Four-Neutrino Mixing, Oscillations and Big-Bang Nucleosynthesis

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Abstract

We investigate the implications of the standard Big-Bang Nucleosynthesis constraint on the number of light neutrinos in the framework of the two four-neutrino schemes that are favored by the results of neutrino oscillation experiments.

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1. Introduction

Forty years after Pontecorvo’s discovery of the theory of neutrino oscillations [1], there are today three experimental indications in favor of the existence of this phenomenon. The strongest evidence is believed to come from the atmospheric neutrino anomaly observed by the Super-Kamiokande experiment [2], following the previous measurements of the Kamiokande, IMB and Soudan atmospheric neutrino experiments [3]. The atmospheric neutrino anomaly consists in a suppression of the ratio of measured atmospheric $\nu_\mu$ and $\nu_e$ with respect to the calculated one (see, for example, Ref. [4]). Furthermore, the new data of the Super-Kamiokande experiment [2] indicate a variation of the atmospheric $\nu_\mu$ flux as a function of the zenith angle, which correspond to a variation as a function
of the distance between the region in the atmosphere where the neutrino is produced and the Super-Kamiokande detector where the neutrino is detected. This is a strong evidence in favor of neutrino oscillations and seems to be supported by the measurement of the flux of neutrino-induced upward-going muons of the MACRO experiment [5].

The oldest indication in favor of neutrino oscillations comes from the solar neutrino problem (see, for example, Ref. [6]), i.e. the suppression of the flux of solar $\nu_e$'s observed in five solar neutrino experiments (Homestake, Kamiokande, GALLEX, SAGE and Super-Kamiokande [7]) with respect to the calculated one [8].

Finally, the LSND accelerator experiment has reported evidence in favor of $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\nu_\mu \rightarrow \nu_e$ oscillations [9].

Neutrino oscillations can occur only if neutrinos are massive particles, if their masses are different and if neutrino mixing is realized in nature. In this case, the left-handed flavor neutrino fields $\nu_\alpha L$ ($\alpha = e, \mu, \tau$) are superpositions of the left-handed components $\nu_{kL}$ ($k = 1, \ldots, n$) of the fields of neutrinos with definite masses $m_k$: $\nu_\alpha L = \sum_{k=1}^n U_{\alpha k} \nu_{kL}$, where $U$ is a unitary mixing matrix.

The general expression for the probability of $\nu_\alpha \rightarrow \nu_\beta$ transitions in vacuum is

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \left| \sum_{k=1}^n U_{\beta k} \exp \left( -i \frac{\Delta m_{kj}^2 L}{2p} \right) U_{\alpha k}^* \right|^2,$$

where $\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$, $L$ is the distance between the neutrino source and detector and $p$ is the neutrino momentum.

\section{Four neutrino mixing and oscillations}

The three experimental indications in favor of neutrino oscillations require the existence of three different scales of neutrino mass-squared differences: $\Delta m_{\text{sun}}^2 \sim 10^{-5} \text{eV}^2$ (MSW) or $\Delta m_{\text{sun}}^2 \sim 10^{-10} \text{eV}^2$ (vacuum oscillations), $\Delta m_{\text{atm}}^2 \sim 5 \times 10^{-3} \text{eV}^2$ and $\Delta m_{\text{SBL}}^2 \sim 1 \text{eV}^2$, where $\Delta m_{\text{SBL}}^2$ is the neutrino mass-squared difference relevant for short-baseline (SBL) experiments, whose allowed range is determined by the positive result of the LSND experiment.

Three independent mass-squared differences require at least four massive neutrinos. The number of active light flavor neutrinos is known to be three from the measurement of the invisible width of the $Z$-boson, but there is no experimental upper bound for the number of massive neutrinos (the lower bound is three). In the following we consider the simplest possibility of existence of four massive neutrinos [10, 11, 12, 13, 14, 15, 16]. In this case, in the flavor basis, besides the three light flavor neutrinos $\nu_e, \nu_\mu, \nu_\tau$ that contribute to the invisible
width of the $Z$-boson, there is a light sterile neutrino $\nu_s$ that is a SU(2)$_L$ singlet and does not take part in standard weak interactions.

Two years ago we have shown [13] that among all the possible four-neutrino mass spectra only two are compatible with the results of all neutrino oscillation experiments:

\[
\begin{align*}
(A) & \quad m_1 < m_2 \ll m_3 < m_4, \\
(B) & \quad m_1 < m_2 \ll m_3 < m_4.
\end{align*}
\]  

(2)

In these two schemes the four neutrino masses are divided in two pairs of close masses separated by a gap of about 1 eV, which provides the mass-squared difference $\Delta m^2_{\text{SBL}} = \Delta m^2_{21} \equiv m_1^2 - m_2^2$ that is relevant for the oscillations observed in the LSND experiment. In scheme A $\Delta m^2_{\text{atm}} = \Delta m^2_{21} \equiv m_1^2 - m_2^2$ is relevant for the explanation of the atmospheric neutrino anomaly and $\Delta m^2_{\text{SBL}} = \Delta m^2_{43} \equiv m_4^2 - m_3^2$ is relevant for the suppression of solar $\nu_e$'s, whereas in scheme B $\Delta m^2_{\text{atm}} = \Delta m^2_{43}$ and $\Delta m^2_{\text{SBL}} = \Delta m^2_{21}$.

Let us define the quantities $c_\alpha$, with $\alpha = e, \mu, \tau, s$, in the schemes A and B as

\[
\begin{align*}
(A) & \quad c_\alpha \equiv \sum_{k=1,2} |U_{\alpha k}|^2, \\
(B) & \quad c_\alpha \equiv \sum_{k=3,4} |U_{\alpha k}|^2.
\end{align*}
\]  

(3)

Physically $c_\alpha$ quantify the mixing of the flavor neutrino $\nu_\alpha$ with the two massive neutrinos whose $\Delta m^2$ is relevant for the oscillations of atmospheric neutrinos ($\nu_1, \nu_2$ in scheme A and $\nu_3, \nu_4$ in scheme B).

The probability of $\nu_\alpha \to \nu_\beta$ transitions ($\beta \neq \alpha$) and the survival probability of $\nu_\alpha$ in SBL experiments are given by [12]

\[
\begin{align*}
P_{\nu_\alpha \to \nu_\beta} & = A_{\alpha;\beta} \sin^2 \frac{\Delta m^2_{\text{SBL}} L}{4p}, \\
P_{\nu_\alpha \to \nu_\alpha} & = 1 - B_{\alpha;\alpha} \sin^2 \frac{\Delta m^2_{\text{SBL}} L}{4p},
\end{align*}
\]  

(4)

with the oscillation amplitudes

\[
\begin{align*}
A_{\alpha;\beta} & = 4 \left| \sum_k U_{\beta k} U_{\alpha k}^* \right|^2, \\
B_{\alpha;\alpha} & = 4c_\alpha (1 - c_\alpha),
\end{align*}
\]  

(5)

where the index $k$ runs over the values 1, 2 or 3, 4. The probabilities (4) have the same form as the corresponding probabilities in the case of two-neutrino mixing, $P_{\nu_\alpha \to \nu_\beta} = \sin^2(2\theta) \sin^2(\Delta m^2 L/4p)$ and $P_{\nu_\alpha \to \nu_\alpha} = 1 - \sin^2(2\theta) \sin^2(\Delta m^2 L/4p)$, which have been used by all experimental collaborations for the analysis of the data in order to get information on the parameters $\sin^2(2\theta)$ and $\Delta m^2$ ($\theta$ and $\Delta m^2$ are, respectively, the mixing angle and the mass-squared difference in the case of
two-neutrino mixing). Therefore, we can use the results of their analyses in order
to get information on the corresponding parameters $A_{\alpha;\beta}$, $B_{\alpha;\alpha}$ and $\Delta m_{SBL}^2$.

The results of all neutrino oscillation experiments are compatible with the
schemes A and B only if \[ c_e \leq a_0^e \quad \text{and} \quad c_\mu \geq 1 - a_0^\mu, \]
where \[ a_0^\alpha \equiv \frac{1}{2} \left(1 - \sqrt{1 - B_{\alpha;\alpha}^0}\right) \quad (\alpha = e, \mu) \]
and $B_{\alpha;\alpha}^0$ is the upper bound for the amplitude of \((^{(-)}\nu_\alpha \rightarrow ^{(-)}\nu_\alpha)^\ast\) oscillations obtained
from the exclusion plots of SBL reactor and accelerator disappearance experiments. Hence, the quantities $a_0^e$ and $a_0^\mu$ depend on $\Delta m^2$. The exclusion curves
obtained in the Bugey reactor experiment \[17\] and in the CDHS and CCFR ac-
ccelerator experiments \[18\] imply that both $a_0^e$ and $a_0^\mu$ are small \[19\]: \[ a_0^e \lesssim 4 \times 10^{-2} \quad \text{and} \quad a_0^\mu \lesssim 2 \times 10^{-1} \]
for any value of $\Delta m^2$ in the range $0.3 \lesssim \Delta m^2 \lesssim 10^3 \text{eV}^2$.

The smallness of $c_e$ in both schemes A and B is a consequence of the
solar neutrino problem \[13\]. It implies that the electron neutrino has a small
mixing with the neutrinos whose mass-squared difference is responsible for the
oscillations of atmospheric neutrinos ($\nu_1, \nu_2$ in scheme A and $\nu_3, \nu_4$ in scheme B).
Therefore, the transition probability of electron neutrinos and antineutrinos into
other states in atmospheric and long-baseline (LBL) experiments is suppressed.
Indeed, it can be shown \[14\] that the transition probabilities of electron neutrinos
and antineutrinos into all other states and the probability of \((^{(-)}\nu_\mu \rightarrow ^{(-)}\nu_e)^\ast\) transitions
in vacuum are bounded by
\[ 1 - P_{\nu_e \rightarrow \nu_e}^{(}\text{LBL}) \leq a_0^e \left(2 - a_0^e\right) \]
\[ P_{\nu_\mu \rightarrow \nu_e}^{(\text{LBL})} \leq \min\left(a_0^e \left(2 - a_0^e\right), a_0^e + \frac{1}{4} A_{\mu;e}^0\right) \]
where $A_{\mu;e}^0$ is the upper bound for the amplitude of \((^{(-)}\nu_\mu \rightarrow ^{(-)}\nu_e)^\ast\) transitions measured
in SBL experiments with accelerator neutrinos.

The upper bound \[8\] obtained from the 90\% CL exclusion plot of the
Bugey experiment \[17\] is shown in Fig. \[1\] (solid line). The shadowed region in
Fig. \[1\] corresponds to the range of $\Delta m^2$ allowed at 90\% CL by the results of the
LSND and all the other SBL experiments. The upper bound for the transition probability of $\bar{\nu}_e$’s into other states obtained from the recently published exclusion
plot of the CHOOZ \[20\] experiment is shown in Fig. \[1\] (dash-dotted line). One
can see that the CHOOZ upper bound agrees with that obtained from Eq.\[8\]. In
Fig. \[1\] we have also drawn the curve corresponding to the expected final sensitivity
of the CHOOZ experiment (dash-dot-dotted line). One can see that the LSND-allowed region imply an upper bound for $1 - P_{\nu_e \rightarrow \nu_e}^{(LBL)}$ of about $5 \times 10^{-2}$, which is smaller than the expected final sensitivity of the CHOOZ experiment.

The bound (9) obtained from the 90% CL exclusion plots of the Bugey experiment and of the BNL E734, BNL E776 and CCFR experiments [21] is depicted by the dashed line in Fig. 2. The solid line in Fig. 2 shows the upper bound on $P_{\nu_{\mu} \rightarrow \nu_e}^{(LBL)}$ taking into account matter effects. The expected sensitivities of the K2K long-baseline accelerator neutrino experiment [22] is indicated in Fig. 2 by the dash-dotted vertical line. The shadowed region in Fig. 2 corresponds to the range of $\Delta m^2$ allowed at 90% CL by the results of the LSND and all the other SBL experiments. It can be seen that the results of SBL experiments indicate an upper bound for $P_{\nu_{\mu} \rightarrow \nu_e}^{(LBL)}$ smaller than $4 \times 10^{-2}$ and smaller than the expected sensitivity of the K2K experiment.

The two schemes A and B have identical implications for neutrino oscillation experiments, but very different implications for neutrinoless double-$\beta$ decay experiments and for tritium $\beta$-decay experiments. Indeed, in scheme A

$$|\langle m \rangle| \leq m_4 \quad \text{and} \quad m(^3\text{H}) \simeq m_4,$$

(10)

whereas in scheme B

$$|\langle m \rangle| \leq a^0_\alpha m_4 \ll m_4 \quad \text{and} \quad m(^3\text{H}) \ll m_4,$$

(11)
where $\langle m \rangle = \sum_{i=1}^{4} U_{ei}^2 m_i$ is the effective Majorana mass that determines the matrix element of neutrinoless double-$\beta$ decay and $m(^{3}\text{H})$ is the neutrino mass measured in tritium $\beta$-decay experiments. Therefore, in scheme B $|\langle m \rangle|$ and $m(^{3}\text{H})$ are smaller than the expected sensitivity of the next generation of neutrinoless double-$\beta$ decay and tritium $\beta$-decay experiments. The observation of a positive signal in these experiments would be an indication in favor of scheme A.

Summarizing, the results of neutrino oscillation experiments indicate that only the two four-neutrino schemes (2) are allowed and the electron neutrino has a very small mixing with the two massive neutrinos that are responsible for the oscillations of atmospheric neutrinos ($\nu_1, \nu_2$ in scheme A and $\nu_3, \nu_4$ in scheme B). Hence, the two schemes have the form shown in Fig. 3, where $\nu_e$ is associated with the two massive neutrinos neutrinos that are responsible for the oscillations of solar neutrinos ($\nu_3, \nu_4$ in scheme A and $\nu_1, \nu_2$ in scheme B), with which it has a large mixing, whereas $\nu_\mu$ is associated with the two massive neutrinos neutrinos that are responsible for the oscillations of atmospheric neutrinos, with which the muon neutrino has a large mixing. The results of neutrino oscillation experiments do not provide yet an indication of where $\nu_\tau$ and $\nu_s$ have to be placed in the two schemes represented in Fig. 3. In the following Section we will show that the standard Big-Bang Nucleosynthesis constraint on the number of light neutrinos provide a stringent limit on the mixing of the sterile neutrino with the two massive neutrinos that are responsible for the oscillations of atmospheric neutrinos \cite{11, 16}.

3. BBN constraints on 4-neutrino mixing

It is well known that the observed abundance of primordial light elements is predicted with an impressive degree of accuracy by the standard model of Big-Bang Nucleosynthesis if the number $N_\nu$ of light neutrinos (with mass much smaller than 1 MeV) in equilibrium at the neutrino decoupling temperature ($T_{\text{dec}} \simeq 2$ MeV...
for $\nu_e$ and $T_{\text{dec}} \approx 4 \text{ MeV}$ for $\nu_\mu, \nu_\tau$) is not far from three [23, 24].

The value of $N_\nu$ is especially crucial for the primordial abundance of $^4\text{He}$. This is due to the fact that $N_\nu$ determines the freeze-out temperature of the weak interaction processes $e^+ + n \rightleftharpoons p + \bar{\nu}_e$, $\nu_e + n \rightleftharpoons p + e^-$ and $n \rightleftharpoons p + e^- + \bar{\nu}_e$ that maintain protons and neutrons in equilibrium, i.e. the temperature at which the rate $\Gamma_W(T) \approx G_F T^5$ ($G_F$ is the Fermi constant) of these weak interaction processes becomes smaller than the expansion rate of the universe

$$\frac{\dot{R}(T)}{R(T)} = \sqrt{\frac{8\pi^3}{90} g_* T^2 / M_P}, \quad (12)$$

($M_P$ is the Planck mass), where $R(T)$ is the cosmic scale factor and $g_* = 5.5 + 1.75 N_\nu$ for $m_e \lesssim T \lesssim m_\mu$. If $N_\nu = 3$ the primordial mass fraction of $^4\text{He}$, $Y_P \equiv \text{mass density of }^4\text{He} / \text{total mass density}$, is $Y_P \approx 0.24$, which agrees very well with the observed value $Y_P^{\text{obs}} = 0.238 \pm 0.002$. Since $Y_P$ is very sensitive to variations of $N_\nu$, it is clear that the observed value of $Y_P$ puts stringent constraints on the possible deviation of $N_\nu$ from the Standard Model value $N_\nu = 3$.

In the four-neutrino schemes (2) standard BBN gives a constrain on neutrino mixing if the upper bound for $N_\nu$ is less than four. In this case the mixing of the sterile neutrino is severely constrained because otherwise neutrino oscillations would bring the sterile neutrino in equilibrium before neutrino decoupling [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 11], leading to $N_\nu = 4$. In particular, we will show that standard BBN with $N_\nu < 4$ implies that $c_s$ is extremely small.

The amount of sterile neutrinos present at nucleosynthesis can be calculated using the differential equation [23]

$$\frac{dn_\nu_s}{dT} = -\frac{1}{2HT} \sum_{\alpha=e,\mu,\tau} \langle P_{\nu_\alpha \rightarrow \nu_s} \rangle_{\text{coll}} \Gamma_{\nu_\alpha} \left( 1 - n_{\nu_s} \right), \quad (13)$$

where $n_{\nu_s}$ is the number density of the sterile neutrino relative to the number density of an active neutrino in equilibrium and $\Gamma_{\nu_\alpha}$ are the collision rates of the active neutrinos, including elastic and inelastic scattering [24], $\Gamma_{\nu_e} = 4.0 G_F^2 T^5$ and $\Gamma_{\nu_\mu} = \Gamma_{\nu_\tau} = 0.7 \Gamma_{\nu_e}$. The quantities $\langle P_{\nu_\alpha \rightarrow \nu_s} \rangle_{\text{coll}}$ are the probabilities of $\nu_\alpha \rightarrow \nu_s$ transitions averaged over the collision time $t_{\text{coll}} = 1 / \Gamma_{\nu_\alpha}$. Hence, also $n_{\nu_s}$ has to be considered as a quantity averaged over the collision time.

Equation (13) describes non-resonant and adiabatic resonant neutrino transitions if $t_{\text{osc}} \ll t_{\text{coll}} \ll t_{\text{exp}}$. The condition $t_{\text{osc}} \ll t_{\text{coll}}$ means that neutrino oscillations have to be fast relative to the collision time. The characteristic expansion time of the universe $t_{\text{exp}}$ is given by $t_{\text{exp}} = 1 / H$ where $H$ is the Hubble parameter $H \equiv \dot{R} / R$, which is related to the temperature $T$ by $H = -\dot{T} / T \approx 0.7 \left( T / 1 \text{ MeV} \right)^2 \text{s}^{-1}$ (this value can be obtained from Eq.(12) with $m_e \lesssim T \lesssim m_\mu$.
and \( N_\nu \simeq 3 \). The relation \( \Gamma_\nu / H \simeq 1.2 (T/1 \text{ MeV})^3 \) shows that for temperatures larger than 2 MeV the collision time is always much smaller than the expansion time \([25]\).

Since by definition \( N_\nu \) is the effective number of massless neutrino species at \( T_{\text{dec}} \), in order to get a constraint on the mixing of sterile neutrinos we need to calculate the value of \( n_{\nu_s} \) at \( T_{\text{dec}} \) produced by neutrino oscillations. With the initial condition \( n_{\nu_s}(T_i) = 0 \) \((T_i \sim 100 \text{ MeV})\), the integration of Eq. (13) gives \([30]\) \( n_{\nu_s}(T_{\text{dec}}) = 1 - e^{-F} \) with

\[
F = \int_{T_{\text{dec}}}^{T_i} \frac{1}{2HT} \sum_{\alpha=e,\mu,\tau} \langle P_{\nu_{\alpha} \rightarrow \nu_{\nu_{s}}} \rangle_{\text{coll}} \Gamma_{\nu_{\alpha}} dT. \tag{14}
\]

Imposing the upper bound \( n_{\nu_s}(T_{\text{dec}}) \leq \delta N \equiv N_\nu - 3 \) one obtains the condition \( F \leq |\ln(1 - \delta N)| \).

For the calculation of \( F \) the averaged transition probabilities \( \langle P_{\nu_{\alpha} \rightarrow \nu_{\nu_{s}}} \rangle_{\text{coll}} \) must be evaluated and the effective potentials of neutrinos and antineutrinos due to coherent forward scattering in the primordial plasma \([32]\)

\[
V_e = -6.02 G_F p \frac{T^4}{M_W^2} \equiv V, \quad V_{\mu,\tau} = \xi V \quad \text{and} \quad V_s = 0, \tag{15}
\]

must be taken into account (in the absence of a lepton asymmetry the effective potentials of neutrinos and antineutrinos are equal). Here \( p \) is the neutrino momentum, which we approximate with its temperature average \( \langle p \rangle \simeq 3.15 T \), \( G_F \) is the Fermi constant, \( M_W \) is the mass of the \( W \) boson and \( \xi = \cos^2 \theta_W / (2 + \cos^2 \theta_W) \simeq 0.28 \), where \( \theta_W \) is the weak mixing angle. The propagation of neutrinos and antineutrinos is governed by the effective hamiltonian in the weak basis

\[
H_W = p + \frac{1}{2p} U \begin{bmatrix} m_1^2, m_2^2, m_3^2, m_4^2 \end{bmatrix} U^\dagger + \begin{bmatrix} V, \xi V, \xi V, 0 \end{bmatrix}. \tag{16}
\]
It is convenient to subtract from $H_W$ the constant term $p + m_1/2p + \xi V$, which does not affect the relative evolution of the neutrino flavor states, in order to get

$$H'_W = \frac{1}{2p} U \text{diag} \left[ 0, \Delta m^2_{21}, \Delta m^2_{31}, \Delta m^2_{41} \right] U^\dagger + \text{diag} \left[ (1 - \xi) V, 0, 0, -\xi V \right]. \quad (17)$$

From this expression it is clear that the relative evolution of the flavor neutrino states depends on the three mass-squared differences and not on the absolute scale of the neutrino masses. The effective hamiltonian in the mass basis is given by $H'_M = U^\dagger H'_W U$:

$$H'_M = \frac{1}{2p} \text{diag} \left[ 0, \Delta m^2_{21}, \Delta m^2_{31}, \Delta m^2_{41} \right] + U^\dagger \text{diag} \left[ (1 - \xi) V, 0, 0, -\xi V \right] U. \quad (18)$$

In the mass basis the mixing has been transferred from the mass term to the potential term. In order to calculate the evolution of the neutrino flavors it is necessary to parameterize the $4 \times 4$ neutrino mixing matrix $U$. However, since the second and third rows and the second and third columns of the diagonal potential matrix in Eq.\,(18) are equal to zero, it is clear that the values of the second and third rows of $U$, corresponding to $\nu_\mu$ and $\nu_\tau$, are irrelevant and do not need to be parameterized. Furthermore, since $c_e$ is small in both schemes A and B, it does not have any effect on neutrino oscillations before BBN and the approximation $c_e = 0$ is allowed. Hence, the $4 \times 4$ neutrino mixing matrix can be partially parameterized as

$$U = \begin{pmatrix} 0 & 0 & \cos \theta & \sin \theta \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \sin \varphi \sin \chi & -\sin \varphi \cos \chi & -\cos \varphi \sin \theta & \cos \varphi \cos \theta \end{pmatrix} \quad (19)$$

in scheme A and

$$U = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ -\cos \varphi \sin \theta & \cos \varphi \cos \theta & \sin \varphi \sin \chi & -\sin \varphi \cos \chi \end{pmatrix} \quad (20)$$

in scheme B, with $0 \leq \varphi \leq \pi/2$. Notice that $c_s = \sin^2 \varphi$ in both schemes A and B. The dots in Eqs.\,(19) and (20) indicate the elements of the mixing matrix belonging to the $\nu_\mu$ and $\nu_\tau$ rows ($U_{\mu i}$ and $U_{\tau i}$ with $i = 1, \ldots, 4$), which do not need to be parameterized. In Eqs.\,(19) and (20) we have parameterized only the elements of the mixing matrix belonging to the $\nu_e$ and $\nu_s$ lines ($U_{ei}$ and $U_{si}$ with $i = 1, \ldots, 4$) in terms of the three mixing angles $\theta, \chi, \varphi$. It is clear that this partial
parameterization of the mixing matrix (with the approximation $U_{e1} = U_{e2} = 0$) is much easier to manipulate than a complete parameterization, which would require the introduction of 6 mixing angles and 3 complex phases.

Notice that no complex phase is needed for the partial parameterization of the mixing matrix in Eqs. (19) and (20), because the elements $U_{ei}$ and $U_{si}$ with $i = 1, \ldots, 4$ can be chosen real. Indeed, if we consider, for example, the scheme A, the line $U_{si}$ with $i = 1, \ldots, 4$ and the element $U_{e4}$ can be chosen real because all observable transition probabilities are invariant under the phase transformation $U_{\alpha j} \rightarrow e^{ix_{\alpha}} U_{\alpha j} e^{iy_{j}}$, where $x_{\alpha}$ and $y_{j}$ are arbitrary parameters. In order to show that also $U_{e3}$ can be chosen real, we multiply the unitarity relation $\sum_{k=1}^{4} U_{ek} U_{sk}^* = 0$ with $U_{e3}^* U_{s3}$. The imaginary part of the resulting relation gives

$$\text{Im}[U_{e3} U_{s3}^* U_{e4} U_{s4}] = \text{Im}[U_{e1} U_{s1}^* U_{e3} U_{s3}] + \text{Im}[U_{e2} U_{s2}^* U_{e3}^* U_{s3}].$$  

(21)

In the approximation $U_{e1} = U_{e2} = 0$ the right-hand part of Eq. (21) vanishes. Therefore, since $U_{s3}, U_{e4}, U_{s4}$ have been chosen to be real, also $U_{e3}$ must be real.

Since the mass-squared differences have a hierarchical structure, $\Delta m_{43}^2 \ll \Delta m_{21}^2 \ll \Delta m_{41}^2$ in scheme A and $\Delta m_{21}^2 \ll \Delta m_{43}^2 \ll \Delta m_{41}^2$ in scheme B, the effective hamiltonian $H'_{M}$ can be diagonalized approximately taking into account only one of the three $\Delta m^2$’s for different ranges of the temperature $T$. Then, it can be shown that [11, 16] the condition $F \leq |\ln(1 - \delta N)|$ gives the bound

$$920 \left( \frac{\Delta m_{SBL}^2}{1 \text{ eV}^2} \right)^{1/2} d_s \sqrt{1 - d_s} + 33 \left( \frac{\Delta m_{atm}^2}{10^{-2} \text{ eV}^2} \right)^{1/2} \frac{\sin^2 2\chi}{\sqrt{1 + \cos 2\chi}} c_s^{3/2} \leq |\ln(1 - \delta N)|,$$

(22)

with $d_s \equiv c_s$ in scheme A and $d_s \equiv 1 - c_s$ in scheme B.

Both terms in the left-hand side of Eq. (22) are positive and must be small if $\delta N < 1$. The SBL term, depending on $\Delta m_{SBL}^2$, is small if either $c_s$ is small...
or large, but the atmospheric term, which depends on \( \Delta m^2_{\text{atm}} \), is small only if \( c_s \) is small. Indeed, if \( c_s \) is close to one we have \((U_{\mu 1}, U_{\mu 2}) \sim (\cos \chi, \sin \chi)\) in scheme A and \((U_{\mu 3}, U_{\mu 4}) \sim (\cos \chi, \sin \chi)\) in scheme B. This means that, in order to accommodate the atmospheric neutrino anomaly, \( \sin^2 2 \chi \) cannot be small. This is in contradiction with the inequality (22) and we conclude that the bound (22) implies that \( c_s \) is small.

Since \( c_s \) is small only non-resonant transitions of active into sterile neutrinos due to \( \Delta m^2_{\text{SBL}} \) are possible in scheme A, as illustrated in Fig. 4 where we have plotted the effective squared masses (obtained from a numerical diagonalization of the Hamiltonian (16)) as functions of \( T_6 \left( \nu_e \right) \) does not have resonant transitions into \( \nu_\mu \) or \( \nu_\tau \) because we have chosen \( c_e = 0 \). Hence, the conditions for the validity of Eq. (13) are satisfied and the SBL term in Eq. (22) gives the bound

\[
c_s \leq 1.1 \times 10^{-3} \left( \frac{\Delta m^2_{\text{SBL}}}{1 \text{ eV}^2} \right)^{-1/2} |\ln(1 - \delta N)|. \tag{23}
\]

On the other hand, since \( c_s \) is small, a resonance occurs in scheme B at the temperature \( T_{\text{res}} = 16(\Delta m^2_{\text{SBL}}/1 \text{ eV}^2)^{1/6}|1 - 2c_s|^{1/6} \text{ MeV}, \) as illustrated in Fig. 5. The condition \( \delta N < 1 \) implies that this resonance must not be passed adiabatically. In this case the conditions for the validity of Eq. (13) are not fulfilled and the SBL term of Eq. (22) does not apply. Using an appropriate formula [27] for the calculation of the amount of sterile neutrinos produced at the resonance through non-adiabatic transitions one can show [16] that the BBN bound on \( c_s \) in scheme B is given by

\[
c_s \leq 1.1 \times 10^{-5} \left( \frac{\Delta m^2_{\text{SBL}}}{1 \text{ eV}^2} \right)^{-1/2} |\ln(1 - \delta N)|. \tag{24}
\]

Figure 6 shows the values of the bounds (23) and (24) obtained from the LSND [9] lower bound \( \Delta m^2_{\text{SBL}} \gtrsim 0.27 \text{ eV}^2 \) for \( 0.2 \leq \delta N < 1 \). One can see that standard BBN implies that \( c_s \) is extremely small.
4. Conclusions

The BBN constraint on $c_s$ derived in the previous Section implies that $\nu_s$ is mainly mixed with the two massive neutrinos that contribute to solar neutrino oscillations ($\nu_3$ and $\nu_4$ in scheme A and $\nu_1$ and $\nu_2$ in scheme B) and the unitarity of the mixing matrix implies that $\nu_\tau$ is mainly mixed with the two massive neutrinos that contribute to the oscillations of atmospheric neutrinos. Adding this information to the two schemes of neutrino mixing depicted in Fig. 3 we obtain the schemes shown in Fig. 8.

The two neutrino mixing schemes shown in Fig. 8 have the following testable implications for solar, atmospheric, long-baseline and short-baseline neutrino oscillation experiments:

- The solar neutrino problem is due to $\nu_e \to \nu_s$ oscillations. This prediction will be checked by future solar neutrino experiments that can measure the ratio of neutral-current and charged-current events [33].

- The atmospheric neutrino anomaly is due to $\nu_\mu \to \nu_\tau$ oscillations. This prediction will be checked by LBL experiments. Furthermore, the absence of $\nu_\mu \to \nu_s$ atmospheric neutrino oscillations may be checked in the future in the Super-Kamiokande atmospheric neutrino experiment [34].

- $\nu_\mu \to \nu_\tau$ and $\nu_\tau \to \nu_s$ oscillations are strongly suppressed in SBL experiments. With the approximation $c_s \simeq 0$, for the amplitude of $\nu_\mu \to \nu_\tau$ oscillations we have the upper bound $A_{\mu\tau} \leq (a_\tau^0)^2$, that is shown in Fig. 7 (solid curve) together with a recent exclusion curve obtained in the CHORUS experiment (dash-dotted curve) and the final sensitivity of the CHORUS and NOMAD experiments (dash-dot-dotted curve) [35].

If these prediction will be falsified by future experiments it could mean that some of the indications given by the results of neutrino oscillations experiments are wrong and neither of the two four neutrino schemes A and B is realized in nature, or that Big-Bang Nucleosynthesis occurs with a non-standard mechanism [36].

In conclusion, we would like to emphasize that if the analysis presented here is correct and one of the two four neutrino schemes depicted in Fig. 8 is realized in nature, at the zeroth-order in the expansion over the small quantities $c_e$ and $c_s$ the $4 \times 4$ neutrino mixing matrix has an extremely simple structure in
which the $\nu_e, \nu_s$ and $\nu_\mu, \nu_\tau$ sectors are decoupled: in scheme A

$$U \simeq \begin{pmatrix}
0 & 0 & \cos \theta & \sin \theta \\
\cos \gamma & \sin \gamma & 0 & 0 \\
-\sin \gamma & \cos \gamma & 0 & 0 \\
0 & 0 & -\sin \theta \cos \theta & 0
\end{pmatrix}$$

(25)

and in scheme B

$$U \simeq \begin{pmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & \cos \gamma & \sin \gamma \\
0 & 0 & -\sin \gamma \cos \gamma & 0 \\
-\sin \theta \cos \theta & 0 & 0 & 0
\end{pmatrix},$$

(26)

where $\theta$ and $\gamma$ are, respectively, the two-generation mixing angles relevant in solar and atmospheric neutrino oscillations. Therefore, the oscillations of solar and atmospheric neutrinos are independent and the two-generation analyses of solar and atmospheric neutrino oscillations yield correct information on the mixing of four-neutrinos.

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References

[1] B. Pontecorvo, J. Exptl. Theoret. Phys. 33, 549 (1957) [Sov. Phys. JETP 6, 429 (1958)]; J. Exptl. Theoret. Phys. 34, 247 (1958) [Sov. Phys. JETP 7, 172 (1958)].

[2] T. Kajita, Talk presented at Neutrino ’98 [37]; Y. Fukuda et al., hep-ex/9807003.

[3] Y. Fukuda et al., Phys. Lett. B 335, 237 (1994); R. Becker-Szendy et al., Nucl. Phys. B (Proc. Suppl.) 38, 331 (1995); W.W.M. Allison et al., Phys. Lett. B 391, 491 (1997).

[4] T. Gaisser, Proc. of Neutrino ’96, Helsinki, June 1996, edited by K. Enqvist et al., p. 211 (World Scientific, Singapore, 1997); Talk presented at Neutrino ’98 [37].

[5] F. Ronga, Talk presented at Neutrino ’98 [37]; The MACRO Collaboration, M. Ambrosio et al., preprint hep-ex/9807003.

[6] J.N. Bahcall, P.I. Krastev and A.Yu. Smirnov, preprint hep-ph/9807216.
[7] B.T. Cleveland et al., Nucl. Phys. B (Proc. Suppl.) 38, 47 (1995); K.S. Hirata et al., Phys. Rev. D 44, 2241 (1991); GALLEX Coll., W. Hampel et al., Phys. Lett. B 388, 384 (1996); J.N. Abdurashitov et al., Phys. Rev. Lett. 77, 4708 (1996); Y. Suzuki, Talk presented at Neutrino ’98 [37].

[8] J.N. Bahcall, S. Basu and M.H. Pinsonneault, preprint astro-ph/9805135.

[9] C. Athanassopoulos et al., Phys. Rev. Lett. 77, 3082 (1996).

[10] J.T. Peltoniemi and J.W.F. Valle, Nucl. Phys. B 406, 409 (1993); D.O. Caldwell and R.N. Mohapatra, Phys. Rev. D 48, 3259 (1993); Z. Berezhiani and R.N. Mohapatra, ibid 52, 6607 (1995); E. Ma and P. Roy, ibid 52, R4780 (1995); R. Foot and R.R. Volkas, ibid 52, 6595 (1995); S. Goswami, ibid 55, 2931 (1997); A.Yu. Smirnov and M. Tanimoto, ibid 55, 1665 (1997); J.R. Primack et al., Phys. Rev. Lett. 74, 2160 (1995); K. Benakli and A.Yu. Smirnov, ibid 79, 4314 (1997); E.J. Chun et al., Phys. Lett. B 357, 608 (1995); J.J. Gomez-Cadenas and M.C. Gonzalez-Garcia, Z. Phys. C 71, 443 (1996); E. Ma, Mod. Phys. Lett. A 11, 1893 (1996); Q.Y. Liu and A.Yu. Smirnov, hep-ph/9712493; V. Barger, T.J. Weiler and K. Whisnant, hep-ph/9712495; E.J. Chun, C.W. Kim and U.W. Lee, hep-ph/9802209; S.C. Gibbons et al., hep-ph/9803299; N. Gaur et al., hep-ph/9806272; V. Barger et al., hep-ph/9806328.

[11] N. Okada and O. Yasuda, Int. J. Mod. Phys. A 12, 3669 (1997).

[12] S.M. Bilenky, C. Giunti, C.W. Kim and S.T. Petcov, Phys. Rev. D 54, 4432 (1996).

[13] S.M. Bilenky, C. Giunti and W. Grimus, Proc. of Neutrino ’96, Helsinki, June 1996, edited by K. Enqvist et al., p. 174 (World Scientific, Singapore, 1997); Eur. Phys. J. C 1, 247 (1998).

[14] S.M. Bilenky, C. Giunti and W. Grimus, Phys. Rev. D 57 (1998) 1920.

[15] S.M. Bilenky, C. Giunti and W. Grimus, Phys. Rev. D 58, 033001 (1998).

[16] S.M. Bilenky, C. Giunti, W. Grimus and T. Schwetz, hep-ph/9804421.

[17] B. Achkar et al., Nucl. Phys. B 434, 503 (1995).

[18] F. Dydak et al., Phys. Lett. B 134, 281 (1984); I.E. Stockdale et al., Phys. Rev. Lett. 52, 1384 (1984).

[19] See Fig.1 of: S.M. Bilenky, A. Bottino, C. Giunti and C.W. Kim, Phys. Rev. D 54, 1881 (1996).
[20] M. Apollonio et al., Phys. Lett. B 420, 397 (1998); C. Bemporad, Talk presented at the Ringberg Euroconference New Trends in Neutrino Physics, 24–29 May 1998, Ringberg Castle, Tegernsee, Germany.

[21] L.A. Ahrens et al., Phys. Rev. D 36, 702 (1987); L. Borodovsky et al., Phys. Rev. Lett. 68, 274 (1992); A. Romosan et al., Phys. Rev. Lett. 78, 2912 (1997).

[22] Y. Oyama (for the K2K collaboration), hep-ex/9803014.

[23] D.N. Schramm and M.S. Turner, Rev. Mod. Phys. 70, 303 (1998).

[24] K.A. Olive, astro-ph/9707212, astro-ph/9712160.

[25] K. Kainulainen, Phys. Lett. B 224, 191 (1990).

[26] R. Barbieri and A. Dolgov, Phys. Lett. B 237, 440 (1990); Nucl. Phys. B 349, 743 (1991).

[27] K. Enqvist, K. Kainulainen and J. Maalampi, Phys. Lett B 249, 531 (1990).

[28] K. Enqvist, K. Kainulainen and J. Maalampi, Nucl. Phys. B 349, 754 (1991).

[29] K. Enqvist, K. Kainulainen and M. Thomson, Nucl. Phys. B 373, 498 (1992).

[30] J.M. Cline, Phys. Rev. Lett. 68, 3137 (1992).

[31] X. Shi, D. N. Schramm and B. D. Fields, Phys. Rev. D 48, 2563 (1993).

[32] D. Nötzold and G. Raffelt, Nucl. Phys. B 307, 924 (1988).

[33] S.M. Bilenky and C. Giunti, Phys. Lett. B 320, 323 (1994); Astrop. Phys. 2, 353 (1994); Z. Phys. C 68, 495 (1995).

[34] Y. Suzuki, Proc. of Neutrino ’96, Helsinki, June 1996, edited by K. Enqvist et al., p. 237 (World Scientific, Singapore, 1997); F. Vissani and A.Yu. Smirnov, Phys. Lett. B 432, 376 (1998); J.G. Learned, S. Pakvasa and J.L. Stone, hep-ph/9805343; F. Vissani, Talk presented at the Ringberg Euroconference New Trends in Neutrino Physics, 24–29 May 1998, Ringberg Castle, Tegernsee, Germany.

[35] E. Eskut et al., CERN-EP/98-073.

[36] R. Foot, M.J. Thomson and R.R. Volkas, Phys. Rev. D 53, 5349 (1996); X. Shi, ibid. 54, 2753 (1996); R. Foot and R.R. Volkas, ibid. 55, 5147 (1997).

[37] Neutrino ’98 Takayama, Japan, June 1998. WWW page: http://www-sk.icrr.u-tokyo.ac.jp/mu98.