Analytical solutions of the problems for equations of one-dimensional hemodynamics

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Abstract. The paper is devoted to the construction of an analytical solution of the initial problem for the one-dimensional system of equations of non-Newtonian hemodynamics. In the model, the blood is considered as the power-law fluid. The perturbation method for the solution is proposed. The cases of the first and second perturbations are considered. The obtained solutions can be used for the comparison of different rheological models of blood or for the testing of programs, which realize the numerical algorithms.

1. Introduction
In the last decades, mathematical models play an important role in the investigations of blood flow dynamics. The main purpose of the models is the prediction of the effects of some changes in the cardiovascular system, caused by vessel pathologies, surgeries, injuries, etc. In many papers, the one-dimensional (1D) models of blood dynamics are successfully used for simulation of processes in vascular systems of human organs [1].

The 1D nonlinear system, which describes the blood flow dynamics, is constructed by the averaging of a three-dimensional hydrodynamical system. In general case, the problems for this system can be solved only numerically. But in some special cases, the analytical solutions can be obtained. In biomechanics, these solutions can be used for some theoretical investigation of blood flow, e.g. for the comparison of different blood rheological models. Another important application of analytical solutions is their use for the testing of the programs, which realized the algorithms of the numerical methods.

In most of the papers, where the analytical solutions are obtained, the attention is focused on the solution of the Riemann problem for the shock wave propagation [2, 3, 4, 5]. For this problem, the viscosity of blood is neglected, the blood is considered as an ideal fluid with a flat profile and the case of infinite [2, 3, 4] or semi-infinite [5] vessel is considered. For the case of the Newtonian model of blood, the analytical solutions are obtained in [6, 7, 8]. S. I. Mukhin et al [7] proposed a perturbation method for the solution of the Cauchy problem for the case of the infinite vessel. In [6] the initial-boundary-value problem for the linearized system of hemodynamical equations is solved. Wang et al [8] used the perturbation method for the solution of the problem with the vessel wall viscosity taken into account. The case of the nonlinear viscosity, dependent on the blood velocity is considered in [9]. Some analytical investigation of the time-dependent non-Newtonian blood flow model is realized in [10].
The presented paper is devoted to the analytical solution of the initial problem for the 1D system in the case of the non-Newtonian approximation of blood. The obtained solution is compared with the case of the model, where the blood viscosity is neglected and the Newtonian model.

The paper has the following structure. In Section 2 the 1D non-Newtonian model for blood flow is considered. In Section 3 the perturbation method for the solution of the nonlinear Cauchy problem is proposed. In Section 4 the numerical example is considered. Some concluding remarks are made in Section 5.

2. The one-dimensional model of blood flow

The 1D model of hemodynamics, where the blood is considered as non-Newtonian power-law fluid, is based on the following equations:

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \\
\frac{\partial Q}{\partial t} + \alpha \frac{\partial}{\partial z} \left( \frac{Q^2}{A} \right) + \frac{A \partial p}{\rho \partial z} + K Q |Q|^{n-1} A^{\frac{3n-1}{2}} = 0,
\]

where

\[
\alpha = \frac{3n + 1}{2n + 1}, \quad K = -\frac{2k s'(1) |s'(1)|^{n-1} \pi^{\frac{n+1}{2}}}{\rho}, \quad s(y) = \frac{3n + 1}{n + 1} \left(1 - y^{1+\frac{1}{n}}\right),
\]

where \(A = A(t, z)\) is the cross-section area of the vessel, \(Q = Q(t, z)\) is a flow rate, \(t\) is a time, \(z\) is a space variable, \(p = p(t, z)\) is the pressure, \(\rho\) is a constant density, \(\alpha\) is a Boussinesq coefficient, \(k, n\) are the parameters of the power-law model, \(s(y)\) is the dimensionless velocity profile, \(y\) is the dimensionless radius.

The model (1)–(2) is obtained by the procedure of the averaging of the equations of hydrodynamics, presented in [1]. Eq. (1) is the averaged incompressibility condition and eq. (2) is the averaged equation of motion.

The system (1)–(2) is closed by the equation-of-state \(p = p(A)\). In most of the works the following dependence is used:

\[
p(A) = p_{\text{min}} + \frac{\beta}{A_{\text{min}}} \left(\sqrt{A} - \sqrt{A_{\text{min}}}\right), \quad \beta = \frac{4}{3} \sqrt{\pi} Eh,
\]

where \(p_{\text{min}}\) and \(A_{\text{min}}\) are the diastolic pressure and cross-section area, \(E\) is the elastic modulus, \(h\) is a vessel wall thickness. In the presented paper, the case of the uniform vessel is considered, where these characteristics are the constants. So, eq. (2) is rewritten as:

\[
\frac{\partial Q}{\partial t} + \alpha \frac{\partial}{\partial z} \left( \frac{Q^2}{A} \right) + \chi \sqrt{A} \frac{\partial A}{\partial z} + K Q |Q|^{n-1} A^{\frac{3n-1}{2}} = 0,
\]

where \(\gamma = \beta/(2A_{\text{min}} \rho)\).

Let the following dimensionless variables are introduced:

\[
\tilde{z} = \frac{z}{L_m}, \quad \tilde{t} = \frac{t}{T_m}, \quad \tilde{A} = \frac{A}{A_m}, \quad \tilde{p} = \frac{p}{\rho U_m^2}, \quad \tilde{Q} = \frac{Q}{A_m U_m}, \quad U_m = \frac{L_m}{T_m},
\]

where \(L_m\) is the typical length, \(T_m\) is the typical time, \(A_m\) is the typical cross-section area, \(U_m\) is the typical velocity. In all notations, presented below, the tilde sign will be ignored.

In dimensionless variables, eq. (3) is rewritten as:

\[
\frac{\partial Q}{\partial \tilde{t}} + \alpha \frac{\partial}{\partial \tilde{z}} \left( \frac{Q^2}{\tilde{A}} \right) + \chi \sqrt{\tilde{A}} \frac{\partial \tilde{A}}{\partial \tilde{z}} + \varepsilon \frac{Q |Q|^{n-1}}{\tilde{A}^{\frac{3n-1}{2}}} = 0,
\]
where $\chi = \gamma \sqrt{A_m / U_m^2}$, $\varepsilon = K U_m^{n-1} T_m A_m^{-(1+n)}$.

Let the infinite vessel is considered, so only the initial conditions are stated for the system (1),(4):

$$A(0, z) = A^0(z), \quad Q(0, z) = Q^0(z), \quad z \in (-\infty, +\infty),$$

where $|A^0(z)|, |Q^0(z)| < +\infty$ at $z \to \pm\infty$.

3. Perturbation method

Dimensionless parameters $\chi$ and $\varepsilon$ can be estimated from the physiological data. In the presented paper, the values from [11] are used. The value of $\chi$ for large arteries can be estimated as $\chi \sim 10 - 30$. But the estimated value of $\varepsilon$ is varied in different vessels: $\varepsilon \sim 0.01 - 0.02$ for the ascending aorta, $\varepsilon \sim 0.1 - 0.3$ for the descending thoracic aorta, $\varepsilon \sim 0.3 - 0.7$ for the femoral artery. So, at some vessels, the value of $\varepsilon$ is relatively small and the perturbation method can be applied for the solution of the nonlinear initial problem.

Let $Q \geq 0$, so eq. (4) is rewritten as:

$$\frac{\partial Q}{\partial t} + \alpha \frac{\partial}{\partial z} \left( \frac{Q^2}{A} \right) + \chi \sqrt{A} \frac{\partial A}{\partial z} + \varepsilon \frac{Q^n}{A^{3n-1}} = 0. \quad (6)$$

Let the initial functions $A^0(z)$ and $Q^0(z)$ are presented as:

$$A^0(z) = A_0 + \varepsilon \varphi_1(z) + \varepsilon^2 \varphi_2(z) + \ldots, \quad Q^0(z) = Q_0 + \varepsilon \psi_1(z) + \varepsilon^2 \psi_2(z) + \ldots,$$

where $A_0, Q_0$ are the constants.

The solution of initial problem is obtained in the following form:

$$A(t, z) = A_0 + \varepsilon A_1(t, z) + \varepsilon^2 A_2(t, z) + \ldots, \quad Q(t, z) = Q_0 + \varepsilon Q_1(t, z) + \varepsilon^2 Q_2(t, z) + \ldots \quad (7)$$

3.1. First order perturbation

After the substitution of (7) into (1) and (6), the following system is obtained for $A_1(t, z)$ and $Q_1(t, z)$:

$$\frac{\partial A_1}{\partial t} + \frac{\partial Q_1}{\partial z} = 0, \quad (8)$$

$$\frac{\partial Q_1}{\partial t} + \left( \chi \sqrt{A_0} - \alpha \frac{Q_0^2}{A_0^2} \right) \frac{\partial A_1}{\partial z} + \frac{2 Q_0}{A_0} \frac{\partial Q_1}{\partial z} = - \frac{Q_0^n}{A_0^{3n-1}}, \quad (9)$$

with the initial condition $A_1(0, z) = \varphi_1(z)$, $Q_1(0, z) = \psi_1(z)$.

The system (8)–(9) can be rewritten as:

$$\frac{\partial \bf{u}}{\partial t} + \bf{M} \frac{\partial \bf{u}}{\partial z} = \bf{f}, \quad (10)$$

where $\bf{u} = (A_1, Q_1)^T$.

The system (10) can be rewritten in the Riemann invariants $\bf{w} = \bf{R}^{-1} \bf{u}$, where $\bf{R}$ is the right eigenvectors matrix for the matrix $\bf{M}$:

$$\frac{\partial \bf{w}}{\partial t} + \Lambda \frac{\partial \bf{w}}{\partial z} = \bf{G}, \quad (11)$$

where $\Lambda$ is the diagonal matrix of eigenvalues of the matrix $\bf{M}$ and $\bf{G} = \bf{R}^{-1} \bf{f}$. System (11) is solved with the following initial condition:

$$\bf{w}(0, z) = \bf{w}^0(z) = \bf{R}^{-1} \bf{u}^0(z), \quad (12)$$
where \( u^0(z) = (\phi_1(z), \psi_1(z))^T \). System (11) is formed by two independent inhomogeneous linear equations, presented as:

\[
\frac{\partial c}{\partial t} + \lambda \frac{\partial c}{\partial z} = B(t, z).
\] (13)

This equation is solved by the introducing of new variables \( \tau = t, v(t, z) = \kappa(z - \lambda t) \), where \( v \) is the solution of the homogeneous equation \( (B \equiv 0) \). In new variables, eq. (13) is rewritten as:

\[
\frac{\partial c}{\partial \tau} = \tilde{B}(\tau, v).
\]

After the integration of this equation on \( \tau \) with initial condition taken into account and transition to the variables \((t, z)\), the solution of the Cauchy problem for (13) can be obtained. The vector \( u \) is obtained as \( u = Rw \).

3.2. Second order perturbation

The equations for \( A_2(t, z) \) and \( Q_2(t, z) \) are written as:

\[
\frac{\partial A_2}{\partial t} + \frac{\partial Q_2}{\partial z} = 0,
\] (14)

\[
\frac{\partial Q_2}{\partial t} + \left( \chi \sqrt{A_0} - \alpha \frac{Q_0^2}{A_0^2} \right) \frac{\partial A_2}{\partial z} + \alpha \frac{2Q_0}{A_0} \frac{\partial Q_2}{\partial z} = F(t, z),
\] (15)

where

\[
F(t, z) = \alpha \left( \frac{2Q_0}{A_0} \frac{\partial (Q_1 A_1)}{\partial z} - \frac{1}{A_0} \frac{\partial Q_1^2}{\partial z} - \frac{Q_0^2}{A_0^2} \right) - \frac{\chi}{2} \frac{A_1}{\sqrt{A_0}} \frac{\partial A_1}{\partial z} +
\]

\[+ \frac{3n - 1}{2} \frac{Q_0^n}{A_0^{3n-1}} A_1 - \frac{n Q_0^{n-1}}{A_0^{3n-1}} Q_1.\]

System (14)–(15) is solved with the following initial conditions:

\[
A_2(0, z) = \varphi_2(z), \quad Q_2(0, z) = \psi_2(z).
\] (16)

The initial problem (14)–(16) is solved in the same way, as for the case of first perturbation.

3.3. The case of the ideal fluid

For the comparison, the case of the ideal fluid, when the blood viscosity is neglected, is considered. Eq. (9) in the system for the first perturbation is presented as:

\[
\frac{\partial Q_1}{\partial t} + \left( \chi \sqrt{A_0} - \alpha \frac{Q_0^2}{A_0^2} \right) \frac{\partial A_1}{\partial z} + \alpha \frac{2Q_0}{A_0} \frac{\partial Q_1}{\partial z} = 0,
\] (17)

so the system (11) is homogeneous:

\[
\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial z} = 0.
\] (18)

The solution of initial problem for (18) is obtained as:

\[
w_1(t, z) = w_1^0(z - \lambda_1 t), \quad w_2(t, z) = w_2^0(z - \lambda_2 t).
\]
Figure 1. The plots of $Q(t,5)$ for different rheological models of blood: 1 — Newtonian fluid ($n = 1$); 2 — non-Newtonian fluid ($n = 0.83$); 3 — ideal fluid

4. Numerical example
The procedure, described above, can be illustrated by an example, defined by the following choice of data: $A_0 = \pi$, $Q_0 = 1$, $\varepsilon = 0.01$, $\chi = 18$, $\varphi_1(z) = 0$, $\varphi_2(z) = 0$, $\psi_1(z) = \sin(z)$, $\psi_2(z) = 0$. Such choice of the initial conditions is based on their smoothness and boundedness at $z \to \pm\infty$. The value of $\alpha$ for the Newtonian fluid ($n = 1$) is equal to $4/3$, for the blood-type non-Newtonian fluid ($n = 0.83$) it is approximately equal to 1.312. For the numerical computations the case of $z \in [0,10]$ and $t \in [0,10]$ is considered. The following solution of homogeneous equation is used for the solution of eq. (13): $v(t,z) = \sin(z-\lambda t)$. The formulas for the solutions are not presented, due to their cumbersome structure, but they can be easily obtained in modern systems of computer algebra.

At fig. 1, the plots of $Q(t,z)$ at a fixed value of $z$ are presented. The cases of an ideal, Newtonian and non-Newtonian fluid are considered. As can be seen, the case of the ideal fluid corresponds to the absence of the damping. For the cases of the viscous fluid model, the damping takes place and results for Newtonian and non-Newtonian models differ. So, even for this simple example, it is demonstrated, that the non-Newtonian effects play an important role and must be taken into account in models for blood flow simulation.

5. Conclusion
In the presented paper, the procedure for the obtaining of analytical solutions of the Cauchy problem for the 1D system of non-Newtonian hemodynamics is presented. The obtained solutions can be used for the comparison of different rheological models of blood and the realization of testing of programs of numerical method algorithms. The considered perturbation method can be extended for the solution of initial-boundary-value problems for semi-infinite or finite vessels.
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