Sharp upper diameter bounds for compact shrinking Ricci solitons

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Abstract
We give a sharp upper diameter bound for a compact shrinking Ricci soliton in terms of its scalar curvature integral and the Perelman’s entropy functional. The sharp cases could occur at round spheres. The proof mainly relies on a sharp logarithmic Sobolev inequality of gradient shrinking Ricci solitons and a Vitali-type covering argument.

Keywords Shrinking Ricci soliton · Diameter · Logarithmic Sobolev inequality

Mathematics subject classification Primary 53C20 · Secondary 53C25

1 Introduction
A complete Riemannian metric $g$ on a smooth $n$-dimensional manifold $M$ is called a Ricci soliton if there exists a smooth vector field $V$ on $M$ such that the Ricci curvature $\text{Ric}$ of the metric $g$ satisfies

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = \lambda g$$

for some real constant $\lambda$, where $\mathcal{L}_V$ denotes the Lie derivative in the direction of $V$. A Ricci soliton is called shrinking, steady or expanding, if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. When $V = \nabla f$ for some smooth function $f$ on $M$, then the Ricci soliton becomes the gradient Ricci soliton

$$\text{Ric} + \text{Hess} f = \lambda g,$$

where $\text{Hess} f$ denotes the Hessian of $f$. The function $f$ is often called a potential function. Perelma [39] proved that every compact Ricci soliton is necessarily gradient. For $\lambda > 0$, scaling metric $g$, we can normalize $\lambda = \frac{1}{2}$ so that

$$\text{Ric} + \text{Hess} f = \frac{1}{2} g. \quad (1.1)$$
We set $\text{Ric}_f := \text{Ric} + \text{Hess} f$, which is customarily called the Bakry-Émery Ricci tensor [2]. $\text{Ric}_f$ is an important geometric quantity, which can be used to show that the Ricci flow is a gradient flow of the Perelman’s $\mathcal{F}$-functional [39]. In the whole paper, we let a triple $(M, g, f)$ denote an $n$-dimensional complete gradient shrinking Ricci soliton. As in [29], normalizing $f$ by adding a constant in (1.1), without loss of generality, we may assume (1.1) simultaneously satisfies

\[
R + |\nabla f|^2 = f \quad \text{and} \quad (4\pi)^{-\frac{n}{2}} \int_M e^{-f} dv = e^\mu,
\]

where $R$ is the scalar curvature of $(M, g)$ and $\mu = \mu(g, 1)$ is the entropy functional of Perelman [39]; see the explanation in Sect. 2. Note that $\mu$ is a finite constant for a fixed complete gradient shrinking Ricci soliton. From Lemma 2.5 in [29], we know that $e^\mu$ is almost equivalent to the volume of the geodesic ball $B(p_0, 1)$ with radius 1 and center $p_0$. That is,

\[
\frac{(4\pi)^{\frac{n}{2}}}{e^{2\mu + \gamma}} \leq V(p_0, 1) e^\mu \leq (4\pi)^{\frac{n}{2}} e^\mu,
\]

where $V(p_0, 1)$ denotes the volume of $B(p_0, 1)$. Here, $p_0 \in M$ is a point where $f$ attains its infimum, which always exists on the complete (compact or not-compact) gradient shrinking Ricci soliton $(M, g, f)$; see [26]. By the Chen’s argument [15], we know that $R \geq 0$. By the Pigola-Rimoldi-Setti work [40], we further know that $R > 0$ unless $(M, g, f)$ is the Euclidean Gaussian shrinking Ricci soliton $(\mathbb{R}^n, g_E, \frac{1}{4}r^2)$.

Gradient shrinking Ricci solitons can be regarded as a natural extension of Einstein manifolds. They play an important role in the Ricci flow as they correspond to some self-similar solutions and often rise as singularity models of the Ricci flow [25]. They are also viewed as critical points of the Perelman’s entropy functional [39]. At present, one of most important project is the classification of complete gradient shrinking Ricci solitons. For dimension 2, the classification is complete [24]. In particular, Hamilton proved that every 2-dimensional compact shrinking Ricci soliton must be Einstein. For dimension 3, Ivey [27] proved that any compact shrinking Ricci solitons are still Einstein; the non-compact case is a little complicated and has been completely classified by the work of [8, 37, 39]. However, for the higher dimensions, even $n = 4$, the classification remains open, though much progress has been made; see, e.g., [9, 13, 16, 20, 34–38, 49, 52].

On the other hand, Ivey [27] confirmed that any compact gradient steady or expanding Ricci solitons are Einstein. Therefore, the shrinking cases are the only possible non-Einstein compact Ricci solitons. In fact for $n = 4$, Cao [6], Koiso [28], Wang and Zhu [45] successfully constructed non-Einstein examples of compact Kähler shrinking Ricci solitons. At present, all of known compact shrinking Ricci solitons are Kähler. It remains an interesting question whether there exists a non-Kähler compact shrinking Ricci soliton. Derdziński [17] proved that every compact shrinking Ricci soliton has a finite fundamental group (the non-compact case due to Wylie [47]). We refer to further related work in [5, 7, 11, 14] and references therein.

In this paper, we will study the diameter estimate for a compact (without boundary) gradient shrinking Ricci soliton. We will give a sharp upper diameter bound in terms of the $L^{\frac{4}{n-2}}$-norm of the scalar curvature and the Perelman’s entropy functional. On a compact shrinking Ricci soliton $(M, g, f)$, the diameter of $M$ is defined by

\[
diam(M) := \max \{ \text{dist}(p, q) | \forall p, q \in M \},
\]
where \( \text{dist}(p, q) \) denotes the geodesic distance between points \( p \) and \( q \). Recently, there has been lots of effort to estimate the diameter of gradient shrinking Ricci solitons. In [23], Futaki and Sano got a lower diameter bound for non-Einstein compact shrinking Ricci solitons, which was then sharpened by Andrews and Ni [1], and Futaki, Li and Li [22]. In [19], Fernández-López and García-Río studied some properties of geodesics on Ricci solitons and obtained many lower diameter bounds for compact gradient solitons in terms of extremal values of the potential function, the scalar curvature and the Ricci curvature on unit tangent vectors. Motivated by the classical Myers’ theorem, Fernández-López and García-Río [18] proved a Myers’ type theorem on Riemannian manifolds when \( \text{Ric}_f \) is bounded below by a positive constant and \( |\nabla f| \) is bounded. Later, Limoncu [31] and Tadano [42], respectively, gave an explicit upper diameter bound for such manifolds, which was sharpened by the author [48]. When the bound of \( |\nabla f| \) is replaced by the bound of \( f \), many upper diameter bounds were studied by Wei and Wylie [46], Limoncu [32] and Tadano [41], etc. For more related results, the interested reader can refer to [4, 21, 33] and the references therein.

In another direction, Bakry and Ledoux [3] applied a sharp Sobolev inequality of manifolds to give an alternative proof of the Myers’ diameter estimate, which indicates that some functional inequalities of manifolds may suffice to produce an upper diameter bound of manifolds. Similar idea also appeared in the other literatures. For example, Topping [44] applied the Michael-Simon Sobolev inequality to obtain an upper diameter bound for a closed connected manifold immersed in \( \mathbb{R}^n \) in terms of its mean curvature integral. Topping’s result was later generalized by Zheng and the author [50] to a general ambient space.

The above method is also suitable to the Ricci flow setting. In [43], Topping applied the Perelman’s \( \mathcal{W} \)-functional to get an upper diameter bound for a compact manifold evolving under the Ricci flow. Here, the upper bound depends on the scalar curvature integral under the evolving metric and some geometric quantities with the initial metric. Inspired by Topping’s argument, Zhang [51] applied the uniform Sobolev inequality along the Ricci flow to obtain an upper diameter bound in terms of the scalar curvature integral, volume and Sobolev constants (or positive Yamabe constants) under the Ricci flow. Meanwhile, he proved a sharp lower bound for the diameters, which depends on the initial metric, time and the scalar curvature integral. We would like to mention that Zhang’s argument is also suitable to stationary manifolds.

Inspired by the work of Topping [43] and Zhang [51], in this paper we are able to prove a sharp upper diameter bound for a compact shrinking Ricci soliton without any assumption. Our result gives an explicit coefficient of the diameter estimate in terms of the scalar curvature integral and the Perelman’s entropy functional.

**Theorem 1.1** Let \((M, g, f)\) be an \( n \)-dimensional \((n \geq 3)\) compact gradient shrinking Ricci soliton satisfying (1.1) and (1.2). Then, there exists a constant \( c(n, \mu) \) depending on \( n \) and \( \mu \) such that

\[
\text{diam}(M) \leq c(n, \mu) \int_M \frac{R^{n-1}}{\nu} \, dv,
\]

where \( R \) is the scalar curvature of \((M, g, f)\) and \( \mu = \mu(g, 1) \) is the Perelman’s entropy functional. In particular, we can take

\[
c(n, \mu) = 4 \max \left\{ w_n^{-1}, (4\pi)^{-\frac{n}{2}} e^{2n-17-\mu-n} \right\},
\]
where $w_n$ is the volume of the unit $n$-dimensional ball in $\mathbb{R}^n$.

**Remark 1.2** The theorem is also suitable to positive Einstein manifolds. The exponent $\frac{n-1}{2}$ of the scalar curvature is sharp. Indeed, we consider the round $n$-sphere $S^n(r)$ of radius $r$ with the canonical metrics $g_0$ and let $f = \text{constant}$. Then, $(S^n(r), g_0, f)$ is a trivial compact gradient shrinking Ricci soliton. Its diameter is almost equivalent to $r$, i.e., $\text{diam}_{g_0}(M) \approx r$; while the scalar curvature $R(g_0) \approx r^{-2}$. If we scale metric $g_0$ to be $g \approx r^{-2} g_0$ such that $\text{Ric}(g) = \frac{1}{2} g$, then by (1.3),

$$e^{\mu(g, 1)} \approx V_g(p, 1) = \frac{V_{g_0}(p, r)}{r^n} = c(n),$$

where $V_g(p, 1)$ denotes the volume of ball $B(p, 1)$ with respect to metric $g$. This indicates that coefficient $c(n, \mu)$ only depends on $n$ and the right-hand side of the diameter estimate in the theorem can be easily computed to be $c(n)r$.

**Remark 1.3** We omit the discussion about the optimal choice of $c(n, \mu)$. One might get a sharper constant $c(n, \mu)$ by choosing a better cut-off function in Sect. 3.

We would like to point out that previous diameter estimates for gradient shrinking Ricci solitons mainly relies on pointwise conditions of geometric quantities; see, e.g., [31, 32, 41, 42, 46, 48]. Our estimate is valid in the integral sense and it seems to be weaker than before. In [35], Munteanu and Wang proved an upper diameter bound for a compact shrinking Ricci soliton in terms of its injectivity radius. Our estimate depends on the scalar curvature integral and the Perelman’s entropy functional, and it may be a more feasible dependence on geometric quantities.

The trick of proving Theorem 1.1 stems from [43], but we need to carefully examine the explicit coefficient of the diameter bound in terms of the scalar curvature integral. Our argument is divided into three steps. First, we apply a sharp logarithmic Sobolev inequality and a proper cut-off function to get a new functional inequality, which is related to the maximal function of scalar curvature and the volume ratio (see Theorem 2.4). We mention that the sharp logarithmic Sobolev inequality is a key inequality in our paper, which was proved by Li, Li and Wang [29] for compact Ricci solitons and then extended by Li and Wang [30] to the non-compact case. Second, we use the functional inequality to give an alternative theorem, which states that the maximal function of scalar curvature and the volume ratio cannot be simultaneously smaller than a fixed constant on a geodesic ball of shrinking Ricci soliton (see Theorem 3.1). Third, we apply the alternative theorem and a Vitali-type covering lemma to give the diameter estimate.

The structure of this paper is as follows. In Sect. 2, we recall some basic results about gradient shrinking Ricci solitons. In particular, we rewrite the Li-Wang’s logarithmic Sobolev inequality [30] as a functional inequality by choosing a proper cut-off function. In Sect. 3, we use the functional inequality to give an alternative theorem. In Sect. 4, we apply the alternative theorem to prove Theorem 1.1.
2 Background

In this section, we recall some basic results about gradient shrinking Ricci solitons and give an explanation why (1.2) can be suitable to (1.1). We also rewrite the Li-Wang’s logarithmic Sobolev inequality [30] to a new functional inequality relating the maximal function of scalar curvature and the volume ratio. For more properties about Ricci solitons, the interested reader refer to the survey [7].

In this paper, we concentrate on compact shrinking Ricci solitons; however the following results are also suitable to the non-compact case. By Hamilton [25], (1.1) gives that

\[ R + \Delta f = \frac{n}{2} - 2\text{Ric}(\nabla f) = \nabla R \]

and

\[ \nabla (R + |\nabla f|^2 - f) = 0. \]

Adding \( f \) by a constant if necessary, we have that

\[ R + |\nabla f|^2 = f. \] (2.1)

Combining the above equalities gives

\[ 2\Delta f - |\nabla f|^2 + R + f - n = 0. \] (2.2)

By Cao-Zhou [10] and Haslhofer-Müller [26], we have a precise asymptotic estimate of \( f \).

Lemma 2.1 Let \((M, g, f)\) be an \( n \)-dimensional complete non-compact gradient shrinking Ricci soliton satisfying (1.1) and (2.1). Then, there exists a point \( p_0 \in M \) where \( f \) attains its infimum (may be not unique). Moreover, \( f \) satisfies

\[ \frac{1}{4} \left[ (r(x, p_0) - 5n)_+ \right]^2 \leq f(x) \leq \frac{1}{4} \left( r(x, p_0) + \sqrt{2n} \right)^2, \]

where \( r(x, p_0) \) is a distance function from \( p_0 \) to \( x \), and \( a_+ = \max\{a, 0\} \) for \( a \in \mathbb{R} \).

Remark 2.2 In view of the flat Euclidean space \((\mathbb{R}^n, \delta_i^j)\) with \( f = |x|^2/4 \), the above leading term \( \frac{1}{4} r^2(x, p_0) \) is optimal.

For an \( n \)-dimensional complete Riemannian manifold \((M, g)\), the definition of the Perelman’s \( W \)-entropy functional [39] is

\[ W(g, \varphi, \tau) := \int_M \left[ \tau (|\nabla \varphi|^2 + R) + \varphi - n \right] (4\pi \tau)^{-n/2} e^{-\varphi} dv \]

for some \( \varphi \in C^\infty(M) \) and \( \tau > 0 \), provided this functional is finite. The Perelman’s \( \mu \)-entropy functional [39] is defined by

\[ \mu(g, \tau) := \inf \left\{ W(g, \varphi, \tau) \mid \varphi \in C^\infty(M) \text{ with } \int_M (4\pi \tau)^{-n/2} e^{-\varphi} dv = 1 \right\}. \]

In general, the minimizer of \( \mu(g, \tau) \) may not exist on non-compact manifolds. However, by Lemma 2.1, the above definitions are both well-defined on non-compact gradient shrinking...
Ricci solitons and many integrations by parts still hold; see the explanation in [26]. Moreover, Carrillo and Ni [12] proved that potential function $f$ is always a minimizer of $\mu(g, 1)$, up to adding $f$ by a constant. That is, for a constant $c$ with

$$\int_M (4\pi)^{-n/2} e^{-(f+c)} dv = 1,$$

we have

$$\mu(g, 1) = \mathcal{W}(g, f + c, 1) = \int_M \left( |\nabla f|^2 + R + (f + c) - n \right) (4\pi)^{-n/2} e^{-(f+c)} dv$$

$$= \int_M \left( 2\Delta f - |\nabla f|^2 + R + (f + c) - n \right) (4\pi)^{-n/2} e^{-(f+c)} dv$$

$$= c.$$

Here, we used the integration by parts in the above third line, because $f$ is uniformly equivalent to the distance function squared and it guarantees the integration by parts on non-compact manifolds; see [26]. We also used (2.2) in the above last line. Therefore, we can assume that (1.1) satisfies (1.2) in the introduction.

Carrillo and Ni [12] proved that $\mu(g, 1)$ is the optimal logarithmic Sobolev constant on complete shrinking Ricci soliton $(M, g, f)$ for scale one. Later, Li, Li and Wang [29] showed that $\mu(g, 1)$ is in fact the optimal logarithmic Sobolev constant on compact shrinking Ricci soliton $(M, g, f)$ for all scales and $\mu(g, \tau)$ is a continuous function on $(0, \infty)$. Shortly after, the same conclusion for the non-compact case was confirmed by Li and Wang [30]. In summary, we have the following sharp logarithmic Sobolev inequality on complete gradient shrinking Ricci solitons for all scales without any curvature assumption.

**Lemma 2.3** Let $(M, g, f)$ be an $n$-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2). For any compactly supported locally Lipschitz function $\varphi$ in $M$ with

$$\int_M \varphi^2 dv = 1$$

and any real number $\tau > 0$

$$\mu + n + \frac{n}{2} \ln(4\pi) \leq \tau \int_M (4|\nabla \varphi|^2 + R \varphi^2) dv - \int_M \varphi^2 \ln \varphi^2 dv - \frac{n}{2} \ln \tau,$$  \hspace{1cm} (2.3)

where $R$ is the scalar curvature of $(M, g, f)$ and $\mu = \mu(g, 1)$ is the Perelman’s entropy functional.

Lemma 2.3 implies a functional inequality, which is closed linked with the maximal function of scalar curvature and the volume ratio.

**Theorem 2.4** Let $(M, g, f)$ be an $n$-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2). For any point $p \in M$ and for any $r > 0$,
where $R$ is the scalar curvature of $(M, g, f)$ and $\mu = \mu(g, 1)$ is the Perelman’s entropy functional.

**Proof of Theorem 2.4** Let $\psi : [0, \infty) \to [0, 1]$ be a smooth cut-off function, which is supported in $[0, 1]$ satisfying $\psi(t) = 1$ on $[0, 1/2]$ and $|\psi' | \leq 2$ on $[0, \infty)$. For any point $p \in M$, we also let

$$
\phi(x) := e^{-\frac{c}{r}} \psi \left( \frac{d(p, x)}{r} \right),
$$

where $c$ is some constant determined by the constraint condition $\int_M \phi^2 dv = 1$. Obviously, constant $c$ satisfies

$$
V \left( p, \frac{r}{2} \right) \leq e^c \int_M \phi^2 dv = e^c
$$

and

$$
e^c = e^c \int_M \phi^2 dv = \int_M \psi^2 (d(p, x)/r) dv \leq V(p, r).
$$

That is, $c$ satisfies

$$
V \left( p, \frac{r}{2} \right) \leq e^c \leq V(p, r).
$$

In the following, we will apply the above cut-off function to simplify the sharp logarithmic Sobolev inequality in Lemma 2.3. Notice that $\phi$ satisfies

$$
|\nabla \phi| \leq \frac{2}{r} \cdot e^{-\frac{c}{r}}
$$

and it is supported in $B(p, r)$. For the first term of the right-hand side of (2.3), we estimate that

$$
4\tau \int_M |\nabla \phi|^2 dv = 4\tau \int_{B(p, r) \setminus B(p, \frac{r}{2})} |\nabla \phi|^2 dv
\leq 4\tau V(p, r) \frac{4}{r^2} e^{-c}
\leq 16\tau \frac{V(p, r)}{V \left( p, \frac{r}{2} \right)}.
$$

For the second term of the right-hand side of (2.3), we have
Then, we estimate the third term of the right-hand side of (2.3). Notice that continuous function \( H(t) := -t \ln t \) is concave with respect to \( t > 0 \) and the Riemannian measure \( dv \) is supported in \( B(p, r) \). Using the Jensen’s inequality and the definition of \( H \), we have that

\[
\frac{\int H(\varphi^2)dv}{\int dv} \leq H\left( \frac{\int \varphi^2 dv}{\int dv} \right)
\]

and the definition of \( H \), we have that

\[
-\frac{\int_{B(p,r)} \varphi^2 \ln \varphi^2 dv}{\int_{B(p,r)} dv} \leq -\frac{\int_{B(p,r)} \varphi^2 dv}{\int_{B(p,r)} dv} \ln \left( \frac{\int_{B(p,r)} \varphi^2 dv}{\int_{B(p,r)} dv} \right).
\]

Since \( \int_{B(p,r)} \varphi^2 dv = 1 \), the above estimate becomes

\[
-\int_{B(p,r)} \varphi^2 \ln \varphi^2 dv \leq \ln V(p,r).
\]

By the definition of \( \varphi(x) \), we therefore get

\[
-\int_{M} \varphi^2 \ln \varphi^2 dv = -\int_{B(p,r)} \varphi^2 \ln \varphi^2 dv \leq \ln V(p,r).
\]

(2.7)

Substituting (2.5), (2.6) and (2.7) into (2.3) gives

\[
\mu + n + \frac{n}{2} \ln(4\pi) \leq \frac{16\tau}{r^2} \cdot \frac{V(p,r)}{V(p,\frac{r}{2})} + \frac{\tau}{V(p,\frac{r}{2})} \int_{B(p,r)} Rdv + \ln \frac{V(p,r)}{\tau^\frac{n}{2}}
\]

for any \( \tau > 0 \). The conclusion follows by letting \( \tau = r^2 \).

\[
\square
\]

3 Maximal function and volume ratio

In this section, we will apply Theorem 2.4 to obtain an alternative theorem about the lower bound for the maximal function of scalar curvature and the volume ratio in the gradient shrinking Ricci soliton.

Following Topping’s argument, given a Riemannian manifold \((M, g)\), for any point \( p \in M \) and \( r > 0 \), we introduce the maximal function

\[
Mh(p, r) := \sup_{s \in (0, r]} s^{-1} \left[ V(p,s) \right]^{-\frac{n-1}{2}} \left( \int_{B(p,s)} |h|dv \right)^{\frac{n-1}{2}}
\]

for any smooth function \( h \) on \((M, g)\), and the volume ratio 

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\]

\[
\tau \int_{M} R\varphi^2 dv \leq \tau e^{-c} \int_{B(p,r)} Rdv
\]

\[
\leq \frac{\tau}{V(p,\frac{r}{2})} \int_{B(p,r)} Rdv.
\]

(2.6)
Now, we give an alternative theorem. It says that the maximal function of scalar curvature and the volume ratio in gradient shrinking Ricci solitons cannot be simultaneously smaller than a fixed constant.

**Theorem 3.1** Let \((M, g, f)\) be an \(n\)-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2). Then, there exits a constant \(\delta > 0\) depending only on \(n\) and \(\mu\) such that for any point \(p \in M\) and for any \(r > 0\), at least one of the following is true:

1. \(\text{MR}(p, r) > \delta\);
2. \(\kappa(p, r) > \delta\).

Here, \(\text{R}(p, r)\) denotes the scalar curvature in the geodesic ball \(B(p, r)\). In particular, we can take

\[
\delta = \min \left\{ w_n, (4\pi)^{\frac{n}{2}} e^{\mu+n-2n+17} \right\},
\]

where \(\mu = \mu(g, 1)\) is the Perelman’s entropy functional and \(w_n\) is the volume of the unit \(n\)-dimensional ball in \(\mathbb{R}^n\).

**Proof of Theorem 3.1** Suppose that there exist a point \(p \in (M, g, f)\) and \(r > 0\) such that \(\text{MR}(p, r) \leq \delta\) for some constant \(\delta > 0\). For any \(0 < \varepsilon < 1\), constant \(\delta\) is defined by

\[
\delta := \min \left\{ (1 - \varepsilon)w_n, (4\pi)^{\frac{n}{2}} e^{\mu+n-2n+17} \right\},
\]

where \(w_n\) is the volume of the unit \(n\)-dimensional ball in \(\mathbb{R}^n\). In the following we will show that \(\kappa(p, r) > \delta\). If this conclusion is not true, then we make the following

**Claim** If there exist a point \(p \in M\) and \(r > 0\) such that \(\text{MR}(p, r) \leq \delta\) for some constant \(\delta > 0\), then for any \(s \in (0, r]\), \(\kappa(p, s) \leq \delta\) implies \(\kappa(p, s/2) \leq \delta\).

This claim will be proved later. We now continue to prove Theorem 3.1. We can use the claim repeatedly and finally get that for any \(m \in \mathbb{N}\),

\[
\kappa(p, \frac{r}{2^m}) \leq \delta \leq (1 - \varepsilon)w_n,
\]

where \(\varepsilon\) is the sufficiently small positive constant. But if we let \(m \to \infty\), then

\[
\kappa(p, \frac{r}{2^m}) \to w_n,
\]

which contradicts the preceding inequality. So \(\kappa(p, r) > \delta\) and the theorem follows. The desired constant \(\delta\) is obtained by letting \(\varepsilon \to 0+\).

\(\square\)

In the rest, we only need to check the above claim.
Proof of Claim  We prove the claim by two cases according to the relative sizes of $V(p, s/2)$ and $V(p, s)$.

Case one. Suppose that

$$
V\left(p, \frac{s}{2}\right) \leq \delta \frac{2^n}{s^{n-1}} 2^{-n} s^{2n} \left[ V(p, s) \right]^{\frac{n-1}{n-1}}.
$$

Then,

$$
\kappa\left(p, \frac{s}{2}\right) := \frac{2^n}{s^{n}} V\left(p, \frac{s}{2}\right) \\
\leq \delta \frac{2^n}{s^{n-1}} 2^{-n} s^{2n} \left[ V(p, s) \right]^{\frac{n-1}{n-1}} \\
= \delta \frac{2^n}{s^{n-1}} \kappa(p, s) \\
\leq \delta \frac{2^n}{s^{n-1}} \delta \frac{2^n}{s^{n-1}} \\
= \delta,
$$

which gives the claim.

Case two. Suppose that

$$
V\left(p, \frac{s}{2}\right) > \delta \frac{2^n}{s^{n-1}} 2^{-n} s^{2n} \left[ V(p, s) \right]^{\frac{n-1}{n-1}}.
$$

Since $MR(p, r) \leq \delta$, by the definition of $MR(p, r)$ and the scalar curvature $R > 0$, we get

$$
\int_{B(p, s)} R dv \leq \delta \frac{2^n}{s^{n-1}} 2^{-n} s^{2n} \left[ V(p, s) \right]^{\frac{n-1}{n-1}}
$$

for all $s \in (0, r]$. Using the assumption of Case Two, we further get

$$
\int_{B(p, s)} R dv \leq 2^n s^{-2} V\left(p, \frac{s}{2}\right)
$$

for all $s \in (0, r]$. Substituting this into (2.4) and using $\kappa(p, s) \leq \delta$, we have

$$
\mu + n + \frac{n}{2} \ln(4\pi) \leq 16 \frac{V(p, s)}{V\left(p, \frac{s}{2}\right)} + \frac{s^2}{V\left(p, \frac{s}{2}\right)} \int_{B(p, s)} R dv + \ln \kappa(p, s) \\
\leq 16 \frac{V(p, s)}{V\left(p, \frac{s}{2}\right)} + 2^n + \ln \delta
$$

for all $s \in (0, r]$. By the definition of $\delta$, we notice that

$$
\ln \delta \leq \mu + n + \frac{n}{2} \ln(4\pi) - 2^n \cdot 17.
$$

Substituting this into the above inequality yields

$$
\frac{V(p, s)}{V\left(p, \frac{s}{2}\right)} \geq 2^n
$$

for all $s \in (0, r]$. Therefore,
for any $s \in (0, r]$. This completes the proof of the claim. \qed

4 Diameter control

In this section, we will apply Theorem 3.1 to finish the proof of Theorem 1.1. The proof uses Topping’s argument in [43]; however more delicate analysis is required to get accurate coefficient dependence on the dimension of manifold and the Perelman’s entropy functional.

**Proof of Theorem 1.1** We choose $r_0 > 0$ sufficiently large so that the total volume of the compact shrinking soliton is less than $\delta r_0^n$. This choice can be achieved, because the soliton is compact. Here, $\delta$ is defined as in Theorem 3.1. Hence for any point $p \in M$, we have

$$\kappa(p, r_0) = \frac{V(p, r_0)}{r_0^n} \leq \frac{V(M)}{r_0^n} \leq \delta,$$

where $V(M)$ denotes the volume of $M$. By Theorem 3.1, we conclude that $MR(p, r_0) > \delta$. By the definition of $MR(p, r_0)$, there exists $s = s(p) > 0$ such that

$$\delta < s^{-1} \left[ V(p, s) \right]^{\frac{n-3}{2}} \left( \int_{B(p, s)} R dv \right)^{\frac{n+1}{n-1}}.$$

(4.1)

By the Hölder inequality

$$\int_{B(p, s)} R dv \leq \left( \int_{B(p, s)} R^{\frac{n+1}{2}} dv \right)^{\frac{2}{n+1}} \cdot \left( \int_{B(p, s)} dv \right)^{\frac{n-1}{n+1}},$$

estimate (4.1) can be reduced to

$$\delta < s^{-1} \int_{B(p, s)} R^{\frac{n+1}{2}} dv.$$

Therefore,

$$s(p) < \delta^{-1} \int_{B(p, s(p))} R^{\frac{n+1}{2}} dv.$$  \hspace{1cm} (4.2)

Now, we pick appropriate points $p$ at which to apply the inequality (4.2). Since $M$ is compact, we can choose $p_1, p_2 \in M$ are two extremal points in $M$ such that $\text{diam}(M) = \text{dist}(p_1, p_2)$. Let $\Sigma$ be a shortest geodesic connecting $p_1$ and $p_2$. Obviously, $\Sigma$ is
covered by the geodesic balls \{B(p, s(p)) \mid p \in \Sigma\}. By a modification of the Vitali-type covering lemma (see Lemma 5.2 in [43], or [50]), there exists a countable (possibly finite) set of points \{p_i \in \Sigma\} such that the geodesic balls \{B(p_i, s(p_i))\} are disjoint, and cover at least a fraction \(\rho\), where \(\rho \in (0, \frac{1}{2})\) of \(\Sigma\):

\[ \rho \text{diam}(M) \leq \sum_i 2s(p_i). \]

Substituting (4.2) into the above inequality,

\[ \text{diam}(M) \leq \frac{2}{\rho} \sum_i s(p_i) \]

\[ \leq \frac{2}{\rho} \delta^{-1} \int_{B(p, s(p))} R^\frac{n-1}{2} \, dv \]

\[ \leq \frac{2}{\rho} \delta^{-1} \int_M R^\frac{n-1}{2} \, dv, \]

where \(\delta > 0\) is a constant, depending on \(n\) and \(\mu\). Letting \(\rho \nearrow \frac{1}{2}\),

\[ \text{diam}(M) \leq 4\delta^{-1} \int_M R^\frac{n-1}{2} \, dv, \]

where \(\delta\) is defined in Theorem 3.1. This proves the desired estimate. \(\square\)

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