Hybrid Block Successive Approximation for One-Sided Non-Convex Min-Max Problems: Algorithms and Applications

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Abstract

The min-max problem, also known as the saddle point problem, is a class of optimization problems in which we minimize and maximize two subsets of variables simultaneously. This class of problems can be used to formulate a wide range of signal processing and communication (SPCOM) problems. Despite its popularity, existing theory for this class has been mainly developed for problems with certain special convex-concave structure. Therefore, it cannot be used to guide the algorithm design for many interesting problems in SPCOM, where some kind of non-convexity often arises.

In this work, we consider a general block-wise one-sided non-convex min-max problem, in which the minimization problem consists of multiple blocks and is non-convex, while the maximization problem is (strongly) concave. We propose a class of simple algorithms named Hybrid Block Successive Approximation (HiBSA), which alternatingly performs gradient descent-type steps for the minimization blocks and one gradient ascent-type step for the maximization problem. A key element in the proposed algorithm is the introduction of certain properly designed regularization and penalty terms, which are used to stabilize the algorithm and ensure convergence. For the first time, we show that such simple alternating min-max algorithms converge to first-order stationary solutions, with quantifiable global rates. To validate the efficiency of the proposed algorithms, we conduct numerical tests on a number of information processing and wireless communication problems, including the robust learning problem, the non-convex min-utility maximization problems, and certain wireless jamming problem arising in interfering channels.

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I. INTRODUCTION

Consider the min-max (a.k.a. saddle point) problem below:

\[
\begin{align*}
\min_{x} \max_{y} & \quad f(x_1, x_2, \cdots, x_K, y) + \sum_{i=1}^{K} h_i(x_i) - g(y) \\
\text{s.t.} & \quad x_i \in \mathcal{X}_i, \ y \in \mathcal{Y}, \ i = 1, \cdots, K
\end{align*}
\]  

where \( f : \mathbb{R}^{NK+M} \rightarrow \mathbb{R} \) is a continuously differentiable function; \( h_i : \mathbb{R}^{N} \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^{M} \rightarrow \mathbb{R} \) are some convex possibly non-smooth function; \( x := [x_1; \cdots; x_K] \in \mathbb{R}^{N \cdot K} \) and \( y \in \mathbb{R}^{M} \) are the block optimization variables; \( \mathcal{X}_i \)'s and \( \mathcal{Y} \) are some convex feasible sets. We call the problem one-sided non-convex problem because we assume that \( f(x, y) \) is non-convex w.r.t. \( x \), and (strongly) concave w.r.t. \( y \).

For notational simplicity, we will use \( \ell(x_1, x_2, \cdots, x_K, y) \) to denote the overall objective function for problem (1).

Problem (1) is quite generic, and it arises in a wide range of signal processing and communication (SPCOM) applications. We list of few of these applications below.

A. Motivating Examples in SPCOM

**Distributed non-convex optimization:** Consider a network of \( K \) agents defined by a connected graph \( G = \{\mathcal{V}, \mathcal{E}\} \) with \(|\mathcal{V}| = K\), where each agent \( i \) can communicate with its neighbors. A generic problem formulation that captures many problems that appear in distributed machine learning and signal processing (e.g., [2]–[5]) can be formulated as follows:

\[
\min_{\{x_i\}} \sum_{i=1}^{K} f_i(x_i) + h_i(x_i), \quad \|x_i - x_j\| \leq c_{i,j}, \ (i, j) \text{ are neighbors}
\]

where each \( f_i : \mathbb{R}^{N} \rightarrow \mathbb{R} \) is a non-convex, smooth function, \( h_i : \mathbb{R}^{N} \rightarrow \mathbb{R} \) is convex non-smooth function that plays the role of the regularizer and \( x_i \in \mathbb{R}^{N} \) is agent \( i \)'s local variable. Each agent \( i \) has access only to \( f_i \) and \( h_i \). The non-negative constants \( c_{i,j} \) are predefined, and they can be selected to represent different requirements on agent consensus. A common practice is to set \( c_{i,j} \equiv 0, \ \forall \ i, j \), and in this way the above problem reduces to the popular consensus-type problem [6]. If \( c_{i,j} \)'s are strictly non-negative, the problem becomes the partial consensus problem which often arises in distributed estimation [7].

Despite the fact that there have been a number of recent works on distributed non-convex optimization [8]–[12], the above problem formulation cannot be covered by any of these due to two main reasons: 1) the nonsmooth regularizers \( h_i \)'s can be different across different local variables, invalidating the assumptions made in, e.g., [8] (which requires uniform regularizer across the nodes), and 2) the partial consensus constraints are considered rather than the exact consensus where \( c_{i,j} \equiv 0, \ \forall \ i, j \), and such general model cannot be covered in any of the works discussed above.
The above problem can be equivalently expressed as:

\[
\min_{x, \tilde{x}} \ f(x) + h(x) := \sum_{i=1}^{K} (f_i(x_i) + h_i(x_i)) \\
\text{s.t.} \quad (A \otimes I_N)x - \tilde{x} = 0, \quad \tilde{x} \in Z \subseteq \mathbb{R}^{|E| \cdot N}
\]

where \( x := [x_1; \ldots; x_K] \in \mathbb{R}^{KN} \); \( A \in \mathbb{R}^{|E| \times K} \) is the incidence matrix, i.e., assuming that the edge \( e \) is incident on vertices \( i \) and \( j \), with \( i > j \) we have that \( A_{ei} = 1, A_{ej} = -1 \) and \( A_{e.} = 0 \) for all other vertices; \( \otimes \) denotes the Kronecker product. \( \tilde{x} \in \mathbb{R}^{|E|N} \) is the auxiliary variable representing the difference between two neighboring local variables; the feasible set \( Z \) represents the bounds on the size of the differences. Using duality theory we can introduce the Lagrangian multiplier vector \( y \) and rewrite the above problem as:

\[
\min_{x \in \mathbb{R}^{KN}, \tilde{x} \in Z} \max_{y \in \mathbb{R}^{|E| \times N}} f(x) + h(x) + \langle y, (A \otimes I_N)x - \tilde{x} \rangle.
\]

See Sec. III-A for detailed discussion on this reformulation. Clearly (3) is in the form of (1). Despite the fact that there have been a number of recent works on distributed non-convex optimization, to the best of our knowledge there has been no analysis on problems that takes the general forms (2) and (3).

**Robust learning over multiple domains:** In [13] the authors introduce a robust learning framework, in which the training sets from \( M \) different domains are used to train a machine learning model. Let \( S_m = \{(s_i^m, t_i^m)\}, 1 \leq m \leq M \) be the individual training sets with \( s_i^m \in \mathbb{R}^N, t_i^m \in \mathbb{R} \); \( x \) be the parameter of the model we intent to learn, \( l(\cdot) \) a non-negative loss function, and \( f_m(x) = \frac{1}{|S_m|} \sum_{(s_i^m, t_i^m) \in S_m} l(s_i^m, t_i^m, x) \) is the (possible) non-convex empirical risk in the \( m \)-th domain. The following problem formulates the task of finding the parameter \( x \) that minimizes the empirical risk, while taking into account the worst possible distribution over the \( M \) different domains:

\[
\min_{x} \max_{y \in \Delta} y^T F(x) - \frac{\lambda}{2} D(y||q)
\]

where \( F(x) := [f_1(x); \ldots; f_M(x)] \in \mathbb{R}^{M \times 1} \); \( y \) describes the adversarial distribution over the different domains; \( \Delta := \{y \in \mathbb{R}^M \mid 0 \leq y_i \leq 1, i = 1, \ldots, M, \sum_{i=1}^{M} y_i = 1\} \) is the standard simplex; \( D(\cdot) \) is some some distance between probability distributions, \( q \) is some prior probability distribution, and \( \lambda > 0 \) is some constant. The last term in the objective function represents some regularizer that imposes structures on the adversarial distribution.

**Power control and transceiver design problem:** Consider a problem in wireless transceiver design, where \( K \) transmitter-receiver pairs transmit over \( N \) channels to maximize their minimum rates. User \( k \) transmits with power \( x_k := [x_1^k; \ldots; x_N^k] \), and its rate is given by (assuming Gaussian signaling):

\[
R_k(x_1, \ldots, x_K) = \sum_{n=1}^{N} \log \left( 1 + \frac{a_{kk}^n x_k^n}{\sigma^2 + \sum_{l=1, l \neq k}^{K} a_{lk}^n x_l^n} \right),
\]
which is a non-convex function on \( x := [x_1; \cdots; x_K] \). Here \( a_{\ell k}^n \)'s denote the channel gain between the pair \((\ell, k)\) on the \( n \)th channel, and \( \sigma^2 \) is the noise power. Let \( \bar{x} \) denote the power budget for each user, then the classical max-min fair power control problem is: \( \max_{x \in \mathcal{X}} \min_k R_k(x) \), where \( \mathcal{X} := \{ x \mid 0 \leq \sum_n x_n^k \leq \bar{x}, \forall k \} \) denotes the feasible power allocations. The above max-min rate problem can be equivalently formulated as \( \mathbf{1} \) (see Sec. III-A for details):

\[
\min_{x \in \mathcal{X}} \max_{y \in \Delta} \sum_{k=1}^{K} -R_k(x_1, \cdots, x_K) \times y_k,
\]

where the set \( \Delta \subseteq \mathbb{R}^K \) is again the standard simplex.

A closely related problem is the coordinated beamforming design in an (multiple input single output) MISO interference channel. In this case the target is to find the optimal beamforming vector for each user in order to maximize some system utility function under the total power and outage probability constraints \( \mathbf{14} \). When the min-rate utility is used, this problem can be formulated as

\[
\max_{x_i \in \mathbb{C}^{N_t}, i} \min R_i(\{x_k\}) \quad \text{s.t.} \quad \|x_i\|^2 \leq \bar{p}, \forall i
\]

where \( x_i \) is transmit beamformer, \( N_t \) denotes the number of antennas. Also, \( R_i(\{x_k\}) = \log_2(1 + \xi_i(\{x_k\}_{k \neq i}) x_i^H Q_{ii} x_i) \), where \( \xi_i \) is introduced to incorporate the outage constraints and the cross-link interference, while \( Q_{ii} \) denotes the covariance matrix of the channel between the \( i \)th transmitter-receiver pair.

For other setups, similar min-max problems can be formulated, some of which can be solved optimally (e.g., power control \( \mathbf{15}-\mathbf{17} \), transmitter density allocation \( \mathbf{18} \), or certain MISO beamforming \( \mathbf{19}, \mathbf{20} \)). But for general multi-channel and/or MIMO interference channel, the corresponding problem is NP-hard \( \mathbf{21} \). Many heuristic algorithms are available for these problems \( \mathbf{21}-\mathbf{24} \), but they are all designed for special problems, and often requires repeatedly invoking general purpose solvers (which can be computationally expensive). For computational tractability, a common approach is to perform the following approximations of the min-rate utility \( \mathbf{25} \):

\[
\min_i r_i \approx -1/\gamma \log_2 \left( \sum_{i=1}^{N} 2^{-\gamma r_i} \right).
\]

However such a procedure can introduce significant rate losses.

**Power control in the presence of a jammer:** Consider an extension of the above scenario (which is first described in \( \mathbf{26} \)), where a jammer participates in a \( K \)-user \( N \)-channel interference channel transmission. Differently from a regular user, the jammer’s objective is to reduce the total sum-rate of the other users

\[1\] A minus sign is added to equivalently transform to the min-max problem.
by properly transmitting noises. Let us use $y^n$ to denote the jammer’s transmission on the $n$th channel, then the corresponding sum-rate maximization-minimization problem can be formulated as:

$$
\min_{x \in X} \max_{y \in Y} \sum_{(k,n)} - \log \left( 1 + \frac{a_{kk}^n x_k^n}{\sigma^2 + \sum_{j=1,j \neq k}^K a_{jk}^n x_j^n + a_{0k}^n y^n} \right),
$$

(8)

where $x_k$ and $y$ are the power allocation of user $k$ and the jammer, respectively; the set $X := X_1 \times \cdots \times X_K$, where $X_k$ are defined similarly as before.

B. Related Work

Motivated by these applications, it is of interest to develop efficient algorithms for solving these problems with theoretical convergence guarantees. In the optimization community, there has been a long history of studying min-max optimization problems. When the problem is convex in $x$ and concave in $y$, previous works [27]–[30] and the references therein have shown that certain primal-dual type algorithms, which alternate between the update of $x$ and $y$ variables, can solve the convex-concave saddle problem optimally. However, when the problem is non-convex, the convergence behavior of such alternating type algorithms has not been well understood.

Although there are many recent works on the non-convex minimization problems [31], only a few works have been focused on the non-convex min-max problems. In a recent line of work [32], [33] the authors study the convergence of vanilla gradient descent/ascent (GDA), where it is established that convergence is not guaranteed even for bilinear problems. An extra term (optimism) is then added to the GDA iterations, leading to an algorithm termed as OGDA, which converges provably to an optimal point in bilinear problems. An optimistic mirror descent algorithm is proposed in [34], and its convergence to a saddle point is established under certain strong coherence assumption. In [13], algorithms for robust optimization have been proposed, where the $x$ problem is unconstrained, and $y$ linearly couples with a non-convex function of $x$ [cf. (4)]. In [35], a proximally guided stochastic mirror descent method (PG-SMD) is proposed, which also updates $x, y$ simultaneously, and provably converges to an approximate stationary point of the problem. Recently, an oracle based non-convex stochastic gradient descent for generative adversarial networks was proposed in [36], [37], where the algorithm solves the maximization subproblem up to some small error. Finally, in [38] the convergence of a primal-dual algorithm to a first-order stationary point is established for a class of generative adversarial networks (GAN) problems formulated as min-max optimization tasks with a coupling term linear w.r.t the discriminator.

C. Contribution of this work

In this work, we design effective algorithms for the min-max problem by adopting the popular block alternating minimization/maximization strategy. The studied problems are quite general, allowing non-
### Table I: Summary of algorithms for the min-max optimization problem

| Algorithm             | Solution concept | Det/ St. | Assumptions                                                                 | Iteration Complexity |
|-----------------------|------------------|----------|------------------------------------------------------------------------------|----------------------|
| OGDA [33], [39]       | Saddle point     | Det.     | \( f(x, y) = x^T Ay, \) A square full rank \( h(x) = 0, g(y) = 0 \) | \( \mathcal{O}(1/e^T) \) |
| Multi-Step GDA [37]   | 1st order SP     | Det.     | \( f \) NC in \( x \)/Polyak-Lojasiewicz in \( y \) \( h(x) = 0, g(y) = 0 \) | \( \mathcal{O}(1/T) \) |
| Robust optim. [13]    | 1st order SP     | Det.     | \( f \) NC in \( x \)/linear in \( y \) \( h(x) = 0, \) \( g \) convex | \( \sqrt{T} \) |
| PG-SMD/ PGSVRG [35]   | 1st order SP     | Det.     | \( f(x, y) = \mathbb{E}[F(x, y, \xi)], \) \( \xi \) random var. \( F \) NC in \( x \)/ (str.) concave in \( y \) | \( 2, 3 \) \( \mathcal{O}(1/T^{1/4}) / \mathcal{O}(1/T^{1/6}) \) |
| HiBSA (our work)      | 1st order SP     | Det.     | \( f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y) \) \( F \) NC in \( x \)/ str. concave in \( y \) | \( 2 \) \( \mathcal{O}(1/T) \) |

1 The rate here is w.r.t the minimization variable \( x \);
2 Note that this algorithm has nested loops, so the total number of iterations are counted;
3 The two results refer to the case where \( F \) is strongly concave and concave, respectively, w.r.t \( y \).

The main contributions of this paper are listed as follows. First, a number of applications in SPCOM have been formulated in the framework of non-convex, one-sided min-max problem (1). Second, based on different assumptions on how \( x \) and \( y \) variables are coupled, as well as whether the \( y \) problem is convex and non-smoothness in the objective, as well as non-linear coupling between variables. The algorithm proposed in this work is named the Hybrid Block Successive Approximation (HiBSA) algorithm, because it updates the variables block by block, where each block is optimized using a strategy similar to the idea of successive convex approximation (SCA) [40] – except that to update the \( y \) block, a concave approximation is used (hence the name “hybrid”). However, despite the fact that such a block-wise alternating optimization strategy is simple and easy to implement [for example it has been used in the popular block successive upper bound minimization (BSUM) framework [40], [41] for minimization-only problem], it turns out that having the maximization subproblem invalidates all the previous analysis for minimization-only problems. In particular, a naive implementation of such a strategy can fail to compute any meaningful solution of problem (1); see Sec. [11] for a simple example illustrating this fact.

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strongly concave or merely concave, three different types of min-max problems are studied. For each of the problem class, a simple algorithm is presented, together with its convergence guarantees. The major benefits of using the block successive approximation strategy are twofold: 1) each subproblem can be solved effectively, and 2) it is relatively easy to integrate many existing algorithms that are designed for only solving minimization problems (such as those based on the BSUM framework [40], [41]). Third, compared with existing works, the proposed algorithms achieve comparable, or sometimes better, convergence rates, while being able to cover a larger class of problems; see Table I for detailed comparison with existing results. Finally, extensive numerical experiments are conducted using selected applications from SPCOM to validate the proposed algorithms.

Overall, to the best of our knowledge this is the first time that the convergence of the block successive approximation type algorithm is rigorously analyzed for the (one-sided) non-convex min-max problem (1).

**Notation** The notation \( \| \cdot \| \) denotes the vector 2-norm \( \| \cdot \|_2 \); \( \otimes \) denotes the Kronecker product; \( I_N \) is the \( N \times N \) identity matrix; \( \langle \cdot , \cdot \rangle \) is the Euclidean inner product; \( I_{\mathcal{X}}(x) \) denotes the indicator function on set \( \mathcal{X} \); in case the subscript is missing the set is implied by the context; \( [K] := \{1, \cdots , K\} \).

II. THE PROPOSED ALGORITHMS AND ANALYSIS

In this section, we present our main algorithm. Towards this end, we will first make a number of blanket assumptions on problem (1), and then present the HiBSA algorithm in its generic form. We will then provide detailed discussion on various algorithmic choices, and discuss the major challenge in analyzing the proposed algorithm.

Let the superscript \( r \) denote iteration number. For notational simplicity, we will define the following:

\[
\begin{align*}
w^{r+1}_i & := [x^{r+1}_1; x^{r+1}_2; \cdots ; x^{r+1}_{i-1}, x^r_i, \cdots x^r_K] \in \mathbb{R}^{NK}, \quad (9a) \\
w^{r+1}_{i-1} & := [x^{r+1}_1; x^{r+1}_2; \cdots ; x^{r+1}_{i-1}, x^r_{i+1}, \cdots x^r_K] \in \mathbb{R}^{N(K-1)}, \quad (9b) \\
x_{i-1} & := [x_1; x_2; \cdots ; x_{i-1}, x_{i+1}, \cdots x_K] \in \mathbb{R}^{N(K-1)}. \quad (9c)
\end{align*}
\]

Throughout the paper, we will assume that problem (1) satisfies the following blanket assumption.

**Assumption A.** The following conditions hold for (1):

A.1 \( f : \mathbb{R}^{KN+M} \rightarrow \mathbb{R} \) is continuously differentiable; The feasible sets \( \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_K \) and \( \mathcal{Y} \subseteq \mathbb{R}^M \) are convex and compact;

A.2 \( h_i(\cdot) \)'s and \( g(\cdot) \) are convex and non-smooth functions;

A.3 \( f \) has Lipschitz continuous gradient with respect to (w.r.t.) \( x_i \) for every \( i \) with constant \( L_{x_i} \), that is:

\[
\| \nabla_{x_i} f(\bar{x}, x) - \nabla_{x_i} f(x, x) \| \leq L_{x_i} \| \bar{x} - x \|, \forall \bar{x}, x \in \mathcal{X}; \quad (10)
\]
Furthermore, \( f \) has Lipschitz continuous gradient w.r.t. \( y \) with constant \( L_y \), that is:

\[
\| \nabla_y f(\tilde{z}) - \nabla_y f(z) \| \leq L_y \| \tilde{z} - z \|, \forall \tilde{z}, z \in \mathcal{X} \times \mathcal{Y}. \tag{11a}
\]

We note that the above set of conditions are quite standard in the optimization literature; see e.g., \([42],[43]\).

Next we describe the proposed HiBSA algorithm.

| The Hybrid Block Successive Approximation (HiBSA) Algorithm |
|-----------------------------------------------------------|
| **At each iteration** \( r = 1, 2, 3, \cdots \) |
| **[S1]** For \( i = 1, \cdots, K \), perform the following update for the \( x_i \)'s: |
| \[
x_i^{r+1} = \arg \min_{x_i \in \mathcal{X}_i} U_i(x_i; w_i^{r+1}, y^r) + h_i(x_i) + \frac{\beta^r}{2} \| x_i - x_i^r \|^2;
\]
| **[S2]** Perform the following update for the \( y \)-block: |
| \[
y^{r+1} = \arg \max_{y \in \mathcal{Y}} U_y(y; x^{r+1}, y^r) - g(y) - \frac{\gamma^r}{2} \| y \|^2;
\]
| **[S3]** If converges, stop; otherwise, set \( r = r + 1 \), go to **[S1]** |

In the description, \( \{\beta^r \geq 0\} \), and \( \{\gamma^r \geq 0\} \) are some parameters, whose values will be specified shortly in the next section. It is worth noting that, properly designing the regularization sequence \( \{\gamma^r\} \) is the key to ensure that the algorithm works when the \( y \) problem is concave but not strongly concave. Further, each function \( U_i(\cdot; w, y) : \mathbb{R}^N \to \mathbb{R} \) [resp. \( U_y(\cdot, w, y) \)] is some approximation function of \( f(\cdot, x_i, y) \) [resp. \( f(x, \cdot) \)]. These functions satisfy the following assumption.

**Assumption B.** The functions \( U_i(\cdot) \)'s satisfy the following conditions [similar conditions also assumed for \( U_y(\cdot) \)]:

**B.1 (Strong convexity).** Each \( U_i(\cdot; w, y) \) is strongly convex with modulus \( \mu_i > 0 \):

\[
U_i(x_i; w, y) - U_i(z_i; w, y) \geq \langle \nabla_{z_i} U_i(z_i; w, y), x_i - z_i \rangle + \frac{\mu_i}{2} \| x_i - z_i \|^2, \quad \forall w, y \in \mathcal{Y}, x_i, z_i \in \mathcal{X}_i.
\]

**B.2 (Gradient consistency).** Each \( U_i(\cdot; w, y) \) satisfies:

\[
\nabla_z U_i(u_i; x, y) \big|_{z_i = x_i} = \nabla_x f(x, y), \quad \forall i, \forall x \in \mathcal{X}, y \in \mathcal{Y}.
\]

**B.3 (Tight upper bound).** Each \( U_i(\cdot; w, y) \) satisfies:

\[
U_i(z_i; x, y) \geq f(x, y), \quad \text{and} \quad U_i(x_i; x, y) = f(x, y), \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}, z_i \in \mathcal{X}_i.
\]

**B.4 (Lipschitz gradient).** Each \( U_i(\cdot; w, y) \) satisfies:

\[
\| \nabla U_i(z_i; w, y) - \nabla U_i(v_i; w, y) \| \leq L_i, \| v_i - z_i \|, \quad \forall w \in \mathcal{X}, y \in \mathcal{Y}, v_i, z_i \in \mathcal{X}_i.
\]
Clearly, the $x$ update step $[\text{S}1]$ closely resembles the popular BSUM algorithm $[40],[44]$, which is designed for multi-block non-convex minimization problems. Similarly as in BSUM, some kind of approximation functions are used to simplify the update for each subproblem; see $[40]$ for a number of such functions that are popular for signal processing applications.

However, the key difference from the BSUM-type algorithm, or for that matter, all successive convex approximation (SCA) based algorithms such as the inexact flexible parallel algorithm (FLEXA) $[43],[45],[46]$, the concave-convex procedure (CCCP) $[47]$ and so on, is the presence of the ascent step in $[\text{S}2]$. This step is needed to deal with the inner maximization problem, but unfortunately the use of it invalidates the existing analysis for SCA-type algorithms such as BSUM, CCCP, FLEXA, all of which critically depend on consistently achieving some form of descent as the algorithms progress. Consequently, how to properly implement and analyze the proposed algorithm represents a major challenge. To see where the issue is, consider the following simple example.

**Example 1.** Consider a special case of problem (1), where $K = 1$ (a single block variable), and $A$ is a randomly generated matrix of size $N \times M$:

$$\min_{x \in \mathbb{R}^N} \max_{y \in \mathbb{R}^M} y^T Ax.$$  

Let us apply a special case of HiBSA algorithm by utilizing the following approximation function:

$$U_1(v; w, y) = y^T Av + \frac{\eta}{2} \|v - w\|^2$$

$$U_y(u; x, y) = u^T Ax - \frac{1}{2\lambda} \|u - y\|^2.$$  

Letting $\gamma^r = 0$ and $\beta^r = 0$ for all $r$, the HiBSA becomes an alternating gradient descent-ascent algorithm

$$x^r = x^{r-1} - \frac{1}{\eta} A^T y^{r-1}, \quad y^r = y^{r-1} + 2\lambda Ax^r, \quad \forall r.$$  

Unfortunately, one can verify that for almost any $A$, regardless the choices of $\eta, \lambda$, (14) will not converge to the desired solution satisfying: $A^T y^* = 0$ and $Ax^* = 0$; see Fig. $[\text{I}]$ This is because the linear system describing the dynamics of the vector $(x^r, y^r)$ is always unstable.

This example suggests that even for the simplest two-block linearly coupling problem, when it involves both minimization and maximization blocks, one cannot directly extend the BSUM or SCA-type methods...
to obtain a convergent algorithm. This represents a significant challenge to design algorithms for the Min-Max problem (1).

In fact, the above example motivates us to introduce both the proximal term $\beta^r/2\|x - x^r\|^2$ in [Step 1] of HiBSA, and the penalty term $-\gamma^r/2\|y\|^2$ in [Step 2]. By properly selecting the coefficient sequences $\{\beta^r, \gamma^r\}$, one can show that the HiBSA will converge for a wide class of problems (which include the linear coupling problem in Example 1 as a special case).

To summarize this subsection, we remark that the major novelty of the proposed algorithm framework is the combination of block successive approximation schemes to minimize and maximize the $x_i$'s and $y$ blocks, together with the use of a sequence of penalization terms $\{-\gamma^r/2\|y\|^2\}$ and the sequence of proximal terms $\{\frac{\beta^r}{2}\|x - x^r\|^2\}$ with properly chosen penalization parameters.

III. Theoretical Properties of HiBSA

We start to present our main convergence results for the HiBSA. Our analysis will be divided into three cases according to the structure of couple term $f(x, y)$. Separately considering different cases of (1) is necessary, since the analysis and convergence guarantees could be different. Note that throughout this section, we will assume that $g(y)$ is convex, but not strongly convex. In case $\ell(x, y)$ is strongly concave in $y$, the strong concave term will be absorbed into $f(x, y)$.

A. Optimality conditions

Before delving into the analysis, we will elaborate on the type of solutions we would like to obtain for problem (1). Because of the non-convexity involved in the minimization problem, we will not be able to use the classical measure of optimality for saddle point problems (i.e., the distance to a saddle point). Instead, we will adopt some kind of first-order stationarity conditions. To precisely state our condition, let us define the proximity operator for $x$ and $y$ blocks as follows:

$$
\text{Px}^\alpha_i(v_i) := \arg\min_{x_i \in \mathcal{X}} h_i(x_i) + \frac{\beta}{2}\|x_i - v_i\|^2, \quad \forall i \in [K]
$$

$$
\text{Py}^{1/\rho}(w) := \arg\max_{y \in \mathcal{Y}} -g(y) - \frac{1}{2\rho}\|y - w\|^2.
$$

Moreover, we define the stationarity gap for problem (1) as:

$$
\nabla G(x, y) := \left[ \begin{array}{c}
\beta(x_1 - \text{Px}_1^\alpha(x_1 - 1/\beta\nabla_x f(x, y))) \\
\vdots \\
\beta(x_K - \text{Px}_K^\alpha(x_K - 1/\beta\nabla_x f(x, y))) \\
1/\rho(y - \text{Py}^{1/\rho}(y + \rho\nabla_y f(x, y)))
\end{array} \right].
$$
Based on these conditions, we say that a tuple \((x^*, y^*)\) is a first-order stationary solution for problem (1) if it holds that:

\[
\|\nabla G(x^*, y^*)\| = 0.
\] (17)

To see that this condition makes sense, first note that if \(h \equiv 0, g \equiv 0, \mathcal{V} = \mathbb{R}^M, \mathcal{X} = \mathbb{R}^{NK}\), then it reduces to the condition \(\|[\nabla_x f(x^*, y^*); \nabla_y f(x^*, y^*)]\| = 0\), which is independent of the algorithm parameters \((\beta, \rho)\). Further, we can check that if \(y\) is not present, then condition (17) is equivalent to the first-order stationary condition for the resulting non-convex minimization problem (see, e.g., [31]). Further, if \(x\) is not present, then condition (17) simply says that \(y \in \arg\max_{y \in \mathcal{V}} f(y) - g(y)\). Based on the above definition of first-order stationarity, we establish the equivalence between a few optimization formulations discussed in Section I.

**Proposition 3.1:** Problems (2) and (3) are equivalent, in the sense that every KKT point for problem (2) is a first-order stationary solution of (3), and vice versa.

**Proof.** For simplicity of notation we assume \(N = 1\). Consider the following KKT conditions for problem (2)

\[
\langle \nabla_x f(x^*) + \xi^* + A^T y^*, x - x^* \rangle - \langle y^*, \tilde{x} - \tilde{x}^* \rangle \geq 0, \forall \text{ feasible } (x, \tilde{x})
\]

\[
Ax^* = \tilde{x}^*
\]

(18)

where \(\xi^* \in \partial h(x^*)\) and \(y\) is the Lagrange multiplier. Now consider a stationary point \((x^*, \tilde{x}^*, y^*)\) of problem (3). Then the stationarity condition (17) implies that

\[
x^* = \arg\min_x \langle A^T y^* + \nabla_x f(x^*), x - x^* \rangle + h(x) + \frac{\beta}{2}\|x - x^*\|^2
\]

(19a)

\[
\tilde{x}^* = \arg\min_{\tilde{x} \in \tilde{Z}} \langle -y^*, \tilde{x} - \tilde{x}^* \rangle + \frac{\beta}{2}\|\tilde{x} - \tilde{x}^*\|^2
\]

(19b)

\[
y^* = \arg\max_y \langle Ax^* - \tilde{x}^*, y - y^* \rangle - \frac{1}{2\rho}\|y - y^*\|^2
\]

(19c)

The optimality conditions for these problems imply

\[
A^T y^* + \nabla_x f(x^*) + \xi(x^*) = 0
\]

(20)

\[
\langle -y^*, \tilde{x} - \tilde{x}^* \rangle \geq 0, \forall \tilde{x} \in \tilde{Z}, \quad Ax^* - \tilde{x}^* = 0.
\]

(21)

Clearly, the conditions (20) – (21) imply (18).

Conversely, suppose (18) is true. Set \(x = x^*\) in (18) we obtain condition (21). Moreover, in order to obtain condition (20) we set \(\tilde{x} = \tilde{x}^*\) in (18) and take into account the fact that (18) holds \(\forall x \in \mathbb{R}^{KN}\). The proof is complete.

**Q.E.D.**
Proposition 3.2: Consider the problem: \( \max_{x \in \mathcal{X}} \min_{k} R_k(x) \), and its reformulation (5). They are equivalent in the sense that, an equivalent smooth reformulation of the former has the same first-order stationary solutions as that of the latter.

**Proof.** A well-known equivalent smooth formulation of the min-utility maximization problem is given below (equivalent in that the global optimal of these two problems are the same)

\[
\max_{\lambda, x \in \mathcal{X}} \lambda, \quad \text{s.t.} \quad R_k(x) \geq \lambda, \quad \forall k.
\]  

(22)

The partial KKT conditions of the above problem are

\[
\left\langle \sum_{i=1}^{K} y_i^* \nabla_x R_i(x^*), x - x^* \right\rangle \leq 0, \quad \forall \ x \in \mathcal{X},
\]

(23)

\[
\sum_{i} y_i^* = 1, \ y_i^* \geq 0, \ y_i^*(\lambda^* - R_i(x^*)) = 0, \ R_i(x^*) \geq \lambda^*, \ \forall \ i,
\]

where \( \{y_i\}_{i=1}^{K} \) are the respective Lagrange multipliers.

Now consider a stationary point \( (x^*, y^*) \) of problem (5). Then the optimality conditions (17) imply that

\[
x^* = \arg \min_{x \in \mathcal{X}} \left( \sum_{i=1}^{K} -y_i^* \nabla_x R_i(x^*), x - x^* \right) + \frac{\beta}{2} \|x - x^*\|^2
\]

(24a)

\[
y^* = \arg \max_{y \in \Delta} \left( -R(x^*), y - y^* \right) - \frac{1}{2\rho} \|y - y^*\|^2,
\]

(24b)

where \( R(x^*) := [R_1(x^*); \ldots; R_K(x^*)] \). Also, we pick \( \lambda^* = \min_{i=1,\ldots,K} \{R_i(x^*)\} \) and so it holds that \( R_i(x^*) \geq \lambda^*, \ \forall \ i \).

The points \( x^*, y^* \) are the solutions of the optimization problems in (24a), (24b). Plugging them into the respective optimality conditions results to the equivalent conditions

\[
\left\langle \sum_{i=1}^{K} -y_i^* \nabla_x R_i(x^*), x - x^* \right\rangle \geq 0, \ \forall \ x \in \mathcal{X}
\]

(25a)

\[
\langle -R(x^*), y - y^* \rangle \leq 0, \ \forall \ y \in \Delta, \ y^* \in \Delta
\]

(25b)

For all \( i \) such that \( R_i(x^*) = \lambda^* \) obviously it holds that \( y_i^*(\lambda^* - R_i(x^*)) = 0 \). Let \( i, j \) be indices such that \( R_i(x^*) > \lambda^* \) and \( R_j(x^*) = \lambda^* \). Then, plugging \( y_i = 0, y_j = y_i^* + y_j^* \) and \( y_k = y_k^*, k \neq i, j \) into (25b) yields \( y_i^*(R_j(x^*) - R_i(x^*)) \geq 0 \). Because \( R_j(x^*) - R_i(x^*) < 0 \) and \( y_i^* \geq 0 \) it must necessarily hold \( y_i^* = 0 \) and thus \( y_i^*(\lambda^* - R_i(x^*)) = 0 \). As a result the conditions (23) are satisfied.

Conversely, assume \( (x^*, y^*) \) satisfies conditions (23). Note that \( R_i(x^*)y_i \geq \lambda^*y_i \) for any \( y \in \Delta \), so

\[
\langle R(x^*), y - y^* \rangle = \sum_{i=1}^{K} R_i(x^*)(y_i - y_i^*) \geq \sum_{i=1}^{K} \lambda^*(y_i - y_i^*) = 0,
\]

for all \( y \in \Delta, y^* \in \Delta \). It is not difficult to see that \( (x^*, y^*) \) satisfy the rest of the conditions and as a result the opposite direction also holds.

Q.E.D.
B. Convergence analysis: $f(x, y)$ strongly concave in $y$

We will first consider a subset of problem (1), where $f(x, y)$ is strongly concave in $y$. Specifically, we assume the following.

**Assumption C-1.** For any $x \in X$, $f(\cdot)$ satisfies the following:

\[ f(x, z) - f(x, y) \leq \langle \nabla_y f(x, y), z - y \rangle - \frac{\theta}{2}\|z - y\|^2, \quad \forall \; y, z \in \mathcal{Y}, \]

where $\theta > 1$ is the strong concavity constant. Further assume:

\[ U_y(u; x, y) = \langle \nabla_y f(x, y), u - y \rangle - \frac{1}{2\rho}\|u - y\|^2. \]  

(26)

We note that it can be verified that the jamming problem (8) satisfies Assumption C-1. Next we will present a series of lemmas which lead to our main result in this subsection.

**Lemma 1:** (Descent Lemma on $x$) Suppose that Assumptions A, B and C-1 hold. Let $(x^r, y^r)$ be a sequence generated by HiBSA, with $\gamma^r = 0$, and $\beta^r = \beta > 0$, $\forall$ $r$. Then we have the following descent estimate:

\[ \ell(x^{r+1}, y^r) - \ell(x^r, y^r) \leq -\sum_{i=1}^{K} \left( \beta + \mu_i - \frac{L_{x_i}}{2} \right) \|x^{r+1}_i - x^r_i\|^2. \]

**Proof.** By using the assumption that $f$ has Lipschitz gradient, $h_i$ is convex (cf. Assumption A), and by noticing that $w^{r+1}_i = (x^r_i, w^{r+1+1}_i)$, we obtain the following:

\[ l(x^{r+1}, w^{r+1}_i, y^r) - l(x^r, w^{r}_i, y^r) \leq \langle \nabla_x f(w^{r+1}_i, y^r) + \partial_i^{r+1} + \partial_i^r, x^{r+1}_i - x^r_i \rangle + \frac{L_{x_i}}{2}\|x^{r+1}_i - x^r_i\|^2 \]  

(27)

for some $\partial_i^{r+1} \in \partial h_i(x^{r+1}_i)$.

Second, the optimality condition for $x_i$ update step in (12) is

\[ \langle \nabla_x, U_i(x^r_i; x^{r+1}_i, w^{r+1}_i, y^r), x^{r+1}_i - x^r_i \rangle \geq 0. \]  

(28)

So adding and subtracting $\langle \nabla_x, U_i(x^r_i; x^{r+1}_i, w^{r+1}_i, y^r), x^{r+1}_i - x^r_i \rangle$ in (28), and by applying assumptions B.1 and B.2, we obtain the following:

\[ \langle \nabla_x, f(w^{r+1}_i, y^r), x^{r+1}_i - x^r_i \rangle + \langle \partial_i^{r+1}, x^{r+1}_i - x^r_i \rangle \leq -\mu_i\|x^{r+1}_i - x^r_i\|^2 - \beta\|x^{r+1}_i - x^r_i\|^2. \]

Then, combining the above expression with (27) results in

\[ l(x^{r+1}, w^{r+1}_i, y^r) - l(x^r, w^{r}_i, y^r) \leq (-\beta - \mu_i + \frac{L_{x_i}}{2})\|x^{r+1}_i - x^r_i\|^2. \]

Summing over $i \in [K]$ we obtain the desired result.  

Q.E.D.
**Lemma 2:** (Descent Lemma on $y$) Suppose that Assumptions A, B and C-1 hold. Let $(x^r, y^r)$ be a sequence generated by HiBSA, with $\gamma^r = 0$, and $\beta^r = \beta > 0$, $\forall \ r$. Then we have the following descent estimate:

$$\ell(x^{r+1}, y^{r+1}) - \ell(x^{r+1}, y^r) \leq \frac{1}{\rho} \|y^{r+1} - y^r\|^2 - \left( \theta - \left( \frac{1}{2\rho} + \frac{\rho L_2^2}{2} \right) \right) \|y^r - y^{r-1}\|^2 + \frac{\rho L_2^2}{2} \|x^{r+1} - x^r\|^2.$$

**Proof.** For notational simplicity, let us define

$$\ell'(x^{r+1}, y) = f(x^{r+1}, y) + \sum_{i=1}^{K} h_i(x_i^{r+1}) - I_Y(y) - g(y).$$

Notice that for any $y \in Y$, we have $\ell'(x^{r+1}, y) = \ell(x^{r+1}, y)$. The optimality condition of the $y$-step becomes

$$0 = \nabla_y f(x^{r+1}, y^r) - \frac{1}{\rho} \left( y^{r+1} - y^r \right) - \xi^{r+1},$$

where $\xi^r$ denotes the subgradient of $I_Y(y^r) + g(y^r)$. Since $\ell'(x, y)$ is concave with respect to $y$, we have

$$\ell'(x^{r+1}, y^{r+1}) - \ell'(x^{r+1}, y^r) \leq \langle \nabla_y f(x^{r+1}, y^r) - \xi^r, y^{r+1} - y^r \rangle$$

$$= \frac{1}{\rho} \|y^{r+1} - y^r\|^2 - \langle \xi^r - \xi^{r+1}, y^{r+1} - y^r \rangle$$

$$(a) \frac{1}{\rho} \|y^{r+1} - y^r\|^2 + \langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^{r-1}), y^{r+1} - y^r \rangle$$

$$- \frac{1}{\rho} \langle y^{r+1} - y^r - (y^r - y^{r-1}), y^{r+1} - y^r \rangle$$

$$(b) \frac{1}{\rho} \|y^{r+1} - y^r\|^2 + \langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^{r-1}), y^{r+1} - y^r \rangle$$

$$+ \frac{1}{2\rho} \|y^r - y^{r-1}\|^2 - \frac{1}{2\rho} \|y^{r+1} - y^r\|^2 - \frac{1}{2\rho} \|v^{r+1}\|^2$$

$$(c) \frac{1}{\rho} \|y^{r+1} - y^r\|^2 + \frac{\rho L_2^2}{2} \|x^{r+1} - x^r\|^2 + \frac{1}{2\rho} \|y^r - y^{r-1}\|^2 - \frac{1}{2\rho} \|v^{r+1}\|^2$$

$$+ \langle \nabla_y f(x^r, y^r) - \nabla_y f(x^r, y^{r-1}), y^{r+1} - y^r \rangle$$

where (a) is from (29), in (b) we apply the following identity:

$$\langle v^{r+1}, y^{r+1} - y^r \rangle = \frac{1}{2} (\|y^{r+1} - y^r\|^2 + \|v^{r+1}\|^2 - \|y^{r+1} - y^{r-1}\|^2);$$

in (c) we add and subtract a term $\langle \nabla_y f(x^r, y^r), y^{r+1} - y^r \rangle$, and apply the Young’s inequality and obtain

$$\langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^r), y^{r+1} - y^r \rangle \leq \frac{\rho L_2^2}{2} \|x^{r+1} - x^r\|^2 + \frac{1}{2\rho} \|y^{r+1} - y^r\|^2,$$
where $L_y$ is defined in (11a). By applying the strong concavity of $f(x,y)$ in $y$, the Young’s inequality and the Lipschitz condition w.r.t $y$, we can have the following bound for the inner product term in (31):

\[
\langle \nabla_y f(x^r, y^r) - \nabla_y f(x^{r'}, y^{r'-1}), y^{r'+1} - y^r \rangle \\
\leq \langle \nabla_y f(x^r, y^r) - \nabla_y f(x^{r'}, y^{r'-1}), y^{r'+1} + y^r - y^{r'-1} \rangle \\
\leq \frac{\rho L_y^2}{2} \|y^r - y^{r'-1}\|^2 + \frac{1}{2\rho} \|v^{r'+1}\|^2 - \theta \|y^r - y^{r'-1}\|^2. \tag{34}
\]

Combining the above with (31) completes the proof. Q.E.D.

At this point, by simply combining Lemmas 1-2, it is not clear how the objective value behaves after each $x$ and $y$ update. To capture the essential dynamics of the algorithm, the key is to identify a proper potential function, which decreases after each round of $x$ and $y$ updates.

**Lemma 3:** Suppose that Assumptions A, B and C-1 hold. Let $(x^r, y^r)$ be a sequence generated by HiBSA, with $\gamma^r = 0\text{, and } \beta^r = \beta > 0, \forall r$. Let us define a potential function as

\[
\mathcal{P}^{r+1} := \ell(x^{r+1}, y^{r+1}) + \left(\frac{2}{\rho^2\theta} + \frac{1}{2\rho} - 4\left(\frac{1}{\rho} - \frac{L_y^2}{2\theta^2}\right)\right) \|y^{r+1} - y^r\|^2.
\]

When the following conditions are satisfied:

\[
\rho < \frac{\theta}{4L_y^2}, \quad \beta > \frac{L_y^2}{\theta^2\rho} + \frac{\rho}{2} + \frac{L_x}{2} - \mu_i, \quad \forall i \tag{35}
\]

then there exist positive constants $c_1, \{c_{2i}\}_{i=1}^N$ such that:

\[
\mathcal{P}^{r+1} - \mathcal{P}^r < -c_1 \|y^{r+1} - y^r\|^2 - \sum_{i=1}^K c_{2i} \|x_i^{r+1} - x_i^r\|^2. \tag{36}
\]

**Proof.** According to (29), the optimality condition of $y$-problem (13) at iteration $r+1$ and $r$ are given by

\[
- \nabla_y f(x^{r+1}, y^r) + \xi^{r+1} + \frac{1}{\rho}(y^{r+1} - y^r) = 0, \tag{37}
\]

\[
- \nabla_y f(x^r, y^{r-1}) + \xi^r + \frac{1}{\rho}(y^r - y^{r-1}) = 0, \tag{38}
\]

where $\xi^r$ denotes one subgradient of the nonsmooth function $\mathcal{I}_y(y^r) + g(y^r)$ as above. We subtract these two equalities, multiply both sides by $y^{r+1} - y^r$, utilize the defining property of subgradient vectors $\langle \xi^{r+1} - \xi^r, y^{r+1} - y^r \rangle \geq 0$, and we obtain [where $v^{r+1}$ is defined in (30)]

\[
\frac{1}{\rho} \langle v^{r+1}, y^{r+1} - y^r \rangle \leq \langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^r), y^{r+1} - y^r \rangle \\
\quad + \langle \nabla_y f(x^r, y^r) - \nabla_y f(x^r, y^{r-1}), y^{r+1} - y^r \rangle.
\]
Applying (32) to the LHS to the above expression, and using similar techniques as in (33), (34), we obtain

\[ F_{r+1} := \frac{1}{2\rho} \| y^{r+1} - y^r \|^2 \]

\[ \quad \leq \frac{1}{2\rho} \| y^r - y^{r-1} \|^2 - \frac{1}{2\rho} \| v^{r+1} \|^2 + \frac{L^2_y}{2\theta} \| x^{r+1} - x^r \|^2 + \frac{\theta}{2} \| y^{r+1} - y^r \|^2 \]

\[ + \frac{\rho L^2_y}{2} \| y^r - y^{r-1} \|^2 + \frac{1}{2\rho} \| v^{r+1} \|^2 - \theta \| y^r - y^{r-1} \|^2 \]

\[ = F^r + \frac{L^2_y}{2\theta} \| x^{r+1} - x^r \|^2 + \frac{\theta}{2} \| y^{r+1} - y^r \|^2 - \left( \theta - \frac{\rho L^2_y}{2} \right) \| y^r - y^{r-1} \|^2. \]

Multiplying both sides of (39) by \(4/\theta\rho\), and combining (31) and Lemma 1-2, we can estimate the descent of the potential function as follows

\[ P_{r+1} \leq P^r + \left( \frac{3}{\rho} + \frac{1}{2\rho} - 4 \left( \frac{1}{\rho} - \frac{L^2_y}{2\theta} \right) \right) \| y^{r+1} - y^r \|^2 \]

\[ - K \sum_{i=1}^K \left( \beta + \mu_i - \frac{L_{xi}}{2} - \frac{2L^2_y}{\theta^2\rho} + \frac{\rho L^2_y}{2} \right) \| x^{r+1}_i - x^r_i \|^2. \]

In the inequality above we do not include \(\theta - \rho L^2_y/2\) (from RHS of the descent estimate in Lemma 1) because by the choice of \(\rho\) this term is positive. Therefore, when

\[ \rho < \frac{\theta}{2L^2_y}, \quad \beta > L^2_y \left( \frac{2}{\theta^2\rho} + \frac{\rho}{2} \right) + \frac{L_{xi}}{2} - \mu_i, \quad \forall i \]

we have sufficient descent of the potential function \(P_{r+1}\). In other words, there exist positive \(c_1\) and \(c_2i\)'s such that

\[ P_{r+1} - P^r < -c_1 \| y^{r+1} - y^r \|^2 - \sum_{i=1}^K c_2i \| x^{r+1}_i - x^r_i \|^2, \]

which completes the proof. Q.E.D.

Combining the above analysis, we can obtain the following convergence guarantee for the HiBSA algorithm.

**Theorem 1:** Suppose that Assumptions A, B and C-1 hold. Let \((x^r, y^r)\) be a sequence generated by HiBSA, with \(\gamma^r = 0\), and \(\beta^r = \beta, \; \forall \; r\), satisfying conditions (35). For a given small constant \(\epsilon\), let \(T(\epsilon)\) denote the first iteration index, such that the following holds: \(T(\epsilon) = \min \{ r \mid \| \nabla G(x^r, y^r) \|^2 \leq \epsilon, \; r \geq 1 \}\). Then there exists some constant \(C > 0\) such that \(\epsilon \leq C \frac{P_1 - P_{T(\epsilon)}}{T(\epsilon)}\), where \(P\) denotes some lower bound of \(P^r\).
Proof. We first bound the elements of the gap \([16]\) by
\[
\|(\nabla G(x^r, y^r))_i\| \leq \beta \|x^r_{i+1} - x^r_i\| + \beta \|x^r_i - \mathbf{P}_{i}^\beta (x^r_i - 1/\beta \nabla_x f(x^r, y^r))\|
\]
\[
\overset{(a)}{\leq} \beta \|x^r_{i+1} - x^r_i\| + \beta \|\mathbf{P}_{i}^\beta (x^r_i - (1/\beta \nabla_x U_i(x^r_{i+1}; w^r_{i+1}, y^r))) - \mathbf{P}_{i}^\beta (x^r_i - 1/\beta \nabla_x f(x^r, y^r))\|
\]
\[
\overset{(b)}{\leq} \beta \|x^r_{i+1} - x^r_i\| + \sum_{u \in U_i} \|x^r_{i+1} - x^r_i\| + \sum_{x \in X_i} \|w^r_i - x^r\|
\]
\[
\leq (\beta + \sum_{u \in U_i} \|w^r_i - x^r\|) \|x^r_{i+1} - x^r\|,
\]
where in (a) we use the optimality conditions w.r.t \(x_i\) in \([12]\); in (b) we use the nonexpansiveness of the proximal operator, \(\nabla_x U_i(x^r_{i+1}; w^r_{i+1}, y^r) = \nabla_x f_i(w^r_{i+1}, y^r)\) (Assumption B2), Assumption B4, as well as the following identity
\[
\nabla_x U_i(x^r_{i+1}; w^r_{i+1}, y^r) - \nabla_x f(x^r, y^r) = \nabla_x U_i(x^r_{i+1}; w^r_{i+1}, y^r)
\]
\[
- \nabla_x U_i(x^r_i; w^r_i, y^r) + \nabla_x U_i(x^r_i; w^r_i, y^r) - \nabla_x f(x^r, y^r).
\]
Moreover, utilizing the same argument for the optimality condition w.r.t \(y\) we obtain
\[
\|(\nabla G(x^r, y^r))_{K+1}\|
\]
\[
\overset{(a)}{\leq} \frac{1}{\rho} \|y^r_{i+1} - y^r\| + \frac{1}{\rho} \|y^r_{i+1} - \mathbf{P}^1/\rho (y^r + \rho \nabla_y f(x^r, y^r))\|
\]
\[
\overset{(b)}{\leq} \frac{1}{\rho} \|y^r_{i+1} - y^r\| + \|\nabla_y f(x^r_{i+1}, y^r) - \nabla_y f(x^r, y^r)\|
\]
\[
\overset{(c)}{\leq} \frac{1}{\rho} \|y^r_{i+1} - y^r\| + \frac{1}{\rho} \|x^r_{i+1} - y^r\|,
\]
where in (a) we use the optimality conditions w.r.t \(y\), in (b) we use the nonexpansiveness of the proximal operator and finally in (c) the Assumption A.3. Combining (41) and the above two inequalities, we see that there exist constants \(\sigma_1 > 0\) and \(\sigma_2 > 0\) such that the following holds:
\[
\|(\nabla G(x^r, y^r))\|^2 \leq \frac{\sigma_2}{\sigma_1} (P^r - P^{r+1}).
\]
Summing the above inequality over \(r \in [T]\), we have
\[
\sum_{r=1}^{T} \|(\nabla G(x^r, y^r))\|^2 \leq \frac{\sigma_2}{\sigma_1} (P^1 - P^{T+1}) \leq \frac{\sigma_2}{\sigma_2} (P^1 - P),
\]
where in the last inequality we have used the fact that \(P^r\) is decreasing (by Lemma 3) and lower bounded by \(P\) (since \(x^r, y^r\) are within the compact sets). By utilizing the definition \(T(\epsilon)\), the above inequality becomes \(T(\epsilon) \leq \frac{\sigma_1}{\sigma_2} (P^1 - P)\).

Dividing both sides by \(T(\epsilon)\), and by setting \(C := \sigma_1/\sigma_2\), the desired result is obtained. Q.E.D.
C. Convergence analysis: $f(x, y)$ concave in $y$

Next, we consider the following assumptions for (1).

**Assumption C-2.** Assume that $f(\cdot)$ in (1) satisfies:

$$f(x, y) - f(x, z) \leq \langle \nabla_y f(x, z), y - z \rangle, \quad \forall \ y, z \in Y, x \in X.$$  

That is, it is concave in $y$. Further, assume that $$U_y(u; x, y) = f(x, u) - \frac{1}{2\rho} \|u - y\|^2.$$  

That is, the $y$ update directly maximizes a regularized version of the objective function. Note that $U_y(u; x, y)$ is strongly concave in $u$, which satisfies the counterpart of Assumption B.1 for $U_y(\cdot)$.

The fact that the $y$ problem is no longer strongly concave poses significant challenge in the analysis. In fact, from Example 1 it is clear that directly utilizing the alternating gradient type algorithm may fail to converge to any interesting solutions. Towards resolving this issue, we specialize the HiBSA algorithm, by using a novel diminishing regularization plus increasing penalty strategy to regularize the $y$ and $x$ update, respectively (by using a sequence of diminishing $\{\gamma^r\}$, and increasing $\{\beta^r\}$).

We have the following convergence analysis.

**Lemma 4:** (Descent lemma) Suppose that Assumptions A, B and C-2 hold. Let $(x^r, y^r)$ be a sequence generated by HiBSA, with $\gamma^r > 0$ and $\beta^r > L_{x_i}, \forall r, i$. Then we have:

$$l(x^{r+1}, y^{r+1}) - l(x^r, y^r) \leq \frac{1}{2\rho} \|y^r - y^{r-1}\|^2$$

$$- \left( \frac{\beta^r}{2} + \mu - \frac{\rho L_y^2}{2} \right) \|x^{r+1} - x^r\|^2 - \left( \frac{\gamma^{r-1}}{2} - \frac{1}{\rho} \right) \|y^{r+1} - y^r\|^2$$

$$+ \frac{\gamma^r}{2} \|y^r\|^2 - \frac{\gamma^{r-1}}{2} \|y^r\|^2 + \frac{\gamma^{r-1} - \gamma^r}{2} \|y^{r+1}\|^2.$$  

(45)

**Proof.** Following similar steps as in Lemma [1] and using the assumption $\beta^r > L_{x_i}, \forall i$ we obtain

$$\ell(x^{r+1}, y^r) - \ell(x^r, y^r) \leq - \left( \frac{\beta^r}{2} + \mu \right) \|x^{r+1} - x^r\|^2,$$  

(46)

where $\mu := \min_{i \in [K]} \mu_i$. To analyze the $y$ update, define

$$\ell'(x^{r+1}, y) = f(x^{r+1}, y) + \sum_{i=1}^{K} h_i(x_i^{r+1}) - I_Y(y) - g(y).$$

The optimality condition for the $y$ update is

$$\xi^{r+1} - \nabla_y f(x^{r+1}, y^{r+1}) + \frac{1}{\rho} (y^{r+1} - y^r) + \gamma^r y^{r+1} = 0,$$  

(47)
where $\xi^r \in \partial(\mathcal{I}_y(y^r) + g(y^r))$. Using this, we have the following inequalities:

\begin{align*}
&l'(x^{r+1}, y^{r+1}) - l'(x^{r+1}, y^r) \\
&\leq (a) \langle \nabla_y f(x^{r+1}, y^r), y^{r+1} - y^r \rangle - \langle \xi^r, y^{r+1} - y^r \rangle \\
&\leq (b) \langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^r), y^{r+1} - y^r \rangle + \frac{1}{\rho} ||y^{r+1} - y^r||^2 \\
&\quad + \gamma^r \langle y^{r+1} - y^r \rangle + \langle \xi^{r+1} - \xi^r, y^{r+1} - y^r \rangle \\
&\leq (c) \gamma^r - l(y^r, y^{r+1} - y^r) + \langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^r), y^{r+1} - y^r \rangle \\
&\quad + \frac{1}{\rho} ||y^{r+1} - y^r||^2 - \frac{1}{\rho} \langle y^{r+1} - y^r \rangle \\
&\leq (d) \frac{1}{2\rho} ||y^r - y^{r-1}||^2 + \frac{\rho L_y^2}{2} ||x^{r+1} - x^r||^2 - \left(\frac{2^{r-1}}{2} - \frac{1}{\rho}\right) ||y^{r+1} - y^r||^2 \\
&\quad - \gamma^r \langle y^r \rangle + \frac{\gamma^r - 1}{2} ||y^r||^2 + \frac{\gamma^r - 1}{2} ||y^{r+1}||^2,
\end{align*}

where $(a)$ uses the concavity of $l'(x, y)$; in $(b)$ we use $(47)$; $(c)$ is from subtracting $(47)$ with the same condition at iteration $r - 1$, and plugging the resulting $\xi^{r+1} - \xi^r$; in $(d)$ we use the quadrilateral identity $(32)$ for the term involving $v$, and the Lipschitz continuity of $\nabla_y f$ (cf. A. 3), the Young’s inequality, as well as the following identity:

\begin{align*}
\gamma^r - l(y^r, y^{r+1} - y^r) &= \frac{\gamma^{r-1}}{2} \left(||y^{r+1}||^2 - ||y^r||^2 - ||y^{r+1} - y^r||^2\right) \\
&= \frac{\gamma^r}{2} ||y^{r+1}||^2 - \left(||y^r||^2 + ||y^{r+1} - y^r||^2\right) + \frac{\gamma^{r-1} - \gamma^r}{2} ||y^{r+1}||^2.
\end{align*}

Combining $(46)$ and $(48)$, we obtain the desired result. \textbf{Q.E.D.}

Next we show that there exists a potential function, given below, which decreases consistently

\begin{align*}
\mathcal{P}^{r+1} &= \left(\frac{1}{2\rho} + \frac{2}{\rho^2 \gamma^r} + \frac{2}{\rho} \left(\frac{1}{\rho \gamma^{r+1}} - \frac{1}{\rho \gamma^r}\right)\right) ||y^{r+1} - y^r||^2 \\
&\quad + l(x^{r+1}, y^{r+1}) - \frac{\gamma^r}{2} ||y^{r+1}||^2 - \frac{2}{\rho} \left(\frac{\gamma^{r-1}}{\gamma^r} - 1\right) ||y^{r+1}||^2.
\end{align*}

\textbf{Lemma 5:} Suppose that Assumptions A, B and C-2 are satisfied. Let $(x^r, y^r)$ be a sequence generated by HiBSA. Suppose the following conditions are satisfied for all $r$,

$$\beta^r > \rho L_y^2 + \frac{4L_y^2}{\rho (\gamma^r)^2} - 2\mu, \quad \beta^r > L_{z_i}, \forall i, \quad \frac{1}{\gamma^{r+1}} - \frac{1}{\gamma^r} \leq \frac{1}{5},$$

February 25, 2019 DRAFT
then the change of potential function can be bounded through
\[
P^{r+1} \leq P^r - \left( \frac{\beta}{2} + \mu - \left( \frac{\rho L_y^2}{2} + \frac{2L_y^2}{\rho (\gamma^r)^2} \right) \right) \|x^{r+1} - x^r\|^2
\]
\[
- \frac{1}{10\rho} \|y^{r+1} - y^r\|^2 + \frac{\gamma^{r-1} - \gamma^r}{2} \|y^{r+1}\|^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2} - \gamma^{r-1}}{\gamma^{r-1} - \gamma^r} \right) \|y^r\|^2.
\]

**Proof.** To simplify notation, define \( f^{r+1} := f(x^{r+1}, y^{r+1}) \). The optimality conditions of \( y \) problem are given by
\[
- \langle \nabla_y f^{r+1}, \frac{1}{\rho}(y^{r+1} - y^r) \rangle - \gamma^r y^{r+1} - \partial^{r+1}, y^{r+1} y \rangle \leq 0
\]
\[
- \langle \nabla_y f^r, \frac{1}{\rho}(y^r - y^{r-1}) \rangle - \gamma^{r-1} y^r - \partial^r, y^r - y \rangle \leq 0,
\]
for all \( y \in \mathcal{Y} \), where \( \partial^{r+1} \in \partial g(y^{r+1}) \).

Plugging in \( y = y^r \) in (51a), \( y = y^{r+1} \) in (51b), adding them together and utilizing the defining property of subgradient vectors, i.e \( \langle y^{r+1} - \partial^r, y^{r+1} - y^r \rangle \geq 0 \), we obtain
\[
\frac{1}{\rho} \langle v^{r+1}, y^{r+1} - y^r \rangle + \langle \gamma^r y^{r+1} - \gamma^{r-1} y^r, y^{r+1} - y^r \rangle \leq \langle \nabla_y f^{r+1} - \nabla_y f^r, y^{r+1} - y^r \rangle.
\]

In the following, we will use this inequality to analyze the recurrence of the size of the difference between two consecutive iterates. First, we have
\[
\langle \gamma^r y^{r+1} - \gamma^{r-1} y^r, y^{r+1} - y^r \rangle
\]
\[
= \langle \gamma^r y^{r+1} - \gamma^r y^r + \gamma^r y^r - \gamma^{r-1} y^r, y^{r+1} - y^r \rangle
\]
\[
= \gamma^r \|y^{r+1} - y^r\|^2 + (\gamma^r - \gamma^{r-1}) \langle y^r, y^{r+1} - y^r \rangle
\]
\[
= \gamma^r \|y^{r+1} - y^r\|^2 + \frac{\gamma^r - \gamma^{r-1}}{2} \left( \|y^{r+1}\|^2 - \|y^r\|^2 - \|y^{r+1} - y^r\|^2 \right)
\]
\[
= \frac{\gamma^r + \gamma^{r-1}}{2} \|y^{r+1} - y^r\|^2 - \frac{\gamma^{r-1} - \gamma^r}{2} \left( \|y^{r+1}\|^2 - \|y^r\|^2 \right).
\]

Substituting (53) and (32) into (52), we have
\[
\frac{1}{2\rho} \|y^{r+1} - y^r\|^2 - \frac{\gamma^{r-1} - \gamma^r}{2} \|y^{r+1}\|^2
\]
\[
\leq \frac{1}{2\rho} \|y^r - y^{r-1}\|^2 - \frac{1}{2\rho} \|v^{r+1}\|^2 - \frac{\gamma^{r-1} - \gamma^r}{2} \|y^r\|^2
\]
\[
- \frac{\gamma^{r-1} + \gamma^r}{2} \|y^{r+1} - y^r\|^2 + \langle \nabla_y f^{r+1} - \nabla_y f^r, y^{r+1} - y^r \rangle
\]
\[
\leq \frac{1}{2\rho} \|y^r - y^{r-1}\|^2 - \gamma^r \|y^{r+1} - y^r\|^2 - \frac{\gamma^{r-1} - \gamma^r}{2} \|y^r\|^2 + \langle \nabla_y f(x^{r+1}, y^r) - \nabla_y f(x^r, y^r), y^{r+1} - y^r \rangle
\]
\[
\leq \frac{1}{2\rho} \|y^r - y^{r-1}\|^2 - \frac{\gamma^{r-1} - \gamma^r}{2} \|y^r\|^2 + \frac{L_y^2}{2\gamma^r} \|x^{r+1} - x^r\|^2 - \frac{\gamma^r}{2} \|y^{r+1} - y^r\|^2
\]
where (a) is true because of the fact that \(0 < \gamma^r < \gamma^{r-1}\) and the concavity of function \(f(x,y)\) in \(y\); in (b) we use the Young’s inequality. Then we have

\[
\frac{4\mathcal{F}^{r+1}}{\rho \gamma^r} \leq \frac{2}{\rho^2 \gamma^r} \| y^r - y^{r-1} \|^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-1}}{\gamma^r} - 1 \right) \| y^r \|^2 \\
- \frac{2}{\rho} \| y^{r+1} - y^r \|^2 + \frac{2L_y^2}{\rho (\gamma^r)^2} \| x^{r+1} - x^r \|^2
\]

\[
\leq \frac{4\mathcal{F}^r}{\rho \gamma^r - 1} + \frac{2}{\rho^2} \left( \frac{1}{\gamma^r} - \frac{1}{\gamma^{r-1}} \right) \| y^r - y^{r-1} \|^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2}}{\gamma^{r-1}} - \frac{\gamma^{r-1}}{\gamma^r} \right) \| y^r \|^2 \\
- \frac{2}{\rho} \| y^{r+1} - y^r \|^2 + \frac{2L_y^2}{\rho (\gamma^r)^2} \| x^{r+1} - x^r \|^2.
\]

(54)

where \(\mathcal{F}^{r+1} := \frac{1}{2\rho} \| y^{r+1} - y^r \|^2 - \frac{\gamma^{r-1} - \gamma^r}{2} \| y^{r+1} \|^2\). Furthermore, combining (45) and (54), we have

\[
l(x^{r+1}, y^{r+1}) - \frac{\gamma^r}{2} \| y^{r+1} \|^2 + \frac{4\mathcal{F}^{r+1}}{\rho \gamma^r}
\]

\[
\leq l(x^r, y^r) - \frac{\gamma^{r-1}}{2} \| y^r \|^2 + \frac{4\mathcal{F}^r}{\rho \gamma^{r-1} - 1} - \frac{1}{\rho} \| y^{r+1} - y^r \|^2 \\
- \left( \frac{\beta^r}{2} + \mu - \left( \frac{\rho L_y^2}{2} + \frac{2L_y^2}{\rho (\gamma^r)^2} \right) \right) \| x^{r+1} - x^r \|^2 + \frac{\gamma^{r-1} - \gamma^r}{2} \| y^{r+1} \|^2 \\
+ \frac{2}{\rho} \left( \frac{1}{\gamma^r} - \frac{1}{\gamma^{r-1}} \right) \| y^r - y^{r-1} \|^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2}}{\gamma^{r-1}} - \frac{\gamma^{r-1}}{\gamma^r} \right) \| y^r \|^2.
\]

Finally, by moving the terms related to \(\| y^r - y^{r-1} \|^2\) to the LHS, using the definition \(\{ \mathcal{P}^r \} \) in (49), we obtain

\[
\mathcal{P}^{r+1} - \mathcal{P}^r \leq -\frac{1}{2\rho} \| y^{r+1} - y^r \|^2 + \frac{\gamma^{r-1} - \gamma^r}{2} \| y^{r+1} \|^2 \\
- \left( \frac{\beta^r}{2} + \mu - \left( \frac{\rho L_y^2}{2} + \frac{2L_y^2}{\rho (\gamma^r)^2} \right) \right) \| x^{r+1} - x^r \|^2 \\
+ \frac{2}{\rho} \left( \frac{1}{\rho \gamma^{r+1}} - \frac{1}{\rho \gamma^r} \right) \| y^{r+1} - y^r \|^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2}}{\gamma^{r-1}} - \frac{\gamma^{r-1}}{\gamma^r} \right) \| y^r \|^2.
\]

According to the above, to achieve descent in \(\| y^{r+1} - y^r \|^2\) we need to ensure that the following holds:

\[
-1/2\rho + 2/\rho^2 (1/\gamma^{r+1} - 1/\gamma^r) < 0.
\]

(55)

Note that, (55) is equivalent to the condition \(1/\gamma^r - 1/\gamma^r \leq \rho/4\). which holds by assumption. This completes the proof.

**Q.E.D.**

Before proving the main result in this section, we make the following assumptions on the parameter choices.

**Assumption C-3.** Suppose that the following conditions hold:
(1) The sequence \( \{\gamma^r\} \) satisfies
\[
\gamma^r - \gamma^{r+1} \geq 0, \gamma^r \to 0, \sum_{r=1}^{\infty} (\gamma^r)^2 = \infty, \frac{1}{\gamma^r + 1} - \frac{1}{\gamma^r} \leq \frac{\rho}{5}. \tag{56}
\]

(2) The sequence \( \beta^r \) satisfies
\[
\beta^r > \rho L_y^2 + \frac{4L_y^2}{\rho(\gamma^r)^2} - 2\mu, \beta^r > L_x, \forall i. \tag{57}
\]

The above assumption on \( \{\gamma^r\} \) can be satisfied, for example, when \( \gamma^r = \frac{1}{\rho^{1/r}} \); see the discussion after (54).

**Theorem 2:** Suppose that Assumptions A, B, C-2 and C-3 hold. Let \((x^r, y^r)\) be a sequence generated by HiBSA. For a given \( \epsilon > 0 \), let \( T(\epsilon) \) be defined similarly as in Theorem [1] Then there exists a constant \( C > 0 \) such that \( \epsilon \leq C \log(T(\epsilon)) \sqrt{T(\epsilon)} \).

**Proof.** For simplicity, let \( G^r := G(x^r, y^r) \). Similarly as in the proof of Theorem [1] we have
\[
\|G^r_i\| \leq (\beta^r + L_{u_i} + L_{x_i})\|x^{r+1} - x^r\|, \forall i \in [K].
\]

For the corresponding bound for \( y \) we have
\[
\|(\nabla G(x^r, y^r))_{K+1}\|
\leq \frac{1}{\rho} \|y^{r+1} - y^r\| + \frac{1}{\rho} \|Py^{1/\rho}(y^r + \rho \nabla y f(x^r, y^r))\|
\leq \frac{1}{\rho} \|y^{r+1} - y^r\| + \frac{1}{\rho} \|Py^{1/\rho}(y^r + \rho \nabla y f(x^{r+1}, y^{r+1}) - \rho \gamma^r y^{r+1}) - Py^{1/\rho}(y^r + \rho \nabla y f(x^r, y^r))\|
\leq L_y \|x^{r+1} - x^r\| + \left(\frac{1}{\rho} + L_y\right) \|y^{r+1} - y^r\| + \gamma^r \|y^{r+1}\|,
\]
where in (a) we use the optimality conditions w.r.t \( y \); in (b) we use the nonexpansiveness of the proximal operator, as well as the the Lipschitz gradient condition w.r.t \( y \) two times. Combining the above two bounds we obtain
\[
\|\nabla G^r\|^2 \leq \sum_{i=1}^{K} (\beta^r + L_{u_i} + L_{x_i})^2 \|x^{r+1} - x^r\|^2
+ 3(\gamma^r)^2 \|y^{r+1}\|^2 + 3L_y^2 \|x^{r+1} - x^r\|^2 + 3 \left(\frac{1}{\rho} + L_y\right)^2 \|y^{r+1} - y^r\|^2
\leq \left(K(L + \beta^r)^2 + 3L_y^2\right) \|x^{r+1} - x^r\|^2 + 3 \left(\frac{1}{\rho} + L_y\right)^2 \|y^{r+1} - y^r\|^2 + 3(\gamma^r)^2 \|y^{r+1}\|^2, \tag{58}
\]

February 25, 2019 DRAFT
where we defined \( L := \max_{i \in [K]} (L_{u,i} + L_{x,i}) \). Moreover, we choose
\[
\beta^r = \rho L_y^2 + \frac{2\kappa L_y^2}{\rho (\gamma^r)^2} - 2\mu,
\]
where \( \kappa \) is chosen to satisfy \( \kappa > 2, \quad \beta^0 > L_{x,i}, \forall i \).

By condition (56), it is clear that \( \beta^{r+1} \geq \beta^r \). Combining this with the choice of \( \kappa \) we have: \( \beta^r \geq \beta^0 > L_{x,i}, \forall i, r \). Thus, this choice of \( \beta^r \) satisfies Assumption C-3.

Moreover, such a choice implies that
\[
\alpha^r := \frac{\beta^r}{2} + \mu - \left( \frac{\rho L_y^2}{2} + \frac{2\kappa L_y^2}{\rho (\gamma^r)^2} \right) = \frac{(\kappa - 2)L_y^2}{\rho (\gamma^r)^2}.
\]

Using these properties in (58), the constants in front of \( \|x^{r+1} - x^r\|^2 \) becomes
\[
K (L + \beta^r)^2 + 3L_y^2 = K \left( L + \rho L_y^2 + \frac{2\kappa L_y^2}{\rho (\gamma^r)^2} - 2\mu \right)^2 + 3L_y^2
\]
\[
\overset{(a)}{=} \left( K^2 L + \rho K^2 L_y^2 - 2\mu K^2 + K^2 \frac{2\kappa}{\kappa - 2} \alpha^r \right)^2 + 3L_y^2
\]
\[
\overset{(b)}{\leq} \left( d_1 \alpha^r \right)^2
\]

in (a) we use the identity shown in (60); (b) always hold for some \( d_1 > 1 \) (which are both independent of \( r \)), since \( \alpha^r \) is an increasing sequence, and \( \alpha^0 \) is bounded away from zero. Note that \( y \) lies in a bounded set, there exists \( \sigma_y \) such that \( \|y^{r+1}\|^2 \leq \sigma_y^2, \forall r \). Using (61), setting \( z := 3 \left( L_y + \frac{1}{\mu} \right)^2 \), we obtain
\[
\|\nabla G^r\|^2 \leq (d_1 \alpha^r)^2 \|x^{r+1} - x^r\|^2 + z\|y^{r+1} - y^r\|^2 + 3(\gamma^r)^2 \sigma_y^2
\]

Furthermore, when \( \beta^r = \rho L_y^2 + \frac{2\kappa L_y^2}{\rho (\gamma^r)^2} - 2\mu \), the bound of the potential function (50) becomes
\[
\mathcal{P}^{r+1} \leq \mathcal{P}^r - \frac{1}{10\rho} \|y^{r+1} - y^r\|^2 - \alpha^r \|x^{r+1} - x^r\|^2 + \frac{\gamma^{r-1} - \gamma^r}{2} \|y^{r+1}\|^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2} - \gamma^{r-1}}{\gamma^{r-1} - \gamma^r} \right) \|y^r\|^2.
\]

Because \( \{\alpha^r\} \) is increasing and \( \|y^r\|^2 \leq \sigma_y^2 \), the above relation implies the following
\[
\frac{1}{10\rho} \|y^{r+1} - y^r\|^2 + \alpha^r \|x^{r+1} - x^r\|^2 \leq \mathcal{P}^r - \mathcal{P}^{r+1} + \frac{\gamma^{r-1} - \gamma^r}{2} \sigma_y^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2} - \gamma^{r-1}}{\gamma^{r-1} - \gamma^r} \right) \sigma_y^2.
\]

Let us define
\[
d_2^r := \min \left\{ \frac{1}{10\rho}, 1 \right\} / \max \left\{ z, d_1^2 \alpha^r \right\}.
\]

Then by combining (63) and (62), we obtain
\[
\|\nabla G^r\|^2 \times d_2^r \leq \mathcal{P}^r - \mathcal{P}^{r+1} + \frac{\gamma^{r-1} - \gamma^r}{2} \sigma_y^2 + \frac{2}{\rho} \left( \frac{\gamma^{r-2} - \gamma^{r-1}}{\gamma^{r-1} - \gamma^r} \right) \sigma_y^2 + 3(\gamma^r)^2 \sigma_y^2 \times d_2^r
\]
Summing both sides from \( r = 1 \) to \( T \), and noting that condition (56) implies \( \gamma r^\alpha \leq 1.2, \forall r \), we obtain

\[
\sum_{r=1}^{T} d_2^r \|\nabla G_r\|^2 \leq \sum_{r=1}^{T} d_2^r \frac{3(k-2)L_2^y\sigma_y^2}{\rho \alpha^r} + \mathcal{P}^1 - \mathcal{P} + \sigma_y^2 \left( \frac{\gamma_0 - \gamma T}{2} + \frac{2}{\rho} \left( \frac{\gamma_1 - \gamma_0}{\gamma T} \right) \right).
\]

Notice that since \( d_1 > 1 \), we have \( d_2^r \leq \frac{d_4}{d_1^{\alpha^r}} \), where \( d_4 := \min \{ 1/10 \rho, 1 \} \). Also, there exists \( d_5 > \max \left\{ \frac{d_2^r}{d_1 \alpha^r} \right\} \) such that \( d_2^r \geq \frac{1}{d_5 \alpha^r} \).

By utilizing the definition of \( T(\epsilon) \) and the above bounds, we know that

\[
\epsilon \leq d_3 d_5 + \frac{3d_4 d_5 (k-2) L_2^y \sigma_y^2}{\rho} \sum_{r=1}^{T(\epsilon)} \frac{1}{\alpha^r}.
\]

Moreover, when \( \gamma^r = \frac{1}{\rho r^{1/4}} \), it can be verified that \( \frac{1}{\gamma^r} - \frac{1}{\gamma^{r+1}} \leq 0.19 \rho, \forall r \geq 1 \), since function \( (r+1)^{1/4} - (r)^{1/4} \) is a monotonically decreasing function and its maximum value is achieved at \( r = 1 \).

In this case we have \( \alpha^r = (k-2) \rho L_2^y \sqrt{T} \). Using these choices of \( \{\gamma^r, \alpha^r\} \), and by utilizing the bounds that \( \sum_{r=1}^{T} 1/r \leq c \ln(T) \) (for some \( c > 0 \)), and \( \sum_{r=1}^{T} 1/\sqrt{T} \geq \sqrt{T} \), inequality (64) becomes:

\[
\epsilon \leq C \log(\sqrt{T(\epsilon)}) \sqrt{T(\epsilon)},
\]

where \( C > 0 \) is some constant independent on the iteration.

**D. Convergence analysis: \( f(x,y) \) linear in \( y \)**

Finally, we briefly discuss the case where the coupling term in (1) is linear in \( y \). The results of this section mostly follow from those of Section III-C, therefore will be mostly omitted.

**Assumption C-4.** Assume that problem (1) takes the following form

\[
\min_x \max_y y^T F(x_1, x_2, \ldots, x_K) + \sum_{i=1}^{K} h_i(x_i) - g(y)
\]

\[
\text{s.t. } x_i \in \mathcal{X}_i, \ y \in \mathcal{Y}, \ i = 1, \ldots, K
\]

where \( F(\cdot): \mathbb{R}^{NK} \rightarrow \mathbb{R}^M \) is a vector function. Further assume that (26) holds for \( U_y(\cdot) \) ■

Note that (66) contains the robust learning problem (4), the min utility maximization problem (5), and Example 1 as special cases. It is worth noting that, due to the use (26), we are able to perform a simple gradient step (or gradient projection step when \( \mathcal{Y} \) is not the full spaces, or proximal gradient step when \( g(y) \) is present) to update \( y \), while in the algorithm proposed in the previous section, each iteration has to solve an optimization problem involving \( y \).

It is worth mentioning that, in this case the analysis steps are similar to those in Sec. III-C. In particular, we can show that the potential function (49) has the same behavior as in Lemma 5. Therefore, we state our convergence result in the following corollary.
Corollary 3.1: Suppose that Assumptions A, B, C-3 and C-4 hold. Let \((x^r, y^r)\) be a sequence generated by HiBSA. For a given \(\epsilon > 0\), let \(T(\epsilon)\) be defined similarly as in Theorem 1. Then there exists a constant \(C > 0\) such that \(\epsilon \leq \frac{C \log(T(\epsilon))}{\sqrt{T(\epsilon)}}\).

IV. NUMERICAL RESULTS

We test our algorithms on three applications: a robust learning problem, a rate maximization problem in the presence of a jammer and a coordinated beamforming problem.

Robust learning over multiple domains. Consider a scenario where we have datasets from two different domains and adopt a neural network model in order to solve a multi-class classification problem. The neural network consists of two hidden layers with 25 neurons, each endowed with ReLU activations, except from the output layer where we adopt the softmax activation. We aim to learn the model parameters using the following two approaches:

[1] Robust Learning: Apply the robust learning model \(^4\) and optimize the cost function using the HiBSA algorithm with \(\gamma^r = \frac{1}{\sqrt{T(\epsilon)}}\). Note that we treat the minimization variable as one block and use the 1st order Taylor expansion of the cost function as the approximation function.

[2] Multitask Learning: Apply a multitask learning model \(^48\), where the weights associated with each loss function/task are fixed to 1/2. The problem is optimized using gradient descent.

Moreover, we evaluate the accuracy of the above algorithms as the worst hit rate across the two domains, i.e., accuracy = \(\min\{\text{hit rate on domain 1, hit rate on domain 2}\}\).

In our experiments we use the MNIST \(^49\) dataset whose data points are images of handwritten digits of dimensions \(28 \times 28\). We select two different parts of the MNIST dataset as the two different domains we mentioned above. The first part consists of the digits from 0 to 4, while the second one contains the rest. Moreover, for the 1st domain we use 800 images for training and 160 for testing, while in the second one we employ 4000 and 800 images respectively. Finally, we average the results over 5 iterations.

Note that we do not perform extensive parameter tuning, since the purpose of this experiment is not to support the superiority of the robust model, but merely to illustrate that both models achieve comparable performance. If true, then one can conclude that the proposed algorithm works well as it is capable of attaining good solutions of min-max optimization problems. Indeed, the results presented in Fig. 2 support this view, since the two approaches achieve approximately the same accuracy on the test set.

Power control in the presence of a jammer. Consider the multi-channel and multi-user formulation \(^8\) where there are \(N\) channels, \(K\) collaborative users and one jammer. We can verify that the jammer
problem (i.e., the maximization problem over $y$) has a strongly concave objective function over the feasible set.

We compare HiBSA with the classic interference pricing algorithm [50], [51], and the WMMSE algorithm [52], which are designed for solving sum-rate optimization problem without the jammer. Our problem is tested using the following setting. We construct a network with $K = 10$, and the interference channel among the users and the jammer is generated using uncorrelated fading channel model with channel coefficients generated from the complex zero-mean Gaussian distribution with unit covariance [52]. All users’ power budget is fixed at $P = 10^{S_{\text{SNR}}/10}$. For test cases without a jammer, we set $\sigma_k^2 = 1$ for all $k$. For test cases with a jammer, we set $\sigma_k^2 = 1/2$ for all $k$, and let the jammer have the rest of the noise power, i.e., $p_{0_{\text{max}}} = N/2$. Note that by splitting the noise power we intend to achieve some fair comparison between the cases with and without the jammer. However, it is not possible to be completely fair because even though the total noise budgets are the same, the noise power transmitted by the jammer has to go through the random channel, so the total received noise power could be different. Nevertheless, this setting is sufficient to demonstrate the behavior of the HiBSA algorithm.

From the Fig. 3 (top), it is clear that the pricing algorithm monotonically increases the sum rate (as is predicted by theory), while HiBSA behaves differently: after some initial oscillation, the algorithm converges to a value that has lower sum-rate. Further in Fig. 3 (bottom), we do see that by using the proposed algorithm, the jammer is able to effectively reduce the total sum rate of the system.

**Coordinated MISO beamforming design.** Consider the coordinated beamforming design problem [14] described in Sec. II over a MISO interference channel. In this problem we experiment with the scenario in which we have $K = 10$ users/transmitter-receiver pairs, each transmitter is equipped with $N = 6$ antennas and we adopt the min-rate utility, i.e $U(\{R_i(x)\}_{i=1}^{K}) = \min_{i=1,\ldots,K} R_i(x)$. Moreover, the users’ transmission is performed over a complex Gaussian channel and we set the power constraints for
The problem of interest is to design the users’ beamformers in order to maximize the system’s utility function under constraints in power and outage probability. We approach the solution of the problem using two different algorithms:

1) **BSUM-LSE** [14] Substitute the min-rate utility function with a popular log-sum-exp approximation, i.e.

\[
\min_{i=1,\ldots,K}\left\{R_i(x)\right\} \approx \frac{1}{\nu}\log_2\left(\sum_{i=1}^{K}2^{-\nu R_i}\right).
\]

Note that \(\nu\) specifies the accuracy of approximation with higher \(\nu\)’s corresponding to better approximation. Then following what is suggested in Sec. C of [14], we formulate the respective problem using the surrogate function, and solve the resulting problem iteratively using the projected gradient descent.

2) **HiBSA** For the HiBSA algorithm we consider the min-max formulation in (5) and adopt for the minimization problem the same surrogate function we employed in BSUM-LSE. Moreover, in the maximization problem we use \(\gamma^r = 1/\nu^{1/4}\).

We run both algorithms for 1000 complete iterations (one complete iteration involves one update of all the block variables, and 1000 iterations are sufficient for both algorithms to converge in all scenarios), set the stepsize of the gradient descent/ascent iterations of HiBSA and the respective iterations of BSUM-LSE equal to \(10^{-2}\) and average the final results over 10 independent random problem instances. Moreover, in order to evaluate the effect of the log-sum approximation we show the achieved min-rate utility of BSUM-LSE, by using 3 different values of \(\nu \in \{1, 5, 7\}\).

In Fig. 4 we plot the min-rate utility for 7 different values of the noise variance and 2 different levels of interference. Notice that the HiBSA algorithm achieves larger utility than BSUM-LSE, while as expected the larger the value of \(\nu\) the higher the utility achieved by the latter algorithm. Since we set the parameters...
Fig. 4: The min-rate utility achieved using the HiBSA and the BSUM-LSE algorithm [14] for two different interference levels in a scenario where we have $K = 10$ users equipped with $N = 6$ antennas. Level 1 (top) corresponds to lower interference than level 2 (bottom). For each interference level we experiment with 3 different values of approximation accuracy ($\nu$). Note that higher $\nu$ corresponds to higher accuracy.

of the two algorithms as close as possible it is implied that the formulation of the problem at hand as a min-max optimization problems rather than using an approximation of the inner problem is beneficial, since it results in higher achievable utility.

V. CONCLUSIONS

In this paper, motivated by the min-max problems appeared in the areas of signal processing and wireless communications, we propose a relatively simple algorithm called HiBSA. By leveraging the (strong) concavity of the maximization problem, we conduct analysis on the convergence behavior of the proposed algorithm. Numerical results show the effectiveness of the proposed algorithms of solving the min-max problems in robust machine learning and wireless communications.

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