Mutual Chern-Simons theory for $Z_2$ topological order

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We study several different $Z_2$ topological ordered states in frustrated spin systems. The effective theories for those different $Z_2$ topological orders all have the same form – a $Z_2$ gauge theory which can also be written as a mutual $U(1) \times U(1)$ Chern-Simons theory. However, we find that the different $Z_2$ topological orders are reflected in different projective realizations of lattice symmetry in the same effective mutual Chern-Simons theory. This result is obtained by comparing the ground-state degeneracy, the ground-state quantum numbers, the gapless edge state, and the projective symmetry group of quasi-particles calculated from the slave-particle theory and from the effective mutual Chern-Simons theories. Our study reveals intricate relations between topological order and symmetry.

Keywords: topological order, mutual Chern-Simons theory, spin liquid

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I. INTRODUCTION

After the discovery of fractional quantum Hall effect,$^1$ we realized that new kind of orders beyond Landau’s symmetry breaking paradigm is possible. This new kind order is called topological order$^{2,3}$ for gapped states and quantum order$^4$ for general states. The new orders reflect patterns of long range entanglements in the ground state.

Gapped $Z_2$ spin liquids have the simplest kind of topological order – $Z_2$ topological order.$^5,6$ Those topological ordered states may appear in frustrated spin systems or dimmer models.$^5–11$ Physically, the topological orders can be (partially) characterized by robust ground-state degeneracy.$^6,12$ The low energy effective theory for those $Z_2$ topologically ordered states is a $Z_2$ gauge theory.

Topological order is a property of a many-body ground state that is robust against any perturbations, even those perturbations that break all the symmetries. In this paper, we like to study the interplay between topological order and symmetry. We like to find out how to characterize topological ordered states that also have certain symmetries.

Recently, it was found that for spin liquids with all the lattice symmetries (such as lattice translation and rotation symmetry), there can be hundreds different $Z_2$ topological orders.$^4,9$ We call those topological orders symmetric topological orders. It is shown that the different symmetric $Z_2$ topological orders can be characterized by different project symmetry groups (PSG). So those symmetric topological orders are good examples to study the relation between topological order and symmetry.

Here, we would like to study the low energy effective theories for those different $Z_2$ topological orders and ask how different symmetric $Z_2$ topological orders are reflected in low energy effective theories. We find that all different $Z_2$ topological orders can be described by the same effective mutual $U(1) \times U(1)$ Chern-Simons (CS) theories.$^{13}$ The lattice symmetry is realized projectively in the effective mutual CS theories. It turns out that different symmetric $Z_2$ topological orders have different projective realizations of the lattice symmetries. To confirm our results, the projective construction (the slave-particle theory)$^{14,15}$ is used to calculate the ground-state degeneracies, the ground-state quantum numbers, and the PSGs of quasi-particles. Those results agree with those obtained from the effective mutual CS theories.

Furthermore, we also used the effective mutual CS theories to study gapless edge states for some $Z_2$ topologically ordered states.

II. PROJECTIVE CONSTRUCTION OF MANY-SPIN WAVE FUNCTIONS

The key to understand topological orders is to construct states that can have long range quantum entanglements. The projective construction introduced in the study of high $T_c$ superconductors is a powerful way to construct such states.$^{14–17}$ In this section, we will briefly review the projective construction of $Z_2$ topologically ordered states.

A spin-1/2 model can be viewed as a hard-core-boson model, if we identify $| \downarrow \rangle$ state as a zero-boson state $|0 \rangle$ and $| \uparrow \rangle$ state as a one-boson state $|1 \rangle$. In the follow we will use the boson-picture to describe our model.

We first introduce a “mean-field” fermion Hamiltonian:$^4$

$$H_{\text{mean}} = \sum_{\langle i,j \rangle} \left( \psi_{i,J}^\dagger u_{ij}^I \psi_{J,j}^I + \psi_{i,J}^\dagger \eta_{ij}^I \psi_{J,j}^I + \text{h.c.} \right) \quad (1)$$

where $I,J=1,2$. We will use $u_{ij}$ and $\eta_{ij}$ to denote the $2 \times 2$ complex matrices whose elements are $u_{ij}^I$ and $\eta_{ij}^I$. Let $| \Psi_{\text{mean}}^{(\eta_{ij},\eta_{ij})} \rangle$ be the ground state of the above free fermion Hamiltonian (ie the lowest energy state obtained.
by filling all the negative energy levels). Then a manyboson wave function can be obtained through

$$
\Phi_{\text{spin}}^{(u_{ij}, \eta_{ij})}(i_1, i_2, \ldots) = \langle 0 | \prod_{n=1}^{N_{\text{site}}/2} b(i_n) | \Phi_{\text{mean}}^{(u_{ij}, \eta_{ij})} \rangle
$$

where $N_{\text{site}}$ is the number of lattice sites,

$$
b(i) = \psi_{1,i} \psi_{2,i}
$$

and $i_1, i_2, \cdots$, label the location of bosons (up-spins). Here, we have assumed that there are $N_{\text{site}}/2$ up-spins and $N_{\text{site}}/2$ down-spins.

We may view $(u_{ij}, \eta_{ij})$ as variational parameters and the physical spin wave function $\Phi_{\text{spin}}^{(u_{ij}, \eta_{ij})}(i_1, i_2, \ldots)$ as a trial wave function. The trial ground state of a spin Hamiltonian can be obtained by minimizing the average energy $\langle H \rangle$.

First let us consider the following spin Hamiltonian

$$
H_{\text{exact}} = g \sum_{i} \tilde{F}_i, \quad \tilde{F}_i = \sigma^y_{i+\hat{x}} \sigma^x_{i+\hat{y}} + \sigma^y_{i+\hat{x}} \sigma^x_{i+\hat{y}}
$$

where $\sigma^x, y, z$ are the Pauli matrices and $i = (i_x, i_y)$ labels the site of a square lattice. We find that if we choose the variational parameters to be

$$
-\eta_{i, i+\hat{x}} = u_{i, i+\hat{x}} = 1 + \tau^2,
-\eta_{i, i+\hat{y}} = u_{i, i+\hat{y}} = 1 - \tau^2,
$$

then the spin wave function Eq. (2) minimize the average energy. In fact the wave function is the exact ground state of Hamiltonian $H_{\text{exact}}$. It was found that all the excitations above the ground state are gapped and the ground state contains a non-trivial topological order described by a $Z_2$ effective gauge theory. We will call such a state $Z2E$ state.

Ref. 6 introduced another many-spin state on square lattice which is described by

$$
u_{i,i+\hat{x}} = u_{i,i+\hat{y}} = \chi \tau^3,

\eta_{i,i+\hat{x}} = \eta_{i,i+\hat{y}} = \eta \tau^2,
u_{i,i} = \eta \tau^1.
$$

and $\eta_{ij} = 0$. However, it is not clear what kind of spin Hamiltonian gives rise to the spin state described by the above variational parameters. Despite of this, some physical properties of the spin state were obtained under the assumptions that the state is stable for a certain local spin Hamiltonian. Again, all excitations above the spin state have finite energy gaps. The spin state is a spin liquid with no spin order. But it contains a non-trivial topological order described by an effective $Z_2$ gauge theory. So we will call such a spin state $Z2A$ state.

Naively, one may expect the Z2A and the Z2E states to be the same state since both have $Z_2$ gauge theory as their low energy effective theory. In the following, we will show that they are different quantum states with different topological orders.

FIG. 1: The links crossing the $x$ line and the $y$ line get an additional minus sign.

### III. GROUND STATE DEGENERACY

One way to study a topological order is to study its ground state degeneracy on a torus. Naively, we expect the Z2A and the Z2E state to have 4 degenerate ground states, as implied by the effective $Z_2$ gauge theory. The argument goes as the following.

First, we note that the physical boson wave function $\Phi^{(u_{ij}, \eta_{ij})}(\{(i_n)\})$ is invariant under the following $SU(2)$ gauge transformations

$$
(\psi_{i, u_{ij}, \eta_{ij}}) \rightarrow (G_{i}\psi_{i, G_{ij}u_{ij}} G_{ij}^{-1}, G_{ij}^{-1} G_{ij}^{-1} T)
$$

where $G_{i} \in SU(2)$. So the average energy $E(u_{ij}, \eta_{ij}) = \langle \Phi^{(u_{ij}, \eta_{ij})} | H | \Phi^{(u_{ij}, \eta_{ij})} \rangle$ satisfies

$$
E(u_{ij}, \eta_{ij}) = E(G_{ij} u_{ij} G_{ij}^{-1}, G_{ij}^{-1} G_{ij}^{-1} T).
$$

Next we assume $(\tilde{u}_{ij}, \tilde{\eta}_{ij})$ give rise to a (variational) ground state of a Hamiltonian. We would like to show that the following four ansatz

$$
u_{ij}^{(m,n)} = (-)^{\nu s_{ij}(ij)}(-)^{\nu s_{ij}(ij)} \tilde{u}_{ij},

\eta_{ij}^{(m,n)} = (-)^{\nu s_{ij}(ij)}(-)^{\nu s_{ij}(ij)} \tilde{\eta}_{ij}
$$

produce four degenerate ground states. Here $m, n = 0, 1$. $s_{ij}(ij)$ and $s_{ij}(ij)$ have values 0 or 1. $s_{ij}(ij) = 1$ if the link $ij$ crosses the $x$ line (see Fig. 1) and $s_{ij}(ij) = 0$ otherwise. Similarly, $s_{ij}(ij) = 0$ if the link $ij$ crosses the $y$ line and $s_{ij}(ij) = 0$ otherwise. Physically, the degenerate states arise from adding $\pi$ flux through the two holes of the torus. The values of $m, n = 0, 1$ reflect the presence or the absence of the $\pi$ flux in the two holes.

We note that $(\tilde{u}_{ij}, \tilde{\eta}_{ij})$ represents the ground state. We also note that $(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})$ with different $m$ and $n$ are locally gauge equivalent. This is because, on an infinite system, the change, say, $u_{ij} \rightarrow (-)^{m s_{ij}(ij)}(-)^{n s_{ij}(ij)} u_{ij}$ can be generated by an $SU(2)$ gauge transformation $u_{ij} \rightarrow W_{ij} u_{ij} W_{ij}^{-1}$, where $W_{i} = (-)^{m \Theta(i)}(-)^{n \Theta(i)}$, and $\Theta(n) = 1$ if $n > 0$ and $\Theta(n) = 0$ if $n \leq 0$. As a result, $E(\tilde{u}_{ij}, \tilde{\eta}_{ij}) = E(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})$. On
the other hand, on a torus, \((u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})\) with different \(m\) and \(n\) are not gauge equivalent in the global sense. There is no \(SU(2)\) gauge transformation defined on the torus that connects those ansatz. So the four ansatz give rise to four different degenerate states. This is how we obtain the four-fold ground state degeneracy for the \(Z_2\) states.

However, the above argument is valid only for even by even lattice. For odd by odd lattice, the argument breaks down. To understand the failure of the above argument, let us construct the mean-field ground state more carefully.

Let us start with a simple case of the \(Z_2\) state. For the ansatz Eq. (6), the “mean-field” Hamiltonian in momentum space becomes

\[
H_{\text{mean}}(\mathbf{k}) = \sum_{\mathbf{k}} \langle \psi^\dagger_{1\mathbf{k}} \psi^\dagger_{2\mathbf{k}} \rangle M \left( \frac{\psi^\dagger_{1\mathbf{k}}}{\psi^\dagger_{2\mathbf{k}}} \right) = \sum_{\mathbf{k}} \varepsilon(\mathbf{k}) \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} - \sum_{\mathbf{k}} \varepsilon(\mathbf{k}) \beta_{\mathbf{k}}^\dagger \beta_{\mathbf{k}}
\]

where

\[
M = \frac{2}{\eta^3} \left( \eta \cos(k_x + k_y) + 2\eta \cos(k_x - k_y) + v \right) + \frac{2}{\lambda^3} \left( \lambda \cos(k_x + k_y) - 2 \lambda \cos(k_x - k_y) \right)^2,
\]

and

\[
\varepsilon(\mathbf{k}) = \sqrt{4\eta^2 \cos^2(k_x + k_y) + 2\eta \cos(k_x + k_y) + 2\eta \cos(k_x - k_y) + v^2 + 2\lambda \cos(k_x + k_y) - 2\lambda \cos(k_x - k_y)}^2.
\]

Here \(\alpha_{\mathbf{k}}\) and \(\beta_{\mathbf{k}}\) are diagonalized quasiparticles operators

\[
\alpha_{\mathbf{k}} = a \psi^\dagger_{1\mathbf{k}} + b \psi^\dagger_{2\mathbf{k}} / \sqrt{1 + a^2},
\]

\[
\beta_{\mathbf{k}} = b \psi^\dagger_{1\mathbf{k}} + a \psi^\dagger_{2\mathbf{k}} / \sqrt{1 + b^2},
\]

where \(a\) and \(b\) are the functions of \(k_x\) and \(k_y\). The mean-field ground state is obtained by filling all the negative levels and is given by

\[
|\Psi_{\text{mean}}\rangle = \prod_{\mathbf{k}} \beta_{\mathbf{k}}^\dagger |0\rangle_{\psi}.
\]

where the state \(|0\rangle_{\psi}\) is defined through \(\psi_{\mathbf{k}} |0\rangle_{\psi} = 0\). (Note that all the particles \(\alpha_{\mathbf{k}}\) has positive energy and all the particles \(\beta_{\mathbf{k}}\) has negative energy.) Since \(\beta_{\mathbf{k}}^\dagger\) is linear combination of \(\psi^\dagger_{1\mathbf{k}}\) and \(\psi^\dagger_{2\mathbf{k}}\) and there are \(L_x \times L_y\) different \(\mathbf{k}\)-levels, the mean-field state \(|\Psi_{\text{mean}}\rangle\) contains \(L_x \times L_y\) number of fermions. Here \(L_x, L_y\) are sizes of the lattice in the \(x\)- and \(y\)-directions.

Clearly, when both \(L_x\) and \(L_y\) are odd, \(|\Psi_{\text{mean}}\rangle\) contains an odd number of fermions. Such a mean-field state does not correspond to any physical spin state since the corresponding spin wave function Eq. (2) vanishes. (Note that Eq. (2) is a projection to the subspace with 0 or 2 fermions per site.) To get a non-zero physical spin wave function we need to start with a mean-field state with one extra fermion in the empty \(\alpha\)-band (or a hole in the filled \(\beta\)-band). But by choosing different states for the extra fermion (or the hole), we can obtain many different spin wave functions which are nearly degenerate. So when both \(L_x\) and \(L_y\) are odd, the excitations in the \(Z_2\) state are gapless, or we may say that the \(Z_2\) state has infinite degeneracy. Physically, the \(Z_2\) state on odd
TABLE I: Crystal momenta \((K_x, K_y)\) of the four ground states, \((m, n) = (0, 0), (0, 1), (1, 0), (1, 1)\) of the Z2A spin liquid on three different lattices, \((L_x, L_y) = (\text{even, even}), (\text{even, odd}), (\text{odd, odd}), (\text{odd, even})\). The crystal momentum is given by

\[
\begin{align*}
K_x &= \frac{L_x}{2} \left( n_x + \frac{1}{2} \right), \\
K_y &= \frac{L_y}{2} \left( n_y + \frac{1}{2} \right)
\end{align*}
\]

The results are summarized in the table I.

IV. TOPOLOGICAL PROPERTIES FOR THE EXACT SOLUBLE MODEL

To understand the topological order in the Z2E state of the exact soluble model, we would like to calculate the ground state degeneracy and ground state crystal momenta of the Z2E state. Just like the Z2A state discussed in the last section, one can construct many-spin wave functions of the degenerate ground states from the mean-field ansatz Eq. (8) with \((u_{ij}, \eta_{ij})\) given by Eq. (5). The four mean-field ansatz \((u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})\) can potentially give rise to four degenerate ground states. But some time, the mean-field ground state contains odd numbers of fermions. In this case, the corresponding mean-field ansatz does not lead to physical spin wave function.

To calculate the fermion number in the mean-field ground state, one can write down the “mean-field” fermion Hamiltonian in momentum space

\[
H_{\text{mean}}(k) = \sum_{k>0} \left( \psi_{1k}^{\dagger} \psi_{1\bar{k}} + \psi_{2k}^{\dagger} \psi_{2\bar{k}} \right) \left( \begin{array}{c} \cos k_x - i \sin k_x \\ \sin k_x \cos k_x \end{array} \right) \left( \begin{array}{c} \psi_{1k} \\ \psi_{2k} \end{array} \right) \]

with

\[
\begin{align*}
\alpha_k &= \exp(-ik_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} \psi_{1k} \\ \psi_{1\bar{k}} \end{pmatrix} \\
\alpha_k^{-1} &= \exp(-ik_y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} \psi_{2k} \\ \psi_{2\bar{k}} \end{pmatrix}
\end{align*}
\]

Here \(k = 0\) means that \((k_x, k_y) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\), and \(k > 0\) means that \(k_x > 0\) or \(k_y > 0\), \(k_x > 0\) and \(k \neq 0\).

We note that both \(\alpha\) band and \(\beta\) band have a positive energy \(E_k = 1\). \(\alpha_{\pm k}, \beta_{\pm k}\) will annihilate the mean-field ground state \(|\Psi_{\text{mean}}\rangle\),

\[
\alpha_{\pm k} |\Psi_{\text{mean}}\rangle = 0, \quad \beta_{\pm k} |\Psi_{\text{mean}}\rangle = 0.
\]
\[
\begin{array}{c|cccc}
(K_x, K_y) & (ee) & (eo) & (oe) & (oo) \\
\hline
(00) & \pi, \pi & - & - & - \\
(01) & (0, 0) & (0, 0) & (0, 0) & (0, 0) \\
(10) & (0, 0) & (0, 0) & - & - \\
(11) & (0, 0) & (0, 0) & (0, 0) & - \\
\end{array}
\]

TABLE II: Crystal momenta of the degenerate ground states, \((m, n) = (0, 0), (0, 1), (1, 0), (1, 1)\), of the Z2E spin liquid on four different lattices, \((L_x, L_y) = (\text{even,even}), (\text{even,odd}), (\text{odd,even}), (\text{odd,odd})\).

We also note that, for \(k \neq 0\),

\[
\alpha_k = u_k \psi_{1,k} + v_k \psi_{1,-k}^\dagger,
\]

\[
\alpha_{-k} = -v_k^* \psi_{1,k} + u_k^* \psi_{1,-k}^\dagger.
\]

The condition \(\alpha_k \Phi_{\text{mean}} = \alpha_{-k} \Phi_{\text{mean}}\) = 0 implies that (if we only consider the \(k\) and \(-k\) levels)

\[
|\Phi_{\text{mean}}\rangle = (u_k + u_k^* \psi_{1,-k} \psi_{1,-k})|0\rangle
\]

We see that \(k \neq 0\) levels always contribute even numbers of fermions. Also, since \(v_k + u_k \psi_{1,-k}^\dagger \psi_{1,-k}\) carries 0 momentum, we see that the contribution to the total momentum from the \(k \neq 0\) levels is zero.

Thus to determine if the mean-field ground state contains even or odd number of \(\psi\) fermions, we only need to examine the occupation on the four \(k = 0\) momentum points: \(k = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\). The Hamiltonian on those four points is contained in Eq. (10).

All the negative energy levels are filled in the mean-field ground state. On an even by even lattice and for the \((m, n) = (0, 0)\) ansatz, all the momenta \((\pi, 0), (0, \pi), (\pi, \pi)\) are allowed. Thus the \((0, 0)\) level and the \((\pi, \pi)\) level each is occupied by a \(\psi_1\) fermion, and the \((0, \pi)\) level and the \((\pi, 0)\) level each is occupied by a \(\psi_2\) fermion. The total momentum of the ground state is \((\pi, \pi)\). Such a mean-field ground state has even numbers of fermions. It will survive the projection and lead to a physical spin ground state. Other situations can be calculated in the same way. Here we only summarize the result: on an even by even lattice, there exist four different degenerate ground states. However, on other kinds of lattice (even by odd, odd by even and odd by odd), there exist only two different ground states. The other two states are projected out since the mean-field ground states contain odd numbers of fermions. The crystal momenta of the degenerate ground states can also be calculated which are summarized in table II.

V. THE MUTUAL U(1)×U(1) CS THEORY

In the above sections we have calculated the topological properties for the Z2A and the Z2E states. Due to their different topological properties, we find that the two states have different topological orders. Then an important issue is to find the low energy effective theories that describe the two different topological orders. We find that a mutual U(1)×U(1) CS theory with different projective realizations of the lattice symmetry can describe the two kind of topological orders. We reach the conclusion by comparing the topological properties of the mutual U(1)×U(1) CS theory with those of the Z2A and the Z2E states. All the topological properties, including topological degeneracy, quantum numbers, and edge states, agree, indicating the equivalence between the \(Z_2\) topological states on lattice and the mutual U(1)×U(1) CS theory.

1. Mutual U(1)×U(1) CS theory

First we introduce the Lagrangian for the mutual U(1)×U(1) CS theory:

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4e_a^2} (f_{\mu\nu})^2 - \frac{1}{4e_A^2} (F_{\mu\nu})^2 + \frac{1}{\pi} \epsilon^{\mu\nu\lambda\rho} A_\mu \partial_\nu A_\lambda + i \epsilon^{\mu\nu} j_\mu + i A_\mu J_\mu
\]

where \(f_{\mu\nu}\) is the gauge field strength for gauge field \(a_\lambda\) and \(F_{\mu\nu}\) is the gauge field strength for gauge field \(A_\mu\). The excitations are described by the currents which are defined as \(j_\mu = (j_i, \rho_a)\) and \(J_\mu = (J_i, \rho_A)\). The gauge charges of \(a_\mu\) and \(A_\mu\) are quantized as integers.

From the equation motions for \(a_\lambda\) and \(A_\lambda\),

\[
-\frac{1}{2e_a} (\partial_\mu f_{\mu\lambda}) + \frac{1}{\pi} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} = -i j_\mu,
\]

\[
-\frac{1}{2e_A} (\partial_\mu F_{\mu\lambda}) + \frac{1}{\pi} \epsilon^{\mu\nu\lambda\rho} f_{\mu\nu} = -i J_\mu,
\]

we find that a U(1) charge for gauge field \(A_\mu\) induces flux of gauge field \(a_\lambda\). As a result, the U(1) charge for gauge field \(A_\mu\) and the U(1) charge for gauge field \(a_\mu\) have a semionic mutual statistics. That is, moving an \(A_\mu\)-charge around an \(a_\mu\)-charge generates a phase \(\pi\). This catches the key topological property for the \(Z_2\) spin liquid. It is well known that the \(Z_2\) spin liquid states contain \(Z_2\) vortex and \(Z_2\) charge excitations. And the \(Z_2\) vortex and the \(Z_2\) charge have semionic mutual statistics between them. So we will propose that the mutual Chern-Simons theory 11 describes a \(Z_2\) gauge theory. The \(A_\mu\)-charge can be identified as the \(Z_2\) charge and the \(a_\mu\)-charge as the \(Z_2\) vortex.

Furthermore, the energy gap for both of the gauge fields comes from the mutual CS term

\[
m_a \sim e_a^2, \quad m_A \sim e_A^2.
\]

The mutual U(1)×U(1) CS theory describes a gapped topological state. This also agrees with the \(Z_2\) topological states where all excitations are gapped.

However, we have two kinds of \(Z_2\) topological orders Z2A and Z2E. How can the two different \(Z_2\) topological orders be described by the same U(1)×U(1) CS theory?
In the following we will show that two different \( Z_2 \) topological orders are described by the same \( U(1) \times U(1) \) CS theory but with different realizations of the lattice symmetry.

To obtain two different realizations of lattice symmetry, we note that \( Z_2 \) vortices for the exactly soluble model (the Z2E state) live on the even plaquettes. The vortices on the odd plaquettes are actually the \( Z_2 \) charge.\(^{9,14} \) So under a translation by one lattice spacing, a \( Z_2 \) vortex is changed into a \( Z_2 \) charge! So in the mutual \( U(1) \times U(1) \) CS theory that describes the Z2E state, \( a_\mu \) and \( A_\mu \) must exchange under the translation by one lattice spacing.

Also, the Z2A state contains \( \pi \) flux through each square. This \( \pi \) flux also affects how \( a_\mu \) is transformed under translation. To see this, let us consider two Wilson loop operators \( W_1 = e^{i \sum_{C_1} d g a_y} \) and \( W_2 = e^{i \sum_{C_2} d g a_y} \) along two loops \( C_1 \) and \( C_2 \). Both loops wrap around the torus in \( y \)-direction. However the loop \( C_2 \) is displaced from the loop \( C_1 \) by one lattice constant in the \( x \)-direction. In the following, we will assume the lattice constant is \( a = 1 \). Due to the \( \pi \) flux through each square, we see that \( W_2 = (-1)^L y W_1 \), where \( L_y \) is the length of the torus in the \( y \)-direction. So under a translation by one lattice constant in the \( x \)-direction, \( a_y \) must change to \( a_y + \pi \), to account for the change in the Wilson loop.

The above discussion motivates us to define two types of mutual \( U(1) \times U(1) \) CS theories which have different realizations of translation symmetries. Let \( T_x \) and \( T_y \) be the translations by one lattice spacing in the \( x \) and \( y \) directions respectively. The first type of the mutual \( U(1) \times U(1) \) CS theory is denoted as Z2A type which describes the Z2A state. The \( \pi \) flux makes the gauge fields transform non-trivially under translations:

\[
\begin{align*}
T_x^{-1} A_x T_x &= A_x, & T_y^{-1} A_y T_y &= A_x + \pi, \\
T_x^{-1} A_y T_x &= A_y + \pi, & T_y^{-1} A_y T_y &= A_y, \\
T_x^{-1} a_x T_x &= a_x, & T_y^{-1} a_x T_y &= a_x + \pi, \\
T_x^{-1} a_y T_x &= a_y + \pi, & T_y^{-1} a_y T_y &= a_y.
\end{align*}
\]

Since the translation \( T_x \) (\( T_y \)) may shift \( A_y \) (\( a_y \)) by \( \pi \), this reproduces the different patterns of crystal momenta of the degenerate ground states on different lattices.

The other type of the mutual CS theory is denoted as Z2E type that describes the Z2E state. It has no flux. However, the gauge fields still transform non-trivially under translations:

\[
\begin{align*}
T_x^{-1} A_x T_i &= A_j, & T_i^{-1} a_j T_i &= A_i, & i, j = x, y
\end{align*}
\]

\( A_i \) and \( a_i \) will exchange under a translation operation by one lattice spacing.

2. The topological degeneracy

In the next few sections, we will calculate the topological properties of the above two types of mutual CS theory. First, we calculate the topological degeneracy for the ground states. In the temporal gauge, \( A_0 = 0 \), and on an even-by-even lattice, the fluctuations \( A_i \) and \( a_i \) are periodic. We can expand them as

\[
\begin{align*}
(A_x, A_y) &= \left( \frac{1}{L_x} \Theta_x + \sum_k A_{x,k} e^{i \pi \cdot k}, \frac{1}{L_y} \Theta_y + \sum_k A_{y,k} e^{i \pi \cdot k} \right), \\
(a_x, a_y) &= \left( \frac{1}{L_x} \theta_x + \sum_k a_{x,k} e^{i \pi \cdot k}, \frac{1}{L_y} \theta_y + \sum_k a_{y,k} e^{i \pi \cdot k} \right)
\end{align*}
\]

where \( k = (k_x, k_y) = (\frac{2\pi}{L_x} n_x, \frac{2\pi}{L_y} n_y) \) where \( n_{x,y} \) are integers. \( (A_{x,k}^x, A_{y,k}^y) \) and \( (a_{x,k}^x, a_{y,k}^y) \) are the gauge fields with non-zero momentum and \( (\Theta_x, \Theta_y) \) and \( (\theta_x, \theta_y) \) are the zero modes with zero momentum for the gauge fields \( A_i \) and \( a_i \). Because the existence of the mass gap, the degree freedoms for gauge fields with non-zero momentum \( (A_{x,k}^x, A_{y,k}^y) \) and \( (a_{x,k}^x, a_{y,k}^y) \) have nothing to do with the low energy physics. It is the degree freedoms of zero momentum \( (\Theta_x, \Theta_y) \) and \( (\theta_x, \theta_y) \) that determine the low energy physics. The effective Lagrangian Eq.\((11)\) determines the dynamics of \( (\Theta_x, \Theta_y) \) and \( (\theta_x, \theta_y) \) which corresponds to two particles on a plane with a finite magnetic field. \( (\Theta_x, \Theta_y) \) are the coordinates of the first particle, and \( (\theta_x, \theta_y) \) are the coordinates of the second particle. Thus we map the original mutual \( U(1) \times U(1) \) CS theory to a quantum mechanics model of two particles (see appendix). The energy spectrum for the quantum mechanics model can be solved easily. The lowest energy levels for above model reveal the topological characters for the ground states. The degeneracy for \( (\Theta_x, \Theta_y) \), degrees of freedom and the degeneracy for \( (\theta_x, \theta_y) \), degrees of freedom are given as \( D(\Theta_x, \Theta_y) = 2 \) and \( D(\theta_x, \theta_y) = 2 \). For both the Z2A type and the Z2E type CS theories, there exist four degenerate ground states

\[
D = D(\Theta_x, \Theta_y) = D(\Theta_x, \Theta_y) = 2 \times 2 = 4.
\]

However, the above result only applies to even-by-even lattice. For other cases, even-by-odd, odd-by-even and odd-by-odd, the situations are changed. We will discuss those more complicated cases in appendix.. We find that for the Z2A type mutual CS theory, the ground state degeneracy remain to be 4 for even-by-odd and odd-by-even lattices. For the Z2E type mutual CS theory, the ground state degeneracy becomes 2 for even-by-odd, odd-by-even, and odd-by-odd lattices.

One way to understand the later result is to note that if \( L_x \) is odd then one gauge field will turn into the other one as we go around the lattice along \( x \)-direction. Thus the gauge fields have a twisted boundary condition:

\[
A_i(x + L_x, y) = a_i(x, y), \quad a_i(x + L_x, y) = A_i(x, y).
\]

This twisted boundary condition means that \( A_\mu \) and \( a_\mu \) can be viewed as a single gauge field on a lattice whose size is doubled in the \( x \)-direction. There are only two zero
modes in the mode expansion. As a result the ground-state degeneracy on even-by-odd, odd-by-even and odd-by-odd is reduced to 2. We can also use the CS theories to calculate the crystal momenta of the ground states (see appendix). The results agree with those in tables I and II.

3. The edge states

We can also use the mutual $U(1) \times U(1)$ CS theories to study edge excitations. First, let us consider the exact soluble model (4) on a finite $L_x \times L_y$ lattice with a periodic boundary condition only along $y$-direction. The lattice has two edges along $y$-direction located at $i_x = 0$ and $i_x = L_x$. Such a lattice model can be obtained from the periodic lattice model (4) by setting $g = 0$ for a column of plaquettes. The resulting model is still exactly soluble. We find that the ground states have $2^L$-fold degeneracy which arise from $\sigma^y_i \sigma^y_j \sigma^y_k \sigma^y_l = \pm 1$ on the column of plaquettes with $g = 0$. Those degenerate states can be viewed as gapless edge excitations on the two boundaries. Since there are $2L_y$ edge sites, we find that there are $\sqrt{2}$ edge states per edge site, indicating that the gapless edge states are described by Majorana fermions. Indeed, the gapless edge excitations can be mapped to a Majorana fermion system exactly.

To obtain the gapless edge states from the mutual CS theories, we introduce

$$a_{+, \mu} = A_\mu + a_\mu, \quad a_{-\mu} = A_\mu - a_\mu$$

and rewrite the mutual $U(1) \times U(1)$ CS effective theory as

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} a_{+, \mu} \partial_\nu a_{+, \lambda} \epsilon^{\mu\nu\lambda} - \frac{1}{4\pi} a_{-, \mu} \partial_\nu a_{-, \lambda} \epsilon^{\mu\nu\lambda} + \ldots \quad (16)$$

The charges of $A_\mu$ and $a_\mu$ are quantized as integers. Converting the $A_\mu$ and $a_\mu$ charges to the $a_{+, \mu}$ and $a_{-\mu}$ charges, we find that the $a_{+, \mu}$ and $a_{-, \mu}$ charges are still quantized as integers. However, $(1/2, 1/2)$ charge for the $a_{+, \mu}$ and $a_{-, \mu}$ field is also allowed.

The mutual CS theory (16) has one right-moving and one left-moving branches of edge excitations. The two branches of the edge excitations are described by the following 1D fermion theory

$$\mathcal{L}_{\text{edge}} = \psi_R^\dagger (\partial_\mu - \imath v \partial_x) \psi_R + \psi_L^\dagger (\partial_\mu + \imath v \partial_x) \psi_L + \ldots$$

at low energies, where $(\ldots)$ represent terms that are consistent with the underlying symmetries of the lattice model. $\psi_R$ carries a unit of $a_+$ charge and $\psi_-$ a unit of $a_-$ charge. We note that the $A_\mu$ and $a_\mu$ charges, as the $Z_2$ charge and the $Z_2$ vortex, are conserved only mod 2. So $(\ldots)$ may contain terms that change $(a_+, a_-)$ charge by $(1, 1)$ and $(1, -1)$. Thus, the following terms

$$a \psi_R \psi_L^\dagger + b \psi_R^\dagger \psi_L^\dagger + h.c.$$}

are allowed in the low energy effective Lagrangian. The additional terms will open an energy gap for the edge excitations and one may conclude that the Z2E state in the exactly soluble model (4) has no gapless edge excitations in general.

However, the above conclusion is not quite correct. We see that although the presence of the edge breaks the translation symmetry in the $x$-direction, the finite system still has the translation symmetry in the $y$-direction. Under the translation in the $y$-direction by lattice spacing, $A_\mu$ and $a_\mu$ are exchanged, or $(a_{+, \mu}, a_{-, \mu})$ are changed into $(a_{-, \mu}, -a_{+, \mu})$. So the translation in the $y$-direction changes the sign of the $a_-$ charge and hence changes $\psi_L$ to $\psi_L^\dagger$. As a result, only the following term

$$a \psi_R (\psi_L + \psi_L^\dagger) + h.c.$$}

can be added to the edge effective Lagrangian, which do not break the translation symmetry along the edge.

Introducing Majorana fermions

$$\psi_R = \lambda_R + \imath \eta_R, \quad \psi_L = \lambda_L \pm \eta_L,$$

we can rewrite the edge effective Lagrangian as

$$\mathcal{L}_{\text{edge}} = \lambda_R (\partial_\mu - \imath v \partial_x) \lambda_R + \eta_R (\partial_\mu - \imath v \partial_x) \eta_R$$

$$+ \lambda_L (\partial_\mu + \imath v \partial_x) \lambda_L + \eta_L (\partial_\mu + \imath v \partial_x) \eta_L$$

$$+ 2 (a \lambda_R \lambda_L + b \lambda_L \eta_L + c \eta_L \eta_L + h.c.).$$

The $a \lambda_R \lambda_L$ term gaps a pair of Majorana fermions and leave the other pair gapless. So the Z2E state has right-moving and left-moving gapless edge excitations described by Majorana fermions, provided that the edge is in the $x$- or $y$-direction. The presence of the translation symmetry in the $x$- or $y$-direction is crucial for the existence of the gapless edge excitations for the Z2E type mutual $U(1) \times U(1)$ CS theory and the exact soluble model.

For the Z2A state, although the low energy effective theory has the same form as the exactly soluble model, the translation does not induce the exchange between $A_\mu$ and $a_\mu$. As a result, in general, there are no gapless edge excitations for the Z2A type mutual $U(1) \times U(1)$ CS theory and the Z2A state.

VI. CONCLUSION

In this paper, two kinds of $Z_2$ topological ordered states for frustrated spin systems, Z2A state and Z2E state, are studied. Using the SU(2) slave-particle theory, we calculate their ground-state degeneracy, their ground-state quantum numbers, their gapless edge state, and the projective symmetry group of their quasi-particles. We propose a mutual $U(1) \times U(1)$ Chern-Simons theory with two different realizations of lattice symmetry as the effective field theories that describe the two types of topological orders. We show that the effective theories produce the same low energy physics, including the degeneracy of the ground state, the quantum number for the
ground state and the edge states. It turns out that the different $Z_2$ topological orders are reflected in different realizations of the lattice symmetry in the same effective mutual Chern-Simons theory.

We like to mention that the Z2A phase appears to be an example of “weak symmetry breaking in dimension 2”, while the Z2E phase appears to be an example of “weak symmetry breaking in dimension 1” discussed in Ref. 18. So these two phases are examples of the two basic ways that lattice symmetries and topological structure can be entangled.

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**APPENDIX A: APPENDIX**

**a. Topological degeneracy for the Z2E state**

We have used ansatz \( \psi^{(m,n)}_{ij} \equiv (\psi_1^{0,0}, \psi_0^{0,0}) \) to describe the four degenerate ground states for the Z2E state. Here \( m, n = 0, 1 \). \( s_{x,y}(ij) \) have values 0 or 1, with \( s_{x,y}(ij) = 1 \) if the link \( ij \) crosses the \( x \) or \( y \) line (see Fig. 1) and \( s_{x,y}(ij) = 0 \) otherwise.

It is pointed out that the above result of four degenerate ground states is right only for the Z2E state on an even-by-even lattice. On other kinds of lattices (even by odd, odd by even and odd by odd), there exist only two different ground states. The other two states are projected out since the mean-field ground states contain odd numbers of fermions.

Let’s calculate the topological degeneracy for the Z2E state on different lattices in detail. It was pointed out that the total number of the \( \psi \) fermions on \( |k \rangle \) and \( -|k \rangle \) is always even even if \( k \neq 0 \). To determine if the mean-field ground state contains even or odd number of \( \psi \) fermions, we will only pay attention to the occupation on the following four momentum points: \( k = (0,0), (0,\pi), (\pi,0), (\pi,\pi) \).

Firstly, we discuss the topological degeneracy for Z2E state on an even by even lattice. For the ground state described by \( (m,n) = (0,0) \), the energy levels for both \( \psi_1 \) and \( \psi_2 \) have positive energies at \( k = (0,0) \) (see Eq. 10). Thus the \( k = (0,0) \) level is not occupied. We also see from Eq. 10 that, at \( k = (0,\pi) \), \( \psi_1 \) has a positive energy \( \psi_2 \) has a negative energy. Thus the \( k = (0,\pi) \) level is occupied by a \( \psi_1 \) particle. Similarly, we find that the \( k = (\pi,0) \) level is occupied by a \( \psi_2 \) particle, the \( k = (\pi,\pi) \) level is occupied by a \( \psi_1 \) particle and a \( \psi_2 \) particle. Therefore, four particles occupy the points \( (0,0), (0,\pi), (\pi,0), (\pi,\pi) \). Because the meanfield ground state \( |\Psi_{\text{mean}}^{(0,0)}\rangle \) has even number particles, it survives the even-particle-per-site projection.

Also, the total contribution to the crystal momentum from the \( k \neq 0 \) levels is zero. Thus the total crystal momentum is determined by the particles that occupy the \( (0,0), (0,\pi), (\pi,0), (\pi,\pi) \) levels. We find that the total crystal momentum of the above state is \( 0 \times (0,0) + 1 \times (0,\pi) + 1 \times (\pi,0) + 2 \times (\pi,\pi) = (\pi,\pi) \).

For the ground states described by \( (m,n) = (1,0), (m,n) = (0,1), \) and \( (m,n) = (1,1) \), non of the high-symmetry points \( (0,0), (0,\pi), (\pi,0), (\pi,\pi) \) exist. Thus the ground states have even number particles, so they are all permitted under the even-particle-per-site projection. The total crystal momenta of the above states are all zero.

Therefore, there are four degenerate ground states on even-by-even lattice. One carries crystal momentum \( (\pi,\pi) \) and other three carry crystal momentum \( (0,0) \). This corresponds to the first column of table II.

Secondly, we discuss the topological degeneracy for Z2E state on an even by odd lattice. For the ground state described by \( (m,n) = (0,0) \), the \( k = (0,0) \) level is not occupied; the \( k = (\pi,0) \) level is occupied by one \( \psi_1 \) particle, as before. The points \( (0,\pi) \) and \( (\pi,\pi) \) do not exist. As a result, only one particle occupies the high-symmetry points. Because the ground state \( |\Psi_{\text{mean}}^{(0,0)}\rangle \) has odd number particles, it is forbidden by the even-particle-per-site projection.

For the ground state described by \( (m,n) = (1,0) \), the \( k = (0,\pi) \) level is occupied by a \( \psi_2 \) particle, and the \( k = (\pi,\pi) \) level is occupied by a \( \psi_1 \) and a \( \psi_2 \) particles. The points \( (0,0) \) and \( (0,\pi) \) points do not exist. As a result, three particles occupy the high-symmetry points. The state is forbidden by the projection.

For the ground state noted by \( (m,n) = (1,1) \), the points \( (0,0), (0,\pi), (\pi,0), (\pi,\pi) \) do not exist. Because the ground state \( |\Psi_{\text{mean}}^{(1,1)}\rangle \) has even number particles, it is also permitted by the projection. Such a state also carries a \( (0,0) \) crystal momentum.

There are therefore two degenerate ground states on an even by odd lattice. Similarly topological degeneracy for Z2E state on an odd by is also two. All those states carry a \( (0,0) \) crystal momentum. This corresponds to the second and third columns of table II.

Lastly, let us discuss the topological degeneracy for Z2E state on an odd by odd lattice. For the ground state described by \( (m,n) = (0,0) \), the \( k = (0,0) \) level is not occupied. The points \( (0,\pi) \) \( (0,\pi) \) and \( (\pi,\pi) \pi \) do not exist. As a result, no particle occupies the high-symmetry points. The ground state \( |\Psi_{\text{mean}}^{(0,0)}\rangle \) has even number particles which is permitted by the projection.

For the ground state described by \( (m,n) = (1,0) \), the \( k = (\pi,0) \) level is occupied by a \( \psi_1 \) particle. The points \( (0,0) \) \( (0,\pi) \) and \( (\pi,\pi) \) do not exist. As a result, one particle occupies the high-symmetry points. Because the ground state \( |\Psi_{\text{mean}}^{(1,0)}\rangle \) has odd number particles, it
is not permitted by the projection.

For the ground state described by \((m, n) = (0, 1)\), the \(k = (0, \pi)\) level is occupied by a \(\psi_2\) particle. The points \((0, 0)\), (\(\pi, 0\)) and (\(\pi, \pi\)) do not exist. As a result, one particle occupies the high-symmetry points. Because the ground state \(|\Psi_\text{mean}^{(1,0)}_{\mu n}, \eta_\theta^{(1,0)}\rangle\) has odd number particles, it is not permitted by the projection.

For the ground state described by \((m, n) = (1, 1)\), at the \(k = (\pi, \pi)\) level is occupied by a \(\psi_1\) and a \(\psi_2\) particle. The points (\(\pi, 0\)), (\(0, \pi\)), and (\(0, 0\)) do not exist. As a result, two particles occupy the high-symmetry points.

The ground state \(|\Psi_\text{mean}^{(1,1)}_{\mu n}, \eta_\theta^{(1,1)}\rangle\) has even number particles, so the state is permitted by the projection.

In conclusion, Z2E state has four-fold degeneracy on an even by even lattice and two-fold degeneracy on an odd by odd lattice, odd by even lattice or odd by odd lattice. The crystal momenta of those ground states are given by the table II.

### b. Quantization for the mutual \(U(1)\times U(1)\) CS theory

To calculate the topological properties for the ground states of the mutual \(U(1)\times U(1)\) CS theories, one needs to quantize the gauge fields. We will choose the temporal gauge \(A_0 = 0\). In the temporal gauge, the physical degrees of freedom are described by \((A_x, A_y)\) and \((a_x, a_y)\). After the mode expansion, the effective Lagrangian can be written as

\[
L = \frac{1}{2} M_x \dot{\theta}_x^2 + \frac{1}{2} M_y \dot{\theta}_y^2 + \frac{1}{2} m_x \dot{\phi}_x^2 + \frac{1}{2} m_y \dot{\phi}_y^2 - \frac{1}{2\pi} \Theta_x \dot{\theta}_y - \frac{1}{2\pi} \Theta_y \dot{\theta}_x + \frac{1}{2\pi} \theta_x \dot{\Theta}_y + \frac{1}{2\pi} \theta_y \dot{\Theta}_x + \ldots
\]

where \((A^k_x, A^k_y)\) and \((a^k_x, a^k_y)\) represent the terms that certain only the \(k \neq 0\) modes. The masses are given as

\[
M_x = \frac{1}{e^2} \frac{L_y}{L_x}, \quad M_y = \frac{1}{e^2} \frac{L_x}{L_y}, \quad m_x = \frac{1}{e^2} \frac{L_y}{L_x}, \quad m_y = \frac{1}{e^2} \frac{L_x}{L_y}.
\]

Because the existence of the mass gap, the degeneracy for gauge fields with non-zero momentum \((A^k_x, A^k_y)\) and \((a^k_x, a^k_y)\) have nothing to do with the low energy physics. So we will concentrate on the dynamics of \(\theta_x, \theta_y\) and \(\Theta_x, \Theta_y\).

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4e^2}(F_{\mu\nu})^2 - \frac{1}{4e^2}(F_{\mu\nu})^2 + \frac{1}{\pi} \epsilon^{\mu\nu\lambda\alpha} A_\mu \partial_\nu a_\lambda.
\] (A1)

From the effective Lagrangian, one can define the conjugate momentum for \((\Theta_x, \Theta_y)\) and \((\theta_x, \theta_y)\),

\[
P_{\Theta_x} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\Theta}_x} = M_x \dot{\Theta}_x + \frac{\theta_y}{2\pi},
\]

\[
P_{\Theta_y} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\Theta}_y} = M_y \dot{\Theta}_y + \frac{\theta_x}{2\pi},
\]

\[
p_{\theta_x} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_x} = m_x \dot{\theta}_x + \frac{\Theta_y}{2\pi},
\]

\[
p_{\theta_y} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_y} = m_y \dot{\theta}_y - \frac{\Theta_x}{2\pi}.
\]

Using the conjugate momentum we write down the following effective Hamiltonian to describe the low energy physics of the mutual \(U(1)\times U(1)\) CS theory

\[
H_{\text{eff}} = \left(\frac{P_{\Theta_x} - \frac{\theta_y}{2\pi}}{2M_x}\right)^2 + \left(\frac{P_{\Theta_y} + \frac{\theta_x}{2\pi}}{2M_y}\right)^2 + \left(\frac{p_{\theta_x} + \frac{\Theta_y}{2\pi}}{2m_x}\right)^2 + \left(\frac{p_{\theta_y} - \frac{\Theta_x}{2\pi}}{2m_y}\right)^2.
\]

By choosing different Landau gauges, the above can rewritten as

\[
H_{\text{eff}} = \frac{(P_{\Theta_x} - \frac{\theta_y}{2\pi})^2}{2M_x} + \frac{(p_{\theta_x} + \frac{\Theta_y}{2\pi})^2}{2m_x} + \frac{(P_{\Theta_y} + \frac{\theta_x}{2\pi})^2}{2M_y} + \frac{(p_{\theta_y} - \frac{\Theta_x}{2\pi})^2}{2m_y}.
\]

The low energy properties of the \(U(1)\times U(1)\) Chern-Simons theory is described by the above Hamiltonian.

### c. Topological degeneracy and crystal momenta for the Z2E type mutual \(U(1)\times U(1)\) CS theory

Let us first use the Hamiltonian to calculate the ground state degeneracy of the Z2E state on an even by even lattice.

We note that \(a_x\) and \(a_x + \frac{2\pi}{L_x}\) are related by a \(U(1)\) gauge transformation. Thus \(\theta_x = 0\) and \(\theta_x = 2\pi\) are also related by a \(U(1)\) gauge transformation, which implies that \(\theta_x = 0\) and \(\theta_x = 2\pi\) should be viewed as the same point. Similarly each of the three pairs \(\theta_y = 0\) and \(\theta_y = 2\pi\), \(\Theta_x = 0\) and \(\Theta_x = 2\pi\), also should be viewed as the same point. Thus the above Hamiltonian describes two particles, each moves on a \(2\pi \times 2\pi\) torus. Each particle also see \(4\pi\) flux through the torus.

The first particle is described by \((\Theta_x, \theta_y)\). Since there are two units of flux through the torus, the ground states for the first particle has a degeneracy \(D_{(\Theta_x, \theta_y)} = 2\). Similarly, the ground states for the second particle also has a degeneracy \(D_{(\Theta_y, \theta_x)} = 2\).
For other cases, on an even-by-odd, odd-by-even or odd-by-odd lattice, the situations are changed. Because for odd number rows along x-axis or y-axis, one gauge field \( A_{\mu} (a_{\mu}) \) will turn into the other one \( a_{\mu} (A_{\mu}) \). For example, on a \( L_x \times L_y \) even-by-odd lattice (\( L_x \) is an even number and \( L_y \) is an odd number), under such a twisted boundary condition for odd number \( L_y \), one has

\[
A_{\mu}(x,y + L_y) = a_{\mu}(x,y), \quad a_{\mu}(x,y + L_y) = A_{\mu}(x,y), \quad A_{\mu}(x + L_x,y) = A_{\mu}(x,y), \quad a_{\mu}(x + L_x,y) = a_{\mu}(x,y). \quad (A4)
\]

The quantization for gauge fields in Eq.14 cannot be applied to the gauge fields under a twisted boundary condition.

Now after putting the mutual U(1)×U(1) CS theory on a \( L_x \times (2L_y) \) even-by-even lattice, we have a periodic boundary condition,

\[
A_{\mu}(x,y + 2L_y) = A_{\mu}(x,y), a_{\mu}(x,y + 2L_y) = a_{\mu}(x,y).
\]

In the temporal gauge, \( A_{0} = 0 \), and on such even-by-even lattice, we can expand the fluctuations for the gauge fields as

\[
(A_x, A_y) = \left( \frac{1}{L_x} \Theta_x + \sum_{k} A_k^x e^{i \pi k} \frac{1}{2L_y} \Theta_y + \sum_{k} A_k^y e^{i \pi k} \right),
\]

\[
(a_x, a_y) = \left( \frac{1}{L_x} \Theta_x + \sum_{k} a_k^x e^{i \pi k} \frac{1}{2L_y} \Theta_y + \sum_{k} a_k^y e^{i \pi k} \right)
\]

where \( k = (k_x, k_y) = \left( \frac{2\pi}{L_x}, \frac{\pi}{L_y} n_y \right) \) where \( n_{x,y} \) are integers. \( A_k^x, A_k^y \) and \( a_k^x, a_k^y \) are the gauge fields with non-zero momentum and \((\Theta_x, \Theta_y)\) and \((\theta_x, \theta_y)\) are the zero modes with zero momentum for the gauge fields \( A_i \) and \( a_i \). However, \( A_k^i (\Theta_i) \) and \( a_k^i (\theta_i) \) are not independent and have constraints to obey the original twisted boundary condition in Eq.A4, we must have

\[
A_k^i = a_k^i e^{i \pi n_y}, \quad \Theta_i = \theta_i.
\]

To calculate the topological degeneracy, we map the original mutual U(1)×U(1) CS theory on even-by-odd lattice to two-particle quantum mechanics model on a torus in a magnetic field \( \frac{1}{2} \). In the "Landau gauge", the effective Hamiltonian of the two-particle quantum mechanics model is given as

\[
H_{\text{eff}} = \frac{(P_{\theta_x} - \frac{\theta_x}{2})^2}{2M_x} + \frac{(P_{\theta_y} + \frac{\theta_y}{2})^2}{2m_y} + \frac{(P_{\phi_x} + \frac{\phi_x}{2})^2}{2M_x} + \frac{(P_{\phi_y} - \frac{\phi_y}{2})^2}{2m_y}
\]

where \( M_x = \frac{1}{4 \alpha_s}, \quad M_y = \frac{1}{\alpha_s} \frac{b_{min}}{2} \) and \( m_x = \frac{1}{\alpha_s} \frac{2b_{min}}{2} \), \( m_y = \frac{1}{\alpha_s} \frac{L_y}{2} \). However, because of the constraint in
Eq. A7, the two particles (θx, θy) and (Θx, Θy) are bound into a single particle! As a result, there are two degenerate ground states instead of four. In addition, we can write down the wave-functions for the two ground states in the Landau gauge with topological degeneracy: for the wave-function [1],

\[ \Psi_1 \simeq e^{-i\theta^2} = e^{-\frac{i\theta^2}{4}}, \]

and the wave-function [2],

\[ \Psi_2 \simeq e^{-i\Theta y}e^{-\frac{i\Theta y}{4}(\theta_y - \pi)^2} = e^{-i\theta y}e^{-\frac{i\Theta y}{4}(\theta_y - \pi)^2}. \]

Now let’s calculate the crystal momentum for the two-fold degenerate ground states. The ground states are invariant under the translation operations

\[ T_x |j\rangle = |j\rangle, \]
\[ T_y |j\rangle = |j\rangle, \]
\[ j = 1, 2. \]

Then the crystal momentum \((K_x, K_y)\) is \((0, 0)\) for the E type mutual U(1)×U(1) CS theory on an even-by-odd lattice.

Furthermore, using the same method, we calculated the topological degeneracies and the crystal momenta for the ground states of the Z2E type mutual U(1)×U(1) CS theory on an odd-by-even or odd-by-odd lattices. The results are similar to those on an even-by-odd lattice: the ground states have two-fold degeneracy and \((K_x, K_y) = (0, 0)\).

In summary, all the low energy physical properties for the Z2E type U(1)×U(1) Chern-Simons theory match that for the Z2E topological ordered state.

d. Topological degeneracy and crystal momenta for the Z2A type mutual U(1)×U(1) CS theory

In this part, we will calculate the topological degeneracy and crystal momenta for the Z2A type mutual U(1)×U(1) CS theory. The effective Hamiltonian to describe the low energy physics of the Z2A type the mutual U(1)×U(1) CS theory can be written in the "Landau gauge" as

\[ H_{eff} = \frac{(p_{\theta x} - \frac{\theta x}{\pi})^2}{2M_x} + \frac{p_{\theta y}^2}{2m_y} + \frac{(p_{\theta x} - \frac{\theta x}{\pi})^2}{2m_x} + \frac{P_{\theta y}^2}{2M_y}. \]

It is noted that there exists the Heisenberg Algebra for zero modes of the gauge fields. The "magnetic" translation operators \(U_{\theta x} = e^{i\pi(p_{\theta x} + \frac{\theta x}{\pi})}\) and \(U_{\theta y} = e^{i\pi(p_{\theta y} + \frac{\theta y}{\pi})}\) consist of the Heisenberg algebra

\[ U_{\theta x}U_{\theta y} = e^{i\pi}U_{\theta y}U_{\theta x}. \]

Because the Hamiltonian is invariant for the operations \(U_{\theta x}\) and \(U_{\theta y}\),

\[ U_{\theta x}^{-1}HU_{\theta x} = H, \]
\[ U_{\theta y}^{-1}HU_{\theta y} = H, \]

the ground states are the eigenstates of \(U_{\theta x}\) and \(U_{\theta y}\). So one can draw a conclusion from the Heisenberg algebra that the ground states have two-degeneracy for \((\theta x, \theta y)\). On the other hand, for \((\Theta x, \Theta y)\), one can do the same calculation. So the ground states have two-degeneracy for \((\theta y, \theta x)\) which is also characterized by the eigenstates of \(U_{\theta y} = e^{i\pi(p_{\theta y} + \frac{\theta y}{\pi})}\) and \(U_{\theta x} = e^{i\pi(p_{\theta x} + \frac{\theta x}{\pi})}\). As a result, for the Z2A type mutual U(1)×U(1) CS theory, the ground states have four-fold degeneracy:

\[ D = D_{(\theta x, \theta y)}D_{(\theta y, \theta x)} = 2 \times 2 = 4. \]

We denote the four ground states with topological degeneracy as [1], [2], [3] and [4],

\[ U_{\theta x}|1\rangle = |1\rangle, \]
\[ U_{\theta x}|2\rangle = |2\rangle, \]
\[ U_{\theta x}|3\rangle = e^{i\pi}|3\rangle, \]
\[ U_{\theta x}|4\rangle = e^{i\pi}|4\rangle, \]

and

\[ U_{\theta y}|1\rangle = |1\rangle, \]
\[ U_{\theta y}|2\rangle = e^{i\pi}|2\rangle, \]
\[ U_{\theta y}|3\rangle = |3\rangle, \]
\[ U_{\theta y}|4\rangle = e^{i\pi}|4\rangle. \]

Now let’s calculate the crystal momentum for the four-fold degenerate ground states. For the Z2A type mutual U(1)×U(1) Chern-Simons theory, the translation operations for the gauge fields are given by Eq. (13). The translation operations for zero modes of the gauge fields are given as (13)

\[ T_x^{-1}\Theta y T_x = \Theta y, \]
\[ T_y^{-1}\Theta x T_y = \Theta x, \]
\[ T_x^{-1}\Theta y T_x = \Theta y + L_y\pi, \]
\[ T_y^{-1}\Theta x T_y = \Theta x, \]

and

\[ T_x^{-1}\theta y T_x = \theta y, \]
\[ T_y^{-1}\theta x T_y = \theta x + L_y\pi, \]
\[ T_x^{-1}\theta y T_x = \theta x, \]
\[ T_y^{-1}\theta y T_y = \theta y. \]

As a result, the real ground states can be labeled by the eigenvalues of \(U_{\theta x}\) (or \(U_{\theta y}, U_{\Theta x}, U_{\Theta y}\)) which are 1 and \(-1\). We denote the four ground states with topological degeneracy as [1], [2], [3] and [4],

\[ U_{\theta x}|1\rangle = |1\rangle, \]
\[ U_{\theta x}|2\rangle = |2\rangle, \]
\[ U_{\theta x}|3\rangle = e^{i\pi}|3\rangle, \]
\[ U_{\theta x}|4\rangle = e^{i\pi}|4\rangle. \]
Then the crystal momentum ($K_x, K_y$) of the four-fold degenerate ground states $|j\rangle$ is $(0,0)$.

Secondly on an odd-by-even lattice ($L_x$ is odd number and $L_y$ is even number), the translation operations are given as

$$
T^{-1}_x \theta_x T_x = \theta_y,
T^{-1}_y \theta_y T_y = \theta_x + \pi,
T^{-1}_x \theta_x T_x = \theta_x,
T^{-1}_y \theta_y T_y = \theta_y,
$$

and

$$
T^{-1}_x \theta_y T_x = \theta_y,
T^{-1}_y \theta_x T_y = \theta_x,
T^{-1}_x \theta_x T_x = \theta_x,
T^{-1}_y \theta_y T_y = \theta_y.
$$

Now the translation operator $T_y$ turns into the "magnetic" translation operator $U_{\Theta_y} = e^{\pi i (p_{dy} + \Theta_y)}$,

$$
T_y |i\rangle = U_{\Theta_y} |i\rangle = e^{\pi i (p_{dy} + \Theta_y)} |i\rangle, \quad i = 1, 2, 3, 4.
$$

Under the translation operations on the wave functions in Eq.A3, we have

$$
T_x |1\rangle = |1\rangle,
T_x |2\rangle = |2\rangle,
T_x |3\rangle = |3\rangle,
T_x |4\rangle = |4\rangle,
$$

and

$$
T_y |1\rangle = U_{\Theta_y} |1\rangle = |1\rangle,
T_y |2\rangle = U_{\Theta_y} |2\rangle = e^{i\pi} |2\rangle,
T_y |3\rangle = U_{\Theta_y} |3\rangle = |3\rangle,
T_y |4\rangle = U_{\Theta_y} |4\rangle = e^{i\pi} |4\rangle.
$$

Using the same method, we can obtain that the crystal momentum of the two ground states $|1\rangle$ and $|2\rangle$ is $(0,0)$. The crystal momentum of the other two ground states $|3\rangle$ and $|4\rangle$ is $(0, \pi)$.

Thirdly on an even-by-odd lattice ($L_x$ is even number and $L_y$ is odd number), the translation operations for its zero modes lead to non-trivial results

$$
T^{-1}_x \theta_y T_x = \theta_y,
T^{-1}_y \theta_x T_y = \theta_x + \pi,
T^{-1}_x \theta_y T_x = \theta_y + \pi,
T^{-1}_y \theta_x T_y = \theta_x.
$$

and

$$
T^{-1}_x \theta_y T_x = \theta_y,
T^{-1}_y \theta_x T_y = \theta_x + \pi,
T^{-1}_x \theta_y T_x = \theta_y + \pi,
T^{-1}_y \theta_x T_y = \theta_x.
$$

Then the translation operator $T_x$ turns into the "magnetic" translation operator $U_{\Theta_y} = e^{\pi i (p_{dy} + \Theta_y)}$,

$$
T_x |i\rangle = U_{\Theta_y} |i\rangle = e^{\pi i (p_{dy} + \Theta_y)} |i\rangle, \quad i = 1, 2, 3, 4.
$$

From them, we have

$$
T_x |1\rangle = U_{\Theta_y} |1\rangle = |1\rangle,
T_x |2\rangle = U_{\Theta_y} |2\rangle = e^{i\pi} |2\rangle,
T_x |3\rangle = U_{\Theta_y} |3\rangle = |3\rangle,
T_x |4\rangle = U_{\Theta_y} |4\rangle = e^{i\pi} |4\rangle.
$$

and

$$
T_y |1\rangle = |1\rangle,
T_y |3\rangle = |3\rangle,
T_y |2\rangle = |2\rangle,
T_y |4\rangle = |4\rangle.
$$

The crystal momentum of two ground states $|1\rangle$ and $|3\rangle$ is $(0,0)$. The crystal momentum of the other two ground states $|4\rangle$ and $|2\rangle$ is $(\pi,0)$.

Fourthly for $L_x$ and $L_y$ are all odd numbers (on an odd-by-odd lattice), the translation operations become

$$
T^{-1}_x \theta_y T_x = \theta_y + \pi,
T^{-1}_y \theta_x T_y = \theta_x,
T^{-1}_x \theta_y T_x = \theta_y + \pi,
T^{-1}_y \theta_x T_y = \theta_x.
$$

and

$$
T^{-1}_x \theta_y T_x = \theta_y,
T^{-1}_y \theta_x T_y = \theta_x + \pi,
T^{-1}_x \theta_y T_x = \theta_y,
T^{-1}_y \theta_x T_y = \theta_y.
$$
Then the translation operators $T_x$ and $T_y$ turn into the "magnetic" translation operator $U_{\Theta y} = e^{\pi i (p_{\Theta y} + \frac{\Theta y}{\pi})}$ and $U_{\Theta x} = e^{\pi i (p_{\Theta x} + \frac{\Theta x}{\pi})}$, 

\[
T_x |i\rangle = U_{\Theta y} |i\rangle = e^{\pi i (p_{\Theta y} + \frac{\Theta y}{\pi})} |i\rangle, \\
T_y |i\rangle = U_{\Theta x} |i\rangle = e^{\pi i (p_{\Theta x} + \frac{\Theta x}{\pi})} |i\rangle, \quad i = 1, 2, 3, 4.
\]

Now $T_x$ and $T_y$ must obey the Heisenberg algebra for $U_{\Theta y}$ and $U_{\Theta x}$

\[
T_x T_y = e^{i\pi T_y T_x}. \quad (A9)
\]

On the other hand, the translation symmetry of the system leads to the commutation relationship between $T_x$ and $T_y$

\[
T_x T_y = T_y T_x. \quad (A10)
\]

The only solve to the Eq.A9 and Eq.A10 is $|i\rangle \equiv 0$. That is, there don’t exist the four degenerate ground states at all. We can see that for the real ground states, the $A_{\mu}$ and $a_{\mu}$ charges for the excitations cannot be zero on an odd by odd lattice. So the non zero background charge leads to an infinity degeneracy on odd by odd lattice for the Z2A type mutual $U(1) \times U(1)$ CS theory.

As a result, all the low energy physical properties for the Z2A type $U(1) \times U(1)$ Chern-Simons theory match that for the Z2A topological ordered state.

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