to be the connected component of $F_a^{-1}(\mathbb{C}) \setminus \gamma_a$ containing the limit set $\Lambda_a$, and in Section 3.3 we construct a holomorphic motion of the complement $\hat{C} \setminus O_a$ of this set. In Section 3.4.1 we choose a base point $a_0 \in K \cap M_{1\Gamma}$ for our holomorphic motion and construct an ‘external map’ $g$ for $F_{a_0}$, by conjugating the dynamics of $F_{a_0}$ by the uniformization map $\alpha$ from the complement of the limit set $\Lambda_{a_0}$ to the complement of the closed unit disc, and we define the sets $O_g$ and $\gamma_g$ for $g$ by taking the image under $\alpha$ of the respective sets for $F_{a_0}$. Using Fatou coordinates (see Lemma 3.3) we construct an equivariant quasi-symmetric map between $\gamma_g$ and
the corresponding forward invariant arc $\gamma_h$ for $h$: extending this we define a quasiconformal map $f$ between the complement of $O_g$ and the complement of the corresponding set $O_h$ for $h$. Then, precomposing $f$ by $\alpha$, we glue the dynamics of $h$ on the complement of $O_h$ to the dynamics of $F_{a_0}$ on $O_{a_0}$. Composing $f$ with our holomorphic motion of the complement of $O_{a_0}$, we obtain the surgery construction for the whole family $F_a$ (Section 3.4.3).

In Section 4 we define the map $\chi : M_\Gamma \to M_1$, using the fact that when $a \in M_\Gamma$ the surgery yields a unique $P_a \in M_1$ hybrid equivalent to $F_a$, and that by Proposition 3.4 this $P_a$ does not depend on the choice of dividing curves or the various choices of quasiconformal homeomorphism made during the surgery construction. We then prove that $\chi : M_\Gamma \to M_1$ is injective (Proposition 4.1).

While $\chi$ is also well-defined on $\hat{K} \setminus M_\Gamma$, its definition there depends on choices made during the surgery, and it is by no means obvious a priori that $\chi$ is injective on the whole of $\hat{K}$, nor that it is continuous there. But in Section 4.2 we show that on $\hat{K} \setminus M_\Gamma$ the map $\chi$ can be interpreted as the same map (Proposition 4.2) as that obtained from the position of the critical value of $F_a|_{V_a}$ in analogy to Douady and Hubbard’s map in their analysis of families of polynomial-like maps [DH]. By applying Lyubich’s formulation (Chapter 6 of [Lyu]) of the methods introduced by Douady and Hubbard, we prove that with suitable choices made in the initial surgery at $a = a_0$ the extension of $\chi$ from $M_\Gamma$ to $\hat{K} \setminus M_\Gamma$ is locally quasiregular (Proposition 4.3), that $\chi$ is continuous on the boundary of $M_\Gamma \cap \hat{K}$ (Proposition 4.5), and that $\chi$ is holomorphic on the interior of $M_\Gamma \cap \hat{K}$ (Propositions 4.6 and 4.7). Finally we deduce that $\chi$ maps $\hat{K}$ homeomorphically onto its image in $\text{Per}_1(1)$ (Proposition 4.8).

Since the surgery preserves limit sets, and hence their connectedness or otherwise, $\chi(M_\Gamma \setminus \{7\})$ is contained in $M_1 \setminus \{1\}$. However it is not obvious that $M_1 \setminus \{1\} \subset \chi(M_\Gamma \setminus \{7\})$. If the root $a = 7$ of $M_\Gamma$ were contained in the domain $\hat{K}$ of our holomorphic motion, we could apply $\chi$ to a loop encircling $M_\Gamma$ and it would follow that $M_1 \subset \chi(M_\Gamma)$, from the known connectedness of $M_1$. But $\hat{K}$ does not contain the root of $M_\Gamma$. We get around this problem by decomposing $M_\Gamma$ into its main hyperbolic component and limbs $L_{p/q}$ (Proposition 4.9), and encircling each limb with a loop starting and ending at the root of the limb (see Lemma 4.3). In Section 4.6 we complete the proof of the Main Theorem by proving (Corollary 4.3) that $\chi : M_\Gamma \setminus \{7\} \to M_1 \setminus \{1\}$ extends continuously to the respective root points $a = 7$ and $B = 1$. In Section 5 we deduce the Corollary that $M_\Gamma$ is the whole of the connectedness locus of the family $F_a$.

Acknowledgments. This research has been partially supported by the Fundação de Amparo à Pesquisa do Estado de São Paulo (Fapesp, processes 2016/50431-6, 2017/03283-4), the Cnpq (406575/2016-19), the prize L’ORÉAL-UNESCO-ABC Para Mulheres na Ciência and the Serrapilheira Institute (grant number Serra-1811-26166).
The families $F_a$ and $\text{Per}_1(1)$

2.1 The family $F_a$

A holomorphic correspondence on $\hat{\mathbb{C}}$ is a multivalued map defined by a polynomial relation $P(z, w) = 0$. The correspondence is $(n : m)$ when the polynomial $P(z, w)$ which defines it has degree $n$ in $z$ and $m$ in $w$, meaning that each $z$ has $m$ corresponding (images) $w$ and each $w$ has $n$ corresponding (pre-images) $z$. In particular, a holomorphic $(2 : 2)$ correspondences on $\hat{\mathbb{C}}$ is a 2-valued map (with a 2-valued inverse map)

$$\mathcal{F}: z \rightarrow w$$

defined implicitly by an equation $P(z, w) = 0$ where $P$ has the form

$$P(z, w) = (az^2 + bz + c)w^2 + (dz^2 + ez + f)w + gz^2 + hz + j.$$ 

The family of holomorphic $(2 : 2)$ correspondences introduced in \cite{BP} are the implicit functions $F_a: z \rightarrow w$ defined by the equation (1.1) (Section 1), repeated here for the convenience of the reader:

$$\left(\frac{aw - 1}{w - 1}\right)^2 + \left(\frac{aw - 1}{w - 1}\right)\left(\frac{az + 1}{z + 1}\right) + \left(\frac{az + 1}{z + 1}\right)^2 = 3,$$

where $a \in \mathbb{C}$. In the current paper such a correspondence will usually either be expressed in terms of the coordinate $z$, as in equation (1.1), or in terms of the coordinate

$$Z = \frac{az + 1}{z + 1},$$

in which the equation of the correspondence becomes

$$J_a(W)^2 + Z.J_a(W) + Z^2 = 3,$$

where $J_a$ is the unique conformal involution which has fixed points at $Z = 1$ and $Z = a$. In the coordinate $z$, the conformal involution has fixed points $z = 0$ and $z = \infty$, and there it is $J_a(z) \equiv J(z) = -z$. To simplify computations, we will also use the coordinates $\zeta = Z - 1$ in Proposition 3.2 and the coordinates $z' = (a - 1)z$ in Propositions 3.1 and 3.2 and for Lemma 3.3.

When written in terms of the coordinate $Z$, it is apparent that $F_a: Z \rightarrow W$ can also be written in the form

$$W = J_a \circ \text{Cov}_Q^0(Z),$$

where $\text{Cov}_Q^0$ is the deleted covering correspondence of the cubic polynomial $Q(Z) = Z^3 - 3Z$, that is to say (see \cite{BL1}) $\text{Cov}_Q^0$ is the correspondence

$$Z \rightarrow W : \frac{Q(W) - Q(z)}{W - z} = 0.$$
Why investigate this particular family of correspondences? The reason is that every $(2 : 2)$ holomorphic correspondence $F$ with the property that $\hat{\mathbb{C}}$ is partitioned into completely invariant $\Omega$ and $\Lambda$ as above, with a conjugacy between $F$ on $\Omega$ and $PSL(2, \mathbb{Z})$ on $\mathbb{H}$, is conformally conjugate to some $F_a$. To see why this is so, observe that the $(2 : 2)$ correspondence defined by $\alpha: z \to z + 1$ and $\beta: z \to z/(z + 1)$ on $\mathbb{H}$ satisfies the 'diagram condition' below, since $\alpha \circ \beta^{-1} \circ \alpha = \beta \circ \alpha^{-1} \circ \beta$.

This diagram condition says that the two 'zig-zag' orbits (forwards, then backwards, then forwards) starting at any initial $z_i$ arrive at the same destination $w_i$. An alternative description is given by regarding an 'arrow' in the diagram as a point of the curve (surface) $P(z, w) = 0$, for example the arrow $z_1 \to w_2$ corresponds to the point $(z_1, w_2)$. The curve, the graph of the correspondence, comes equipped with covering involutions, $I_+$ and $I_-$:

$$I_+(z, w) = (z, w') \text{ where } P(z, w') = 0,$$
$$I_-(z, w) = (z', w) \text{ where } P(z', w) = 0.$$

The diagram condition above is equivalent to asking that

$$(I_- \circ I_+)^3 = (I_- \circ I_+) \circ (I_- \circ I_+) \circ (I_- \circ I_+) = Id.$$ 

For this condition to hold on $\Omega$, it must (by analytic continuation) hold on the whole of $\hat{\mathbb{C}}$. This implies that our correspondence factorises as a deleted covering correspondence $Cov^Q_0$ of a degree 3 rational map $Q$, (the rational map identifying together the points $z_1, z_2$ and $z_3$ in the diagram condition) followed by a Möbius transformation $M$ (the zig-zag map sending each $z_i$ to the corresponding $w_i$).

Thus

$$\mathcal{F} = M \circ Cov^Q_0.$$

Moreover on $\Omega$ the Möbius transformation $M$ restricts to an involution (since $\alpha \circ \beta^{-1} \circ \alpha$ is an involution on $\mathbb{H}$), so, again by analytic continuation, $M$ must be an involution (which we shall denote by $J$ in this paper) on $\hat{\mathbb{C}}$, and thus

$$\mathcal{F} = J \circ Cov^Q_0.$$

Every degree 3 rational map has 4 critical points (counted with multiplicity). Our cubic $Q$ must have a double critical point (this corresponds to the point of $\mathbb{H}$ where $z \to z + 1$ and $z \to z/(z + 1)$ coincide), and two single critical points.
else the correspondence would factorise into a pair of Möbius transformations), and as we can post-compose $Q$ by any Möbius transformation without altering $Cov_Q^0$, we can normalise $Q$ to the specific polynomial

$$Q(Z) = Z^3 - 3Z$$

which is what it will be for the rest of this paper. Finally, we note that one consequence of the conditions we have imposed on the restriction of $F$ to $\Lambda_+$ and $\Lambda_-$ is that $\Lambda_- \cap \Lambda_+ = \{P\}$ is a fixed point of $F$ (that is, $P \in F(P)$), and as $P$ is also a fixed point of $J$ (since $J$ is the zig-zag map, and it is easily seen that our conditions imply that the zig-zag map sends $\Lambda_-$ to $\Lambda_+$) we deduce that $P$ must be fixed by $Cov_Q^0$, and it is therefore one of the critical points $Z = \pm 1$ of $Q$. Replacing the $Z$-coordinate by $-Z$ if necessary, we can choose this critical point to be $Z = +1$. So $J = J_a$ has fixed points $Z = 1$ and $Z = a$.

Having answered the question of why consider the family $F_a$, our next task is to show that for some $a \in \mathbb{C}$ the Riemann sphere $\hat{\mathbb{C}}$ can indeed be partitioned into a completely invariant open set $\Omega$ and a completely invariant closed set $\Lambda$ with the properties we would like. By construction, for certain $a \in \mathbb{C}$, there exists a fundamental domain for $F_a$, which we can obtain by intersecting fundamental domains for $Cov_Q^0$ and $J_a$, constructed as follows. The point $Z = 1$ is a critical point for $Q$ with co-critical point $Z = -2$ (that is $Q(1) = Q(-2)$). The three blue half-lines in Figure 2 are the inverse image $Q^{-1}((-\infty, -2])$ of the half line $(-\infty, -2]$ on the quotient (image) Riemann sphere. Thus the deleted covering correspondence $Cov_Q^0$ of $Q$ sends each of these half-lines to the other two. The open set $\Delta_{Cov}^f$ bounded by the two half-lines starting at $Z = 1$ and running off to $\infty$ at angles $\pm \pi/3$ with the positive real axis, and containing the point $Z = 2$, is the standard fundamental domain for $Cov_Q^0$. The complement of the round disc with boundary passing through $Z = 1$ and $Z = a$ (illustrated in Figure 3, on the left) is the standard fundamental domain for $J_a$, which we denote $\Delta_J^f$.

When $a \in \mathbb{C}(4,3) \setminus \{1\}$ the complements of $\Delta_{Cov}^f$ and $\Delta_J^f$ intersect in the single point $Z = 1$ and $\Delta_{corr}^f := \Delta_J^f \cap \Delta_{Cov}^f$ is a fundamental domain for $F_a$ on the union $\Omega$ of all images of $\Delta_{corr}^f$ under (mixed) iteration of $F_a$ and $F_a^{-1}$ (the correspondence $F_a$ is undefined at $a = 1$).
Figure 3: On the left, standard fundamental domains for $Cov_Q$ and $J_a$. On the right, images and preimages of fundamental domains.

More generally, the *Klein combination locus* $K$ is the set of parameters $a \in \mathbb{C}$ for which there exist fundamental domains $\Delta_J$ and $\Delta_{Cov}$, bounded by Jordan curves and such that $\Delta_J \cup \Delta_{Cov} = \hat{\mathbb{C}} \setminus \{1\}$. For every $a \in K$ the set

$$\Delta_{corr} := \Delta_J \cap \Delta_{Cov}$$

is a fundamental domain for $\mathcal{F}_a$ acting on the union of all images of $\Delta_{corr}$ (see [BL1]). Note that $K$ contains $D(4,3) \setminus \{1\}$.

By construction, for every $a \in K$ we have the following behaviour: $Z = 1$ is a parabolic fixed point for $\mathcal{F}_a$ of multiplier 1; both images of the complement of $\Delta_J$ belong to the complement of $\Delta_J$, that is: $\mathcal{F}_a((\Delta_J)^c) \subset (\Delta_J)^c$ and the restriction of $\mathcal{F}_a$ to $\Delta_J$ is a $(1 : 2)$ correspondence; while both pre-images of $\Delta_J$ lie inside $\Delta_J$, so the restriction of $\mathcal{F}_a$ to the preimage of $\Delta_J$ is a $(2 : 1)$ map (see Proposition 3.4 in [BL1] and Figure 3 on the right). The *forward limit set* $\Lambda_{a,+}$ for $\mathcal{F}_a$ is defined as

$$\Lambda_{a,+} = \bigcap_{n=0}^{\infty} \mathcal{F}_a^n(\hat{\mathbb{C}} \setminus \Delta_J),$$

the *backward limit set* $\Lambda_{a,-}$ for $\mathcal{F}_a$ is defined to be

$$\Lambda_{a,-} = \bigcap_{n=0}^{\infty} \mathcal{F}_a^{-n}(\Delta_J) = J(\Lambda_{a,+})$$

and the *limit set* $\Lambda_a$ for $\mathcal{F}_a$ is defined to be $\Lambda_a = \Lambda_{a,+} \cup \Lambda_{a,-}$ (by Proposition 3.4 in [BL1] we have $\Lambda_{a,+} \cap \Lambda_{a,-} = \{1\}$). The regular set $\Omega_a$ is defined to be $\hat{\mathbb{C}} \setminus \Lambda_a$: it is tiled by the images of $\Delta_{corr}$. The limit set of $\mathcal{F}_a$ is shown in grey in Figure 4 for three different values of $a$: the pictures are plots in the plane of the coordinate $z$, the red and blue lines are the boundaries of the standard domains (transferred from the $Z$-coordinate to the $z$-coordinate), and their images under mixed iteration of $\mathcal{F}_a$.

The correspondence $\mathcal{F}_a$ has certain associated *characteristic points*: a conformal conjugacy between $\mathcal{F}_a$ and $\mathcal{F}_{a'}$ necessarily sends each of these to the
corresponding characteristic point of $F_{a'}$. The first such point is the persistent parabolic fixed point $P = P_a$ of $F_a$, characterised by the fact that it is both a fixed point of $J$ and a critical point of $Q$: in the $Z$-coordinate it is the point $Z = 1$, and in the $z$-coordinate $P$ is $z = 0$. The branch of $F_a$ which fixes $P$ has derivative 1 at $P$. Further characteristic points are the other critical points $Z = -1$ and $Z = \infty$ of $Q$, and the other fixed point, $Z = a$ of $J_a$: we remark that $Z = -1$ can also be regarded as the critical point $c_a$ of the $(2 : 1)$ restriction of $F_a$ to $\Lambda_{a,-}$. The cross-ratio of any four distinct characteristic points is preserved by every conformal conjugacy between $F_a$ and $F_{a'}$. Applying this to the four points $Z = 1$, $-1$, $\infty$ and $a$ we have:

**Lemma 2.1.** If $F_a$ is conformally conjugate to $F_{a'}$ then $a = a'$. □

### 2.2 $Per_1(1)$

$Per_1(1)$ is Milnor’s notation for the space of conformal conjugacy classes of quadratic rational maps having a parabolic fixed point with multiplier 1. Normalising such a map to put the parabolic fixed point at $z = \infty$, and critical points at $\pm 1$, $Per_1(1)$ is the family

$$Per_1(1) := \{ [P_A] | P_A(z) = z + 1/z + A, \ A \in \mathbb{C} \}$$

of conformal conjugacy classes $[P_A]$. Here $P_A \sim P_{A'}$ if and only if $A' = \pm A$ (note that $P_A \sim P_{-A}$, by the involution $z \rightarrow -z$, which interchanges the critical points). For every $A \in \mathbb{C}$, the parabolic basin of attraction of infinity, $A_A(\infty)$, is completely invariant, and we can define the filled Julia set $K_A$ to be its complement, that is,

$$K_A := \hat{\mathbb{C}} \setminus A_A(\infty).$$

This is the parabolic counterpart of the definition of filled Julia set for any polynomial $P$ on $\hat{\mathbb{C}}$. Note that the filled Julia set $K_A$ is well defined for every $A \neq 0$. The map $P_0(z) = z + 1/z$ is the unique member of $Per_1(1)$ for which the multiplicity of the parabolic fixed point at $\infty$ is equal to 3, which means that for this map there exist two attracting petals, and both are completely
invariant Fatou components: the Julia set of $P_0$ is the imaginary axis, and the two attracting petals are the positive and the negative half planes $\{z \in \mathbb{C} | \text{Re}(z) > 0\}$ and $\{z \in \mathbb{C} | \text{Re}(z) < 0\}$ respectively. For consistency with [L1], we set $K_0 = \{z \in \mathbb{C} | \text{Re}(z) < 0\}$.

The parabolic Mandelbrot set $\mathcal{M}_1$ (Figure 1 center) is the connectedness locus for $\text{Per}_1(1)$, which is usually parametrised by

$$B = 1 - A^2,$$

the multiplier of the fixed point of $P_A$ other than $z = \infty$. In Figure 5 we show the filled Julia set $K_A$ of $P_A$, for $B = 1 - A^2$ at the centres of the main hyperbolic component of $\mathcal{M}_1$, the period 2 component and a period 3 component.

For $A = 0$, the map $P_0(z) = z + 1/z$ is conformally conjugate on the Riemann sphere to the map

$$h(z) = \frac{z^2 + 1/3}{z^2/3 + 1}$$

via the conjugating function $\phi(z) = \frac{z + 1}{z - 1}$. Using Fatou coordinates, it is easy to see that, for every $A$ such that $B = 1 - A^2 \in \mathcal{M}_1$, the dynamics of the map $P_A$ is conformally conjugate on $\mathcal{A}(\infty)$ to the dynamics of $P_0$ on $\mathcal{A}_0(\infty) = \mathbb{H} = \{x + iy | x > 0\}$. The second author proved that this conjugacy still exists in the disconnected case, although restricted; more precisely, when $P_A$ has disconnected Julia set, there exists a conformal conjugacy between $P_0$ and $P_A$ defined between fundamental annuli about the respective filled Julia sets (see Proposition 4.2 in [L1]). Roughly speaking, this means that the map $h$ encodes the dynamics of the $P_A$ on its basin of attraction of infinity. The Main Theorem will be obtained by gluing the map $h$ to the outside of the backward limit set $\Lambda_{a,-}$ of $\mathcal{F}_a$, using quasiconformal surgery (see Section 3.4). We observe that $h|\mathbb{S}^1$ is topologically conjugate to the map $z \to z^2$, and hence via the Minkowski question mark function to the union of the generators $\alpha, \beta$ of the modular group.
on $[-\infty,-1]$ and $[-1,0]$ respectively (once we have identified the ends $-\infty$ and 0 of the negative real axis). So with hindsight the construction in [BP] could have started with the family $\text{Per}_1(1)$ in place of quadratic polynomials.

## 3 The surgery

In this Section we develop a surgery construction which, for $a$ in a certain open subset $K$ of parameter space, will convert the correspondence $F_a$ into a rational map in $\text{Per}_1(1)$, uniformly with respect to the parameter $a$. This construction is at the heart of our definition of the map $\chi: \mathcal{M}_T \to \mathcal{M}_1$ (see section 3.4), and of its extension to $K \setminus \mathcal{M}_T$ (see 4.2).

### 3.1 The parameter space lune $\mathcal{L}_\theta$ and dynamical space lune $V_a$

A lune is the name given in Euclidean geometry to the region trapped between two intersecting circular arcs. All our lunes will be open sets, in other words a lune will not include its boundary arcs. The vertex angle of a lune is the angle between the arcs at their intersection points (the vertices). It will be crucial to our construction that we restrict the correspondence $F_a$ to a lune $V_a$ of vertex angle strictly less than $\pi$, such that the parabolic fixed point $P_a$ of $F_a$ is one of the vertices of $V_a$, and $V_a$ contains $\Lambda_{-a} \setminus \{P_a\}$. Moreover we will need the boundary $\partial V_a$ to move holomorphically with respect to $a$. In [BL2], as an application of a new Yoccoz-type inequality derived there for family $F_a$, we proved that there exists an angle $\theta \in [\pi/3,\pi/2)$ such that $\mathcal{M}_T$ is contained in the closure of the lune (of vertex angle $2\theta$)

$$\mathcal{L}_\theta := \{ a : |\arg(\frac{a-1}{1-a})| < \theta \}.$$  

More precisely, we proved in Theorem 3 of [BL2] that $\mathcal{M}_T \subset \mathcal{L}_\theta \cup \{7\}$ and $\mathcal{M}_T \cap \partial \mathcal{L}_\theta = \{7\}$.

We now define the lune $L_a$ in the $Z$-coordinate to be the open set bounded by the two arcs intersecting at $Z = 1$ (the point $P_a$) and $Z = a$ which pass through $Z = 1$ at angles $\pm \theta$ to the positive real axis. In the $z$-coordinate $L_a$ is a sector (as arcs of circles through $z = 0$ and $z = \infty$ are straight lines); this sector varies with $a$ but its motion is holomorphic with respect to $a$: indeed in the new coordinate defined by $z' = (a-1)z$ the lune $L_a$ is independent of $a$, so stationary. In the Appendix of the current paper (Proposition A.1 and Corollary A.1) we show that for every $a \in \mathcal{L}_\theta \cup \{7\}$, the forward image $F_a(L_a)$ is contained in $L_a \cup \{P_a\}$. It follows that the forward limit set $\Lambda_{+,a}$ is contained in $L_a \cup \{P_a\}$.

A computation (see Proposition 3.5 in [BL1]) shows that when $a \neq 7$, in the coordinate $\zeta = Z - 1$ the power series expansion of the branch of $F_a$ which fixes $\zeta = 0$ has the form:

$$\zeta \to \zeta + \frac{a-7}{3(a-1)}\zeta^2 + \ldots,$$
so the repelling direction is $\arg\left(\frac{z-a}{z'}\right)$. For technical reasons which will become apparent shortly, we now choose $\theta \in (\theta, \pi/2)$, and let $\hat{L}_a$ denote the lune which has vertices at $Z = 1$ and $Z = a$, bounded by arcs which meet the real Z-axis at angles $\pm \theta$ (so $L_a \subset \hat{L}_a$), and define

$$V_a := J(\hat{L}_a).$$

For all $a \in \mathcal{L}_\theta$ the repelling parabolic direction is compactly contained in $V_a$ since $\theta > \theta$ (this will be important for the proof of Proposition 3.2). By construction $V_a$ moves holomorphically with $a$. Moreover $\mathcal{F}_a^{-1}(V_a) \subset V_a \cup \{P_a\}$ and hence $\Lambda_{a,-} \subset V_a \cup \{P_a\}$.

**Lemma 3.1.** $\mathcal{M}_\Gamma$ is the set of parameters $a \in \mathcal{L}_\theta \cup \{\hat{L}\}$ for which the critical point $c_a$ is in $\Lambda_{\cdot}(\mathcal{F}_a)$. In particular, $\mathcal{M}_\Gamma$ is closed.

**Proof.** From the definition of $\mathcal{M}_\Gamma$, for each $a \in \mathcal{M}_\Gamma$ the set $\Lambda_{\cdot}(\mathcal{F}_a)$ is connected. If the critical point $c_a$ is not in $\Lambda_{\cdot}(\mathcal{F}_a)$, then some $\mathcal{F}_a^{-n}(V_a)$ is a pair of topological discs. Hence $\Lambda_{\cdot}(\mathcal{F}_a)$ is disconnected (indeed a Cantor set). Conversely, if $c_a \in \Lambda_{\cdot}(\mathcal{F}_a)$ then each $\mathcal{F}_a^{-n}(V_a)$ is connected and hence so is $\Lambda_{\cdot}(\mathcal{F}_a)$. \qed

### 3.2 The parameter space $\hat{K} \subset \mathcal{L}_\theta$

Define $V'_a := \mathcal{F}_a^{-1}(V_a)$. For our surgery construction we will need the critical point $c_a$ of $\mathcal{F}_a|_{V_a}$ to be in $V'_a$, so we first restrict our parameter space to the subset $\mathcal{L}'_\theta \subset \mathcal{L}_\theta$ for which the critical value $v_a$ of $\mathcal{F}_a|_{V_a}$ is in $V_a$.

**Proposition 3.1.** $\mathcal{L}'_\theta$ is a simply connected open subset of $\mathcal{L}_\theta$, with the properties that $\mathcal{M}_\Gamma \subset \mathcal{L}'_\theta \cup \{\hat{L}\}$ and $\partial \mathcal{L}'_\theta \cap \mathcal{M}_\Gamma = \{\hat{L}\}$.

**Proof.** Write $\mathcal{F}_a$ as $J_a \circ \text{Cov}_Q^Q$ (see Section 2.4). In $Z$ coordinates, the critical point of $\text{Cov}_Q^Q$ which belongs to $V_a$ is $Z = -1$, and the corresponding critical branch of $\text{Cov}_Q^Q$ sends $-1 \to 2$, so the critical value of $\mathcal{F}_a$ in $V_a$ is $J(2)$.

For the remainder of the proof it will be convenient to change to the coordinate

$$z' = (a-1)z = (a-1)(Z-1)/(a-Z),$$

where $\hat{L}_a$ and $V_a$ are independent of $a$. In this new coordinate $\hat{L}_a$ is the subset of the right-hand half-place bounded by the straight lines through at the origin at angles $\pm \theta$ to the positive real axis, the involution is $J(z) = -z$, and the critical value is $v_a = -(a-1)/(a-2)$.

By definition $\mathcal{L}'_\theta$ is the subset of $\mathcal{L}_\theta$ for which $v_a \in V_a$, that is to say for which $J(v_a) \in \hat{L}_a$, equivalently $(a-1)/(a-2) \in \hat{L}_a$: the $a \in \mathcal{L}_\theta$ for which $\arg((a-1)/(a-2)) \in (-\theta, \theta)$. Thus $\mathcal{L}'_\theta$ is obtained from $\mathcal{L}_\theta$ by excising its intersection with the pair of closed round discs which have boundary circles passing through both $a = 1$ and $a = 2$ at angles $\pm \theta$ to the real axis. The ‘truncated lune’ $\mathcal{L}'_\theta$, union the point $a = 7$, certainly contains $\mathcal{M}_\Gamma$ since when $a \in \mathcal{M}_\Gamma$ we know that $v_a \in \Lambda_{\cdot} \subset V_a$, and the boundary of $\mathcal{L}'_\theta$ can only meet that of $\mathcal{M}_\Gamma$ at $a = 7$ since $\mathcal{L}'_\theta$ is open and $\mathcal{M}_\Gamma$ is closed. \qed
Lemma 3.2. For $a \in \mathcal{L}_\theta' \cup \{7\}$ the restriction $F_a|_{V'_a} : V'_a \to V_a$ is a branched covering map of degree 2, which is holomorphic in both variables and has a persistent parabolic fixed point at $P_a \in \partial V_a$.

Proof. For every $a \in \mathcal{L}_\theta' \cup \{7\}$ we have $V_a \subset \Delta_J$, and by Proposition 3.4 in [BL1] (see also the discussion in Section 2.1 and Figure 3) the restriction of $F_a$ to domain $F_a^{-1}(\Delta_J)$ and codomain $\Delta_J$ is a degree 2 holomorphic map, so that $F_a|_{V'_a} : V'_a \to V_a$ is also holomorphic and of degree 2 in the dynamical variable. The fact that it is also holomorphic in the parameter $a$ follows from its explicit representation as a polynomial relation.

Lemma 3.3. For $a \in \mathcal{L}_\theta'$, the boundary $\partial V_a$ of the lune $V_a$ and its inverse image $\partial V'_a := F_a^{-1}(\partial V_a)$ move holomorphically.

Proof. Choose a base point $a_0 \in \mathcal{L}_\theta'$. To say that $\partial V_a$ moves holomorphically is to say that there exists a homeomorphism $\psi_a : \partial V_a \to \partial V_a$ depending holomorphically on the parameter $a$. But in the coordinate $z' = (a - 1)z$ we may take $\psi_a$ to be the identity as in this coordinate $V_a$ is independent of $a$: hence $V_a$ moves holomorphically in any coordinate varying holomorphically with $a$. The correspondence $F_a : V'_a \to V_a$ is holomorphic in both variables, and for $a \in \mathcal{L}_\theta'$ the critical point belongs to $V'_a$, so by lifting the holomorphic motion of $\partial V_a$ we obtain a holomorphic motion of $\partial V'_a$.

3.2.1 The doubly truncated lune $\check{K} \subset \mathcal{L}_\theta'$

Let $N$ be a small round closed disc neighbourhood of $a = 7$ in the plane of the parameter $a$, and let

$$K = \mathcal{L}_\theta' \setminus N$$

(see Figure 6).

Then $K$ is a compact set and its interior $\check{K}$ is a topological disc. The doubly truncated lune $\check{K}$ is the set of parameter values for which we will perform the surgery and which will ultimately be the domain on which shall define the map $\chi$. For the surgery construction, the existence of forward invariant arcs $\gamma_{a,i}, i \in \{1, 2\}$ for $F_a$, with the properties listed in the following Proposition, will be essential (see Section 3.4.2).

Proposition 3.2. There exists a family of forward invariant arcs $\gamma_{a,i}, i \in \{1, 2\}$ for $F_a$, parametrised by $a \in \check{K}$, such that each $\gamma_a := \gamma_{a,1} \cup \gamma_{a,2}$ is connected, moves holomorphically with $a \in \check{K}$, and meets $\partial V_a$ transversally.

Proof. For the proof of this proposition we will work in the $z' = (a - 1)z$ coordinate, in which the lune $V_a$ is always the subset of the left-hand half-plane bounded by the straight lines through the origin at angles $\pm \theta$ to the negative real axis, and the parabolic fixed point $P_a$ is the origin. So we will write $V$ for $V_a$, and $P$ for $P_a$.

The idea of the proof is the following. Using pre-Fatou coordinates (denoted by $\psi_a$) and compactness, we will construct a subset $\check{K}$ of $\mathcal{L}_\theta'$ and repelling petals $\Xi_a$ of $F_a$ at $P$ moving holomorphically with the parameter sufficiently
Figure 6: The doubly truncated lune $K$ in parameter space. Excising the left-hand end of the lune $L_\theta$ gives a truncated lune $L'_\theta$ with the property that $\forall a \in L'_\theta$ the critical value $v_a$ of $F_a$ lies in the dynamical lune $V_a$ (Proposition 3.1). Removing the intersection of $L'_\theta$ with an arbitrarily small disc $N$ at the right-hand end gives $K$ with the property that there exists $\gamma_a$ in $V_a$ moving holomorphically with $a \in \tilde{K}$ (Proposition 3.2).

Figure 7: For $a \in L_\theta$, the repelling direction is contained in $V$, and so $\psi_a([w_{a,i}, P]), i \in \{1, 2\}$ (represented as the green half lines on the right) are not parallel to the real axis.

large enough that there exist a pair of points $W^i \in \partial V, i \in \{1, 2\}$ belonging to $\Xi_a$ for all $a \in \tilde{K}$. Then, using Fatou coordinates (denoted by $\phi_a$) sending the repelling direction to the real axis and depending holomorphically on the parameter $a$, we can define $\gamma_{a,i}$ to be the pre-image under Fatou coordinates of the horizontal straight line between the image of $W^i$ under Fatou coordinates and $-\infty$. By construction, $\gamma_{a,i}, i \in \{1, 2\}$ are forward invariant under $F_a$, the path $\gamma_a := \gamma_{a,1} \cup \gamma_{a,2}$ is connected and, since $\phi_a(W^1)$ and $\phi_a(W^2)$ move holomorphically with $a$, the whole of $\gamma_a$ also moves holomorphically with $a$.

We proceed to the details. As remarked in Section 3.1 in $\zeta = Z - 1$ coordinates, when $a \neq 7$, the power series expansion of the branch of $F_a$ which fixes $\zeta = 0$ has the form:

$$\zeta \rightarrow \zeta + \frac{a - 7}{3(a - 1)} \zeta^2 + \ldots$$

Since the change of coordinates from $\zeta = Z - 1$ to $z' = (a - 1)(Z - 1)/(a - Z)$
has derivative equal to $+1$ at $\zeta = 0$, the power series expansion in $z'$ of the branch of $F_a$ fixing $P_a$ has the same coefficients in its terms of degree $\leq 2$ as those in the $\zeta$ coordinate, so this power series has the form:

$$z' \rightarrow z' + \frac{a - 7}{3(a - 1)} z'^2 + \ldots$$

Let $\psi_a(\zeta) = -1/(b(a)\zeta)$, where $b(a) = \frac{a - 7}{3(a - 1)}$, be pre-Fatou coordinates, and define the map

$$F_a = \psi_a \circ F_a \circ \psi_a^{-1}.$$ 

Then a computation shows that

$$F_a(w) = w + 1 + d(a)/w + O(1/w^2)$$

for a constant $d(a)$ (see Section 2.1.2 in [Sh]). Let $c \geq 2$ be sufficiently large that, setting $\tau := \arctan(1/c)$, we have $(\pi/2 - \hat{\theta})/2 < \tau < \pi/2 - \hat{\theta}$ (where $2\theta$ is the angle of $V$ at $P$), and let $R = R(a)$ be sufficiently large that whenever $|w| > R/(2c)$ we have $|F_a(w) - (w + 1)| < 1/(2c)$. Now, setting and so defining

$$X_a := \{ w \in \mathbb{C} : Re(w) < c|Im(w)| - R \},$$

$$F_a$$

is defined and injective on $X_a$. Recalling that $K = \mathcal{Z}_a \setminus \hat{N}$ for a small neighbourhood of $a = 7$, let $\hat{R} = \max_{a \in K} R(a)$ and define

$$X := \{ w \in \mathbb{C} : Re(w) < c|Im(w)| - \hat{R} \}.$$ 

Then the set $\Xi_a := \psi_a^{-1}(X)$ is a repelling petal for $F_a$ at $P$, that moves holomorphically with $a \in K$, and that contains a neighbourhood of $P$ in $\partial V$ for all $a \in K$. Hence, for every $a \neq 7$, there exist $w_{a,i} \in \partial V$, $i \in \{1,2\}$ such that $\psi_a([w_{a,i}, P)) \in X$, $i \in \{1,2\}$, and since the repelling direction is contained in $V$, for $w_{a,i}$ sufficiently close to $P$ the half-lines $\psi_a([w_{a,i}, P))$ are nowhere parallel to the real axis (see Figure 7). By the compactness of $K$, there exists a pair of points $W^i \in \partial V$, $i \in \{1,2\}$, one on each side of $P$, which lie in $\Xi_a$ for all $a \in K$ and have the property that the half-lines $\psi_a([W^i, P)), i \in \{1,2\}$ are nowhere parallel to the real axis. 

Now let $\phi_a : \Xi_a \rightarrow \hat{X} := \phi_a(\Xi_a) \subset \mathbb{C}$ be Fatou coordinates depending holomorphically on the parameter $a$ and sending the repelling direction to the real axis (this is, $\phi_a = \Phi_a \circ \psi_a$, where $\Phi(w) = w - d(a) + c + O(1/w^2)$ and $c$ is a constant, see Proposition 2.2.1 in [Sh]). Thus in particular $W_{a,i} := \phi_a(W^i), i \in \{1,2\}$ move holomorphically. Let $\hat{\gamma}_{a,i}, i \in \{1,2\}$ denote the horizontal straight lines between $W_{a,i}$ and $-\infty$. Then $\hat{\gamma}_{a,i}, i \in \{1,2\}$ are contained in $\hat{X}$, are forward invariant, are transversal to $\phi_a(\partial V)$, and move holomorphically with $a \in K$. Setting $\gamma_{a,i} := \phi_a^{-1}(\hat{\gamma}_{a,i}), i \in \{1,2\}$ and $\gamma_a := \gamma_{a,1} \cup \gamma_{a,2}$, the family of $\gamma_a$ for $a \in K$ has the desired properties. 

\[\Box\]
3.3 Holomorphic motions

The curves $\gamma_{a,i}, i = \{1, 2\}$ divide $V'_a$ into three topological discs. Two of these are in the repelling petal $\Xi_a$ for $F_a$ at $P_a$. The third contains the critical point $c_a$ and the limit set $\Lambda_{a,-}$ of $F_a$, and it will play a central role in our surgery: we call this set the central set (Figure 8), and we denote it by $O_a$. Later, we will define a corresponding set $O_h$ for the Blaschke product $h$; our surgery construction will glue $F_a$ on $O_a$ to $h$ on $\hat{\mathbb{C}} \setminus O_h$. We define the fundamental pinched annulus to be

$$\mathcal{PA}_a := V_a \setminus O_a.$$ 

Then $\mathcal{PA}_a$ does not contain the critical point, nor does it intersect the backward limit set. Recall that the set $\mathring{K} \subset \mathcal{L}'_\theta$ is a topological disc, and that by Lemma 3.3 for $a \in \mathcal{L}'_\theta$ both $\partial V_a$ and $\partial V'_a$ move holomorphically. Since their intersection is precisely the parabolic fixed point, their union also moves holomorphically. By Proposition 3.2, the curve $\gamma_a := \gamma_{a,1} \cup \{P_a\} \cup \gamma_{a,2}$ moves holomorphically for $a \in \mathring{K} \subset \mathcal{L}'_\theta$. Hence, choosing a base point $a_0$ in the set $\mathring{M}_\Gamma \setminus \mathcal{N} \subset \mathring{K}$, we can define the holomorphic motion

$$\tau : \mathring{K} \times (\partial V_{a_0}) \cup (\partial V'_{a_0}) \cup \gamma_{a_0} \to \mathbb{C}$$

holomorphic in $a$ (fixing $z$) and quasiconformal in $z$ (fixing $a$), such that

$$\tau_a(\partial V_{a_0}) \cup (\partial V'_{a_0}) \cup \gamma_{a_0} = \partial V_a \cup \partial V'_a \cup \gamma_a.$$ 

Since $\mathring{K}$ is conformally homeomorphic to a disc, by Slodkowski’s Theorem the motion $\tau$ extends to a holomorphic motion of the complex plane:

$$\mathbf{\tau} : \mathring{K} \times \mathbb{C} \to \mathbb{C},$$

which we can then restrict to a holomorphic motion of the exterior of the central set $O_a$:

$$\mathring{\tau} : \mathring{K} \times \mathring{\mathbb{C}} \setminus O_{a_0} \to \mathring{\mathbb{C}},$$

and by construction $\mathring{\tau}_a(\mathcal{PA}_{a_0}) = \mathcal{PA}_a$.

3.4 Surgery construction

We showed in [BL1] that for every $a$ in the Klein combination locus $\mathcal{K}$, the correspondence $F_a$ can be modified by surgery to become a parabolic-like map, with the consequence that $F_a$ is hybrid equivalent to a member of $\text{Per}_1(1)$ for every $a \in \mathcal{K}$, a unique such member if $\Lambda_{a,-}(F_a)$ is connected.

We now perform the surgery in a different fashion, going directly from the correspondences $F_a$ to rational maps in $\text{Per}_1(1)$, by adapting the methods of [L1]. The map

$$h(z) = \frac{z^2 + 1/3}{1 + z^2/3}$$

is an external map for $P_A(z) = z + 1/z + A$ for all $A \in \mathbb{C}$ ([L1]), and the idea of the surgery is to glue this external map acting on $\mathring{\mathbb{C}} \setminus \mathcal{D}$ onto $\mathring{\mathbb{C}} \setminus \Lambda_{a,-}$ by
quasiconformal surgery. In order to do this we first fix \(a_0 \in \mathcal{M}_\Gamma\), we uniformize the complement of \(\Lambda_{a_0}\) to the complement of the closed unit disc and we conjugate \(F_{a_0}\) with the uniformization, constructing an ‘external map’ for \(F_{a_0}\). We use this external map to construct a quasiconformal map \(\hat{f}\) from \(\hat{\mathbb{C}} \setminus O_{a_0}\) to the corresponding set for \(h\), which we use to convert \(F_{a_0}\) into a member of \(\text{Per}_1(1)\). Finally, composing with the holomorphic motion of \(\hat{\mathbb{C}} \setminus O_{a}\) moves this to a surgery on \(F_a\).

### 3.4.1 An ‘external map’ \(g\) for \(F_{a_0}\)

Let \(a_0 \in \mathcal{M}_\Gamma\), and let \(\alpha : \hat{\mathbb{C}} \setminus \Lambda_{a_0,-} \to \hat{\mathbb{C}} \setminus \mathbb{D}\) be the Riemann map, normalized by \(\alpha(\infty) = \infty\) and \(\alpha(\gamma_{a_0}(t)) \to 1\) as \(t \to 0\). Define \(\check{V}_g := \alpha(V \setminus \Lambda_{a_0,-})\), and \(\check{V}'_g := \alpha(V' \setminus \Lambda_{a_0,-})\) (see Figure 9). Then the map

\[
\tilde{g} := \alpha \circ F_{a_0} \circ \alpha^{-1} : \check{V}_g \to \check{V}_g
\]

is a holomorphic degree 2 covering by construction. Let \(\tau(z) = 1/\bar{z}\) be the reflection with respect to the unit circle, and define the sets \(\check{V}_g := \tau(\check{V}_g)\), \(\check{V}'_g := \tau(\check{V}'_g)\), \(V_g := \check{V}_g \cup S^1 \cup \check{V}_g\) and \(V'_g := \check{V}'_g \cup S^1 \cup \check{V}'_g\). Applying the strong reflection principle with respect to \(S^1\) we can analytically extend the map \(\tilde{g} : \check{V}'_g \to \check{V}_g\) to \(g : V'_g \to V_g\).

By construction, \(g\) has a parabolic fixed point at \(z = 1\) with two repelling petals \(\Xi_{g,1}, \Xi_{g,2}\), which intersect the unit circle \(S^1\). Define \(\gamma_{g,i} := \alpha(\gamma_{a_0,i})\), for \(i = 1, 2\). Then by construction \(\gamma_{g,i} \in \Xi_{g,i}, i = 1, 2\). Let \(\phi_{g,1}, \phi_{g,2}\) be repelling Fatou coordinates defined in \(\Xi_{g,1}, \Xi_{g,2}\) respectively, with axis tangent to the unit circle at the parabolic fixed point. Define \(\gamma_g := \gamma_{g,1} \cup \{1\} \cup \gamma_{g,2}\), and \(O_g := V_g \setminus \alpha(\mathcal{P}A_{a_0})\). Then \(\partial O_g := \alpha(\partial O_{a_0}) \cup \{1\}\) is a quasicircle, as it is a piecewise \(C^1\)-curve with no zero angle.
\[
\begin{align*}
\alpha &:= F_{a,0} \circ \alpha^{-1} \\
g &:= \alpha \circ F_{a,0} \circ \alpha^{-1} \\
\psi_1 &:= \partial V_g \cap \gamma_g, i = \{1, 2\} \\
z_{g,i} &:= \partial V_g \cap \gamma_g, i = \{1, 2\} \\
z_{a,i} &:= \partial V_g \cap \gamma_a, i = \{1, 2\} \\
\gamma_a, 1 &:= \phi^{-1}_a \circ \phi_a(\gamma_a, 1) \\
\gamma_a, 2 &:= \phi^{-1}_a \circ \phi_a(\gamma_a, 2) \\
\gamma_g, 1 &:= \phi^{-1}_g \circ \phi_g(\gamma_g, 1) \\
\gamma_g, 2 &:= \phi^{-1}_g \circ \phi_g(\gamma_g, 2) \\
\end{align*}
\]

Figure 9: Upper left, we have \( F_a : V'_a \to V_a \) and the dividing arcs \( \gamma_{a,i}, i = \{1, 2\} \). Below, we have \( g : V'_g \to V_g \) and its dividing arcs \( \gamma_{g,i} = \alpha(\gamma_{a,i}) \). On the right, \( h : V'_h \to V_h \) and its dividing arcs \( \gamma_{h,i} = \phi^{-1}_h \circ \phi_g(\gamma_{g,i}) \).

3.4.2 Gluing \( F_a \) on \( O_{a_0} \) to \( h \) on \( \hat{C} \setminus O_h \)

Choose \( \epsilon > 0 \), let \( V_h := D(-\epsilon, 1 + \epsilon) \), and \( V'_h := h^{-1}(V_h) \). The map \( h \) has two repelling petals \( \Xi_{h,i}, i = 1, 2 \) which intersect the unit circle; let \( \phi_{h,1}, \phi_{h,2} \) be repelling Fatou coordinates defined on \( \Xi_{h,1}, \Xi_{h,2} \) respectively and with axis tangent to the unit circle at the parabolic fixed point. On \( V'_h \) define the arcs \( \gamma_{h,i} := \phi^{-1}_{h,i} \circ \phi_{g,i}(\gamma_{g,i}) \), and \( \gamma_h := \gamma_{h,1} \cup \{1\} \cup \gamma_{h,2} \). Let \( \phi = \phi^{-1}_h \circ \phi_g : \gamma_g \to \gamma_h \) be

\[
\phi := \begin{cases} 
\phi^{-1}_{h,1} \circ \phi_{g,1} & \text{on } \gamma_{g,1} \\
\phi^{-1}_{h,2} \circ \phi_{g,2} & \text{on } \gamma_{g,2}
\end{cases}
\]

Then \( \phi \) is a conjugacy between \( g|\gamma_g \) and \( h|\gamma_h \). This conjugacy is quasisymmetric by Proposition 3.3, which we delay until the end of this subsection to avoid interrupting the construction of the surgery.

Remark 3.1. Other choices are possible for \( V_h \) (see Proposition 4.2).

Let \( z_{g,i} := \partial V_g \cap \gamma_g, i = \{1, 2\} \) and \( z_{h,i} := \partial V_h \cap \gamma_h, i = \{1, 2\} \). Let \( \psi_1 : \partial V_g \to \partial V_h \) be a diffeomorphism sending the points \( z_{g,1} \) and \( z_{g,2} \) to \( z_{h,1} \) and \( z_{h,2} \) respectively, and let \( \psi_2 : \partial V'_g \to \partial V'_h \) be a lift of \( \psi_1 \) to double covers (so that \( \psi_1 \circ g = h \circ \psi_2 \)). Note that \( \gamma_g \) divides \( V_g \setminus V'_g \) into three connected components, two of which have the parabolic fixed point 1 on the boundary. Call the third one \( Q_g \) (so \( \partial Q \cap \{1\} = \emptyset \)). Similarly, let \( Q_h \) be the connected
component of \((V_h \setminus V'_h) \setminus \gamma_h\) with \(\partial Q_h \cap \{1\} = \emptyset\). Then both \(\partial Q_g\) and \(\partial Q_h\) are quasicircles, as they are piecewise \(C^1\) curves without zero angles, so that the quasisymmetric map

\[ \Phi_Q : \partial Q_g \to \partial Q_h \]

defined as:

\[
\Phi_Q := \begin{cases} 
\psi_1 & \text{on } \partial V_g \\
\psi_2 & \text{on } \partial V'_g \\
\phi_h^{-1} \circ \phi_g & \text{on } \gamma_{g,i}, i = [1,2] \cap \partial Q_g 
\end{cases}
\]

extends to a quasiconformal map

\[ \Psi_Q : \overline{Q}_g \to \overline{Q}_h. \]

The arc \(\gamma_h\) divides \(V'_h\) into three connected components: the set \(O_h\) is the one containing the unit disc. Define \(U_g := O_g \cup Q_g\), and similarly \(U_h := O_h \cup Q_h\). As both \(\partial U_g\) and \(\partial U_h\) are piecewise \(C^1\) curves without zero angles, the quasisymmetric map \(\Phi_U : \partial U_h \to \partial U_h\) defined as:

\[
\Phi_U := \begin{cases} 
\psi_1 & \text{on } \partial V_g \\
\phi_h^{-1} \circ \phi_g & \text{on } \gamma_{g,i}, i = [1,2] 
\end{cases}
\]

extends to a quasiconformal map \(\Psi_U : \hat{\mathbb{C}} \setminus U_g \to \hat{\mathbb{C}} \setminus U_h\). Define the quasiconformal map \(\Psi_{O_h} : \hat{\mathbb{C}} \setminus O_g \to \hat{\mathbb{C}} \setminus O_h\) to be

\[
\Psi_{O_h} := \begin{cases} 
\Psi_Q & \text{on } Q_g \\
\Psi_{U'} & \text{on } \hat{\mathbb{C}} \setminus U_g. 
\end{cases}
\]

We can now define the quasiconformal map \(f : \hat{\mathbb{C}} \setminus O_{a_0} \to \hat{\mathbb{C}} \setminus O_h\) to be

\[ f := \Psi_{O_h} \circ \alpha. \]

Putting everything together, define \(F_{a_0} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) to be

\[
F_{a_0} := \begin{cases} 
F_{a_0} & \text{on } O_{a_0} \\
f^{-1} \circ h \circ f & \text{on } \hat{\mathbb{C}} \setminus O_{a_0}. 
\end{cases}
\]

Then \(F_{a_0}\) is continuous, so quasiregular. On \(\hat{\mathbb{C}} \setminus O_{a_0}\) define the Beltrami form

\[ \hat{\mu}_{a_0} := f^*(\mu_0), \]

and on \(\hat{\mathbb{C}}\) the Beltrami form

\[
\mu_{a_0} := \begin{cases} 
\hat{\mu}_{a_0} & \text{on } \hat{\mathbb{C}} \setminus O_{a_0} \\
(F_{a_0}^{-1})^*(\hat{\mu}_{a_0}) & \text{on } F_{a_0}^{-1}(C_{a_0}) \cap O_{a_0} \\
\mu_0 & \text{on } \Lambda_{a_0}. 
\end{cases}
\]

and note that, by construction, \(\mu_{a_0}\) is \(F_{a_0}\)-invariant and has \(|\mu_{a_0}|_{\infty} < 1\).
It follows by the Measurable Riemann Mapping Theorem that there exists a quasiconformal homeomorphism \( \phi_{a_0} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \), integrating \( \mu_{a_0} \), and sending the parabolic fixed point \( P_a \) to \( \infty \), its preimage \( S_a \) to \( 0 \), and the critical point \( c_{a_0} \) to \(-1\). Define \( \bar{F}_{a_0} \) to be the holomorphic map of degree two

\[
\bar{F}_{a_0} := \phi_{a_0} \circ F_{a_0} \circ \phi_{a_0}^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}},
\]

and observe that \( \bar{F}_{a_0} \) is a quadratic rational map, with a parabolic fixed point at \( \infty \) having multiplier 1, with the other preimage of this fixed point at 0, and with a critical point at \(-1\), concluding our surgery construction converting the correspondence \( F_{a_0} \) into an element of \( \text{Per}_1(1) \).

As promised in the first paragraph of this subsection, we now present a proof that the map \( \phi : \gamma_g \to \gamma_h \) defined there is quasisymmetric. This employs a similar argument to that in Proposition 5.3(ii) of [L1]: we give details for the convenience of the reader.

**Proposition 3.3.** The map \( \phi := \phi_{h}^{-1} \circ \phi_{g} : \gamma_g \to \gamma_h \) defined as

\[
\phi = \begin{cases} 
\phi_{h,1}^{-1} \circ \phi_{g,1} & \text{on } \gamma_g,1 \\
\phi_{h,2} \circ \phi_{g,2} & \text{on } \gamma_g,2
\end{cases}
\]

is a quasisymmetric conjugacy between \( g|\gamma_g \) and \( h|\gamma_h \).

**Proof.** It is clear that \( \phi \) is a conjugacy and that it is quasisymmetric on each of the two halves \( \gamma_{g,i} \) \( i = 1, 2 \), of \( \gamma_g \), since \( \phi_{g,i} \) and \( \phi_{h,i} \) are Fatou coordinates for a repelling petal of \( g \) and a repelling petal of \( h \) respectively, hence they are diffeomorphisms, and since the curves \( \gamma_{g,i} \) and \( \gamma_{h,i} i = 1, 2 \), are quasicircles (arcs of quasicircles). So it just remains to check that \( \phi \) is quasisymmetric at the parabolic point, where the curves \( \gamma_{g,i} i = 1, 2 \), join; we will show this by looking at the asymptotics of Fatou coordinates at the parabolic point, following the proof of Proposition 5.3(ii) in [L1].

Let \( T(z) = z + 1 \), then \( \hat{g}(z) := T \circ \bar{g} \circ T^{-1}(z) = z + a z^3 + h.o.t. \) for some \( 0 \neq a \in \mathbb{C} \). Let \( \hat{\Xi}_{g,i} := T^{-1}(\Xi_{g,i}) \) be repelling petals for \( \hat{g} \), set \( \hat{\gamma}_{g,i} := T^{-1}(\gamma_{g,i}) \), and \( \hat{\gamma}_g := \hat{\gamma}_{g,1} \cup \{0\} \cup \hat{\gamma}_{g,2} \). Let \( \hat{\phi}_{g,i} : \hat{\Xi}_{g,i} \to \hat{H}_g \) be repelling Fatou coordinates for \( \hat{g} \) with axis tangent to the imaginary axis at the parabolic fixed point. Then

\[
\hat{\phi}_{g,i} = \Phi_{g,i} \circ I_g,
\]

where \( I_g(z) := -\frac{1}{2a z^2} \) conjugates \( \hat{g} \) to the map \( \hat{G}(w) = w + 1 + \frac{a}{w} + O\left(\frac{1}{w^2}\right) \) on \( \Xi_{g,i}, i = 1, 2 \), and the map \( \Phi_{g,i}(w) = w - \hat{a} \log(w) + c_i + o(1) \) conjugates \( \hat{G}(w) \) to the map \( T(z) = z + 1 \) on \( I_g(\Xi_{g,i}), i = 1, 2 \), (see Proposition 2.2.1 in [SI]). Define \( \Gamma_{g,1} := I_g(\hat{\gamma}_{g,1}), \Gamma_{g,2} := -I_g(\hat{\gamma}_{g,2}) \) (see Figure [10]), and

\[
\Gamma_g := \Gamma_{g,1} \cup \{\infty\} \cup \Gamma_{g,2}.
\]

The map \( \tilde{I}_g : \hat{\gamma}_g \to \Gamma_g \) (see Figure [10] top right), defined as:
is quasisymmetric on a neighborhood of 0. Set \( \gamma_1 := (\Phi_g, 1 \circ I_g)(\hat{\gamma}_g, 1) \), \( \gamma_2 := (\Phi_g, 2 \circ I_g)(\hat{\gamma}_g, 2) \) and \( \gamma := \gamma_1 \cup \{ \infty \} \cup -\gamma_2 \). Define the map \( \Phi_g : \Gamma_g \to \gamma \) as follows (see Figure 10, bottom right):

\[
\bar{\Phi}_g(w) = \begin{cases} 
\Phi_{g,1}(w) & \text{on } \Gamma_{g,1} \\
-\Phi_{g,2}(-w) & \text{on } \Gamma_{g,2}
\end{cases}
\]

Note that the map \( \bar{\Phi}_g \) is conformal on \( \Gamma_g \setminus \infty \). Since \( \Phi_{g,i}(w) = w - \hat{a} \log(w) + c_i + o(1) \), the maps \( \Phi_{g,i} \) have derivatives \( \Phi'_{g,i} = 1 + o(1) \) at \( \infty \), hence the map \( \bar{\Phi}_g : \Gamma_g \to \gamma \) is a diffeomorphism.

Repeating the process for the map \( h \), we can write the map \( \hat{\phi}_h^{-1} \circ \hat{\phi}_g \) as

\[
\hat{\phi}_h^{-1} \circ \hat{\phi}_g = T \circ \phi_h^{-1} \circ \phi_g \circ T^{-1} = \hat{I}_h^{-1} \circ \hat{\Phi}_h^{-1} \circ \hat{\Phi}_g \circ \hat{I}_g : \hat{\gamma}_g \to \hat{\gamma}_h
\]

which is quasisymmetric.

### 3.4.3 Surgery for the whole family

The surgery so far has been to convert a single correspondence \( F_{a_0} \) into a rational map in \( \text{Per}_1(1) \). We now move this holomorphically to make a surgery construction for the whole family \( F_a, a \in \hat{K} \). In Section 3.3, we constructed a holomorphic motion

\[
\tau : \hat{K} \times \hat{\mathbb{C}} \setminus O_{a_0} \to \hat{\mathbb{C}},
\]
with restriction $\tilde{\tau}_a(\mathcal{P}A_a) = \mathcal{P}A_a$. Let $f : \hat{\mathcal{C}} \setminus O_{a_0} \to \hat{\mathcal{C}} \setminus O_h$ be the quasiconformal map we constructed in Section 3.4.2. We now define the map

$$T_a := f \circ \tau_a^{-1} : \hat{\mathcal{C}} \setminus O_a \to \hat{\mathcal{C}} \setminus O_h$$

that will play the role of a tubing (see Figure 11).

For $a \in \hat{K}$ define the map $F_a : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ to be

$$F_a := \begin{cases} F_a & \text{on } O_a \\ (T_a)^{-1} \circ h \circ T_a & \text{on } \hat{\mathcal{C}} \setminus O_a \end{cases}$$

and note that this map is continuous by construction, and hence quasiregular of degree 2. Define the Beltrami form $\hat{\mu}_a = T_a^*(\mu_0)$ on $\hat{\mathcal{C}} \setminus O_a$, and on $\hat{\mathcal{C}}$ define the Beltrami form

$$\mu_a := \begin{cases} \hat{\mu}_a & \text{on } \hat{\mathcal{C}} \setminus O_a \\ (F^n)^*(\hat{\mu}_a) & \text{on } O_a \setminus \Lambda_{a,-} \\ \mu_0 & \text{on } \Lambda_{a,-} \end{cases}$$

By construction, $\mu_a$ is $F_a$-invariant and $|\mu_a|_\infty < 1$. Hence, by the Measurable Riemann Mapping Theorem, there exists a unique quasiconformal homeomorphism $f_a : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$, integrating $\mu_a$, and sending the parabolic fixed point $P_a$ to $\infty$, the critical point $c_a$ to $+1$, and $T_a^{-1}(\infty)$ to $-1$. The map

$$F_a := f_a \circ F_a \circ f_a^{-1} : \hat{\mathcal{C}} \to \hat{\mathcal{C}},$$

is a holomorphic degree 2 map, and hence a quadratic rational map, with a parabolic fixed point at $\infty$ having multiplier 1, and critical points at $-1$ and $+1$. Thus $F_a(z) = z + 1/z + A$ for some (unique) $A \in \mathbb{C}$. 

![Figure 11: Construction for the whole family](image)
Remark 3.2. Note that, for each $a \in \hat{K}$, the dilatation of the integrating map $\phi_a$ is the same as that of the tubing $T_a$. Indeed, on $\hat{C} \setminus O_a$ we have $\mu_a = T_a^* (\mu_0)$, and on the preimages of $\mathcal{P} \mathcal{A} a \in O_a$ the Beltrami form $\mu_a$ is obtained by spreading $T_a^* (\mu_0)$ by the dynamics of $\mathcal{F} a | \mathcal{O} a$ (a holomorphic map, so it does not change the dilatation of $\mu_a$), while on $\Lambda_{a,-}$ we have $\mu_a = \mu_0$.

Remark 3.3. By the Measurable Riemann Mapping Theorem, there also exists a unique quasiconformal map $\psi_a : \hat{C} \to \hat{C}$ integrating $\mu_a$ and sending $P$ to $\infty$, $c_a \to -1$ and $T_a^{-1} (\infty) \to +1$. If $\phi_a \circ F_a \circ \phi_a^{-1} (z) = z + 1/z + A$, for some $A \in \mathbb{C}$, we have that $\psi_a \circ F_a \circ \psi_a^{-1} (z) = z + 1/z - A$.

Proposition 3.4. For each $a \in \hat{K} \cap \mathcal{M}_1$ there exists a unique $B \in \mathcal{M}_1$ such that $P_A (z) = z + 1/z + A$, where $B = 1 - A^2$, is hybrid equivalent to $\mathcal{F} a$ on a doubly pinched neighbourhood of $\Lambda_{a,-}$.

Proof. Assume that there exist $\phi_1 : \hat{C} \to \hat{C}$ and $\phi_2 : \hat{C} \to \hat{C}$ quasiconformal maps such that $F_{a,1} = \phi_1 \circ F_a \circ \phi_1^{-1} = P_{A_1}$, and $F_{a,2} = \phi_2 \circ F_a \circ \phi_2^{-1} = P_{A_2}$. Then $P_{A_1}$ is hybrid equivalent to $P_{A_2}$ on a doubly pinched neighbourhood of the filled Julia set $K_{A_1}$, and so, by Proposition 6.5 in [1], $P_{A_1}$ is Möbius conjugate to $P_{A_2}$. Hence $(A_1)^2 = (A_2)^2$, so $B_1 = B_2$.

4 The map $\chi : \mathcal{M}_1 \to M_1$ is a homeomorphism

Our surgery construction converts a correspondence $\mathcal{F}_a$, $a \in \hat{K}$, into a rational map $P_A : a \to z + 1/z + A$. A priori this rational map might depend on all the choices made along the way in the construction, but if $a \in \mathcal{M}_1 \cap \hat{K}$ then Proposition 3.3 guarantees that $P_A$ is unique up to $A \sim -A$. Moreover as $\mathcal{F}_a$ has connected limit set $\Lambda_{a,-}$, the hybrid equivalent map $P_A$ has connected filled Julia set $K_A$ and so we may define a map

$$\chi : \mathcal{M}_1 \cap \hat{K} \to \mathcal{M}_1$$

by $\chi(a) = B$, where $B = 1 - A^2$. We shall first prove that $\chi$ is injective, using the Rickmann Lemma (see Proposition 4.1). We will then define a quasiregular extension of $\chi | \mathcal{M}_1 \cap \hat{K}$ to $\hat{K} \setminus \mathcal{M}_1$ (Section 4.2 and Proposition 4.3), which will turn out to be identical to the function $\chi$ that we have already defined via surgery (Proposition 4.4). In Section 4.3 we prove that $\chi$ is continuous on the whole of $\hat{K}$, using the Mañé-Sad-Sullivan partition of $\mathcal{L}_0'$ into $\mathcal{M}_1$, $\partial \mathcal{M}_1 \setminus \{T\}$ and $\mathcal{L}_0' \setminus \mathcal{M}_1$ (Section 4.3.1), and the now classical method of Douady and Hubbard [DH] in the formulation presented by Lyubich [15]. In Section 4.4 we show that $\chi : \hat{K} \to \chi(\hat{K})$ is a homeomorphism (Proposition 4.5). Finally, an analysis of the positions and diameters of the limbs of $\mathcal{M}_1$ in a neighbourhood of its root point $a = 7$ and those of the corresponding limbs of $\mathcal{M}_1$ will enable us to prove that every $B \in \mathcal{M}_1 \setminus \{1\}$ is in the image of $\chi(\hat{K})$ (for $K = \mathcal{L}_0' \setminus N$) with the neighbourhood $N$ of $a = 7$ sufficiently small (Section 4.5), and to deduce that $\chi$ is a homeomorphism from $\mathcal{M}_1$ to $\mathcal{M}_1$ (Section 4.6).
4.1 Injectivity of $\chi : \mathcal{M}_R \cap \hat{K} \to \mathcal{M}_1$

**Lemma 4.1.** If $\chi(a_1) = \chi(a_2)$, then $F_{a_1}$ and $F_{a_2}$ are hybrid equivalent on quadruply pinched neighbourhoods of $\Lambda_1 = \Lambda_{a_1,-} \cup \Lambda_{a_1,+}$ and $\Lambda_2 = \Lambda_{a_2,-} \cup \Lambda_{a_2,+}$ respectively.

**Proof.** Assume $\chi(a_1) = \chi(a_2)$. Then the composition $\phi_{a_1,a_2} := \phi_{a_2}^{-1} \circ \phi_{a_1}$ is a hybrid conjugacy between $F_{a_1}$ and $F_{a_2}$ on $O_{a_1}$.

Note that $O_{a_1}$ is a doubly pinched neighbourhood of $\Lambda_{a_1,-}$, pinched at $P_{a_1}$ and $S_{a_1}$ (where $S_{a_1}$ is the preimage of $P_{a_1}$), and that $O_{a_1} \subset V_{a_1} \subset \Delta_J$ (where $\Delta_J$ is the fundamental domain for the involution, which in $z$ coordinates is $J(z) = -z$). So the set

$$\hat{O}_{a_1} = O_{a_1} \cup J(O_{a_1})$$

is a quadruply pinched neighbourhood of $\Lambda_{a_1}$, pinched at $S_{a_1}$, at $J(S_{a_1})$, and doubly pinched (from both sides) at the parabolic fixed point $P_{a_1}$. The map $\overline{\phi_{a_1,a_2}} : \hat{O}_{a_1} \to \hat{O}_{a_2}$ defined as:

$$\overline{\phi_{a_1,a_2}}(z) := \begin{cases} 
\phi_{a_1,a_2}(z) & \text{if } z \in O_{a_1} \\
J(\phi_{a_1,a_2}(J(z))) & \text{if } z \in J(O_{a_1})
\end{cases}$$

is a hybrid conjugacy between $F_{a_1}$ and $F_{a_2}$ on a quadruply pinched neighbourhood of $\Lambda_{a_1}$.

**Proposition 4.1.** The straightening map $\chi : \mathcal{M}_R \cap \hat{K} \to \mathcal{M}_1$ is injective.

**Proof.** If $\chi|_{\mathcal{M}_R}$ is not injective, there exist two different correspondences $F_{a_1}$ and $F_{a_2}$, both with connected limit set, hybrid equivalent to the same map $P_A$. Then, by Lemma 4.1 above there exists a hybrid conjugacy $\phi_{a_1,a_2}$ between $F_{a_1}$ and $F_{a_2}$ defined on the quadruply pinched neighbourhood $O_{a_1}$ of $\Lambda_{a_1}$.

On the other hand, for every $a \in \mathcal{M}_R$, the Riemann map $R_a : \Omega_a \to \mathbb{H}$ conjugates the action of $F_a$ on the regular set $\Omega_a = \hat{C} \setminus \Lambda_a$ to the action of the modular group on the upper half plane $\mathbb{H}$ (see [BL1], Theorem A). Hence, the map $R_{a_1,a_2} := R_{a_2}^{-1} \circ R_{a_1} : \Omega_{a_1} \to \Omega_{a_2}$ is a holomorphic conjugacy between $F_{a_1}$ and $F_{a_2}$ on their regular sets, and the map $\Psi : \hat{C} \to \hat{C}$ defined by:

$$\Psi := \overline{\phi_{a_1,a_2}} \text{ on } \Lambda_{a_1}$$

$$\Psi := R_{a_1,a_2} \text{ on } \Omega_{a_1}$$

is a conjugacy between $F_{a_1}$ and $F_{a_2}$ on the whole Riemann sphere, holomorphic on $\Lambda_a$ and on $\Omega_a$. If $\Psi$ is continuous, then by the Rickman Lemma it is holomorphic on $\hat{C}$, and then by Lemma 4.1 $F_{a_1} \overline{\phi_{a_1,a_2}} = F_{a_2}$. So we need to show that $\Psi$ is continuous, more specifically that both $\overline{\phi_{a_1,a_2}}(z)$ and $R_{a_1,a_2}(z)$ tend to the same point when $z \in \Omega_{a_1}$ tends to $\Lambda_{a_1}$.

Let $S_{a_1} := R_{a_1}(O_{a_1} \cap \Omega_{a_1}) \subset \mathbb{H}$, $S_{a_2} := R_{a_2}(\overline{\phi_{a_1,a_2}}(O_{a_1} \cap \Omega_{a_1})) \subset \mathbb{H}$, and let $\overline{\Psi}$ denote the quasiconformal homeomorphism

$$\overline{\Psi} := R_{a_2} \circ \overline{\phi_{a_1,a_2}} \circ R_{a_1}^{-1} : S_{a_1} \to S_{a_2}.$$
Foliate \( S_{a_1} \subset \mathbb{H} \) by vertical line segments \( L_x \), one for each \( x \in \mathbb{R} \). Its image \( S_{a_2} = \nabla(S_{a_1}) \subset \mathbb{H} \) is foliated by the paths \( \nabla(L_x) \). When \( x \in \mathbb{Q} \), the path \( R_{a_1}^{-1}(L_x) \) in \( \Omega_{a_1} \) lands on \( \Lambda_{a_1} \) (see [12], Proposition 1 and Corollary 1). The landing point is a preperiodic, indeed pre-fixed, point \( z_0 \) of \( \mathcal{F}_{a_1} \) (or \( \mathcal{F}_{a_1}^{-1} \) if \( z_0 \in \Lambda_{a_1} \)). The path \( \nabla_{a_1,a_2}(R_{a_1}^{-1}(L_x)) \) in \( \Omega_{a_2} \) lands on \( \Lambda_{a_2} \) at the corresponding pre-fixed point \( \phi_{a_1,a_2}(z_0) \) of \( \mathcal{F}_{a_2} \). But \( \phi_{a_1,a_2}(R_{a_1}^{-1}(L_x)) = R_{a_2}^{-1}(\nabla(L_x)) \), so we deduce that the path \( \nabla(L_x) \) in \( \mathbb{H} \) lands on the real axis at \( x \). Since this holds for every \( x \in \mathbb{Q} \), and \( \nabla \) preserves the order of the leaves of the foliation of \( S_{a_1} \), it follows that for every \( x \in \mathbb{R} \) the path \( \nabla(L_x) \) lands on the real axis at \( x \), and thus that \( \nabla : S_{a_1} \to S_{a_2} \) extends continuously to the identity map on \( \partial \mathbb{H} = \mathbb{R} \).

We deduce that for every sequence \( z_n \in \Omega_{a_1} \) converging to a point \( \hat{z} \in \Lambda_{a_1} \), the sequences \( R_{a_2}(\phi_{a_1,a_2}(z_n)) \) and \( R_{a_1}(z_n) \) in \( \mathbb{H} \) converge to the same point of \( \partial \mathbb{H} \). Hence the sequences \( \phi_{a_1,a_2}(z_n) \) and \( R_{a_1,a_2}(z_n) = R_{a_2}^{-1} \circ R_{a_1}(z_n) \) in \( \Omega_{a_2} \) converge to the same point (necessarily \( \phi_{a_1,a_2}(\hat{z}) \)) in \( \Lambda_{a_2} \).

### 4.2 The map \( \chi \) is locally quasiregular on \( \overset{\circ}{\mathbb{K}} \setminus \mathcal{M}_\Gamma \)

In this section, we show that with the right choice of \( V_h \) we can write \( \chi \) on \( \overset{\circ}{\mathbb{K}} \setminus \mathcal{M}_\Gamma \) as the composition of iterated lifted extended tubings and two conformal maps (see Section 4.2.1 and Proposition 4.2). The local quasiregularity of \( \chi \) on \( \overset{\circ}{\mathbb{K}} \setminus \mathcal{M}_\Gamma \) follows (see Proposition 4.3).

#### 4.2.1 Extending the holomorphic motion \( \tau \) by iterated lifting

Define

\[
\mathcal{P}A^1_a := \mathcal{F}_a^{-1}(\mathcal{P}A_a) \cap O_a
\]

and for each \( k > 1 \) set

\[
\mathcal{P}A^k_a := \mathcal{F}_a^{-(k-1)}(\mathcal{P}A^1_a).
\]

Note that \( \mathcal{F}_a : \mathcal{P}A_a \to \mathcal{P}A_a \) is a double covering over \( \mathcal{P}A_a \cap O_a \) but only a single covering over \( \mathcal{P}A_a \cap (\overset{\circ}{\mathbb{C}} \setminus O_a) \), and that all the \( \mathcal{F}_a : \mathcal{P}A^{k+1}_a \to \mathcal{P}A^k_a \) are double-sheeted coverings. The sets \( \mathcal{P}A_a, \mathcal{P}A^k_a (k \geq 1) \) and \( \Lambda_{a,-} \) are disjoint, and their union is \( V_a \). Define corresponding subsets of the parameter space \( \overset{\circ}{\mathbb{K}} \):

\[
K^0 := \{ a \in \overset{\circ}{\mathbb{K}} | v_a \in \mathcal{P}A_a \} \text{ and } K^k := \{ a \in \overset{\circ}{\mathbb{K}} | v_a \in \mathcal{P}A^k_a \} (k \geq 1),
\]

and set

\[
K_k := \overset{\circ}{\mathbb{K}} \setminus \bigcup_{j=0}^{k-1} K^j,
\]

noting that \( (\mathcal{M}_\Gamma \cap \overset{\circ}{\mathbb{K}}) \subset K_k \subset K_{k-1} \subset \ldots \subset K_0 = \overset{\circ}{\mathbb{K}} \), and that each \( K_k \) is the disjoint union of \( K_{k-1} \) and \( K^k \). Our base point \( a_0 \) is in \( \mathcal{M}_\Gamma \cap \overset{\circ}{\mathbb{K}} \), so \( a_0 \in K_k \) for all \( k \geq 0 \).

When \( a \in K_1 \), the critical value \( v_a \) is not in \( \mathcal{P}A_{a_0} \), so we can lift the qc homeomorphism \( \tau_a : \mathcal{P}A_{a_0} \to \mathcal{P}A_a \) to a qc homeomorphism of covers:

\[
\tau^1_a := \mathcal{F}_a^{-1} \circ \tau_a \circ \mathcal{F}_{a_0} : \mathcal{P}A^1_{a_0} \to \mathcal{P}A^1_a.
\]
Despite the fact that $PA_{a_0}^1$ is defined as the intersection of $F_a^{-1}(PA_{a_0}^1)$ with the complement of $O_{a_0}$, and the cover $F_a : PA_{a_0}^1 \to PA_{a_0}$ is therefore double-sheeted in one part and single-sheeted in another, the lift exists since the covering $F_a : PA_{a_0}^1 \to PA_a$ is partitioned in exactly the corresponding way. When $a \in K_k$, $k \geq 1$, we can iterate the process and lift $\tau_a$ to $2^{k-1}$-fold covers (this time they have $2^{k-1}$ sheets everywhere):

$$\tau_a^k := F_a^{- (k-1)} \circ \tau_a \circ F_a : PA_{a_0}^1 \to PA_a^1.$$  

Denoting by $\tau_k$ the union over all $a \in K_k$ of $\tau_a$ and $\tau_a^j$, $1 \leq j \leq k$, we have:

$$\tau_k : K_k \times (PA_{a_0}^1 \cup PA_{a_0}^1 \cup \ldots \cup PA_{a_0}^k) \to C.$$  

This union $\tau_k$ of lifts of $\tau$ is well-defined for all $a \in K_k$. Moreover it is continuous at the boundary between $PA_{a_0}^*$ and $PA_{a_0}$ because by definition the motion of the inner boundary $\partial V_{a_0}$ of $PA_{a_0}$ is the lift (via $F_a^{-1}$) of that of the outer boundary $\partial V_{a_0}$. Similarly $\tau_k$ is continuous at the boundary between each $PA_{a_0}^j$ and $PA_{a_0}^{j+1}$.

For $a_0$ and $a \in M \cap K$, we can repeat the iterative process indefinitely, obtaining in the limit a well-defined lift:

$$\hat{\tau} : (M \cap K) \times (V_{a_0} \setminus \Lambda_{a_0, -}) \to C,$$

which is a holomorphic motion on each connected component of $M \cap K$.

### 4.2.2 Extending the tubing

In Section 3.4.3 we defined the tubing $T_a$ as the family of qc homeomorphisms

$$T_a := f \circ \tau_a^{-1} : \hat{C} \setminus O_a \to \hat{C} \setminus O_h,$$

one for each $a \in K$, where $f := \Psi_{O^c} o \alpha : \hat{C} \setminus O_a \to \hat{C} \setminus O_h$ is a quasiconformal homeomorphism chosen during the surgery construction on $F_a$. Denote $f(\partial V_{a_0}) \subset \hat{C} \setminus O_h$ by $PA_h$. As $F_a$ has connected limit set, there is no obstruction to lifting $f_a : PA_{a_0} \to PA_h$ to $2^j$-fold covers for all $j$. Denoting this lift by $f_a^j$, and the lift of $\tau_a : PA_{a_0} \to PA_a$ to $2^j$-fold covers constructed in the previous subsection 4.2.1 by $\tau_a^j$, we extend the tubing $T_a$ by defining the qc homeomorphisms:

$$T_a^j := f_a^j \circ (\tau_a^j)^{-1} : PA_{a_0}^j \to PA_{h}^j := h^{-j}(PA_{h}) \cap O_h \ (1 \leq j < \infty)$$

for each $a \in K_k \subset K$. We note that $\bigcup_{j=0}^k T_a^j$ is continuous at the boundary between each $PA_{a_0}^j$ and $PA_{a_0}^{j+1}$, as $\bigcup_{j=0}^k (\tau_a^j)^{-1}$ has this property. We have the following commuting diagram of qc homeomorphisms, where the bottom row (read left to right) is the composition $T_a$, and the top row is its lift $T_a^2$:
If \( a \in \mathcal{M}_\Gamma \), we can set \( T_a = T_a \cup \bigcup_{j=1}^\infty T_a^j \), and observe that \( T_a \) is a qc homeomorphism
\[
T_a : \hat{C} \setminus \Lambda_{a,-} \to \hat{C} \setminus \mathcal{B}.
\]
On the other hand, if \( a \not\in \mathcal{M}_\Gamma \), we can only iterate the double covering procedure until we reach the critical value \( v_a \) of \( F_a \). To be precise, if \( v_a \in \mathcal{P}_A^0 \), then the last level to which we can lift the tubing is \( T_a^n : \mathcal{P}_A^n \to \mathcal{P}_A^n \).

### 4.2.3 Milnor’s model for \( \mathbb{C} \setminus \mathcal{M}_1 \)

In the paper [Mi], Milnor constructs a conformal isomorphism \( \Psi \) between \( \mathbb{C} \setminus \mathcal{M}_1 \) and the punctured disc. We now briefly review this construction, as it is at the heart of the new characterisation of \( \chi \) on \( \mathbb{C} \setminus \mathcal{B} \).

Let \( Q_{1/4}(z) = z^2 + 1/4 \), let \( \mathcal{B} \) denote the interior of its filled Julia set, in other words the parabolic basin of attraction of \( z = 1/2 \), and let \( \mathcal{P}_0 \subset \mathcal{B} \) be the largest attracting petal of \( z = 1/2 \) such that the Fatou coordinate carries \( \mathcal{P}_0 \) diffeomorphically onto a right half-plane. Then the critical point \( z = 0 \) belongs to the boundary of \( \mathcal{P}_0 \), and hence the critical value \( z = 1/4 \) lies on the boundary of \( \mathcal{P}_{-1} = Q_{1/4}(\mathcal{P}_0) \). We normalise the Fatou coordinate to send the critical value to 1. Let \( \mathcal{P}_1 = Q_{1/4}^{-1}(\mathcal{P}_0) \), then \( \mathcal{P}_{-1} \subset \mathcal{P}_0 \subset \mathcal{P}_1 \), and recursively defining \( \mathcal{P}_{k+1} = Q_{1/4}^{-1}(\mathcal{P}_k) \), we obtain that the parabolic basin \( \mathcal{B} \) is the union
(see Figure 5 on page 497 of [Mi])
\[
\mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_k \subset \ldots
\]

A model space for \( \mathbb{C} \setminus \mathcal{M}_1 \) is provided by the disc (punctured at the critical value \( z = 1/2 \)):
\[
(\mathcal{B} \setminus \mathcal{P}_{-1})/\alpha
\]
where \( \alpha \) is the identification \( z \sim \bar{z} \) on \( \partial \mathcal{P}_{-1} \). To each \( B \) in the ‘shift locus’ \( \text{Per}_1(1) \setminus \mathcal{M}_1 \) Milnor associates a point in this model space by the following recipe. Let \( Q_{A,0} \) be the largest attracting petal of the parabolic fixed point of \( P_A \), \( B = 1 - A^2 \), such that the Fatou coordinate carries \( Q_{A,0} \) diffeomorphically onto a right half-plane. We normalise the Fatou coordinate to send the first critical value \( z = -2 + A \) of \( P_A \) to 1. There exists a unique conformal map isomorphism \( \varphi_0 : P_A \to Q_{A,0} \) conjugating \( Q_{1/4} \) to \( Q_{A,0} \) and sending 0 to the first critical point \( z = -1 \) of \( P_A \), namely the composition of the Fatou coordinate for \( Q_{1/4} \) (normalised by sending 1/4 to 1) and the inverse Fatou coordinate for \( P_A \) (normalised by sending the \( A - 2 \) to 1). Note that one can lift
\( \varphi_0 \) back to \( \varphi_k : P_k \rightarrow Q_{A,k} = P_A^{-k}(Q_{A,0}) \) up to and including the first value \( k \) for which \( Q_{A,k} \) contains the second critical value \( v_A = 2 + A \) of \( P_A \). In the second proof of Theorem 4.2 in [M1], using a method suggested by Shishikura, Milnor shows that the assignment \( B = 1 - A^2 \rightarrow \varphi_k^{-1}(v_A) \) defines a conformal isomorphism \( \Psi \) from \( \mathbb{C} \setminus M_1 \) to \( (\mathbb{B} \setminus P_-)/\alpha \).

### 4.2.4 The map \( \chi \) on \( \hat{K} \setminus M_1 \)

Let \( \Phi : \mathcal{B} \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D} \) be the conformal conjugacy between \( Q_{1/4}\mid \mathcal{B} \) and \( h|_{\hat{\mathbb{C}}\setminus \mathbb{D}} \), given by Fatou coordinates (where \( h(z) \), as always, denotes \( \frac{z^2 + 1/3}{2z^2 + 1} \)). More specifically, let \( \mathcal{H}_0 \subset \hat{\mathbb{C}} \setminus \mathbb{D} \) be the largest attracting petal of the parabolic fixed point 1 of \( h \) such that the Fatou coordinate (normalized to send the critical value \( z = 3 \) of \( h \) to the point 1) carries \( \mathcal{H}_0 \) diffeomorphically onto a right half-plane. There exists a unique conformal isomorphism \( \phi_0 : \mathcal{H}_0 \rightarrow \mathcal{P}_0 \) conjugating \( h|\mathcal{H}_0 \) to \( K_{1/4}|\mathcal{P}_0 \) and sending the critical value 3 of \( h \) in \( \mathcal{H}_0 \) to the critical value 1/4 of \( Q_{1/4} \) in \( \mathcal{P}_0 \) (namely the composition of Fatou coordinates for \( h \) normalized by sending 3 to 1 and the inverse Fatou coordinates for \( Q_{1/4} \) sending 1/4 to 1). As \( h \) and \( Q_{1/4} \) both have connected Julia sets, we can lift \( \phi_0 \) to the whole parabolic basin, obtaining a conformal conjugacy \( \Phi : \mathcal{B} \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D} \) between \( Q_{1/4}\mid \mathcal{B} \) and \( h|_{\hat{\mathbb{C}}\setminus \mathbb{D}} \).

For \( a \in \hat{K} \setminus M_1 \), let \( n(a) \) be the entry time of the critical value \( v_a \) into the pinched fundamental annulus, that is \( F_a^{n(a)}(v_a) \in \mathcal{P}_a \), or equivalently \( v_a \in \mathcal{P}_a|_{\mathcal{M}_1} = F_a^{-n(a)}(\mathcal{P}_a \cap O_a) \). Then we can write \( \chi|_{\mathcal{M}_1 \cap \hat{K}} \) as (see Figure 12):

\[
\chi : \hat{K} \setminus M_1 \rightarrow \hat{\mathbb{C}} \setminus \mathcal{M}_1 \\
a \rightarrow \Psi^{-1} \circ \Phi^{-1} \circ T_a^{n(a)}(v_a).
\]

**Proposition 4.2.** With a suitable choice of \( V_h \) in the initial surgery on \( \mathcal{F}_a \), the map \( \chi : \hat{K} \setminus M_1 \rightarrow Per_1(1) \setminus M_1 \) becomes the composition

\[
a \rightarrow \Psi^{-1} \circ \Phi^{-1} \circ T_a^{n(a)}(v_a).
\]

**Proof.** In our surgery construction for \( \mathcal{F}_a \) (and hence for \( \mathcal{F}_a \), \( a \in \hat{K} \)), we defined \( V_h \) to be the disc of radius \((1 + \epsilon)\) tangent to the unit circle at the parabolic fixed point 1, and \( V_h' = h^{-1}(V_h) \). But we could equally well have defined it to be \( \mathcal{H}_0 \subset \hat{\mathbb{C}} \setminus \mathbb{D} \) (coloured green in Figure 13), the maximal attracting petal for the parabolic fixed point \( z = 1 \) of \( h \), with boundary passing through the critical point, i.e., \( \infty \in \partial \mathcal{H}_0 \), and \( \mathcal{H}_1 = h^{-1}(\mathcal{H}_0) \). In this case, \( O_h \) is the connected component of \( \mathcal{H}_1 \setminus \gamma_h \) containing the unit disc, and \( \mathcal{P}_a \cap \mathcal{H}_1 = V_h \setminus O_h \). Then, when the map \( F_a \) resulting from surgery to the correspondence \( \mathcal{F}_a \) is straightened to become \( \mathcal{P}_a \), the pinched fundamental annulus \( \mathcal{P}_a \) is identified with the pinched fundamental annulus \( \mathcal{P}_a \), and so each tile \( \mathcal{Q}_{P_a,k} \setminus \mathcal{Q}_{P_a,k} \) is identified with the corresponding \( \mathcal{H}_{k+1} \setminus \mathcal{H}_k \), where \( \mathcal{H}_k = h^{-k}(\mathcal{H}_0) \).
Let $\phi_a$ be the straightening map for $\mathcal{F}_a$ constructed in Section 4.4.3 and let $\mathcal{Q}_{A,n} = P_A^{-n}(\mathcal{Q}_{A,0})$. Then for each fixed $a$ the map $\Phi^{-1} \circ T_a^{n(a)} \circ \phi_a^{-1} : \mathcal{Q}_{A,n(a)} \to \mathcal{P}_{n_a} \subset B$ is conformal, as by construction $(\Phi^{-1} \circ T_a^{n(a)} \circ \phi_a^{-1})^* \mu_0 = \mu_0$. This composition conjugates $P_A|_{\mathcal{Q}_{A,n(a)}}$ to $Q_{1/4}|_{\mathcal{P}_{n(a)}}$, and it sends the first critical value $z = -2 + A$ of $P_A$ to the critical value $1/4$ of $Q_{1/4}$. Therefore, by unicity, $\Phi^{-1} \circ T_a^{n(a)} \circ \phi_a^{-1} = \varphi_a^{-1}$, where $\varphi_a : \mathcal{P}_0 \to \mathcal{Q}_{A,0}$ is the unique conformal isomorphism used by Milnor to define the conformal isomorphism $\Psi : \mathbb{C} \setminus M_1 \to (B \setminus \mathcal{P}_{-1})/\alpha$ (see Section 4.2.3). Hence, for every $a \in \mathcal{K} \setminus \mathcal{M}_\Gamma$,

$$\Psi^{-1} \circ \Phi^{-1} \circ T_a^{n(a)}(v_a) = \phi_a(v_a),$$

and the map $\chi|_{\mathcal{M}_\Gamma \cap \mathcal{K}}$ can be written as

$$\chi : \bar{K} \setminus \mathcal{M}_\Gamma \to \mathbb{C} \setminus M_1$$

$$a \to \Psi^{-1} \circ \Phi^{-1} \circ T_a^{n(a)}(v_a).$$

\[\square\]

**Proposition 4.3.** $\chi : \bar{K} \setminus \mathcal{M}_\Gamma \to \mathbb{C} \setminus M_1$ is locally quasiregular.

**Proof.** As $\Phi$ and $\Psi$ are conformal, to prove that $\chi : \bar{K} \setminus \mathcal{M}_\Gamma \to \mathbb{C} \setminus M_1$ is locally quasiregular, it will suffice to show that $a \to T_a^{n(a)}(v_a)$ is locally quasiregular. As the tubing is defined as the composition $T_a = f \circ \tilde{\tau}_a^{-1}$, where $f$ is quasiconformal, it is enough to show that the map $a \to \tilde{\tau}_a^{-1}(v_a)$ is locally quasiregular.

Consider the union of the holomorphic motions of $\mathcal{P}_A_{a_0}$ and its lifts as a map

$$\tau \cup \tau_1 \cup \ldots \cup \tau_k : \bar{K} \times (\mathcal{P}_A_{a_0} \cup \mathcal{P}_A_{a_1} \cup \ldots \cup \mathcal{P}_A_{a_k}) \to \bar{K} \times \mathbb{C}$$

30
defined at the points where the various lifts exist. For each \( j \) and \( z \in \mathcal{P}_a \), let \( \ell_z \) denote the image of \( z \) under the lifted motion \( \tau_j \). The image of the map \( \tau \cup \tau_1 \cup \ldots \cup \tau_k \) is foliated by the leaves \( \ell_z \).

The set \( \{(a_0, z) : z \in V_{a_0} \setminus \Lambda_{a_0}\} \) is a curve transversal to this foliation (by definition of the holomorphic motion), and the set \( \{(a, v_a) : a \in \hat{K}\} \) is a holomorphic curve in the domain of the motion, so by Lemma 17.9 in [Lyu] the holonomy along the leaves from the curve \( \{(a, v_a) : a \in \hat{K}\} \) to the transversal \( \{(a_0, z) : z \in V_{a_0} \setminus \Lambda_{a_0}\} \) is locally quasiregular. In the case that each leaf in the foliation is connected, the statement of the Proposition follows.

However, we have not yet proved that \( \mathcal{M}_\Gamma \) is connected, so we must allow for the possibility that there exist points \( a \in \hat{K} \) for which the leaf component through \( (a, v_a) \) does not meet the transversal \( \{(a_0, z) : z \in V_{a_0} \setminus \Lambda_{a_0}\} \), and so the holonomy map is not defined directly. But in this case we can use equivariance to define it indirectly: we send \( v_a \) to \( \mathcal{F}_a^n(v_a) \in \mathcal{P}_a \), we then move \( \mathcal{F}_a^n(v_a) \) to \( \mathcal{P}_a \subset (V_{a_0} \setminus \Lambda_{a_0}) \) by holonomy, and finally apply the appropriate branch of \( \mathcal{F}_a^{-n(a)} \) within \( V_{a_0} \setminus \Lambda_{a_0} \). The result now follows as before.

\[ \square \]

4.3 Continuity of \( \chi \) on \( \mathcal{M}_\Gamma \cap \hat{K} \) and holomorphicity on \( \mathcal{M}_\Gamma \)

We know from the previous section that \( \chi \) is continuous on \( \hat{K} \setminus \mathcal{M}_\Gamma \) (since it is locally quasiregular). We next prove that \( \chi \) is continuous on \( \partial \mathcal{M}_\Gamma \cap \hat{K} \) and holomorphic on \( \mathcal{M}_\Gamma \), thus proving continuity everywhere in \( \hat{K} \). The proofs in this section follow those formulated by Douady and Hubbard [DH], and refined by Lyubich [Lyu], and subsequent authors: we present details to make this article as self-contained as possible.
4.3.1 Indifferent periodic points

We start by considering the Mañé-Sad-Sullivan decomposition of the parameter space $L'_0$. Recall that for all $a \in L'_0$, $F_a : V'_a \to V'_a$ is a degree 2 holomorphic map depending holomorphically on the parameter, with a persistent parabolic fixed point at $P_a$. Recall also that by Corollary 1.2 in [BL1], the boundary of the backward limit set $\partial \Lambda_{a,-}$ is the closure of the set of repelling periodic points of $\mathcal{F}_a$.

We define $I$ to be the set of parameters in $L'_0$ for which $F_a$ has a non-persistent indifferent periodic point, and set $R = L'_0 \setminus I$. The set $R$ is open, and there $\partial \Lambda_{a,-}$ moves holomorphically (these results follow from the implicit function theorem and the $\lambda$-Lemma, see [MSS]).

**Proposition 4.4.** $R = L'_0 \setminus \partial M_F$.

**Proof.** As on $R$ we can construct a holomorphic motion of the set $\partial \Lambda_{a,-}$ (using the implicit function theorem on the repelling periodic points and then the $\lambda$-Lemma, see [MSS]), and since we cannot map a connected set homeomorphically to a disconnected one and vice versa, $R$ cannot intersect $\partial M_F$, so $R \subset L'_0 \setminus \partial M_F$.

Therefore, to prove the statement, it is enough to prove that $L'_0 \setminus \partial M_F \subset R$. It is easy to show that $L'_0 \setminus M_F \subset R$: for all $a \in L'_0 \setminus M_F$ the critical point $c_a$ of $F_a\vert_{V'_a}$ is outside $\Lambda_{a,-}$, hence $F_a\vert_{V'_a}$ cannot have an indifferent periodic point in addition to $P_a$, so $I \cap (L'_0 \setminus M_F) = \emptyset$, which implies $L'_0 \setminus M_F \subset R$ as $L'_0 \setminus M_F$ is open. Hence, it remains to prove that $M_F \subset R$, or equivalently (as $M_F$ is open), that $I \cap M_F = \emptyset$, which we will do by contradiction.

So, let us assume that there exists $a_0 \in M_F$ for which $F_{a_0}\vert_{V'_{a_0}}$ has an indifferent periodic point $z_0$ of period $k$, and assume first $(F_{a_0}^k\vert_{V'_{a_0}})'(z_0) \neq 1$. It follows by the Implicit Function Theorem that there exist a neighbourhood $W(a_0)$ of $a_0$ in $M_F$ and a neighbourhood $O(z_0)$ of $z_0$ where the cycle $\{z^1(a), ..., z^k(a)\}$, its multiplier $\rho(a) = (F_{a_0}^k\vert_{V'_{a_0}})'(z_0)$, and the critical point $c(a)$ move holomorphically with the parameter, and $a_0$ is the unique parameter in $W(a_0)$ for which the cycle is indifferent with multiplier $\rho(a_0)$. Let $(a_n) \in W(a_0)$ be a sequence converging to $a_0$ such that, for all $n$, $|\rho(a_n)| < 1$: then for every $n$ there exists $z^i(a_n) \in \{z^1(a_n), ..., z^k(a_n)\}$ such that

$$F_{a_n}^{i+k}\vert_{V'_{a_n}}(c_{a_n}) \to z^i(a_n) \text{ as } n \to \infty$$

(we can assume $i$ independent of $a$ by choosing a subsequence). On the other hand, the family

$$F_p(a) = F_{a_0}^{i+k}\vert_{V'_{a_0}}(c_{a_0}), \quad a \in W(a_0)$$

is a normal family (as it is analytic and bounded: bounded because $W(a_0) \subset M_F$), so there exists a subsequence $F_{p_n}$ converging to some function $h$, and since $h(a_n) = z^i(a_n)$ for all $n$, $h(a) = z^i(a)$ for all $a \in W(a_0)$. This implies that $F_{p_n}(a) \to z^i(a)$ for all $a \in W(a_0)$, which is impossible as $W(a_0)$ contains parameters $a^*$ for which the cycle is repelling, hence $z^i(a^*)$ cannot attract the sequence $F_p(a^*)$.

32
If \( F^k_{a_n} | V_{a_n} \) \( (z_0) = 1 \), let \( U(a_0) \) be a neighbourhood of \( a_0 \in \hat{M}_1 \), \( a : W(0) \to U(a_0), t \to t^2 + a_0 \), be a branched covering of \( U(a_0) \) branched at 0 for some neighbourhood \( W(0) \) of 0, and repeat the previous argument. \( \square \)

4.3.2 Continuity on \( \partial M_1 \cap \hat{K} \)

To prove that \( \chi \) is continuous on \( \partial M_1 \cap \hat{K} \) we will use precompactness of sequences of quasiconformal maps, in the topology of uniform convergence on compact subsets, and quasiconformal rigidity on \( \partial M_1 \).

Proposition 4.5. The map \( \chi : \hat{K} \to \mathbb{C} \) is continuous at every point \( a \in \partial M_1 \cap \hat{K} \), and \( \chi(\partial M_1 \cap \hat{K}) \subset \partial M_1 \).

Proof. The map \( \chi \) is continuous at \( \hat{a} \in \partial M_1 \) if and only if given any sequence \( a_n \) converging to \( \hat{a} \) the sequence \( \chi(a_n) \) has a subsequence converging to \( \chi(\hat{a}) \). We will first prove that \( \chi(\hat{a}) \in \partial M_1 \). Let \( a_n \to \hat{a} \in \partial M_1 \) with \( a_n \in I \) for all \( n \). For every \( n \), our surgery construction provides us with a quasiconformal conjugacy \( \phi_{a_n} \) between \( F_{a_n} \) and \( P_{\chi(a_n)} \). By construction, for every \( a \), the dilatation of the quasiconformal conjugacy \( \phi_a \) is the dilatation of the tubing \( T_a \) (see Remark 3.3.2; hence by the second \( \lambda \)-Lemma this dilatation is locally bounded (see the second \( \lambda \)-Lemma in Chapter 17.4 in [Lv]). So, by precompactness, \( \phi_{a_n} \) has a convergent subsequence, say \( \phi_{a_{nk}} \to \phi \) as \( k \to \infty \) and this qc homeomorphism \( \phi \) conjugates \( F_{\hat{a}} \) to a rational map \( P_{\hat{A}} \in Per_1(1) \). Since \( a_n \in I \) for all \( n \), we have that for every \( n \) the map \( P_{\chi(a_n)} \) has an indifferent periodic point, as \( F_{a_n} \) has. Hence for every \( n \), \( \chi(a_n) \in \partial M_1 \), and so its limit too: \( \hat{B} \in \partial M_1 \), \( \hat{B} = 1 - \hat{A}^2 \).

Therefore, \( F_{\hat{a}} \) is quasiconformally conjugate to both \( P_{\chi(\hat{a})} \) and to \( P_{\hat{A}} \), and since we have rigidity on \( \partial M_1 \), that is, quasiconformal conjugacy implies conformal conjugacy (see [L2], Proposition 4.2), we obtain \( \chi(\hat{a}) = \hat{B} \). This shows that \( \chi(\hat{a}) \in \partial M_1 \), and more generally that \( \chi(\partial M_1 \cap \hat{K}) \subset \partial M_1 \).

Let us now show continuity. Let \( (a_n) \) be a sequence in \( \hat{K} \) converging to \( \hat{a} \in \partial M_1 \). As we saw above, for all \( n \) the surgery construction provides us with a quasiconformal conjugacy \( \phi_{a_n} \) between \( F_{a_n} \) and \( P_{\chi(a_n)} \), which has locally bounded dilatation, and hence the sequence \( \phi_{a_n} \) has a converging subsequence, say \( \phi_{a_{nk}} \to \phi \) as \( k \to \infty \), and \( \phi \) conjugates \( F_{\hat{a}} \) to \( P_{\hat{A}} = \phi \circ F_{\hat{a}} \circ \phi^{-1} \). Hence \( P_{\chi(\hat{a})} \) and \( P_{\hat{A}} \) are quasiconformally conjugate, and as we showed above that \( \chi(\hat{a}) \in \partial M_1 \), by rigidity we have that \( \chi(\hat{a}) = \hat{B} \). \( \square \)

4.3.3 Holomorphicity on \( \hat{M}_1 \)

Let \( W \) be a connected component of the interior \( \hat{M}_1 \) of \( M_1 \). We call \( W \) hyperbolic if there exists (at least) one parameter \( a_0 \in W \) for which \( F_{a_0} \) has an attracting cycle. On the other hand, we call \( W \) queer if there is no parameter \( a_0 \in W \) for which \( F_{a_0} \) has an attracting cycle. We will denote hyperbolic components by \( H \), and queer components by \( Q \), and we will prove that \( \chi : \hat{M}_1 \cap \hat{K} \to \hat{M}_1 \) is holomorphic first on hyperbolic components (Proposition 4.3.3) and then on queer components (Proposition 4.3.4).
Proposition 4.6. On hyperbolic components, the map $\chi$ is holomorphic and proper.

Proof. Let $H$ be a hyperbolic component of $\mathcal{M}_1$. We are going to prove that there exists a hyperbolic component $C \subset \mathcal{M}_1$ such that $\chi|_H : H \to C$ is holomorphic. Since $H$ is a hyperbolic component, there exists $a_0 \in H$ for which $F_{a_0}$ has an attracting cycle. Since $F_a$ has no indifferent parameters in $H$ (see Prop. 4.4), by the Implicit Function Theorem for all $a \in H$, $F_a$ has an attracting cycle. Therefore, for all $a \in H$ the hybrid conjugate member of $\text{Per}_1(1)$ has an attracting cycle, and hence $C$ is a hyperbolic component of $\mathcal{M}_1$. Since the attracting cycle with its basin belongs to the filled Julia set $K_a$ of $P_A$, and the hybrid conjugacy is conformal on the interior of the limit set $\Lambda_{a,-}$, the multipliers of the conjugated attracting cycles must coincide. Hence, denoting by $\rho_H(a)$ the multiplier map for $F_a$ on $H$, and denoting by $\rho_C$ the multiplier map for the family $\text{Per}_1(1)$ on $C$, we have that

$$\rho_H(a) = \rho_C(\Lambda_a),$$

and we can write the restriction of the map $\chi$ on $H$ as

$$\chi|_H = \rho_C^{-1} \circ \rho_H.$$

Since $\rho_H(a)$ is holomorphic, and is holomorphic with degree 1 (see [PT]), the composition $\chi|_H$ is holomorphic.

By Proposition 4.5 $\chi : H \to C$ extends continuously to the boundary, and $\chi(\partial H) \subset \partial C$, so that the map $\chi|_H$ is proper. \qed

Note that, if $a \in \mathcal{M}_1$ and $F_a$ does not have an attracting cycle, then $\Lambda_{a,-}$ is connected with empty interior. Indeed, $F_a$ does not have an attracting nor an indifferent cycle (by the assumption and Proposition 4.4), and so the hybrid equivalent $P_A$ has no attracting or indifferent cycle. Hence, $K_a$ is connected with empty interior, and so is $\Lambda_{a,-}$, as these sets are quasiconformally homeomorphic (since $F_a$ is hybrid conjugate to $P_A$ on a doubly pinched neighbourhood of $\Lambda_{a,-}$).

Hence, if $Q$ is a queer component of $\mathcal{M}_1$, for all $a \in Q$, the set $\Lambda_{a,-}$ is connected and has no interior.

Proposition 4.7. On queer components the map $\chi$ is holomorphic and proper.

Proof. Assume $Q$ is a queer component of $\mathcal{M}_1$, and take the base point $a_0 \in Q$. Since for every $a \in Q$ the critical point $c_a$ belongs to $\Lambda_{a,-}$, the holomorphic motion on $Q$ extends to

$$\tau_a : Q \times (V_{a_0} \setminus \Lambda_{a_0,-}) \to V_a \setminus \Lambda_{a,-},$$

since $\partial V_{a_0}$ and $\partial V_a$ are quasicircles, it extends to

$$\tau_a : Q \times (\hat{C} \setminus \Lambda_{a,-}) \to \hat{C} \setminus \Lambda_{a,-},$$

34
and since on $Q$ the limit set $\Lambda_{a,-}$ is nowhere dense, by the $\lambda$-Lemma we obtain a dynamical holomorphic motion

$$\tau_a : Q \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$  

The tubing $T_a = f \circ \tau_a^{-1}$, where $f : \overline{\mathbb{C}} \setminus O_{a_0} \to \mathbb{C} \setminus \overline{D}$ is quasiconformal (see Section 3.4.2), also extends to (see Section 4.2.2)

$$T_a : \overline{\mathbb{C}} \setminus \Lambda_{a,-} \to \mathbb{C} \setminus \overline{D}.$$  

Let $\phi_{a_0}$ be the quasiconformal conjugacy between $F_{a_0} = \{F_a|_{O_{a_0}}\}$ and $P_{A_0}(z) = z + 1/z + A_0$ constructed in Section 3.4.2, so in particular $\phi_{a_0}$ is a quasiconformal conjugacy between $F_{a_0}$ and $P_{A_0}$ on $O_{a_0}$. Hence, for all $a \in Q$, the map

$$\Phi_a := \tau_a \circ \phi_{a_0} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

is a quasiconformal conjugacy between $P_{A_0}$ and

$$F_a = \left\{ \begin{array}{ll} F_{a_0} & \text{on } O_{a_0} \\ f^{-1} \circ h \circ f & \text{on } \hat{\mathbb{C}} \setminus O_a \end{array} \right.$$  

on $\hat{\mathbb{C}}$, and between $P_{A_0}$ on $\Phi_a^{-1}(O_0)$ and $F_a$ on $O_a$. So, in particular it is a quasiconformal conjugacy between $P_{A_0}$ on $K_0 = K_{A_0}$ and $F_a$ on $\Lambda_{a,-}$. As $a$ belongs to a queer component of $\mathcal{M}_\Gamma$, $A_0$ must belong to a queer component of $\mathcal{M}_1$, hence $K_0 = J_{P_{A_0}}$, which by Proposition 4.4 in [L2] has positive area.

Let $\nu_a$ be the family of Beltrami forms on $\hat{\mathbb{C}}$ defined as follows:

$$\nu_a(z) := \left\{ \begin{array}{ll} (\Phi_a)^* \mu_0 & \text{on } K_0 \\ \mu_0 & \text{on } \hat{\mathbb{C}} \setminus K_0 \end{array} \right.$$  

Then, by construction, $\text{area}(\text{supp}(\nu_a)) > 0$, $\nu_a$ depends holomorphically on $a$, and it is invariant under $P_{A_{a_0}}$. Let $\psi_a : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the family of integrating maps fixing $-1, 1$ and $\infty$. Then the family $P_{A(a)} = \psi_a \circ P_{A_{a_0}} \circ (\psi_a)^{-1}$ consists of holomorphic maps of degree 2 acting on $\hat{\mathbb{C}}$ and with a persistent parabolic fixed point at $\infty$ and critical points at $\pm 1$, and hence it has the form $P_{A(a)}(z) = z + 1/z + A(a)$, where $A(a)$ depends holomorphically on the parameter.

For every $a \in Q$, the map $\psi_a \circ \phi_{a_0} \circ \tau_a^{-1}$ is a hybrid conjugacy between $F_a|_{\Lambda_{a,-}}$ and $P_{A(a)}$, and $P_{A_{a_0}}$ on a doubly pinched neighbourhood of $\Lambda_{a,-}$, hence by unicity, $P_{A_a}$ is the member of $\text{Per}_{1}(1)$ hybrid equivalent to $F_a|_{\Lambda_{a,-}}$. Hence the map $\chi|_Q$ is holomorphic.

As the map $\chi : \mathcal{M}_\Gamma \cap \hat{K} \to \mathcal{M}_1$ is injective on $\mathcal{M}_\Gamma$ (see Proposition 4.1), the restriction $\chi|_Q$ cannot be constant, and as it extends continuously to the boundary and the extension sends boundaries to boundaries (see Proposition 4.5), the map $\chi|_Q$ is proper.
4.4 The straightening map \( \chi \) is a homeomorphism on \( \hat{K} \)

So far we know that \( \chi \) is well-defined and continuous everywhere on the (open) doubly truncated lune \( \hat{K} \), that it is injective on \( M_\Gamma \), and that it is quasiregular on \( \hat{K} \setminus M_\Gamma \). We will use elementary degree theory from algebraic topology to deduce that \( \chi \) is injective on \( \hat{K} \), hence a homeomorphism from \( \hat{K} \) onto its image.

**Proposition 4.8.** \( \chi \) is a homeomorphism from \( \hat{K} \) onto \( \chi(\hat{K}) \subset \text{Per}_1(1) \).

*Proof.* First observe that \( \chi \) is proper, since it extends continuously to \( \partial K \) (we could have taken a larger closed subset \( K \) of \( L_{\theta}^\prime \) when we were defining our extension to \( \chi \) in the first place). Hence \( \chi \) induces a homomorphism of cohomology with compact supports:

\[
\chi^* : H^2_c(C) \to H^2_c(\hat{K}).
\]

Since \( \hat{K} \) and \( C \) are path-connected surfaces, these cohomology groups are both isomorphic to \( \mathbb{Z} \), generated by the respective fundamental classes. The element \( 1 \in \mathbb{Z} = H^2_c(C) \) is mapped to a non-zero \( d \in \mathbb{Z} = H^2_c(\hat{K}) \) known as the degree, \( d(\chi) \), of \( \chi \). Since we have proved that \( \chi \) is injective on \( M_\Gamma \), and \( \hat{M}_\Gamma \cap \hat{K} \) is a non-empty open subset of \( \hat{K} \), we know that \( d(\chi) = 1 \).

But \( \chi : (\hat{K} \setminus M_\Gamma) \to \chi(\hat{K} \setminus M_\Gamma) \) is locally quasiregular, so it is a branched covering, and a branched covering of degree 1 is a homeomorphism. Thus \( \chi \) is injective on \( \hat{K} \setminus M_\Gamma \) as well as on \( M_\Gamma \), and, as their images are disjoint, \( \chi \) is therefore injective on their union \( \hat{K} \). As \( \chi \) is continuous on \( \hat{K} \) the result follows. \( \square \)

**Remark 4.1.** We do not know if there exists a continuous extension of \( \chi : M_\Gamma \setminus \{7\} \to M_1 \setminus \{1\} \) to the whole of \( L_{\theta}^\prime \), not just to \( \hat{K} \), but such an extension would be a homeomorphism from the whole of \( L_{\theta}^\prime \) onto its image \( \chi(L_{\theta}^\prime) \).

4.5 Every \( B \in M_1 \setminus \{1\} \) is in \( \chi(\hat{K}) \) for \( K = L_{\theta}^\prime \setminus N \) with \( N \) sufficiently small

We remind the reader that our map \( \chi \) is only defined on the doubly-truncated lune \( K = L_{\theta}^\prime \setminus N \), and that this definition of \( \chi \) is dependent on \( K \), and hence on \( N \), but that on \( M_\Gamma \setminus \{7\} \subset L_{\theta}^\prime \) the definition is independent of \( N \). We start the section with a lemma that we could have stated and proved earlier, but which is now an easy consequence of Proposition 4.8. We continue by showing that we can decompose \( M_\Gamma \) into a main component and rational limbs (Proposition 4.9), and we prove surjectivity limb by limb (Proposition 4.10).

**Lemma 4.2.** \( M_\Gamma \setminus \{7\} \) is connected, and so is \( M_\Gamma \).

*Proof.* If \( M_\Gamma \setminus \{7\} \) is not connected then there exists a point \( a \in M_\Gamma \) and loop \( u \) in \( L_{\theta}^\prime \setminus M_\Gamma \) winding once around \( a \). This loop lies in \( K = L_{\theta}^\prime \setminus N \) for any sufficiently small neighbourhood \( N \), and since \( \chi \) is a homeomorphism the loop \( \chi(u) \) in \( \chi(\hat{K} \setminus M_\Gamma) \) winds once around \( \chi(a) \in M_1 \setminus \{1\} \). But there can be no such loop since the complement of \( M_1 \) in \( \hat{C} \) is simply-connected \([\mathbb{M}1]\). So
$\mathcal{M}_\Gamma \setminus \{7\}$ is connected. That $\mathcal{M}_\Gamma$ is also connected follows since the point $a = 7$ is in the closure of $\mathcal{M}_\Gamma \setminus \{7\}$.

We next recall from Theorem 1 in [BL2] that for every $a \in \mathcal{M}_\Gamma$ for which the alpha-fixed-point of $F_a$ is repelling, this fixed point has a well-defined combinatorial rotation number $\rho_a$ which is always a non-zero rational. Define the $p/q$-limb $L_{p/q}$ of $\mathcal{M}_\Gamma$ to be the set of $a \in \mathcal{M}_\Gamma$ for which the $\alpha$-fixed point of $F_a$ is repelling, and has rotation number $\rho_a = p/q$, together with the parameter value $a_{p/q}$ at which the derivative at the fixed point is $e^{2\pi ip/q}$ (this additional point is called the root of $L_{p/q}$). Our reason for choosing this definition of $L_{p/q}$, rather than defining it as the part of $\mathcal{M}_\Gamma$ which lies between the two external parameter rays which land at $a_{p/q}$, is simply to avoid a diversion into proving that these parameter rays are well defined and do indeed land.

**Proposition 4.9.** (i) $\mathcal{M}_\Gamma$ is the disjoint union of the closure of the main component of $\mathcal{M}_\Gamma$ and the sets $L_{p/q} \setminus \{a_{p/q}\}$, $p/q \in \mathbb{Q}$, $0 < p/q < 1$;

(ii) for each $p/q$ the intersection of $L_{p/q}$ with the closure of the main component of $\mathcal{M}_\Gamma$ is the single point $\{a_{p/q}\}$;

(iii) for each $p/q$ the limb $L_{p/q}$ is closed and connected.

**Proof.** Parts (i) and (ii) follow from the fact that the main component of $\mathcal{M}_\Gamma$ is the set of parameter values for which the alpha-fixed-point of $F_a$ is an attractor (the analytic expression of the derivative of $F_a$ at its alpha-fixed-point shows we have a unique main component, see Section 7 in [BL2]), the fact that the derivative of $F_a$ at its alpha-fixed-point is a holomorphic function of $a$, and the fact that $\mathcal{M}_\Gamma$ does not have irrational limbs (see [BL2]). Part (iii) follows from parts (i) and (ii) since $\mathcal{M}_\Gamma$ is connected (Lemma 4.2 above).

To prove the main results of this subsection and the next, we shall need to use properties of the parabolic Mandelbrot set $\mathcal{M}_1$ analogous to properties of $\mathcal{M}_\Gamma$ and the classical Mandelbrot set $\mathcal{M}$. We shall refer to [PR] for proofs of these: alternatively they may be proved by the methods of [BL2]. The first concerns the existence of a rational combinatorial rotation number for a repelling alpha-fixed-point.

**Proposition 4.10.** Every repelling fixed point of a member of the family $P_A$, $B \in \mathcal{M}_1$ (where $B = 1 - A^2$), is the landing point of a periodic ray.

**Proof.** [PR], Theorem 3.7.

We can now define the $p/q$-limb $L'_{p/q}$ of $\mathcal{M}_1$ to be the set of $B \in \mathcal{M}_1$ ($B = 1 - A^2$), for which the combinatorial rotation number of the fixed point of $P_A$ is $p/q$, together with the parameter value $B_{p/q}$ for which the derivative at the fixed point of $P_A$ is $e^{2\pi ip/q}$.

**Corollary 4.1.** (i) $\mathcal{M}_1$ is the disjoint union of the closure of the main component of $\mathcal{M}_1$ and the sets $L'_{p/q} \setminus \{a_{p/q}\}$, $p/q \in \mathbb{Q}$, $0 < p/q < 1$;
has connected filled Julia set and $\chi$ main component of the interior of $\hat{\mathcal{M}}$ is the single point $\{B_{p/q}\}$;

(iii) for each $p/q$ the limb $L'_{p/q}$ is closed and connected.

Proof. [PR], Section 3.2.3 and in particular Theorem 3.14.

Lemma 4.3. For each integer $n > 1$ there exists a loop $u_n \subset \mathcal{L}'_\theta$ enclosing $\bigcup\{L_{p/q} : 1/n \leq p/q \leq (n-1)/n\}$. These loops can be chosen in such a way that every point of $\mathcal{M}_\Gamma \setminus \{T\}$ is enclosed by $u_n$ for some $n$.

Proof. This would be straightforward if we had established the existence of parameter space external rays landing at the roots of limbs, but we shall sidestep this question and rely instead on the fact that since $\hat{\mathcal{C}} \setminus \mathcal{M}_\Gamma$ is a simply-connected open set there exists a dense set of points in $\partial \mathcal{M}_\Gamma$ which are accessible from the complement of $\mathcal{M}_\Gamma$ in $\mathbb{C}$. First we observe that there exists a neighbourhood $N_{n+1}$ of $a_{1/(n+1)}$ (the root point of $L_{1/(n+1)}$), in $\mathcal{L}'_\theta$, such that $N_{n+1}$ does not meet any $L_{p/q}$ having $1/n \leq p/q \leq 1 - 1/n$. This follows from the restrictions that the Yoccoz inequality proved in [BL2] puts on the derivative at the fixed point of $F_a$ for $a \in L_{p/q}$ (see Theorem 2 and Figure 3 in [BL2]), together with the continuity of this derivative with respect to $a$. Now let $\alpha'_{n+1}$ be some point of $N_{n+1} \cap \partial \mathcal{M}_\Gamma$ accessible from outside $\mathcal{M}_\Gamma$, and define a loop $u_n$ by joining $a_{1/(n+1)}$ to $a'_{1/(n+1)}$ by any path within $N_{n+1}$, then from $a'_{1/(n+1)}$ to its complex conjugate $\alpha'_{n/(n+1)}$ by any path in $\mathcal{L}'_\theta \setminus \mathcal{M}_\Gamma$, then from $\alpha'_{n/(n+1)}$ to $a_{n/(n+1)}$ in the obvious way, and finally from $a_{n/(n+1)}$ to $a_{1/(n+1)}$ by any path within the main component of $\mathcal{M}_\Gamma$, noting that we may choose these last paths in such a way that every point of the main component is enclosed by at least one of the loops $u_n$. Since $u_n$ encloses all the roots $\{a_{p/q} : 1/n \leq p/q \leq (n-1)/n\}$, it must also enclose the corresponding limbs.

Proposition 4.11. Every $B \in \mathcal{M}_1 \setminus \{1\}$ is in $\chi(\hat{K})$ for $K = \mathcal{L}'_\theta \setminus N$ with $N = \mathbb{D}(7, r)$ for $r$ sufficiently small.

Proof. For a contradiction, suppose that there exists $B \in \mathcal{M}_1 \setminus \{1\}$ and a sequence of positive real numbers $r_m$ converging to zero such that for every $K = \mathcal{L}'_\theta \setminus \mathbb{D}(7, r_m)$ we have $B \notin \chi(\mathcal{M}_\Gamma \cap K)$. We prove that the existence of any $B \in \mathcal{M}_1 \setminus \chi(\mathcal{M}_\Gamma)$ leads to a contradiction. Let $B$ be such a parameter value. The main component of $\mathcal{M}_1$ is the unit disc and the main component of $\mathcal{M}_\Gamma$ is also a Jordan disc (bounded by a simple closed curve the equation of which can be readily computed from the information in Section 7 of [BL2], in particular Remark 4). So our homeomorphism $\chi$ between these components extends to a homeomorphism between their closures. Thus $B$ cannot lie in the closure of the main component of the interior of $\mathcal{M}_1$ and must therefore lie in some limb $L'_{p/q}$ of $\mathcal{M}_1$. Choose $1/n < p/q$ and let $D_n$ denote the open topological disc in $\mathcal{L}'_\theta$ bounded by the Jordan curve $u_n$ constructed in the preceding lemma. Let $r_m$ be small enough that $\mathbb{D}(7, r_m)$ does not intersect $D_n$, and let $K = \mathcal{L}'_\theta \setminus \mathbb{D}(7, r_m)$, so $D_n \subset K$. Now $B$ cannot lie in $\chi(D_n)$, since if it did then as $P_A$ (for $B = 1 - A^2$) has connected filled Julia set and $\chi$ is defined by a surgery preserving filled
Julia sets, we would have $B = \chi(a)$ for some $a \in \mathcal{M}_1$. Hence $B \notin \chi(D_n \setminus L_{p/q})$ disconnects the limb $L_{p/q}$ of $\mathcal{M}_1$, contradicting Corollary 4.1(iii).

Since $\chi(\mathcal{M}_1) \subset \mathcal{M}_1$ and $\chi(\hat{K} \setminus \mathcal{M}_1) \subset \text{Per}_1(1) \setminus \mathcal{M}_1$, we deduce from Propositions 4.8 and 4.11:

**Corollary 4.2.** $\chi : \mathcal{M}_1 \setminus \{7\} \rightarrow \mathcal{M}_1 \setminus \{1\}$ is a homeomorphism.

### 4.6 Completing the proof of the Main Theorem: $\mathcal{M}_1$ is homeomorphic to $\mathcal{M}_1$

We use a further property of $\mathcal{M}_1$ analogous to properties of $\mathcal{M}_\Gamma$ and $\mathcal{M}$:

**Proposition 4.12.** The limbs $L_{p/q}$ of $\mathcal{M}_1$ have diameters which converge to 0 as $p/q$ converges to 0, and attaching points $B_{p/q}$ which converge to $B = 1$ as $p/q$ converges to 0.

**Proof.** [PR], Corollary 3.23. The fact that the roots of the limbs tend to $B = 1$ is an immediate consequence of the continuity at $B = 1$ of the derivative of $P_a$ at its alpha-fixed-point.

**Corollary 4.3.** The map $\chi : \mathcal{M}_\Gamma \rightarrow \mathcal{M}_1$ is a homeomorphism.

**Proof.** We have already established that $\chi$ is a homeomorphism between $\mathcal{M}_\Gamma \setminus \{7\}$ and $\mathcal{M}_1 \setminus \{1\}$. Setting $\chi(7) = 1$ extends $\chi$ restricted to the main components of $\mathcal{M}_\Gamma$ and $\mathcal{M}_1$ to a homeomorphism between these components union the points $a = 7$ and $B = 1$ respectively. It remains to consider sequences in the limbs of $\mathcal{M}_\Gamma$ and $\mathcal{M}_1$ converging to $a = 7$ and $B = 1$ respectively. But when $p/q$ converges to zero, the diameters of the $p/q$-limbs converge to zero, and their roots converge to $a = 7$ and $B = 1$ respectively, so we deduce that $\chi$ is a homeomorphism from $\mathcal{M}_\Gamma$ to $\mathcal{M}_1$.

This completes the proof of the Main Theorem, since we have already proved that $\chi$ has properties (i), (ii) and (iii) of the statement of the theorem. Property (i) is Proposition 3.3, property (ii) is Propositions 4.6 and 4.7, and property (iii) is Proposition 4.8.

### 5 Proof of the Corollary to the Main Theorem

In [BL1] we defined $\mathcal{C}_\Gamma$ to be the connectedness locus for the family $F_a$, where we let $a$ vary over the whole of the Klein combination locus $\mathcal{K} \subset \mathbb{C}$. We defined the modular Mandelbrot set $\mathcal{M}_\Gamma$ to be the intersection of $\mathcal{C}_\Gamma$ with the closed disc of centre 4 and radius 3 (which we know to be contained in $\mathcal{K}$ since for $a \in \overline{\mathbb{D}}(4,3)$ the standard fundamental domains $\Delta_{\text{cov}}^a, \Delta_{\text{f}}^a$ are a Klein combination pair):

$$\mathcal{M}_\Gamma := \mathcal{C}_\Gamma \cap \overline{\mathbb{D}}(4,3).$$

We can now prove that $\mathcal{M}_\Gamma$ is the whole of $\mathcal{C}_\Gamma$ (in other words $\mathcal{C}_\Gamma \subset \overline{\mathbb{D}}(4,3)$).
Corollary. The modular Mandelbrot set $\mathcal{M}_\Gamma$ is the whole connectedness locus $\mathcal{C}_\Gamma$ of the family $\mathcal{F}_a$.

Proof. Suppose there exists $a \in \mathcal{C}_\Gamma \setminus \mathcal{M}_\Gamma$. Then the corresponding $\mathcal{F}_a$ is a mating between the modular group and a rational map $P_A(z) = z + 1/z + A$, $A \in \mathbb{C}$ (see the Main Theorem in [BL1]). Since $a \in \mathcal{C}_\Gamma$, the correspondence $\mathcal{F}_a$ has connected limit set, therefore $P_A$ has connected Julia set, and so $B = 1 - A^2 \in \mathcal{M}_1$. Moreover $\mathcal{F}_a$ is hybrid conjugate to $P_A$ on doubly pinched neighbourhoods of $\Lambda_-(\mathcal{F}_a)$ and $K(P_A)$ respectively, by the Main Theorem in [BL1]. Since $\chi : \mathcal{M}_\Gamma \to \mathcal{M}_1$ is a homeomorphism, there exists $a' \in \mathcal{M}_\Gamma$ such that $\chi(a') = B$, and $\mathcal{F}_{a'}$ is hybrid equivalent to $P_A$ on doubly pinched neighbourhoods of $\Lambda_-(\mathcal{F}_a)$ and $K(P_A)$ respectively. But then $\mathcal{F}_a$ is hybrid equivalent to $\mathcal{F}_{a'}$ on quadruply pinched neighbourhoods of $\Lambda_a = \Lambda_{a,-} \cup \Lambda_{a,+}$ and $\Lambda_{a'} = \Lambda_{a',-} \cup \Lambda_{a',+}$ respectively by Proposition A.1, and the proof of Proposition A.1 shows that $\mathcal{F}_a$ is conformally conjugate to $\mathcal{F}_{a'}$, and hence that $a = a'$ by Lemma 2.1.

A Appendix: Dynamical Lunes

Recall that in the parameter plane $\mathcal{L}_\theta$ denotes the open lune bounded by the two arcs of circles passing through $a = 1$ and $a = 7$ which at $a = 1$ have tangents at angles $\pm \theta$ to the positive real axis. Analogously, in the dynamical plane, coordinatised by $Z$, we let $L_a$ denote the open lune bounded by the two arcs of circles passing through $Z = 1$ and $Z = a$ which at $Z = 1$ have tangents at angles $\pm \theta$ to the positive real axis. Recall that $Z = 1$ is the persistent parabolic fixed point, and that our coordinate-free notation for this point is $P_a$. In this Appendix we prove:

Proposition A.1. Given any $\theta$ in the range $\pi/3 \leq \theta \leq \pi/2$, for every parameter value $a \in \mathcal{L}_\theta \cup \{7\}$ the image $\mathcal{F}_a(T_a)$ of $T_a$ is contained in $L_a \cup \{P_a\}$.

Note that here $\mathcal{F}_a : T_a \to \mathcal{F}_a(T_a)$ is a 1-to-2 correspondence. Note also that in the coordinate $z' = (a - 1)(Z - 1)/(a - Z)$, the points $Z = 1$ and $Z = a$ become $z' = 0$ and $z' = \infty$ respectively and that as the coordinate change from $Z$ to $z'$ has derivative 1 at $Z = 1$, the boundary of the lune $L_a$ is carried to a pair of fixed straight lines through the origin in the $z'$-plane at angles $\pm \theta$ to the positive real axis. Thus in the coordinate $z = (Z - 1)/(a - Z)$ the lune $L_a$ is no longer independent of $a$, but the points in it move holomorphically with $a$.

Consider $J(L_a)$, and observe that since $J : z \leftrightarrow -z$ in the $z$-coordinate, $J(L_a) = -L_a$ in this coordinate. We remark that $J(L_a)$ is contained in the standard fundamental domain $\Delta_J$ for $J$, which is defined in the $Z$-coordinate [BL1] as the complement of the round disc which has centre on the real $Z$-axis and boundary circle passing through the points $Z = 1$ and $Z = a$. The following is an immediate consequence of Proposition A.1.

Corollary A.1. For $\pi/3 \leq \theta \leq \pi/2$,
\(a \in \mathcal{L}_\theta \cup \{7\} \Rightarrow \mathcal{F}_a^{-1}(J(L_a)) \subset J(L_a) \cup \{P_a\},\) and hence

(i) \(\Lambda_{-a} \subset J(L_a) \cup \{P_a\};\)

(ii) \(\Lambda_{-a} \cap \partial J(L_a) = \{P_a\}.\)

In [BL2] we proved that there exists \(\theta\) in the half-open interval \([\pi/3, \pi/2)\) such that \(\mathcal{L}_\theta\) is a neighbourhood of \(\mathcal{M}_1 \setminus \{7\}\), pinched at \(a = 7\). Corollary A.1 tells us that for all \(a\) in this pinched neighbourhood, the holomorphically moving dynamical lune \(J(L_a)\) contains \(\Lambda_{-a} \setminus \{P_a\}\).

A.0.1 Proof of Proposition A.1

Working in the \(Z\)-coordinate, let \(C_d\) denote the circle in the dynamical plane which passes through the points \(Z = 1\) and \(Z = 7\) and which has centre at the point \(4 - di\) (where \(d\) is real). As usual let \(Q(Z) = Z^3 - 3Z\).

Lemma A.1. For every \(-\sqrt{3} \leq d \leq \sqrt{3}\), the circle \(C_d\) meets its image under \(\text{Cov}^Q_0\) uniquely at the point \(Z = 1\).

Assuming this lemma (proved below), the proof of Proposition A.1 proceeds as follows. Let \(\pi/3 \leq \theta \leq \pi/2\). Setting \(d = 3 \cot \theta\), we note that the upper boundary arc of the parameter space lune \(\mathcal{L}_\theta\) is the upper arc of \(C_d\) from \(a = 1\) to \(a = 7\), and the lower boundary arc of \(\mathcal{L}_\theta\) is the lower arc of \(C_{-d}\) between these two points.

Now let \(a \in \mathcal{L}_\theta \cup \{7\}\), and let \(C_{a,\theta}\) denote the circle in dynamical space through \(Z = 1\) and \(Z = a\) which has tangent at angle \(\theta\) to the real axis at \(Z = 1\). The upper boundary of the dynamical space lune \(L_a\) is the upper arc of \(C_{a,\theta}\) from \(Z = 1\) to \(Z = a\) and the lower boundary is the lower arc of \(C_{a,-\theta}\) between these two points. If we let \(D_d\) denote the open disc bounded by \(C_d\), and \(D_{a,\theta}\) denote the open disc bounded by \(C_{a,\theta}\), we see that

(i) \(D_{a,\theta} \subseteq D_d\) and \(D_{a,-\theta} \subseteq D_{-d}\);

(ii) \(L_a = D_{a,\theta} \cap D_{a,-\theta}\).

But from Lemma A.1 we know that \(\text{Cov}^Q_0(D_d)\) meets \(D_d\) at the single point \(Z = 1\), and since the involution \(J\) sends the exterior of \(D_a\) to its interior, we deduce from (i) that \(\mathcal{F}_a(D_{a,\theta}) \subset D_{a,\theta} \cup \{1\}\). Similarly \(\mathcal{F}_a(D_{a,-\theta}) \subset D_{a,-\theta} \cup \{1\}\) and the observation (ii) yields the statement of the Proposition.

A.0.2 Proof of Lemma A.1

The correspondence \(\text{Cov}^Q_0\) sends \(Z \to Z'\) where

\[
\frac{Q(Z) - Q(Z')}{Z - Z'} = 0 \quad \text{i.e.} \quad Z'^2 + ZZ' + Z'^2 = 3
\]

\(\text{i.e.} \quad Z' = \frac{-Z}{2} \pm \sqrt{3 \left(1 - \left(\frac{Z}{2}\right)^2\right)}\)
The image of $C_d$ under $Cov_0^Q$ is a topological circle which wraps twice around $C_d$ under the inverse of $Cov_0^Q$, a two-to-one map (see Figure 14(i)). This topological circle has a cusp at $Z = -2$ (the ‘other’ inverse image of the fixed point $Z = 1$) and it has a self-intersection near to the cusp if and only if it intersects $C_d$ near $Z = 1$. The tangent and curvature of $Cov_0^Q(C_d)$ at $Z = 1$ are identical to those of $C_d$ at $Z = 1$, whatever the value of $d$, and the transition of intersection behaviour at the point $Z = 1$ when $d = \sqrt{3}$ is associated with the values of the third (or higher) derivatives of the two curves at this point.

The full picture is best seen in the $W$-plane double-covering the $Z$-plane via $W \rightarrow Z = W + 1/W$ (Figure 14(ii)). To investigate the intersections we pass to the $W$-plane, where $Cov_0^Q$ lifts to the 2-valued map defined by multiplication by $e^{\pm 2\pi i/3}$, and where we can write explicit polynomial equations for the lifts of the various curves.

Writing $Z = X + iY$ and $W = U + iV$, the equation $W + 1/W = Z$ gives:

$$U + iV + \frac{U - iV}{U^2 + V^2} = X + iY.$$ 

Thus

$$X = U \left( \frac{U^2 + V^2 + 1}{U^2 + V^2} \right) \quad \text{and} \quad Y = V \left( \frac{U^2 + V^2 - 1}{U^2 + V^2} \right).$$

The equation for $C_d$ is:

$$(X - 4)^2 + (Y + d)^2 = 9 + d^2$$

and the equation of the lift $\tilde{C_d}$ of $C_d$ to the $W$-plane is therefore:

$$\left( U \left( \frac{U^2 + V^2 + 1}{U^2 + V^2} \right) - 4 \right)^2 + \left( V \left( \frac{U^2 + V^2 - 1}{U^2 + V^2} \right) + d \right)^2 = 9 + d^2.$$
Multiplying through by \((U^2 + V^2)^2\) and re-arranging terms we find an expression which has \(U^2 + V^2\) as a factor, and when this factor is divided out we obtain as equation for \(\tilde{C}_d\):
\[
(U^2 + V^2)(U^2 + V^2 - 8U + 7) + 2dV(U^2 + V^2 - 1) + 2U^2 - 2V^2 - 8U + 1 = 0 \quad (1)
\]
The map \(x e^{2\pi i/3}\) sends \((U, V)\) to \((U', V')\) where
\[
\begin{pmatrix}
U' \\
V'
\end{pmatrix} = \begin{pmatrix}
-U/2 - \sqrt{3V}/2 \\
\sqrt{3}U/2 - V/2
\end{pmatrix}.
\]
That is,
\[
\begin{pmatrix}
U \\
V
\end{pmatrix} = \begin{pmatrix}
-U'/2 + \sqrt{3}V'/2 \\
-\sqrt{3}U'/2 - V'/2
\end{pmatrix}.
\]
Substituting these expressions into equation \((1)\) gives us
\[
(U'^2 + V'^2) \left(U'^2 + V'^2 + 4(U' - \sqrt{3}V') + 7\right) - d(\sqrt{3}U' + V')(U'^2 + V'^2 - 1)
\]
\[
-U'^2 + V'^2 - 2\sqrt{3}U'V' + 4(U' - \sqrt{3}V') + 1 = 0.
\]
Thus after rotation through \(2\pi/3\) about \((0, 0)\) in the \((U, V)\)-plane, \(\tilde{C}_d\) becomes
\[
(U^2 + V^2) \left(U^2 + V^2 + 4(U - \sqrt{3}V) + 7\right) - d(\sqrt{3}U + V)(U^2 + V^2 - 1)
\]
\[
-U^2 + V^2 - 2\sqrt{3}UV + 4(U - \sqrt{3}V) + 1 = 0 \quad (2)
\]
and after rotation through \(-2\pi i/3\) the curve \(\tilde{C}_d\) becomes:
\[
(U^2 + V^2) \left(U^2 + V^2 + 4(U + \sqrt{3}V) + 7\right) - d(-\sqrt{3}U + V)(U^2 + V^2 - 1)
\]
\[
-U^2 + V^2 + 2\sqrt{3}UV + 4(U + \sqrt{3}V) + 1 = 0 \quad (3)
\]
(which is just \(2\) with \(\sqrt{3}\) replaced by \(-\sqrt{3}\)). We now show that provided \(|d| \leq \sqrt{3}\) the curves \((1)\) and \((2)\) meet at the unique point \((U, V) = (1/2, \sqrt{3}/2)\).
By rotational symmetry an equivalent problem is to show that the curves \((2)\) and \((3)\) have unique real intersection point \((U, V) = (-1, 0)\).

The \(U\)-coordinates of the points of intersection of \((2)\) and \((3)\) are the zeros of the resultant of \((2)\) and \((3)\), where these are regarded as polynomials in \(V\) with coefficients polynomials in \(U\).

Recall that the resultant of the polynomials \(a_4 V^4 + a_3 V^3 + a_2 V^2 + a_1 V + a_0\) and \(b_4 V^4 + b_3 V^3 + b_2 V^2 + b_1 V + b_0\) is the determinant
\[
\begin{vmatrix}
a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\
0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\
0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\
0 & 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\
b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\
0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\
0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\
0 & 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0
\end{vmatrix}
\]

43
The resultant of (2) and (3) is the polynomial $P(U)$ given by the determinant above with

\[
\begin{align*}
a_4 &= 1, & a_3 &= -4\sqrt{3} - d, & a_2 &= 2U^2 + (4 - d\sqrt{3})U + 8 \\
a_1 &= (-4\sqrt{3} - d)U^2 - 2\sqrt{3}U + (d - 4\sqrt{3}) \\
a_0 &= U^4 + (4 - d\sqrt{3})U^3 + 6U^2 + (4 + d\sqrt{3})U + 1
\end{align*}
\]

and with $b_j (0 \leq j \leq 4)$ obtained by substituting $-\sqrt{3}$ for $\sqrt{3}$ in $a_j$.

A computation using MAPLE gives:

\[
P(U) = 2304(d^2 + 9)(U + 1)^4Q(U)
\]

where

\[
Q(U) = (d^2 + 25)U^4 + 40U^3 + (96 - 12d^2)U^2 + (64 + 16d^2)U + 64.
\]

When $d = 0$, $Q(U)$ is a quartic which takes values $> 0$ for all $U \in \mathbb{R}$, since then:

\[
Q(U) = 25U^4 + 40U^3 + 96U^2 + 64U + 64 = U^2(5U + 4)^2 + 80(U + 2/5)^2 + 256/5.
\]

When $d = \pm \sqrt{3}$ we have

\[
Q(U) = 28U^4 + 40U^3 + 60U^2 + 112U + 64 = (U + 1)^2(28U^2 - 16U + 64)
\]

which has unique real zero $U = -1$.

As we increase $|d|$ from 0 to $\sqrt{3}$, the only way that a real zero of $Q(U)$ can be born is as a repeated zero, thus at a value of $d$ where the resultant of $Q(U)$ and its derivative $Q'(U)$ is zero. Another appeal to MAPLE tells us that this resultant is equal to:

\[
-143327232d^4(d^2 + 25)(d^2 - 3)(d^2 + 24)^2
\]

\[
= -2^{16}3^7d^4(d^2 + 25)(d^2 - 3)(d^2 + 24)^2
\]

and therefore that $Q(U)$ has no real zero for any value of $|d|$ in the interval $0 < |d| < \sqrt{3}$, completing the proof of the lemma.

\[\blacksquare\]

**Remark A.1.** By considering the expression for $P(U)$ for values of $d$ with $|d|$ equal to $\sqrt{3} + \varepsilon$ one can show that new real intersections of (2) and (3) bifurcate out of the intersection point $(U, V) = (-1, 0)$ as $|d|$ passes through $\sqrt{3}$. Thus the value $\sqrt{3}$ in the statement of Lemma A.1 (and hence the value $\pi/3$ in the statement of Proposition A.1) is best possible.

**References**

[BH] S. Bullett, P. Haissinsky, *Pinching holomorphic correspondences*, Conform. Geom. Dyn. 11 (2007), 65–89.
[BL1] S. Bullett, L. Lomonaco, *Mating quadratic maps with the modular group II*, Invent. math., Vol 220, (2020), 185–210.

[BL2] S. Bullett, L. Lomonaco, *Dynamics of Modular Matings*, Adv. Math. 410, Part B (2022), 108758.

[BP] S. Bullett, C. Penrose, *Mating quadratic maps with the modular group*, Invent. math., Vol 115, (1994), 483–511.

[DH] A. Douady & J. H. Hubbard, *On the Dynamics of Polynomial-like Mappings*, Ann. Sci. École Norm. Sup.,(4), Vol.18 (1985), 287–343.

[L1] L. Lomonaco, *Parabolic-like maps*, Erg. Theory and Dyn. Syst. 35 (2015), 2171–2197.

[L2] L. Lomonaco, *Parameter space for families of parabolic-like maps*, Adv. Math. 261 (2014), 200–219.

[LLMM] S. Lee, M. Lyubich, N. Makarov, S. Mukherjee, *Schwarz reflections and anti-holomorphic correspondences*, Adv. Math. 385 (2021), Article 107766.

[Lyu] M. Lyubich, *Conformal Geometry and Dynamics of Quadratic Polynomials*, www.math.sunysb.edu/~mlyubich/book.pdf

[M1] J. Milnor, *On Rational Maps with Two Critical Points*, Experimental Mathematics, (4), Vol. 9 (2000), 481–522.

[MSS] R. Mañé, P. Sad & D. Sullivan, *On the Dynamics of Rational maps*, Ann. Sci. École Norm. Sup.,(4), Vol.16 (1983), 193–217.

[PR] C. Petersen, P. Roesch, *The parabolic Mandelbrot set*, https://arxiv.org/pdf/2107.09407.

[PT] C. Petersen, L. Tan, *Branner-Hubbard motions and attracting dynamics*, Dynamics on the Riemann sphere (eds. P Hjorth, C. Petersen), Eur. Math. Soc. (2006), 45–70.

[Sh] M. Shishikura, *Bifurcation of parabolic fixed points*, The Mandelbrot set, Theme and Variations (ed. Tan Lei), London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, (2000), 325–363.

[S] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou–Julia problem on wandering domains*, Ann. of Math. 122 (1985), 401–418.

School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK s.r.bullett@qmul.ac.uk

Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Jardim Botânico, Rio de Janeiro, RJ, 22460-320, Brasil luna@impa.br