ESTIMATES FOR THE DERIVATIVE OF DIFFUSION SEMIGROUPS

L.A. RINCON
Department of Mathematics
University of Wales Swansea
Singleton Park
Swansea SA2 8PP, UK
e-mail: L.Rincon@swansea.ac.uk

submitted April 15, 1998; revised August 18, 1998

AMS 1991 Subject classification: 47D07, 60J55, 60H10.
Keywords and phrases: Diffusion Semigroups, Diffusion Processes, Stochastic Differential Equations.

Abstract
Let \( \{P_t\}_{t \geq 0} \) be the transition semigroup of a diffusion process. It is known that \( P_t \) sends continuous functions into differentiable functions so we can write \( DP_t f \). But what happens with this derivative when \( t \rightarrow 0 \) and \( P_0 f = f \) is only continuous \( ? \). We give estimates for the supremum norm of the Fréchet derivative of the semigroups associated with the operators \( A + V \) and \( A + Z \cdot \nabla \) where \( A \) is the generator of a diffusion process, \( V \) is a potential and \( Z \) is a vector field.

1 Introduction

Consider the following stochastic differential equation on \( \mathbb{R}^n \)

\[
\begin{align*}
    dX_t &= X(X_t) dB_t + A(X_t) dt, \\
    X_0 &= x \in \mathbb{R}^n,
\end{align*}
\]

for \( t \geq 0 \), where the first integral is an Itô stochastic integral and the second is a Riemann integral. Here \( \{B_t\}_{t \geq 0} \) is Brownian motion on \( \mathbb{R}^m \) and the equality holds almost everywhere. The coefficients of this equation are the mapping \( X: \mathbb{R}^n \rightarrow L(\mathbb{R}^m; \mathbb{R}^n) \) and the vector field \( A: \mathbb{R}^n \rightarrow \mathbb{R}^m \). Assume standard regularity conditions on these coefficients so that there exists a strong solution \( \{X_t\}_{t \geq 0} \) to our equation. We write \( X^x_t \) for \( X_t \) when we want to make clear its dependence on the initial value \( x \). It is known that under further assumptions on the coefficients of our equation, the mapping \( x \mapsto X^x_t \) is differentiable (see for instance [1]).

1Supported by a grant from CONACYT (México).
Assume coefficients $X$ and $A$ are smooth enough and consider the associated derivative equation

$$dV_t = DX(X_t)(V_t) dB_t + DA(X_t)(V_t) dt, \quad V_0 = v \in \mathbb{R}^n,$$

whose solution $\{V_t\}_{t \geq 0}$ is the derivative of the mapping $x \mapsto X^*_t$ at $x$ in the direction $v$. We will assume that there exists a smooth map $\gamma: \mathbb{R}^n \to \mathbb{L}(\mathbb{R}^n; \mathbb{R}^{mn})$ such that $\gamma(x)$ is the right inverse of $X(x)$. That is, $X(x)\gamma(x) = I_{\mathbb{R}^n}$ for all $x$ in $\mathbb{R}^n$. We shall also assume that the process $\{\gamma(X_t)(V_t)\}_{t \geq 0}$ belongs to $L^2([0, t])$ for each $t > 0$, that is, $\int_0^t |\gamma(X_s)(V_s)|^2 \, ds < \infty$ and thus we can write $\int_0^t \gamma(X_s)(V_s) \, dB_s$.

With all above assumptions, we have from [6], the following result (see Appendix for a proof)

**Theorem 1** For every $t > 0$ there exist positive constants $k$ and $a$ such that

$$\mathbb{E} \left| \int_0^t \gamma(X_s)(V_s) \, dB_s \right| \leq k \sqrt{e^{at} - 1}. \quad (1)$$

We are interested in small values for $t$. Thus, since $\sqrt{e^{at} - 1} = O(\sqrt{t})$ as $t \to 0$, we have that there exist a positive constant $N$ such that $\sqrt{e^{at} - 1} \leq N \sqrt{t}$ for small $t$. Hence for sufficiently small $t$, we have the following estimate

$$\mathbb{E} \left| \int_0^t \gamma(X_s)(V_s) \, dB_s \right| \leq c \sqrt{t}, \quad (2)$$

where $c$ is a positive constant.

Let now $BC^r(\mathbb{R}^n)$ be the Banach space of bounded measurable functions on $\mathbb{R}^n$ which are $r$-times continuously differentiable with bounded derivatives. The norm of this space is given by the supremum norm of the function plus the supremum norm of each of its $r$ derivatives. In particular $B(\mathbb{R}^n)$ is the Banach space of bounded measurable functions on $\mathbb{R}^n$ with supremum norm $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$. Suppose our diffusion process $\{X_t\}_{t \geq 0}$ has transition probabilities $P(t, x, \Gamma)$. Then this induces a semigroup of operators $\{P_t\}_{t \geq 0}$ as follows. For every $t \geq 0$ we define on $B(\mathbb{R}^n)$ the bounded linear operator

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(y) P(t, x, dy) = \mathbb{E}(f(X^*_t)) \quad (3)$$

The semigroup $\{P_t\}_{t \geq 0}$ is a strongly continuous semigroup on $BC^0(\mathbb{R}^n)$. Denote by $A$ its infinitesimal generator.

It is known that $\{P_t\}_{t \geq 0}$ is a strong Feller semigroup, that is, $P_t$ sends continuous functions into differentiable functions. In fact, under above assumptions, a formula for the derivative of $P_t f$ is known (see [4] or [5]).

**Theorem 2** If $f \in BC^2(\mathbb{R}^n)$ then the derivative of $P_t f: \mathbb{R}^n \to \mathbb{R}$ is given by

$$D(P_t f)(x)(v) = \frac{1}{t} \mathbb{E}\{f(X_t) \int_0^t \gamma(X_s)(V_s) \, dB_s\}. \quad (4)$$
Higher derivatives in a more general setting are given in [4]. See also [2] for a general formula of this derivative in the context of a stochastic control system. Observe the mapping \( f \mapsto \frac{1}{t} \mathbb{E}\{f(X_t) \int_0^t \mathbb{V}(X_s)(V_s) \, dB_s\} \) defines a bounded linear functional on \( BC^2(\mathbb{R}^n) \). Hence there exists a unique extension on \( BC^0(\mathbb{R}^n) \). Since the expression of this linear functional does not depend on the derivatives of \( f \), it has the same expression for any \( f \) in \( BC^0(\mathbb{R}^n) \).

From last theorem we obtain

\[
|DP_t f(x)(v)| \leq \frac{1}{t} \|f\|_\infty \mathbb{E}\left| \int_0^t \mathbb{V}(X_s)(V_s) \, dB_s \right|
\leq \frac{1}{t} \|f\|_\infty k \sqrt{e^{at} - 1}.
\]

And hence for small \( t \)

\[
\|DP_t f\|_\infty \leq \frac{c\|f\|_\infty}{\sqrt{t}}.
\]

Observe that, as expected, our estimate goes to infinity as \( t \) approaches 0 since \( P_0 f = f \) is not necessarily differentiable. Also the rate at which it goes to infinity is not faster than \( \frac{1}{\sqrt{t}} \) does.

### 2 Potential

Let \( V: \mathbb{R}^n \to \mathbb{R} \) be a bounded measurable function. We shall perturb the generator \( \mathcal{A} \) by adding to it the function \( V \). We define the linear operator

\[
\mathcal{A}^V = \mathcal{A} + V,
\]

with the same domain as for \( \mathcal{A} \). A semigroup \( \{P_t^V\}_{t \geq 0} \) having \( \mathcal{A}^V \) as generator is given by the Feynman-Kac formula \( P_t^V f = \mathbb{E}\{f(X_t)e^{\int_0^t V(X_s) \, ds}\} \). We will find a similar estimate as (5) for \( \|DP_t^V f\|_\infty \). We first derive a recursive formula that will help us calculate the derivative of \( P_t^V f \). We have

\[
P_t^V f = P_t f + \left[ P_t P_s^V f \right]_{s=0}^{s=t} = P_t f + \int_0^t \frac{\partial}{\partial s}(P_{t-s} P_s^V f) \, ds = P_t f + \int_0^t [-\mathcal{A}(P_{t-s} P_s^V f) + P_{t-s}((\mathcal{A} + V) P_s^V f)] \, ds.
\]

Hence

\[
P_t^V f = P_t f + \int_0^t P_{t-s} (V P_s^V f) \, ds.
\]
Now we use our formula for differentiation (4) to calculate the derivative of this semigroup. We have
\[
DP_t^V f(x)(v) = DP_t f(x)(v) + \int_0^t DP_{t-s}^V (VP_s^V f)(x)(v) \, ds
\]
\[
= \frac{1}{t} E\{f(X_t) \int_0^t \mathcal{Y}(X_s)(V_s) \, dB_s \} + \int_0^t \frac{1}{t-s} E\{V(X_{t-s})P_s^V f(X_{t-s}) \int_0^{t-s} \mathcal{Y}(X_u)(V_u) \, dB_u \} \, ds.
\]
Then, by the Feynman-Kac formula and the Markov property we have
\[
DP_t^V f(x)(v) = \frac{1}{t} E\{f(X_t) \int_0^t \mathcal{Y}(X_s)(V_s) \, dB_s \} + \int_0^t \frac{1}{t-s} E\{V(X_{t-s})E\{f(X_{t-s})E\{e^{\int_{t-s}^t V(X_u) \, du} \} \int_0^{t-s} \mathcal{Y}(X_u)(V_u) \, dB_u \} \, ds,
\]
from which we obtain
\[
\|DP_t^V f\|_\infty \leq \frac{1}{t} \|f\|_\infty E\{f(X_t) \int_0^t \mathcal{Y}(X_s)(V_s) \, dB_s \} + \|V\|_\infty \|f\|_\infty \int_0^t e^{s\|V\|_\infty} E\{f(X_u)(V_u) \, dB_u \} \, ds.
\]
And hence for small \( t \)
\[
\|DP_t^V f\|_\infty \leq \frac{c\|f\|_\infty}{\sqrt{t}} + \|V\|_\infty \|f\|_\infty e^{t\|V\|_\infty} \int_0^t \frac{c}{\sqrt{t-s}} \, ds
\]
\[
= \frac{c\|f\|_\infty}{\sqrt{t}} + 2c\|V\|_\infty \|f\|_\infty e^{t\|V\|_\infty}.
\]
Observe again that our estimate goes to infinity as \( t \to 0 \).

## 3 Bounded Smooth Drift

Let \( Z : \mathbb{R}^n \to \mathbb{R}^n \) be a bounded smooth vector field. We shall consider another perturbation to the generator \( \mathcal{A} \). This time we define the linear operator
\[
\mathcal{A}^Z = \mathcal{A} + Z \cdot \nabla.
\]
The existence of a semigroup \( \{P^Z_t\}_{t \geq 0} \) having \( \mathcal{A}^Z \) as infinitesimal generator is guaranteed by the regularity of \( Z \). Indeed, if we write \( Z(x) = (Z^1(x), \ldots, Z^n(x)) \), then the operator \( \mathcal{A}^Z \) can be written as
\[
\mathcal{A}^Z = \frac{1}{2} \sum_{i,j=1}^n \partial_x Z_i(x)^* \partial_x Z_j(x) + \sum_{i=1}^n (\mathcal{A}_i(x) + Z(x)) \cdot \partial_x Z_i(x),
\]
and this operator is the infinitesimal generator associated with the equation
\[
\begin{align*}
dX_t &= \mathcal{X}(X_t)dB_t + [\mathcal{A}(X_t) + Z(X_t)]dt, \\
X_0 &= x \in \mathbb{R}^n.
\end{align*}
\]
Thanks to the smoothness of $Z$, this equation yields a diffusion process $(X^x_t)_{t \in T}$ and hence the semigroup $P_t^Z f(x) = \mathbb{E}(f(X^x_t))$. Then previous estimate applies also to $\|DP_t^Z f\|_\infty$. But we can do better because we can find the explicit dependence of the estimate upon $Z$ as follows. As before, we first find a recursive formula for this semigroup.

$$P_t^Z f = P_t f + \left[ P_{t-s} P_s^Z f \right]_{s=0}^{s=t}$$

$$= P_t f + \int_0^t \frac{\partial}{\partial s} (P_{t-s} P_s^Z f) \, ds$$

$$= P_t f + \int_0^t \left[ -A(P_{t-s} P_s^Z f) + P_{t-s} ((A + Z \cdot \nabla) P_s^Z f) \right] \, ds.$$  

Hence

$$P_t^Z f = P_t f + \int_0^t P_{t-s} (Z \cdot \nabla P_s^Z f) \, ds. \quad (6)$$

We can now calculate its derivative as follows

$$DP_t^Z f(x)(v) = DP_t f(x)(v) + \int_0^t DP_{t-s} (Z \cdot \nabla P_s^Z f)(x)(v) \, ds$$

$$= \frac{1}{t} \mathbb{E}\{f(X_t) \int_0^t \Upsilon(X_s)(V_s) \, dB_s\}$$

$$+ \int_0^t \frac{1}{t-s} \mathbb{E}\{Z(X_{t-s}) \cdot \nabla P_s^Z f(X_{t-s}) \int_0^{t-s} \Upsilon(X_u)(V_u) \, dB_u\} \, ds.$$  

We now find an estimate for the supremum norm of this derivative. Taking modulus we obtain

$$|DP_t^Z f(x)(v)| \leq \frac{1}{t} \|f\|_\infty \mathbb{E}\{\int_0^t \Upsilon(X_s)(V_s) \, dB_s\}$$

$$+ \int_0^t \frac{1}{t-s} \mathbb{E}\{DP_s^Z f(X_{t-s})(Z(X_{t-s})) \| \int_0^{t-s} \Upsilon(X_u)(V_u) \, dB_u\} \, ds.$$  

Hence for small $t$

$$\|DP_t^Z f\|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty + c \|Z\|_\infty \int_0^t \frac{\|DP_s^Z f\|_\infty}{\sqrt{t-s}} \, ds.$$  

We now solve this inequality. If we iterate once we obtain

$$\|DP_t^Z f\|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty + c^2 \|Z\|_\infty \|f\|_\infty \int_0^t \frac{ds}{\sqrt{s(t-s)}}$$

$$+ c^2 \|Z\|_\infty^2 \int_0^t \int_0^s \frac{\|DP_u^Z f\|_\infty}{\sqrt{(t-s)(s-u)}} \, du \, ds.$$  

By Fubini’s theorem, the double integral becomes

$$\int_0^t \|DP_u^Z f\|_\infty \int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} \, du.$$
and then we observe that
\[ \int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} = 2\tan^{-1}\sqrt{\frac{s-u}{t-s}} \bigg|_u^t = \pi. \]

The case \( u = 0 \) solves also the first integral. Hence our inequality reduces to
\[ \|DP_t^Z f\|_\infty \leq c\sqrt{t} \|f\|_\infty + c^2\pi \|Z\|_\infty \|f\|_\infty + c^2\pi \|Z\|_\infty^2 \int_0^t \|DP_u^Z f\|_\infty \, du. \]

We now apply Gronwall’s inequality. After some simplifications (extending the integral up to infinity) we finally obtain the estimate
\[ \|DP_t^Z f\|_\infty \leq c\sqrt{t} \|f\|_\infty + 2c^2\pi \|Z\|_\infty \|f\|_\infty e^{t(c^2\pi \|Z\|_\infty)} \] (7)

As expected, our estimate goes to infinity as \( t \to 0 \) since \( P_0^Z f = f \) is not necessarily differentiable.

4 Bounded Uniformly Continuous Drift

We now find a similar estimate when \( Z : \mathbb{R}^n \to \mathbb{R}^n \) is only bounded and uniformly continuous. We look again at the operator \( A^Z = A + Z \cdot \nabla \). The problem here is that in this case we do not have the semigroup \( \{P_t^Z f\}_{t \geq 0} \) since the stochastic equation with the added nonsmooth drift \( Z \) might not have a strong solution. So we cannot even talk about its derivative. To solve this problem we proceed by approximation.

4.1 Existence of Semigroup

Since \( Z \in BC^0(\mathbb{R}^n; \mathbb{R}^n) \) is uniformly continuous, and \( BC^\infty(\mathbb{R}^n; \mathbb{R}^n) \) is dense in \( BC^0(\mathbb{R}^n; \mathbb{R}^n) \), there exists a sequence \( \{Z_i\}_{i=1}^\infty \) in \( BC^\infty(\mathbb{R}^n; \mathbb{R}^n) \) such that \( Z_i \) converges to \( Z \) uniformly. Thus, for every \( i \in \mathbb{N} \), we have the semigroup \( \{P_t^{Z_i}\}_{t \geq 0} \) since our stochastic equation with the added smooth drift \( Z_i \) has a strong solution.

For every \( t \geq 0 \) and \( f \in BC^0(\mathbb{R}^n) \) fixed, the sequence of functions \( \{P_t^{Z_i} f\}_{i=1}^\infty \) is a Cauchy sequence in the Banach space \( BC^0(\mathbb{R}^n) \). We will prove this fact later. Let us denote its limit by \( P_t^Z f \). All properties required for \( P_t^Z f \) are inherited from those of the semigroup \( P_t^{Z_i} f \). Indeed by simply writing \( P_t^Z f = \lim_{i \to \infty} P_t^{Z_i} f \) and using an interchange of limits we can prove

1. \( f \mapsto P_t^Z f \) is a bounded linear operator.
2. \( t \mapsto P_t^Z f \) is a contraction semigroup of operators.
3. \( \{P_t^Z\}_{t \geq 0} \) has generator \( A^Z = A + Z \cdot \nabla \).

Now, suppose for a moment that the sequence of derivatives \( \{DP_t^{Z_i} f\}_{i=1}^\infty \) converges uniformly. Then we would have \( D(\lim_{i \to \infty} P_t^{Z_i} f) = \lim_{i \to \infty} DP_t^{Z_i} f \). This proves that \( P_t^Z f \) is differentiable. Then, for any \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( i \geq N \), \( \|DP_t^{Z_i} f\|_\infty \leq \|DP_t^{Z_i} f\|_\infty + \epsilon \) and therefore our estimate also applies to \( \|DP_t^Z f\|_\infty \).
4.2 Uniform Convergence of Derivatives

We now prove that the sequence \( \{DP_t^Z f\}_{t=1}^\infty \) converges uniformly. We use again our recursive formula (6) to obtain

\[
P_t^Z f - P_t^Z f = \int_0^t P_{t-s}(Z_i \cdot \nabla P_s^Z f - Z_j \cdot \nabla P_s^Z f) \, ds.
\]

We now use our formula for differentiation (4). After differentiating and taking modulus we obtain

\[
|D(P_t^Z f - P_t^Z f)(x)(v)| \\
\leq \int_0^t \frac{1}{t-s} \mathbb{E}\{|DP_s^Z f(X_{t-s})(Z_i(X_{t-s})) - DP_s^Z f(X_{t-s})(Z_j(X_{t-s}))| \\
+ \int_0^{t-s} \mathbb{Y}(X_u)(V_u) \, dB_u| \, ds.
\]

Thus

\[
\|D(P_t^Z f - P_t^Z f)\|_\infty \leq \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{t-s} \sqrt{e^{a(t-s)} - 1} \|DP_s^Z f\|_\infty \, ds \\
+ \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \sqrt{e^{a(t-s)} - 1} \|DP_s^Z f - DP_s^Z f\|_\infty \, ds.
\]

And hence for small \( t \)

\[
\|D(P_t^Z f - P_t^Z f)\|_\infty \leq \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \|DP_s^Z f\|_\infty \, ds \\
+ \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \|DP_s^Z f - DP_s^Z f\|_\infty \, ds.
\]

Now write estimate (7) as

\[
\|DP_t^Z f\|_\infty \leq \frac{A}{\sqrt{t}} + B e^{C t},
\]

for some positive constants \( A, B \) and \( C \) depending on \( \|f\|_\infty \) and \( \|Z_i\|_\infty \). Thus, substituting this in our last estimate gives

\[
\|D(P_t^Z f - P_t^Z f)\|_\infty \leq \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \frac{A}{\sqrt{t-s}} \, ds \\
+ \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} B e^{C s} \, ds \\
+ \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \|DP_s^Z f - DP_s^Z f\|_\infty \, ds.
\]

The first two integrals are bounded if we allow \( t \) to move within a finite interval \( (0, T] \). Thus there exist positive constants \( M_1 \) and \( M_2 \) such that

\[
\|D(P_t^Z f - P_t^Z f)\|_\infty \leq M_1 \|Z_i - Z_j\|_\infty \\
+ M_2 \int_0^t \frac{1}{\sqrt{t-s}} \|DP_s^Z f - DP_s^Z f\|_\infty \, ds.
\]
Now iterate this to obtain
\[
\| D(P_Z^t, f - P_Z^t, f) \|_\infty \leq M_1 \| Z_i - Z_j \|_\infty \\
+ M_2 M_1 \| Z_i - Z_j \|_\infty \int_0^t \frac{1}{\sqrt{t-s}} \, ds \\
+ M_2^2 \int_0^t \int_0^s \| D(P_u^Z, f - P_u^Z, f) \|_\infty \, du \, ds.
\]

As before the double integral reduces to
\[
\pi M_1^2 \int_0^t \| D(P_u^Z, f - P_u^Z, f) \|_\infty \, du.
\]

Collecting constants into new constants $M$ and $N$ we arrive at
\[
\| D(P_Z^t, f - P_Z^t, f) \|_\infty \leq M \| Z_i - Z_j \|_\infty + N \int_0^t \| D(P_u^Z, f - P_u^Z, f) \|_\infty \, du.
\]

Now we apply again Gronwall’s inequality to obtain
\[
\| D(P_Z^t, f - P_Z^t, f) \|_\infty \leq M \| Z_i - Z_j \|_\infty + N M \| Z_i - Z_j \|_\infty \int_0^t e^{N(t-u)} \, du.
\]

The right hand side goes to zero as $\| Z_i - Z_j \|_\infty \to 0$. This proves the sequence $\{DP_Z^t\}_{i=1}^\infty$ is uniformly convergent.

### 4.3 Uniform Convergence of Semigroups

We finally prove that the sequence of functions $\{P_Z^t, f\}_{i=1}^\infty$ is a Cauchy sequence. From our recursive formula (6) we obtain
\[
\| P_Z^t, f - P_Z^t, f \|_\infty \leq \int_0^t \| Z_i \cdot \nabla P_s^Z, f - Z_j \cdot \nabla P_s^Z, f \|_\infty \, ds \\
\leq \int_0^t \| Z_i - Z_j \|_\infty \| DP_s^Z, f \|_\infty \, ds \\
+ \int_0^t \| Z_j \|_\infty \| DP_s^Z, f - DP_s^Z, f \|_\infty \, ds \\
\leq \| Z_i - Z_j \|_\infty \int_0^t k \sqrt{e^{as}} - 1 \, ds \\
+ \| Z_j \|_\infty \int_0^t \| DP_s^Z, f - DP_s^Z, f \|_\infty \, ds.
\]

The first integral is bounded so the first part goes to zero as $\| Z_i - Z_j \|_\infty$ approaches 0. The second part also goes to zero since we just proved the sequence of derivatives is a Cauchy sequence. Hence $\{P_Z^t, f\}_{i=1}^\infty$ is uniformly convergent. Thus, with this approximating procedure we found a semigroup for the operator $A + Z \cdot \nabla$ for $Z$ bounded and uniformly continuous and we proved it is differentiable and that our estimate also applies to its derivative.
Observe that the boundedness and uniform continuity of $Z$ are required in order to ensure the existence of a sequence of smooth vector fields $Z_i$ uniformly convergent to $Z$. Alternative assumptions on $Z$ that guarantee the existence of such approximating sequence may be used.

Appendix

We here give a proof of inequality (1) which is taken from [6]. By using the Cauchy-Schwarz inequality and then the isometric property, we have

$$\mathbb{E} \left| \int_0^t \mathcal{Y}(x_s)(v_s) dB_s \right| \leq \left( \mathbb{E} \int_0^t \left\| \mathcal{Y}(x_s)(v_s) \right\|^2 ds \right)^{1/2} \leq \|\mathcal{Y}\|_\infty^2 \left( \int_0^t \mathbb{E}\|v_s\|^2 ds \right)^{1/2}. \quad (8)$$

We will estimate the right-hand side of the last inequality. Itô’s formula applied to the function $f(\cdot) = \|\cdot\|^2 : \mathbb{R}^n \to \mathbb{R}$ and the semimartingale $(v_t)_{t \geq 0}$ yields

$$\|v_s\|^2 = \|v_0\|^2 + 2 \int_0^s v_u \, dv_u + \sum_{i=1}^n \int_0^s d\langle v^i \rangle_u.$$ 

Therefore

$$\mathbb{E} \|v_s\|^2 = \|v_0\|^2 + \mathbb{E}\langle v \rangle_s$$
$$= \|v_0\|^2 + \mathbb{E} \int_0^s [D\mathcal{X}(x_u)(v_u)] \, [D\mathcal{X}(x_u)(v_u)]^* \, du$$
$$= \|v_0\|^2 + \mathbb{E} \int_0^s \|D\mathcal{X}(x_u)(v_u)\|^2 du$$
$$\leq \|v_0\|^2 + \|D\mathcal{X}\|_\infty^2 \int_0^s \mathbb{E}\|v_u\|^2 du.$$

We now use Gronwall’s inequality to obtain

$$\mathbb{E} \|v_s\|^2 \leq \|v_0\|^2 e^{\|D\mathcal{X}\|_\infty s}.$$ 

Integrating from 0 to $t$ yields

$$\int_0^t \mathbb{E} \|v_s\|^2 ds \leq \frac{\|v_0\|^2}{\|D\mathcal{X}\|_\infty} (e^{\|D\mathcal{X}\|_\infty t} - 1),$$

and thus, substituting in (8) we obtain

$$\mathbb{E} \left| \int_0^t \mathcal{Y}(x_s)(v_s) dB_s \right| \leq \|\mathcal{Y}\|_\infty^2 \frac{\|v_0\|}{\|D\mathcal{X}\|_\infty^{1/2}} \sqrt{e^t \|D\mathcal{X}\|_\infty - 1}.$$ 

This proves inequality (1).
Acknowledgments

I would like to thank Prof. K. D. Elworthy at Warwick from whom I have enormously benefitted. Also thanks to Dr. H. Zhao and Prof. A. Truman at Swansea for very helpful suggestions. CONACYT has financially supported me for my postgraduate studies at Warwick and Swansea.

References

[1] Da Prato, G. and Zabczyk, J. (1992) Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications 44. Cambridge University Press.

[2] Da Prato, G. and Zabczyk, J. (1997) Differentiability of the Feynman-Kac Semigroup and a Control Application. Rend. Mat. Acc. Lincei s. 9, v. 8, 183–188.

[3] Elworthy, K. D. (1982) Stochastic Differential Equations on Manifolds. Cambridge University Press.

[4] Elworthy, K. D. and Li, X. M. (1994) Formulae for the Derivative of Heat Semigroups. Journal of Functional Analysis 125, 252–286.

[5] Elworthy, K. D. and Li, X. M. (1993) Differentiation of Heat Semigroups and applications. Warwick Preprints 77/1993.

[6] Li, X.-M. (1992) Stochastic Flows on Noncompact Manifolds. Ph.D. Thesis. Warwick University.

[7] Pazy, A. (1983) Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York.

[8] Rincón, L. A. (1994) Some Formulae and Estimates for the Derivative of Diffusion Semigroups. Warwick MSc. Dissertation.