MIXED TENSORS OF THE GENERAL LINEAR SUPERGROUP

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ABSTRACT. We describe the image of the canonical tensor functor from Deligne’s interpolating category $\text{Rep}(Gl_{m-n})$ to $\text{Rep}(Gl(m|n))$ attached to the standard representation. This implies explicite tensor product decompositions between any two projective modules and any two Kostant modules of $Gl(m|n)$, covering the decomposition between any two irreducible $Gl(m|1)$-representations. For $m > n$ we classify the mixed tensors with non-vanishing superdimension. For $m = n$ we characterize the maximally atypical mixed tensors and show some applications regarding tensor products.

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INTRODUCTION

This article deals with a special class of indecomposable representations - the mixed tensors - of the General Linear Supergroup $Gl(m|n)$ over an algebraically closed field of characteristic zero. In the category of finite-dimensional representations $Rep(Gl(m|n))$ the decomposition of the tensor product of two irreducible modules is known for a very small class of representations, the covariant and contravariant modules.

Let $\mathcal{A}$ be a pseudoabelian $k$-linear tensor category and $V \in \mathcal{A}$. For a partition $\lambda$ denote the associated Schur functor by $S_\lambda$. For the special elements $S_\lambda(V)$ the following decomposition formula holds [Del02], prop 1.6:

$$S_\mu(V) \otimes S_\nu(V) = \bigoplus c^{\lambda}_{\mu,\nu} S_\lambda(V)$$

where the sum runs over the partitions of $r = |\mu| + |\nu|$. The coefficients $c^{\lambda}_{\mu,\nu}$ are known as Littlewood-Richardson coefficients.

If $\mathcal{A}$ is the category of finite-dimensional representations of $Sl(n)$ and $V$ the standard representation of dimension $n$, $S_\lambda(V)$ is the irreducible representation $L(\lambda')$ with highest weight $\lambda' = (\lambda_1, \ldots, \lambda_n)$. In particular every irreducible representation of $Sl(n)$ is of the form $S_\lambda(V)$ for some partition $\lambda$. Hence the formula above solves the problem of decomposing tensor products in the classical case. The $Gl(n)$-case can be reduced to this setting by a suitable determinant twist.

If $\mathcal{A}$ is on the other hand the category of representations of $Sl(m|n)$ and $V$ the standard representation, the representation $V^{\otimes r}$ is again completely reducible for every $r$. The irreducible representations obtained in this way - the covariant representations - can be parametrized by $(m,n)$-hook partitions, and their highest weights can be explicitly determined [Ser85] [BR87]. It turns out that these modules form only a very small subset of the irreducible $Sl(m|n)$-modules. The classical approach is therefore of very limited use. Instead of considering the space of covariant tensors $V^{\otimes r}$ one should look at the larger space of mixed tensors $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$, $r, s \in \mathbb{N}$. However the space of mixed tensors is no longer fully reducible. Accordingly the tensor product decomposition of two mixed tensors is not understood. This problem can be solved using a construction of Deligne.

In [Del07] Deligne constructs for any $\delta \in k$ a tensor category $Rep(Gl_\delta)$ which interpolates the classical representation categories $Rep(Gl(n))$ in the sense that for $\delta = n \in \mathbb{N}$ we have an equivalence of tensor categories $Rep(Gl_{\delta=n})/\mathcal{N} \simeq Rep(Gl(n))$ where $\mathcal{N}$ denotes the tensor ideal of negligible morphisms. These interpolating categories possess a canonical element.
of dimension \( \delta \) which we call the standard representation \( st \). Deligne’s family of tensor categories are the universal tensor categories on an object of dimension \( \delta \) in the sense of the following universal property.

**0.1 Theorem.** [Del07] Let \( A \) be a \( k \)-linear tensor category such that \( \text{End}(\mathbb{1}) = k \). The functor \( F \mapsto F(st) \) is an equivalence \( \text{Hom}^\otimes_{\mathcal{A}}(\text{Rep}(\text{Gl}_\delta), \mathcal{A}) \) of the tensor functors of \( \text{Rep}(\text{Gl}_\delta) \to A \) with the category of objects in \( \mathcal{A} \) which are dualisable of dimension \( \delta \) and their isomorphisms.

In particular for \( d = m - n \in \mathbb{N}_{\geq 0} \) we have two tensor functors starting from the Deligne category \( \text{Rep}(\text{Gl}_d) \): One into \( \text{Rep}(\text{Gl}(m - n)) \), the other one into \( \text{Rep}(\text{Gl}(m|n)) \) (both determined by the choice of the standard representation). This suggests a new approach to study the tensor product decomposition in \( \text{Rep}(\text{Gl}(m|n)) \): We should understand the tensor product decomposition in Deligne’s category. If we are then able to understand the functor \( F_{mn} : \text{Rep}(\text{Gl}_{m-n}) \to \text{Rep}(\text{Gl}(m|n)), st \mapsto st \), we will be able to decompose tensor products in its image. The tensor product decomposition in Deligne’s category has been determined by Comes and Wilson [CW11]. They also determine the kernel of the functor \( F_{mn} \) and show that its image is precisely the space of mixed tensors \( T \): The full subcategory of \( \text{Rep}(\text{Gl}(m|n)) \) of objects which are direct summands in a tensor product \( st \otimes r \otimes (st \lor) \otimes s \) for some \( r, s \in \mathbb{N} \). However Comes and Wilson do not describe the image \( F_{mn}(X) \) of an individual element \( X \).

The space of mixed tensors has also been studied by Brundan and Stroppel [BS11]. In both approaches the indecomposable mixed tensors are described by certain pairs \( \lambda = (\lambda^L, \lambda^R) \) of partitions, so-called \( (m,n) \)-cross bipartitions. The advantage of Brundan and Stroppel’s results is that it permits to analyse the Loewy structures of the mixed tensors and gives conditions on their highest weights. This allows to identify the image of the tensor functor \( \text{Rep}(\text{Gl}_{m-n}) \to \text{Rep}(\text{Gl}(m|n)) \). In part 1 we define two invariants \( d(\lambda) \) and \( k(\lambda) \) of a bipartition.

**0.2 Theorem.** A mixed tensor is irreducible if and only if \( d(\lambda) = 0 \). A mixed tensor is projective if and only if \( k(\lambda) = n \). Every projective module is a mixed tensor. We have an explicit bijection \( \theta_n \) between the bipartitions with \( k(\lambda) = n \) and the projective covers of irreducible modules. Similarly we have an explicit bijection \( \theta_0 \) between the bipartitions with \( d(\lambda) = 0 \) and the irreducible mixed tensors.

Hence we obtain an explicit decomposition law for tensor products between projective representations of \( \text{Gl}(m|n) \).
0.3 Theorem. Every irreducible mixed tensor is a Kostant module. Conversely, every Kostant module of atypicality $< m$ is a Berezin twist of an irreducible mixed tensor.

This result gives the tensor product decomposition between two Kostant modules (For $m = n$ the maximally atypical Kostant modules are the Berezin powers). Since every single atypical module is a Kostant-module and every typical module is projective, the last two theorems solve the problem of decomposing tensor products between any two irreducible $Gl(m|1)$-representations. The result on the maximally atypical modules imply also the tensor equivalence $(m > n)$

$$T/N \simeq Rep(Gl(m - n)).$$

Since we have a nice character formula and a nice dimension formula for every mixed tensor by $[CW11]$, thm 8.5.2, we get character and dimension formulas for any Kostant modules and any projective modules.

In the $m = n$ case no maximal atypical irreducible representation of $Rep(GL(n|n))$ is in the image of $F_{nn} : Rep(GL_0) \rightarrow Rep(GL(n|n))$. In the third part we study the class of the smallest maximally atypical tensors (the ones of minimal Loewy length) which we call the symmetric powers $A_{S^i}, i \in \mathbb{N}$. We derive a closed formula for their tensor products $A_{S^i} \otimes A_{S^j}$ (a generalized Pieri rule). As all mixed tensors these have a simple socle which we denote by $S^{i-1}$. One might hope to infer back from the $A_{S^i} \otimes A_{S^j}$-tensor product to the $S^{i-1} \otimes S^{j-1}$-tensor product. This is indeed true as it will enable us in $[HWng]$ to decompose tensor products of the form $S^i \otimes S^j$.

We remark that $A = A_{S^1}$ is the adjoint representation. We end the article with some general remarks about composition factors and projective module in tensor products in section 17.

Preliminaries

Representations. For the linear supergroup $G = GL(m|n)$, $m \geq n$, over $k$ let $F$ be the category of super representations $\rho$ of $GL(m|n)$ on finite dimensional super vectorspaces over $k$. The morphisms in the category $F$ are the $G$-linear maps $f : V \rightarrow W$ between super representations, where we allow even and odd with respect to the gradings on $V$ and $W$. Let $F^{ev} = sRep_\Lambda(G)$ be the subcategory of $F$ with the same objects as $F$ and $Hom_{F^{ev}}(M, N) = Hom_F(M, N)_{\mathbb{Z}/2\mathbb{Z}}$.

The category $\mathcal{R}$. Fix the morphism $\epsilon : \mathbb{Z}/2\mathbb{Z} \rightarrow GL_0 = GL(m) \times GL(n)$ which maps $-1$ to the element $diag(E_m, -E_n) \in GL(m) \times GL(n)$ denoted $\epsilon_n$. Notice that $Ad(\epsilon_n)$ induces the parity morphism on the Lie superalgebra $gl(m|n)$ of
G. We define the abelian subcategory $\mathcal{R}_{mn} = \mathcal{R} = s\text{Rep}(G, \varepsilon)$ of $F^{ev}$ as the full subcategory of all objects $(V, \rho)$ in $F^{ev}$ with the property $p_V = \rho(\varepsilon)_n$; here $\rho$ denotes the underlying homomorphism $\rho : \text{Gl}(m) \times \text{Gl}(n) \to \text{Gl}(V)$ of algebraic groups over $k$. The subcategory $\mathcal{R}$ is stable under the dualities $\vee$ (the ordinary dual) and $\ast$ (the graded dual [Ger98]). For $G = \text{Gl}(n|m)$ we usually write $F^{ev}_n$ instead of $F^{ev}$, and $\mathcal{R}_n$ instead of $\mathcal{R}$ or $\mathcal{R}_{nn}$. The abelian category $F^{ev}$ decomposes as

$$F^{ev} = \mathcal{R}_{mn} \oplus \Pi(\mathcal{R}_{mn})$$

by [Bru03], Cor. 4.44 where $\Pi$ denotes the parity shift functor.

The irreducible representations in $\mathcal{R}$ are parametrized by the (integral dominant) highest weights $X^+$

$$\lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=m+1}^{m+n} \lambda_j \delta_j = (\lambda_1, \ldots, \lambda_m|\lambda_{m+1}, \ldots, \lambda_{m+n})$$

with respect to the choice of the standard Borel group and the usual basis elements $\varepsilon_i, \delta_j$ [Ger98]. Here $\lambda_1 \geq \ldots \geq \lambda_m$ and $\lambda_{m+1} \geq \ldots \geq \lambda_{m+n}$ are integers and every $\lambda \in \mathbb{Z}^{m+n}$ with these properties parametrises a highest weight of an irreducible representation. The irreducible representations in $F^{ev}$ are given by the set

$$\{L(\lambda), \Pi L(\lambda) \mid \lambda \in X^+\}.$$

We denote by $K(\lambda)$ the Kac-module of the weight $\lambda$ and by $P(\lambda)$ the projective cover of the irreducible representation $L(\lambda)$.

Atypicality. If $K(\lambda)$ is irreducible the weight $\lambda$ is called typical. If not, $\lambda$ is called atypical. $K(\lambda)$ is irreducible if and only if $K(\lambda)$ is projective. The atypicality of a weight can be measured by a number between 0 and $\min(m, n)$. If the atypicality is $n$, we say the weight is maximal atypical. Examples are the trivial module $1$ and the standard representation $st$ of highest weight $\lambda = (1, \ldots, 0|0, \ldots, 0)$ for $m \neq n$. Another example is the Berezin determinant

$$B = Ber = L(1, \ldots, 1 \mid -1, \ldots, 1)$$

of dimension 1. The abelian categories $F^{ev}$ and $\mathcal{R}$ decompose into blocks and the degree of atypicality is a block-invariant. Hence we can define the degree of atypicality of an arbitrary indecomposable module to be the degree of atypicality of its composition factors. The full subcategory of modules of atypicality $i$ is denoted $\mathcal{A}_i$.

Khovanov algebras. We review some facts from the articles by Brundan and Stroppel [BS08], [BS10a], [BS08], [BS10b], [BS11]. We denote the
Khovanov-algebra of \([BS10b]\) associated to \(Gl(m|n)\) by \(K(m|n)\). These algebras are naturally graded. For \(K(m|n)\) we have a set of weights or weight diagrams which parametrise the irreducible modules (up to a grading shift). This set of weights is again denoted \(X^+\). For each weight \(\lambda \in X^+\) we have the irreducible module \(L(\lambda)\), the indecomposable projective module \(P(\lambda)\) with top \(L(\lambda)\) and the standard or cell module \(V(\lambda)\). If we forget the grading structure on the \(K(m|n)\)-modules, the main result of \([BS10b]\) is:

**0.4 Theorem.** There is an equivalence of categories \(E\) from \(R_{mn}\) to the category of finite-dimensional left-\(K(m|n)\)-modules such that \(EL(\lambda) = L(\lambda)\), \(EP(\lambda) = P(\lambda)\) and \(EK(\lambda) = V(\lambda)\) for \(\lambda \in X^+\).

More precisely \(K(m|n)\) is isomorphic to the locally finite endomorphism algebra \(\text{End}^{\text{fin}}_{G(P)}\) of a canonical minimal projective generator \(P \simeq \bigoplus_{\lambda \in X^+} P(\lambda)\) for \(R_{mn}\). In particular \(E\) is a Morita equivalence. Hence \(E\) will preserve the Loewy structure of indecomposable modules. This will enable us to study questions regarding extensions or Loewy structures in the category of Khovanov modules.

**Weight diagrams.** To each highest weight \(\lambda \in X^+\) we associate, following \([BS10b]\), two subsets of cardinality \(m\) respectively \(n\) of the numberline \(\mathbb{Z}\)

\[
I_\times(\lambda) = \{\lambda_1, \lambda_2 - 1, ..., \lambda_m - m + 1\}
\]

\[
I_\circ(\lambda) = \{1 - m - \lambda_{m+1}, 2 - m - \lambda_{m+2}, ..., n - m - \lambda_{m+n}\}.
\]

The integers in \(I_\times(\lambda) \cap I_\circ(\lambda)\) are labeled by \(\vee\), the remaining ones in \(I_\times(\lambda)\) respectively \(I_\circ(\lambda)\) are labeled by \(\times\) respectively \(\circ\). All other integers are labeled by a \(\wedge\). This labeling of the numberline \(\mathbb{Z}\) uniquely characterizes the weight \(\lambda\). If the label \(\vee\) occurs \(r\) times in the labeling, then \(r\) is called the degree of atypicality of \(\lambda\). Notice that \(0 \leq r \leq n\), and \(\lambda\) is called maximal atypical if \(r = n\). This notion of atypicality agrees with the previous one.

**Blocks.** Two irreducible representations \(L(\lambda)\) and \(L(\mu)\) in \(R_{mn}\) are in the same block if and only if the weights \(\lambda\) and \(\mu\) define labelings with the same position of the labels \(\times\) and \(\circ\). The degree of atypicality is a block invariant, and the blocks \(\Lambda\) of atypicality \(r\) are in 1-1 correspondence with pairs of disjoint subsets of \(\mathbb{Z}\) of cardinality \(m - r\) respectively \(n - r\).

**Bruhat order.** The Bruhat order \(\geq\) is the partial order on the set of weight diagrams generated by the operation of swapping a \(\vee\) and a \(\wedge\), so that getting bigger in the Bruhat order means moving \(\vee\)'s to the right.

**Cups and Caps.** To each such weight diagram with \(r\) vertices labelled \(\vee\) we associate its cup diagram as in \([BS08]\). Here a cup is a lower semi-circle
joining two vertices. To construct the cup diagram go from left to right through the weight diagram until one finds a pair of vertices $\lor \land$ such that there are only $\times$’s, $\circ$’s or vertices which are already joined by cups between them. Then join $\lor \land$ by a cup. This procedure will result in a diagram with $r$ cups. Now remove all the labels of the vertices and draw rays down to infinity at all vertices which are not part of a cup. If we draw the picture of a cup diagram we will not draw the rays. As an example consider the trivial weight $(0, \ldots, 0)\vert 0, \ldots, 0)$ in $Gl(n|n)$. Its weight diagram is given by

\[
\begin{array}{cccccccc}
\wedge & \lor & \lor & \lor & \wedge & \wedge & \wedge & \wedge \\
\wedge & \lor & \lor & \lor & \wedge & \wedge & \wedge & \wedge
\end{array}
\]

with $n \lor$’s at the vertices $-n + 1, \ldots, 0$. Its cup diagram is given by

Analogously we define a cap to be an upper semi-circle joining two vertices. The cap diagram is build in the same way as the cup diagram. It is obtained from the latter by reflecting along the numberline. As with the cup diagram we will not draw the rays in pictures.

**PART 1. DELIGNE’S INTERPOLATING CATEGORIES AND MIXED TENSORS**

We introduce Deligne’s interpolating categories and explain how to decompose tensor products in them. Then we describe the image of a canonical functor from Deligne’s category for the parameter $\delta \in \mathbb{N}$ into $Rep(GL(m\vert n))$, $m - n = \delta$. As a result we get rules to decompose the tensor product of two representations in the image.

**1. BIPARTITIONS AND INDECOMPOSABLE MODULES**

For every $\delta \in k$ we dispose over Deligne’s interpolating category [Del07] [CW11] denoted $Rep(GL_{\delta})$. This is a $k$-linear abelian rigid tensor category. By construction it contains an object of dimension $\delta$, called the standard representation. Given any $k$-linear pseudoabelian tensor category $C$ with unit object and a tensor functor

\[ F : Rep(GL_{\delta}) \rightarrow C, \]
the assignment $\lambda \mapsto R(\lambda)$ in $\text{Rep}(GL_\delta)$. By [CW11] the assignment $\lambda \mapsto R(\lambda)$ defines a bijection between the set of bipartitions of arbitrary size and the set of isomorphism classes of nonzero indecomposable objects in $\text{Rep}(GL_\delta)$. We sometimes write $(\lambda)$ instead of $R(\lambda)$. By the universal property of Deligne’s category there exists for $\delta = d \in \mathbb{N}$ a full tensor functor

$$F_d : \text{Rep}(GL_d) \to \text{Rep}(GL(d)).$$

Given a bipartition $\lambda = (\lambda^L, \lambda^R)$ of length $\leq d$, $\lambda^L = (\lambda_1^L, \ldots, \lambda_s^L, 0, \ldots)$, $\lambda^R_1 > 0$, $\lambda^R = (\lambda_1^R, \ldots, \lambda_s^R, 0, \ldots)$, $\lambda^R > 0$, put

$$\text{wt}(\lambda) = \lambda_1^L \epsilon_1 + \ldots + \lambda_s^L \epsilon_s - \lambda_1^R \epsilon_{d+1} - \ldots - \lambda_s^R \epsilon_d.$$ 

This defines the irreducible $GL(d)$-module $L(\text{wt}(\lambda))$ with highest weight $\text{wt}(\lambda)$. By [CW11]

$$F_d(R(\lambda)) = \begin{cases} L(\text{wt}(\lambda)) & l(\lambda) \leq d \\ 0 & l(\lambda) > d. \end{cases}$$

This defines a bijection between bipartitions of length $\leq d$ with highest weights of $GL(d)$. Similarly we dispose over a tensor functor $F_{mn} : \text{Rep}(GL_d) \to \mathcal{R}_{mn}$ for $d = m - n$ given by standard representation of superdimension $m - n$.

1.1 Theorem. [CW11] The image of $F_{mn}$ is the space of mixed tensors, the full subcategory of objects which appear as a direct summand in a decomposition of

$$T(r, s) := V^\otimes r \otimes (V^\vee)^\otimes s$$

for some $r, s \in \mathbb{N}$. The functor $F_{mn}$ is full. If $\lambda \neq \mu$, we have $F_{mn}(R(\lambda)) \neq F_{mn}(R(\mu))$.

A bipartition is said to be $(m, n)$-cross if there exists some $1 \leq i \leq m+1$ with $\lambda_i^L + \lambda_{m+2-i}^R < n + 1$. The set of $(m, n)$-cross bipartitions is denoted $\Lambda_{mn}$, or simply $\Lambda^{m,n}$. By [CW11] the modules $R(\lambda) := F_{mn}(L(\lambda))$ are \( \neq 0 \) if and only
if \( \lambda \) is an \((m,n)\)-cross bipartition. Up to isomorphism the indecomposable nonzero summands of \( V^r \otimes W^s \) are the modules \([BS11]\), Thm 8.19,
\[ \{ R(\lambda) \mid \lambda \in \hat{\Lambda}_{r,s} (m,n) - \text{cross} \}\]
where \( (\delta = m - n) \)
\[ \Lambda_{r,s} := \{ \lambda \in \Lambda^x \mid |\lambda^L| = r - t, |\lambda^R| = s - t \text{ for } 0 \leq t \leq \min(r,s) \} \]
\[ \hat{\Lambda}_{r,s} := \begin{cases} \Lambda_{r,s} & \text{if } \delta \neq 0, \text{ or } r \neq s \text{ or } r = s = 0 \\ \Lambda_{r,s} \setminus (0,0) & \text{if } \delta = 0 \text{ and } r = s > 0. \end{cases} \]
For any bipartition define the two sets \( \lambda \)
\[ I_\wedge(\lambda) := \{ \lambda^L_1, \lambda^L_2 - 1, \lambda^L_3 - 2, \ldots \} \]
\[ I_\vee(\lambda) := \{ 1 - \delta - \lambda^R_1, 2 - \delta - \lambda^R_2, \ldots \}. \]
Here we use the convention that a partition is always continued by an infinite number of zeros. To these two sets one can attach a weight diagram in the sense of \([BS08]\) as follows: Label the integer vertices \( i \) on the number-line by the symbols \( \wedge, \vee, \circ, \times \) according to the rule
\[ \begin{cases} \circ & \text{if } i \notin I_\wedge \cup I_\vee, \\ \wedge & \text{if } i \in I_\wedge, i \notin I_\vee, \\ \vee & \text{if } i \in I_\vee, i \notin I_\wedge, \\ \times & \text{if } i \in I_\wedge \cap I_\vee. \end{cases} \]
To any such weight diagram one attaches a cap-diagram as in \([BS08]\). For integers \( i < j \) one says that \( (i,j) \) is a \( \vee\wedge \)-pair if they are joined by a cap. For \( \lambda, \mu \in \Lambda \) one says that \( \mu \) is linked to \( \lambda \) if there exists an integer \( k \geq 0 \) and bipartitions \( \nu^{(n)} \) for \( 0 \leq n \leq k \) such that \( \nu^{(0)} = \lambda, \nu^{(k)} = \mu \) and the weight diagram of \( \nu^{(n)} \) is obtained from the one of \( \nu^{(n-1)} \) by swapping the labels of some pair \( \vee\wedge \)-pair. Then put
\[ D_{\lambda,\mu} = \begin{cases} 1 & \text{if } \mu \text{ is linked to } \lambda \\ 0 & \text{otherwise.} \end{cases} \]
One has \( D_{\lambda,\lambda} = 1 \) for all \( \lambda \). Further \( D_{\lambda,\mu} = 0 \) unless \( \mu = \lambda \) or \( |\mu| = (|\lambda^L| - i, |\lambda^R| - i) \) for some \( i > 0 \). Let \( t \) be an indeterminate and \( R_\delta \) respective \( R_t \) the Grothendieck rings of \( \text{Rep}(GL_\delta) \) over \( k \) respective of \( \text{Rep}(GL_t) \) over \( k(t) \). Now define \( \text{lift}_\delta : R_\delta \to R_t \) as the \( \mathbb{Z} \)-linear map defined by
\[ \text{lift}_\delta(\lambda) = \sum_{\mu} D_{\lambda,\mu} \mu. \]
By \([CW11]\), Thm. 6.2.3, \( \text{lift}_\delta \) is a ring isomorphism for every \( \delta \in k \).

Tensor products. We recall the results of Comes and Wilson about the decomposition of tensor products of the indecomposable modules \( R(\lambda) \) in
Deligne’s category. To get the tensor product in $\mathcal{R}_{mn}$ the tensor product is computed in $\text{Rep}(GL_d)$ and then pushed to $\text{Rep}(GL(m,n))$ by means of the tensor functor $F_{mn}$. By [CW11], Thm 7.1.1, the following decomposition holds for arbitrary bipartitions in $R_t$:

$$\lambda \mu = \sum_{\nu \in \Lambda} \Gamma_{\lambda \mu}^\nu \nu$$

with the numbers

$$\Gamma_{\lambda \mu}^\nu = \sum_{\alpha,\beta,\eta,\theta \in P} \left( \sum_{\kappa,\rho \in P} c_{\kappa \alpha}^{L \lambda \mu} c_{\rho \beta}^{R \lambda \mu} \right) \left( \sum_{\gamma,\delta \in P} c_{\gamma \eta}^{L \gamma \delta} c_{\delta \theta}^{R \gamma \delta} \right) c_{\alpha \theta}^{L \rho \beta} c_{\beta \eta}^{R \rho \beta},$$

see [CW11], Thm 5.1.2. In particular if $\lambda \vdash (r,s)$, $\mu \vdash (r',s')$, then $\Gamma_{\lambda \mu}^\nu = 0$ unless $|\nu| \leq (r + r', s + s')$. As a special case we obtain

$$(\lambda^L; 0) (0; \mu^R) = \sum_{\nu \in \Lambda} \sum_{\kappa,\rho \in P} c_{\kappa \nu}^{L \lambda \nu} c_{\rho \nu}^{R \mu \nu}$$

in $R_t$. So to decompose tensor products in $\text{Rep}(GL_\delta)$ apply the following three steps: Determine the image of the lift $\text{lift}_\delta(\lambda \mu)$ in $R_t$, use the formula above and then take $\text{lift}_\delta^{-1}$.

2. THE MODULES $R(\lambda)$

The mixed tensors can be interpreted as the images of certain Khovanov-modules under the equivalence of categories $E^{-1} : K(m|n) \to \mathcal{R}_{mn}$. This will give a way to identify the image $F_{mn}(R(\lambda))$.

Some terminology of Brundan and Stroppel. Let $\alpha, \beta$ be weight diagrams for $K(m|n)$. Let $\alpha \sim \beta$ mean that $\beta$ can be obtained from $\alpha$ by permuting $\vee$’s and $\wedge$’s. The equivalence classes of this relation are called blocks. Given $\lambda, \mu \sim \alpha$ one can label the cup diagram $\lambda$ respectively the cap diagram $\mu$ with $\alpha$ to obtain $\lambda_\alpha$ resp. $\mu_\bar{\alpha}$. These diagrams are by definition consistently oriented if and only if each cup respectively cap has exactly one $\vee$ and one $\wedge$ and all the rays labelled $\wedge$ are to the left of all rays labelled $\vee$. Put $\lambda \subset \alpha$ if and only if $\lambda \sim \alpha$ and $\lambda_\alpha$ is consistently oriented.

A crossingless matching is a diagram obtained by drawing a cap diagram underneath a cup diagram and then joining rays according to some order-preserving bijection between the vertices. Given blocks $\Delta, \Gamma$, a $\Delta \Gamma$-matching is a crossingless matching $t$ such that the free vertices (not part of cups, caps or lines) at the bottom are exactly at the position as the vertices labelled $\circ$ or $\times$ in $\Delta$; and similarly for the top with $\Gamma$. Given a $\Delta \Gamma$-matching $t$ and $\alpha \in \Delta$ and $\beta \in \Gamma$, one can label the bottom line with $\alpha$ and the upper line with $\beta$ to obtain $\alpha t \beta$. $\alpha t \beta$ is consistently oriented if each cup
respectively cap has exactly one ∨ and one ∧ and the endpoints of each line segment are labelled by the same symbol. Notation: \( \alpha \rightarrow^t \beta \).

For a crossingless \( \Delta \Gamma \)-matching \( t \) and \( \lambda \in \Delta, \, \mu \in \Gamma \), label the bottom and the upper line as usual. The lower reduction \( \text{red}(\lambda t) \) is the cup diagram obtained from \( \lambda t \) by removing the bottom number line and all connected components that do not extend up to the top number line. The upper reduction \( \text{red}(\mu \bar{t}) \) is the cap diagram obtained from \( \mu \bar{t} \) by removing the top line.

If \( M = \bigoplus_{j \in \mathbb{Z}} M_j \) is a graded \( K(m|n) \)-module, write \( M < j > \) for the same module with new grading \( M < j > := M_{i-j} \). The modules \( \{L(\lambda) < j > \mid \lambda \in X^+, \, j \in \mathbb{Z} \} \) give a complete set of representatives for the isomorphism classes of irreducible graded \( K(m|n) \)-modules. The Grothendieck group is the free \( \mathbb{Z} \)-module with basis the \( L(\lambda) < j > \)’s. Viewing it instead as a \( \mathbb{Z}[q, q^{-1}] \)-module so that \( q^j[M] := [M < j >] \), \( K_0(\text{Rep}(K(m|n))) \) becomes the free \( \mathbb{Z}[q, q^{-1}] \)-module with basis \( \{L(\lambda) \mid \lambda \in X^+ \} \).

For any \( \Delta \Gamma \)-matching \( t \) we have the special projective functors \( G^t_{\Delta \Gamma} \) in the category of graded \( K(m|n) \)-modules [BS10a]. The mixed tensors \( R(\lambda) \) will be the images of certain special cases of the projective functors of the theorem. Given a bipartition \( \lambda \) we denote by the defect \( d(\lambda) \) of \( \lambda \) the number of caps in the cap diagram and by rank of \( \lambda \) \( rk(\lambda) = \min(\# \times, \# \circ) \). For \( \delta \geq 0 \) one has \( rk(\lambda) = \# \circ \)'s. Then put

\[
k(\lambda) := d(\lambda) + rk(\lambda).
\]

Denote by \( \eta \) the weight diagram [BS11], 6.1

```
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Let \( \Gamma \) be the block of \( \zeta \), \( \Delta \) be the block containing \( \lambda \) and put
\[
R(\lambda) = G_{\Delta \Gamma}^t L(\zeta)
\]
where \( G_{\Delta \Gamma}^t \) is a special projective functor from \([BS10a] \) \([BS11] \). We transport \( R(\lambda) \) by the equivalence of categories \( E : \mathcal{R}_{mn} \rightarrow K(m,n) - \text{mod.} \). By Morita equivalence the Loewy layers are preserved. We denote by \( \lambda^\dagger \) the highest weight of the irreducible socle of \( R(\lambda) \). This defines a map \( \theta : \Lambda^\times \rightarrow X^+, \lambda \mapsto \lambda^\dagger \).

### 3. Irreducible Modules and Projective Covers

We describe the \( R(\lambda) \) which are irreducible and those which are the projective cover of some atypical representation.

#### 3.1 Theorem. \([BS11]\), Thm 3.4. and \([BS10a]\), Thm 4.11:

(i) Given a \( \Delta \Gamma \)-matching \( t \) as above. Then \( G_{\Delta \Gamma}^t L(\mu) \) is an indecomposable module with irreducible head and socle which differ only by a grading shift.

(ii) In the graded Grothendieck group
\[
[G_{\Delta \Gamma}^t L(\mu)] = \sum_{\gamma} (q + q^{-1})^{n_\gamma} [L(\gamma)]
\]
where \( n_\gamma \) denotes the number of lower circles in \( \gamma t \) and the sum is over all \( \gamma \in \Lambda \) such that a) \( \mu \) is the lower reduction of \( \gamma t \) and b) the rays of each lower line in \( \gamma t \) are oriented so that exactly one is \( \lor \) and one is \( \land \).

(iii) If we forget the grading then
\[
[G_{\Delta \Gamma}^t L(\mu)] = \sum_{\lambda \subset \alpha \rightarrow \mu, \text{red}(\Delta \alpha) = \mu} [L(\lambda)].
\]

The information about the graded composition multiplicities is finer than the mere information about the composition factors since it gives rise to a grading filtration with semisimple quotients.

#### 3.2 Corollary.
\( R(\lambda) \) has Loewy length \( 2d(\lambda) + 1 \). It is rigid.

**Proof:** Let \( R(j) \) be the submodule of \( R(\lambda) \) spanned by all graded pieces of degree \( \geq j \). Then
\[
R(\lambda) = R(-d(\lambda)) \supset R(-d(\lambda) + 1) \supset \ldots \supset R(d(\lambda))
\]
with successive semisimple quotients \( R(j)/R(j + 1) \) of degree \( j \). By \([BS10b]\) every block of \( \mathcal{R}_n \) is Koszul. We already know that the top and socle are simple. Since Koszul algebras are quadratic, the following proposition finishes the proof. \( \square \)
3.3 Proposition. [BGS96], prop. 2.4.1. Let $A$ be a graded ring such that i) $A_0$ is semisimple, ii) $A$ is generated by $A_1$ over $A_0$. Let $M$ be a graded $A$-module of finite length. If $\text{soc}(M)$ (resp. $\text{top}(M)$) is simple, the socle (resp. the radical) filtration on $M$ coincides with the grading filtration (up to a shift).

3.4 Corollary. Every indecomposable module in $\mathcal{R}$ with irreducible top and socle is rigid.

3.5 Corollary. $R(\lambda)$ is irreducible if and only if $d(\lambda) = 0$.

3.1. Tensor generators. A representation $X$ of a supergroup $G$ is a tensor generator if every representation is a quotient of a finite direct sum of representations $X^\otimes r \otimes (X^\vee)^\otimes s$ for some $r, s \geq 0$. If $G$ is an algebraic group, every faithful representation is a tensor generator. In the $\text{Gl}(m|n)$-case it is easily seen that $st \otimes st^\vee$ is a tensor generator of $\mathcal{R}_{mn}$ either by adapting the classical proof [Del82] [Mil12] or by reducing the proof to the classical case. This can be done using the splitting theorem of Weissauer [Wei09] or Masuoka [Mas13] stating that $k[\text{Gl}(m|n)] = k[\text{Gl}(m) \times \text{Gl}(n)] \otimes (g_-)^*$ where $\text{Gl}(m) \times \text{Gl}(n)$ is the underlying classical group and $(g_-)^*$ is the $k$-dual of the odd part of the underlying Lie superalgebra associated to $G$. Note that we have the same equivalence of categories between $k[G]$-comodules and representations of $G$ as in the classical case [Wei09] [Mas13]. More generally consider immersive representations $\rho : G \rightarrow \text{Gl}(V)$, $V \simeq k^{m|n}$, i.e. $\rho$ is injective on the level of the underlying classical groups and on the Lie superalgebra level. The following theorem can be easily proven using the splitting theorem.

3.6 Theorem. (Weissauer) Let $\rho : G \rightarrow \text{Gl}(V)$ be an immersive representation. Then any finite dimensional $k[G]$-comodule is a quotient of a finite multiple of some iterated tensor product of the $k[G]$-comodules $V$ and $V^\vee$.

3.2. Projective covers. Recall that the indecomposable projective modules in $\text{Rep}(\text{Gl}(m, n))$ are precisely the irreducible typical modules by [Kac78] and the projective covers of the irreducible atypical modules.

3.7 Lemma. Every indecomposable projective module appears as some $R(\lambda)$.

Proof: The module $st \otimes st^\vee$ is a tensor generator of $\mathcal{R}_{mn}$. Hence every module $M \in \mathcal{R}$ appears as a subquotient of some direct sum of $T(r, s)$. If
$M$ is indecomposable projective the surjection will split, hence $M$ appears as a direct summand.

Since every atypical weight appears in the socle and top of its projective cover we obtain also

**3.8 Corollary.** The map $\theta : \Lambda^\tau \rightarrow X^+$ is surjective.

Now that we know that every projective cover appears as some $R(\lambda)$, we characterize the projective covers in this part.

**3.9 Lemma.** The crosses and circles of the bipartition $\lambda$ are at the same vertices as the crosses and circles of the highest weight $\lambda^\dagger$. In particular $at(R(\lambda)) = n - rk(\lambda)$.

**Proof:** This is clear since only the labels of $\lambda$ which have a $\vee$ or a $\wedge$ are changed when applying $\theta$.

We use the following notation: If $\lambda$ is a weight or weight diagram, we write $\lambda(i)$ for the $i$-th vertex.

**3.10 Theorem.** A mixed tensor $R(\lambda)$ is projective if and only if $k(\lambda) = n$. In this case $R(\lambda) = P(\lambda^\dagger)$.

**Proof:** For every indecomposable module $M$ with $head(M) = L(\lambda^\dagger)$ there exists a surjection $P(\lambda^\dagger) \rightarrow M$ by [Zou96], lemma 3.4. If $M$ has the same composition factors as $P(\lambda)$, this surjection has to have trivial kernel and gives an isomorphism. By [BS10b] the following formulas hold in the Grothendieck group:

$$[P(\lambda^\dagger)] = \sum_{\mu \supset \lambda^\dagger} [K(\mu)]$$

$$K(\mu) = \sum_{\rho \subset \mu} [L(\rho)].$$

On the other hand

$$R(\lambda) = G^t_{\Delta \Gamma} L(\zeta) = \sum_{\mu \subset \alpha \rightarrow^t \zeta, red(\mu) = \zeta} L(\mu).$$

We will show that the second formula is equal to the first one. Since $\zeta$ and $t$ are fixed, the conditions $\alpha \rightarrow^t \zeta$ and $\alpha \sim \zeta$ imply that $\alpha(i)$ is fixed up to the choice of the position of $\vee$ and $\wedge$ in each cup: All other coordinates are determined by the condition that the endpoints of line segments of $t$ must be labelled by the same symbol (and implies that $\alpha$ has $m$ cups and no free $\vee$’s). Hence any such $\alpha$ differs from $\lambda^\dagger$ only by the position of $\vee$ and $\wedge$ in
each cup. The set of $\alpha$ so obtained is precisely the set of $\alpha$ with $\alpha \supset \lambda^\dagger$: the condition that there cannot be free $\lor$’s to the left of free $\land$’s forces all $m \lor$’s to be bound in cups. Hence

$$R(\lambda) = G_{\Delta \Gamma}^t L(\zeta) = \sum_{\mu \in \alpha \supset \lambda^\dagger, \text{red}(\mu \lambda) = \zeta} L(\mu).$$

It is easy to see that the condition $\text{red}(\mu \lambda) = \zeta$ is always satisfied for $k(\lambda) = n$, hence we know $R(\lambda) = P(\lambda^\dagger)$ for $k(\lambda) = n$. For maximal defect the condition $\alpha \leftrightarrow^t \zeta$ is equivalent to $\alpha \supset \lambda^\dagger$. For $k(\lambda) = n - r$, $r > 0$, the condition $\alpha \leftrightarrow^t \zeta$ is stricter than the condition $\alpha \supset \lambda^\dagger$. Hence the composition factors of $R(\lambda)$ are just a proper subset of the ones of $P(\lambda^\dagger)$. □

**Example:** The module $R((3, 2, 1), (3, 2, 1))$ is the projective cover $P([2, 1, 0])$ in $R_3$.

4. **The map $\theta$**

As noted by [BS11] the map $\theta : \Lambda^x \to X^+$ is in general not injective. It is not even injective if one fixes the defect and the rank of the bipartition.

**4.1 Lemma.** $\theta$ is injective if $d(\lambda) = 0$ and in the case $k(\lambda) = n$.

**Proof:** If $k(\lambda) = n$, then $R(\lambda) = P(\lambda^\dagger)$. If $d(\lambda) = 0$, then $R(\lambda) = L(\lambda^\dagger)$. Both $P(\lambda^\dagger)$ and $L(\lambda^\dagger)$ are determined by their socle. We are done since $R(\lambda) = R(\mu)$ if and only if $\lambda = \mu$. □

Since $\theta$ is injective for minimal and maximal defect we can describe its inverse $\theta^{-1} : X^+ \to \Lambda$ in these fixed situations. Here and in the following we use implicitly the following obvious lemma.

**4.2 Lemma.** The labelled matching $\lambda t \alpha$ is consistently oriented.

**Proof:** We have to show that line segments of $t$ starting with a $\lor$ connect with line segments labelled by a $\lor$ and likewise for the $\land$’s. After removing all crosses, circles and cups $\alpha$ and $\lambda$ look both like

```
\land \land \land \land \land \land \lor \lor \lor \lor \lor \lor
```

We will choose specific points $T^+$, $T^-$ such that the matching is the identity for labels $\geq$ resp. $\leq T^+$ resp $T^-$. Then we just have to count the numbers of $\land$’s and $\lor$’s occurring in $\alpha$ and $\lambda$ between $T^+$ and $T^-$. If the numbers agree
we are done. We choose the minimal positions $T^+, T^-$ from which on $t$ is the identity. We put

\[ T_\lambda^+ = \max(l(\lambda^R) + 1 - (m - n), \lambda^L_t + 1) \]

\[ T^- = \max(l(\lambda^R) + 1 - (m - n), \lambda^L_t + 1, k(\lambda) + 1). \]

Then $T_\lambda^+$ is the label left to the first position coming from $\lambda^R_{l(\lambda)}$ or the position of the rightmost $x$. Similarly put

\[ T^- = \min(-l(\lambda^L), -(m - n) - \lambda^R_t) \]

\[ T^- = \min(-l(\lambda^L), -(m - n) - \lambda^R_t, -(m - n) - k(\lambda)). \]

We want to count the $\lor$ and $\land$ between $T^+$ and $T^-$. For $\alpha$ we count the $\land$’s $> T^-_{\lambda}$ and $\leq -(m - n) - k(\lambda)$. There are

\[ (-1)(T^-_{\lambda} - (-(m - n) - k(\lambda))) = -T^-_{\lambda} (m - n) - k(\lambda). \]

The number of $\lor \geq k(\lambda) + 1$ and $< T^+_\lambda$ is

\[ T^+_\lambda - k(\lambda) - 1. \]

One can check that the number of $\lor$’s and $\land$’s in the weight diagram of $\lambda$ is the same. \hfill \square

As a consequence any $\alpha(i) \neq \zeta(i)$ will result in a switch of a label in $\lambda$ when passing from $\lambda \to \lambda^\dagger$. This results in the following simplified description for $\lambda \mapsto \lambda^\dagger$.

4.1. **An algorithm.** The weight diagram $\alpha$ differs from $\zeta$ in the following way: To the left of the $m - n$ crosses we have $n - k(\lambda)$ different labels and to the right infinitely many. Define $M = \text{maximal vertex labelled with a } \times \text{ or } \circ \text{ or part of a cup in } \lambda$. The matching $t$ will be the identity (meaning $t$ connects the $i$-th vertex of $\alpha$ with the $i$-th vertex of $\lambda$) from vertices greater or equal to

\[ T = \max(k(\lambda) + 1, M + 1). \]

Since $\alpha(i) \neq \zeta(i)$ for all $i \geq T$, all labels in $\lambda$ at vertices greater or to $T$ will be switched. Now define

\[ X = \begin{cases} 
0 & M + 1 \leq k(\lambda) + 1 \\
M - k(\lambda) & \text{else.} 
\end{cases} \]

A free vertex is one which does not have a cross, or a circle or is not part of a cup.

4.3 **Corollary.** The weight diagram of $\lambda^\dagger$ is obtained from the weight diagram of $\lambda$ by switching all labels at vertices $\geq T$ and switching the first $X + n - k(\lambda)$ free vertices $< T$. 

Example: Typical weights. Say the $\times$ are at position $v_1 > v_2 > \ldots > v_m$ and the circles at position $w_1 > \ldots > w_n$. Then

$$\lambda_1^\dagger = v_1, \lambda_2^\dagger = v_2 + 1, \ldots, \lambda_m^\dagger = v_m + m - 1, \lambda_{m+1}^\dagger = w_n + m - 1, \ldots,$$

$$\lambda_{m+n}^\dagger = w_1 + m - n.$$ 

The inverse $\lambda^\dagger \mapsto \lambda$: Given any typical weight $\lambda^\dagger$ we distinguish two cases: Either $T = M + 1$ where $M$ is the rightmost vertex labelled with $\times$ or $\circ$ or $T = n + 1$. If $T = n + 1$, all the free entries up to $n$ are labelled with $\wedge$’s and the remaining ones to the right with $\vee$’s. Otherwise there will be $\vee$’s in the $T - n - 1$ free positions to the left of the rightmost cross or circle. After that (to the left) there will be $\wedge$’s. If $T = n + 1$ we switch all the labels at vertices $\geq M + 1$ as well as the labels at the first $M - n$ free vertices left of $M$. This describes $\theta$ and $\theta^{-1}$ for $\lambda^\dagger$ typical.

4.2. The map $\theta$ in the typical case. If $\lambda^\dagger$ is typical, an explicit expression for the two maps $\theta$ and $\theta^{-1}$ can be given in terms of the coordinates of the bipartition using [MVdJ04], [MVdJ06]. The authors define a subset $\Lambda^st \subset \Lambda^x$ and attach to such a bipartition (called $gl(m|n)$-standard) the highest weight $\tilde{\theta}(\mu, \nu)$. Conversely to any typical weight $\lambda^\dagger$ we have an attached bipartition $(\mu, \nu)$ [Moe06], lemma 3.15.

4.4 Lemma. Let $\lambda$ be such that $R(\lambda) = L(\lambda^\dagger)$ is typical. Then $\lambda^\dagger = \lambda_{\lambda^\dagger, \lambda^\dagger}$ and the inverse $\theta^{-1}(\lambda^\dagger)$ is given by the rule above.

Proof: The set $\Lambda^st$ is a subset of $\Lambda^x$. Hence both $\lambda^\dagger$ and $\lambda_{\mu, \nu}$ are defined on $\Lambda^st$. Every typical weight in $X^+$ is in the image of $\tilde{\theta}$ by [Moe06], lemma 3.15. The character of $L(\lambda_{\mu, \nu})$ is computed in [MVdJ06] and is given by the supersymmetric Schur function $s_{\mu, \nu}$. Similarly the character of $R(\lambda) = L(\lambda^\dagger)$ is computed in [CW11]. The two characters are equal. Since the character determines the irreducible representation the result follows.

Note that the condition $gl(m|n)$-standard of loc.cit is not equivalent to the condition $(m, n)$-cross. Furthermore the map which associates to any bipartition the weight $\lambda_{\mu, \nu}$ does in general not agree with $\lambda \mapsto \lambda^\dagger$.

4.3. Kostant weights. A weight $\mu$ is called a Kostant weight if the cup diagram of $L(\mu)$ is completely nested. In other words if its weight diagram is $\wedge \vee \wedge \vee$-avoiding in the sense that there are no vertices $i < j < k < l$ labelled in this order by $\wedge \vee \wedge \vee$.

4.5 Lemma. Every irreducible mixed tensor is a Kostant module.
Proof: This follows from the simplified algorithm since the weight diagram of a bipartition with \(d(\lambda) = 0\) looks like

\[
\begin{array}{cccccccc}
& \wedge & \wedge & \wedge & \wedge & \wedge & \vee & \vee & \vee & \vee & \vee
\end{array}
\]

after removing the crosses and circles. Applying \(\theta\) means specifying a vertex, say \(V\), and switching all free labels at vertices \(\geq V\). This will not create any neighbouring vertices labelled \(\vee \wedge \vee \wedge\). □

4.6 Corollary. If \(L(\mu)\) is an irreducible mixed tensor then:

1. The Kazhdan-Lusztig polynomials are multiplicity free: \(p_{\lambda,\mu}(q) = q^{l(\lambda,\mu)}\) for all \(\lambda \leq \mu\).
2. \(\sum_{i \geq 0} \dim \text{Ext}^i(K(\lambda), L(\mu)) \leq 1\) for all \(\lambda \in X^+\).
3. \(L\) possesses a resolution by multiplicity free direct sums of Kac modules (BGG-resolution).

Proof: This are properties of Kostant weights [BS10a], lemma 7.2 and theorem 7.3. □

4.4. Tensor products of projective modules. We obtain an algorithm to decompose tensor products of projective modules. Note that \(Proj\) is a tensor ideal, i.e. the tensor product of a projective module with any other module will split in a direct sum of irreducible typical representations and projective covers of atypical modules

\[
P \otimes M = \bigoplus P_i \oplus \bigoplus L(\lambda).
\]

Since every projective module is in the image of \(F_m\) and we have an explicit bijection \(\theta\) between the projective modules and bipartitions with \(k(\lambda) = n\), the tensor product formula in the Deligne category gives us an explicit algorithm for the decomposition.

Example: We compute the tensor product \(P(1, 1, 1, 0|0) \otimes P(1, 1, 1, 0|0)\) in \(Rep(GL(4|1))\). The corresponding bipartition is \(\theta^{-1}(1, 1, 1, 0|0) = (1^4; 1)\). We have

\[
\text{lift}(1^4; 1) = (1^4; 1) \oplus (1^3; 0).
\]

So we have to compute the tensor product

\[
((1^4; 1) + (1^3; 0)) \otimes ((1^4; 1) + (1^3; 0))
\]
in $R_i$. This decomposes in $R_i$ as

$$(2^4; 2) + (2^4; 1^2) + ((2^3, 1^2); 1^2) + 4((2^3, 1); 1) + 2(2^3; 0) + ((2^2, 1^4); 2)$$

$$+ ((2, 1^4); 1^2) + 4((2^2, 1^3); 1) + 4((2^2, 1^2); 0) + ((2, 1^6); 2) + ((2, 1^6); 1^2)$$

$$+ 4((2, 1^5); 1) + 4((2, 1^4); 0) + (1^8; 2) + (1^8; 1^2) + 4(1^7; 1) + 4(1^6; 0).$$

This gives in $\mathcal{R}_{41}$ the decomposition

$$P(1, 1, 1, 0|0) \otimes P(1, 1, 1, 0|0) =$$

$$P(2, 2, 2, 1| − 1) \oplus P(2, 2, 2, −1|1) \oplus 2P(2, 2, 2, 0|0)$$

$$\oplus L(2, 2, 1, −1|2) \oplus L(2, 1, 0, 0|1) \oplus 4L(2, 2, 1, 0|1) \oplus 4L(2, 2, 0, 0|0)$$

$$\oplus L(2, 1, 1, −1|3) \oplus L(2, 1, 0, 0|3) \oplus 4L(2, 1, 1, 0|2) \oplus 4L(2, 1, 1, 1|1)$$

$$\oplus L(1, 1, 1, −1|4) \oplus L(1, 1, 0, 0|4) \oplus L(1, 1, 1, 0|3) \oplus L(1, 1, 1, 1|2).$$

4.5. **Tannaka duals.** We also obtain an explicit description of the Tannaka dual of any irreducible module. Brundan [Bru03] gave an algorithm using certain operators on crystal graphs. For an algorithm on the cup diagram $\lambda$ see [BS10a].

Any irreducible module occurs as socle and head in its projective cover. Clearly

$$P(\lambda^\dagger)^\vee = P((\lambda^\dagger)^\vee).$$

On the other hand $P(\lambda^\dagger)^\vee = R(\lambda^L, \lambda^R)^\vee = R(\lambda^R, \lambda^L) = P((\lambda^\dagger)^\vee)$. So to compute the Tannaka dual of an irreducible module, take its highest weight and associate to it the unique $(m, n)$-cross bipartition $(\lambda^L, \lambda^R)$ of maximal defect as given above (labelling the projective cover of the irreducible module), switch it to $\tilde{\lambda} = (\lambda^R, \lambda^L)$ and then compute $\tilde{\lambda}^\dagger$. Then

$$L(\lambda^\dagger)^\vee = L(\tilde{\lambda}^\dagger).$$

For a description of the Tannaka dual of an irreducible maximally atypical $GL(m|m)$-module see [HWng].

**Example.** We compute the duals of the irreducible modules in the maximal atypical block of $\mathcal{R}_2$. Since every such module is a Berezin-twist of one of the $S^i := [i, 0], \ i \in \mathbb{N}$ we may restrict to this case. The projective cover of $S^i = [i, 0]$ is the module $R((i + 1, 1); (2, 1^1))$. Hence the dual of the projective cover $P[i, 0]$ is the module $R((2, 1^1), (i + 1, 1))$. The irreducible module in the socle has weight $[1, 1 − i]$, hence

$$(S^i)^\vee = [1, 1 − i],$$

ie. $S^i = Ber^{i-1}(S^i)^\vee$. In particular the representations $Ber^{-l}S^{2l+1}$ are self-dual.
4.6. Contravariant modules for \( m = n \). The contravariant modules are the modules in the decomposition \( T(0, r) = (V^*)^\otimes r \). Hence they are the duals of the covariant modules \( \{ \lambda \} \). Recall that the highest weight of \( \{ \lambda \} =: L(\mu) \) is obtained as follows: Put \( \mu_i = \lambda_i \) for \( i = 1, \ldots, m \) and \( \mu_{m+i} = \max(0, \lambda^*_i - m) \) for \( i = 1, \ldots, m \) where \( \lambda^* \) is the conjugate partition and \( \lambda \) is an \((m, m)\)-hook partition. The set of this partitions is denoted by \( H(m, m) \). Put further \( (\lambda_1, \ldots, \lambda_r)^v = (-\lambda_r, \ldots, -\lambda_1) \). Recall that for \( \lambda \in H(m, m) \) we have \( \lambda^* \in H(m, m) \).

4.7 Lemma. \( \{ \lambda \}^v \) has highest weight \( \mu^v \) where \( \mu \) is the highest weight of \( \{ \lambda^* \} \).

Proof: We determine the weight diagram of the highest weight \( \lambda^\dagger \) in the socle of the mixed tensor \( R(0, \lambda^R) \) using the description of \( \theta \). The highest weight of \( \{ \lambda \} \) is given by \( \mu_i = \lambda_i \) for \( i = 1, \ldots, m \) and \( \mu_{m+i} = \max(0, \lambda^*_i - m) \) for \( i = 1, \ldots, m \). For the transposed partition \( \lambda_i^* = \sharp \lambda_i : \lambda_i \geq i \). Hence the highest weight of \( \lambda^* \) is given by \( \mu_i = \lambda_i^* = \sharp \lambda_i : \lambda_i \geq i \) for \( i = 1, \ldots, m \) and \( \mu_{m+i} = \max(0, \lambda_i - m) \) for \( i = 1, \ldots, m \). Applying \( ()^v \) yields the proposed highest weight of \( \{ \lambda \}^v \)

\[
\mu = (-\max(0, \lambda_m - m), \ldots, -\max(0, \lambda_1 - m)) |
-(\sharp \lambda_m : \lambda_m \geq m), \ldots, -(\sharp \lambda_1 : \lambda_1 \geq 1).
\]

Now we determine \( I_X \) and \( I_o \) according to the rules of Brundan-Stroppel. It is easy to see that one obtains the same weight diagram. □

5. THE CONSTITUENT OF HIGHEST WEIGHT

We have seen that the irreducible modules in \( T \) are the ones with \( d(\lambda) = 0 \). We describe the constituent of highest weight of \( R(\lambda) \) for \( d(\lambda) > 0 \). The constituents of \( R(\lambda) \) are given by \( [R(\lambda)] = [G^t_{\Delta^t} L(\zeta)] = \sum_{\mu \subseteq \alpha \rightarrow ^t \zeta, \text{red}(\mu) = \zeta} [L(\mu)] \). The condition \( \mu \subseteq \alpha \) implies \( \alpha \geq \mu \) in the Bruhat order, hence the constituent of highest weight must be among the \( \alpha \rightarrow ^t \zeta \).

We define \( A_\lambda \) by taking the weight diagram of \( \lambda^\dagger \) and by labelling all caps in the matching \( t \) by \( \wedge \vee \). This is the maximal element in the Bruhat order among all the possible \( \alpha \). It will give the constituent of highest weight if \( A_\lambda \) satisfies the condition \( \text{red}(A_\lambda t) = \zeta \).

5.1 Lemma. \( A_\lambda \) is the constituent of highest weight of \( R(\lambda) \). It occurs with multiplicity 1 in the middle Loewy layer.
If we have a cup diagram we group all cups which are adjacent to each other (possibly separated by $\times$ and $\circ$) into segments. If we have one segment it consists of adjacent outer cups which enclose some other cups. We call the interval on the numberline enclosed by the outer cup a sector so that the segment is a disjoint union of adjacent sectors. For the precise notion of sector and segment of a cup diagram we refer to [HW14].

**Proof:** If $k(\lambda) = n$ the assertion is clear (see the section on projective covers). So assume $k(\lambda) < n$. The cup diagram of $\alpha$ is completely nested with $k(\lambda)$ cups with the innermost cup at position $(-1,0)$. After the change from $\alpha$ to $\zeta$ the upper line in the matching $t$ looks like

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (2,0) -- (2,1);
\draw (3,0) -- (3,1);
\end{tikzpicture}
\end{center}

with $n - k(\lambda)$ free $\lor$ to the left of the nested cups and $n - k(\lambda)$ free $\land$’s to the right of the nested cups. We call the ones to the left $k_1^-, k_2^-, \ldots, k_{n-k(\lambda)}^-$, the ones to the right $k_1^+, k_2^+, \ldots, k_{n-k(\lambda)}^+$ We have $\text{red}(A_\lambda t) = \zeta$ if and only if $k_1^-$ will be connected with $k_1^+$ via $t$ when performing the lower reduction, $k_2^-$ with $k_2^+$ and so forth. Under $t$ $k_1^-$ is connected to a position in $A_\lambda$ which we call again $k_1^-$, $k_2^-$ to a position which we call $k_2^-$ etc. Since $t$ is oriented the $-$-positions are labelled by a $\lor$, the $+$-positions by a $\land$. Assume first that $k(\lambda) = n - 1$. If $k_1^- = k_1^+ - 1$ then we are done. If not, we look at the cup diagram in the interval $I = [k_1^- + 1, k_1^+ - 1]$. By construction of $t$ there are no free $\lor$ or $\land$ in $I$. We may ignore $\times$ and $\circ$’s and assume that the cup diagram consists of one segment and $r$ different sectors $C_1, \ldots, C_r$. If $r = 1$ the cup diagram is completely nested and we get

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (2,0) -- (2,1);
\draw (3,0) -- (3,1);
\end{tikzpicture}
\end{center}

The situation generalizes immediately if the cup diagram is a union of $r > 1$ sectors, eg
Hence the assertion is true for \( k(\lambda) = n - 1 \). In case \( k(\lambda) < n - 1 \) we may connect \( k_1^- \) to \( k_1^+ \) as above. We may then remove the part of the cup diagram connected to \( k_1^- \) and \( k_1^+ \) and obtain a diagram with one \( k_1^\pm \) less. We can then connect \( k_2^- \) to \( k_2^+ \) as above and iterate this procedure to finish the proof.

5.2 Corollary. Two direct sums \( \bigoplus P_i, \bigoplus Q_j \) of projective modules are equal if and only if they are equal in \( K_0 \).

Proof: It suffices to test this for a single block \( \Gamma \). It is easy to see that \( A_\lambda \) and \( R(\lambda) = P(\lambda^\dagger) \) determine each other. Hence it is equivalent to give the direct sum \( \bigoplus_{i \in I} P_i \) in \( K_0 \) and the set \( \{ A_i \}_{i \in I} \). Hence \( \bigoplus P_i = \bigoplus Q_j \) if and only if \( \{ A_i \}_{i \in I} = \{ A_j \} - j \in J \). We are done if we can determine the set \( \{ A_i \} \) uniquely from the decomposition \( [\bigoplus P_i] \) in \( K_0 \). We will give an algorithm to do so. The block will be represented by the numberline with \( k \) \( \vee \)'s (with variable position) and \( m - k \times \) and \( n - k \odot \) (with fixed position). Let \( P \) be the set of composition factors of \( \bigoplus P_i \). It may be identified with the set of the corresponding weight diagrams. We go from the right to the left through these diagrams. Let \( i_1 \) be the rightmost position with a \( \vee \) in \( P \). We restrict to the subset \( P_{i_1} \) of \( P \) of diagrams with a \( \vee \) at position \( i_1 \). From \( i_1 \) we move to the left. Let \( i_2 \) be the next position with a \( \vee \) among the diagrams in \( P_{i_1} \). Let \( P_{i_1,i_2} \) the set of weight diagrams with a \( \vee \) at position \( i_1 \) and \( i_2 \). Iterating this procedure we obtain \( P_{i_1,i_2,...,i_k} \). This set consists of the weight diagram of a unique weight, possibly with multiplicity \( \geq 1 \) (since \( \times, \odot \) and \( \vee \)'s are fixed). We claim that this weight is of the form \( A_i \) for some \( P_i \). This is clear: The weight determines a composition factor of some \( P(a) \). If \( L(...) \neq A_a \), then \( A_a > L(...) \) in contradiction to the construction above. The factor \( A_i \) determines the corresponding projective module \( P_i \). We remove all the composition factors of the copies of \( P_i \) from \( P \). Now we apply the same algorithm again to the set \( P \setminus r[P_i] \) to obtain again a weight of the form \( A_i \) with corresponding projective module \( P_i \). We remove its composition factors etc until there are no weights left in \( P \). Hence we have constructed all the weights \( A_i \) from the \( K_0 \)-decomposition.
6. THE DUFLO-SERGANOVA FUNCTOR

Let $M$ be any $g = gl(m|n)$-module. For any $\xi \in X = \{x \in g_1 \mid [x, x] = 0\}$ there exists $g \in Gl(m) \times Gl(n)$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \ldots, \alpha_k$ such that $Ad_g(\xi) = \xi_1 + \ldots + \xi_k$ with $\xi_i \in g_{\alpha_i}$. The number $k$ is called the rank of $\xi$ [Ser10]. For any $x$ of $rk(x) = k$ we dispose over the cohomological tensor functor - the fibre functor - $M \to M_x$ from $\mathcal{R}_{mn} \to \mathcal{R}_{mn} \oplus \Pi \mathcal{R}_{mn}$ [HW14] [Ser10]. We quote [Ser10], thm 2.1, cor 2.2.

6.1 Theorem. If $at(M) < rk(M)$, then $M_x = 0$. If $at(M) = rk(x)$, then $M_x$ is a typical module. If $rk(x) = r$, then $at(M_x) = at(M) - r$.

From now on we will study the Duflo-Serganova tensor functor for special $x$. We define

$$x_r = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_r \end{pmatrix}, \quad \epsilon_r = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$$

with $r$ 1’s on the diagonal. We denote the corresponding tensor functor by $DS_x$. If $r = 1$ we simply write $DS$. An easy computation shows the next lemma.

6.2 Lemma. $DS_x$ maps $st$ to the standard representation of $Gl(m-r|n-r)$.

6.3 Proposition. Under $DS_x$,

$$R(\lambda) \mapsto \begin{cases} 0 & \text{if } k(\lambda) > n - r \\ R(\lambda) & \text{otherwise} \end{cases}$$

In the case $r = 1$ this specialises to

$$R(\lambda) \mapsto \begin{cases} 0 & \text{if } R(\lambda) \text{ projective} \\ R(\lambda) & \text{otherwise} \end{cases}$$

Proof: This follows from the diagram

\[
\begin{array}{c}
\text{Rep}(Gl_{m-n}) \\
\downarrow F_{m,n}^{ev} \\
F_{m-r,n-r}^{ev} \\
\downarrow DS_x \\
F_{m-r,n-r}^{ev}
\end{array}
\]

Since $DS_x$ maps the standard representation to the standard representation the universal property of Deligne’s category implies that the diagram is commutative. In the case $r = 1$ the kernel of $DS_x$ consists of the
(m − 1, n − 1)-cross bipartitions which are not (m, n)-cross. This is equivalent to \( k(\lambda) = n \) which is equivalent to \( R(\lambda^L, \lambda^R) \) projective.

Remark: This is a special case of a more general result [BKN09], page 16: If \( M \) is \(*\)-invariant, then \( M \) is projective if and only if \( M_x = 0 \) for some \( x \) of rank 1.

Example: If \( M := R((n − 1, n − 2, \ldots, 1); (n − 1, n − 2, \ldots, 1)^* \) in \( GL(n|n) \) then the socle is \( L[n − 2, n − 3, \ldots, 1, 0, 0] \). We obtain \( M_x = P[n − 2, n − 3, \ldots, 1, 0] \) in \( Rep(GL(n − 1|n − 1)) \) for \( x \) of rank 1.

6.4 Lemma. Let \( y \in X \) of rank \( r \) such that \( DS_y \) maps the standard representation to the standard representation. Then \( DS_y = DS_x \), when restricted to \( T \).

Proof: This follows from the diagram above and the universal property of Deligne’s category.

6.5 Lemma. If \( R(\lambda^L, \lambda^R) \) is irreducible, so is \( DS_x(R(\lambda^L, \lambda^R)) \).

Proof: \( R(\lambda^L, \lambda^R) \) is irreducible if and only if \( d(\lambda^L, \lambda^R) = 0 \). The defect of a bipartition only depends on the difference \( m − n = (m − r) − (n − r) \).

7. IRREDUCIBLE REPRESENTATIONS IN THE IMAGE

7.1 Lemma. Let \( \Gamma \) be a block of atypicality \( k < n \). Then \( \Gamma \) contains a unique irreducible mixed tensor.

Proof: The block is characterized by the position of the \( m − k \) crosses and \( n − k \) circles on the number line. Denote by \( L^{core} \) the typical \( GL(m − k|n − k) \)-module which is given by this position of the circles and crosses. Then \( L^{core} = R(\lambda^\Gamma) \) for a unique bipartition \( \lambda^\Gamma \) of \( rk(\lambda^\Gamma) = n − k \) and \( d(\lambda^\Gamma) = 0 \). This bipartition defines also an irreducible mixed tensor in \( \Gamma \subset R_{mn} \) since the weight diagram of a bipartition depends only on \( m − n = m − k − (n − k) \). Assume that we would have two irreducible mixed tensors \( R(\lambda^\Gamma) \) and \( R(\lambda^\Gamma') \) in \( \Gamma \). Then both map to \( L^{core} \) when applying \( DS \) \( k \) times or \( DS_x \) one time. Since \( DS(R(\lambda)) = R(\lambda) \) this implies \( R(\lambda^\Gamma) = R(\lambda^\Gamma') \).

7.2 Theorem. Every Kostant module of atypicality \( k < n \) is a Berezin-twist of an irreducible mixed tensor.
We will prove that this is also true in the maximally atypical case provided $m > n$ in section 10.

**Proof:** We describe the Berezin twist explicitly. We use the description how to obtain $\lambda^\dagger$ from $R(\lambda\Gamma)$ in the typical $GL(m-k|n-k)$-case from section 4. The highest weight $\lambda^\dagger$ of the mixed tensor $R(\lambda\Gamma)$ in $\mathcal{R}_{mn}$ is then obtained as follows: If $M$ is the rightmost vertex labelled $\times$ or $\circ$ we distinguish the two cases a) $n - k \geq M$ or b) $M \geq n - k$. If $n - k \geq M$ we switch exactly the labels at vertices $> n - k$. In case b) we switch all labels at vertices $> M$ and the first $M - n + 2k$ free labels $< M$. If $L$ is a $k$-fold atypical Kostant module, we denote by $\nabla^{\min}$ the vertex with the leftmost label $\nabla$, by $z$ the number of crosses and circles at vertices $> \nabla^{\min}$ and $< M$, and by $\nabla^{\max}$ the vertex with the leftmost label $\nabla$. If $\nabla^{\max} > M$, we move $\nabla^{\max}$ with a Berezin-twist to the vertex $n - k$. If $\nabla^{\max} < M$ we move $\nabla^{\min}$ with a Berezin twist to the position $M - (M - n + 2k) - z = n - 2k - z$. In both cases we get an irreducible mixed tensor. Indeed we find a typical $GL(m-k|n-k)$-module $L^{\text{core}'} = R(\lambda\Gamma')$ with the same $\times$, $\circ$-labeling as $Ber\cdots \otimes L$. By the rules of $\lambda\Gamma' \mapsto \lambda^\dagger$, we get $R(\lambda\Gamma') \simeq Ber\cdots \otimes L \in \mathcal{R}_{mn}$. $\square$

In the maximally atypical $GL(m|m)$-case the Kostant modules are the Berezin powers. In particular we dispose now over an algorithm to decompose the tensor product between any two Kostant-modules in $\mathcal{R}_{mn}$.

**Example:** The irreducible module with weight $(6, 4, 2, 1, 1, 0| - \ 2, -2, -2, -2)$ is a 3-fold atypical Kostant module in $\mathcal{R}_{64}$. Twisting with $B^{-1}$ gives the mixed tensor $R(\lambda\Gamma') = R((5, 3, 1); 5)$.

**7.1. Twisted symmetric powers.** We classify the irreducible mixed tensors of atypicality $m - 1$ in $\mathcal{R}_m$. Since $\theta$ preserves the $\times$ and $\circ$ positions, $I_\times \cap I_\circ = \{\text{point}\}$ and $I_\times \cup I_\circ = \mathbb{Z} \setminus \{\text{point}\}$. Further $I_\circ < I_\times$ with the exception of a single point. We determine the possible $\lambda^L$. Every jump $\lambda^L_i > \lambda^L_{i+1}$ in $\lambda^L$ will give a gap in the numberline. Exactly one gap (one $\circ$) has to appear. Since $d(\lambda) = 0$, no $\nabla$ may fill the resulting gaps in the numberline. Hence there can be either at most one jump of size 1 in $\lambda^L$, leading to $\lambda^L = (1^i)$ for some $i \geq 0$, or the $\lambda^L_i$ position is given by a cross, leading to $\lambda^L = (i, 0, \ldots)$.

**7.3 Lemma.** The $(m - 1)$-times atypical irreducible mixed tensors in $\mathcal{R}_m$ are the $R(i; 1^j), i \geq 0, j \neq i, (i, j) \neq (0, 0)$ and their duals $R(1^j; i)$. 

We call the \((i; 1^j)\) twisted symmetric powers. We compute their highest weights. Since \(k(\lambda) = 1\) the resulting matching looks like

with the \(\times\) at position \(i\) and the \(\circ\) at position \(j\). We obtain \(\lambda^{\dagger}\) by switching all free positions \(\geq -m + 1\), hence

\[ R(\lambda) = L(\lambda^{\dagger}) = L(i, 0, \ldots, 0|0, \ldots, 0, -j). \]

7.2. **Character and dimension formula.** By Comes and Wilson [CW11], thm 8.5.2, we have a character and dimension formula for mixed tensors which is a lot nicer than the general formulas of [SZ07]. By loc.cit the character of a mixed tensor is given as

\[ ch R(\lambda) = \sum s_\mu \]

where the sum runs over the bipartition’s occurring in \(lift(\lambda)\) and \(s_\mu\) is the composite supersymmetric Schur polynomial associated to \(\lambda\). Given an arbitrary Kostant module \(L(\lambda)\) (which is not a Berezin power) and the unique Berezin-twist \(Ber^r\) with \(Ber^r \otimes L(\mu) = R(\lambda^{\Gamma_r})\), the character of \(L(\lambda)\) is

\[ ch L(\lambda) = ch Ber^{-r} \cdot ch R(\lambda^{\Gamma_r}) = ch Ber^{-r} \cdot s_{\lambda^{\Gamma_r}}. \]

A similar formula has been obtained before in [CHR13]. Since the dimension does not change after tensoring with \(Ber^r\) we get

\[ \text{dim} \ L(\lambda) = \text{dim} \ R(\lambda^{\Gamma_r}) = d_{\lambda^{\Gamma_r}}. \]

8. **Elementary properties of the \(R(\lambda)\)**

Given two \((m, n)\)-Hook partitions \(\lambda^L, \lambda^R\) we form the bipartition \((\lambda^L, \lambda^R)\). It is in general not \((m, n)\)-cross. We will assume this in this section.

**8.1 Lemma.** Given two \((m, n)\)-Hook partitions \(\lambda^L, \lambda^R\) such that \((\lambda^L, \lambda^R)\) is \((m, n)\)-cross. Then \(\{\lambda^L\} \otimes \{\lambda^R\}^\vee\) contains \(R(\lambda^L, \lambda^R)\) as a direct summand.

In the decomposition

\[ \{\lambda^L\} \otimes \{\lambda^R\}^\vee = R(\lambda^L, \lambda^R) \oplus \bigoplus R(\mu^j) \]
all $\mu^j$ satisfy $(\mu^j)_i^L \leq \lambda_i^L$ and $(\mu^j)_i^R \leq \lambda_i^R$ for all $i$ and $\text{deg}(\mu^j) < \text{deg}(\lambda^L, \lambda^R)$. 

**Proof:** Recall that in $R_t$

$$(\lambda^L, 0) \otimes (0, \lambda^R) = \sum_{\nu} \sum_{\kappa \in P} c_{\kappa, \nu}^L c_{\kappa, \nu}^R \nu.$$ 

Putting $\kappa = 0$ yields $\nu^L = \lambda^L$, $\nu^R = \lambda^R$. Hence

$$(\lambda^L, 0) \otimes (0, \lambda^R) = (\lambda^L, \lambda^R) + \sum_{\nu} \sum_{\kappa \in P, \kappa \neq 0} c_{\kappa, \nu}^L c_{\kappa, \nu}^R \nu.$$ 

All other bipartitions $\nu = (\nu^L, \nu^R)$ will have degree strictly lower than $(\lambda^L, \lambda^R)$ and length $\geq l(\lambda^L, \lambda^R)$. By Comes-Wilson lift $\lambda + \ldots$ where the other bipartitions are obtained by swapping successively $\forall \land$-pairs, i.e. decreasing the coefficients of the bipartition. Since $(\lambda^L, \lambda^R)$ is the largest bipartition, $R(\lambda^L, \lambda^R)$ will occur with multiplicity one in the decomposition. 

For any two partitions $\lambda^L, \lambda^R$ such that the pair $(\lambda^L, \lambda^R)$ is $(m, n)$-cross we define

$A_{\lambda^L, \lambda^R} := \{\lambda^L\} \otimes \{\lambda^R\}^\vee.$ 

**8.2 Proposition.** $R(\lambda^L, \lambda^R)$ is $\ast$-invariant 

**Proof:** Clearly $A_{\lambda^L, \lambda^R}$ is $\ast$-invariant since irreducible modules are $\ast$-invariant. In the decomposition

$$R(\lambda^L, \lambda^R) \ast = R(\lambda^L, \lambda^R) \oplus \bigoplus_i R(\mu_i)$$

$R(\lambda^L, \lambda^R)$ occurs as a direct summand with multiplicity 1; and $\text{deg}(\lambda^L, \lambda^R) > \text{deg}(\mu_i)$. Assume $R(\lambda^L, \lambda^R)$ would not be $\ast$-invariant. Then there exists a $\mu_i$ occuring with multiplicity 1 in the decomposition with $R(\lambda^L, \lambda^R)^\ast = R(\mu_i)$. Write $\mu_i = (\mu_i^L, \mu_i^R)$. As for $(\lambda^L, \lambda^R)$, $R(\mu_i)$ occurs with multiplicity 1 in the decomposition of the $\ast$-invariant

$$A_{\mu_i^L, \mu_i^R} = R(\mu_i) \oplus \bigoplus_j R(\nu_j)$$

with degree strictly larger then the other bipartitions $\nu_j$. Hence there exists a $\nu_j$ with $R(\mu_i)^\ast = R(\nu_j)$. Since $\ast^2 = id$ this forces $\nu_j = (\lambda^L, \lambda^R)$. However $\text{deg}(\lambda^L, \lambda^R) > \text{deg}(\mu_i) > \text{deg}(\nu_j).$ 

Hence by the lemma the $R(\lambda)$ are the modules with largest bipartition in the decomposition $\{\lambda^L\} \otimes \{\lambda^R\}^\vee = R(\lambda^L, \lambda^R) \oplus \bigoplus R(\mu^j)$. Can $R(\lambda)$ be
characterised intrinsically as a certain direct summand in the decomposition \( \{\lambda^L\} \otimes \{\lambda^R\} \)?

8.3 Lemma. Assume \( \mu \leq \lambda \). Then \( k(\lambda) \geq k(\mu) \).

Proof: By [BS11] \( \lambda \) is \((m,n)\)-cross if and only if \( k(\lambda) \leq n \). Choose \( \tilde{n} \) minimal such that \( \lambda \) is \((m,\tilde{n})\)-cross. Then \( k(\lambda) = \tilde{n} \). Since \( \mu \leq \lambda \mu \) is \((m,\tilde{n})\)-cross, hence \( k(\mu) \leq \tilde{n} = k(\lambda) \).

8.4 Lemma. If \( R(\lambda) \) is maximally atypical then \( d(\lambda) \geq d(\mu_j) \) for all \( j \). If \( R(\lambda) \) is maximally atypical and irreducible then \( \{\lambda^L\} \otimes \{\lambda^R\} \) is completely reducible and splits into maximally atypical irreducible summands.

Proof: \( R(\lambda) \) is maximally atypical if and only if \( rk(\lambda) = 0 \). Hence \( k(\lambda) \geq k(\mu_j) \) implies the first statement. If \( R(\lambda) \) is additionally irreducible, then \( d(\mu_j) = 0 \) for all \( j \).

8.5 Lemma. In the tensor product
\[
R(\lambda) \otimes R(\mu) = \sum_i \kappa_{\lambda \mu}^{\nu_i} R(\nu_i)
\]
all \( \nu_i \) satisfy
\[
k(\nu_i) \geq \max(k(\lambda),k(\mu)).
\]

Proof: Let \( n' = \max(k(\lambda),k(\mu)) \). Apply \( DS_{n-n'}: \mathcal{R}_{mn} \to \mathcal{R}_{m'n'} \oplus \Pi \mathcal{R}_{m'n'} \). Without loss of generalisation \( n' = k(\lambda) \). Then \( R(\lambda) \) is projective in \( \mathcal{R}_{m'n'} \). The projective modules form a tensor ideal, hence \( R(\lambda) \otimes R(\mu) \) decomposes in \( \mathcal{R}_{m'n'} \) into indecomposable projective modules. Since the tensor product comes from the Deligne category
\[
\begin{array}{ccc}
\text{Rep}(Gl_{m-n}) & \xrightarrow{F_{m,n}} & \mathcal{R}_{mn} \\
\downarrow & & \downarrow \text{DS}_{n-n'} \\
\mathcal{R}_{m'n'} \oplus \Pi \mathcal{R}_{m'n'} & \xrightarrow{F_{m',n'}} & \mathcal{R}_{mn} \end{array}
\]
we have in \( \mathcal{R}_{mn} \)
\[
\sum_i \kappa_{\lambda \mu}^{\nu_i} R(\nu_i) \oplus \ker(DS_{n-n'} \)
\]
with \( k(\nu_i) \geq n' \) for all \( i \). Further \( \ker(DS_{n-n'}) \) are the mixed tensors \( R(\gamma) \) with \( n' < k(\gamma) \leq n \).

Example: Any irreducible summand in \( R(\lambda) \otimes R(\mu) \) has atypicality \( \leq n - \max(k(\lambda),k(\mu)) \).
We denote by $T_{mn}^i$ the subset of mixed tensors with $k(\lambda) \geq i$.

**8.6 Corollary.** The $T^i$ are tensor ideals in $\mathcal{R}_{mn}$ for $m > n$ and tensor ideals in $\mathcal{R}_{mn} \cup 1$. We have strict inclusion

$$T^0 \supsetneq T^1 \supsetneq \ldots \supsetneq T^n$$

with $T^0 = T$ and $T^n = \text{Proj}$.

By [Ser10] any two irreducible objects of atypicality $k$ generate the same tensor ideal in $\mathcal{R}_{mn}$. Therefore write $I_k$ for the tensor ideal generated by an irreducible object of atypicality $k$. Clearly $I_0 = \text{Proj}$ and $I_n = T_n$ since it contains the identity. This gives the following filtration of $\mathcal{R}$

$$\text{Proj} = I_0 \subsetneq I_1 \subsetneq \ldots I_{n-1} \subsetneq I_n = \mathcal{R}_{mn}$$

with strict inclusions by [Ser10] and [Kuj11].

**8.7 Lemma.** $I_k|_T = T^{n-k}$ for $m > n$ for all $k = 0, \ldots, n$. For $m = n$ $I_k|_T = T^{n-k}$ for all $k < n$.

**Proof:** For any atypicality $k$ there exists an irreducible mixed tensor with that atypicality (except for $m = n$ and $k = n$), hence $I_k|_T \subset T^{n-k}$. Conversely let $R(\lambda) \in T^{n-k}$. It occurs as a direct summand in $R(\lambda^L, 0) \otimes R(0, \lambda^R)$. Then $\max(k(\lambda^L, 0), k(0, \lambda^R)) \leq n - k$, hence $rk(\lambda^L, 0), rk(0, \lambda^R) \leq n - k$, hence $at(R(\lambda^L, 0), at(R(0, \lambda^R)) \geq k$, hence $R \in I_l$ for any $l \geq k$. 

**8.8 Lemma.** For $m > n$ $I_{n-1}|_T = N|_T$. For $m = n$ $N|_T = T$.

**Proof:** Clearly $T_1 \subset N|_T$. Ket $m > n$. If $R \in N|_T$, then $k(\lambda) \geq 1$. Indeed $k(\lambda) = 0$ implies $R(\lambda)$ is maximally atypical irreducible, hence $sdimR(\lambda) \neq 0$.

**Part 2. Maximally atypical modules in the space of mixed tensors**

**9. Multiplicities and tensor quotients**

For $d = m - n > 0$ we have the two tensor functors

$$\text{Rep}(Gl_{m-n}) \xrightarrow{F_{mn}} \mathcal{R}_{mn} \xrightarrow{F_{m-n}} \text{Rep}(Gl(m-n))$$
given by mapping the standard representation to the two standard representations. We also dispose over Weissauer’s tensor functor: By [Wei10] there exists a purely transcendental field extension $K/k$ of transcendence degree $n$ and a $K$-linear exact tensor functor

$$\rho: \mathcal{R}_{mn} \otimes_K K \to \text{Rep}(\text{Gl}(m - n)) \otimes \text{svec}_K.$$ 

By [Wei10] each simple maximal atypical object $L(\mu)$ maps to the isotypic representation $m(\mu)\rho(V)[\rho(\mu)]$ where $m(\mu)$ is a positive integer, $V$ is the ground state (see loc.cit) of the block of $\mu$ and $p(\mu)$ is the parity of $\mu$. After a suitable specialisation of $\rho$ we may assume that $\rho$ is defined over $k$ and maps the standard to the standard representation. Hence we get the commutative diagramm of tensor functors (due to Deligne’s universal property)

Here the functor $F_{m-n} \otimes \text{svec}$ maps $R(\lambda)$ to the even representation

$$L(\text{wt}(\lambda)) \in \text{Rep}(\text{Gl}(m - n)) \subset \text{Rep}(\text{Gl}(m - n)) \otimes \text{svec}.$$ 

9.1 Lemma. Let $m > n$ and $d = m - n$. Then $R(\lambda)$ has superdimension $\neq 0$ if and only if $l(\lambda) \leq d$.

Proof: This follows from the commutative diagram above. Use the bijection between the highest weights of $\text{GL}(d)$ and bipartitions of length $\leq d$ to choose for any $(m,n)$-cross bipartition $\lambda$ the irreducible highest weight module $L(\text{wt}(\lambda))$. By the commutativity the indecomposable module $R(\lambda)$ has to map to $L(\text{wt}(\lambda))$. Its superdimension is the dimension of $L(\text{wt}(\lambda))$. □

Assume $m > n$. The mixed tensors form a pseudoabelian tensor subcategory of $\mathcal{R}_{mn}$. It is closed under duals $(T(r,s))^\vee = T(s,r)$ and contains the identity. The functor of Weissauer

$$\rho: \mathcal{R}_{mn} \to \text{Rep}(\text{Gl}(m - n)) \otimes \text{svec}$$

can be restricted to $T$. Let us denote by $\mathcal{N}$ the tensor ideal of negligible morphisms [Hei12] [KA02].
9.2 Theorem. The functor $\rho_T : T \rightarrow \text{Rep}(Gl(m - n)) \otimes \text{svec}$ factorises over $T/N$ and defines an equivalence of tensor categories

$$T/N \simeq \text{Rep}(Gl(m - n)).$$

It maps the element $R(\lambda)$ to the irreducible element $L(wt(\lambda))$.

Proof: The functor will factorize if $\rho_T$ is full [Hei12]. This follows from the commutative diagram since an indecomposable module maps to an irreducible module. $R(\lambda) \mapsto L(wt(\lambda))$ is forced by the commutativity of the diagram. By the bijection between highest weights of $Gl(m - n)$ and bipartitions of length $\leq m - n$ the functor is one-to-one on objects. Fully faithful follows from Schur’s lemma in the semisimple tensor category $T/N$. □

Remark: Pulling back to $T$ gives the tensor product of the modules in $T$ up to superdimension zero. We will see that the modules $R(\lambda)$ of non-vanishing super dimension are essentially the maximally atypical Kostant modules.

9.1. An alternative approach. Assume $m > n$. All bipartitions are $(m, n)$-cross. We provide an alternative proof that $\rho : T/N \simeq \text{Rep}(Gl(m - n))$ which does not use the existence of a tensor functor $\text{Rep}(Gl(m, n)) \rightarrow \text{Rep}(Gl(m - n)) \otimes \text{svec}$.

9.3 Proposition. Let $\lambda$ be a bipartition of length $\leq m - n$. Then $d(\lambda) = 0$.

Proof: Let $k$ be the length of $\lambda^L = (a_1, \ldots, a_k)$, hence length of $\lambda^R \leq m - n - k$. We use the notation $\lambda^R = (b_1, b_2, \ldots)$. Define the sets

$$I_\wedge = I_{\wedge}^{\leq k} \cup I_{\wedge}^{>k} = \{a_1, \ldots, a_k - k + 1\} \cup I_{\wedge}^{>k}$$

$$I_\vee = I_{\vee}^{\leq m-n-k} \cup I_{\vee}^{>m-n-k}$$

$$= \{1 - m - n - b_1, \ldots, m - n - k - (m - n) - b_{m-n-k}\} \cup I_{\vee}^{>k}.$$  

We have

$I_{\wedge}^{>k} = [-k, -\infty), I_{\vee}^{>m-n-k} = [-k + 1, \infty), \text{ hence } I_{\wedge}^{>k} \cap I_{\vee}^{>m-n-k} = \emptyset.$

Hence crosses can only appear by the intersections

$I_1 = I_{\wedge}^{\leq k} \cap I_{\vee}^{\leq m-n-k}, I_2 = I_{\wedge}^{>k} \cap I_{\vee}^{\leq m-n-k}, I_3 = I_{\wedge}^{>m-n-k} \cap I_{\wedge}^{\leq k}.$

Note that

$I_1 \cup I_2 \cup I_3 \subseteq (I_{\wedge}^{\leq k} \cup I_{\vee}^{\leq m-n-k}).$

However any $\lambda$ has at least $m - n$ crosses. Since $|I_{\wedge}^{\leq k} \cup I_{\vee}^{\leq m-n-k}| = m - n$ we obtain that the crosses are at the positions

$I_{\wedge}^{\leq k} \cup I_{\vee}^{\leq m-n-k}.$
This implies \( d(\lambda) = 0 \): Since \( I^{\geq m-n-k} \) \( I^{\geq k} \) a \( \vee \wedge \)-pair is not possible. \( \square \)

### 9.4 Corollary
If \( l(\lambda) \leq m - n \) then \( \text{lift}_d(\lambda) = \lambda \) for all \( d \).

### 9.5 Corollary
Let \( \lambda, \nu \) be bipartitions of length \( \leq m - n \). Then their tensor product is given by the Littlewood-Richardson rule for \( \text{Gl}(m - n) \) up to superdimension 0. More precisely

\[
R(\lambda) \otimes R(\nu) = \bigoplus_{\nu, l(\nu) \leq m-n} c_{\text{wt}(\lambda), \text{wt}(\nu)}^{\text{wt}(\nu)} R(\nu) \mod \mathcal{N}
\]

where \( c_{\text{wt}(\lambda), \text{wt}(\nu)}^{\text{wt}(\nu)} \) denotes the multiplicity of the \( \text{Gl}(n) \)-representation \( L(\text{wt}(\nu)) \) in the decomposition \( L(\text{wt}(\lambda)) \otimes L(\text{wt}(\mu)) \).

**Proof:** (cf the proof of 7.1.1 in [CW11]) Let \( \nu_1, \ldots, \nu_k \) bipartition such that

\[
\lambda \mu = \nu_1 + \ldots + \nu_k
\]

in \( R_t \). Since \( \text{lift}(\lambda) = \lambda \), \( \text{lift}(\mu) = \mu \) we may assume mod \( \mathcal{N} \) that all \( \nu_i \) have length \( \leq m - n = d \). So \( d \) fulfills \( d \geq l(\nu_i) \) for all \( i \) and \( \text{lift}_d \) fixes \( \lambda, \mu, \nu_1, \ldots, \nu_k \). Hence \( \lambda \mu = \nu_1 + \ldots + \nu_k \) holds in \( R_d \) as well. Using the tensor functor \( F_d : \text{Rep}(\text{Gl}_d) \rightarrow \text{Rep}(\text{Gl}(d)) \) which maps \( \lambda \) to \( L(\text{wt}(\lambda)) \) we obtain

\[
L(\text{wt}(\lambda)) \otimes L(\text{wt}(\mu)) = L(\text{wt}(\nu_1)) \oplus \ldots \oplus L(\text{wt}(\nu_k))
\]

by the Littlewood-Richardson rule in \( \text{Rep}(\text{Gl}(d)) \). Taking the preimage one obtains modulo \( \mathcal{N} \) the result. \( \square \)

### 9.6 Corollary
Let \( m > n \) and \( l(\lambda) \leq m - n \). Then \( R(\lambda) \) is irreducible.

### 9.7 Corollary
Let \( \lambda \) and \( \mu \) be such that \( l(\lambda) + l(\mu) \leq m - n \). Then \( R(\lambda) \otimes R(\mu) \) splits completely into irreducible maximally atypical modules. The decomposition rule is given by the Littlewood-Richardson rule for \( \text{Gl}(m - n) \).

**Example:** Consider the irreducible representation \( \Lambda^{m-n}(st) = R(1^{m-n}; 0) \) and tensor products \( R(1^{m-n}, 0) \otimes R(\lambda) \) for \( l(\lambda) \leq m - n \). The weight of \( (1^{m-n}) \) for \( \text{Gl}(m - n) \) is \((1, \ldots, 1)\), so \( \Lambda^{m-n}(st) \otimes L(\lambda) = L(\lambda_1 + 1, \ldots, \lambda_{m-n} + 1) \) in \( \text{Rep}(\text{Gl}(m - n)) \). If \( R(\lambda) = R(a_1, \ldots, a_k; b_{k+1}, \ldots, b_{k+r}) \) for \( k + r \leq m - n \), then tensoring with \( \Lambda^{m-n} \) gives \( R((a_1 + 1, \ldots, a_k + 1, 1^{(m-n)-(k+r)}); (b_{k+1} - 1, \ldots, b_{m-n} - 1)) \).
It is now easy to recover the theorem from the previous section. Since
\[ F_{mn} : \text{Rep}(\text{Gl}_{m-n}) \to \mathcal{R}_{mn} \]
has its image in \( T \) we can consider the diagram

\[
\begin{array}{c}
\text{Rep}(\text{Gl}_{m-n}) \\
\downarrow F_{mn} \\
T \rightarrow T/N \rightarrow \text{Rep}(\text{Gl}(m-n)).
\end{array}
\]

Using the bijection between the irreducible elements \( R(\lambda) \) and the irreducible elements in \( \text{Rep}(\text{GL}(m-n)) \), we define the lower horizontal functor by putting \( R(\lambda) \mapsto L(\text{wt}(\lambda)) \) on objects. Since both categories are semisimple tensor categories, Schur’s lemma holds and the functor sends the morphism \( \text{id} : R(\lambda) \to R(\lambda) \) to \( \text{id} : L(\text{wt}(\lambda)) \to L(\text{wt}(\lambda)) \). The results on the tensor products show that this defines a tensor functor. It is clearly fully faithful.

10. Maximal atypical irreducible modules

10.1 Proposition. Let \( m > n \). Every maximally atypical Kostant module is a Berezin twist of an irreducible mixed tensor.

Proof: Assume \( m > n, l(\lambda) \leq m - n \). Assume that

\[ \lambda = ((a_1, \ldots, a_k); (b_{k+1}, \ldots, b_{m-n})) \]

and assume additionally \( a_1 \) and \( b_{k+1} \) to be greater zero (otherwise we have covariant or contravariant modules). Recall that the crosses are at the positions

\[ I_{\wedge}^k \cup I_{\vee}^{m-n-k}, \]

hence at the vertices

\[ a_1, a_2 - 1, \ldots, a_k - (k - 1), 1 - (m - n) - b_{k+1}, \ldots, -k - b_{m-n}. \]

Since \( a_1, \ldots, a_k, b_{k+1}, \ldots, b_{m-n} \) are arbitrary, the position of the crosses is arbitrary. Note that the crosses coming from the \( a_i \) are to the right of the \( b_i \)-crosses: \( a_k - (k - 1) > -k - b_{m-n} \). The position of the \( \vee \)'s: We have \( a_1 = \lambda_1^\wedge \), hence there are \( a_1 + n \) switches in the free positions left from the cross at \( a_1 \). To know the position of the \( \vee \)'s, the change from the \( \wedge \) to the \( \vee \)'s has to be known: In fact \( I_{\wedge}^k = [-k, -\infty), \ I_{\vee}^{m-n-k} = [-k + 1, \infty), \) hence the free positions \( \leq -k \) have \( \wedge \)'s, the free ones \( \geq -k + 1 \) have \( \vee \)'s. In the free vertices \( \geq -k + 1 \lambda \) has \( \vee \)'s. These get turned into \( \wedge \)'s. This
are precisely \( a_1 \) free vertices since there are \( k \)-crosses between \( a_1 \) and \(-k\). The next \( n \) free vertices \( \leq -k \) contain \( \land \)'s. These get turned into \( \lor \)'s. After that all free vertices are labelled by a \( \land \). The cup diagram is completely nested: All the \( \lor \)'s are at the first \( n \) free vertices to the left of \(-k\). Given a maximally atypical Kostant module, let \( \lor_m \) be the rightmost \( \lor \). Count the crosses with labels to the right of \( \lor_m \). Name that number \( k \). Then move \( \lor_m \) with a Berezin twist to the position \(-k\). An inspection of the algorithm above shows that this is an irreducible module in \( T \). \( \square \)

**Example:** The highest weight \( \mu = (12, 12, 10, 10, 10, 10, 0) \) of \( Gl(7|3) \) is maximal atypical with rightmost \( \lor \) at position 8 and two crosses at position 11 and 12 to the right. Hence twist \( L(\mu) \) with \( Ber^{-10} \) to move \( \lor \) to position -2 and obtain \( \tilde{\mu} = (2, 2, 0, 0, 0, 0, -10) \) with \( k = 8 \). The algorithm above gives a Berezin twist to the position \(-k\) with \( k = 8 \). An inspection of the algorithm above shows that this is an irreducible module in \( T \).

**10.2 Corollary.** Any Kostant module of atypicality \( < m \) is a Berezin twist of a mixed tensor.

Given two \( \lambda, \mu \) Kostant weights we shift both into \( T \)

\[
L(\lambda) \otimes Ber^{\lambda'} = L(\tilde{\lambda}) \in T, \quad L(\mu) \otimes Ber^{\mu'} = L(\tilde{\mu}) \in T
\]

where \( \lambda', \mu' \) only depend on the position of the unique segment. Therefore

\[
L(\lambda) \otimes L(\mu) = (L(\tilde{\lambda}) \otimes L(\tilde{\mu})) \otimes (Ber^{\lambda'} \otimes Ber^{\mu'}) = \bigoplus_{\nu} c_{\lambda,\tilde{\mu}}^{\nu} L(\nu) \otimes Ber^{\lambda'+\mu'}
\]

for certain coefficients \( c_{\lambda,\mu}^{\nu} \) which can be calculated explicitly from [CW11]. In particular the tensor product of two such modules can be decomposed explicitly.

For two weights \( \lambda = (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n}) \) and \( \mu = (\mu_1, \ldots, \mu_m \mid \mu_{m+1}, \ldots, \mu_{m+n}) \) say that \( \lambda \succeq \mu \) if there exists \( i \in \{1, \ldots, m\} \) with the property \( \lambda_j = \mu_j \) for all \( j < i \) and \( \lambda_i > \mu_i \). Recall that \( \{\lambda^L\} \otimes \{\lambda^R\} = R(\lambda) \oplus \bigoplus R(\mu_j) \) with \( \text{deg}(\mu_j) < \text{deg}(\lambda) \).

**10.3 Lemma.** Let \( R(\lambda) \) be maximally atypical irreducible. Then \( R(\lambda) = L(\lambda^1) \) with \( L(\lambda^1) \succeq L(\mu_j^1) \) for all \( j \).

**Proof:** Define \( I^{\max}_x(\lambda) \) as largest label with a \( \times \) or \( \lor \). We claim \( I^{\max}_x(\lambda) \geq I^{\max}_x(\mu_j) \) for all \( j \). The position of the crosses is given by the elements in \( I^{L,k}_\lambda \cup I^{R,-n-k}_\lambda \). Since \( \lambda^L_i \geq \mu_j^L, I^{L,k}_\lambda(\mu_j) \leq I^{L,k}_{\lambda_i}(\lambda) \). There are \( k \) crosses to the right of \(-k \) (meaning for \( k \lambda \) and \( k \mu_j \)). Hence for the first \( k \mu_j, \lambda_i \geq \mu_{j,i} \).
for all \( i \in \{1, \ldots, k_\lambda\} \). This holds in fact for the first \( k_\lambda \)-coordinates: There are \( k_\lambda \) crosses at positions \( > -k_\lambda \), \( k_\mu \) crosses at positions \( > -k_\mu \). The next \( k_\lambda - k_\mu \) positions with crosses or \( \vee \)'s in \( \mu \) are then at the positions 

\[ -k_\mu, -k_\mu - 1, \ldots, -k_\lambda + 1 \]. Since there exists at least one \( i \) with \( \lambda_i^L > \mu_{j,i} \), the claim follows. \( \Box \)

So the maximally atypical \( R(\lambda) \) for \( m > n \) of \( \text{sdim} \neq 0 \) could be characterized as follows: Take all the tensor products of two \((m, n)\)-Hook partitions \( \lambda^L, \lambda^R \) such that \( (\lambda^L, \lambda^R) \) is \((m, n)\)-cross. Then the \( R(\lambda) \) are the indecomposable modules in the decomposition \( \{ \lambda^L \} \otimes \{ \lambda^R \}^\vee \) which satisfy

\[ R(\lambda) = L(\lambda^\dagger) \supset L(\mu_j^\dagger) \] for all \( j \).

10.1. **The case \( Gl(m|1) \).** In the case \( Gl(m|1) \) and \( Sl(m|1) \) every weight is a Kostant weight. Since \( Ber \) is trivial in the \( Sl \)-case we obtain:

10.4 Proposition. Up to a twist of a suitable power of \( Ber \) every irreducible module of \( Gl(m|1) \) is in \( T \). Every irreducible module of \( Sl(m|1) \) is in \( T \).

**Example:** The \( Gl(2|1) \)-case. Since \( l(\lambda) \leq 1 \), the irreducible atypical mixed tensors are the covariant and contravariant tensors. The highest weight \( (\lambda_1, \lambda_2 | \lambda_3) \) is atypical if and only if either \( \lambda_2 = -\lambda_3 \) or \( \lambda_3 = -\lambda_1 - 1 \). The covariant module \( R(a; 0) \) has highest weight \( (a, 0, 0) \) and the contravariant module \( R(0; b) \) has highest weight \( (0, -b + 1, -1) \). The modules with highest weights \( (\lambda_1, \lambda_2 - \lambda_2) \) are Berezin twists of covariant modules and the modules with highest weights \( (\lambda_1, \lambda_2 | -\lambda_1 - 1) \) are Berezin twists of contravariant modules.

By [Ger98] the indecomposable modules in \( \mathcal{R}_m \) are the (Anti-)ZigZag-modules and the projective hulls of the irreducible atypical representations.

10.5 Corollary. \( Gl(m|1) \)-case: If \( l(\lambda) \leq m - 1 \), then \( R(\lambda^\dagger) \) is irreducible singly atypical. If \( l(\lambda) > m - 1 \) and \( d(\lambda) = 0 \) then \( R(\lambda) = L(\lambda^\dagger) \) is typical. If \( \lambda \) is any bipartition with \( d(\lambda) = 1 \) then \( R(\lambda) = P(\lambda^\dagger) \).

10.6 Corollary. In the decomposition \( L(\lambda) \otimes L(\mu) \) between two irreducible \( Gl(m|1) \)-modules no ZigZag module \( Z_l(a) \) with \( l \geq 2 \) appears.

10.2. **Tensor products.** Since any irreducible \( Gl(m|1) \)-module is up to an explicit Berezin-Twist in \( T \), the tensor product formula in Deligne’s category and the description of the image of \( F_m \) solves the problem of decomposing any two irreducible \( Gl(m|1) \)-representations.
Example 1: We compute $L(2,0,0,0|0) \otimes L(1,0,0,0|−1)$ in $R_{41}$. Applying $\theta^{−1}$ we see that the corresponding bipartitions are $(2;0)$ and $(1;1)$. Since the defect is zero, we only have to compute $(2;0) \otimes (1;1)$ in $R_{3}$. By [CW11], p.35 we have

$$(2;0) \otimes (1;1) = ((2,1); 1) + (3;1) + (1^2;0) + (2;0)$$

for $\delta = 3$ in $R_{3}$. Hence

$$L(2,0,0,0|0) \otimes L(1,0,0,0|−1) =$$

$$L(1,1,0,0|0) + L(2,0,0,0|0) \otimes L(3,0,0,0|−1) \oplus L(2,1,0,0|−1)$$

in $Rep(GL(4|1))$.

Example 2: One could hope that the tensor product of two atypical irreducible modules splits into a sum of irreducible atypical and typical modules. This is wrong: Take $GL(4|1)$, $\lambda^L = (3,2,1)$, $\lambda^R = (1,1)$. Then $R((3,2,1); (1,1))$ is projective.

ZigZag modules [Ger98] [GQS07] of length greater than 1 never occur in the image of $F_{m1}$. However, the tensor product between an indecomposable projective module with a ZigZag-module is easily reduced to the known cases by the following well-known fact:

10.7 Proposition. Let $P$ be projective and $M$ any module. Then $P \otimes M = \bigoplus_i P \otimes M_i$ where the sum runs over the composition factors $M_i$ of $M$.

Proof: Use induction on the length of $M$. If $M$ is of length $n$ consider an sequence

$$0 \rightarrow M_i \rightarrow M \rightarrow M' \rightarrow 0$$

with $\text{length}(M') = n-1$. Tensoring with $P$ and using that $Proj$ is a tensor ideal we see that the sequence splits. □

10.8 Lemma. Let $P$ be an indecomposable projective $GL(m|1)$-Module. Then

$$P \otimes Z^r(a) = \bigoplus_{a_i} P \otimes L(a_i), \quad P \otimes \overline{Z}^r(a) = \bigoplus_{a_i} P \otimes L(a_i)$$

where the sums run over the composition factors $L(a_i)$ of $Z^r(a)$ respectively $\overline{Z}^r(a)$.

All in all the only remaining unknown tensor products in the $GL(m|1)$-case are the tensor products $Z^r(a) \otimes Z^s(b)$ and vice versa for the Anti-ZigZag-modules. If $r,s$ are odd their tensor product decomposes as given by the Littlewood-Richardson Rule for $GL(m−n)$ modulo $\mathcal{N}$ [Hei12].
11. Maximal atypical $R(\lambda)$ for $m = n$

For $m = n$ no maximally atypical irreducible modules are in $T$ because their superdimension does not vanish. In this section we characterise the maximally atypical modules for $m = n$. Assume from now on that $\lambda^t$ is in the maximal atypical block $\Gamma$, i.e. the weight diagram has no $\times$, $\circ$ and exactly $m \lor$‘s.

11.1 Lemma. $R(\lambda^L, \lambda^R)$ is maximal atypical if and only if $\lambda^R = (\lambda^L)^*$.

Proof: Since there are no $\circ$ and no $\times$ $I_\lor \cup I_\land = \mathbb{Z}, I_\lor \cap I_\land = \emptyset$.

Hence $\lambda^L$ and $\lambda^R$ determine each other uniquely. The biggest $\land$ is at position $\lambda_1$. If

$$\lambda^L_1 = \ldots = \lambda^L_{s_1} > \lambda^L_{s_1+1} = \ldots = \lambda^L_{s_2} > \lambda^L_{s_2+1} = \ldots$$

put $\delta_1 = s_1$ and $\delta_i = s_i - s_{i-1}$ and $\Delta_i = \lambda^L_{s_i} - \lambda^L_{s_{i-1}+1}$:

$$\begin{align*}
\Delta_3 & \vdash \Delta_2 \vdash \Delta_1 \\
\vdash \ldots \vdash & \ldots \lor \ldots \lor \ldots \lor \ldots \lor \ldots \lor \ldots \lor \land
\end{align*}$$

Then $\delta_i = \Delta_{t-i}^*$ and $\Delta_i = \delta_i^*$ where $(\cdot)^*$ denotes the corresponding number for the conjugate partition. Note further that the leftmost $\lor$ is at the vertex $\lambda^L_1 - \sum \delta_i - \sum \Delta_i + 1 = \lambda^L_1 - l(\lambda^L) - \lambda^L_1 + 1 = 1 - (\lambda^L)_1^*$.

A counting argument finishes the proof.

11.2 Corollary. $T(r, s)$ contains a maximally atypical summand only for $r = s$.

Proof: By [BS11] and the characterisation of maximally atypical $R(\lambda)$

$$pr_\Gamma T(r, s) = \bigoplus R(\lambda, \lambda^*)$$

where $|\lambda| = r - t$, $|\lambda^*| = s - t$. Since $|\lambda| = |\lambda^*|$ this can only happen for $r = s$.

Notation: From now on we always write $R(\lambda)$ where $\lambda$ is a partition such that $(\lambda, \lambda^*)$ is $(m, m)$-cross.

11.3 Lemma. Assume $l(\lambda) \leq m$ and $d(\lambda) = m$. Then $\lambda^t = [\lambda]^0$.

Proof: This is easily seen using the algorithm of determining $[\lambda]^0$ given in [BS10a], page 36, and the fact that the positions of all $\lor$‘s is determined due to maximal defect.
Remark. Let $\lambda$ be any partition and let $\beta$ the intersection of the Young diagram with the box of length $m$ and width $m$ with upper left corner at the position $(0,0)$. Then the Young diagram has the following shape

$$\lambda = \begin{pmatrix} \beta \\ \gamma \\ \alpha \end{pmatrix}.$$ 

Hence if $l(\lambda) \leq m$ then $\gamma = 0$. Define the weight $A_\lambda := [\alpha + \beta + (\gamma)^v]$ where $(\gamma_1, \ldots, \gamma_r)^v = (-\gamma_r, \ldots, -\gamma_1)$. For $l(\lambda) \leq m$ this is nothing but the weight $[\lambda]$. $A_\lambda = \{\lambda\} \otimes \{\lambda^v\}$ always contains the maximal atypical constituent $[\alpha + \beta + (\gamma)^v]$ as highest weight representation with multiplicity 1 as can be seen from a restriction to $Gl(m) \times Gl(m)$ as was observed by Weissauer. Since the restriction of $A_\lambda$ to the maximal atypical block decomposes as

$$pr_T A_\lambda = R(\lambda) \oplus \bigoplus R(\lambda^i)$$

for partitions $\lambda^i$ with $\lambda_j \geq \lambda^i_j$ for all $i, j$, it seems likely to assume that the unique constituent of highest weight in $R(\lambda)$ is given by $A_\lambda := [\alpha + \beta + (\gamma)^v]$. This is wrong, as the following example shows: Take $Gl(4|4)$ and choose $\lambda = (3^4, 1^2)$. Then $\lambda^1 = [1, 1, 1, 0]$, $A_\lambda = [3, 3, 1, 1]$ but $[\alpha + \beta + (\gamma)^v] = [3, 3, 3, 1]$. In particular $R(\lambda)$ cannot be characterised as the constituent of highest weight in the $A_\lambda$-decomposition. This is however correct if one restricts to partitions $\lambda$ of length $\leq n$.

11.1. The involution $I$. Recall that the Tannaka dual of an indecomposable element in $T$ is given by $R(\lambda^L, \lambda^R)^\vee = R(\lambda^R, \lambda^L)$. Similarly we define

$$IR(\lambda^L, \lambda^R) := R((\lambda^R)^*, (\lambda^L)^*).$$

11.4 Lemma. This is a well-defined operation on $T$ for $m = n$ (ie. $(\lambda^R)^*, (\lambda^L)^*$ is again $(m, m)$-cross). $I$ is an involution and commutes with Tannaka duality. $I$ is the identity if and only if $R(\lambda)$ is maximally atypical.

Proof: Let $i \in 1, \ldots, m$ have the property $\lambda^L_{i+1} + \lambda^R_{m-i+1} \leq m$, so $\lambda^L_{i+1} \leq k$ and $\lambda^R_{m-i+1} \leq m-k$ for some $k$. Then $(\lambda^L_{k+1})^* \leq i$ and $(\lambda^R_{m-k+1})^* \leq m-i$, hence $(\lambda^L_{k+1})^* + (\lambda^R_{m-k+1})^* \leq m$. The other statements are clear. □

Remark: For $m > n$ the bipartition $((\lambda^R)^*, (\lambda^L)^*)$ may fail to be $(m, n)$-cross.

11.5 Lemma. $I$ preserves dimensions.

Proof: Since the dimension is preserved under dualising $(\lambda^L, \lambda^R) \mapsto (\lambda^R, \lambda^L)$, we only have to take care of $(\lambda^L, \lambda^R) \mapsto (\lambda^{L^*}, \lambda^{R^*})$. By [CW11],
Lemma. \(1\) For every positive weight \(\lambda\), let \(\lambda^\dagger = [\lambda_1^\dagger, \ldots, \lambda_n^\dagger]\) be maximally atypical in \(\mathbb{R}_n\). We normalize \([\lambda]\) so that \(\lambda_n = 0\). More generally a weight with \(\lambda_n \geq 0\) will be called positive. If \(k \in \{1, \ldots, n\}\) is the biggest index with \(\lambda_k \neq 0\) we say that the weight is of length \(k\). Such a weight defines a partition \(\lambda\) of length \(k\).

12.1 Lemma. If \(l(\lambda) \leq k\), then \(d(\lambda) \leq k\). If \(\lambda_1 \leq k\), then \(d(\lambda) \leq k\). In particular a mixed tensor can be projective only if \(l(\lambda) \geq n\) and \(\lambda_1 \geq n\).

Example: There is a unique projective mixed tensor \(R(\lambda)\) of smallest degree. It is given by \(\lambda = (n, n-1, \ldots, 1)\) and gives the projective cover of \([\lambda^\dagger] = [n-1, n-2, \ldots, 1, 0]\).

12.2 Theorem. \(1\) For every positive weight \(\lambda^\dagger = [\lambda_1^\dagger, \ldots, \lambda_n^\dagger]\) of length \(k\) exists a unique mixed tensor of defect \(k\) \(R(\lambda) = R(\lambda^L, \lambda^R)\) and length \(\lambda^L = k\) and \(\text{socle}(R(\lambda)) = [\lambda^\dagger]\). \(2\) For every positive weight \([\lambda]\) of length \(k\) the mixed tensor \(R(\lambda)\) has defect \(\leq k\) and contains \([\lambda]\) with multiplicity 1 in the middle Loewy layer. \([\lambda]\) is the constituent of highest weight in \(R(\lambda)\).

In particular \([\lambda] \to R(\lambda)\) gives a bijection between the positive weights of length \(k\) and mixed tensors given by partitions of length \(k\).
Proof. Proof of 1) We construct $R(\lambda)$ explicitly. To an irreducible highest weight we associate its cup diagram with $n$ cups. Since the length of $[\lambda^\dagger]$ is $k$, exactly $k \wedge$’s are are bound in a cup with a $\vee$ associated to one of the $\lambda_1^\dagger, \ldots, \lambda_k^\dagger$. Label the $k \wedge$’s from the rightmost to the leftmost position by $\{v_1, v_2, \ldots, v_k\}$. Then define the partition $\lambda = (v_1, v_2 + 1, v + 3 + 2, \ldots, v_k + k - 1)$. Then $l(\lambda) = d(\lambda) = k$. The $k$ cups of $\lambda$ agree with the $k$ cups of $[\lambda^\dagger]$ associated to the nontrivial $\lambda_i^\dagger$. By construction (and the positivity of $[\lambda^\dagger]$) the largest label in a cup in the cup diagram of $\lambda$ is at a vertex $\geq k$.

We obtain the highest weight in $R(\lambda)$ according to the rules of section 4 by switching all labels at vertices $\geq v_1$ and the first $v_1 + n - 2k$ labels at vertices $\leq v_1$ which are not part of a cup. Since the leftmost cup has its leftmost label at a vertex $\geq 2 - k$, this means switching exactly all the free (not part of a cup) labels at vertices $\geq n + 1$. This switches all labels $\vee$ at vertices $\leq v_1$. Since the length is $k$ we have $\lambda$’s at all vertices $k$.

The $n - k$ rightmost $\wedge$’s at the positions $-n + 1, -n + 2, \ldots, -n + (n - k)$ will be switched to $\vee$’s. These $n - k \wedge$’s give the $n - k$ zeros in $\lambda^\dagger$.

Uniqueness: We apply $DS' := DS^{n-k} : \mathcal{R}_n \rightarrow \mathcal{R}_k$. Then $DS'(R(\lambda))$ is the projective cover of a unique irreducible module and the mixed tensors of defect $k$ are in bijection with the projective covers of irreducible modules.

We show $soc(DS'(R(\lambda))) = [\lambda_1^\dagger, \ldots, \lambda_k^\dagger]$ which implies our assertions about the uniqueness of the mixed tensor of length $k$ with prescribed socle. Since $\lambda$ does not depend on $n$ we get the same weight and cup diagram of $\lambda$. The highest weight of the socle is obtained as above by switching all labels at vertices $\geq v_1$ and the first $v_1 + k - 2k$ labels at vertices $\leq v_1$ which are not part of a cup. This means that we do not switch the $n - k$ leftmost labels at the vertices $-n + 1, -n + 2, \ldots, -n + (n - k)$.

Proof of 2): We use 1). We start with the partition $\lambda$. We have seen that $l(\lambda) \leq n$ means switching the free $\wedge$’s at vertices $\geq -n + 1$ and $\geq max(k(\lambda), M)$ to $\wedge$’s and vice versa. Similarly all the free vertices $\geq max(k(\lambda), M)$ are labelled by $\vee$’s which are switched to $\wedge$’s. We obtain $A_\lambda$ from $[\lambda^\dagger]$ by interchanging the $\vee$’s with the $\wedge$’s in the $d(\lambda)$ cups of $\lambda$. Hence to get the weight diagram of $A_\lambda$ from the weight diagram of $\lambda$ means switching all labels at vertices $\geq -n + 1$. The $\wedge$’s at the vertices $\geq -n + 1$ are at the vertices $\lambda_1, \lambda_2 - 1, \ldots, \lambda_k - k + 1, -k, \ldots, -n + 1$, hence the $\vee$’s in the weight diagram of $A_\lambda$ are at the vertices $\lambda_1, \lambda_2 - 1, \ldots, \lambda_k - k + 1, -k, \ldots, -n + 1$. \[ \square \]

Remark: Note that different partitions of the same defect but different length can give the same highest weight in the socle.

Example: Assume $[\lambda^\dagger] = [\lambda_1^\dagger, \ldots, \lambda_k^\dagger, 0, \ldots, 0]$ with $\lambda_1^\dagger > \lambda_2^\dagger > \ldots > \lambda_k^\dagger$. The $k \wedge$’s in the $k$ cups are at the vertices $\lambda_1^\dagger + 1, \lambda_2^\dagger, \lambda_3^\dagger - 1, \ldots, \lambda_k^\dagger - k + 2$, hence $\lambda = (\lambda_1^\dagger + 1, \lambda_2^\dagger, \lambda_3^\dagger - 1, \ldots, \lambda_k^\dagger - k + 2)$. \[ 40 \]
Remark: If the ∧’s of [λ†] in the k cups are at the vertices v₁, ..., vₖ so that λ = (v₁, v₂ + 1, ..., vₖ + k - 1), Aₜ = [v₁, v₂ + 1, ..., vₖ + k - 1, 0, ..., 0].

12.3 Lemma. Let λ be an (n, n)-cross partition. The socle of R(λ) is positive if and only if l(λ) ≤ n. The highest weight constituent Aₜ is positive if l(λ) ≤ n.

Proof. Let M be the largest vertex which is part of a cup in λ. We distinguish two cases: Either k(λ) ≥ M or k(λ) ≤ M. Let us assume k(λ) ≤ M. Then λ† is obtained from switching all the labels in the weight diagram of λ at vertices ≥ M + 1 and switching the first M + n - 2k(λ) labels in vertices which are not part of a cup ≤ M. If the length of λ is ≤ n, all the k(λ) cups are at positions ≥ −n + 1 since the leftmost label ∧ is at a vertex ≥ 2 − n. All the free labels at vertices ≥ −n + 1 are switched and no labels at vertices ≥ −n are switched. Since we have only ∧’s at vertices ≥ −n this proves the positivity of λ†. If k(λ) > M, we obtain λ† from λ by switching all the labels at vertices ≥ k(λ) + 1 and then switching the first n - k(λ) free labels at vertices ≤ k(λ) + 1. Again by l(λ) ≤ n we only have ∧’s at vertices ≥ −n which do not get switched, showing again the positivity of λ†. If on the other hand l(λ) = r = n + i > n, then the leftmost ∧ is at the vertex λᵣ−r+1 and it is easy to see that at least one of the λᵣ ∧’s at vertices −r+1, −r+2, ..., λᵣ−r is part of a cup. This ∨ will give a label in λ† smaller then zero.

We define the degree deg[λ] of an arbitrary maximally atypical highest weight as ∑ₙᵢ₌₁ λᵢ. With this definition the constituent of highest weight in R(λ) is the constituent of largest degree.

12.4 Lemma. We have degₜ Aₜ ≤ deg(λ) with equality if and only if l(λ) ≤ n.

Proof. If l(λ) ≤ n we have seen this in 12.2. If l(λ) > n, then for the n ∨ in the weight diagram of [λ] there are at least n ∧’s {∧₁, ..., ∧ₙ} corresponding to n non-trivial λᵢ {λᵢ₁, ..., λᵢₙ} at vertices greater or equal to the vertices of the n ∨. Then degₜ Aₜ ≤ ∑ₙᵢ₌₁ λᵢᵢ. Since the length of λ is larger then n, deg(λ) > ∑ₙᵢ₌₁ λᵢᵢ.

12.5 Lemma. The mixed tensors with soc(R(λ)) = Bᵏ, k ≠ 0, are precisely the projective covers P(Bᵏ). We have

\[
P(Bᵏ) = R((n + k)ⁿ) \quad k \in [-n + 1, \infty)
\]
\[
P(Bⁿ⁻ʳ) = R(nʳ) \quad r > n.
\]
The mixed tensors with socle 1 are the modules

\[ R(k^k), \quad k \in \{1, \ldots, n\}. \]

We remark that the constituent of highest weight in \( R(k^k) \) is \([k, \ldots, k, 0, \ldots, 0]\) and the constituent of highest weight in \( R((n + k)^n) \) is \([n + k, n + k, \ldots, n + k]\) for \( k \in [-n + 1, \infty) \).

**Proof.** For \( d(\lambda) = n \) one can easily check the claims of the lemma. Let us assume \( 1 \leq k < n \) with \( k = d(\lambda) \). Then \( \lambda \) must be completely nested with \( k \) cups. The \( n - k \) vertices left of the \( k \) cups have to be labelled by i) either \( \vee \)'s which remain stable when applying \( \theta \) or ii) must be all labelled with a \( \wedge \) and get switched under \( \theta \). Assume i), hence the \( \vee \) are to the left of the interval where labels are switched, hence \( k > M \). To obtain \( \lambda^\dagger \) from \( \lambda \) we switch the first \( n - k \) free places left of \( k + 1 \). Hence at these vertices the labels cannot be \( \wedge \)'s. Hence the rightmost \( \wedge \) can only appear at a vertex \( \leq 0 \). Contradiction, hence let us assume ii) holds. If the weight diagram of \( \lambda \) has a vertex labelled \( \vee \) left of the \( n - k \vee \)'s in the cups, we would get an additional cup, hence all vertices left of the cups must be labelled by \( \wedge \)'s. Hence

\[ \lambda = (k^k) \]

for some \( k > 0 \). If \( k > n \) we would get more then \( n \) cups, hence \( k \in \{1, \ldots, n - 1\} \). In all these cases the \( n - k \vee \)'s in the cups are at the vertices \( 0, -1, \ldots, -k + 1 \), hence \([\lambda^\dagger] = 1\). \( \square \)

**12.6 Lemma.** The mixed tensors \( R(\lambda) \) with \( A_\lambda = B^k \) are the projective covers \( P(B^k) \) with constituent of highest weight \( B^{k+n} \) and the \( R(k^n) \) for \( 1 < k < n \) with highest weight constituent \( B^k \) and defect \( k \).

**Proof.** We obtain \( A_\lambda \) from \([\lambda^\dagger] \) by switching the labels in the \( k \) cups in the weight diagram of \( \lambda \). Hence \( \lambda \) must be completely nested. If \( d(\lambda) = n \) we can easily check the claim. Let us assume \( 1 \leq k \leq n - 1 \) for \( k = d(\lambda) \). As for the socle it is easy to see that we have either i) \( n - k \vee \)'s to the right of the \( k \) cups which do not get switched under \( \theta \) or ii) \( n - k \wedge \)'s to the right of the \( k \) cups which get switched under \( \theta \). Assume i) Then all vertices labelled to the right of cups are labelled \( \vee \). To the left of the cups we cannot have vertices labelled \( \wedge \). These would be switched to \( \vee \)'s, hence all vertices to the left of the cups which are switched must be labelled by a \( \vee \). We obtain \([\lambda^\dagger] \) by switching the labels in the first \( \lambda_1 + n - 2k \) vertices left of the cups. Hence we must have \( \lambda_1 = \lambda_1 + n - k \) for \( k < n \), a contradiction. Hence the \( n - k \) labels at the vertices to the right of the cups must be \( \wedge \)'s. More then
\(n - k \land \)'s would give too many \(\lor\)'s when applying \(\theta\). Hence
\[
\lambda = (k^n), \quad k \in \{1, \ldots, n - 1\}.
\]

\[\square\]

**Example:** \(P(1) = R(n^n)\) has \(B^n\) as highest weight constituent.

### 13. Appendix: The Orthosymplectic Case

In this section we study a toy model: We divide the space of mixed tensors of an orthosymplectic Lie superalgebra by the ideal \(\mathcal{N}\). Recall that for \(m > n\) we have \(T/\mathcal{N} \simeq \text{Rep}(\text{Gl}(m - n))\).

Here we prove the analogous result in the orthosymplectic case. As for \(\text{Gl}(n)\) there exists an interpolating category \(\text{Rep}(O_t)\), \(t \in k\) with a standard representation \(st\). Following Deligne [Del07] we define for \(t = n \in \mathbb{Z}\) the following triples \((G, \epsilon, X)\) where \(G\) is a supergroup, \(\epsilon\) an element of order 2 such that \(\text{int}(\epsilon)\) induces on \(O(G)\) its grading modulo 2 and \(X \in \text{Rep}(G, \epsilon)\):

- \(n \geq 0\) : \((O(n), \text{id}, st)\)
- \(n = -2m \leq 0\) : \((\text{Sp}(2m), -1, st\text{ seen as odd })\)
- \(n = 1 - 2m \leq 0\) : \((\text{OSp}(1, 2m), \text{diag}(1, -1, \ldots, -1), st)\)

By the universal property [Del07], prop 9.4 the assignment \(st \mapsto X\) defines a tensor functor \(\text{Rep}(O_t) \rightarrow \text{Rep}(G, \epsilon)\).

**13.1 Theorem.** [Del07], thm 9.6 *The functor \(st \mapsto X\) of \(\text{Rep}(O_t) \rightarrow \text{Rep}(G, \epsilon)\) defines an equivalence of \(\otimes\)-categories*

\[
\text{Rep}(O_t)/\mathcal{N} \rightarrow \text{Rep}(G, \epsilon).
\]

By the universal property we also have a tensor functor \(\text{Rep}(O_t) \rightarrow \text{Rep}(\text{OSp}(n, m))\) for \(t = n - m\).

**13.2 Proposition.** *For \(t = n - m\) we have a commutative diagram of tensor functors*

\[
\begin{array}{ccc}
\text{Rep}(O_t) & \rightarrow & \text{Rep}(G, \epsilon) \\
\downarrow & & \downarrow \\
\text{Rep}(\text{OSp}(n, m)) & \rightarrow & \text{Rep}(G, \epsilon) \otimes \text{svect.}
\end{array}
\]
Proof: We construct $S_x$. Take

$$x = \begin{pmatrix} 0 & \text{diag}(1, \ldots, 1) \\ 0 & 0 \end{pmatrix}.$$

Then $x \in g_1$ and $[x, x] = 0$. An easy computation shows $rk(x) = def(g)$. For any such $x$ the formalism of [Ser10] gives a tensor functor $M \mapsto M_x$ from $Rep(OSp(m, n)) \to Rep(G, \epsilon)$. A second calculation shows that it maps the standard representation to the standard representation. Hence $st \mapsto st$ on both sides of the diagram. By the universal property a tensor functor from Deligne’s category is already determined by the image of the standard representation.

Let $T$ denote the image of $F_{m,n} : \text{Rep}(O_t) \to \text{Rep}(OSp(m, n))$. Instead of the above diagram we consider the commutative diagram

$$
\begin{array}{ccc}
\text{Rep}(O_t) & \xrightarrow{N} & \text{Rep}(G, \epsilon) \\
\downarrow T & & \downarrow S_x \\
\text{Rep}(O_t) & \xrightarrow{\text{id}} & \text{Rep}(O_t)
\end{array}
$$

13.3 Theorem. We have $T/N \simeq \text{Rep}(G, \epsilon)$.

Proof: The functor $S_x$ is full when restricted to $T$, hence $T \rightarrow \text{Rep}(G, \epsilon)$ factorises over $T/N$. The equivalence $\text{Rep}(O_t)/N \simeq \text{Rep}(G, \epsilon)$ gives us a bijection between the irreducible elements of $\text{Rep}(G, \epsilon)$ and the indecomposable modules $X$ in $\text{Rep}(O_t)$ with $id_X \notin N$. Any $X$ in $\text{Rep}(O_t)$ with $id_X \in N$ maps to zero in $T/N$. Note that the image of an indecomposable element of $\text{Rep}(O_t)$ in $T/N$ is indecomposable by [CW11], lemma 2.7.4 since $F_{mn}$ is full. This shows that the functor $T/N \rightarrow \text{Rep}(G, \epsilon)$ is one-to-one on objects. Fully faithfulness follows trivially from Schur’s lemma.

Similarly to the $Gl(m|n)$-case the maximally atypical modules of non-vanishing superdimension in $T$ are those which are parametrized by partitions of length $\leq t$. 44
PART 3. SYMMETRIC POWERS AND THEIR TENSOR PRODUCTS

We study a class of indecomposable mixed tensors living in the maximal atypical block of $R_n$ for any $n \geq 1$ of Loewy length 3. They are the smallest indecomposable modules in $T$ with these properties. We then compute their tensor products. This will be crucial for the evaluation of the tensor products between the irreducible maximally atypical modules $S^i := [i, 0, \ldots, 0]$.

14. THE SYMMETRIC AND ALTERNATING POWERS

We define $A_{S^i} := R(i; 1^i) = R(i)$ and $A_{\Lambda^i} := (A_{S^i})^\vee = R(1^i; i) = R(1^i)$.

14.1 Lemma. If $d(\lambda) = 1$, then $R(\lambda) = A_{S^i}$ or $A_{\Lambda^i}$ for some $i > 0$.

Proof: For $d(\lambda) = 1$ there can be at most one jump $\lambda_j > \lambda_{j+1}$ in the bipartition, hence $\lambda = (a, 0, \ldots)$ or $\lambda = (b, b, \ldots, b, 0, \ldots)$ for $n > 1$. For $b > 1$ two $\vee$ will occur, hence $d(\lambda) > 1$. □

We want to compute $((i); 0) \otimes (0; 1^i)$ in $R_t$, hence the sum $\sum_\nu \sum_{\kappa} c^{(i)}_{\kappa, \nu} c^{(1^i)}_{\kappa, \nu^*}$, hence we search the pairs $(\kappa, \nu)$, $(\kappa, \nu^*)$ in $\lambda^{-1}$ resp $\lambda^*-1$. The Pieri rules tells one that the only such pairs are the pairs $((0), (i)) \leftrightarrow ((0), (1^i))$ and $((1), (i - 1)) \leftrightarrow ((1), (1^{i-1}))$.

Hence $$(i; 0) \otimes (0; 1^i) = (i) \oplus (i - 1).$$

in $R_t$. Now clearly $\text{lift}(i) = (i) \oplus (i - 1)$, hence

14.2 Lemma. $A_{S^i} = \{(i)\} \otimes \{(1^i)\}^\vee$. Dito for $A_{\Lambda^i}$.

We define $S^i = [i, 0, \ldots, 0]$ for integers $i \geq 1$.

14.3 Lemma. The Loewy structure of the $A_{S^i}$ is given by $(n \geq 2)$

$A_{S^1} = (1, S^1, 1)$

$A_{S^i} = (S^{i-1}, S^i \oplus S^{i-2}, S^{i-1}) \quad 1 < i \neq n$

$A_{S^n} = (S^{n-1}, S^n \oplus S^{n-2} \oplus B^{n-1}, S^{n-1})$.

Remark: For $n = 1$ we get $A = P(1)$. 45
Proof: We sketch the computation for \( A_{S_i}, \ 1 < i < m \). The module in the socle can be computed by applying \( \theta \). The matching \( t \) looks schematically like (picture for \( i = 4 \))

with the upper cup at the vertices \((0,1)\) and the lower one at the vertices \((i-1,i)\). To determine the remaining composition factors we search the \( \mu \) with \( \mu \subset \alpha \to t \), \( \text{red}(\mu t) = \zeta \). Since \( t \) and \( \zeta \) are fixed and the matching has to be consistently oriented this determines \( \alpha \) up to the position at the unique cup in \( t \) at position \((i-1,i)\). Now consider \( \mu \) where \( \mu \) is obtained from \( \lambda^\dagger = S_{i-1} \) by moving the \( \lor \) at position \( i-1 \) to position \( i-2 \). This gives a cup at position \((i-2,i-1)\). The lower reduction property is satisfied and gives the weight \( S_{i-2} \). No other \( \mu \subset \lambda^\dagger \) fulfill the summation conditions.

The second possible case for \( \alpha \) (switching the \( \land \) with the \( \lor \) in the rightmost cup, hence moving \( \lor \) one to the right) gives the module \([S^i] = [i,0,\ldots,0]\). As in the case of \( \alpha = \lambda^\dagger \) a second \( \mu \subset [S^i] \) may be obtained by moving the rightmost \( \lor \) one to the left. The corresponding module is \([S^{i-1}]\) and gives the second copy of \([S^{i-1}]\). One can check that no other weight diagrams fulfill the summation conditions. The Loewy layers can be determined from the number of lower circles in \( \text{red}(\mu t) = \underline{1} \). The remaining cases can be treated in the same way. \( \square \)

Example: Typical \( \otimes S^i \). We obtain a recursive algorithm to compute the tensor product \( L(v) \otimes S^i \) where \( L(v) \) is a typical module in the \( m = n \)-case.

The tensor product \( L(v) \otimes A \) (where \( A = A_{S^i} \)) is known since both modules are in the image of \( F_{mm} \). Since \( L(v) \) is projective and \( A = (1,S^1,1) \), it splits into \( 2L(v) \oplus L(v) \otimes S^1 \). Removing the two \( L(v) \) we obtain \( L(v) \otimes S^1 \).

Similarly \( L(v) \otimes A_{S^1} = L(v) \otimes S^2 \oplus 2L(v) \otimes S^1 \oplus L(v) \) which gives a formula for \( L(v) \otimes S^2 \). Iterating this procedure gives the decomposition of \( L(v) \otimes S^i \) for any \( i \). In particular it gives an algorithm to decompose \( L(v) \otimes L[a,b] \) where \( L(v) \) is a typical \( Gl(2|2) \)-module and \( L[a,b] \) is a maximal atypical weight of \( Gl(2|2) \). For the \( psl(2|2) \)-case see also [GQS05].

Example: We want to compute \( L(2,2|1,1) \otimes L(2,1|-1,-2) \) in \( R_2 \). We have \( L(2,2|1,1) = R(2^4;0) \), so we compute \( (2^3;0) \otimes ((1;1) + (0;0)) \) in \( R_1 \). This
Proof: This follows since the defect of \((i, 0, \ldots)\) and \((1^i, 0, \ldots)\) is maximal for \(Gl(1|1)\).

15.2 Corollary. In \(Gl(1|1)\)
\[
\mathbb{A}_{S_i} \otimes \mathbb{A}_{S_j} = \mathbb{A}_{S_i}^{[-i+j]+1} \oplus 2 \mathbb{A}_{S_i}^{[-i+j]} \oplus \mathbb{A}_{S_i}^{[-i+j]+2}
\]
\[
\mathbb{A}_{S_i} \otimes \mathbb{A}_{S_j} = \mathbb{A}_{S_i}^{[i+j]} \oplus 2 \cdot \mathbb{A}_{S_i}^{[i+j]-1} \oplus \mathbb{A}_{S_i}^{[i+j]-2}
\]

Proof: This is just rewriting the known formula \((a, b \in \mathbb{Z})\)
\[
P(a) \otimes P(b) = P(a + b + 1) \oplus 2P(a + b) \oplus P(a + b - 1)
\]

from [GQS07].
Let us assume from now on \( m, n \geq 2 \).

**15.3 Lemma.** After projection to the maximal atypical block \( (n \geq 2) \)

\[
\mathbb{A}_{S^i} \otimes \mathbb{A}_{M^j} = \mathbb{A}_{S^i-I+j+2} \oplus 2 \mathbb{A}_{S^i-I+j+1} \oplus \mathbb{A}_{S^i-I+j} \oplus R_1
\]

\[
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} = \mathbb{A}_{S^i+j} \oplus 2 \mathbb{A}_{S^i+j-1} \oplus \mathbb{A}_{S^i+j-2} \oplus R_2
\]

where \( R_1 \) and \( R_2 \) are direct sums of modules which do not contain any \( \mathbb{A}_{S^i} \) or \( \mathbb{A}_{M^j} \).

**Proof:** This follows from the \( GL(1|1) \)-case and the identification between the projective covers and the symmetric and alternating powers. In \( GL(1|1) \) \[GQS07\]

\[
P(a) \otimes P(b) = P(a+b-1) \oplus 2P(a+b) \oplus P(a+b+1).
\]

Hence this formula holds for the corresponding \( \mathbb{A}_{S^i} \), respectively \( \mathbb{A}_{M^j} \). It then holds in \( Rep(GL_0) \) and hence in any \( Rep(GL(m|m)) \) up to contributions which lie in the kernel \( F_{mm} : Rep(GL_0) \rightarrow Rep(GL(m|m)) \) and which are not \((1,1)-cross\).

We carry out the tensor product decomposition in \( Rep(GL_0) \). Recall that this consists of three steps: i) take the lift \( R_0 \rightarrow R_t \); ii) decompose the lift in \( R_t \) according to Comes-Wilson, iii) take \( lift^{-1} \). From the resulting sum in \( Rep(GL_0) \) we remove the terms in \( ker(F_0) \) and get the result in \( R_n \).

**Lifts:** Clearly \( lift(i) = (i) + (i-1) \), \( lift(1^i) = (1^i) + (1^{i-1}) \). In order to compute the tensor product \( \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} \) we have to compute the tensor product \( (i) \otimes (j) \oplus (i) \otimes (j-1) \oplus (i-1) \otimes (j) \oplus (i-1) \otimes (j-1) \) in \( R_t \).

We derive first a closed formula for \( (i) \otimes (j) \) in \( R_t \), i.e. \((i,0,\ldots),(1^i)) \otimes (j,0,\ldots),(1^j)\). Without saying we often restrict to the maximal atypical case where \( \nu^L = (\nu^R)^* \) and omit the other factors.

- The contribution \( \sum_{\gamma \in P} c^L_{\alpha,\beta} c^R_{\beta,\eta} \): Here \( \lambda^R = (1^i) \) and \( \mu^L = (j,0,\ldots) \).
  Now the Pieri rule gives \( (\mu^L)^{-1} = (0,j),(1,j-1),\ldots,(j-1,1),(j,0) \) and \( (\lambda^R)^{-1} = (0,1^j),(1,1^{i-1}),\ldots,(1^j,0) \). In the sum over all bipartitions \( \nu \) we consider only those with \( \nu^L = (\nu^R)^* \). This condition permits only the pairs \((0,i) \leftrightarrow (0,1^j)\) and \((1,i-1) \leftrightarrow (1,1^{i-1}) \) (to have same \( \gamma \)).
- The contribution \( \sum_{\kappa \in P} c^L_{\kappa,\alpha} c^R_{\kappa,\beta} \): Here \( \mu^R = (1^j), \lambda^L = (i) \). As in the previous case this gives only the possibilities \( c^L_{0,1} c^R_{0,1} \) and \( c^L_{1,i-1} c^R_{1,i-1} \).
Hence the sum
\[ \sum_{\alpha,\beta,\eta,\theta} \left( \sum_{\kappa} c^{\lambda}_{\kappa,\alpha} c^{\mu}_{\kappa,\beta} \right) \left( \sum_{\gamma} c^{\nu}_{\gamma,\theta} c^{\rho}_{\gamma,\eta} \right) \]
collapses to
\[ (c^i_{0,i} c^1_{0,1} + c^i_{1,i-1} c^1_{1,i-1}) (c^i_{0,i} c^j_{0,j} + c^i_{1,i-1} c^j_{1,j-1}). \]
This corresponds to the choices
- (A) \( \alpha = i, \beta = 1^j \)
- (B) \( \alpha = i - 1, \beta = 1^{j-1} \)
- (C) \( \eta = 1^i, \theta = j \)
- (D) \( \eta = 1^{i-1}, \theta = j - 1. \)

Only for these choices AC, AD, BC, BD can there be a non-vanishing contribution \( c^{\nu}_{\alpha,\theta} c^{\rho}_{\beta,\eta}. \) We assume always \( \nu^L = (\nu^R)^* \).

- The AC-case: \( c^{\nu^L}_{i,j} c^{\nu^R}_{1^i,1^j} (\nu^L, \nu^R). \) By the Pieri rule \( \nu^L \) can be any of \( (i + j), (i + j - 1, 1), (i + j - 2, 2), \ldots \) and \( \nu^R \) any of \( (1^{i+j}), (2, 1^{i+j-2}), \ldots, (i, |i-j|) \). Hence the following bipartitions \( \nu \) appear with multiplicity 1:
  \[ (i + j), (i + j - 1, 1), \ldots, ((\max(i, j), \min(i, j)). \]
- The AD-case: \( c^{\nu^L}_{i,j-1} c^{\nu^R}_{1^i,1^{j-1}}. \) Restricting to \( \nu^L = (\nu^R)^* \) we obtain
  \[ \nu \in \{(i + j - 1), (i + j - 2, 1), \ldots, ((\max(i, j), \min(i, j) - 1))\}. \]
- The BC-case: \( c^{\nu^L}_{i-1,j} c^{\nu^R}_{1^{i-1},1^j}. \) Here \( \nu \) is any of
  \[ \nu \in \{(i + j - 1), (i + j - 2, 1), \ldots, ((\max(i, j), \min(i, j) - 1))\}. \]
- The BD-case: \( c^{\nu^L}_{i-1,j-1} c^{\nu^R}_{1^{i-1},1^{j-1}}. \) Here
  \[ \nu \in \{(i + j - 2), (i + j - 3, 1), \ldots, (\max(i - 1, j - 1), \min(i - 1, j - 1))\}. \]

Hence
\[ (i) \otimes (j) = \]
\[ (i + j) \oplus (i + j - 1, 1) \oplus \ldots \oplus ((\max(i, j), \min(i, j)) \]
\[ \oplus (i + j - 1) \oplus (i + j - 2, 1) \oplus \ldots \oplus ((\max(i, j), \min(i, j) - 1)) \]
\[ \oplus (i + j - 1) \oplus (i + j - 2, 1) \oplus \ldots \oplus ((\max(i, j), \min(i, j) - 1)) \]
\[ \oplus (i + j - 2) \oplus (i + j - 3, 1) \oplus \ldots \oplus (\max(i - 1, j - 1), \min(i - 1, j - 1)). \]

We want to compute \( R((i)) \otimes R((j)). \) We know \( \text{lift}(i) = (i) \oplus (i - 1). \) This gives in \( R_t ((i) \oplus (i - 1)) \cdot ((j) \oplus (j - 1)) = (i)(j) \oplus (i)(j - 1) \oplus (i - 1)(j) \oplus (i - 1)(j - 1). \)
The special case $j = 1, i > 1$: Then $(j - 1) = 0$. In this case $\text{lift}((i) \otimes (1)) = (i) \otimes (1) \oplus (i) \oplus (i - 1) \oplus (i - 1) \otimes (1)$. In $R_t$ we have

$$(i) \otimes (1) = (i + 1) \oplus (i, 1) \oplus 2(i) \oplus (i - 1)$$

so that

$$\text{lift}((i) \otimes (1) = (i + 1) \oplus (i, 1) \oplus 4(i) \oplus (i - 1, 1) \oplus 4(i - 1) \oplus (i - 2).$$

After removing the contributions which will lead to $A_{S_{i+1}} \oplus 2A_{S_i} \oplus A_{S_{i-1}}$ we are left with $(i, 1) \oplus (i) \oplus (i - 1, 1) \oplus (i - 1)$. Hence

**15.4 Lemma.** For $i \geq 2$

$$A_{S_i} \otimes A_{S_1} = A_{S_{i+1}} \oplus 2A_{S_i} \oplus A_{S_{i-1}} \oplus R(i, 1).$$

In the general case we add up the contributions $((i) \otimes (i - 1)) \cdot ((j) \otimes (j - 1)) = (i)(j) \oplus (i)(i - 1) \oplus (i - 1)(j) \oplus (i - 1)(j - 1)$. All the summands are of the following type: $(a, 0), (a, b), a > b > 0, (a, a), a > 0$. We have

$$\text{lift}(a, b) = (a, b) \oplus (a, b - 1) \oplus (a - 1, b) \oplus (a - 1, b - 1), \quad a > b > 0$$

$$\text{lift}(a, a) = (a, a) \oplus (a, a - 1) \oplus (a - 1, a - 2) \oplus (a - 2, a - 2).$$

After removing the contributions in $R_t$ which will give the $A_{S_{i+1}} \oplus 2 \cdot A_{S_{i+1}} \oplus A_{S_{i+1}} \oplus A_{S_{i+1}}$ and applying successively the liftings from above we get the following decompositions. We assume $m = n \geq 2, i > j$.

For $i > 2, j = 2$ we get

$$A_{S_i} \otimes A_{S_2} = A_{S_{i+2}} \oplus 2 \cdot A_{S_{i+1}} \oplus A_{S_i} \oplus R(i, 1) \oplus R(i, 1) \oplus R(i - 1, 1)$$

Assume now $i > 2, j \geq 2$ and $i \neq j$ (for $i = j$ see below) and $i > j$. Then

$$A_{S_i} \otimes A_{S_j} = A_{S_{i+j}} \oplus 2 \cdot A_{S_{i+j-1}} \oplus A_{S_{i+j-2}} \oplus R(i, j - 1, 1)$$

$$\oplus R(i + j - 2, 2) \oplus 2 \cdot R(i + j - 2, 1)$$

$$\oplus R(i + j - 3, 3) \oplus 2 \cdot R(i + j - 3, 2) \oplus R(i + j - 3, 1)$$

$$\oplus R(i + j - 4, 4) \oplus 2 \cdot R(i + j - 4, 3) \oplus R(i + j - 4, 2)$$

$$\oplus R(i + j - 5, 5) \oplus 2 \cdot R(i + j - 5, 4) \oplus R(i + j - 5, 3)$$

$$\oplus R(i + j - 6, 6) \oplus \ldots$$

$$\oplus R(i, j) \oplus 2 \cdot R(i, j - 1) \oplus R(i, j - 2)$$

$$\oplus R(i - 1, j - 1).$$
Now assume \( i = j \). For \( i = j = 2 \) we get

\[
\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} = \mathbb{A}_{S^4} \oplus 2 \cdot \mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \\
\oplus R(3,1) \oplus R(2,2) \oplus 2 \cdot R(2,1).
\]

For \( i = j > 2 \) we get

\[
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^i} = \mathbb{A}_{S^{i+1}} \oplus 2 \cdot \mathbb{A}_{S^{i+1-1}} \oplus \mathbb{A}_{S^{i+2-2}} \\
\oplus R(i+j-1,1) \\
\oplus R(i+j-2,2) \oplus 2 \cdot R(i+j-1,1) \\
\oplus R(i+j-3,3) \oplus 2 \cdot R(i+j-3,2) \oplus R(i+j-3,1) \\
\oplus R(i+j-4,4) \oplus 2 \cdot R(i+j-4,3) \oplus R(i+j-4,2) \\
\oplus R(i+j-5,5) \oplus 2 \cdot R(i+j-5,4) \oplus R(i+j-5,3) \\
\oplus R(i+j-6,6) \oplus \ldots \\
\oplus R(i,j) \oplus 2 \cdot R(i,j-1) \oplus R(i,j-2).
\]

We get the same result as for \( i \neq j \) with omitting the last factor \( \oplus R(i+j-\min(i,j)-1,\min(i,j)-1) \).

**Example.** We then obtain the following formulas

\[
\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} = \mathbb{A}_{S^4} \oplus 2 \mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \oplus R(3,1) \oplus R(2,2) \oplus 2 \cdot R(2,1) \\
\mathbb{A}_{S^3} \otimes \mathbb{A}_{S^3} = \mathbb{A}_{S^5} \oplus 2 \mathbb{A}_{S^4} \oplus \mathbb{A}_{S^3} \oplus R(4,1) \oplus R(3,2) \oplus 2 \cdot R(3,1) \oplus R(2,1).
\]

The highest weights appearing in the socle and head of these indecomposable modules are \( [3,0,\ldots,0] \) for \( \lambda = (4,1) \), \( [2,1,0,\ldots,0] \) for \( \lambda = (3,2) \), \( [2,0,\ldots,0] \) for \( \lambda = (3,1) \), \( [0,0,\ldots,0] \) for \( \lambda = (2,2) \) and \( [1,0,\ldots,0] \) for \( \lambda = (2,1) \).

**Remark:** These formulas can be used to compute the tensor product \( S^i \otimes S^j \) in \( \mathcal{R}_n \) as in [HWng]. In the \( \text{Gl}(2|2) \)-case the \( R(a,b) \) are projective covers. The composition factors of these were worked out in [Dro09]. If we compare the two \( K_0 \)-decomposition of \( \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} \) given by our formula above, we can use this to determine the composition factors of \( S^i \otimes S^j \) recursively starting from \( S^1 \otimes 1 \). This will enable us to prove a closed formula for the \( S^i \otimes S^j \) tensor product in \( \mathcal{R}_2 \) and by means of the cohomological tensor functors from [HW14] for general \( n \). It is easy to derive a closed formula for the not maximal atypical part of \( \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} \) as well [HWng].
16. THE TENSOR PRODUCTS $A_{S^i} \otimes A_{\Lambda^j}$

We derive a closed formula for projection on the maximal atypical block of the tensor product $A_{S^i} \otimes A_{\Lambda^j}$. We have

$$\text{lift}((i) \otimes (1^j)) = (i) \otimes (1^j) \oplus (i - 1) \otimes (1^j) \oplus (i) \otimes (1^{j - 1}) \oplus (i - 1) \otimes (1^{j - 1}).$$

in the Grothendieck ring $R_t$. We may assume that $j > 1$ since $A_{S^i} \otimes A_{\Lambda^j} = A_{S^i} \otimes A_{S^i}$. We may also assume that $i \geq j$ since $(A_{S^i} \otimes A_{\Lambda^j})^\vee = A_{\Lambda^j} \otimes A_{S^i}$.

We compute $(i) \otimes (1^j)$ in $R_t$. Recall the classical Pieri rule $(i) \otimes (1^j) = (i + 1, 1^{j - 1}) \oplus (i, 1^j)$.

- $\sum_{\gamma \in P} c^L_{\alpha, \theta} c^R_{\beta, \eta}$: We evaluate this for $\lambda^R = (1^i)$, $\mu^L = (1^j)$. $(\lambda^R)^{-1} = (0, 1^i)$, $(1, 1^{i-1}), \ldots, (1^i, 0)$ and $(\mu^L)^{-1} = (0, 1^j)$, $(1, 1^{j-1}), \ldots, (1^j, 0)$. Pairs with the same $\gamma$ are
  
  $$(0, 1^i) \leftrightarrow (0, 1^j),$$
  $$(1, 1^{i-1}) \leftrightarrow (1, 1^{j-1}),$$
  $$(1^{min(i,j)}, 1^{|i-j|}) \leftrightarrow (1^{min(i,j)}, 1^{|i-j|}).$$

- $\sum_{\kappa \in P} c^L_{\kappa, \alpha} c^R_{\kappa, \beta}$: Here $\mu^R = (j)$, $\lambda^L = (i)$. Here the permitted pairs are the
  $$(0, i) \leftrightarrow (0, j),$$
  $$(1, i - 1) \leftrightarrow (1, j - 1),$$
  $$(min(i,j), (i - |i-j|)) \leftrightarrow (min(i,j), (j - |i-j|)).$$

The big sum collapses to

$$c^L_{i,0} c^R_{0,j} + \ldots + c^L_{i,|i-j|} c^R_{0,j-|i-j|}$$

$$c^L_{0,i} c^R_{1,j} + \ldots + c^L_{0,|i-j|} c^R_{1,j-|i-j|}$$

We have to evaluate $\sum_{\alpha, \beta, \eta, \theta} c^L_{\alpha, \theta} c^R_{\beta, \eta}$. The following values for these for $\alpha, \beta, \eta, \theta$ give non-vanishing coefficients (let $t = min(i,j)$):

- a) $\alpha = i$, $\beta = j$
  - a') $\eta = 1^i$, $\theta = 1^j$
- b) $\alpha = i - 1$, $\beta = j - 1$
  - b') $\eta = 1^{i-1}$, $\theta = 1^{j-1}$
- ... 
- t) $\alpha = i - t$, $\beta = j - t$
  - t) $\eta = 1^{i-t}$, $\theta = 1^{j-t}$.

This gives $(t + 1)^2$ non-vanishing products, namely $aa'$, $ab'$, ..., at, $ba'$, $bb'$, ..., tt. Now we use $(i) \otimes (1^j) = (i + 1, 1^{j-1}) \oplus (i, 1^j)$ in order so see which ones will give maximally atypical $\nu$. 

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Now $\Gamma_{\alpha,\beta,\eta}^{\nu} \cdots = 0$ unless the indices form one of the tuples $aa', ab', \ldots, at', ba', bb', \ldots, tt'$. A bipartition $\nu$ will appear if and only if there exists a tuple such that $c_{\alpha,\beta,\eta}^{\nu} \neq 0$. The classical formula $(i) \otimes (i') = (i + 1, 1^{j-1}) \oplus (i, 1^j)$ tells us that such a $\nu$ is necessarily of the form 

$$\nu = [(n, 1^{n}), (\tilde{n} + 1, 1^{n-1})]$$

for $n, \tilde{n}$ in a suitable range. We have

$$(\nu^L)^{-1} = (n, 1^{\tilde{n}}) \text{ resp. } (n - 1, 1^{\tilde{n}+1})$$

$$(\nu^R)^{-1} = (\tilde{n} + 1, 1^{n-1}) \text{ resp. } (\tilde{n}, 1^{n})$$

Hence a given $\nu$ can be realised in maximally 4 different ways: Through either one of

- i) $\alpha = n, \theta = 1^{\tilde{n}}, \beta = \tilde{n} + 1, \eta = 1^{n-1}$
- ii) $\alpha = n, \theta = 1^{\tilde{n}}, \beta = \tilde{n}, \eta = 1^n$
- iii) $\alpha = n - 1, \theta = 1^{\tilde{n}+1}, \beta = \tilde{n} + 1, \eta = 1^{n-1}$
- iv) $\alpha = n - 1, \theta = 1^{\tilde{n}+1}, \beta = \tilde{n}, \eta = 1^n$

We carry out the summation $\sum_{l=0}^{t} \sum_{k=0}^{t} lk'$. We first treat the partial sum $aa' + ab' + \ldots + at'$. In that case only $aa'$ and $ab'$ give a contribution. $aa'$ yields $(i + 1, 1^{j-1})$ and $(i, 1^j)$ and $ab'$ yields $(i, 1^{j-1})$. Now consider a generic summand $lk', i \neq a, t$. The corresponding product of the Littlewood-Richardson coefficients is

$$c_{i-l,1^{j-k}}^{\nu^L} \cdot c_{j-k,1^{i-1}}^{\nu^R}.$$ 

The possible $\nu^L$ are of the form

$$\nu^L_i = (i - l + 1, 1^{j-k-1}), \quad \nu^L_2 = (i - l, 1^{j-k})$$

and the possible $\nu^R$ are of the form

$$\nu^R_1 = (j - k + 1, 1^{i-l-1}), \quad \nu^R_2 = (j - k, 1^{i-l}).$$

We only consider $\nu$ with $\nu^R = (\nu^L)^*$. We have

$$(\nu^L_1)^* = (j - k, 1^{i-l}).$$

This is equal to one of the two $\nu^R$ for $k = l$ in which case we get $(\nu^L_1)$ and $(\nu^L_2)$ as a contribution. The pair $lk$ will not give any contribution for $k \notin \{l - 1, l, l + 1\}$. For $l = k + 1$ we get the contribution $(\nu^L_1)$ and for $l = k - 1$ we get the contribution $(\nu^L_2)$.

The sum $ta' + \ldots + tt'$ gives the contribution

$$\begin{cases} 
(i - j + 1) \oplus (i - j) & i > j \\
(1^{2-i+1}) \oplus (1^{j-i}) & j > i \\
(1) \oplus (0) & i = j
\end{cases}$$
Hence we obtain the following closed formula:

\[(i) \otimes (1^j) = (i + 1, 1^{j-1}) \oplus (i, 1^j) \oplus (i, 1^{j-1})\]

\[
\oplus \bigoplus_{l=1}^{l-1} \left[ (i - l, 1^{j-l-1}) \oplus (i - l, 1^{j-l}) \oplus (i - l + 1, 1^{j-l-1}) \oplus (i - l + 1, 1^{j-l}) \right]
\]

\[
\begin{cases}
(i - j + 1) \oplus (i - j) & \text{if } i > j \\
(1^{j-i+1}) \oplus (1^{j-i}) & \text{if } j > i \\
(1) \oplus (0) & \text{if } i = j
\end{cases}
\]

We apply this formula to the four summands of \(\text{lift}((i) \otimes (1^j)), (i) \otimes (1^j), (i-1) \otimes (1^j), (i) \otimes (1^{j-1}), (i-1) \otimes (1^{j-1})\). The contributions in the total sum are either of the form \((i)\) or \((1^j)\) or \((i,1^j)\). We have

\[\text{lift}(i, 1^j) = (i, 1^j) \oplus (i - 1, 1^j) \oplus (i, 1^{j-1}) \oplus (i - 1, 1^{j-1}).\]

From the \(G(1|1)\)-case we know that the contribution of the alternating and symmetric powers will be given by \((i > j)\)

\[A_{S^i} \otimes A_{\Lambda^j} = A_{S^{i+j}+2} \oplus 2A_{S^{i+j}+1} \oplus A_{S^{i+j}} \oplus R\]

and by

\[A_{S^i} \otimes A_{\Lambda^j} = A_{S^i+2} \oplus 2A \oplus A_{\Lambda^j} \oplus R\]

for \(i = j\) for some \(R\)-term which does not involve any alternating or symmetric powers. Removing all the corresponding bipartitions from the total sum and working downwards as in the \(A_{S^i} \otimes A_{S^j}\)-case we obtain the final result. For \(i = j = 2\) we obtain:

\[A_{S^2} \otimes A_{\Lambda^2} = A_{S^4+2} \oplus 2A \oplus A_{\Lambda^2} \oplus R(3, 1) \oplus R(2, 1^2) \oplus 2R(2, 1)\]

and for \(i > j = 2\) we obtain

\[A_{S^i} \otimes A_{\Lambda^2} = \]

\[A_{S^i} \oplus 2A_{S^{i-1}} \oplus A_{S^{i-2}} \oplus R(i + 1, 1) \oplus R(i, 1^2) \oplus 2R(i, 1) \oplus R(i - 1, 1)\]

The general formula is for \(i > j > 2\) as follows

\[A_{S^i} \otimes A_{\Lambda^j} = A_{S^{i+j}+2} \oplus 2A_{S^{i+j}+1} \oplus A_{S^{i+j}}\]

\[\oplus R(i + j - (j - 1), 1^{j-1})\]

\[\oplus R(i + j - j, 1^j) \oplus 2R(i, 1^{j-1}) \oplus R(i, 1^{j-2})\]

\[\ldots\]

\[\oplus R(i + j - k, 1^k) \oplus 2 \cdot R(i + j - k, 1^{k-1}) \oplus R(i + j - k, 1^{k-2})\]

\[\oplus \ldots\]

\[\oplus R(i - j + 2, 1^2) \oplus 2 \cdot R(i - j + 2, 1)\]

\[\oplus R(i - j + 1, 1).\]
For $i = j > 2$ one has to remove the last term $R(i - j + 1, 1)$.

**Example.**

\[ A_{S^3} \otimes A_{\Lambda^2} = A_{S^3} \oplus 2A_{S^2} \oplus A_{\Lambda^1} \oplus R(4, 1) \oplus R(3, 1^2) \oplus 2R(3, 1) \oplus R(2, 1) \]

\[ A_{S^3} \otimes A_{\Lambda^3} = A_{S^2} \oplus 2A \oplus A_{\Lambda^2} \oplus R(4, 1^2) \oplus R(3, 1^3) \oplus 2R(3, 1^2) \oplus R(3, 1) \]

\[ \oplus R(2, 1^2) \oplus 2R(2, 1). \]

17. **Remarks on tensor products**

We can embed any positive $[\lambda]$ of length $k$ in the socle of a mixed tensor of defect $k$ or as highest weight constituent. In the $S^i \otimes S^j$-case this permits us to obtain the decomposition of $S^i \otimes S^j$ [HWng]. Copying the approach in the $A_{S^i} \otimes A_{S^j}$-case seems to be hopeless because general $R(\lambda)$ have lots of composition factors which are difficult to determine. We content ourselves with the following observations. The estimate on the composition factors is trivial and could be obtained from a restriction to $Gl(m) \times Gl(m)$.

17.1. **Composition factors.** As before we consider only bipartitions of the form $(\nu, \nu^*)$ and we identify such a bipartition with the partition $\nu$.

17.1 **Lemma.** $\Gamma_{\lambda \mu}^\nu$ is zero unless $l(\nu) \leq l(\lambda) + l(\mu)$.

**Proof.** The Littlewood-Richardson coefficients $c_{\lambda \mu}^\nu$ are zero unless $l(\nu) \leq l(\lambda) + l(\mu)$ and $l(\nu) \geq \max(l(\lambda), l(\mu))$. In the sum

\[ \sum_{\kappa \in \mathcal{P}} c_{\kappa \lambda}^\nu c_{\kappa \beta}^\mu \]

$c_{\kappa \alpha}^\lambda = 0$ unless $l(\alpha) \leq l(\lambda^L)$. Similarly $c_{\gamma \theta}^\mu = 0$ unless $l(\theta) \leq l(\mu^L)$. Hence any $\nu^L$ with non-vanishing $c_{\alpha \beta}^{\mu L}$ satisfies

\[ l(\nu^L) \leq l(\alpha) + l(\theta) \leq l(\lambda^L) + l(\mu^L). \]

\[ \square \]

17.2 **Lemma.** If $\Gamma_{\lambda \mu}^\nu \neq 0$, then $\nu_1 \leq (\lambda + \mu)_1$.

**Proof.** Follows at once from the corresponding property of the $c_{\lambda \mu}^\nu$. \[ \square \]

17.3 **Lemma.** If $c_{\lambda \mu}^\nu \neq 0$, then $\Gamma_{\lambda \mu}^\nu = (c_{\lambda \mu}^\nu)^2$. These $\nu$ are exactly the $\nu$ with degree $\deg(\lambda) + \deg(\mu)$. If $\nu$ is any other partition with $\Gamma_{\lambda \mu}^\nu \neq 0$, then $\deg(\nu) < \deg(\lambda) + \deg(\mu)$. 

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**Lemma 17.5.**

If $R$ is a mixed tensor, we note that in general a classical solution to degree decreasing, none of the partitions obtained from maximal degree cannot occur in any other tensor product if and only if 

$$\nu = \Gamma_{\lambda,\mu}.$$ 

Proof. We get $\Gamma_{\lambda,\mu} = (c_{\lambda,\mu}^\kappa)^2$ by putting $\kappa = 0$ and $\gamma = 0$ in the expression for $\Gamma_{\lambda,\mu}$. If $\kappa = 0$, then $\alpha = \lambda^L$ and $\beta = \mu^R$. If $\gamma = 0$, then $\eta = \lambda^R$ and $\theta = \mu^L$. Then $\Gamma_{\lambda,\mu} = c_{\lambda,\mu}^\kappa c_{\lambda,\mu}^{\kappa'}$. Since $c_{\lambda,\mu}^\kappa = c_{\lambda,\mu}^{\kappa'}$, we get $\Gamma_{\lambda,\mu} = (c_{\lambda,\mu}^\kappa)^2$ if we only consider maximally atypical contributions $\nu = (\nu, \nu^*)$. In general $c_{\lambda,\mu}^\kappa \neq 0$ implies $\deg(\lambda) + \deg(\mu) = \deg(\mu)$, hence

$$\deg(\nu^L) = \deg(\lambda) + \deg(\mu) - \deg(\gamma)$$

and for non-trivial $\kappa$ or $\gamma$ the partition $\nu$ cannot satisfy $c_{\lambda,\mu}^\kappa \neq 0$. \hfill \Box

We will call any $\nu$ with $c_{\lambda,\mu}^\kappa \neq 0$ a classical solution of $\Gamma_{\lambda,\mu}$.

**Lemma 17.6.**

**Lemma.** In $R_0$

$$R(\lambda) \otimes R(\mu) = \bigoplus_{\deg(\nu) = \deg(\lambda) + \deg(\mu)} (c_{\lambda,\mu}^\kappa)^2 R(\nu) \oplus \bigoplus_{\deg(\nu) < \deg(\lambda) + \deg(\mu)} R(\nu).$$

Proof. To calculate the tensor product in $R_t$ we have to compute $lift(\lambda) \otimes lift(\mu)$ in $R_t$. Now $lift(\lambda) = \lambda + \sum_i \lambda^i$ with partitions $\lambda^i$ of degree strictly smaller then the degree of $\lambda$. Likewise for $\mu$. Hence the partitions $\nu$ of maximal degree cannot occur in any other tensor product from the partitions obtained from $lift(\lambda)$ respectively $lift(\mu)$ other then $\lambda \otimes \mu$. To pass from $R_t$ to $R_0$ we have to take $lift^{-1}$ of the tensor product. Since the lift is strictly degree decreasing, none of the partitions $\nu$ can occur in in the lift of another partition. \hfill \Box

Note that in general a classical solution $\nu$ will not be $(n, n)$-cross. Hence in $R_n$ the sum above only incorporates $\nu$ which are $(n, n)$-cross. However the mixed tensor $R(\lambda + \mu)$ occurs always in the decomposition $R(\lambda) \otimes R(\mu)$ in $R_n$ due to following lemma.

**Lemma 17.5.** If $l(\lambda) \leq n$, then $\lambda$ is $(n, n)$-cross.

Proof. A bipartition $\lambda$ is $(n, n)$-cross if and only if at least one of inequalities $\lambda_i + \lambda^*_n + 2 - i \leq n$ for $i = 1, \ldots, n + 1$ is satisfied. If $l(\lambda) \leq n$, then $\lambda^*_i \leq n$, hence $\lambda_{n+1} + \lambda^*_1 \leq n$. \hfill \Box

**Lemma 17.6.** Let $\nu$ be a classical solution of length $\leq n$ in $R(\lambda) \otimes R(\mu)$. Let $[\nu']$ be a constituent in $R(\lambda) \otimes R(\mu)$. Then $\deg[\nu'] \leq \deg A_{\nu}$ with equality if and only if $[\nu'] = A_{\nu^*}$ with $\nu^*$ a classical solution of length $\leq n$.

Proof. This follows from the degree estimates in section 12. \hfill \Box

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17.7 Proposition. Assume $\nu$ is a classical solution with $l(\nu) \leq n$. Then $[\lambda] \otimes [\mu]$ contains the composition factor $[\nu]$ with multiplicity $(c_{\lambda\mu}^\nu)^2$.

Proof. We know that $[\lambda]$ and $[\mu]$ are the constituents of highest weight in $R(\lambda)$ and $R(\mu)$. Let

$$[R(\lambda)] = \sum_i [\lambda^i] + [\lambda]$$

$$[R(\mu)] = \sum_j [\mu^j] + [\mu]$$

with $[\lambda] > [\lambda^i]$ and $[\mu] > [\mu^j]$ for all $i, j$ in the Bruhat order. Assume first that all $[\lambda^i]$ and $[\mu^j]$ are positive. Then they define the mixed tensors $R(\lambda^i)$ and $R(\mu^j)$ and $\text{deg}(\lambda) > \text{deg}(\lambda^i)$ and $\text{deg}(\mu) > \text{deg}(\mu^j)$ for all $i, j$. Accordingly none of the mixed tensors $R(\nu)$ with $\text{deg}(\nu) = \text{deg}(\lambda) + \text{deg}(\mu)$ can appear in a tensor product

$$R(\lambda) \otimes R(\mu), R(\lambda^i) \otimes R(\mu), R(\lambda) \otimes R(\mu^j).$$

Now

$$[R(\lambda) \otimes R(\mu)] = [\lambda] \otimes [\mu]$$

$$+ \sum_j [\lambda] \otimes [\mu^j] + \sum_i [\lambda^i] \otimes [\mu] + \sum_{i,j} [\lambda^i] \otimes [\mu^j]$$

and similarly for $R(\lambda) \otimes R(\mu^j)$, $R(\lambda^i) \otimes R(\mu)$ and $R(\lambda^i) \otimes R(\mu^j)$. We claim: Since the $R(\nu)$ do not appear in any of these tensor products, their constituent of highest weights does not appear as a composition factor in any of these tensor products. If this claim is true, none of the tensor products $[\lambda] \otimes [\mu^j]$, $[\lambda^i] \otimes [\mu]$ and $[\lambda^i] \otimes [\mu^j]$ can contain $[\nu]$ as a composition factor, hence $[\nu]$ must be a composition factor of $[\lambda] \otimes [\mu]$. For the proof of the claim we distinguish two cases. Since $l(\nu) \leq n$, $[\nu]$ is positive. Consider first a summand $R(\theta)$ with $l(\theta) \leq n$. Then $[\theta]$ is positive and $\sum \theta_i < \sum \nu_i$. Since $[\theta]$ is the constituent of highest weight of $R(\theta)$, all constituents are smaller then $[\nu]$. If $R(\theta)$ is a summand with $l(\theta) > n$, $\text{deg} A_{\theta} < \text{deg}[\nu]$. This proofs the claim. Finally we remove the assumption that all $[\lambda^i]$ and $[\mu^j]$ are positive. If $[\lambda^i]$ is not positive we calibrate it with a twist with $Ber^{-\lambda^i}$. Similarly for $[\mu^j]$. Call these modules $[\tilde{\lambda}]$ respectively $[\tilde{\mu}]$. Then $\text{deg}(\tilde{\lambda}) = \text{deg}(\lambda^i) + n(-1)\lambda_i^\theta$ and $\text{deg}(\tilde{\mu}) = \text{deg}(\mu^j) + n(-1)\mu_i^\theta$. Embed the modules $[\tilde{\lambda}]$ respectively $[\tilde{\mu}]$ as constituents of highest weight in $R(\tilde{\lambda})$ respectively $R(\tilde{\mu})$ as in 12.2. In $R(\tilde{\lambda}) \otimes R(\tilde{\mu}) = \bigoplus R(\tilde{\nu})$ all constituents $[\nu]$
have degree
\[
\deg [\nu] \leq \deg (\tilde{\lambda}) + \deg (\tilde{\mu})
\]
\[
= \deg (\lambda^i) + n(-1)\lambda_n^i + \deg (\mu^j) + n(-1)\mu_n^j
\]
\[
< \deg (\lambda) + n(-1)\lambda_n^i + \deg (\mu) + n(-1)\mu_n^j
\]
\[
= \deg (\nu) + n(-1)\lambda_n^i + n(-1)\mu_n^j.
\]
Since \([\lambda^i] \otimes [\mu^j] = B^{-\lambda_n^i} \otimes B^{-\mu_n^j}([\tilde{\lambda}] \otimes [\tilde{\mu}])\), every constituent in \([\lambda^i] \otimes [\mu^j]\) has degree \(\leq \deg (\lambda^i) + \deg (\mu^j) < \deg (\lambda) + \deg (\mu)\).

\begin{example}
S^i \otimes S^j in \mathcal{R}_2 with i > j: In this case
\[
\mathcal{A}_{S^i} \otimes \mathcal{A}_{S^j} = \mathcal{A}_{S^{i+j}} + R(i + j - 1, 1) \oplus R(i + j - 2, 2) \oplus \ldots \oplus R(i, j) \oplus \tilde{R}
\]
where \(\tilde{R}\) represents the summands with \(\deg < \deg (\mathcal{A}_{S^i}) + \deg (\mathcal{A}_{S^j}) = i + j\). All the classical solutions have length \(\leq 2\), hence their highest weights occur in the \(S^i \otimes S^j\) tensor product. The highest weights are
\[
S^{i+j}, B S^{i+j-2}, \ldots, B^3 S^i.
\]
In fact [HWng] these constituents give half of the constituents in the middle Loewy layer of \(M = S^i \otimes S^j\), the other half given by their twists with \(B^{-1}\):
\[
B^{-1}(S^{i+j} + B S^{i+j-2} + \ldots + B^3 S^i).
\]
\end{example}

17.2. Projective covers.

17.8 Lemma. If \(P\) is a projective cover occurring as a direct summand in the decomposition \([\lambda^i] \otimes [\mu^j]\) with multiplicity \(k\), then \(R(\lambda) \otimes R(\mu)\) contains \(P\) as a direct summand with multiplicity at least \(k\).

\begin{proof}
We embed \([\lambda^i]\) and \([\mu^j]\) as the socles of the mixed tensors \(R(\lambda)\) and \(R(\mu)\). Projection of these modules on the top gives
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \ker (\varphi) & \longrightarrow & R(\lambda) & \varphi & [\lambda^i] & \longrightarrow & 0 \\
0 & \longrightarrow & \ker (\psi) & \longrightarrow & R(\mu) & \psi & [\mu^j] & \longrightarrow & 0
\end{array}
\]
This gives the surjection \(R(\lambda) \otimes R(\mu) \rightarrow [\lambda^i] \otimes [\mu^j]\). If \([\lambda^i] \otimes [\mu^j] = \bigoplus M_i \oplus \bigoplus P_i\) we get a surjection \(R(\lambda) \otimes R(\mu) \rightarrow \bigoplus P_i\). Since the \(P_i\) are projective this surjection has to split and hence the \(\bigoplus P_i\) are direct summands in \(R(\lambda) \otimes R(\mu)\).

This result implies that some tensor products \([\lambda^i] \otimes [\mu^j]\) do not have maximally atypical projective summands. Indeed if \(\deg (\lambda) + \deg (\mu) < n(n + 1)/2\) or \(l(\lambda) + l(\mu) < n\), no projective cover can occur in \(R(\lambda) \otimes R(\mu)\) since the
smallest degree of a partition defining a projective cover is \( n(n+1)/2 \) and the smallest length is \(< n \) (the minimal projective cover is \( R(n, n-1, \ldots, 1) = P[n-1, n-2, \ldots, 1, 0] \)). More generally the Loewy length of any subquotient of a module \( M \) is smaller or equal to the one of \( M \), hence we have

\[
ll(R(\lambda) \otimes R(\mu)) \geq ll(L(\lambda^\dagger \otimes L(\mu^\dagger)).
\]

**Example:** \( S^i \otimes S^j \) does not contain any atypical projective summands. Indeed for \( n = 2 \) this follows from [HWng]. For \( n \geq 3 \) none of the mixed tensors in the decomposition of \( A_{S_{i+1}} \otimes A_{S_{j+1}} \) is projective.

**Example:** If \( l(\lambda) + l(\mu) < n \), \( [\lambda] \otimes [\mu] \) does not have a projective summand.

**17.9 Lemma.** Suppose \( \nu \) is a classical solution and \( l(\nu) \leq n \). If \([\nu]\) is a composition factor in an indecomposable projective module \( P = R(\theta) \), then \([\nu] = A_\theta\).

**Proof.** By definition \((A_\theta)_i \geq \nu_i\) for all \( i \in \{1, \ldots, n\} \). In particular \([\theta]\) is positive. Hence \( l(\theta) \leq n \). Hence

\[
\text{deg}(\theta) = \sum_{i=1}^{n}(\theta_{\text{max}})_i.
\]

Since \( P = R(\theta) \) is a summand in \( R(\lambda) \otimes R(\mu) \), \( \text{deg}(\theta) \leq \text{deg}(\nu) \). But if \([\nu] \neq A_\theta\), then \( \sum_{i=1}^{n}(A_\theta)_i > \sum_{i=1}^{n}\nu_i \), a contradiction. \( \square \)

**17.10 Corollary.** If \([\nu]\) is a classical solution of length \( \leq n \) and \( d(\nu) < n \), \([\nu]\) is not a composition factor of a projective module \( P \).

**Proof.** By the last lemma we have \( P = R(\nu) \). But \( R(\nu) \) is projective if and only if \( d(\nu) < n \). \( \square \)

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