PRIKRY-TYPE FORCING AND THE SET OF POSSIBLE COFINALITIES

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Abstract. It is known that the set of possible cofinalities pcf(A) has good properties if A is a progressive interval of regular cardinals. In this paper, we give an interval of regular cardinals A such that pcf(A) has no good properties in the presence of a measurable cardinal, or in generic extensions by Prikry-type forcing.

1. Introduction

Cardinal arithmetic has been one of the most important areas in set theory. Shortly after Cohen devised the method of forcing, Easton [3] proved that the powers of regular cardinals is subject only to König’s theorem in ZFC. Easton’s theorem left the behavior of the powers of singular cardinals as the Singular Cardinal Problem. Some time later, Silver [11] proved the first nontrivial result around the problem: Singular cardinals of uncountable cofinality cannot be the least cardinal at which the Generalized Continuum Hypothesis (GCH) fails. Still later, Shelah [10] developed pcf theory and established a result that supersedes Silver’s theorem:

Theorem 1.1 (Shelah). \( N_\omega^\kappa < (2^{\omega_\kappa})^+ + N_{\omega_\kappa}. \)

An outline of the proof is as follows. First we have \( N_\omega^\kappa = 2^{\omega_\kappa} + \text{cf}(\{N_\omega^\kappa \mid n < \omega\}). \) The crucial claim is that \( \text{cf}(\{N_\omega^\kappa \mid n < \omega\}) = \max \text{pcf}(\{N_\omega^\kappa \mid n < \omega\}) < N_{\omega_\kappa}. \) Shelah proved it by analyzing the structure of \( \text{pcf}(\{N_\omega^\kappa \mid n < \omega\}). \) More specifically, he obtained the latter inequality by showing the following results for a progressive interval of regular cardinals A:

- \( \text{pcf}(A) \) is an interval of regular cardinals with a largest element.
- \( |\text{pcf}(A)| < |A|^+. \)

Theorem 1.1 can be generalized for a non-fixed point of the \( N \) function. Let \( \kappa \) be a singular cardinal with \( \kappa = N_\kappa > \mu. \) Shelah proved that \( \text{cf}(\{\kappa \mid n < \omega\}, \subseteq) = \max \text{pcf}(A) \) for some progressive interval of regular cardinals A with \( \sup A = \kappa. \) As before, this reduces the investigation of the power of \( \kappa \) to that of the structure of \( \text{pcf}(A). \) Note that we can take A to be progressive because \( \kappa \) is a non-fixed point of the \( N \) function. Thus the assumption of A being progressive seems essential in pcf theory. Now one may ask

Question 1.2. What if A is a non-progressive interval of regular cardinals?

Motivated by the question, we prove in this paper

Theorem 1.3. Suppose \( \kappa \) is a measurable cardinal. Then the following hold:

1. \( \text{pcf}(\kappa \cap \text{Reg}) = (2^\kappa)^+ \cap \text{Reg}. \)
Prikry forcing over $\kappa$ forces that $\text{pcf}(\kappa \cap \text{Reg}) = (2^\kappa)^+ \cap \text{Reg}$.

From Theorem 1.3(1) we get

**Corollary 1.4.** Suppose $\kappa$ is a measurable cardinal. Then the following hold:

1. $\text{pcf}(\kappa \cap \text{Reg})$ has no largest element if $2^\kappa$ is singular.
2. $|\text{pcf}(\kappa \cap \text{Reg})| > |\kappa \cap \text{Reg}|^{+4}$ if $2^\kappa > \kappa^{(+\kappa)}$.

The following corollary of Theorem 1.3(2) answers Question 1.2:

**Corollary 1.5.** Suppose there is a supercompact cardinal. Then in some forcing extension there is a non-progressive interval of regular cardinals $A$ such that $\text{sup}(A)$ is singular, $\text{pcf}(A)$ has no largest element and $|\text{pcf}(A)| > |A|^{+4}$.

The proof of Corollary 1.5 is as follows. Let $\kappa$ be a supercompact cardinal. We may assume that $\kappa$ is indestructibly supercompact in the sense of Laver [6]. This enables us to get a model in which $\kappa$ is supercompact and $2^\kappa$ is a singular cardinal $> \kappa^{(+\kappa)}$. Finally, Prikry forcing gives a model in which $A = \kappa \cap \text{Reg}$ is as desired by Theorem 1.3 (2). See Corollary 3.1 for an additional property of the final model.

Some large cardinal hypothesis is necessary in Theorem 1.3. Assume on the contrary that GCH holds and there is no weakly inaccessible cardinal. A simple argument shows that if $A \subseteq \text{Reg}$, then $\text{pcf}(A) = A \cup \{(\sup B)^+ \mid B \subseteq A \text{ has no maximal element}\}$, so that $\text{pcf}(A)$ has a largest element and $|\text{pcf}(A)| < |A|^{+4}$.

The structure of this paper is as follows. In Section 2, we recall basic facts of $\text{pcf}$ theory and Prikry forcing. Theorem 1.3 is proved in Section 3. We also consider the problem whether $\text{pcf}(\text{pcf}(A)) = \text{pcf}(A)$ holds. In Section 4, we prove an analogue of Theorem 1.3 (2) for Magidor forcing.

2. Preliminaries

In this section, we recall basic facts of $\text{pcf}$ theory and Prikry forcing. For more on the topics, we refer the reader to [11] and [4] respectively. We also use [5] as a reference for set theory in general.

Our notation is standard. We let Reg denote the class of all regular cardinals. Let $A \subseteq \text{Reg}$. Then $\prod A$ is the set $\{f : A \rightarrow \bigcup A \mid \forall \gamma \in A (f(\gamma) < \gamma)\}$. Let $F$ be a filter over $A$. We define a strict order $<_F$ on $\prod A$ by $f <_F g$ if $\{\gamma \in A \mid f(\gamma) < g(\gamma)\} \in F$.

**Definition 2.1.** For $A \subseteq \text{Reg},$

$$\text{pcf}(A) = \left\{\text{cf} \left(\prod A, <_D\right) \mid D \text{ is an ultrafilter over } A\right\}.$$ Note that $A \subseteq \text{pcf}(A) \subseteq (2^{\sup A})^+ \cap \text{Reg}$. If there is an increasing and cofinal sequence in $(\prod A, <_F)$ of length $\theta$ for some filter $F$ over $A$, then $\text{cf}(\theta) \in \text{pcf}(A)$.

A set $A \subseteq \text{Reg}$ is progressive if $\text{min} A > |A|$. An interval of regular cardinals is a set of the form $[\lambda, \kappa) \cap \text{Reg}$ for a pair of cardinals $\lambda < \kappa$. Here is the fundamental theorem on progressive intervals of regular cardinals.

**Theorem 2.2** (Shelah). If $A \subseteq \text{Reg}$ is a progressive interval, then we have

1. $\text{pcf}(A)$ has a largest element.
2. $|\text{pcf}(A)| < |A|^{+4}$.
3. $\text{pcf}(\text{pcf}(A)) = \text{pcf}(A)$.

Theorem 2.2 is known as the scale theorem.
Theorem 2.3 (Shelah). Suppose $\kappa$ is a singular cardinal. Then there is a set $A \in [\kappa \cap \text{Reg}]^{\text{cf}(\kappa)}$ such that $\sup A = \kappa$ and $(\prod A, \leq_F)$ has an increasing and cofinal sequence of length $\kappa^+$. Here, $F$ is the cobounded filter over $A$. In particular, $\kappa^+ \in \text{pcf}(\kappa \cap \text{Reg}).$

Next, we recall basic facts of Prikry forcing. Let $\kappa$ be a measurable cardinal and $U$ a normal ultrafilter over $\kappa$. Prikry forcing $\mathbb{P}$ is the set $[\kappa]^{\omega_1} \times U$ ordered by $(b, Y) \leq (a, X)$ iff $a \subseteq b$ (i.e. $a = b \cap (\text{max}(a) + 1)$) and $b \subseteq X$ and $b \setminus a \subseteq X$.

$\mathbb{P}$ has the $\kappa^+$-c.c. and size $2^\kappa$. Thus, $\mathbb{P}$ does not change the value of $2^\kappa$ for any $\theta \geq \kappa$. $\mathbb{P}$ preserves all cardinals above $\kappa$ but changes the cofinality of $\kappa$. Let $\dot{g}$ be a $\mathbb{P}$-name such that $\mathbb{P} \Vdash \dot{g} = \bigcup \{a \mid \exists X((a, X) \in G)\}$, where $G$ is the canonical $\mathbb{P}$-name for a generic filter. Then $\dot{g}$ is forced to be a cofinal subset of $\kappa$ of order type $\omega$. Moreover, we need

- $(a, X) \Vdash a \subseteq \check{g} \land \check{g} \setminus a \subseteq X$. In particular, $\mathbb{P} \Vdash \check{g} \subseteq^* X$ for every $X \in U$.
- If $(a, X) \Vdash \xi \in \dot{g}$, then $\check{\xi} \in a$.

The latter property follows by $(a, X \setminus (\xi + 1)) \leq (a, X)$ forces $\check{\xi} \setminus a \subseteq X \setminus (\xi + 1)$.

For subsequent purposes, we present a direct proof of Prikry lemma. Suppose $(X_b \mid b \in [\kappa]^{\omega_1}) \subseteq U$. The diagonal intersection $\Delta_b X_b$ is defined to be the set $\{\xi < \kappa \mid \forall b \in [\xi]^{\omega_1} (\xi \in X_b)\}$. Since $U$ is normal, we have $\Delta_b X_b \in U$.

**Lemma 2.4.** Suppose $[X_b \mid b \in [\kappa]^{\omega_1}] \subseteq U$ and $a \in [\kappa]^{\omega_1}$. Then any extension of $\langle a, \Delta_b X_b \rangle$ is compatible with $\langle a, X_a \rangle$.

**Proof.** Let $\langle c, Y \rangle \leq (a, \Delta_b X_b)$. Then $c \setminus a \subseteq X_b$ by $a \subseteq c$ and $c \setminus a \subseteq \Delta_b X_b$. Thus $\langle c, Y \cap X_b \rangle$ is a common extension of $\langle c, Y \rangle$ and $\langle a, X_a \rangle$, as desired. \hfill $\Box$

**Lemma 2.5** (Prikry lemma). Let $a \in [\kappa]^{\omega_1}$ and $\sigma$ be a statement of the forcing language. Then there is an $X \in U$ such that $\langle a, X \rangle$ decides $\sigma$, i.e. $\langle a, X \rangle \Vdash \sigma$ or $\langle a, X \rangle \Vdash \neg \sigma$.

**Proof.** For each $b \in [\kappa]^{\omega_1}$ define $X_b \in U$ as follows: If $a \subseteq b$, let $X_b$ be the unique set from the following mutually disjoint sets

- $X_b^+ = \{\xi < \kappa \mid b \subseteq \xi \land \exists Y \in \langle b \cup \{\xi\}, Y \rangle \Vdash \sigma\}.$

- $X_b^- = \{\xi < \kappa \mid b \subseteq \xi \land \exists Y \in \langle (b \cup \{\xi\}, Y \rangle \Vdash \neg \sigma\}.$

- $X_b^0 = \kappa \setminus (X_b^+ \cup X_b^-)$.

Otherwise, let $X_b = \kappa$. For each $b \in [\kappa]^{\omega_1}$ define $Y_b \in U$ as follows: If there is a $Y \in U$ such that $\langle b, Y \rangle$ decides $\sigma$, let $Y_b$ be one such $Y$. Otherwise, let $Y_b = \kappa$. We claim that $X = \Delta_b (X_b \cap Y_b) \in U$ is as desired. Take an arbitrary extension $\langle c, Y \rangle \leq (a, X)$ that decides $\sigma$. We may assume $c = b \cup \{\xi\}$ with $a \subseteq b \subseteq \xi$. Note that $\langle c, Y_c \rangle$ decides $\sigma$. We may assume $\langle c, Y_c \rangle \Vdash \sigma$. Then $\langle c, \Delta_b Y_b \rangle \Vdash \sigma$ by Lemma 2.4. Thus $\langle c, X \rangle \leq \langle c, \Delta_b Y_b \rangle$ forces $\sigma$. We claim that $\langle b, X \rangle \Vdash \sigma$, which completes the proof by repeating the argument.

It suffices to show that any extension of $\langle b, X \rangle$ is compatible with a condition forcing $\sigma$. Let $\langle d, Z \rangle \leq (b, X)$. We may assume $b \subseteq d$. Note that $\xi \in X_b$ by $\xi \in X$, and hence $X_b = X_b^+$ by $\langle b \cup \{\xi\}, X \rangle \Vdash \sigma$. Let $\eta = \text{min}(d \setminus b) \in X$. Then $\eta \in X_b^+$, so $\langle b \cup \{\eta\}, Y \rangle \Vdash \sigma$ for some $Y$, and hence $(b \cup \{\eta\}, Y_{b \cup \{\eta\}}) \Vdash \sigma$. Note that $d \setminus (b \cup \{\eta\}) \subseteq Y_{b \cup \{\eta\}}$ by $b \cup \{\eta\} \subseteq d$ and $d \setminus (b \cup \{\eta\}) \subseteq d \setminus b \subseteq X$. Thus $\langle d, Z \cap Y_{b \cup \{\eta\}} \rangle$ is a common extension of $\langle d, Z \rangle$ and $\langle b \cup \{\eta\}, Y_{b \cup \{\eta\}} \rangle$, as desired. \hfill $\Box$

**Corollary 2.6.** $\mathbb{P}$ adds no new bounded subsets of $\kappa$. In particular, $\mathbb{P}$ preserves all cardinals below $\kappa$.

3. **Prikry Forcing and a Non-Progressive Interval**

The first half of this section is devoted to
Proof of Theorem \([2.2]\) Let \(\kappa\) be a measurable cardinal. Take a normal ultrafilter \(U\) over \(\kappa\) and form \(j: V \to M \simeq \text{Ult}(V, U)\). For each \(\alpha \leq 2^\kappa\), we can choose \(f_\alpha \in {}^\kappa \kappa\) such that \(\alpha = [f_\alpha]_U\) by \(2^\kappa \leq (2^\kappa)^\kappa\).

Note that \(\kappa \cap \text{Reg} \subseteq \text{pcf}(\kappa \cap \text{Reg}) \subseteq (2^\kappa)^\kappa \cap \text{Reg}\). To complete the proof, it suffices to show that \([\kappa, (2^\kappa)^\kappa \cap \text{Reg}] \subseteq \text{pcf}(\kappa \cap \text{Reg})\) in both cases, (1) and (2).

(1) Let \(\theta \in \kappa \cap \text{Reg}\). Then we may assume \(f_\theta \in {}^\kappa \kappa\). Since \(\kappa = [\text{id}]_U \leq [f_\theta]_U\), we have

\[X = \{\xi < \kappa \mid \forall \eta < \xi (f_\theta(\eta) < \xi) \land \xi \leq f_\theta(\xi)\} \in U.\]

Note that \(f_\theta \upharpoonright X\) is strictly increasing. Define an ultrafilter \(U_\theta\) over \(\kappa \cap \text{Reg}\) by \(Y \in U_\theta\) iff \(f_\theta^{-1}Y \in U\). Then we have \(\prod_{\xi \in X} f_\theta(\xi), <_U \simeq (\prod_{\xi \in X} f_\theta(\xi), <_U) = \prod_{\xi \in X} f_\theta(\xi), <_U\).

Since \((f_\alpha \upharpoonright X \mid \alpha < \theta)\) is increasing and cofinal in \(\prod_{\xi \in X} f_\theta(\xi), <_U\), we have \(\theta = \text{cf}(\prod_{\xi \in X} f_\theta(\xi), <_U) = \text{cf}(\prod \kappa \cap \text{Reg}, <_{U_\theta})\) in \(\text{pcf}(\kappa \cap \text{Reg})\), as desired.

(2) Let \(\mathbb{P}\) be Prikry forcing defined by \(U\). Note that the set \((\kappa, (2^\kappa)^\kappa \cap \text{Reg})\) remains the same after forcing with \(\mathbb{P}\) and \(\kappa\) is singular. Let \(\theta \in (\kappa, (2^\kappa)^\kappa \cap \text{Reg})\). It suffices to prove that \(\mathbb{P} \vdash \theta \in \text{pcf}(\kappa \cap \text{Reg})\).

First, note that

\[X = \{\xi < \kappa \mid \forall \eta < \xi (f_\theta(\eta) < \xi) \land \xi < f_\theta(\xi)\} \in U.\]

Since \(\mathbb{P} \vdash g \subseteq^* X\), we have

\[\mathbb{P} \vdash (\prod_{\xi \in X} f_\theta(\xi), <^*) \simeq (\prod_{\xi \in X} f_\theta(\xi), <_U).\]

Here \(<^*\) and \(\hat{\cdot}\) are \(\mathbb{P}\)-names for the order on \(\prod_{\xi \in X} f_\theta(\xi)\) defined by the cobounded filter over \(g\), and the cobounded filter over \(f_\theta \upharpoonright g\) respectively. Thus it suffices to prove

(i) \(\mathbb{P} \vdash \langle f_\alpha \upharpoonright g \mid \alpha < \theta \rangle\) is increasing in \((\prod_{\xi \in X} f_\theta(\xi), <^*)\).

(ii) \(\mathbb{P} \vdash \langle f_\alpha \upharpoonright g \mid \alpha < \theta \rangle\) is cofinal in \((\prod_{\xi \in X} f_\theta(\xi), <^*)\).

Let \(\alpha < \beta\). Then \(Y = \{\xi < \kappa \mid f_\alpha(\xi) < f_\beta(\xi)\} \in U\). If \(\langle a, Z \rangle \in \mathbb{P}\), then \(\langle a, Y \cap Z \rangle \vdash \forall \xi \in g \setminus a(f_\alpha(\xi) < f_\beta(\xi))\), as desired.

(ii) By the proof of (i), it suffices to show that \(\langle h \upharpoonright g \mid h \in \prod_{\xi \in X} f_\theta(\xi)\rangle\) is forced to be cofinal in \((\prod_{\xi \in X} f_\theta(\xi), <^*)\).

Assume \(\mathbb{P} \vdash h \in \prod_{\xi \in X} f_\theta(\xi)\). For each \(b \in [\kappa]^{< \omega}\) define \(Y_b \in U\) and \(\eta_b < \kappa\) as follows. Note that \(h \upharpoonright \eta_b \in g\) and hence \(h(\max b) < f_\theta(\max b)\). By Prikry lemma, there is a \(b, Y_b \leq \langle b, X \rangle\) that decides \(h(\max b) = \eta\) for every \(\eta < f_\theta(\max b)\). Then we can take an \(\eta_b < f_\theta(\max b)\) such that

\[\langle b, Y_b \rangle \vdash h(\max b) = \eta_b.\]

For each \(\xi \in X\) define

\[h(\xi) = \sup(\eta_b + 1 \mid b \in [\xi + 1]^{< \omega}).\]

Since \(f_\theta(\xi) > \xi\) is regular, we have \(h \in \prod_{\xi \in X} f_\theta(\xi)\) in \(V\). Let \(Y = \Delta_b Y_b \in U\). We claim that \(\langle a, Y \rangle \vdash \forall \xi \in g \setminus a(h(\xi) < h(\xi))\) for every \(a \in [\kappa]^{< \omega}\), which completes the proof. It suffices to show that any extension of \(\langle a, Y \rangle\) forcing \(\xi \in g \setminus a\) is compatible with a condition forcing \(h(\xi) < h(\xi)\).

Suppose \(b, Z \leq (a, Y)\) forces \(\xi \in g \setminus a\). By the property we saw in Section 2, we have \(\xi \in b \setminus a\). \(b, Z\) is compatible with \(b \cap (\xi + 1), Y_b \cap (\xi + 1)\) forcing \(h(\xi) = \eta_b(\xi + 1) < h(\xi)\), as in the proof of Prikry lemma.

\(\square\)

Corollary [1.5] shows that the assumption of \(A\) being progressive is necessary in Theorem [2.2](1) and (2). Corollary [3.1] does the same for Theorem [2.2](3).
Corollary 3.1. One can add “$\kappa \subseteq pcf(A) \subseteq pcf(pcf(A))$” to the list of properties of $A$ in Corollary 1.5.

Proof. Let $A = \kappa \cap \text{Reg}$ in the final model for Corollary 1.5 where $2^\kappa$ is singular. By Theorem 1.3 (2) we have $pcf(A) = (2^\kappa)^+ \cap \text{Reg} = 2^\kappa \cap \text{Reg} \neq A$, which in turn implies that $(2^\kappa)^+ \in pcf(pcf(A)) \setminus pcf(A)$ by Theorem 2.3. □

The rest of this section is devoted to improving Corollary 3.1. Define $pcf^n(A)$ for $n < \omega$ by $pcf^0(A) = A$ and $pcf^{n+1}(A) = pcf(pcf^n(A))$.

Theorem 3.2. Suppose $(\kappa_i : i < \omega)$ is an increasing sequence of supercompact cardinals. Then the following hold in some forcing extension:

1. $\kappa_\omega$ is a singular cardinal of cofinality $\omega$.
2. $pcf^n(\kappa_i \cap \text{Reg}) \subseteq pcf^{n+1}(\kappa_0 \cap \text{Reg})$ for every $n < \omega$.

Lemma 3.3 ensures that sets of the form $pcf(\theta \cap \text{Reg})$ remain the same throughout forcing extensions for Theorem 3.2.

Lemma 3.3. Suppose $A \subseteq \text{Reg}$, and $Q$ has the $\kappa$-c.c. with $\kappa = \text{min}(A)$. Then $\text{Q} \vdash pcf^{\kappa}(A) \subseteq pcf(A)$.

Proof. In $V$, let $\theta \in pcf(A)$ be arbitrary. Then there are an ultrafilter $D$ over $A$ and an increasing and cofinal sequence $(\langle f_\alpha \mid \alpha < \theta \rangle)$ in $(\prod A, \triangleleft_D)$. Let $\dot{E}$ be a $Q$-name for the filter generated by $D$. Since $\theta \geq \kappa$ remains regular after forcing with $Q$, it suffices to prove that $(\langle f_\alpha \mid \alpha < \theta \rangle)$ is forced to be increasing and cofinal in $(\prod A, \leq_{\dot{E}})$.

It is easy to see the former. For the latter, it suffices to prove that $\prod A$ is forced to be cofinal in $(\prod A, \leq_{\dot{E}})$. Assume $p \vdash \dot{h} \in \prod A$. For each $\gamma \in A$, define

$$h^*(\gamma) = \sup(\xi + 1 \mid \exists q \leq p(q \vdash \dot{h}(\gamma) = \xi)).$$

Then $p \vdash \dot{h}(\gamma) < h^*(\gamma)$ for every $\gamma \in A$. Since $Q$ has the $\kappa$-c.c., and $\gamma \geq \kappa$ is regular, we have $h^* \in \prod A$ in $V$, as desired. □

Proof of Theorem 3.2. We may assume that each $\kappa_i$ is indestructibly supercompact in the sense of Laver [6] and $2^\kappa = \kappa^+$. We refer the reader to [2] for more details.

Let $Q$ be the full support product $\prod_{i \in \omega} \text{Add}(\kappa_i, \kappa_{i+1})$, where $\text{Add}(\kappa_i, \kappa_{i+1})$ is the poset adding $\kappa_{i+1}$ many Cohen subsets of $\kappa_i$. Standard arguments show that $Q$ preserves cofinalities and forces $2^\kappa = \kappa_{n+1}$ for every $n < \omega$. We claim that $Q$ forces $pcf((\kappa^+, \kappa^+_{n+1}) \cap \text{Reg}) \supseteq (\kappa^+, \kappa^+_{n+2}) \cap \text{Reg}$ for every $n < \omega$.

Let $G \subseteq Q$ be generic. Since $Q \simeq \prod_{i \in \omega} \text{Add}(\kappa_i, \kappa_{i+1}) \times \prod_{i \in \omega} \text{Add}(\kappa_i, \kappa_{i+1})$ in $V$, we have $G = G_n \times H_n$ in $V[G]$. By Theorem 1.3 (1), $pcf(\kappa^+_n \cap \text{Reg}) = pcf(\kappa_n \cap \text{Reg} \cup \{\kappa_n\} = (2^\kappa)^+ \cap \text{Reg} = \kappa^+_n \cap \text{Reg}$ in $V$. This remains true in $V[G_n]$ by the $\kappa_{n+1}$-closure of the corresponding poset. Now we work in $V[G_n]$. Note that $\kappa_{n+1}$ is supercompact and $2^{\kappa_n} = \kappa_{n+1}$. By Theorem 1.3 (1) we have $pcf(\kappa^+_n \cap \text{Reg}) = pcf(\kappa_n \cap \text{Reg} \cup \{\kappa_n\} = (2^\kappa)'^+ \cap \text{Reg} = \kappa^+_n \cap \text{Reg}$ in $V[G_n]$. Therefore $pcf((\kappa^+_n, \kappa^+_{n+1}) \cap \text{Reg}) = (\kappa^+_n, \kappa^+_{n+2}) \cap \text{Reg}$. Note that $(\prod_{i \in \omega} \text{Add}(\kappa_i, \kappa_{i+1}))^V = \prod_{i \in \omega} \text{Add}(\kappa_i, \kappa_{i+1})$ has the $\kappa^+_n$-c.c. By Lemma 3.3 we have $pcf((\kappa^+_n, \kappa^+_{n+1}) \cap \text{Reg}) \supseteq pcf((\kappa^+_n, \kappa^+_{n+2}) \cap \text{Reg})$ in $V[G] = [\kappa^+_n, \kappa^+_{n+2}) \cap \text{Reg}$ in $V[G] = V[G_n][H_n]$, as desired.

Since $Q$ is $\kappa_0$-directed closed in $V$, $\kappa_0$ remains supercompact in $V[G]$. So we can define Prikrý forcing $\mathbb{P}$ over $\kappa_0$. By Theorem 1.3 (2), $\mathbb{P}$ forces $pcf(\kappa_0 \cap \text{Reg}) = (2^\kappa)^+ \cap \text{Reg} = \kappa^+_0 \cap \text{Reg}$. By Lemma 3.3 $\mathbb{P}$ forces $pcf((\kappa^+_n, \kappa^+_{n+2}) \cap \text{Reg}) \subseteq pcf((\kappa^+_n, \kappa^+_{n+2}) \cap \text{Reg}) \subseteq (\kappa^+_n, (2^\kappa)^+)^+ \cap \text{Reg} = (\kappa^+_n, \kappa^+_{n+2}) \cap \text{Reg}$ for every $n < \omega$. Let $H \subseteq \mathbb{P}$ be generic. In $V[G][H]$, we have $pcf^{n+1}(\kappa_0 \cap \text{Reg}) = \kappa^+_{n+1} \cap \text{Reg}$ by induction on $n < \omega$. □
4. An Analogue for Magidor Forcing

Prikry forcing is known for a wealth of variations. In this section, we give an analogue of Theorem 1.3 (2) for one of them. Here we take up Magidor forcing from [7], but the argument works equally well for other variations, e.g. the diagonal Prikry forcing as defined in [8].

Magidor forcing uses a sequence of ultrafilters rather than a single ultrafilter, and makes a hypermeasurable cardinal into a singular cardinal of uncountable cofinality. For normal ultrafilters \( U, U' \) over \( \kappa \), \( U \prec U' \) iff \( U \in M \equiv \text{Ult}(V, U') \). Let \( \langle U_\alpha \mid \alpha < \lambda \rangle \) be a \( \prec \)-increasing sequence with \( \lambda < \kappa \). Note that there is a such sequence if \( \kappa \) is supercompact. For any \( \beta < \alpha < \lambda \), we fix a function \( F^\alpha_\beta \in \mathcal{V} \) such that \( [F^\alpha_\beta]_{U_\alpha} = U_\beta \). For each \( \alpha < \lambda \), define

\[
A_\alpha = \{ \delta < \kappa \mid \forall \beta < \alpha \forall \gamma < \beta (F^\alpha_\gamma(\delta) \prec F^\alpha_\beta(\delta) \text{ are normal ultrafilters over } \delta) \}.
\]

\[
B_\alpha = \{ \delta \in A_\alpha \setminus (\lambda + 1) \mid \forall \beta < \alpha \forall \gamma < \beta (F^\alpha_\gamma(\delta) \prec F^\alpha_\beta(\delta)) \}.
\]

Note that \( B_\alpha \in U_\alpha \). Magidor forcing \( \mathbb{M} \) is the set of pairs \( \langle a, X \rangle \) such that

- \( a \) is an increasing function such that
  - \( \text{dom}(a) \in [\lambda]^{<\omega} \) and \( \forall \alpha \in \text{dom}(a)(a(\alpha) \in B_\alpha) \).
- \( X \) is a function such that
  - \( \text{dom}(X) = \lambda \setminus \text{dom}(a) \text{ and } \forall \alpha \in \text{dom}(X)(X(\alpha) \subseteq B_\alpha) \).
  - For every \( \alpha \in \text{dom}(X) \), if \( \text{dom}(a) \setminus (\alpha + 1) = \emptyset \), \( X(\alpha) \in U_\alpha \). Otherwise, \( X(\alpha) \in F^\alpha_\beta(a(\alpha)) \) where \( \beta = \min(\text{dom}(a) \setminus (\alpha + 1)) \).

\( \mathbb{M} \) is ordered by \( \langle a, X \rangle \leq \langle b, Y \rangle \) iff \( b \subseteq a \), \( \forall \alpha \in \text{dom}(X)(X(\alpha) \subseteq Y(\alpha)) \) and \( \forall \alpha \in \text{dom}(a) \setminus \text{dom}(b)(a(\alpha) \in Y(\alpha)) \). \( \mathbb{M} \) has the \( \kappa^+ \)-c.c. and size \( 2^\kappa \). Thus, \( \mathbb{M} \) does not change the value of \( 2^\theta \) for any \( \theta \geq \kappa \). \( \mathbb{M} \) preserves all cardinals above \( \kappa \) but changes the cofinality of \( \kappa \) like Prikry forcing. Let \( \dot{g} \) be an \( \mathbb{M} \)-name such that \( \mathbb{M} \models \dot{g} = \bigcup \{ a \mid \exists X(a, X) \in G \} \), where \( G \) is the canonical \( \mathbb{M} \)-name for a generic filter. \( \dot{g} \) is forced to be an increasing sequence of length \( \lambda \) which converges to \( \kappa \). As in Prikry forcing, we also have

- \( \langle a, X \rangle \models \dot{g} \upharpoonright \text{dom}(a) = a \land \forall \alpha \in \lambda \setminus \text{dom}(a)(\dot{g}(\alpha) \in X(\alpha)) \).

For each \( \beta < \lambda \), we let \( \mathbb{M}_\beta = \{ \langle a, X \rangle \mid \langle a, X \rangle \in \mathbb{M} \} \) and \( \mathbb{M}^\beta = \{ \langle a, X \rangle^\beta \mid \langle a, X \rangle \in \mathbb{M} \} \). Here, \( \langle a, X \rangle^\beta \) and \( \langle a, X \rangle \) are \( \langle a \upharpoonright (\beta + 1), X \upharpoonright (\beta + 1) \rangle \) and \( \langle a \upharpoonright (\lambda \setminus (\beta + 1)), X \upharpoonright (\lambda \setminus (\beta + 1)) \rangle \) respectively. The orders on \( \mathbb{M}_\beta \) and \( \mathbb{M}^\beta \) are naturally defined by that on \( \mathbb{M} \). \( \mathbb{M} \) can be factored as follows.

**Lemma 4.1.** For every \( \langle a, X \rangle \in \mathbb{M} \) and \( \beta \in \text{dom}(a) \), we have

\[
\mathbb{M} / \langle a, X \rangle = \mathbb{M}_\beta / \langle a, X \rangle_{\beta} \times \mathbb{M}^\beta / \langle a, X \rangle^\beta.
\]

Note that \( \mathbb{M}_\beta / \langle a, X \rangle_{\beta} \) has the \( a(\beta)^+ \)-c.c. Lemmas 4.2 and 4.3 are analogues of Lemmas 2.4 and 2.5 for Magidor forcing respectively. See [7] for proofs.

**Lemma 4.2.** Suppose that \( \langle a, X \rangle \in \mathbb{M} \) and \( \{ \langle b, X_b \rangle \mid b \in LP \} \) is a set of extensions of \( \langle a, X \rangle \) where \( LP = \{ b \mid \exists Y((b, Y) \leq \langle a, X \rangle) \} \). Then there is a \( Z \) such that \( \langle a, Z \rangle \in \mathbb{M} \) and every extension of \( \langle b, Y \rangle \) is compatible with \( \langle b, X_b \rangle \) if \( (b, Y) \not\leq \langle a, Z \rangle \).

**Lemma 4.3** (Prikry lemma). For every \( \langle a, X \rangle \in \mathbb{M} \) and statement \( \sigma \) of the forcing language, \( \beta \in \text{dom}(a) \), there is a \( Z \) such that

- \( \langle a, Z \rangle \leq \langle a, X \rangle \) and \( \langle a, Z \rangle_{\beta} = \langle a, X \rangle_{\beta} \).
- If \( (b, Y) \leq \langle a, Z \rangle \) decides \( \sigma \), then \( \langle b, Y \rangle_{\beta} \langle a, Z \rangle^\beta \) decides \( \sigma \).

Here is the fundamental theorem of Magidor forcing:
Theorem 4.4 (Magidor). The following hold:

1. $M$ adds no new subsets of $\lambda$. In particular, $\lambda^+ \cap \text{Reg}$ remains the same by $M$.
2. $M$ preserves all cardinals.
3. $M$ forces that $\kappa$ is a strong limit singular cardinal of cofinality $\lambda$.

Now we get an analogue of Theorem 1.3(2) for Magidor forcing.

Theorem 4.5. $M$ forces that $\text{pcf}(\kappa \cap \text{Reg}) = (\lambda^+) \cap \text{Reg}$.

Proof. By the proof of Theorem 1.3, it suffices to show that $M \models (\kappa, (\lambda^+) \cap \text{Reg}) \subseteq \text{pcf}(\kappa \cap \text{Reg})$. Note that $(\kappa, (\lambda^+) \cap \text{Reg})$ remains the same after forcing with $M$. Let $\theta \in (\kappa, (\lambda^+) \cap \text{Reg})$. Let us see that $M \models \theta \in \text{pcf}((\kappa, (\lambda^+) \cap \text{Reg})$. For every $\gamma \leq \theta$ and $\alpha < \lambda$, we fix a function $f^\alpha_\gamma \in {}^\kappa \kappa$ such that $[f^\alpha_\gamma]_{U_\alpha} = \gamma$. We may assume $f^\alpha_\gamma \in {}^\kappa \kappa$ such that $X' \subseteq \prod_{\alpha < \lambda} U_\alpha$ be a function in $V$ such that $X'(\alpha) = \{\xi \in B_\alpha | \forall \eta \prec \xi \in \gamma \wedge \exists \xi \prec f^\alpha_\gamma(\eta)\}$ for any $\alpha < \lambda$.

We will show that $M$ forces $\left(\prod_{\alpha < \lambda} f^\alpha_\gamma(\gamma(\alpha)), \prec\right)$ has an increasing and cofinal sequence of length $\theta$. Here, $\prec$ is an $M$-name for the order on $\prod_{\alpha < \lambda} f^\alpha_\gamma(\gamma(\alpha))$ defined by the cobounded filter over $\lambda$. This gives the desired result, as shown by the following argument:

By a usual density argument, we can find an $M$-name $\dot{A}$ such that $M$ forces the following properties:

- $\dot{A} \in [\lambda]^\lambda$.
- $\forall \alpha, \beta \in \dot{A}(\alpha < \beta \to f^\alpha_\gamma(\gamma(\alpha)) < f^\beta_\gamma(\gamma(\beta)))$.

And thus, by the proof of Theorem 1.3, we have $M \models \left(\prod_{\alpha \in \dot{A}} f^\alpha_\gamma(\gamma(\alpha)), \prec\right) \approx \left(\prod_{\alpha \in \dot{A}} f^\alpha_\gamma(\gamma(\alpha)) | \alpha \in \dot{A}, \prec\right)$.

Here, $\dot{F}$ is an $M$-names for the cobounded filter over $\{f^\alpha_\gamma(\gamma(\alpha)) | \alpha \in \dot{A}\}$. It follows that $M$ forces $\left(\prod_{\alpha \in \dot{A}} f^\alpha_\gamma(\gamma(\alpha)) \prec\right)$ has an increasing and cofinal sequence of length $\theta$.

For every $\gamma < \theta$, let $\dot{f}_\gamma$ be an $M$-name for a function $\alpha \mapsto f^\alpha_\gamma(\gamma(\alpha))$. It suffices to prove

(i) $M \models \langle \dot{f}_\gamma | \gamma < \theta \rangle$ is increasing in $\left(\prod_{\alpha \in \dot{A}} f^\alpha_\gamma(\gamma(\alpha)), \prec\right)$.
(ii) $M \models \langle \dot{f}_\gamma | \gamma < \theta \rangle$ is cofinal in $\left(\prod_{\alpha \in \dot{A}} f^\alpha_\gamma(\gamma(\alpha)), \prec\right)$.

(i) Let $\gamma < \delta < \theta$. Note that we have $Y(\alpha) = \{\xi \in \kappa | f^\alpha_\gamma(\xi) < f^\alpha_\delta(\xi)\} \subseteq U_\alpha$ for each $\alpha < \lambda$. Let $(a, X) \in M$ be arbitrary. Define $Z = (X \upharpoonright \beta_\alpha) \upharpoonright (X(\alpha) \cap Y(\alpha) | \alpha \geq \beta_\alpha)$. Here, $\beta_\alpha = \text{max}\text{dom}(a)$. Then $(a, Z) \subseteq (a, X)$ forces $f^\alpha_{\gamma}(\gamma(\alpha)) < f^\beta_\delta(\alpha)$ for every $\alpha > \beta_\alpha$.

(ii) Let $(a, X) \in \dot{M}$ and $\dot{h}$ be arbitrary. Suppose $(a, X) \models \dot{h} \in \prod_{\alpha < \lambda} f^\alpha_\gamma(\gamma(\alpha))$. By the proof of (i), we may assume that $(X(\alpha) \subseteq X'(\alpha))$ for all $\alpha > \beta_\alpha$. For each $b \in \text{LP} = \{b \mid \exists Y((b, Y) \leq (a, X))\}$ define $Y_b$ and $\eta_b < \kappa$ as follows. If $\beta_\beta > \beta_\delta$, by Lemma 4.3 and $f^\beta_\delta(b(\beta_\delta)) < \kappa$, there is a $Y_b$ such that

- $\langle b, Y_b \rangle \leq (a, X)$.
- if $(c, Z) \leq (b, X)$ forces $h(\beta_\delta) = \zeta$, then $\langle c, Z^\zeta(\delta) \rangle_{\beta_\delta}(b, Y_b)^{\beta_\delta} \models h(\beta_\delta) = \zeta$.

Define $\eta_b$ by

$\eta_b = \sup(\zeta + 1 < f^{\beta_\delta}_\delta(b(\beta_\delta)) | \exists p \in \dot{M}_{\beta_\delta}/(b, Y_b)^{\beta_\delta}(p \upharpoonright (b, Y_b)^{\beta_\delta} \models h(\beta_\delta) = \zeta)).$

Then,

$\langle b, Y_b \rangle \models h(\beta_\delta) < \eta_b.$
For \( b \in \text{LP} \) with \( \beta_b \leq \beta_a \), \( Y_b = X \) and \( \eta_b = 0 \).

For each \( b \in \text{LP} \), since \( \mathcal{M}_{b_{\beta_b}}(b, Y_{b_{\beta_b}}) \) has the \( b(\beta_{b})^+\)-c.c., \( \eta_b < f_{\beta_b}^b(b(\beta_b)) \). For every \( \alpha > \alpha_b \), define \( h^b(\xi) = \sup\{\eta < f_{\beta_b}^b(\xi) \mid b \in \text{LP} \land b(\beta_b) = \xi \land \beta_b = \alpha \} \). Because of \( ||b \in \text{LP} \mid b(\beta_b) = \xi \land \beta_b = \alpha|| = |\alpha| \cdot |\xi| \), we have \( h^b(\xi) < f_{\beta_b}^b(\xi) \) for every \( \xi \in X(\alpha) \).

Let \( y = \sup_{\alpha > \beta_a} [h^\alpha]_{\alpha} + 1 < \theta \). By Lemma 4.2 and the proof of (i), there is an extension \( \langle a, Z \rangle \leq \langle a, X \rangle \) such that

- every extension of \( \langle b, Y \rangle \) is compatible with \( \langle b, Y_b \rangle \) if \( \langle b, Y \rangle \leq \langle a, Z \rangle \).
- \( \forall \alpha > \beta_a \forall \xi \in Z(\alpha)(h^\alpha(\xi) < f_{\beta_b}^\alpha(\xi)) \).

Lastly, we claim that \( \langle a, Z \rangle \not\vDash \dot{h}(\alpha) < \dot{f}_\alpha(\alpha) \) for all \( \alpha > \beta_a \). Let \( \langle b, Y \rangle \leq \langle a, Z \rangle \) and \( \alpha > \beta_a \) be arbitrary. Extending \( \langle b, Y \rangle \) we may assume that \( \alpha \in \text{dom}(b) \setminus (\beta_a + 1) \). Now, we can find \( \langle c, Y' \rangle \) such that \( b(\beta_b) \leq \langle c, Y' \rangle \leq \langle a, Z \rangle \) and \( \beta_c = \alpha \). By the certain property of \( \langle a, Z \rangle \), \( \langle c, Y_c \rangle \) and \( \langle b, Y \rangle \) have a common extension forcing \( h(\alpha) < \eta_c < h(\xi(\alpha)) < f_{\beta_b}^\alpha(\xi(\alpha)) = \dot{f}_\alpha(\alpha) \), as desired.

Theorem 4.5 enables us to generalize Theorem 3.2 as follows, including the case of uncountable cofinality.

**Theorem 4.6.** Suppose \( \langle \kappa_i \mid i < \omega \rangle \) is an increasing sequence of supercompact cardinals greater than a regular cardinal \( \lambda \). Then in some forcing extension the following hold:

1. \( \kappa_0 \) is a singular cardinal of cofinality \( \lambda \).
2. \( \text{pcf}(\kappa_0 \cap \text{Reg}) \subseteq \text{pcf}^{n+1}(\kappa_0 \cap \text{Reg}) \) for all \( n < \omega \).
3. \( \dot{\lambda}^+ \cap \text{Reg} = (\dot{\lambda}^+ \cap \text{Reg})^V \).

For \( A \subseteq \text{Reg} \), define

\[
\text{pcf}^\alpha(A) = \begin{cases} 
A & \alpha = 0 \\
\text{pcf}(\text{pcf}^\beta(A)) & \alpha = \beta + 1 \\
\bigcup_{\beta < \alpha} \text{pcf}^\beta(A) & \alpha \in \text{Lim}
\end{cases}
\]

Note that GCH implies \( \text{pcf}(\text{pcf}(A)) = \text{pcf}(A) \) for every \( A \subseteq \text{Reg} \). By Theorem 4.6, it is consistent that \( \langle \text{pcf}^\alpha(A) \mid n < \omega \rangle \) is \( \subseteq \)-increasing for some \( A \subseteq \text{Reg} \). We conclude this paper with the following

**Question 4.7.** Is it a theorem of ZFC that for every \( A \subseteq \text{Reg} \) there is an \( \alpha \) such that \( \text{pcf}^{n+1}(A) = \text{pcf}^\alpha(A) \)?

**References**

[1] Abraham, U. and Magidor, M. Cardinal arithmetic. In Handbook of set theory. Vols. 1, 2, 3, pages 1149–1227. Springer, Dordrecht, 2010.

[2] Apter, A. W. Some results on consecutive large cardinals. Ann. Pure Appl. Logic 25(1983), no.1, 1–17.

[3] Easton, W. B. Powers of regular cardinals. Ann. Math. Logic 1 (1970), 139–178.

[4] Gitik, M. Prikry-type forcings. In Handbook of set theory. Vols. 1, 2, 3, pages 1351–1447. Springer, Dordrecht, 2010.

[5] Kanamori, A. The higher infinite. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, Large cardinals in set theory from their beginnings, Paperback reprint of the 2003 edition, 2009.

[6] Laver, R. Making the supercompactness of \( \kappa \) indestructible under \( \kappa \)-directed closed forcing. Israel J. Math. 29(1978), no.4, 385–388.

[7] Magidor, M. Changing cofinality of cardinals. Fund. Math. 99(1978), no. 1, 61–71.

[8] Neeman, I. and Unger, S. Aronszajn trees and the SCH. In Appalachian set theory 2006–2012, volume 406 of London Math. Soc. Lecture Note Ser., pages 187–206. Cambridge Univ. Press, Cambridge, 2013.

[9] Prikry, K. L. Changing measurable into accessible cardinals. Dissertationes Math. (Rozprawy Mat.) 68(1970), 55.

[10] Shelah, S. Cardinal arithmetic. volume 29 of Oxford Logic Guides. The Clarendon Press, Oxford University Press, New York, Oxford Science Publications, 1994.
[11] Silver, J. On the singular cardinals problem. In Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pages 265–268, 1975.

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