Numerical model of biological tissue heating using the models of bio-heat transfer with delays

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Abstract. The numerical model of thermal processes in domain of biological tissue subjected to an external heat source is discussed. The model presented is based on the second order dual-phase-lag equation (DPLE) in which the relaxation time and thermalization time \( \tau_r \) and \( \tau_T \) are taken into account. In this paper the homogeneous, cylindrical skin tissue domain is considered. The most important aim of the research is to compare the results obtained using the classical model (the first-order DPLE) with the numerical solution resulting from the higher order form of this equation. At the stage of numerical computations the Finite Difference Method (FDM) is applied. In the final part of the paper the examples of computations are shown.

1 Introduction

The problem of thermal processes occurring in the domain of skin tissue subjected to an external heat source is discussed. Recently there is the view that, taking into account the specific internal structure of the tissue, the hyperbolic type equations better than parabolic ones reproduce the actual course of the thermal processes taking place in the domain considered, e.g. [1-4]. In this paper the heat transfer in the tissue is described by the single, second-order DPLE (a homogeneous domain). In the further stages of the research, the system of these equations (the multi-layered skin tissue domain) will be considered. The energy equation contains the additional components (the source functions) related to the blood perfusion and metabolism [5-7] just like the classic Pennes equation [8]. The starting point for the considerations concerning the DPLE is as (one knows) the generalized form of the Fourier law in which the lag times (the relaxation and thermalization times) are introduced. The left and right hand sides of generalized Fourier law are developed into the Taylor series with accuracy to the first derivative and time.

2 Governing equations

The starting point for the formulation of the energy equation with delays is the generalized Fourier law. In particular, the relationship between heat flux \( q \) and the temperature gradient \( VT \) is given in the form

\[
X \in \Omega: \quad q(X,t + \tau_q) = -\lambda VT(X,t + \tau_T) \quad (1)
\]

where \( \lambda \) is a thermal conductivity, \( \tau_q \) and \( \tau_T \) are the relaxation time and thermalization time, respectively, \( X = \{r, z\} \) and \( t \) denote spatial co-ordinates (cylindrical co-ordinates) and time.

Using the Taylor series expansions, the following second-order approximation of the formula (1) can be taken into account

\[
q(X,t) + \tau_q \frac{\partial q(X,t)}{\partial t} + \frac{\tau_T^2}{2} \frac{\partial^2 q(X,t)}{\partial t^2} = -\lambda \left[ VT(X,t) + \tau_T \frac{\partial VT(X,t)}{\partial t} + \frac{\tau_T^2}{2} \frac{\partial^2 VT(X,t)}{\partial t^2} \right] \quad (2)
\]

Depending on the assumed number of the components, the different forms of DPLE can be obtained. Using the known diffusion equation (in the case of the constant values of thermophysical parameters) one obtains

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As one knows, in the case of axially symmetrical problem

\[ \nabla^2 T(r, z, t) = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial T(r, z, t)}{\partial r} \right] + \frac{\partial^2 T(r, z, t)}{\partial z^2} = \] (7)

The energy equation (6) must be supplemented by the boundary and initial conditions. In the case of the problem considered, the boundary conditions take a form of the Dirichlet and Neumann ones. Thus, on the bottom of the cylinder \((z=Z)\) the body core temperature \(T_b\) is given. On the top of the cylinder \((z=0)\) outside the domain of external heat source action and on the lateral surface \((r=R)\) the no-flux condition is assumed. For \(r \leq R_0\) (radius of external heat flux action) the boundary heat flux is equal to \(q_b\) (Figure 1). The condition discussed is of the form

\[ q_b(r, z, t) + \tau_r \frac{\partial q_b(r, z, t)}{\partial t} + w_q \frac{\tau_t^2}{2} \frac{\partial^2 q_b(r, z, t)}{\partial t^2} = \] (8)

where \(\nabla T(r, z, t)\) denotes a normal derivative. One can see, that for the constant value of \(q_b(r, z, t)\) and also for the no-flux condition the equation (8) takes a simpler form. The initial conditions are also given

\[ t = 0: \quad T(r, z, 0) = T_r, \] (9)

\[ \left. \frac{\partial T(r, z, t)}{\partial t} \right|_{t=0} = u(r, z), \]

\[ \left. \frac{\partial^2 T(r, z, t)}{\partial t^2} \right|_{t=0} = v(r, z) \]

where \(T_r\) is an initial temperature, while \(u(r, z)\) and \(v(r, z)\) are the known functions. For the first-order DPLE the function \(v\) is not defined.

### 3 Numerical solution

The algorithm presented below is based on the implicit scheme of the finite difference method (FDM).

Let \(T_{ij}^m = T(r_i, z_j, f \Delta t)\) where \(\Delta t\) is the time step, \(r_i = ih, z_j = jh\) (\(h\) is the mesh step in \(r\) and \(z\) directions) and \(f=0, 1, \ldots, F\). Taking into account the initial conditions (9), under the assumption that \(u(r, z)\) and \(v(r, z)\) are known functions, the approximate form of equation (6) resulting from the introduction of the adequate differential quotients is as follows
where (c.f. formula (7))

\[
\begin{align*}
\tau_{ij}^2 T_{ij}^/-3T_{ij}'^/-3+3T_{ij}''^/-3-T_{ij}'^/=\frac{w_r}{2} \left(\Delta t\right)^3,
\end{align*}
\]

\[
\begin{align*}
a(\nabla^2 T)^/-i_j + \frac{\alpha_r}{\Delta t} \left[ (\nabla^2 T)^/-i_j - (\nabla^2 T)^/-i_{j+1} \right] +
\end{align*}
\]

\[
\begin{align*}
w_r \frac{\alpha_r}{2(\Delta t)} \left[ (\nabla^2 T)^/-i_j - 2(\nabla^2 T)^/-i_{j+1} + (\nabla^2 T)^/-i_{j+1} \right] -
\end{align*}
\]

\[
\begin{align*}
G_c T_{ij}^/-c + G_c T_B + Q_{out}^/-c \left(\Delta t\right)^3
\end{align*}
\]

where (c.f. formula (7))

\[
\begin{align*}
(\nabla^2 T)^/-i_j = & \frac{1}{h^2} T_{ij}'^/-i_j - T_{ij}'^/-i_{j+1} +
\end{align*}
\]

\[
\begin{align*}
T_{ij}'^/-i_j - 2T_{ij}'^/-i_{j+1} + T_{ij}'^/-i_{j+1} +
\end{align*}
\]

\[
\begin{align*}
\frac{h^2}{2 h^2 n_{ij}} = \frac{2}{h^2 n_{ij}} T_{ij}'^/-i_{j+1} +
\end{align*}
\]

\[
\begin{align*}
\frac{2 r_{ij} + h}{2 h^2 n_{ij}} T_{ij}'^/-i_{j+1} + \frac{1}{h^2} T_{ij}'^/-i_{j+1} + \frac{1}{h^2} T_{ij}'^/-i_{j+1} - 4 T_{ij}'^/-i_{j+1}
\end{align*}
\]

while \(s = f, s = f - 1\) or \(s = f - 2\). After arduous transformations one obtains

\[
\begin{align*}
\left(B_i B_j + 4B_i \frac{\lambda \Delta t}{h^2} \right) T_{ij}'^/-i_j = \left(B_i c + 2B_i B_j \right) T_{ij}'^/-i_{j+1} -
\end{align*}
\]

\[
\begin{align*}
(2B_i c + B_i w_r \tau_{ij}^2) T_{ij}'^/-i_{j+1} +
\end{align*}
\]

\[
\begin{align*}
B_i \lambda \Delta t \left\{ \frac{2 r_{ij} + h}{2 h^2 n_{ij}} T_{ij}'^/-i_{j+1} + \frac{1}{h^2} T_{ij}'^/-i_{j+1} + \frac{1}{h^2} T_{ij}'^/-i_{j+1} \right\}
\end{align*}
\]

\[
\begin{align*}
2B_i \lambda \Delta t \left[ (\nabla^2 T(r, z, t))^/-i_{j+1} + \lambda \Delta n_{w_r} \tau_{ij}^2 \left(\nabla^2 T(r, z, t)^/-i_{j+1} +
\right.ight.
\end{align*}
\]

\[
\begin{align*}
2 \left(G_c c B_B + Q_{out}^/-c \left(\Delta t\right)^3 \right)
\end{align*}
\]

\[
\begin{align*}
B_1 = 2(\Delta t)^2 + 2 \tau_{ij} \Delta t + w_r \tau_{ij}^2,
B_2 = \tau_{ij} \Delta t + w_r \tau_{ij}^2,
B_3 = 2(\Delta t)^2 + 2 \tau_{ij} \Delta t + w_r \tau_{ij}^2.
\end{align*}
\]

The final form of the FDM equation for transition \(i' \rightarrow t' (f \geq 3)\) is as follows

\[
\begin{align*}
T_{ij}'^/-A_j = & \frac{2 r_{ij} + h}{2 h^2 n_{ij}} T_{ij}'^/-i_{j+1} +
\end{align*}
\]

\[
\begin{align*}
A_i T_{ij}'^/-i_j + A_i T_{ij}'^/-i_{j+1} - A_i \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j -
\end{align*}
\]

\[
\begin{align*}
A_i \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + A_i T_{ij}'^/-i_{j+1} - A_i T_{ij}'^/-i_{j+1} +
\end{align*}
\]

\[
\begin{align*}
A_i T_{ij}'^/-i_j
\end{align*}
\]

where

\[
\begin{align*}
D = B_i B_j + 4B_i \frac{\lambda \Delta t}{h^2},
A_1 = \frac{B_i \lambda \Delta t}{D h^2},
A_2 = \frac{2B_i \lambda \Delta t}{D},
A_3 = \frac{\lambda \Delta n_{w_r} \tau_{ij}^2}{D},
A_4 = \frac{B_i c + 2B_i B_j}{D},
A_5 = \frac{\lambda \Delta n_{w_r} \tau_{ij}^2}{D},
A_6 = \frac{c w_r \tau_{ij}^2}{D},
A_7 = \frac{2 \left(G_c c B_B + Q_{out}^/-c \left(\Delta t\right)^3 \right)}{D},
\end{align*}
\]

\[
\begin{align*}
\lambda \tau_{ij}^2 \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j - (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] -
\end{align*}
\]

\[
\begin{align*}
\frac{\lambda \Delta n_{w_r} \tau_{ij}^2}{2} \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] -
\end{align*}
\]

\[
\begin{align*}
\frac{\lambda \Delta n_{w_r} \tau_{ij}^2}{2} \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] +
\end{align*}
\]

\[
\begin{align*}
\frac{\lambda \Delta n_{w_r} \tau_{ij}^2}{2} \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] +
\end{align*}
\]

\[
\begin{align*}
\lambda \frac{w_r \tau_{ij}^2}{2} \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] -
\end{align*}
\]

\[
\begin{align*}
\left(q_{ij}\right)^/-i_{j+1} + \tau_{ij} \left(\frac{\partial q_{ij}}{\partial t} + \frac{w_r \tau_{ij}^2}{2} \left(\frac{\partial^2 q_{ij}}{\partial t^2}\right)_{ij}\right)
\end{align*}
\]

Fig. 1. The mesh.

In the model presented, on the external surface of the system the Neumann boundary condition (8) is taken into account. The FDM form of this condition is the following

\[
\begin{align*}
-n \cdot \nabla T(r, z, t)_{ij}^/-i_j -
\end{align*}
\]

\[
\begin{align*}
\lambda \frac{w_r \tau_{ij}^2}{2} \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] +
\end{align*}
\]

\[
\begin{align*}
\lambda \frac{w_r \tau_{ij}^2}{2} \left[ (\nabla^2 T(r, z, t))_{ij}^/-i_j + (\nabla^2 T(r, z, t))_{ij}^/-i_{j+1} \right] -
\end{align*}
\]

\[
\begin{align*}
\left(q_{ij}\right)^/-i_{j+1} + \tau_{ij} \left(\frac{\partial q_{ij}}{\partial t} + \frac{w_r \tau_{ij}^2}{2} \left(\frac{\partial^2 q_{ij}}{\partial t^2}\right)_{ij}\right)
\end{align*}
\]
or
\[
\lambda \left[ 2(\Delta t)^2 + 2\tau_x \Delta t + w_{r \tau}^2 \right]\frac{n \cdot \nabla T(r, z, t)}{2(\Delta t)^2} - 2\lambda \tau_x \Delta t \frac{n \cdot \nabla T(r, z, t)}{2(\Delta t)^2} - \frac{\lambda w_{r \tau}^2}{2(\Delta t)^2} - \frac{n \cdot \nabla T(r, z, t)}{2(\Delta t)^2} \right]_{i,j}
\]

Using introduced previously denotations one obtains
\[
B \left( \frac{n \cdot \nabla T(r, z, t)}{\Delta t} \right)_{i,j} =
\]

\[
(\Delta t)^2 \left[ (q_b)_{i,j} + \tau_s \left( \frac{\partial q_b}{\partial t} \right)_{i,j} + \frac{w_{r \tau}^2}{2} \left( \frac{\partial \Delta q_b}{\partial t} \right)_{i,j} \right]
\]

For example, on the lateral surface \( r=R \) one has
\[
B \left( \frac{n \cdot \nabla T(r, z, t)}{h} \right)_{i,j} = 2B \left( \frac{n \cdot \nabla T(r, z, t)}{h} \right)_{i,j} + \frac{w_{r \tau}^2}{h} \left( \frac{\partial T}{\partial t} \right)_{i,j} = 0
\]

or
\[
T_{i,n}^{f-1} = T_{i,n}^{f-1} + \frac{2B}{B} \left( T_{i,n}^{f-1} - T_{i,n}^{f-1} \right) - \frac{w_{r \tau}^2}{B} \left( T_{i,n}^{f-2} - T_{i,n}^{f-2} \right)
\]

The others equations corresponding to the Neumann boundary condition are also simple and similar to equation above presented.

4 Results of computations

The cylindrical domain of dimensions \( R=0.03 \) m, \( Z=0.03 \) m is considered. Radius of the surface on which the external heat source operates is equal to \( R_0=R/4 \) (Figure 1). Thermophysical parameters of the biological tissue are the following [4]: thermal conductivity \( \lambda = 0.5 \) W/(mK), volumetric specific heat of tissue \( c = 3 \) MW/(m³ K), blood perfusion rate \( G_B =0.002 \) 1/s, volumetric specific heat of blood \( c_B = 3.9962 \) MW/(m³ K), blood temperature \( T_B = 37ºC \), metabolic heat source \( Q_m =245 \) W/m³, relaxation time \( \tau_m = 15 \) s, thermalization time \( \tau_T = 10 \) s. For \( z = Z \) the Dirichlet condition \( T_B = 37ºC \) is assumed. Three variants of tissue heating corresponding to thermal dose 50 kJ/m² are collected in Table 1. On the other surfaces of the cylinder the no-flux condition is accepted. The initial temperature of biological tissue is equal to \( T_0 = 37 ºC \).
For the assumed input data the maximum values of temperature are equal to
- first variant: $T_{\text{max}}=42.6\, ^\circ\text{C}$ (after 50 seconds) and
  $T_{\text{max}}=42.9\, ^\circ\text{C}$ (after 53.6 seconds),
- second variant: $T_{\text{max}}=44\, ^\circ\text{C}$ (after 22 seconds) and
  $T_{\text{max}}=44.8\, ^\circ\text{C}$ (after 28.8 seconds),
- third variant: $T_{\text{max}}=44.4\, ^\circ\text{C}$ (after 14.3 seconds) and
  $T_{\text{max}}=45.3\, ^\circ\text{C}$ (after 22.3 seconds).
Generally speaking, the differences between solutions grow with the increase of the external source efficiency and the shortening of its exposure time. The testing calculations show that for the small values of $q_0$ (less than 1kW/m²) the solutions resulting from the models discussed are practically the same and close to the classical Pennes model solution. In Figures 5, 6, 7 the courses of isotherms for $t=t_{\text{exp}}$, $0 \leq r \leq 0.01\, \text{m}$ and $0 \leq z \leq 0.005$ are shown.

The computations above presented have been repeated for the other values of the lag times, namely $\tau_q=\tau_T=2\, \text{s}$ (Figures 8, 9, 10). These significantly different from the previously assumed values of the delay times result from the considerations presented in the works [11, 12] in which $\tau_q$ and $\tau_T$ are dependent mainly on the porosity of the tissue. Delays determined in this way are shorter than the most frequently reported in the literature (e.g. [5, 13]) and are similar to each other. The differences between both solutions are visible, although all geometric and
thermophysical parameters have been retained (except for delay times).

Fig. 10. Temperature history - variant 3, $\tau_q=\tau_T=2$ s.

5 Conclusions

The basic goal of the paper was to compare the results of numerical solutions relating to the heating of biological tissue subjected to an external heat source. Two variants of dual-phase-lag equation have been taken into account, in particular the first- and the second-order DPLE have been considered. To achieve this aim, an algorithm based on the implicit scheme of the Finite Difference Method has been developed. The 3D axially-symmetrical tissue domain has been considered. A large external radius of the domain was assumed and then on the lateral surface of the cylinder the adiabatic condition could be accepted. The system of equations associated with the transition from time $t$ to time $t+\Delta t$ was solved using the Gaussian iteration method.

Differences between solutions corresponding to the first- and the second-order DPLE are clearly visible. In particular, they have a place in the case of the significant external source capacity (more than 1 kW/m$^2$) and the rather short exposure time.

Typical heating / cooling curves for the process under consideration and the small values of delays reach the maximum values on the heated surface (e.g. at the point (0, 0)) for $t=t_{exp}$. In the inner points (e.g. (0, 0.0005 m)) of the cylinder, the extreme value appears after a slightly longer time. The similar situation takes place for the larger values of delay times.

The assumption concerning the values of delay times is very important for the results of numerical simulations.

Comparison of the solutions obtained for $\tau_q=15$ s and $\tau_T=5$ s and the results corresponding to $\tau_q=2$ s and $\tau_T=2$ s confirms the qualitative consistency of the results, but the numerical values are strongly different for both the first- and second-order DPLE. It can also be seen that for similar values of delay times the differences between solutions (see: Figures 8 - 10) are negligible.

Summing up, it should be emphasized that the developed computer programs work correctly but their practical usefulness is determined by the adoption of the appropriate values of delay times.

Further research will be carried out to take into account the layered structure of the skin tissue[12, 14] and, as a consequence, the modeling of boundary conditions at the interface between the skin layers.

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