Weighted Well-Covered Claw-Free Graphs

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Abstract
A graph $G$ is well-covered if all its maximal independent sets are of the same cardinality. Assume that a weight function $w$ is defined on its vertices. Then $G$ is $w$-well-covered if all maximal independent sets are of the same weight. For every graph $G$, the set of weight functions $w$ such that $G$ is $w$-well-covered is a vector space. Given an input claw-free graph $G$, we present an $O(n^6)$ algorithm, whose input is a claw-free graph $G$, and output is the vector space of weight functions $w$, for which $G$ is $w$-well-covered.

A graph $G$ is equimatchable if all its maximal matchings are of the same cardinality. Assume that a weight function $w$ is defined on the edges of $G$. Then $G$ is $w$-equimatchable if all its maximal matchings are of the same weight. For every graph $G$, the set of weight functions $w$ such that $G$ is $w$-equimatchable is a vector space. We present an $O(m \cdot n^4 + n^5 \log n)$ algorithm which receives an input graph $G$, and outputs the vector space of weight functions $w$ such that $G$ is $w$-equimatchable.

1 Introduction

1.1 Basic Definitions and Notation

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

Cycles of $k$ vertices are denoted by $C_k$, and paths of $k$ vertices are denoted by $P_k$. When we say that $G$ contains a $C_k$ or a $P_k$ for some $k \geq 3$, we mean that $G$ admits a subgraph isomorphic to $C_k$ or to $P_k$, respectively. It is important to mention that these subgraphs are not necessarily induced.

Let $u$ and $v$ be two vertices in $G$. The distance between $u$ and $v$, denoted $d(u, v)$, is the length of a shortest path between $u$ and $v$, where the length of a path is the number of its edges. If $S$ is a non-empty set of vertices, then the distance between $u$ and $S$, denoted $d(u, S)$, is defined by

$$d(u, S) = \min\{d(u, s) : s \in S\}.$$
For every positive integer \( i \), denote 
\[
N_i(S) = \{ x \in V : d(x, S) = i \}, 
\]
and 
\[
N_i[S] = \{ x \in V : d(x, S) \leq i \}. 
\]

We abbreviate \( N_1(S) \) and \( N_1[S] \) to be \( N(S) \) and \( N[S] \), respectively. If \( S \) contains a single vertex, \( v \), then we abbreviate 
\[
N_i(\{v\}), N_i[\{v\}], N(\{v\}), \text{ and } N[\{v\}] 
\]
to be 
\[
N_i(v), N_i[v], N(v), \text{ and } N[v], 
\]
respectively. We denote by \( G[S] \) the subgraph of \( G \) induced by \( S \). For every two sets, \( S \) and \( T \), of vertices of \( G \), we say that \( S \) dominates \( T \) if \( T \subseteq N[S] \).

### 1.2 Well-Covered Graphs

Let \( G = (V, E) \) be a graph. A set of vertices \( S \) is independent if its elements are pairwise nonadjacent. An independent set of vertices is maximal if it is not a subset of another independent set. An independent set of vertices is maximum if the graph does not contain an independent set of a higher cardinality.

The graph \( G = (V, E) \) is well-covered if every maximal independent set is maximum. Assume that a weight function \( w : V \rightarrow \mathbb{R} \) is defined on the vertices of \( G \). For every set \( S \subseteq V \), define 
\[
w(S) = \sum_{s \in S} w(s). 
\]

Then \( G \) is \( w \)-well-covered if all maximal independent sets of \( G \) are of the same weight.

The problem of finding a maximum independent set in an input graph is \( \text{NP} \)-complete. However, if the input is restricted to well-covered graphs, then a maximum independent set can be found polynomially using the greedy algorithm. Similarly, if a weight function \( w : V \rightarrow \mathbb{R} \) is defined on the vertices of \( G \), and \( G \) is \( w \)-well-covered, then finding a maximum weight independent set is a polynomial problem.

The recognition of well-covered graphs is known to be \( \text{co-NP} \)-complete. This was proved independently in [4] and [20]. In [3] it is proven that the problem remains \( \text{co-NP} \)-complete even when the input is restricted to \( K_{1,4} \)-free graphs. However, the problem is polynomially solvable for \( K_{1,3} \)-free graphs [21] [22], for graphs with girth at least 5 [5], for graphs with a bounded maximal degree [2], for chordal graphs [18], for bipartite graphs [7] [17] [19], and for graphs without cycles of length 4 and 5 [9]. It should be emphasized that the forbidden cycles are not necessarily induced.

For every graph \( G \), the set of weight functions \( w \) for which \( G \) is \( w \)-well-covered is a vector space [2]. That vector space is denoted \( WCW(G) \) [1].
Clearly, $w \in WCW(G)$ if and only if $G$ is $w$-well-covered. Since recognizing well-covered graphs is co-NP-complete, finding the vector space $WCW(G)$ of an input graph $G$ is co-NP-hard. In [14] there is a polynomial algorithm which receives as its input a graph $G$ without cycles of lengths 4, 5, and 6, and outputs $WCW(G)$.

This article presents a polynomial algorithm whose input is a $K_{1,3}$-free graph $G$, and the output is $WCW(G)$. Thus we generalize [21, 22], which supply a polynomial time algorithm for recognizing well-covered $K_{1,3}$-free graphs.

1.3 Generating Subgraphs and Relating Edges

We use the following notion, which has been introduced in [13]. Let $B$ be an induced complete bipartite subgraph of $G$ on vertex sets of bipartition $B_X$ and $B_Y$. Assume that there exists an independent set $S$ such that each of $S \cup B_X$ and $S \cup B_Y$ is a maximal independent set of $G$. Then $B$ is a generating subgraph of $G$, and it produces the restriction: $w(B_X) = w(B_Y)$. Every weight function $w$ such that $G$ is $w$-well-covered must satisfy the restriction $w(B_X) = w(B_Y)$. The set $S$ is a witness that $B$ is generating. In the restricted case that the generating subgraph $B$ is isomorphic to $K_{1,1}$, call its vertices $x$ and $y$. In that case $xy$ is a relating edge, and $w(x) = w(y)$ for every weight function $w$ such that $G$ is $w$-well-covered.

The decision problem whether an edge in an input graph is relating is NP-complete [1]. Therefore, recognizing generating subgraphs is NP-complete as well. In [15] it is proved that recognizing relating edges and generating subgraphs is NP-complete even in graphs without cycles of lengths 4 and 5. However, recognizing relating edges can be done polynomially if the input is restricted to graphs without cycles of lengths 4 and 5 [12], and recognizing generating subgraphs is a polynomial problem when the input is restricted to graphs without cycles of lengths 4, 6 and 7 [13].

Generating subgraphs play an important role in finding the vector space $WCW(G)$. In this article we use generating subgraphs in the algorithm which receives as its input a $K_{1,3}$-free graph $G$, and outputs $WCW(G)$.

1.4 Equimatchable Graphs

Let $G = (V, E)$ be a graph. The line graph of $G$, denoted $L(G)$ is a graph such that every vertex of $L(G)$ represents an edge in $G$, and two vertices of $L(G)$ are adjacent if and only if they represent two edges in $G$ with a common endpoint.

Every independent set of vertices in $L(G)$ defines a set of pairwise non-adjacent edges in $G$. A set of pairwise non-adjacent edges is called a matching. A matching $M$ dominates a set $S$ of vertices if every vertex of $S$ is an endpoint of an edge of $M$.

The size of a matching $M$, denoted $|M|$, is the number of its edges. A matching $M$ is maximum if the graph does not admit a matching with size bigger than $|M|$.
A graph is called equimatchable if all its maximal matchings are maximum. Clearly, \( G \) is equimatchable if and only if \( L(G) \) is well-covered.

Line graphs are characterized by a list of forbidden induced subgraphs \([11]\). One of these subgraphs is \( K_{1,3} \), called a claw.

Hence, every line graph is claw-free. Thus the existence of a polynomial algorithm for recognizing well-covered claw-free graphs \([21, 22]\), implies a polynomial algorithm for recognizing equimatchable graphs.

Assume that a weight function \( w : E \rightarrow \mathbb{R} \) is defined on the edges of \( G \). For every set \( S \subseteq E \), define

\[ w(S) = \sum_{s \in S} w(s). \]

Then \( G \) is \( w \)-equimatchable if all its maximal matchings are of the same weight.

It is easy to see that for every graph \( G \), the set of weight functions \( w \) such that \( G \) is \( w \)-equimatchable is a vector space. We denote that vector space by \( EVS(G) \).

In this paper we present a polynomial algorithm whose input is a graph \( G \), and the output is the vector space \( EVS(G) \).

### 2 Weighted Hereditary Systems

A hereditary system is a pair \( H = (S, \Psi) \), where \( S \) is a finite set and \( \Psi \) is a family of subsets of \( S \), where \( f \in \Psi \) and \( f' \subseteq f \) implies \( f' \in \Psi \). The members of \( \Psi \) are called the feasible sets of the system.

A feasible set is maximal if it is not contained in another feasible set. A feasible set is maximum if the hereditary system does not admit a feasible set with higher cardinality.

A hereditary system is greedy if and only if its maximal feasible sets are all of the same cardinality. Equivalently, a greedy hereditary system is a hereditary system for which the greedy algorithm for finding a maximal feasible set always produces a maximum cardinality feasible set.

Assume that a weight function \( w : S \rightarrow \mathbb{R} \) is defined on the elements of a hereditary system. The hereditary system is greedy if and only if all its maximal feasible sets are of the same weight, and equivalently, the greedy algorithm for finding a maximal feasible set always produces a feasible set of maximum weight.

An example of the above is a hereditary system \( H = (S, \Psi) \), where \( S = V \) is the set of vertices of a given graph \( G = (V, E) \), and \( \Psi \) is the family of all independent sets of \( G \). Clearly, the hereditary system \( H = (V, \Psi) \) is greedy if and only if \( G \) is well-covered. Similarly, if a weight function \( w : V \rightarrow \mathbb{R} \) is defined, then the hereditary system \( H = (V, \Psi) \) is greedy if and only if \( G \) is \( w \)-well-covered.

Another example of a hereditary system is a pair \( H = (S, \Psi) \), where \( S = E \) is the set of edges of a graph \( G = (V, E) \), and \( \Psi \) is the family of its matchings. Clearly, the hereditary system \( H = (E, \Psi) \) is greedy if and only if the graph is
equimatchable. Similarly, if a weight function $w : E \rightarrow \mathbb{R}$ is defined, then the hereditary system $H = (E, \Psi)$ is greedy if and only if $G$ is $w$-equimatchable.

**Theorem 1** [23] Let $H = (S, \Psi)$ be a hereditary system. Then $H$ is not greedy if and only if there exist two maximal feasible sets, $F_1$ and $F_2$, of $S$ with different cardinalities, $|F_1| \neq |F_2|$, such that for each $f_1 \in F_1 \setminus F_2$, and for each $f_2 \in F_2 \setminus F_1$, the set $(F_1 \cap F_2) \cup \{f_1, f_2\}$ is not feasible.

The following is a generalization of Theorem 1.

**Theorem 2** Let $$(H = (S, \Psi), w : S \rightarrow \mathbb{R})$$ be a hereditary system with a weight function defined on its elements. Then $$(H, w)$$ is not greedy if and only if there exist two maximal feasible sets, $F_1$ and $F_2$, of $S$ with different weights, $w(F_1) \neq w(F_2)$, such that for each $f_1 \in F_1 \setminus F_2$, and for each $f_2 \in F_2 \setminus F_1$, the set $(F_1 \cap F_2) \cup \{f_1, f_2\}$ is not feasible.

**Proof.** Clearly, if there exist two maximal feasible sets with different weights then the hereditary system is not greedy.

Suppose $(H, w)$ is not greedy. There exist two maximal feasible sets, $F_1$ and $F_2$, of $S$ with the following two properties:

1. $w(F_1) \neq w(F_2)$.

2. For every two maximal feasible sets, $F'_1$ and $F'_2$, of $S$, if $w(F'_1) \neq w(F'_2)$ then $|F_1 \cap F_2| \geq |F'_1 \cap F'_2|$.  

Assume on the contrary that there exist $f_1 \in F_1 \setminus F_2$, and $f_2 \in F_2 \setminus F_1$, such that the set $F_3 = (F_1 \cap F_2) \cup \{f_1, f_2\}$ is feasible. Clearly,  

$$\min\{|F_1 \cap F_3|, |F_2 \cap F_3|\} > |F_1 \cap F_2|.$$  

Therefore, $w(F_1) = w(F_3)$ and $w(F_2) = w(F_3)$. Hence, $w(F_1) = w(F_2)$, which is a contradiction.

We proved that for every $f_1 \in F_1 \setminus F_2$, and for every $f_2 \in F_2 \setminus F_1$, the set $(F_1 \cap F_2) \cup \{f_1, f_2\}$ is not feasible. $\blacksquare$
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The following is an instance of Theorem 2.

**Theorem 3** Let

\[(G = (V, E), w : V \rightarrow \mathbb{R})\]

be a graph with a weight function defined on its vertices. Then \(G\) is not \(w\)-well-covered if and only if there exist two maximal independent sets, \(S_1\) and \(S_2\), with different weights, \(w(S_1) \neq w(S_2)\), such that the subgraph induced by \(S_1 \triangle S_2\) is complete bipartite.

**Corollary 4** Let \(G = (V, E)\) be a graph, and let \(B\) be an induced complete bipartite subgraph of \(G\) on vertex sets of bipartition \(B_X\) and \(B_Y\). Then the following two conditions are equivalent:

1. There exist two maximal independent sets, \(S_1\) and \(S_2\), of \(G\) such that \(B_X = S_1 \setminus S_2\) and \(B_Y = S_2 \setminus S_1\).

2. \(B\) is generating.

**Proof.** If the first condition holds then \(S_1 \cap S_2\) is a witness that \(B\) is generating. If \(B\) is generating, let \(S\) be a witness of \(B\). The first condition holds for \(S_1 = S \cup B_X\) and \(S_2 = S \cup B_Y\). \(\blacksquare\)

The main result of this section is the following.

**Theorem 5** There exists an \(O(|V|^6)\) algorithm, which receives as its input a claw-free graph \(G\), and finds \(WCW(G)\).

**Proof.** Let \(G = (V, E)\) be a graph. The following algorithm finds \(WCW(G)\).

1. For every induced complete bipartite subgraph \(B\) of \(G\)
   
   (a) Denote its vertex sets of bipartition \(B_X\) and \(B_Y\).
   
   (b) Decide whether \(B\) is generating.
   
   (c) If \(B\) is generating
       
       i. List the restriction \(w(B_X) = w(B_Y)\).

2. \(w \in WCW(G)\) if and only if \(w\) satisfies all listed restrictions.

In the general case, this algorithm is not polynomial, because the number of induced complete bipartite subgraphs is not polynomial, and the time needed to decide whether one induced subgraph is generating is not polynomial. However, we show that the algorithm can be implemented polynomially if the input graph is claw-free.

Suppose \(G\) is claw-free. Then every induced complete bipartite subgraph is isomorphic to one of the following graphs: \(K_{1,1}\), \(K_{1,2}\), and \(K_{2,2}\). Hence, the number of subgraphs the algorithm checks is polynomial. It remains to prove
that it is possible to decide polynomially for a single subgraph whether it is generating.

Let $B$ be an induced complete bipartite subgraph of $G$ on vertex sets of bipartition $B_X$ and $B_Y$. Define

$$M_1 = (N(B_X) \cap N_2(B_Y)) \cup (N_2(B_X) \cap N(B_Y)),$$

and

$$M_2 = (N_2(B_X) \cap N_3(B_Y)) \cup (N_3(B_X) \cap N_2(B_Y)).$$

Clearly, $B$ is generating if and only if there exists an independent set in $M_2$ which dominates $M_1$.

If $B = K_{2,2}$ then the fact that the graph is claw-free implies that $M_1$ and $M_2$ are empty sets. Hence, $M_2$ dominates $M_1$, and $B$ is generating.

Assume $B \neq K_{2,2}$. In order to decide whether $B$ is generating, define a weight function

$$w: M_2 \rightarrow \mathbb{R} \text{ by } w(x) = |N(x) \cap M_1|,$$

i.e. the weight of every vertex in $M_2$ is the number of vertices it dominates in $M_1$. The fact that the graph is claw-free implies that a vertex of $M_1$ can not be dominated by two non-adjacent vertices of $M_2$. Therefore, if $S \subseteq M_2$ is independent then

$$w(S) = \sum_{s \in S} w(s) = \sum_{s \in S} |N(s) \cap M_1| = |N(S) \cap M_1|,$$

i.e., the weight of $S$ is the number of vertices it dominates in $M_1$.

The next step is to invoke an algorithm finding the maximum weighted independent set in claw-free graphs. First such algorithm is due to Minty [16], while the best known one with the complexity $O(|V|^3)$ may be found in [6]. Let $S^*$ be a maximum weight independent set of $G[M_2]$. Clearly, $w(S^*) \leq |M_1|$. If $w(S^*) = |M_1|$ then $S^*$ dominates $M_1$, and $B$ is generating. Otherwise, there does not exist an independent set of $M_2$ which dominates $M_1$, and $B$ is not generating.

The number of induced complete bipartite subgraphs which are isomorphic to $K_{1,1}$ or $K_{1,2}$ is $O(|V|^2)$. Hence, the complexity of the algorithm is $O(|V|^6)$.

4 $w$-Equimatchable Graphs

Let $G = (V, E)$ be a graph and $w: E \rightarrow \mathbb{R}$ a weight function defined on its vertices. Since there is a 1-1 mapping between the edges of $G$ and the vertices of $L(G)$, the function $w$ can be viewed as a weight function on the vertices of $L(G)$. Therefore, $G$ is $w$-equimatchable if and only if $L(G)$ is $w$-well-covered. Hence, $EV_S(G) = WCW(L(G))$. Obviously, $EV_S(G)$ can be found polynomially by constructing the line-graph, $L(G)$, and then applying the algorithm of the proof of Theorem 5 to find $WCW(L(G))$. The number of vertices in $L(G)$ is $|E|$. 


Hence, $WCW(L(G))$ can be found in $|E|^6$ time. However, the main result of this section is an algorithm which finds $EV S(G)$ in $O \left( |E| \cdot |V|^4 + |V|^5 \log |V| \right)$ time.

The following is an instance of Theorem 2.

**Theorem 6** Let

$$(G = (V, E), w : E \rightarrow \mathbb{R})$$

be a graph with a weight function defined on its edges. Then $G$ is not $w$-equimatchable if and only if there exist two maximal matchings, $M_1$ and $M_2$, with different weights, $w(M_1) \neq w(M_2)$, such that $M_1 \triangle M_2$ is one of the following.

1. Two adjacent edges, $v_1v_2 \in M_1$ and $v_2v_3 \in M_2$.
2. Three edges, $\{v_1v_2, v_3v_4\} \subseteq M_1$ and $v_2v_3 \in M_2$.
3. Four edges, $\{v_1v_2, v_3v_4\} \subseteq M_1$, and $\{v_2v_3, v_1v_4\} \subseteq M_2$.

We need the following three lemmas to prove the main result of this section.

**Lemma 7** The following problem can be solved in $O \left( |E| \cdot |V|^4 + |V|^5 \log |V| \right)$ time.

**Instance:** A graph $G = (V, E)$ and a path $P = (v_1v_2, v_2v_3, ..., v_{k-1}v_k)$ in $G$ for some $k \geq 3$.

**Question:** Do there exist two maximal matchings, $M_1$ and $M_2$, of $G$ such that $P = M_1 \triangle M_2$?

**Proof.** If $k$ is even and $v_1v_k \in E$ then the instance is obviously negative. Hence, we assume that $k$ is odd or $v_1v_k \notin E$.

Define

$$V' = V \setminus \{v_1, ..., v_k\} \text{ and } D = N(\{v_1, v_k\}) \cap V'.$$

Let $G'$ be the induced subgraph of $G$ on vertex set $V'$, and denote the set of its edges by $E'$.

Define a weight function $w : E' \rightarrow \mathbb{R}$ by:

$$\forall xy \in E' \quad w(xy) = |\{x, y\} \cap D|.$$

For every matching $M$ in $G'$, its weight, $w(M)$, equals to the number of vertices of $D$ which are dominated by $M$. We now invoke the algorithm of [10] for finding a maximum weight matching in a graph, and denote the output of the algorithm by $M^*$. Clearly, $w(M^*) \leq |D|$.

Suppose $w(M^*) = |D|$. Then $M^*$ dominates $D$. Let $M^{**}$ be any maximal matching in $G'$ which contains $M^*$, and define

$$M_1 = M^{**} \cup \{u_{2i-1}u_{2i} : 1 \leq 2i \leq k\},$$

$$M_2 = M^{**} \cup \{u_{2i}u_{2i+1} : 3 \leq 2i + 1 \leq k\}.$$
Obviously, $M_1$ and $M_2$ are two maximal matchings of $G$ and $P = M_1 \triangle M_2$.

On the other hand, suppose $w(M^*) < |D|$. There does not exist a maximal matching of $G'$ which dominates $D$, and therefore the instance at hand is negative.

The complexity of the algorithm of [10] is $O \left( |E| \cdot |V| + |V|^2 \log |V| \right)$. This is also the complexity of this algorithm. ■

Lemma 8 Let $G = (V, E)$ be a graph, and let $C = (v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1)$ be an even cycle in $G$, for some $k \geq 4$. Then there exist two maximal matchings, $M_1$ and $M_2$, of $G$ such that $C = M_1 \triangle M_2$.

Proof. Let $M$ be any maximal matching in $G \setminus \{v_1, \ldots, v_k\}$. Define

$$M_1 = M \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq \frac{k}{2}\}$$

and

$$M_2 = M \cup \{v_{2i}v_{2i+1} : 1 \leq i \leq \frac{k}{2} - 1\} \cup \{v_kv_1\}.$$ 

Obviously, $M_1$ and $M_2$ are two maximal matchings of $G$ and $C = M_1 \triangle M_2$. ■

The naive algorithm for finding $EVS(G)$ checks all structures described in Theorem 3, i.e., all paths of lengths 2 and 3, and cycles of length 4. For each of these structures, the algorithm decides whether it is the symmetric difference of two maximal matchings. If so, an appropriate equation is added to the list of restrictions. A weight function $w : E \to \mathbb{R}$ satisfies all the restrictions found by the algorithm if and only if $w \in EVS(G)$. For each path of lengths 2 or 3, the naive algorithm invokes the algorithm of Lemma 7. By Lemma 8, every cycle of length 4 is the symmetric difference of two maximal matchings. Hence, the total complexity of the naive algorithm is

$$O \left( |E| \cdot |V|^5 + |V|^6 \log |V| \right).$$

However, we present a more efficient algorithm.

Lemma 9 The following problem can be solved in $O \left( |E| \cdot |V|^2 + |V|^3 \log |V| \right)$ time:

Input: A graph $G = (V, E)$, and two non-adjacent vertices, $v_1$ and $v_4$, in $G$.

Output: All paths $P = (v_1v_2, v_2v_3, v_3v_4)$, such that there exist two maximal matchings, $M_1$ and $M_2$, in $G$, and $M_1 \triangle M_2 = P$?

Proof. Let $G' = (V', E')$ be the subgraph of $G$ induced by $V' = V \setminus \{v_1, v_4\}$, and let $\epsilon = \frac{1}{|V'|}$. Define $w : E' \to \mathbb{R}$ as follows:

$$w(xy) = \begin{cases} 
2 + \epsilon & \text{if } x \in N(v_1) \text{ and } y \in N(v_4) \\
|\{x, y\} \cap N(\{v_1, v_4\})| & \text{otherwise}
\end{cases}$$

(see Figure 11)
Figure 1: The weight function $w$. Note that $w$ is not defined on edges which dominate $v_1$ and $v_4$.

For every matching $M$ in $G'$. There exist two integers, $0 \leq A \leq |V|$ and $0 \leq B \leq |V|$, such that $w(M) = A + B\epsilon$, where $A$ is the number of vertices of $N(\{v_1, v_4\})$ dominated by $M$, and $B$ is the number of edges of $M$ with one endpoint in $N(v_1)$ and another endpoint in $N(v_4)$.

Let $M^*$ be a maximum weight matching in $G'$. Let $0 \leq A \leq |V|$ and $0 \leq B \leq |V|$ such that $w(M^*) = A + B\epsilon$. Among all maximal matchings in $G'$, the matching $M^*$ dominates maximum possible number of vertices in $N(\{v_1, v_4\})$. Among all maximal matchings in $G'$, which dominate $A$ vertices of $N(\{v_1, v_4\})$, the matching $M^*$ contains maximum number of edges with one endpoint in $N(v_1)$ and another endpoint in $N(v_4)$.

Clearly, $A \leq |N(\{v_1, v_4\})|$. If $A = |N(\{v_1, v_4\})|$ and $B > 0$ then $M^*$ dominates $N(\{v_1, v_4\})$, and contains at least one edge $v_2v_3$ where $v_2 \in N(v_1)$ and $v_3 \in N(v_4)$. Hence, $M^*$ and

$$M^{**} = (M^* \cup \{v_1v_2, v_3v_4\}) \setminus \{v_2v_3\}$$

are two maximal matchings in $G$, and $M^* \Delta M^{**} = P$.

If $A = |N(\{v_1, v_4\})|$ and $B = 0$ then there exist matchings of $G'$ which dominate $N(\{v_1, v_4\})$, but non of them contains an edge $v_2v_3$ such that $v_2 \in N(v_1)$ and $v_3 \in N(v_4)$. Therefore, there does not exist a path $(v_1v_2, v_2v_3, v_3v_4)$, which is the symmetric difference of two maximal matchings.

If $A < |N(\{v_1, v_4\})|$ then there does not exist a matching of $G'$ which dominates $N(\{v_1, v_4\})$, and therefore there does not exist a path $(v_1v_2, v_2v_3, v_3v_4)$ which is the symmetric difference of two maximal matchings.

The following algorithm solves the problem.

1. Define $G'$, $\epsilon$ and $w$ as above.

2. Invoke the algorithm of [10] to find a maximum weight matching $M'$ in $G'$.
3. While \( w(M') > |N(\{v_1, v_4\})| \)
   \[
   \begin{align*}
   (a) & \text{ For every edge } v_2v_3 \in M' \text{ such that } w(v_2v_3) = 2 + \epsilon \\
   & \quad \text{i. List the path } (v_1, v_2, v_3, v_4). \\
   & \quad \text{ii. Set } w(v_2v_3) = 2.
   \\
   (b) & \text{ Invoke again the algorithm of [10] with the modified definition of } w,
   \quad \text{and get a new maximum weight matching } M' \text{ in } G'.
   \end{align*}
   \]

   The complexity of the algorithm of [10] is \( O(|E| \cdot |V| + |V|^2 \log |V|) \), and it
   is invoked at most \( O(|V|) \) times. Hence, the total complexity of this algorithm
   is \( O \left( |E| \cdot |V|^2 + |V|^3 \log |V| \right) \). Note that if \( v_1 \) and \( v_4 \) are not the endpoints of a
   path of length 3, which is the symmetric difference of two maximal matchings,
   then the algorithm of [10] is invoked only once. In this restricted case the
   complexity of the algorithm is \( O \left( |E| \cdot |V| + |V|^2 \log |V| \right) \). ■

   The next theorem is the main result of this section.

   **Theorem 10** The following problem can be solved in \( O \left( |E| \cdot |V|^4 + |V|^5 \log |V| \right) \)
   time:
   
   **Input:** A graph \( G = (V, E) \).
   **Output:** \( EVS(G) \).

   **Proof.** The following algorithm solves the problem:

   1. For each subgraph \( H \) (not necessarily induced) isomorphic to \( P_3 \) on vertex
      set \( \{v_1, v_2, v_3\} \):
         \[
         \begin{align*}
         (a) & \text{ Invoke the algorithm described in the proof of Lemma 7 to decide } \\
         & \quad \text{whether } (v_1v_2, v_2v_3) \text{ is the symmetric difference of two maximal match-} \\
         & \quad \text{ings.}
         \end{align*}
         \]
         \[
         (b) & \text{ If so, add the restriction: } w(v_1v_2) = w(v_2v_3).
         \]

   2. For each pair of non-adjacent vertices, \( v_1 \) and \( v_4 \):
      \[
      \begin{align*}
      (a) & \text{ Invoke the algorithm of Lemma 9 } \\
      (b) & \text{ For each path } (v_1v_2, v_2v_3, v_3v_4) \text{ found by the algorithm: }
      \end{align*}
      \]
      \[
      \begin{align*}
      & \quad \text{i. Add the restriction: } w(v_1v_2) + w(v_3v_4) = w(v_2v_3).
      \end{align*}
      \]

   3. For each subgraph (not necessarily induced) isomorphic to \( C_4 \) on vertex
      set \( \{v_1, v_2, v_3, v_4\} \):
      \[
      \begin{align*}
      (a) & \text{ Add the restriction: } w(v_1v_2) + w(v_3v_4) = w(v_2v_3) + w(v_1v_4).
      \end{align*}
      \]
The complexity of the algorithm of Lemma \([7]\) is \(O\left(|E| \cdot |V| + |V|^2 \log |V|\right)\), and it is invoked \(O(|V|^3)\) times in step 1. Hence, the complexity of step 1 is \(O\left(|E| \cdot |V|^4 + |V|^5 \log |V|\right)\). The complexity of the algorithm of Lemma \([9]\) is \(O\left(|E| \cdot |V|^2 + |V|^3 \log |V|\right)\), and it is invoked \(O(|V|^2)\) times in step 2. Hence, the complexity of step 2 is \(O\left(|E| \cdot |V|^4 + |V|^5 \log |V|\right)\). The complexity of step 3 is \(O(|V|^4)\). Thus the total complexity of this algorithm is

\[ O\left(|E| \cdot |V|^4 + |V|^5 \log |V|\right). \]

5 Conclusion and Future Work

A graph \(G\) is equimatchable if and only if \(EV S(G)\) contains the function \(w \equiv 1\). It follows from Theorem \([6]\) that \(G\) is equimatchable if and only if there do not exist two maximal matchings, \(M_1\) and \(M_2\), such that \(M_1 \Delta M_2\) is a path of length 3.

Hence, the following algorithm decides whether \(G\) is equimatchable: For every pair of non-adjacent vertices, \(v_1\) and \(v_4\), in \(G\), invoke the algorithm of Lemma \([9]\) with input \((G, v_1, v_4)\). Once the algorithm of Lemma \([9]\) yields a non-empty list of paths, this algorithm outputs that \(G\) is not equimatchable. If all calls of the algorithm of Lemma \([9]\) yielded empty lists of paths, then \(G\) is equimatchable.

The algorithm of Lemma \([9]\) is called at most \(O(|V|^2)\) times. However, all of these calls, except maybe the last one, yielded empty lists. The complexity invoking the algorithm of Lemma \([9]\) and receiving an empty output is \(O\left(|E| \cdot |V| + |V|^2 \log |V|\right)\), while the complexity invoking the algorithm of Lemma \([9]\) and receiving a non-empty output is \(O\left(|E| \cdot |V|^2 + |V|^3 \log |V|\right)\). Hence, the total complexity of this algorithm is \(O\left(|E| \cdot |V|^3 + |V|^4 \log |V|\right)\) time.

However, for this restricted case a more efficient algorithm has been found in \([5]\). That algorithm decides whether an input graph is equimatchable in \(O(|E| \cdot |V|^2)\) time. It seems worth trying to improve on our algorithm returning \(EV S(G)\) using the technique presented in \([5]\).

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