FEEDING AND KILLING END POINTS IN CHAINABLE CONTINUA

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Dedicated to Professor Jerzy Mioduszewski, my amazing mathematical grandfather

Abstract. Using the classical technique of condensation of singularities, we prove that, for every zero-dimensional, complete separable metric space $G$, there exists a Suslinian, chainable metric continuum whose set of end points is homeomorphic to $G$. This answers a question posed by R. Adikari and W. Lewis in [Houston J. Math. 45 (2019), no. 2, pp. 609–624].

All spaces considered in this paper are metric and separable, and all maps are continuous. The terminology follows R. Engelking [8, 9] and S. B. Nadler [16].

A continuum (= non-empty, connected compact space) $X$ is said to be chainable if, for every $\varepsilon > 0$, $X$ has a finite covering by open sets $U_1, \ldots, U_n$ such that $\text{diam} U_i < \varepsilon$ and $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$ for $i, j = 1, \ldots, n$. An element $p \in X$ is called an end point of $X$ if we can moreover assume that $p \in U_1$.

R. H. Bing initiated investigation of end points of chainable continua in the seminal 1951 paper [3]. In the recent detailed study [1] R. Adikari and W. Lewis remark inter alia that the set of end points of a non-degenerate, hereditarily decomposable chainable continuum is non-empty, nowhere dense, $G_\delta$ in the continuum, and does not contain any non-degenerate connected subset. On the other hand, given any zero-dimensional compact space $F$, they use inverse limits and construct an hereditarily decomposable chainable continuum whose set of end points is homeomorphic to $F$. In the present note we follow this trail and answer their Questions 4 and 5 by proving

**Theorem 1.** For every zero-dimensional complete space $G$, there exists a Suslinian chainable continuum whose set of end points is homeomorphic to $G$.

**Corollary.** There is a Suslinian chainable continuum whose set of end points is homeomorphic to the set of irrationals. □

1. Preliminaries: Atomic maps, chainable continua, and condensation of singularities

The following characterisation elucidates the notion of an end point.

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1A continuum is decomposable if it is the union of two proper subcontinua, otherwise it is indecomposable. A continuum is hereditarily decomposable if each of its non-degenerate subcontinua is decomposable.

2A continuum is said to be Suslinian if every family of its pairwise disjoint non-degenerate subcontinua is countable. Since any non-degenerate indecomposable continuum has $2^{\aleph_0}$ pairwise disjoint composants, every Suslinian continuum is hereditarily decomposable.
Theorem 2 (Bing [3, Theorem 13]). For a point \( p \) in a chainable continuum \( X \), the following statements are equivalent.

(a) \( p \) is an end point of \( X \).
(b) For every pair of subcontinua \( P, Q \supseteq p \) of \( X \), either \( P \subseteq Q \) or \( Q \subseteq P \). □

Let us adopt the natural generalisation: If a point \( p \) of a continuum \( X \), not necessarily chainable, satisfies the statement (b) of Bing’s Theorem 2, then we shall call \( p \) an end point of \( X \).

A continuum \( X \) is said to be unicoherent if, for each pair of its subcontinua \( P, Q \) such that \( X = P \cup Q \), the intersection \( P \cap Q \) is connected. \( X \) is hereditarily unicoherent if each of its subcontinua is unicoherent. \( X \) is called a triod if it contains a subcontinuum \( P \) such that \( X \setminus P \) if the union of three non-empty sets each two of which are mutually separated in \( X \). A continuum is atriodic if it does not contain any triod. A subcontinuum \( P \) of a space \( X \) is terminal in \( X \) if, for every subcontinuum \( Q \subset X \) that meets \( P \), either \( P \subset Q \) or \( Q \subset P \). A map \( f: X \to Y \) is said to be monotone [respectively: atomic] if each of its point-inverses \( f^{-1}(y), y \in Y \), is a subcontinuum [respectively: terminal subcontinuum] of \( X \) (hence, \( f \) is a map onto \( Y \) as any continuum is non-empty). The following well-known proposition is easily proved.

**Proposition 1.** Suppose that \( f: X \to Y \) is an atomic map, and \( P \) is a subcontinuum of \( X \) with \( f(P) \) non-degenerate. Then

(a) The formula \( P = f^{-1}(f(P)) \) holds.
(b) If \( f(P) \) is a decomposable continuum [respectively: indecomposable continuum, triod, unicoherent continuum], then so is \( P \).
(c) If \( f(P) \) and each of the point-inverses \( f^{-1}(y), y \in f(P) \), is an atriodic [respectively: hereditarily decomposable, hereditarily unicoherent] continuum, then so is \( X \).
(d) If \( f(P) \) is a Suslinian continuum and there are at most countably many non-degenerate point-inverses \( f^{-1}(y), y \in f(P) \), which are moreover Suslinian, then \( P \) itself is Suslinian.
(e) An element \( p \in P \) is an end point of \( P \) if, and only if \( p \) is an end point of the continuum \( f^{-1}(f(p)) \) and the image \( f(p) \) is an end point of \( f(P) \). □

We shall use the following characterisation of chainable continua.

**Theorem 3** (J. B. Fugate [3, Theorem 2]). A continuum \( X \) is chainable if, and only if \( X \) is atriodic, hereditarily unicoherent, and each indecomposable subcontinuum of \( X \) is chainable. □

**Corollary** (T. Mackowiak [13, Proposition 11(iii)]). Suppose that \( f: X \to Y \) is an atomic map, and \( P \) is a subcontinuum of \( X \) with \( f(P) \) non-degenerate. Then

(f) If \( f(P) \) is chainable and hereditarily decomposable, and each of the point-inverses \( f^{-1}(y), y \in f(P) \), is chainable, then \( P \) is chainable.

\[^{3}\text{In} \ [7, \text{p. 384}] \text{Fugate has a notion of a terminal subcontinuum, which is different from ours. We would prefer to name his terminal subcontinuum an end subcontinuum because its definition is rather a generalisation of the statement (b) of Theorem 2.} \]
Proof. $P = f^{-1}(f(P))$ by Proposition 1(a). If $f(P)$ is hereditarily decomposable, then by Proposition 1(a, b) every indecomposable subcontinuum of $P$ is contained in a point-inverse under $f$. Since every chainable continuum is atriodic and hereditarily unicoherent (see [16] Theorems 12.2 and 12.4), Proposition 1(c) and Fugate’s theorem imply (f). □

Condensation of singularities has its origin in Z. Janiszewski’s note [11], where he constructed a non-degenerate continuum that contains no arc. The technique was developed by G. T. Whyburn [19], R. D. Anderson and G. Choquet [2], who put inverse limits to work, and was further augmented by Maćkowiak [13] and E. Pol and M. Reńska [18].

Janiszewski’s arcless continuum, L. E. J. Brouwer’s, Janiszewski’s, and K. Yoneyama’s indecomposable continua (let us think of the bucket-handle continuum), or B. Knaster’s hereditarily indecomposable continuum (called by E. E. Moise the pseudo-arc) were only possible in Cantor’s paradise. It is interesting to see how these examples were simultaneously represented as inverse limits: Anderson and Choquet, the tent map for the bucket-handle, J. R. Isbell’s proof [10] that every chainable continuum is an inverse limit with bonding maps from $[0, 1]$ to itself, and J. Mioduszewski’s functional approach to chainable continua and the pseudo-arc [14, 15]. All the mentioned papers by Anderson, Choquet, Isbell, and Mioduszewski were published between 1959 and 1964, and they displayed categorical thinking.

Theorem 4 (Pol and Reńska [18] Theorem 3.2]). Suppose that $X$ is a continuum, $(A_i)_{i=1}^{\infty}$ is a sequence of pairwise disjoint, at most zero-dimensional closed subsets of $X$, and $(Z_i)_{i=1}^{\infty}$ is any sequence of compact spaces. If each $Z_i$ admits a monotone map onto $A_i$, then there exists a continuum $L(X, Z_i, A_i)$ with a surjective map $\text{pr}: L(X, Z_i, A_i) \rightarrow X$ which have the following properties.

(a) $\text{pr}$ is atomic.
(b) The restriction $\text{pr}|\text{pr}^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i): \text{pr}^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) \rightarrow X \setminus \bigcup_{i=1}^{\infty} A_i$ is a homeomorphism.
(c) $\text{pr}^{-1}(A_i)$ is homeomorphic to $Z_i$ for each $i$ (then for every point $a \in A_i$, the pre-image $\text{pr}^{-1}(a)$ must be homeomorphic to a component of $Z_i$). □

When $A_1 = \{a\}$ is only one ($A_2, A_3, \ldots$ are empty) and $Z_1 = Z$ is a continuum, we can say the continuum $L(X, Z, a)$ is built by replacing the point $a \in X$ with $Z$. The atomic map $\text{pr}$ compresses the terminal subcontinuum $Z$ of $L(X, Z, a)$ into $a$.

2. Two strategies of proof

Every diligent schoolboy may prove the following claim.

Proposition 2. For every zero-dimensional complete space $G$, there is an inverse sequence $(X_n, f_{n+1}^{n})_{n=1}^{\infty}$ such that each space $X_n$ is countable (finite or infinite) and discrete, each bonding map $f_{n+1}^{n}: X_{n+1} \rightarrow X_n$ is a surjection, and the inverse limit $\lim_{\leftarrow}(X_n, f_{n+1}^{n})_{n=1}^{\infty}$ is homeomorphic to $G$. □

Let $D_0$ be the union of two homeomorphic copies of the bucket-handle continuum, joint at their common end point; $D_0$ is chainable and has no end point. Let $D_1$ denote a singleton; the point is trivially an end point of $D_1$. Adikari and Lewis
Theorem 8] have proved that every non-degenerate, hereditarily decomposable chainable continuum has at least one pair of opposite end points. It is a good heuristic exercise to construct, for each natural number \( k \geq 2 \), a Suslinian chainable continuum \( D_k \) with exactly \( k \) end points (try to modify the \( \sin(1/x) \)-curve). Adikari and Lewis [1, pp. 621–622] have also constructed Suslinian chainable continua \( D_\infty \) and \( D_C \) whose sets of end points are homeomorphic, respectively, to the countably infinite discrete space and the Cantor set. In Remark (A) below we shall see more explicitly what \( D_k \) and \( D_\infty \) look like.

**Proof of Theorem [1]**. For a complete zero-dimensional space \( G \), let \( (X_n, f_n^{n+1})_{n=1}^{\infty} \) be the inverse sequence of Proposition 2. Write \( |X_n| \) for the number of elements in \( X_n \) or \( \infty \) if \( X_n \) is infinite. Take the continuum \( Y_1 = D_{|X_1|} \); and let \( h_1: X_1 \to Y_1 \) be a bijection onto the end point set \( E_1 \) of \( Y_1 \).

Assume that, for \( n = 1, \ldots, m \), we have defined a Suslinian chainable continuum \( Y_n \) with a one-to-one function \( h_n: X_n \to Y_n \) onto the end point set \( E_n \) of \( Y_n \). Consider the composition \( \varphi = h_m f_m^{m+1}: X_{m+1} \to E_m \) and, for each \( a \in E_m \), put \( Z_a = D_{|\varphi^{-1}(a)|} \). We are ready to apply Theorem 4. Let \( Y_{m+1} \) be \( L(Y_m, Z_a, a) \), and let \( g_{m+1} = pr: Y_{m+1} \to Y_m \) be the atomic map of Theorem 4; for each \( a \in E_m \), \( pr^{-1}(a) \) is homeomorphic to \( D_{|\varphi^{-1}(a)|} \). By Proposition 1(d, e) and Maćkowski’s Corollary, \( Y_{m+1} \) is Suslinian, chainable, and its end point set \( E_{m+1} \) consists of all the end points of \( pr^{-1}(a) \), \( a \in E_m \). Thus, there is a bijection \( h_{m+1}: X_{m+1} \to E_{m+1} \) such that \( h_{m+1}(\varphi^{-1}(a)) = E_{m+1} \cap pr^{-1}(a) \) for each \( a \in E_m \). Therefore \( g_{m+1} h_{m+1} = h_m f_m^{m+1} \).

We have defined an inverse sequence \( (Y_n, g_n^{n+1})_{n=1}^{\infty} \), and the one-to-one functions \( h_n: X_n \to Y_n \) form a map from the inverse sequence \( (X_n, f_n^{n+1})_{n=1}^{\infty} \) to \( (Y_n, g_n^{n+1})_{n=1}^{\infty} \); see [8] pp. 101–104 or [16, Exercise 2.22]. Take the limit \( D_G = \lim_{\leftarrow}(Y_n, g_n^{n+1})_{n=1}^{\infty} \) and the induced homeomorphical embedding \( h: \lim_{\leftarrow}(X_n, f_n^{n+1})_{n=1}^{\infty} \to D_G \). The following facts are well-known or easily checked. (1) An inverse limit of chainable continua is chainable, too. (2) If bonding maps \( g_n^{n+1} \) are monotone/atomic, then so are projections \( q_n: D_G \to Y_n \). This and Proposition 1(a) yield two more facts. (3) Since \( Y_n \) are Suslinian, \( D_G \) is Suslinian, too. (4) The set

\[
E = \bigcap_{n=1}^{\infty} q_n^{-1}(E_n) = \lim_{\leftarrow}(E_n, g_n^{n+1}|E_{n+1})_{n=1}^{\infty} = h(\lim_{\leftarrow}(X_n, f_n^{n+1})_{n=1}^{\infty})
\]

consists of end points of \( D_G \).

We shall take a closer look at the simplest continua \( D_k \), the continua \( Y_n \) of the foregoing proof, and the location of their end points. The continua will turn out to have a nice three-storey structure, and this will enable us to improve the resulting continuum \( D_G \).

**Remarks. (A)** The continuum \( D_3 \), the \( \sin(1/x) \)-curve, is not arcwise connected, and the complement of its end point set consists of two arc-components, which are homeomorphic to \( (0, 1) \). Let \( j^2_3: D_3 \to [0, 1] \) be the atomic map which sends the only terminal subarc of \( D_3 \) to 0. Let \( e_2 = 0, e'_2 = 1 \), and write \( e_3, e'_3 \) for the ends of \( (j^2_3)^{-1}(e_2) \). We obtain \( D_{k+1} \) by replacing an end point of \( D_k \) with an arc. Thus, there is an atomic map \( j_{k+1}^k: D_k \to D_k \) which compresses a terminal subarc of \( D_{k+1} \), sends it to an end point \( e_k \) of \( D_k \), and \( e_{k+1}, e'_{k+1} \in D_{k+1} \) are the
ends of the arc \((j_k^{k+1})^{-1}(e_k)\). Write
\[
D_\omega = \lim_{\to} (D_k, j_k^{k+1})_{k=2}^\infty, \quad e_\omega = (e_k)^{\infty}_{k=2} \in D_\omega, \quad \text{and} \quad r_k : D_\omega \to D_k
\]
for the projections of the inverse limit. \(r_k\) are atomic maps for \(k = 2, 3, \ldots\). The end point set of \(D_\omega\) consists of the points \(r_k^{-1}(e_k^k), k = 2, 3, \ldots\), that converge to the end point \(e_\omega\). All the continua lie on the plane. Finally, we obtain \(D_\omega\)
when we replace \(e_\omega \in D_\omega\) with an arc, i.e. there is an atomic map \(j_\omega : D_\infty \to D_\omega\) that compresses a terminal subarc of \(D_\infty\) and sends it to \(e_\omega\). We can ensure that the sequence of end points \((r_k j_\omega)_{k=1}^\infty\) converge to a non-end point of the arc \((j_\omega)_{k=1}^\infty\) of \(e_\omega\). Therefore, the closure of the end point set of \(D_\infty\) can be a sequence with its limit—this (the zero-dimensional closure of the end point set) is the first storey of the structure. What will be important is that, for every atomic map above and below, the pre-image of any non-end point is a singleton.

For each of \(D_k, D_\omega, D_\infty\), the complement of the end point set consists of pairwise disjoint arc-components which are arcs without ends. Write \(\mathcal{I}(D_k), \mathcal{I}(D_\omega), \mathcal{I}(D_\infty)\) for the arc-component families.

The third structural element is the following somewhat idiosyncratic property for \(X = D_k, D_\omega, D_\infty\):

**(†)** For every non-degenerate subcontinuum \(P\) of \(X\), there is an endless arc \(I \in \mathcal{I}(X)\) such that \(I \cap P\) is connected and dense in \(P\).

To prove this, see at first that \([0, 1]\) with \(\mathcal{I}([0, 1]) = \{(0, 1)\}\) has property (†). Note that \(\mathcal{I}\{s\}\) is a singleton \(s\) as \(s\) is its end point.

The reader may treat the following as a lemma.

**Claim.** Since \(j_k^{k+1} : D_{k+1} \to D_k\) is an atomic map, every point-inverse \((j_k^{k+1})^{-1}(y)\) of a non-end point \(y \in D_k\) is degenerate, and \(D_k\) and all point-inverses \((j_k^{k+1})^{-1}(y)\) have property (†)—this all implies the equality
\[
\mathcal{I}(D_{k+1}) = \{(j_k^{k+1})^{-1}(I) : I \in \mathcal{I}(D_k)\} \cup \bigcup_{y \in D_k} \mathcal{I}((j_k^{k+1})^{-1}(y))
\]
and property (†) for \(D_{k+1}\).

**Proof.** Take a non-end point \(x \in D_{k+1}\), and write \(y = j_k^{k+1}(x)\). By Proposition 1(c) there are two cases: (1) \(x\) is not an end point of \((j_k^{k+1})^{-1}(y)\). Then \((j_k^{k+1})^{-1}(y)\) is non-degenerate, and a certain arc-component \(J\) of the complement of the end point set of \((j_k^{k+1})^{-1}(y)\) contains \(x\), i.e. \(x \in J \in \mathcal{I}((j_k^{k+1})^{-1}(y))\). From Proposition 1(a) and the fact that an arc does not contain a non-trivial terminal subcontinuum we infer that every arc \(A \ni x\) is contained in \((j_k^{k+1})^{-1}(y)\). It follows \(J \in \mathcal{I}(D_{k+1})\).

(2) \(y\) is not an end point of \(D_k\). Then \((j_k^{k+1})^{-1}(y)\) is a singleton, and \(y \in I \in \mathcal{I}(D_k)\) for a certain \(I\). Hence, for every arc \(A \ni x\), the restriction \(j_k^{k+1}|A\) is an embedding, and \(J = (j_k^{k+1})^{-1}(I) \in \mathcal{I}(D_{k+1})\).

To prove that \(D_{k+1}\) has property (†), take a non-degenerate subcontinuum \(P\) of \(D_{k+1}\). By Proposition 1(a) there are two cases: (1) If \(P \subset (j_k^{k+1})^{-1}(y)\), then there is an arc-component \(I \in \mathcal{I}((j_k^{k+1})^{-1}(y))\) which witnesses property (†).

(2) If \(P = (j_k^{k+1})^{-1}(j_k^{k+1}(P))\) and this image of \(P\) is non-degenerate, then there is an arc-component \(I \in \mathcal{I}(D_k)\) such that \(I \cap j_k^{k+1}(P)\) is connected and dense in
Then \( J = (j_k^{k+1})^{-1}(I) \in \mathcal{I}(D_{k+1}) \) is homeomorphic with \( I \), \( j_k^{k+1} \) homeomorphically maps \( J \cap P \) onto \( I \cap j_k^{k+1}(P) \), and hence \( J \cap P \) is connected. The set \( cl(J \cap P) \) is a continuum with \( j_k^{k+1}(cl(J \cap P)) = cl(j_k^{k+1}(J \cap P)) = j_k^{k+1}(P) \), and, as \( j_k^{k+1} \) is atomic, we obtain \( cl(J \cap P) = P \). Therefore, \( D_{k+1} \) has property (†).

We have shown by induction that \( D_k, k = 2, 3, \ldots, \) satisfy (†). Now, we shall prove (†) for an inverse limit.

Claim. Since, for \( k = 2, 3, \ldots, \) continua \( D_k \) and maps \( j_k^{k+1} : D_{k+1} \to D_k \) satisfy the assumptions of the previous Claim, we obtain for \( D_\omega = \lim_\omega (D_k, j_k^{k+1})_{k=2}^\infty \) the equality

\[
\mathcal{I}(D_\omega) = \bigcup_{k=2}^\infty \{ r_k^{-1}(I) : I \in \mathcal{I}(D_k) \}
\]

and property (†).

**Proof.** Using the statement (b) of Theorem 2, we can easily check for a point \( x = (x_k)_{k=2}^\infty \in D_\omega \) that, if \( x_k \) is an end point of \( D_k \) for \( k = 2, 3, \ldots, \) then \( x \) is an end point of \( D_\omega \). Therefore, if \( x \) is not an end point of \( D_\omega \), then there is a \( k \) such that \( x_k, x_{k+1}, \ldots \) are not end points of \( D_k, D_{k+1}, \ldots \), respectively. Then \( x_k \in I \in \mathcal{I}(D_k) \) and, by the same argument as in the previous proof applied to \( r_k : D_\omega \to D_k \), we obtain that \( x \in J = r_k^{-1}(I) \in \mathcal{I}(D_\omega) \).

To prove that \( D_\omega \) satisfies (†), take a non-degenerate subcontinuum \( P \) of \( D_\omega \) and a \( k \) such that \( r_k(P) \) is non-degenerate. There is an endless arc \( I \in \mathcal{I}(D_k) \) such that \( I \cap r_k(P) \) is connected and \( r_k(P) \subset cl(I \cap r_k(P)) \). By the same argument as in the previous proof, \( J = r_k^{-1}(I) \in \mathcal{I}(D_\omega) \) witnesses property (†) for \( D_\omega \). \( \square \)

In the same way as above, since \( j_\omega^\infty \) is an atomic map, the complement of the end point set of \( D_\infty \) is a disjoint union of arc-components in \( \mathcal{I}(D_\infty) \) homeomorphic to \((0,1)\), and \( D_\infty \) has property (†).

**B** Let us come back to the proof of Theorem 1 and observe that we can make the continua \( Y_n \) and its inverse limit so that they have the three-storey structure described above. Let us take the simplest \( D_\infty \), whose end points form a sequence convergent to a non-end point. Also note that each restriction \( g_n^{n+1}(g_n^{n+1})^{-1}(Y_n \setminus E_n) : (g_n^{n+1})^{-1}(Y_n \setminus E_n) \to Y_n \setminus E_n \) is a homeomorphism. It is clear by induction that the accumulation points of \( cl\ E_n \) are not end points of \( Y_n \). Hence, \( cl\ E_n \) is countable for each \( n \). It follows that

\[
cl\ E = \bigcap_{n=1}^\infty q_n^{-1}(cl\ E_n) = \lim_\omega (cl\ E_n, g_n^{n+1} | cl\ E_{n+1})_{n=1}^\infty
\]

is zero-dimensional. Secondly, the arc-components of \( Y_n \setminus E_n \) and \( D_G \setminus E \) are homeomorphic to \((0,1)\) by induction and the same argument as in Remark (A), and they form countable families \( \mathcal{I}(Y_n) \) and \( \mathcal{I}(D_G) \). Therefore, \( D_G \) is the union of a closed zero-dimensional set \( cl\ E \) and the countable family \( \mathcal{I}(D_G) \) of pairwise disjoint subsets homeomorphic to \((0,1)\). Finally, by induction and the same argument as in Remark (A) each \( Y_n \) and \( D_G \) have property (†).

**C** Having noted the structure of the continuum \( D_G \), we can repeat Janiszewski’s procedure [π] exactly without changes: into each endless arc of \( D_G \) we insert an arc as a terminal subcontinuum, in this way we damage all endless arcs of \( D_G \), and then, proceeding to infinity, we damage all newly appearing arcs in
order to obtain an arcell, Suslinian, chainable continuum $D'_G$ whose end points form a set homeomorphic to $G$. This can be readily formalised as an inverse limit.

(D) A [weak] Cook continuum is one, call it $X$, such that each non-constant map from a subcontinuum $P$ of $X$ to $X$ is the identity $\id_P: P \to X$ [respectively: has $P \cap f(P) \neq \emptyset$]. Maćkowiak’s aim [13] was to construct a Suslinian, chainable weak Cook continuum (his terminology was somewhat different). Let us consider the continuum $D_G$ of Remark (B), and let $C = \{c_i\}_{i=1}^\infty \subseteq D_G \setminus E$ be a set such that, for each endless arc $I \in \mathcal{I}(D_G)$, the intersection $I \cap C$ is dense in $I$. Let $Z_i$, $i = 1, 2, \ldots$, be pairwise disjoint non-degenerative subcontinua of Maćkowiak’s weak Cook continuum. Finally, let $D''_G$ be the continuum $L(D_G, Z_i, c_i)$ of Theorem 4 and let $pr: D''_G \to D_G$ be that atomic map. By Proposition 1(d, e) and the Corollary to Theorem 3, $D''_G$ is chainable, Suslinian, and its end point set is $pr^{-1}(E)$, which is homeomorphic to $G$.

Claim. $D''_G$ is a weak Cook continuum.

Proof. Let us start with showing that every map $f: P \to D''_G$ from a subcontinuum $P$ of $D_G$ is constant. If $P$ is non-degenerate, then by property ($\dagger$) there is an endless arc $I \in \mathcal{I}(D_G)$ such that $I \cap P$ is connected and $P \subset \cl (I \cap P)$. Let $A \subset I \cap P$ be an arc with ends. There are two cases. (1) $Q = f(A) \subset pr^{-1}(y)$ for a point $y \in D_G$. If $f(A)$ were non-degenerate, then $y$ would be a $c_i$ and $Z_i$ would contain an arc, which is impossible for a weak Cook continuum. Thus, $f|A$ is constant. (2) The image $pr(Q)$ is non-degenerate and $Q = pr^{-1}(pr(Q))$ by Proposition 1(a). It follows from property ($\dagger$) applied to $pr(Q)$ that a certain $c_i \in pr(Q)$. Thus, $pr^{-1}(c_i) \subset Q$. Theorem 12.46 in [16] says that each map onto a chainable continuum is weakly confluent, i.e. there is a continuum $R \subset A$ such that $f(R) = pr^{-1}(c_i)$. Hence, again $Z_i$ contains an arc. A contradiction. Therefore, the case (2) does not hold, and $f(A)$ is degenerate. It follows that $f|(I \cap P)$ is constant, and $f$ is constant as $I \cap P$ is dense in $P$.

Now, take a subcontinuum $P$ of $D''_G$ and a map $f: P \to D''_G$ with $P \cap f(P) = \emptyset$. We shall repeat the above schema to prove that, if $pr(P)$ is a singleton, then $f$ is a constant map. Assume $pr|P$ is constant. There are two cases. (1) If $Q = f(P)$ is contained in a point-inverse $pr^{-1}(y)$, $y \in D_G$, then either $f$ is constant or $f(P) \subset pr^{-1}(c_i)$. In case $f(P) \subset pr^{-1}(c_i)$, $f$ induces a map into the subcontinuum $Z_i$ of Maćkowiak’s continuum, and hence $f$ is constant. (2) The image $pr(Q)$ is non-degenerate and $pr^{-1}(pr(Q)) = Q$. Since $D_G$ has property ($\dagger$), there is an endless arc $I \in \mathcal{I}(D_G)$ such that $I \cap pr(Q)$ is dense in $pr(Q)$. Thus, there is a $c_i \in I \cap pr(Q)$ and $pr^{-1}(c_i) \subset Q = f(P)$. By [16] Theorem 12.46, $f$ is weakly confluent, i.e. there is a continuum $R \subset P$ such that $f(R) = pr^{-1}(c_i)$. As $R$ and $pr^{-1}(c_i)$ are disjoint subcontinua of Maćkowiak’s continuum, $f|R$ is constant, and the case (2) does not hold, in fact.

To continue assume $P = pr^{-1}(pr(P))$. Then, for every $y \in pr(P)$, the restriction $f|pr^{-1}(y)$ is constant. It follows that $f$ has a factorisation $f = g pr$, where $g: pr(P) \to D''_G$ is defined by the correct formula $g(y) = f(pr^{-1}(y))$. By the first paragraph of this proof, $g$ is constant, and so is $f$. □

A map $f: X \to Y$ is **weakly confluent** if, for every subcontinuum $Q$ of $Y$, there is a subcontinuum $P$ of $X$ with $f(P) = Q$.
Therefore, $D''_G$ is a weak Cook, Suslinian chainable continuum whose set of end points is homeomorphic to $G$.

(E) In [12, Footnote 6] we observed that no non-degenerate Suslinian continuum is a Cook continuum.

In our proof of Theorem 1 we have multiplied end points in $E_n$ in order to breed the $E_n$'s' inverse limit. There is another strategy, although it will not be entirely effective. We can take Adikari and Lewis' continuum $D_C$ whose end points form a Cantor set, and destroy those unwanted. Again the next proof is left to the reader.

**Proposition 3.** Any $\sigma$-compact zero-dimensional space is the union of a sequence of pairwise disjoint compact subsets $F_1, \ldots, F_i, \ldots$ with $\lim_{i \to \infty} \text{diam } F_i = 0$. □

**Unsuccessful attempt of proof of Theorem 7**. The end points of the chainable continuum $D_C$ are in the Cantor set $C \subset D_C$. Given a complete zero-dimensional space $G$, we can assume that $G$ is a $G_\delta$ subset of $C$. Thus, $C \setminus G$ is the union $\bigcup_{i=1}^{\infty} F_i$ of Proposition 3. We use $D_0$, a chainable continuum without end points, and we take Pol and Reńska’s continuum $D_G = L(D_C, F_i \times D_0, F_i)$ with the atomic map $pr: D_G \to D_C$. Since $D_0$ has no end point, $pr^{-1}(G)$ is the end point set of $D_G$ by Proposition 1(e). The restriction $pr|pr^{-1}(G): pr^{-1}(G) \to G$ is a homeomorphism. $D_G$ is chainable by Maćkowiak’s Corollary. Unfortunately, $D_G$ contains homeomorphic copies of the indecomposable bucket-handle since $D_0$ does. □

3. Questions and reflections

Let us restate Adikari and Lewis’ problem. (Note that every point of a pseudo-arc is its end point.)

**Problem 1.** Construct chainable continua with interesting end point sets or disprove their existence.

In particular, does there exist a hereditarily decomposable [respectively: Suslinian] chainable continuum whose set of end points is one-dimensional? Can Kuratowski’s set or a complete Erdős space be the set of end points of a hereditarily decomposable [respectively: Suslinian] chainable continuum?

If we take the Knaster-Kuratowski fan and remove its dispersion point, then we obtain a one-dimensional hereditarily disconnected space $M$ that cannot be contained in the end point set of an hereditarily decomposable chainable continuum. Indeed, E. Pol [17] proved that every completely metrisable space which contains $M$ also contains an arc, and thus the end point set would contain an arc.

Maćkowiak’s Corollary to Theorem 3 may be considered as the inverse invariance of chainability under atomic maps. But it is not entirely general. Many authors (the present one in this number) construct their examples by inserting a chainable continuum into another chainable continuum instead of its point. It is

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5See [9, Problem 1.2.E].
6See J. J. Dijkstra and J. van Mill [6], [8, Problem 6.3.24], and cf. [9, Example 1.2.15].
7See [8, Problem 6.3.23] and [9, Problem 1.4.C].
intriguing that we do not know whether we always obtain a chainable continuum. When we insert the continuum into a hereditarily decomposable continuum, that Corollary says ‘Yes’! In a joint paper [4] with M. G. Charalambous we have proved that, when we insert a pseudo-arc into a pseudo-arc, we again obtain a pseudo-arc. Therefore, we consider the following problem as most important.

**Problem 2.** Suppose that $X$ is a continuum, it contains a terminal chainable subcontinuum $P$, and the quotient continuum $X/P$ is chainable. Under what circumstances is $X$ itself chainable? If moreover $X/P$ is the bucket-handle continuum?

We have to assume that the quotient map is atomic as the simple example of a monotonic map from the figure T continuum to $[0,1]$ shows. Much less obvious example of a non-terminal subcontinuum is that of J. F. Davis and W. T. Ingram [5]; in that case the quotient continuum is indeed the bucket-handle. In light of Fugate’s theorem it is sufficient to solve the problem for indecomposable quotients $X/P$.

* The author would never expect that property (†) might be advantageous. I found it when I tried to prove that the continuum $D''$ of Remark (D) is weak Cook. That was analysis, the method to find cause or to describe conducive circumstances when we know the result. Then, that was only formalism which led me to find appropriate hypotheses for the claims on atomic maps of Remark (A) and simpler and simpler proofs. These are not deep remarks, but we should illustrate to our students that analysis, not formal logic, is a method of the *ars inveniendi*. Synthesis is a method of the *ars iudicandi*. Between them there is the art of establishing convenient notation and terminology, and the three: heuristic, logic, and semantics, play complementary and supplementary roles. This is, in fact, the old division of the trivium: rhetoric, dialectic, and grammar. And it is that disfavoured rhetoric that is pertinent to the quest for the shape of the universe or for the nature of the human brain’s functioning. *Matematyka prowadzi nas ku światu* (Polish) [Mathematics guides us towards the world]—this is a dictum of Professor Mioduszewski.

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