Fuzzy Discrete Event Systems under Fuzzy Observability and a Test-Algorithm
Daowen Qiu and Fuchun Liu

Abstract—In order to more effectively cope with the real-world problems of vagueness, impreciseness, and subjectivity, fuzzy discrete event systems (FDESs) were proposed recently. Notably, FDESs have been applied to biomedical control for HIV/AIDS treatment planning and sensory information processing for robotic control. Qiu, Cao and Ying independently developed supervisory control theory of FDESs. We note that the controllability of events in Qiu’s work is fuzzy but the observability of events is crisp, and the observability of events in Cao and Ying’s work is also crisp although the controllability is not completely crisp since the controllable events can be disabled with any degrees. Motivated by the necessity to consider the situation that the events may be observed or controlled with some membership degrees, in this paper, we establish the supervisory control theory of FDESs with partial observations, in which both the observability and controllability of events are fuzzy instead. We formalize the notions of fuzzy controllability condition and fuzzy observability condition. And Controllability and Observability Theorem of FDESs is set up in a more generic framework. In particular, we present a detailed computing flow to verify whether the controllability and observability conditions hold. Thus, this result can decide the existence of supervisors. Also, we use this computing method to check the existence of supervisors in the Controllability and Observability Theorem of classical discrete event systems (DESs), which is a new method and different from classical case. A number of examples are elaborated on to illustrate the presented results.

Index Terms—Discrete event systems, fuzzy logic, observability, supervisory control, fuzzy finite automata.

I. INTRODUCTION

Discrete event systems (DESs) are dynamical systems whose evolution in time is governed by the abrupt occurrence of physical events at possibly irregular time intervals. Event though DESs are quite different from traditional continuous variable dynamical systems, they clearly involve objectives of control and optimization. A fundamental issue of supervisory control for DESs is how to design a controller (or supervisor), whose task is to enable and disable the controllable events such that the resulting closed-loop system obeys some prespecified operating rules [1]. Up to now, the supervisory control theory of DESs has been significantly applied to many technological and engineering systems such as automated manufacturing systems, interaction telecommunication networks and protocol verification in communication networks [2-9].

In most of engineering applications, the states of a DES are crisp. However, this is not the case in many other applications in complex systems such as biomedical systems and economic systems, in which vagueness, impreciseness, and subjectivity are typical features. For example, it is vague when a man’s condition of the body is said to be “good”. Moreover, it is imprecise to say at what point exactly a man has changed from state “good” to state “poor”. It is well known that the fuzzy set theory first proposed by Zadeh [10] is a good tool to cope with those problems. Indeed, up to now, fuzzy control systems have been well developed by many authors, and we may refer to [11] (and these references therein) regarding a survey on model-based fuzzy control systems. Notably, Lin and Ying [12, 13] recently initiated significantly the study of fuzzy discrete event systems (FDESs) by combining fuzzy set theory [14] with classical DESs. Notably, FDESs have been applied to biomedical control for HIV/AIDS treatment planning [15, 16] and decision making [17]. More recently, R. Huq et al [18, 19] have proposed an intelligent sensory information processing technique using FDESs for robotic control in the field of mobile robot navigation.

Just as Lin and Ying [13] pointed out, a comprehensive theory of FDESs still needs to be set up, including many important concepts, methods and theorems, such as controllability, observability, and optimal control. These issues have been partially investigated in [20-23]. It is worthy to mention that Qiu [20], Cao and Ying [21] independently developed the supervisory control theory of FDESs. The similarity between the two theories is that the fuzzy systems considered in both [20] and [21] are modeled by max-min automata instead of max-product automata adopted in [13], and the controllability theorem was established in their respective frameworks. However, there are great differences between them. For the purpose of control, the set of events in [21] is partitioned into two disjoint subsets of controllable and uncontrollable events, as usually done in classical DESs, but the controllability of events is not completely crisp since the controllable events can be disabled by supervisors with any degrees. In contrast with [21], the controllable set and uncontrollable set of events in [20] are two fuzzy subsets of the set of events. That is, each event not only belongs to the uncontrollable set but also belongs to the controllable set; only its degree of belonging to those sets may be different. In particular, Qiu [20] presented an algorithm to check the existence of fuzzy supervisors for FDESs. As a continuation of the supervisory control under full observations [20, 21], this paper is to deal with the supervisory
control of FDESs with fuzzy observations (generalizing partial observations).

We notice that the observability in Qiu’s work [20] and Cao and Ying’s work [21-23] is crisp, that is, each fuzzy event is either completely observable or completely unobservable, although the controllability is fuzzy in [20] and not completely crisp in [21-23] where the controllable events can be disabled with any degrees. However, in real-life situation, each event generally has a certain degree to be observable and unobservable, and, also, has a certain degree to be controllable and uncontrollable. In fact, this idea of fuzziness of observability and controllability was originally proposed by Lin and Ying [13], and Qiu [20], and then it has been subsequently applied to robot sensory information processing by Huq et al [18, 19]. For example, in the cross process for a patient having cancer via either operation or drug therapy [24], some treatments (events) can be clearly seen by supervisors (viewed as a group of physicians), while some therapies (such as some operations), while some therapies (such as some operations)

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III. CONTROLLABILITY AND OBSERVABILITY THEOREM

Let $\bar{G} = (\bar{Q}, \bar{E}, \bar{\delta}, \bar{q}_0, \bar{Q}_m)$ be a fuzzy finite automaton. As mentioned in Section I, each fuzzy event may be observable or controllable with a certain membership degree. Thus, the uncontrollable set $\Sigma_{uc}$ and controllable set $\Sigma_{c}$, as well as the unobservable set $\Sigma_{uo}$ and observable set $\Sigma_{o}$, are thought of as four fuzzy subsets of $\bar{E}$, which are defined formally as follows.

Definition 2: The uncontrollable set $\Sigma_{uc} \subseteq \mathcal{F}(\bar{E})$ and controllable set $\Sigma_{c} \subseteq \mathcal{F}(\bar{E})$ are respectively defined as a function $\Sigma_{uc}: \bar{E} \rightarrow [0,1]$ and a function $\Sigma_{c}: \bar{E} \rightarrow [0,1]$ which satisfy: for any $\bar{\sigma} \in \bar{E}$,

$$\Sigma_{uc}(\bar{\sigma}) + \Sigma_{c}(\bar{\sigma}) = 1.$$  
(4)

Similarly, the unobservable set $\Sigma_{uo} \subseteq \mathcal{F}(\bar{E})$ and observable set $\Sigma_{o} \subseteq \mathcal{F}(\bar{E})$ are respectively defined as $\Sigma_{uo}: \bar{E} \rightarrow [0,1]$ and $\Sigma_{o}: \bar{E} \rightarrow [0,1]$ which satisfy: for any $\bar{\sigma} \in \bar{E}$,

$$\Sigma_{uo}(\bar{\sigma}) + \Sigma_{o}(\bar{\sigma}) = 1.$$  
(5)

Remark 2: The degrees of observability and unobservability for FDESs were originally proposed by Lin and Ying ([13], pp. 412), and the degrees of controllability and uncontrollability were introduced by Qiu ([20]), pp. 76). Intuitively, $\Sigma_{uc}(\bar{\sigma})$ and $\Sigma_{c}(\bar{\sigma})$ represent the degree of fuzzy event $\bar{\sigma}$ to be uncontrollable and the degree of $\bar{\sigma}$ to be controllable, respectively. And, $\Sigma_{uo}(\bar{\sigma})$ and $\Sigma_{o}(\bar{\sigma})$ represent the degree of $\bar{\sigma}$ to be unobservable and the degree of $\bar{\sigma}$ to be observable, respectively.

Definition 3: The projection $P: \bar{E} \rightarrow \bar{E}$ is defined as:

$$P(\bar{\sigma}) = \begin{cases} \bar{\sigma}, & \text{if } \Sigma_{uo}(\bar{\sigma}) > 0, \\ \epsilon, & \text{otherwise}. \end{cases}$$
(6)

And it can be extended to $\bar{E}^*$ by $P(\epsilon) = \epsilon$ and $P(\bar{s}\bar{\sigma}) = P(\bar{s})P(\bar{\sigma})$ for $\bar{s} \in \bar{E}^*$ and $\bar{\sigma} \in \bar{E}$.

Remark 3: The purpose of projection is to erase the completely unobservable fuzzy events in the strings.

In order to emphasize the observability degree of fuzzy event strings by means of projection $P$ associated with the fuzzy observable subset $\Sigma_{o}$, we define the factor of observable projection $\tilde{D}$ as a fuzzy subset of $P(\bar{E})$: for any $\bar{\sigma} \in P(\bar{E})$, $\tilde{D}(\bar{\sigma}) = \Sigma_{o}(\bar{\sigma})$, and

$$\tilde{D}(\bar{\sigma}_1 \bar{\sigma}_2 \cdots \bar{\sigma}_n) = \min(\tilde{D}(\bar{\sigma}_i) : i = 1,2,\ldots,n),$$

where $\bar{\sigma}_i \in P(\bar{E})$, $i = 1, 2, \ldots, n$. Especially, $\tilde{D}(\epsilon) = 0$. Intuitively, $\tilde{D}(P(\bar{s})) \cdot L_{\bar{G}}(\bar{s})$ represents the possibility for the fuzzy event string $\bar{s} \in \bar{E}^*$ being possible under the effect of observable projection. And, $\tilde{D}(P(\bar{s}\bar{\sigma})) \cdot \Sigma_{uc}(\bar{\sigma})$ and $\tilde{D}(P(\bar{s})) \cdot pr(K)(\bar{s})$ respectively denote the degree of $\bar{\sigma}$ in $\bar{E}$, as a continuation of the string $\bar{s}$, being uncontrollable, and the possibility of string $\bar{s}$ belonging to the prefix-closure of sublanguage $\bar{K}$ under the effect of observable projection. Furthermore, for the sake of convenience, in what follows we use the following notation:

$$L_{\bar{G}}^f(\bar{s}) = \begin{cases} 1, & \text{if } \bar{s} = \epsilon, \\ \tilde{D}(P(\bar{s})) \cdot L_{\bar{G}}(\bar{s}), & \text{otherwise}; \end{cases}$$  
(7)

$$pr(K)^f(\bar{s}) = \begin{cases} 1, & \text{if } \bar{s} = \epsilon, \\ \tilde{D}(P(\bar{s})) \cdot pr(K)(\bar{s}), & \text{otherwise}; \end{cases}$$  
(8)

$$\tilde{D}(\bar{\sigma}) = \tilde{D}(P(\bar{s}\bar{\sigma})) \cdot \tilde{D}(\bar{\sigma}),$$

where $\bar{\sigma}$ is the continuation of string $\bar{s}$.

Definition 4: For any FDES $G$, a supervisor under the projection $P$ is said a fuzzy supervisor, denoted by $\tilde{S}_P$, that is formally defined as a function

$$\tilde{S}_P: P(\bar{E}^*) \rightarrow \mathcal{F}(\bar{E})$$

where for each $\bar{s} \in \bar{E}^*$ and $\bar{\sigma} \in \tilde{D}(\bar{s})$, $\tilde{S}_P(P(\bar{s})(\bar{\sigma}))$ represents the possibility of fuzzy event $\bar{\sigma}$ being enabled after the occurrence of the string $P(\bar{s})$.

The supervisors $\tilde{S}_P$ are usually required to satisfy the following admissibility condition.

Definition 5: The fuzzy admissibility condition for fuzzy supervisor $\tilde{S}_P$ is characterized as follows: for each $\bar{s} \in \bar{E}^*$ and each continuation $\bar{\sigma} \in \bar{E}$, the following inequality holds

$$\min(\Sigma_{uc}(\bar{\sigma}), L_{\bar{G}}^f(\bar{s}\bar{\sigma})) \leq \tilde{S}_P(P(\bar{s})(\bar{\sigma})).$$  
(10)

Intuitively, the fuzzy admissibility condition (10) means that, under the effect of observable projection, the degree of any fuzzy event $\bar{\sigma}$ following any fuzzy event string $\bar{s}$ being possible together with $\bar{\sigma}$ being uncontrollable is not larger than the possibility for $\bar{\sigma}$ being enabled by the fuzzy supervisor $\tilde{S}_P$ after string $P(\bar{s})$ occurring.

The fuzzy controlled system by means of $\tilde{S}_P$, denoted by $\tilde{S}_P/G$, is an FDES, and the behavior of $\tilde{S}_P/G$ when $\tilde{S}_P$ is controlling $G$ is defined as follows.

Definition 6: The fuzzy languages $L_{\tilde{S}_P/G}$ and $L_{\tilde{S}_P/G,m}$ generated and marked by $\tilde{S}_P/G$, respectively, are defined as follows: for any $\bar{s} \in \bar{E}^*$ and any $\bar{\sigma} \in \bar{E}$,

1) $L_{\tilde{S}_P/G}(\epsilon) = 1$;

2) $L_{\tilde{S}_P/G}(\bar{s}\bar{\sigma}) = \min(L_{\tilde{S}_P/G}(\bar{s}), L_{\tilde{S}_P/G}(\bar{s}\bar{\sigma}))$;

3) $L_{\tilde{S}_P/G,m} = \tilde{S}_P/G \cap \tilde{S}_P/G$, where symbol $\cap$ denotes Zadeh fuzzy AND operator, i.e., $(A\cap B)(x) = \min(A(x), B(x))$.

Definition 6 indicates that the degree of $\bar{s}\bar{\sigma}$ being physically possible in the controlled system $\tilde{S}_P/G$ is the smallest one among the degree of $\bar{s}$ being possible in $\tilde{S}_P/G$, the degree of $\bar{s}\bar{\sigma}$ being possible in $G$ under the effect of observable projection, and the possibility of $\bar{\sigma}$ being enabled by the supervisor after the occurrence of $P(\bar{s})$. It is clear that Definition 6 generalizes the corresponding concepts from full observations ([20], pp. 6) to partial observations.

In supervisory control of DESS, nonblockingness is usually required, and it means that the controlled system does not produce deadlocks [1, 20].

Definition 7: A fuzzy supervisor $\tilde{S}_P$ of $G$ is said to be nonblocking, if for any $\bar{s} \in \bar{E}^*$, the following equation holds:

$$L_{\tilde{S}_P/G,m}(\bar{s}) = \begin{cases} 1, & \text{if } \bar{s} = \epsilon, \\ 1 - \tilde{D}(P(\bar{s})) \cdot pr(K)(\bar{s}), & \text{otherwise}. \end{cases}$$  
(11)

Intuitively, if $\tilde{S}_P$ is nonblocking, then for any string $\bar{s}$, the possibility that $\bar{s}$ is one of the behaviors of the supervised fuzzy system $\tilde{S}_P/G$ equals the degree of $\bar{s}$ belonging to the prefix-closure of the fuzzy language marked by the supervised fuzzy system $\tilde{S}_P/G$ under the effect of observable projection.
Definition 8: A fuzzy sublanguage $\bar{K}$ is said to be $\mathcal{L}_{\bar{G},m}$-closed, if for any $\bar{s} \in \bar{E}^*$,
\[
\bar{K}(\bar{s}) = \begin{cases} 
1, & \text{if } \bar{s} = \epsilon, \\
\min\{pr(\bar{K})^f(\bar{s}), \mathcal{L}_{\bar{G},m}(\bar{s})\}, & \text{otherwise.}
\end{cases}
\]

(12)

Obviously, if all fuzzy events can be observed fully [20], that is to say, $\Sigma_o(\bar{\sigma}) = 1$ for any fuzzy event $\bar{\sigma}$, then Eq. (12) reduces to $\bar{K} = pr(\bar{K})$ introduced in [1, 4, 7, 20], where all events are supposed to be observable.

Definition 9: Let $\bar{K} \subseteq \mathcal{L}_{\bar{G}}$. If for any $\bar{s} \in \bar{E}^*$ and its continuation $\bar{\sigma} \in \bar{E}$, the following inequality holds:

$$\min\{pr(\bar{K})^f(\bar{s}), \Sigma_{uc}(\bar{\sigma}), \mathcal{L}_{\bar{G}}(\Sigma_o(\bar{\sigma}))\} \leq pr(\bar{K})^f(\bar{s})$$

then we call $\bar{K}$ satisfying fuzzy controllability condition with respect to $\bar{G}$, $P$ and $\Sigma_{uc}$.

Intuitively, (13) means that under the effect of observable projection, the degree to which any fuzzy event string $\bar{s}$ belongs to the prefix-closure of $\bar{K}$ and fuzzy event $\bar{\sigma}$ following string $\bar{s}$ is physically possible together with $\bar{\sigma}$ being uncontrollable, is not larger than the possibility of string $\bar{s}\bar{\sigma}$ belonging to the prefix-closure of $\bar{K}$.

Remark 4: Definition 9 generalizes the corresponding concepts concerning controllability in [1, 20]. If all fuzzy events can be observed fully, then Ineq. (13) reduces to the fuzzy controllability condition introduced in [20]. If we further assume that the events and states are crisp, then it reduces to the controllability condition introduced in [1].

To illustrate the application of fuzzy controllability condition, we provide an example.

Example 1. Consider a fuzzy automaton $\bar{G} = (\bar{Q}_1, \bar{E}, \bar{\delta}, \bar{q}_0)$, where $\bar{E} = \{\bar{a}, \bar{b}, \bar{c}\}$, $\bar{q}_0 = [0.8, 0]$, and

$$\bar{a} = [0.8, 0.2], \bar{b} = [0.2, 0.8], \bar{c} = [0.2, 0.8].$$

Let $pr(\bar{K})$ be generated by a fuzzy automaton $\bar{H} = (\bar{Q}_2, \bar{E}, \bar{\delta}, \bar{p}_0)$, where $\bar{p}_0 = [0.5, 0.5]$, $\bar{E} = \{\bar{a}, \bar{b}, \bar{c}\}$, and $\bar{a}, \bar{b}$ are the same as those in $\bar{G}$, but $\bar{c}$ is changed as follows:

$$\bar{c} = [0.1, 0.4].$$

Suppose that $\Sigma_{uc}$ and $\Sigma_o$ are defined as: $\Sigma_{uc}(\bar{a}) = 0.3$, $\Sigma_{uc}(\bar{b}) = 0.5$, and $\Sigma_{uc}(\bar{c}) = 0.8$; $\Sigma_o(\bar{a}) = \Sigma_o(\bar{b}) = 0.7$, and $\Sigma_o(\bar{c}) = 0.5$.

In the following, we show that $\bar{K}$ is not fuzzy controllable. Take $\bar{s} = \bar{b}$ and $\bar{\sigma} = \bar{c}$. Then

$$\min\{pr(\bar{K})^f(\bar{s}), \Sigma_{uc}(\bar{\sigma}), \mathcal{L}_{\mathcal{L}_{\bar{G}}}(\Sigma_o(\bar{\sigma}))\} = \min\{0.7 \times 0.5, 0.5 \times 0.8, 0.5 \times 0.8\} = 0.35.$$

However,

$$pr(\bar{K})^f(\bar{s}) = 0.5 \times 0.4 = 0.2.$$

Therefore, the fuzzy controllability condition does not hold.

If $\Sigma_{uc}$ is changed into $\Sigma_{uc}(\bar{\sigma}) \leq 0.05$ for any $\bar{\sigma} \in \bar{E}$, then we can check that the fuzzy controllability condition holds.

Before setting up the Controllability and Observability Theorem of FDESs, we need a characterization of the observability of fuzzy sublanguage.

Definition 10: Let $\bar{K} \subseteq \mathcal{L}_{\bar{H}}$. If for any $\bar{s} \in \bar{E}^*$ and $\bar{\sigma} \in \bar{E}$, then the following inequality holds:

$$\min\{pr(\bar{K})^f(\bar{s}), pr(\bar{K})^f(\bar{\bar{\sigma}}), \mathcal{L}_{\bar{G}}(\Sigma_o(\bar{\bar{\sigma}}))\} \leq pr(\bar{K})^f(\bar{s})$$

(14)

for any $\bar{\bar{\sigma}} \in \bar{E}^*$, where $P(\bar{s}) = P(\bar{\bar{\sigma}})$, then $\bar{K}$ is said satisfying fuzzy observability condition with respect to $\bar{G}$ and $P$.

Intuitively, (14) means that if there is another string $\bar{\bar{\sigma}}$ possessing the same projection as $\bar{s}$, then under the effect of observable projection, the degree to which string $\bar{s}$ belongs to the prefix-closure of $\bar{K}$ and fuzzy event $\bar{\bar{\sigma}}$ following string $\bar{s}$ is physically possible together with $\bar{\bar{\sigma}}$ belonging to the prefix-closure of $\bar{K}$, is not larger than the possibility of string $\bar{s}\bar{\bar{\sigma}}$ belonging to the prefix-closure of $\bar{K}$.

Example 2. Consider a fuzzy automaton $\bar{G} = (\bar{Q}_1, \bar{E}, \bar{\delta}, \bar{q}_0)$, where $\bar{E} = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$, $\bar{q}_0 = [0.8, 0]$, and

$$\bar{a} = [0.8, 0.2], \bar{b} = [0.2, 0.8], \bar{c} = [0.5, 0.5], \bar{d} = [0.2, 0.4].$$

Let $pr(\bar{K})$ be generated by a fuzzy automaton $\bar{H} = (\bar{Q}_2, \bar{E}, \bar{\delta}, \bar{p}_0)$, where $\bar{p}_0 = [0.5, 0]$, $\bar{E} = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$, and $\bar{a}, \bar{b}$ are the same as those in $\bar{G}$, but $\bar{c}$ and $\bar{d}$ are changed as follows:

$$\bar{c} = [0.3, 0.4], \bar{d} = [0.2, 0.4].$$

Suppose that $\Sigma_{uc}$ is defined as: $\Sigma_{uc}(\bar{a}) = 0.5$, $\Sigma_{uc}(\bar{b}) = 0.7$, $\Sigma_o(\bar{c}) = 0.4$, and $\Sigma_o(\bar{d}) = 0$.

If we take $\bar{s} = \bar{b}$, $\bar{\bar{\sigma}} = \bar{c}$ and $\bar{\bar{\bar{\sigma}}} = \bar{d}$, then $P(\bar{s}) = P(\bar{\bar{\bar{\sigma}}}$, and

$$\min\{pr(\bar{K})^f(\bar{s}), pr(\bar{K})^f(\bar{\bar{\sigma}}), \mathcal{L}_{\bar{G}}(\Sigma_o(\bar{\bar{\sigma}}))\} = \min\{0.7 \times 0.4, 0.4 \times 0.4, 0.4 \times 0.5\} = 0.16.$$

However,

$$pr(\bar{K})^f(\bar{s}) = 0.4 \times 0.3 = 0.12.$$

Therefore, the fuzzy observability condition does not hold.

On the basis of the preliminaries, we are ready to present the main theorem of the paper.

Theorem 1: (Controllability and Observability Theorem of FDESs). Let $\bar{G} = (\bar{Q}, \bar{E}, \bar{\delta}, \bar{q}_0, \bar{Q}_m)$ be a fuzzy automaton with a projection $P$. Suppose that fuzzy language $\bar{K} \subseteq \mathcal{L}_{\bar{G},m}$ satisfies $\bar{K}(\epsilon) = 1$ and $pr(\bar{K}) \subseteq \mathcal{L}_{\bar{G},m}$. Then there exists a nonblocking fuzzy supervisor $\bar{S}_P : P(\bar{E}^*) \rightarrow \mathcal{F}(\bar{E})$, such that $S_P$ satisfies the fuzzy admissibility condition, and

$$\mathcal{L}_{\bar{S}_P,G}(\bar{s}) = pr(\bar{K})^f(\bar{s})$$

and

$$\mathcal{L}_{\bar{S}_P,G,m}(\bar{s}) = \bar{K}(\bar{s})$$

for any $\bar{s} \in \bar{E}^*$, if, and only if the following conditions hold:

1. $\bar{K}$ satisfies fuzzy controllability condition w.r.t. $\bar{G}$, $P$ and $\Sigma_{uc}$.
2. $\bar{K}$ satisfies fuzzy observability condition w.r.t. $\bar{G}$ and $P$.
3. $\bar{K}$ is $\mathcal{L}_{\bar{G},m}$-closed.

Proof: See Appendix.
IV. REALIZATION OF SUPERVISORS IN CONTROLLABILITY AND OBSERVABILITY THEOREM OF FDES

In this section, we present a detailed computing method to verify the controllability and observability conditions. Thus, this method can decide the existence of supervisors in Controllability and Observability Theorem of FDESs. As applications, two examples are elaborated to illustrate that this computing method is suitable to check the existence of supervisors not only for FDESs but also for classical DESs.

A. Method of Checking the Existence of Supervisors for FDESs

Clearly, the existence of supervisor is associated with both fuzzy controllability condition and fuzzy observability condition. Therefore, testing the two conditions described by Ineqs. (13, 14) is of great importance. In classical DESs, for a given automaton $G$ and a language $K$, the controllability condition is checked by comparing the active event set of each state of $H \times G$ with the active event set of each state of $G$, where automaton $H$ generates $pr(K)$. And the observability condition is checked by building an observer of an automaton with unobservable events at each site [1].

For FDESs, a computing method of checking the fuzzy controllability condition was given by Qiu [20]. Based on the main idea of the finiteness of fuzzy states in FDESs modeled by max-min automata, we present a detailed approach for testing the fuzzy observability condition by means of computing trees. Let $G = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0, Q_m)$ be a fuzzy automaton with partial observations and $\tilde{E} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$. Assume that the prefix-closure of fuzzy language $\tilde{K} \subseteq L_{G,m}$ is generated by a fuzzy automaton $\tilde{H} = (\tilde{Q}_1, \tilde{E}, \tilde{\delta}, \tilde{p}_0)$. We describe the computing process via three steps as follows.

The first step gives a computing tree for deriving the set of all fuzzy states reachable from the initial state $\tilde{q}_0$, and the sets of strings respectively corresponding to each accessible fuzzy state are also obtained. The basic idea is based on the following two points:

- $\tilde{p}_0 \circ \tilde{s} = \tilde{p}_0 \circ \tilde{s} \circ (\tilde{t})^k$ for any $k \geq 0$ if $\tilde{p}_0 \circ \tilde{s} = \tilde{p}_0 \circ \tilde{s} \circ \tilde{t}$ for $\tilde{t} \in \tilde{E}^*$, where $(\tilde{t})^k$ denotes the $\circ$ product of $k$’s $\tilde{t}$.
- The set of fuzzy states $\{\tilde{p}_0 \circ \tilde{s} : \tilde{s} \in \tilde{E}^*\}$ is always finite since $\tilde{E}$ is finite [20].

Without loss of generality, we present the computing tree for $\tilde{E} = \{\tilde{a}_1, \tilde{a}_2\}$ of two fuzzy events via Fig. 1, and the case of more than two fuzzy events is analogous.

Step 1: For a fuzzy automaton $\tilde{H} = (\tilde{Q}_1, \tilde{E}, \tilde{\delta}, \tilde{p}_0)$, we search for all possible fuzzy states $\tilde{r}_i$ reachable from $\tilde{p}_0$ in $\tilde{H}$, $i = 1, 2, \ldots, m_1$; also, we can obtain the sets $C(\tilde{r}_i)$ of all fuzzy event strings whose inputs lead $\tilde{p}_0$ to $\tilde{r}_i$, $i = 1, 2, \ldots, m_1$. This process can be realized by the finite computing tree that is visualized by Fig. 1 as follows.

In the computing tree, the initial fuzzy state $\tilde{p}_0$ is its root; each vertex, say $\tilde{p}_0 \circ \tilde{s}$, may produce $n$’s sons, i.e., $\tilde{p}_0 \circ \tilde{s} \circ \tilde{a}_i$, $i = 1, 2, \ldots, n$. However, if $\tilde{p}_0 \circ \tilde{s} \circ \tilde{a}_i$ equals some its father, then $\tilde{p}_0 \circ \tilde{s} \circ \tilde{a}_i$ is a leaf, that is marked by a underline. The computing ends with a leaf at the end of each branch.

For two fuzzy automata $\tilde{G}$ and $\tilde{H}$, our purpose is to search for the all different fuzzy state pairs reachable from the initial fuzzy state pair $(\tilde{q}_0, \tilde{p}_0)$. The method is similar to Step 1, which is also carried out by a computing tree. In this computing tree, the root is labelled with pair $(\tilde{q}_0, \tilde{p}_0)$, and each vertex, say $(\tilde{q}_0 \circ \tilde{s}, \tilde{p}_0 \circ \tilde{s})$ for $\tilde{s} \in \tilde{E}^*$, may produce $n$’s sons, i.e., $(\tilde{q}_0 \circ \tilde{s} \circ \tilde{a}_i, \tilde{p}_0 \circ \tilde{s} \circ \tilde{a}_i)$, $i = 1, 2, \ldots, n$. But if a pair $(\tilde{q}_0 \circ \tilde{s} \circ \tilde{a}_i, \tilde{p}_0 \circ \tilde{s} \circ \tilde{a}_i)$ is the same as some of its fathers, then this pair will be treated as a leaf, that is marked with a underline. Such a computing tree is depicted by Fig. 2. Since the set of all fuzzy state pairs is finite, the computing tree ends with a leaf at the end of each branch.

Step 2: For fuzzy automata $\tilde{G}$ and $\tilde{H}$, we search for all possible pairs of fuzzy states $(\tilde{q}_i, \tilde{p}_i)$, $i = 1, 2, \ldots, m_2$, reachable from $(\tilde{q}_0, \tilde{p}_0)$ by a finite computing tree (Fig. 2), and, in the same time, we can decide the sets $C(\tilde{q}_i, \tilde{p}_i)$ of all fuzzy event strings each of which makes $(\tilde{q}_0, \tilde{p}_0)$ become $(\tilde{q}_i, \tilde{p}_i)$, $i = 1, 2, \ldots, m_2$.
we try to choose a shorter string, and this will decrease our computing complexity in what follows). Given any \( i \in \{1, 2, \ldots, m_2\} \), by \( FR(\tilde{q}_i, \tilde{p}_i) \) we mean the set of all strings \( \tilde{t}_{ij} \) we have chosen, say
\[
FR(\tilde{q}_i, \tilde{p}_i) = \{\tilde{t}_{i1}, \tilde{t}_{i2}, \ldots, \tilde{t}_{ik}\}.
\]
If Ineq. (14) holds for each \( \tilde{s}_i \in C(\tilde{q}_i, \tilde{p}_i) \) and each \( \tilde{t}_{ij} \in FR(\tilde{q}_i, \tilde{p}_i) \) where \( i \in \{1, 2, \ldots, m_2\} \) and \( j \in \{1, 2, \ldots, k_i\} \), then the fuzzy observability condition (14) holds; otherwise it does not hold. This is further verified by the following Proposition 2.

**Proposition 2:** Let \( \tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{s}, \tilde{q}_0, \tilde{Q}_m) \) and \( \overline{H} = (\tilde{Q}_1, \tilde{E}, \tilde{\delta}, \tilde{p}_0) \) be two fuzzy automata. Suppose that fuzzy sublanguage \( \tilde{K} \) satisfies \( pr(\tilde{K}) = L_{\overline{H}} \subseteq L_{\tilde{G}} \).

**Proof:** For any \( \tilde{t} \in \tilde{E}^* \), without loss of generality, suppose that \( \tilde{t} \in C(\tilde{q}_0, \tilde{p}_0) \) for some \( i_0 \in \{1, 2, \ldots, m_2\} \), since \( \tilde{E}^* = \bigcup_{i=1}^{m_2} C(\tilde{q}_i, \tilde{p}_i) \). For any \( \tilde{t} \in \tilde{E}^* \) satisfying \( P(\tilde{t}) = P(\tilde{t}) \), then \( \tilde{t} \in P(\tilde{q}_0, \tilde{p}_0) \), and we can further assume \( \tilde{t} \in C(\tilde{t}_{i_0}) \) for some \( j_0 \in \{1, 2, \ldots, m_1\} \), due to \( \tilde{E}^* = \bigcup_{j=1}^{m_1} C(\tilde{t}_j) \). Therefore, there is \( \tilde{t}_{i_0j_0} \in FR(\tilde{q}_i, \tilde{p}_i) \).

Now we have the following relations:
\[
pr(\tilde{K})(\tilde{t}) = [\tilde{p}_0 \circ \tilde{t}] = [\tilde{p}_0 \circ \tilde{s}_{i_0}], \quad (16)
\]
\[
pr(\tilde{K})(\tilde{t} \circ \tilde{\sigma}) = [\tilde{p}_0 \circ \tilde{t} \circ \tilde{\sigma}] = [\tilde{p}_0 \circ \tilde{t}_{i_0j_0} \circ \tilde{\sigma}], \quad (17)
\]
\[
pr(\tilde{K})(\tilde{t} \circ \tilde{\sigma}) = [\tilde{p}_0 \circ \tilde{t} \circ \tilde{\sigma}] = [\tilde{p}_0 \circ \tilde{t}_{i_0j_0} \circ \tilde{\sigma}], \quad (18)
\]
\[
L_{\tilde{G}}(\tilde{t} \circ \tilde{\sigma}) = [\tilde{q}_0 \circ \tilde{t} \circ \tilde{\sigma}] = [\tilde{q}_0 \circ \tilde{t}_{i_0j_0} \circ \tilde{\sigma}]. \quad (19)
\]

By means of the existing condition in this proposition, we know that
\[
\min\{pr(\tilde{K})^f(\tilde{s}_{i_0}), pr(\tilde{K})^f(\tilde{t}_{i_0j_0} \circ \tilde{\sigma}), L_{\tilde{G}}^f(\tilde{s}_{i_0} \circ \tilde{\sigma})\} \leq pr(\tilde{K})^f(\tilde{s}_{i_0} \circ \tilde{\sigma}). \quad (20)
\]

In terms of Eqs. (16-19) and Ineq. (20) we therefore obtain
\[
\min\{pr(\tilde{K})^f(\tilde{t}), pr(\tilde{K})^f(\tilde{t} \circ \tilde{\sigma}), L_{\tilde{G}}^f(\tilde{t} \circ \tilde{\sigma})\} \leq pr(\tilde{K})^f(\tilde{t} \circ \tilde{\sigma}). \quad (21)
\]

and this completes the proof of proposition.

Based on the above Proposition 2, we can check the fuzzy observability condition described by Ineq. (14) by the above computing flow (Steps 1–3). Furthermore, the fuzzy controllability condition described Ineq. (13) also can be clearly tested by similar computing flow with slight changes (\( pr(\tilde{K})^c(\tilde{s}) \) is replaced by \( \tilde{\Sigma}_{mc}(\tilde{\sigma}) \)), besides using the approach proposed by Qiu [20].

**Remark 6.** To conclude this section, we roughly analyze the complexity of the above computing flow. Suppose that the number of all different fuzzy states \( \{\tilde{n}_0 \circ \tilde{s} : \tilde{s} \in \tilde{E}^*\} \) is \( m_1 \), and the number of all different fuzzy state pairs reachable from the initial fuzzy state pair (\( \tilde{q}_0, \tilde{p}_0 \)), namely, \( \{(\tilde{q}_0 \circ \tilde{s}, \tilde{p}_0 \circ \tilde{s}) : \tilde{s} \in \tilde{E}^*\} \), is \( m_2 \). Then, in Step 1, by means of Figure 1 the number of computing steps is \( O(m_1) \) and, in Step 2, in terms of Figure 2 the number of computing steps is \( O(m_2) \). As to Step 3, we can see that the computing complexity is
\[
O(m_1m_2|E|), \quad \text{where } |E| \text{ is the cardinal number of alphabet } E. \quad \text{Thus, if we avoid the cost regarding the operation } \circ, \text{ the computing steps of the above flow is } O(m_1m_2|E|).
\]

**B. Applications to Supervisory Control of Classical DESs and FDESs**

In this subsection, we present two examples to illustrate the applications of the supervisory control theory for FDESs presented above. Example 3 will indicate that the computing approach given in Section IV-A can be applied to check the existence of supervisors for classical DESs. Example 4 arising from a medical treatment will describe a detailed computing processing for FDESs, which may be viewed as an applicable background of supervisory control of FDESs under partial observations.

We first recall some notions of classical DESs. Let \( G \) be a classical DES. Suppose that \( \Sigma_c \) and \( \Sigma_o \) are designated as controllable and observable event sets, respectively. \( P \) is the corresponding projection. A language \( K \) is said to be observable with respect to \( G \) and \( P \), if for all \( s, t \in pr(K) \) and all \( \sigma \in \Sigma_c \), if \( P(s) = P(t) \), then \( s \sigma \in L_G \), \( t \sigma \in pr(K) \) \( \Rightarrow \) \( s \sigma \in pr(K) \). \quad (22)

In classical DESs [1], for a given automaton \( G \) and a language \( K \), the controllability condition is checked by comparing the active event set of each state of \( H \times G \) with the active event set of each state of \( G \), where automaton \( H \) generates \( pr(K) \). And the observability condition is checked by building an observer of an automaton with unobservable events at each site [1].

**Example 3:** Consider the example presented in Section 3.7 of [1] (Example 3.18, page 196) to illustrate the method of testing the observability condition (22). \( G \) and \( H \) are two automata of classical DESs with crisp state set \( E = \{u, b\} \) shown in Fig. 3. Language \( K \) satisfies \( pr(K) = L_H \). Assume that \( \Sigma_c = \{b\} \) and \( \Sigma_o = \{u, b\} \). In order to test \( K \) being unobservable, an observer automaton \( H_{obs} \) is constructed in [1]. In fact, the observability condition (22) cannot be satisfied when \( s = \epsilon, t = u \) and \( \sigma = b \).

![Fig. 3. Automata G and H of classical DESs in Example 3.](image_url)

In the following, we verify the above conclusion by means of the computing method we presented in Section IV-A. Firstly, classical DES \( G \) can be viewed as a fuzzy automaton \( \tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0) \), where the fuzzy states are
\[
\tilde{q}_0 = [1, 0, 0, 0], \quad \tilde{q}_1 = [0, 1, 0, 0],
\]
\[
\tilde{q}_2 = [0, 0, 1, 0], \quad \tilde{q}_3 = [0, 0, 0, 1],
\]
and the fuzzy events are
\[
\bar{u} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Similarly, the automaton \( H \) can be viewed as a fuzzy automaton \( \tilde{H} = (\tilde{Q}_2, \tilde{E}, \tilde{\delta}, \tilde{p}_0) \), where the fuzzy states are
\[
\tilde{p}_0 = [1, 0, 0, 0], \quad \tilde{p}_1 = [0, 1, 0, 0], \quad \tilde{p}_2 = [0, 0, 1, 0],
\]
and the fuzzy events are
\[
\bar{u} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

The fuzzy subsets \( \tilde{\Sigma}_o \) and \( \tilde{\Sigma}_c \) are determined by \( \Sigma_o = \{ b \} \) and \( \Sigma_c = \{ u, b \} \), which are listed as follows:
\[
\tilde{\Sigma}_o(\tilde{u}) = 0, \quad \tilde{\Sigma}_o(\tilde{b}) = 1; \quad \tilde{\Sigma}_c(\tilde{u}) = \tilde{\Sigma}_c(\tilde{b}) = 1.
\]

By constructing the computing trees of \( H, \tilde{G} \) and \( \tilde{H} \), we know that there are three fuzzy states \( \tilde{p}_0, \tilde{p}_1, \tilde{p}_2 \) reachable from \( \tilde{p}_0 \), and three fuzzy states pairs \((\tilde{q}_0, \tilde{p}_0), (\tilde{q}_1, \tilde{p}_1), (\tilde{q}_2, \tilde{p}_2)\) reachable from \((\tilde{q}_0, \tilde{p}_0)\), and the corresponding fuzzy event strings are \( \epsilon, \tilde{u}, \tilde{v} \) and \( \tilde{u} \tilde{b} \). Therefore, we should necessarily check the fuzzy observability condition in term of whether the all elements in the rightmost column of the following Table I are “T” (True) when \( \tilde{s} = \epsilon, \tilde{s} = \tilde{u}, \tilde{s} = \tilde{u} \tilde{b} \), where
- \( x_1 = [\tilde{p}_0 \circ \tilde{s}], x_2 = [\tilde{p}_0 \circ \tilde{u} \circ \tilde{s}], x_3 = [\tilde{q}_0 \circ \tilde{s} \circ \tilde{\sigma}], \)
- \( y = [\tilde{p}_0 \circ \tilde{s} \circ \tilde{\sigma}], \)
- \( V = \min\{pr(K) f(\tilde{s}), pr(\tilde{K}) f(\tilde{\sigma}), L_G^f(\tilde{s}, \tilde{\sigma})\}, \)
- \( W = pr(K) f(\tilde{\sigma}). \)

From Table I we see that the fuzzy observability condition does not hold since an “F” (False) has been found out in the rightmost column when \( \tilde{s} = \epsilon, \tilde{t} = \tilde{u} \) and \( \tilde{\sigma} = \tilde{b} \).

Example 3 indicates that our method also can be applied to testing the existence of supervisors for classical DESs [1]. Next we apply our results to an applicable example arising from a medical treatment problem.

Example 4: Suppose that there is a patient sickening for a new disease. For simplicity, it is assumed that the doctors consider roughly the patient’s condition to be two states, say “poor” and “good”. For the new disease, the doctors have no complete knowledge about it, but they believe by their experience that these drugs such as theophylline, Erythromycin Ethylsuccinate and dopamine may be useful to the disease.

As mentioned in Introduction, considering the features of vagueness, patient’s condition can simultaneously belong to “poor” and “good” with respective memberships; also, an event occurring (i.e., treatment) may lead a state to multistates with respective degrees. Therefore, the patient’s conditions and their changes after the treatments can be modeled by an FDES \( \tilde{G} = (\tilde{Q}_1, \tilde{E}, \tilde{\delta}, \tilde{q}_0, \tilde{Q}_m) \), in which each fuzzy state, denoted as a two-dimensional vector \( \tilde{q} = [a_1, a_2] \), is represented as the possibility distribution of the patient’s condition over the two crisp states “poor” and “good”; each fuzzy event, denoted as a \( 2 \times 2 \) matrix \( \tilde{\sigma} = [a_{ij}]_{2 \times 2} \), means the possibility for patient’s condition to transfer from one crisp state to another crisp state when a certain drug treatment is adopted.

Suppose that the patient’s initial condition is \( \tilde{q}_0 = [0.9, 0] \). The drug events \( \tilde{a}, \tilde{b}, \tilde{c} \), namely, theophylline, Erythromycin Ethylsuccinate and dopamine, respectively, may be evaluated according to doctors’ experience as follows:
\[
\tilde{a} = \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}.
\]

We specify a fuzzy set of control specifications \( \tilde{K} \) that are desired for the doctors. For the sake of simplicity, it is assumed that \( \tilde{K} \) is \( L_{\tilde{G}, m} \)-closed. As usual, let \( pr(\tilde{K}) \) be generated by a fuzzy automaton \( \tilde{H} = (\tilde{Q}_2, \tilde{E}, \tilde{\delta}, \tilde{p}_0) \), where \( \tilde{p}_0 = [0.9, 0], \tilde{E} = \{\tilde{a}, \tilde{b}, \tilde{c}\} \), with \( \tilde{a}, \tilde{b}, \tilde{c} \) being the same as those in \( G \), except that \( \tilde{c} \) is changed as follows:
\[
\tilde{c} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}.
\]

For these drug events, some effects such as headache disappears are clearly observed, but some effects may be observed only by means of medical instruments; also, some effects such as alleviation of pain can be controlled, but some potential side effects may be uncontrolled. Therefore, each drug event may be observed or controlled with some membership degrees.

Suppose that \( \tilde{\Sigma}_{uc} \) and \( \tilde{\Sigma}_o \) are defined as follows:
\[
\tilde{\Sigma}_{uc}(\tilde{a}) = \tilde{\Sigma}_{uc}(\tilde{b}) = 0.1, \quad \tilde{\Sigma}_{uc}(\tilde{c}) = 0.2; \quad \tilde{\Sigma}_o(\tilde{a}) = 0.4, \quad \tilde{\Sigma}_o(\tilde{b}) = 0.6, \quad \tilde{\Sigma}_o(\tilde{c}) = 0.
\]

In supervisory control of FDESs, the purpose of non-blocking fuzzy supervisors is to disable the fuzzy events with respective degrees such that the generated and marked behaviors of the supervised system satisfy some prespecified specifications, and the controlled system does not produce deadlocks. Therefore, for this example, the problem is whether there exists such a nonblocking fuzzy supervisor \( \tilde{S}_P : \tilde{F}(\tilde{E}^*) \rightarrow \tilde{F}(\tilde{E}) \). In the following, we will answer the problem by proving \( \tilde{K} \) to be fuzzy controllable and fuzzy observable by means of computing approach presented in Section IV-A.

| \( a \) | \( t \) | \( \sigma \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( y \) | \( V \) | \( W \) | \( V < W \) |
|------|------|------|------|------|------|------|------|------|------|
| \( e \) | \( \epsilon \) | \( u \) | 1 | 1 | 1 | 1 | 0 | 0 | T |
| \( u \) | \( u \) | 0 | 1 | 0 | 0 | 0 | 0 | T |
| \( \bar{u} \) | \( \epsilon \) | \( u \) | 1 | 1 | 1 | 0 | 0 | F |
| \( u \) | \( u \) | 0 | 1 | 0 | 1 | 0 | T |
| \( \bar{ub} \) | \( u \) | 0 | 1 | 0 | 0 | 0 | 0 | T |
| \( ub \) | \( u \) | 1 | 0 | 0 | 0 | 0 | T |
| \( \bar{ub} \) | \( u \) | 1 | 0 | 0 | 0 | 0 | T |

Table I

TESTING THE FUZZY OBSERVABILITY CONDITION IN EXAMPLE 3
For $\tilde{G}$ and $\tilde{H}$, we search for all possible fuzzy state pairs $(\tilde{q_i}, \tilde{p_i})$ reachable from $(\tilde{q_0}, \tilde{p_0})$ by the finite computing tree shown in Fig.4, which is followed by the other three subtrees visualized by Figs. 5, 6, 7, respectively.

\[
\tilde{(q_0, p_0)} = (0.9, 0) \quad \tilde{(q_0, p_0)}
\]

![Diagram](Fig. 4. Computing tree of all state pairs reachable from $(\tilde{q_0}, \tilde{p_0})$)

\[
\tilde{(q_0, p_0)} = (0.9, 0) \quad \tilde{(q_0, p_0)}
\]

![Diagram](Fig. 5. Subtree $T_1$)

\[
\tilde{(q_0, p_0)} = (0.9, 0) \quad \tilde{(q_0, p_0)}
\]

![Diagram](Fig. 6. Subtree $T_2$)

\[
\tilde{(q_0, p_0)} = (0.9, 0) \quad \tilde{(q_0, p_0)}
\]

![Diagram](Fig. 7. Subtree $T_3$)

From above computing trees, it follows that there are only eight different fuzzy state pairs and eight different fuzzy states reachable from $(\tilde{q_0}, \tilde{p_0})$ and $\tilde{p_0}$, respectively, which together with the corresponding fuzzy event strings are listed in Table II. Therefore, we should necessarily check the fuzzy observability condition only when $s = \epsilon$, or $\tilde{a}$, or $\tilde{b}$, or $\tilde{c}$, or $\tilde{ba\tilde{c}}$, or $\tilde{ba}$, or $\tilde{bc}$, or $\tilde{ca}$.

| $s$ | $(q_0 \circ s, p_0 \circ s)$ |
|-----|-----------------------------|
| $\epsilon$ | $(0.9, 0), (0.9, 0)$ |
| $a$ | $(0.9, 0.4), (0.9, 0.4)$ |
| $b$ | $(0.4, 0.9), (0.4, 0.9)$ |
| $c$ | $(0.4, 0), (0.2, 0)$ |

![Table](Table II: Eight different state pairs reachable from $(\tilde{q_0}, \tilde{p_0})$)

| $c$ | $(0.9, 0), (0.9, 0)$ |
|-----|-----------------------------|
| $b$ | $(0.4, 0.9), (0.4, 0.9)$ |
| $c$ | $(0.4, 0), (0.2, 0)$ |

![Table](Table III: Testing the fuzzy observability condition for $c$ and $\tilde{ca}$)

| $s$ | $(q_0 \circ s, p_0 \circ s)$ |
|-----|-----------------------------|
| $\epsilon$ | $(0.9, 0), (0.9, 0)$ |
| $a$ | $(0.9, 0.4), (0.9, 0.4)$ |
| $b$ | $(0.4, 0.9), (0.4, 0.9)$ |
| $c$ | $(0.4, 0), (0.2, 0)$ |

(1) If $s = \epsilon$, from Fig. 4 we know that $[\tilde{p_0} \circ \tilde{\sigma}] = [\tilde{q_0} \circ \tilde{\sigma}]$ for $\tilde{\sigma} = \tilde{a}$ and $\tilde{\sigma} = \tilde{b}$, so the fuzzy observability condition holds for $\tilde{\sigma} = \tilde{a}$ and $\tilde{\sigma} = \tilde{b}$. For $\tilde{\sigma} = \tilde{c}$, it is clear to know that the fuzzy observability condition holds since $\tilde{D}(P(\tilde{\sigma})) = 0$.

(2) If $s = \tilde{a}$, or $\tilde{b}$, or $\tilde{ba\tilde{c}}$, or $\tilde{ba}$, or $\tilde{bc}$, we check the fuzzy observability condition via Figs. 4, 5, 6. The fuzzy observability condition holds obviously since in subtrees $T_1$ and $T_2$, for any $s$ and any $\tilde{\sigma}$, $[\tilde{p_0} \circ s \circ \tilde{\sigma}] = [\tilde{q_0} \circ s \circ \tilde{\sigma}]$.

(3) We consider the last cases of $s = \tilde{c}$, or $s = \tilde{ca}$. If $s = \tilde{c}$, then $t = \epsilon$, or $t = \tilde{c}$ such that $P(\tilde{s}) = P(t)$. If $s = \tilde{ca}$, then $t = \tilde{a}$, or $t = \tilde{ac}$, or $t = \tilde{ca}$ such that $P(\tilde{s}) = P(t)$. According to Fig. 7, we can test that the fuzzy observability condition holds when $s \in \{\tilde{c}, \tilde{ca}\}$ by means of the following Table III.

In light of the above computing process, we have verified that $K$ satisfies the fuzzy observability condition.

On the other hand, we notice that $\Sigma_{we}(\tilde{\sigma}) \leq 0.2$ and $pr(K)(\tilde{s}\tilde{\sigma}) \geq 0.2$ for any $\tilde{\sigma} \in \tilde{E}$ and any $\tilde{s} \in \tilde{E}^*$, so $K$ satisfies the fuzzy controllability condition clearly.

Therefore, from $K$ being fuzzy observable and fuzzy controllable together with the assumption of $\tilde{K}$ being $\mathcal{L}_{\tilde{G}, \tilde{m}}$-closed, by Theorem 1, we know that there exists a nonblocking fuzzy supervisor $\tilde{S}_p : P(\tilde{E}^*) \rightarrow \tilde{F}(\tilde{E})$ that can disable the fuzzy events with respective degrees such that

\[
\mathcal{L}_{\tilde{S}_p/G}(\tilde{s}) = pr(K)(\tilde{s})
\]

In fact, $\tilde{S}_p$ may be constructed as the proof of Theorem 1 in Appendix.

V. CONCLUDING REMARKS

Since FDES was introduced by Lin and Ying [12, 13], it has been successfully applied to biomedical control for
HIV/AIDS treatment planning [15, 16], decision making [17] and intelligent sensory information processing for robotic control [18, 19]. In view of the impreciseness for some events being observable and controllable in practice, in this paper we dealt with Controllability and Observability Theorem, in which both the observability and the controllability of events are considered to be fuzzy. In particular, we have presented a computing method for deciding whether or not the fuzzy observability and controllability conditions hold, and thus, can further test the existence of supervisors in Controllability and Observability Theorem of FDESs. As some examples (Example 3) presented show, this computing method is clearly applied to testing the existence of supervisors in the Controllability and Observability Theorem of classical DESs [1], and this is a different method from classical case [1].

As pointed out in [1], in supervisory control theory there are three fundamental theorems: Controllability Theorem, Nonblocking Controllability Theorem, and Controllability and Observability Theorem. This paper, together with [20-23], has primarily established supervisory control theory of FDESs. An further issue is regarding the diagnosis of FDESs, as the diagnoses of classical and probabilistic DESs [32, 33]. Also, it is worth further considering to apply the supervisory control theory of FDESs to practical control issues, particularly in biomedical systems and traffic control systems [34, 35]. Moreover, dealing with FDESs modelled by fuzzy petri nets [36] is of interest, as the issue of DESs modelled by Petri nets [37-39].

**APPENDIX A**

**Proof of Theorem 6**

We construct a fuzzy supervisor $\tilde{S}_P : P(\tilde{E}^*) \rightarrow \mathcal{F}(\tilde{E})$ as follows: $\tilde{S}_P(e)(\tilde{\sigma}) = pr(\tilde{K})f(\tilde{\sigma})$, and for $\tilde{s} \in \tilde{E}^*$, $\tilde{S}_P(P(\tilde{s}))(\tilde{\sigma})$ is defined by the following two cases:

**Case 1:** If there exists another string $\tilde{s}' \in \tilde{E}^*$ such that $P(\tilde{s}) = P(\tilde{s}')$, then

$$\begin{align*}
\tilde{S}_P(P(\tilde{s}))(\tilde{\sigma}) &= \left\{ \begin{array}{ll}
\max\{\tilde{\Sigma}_uc(\tilde{\sigma}), \min\{pr(\tilde{K})(\tilde{s}, \tilde{\sigma}), \tilde{\Sigma}_uc(\tilde{\sigma}) \}\}, & \text{if } pr(\tilde{K})(\tilde{s}, \tilde{\sigma}) \leq pr(\tilde{K})(\tilde{s}', \tilde{\sigma}); \\
\max\{\tilde{\Sigma}_uc(\tilde{\sigma}), pr(\tilde{K})(\tilde{s}', \tilde{\sigma}) \}, & \text{if } pr(\tilde{K})(\tilde{s}', \tilde{\sigma}) > pr(\tilde{K})(\tilde{s}, \tilde{\sigma}).
\end{array} \right. \\
\end{align*}$$

(23)

**Case 2:** If there does not exist another string $\tilde{s}' \in \tilde{E}^*$ such that $P(\tilde{s}) = P(\tilde{s}')$, then

$$\begin{align*}
\tilde{S}_P(P(\tilde{s}))(\tilde{\sigma}) &= \left\{ \begin{array}{ll}
\min\{\tilde{\Sigma}_uc(\tilde{\sigma}), \tilde{\Sigma}_uc(\tilde{\sigma}) \}, & \text{if } pr(\tilde{K})(\tilde{s}, \tilde{\sigma}) \leq \tilde{\Sigma}_uc(\tilde{\sigma}), \\
pr(\tilde{K})(\tilde{s}, \tilde{\sigma}), & \text{if } pr(\tilde{K})(\tilde{s}, \tilde{\sigma}) > \tilde{\Sigma}_uc(\tilde{\sigma}).
\end{array} \right. \\
\end{align*}$$

(24)

Firstly we prove the sufficiency.

1. We check the fuzzy admissibility condition. Let $\tilde{s} \in \tilde{E}^*$ and $\tilde{\sigma} \in \tilde{E}$. If $\tilde{s} = \epsilon$, then by the fuzzy controllability condition, we have

$$\min\{\tilde{\Sigma}_uc(\tilde{\sigma}), \tilde{\Sigma}_uc(\tilde{\sigma}) \} \leq pr(\tilde{K})(\tilde{s}, \tilde{\sigma})$$

Therefore, the fuzzy admissibility condition holds when $\tilde{s} = \epsilon$. For $\tilde{s} \neq \epsilon$, we check the fuzzy admissibility condition by the following two cases. (i) If there exists $\tilde{s} \in \tilde{E}^*$ such that $P(\tilde{s}) = P(\tilde{s}')$, then from (23), we have

$$\min\{\tilde{\Sigma}_uc(\tilde{\sigma}), \tilde{\Sigma}_uc(\tilde{\sigma}) \} \leq pr(\tilde{K})(\tilde{s}, \tilde{\sigma})$$

(ii) If there does not exist $\tilde{s} \in \tilde{E}^*$ such that $P(\tilde{s}) = P(\tilde{s}')$, then from (24), we have

$$\min\{\tilde{\Sigma}_uc(\tilde{\sigma}), \tilde{\Sigma}_uc(\tilde{\sigma}) \} = pr(\tilde{K})(\tilde{s}, \tilde{\sigma})$$

when $pr(\tilde{K})(\tilde{s}, \tilde{\sigma}) \leq \tilde{\Sigma}_uc(\tilde{\sigma})$, and

$$\min\{\tilde{\Sigma}_uc(\tilde{\sigma}), \tilde{\Sigma}_uc(\tilde{\sigma}) \} = pr(\tilde{K})(\tilde{s}, \tilde{\sigma}) = pr(\tilde{K})(\tilde{s}, \tilde{\sigma})$$

when $pr(\tilde{K})(\tilde{s}, \tilde{\sigma}) > \tilde{\Sigma}_uc(\tilde{\sigma})$.

2. We check $\tilde{L}_{sp}/\tilde{G}(\tilde{s}) = pr(\tilde{K})(\tilde{s}, \tilde{\sigma})$ for any $\tilde{s} \in \tilde{E}^*$, where $\tilde{\Sigma}_uc(\tilde{s}) > 0$. We proceed by induction on the length of $\tilde{s}$. If $|\tilde{s}| = 1$, by Definition 6,

$$\tilde{L}_{sp}/\tilde{G}(\tilde{s}) = \min\{pr(\tilde{K})(\tilde{s}, \tilde{\sigma}), \tilde{L}_{G}(\tilde{s}, \tilde{\sigma}), \tilde{S}_P(P(\tilde{s}))(\tilde{\sigma}) \}.$$
(3) If there does not exist another string $\tilde{s} \in E^*$ such that $P(\tilde{s}) = P(s')$, then with the definition of $\tilde{S}_P(P(\tilde{s}))(\tilde{\sigma})$, i.e., Eq. (24)], we obtain that
\[
L_{SP/G}(\tilde{\sigma}) = \min\{pr(K)^f(\tilde{\sigma}), L^f_G(\tilde{\sigma}), \Sigma^f_{uc}(\tilde{\sigma})\}
\]
when $\Sigma_{uc}(\tilde{\sigma}) \leq pr(K)(\tilde{\sigma})$; and
\[
L_{SP/G}(\tilde{\sigma}) = \min\{L^f_G(\tilde{\sigma}), pr(K)^f(\tilde{\sigma})\}
\]
when $\Sigma_{uc}(\tilde{\sigma}) > pr(K)(\tilde{\sigma})$.

We can analogously verify $L_{SP/G}(\tilde{\sigma}) = pr(K)^f(\tilde{\sigma})$ from the fuzzy controllability condition.

3. We show that $L_{SP/G} = K$ and $S_P$ is nonblocking as follows. Since $K$ is $L_{G,m}$ closed and $L_{SP/G} = pr(K)^f(\tilde{s})$ has been proved above, by Definition 6,
\[
L_{SP/G,m}(\tilde{s}) = \min\{L_{SP/G}(\tilde{s}), L_{G,m}(\tilde{s})\}
\]
We have completed the proof of sufficiency. The remainder is to demonstrate the necessity.

1. We prove that $K$ satisfies the fuzzy controllability condition. Obviously, the fuzzy controllability condition holds for $\tilde{s} = \epsilon$. For any $\tilde{s} e E^*$, by the fuzzy admissibility condition, we have
\[
\min\{pr(K)^f(\tilde{s}), \Sigma^f_{uc}(\tilde{\sigma}), L^f_G(\tilde{\sigma})\}
\]
and $\frac{pr(K)^f(\tilde{\sigma})}{\Sigma^f_{uc}(\tilde{\sigma})} = L^f_G(\tilde{\sigma})$.

2. $K$ is $L_{G,m}$ closed obviously. In fact, from $L_{SP/G,m} = K$, we have
\[
K(\tilde{s}) = L_{SP/G,m}(\tilde{s}) = \min\{L_{SP/G}(\tilde{s}), L_{G,m}(\tilde{s})\} = \min\{pr(K)^f(\tilde{s}), L_{G,m}(\tilde{s})\}.
\]

3. We check that $K$ satisfies the fuzzy observability condition. For any $\tilde{s} e E^*$ and $\tilde{\sigma} e \tilde{E}$, if there exists another string $\tilde{s} e E^*$ such that $P(\tilde{s}) = P(s')$, then the fuzzy observability condition holds obviously if $pr(K)(\tilde{\sigma}) > pr(K)(\tilde{s})$. If $pr(K)(\tilde{\sigma}) \leq pr(K)(\tilde{\sigma})$, we have
\[
\min\{pr(K)^f(\tilde{s}), L^f_G(\tilde{\sigma})\}
\]
and $\frac{pr(K)^f(\tilde{s})}{L^f_G(\tilde{\sigma})} = L^f_G(\tilde{\sigma})$.

Therefore, $K$ satisfies the fuzzy observability condition. And the proof of necessity is completed.

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