Quantum walk as generalized evolution and measurement device

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Abstract
We show that arbitrary quantum evolution, both unitary and non-unitary ones, can be efficiently realized via one-dimensional quantum walk. This is done based on the equivalence between quantum evolution and positive-operated valued measurements, and an algorithm of properly controlling one-dimensional discrete-time quantum walk to realize arbitrary positive-operated valued measurements. Furthermore, our method based on one-dimensional quantum walk can be easily generalized to other platforms. Our method enriches technics for quantum information processes, and the applications of quantum walk.

1. Introduction
Quantum evolution, which maps the initial state of a quantum system to the final state, is integral to quantum information processes. Therefore, how to realize quantum evolution is a question with long-standing interests, and the solutions cover from mathematical decomposition and machine learning [1–3]. Recently, the rapid development of non-unitary physics initiates new requests for the method to realize non-unitary quantum evolution [4–14]. In experiments, non-unitarity of quantum evolution is always realized via introducing specific loss to the system [7, 8]. However, a universal method to realize arbitrary non-unitary evolution, i.e., when and how to introduce loss, is imperative for studying non-unitary physics. In this work, we investigate such a method with quantum walk (QW). For a unitary evolution, projective measurements in the basis of the final state are von Neumann measurements of the initial state. Whereas, for non-unitary evolution, it turns to be positive-operated valued measurement (POVM). Thus, the issue of realizing quantum evolution is highly related to the one of implementing POVM. In this work, we show such relation allows for realizing arbitrary quantum evolution based on POVM. A POVM $\mathcal{E}$ is generalized quantum measurement described by a set of positive operators $E_i$, which satisfy the completeness condition $\sum_i E_i = 1$, where $I$ denotes identity operator. POVM offers an efficient method for generalized acquisition of information, and thereby widely used in quantum information processes [15–21]. Thus, the realization of POVM is an important issue for both the target of realizing arbitrary quantum evolution and other quantum information processes based on POVM. POVM can be realized via a number of systems [22–26]. Especially, Kurzyński and Wójcik proposed a universal method of realizing POVM for qubit system based on discrete-time QW [27–29]. In this paper, we initialize the QW in multi positions, which plays the role of a high dimensional system, and generalize the method of realizing arbitrary POVM of qubit system to the case of arbitrary dimensional systems. Based on the device of POVM, we provide a method to realize arbitrary quantum evolution. Thus, we conclude that QW serves as simulation tool for a generalized quantum evolution and quantum measurement device.
2. One-dimensional discrete-time QW

Discrete-time QW is a standard quantum evolution process of a quantum particle (walker) in position space controlled by the state of another degree of freedom (coin) [30, 31]. The state of a walker is described by the basis \(|x, c\rangle = |x\rangle \otimes |c\rangle\), which is composed by the position state \(|x\rangle\) and coin state \(|c\rangle\). For one-dimensional QW, we have \(x = ..., -1, 0, 1, ...\) and \(c=0,1\). The discrete dynamic of QW is described as time-step operators \(U_t = TC\) for \(t\)-th step, where

\[
T = \sum_x |x + 1\rangle \langle x| \otimes |0\rangle \langle 0| + |x - 1\rangle \langle x| \otimes |1\rangle \langle 1|
\]

(1)
is the coin dependent translation operator, and

\[
C = \sum_x |x\rangle \langle x| \otimes C(x, t),
\]

(2)
is the coin flip operator with \(C(x, t)\) the qubit operation applied to the coin state at position \(x\) in \(t\)-th step. The probability distribution of finding the walker in position space cannot be reproduced by its classical counterpart, which makes it widely used in designing quantum algorithms and simulate various quantum dynamics [8, 32, 33], and the dynamics of QW can be properly engineered by the coin operations which depend on both position and time-step, which makes QW an efficient platform for various quantum information processes including state transfer [34, 35], qubit POVM [27–29] and our method for arbitrary evolution and POVM.

3. Implementation of quantum evolution via POVM

A quantum evolution

\[
U = \sum_{i=1}^{2n} |i\rangle \langle \psi_i|,
\]

(3)
of an \(2n\)-dimensional system, maps the initial state of the system \(|\psi_{\text{in}}\rangle\) to final state \(|\psi_{\text{final}}\rangle = U|\psi_{\text{in}}\rangle\). Measurements \(|i\rangle \langle i|\) on the final state correspond to measurements \(|\psi_i\rangle \langle \psi_i|\) on the initial state. For unitary evolution \(U\), \(UU^\dagger = 1\), measurements \(|\psi_i\rangle \langle \psi_i|\) are projective von Neumann measurements. Whereas for non-unitary evolution, \(UU^\dagger \neq 1\), measurements \(|\psi_i\rangle \langle \psi_i|\) are not projective measurements since the states \(|\psi_i\rangle\) are not complete orthogonal basis. However, one can always find a maximal real number \(0 < \alpha < 1\) to rescale the states as \(|\tilde{\psi}_i\rangle = \sqrt{\alpha}|\psi_i\rangle\) which guarantees \(1 - \sum_i |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|\) a positive-semi-definite operator. Thus, we get a rescaled evolution

\[
\tilde{U} = \sum_{i=1}^{2n} |i\rangle \langle \tilde{\psi}_i|,
\]

(4)
after which the measurements \(|i\rangle \langle i|\) on the final state correspond to POVM elements \(|\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|\) implemented to the initial state. Thus, one can define a POVM \(\mathcal{E}\) with \(2n + 1\) elements, \(E_i = a|\psi_i\rangle \langle \psi_i| = |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|\) for \(i = 1, 2, ..., 2n\) and \(E_{2n+1} = 1 - \sum_{i=1}^{2n} a|\psi_i\rangle \langle \psi_i|\), where \(a = 1\) and \(E_{2n+1} = 0\) for unitary case. Thus, realization of the rescaled evolution allows the realization of a corresponding POVM.

Let us consider a measurement device realizing such a POVM \(\mathcal{E}\). The device always works as entangling the measured system with \(2n + 1\) output ports \(D_i\) of the device, and the state of the first \(2n\) ports is \(\sum_{i=1}^{2n} |(D_i) \otimes |\psi_i\rangle\langle \psi_i|\langle \psi_{\text{in}}|\rangle\), where \(|D_i\rangle\) denote the state of the output ports. That is to say that detecting the quantum system in port \(D_i\) corresponds to the measurement result of POVM element \(E_i\) and when the quantum system is detected at the port \(D_i\) for \(i = 1, 2, ..., 2n\), the state of the quantum system is definitely proportional to \(|\psi_i\rangle\).

4. Generation of arbitrary POVM via QW

In this paper, we focus on rank 1 POVMs for \(2n\)-dimensional system, which can be easily extended to the case of arbitrary POVM for arbitrary dimensional system. This is because any system with odd dimensions \(2n - 1\) is subsystem of a \(2n\)-dimensional system, and higher rank POVM elements are able to be constructed directly as a
convex combination of rank 1 elements [27]. Here rank 1 elements are of the form \( E_i = a_i |\psi_i\rangle \langle \psi_i| \) with \( 0 \leq a_i \leq 1 \), which is exactly the property of the POVM elements required for realizing quantum evolution.

In order to realize POVM of 2n-dimensional system via one-dimensional QW, we first encode the system in hybrid position and coin state of the walker [2]. The state \( |\psi\rangle \) with basis \([|1\rangle, |2\rangle, \ldots, |2n\rangle]\) can be encoded as the QW state \( |\Psi(\psi)\rangle = \sum_{x=0}^{2n-1} \sum_{c=0}^{n-1} (2x + c \ 1 |\psi\rangle \langle |\psi| - 2x\rangle |c\rangle \) at \( n \) positions with the basis \([2x + c + 1]\) encoded as \(-2x\rangle |c\rangle \).

### 4.1. Preparation of high dimensional state with one-dimensional QW

Here we briefly discuss a method of preparing one-dimensional QW system into a \( 2n \)-dimensional state. Consider a desired state \( |\psi\rangle \) of the \( 2n \)-dimensional system with basis \([|1\rangle, |2\rangle, \ldots, |2n\rangle]\). We begin the preparation with the walker initialized as \( |\psi_p\rangle = |x_0, 0\rangle \), where the subscript \( p \) denotes preparation and superscript (0) denotes the number of time-steps. Our method to prepare the desired state includes two procedures: Firstly, applying \( n - 1 \) steps of QW with the coin operations of the \( t \)-th step are set to be \( C_{i(t)} \) at position \( x_0 + (t - 1) \) and \( 1 \) elsewhere, and secondly, applying coin operations \( C_{i(2)} \) at positions \( x_0 + n - 1 = -2x \) for \( x = 0, 1, \ldots, n - 1 \). Here the subscripts (1) and (2) of \( C \) denote the first and second procedures, respectively.

The state of walker after \( t - 1 \) steps, where \( t < n - 1 \), is \( |\psi_p(t-1)\rangle \) with the coin state at the rightmost position \( x_0 + (t - 1) \) is \( |0\rangle \). The operation \( C_{i(1)} \) flips the coin state at position \( x_0 + (t - 1) \), and then the walker with coin state \( |1\rangle \) walks to the left in the following steps. Thus after \( n - 1 \) steps, it appears in position \( x_0 + n + 2t - 2 \) with coin state \( |1\rangle \) and amplitude \( x_0 + n + 2t - 2 = 1 |\psi_p(n-1)\rangle \)

\[
|2n - 2t + 1 |\psi_p(n-1)\rangle^2 + |2n - 2t + 2 |\psi_p(n-1)\rangle^2 \text{ starting from } i = 1 \text{ to } i = n - 1. \]

The normalization condition implies that \( |x_0 + n - 1, 0 |\psi_p(n-1)\rangle^2 = |1 |\langle 1| \psi_p^2 + |2 |\psi_p^2 \text{. That is to say the amplitudes of } \}

\[
|x_0 + n - 1 - 2x, 1 |\psi_p(n-1)\rangle^2 = |2x + 1 |\psi_p^2 + |2x + 2 |\psi_p^2 \text{ for } x = 0, 1, \ldots, n - 1. \] The second procedure gives the state \( |\psi_p(n)\rangle \). By properly choosing the coin operations \( C_{i(2)} \), the amplitudes \( |\psi_p(n-1)\rangle \) are \( |x_0 + n - 1 - 2x, c |\psi_p(n-1)\rangle = |2x + c \ 1 |\psi_p\rangle \), where \( x = 0, 1, \ldots, n - 1 \). Thus the final state \( |\psi_p(n)\rangle \) encodes the \( 2n \)-dimensional state \( |\psi\rangle \). Noting that the state \( |\Psi(\psi)\rangle \), which encodes the system in positions \(-2x \) with \( x = 0, 1, \ldots, n - 1 \), is prepared with the method by choosing initial position as \( x_0 = -n + 1 \). The method for choosing the coin operations \( C_{i(1)} \) and \( C_{i(2)} \) is given in Appendix.

### 4.2. Algorithm to generate POVM via QW

After the state to be measured \( |\psi\rangle \) is initialized in walker at positions \( |\Psi(\psi)\rangle \), the walker is distributed in \( n \) positions \(-2x \) with \( x = 0, 1, \ldots, n - 1 \). As shown in figure 1(a), our algorithm to generate an arbitrary rank 1 POVM controls the quantum walk in two regions, the inner region with positions between 1 and \(-2n + 1 \), and the right region with positions larger than 1. This is realized by applying the coin operations at position \(-2n + 1 \) as \( \sigma_x \), so that the walker initialized between positions 1 and \(-2n + 1 \) never travels to the positions less than \(-2n - 1 \). Thus, we denote the state of the system as \( |\Psi \rangle \) \( |\Phi \rangle \), where \( |\Psi \rangle \) and \( |\Phi \rangle \) are the state of inner and right region, respectively. The initial state is \( |\Psi_0 \rangle + |\Phi_0 \rangle \), with \( |\Phi_0 \rangle = 0 \). Moreover, the coin operations in the right region are always \( I \), so once part of the state travels to the right region, it walks to right and never goes back to the inner region.

Our algorithm to generate an arbitrary rank 1 POVM \( \{E_1, E_2, \ldots, E_m\} \) works as follows:

(1) Set the iteration number \( i = 1 \).

(2) While \( i \leq m \) do the following:

(a) Apply coin operations \( C_i \) at position \(-2x \) for \( x = 0, 1, \ldots, n - 1 \), and identity \( I \) elsewhere.

(b) If \( n \geq 2 \), set subiteration number \( j = 1 \).

(c) While \( j \leq n - 1 \), do the following:

(i) Firstly apply translation operator \( T \), secondly apply coin operations \( \sigma_x \) at position \(-2n + 1 \) and \( I \), thirdly apply translation operator \( T \) again, and finally apply coin operation \( C_i^{n-1+j} \) at position \( 0 \) and \( I \) elsewhere.

(ii) Set \( j := j + 1 \) and return to step (c).

(d) Firstly apply translation operator \( T \), and then apply coin operation \( C_i^{2n-1} \) at position \( 1 \), \( \sigma_x \) at position \(-2n + 1 \) and \( I \) elsewhere, and finally apply translation operator \( T \).


Let us analyze the action of the algorithm and introduce the settings of coin operations \( C^+_i \), \( C^{n-1+i} \), and \( C^{2n-1} \). We first consider a single iteration \((i := k)\) of the algorithm, which begins with the state \(|\Psi_{k-1,0}\rangle\) generated by previous iteration with \(i = k - 1\). In step (2a), the coin operation evolves the state \(|\Psi_{k-1,0}\rangle\) into \(|\Psi_{k-1,0}\rangle\), where the second subscript 0 denotes the number of subiterations, with the amplitudes

\[
\langle -2x, c|\Psi_{k-1,0}\rangle = \sum_{c'} \langle c| C^+_k|c'\rangle \langle -2x, c'|\Psi_{k-1,0}\rangle.
\]

The amplitudes are dependent on both the state \(|\Psi_{k-1,0}\rangle\) and the coin operations \( C^+_k \).

Next, we consider a single subiteration of step (2c) with \(j = l\). The subiteration evolves the walker from state \(|\Psi_{k-1,0}\rangle\) into \(|\Psi_{k-1,0}\rangle\), with the amplitudes \(-2x, c|\Psi_{k-1,0}\rangle\) as

\[
\begin{align*}
\langle -2x - 2, c|\Psi_{k-1,0}\rangle & \quad n - 2 \leq x \leq 1, c = 0; \\
\langle -2x + 2, c|\Psi_{k-1,0}\rangle & \quad n - 1 \leq x \leq 1, c = 1; \\
\langle -2(n - 1), c|\Psi_{k-1,0}\rangle & \quad x = n - 1, c = 0;
\end{align*}
\]

Thus, after all the subiterations of step (2c), the state of inner region is \(|\Psi_{k-1,n-1}\rangle\), with the amplitudes satisfy equation (6).

Next, we apply step (2d) and the state \(|\Psi_{k-1,n-1}\rangle\) evolves to

\[
|\Psi_k\rangle = \langle 0| C^{2n-1}_k|0\rangle \langle 0, 0|\Psi_{k-1,n-1}\rangle |2, 0\rangle,
\]

where the amplitudes \(-2x, c|\Psi_k\rangle\) are

\[
\begin{align*}
\langle 1| C^{2n-1}_k|0\rangle \langle 0, 0|\Psi_{k-1,n-1}\rangle & \quad x = 0, c = 1 \\
\langle -2x - 2, c|\Psi_{k-1,n-1}\rangle & \quad n - 2 \leq x \leq 0, c = 0; \\
\langle -2x + 2, c|\Psi_{k-1,n-1}\rangle & \quad n - 1 \leq x \leq 1, c = 1; \\
\langle -2(n - 1), c|\Psi_{k-1,n-1}\rangle & \quad x = n - 1, c = 0;
\end{align*}
\]

The left term \(|\Psi_k\rangle\) of equation (7) is the state of inner region generated by this iteration and serves as the beginning of next iteration, and the right term in equation (7) is the new part of the walk travels in the right region, which belongs to \(|\Psi_k\rangle\). Other parts of \(|\Psi_k\rangle\) are contributed by \(|\Psi_{k-1}\rangle\), which propagate directly to the right during the iteration. Thus, after the iteration, the state of right region is

\[
|\Psi_k\rangle = \sum_x \langle x + 2n| x|\Psi_{k-1}\rangle + \langle 0| C^{2n-1}_k|0\rangle \langle 0, 0|\Psi_{k-1,n-1}\rangle |2, 0\rangle.
\]

(c) Set \(i := i + 1\) and return to step (2).
Considering the initial state of right region $|\Phi_0\rangle = 0$, we have
\[ |\Psi_k\rangle = \sum_{i=1}^k \langle 2, 0|\Phi_i\rangle |2 + 2n(k - i), 0\rangle, \] (10)
by repeatedly using equation (9).

After all the iterations of the algorithm, we have $|\Psi_m\rangle + |\Phi_m\rangle$ with the amplitudes described in a recursive way using equations (5), (6), (8), (9), and (10). Thus, the amplitudes equal to the inner product between the initial state and an unnormalized state. Thus at the end of the algorithm, the probability $p_i$ of finding the walker at position $2n(m - i) + 2$ is
\[ p_{n} = |\langle 2n(m - i) + 2, 0|\Phi_{m}\rangle|^2 = |\langle 2, 0|\Phi_{m}\rangle|^2 = |\langle 0| C_{j}^{2n-1}|0\rangle |^2 |\langle 0, 0|\Phi_{i-n-1}\rangle|^2. \] (11)

We now show the proper settings of coin operations to make the probabilities $p_i$ equal to the probabilities of desired POVM elements $E_i$. For this purpose, we choose the coin operations $C_i^n$ to make
\[ -2x_i, 1|\Psi_{i-j}\rangle = 0, \] (12)
for $i = 1, 2, \ldots, m$, and $x = 0, 1, \ldots, n - 1$, and $C_{n-i}^{n+i}$ to make
\[ \langle 0, 1|\Psi_{i-j}\rangle = 0, \] (13)
for $j = 1, 2, \ldots, n - 1$. An example of such choice is
\[ C_i^n = \frac{1}{\sqrt{\langle -2x, 0|\Psi_{i-j}\rangle^2 + \langle -2x, 1|\Psi_{i-j}\rangle^2}} \bigg\langle \langle -2x, 1|\Psi_{i-j}\rangle - \langle -2x, 0|\Psi_{i-j}\rangle \bigg\rangle, \]
and
\[ C_{n-i}^{n+i} = \frac{1}{\sqrt{\langle 0, 0|\Psi_{i-j}\rangle^2 + \langle 0, 0|\Psi_{i-j}\rangle^2}} \bigg\langle \langle 0, 0|\Psi_{i-j}\rangle - \langle -2, 0|\Psi_{i-j}\rangle \bigg\rangle. \]

With the settings of equations (12) and (13), the walker initially in state $|\Psi_{0}\rangle$ evolves to state $|\Psi_{i-n-1}\rangle$ with the amplitudes $(-2x, c|\Psi_{i-n-1}\rangle = 0$ except $x = 0$ and $c = 0$. That is to say the amplitude
\[ \langle 0, 0|\Psi_{i-n-1}\rangle = \sqrt{a_i} |\Psi_{i}\rangle, \]
where $0 \leq a_i' \leq 1$, and $a_i' = 1 - |\langle \Psi_{i-n-1}|\Psi_{i}\rangle|$. The coin operations $C_{i}^{2n-1}$ are set as
\[ \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}, \]
(14)
For the first iteration, we have $\langle 0, 1|\Phi_{i-n-1}\rangle = \langle \Psi_{i}\rangle = 1$ with $a_i' = 1$. With properly setting $C_{i}^{2n-1}$ as $\cos^2 \theta_i = a_i$, the probability $p_i = a_i |\Psi_{i}\rangle |^2$ equals to the result of POVM element $E_i$. We assume that the probabilities $p_i$ for $i = 1, 2, \ldots, n - 1$ equals to the probability of POVM elements $E_i$. Then after the iteration $i = k$, the amplitudes $\langle 0, 0|\Psi_{i-n-1}\rangle = \sqrt{a_k} |\Psi_{i}\rangle$ with $a_k = \text{Tr}[\sum_{j=1}^{n-1} E_j |\Psi_{i}\rangle \langle \Psi_{i}|] \geq a_k$. Thus, via properly setting $C_{i}^{2n-1}$ as $\cos^2 \theta_i = a_k/a_k'$, the probability $p_k$ equals to the probability of POVM elements $E_k$.

If one only interests in the measured probabilities, the algorithm together with previous settings of coin operations successfully generate the measured probability of the desired POVM. However, standard POVM devices collapse the measured system to postmeasurement state for each measurement result. Noting that the state of the system correspond to each element after the algorithm are pure state $|2n(m - i) + 2, 0\rangle$ at positions $2n(m - i) + 2$. The proper postmeasurement state can be realized by applying state preparation processes to those states. For example, we continue the QW for $(n - 1)$ steps, then each state $|2n(m - i) + 2, 0\rangle$ evolves to $|2n(m - i) + n + 1, 0\rangle$. After that, state preparation process introduced in section 4.1 can prepare the proper postmeasurement states $|\psi_i\rangle$ encoded in QW state as $|2n(m - i) + 2x, c\rangle$ for $x = 0, 1, \ldots, n - 1$ and $c = 1, 2$.

After preparing the proper postmeasurement states, the measured probability of element $E_i$ corresponds to the probability of finding the walker in the region between positions $2n(m - i + 1)$ and $2n(m - i) + 2$ (see figure 1(b)), and the postmeasurement state is encoded by the position and coin state within this region. Suppose we define the positions $|2n(m - i + 1) - 2x\rangle = |i\rangle_R \otimes |2x\rangle_I$, where $|i\rangle_R$ denotes the state of region, and $|2x\rangle_I$ denotes inherent positions of each region. Thus, in the POVM device simulated by QW, the state of region $|i\rangle_R$ corresponding to the output port-state $|D_i\rangle$ of POVM device, and the inherent state $|2x\rangle_I$ together with coin state $|c\rangle$ encode the postmeasurement state.

### 4.3. Realization of higher ranker POVM

For higher rank POVMs, the elements can be constructed as a convex combination of orthogonal rank 1 elements [27]. Thus, our algorithm can be modified to realize higher rank POVMs by simply combining orthogonal elements together. For example, a rank $r$ POVM element is a combination of $r$ mutually orthogonal
rank 1 elements \( \sum_{i=1}^{n-1} b_i |\psi_i\rangle \langle \psi_i | \), where \( \langle \psi_l | \psi_l' \rangle = 0 \) for \( l \neq l' \). We can generate rank 1 POVM which includes the elements \( b_i |\psi_i\rangle \langle \psi_i | \) with previous algorithm, and then the elements are correspond to \( r \) positions. The probability to find the walker in those positions corresponds to the desired rank \( r \) POVM elements.

Other than the measurement probabilities, proper postmeasurement states of higher rank POVM are also allowed by our method. After the algorithm, the states corresponding to each rank 1 POVM are mutually orthogonal, so they can be merged into \( 2n \) neighbored positions with the method of state transfer\(^{34}\). Then proper postmeasurement state can be realized via a \( 2n \)-dimensional unitary operation applied to that region. For details of transfer the states to neighbored positions, we refer the interested readers to the appendix, and the method for realizing \( 2n \)-dimensional unitary operation is discussed in section 5.

4.4. Simulation of POVM without one-dimensional U(2) coined discrete-time QW

Our method to generate POVM is based on an algorithm with one-dimensional QW. However, the algorithm can be directly modified to generate POVM with other platforms other than one-dimensional QW. Since the dimension of state in inner region is unchanged throughout the algorithm, steps (a-c) form unitary operation of the \( 2n \)-dimensional system. Step (d) is state-dependent loss of the system. Thus, our algorithm can be easily generalized into high dimensional QWs, with step (a-c) serves as high dimensional coin operations, and step (d) loss part of the system via let it travels to the outside of the original positions. Moreover, our algorithm can also be generalized to platforms other than QW. Such a platform is required to be able to realize the \( 2n \)-dimensional unitary operation corresponding to steps (a-c), and state-dependent loss corresponding to step (d). For example, optical system where the unitary operation can be realized via linear optical elements, and the state-dependent loss can be realized via partially polarization-dependent beam splitter\(^{20}\).

5. Quantum evolution based on QW

Let us consider the realization of evolution \( \tilde{U} \) in equation (4). We first design the POVM \( \mathcal{E} \) corresponding to the evolution, and the \( 2n \) POVM elements \( E_i \) can be generated through the algorithm above. After \( 2n \) iterations, the state of the right region is

\[
|\Phi_{2n} \rangle = \sum_{i=1}^{2n} (\tilde{\psi}_i | \psi_i \rangle | 2n(2n - i) + 2, 0 \rangle.
\]

Thus, the evolution

\[
\tilde{U} = \sum_{i=1}^{2n} | 2n(2n - i) + 2, 0 \rangle \langle \tilde{\psi}_i |,
\]

is successfully realized with the basis \(|i\rangle\) of final state encoded as \(| 2n(2n - i) + 2, 0 \rangle\) of the \( 2n \)-dimensional system.

One may be interested in encoding the basis in \( n \) neighbored positions, which is same as the initial state and feasible with further operations. We note that such a request can be easily satisfied by changing some coin operations in the right region via the method of state transfer\(^{34,35}\). That is to say we transform state \(| 2n(2n - i) + 2, 0 \rangle\) into \(| 2n - 2x, c \rangle\), where \( x = 0, 1, \ldots, n - 1 \), \( c = 0, 1 \) and \( i = 2x + c + 1 \), so that the state of right region becomes

\[
|\Phi_{2n} \rangle = \sum_{x=0}^{n-1} \sum_{c=0,1} \langle \psi_{x+1} | \tilde{\psi} \rangle | 2n - 2x, c \rangle.
\]

Thus, the evolution is successfully realized with the basis of neighbored positions \(| 2n - 2x, c \rangle\) encodes the basis of final state \(| 2x + c + 1 \rangle\). Such process combines the \( 2n \) ports into the one corresponding to element \( E_{2n} \) (see figure 1(c)). For details of how to realize such process, we refer the interested reader to the appendix.

6. Discussion

We demonstrated that one-dimensional discrete-time QW, is capable of realizing arbitrary POVM and evolution for arbitrary dimensional system. The method to realize quantum evolution is based on POVM device but not limited to QW architect, which makes the method compatible with any device that can realize POVM. For more complex quantum maps, e.g. quantum channel, one can decompose the map into Kraus operator, and then via time-sharing\(^{36-39}\) or space-multiplexed\(^{40,41}\) method, the quantum map can be realized via our algorithm by implementing the decomposed evolutions. Thus, our algorithm provides a method to realize an arbitrary quantum map. It would be interesting not only experimentally verifying our results, but also applying
our results to design experiments for various quantum information processes, especially the one with non-unitary evolutions.

Moreover, our method to realize POVM and evolution is not limited to one-dimensional QW. For example, via encoding the measured system into the high dimensional coin state, our algorithm can be directly generalized to high dimensional QW [42], which will highly suppress the required number of QW steps. Furthermore, the architect of recursively implementing unitary operation and state-dependent loss is very similar to optical POVM devices [20].

Our method employs the physical effect of the interference between the probability amplitudes of the quantum walker, and provides a positive answer to the question left in [27], which is whether the POVM based on QW can be generalized to the case that the measured system is more than two-dimensional. Thus, the process provides another generalized example of extending the von Neumann measurement to arbitrary generalized measurement. Considering that QW is a unitary process, the non-unitarity of realized evolution is introduced by focusing on a subspace of the whole system (walker at the right region). By generating non-unitary evolution via controlling the unitary evolution of QW, the process also provides a new model of engineering unitary system into non-unitary ones.

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Appendix A. Preparation of high dimensional state with one-dimensional QW

Here we give the details for the method of preparing one-dimensional QW system into a 2n-dimensional state. Consider the walker initialized as $|\Psi_0^{|1}\rangle = |x_0, 0\rangle$, our method to prepare the desired state includes two procedures. Firstly, applying $n-1$ steps of QW with the coin operations of the $t$-th step are set to be $C|_{(1)}^t$ at position $x_0 + (t - 1)$ and 1 elsewhere, and secondly, performing coin operations $C|_{(2)}^n$ at positions $x_0 + n - 2x - 1$ for $x = 0, 1, \ldots, n - 1$.

We now show the settings of those coin operations, which leads to desired state preparation. We choose the coin operation $C|_{(1)}^1$ as

$$C|_{(1)}^1 = \begin{pmatrix} \cos \theta_t & \sin \theta_t \\ -\sin \theta_t & \cos \theta_t \end{pmatrix},$$

so that the state of walker after first step is

$$|\Psi_1^{|1}\rangle = \langle 0 | C|_{(1)}^1 |0\rangle \langle x_0, 0|\Psi_0^{|0}\rangle |x_0 + 1, 0\rangle + \langle 1 | C|_{(1)}^1 |0\rangle \langle x_0, 0|\Psi_0^{|0}\rangle |x_0 - 1, 1\rangle = \cos \theta_t |x_0 + 1, 0\rangle - \sin \theta_t |x_0 - 1, 1\rangle.$$ (A.2)

Let us assume that the state of walker after $s$ steps, where $2 \leq s \leq n - 1$ is of the form

$$|\Psi_s^{|1}\rangle = \prod_{t'=1}^s \cos \theta_{t'} |x_0 + s, 0\rangle - \sin \theta_{t'} |x_0 - s, 1\rangle + \sum_{t'=2}^s (-\sin \theta_{t'}) \prod_{t''=1}^{t'-1} \cos \theta_{t''} |x_0 - s - 2 + 2t', 1\rangle,$$ (A.3)

which agrees with the state $|\Psi_s^{|2}\rangle$ as a direct consequence of equation (A.2). The state after one more step is

$$|\Psi_{s+1}^{|1}\rangle = \langle 0 | C|_{(1)}^{s+1} |0\rangle \prod_{t'=1}^s \cos \theta_{t'} |x_0 + s + 1, 0\rangle + \langle 1 | C|_{(1)}^{s+1} |0\rangle \prod_{t'=1}^s \cos \theta_{t'} |x_0 + s - 1, 1\rangle$$

$$- \sin \theta_{t+1} |x_0 - s - 1, 1\rangle = \prod_{t'=1}^{s+1} \cos \theta_{t'} |x_0 + s + 1, 0\rangle - \sin \theta_{t+1} \prod_{t'=1}^{s} \cos \theta_{t'} |x_0 + s - 1, 1\rangle$$

$$- \sin \theta_{t+1} |x_0 - s - 1, 1\rangle + \sum_{t'=2}^{s+1} (-\sin \theta_{t'}) \prod_{t''=1}^{t'-1} \cos \theta_{t''} |x_0 - s - 3 + 2t', 1\rangle,$$

$$= \prod_{t'=1}^{s+1} \cos \theta_{t'} |x_0 + s + 1, 0\rangle - \sin \theta_{1} |x_0 - s - 1, 1\rangle$$

$$+ \sum_{t'=2}^{s+1} (-\sin \theta_{t'}) \prod_{t''=1}^{t'-1} \cos \theta_{t''} |x_0 - s - 3 + 2t', 1\rangle,$$
which also agrees with the form assumed in equation (A.3). Thus, the state of QW after the first procedure is

\[
|\Psi_p^{(n-1)}\rangle = \prod_{i=0}^{n-1} \cos \theta_i |x_0 + n - 1, 0\rangle - \sin \theta_i |x_0 - n + 1, 1\rangle + \sum_{i=2}^{n-1} (-\sin \theta_i) \prod_{r=1}^{i-1} \cos \theta_r |x_0 - n + 2r, 1\rangle,
\]

(A.4)

which also satisfies the form as equation (A.3).

The parameters \(\theta_i\) of coin operations are chosen as

\[
\sin^2 \theta_i = \begin{cases} \frac{|(2n - 1)\psi|^2 + |(2n|\psi|)^2|}{\prod_{r=1}^{i-1} \cos^2 \theta_r}, & t = 1 \\ \frac{|(2n - 2t + 1)\psi|^2 + |(2n - 2t + 2|\psi|)^2|}{\prod_{r=1}^{i-1} \cos^2 \theta_r}, & t \geq 2 \end{cases}
\]

(A.5)

so that the amplitudes of \(|\Psi_p^{(n-1)}\rangle\) are

\[
\langle x_0 - n + 1 + 2t, 1|\Psi_p^{(n-1)}\rangle^2 = |(2n - 2t + 1|\psi|)^2| + |(2n - 2t + 2|\psi|)^2|,
\]

(A.6)

for \(x = 1, 2, \ldots, n - 1\). Then the amplitude

\[
\langle x_0 - n + 1, 0|\Psi_p^{(n-1)}\rangle^2 = |\langle 1|\psi\rangle|^2 + |\langle 2|\psi\rangle|^2,
\]

(A.7)

due to normalization. This can be shown by calculating the \(\theta_i\) from equation (A.5) starting from \(t = 1\) to \(t = n - 1\).

Next, we consider the second procedure with applying coin operations \(C^x_{(i)}\) at positions \(x_0 + n - 1 - 2x\) for \(x = 0, 1, \ldots, n - 1\). For simplicity, we rewrite the state \(|\Phi_p^{(n-1)}\rangle\) as

\[
|\Phi_p^{(n-1)}\rangle = \alpha |x_0 + n - 1, 0\rangle + \sum_{x=1}^{n-1} \beta_x |x_0 + n - 1 - 2x, 1\rangle,
\]

(A.8)

with \(x = n - t\), where \(\alpha\) and \(\beta_x\) denote corresponding amplitudes. The state of QW after the second procedure is

\[
|\Psi_p^{(n-1)}\rangle = \sum_{c=0,1} \alpha |c| C^0_{(2)}|0\rangle |x_0 + n - 1, c\rangle + \sum_{x=1}^{n-1} \sum_{c=0,1} \beta_x |c| C^x_{(2)}|1\rangle |x_0 + n - 1 - 2x, c\rangle.
\]

(A.9)

After choosing coin operation \(C^0_{(2)}\) as

\[
\langle c| C^0_{(2)}|0\rangle = \langle c + 1|\psi\rangle / \alpha,
\]

(A.10)

the amplitudes of \(|\Psi_p^{(n-1)}\rangle\) in position \(x_0 + n - 1\) are

\[
\langle x_0 + n - 1, c|\Psi_p^{(n-1)}\rangle = \langle c + 1|\psi\rangle,
\]

(A.11)

for \(c=0,1\). An example of such coin operation is

\[
C^0_{(2)} = \begin{pmatrix} \langle \psi|2\rangle / \alpha^* & \langle 1|\psi\rangle / \alpha \\ -\langle \psi|1\rangle / \alpha^* & \langle 2|\psi\rangle / \alpha \end{pmatrix}
\]

(A.12)

The other coin operations \(C^x_{(2)}\) for \(x = 1, 2, \ldots, n - 1\) are chosen as

\[
\langle c| C^x_{(2)}|1\rangle = \langle 2x + c + 1|\psi\rangle / \beta_x,
\]

(A.13)

and then the amplitudes

\[
\langle x_0 + n - 1 - 2x, c|\Psi_p^{(n-1)}\rangle = \langle 2x + c + 1|\psi\rangle.
\]

(A.14)

An example of such coin operation is

\[
C^x_{(2)} = \begin{pmatrix} \langle \psi|2x + 2\rangle / \beta_x^* & \langle 2x + 1|\psi\rangle / \beta_x \\ -\langle \psi|2x + 1\rangle / \beta_x^* & \langle 2x + 2|\psi\rangle / \beta_x \end{pmatrix}
\]

(A.15)

for \(x = 1, 2, \ldots, n - 1\). Thus, we finally get state

\[
|\Psi_p^{(n-1)}\rangle = \sum_{x=0}^{n-1} \sum_{c=0,1} \langle 2x + c + 1|\psi\rangle |x_0 + n - 1 - 2i, c\rangle,
\]

(A.16)

which is QW state encoding state \(|\psi\rangle\) with the basis \(|2x + c + 1\rangle\) encoded in \(|x_0 + n - 1 - 2i, c\rangle\) of QW system. If one choose \(x_0 = -n + 1\), the prepared QW walk state \(|\Psi_p^{(n-1)}\rangle\) is exactly the state \(|\Psi_0\rangle\) in the main text.
Appendix B. Quantum evolution with basis in neighbored positions

In the main text, we have demonstrated that after 2\(n\) iterations of the algorithm to generate POVM, one can realize evolution with the new basis \([2x + c + 1]\) of final state, for \(x = 0, \ldots, n - 1\) and \(c = 0, 1\), encoded in the QW basis \([2n(n - i) + 2, 0]\). We now show how to use the method of state transfer to encode the basis in neighbored positions of QW.

The whole process generating such POVM contains \((2n)^2\) steps of QW. We denote the state of the right region after \(s\)-th step as \(|\Phi_i^s\rangle\). The state is initially zero, \(|\Phi_i^0\rangle = 0\) and new parts \(|\psi_i\rangle\) \([2, 0]\) added in after \((2n)^2\) steps for \(i = 1, \ldots, 2n\).

The process is as following: Firstly set and keep the coin operation at position \(2n + 1\) as \(\sigma_x\). Thus the state is blocked within the positions not larger than \(2n + 1\). Then, we set and keep the coin operation at position \(2n + 1 + i\) as \(\sigma_x\), since step \(2n^2(i + 1) + 2 - i\) for \(i = 1, \ldots, 2n\). Then the state \(|\psi_k\rangle\) \([2, 0]\) after step \(2n^2k\) will propagate to position \((2n + 1 - k) = (2n + 1 - k, 1)\) after steps \(2n^2(k + 1) + 1 - k\), where the coin state is flipped by the coin operation \(\sigma_x\) at step \(2n^2(k + 1) - k\) kept from step \(2n^2k + 2 - k\). After setting the coin operation at position \(2n + 1 + k\) as \(\sigma_x\), step \(2n^2(k + 1) + 2 - i\) changes the state into \(|\psi_k\rangle\) \([2n + 2 - k, 0]\). Since then, the coin operations at position \(2n + 2 - k\) and \(2n + 1 - k\) are both kept as \(\sigma_x\). Thus, the state oscillates between states \(|\psi_k\rangle\) \([2n + 1 - k, 1]\) and \(|\psi_k\rangle\) \([2n + 2 - k, 0]\). Thus, the final state is

\[
|\Phi_i^{(2n)}\rangle = \sum_{i=1}^{2n} \left( |\psi_i\rangle |2n + 2 - 2i, 1\rangle + |\psi_{2n-i}\rangle |2n + 2 - 2i, 0\rangle \right)
= \sum_{x=0}^{n-1} \sum_{c=0,1} \left( |\psi_{2n+x+c}\rangle |2n - 2x, 1 - c\rangle \right),
\]

where \(x = i - 1\).

Finally, after coin operations \(\sigma_x\) at all the positions, we have

\[
|\Phi_i^{(2n)}\rangle = \sum_{x=0}^{n-1} \sum_{c=0,1} \left( |\psi_{2n+x+c}\rangle |2n - 2x, c\rangle \right)
\]

which corresponds to the final state of desired evolution, with the basis \([2x + c + 1]\) encoded as \([2n - 2x, c]\), which are distributed in neighbored positions \(2n - 2x\). Further evolution and measurement can be applied to the state by directly using our method in the main text.

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