Fundamental solutions of the Dirac operator in the Friedmann-Lemaître-Robertson-Walker spacetime

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Abstract

The equation of the spin-\(\frac{1}{2}\) particles in the Friedmann-Lemaître-Robertson-Walker spacetime is investigated. The retarded and advanced fundamental solutions to the Dirac operator and generalized Dirac operator as well as the fundamental solutions to the Cauchy problem are written in explicit form via the fundamental solution of the wave equation in the Minkowski spacetime.

0 Introduction

In this article we derive fundamental solutions of the Dirac operator in the curved spacetime of the Friedmann-Lemaître-Robertson-Walker (FLRW) models of cosmology. More precisely, we derive in explicit form the retarded and advanced fundamental solutions to the Dirac operator as well as the fundamental solutions to the Cauchy problem via the fundamental solution of the wave equation in the Minkowski spacetime.

The metric tensor in the spatially flat FLRW spacetime is

\[
(g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a^2(t) & 0 & 0 \\
0 & 0 & -a^2(t) & 0 \\
0 & 0 & 0 & -a^2(t)
\end{pmatrix}, \quad \mu, \nu = 0, 1, 2, 3.
\]

We will focus on the de Sitter space with the scale factor \(a(t) = e^{Ht}\) (see, e.g., [18]) that is modeling the expanding or contracting universe if \(H > 0\) or \(H < 0\), respectively. The curvature of this space is \(-12H^2\).

The Dirac equation in the de Sitter space is (see, e.g., [2])

\[
(i\gamma^0 \partial_0 + ie^{-Ht}\gamma^1 \partial_1 + ie^{-Ht}\gamma^2 \partial_2 + ie^{-Ht}\gamma^3 \partial_3 + i\frac{3}{2}H\gamma^0 - m_4)\psi = f,
\]

where the contravariant gamma matrices are (see, e.g., [5, p. 61])

\[
\gamma^0 = \begin{pmatrix}
\mathbb{I}_2 & \mathbb{O}_2 \\
\mathbb{O}_2 & -\mathbb{I}_2
\end{pmatrix}, \quad \gamma^k = \begin{pmatrix}
\mathbb{O}_2 & \sigma^k \\
-\sigma^k & \mathbb{O}_2
\end{pmatrix}, \quad k = 1, 2, 3.
\]

Here \(\sigma^k\) are Pauli matrices

\[
\sigma^1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma^3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

and \(\mathbb{I}_n, \mathbb{O}_n\) denote the \(n \times n\) identity and zero matrices, respectively. In this article we present in explicit forms the fundamental solutions of the Dirac operator in the de Sitter spacetime.
The general approach to the study of the Dirac operator in the curved spacetime can be described as follows. (See, e.g., [19, Sec.5.6].) Denote the Lorenzian metric tensor

$$\langle \eta_{\mu\nu} \rangle = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}$$

and by $\gamma^\mu(x)$ the matrices, which are defined by

$$\gamma^\mu(x) \gamma^\nu(x) + \gamma^\nu(x) \gamma^\mu(x) = 2 g^{\mu\nu}(x),$$

while $\gamma^\mu(x) = g_{\mu\nu}(x) \gamma^\nu(x)$. Here and henceforth, Einstein summation convention over repeated indexes is employed. The covariant derivative of a spinor field $\psi$ is

$$\nabla_\mu \psi \equiv (\partial_\mu - \Gamma_\mu) \psi,$$

where the spinorial affine connections $\Gamma_\mu(x)$ are matrices, which are defined by the annihilating of the covariant derivative of the $\gamma$-matrices,

$$\nabla_\mu \gamma_\nu \equiv \partial_\mu \gamma_\nu - \Gamma^\lambda_{\mu\nu} \gamma_\lambda - \Gamma^\lambda_{\nu\mu} \gamma_\lambda + \gamma_\mu \Gamma_\nu = 0,$$

and $\Gamma^\lambda_{\mu\nu}$ are affine connections determined by the metric $g$. Then the covariant Dirac equation in the curved spacetime reads

$$(i \gamma^\mu(\nabla_\mu - m) \psi(x) = 0.$$  

The following identity

$$\gamma^\mu \nabla_\mu \left( \sqrt{|g|} g^\mu\nu \nabla_\nu \psi \right) = \frac{1}{\sqrt{|g|}} \nabla_\mu \left( \sqrt{|g|} g^\mu\nu \nabla_\nu \psi \right) - \frac{1}{4} R \psi,$$  

where $|g| = |\det (g_{\mu\nu})|$, establishes a relation between the Dirac operator, spinorial D’Alembert operator and the scalar curvature $R$. (See, e.g., [23, Eq. (74)], [19, Sec. 3.9, 5.6].) Hence, if $E_{KG}(x, x')$ is a fundamental solution to the spinorial Klein-Gordon operator,

$$(\frac{1}{\sqrt{|g|}} \nabla_\mu \left( \sqrt{|g|} g^\mu\nu \nabla_\nu \psi \right) - \frac{1}{4} R + m^2) E_{KG} (x, x') = \delta(x, x') I_4,$$  

then

$$E(x, x') := - \left( i \gamma^\mu \nabla_\mu + m \right) E_{KG}(x, x')(x, x')$$

is the fundamental solution to the Dirac operator:

$$(i \gamma^\mu \nabla_\mu - m) E (x, x') = \delta(x, x') I_4.$$  

In this paper we modify the factorization (0.2). For the first factor we put the operator of (0.1) and then choose an appropriate second factor that leads to the diagonal $4 \times 4$ operator matrix containing scalar Klein-Gordon operators with the modified complex-valued mass terms. Thus, in order to construct fundamental solution to the Dirac operator the only things that remain are to find an explicit form for the fundamental solution to the Klein-Gordon operator in the curved spacetime and an explicit form for the spin connection. Obviously, the Klein-Gordon operator of (0.3) has variable coefficients with the spinorial structure that makes difficult finding of the explicit representation of the fundamental solutions. The explicit form of the fundamental solutions to the scalar Klein-Gordon operator in the curved spacetime is an interesting and difficult problem in its own right. For some FLRW models such fundamental solutions were recently written via special integral transform in terms of the solution to the scalar wave equation in the Minkowski spacetime. For more details we refer the reader to [29, 30].

At the same time a straightforward approach to the Dirac operator based on the separation of variables that was refraining from an explicit factorization led to some important sets of the exact solutions to the Dirac equation in FLRW spaces. (See, e.g., [2, 7, 10, 14, 26].) In particular, in [7] are investigated the hydrogen atom, the Dirac-Morse oscillator, and the Dirac particle in a curved spacetime with the metric

$$ds^2 = e^{2f(r)} dt^2 - e^{2g(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2.$$
In fact, the function $u$ cones are defined as follows:

Scalar field with the mass $m$ of the Klein-Gordon equation in the de Sitter spacetime from [29, 30]. The Klein-Gordon equation for the

in different direction of time.

(E.1) is a matrix operator de Sitter metric. Recall that a retarded fundamental solution (a retarded inverse) for the Dirac operator

paper to the Huygens’ principle will be given in the forthcoming paper.

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this mechanism appeals to some additional “time” variable.

As it is shown in Section 2 this mechanism exists even in the case of the vanishing cosmological constant

that from the massless field in the Minkowski spacetime generates massive particle in the curved spacetime.

hypergeometric function in the kernel. One can regard the integral transform as an analytical mechanism

explicit formulas for all solutions and the fundamental solutions were remaining open. Our paper fills that

questions of physics (see, e.g., [2, 6, 10, 14, 16] and bibliography therein) in the de Sitter spacetime, the

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of the comoving frame.

The construction of a quantum field theory in curved spacetimes and the definition of a quantum vacuum

demand a detailed investigation of the solutions of relativistic equations in curved backgrounds. (See, e.g., [4].) The explicit formulas for all solutions and, in particular, for the fundamental solutions of those equations may contribute in the resolving of that challenging problem.

Even though nowadays, numerical solutions of differential equations are available, in some cases a deep understanding of properties of the solutions is possible only by the examination of the explicit formulas when they are known. This is the case with the Huygens’ principle. Some known results on the Huygens’ principle for the Dirac equation one can find in [8, 9, 13, 17, 20, 25, 27]. The application of the results of the present paper to the Huygens’ principle will be given in the forthcoming paper.

We start with the the fundamental solution to the Klein-Gordon operator in the FLRW model with the de Sitter metric. Recall that a retarded fundamental solution (a retarded inverse) for the Dirac operator (0.1) is a matrix operator $E_{\text{ret}} = E_{\text{ret}}(x, t; x_0, t_0; m)$ that solves the equation

$$
\left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^t \partial_t + i\frac{3}{2}H\gamma^0 - mI_4 \right) E(x, t; x_0, t_0; m) = \delta(x - x_0, t - t_0)I_4,
$$

(x, t, x_0, t_0) \in \mathbb{R}^8, \quad (0.4)

and with the support in the chronological future (“forward light cone”) $D_+(x_0, t_0)$ of the point $(x_0, t_0) \in \mathbb{R}^4$. The advanced fundamental solution (propagator) $E_{\text{adv}} = E_{\text{adv}}(x, t; x_0, t_0; m)$ solves the equation (0.4) and has a support in the chronological past (“backward light cone”) $D_-(x_0, t_0)$. The forward and backward light cones are defined as follows:

$$
D_\pm(x_0, t_0) := \{(x, t) \in \mathbb{R}^{3+1}; |x - x_0| \leq \pm(\phi(t) - \phi(t_0))\}
$$

where $\phi(t) := (1 - e^{-Ht})/H$ is a distance function. In fact, any intersection of $D_-(x_0, t_0)$ with the hyperplane $t = \text{const} < t_0$ determines the so-called dependence domain for the point $(x_0, t_0)$, while the intersection of $D_+(x_0, t_0)$ with the hyperplane $t = \text{const} > t_0$ is the so-called domain of influence of the point $(x_0, t_0)$. The Dirac equation (0.1) is non-invariant with respect to time inversion and its solutions have different properties in different direction of time.

For the construction of the fundamental solutions of the Dirac equation we use the fundamental solutions of the Klein-Gordon equation in the de Sitter spacetime from [29, 30]. The Klein-Gordon equation for the scalar field with the mass $m$ in the de Sitter universe in the physical variables is:

$$
\frac{1}{c^2} \psi_{tt} + \frac{1}{c^2} 3H\psi_t - e^{-2Ht} \Delta \psi + \frac{c^2 m^2}{h^2} \psi = 0.
$$

In fact, the function $u = e^{\frac{3}{2}Ht}\psi$ solves the Klein-Gordon non-covariant equation

$$
\frac{1}{c^2} u_{tt} - e^{-2Ht} \Delta u - \left( \frac{9H^2}{4c^2} - \frac{c^2 m^2}{h^2} \right) u = 0.
$$

3
We remind that (see, e.g., [24]) if \(n\) Minkowski spacetime in the backward cone \(D\) equation in the de Sitter spacetime. For \((x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}, M \in \mathbb{C}\), we define the function

\[
E(x, t; x_0, t_0; M) := 4^{-\frac{n}{2}} e^{M(t_0 + t)} \left( (e^{-Ht_0} + e^{-Ht})^2 - (x-x_0)^2 \right)^{\frac{n}{2} - \frac{1}{2}} \times F \left( \frac{1}{2}, -\frac{M}{H}, \frac{1}{2}, -\frac{M}{H}; 1; \frac{(-e^{-Ht} + e^{-Ht_0})^2 - (x-x_0)^2}{(e^{-Ht} + e^{-Ht_0})^2 - (x-x_0)^2} \right),
\]

where \((x, t) \in D_+(x_0, t_0) \cup D_-(x_0, t_0)\) and \(F(a, b; c; \zeta)\) is the hypergeometric function (see, e.g., [3]). When no ambiguity arises, we use the notation \(x^2 := |x|^2\) for \(x \in \mathbb{R}^n\). Thus, the function \(E\) depends on \(r^2 = (x-x_0)^2/H^2\), and we will write \(E(r, t; 0, t_0; M)\) for \(E(x, t; x_0, t_0; M)\):

\[
E(r, t; 0, t_0; M) := 4^{-\frac{n}{2}} e^{M(t_0 + t)} \left( (e^{-Ht_0} + e^{-Ht})^2 - (Ht)^2 \right)^{\frac{n}{2} - \frac{1}{2}} \times F \left( \frac{1}{2}, -\frac{M}{H}, \frac{1}{2}, -\frac{M}{H}; 1; \frac{(-e^{-Ht} + e^{-Ht_0})^2 - (rH)^2}{(e^{-Ht} + e^{-Ht_0})^2 - (rH)^2} \right). \tag{0.5}
\]

Let \(\Delta\) be the Laplace operator in \(\mathbb{R}^n\). For the Klein-Gordon non-covariant operator in the de Sitter spacetime

\[
S_{KG} = \partial_t^2 - e^{-2Ht} \Delta - M^2
\]

we define two fundamental solutions \(\mathcal{E}_{\pm, KG}(x, t; x_0, t_0; M) = (\mathcal{E}_{\pm, KG}(x-x_0, t; 0, t_0; M))\) as the distributions \(\mathcal{E}_{\pm, KG} \in \mathcal{D}'(\mathbb{R}^{2n+2})\) with supports in the cones \(D_\pm(x_0, t_0), x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}, \supp \mathcal{E}_{\pm, KG} \subseteq D_\pm(x_0, t_0)\), by

\[
\left(\partial_t^2 - e^{-2Ht} \Delta - M^2\right) \mathcal{E}_{\pm, KG}(x, t; x_0, t_0; M) = \delta(t-t_0)\delta(x-x_0).
\]

Since all formulas for the contracting universe are evident modifications of ones for the expanding universe, in order to avoid unnecessary complications in the formulas, henceforth we restrict ourselves to the case of \(H > 0\). According to [29, 30], if \(x \in \mathbb{R}^n\) and \(M \in \mathbb{C}\), then for the operator \(S_{KG}\) (0.6) the retarded fundamental solution (retarded propagator) \(\mathcal{E}_{+, KG}(x, t; x_0, t_0; M) = (\mathcal{E}_{+, KG}(x-x_0, t; 0, t_0; M))\) with support in the forward cone \(D_+(x_0, t_0), x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}, \supp \mathcal{E}_{+, KG} \subseteq D_+(x_0, t_0)\), is given by the following integral

\[
\mathcal{E}_{+, KG}(x, t; x_0, t_0; M) = 2 \int_0^{\phi(t)-\phi(t_0)} E(r, t; 0, t_0; M)E^w(x-x_0, r) dr, \quad t > t_0.
\]

Here the distribution \(E^w(x, t)\) is a fundamental solution to the Cauchy problem for the wave equation in the Minkowski spacetime

\[
E^w_t - \Delta E^w = 0, \quad E^w(x, 0) = \delta(x), \quad E^w_t(x, 0) = 0.
\]

The fundamental solution (advanced propagator) \(\mathcal{E}_{-, KG}(x, t; x_0, t_0) = (\mathcal{E}_{-, KG}(x-x_0, t; 0, t_0))\) with support in the backward cone \(D_-(x_0, t_0), x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}, \supp \mathcal{E}_{-, KG} \subseteq D_-(x_0, t_0)\), is given by the following integral

\[
\mathcal{E}_{-, KG}(x, t; x_0, t_0; M) = -2 \int_0^{\phi(t)-\phi(t_0)} E(r, t; 0, t_0; M)E^w(x-x_0, r) dr, \quad t < t_0.
\]

We remind that (see, e.g., [24]) if \(n\) is odd, then

\[
\mathcal{E}^w(x, t) := \frac{1}{\omega_{n-1} \cdot 3 \cdot 5 \cdots (n-2)} \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{1}{t} \delta(|x|-t),
\]

while for the even \(n\) we have

\[
\mathcal{E}^w(x, t) := \frac{2}{\omega_{n-1} \cdot 3 \cdot 5 \cdots (n-1)} \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \frac{1}{\sqrt{t^2 - |x|^2}} \chi_{B_t}(x).
\]
Here \( \chi_{B_t(x)} \) denotes the characteristic function of the ball \( B_t(x) := \{ x \in \mathbb{R}^n ; |x| \leq t \} \). The constant \( \omega_{n-1} \) is the area of the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). The distribution \( \delta(|x| - t) \) is defined by

\[
\langle \delta(|x| - t), \psi(\cdot) \rangle = \int_{|x|=t} \psi(x) \, dx \quad \text{for} \quad \psi \in C_0^\infty(\mathbb{R}^n).
\]

Denote

\[
M_+ = \frac{1}{2} H + im, \quad M_- = \frac{1}{2} H - im.
\]

Let \( \mathcal{E}_{ret,KG} \) be a matrix with the operator-valued entries \( (\mathcal{E}_{ret,KG}(x,t;x_0,t_0;0))_{ij} \) defined as follows

\[
\mathcal{E}_{ret,KG}(x,t;x_0,t_0;m) := \left( \begin{array}{cc}
\mathcal{E}_{+,KG}(x,t;x_0,t_0;M_+) \mathbb{I}_2 & \mathcal{E}_{0,KG}(x,t;x_0,t_0;M_-) \mathbb{I}_2 \\
\mathcal{E}_{-,KG}(x,t;x_0,t_0;M_+) & \mathcal{E}_{+,KG}(x,t;x_0,t_0;M_-) \mathbb{I}_2
\end{array} \right).
\]

Let \( e^{Ht} \) be the operator multiplication by \( e^{Ht} \). The next theorem gives in the explicit form fundamental solutions of the Dirac operator in the de Sitter spacetime via the fundamental solutions of the wave operator in the Minkowski spacetime.

**Theorem 0.1** The fundamental solution \( \mathcal{E}^{ret} = \mathcal{E}^{ret}(x,t;x_0,t_0;0) \) of the Dirac operator in the de Sitter spacetime

\[
\mathcal{D} = i \gamma^0 \partial_0 + ie^{-Ht} \gamma^1 \partial_1 + ie^{-Ht} \gamma^2 \partial_2 + ie^{-Ht} \gamma^3 \partial_3 + \frac{3}{2} H \gamma^0 - m \mathbb{I}_4,
\]

\( m \in \mathbb{C} \), is given by the following formula

\[
\mathcal{E}^{ret}(x,t;x_0,t_0;0) = -e^{-Ht} \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k \partial_k - i \frac{H}{2} \gamma^0 + m \mathbb{I}_4 \right) \mathcal{E}_{ret,KG}(x,t;x_0,t_0;0) [e^{Ht}].
\]

The similar representation holds for the advanced propagator. The next theorem gives the representation formulas for the solutions of the Cauchy problem. We introduce the operator \( \mathcal{G}(x,t,D_x;M) \) by

\[
\mathcal{G}(x,t,D_x;M)[f] = 2 \int_0^t \int_0^b E(r,t;0,b;M) \int_{\mathbb{R}^n} \mathcal{E}^{w}(x-y,r)f(y,b) \, dy \, dr, \quad f \in C_0^\infty(\mathbb{R}^{n+1}).
\]

Next, we define the kernel function

\[
K_1(r,t;M) := E(r,t;0,0;M)
\]

and the operator \( \mathcal{K}_1(x,t,D_x;M) \) as follows:

\[
\mathcal{K}_1(x,t,D_x;M) \varphi(x) = 2 \int_0^{\phi(t)} K_1(s,t;M) \int_{\mathbb{R}^n} \mathcal{E}^{w}(x-y,s) \varphi(y) \, dy \, ds, \quad \varphi \in C_0^\infty(\mathbb{R}^n).
\]

**Theorem 0.2** A solution to the Cauchy problem

\[
\begin{cases}
(i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k \partial_k + \frac{3}{2} H \gamma^0 - m \mathbb{I}_4) \Psi(x,t) = F(x,t), \\
\Psi(x,0) = \Phi(x),
\end{cases}
\]

\( m \in \mathbb{C} \), is given by the following formula

\[
\Psi(x,t) = -e^{-Ht} \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k \partial_k + i \frac{H}{2} \gamma^0 + m \mathbb{I}_4 \right) \times \left( \begin{array}{ccc}
\mathcal{G}(x,t,D_x;M_+) \mathbb{I}_2 & 0 & 0 \\
0 & \mathcal{G}(x,t,D_x;M_-) \mathbb{I}_2 & 0 \\
0 & 0 & \mathcal{K}_1(x,t,D_x;M_+) \mathbb{I}_2
\end{array} \right) [e^{Ht} F] + i \gamma^0 \left( \begin{array}{ccc}
K_1(x,t,D_x;M_+)(x,t) & 0 & 0 \\
0 & \mathcal{K}_1(x,t,D_x;M_-) \mathbb{I}_2 & 0 \\
0 & 0 & \mathcal{K}_1(x,t,D_x;M_+) \mathbb{I}_2
\end{array} \right) [\Phi].
\]
Corollary 0.3 In particular, for the massless particle (e.g., neutrino \[5, 6, 14\]) \(M_+ = M_- = \frac{1}{2}H\) the last formula simplifies to the following one

\[
\Psi(x, t) = -e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht} \gamma^k \partial_k - i\frac{H}{2} \gamma^0 \right) 
\times e^{\frac{H}{2}t} \left[ \int_0^t \left( \int_0^t \int_{\mathbb{R}^n} \mathcal{E}^u(x - y, r)F(y, b) \, dy \, dr \right) e^{\frac{2H}{b}} db 
\right.

\left. + i\gamma^0 \int_0^t \int_{\mathbb{R}^n} \mathcal{E}^u(x - y, r) \Phi(y) \, dy \, dr \right].
\]

In fact, for the massless field the kernel functions are simplified to

\[
E \left( r, t; 0, t_0; \frac{H}{2} \right) = \frac{1}{2} e^{\frac{H}{2}(t_0 + t)}, \quad K_1 \left( r, t; \frac{H}{2} \right) = \frac{1}{2} e^{\frac{H}{2}t}
\]

and that proves the corollary.

The paper is organized as follows. In Section 1 we introduce the formulas for the solution of the generalized Klein-Gordon equation in the de Sitter and Minkowski spacetimes. In Section 3 we introduce the generalized Dirac operator and show how the generalized Klein-Gordon operator in the de Sitter and Minkowski spacetimes can be factorized into a product of two first-order matrix coefficients operators with the generalized Dirac operator in the same spacetimes as the first factor. Examples of such factorizations are also provided in that section. Section 4 is devoted to the completion of the proof of Theorem 0.1.

## 1 Representation of the Solution of the Dirac equation in de Sitter spacetime

Additional to (0.5) for \(M \in \mathbb{C}\) we recall two more kernel functions from [29, 30]

\[
K_0(r, t; M) := -\left[ \frac{\partial}{\partial b} E(r, t; 0, b; M) \right]_{b=0}, \quad (1.7)
\]

\[
K_1(r, t; M) := E(r, t; 0, 0; M). \quad (1.8)
\]

Then according to [30] the solution operator for the Cauchy problem for the scalar generalized Klein-Gordon equation in the de Sitter spacetime

\[
\left( \partial_0^2 - e^{-2Ht} \mathcal{A}(x, \partial_x) - M^2 \right) \psi = f, \quad \psi(x, 0) = \varphi_0(x), \quad \psi_t(x, 0) = \varphi_0(x), \quad (1.9)
\]

is given as follows

\[
\psi(x, t) = \mathcal{G}(x, t, D_x; M)[f] + \mathcal{K}_0(x, t, D_x; M)[\varphi_0] + \mathcal{K}_1(x, t, D_x; M)[\varphi_1].
\]

Here \(\mathcal{A}(x, \partial_x)\) is the differential operator \(\mathcal{A}(x, \partial_x) = \sum_{|\alpha| \leq m} a_\alpha(x)D_x^\alpha\) and the coefficients \(a_\alpha(x)\) are \(C^\infty\)-functions in the open domain \(\Omega \subseteq \mathbb{R}^n\), that is \(a_\alpha \in C^\infty(\Omega)\). The kernels \(K_0(z, t; M)\) and \(K_1(z, t; M)\) can be
written in the explicit form as follows

\[ K_0(r, t; M) = -4^{\frac{H}{2}} \left( 1 + e^{-Ht} \right)^2 - H^2 r^2 \left( 1 + e^{-Ht} \right)^2 4 \left( 1 - e^{Ht} \right)^2 - H^2 r^2 \]

\[ \times \left\{ e^{-2Ht} \left( (1 + e^{-Ht})^2 - H^2 r^2 \right) \left( (1 - e^{-Ht})^2 - H^2 r^2 \right) \right\} \]

\[ \times F \left( \frac{1}{2} - \frac{M}{H}, \frac{1}{2} - \frac{M}{H}; 1; \frac{1 - e^{-Ht}}{1 - e^{-Ht}} - H^2 r^2 \right) \]

\[ + \frac{1}{H} (H - 2M)^2 e^{2Ht} \left( (1 - e^{-Ht})^2 - H^2 r^2 \right) \]

\[ \times F \left( \frac{3}{2} - \frac{M}{H}, \frac{3}{2} - \frac{M}{H}; 2; \frac{1 - e^{-Ht}}{1 - e^{-Ht}} - H^2 r^2 \right) \right\}, \]

\[ K_1(r, t; M) = 4^{\frac{H}{2}} e^{Mt} \left( (1 + e^{-Ht})^2 - (Hr)^2 \right)^{\frac{H}{2} - \frac{1}{2}} \]

\[ \times F \left( \frac{1}{2} - \frac{M}{H}, \frac{1}{2} - \frac{M}{H}; 1; \frac{1 - e^{-Ht}}{1 - e^{-Ht}} - (Hr)^2 \right). \]

To describe the operators $G, K_1$ we recall the results of Theorem 1.1 [30]. For $f \in C(\Omega \times I), I = \{0, T\}$, $0 < T \leq \infty$, and $\varphi_0, \varphi_1 \in C(\Omega)$, let the function $v_f(x, t; b) \in C^{1+2,0}_{x,t} (\Omega \times [0, (1 - e^{-HT})/H] \times I)$ be a solution to the problem

\[ \begin{cases} v_{tt} - A(x, \partial_x) v = 0, & x \in \Omega, \quad t \in [0, (1 - e^{-HT})/H], \\ v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0, & b \in I, \quad x \in \Omega, \end{cases} \]

(1.10)

and the function $v_\omega(x, t) \in C^{1+2,0}_{x,t} (\Omega \times [0, (1 - e^{-HT})/H])$ be a solution of the problem

\[ \begin{cases} v_{tt} - A(x, \partial_x) v = 0, & x \in \Omega, \quad t \in [0, (1 - e^{-HT})/H], \\ v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0, & x \in \Omega. \end{cases} \]

(1.11)

Then the function $u = u(x, t)$ defined by

\[ u(x, t) = 2 \int_0^t db \int_0^{\phi(t)} E(r, t; 0, b; M) v_f(x, r; b) dr + e^{\frac{H}{2}} v_{\varphi_0}(x, \phi(t)) \]

\[ + \left| 2 \int_0^{\phi(t)} K_0(s, t; M) v_{\varphi_0}(x, s) ds + 2 \int_0^{\phi(t)} v_{\varphi_1}(x, s) K_1(s, t; M) ds \right|, \]

$x \in \Omega, t \in I$,

where $\phi(t) := (1 - e^{-HT})/H$, solves the problem

\[ \begin{cases} u_{tt} - A(x, \partial_x) u - M^2 u = f, & x \in \Omega, \quad t \in I, \\ u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x), & x \in \Omega. \end{cases} \]

Remark 1.1 We stress here that the existence of the solutions in the problems (1.10) and (1.11) is assumed. For the case of second-order elliptic operator $A(x, \partial_x)$ in order to guarantee existence of the solutions in the problems (1.10) and (1.11) the initial data must be given in the domain $\{x \in \mathbb{R}^n | \text{dist}(x, \Omega) \leq c\}$, where $c > 0$ due to the propagation phenomena.

Moreover, if $E^u_\omega(x, t; x_0)$ is a fundamental solution of the Cauchy problem for the massless equation (1.11) in the spacetime without expansion, that is,

\[ \begin{cases} \left( \partial^2 - A(x, \partial_x) \right) E^u_\omega(x, t; x_0) = 0, & x \in \mathbb{R}^n, \quad t \in I, \\ E^u_\omega(x, 0; x_0) = \delta(x - x_0), \quad \partial_t E^u_\omega(x, 0; x_0) = 0, & x \in \mathbb{R}^n, \end{cases} \]
then, the fundamental solutions $E_{\pm,KG,A}$ of the operator $\partial_t^2 - e^{-2HT}A(x,\partial_x) - M^2$ can be written as follows

$$E_{\pm,KG,A}(x,t;x_0,t_0;M) = 2 \int_0^\phi(t)_{(t_0)} E(r,t;0,t_0;M)E_{\pm}^+(x,r;x_0) dr,$$

when $\pm (t-t_0) > 0$

on their supports.

We mention here two important examples. The first one has $A(x,\partial_x) = \Delta$ and it is related to the problem written in the Cartesian coordinates. The second one has the equation written in the spherical coordinates $(r,\theta,\phi)$. In the FLRW spacetime with the line element

$$ds^2 = dt^2 - e^{2HT}\left(\frac{1}{1-Kr^2}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right)$$

the Klein-Gordon equation is

$$\partial_t^2 \psi + 3H\partial_r \psi - e^{-2HT}A(x,\partial_x)\psi + m^2 \psi = 0 ,$$

where

$$A(x,\partial_x) := \frac{\sqrt{1-Kr^2}}{r^2} \frac{\partial}{\partial r} \left( r^2 \sqrt{1-Kr^2} \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial \psi}{\partial \phi} \right)$$

is the Laplace-Beltrami operator in the spatial variables, where $K = -1,0$, or $+1$, for a hyperbolic, flat, or spherical spatial geometry, respectively. In the Cartesian coordinates in the metric (1.12) the covariant Klein-Gordon equation reads (1.13), where (see [12, Example 2])

$$A(x,\partial_x) = (1 - Kx_1^2)\partial_{x_1}^2 + (1 - Kx_2^2)\partial_{x_2}^2 + (1 - Kx_3^2)\partial_{x_3}^2 - 2Kx_1x_2\partial_{x_1}\partial_{x_2} - 2Kx_1x_3\partial_{x_1}\partial_{x_3} - 2Kx_2x_3\partial_{x_2}\partial_{x_3} - 3Kx_1\partial_{x_1} - 3Kx_2\partial_{x_2} - 3Kx_3\partial_{x_3}$$

and $x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$ is the radial vector field.

We note here that the transition from the Laplace operator in $\mathbb{R}^n$ to more general operator $A(x,\partial_x)$ allows us to consider more general equations than ones generated by the change of coordinates.

The function $u = e^{\frac{t}{2}Ht} \psi$ solves the generalized scalar Klein-Gordon equation

$$\partial_t^2 u - e^{-2HT}A(x,\partial_x)u + \left( m^2 - \frac{9}{4}H^2 \right) u = 0 .$$

The next theorem gives a representation of the solution of the Cauchy problem for the $4 \times 4$ system of the generalized non-covariant Klein-Gordon equations with complex valued mass in the de Sitter spacetime.

**Theorem 1.2** Let a vector-valued function $\Psi$ be a solution to the equation

$$\left( \partial_0^2 - e^{-2HT}A(x,\partial_x) \right) L_4 \Psi + \left( mL_4 - \frac{H}{2} \right)^2 \Psi = F .$$

Then its components $\Psi_i(x,t), \Psi_1(x,t), \Psi_2(x,t), \Psi_3(x,t)$ can be written as follows: for $i = 0,1$

$$\Psi_i(x,t) = G(x,t,D_x;M_+)[F_i] + K_0(x,t,D_x;M_+)[\Psi_i(x,0)] + K_1(x,t,D_x;M_+)[\partial_0\Psi_i(x,0)],$$

while for $k = 2,3$

$$\Psi_k(x,t) = G(x,t,D_x;M_-)[F_k] + K_0(x,t,D_x;M_-)[\Psi_k(x,0)] + K_1(x,t,D_x;M_-)[\partial_0\Psi_k(x,0)].$$
Proof. Equation (1.14) can be written as follows
\[
\left( \frac{\partial_0^2}{2} - e^{-2Ht} A(x, \partial_x) + m^2 - \frac{1}{4} H^2 \right) \mathbb{1}_4 \Psi - imH\gamma^0 \Psi = F.
\]
We write the last equation in the components \( \Psi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3)^T \) (\( A^T \) means transposition of the matrix \( A \)) as follows
\[
\left( \frac{\partial_0^2}{2} - e^{-2Ht} A(x, \partial_x) + m^2 - \frac{1}{4} H^2 - imH \right) \Psi_j = F_j, \quad j = 0, 1,
\]
\[
\left( \frac{\partial_0^2}{2} - e^{-2Ht} A(x, \partial_x) + m^2 - \frac{1}{4} H^2 + imH \right) \Psi_k = F_k, \quad k = 2, 3.
\]
It remains to apply Theorem 1.1 \cite{30}. Theorem is proved. \( \Box \)

The last theorem can be easily extended to the diagonal system of Klein-Gordon equations
\[
\left( \frac{\partial_0^2}{2} - e^{-2Ht} A(x, \partial_x) \right) \mathbb{1}_4 \Psi + M \Psi = F, \tag{1.15}
\]
where \( A(x, \partial_x) \) is a diagonal matrix of operators, while \( M \) is a constant diagonal matrix \( M = \text{diag}(M_i) \), \( i = 1, \ldots, m \), with the entries \( M_i \in \mathbb{C} \) possibly depending on some parameters. Indeed, if we introduce the diagonal matrices of the operators
\[
(G(x, t, D_x; M))_{jk} = \delta_{jk} G(x, t, D_x; M),
\]
\[
(K_0(x, t, D_x; M))_{jk} = \delta_{jk} K_0(x, t, D_x; M), \quad (K_1(x, t, D_x; M))_{jk} = \delta_{jk} K_1(x, t, D_x; M),
\]
then the solution to the equation (1.15) is given by
\[
\Psi(x, t) = G(x, t, D_x; M)[F] + K_0(x, t, D_x; M)[\Psi(x, 0)] + K_1(x, t, D_x; M)[\partial_t \Psi(x, 0)].
\]

Theorem 1.2 allows us to write solution of the \textit{generalized Dirac equation} in the de Sitter spacetime. In order to define that class of Dirac operators we note that the Dirac operator in the Minkowski spacetime written in the Cartesian or curvilinear coordinates is a member of the family of operators of the form
\[
i\gamma^0 \partial_0 + i\gamma^k A_k(x, \partial_x) - m \mathbb{1}_4,
\]
where the operators \( A_k(x, \partial_x) = \sum_{i=1}^3 a_{ki}(x) \partial_{x_i} \), \( k = 1, 2, 3 \), have variable coefficients depending on the spatial variables. Having in mind the diagonal system of Klein-Gordon equations (1.15) and four diagonal linearly independent matrices
\[
\mathbb{1}_4, \quad \gamma^0, \quad \gamma^1 \gamma^2 = -i \left( \begin{array}{cc} \sigma^3 & \sigma_2 \\ \sigma_2 & \sigma^3 \end{array} \right), \quad -i\gamma^3 \gamma^0 \gamma^1 \gamma^2 = \left( \begin{array}{cc} -\sigma^3 & \sigma_2 \\ \sigma_2 & \sigma^3 \end{array} \right),
\]
we can then define \textit{generalized Dirac operator}
\[
i\gamma^0 \partial_0 + i\gamma^k A_k(x, \partial_x) - m \mathbb{1}_4
\]
by the condition
\[
\gamma^k A_k(x, \partial_x) \gamma^j A_j(x, \partial_x) = -A(x, \partial_x) \mathbb{1}_4 + B(x, \partial_x) \gamma^0 + C(x, \partial_x) \gamma^1 \gamma^2 + D(x, \partial_x) \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3
\]

imposed on, in general, the scalar pseudo-differential operators \( A_k(x, \partial_x) \), \( A(x, \partial_x) \), \( B(x, \partial_x) \), \( C(x, \partial_x) \), and \( D(x, \partial_x) \). For the purpose of this paper it suffices to consider the case of \( D(x, \partial_x) = 0 \). Accordingly, we define \textit{generalized Dirac operator in the de Sitter spacetime}
\[
i\gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) + \frac{3}{2} H \gamma^0 - m \mathbb{1}_4,
\]
where \( A(x, \partial_x) \), \( A_k(x, \partial_x) \), \( k = 1, 2, 3 \), are the scalar operators with the property
\[
\gamma^k A_k(x, \partial_x) \gamma^j A_j(x, \partial_x) = -A(x, \partial_x) \mathbb{1}_4.
\]

Theorem 0.2 is a particular case of the next theorem.
\textbf{Theorem 1.3} Assume that $A(x, \partial_x)$, $A_k(x, \partial_x)$, $k = 1, 2, 3$, are the scalar operators with the properties (1.17). Then the solution to the Cauchy problem

\[
\begin{align*}
&\left\{ \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{3}{2}H\gamma^0 - m\mathbb{1}_4 \right) \Psi(x, t) = F(x, t), \\
&\Psi(x, 0) = \Phi(x) \right. \end{align*}
\] (1.18)

is given as follows

\[
\Psi(x, t) = -e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) - i\frac{H}{2}\gamma^0 + m\mathbb{1}_4 \right) X, \] (1.19)

where

\[
X = - \left( \frac{G(x, t, D_x; M_+)}{\mathbb{1}_2} \frac{\partial_0}{\mathbb{1}_2} G(x, t, D_x; M_-) \right) [e^{Ht} F] - \left( \frac{K_1(x, t, D_x; M_+)}{\mathbb{1}_2} \frac{\partial_0}{\mathbb{1}_2} K_1(x, t, D_x; M_-) \right) [\partial_0 X(x, 0)]
\]

and

\[
X(x, 0) = 0, \quad (\partial_0 X)(x, 0) = -i\gamma^0 \Phi(x).
\]

Then the substitution of (1.19) into the equation of (1.18) and the relation (1.17) together with Proposition 3.1 imply

\[
\left( \partial_0^2 - e^{-2Ht} A(x, \partial_x) + m^2 - \frac{1}{4}H^2 \right) \mathbb{1}_4 + im\gamma^0 X = -e^{Ht} F.
\]

Consequently,

\[
\left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{3}{2}H\gamma^0 - m\mathbb{1}_4 \right) \Psi(x, t) =
\]

\[
\left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{3}{2}H\gamma^0 - m\mathbb{1}_4 \right) \times e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) - i\frac{H}{2}\gamma^0 + m\mathbb{1}_4 \right) X =
\]

\[
= -e^{-Ht} \left( \partial_0^2 - e^{-2Ht} A(x, \partial_x) + m^2 - \frac{1}{4}H^2 + imH\gamma^0 \right) X =
\]

\[
= F.
\]

For the initial value of $\Psi(x, 0)$ we have

\[
\Psi(x, 0) = \lim_{t \to 0} e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) - i\frac{H}{2}\gamma^0 + m\mathbb{1}_4 \right) X = (i\gamma^0)(-i\gamma^0)\Phi(x) = \Phi(x).
\]

Theorem is proved. \hfill \Box

In Example 5 of Section 3 we can apply Theorem 1.3 to Dirac field in the de Sitter spacetime in the presence of constant magnetic field by using the further generalization of the relation (1.16).
2 The case of vanishing $H$

Now we turn to the limit case of Theorem 1.3 as $H$ approaches zero. This includes, in particular, the Dirac equation in the Minkowski spacetime. That theorem shows how massless field via the integral transform provides mass to the massive field. In fact, the integral transform approach leads to the formulas for the solutions also of the generalized Dirac equation. We start with the Klein-Gordon equation by the following theorem. We skip a proof that it can be done by straightforward calculations.

**Theorem 2.1** The function $u = u(x, t)$ defined by

$$u(x, t) = \int_{t_0}^{t} \int_{0}^{t-b} I_{0} \left( M \sqrt{(t-b)^2 - r^2} \right) \nu_{f}(x, r; b) \, dr$$

$$+ v_{\varphi_0}(x, t) - \int_{0}^{t} \frac{i M t}{\sqrt{t^2 - r^2}} J_1 \left( i M \sqrt{t^2 - r^2} \right) \nu_{\varphi}(x, r) \, dr$$

$$+ \int_{0}^{t} I_{0} \left( M \sqrt{t^2 - r^2} \right) \nu_{\varphi_1}(x, r) \, dr, \quad x \in \Omega, \ t \in [0, T],$$

where $M \in \mathbb{C}$, $v_{f}(x, r; b)$ is a solution of (1.10), while $v_{\varphi_0}(x, s)$ and $v_{\varphi_1}(x, s)$ solve the problem (1.11), is a solution of the problem

$$\begin{aligned}
&u_{tt} - \mathcal{A}(x, \partial_x)u - M^2 u = f, \quad x \in \Omega, \ t \in [0, T], \\
u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x), \quad x \in \Omega.
\end{aligned} \tag{2.20}$$

Here $I_v$ and $J_v$ are the Bessel functions.

Moreover, similar statement true for the fundamental solution of the generalized Klein-Gordon operator. We skip the proof of the next statement since it similar to one of the previous theorem.

**Theorem 2.2** If $\mathcal{E}_{x,M}^w = \mathcal{E}_{A,x,M}^w(x, t; x_0)$ is a fundamental solution of the Cauchy problem for the massless equation that is,

$$\begin{cases}
(\partial_t^2 - \mathcal{A}(x, \partial_x)) \mathcal{E}_{x,M}^w(x, t; x_0) = 0, & x, x_0 \in \mathbb{R}^n, \ t \in \mathbb{R}, \\
\mathcal{E}_{x,M}^w(x, 0; x_0) = \delta(x - x_0), & \partial_t \mathcal{E}_{x,M}^w(x, 0; x_0) = 0, \quad x, x_0 \in \mathbb{R}^n,
\end{cases}$$

then, the fundamental solutions $\mathcal{E}_{\pm, KG, A}(x, t; x_0, t_0; M)$ for the operator of (2.20) can be written as follows

$$\mathcal{E}_{\pm, KG, A}(x, t; x_0, t_0; M) = 2 \int_{t_0}^{t-t_0} I_{0} \left( M \sqrt{(t-t_0)^2 - r^2} \right) \mathcal{E}_{x,M}^w(x, r; x_0) \, dr, \quad \text{when } \pm (t - t_0) > 0$$

on their supports.

The fundamental solutions $\mathcal{E}_{x,M}^w = \mathcal{E}_{x,M}^w(x, t; x_0)$ for a wide class of hyperbolic operators are constructed in [28, Ch.1, Ch.4] (see also referenced therein). The application of the previous two theorems leads to the following result. Define the operators $\mathcal{G}$ and $\mathcal{K}$ as follows

$$\mathcal{G}^{Min}(x, t, D_x; M)[f] := \int_{t_0}^{t} \int_{0}^{t-b} I_{0} \left( M \sqrt{(t-b)^2 - r^2} \right) \nu_f(x, r; b) \, dr,$$

$$\mathcal{K}^{Min}_1(x, t, D_x; M)[\varphi(x)] := \int_{0}^{t} I_{0} \left( M \sqrt{t^2 - r^2} \right) \nu_{\varphi}(x, r) \, dr, \quad x \in \Omega, \ t \in \mathbb{R}_+,$$

while $\nu_f(x, r; b)$ is a solution of

$$\begin{cases}
v_{tt} - \mathcal{A}(x, \partial_x)v = 0, & x \in \Omega, \ t \in \mathbb{R}_+, \\
v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0, & b \in \mathbb{R}_+, \quad x \in \Omega,
\end{cases}$$

and $\nu_{\varphi}(x, s)$ solves the problem

$$\begin{cases}
v_{tt} - \mathcal{A}(x, \partial_x)v = 0, & x \in \Omega, \ t \in \mathbb{R}_+, \\
v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0, & x \in \Omega.
\end{cases}$$
Theorem 2.3 Assume that $A(x, \partial_x)$, $A_k(x, \partial_x)$, $k = 1, 2, 3$, are the scalar operators with the properties (1.17). Then the solution $\Psi = \Psi(x, t)$ to the Cauchy problem

$$\begin{align*}
\begin{cases}
(i \gamma^0 \partial_0 + i \gamma^k A_k(x, \partial_x) - m \mathbb{I}_4) \Psi(x, t) = F(x, t), & x \in \Omega, \ t \in \mathbb{R}_+ , \\
\Psi(x, 0) = \Phi(x), & x \in \Omega,
\end{cases}
\end{align*}$$

is given as follows

$$\Psi(x, t) = -(i \gamma^0 \partial_0 + i \gamma^k A_k(x, \partial_x) + m \mathbb{I}_4) \left( \left( G_{\text{Min}}(x, t, D_x; im) \mathbb{I}_2 \bigg/ \mathbb{O}_2 \bigg/ G_{\text{Min}}(x, t, D_x; -im) \mathbb{I}_2 \right)^\gamma [F] + i \gamma^0 \left( K_{\text{Min}}^1(x, t, D_x; im) \mathbb{I}_2 \bigg/ \mathbb{O}_2 \bigg/ K_{\text{Min}}^1(x, t, D_x; -im) \mathbb{I}_2 \right) [\Phi] \right).$$

3 Factorization of the generalized Klein-Gordon operator in de Sitter spacetime

In this section we introduce some factorization of the generalized Klein-Gordon operator in de Sitter spacetime with scalar diagonal mass matrix. We believe that that factorization is interesting in its own right and therefore we illustrate it with several examples.

In the next proposition, by appealing to the complex-valued mass matrix, we involve the Dirac operator in the de Sitter spacetime in the factorization of the generalized Klein-Gordon operator. In view of Pauli’s fundamental theorem [21] for the given generalized Klein-Gordon operator the next factorization is not unique.

Proposition 3.1 Let $A(x, \partial_x)$, $A_k(x, \partial_x)$, $k = 1, 2, 3$, be scalar operators with the property (1.17) and $a, b, c \in \mathbb{C}$. Then

$$e^{\alpha H_t} \left( i \gamma^0 \partial_0 + ie^{-H_t} \gamma^k A_k(x, \partial_x) + i b \frac{H}{2} \gamma^0 - m \mathbb{I}_4 \right) \times e^{-\alpha H_t} \left( i \gamma^0 \partial_0 + ie^{-H_t} \gamma^j A_j(x, \partial_x) - i (b - c) \frac{H}{2} \gamma^0 + m \mathbb{I}_4 \right)$$

$$= -i \mathbb{I}_4 \partial_0^2 + e^{-2H_t} A(x, \partial_x) \mathbb{I}_4 - \left( b - a - 1 - \frac{c}{2} \right) H e^{-H_t} \gamma^0 \gamma^k A_k(x, \partial_x)$$

$$- \left( m \mathbb{I}_4 - i \frac{bH}{2} \gamma^0 \right) \partial_0^2 - i \left( a + \frac{c}{2} \right) H m \gamma^0 + \left( a - \frac{c}{2} \right) H \mathbb{I}_4 \partial_0 - (bc + 2ab - 2ac) \frac{H^2}{4} \mathbb{I}_4.$$

In particular, (i) If $a = c = 0 = m$ and $b = 1$ the second order operator has the diagonal imaginary mass $i H/2 \mathbb{I}_4$.

$$\left( i \gamma^0 \partial_0 + ie^{-H_t} \gamma^k A_k(x, \partial_x) + i \frac{H}{2} \gamma^0 \right) \left( i \gamma^0 \partial_0 + ie^{-H_t} \gamma^j A_j(x, \partial_x) - i \frac{H}{2} \gamma^0 \right)$$

$$= -\partial_0^2 + e^{-2H_t} A(x, \partial_x) + \frac{H^2}{4} \mathbb{I}_4.$$

(ii) If $a = 1$, $b = 3$ and $c = 2$, then

$$e^{H_t} \left( i \gamma^0 \partial_0 + ie^{-H_t} \gamma^k A_k(x, \partial_x) + i \frac{H}{2} \gamma^0 - m \mathbb{I}_4 \right) \times e^{-H_t} \left( i \gamma^0 \partial_0 + ie^{-H_t} \gamma^j A_j(x, \partial_x) - i \frac{H}{2} \gamma^0 + m \mathbb{I}_4 \right)$$

$$= -\mathbb{I}_4 \partial_0^2 + e^{-2H_t} \mathbb{I}_4 A(x, \partial_x) - \left( m \mathbb{I}_4 - i \frac{H}{2} \gamma^0 \right) \partial_0^2,$$
where the last second order scalar operator represents the generalized non-covariant Klein-Gordon operators in de Sitter spacetime with the complex masses, while the first factor is the generalized Dirac operator in de Sitter spacetime. 

(iii) If \( a = -1/2 \) and \( b = 3 \), \( c = 5 \) then

\[
e^{-Ht/2} \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{H}{2}\gamma^0 - m\mathbb{I}_4 \right)
\]

\[
\times e^{-Ht/2} \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^j A_j(x, \partial_x) + i\frac{H}{2}\gamma^0 + m\mathbb{I}_4 \right)
\]

\[
= -\mathbb{I}_4 \partial_0^2 - 3H\mathbb{I}_4 \partial_0 + e^{-2Ht}A(x, \partial_x) - \left( m\mathbb{I}_4 - i\frac{H}{2}\gamma^0 \right)^2 - \frac{9H^2}{4}\mathbb{I}_4 ,
\]

where the second scalar order operators are the generalized covariant Klein-Gordon operators in de Sitter spacetime and the diagonal complex valued mass matrix is \( (m\mathbb{I}_4 - \frac{iH}{2}\gamma^0)^2 + \frac{9H^2}{4}\mathbb{I}_4 \).

Proof. We start with the case of diagonal imaginary mass. First we verify

\[
- \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{H}{2}\gamma^0 \right) \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^j A_j(x, \partial_x) - i\frac{H}{2}\gamma^0 \right)
\]

\[
= \mathbb{I}_4 \partial_0^2 - e^{-2Ht}A(x, \partial_x) + (b - 1)He^{-Ht}\gamma^0 / b H^2 - \frac{b^2 H^2}{4}\mathbb{I}_4 .
\]

Indeed, straightforward calculations lead to the claimed statement:

\[
- \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{H}{2}\gamma^0 \right) \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^j A_j(x, \partial_x) - i\frac{H}{2}\gamma^0 \right)
\]

\[
= \left( \gamma^0 \partial_0 + e^{-Ht}\gamma^k A_k(x, \partial_x) \right) \left( \gamma^0 \partial_0 + e^{-Ht}\gamma^j A_j(x, \partial_x) \right)
\]

\[
- \left( \gamma^0 \partial_0 + e^{-Ht}\gamma^k A_k(x, \partial_x) \right) b\frac{H}{2}\gamma^0 + b\frac{H}{2}\gamma^0 \left( \gamma^0 \partial_0 + e^{-Ht}\gamma^j A_j(x, \partial_x) \right) - \left( b\frac{H}{2}\gamma^0 \right)^2
\]

\[
= \mathbb{I}_4 \partial_0^2 + \gamma^0 \partial_0 e^{-Ht}\gamma^k A_k(x, \partial_x) + e^{-Ht}\gamma^k A_k(x, \partial_x) \gamma^0 \partial_0 + e^{-Ht}\gamma^k A_k(x, \partial_x) e^{-Ht}\gamma^j A_j(x, \partial_x)
\]

\[
- \gamma^0 \partial_0 b\frac{H}{2}\gamma^0 - e^{-Ht}\gamma^k A_k(x, \partial_x) b\frac{H}{2}\gamma^0 + b\frac{H}{2}\gamma^0 \left( \gamma^0 \partial_0 + e^{-Ht}\gamma^j A_j(x, \partial_x) \right) - \frac{b^2 H^2}{4}\mathbb{I}_4
\]

\[
= \mathbb{I}_4 \partial_0^2 - e^{-2Ht}A(x, \partial_x) + (b - 1)He^{-Ht}\gamma^0 / b H^2 - \frac{b^2 H^2}{4}\mathbb{I}_4 .
\]

If \( b = 1 \) the last equation implies (i). Since \( m\mathbb{I}_4 \) commutes with any operator, we derive

\[
\left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^k A_k(x, \partial_x) + i\frac{H}{2}\gamma^0 - m\mathbb{I}_4 \right) \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^j A_j(x, \partial_x) - i\frac{H}{2}\gamma^0 + m\mathbb{I}_4 \right)
\]

\[
= -\mathbb{I}_4 \partial_0^2 + e^{-2Ht}A(x, \partial_x) - (b - 1)He^{-Ht}\gamma^0 \gamma^k A_k(x, \partial_x) - \left( m\mathbb{I}_4 - i\frac{b H}{2}\gamma^0 \right)^2 .
\]
Then, for every $c \in \mathbb{C}$ we have
\[
\left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) + ib \frac{H}{2} \gamma^0 - m \mathbb{1}_4 \right)
\times \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) - i(b - c) \frac{H}{2} \gamma^0 + m \mathbb{1}_4 \right)
= -\mathbb{I}_4 \partial_0^2 - e^{-2Ht} \mathcal{A}(x, \partial_x) - \left( b - 1 - \frac{c}{2} \right) He^{-Ht} \gamma^0 \gamma^k A_k(x, \partial_x) - \left( m \mathbb{1}_4 - i \frac{bH}{2} \gamma^0 \right)^2.
\]

Furthermore,
\[
e^{aHt} \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) + ib \frac{H}{2} \gamma^0 - m \mathbb{1}_4 \right)
\times e^{-aHt} \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^j A_j(x, \partial_x) - i(b - c) \frac{H}{2} \gamma^0 + m \mathbb{1}_4 \right)
= \left[ \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) + ib \frac{H}{2} \gamma^0 - m \mathbb{1}_4 \right) - i \gamma^0 aH \right]
\times \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^j A_j(x, \partial_x) - i(b - c) \frac{H}{2} \gamma^0 + m \mathbb{1}_4 \right)
= \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) + ib \frac{H}{2} \gamma^0 - m \mathbb{1}_4 \right)
\times \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^j A_j(x, \partial_x) - i(b - c) \frac{H}{2} \gamma^0 + m \mathbb{1}_4 \right)
- i \gamma^0 aH \left( i \gamma^0 \partial_0 + ie^{-Ht} \gamma^k A_k(x, \partial_x) - i(b - c) \frac{H}{2} \gamma^0 + m \mathbb{1}_4 \right).
\]

The last expression can be written as follows
\[
-\mathbb{I}_4 \partial_0^2 - e^{-2Ht} \mathcal{A}(x, \partial_x) - \left( b - 1 - \frac{c}{2} \right) He^{-Ht} \gamma^0 \gamma^k A_k(x, \partial_x) - \left( m \mathbb{1}_4 - i \frac{bH}{2} \gamma^0 \right)^2.
\]

The proposition is proved. \qed
Proposition 3.2 The following identity holds
\[
\left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^k A_k(x, \partial_x) + i \frac{3H}{2} \gamma^0 - m I_4 \right) \times \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^j A_j(x, \partial_x) + i \frac{3H}{2} \gamma^0 + m I_4 \right) = -I_4 \partial_0^2 - 3H I_4 \partial_0 + e^{-2Ht} I_4 A(x, \partial_x) - H e^{-Ht} \gamma^k \gamma^0 A_k(x, \partial_x) - m^2 I_4 - \frac{9H^2}{4} I_4.
\]

Proof. It suffice to consider the case of \( m = 0 \). According to Proposition 3.1 case (i):
\[
\left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a A_a(x, \partial_x) + i \frac{3H}{2} \gamma^0 \right) \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a A_a(x, \partial_x) + i \frac{3H}{2} \gamma^0 \right)
\]
\[
= \left( -\partial_t^2 + e^{-2ht} A(x, \partial_x) + \frac{H^2}{4} \right) I_4 + \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a A_a(x, \partial_x) + i \frac{3H}{2} \gamma^0 \right) i2H \gamma^0 + iH \gamma^0 \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a A_a(x, \partial_x) - i \frac{H}{2} \gamma^0 \right) - 2H^2 I_4
\]
\[
= \left( -\partial_t^2 + e^{-2ht} A(x, \partial_x) + \frac{H^2}{4} \right) I_4 - H e^{-Ht} \gamma^a A_a(x, \partial_x) \gamma^0 - 3H I_4 \partial_0 - \frac{5}{2} H^2 I_4.
\]
Proposition is proved. \( \square \)

Corollary 3.3 If we denote the generalized spinorial covariant D’Alembert operator
\[
\Box_g = I_4 \partial_0^2 + 3H I_4 \partial_0 - e^{-2Ht} I_4 A(x, \partial_x) + H e^{-Ht} \gamma^k \gamma^0 A_k(x, \partial_x) - \frac{3H^2}{4} I_4
\]
and the curvature of the de Sitter spacetime \( R = -12H^2 \), then
\[
\left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^k A_k(x, \partial_x) + i \frac{3H}{2} \gamma^0 \right) \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^j A_j(x, \partial_x) + i \frac{3H}{2} \gamma^0 \right)
\]
\[
= -\Box_g - \frac{R}{4} I_4.
\]
Equation (3.21)

The matrices \( \gamma^k \gamma^0 \) in the expression of \( \Box_g \) are not diagonal. The formula (3.21) is an extension of the relation (0.2) to the generalized Dirac operators in the de Sitter spacetime.

Example 1. For the Dirac operator in the Cartesian coordinates with the operators
\[
A_k(x, \partial_x) = \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3, \quad A(x, \partial_x) = \Delta
\]
in the spatial part the factorization (1.17) holds.
Example 2. Consider the operators

\[
A_1(x, \partial_x) = a(x, y) \frac{\partial}{\partial x}, \quad A_2(x, \partial_x) = b(x, y) \frac{\partial}{\partial y}, \quad A_3(x, \partial_x) = c(z) \frac{\partial}{\partial z},
\]

\[
A(x, \partial_x) = \left( a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2} + c^2 \frac{\partial^2}{\partial z^2} \right) + \frac{1}{2} \left( \frac{\partial a^2(x, y)}{\partial x} \frac{\partial}{\partial x} + \frac{\partial b^2(x, y)}{\partial y} \frac{\partial}{\partial y} + \frac{\partial c^2(z)}{\partial z} \frac{\partial}{\partial z} \right),
\]

\[
C(x, \partial_x) = \left( -a_x(x, y) b(x, y) \frac{\partial}{\partial x} - a(x, y) b_x(x, y) \frac{\partial}{\partial y} \right)
\]

with the real-valued smooth coefficients \(a(x, y), b(x, y), c(z)\). They form the generalized Dirac operator. Indeed, using the properties of the Dirac matrices we verify the condition (1.16) as follows

\[
\left( \gamma^1 a(x, y) \frac{\partial}{\partial x} + \gamma^2 b(x, y) \frac{\partial}{\partial y} + \gamma^3 c(z) \frac{\partial}{\partial z} \right)^2 = -A(x, \partial_x) \mathbb{I}_4 + C(x, \partial_x) \gamma^1 \gamma^2.
\]

If \(a = a(x)\) and \(b = b(y)\), then \(C(x, \partial_x)\) vanishes. If the functions \(a(x, y), b(x, y), c(z)\), have zeros, then the Klein-Gordon operator of (1.9) is weakly hyperbolic and, in general, for the well-posedness of the Cauchy problem the so-called Levi conditions are necessary. For this example they are fulfilled. The constructions of the parametrix and the fundamental solutions (propagators) for such operators are given in [28, Ch.4].

Example 3. The factorization of the Klein-Gordon operator in the de Sitter spacetime in the cylindrical coordinates. In the cylindrical coordinates

\[
x = \rho \cos(\varphi), \quad y = \rho \sin(\varphi), \quad z = z
\]

one can choose

\[
A_1(x, \partial) = \cos(\varphi) \partial_\rho - \frac{\sin(\varphi)}{\rho} \partial_\varphi, \quad A_2(x, \partial) = \sin(\varphi) \partial_\rho + \frac{\cos(\varphi)}{\rho} \partial_\varphi, \quad A_3(x, \partial) = \partial_z.
\]

Here \(x_1 = r, x_2 = \phi,\) and \(x_3 = z\). It is easy to verify that with

\[
A(x, \partial_x) = \partial^2_\rho + \frac{1}{\rho} \partial_\varphi + \frac{1}{\rho^2} \partial_\varphi^2 + \partial^2_z
\]

the condition (1.17) of Proposition 3.1 is fulfilled. The Dirac operator is

\[
i \gamma^0 \partial_t + ie^{-Ht} \gamma^1 A_1(x, \partial) + ie^{-Ht} \gamma^2 A_2(x, \partial) + ie^{-Ht} \gamma^3 A_3(x, \partial) + i \frac{3}{2} H \gamma^0 - m \mathbb{I}_4.
\]

Example 4. The factorization of the Klein-Gordon operator in the de Sitter spacetime in the spherical coordinates. For the Laplace operator in the spherical coordinates in \(\mathbb{R}^3\)

\[
x(r, \theta, \phi) := r \cos(\phi) \sin(\theta), \quad y(r, \theta, \phi) := r \sin(\phi) \sin(\theta), \quad z(r, \theta, \phi) := r \cos(\theta)
\]

one can choose

\[
A_1(x, \partial) = \cos(\phi) \sin(\theta) \partial_r + \frac{\cos(\phi) \cos(\theta)}{r} \partial_\theta - \frac{\sin(\phi)}{r \sin(\theta)} \partial_\phi,
\]

\[
A_2(x, \partial) = \sin(\phi) \sin(\theta) \partial_r + \frac{\sin(\phi) \cos(\theta)}{r} \partial_\theta + \frac{\cos(\phi)}{r \sin(\theta)} \partial_\phi,
\]

\[
A_3(x, \partial) = \cos(\theta) \partial_r - \frac{\sin(\theta)}{r} \partial_\theta.
\]

Here \(x_1 = r, x_2 = \theta,\) and \(x_3 = \phi\). It is easy to verify that with

\[
A(x, \partial_x) = \partial^2_r + \frac{2}{r} \partial_r + \frac{1}{r^2} \left( \partial_\theta^2 + \cot(\theta) \partial_\theta + \csc^2(\theta) \partial_\phi^2 \right)
\]
the condition (1.17) of Proposition 3.1 is fulfilled. The Dirac operator is

\[ i\gamma^0 \partial_t + ie^{-Ht}\gamma^1 A_1(x, \partial) + ie^{-Ht}\gamma^2 A_2(x, \partial) + ie^{-Ht}\gamma^3 A_3(x, \partial) + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4. \]

We can write the Dirac equation

\[ \left( i\gamma^0 \partial_0 + ie^{-Ht}\gamma^0 \partial_0 + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4 \right) \Psi(x, t) = F(x, t) \]
as follows

\[ \left( i\gamma^0 \partial_0 + ie^{-Ht} (\gamma^c \partial_r + \gamma^0 \partial_0 + \gamma^0 \partial_\phi) + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4 \right) \Psi(x, t) = F(x, t), \]

where in this Cartesian tetrad gauge the gamma matrices will be given by (see, e.g., [22])

\[
\begin{align*}
\gamma^c_\tau &= \gamma^1 \cos(\phi) \sin(\theta) + \gamma^2 \sin(\theta) \sin(\phi) + \gamma^3 \cos(\theta), \\
\gamma^c_\phi &= -\gamma_1 \sin(\phi) + \gamma_2 \cos(\phi) = \frac{1}{r \sin(\theta)} (-\gamma_1 \sin(\phi) + \gamma_2 \cos(\phi)), \\
\gamma^\theta_c &= \frac{1}{r} \left( \gamma^1 \cos(\theta) \cos(\phi) + \gamma^2 \sin(\phi) \cos(\theta) - \gamma^3 \sin(\theta) \right).
\end{align*}
\]

We have used the subscript \( c \) for Cartesian. We can also write

\[
\left( i\gamma^0 \partial_0 + ie^{-Ht} \left( \tilde{\gamma}^c_\tau \partial_r + \tilde{\gamma}^\phi_c \frac{1}{r \sin(\theta)} \partial_\phi + \tilde{\gamma}^\theta_c \frac{1}{r} \partial_\theta \right) + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4 \right) \Psi(x, t) = F(x, t),
\]

where \( \tilde{\gamma}^c_\tau = \gamma^0 \) and

\[
\begin{align*}
\tilde{\gamma}^c_\tau &= \gamma^1 \cos(\phi) \sin(\theta) + \gamma^2 \sin(\theta) \sin(\phi) + \gamma^3 \cos(\theta), \\
\tilde{\gamma}^\phi_c &= -\gamma_1 \sin(\phi) + \gamma_2 \cos(\phi), \\
\tilde{\gamma}^\theta_c &= \gamma^1 \cos(\theta) \cos(\phi) + \gamma^2 \sin(\phi) \cos(\theta) - \gamma^3 \sin(\theta).
\end{align*}
\]

and with the Lorenzian metric \( \eta \) in the Minkowski space time, we have

\( \left\{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \right\} = 2\eta^{\mu\nu}, \quad \mu, \nu = t, r, \theta, \phi, \quad \eta^{rr} = \eta^{\theta\theta} = \eta^{\phi\phi} = -1, \quad \eta^{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu. \)

**Proposition 3.4** In the spherical coordinates the following factorization of the Klein-Gordon operator in the de Sitter spacetime holds

\[
\left( \partial_0^2 - e^{-2Ht} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot(\theta)}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \right) \right) \mathbb{I}_4
\]

\[
\quad \quad \quad + \left( m\mathbb{I}_4 - \frac{1}{2}H\gamma^0 \right)^2
\]

\[ = -e^{Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht} \left( \tilde{\gamma}^c_\tau \partial_r + \tilde{\gamma}^\phi_c \frac{1}{r \sin(\theta)} \partial_\phi + \tilde{\gamma}^\theta_c \frac{1}{r} \partial_\theta \right) + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4 \right)
\]

\[ \times e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht} \left( \tilde{\gamma}^c_\tau \partial_r + \tilde{\gamma}^\phi_c \frac{1}{r \sin(\theta)} \partial_\phi + \tilde{\gamma}^\theta_c \frac{1}{r} \partial_\theta \right) - i\frac{H}{2}\gamma^0 + m\mathbb{I}_4 \right). \]

In particular, for the Dirac equation with the source term \( F \) we have

\[ i\partial_0 \Psi + ie^{-Ht}\gamma^0 (\gamma^\tau \partial_r + \gamma^0 \partial_0 + \gamma^\phi \partial_\phi) + i\frac{3}{2}H\mathbb{I}_4 - m\gamma^0 \Psi = \gamma^0 F, \]

where in this Cartesian tetrad gauge the gamma matrices are given by (3.22),(3.23),(3.24).
Example 5. Let the functions \(a(t, x, y, z), b(t, x, y, z), c(t, x, y, z), d(t, x, y, z)\) be such that

\[
\begin{align*}
    a(t, x, y, z) &= c_y(x, y, z), \\
    b(t, x, y, z) &= c_y(x, y, z), \\
    a(t, x, y, z) &= d_x(x, y, z), \\
    b(t, x, y, z) &= d_y(x, y, z), \\
    c(t, x, y, z) &= d_z(x, y, z).
\end{align*}
\]

It is easy to verify in the Cartesian coordinates with

\[
\begin{align*}
    A_0(t, x, y, z, \partial_t) &= \frac{\partial}{\partial t} + d(t, x, y, z), \\
    A_1(t, x, y, z, \partial_x) &= \frac{\partial}{\partial x} + a(t, x, y, z), \\
    A_2(t, x, y, z, \partial_y) &= \frac{\partial}{\partial y} + b(t, x, y, z), \\
    A_3(t, x, y, z, \partial_z) &= \frac{\partial}{\partial z} + c(t, x, y, z),
\end{align*}
\]

for the Dirac operator

\[
i\gamma^0 A_0(t, x, y, z, \partial_t) + i\gamma^1 A_1(t, x, y, z, i\partial_x) + \gamma^2 A_2(t, x, y, z, \partial_y) + i\gamma^3 A_3(t, x, y, z, \partial_z) - mI_4,
\]

(3.25)

the following identity

\[
\begin{align*}
    \left(\gamma^0 A_0(t, x, y, z, \partial_t) + \gamma^1 A_1(t, x, y, z, \partial_x) + \gamma^2 A_2(t, x, y, z, \partial_y) + \gamma^3 A_3(t, x, y, z, \partial_z)\right)^2 = & \\
    \left\{ \partial_t^2 - \Delta - 2 \left( a(t, x, y, z) \frac{\partial}{\partial x} + b(t, x, y, z) \frac{\partial}{\partial y} + c(t, x, y, z) \frac{\partial}{\partial z} \right) + 2d(t, x, y, z) \frac{\partial}{\partial t} \\
    - a(t, x, y, z)^2 - b(t, x, y, z)^2 - c(t, x, y, z)^2 + d(t, x, y, z)^2 \\
    - \frac{\partial a(t, x, y, z)}{\partial x} - \frac{\partial b(t, x, y, z)}{\partial y} - \frac{\partial c(t, x, y, z)}{\partial z} + \frac{\partial d(t, x, y, z)}{\partial t} \right\} I_4 + \\
    + \left( \frac{\partial a(t, x, y, z)}{\partial x} - \frac{\partial b(t, x, y, z)}{\partial y} \right) \gamma^1 \gamma^2.
\end{align*}
\]

If \(id(t, x, y, z), ia(t, x, y, z), ib(t, x, y, z), ic(t, x, y, z)\) are real-valued functions, then the operator (3.25) is the Dirac operator in the presence of an electromagnetic potential \(i(d(t, x, y, z), a(t, x, y, z), b(t, x, y, z), c(t, x, y, z))\) for the particle with charge \(e = 1\). Thus, the square of this operator is a diagonal matrix of differential operators. It allows us to reduce solution of the corresponding equation to the scalar Klein-Gordon equation. This reduction is effective to produce the explicit formulas of solution.

In particular, if \(i\tilde{A}(x, y, z) = i(a(x, y, z), b(x, y, z), c(x, y, z))\) is a magnetic potential, then the magnetic field \(\tilde{H} = i\text{curl}\tilde{A}(x, y, z)\) is in the direction of \(z\) and time independent. The case of \(a(x, y, z) = c(x, y, z) = 0\) and \(b(x, y, z) = Hx\), where \(H\) is constant, while

\[
\left( i\gamma^0 \partial_t + i\gamma^1 \partial_x + \gamma^2 (i\partial_y - Hx) + i\gamma^3 \partial_z \right)^2 = - \left( \partial_t^2 - \Delta - 2Hx \frac{\partial}{\partial y} - H^2 x^2 \right) I_4 + H \gamma^1 \gamma^2,
\]

is of particular interest and thoroughly studied in the literature, see, e.g., [1, Sec. 1.6.2]. Using the last formula we can write the explicit formulas for the general solution of the Dirac equation in the de Sitter spacetime

\[
\left( i\gamma^0 \partial_t + ie^{-Ht} \gamma^1 \partial_x + e^{-Ht} \gamma^2 (i\partial_y - Hx) + ie^{-Ht} \gamma^3 \partial_z + i\frac{3}{2} H \gamma^0 - mI_4 \right) \psi = 0.
\]

This will be done in the forthcoming paper. For the irrotational vector field the term with \(\gamma^1 \gamma^2\) vanishes.

4 Proof of main Theorem 0.1

Proposition 4.1 The following factorization of the Klein-Gordon operator with the matrix-valued mass holds

\[
\left( \partial^2_t - e^{-2Ht} \Delta \right) I_4 + \left( mI_4 - \frac{1}{2} iH \gamma^0 \right)^2 = -e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht} \gamma^a \partial_a + i\frac{3}{2} H \gamma^0 - mI_4 \right) e^{-Ht} \left( i\gamma^0 \partial_0 + ie^{-Ht} \gamma^a \partial_a - \frac{H}{2} \gamma^0 + mI_4 \right).
\]

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Proof. Indeed, by simple calculations we check the following identity
\[
\left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a \partial_a + i \frac{H}{2} \gamma^0 - m \mathbb{I}_4 \right) \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a \partial_a - i \frac{H}{2} \gamma^0 + m \mathbb{I}_4 \right) = \left( -\Box + \frac{1}{4} H^2 \right) \mathbb{I}_4 + i H m \gamma^0 - m^2 \mathbb{I}_4
\]
\[
= -\Box \mathbb{I}_4 - \left( m \mathbb{I}_4 - \frac{1}{2} i H \gamma^0 \right)^2 ,
\]
where \( \Box = \partial_0^2 - e^{-2Ht} \Delta = \partial_0^2 - e^{-2Ht} \left( \partial_1^2 + \partial_2^2 + \partial_3^2 \right) \).

Proof of Theorem 0.1. Indeed we can write
\[
\left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a \partial_a + i \frac{3H}{2} \gamma^0 - m \mathbb{I}_4 \right) \mathcal{E}^{ret} (x,t;x_0,t_0;m) = \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a \partial_a + i \frac{3H}{2} \gamma^0 - m \mathbb{I}_4 \right)
\times e^{-Ht} \left( i\gamma^0 \partial_0 + i e^{-Ht} \gamma^a \partial_a - i \frac{H}{2} \gamma^0 + m \mathbb{I}_4 \right) \mathcal{E}^{ret,KG} (x,t;x_0,t_0;m) [e^H] =
\]
\[
e^{-Ht} \left( \partial_0^2 - e^{-2Ht} \Delta + m^2 - \frac{1}{4} H^2 + i m H \gamma^0 \right) \mathcal{E}^{ret,KG} (x,t;x_0,t_0;m) [e^H] =
\]
\[
e^{-Ht} \left( e^H \delta(x-x_0) \delta(t-t_0) \right) \mathbb{I}_4 =
\]
\[
\delta(x-x_0) \delta(t-t_0) \mathbb{I}_4 .
\]

Theorem is proved.

5 Conclusions

In this paper, we have used the integral transform approach to write various fundamental solutions to the Dirac equation in the de Sitter space. This new integral transform one can regard as an analytical mechanism that generates a spin-1/2 (massive or massless) field in the curved spacetime from the massless scalar field in the Minkowski spacetime. We also have shown that this mechanism exists even in the case of the vanishing cosmological constant; it provides spin-1/2 particles with the mass due to massless scalar field. In fact, this mechanism appeals to some additional “time” variable.

References

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