Quantum Gate for Kerr-Nonlinear Parametric Oscillator Using Effective Excited States

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A Kerr-nonlinear parametric oscillator (KPO) can stabilize a quantum superposition of two coherent states with opposite phases, which can be used as a qubit. In a universal gate set for quantum computation with KPOs, an $R_x$ gate, which interchanges the two coherent states, is harder to perform because these states are stabilized by an effective double-well potential due to a parametric drive \[4\]. An $R_x$ gate by controlling a detuning frequency of the KPO has been proposed \[6, 7\]. Another $R_x$ gate has been demonstrated experimentally \[24\], where the parametric drive is turned off for a certain time and the state evolves with Kerr nonlinearity \[28\]. The former method can execute an $R_x(\theta)$ gate with an arbitrary angle $\theta$, while the latter implements only an $R_x(\pi/2)$ gate. Although the $R_x(\pi/2)$ gate combined with arbitrary $R_z$ and $R_{zz}$ gates is sufficient for a universal gate set \[29\], the continuous $R_x$ gate will be useful for noisy intermediate-scale quantum (NISQ) algorithms such as variational quantum algorithms (VQAs) \[30, 31\]. However the continuous $R_x$ gate has not been realized experimentally, and thus a method for its simple implementation is desired.

In this paper we propose an alternative method for a continuous $R_x$ gate by intentionally exciting the KPO outside the qubit space spanned by the two stable coherent states. Such effective excited states in a rotating frame have not been utilized so far. This method can realize a continuous $R_x$ gate by only adding a single-photon or two-photon driving field. We numerically show that the $R_x$ gate can be performed with a high fidelity by utilizing parity selectivity of the excitation. In this method, we obtain a faster $R_x$ gate when we use effective excited states rather than those nearest to the qubit space. Numerical results indicate that the two-photon drive gives better performance than the single-photon one. The proposed method can offer a simple implementation of a continuous $R_x$ gate for a KPO qubit in a superconducting circuit, which will enable flexible designs of, e.g., VQAs.

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I. INTRODUCTION

Quantum computation using two coherent states as a qubit has originally been proposed in an optical circuit \[1\]. A superposition of two coherent states, which is called a Schrödinger cat state, has been an important resource state. Recently, generation of such a superposition has been designed \[2\] and realized experimentally \[3\] in a superconducting circuit by engineering a two-photon drive and two-photon loss. Also by this approach, a coherent state has experimentally been stabilized, where a dominant error source is single-photon loss \[4\].

In a superconducting circuit, another method to stabilize a superposition of coherent states has been proposed by using a Kerr-nonlinear parametric oscillator (KPO), which is based on a two-photon drive and Kerr nonlinearity \[5, 8\]. KPOs have first been studied for application to quantum annealing \[5, 9, 10\], and then universal quantum computation \[6, 7\]. Recently, a set of gate operations for KPOs has been proposed such that a type of error is suppressed \[17\], and has been developed toward fault-tolerant quantum computation \[18, 20\].

A KPO can be implemented by using a superconducting circuit similar to a Josephson parametric oscillator (JPO) \[21, 22\], and has recently been realized in the following experiments. A Schrödinger cat state has first been observed \[23\], and then single-qubit gate operations have been demonstrated \[24\]. Also a crossover from a JPO to a KPO has been observed \[25\]. Related theoretical analyses have been reported \[26, 27\].

For the KPO qubit, which consists of two coherent states with opposite phases and sufficiently large amplitudes, the following universal gate set can be implemented: $X$ rotation ($R_x$ gate), $Z$ rotation ($R_z$ gate), and $ZZ$ rotation ($R_{zz}$ gate) \[2, 6, 7\]. $R_z$ and $R_{zz}$ gates can be realized relatively easily by an external driving field with a frequency equal to the resonance frequency of the KPO (single-photon drive), and linear coupling between two KPOs, respectively.

In contrast, an $R_x$ gate, which exchanges populations between the two coherent states, is harder to perform because these states are stabilized by an effective double-well potential due to a parametric drive \[4\]. An $R_x$ gate by controlling a detuning frequency of the KPO has been proposed \[6, 7\]. Another $R_x$ gate has been demonstrated experimentally \[24\], where the parametric drive is turned off for a certain time and the state evolves with Kerr nonlinearity \[28\]. The former method can execute an $R_x(\theta)$ gate with an arbitrary angle $\theta$, while the latter implements only an $R_x(\pi/2)$ gate. Although the $R_x(\pi/2)$ gate combined with arbitrary $R_z$ and $R_{zz}$ gates is sufficient for a universal gate set \[29\], the continuous $R_x$ gate will be useful for noisy intermediate-scale quantum (NISQ) algorithms such as variational quantum algorithms (VQAs) \[30, 31\]. However the continuous $R_x$ gate has not been realized experimentally, and thus a method for its simple implementation is desired.

In this paper we propose an alternative method for a continuous $R_x$ gate by intentionally exciting the KPO outside the qubit space spanned by the two stable coherent states. Such effective excited states in a rotating frame have not been utilized so far. This method can realize a continuous $R_x$ gate by only adding a single-photon or two-photon driving field. We numerically show that the $R_x$ gate can be performed with a high fidelity by utilizing parity selectivity of the excitation. In this method, we obtain a faster $R_x$ gate when we use effective excited states rather than those nearest to the qubit space. Numerical results indicate that the two-photon drive gives better performance than the single-photon one. The proposed method can offer a simple implementation of a continuous $R_x$ gate for a KPO qubit in a superconducting circuit, which will enable flexible designs of, e.g., VQAs.
The paper is organized as follows. In Sec. III we introduce the proposed method, and in Sec. III we present numerical simulation results. \( R_x \) gates with \(|\theta| = \pi/2\) are shown in detail in Sec. IIIA and continuous \( R_x \) gates are investigated in Sec. IIIB. In Sec. IIIC we evaluate \( R_x \) gates faster than the gates in the previous sections. Finally, effect of single-photon loss is reported in Sec. III D.

The paper is concluded with a summary and a brief outlook in Sec. IV.

II. \( R_x \) GATE USING EXCITED STATES OF KPO

We first describe a Hamiltonian of a KPO and its eigenstates, and then propose a method for an \( R_x \) gate utilizing effective excited states. The Hamiltonian is given by \[ \frac{H_{KPO}}{\hbar} = -\frac{K}{2}a^2a^\dagger + \frac{p_0}{2}(a^2 + a^\dagger2), \] in a frame rotating at the resonance frequency, \( \omega_0 \), of the KPO and within the rotating-wave approximation. Here, \( a \) and \( a^\dagger \) are respectively the annihilation and creation operators for the KPO, and \( \hbar, K, \) and \( p_0 \) are the reduced Planck constant, the Kerr coefficient, and the amplitude of the parametric drive, respectively. We choose \( K > 0 \) as in KPOs with superconducting circuits \[ 23, 25 \]. In the following numerical calculations, we represent operators and states in the photon-number basis with the largest photon number of 30.

A. Effective ground and excited states of KPO

Figure 1 shows eigenvalues of \( H_{KPO} \) as functions of \( p_0/K \). Although \( E_0 \) and \( E_1 \) are the degenerate highest eigenvalues in the rotating frame, the corresponding eigenstates can be regarded as effective ground states, because at \( p_0/K = 0 \) these states are the eigenstates with the smallest photon numbers, that is, the vacuum and single-photon states, respectively \[ 5 \]. These states are the “effective” ground states because for \( p_0/K \neq 0 \) the Hamiltonian in the original laboratory frame depends on the time owing to the parametric drive. In the following, we refer to the states \( |E_0\rangle \) and \( |E_1\rangle \) as the ground states.

The eigenstates of \( H_{KPO} \) can be given by parity eigenstates, because \( H_{KPO} \) commutes with the parity operator \( e^{i\pi a^\dagger a} \). The degenerate ground states can be expressed as \[ 5, 7, 32 \].

\[ |E_0\rangle = N_+ (|\alpha\rangle + |-\alpha\rangle), \]
\[ |E_1\rangle = N_- (|\alpha\rangle - |-\alpha\rangle), \]

which have the even and odd parities, respectively. \(|\pm \alpha \rangle \) are coherent states with \( \alpha = \sqrt{p_0/K} \), satisfying \( a|\pm \alpha \rangle = \pm \alpha |\pm \alpha \rangle \), and \( N_\pm = 1/\sqrt{2} (1 \pm e^{-2\alpha^2}) \) are normalization factors. We define the computational basis by

\[ |\tilde{\emptyset}\rangle = \frac{1}{\sqrt{2}}(|E_0\rangle + |E_1\rangle), \]
\[ |\tilde{1}\rangle = \frac{1}{\sqrt{2}}(|E_0\rangle - |E_1\rangle), \]

which can be approximated by \(|\alpha \rangle \), respectively, for sufficiently large \( \alpha \) \[ 6, 7, 17 \]. These basis states are exactly orthogonal, namely, \( \langle \tilde{\emptyset} | \tilde{1} \rangle = 0 \), while \(|\pm \alpha \rangle \) are approximately orthogonal with an exponentially small error of \( \langle |\alpha\rangle - |\alpha\rangle \rangle^2 = e^{-4\alpha^2} = 9.2 \times 10^{-6} \) even for the smallest \( p_0/K = 2.9 \) in this study.

The rest of the eigenstates can be considered as effective excited states in the rotating frame, which we refer to as the excited states in the following. These excited states can be detected spectroscopically \[ 25, 27 \]. The parity of an excited state \(|E_k\rangle \) is the same as that of the integer \( k \), because \( k \) coincides with the photon number of the eigenstate at \( p_0/K = 0 \), and the parametric drive preserves the parity.

B. Proposed method

To utilize the excited states for an \( R_x \) gate, we add a single-photon or two-photon driving term \( H_d \) as

\[ H = H_{KPO} + H_d, \]
\[ \frac{H_d}{\hbar} = Ae^{-i\omega_d t} + A^\dagger e^{i\omega_d t}, \]

where \( t, \omega_d, \) and \( A \) are, respectively, the time, the driving frequency, and the driving operator given by

\[ A = p_d(t) a, \quad \text{for the single-photon drive}, \]
\[ A = p_d(t) \frac{a^2}{2}, \quad \text{for the two-photon drive}, \]

with a time-dependent amplitude \( p_d(t) \). These single-photon and two-photon drives can be realized experimentally by applying an external driving field with the frequency \( \omega_0 - \omega_d \) and by adding a two-photon driving field with the frequency \( 2\omega_0 - \omega_d \), respectively.

\( H_d \) can induce parity-selective transitions as follows. The single-photon drive leads to transitions between

![FIG. 1. Eigenvalues of \( H_{KPO} \) as functions of \( p_0/K \). \( E_0 \) and \( E_1 \) are degenerate.](image-url)
states with different parities because the parity of $H_d$ is odd. For the two-photon drive, $H_d$ has an even parity and induces only parity-conserving transitions.

Figure 2 shows matrix elements of $a$ and $a^2/2$ connected to the ground states, as functions of $p_0/K$. Note that these matrix elements are finite even for higher excited states such as $k = 4$, 5, and 6, which are essential for our proposed method. For the single-photon drive [Figs. 2(a) and 2(b)], large matrix elements are present between the ground states, which are approximated by

$$
(E_0|a|E_1) \approx (E_1|a|E_0) \approx \frac{p_0}{K}, \tag{10}
$$

for large $p_0/K$. In contrast, for the two-photon drive such matrix elements are exactly zero, $(E_0|a^2|E_1) = (E_1|a^2|E_0) = 0$, because $a^2$ conserves the parity. Thus, by tuning $\omega_d$, the two-photon drive can resonantly transfer the population of only one of $|E_0\rangle$ and $|E_1\rangle$ to an excited state, keeping the other ground state almost unchanged. For the single-photon drive, such an ideal selective transition is prevented by the coupling between the ground states due to the large matrix elements Eq. (10).

We explain the proposed $R_x$ gate first by the ideal selective transitions, and then by more realistic transitions. $R_x(\theta)$ with a rotation angle $\theta$ is expressed in $\{|0\rangle, |1\rangle\}$ as

$$
R_x(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}. \tag{11}
$$

Ideally, an $R_x$ gate can be designed as follows. An initial state is written in $\{|E_0\rangle, |E_1\rangle\}$ as

$$
|\psi_0\rangle = c_0|0\rangle + c_1|1\rangle = c_0|E_0\rangle + c_-|E_1\rangle, \tag{12}
$$

with $c_{\pm} = (c_0 \pm c_1)/\sqrt{2}$. We transfer a part of the population of one of the ground states, e.g., $|E_0\rangle$, to an excited state and then return it. As a result, $|E_0\rangle$ acquires a phase of an angle $\theta_0$ relative to $|E_1\rangle$, generating a final state of

$$
|\psi_1\rangle = c_+e^{-i\theta_0}|E_0\rangle + c_-|E_1\rangle = e^{-i\theta_0/2}R_x(\theta_0)|\psi_0\rangle. \tag{13}
$$

Hence $R_x(\theta_0)$ is performed (the factor $e^{-i\theta_0/2}$ can be neglected). Similarly, when $|E_1\rangle$ is selectively coupled with an excited state, $R_x(-\theta_1)$ is realized,

$$
|\psi_1\rangle = c_+|E_0\rangle + c_-e^{-i\theta_1}|E_1\rangle = e^{-i\theta_1/2}R_x(-\theta_1)|\psi_1\rangle. \tag{14}
$$

By assuming that the driving operator $A$ varies adiabatically, the angle $\theta_g$ with $g = 0$ or 1 can be expressed as

$$
\theta_g = \text{sgn}(\delta_{ge}) \int_0^T dt \left( \sqrt{2} \delta_{ge} + (E_g|A|E_e)^2 - |\delta_{ge}| \right), \tag{15}
$$

$$
\delta_{ge} = \frac{1}{2} \left( \frac{E_g - E_e}{\hbar} - \omega_d \right), \tag{16}
$$

where $T$ is the gate time, and $e$ denotes the utilized excited state (see Appendix A for details). Equation (15) clarifies that $\theta_g$ originates from a nonzero $(E_g|A|E_e)$. The sign of $\theta_g$ is determined by that of $\delta_{ge}$ because the integrand is positive.

More realistically, one can perform an $R_x$ gate by choosing $\omega_d$ near two excited states. Populations of $|E_0\rangle$ and $|E_1\rangle$ are separately transferred to these excited states owing to the parity selectivity (when the coupling between the ground states can be neglected). Then both $|E_0\rangle$ and $|E_1\rangle$ obtain phases, giving $R_x(\theta_g - \theta_0)$,

$$
|\psi_1\rangle = c_+e^{-i\theta_0}|E_0\rangle + c_-e^{-i\theta_1}|E_1\rangle \tag{17}
$$

$$
e^{-i(\delta_{0g}+\delta_{1e})/2}R_x(\theta_0 - \theta_1)|\psi_1\rangle. \tag{18}
$$

Although Eqs. (15) and (18) provide a picture of the proposed method, we estimate the rotation angle $\theta$ by another way in numerical simulations in Sec. III, because the precise value of $\theta$ can be affected by transitions to other states as can be seen shortly.
III. NUMERICAL SIMULATIONS

Here we present our numerical simulation results of the proposed method. The gate time $T$ in Sec. IIIA and IIIB is $KT=10$, which is the same as the previous study [6]. (Shortening $T$ is discussed in Sec. IIIC and IIID.) We use a pulse-shaped $p_{d}(t)$ characterized by a maximum value $p_{d1}$ and a rise time $\tau$ (see Appendix B for details).

We set the initial state to $|\psi_{i}\rangle = |0\rangle$ in order to demonstrate that a part of the population is transferred from $|0\rangle$ to $|1\rangle$. In Sec. IIIA and IIIC we calculate the time evolution of a state $|\psi\rangle$ by solving the Schrödinger equation,

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$ \hfill (19)

From a final state $|\psi_{f}\rangle$, the fidelity of an $R_{\theta}(\theta)$ gate is calculated by

$$F = |\langle \psi_{f}|R_{\theta}(\theta)|\psi_{f}\rangle|^{2},$$ \hfill (20)

and the error is defined by $1 - F$. The rotation angle $\theta$ corresponding to parameters $\omega_{d}$ and $p_{d}(t)$ is defined by $\theta$ that maximizes $F$ [6], instead of predetermining $\theta$ with Eqs. (15) and (18), as mentioned above. Once such a correspondence is known, then a desired $R_{\theta}(\theta)$ gate can be executed by using corresponding $\omega_{d}$ and $p_{d}(t)$. We set a criterion for high fidelity by $1 - F < 10^{-3}$, and optimize $p_{0}/K$ and $K\tau$ so that $R_{\theta}(\theta)$ gates with $1 - F < 10^{-3}$ and $|\theta| \geq \pi/2$ are achieved in larger areas of $(\omega_{d}/K, p_{d1}/K)$. By these parameters, high-fidelity $R_{\theta}(\theta)$ gates with $|\theta| < \pi/2$ are also obtained as shown in Sec. IIIIB.

A. $R_{\theta}(\theta)$ gates with $|\theta| = \pi/2$

First, we show high-fidelity $R_{\theta}(\theta)$ gates with $|\theta| = \pi/2$, which are sufficient for a universal gate set as described in Sec. I. By the single-photon drive, we obtain an $R_{\theta}(-\pi/2)$ with an error of $1 - F = 5.1 \times 10^{-4}$, despite the imperfect selectivity in transitions. Figure 3(a) shows that, although the populations oscillate between $|E_{0}\rangle$ and $|E_{1}\rangle$ owing to the large matrix elements between these states [Eq. (10)], the populations almost return to the
initial values at the final time. Also, because the chosen \( \omega_d/K \) is close to both \( \xi_4 \) and \( \xi_3 \) with
\[
\xi_k = \frac{E_0 - E_k}{\hbar K},
\] (21)
two excited states \( |E_4\rangle \) and \( |E_3\rangle \) are largely populated. As a result, both \( |E_0\rangle \) and \( |E_1\rangle \) acquire phases with \( \theta_{g} = -\arg \langle \{E_g|\psi\rangle \rangle \). These phases then contribute constructively to the total rotation angle \(-\pi/2\), as can be seen in Fig. 3(b), where the signs of \( \theta_{g} \) are consistent with Eq. (15). Figure 3(c) shows that this operation leads to a desired population transfer from \( |0\rangle \) to \( |1\rangle \), as well as a correct rotation with a relative angle \( \phi = \arg \langle \{1|\psi\rangle/\langle 0|\psi\rangle \rangle = \pi/2 \).

For the single-photon drive, the value of \( p_0/K = 2.9 \) optimized as above is rather small, because the small \( p_0/K \) reduces the matrix elements between \( |E_0\rangle \) and \( |E_1\rangle \), hence the population oscillation between these states. In addition, \( p_0/K = 2.9 \) is optimal in that \( |E_1|a|E_4\rangle \) is the largest near this value [Fig. 2(b)], which makes the transition easier.

By the two-photon drive, we obtain a high-fidelity \( R_x(\pi/2) \) \( (1 - F = 5.4 \times 10^{-4}) \). The time evolution is close to the one designed in Sec. II B \( |E_0\rangle \) is populated almost selectively [Fig. 3(d)], and the phase of \( |E_1\rangle \) dominantly rotates [Fig. 3(e)], leading to the desired operation [Fig. 3(f)].

For the two-photon drive, the optimal \( p_0/K = 4.7 \) is larger, firstly because no direct transition occurs between \( |E_0\rangle \) and \( |E_1\rangle \), in contrast to the single-photon drive. Secondly, the larger \( p_0/K \) widens the separations between the eigenvalues of the lower excited states \( (E_2 \text{ and } E_3) \) and the higher ones \( (E_4 \text{ and } E_6) \), as can be seen in Fig. 4. These larger separations improve the selectivity of the transition. Also, the larger separations allow us to use larger \( \delta_{gr} \) [Eq. (16)], which can suppress nonadiabatic errors. When \( p_0/K \) is too large, however, the excited states become degenerate in pairs \( (23, 24, 33) \), as in Fig. 1. These degenerate states are localized near \( \pm \sqrt{p_0/K} \) in the quadrature phase space owing to a double-well potential by the parametric drive \( (23, 34, 35) \). In other words, the states with the large separations of the eigenvalues are the states extended over the double wells, which we utilize in this method.

Here the errors are mostly due to leakage, that is, residual populations in excited states. For KPOs, a method to correct the leakage has been proposed, which applies an artificial two-photon loss \( (17, 19, 35) \). In this paper, we regard the leakage as an error for simplicity.

### B. Continuous \( R_x(\theta) \) gates

Next, we present the results of \( R_x(\theta) \) gates with arbitrary rotation angles \( \theta \) by changing \( \omega_d \) and \( p_{11} \). Figure 4 shows the values of \( \theta \) that are obtained with high fidelities, as functions of \( \omega_d/K \). The vertical dashed lines indicate \( \xi_k \) in Eq. (21). For the single-photon drive, continuous rotations between \( \theta = 0 \) and \( -\pi/2 \) are possible with high fidelities of \( 1 - F < 10^{-3} \) by \( \omega_d/K \) around \( \xi_4 \). The high-fidelity two-photon drive, such high-fidelity continuous \( R_x(\theta) \) are possible by \( \omega_d/K \) around \( \xi_5 \) and \( \xi_6 \), where \( \theta \) can be varied from 0 to \( \pi \), and from \(-\pi/2 \) to \( \pi/2 \), respectively. The signs of \( \theta \) can again be understood from Eqs. (15) and (18).

For both the drives, the high-fidelity \( R_x(\theta) \) with large \( |\theta| \) are obtained by \( \omega_d/K \) near the higher excited states \( (\xi_4, \xi_5, \text{ and } \xi_6) \), rather than the lower ones \( (\xi_2 \text{ and } \xi_3) \), because the separations of the eigenvalues are larger between the higher excited states, as mentioned above. This feature is robust against changes in \( p_{0}/K \) in a certain range. In comparison of the two drives in Fig. 4, the high-fidelity area is larger for the two-photon drive.

### C. Faster \( R_x \) gates

Faster gate operations are favorable in suppressing effect of photon loss. Here we evaluate the shortest gate time in the proposed method, keeping the high fidelities of \( 1 - F < 10^{-3} \). As far as we searched, by the single-
photon drive, \( R_x(-\pi/2) \) within \( KT = 9.1 \) is obtained with \( \rho_0/K = 2.9, \omega_1/K = 7.78, \rho_1/K = 0.848, \) and \( K \tau = 2.3. \) The by the two-photon drive, \( R_x(-\pi/2) \) can be accelerated further to \( KT = 6.4 \) with \( \rho_0/K = 4.2, \omega_1/K = 20.51, \rho_1/K = 0.732, \) and \( K \tau = 1. \) At this \( \omega_1/K, |E_0\rangle \) is populated, instead of \(|E_3\rangle \) used in Sec. IIIA. The faster \( R_x \) gate is obtained by the two-photon drive, again because of the absence of the direct transition between \(|E_0\rangle \) and \(|E_1\rangle \), and the larger separations of the eigenvalues.

D. Effect of single-photon loss

Finally we study effect of single-photon loss by solving the master equation for a density operator \( \rho \),

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \frac{\kappa}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a a^\dagger),
\]

where \( \kappa \) is the loss rate. We use the parameters that give the shortest gate times for \( R_x(-\pi/2) \) in Sec. IIIA. Figure 5 shows the errors \( 1 - F \) of the \( R_x \) gates as functions of \( \kappa/K \), where the fidelity is calculated from the density operator at the final time, \( \rho_t \), by

\[
F = \langle \psi_i | R_x^t(\theta) \rho_t R_x(\theta) | \psi_i \rangle.
\]

IV. SUMMARY

We have proposed a method for a continuous \( R_x \) gate for a KPO by using effective excited states in a rotating frame, which have not been utilized before. We have numerically demonstrated that a high-fidelity \( R_x \) gate is obtained with parity-selective transitions. In this method, the use of higher excited states, rather than states adjacent to ground states, leads to faster execution of the \( R_x \) gate. Although both single-photon and two-photon drives can implement the \( R_x \) gate, the latter shows better performance due to the absence of direct transition between the ground states and due to larger separations of the eigenvalues of the excited states. This continuous \( R_x \) gate will be useful for not only universal quantum computation but also NISQ applications.

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Appendix A: Angle \( \theta_g \)

The angle \( \theta_g \) in Eq. (15) is derived from the Schrödinger equation for a subsystem spanned by relevant two eigenstates of \( H_{\text{KPO}} \). In the interaction picture |\( \psi \rangle_\text{i} = e^{iH_{\text{KPO}}t/\hbar} |\psi \rangle \), the Schrödinger Eq. (19) is written as

\[
\frac{i}{\hbar} \frac{d\psi_k}{dt} = \sum_l \left[ A_{kl} e^{i2\delta_{kl}t} + (A^{\dagger})_{kl} e^{-i2\delta_{kl}t} \right] \psi_l,
\]

where \( \psi_k = \langle E_k | \psi \rangle_\text{i}, A_{kl} = \langle E_k | A | E_l \rangle, \) and \( \delta_{kl} = \left( (E_k - E_l)/\hbar - \omega_\text{d} \right)/2. \left| \delta_{kl} \right| \) is supposed to be the smallest for \( k = g \) and \( l = e \), where \( g \) and \( e \) represent ground and excited states, respectively. Then, by the rotating wave approximation, rapidly oscillating terms are dropped. (As described in Sec. IIIA, this approximation is valid for the two-photon drive, but not so good for the single-photon drive owing to large \( A_{0g} \) and \( A_{10} \).

According to the parity-selection rules, \( A_{ge} = \gamma \) is supposed to be finite for a ground state \( g \), while \( A_{ge} = 0 \) for the other ground state \( \bar{g} \). \( \gamma \) can be made positive by choosing the phases of \( |E_0\rangle \) and \( |E_i\rangle \), and \( \gamma \geq 0 \) is assumed in the following. In the relevant two states,
Eq. (A1) is expressed as
\[ i \frac{d}{dt} \begin{pmatrix} \psi_g \\ \psi_e \end{pmatrix} = \begin{pmatrix} 0 & \delta e^{i2\delta t} \\ \gamma e^{-i2\delta t} & 0 \end{pmatrix} \begin{pmatrix} \psi_g \\ \psi_e \end{pmatrix}. \] (A2)
where \( \delta = \delta_{ge} \). Then, \( \tilde{\psi}_g = e^{-i\delta t} \psi_g \) and \( \tilde{\psi}_e = e^{i\delta t} \psi_e \) yield
\[ i \frac{d}{dt} \begin{pmatrix} \tilde{\psi}_g \\ \tilde{\psi}_e \end{pmatrix} = \begin{pmatrix} \delta & \gamma \\ \gamma & -\delta \end{pmatrix} \begin{pmatrix} \tilde{\psi}_g \\ \tilde{\psi}_e \end{pmatrix}. \] (A3)
The eigenvalues of the matrix in Eq. (A3) are \( \pm \Omega = \sqrt{\delta^2 + \gamma^2} \), and the corresponding eigenvectors are
\[ u_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm \sqrt{1 + \delta/\Omega} \end{pmatrix}, \] and \( u_- = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ \pm \sqrt{1 - \delta/\Omega} \end{pmatrix} \). (A4)

We set \( \gamma = 0 \) at the initial and final times, which determines the initial and final states to be \( u_\pm \) for \( \delta \geq 0 \). When \( \gamma \) varies adiabatically, the state is kept in \( u_\pm \) and a dynamical phase due to \( \pm \Omega \) is accumulated. As a result, \( |E_g\rangle \) obtains a phase factor \( e^{-i\theta_g} \) relative to \( |E_g\rangle \), leading to Eq. (15),
\[ \theta_g = \text{sgn}(\delta) \int_0^T dt \left( \sqrt{\delta^2 + \gamma^2} - |\delta| \right). \] (A5)

Appendix B: Pulse shape

We choose a pulse shape
\[ p_\delta(t) = p_{\delta1} \left( \frac{\tanh(t/\tau) \tanh[(T-t)/\tau]}{\tanh^2(T/(2\tau))} \right)^2, \] (B1)
where \( T \) is the gate time, and \( p_{\delta1} \) and \( \tau \) are the maximum value and the rise time of the pulse, respectively. By Eq. (B1), one can generate both a single-peak pulse and a trapezoid-like pulse by changing \( \tau \), which is difficult by, e.g., a Gaussian pulse shape. Also, Eq. (B1) ensures that \( p_\delta(t) \) and \( dp_\delta(t)/dt \) vanish at \( t = 0 \) and \( T \), reducing nonadiabatic errors [33]. Figure 6 shows the pulses used in Sec. III A.

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