The Use of Mixed Variable Complementary Energy Principle for Solving Rectangular Plate Bending with the Action of Uniformly Distributed Loads

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ABSTRACT

In this paper, the minimum complementary energy principle of mixed variables was applied to get all the complementary energy expression of the rectangular plate under complex boundary conditions. When its variation extremum is taken, the boundary deflection will be more convenient to get. The principle was generalized to solve the problem of the rectangular bending plate under mixed boundary conditions with the action of uniformly distributed loads. Setting four corners of the fixed support as an example to calculate and then obtains a surface deflection equation.

INTRODUCTION

The application principle of potential energy principle and complementary methods shows many advantages in the analysis of the bending of rectangular plates with different boundary constraints\(^{(1,2)}\), especially in the case of multiple factors involved in calculation, and it makes the analysis of deflection and vibration\(^{(3,4)}\) more simple and convenient. At the same time, the use of stress variational method\(^{(5)}\) relationship between force and displacement can avoid the force balance, the relationship between force and displacement problem solving. For complex structures, especially for the flat, thin shell, such as elastic problems, its superiority is fully displayed. The emergence of modern electronic computer makes it possible to solve the high-order
linear algebra equations, for those problems of sheet with multiple functions which are difficult to solve, you can use the finite element\cite{6} to obtain the discrete solution represented by the node function, So with the principle of minimum complementary energy calculation method, the solution of rectangular plate bending problems can be widely used.

In this paper, rectangular plates by uniformly distributed load balancing in a complex boundary conditions\cite{and} and bending problems were studied. By Reciprocal theorem established a curved rectangular plate with mixed variables of complementary energy principle, it weakens the constraint boundary conditions, then the equivalent equation is increased. So no matter how constrained conditions and load cases is, you can more easily obtain the numerical solution\cite{7}.

**EXTERIOR ZERO VARIATION RELATIONSHIP WORK**

Consider the first status to a line of elastomer

| V | S_p | S_u |
|---|-----|-----|
| F_i | p_i + δp_i | p_i + δp_i |
| u_i | u_i | u_i |

The corresponding to the second status

| V | S_p | S_u |
|---|-----|-----|
| F_i | p_i | p_i + δp_i |
| u_i | u_i + δu_i | u_i |

![Figure 1. Actual system of rectangular plate with four corner points supported under p.](image)

Among tem, , In a state between the above two applications the reciprocal theorem of work, omit the second order item, it is

$$\int_{\partial V} \delta(p \cdot u) ds = 0$$

(1) is defined as known outside surface zero variation relationships. And here are all the variational variables.

**DEFLECTION SURFACE EQUATION**

About solving more complex boundary conditions rectangular plate bending problem, we can simplify calculation model into four edges simply supported rectangular plate with a mix of different boundary constraints.

As shown in figure 1, under the action of uniformly distributed load support at four corners of the rectangular plate. Calculation model can be simplified into four edges simply supported rectangular plate with four edges on the longitudinal \(w_{x0}, w_{xh}, w_{y0} \) and combination of displacement. In fact, it is only under the action of
uniformly distributed load $q$, at $x = 0, y = 0$ on the border deflection is known, and no displacement on four corners rectangular plate.

The total remaining energy of the mixed variables corresponding with the rectangular plate in Figure 1 can be written as

$$
\Pi_{mc} = \frac{1}{2D(1-\nu^2)} \int_0^a \int_0^b [(M_x + M_y)^2 - 2(1+\nu)(M_x M_y - M_{xy}^2)]
$$

$$
dx dy + \int_0^b w_{x0} V_x dx - \int_0^b w_{x0} V_x dx + \int_0^a w_{y0} V_y dy - \int_0^a w_{y0} V_y dy
$$

(2)

Define meet plate differential equilibrium equations of bending moment and torque for weak permissible bending moment and torque. For weak $\Pi_{mc}$ allow variable $M_x, M_y, M_{xy}$ variational extremum and $w$, because this is a symmetric bending, you can get the relationship of bending moment-curvature and bending moment-torsion for the euler’s equation and containing $w_{x0}, w_{y0}, w_{y0}$ and $w_{y0}$ the border of the deflection

$$
\bar{w}_{x0} = w_{x0} = \sum_{n=1,3}^\infty a_n \sin \beta_n y, \quad \bar{w}_{y0} = w_{y0} = \sum_{n=1,3}^\infty c_n \sin \alpha_n x
$$

(3-4)

Among them,

$$
\alpha_n = \frac{m\pi}{a}, \beta_n = \frac{n\pi}{b}, \quad D_{mn} = \frac{1}{2\alpha_n \beta_n} \left[ (\alpha_n^2 b_{mn} + \beta_n^2 c_{mn}) - \frac{16q}{ab} \frac{1}{\alpha_n \beta_n} \right]
$$

(5)

Flexural equation[27] was

$$
w(\xi, \eta) = \frac{4q}{Da} \sum_{m=1,3}^\infty \left\{ 1 + \frac{1}{2 \cosh \frac{1}{\alpha_n b}} [\alpha_n (\eta - \frac{b}{2}) \sinh \alpha_n (\eta - \frac{b}{2}) - (2 + \frac{1}{2} \alpha_n b \tanh \frac{1}{2} \alpha_n b) \right\}
$$

$$
\cosh \alpha_n (\eta - \frac{b}{2}) \sinh \alpha_n \xi + \frac{1-\nu}{\cosh \frac{1}{2\alpha_n b}} \sum_{n=1,3}^\infty \left\{ \left( \frac{\beta_n a}{\sinh \beta_n a} - \frac{2}{1-\nu} \right) \sinh \beta_n \xi + \beta_n \xi \cosh \beta_n \xi \right\}
$$

$$
\tanh \frac{\beta_n a}{2} + \frac{2}{1-\nu} \cosh \beta_n \xi - \beta_n \xi \sinh \beta_n \xi \sin \beta_n \eta(a_n) + \frac{1-\nu}{2} \sum_{m=1,3}^\infty \left\{ \left( \frac{\alpha_m b}{\sinh \alpha_m b} - \frac{2}{1-\nu} \right) \sinh \alpha_m \xi \sin \alpha_m \eta(c_m)
$$

(6a)

or

$$
w(\xi, \eta) = \frac{4q}{Da} \sum_{n=1,3}^\infty \left\{ 1 + \frac{1}{2 \cosh \frac{1}{2\beta_n b}} [\beta_n (\xi - \frac{a}{2}) \sinh \beta_n (\xi - \frac{a}{2}) - (2 + \frac{1}{2} \beta_n a \tanh \frac{1}{2} \beta_n a) \right\}
$$

$$
\cosh \beta_n (\xi - \frac{a}{2}) \sinh \beta_n \eta + \frac{1-\nu}{\cosh \frac{1}{2\beta_n b}} \sum_{n=1,3}^\infty \left\{ \left( \frac{\beta_n a}{\sinh \beta_n a} - \frac{2}{1-\nu} \right) \sinh \beta_n \xi + \beta_n \xi \cosh \beta_n \xi \right\}
$$

$$
\tanh \frac{\beta_n a}{2} + \frac{2}{1-\nu} \cosh \beta_n \xi - \beta_n \xi \sinh \beta_n \xi \sin \beta_n \eta(a_n) + \frac{1-\nu}{2} \sum_{m=1,3}^\infty \left\{ \left( \frac{\alpha_m b}{\sinh \alpha_m b} - \frac{2}{1-\nu} \right) \sinh \alpha_m \xi \sin \alpha_m \eta(c_m)
$$

(6b)

$$
0 \leq \xi \leq a, 0 \leq \eta \leq b
$$

(7)
BOUNDARY CONDITIONS

(6) Should satisfy the boundary conditions respectively was

\[-D_l \left( \frac{\partial^3 w}{\partial \eta^3} + (2 - \nu) \frac{\partial^3 w}{\partial \xi^2 \partial \eta} \right)_{\eta=0} = 0, \quad -D_l \left[ \frac{\partial^3 w}{\partial \xi^3} + (2 - \nu) \frac{\partial^3 w}{\partial \xi \partial \eta^2} \right]_{\xi=0} = 0 \]  

(8-9)

THE NUMERICAL

As a numerical example, the use of boundary conditions (8-9) are implemented in the execution of boundary condition equation \(a_n\) and \(c_m\) are 11, for rectangular plate with \(H=0.5\), take \(a=1\) m, \(b=0.5\) m, \(a_n\) and \(c_m\) also take 11, by solving the equations of deflection and bending moment value, as shown in table 1 and table 2.

ANSYS\(^8\) finite element software to simulate. Under the environment of ANSYS to establish four-point supported rectangular plate model. To rectangular thin plate which has four corners to support the plate were taken long \(x_a = y_b = 1m\) and \(x_a = 1m, y_b = 0.5m\). \(z\) as the thickness direction, thickness \(t = 0.01m\), the elastic modulus \(E = 2.1E11Pa\), Poisson's ratio \(\nu = 0.3\). After the modeling and the loading is completed, the read-out in the \(z\) direction of the displacement of each node of the bending moment and the cell extract, each node of the bending moment obtained by subtraction, the statistics of the data in Table 1 and Table 2 finally through the MATLAB programming, you can get the deflection and bending moment distribution graph.

Table 1. Deflection values along \(z\) direction of rectangular plate with four corner points supported under uniform load \((10^{-2}m)\).

| \(x/a\) | \(y/b=0.0\) | \(y/b=0.4\) | \(y/b=0.5\) |
|-------|-------------|-------------|-------------|
|       | This paper | ANSYS       | This paper  | ANSYS       | This paper | ANSYS       |
| 0.0   | 0.0000     | 0.0000      | 1.6925      | 1.6900      | 1.7747     | 1.7721      |
| 0.1   | 0.5694     | 0.5681      | 1.9182      | 1.9154      | 1.9869     | 1.9841      |
| 0.2   | 1.0675     | 1.0656      | 2.1459      | 2.1430      | 2.2034     | 2.2005      |
| 0.3   | 1.4513     | 1.4490      | 2.3374      | 2.3343      | 2.3866     | 2.3835      |
| 0.4   | 1.6925     | 1.6900      | 2.4640      | 2.4608      | 2.5082     | 2.5050      |
| 0.5   | 1.7747     | 1.7721      | 2.5082      | 2.5050      | 2.5507     | 2.5474      |
| 0.6   | 1.6925     | 1.6900      | 2.4640      | 2.4608      | 2.5082     | 2.5050      |
| 0.7   | 1.4513     | 1.4490      | 2.3374      | 2.3343      | 2.3866     | 2.3835      |
| 0.8   | 1.0675     | 1.0656      | 2.1459      | 2.1430      | 2.2034     | 2.2005      |
| 0.9   | 0.5694     | 0.5681      | 1.9182      | 1.9154      | 1.9869     | 1.9841      |
| 1.0   | 0.0000     | 0.0000      | 1.6925      | 1.6900      | 1.7747     | 1.7721      |
Table 2. Deflection values along z direction of rectangular plate with two adjacent edges free and the other two adjacent edges fixed under uniform load(10^{-2}m).

| x/a | b/a =1, y=0 Mx | b/a =0.5, y=0 Mx | b/a =0.5, x=0 Mx |
|-----|----------------|------------------|------------------|
|     | This paper  | ANSYS  | This paper  | ANSYS  | This paper  | ANSYS  |
| 0   | 0            | 0      | 0            | 0      | 0            | 0      |
| 0.1 | 0.6626     | 0.6383 | 0.5268       | 0.5205 | 0.2545       | 0.2297 |
| 0.2 | 1.0494     | 1.0275 | 0.8694       | 0.8633 | 0.3913       | 0.3692 |
| 0.3 | 1.3068     | 1.2855 | 1.1095       | 1.1034 | 0.4787       | 0.4576 |
| 0.4 | 1.4566     | 1.4356 | 1.2535       | 1.2473 | 0.5285       | 0.5077 |
| 0.5 | 1.506      | 1.4726 | 1.3015       | 1.2833 | 0.5447       | 0.52   |
| 0.6 | 1.4566     | 1.4356 | 1.2535       | 1.2473 | 0.5285       | 0.5077 |
| 0.7 | 1.3068     | 1.2855 | 1.1095       | 1.1034 | 0.4787       | 0.4576 |
| 0.9 | 0.6626     | 0.6383 | 0.5268       | 0.5205 | 0.2545       | 0.2297 |
| 1   | 0           | 0      | 0            | 0      | 0            | 0      |

Figure 2. Deflection curves.  
Figure 3. Bending moment curves at free edge.

CONCLUSION

In this paper, by using the principle of minimum complementary energy mix variables, it gives theoretical derivation and calculation to the rectangular plate supported on the four corners of the point, and then the boundary conditions, the deflection surface and moment equations of rectangular plate are given who is under uniformly distributed load.

Through the calculation and analysis of the chart, it shows that more than a small difference in energy principle of analog values in the calculation of mixed variables rectangular plate bending deflection and strain problems derived from the calculated values and ANSYS. When the load closes to the final deformation load deflection, the error still hasn't changed much. Furthermore, there are some limitations of ANSYS in the solution process, such as: the degree of fine mesh on the result of analysis, etc. So the method in this paper to solve complicated boundary is fully applicable, and the calculation process is simple, convenient and accurate. It can also provide a more reliable theoretical basis for engineering practice.
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