The Ihara-Selberg zeta function for PGL3 and hecke operators

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The Ihara-Selberg zeta function for $\text{PGL}_3$ and Hecke operators

Anton Deitmar & J. William Hoffman

Abstract. A weak version of the Ihara formula is proved for zeta functions attached to quotients of the Bruhat-Tits building of $\text{PGL}_3$. This formula expresses the zeta function in terms of Hecke-Operators. It is the first step towards an arithmetical interpretation of the combinatorially defined zeta function.

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Introduction

Y. Ihara [9] extended the theory of Selberg type zeta functions to $p$-adic settings. His work was later generalized by K. Hashimoto [6, 7, 8], H. Bass [1], H. Stark and A. Terras [10], and others. Ihara defined the zeta function in group theoretical terms first, but it can be described geometrically as follows. Let $\Gamma \backslash X$ be a finite quotient of the Bruhat-Tits tree of a rank one $p$-adic group modulo an arithmetic group $\Gamma$. Then define the zeta function by

$$Z(u) = \prod_c (1 - u^{l(c)}),$$

where the product runs over all primitive closed loops in $\Gamma \backslash X$. Ihara proved the remarkable formula

$$Z(u) = \chi \det(1 - Au + qu^2),$$

where $A$ is the adjacency operator on $\Gamma \backslash X$ which can be interpreted as the canonical generator of the unramified Hecke-algebra. Further, $\chi < 0$ is the Euler-characteristic of $\Gamma \backslash X$, and $q$ is the order of the residue class field.

For $\Gamma$ being the unit group of the maximal order in a quaternion algebra, this formula allowed Ihara to relate $Z(u)$ to the Hasse-Weil zeta function of the Shimura curve attached to $\Gamma$. This is the only proven link between Selberg-type zeta functions and arithmetical zeta functions.

In [5] the author gave a definition of an Ihara-type zeta function $Z(u)$ for a higher rank group. There is no Ihara-formula for higher rank up to date. In this paper we give an approximation to an Ihara formula in the case of the group $\text{PGL}_3$. For this group the unramified Hecke-algebra has two generators $\pi_1, \pi_2$. The canonical replacement of the determinant factor in Ihara’s formula is

$$\det(1 - u\pi_1 + u^2q\pi_2 - u^3q^3).$$

The main result of the present paper is

**Theorem 0.1** There are a natural number $n$ and a polynomial $P(u)$ such that

$$Z(u) = \frac{\det(1 - u\pi_1 + u^2q\pi_2 - u^3q^3)^n}{P(u)}.$$

I thank A. Setyadi for pointing out an error in an earlier version.
1 The building

Let $F$ be a non-archimedean local field. Let $\mathcal{O}$ be its valuation ring with maximal ideal $m \subset \mathcal{O}$. Fix a generator $\varpi$ of $m$ and let $q$ be the cardinality of the residue class field $k = \mathcal{O}/m$.

Consider the locally compact group $G = \text{PGL}_3(F) = \text{GL}_3(F)/F^\times$. It is totally disconnected and every maximal compact subgroup is conjugate to $K = \text{PGL}_3(\mathcal{O}) = \text{GL}_3(\mathcal{O})/\mathcal{O}^\times$. Let $X$ be the Bruhat-Tits building of $G$. In this particular case the Bruhat-Tits building can be described nicely. The vertex set $X_0$ of $X$ is the set of homothety classes of $\mathcal{O}$-lattices in $F^3$. Recall that an $\mathcal{O}$-lattice in $F^3$ is a finitely generated $\mathcal{O}$-submodule $\Lambda$ of $F^3$ such that $F\Lambda = F^3$. Two lattices $\Lambda, \Lambda'$ are homothetic, if there exists $\alpha \in F^\times$ such that $\Lambda' = \alpha \Lambda$. Every lattice $\Lambda$ is the image under some $g \in \text{GL}_3(F)$ of the standard lattice $L_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}e_3$, where $e_1, e_2, e_3$ is the standard basis of $F^3$. The set of all lattices thus can be identified with $\text{GL}_3(F)/\text{GL}_3(\mathcal{O})$ and the set $X_0$ of homothety classes of lattices with $G/K$. Let $G'$ denote the image of $\text{SL}_3(F)$ in $G$. The set $X_0$ splits into three orbits under the action of $G'$. These orbits are given by $L_0$ as above, $L_1 = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \varpi \mathcal{O}e_3$, and $L_2 = \mathcal{O}e_1 \oplus \varpi \mathcal{O}e_2 \oplus \varpi \mathcal{O}e_3$. For a given vertex $x \in X_0$ we say $x$ is of type $j$ if $G'x \subseteq G'L_j$ for $j = 0, 1, 2$. Two vertices $x \neq y$ are joined by an edge if and only if there are representatives $\Lambda_1$ and $\Lambda_2$ for $x$ and $y$ such that $\varpi \Lambda_1 \subset \Lambda_2 \subset \Lambda_1$. It follows that $x$ and $y$ must be of different type. This describes the 1-skeleton $X_1$ of $X$. The following Lemma gives further properties of the graph $X_1$.

**Lemma 1.1** (a) Every vertex in $X$ has $2(q^2 + q + 1)$ neighbours.

(b) Two neighboured vertices have $q + 1$ common neighbours.

(c) Any three distinct vertices have at most one common neighbour.

**Proof:** For (a) it suffices to consider the vertex given by $L_0$. Every neighbour has a representative lattice $L$ with

$$\varpi L_0 \subset L \subset L_0.$$ 

Now $L_0/\varpi L_0 \cong \mathbb{F}_q^3$ as a vector space over $\mathbb{F}_q$, and $L$ defines a sub vector space. Thus the set of all neighbours of $[L_0]$ is in bijection with the set of all non-trivial sub vector spaces of $\mathbb{F}_q^3$ which are $2(1 + q + q^2)$ in number. Part (b) and (c) are similar. $\square$
Whenever three vertices $x, y, z$ are mutually connected by edges, then this triangle forms the boundary of a 2-cell of $X$, called a chamber. This describes $X$ as a CW-complex. There is, however, more structure through the geometry of the apartments. For instance, whenever two edges meet in a vertex, there is an angle between them which can be $\pi/3, 2\pi/3$, or $\pi$. A geodesic $c$ in $X$ is a straight oriented line in one apartment. If $c$ happens to lie inside $X_1$, then it gives rise to a sequence of edges $(\ldots, e_{-1}, e_0, e_1, \ldots)$ such that $e_k$ and $e_{k+1}$ have angle $\pi$ for every $k \in \mathbb{Z}$. In this case we say that $c$ is a rank-one geodesic.

For each edge $e$ with vertices $\{x, y\}$ we fix an orientation, i.e., an ordering of the vertices $(x, y)$ such that if $x$ is of type $j$, then $y$ is of type $j + 1 \text{ mod}(3)$. An edge equipped with this orientation will be called positively oriented. Likewise, for the chambers we fix a positive orientation by ordering the vertices by type.

**Lemma 1.2** The action of $G$ on the edges and the chambers preserves the positive orientation.

**Proof:** Let $g \in G$. It suffices to show that if $gL_0$ is of type $j$, then $gL_1$ is of type $j + 1 \text{ mod}(3)$. First note that the double quotient $G'\backslash G/K$ has three elements given by the class of 1, diag$(1, 1, \varpi)$ and diag$(1, \varpi, \varpi)$. Next note that the action of $K$ preserves the positive orientation on edges that contain the base point $L_0$. Thus it suffices to prove the claim for the three given elements which is easily done. \qed

## 2 The zeta function

Let $\Gamma \subset G$ be a discrete cocompact and torsion-free subgroup. Then $\Gamma$ acts without fixed points on $X$ and thus $\Gamma$ is the fundamental group of the quotient $\Gamma \backslash X$.

### 2.1 Definition

A geodesic $c$ in the quotient $\Gamma \backslash X$ is the image of a geodesic $\tilde{c}$ in $X$ under the projection map $X \rightarrow \Gamma \backslash X$. The geodesic $c$ is called rank-one if $\tilde{c}$ is. For a geodesic $c$ we denote by $c^{-1}$ the geodesic with the reversed orientation. When speaking about closed geodesics in $\Gamma \backslash X$ we adopt the convention that a closed geodesic comes with a multiplicity (going round more then once).
A closed geodesic with multiplicity one is called a *primitive* closed geodesic. To a given closed geodesic $c$ there is a unique primitive one $c_0$ such that $c$ is a power of $c_0$. For a closed geodesic $c$ in $\Gamma \backslash X$ let $l(c)$ denote its length. Here the length is normalized such that any edge gets the length 1. We define the zeta function

$$Z(u) = \prod_c \left(1 - u^{l(c)}\right)$$

as a formal power series at first, where $c$ ranges over the set of all primitive rank-one closed geodesics in $\Gamma \backslash X$ modulo homotopy and modulo change of orientation. It is easy to show that the Euler product defining $Z(u)$ actually converges for $u \in \mathbb{C}$ with $|u|$ small enough.

2.2 A comparison

An element $g$ of $G$ is called *neat* if for every rational representation $\rho: G \to \text{GL}_n(F)$ over $F$ the matrix $\rho(g)$ has the following property: the subgroup of $F^\times$ generated by all eigenvalues of $\rho(g)$ is torsion-free. Here $F$ is an algebraic closure of $F$. The element $g$ is called *weakly neat* if the adjoint $\text{Ad}(g) \in \text{GL}(\text{Lie}(G))$ has no non-trivial root of unity as eigenvalue. Obviously neat implies weakly neat. A subgroup $\Gamma \subset G$ is called neat/weakly neat if every $\gamma \in \Gamma$ is neat/weakly neat in $G$. Every arithmetic group $\Gamma$ has a subgroup of finite index which is neat [2].

An element $g$ of $G$ is called *regular* if its centralizer is a torus. A subgroup $\Gamma$ of $G$ is called regular if every $\gamma \in \Gamma$, $\gamma \neq 1$ is regular in $G$. A regular group is weakly neat.

In [5], the author defined for $\Gamma$ being discrete, cocompact, and weakly neat a zeta function $Z_P(u)$ attached to a parabolic subgroup $P \subset G$ of split rank one. It is shown that $Z_P(u)$ is a rational function and that its poles and zeros can be described in terms of certain cohomology groups.

**Proposition 2.1** Suppose the group $\Gamma$ is discrete, cocompact and regular. Then $Z_P(u) = Z(u)$ for every parabolic $P$ of split rank one.

**Proof:** We will recall the definition of $Z_P(u)$. Let $P = LN$ be a Levi decomposition of $P$ and $A \subset L$ be a maximal split torus. The dimension of $A$ is one. Let $A^+$ be the set of all $a \in A$ that act on the Lie algebra of $N$ by eigenvalues $\mu$ with $|\mu| > 1$. Fix an isomorphism $\varphi: A \cong F^\times$ that maps $A^+$ to the set of $x \in F^\times$ with $v(x) > 0$, where $v$ is the valuation of $F$. For $a \in A^+$
let $l(a) = v(\varphi(a))$. Let $M$ be the derived group of $M$ and let $M_{\text{ell}}$ be the set of all elliptic elements of $M$. Let $\mathcal{E}_P(\Gamma)$ denote the set of all conjugacy classes $[\gamma]$ in $\Gamma$ such that $\gamma$ is in $G$ conjugate to an element $a_\gamma m_\gamma \in A^+ M_{\text{ell}}$. An element $\gamma \in \Gamma$ is called primitive if $\gamma = \sigma^n m n \in \mathbb{N}$, $\sigma \in \Gamma$ implies $n = 1$. Let $\mathcal{E}_P^\Gamma(\Gamma)$ denote the set of primitive elements in $\mathcal{E}_P(\Gamma)$. The zeta function $Z_P$ is defined as

$$Z_P(u) = \prod_{[\gamma] \in \mathcal{E}_P^\Gamma(\Gamma)} (1 - u^{l(a_\gamma)}) \chi_1(\Gamma_\gamma),$$

where $\Gamma_\gamma$ is the centralizer of $\gamma$ in $\Gamma$ and

$$\chi_1(\Gamma_\gamma) = \sum_{p=0}^{\dim X} p(-1)^{p+1} \dim H^p(\Gamma_\gamma, \mathbb{Q}).$$

First we remark that since $\Gamma$ is regular, we have that $\Gamma_\gamma \cong \mathbb{Z}$ for every $\gamma \in \mathcal{E}_P(\Gamma)$ and thus the Euler numbers $\chi_1(\Gamma_\gamma)$ are all equal to 1. Next let $[\gamma] \in \mathcal{E}_P^\Gamma(\Gamma)$. The function $d_\gamma(x) = \text{dist}(\gamma x, x)$ on $X$ attains its minimum on a unique apartment of $X$. On this apartment, $\gamma$ acts by a translation along a rank-one geodesic $\tilde{c}$ by the amount $l(a_\gamma)$. So $\gamma$ closes this geodesic and its image $c$ in $\Gamma \backslash X$ has length $l(c) = l(a_\gamma)$. The other way round, every rank-one closed geodesic $c$ must be closed by one primitive element $\gamma$ of $\Gamma$. Then $\gamma$ either lies in $\mathcal{E}_P(\Gamma)$ or has splitrank two in which case the geodesic it closes cannot be rank-one. This shows that the Euler products defining $Z(u)$ and $Z_P(u)$ coincide. □

2.3 A factorization

Two rank-one geodesics in $X$ are called adjacent if they lie in the same apartment, they are parallel, and there is only one row of chambers between them. Recall a gallery \cite{3} in $X$ is a sequence $g = (C_0, \ldots, C_n)$ of chambers such that $C_j$ and $C_{j+1}$ are adjacent for every $j$. We say that a gallery $g$ is rank-one if $C_{j-1} \neq C_{j+1}$ for every $j$ and the gallery is located between two rank-one geodesics. The next picture shows an example of a rank-one gallery.
A rank-one gallery in $\Gamma \backslash X$ is the image of a rank-one gallery in $X$ under the projection map. In $\Gamma \backslash X$ it may happen for a rank-one gallery $g = (C_0, \ldots, C_n)$ that $C_0 = C_n$ in which case we say that $g$ is closed. In this case the number $n$ is even and we define the length of $g$ to be $l(g) = n/2$. We say that $g$ is primitive if furthermore $C_0 \neq C_j$ for $0 < j < n$. Two closed galleries $(C_0, \ldots, C_n)$ and $(E_0, \ldots, E_n)$ are equivalent if there is $k \in \mathbb{Z}$ with $C_j = E_{j+k}$, where the indices run modulo $n$. An equivalence class of closed rank-one galleries is called a loop of galleries.

Let $C_1$ denote the set of all primitive closed rank-one geodesics in $\Gamma \backslash X$ modulo reversal of orientation. Let $C_2$ denote the set of all primitive loops of galleries in $\Gamma \backslash X$ modulo reversal of orientation. Let

$$Z_j(u) \overset{\text{def}}{=} \prod_{c \in C_j} (1 - u^{l(c)})$$

for $j = 1, 2$.

**Proposition 2.2** For the zeta function $Z$ we have $Z(u) = Z_1(u) Z_2(u)$. Moreover, if $\Gamma$ is regular, then $Z_2(u) = 1$.

**Proof:** For any two topological spaces $X, Y$ let $[X, Y]$ be the set of homotopy classes of continuous maps from $X$ to $Y$. Let $S^1$ be the 1-sphere and consider the natural bijection

$$\Gamma/\text{conjugation} \rightarrow [S^1, \Gamma \backslash X]$$

given by the identification $\Gamma \cong \pi_1(\Gamma \backslash X)$. If two closed geodesics $c_1, c_2$ are homotopic, then they are closed by conjugate elements of $\Gamma$. So they have
preimages $\tilde{c}_1, \tilde{c}_2$ in $X$ which are closed by the same element $\gamma$. Hence $\tilde{c}_1$ and $\tilde{c}_2$ lie both in the apartment $\mathfrak{a}$ where $d_\gamma(x)$ is minimized. Since $\langle \gamma \rangle \backslash \mathfrak{a}$ is a cylinder, $c_1$ and $c_2$ are homotopic through closed geodesics of the same length passing through loops of galleries or intermediate rank-one geodesics.

On the other hand, each closed loop of galleries in $\Gamma \backslash X$ induces a homotopy between two closed geodesics of the same length: the two boundary components of the gallery. Thus we see that the overcounting in $Z_1(u)$ is remedied by dividing by $Z_2(u)$ to result in $Z(u)$.

For the second part assume there is a closed loop $l$. Let $(C_0, \ldots, C_n)$ be a gallery in $X$ being mapped to $l$. Then there is $\gamma \in \Gamma$ with $\gamma C_0 = C_n$ and $C_0 \subset M_\gamma$, where

$$M_\gamma = \{ x \in X : d_\gamma(x) = \min \}.$$ 

For any $\gamma$ the set $M_\gamma$ is either a geodesic line or an apartment. Since $C_0 \subset M_\gamma$ is our given case, it follows that $M_\gamma$ is an apartment attached to a maximal split torus $A$ which contains $\gamma$. But since $\gamma$ translates along a rank-one geodesic it must lie in a one-dimensional standard subtorus of $A$, which means it is not regular. A contradiction. The claim follows. □

3 The zeta function for the 1-skeleton

Let $E(X)$, resp. $E(\Gamma \backslash X)$ denote the set of positively oriented edges in $X$, resp. $\Gamma \backslash X$.

Consider the vector spaces

$$C_1(X) = \prod_{e \in E(X)} \mathbb{C}e, \quad C_1(\Gamma \backslash X) = \prod_{e \in E(\Gamma \backslash X)} \mathbb{C}e.$$ 

The second space is finite dimensional. This notion actually makes sense due to Lemma 1.2. Define a linear operator $T$ on $C_1(\Gamma \backslash X)$ by $Te = \sum_{e' : e \rightarrow e'} e'$, where the sum runs over all positively oriented edges $e'$ such that the endpoint of $e$ is the starting point of $e'$ and $e, e'$ lie on a rank-one geodesic, i.e., have angle $\pi$. By the same formula, we define an operator $\tilde{T}$ on $C_1(X)$. Note that $\Gamma$ acts on $C_1(X)$ and that $\tilde{T}$ is $\Gamma$-equivariant. One has a natural identification $C_1(\Gamma \backslash X) \cong C_1(X)^\Gamma$, and $T \cong \tilde{T}|_{C_1(X)^\Gamma}$.

Theorem 3.1 We have $Z_1(u) = \det(1 - uT)$. In particular, $Z_1(u)$ is a polynomial of degree equal to the number of edges of $\Gamma \backslash X$, or, equivalently,

$$\deg Z_1(u) = \frac{(q + 1)N}{2}.$$
where \( N \) is the number of vertices in \( \Gamma \setminus X \).

**Proof:** One computes

\[
\text{tr} \, T^n = \sum_e \langle Te, e \rangle = \sum_{c: l(c) = n} l(c_0),
\]

where the second sum runs over all closed geodesics of length \( n \) and \( c_0 \) is the underlying primitive of \( c \). In the next computation, we will use the letter \( c \) for an arbitrary closed geodesic, \( c_0 \) for a primitive one, and if both occur, it will be understood that \( c_0 \) is the primitive underlying \( c \). We compute

\[
Z_1(u) = \exp \left( - \sum_{c_0} \sum_{m=1}^{\infty} \frac{u^{l(c_0)m}}{m} \right)
\]

\[
= \exp \left( - \sum_c \frac{u^{l(c)}}{l(c)} l(c_0) \right)
\]

\[
= \exp \left( - \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{c: l(c) = n} l(c_0) \right)
\]

\[
= \exp \left( - \sum_{n=1}^{\infty} \frac{u^n}{n} \text{tr} \, T^n \right)
\]

\[
= \det(1 - uT).
\]

For the last line we used the fact that for a matrix \( A \) we have \( \exp(\text{tr} \, (A)) = \det(\exp(A)) \). To prove the final assertion of the Theorem it suffices to show that \( T \) is invertible on \( C_1(\Gamma \setminus X) \). For this in turn it suffices to show that \( \tilde{T} \) has a right-inverse on \( C_1(X) \). So let \( e \) be a positively oriented edge with endpoint \([L_0]\), and let \( e' \) be a positively oriented edge with start point \([L_0]\) such that \( e, e' \) lie on a geodesic. Let \([A_2]\) be the start point of \( e \) and \([A_1]\) the end point of \( e' \). The situation is this:

\[
[A_2] \xrightarrow{e} [L_0] \xrightarrow{e'} [A_1],
\]

where \([A_j]\) is of type \( j \) for \( j = 1, 2 \). We can choose representatives satisfying

\[
\varpi L_0 \subset A_1, A_2 \subset L_0.
\]
The condition on the types translates to the $\mathbb{F}_q$-vector space $\Lambda_j/\varpi L_0$ being of dimension $j$. The condition that $e, e'$ lie on a geodesic is equivalent to $\Lambda_1 \nsubseteq \Lambda_2$.

For $j = 1, 2$ let $W_j$ be the complex vector space formally spanned by the set of all $j$-dimensional sub vector spaces of $\mathbb{F}_q^3$. Let

$$T: W_2 \to W_1$$

be given by

$$T(\Lambda_2) = \sum_{\Lambda_1 \nsubseteq \Lambda_2} \Lambda_1.$$ 

Define $T': W_1 \to W_2$ by

$$T'(\Lambda_1) = \frac{-1}{q + 1} \sum_{\Lambda_2 \supset \Lambda_1} \Lambda_2 + \frac{1}{q^2 - q - 1} \sum_{\Lambda_2 \not\supset \Lambda_1} \Lambda_2.$$ 

Then

$$\tilde{T}T'(\Lambda_1) = \frac{-1}{q + 1} \sum_{\Lambda_2 \supset \Lambda_1} \sum_{\Lambda'_1 \nsubseteq \Lambda_2} \Lambda'_1 + \frac{1}{q^2 - q - 1} \sum_{\Lambda_2 \not\supset \Lambda_1} \sum_{\Lambda'_1 \nsubseteq \Lambda_2} \Lambda'_1$$

$$= \sum_{\Lambda'_1} c(\Lambda'_1) \Lambda'_1,$$

where

$$c(\Lambda'_1) = \frac{-\#\{\Lambda_2 \supset \Lambda_1, \Lambda_2 \not\supset \Lambda'_1\}}{q + 1} + \frac{\#\{\Lambda_2 \not\supset \Lambda_1, \Lambda_2 \not\supset \Lambda'_1\}}{q^2 - q - 1}$$

$$= \begin{cases} 1 & \text{if } \Lambda'_1 = \Lambda_1 \\ 0 & \text{if } \Lambda'_1 \neq \Lambda_1. \end{cases}$$

This calculation shows that the operator $T'$ on $C_1(X)$ given by

$$T'(e) = \frac{-1}{q + 1} \sum_{e' \rightarrow e \text{ non geodesic}} e' + \frac{1}{q^2 - q - 1} \sum_{e' \rightarrow e \text{ geodesic}} e'$$

is a right-inverse to $\tilde{T}$. The claim follows. \qed
3.1 A combinatorial computation

In the following, we will write $c$ for an arbitrary closed rank-one geodesic in $\Gamma \backslash X$ and $c_0$ for a primitive one. If $c$ and $c_0$ both occur, it will be understood that $c_0$ is the underlying primitive of $c$. We compute

\[
\frac{Z_1'(u)}{Z_1(u)} = (\log Z_1(u))' = - \sum_{c_0} \sum_{n=1}^{\infty} l(c_0) u^{l(c_0)n-1} = - \sum_{n=1}^{\infty} u^{n-1} \sum_{c:l(c)=n} l(c_0).
\]

Note that the sums run modulo reversal of orientation.

There is a natural orientation on each rank-one geodesic in $X$ given as follows. We say that a rank-one geodesic $C$ in $X$ is positively oriented if it runs through the vertices in the order of types: 0, 1, 2, 0, 1, 2, ... The image $c_\Gamma$ of $c$ in $\Gamma \backslash X$ is isomorphic to the image in $\langle \gamma \rangle \backslash X$, where $\gamma \in \Gamma$ is the element in $\Gamma$ that closes $c$. Since $\gamma c = c$ and $\gamma$ acts on $c$ by a translation it preserves the orientation of $c$ and so it does make sense to speak of positive or negative orientation for $c_\Gamma$.

A line segment in $X$ is a sequence of vertices $s = (x_0, \ldots, x_k)$ such that they are consecutive vertices on a rank-one geodesic. A line segment in $\Gamma \backslash X$ is the image of one in $X$. The length of a line segment $s = (x_0, \ldots, x_n)$ is $l(s) = n$. On the vector spaces

\[
C_0(X) \overset{\text{def}}{=} \bigoplus_{x \text{ vertex in } X} \mathbb{C} x, \quad C_0(\Gamma \backslash X) \overset{\text{def}}{=} \bigoplus_{x \text{ vertex in } \Gamma \backslash X} \mathbb{C} x,
\]

we define an operator $A_n$ for each $n \in \mathbb{N}$ by

\[
A_n x = \sum_{s:l(s)=n, o(s)=x} e(s),
\]

where the sum runs over all positively oriented line segments $s$ in $\Gamma \backslash X$ with starting point $x$ and length $n$, and $e(s)$ denote the endpoint of $s$.

**Lemma 3.2** The operator $A_n$ has the trace

\[
\text{tr } A_n = \sum_{c:l(c)=n} l(c_0),
\]
where the sum runs over all closed rank-one geodesics in \( \Gamma \setminus X \) modulo reversal of orientation.

**Proof:** Instead of summing modulo reversal of orientation one can as well sum over all positively oriented geodesics. Recall \( \text{tr} A_n = \sum_x \langle A_n x, x \rangle \), where the sum runs over all vertices of \( \Gamma \setminus X \) and the pairing \( \langle , \rangle \) is the one given by \( \langle x, y \rangle = \delta_{x,y} \) for vertices \( x, y \). A vertex \( x \) can only have a non-zero contribution \( \langle A_n x, x \rangle \) if it lies on a close geodesic of length \( n \). The contribution of each given geodesic \( c \) equals \( l(c_0) \).

\[ \square \]

### 3.2 The unramified Hecke algebra

Recall that \( G \) is a unimodular group, so any Haar-measure will be left- and right-invariant. We normalize the Haar measure so that the compact open subgroup \( K \) gets volume 1. For a subset \( A \) of \( G \) we write \( 1_A \) for its indicator function. Let \( \mathcal{H}_K \) denote the space of compactly supported functions \( f: G \to \mathbb{C} \) with \( f(k_1 x k_2) = f(x) \) for all \( k_1, k_2 \in K, x \in G \). This is an algebra under convolution,

\[ f * g(x) = \int_G f(y) g(y^{-1}x) \, dx. \]

It is known \[4\], that \( \mathcal{H}_K \) is a commutative algebra. It has a unit element given by \( 1_K \).

We will also write \( KgK \) for the function \( 1_{KgK} \in \mathcal{H}_K \). So a typical element of \( \mathcal{H}_K \) is written as

\[ f = \sum_j c_j Kg_j K, \quad \text{finite sum}, \]

and

\[ I(f) = \sum_j c_j \text{vol}(Kg_j K). \]

The space \( C_c(G/K) \) can be identified with \( C_0(X) \) since \( G/K \) can be identified with the set of vertices via \( gK \mapsto gL_0 \). Likewise, \( C_c(\Gamma \setminus G/K) \) identifies with \( C_0(\Gamma \setminus X) \). The Hecke algebra \( \mathcal{H}_K \) acts on \( C_c(G/K) \) and \( C_c(\Gamma \setminus G/K) \) via \( g \mapsto g * f, g \in C_c(G/K), f \in \mathcal{H}_K \). This will be considered as a left action as is possible since \( \mathcal{H}_K \) is commutative. In \( \mathcal{H}_K \) we consider the elements

\[ \pi_1 = K \text{diag}(1, 1, \varpi) K, \quad \pi_2 = K \text{diag}(1, \varpi, \varpi) K. \]
Lemma 3.3 For $j = 1, 2$,
\[ \pi_j L_0 = \sum_{x \text{ adjacent to } L_0} x. \]

Proof: Clear. \qed

Proposition 3.4 As operators on $C_0(X)$ or $C_0(\Gamma \setminus X)$ respectively,

(a) $A_1 = \pi_1$,
(b) $A_2 = \pi_1^2 - (q + 1)\pi_2$,
(c) $A_3 = \pi_1^3 - (2q + 1)\pi_1\pi_2 + (1 + q + q^2)q$,
(d) For $n \geq 3$,
\[ A_{n+1} = A_n\pi_1 - qA_{n-1}\pi_2 + q^3A_{n-2}. \]

Proof: Part (a) follows from Lemma 3.3. It is clear that $\pi_1^2 = A_2 + c\pi_2$ for some number $c$. From (b) in Lemma 3.1 it follows that $c = q + 1$ which implies part (b). The rest follows similarly. \qed

Let $F(u)$ be the following formal powers series with values in the space $\text{End}(C_0(\Gamma \setminus X))$,
\[ F(u) = \sum_{n=1}^{\infty} u^{n-1}A_n. \]

Then $\text{tr} F(u) = \frac{Z_1'}{Z_1}(u)$. The relations in Proposition 3.4 imply the following Lemma.

Lemma 3.5 We have
\[ F(u) = H(u) \left( 1 - u\pi_1 + u^2q\pi_2 - u^3q^3 \right)^{-1}, \]
where $H(u)$ is the polynomial
\[ H(u) = (\pi_2 - \pi_1^2) + u(\pi_1^3 - \pi_1\pi_2 + \pi_2^3 - (q + 1)\pi_2) + u^2(\pi_1^3 - (2q + 1)\pi_1\pi_2 + (1 + q + q^2)q). \]

Proof: This follows from Proposition 3.4 by a straightforward computation. \qed
Theorem 3.6 There is \( m \in \mathbb{N} \) and a polynomial \( Q(u) \) such that
\[
Z_1(u) = \frac{\det(1 - u\pi_1 + u^2 q\pi_2 - u^3 q^3)^m}{Q(u)}.
\]

Proof: We have \( \frac{Z_1'}{Z_1}(u) = \text{tr} F(u) \), so the poles of \( \frac{Z_1'}{Z_1}(u) \) must be singularities of \( F(u) \), which form a subset of the set of zeros of the polynomial \( \det(1 - u\pi_1 + u^2 q\pi_2 - u^3 q^3) \). This implies the claim. \( \square \)

4 The zeta function on galleries

We now will show that the zeta function on galleries, \( Z_2(u) \), also is a polynomial. Recall that every chamber \( C \) of \( X \) or \( \Gamma \setminus X \) has three vertices, one of each type 0,1,2. Accordingly, it has three edges of types (0,1), (1,2), and (2,0) respectively. So let
\[
C_2(X) = \prod_C \mathbb{C}C, \quad C_2(\Gamma \setminus X) = \prod_{C \mod \Gamma} \mathbb{C}C,
\]
where the product runs over all chambers of the buildings \( X \) and \( \Gamma \setminus X \).

On \( C_1(\Gamma \setminus X) \) we define a linear operator \( L_1 \) mapping a chamber \( C \) to the sum of all chambers \( C' \) such that the (1,2)-edge of \( C' \) is the direct geodesic prolongation of the (0,1)-edge of \( C \) as in the following picture.

Similarly, define \( L_2 \) and \( L_3 \) by replacing \( (0,1,2) \) by \( (1,2,0) \) and \( (2,0,1) \) respectively. Then let \( L \stackrel{\text{def}}{=} L_3L_2L_1 \).
Proposition 4.1 We have

\[ Z_2(u) = \det(1 - u^3L). \]

In particular, \( Z_2(u) \) is a polynomial of degree at most 3 times the number of chambers of \( \Gamma\backslash X \).

**Proof:** It is easy to see that

\[ \text{tr } L^n = \sum_{c: l(c)=3n} \frac{l(c_0)}{3}, \]

where the sum runs over all loops of galleries in \( \Gamma\backslash X \). From this, the proposition follows by the same computation as before. \( \square \)

Finally, Theorem 0.1 follows from Proposition 2.2, Theorem 3.6 and Proposition 4.1.

**References**

[1] Bass, H.: *The Ihara-Selberg zeta function of a tree lattice.* Int. J. Math. 3, No.6, 717-797 (1992).

[2] Borel, A.: *Introduction aux groupes arithmétiques.* Hermann, Paris 1969.

[3] Brown, K.: *Buildings.* Springer-Verlag, New York, 1989.

[4] Cartier, P.: *Representations of \( p \)-adic groups: A survey.* Automorphic forms, representations and L-functions, Proc. Symp. Pure Math. Am. Math. Soc., Corvallis/Oregon 1977, Proc. Symp. Pure Math. 33, 1, 111-155 (1979).

[5] Deitmar, A.: *Geometric zeta-functions on \( p \)-adic groups.* Math. Japon. 47, No. 1, 1-17 (1998).

[6] Hashimoto, K.: *Zeta functions of finite graphs and representations of \( p \)-adic groups.* Automorphic forms and geometry of arithmetic varieties. Adv. Stud. Pure Math. 15, 211-280 (1989).
[7] Hashimoto, K.: *On zeta and L-functions on finite graphs.* Int. J. Math. 1. no 4, 381-396 (1990)

[8] Hashimoto, K.: *Artin type L-functions and the density theorem for prime cycles on finite graphs.* Int. J. Math. 3, no 6, 809-826 (1992).

[9] Ihara, Y.: *Discrete subgroups of PL(2, k_v).* Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 272–278 Amer. Math. Soc., Providence, R.I. (1966).

[10] Stark, H. M.; Terras, A. A.: *Zeta functions of finite graphs and coverings.* Adv. Math. 121 (1996), no. 1, 124–165.

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