The Numerical Range of $6 \times 6$ Irreducible Matrices

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ABSTRACT

In this paper, we consider the problem of characterizing the numerical range of $6 \times 6$ irreducible matrices which have line segments on their boundary.

Keywords: numerical range, irreducible matrices.

1. Introduction

Let $M_n(C)$ be the algebra of $n \times n$ complex matrices. The numerical range of $A \in M_n$ is defined by $W(A)=\{x^*Ax: x \in C^n, x^*x=1\}$ [4], where $x^*$ the adjoint of $x \in C$ is defined by $x^*=(\bar{x})^T$ where $\bar{x}$ is the component-wise conjugate, and $x^T$ is the transpose of $x$ [4]. As pointed out by many authors, for $2 \times 2$ matrices $A$ a complete description of the numerical range $W(A)$ is well-known. Namely, $W(A)$ is an ellipse with foci at the eigenvalues $\lambda_1$, $\lambda_2$ of $A$ and a minor axis of the length $s=(\text{trace}(A^*A)-|\lambda_1|^2-|\lambda_2|^2)^{1/2}$. In [4], of course, $s=0$ for normal $A$, and the ellipse in this case degenerates into a line segment connecting $\lambda_1$ with $\lambda_2$. On the other hand, for $2 \times 2$ matrices $A$ with coinciding eigenvalues the ellipse $W(A)$ degenerates into a disk. For $3 \times 3$ matrix $A$, this was first done by Kippenhahn. In [6], his characterization is based on the factorability of the associated polynomial $P_A(x,y,z)=\text{det}(xReA+yImA+zI_3)$. This was improved in [5] by expressing the condition in terms of entries of $A$, also for $4 \times 4$ and $5 \times 5$ matrices $A$, this was improved in [2] by expressing the conditions in terms of entries of $A$. The aim of this paper is to give a sufficient and necessary condition for numerical range of $6 \times 6$ matrix with a line segment on its boundary.
2. Preliminaries

In the following, we give some definitions and results on $W(A)$ that are useful in this study.

**Definition 2.1** [4] A matrix $A \in M_n(C)$ is said to be irreducible if either $n=1$ or $n \geq 2$ and there does not exist a permutation matrix $P \in M_n(C)$ such that

$$P^TAP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where $B, D$ are nonempty square matrices.

**Definition 2.2** [4] A matrix $B \in M_n(c)$ such that $x^*Bx \geq 0$ for all $x \in C^n$, is said to be positive semidefinite.

**Proposition 2.3** [4] The numerical range of $A \in M_n$ is always a compact convex set in $C$. It contains the spectrum $\sigma(A)$ of $A$ and is equal to the convex hull of $\sigma(A)$ if $A$ is normal.

**Proposition 2.4** [4]

a) $W(A) = W(A^*)$

b) $W(A) = W(U^*AU)$ for any unitary $U$.

c) $W(\lambda A) = \lambda W(A)$ for any $\lambda \in C$.

d) $W(\lambda I + A) = \lambda + W(A)$ for any $\lambda \in C$.

e) $W(A^*) = \{ z : z \in W(A) \}$

f) $W(ReA) = ReW(A)$ and $W(ImA) = ImW(A)$. Here $ReA = (A^* + A)/2$ and $ImA = (A^* - A)/2i$ are the real and imaginary part of $A$ respectively.

**Proposition 2.5** [4] Let $A \in M_n(C)$ and $a,b$ be scalars

a) $W(A) = \{ \lambda \}$ if and only if $A = \lambda I$.

b) $W(A)$ is contained in a straight line of the plane if and only if $A = aB + bI$ for some Hermitian matrix $B$. In particular in this case $A$ is normal.

c) $W(A) \geq 0$ if and only if $A$ is positive semidefinite.

**Proposition 2.6** [4] Suppose that $B$ is a principal submatrix of $A \in M_n(C)$. Then $W(B) \subseteq W(A)$.

We now relate the numerical range of an $n \times n$ matrix to an algebraic curve of class $n$. The next proposition indicates how the characteristic polynomial of some pencil associated with the matrix arises in this connection.

**Proposition 2.7** [7] Let $A \in M_n(C)$. If $ax + by + c = 0$ is a supporting line of $W(A)$, then $det(aReA + bImA + cI_n) = 0$
It follows from the above proposition that when studying the numerical range of $A$ it is sensible to consider the algebraic curve $C(A)$ which is dual to the one given by $P_A(x \text{Re}A + y \text{Im}A + z I_n) = 0$.

Note that $P_A(-x,-y,z) = \det(zI_n - x \text{Re}A - y \text{Im}A)$ is nothing but the characteristic polynomial of the pencil $x \text{Re}A + y \text{Im}A$. It is easily to see that $P_A$ is a homogenous polynomial of degree $n$ with real coefficients. Thus, in particular, the curve $C(A)$ is of class $n$ and has $n$ real eigenvalues of $A$.

**Proposition 2.8** [7] If the eigenvalues of the $n \times n$ matrix $A$ are $a_j + ib_j$, $j=1, \ldots, n$ where the $a_j$ and $b_j$ are real, then the real foci of the algebraic curve $C(A)$ are exactly the points $(a_j, b_j)$, $j=1, \ldots, n$.

Note that the proposition (2.7 together with the duality, implied that any supporting line of $W(A)$ is tangent to $C(A)$. The following proposition gives a more precise relation between $W(A)$ and $C(A)$.

**Proposition 2.9** [6] If $A$ is a $n \times n$ matrix, then its numerical range $W(A)$ is the convex hull of the real points of the curve $C(A)$. The real part of the curve $C(A)$ in the complex plane namely the set $\{a + ib \in \mathbb{C} : a, b \in \mathbb{R} \text{ and } ax + by + z = 0 \text{ is tangent to } P_A(x,y,z) = 0\}$, will be denoted by $C_R(A)$ and is called the Kippenhahn curve of $A$.

### 3. Line segments of the Boundary of Numerical Range

In the following we will restrict ourselves to the irreducible matrix $A$. The next theorem gives conditions for the numerical range to have a line segments on its boundary.

**Theorem 3.1** Let $A$ be an irreducible matrix. Then the following statements are equivalent:

(a) $W(A)$ has a line segments on its boundary.

(b) $0 < \text{rank}(u \text{Re}A + v \text{Im}A + w I_6) \leq 4$ for some real numbers $u, v$ and $w$ ( $u$ and $v$ not both zero).

(c) $u \text{Re}A + v \text{Im}A$ has a multiple eigenvalue for some real $u$ and $v$ which are not both zero.

**Proof:** $W(A)$ has a line segments on the boundary of $C_R(A)$ has a double or triple or quadrupartite or tangent $ux + vy + w = 0$ ( $u$ and $v$ not both zero). This corresponds to a root $w$ of the equation $\det(u \text{Re}A + v \text{Im}A + z I_6) = 0$ with multiplicity 2 or 3 or 4 or 5, which is the same as saying that $u \text{Re}A + v \text{Im}A + z I_6$ has rank 4 or 3 or 2 or 1. This proves the equivalence of (a) and (b).
(b) $\Rightarrow$ (c): Suppose (b) holds then $-w$ is an eigenvalue of $u\text{Re}A+v\text{Im}A$ with multiplicity 5, and we get double or triple or quadripartite or tangent. Thus (b) implies (c). To prove (c) $\Rightarrow$ (b), assume that $-w$ is a multiple eigenvalue of $u\text{Re}A+v\text{Im}A$. If $-w$ is of multiplicity 6, then the Hermitian $u\text{Re}A+v\text{Im}A$ would be a scalar matrix. In this case, $\text{Re}A$ and $\text{Im}A$ would commute and hence $A$ would be normal, contradicting the irreducibility of $A$. Thus (c) implies (b).

**Corollary 3.2** Let $A$ be an irreducible matrix and unitary similar to real matrix. Then $W(A)$ has a line segments on its boundary if and only if $\text{Re}A$ has a multiple eigenvalue.

**Proof:** If $A$ is a real matrix. Then $W(A)$ is symmetric about the real axis. Hence the line segment of the boundary of $W(A)$ must be a vertical line, by theorem(3.1) the matrix $1.\text{Re}A+0.\text{Im}A=\text{Re}A$ has a multiple eigenvalue.

We begin by deriving a canonical form for an irreducible form for an irreducible $6 \times 6$ matrix with a line segment on the boundary of it is numerical range, if $W(A)$ has a line segment on its boundary. After rotation, Shifting, and multiplication by a positive number, we may assume that a line segment stretches from 0 to $i$. Since $W(A)$ is convex, it must be contained entirely in the right or left half-plane. Applying yet another rotation and translation, if necessary we may assume that $W(A)$ is in the right half-plane. By theorem(3.1) we have rank$A=1$ or 2 or 3. Therefore we will discuss these two cases, respectively.

**Theorem 3.3** Let $A$ be an irreducible matrix. Then $W(A)$ has a line segment extending from 0 to $i$ on its boundary and rank$A=1$ if and only if $A$ may be written in the form $A=\begin{bmatrix} i & 0 & 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & e_1 & -e_2 \\ 0 & 0 & e_1 & e_2 & -e_3 \\ 0 & e_1 & e_2 & e_3 & -e_4 \\ c_1 & c_2 & e_2 & e_3 & e_4 & -e_6 \end{bmatrix}$, where $c_1,c_2,e_2,e_3,e_4,$ and real part of $e_6$ are positive and $c_1,e_2,e_3 \in [0,i]$.

**Proof:** Since $W(A)$ is contained in the closed right half-plane, $\text{Re}A$ is positive semi definite. By assumption, $\ker \text{Re}A$ is 5-dimensional subspace, we may represent the line transformation of $A$ restricted to $\ker \text{Re}A$, $A|_{\ker \text{Re}A}=A'$ by a $5 \times 5$ matrix. By choosing a proper basis for $A,A'$ is the
The Numerical Range of $A$.

Since $0$ and $i$ are in $W(A)$, there exists unit vectors $x_1, x_2 \in \mathbb{C}^6$ such that $<Ax_1, x_1> = 0$ and $<Ax_2, x_2> = 0$. It is clear that $x_1, x_2 \in \ker Re A$, since $Re A \succeq 0$ and $<(Re A)x_j, x_j> = 0$, $j = 1, 2$. It follows that the line segment $[0, i]$, is contained in $W(A')$. Also $W(A') \subseteq W(A)$. Since $A_{\ker Re A} = A'$, we have $Re A' = 0$ and $Re W(A') = W(Re A') = \{0\}$. We thus get $W(A') = [0, i]$. This implies that $A'$ is normal with eigenvalues $0$ and $i$ so

$$A' = \begin{bmatrix} i & 0 & 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_1 & e_2 & e_3 & e_4 \\ c_1 & c_2 & c_3 & e_4 & e_5 & 2Re e_6 \end{bmatrix}$$

with proper basis $A'$. And $A'$ is normal with eigenvalues $0$ and $i$ so

$$2Re A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & v_1 + \bar{v}_1 \\ 0 & 0 & 0 & 0 & 0 & v_2 + \bar{v}_2 \\ 0 & 0 & 0 & 0 & e_1 + \bar{e}_1 & e_2 + \bar{e}_2 \\ 0 & 0 & e_1 + \bar{e}_1 & e_2 + \bar{e}_2 & e_3 + \bar{e}_3 \\ 0 & 0 & e_1 + \bar{e}_1 & e_2 + \bar{e}_2 & e_3 + \bar{e}_3 & e_4 + \bar{e}_4 \\ c_1 + \bar{v}_1 & c_2 + \bar{v}_2 & e_3 + \bar{e}_3 & e_4 + \bar{e}_4 & 2Re e_6 \end{bmatrix}$$

Moreover, $Re e_6 \geq 0, v_1 = -\bar{c}_1, v_2 = -\bar{c}_2, e_2 = -\bar{e}_3, e_3 = -\bar{e}_4$ and $e_4 = -\bar{e}_5$, $A'$ is normal with eigenvalues $0$ and $i$ so

$$\alpha = <Ax, x> = <(Re A)x, x> + i<(Im A)x, x>$$. Then $<(Re A)x, x> = 0$. Hence
x ∈ ker Re A = span{[1, 0, 0, 0, 0]T, [0, 1, 0, 0, 0]T, [0, 0, 1, 0, 0]T, [0, 0, 0, 1, 0]T, [0, 0, 0, 0, 1]T} if \( \|x\| = 1 \), then \( x = [v_1, v_2, v_3, v_4, v_5]^T \)

with \( |v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 = 1 \), and

\[
0 \leq \langle (\text{Im } A)x, x \rangle = |v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 \leq |v_1|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 \leq 1.
\]

**Theorem 3.4** Let A be an irreducible matrix written in the form

\[
A = \begin{bmatrix}
h_1 & h_12 & h_13 & h_14 & h_15 & h_16 \\
h_12 & h_2 & h_23 & h_24 & h_25 & h_26 \\
h_13 & h_23 & h_3 & h_34 & h_35 & h_36 \\
h_14 & h_24 & h_34 & h_4 & h_45 & h_46 \\
h_15 & h_25 & h_35 & h_45 & h_5 & h_56 \\
h_16 & h_26 & h_36 & h_46 & h_56 & h_6
\end{bmatrix} + i \begin{bmatrix}
k_1 & 0 & 0 & 0 & 0 & 0 \\
0 & k_2 & 0 & 0 & 0 & 0 \\
0 & 0 & k_3 & 0 & 0 & 0 \\
0 & 0 & 0 & k_4 & 0 & 0 \\
0 & 0 & 0 & 0 & k_5 & 0 \\
0 & 0 & 0 & 0 & 0 & k_6
\end{bmatrix}
\]

where \( k_1, k_2, k_3, k_4, k_5, k_6 \) are distinct. Then W(A) has a line segment on its boundary if and only if

\[
h_6(k_4 - k_5) + h_5(k_6 - k_4) + h_4(k_5 - k_6) = (k_4 - k_5)(\frac{h_{36}h_{46}}{h_{35}}) + (k_6 - k_4)(\frac{h_{35}h_{45}}{h_{36}}) + (k_5 - k_6)(\frac{h_{36}h_{56}}{h_{35}}) \quad (3.1)
\]

\[
h_6(k_3 - k_4) + h_4(k_6 - k_4) + h_3(k_4 - k_6) = (k_3 - k_4)(\frac{h_{36}h_{46}}{h_{35}}) + (k_6 - k_4)(\frac{h_{46}h_{45}}{h_{36}}) + (k_4 - k_6)(\frac{h_{46}h_{56}}{h_{35}}) \quad (3.2)
\]

\[
h_6(k_2 - k_3) + h_3(k_6 - k_2) + h_2(k_3 - k_6) = (k_2 - k_3)(\frac{h_{36}h_{46}}{h_{35}}) + (k_6 - k_2)(\frac{h_{46}h_{45}}{h_{36}}) + (k_3 - k_6)(\frac{h_{46}h_{56}}{h_{35}}) \quad (3.3)
\]

\[
h_1(k_5 - k_6) + h_5(k_6 - k_1) + h_6(k_1 - k_5) = (k_5 - k_6)(\frac{h_{36}h_{46}}{h_{35}}) + (k_6 - k_1)(\frac{h_{16}h_{15}}{h_{56}}) + (k_1 - k_5)(\frac{h_{36}h_{56}}{h_{35}}) \quad (3.4)
\]

\[
h_{12}h_{26}h_{16} + h_{13}h_{36}h_{16} + h_{14}h_{46}h_{16} + h_{15}h_{56}h_{16} + h_{35}h_{56}h_{36} + h_{34}h_{43}h_{36} + h_{24}h_{35}h_{26} + h_{23}h_{34}h_{26}.
\]

where \( h_{23}h_{34}h_{26} \) and \( h_{12}h_{13}h_{14}h_{15}h_{16} \) are real with \( h_{12}h_{13}h_{14}h_{15}h_{16}h_{23}h_{24}h_{25}h_{26}h_{34}h_{35}h_{36}h_{43}h_{45}h_{46} = 0 \). \quad (3.5)

**Proof:** suppose rank B = rank(u Re A + v Im A + w I_6) = 1, since the eigenvalues of Im A are all distinct, it is possible only when u is non zero. With-out loss of generality we may assume that u = 1. To simplify further calculations, rewrite B in the form
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For convenience, let \( w' = \frac{\bar{h}_{56} h_{46}}{h_{45}} \) in the remaining proof, since
\[ h_{12} h_{13} h_{14} h_{15} h_{16} h_{23} h_{24} h_{25} h_{26} h_{34} h_{35} h_{36} h_{45} h_{46} h_{56} \neq 0 \] and \( w' \) is real, the equality (3.11) yields, (3.5) and (3.6) from (3.12) we have
\[ k'_4(\frac{h_{45} h_{56}}{h_{46}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_{5}) = k'_5(\frac{\bar{h}_{45} h_{46}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_{4}) \quad \text{...}(3.13) \]
\[ k'_3(\frac{\bar{h}_{45} h_{46}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_{4}) = k'_4(\frac{\bar{h}_{45} h_{46}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_{3}) \quad \text{...}(3.14) \]
\[ k'_2(\frac{\bar{h}_{35} h_{36}}{h_{46}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_{3}) = k'_3(\frac{\bar{h}_{25} h_{26}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_{2}) \quad \text{...}(3.15) \]
\[ k'_1(\frac{h_{56} h_{35}}{h_{46}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - \bar{h}_{1}) \quad \text{...}(3.16) \]

It is easy to check that (3.1), (3.2), (3.3), (3.4) and (3.5) follow from (3.13), (3.14) and (3.15) and (3.16).

To prove the converse assume that (3.1), (3.2), (3.3), (3.4), (3.5) hold, \( w' = \frac{\bar{h}_{56} h_{46}}{h_{45}} \) by (3.1), (3.2), (3.3), (3.4) and (3.5) we obtain (3.13), (3.14), (3.15) and (3.16) where \( h'_i = h_i - h_6 \), \( k'_i = k_i - k_6 \), \( i = (1, 2, 3, 4, 5) \), moreover we set
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\[ v = \frac{1}{k_1'} \left( \frac{h_{15} h_{16}}{h_{56}} - w' - h_1' \right) = \frac{1}{k_2'} \left( \frac{h_{25} h_{26}}{h_{56}} - w' - h_2' \right) = \frac{1}{k_3'} \left( \frac{h_{35} h_{36}}{h_{56}} - w' - h_3' \right) \]

\[ = \frac{1}{k_4'} \left( \frac{h_{45} h_{46}}{h_{56}} - w' - h_4' \right) = \frac{1}{k_5'} \left( \frac{h_{45} h_{56}}{h_{46}} - w' - h_5' \right) \quad \text{.................................(3.17)}\]

By taking \( w' = \frac{h_{56} h_{46}}{h_{45}} \) and (3.17), we conclude that (3.6),(3.7),(3.8),(3.9) and (3.10) hold.

Finally, choose \( u = 1 \), \( w = w' - v k_6 \), then \( u \Re A + v \Im A + w I_6 \) is in the form (M) and has rank 1, hence by theorem (3.1) \( W(A) \) has a line segment on its boundary.
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