Spectral Theory and Numerical Approximation for Singular Fractional Sturm-Liouville eigen-Problems on Unbounded Domain

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Abstract

In this article, we first introduce a singular fractional Sturm-Liouville eigen-problems (SFSLP) on unbounded domain. The associated fractional differential operators in these problems are both Weyl and Caputo type. The properties of spectral data for fractional operators on unbounded domain have been investigated. Moreover, it has been shown that the eigenvalues of the singular problems are real-valued and the corresponding eigenfunctions are orthogonal. The analytical eigensolutions to SFSLP is obtained and defined as generalized Laguerre fractional-polynomials. The optimal approximation of such generalized Laguerre fractional-polynomials in suitably weighted Sobolev spaces involving fractional derivatives has been derived, which is also available for approximated fractional-polynomials growing fast at infinity. The obtained results demonstrate that the error analysis beneficial of fractional spectral methods for fractional differential equations on unbounded domains. As a numerical example, we employ the new fractional-polynomials bases to demonstrate the exponential convergence of the approximation in agreement with the theoretical results.

Keywords: singular fractional Sturm-Liouville operators, fractional-polynomials eigenfunctions, weighted Sobolev spaces, approximation results, spectral accuracy.

1 Introduction

A large number of problems are governed by differential equations defined on unbounded domains. Several spectral methods were considered for their numerical solutions, see [1–5]. The Sturm-Liouville theory have been played an important role developing them and the theory of self-adjoint operators [6]. However, over the last decade, many applications arising in science and engineering fractional derivatives based models provide more accurate solutions of the systems than the integer order derivation bases do [7–11].

In recent years, more attentions were paid to Fractional Sturm-Liouville problems defined on bounded domains. In most of the fractional Sturm-Liouville formulations presented recently, fractional differential transform method proved to be efficient for computing the eigen-elements of Sturm-Liouville problems of fractional order [12] and Q.M. Al-Mdalla [13] proposed the numerical solution of the fractional Sturm-Liouville problems. The domain decomposition method proved to be very efficient for computing the eigen-elements of the present problem. A. Neamaty et al. [14] derived Haar wavelet operational matrix of the fractional integration and use it to obtain eigenvalues of fractional Sturm-Liouville problem. Zayernouri and Karniadakis [15] have proficiently defined a fractional Sturm-Liouville operators to investigate the properties of the eigenfunctions and the eigenvalues of this operator. Klimek, Agrawa [16] defined a Fractional Sturm-Liouville Operator (FSLO), introduced a regular Fractional Sturm-Liouville Problem (FSLP), and investigated the properties of the eigen-functions and the eigenvalues of the operator. The exact eigen-solutions to regular and singular fractional Sturm-Liouville problems are obtained explicitly in terms of non-polynomials functions,known as Jacobi Polyfractonomials family. This Jacobi Polyfractonomials family has been successfully employed in [17] and [18] as new basis functions in solving fractional ordinary and partial differential equations. It is likely that future models of nature may lead

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to fractional Sturm-Liouville type problems on unbounded domain. The recent progress in Singular Fractional Sturm-Liouville Problems on a semi-infinite interval is promising for developing new spectral methods for fractional PDEs, however, the eigensolutions have not been obtained explicitly and no numerical approximation results have been published so far.

Our fundamental goal of this paper is to develop a spectral theory and numerical approximation for Singular Fractional Sturm-Liouville Problems (SFSLP) on a semi-infinite interval. As a consequence, we introduce fractional Sturm-Liouville problems on half line. We prove that the eigenvalues of these singular problems are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal; these too are computed analytically. We also show that these eigenfunctions are dense and forming a complete basis in the Hilbert space. In addition, these eigenfunctions are showing to be hierarchical. Next, we establish the analysis of approximations by generalized Laguerre fractional polynomials in weighted Sobolev spaces involving fractional derivatives. These results will be useful for error analysis of fractional spectral methods for fractional differential equations on unbounded domains. The suggested basis also work well, even if the approximated functions grow up at infinity.

In the following, we first present some preliminary of fractional calculus in Section 2, and we proceed with the theory on SFSLP on half line in Sections 3. In Section 4, we establish the approximation results for these generalized Laguerre fractional polynomials. In Sections 5, we present numerical approximations of selected functions and we summarize our results in Section 6.

2 Fractional preliminaries

We start with some preliminary definitions of fractional calculus [19].

2.1 Liouville-Caputo Fractional Calculus

The left-sided and right-sided Riemann-Liouville integrals of order \( \mu \), when \( 0 < \mu < 1 \), are defined, respectively, as

\[
(\mathcal{RL}_{x}^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{x_L} \frac{f(s)ds}{(x-s)^{1-\mu}}, \quad x > x_L, \tag{2.1}
\]

and

\[
(\mathcal{RL}_{x}^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{x_R} \frac{f(s)ds}{(s-x)^{1-\mu}}, \quad x < x_R, \tag{2.2}
\]

where \( \Gamma \) represents the Euler gamma function. The corresponding inverse operators, i.e., the left-sided and right-sided fractional derivatives of order \( \mu \), are then defined based on (2.1) and (2.2), as

\[
(\mathcal{RL}_{x}^{\mu} D^{\mu}_{x} f)(x) = \frac{d}{dx}(\mathcal{RL}_{x}^{\mu} T^{1-\mu}_{x} f)(x) = \frac{1}{\Gamma(1-\mu)} \frac{df}{dx} \int_{x}^{x_L} \frac{f(s)ds}{(x-s)^{\mu}}, \quad x > x_L, \tag{2.3}
\]

and

\[
(\mathcal{RL} D^{\mu}_{x} f)(x) = \frac{-d}{dx}(\mathcal{RL} T^{1-\mu}_{x} f)(x) = \frac{1}{\Gamma(1-\mu)} \frac{-df}{dx} \int_{x}^{x_R} \frac{f(s)ds}{(s-x)^{\mu}}, \quad x < x_R. \tag{2.4}
\]

Furthermore, the corresponding left-sided and right-sided Caputo derivatives of order \( \mu \in (0,1) \) are obtained as

\[
(\mathcal{C} D^{\mu}_{x} f)(x) = (\mathcal{RL}_{x}^{\mu} T^{1-\mu}_{x} f)(x) = \frac{1}{\Gamma(1-\mu)} \int_{x}^{x_L} f'(s)ds \frac{1}{(x-s)^{\mu}}, \quad x > x_L, \tag{2.5}
\]

and

\[
(\mathcal{C} D^{\mu}_{x} f)(x) = (\mathcal{RL} T^{1-\mu}_{x} f)(x) = \frac{1}{\Gamma(1-\mu)} \int_{x}^{x_R} f'(s)ds \frac{1}{(x-s)^{\mu}}, \quad x < x_R. \tag{2.6}
\]

The two definitions of the left- and right-sided fractional derivatives of both Riemann-Liouville and Caputo type are linked by the following relationship, which can be derived by a direct calculation

\[
(\mathcal{RL}_{x}^{\mu} D^{\mu}_{x} f)(x) = \frac{f(x_L)}{(\Gamma(1-\mu)(x-x_L)^{\mu}))} + (\mathcal{C} D^{\mu}_{x} f)(x), \tag{2.7}
\]
\[(^{RL}D^\mu_x f)(x) = \frac{f(x_R)}{\Gamma(1-\mu)(x_R-x)^\mu} + (^C D^\mu_x f)(x).\]  

(2.8)

### 2.2 Weyl Fractional Calculus

The Weyl fractional integrals of order \(\mu, 0 < \mu < 1\) are defined as

\[(^W_{-\infty} I^\mu_x f)(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x f(s)\frac{ds}{(x-s)^{1-\mu}}.\]  

(2.9)

and

\[(^W_x I^\mu_{\infty} f)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty f(s)\frac{ds}{(s-x)^{1-\mu}}.\]  

(2.10)

The Weyl fractional derivative of order \(0 < \mu < 1\) are defined as the left-inverse operators of the corresponding Weyl fractional integrals

\[(^W_{-\infty} D^\mu_x f)(x) = \frac{d}{dx}(^W_{-\infty} I^{1-\mu}_x f)(x),\]  

(2.11)

and

\[(^W_x D^\mu_{\infty} f)(x) = -\frac{d}{dx}(^W_x I^{1-\mu}_{\infty} f)(x).\]  

(2.12)

**Proposition 2.1** The fractional differential operator defined in (2.9)-(2.12) satisfy the following identity:

\[
\int_0^\infty g(x)^W_x D^\mu_{\infty} f(x)dx = \int_0^\infty f(x)^C_0 D^\mu_x g(x)dx - \lim_{a \to \infty} \{g(x)^W_x I^{1-\mu}_{\infty} f(x)\}_a^a. \tag{2.13}
\]

### 3 Singular fractional Sturm-Liouville problems on unbounded domain

Let’s denote a singular fractional Sturm-Liouville problem on unbounded domain for differential operator with the differential part containing the left- and right-sided derivatives. Let’s use the form of the integration by parts formulas (2.13) for this new approximation. Main properties of eigenfunctions and eigenvalues in the theory of classical Sturm-Liouville problems are related to the integration by parts formula for the first order derivatives. In The associated fractional version we note that both left and right derivatives appear and the essential pair are the left Weyl derivative with the right Caputo derivative.

**Definition 3.1** Let \(L^\mu\) be the singular fractional differential operator is written as

\[L^\mu := W_x D^\mu \{p(x)^C D^\mu(\_\_\_\_),\}.\]

(3.1)

Consider the singular fractional Sturm-Liouville equation

\[L^\mu f(x) + \Lambda W(x)f(x) = 0, \tag{3.2}\]

where the fractional order \(\mu \in (0, 1)\) and \(W_x D^\mu = W_x D^\mu (\text{i.e., Weyl fractional derivative of order } \mu)\) and \(^C D^\mu = ^C D^\mu (\text{i.e., left-sided Caputo fractional derivative of order } \mu), p(x) \neq 0, W(x) > 0 \forall x \in [0, \infty)\) and \(p, W\) are real valued continuous functions in interval \([0, \infty)\).

**Theorem 3.1** Fractional differential operator is self-adjoint on half line.
Proof: Let us consider the following equation

\[ < L^\mu f, g > - < f, L^\mu g > = \int_0^\infty L^\mu f(x), g(x) \, dx - \int_0^\infty f(x), L^\mu g(x) \, dx \]

\[ = \int_0^\infty W D^\mu \{ p(x)^C D^\mu f(x) \}, g(x) \, dx - \int_0^\infty W D^\mu \{ p(x)^C D^\mu g(x) \}, f(x) \, dx. \]  

(3.3)

By means of property 2.1, we obtain

\[ = \int_0^\infty p(x)^C D^\mu D_x (f(x)). D^\mu g(x) \, dx - \lim_{a \to \infty} \{ g(x)^W T^{1-\mu} \{ p(x)^C D^\mu f(x) \} \}_0^a \]

\[ - \int_0^\infty p(x)^C D^\mu g(x), D^\mu f(x) \, dx + \lim_{a \to \infty} \{ f(x)^W T^{1-\mu} \{ p(x)^C D^\mu g(x) \} \}_0^a. \]  

(3.4)

We impose boundary conditions

\[ \lim_{a \to \infty} \{ g(x)^W T^{1-\mu} \{ p(x)^C D^\mu f(x) \} \}_0^a = 0, \]  

(3.5)

and

\[ \lim_{a \to \infty} \{ f(x)^W T^{1-\mu} \{ p(x)^C D^\mu g(x) \} \}_0^a = 0. \]  

(3.6)

Hence \( L^\mu \) to be self-adjoint.

\[ < L^\mu f, g > = < f, L^\mu g >. \]  

\[ \square \]

Theorem 3.2 The eigenvalues of SFSLP are real valued on a semi-infinite interval.

Proof: Let \( L^\mu \) be the fractional differential operator as

\[ L^\mu := W D^\mu \{ p(x)^C D^\mu (.) \}. \]

and assume that \( \lambda \) is the eigenvalue of (3.2) corresponding the eigenfunction \( f(x) \).

\[ L^\mu f(x) + \lambda W(x) f(x) = 0. \]  

(3.7)

and its complex conjugate \( \hat{f}(x) \) and let \( \hat{\lambda} \) is the eigenvalue of \( \hat{f}(x) \)

\[ L^\mu \hat{f}(x) + \hat{\lambda} W(x) \hat{f}(x) = 0. \]  

(3.8)

Now, we multiply (3.7) by \( \hat{f}(x) \), and (3.8) by \( f(x) \) and subtract them as

\[ (\lambda - \hat{\lambda}) W(x) f(x) \hat{f}(x) = f(x)L^\mu \hat{f}(x) - \hat{f}(x)L^\mu f(x). \]

Integrating over the interval \([0, \infty)\) and utilizing the fractional integration-by-parts (2.13), we obtain

\[ (\lambda - \hat{\lambda}) \int_0^\infty W(x)|f|^2 \, dx = - \lim_{a \to \infty} \{ f(x)^W L^{1-\mu} \{ p(x)^C D^\mu \hat{f}(x) \} \}_0^a \]

\[ + \lim_{a \to \infty} \{ \hat{f}(x)^W T^{1-\mu} \{ p(x)^C D^\mu f(x) \} \}_0^a. \]  

(3.9)

We impose the boundary conditions for \( f(x) \) and \( \hat{f}(x) \)

\[ \lim_{a \to \infty} \{ \hat{f}(x)^W T^{1-\mu} \{ p(x)^C D^\mu f(x) \} \}_0^a = 0, \]  

(3.10)

\[ \lim_{a \to \infty} \{ f(x)^W T^{1-\mu} \{ p(x)^C D^\mu \hat{f}(x) \} \}_0^a = 0. \]  

(3.11)
We obtain

\[(\lambda - \hat{\lambda}) \int_0^\infty W(x)|f|^2\,dx = 0. \tag{3.12}\]

Therefore, \(\lambda = \hat{\lambda}\) because \(f\) is a non-trivial solution of the problem, and \(W(x)\) is non-negative in interval \([0, \infty)\).

\[\blacksquare\]

**Theorem 3.3** The eigenfunctions corresponding to distinct eigenvalues of SFSLP (3.2) are orthogonal w.r.t. weight function \(W(x)\) on \([0, \infty)\) that is

\[\int_0^\infty W(x)f_{\lambda_1}(x)f_{\lambda_2}(x)\,dx = 0, \quad \lambda_1 \neq \lambda_2,\]

where function \(f_{\lambda_j}\) corresponds to the eigenvalue \(\lambda_j\).

**Proof:** Assume that \(f_{\lambda_1}(x)\) and \(f_{\lambda_2}(x)\) are two eigenfunctions corresponding to two distinct eigenvalues \(\lambda_1\) and \(\lambda_2\), respectively:

\[L^\mu f_{\lambda_j}(x) + \lambda_j W(x)f_{\lambda_j}(x) = 0. \tag{3.13}\]

Now, we obtain from Theorem 3.1

\[<L^\mu f_{\lambda_1}, f_{\lambda_2} >= <f_{\lambda_1}, L^\mu f_{\lambda_2}>, \tag{3.14}\]

\[< -\lambda_1 W(x)f_{\lambda_1}(x), f_{\lambda_2}(x) >= <f_{\lambda_1}(x), -\lambda_1 W(x)f_{\lambda_2}(x)>, \tag{3.15}\]

\[-\lambda_1 \int_0^\infty W(x)f_{\lambda_1}(x)f_{\lambda_2}(x)\,dx = -\lambda_2 \int_0^\infty W(x)f_{\lambda_1}(x)f_{\lambda_2}(x)\,dx. \tag{3.16}\]

and since \(\lambda_1 \neq \lambda_2\), we obtain

\[\int_0^\infty W(x)f_{\lambda_1}(x)f_{\lambda_2}(x)\,dx = 0. \tag{3.17}\]

### 3.1 Analytical eigensolutions to Singular Fractional Sturm-Liouville Problems on a semi-infinite interval

Here, we obtain the analytical solution \(f(x)\) to SFSLP (3.2). Before that, we recall the following lemmas for the standard Laguerre polynomials \(L^\alpha(x)\)

**Lemma 3.1** (See [20].) For \(\mu > 0\), \(\alpha > -1\), and \(x \in [0, \infty)\)

\[
\frac{x^{\alpha+\mu} L_n^{\alpha+\mu}(x)}{\Gamma(n+\alpha+\mu+1)} = \frac{1}{\Gamma(\mu)} \int_0^x \frac{(x-y)^{\mu-1} y^\alpha L_n^\alpha(y)}{\Gamma(n+\alpha+1)}\,dy. \tag{3.18}
\]

By the left-sided Riemann-Liouville integral [21], we can rewrite (3.18) as

\[\int_0^\infty L_n^\alpha(x)\,dx = \frac{\Gamma(n+\mu+1)}{\Gamma(n+\alpha+\mu+1)} x^{\alpha+\mu} L_n^{\alpha+\mu}(x). \tag{3.19}\]
Lemma 3.2 (See [20].) For \( \mu > 0, \alpha > -1, \) and \( x \in [0, \infty) \)
\[
e^{-x} L_n^\alpha(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (x - y)^{\mu-1} e^{-y} L_n^{\alpha+\mu}(y) dy. \tag{3.20}
\]

By the right-sided Weyl integral (2.10), we can re-write (3.20) as
\[
W x \mathcal{I}_\infty e^{-x} L_n^{\alpha+\mu}(x) = e^{-x} L_n^\alpha(x). \tag{3.21}
\]

In [21] introduced a new family of generalized Laguerre polynomials, which is mutually orthogonal on the half-line, that we can use it to prove that the following two lemmas:

Lemma 3.3 For \( \mu > 0, \alpha > -1, \beta \geq 1, \) and \( x \in [0, \infty) \)
\[
x^\alpha y L_n^{\alpha+\mu}(x) = \frac{1}{\Gamma(n+\alpha+\mu+1)} \int_0^x (x - y)^{\mu-1} y^{\alpha+\mu} L_n^{\alpha+\mu}(y) dy. \tag{3.22}
\]

By the left-sided Riemann-Liouville integral (2.1), we can re-write (3.22) as
\[
\mathcal{R} L x^\alpha y L_n^{\alpha+\mu}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)} x^{\alpha+\mu} L_n^{\alpha+\mu}(x). \tag{3.23}
\]

Lemma 3.4 For \( \mu > 0, \alpha > -1, \beta \geq 1, \) and \( x \in [0, \infty) \)
\[
e^{-\beta x} L_n^{\alpha+\beta}(x) = \frac{\beta^\mu}{\Gamma(\mu)} \int_x^\infty (x - y)^{\mu-1} e^{-\beta y} L_n^{\alpha+\beta}(y) dy. \tag{3.24}
\]

By the right-sided Weyl integral (2.10), we can re-write (3.24) as
\[
W x \mathcal{I}_\infty e^{-\beta x} L_n^{\alpha+\beta}(x) = \frac{e^{-x} L_n^{\alpha+\beta}(x)}{\beta^\mu}. \tag{3.25}
\]

We begin with our definition of the singular fractional Sturm-Liouville generalized laguerre fraction polynomial with parameters \(-1 < \alpha < \mu - 1\) and \(\beta \geq 1\) for \( x \in [0, \infty) \) as
\[
W D^\mu \{ p(x)^C D^\mu p(x) \} + \Lambda x^{\alpha+\mu+1} e^{-\beta x} p(x) = 0, \tag{3.26}
\]
where \( p(x) = x^{\alpha+1} e^{-\beta x} \) used in the fractional differential operator in (3.25). We also note that the weight function \( W^{\alpha,\beta}(x) = x^{\alpha+\mu+1} e^{-\beta x} \) is a non-negative function.

Theorem 3.4 The eigenvalues of SFSLP (3.26) are real-valued, moreover, the eigenfunctions corresponding to distinct eigenvalues of SFSLP are orthogonal on a semi-infinite interval with respect to the weight function.
\[
W^{\alpha,\beta}(x) = x^{\alpha+\mu+1} e^{-\beta x}.
\]

Proof: Part a: Let \( L^{\alpha,\beta;\mu} \) be the fractional differential operator of order 2\( \mu \) as
\[
L^{\alpha,\beta;\mu} := W D^\mu \{ p(x)^C D^\mu (.) \}, \tag{3.27}
\]
and assume that \( \Lambda \) is the eigenvalue of (3.25) corresponding the eigenfunction \( \psi(x) \).
\[
L^{\alpha,\beta;\mu} \psi(x) + \Lambda W^{\alpha,\beta;\mu}(x) \psi(x) = 0. \tag{3.28}
\]
and its complex conjugate \( \hat{\psi}(x) \) and \( \hat{\Lambda} \) is the eigenvalue of \( \hat{\psi}(x) \)
\[
L^{\alpha,\beta;\mu} \hat{\psi}(x) + \hat{\Lambda} W^{\alpha,\beta;\mu}(x) \hat{\psi}(x) = 0. \tag{3.29}
\]
Now, we multiply (3.28) by $\hat{\psi}(x)$, and (3.29) by $\psi(x)$ and subtract them as

$$(\Lambda - \hat{\Lambda}) W^{\alpha,\beta,\mu}(x)\psi(x)\hat{\psi}(x) = \psi(x)L^{\alpha,\beta,\mu}\hat{\psi}(x) - \hat{\psi}(x)L^{\alpha,\beta,\mu}\psi(x).$$

Integrating over the interval $[0, \infty)$ and utilizing the fractional integration-by-parts (2.13), we obtain

$$(\Lambda - \hat{\Lambda}) \int_0^\infty W^{\alpha,\beta,\mu}(x)|\psi|^2 dx = \int_0^\infty p(y)C^{\mu}\psi(y)C^{\mu}\hat{\psi}(y)dy - \lim_{a \to \infty} \left\{ \psi(y)W^{1-\mu}\{p(y)C^{\mu}\hat{\psi}(y)\} \right\}_0^a$$

$$- \int_0^\infty p(y)C^{\mu}\hat{\psi}(y)C^{\mu}\psi(y)dy + \lim_{a \to \infty} \left\{ \hat{\psi}(y)W^{1-\mu}\{p(y)C^{\mu}\psi(y)\} \right\}_0^a,$$

$$(\Lambda - \hat{\Lambda}) \int_0^\infty W^{\alpha,\beta,\mu}(x)|\psi|^2 dx = \lim_{a \to \infty} \left\{ \psi(y)W^{1-\mu}\{p(y)C^{\mu}\hat{\psi}(y)\} \right\}_0^a$$

$$+ \lim_{a \to \infty} \left\{ \hat{\psi}(y)W^{1-\mu}\{p(y)C^{\mu}\psi(y)\} \right\}_0^a. \tag{3.30}$$

We impose the boundary conditions for $\psi(x)$ and $\hat{\psi}(x)$

$$\lim_{a \to \infty} \left\{ \hat{\psi}(x)W^{1-\mu}\{x^{\alpha+1}e^{-\beta x}C^{\mu}\hat{\psi}(x)\} \right\}_0^a = 0, \tag{3.32}$$

and

$$\lim_{a \to \infty} \left\{ \psi(x)W^{1-\mu}\{x^{\alpha+1}e^{-\beta x}C^{\mu}\psi(x)\} \right\}_0^a = 0. \tag{3.33}$$

we obtain

$$(\Lambda - \hat{\Lambda}) \int_0^\infty W^{\alpha,\beta,\mu}(x)|\psi|^2 dx = 0. \tag{3.34}$$

Therefore, $\Lambda = \hat{\Lambda}$ because $\psi(x)$ is a non-trivial solution of the problem, and $W^{\alpha,\beta,\mu}(x)$ is non-negative in interval $[0, \infty)$.

Part b: Now, we prove the second statement on the orthogonality of the eigenfunctions with respect to the weight function $W^{\alpha,\beta,\mu}(x)$. Assume that $\psi_1(x)$ and $\psi_2(x)$ are two eigenfunctions corresponding to two distinct eigenvalues $\Lambda_1$ and $\Lambda_2$, respectively. Then they both satisfy (3.26) as

$$L^{\alpha,\beta,\mu}\psi_1(x) + \Lambda_1 W^{\alpha,\beta,\mu}(x)\psi_1(x) = 0, \tag{3.35}$$

and

$$L^{\alpha,\beta,\mu}\psi_2(x) + \Lambda_2 W^{\alpha,\beta,\mu}(x)\psi_2(x) = 0. \tag{3.36}$$

It can be shown that

$$(\Lambda_1 - \Lambda_2) W^{\alpha,\beta,\mu}(x)\psi_1(x)\psi_2(x) = \psi_1(x)L^{\alpha,\beta,\mu}\psi_2(x) - \psi_2(x)L^{\alpha,\beta,\mu}\psi_1(x).$$

Integrating over the interval $[0, \infty)$ and utilizing the fractional integration-by-parts (2.13), we obtain

$$(\Lambda_1 - \Lambda_2) \int_0^\infty W^{\alpha,\beta,\mu}(x)\psi_1(x)\psi_2(x)dx = - \lim_{a \to \infty} \left\{ \psi_1(y)W^{1-\mu}\{p(y)C^{\mu}\psi_2(y)\} \right\}_0^a$$

$$+ \lim_{a \to \infty} \left\{ \psi_2(y)W^{1-\mu}\{p(y)C^{\mu}\psi_1(y)\} \right\}_0^a. \tag{3.37}$$

We impose the boundary conditions for $\psi_1(x)$ and $\psi_2(x)$

$$\lim_{a \to \infty} \left\{ \psi_2(x)W^{1-\mu}\{x^{\alpha+1}e^{-\beta x}C^{\mu}\psi_1(x)\} \right\}_0^a = 0, \tag{3.38}$$

and

$$\lim_{a \to \infty} \left\{ \psi_1(x)W^{1-\mu}\{x^{\alpha+1}e^{-\beta x}C^{\mu}\psi_2(x)\} \right\}_0^a = 0. \tag{3.39}$$
and since $\Lambda_1 \neq \Lambda_2$, we obtain
\[
\int_0^\infty W^{\alpha,\beta,\mu}(x)\psi_1(x)\psi_2(x)dx = 0. \tag{3.40}
\]

This ends the proof. \(\square\)

**Theorem 3.5** The exact eigenfunctions of SFSLP \((3.20)\) are given as
\[
\mathcal{P}_n^{(\alpha,\beta,\mu)}(x) = x^{-\alpha+\mu-1}L_{\alpha-1}^{-\alpha+\mu-1,\beta}(x), \tag{3.41}
\]
and the corresponding distinct eigenvalues are
\[
\Lambda_n = -\frac{\Gamma(n - \alpha + \mu - 1)\beta^\mu}{\Gamma(n - \alpha - 1)}. \tag{3.42}
\]

**Proof.** We show that \((3.41)\) satisfies \((3.20)\) with eigenvalues \((3.42)\). First, from \((3.41)\), it is clear that $\mathcal{P}_n^{(\alpha,\beta,\mu)}(0) = 0$, we take a fractional integration of order $\mu$ on both sides of \((3.20)\) and substitute \((3.41)\). Then, again by replacing the Caputo derivative by the Riemann-Liouville one, thanks to \((2.7)\), we obtain
\[
x^{\alpha+1}e^{-\beta x}D^{\alpha+1,\mu}x^{-\alpha+\mu-1}L_{\alpha-1}^{-\alpha+\mu-1,\beta}(x) = -\Lambda_n W^\mu e^{-\beta x}L_{\alpha-1}^{-\alpha+\mu-1,\beta}(x). \tag{3.43}
\]

Finally, the fractional derivative on the left-hand side and the fractional integration on the right-hand side is worked out using \((3.23)\) and \((3.25)\) as
\[
\frac{\Gamma(n - \alpha + \mu)}{\Gamma(n - \alpha)}e^{-\beta x}L_{\alpha-1}^{-\alpha+\mu-1,\beta}(x) = -\Lambda_n \frac{\beta^\mu}{\mu}e^{-\beta x}L_{\alpha-1}^{-\alpha+\mu-1,\beta}(x).
\]

By a similar argument on the $e^{-\beta x}L_{\alpha-1}^{-\alpha+\mu-1,\beta}(x)$ being non-zero almost everywhere, we can cancel this term out on both sides and obtain
\[
\Lambda_n = -\frac{\Gamma(n - \alpha + \mu - 1)\beta^\mu}{\Gamma(n - \alpha - 1)}.
\]

Now, we need to check Theorem 3.4 to see if \((3.42)\) verifies that the eigenvalues are indeed real-valued and distinct, and the orthogonality of the eigenfunctions with respect to $W^{\alpha,\beta,\mu}(x) = x^{\alpha-\mu+1}e^{-\beta x}$ is valid:
\[
\int_0^\infty W^{\alpha,\beta,\mu}(x)\psi_k^{\alpha,\beta,\mu}(x)\psi_j^{\alpha,\beta,\mu}(y)dx = \int_0^\infty W^{\alpha,\beta,\mu}(x)(x^{\alpha-\mu+1})^2 L_k^{\alpha-\mu+1,\beta}(x)L_j^{\alpha-\mu+1,\beta}(x)dx
\]
\[
= \int_0^\infty e^{-\beta x}L_k^{\alpha-\mu+1,\beta}(x)L_j^{\alpha-\mu+1,\beta}(x)dx \tag{3.44}
\]
\[
= \kappa_{(k-1)}^{-\alpha+\mu-1,\beta}\delta_{kj},
\]
where
\[
\kappa_{(k-1)}^{-\alpha+\mu-1,\beta} = \frac{\Gamma(k - \alpha + \mu - 1)}{\beta^{\alpha+\mu}(k - 1)!}, \tag{3.45}
\]

where $\kappa_{(k-1)}^{-\alpha+\mu-1,\beta}$ is the orthogonality constant of the family of Laguerre polynomials. The simplicity of the eigenvalues can be also shown in next proof of theorem , and this completes the proof. \(\square\)

**Theorem 3.6** eigen-solutions is dense in the Hilbert space and it forms a basis for $L^2_W[0,\infty)$.}

8
Let $P(x) \in L^2_W[0, \infty)$ and then clearly $f(x) = x^{-\alpha+\mu-1}P(x) \in L^2_W[0, \infty)$, as well when $\mu \in (0, 1)$. Hence

$$
\| \sum_{k=1}^{N} a_k \tilde{P}_k(x) - P(x) \|_{L^2_W[0, \infty)} = \| \sum_{k=1}^{N} a_k x^{-\alpha+\mu-1}L^{-\alpha+\mu+1,\beta}_{k-1}(x) - P(x) \|_{L^2_W[0, \infty)}
$$

$$
= \| x^{-\alpha+\mu-1} \sum_{k=1}^{N} a_k L^{-\alpha+\mu+1,\beta}_{k-1}(x) - x^{\alpha+\mu+1}P(x) \|_{L^2_W[0, \infty)}
$$

$$
= \| x^{-\alpha+\mu-1} \sum_{k=1}^{N} a_k L^{-\alpha+\mu+1,\beta}_{k-1}(x) - f(x) \|_{L^2_W[0, \infty)} \leq \| x^{-\alpha+\mu-1} \|_{L^2_W[0, \infty)} \| \sum_{k=1}^{N} a_k L^{-\alpha+\mu+1,\beta}_{k-1}(x) - f(x) \|_{L^2_W[0, \infty)}
$$

$$
\leq c \| \sum_{k=1}^{N} a_k L^{-\alpha+\mu+1,\beta}_{k-1}(x) - f(x) \|_{L^2_W[0, \infty)}
$$

by using Weierstrass theorem, we obtain

$$
\lim_{N \to \infty} \| \sum_{k=1}^{N} a_k \tilde{P}_k(x) - P(x) \|_{L^2_W[0, \infty)} \leq \lim_{N \to \infty} c \| \sum_{k=1}^{N} a_k L^{-\alpha+\mu+1,\beta}_{k-1}(x) - f(x) \|_{L^2_W[0, \infty)} = 0. \quad (3.47)
$$

Moreover $\sum_{k=1}^{N} a_k \tilde{P}_k(x) \to P(x)$ implying that $\tilde{P}_k(x), k = 1, 2, \ldots$ is dense and it forms a basis for $L^2_W[0, \infty)$. We need to show the simplicity of the eigenvalues, let that corresponding to the eigenvalue $\Lambda_j$ there exits another eigenfunction $\tilde{P}_k(x) \in L^2_W[0, \infty)$ in addition to $\tilde{P}_k(x)$, which is by Theorem 3.4 orthogonal to the rest of the eigenfunctions $\tilde{P}_n(x), n \neq k$ By the density of the eigenfunctions set, i.e., (3.47), we can represent $\tilde{P}_k(x)$ as

$$
\tilde{P}_k(x) = \sum_{n=1}^{\infty} \hat{v}_n \tilde{P}_n(x), \quad (3.48)
$$

Now, by multiplying both sides by $\tilde{P}_j(x), k = 1, 2, \ldots$ and $j \neq k$, and integrating with respect to the weight function $W(x)$, we obtain

$$
\int_0^{\infty} W(x) \tilde{P}_k(x) \tilde{P}_j(x) dx = \sum_{n=1}^{\infty} \hat{v}_n \int_0^{\infty} W(x) \tilde{P}_n(x) \tilde{P}_j(x) dx = \hat{v}_j D_j, \quad (3.49)
$$

this is contradicts to Theorem 3.4. Moreover, the eigenvalues $\Lambda_j$ are simple, and this ends the proof. $\Box$

**Theorem 3.7** Fractional differential operator in (3.21) is self-adjoint.

**Proof:** We let $y_1$ and $y_2$ be generalized Laguerre fractional-polynomials and let consider the following equation

$$
< L^{\alpha,\beta;\mu} y_1, y_2 > - < y_1, L^{\alpha,\beta;\mu} y_2 > = \int_0^{\infty} L^{\alpha,\beta;\mu} y_1, y_2 dx - \int_0^{\infty} y_1, L^{\alpha,\beta;\mu} y_2 dx
$$

$$
= \int_0^{\infty} W D^{\mu} \{ x^{\alpha+1} e^{-\beta x C} D^{\mu} y_1 \}, y_2 dx
$$

$$
- \int_0^{\infty} W D^{\mu} \{ x^{\alpha+1} e^{-\beta x C} D^{\mu} y_2 \}, y_1 dx. \quad (3.50)
$$
By means of property 2.1 we obtain

$$\int_0^\infty x^{\alpha+1}e^{-\beta x} C D^\mu y_1.C D^\mu y_2 dx - \lim_{\alpha \to \infty} \{ y_2 W T^{1-\mu} x^{\alpha+1} e^{-\beta x} D^\mu y_1 \}^0_0$$

$$- \int_0^\infty x^{\alpha+1}e^{-\beta x} C D^\mu y_2.C D^\mu y_1 dx + \lim_{\alpha \to \infty} \{ y_1 W T^{1-\mu} x^{\alpha+1} e^{-\beta x} D^\mu y_2 \}^0_0 = 0. \quad (3.51)$$

We know that $y_1$ and $y_2$ are continuous at 0, now we consider $\psi_{n-1}^{\alpha,\beta,\mu}(y)$ generalized Laguerre fractional-polynomials, we also see that the weighting function $W^{\alpha,\beta,\mu}(x) = x^{\alpha+1}e^{-\beta x}$ and the property 2.7 helps in replacing $C D^\mu$ by $RL D^\mu$. Consequently

$$\lim_{y \to \infty} \{ W T^{1-\mu} y^{\alpha+1} e^{-\beta y} C D^\mu \psi_{n-1}^{\alpha,\beta,\mu}(y) \} = \lim_{y \to \infty} \{ W T^{1-\mu} y^{\alpha+1} e^{-\beta y} RL D^\mu \psi_{n-1}^{\alpha,\beta,\mu}(y) \}$$

$$= \lim_{y \to \infty} \{ W T^{1-\mu} y^{\alpha+1} e^{-\beta y} RL D^\mu y^{\alpha+1} \mu - \alpha + 1 \beta L_{n-1}^{\alpha+1,\beta}(y) \}. \quad (3.52)$$

and by carrying out the fractional $RL$ derivative using Lemma 3.3

$$= \lim_{y \to \infty} \{ W T^{1-\mu} y^{\alpha+1} e^{-\beta y} C D^\mu \psi_{n-1}^{\alpha,\beta,\mu}(y) \}$$

$$= \frac{\Gamma(n - \alpha + \mu)}{\Gamma(n + \alpha + 1)} \lim_{y \to \infty} \{ W T^{1-\mu} e^{-\beta y} L_{n-1}^{\alpha+1,\beta}(y) \}. \quad (3.53)$$

and by working out the fractional integration using Lemma 3.4 we obtain

$$= \frac{\Gamma(n - \alpha + \mu)}{\Gamma(n + \alpha + 1)} \lim_{y \to \infty} \{ e^{-\beta y} L_{n-1}^{\alpha+1,\beta}(y) \} = 0. \quad (3.54)$$

Hence

$$\lim_{\alpha \to \infty} \{ y_2 W T^{1-\mu} x^{\alpha+1} e^{-\beta x} C D^\mu y_1(x) \}^0_0 = 0. \quad (3.55)$$

and

$$\lim_{\alpha \to \infty} \{ y_1 W T^{1-\mu} x^{\alpha+1} e^{-\beta x} C D^\mu y_2(x) \}^0_0 = 0. \quad (3.56)$$

Hence $L^{\alpha,\beta;\mu}$ to be self-adjoint. \(\square\)

Here we do not need to impose boundary conditions. Also by Theorem 3.4 we could see that the fractional-polynomials in (3.41) solutions to the Fractional generalized Laguerre equations (3.26) are orthogonal on the interval $[0, \infty)$ with respect to the weight function $W^{\alpha,\beta,\mu}(x) = x^{\alpha+1}e^{-\beta x}$.

### 3.2 Properties of the eigen-solutions to SFSLP.

The basic properties of the generalized Laguerre fractional-polynomials are summarized below:

- **Fractional derivatives:**

  $$RL D^{\alpha+\mu+1} P_n^{(\alpha,\beta,\mu)}(x) = C D^{\alpha+\mu+1} P_n^{(\alpha,\beta,\mu)}(x) = \frac{\Gamma(n - \alpha + \mu - 1)}{\Gamma(n)} L_{n-1}^{\alpha,\beta}(x). \quad (3.57)$$
• First derivatives:
\[
\frac{d}{dx} P_n^{(\alpha,\beta,\mu)}(x) = (-\alpha + \mu - 1)x^{-\alpha+\mu-2}L_{n-1} - \alpha + \mu - 1, \beta(x) - \beta x^{-\alpha+\mu-2}L_{n-2}(x) .
\] (3.58)
we introduce the derivative operator:
\[
\hat{\partial}_x = \partial_x + (-\alpha + \mu - 1)x^{-1},
\] (3.59)
It is clear that
\[
\partial_x L_{n-1} - \alpha + \mu - 1, \beta = \hat{\partial}_x P_n^{(\alpha,\beta,\mu)}(x).
\] (3.60)
such that
\[
\partial_x^k L_n^{\alpha,\beta} = (-1)^k L_{n-k}^{(\alpha+k,\beta)}(x), \quad n > k
\] (3.61)
• Recurrence relations:
A recurrence relations is obtained for the generalized Laguerre fractional-polynomials
\[
P_n^{(\alpha,\beta,\mu)}(x) = x^{-\alpha+\mu-1},
P_2^{(\alpha,\beta,\mu)}(x) = x^{-\alpha+\mu-1}(-x - \alpha + \mu),
\]
\[
\vdots
\]
\[
P_{n+1}^{(\alpha,\beta,\mu)}(x) = (a_n - b_n x)P_n^{(\alpha,\beta,\mu)}(x) - c_n P_n^{(\alpha,\beta,\mu)}(x),
\]
a_n = \frac{2n - \alpha + \mu - 2}{n}, \quad b_n = \frac{\beta}{n}, \quad c_n = n - \alpha + \mu - 2.
\] (3.62)
\[
P_n^{(\alpha,\beta,\mu)}(x) = \hat{\partial}_x P_n^{(\alpha,\beta,\mu)}(x) - \hat{\partial}_x P_{n+1}^{(\alpha,\beta,\mu)}(x),
\]
x\hat{\partial}_x P_n^{(\alpha,\beta,\mu)}(x) = \frac{n-1}{\beta} P_n^{(\alpha,\beta,\mu)}(x) - \frac{(n + \alpha - 1)}{\beta} P_{n-1}^{(\alpha,\beta,\mu)}(x).
\] (3.63)
• Orthogonality:
\[
\int_0^\infty W^{(\alpha,\beta,\mu)}(x) P_k^{(\alpha,\beta,\mu)}(x) P_j^{(\alpha,\beta,\mu)}(x) dx = \Theta_{k}^{\alpha,\beta,\mu} \delta_{kj}.
\] (3.64)
where
\[
\Theta_{k}^{\alpha,\beta,\mu} = \frac{\Gamma(k - \alpha + \mu)}{\beta^{\alpha+\mu}(k - 1)!}.
\] (3.65)
• Special values:
\[
P_n^{(\alpha,\beta,\mu)}(0) = 1.
\] (3.66)

Lemma 3.5 The eigensolutions to SFSLP denoted by \(P_n^{(\alpha,\beta,\mu)}(x)\) are given by
\[
P_n^{(\alpha,\beta,\mu)}(x) = \sum_{j=0}^n \frac{(-\beta)^j}{j!} \left( \frac{n - \alpha + \mu - 2}{n - j - 1} \right) x^{j - \alpha + \mu - 1}.
\] (3.67)
Definition 3.2 We define the finite-dimensional fractional-polynomial space:

\[ F^{(\mu,\beta)}_N(\mathbb{R}_+) = \{ \varphi = x^{\tilde{\mu}} \psi, \psi \in \mathcal{P}_N, \tilde{\mu} \in (0, 1) \} . \]  

(3.68)

Taking \( \tilde{\mu} = -\alpha + \mu - 1 \), then fractional-polynomial space \( F^{(\mu,\beta)}_N \) can be rewritten as

\[ F^{(\mu,\beta)}_N(\mathbb{R}_+) = F^{(\alpha,\beta,\mu)}_N(\mathbb{R}_+) = \{ \varphi = x^{\tilde{\mu}} \psi, \psi \in \mathcal{P}_N \} = \text{span}\{ \mathcal{P}^{(\alpha,\beta,\mu)}_n : 0 \leq n \leq N \} . \]  

(3.69)

Theorem 3.8 The eigensolutions to (3.20), \( \mathcal{P}_n(x) \), \( n \in \mathbb{N} \) and \( n < \infty \), form a complete hierarchical basis for the finite dimensional space of fractional-polynomials \( F^{(\alpha,\beta,\mu)}_N \), where \( \mu \in (0, 1) \).

Proof. From Definition 4.1, we can re-write (3.67) as

\[ M \tilde{x} = \tilde{\mathcal{P}} , \]  

(3.70)

where

\[ \tilde{x} = \begin{pmatrix} x^{\mu-\alpha-1} \\ x^{\mu-\alpha} \\ \vdots \\ x^{n-1+\mu-\alpha-1} \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{P}} = \begin{pmatrix} \tilde{\mathcal{P}}_1(x) \\ \tilde{\mathcal{P}}_2(x) \\ \vdots \\ \tilde{\mathcal{P}}_2(x) \end{pmatrix} , \]

and finally, \( M = M_{j,k} \) is an \( n \times n \) matrix obtained as

\[ M = \{ M_{j,k} \}_{j,k=1}^n = \begin{pmatrix} n - \alpha + \mu - 2 \\ n - j - 1 \end{pmatrix} , \]

(3.71)

which is a lower-triangular matrix. Thanks to the orthogonality of the \( \tilde{\mathcal{P}}_n(x) \), the eigenfunctions are linearly independent, therefore, the matrix \( M \) is invertible. Let \( \mathcal{M} = M^{-1} \), which is also lower triangular. Hence,

\[ \tilde{x} = \mathcal{M} \tilde{\mathcal{P}} . \]

(3.71)

In other words, each element in the poly-fractonomial space \( F^{(\alpha,\beta,\mu)}_N \), say \( x^{m+\mu-\alpha-1} \), \( 0 \leq m \leq n - 1 \), can be uniquely represented through the following expansion

\[ x^{m+\mu-\alpha-1} = \sum_{k=1}^{n} c_k \mathcal{P}_n(x) = \sum_{k=1}^{n} \{ \mathcal{M}_{mk} \} \mathcal{P}_n(x) = \sum_{k=1}^{m} \{ \mathcal{M}_{mk} \} \mathcal{P}_n(x) , \]

(3.72)

where the last equality holds since \( \mathcal{M} \) is a lower-triangular matrix. As seen in (3.69), the fractal expansion set \( F^{(\alpha,\beta,\mu)}_N \subset F^{(\alpha,\beta,\mu)}_{N+1} \), which indicates that the eigen-solutions \( \tilde{\mathcal{P}}_n \) form a hierarchical expansion basis set.

In Fig. 1, The growth of the magnitude in the eigenvalues of SFSLP, the optimal highest magnitude is achieved when \( \alpha \rightarrow -1, \forall \mu \in (0, 1) \). The growth of the \(|\Lambda_n|\) corresponding to two values of \( \mu = 0.5 \), and \( \mu = 0.99 \) is shown in Fig. 1. The case \( \mu = 0.99 \) leads to an exactly linear growth mode. We emphasize that \( \mu = 0.99 \) grows linearly as opposed to quadratical growth in the Jacobi poly-fractonomials case [13]. In Fig.2, we plot the eigenfunctions of SFSLP, \( \mathcal{P}_n^{(\alpha,\beta,\mu)}(x) \), of different orders and corresponding to different values of \( \mu \) used in Fig.1. In a similar fashion, we compare the eigensolutions with the corresponding standard generalized Laguerre \( L_{\alpha+\mu-1,\beta}(x) \) in each plot.
Figure 1: Magnitude of the eigenvalues of SFSLP, versus $n$, corresponding to $\alpha = -0.9$ and $\beta = 3$, corresponding to different fractional-order $\mu = 0.5$ left: sublinear growth, and $\mu = 0.99$ right: linear growth.

4 Approximation by generalized Laguerre fractional-polynomials

The main concern of this section is to study the approximation properties of the family of generalized Laguerre fractional-polynomials and show that approximation by generalized Laguerre fractional-polynomials leads to typical spectral convergence for functions in appropriate weighted Sobolev spaces involving fractional derivatives. Such approximation results play a crucial role in the analysis of fractional spectral methods for fractional differential equations.

We can study the approximation properties of the family of generalized Laguerre fractional-polynomials. In
such setting, we can expand any \( v \in L^2_{W(\alpha, \beta, \mu)}(\mathbb{R}^+) \) as

\[
v(x) = \sum_{n=1}^{\infty} \hat{v}_n^{(\alpha, \beta, \mu)} P_n^{(\alpha, \beta, \mu)}(x),
\]

where

\[
\hat{v}_n^{(\alpha, \beta, \mu)} = \frac{1}{\Theta_n^{\alpha, \beta, \mu}} \int_0^{\infty} v(x) P_n^{(\alpha, \beta, \mu)}(x) W^{(\alpha, \beta, \mu)}(x) dx.
\]

**Theorem 4.1** the expansion coefficients \( \hat{v}_n \) in (4.1) decay.

**Proof:** By multiplying (4.1) by \( L^{\alpha, \beta, \mu} P_k^{(\alpha, \beta, \mu)}(x) \), \( k = 1, 2, \ldots, N \), and integrating in the interval \([0, \infty)\), we
obtain

\[ \int_0^\infty v(x)L^{\alpha,\beta;\mu}\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx = \int_0^\infty \left( \sum_{n=1}^N \hat{v}_n \mathcal{P}^{(\alpha,\beta,\mu)}_n(x) \right) L^{\alpha,\beta;\mu}\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx. \]  

(4.3)

Hence

\[ \int_0^\infty v(x)L^{\alpha,\beta;\mu}\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx = \int_0^\infty \left( \sum_{n=1}^N \hat{v}_n \mathcal{P}^{(\alpha,\beta,\mu)}_n(x) \right) L^{\alpha,\beta;\mu}\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx, \]  

(4.4)

\[ \int_0^\infty v(x)L^{\alpha,\beta;\mu}\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx = - \sum_{n=1}^N \hat{v}_n \Lambda_n \int_0^\infty x^{\alpha-\mu+1} e^{-\beta x} \mathcal{P}^{(\alpha,\beta,\mu)}_n(x) \mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx, \]  

(4.5)

and thanks to the orthogonality property (3.64) we get

\[ \hat{v}_k = - \frac{1}{\Lambda_k \Theta_k^{\alpha,\beta,\mu}} \int_0^\infty v(x)L^{\alpha,\beta;\mu}\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)dx, \]  

(4.6)

or equivalently by (3.37),

\[ \hat{v}_k = - \frac{1}{\Lambda_k \Theta_k^{\alpha,\beta,\mu}} \int_0^\infty v(x)^W D^{\mu} \{ x^{\alpha+1} e^{-\beta x} D^{\mu} (\mathcal{P}^{(\alpha,\beta,\mu)}_k(x)) \} dx. \]  

(4.7)

Now, by carrying out the fractional integration-by-parts (2.13), we get

\[ \hat{v}_k = - \frac{1}{\Lambda_k \Theta_k^{\alpha,\beta,\mu}} \int_0^\infty x^{\alpha+1} e^{-\beta x} D^{\mu} v(x)^C D^{\mu} (\mathcal{P}^{(\alpha,\beta,\mu)}_k(x))dx. \]  

(4.8)

Again, by the fractional integration-by-parts (2.13), we obtain

\[ \hat{v}_k = - \frac{1}{\Lambda_k \Theta_k^{\alpha,\beta,\mu}} \int_0^\infty (\mathcal{P}^{(\alpha,\beta,\mu)}_k(x))^W D^{\mu} \{ x^{\alpha+1} e^{-\beta x} D^{\mu} (v(x)) \} dx, \]  

(4.9)

Figure 2: Eigenfunctions of SFSLP, \( \mathcal{P}^{(\alpha,\beta,\mu)}_n(x) \), versus \( x \), for \( n = 1 \) (first row), \( n = 2 \) (second row), \( n = 5 \) (third row), and \( n = 10 \) (last row), corresponding to the fractional-order \( \mu = 0.5 \) (left column) and \( \mu = 0.99 \) (right column). Here, we take the same values \( \alpha = -0.9 \) and \( \beta = 3 \), as shown in Fig. 1.
or equivalently
\[
\hat{v}_k = \frac{-1}{\Lambda_k \Theta_k^{\alpha,\beta,\mu}} \int_0^\infty P_k^{(\alpha,\beta,\mu)}(x) L^{\alpha,\beta,\mu}v(x)dx,
\]
(4.10)
when
\[
|\hat{v}_k| \leq \frac{1}{\Lambda_k \Theta_k^{\alpha,\beta,\mu}} \left( \int_0^\infty (P_k^{(\alpha,\beta,\mu)}(x))^2 W^{(\alpha,\beta,\mu)}(x)dx \right)^{1/2} \left( \int_0^\infty |v_1(x)|^2 W(x)dx \right)^{1/2} = \frac{\rho}{|\Lambda_n|} \|v_1(x)\|_{L^2_W},
\]
(4.11)
where \(v_1(x) \equiv L^{\alpha,\beta,\mu}v(x)/W^{(\alpha,\beta,\mu)}(x)\). By carrying out the fractional integration-by-parts another \((m-1)\) times, and setting \(v_m(x) \equiv L^{\alpha,\beta,\mu}f_{m-1}(x)/W^{(\alpha,\beta,\mu)}(x)\), we obtain
\[
|\hat{v}_k| \leq \frac{\rho}{|\Lambda_k|} \|v_m(x)\|_{L^2_W}.
\]
(4.12)
Consequently, if the function \(v(x) \in C^\infty[0, \infty)\), we recover the exponential decay of the expansion coefficients \(\hat{v}_k\).

\[\square\]

4.1 Optimal approximation for the generalized Laguerre fractional-polynomials

We now turn to several orthogonal projections which are frequently using in the generalized Laguerre fractional-polynomials spectral method.

Consider the \(L^2_W^{(\alpha,\beta,\mu)}(\mathbb{R}_+)^m\)-orthogonal projection upon \(F_N^{(\alpha,\beta,\mu)}(\mathbb{R}_+)^m\), defined by
\[
\left( I_N^{(\alpha,\beta,\mu)}v - v, u_N \right)_{W^{(\alpha,\beta,\mu)}} = 0, \quad u_N \in F_N^{(\alpha,\beta,\mu)}(\mathbb{R}_+).\]
(4.13)
By the orthogonality (3.55),
\[
\int_0^\infty C^\alpha+k P_n^{(\alpha,\beta,\mu)}(x) C^\alpha+k P_m^{(\alpha,\beta,\mu)}(x) w^{\alpha,\beta,\mu} dx = \chi^{(\alpha,\beta,\mu)}_{n,k} \delta_{n,m},
\]
(4.14)
where
\[
\chi^{(\alpha,\beta,\mu)}_{n,k} = \frac{\beta^2 \Gamma(n - \alpha + \mu - 1)^2}{\Gamma(n)^2} \kappa_{n-k}^{\alpha,\beta,\mu} = \frac{\beta^{k-1} \Gamma(n - \alpha + \mu - 1)^2}{\Gamma(n)(n - k)!},
\]
and there holds the Parseval identity:
\[
\|v\|_{W^{(\alpha,\beta,\mu)}}^2 = \sum_{n=1}^\infty \Theta_n^{(\alpha,\beta,\mu)} \|\hat{v}_n^{(\alpha,\beta,\mu)}\|^2.
\]
(4.16)

**Remark 4.1** It is worthwhile to point out that for \(\alpha > -1\) and \(\tilde{\alpha} = -\alpha + \mu - 1\), we have
\[
\left( C^\alpha+k I_N^{(\alpha,\beta,\mu)}v - v, D^k Q_N \right)_{\alpha,\beta,\mu} = 0, \quad \forall Q_N \in \mathcal{P}_N.
\]
(4.17)
Notice that
\[
\left( I_N^{(\alpha,\beta,\mu)}v - v \right)(x) = \sum_{n=N+1}^\infty \hat{v}_n^{(\alpha,\beta,\mu)} P_n^{(\alpha,\beta,\mu)}(x),
\]
(4.18)
and \(\mathcal{P}_N = \text{span}\{L_n^{(\alpha,\beta)}(x) : 0 \leq n \leq N\}\) and \(C^\alpha+k = C^\alpha D^k\), we obtain (4.17) from (3.57), (3.61) and the orthogonality of the classical generalized Laguerre polynomials, we notice that \(W^{\alpha,\beta,\mu}(x) = W^{\alpha,\beta,\mu+1,\mu}(x)\), \(w^{\alpha,\beta}(x)\) is the weight function of the family of Laguerre polynomials.

To characterize the regularity of \(v\), we introduce the non-uniformly weighted space involving fractional derivatives:
We find that

\[ A_{\alpha,\beta}^m(\mathbb{R}+) := \{ v \in L^2_{W(\alpha,\beta)}(\mathbb{R}+) : C \hat{D}^{\alpha+k} v \in L^2_{W(k,\beta)}(\mathbb{R}+) \text{ for } 0 \leq k \leq m \}, \quad m \in \mathbb{N}_0. \]  

By (4.1), (4.14) and (4.18), we have that

\[ C \hat{D}^{\alpha+k}(I_N^{(\alpha,\mu)} v - v)(x) = \sum_{n=N+1}^{\infty} \frac{(-\beta)^k \Gamma(n - \alpha + \mu - 1)}{\Gamma(n)} \hat{D}^{(\alpha,\beta)} I_n^{k,\beta}(x), \quad \text{whence} \]

\[ \| C \hat{D}^{\alpha+k}(I_N^{(\alpha,\mu)} v - v)(x) \|_{W^{(\alpha,\beta)}}^2 = \sum_{j=N+1}^{\infty} \chi_{\alpha,\beta,\mu}^{(j)} \chi_{j,k}^{(\alpha,\beta,\mu)} \chi_{j,m}^{(\alpha,\beta,\mu)} \| \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta)}}^2. \]

On the other hand,

\[ \| C \hat{D}^{\alpha+k} v(x) \|_{W^{(\alpha,\beta)}}^2 = \sum_{j=k}^{\infty} \chi_{\alpha,\beta,\mu}^{(j)} \chi_{j,k}^{(\alpha,\beta,\mu)} \| \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta)}}^2. \]

We have the following basic result.

**Theorem 4.2**

- For \(0 \leq k \leq m \leq N\),

\[ \| C \hat{D}^{\alpha+k}(I_N^{(\alpha,\mu)} v - v)(x) \|_{W^{(\alpha,\beta)}} \leq c\beta^{(k-m)/2} \sqrt{\frac{(N-m+1)!}{(N-k+1)!}} \| C \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta)}}. \]  

In particular, if \(m\) is fixed, then

\[ \| C \hat{D}^{\alpha+k}(I_N^{(\alpha,\beta,\mu)} v - v)(x) \|_{W^{(\alpha,\beta)}} \leq c(\beta N)^{(k-m)/2} \| C \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta)}}. \]  

- For \(0 \leq m \leq N\), we also have the \(L^2_{W(\alpha,\beta,\mu)}\) estimates:

\[ \| (I_N^{(\alpha,\beta,\mu)} v - v)(x) \|_{W^{(\alpha,\beta,\mu)}} \leq c\beta^{(\alpha-m+1)/2} \| C \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta,\mu)}}. \]  

In particular, if \(m\) is fixed, then

\[ \| (I_N^{(\alpha,\beta,\mu)} v - v)(x) \|_{W^{(\alpha,\beta,\mu)}} \leq c(\beta N)^{(\alpha-m+1)/2} \| C \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta,\mu)}}^2. \]

**Proof.** By (4.1), (4.13) and (4.21),

\[ \| C \hat{D}^{\alpha+k}(I_N^{(\alpha,\beta,\mu)} v - v)(x) \|_{W^{(\alpha,\beta,\mu)}} \|_{W^{(\alpha,\beta,\mu)}} \]

\[ = \sum_{j=N+1}^{\infty} \chi_{\alpha,\beta,\mu}^{(j)} \chi_{j,k}^{(\alpha,\beta,\mu)} \chi_{j,m}^{(\alpha,\beta,\mu)} \| \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta,\mu)}}^2, \]

We find that

\[ \frac{\chi_{\alpha,\beta,\mu}^{(j+1)} \chi_{j,k}^{(\alpha,\beta,\mu)} \chi_{j,m}^{(\alpha,\beta,\mu)}}{\chi_{\alpha,\beta,\mu}^{(j)} \chi_{j+1,k}^{(\alpha,\beta,\mu)} \chi_{j+1,m}^{(\alpha,\beta,\mu)}} \leq \beta^{-m} \frac{(N-m+1)!}{(N-k+1)!}. \]

Therefore, a combination of (4.22), (4.23) and (4.24) yields that

\[ \| C \hat{D}^{\alpha+k}(I_N^{(\alpha,\mu)} v - v)(x) \|_{W^{(\alpha,\beta,\mu)}} \leq c\beta^{(k-m)/2} \| C \hat{D}^{\alpha+m} v(x) \|_{W^{(\alpha,\beta,\mu)}}. \]
which implies result (4.23). If \( m \) is fixed, then we find that

\[
\frac{\chi_{N+1,k}}{\chi_{N+1,m}} \leq \beta^{m-k} \frac{(N - m + 1)!}{(N - k + 1)!} \leq (\beta)^{(m-k)}(N)^{(k-m)},
\]  

(4.30)

Hence

\[
\| {C} D^\alpha + k (f_\alpha^N - v)(x) \|_{w^{(k,\beta)}} \leq c(\beta N)^{(k-m)/2} \| {C} D^\alpha + m v(x) \|_{w^{(m,\beta)}},
\]

(4.31)

which implies result (4.24). The \( L^2_{w^{(\alpha,\beta,\mu)}} \) estimates can be obtained by using the same argument. We sketch the derivation below. By (4.16) and (4.21),

\[
\| (f_\alpha^N - v)(x) \|_{W^{(\alpha,\beta,\mu)}} = \sum_{j=N+1}^{\infty} \theta^\alpha_{\alpha,\beta,\mu} \| \psi_j^{(\alpha,\beta,\mu)} \| = \sum_{j=N+1}^{\infty} \frac{\theta^\alpha_{\alpha,\beta,\mu}}{\chi_{N+1,m}} x_j^{\alpha,\beta,\mu} \| \psi_j^{(\alpha,\beta,\mu)} \|^2 \\
\leq \frac{\theta^\alpha_{\alpha,\beta,\mu}}{\chi_{N+1,m}} \sum_{j=N+1}^{\infty} x_j^{\alpha,\beta,\mu} \| \psi_j^{(\alpha,\beta,\mu)} \|^2.
\]

(4.32)

By the Stirling formula, \( \Gamma(k + 1) = \sqrt{2\pi k} k^k e^{-k} (1 + O(k^{-1}) \) , we find that

\[
\frac{\theta^\alpha_{\alpha,\beta,\mu}}{\chi_{N+1,m}} = \frac{\Gamma(N - \alpha + \mu)}{\beta - \alpha + \mu} \frac{\Gamma(N + 1)}{\Gamma(N - \alpha + \mu)^2} \frac{\beta^{1-m}(N - m + 1)!}{\Gamma(N + 1)} \leq C\beta^{(\alpha - m + 1)} \frac{(N - m + 1)!}{(N - \alpha + \mu)}.
\]

(4.33)

Hence

\[
\| (f_\alpha^N - v)(x) \|_{W^{(\alpha,\beta,\mu)}} \leq c\beta^{(\alpha - m + 1)/2} \sqrt{\frac{(N - m + 1)!}{(N - \alpha + \mu)!}} \| {C} D^\alpha + m v(x) \|_{w^{(m,\beta)}},
\]

(4.34)

which implies result (4.24). In particular, if \( m \) is fixed, then

\[
\frac{\theta^\alpha_{\alpha,\beta,\mu}}{\chi_{N+1,m}} \leq c(\beta N)^{(\alpha - m + 1)/2}.
\]

(4.35)

This ends the proof. □

5  Numerical Results

In the following examples, we test the convergence rate in approximating some fractional-polynomials. By Theorem 4.3 we can exactly represent any fractional-polynomials of order \( N + \mu \) in terms of the first \( N \) singular generalized Laguerre fractal basis functions. We introduce some fractional-polynomials decay algebraically or exponentially at infinity in example 1. The fast (super) spectral convergence of the our fractal basis functions shown in Fig. 3, compared to that of the generalized Laguerre expansion, highlights the efficiency of generalized Laguerre fractal-polynomials basis functions in approximating fractional-polynomial functions. Clearly, the errors decay very fast as \( N \) increases. Finally, we consider fractal functions such as growing fast at infinity in example 2. In Fig. 4, we plot the values of global errors with different \( N \). In particular, new fractal basis works well for test functions growing up as \( x \) increases.

**Example 1.** We consider some fractional-polynomials decay algebraically or exponentially at infinity
(a). Exponential decay without oscillations at infinity.

\[ f(x) = x^{1+\frac{3}{5}}e^{-x}. \]

(b). Algebraic decay without oscillations at infinity.

\[ f(x) = \frac{x^{1+\frac{1}{2}}}{(1+x)^h}, \quad h > 1. \]

(c). Algebraic decay with oscillations at infinity.

\[ f(x) = \frac{\sin(kx^{\frac{3}{4}})}{(1+x)^h}, \quad h > 1. \]

(d). Exponential decay with oscillations at infinity.

\[ f(x) = \sin(kx^{\frac{3}{4}})e^{-x}. \]

Example 2. Grows up with oscillations at infinity.

\[ f(x) = x^{1+\frac{3}{4}}\sin(kx). \]

6 Summary

In this article, we have considered singular fractional Sturm-Liouville eigen-problems (SFSLP) of order 2\(\mu\) with fractional differential operators associated to Weyl and Caputo type of the same fractional-order \(\mu\). This choice is motivated by a proper fractional integration-by-parts. Also, we have obtained the analytical eigensolutions, and their recursive relation, to SFSLP which is defined as generalized Laguerre fractional-polynomials. These eigenfunctions were shown to be orthogonal with respect to the weight function, which is also hierarchical. The analysis of approximations by generalized Laguerre fractional-polynomials is derived. Such approximation plays an important role in the analysis of fractional spectral methods for fractional differential equations. The exponential convergence in approximating fractal functions in addition to some other fractal functions (such as growing fast at infinity) highlighted the efficiency of the new fractal basis functions compared to standard generalized polynomials.
Figure 3: Global errors in Theorem 4.2 versus $N$ with $\beta = 3$ of expansion terms in (4.1) to the functions (a,b,c,d)

Figure 4: Global errors in Theorem 4.2 versus $N$ with $\beta = 3$ of expansion terms in (4.1) to example 2.

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