On 2-Parameter Estimation of Lomax Distribution

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Abstract. Preference among the three methods which are (Moment Method, Maximum Likelihood Method, Term Omission Method) that selected to estimate parameters of Lomax distribution is the purpose of this paper, where we using simulation of different sample sizes (n=25, 150, 250) and different values of parameters using Mean Square Error and Total Deviation criterions.

Keywords: Lomax Distribution, Moment Method, Maximum Likelihood Method, Term Omission Method, Mean Square Error, Total Deviation.

1. Introduction:
"Pareto Type II" also called on Lomax distribution which is a particular case of Generalized Pareto Distribution (GPD), it is application has found widely in a variety of fields like: biological sciences, life testing and actuarial and reliability modeling (Hassan and Al-Ghamdi, 2009), firm size (Corbelini et al., 2007), queuing problems, modeling the distribution of the sizes of files on servers of computer (Holland et al., 2006) and economics.

In this research we will introduce three methods which are: (Moments, Maximum Likelihood and Term Omission) for a different sample sizes (n=25, 150, 250) using simulation to estimate parameters of Lomax distribution. The Mean Square Error (MSE) and Total Deviation (TD) criterions were used to compare between the three methods.

2. Lomax Distribution:
The probability density function of the Lomax distribution is given by: (Para et al., 2018)
\[ f(x;\alpha,\beta) = \frac{\alpha}{\beta} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} \quad x > 0, \alpha, \beta > 0 \]
where \(\alpha, \beta\) are the shape and scale parameter respectively, and the cumulative distribution function is given by (Giles et al., 2011)
\[ F(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha} \]
Among the research needs is to generate a sample data from the Lomax distribution by using the following equation:
\[ x = \beta \left[ \left(1 - y\right)^{1/\alpha} - 1 \right] \]
where \(y\)~uniform(0,1), and the parameters \(\alpha, \beta\) are known.
3. Estimation Methods:

We'll use the three methods (Moments, Maximum Likelihood, Term Omission) to found the estimations of parameters of Lomax distribution as follows:

3.1. Moment Estimation:

The Method of Moment Estimation can be clarified as equality moment of population with moment of sample. 

The first moment of population of Lomax distribution is: (Giles et al., 2011)

\[ E(X) = \frac{\beta}{\alpha - 1}, \quad \alpha > 1 \]

\[ Var(X) = \frac{\beta^2 \alpha}{(\alpha - 1)^2 (\alpha - 2)}, \quad \alpha > 2 \]

And the first moment of sample of Lomax distribution is:

\[ m_1 = \frac{1}{n} \sum X_i = \bar{X} \]

\[ m_2 = \frac{1}{n} \sum X_i^2 \]

And with equality we have:

\[ \frac{\beta}{\alpha - 1} = \bar{X}, \quad \sum \frac{X_i^2}{n} = \frac{2\beta^2}{(\alpha - 1)^2 (\alpha - 2)} \]

With simplification, we get: (Para and Jan, 2018)

\[ \hat{\alpha}_{MME} = \frac{2n\bar{X}^2}{\sum X_i^2} + 2 \]

\[ \hat{\beta}_{MME} = \bar{X} (\hat{\alpha} - 1) \]

3.2. Maximum Likelihood Estimation:

The likelihood function of Lomax distribution can be written as follows:

\[ L(x; \alpha, \beta) = \frac{\alpha^n}{\beta^n} \prod (1 + \frac{x_i}{\beta})^{-(\alpha + 1)} \]

Then we can write the log-likelihood function as follows:

\[ \ln(L(x; \alpha, \beta)) = n \ln \alpha - n \ln \beta - (\alpha + 1) \sum \ln[(1 + \frac{x_i}{\beta})] (*) \]

By maximize equation (*) by taking first partial derivative over \( \alpha \) we have:

\[ \frac{\partial}{\partial \alpha} \ln(L(x; \alpha, \beta)) = n \frac{1}{\alpha} - \sum \ln[(1 + \frac{x_i}{\beta})] \quad (*1) \]

Then the maximum likelihood estimation can be obtained from equation (*1) as

\[ n \frac{1}{\alpha} - \sum \ln[(1 + \frac{x_i}{\beta})] = 0 \]

and

\[ \hat{\alpha}_{MLE} = \frac{n}{\sum \ln[(1 + \frac{x_i}{\beta})]} \]

And again by maximize equation (*) by taking first partial derivative over \( \beta \) we have:
\[
\frac{\partial \ln L_f (x; \alpha, \beta)}{\partial \beta} = \frac{n}{A} A' - \frac{n}{\beta} - AB' - A' B - B' \quad (*)2
\]

where
\[
A = \frac{n}{B} \quad \quad B = \sum \ln(1 + \frac{x_i}{\beta})
\]
\[
A' = -\frac{n B'}{B^2} \quad \quad B' = -\frac{1}{\beta} \sum \frac{x_i}{(\beta + x_i)}
\]

Then the maximum likelihood estimation can be obtained from equation (*2) as
\[
\frac{n}{A} A' - \frac{n}{\beta} - AB' - A' B - B' = 0 \quad (*)3
\]

Since there is no closed-form solution to the equation (*3), and so we can using the suitable numerical algorithm to obtain maximum likelihood estimation of \( \beta \) which is Newton-Raphson method
\[
\beta_{n+1} = \beta_n - \frac{g'(\beta)}{g''(\beta)}
\]

where
\[
g'(\beta) = \frac{n}{A} A' - \frac{n}{\beta} - AB' - A' B - B'
\]

and
\[
g''(\beta) = \frac{n}{A} A'' A - \frac{A' A'}{A^2} + \frac{n}{\beta^2} - AB'' - 2A' B - A'' B - B''
\]

where
\[
A'' = -\frac{n B'' B^2 - 2 B B''}{[B']^2} \quad \quad B'' = \sum \frac{x_i (2 \beta + x_i)}{\beta^2 (\beta + x_i)^2}
\]

3.3. Term Omission Estimation:[5#][6]

Here, we will use the cumulative probability of Lomax distribution, below are illustration steps of Term Omission method: (Labban, 2005)

a) \[x_i = 1 - (1 + \frac{x_i}{\beta})^{-a}\]

b) Subtracting each from 1, we get \[x_i = (1 + \frac{x_i}{\beta})^{-a}\]

c) Taking logarithm for each, we get \[x_i = -\alpha \ln(1 + \frac{x_i}{\beta}) = k_i\]

\[x_j = -\alpha \ln(1 + \frac{x_j}{\beta}) = k_j\]

d) Dividing one over other we get \[k \ln((1 + \frac{x_j}{\beta}) - \ln((1 + \frac{x_i}{\beta})] = 0 \quad (*)4\]
where \( k = \frac{k_i}{k_2} \)

Here, there is no closed-form solution to the equation (*4), and so we'll use Newton-Raphson method to find values of \( \beta \) say \( \hat{\beta}_{TOE} \), 1 \( \leq \ell \leq n-1 \), were the estimation of \( \beta \), say \( \hat{\beta}_{TOE} \) can be taken by:

\[
\hat{\beta}_{TOE} = \text{Min}_{1 \leq i \leq n-1} \left[ \sum_{i=1}^{n} \left( F(x_i; \alpha^{'}, \hat{\beta}^{'}_{TOE}) - F(x_i; \alpha, \hat{\beta}) \right)^2 \right]
\]

Where \( \hat{\alpha}_{TOE} = \frac{k_i}{\ln((1 + \frac{x_i}{\hat{\beta}_{TOE}}))} \), 1 \( \leq i \leq n-1 \)

4. Simulation

We will use the simulation to generate sample data with contaminated of Lomax distribution with different ratio of contaminated \( \delta \) (10\%, 20\%, 30\%) as the following:

\( f(x) = (1-\delta)L(x; \alpha, \beta) + \delta L(x; \alpha, \beta) \)

where \( L(x; \alpha, \beta) \) and \( L(x; \alpha, \beta) \) denote the distribution of Lomax distribution with \( \tau > 1 \).

5. Results:

In this section we will calculate the estimation of parameters of Lomax Distribution with selected sample size (n=25, 150, 250), with different values of parameters. All results are shown in the tables, as follows.

| Table 1: Estimation Lomax Distribution parameters using the three methods, \( (\alpha=2, \beta=3, \tau=3) \) with contaminate \( \delta=10\% \). |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| **Method** | **n** | **\( \alpha \)** | **\( \beta \)** | **TD** | **MSE** |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| MM | 25 | 2.687005 | 4.854449 | 1.020275 | 0.007217 |
| MLE | 25 | 86.04058 | 172.7312 | 96.88805 | 0.005011 |
| TOM | 25 | 2.4 | 3.6 | 1.18E-09 | 9.34E-22 |
| MM | 150 | 2.468303 | 4.300415 | 0.709249 | 0.002435 |
| MLE | 150 | 2.224582 | 3.472509 | 0.673972 | 0.000983 |
| TOM | 150 | 2.36 | 3.54 | 3.38E-10 | 9.73E-23 |
| MM | 250 | 2.421141 | 4.183214 | 0.652395 | 0.001749 |
| MLE | 250 | 2.152071 | 3.328932 | 0.530986 | 0.000563 |
| TOM | 250 | 2.352 | 3.528 | 2.41E-10 | 8.04E-23 |

| Table 2: Estimation Lomax Distribution parameters using the three methods, \( (\alpha=4, \beta=10, \tau=5) \) with contaminate \( \delta=10\% \). |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| **Method** | **n** | **\( \alpha \)** | **\( \beta \)** | **TD** | **MSE** |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| MM | 25 | 2.872705 | 6.264152 | 0.770728 | 0.004486 |
| MLE | 25 | 165.0634 | 461.7584 | 80.40879 | 0.00374 |
| TOM | 25 | 5.696 | 14.24 | 2.38E-11 | 2.2E-25 |
| MM | 150 | 2.745089 | 5.776275 | 0.840763 | 0.001535 |
| MLE | 150 | 43.75898 | 130.3695 | 20.06716 | 0.000812 |
| TOM | 150 | 5.632 | 14.08 | 8.99E-12 | 1.7E-26 |
| MM | 250 | 2.726625 | 5.726821 | 0.847182 | 0.001207 |
| MLE | 250 | 28.48336 | 80.89191 | 13.64403 | 0.000502 |
| TOM | 250 | 5.632 | 14.08 | 7.93E-12 | 1.59E-26 |
Table 3: Estimation Lomax Distribution parameters using the three methods, \((\alpha=2, \beta=3, \tau=3)\) with contaminate \(\delta=20\%\).

| Method | n  | \(\alpha\)   | \(\beta\)   | TD     | MSE     |
|--------|----|-------------|-------------|--------|---------|
| MM     | 25 | 2.686375    | 4.916014    | 1.027437 | 0.007996 |
| MLE    | 25 | 101.8749    | 192.7497    | 100.3421 | 0.004413 |
| TOM    | 25 | 2.736       | 4.104       | 1.02E-09 | 7.26E-22 |
| MM     | 150| 2.465142    | 4.257649    | 0.762562 | 0.002426 |
| MLE    | 150| 2.273181    | 3.570902    | 0.733649 | 0.001083 |
| TOM    | 150| 2.872       | 4.308       | 3.04E-10 | 8.7E-23  |
| MM     | 250| 2.423836    | 4.186019    | 0.718383 | 0.002138 |
| MLE    | 250| 2.137849    | 3.291125    | 0.636197 | 0.000852 |
| TOM    | 250| 2.792       | 4.188       | 2.15E-10 | 6.93E-23 |

Table 4: Estimation Lomax Distribution parameters using the three methods, \((\alpha=4, \beta=10, \tau=5)\) with contaminate \(\delta=20\%\).

| Method | n  | \(\alpha\)   | \(\beta\)   | TD     | MSE     |
|--------|----|-------------|-------------|--------|---------|
| MM     | 25 | 2.87852     | 6.195984    | 0.85191 | 0.004404 |
| MLE    | 25 | 169.1024    | 485.7671    | 78.55923 | 0.003844 |
| TOM    | 25 | 6.912       | 17.28       | 2.04E-11 | 1.3E-25  |
| MM     | 150| 2.741175    | 5.74257     | 0.923997 | 0.001459 |
| MLE    | 150| 45.75036    | 137.3368    | 22.98608 | 0.000806 |
| TOM    | 150| 6.944       | 17.36       | 8.45E-12 | 1.53E-26 |
| MM     | 250| 2.725466    | 5.762652    | 0.905216 | 0.00113  |
| MLE    | 250| 4.871569    | 12.68361    | 0.977281 | 0.000528 |
| TOM    | 250| 6.592       | 16.48       | 7.63E-12 | 1.49E-26 |

6. Conclusions:

From the previous tables we can conclude that the Term Omission is the preference method in all tables among the methods, although different sample size and contaminate using mean square error and total deviation.

7. References:

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