REGIONS OF THE TYPE C CATALAN ARRANGEMENT

Anne Micheli and Vu Nguyen Dinh

Abstract. In this paper, we determine the number of regions of the type C Catalan arrangement which is $2^n n! \sum_{s=1}^{n-1} s^2 \frac{2^{n-s}}{n-s} + 4^n n!$. Moreover, we exhibit a bijection between rooted labeled ordered forests with a subset of their leaves and the regions of the type C Catalan arrangement.

The type C Catalan arrangement $C_{(-1,0,1)}(n)$ in $\mathbb{R}^n$ is the set of the hyperplanes $\{x_i - x_j = s, x_i + x_j = s, 2x_i = s, \forall s \in \{-1, 0, 1\}, \forall 1 \leq i < j \leq n\}$. The regions of $C_{(-1,0,1)}(n)$ are the connected components of $\mathbb{R}^n \setminus \bigcup_{H \in C_{(-1,0,1)}(n)} H$. In this paper, we determine that the number of regions of $C_{(-1,0,1)}(n)$ is $2^n n! \sum_{s=1}^{n-1} s^2 \frac{2^{n-s}}{n-s} + 4^n n!$. Moreover, we exhibit a bijection between rooted labeled ordered forests with a subset of their leaves and the regions of $C_{(-1,0,1)}(n)$.

Type A arrangements have been and are still vastly studied in combinatorics, in particular the problem of bijectively enumerating the regions of type A arrangements. The reader can find an introduction to hyperplane arrangement and its connexions to combinatorics by R.P. Stanley [9]. The equation of an hyperplane of a type A arrangement is of the form $x_i - x_j = s$ with $s$ in $\mathbb{Z}$ and $i, j$ in $\{1, n\} = \{1, 2, \ldots, n\}$. The case of the braid $(s = 0)$ arrangement is easy to understand, the Shi $(s = 0, 1)$ and Catalan $(s = -1, 0, 1)$ cases have nice and simple formulas which have been bijectively interpreted [3, 5, 6, 8, 9].

The number of regions of the Linial $(s = 1)$ arrangement was known but it is only recently that O. Bernardi gave a bijective interpretation [4]. His bijection extends to the regions of many type A arrangements [4], including Catalan, Shi and semi-order type A arrangements. Our bijections between orders, families of forests and regions of the type C Catalan arrangement were inspired by the Bernardi bijection.

The results on type C arrangement are less extensive. In 1996, C.A. Athanasiadis computed the number of regions of the type C Shi arrangement [1]. The formula obtained is very simple $(2n+1)^n$ and a bijective proof was given by K. Mészáros [7] in 2013, which was a generalization of the bijection exhibited by C.A. Athanasiadis and S. Linusson in the type A case [3]. C.A. Athanasiadis also computed among others the number of regions of the Linial arrangement of type C [2]. No bijective proof of this enumeration has yet emerged.

The paper is divided in three sections. In section 1, we explain how to go bijectively from regions of the type C Catalan arrangement to some orders. Then in section 2, we exhibit a bijection between these orders and rooted labeled ordered forests. Finally, we compute in section 3 the number of regions of the type C Catalan arrangement.

1. From regions to orders

In this section we show that each region $R$ of the type C Catalan arrangement corresponds bijectively to a specific order between the variables $x_i$ and $1 + x_i$ for any $i$ in $[-n, n] \setminus \{0\}$ where $(x_1, \ldots, x_n)$ denotes the coordinates of any point of $R$ and $x_{-i} = -x_i$ for all $i$ in $[1, n]$.

In the sequel, for any $i$ in $[-n, n] \setminus \{0\}$, we denote by:

- $\alpha_i(0)$ the variable $x_i$,
- $\alpha_i(1)$ the variable $1 + x_i$.

These notations are derived from the paper of O. Bernardi [4]. We also denote by $\mathcal{A}_{2n}$ the alphabet $\{\alpha_i^{(0)}, \alpha_i^{(1)}, \forall i \in [-n, n] \setminus \{0\}\}$.

We first define a symmetric annotated 1-sketch and explain its symmetries. Then, in a second time, we will show that the regions of the type C Catalan arrangement are in one-to-one correspondance with symmetric annotated 1-sketches.

1.1. Symmetric annotated 1-sketch.

Definition 1.1. A symmetric annotated 1-sketch of size $2n$ is a word $\omega = w_1 \ldots w_{4n}$ that satisfies for all $i, j \in [-n, n] \setminus \{0\}$:


\( (i) \) \( \{w_1, ..., w_{2n}, ..., w_{4n}\} = \mathcal{A}_{2n} \),

\( (ii) \alpha_i^{(0)} \) appears before \( \alpha_j^{(1)} \);

\( (iii) \) If \( \alpha_i^{(0)} \) appears before \( \alpha_j^{(0)} \) then \( \alpha_i^{(1)} \) appears before \( \alpha_j^{(1)} \);

\( (iv) \) If \( \alpha_i^{(0)} \) appears before \( \alpha_j^{(0)} \) then \( \alpha_i^{-j} \) appears before \( \alpha_j^{-j} \), \( \forall s \in \{0, 1\} \).

Let \( D^{(1)}(2n) \) be the set of symmetric annotated 1-sketches of size 2n.

**Example 1.1.** \( \omega = \alpha_2^{(0)} \alpha_0^{(0)} \alpha_3^{(1)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_1^{(0)} \in D^{(1)}(6) \).

**Remark 1.1.**

1. Condition (ii) of Definition 1.1 implies that a symmetric annotated 1-sketch starts with a sequence of \( \alpha_i^{(0)} \) letters and ends with a sequence of \( \alpha_i^{(1)} \) letters.

2. Condition (iv) of Definition 1.1 implies that the subword of \( \omega \) composed of the \( \alpha^{(0)} \) letters has the form \( \alpha_i^{(0)} \alpha_i^{(0)} \alpha_{i+n}^{(0)} \alpha_{2i}^{(0)} \alpha_{2i}^{(0)} \alpha_{2i}^{(0)} \) with \( \{i_1, ..., i_n\} = [1, n] \). Moreover, the subword of \( \omega \) composed of the \( \alpha^{(1)} \) letters is exactly \( \alpha_{1+1}^{(1)} \alpha_{1-n}^{(1)} \alpha_{-i+1}^{(1)} \alpha_{-i}^{(1)} \).

Furthermore, a symmetric annotated 1-sketch is the result of a specific shuffle between two words on the alphabet \( \mathcal{A}_{2n} \) where one is the symmetric of the other in the following sense:

**Definition 1.2.** Let \( \omega_1 \) be a word on \( \mathcal{A}_{2n} \) that ends with letter \( u \), i.e. \( \omega_1 = \omega_0 u \). We define the symmetric of \( \omega_1 \) as a word \( \mathcal{P} \mathcal{P}_1 = \mathcal{P}_1 \mathcal{P}_0 \) where \( \mathcal{P}_1 = \alpha_1^{(1-s)} \) if \( u = \alpha_k^{(s)} \), \( s \in \{0, 1\} \) and \( \mathcal{P}_0 \) is recursively defined in the same way.

**Example 1.2.** The symmetric of \( \omega_1 = \alpha_3^{(0)} \alpha_0^{(0)} \alpha_3^{(1)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_1^{(0)} \) is \( \mathcal{P}_1 = \alpha_3^{(0)} \alpha_0^{(0)} \alpha_3^{(1)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_0^{(0)} \alpha_1^{(0)} \).

Now, a symmetric annotated 1-sketch is the combination of two symmetric words \( \omega_1 \) and \( \omega_2 = \mathcal{P}_1 \). As a matter of fact, we will now explain how we obtain \( \omega_1 \) and \( \omega_2 \) from \( \omega \). We call words of the form \( \omega_1 \), annotated 1-sketches which formal definition is:

**Definition 1.3.** An annotated 1-sketch of size \( n \) is defined by \( 2n \) letters \( \alpha_i^{(0)} \) and \( \alpha_i^{(1)} \), \( k \) in \([1, n]\) such that \( \{j_1, ..., j_n\} = [1, n] \) and which satisfies conditions (ii) and (iii) of Definition 1.1.

We denote by \( A_{n,s} \), \( n \leq s \leq 2n - 1 \), the set of annotated 1-sketches where the rightmost letter \( \alpha_i^{(0)} \) is at position \( s \).

Thus we get that:

**Proposition 1.1.** Any symmetric annotated 1-sketch \( \omega \) is the composition of an annotated 1-sketch \( \omega_1 \) and its symmetric \( \mathcal{P}_1 \).

**Proof.** We define \( \omega_1 \) as the subword of \( \omega \) composed of the \( n \) leftmost \( \alpha_i^{(0)} \) letters and the corresponding \( \alpha_i^{(1)} \) letters (if \( \alpha_i^{(0)} \) appears in \( \omega \) then \( \alpha_i^{(1)} \) appears in \( \omega_1 \)). Remark 1.2 implies that \( \alpha_i^{(0)} \) and \( \alpha_i^{(1)} \) cannot both belong to the set of the \( n \) leftmost \( \alpha_i^{(0)} \) letters of \( \omega \). Thus, it is easy to see that \( \omega_1 \) is an annotated 1-sketch.

This remark and condition (iv) of Definition 1.1 also imply that \( \omega_2 \) the subword of \( \omega \) composed of the letters not in \( \omega_1 \) is the symmetric of \( \omega_1 \).

**Example 1.3.** \( \omega \) of Example 1.1 is composed of \( \omega_1 = \alpha_3^{(0)} \alpha_0^{(0)} \alpha_3^{(1)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_0^{(0)} \alpha_1^{(0)} \alpha_1^{(0)} \in A_{3,4} \) and \( \mathcal{P}_1 \).

Conversely, for any annotated 1-sketch \( \omega_1 \), we can construct a symmetric annotated 1-sketch, the result of shuffles between \( \omega_1 \) and \( \mathcal{P}_1 \). We first give the definition of these shuffles and then prove the assertion.

**Definition 1.4.** Let \( \psi = \alpha_1^{(1)} ... \alpha_k^{(1)} \). We define the set of shuffles \( \psi \bowtie \bar{\psi} \) recursively with \( \psi \bowtie \bar{\psi} = \{\epsilon\} \) if \( \psi \) is the empty word \( \epsilon \), as the set of following words:

- \( \alpha_i^{(0)} \psi \bowtie \bar{\psi} \alpha_j^{(1)} \) with \( \psi' = \alpha_{j_1}^{(1)} ... \alpha_{j_{k-1}}^{(1)} \psi' = \epsilon \) if \( k = 1 \),

- \( \alpha_{j_1}^{(1)} ... \alpha_{j_k}^{(1)} \psi \bowtie \bar{\psi} \alpha_{k_1}^{(1)} ... \alpha_{k_{l-1}}^{(1)} \) with \( \psi' = \alpha_{j_{k+1}}^{(1)} ... \alpha_{j_{l-1}}^{(1)} \psi' = \epsilon \) if \( i = k+1, ... \leq l \leq k - 1, \alpha_{j_{k+1}}^{(1)} ... \alpha_{j_{l-1}}^{(1)} \) \( \epsilon \).

**Example 1.4.** The set of shuffles \( \psi \bowtie \bar{\psi} \) with \( \psi = \alpha_{j_1}^{(1)} \alpha_{j_2}^{(1)} \) is composed of the four words \( \alpha_{-j_1}^{(0)} \alpha_{-j_2}^{(0)} \alpha_{-j_1}^{(1)} \alpha_{-j_2}^{(1)} \), \( \alpha_{-j_1}^{(1)} \alpha_{-j_2}^{(1)} \alpha_{-j_1}^{(0)} \alpha_{-j_2}^{(0)} \), \( \alpha_{j_1}^{(1)} \alpha_{j_2}^{(1)} \alpha_{j_1}^{(0)} \alpha_{j_2}^{(0)} \) and \( \alpha_{j_1}^{(1)} \alpha_{j_2}^{(1)} \alpha_{j_1}^{(0)} \alpha_{j_2}^{(0)} \).
Definition 1.5. Let \( \omega_1 = \omega_0 \alpha^{(0)}_j \psi \) with \( \psi = \alpha^{(1)}_{j-n+1} \cdots \alpha^{(1)}_{j-2} \alpha^{(1)}_{jn} \), be an annotated 1-sketch. Then \( \omega_1 \bowtie \varpi_1 = \omega_0 \alpha^{(0)}_j \psi \bowtie \varpi \alpha^{(1)}_{jn} \omega_0 = \{ \omega_0 \alpha^{(0)}_j \varpi \omega_0 \alpha^{(1)}_{jn}, \varpi \in \psi \bowtie \varpi \} \).

Proposition 1.2. For any annotated 1-sketch \( \omega_1 \) of size \( n \), \( \omega_1 \bowtie \varpi_1 \subset D^{(1)}(2n) \).

Example 1.5. \( \omega_1 = \alpha^{(0)}_2 \alpha^{(0)}_2 \alpha^{(1)}_3 \alpha^{(1)}_3 \alpha^{(1)}_3 \in A_{3,4} \). Then \( \omega_1 \bowtie \varpi_1 \) is the set of 4 elements:

\[
\alpha^{(0)}_1 \alpha^{(0)}_2 \alpha^{(0)}_3 \alpha^{(0)}_1 \alpha^{(1)}_2 \alpha^{(1)}_2 \alpha^{(1)}_2 \alpha^{(1)}_3, \alpha^{(0)}_2 \alpha^{(0)}_2 \alpha^{(0)}_3 \alpha^{(0)}_1 \alpha^{(1)}_3 \alpha^{(1)}_3 \alpha^{(1)}_2 \alpha^{(1)}_2, \alpha^{(0)}_2 \alpha^{(0)}_2 \alpha^{(0)}_3 \alpha^{(0)}_1 \alpha^{(1)}_3 \alpha^{(1)}_3 \alpha^{(1)}_2 \alpha^{(1)}_2, \alpha^{(0)}_2 \alpha^{(0)}_2 \alpha^{(0)}_3 \alpha^{(0)}_1 \alpha^{(1)}_3 \alpha^{(1)}_3 \alpha^{(1)}_2 \alpha^{(1)}_2.
\]

Proof. We must prove that any word of \( \omega_1 \bowtie \varpi_1 \) is a symmetric annotated 1-sketch, meaning that it verifies conditions (i) to (iv) of Definition 1.1.

Conditions (i), (ii) and (iii) are straightforward, since \( \omega_1 \) and \( \varpi_1 \) are annotated 1-sketches, each one the symmetric of the other, and their letters are not permuted.

Let \( \omega_1 = \omega_0 \alpha^{(0)}_j \psi \) with \( \psi = \alpha^{(1)}_{j-n+1} \cdots \alpha^{(1)}_{j-2} \alpha^{(1)}_{jn} \). A word of \( \omega_1 \bowtie \varpi_1 \) which obviously verifies condition (iv), or has one of the following form and we can thus check recursively that it verifies condition (iv):

\[
\begin{align*}
\omega_0 \alpha^{(0)}_j \alpha^{(0)}_j \alpha^{(0)}_j \alpha^{(0)}_j \varpi \alpha^{(1)}_{jn} \varpi, \varpi' &= \alpha^{(1)}_{j-n+1} \cdots \alpha^{(1)}_{j-2} \alpha^{(1)}_{jn}, \\
\omega_0 \alpha^{(0)}_j \alpha^{(0)}_j \cdots \alpha^{(0)}_j \alpha^{(0)}_j \varpi \varpi_0, \varpi' &= \alpha^{(1)}_{j-n+1} \cdots \alpha^{(1)}_{j-2} \alpha^{(1)}_{jn}, \varpi &= \alpha^{(1)}_{j-n+1} \cdots \alpha^{(1)}_{j-2} \alpha^{(1)}_{jn}, \varpi' &= \alpha^{(1)}_{j-n+1} \cdots \alpha^{(1)}_{j-2} \alpha^{(1)}_{jn}.
\end{align*}
\]

\[\square\]

1.2. Bijection between regions and symmetric annotated 1-sketches.

A symmetric annotated 1-sketch corresponds to a specific order between the variables \( x_1 \) and \( x_j \) for any \( j \) in \([-n,n]\). We show here that these orders are bijectively related to the coordinates of the points of the regions of the type C Catalan arrangement.

Proposition 1.3. There is a one to one correspondence between regions of the type C Catalan arrangement in \( R^n \) and the symmetric annotated 1-sketches of size \( 2n \).

Proof. Let \( x \in R^n \) be a point such that \( x_i + s = x_j + t \) then \( x \in \bigcup_{H \in \mathcal{C}(1,0,1)} H \). Therefore, for any \( x = \{x_1, \ldots, x_n\} \) that belongs to \( R^n \setminus \bigcup_{H \in \mathcal{C}(1,0,1)} H \), the elements of \( \{x_i + s : i \in [-n,n] \setminus \{0\}, s \in \{0,1\}\} \) are all distinct, with \( x_i = x_{i+1} = x_{i+2} = \cdots = x_{i+s} \) for all \( i \). We define \( \sigma(x) = w_1 w_2 \cdots w_{4n} \), where \( w_p = \alpha_1^{(s)} \) if \( z_p = x_i + s \) with \( z_1 < z_2 < \cdots < z_{4n} \).

\( \sigma(x) \) satisfies conditions (i) - (iii) of Definition 1.1. We now prove that \( \sigma(x) \) satisfies condition (iv) of Definition 1.1. Indeed, if \( \alpha_1^{(s)} \) appears before \( \alpha_2^{(s)} \) with \( s \in \{0,1\} \) then \( x_i < x_{i+s} \), hence \( x_{i+s} < x_i \). It induces that \( \alpha_1^{(s)} \) appears before \( \alpha_2^{(s)} \). Therefore \( \sigma(x) \) is a symmetric annotated 1-sketch of size \( 2n \).

The mapping \( \sigma \) is constant over each region of \( \mathcal{C}(1,0,1) \). Thus, \( \sigma \) is a mapping from the regions of \( \mathcal{C}(1,0,1) \) to \( D^{(1)}(2n) \).

2. From orders to forests.

In this section, we present a bijection between the symmetric annotated 1-sketches and some rooted labeled ordered forests that we call symmetric forests. We will first define these forests and then expose the bijection.

2.1. Symmetric forests.

In order to define a symmetric forest, we need to introduce the notion of sub-descendant in a forest.

For any rooted labeled ordered forest \( F \), we say that we read the nodes of \( F \) in BFS order if we list the labels of the nodes of \( F \) in a breadth-first search starting from the root.
**Definition 2.1.** Let $i$ and $j$ be two nodes in a rooted ordered forest. We say that $i$ is a sub-descendant of $j$ if $i$ appears after $j$ and strictly before any child of $j$ in the BFS order. We also say that $i$ and $j$ satisfy the sub-descendant property (SDP) if $i$ is a sub-descendant of $j$ implies that $-j$ is a sub-descendant of $-i$.

**Definition 2.2.** A symmetric forest with $2n$ nodes is a rooted labeled ordered forest that satisfies:

(i) the first $n$ nodes read in BFS order are labeled $e_1, \ldots, e_n$ such that $\{|e_1|, \ldots, |e_n|\} = [1, n]$,

(ii) $e_{n+j} = -e_{n-j+1}$ with $j \in [1, n]$,

(iii) for every two nodes $i, j$, $i$ and $j$ satisfy the sub-descendant property.

We denote by $F_{S}(2n)$ the symmetric forests with $2n$ nodes.

**Example 2.1.** For the symmetric forest $G$ in Figure 1, $1$ is a sub-descendant of $-2$ and $2$ is a sub-descendant of $-1$, hence $\{1, -2\}$ satisfy the sub-descendant property. Moreover, $G \in F_{S}(6)$.

As a matter of fact, a symmetric forest is composed of two sub-forests where one is the symmetric of the other in the following sense:

**Definition 2.3.** Let $F$ be a rooted ordered forest defined on $n$ labeled nodes $e_1, \ldots, e_n$. We define the symmetric of $F$ as a rooted ordered forest $\overline{F}$ with $n$ labeled nodes $-e_1, \ldots, -e_n$ such that for all $i \neq j \in [1, n]$, $-e_i$ is a sub-descendant of $-e_j$ in $\overline{F}$ if and only if $e_j$ is a sub-descendant of $e_i$ in $F$.

We now explain how to decompose a symmetric forest $G$ into a forest $F$ and its symmetric. $F$ is the sub-forest of $G$ defined on the first $n$ nodes read in BFS order.

![Figure 1](image)

**Thus we have that:**

**Proposition 2.1.** A symmetric forest with $2n$ nodes is the composition of a rooted labeled ordered forest with $n$ nodes and its symmetric.

**Proof.** $F$ is the sub-forest of $G$ defined by the first $n$ nodes read in BFS order. Then the sub-forest of $G$ corresponding to the $n$ last nodes read in BFS order is the forest, symmetric of $F$, by Definition 2.2 and Definition 2.3.

Conversely, any shuffle between any rooted labeled ordered forest $F$ and its symmetric, is in bijection with a symmetric forest. We first give the definition of a special leaf, then the definition of the shuffles between a forest and its symmetric (see Figure 2) and finally we prove the assertion.

**Definition 2.4.** In a rooted ordered forest with $n$ labeled nodes $e_1, \ldots, e_n$ such that $\{|e_1|, \ldots, |e_n|\} = [1, n]$, the special leaves are the leaves which are after the last internal node in the BFS order. If a forest $F$ has only leaves, we consider that its last internal node is a fictif node, parent of the leaves of $F$. Let us call $F_{n,s}, 1 \leq s \leq n$, the set of rooted labeled ordered forests of size $n$ with $s$ special leaves.

**Example 2.2.** The rooted labeled ordered forest $F$ of Figure 2 has two special leaves, 1 and 3.
Definition 2.5. Let $F$ be a rooted ordered forest defined on $n$ labeled nodes $e_1, \ldots, e_n$, ordered in BFS order and such that $\{[e_1], \ldots, [e_n]\} = [1, n]$. with $s$ special leaves. The set of shuffles between $F$ and its symmetric $\overline{F}$, $F \bowtie \overline{F}$, is the set of forests obtained when we connect $s$ edges from $\{-e_n, -e_{n-1}, \ldots, -e_{n+s+1}\}$ to $\{e_{n-s}, e_{n-s+1}, \ldots, e_{n-1}, e_n\}$ such that any pair $(u, v)$, $u$ in $\{-e_n, -e_{n-1}, \ldots, -e_{n+s+1}\}$ and $v$ in $\{e_{n-s}, \ldots, e_{n-1}, e_n\}$, satisfies the sub-descendant property and the sequence of the nodes read in BFS order is $e_1, \ldots, e_n, -e_n, -e_{n-1}, \ldots, -e_1$. We say that $\{-e_n, -e_{n-1}, \ldots, -e_{n-s+1}\}$ and $\{e_{n-s}, \ldots, e_{n-1}, e_n\}$ satisfy the sub-descendant property.

Proposition 2.2. For any rooted ordered forest $F$ with $n$ nodes labeled with $e_1, \ldots, e_n$ such that $\{[e_1], \ldots, [e_n]\} = [1, n]$, the set $F \bowtie \overline{F}$ is a set of symmetric forests with $2n$ nodes.

Proof. Conditions (i) and (ii) of Definition 2.2 are verified by definition of the shuffle. Notice that for any connection of $s$ edges from $\{-e_n, -e_{n-1}, \ldots, -e_{n+s+1}\}$ to $\{e_{n-s}, e_{n-s+1}, \ldots, e_{n-1}, e_n\}$, $\{-e_n, -e_{n-s-1}, \ldots, -e_1\}$ always satisfies the sub-descendant property with $\{e_1, e_2, \ldots, e_n\}$, and $\{-e_n, -e_{n-1}, \ldots, -e_{n-s+1}\}$ always satisfies the sub-descendant property with $\{e_1, e_2, \ldots, e_{n-s-1}\}$. By Definition 2.5, $\{-e_n, -e_{n-1}, \ldots, -e_{n-s+1}\}$ and $\{e_{n-s}, \ldots, e_{n-1}, e_n\}$ satisfy the sub-descendant property. Therefore, for any forest $G$ in $F \bowtie \overline{F}$, for every two nodes $e_i, e_j, e_i$ and $e_j$ satisfy the sub-descendant property. Thus, $G$ is a symmetric forest. □

![Figure 2. the set of shuffles between the forests $F$ and $\overline{F}$ of Figure 1.](image)

2.2. Bijection between symmetric annotated 1-sketches and symmetric forests. We will show here that a symmetric annotated 1-sketch corresponds bijectively to a symmetric forest. Moreover, the decomposition of a symmetric annotated 1-sketch (see Proposition 2.1) corresponds to the decomposition of a symmetric forest (see Proposition 2.3).

Proposition 2.3. There is a one to one correspondence between symmetric annotated 1-sketches of size $2n$ and symmetric forests of size $2n$.

Proof. We now prove the proposition in 3 steps:
Step 1: we first present an algorithm to get the symmetric forest from a symmetric annotated 1-sketch $\omega$ of $D^1(2n)$. We define the map $\phi$ of $D^1(2n)$ and $F_S(2n)$ by the following algorithm (see Figure 3):
(i) Read $\omega$ from left to right.
(ii) When $\alpha_i^{(0)}$ is read, create a node $i$ such that, if $\alpha_i^{(0)}$ is not the first letter, if the preceding letter is $\alpha_j^{(0)}$ then $i$ becomes the next right sibling of $j$, and if the preceding letter is $\alpha_j^{(1)}$, then $i$ becomes the leftmost child of $j$.

First note that $\alpha_i^{(0)}$ and $\alpha_i^{(0)}$ cannot both be in the first $n$ $\alpha^{(0)}$ letters. By definition, the forest $\phi(\omega)$ has $n$ first nodes labeled by the first $n$ $\alpha^{(0)}$-letters. And the last $n$ nodes are defined symmetrically as in the symmetric annotated 1-sketches $D^1(2n)$.

Second, remark that:

Remark 2.1. (1) If $\alpha_i^{(1)}$ is not followed by an $\alpha^{(0)}$-letter then node $i$ is a leaf.
(2) If $\alpha_i^{(0)}$ appears before $\alpha_j^{(0)}$ in $\omega$ then node $i$ appears before node $j$ in the BFS order of the nodes of the obtained forest.
(3) The property “$\alpha_i^{(0)}$ appears before $\alpha_j^{(0)}$ then $\alpha_i^{(0)}$ appears before $\alpha_j^{(0)}$, implies that “$i$ appears before $j$ then $-j$ appears before $-i$ in the BFS order of the nodes of the obtained forest”.


From Propositions 1.3 and 2.3, we get that:

(4) The property \(\alpha_i^{(0)}\) appears before \(\alpha_j^{(1)}\) then \(\alpha_{-j}^{(0)}\) appears before \(\alpha_{-i}^{(1)}\) is equivalent to \(\alpha_i^{(0)}\) \(\alpha_j^{(0)}\) \(\alpha_j^{(1)}\) then \(\alpha_{-i}^{(0)}\) \(\alpha_{-j}^{(0)}\) \(\alpha_{-j}^{(1)}\). So, if \(i\) is a sub-descendant of \(j\) then \(-j\) is a sub-descendant of \(-i\).

From these last two remarks, it is clear that \(\phi\) is a symmetric forest.

Step 2: before showing that \(\phi\) is a bijection, we describe the inverse mapping \(\psi\). Let \(G \in F_S(2n)\) and \(e_1,e_2,...,e_{2n}\) be the \(2n\) nodes in \(G\) read in BFS order. Let \(\psi(G)\) be the word \(\omega\) defined inductively as follow:

- Read the vertices in BFS order. \(\omega_1 = \alpha_{e_1}\).
- For any \(2 \leq j \leq 2n\), if \(e_j\) is the next right sibling of \(e_{j-1}\) then \(\omega_j = \omega_{j-1}\alpha_{e_j}^{(0)}\), if \(e_j\) is the leftmost child of \(e_i\) then \(\omega_j = \omega_{j-1}\alpha_{e_i}^{(1)}\alpha_{e_j}\).
- \(\omega = \omega_{2n}\alpha_{e_{2n-1}}\alpha_{e_{2n-2}}...\alpha_{e_2}\alpha_{e_1}\) if \(G\) has \(s\) special leaves.

Note that for all \(G \in F_S(2n)\), the word \(\psi(G)\) satisfies the properties (i)-(iv) of symmetric annotated 1-sketches. Hence \(\psi\) is a mapping from \(F_S(2n)\) to \(D^1(2n)\).

Step 3: it is easy to prove that \(\psi(\phi(D^1(2n))) = D^1(2n)\) and \(\phi(\psi(F_S(2n))) = F_S(2n)\). 

\[\square\]

From Propositions 1.3 and 2.3 we get that:

**Corollary 2.1.** \(\Phi = \phi \circ \sigma\) is a bijection from the regions of the Catalan arrangement \(C_{\{1,0,1\}}(n)\) to the symmetric forests \(F_S(n)\).

The bijection \(\phi\) induces a bijection between the annotated 1-sketches of size \(n\) with rightest \(\alpha^{(0)}\)-letter at position \(s\) and the rooted labeled ordered forests with \(2n - s\) special leaves.

**Proposition 2.4.** The mapping \(\phi\) induces a bijection between \(A_{n,s}\) and \(F_{n,2n-s}, n \leq s \leq 2n - 1\).

**Proof.** Let \(\omega_1 \in A_{n,s}\). It means that \(\omega_1 = \omega_0\alpha_{j_0}^{(0)}\alpha_{j_1}^{(1)}\alpha_{j_2}^{(0)}...\alpha_{j_{s-1}}^{(1)}\alpha_{j_s}^{(0)}\alpha_{j_{s+1}}^{(1)}...\alpha_{j_{2n-s}}^{(0)}\alpha_{j_{2n-s+1}}^{(1)}...\alpha_{j_{2n-1}}^{(0)}\alpha_{j_{2n}}^{(1)}\).

From the first step of the proof of Proposition 2.3 we have that:

- \(\alpha_{j_1}^{(0)}...\alpha_{j_n}^{(0)}\) represent the nodes \(j_1,...,j_n\) read in BFS order in \(\phi(\omega_1)\),
- If \(\alpha_{i}^{(1)}\) is not followed by an \(\alpha^{(0)}\)-letter then node \(i\) is a leaf,
- The last \(\alpha^{(1)}\)-letter followed by an \(\alpha^{(0)}\)-letter is \(\alpha_{j_{2n-s}}^{(1)}\). This implies that the last internal node in the BFS order of the nodes of \(\phi(\omega_1)\) is \(j_{2n-s}\).

Thus, \(\phi(\omega_1)\) is a rooted ordered forest with \(n\) labeled nodes \(j_1,...,j_n\) where the nodes \(j_{s-n+1},j_{s-n+2},...,j_n\) are \(2n-s\) special leaves and \(\phi(\omega_1) \in F_{n,2n-s}\). Conversely, let \(F \in F_{n,2n-s}\) with \(2n-s\) special leaves \(j_{s-n+1},j_{s-n+2},...,j_n\). Then \(\alpha_{j_n}^{(0)}\) is at the \(s^{th}\) position in \(\psi(F)\), hence it belongs to \(A_{n,s}\).

It is easy to prove that \(\psi(\phi(A_{n,s})) = A_{n,s}\). Similarly, \(\phi(\psi(F_{n,2n-s})) = F_{n,2n-s}\). 

\[\square\]
We now show that the different possible shuffles between an annotated 1-sketch and its symmetric correspond by \( \phi \) to the different possible shuffles between a rooted labeled ordered forest and its symmetric.

**Proposition 2.5.** The bijections \( \phi \) and \( \psi \) are compatible with shuffles and symmetrics. Indeed, let \( \omega_1 \) be annotated 1-sketch, then \( \phi(\omega_1) = \phi(\omega_1) \) and \( \phi(\omega_1 \bowtie \omega_1) = \phi(\omega_1) \bowtie \phi(\omega_1) \).

**Proof.** Let \( \omega_1 = \omega_0 \phi(\omega_n) \alpha_{j-1}^{(1)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \in A_n \).

From Proposition 2.4 we get \( \phi(\omega_1) \in F_{n,2n-2} \) and \( \phi(\omega_1) \in F_{n,2n-2} \).

Remark that if \( \omega_1 \) is of the form \( \alpha_{i}^{(0)} \ldots \alpha_{i}^{(0)} \ldots \alpha_{i}^{(1)} \ldots \alpha_{i}^{(1)} \ldots \), then \( \omega_1 \) is of the form \( \alpha_{i}^{(0)} \ldots \alpha_{i}^{(0)} \ldots \alpha_{i}^{(1)} \ldots \alpha_{i}^{(1)} \ldots \)

It means that in \( \phi(\omega_1) \), \( j \) is a sub-descendant of \( i \) and in \( \phi(\omega_1) \), \( -i \) is a sub-descendant of \( -j \). Thus \( \phi(\omega_1) = \phi(\omega_1) \).

Moreover a shuffle between \( \alpha_{j-1}^{(1)} \ldots \alpha_{j-1}^{(1)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \) corresponds by \( \phi \) to a shuffle between the special leaves of \( \phi(\omega_1) \), \( j_{s-1} \ldots j_{s-2} \ldots j_{s-1} \), and the nodes of \( \phi(\omega_1) \), \( -j_{s-1} \ldots -j_{s-2} \ldots -j_{s-1} \).

Since \( \omega_1 \bowtie \omega_1 = \omega_0 \phi(\omega_n) \phi(\omega_n) \alpha_{j-1}^{(1)} \ldots \alpha_{j-1}^{(1)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \bowtie \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(0)} \ldots \alpha_{j-1}^{(1)} \ldots \alpha_{j-1}^{(1)} \ldots \), it is straigntforward to conclude that \( \phi(\omega_1 \bowtie \omega_1) = \phi(\omega_1) \bowtie \phi(\omega_1) \).

\( \square \)

### 3. The number of regions of the type C Catalan arrangement

We are now able to compute the number of regions of the type C Catalan arrangement. We first compute the number of rooted ordered forests of size \( n \) with \( s \) special leaves.

**Proposition 3.1.** The number of rooted ordered forests of size \( n \) with \( s \) special leaves, \( C_{n,s} \), verifies the following formula : \( C_{n,s} = \frac{s^{2n+1}}{n!} \) for \( 1 \leq s \leq n-1 \) and \( C_{n,n} = 1 \).

**Proof.** Let \( F \) be a rooted ordered forest of size \( n \) with \( s \) special leaves \( e_{n-s+1}, \ldots, e_{n-1}, e_n \). If we cancel the last special leaf \( e_n \) then we get a rooted ordered forest of size \( n-1 \) with \( s \) special leaves, \( s+1 \leq s \leq n-1 \). Hence, \( C_{n,s} = C_{n-1,s-1} + C_{n-1,s} + \ldots + C_{n-1,n-1}, 1 \leq s \leq n-1 \). Moreover \( C_{n,n} = 1 \) since a rooted ordered forest of size \( n \) with no special leaves is the forest with only leaves.

Then \( C_{n,s} = C_{n-1,s-1} + C_{n-1,s-1}, 1 \leq s \leq n-1 \) and \( C_{n,n} = 1 \).

The number of rooted ordered forests with \( s \) special leaves in the case \( n = 2,3 \) are given by \( C_{2,1} = C_{2,2} = 1, C_{3,1} = C_{3,2} = 2 \).

Thus, by induction, we conclude.

\( \square \)

Now we are able to enumerate the regions of the type C Catalan arrangement.

**Proposition 3.2.** The number of regions of the type C Catalan arrangement is

\[
r(C_{-1,0,1}(n)) = 2^n n! \sum_{s=1}^{n-1} \frac{s^{2n}}{n!} + 4^n n!.
\]

**Proof.** The number of labeling of a rooted ordered forest of size \( n \) with labels \( e_1, \ldots, e_n \) such that \( \{|e_1|, \ldots, |e_n|\} = [1, n] \) is \( 2^n n! \). Thus, from Corollary 2.1 we can compute the number of regions of the type C Catalan arrangement :

\[
r(C_{-1,0,1}(n)) = 2^n n! \sum_{s=1}^{n} C_{n,s} D_{n,s}
\]

where \( C_{n,s} \) is given by Proposition 3.1 and \( D_{n,s} \) is the number of shuffles between any rooted labeled ordered forest of size \( n \) with \( s \) special leaves and its symmetric.

Now we compute \( D_{n,s} \). By Proposition 2.5 and Proposition 2.4 this is equal to the number of shuffles between an annotated 1-sketch \( \omega_1 = \omega_0 \phi(\omega_n) \alpha_{e_{n-s+1}}^{(0)} \alpha_{e_{n-s+2}}^{(1)} \ldots \alpha_{e_{n-s+1}}^{(0)} \ldots \alpha_{e_{n-s+1}}^{(0)} \) and its symmetric \( \omega_1 = \alpha_{e_{n-s+1}}^{(0)} \alpha_{e_{n-s+2}}^{(0)} \alpha_{e_{n-s+1}}^{(0)} \ldots \alpha_{e_{n-s+1}}^{(0)} \).

Assume that \( \phi(\omega_1) \) is followed immediately by \( \alpha_{e_{n-s}}^{(0)} \), hence the set \( \omega_1 \bowtie \omega_1 \) is equal to the set \( \omega_0 \alpha_{e_{n-s}}^{(0)} \alpha_{e_{n-s}}^{(0)} \alpha_{e_{n-s}}^{(1)} \ldots \alpha_{e_{n-s}}^{(0)} \ldots \alpha_{e_{n-s}}^{(0)} \). Let \( a_k \) be the number of possible shuffles between \( \alpha_{e_{n-s}}^{(0)} \ldots \alpha_{e_{n-s}}^{(0)} \) and \( \alpha_{e_{n-s}}^{(0)} \ldots \alpha_{e_{n-s}}^{(0)} \). Thus, in this case we have \( a_{n-s-1} \) possible shuffles.

Now we assume that \( \alpha_{e_{n-s}}^{(0)} \) appears immediately after \( \alpha_{e_{k}}^{(1)} \), \( n-s+1 \leq k \leq n-1 \), then we have \( a_{n-k-1} \) possible shuffles. And if \( \alpha_{e_{n}}^{(0)} \) appears immediately after \( \alpha_{e_{n}}^{(1)} \), we get exactly one word.
Hence \( D_{n,s} = a_{s-1} + a_{s-2} + \ldots + a_1 + a_0 + 1, \ s \geq 1. \) Notice that in the case \( \alpha^{(0)}_{-e_n} \) appears immediately after \( \alpha^{(1)}_{e_{n-1}} \), \( a_0 = 1. \)

Moreover, in a shuffle between \( \alpha^{(1)}_{e_1} \ldots \alpha^{(1)}_{e_k} \) and \( \alpha^{(0)}_{-e_k} \ldots \alpha^{(0)}_{-e_1} \), if \( \alpha^{(1)}_{e_1} \) appears at position 1 then \( \alpha^{(0)}_{-e_1} \) appears at position \( 2k \), and if \( \alpha^{(0)}_{-e_k} \) appears at position 1 then \( \alpha^{(1)}_{e_k} \) appears at position \( 2k \). This implies that \( a_k = 2a_{k-1} \). Therefore, \( D_{n,s} = 2^s. \)

\[ \square \]

References

[1] Christos A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields. Advances in Mathematics, 122(2):193 – 233, 1996.
[2] Christos A. Athanasiadis. Extended linial hyperplane arrangements for root systems and a conjecture of postnikov and stanley. Algebraic Combin., 10:207–225, 1999.
[3] Christos A. Athanasiadis and Svante Linusson. A simple bijection for the regions of the shi arrangement of hyperplanes. Discrete Mathematics, 204(1):27 – 39, 1999. Selected papers in honor of Henry W. Gould.
[4] Olivier Bernardi. Deformations of the braid arrangement and trees. Advances in Mathematics, 335:466 – 518, 2018.
[5] Sylvie Corteel, David Forge, and Véronique Ventos. Bijections between affine hyperplane arrangements and valued graphs. European Journal of Combinatorics, 50:30 – 37, 2015. Combinatorial Geometries: Matroids, Oriented Matroids and Applications. Special Issue in Memory of Michel Las Vergnas.
[6] Rui Duarte and António Guedes de Oliveira. Between shi and ish. Discrete Mathematics, 341:388 – 399, 2018.
[7] Karola Mészáros. Labeling the regions of the type \( c_n \) shi arrangement. the Elec. J. of Combinatorics, 20(2), 2013.
[8] Alexander Postnikov and Richard P. Stanley. Deformations of coxeter hyperplane arrangements. Journal of Combinatorial Theory, Series A, 91(1):544 – 597, 2000.
[9] Richard P. Stanley. An introduction to hyperplane arrangements. 13:389–496, 2007.

IRIF, CNRS and University Paris Diderot, Case 7014, 75205 Paris Cedex 13 France
E-mail address: amicheli@irif.fr
ndvu@math.ac.vn, Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam
E-mail address: ndvu@math.ac.vn