The ideal of Lesieur-Croisot elements of Jordan pairs

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ABSTRACT
We study the sets of elements of Jordan pairs whose local Jordan algebras are Lesieur-Croisot algebras, that is, classical orders in nondegenerate Jordan algebras with finite capacity. It is then proved that, if the Jordan pair is nondegenerate, the set of its Lesieur-Croisot elements is an ideal of the Jordan pair.

1. Introduction
Two obstacles to be overcome in order to embed a semiprime associative algebra into a semisimple artinian algebra are singularity and uniform dimension. Once we are able to ensure both nonsingularity and finite uniform dimension for a semiprime associative algebra, Goldie theory gives us the existence of its classical algebra of quotients. Nevertheless there is a number of conditions, equivalent to being nonsingular and having finite uniform dimension, that also imply that any semiprime associative algebra satisfying any of these equivalent conditions is an order in a semisimple artinian algebra. One of these equivalent conditions, more precisely, that given in terms of the essentiality of one-sided ideals of semiprime Goldie algebras as exactly those containing regular elements, becomes particularly outstanding in the extension of Goldie’s Theorems to Jordan algebras.

First Zelmanov [28, 29] for linear Jordan algebras and later Fernández López, García Rus and Montaner [8] for quadratic Jordan algebras considered the extension of Goldie theory to Jordan algebras. In [8], besides characterizing nodegenerate Jordan Goldie algebras as those having a nondegenerate artinian classical algebra of quotients, the authors translated to the Jordan algebras the above mentioned equivalent characterizations of semiprime associative Goldie algebras. However it turned out that the previously mentioned condition involving essential one-sided ideals did not hold for Jordan algebras. Indeed, nondegenerate Jordan algebras whose essential inner ideals can be characterized as those containing injective elements are not Goldie, since they cannot be always imbedded into nondegenerate artinian algebras, but into nondegenerate Jordan algebras having finite capacity. These algebras were named Lesieur-Croisot Jordan algebras by Montaner and Tocón in [25].

In [25] the authors studied not only Jordan algebras having the property of being Lesieur-Croisot, but also the set of elements of a nondegenerate Jordan algebra whose local algebras are Lesieur-Croisot, thus introducing what was called in that reference a local Lesieur-Croisot theory of Jordan algebras.

Recall that since they were introduced by Meyberg in [18], local algebras have been related to some of the most important problems arising in Jordan theory since, as a result of a series of local-to-global
inheritance procedures, local properties of local algebras can be carried back to the original Jordan algebra, pair or triple system. For instance, the study of Jordan systems with nonzero local PI-algebras has been a recurrent topic appearing in Jordan theory since Zelmanov’s work on pairs and triple systems.

In the associative pair setting Fernández López, García Rus, Gómez Lozano, and Siles Molina established in [6] an analogue of the classical Goldie theory for semiprime associative pairs, where orders are defined locally. The local approach to Goldie theory for linear Jordan algebras was initiated by Fernández López and García Rus in [4] for strongly prime algebras and extended to the nondegenerate case in [5]. Later Fernández López and García Rus together with Jaa introduced a notion of order in nondegenerate linear Jordan pairs with descending chain condition on principal inner ideals in [7].

Following this local approach, and taking into account [25], in this paper we study the set of elements of a Jordan pair which give rise to local algebras that are classical orders in nondegenerate Jordan algebras of finite capacity, i.e., local algebras which are Lesieur-Croisot algebras. The main result we prove in the present paper is that this set is an ideal when the Jordan pair is nongenerate.

The paper is organized as follows. After this introduction, we devote the first section to basic notions and preliminaries. Then, in Section 2, we include a series of results on the set of elements of a Jordan pair having essential annihilator. This set was proved to be an ideal of any nondegenerate Jordan pair by Fernández López, García Rus and Montaner. In this section besides formulating the above mentioned result, we also include some properties relating the singular ideal of a nondegenerate Jordan pair to that of any of its local algebras.

Section 3 is devoted to Lesieur Croisot Jordan algebras. These Jordan algebras were characterized by Fernández López, García Rus and Montaner in [8]. We also recall here the notion of Lesieur Croisot element as introduced by Montaner and Tocón in [25].

Sections 4 and 5 contain some results that will be instrumental later. More precisely, Section 4 focuses on associative pairs and their left and right singular ideals. Recall that associative pairs naturally arise as associative envelopes of special Jordan pairs, and therefore they are of great relevance in the study strongly prime Jordan pairs of hermitian type. We revisit some results introduced in [6] and [10] focusing on the relationship between the left or right singular ideal of a semiprime associative pair and the ones of its local algebras. We also consider the relationship between the sets of elements of a semiprime associative pair having finite left or right Goldie (or uniform) dimension and the annihilator of its left and right singular ideals. Then, as a first approach to study the set of Lesieur-Croisot elements of nondegenerate Jordan pairs we consider in Section 5 the set of PI-elements of strongly prime Jordan pairs of hermitian type. More precisely, we consider the relationship between the set of PI-elements of a strongly prime Jordan pair of hermitian type and those of any of its \( * \)-tight associative pair envelopes, thus extending to Jordan pairs a result previously given by Montaner in [19] for Jordan algebras.

Finally, in the sixth and last section, we address the proof that the set of LC-elements of a nondegenerate Jordan pair is an ideal. Here we follow the pattern given in [25], and after dealing with strongly prime Jordan pairs having non-zero PI-elements, we study the strongly prime PI-less case. Though the steps to follow are rather similar to those in [25], there are still some intermediate results on Jordan pairs that need to be proved since in [25] the authors made use of a series of facts proved [8] only for Jordan algebras. Finally, the extension from strongly prime to nondegenerate Jordan pairs follows the main guidelines considered in [25] in the algebra case.

2. Preliminaries

2.1. We will work with associative and Jordan systems (algebras and pairs) over a unital commutative ring of scalars \( \Phi \) which will be fixed throughout. We refer to [11, 17, 18] for notation, terminology and basic results. For us, Jordan system will mean a quadratic Jordan system. We will use identities for Jordan pairs proved in [11], which will be quoted with the labellings J\( \Phi \)n.

2.2. A Jordan algebra \( J \) has products \( U_{xy} \) and \( x^2 \), quadratic in \( x \) and linear in \( y \), whose linearizations are \( U_{x,y} = V_{x,y} = \{ x, y, z \} = U_{x+z,y} - U_{x,y} - U_{z,y} \) and \( x \circ y = V_{x,y} = (x + y)^2 - x^2 - y^2 \). A Jordan pair \( V = (V^+, V^-) \) has products \( Q_{x,y} \) for \( x \in V^\sigma, y \in V^{-\sigma}, \sigma = \pm \), with linearizations
Q_{x,z}y = D_{x,y}z = \{x,y,z\} = Q_{x+z}y - Q_xy - Q_zy. This introductory section mainly focuses on Jordan pairs. The reader is referred to [23, Section 0] for the corresponding basic preliminary results on Jordan algebras.

2.3. An associative pair is a pair \( A = (A^+, A^-) \) of \( \Phi \)-modules together with trilinear maps \( A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma, \sigma = \pm \), defined by \((x,y,z) \mapsto xyz\) and such that \((xyz)uv = x(yzu)v = xy(zuv)\) for all \( x, z, v \in A^+, y \in A^{-\sigma}, \sigma = \pm \). An (polarized) involution \(*\) in an associative pair \( A \) consists of two linear maps \(*: A^\sigma \to A^\sigma\) such that \((x^*)^* = x\) and \((xyz)^* = z^*y^*x^*\) for all \( x, z \in A^\sigma, y \in A^{-\sigma}, \sigma = \pm \). Associative pairs with polarized involutions were studied in [9, Section 3].

2.4. Similarly to the algebra case, where every associative algebra \( A \) gives rise to a Jordan algebra \( A^{(+)\sigma} \) by taking \( U_{xy} = xyy\) and \( x^2 = xx\) for \( x, y \in A\), any associative pair \( A = (A^+, A^-) \) provides a Jordan pair \( A^{(+)\sigma} \) taking \( Q_{xy} = xyy\) for all \( x \in A^\sigma, y \in A^{-\sigma}, \sigma = \pm \). A Jordan system (algebra or pair) is special if it is isomorphic to a Jordan sub-system of \( A^{(+)\sigma}\) for some associative system \( A \) and \( i\)-special if it satisfies all identities satisfied by all special Jordan systems.

2.5. If \( A \) is an associative algebra with involution \(*\), then \( H(A, * ) = \{ a \in A \mid \sigma a \ast = a \} \) is a Jordan subalgebra of \( A^{(+)\sigma}\) and so are ample subalgebras \( H_0(A, \ast) \) of symmetric elements of \( A \) i.e. subspaces of \( H(A, \ast) \) such that \( \{ a \} = a + a^\ast, a^a^\ast \) and \( a^a^\ast \) are in \( H_0(A, \ast) \) for all \( a \in A \) and all \( h \in H_0(A, \ast) \). If \( A = (A^+, A^-) \) is an associative pair with polarized involution \(*\), then \( H(A, \ast) = (H(A^+, \ast), H(A^-, \ast)) \), where \( H(A^\sigma, \ast) = \{ a \in A^\sigma \mid a^a^\ast = a \} \) is a Jordan subpair of \( A^{(+)\sigma}\). Ample subpairs \( H_0(A, \ast) \subseteq H(A, \ast) \) of symmetric elements containing the traces \( \{ a \} = a + a^\ast \) of elements of \( A^\sigma \) and satisfying \( a^a^\ast H_0(A^{-\sigma}, \ast) \subseteq H_0(A^\sigma, \ast) \) for all \( a \in A^\sigma, \sigma = \pm \) are also Jordan subpairs of \( A^{(+)\sigma}\). Ample subpairs of associative pairs with involution were studied in [9, Section 4].

2.6. A \( \Phi \)-submodule \( K \) of \( V^\sigma \) is an inner ideal of \( V \) if \( Q_k V^{-\sigma} \subseteq K \) for all \( k \in K \). A \( \Phi \)-submodule \( I = (I^+, \Gamma) \) is an ideal if \( Q_{V^\sigma} V^{-\sigma} + Q_{V^\sigma} \Gamma^{-\sigma} + \{ I^\sigma, V^{-\sigma}, V^\sigma \} \subseteq I^\sigma \) for \( \sigma = \pm \). An ideal of \( V \) is essential if it intersects nontrivially any nonzero ideal of \( V \). Essentiality of an inner ideal \( K \) of \( V^\sigma \) implies that it hits nontrivially any other nonzero inner ideal of \( V \), contained in \( V^\sigma \).

2.7. A Jordan pair \( V \) is nondegenerate if \( Q_k V^\sigma \neq 0 \) for any nonzero \( x \in V^{-\sigma}, \sigma = \pm \), and prime if \( Q_k L \neq 0 \) for any nonzero ideals \( I \) and \( L \) of \( V \). A Jordan pair is strongly prime if \( V \) is both nondegenerate and prime. A Jordan pair is semiprime if \( V \) has no nonzero trivial ideals i.e. if \( Q_k I \neq 0 \) for any nonzero ideal \( I \) of \( V \).

2.8. The product \( Q_k L \) of two ideals \( I \) and \( L \) of a Jordan pair \( V \) is not an ideal of \( V \), but a semi-ideal. Recall that a \( \Phi \)-submodule \( I = (I^+, \Gamma) \) of \( V \) is a semi-ideal if \( Q_k V + D_{V,y} I + Q_{V^\sigma} Q_{V^\sigma} \subseteq I \). The product \( Q_k L \) of semi-ideals is again a semi-ideal [15]. For any semi-ideal \( I \) we can form a chain of semi-ideals (called intrinsic derived spaces) given by \( I^{(0)} = I, I^{(1)} = Q_k I \) and \( I^{(n)} = Q_k I^{(n-1)} \), for all \( n \geq 1 \). On the other hand, if \( I \) and \( L \) are ideals of \( V \), then so is \( I \ast L = Q_k L + Q_{V^\sigma} Q_k L \) and, in particular, \( I^{(1)} = I \ast I \) the derived ideal of \( I \) and the nth derived ideals \( I^{(n)} = I^{(n-1)} \ast I^{(n-1)} \) for all \( n \geq 1 \).

We remark that given a semi-ideal \( I \) of \( V \), then \( I + Q_{V^\sigma} I \) is an ideal of \( V \) called the hull of \( I \) and denoted \( I^{(1)} \). Similarly we can consider \( I^{(n)} = I^{(n)} + Q_{V^\sigma} I^{(n)} \) which are ideals of \( V \) for any semi-ideal \( I \). For any ideal \( I \) of \( V \), it holds that \( I^{(n+1)} \subseteq I^{(n)} \subseteq I^{(n)} \subseteq I^{(n)} \) for any \( n \geq 0 \) [3, p. 210-211]. Moreover the following conditions are equivalent for any ideal \( I \) of \( V \) and any \( n \geq 1 \):

(i) \( V \) is semiprime, that is, \( I^{(1)} = 0 \) implies \( I = 0 \) for any ideal \( I \) of \( V \).
(ii) \( I^{(n)} = 0 \Rightarrow I = 0 \) for any ideal \( I \) of \( V \).
(iii) \( I^{(n)} = 0 \Rightarrow I = 0 \) for any ideal \( I \) of \( V \).
(iv) \( I^{(n)} = 0 \Rightarrow I = 0 \) for any ideal \( I \) of \( V \).
2.9. The annihilator of a subset $X \subseteq V^\sigma$ is the set $\text{ann}_V(X)$ of all $a \in V^{-\sigma}$ such that $Q_ax = Q_\sigma a = 0$, $Q,aQ_2V^{-\sigma} = D_{ax}V^{-\sigma} = 0$ and $Q_\sigma Q_aV^\sigma = D_{xa}V^\sigma = 0$ for all $x \in X$. Annihilators are always inner ideals. If $I = (I^+, I^-)$ is an ideal of $V$, then $\text{ann}_V(I) = (\text{ann}_V(I^-), \text{ann}_V(I^+))$ is again an ideal of $V$. If $V$ is nondegenerate, the annihilator of an ideal $I$ of $V$ is characterized as follows (see [15, Proposition 1.7(i)] and [20, Lemma 1.3]):

$$\text{ann}_V(I^\sigma) = \{ a \in V^{-\sigma} | Q_\sigma a = 0 \} = \{ a \in V^{-\sigma} | Q_\sigma a = 0 \}.$$

An ideal of a nondegenerate Jordan pair is essential if and only if it has zero annihilator.

2.10. Let $V$ be a special Jordan pair. An associative pair with a (polarized) involution $(A, \ast)$ is an associative $\ast$-envelope of $V$ if $A$ is generated by $V \subseteq H(A, \ast)$, and it is said to be $\ast$-tight over $V$ if any nonzero $\ast$-ideal $I$ of $A$ hits $V$ nontrivially: $I \cap V \neq 0$ for any nonzero $\ast$-ideal $I$ of $A$.

2.11. An $i$-special Jordan pair $V$ is of hermitian type if its hermitian parts are not annihilated, i.e., $\text{ann}_V(\bigcup_{\mathcal{H}} \mathcal{H}(V)) = \bigcap_{\mathcal{H}} \text{ann}_V(\mathcal{H}(V)) = 0$. Clearly a prime Jordan pair is of hermitian type if and only if $\mathcal{H}(V) \neq 0$ for some hermitian ideal $\mathcal{H}(X)$ of the free special Jordan pair $FSJP[X]$.

2.12. Given a Jordan pair $V = (V^+, V^-)$ and an element $a \in V^{-\sigma}$ the $\Phi$-module $V^\sigma$ becomes a Jordan algebra denoted by $(V^\sigma)^{(a)}$ and called the $a$-homotope of $V$ defining $U_x^{(a)}y = Q_\sigma Q_\sigma x\sigma y$ and $x^{(2\sigma)} = Q_\sigma a$ for all $x, y \in V^\sigma$. The set $\text{Ker}_V(a) = \text{Ker} a = \{ x \in V^\sigma | Q_\sigma x = Q_a Q_\sigma a = 0 \}$ is an ideal of $(V^\sigma)^{(a)}$ and the quotient $V^\sigma_a = (V^\sigma)^{(a)}/\text{Ker} a$ is a Jordan algebra called the local algebra of $V$ at $a$. If $V$ is nondegenerate, then $\text{Ker} a = \{ x \in V^\sigma | Q_\sigma x = 0 \}$.

2.13. The socle $\text{Soc}(V) = (\text{Soc}(V^+), \text{Soc}(V^-))$ of a Jordan pair was studied by Loos in [12]. The socle of a nondegenerate Jordan pair $V$ is the sum of all minimal inner ideals of $V$, it is a direct sum of simple ideals of $V$, consists of von Neumann regular elements (i.e. elements $x \in V^\sigma$ such that $x = Q_\sigma y$ for some $y \in V^{-\sigma}$), and satisfies the dcc (descending chain condition) on principal inner ideals [13]. The elements of the socle of a nondegenerate Jordan pair are exactly those whose local algebras have finite capacity [19, Lemma 0.7(b)], that is, Jordan algebras satisfying the acc (ascending chain condition) and dcc on principal inner ideals. A characterization of the elements of the socle of a nondegenerate Jordan pair in terms of the descending chain condition of principal inner ideals can be found in [12, Theorem 1]. Strongly prime Jordan pairs with nonzero socle were determined in [9].

2.14. An element $a$ of a Jordan pair $V$ is a PI-element if the local algebra of $V$ at $a$ is a PI-algebra. If $V$ is a Jordan pair, we write $\text{PI}(V) = (\text{PI}(V^+), \text{PI}(V^-))$ where $\text{PI}(V^\sigma)$ is the set of PI-elements of $V^\sigma$, $\sigma = \pm$. It was proved in [19, Theorem 5.4] that the set of PI-elements of a nondegenerate Jordan pair is an ideal. Jordan pairs with $\text{PI}(V) = 0$ are called PI-less Jordan pairs. PI-less strongly prime Jordan pairs are special of hermitian type.

3. The singular ideal of a Jordan pair

In this section we recall the definition of the singular set for Jordan pairs, which is analogous to that given in [8] for Jordan algebras. The contents of this section correspond to an unpublished work by the first author together with Fernández López and García Rus. The main result we provide here is that the set of elements of a Jordan pair having essential annihilator forms an ideal when the Jordan pair is nondegenerate.

3.1. The singular set of a Jordan pair $V$ is $\Theta(V) = (\Theta(V^+), \Theta(V^-))$, where

$$\Theta(V^\sigma) = \{ x \in V^\sigma | \text{ann}_V(x) \text{ is essential} \}.$$
A Jordan pair \( V \) will be called nonsingular if \( \Theta(V) = 0 \). If \( \Theta(V) = V \), then the Jordan pair \( V \) is called singular.

3.2. For all \( x, y \in V^\sigma \), \( \text{ann}_V(x) \cap \text{ann}_V(y) \subseteq \text{ann}_V(x + y) \) and \( \text{ann}_V(x) \subseteq \text{ann}_V(Q_x V^{-\sigma}) \). This implies that \( \Theta(V) \) consists of a pair of inner ideals of \( V \).

3.3 Proposition. Let \( V \) be a nondegenerate Jordan pair and \( b \in V^{-\sigma} \).

(i) \( (\Theta(V^\sigma) + \text{Ker } b)/\text{Ker } b \) is an inner ideal of \( V^\sigma_b \) contained in \( \Theta(V^\sigma_b) \).

(ii) If \( x \in \Theta(V^\sigma_b) \), then \( Q_b x \in \Theta(V^{-\sigma}) \).

Proof. (i) Let \( x \in \Theta(V^\sigma) \) and take \( K \) a nonzero inner ideal of \( V^\sigma_b \), where \( K \supseteq \text{Ker } b \) is an inner ideal of \( (V^\sigma)^{(b)} \) and \( \sigma \) denotes the projection of \( (V^\sigma)^{(b)} \) onto \( V^\sigma_b \). The proof of [8, Proposition 6.4] works here with minor changes to prove \( \text{ann}_V(x) \cap K = 0 \), which implies that \( x \in \Theta(V^\sigma_b) \).

(ii) Following [8, Theorem 6.1] and [25, Lemma 4.1] the proof consists of a series of successive steps leading to contradiction. Suppose there exists \( x \in \Theta(V^\sigma_b) \) such that \( Q_b x \notin \Theta(V^{-\sigma}) \). Then, there exists a nonzero inner ideal \( K \subseteq V^\sigma \) with \( K \cap \text{ann}_V(Q_b x) = 0 \).

We may also assume \( Q_b K = 0 \). Consider otherwise that \( Q_b K \neq 0 \). Then \( (K + \text{Ker } b)/\text{Ker } b \) is a nonzero inner ideal of \( V^\sigma_b \) and, since \( x \in \Theta(V^\sigma_b) \), there exists \( k \in K \) such that \( 0 \neq K \in \text{ann}_V(x) \).

Then, see 2.12, \( Q_b x k = Q_b U^{(b)}_x k = 0 \), and by JP2 it holds that \( D(Q_b x, Q_b x) = D(k, Q_b x) = 0 \). Similarly one obtains \( D(Q_b x, Q_b Q_b x) = 0 \). Next, again using \( K \in \text{ann}_V(x) \), we have \( Q_b Q_b x = Q_b U^{(b)}_x U^{(b)}_k = Q_b k = 0 \), and we obtain:

\[
Q_b Q_b x \in K \cap \text{ann}_V(Q_b x) = 0. \tag{1}
\]

By JP8, for any \( v \in V^\sigma \) we have:

\[
D(Q_b Q_b v, Q_b x) = D(k, \{Q_b v, k, Q_b x\}) - D(Q_b Q_b x, Q_b v) = D(k, \{Q_b v, k, Q_b x\}).
\]

Linearizing JP3 and denoting by \( (-,-,-)^{(b)} \) the triple product at the \( b \)-homotope algebra \( (V^\sigma)^{(b)} \), since \( K \in \text{ann}_V(x) \),

\[
\{Q_b v, k, Q_b x\} = Q_b \{v, k, Q_b x\} = Q_b \{v, k, x\}^{(b)} = 0,
\]

which implies that \( D(Q_b Q_b v, Q_b x) = 0 \). Analogously, using JP7, we prove \( D(Q_b x, Q_b Q_b v) = 0 \). It also holds \( Q_b Q_b Q_b v = Q_b U^{(b)}_x U^{(b)}_k Q_b = 0 \), and similarly \( Q_b Q_b Q_b x = 0 \). Since \( Q_b Q_b Q_b v = 0 \), this results into \( Q_b Q_b v \in \text{ann}_V(Q_b x) \cap K = 0 \). Finally, since \( V \) is nondegenerate, we obtain \( Q_b Q_b V^{-\sigma} = 0 \), and therefore replacing \( K \) by \( Q_b V^{-\sigma} \) we can assume \( Q_b K = 0 \).

Next we claim:

\[
\{k, b, x\} \neq 0 \text{ for every nonzero } k \in K. \tag{2}
\]

Assume on the contrary \( \{k, b, x\} = 0 \) for some \( 0 \neq k \in K \). Then, since \( Q_b K = 0 \), by JP7, we have \( D(k, Q_b x) = D(k, b, x) = 0 \) and using JP8, it follows that \( D(Q_b x, k) = D(b, k, b, x) = 0 \). Again using \( Q_b K = 0 \), we obtain \( Q_b Q_b Q_b k = Q_b Q_b Q_b k = 0 \), which implies, by the nondegeneracy of \( V \), that \( k \in K \cap \text{ann}_V(Q_b x) = 0 \), contradicting the choice of \( k \). This proves (2).

Take now \( 0 \neq k \in K \). Since \( V \) is nondegenerate, by (2), \( Q_b \{k, b, x\} u \neq 0 \) for some \( u \in V^{-\sigma} \). Then, by JP20, using that \( Q_b K = 0 \) and the nondegeneracy of \( V \),

\[
Q_{\{k, b, x\}} u = \{k, Q_b \{x, u, k\}, x\} - \{Q_b b, u, Q_b x\} + Q_b Q_b Q_b u + Q_b Q_b Q_b u = \{k, Q_b \{x, u, k\}, x\},
\]

which implies that \( Q_b \{x, u, k\} \neq 0 \), or equivalently, \( 0 \neq \{x, u, k\} \in V^\sigma_b \). Therefore, since \( x \in \Theta(V^\sigma_b) \), there exists \( t \in V^\sigma \) such that

\[
0 \neq U^{(b)}_{\{x, u, k\}} t \in U^{(b)}_{\{x, u, k\}} V^\sigma_b \cap \text{ann}_V(x). \tag{3}
\]
By JP21, we note that for any \( k \in K, v \in V^\sigma \) and \( w \in V^{-\sigma} \),
\[
Q_bQ_{\{x,w,k\}}Q_bv = Q_b\{x,w, Q_k\{Q_bv, x, w\}\} - Q_b\{Q_xQ_wk, Q_bv, k\} + \\
+ Q_bQ_xQ_yQ_kQ_bv + Q_bQ_kQ_wQ_xQ_bv.
\]
Since \( Q_b\{Q_xQ_wk, Q_bv, k\} = \{Q_bQ_xQ_wQ_k, v, Q_bk\} \), \( V \) is nondegenerate and \( Q_bK = 0 \), it follows that:
\[
Q_{Q_b\{x,w,k\}}v = Q_b\{x, w, Q_k\{Q_bv, x, w\}\}. \quad (4)
\]
Write now \( l = Q_k\{Q_b t, x, w\} \in K \), with \( t \in V^\sigma \) as in (3). By (4),
\[
Q_{b\{x,u,l\}} = Q_{b\{x,u, Q_k\{Q_b t, x, u\}\}} = Q_{Q_b\{x,u,k\}}t
\]
and we have
\[
0 \neq \overline{[v, x, u]} = U^{(b)}_{\{x,u,k\}} \in \text{ann}_{V_b}^\sigma (\overline{V}). \quad (5)
\]
Thus we have proven the following assertion:
There exists \( l \in K \) and \( u \in V^{-\sigma} \) such that \( 0 \neq \overline{[x, u, l]} \in \text{ann}_{V_b}^\sigma (\overline{V}). \quad (6)
\]
Let now \( k \in K \) be such that \( 0 \neq \overline{[x, u, k]} \in \text{ann}_{V_b}^\sigma (\overline{V}). \) Since \( V \) is nondegenerate, by (4), there exists \( t \in V^\sigma \) such that:
\[
Q_{Q_b\{x,u,k\}}t = Q_{b\{x, u, Q_k\{Q_b t, x, u\}\}} \neq 0. \quad (7)
\]
We claim that (7) leads to contradiction. By JP12 and JP7, it holds:
\[
D(Q_k\{Q_b x, x, u\}, Q_bx) - D(\{k, Q_bx, \{x, u, k\}\}, Q_bx) = \\
D(\{x, u, Q_kQ_bx\}, Q_bx) - D(\{x, u, k\}, Q_bx).
\]
Since \( V \) is nondegenerate and \( Q_bK = 0 \), we have \( Q_kQ_bz = 0 \) and \( Q_{Q_bk}k = Q_bQ_zk = 0 \). Moreover, since \( 0 \neq \overline{[x, u, k]} \in \text{ann}_{V_b}^\sigma (\overline{V}), \) it holds that \( Q_{Q_bx}Q_{Q_b\{x,u,k\}} = Q_bQ_xQ_{Q_b\{x,u,k\}} = 0 \), which finally implies:
\[
Q_{k\{Q_b x, x, u\}} \in \text{ann}_{V_b}^\sigma (\overline{V}) \cap K = 0. \quad (8)
\]
By JP12, \( Q_{k\{u, x, Q_b t\}} = \{k, Q_b t, \{x, u, k\}\} \) and then:
\[
D(Q_{k\{u, x, Q_b t\}}, Q_bx) = D(\{k, Q_b t, \{x, u, k\}\}, Q_bx) = \\
D(\{x, u, Q_kQ_b t\}, Q_bx) = \\
D(\{k, \{Q_b t, \{x, u, k\}, Q_bx\} = D(\{x, u, k\}, Q_bx) = \\
D(\{Q_b t, \{x, u, k\}, Q_bx\} + D(\{x, u, k\}, Q_bx).
\]
Now, using \( Q_bK = 0 \) and (8), by the nondegeneracy of \( V \) and taking into account that \( \overline{[x, u, k]} \in \text{ann}_{V_b}^\sigma (\overline{V}) \) implies
\[
\{Q_b t, \{x, u, k\}, Q_bx\} = Q_b\{t, \{x, u, k\}, x\}^{(b)} = 0,
\]
which together with \( D(\{Q_b x, Q_{k\{Q_b t, x, u\}} = 0 \) and \( Q_{Q_b x}Q_{Q_b\{Q_b t, x, u\}} = 0 \), imply that \( Q_k\{Q_b t, x, u\} \in \text{ann}_{V_b}^\sigma (\overline{V}) \cap K = 0 \), contradicting (7).

3.4 Theorem. Let \( V \) be a nondegenerate Jordan pair. Then \( \Theta(V) \) is an ideal of \( V \), called the singular ideal of \( V \).
Proof. By 3.2, \( \Theta(V) = (\Theta(V^+), \Theta(V^-)) \) consists of a pair of inner ideals, and it follows from Proposition 3.3 that \( Q_{V^\sigma} \Theta(V^{-\sigma}) \subseteq \Theta(V^\sigma) \).

We claim \( D(V^\sigma, V^{-\sigma}) \Theta(V^\sigma) \subseteq \Theta(V^\sigma) \). To prove this claim let \( x \in V^\sigma, y \in V^{-\sigma} \) and \( z \in \Theta(V^\sigma) \), and assume that there exists a nonzero inner ideal \( K \subseteq V^{-\sigma} \) such that \( K \cap \text{ann}_V(D(x, y)z) = 0 \). By the essentiality of \( \text{ann}_V(z) \), using a standard argument involving structural transformations [11], we can suppose that the inner ideals \( K, Q_yQ_xK \) and \( B(y, x)K \) are all contained in \( \text{ann}_V(z) \), which implies also that \( D(y, x)K = (\text{Id}_{V^{-\sigma}} + Q_yQ_x - B(y, x))K \subseteq \text{ann}_V(z) \). (Here \( B(\cdot, \cdot) \) denotes the Bergman operator.)

Let \( k \in K \). Taking into account that \( K, Q_yQ_xQ_k \) and \( D(y, x)K \) are all assumed to be contained into \( \text{ann}_V(z) \), by JP21,

\[
Q_{D(y, x)z}Q_k = Q_{(x, y)z}Q_k = Q_xQ_yQ_k + Q_zQ_yQ_k + D(y, x)Q_zD(y, x)Q_k - Q(z, Q_xQ_y)Q_k = 0,
\]

and by JP15, \( D(k, D(y, x)z) = D(D(y, x)k, z) + [D(k, z)D(y, x)] = 0 \). Similarly one obtains \( D(D(y, x)z, k) = 0 \), which results into \( K \subseteq \text{ann}_V(D(x, y)z) \), contradicting therefore that \( K \cap \text{ann}_V(D(x, y)z) = 0 \). Hence, it holds that \( D(V^\sigma, V^{-\sigma}) \Theta(V^\sigma) \subseteq \Theta(V^\sigma) \). \( \square \)

Next we bring together, for further use and citation, different results involving the singular ideal \( \Theta(V) \) of a nondegenerate Jordan pair \( V \).

### 3.5 Corollary.
Let \( V \) be a nondegenerate Jordan pair. Then:

(i) Every \( b \in V^{-\sigma} \) such that \( V^\sigma_b \) is nonsingular is contained in \( \text{ann}_V(\Theta(V^\sigma)) \).

(ii) \( V \) is nonsingular if and only if for every \( b \in V^{-\sigma} \) the local algebra \( V^\sigma_b \) of \( V \) at \( b \) is nonsingular.

(iii) If \( V \) is strongly prime and \( V^\sigma_b \) is nonsingular for a nonzero \( b \in V^{-\sigma} \), then \( V \) is nonsingular.

(iv) \( \Theta(V) \cap \pi(V) = 0 \). In particular, if \( V \) is strongly prime and \( \pi(V) \neq 0 \) then \( V \) is nonsingular.

(v) For any ideal \( I \) of \( V \), \( \Theta(V) \cap I = \Theta(I) \). Thus, if \( I \) is essential, \( V \) is nonsingular if and only if \( I \) is nonsingular.

(vi) \( \Theta(V) \) does not contain nonzero von Neumann regular elements. Thus, any nondegenerate Jordan pair with essential socle is nonsingular.

**Proof.** (i) It follows from Proposition 3.3.

(ii) If \( V \) is nonsingular, by Proposition 3.3, so are all its local algebras. Assume now that \( V^\sigma_b \) is nonsingular for all \( b \in V^{-\sigma} \). Then, by (i), \( V^{-\sigma} \subseteq \text{ann}_V(\Theta(V^\sigma)) \), which implies \( \Theta(V) = 0 \), by the nondegeneracy of \( V \).

(iii) Suppose \( V \) is strongly prime with a nonsingular local algebra \( V^\sigma_b \) for some element \( b \in V^{-\sigma} \). By (i), \( b \in \text{ann}_V(\Theta(V^\sigma)) \), and the result now follows from Theorem 3.4 (see 2.9).

(iv) Nondegenerate Jordan PI-algebras were proven to be nonsingular in [8, Proposition 6.6]. Thus, if \( V \) is nondegenerate, by (i), \( \pi(V^{-\sigma}) \subseteq \Theta(V^\sigma) \). If \( V \) is strongly prime with nonzero PI-elements, by (iii), \( \Theta(V) = 0 \).

(v)-(vi) The proofs of [8, Proposition 6.2] and [8, Proposition 6.5] work here with minor changes. \( \square \)

### 4. Lesieur-Croisot Jordan algebras

This section contains some of the main basic definitions and results relative to Goldie theory for quadratic Jordan algebras as well as the notion of Lesieur Croisot Jordan algebra. We refer the reader to [8] and [25, 26] for more accurate results on this theory.

**4.1.** Let \( \tilde{J} \) be a Jordan algebra, \( J \) be a subalgebra of \( \tilde{J} \), and \( \tilde{a} \in \tilde{J} \). We recall from [20] that an element \( x \in J \) is a \( J \)-denominator of \( \tilde{a} \) if the following multiplications take \( \tilde{a} \) back into \( J \):

\[
\begin{align*}
(D_i) & \ U_x\tilde{a} \\
(D_{ii}) & \ U_{\tilde{x}}x \\
(D_{iii}) & \ U_{\tilde{x}}U_{\tilde{a}}J \\
(D_{iii'}) & \ U_{\tilde{x}}U_{\tilde{a}}J \\
(D_{iv}) & \ V_{x,\tilde{a}}J \\
(D_{iv'}) & \ V_{\tilde{a},x}J,
\end{align*}
\]
where $\mathcal{T}$ denotes the unital hull $\mathcal{T} = \Phi 1 \oplus J$ of $J$. We will denote the set of $J$-denominators of $\tilde{a}$ by $D_J(\tilde{a})$. It has been proved in [20] that $D_J(\tilde{a})$ is an inner ideal of $J$.

4.2. A nonempty subset $S \subseteq J$ is a monad if $U_t$ and $s^2$ are in $S$ for all $s, t \in S$. A subalgebra $J$ of a unital Jordan algebra $Q$ is an $S$-order in $Q$ or equivalently $Q$ is an algebra of fractions (of $J$ relative to $S$) if:

(CLQ1) every element $s \in S$ is invertible in $Q$.
(CLQ2) each $q \in Q$ has a $J$-denominator in $S$.
(CLQ3) for all $s, t \in S$, $U_s S \cap U_t S \neq \emptyset$.

4.3. An element $s$ of a Jordan algebra $J$ is said to be injective if the mapping $U_s$ is injective over $J$. We will denote by $\text{Inj}(J)$ the set of injective elements of $J$ [8].

4.4. A Jordan algebra $Q$ containing $J$ as a subalgebra is a classical algebra of quotients of $J$, or an algebra of fractions of $J$ (and $J$ is a classical order in $Q$) if all injective elements of $J$ are invertible in $Q$ and $D_J(q) \cap \text{Inj}(J) \neq \emptyset$ for all $q \in Q$. That is, classical algebras of quotients are $S$-algebras of quotients (or algebras of fractions relative to $S$) for $S = \text{Inj}(J)$. Moreover, they are algebras of quotients in the sense of [21, 22] (see for instance [21, Examples 2.3.5]).

4.5. A nondegenerate Jordan algebra is Goldie if and only if it is nonsingular and has finite uniform dimension. Nondegenerate Goldie Jordan algebras were characterized in [8, Theorem 9.3]. We only recall here that although nondegenerate Goldie Jordan algebras satisfy the property that essential inner ideals contain injective elements, such Jordan algebras are not necessarily Goldie in contrast to the analogous assertion for associative rings. Indeed it suffices to consider any Jordan algebra of a quadratic form on a vector space with infinite dimensional totally isotropic subspaces.

4.6 Theorem. [8, Theorem 10.2] A Jordan algebra $J$ is a classical order in a nondegenerate unital Jordan algebra $Q$ with finite capacity if and only if it is nondegenerate and satisfies the property: an inner ideal $K$ of $J$ is essential if and only if $K$ contains an injective element. Moreover, $Q$ is simple if and only if $J$ is prime.

4.7. Jordan algebras satisfying the equivalent conditions of [8, Theorem 10.2] are named Lesieur-Croisot Jordan algebras, LC-algebras for short, in [25, Definition 3.2]. An element $a$ of a Jordan algebra $J$ is said to be a Lesieur-Croisot element (LC-element, for short) if the local algebra of $J$ at $a$ is a LC-algebra. The set $\text{LC}(J)$ of LC-elements of a Jordan algebra $J$ is an ideal if the Jordan algebra $J$ is nondegenerate [25, Theorem 5.13]. It follows straightforwardly from [8, Theorem 9.3] that nondegenerate Goldie Jordan algebras are LC-algebras. LC-Jordan algebras were studied by Montaner and Tocón in [25] (see also [26]).

5. The singular ideal of semiprime associative pairs

In this section we study the set of elements of a semiprime associative pair with essential left (or right) annihilator.

5.1. Let $A = (A^+, A^-)$ be an associative pair. Following [11] we denote by $(E, e)$ the standard imbedding of $A$, a unital associative algebra having an idempotent $e$ which produces a Peirce decomposition $E = E_{11} \oplus E_{12} \oplus E_{21} \oplus E_{22}$ where $E_{ij} = e_i E e_j$ with $e_1 = e$ and $e_2 = 1 - e$, such that $A = (E_{12}, E_{21})$. Recall that $E_{11}$ is spanned by the idempotent $e$ and all the elements $x_{12} y_{21}$ where $x_{12} \in E_{12}, y_{21} \in E_{21}$. Similarly $E_{22}$ is spanned by the element $1 - e$ and all the elements $y_{21} x_{12}$. We will denote by $A$ the subalgebra of $E$ generated by the elements $x_{12} \in E_{12}$ and $y_{21} \in E_{21}$. We will say that $A$ is the associative envelope of $A$. Note that $A$ is an essential ideal of $E$ and $A = (A_{12}, A_{21})$ where $A_{ij} = e_i A e_j, i, j = 1, 2$. 
5.2 Remark. Notation for the associative envelope $E$ and the standard imbedding $A$ of an associative pair $A$, as used here, follows that considered in [24]. We refer the reader to [24, Remark 2.1] for discussion about the different notation and terminology used for $E$ and $A$. See, for instance, [6, 9, 10, 14].

5.3. Given an associative pair $A = (A^+, A^-)$, we will say that a $\Phi$-submodule $L$ of $A^\sigma$ is a left ideal of $A$ if $A^\sigma A^- L \subseteq L$, $\sigma = \pm$. Right ideals are defined similarly. An ideal $I = (I^+, I^-)$ of $A$ is a pair of two-sided (i.e. left and right) ideals $I^\sigma \subseteq A^\sigma$, such that $A^\sigma I^- A^\sigma \subseteq I^\sigma$, $\sigma = \pm$.

5.4. A left ideal $L$ of $A$, contained in $A^\sigma$, is essential if it has nonzero intersection with any nonzero left ideal of $A$ contained in $A^\sigma$. Essential right ideals are defined similarly.

5.5. An associative pair $A$ is semiprime if and only if $I^\sigma A^- I^\sigma = 0$, $\sigma = \pm$, implies $I = 0$ for any ideal $I$ of $A$ and prime if and only if $I^\sigma A^- I^\sigma = 0$, $\sigma = \pm$, implies $I = 0$ or $J = 0$ for any ideals $I$ and $J$ of $A$.

An associative pair $A$ is (semi)prime if and only if so is its standard embedding $E$ if and only if so is its associative envelope $A [6, Proposition 4.2].$

5.6. Let $A = (A^+, A^-)$ be an associative pair. Any element $a \in A^-$, endows the $\Phi$-module $A^\sigma$ with a multiplication $x \cdot y = x ay$ for any $x, y \in A^\sigma$, and defines an associative $\Phi$-algebra $(A^\sigma)^{(a)}$ called the $a$-homotope algebra of $A$. The local algebra $A^\sigma_a$ of $A$ at $a$ is defined as the quotient of the $a$-homotope algebra $(A^\sigma)^{(a)}$ by the ideal $\text{Ker } a = \{x \in A^\sigma | axa = 0\}$. Moreover, for all $a \in A^-$, we have $A^\sigma_a \cong E_a \cong A^\sigma_a [9, Proposition 3.4(3)].$

5.7. An element $a \in A^-$ of an associative pair $A$ is a PI-element if the local algebra $A^\sigma_a$ of $A$ at $a$ satisfies a polynomial identity. We write $PI(A) = (PI(A^+), PI(A^-))$, where $PI(A^\sigma)$ is the set of PI-elements of $A^\sigma$. The set of PI-elements of any associative pair was proved to be an ideal in [19, Proposition 1.6]. Moreover, if $A$ is semiprime $PI(A) = PI(E) \cap A [24, Proposition 2.3].$

5.8. Let $A$ be an associative pair. For any subset $X \subseteq A^\sigma$, $\sigma = \pm$, the sets

\[ lann_A(X) = \{b \in A^- | bXA^- = A^\sigma bX = 0\} \]
\[ rann_A(X) = \{b \in A^- | XbA^- = A^- Xb = 0\} \]

are the left and right annihilators of $X$ in $A$ and are left and right ideals of $A$ respectively. If $A$ is semiprime, then

\[ lann_A(X) = \{b \in A^- | A^- bX = 0\} \]
\[ rann_A(X) = \{b \in A^- | A^- Xb = 0\} \]

If $X = \{x\}$ we will denote by $lann_A(x)$ and $rann_A(x)$ the left and right annihilators of the element $x$. The annihilator of $X$ is $ann_A(X) = lann_A(X) \cap rann_A(X)$. If $A$ is semiprime, for any ideal $I = (I^+, I^-)$ of $A$, $ann_A(I) = (ann_A(I^+), ann_A(I^-))$, where $ann_A(I^\sigma) = \{x \in A^- | xI^\sigma x = 0\}$, is an ideal of $A [6, Proposition 2.2].$

5.9. Let $A$ be a semiprime associative pair. Then $Z_i(A) = (Z_i(A^+), Z_i(A^-))$, where $Z_i(A^\sigma) = \{a \in A^\sigma | lann_A(a)$ is an essential left ideal of $A\}$, is an ideal of $A$ called the left singular ideal of $A [6, Theorem 3.1]$. Analogously we define $Z_i(A)$ to be the right singular ideal of $A$. We will say that $A$ is left nonsingular if $Z_i(A) = 0$. Right nonsingularity is defined similarly. An associative pair $A$ is said to be nonsingular if it is both left and right nonsingular.

5.10 Proposition. Let $A$ be a semiprime associative pair, and $a \in A^-$. Then:

(i) If $\overline{x} = x + \text{Ker } a \in Z_i(A^\sigma_a)$, then $axa \in Z_i(A^\sigma)$.
(ii) If $a \in Z_i(A^\sigma)$, then $Z_i(A^\sigma_a) = A^\sigma_a$.
(iii) $\overline{Z_i(A^\sigma)} = (Z_i(A^\sigma) + \text{Ker } a) / \text{Ker } a \subseteq Z_i(A^\sigma_a)$. Thus $Z_i(A^\sigma_a) = 0$ implies that $a \in ann_A(Z_i(A^\sigma))$. 
Proof. (i) and (ii) follow from [6, Proposition 5.3(i)-(ii)]. To prove (iii), take \( x \in Z_l(A^\sigma) \) with \( \bar{x} = x + \text{Ker} \sigma \neq 0 \), so that \( axa \neq 0 \) and let \( \bar{L} \) be a nonzero left ideal of \( A^\sigma_a \), where \(-\) denotes the projection of \((A^\sigma_a)^{(-)}\) onto \( A^\sigma_a \). We can assume that \( L \) is a nonzero left ideal of \( A \), contained in \( A^\sigma \) (see the proof of [8, Lemma 6.3]). Moreover, as \( Z_l(A) \) is an ideal of \( A \), and therefore \( axa \in Z_l(A) \), by essentiality of \( \text{ann}_A(axa) \) in \( A \), we can assume that \( L \subseteq \text{ann}_A(axa) \).

Since \( \bar{L} \neq 0 \), we have \( aL \neq 0 \) and then, by semiprimeness of \( A \), \( aLa^\sigma aL \neq 0 \). Hence \( 0 \neq La^\sigma a \subseteq \text{ann}_A(axa) \). Take now \( l_1, l_2 \in L, r \in A^\sigma \) with \( a_l l_1 a_l l_2 a \neq 0 \). Then \( 0 \neq a_l l_1 a_l l_2 a \subseteq L \cap \text{ann}_A(axa) \). Therefore we have \( a_l l_1 a_l l_2 a x = 0 \) which implies that \( l_1 \cdot \bar{r} \cdot l_2 \cdot \bar{x} = 0 \), hence \( l_1 \cdot \bar{r} \cdot l_2 \in \bar{L} \cap \text{ann}_A(\bar{x}) = \bar{L} \cap \text{ann}_A(axa) \).

Finally, if \( Z_l(A^\sigma_a) = 0 \), then we have \( aZ_l(A^\sigma)a = 0 \) and therefore, by [6, Proposition 2.2(ii)], \( a \in \text{ann}_A(Z_l(A^\sigma)) \).

5.11. The left Goldie (or uniform) dimension of a left ideal \( L \) of an associative pair \( A \) is a nonnegative integer \( n \) such that \( L \) contains a direct sum of \( n \) nonzero left ideals and any direct sum of nonzero left ideals contained in \( L \) has at most \( n \) nonzero left ideals. The left Goldie (or uniform) dimension of an element \( a \in A^\sigma \) is the left Goldie dimension of the principal left ideal \( A^\sigma_\langle a \rangle \) generated by \( a \). A left ideal \( L \) of a semiprime associative pair \( A \), contained in \( A^\sigma \), has finite left Goldie (or uniform) dimension if it contains no infinite direct sums of nonzero left ideals of \( A \), contained in \( A^\sigma \). If any element of \( A \) has finite left Goldie (or uniform) dimension, then it is said that \( A \) has finite left local Goldie (or uniform) dimension. Following [10] we write \( I_l(A) = (I_l(A^+), I_l(A^-)) \), where \( I_l(A^\sigma) \) denotes the set of elements of \( A^\sigma \) having finite left Goldie (or uniform) dimension. Similarly, we denote the set of elements of \( A \) with finite right Goldie (or uniform) dimension as \( I_r(A) \).

5.12 Proposition. Let \( A \) be a semiprime associative pair. If \( Z_l(A) \cap Z_r(A) = 0 \), then \( I_l(A) \cap I_r(A) \) is an ideal of \( A \) contained in \( \text{ann}_A(Z_l(A) + Z_r(A)) \).

Proof. The proof of [25, Proposition 4.6] can be used here with only slight modifications for adapting it to the pair setting. Indeed, it suffices to replace references for associative algebras [25, Lemma 4.5(i)-(ii)] by Proposition 5.10(i)-(iii), and use [6, Proposition 3.2] and [10, Proposition 3.3(iv)].

6. PI-elements of special Jordan pairs

PI-elements play an important role in the local LC-theory of Jordan algebras developed in [25]. To tackle the local LC-theory for Jordan pairs, we study in this section the relationship between the PI-ideal of a strongly prime Jordan pair of hermitian type and that of its \(*\)-tight associative pair envelope, thus extending to Jordan pairs results originally given for special Jordan algebras in [19, Theorem 6.5].

6.1 Lemma. Let \( V \) be a special Jordan pair, subpair of \( H(A, *) \) for an associative pair \( A \) with involution \(*\), and let \( H(X) \) be a hermitian ideal of the free special Jordan pair \( FSJP[X] \). Then:

(i) for any ideal \( I \) of \( V \) the \( n \)-tads

\[ \{ \ldots, H(V)^{\sigma}, I^{-\sigma}, (H(V)^{\langle 2 \rangle})^{-\sigma}, (H(V)^{\langle 2 \rangle})^{-\sigma}, H(V)^{\sigma}, \ldots \} \]

are in \( I \) for all odd \( n \).

(ii) if \( I = H(V)^{\langle 2 \rangle} \), for any \( a \in V^\sigma \) the \( n \)-tads \( \{ I^{\sigma}, \ldots, I^{-\sigma}, a \} \) and \( \{ I^{-\sigma}, \ldots, I^{-\sigma}, a, I^{-\sigma} \} \) are in \( I \) for all odd \( n \).

Proof. Take \( s \in (H(V)^{\langle 1 \rangle})^{\sigma}, t \in (H(V)^{\langle 1 \rangle})^{-\sigma}, r \in (H(V)^{\langle 2 \rangle})^{-\sigma} \) and variables \( y \notin X^{-\sigma}, z \notin X^\sigma \) (where \( X \) is considered to be a polarized set of variables). Note that, since \( H(V)^{\langle 2 \rangle} \subseteq H(V)^{\langle 1 \rangle} \subseteq H(V)^{\langle 1 \rangle} \), we can write \( r = Q_{s\bar{t}} \in H(V)^{\langle 1 \rangle} = Q_{H(V)}H(V) \). Now:

\[ \{ \ldots, H(X)^{\sigma}, y, Q_{s\bar{t}}, (H(V)^{\langle 2 \rangle})^{-\sigma}, \ldots \} = \]
and

\[
\{ \ldots, \mathcal{H}(X)\sigma, y, Q_z Q_t s, r, \mathcal{H}(X)\sigma, \ldots \} = \\
= \{ \ldots, \mathcal{H}(X)\sigma, y, z, Q_t s, z, Q_p q, \mathcal{H}(X)\sigma, \ldots \} - \\
- \{ \ldots, \mathcal{H}(X)\sigma, Q_t s, Q_z y, Q_p q, \mathcal{H}(X)\sigma, \ldots \} = \\
= \{ \ldots, \mathcal{H}(X)\sigma, y, z, Q_t s, \{ z, p, q \}, p, \mathcal{H}(X)\sigma, \ldots \} - \\
- \{ \ldots, \mathcal{H}(X)\sigma, y, z, Q_t s, q, Q_p z, \mathcal{H}(X)\sigma, \ldots \} - \\
- \{ \ldots, \mathcal{H}(X)\sigma, Q_t s, \{ z, y, p, q \}, p, \mathcal{H}(X)\sigma, \ldots \} + \\
+ \{ \ldots, \mathcal{H}(X)\sigma, Q_t s, q, Q_p Q_z y, \mathcal{H}(X)\sigma, \ldots \}
\]

are sums of $n$-tads ($n$ odd) elements of $\mathcal{H}(X \cup \{ y, z \})$, so they are Jordan polynomials in $FSJP(X \cup \{ y, z \})$, and the proof of [1, Lemma 2.1] applies here yielding (i).

Finally, we note that if $n < 5$, then (ii) is trivial, since $I$ is an ideal of $V$, and for $n \geq 5$ it follows from (i) above. 

The next result states that local algebras of ample subpairs of associative pairs with involution retain amleness in the set of symmetric elements of the local algebra with the induced involution.

**6.2 Corollary.** Let $V = H_0(A, *)$ be an ample subpair of an associative pair $A$ with an involution $\ast$. Then for any $a \in V^{-\sigma}$ the local algebra $V_a^\sigma$ is an ample subspace of $H(A_a^\sigma, *)$, $\sigma = \pm$.

**Proof.** The proof is straightforward from Lemma 6.1 (see [1, Lemma 2.3] for the algebra case). 

The proof of the next result follows that of [19, Theorem 6.5]. Here however we need to tackle the fact that, as noted in 2.8, products of ideals of Jordan pairs are not again ideals, merely semi-ideals. We will use Lemma 6.1 to overcome this additional difficulty on the generation of powers of ideals.

**6.3 Theorem.** Let $V$ be a strongly prime Jordan pair with a $\ast$-tight associative envelope $A$. If $V$ is of hermitian type, then $PI(V) = PI(A) \cap V$.

**Proof.** The containment $PI(A) \cap V \subseteq PI(V)$ was proved in [19, Lemma 6.4(b)] for special Jordan systems (including Jordan pairs).

To prove the reverse inclusion take $a \in PI(V^\sigma)$ and suppose $\mathcal{H}(V) \neq 0$ for a hermitian ideal $\mathcal{H}(X)$ of $FSJP[X]$. Set $I = \mathcal{H}(V)^{(2)}$ (see 2.8) and denote by $S$ the Jordan subpair of $V$ generated by $I$ and the element $a$, by $R$ the associative subpair of $A$ generated by $S$ and by $B$ the subpair of $R$ generated by $I$. Then $S$ is strongly prime, since it contains the nonzero ideal $I$ of $V$ (the proof of [1, Lemma 2.4] applies here), and it follows from Lemma 6.1 that $S = H_0(R, \ast)$ and $I = H_0(B, \ast)$. Moreover, by [3, Lemma 2.31], and as a result of the correspondence between Jordan pairs and polarized triple systems (see [3, Section 5] for further details), we have that $B$ is $\ast$-tight over $I$.

Now by [16, Jordan Memory Theorem 4.2] we have $I = H_0(B, \ast) \triangleleft V \subseteq H(Q_s(B), \ast)$, where $Q_s(B)$ is the pair of Martindale symmetric quotients of $B$. Moreover we have $B \subseteq R \subseteq Q_s(B)$. Hence $R$ is semiprime by [10, Lemma 2.13(ii)].

Next, since $a \in PI(V)$, the local algebra $S_a^{\ast\sigma}$ which is a subalgebra of the PI-algebra $V_a^{-\sigma}$, is itself a Jordan PI-algebra, and since by Corollary 6.2 we have $S_a^{\ast\sigma} = H_0(R_a^{-\sigma}, \ast)$, it follows from [27, Corollary 7.4.13] that $R_a^{-\sigma}$ is an associative PI-algebra.

Consider now the standard imbedding $\mathcal{E}_R$ of the associative pair $R$ (see 5.1). Since $(\mathcal{E}_R)_a \cong R_a^{-\sigma}$ by [9, Proposition 3.4(3)], we have that the associative algebra $\mathcal{E}_R$ satisfies a GPI. (Note also that $a \in PI(R) =$
7. Lesieur-Croisot elements of nondegenerate Jordan pairs

This section is devoted to study the set of LC-elements of a Jordan pair, concretely to prove that this set is an ideal if \( V \) is nondegenerate. We first focus on the strongly prime case, since it requires new intermediate results for Jordan pairs. The nondegenerate case, as it closely follows [25] for Jordan algebras, is just outlined. Yet, all necessary references relative to (associative and Jordan) pairs are however provided.

7.1. Let \( V \) be a Jordan pair. We will say that an element \( x \in V^\sigma, \sigma = \pm \) is a Lesieur Croisot element of \( V \) (and LC-element for short) if the local algebra \( V_x^{-\sigma} \) of \( V \) at \( x \) is a Lesieur-Croisot Jordan algebra (see 4.7). We will denote by \( LC(V) = (LC(V^+), LC(V^-)) \) the set of LC-elements of a Jordan pair \( V \).

We start stating that the set of LC-elements of any strongly prime Jordan pair with nonzero PI-elements coincides with its PI-ideal.

7.2 Proposition. Let \( V \) be a strongly prime Jordan pair having nonzero PI-elements. Then \( LC(V) = PI(V) \), and in particular, \( LC(V) \) is an ideal of \( V \).

Proof. The proof of [25, Proposition 3.3] works here.

7.3 Proposition. Let \( V \) be a strongly prime Jordan pair such that \( PI(V) = 0 \) and \( LC(V) \neq 0 \). Then \( V \) is nonsingular and \( LC(V) = F(V) \), where \( F(V) = (F(V^+), F(V^-)) \) with \( F(V^\sigma) = \{ x \in V^\sigma | u\dim(V_x^{-\sigma}) < \infty \} \).

Proof. Following the proof of [25, Proposition 4.2], it suffices to replace [8, Proposition 6.4] by Proposition 3.3(i) and Corollary 3.5(i), and recall that local algebras of nonsingular Jordan pairs inherit nonsingularity by Corollary 3.5(ii).

Next we prove a pair analogue of a result contained into the proof of [19, Theorem 6.5] (see also [25, Lemma 4.3]). To prove the following lemma we make use of a result from [2] on the \(*\)-tightness of any \(*\)-prime associative system over its nonzero ample subsystems together to some other intermediate results appearing in the proof of Theorem 6.3.

7.4 Lemma. Let \( V \) be a strongly prime Jordan pair such that \( PI(V) = 0 \) and let the associative pair \( A \) be a \(*\)-tight associative envelope of \( V \). Then for each \( a \in V \), the subpair \( S \) of \( V \) generated by the ideal \( I = \mathcal{H}(V)^{(2)} \), where \( \mathcal{H}(V) \neq 0 \) for some nonzero hermitian ideal \( \mathcal{H}(X) \) and the element \( a \) is strongly prime of hermitian type. Moreover \( S = H_0(R, *) \), where \( R \), the associative subpair of \( A \) generated by \( I \cup \{ a \} \), is a \(*\)-tight associative envelope of \( S \).

Proof. It follows from the proof of Theorem 6.3 that \( S \) is a strongly prime Jordan pair and an ample subpair \( S = H_0(R, *) \) of \( R \), where \( R \) denotes the associative subpair of \( A \) generated by \( I \) and the element \( a \). Now, since \( R \) is semiprime (indeed, it is \(*\)-prime), the \(*\)-tightness of \( R \) over \( S \) follows from [2, Proposition 3.1].

7.5 Theorem. Let \( V \) be a strongly prime Jordan pair with \( PI(V) = 0 \) and let the associative pair \( A \) be a \(*\)-tight associative envelope of \( V \). Then we have \( F(V) = I_1(A) \cap V = I_1(A) \cap V \), where \( I_1(A) = (I_1(A^+), I_1(A^-)) \) with \( I_1(A^\sigma) = \{ a \in A^\sigma | u\dim(A_a^{-\sigma}) < \infty \} \) and \( I_1(A) = (I_1(A^+), I_1(A^-)) \) with \( I_1(A^\sigma) = \{ a \in A^\sigma | u\dim(A_a^{-\sigma}) < \infty \} \). Moreover if \( A \) is prime, then \( u\dim(V_a^{-\sigma}) = u\dim(A_a^{-\sigma}) = u\dim(A_a^{-\sigma}) \) for all \( a \in F(V^\sigma), \sigma = \pm \).
**Proof.** Let \( a \in V^\sigma \). We first note that it suffices to prove the left version of the claimed statement as then the right one will follow symmetrically, that is, it suffices to prove that the local algebra \( V_a^{-\sigma} \) of \( V \) at \( a \) has finite uniform dimension if and only if so has \( A_a^{-\sigma} \), and that both uniform dimensions coincide if \( A \) is prime. It is straightforward to check that the arguments given in the proof of [25, Theorem 4.4] work here applied now to Jordan pairs. Indeed, it suffices to replace [25, Lemma 4.3] by Lemma 7.4 and recall that the set \( PI(A) \) of PI-elements of an associative pair \( A \) is an ideal of \( A \) by [19, Proposition 1.6]. Finally we recall the relationship between the set of PI-elements of any strongly prime Jordan pair of hermitian type and those of any of its \(*\)-tight associative envelopes given in Proposition 6.3 and also that between local algebras of associative pairs and those of their standard imbeddings (see 5.7). \( \square \)

**7.6 Proposition.** Let \( V \) be a strongly prime PI-less Jordan pair and let the associative pair \( A \) be a \(*\)-tight associative envelope of \( V \). If \( V \) is nonsingular, then \( Z_l(A) \cap V = 0 \).

**Proof.** We first note that \( V \) is special of hermitian type since it is PI-less. Take a hermitian ideal \( H(V) \neq 0 \), and let \( R \) be the associative subpair of \( A \) generated by \( H(V) \). Then \( H(V) = H_0(R, \ast) \) is an ample subpair of \( H(R, \ast) \) and \( R \) is \(*\)-tight over \( H(V) \) [3, Lemma 2.32]. Therefore \( R \) is \(*\)-prime. Moreover since \( R \subseteq A \subseteq Q_s(R) \), where \( Q_s(R) \) denotes the Martindale pair of symmetric quotients of \( R \) we have \( Z_l(A) \cap R = Z_l(R) \) [10, Proposition 2.14(iii)]. Thus by Corollary 3.5(v) the ideal \( H(V) \) is nonsingular, we may replace \( V \) by \( H(V) \) and assume that \( V = H_0(A, \ast) \) is an ample subpair of \( A \).

Take now \( a \in Z_l(A) \cap V^\sigma \). By Corollary 6.2 we have \( V_a^{-\sigma} = H_0(A_a^{-\sigma}, \ast) \) and, since \( A_a^{-\sigma} \) is \(*\)-prime, it follows from [2, Theorem 1.2] that \( A_a^{-\sigma} \) is \(*\)-tight over \( V_a^{-\sigma} \). Therefore, by [8, Theorem 6.14], \( Z_l(A_a^{-\sigma}) \cap V_a^{-\sigma} = \Theta(V_a^{-\sigma}) \), where \( Z_l(A_a^{-\sigma}) = A_a^{-\sigma} \) by Proposition 5.10(ii). Hence we have \( \Theta(V_a^{-\sigma}) = V_a^{-\sigma} \) which implies \( Q_aV_a^{-\sigma} \subseteq \Theta(V) = 0 \) by Proposition 3.3(ii), and we finally obtain \( a = 0 \) by nondegeneracy of \( V \). \( \square \)

**7.7 Theorem.** Let \( V \) be a strongly prime Jordan pair. Then \( LC(V) \) is an ideal of \( V \).

**Proof.** Obviously we can assume that \( LC(V) \neq 0 \). If \( PI(V) \neq 0 \) it was already proved in Proposition 7.2 that \( LC(V) \) is an ideal of \( V \), we can therefore suppose that \( V \) is PI-less. Let \( A \) be a \(*\)-tight associative envelope of \( V \), we have that \( V \) is nonsingular, and that \( LC(V) = F(V) \) by Proposition 7.3. Moreover \( (Z_l(A) \cap Z_r(A)) \cap V = 0 \) by Proposition 7.6. Thus, since \( Z_l(A) \cap Z_r(A) = Z_l(A) \cap Z_l(A)^* \) is a \(*\)-ideal of \( A \), we get \( Z_l(A) \cap Z_r(A) = 0 \) by \(*\)-tightness of \( A \) over \( V \). Now, by Proposition 5.12, \( I_l(A) \cap I_r(A) = I_l(A) \cap I_l(A)^* \) is a \(*\)-ideal of \( A \) contained in \( ann_A(Z_l(A) + Z_r(A)) \). (The proof that \( I_l(A) \cap I_r(A) = I_l(A) \cap I_l(A)^* \) is an \(*\)-ideal of \( A \) works as in [25, Proposition 4.6].) Hence \( LC(V) \) is an ideal of \( V \) by Theorem 7.5. \( \square \)

Finally, to extend the above result to nondegenerate Jordan pairs it suffices to restate adapted to pairs the results of [25, Section 5] to prove that the set of LC-elements of a nondegenerate Jordan algebra is an ideal.

**7.8 Theorem.** Let \( V \) be a nondegenerate Jordan pair. Then the set \( LC(V) = (LC(V^+), LC(V^-)) \) where \( LC(V^\sigma) = \{ a \in V^\sigma \mid V_a^{-\sigma} \text{ is an LC-algebra} \} \) is an ideal of \( V \).

**Proof.** The proof of [25, Theorem 5.13] works here to reduce the nondegenerate case to that of strongly prime Jordan pairs, which in turn follows from Theorem 7.7. \( \square \)

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**References**

[1] Anquela, J. A., Cortés, T., Montaner, F. (1995). Local inheritance in Jordan algebras. *Arch. Math.* 64:393–401.
[2] Anquela, J. A., Cortés, T., McCrimmon, K., Montaner, F. (1997). Strong primeness of Hermitian Jordan systems. *J. Algebra* 198:311–326.
[3] D'Amour, A. (1992). Quadratic Jordan systems of Hermitian type. *J. Algebra* 149:197–233.
[4] Fernández López, A., García Rus, E. (1995). Prime Jordan algebras satisfying local Goldie conditions. *J. Algebra* 174:1024–1048.
[5] Fernández López, A., García Rus, E. (1996). Nondegenerate Jordan algebras satisfying local Goldie conditions. *J. Algebra* 182:52–59.
[6] Fernández López, A., García Rus, E., Gómez Lozano, M., Siles Molina, M. (1998). Goldie theorems for associative pairs. *Commun. Algebra* 26(9):2987–3020.
[7] Fernández López, A., García Rus, E., Jaa, O. (1998). Towards a Goldie theory for Jordan pairs. *Manuscripta Math.* 95:79–90.
[8] Fernández López, A., Garcia Rus, E., Montaner, F. (2002). Goldie Theory for Jordan algebras. *J. Algebra* 248:397–471.
[9] Fernández López, A., Tocón, M. I. (2003). Strongly prime Jordan pairs with nonzero socle. *Manuscripta Math.* 111:321–340.
[10] Gómez Lozano, M., Siles Molina, M. (2004). Left quotient associative pairs and Morita invariant properties. *Commun. Algebra* 32(7):2841–2862.
[11] Loos, O. (1975). *Jordan Pairs*. Lecture Notes in Mathematics, Vol. 460. New York: Springer.
[12] Loos, O. (1989). On the socle of a Jordan pair. *Collect. Math.* 40:109–125.
[13] Loos, O. (1991). Finiteness conditions in Jordan pairs. *Math. Z.* 206:577–587.
[14] Loos, O. (1995). Elementary groups and stability for Jordan pairs. *K. Theory* 9:77–116.
[15] McCrimmon, K. (1984). Strong prime inheritance in Jordan systems. *Algebras Groups Geom.* 1:217–234.
[16] McCrimmon, K. (1998). Martindale systems of symmetric quotients. *Algebras Groups Geom.* 6:153–237.
[17] McCrimmon, K., Zelmanov, E. (1988). The structure of strongly prime quadratic Jordan algebras. *Adv. Math.* 69:133–222.
[18] Meyberg, K. (1972). *Lectures on Jordan Algebras and Triple Systems*. Lecture Notes. Charlottesville: University of Virginia.
[19] Montaner, F. (1999). Local PI theory of Jordan systems. *J. Algebra* 216:302–327.
[20] Montaner, F. (2001). Local PI theory of Jordan systems II. *J. Algebra* 241:473–514.
[21] Montaner, F. (2010). Algebras of quotients of Jordan algebras. *J. Algebra* 323:2638–2670.
[22] Montaner, F., Paniello, I. (2007). Algebras of quotients of nonsingular Jordan algebras. *J. Algebra* 312(2):963–984.
[23] Montaner, F., Paniello, I. (2017). Local orders in Jordan algebras. *J. Algebra* 485:45–76.
[24] Montaner, F., Paniello, I. (2021). PI theory for associative pairs. *Linear Multilinear Algebra* 69(14):2586–2606.
[25] Montaner, F., Tocón, M. (2006). Local Lesieur-Croisot theory of Jordan algebras. *J. Algebra* 301:256–273.
[26] Montaner, F., Tocón, M. I. (2009). The ideal of Lesieur-Croisot elements of a Jordan algebra. II. In: *Algebras, Representations and Applications*, Contemporary Mathematics, Vol. 483. Providence, RI: American Mathematical Society, pp. 199–203.
[27] Rowen, L. H. (1980). *Polynomial Identities in Ring Theory*. New York: Academic Press.
[28] Zelmanov, E. (1987). Goldie’s theorem for Jordan algebras. *Siberian Math. J.* 28:44–52.
[29] Zelmanov, E. (1988). Goldie’s theorem for Jordan algebras II. *Siberian Math. J.* 29:68–74.