Gravity and Random Surfaces on the Lattice: A Review

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We review recent work in the lattice approach to random surfaces and quantum gravity. Our task is made somewhat easier by some very interesting results, particularly in four dimensions, that have appeared recently and which are reported elsewhere in these proceedings. Inevitably, given the scope of the review and the limitations of space, the presentation will omit work of importance and be telegraphic in discussing work that is included, for which apologies are offered in advance. After the customary brief historical introduction we work our way in dimensional order from one up to four dimensions before closing with some remarks on the relation, if any, between the various lattice models and “real” 4D gravity.

1. INTRODUCTION

It is probably fair to say that lattice gauge theory in the large is now a mature subject. The theoretical underpinnings are understood and the numerics are under control. The subject of this review is a much more speculative application of lattice methods, namely to the study of quantum gravity in various dimensions. Geometry itself becomes dynamical in gravitation, so it is clear that unlike standard lattice theories we will be dealing with dynamical lattices of some sort in which the geometry and the matter living on the lattice, if any, are in interaction.

The first question to ask before embarking on the numerical investigation of any theory is why bother? In the case of gravity, the answer is quite clear: the Einstein-Hilbert action for general relativity is perturbatively non-renormalizable, essentially because of the dimensionful coupling. Although this does not exclude a valid theory \(^1\), it does force us to employ methods other than those of perturbative quantum field theory to investigate the model. There are further problems lurking even after the rotation to Euclidean signature which is generically necessary to obtain tractable simulations on the lattice. These are due to the unboundedness below of the Euclidean action, due to conformal mode fluctuations,

\[
S = \int_M d^4x \sqrt{g} \left( \Lambda - \frac{1}{16\pi G} R \right)
\]

where \(M\) is our spacetime manifold. At first sight this might seem to render even the Euclidean partition function

\[
Z = \int [Dg_{ab}] \exp (-S),
\]

ill-defined. We might be saved, however, by the measure in the path integral giving negligible weights to such configurations. In order to see if such a happy occurrence does take place we need the framework of some regulated lattice theory where the integral/sum over configurations can be clearly defined.

It is perhaps worth strengthening the case for numerical investigations by noting that minimal tinkerings with the quantum field theory don’t work. Introducing higher derivative terms gives a renormalizable theory with action

\[
\int_M d^4x \sqrt{g} \left( \Lambda - \frac{1}{16\pi G} R + \alpha R_{\mu \nu} R^{\mu \nu} + \beta R^2 \right)
\]

which is is even asymptotically free for suitably chosen \(\alpha\) and \(\beta\) but, as might be expected, one runs into problems with unitarity. More radical solutions, SUSY, SUGRA, strings, p-branes, M-theory, non-commutative differential geometry have all generated a lot of beautiful mathematics (and even more preprints) but getting

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\(^{1}\) An example of a perturbatively non-renormalizable theory that does have a well-defined continuum limit is the 4D O(4) non-linear sigma model \[.\]
back to boring old four dimensions and a non-supersymmetric, or at least broken supersymmetric particle spectrum, poses problems of varying degrees for all of them.

A more modest line of attack is to attempt to formulate a non-perturbative approach to the Einstein-Hilbert action and its close relatives. There are two complementary schemes that have been used in most recent work, Regge Calculus (RC) and Dynamical Triangulations (DT), which we now describe briefly in turn.

**Regge calculus (RC):** was originally suggested by Regge in 1961 as a means of discretizing classical GR. The recipe for discretizing the D-dimensional spacetime manifold $M$ was to replace it by D-simplices. Curvature then resides on the $D-2$ dimensional hinges between the simplices (i.e., vertices in 2D gravity, links in 3D gravity and faces in 4D gravity). In the RC approach the discretized equivalent of the integration over metrics is usually implemented by taking a fixed connectivity simplicial complex with varying edge lengths. Regge and Ponzano came back for a second bite with a theory of quantum 3D gravity in 1968 and more recently there has been much numerical work on the quantum theory in 4D by Hamber and the Vienna group.

**Dynamical Triangulations (DT):** The recent work with this approach is an outgrowth of ideas that had their genesis in the theory of random surfaces and 2D gravity but this variant also has a long pedigree, in that the basic ideas were formulated Weingarten in 1982 using hypercubes. As with RC, one replaces $M$ by D-simplices, but now we take varying connectivity in our simplicial complex and fixed edge lengths in order to implement a discretized version of the integral over metrics. In two dimensions this formulation is essentially a recipe for the numerical evaluation of matrix model partition functions and has been an unqualified success. As a cautionary remark it should be noted that one does not have the same theoretical underpinning in three and four dimensions.

From the statistical mechanical viewpoint in both RC and DT we are dealing with theories with a particular sort of annealed disorder coming from the geometric fluctuations. This is as we might expect - the dynamics of the lattice should be taking place on the same timescale as any matter, so we have annealed rather than quenched averages to contend with. In RC we have annealed bond length disorder whereas in DT we have annealed connectivity disorder.

In both RC and DT the discretized actions are appealingly simple, being for Regge Calculus in 4D

$$S = \beta \sum_t A_t \delta_t - \lambda \sum_s V_s$$

where $A_t$ is the area of triangular faces $t$, $V_s$ is the volume of simplices $s$ and the deficit angle is given by

$$\delta_t = 2\pi - \sum_{\text{tes}} \theta_{s,t}$$

and for dynamical triangulations in various dimensions

$$S = \kappa_4 N_4 - \kappa_2 N_2 \quad (4D)$$
$$S = \kappa_3 N_3 - \kappa_1 N_1 \quad (3D)$$
$$S = \kappa_2 N_2 \quad (2D)$$

where the $N_k$ are the numbers of $k$-simplices and the various $\kappa$'s are the coupling constants.

There are various choices to be made in the two schemes. In RC the measure for the integration over the edge lengths must be chosen according to some prescription and a fixed triangulation decided upon. In DT one must decide which class of simplicial complexes to employ - for instance in 2D the finite size effects are minimized, rather counterintuitively, by allowing various degenerate triangulations. In both schemes one must decide which topology to fix or whether, indeed, to let it vary.

In DT the moves that change the connectivity in $D$ dimensions can be obtained from “slicing” $D+1$ dimensional simplices appropriately. In two dimensions there is a “flip” move that preserves the number of vertices in a triangulation as well as an insert/delete move, so it is possible to carry out either canonical fixed number of nodes simulations or grand canonical varying number of nodes simulations. The moves in two dimensions are shown overleaf:
This option is no longer open in three and four dimensions where the moves change the number of simplices. These moves are represented in their simplest form as so-called \((k, l)\) moves \([9]\) which take \(k\) subsimplices to \(l\) subsimplices, and may be shown to be ergodic. The moves in the three dimensional case are shown below:

being \((1, 4)\) and \((2, 3)\) moves respectively. My very limited skills as an illustrator defeat me in the four dimensional case where the \((k, l)\) moves are \((1, 5)\), \((2, 4)\) and \((3, 3)\). In addition cluster-like moves have also been defined, which have the effect of cutting off large chunks of the simplicial complex and gluing them back on elsewhere.

The partition function of interest in the 4D simulations carried out using the moves described above is

\[
Z(\kappa_2, \kappa_4) = \sum_{N_2, N_4} Z_{N_2,N_4} \exp (-S)
\]

where

\[
Z_{N_2,N_4} = \sum_{T(N_2,N_4)} W(T)
\]

with \(W(T)\) being the symmetry factor for a simplicial complex and \(S\) being the 4D DT action. To discuss possible critical behaviour it is convenient to define a fixed volume partition function

\[
Z(N_4, \kappa_2) = \sum_{T(N_4)} \exp (\kappa_2 N_2)
\]

so

\[
Z(\kappa_2, \kappa_4) = \sum_{N_4} Z(N_4, \kappa_2) \exp (-\kappa_4 N_4)
\]

As we have noted it is impossible to carry out a fixed volume simulation in 4D because an ergodic set of \((k, l)\) moves entails changes in the number of simplices, so we perform a fudge to stay “close” to given \(N_4\) by modifying the action with a gaussian term

\[
S = \kappa_4 N_4 - \kappa_2 N_2 + \gamma (N_4 - V)^2
\]

The observables in our simulations are the discretized versions of curvature and the associated susceptibilities

\[
< R > \simeq \left( \frac{N_2}{N_4} \right)
\]

\[
\chi_R \simeq \langle N_4 \rangle \left( < R^2 > - < R >^2 \right)
\]

as well as other geometrical properties such mean geodesic distances, hausdorff dimensions and curvature correlators.

A schematic phase diagram for the theory is shown below, where moving along the bold line representing the infinite volume limit we may
reach a critical point at some value of $\kappa_2$ at which correlation lengths may diverge and we can hope to take a continuum limit.

Extensive simulations by various groups both in DT \cite{10} and RC \cite{5,6} over the past few years suggested the following picture:

- In DT one sees a low $\kappa_2$ “crumpled” phase and a large $\kappa_2$ branched polymer like phase in both 3D and 4D.
- A fractal baby universe structure was manifest in 3D and 4D (and also in 2D) with DT.
- With DT in 3D there was a first order transition between the crumpled and branched polymer phases.
- With DT in 4D the transition was second order.
- RC saw a similar phase structure (but not necessarily order of transitions).
- Issues of an exponential bound on the number of configurations \cite{11} and computability in DT appeared to have been settled to (almost!) everyone’s satisfaction.

The schema above for DT has the great merit of conceptual simplicity - starting with a very simple action in both 3D and 4D one has found no continuum limit in 3D (fine, there is no graviton in 3D) but a continuum limit in 4D (also fine, we want to get a graviton here). It has, unfortunately, the great demerit of apparently being wrong, as the latest numerical work presented in these proceedings indicates.

In the rest of this article we will review the recent work in the field in one, two, three and four dimensions. The lower dimensional work is included not just for completeness, a lot of it has a direct bearing on the understanding of the 4D theory. The 1D work on polymers, of course, stands on its own merits as does the 2D work on random surfaces which have a wide application outside the rather esoteric world of quantum gravity simulations and string theory.

2. 1D STRUCTURES AND POLYMERS

It seems to be a general principle that models of dynamical geometry in higher dimensions will, given half a chance, collapse to lower dimensional structures - the large $\kappa_2$ branched polymer like phase in 3D and 4D DT being a prime example. It is therefore of some interest to study simplified models of branched polymer like objects in their own right. Indeed, it has been suggested that in 4D DT the entire branched polymer phase may be critical because of power law correlations observed there \cite{12}.

A note of caution on these observations has been sounded recently by Bialas \cite{13} who considered a model of planar rooted planted trees as shown above. The partition function for these is just

$$Z = \sum_{T \text{trees}} \rho(T)$$

where the weight, $\rho(T)$, of a given tree depends on how many vertices $n_k$ of order $k$ are present

$$\rho(T) = t_0^{n_0} t_1^{n_1} t_2^{n_2} \ldots t_k^{n_k} \ldots$$

with the $t_k$’s being the weights for order $k$ vertices.

He considered correlations of the form

$$G(\mu, t, r) = \sum_{T} \exp(-\mu n) \rho(T)$$

where $\mu$ is the “cosmological constant” with a marked point a distance $r+1$ from the root, more specifically Root-Tail correlations

$$G(\mu, t, r) = \sum_{T_r} d(v_1)d(v_{r+1}) e^{-\mu n} \rho(T)$$
where the \( d(v) \) are the degrees of the vertices in question. He found, with some subtleties in the definitions of correlators, \( \tilde{G} = 0 \). However, in a canonical (fixed number of vertices) ensemble there were negative \( \sim 1/r^2 \) correlations due to the absence of short trees. The moral is that the almost fixed volume constraint in 4D gravity simulations might be responsible for the power law behaviour seen in the branched polymer phase rather than being a sign of true critical behaviour. The issue awaits further clarification.

Planar rooted planted trees were also used by Ambjørn et al. in an analytical implementation of fractal blocking [4]. Various MCRG schemes have been proposed for two and higher dimensional DT simulations. The new feature here, compared with MCRG on a regular lattice, is that the geometry is both irregular and dynamical so the choice of basic blocking move for the lattice itself is not immediately apparent. One possibility is the so-called fractal blocking or baby universe renormalization group which makes use of the fractal structure of the DT simplicial manifolds [13,14]. In all dimensions these contain regions where the lattice necks down, which can be visualized as baby universes budding off a mother universe. This can be used to define a blocking move as shown above right - extremal baby universes (those with no babies themselves) are lopped off and the couplings rescaled.

The simplicity of the tree model allows an analytical implementation of this program and shows, surprisingly that fractal blocking moves you away from a branched structure directly to a linear chain. Whether this signifies a possible pathology of the blocking scheme or is merely a quirk of the model is not entirely clear. The fact that fractal blocking gives good results numerically in higher dimensions argues for the latter.

The last result I want to mention in 1D concerns planar rooted trees [3] where the restriction on having only one branch emerging from the root node is relaxed. This technical change gives a soluble model

\[
Z = \sum_T \exp (-\mu n(T)) \exp (-\beta E(T))
\]

where \( n(T) \) is the number of vertices, \( E(T) = \sum_v \ln k(v) \) and \( k(v) \) is the vertex degree. There are two phases: a large \( \beta \) branched polymer phase with lots of short “bushes” ; and a small \( \beta \) elongated branched polymer phase. The transition between these is fourth order

\[
\phi - \phi_0 = (\mu - \mu_0)^{1 - \gamma_{\text{string}}}
\]

where the free energy is \( \phi = -\ln Z \), and \( \gamma_{\text{string}} = 0.3237... \). The result is interesting because it provides a counterexample to the belief that only values of \( \gamma_{\text{string}} = 1/n \) for \( n = 2, 3, 4, \ldots \) were possible in branched polymer models. Indeed, by tinkering with the vertex weights a range of \( \gamma_{\text{string}} \) values can be extracted from the model.

3. 2D RANDOM SURFACES

We make an extended stop in 2D also as the DT approach has its genesis here. The usual model of interest is the Polyakov String

\[
S = \int d^2 \sigma \sqrt{g} g^{ab} \partial_a \vec{X} \partial_b \vec{X} + \lambda \int d^2 \sigma K^2
\]

where \( K^2 \) is the extrinsic curvature squared and acts as a stiffness or rigidity term. This discretizes neatly to

\[
S = \frac{1}{2} \sum_{<ij>} (\vec{X}_i - \vec{X}_j)^2 + \lambda \sum_{\Delta} (1 - \vec{n}_\Delta \cdot \vec{n}_\Delta)
\]
where the \( n_\Delta \) are normals to triangles. On a dynamical triangulation this action can serve as a model for a fluid surface, and on fixed lattice (so there is no gravity at all) as a model for a polymerized or crystalline surface. Analytically the fluid model appears to be always in a rough phase but simulations provide evidence for a crumpling transition as \( \lambda \) is increased.

More recently an alternative action also based on geometrical ideas, the gonihedric string, has been proposed by Savvidy et.al. \[20\].

The aim of showing conclusively that a flat phase does exist in the crystalline model (and hence a crumpling transition) was pursued by the Syracuse group \[21\]. The answer is not immediately obvious because considering an action of the form

\[
S = \sum_{<ij>} \left( |\vec{X}_i - \vec{X}_j| - a \right)^2
\]

where the \( a \) represents an intrinsic length for each edge gives an effective action for the stress-strain tensor \( u_{\alpha\beta} \)

\[
S \approx \int d^2 \sigma \left( \frac{\sqrt{3}}{2} a^2 u_{\alpha\beta} u_{\beta\alpha} + \frac{\sqrt{3}}{4} a^2 u_{\gamma\gamma} \right)
\]

in which the elastic constants apparently vanish as \( a \to 0 \) \[3\] so it is difficult to see how a flat phase might be stabilized.

However, measurement of the various scaling exponents for the running bending stiffness \((\kappa\text{ in the notation of }[20], \text{ }\lambda \text{ elsewhere in this review!})\) and the running elastic “constants”, \( \mu \) and \( \lambda \)

\[
\kappa(q) \sim q^{-\eta}, \quad \mu(q) \sim \lambda(q) \sim q^{\eta_a}
\]

as well as the roughness exponent \( \zeta \)

\[
< h^2 > \sim L^{2\zeta}
\]

all of which are related by scaling relations \( \zeta = 1 - \frac{\gamma}{2}, \quad \eta = 1 - \frac{\beta}{2} \) gave \( \eta_a = 0.50(1) \) (so from the scaling relation \( \eta = 0.75 \)), \( \zeta = 0.64(2) \) (thus \( \eta = 0.72 \) from the other scaling relation). These consistent estimates of a positive \( \eta \) confirm the existence of a flat phase.

Having surreptitiously slipped in a discussion of surfaces without gravity, we now return to our remit to look at work on dynamically triangulated surfaces - in other words, matter coupled to two dimensional gravity. One of the most interesting recent developments has been the formulation of MCRG methods for dynamical lattices. The fractal blocking scheme has already been described, but another scheme, node decimation, has been pioneered by the Syracuse group \[21\]. The idea is again simple: we pick a node, remove it, and then retriangulate the gap to make sure that we still have a triangulation as shown overleaf.

In practice it is simpler to pick a node at random, flip the surrounding links until it has only three neighbours and then remove it. Work with this method has concentrated on 2D simulations, looking at the flow of geometric operators such as

\[
\frac{1}{N} \sum_{i} (q_i - 6)^2, \quad \frac{1}{N} \sum_{<ij>} (q_i - 6)(q_j - 6)
\]

\[2\] Although their fluctuations diverge.
where \( q_i \) is the number of neighbours of site \( i \), when a measure factor \( \alpha \sum_i \log q_i \) is varied. The Ising model

\[
S = \beta \sum_{\langle ij \rangle} \sigma_i \sigma_j
\]

coupled to 2D gravity has also been investigated and measurements of \( \gamma_{\text{string}} \)

\[
Z(N) \sim N^{\gamma_{\text{string}} - 3} \exp (\mu_c N)
\]

after blocking moves confirm a flow to the Ising value \( \gamma_{\text{string}} = -1/3 \).

The other main thrust of recent work in 2D gravity has been the clarification of scaling behaviour. This was discussed in some detail by Simon Catterall in Lattice95 [22], so I will be very brief here. The canonical object of interest in such measurements for the geometric sector of the theory is the distribution of geodesic distances, given in a grand canonical ensemble with cosmological constant \( \mu \) by

\[
G_\mu(R) = \left\langle \int \int \sqrt{g} \sqrt{\delta((g_{\xi,\xi'} - R)} \right\rangle
\]

which can be shown to scale as

\[
G_\mu(R) \sim \mu^{-\gamma_{\text{string}}} F(\mu R), \quad \nu = 1/d_H
\]

where \( d_H \) is the hausdorff dimension of the triangulation and \( F \) is some scaling function. The other scaling exponent \( \eta \) is related to \( \gamma_{\text{string}} \) by \( \gamma_{\text{string}} = \nu(2 - \eta) \). The number of points in a radial shell at distance \( R \) in a microcanonical ensemble of volume \( V \) can be shown to scale as

\[
\langle S(R) \rangle_V = R^{d_H - 1} F \left( \frac{R}{V^{\nu}} \right)
\]

and is perhaps a more amenable observable to measure.

Simulations [23] have shown that \( d_H = 4 \) when the central charge \( c \) of the matter living on the triangulation is less than one, so rather surprisingly the back reaction of the matter does not affect \( d_H \). As the triangulations degenerate to branched polymers for large \( c \) it is also known that \( d_H = 2 \) in this regime.

What was unclear until very recently was whether a diverging matter correlation length could be successfully defined on dynamical triangulations. Indeed, a toy model of spins on a restricted class of triangulations [24] displayed a transition but no such divergence. However, simulations presented at this conference [25] have shown that such a diverging correlation length does exist in the Ising and 3-state Potts models on the full dynamical triangulations and also measured the effect of the matter back reaction on the scaling functions - a delicate task as it is the nose and tail of the distributions where the effects are seen.

There have also been simulations [26] of variants of the standard action including \( R^2 \) terms on dynamical triangulations with actions of the form

\[
S = \beta \sum_i \frac{(6 - q_i)^2}{q_i} + \alpha \sum_{\langle ij \rangle} \frac{(6 - q_i)(6 - q_j)}{q_i q_j}
\]

A recent analytical solution of dually weighted matrix models [27] has shown that \( R^2 \) gravity is in the same universality class as ordinary gravity, which is what is seen in these simulations. This is rather similar to particles paths weighted by curvature, which behave as random walks unless the curvature coupling is taken to infinity.

One enduring lacuna in our understanding of 2D gravity is the nature of the so-called \( c = 1 \) barrier. Matrix models and continuum methods both fail to get beyond \( c = 1 \), but simulations see no immediate evidence of pathologies for \( c \) values not much larger than one. It is only at large
c that triangulations collapse to branched polymers. One simulation that casts some light on this problem appeals to methods from electrical engineering to probe the nature of the random surface \[28\]. One treats a triangulation of spherical topology as a random network of unit resistors and puts current in and out at two vertices, measuring the voltage drop across another two. The voltage-current relation is

\[
V(z_1) - V(z_2) = R(z_1, z_2; z_3, z_4) \quad I
\]

where the resistance is

\[
R = -\frac{r}{2\pi} \ln \left| [z_1, z_2, z_3, z_4] \right|
\]

\[r\] is the resistivity constant and \([z_1, z_2, z_3, z_4] \] is the cross-ratio, which appears because of the conformal invariance. If the theory is to have a sensible continuum limit the distribution of measured \(r\) values should become sharper as the lattice size is increased, which happens for \(c \leq 1\) but not, apparently, for \(c > 1\). This suggests that the conformal structure of the surface is breaking down at \(c = 1\). It is also possible to extract the distribution of modular parameters on a toroidal triangulation, but finite size effects are rather stronger here, so the evidence for something nasty happening just above \(c = 1\) is weaker.

I have so far relegated RC to a few cursory sentences, which I now rectify. Some older simulations of the Ising model on a 2D RC lattice gave an unpleasant surprise \[29\], finding results consistent with the Onsager (flat 2D lattice) exponents

\[
\begin{array}{cccccccc}
q & c & \alpha & \beta & \gamma & \delta & \nu & \eta \\
2 & \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{4} & 15 & 1 & \frac{1}{4}
\end{array}
\]

rather than the expected KPZ/DDK exponents for the Ising model coupled to 2D gravity. If they were to be believed RC was failing to simulate gravity in 2D. Simulations of the Ising and Potts models with DT do find the correct KPZ/DDK exponents.

\[
\begin{array}{cccccccc}
q & c & \alpha & \beta & \gamma & \delta & \nu & \eta \\
2 & \frac{1}{2} & -1 & \frac{1}{2} & 2 & 5 & \frac{1}{4} & 2 - \frac{4d}{8}
\end{array}
\]

An obvious first reaction is that there is a problem with the measure for integrating over the edge lengths in RC, as this is not prescribed at the outset. In general a local measure is used, one common choice being

\[
\prod_{ij} \frac{dl_{ij}^2}{r_{ij}^2}
\]

for various \(\sigma\). It has been pointed out that if one thinks of the lattice model as being a discretization of an exact diffeomorphism invariant theory then gauge fixing and the resulting Fadeev-Popov determinant are obligatory \[30\]. The resulting non-local action would be very difficult to simulate even in 2D and things get worse in higher dimensions. An alternative point of view is that exact diffeomorphism invariance is recovered in the continuum limit even when broken on the lattice, so we can get away with local measure \[31\].

There have been a series of very careful investigations by Janke and Holm of both pure gravity and gravity plus matter in 2D RC with local measures \[32\]. In the pure gravity case they looked at an \(R^2\) action

\[
S = \int d^2\sigma \sqrt{g} \left( \lambda + \frac{a}{4} R^2 \right), \quad \hat{A} = \frac{A}{a}
\]

which has two scaling regimes

\[
\hat{A} << 1
\]

\[
Z(A) \sim A^{\gamma_{st} - 3} \exp \left( -\frac{S_c}{\hat{A}} \right) \exp \left( -\lambda R A - \xi \hat{A} \right)
\]

giving \(\gamma_{st} = 2 - 2(1 - g)\), \(S_c = 16\pi^2(1 - g)^2\) and the standard 2D gravity regime

\[
\hat{A} >> 1
\]

\[
Z(A) \sim A^{\gamma_{st} - 3} \exp(-\lambda R A)
\]

where \(\gamma_{st} = 2 - 5/2(1 - g)\). They found that the case for the failure of RC was not proven within the limits of accuracy of the simulation, contrary to some earlier claims.

They also considered Ising matter with a \(dl/l\) measure and found that the exponents were definitely Onsager. It was found, for instance, from
the finite size scaling of the susceptibility maxima that
\[ \frac{\gamma}{\nu} = 1.745(6) \]
so we can say with some certainty that 2D RC fails to reproduce gravitational effects when matter is coupled in. Whether this is simply a failure of the scheme used in the coupling rather than the RC itself remains to be seen.

The discussion of the RC results leads on to the broader question of what constitutes the universality class of 2D gravity. There have been various results for Ising and Potts model simulations that have a bearing on this:

• [33] Ising spins living in a flat geometry but with fluctuating connectivity, in effect a mixture of DT and RC approaches, give KPZ/DDK exponents.
• [35] Ising spins on quenched ensembles of Poissonian random lattices give Onsager exponents.
• [36] Spins on quenched ensembles of 2D gravity graphs may give "quenched" KPZ/DDK exponents.
• [37] Squeezing 2D gravity (truncating the neighbour distribution in a triangulation to have just 5, 6, 7 neighbours) does not affect the KPZ/DDK values of the exponents.
• [38] Spins on any, not just planar, $\phi^3$ graphs (the duals of triangulations) give mean field exponents.

In the above list [33] appears to show that it is fluctuating connectivity that is important rather than curvature fluctuations in obtaining the KPZ/DDK exponents, whereas [34] suggests that even with a quenched ensemble of graphs, and hence no fluctuating connectivities, gravitational effects are still present. On the other hand, any collection of random graphs does not guarantee gravitational effects: Poissonian random graphs give flat space [35] exponents and generic $\phi^3$ graphs give mean field exponents [38].

Nonetheless, the vertex number distribution of 2D DT triangulations can be quite brutally truncated and the effects of gravity still observed [37]. Finally, the Onsager exponents seen in Regge calculus suggest that the fractal structure of the 2D DT lattices (apparently absent in RC, but still present in [36,37]) may be an important factor in determining the universality class. Exactly what determines the universality class of the 2D gravity exponents is thus still not clear.

We close this section with a quick word on plaquette surface simulations. Such models are closer to original Nambu-Goto idea in which the target space is discretized
\[ S = \int d^2\sigma \sqrt{\det |\partial_a \vec{X} \partial_b \vec{X}|} \]
so strictly speaking gravity has again disappeared from the picture. For a 2D surface embedded in 3D the general action $S = \beta_s Area + \beta_{\text{Intersections}} + \beta_{\text{bends}}$ may be written as an Ising like model
\[ S_{\text{Ising}} = J_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \sigma_i \sigma_j + J_3 \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l \]
where the Ising couplings are related directly to the surface couplings $J_1 = \frac{\beta_+ + \beta_{\text{c}}}{2}$, $J_2 = -\frac{\beta_+ - \beta_{\text{c}}}{2}$, $J_3 = \frac{\beta_+ - \beta_{\text{c}}}{4}$. Following this approach for the Savvidy gonihedric string action of [18] gives an Ising model with interesting properties, including an unusual semi-global symmetry and Onsager-like exponents [39]. It has been suggested very recently [40] that the gonihedric ideas that went into this string action may also be of relevance for 3D and 4D DT models, which we now move on to discuss.

4. HIGHER D

Some features of discretized 3D and 4D gravity in the DT approach appear rather similar to 2D. There is a baby universe fractal structure and it seems that various scaling distributions can be
defined in both the crumpled and branched polymer-like phases. One feature of the 4D model not shared by its 2D counterpart was pointed out in: The vertex order distribution in the crumpled phase is very strange. The distribution \( \rho(o) \) of the vertex orders \( o(v) \), display a rather singular behaviour.

In the branched polymer phase at \( \kappa_2 = 2.0 \) there is a smooth distribution, as one might have naively expected, but the crumpled phase at \( \kappa_2 = 0.0 \) has a detached “spike” at very large vertex order. Extensive simulations showed that this was not a finite size effect, it was not a thermalization problem and that it also appeared for other topologies.

These observations were elaborated on in where it was demonstrated numerically that the generic triangulation of the \( D \) sphere for \( D > 3 \) contains one singular \( D - 3 \) simplex (\( \kappa_2 = 0 \) for simplicity, but this is true throughout the crumpled phase). This singular structure is attached to an extensive fraction of the rest of the triangulation. The geometric picture is shown above right for 4D and 5D respectively.

In general one finds that the local volume associated with the singular \( D - 3 \) simplex grows as \( \sim V^{2/3} \) whereas the local volumes associated with the secondary simplices attached to this grow as \( \sim V \). Some counting arguments were presented in as to why this should be so. It is surprising, and perhaps rather disturbing, that such singular features are present in the theory, especially given that the observed phase transition in 4D is apparently tied up with their appearance and disappearance.

The nature of the 4D transition in DT has also been the focus of further work using the fractal blocking MCRG scheme. As in 2D it is the distribution of geodesic distances that is of interest, which in discretized form is

\[
< r > = \left\langle \frac{1}{N^2} \sum_{ij} r_{ij} \right\rangle
\]

In 4D, setting \( \kappa_2 = \kappa \) and \( N_4 = N \) for conciseness, one has to take account of the running of the coupling under the blocking move

\[
\delta r = r_N \delta \ln N + r_c \delta \kappa
\]

so with the physical volume \( N a^4 \) fixed, with \( a \) the lattice spacing, we have \( \delta \ln \frac{1}{a} = \frac{1}{4} \delta \ln N \). We can thus deduce \( \delta \kappa \) from measurements and extract the \( \beta \) function

\[
\beta(\kappa) \sim \frac{\delta \kappa}{\delta \ln \frac{1}{a}}
\]

The picture that emerged from doing this was of an ultraviolet fixed point, not an infrared one as had been suggested by some continuum treatments. One puzzling feature of the simulations in was that the slope of the \( \beta \) function at the fixed point did not appear to be consistent with
a second order transition, which was the generally accepted picture for DT in 4D. Further work looking at the scaling of cumulants such as

$$c_2(N_4) = \frac{1}{N_4} [< N_0^2 > - < N_0 >^2]$$

where $N_0 = \frac{N_4}{2} - N_4 + 2$ was carried out to check this \[14\]. The finite size scaling form for this is

$$c_2(N_4) \sim N_0^b f ((\kappa_2 - \kappa_0^2)N_0^4)$$

where $b = c = 1$ for a first order transition and take non-trivial values for a second order transition. The simulations found the first order values $b = c = 1$. Further evidence for the first order nature of the transition was found by deBakker \[13\] who saw time series flips characteristic of first order transitions, as well as a double peak histogram structure emerging on larger lattices.

Simulations discussed at this conference \[14\] show that the first order transition is not an artifact stemming from the inclusion of the $\gamma(N_4 - V)^2$ term in the action - relaxing the constraint to allow huge volume excursions has no effect, so we are left with the task of deciding whether the model should be patched up to give a second order transition, is still useful as it stands, or should be consigned to the bin.

The optimists argue that some things are right and in any case there are lattice theories with first order transitions and a Coulomb phase for an entire range of couplings. The power law correlations in the branched polymer phase might be argued to be just such a case \[16\]. The DT model also appears to display gravitational binding \[15\] as defining one and two particle connected propagators for scalar test particles shows \[17\].

If we insist on getting a continuous transition, various alternatives suggest themselves. Ray Renken presented evidence \[18\] using a generalization of the node decimation scheme to higher dimensions that suppressing vertices of high order with a measure term could weaken the 3D DT transition. A similar effect may also appear in 4D.

It might also be possible to use the ideas of RC stemming from the gonihedric string, where new integral invariants for simplicial complexes corresponding to higher derivative terms were derived. These would have the effect of “stiffening” the lattices and possibly changing the nature of the transition. One might, of course, lose the transition completely with either of these approaches.

If we are willing to overlook the problems experienced in 2D by Regge Calculus as an aberration of lower dimensional gravity alone it is interesting to ask what what Regge calculus says about the 4D transition. This is not such a leap of faith as it might appear at first sight - in 2D there is no classical action at all and it is the quantum effects that contain the dynamics of the theory, whereas in 4D there is a non-trivial classical dynamics. It is thus conceivable that a model like RC which starts out “close” to smooth manifolds might do a better job in 4D. The Vienna group \[19\] have looked at various models in 4D including an Ising-link model in which the edge lengths are restricted to two values $q_l = b_l(1 + \sigma_1)$ where $\epsilon$ is a small constant and $\sigma_1$ is an ising spin living on the link. They found a first order transition at positive $\beta$ (Newton coupling) as well as another second order transition at negative $\beta$.

If one couples in $SU(2)$ gauge matter \[19\]

$$S = \beta \sum_i A_i \alpha_i - \frac{\beta_2}{2} \sum_i W_i \text{Re}[\text{Tr}(1 - U_i)]$$

the gauge field string tension scaling at the negative $\beta$ transition is still what would be expected of a physical theory. The suggestion is then that somehow DT is looking at the “wrong” first order transition - the physical theory being the one that resides at the negative $\beta$ second order transition.

5. IN CLOSING

In summary: the 1D results shed an interesting light on polymer phases in higher dimensional models, 2D DT is in very good shape, 2D RC less so, in 4D given the recent results on the order of the transition the big question is whether either the DT or RC theories are telling us anything about Euclidean quantum gravity. There will no doubt be much of interest to report at Lattice97.
Lest the reader be too despondent about the implications of the 4D results I would like to finish with a couple of sentences on recent purely analytical work in Ashtekhar’s approach to quantum gravity that suggest a model not far removed from Regge Calculus or DT emerging from a completely different perspective. If we make $3+1$ split, and use the canonical variables

$$E_a^a K^b = \frac{1}{\sqrt{g}} K_{ab} E^{bi}$$

where $q^{ab}$ is the 3-metric, $E_a^a$ a dreibein and $K$ the extrinsic curvature, then a canonical transformation takes us to Ashkekhar’s “new” variables

$$(E_a^a, G^{-1} K^i) \rightarrow (A^i_a = G^{-1}(\gamma^i_a - iK^i_a), E_a^a)$$

where $G$ is the Newton coupling. The constraints, a major stumbling block in the canonical approach for many years, are greatly simplified with these variables at the expense of a complex connection $A$.

As in lattice gauge theory a loop representation exists and it was recently realized that a spin network basis resolves the Mandelstam constraints that had frustrated progress in this approach. The picture that one arrives at is, crudely, of a embedded trivalent graph with edges coloured by spins $j$ such that $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$. We can even have real connection $A$ if we can put up with more complicated constraints (i.e. Hamiltonian)\footnote{A non-zero volume actually requires higher than trivalent vertices, but these may be broken down into trivalent parts.}. Continuum spacetime emerges from coarse graining this lattice-like structure. To quote Ashtekar\footnote{\cite{Ashtekar}}, “The fundamental Planck scale excitations of the gravitational field are 1-dimensional and the corresponding geometry is distributional”, so perhaps the next presenter of this review will not be forced to include “on the lattice” in the title if space(time) is a lattice.

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