Rational curves of degree 10 on a general quintic threefold

Ethan Cotterill

December, 2004

Abstract

We prove the “strong form” of the Clemens conjecture in degree 10. Namely, on a general quintic threefold $F$ in $\mathbb{P}^4$, there are only finitely many smooth rational curves of degree 10, and each curve $C$ is embedded in $F$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Moreover, in degree 10, there are no singular, reduced, and irreducible rational curves, nor any reduced, reducible, and connected curves with rational components on $F$.

Introduction

Almost twenty years ago, Clemens conjectured that the number of smooth rational curves of fixed degree on a general quintic threefold in $\mathbb{P}^4$ is finite. In 1986, Sheldon Katz [Ka] proved Clemens’ conjecture in degrees at most 7. His method of proof was in two steps. First he used a deformation-theoretic argument to show that

Clemens’ conjecture holds for smooth rational curves $C$ of degree $d$ provided the incidence scheme

$$\Phi_d := \{(C, F) \mid F \text{ a quintic containing } C\}$$

is irreducible.

He then proved that $\Phi_d$ is irreducible for all $d \leq 7$, by arguing that the fibres of its projection onto the Hilbert scheme of smooth degree-$d$ curves are equidimensional projective spaces. Indeed, equidimensionality follows immediately from a vanishing result for ideal-sheaf cohomology of Gruson, Lazarsfeld, and Peskine.
Building on Katz’s work, Johnsen and Kleiman \cite{JK1} next showed that the only reduced connected curves of degree \(d\) at most 9 on \(F\) with rational components are irreducible and either smooth or six-nodal plane quintics. (In the meantime, Vainsencher \cite{Va} had discovered that there are 17,601,000 six-nodal plane quintics on \(F\). For a proof of the latter assertion, see \cite{KIP2}.) To establish that the generic threefold \(F\) contains curves of the asserted sort, Johnsen and Kleiman first observe that the space

\[ M_d := \{ \text{degree-}d \text{ morphisms } f : \mathbb{P}^1 \to \mathbb{P}^4 \} \]

is stratified by locally-closed subsets \(M_{d,i}\), where \(i := h^1(I_C/\mathbb{P}^4(5))\) and \(C\) is the image of \(f\). Pulling back by \(\pi_d : \Phi_d \to M_d\), they obtain a corresponding stratification of the incidence scheme \(\Phi_d\) into loci \(I_{d,i}\), and they show that the projection \(I_{d,i} \to M_{d,i}\) is dominant exactly over \(M_{d,0}\), where all the fibres are equidimensional.

Johnsen and Kleiman apply the same result of Gruson–Lazarsfeld–Peskine used by Katz, as well as a related result of d’Almeida, to show that whenever the fibre dimension \(h^0(I_C/\mathbb{P}^4(5)) - 1\) is not the expected one, then the curve \(C\) necessarily admits a highly-incident secant line. Smooth curves with high-incidence secants comprise a locally closed locus of high codimension inside the Hilbert scheme; Johnsen and Kleiman settle Clemens’ conjecture for smooth curves by obtaining appropriate bounds on the dimensions of the fibres \(\pi_d^{-1}(C)\) over this locus.

To handle singular irreducible and reduced rational curves \(C\), Johnsen and Kleiman use the same basic approach, but are forced to work harder, because for singular curves the arithmetic genus \(g(C)\) varies over a finite nonzero range. To bound this variation, they apply a classical result of Castelnuovo–Halphen giving the maximal possible arithmetic genus of each curve in terms of the dimension of the minimal linear space it spans. Next, they stratify low-genus curves according to the dimension of their spans, as well as the types of their singularities, and analyze the number of linear independent conditions imposed on curves in each case. By controlling the dimension of each stratum inside the appropriate mapping space of non-degenerate rational curves, they are able to conclude that the projection \(\Phi_d \to M_d\) is never dominant over the locus of singular curves.

Johnsen and Kleiman also show that no reducible curves lie on the general quintic. To do so, they bound the length of the intersections of curve components, and thereby obtain lower bounds for the codimension of such reducible curves inside the Hilbert scheme. Those bounds allow them to conclude.

The next case to check, which is the subject of this work, is that of rational curves of degree 10. It is interesting in its relation to mirror symmetry.
In fact (see [CK, p.206]), the finiteness of the Hilbert scheme of degree-10 rational curves on the general quintic smooth implies that the instanton number \( n_{10} \) is given by

\[ n_{10} = 6 \times 17,601,000 + \#\{\text{smooth rational curves of degree 10 in } F\}. \]

A couple of comments are in order. First of all, Clemens’ original conjecture predicted that only smooth rational curves lie on a general quintic \( F \). Clemens’ conjecture in its revised form predicts that the six-nodal plane quintics are the only singular, reduced, and irreducible rational curves on \( F \). On the other hand, mirror symmetry includes Vainsencher’s singular quintics in its count of rational curves of degree five, but fails to count six double covers corresponding to each of these. So we see both that 10 is the first \( d \) for which the instanton number \( n_d \) fails to count smooth rational curves of degree \( d \) on the general quintic \( F \), and that the discrepancy between \( n_{10} \) and the actual number of smooth rational curves is accounted for by double covers of nodal plane quintics.

In this paper, we extend the results of Johnsen and Kleiman to include curves of degree 10. In doing so we adopt their basic strategy to show that the only rational, reduced, irreducible curves of degree 10 on a general quintic hypersurface \( F \) are smooth, and that there are only finitely many such smooth curves. Since the results of [GLP] are less useful than before, however, we are forced to use a different technique to estimate the dimensions of the fibres of the projection \( I_{10} \to M_{10} \). In order to make our dimension estimates, we study the generic initial ideals which result from degenerations of rational curves. By a combinatorial analysis, we are able to control the cohomology of generic initial ideals, and deduce uniform bounds on \( h^0(\mathcal{I}_C/P^4(5)) \), where \( C \) is a rational curve of degree 10.

There are several reasons why studying the combinatorics of the generic initial ideal, or \( gin \), is useful (and feasible). First, it is a monomial ideal fixed by the action of the group of upper triangular matrices, so it has a rather simple structure. In addition, its Castelnuovo–Mumford regularity is the same as that of (the ideal sheaf of) the curve \( C \). Finally, the presentation of the generic initial ideal \( \mathcal{I}_C \) for the reverse-lexicographic term order is closely related to the presentation of \( \mathcal{I}_{C\cap H/H} \), where \( H \) is a general hyperplane. Since the general hyperplane section of a curve \( C \) is a collection of points in uniform position (see, for example, [EH, p. 85]), there are natural restrictions on the Hilbert functions and regularity properties of the corresponding ideals.

By systematically using all of the preceding considerations, we are able to obtain relatively robust bounds on the higher cohomology of \( \mathcal{I}_C/P^4(5) \),
when $C$ is the image of any map $f: \mathbb{P}^1 \to \mathbb{P}^4$ outside a certain special locus inside the mapping space $M_{10}$. By combining our bounds on cohomology with the estimates on parameter-space dimensions given in [JK1], we are able to extend the results of [JK1] to degree 10. To our knowledge, our method constitutes a new application of generic initial ideals to problem-solving in algebraic geometry; hopefully, it will be a useful tool for work on other problems, too.

Hereafter, a “generic” quintic hypersurface in $\mathbb{P}^4$ means one cut out by a polynomial $F$ belonging to a Zariski-dense open subset of the family $\mathbb{P}_{125}^4$ of degree-5 polynomials on $\mathbb{P}^4$. We always work over the complex numbers $\mathbb{C}$. Moreover, the abbreviations $I_C$ and $I_\Gamma$ are used in place of $I_{C/\mathbb{P}^4}$ and $I_{\Gamma/\mathbb{P}^3}$, respectively, for ideal sheaves of curves $C \subset \mathbb{P}^4$ and their hyperplane sections. When we refer to the regularity of a variety $X$, we mean the Castelnuovo–Mumford regularity of the ideal sheaf, $\text{reg } X := \text{reg } I_X$. Similarly, whenever $I$ is a homogeneous ideal we take $H^i(I)$ and $\text{reg } I$ to mean the $i$th cohomology group and the regularity, respectively, of the sheaf associated to $I$. The notation $I_{X,m}$ denotes the $m$th graded piece of the homogeneous ideal $I_X$. Similarly, $I_X|_H := I_X \otimes \mathcal{O}_H$ denotes the restriction of $I_X$ to the hyperplane $H$. Likewise, $(I_X)_{x_n}$ denotes $\bigcup_{i=1}^\infty I_X : (x_n^i)$, the saturation of $I_X$ with respect to the element $x_n$.

A minimal generator of a monomial ideal $I$ means a monomial that is minimal for the partial order defined by divisibility. (A basic fact from commutative algebra is that every monomial ideal $I$ admits a unique set of minimal generators, and these generate $I$.) The term genus $g(C)$ of a one-dimensional scheme $C$ always denotes arithmetic genus, that is, $1 - \chi(O_C)$, where $\chi$ is the Euler characteristic of $O_C$. The degree or genus of a homogeneous ideal $I$ refers to the degree or genus of the projective scheme defined by $I$. Finally, we only consider initial ideals for the reverse-lexicographic, or revlex, term order, so henceforth $\text{gin } I$ denotes the generic initial ideal of $I$ with respect to reverse-lexicographic order. Revlex is distinguished among term orders for being “best-behaved” with regard to intersections with hyperplanes. (For a discussion of term orders, see [Ei, Ch. 15] and [Gre].)

The paper is in three sections. In section II we recall the basic theory of generic initial ideals, in order to show how to compute the basic cohomological invariants of such ideals. To bound $h^1(I_C(5))$, we first describe the initial ideals of general hyperplane sections of irreducible curves of degree 10, via a result of [Bal] that bounds their Castelnuovo–Mumford regularity. Next we use a result of [BS] to relate the initial ideals of curves to those of their hyperplane sections.
In section 2, the strong form of Clemens’ conjecture for irreducible rational curves of degree 10 in $\mathbb{P}^4$ is proved. To do so, we establish bounds on $h^1(I_C(5))$ case by case for rational curves $C$ of degree 10. To conclude that the bounds we obtain are adequate, we appeal to a result of Verdier (see [Ve, Thm., p.139] and [Ra, Thm.1, p.181]) that describes the stratification of the space of all morphisms $f : \mathbb{P}^1 \to \mathbb{P}^4$ of a given degree according to the splitting of the restricted tangent bundle $f^*T_{\mathbb{P}^4}$.

Section 3 extends Johnsen and Kleiman’s analysis of reduced, reducible, and connected rational curves to cover those of degree 10. As in [JK1], we stratify curves according to the length of the intersection of their components. To bound the dimensions of these loci, we use a simple argument involving the relative Hilbert scheme of morphisms, in place of the arguments in local coordinates given in [JK1]. Regularity arguments from the latter paper carry over, with the exception of a regularity lemma for unions of nodal quintics. We prove that such curves are 6-regular (so, in particular, they verify $h^1(I_C(5)) = 0$) by showing their initial ideals are generated by polynomials of degree 6 or less.

Many of the methods discussed in this paper carry over to degree 11; however, one needs to work even harder to bound $h^1IC(5)$. In a preprint to be distributed shortly, the author verifies Clemens’ conjecture in degree 11, by showing that those curves which do not satisfy suitable bounds on $h^1IC(5)$ necessarily lie on a large number of linearly independent hypercubics in $\mathbb{P}^4$. A liaison-theoretic argument shows that such curves necessarily lie on surfaces of degree at most 8, and the author concludes by showing that the locus of rational curves lying on such surfaces has small dimension.

Acknowledgements.

I am grateful to I. Coskun, J. Harris, and A. Iarrobino, for providing me with valuable advice during the preparation of this paper. I am especially grateful to S. Kleiman, for an unending supply of input, encouragement, and patience.

1 Generic initial ideals

Given any homogeneous ideal $I$ in $n + 1$ variables $x_i$, together with a choice of partial order $<$ on the monomials $m(x_i)$, the initial ideal of $I$ with respect to $<$ is, by definition, generated by leading terms of elements in $I$. So for any subscheme $C$ of $\mathbb{P}^n$, upper-semicontinuity implies that the regularity
and values $h^j$ of the ideal sheaf $\mathcal{I}_C$ are majorized by those of the (sheaf associated to the) initial ideal $\text{in}(\mathcal{I}_C)$ with respect to any partial order on the monomials of $\mathbb{P}^n$. On the other hand, Macaulay’s theorem establishes that the Hilbert functions of $\mathcal{I}_C$ and $\text{in}(\mathcal{I}_C)$ agree. If, moreover, we replace $C$ by a general $\text{PGL}(n+1)$-translate of $C$, then

$$\text{reg}(\mathcal{I}_C) = \text{reg}(\text{in}(\mathcal{I}_C)),$$

and $\text{in}(\mathcal{I}_C)$ is called the generic initial ideal of $C$, written $\text{gin}(\mathcal{I}_C)$.

A theorem of Galligo establishes that there is a unique generic initial ideal associated to any ideal. Further, Bayer and Stillman showed that the following three statements are equivalent.

1. The monomial ideal $\text{gin} \mathcal{I}_C$ is saturated, in the ideal-theoretic sense. In other words, for all polynomials $g \in \text{gin} \mathcal{I}_C$, $m \in (x_0, \ldots, x_n)$,

$$g \cdot m \in \text{gin} \mathcal{I}_C \Rightarrow g \in \text{gin} \mathcal{I}_C.$$

2. The homogeneous ideal $\mathcal{I}_C$ is saturated.

3. No minimal generator of $\text{gin} \mathcal{I}_C \subset \mathbb{C}[x_0, \ldots, x_n]$ is divisible by $x_n$.

For proofs, see [Gre, Thm. 2.30]. More generally, let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be any ideal that is Borel-fixed, i.e., fixed under the action of upper triangular matrices $\mathcal{T} \subset \text{PGL}(n+1)$; then $I$ is saturated if and only if none of its minimal generators is divisible by $x_n$. Hereafter, we work exclusively with saturated generic initial ideals. In the subsections to follow, we reduce the problem of bounding $h^1(\mathcal{I}_C(5))$, for reduced curves $C$, to the problem of bounding $h^1(\mathcal{I}_X(5))$, for $X$ belonging to a certain finite set of Borel-fixed ideals. We also give an explicit procedure for computing $h^1(\mathcal{I}_X(5))$.

### 1.1 Hyperplane sections of nondegenerate irreducible curves

In this subsection, we examine monomial ideals that arise as gins of hyperplane sections, or hyperplane gins for short, of nondegenerate integral rational curves $C$ of degree 10. More to the point, we examine monomial ideals corresponding to hyperplane gins that are general in a sense we will now make precise.

Begin by fixing $C$. Choose general homogeneous coordinates $x_0, \ldots, x_4$ for $\mathbb{P}^4$, and set $H := \{x_4 = 0\}$. Then $H$ is a general hyperplane, whose intersection with $C$ is a set of ten points in uniform position: the set’s monodromy is the full symmetric group on ten letters. Associated to the saturated ideal defining $C \cap H$, there is a corresponding Borel-fixed monomial
ideal in \( \mathbb{C}[x_0, \ldots, x_3] \), the saturated generic initial ideal \( \mathcal{I}_{C \cap H} \). This ideal is invariant under general linear transformations of the coordinates \( x_0, \ldots, x_3 \). We therefore call it the hyperplane \( \text{gin} \) of \( C \).

As explained in the paragraphs preceding this subsection, saturatedness and Borel-fixity together imply that the hyperplane \( \text{gin} \) of \( C \) is minimally generated by monomials in the first three variables \( x_0, x_1, \) and \( x_2 \). Moreover, because it cuts out a zero-dimensional scheme, \( \mathcal{I}_{C \cap H} \) has some minimal generator of the form \( x_2^\lambda \) with \( \lambda > 0 \). Indeed, if \( I := \text{gin} \mathcal{I}_{C \cap H} \) contained no such minimal generator, then the vanishing locus \( V(I) \) would contain a 1-dimensional projective subspace of vectors \( \{(0, 0, a_2, a_3) : a_1, a_2 \in \mathbb{C}\} \), which is absurd according to the dimension theorem [CLO, Thm.9.3.11] for monomial ideals.

On the other hand, Borel-fixity implies the hyperplane \( \text{gin} \) is combinatorially simple: Recall that an ideal \( I \) is Borel-fixed if and only if for every monomial \( P \), \[ P^* := \frac{x_i}{x_j} \cdot P \] belongs to \( I \) whenever \( i < j \); see [Ei] Thm.15.23. In constructing minimal generating sets for Borel-fixed monomial ideals, we make systematic use of this fact without mentioning it.

Any set of minimal generators for the general hyperplane \( \text{gin} \) of \( C \) may be represented by a tree \( T \), or directed graph without cycles. Namely, fix an alphabet \( \mathcal{A} := \{\emptyset, x_0, x_1, x_2\} \), and consider the set of all trees with root vertices \( \emptyset \) and all other vertices labeled by either \( x_0, x_1, \) or \( x_2 \). Call terminal vertices leaves. Then the unique path from the root vertex \( \emptyset \) to any leaf labeled \( x_i \), determines a sequence of vertices \( \emptyset, x_{i_1}, x_{i_2}, \ldots, x_{i_l} \). (We will only be concerned with trees associated to nonzero ideals, so \( \emptyset \) will never arise as a leaf.) It’s natural to interpret the string \( x_{i_1} x_{i_2} \cdots x_{i_l} \) as a polynomial of degree \( l \). Likewise, if the unique path from the root vertex to a given vertex \( v \) involves \( d \) edges, then we say that \( v \) has degree \( d \). A rewriting rule applied to a leaf \( v \) is then a formal operation on \( T \) that at \( v \) glues a certain number of new edges \( e_i(v) \) terminating in vertices \( v_i(v) \). The result is a new tree \( T' \) in which the vertices \( v_i(v) \) are leaves of degree \( d + 1 \). Say that a vertex \( V_1 \) dominates a vertex \( V_2 \) if it is closer to the root vertex \( \emptyset \). We also stipulate that, if \( V_i \) with label \( x_i \) dominates \( V_j \) with label \( x_j \), then \( i \leq j \).

For each homogeneous monomial ideal \( I \), the unique minimal generating set of \( I \) determines a tree \( T(I) \) whose leaves correspond to minimal generators of \( I \). The assignment \( I \mapsto T(I) \), moreover, is unique. Therefore, we treat rewriting rules interchangeably as operations on either ideals or trees.

To clarify our terminology, here are a couple of simple examples of zero-dimensional subschemes of \( \mathbb{P}^3 \) and the minimal generating sets of the corre-
Table 1: A-rules for nondegenerate curves in \( \mathbb{P}^4 \)

1. \( x_0^e \mapsto (x_0^{e+1}, x_0^e x_1, x_0^e x_2) \),
2. \( x_0^e x_1^f \mapsto (x_0^e x_1^{f+1}, x_0^e x_1^f x_2) \),
3. \( x_0^e x_1^f x_2^g \mapsto x_0^e x_1^f x_2^{g+1} \), and an initial rule
4. \( \emptyset \mapsto (x_0, x_1, x_2) \).

Corresponding ideals. There is exactly one zero-dimensional Borel-fixed ideal of degree 1, namely \( I_1 = (x_0, x_1, x_2) \). Similarly, \( I_2 = (x_0, x_1, x_2^2) \) is the unique zero-dimensional Borel-fixed ideal of degree 2. Note that a minimal generating set for \( I_2 \) may be obtained from one for \( I_1 \) by exchanging the minimal generator \( x_2 \) for \( I_1 \) with the minimal generator \( x_2 \) for \( I_2 \). In other words, a set of minimal generators for \( I_2 \) may be obtained from a set of minimal generators for \( I_1 \) by replacing \( x_2 \) with \( x_2^2 \) and keeping the other generators.

The corresponding operation on trees adds a new vertex \( v(x_2^2) \) to \( T(I_1) \) and connects \( v(x_2^2) \) with the vertex \( v(x_2) \) corresponding to \( x_2 \in I_1 \) along a new edge. In other words, to obtain \( T(I_2) \) from \( T(I_1) \) it suffices to apply a single rewriting at the vertex \( x_2 \), that we denote by \( x_2 \mapsto x_2^2 \).

In general, as we show in the next subsection, any hyperplane section \( g \) of a curve may be constructed by applying a sequence of rewriting rules of the form \( X_k \mapsto Y_{k+1} \) to the “empty ideal,” \( \emptyset \). Here \( X_k \) is a single monomial and \( Y_{k+1} \) is a set of monomials that “replace” \( X_k \). In other words, \( \emptyset \) and \( I \) fit into a sequence of monomial ideals \( I_0 = \emptyset, \ldots, I_n = I \) where \( I_{k+1} \) is the ideal generated by all monomial minimal generators of \( I_k \) except \( X_k \), together with a set of monomials \( Y_{k+1} \) of degree \( \deg(X_k) + 1 \).

1.2 Rewriting rules for nondegenerate hyperplane gins

Fix a nondegenerate curve \( C \), together with a hyperplane \( H \) that is general in the sense that \( C \cap H \) is a collection of 10 distinct points in uniform position. Letting \( I \) denote the corresponding hyperplane gin, we see that \( I \) is a homogeneous Borel-fixed ideal in the coordinates \( x_i, 0 \leq i \leq 3 \), of \( H \). It follows from the discussion of subsection 1.2 that \( I \) is saturated if and only if none of its minimal generators is divisible by \( x_3 \), and such a saturated ideal defines a zero-dimensional scheme if and only if some minimal generator of \( I \) is of the form \( x_2^\lambda \).

Our next task is to describe a set of rewriting rules for hyperplane gins, or \( \Lambda \)-rules, that accounts for all possible \( I \) arising as above. A candidate for a complete list of \( \Lambda \)-rules is given in Table[1]. Before showing that it is
complete, a few words are in order.

First of all, it is clear that the proposed Λ-rules preserve saturatedness and zero-dimensionality. Moreover, for every Λ-rule $X_k \mapsto Y_{k+1}$ that replaces the ideal $I_k$ by $I_{k+1}$, the set of monomials $Y_{k+1}$ are in fact minimal generators of $I_{k+1}$. Similarly, Λ-rules preserve Borel-fixity because the combinatorial criterion that characterizes the property is satisfied at every step. So applying a sequence of Λ-rules to the set of minimal generators of a hyperplane gin yields a set of minimal generators for a Borel-fixed ideal of dimension zero.

As explained in the preceding subsection, it is also useful to interpret the Λ-rules graphically. Namely, applying the Λ-rule $X_k \mapsto Y_{k+1}$ alters a tree of minimal generators for $I_k$ by gluing Card($Y_{k+1}$) new edges onto the leaf corresponding to $X_k$. Therefore, at the risk of creating additional confusion, we mean to make more transparent the correspondence between gluing operations on trees of minimal generators and replacement operations on sets of minimal generators.

Finally we show that the Λ-rules comprise a complete set of rewriting rules for hyperplane gins.

**Lemma 1.2.1.** The minimal generating set of every nondegenerate hyperplane gin may be realized by applying Λ-rules.

**Proof.** Let $C$ and $\Gamma$ respectively denote a nondegenerate curve $C \subset \mathbb{P}^4$ and a general hyperplane section of the curve. Recall from subsection 2.2 that $\text{gin}(I_\Gamma)$ is Borel-fixed, and has a minimal generator of the form $x_2^\lambda$, which is necessarily of maximal degree among all minimal generators of $\text{gin}(I_\Gamma)$.

Now consider the set of leaves $l$ of maximal degree $\lambda$ in $T(\text{gin}(I_\Gamma))$. Letting $v_d(l)$ denote the vertex dominating $l$, note that $v_d(l)$ verifies this combinatorial property: if $v_d(l)$ is labeled $x_0$ (resp. $x_1, x_2$), then $v_d(l)$ dominates a 3-tuple of vertices $x_0, x_1, x_2$ without omissions (resp., a pair $x_1, x_2$ or a singleton $x_2$). If $v_d(l)$ is labeled $x_2$, then the property holds trivially, by the conventions for trees established in the previous subsection. In the other two cases, the observation follows from Borel-fixity, together with the fact that $x_2^\lambda$ lies in $\text{gin}(I_\Gamma)$.

Note, moreover, that the tree obtained from $T(\text{gin}(I_\Gamma))$ by “pruning”, or removing all leaves of maximal degree and contracting the corresponding edges to their dominating vertices, is also Borel-fixed, saturated, and zero-dimensional. By induction on degree, it follows that every vertex of $T(\text{gin}(I_\Gamma))$ verifies the combinatorial property of the previous paragraph.
Table 2: C-rules for nondegenerate irreducible curves

|   | Rule                                                                 |
|---|----------------------------------------------------------------------|
| 1 | $x_0^e \mapsto x_0^e \cdot (x_0, x_1, x_2, x_3)$                     |
| 2 | $x_0^e x_1^f \mapsto x_0^e x_1^f \cdot (x_1, x_2, x_3)$, and         |
| 3 | $x_0^e x_1^f x_2^g \mapsto x_0^e x_1^f x_2^g \cdot (x_2, x_3)$       |

On the other hand, any tree generated by Λ-rules may be pruned, as pruning is clearly an inverse for rewriting. The same argument by induction on degree invoked in the preceding paragraph now shows that those trees obtained by Λ-rules are exactly those verifying the combinatorial property above.

Now let $X$ be any subscheme of $\mathbb{P}^n$, and let $\Gamma$ be a hyperplane section defined by $\Gamma := X \cap H$ where $H$ is cut out by a general linear form. Make a general choice of coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ with respect to which $H$ is defined by $x_n = 0$. As explained in [Gre, p. 163], the surjectivity of the restriction map

$$I_{X,m} \rightarrow I_{\Gamma,m},$$

for $m$ sufficiently large, implies that

$$\text{sat}(I_{X|H}) = I_{\Gamma}.$$

According to [Gre, Prop. 2.21], the latter equality of ideals may be formulated as the following fact relating the generic initial ideal of $X$ to the generic initial ideal of a general hyperplane $\Gamma$.

**Fact 1.** The saturation of $\text{gin}(I_{X|H})$ with respect to $x_{n-1}$ is equal to $\text{gin}(I_{\Gamma})$.

Note that the transformation $\text{gin}(I_{X|H}) \mapsto \text{gin}(I_{X|H}) x_{n-1}$ induces a graphical transformation $T(\text{gin}(I_{X|H})) \mapsto T(\text{gin}(I_{X|H}) x_{n-1})$ given by iteratively contracting all edges in the tree dominated by vertices that dominate $x_{n-1}$-labeled vertices, until no leaves labeled $x_{n-1}$ remain. On the other hand, the contractions taking $T(\text{gin}(I_{X|H}))$ to $T(\text{gin}(I_{X|H}) x_{n-1})$ may be viewed as transformations of trees of (minimal generators of) homogeneous ideals of $\mathbb{C}[x_0, \ldots, x_n]$ that are inverse to certain rewriting rules, which we call C-rules and display in Table 2. Therefore, Fact 1 may be reformulated in the following useful way.

The tree corresponding to any generic initial ideal defining a nondegenerate subscheme $X \subset \mathbb{P}^4$ of dimension 1 is obtained from the gin of a general hyperplane section of $X$ by applying a sequence of C-rules.
Since hyperplane gins are generated by $\Lambda$-rules, we have proved the following statement, which is essential for what follows and will be used hereafter without comment.

*Every Borel-fixed monomial ideal associated to a curve $C \subset \mathbb{P}^4$ may be represented by a tree obtained from the empty symbol by applying a certain sequence of $\Lambda$-rules, followed by a certain sequence of $C$-rules.*

Likewise, we will make use of the following result of Bayer-Stillman (see [Gre, Thm. 2.27]): *The regularity of any Borel-fixed homogeneous ideal $I$ is equal to*

$$\max_P \{ \deg P \mid P \text{ is a minimal generator of } \text{gin}(I) \}.$$  

Here, as everywhere else in this paper, the underlying term order on monomials is assumed to be revlex; indeed the proposition fails to hold for arbitrary term orders.

The following lemma permits us to identify all Borel-fixed ideals associated to general hyperplane sections of subschemes of $\mathbb{P}^4$.

**Lemma 1.2.2.** The number of nonleaf vertices in the tree defining a zero-dimensional subscheme of $\mathbb{P}^3$ is equal to the scheme’s degree.

*Proof.* We proceed by induction on the degree of the subscheme $\Gamma \subset \mathbb{P}^3$. Without loss of generality, we may assume $\Gamma$ is nondegenerate, since the restriction $\Gamma \mid L$ of $\Gamma$ to its linear span $L$ satisfies $\deg(\Gamma \mid L) = \deg(\Gamma)$. Thus, $\text{gin}(I_\Gamma)$ is a monomial ideal whose minimal generators are polynomials in $x_0, x_1, x_2$. Moreover, since $\Gamma$ is zero-dimensional, $\text{gin}(I_\Gamma)$ has one minimal generator of the form $x_0^\lambda$, for some positive integer $\lambda$. Furthermore, $\lambda$ is the maximal total degree of any minimal generator of $\text{gin}(I_\Gamma)$.

If $\deg(\Gamma) = 1$, the claim is trivial, since the only nondegenerate zero-dimensional subscheme of $\mathbb{P}^3$ of degree one is the point, whose ideal (up to projective linear transformation) is $(x_0, x_1, x_2)$. If $\deg(\Gamma) > 1$, then $\text{gin}(I_\Gamma)$ may be obtained from the gin of the point by applying finitely many $\Gamma$-rules, so by induction it suffices to check the following statement:

*Let $\Gamma$ and $\Gamma'$ be zero-dimensional subschemes of $\mathbb{P}^3$ defined by Borel-fixed ideals $I$ and $I'$ of $\mathbb{C}[x_0, x_1, x_2, x_3]$, respectively. If $I'$ is obtained from $I$ by applying a single $\Lambda$-rule to $I$, then $\deg(\Gamma') = \deg(\Gamma) + 1$.***

To check the latter claim, first note that by the regularity criterion [Gre Thm. 2.27] of Bayer-Stillman cited above, the ideal sheaves $I$ and $I'$ associated to the two homogeneous ideals in question are $\lambda'$-regular, where

$$\lambda' := \max_P \{ \deg(P) \mid P \text{ is a minimal generator of } I' \}.$$
Therefore, there are exact sequences on global sections:

\[ 0 \to H^0(\mathcal{I}(\lambda')) \to H^0(\mathcal{O}_{\mathbb{P}^3}(\lambda')) \to H^0(\mathcal{O}_{\Gamma}(\lambda')) \to 0 \]

and

\[ 0 \to H^0(\mathcal{I}'(\lambda')) \to H^0(\mathcal{O}_{\mathbb{P}^3}(\lambda')) \to H^0(\mathcal{O}_{\Gamma'}(\lambda')) \to 0. \]

Moreover, since \( \Gamma \) and \( \Gamma' \) are zero-dimensional, their degrees are equal to their Euler characteristics. Since \( \mathcal{I}(\lambda'), \mathcal{I}'(\lambda') \), and \( \mathcal{O}_{\mathbb{P}^3}(\lambda') \) have no higher cohomology, \( \chi = h^0 \) for each of these sheaves.

Because \( \chi \) is additive across short exact sequences, it follows that

\[ \chi(\Gamma) = h^0(\mathcal{O}_{\mathbb{P}^3}(\lambda')) - H^0(\mathcal{I}(\lambda')) \] and \[ \chi(\Gamma') = h^0(\mathcal{O}_{\mathbb{P}^3}(\lambda')) - H^0(\mathcal{I}'(\lambda')). \]

But clearly \( h^0(\mathcal{I}'(\lambda')) = h^0(\mathcal{I}(\lambda')) - 1 \), since \( I' \) is a saturated homogeneous ideal with one fewer monomial generator (not necessarily minimal!) in degree \( \lambda' \) than \( I \), which is also saturated and homogeneous. It follows immediately that \( \chi(\Gamma') = \chi(\Gamma) + 1 \), whence \( \deg(\Gamma') = \deg(\Gamma) + 1 \).

**Example.** Figure \( \square \) shows the tree-representation of the hyperplane \( \text{gin} \) of a rational normal quartic in \( \mathbb{P}^4 = \text{Proj} \mathbb{C}[x_0, x_1, x_2, x_3, x_4] \). Note that in this example,

\[ \text{gin}(\mathcal{I}_C) = \text{gin}(\mathcal{I}_{C \cap H})^\text{ext}, \]

where \( H \) is any hyperplane generic with respect to \( C \), and \( \text{gin}(\mathcal{I}_{C \cap H})^\text{ext} \) denotes the extension of \( \text{gin}(\mathcal{I}_{C \cap H}) \) to \( \mathbb{C}[x_0, \ldots, x_4] \). The \( \Lambda \)-rules are marked as \( \Lambda_i \), for \( i = 1, \ldots, 4 \). Exactly four vertices are not leaves.

In this example the hyperplane \( \text{gin} \) is

\[ I = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2), \]

the \( \text{gin} \) of four points in general position in \( \mathbb{P}^3 \). These points arise as the generic hyperplane section of a rational normal quartic, so according to Fact \( \square \) the \( \text{gin} \) of some rational normal quartic must be obtainable by the application of some nonnegative number \( m \) of \( \Lambda \)-rules from \( I \). Indeed, one can check (using, e.g., the computer algebra system Macaulay2 of [GS]) that the \( \text{gin} \) of the rational normal quartic is \( (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2) \); so in this case, \( m = 0 \).

To construct the tree in the example, we may proceed as follows. First, apply the initial \( \Lambda \)-rule to obtain the ideal \( (x_0, x_1, x_2) \). Next, apply a rewriting rule at the vertex corresponding to the generator \( x_2 \), replacing \( x_2 \) with \( x_2^3 \). The corresponding rule is \( x_2 \mapsto x_2^3 \). Similarly, replace \( x_1 \) with \( (x_1^2, x_1x_2) \) via \( x_1 \mapsto (x_1^2, x_1x_2) \) and replace \( x_0 \) with \( (x_0^2, x_0x_1, x_0x_2) \) via \( x_0 \mapsto (x_0^2, x_0x_1, x_0x_2) \).
1.3 Hyperplane gins for nondegenerate curves

In this subsection, $I$ denotes a saturated Borel-fixed ideal defining the gin of some general hyperplane section of some (fixed) nondegenerate, irreducible degree-10 rational curve $C$. Throughout, we use $\text{Borel}(J)$ to denote the smallest Borel-fixed ideal containing the ideal $J$. We shall see that the minimal generating set of $I$ is subject to significant numerical restrictions, which together imply there are very few possibilities for $I$. Four out of the five corresponding possible trees, which we will implicitly refer to in the course of establishing bounds on $h^1(I_C(5))$, are given in figures 2 through 5.

To begin, note that every general hyperplane section $\Gamma$ of $C$ is (the reduced scheme associated to) a set of ten points in uniform position in a three-dimensional projective space. By a result of [Bal] bounding the regularity of ideals of points in uniform position, $I_\Gamma$ is 4-regular, so $I$ is minimally generated in degrees at most 4.

*If $I$ has no quadratic generators, then*

$$I = \text{Borel}(x_2^3)$$

$$= (x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_1x_2, x_0x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3),$$

by saturatedness, Borel-fixity, and Lemma 1.2.2. See Figure 2.

Similarly, *if $I$ has exactly one quadratic generator, then*

$$I = \text{Borel}(x_1x_2^2) + (x_0^2, x_2^4).$$

See Figure 3.

Next say that $I$ has exactly two quadratic generators. Borel-fixity implies these are necessarily $x_0^2$ and $x_0x_1$. Note that $I \neq \text{Borel}(x_3^2) + (x_0x_1, x_0^2)$ since
the corresponding tree has eight nonterminal vertices. Similarly, $x_1 x_2^3$ cannot belong to $I$, since otherwise degree considerations force

$$I = \text{Borel}(x_0 x_1) + \text{Borel}(x_1 x_2^2) + (x_2^5).$$

The latter ideal is not 4-regular, and therefore violates the main result of [Bal]. Therefore, if $I$ has exactly two quadratic generators, then

$$I = \text{Borel}(x_0 x_1) + (x_1^3, x_1^2 x_2, x_0 x_2^2) + (x_1 x_2^3, x_2^4).$$

See Figure 4

On the other hand, if $I$ has exactly three quadratic generators, then these are either $\{x_0^2, x_0 x_1, x_0 x_2\}$ or $\{x_0^2, x_0 x_1, x_1^2\}$, so degree considerations, saturatedness, and Borel-fixity force

$$I = \text{Borel}(x_0 x_2) + \text{Borel}(x_1^3) + (x_1^2)$$

or

$$I = \text{Borel}(x_0 x_1) + (x_0 x_2^3, x_1^2, x_1 x_2^3, x_2^4).$$
Figure 4: Borel($x_0x_1$) + ($x_1^3, x_1^2x_2, x_0x_2^2$) + ($x_1^2, x_2^1$)

Note that no zero-dimensional, saturated, and nondegenerate Borel-fixed ideal $I \subset \mathbb{C}[x_0, \ldots, x_3]$ of degree 10 having at least four quadratic generators is 4-regular. For, by Borel-fixity, if $I$ has at least four quadratic generators then these necessarily include

$$x_0^2, x_0x_1, x_0x_2, \text{ and } x_1^2,$$

so that

$$I = (x_0^2, x_0x_1, x_0x_2, x_1^2) + (x_1^2, x_2^1),$$

for some $e$ and $f$. Borel-fixity implies $f \geq e$, but then Lemma 1.2.2 implies $e + f = 10$, so $I$ has minimal generators in degree 5 or higher.

In sum, we have proved the following result.

Figure 5: Borel($x_0x_1$) + ($x_0x_2^3, x_1^3, x_1x_2^3, x_2^4$)
Proposition 1.3.1. $I$ has at most three quadratic generators, and is one of the five monomial ideals listed above.

1.4 Computing $h^1(I_C(5))$ from a tree of minimal generators

In this subsection we prove a few technical lemmas that will be used, in the proof of Theorem 2.1, to obtain bounds on $h^1(I_C(5))$.

Lemma 1.4.1. Let $C \subset \mathbb{P}^4$ denote any nondegenerate degree-10 integral curve. Then $h^1(O_C(5)) = 0$.

Proof. The Castelnuovo–Hilphen genus bound (see, e.g., [Ci, (1.1), p.27]) implies $g(C) \leq 9$, whence $\deg(K_C) = 2g(C) - 2 \leq 16$. Since $C$ is integral (and in particular Cohen-Macaulay), Grothendieck duality (see [AK]) implies

$$h^1(O_C(5)) = h^0(K_C(-5)),$$

where $K_C$ denotes the dualizing sheaf $C$. It therefore suffices to show that $K_C(-5)$ has no global sections.

Suppose $K_C(-5)$ has a section. Then there is an injection

$$0 \to O_C \to K_C(-5),$$

or equivalently, an injection of invertible sheaves

$$0 \to O_C(5) \to K_C.$$  (1.1)

In particular, we have

$$\chi(K_C) = \chi(O_C(5)) + \chi(Q),$$

where $Q$ denotes the quotient of (1.1). Since (1.1) is generically an isomorphism, $Q$ is supported at finitely many points, so $\chi(Q) = h^0(Q)$ is nonnegative, and therefore

$$\chi(K_C) \geq \chi(O_C(5)).$$  (1.2)

The right side of (1.2) equals $5 \times 10 - g(C) + 1$ and is therefore at least 42, whereas the left equals $2g(C) - 2 - g(C) + 1$ and is therefore at most 8. So $h^0(K_C(-5)) = 0$ after all, and the lemma is proved. \hfill \Box

As explained in Section 1.1, it follows that

$$h^0(I_C(5)) = 126 - (5 \times 10 - g + 1) + i.$$  (1.3)

16
On the other hand, \( \mathcal{I}_C \) and \( \operatorname{gin} \mathcal{I}_C \) have the same regularity, so since

\[
h^2(\mathcal{I}_C(5)) = h^1(\mathcal{O}_C(5)) = 0,
\]
we also have \( h^2(\operatorname{gin} \mathcal{I}_C(5)) = 0 \). So in fact (1.3) remains true when \( \mathcal{I}_C \) is replaced by \( \operatorname{gin}(\mathcal{I}_C) \).

**Lemma 1.4.2.** Let \( I \subset \mathbb{C}[x_0, \ldots, x_4] \) be a Borel-fixed monomial ideal of dimension 1, and \( I' \) the ideal obtained from \( I \) by applying a single \( C \)-rule. Then \( g(I') = g(I) - 1 \).

(Recall from the introduction that \( g(I) \) and \( g(I') \) denote the genera of the subschemes of \( \mathbb{P}^4 \) defined by \( I \) and \( I' \).)

**Proof.** Let \( \mathcal{I} \) and \( \mathcal{I}' \) denote the sheaves associated to \( I \) and \( I' \), and let \( C \) and \( C' \) denote the corresponding subschemes of \( \mathbb{P}^4 \). Let \( m \) be any positive integer for which all four sheaves are \((m-1)\)-regular. Then \( h^1(\mathcal{I}(m)) = h^1(\mathcal{I}'(m)) = 0 \), so there are exact sequences of sections

\[
0 \to H^0(\mathcal{I}(m)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(m)) \to H^0(\mathcal{O}_C(m)) \to 0
\]

and

\[
0 \to H^0(\mathcal{I}'(m)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(m)) \to H^0(\mathcal{O}_{C'}(m)) \to 0,
\]

Moreover, since \( h^1(\mathcal{O}_C(m)) = h^1(\mathcal{O}_{C'}(m)) = 0 \), we have

\[
h^0(\mathcal{I}(m)) = \binom{m + 4}{m} - (10m - g(C) + 1) \tag{1.4}
\]

and

\[
h^0(\mathcal{I}'(m)) = \binom{m + 4}{m} - (10m - g(C') + 1) \tag{1.5}
\]

by the Riemann-Roch formula. On the other hand, \( \mathcal{I} \) and \( \mathcal{I}' \) are both saturated, so \( h^0(\mathcal{I}(m)) \) and \( h^0(\mathcal{I}'(m)) \) are exactly the numbers of linearly independent monomials in the homogeneous ideals \( I \) and \( I' \), respectively, of total degree at most \( m \). Hence

\[
h^0(\mathcal{I}'(m)) = h^0(\mathcal{I}(m)) - 1.
\]

The lemma now follows immediately from equations (1.4) and (1.5). \( \square \)

Now let \( C \subset \mathbb{P}^4 \) be a nondegenerate degree-10 integral curve, and set \( i := h^1(\mathcal{I}_C(5)) \).
Lemma 1.4.3. The number of vertices in the tree representing \( \text{gin} I_C(5) \) dominating vertices of degree greater than 6 (equivalently, the number of rewritings applied to vertices of degree 6 or greater) equals \( i \).

Proof. Let \( v \) denote the number of rewritings applied to vertices of degree 6 or greater. Recall ([Gre Thm. 2.27]) that any generic initial ideal \( I \) for the reverse-lexicographic order is \( m \)-regular if and only if it is minimally generated in degrees \( m \) or less. So if \( v = 0 \), then \( C \) is 6-regular, and therefore \( i = 0 = v \). Now say \( v > 0 \). By Lemma 1.4.2 each successive \( C \)-rewriting \( r_i : T(I_i) \to T(I_{i+1}) \) satisfies

\[
g(C_{i+1}) = g(C_i) - 1,
\]

where \( C_i \) and \( C_{i+1} \) denote the subschemes of \( \mathbb{P}^4 \) defined by \( I_i \) and \( I_{i+1} \), respectively. Now let \( \text{deg}(r_i) \) denote the degree of the vertex of \( T(I_i) \) at which the rewriting \( r_i \) is applied. Note that if \( \text{deg}(r_i) \geq 6 \), then the number of linearly independent quintic polynomials in \( I_{i+1} \) equals the corresponding number in \( I_i \). So, because rewritings preserve saturatedness,

\[
h^0(I_{C_{i+1}}(5)) = h^0(I_{C_i}(5)).
\]

By (1.3), it follows that if \( r_i \geq 6 \), then

\[
h^1(I_{C_{i+1}}(5)) = h^1(I_{C_i}(5)) + 1.
\]

The lemma follows immediately by induction on \( v \).

Note that by Lemma 1.4.2 \( i \) also measures the failure of \( C \) to impose linear independent conditions on quintic hypersurfaces.

1.5 Nondegenerate curve gins associated to hyperplane gins

In this subsection, we obtain restrictions on minimal generating sets of generic initial ideals of irreducible, nondegenerate rational curves of degree 10 in \( \mathbb{P}^4 \). To see how this is possible, fix a choice \( C \) of such a curve, with \( \Gamma \) a hyperplane section of \( C \) defined by

\[
\Gamma := C \cap H,
\]

where \( H \) is a general linear form. Make a general choice of coordinates \( x_0, \ldots, x_4 \) on \( \mathbb{P}^4 \) with respect to which \( H \) is defined by \( x_4 = 0 \). Let \( I := \text{gin}(I_{\Gamma}) \). Then by Fact 11 \( \text{gin} I_C \) is obtained via a sequence of \( C \)-rewriting rules from the extension of \( I \subset \mathbb{C}[x_0, \ldots, x_3] \), to \( \mathbb{C}[x_0, \ldots, x_4] \).
As usual, let $i := h^1(I_C(5))$. The result of [GLP Thm. 3.1, p. 501] implies that the ideal sheaf $I_C$ is 7-regular unless $C$ admits an 8-secant line (here points along the line are counted with multiplicity). So most of the time, [Gre Thm. 2.27] implies that $gin I_C$ is minimally generated by polynomials of degree at most 7. This fact limits the number of $C$-rewritings that occur in degrees at least 6 used to obtain $gin I_C$ from $I$, which in turn measures $i$, according to Lemma 1.4.3.

Let $C_{\Gamma}$ denote the cone with vertex $(0, 0, 0, 0, 1)$ over the zero-dimensional scheme defined by the vanishing of $I$ in $H$. Thus $C_{\Gamma}$ is one-dimensional, and its minimal generators are exactly those of $I$; that is, $I_{C_{\Gamma}}$ is the extension of $I$ to $\mathbb{C}[x_0, \ldots, x_4]$. Let $g_{\Gamma} := g(C_{\Gamma})$. In the proof of Theorem 2.1 we will repeatedly use the following two technical results.

**Lemma 1.5.1.** $g_{\Gamma} - g$ is the number of $C$-rewritings applied to yield $gin(I_C)$ from $I$, and $g + i \leq g_{\Gamma}$.

**Proof.** The first statement is an immediate consequence of Lemma 1.4.2. The second statement therefore follows immediately whenever $i = 0$. Note that $gin I_C$ may be obtained in two steps:

1. Perform a number $r_1$ of rewritings in degrees less than six.
2. Perform a number $r_2$ of rewritings in degrees six or greater.

Let $\widetilde{C}$ denote the scheme defined by the ideal that is the outcome of step 1. Clearly, we have

$$g_{\Gamma} - g(\widetilde{C}) = r_1$$

and

$$g(\widetilde{C}) - g = r_2.$$

But Lemma 1.4.3 implies that

$$i = r_2,$$

so the second statement of the Lemma follows immediately.

Now, let $m$ be any positive integer such that $C$ is $m$-regular.

**Lemma 1.5.2.** $g_{\Gamma} = 10m + 1 - (m+4) + h^0(I_{C_{\Gamma}}(m))$.

**Proof.** Because $C_{\Gamma}$ is $m$-regular, the long exact sequence in cohomology associated to

$$0 \to I_{C_{\Gamma}}(m) \to O_{\mathbb{P}^4}(m) \to O_{C_{\Gamma}}(m) \to 0$$

19
shows that $\mathcal{O}_C$ is $m$-regular, too. Therefore, there is an exact sequence

$$0 \to H^0(\mathcal{I}_C(m)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(m)) \to H^0(\mathcal{O}_C(m)) \to 0,$$

and $h^1(\mathcal{O}_C(m)) = 0$. It follows that

$$h^0(\mathcal{O}_C(m)) = 10 \times m - g + 1,$$

(1.6)

and also

$$h^0(\mathcal{O}_C(m)) = \binom{m+4}{4} - h^0(\mathcal{I}_C(m)).$$

(1.7)

To conclude, simply compare (1.6) and (1.7). □

2 Irreducible rational curves of degree 10 in $\mathbb{P}^4$

In this section, we’ll prove the following theorem, which extends to degree 10 an earlier result ([JK1, Thm.3.1]) of Johnsen and Kleiman’s.

**Theorem 2.1.** The incidence scheme $\Phi_d$ of smooth rational curves of degree $d$ at most 10 on quintic hypersurfaces $F \subset \mathbb{P}^4$ is irreducible. Moreover, any smooth curve $C$ lying on a general quintic $F$ is embedded with normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Furthermore, there are no rational and singular, reduced, and irreducible curves of degree at most 10 lying on a general quintic in $\mathbb{P}^4$, other than the six-nodal plane quintics.

As noted in the introduction, there is an interesting corollary with significance for mirror symmetry.

**Corollary 2.1.** The instanton number $n_{10}$ for a general quintic threefold $F$ does not equal the number of smooth rational curves of degree 10 that lie on $F$.

To prove the theorem, we’ll borrow extensively from Johnsen and Kleiman’s work; they prove the theorem in degree at most 9.

2.1 Initial reductions

Let $M_{10}^{r,g}$ denote the affine space of parameterized mappings of $\mathbb{P}^1$ into $\mathbb{P}^4$ that map birationally onto images of degree 10, of arithmetic genus $g$, and of span an $r$-dimensional projective space. Note that we do not mod out by the PGL(2)-action on $\mathbb{P}^1$; nor do we projectivize. Let $I_{10}^{r,g}$ denote the
incidence locus in $M_{10}^{r,g} \times \mathbb{P}^{125}$. Let $M_{10,i}^{r,g} \subset M_{10}^{r,g}$ denote the sublocus of morphisms with images $C$ satisfying

$$h^1(I_C(5)) = i,$$

and let $I_{10,i}^{r,g} \subset I_{10}^{r,g}$ denote the pullback of $M_{10,i}^{r,g}$ under the canonical projection $I_{10}^{r,g} \to \mathbb{P}^{125}$.

As explained in [JK1], to prove Theorem 2.1 it’s enough to show that the projection $I_{10}^{r,g} \to \mathbb{P}^{125}$ is surjective only over the irreducible component $M_{10}^{r,g}$ of morphisms whose images $C$ are smooth with $h^1(I_C(5)) = 0$. In particular, this statement implies the finiteness of the Hilbert scheme of curves on a general quintic, and the splitting property of the normal bundles $N_C/F$ follows immediately from Verdier’s result [Ve, Thm, p.139]. See the proof of [JK1, Cor 2.5] for more details.

However, since the projection’s fibres all have dimension at least 4, to prove that it’s not surjective over the locus of curves with $i$ nonzero, it suffices to establish that $\dim I_{10,i}^{r,g} < 129$. Using the exact sequence

$$0 \to H^0(I_C(5)) \to H^0(O_{\mathbb{P}^4}(5)) \to H^0(O_C(5)) \to H^1(I_C(5)) \to 0,$$

to relate the Hilbert polynomial of $I_C$ to that of $C$, we see that

$$\dim I_{10,i}^{r,g} \leq \dim M_{10,i}^{r,g} + 126 - (5 \cdot 10 - g + 1) + i,$$

at least provided $H^1(O_C(5)) = 0$. (See [JK1] for more details.) Assuming this vanishing, which we prove in Lemma 1.4.1 below, we are reduced to showing that

$$\dim M_{10,i}^{r,g} < 55 - g - i$$

(2.1)

whenever $i$ is nonzero.

In order to simplify the notation, we let $i := h^1(I_C(5))$ hereafter.

We may also assume $r \geq 3$, since (as noted in [JK1 Section 3]) no planar rational curves of degree greater than 5 lie on a general quintic.

Moreover, by the result of [JK1 Lemma 3.4], we have

$$\dim M_d^{1,g} \leq 5(d + 1) - 1 - \min(2g, 8) \text{ for all } d \geq 7, \text{ and}$$

$$\dim M_d^{3,g} \leq 4(d + 1) - 1 - \min(g, 5) \text{ for all } d \geq 9.$$

(2.2a)

To establish the validity of (2.2a), we treat nondegenerate curves and curves with 3-planar images separately. It’s interesting to note that in this paper, unlike in [JK1], the main technical work lies in the proof of (2.2a) in the nondegenerate case. Accordingly, we first consider nondegenerate curves in $\mathbb{P}^4$. Applying (2.2a), it’s easy to see that Theorem 2.1 for nondegenerate curves amounts to the following.
**Theorem 2.2.** The nondegenerate reduced irreducible rational curves verifying
\[ g + i \geq \min(2g, 8) \]
define a sublocus of the parameter space \( M_{10} \) of degree-10 rational maps of codimension greater than \( g + i \).

### 2.2 Special subloci of the mapping space

In this subsection, we obtain lower bounds on the codimensions of certain special subloci of \( M_{10} \). These estimates effectively allow us to ignore certain curves \( C \) corresponding to generic initial ideals with large \( h^1(I_C(5)) \).

**Lemma 2.2.1.** Degree-10 morphisms determining nondegenerate degree-10 rational curves with 8-secant lines have codimension at least 10 in the space of rational mappings of degree 10.

**Proof.** Every rational map \( f : \mathbb{P}^1 \to \mathbb{P}^4 \) of degree 10 is parametrized by 5 equations \( x_i = f_i(t, u) \) for \( i = 0, \ldots, 4 \), where \( (t, u) \) are coordinates on \( \mathbb{P}^1 \), the \( x_i \) are coordinates on \( \mathbb{P}^4 \), and \( f_i \) is a polynomial of degree 10 in \( t \) and \( u \). To say that the image of \( f \) intersects the line \( L = \{x_0 = x_1 = x_2 = 0\} \) in distinct points \( f(t_j, u_j) \) for \( 1 \leq j \leq 8 \), means precisely that
\[ f_i(t_j, u_j) = 0 \quad \text{for} \quad 0 \leq i \leq 2 \quad \text{and} \quad 1 \leq j \leq 8. \quad (2.3) \]
In other words, 24 linearly independent equations must be satisfied. So requiring a map \( f \) of degree 10 to map any fixed degree-8 divisor onto distinct points along any fixed line \( L \) is a codimension-24 condition. Moreover, if we allow the image points to coalesce and replace the equations (2.3) by their higher-contact analogues, then the same argument shows that requiring \( f \) of degree 10 to map any fixed degree-8 divisor along any fixed line \( L \) is a codimension-24 condition. On the other hand, the degree-8 divisors on \( \mathbb{P}^1 \) determine a \( \mathbb{P}^8 \) and the Grassmannian of lines in \( \mathbb{P}^4 \) has dimension 6, so by varying our choice of eight points in \( \mathbb{P}^1 \) along with the choice of the line \( L \subset \mathbb{P}^4 \), we obtain 10 conditions and the result follows.

**Lemma 2.2.2.** Degree-10 morphisms defining rational curves in reduced, irreducible hyperquadrics determine a locus of codimension at least 7 in \( M_{10} \).

**Proof.** By the main result of [KiPa], the space of rational curves of any fixed degree \( d \) inside a projective homogeneous space is irreducible and of the expected dimension. On the other hand, every smooth hyperquadric is a homogeneous space, and any two hyperquadrics are isomorphic, so the
codimension of curves lying on smooth hyperquadrics is the expected one. Note that hyperquadrics $F \subset \mathbb{P}^4$ comprise a $\mathbb{P}^{14}$ and that the equation defining the incidence scheme $\{ C \subset F : C \in M_{10}, F \in \mathbb{P}^{14} \text{ in } M_{10} \times \mathbb{P}^{14} \}$ is a polynomial in the coordinates of $\mathbb{P}^1$ in 21 parameters. So 21 linearly independent conditions must be met for a degree-10 curve to lie on a fixed smooth hyperquadric. By varying the choice of hyperquadric, we find that the codimension of (morphisms corresponding to) curves inside smooth hyperquadrics is 7.

Now let $Q$ be a singular, reduced, and irreducible hyperquadric. Since it contains a nondegenerate curve $C$, it must be a cone over a quadric $Q$ of rank 2 or 3.

Say rank$(Q) = 3$. Let $\text{Hom}_{10}(\mathbb{P}^1, Y)$ denote the (affine) parameter space of morphisms $\mathbb{P}^1 \to Y$. When $Y = \mathbb{P}^4$, a morphism $f : \mathbb{P}^1 \to Y$ is given by 5 sections $f_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(\deg(f)))$ for $i = 0, \ldots, 4$. Projection from the vertex of $Q$ defines a rational map $\pi : \text{Hom}_{10}(\mathbb{P}^1, \mathbb{P}^4) \to \text{Hom}_{10}(\mathbb{P}^1, \mathbb{P}^3)$. Now assume that the vertex $p$ of $Q$ has coordinates $(0, 0, 0, 0, 1)$. Then $\pi : \text{Hom}_{10}(\mathbb{P}^1, \mathbb{P}^4)$ is the rational map that drops the fifth parameterizing polynomial: $(f_0, f_1, f_2, f_3, f_4) \mapsto (f_0, f_1, f_2, f_3)$. Since $f_4$ has 11 coefficients, the (affine) fibre dimension of $\pi$ is 11 over any degree-10 map $f$ whose image curve avoids $p$. Any such map $f \in \text{Hom}_{10}(\mathbb{P}^1, Q)$ is sent under $\pi$ to a map $\tilde{f} \in \text{Hom}_{10}(\mathbb{P}^1, \tilde{Q})$, where $\tilde{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric surface. As explained in [JK1, Lemma 3.2], the scheme associated to $\text{Hom}_{10}(\mathbb{P}^1, \tilde{Q})$ is a disjoint union of 23-dimensional projective spaces.

On the other hand, quadric cones determine a divisor inside the $\mathbb{P}^{14}$ of hyperquadrics in $\mathbb{P}^4$. Since rational maps whose images pass through the vertex of $Q$ clearly comprise a proper closed subset of $\text{Hom}_{10}(\mathbb{P}^1, Q)$, we conclude that rational maps to cones over quadric surfaces vary in a family of dimension at most $13 + 11 + 23 = 47$. In other words, such maps determine a locus of codimension $55 - 47 = 8$ inside $M_{10}^4$.

To complete the proof of Lemma 2.2.2, we must handle rational curves whose images lie in cones $Q'$ with 1-dimensional vertices over plane conics. In fact, these pose no problem. For, by a result of [Gu] quoted in [JK1, proof of Lemma 3.2], $\text{Hom}_{10}(\mathbb{P}^1, Q')$ is at most 23-dimensional. We now conclude by applying the same projection-based argument used in the preceding case. $\square$
2.3 A useful stratification of the mapping space

Let \( f : \mathbb{P}^1 \to \mathbb{P}^4 \) be a map with image \( C \). Note that the \((-1)\)-twist of the “restricted tangent bundle” \( f^*T_{\mathbb{P}^4}(-1) \), has degree 10 and rank 4 over \( \mathbb{P}^1 \), say with splitting

\[
f^*T_{\mathbb{P}^4}(-1) = \bigoplus_{i=1}^{4} O_{\mathbb{P}^1}(a_i).
\]

A result of Verdier’s (see [Ve] and [Ra]) establishes that the scheme of morphisms \( \mathbb{P}^1 \to \mathbb{P}^4 \) of fixed degree corresponding to a particular splitting type \((a_1, a_2, a_3, a_4)\) with \( a_1 \geq a_2 \geq a_3 \geq a_4 \) is irreducible of the expected codimension

\[
\sum_{i \neq j} \max\{0, a_i - a_j - 1\}.
\]

In particular, every special splitting stratum has codimension 4 or more, and the stratum \((4, 3, 2, 1)\) is the unique stratum of codimension 4. If \( f^*T_{\mathbb{P}^4}(-1) \) has the generic splitting, then \( \bigwedge^2 (f^*T_{\mathbb{P}^4}(-1))^* \otimes O_{\mathbb{P}^1}(5) \) has no higher cohomology, and it follows by [GLP, Proposition 1.2] that \( I_C \) is 6-regular. So \( i \neq 0 \) in codimension 4. Therefore, in light of the discussion immediately preceding the statement of Theorem 2.2 we have reduced to showing that for all nondegenerate \( C \) such that \( i \neq 0 \), one of following conditions is verified:

\[
\begin{align*}
g + i < 4, \text{ or} & \quad (2.4a) \\
C \text{ admits an 8-secant line and } g + i < 10, \text{ or} & \quad (2.4b) \\
C \text{ lies on a hyperquadric and } g + i < 7, \text{ or} & \quad (2.4c) \\
g + i < \min(2g, 8). & \quad (2.4d)
\end{align*}
\]

A curve \( C \) meeting any of these conditions will be called nonproblematic. Similarly, any sublocus of \( M_{10} \) comprised of nonproblematic curves will be called nonproblematic.

2.4 Proof of Theorem 2.1 for nondegenerate curves

To prove Theorem 2.1 we obtain bounds on \( h^1(I_C(5)) \), based upon an analysis of possible corresponding Borel-fixed monomial ideals. Each \( \text{gin}(I_C) \) is obtained, by Fact 1, by applying a sequence of rewriting rules to \( \text{gin}(I_{\Gamma}) \). We now argue case by case, based upon the possibilities for \( \text{gin}(I_{\Gamma}) \).

Case 1. Say \( \text{gin}(I_{\Gamma}) = \text{Borel}(x_3^2) \).
Because $I_{C\Gamma}$ is minimally generated by polynomials of degree at most 3, $C\Gamma$ is 3-regular. So Lemma 1.5.2 implies that

$$g_{\Gamma} = h^0(I_{C\Gamma}(3)) - 4.$$  

But, because $I_{C\Gamma}$ is saturated, $h^0(I_{C\Gamma}(3))$ is equal to the number of linearly independent polynomials of degree 3 in $I_{C\Gamma}$. Since there are exactly ten of these, we find $g_{\Gamma} = 6$.

It follows from Lemma 1.5.1 that $g + i \leq 6$. Note that if $C$ as above admits an 8-secant line, then $C$ belongs to a locally-closed subscheme inside $M_{10}$ of codimension at least 10, by Lemma 2.2.1. Thus $C$ is nonproblematic. The remaining curves are 7-regular, by [GLP Thm. 3.1, p.501]; we will see that they, too, are nonproblematic. For this purpose, we begin by showing that each one satisfies

$$i \leq 1. \quad (2.5)$$  

By Lemma 1.5.1 $\text{gin}(I_C)$ may be obtained from $I_{C\Gamma}$ by applying precisely $6 - g$ rewriting rules. On the other hand, by Lemma 1.4.3 $i$ is exactly the number of $C$-rewritings applied in degrees 6 or greater. In particular, $i$ is maximized when the number of $C$-rewritings (of arbitrary degrees) is maximized, which happens when $g = 0$. Therefore, for the purpose of verifying (2.5), we assume $g = 0$.

There are now a large number of possible $C$-rewriting sequences to choose from. For the sake of bounding $i$, however, we only need consider those rewriting sequences that result in the maximal number of minimal generators of degree greater than six. Moreover, certain rewritings are dictated by Borel-fixity. If $i$ is nonzero, for instance, then the minimal generator $x_2^3$ must be rewritten. Without loss of generality, therefore, we assume the first rewriting rule exchanges the minimal generator $x_2^3$ for minimal generators $x_2^4$ and $x_3^3x_3$, i.e., that it is $x_2 \mapsto (x_2, x_3)$ applied at the leaf corresponding to the minimal generator $x_2^3$ of $\text{gin} I_{\Gamma}$.

Similarly, we may assume the next three rewriting rules have the effect of exchanging the minimal generator $x_2^3x_3$ for $x_2^3x_3^4$. But then no further rewriting may be applied to $x_2^3x_3^4$, since $C$ is 7-regular. So rewriting twice more, beginning with minimal generators of degree at most 4, results in minimal generators of degree at most 6; for instance, we might have

$$\text{gin}(I_C) = \text{Borel}(x_1x_2^3) + (x_2^5, x_2^4x_3^2, x_2^3x_3^4).$$  

The latter ideal is obtained from the (extension of the) hyperplane $\text{gin}$ by applying $C$-rules to leaves that are “farthest to the right” in the tree of Figure 2; see Figure 6. It follows immediately by Lemma 1.4.3 that $i \leq 1$.  

25
Figure 6: $\text{gin } I_C = \text{Borel}(x_1x_2^2) + (x_5^5, x_2^4x_3^2, x_2^3x_3^4)$. The $C$-rules are marked as $C_i, i = 1, \ldots, 6$.

It follows that

$$g + i < 4$$

whenever $g \leq 2$. Therefore, if $g \leq 2$, then $C$ is nonproblematic.

It remains to show that 7-regular $C$ with $g \geq 3$ and $g + i \leq 6$ are nonproblematic. By Lemma 1.5.1, every such $C$ is such that $\text{gin}(I_C)$ is obtained from the hyperplane $\text{gin}$ by applying at most 3-rewriting rules, all of which are necessarily in degrees less than six. By Lemma 1.4.3 it follows that $i = 0$. So every such $C$ is indeed nonproblematic.

**Case 2.** Say $\text{gin } I_\Gamma = \text{Borel}(x_1x_2^2) + (x_0^2, x_2^4)$.

Because $I_{C_1}$ is minimally generated by polynomials of degree at most 4, $C_\Gamma$ is 4-regular. So Lemma 1.5.2 implies that

$$g_\Gamma = h^0(I_{C_\Gamma}(4)) - 29.$$  

Because $I_{C_\Gamma}$ is saturated, $h^0(I_{C_\Gamma}(4))$ is equal to the number of linearly independent polynomials of degree 4 in $I_{C_\Gamma}$; there are 36 of these, so $g_\Gamma = 7$.  

26
It follows from Lemma 1.5.1 that $g + i \leq 7$. It follows, as in Case II, that if $C$ admits an 8-secant line, then $C$ is nonproblematic. The remaining curves are 7-regular, by [GLP, Thm. 3.1, p.501]; we now show they are also nonproblematic.

By Lemma 1.4.3, $i$ is exactly the number of $C$-rewritings applied in degrees 6 or greater. But by Lemma 1.5.1, $\text{gin}(I_C)$ may be obtained from $I_C$ by applying precisely $7 - g$ rewriting rules. But because $C$ is 7-regular, it follows there are no rewritings in degree greater than 6. Moreover, there are at most three rewritings in degree 6, because it takes six rewritings total to obtain three rewritings in degree 6, and any seventh rewriting will be in degree less than 6. (Compare Figure 3.) Therefore, $i \leq 3$.

It follows immediately that if $g = 0$, then $g + i \leq 3$ and, therefore, $C$ is nonproblematic.

Similarly, if $g = 1$, then either $g + i \leq 3$, in which case $C$ is nonproblematic, or $g + i = 4$ and $\text{gin}(I_C)$ corresponds to the tree of Figure 4. In the latter situation, however, Figure 7 shows that $C$ lies on a hyperquadric, and therefore, by Lemma 2.2.2, $C$ is nonproblematic.

Similarly, if $g = 2$, then $\text{gin}(I_C)$ is obtained from the hyperplane $\text{gin}$ in 5 rewritings, of which at most 2 may be in degree 6. So $i \leq 2$ by Lemma 1.4.3, i.e., $g + i \leq 4$. Moreover, equality is obtained if and only if $C$ lies on a hyperquadric, in which case $C$ is nonproblematic, by Lemma 2.2.2.

Finally, if $g \geq 3$, then $\text{gin}(I_C)$ is obtained from the hyperplane $\text{gin}$ in at most 4 rewritings, so $i \leq 2$, which implies $g + i < \min(2g, 8)$. So $C$ is nonproblematic in this situation as well, which enables us to conclude.

**Case 3.** Say $\text{gin} I_C = \text{Borel} (x_0 x_1) + (x_1^3, x_1^2 x_2, x_0 x_2^2) + (x_1 x_2^3, x_2^4)$.

To show that the corresponding curves $C$ are nonproblematic, we proceed as follows.

First, $C$ is 4-regular, so

$$g_r = h^0(I_{C_1}(4)) - 29,$$

by Lemma 1.5.2. Since $I_{C_1}$ contains 37 linearly independent polynomials in degree 4, we have $g_r = 8$. Hence, $g + i \leq 8$, by Lemma 1.5.1. Note that applying $r \leq 2$ $C$-rewriting rules to $I_{C_1}$ results in a Borel-fixed ideal that is minimally generated in degrees at most 6 and has genus $g_r - r$, by Lemma 1.4.2. So if $g = 6$ or $g = 7$, then $\text{gin} I_C$ is 6-regular, by [Gre Thm. 2.27], and moreover

$$g + i < \min(2g, 8),$$

27
Figure 7: $\text{gin}(I_C) = (x_0^2) + \text{Borel}(x_1x_2^2) + (x_2^7, x_2^6x_3, x_2^5x_3^2, x_2^4x_3^3)$. The $C$-rules are marked as $C_i, i = 1, \ldots, 6$. Clearly, $g = 1$. Moreover, $i = 3$, since there are exactly three $C$-rules in degree 6.

and therefore $C$ is 6-regular. We now show curves of genus 8 are nonproblematic, too.

Note that any such curve $C$ with $g = 8$ necessarily satisfies

$$\text{gin}(I_C) = I_{C_1},$$

by Lemma 1.5.1. Since $h^0(I_{C_1}(2)) = 2$, it follows that $C$ lies on two linearly independent hyperquadrics $Q_1$ and $Q_2$. These intersect in a surface $S$ of degree 4. In fact, $C$ is also contained in a hypercubic, $K$, that is linearly independent of $Q_1$ and $Q_2$. To see why, assume that no such $K$ exists. Then every hypercubic $K$ containing $C$ satisfies $Q_1 \cap Q_2 \subset K$, so we must have

$$h^0(I_C(3)) \leq h^0(I_{Q_1 \cap Q_2}(3)),$$

and therefore

$$h^0(I_{\Gamma}(3)) \leq h^0(I_\Lambda(3)), \quad (2.6)$$

28
where \( \Gamma \) and \( \Lambda \) denote general hyperplane sections of \( C \) and \( Q_1 \cap Q_2 \), respectively. Note that \( Q_1 \) and \( Q_2 \) are nondegenerate, since they contain \( C \); so \( Q_1 \cap Q_2 \) is a nondegenerate quartic surface, and \( \Lambda \) is a nondegenerate quartic space curve. By [GLP, Thm. 3.1, p.501], \( \Lambda \) is 3-regular, and it follows from the usual long exact sequence in cohomology that \( \mathcal{O}_\Lambda \) is 3-regular, too. So there is an exact sequence

\[
0 \to H^0(I_\Lambda(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_\Lambda(3)) \to 0,
\]

and by the Riemann-Roch theorem, we deduce that

\[
h^0(\mathcal{O}_\Lambda(3)) = 13 - g(\Lambda).
\]

By [Ci], \( \Lambda \) has genus at most 1; it follows that

\[
h^0(I_\Lambda(3)) \leq 8.
\]

On the other hand, we have

\[
h^0(I_\Gamma(3)) = 10,
\]

so (2.6) is violated, which gives a contradiction.

Now let \( S := Q_1 \cap Q_2 \). If \( S \cap K \) contains a surface component \( S^* \), then clearly \( S^* \) is properly contained in \( S \), whence \( \deg(S^*) < 4 \). Note that \( C \) lies on either \( S^* \) or on a component \( S^{**} \) belonging to the residual to \( S^* \) in \( Q_1 \cap Q_2 \). However, because \( C \) is nondegenerate, \( C \) lies on no component of \( S \) degree 1 or 2. So the component of \( S \) on which \( C \) lies is a nondegenerate threefold of degree 3, which is necessarily a cubic scroll by [GH, Prop., p.525]. Moreover, no cubic scroll containing \( C \) meets \( K \) properly, since \( \deg(C) = 10 \). So \( S^{**} \) is a surface component of \( S \cap K \). Therefore, replacing \( S^* \) by \( S^{**} \) if necessary, we may assume \( S^* \) is a cubic scroll containing \( C \).

We handle degree-10 curves on cubic scrolls in \( \mathbb{P}^4 \) as follows. As explained in [GH, pp.519-523], cubic scrolls come in two basic types. Another good reference for scrolls, whose notation we will use and to which we will often refer, is [Co]. Those that are smooth will be denoted \( S_{1,2} \) (in the notation of [GH], these are the scrolls \( S_{1,1} \)). Those that are singular are cones over twisted cubic curves, and will be denoted by \( S_{0,3} \).

Every \( S_{1,2} \) may be realized as the image of the Hirzebruch surface \( F_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \) under the map \( \phi_{1,2} \) defined by the complete linear series \( |\mathcal{O}_{F_1}(e + 2f)| \), where \( e \) is the divisor class of the \(-1\)-curve on \( F_1 \to \mathbb{P}^1 \) and \( f \) is the class of the fibre.
The intersection pairing on $F_1$ is given by

$$e^2 = -1, e \cdot f = 1, f^2 = 0.$$  

Moreover, the canonical class of $F_1$ is

$$K_{F_1} = -2e - 3f.$$  

Let $\phi_{1,2}^*[C] = ae + bf \in \text{Pic } F_1$. The adjunction formula implies that the arithmetic genus of $C$ satisfies

$$2g - 2 = ((a - 2)e + (b - 3)f) \cdot (ae + bf),$$  

i.e.,

$$a^2 - 2ab + a + 2b + 2g - 2 = 0.$$  

(2.7)

Also, since $\deg(C) = 10$, we have

$$(e + 2f) \cdot (ae + bf) = 10,$$  

or equivalently,

$$b = 10 - a.$$  

(2.9)

Substituting (2.9) in (2.8), we obtain

$$3a^2 - 21a + 18 + 2g = 0.$$  

(2.10)

If $g = 8$, then (2.10) has no integral solutions $a$.

Finally, we treat curves $C$ lying on cones $S_{0,3}$ over twisted cubics. Every $S_{0,3}$ is the image of the Hirzebruch surface $F_3 = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(3))$ under the map $\phi_{0,3}$ defined by the complete linear series $[O_{F_3}(e + 3f)]$, where $e$ is the divisor class of the of $-3$-curve on $F_3$ and $f$ is the class of the fibre. Note that $\phi_{0,3}$ is a birational map that blows down the $-3$-curve to the vertex of $S_{0,3}$.

The intersection pairing on $F_3$ is given by

$$e^2 = -3, e \cdot f = 1, f^2 = 0.$$  

The canonical class of $F_3$ is

$$K_{F_3} = -2e - 5f.$$  

(2.11)

Now say that $C$ avoids the vertex $p$ of $S_{0,3}$. Then $[\phi_{0,3}^{-1}C] = ae + bf$, for some integers $a$ and $b$. In fact, since $\deg(C) = 10$, we have

$$(e + 3f) \cdot (ae + bf) = 10,$$  

or equivalently,

$$b = 10 - a.$$  

(2.9)
whence,
\[ b = 10. \]  
(2.12)

On the other hand, the adjunction formula then implies that
\[ 2g - 2 = ((a - 2)e + (b - 5)f) \cdot (ae + bf). \]  
(2.13)

Substituting \( b = 10 \) in (2.13) and solving for \( a \) yields
\[ a = \frac{2}{7}g + 1. \]  
(2.14)

Here \( g = 8 \); by (2.14), it follows \( a \) is not an integer, which is plainly absurd.

Now say that \( C \) passes through the vertex of \( S_{0,3} \) with multiplicity \( m > 0 \).

Then \( [\phi_{S_{0,3}}^{-1}C] = ae + bf \), for some integer \( b \), and
\[ [\tilde{C}] = (a - m)e + bf, \]
where \( \tilde{C} \) denotes the proper transform of \( C \) on \( F_3 \).

Let \( \tilde{g} := g(\tilde{C}) \). Then we have
\[ 2 \tilde{g} - 2 = ((a - m - 2)e + (b - 5)f) \cdot ((a - m)e + bf) \]  
(2.15)

by the adjunction formula,
\[ \tilde{g} \leq g, \]  
(2.16)
and
\[ (e + 3f) \cdot (ae + bf) = 10, \]
since \( \deg(C) = 10 \). The latter equation implies
\[ b = 10, \]
just as before.

Substituting \( b = 10 \) in (2.15) and solving for \( a - m \) yields
\[ a - m = \frac{2}{7} \tilde{g} + 1, \]  
(2.17)
for some \( \tilde{g} \leq 8 \). Because \( a \) and \( m \) are integers it follows from (2.17) that
\[ \tilde{g} = 7 \text{ and } a - m = 3. \]
In particular, the genus discrepancy $g - \tilde{g}$ is equal to 1; whence,

$$m \leq 2,$$

by the result of Rosenlicht quoted in [AK, Ch. VIII, Prop. 1.16]. Therefore, there are two possibilities for the class of $[\phi^{-1}_{0,3}C]$: either

$$[\phi^{-1}_{0,3}C] = 4e + 10f,$$

or

$$[\phi^{-1}_{0,3}C] = 5e + 10f.$$

To handle curves of the latter two types, we argue as follows. Since $F_3$ is rational, we have $\chi(O_{F_3}) = 1$. Therefore, the Riemann-Roch theorem for surfaces implies that

$$\chi(ae + bf) = 1 + \frac{1}{2}((ae + bf)^2 - (ae + bf) \cdot (-2e - 5f)) = -\frac{3}{2}a^2 + ab - a + 2b + 1.$$

Substituting $(a, b) = (4, 10)$ and $(a, b) = (5, 10)$ yields

$$\chi(ae + bf) = 33$$

and

$$\chi(ae + bf) = \frac{57}{2},$$

respectively. However, $\chi(ae + bf)$ is necessarily an integer, so we must have

$$(a, b) = (4, 10).$$

Then (2.17) implies that the proper transform of $C$ on $F_3$ has class

$$[\tilde{C}] = e + 10f.$$

However, the Riemann-Roch formula then implies that $\chi(\tilde{C})$ is not an integer, which is absurd.

Now consider the only remaining possibility, that $S \cap K$ is a complete intersection $X$ of type $(2, 2, 3)$. Thus $C$ is residual in $X$ to a 1-dimensional scheme $R$ of degree 2. Let $L$ denote the linear series of hypercubics containing $C$. By Bertini’s theorem, a general member of $L$ is smooth away from $\text{Sing}(C) \cup \text{Bs}(L)$, where $\text{Bs}(L)$ denotes the base locus of $L$. Assuming, as we
may, that the quadrics and cubics defining $X$ are general among quadrics and cubics containing $C$, it follows that $R$ is reduced (i.e., a plane conic), since otherwise

$$X \subset \text{Bs}(\mathcal{L}),$$

which is absurd. On the other hand, the adjunction formula implies that

$$K_X = \mathcal{O}_X(2) \text{ and } g(X) = 13,$$

So $g(R) = 16$, by the main result of [N], which is also absurd.

It follows that every curve $C$ of genus 8 is nonproblematic.

Next, say $g = 6$ or 7. Then $\text{gin}(\mathcal{I}_C)$ is obtained from the hyperplane $\text{gin}$ in at most two rewritings, by Lemma 1.5.1. Because the hyperplane $\text{gin}$ is 4-regular, $\text{gin}(\mathcal{I}_C)$, and therefore $C$, is 6-regular. Therefore, $i = 0$. It follows immediately from (2.2a) that $C$ is nonproblematic.

Finally, assume $g \leq 5$. Any $C$ with $g \leq 5$ that admits an 8-secant line is nonproblematic, by Lemma 2.2.1 and the remaining curves are 7-regular, by [GLP Thm. 3.1, p.501]. From Figure 7 it’s clear that any sequence of at most eight $C$-rewriting rules applied to the hyperplane $\text{gin}$ that results in a 7-regular ideal involves at most 3 rewritings in degree 6. Therefore, by Lemma 1.4.3 $i \leq 3$. It follows immediately that if $g \geq 4$, then $g + i < \min\{2g, 8\}$, and therefore such curves are nonproblematic.

Now say $g \leq 3$. Since curves on hyperquadrics have codimension 7 in $M_{10}$, by Lemma 2.2.2 we may and therefore shall assume that $\text{gin} \mathcal{I}_C$ has no quadratic generators. Since $\text{gin} \mathcal{I}_C$ has two quadratic generators, the rewriting sequence that yields $\text{gin}(\mathcal{I}_C)$ from the hyperplane $\text{gin}$ necessarily involves two rewritings in degree 2. So the generic initial ideal of $C$ is obtained from the hyperplane $\text{gin}$ via a sequence of rewritings involving at most one rewriting in degree at least six. Therefore, $i \leq 1$, by Lemma 1.4.3. It follows that all curves of genus higher than 1 are nonproblematic. Similarly, if $g = 1$ then $g + i < 4$ so $C$ is nonproblematic. Finally, smooth curves are nonproblematic because $i \leq 1$ and $i \neq 0$ in codimension 4.

**Case 4.** Say $\text{gin} \mathcal{I}_C$ admits exactly three quadratic minimal generators.

Then either

$$\text{gin} \mathcal{I}_C = \text{Borel}(x_0 x_2) + \text{Borel}(x_2^4) + (x_1^3)$$

or

$$\text{gin} \mathcal{I}_C = \text{Borel}(x_0 x_1) + (x_0 x_2^3, x_1^3, x_1 x_2^3, x_2^4).$$

By Lemma 1.5.2 $g_\mathcal{I} = 9$ in either case. Therefore, $g + i \leq 9$, by Lemma 1.5.1.
The proof that the corresponding locus of morphisms is nonproblematic follows the same lines as in Case 3 above, so we merely sketch it. We may and shall assume \( C \) is 8-regular. Otherwise by [GLP, Thm. 3.1, p.501], \( C \) has an 8-secant line; therefore, \( C \) belongs to a proper sublocus of \( M_{10} \) of codimension at least 10, by Lemma 2.2.1. As \( g + i < 10 \), the latter sublocus is nonproblematic.

Next we show that the locus of morphisms corresponding to \( C \) with \( g \leq 7 \) is nonproblematic. Since we are assuming \( C \) to be 7-regular, and since \( \text{gin}(\mathcal{I}_C) \) is obtained from the hyperplane \( \text{gin} \) in at most \( g_7 \) rewritings, it’s not hard to see that at most 5 rewritings occur in degree 6, and therefore \( i \leq 5 \), by Lemma 1.4.3. Moreover, our upper bound on \( i \) improves to \( i \leq 3 \) if we also assume \( \text{gin}(\mathcal{I}_C) \) has no quadratic minimal generators. If the assumption fails, then the morphism defining \( C \) lies in a proper sublocus of codimension 7, by Lemma 2.2.2 but in such a situation \( C \) is nonproblematic. So our second assumption is justified, and we shall make it. The remainder of the argument when \( g \leq 7 \) is completely analogous to the one given in Case 3 above.

Finally, say \( g = 8 \) or 9. The liaison argument given in Case 3 implies that \( C \) necessarily lies on a (possibly singular) cubic scroll. Since rational degree-10 curves of genus 8 on cubic scrolls do not exist, as shown by the analysis of Case 3 above, we may assume \( g = 9 \). Then \( C \) is a Castelnuovo curve in the sense of [Ci, p.27] and [GH, p.527]; namely, \( C \) is of maximal genus among nondegenerate, irreducible and nondegenerate curves of degree 10 in \( \mathbb{P}^4 \).

Say that \( C \) lies on a cubic scroll of type \( S_{1,2} \). The adjunction formula implies that (2.7) holds, with \( g = 9 \). Solving, we obtain \( a = 3 \) or \( a = 4 \), which correspond to the classes \( 3a + 7f \) and \( 4a + 6f \), respectively. We next compute \( h^0(\mathcal{O}_{\mathbb{P}^3}(ae + bf)) \) for the pairs \((3,7)\) and \((4,6)\), respectively.

Note that the dimension of the space of cubic scrolls \( S_{1,1} \subset \mathbb{P}^4 \) equals \( h^0(\mathcal{O}_{\mathbb{P}^3}(e + f)) - 1 \), while the dimension of the space of curves of class \( ae + bf \) on a given scroll equals \( h^0(\mathcal{O}_{S_1}(ae + bf)) - 1 \).

Using the Riemann-Roch formula for surfaces, together with the intersection pairing, we deduce
\[
\chi(\mathcal{O}_{\mathbb{P}^3}(e + f)) = 4,
\chi(\mathcal{O}_{\mathbb{P}^3}(3e + 7f)) = 23, \text{ and}
\chi(\mathcal{O}_{\mathbb{P}^3}(4e + 6f)) = 40.
\]

Moreover, a straightforward cohomological calculation (see [Co, Section 34]...
2]) shows that
\[ h^i(\mathcal{O}_{\mathbb{P}^1}(ae + bf)) = 0, \ i = 1, 2, \]
when \((a, b) = (1, 1), (3, 7), \) or \((4, 6).\)

If follows that Castelnuovo curves of degree 10 on cubic scrolls of type \(S_{1, 2}\) in \(\mathbb{P}^4\) determine a locally closed sublocus of \(M_{10}\) of dimension at most 44. On the other hand, each Castelnuovo curve under our consideration satisfies \(i = 0,\) because each admits a 6-regular generic initial ideal \(\text{gin}(I_C).\)

Therefore, because
\[ 55 - 44 > g, \]
we deduce immediately that the sublocus of \(M_{10}\) corresponding to Castelnuovo curves on scrolls \(S_{1, 2}\) is nonproblematic.

Finally, say that \(C\) lies on a cubic scroll of type \(S_{0, 3}.\) Note that \(C\) cannot pass through the vertex of \(S_{0, 3}.\) To see why, note that adjunction applied to the proper transform \(\tilde{C}\) of \(C\) in \(\mathbb{P}_3\) yields
\[ \tilde{g} = 7 \text{ and } a - m = 3, \]
just as in Case 3 where \(a\) is the coefficient of \(e\) in the proper transform \([\tilde{C}]\) of \(C\) in \(\mathbb{P}_3,\) \(m\) is the multiplicity with which \(C\) passes through the vertex of the scroll. Because \(a - m\) and \(a\) are of opposite parity, the Riemann-Roch formula implies that either \(\chi(\phi_{0,3}^{-1}C)\) or \(\chi(\tilde{C})\) is not an integer, which is absurd.

We conclude that all Castelnuovo curves are nonproblematic. The proof of Theorem 2.2 is now complete.

### 2.5 Curves spanning hyperplanes

Next, we consider rational curves \(C\) whose linear spans are 3-dimensional hyperplanes \(H \subset \mathbb{P}^4.\) By (2.2b), in order to prove Theorem 2.1 for such curves it suffices to show the following.

**Theorem 2.3.** The reduced irreducible rational curves verifying
\[ g + i \geq 11 + \min(g, 5) \]
define a sublocus of \(M_{10}\) of codimension greater than \(g + i.\)

We now argue much like we did in our analysis of nondegenerate curves. The analysis here is simpler, though, because the required bound on \(g + i\) is easier to obtain. Therefore, we give the argument, while omitting the proofs of results that generalize immediately from the nondegenerate case.

35
Fix $C$, and let $H_1$ denote the linear span of $C$. Let $H_2$ be a hyperplane in $H$ that is general with respect to $C$ in the sense that $\Gamma := C \cap H_1$ is a collection of ten points in uniform position in $H$. Choose coordinates for $\mathbb{P}^4$ in such a way that $H$ and $H_1$ are defined by $x_4 = 0$ and $x_3 = x_4 = 0$, respectively. Just as we did for nondegenerate curves, we define the hyperplane gin of $C$ to be the (saturated) generic initial ideal of the saturation of $\mathcal{I}_{C \cap H_1}/H_1$, and we abusively denote it by gin($I_\Gamma$). Just as before, because the hyperplane gin is saturated and Borel-fixed, it follows that gin $I_\Gamma$ is a monomial ideal in $x_0$ and $x_1$ having a minimal generating set of the form
\[ (x_0^k, x_0^{k-1}x_1^{\lambda_{k-1}}, \ldots, x_0x_1^{\lambda_1}, x_1^{\lambda_0}). \]

By a result of Ellia and Peskine [Gre, Cor. 4.8], the invariants $\lambda_i$ of the above generating set satisfy
\[ \lambda_i - 1 \geq \lambda_{i+1} \geq \lambda_i - 2 \]
for all $i = 0, \ldots, k - 2$.

Every Borel-fixed ideal with minimal generators that are monomials $x_0^i x_1^j$ has a unique tree-representation analogous to the tree representations for the hyperplane gins of nondegenerate curves in $\mathbb{P}^4$ introduced previously. Moreover, every tree may be obtained from an empty tree with a single vertex by applying a sequence of rules that we denote as before by $\Lambda$-rules (see Table 3). A straightforward inductive argument analogous to those already carried out in our analysis of nondegenerate hyperplane gins shows that $\sum_{i=0}^{k-2} \lambda_i$ is equal to the number of nonterminal vertices in a minimal generating tree for the hyperplane gin, which in turn is equal to the degree of the curve $C$. Whence,
\[ \sum_{i=0}^{k-2} \lambda_i = 10. \tag{2.18} \]

On the other hand, by the main result of [Bal] implies that $I_\Gamma$ is 5-regular. It follows from (2.18) and the combinatorial characterization of Borel-fixity in [El, Thm. 15.23] that the gin of a general hyperplane section $\Gamma$ of a reduced, irreducible degree-10 curve $C$ in $\mathbb{P}^3$ is either $\text{Borel}(x_1^4)$ or $(x_1^5, x_0 x_1^3, x_0^2 x_1^2, x_0^3)$. Define $g_\Gamma$ for these ideals as before; then a calculation yields $g_\Gamma = 11$ and $g_\Gamma = 12$, respectively. It follows that
\[ g + i \leq 11 \text{ and } g + i \leq 12, \tag{2.19} \]
respectively.
Table 3: Λ-rules for curves spanning hyperplanes

1. $x_e^0 \mapsto (x_e^{e+1}, x_0 x_1)$,
2. $x_e^0 x_1^f \mapsto x_e^0 x_1^{f+1}$, and an initial rule
3. $\emptyset \mapsto (x_0, x_1, x_2)$.

Table 4: C-rules for irreducible curves spanning hyperplanes

1. $x_e^0 \mapsto x_e^0 \cdot (x_0, x_1, x_2)$,
2. $x_e^0 x_1^f \mapsto x_e^0 x_1^f \cdot (x_1, x_2)$, and
3. $x_e^0 x_1^f x_2^g \mapsto x_e^0 x_1^f x_2^g \cdot x_2$

Just as in the case of nondegenerate curves, a minimal generating set for the generic initial ideal $\text{gin}(I_{\mathcal{C}/H})$ may be represented in a tree obtainable from the tree of corresponding hyperplane $\text{gin}$ by applying a sequence of rules; these we denote as before by $C$-rules. The $C$-rules are given in Table 4.

The same argument used for nondegenerate curves shows that for every $C$, $h^1(I_{\mathcal{C}/H}(5))$ is equal to the number of $C$-rewritings applied in degrees greater than 6. Finally, the long exact sequence in cohomology associated to

$0 \to I_{\mathbb{P}^4}(4) \to I_{\mathcal{C}/\mathbb{P}^4}(5) \to I_{\mathcal{C}/H}(5) \to 0$,

shows that

$h^1(I_{\mathcal{C}/H}(5)) = h^1(I_{\mathcal{C}/\mathbb{P}^4}(5))$.

With these preliminaries in hand, the proof of Theorem 2.3 is almost immediate. For, note that applying a single rewriting rule to either $\text{Borel}(x_1^4)$ or $(x_5^5, x_0 x_4, x_0^2 x_1, x_0^3)$ produces a saturated ideal that is minimally generated in degrees at most six, and is, therefore, 6-regular. Accordingly, our estimates improve to $g + i \leq 10$ and $g + i \leq 12$, respectively. Therefore, Theorem 2.3 is verified in every case except possibly if $g = 0$.

On the other hand, by [GLP, Thm. 3.1], $C$ is 8-regular unless $C$ admits an 8-secant line. An inspection of possible generic initial ideals in the second case yields that if $C$ is 8-regular then $i \leq 10$. Therefore, Theorem 2.3 is verified in every case away from the locus of genus-0 curves that admit 8-secant lines. On the other hand, smooth rational curves $C$ of degree 10 that admit 8-secant lines comprise a sublocus of the Hilbert scheme of rational,
smooth, and irreducible curves in $H$ of codimension at least 5, by [JK1, Lemma (2.4)]. In other words, by (2.2b), the sublocus has codimension at least 16 in $M_{10}$. Theorem 2.3 follows immediately.

3 Reducible curves

In this section we’ll prove the following theorem, extending [JK1, Thm.4.1].

Theorem 3.1. On a general quintic threefold in $\mathbb{P}^4$, there is no connected, reduced and reducible curve of degree at most 10 whose components are rational.

Suppose, on the contrary, that such a curve $C$ exists. By the results of [JK1], we may assume $C$ has two components and is of degree 10. Consider one of them. By the result of [JK1, Theorem 3.1], either it’s a six-nodal plane quintic or it’s smooth. If it’s smooth, then, by [JK1, Cor. 2.5(3)], either it’s a rational normal curve of degree $\leq 4$ or it spans $\mathbb{P}^4$. We will prove that there can be no such $C$.

To this end, we follow Johnsen and Kleiman once more. Let $M'_a$ denote the open subscheme of the Hilbert scheme of $\mathbb{P}^4$ parametrizing the smooth irreducible curves of degree $a$ that are rational normal curves if $a \leq 4$ and that span $\mathbb{P}^4$ if $a \geq 4$. Denote the scheme parametrizing six-nodal plane quintics in $\mathbb{P}^4$ by $N_5$. Let $R_{a,b,n}, S_{5,n}$, and $S'_n$ denote the subsets of $M'_a \times M'_b$ (resp., $M_5 \times N_5, N_5 \times N_5$) of pairs $(A, B)$ such that $A \cap B$ has length $n$. Finally, let $I_{a,b,n}$ (resp., $J_n, K_n$) denote the subset of $R_{a,b,n} \times \mathbb{P}^{125}$ (resp., $S_{5,n} \times \mathbb{P}^{125}, S'_n \times \mathbb{P}^{125}$) of triples $(A, B, F)$ such that $A \subset F$ and $B \subset F$. The $F$ that contain a plane form a proper closed subset of $\mathbb{P}^{125}$; form its complement, and the preimages of this complement in the incidence schemes $I_{a,b,n}, J_n, K_n$. Now replace $J_n$ (resp., $K_n$). Replace $S_{5,n}$ and $S'_n$ by the images of those preimages, and replace $I_{a,b,n}, J_n$ and $K_n$ by the new preimages. Then given any pair $(A, B)$ in $S'_n$, there is an $F$ that contains both $A$ and $B$, but not any plane. It remains to show that $I_{a,b,n}$ (resp., $J_n, K_n$) has dimension at most 124 whenever $a + b = 10$ and $n \geq 1$.

We note that the fibre of $I_{a,b,n}$ (resp., $J_n, K_n$) over a pair $(A, B)$ is a projective space of dimension $h^0(\mathcal{I}_C(5)) - 1$, where $C$ is the reducible curve $C = A \cup B$ and $\mathcal{I}_C$ is the ideal sheaf of the corresponding subscheme of $\mathbb{P}^4$. Hence we have

$$\dim I_{a,b,n} \leq \dim R_{a,b,n} + 125 - \min_C \{h^0(\mathcal{O}_C(5)) - h^1(\mathcal{I}_C(5))\}$$
Obviously, we have
\[ h^0(\mathcal{O}_C(5)) \geq \chi(\mathcal{O}_C(5)), \]
which implies
\[
\begin{align*}
h^0(\mathcal{O}_C(5)) &\geq 5(a + b) + 2 - n, \\
h^0(\mathcal{O}_C(5)) &\geq 20 + 5a + 1 - n = 21 + 5a - n, \quad \text{and} \\
h^0(\mathcal{O}_C(5)) &\geq 20 + 20 - n = 40 - n, \\
\end{align*}
\]
respectively.

Our theorem is then a consequence of the following two lemmas.

**Lemma 3.0.1.** For \(a + b = d\) and \(n \geq 1\),
\[
\begin{align*}
\dim R_{a,b,n} &\leq 5(a + b) + 1 - n, \\
\dim S_{a,n} &\leq 20 + 5a - n \\
\text{and} \quad \dim S'_{n} &\leq 39 - n.
\end{align*}
\]

**Lemma 3.0.2.** For \(a + b = d\) and \(n \geq 1\),
\[ h^1(\mathcal{I}_C(5)) = 0. \]

To prove the first lemma, begin by letting \((A, B)\) denote an arbitrary pair in \(R_{a,b,n}\). Fix \(B\), and let \(A\) vary in the fibre of \(R_{a,b,n}\) over \(B\). Assume that \(a \leq b\), so in particular \(a \leq 4\).

If \(a = 1\) or \(a = 3\), then the lemma holds on the basis of the arguments in \([\text{JK1}]\). Moreover, if \(a = 4\) then the argument of \([\text{JK1}]\) carries over except in the case where \(B\) is a sextic meeting \(A\) in twelve points along which the sextic intersects a hyperquadric containing \(A\).

To handle the latter situation, recall that the restricted tangent bundles \(T_{\mathbb{P}^4}|_A\) and \(T_{\mathbb{P}^4}|_B\) have balanced splittings \((5,5,5,5)\) and \((8,8,7,7)\), respectively (\([\text{JK1}]\) Cor 2.5)). Now fix a divisor \(D_1\) of degree 7 along \(\mathbb{P}^1\) by and a divisor \(D_2\) of degree 7 along the sextic. The space of degree-6 morphisms \(\mathbb{P}^1 \rightarrow \mathbb{P}^4\) mapping \(D_1\) to \(D_2\) has dimension \(h^0(T_{\mathbb{P}^4}|_B \otimes \mathcal{I}_{D_2}) = 30 - 4 \cdot 7 = 30 - 28 = 2\), since \(h^1(T_{\mathbb{P}^4}|_B \otimes \mathcal{I}_{D_2}) = 0\) (see \([\text{De}]\) p. 45)). As \(D_1\) and \(D_2\) each vary in a 7-dimensional family, it follows that those sextics \(B\) intersecting \(A\) in seven points cut out a locus of codimension 14 inside \(M_6\). Since 14 is larger than 12, we conclude that \(\dim R_{4,6,12}\) meets the required bound.

Similarly, if \(a = 2\) and \(b = 8\) then by the argument in \([\text{JK1}]\) we may assume \(n \geq 7\) without loss of generality. Note that if \(n \geq 7\), then in fact \(n = 7\). For, denote the plane of \(A\) by \(J\). Note that \(B\) spans \(\mathbb{P}^4\), as \(b > 3\).
Let \( H \) be the hyperplane spanned by \( J \) and a general point of \( B \); then \( l(H \cap B) = b \geq n + 1 \). From \( b = 8 \), we conclude that \( n = 7 \).

To bound the dimension of \( R_{2,8,7} \), the space of unions of irreducible rational conics \( A \) and irreducible rational nondegenerate octics \( B \) intersecting in projective schemes of length 7, we proceed much as we did to bound \( \dim R_{4,6,12} \). Note it suffices to show that rational nondegenerate octics intersecting a conic in at least seven points have codimension at least 8 inside \( M_8 \). But just as in the analysis of \( R_{4,6,12} \), the assertion is clear since [JK1, Cor 2.5] implies that the restricted tangent bundle \( T_{P^4} |_B \) has balanced splitting type \((10, 10, 10, 10)\).

Next we treat \( \dim R_{5,5,n} \), where \( n \geq 1 \). Using [JK1, Cor 2.5], we note that the 3-regular schemes \( A \) and \( B \) are each cut out by hyperquadrics and hypercubics. Therefore,

\[
h^0(I_A(2)) = h^0(O_{P^4}(2)) - h^0(O_A(2)) = 15 - (5(2) + 1) = 4.
\]

For degree reasons, no three linearly independent hyperquadrics containing \( A \) may cut out a complete intersection curve containing \( A \cup B \).

The only remaining possibility is that \( A \) and \( B \) lie on a nondegenerate cubic scroll, defined by three hyperquadrics. But in that case, any “fourth” hyperquadric containing \( A \) intersects the scroll properly, in a scheme of degree at most 6. So we conclude that some hyperquadric \( Q \) containing \( A \) doesn’t contain \( B \). Thus degree \( A \cap B \leq \deg B \cap Q = 10 \). On the other hand, \( T_{P^4} |_A \) has balanced splitting type \((7, 6, 6, 6)\), from which it follows (by the same argument used earlier) that for all \( 1 \leq n \leq 7 \), the codimension of curves \( A \) intersecting curves \( B \) in subschemes of length \( n \) is equal to \( 2n \). As \( \dim R_{5,5,n} \leq \dim R_{5,5,m} \) whenever \( m \geq n \), and \( 10 < 14 \), it follows immediately that no curve in \( R_{5,5,n}, n \geq 1 \) lies on a general quintic hypersurface.

Now consider reducible unions belonging to \( \dim S_{5,n} \), where \( n \geq 1 \). Let \( A \) be a nondegenerate, rational, smooth quintic curve, and let \( B \) be a six-nodal plane quintic with linear span \( J \). The intersection \( A \cap J \) is proper, whence of degree at most 5; it follows that \( \deg(A \cap B) \leq 5 \). Since \( T_{P^4} |_A \) has splitting type \((7, 6, 6, 6)\), the codimension of curves \( A \) intersecting curves \( B \) in subschemes of length \( n \) is equal to \( 2n \), for all \( n \leq 7 \). It follows that no curve in \( S_{5,n}, n \geq 1 \) lies on a general quintic hypersurface.

Finally, consider a pair \((A, B)\) in \( S_{n} \). Let \( J \) denote the plane of \( A \), and \( K \) the plane of \( B \). We may assume \( J \neq K \) without loss of generality, since the general quintic threefold \( F \) intersects \( J \) properly in a quintic curve, which shows that if \( J = K \), then

\[
A \cup B \subset F
\]
is impossible. So \( A \cup B \) spans at least a 3-space.

Moreover, it’s clear that \( A \cap B \subset J \cap B \), so \( A \cap B \) has degree at most 5. On the other hand, the restricted tangent bundle \( T_{\mathbb{P}^2}|_A \) has splitting type \((a_1, a_2)\), where \( a_1 + a_2 = 15 \). Assume \( a_1 \geq a_2 \). As usual, our goal is to bound the codimension of those six-nodal plane quintics that meet other six-nodal plane quintics in \( 1 \leq n \leq 5 \) points using ampleness properties of the restricted tangent bundle of \( A \). In particular, we are done provided \( a_2 \geq 2 \), which is certainly the case. For (cf. [GLP]), there is an exact sequence

\[
0 \to \bigoplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^1}(-a_i + 5) \to H^0(\mathcal{O}_{\mathbb{P}^1}(5)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(5) \to 0
\]

which implies \( a_i - 5 \geq 0 \). The proof of the first lemma is now complete.

We now proceed to the proof of the second lemma, i.e. that all curves \( C \) in rational components \( A \) and \( B \) satisfying our hypotheses are 6-regular. First say \( A \) and \( B \) are smooth. It’s a well-known fact (see, for example, [Gi, Thm 2.1]) that

\[
\text{reg} \ A \cup \ B \leq \text{reg} \ A + \text{reg} \ B,
\]

so we need only establish that \( A \) and \( B \) are 3-regular. But this follows, e.g., from the result of [JK2 Prop. 2.2]. (See also the discussion following the proof of [JK1 Cor. 2.5]; the key point is to observe that the components \( C_i \) of \( C \) are necessarily of maximal rank in every degree for the canonical morphisms \( H^0(\mathcal{O}_{\mathbb{P}^4}(k)) \to H^0(\mathcal{O}_{C_i}(k)) \).

In the only remaining case, \( A \) is a smooth rational quintic spanning \( \mathbb{P}^4 \) and \( B \) is a six-nodal plane quintic. In that situation, \( C = A \cup B \) is 4-regular, by the “Horace lemma” [JK1 Lemma 4.5].

Therefore, we may assume \( A \) and \( B \) are plane quintics. Once more, we treat separately the cases where rank \( C = 3 \) and rank \( C = 4 \).

If \( G \), the linear span of \( A \cup B \), is a 3-space, then \( G \cap H \) is a plane containing \( K \), and is therefore equal to \( K \). Hence

\[
C \cap H = (C \cap G) \cap H = C \cap (G \cap H) = C \cap K.
\]

On the other hand, we have

\[
(A \cup B) \cap K = (A \cap K) \cup B.
\]

Indeed, the latter equality of schemes is equivalent to the following equality of ideal sheaves:

\[
(I_{A/\mathbb{P}^4} \cap I_{B/\mathbb{P}^4}) + I_{K/\mathbb{P}^4} = (I_{A/\mathbb{P}^4} + I_{K/\mathbb{P}^4}) \cap I_{B/\mathbb{P}^4}.
\]
Any element \( l \) belonging to the left side is of the form \( l = a + k = b + k \) where \( a \in \mathcal{I}_{A/\mathbb{P}^4}, b \in \mathcal{I}_{B/\mathbb{P}^4}, \) and \( k \in \mathcal{I}_{K/\mathbb{P}^4}. \) The inclusion \( l \in \mathcal{I}_{A/\mathbb{P}^4} + \mathcal{I}_{K/\mathbb{P}^4} \) follows immediately, and we also have \( l \in \mathcal{I}_{B/\mathbb{P}^4} \) because \( \mathcal{I}_{K/\mathbb{P}^4} \subset \mathcal{I}_{B/\mathbb{P}^4} \) from the inclusion \( B \subset K. \) In the opposite direction, given \( r \equiv a + k = b, \) we see that \( a = b - k \in \mathcal{I}_{B/\mathbb{P}^4} \) because \( \mathcal{I}_{K/\mathbb{P}^4} \subset \mathcal{I}_{B/\mathbb{P}^4}. \) Since a general quintic threefold \( F \) contains no plane, we may assume \( A \cup B \subset F \) but \( K \) lies outside \( F, \) so \( F \cap K = B \) by Bezout’s theorem. Then \( A \cap K \subset B, \) and it follows immediately that \( C \cap H = C \cap K = B. \)

In what follows, we let \( D \) denote \( C \cap H. \) To compute \( h^1(\mathcal{I}_{C/\mathbb{P}^4}(5)) \), we use the exact sequence:

\[
0 \to \mathcal{I}_{A/\mathbb{P}^4}(-1) \to \mathcal{I}_{C/\mathbb{P}^4} \to \mathcal{I}_{D/H} \to 0.
\]

To bound the cohomology of the middle term, we bound cohomology on the right and left. Note there is an exact sequence

\[
0 \to \mathcal{I}_{K/H} \to \mathcal{I}_{D/H} \to \mathcal{I}_{D/K} \to 0
\]

with \( \mathcal{I}_{K/H} = \mathcal{O}_H(-1) = \mathcal{O}_{\mathbb{P}^3}(-1) \) and \( \mathcal{I}_{D/K} = \mathcal{O}_K(-5) = \mathcal{O}_{\mathbb{P}^2}(-5). \) It follows immediately from Serre’s theorem that \( h^1(\mathcal{I}_{K/H}(m)) = h^1(\mathcal{I}_{D/K}(m)) = 0 \) for all integers \( m. \) Similarly, there is an exact sequence

\[
0 \to \mathcal{I}_{J/\mathbb{P}^4} \to \mathcal{I}_{A/\mathbb{P}^4} \to \mathcal{I}_{A/J} \to 0
\]

with \( \mathcal{I}_{A/J} = \mathcal{O}_{\mathbb{P}^2}(-5) \) and \( \mathcal{I}_{J/\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-1)^2 \), and hence \( h^1(\mathcal{I}_{A/\mathbb{P}^4}(m)) = 0 \) for all integers \( m. \)

Finally, say \( C = A \cup B \) is nondegenerate. After having made an appropriate change of coordinates, we may assume that \( J \) and \( K \) intersect in the point \( P = (0,0,1,0,0) \) and that the homogeneous ideals describing the embeddings of \( A \) and \( B \) inside \( \mathbb{P}^4 \) are given in coordinates by

\[
I_A = (x_3, x_4, f(x_0, x_1, x_2)) \quad \text{and} \quad I_B = (x_0, x_1, g(x_2, x_3, x_4))
\]

for some trivariate homogeneous quintic polynomials \( f \) and \( g. \) Since \( A \) and \( B \) pass through \( P = (0,0,1,0,0), \) \( f \) and \( g, \) which vanish at \( P, \) do not contain \( x_5 \) in their expansions.

Note that

\[
(x_3, x_4) \cap I_B + f(x_0, x_1, x_2) \cap I_B \subset I_A \cap I_B.
\]

On the other hand, given any element \( e \in I_A \cap I_B, \) viewed as a combination of \( x_3, x_4, \) and \( f(x_0, x_1, x_2) \) with polynomial coefficients, any terms of \( e \) divisible
by \(f(x_0, x_1, x_2)\) automatically belong to \(I_B\), since \(f\) does not contain \(x_2^2\) in its expansion. It follows immediately that \(e = e_1 + e_2\) with \(e_2 \in f(x_0, x_1, x_2) \cap I_B\) and \(e_1 \in (x_3, x_4) \cap I_B\), and therefore that

\[
(x_3, x_4) \cap I_B + (f(x_0, x_1, x_2)) \cap I_B = (x_3, x_4) \cap I_B + (f(x_0, x_1, x_2)) = I_A \cap I_B.
\]

Continuing in this vein, we deduce that the homogeneous ideal of \(C \subset \mathbb{P}^4\) is

\[
I_C = I_A \cap I_B = (x_3, x_4) \cap (x_0, x_1) + (g(x_2, x_3, x_4)) + (f(x_0, x_1, x_2))
\]

\[
= (x_1x_4, x_0x_4, x_1x_3, x_0x_3, f(x_0, x_1, x_2), g(x_2, x_3, x_4)).
\]

We will now show that

\[
(x_1x_4, x_0x_4, x_1x_3, x_0x_3, \text{lt}(f), \text{lt}(g)) = \text{in}(I_C),
\]

where \(\text{lt}(F)\) denotes the leading term of a homogeneous polynomial \(F\) with respect to the revlex order, and \(\text{in}(I)\) denotes the ideal of leading terms of the homogeneous ideal \(I\).

We will deduce the latter inclusion as a consequence of [Gre, Prop 4.3], which establishes that for any two homogeneous ideals \(I\) and \(J\), if

\[
\text{in}(I) \cap \text{in}(J) \subset \text{in}(I \cap J)
\]

then \(\text{in}(I + J) = \text{in}(I) + \text{in}(J)\).

Now let

\[
f = \sum_{i+j+k=5} a_{ijk}x_0^i x_1^j x_2^k;
\]

since \((i, j) = (0, 0)\) is disallowed, we have

\[
x_3f \in (x_1x_4, x_0x_4, x_1x_3, x_0x_3) \cap (f).
\]

So

\[
\text{lt}(x_3f) = x_3 \text{lt}(f) \in \text{in}((x_1x_4, x_0x_4, x_1x_3, x_0x_3) \cap (f))
\]

and similarly

\[
\text{lt}(x_4f) \in \text{in}((x_1x_4, x_0x_4, x_1x_3, x_0x_3) \cap (f)).
\]

On the other hand, if

\[
\text{lt}(f) = a_{ijk}x_0^i x_1^j x_2^k,
\]

then \(x_3a_{ijk}x_0^i x_1^j x_2^k\) and \(x_4a_{ijk}x_0^i x_1^j x_2^k\) generate the intersection

\[
\text{in}(x_1x_4, x_0x_4, x_1x_3, x_0x_3) \cap \text{in}(f).
\]
(This is clear, since any monomial $e$ belonging to the latter intersection is divisible by one of the monomials $x_1x_4, x_0x_4, x_1x_3, x_0x_3$, hence by $x_3$ or $x_4$, and also $\text{lt } f$. Thus $e$ is divisible by $x_3\text{lt } f$ or $x_4\text{lt } f$.) So in fact

$$\text{in}(x_1x_4, x_0x_4, x_1x_3, x_0x_3) \cap \text{in}(f) \subset \text{in}((x_1x_4, x_0x_4, x_1x_3, x_0x_3) \cap (f)).$$

Therefore,

$$\text{in}(x_1x_4, x_0x_4, x_1x_3, x_0x_3, f) = (x_1x_4, x_0x_4, x_1x_3, x_0x_3, \text{lt}(f)).$$

Essentially the same argument shows that

$$\text{in}(x_1x_4, x_0x_4, x_1x_3, x_0x_3, f) \cap \text{in}(g) \subset \text{in}((x_1x_4, x_0x_4, x_1x_3, x_0x_3, f) \cap (g)).$$

We conclude that

$$\text{in}(x_1x_4, x_0x_4, x_1x_3, x_0x_3, f, g) = (x_1x_4, x_0x_4, x_1x_3, x_0x_3, \text{lt}(f), \text{lt}(g)).$$

Since there is a flat degeneration taking $\mathcal{I}_C$ to in $\mathcal{I}_C$, and the latter ideal is generated in degrees at most 5, we conclude by the basic regularity result of Bayer–Stillman [Gre] Theorem 2.27 that $\mathcal{I}_C$ is 6-regular.

References

[AK] A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Springer Lecture Notes in Math. 146 (1970).

[Bal] E. Ballico, *On singular curves in positive characteristic*, Math. Nachr. 141 (1989), 267-73.

[Bay] D. Bayer, *The division algorithm and the Hilbert scheme*, Harvard Ph. D thesis, 1982.

[BS] D. Bayer and M. Stillman, *A criterion for detecting $m$-regularity*, Invent. Math. 87, 1-11.

[Ci] C. Ciliberto, *Hilbert functions of finite sets of points and the genus of a curve in a projective space*, in “Space Curves; Proceedings, Rocca di Pappa 1985,” F. Ghione, C. Peskine, E. Sernesi (Eds.), Springer Lecture Notes in Math. 1266 (1986), 24-73.

[CK] D.A. Cox and S. Katz, “Mirror symmetry and algebraic geometry,” Math. Survey and Monographs vol. 68, AMS, 1999.
[CLO] D. Cox, J. Little, and D. O'Shea, “Ideals, varieties, and algorithms,” 2nd ed., Springer, New York, 1997.

[Co] I. Coskun, “Degenerations of surface scrolls and the Gromov-Witten invariants of Grassmannians,” arXiv preprint math.AG/0407253.

[d’A] J. d’Almeida, Courbes de l’espace projectif: Series lineaires incompletes et multisecantes, J. reine ang. Math. 370 (1986), 30-51.

[De] O. Debarre, “Higher-dimensional algebraic geometry,” Springer, New York, 2001.

[Ei] D. Eisenbud, “Commutative algebra with a view towards algebraic geometry,” Springer, New York, 1995.

[EH] D. Eisenbud and J. Harris, “Curves in projective space,” les Presses de l’Université de Montreal, 1982.

[EP] P. Ellia and C. Peskine, Groupes de points de P²; caractè re et position uniforme, in “Algebraic Geometry; Proceedings, L’Aquila 1988,” Springer Lecture Notes in Math. (1990).

[Gi] D. Giaimo, On the Castelnuovo–Mumford regularity of connected curves, arXiv preprint math.AG/0309051.

[Gre] M. Green, Generic initial ideals, in “Six lectures on commutative algebra,” J. Elias, J.M. Giral, R.M. Miro-Roig, and S. Zarzuela (Eds.), Birkhauser, Boston, 1998.

[GH] P. Griffiths and J. Harris, “Principles of algebraic geometry”, Wiley, New York, 1978.

[GLP] L. Gruson, R. Lazarsfeld, and C. Peskine, On a theorem of Castelnuovo, and the equations defining space curves, Invent. Math. 72 (1983), pp. 491-506.

[GS] D. Grayson and M. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2

[Gu] M. Guest, On the space of holomorphic maps from the Riemann sphere to the quadric cone, Quart. J. Math. Oxford Ser. (2) 45 (1994), 57-75.

[H] R. Hartshorne, “Algebraic geometry,” Springer-Verlag, New York, 1977.
[JK1] T. Johnsen and S. Kleiman, *Rational curves of degree at most 9 on a general quintic threefold*, Comm. in Alg., 24 (1996), no. 8, 2721-53.

[JK2] T. Johnsen and S. Kleiman, *Towards Clemens’ conjecture in degrees between 10 and 24*, Serdica Math. J. 23 (1997), no. 2, 131-42.

[Ka] S. Katz, *On the finiteness of rational curves on quintic threefolds*, Compositio Math. 60 (1986), no. 2, 151-62.

[KiPa] B. Kim and R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, in “Symplectic geometry and mirror symmetry (Seoul, 2000),” 187-201, World Scientific, River Edge, NJ, 2001.

[KiPi] S. Kleiman and R. Piene, *Node polynomials for families: methods and applications*, Math. Nachr. 271 (2004), 69-90.

[N] U. Nagel, *On Hilbert function under liaison*, Le Matematiche XLVI (1991), 547-58.

[Ra] L. Ramella, *La stratification du schéma de Hilbert des courbes rationnelles de P^n par le fibré tangent restreint*, C.R. Acad. Sci. Paris 311 (1990), 181-4.

[Re] A. Reeves, *Combinatorial structure on the Hilbert scheme*, Cornell Ph. D thesis, 1992.

[Va] I. Vainsencher, *Enumeration of n-fold tangent hyperplanes to a surface*, J. Alg. Geom. 4 (1995), 503-26.

[Ve] J.-L. Verdier, *Two dimensional sigma-models and harmonic maps from S^2 to S^{2n}*, in “Group theoretical methods in physics,” Springer Lecture Notes in Physics 180 (1983), 136-41.