On unimodular tournaments

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Abstract

A tournament is unimodular if the determinant of its skew-adjacency matrix is 1. In this paper, we give some properties and constructions of unimodular tournaments. A unimodular tournament $T$ with skew-adjacency matrix $S$ is invertible if $S^{-1}$ is the skew-adjacency matrix of a tournament. A spectral characterization of invertible tournaments is given. Lastly, we show that every $n$-tournament can be embedded in a unimodular tournament by adding at most $n - \lfloor \log_2(n) \rfloor$ vertices.

Keywords: Unimodular tournament, skew-adjacency matrix, invertible tournament, skew-spectra.

MSC Classification: 05C20, 05C50.

1 Introduction

Let $T$ be a tournament with vertex set $\{v_1, \ldots, v_n\}$. The skew-adjacency matrix of $T$ is the $n \times n$ zero-diagonal matrix $S = [s_{ij}]_{1 \leq i, j \leq n}$ in which $s_{ij} = 1$ and $s_{ji} = -1$ if $v_i$ dominates $v_j$. Equivalently, $S = A - A^t$ where $A$ is the adjacency matrix of $T$. We define the determinant $\det(T)$ of $T$ as the determinant of $S$. As $S$ is skew symmetric, $\det(T)$ vanishes when $n$ is odd. When $n$ is even, the determinant is the square of the Pfaffian of $S$. Moreover, McCarthy and Benjamin [16, Proposition 1] proved that the determinant of an $n$-tournament has the same parity as $n - 1$.

Proposition 1.1. The determinant of a tournament with an even number of vertices is the square of an odd number.

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Let $T$ be a tournament. The converse of $T$, obtained by reversing all its arcs, has the same determinant as $T$. The switching is another operation that preserves the determinant. The switch of a tournament on a vertex set $V$ with respect to a subset $X$, is the tournament obtained by reversing all the arcs between $X$ and $V \setminus X$. It is well-known that if two tournaments are switching equivalent, then their skew-adjacency matrices are $\{\pm 1\}$-diagonally similar [17]. Hence, switching equivalent tournaments have the same determinant.

In this paper, we consider the class of tournaments whose skew-adjacency matrices are unimodular, or equivalently, tournaments whose determinants are equal to one. We call such tournaments unimodular. By the foregoing, this class is closed under the converse and switching operations. Examples of unimodular tournaments are transitive tournaments with an even number of vertices and their switches. The smallest tournaments that are not unimodular consist of a vertex dominating or dominated by a 3-cycle. These tournaments are called diamonds [9]. A tournament contains no diamonds if and only if it is switching equivalent to a transitive tournament [1]. Tournaments without diamonds are known as local orders [5], locally transitive tournaments [15], or vortex-free tournaments [13].

The join of a tournament $T_1$ to a tournament $T_2$, denoted by $T_1 \rightarrow T_2$, is the tournament obtained from $T_1$ and $T_2$ by adding an arc from each vertex of $T_1$ to all vertices of $T_2$. The join of a tournament $T$ to a tournament with one vertex is denoted by $T^+$. Our first main result gives a necessary and sufficient condition on the unimodularity of the join of two tournaments. It follows directly from Theorem 2.1, which will be proved in the next section.

**Theorem 1.2.** Let $T_1$ and $T_2$ be two tournaments with $p$ and $q$ vertices respectively.

i) If $p$ and $q$ are even, then $T_1 \rightarrow T_2$ is unimodular if and only if $T_1$ and $T_2$ are unimodular.

ii) If $p$ and $q$ are odd, then $T_1 \rightarrow T_2$ is unimodular if and only if $T_1^+$ and $T_2^+$ are unimodular.

Let $T$ be a unimodular tournament and let $S$ be its skew-adjacency matrix. The inverse $S^{-1}$ of $S$ is a unimodular skew-symmetric integral matrix, but its off-diagonal entries are not necessarily from $\{-1, 1\}$. We say that $T$ is invertible if $S^{-1}$ is the skew-adjacency matrix of a tournament. We call this tournament the inverse of $T$ and we denote it by $T^{-1}$. For graphs, the inverse was introduced by considering the adjacency matrix and has been studied extensively [6, 11, 12, 3, 20].

We give a spectral characterization of invertible tournaments. Moreover, we prove that every $n$-tournament can be embedded in an invertible, and hence unimodular, $2n$-tournament. The following problem arises naturally.

**Problem 1.3.** For a tournament $T$ on $n$ vertices, what is the smallest number $u^+(T)$ of vertices we must add to $T$ to obtain a unimodular tournament?

We prove that $u^+(T)$ cannot exceed $n - \lfloor \log_2(n) \rfloor$. Moreover, if the skew-adjacency matrix $S$ of $T$ is a skew-conference matrix, that is, $S^2 = (1 - n)I_n$, then $u^+(T)$ is at least $n/2$. Hence, $u^+(T)$ can be arbitrarily large.
2 The determinant of the join of tournaments

Let $T_1$ and $T_2$ be two tournaments, and let $\chi_1(x)$ and $\chi_2(x)$ be the characteristic polynomials of their adjacency matrices. Then, the characteristic polynomial of $T_1 \to T_2$ is $\chi_1(x)\chi_2(x)$. There is no similar result for the skew-adjacency matrix. However, we obtain the following result.

**Theorem 2.1.** Let $T_1$ and $T_2$ be two tournaments with $p$ and $q$ vertices respectively.

1. If $p$ and $q$ are even, then $\det(T_1 \to T_2) = \det(T_1) \cdot \det(T_2)$.
2. If $p$ and $q$ are odd, then $\det(T_1 \to T_2) = \det(T_1^+) \cdot \det(T_2^+)$.

Let $S_1$ and $S_2$ be the skew-adjacency matrices of $T_1$ and $T_2$ respectively. The skew-adjacency matrix $S$ of $T_1 \to T_2$ can be written as follows

$$S = \begin{pmatrix} S_1 & J \\ -J^t & S_2 \end{pmatrix}.$$  

where $J$ is the all ones matrix of order $p \times q$. As mentioned above, if $p$ is even, $S_1$ is non-singular. The first assertion follows from the more general result.

**Lemma 2.2.** Let $M$ be a skew-symmetric matrix of the form

$$M = \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix}.$$  

If $A$ is non-singular and $\text{rank}(B) = 1$, then

$$\det(M) = \det(A) \cdot \det(D).$$

**Proof.** Using Schur’s complement formula, we get

$$\det(M) = \det(A) \cdot \det(D + BA^{-1}B^t).$$

As $\text{rank}(B) = 1$, there exist two column vectors $\alpha$ and $\beta$ such that $B = \alpha\beta^t$. Then, we have

$$BA^{-1}B^t = \alpha\beta^tA^{-1}\beta\alpha^t.$$  

Since the matrix $A^{-1}$ is skew-symmetric and $(\beta^tA^{-1}\beta)$ is a scalar, $\beta^tA^{-1}\beta = 0$, and hence

$$\det(M) = \det(A) \cdot \det(D).$$

**Proof of Theorem 2.1.** The first assertion is already proven. For the second assertion, we prove that $\det(T_1 \to T_2) = \det(T_1^+ \to T_2^+)$. For this, consider the tournament $R$ on two vertices. It is easy to see that $T_1^+ \to T_2^+$ is switching equivalent to $(T_1 \to T_2) \to R$. Hence, $\det(T_1^+ \to T_2^+) = \det((T_1 \to T_2) \to R)$. By the first assertion, $\det((T_1 \to T_2) \to R) = \det(T_1 \to T_2) \cdot \det(R)$. Then, $\det(T_1 \to T_2) = \det(T_1^+ \to T_2^+)$, because $\det(R) = 1$. Furthermore, $\det(T_1^+ \to T_2^+) = \det(T_1^+) \cdot \det(T_2^+)$ by the first assertion.
As we have seen above, the converse and switching operations preserve unimodularity. Together with the join, these operations generate a subclass \( \mathcal{H} \) of unimodular tournaments, defined as follows.

1. The unique 2-tournament is in \( \mathcal{H} \).
2. If \( T_1, T_2 \) are in \( \mathcal{H} \), then the tournament \( T_1 \rightarrow T_2 \) is in \( \mathcal{H} \).
3. The switch of a tournament in \( \mathcal{H} \) is also in \( \mathcal{H} \).

Let \( T \) be a tournament with \( n \geq 4 \) vertices. We say that \( T \) is switching decomposable if there exist two tournaments \( T_1 \) and \( T_2 \), each with at least 2 vertices, such that \( T \) is switching equivalent to \( T_1 \rightarrow T_2 \). Otherwise, we say that \( T \) is switching indecomposable. Switching decomposability coincides with the bijoin decomposability \( [2, 4] \).

Example 2.3. For an odd integer \( n \), consider the well-known circular tournament \( C_n \) whose vertex set is the additive group \( \mathbb{Z}_n = \{0, 1, \cdots, n-1\} \) of integers modulo \( n \), such that \( i \) dominates \( j \) if and only if \( i-j \in \{1, \cdots, (n-1)/2\} \). The tournament \( C_n \) is strongly connected. However, by reversing the arcs between the even and the odd vertices we obtain a transitive tournament. Hence, \( C_n \) is switching decomposable for every odd integer \( n \geq 5 \).

Clearly, every tournament in \( \mathcal{H} \) with more than 2 vertices is switching decomposable. It is easy to check that all unimodular tournaments with at most 6 vertices are in \( \mathcal{H} \). However, we have found a switching indecomposable unimodular tournament with 8 vertices. Its skew-adjacency matrix is \( F_8 \)

\[
\begin{pmatrix}
0 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 0 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 0 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 0 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & 0 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 0
\end{pmatrix}
\]

Problem 2.4. Find an infinite family of unimodular switching indecomposable tournaments.

3 Spectral properties of unimodular tournaments

Let \( S \) be an integral skew-symmetric matrix. The nonzero eigenvalues of \( S \) are purely imaginary and occur as conjugate pairs \( \pm i\lambda_1, \ldots, \pm i\lambda_k \), where \( \lambda_1, \ldots, \lambda_k \) are totally real algebraic integers. Moreover, the norm \( N(\lambda_i) \) of \( \lambda_i \) divides the determinant of \( S \).

If \( S \) is unimodular, then \( \prod_{i=1}^{k} \lambda_i = \pm 1 \) and hence, \( \lambda_i \) are all algebraic units. Conversely, suppose that every \( \lambda_i \) is an algebraic unit. Then, the constant coefficient in the minimal polynomial of every eigenvalue is \( \pm 1 \). Hence, the determinant of \( S \) is \( \pm 1 \). In particular, we have the following result.

Proposition 3.1. A tournament is unimodular if and only if all its eigenvalues are algebraic units.
Godsil [10] proved that every algebraic integer $\lambda$ occurs as an eigenvalue of the adjacency matrix of a digraph. Estes [8] proved that if $\lambda$ is a totally real integer, that is, all its conjugates are real, then it is an eigenvalue of the adjacency matrix of a graph. Recently, Salez [18] proved that the graph may be chosen to be a tree. For tournaments, we can ask the following question.

**Question 3.2.** Let $\lambda$ be a totally real algebraic integer with an odd norm. Are there any other conditions on $\lambda$ so that $i\lambda$ is the eigenvalue of a tournament?

A similar question can also be asked about the determinant of tournaments. By Proposition 1.1, the determinant of a tournament with an even number of vertices is the square of an odd number. Conversely,

**Question 3.3.** Does there exist a tournament whose determinant is $m^2$ for every odd number $m$?

By Theorem 2.1, it is enough to consider Question 3.3 for odd prime numbers.

## 4 Invertible tournaments

Let $T$ be a tournament with skew-adjacency matrix $S$. Let $\phi_S(x) = x^n + \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \cdots + \sigma_{n-1} x + \sigma_n$ be the characteristic polynomial of $S$. Then,

$$\sigma_k = (-1)^k \sum (\text{all } k \times k \text{ principal minors}). \quad (2)$$

Since $S$ is the skew-adjacency matrix of a tournament, we have

1. $\sigma_2 = \binom{n}{2}$.
2. $\sigma_n = \det(S)$.
3. $\sigma_k = 0$ if $k$ is odd.

The determinant of a 4-tournament is 9 if it is a diamond and 1 otherwise. It follows that $\sigma_4 = 8\delta_T + \binom{n}{4}$, where $\delta_T$ is the number of diamonds in $T$. In particular, $T$ has no diamonds if and only if $\sigma_4 = \binom{n}{4}$.

If $T$ is unimodular, then the inverse $S^{-1}$ of $S$ is an integral unimodular skew-symmetric matrix. Furthermore, $\phi_{S^{-1}}(x) = x^n + \sigma_{n-2} x^{n-2} + \cdots + \sigma_2 x^2 + 1$. Hence, a necessary condition for $T$ to be invertible is $\sigma_{n-2} = \binom{n}{2}$. The following proposition shows that this condition is sufficient.

**Proposition 4.1.** Let $T$ be a unimodular $n$-tournament, and let $S$ be its skew-adjacency matrix. Then, the off-diagonal entries of $S^{-1}$ are odd. Moreover, the following assertions are equivalent.

1. $T$ is invertible.
2. Every $(n-2)$-subtournament of $T$ is unimodular.
3. The coefficient of $x^2$ in the characteristic polynomial of $S$ is $\binom{n}{2}$.
Proof. Let \([n] = \{1, \ldots, n\}\), and let \(I\) be a subset of \([n]\). Denote by \(S[I]\) the submatrix of \(S\) whose rows and columns are indexed by \(I\). Let \(i \neq j \in [n]\), it follows from Jacobi’s complementary minors theorem that

\[
\det(S^{-1}[\{i, j\}]) = \det(S[[n] \setminus \{i, j\}]).
\]

Moreover, as \(S^{-1}\) is skew-symmetric, \(\det(S^{-1}[\{i, j\}]) = (S_{ij}^{-1})^2\). Then

\[
(S_{ij}^{-1})^2 = \det(S[[n] \setminus \{i, j\}]).
\]

By Proposition 1.1, \(\det(S[[n] \setminus \{i, j\}])\) is the square of an odd number. Hence, the \((i, j)\)-entry of \(S^{-1}\) is odd.

The equivalence \((i) \Leftrightarrow (ii)\) follows directly from (4). The equivalence \((ii) \Leftrightarrow (iii)\) follows from (2) and Proposition 1.1 which implies that the determinant of tournaments with an even number of vertices is at least 1. \(\square\)

Example 4.2. Let \(T\) be an \(n\)-tournament without diamonds and let \(S\) be its skew-adjacency matrix. Every subtournament of \(T\) with an even number of vertices is unimodular. Hence, \(T\) is invertible and \(\phi_S(x) = \phi_{S^{-1}}(x) = x^n + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-2}x^2 + 1\). It follows that \(S^{-1}\) is the skew-adjacency matrix of a tournament without diamonds.

In the example above, the characteristic polynomial is palindromic, that is, the coefficients of \(x^i\) and \(x^{n-i}\) are equal. We call a tournament palindromic if the characteristic polynomial of its skew-adjacency matrix is palindromic. Let \(T\) be a tournament and let \(S\) be its skew-adjacency matrix. Clearly, if \(\phi_S(x)\) is palindromic, then \(T\) is unimodular, the inverse of \(S\) is the skew-adjacency matrix of a tournament and \(\phi_S(x) = \phi_{S^{-1}}(x)\).

Let \(T\) be an \(n\)-tournament with vertex set \(V = \{v_1, \ldots, v_n\}\), and let \(S\) be its skew-adjacency matrix. Let \(\hat{T}\) be the tournament obtained from \(T\) by adding a copy \(T'\) of \(T\) with vertex set \(\{v'_1, \ldots, v'_n\}\), such that \(v_i\) dominates \(v'_i\) and \(v_i\) dominates \(v'_j\) if and only \(v_i\) dominates \(v_j\). The skew-adjacency matrix \(\hat{S}\) of \(\hat{T}\) can be written as follows.

\[
\hat{S} = \begin{pmatrix}
S & S + I_n \\
S - I_n & S
\end{pmatrix}.
\]

The inverse of \(\hat{S}\) is \(\begin{pmatrix}
S & -(S + I_n) \\
-(S - I_n) & S
\end{pmatrix}\). Then \(\hat{T}\) is invertible. Moreover, \(\hat{T}\) and \(\hat{T}^{-1}\) are switching equivalent. Indeed,

\[
\hat{S}^{-1} = D\hat{S}D.
\]

where \(D = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}\). It follows that \(\phi_{S}(x) = \phi_{(\hat{S})^{-1}}(x)\), and hence \(\phi_{S}(x)\) is palindromic.

Remark 4.3. Let \(I\) be a nonempty proper subset of \([2n]\). By (5), \(\det(\hat{S}[I]) = \det(\hat{S}^{-1}[I])\). Moreover, using Jacobi’s complementary minors theorem, \(\det(\hat{S}^{-1}[I]) = \det(\hat{S}[[2n] \setminus I])\). It follows that \(\det(\hat{S}[I]) = \det(\hat{S}[[2n] \setminus I])\).
5 Embedding of tournaments in unimodular tournaments

In the previous section, we proved that every \( n \)-tournament can be embedded in a unimodular \( 2n \)-tournament. For a tournament \( T \) on \( n \) vertices, let \( u^+(T) \) be the smallest number of vertices we must add to \( T \) to obtain a unimodular tournament. A dual notion of \( u^+(T) \) is to consider the minimum number \( u^-(T) \) of vertices we must remove from \( T \) to obtain a unimodular tournament. It follows from Theorem 4.2 that if \( T_1 \) and \( T_2 \) are two tournaments, then

\[
\begin{align*}
    u^+(T_1 \rightarrow T_2) &\leq u^+(T_1) + u^+(T_2), \\
    u^-(T_1 \rightarrow T_2) &\leq u^-(T_1) + u^-(T_2).
\end{align*}
\]

(6) \hspace{1cm} (7)

It is shown in [7] that every \( n \)-tournament \( T \) contains a transitive subtournament of order at least \( \lceil \log_2(n) \rceil + 1 \). In particular, it contains a unimodular tournament of order at least \( \lceil \log_2(n) \rceil \). Then, \( u^-(T) \leq n - \lceil \log_2(n) \rceil \).

The following proposition provides a relationship between \( u^+(T) \) and \( u^-(T) \).

**Theorem 5.1.** Let \( T \) be an \( n \)-tournament. Then,

\[ u^+(T) \leq u^-(T). \]

In particular, \( u^+(T) \leq n - \lceil \log_2(n) \rceil \).

**Proof.** Let \( V = \{v_1, \ldots, v_n\} \) be the vertex set of \( T \). Consider the \( 2n \)-tournament \( \hat{T} \) obtained from \( T \) and a copy \( T' \) of \( T \) as described in the previous section. There exists \( I \subset V', |I| = u^-(T') \), such that \( T[V' \setminus I] \) is unimodular. By Remark 4.3 \( \det(\hat{T}[V' \setminus I]) = \det(T[V \cup I]) = 1 \). Moreover, the tournament \( \hat{T}[V \cup I] \) contains \( T \), hence \( u^+(T) \leq u^-(T) \). \( \square \)

**Remark 5.2.** Equality in Theorem 5.1 may be strict. Indeed, let \( T \) be the tournament whose skew-adjacency matrix is \( S \).

\[
S = \begin{pmatrix}
0 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & 0 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & 0 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 0
\end{pmatrix}.
\]

By adding a vertex dominating \( T \) we obtain a unimodular tournament, hence \( u^+(T) = 1 \). The tournament \( T \) has no unimodular \((n-1)\)-subtournament. Moreover, removing the last three rows of \( S \) and their corresponding columns yields the skew-adjacency matrix of a unimodular tournament, hence \( u^-(T) = 3 \). This example was found using SageMath [19].

In what follows, we give a lower bound on \( u^+(T) \), using the spectra of the skew-adjacency matrix of \( T \).
Theorem 5.3. Let $T$ be a non-unimodular $n$-tournament and let $\nu(T)$ be the maximum multiplicity among the non-unit eigenvalues of its skew-adjacency matrix. Then,

$$\nu(T) \leq u^+(T).$$

To prove this theorem, we need the following lemma, which is a direct consequence of Cauchy Interlace Theorem.

Lemma 5.4. Let $A$ be a hermitian matrix of order $m$, and let $B$ be a principal submatrix of $A$ of order $n$, with an eigenvalue $\lambda$ of multiplicity $r$. If $m-n < r$, then $\lambda$ is an eigenvalue of $A$.

Proof of Theorem 5.3. Let $T$ be a non-unimodular $n$-tournament and let $i\lambda$ be a non-unit eigenvalue of its skew-adjacency matrix $S$ with multiplicity $\nu(T)$. Let $T'$ be an $m$-tournament containing $T$ such that $m < n + \nu(T)$ and denote by $S'$ its skew-adjacency matrix. Clearly, $\lambda$ is an eigenvalue of $iS$ with multiplicity $\nu(T)$. Then, by Lemma 5.4, $\lambda$ is also an eigenvalue of $iS'$. Hence, $S'$ has a non-unit eigenvalue. It follows from Proposition 3.1 that $T'$ is not unimodular.

Tournaments with large $\nu(T)$ can be obtained from skew-conference matrices. Let $T$ be an $n$-tournament and let $S$ be its skew-adjacency matrix. Assume that $S$ is a skew-conference matrix. It follows that the eigenvalues of $S$ are $\pm i\sqrt{n-1}$ each with multiplicity $n/2$. As $i\sqrt{n-1}$ is not an algebraic unit, then $\nu(T) = n/2$. Hence, by Theorem 5.3, $u^+(T) \geq n/2$.

It is conjectured that skew-conference matrices exist if and only if $n = 2$ or $n$ is divisible by 4 [21]. If this conjecture is true, Lemma 5.4 implies that for every integer $n \geq 4$, there exists an $n$-tournament $T$ such that $u^+(T) \geq \frac{n-3}{2}$. Denote by $u^+(n)$ the maximum $u^+(T)$ among $n$-tournaments. By the forgoing, we have the following theorem.

Theorem 5.5. Assuming the existence of skew-conference matrices of every order divisible by 4, we have

$$\frac{n-3}{2} \leq u^+(n) \leq n - \lceil \log_2(n) \rceil.$$

Examples of tournaments with a skew-conference matrix can be obtained from Paley tournaments. For a prime power $q \equiv 3 \mod 4$, the Paley tournament with $q$ vertices is the tournament whose vertex set is the Galois field $GF(q)$, such that $x$ dominates $y$ if and only if $x - y$ is a nonzero quadratic residue in $GF(q)$. There are many other infinite families of skew-conference matrices, see for example [14].

6 Concluding remarks

The main concern of this paper is the determinant of the skew-adjacency matrix of tournaments. A multiplicative formula for the determinant of the join of two tournaments was given. This formula provides a new construction of unimodular tournaments. Another construction is the blow-up operation, in which every vertex of a tournament is replaced by a tournament with two vertices. This construction shows that every $n$-tournament can be embedded in a $2n$-unimodular tournament for which the inverse of the skew-adjacency matrix is also the skew-adjacency matrix of a tournament. The
minimum number of vertices that must be added to a tournament to be unimodular is considered. We showed that it does not exceed the minimum number of vertices to be removed to obtain a unimodular tournament, and that it is related to the multiplicity of its non-unit eigenvalues.

In addition to the problems presented, many other questions and directions can be considered.

- The construction of the class \( H \), considered in Section \( \ref{section} \) is simple. Nevertheless, this family seems to be rich as it can be proven, by induction, that the blow-up of every tournament is in \( H \). Is a positive proportion of the set of unimodular tournaments in \( H \)?

- Find examples of invertible tournaments that are not palindromic.

- The problem of finding \( u^+(T) \) seems extremely hard. We suspect that there is no polynomial time algorithm to solve this problem. Find a non-brute force algorithm to compute \( u^+(T) \).

- As we have seen above, skew-conference matrices have non-unit eigenvalues with maximum possible multiplicities. Another property of skew-conference matrices is that they have maximum determinant among zero-diagonal \( \{-1, 1\} \)-matrices. Do tournaments with skew-conference adjacency matrices have maximum \( u^+(T) \)?

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