Eigenfunction expansions and scattering theory in rigged Hilbert spaces

F Gómez-Cubillo
Dpt. de Análisis Matemático, Universidad de Valladolid. Facultad de Ciencias, 47011 Valladolid, Spain
E-mail: fgcubill@am.uva.es

Abstract. The work reviews some mathematical aspects of spectral properties, eigenfunction expansions and scattering theory in rigged Hilbert spaces, laying emphasis on Lippmann-Schwinger equations and Schrödinger operators.

1. Introduction
The abstract eigenfunction expansion problem for selfadjoint operators has been widely investigated and there is a rich literature on the subject. See, for instance, [1–5]. Generalized eigenfunctions corresponding to values in the continuous spectrum are not in the Hilbert space \( \mathcal{H} \) the operator is defined on and auxiliary spaces are necessary. The method depends on the introduction of a so-called \textit{rigged Hilbert space} or \textit{Gelfand triplet} 

\[ \Phi \subset \mathcal{H} \subset \Phi^\times. \]

The subspace \( \Phi \) of \( \mathcal{H} \) is provided with its own linear topology \( \tau \), stronger than that one induced by \( \mathcal{H} \), so that its adjoint space \( \Phi^\times \), the set of all continuous antilinear forms on \( (\Phi, \tau) \), includes \( \mathcal{H} \). The eigenfunctions will be elements of \( \Phi^\times \). The method is introduced in Sections 2 and 3 below, from spectral measures to eigenfunction expansions, with certain Radon-Nykodym derivatives as guiding elements.

The main objects in scattering theory [6–8] are the wave operators \( \Omega_{\pm} \). They are defined in the time-dependent theory by (strong, weak, Abelian) limits of the form

\[ \Omega_{\pm} := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}, \]

where \( H_0, H \) are selfadjoint operators acting in a Hilbert space \( \mathcal{H} \). The stationary scattering theory defines wave operators in terms of the resolvents \( (H_0 - z)^{-1}, (H - z)^{-1} \). The central questions of existence, completeness and relationship between the various wave operators have been extensively studied. Section 4 is an introductory review of the subject.

Mathematical reasons suggesting the study of wave operators result from the fact that \( \Omega_{\pm} \) are intertwining operators for \( H_0, H \). The existence of a nontrivial intertwining operator implies the unitary equivalence of nontrivial parts of \( H_0 \) and \( H \). The intertwining property of wave

\(^1\) Dedicated to Professor Arno Bohm on the occasion of his seventieth anniversary
operators is in the basis of the deep connection between scattering theory and spectral analysis. In particular, eigenfunction expansions are main tools in scattering theory and, conversely, wave operators are in the core of abstract Lippmann-Schwinger equations to obtain generalized eigenfunctions. A rigorous description of these deep connections in the framework of rigged Hilbert spaces is given in Section 5 below.

Finally, by way of application, Section 6 deals with Schrödinger pairs $H_0 = -\Delta$, $H = -\Delta + V$ in $L^2(\mathbb{R}^n)$, usual in quantum mechanics. An extensive mathematical work has been devoted to eigenfunction expansions and scattering theory for Schrödinger operators, mainly for short-range multiplicative perturbations $V(x)$. Rigged Hilbert spaces of weighted $L^2$ spaces

$$L^{2,s}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \subseteq L^{2,-s}(\mathbb{R}^n)$$

for suitable $s > 0$ work dealing with such short-range potentials $V(x)$.

2. Spectral measures and spectral forms

Let $H$ be a separable Hilbert space with scalar product $(\cdot, \cdot)$. The classical version of the spectral theorem [9] associates to each normal operator $A$ defined on $H$ a Borel spectral measure $(\sigma(A), \mathcal{B}, P)$, where $\sigma(A)$ is the spectrum of $A$, $\mathcal{B}$ the Borel $\sigma$-algebra of $\sigma(A)$ and $P(E)$ the orthogonal projection corresponding to each set $E \in \mathcal{B}$. Every pair $f, h \in H$ defines the complex measure

$$\mu_{f,h}(E) := (f, P(E)h), \quad (E \in \mathcal{B}).$$

When $f = h$ we shall write $\mu_f := \mu_{f,f}$.

For each element $g$ of $H$ denote by $\mathcal{H}_g$ the closed subspace of $H$ defined by

$$\mathcal{H}_g = \text{adh} \{ f \in H : f = P(E)g \},$$

where $E$ runs over $\mathcal{B}$ and $\text{adh}$ denotes adherence. A family of vectors $\{g_j\}_{j=1}^m$ in $H$, where $m \in \{1, 2, \ldots, \infty\}$, is called a generating system of $H$ with respect to $P$ if $H$ is the orthogonal sum of the spaces $\mathcal{H}_{g_j}$:

$$H = \bigoplus_{j=1}^m \mathcal{H}_{g_j}.$$

By the type $[\mu]$ of a measure $\mu$ defined on $(\sigma(A), \mathcal{B})$ we understand the equivalence class of all measures $\nu$ on $(\sigma(A), \mathcal{B})$ such that $\mu$ and $\nu$ are mutually absolutely continuous, that is, they have the same class of null sets. We shall write $[\nu] \succ [\mu]$ to indicate that a measure $\mu$ is absolutely continuous with respect to a measure $\nu$ and, in this case, $d\mu/d\nu$ shall denote the Radon-Nykodym derivative of $\mu$ with respect to $\nu$.

A nonzero vector $g \in H$ is of maximal type with respect to the spectral measure $P$ if for each $f \in H$, $[\mu_g] \succ [\mu_f]$. In this case, $g$ is called a maximal vector. Such maximal vectors always exist provided that $H$ is separable. The type $[\mu_g]$ of a maximal vector is called the spectral type of $P$ and shall be denoted by $[P]$.

It has been proved in [10] that, given a $\sigma$-finite nonnegative scalar measure $\mu$ on $(\sigma(A), \mathcal{B})$ such that $[\mu] = [P]$ and a generating system $\{g_j\}_{j=1}^m$ of $H$ with respect to $P$, then, for each $f, h \in H$ and $E \in \mathcal{B}$,

$$(f, P(E)h) = \sum_{j=1}^m \int_E \frac{d\mu_{g_j}}{d\mu}(\lambda) \frac{d\mu_{f,g_j}}{d\mu_{g_j}}(\lambda) \frac{d\mu_{g_j}}{d\mu_{g_j}}(\lambda) d\mu(\lambda). \quad (1)$$

The Radon-Nykodym derivatives in Eq. (1) are defined for almost every $\lambda \in \sigma(A)$ with respect to measures $\mu$ and $\mu_{g_j}$, the null sets depending on $f$, $h$ and $g_j$. 2
For each $j \in \{1, 2, \ldots, m\}$ let \( \{\phi_{j,k}\}_{k=1}^{d_j} \) and orthonormal basis of \( \mathcal{H}_{g_j} \). Then

\[
\Phi := \text{span}\{\phi_{j,k} : j \in \{1, 2, \ldots, m\}, k = 1, 2, \ldots, d_j\}
\]

is a dense subspace of \( \mathcal{H} \) and

\[
s(\lambda; \varphi, \phi) := \sum_{j=1}^{m} \frac{d\mu_{g_j}}{d\mu}(\lambda) \frac{d\mu_{g_j,\phi}}{d\mu_{g_j}}(\lambda) \frac{d\mu_{g_j,\varphi}}{d\mu_{g_j}}(\lambda)
\]

is well-defined for every \( \varphi, \phi \in \Phi \) and \( \lambda \in \hat{\Lambda} \), where

\[
\hat{\Lambda} := \{ \lambda \in \sigma(\mathcal{A}) : \frac{d\mu_{g_j}}{d\mu}(\lambda) \text{ and } \frac{d\mu_{g_j,\phi}}{d\mu_{g_j}}(\lambda) \text{ exist and are finite}
\]

for \( j \in \{1, 2, \ldots, m\} \) and \( k = 1, 2, \ldots, d_j \} \).

**Definition 2.1** In general, by a *spectral system* \((\Lambda, \mathcal{B}, \mu, \mathcal{H}, P)\) we mean a spectral measure \( P \) on a measurable space \((\Lambda, \mathcal{B})\), with values in the set of all orthogonal projections in a complex Hilbert space \( \mathcal{H} \), together with a \( \sigma \)-finite nonnegative scalar measure \( \mu \) on \((\Lambda, \mathcal{B})\). By a standard process, \( P \) is decomposed into the absolutely continuous part \( P_{ac} \) and the singular part \( P_s \) with respect to \( \mu \).

The basic elements of the Kato-Kuroda scattering theory [11] are the *spectral forms* for spectral systems:

**Definition 2.2** Let \((\Lambda, \mathcal{B}, \mu, \mathcal{H}, P)\) be an spectral system. A *spectral form* for this system is a complex function on \( \hat{\Lambda} \times \Phi \times \Phi \), where \( \Phi \) is a subspace of \( \mathcal{H} \) and \( \hat{\Lambda} \subseteq \Lambda \) belongs to \( \mathcal{B} \), with the following properties:

(i) For each \( \phi, \varphi \in \Phi \), \( \lambda \mapsto s(\lambda; \varphi, \phi) \) is \( \mu \)-integrable in \( \hat{\Lambda} \) and its integral on each \( E \subseteq \hat{\Lambda} \), \( E \in \mathcal{B} \), is equal to \( (\varphi, P_{ac}(E)\phi) \), i.e.,

\[
(\varphi, P_{ac}(E)\phi) = \int_E s(\lambda; \varphi, \phi) \, d\mu(\lambda).
\]

(ii) For each \( \lambda \in \hat{\Lambda} \), the function \( \varphi, \phi \mapsto s(\lambda; \varphi, \phi) \) is a nonnegative Hermitian form on \( \Phi \times \Phi \).

We shall write \( s(\lambda; \phi) \) for \( s(\lambda; \phi, \phi) \).

The subspace \( \Phi \) is usually called a *spectral subspace* and the subset \( \hat{\Lambda} \) a *spectral core* of the spectral system.

Clearly, if \( \Phi, s \) and \( \hat{\Lambda} \) are respectively defined by (2), (3) and (4), then \( \mu(\sigma(\mathcal{A}) \setminus \hat{\Lambda}) = 0 \) and the function \( s : \hat{\Lambda} \times \Phi \times \Phi \rightarrow \mathbb{C} \) is a *spectral form* of the spectral system \((\sigma(\mathcal{A}), \mathcal{B}, \mu, \mathcal{H}, P)\).

### 3. Eigenfunction expansions

Here the theory of eigenfunction expansions is formulated in an abstract way following ideas by Gelfand-Vilenkin [3], Gelfand-Shilov [2] and Kato-Kuroda [11]. The method depends on the introduction of a rigged Hilbert space \( \Phi \subset \mathcal{H} \subset \Phi^\times \). The eigenfunctions will be elements of \( \Phi^\times \).

In what follows the value of \( \psi^\times \in \Phi^\times \) at \( \phi \in \Phi \) shall be denoted by \( \langle \phi|\psi^\times \rangle \), and we shall write \( \langle \psi^\times|\phi \rangle = \langle \phi|\psi^\times \rangle^* \) (the asterisk * denotes complex conjugation). The dual pair \( \langle \phi|\psi^\times \rangle \) is antilinear in \( \phi \) and linear in \( \psi^\times \).

Let us begin by considering a selfadjoint operator \( A \) in \( \mathcal{H} \) and, for simplicity, assume that \( A \) has simple spectrum. Let \( P \) be the spectral measure of \( A \) and let \( g \) be any generating vector, so that \( \mathcal{H} = \mathcal{H}_g \) and

\[
\mu_g(\lambda) = (g, P(\lambda)g).
\]
Suppose first that the spectrum of $A$ consist only of certain eigenvalues $\sigma(A) = \{\lambda_1, \lambda_2, \ldots\}$. Then, in the sense of norm in $\mathcal{H}$, the limit

$$\frac{dP\,g}{d\mu_g}(\lambda) = \lim_{\delta \to 0} \frac{P[\lambda, \lambda + \delta) g}{\mu_g[\lambda, \lambda + \delta)}$$

exists for each $\lambda \in \mathbb{R}$, is null for $\lambda \notin \sigma(A)$, and the elements $\frac{dP\,g}{d\mu_g} \big|_{\lambda_1}$, $\frac{dP\,g}{d\mu_g} \big|_{\lambda_2}$, ... form a complete system of eigenvectors of the operator $A$.

Suppose now that for the operator $A$ there exists a point $\lambda$ of the spectrum which is not an eigenvalue. Then the derivative $\frac{d\mu_g}{d\mu_g}(\lambda)$ does not exist in the space $\mathcal{H}$. However, if the rigging $\Phi \subset \mathcal{H} \subset \Phi^\times$ is carried out in a suitable manner, then this derivative could exist in the wider space $\Phi^\times$:

$$\frac{dP\,g}{d\mu_g}(\lambda) = \psi^\times(\lambda) \in \Phi^\times.$$ 

Further, it turns out that $\psi^\times(\lambda)$ is a generalized eigenfunctional of the operator $A$ in the following sense:

$$(A\phi|\psi^\times(\lambda)) = \lambda \langle \phi|\psi^\times(\lambda) \rangle = \lambda \frac{d\mu_g,\phi}{d\mu_g}(\lambda), \quad (\phi \in \Phi),$$

(here we assume that $\Phi \subseteq \mathcal{D}(A)$ and $A\Phi \subseteq \Phi$). The system of generalized eigenfunctionals obtained in this way proves to be orthogonal and complete in the sense that a sort of Parseval equation holds:

$$||\phi||^2 = \int |\langle \phi|\psi^\times(\lambda) \rangle|^2 d\mu_g(\lambda), \quad (\phi \in \Phi).$$

In general, eigenfunction expansions are defined to consider more refined representations than the spectral ones constructed above:

**Definition 3.1** An eigenfunction expansion for an spectral system $(\Lambda, \mathcal{B}, \mu, \mathcal{H}, P)$ with spectral form $(\Lambda, \Phi, s)$ is defined in the following terms:

1. There exist a $\sigma$-finite measure space $(\Gamma, \Sigma, \rho)$, a partial isometry $F$ of $\mathcal{H}$ onto $L^2(\rho)$ with initial set $\mathcal{H}^{ac}$ and a measurable function $\lambda : \Gamma \to \Lambda$ such that

$$[F \alpha(P) h](\xi) = \alpha(\lambda(\xi)) [F h](\xi), \quad (\rho \text{-a.e. } \xi \in \Gamma),$$

for each $h \in \mathcal{H}$ and $\alpha \in L^\infty(\mu)$, $\alpha(P) := \int_{\Lambda} \alpha(\lambda) P(d\lambda)$.

2. There is a complex-valued function $\psi$ on $\Gamma \times \Phi$ such that for each fixed $\xi \in \Gamma$, $\phi \leftrightarrow \psi(\xi; \phi)$ is linear and for each fixed $\phi \in \Phi$,

$$\psi(\xi; \phi) = [F \phi](\xi), \quad (\rho \text{-a.e. } \xi \in \Gamma).$$

3. $\Phi$ is a topological vector space (tvs) and $\phi \leftrightarrow \psi(\xi; \phi)$ is continuous on $\Phi$ for each $\xi \in \Gamma$. In this case we shall write $\psi(\xi; \phi) = \langle \psi^\times(\xi)|\phi \rangle$, where $\psi^\times(\xi) \in \Phi^\times$. Each $\psi(\xi)$ shall be called an eigenfunction of $P$.

**Example 1** Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $F$ the Fourier-Plancherel transform from $L^2(\mathbb{R}^n)$ onto itself, so that $\Gamma$ is another copy of $\mathbb{R}^n$, $\Sigma$ the Borel $\sigma$-algebra and $\rho$ the Lebesgue measure. Let $\lambda$ be the map from $\Gamma = \mathbb{R}^n$ into $\Lambda = \Lambda = \mathbb{R}^+$ given by $\lambda(\xi) = |\xi|^2 = \sum_{j=1}^n \xi_j^2$, $B$ the Borel $\sigma$-algebra of $\Lambda$, $\mu$ the Lebesgue measure on $\Lambda$. Let $P_0(E) = F^{-1} \hat{P}_0(E) F$, where $\hat{P}_0(E)$ is the operator...
of multiplication by the characteristic function $\chi_{w^{-1}(E)}$, and $\Phi = L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with the $L^1$-topology. Then $\psi_0(\xi)$ is given by the function $(2\pi)^{-n/2} e^{i\xi \cdot x}$ in the sense that

$$\langle \psi_0^\times(\xi) | \phi \rangle = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(x) e^{-i\xi \cdot x} \, dx , \quad (\phi \in \Phi) ,$$

The spectral measure $P_0$ is the one associated with the selfadjoint realization of $-\Delta = -\sum_{j=1}^n \partial^2 / \partial x_j^2$ in $L^2(\mathbb{R}^n)$ and the $\psi_0(\xi)$ are the eigenfunctions of this operator in the usual sense. In particular, for $n = 3$, a spectral form is (see Section 15.14 in [1])

$$s_0(\lambda; \varphi, \phi) = (4\pi^2)^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\sin(\lambda^{1/2}|x-y|)}{|x-y|} \phi(x) \varphi^*(y) \, dx \, dy , \quad (\lambda \in \Lambda, \varphi, \phi \in \Phi) .$$

In particular, for a spectral system $(\Lambda, \mathcal{B}, \mu, \mathcal{H}, P)$ with $[\mu] = [P]$ one can consider the spectral form $(\tilde{\Lambda}, \tilde{\Phi}, s)$ given by the formulae (2), (3) and (4), and take the space $L^2(\Gamma, \Sigma, \rho)$ in Definition 3.1 to be $L^2(\Lambda \times \{1,2,\ldots,m\}, \mu \times d)$, where $d$ denotes the discrete measure on $\{1,2,\ldots,m\}$. In this situation a complete system of generalized eigenfunctions is given by

$$\langle \psi^\times(\lambda \times k) | \phi \rangle := \sqrt{\frac{d\mu_{\lambda k}}{d\mu}}(\lambda) \frac{d\mu_{\lambda k}}{d\mu_{\lambda k}} \varphi(\lambda) , \quad (\phi \in \Phi, \lambda \in \Lambda, k \in \{1,2,\ldots,m\}) . \quad (5)$$

The minimal topology on $\Phi$ for which each $\psi^\times(\lambda \times k)$ is in $\Phi^\times$ is just the one generated by the functionals $\psi^\times(\lambda \times k)$. See [10] for details.

In this context renewed versions of the nuclear and inductive spectral theorems have been proved in [12]. These results assert that dual pairs $(\Phi, \Phi^\times)$ with nuclear or countable Hilbert-Schmidt inductive topologies on $\Phi$ are universal riggings in the sense that the antiduals spaces $\Phi^\times$ contain complete systems of generalized eigenfunctions for any Vitali spectral measure.

There is another version of the spectral theorem in rigged Hilbert spaces due to Yu.M. Berezanskiĭ [5], the so-called projection spectral theorem. Given a normal operator $A$ on a Hilbert space $\mathcal{H}$, Berezanskiĭ considers a chain of spaces $D \subseteq \mathcal{H}_+ \subseteq \mathcal{H} \subseteq \mathcal{H}_-$ associated with $A$ in a certain sense and proves the existence of a nonnegative operator $S(\lambda) : \mathcal{H}_+ \to \mathcal{H}_-$ such that $T^i P(\Lambda) = \int_T S(\lambda) \rho(\lambda) \, d\lambda$, where $T$ is the injection of $\mathcal{H}_-$ into $\mathcal{H}$, $T^i$ its adjoint, $\rho(\Lambda) = \text{Tr}(T^i P(\Lambda) T)^i$ and $P$ is the spectral measure associated to $A$. The projection spectral theorem assures that, almost everywhere with respect to a special modification of the measure $\rho$, the range of $S(\lambda)$ consists of all generalized eigenvectors with eigenvalue $\lambda$ of $A$. The connections of the projection spectral theorem with the nuclear spectral theorem are discussed in [13].

The construction of eigenfunction expansions for selfadjoint operators given by Poerschke, Stolz and Weidmann in [14] is closely related to the above approaches. For a selfadjoint operator $A$ defined on $\mathcal{H}$ they start from the spectral representation of $A$ and use triplets of the form $\mathcal{H}_+(T) \subset \mathcal{H} \subset \mathcal{H}_-(T)$, where $T$ is a selfadjoint operator in $\mathcal{H}$ with $T \geq 1$ and $\mathcal{H}_\pm(T)$ are the Hilbert spaces obtained by completion of $\mathcal{D}(T)$ for the scalar products $(f, g)_\pm := (T^{\pm 1}f, T^{\pm 1}g)_\mathcal{H}$. If there is a bounded continuous function $\gamma : \mathbb{R} \to \mathbb{C}$ with $|\gamma| > 0$ on $\sigma(A)$ and such that $\gamma(A) T^{-1}$ is a Hilbert-Schmidt operator, then there exist a complete system of generalized eigenfunctions of $A$ in $\mathcal{H}_-(T)$.

4. Wave operators

Given a pair $(H_0, H)$ of selfadjoint operators in a Hilbert space $\mathcal{H}$, the time-dependent scattering theory deals with limits of $e^{itH} e^{-itH_0}$ as $t \to \pm \infty$ in strong, weak or Abelian sense, paying attention to the absolutely continuous subspaces of $\mathcal{H}$ with respect to both operators $H_0$ and $H$ and Lebesgue measure $d\lambda$. In what follows such subspaces shall be denoted by $\mathcal{H}_{ac}^0$ and $\mathcal{H}_{ac}^\times$, and the orthogonal projections from $\mathcal{H}$ onto them by $P_{ac}^0$ and $P_{ac}^\times$. 

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Definition 4.1 The strong wave operators for the pair \((H_0, H)\) are defined as

\[
\Omega^s_{\pm} := \text{s-limit } e^{itH} e^{-itH_0} P^\text{ac}_0. 
\]

In the same way are defined the weak and Abelian wave operators \(\Omega^w_{\pm}\) and \(\Omega^a_{\pm}\), substituting in the above expression strong limits (s-limit) by weak (w-limit) and Abelian (a-limit) ones. Among the Abelian limits one can distinguish between strong, weak and absolute Abelian limits.

The stationary theory define the wave operators in terms of the resolvents \(R_0(z) = (H_0 - z)^{-1}\) and \(R(z) = (H - z)^{-1}\).

Definition 4.2 If there exist dense subspaces \(\Phi_0\) and \(\Phi\) of \(\mathcal{H}\) such that for each \(\phi_0 \in \Phi_0\) and \(\phi \in \Phi\) the limits

\[
\lim_{\epsilon \to 0} \pi^{-1} \epsilon (R(\lambda \pm i\epsilon)\phi, R_0(\lambda \pm i\epsilon)\phi_0)
\]

exist for a.a. \(\lambda \in \mathbb{R}\) (with respect to Lebesgue measure, the corresponding null sets depending on \(\phi_0\) and \(\phi\)), the stationary wave operators \(\Omega^{\text{est}}_{\pm}\) for the pair \((H_0, H)\) are defined in \(\Phi_0 \times \Phi\) by the sesquilinear forms

\[
(\phi, \Omega^{\text{est}}_{\pm} \phi_0) := \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \pi^{-1} \epsilon (R(\lambda \pm i\epsilon)\phi, R_0(\lambda \pm i\epsilon)\phi_0) \, d\lambda. \tag{6}
\]

If they exist, the (strong, weak, abelian, stationary) wave operators \(\Omega_{\pm}\) are bounded, satisfy the intertwining property

\[
P^{\text{ac}}(E) \Omega_{\pm} = \Omega_{\pm} P^{\text{ac}}_0(E), \quad \text{for every Borel set } E \subset \mathbb{R}, \tag{7}
\]

and \(\mathcal{H} \ominus \mathcal{H}^{\text{ac}}_0 \subset \ker(\Omega_{\pm})\) and \(\text{Ran}(\Omega_{\pm}) \subset \mathcal{H}^{\text{ac}}\). In particular the strong and stationary wave operators are partial isometries with initial subspace \(\mathcal{H}^{\text{ac}}_0\). The wave operators \(\Omega_{\pm}\) are said to be complete when

\[
\ker(\Omega_{\pm}) = \mathcal{H} \ominus \mathcal{H}^{\text{ac}}_0 \quad \text{and} \quad \text{rang}(\Omega_{\pm}) = \mathcal{H}^{\text{ac}}. 
\]

The books by Baumgärtel-Wollenberg [7], Yafaev [8] and Reed-Simon [6] deal at length with the problem of determining conditions ensuring the existence and completeness of the various wave operators as well as with the relationship between them. Typical criteria have been given for trace class perturbations or in terms of time-falloff conditions or a factorization of the perturbation and the relative regularity of the factors.

The central result of scattering theory for trace class perturbations is the Kato-Rosenblum theorem [15, 16]: For selfadjoint operators \(H_0\) and \(H\) with trace class difference, \(V = H - H_0 \in \mathcal{S}_1\), the strong wave operators \(\Omega^s_{\pm}\) exist and are complete. In particular, the absolutely continuous parts \(\mathcal{H}^{\text{ac}}_0\) and \(\mathcal{H}^{\text{ac}}\) are unitarily equivalent.

The condition \(V \in \mathcal{S}_1\) is too much restrictive in most applications. There are some less demanding local trace class criteria. For example, if \(P(\Lambda)V P_0(\Lambda) \in \mathcal{S}_1\) for every bounded interval \(\Lambda\) and \(R(z) - R_0(z)\) is a compact operator for some (and then for all) \(z \in \rho(H_0) \cap \rho(H)\), then the strong wave operators \(\Omega^s_{\pm}\) exist and are complete [8, Th.6.4.5].

Additional results with trace class conditions can be found in [17, Section X.4], [6, Section XI.3], [7, Chapter 16] and [8, Chapter 6].

One of the simplest time-falloff criteria is known as Cook’s criterium [8, Th.2.5.1]: Suppose that \(\mathcal{D}(H_0) \subseteq \mathcal{D}(H)\) and that for some set \(D_0 \subset \mathcal{D}(H_0) \cap \mathcal{H}^{\text{ac}}_0\) dense in \(\mathcal{H}^{\text{ac}}_0\) one has

\[
\int_0^{\pm \infty} ||Ve^{-itH_0} f_0|| \, dt < \infty, \quad (f_0 \in D_0). 
\]
Then the strong wave operators $\Omega^\pm_z$ exist. More time-falloff criteria can be found in [7, Chapter 15].

In order to introduce factorization criteria, for each selfadjoint operator $H$ in $\mathcal{H}$, with spectral measure $P$ and resolvent $R(z) = (H - z)^{-1}$, consider $\delta = \delta_H$ the operator-valued function $\delta = \delta_H$ defined by

$$\delta(\lambda, \epsilon) := (2\pi i)^{-1} [R(\lambda + i\epsilon) - R(\lambda - i\epsilon)] = \pi^{-1} \epsilon R(\lambda + i\epsilon) R(\lambda - i\epsilon), \quad (\lambda \in \mathbb{R}, \epsilon > 0).$$

For $f, g \in \mathcal{H}$, one has

$$\lim_{\epsilon \to 0} (g, \delta(\lambda, \epsilon) f) = \frac{d(g, P(\lambda)f)}{d\lambda} = \frac{d\mu_{g,f}}{d\lambda}, \quad \text{for a.a. } \lambda \in \mathbb{R}. \quad (8)$$

The null set depending on $f$ and $g$, relation (8) does not imply the existence of the weak limit of $\delta(\lambda, \epsilon)$ as $\epsilon \to 0$. Actually, $\delta(\lambda, \epsilon)$ has not bounded weak limit for any $\lambda \in \mathbb{R}$ because

$$||\delta(\lambda, \epsilon)|| = \pi^{-1} \epsilon \sup_{u \in \mathbb{R}} [(u - \lambda)^2 + \epsilon^2]^{-1} = (\pi \epsilon)^{-1}.$$

But, for suitable operators $G$, the weak limit of $G\delta(\lambda, \epsilon)$ can exist. This leads to the following definitions.

**Definition 4.3** Let $H$ be a selfadjoint operator in a Hilbert space $\mathcal{H}$. An operator $G$ from $\mathcal{H}$ into a Hilbert space $G$ is called $H$-bounded if $\mathcal{D}(H) \subseteq \mathcal{D}(G)$ and there are positive constants $a$ and $b$ such that $||Gf|| \leq a ||f|| + b ||Hf||$ for every $f \in \mathcal{D}(H)$. This condition is equivalent to the boundedness of $GR(z)$ for every $z \in \rho(H)$. An $H$-bounded operator $G$ is called $H$-regular in weak sense if there exist positive constants $C(\lambda)$ such that

$$||G\delta(\lambda, \epsilon)G^\dagger|| \leq C(\lambda), \quad \text{for a.a. } \lambda \in \mathbb{R}. \quad (9)$$

An $H$-bounded operator $G$ is called $H$-regular when

$$\sup_{\lambda \in \mathbb{R}, \epsilon > 0} ||G\delta(\lambda, \epsilon)G^\dagger|| < \infty. \quad (10)$$

Condition (9) is equivalent to the existence of the weak limit $\text{w-lim}_{\epsilon \to 0} G\delta_H(\lambda, \epsilon)G^\dagger$ for a.a. $\lambda \in \mathbb{R}$, and also implies that $\text{w-lim}_{\epsilon \to 0} G\delta_H(\lambda, \epsilon)$ exists for a.a. $\lambda \in \mathbb{R}$. The $H$-regularity of $G$ can also be given in terms of the group $e^{-itH}$ or the resolvent $R(z)$: An $H$-bounded operator $G$ is $H$-regular if and only if

$$\sup_{f \in \mathcal{D}(H), ||f|| = 1} \int_{-\infty}^{\infty} ||Ge^{-itH}f||^2 \, dt < \infty$$

or

$$\sup_{||f|| = 1, \epsilon > 0} \int_{-\infty}^{\infty} (||GR(\lambda - i\epsilon)f||^2 + ||GR(\lambda + i\epsilon)f||^2) \, d\lambda < \infty.$$

The last condition implies that the function $GR(z)f$, vector-valued in $G$, belongs to the Hardy classes $\mathcal{H}^2_\pm(\mathbb{R}, d\lambda; G)$. Then, for every $f \in \mathcal{H}$, there exists the strong limits $s\text{-lim}_{\epsilon \to 0} GR(\lambda \pm i\epsilon)f$ for a.a. $\lambda \in \mathbb{R}$. More equivalent conditions to (9) and (10) can be found in Sections 5.1 and 4.3 of Yafaev [8].

Assume the perturbation admits a factorization of the form\(^2\)

$$V := H - H_0 = G^\dagger G_0, \quad (11)$$

\(^2\) This relation is understood as equality of the corresponding sesquilinear forms: For $f_0 \in \mathcal{D}(H_0)$ and $f \in \mathcal{D}(H)$, one has $v[f, f_0] := (Hf, f_0) - (f, H_0f_0) = (Gf, G_0f_0).$
where $G_0 : \mathcal{H}_0 \rightarrow \mathcal{G}$, $G : \mathcal{H} \rightarrow \mathcal{G}$ with $\mathcal{G}$ an auxiliary Hilbert space. If $G_0$ is $H_0$-bounded and $G$ is $H$-bounded, then

$$\pi^{-1} \epsilon (R(\lambda \pm i \epsilon) f, R_0(\lambda \pm i \epsilon) f_0) = (f, \delta_H(\lambda, \epsilon) f_0) + (G \delta_H(\lambda, \epsilon) f, G_0 R_0(\lambda \pm i \epsilon) f_0).$$

From (8), the first summand in the right converges to $d(f, P(\lambda) f_0)/d\lambda$ as $\epsilon \rightarrow 0$ for a.a. $\lambda \in \mathbb{R}$. Thus, if the functions $G_0 R_0(\lambda \pm i \epsilon) f_0$ and $G \delta_H(\lambda, \epsilon) f$ converge for $f_0$ and $f$ in dense sets of $\mathcal{H}_0^\alpha$ and $\mathcal{H}^\alpha$, one in strong sense and the other in weak sense, the conditions in Definition 4.2 are satisfied and the stationary wave operators $\Omega^\pm_{\pm}$ exist and in the same way if we consider the functions $G_0 \delta_0(\lambda, \epsilon) f_0$ and $G R(\lambda \pm i \epsilon) f$. In particular, if $G_0$ is $H_0$-regular and $G$ is $H$-regular, one in weak sense, the stationary wave operators $\Omega^\pm_{\pm}$ exist. In this case, if any other wave operators (strong, weak, Abelian) exist, they coincide with the stationary ones. See Sections 5.2 and 5.3 of [8] for details.

When $G_0$ is $H_0$-regular and $G$ is $H$-regular, the strong wave operators $\Omega^\pm_{\pm}$ exist and are complete [8, Th.4.5.1]. In this case the stationary wave operators $\Omega^\pm_{\pm}$ also exist, coincide with $\Omega^\pm_{\pm}$ and relations (6) are satisfied for every $\phi_0, \phi \in \mathcal{H}$.

Standard stuff on relative smoothness can be found in [7, Chapter 17] and [8, Chapters 4, 5].

5. Wave operators vs eigenfunction expansions

There is a close relationship between the wave operators for a pair of selfadjoint operators $H_0$, $H$ and the eigenfunction expansions of $H_0$, $H$. The key property is the intertwining relation (7).

Let $P_0$, $P$ be the spectral measures of $H_0$, $H$ and let $P_0^\alpha$, $P^\alpha$ be their absolute continuous parts with respect to the Lebesgue measure $d\lambda$. Consider a direct integral spectral decomposition of $P_0^\alpha$

$$\mathcal{H}^{\alpha}_{d\lambda,H_0} = \int_\sigma^\oplus \mathcal{H}^0_{\lambda} d\lambda$$

and a structural isomorphism $\mathcal{V}_0 : \mathcal{H}^\alpha_{0} \rightarrow \mathcal{H}^{\alpha}_{d\lambda,H_0}$ performing a spectral representation of $P_0^\alpha$, i.e., a unitary operator $\mathcal{V}_0$ such that

$$\mathcal{V}_0 P_0^\alpha(E) = \chi_E \mathcal{V}_0,$$

where $\chi_E$ denotes the characteristic function of $E$. For any wave operators $\Omega^\pm$, from (7) and (12) it is obvious that the operators $\mathcal{V}^\pm$ defined by

$$\mathcal{V}^\pm := \mathcal{V}_0 \Omega^\dagger_{\pm} : \mathcal{H}^\alpha_{d\lambda,H_0} \rightarrow \mathcal{H}^{\alpha}_{d\lambda,H_0}$$

satisfy the intertwining property

$$\mathcal{V}^\pm P^\alpha(E) = \chi_E \mathcal{V}^\dagger_{\pm}.$$

Thus, if the operators $\Omega^\pm : \mathcal{H}^\alpha_{0} \rightarrow \mathcal{H}^\alpha_{d\lambda,H_0}$ are unitary, then $\mathcal{V}^\pm$ are also unitary and perform direct integral representations of $H^\alpha$ in $\mathcal{H}^\alpha_{d\lambda,H_0}$. Moreover, since $\mathcal{V}_0^\dagger \mathcal{V}_0 = P_0^\alpha$, the operators $\Omega^\pm$ can be reconstructed from $\mathcal{V}^\pm$:

$$\Omega^\pm = \mathcal{V}^\dagger_{\pm} \mathcal{V}_0.$$

Following the work of Birman and Yafaev [8, Th.5.5.1], assume that the perturbation $V = H - H_0$ admits a factorization of the form (11). If $G_0$ is $H_0$-regular in weak sense, the operators

$$Z_0(\lambda; G_0) : \mathcal{G} \rightarrow \mathcal{H}^0_{\lambda}$$

$$\psi \mapsto [\mathcal{V}_0 G_0^\dagger \psi](\lambda)$$
are well-defined bounded operators for a.a. $\lambda \in \mathbb{R}$, for every $f \in \mathcal{H}$ the derivative $dG_0P_0(\lambda)f/d\lambda$ exists in weak sense and $dG_0P_0(\lambda)f/d\lambda = [Z_0(\lambda, G_0)]^V\nu_0f(\lambda)$ for a.a. $\lambda \in \Lambda$. If also the strong limits

$$s\text{-lim}_{\epsilon \to 0} GR(\lambda \pm i\epsilon)\phi =: GR(\lambda \pm i0)\phi$$

exist for a.a. $\lambda \in \mathbb{R}$ for each $\phi$ in a dense subspace $\Phi$ of $\mathcal{H}_{ac}$, then, as we have seen, the stationary wave operators $\Omega^{ac}_{\pm}$ exist. Let us define at first in $\Phi$ the operators $\nu_0 := \nu_0(\Omega^{est})^V : \mathcal{H} \to \mathcal{H}_{ac}$.

Then $\nu_\pm$ are bounded operators with a unique extension to $\mathcal{H}$ and, for each $f \in \mathcal{H}$, they verify (in weak sense):

$$[\nu_\pm f](\lambda) = [\nu_0f](\lambda) - Z_0(\lambda, G_0)GR(\lambda \pm i0)f, \quad \text{for a.a. } \lambda \in \mathbb{R},$$

the null-set depending on $f$.

In general, let $\mu$ be a Borel $\sigma$-finite nonnegative scalar measure $\mu$ on $\mathbb{R}$ and consider respective direct integral decompositions $\mathcal{H}^{ac}_{\mu, H_0} = \int_\sigma^\infty \mathcal{H}_\lambda^0 d\mu$, $\mathcal{H}^{ac}_{\mu, H} = \int_\sigma^\infty \mathcal{H}_\lambda d\mu$ over spectral cores $\hat{\sigma}_0, \hat{\sigma}$ of $P_0^ac, P^{ac}$ with corresponding structural isomorphisms $\nu_0, \nu$ (now the spectral cores and absolutely continuous parts with respect to $\mu$). Then any bounded operator $\Omega : H_0 \to \mathcal{H}$ satisfying the intertwining property (7) diagonalizes in these representations, i.e., the operator

$$\nu_0^\dagger : \mathcal{H}^{ac}_{\mu, H_0} \longrightarrow \mathcal{H}^{ac}_{\mu, H}$$

acts as multiplication by an operator-valued function $U(\lambda) : \mathcal{H}_\lambda^0 \to \mathcal{H}_\lambda$. One has $||\Omega|| = \text{ess sup} ||U(\lambda)||$ and, for $\mu$-a.a. $\lambda \in \hat{\sigma}_0 \setminus \hat{\sigma}$, the operators $U(\lambda)$ are the null-operator. Furthermore, if $\Omega : \mathcal{H}^{ac}_{\mu, H_0} \to \mathcal{H}^{ac}_{\mu}$ is unitary, then the operators $U(\lambda)$ are unitary for $\mu$-a.a. $\lambda \in \hat{\sigma}_0 = \hat{\sigma}$. In this case, each measurable orthonormal basis $\{e_j(\lambda)\}$ of $\mathcal{H}^{ac}_{\mu, H_0}$ leads to a measurable orthonormal basis $\{e_j(\lambda)\}$ of $\mathcal{H}^{ac}_{\mu, H}$ through the relation

$$e_j(\lambda) = U(\lambda)e_j^0(\lambda), \quad \text{for } j \in \{1, 2, \ldots, m\} \text{ and } \mu\text{-a.a. } \lambda \in \hat{\sigma}_0 = \hat{\sigma}. \quad (14)$$

The connection of measurable orthonormal bases $e_j^0(\lambda), e_j(\lambda)$ with generalized eigenfunctions $\psi^0_\lambda(\lambda, j), \psi^\lambda_\lambda(\lambda, j)$ –see Eq. (5)– is established in [10]: There exist respective generating systems $\{g_j^0\}_{j=1}^m, \{g_k\}_{k=1}^m$ of $\mathcal{H}^{ac}_{\mu, H}, \mathcal{H}^{ac}_{\mu}$ with respect to $P_0^ac, P^{ac}$ such that

$$\langle \psi^0_\lambda^*(\lambda, j) | \cdot \rangle = (e_j^0(\lambda), [\nu_0(\cdot) | \lambda)_{\mathcal{H}^0_\lambda} = \sqrt{\frac{d\mu_{g_0}^0}{d\mu}}(\lambda) \frac{d\mu_{g_j^0}}{d\mu}(\lambda),$$

$$\langle \psi^\lambda(\lambda, j) | \cdot \rangle = (e_j(\lambda), [\nu_\lambda(\cdot) | \lambda)_{\mathcal{H}_\lambda} = \sqrt{\frac{d\mu_{g_j}}{d\mu}}(\lambda) \frac{d\mu_{g_0}}{d\mu}(\lambda).$$

One has

$$\langle e_j^0(\lambda), [\nu_0^\dagger(\cdot) | \lambda)_{\mathcal{H}^0_\lambda} = (e_j(\lambda), [\nu\Omega(\cdot) | \lambda)_{\mathcal{H}_\lambda}, \quad (e_j(\lambda), [\nu_\lambda(\cdot) | \lambda)_{\mathcal{H}_\lambda} = (e_j^0(\lambda), [\nu_0(\Omega^\dagger(\cdot) | \lambda)_{\mathcal{H}^0_\lambda},$$

and the relationship between the generating systems $\{g_j^0\}_{j=1}^m, \{g_j\}_{j=1}^m$ is of the form

$$\Omega(\int \phi dP_0^0 g_j^0) = \int \phi dP g_j, \quad \text{for } j \in \{1, 2, \ldots, m\} \text{ and } \phi \in L^{\infty}(\mu).$$

Kato-Kuroda [11] construct direct integral spectral representations in a simple way from the spectral forms as follows: Let $(\Lambda, \mathcal{B}, \mu, \mathcal{H}, P)$ be a spectral system with spectral form $(\hat{\Lambda}, \Phi, s)$. 

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For each \( \lambda \in \hat{\Lambda} \), \( s(\lambda; \cdot, \cdot) \) defines a semi-inner product in \( \Phi \). Let \( \mathcal{N}_\lambda \) be the set of all \( \phi \) with \( s(\lambda; \phi) = 0 \). Then the quotient space \( \Phi/\mathcal{N}_\lambda \) is a pre-Hilbert space with the inner product induced by \( s(\lambda; \cdot, \cdot) \). Denote by \( \Phi_\lambda \) its completion, by \( (\cdot, \cdot)_\lambda \) and \( || \cdot ||_\lambda \) the inner product and norm in \( \Phi_\lambda \) and by \( q_\lambda \) the quotient map of \( \Phi \) onto \( \Phi/\mathcal{N}_\lambda \subset \Phi_\lambda \). Let \( \Phi \) be the vector space of all vector fields \( \tilde{\phi} = \{ \tilde{\phi}(\lambda) \}_{\lambda \in \hat{\Lambda}} \) with \( \tilde{\phi}(\lambda) \in \Phi_\lambda \), identifying elements equal \( \mu \)-a.e. By a quasi-simple function \( \tilde{\phi} \) one means a function of the form (finite sum) \( \tilde{\phi}(\lambda) = \sum \alpha_k(\lambda) \phi_k \) with \( \alpha_k \in L^\infty(\mu) \) and \( \phi_k \in \Phi \). A function \( \tilde{\phi} \in \Phi \) is said to be s-measurable if there is a sequence \( \{ \tilde{\phi}_n \} \) of quasi-simple functions on \( \hat{\Lambda} \) such that

\[
\lim_{n \to \infty} || \tilde{\phi}(\lambda) - q_\lambda \phi_n(\lambda) ||_\lambda = 0 \quad \text{for } \mu \text{-a.a. } \lambda \in \hat{\Lambda}.
\]

The direct integral \( \mathcal{H}_{\mu, \Phi} \) is the Hilbert space of all s-measurable elements \( \tilde{\phi} \in \Phi \) such that

\[
|| \tilde{\phi} ||_{\mathcal{H}_{\mu, \Phi}}^2 = \int_\hat{\Lambda} || \tilde{\phi}(\lambda) ||_\lambda^2 d\mu(\lambda) < \infty
\]

with inner product \( (\tilde{\phi}, \tilde{\phi})_{\mathcal{H}_{\mu, \Phi}} := \int_\hat{\Lambda} (\tilde{\phi}(\lambda), \tilde{\phi}(\lambda))_\lambda d\mu(\lambda) \).

Quasi-simple functions are densely embedded in \( \mathcal{H}_{\mu, \Phi} \) and the structural isomorphism \( \hat{V} : \mathcal{H}_{ac} \to \mathcal{H}_{\mu, \Phi} \) is the unitary extension of the isometry \( \phi \mapsto \{ q_\lambda \tilde{\phi}(\lambda) \}_{\lambda \in \hat{\Lambda}} \) defined for quasi-simple \( \tilde{\phi} \).

In what follows \( \mathcal{H}_\Phi \) shall denote the smallest closed subspace of \( \mathcal{H} \) containing \( \Phi \) an reducing \( P \). Its absolutely continuous part with respect to \( \mu \), \( \mathcal{H}_{ac} : = P_{ac} \mathcal{H}_\Phi \), also reduces \( P \). \( \mathcal{H}_{ac} = \mathcal{H}_\Phi \) if \( \Phi \) generates \( \mathcal{H} \), i.e., if \( \mathcal{H}_\Phi = \mathcal{H} \).

General wave operators \( \Omega : \mathcal{H}_1 \to \mathcal{H}_2 \) are defined between two spectral systems \( (\Lambda, B, \mu, \mathcal{H}_j, P_j) \) with spectral forms \( (\Lambda, \Phi_j, s_j) \) and corresponding direct integrals \( \mathcal{H}_{\mu, \Phi_j}, j = 1, 2 \), where the measure space \( (\Lambda, B, \mu) \) and \( \hat{\Lambda} \subseteq \Lambda \) are common to both systems. A general wave operator \( \Omega \) is a partial isometry with initial subspace \( \mathcal{H}_{ac} \) and final subspace contained in \( \mathcal{H}_{ac} \), which intertwines \( P_1 \) and \( P_2 \). Kato-Kuroda [11] obtain such general wave operators \( \Omega \) going in inverse direction from (14) to (13). They derive \( U(\lambda) \) from converging sequences of operators defined between the auxiliary spaces \( \Phi_j \) and give conditions leading to abstract Lippmann-Schwinger equations between the eigenfunctions in the antidual spaces \( \Phi_j^\times \). To be precise, assume the following conditions:

(A1) \( \Phi_j \) is a topological vector space with its own topology and there is a sequence of approximating spectral forms \( s_{jn}, n = 1, 2, \ldots \), on \( \Lambda \times \Phi_j \times \Phi_j \), in the following sense:

For each \( \lambda \in \Lambda \), \( s_{jn}(\lambda; \cdot, \cdot) \) is an equicontinuous family of nonnegative Hermitian form on \( \Phi_j \times \Phi_j \) and \( \lim_{n \to \infty} s_{jn}(\lambda; \phi, \varphi) = s_j(\lambda; \phi, \varphi) \) for each \( \phi, \varphi \in \Phi_j \).

(A2) For each \( \lambda \in \hat{\Lambda} \), there is an equicontinuous sequence \( \{ K_n(\lambda) \} \) of operators on \( \Phi_1 \) to \( \Phi_2 \) such that \( s_{2n}(\lambda; K_n(\lambda)\phi) = s_{1n}(\lambda; \phi) \) for each \( \phi \in \Phi_1 \); for each \( \phi \in \Phi_1 \) and \( n \), the function \( K_n(\cdot) \phi \) is s-measurable on \( \Lambda \); for each \( \lambda \in \hat{\Lambda} \) and \( \phi \in \Phi_1 \), \( \{ K_n(\lambda)\phi \}_n \) is a Cauchy sequence in \( \Phi_2 \). Moreover, the operators \( K_n(\lambda) \) are surjective and \( K_n(\lambda)^{-1} : \Phi_2 \to \Phi_1 \) satisfy the above conditions with subscripts 1, 2 interchanged.

Under conditions (A1) and (A2) a general wave operator \( \Omega \) exists and is complete on \( \mathcal{H}_{ac} \) to \( \mathcal{H}_{ac} \). Furthermore, the operators \( K_n(\lambda) : \Phi_1 \to \Phi_2 \) being continuous, there exist the operators \( K_n(\lambda)^\times : \Phi_2^\times \to \Phi_1^\times \) defined by the relations

\[
\langle K_n(\lambda)^\times \phi_1, \phi_2^\times \rangle = \langle \phi_1 | K_n(\lambda)^\times \phi_2^\times \rangle, \quad (\phi_1 \in \Phi_1, \phi_2^\times \in \Phi_2^\times).
\]

Since \( \{ K_n(\lambda)\phi \}_n \) is Cauchy for each \( \phi \in \Phi_1 \), \( \lim_n \langle \phi | K_n(\lambda)^\times \phi^\times \rangle = \langle \phi | K(\lambda)^\times \phi^\times \rangle \) exists for each \( \phi \in \Phi_1 \) and \( \phi^\times \in \Phi_2^\times \). Since \( \{ \langle \cdot | K(\lambda)^\times \phi^\times \rangle \}_n \) is equicontinuous, the limit is continuous and can be written as \( \langle \cdot | \phi^\times \rangle \) for a unique \( \phi^\times \in \Phi_1^\times \). The relation \( \phi^\times = K(\lambda)^\times \phi^\times \) defines an operator \( K(\lambda)^\times : \Phi_2^\times \to \Phi_1^\times \). In other words,

\[
w^\times \lim_{n \to \infty} K_n(\lambda)^\times = K(\lambda)^\times,
\]

where \( K(\lambda)^\times \) is the direct integral induced by \( \Phi \). The direct integral

\[
\int_\Lambda K(\lambda)^\times d\mu(\lambda) = K(\lambda)^\times
\]

is the Hilbert space of all \( \phi^\times \in \Phi_2^\times \).
where $\text{w}^*\text{lim}$ denotes weak* limit. Similarly, $\text{w}^*\text{lim}_{n \to \infty} [K_n(\lambda)^{-1}]^* = [K(\lambda)^*]^{-1}$.

If, moreover, the system 1 has an eigenfunction expansion, i.e., satisfies conditions (E1), (E2) and (E3) of Definition 3.1, with partial isometry $\mathcal{F}_1 : \mathcal{H}_1 \to L^2(\rho)$ and eigenfunctions $\psi_1^\times(\xi) \in \Phi_1^\times$, then the system 2 also has an eigenfunction expansion with the same measure space $(\Gamma, \Sigma, \rho)$, partial isometry

$$\mathcal{F}_2 := \mathcal{F}_1 \Omega^\dagger : \mathcal{H}_2 \to L^2(\rho)$$

and eigenfunctions $\psi_2^\times(\xi) \in \Phi_2^\times$ given by

$$\psi_2^\times(\xi) := [K(w(\xi))]^{-1}\psi_1^\times(\xi),$$

the Lippmann-Schwinger equations in abstract form.

Particular methods using an auxiliary Banach space $\Phi$ were proposed by Howland [18,19] and Retjo [20]. In these works complete wave operators of stationary character are defined in a direct way and systems of generalized eigenfunctions for $H_0$ and $H$ exist in $\Phi^\times$ satisfying the original Lippmann-Schwinger equations [21]. These methods are applied to $\Phi$-regular pairs and gentle perturbations.

6. Schrödinger operators

In this Section we shall be concerned with eigenfunction expansions and scattering theory for Schrödinger pairs $[-\Delta, -\Delta + V(x)]$ in $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$. Here $L^2 = L^2(\mathbb{R}^n)$ is the space of square integrable complex-valued functions on $\mathbb{R}^n$ and $\Delta = \sum_j \partial^2 / \partial x_j^2$ is the Laplacian operator, the derivatives taken in distributional sense.

For any real $s$ we shall consider the weighted $L^2$ space

$$L^{2,s} = L^{2,s}(\mathbb{R}^n) := \{ u(x) : (1 + |x|^2)^{s/2} u(x) \in L^2(\mathbb{R}^n) \},$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_j x_j^2$. In $L^{2,s}$ we introduce the norm $\|u\|_{0,s} := \|(1 + |x|^2)^{s/2} u(x)\|_{L^2}$. Obviously, $(L^{2,s})^\times = L^{2,-s}$ and the dual pair is given by $\langle u, v \rangle = \int u(x)^\dagger v(x) \, dx$, $u \in L^{2,s}$ and $v \in L^{2,-s}$.

The weighted Sobolev $L^2$ spaces are given by

$$\mathcal{H}_{m,s} = \mathcal{H}_{m,s}(\mathbb{R}^n) = \{ u(x) : D^\alpha u \in L^{2,s}, 0 \leq |\alpha| \leq m \}.$$  

Here for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of order $|\alpha| = \sum_j \alpha_j$ the derivatives $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial / \partial x_j$, are taken in distributional sense. $\mathcal{H}_{m,s}$ is a Hilbert space under the norm $\|u\|_{m,s}^2 := \sum_{|\alpha| \leq m} \|D^\alpha u\|_{0,s}^2$. The usual Sobolev spaces $\mathcal{H}_{m,0}$ will also be denoted by $\mathcal{H}_m$.

**Definition 6.1** A real function $V(x) \in L^p_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the class SR (short-range) if for some $\epsilon > 0$ the multiplication mapping

$$u(x) \mapsto (1 + |x|)^{1+\epsilon} V(x) u(x)$$

defines a compact operator from $\mathcal{H}_{2}(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$.

A real function $V$ verifying

$$\sup_{x \in \mathbb{R}^n} \left[ (1 + |x|)^{2+2\epsilon} \int_{|y-x| \leq 1} |V(y)|^2 |y-x|^{-n+\mu} \, dy \right] < \infty$$

for some $\epsilon > 0$ and $0 < \mu < 4$ is of class SR. Note that (15) holds in particular for $V \in L^p_{\text{loc}}(\mathbb{R}^n)$, with $p = 2$ for $n \leq 3$, $p > n/2$ for $n \geq 4$, and such that

$$V(x) = O(|x|^{-1-\epsilon}) \quad \text{as } |x| \to \infty.$$
Denote by $H_0$ the selfadjoint realization of the Laplacian $-\Delta$ in $L^2(\mathbb{R}^n)$, with domain $\mathcal{D}(H_0) = \mathcal{H}_2(\mathbb{R}^n)$. For $V(x)$ of class SR the multiplication operator $u \mapsto V(x)u(x)$ is $H_0$-compact. Then $-\Delta + V(x)$ admits a unique selfadjoint realization in $L^2(\mathbb{R}^n)$, which we shall denote by $H$. Both operators $H_0$ and $H$ possess the same domain of definition $\mathcal{H}_2(\mathbb{R}^n)$ and also the same essential spectrum which consists of the interval $[0, \infty)$. The spectrum of $H$ is $\sigma(H) = [0, \infty) \cup \{\lambda_j\}$ with $\{\lambda_j\}$ a discrete set of negative eigenvalues with finite multiplicity and having zero as its only possible limit point. Furthermore, $H$ lacks continuous singular spectrum and the set $e_+ = e_+(H)$ of all positive eigenvalues of $H$ is a discrete set of eigenvalues of finite multiplicity with $0$ and $\infty$ as the only possible limit points. See [17, Section IV.5.6] and [22] for details. Let $e_+^{1/2} = e_+(H)^{1/2} := \{k \in (0, \infty) : k^2 \in e_+(H)\}$ and $\mathcal{N}(H) := \{\xi \in \mathbb{R}^n : |\xi|^2 \in e_+(H)\} \cup \{0\}$.

The eigenfunction expansion for $H_0$ has been given in Example 1. For potentials $V(x)$ satisfying condition (15) Agmon [22] proves the existence of two (essentially unique) families $\psi_\pm(x, \xi)$ of generalized eigenfunctions of $H = -\Delta + V(x)$ defined for every $\xi \in \mathbb{R}^n \setminus \mathcal{N}$ and of class $L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{N})$. For every fixed $\xi$ the functions $\psi_\pm(x, \xi)$ belong to $C(\mathbb{R}^n_x) \cap \mathcal{H}_2^{\text{loc}}(\mathbb{R}^n_x)$ and satisfies the differential equation

$$[-\Delta_x + V(x) - |\xi|^2] \psi_\pm(x, \xi) = 0,$$

(16)

that is, $\psi_\pm(x, \xi)$ are eigenfunctions of $H$ corresponding to $|\xi|^2$.

The eigenfunction expansion theorem is given in the following terms: There exist two bounded linear maps $\mathcal{F}_\pm : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that:

(a) Let $P_{ac}$ and $P_p$ denote the respective projections onto the absolutely continuous (with respect to Lebesgue measure) and point (eigenvalue) parts of $L^2(\mathbb{R}^n)$ with respect to $H$, so that $L^2(\mathbb{R}^n) = P_{ac}L^2(\mathbb{R}^n) \oplus P_pL^2(\mathbb{R}^n)$. Then $\text{Ker} \mathcal{F}_\pm = P_pL^2(\mathbb{R}^n)$ and the restriction of $\mathcal{F}_\pm$ to $P_{ac}L^2(\mathbb{R}^n)$ is a unitary operator from $P_{ac}L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$.

(b) For any $f \in L^2(\mathbb{R}^n)$,

$$[\mathcal{F}_\pm f](\xi) = (2\pi)^{-n/2} \lim_{N \to \infty} \int_{|x| < N} f(x) \psi_\pm(x, \xi)^* \, dx,$$

$$[\mathcal{F}_\pm^* f](x) = (2\pi)^{-n/2} \lim_{j \to \infty} \int_{K_j} f(\xi) \psi_\pm(x, \xi) \, d\xi,$$

where $K_j$ is an increasing sequence of compacts sets such that $\cup_j K_j = \mathbb{R}^n \setminus \mathcal{N}$.

(c) Let $M_{|\xi|^2}$ denote the multiplication operator by $|\xi|^2$. Then for every $f \in \mathcal{D}(H)$

$$P_{ac}Hf = \mathcal{F}_\pm^* M_{|\xi|^2} \mathcal{F}_\pm f.$$

Then the wave operators $\Omega_\pm$ for the pair $(H_0, H)$ exist and are complete ($\text{Rang} \Omega_\pm = P_{ac}L^2(\mathbb{R}^n)$). They are given by

$$\Omega_\pm := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} = \mathcal{F}_\pm^* \mathcal{F},$$

(17)

with $\mathcal{F}$ the ordinary Fourier transform.

The methods in [22] are of stationary nature. Consider the resolvents $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ as analytic operator valued functions on $\mathbb{C} \setminus \mathbb{R}_+$ and $\mathbb{C} \setminus \sigma(H)$, respectively, with values in $B(L^{2,s}(\mathcal{H}_{2,-s}))$ for any $s > 1/2$. For potentials $V$ of class SR the following limits exist in the uniform operator topology of $B(L^{2,s}, \mathcal{H}_{2,-s})$:

$$R_0(\lambda \pm i0) := \lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon), \quad (\lambda \in \mathbb{R}_+),$$

$$R(\lambda \pm i0) := \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon), \quad (\lambda \in \mathbb{R}_+ \setminus e_+(H)).$$
This result is usually known as the limiting absorption principle.

Introduce polar coordinates \( \xi = k \omega \) with \( k = |\xi| \) and \( \omega = \xi/k \), and write: \( \psi_{\pm}(x, k, \omega) = \psi_\pm(x, k, \omega) \). Then for fixed \((x, k) \in \mathbb{R}^n \times \mathbb{R}_+ \setminus \mathbb{R}_+^{1/2} \) the function \( \psi_\pm(x, k, \omega) \) belongs to \( L^2(S^{n-1}) \).

For any function \( g \) in \( L^2(S^{n-1}) \) define

\[
\psi^g_\pm(x, k) := \int_{S^{n-1}} \exp(ik\omega \cdot \tau) g(\tau) dS_\tau, \quad \psi^g_\pm(x, k) := \int_{S^{n-1}} \psi_\pm(x, k, \omega) g(\omega) dS_\omega.
\]

Then for any fixed \( k \in \mathbb{R}_+ \setminus \mathbb{R}_+^{1/2} \) the function \( \psi^g_\pm(x, k) \) lies in \( \mathcal{H}_{2-s}(\mathbb{R}^n) \cap C^{0,-s}(\mathbb{R}^n) \) for any \( s > 1/2 \), satisfies the differential equation \(-\Delta_x + V(x) - k^2 \psi^g_\pm(x, k) = 0 \) and the Lippmann-Schwinger equation

\[
\psi^g_\pm(x, k) = \psi^g_\pm(x, k) - R(k^2 \mp i0)[V(\cdot) \psi^g_\pm(\cdot, k)](x).
\]

Eigenfunction expansions for particular short-range potential Schrödinger operators were already obtained by Povzner [23], who considered potentials \( V(x) \) bounded from below and vanishing outside a compact set in \( \mathbb{R}^3 \). Povzner’s eigenfunctions for \( H = -\Delta + V(x) \) are of the form \( \psi = (2\pi)^{-3/2} \exp(-i\xi x) + \psi_1 \), where \( \psi_1 = O(|x|^{-1}) \) and \( |x|(d\psi_1/d|x| - i)\xi \psi_1 = O(|x|^{-1}) \) as \( |x| \to \infty \). Faddeev [24] generalized Povzner’s results for potentials \( V(x) \) such that \( V(x) \in L^1 \), \( V(x) = O(|x|^{-3-\varepsilon}) \) and \( \text{grad} V(x) = O(1/|x|^{3-\varepsilon}) \) as \( |x| \to \infty \), \( (\varepsilon > 0) \). Later Ikebe [25] considered potentials \( V(x) \) in \( L^2(\mathbb{R}^3) \) satisfying \( V(x) = O(|x|^{-2-\varepsilon}) \) as \( |x| \to \infty \), which, except for a finite number of singular points, are locally Hölder-continuous. Alsholm and Schmidt [26] extended the above works in \( \mathbb{R}^n \), \( n \geq 3 \), requiring integral, rather than pointwise, smoothness and decay conditions on \( V(x) \), and obtained simplified proofs under more general hypotheses. A detailed study of potentials \( V : \mathbb{R}^3 \to \mathbb{R} \) such that \( V(x) = (1 + |x|)^{-s} [B_1(x) + B_2(x)] \) with \( B_1 \in L^2, |B_2(x)| \leq \beta (1 + |x|)^{-\frac{3}{2} - \delta} \) for some \( s > 3/2, \delta > 0 \) and \( 0 \leq \beta < \infty \) (roughly speaking, absolutely square-integrable \( V \) decreasing to zero more rapidly than \( |x|^{-2-\varepsilon} \) for some \( \varepsilon > 0 \) as \( |x| \to \infty \) can be found in [27].

Kuroda [28] studied real valued potentials \( V(x) \) on \( \mathbb{R}^n \), \( n \geq 3 \), such that there exists \( s > n \) with

\[
(1 + |x|)^{s/2} V(x) \in L^p(E_1), \quad E_1 := \{x : |(1 + |x|)^{s/2} V(x)| \leq 1\}, \quad (1 + |x|)^{s/2} V(x) \in L^p(E_2), \quad E_2 := \{x : |(1 + |x|)^{s/2} V(x)| \leq 1\},
\]

for some \( p < 2n \) and \( q > n/2 \) (if \( n \geq 4 \)) or \( q = 2 \) (if \( n = 3 \)). Under the above conditions \( H \) defines a selfadjoint operator on \( \mathcal{D}(H) = \mathcal{D}(H_0) = \mathcal{H}_2(\mathbb{R}^n) \) and the spectrum of \( H \) consists of the absolutely continuous part (with respect to Lebesgue measure) extending from \( 0 \) to \( \infty \), a finite number of nonpositive eigenvalues and, possibly, a positive singular spectrum confined in a closed null set \( e_s \). Moreover, for any \( \xi \in \mathbb{R}^n \) such that \( |\xi|^2 \notin e_s \) the Lippmann-Schwinger equation

\[
\psi_{\pm}(x, \xi) = (2\pi)^{-n/2} \exp(i\xi \cdot x) - \int_{\mathbb{R}^n} R^\pm_n(x, y ; |\xi|) q(y) \psi_{\pm}(y, \xi) dy
\]

has a unique solution in \( L^{2-s}(\mathbb{R}^n) \). Here the kernels \( R^\pm_n(x, y ; k) \) are the outgoing and incoming fundamental solutions of \(-\Delta - k^2 \), which can be expressed in terms of the Hankel functions \( H^{(\pm)}_\nu = H^{(1)}_\nu \) and \( H^{(-)}_\nu = H^{(2)}_\nu \) [1, Formula 13.7.2]:

\[
R^\pm_n(x, y ; k) = \frac{i}{4} \left( \frac{k}{2\pi |x - y|} \right)^{(n-2)/2} H^{(\pm)}_{(n-2)/2}(k |x - y|).
\]

The functions \( \psi_{\pm}(x, \xi) \) satisfy the differential equation (16) in the distribution sense and serve as kernels of the generalized Fourier transforms \( \mathcal{F}_\pm \) associated to the absolutely continuous part.
unitarily equivalent to $H$. Also in this case relations (17) lead to complete wave operators, so that $H^{ac}$ is unitarily equivalent to $H_0$. For each fixed $\xi$, $\psi_{\pm}(x, \xi)$ is a bounded continuous function of $x$ which differs from $(2\pi)^{-n/2}\exp(ix \cdot \xi)$ by a function tending to 0 uniformly as $|x| \to \infty$.

Similar results were obtained by Shenk and Thoe [29] in exterior domains for real valued potentials $V(x)$ on $\mathbb{R}^n$, $n \geq 2$, such that $(1 + |x|^{n+1+\epsilon})/2 V(x)$ ($\epsilon > 0$ fixed but arbitrary) is uniformly $\alpha$-H"older continuous on $\mathbb{R}^n$ ($0 < \alpha \leq 1$) and tends to zero as $|x| \to \infty$.

The abstract results in Poerschke-Stolz-Weidmann [14] (see Section 3) are applied to Schr"odinger operators in $L^2(\mathbb{R}^n)$, $n \leq 3$. For potentials $V : \mathbb{R}^n \to \mathbb{R}$ which are uniformly locally in $L^2(\mathbb{R}^n)$, generalized eigenfunctions for the Schr"odinger operator $H = -\Delta + V(x)$ in $L^2(\mathbb{R}^n)$ are found belonging to $(1 + |x|^2)^{s/2}\mathcal{H}_1 \subset \mathcal{H}^{2,s} \cap \mathcal{H}^{s,loc}$ with $s > n/2$.

Later Poerschke and Stolz [30] reversed the original procedure and established eigenfunction expansions as a result of the existence and intertwining property of (modified (channel)) wave operators. The existence of wave operators $\Omega_{\pm}$ is taken as granted and used to construct expansions on the ranges of $\Omega_{\pm}$. They are able to include singular potentials and even perturbations by first and second order differential operators, the eigenfunctions remaining in weighted Sobolev spaces. This reversed procedure was first adopted by Jensen and Kitada [31] mainly for smooth potentials.

Notably, the spectrum $\sigma(H)$ of $H$ on $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$, may be characterized (up to a set of spectral measure zero) as the set of values for which a polynomially bounded generalized eigenfunction exists. This fact was proved in [32,33] for potentials $V = V_+ - V_-$ with $V_\pm \geq 0$, $V_+ \in K^{loc}_n$ and $V_- \in K_n$, where, for $n \in \mathbb{N}$, $K_n$ is the class defined as follows: If $n \geq 3$, $f \in K_n$ if \[ \limsup_{x \to 0} \frac{1}{s-n/2} \int_{|x-y| \leq \alpha} |x-y|^{-(n-2)} |f(y)| dy = 0 \] if $n = 2$, replace $|x-y|^{-n/2}$ by $\log |x-y|^{-1}$ and take $\alpha \leq 1$; if $n = 1$, $K_1$ is the set of $f$ with $\sup_y \int |x-y| \leq 1 |f(y)| dy < \infty$. We say that $f \in K^{loc}_n$ if $fg \in K_n$ for all $g \in C_0^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support.

For such type of potentials $V = V_+ - V_-$ Simon [34, Th.C.5.4] showed that there exists an eigenfunction expansion for $H$ with eigenfunctions $\psi(x, \lambda)$ satisfying $|\psi(x, \lambda)| \leq C (1 + |x|^2)^{s/2}$ for $s > n/2$ and some constant $C$. Furthermore, the semigroup $\{e^{-tH} : t \geq 0\}$ is defined on $L^2,s$. Denote its generator by $H_s$. One has $\sigma(H_s) = \sigma(H)$ for every $s \in \mathbb{R}$ and the adjoint $H^*_s$ of $H_s$ with respect to the dual pair $(L^2,s, L^{2,-s})$ coincides with $H_{-s}$. Moreover, assume also that $V \in L^2,s$ and let $\psi \in L^2,-s$ for some $s > 0$ such that $(\psi, Hg) = \lambda (\psi, g)$ for all $g \in C_0^\infty$. Then $\psi \in D(H_{-s})$, $H_{-s}\psi = \lambda\psi$ and $\lambda \in \sigma(H_{-s}) = \sigma(H)$. See [33].

A detailed study of the class $K_n$, its connections with $L^p$ properties, functional integrals, and regularity properties of eigenfunctions can be found in [34].

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