Theoretical analysis of stress intensity factor for two unequal collinear cracks subjected to internal pressure and compressive stress

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ABSTRACT

In this paper, a new solution of the stress intensity factors (SIFs) for two unequal collinear cracks is developed considering internal pressure, friction and compressive stress. The complex stress function and elliptic integral function are used to obtain the analytical solution of the SIFs, and it shows a good agreement with the previous researchers’ solutions and numerical results. The theoretical results show that the difference and interaction of the SIFs at cracks’ tips are caused by crack geometry parameters, and they also indicate that the internal pressure leads to the SIFs of a mode I crack and affects the SIFs of a mode II crack because of friction.

KEYWORDS: stress intensity factor, collinear cracks, complex stress function, internal pressure

1. INTRODUCTION

Most brittle solids in nature often contain some kinds of initial defects or intrinsic flaws in the form of multiple cracks. As is well known, these cracks with a specific shape, size and distribution can significantly impact the members’ mechanical behaviors. However, for porous materials containing cracks or flaws and subject to fluid infiltration, there is still a need to better understand how crack-sealed fluid pressure can alter the stress distribution under compression. Some materials, such as concrete \cite{1}, coal \cite{2} and shale \cite{3}, containing a series of pressurized cracks under compression, are often encountered in various engineering applications. From the view of fracture mechanics, the correct knowledge of the stress intensity factors (SIFs) in the crack region vicinity is essential to accurately predict fracture failure of the members \cite{4} and crack propagation behavior \cite{5, 6}.

Collinear cracks, as a type of multiple cracks, exist widely and affect each other in brittle solids. Willmore \cite{7} first obtained the solution of the SIFs at the tips of two collinear cracks. Muskhelishvili \cite{8} used the complex stress functions to solve the elasticity problems of a plane containing collinear cracks, and his results were developed to obtain the analytical expressions for SIFs at each crack tip of three equal collinear cracks under far-field tension \cite{9}. Unlike tension, the compression load may cause the cracks to close, so the friction on the crack surface should be considered \cite{10–12}. Zhu \cite{13} considered the surface friction to obtain the solution of the SIFs for two collinear cracks under compression by using the complex variable function method and proposed a novel crack propagation criterion. Jin et al. \cite{14} applied the complex functions and the least-squares boundary collocation method to calculate the overall displacement field, stress field and the SIFs of collinear cracks. Zheng et al. \cite{15} obtained the analytical solution of the SIFs at the tips of three collinear cracks in an infinite plane under compression.

Different from two equal collinear cracks, the more complex methods were used to obtain the solution of the SIFs at the tips of two unequal collinear cracks due to elliptic integral function. Kastratović et al. \cite{16} obtained the SIFs in two unequal cracks in an infinite plate subject to remote uniform tensile stress and studied the interaction effect between cracks with the increase of SIFs. Liu et al. \cite{17} presented the SIFs of two unequal-length collinear cracks in an infinite sheet for arbitrary load case by weight functions. For the interaction of multiple cracks, Kachanov \cite{18} discussed the state of some fundamental problems in an elastic medium. To better understand their mutual effects under compression, Fan et al. \cite{19} analyzed the analytical expressions of the SIFs of two unequal collinear cracks based on Muskhelishvili’s results. Basista \cite{18} modified the Kachanov method for the brittle solids under compression.

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to obtain and analyze the approximate solutions for SIFs considering the interaction of frictional cracks. Qing and Yang [20] and Li et al. [21] improved the Kachanov method when dealing with strongly interacting multiple cracks in an infinite plate and calculated the SIFs of two collinear cracks. Their research results give us a sufficient understanding of the SIFs at the tips and the interaction of multiple cracks under compression. However, they did not consider the effect of internal pressure on SIFs caused by fluid in cracks.

Rahman et al. [22] formulated approximately the SIFs for internally pressurized and oriented cracks in remotely compressed rocks. Santare [23] developed a model whereby an incompressible fluid penetrated bone-cement cracks and calculated the mode I SIF to test the feasibility of crack growth under compressive loads. Bruno and Nakagawa [24] described the influence of pore pressure and the distribution of pore pressure gradients on tensile fracture initiation and propagation direction theoretically and experimentally. They were successful in investigating the effect of internal pressure on SIF, but only a fracture plane. For an infinite medium containing many collinear cracks, England and Green [25] solved the problem considering the internal pressure on each crack surface. Tranter [26] and Lowengrub and Srivastava [27] applied the Fourier transform method to investigate the stress distribution of two equal cracks subjected to internal pressure. Liu et al. [28] developed a model for describing the SIF evolution of a wing crack subjected to compression–shear load and pore fluid pressure.

Although some research has been carried out on collinear pressurized crack under compression, shear and tension, many important results were presented. However, fewer studies have been reported about the combined actions of internal pressure and compression, which may cause opening or closing of cracks and affect SIFs of mode II crack because of crack surface friction.

This paper attempts to obtain the analytical expressions of the SIFs of two unequal collinear cracks subjected to internal pressure and biaxial compressive stress using a complex stress function method, and the analytical solutions show a good agreement with the previous researchers’ solutions and numerical results using the ABAQUS code. The influences of internal pressure, crack geometry parameters and surface friction on the SIFs are investigated through the analytical results.

2. SIFS AT THE COLLINEAR CRACKS IN AN INFINITE PLATE SUBJECT TO INTERNAL PRESSURE AND COMPRESSION

2.1 Complex representation

Muskhelishvili’s complex variable theory [29] is a convenient way to solve plane crack problems. In this theory, the analyst is provided with insight into the mathematical character of the solution, especially the problem of linear relation expressed by analytic continuation arguments [30], and the general solution for linear elasticity problem about stress \((\sigma_{xx}, \sigma_{yy}, \sigma_{xy})\) and displacement \((u_x, u_y)\) can be written by the complex stress functions, \(\Phi(z)\) and \(\Omega(z)\), such that

\[
\begin{align*}
\sigma_{xx} + \sigma_{yy} &= 4 \text{Re} \Phi(z), \\
\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2\left[ (\dot{z} - z) \Phi'(z) - \Phi(z) + \dot{\Omega}(z) \right], \\
2G(u_x + iu_y) &= \kappa \int \Phi(z) \, dz - \int \Omega(z) \, dz - (z - \bar{z}) \dot{\Phi}(\bar{z}),
\end{align*}
\]

where \(z = x + iy\) denotes a complex variable, \(G = E/2(1 + \nu)\) is the shear modulus of elasticity, \(\kappa = (3 - \nu)/(1 + \nu)\) (generalized plane stress) and \(\kappa = 3 - 4\nu\) (plane strain). \(E\) and \(\nu\) denote Young’s modulus and Poisson’s ratio, respectively. The plane stress condition is assumed.

For an infinite plane under uniform principal stresses, \(\sigma_1\) and \(\sigma_2\) (see Fig. 1), there are a series of cracks along the \(x\)-axis, for \(m = 1, 2, 3, \ldots, n\).

The angle \(\alpha\) between the crack and the \(\sigma_2\)-axis varies from \(0^\circ\) to \(90^\circ\), and the friction coefficient of the crack surface is \(f\).

According to Muskhelishvili’s works [29], the solution of the stress boundary value problem for the infinite plate can be written as follows:

\[
\Phi(z) = \frac{1}{2\pi i \text{Im}(z)} \int_L \frac{X^+(t) \, p(t)}{t-z} \, dt + \frac{1}{2\pi i} \int_L \frac{q(t)}{t-z} \, dt + \frac{P_n(z)}{X^-(z)} \frac{\sigma_1 - \sigma_2}{4} e^{-2\lambda t},
\]

\[
\Omega(z) = \frac{1}{2\pi i \text{Im}(z)} \int_L \frac{X^+(t) \, p(t)}{t-z} \, dt - \frac{1}{2\pi i} \int_L \frac{q(t)}{t-z} \, dt + \frac{P_n(z)}{X^-(z)} \frac{\sigma_1 - \sigma_2}{4} e^{-2\lambda t},
\]

where \(L = L_1 + L_2 + L_3 + \cdots + L_n\) is the union of the \(n\) cracks, \(X(z)\) is the Plemelj function for characterizing multiple cracks shown in Fig. 1 and \(t\) denotes the coordinate of a point on the surface of the crack, which can be written as

\[
X(z) = \sqrt{(z - a_1)(z - b_1) \cdots (z - a_n)(z - b_n)}.
\]

When \(z \to \infty\), \(z^{-n}X(z) \to 1\) and \(X^+(t) = -X^-(t)\), \(a_n\) and \(b_n\) denote the coordinates of the left endpoint and right endpoint of the \(n\)th crack, respectively. \(P_n(z)\) is a polynomial of \(z\), which can be expressed as

\[
P_n(z) = C_0 z^n + C_{n-1} z^{n-1} + \cdots + C_1 z + C_0,
\]

where \(C_0, C_1, C_2, \ldots, C_n\) are constants.
Figure 1 Sketch of an infinite plate containing a series of collinear cracks subjected to internal pressure and compression, where $\sigma_N$ is normal stress, $p$ is internal pressure, $\sigma_s$ is the shear stress and $\sigma_f$ is the friction on the crack surfaces.

From Eq. (2), one can have
\[
\lim_{z \to \infty} \Phi(z) = -\frac{(\sigma_1 + \sigma_2)}{4}. \tag{5}
\]
Combining Eqs (3)–(5), one can get the following equation:
\[
C_n = -\frac{\sigma_1 + \sigma_2}{4} - \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha}. \tag{6}
\]
The constants $C_0, C_1, C_2, \ldots, C_{n-1}$ are determined by the single value condition of displacement, which can be expressed as
\[
\kappa \int_{\Gamma_m} \Phi(z) \, dz - \int_{\Gamma_m} \Omega(\bar{z}) \, d\bar{z} = 0 \quad (m = 1, 2, 3, \ldots, n-1), \tag{7}
\]
where $\Gamma_m$ is the contour of the $m$th crack. If the contour $\Gamma_m$ shrinks to a place around the crack, we can get the following equation:
\[
\kappa \int_{L_m} \left[ \Phi^+(t) - \Phi^-(t) \right] \, dt - \int_{L_m} \left[ \Omega^-(t) - \Omega^+(t) \right] \, dt = 0 \quad (m = 1, 2, 3, \ldots, n-1). \tag{8}
\]
It is not easy to solve the coefficients $C_0, C_1, C_2, \ldots, C_{n-1}$ owing to the elliptic integrals involved because of the Plemelj function $X(z)$.
Here, $p(t)$ and $q(t)$ denote the loads on the upper and lower surfaces of cracks, respectively, which can be written as
\[
p(t) = \frac{\sigma_{yy}^+ + \sigma_{yy}^-}{2} - i \frac{\sigma_{xy}^+ + \sigma_{xy}^-}{2},
\]
\[
q(t) = \frac{\sigma_{yy}^+ - \sigma_{yy}^-}{2} - i \frac{\sigma_{xy}^+ - \sigma_{xy}^-}{2}, \tag{9}
\]
where “+” and “−” denote the upper surface and lower surface, respectively.
The boundary condition can be expressed by a complex variable function:
\[
\int \Phi(z) \, dz + \int \Omega(z) \, d\bar{z} + (z - \bar{z}) \Phi(\bar{z}) = i \int_{AB} (\bar{X} + i\bar{Y}) \, ds + \text{const}, \tag{10}
\]
where $X$ and $Y$ are the surface forces along the boundary in the $x$- and $y$-direction, respectively. The right-hand side of Eq. (10) is a function of the boundary force. Keep the area under consideration on the left when looking along the arc $z_0\bar{z}$ (Fig. 2) in the positive direction. The left-hand side of Eq. (10) is a value when $z$ tends to the region boundary. From Eq. (10), adding or subtracting a constant of the complex variable functions $\Phi(z)$ or $\Omega(z)$ does not affect the stress value, so const $= 0$ in Eq. (10).
The stress boundary conditions can be written as
\[
\sigma_{xx} \cos(n, x) + \sigma_{yy} \cos(n, y) = \overline{X},
\]
\[
\sigma_{yy} \cos(n, x) + \sigma_{xy} \cos(n, y) = \overline{Y}. \tag{11}
\]
From Fig. 1 and Eq. (9), the stress conditions of a crack surface can be expressed as

\[
\begin{align*}
\sigma_{yy}^+ &= -\sigma_N, \quad \sigma_{xy}^+ = -\sigma_t \text{ (upper surface)}, \\
\sigma_{yy}^- &= -\sigma_N, \quad \sigma_{xy}^- = -\sigma_t \text{ (lower surface)},
\end{align*}
\]  

(12)

where \(\sigma_N\) and \(\sigma_t\) are the normal stress and the crack surface friction stress, respectively. Substituting Eq. (12) into Eq. (9), one has

\[
\begin{align*}
p(t) &= -\sigma_N + i\sigma_f = Q, \\
q(t) &= 0,
\end{align*}
\]

(13)

where \(Q\) is a constant that is determined by far-field principal stress and the inner pressure along the crack surface.

From Fig. 1, the infinite plate is subjected to the stresses \(\sigma_1\) and \(\sigma_2\), and the crack surfaces are subjected to internal pressure \(p\). So, one has

\[
\sqrt{x^2 + y^2} \rightarrow \infty : \quad \sigma_{xx} = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha,
\]

\[
\sigma_{yy} = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha,
\]

\[
\sigma_{xy} = \frac{\sigma_1 - \sigma_2}{2}\sin 2\alpha,
\]

\((x, y) \in L : \quad \sigma_{xx} \cos \alpha + \sigma_{xy} \sin \alpha = -p \cos \alpha,
\]

\[
\sigma_{yy} \cos \alpha + \sigma_{xy} \sin \alpha = -p \sin \alpha.
\]

(14)

Combining Eqs (14) and (11), the normal stress and friction stress can be written as

\[
\begin{align*}
\sigma_N &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha - p, \\
\sigma_t &= f\sigma_N = f\left(\frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha - p\right).
\end{align*}
\]

(15)

It should be noted that the shear stress is greater than the friction force on the crack surface; see [13] for a specific explanation. The constant \(Q\) in Eq. (13) can be written as

\[
Q = (-1 + if) \left(\frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha - p\right).
\]

(16)

Therefore, for an infinite plate containing finite collinear cracks, from Eqs (2) and (13), the complex stress functions \(\Phi(z)\) and \(\Omega(z)\) can be reduced to

\[
\begin{align*}
\Phi(z) &= \frac{Q}{2\pi iX(z)} \int_L \frac{X^+(t)}{t-z} dt + \frac{P_n(z)}{X(z)} \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha}, \\
\Omega(z) &= \frac{Q}{2\pi iX(z)} \int_L \frac{X^+(t)}{t-z} dt + \frac{P_n(z)}{X(z)} \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha}.
\end{align*}
\]

(17)
Thus, the problem of determining the expressions of $\Phi_1(z)$ and $\Omega_1(z)$ is reduced to solving the following equation:

$$I(z) = \frac{1}{2\pi i} \int_L \frac{X'(t)}{t-z} \, dt. \quad (18)$$

The integrals along $L$ in Eq. (18) can be expressed by the integral along a lacet $C'$ surrounding these cracks, and Eq. (18) can be expressed as

$$I(z) = \frac{1}{4\pi i} \int_{C'} \frac{X(\varsigma)}{\varsigma-z} \, d\varsigma. \quad (19)$$

The sum can show the integral in Eq. (19) of the residues at infinity and point $z$ by using residue theory, as

$$I(z) = \frac{1}{2} \left\{ X(z) + \text{Res} \left[ \frac{X(\varsigma)}{\varsigma-z}, \infty \right] \right\}. \quad (20)$$

So, the solution of $\Phi(z)$ and $\Omega(z)$ can be rewritten as

$$\Phi(z) = \frac{Q}{2X(z)} \left\{ X(z) + \text{Res} \left[ \frac{X(\varsigma)}{\varsigma-z}, \infty \right] \right\} + \frac{C_n e^n + \cdots + C_0}{X(z)} + \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha},$$

$$\Omega(z) = \frac{Q}{2X(z)} \left\{ X(z) + \text{Res} \left[ \frac{X(\varsigma)}{\varsigma-z}, \infty \right] \right\} + \frac{C_n e^n + \cdots + C_0}{X(z)} - \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha}. \quad (21)$$

### 2.2 Solution for two unequal collinear cracks

The geometry of two collinear cracks in an infinite plate is shown in Fig. 3. When the two unequal collinear cracks are subjected to internal pressure caused by pore fluid, of which physical lengths are identically $a-b$ and $c-d$, respectively, they are subject to remote compressive stresses, $\sigma_1$ and $\sigma_2$. The distance between the outer crack tips is $a-d$ and that between the inner crack tips is $b-c$, as shown in Fig. 3.

For an infinite plate containing two collinear cracks, a primary Plemelj function is defined as

$$X(z) = \sqrt{(z-a)(z-b)(z-c)(z-d)}. \quad (22)$$

The polynomial $P(z)$ can be simplified as $P(z) = C_2 z^2 + C_1 z + C_0$. So, Eq. (21) can be solved, and the solution of $\Phi(z)$ and $\Omega(z)$ can be rewritten as

$$\Phi(z) = \frac{1}{X(z)} \left( S_2 z^2 + S_1 z + S_0 \right) + \frac{Q}{2} + \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha},$$

$$\Omega(z) = \frac{1}{X(z)} \left( S_2 z^2 + S_1 z + S_0 \right) + \frac{Q}{2} - \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha}. \quad (23)$$
where \( S_2 = C_2 - \frac{Q}{2}, \quad S_1 = C_1 + \frac{Q}{4} (a - c), \quad S_0 = C_0 + \frac{Q}{4} \left( b^2 + \left( \frac{a + c}{2} \right)^2 \right). \)

Combining Eqs (23) and (8), one can have

\[
2 (\kappa + 1) \int_{x_m} X (t) \left( S_2 t^2 + S_1 t + S_0 \right) \, dt = 0. \tag{24}
\]

For the two unequal cracks AB and CD shown in Fig. 3, we can have

\[
\int_b^a \frac{S_2 t^2 + S_1 t + S_0}{\sqrt{(t - a) (t - b) (t - c) (t - d)}} \, dt = 0,
\]

\[
\int_c^d \frac{S_2 t^2 + S_1 t + S_0}{\sqrt{(t - a) (t - b) (t - c) (t - d)}} \, dt = 0. \tag{25}
\]

According to the elliptic handbook [31], the elliptic integral equation, Eq. (25), can be solved as

\[
S_0 V_0 + S_1 \left[ (a - d) V_1 + dV_0 \right] + S_2 \left[ \frac{(a - c) (b - d)}{2} E (u) + \frac{ad - ab + bd + d^2}{4} V_0 + \frac{(a - d) (a + b + c + d)}{2} V_1 \right] = 0,
\]

\[
S_0 V_0 + S_1 \left[ (d - a) V_1' + aV_0 \right] + S_2 \left[ \frac{(a - c) (b - d)}{2} E (u) + \frac{ac + ad - cd}{2} V_0 - \frac{(a - d) (a + b + c + d)}{2} V_1' \right] = 0. \tag{26}
\]

where \( E(u) \) is Legendre’s incomplete elliptic integral of the second kind, and

\[
V_0 = \int du = u = E (\varphi, k), \quad V_1 = \int \frac{1}{1 - \alpha^2 \sin^2 u} \, du = \prod \left( \varphi, \alpha^2, k \right), \quad V_1' = \int \frac{1}{1 - \alpha^2 \sin^2 u} \, du = \prod \left( \varphi, \alpha^2, k \right),
\]

where \( F(k) \) is the incomplete elliptic integral of the first kind, and

\[
\alpha^2 = \frac{b - a}{b - d}, \quad \alpha'^2 = \frac{d - e}{a - c}, \quad k^2 = \frac{(a - b) (c - d)}{(a - c) (b - d)}, \quad \varphi = \frac{\pi}{2}.
\]

By solving Eq. (26), we can have

\[
S_1 = -\frac{a + b + c + d}{2} S_2,
\]

\[
S_0 = \left[ \frac{cd + ab}{2} - \frac{(a - c) (b - d) E (k)}{2 F (k)} \right] S_2, \tag{27}
\]

where

\[
k^2 = \frac{(a - b) (c - d)}{(a - c) (b - d)}, \quad E (k) = \frac{1 - \left( \frac{1}{2} \right)^2 k^2 - \left( \frac{1}{2} \times \frac{3}{2} \right)^2 k^2 - \left( \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \right)^2 k^2 - \cdots}{1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1}{2} \times \frac{3}{2} \right)^2 k^2 + \left( \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \right)^2 k^2 + \cdots}.
\]

So, the function \( \Phi (z) \) can be written as

\[
\Phi (z) = S_2 F (z) + \frac{Q}{2} + \frac{\sigma_1 - \sigma_2}{4} e^{-2i\alpha}, \tag{28}
\]

where

\[
F (z) = \frac{2z^2 - (a + b + c + d) z + cd + ab - (a - c) (b - d) E (k)}{2 \sqrt{(z - a) (z - b) (z - c) (z - d)}}.
\]

The SIFs \( K_I \) and \( K_{II} \) can be written as

\[
K_I - iK_{II} = \lim_{z \to \alpha} 2 \sqrt{2 \pi e^{-i\beta} (z - \alpha)} \Phi (z) \Rightarrow \lim_{z \to \alpha} 2 \sqrt{2 \pi e^{-i\beta} (z - \alpha)} S_1 F (z), \tag{29}
\]

where \( \alpha \) is the \( x \)-coordinate of crack tips, \( \beta \) denotes the angle between crack and \( x \)-axis, and for the right crack tip and left crack tip, \( \beta = 0 \) and \( \pi \), respectively; \( F(z) \) is determined by the crack geometry. From the left-hand side of Eq. (29), the mode I factor \( K_I \) and the mode II factor \( K_{II} \) correspond to the real value and the imaginary value in the right-hand side of Eq. (29), respectively. So, the coefficient \( S_1 \) is important to the fracture modes.

From Eq. (29), the dimensionless \( K_I \), i.e. \( Y \), can be expressed as

\[
Y = Y_1 - iY_2 = \frac{K}{K_0} = \frac{(K_I - iK_{II})}{\sigma_1 \sqrt{\pi} (a - b)}, \tag{30}
\]
where the coefficients $a$ and $b$ are shown in Fig. 3.

The dimensionless $Y$s at tips A, B, C and D of unequal collinear cracks are expressed as

$$\begin{align*}
\{Y_1^A, Y_1^B\} &= \frac{F^A}{\sigma_1} \{\text{Re} \ S_2, -\text{Im} \ S_2\}, \\
\{Y_1^B, Y_1^C\} &= \frac{F^B}{\sigma_1} \{\text{Re} \ S_2, -\text{Im} \ S_2\}, \\
\{Y_1^C, Y_1^D\} &= \frac{F^C}{\sigma_1} \{\text{Re} \ S_2, -\text{Im} \ S_2\}, \\
\{Y_1^D, Y_1^E\} &= \frac{F^D}{\sigma_1} \{\text{Re} \ S_2, -\text{Im} \ S_2\},
\end{align*}$$

(31)

where

$$\begin{align*}
F^A &= \sqrt{\frac{2(a-d)(a-c)}{a-b}} \left\{1 - \frac{b-d \ E(k)}{a-d \ F(k)} \right\}, \\
F^B &= \sqrt{\frac{2(b-c)(b-d)}{a-b}} \left\{1 - \frac{a-c \ E(k)}{b-c \ F(k)} \right\}, \\
F^C &= \sqrt{\frac{2(a-b)(b-c)}{(a-b)(c-d)}} \left\{1 - \frac{b-d \ E(k)}{(a-b)(c-d)} \right\}, \\
F^D &= \sqrt{\frac{2(a-b)(b-c)}{(a-b)(c-d)}} \left\{1 - \frac{a-c \ E(k)}{(a-b)(c-d)} \right\},
\end{align*}$$

and

$$S_2 = C_2 - \frac{Q}{2} = \frac{1}{4} \left\{(σ_1 - σ_2) \sin 2α - [σ_1 + σ_2 + (σ_1 - σ_2) \cos 2α - 2p] f \right\} - \frac{p}{2}.$$

From Eq. (31), the dimensionless SIFs, $Y$s, are determined by the crack geometry parameters and stress conditions. The equation shows that some $Y$s are negative, and the negative signs of $Y$ are neglected in the calculation process because it only denotes crack propagation direction. When the shear stress $σ = (σ_1 - σ_2) \sin 2α/2$ on the crack surface is less than the friction force $σ = (σ_1 + σ_2 + (σ_1 - σ_2) \cos 2α - 2p)/2$, which means Eq. (32) is satisfied, the crack tip SIF is zero, i.e. $Y_1^A = Y_1^B = Y_1^C = Y_1^D = 0$:

$$\frac{σ_2}{σ_1} \geq \frac{1 - f \ cot α}{1 + f \ tan α} + \frac{2pf}{(1 + f \ tan α) \ σ_1 \ sin 2α}.$$

(32)

\section{3. VALIDATION OF THE ANALYTICAL SOLUTION}

This section mainly focuses on the numerical validation of SIFs evaluated here using the ABAQUS code to the established analytical results, i.e. Eq. (31). The numerical model consists of two unequal pressurized cracks in a square matrix ($2000 \times 2000 \text{ cm}^2$) under biaxial compression. The ratio of domain size to crack length should be considered for the simulation of an infinite plane. The crack shape is defined by the lengths $a$, $b$, $c$ and $d$ of $5$, $4$, $3$ and $1 \text{ cm}$, respectively (see Fig. 3). The CPS8 and CPS6 are used to calculate the SIFs at the crack tips (see Fig. 4). The analyses are executed with plane stress conditions. To facilitate comparison, the equation and ABAQUS model are under the same boundary conditions that both internal pressure and friction coefficient are zero, and the ratio of stress $σ_2/σ_1$ increases gradually. The results obtained by Eq. (31) are very close to those by software ABAQUS in Fig. 5, and the maximum error of all results is only 4.5%.

As shown in Fig. 5, the values of dimensionless $Y$s generally decrease as the ratio stress $σ_2/σ_1$ increases. The corresponding $Y$ curve coincides with the $x$-axis when the ratio $σ_2/σ_1$ is equal to 1.0, and when the ratio $σ_2/σ_1$ is equal to 0, the curve involved is upper in all the curves in Fig. 5. The values of $Y_1$ at the tips of the short crack are less than those at the tips of the long one.

In addition, SIFs in Eq. (31) agree with the results given by Fan et al. [15, 19] when the axis moves to the middle of two cracks and the internal pressure is zero, namely $c = -b$, $d = -c$ and $p = 0$ in Eq. (31). In [19], the SIFs at crack tips $A'$, $B'$, $C'$ and $D'$ in Fig. 6 are listed as follows:

$$Y_1^{A'} = \frac{K_{0}}{K_0} = \sqrt{\frac{2(b+c)\left(\frac{a+c}{a-b}\right)}{a-b}} \left(1 - \frac{b+c}{a+c} \ E(k) \frac{b+c}{a+c} K(k)\right).$$

Figure 4 The finite element model and meshes of unequal collinear cracks in ABAQUS.
Figure 5 The calculation results of $Y_{II}$ at crack tips A, B, C and D under compression by Eq. (31) and using the ABAQUS code, where $f = 0$ and $p = 0$. The relative error is the percentage of the difference between theoretical calculation results and the calculated values by ABAQUS code for the calculated values by ABAQUS code. (a) $Y_{II}$ at crack tip A; (b) $Y_{II}$ at crack tip B; (c) $Y_{II}$ at crack tip C; and (d) $Y_{II}$ at crack tip D.
Figure 6 Sketch of two unequal collinear cracks subjected to compression [19].

\[
Y_{ii}^{B'} = \frac{K_{ii}^{B'}}{K_0} = \frac{1}{2\sigma_1} \sqrt{b'(b' + c')} \left(1 - \frac{2b'}{a' + b'} \frac{E(k')}{K(k')}\right) S_2',
\]

\[
Y_{ii}^{C'} = \frac{K_{ii}^{C'}}{K_0} = \frac{1}{2\sigma_1} \sqrt{(a' + b') b'} \left(1 - \frac{b' + c'}{a' + c'} \frac{E(k')}{K(k')}\right) S_2',
\]

\[
Y_{ii}^{D'} = \frac{K_{ii}^{D'}}{K_0} = -\frac{\sqrt{2}}{4\sigma_1} \sqrt{(c' + b') (a' + b') (c' - b') (a' - b')} \left(1 - \frac{b' + c'}{a' + c'} \frac{E(k')}{K(k')}\right) S_2',
\]

where

\[
S_2' = \{(\sigma_1 - \sigma_2) \sin 2\alpha - [\sigma_1 + \sigma_2 + (\sigma_1 - \sigma_2) \cos 2\alpha] f\}, \quad k' = \frac{(a' - b') (c' - b')}{(a' + b') (b' + c')},
\]

\[
E(k') = \frac{1 - \left(\frac{1}{2}\right)^2 \frac{\nu}{\nu - 1} - \left(\frac{1}{2} \frac{1}{2}\right)^2 \frac{\nu^2}{\nu^2 - 1} - \left(\frac{1}{2} \frac{1}{2} \frac{1}{2}\right)^2 \frac{\nu^3}{\nu^3 - 1} - \ldots}{1 + \left(\frac{1}{2}\right)^2 \frac{\nu}{\nu - 1} + \left(\frac{1}{2} \frac{1}{2}\right)^2 \frac{\nu^2}{\nu^2 - 1} + \left(\frac{1}{2} \frac{1}{2} \frac{1}{2}\right)^2 \frac{\nu^3}{\nu^3 - 1} + \ldots}.
\]

The location relationships of crack tips between Figs 3 and 6 can be written by

\[
a' - b' = a - b, \quad c' - b' = c - d, \quad b' = \frac{b - c}{2}. \tag{34}
\]

From Eq. (34), we can know that

\[
a' + b' = a - c, \quad a' + c' = a - d, \quad b' + c' = b - d. \tag{35}
\]

and we can get that

\[
k' = \frac{(a' - b') (c' - b')}{(a' + b') (b' + c')} = \frac{(a - b) (c - d)}{(a - c) (b - d)} = k^2, \quad \frac{E(k')}{K(k')} = \frac{E(k)}{K(k)}, \tag{36}
\]

and $S_2'$ quadruples the imaginary part of $S_2$. Substituting Eqs (34) and (35) into Eq. (33), one can obtain

\[
Y_{ii}^{A'} = Y_{ii}^{A}, \quad Y_{ii}^{B'} = Y_{ii}^{B}, \quad Y_{ii}^{C'} = Y_{ii}^{C}, \quad Y_{ii}^{D'} = Y_{ii}^{D}. \tag{37}
\]

It should be noted that the magnitude of SIFs around the crack tip is independent of the location of the coordinate axis. This also confirms the correctness of the theoretical equation, Eq. (31).
4. THE INFLUENCING FACTORS OF SIFs

To study the effect of crack geometry, crack surface friction and inner pressure on SIFs, the values of SIFs under various conditions are calculated by Eq. (31). To study the effect of inner pressure and crack friction, the crack geometry parameters selected are as follows: \( a \) is 5 cm, \( b \) is 4 cm, \( c \) is 3 cm and \( d \) is 1 cm. It is easy to see that \( Y_I \) is a linear correlation with crack geometry coefficient \( F \) and the ratio of stress \( p/\sigma_1 \). So, we focus on the effect of the factors on \( F \) and \( Y_{II} \).

4.1 Collinear crack geometry

From Eq. (31), the dimensionless \( Y_s \) are a linear correlation with crack geometry coefficient \( F \), a function of crack geometry parameters \( a, b, c \) and \( d \). The coefficient \( F \) shows the effect of crack geometry parameters on the SIFs directly, and in a certain sense, it can also reflect the interaction effect between both cracks. To study the influence of \( a, b, c \) and \( d \) on crack geometry coefficient \( F_s \), the values of \( F \) at tips \( A, B, C \) and \( D \) are calculated by Eq. (31), and their results are shown in Fig. 7.

From Fig. 7, the absolute values of \( F \) at the tips of two unequal cracks generally decrease with the increase of the ratio of \( X_L \), and the absolute values generally decrease with the decrease of the ratio of \( Y_L \). Generally, the absolute values of \( F \) at the outer crack tip (e.g. \( A \) and \( D \)) are less than those at the inner crack tip (e.g. \( B \) and \( C \)) for each crack. The difference for the absolute value of crack geometry coefficient \( F \) is controlled by the values of ratios \( X_L \) and \( Y_L \). The difference between two crack tips in the same crack increases with the increase of the ratios \( X_L \) and \( Y_L \). On the one hand, there is no stress superposition around the inner tips between two constant length cracks when two collinear cracks are far enough apart. Therefore, the SIFs at the two tips of a crack are equal. The stress fields at two
inner tips of different cracks are superimposed to increase when two collinear cracks are close to each other, leading to the SIFs of the inner tip being greater than those of the outer tip for the same crack.

On the other hand, for constant spacing between two collinear cracks, the ratios $X_L$ and $Y_L$ increase with the increase of crack lengths, and the increase in both $X_L$ and $Y_L$ causes an increase in the crack geometry coefficient $F$. A slight difference exists between the two crack tips because of different crack lengths. The values of $F$ at the tips of the longer crack are larger than those at the tips of the shorter ones, so are the SIFs. So, the variation of crack geometry coefficient $F$ shows this interaction between two collinear cracks because it is caused by the different locations of crack tips and different crack lengths. It can show the effect of crack geometry on the SIFs because of a linear relationship between SIFs and the geometry coefficient $F$.

4.2 Crack surface friction

To investigate the effect of surface friction on SIFs, we take an infinite plane containing two unequal collinear cracks into consideration and obtain the calculation results of $Y_{II}$ from Eq. (31) when the ratio of stress $\sigma_2/\sigma_1$ and internal pressure $p$ are 0.1 and 0, as shown in Fig. 8. As shown in Fig. 8, the friction coefficient $f$ has a significant influence on SIFs. There is no friction on the crack surface, and $Y_{II}$ values at crack tips are large when the value of friction coefficient $f$ is zero, and the crack dip angle $\alpha$ reaches a certain value. The peak value of $Y_{II}$ decreases sharply with the increase of friction coefficient $f$ because the frictional force on the crack surfaces is not conducive to the sliding of crack surface, and thus it can reduce the values of $Y_{II}$ at crack tips. The crack cannot slide, and $Y_{II}$ values at crack tips would be zero when the friction coefficient $f$ exceeds a particular value, and the crack dip angle $\alpha$ is smaller than a particular value; thus, the corresponding curves coincide with the x-axis. In addition, each curve is symmetrical with respect to angle, and the angle corresponding to the summit of the curves $Y_{II}$ increases and the SIFs decrease as the friction coefficient $f$ increases.

4.3 Internal pressure

The internal pressure can open the crack and has a significant effect on SIFs. It is the most direct manifestation that $Y_I$ increases linearly with respect to the ratio of stress $p/\sigma_1$ when the geometry coefficient $F$ is constant, and its magnitude is independent of the crack dip angle. In addition, as shown in Fig. 9, the calculation results of $Y_{II}$ by using Eq. (31) were plotted according to the different stress ratios $p/\sigma_1$ when the ratio of stress $\sigma_2/\sigma_1$ and the surface friction coefficient are 0.4 and 0.3, respectively. $Y_{II}$ increases as the ratio $p/\sigma_1$ increases. When the ratio of stress $p/\sigma_1$ exceeds a particular value, the crack surface contact can be ignored. The effective stress around the crack tip is in a tensile and shear state, and the shear stress is caused by the far-field compression when the crack dip angle $\alpha$ is not zero or $\pi/2$, and the ratio of stress $\sigma_2/\sigma_1$ is not 1.

The upper and lower surfaces of the crack are compressed to contact each other when the internal pressure is smaller, and the effect of crack surface contact on SIFs, i.e. crack surface friction, will be stronger. From Eq. (31), the internal pressure does not affect $Y_{II}$ when the crack surface friction $f$ is zero. In other words, the internal pressure shows a positive effect on $Y_{II}$ through crack surface friction.

5. CONCLUSIONS

In this paper, the problem for an infinite plane containing two unequal collinear cracks subjected to internal pressure and biaxial compressive loadings has been investigated by the complex variable method, and the analytical expression of SIFs has been obtained. The ABAQUS code confirmed the exactness of analytical results, and the effects of internal pressure, crack geometric parameters and
crack surface friction on SIFs are presented by the theoretical analysis. From the above study on two unequal collinear cracks subjected to compression and internal pressure, the conclusions are listed as follows:

1. The geometry coefficient \( F \) has a linear correction with SIFs, and the difference and interaction of the SIFs at cracks' tips are caused by the geometry coefficient \( F \) due to different crack geometry parameters. The coefficient \( F \) can show the effect of crack geometry parameters \( a, b, c \) and \( d \) on SIFs directly.

2. From Eq. (31), the internal pressure \( p \) combines with crack surface friction \( f \) to affect the SIFs of mode II crack. On the one hand, the internal pressure \( p \) does not affect the SIFs of mode II crack when the crack surface friction \( f \) is zero. On the other hand, as the internal pressure \( p \) increases and reaches a certain value, the crack surface contact can be ignored, the frictional force cannot play a role in the SIFs of mode II crack.

3. The internal pressure \( p \) can open the crack causing the crack tip to be in a tensile state. The SIFs of mode I crack are linearly corrected with the ratio of stress \( p/\sigma_1 \). Meanwhile, it shows a positive effect on the SIFs of mode II crack through crack surface friction. The SIFs of mode II crack increase as the ratio of stress \( p/\sigma_1 \) increases.

4. The crack surface friction \( f \) affects the SIFs of mode II crack when the upper and lower surfaces of the crack are compressed to contact each other. The frictional force on the crack surfaces is not conducive to the sliding of crack surface and can reduce the values of the SIFs of mode II crack at crack tips.

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