Approximate Controllability of Non-autonomous Second Order Impulsive Functional Evolution Equations in Banach Spaces

Sumit Arora · Soniya Singh · Manil T. Mohan · Jaydev Dabas

Received: 24 April 2022 / Accepted: 2 December 2022 / Published online: 7 January 2023
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2023

Abstract
This article investigates the approximate controllability of second order non-autonomous functional evolution equations involving non-instantaneous impulses and nonlocal conditions. First, we discuss the approximate controllability of second order linear system in detail, which lacks in the existing literature. Then, we derive sufficient conditions for approximate controllability of our system in separable reflexive Banach spaces via linear evolution operator, resolvent operator conditions, and Schauder’s fixed point theorem. Moreover, in this paper, we define proper identification of resolvent operator in Banach spaces. Finally, we provide two concrete examples to validate our results.

Keywords Abstract functional evolution equations · Non-instantaneous impulses · Approximate controllability · Evolution operator · Cosine family

Mathematics Subject Classification 34K06 · 34A12 · 37L05 · 93B05

Jaydev Dabas
ejay.dabas@gmail.com
Sumit Arora
sumit@as.iitr.ac.in
Soniya Singh
sonia.iitd.21@gmail.com
Manil T. Mohan
maniltmohan@ma.iitr.ac.in; maniltmohan@gmail.com

1 Department of Applied Mathematics and Scientific Computing, Indian Institute of Technology Roorkee, Roorkee, Uttarakhand 247667, India
2 Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, Uttarakhand 247667, India
1 Introduction

Controllability is one of the fundamental notions in mathematical control theory and plays a crucial role in various control problems such as the time optimal control problems [6], irreducibility of transition semigroups [13], stabilization of unstable systems via feedback control [7] etc. In the finite dimensional settings, the problems of exact and approximate controllability are same, whereas in the infinite dimensions, one has to distinguish these two concepts. Exact controllability refers that the solution of a control system can steer from an arbitrary initial state to a desired final state, while the approximate controllability means that the solution enables to steered an arbitrary small neighborhood of a final state. In the infinite dimensional case, the approximately controllable systems are more adequate and have extensive range of applications (cf. [33, 50, 51, 55], etc). In the past two decades, a good number of publications discussed the problems of existence and approximate controllability of non-linear evolution systems (in Hilbert and Banach spaces), see for instance, [2, 4, 16, 18, 20, 33, 42, 45], etc and the references therein.

There are many dynamical systems with nonlocal initial conditions are more suitable than regular initial conditions. For example, the diffusion equation with boundary conditions of an integral form and Hydratational heat in which the intensity of heat sources depends on the amount of heat already produced, see [36]. Byszewski and Lakshmikantham [10] first discussed the abstract Cauchy problem with nonlocal initial conditions and established the existence and uniqueness of a mild solution for that problem. In the recent time, several papers appeared in the literature dealing with the approximate controllability problem of the non-linear evolution systems with nonlocal initial conditions, see for instance, [3, 5, 46, 52], etc. On the other hand, there are many physical phenomena in which the current state of a system is influenced by the previous states. These types of phenomena are suitably modeled by delay differential equations, which naturally occurs in ecological models, neural network, inferred grinding models, logistic reaction-diffusion model with delay, etc (cf. [11, 32, 37], etc).

Many evolutionary processes such as harvesting, shocks, and natural disasters etc, are experience abrupt changes in their states for negligible time instants. Generally, these short-term perturbations are estimated as instantaneous impulses and such processes are mathematically modeled by impulsive differential and partial differential equations. The theory of impulsive evolution equations has found various applications such as threshold, bursting rhythm models in medicine, optimal control models in economics, paramacokinetics and frequency modulated systems, for more details, one can refer the monographs [8, 27]. In the past few years, many works formulated sufficient conditions for the approximate controllability of the impulsive systems with instantaneous impulses, see for instance, [3, 43, 45, 54], etc and the references therein. Moreover, the evolution processes in pharmacotherapy such as the distribution of drugs in the bloodstream and the consequent absorption of the body are a moderate and continuous process and the dynamics of such process can not be interpret by the instantaneous impulsive models. Therefore, this situation can be analyzed as an impulsive action that starts suddenly and remains active over a finite time interval. Hence,
the dynamics of such phenomena are characterized by non-instantaneous impulsive systems.

Hernández and O’Regan [22] first considered a non-instantaneous impulsive abstract differential equation and studied the global solvability of that system. Later, Feckan et al. in [15] altered the impulsive conditions considered in [22] and investigate the existence and uniqueness of solutions. In [41], Pierri et al. discussed a global solution for a non-instantaneous impulsive evolution equations. Recently, a few developments have been reported on the controllability problems of the non-instantaneous impulsive systems. In [25, 35], Muslim et al., considered a nonlinear second order control system with non-instantaneous impulses and studied the existence and stability of the solution and also proved the exact controllability of the considered system. Ahmed and his co-authors in [1] examined the approximate controllability of the non-instantaneous impulsive Hilfer fractional neutral stochastic integro-differential equations with fractional Brownian motion and nonlocal condition.

In the literature, it has been observed that the problem of approximate controllability for first and second order autonomous systems is extensively studied. However, the approximate controllability results for the non-autonomous semilinear systems remain limited. There are a few publications available on the approximate controllability of the first and second order non-autonomous semilinear systems, see for example, [2, 16, 17, 26, 34, 42], etc. In [34], Mahmudov et al. considered the non-autonomous second order differential inclusions and investigated the approximate controllability results in Hilbert space by applying the Bohnenblust–Karlin’s fixed point theorem. Also, they developed approximate controllability for an impulsive system with nonlocal initial conditions. Kumar et al. [26] investigated the approximate controllability of a second order non-autonomous system with finite delay in Banach spaces by applying a fixed point approach. The resolvent operator considered in that work is well define only if the state space is a Hilbert space (see, Remark 2.2 below). So, it seem to the authors that the results developed in [26] are reasonable only in separable Hilbert spaces. Moreover, to the best of author’s knowledge, the study of the approximate controllability of second order non-autonomous abstract Cauchy problems with non-instantaneous impulses in Banach spaces is leftover in the literature. The main purpose of this paper is to study the approximate controllability of a class of second-order non-autonomous impulsive systems with nonlocal conditions in a separable reflexive Banach space. Our paper suitably modifies the phase space characterization to incorporate Guedda’s observations in the case of impulsive differential equations with delay (cf. [19]). We introduce the integral norm in the delay space $\mathcal{D}$ given in (2.13) below, instead of the uniform norm. Moreover, we also discuss the approximate controllability of second order non-autonomous linear systems in detail, which is lacking in the existing literature.

Let us consider the following non-autonomous second order impulsive system:

$$x''(t) = A(t)x(t) + Bu(t) + f(t, x_t), \quad t \in \bigcup_{i=0}^{N} (s_i, t_{i+1}] \subset J = [0, T], \quad (1.1a)$$

$$x(t) = \rho_i(t, x(t^-)), \quad t \in (t_i, s_i], \quad i = 1, \ldots, N, \quad (1.1b)$$
\[
x'(t) = \rho_i(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, \ldots, N, \tag{1.1c}
\]
\[
x(t) = \phi(t), \quad t \in [-q, 0], \quad q > 0, \tag{1.1d}
\]
\[
x(0) = \phi(0) + g(x), \tag{1.1e}
\]
\[
x'(0) = \eta + h(x), \tag{1.1f}
\]

where

- the state variable \(x(\cdot)\) takes values in a separable reflexive Banach space \(X\) having uniformly convex dual \(X^*\) and \(U\) is a separable Hilbert space,
- the family of linear operators \(\{A(t) : t \in J\}\) on \(X\) is closed and domain is dense in \(X\),
- the operator \(B : U \to X\) is bounded with \(\|B\|_{\mathcal{L}(U; X)} \leq M_B\), where \(\mathcal{L}(U; X)\) denotes the space of bounded linear operators from \(U\) to \(X\), and the control function \(u \in L^2(J; U)\),
- the nonlinear function \(f : \bigcup_{i=0}^N [s_i, t_{i+1}] \times \mathcal{D} \to X\), (see (2.13) below for the definition of \(\mathcal{D}\)),
- the functions \(g, h : \text{PC}(J_q; X) \to X, J_q = [-q, T]\) (see (2.12) below for the definition of the space \(\text{PC}(J_q; X)\)),
- the fixed points \(s_i\) and \(t_i\) satisfy \(0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < t_N \leq s_N \leq t_{N+1} = T\) and \(x(t_i^+)\) and \(x(t_i^-)\) exist for \(i = 1, \ldots, N\) with \(x(t_i^-) = x(t_i)\),
- for every \(t \in J\), \(x_t \in \mathcal{D}\) and we define \(x_t(s) = x(t+s), -q \leq s < 0\),
- the functions \(\rho_i(t, x(t_i^-)) : [t_i, s_i] \times X \to X\) and their derivatives \(\rho_i'(t, x(t_i^-)) : [t_i, s_i] \times X \to X\), for \(i = 1, \ldots, N\) represent the non-instantaneous impulses.

Some of the non-autonomous second order impulsive systems may describe the dynamics of chemotherapeutic drugs. Because, second-order differential equations naturally occur in models where the spatial components play an essential role (for example, cancers), as well as in the so-called two-compartment models (used for drugs which slowly equilibrate with the tissue compartment, for example, vancomycin). The jump at each point of discontinuity may represent the diffusion of the drug in the bloodstream and the non-instantaneous part represents the absorption of the drugs by the cells (cf. [12]). Besides this, the delay can shape the behavior of the affected cells (cf. [40]). Indeed, we stress that in the case of chemotherapy, the cells react to the drugs by committing “suicide” only after the DNA checking cycle, that is, with a delay.

The rest of the manuscript is structured as follows: in Sect. 2, we review some fundamental definitions, important results and provide the assumptions which are useful to develop our results in subsequent sections. In Sect. 3, we formulate sufficient condition for the approximate controllability of the non-autonomous system (1.1a)–(1.1f). To determine this, we first investigate the approximate controllability of the linear control system corresponding to the problem (1.1a)–(1.1f). Further, we prove the existence of a mild solution for the system (1.1a)–(1.1f) via Schauder’s fixed point theorem. Then, we establish the approximate controllability of the considered system. In the final section, two notable examples of wave equation with non-instantaneous impulses and finite delay is discussed.
2 Preliminaries

In present section, we recall some basic definitions, results and assumptions that will be useful in succeeding sections.

Recently, the study of the non-autonomous abstract second order initial value problem is more extensive. Let us consider the following system:

\[
\begin{align*}
    x''(t) &= A(t)x(t) + f(t), \quad 0 \leq s, \quad t \leq T, \\
    x(s) &= v, \quad x'(s) = w,
\end{align*}
\]  

(2.1)

where the operator \( A(t) : D(A(t)) \subseteq X \to X \) is closed and densely defined for each \( t \in J \) and the function \( f : J \to X \) is define suitably. Many works discussed the existence of a solution to the problem (2.1), we refer the interested readers to \([9, 24, 29, 38]\) and the reference therein. In these works, generally the existence of a solution of the system (2.1) depends on the existence of an evolution operator \( S(t, s) \) for the equation

\[
x''(t) = A(t)x(t), \quad t \in J.
\]  

(2.2)

Throughout this work, we consider that the domain of \( A(t) \) is a subspace \( D(A) \), which is independent of \( t \) and dense in \( X \). We also assumed the function \( t \mapsto A(t)x \) is continuous for each \( x \in D(A) \).

2.1 Evolution Operator and Cosine Family

We now introduce the following concept of evolution operator.

**Definition 2.1** (Definition 2.1, \([24]\)) A map \( S : J \times J \to \mathcal{L}(X; X) =: \mathcal{L}(X) \) said to be an evolution operator if the following conditions are satisfied:

(D1) The map \( (t, s) \mapsto S(t, s)x \) is continuously differentiable for each \( x \in X \) and

- \( \partial_S S(t, s)x|_{t=s} = x \) and \( \partial_t S(t, s)x|_{t=s} = -x \).

(D2) If \( x \in D(A) \), then \( S(t, s)x \in D(A) \) for all \( t, s \in J \) and the map \( t \mapsto S(t, s)x \) is of class \( C^2 \) and

- \( \partial^2_S S(t, s)x = A(t)S(t, s)x, \)

- \( \partial^2_S S(t, s)x = S(t, s)A(s)x, \)

- \( \partial^3_S S(t, s)x|_{t=s} = 0. \)

(D3) For all \( t, s \in J \), if \( x \in D(A) \), then \( \partial_S S(t, s)x \in D(A), \) the derivatives \( \partial^3_S S(t, s)x, \partial^3_S S(t, s)x \) exist and
(a) \[
\frac{\partial^3}{\partial t^2 \partial s} S(t, s)x = A(t) \frac{\partial}{\partial s} S(t, s)x,
\]

(b) \[
\frac{\partial^3}{\partial s^2 \partial t} S(t, s)x = \frac{\partial}{\partial t} S(t, s)A(s)x.
\]

Moreover, the map \((t, s) \mapsto \frac{\partial}{\partial s} S(t, s)x\) is continuous.

Let us assume that there exists an evolution operator \(S(t, s)\) associated to the operator \(A(t)\). We define the operator \(C(t, s) = -\frac{\partial}{\partial s} S(t, s)\) and there exists a set of positive constants \(M, \tilde{M}\) and \(N\) such that

\[
\sup_{0 \leq s, t \leq T} \|C(t, s)\|_{L(X)} \leq M,
\]
\[
\sup_{0 \leq s, t \leq T} \|S(t, s)\|_{L(X)} \leq \tilde{M},
\]
\[
\|S(t + \tau, s) - S(t, s)\|_{L(X)} \leq N|\tau|, \quad \text{for all } t, t + \tau, s \in J.
\]

Moreover, various approaches have been discussed in the literature about the existence of the evolution operator \(S(t, s)\) (cf. [9, 24, 29, 38, 39, 44]). A very often studied situation is that the operator \(A(t)\) is the perturbation of an operator \(A\), which generates a strongly continuous cosine family. Therefore, it is necessary to review some properties of the cosine family.

Let \(A : D(A) \subset X \rightarrow X\) be an infinitesimal generator of a strongly continuous cosine family \(\{C_0(t) : t \in \mathbb{R}\}\) of bounded linear operators on \(X\) and the associated sine family \(\{S_0(t) : t \in \mathbb{R}\}\) on \(X\) is defined as

\[
S_0(t)x = \int_0^t C_0(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R}.
\]

Moreover

\[
C_0(t)x - x = A \int_0^t S_0(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R}.
\]

The infinitesimal generator \(A\) of a strongly continuous cosine family \(\{C_0(t) : t \in \mathbb{R}\}\) is defined as

\[
Ax = \frac{d^2}{dt^2} C_0(t)x \bigg|_{t=0}, \quad x \in D(A),
\]

where

\[
D(A) = \{x \in X : C_0(t)x \text{ is twice continuously differentiable function in } t\},
\]

equipped with the graph norm

\[
\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X, \quad x \in D(A).
\]
We also define the set
\[ E = \{ x \in \mathbb{X} : C_0(t)x \text{ is once continuously differentiable function of } t \} , \]
endowed with the norm
\[ \| x \|_1 = \| x \|_{\mathbb{X}} + \sup_{0 \leq t \leq 1} \| AC_0(t)x \|_{\mathbb{X}} , \quad x \in E , \]
which forms a Banach space (see [23]). Moreover, the operator \( A = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \) defined on \( D(A) \times E \) generate a strongly continuous group of bounded linear operators
\[ \mathcal{H}(t) = \begin{pmatrix} C_0(t) & S_0(t) \\ AS_0(t) & C_0(t) \end{pmatrix} \]
on the space \( E \times \mathbb{X} \), (see Proposition 2.6, [47]). From this fact, it follows that \( AS_0(t) : E \to \mathbb{X} \) is a bounded linear operator such that
\[ AS_0(t)x \to 0 \quad \text{as} \quad t \to 0 , \quad \text{for each} \quad x \in E . \]

Travis and Webb [47–49] discussed existence results of the following second order abstract Cauchy problem:
\[
\begin{align*}
  x''(t) &= Ax(t) + f(t), \quad t \in J, \\
  x(s) &= v, \quad x'(s) = w .
\end{align*}
\]
(2.6)

If the function \( f : J \to \mathbb{X} \) is integrable, then a continuous function \( x : [0, T] \to \mathbb{X} \) given by
\[
x(t) = C_0(t-s)v + S_0(t-s)w + \int_s^t S_0(t-\tau)f(\tau)d\tau , \quad (2.7)
\]
is called a mild solution of (2.6). Moreover, when \( v \in E \), the function \( x(\cdot) \) is continuously differentiable and
\[
x'(t) = AS_0(t-s)v + C_0(t-s)w + \int_s^t C_0(t-\tau)f(\tau)d\tau .
\]

Such a solution (2.7) is called a strong solution. Furthermore, if \( v \in D(A) \), \( w \in E \) and \( f \) is a continuously differentiable function, then the function \( x(\cdot) \) given in (2.7) becomes a classical solution of the initial value problem (2.6).

**Remark 2.1** Under the assumptions \( (A1)–(A7) \) on the operators \( A(t) \) for \( t \in J \) given in [24], the author generates an evolution family \( S(t,s) \) for \( t, s \in J \), that satisfy the conditions \( (D1)–(D3) \) in Definition 2.1. The author also arise a question about an
example of a family of operators $A(t), t \in J$ which satisfy the conditions $(A1)$–$(A7)$. In order to overcome this problem, we adopt a construction of an evolution family $S(\cdot, \cdot)$ given in [20] which satisfies the conditions $(D1)$–$(D3)$. The evolution family $S(t, s)$ for $t, s \in J$, considered in [20], is generated by the family of operators $A(t)$ that has the form $A(t) = A + F(t)$ for $t \in J$, where $A$ is the infinitesimal generator of a cosine family and $F : \mathbb{R} \to \mathcal{L}(E; \mathbb{X})$ is a map such that the function $t \mapsto F(t) x$ is continuously differentiable in $\mathbb{X}$ for each $x \in E$.

Let us now assume that $A(t) = A + F(t)$. Serizawa [44] established that, if $v \in D(A), w \in E$, the following non-autonomous system

$$\begin{align*}
\begin{cases}
  x''(t) = (A + F(t))x(t) & t \in J, \\
  x(0) = v, & x'(0) = w,
\end{cases}
\end{align*}$$

has a unique solution $x(\cdot)$ such that the function $t \mapsto x(t)$ is continuously differentiable in $E$. The same argument follows to conclude that the system (2.8) along with the initial condition given in the problem (2.6) has a unique solution $x(t, s)$ such that the function $t \mapsto x(t, s)$ is continuously differentiable in $E$. From (2.7), we infer that the solution $x(t, s)$ can be written as

$$x(t, s) = C_0(t - s)v + S_0(t - s)w + \int_s^t S_0(t - \tau)F(\tau)x(\tau, s)d\tau. \quad (2.9)$$

Particularly, for $v = 0$, we have

$$x(t, s) = S_0(t - s)w + \int_s^t S_0(t - \tau)F(\tau)x(\tau, s)d\tau.$$  

Therefore,

$$
\|x(t, s)\|_1 \leq \|S_0(t - s)\|_{\mathcal{L}(E; \mathbb{X})} \|w\|_{\mathbb{X}} \\
+ \int_s^t \|S_0(t - \tau)\|_{\mathcal{L}(E; \mathbb{X})} \|F(\tau)\|_{\mathcal{L}(E; \mathbb{X})} \|x(\tau, s)\|_1 d\tau,
$$

and by using the Gronwall-Bellman lemma, we obtain

$$\|x(t, s)\|_1 \leq C \|w\|_E. \quad (2.10)$$

Let us now define the operator

$$S(t, s)w = x(t, s).$$

The estimate (2.10) guarantees that the operator $S(\cdot, \cdot)$ is bounded on $E$. Since $E$ is dense in $\mathbb{X}$, the linear operator $S(\cdot, \cdot)$ can be extended to $\mathbb{X}$ and we still denoted it by $S(\cdot, \cdot)$ itself.
Theorem 2.1 (Theorem 1.2, [20]) Under the preceding conditions on $A$ and $F(\cdot)$, $S(\cdot, \cdot)$ is an evolution operator for the system (2.8). Moreover, if the sine family $S_0(t)$ is compact for all $t \in J$ implies that the evolution operator $S(t, s)$ is also compact for all $0 \leq s \leq t \leq T$.

2.2 Resolvent Operator and Assumptions

To study the approximate controllability of the system (1.1a)–(1.1f), we define the following operators:

\[
\begin{cases}
L_T u := \int_0^T S(T, t) Bu(t) \, dt, \\
\Psi_{s_i}^{t_i+1} := \int_0^T S(T, t) B^* S(T, t)^* \, dt, \quad i = 0, 1, \ldots, N, \\
R(\lambda, \Psi_{s_i}^{t_i+1}) := (\lambda I + \Psi_{s_i}^{t_i+1} J)^{-1}, \quad \lambda > 0, \quad i = 0, 1, \ldots, N,
\end{cases}
\]

(2.11)

where $B^*$ and $S(T, t)^*$ denote the adjoint operators of $B$ and $S(T, t)$ respectively. Moreover, the map $J$ stands for the duality mapping.

Definition 2.2 [6] The duality mapping $J : \mathbb{X} \to 2^{\mathbb{X}^*}$ is defined as

\[
J = \{ x^* \in \mathbb{X}^* : \langle x, x^* \rangle = \| x \|_{\mathbb{X}}^2 = \| x^* \|_{\mathbb{X}^*}^2, \ \text{for all} \ x \in \mathbb{X}, \}
\]

where $\langle \cdot, \cdot \rangle$ represents the duality pairing between $\mathbb{X}$ and $\mathbb{X}^*$.

Remark 2.2 (i) If the space $\mathbb{X}$ is reflexive, then the space $\mathbb{X}$ and $\mathbb{X}^*$ become strictly convex (cf. [6]). Moreover, the strict convexity of $\mathbb{X}^*$ guarantees that the mapping $J$ is bijective, strictly monotonic and demicontinuous, that is,

\[
x_k \to x \ \text{in} \ \mathbb{X} \Rightarrow J[x_k] \overset{w}{\to} J[x] \ \text{in} \ \mathbb{X}^* \ \text{as} \ k \to \infty.
\]

(ii) If $\mathbb{X}$ is a Hilbert space (identified with its own dual), then $J = 1$, the identity operator on $\mathbb{X}$.

(iii) The operators $R(\lambda, \Psi_{s_i}^{t_i+1}) : \mathbb{X} \to \mathbb{X}$, $\lambda > 0$ for $i = 0, 1, \ldots, N$, are uniformly continuous in every bounded subset of $\mathbb{X}$.

Let us define the set

\[
\text{PC}(J; \mathbb{X}) := \left\{ x : J \to \mathbb{X} : x|_{t \in I_i} \in C(I_i; \mathbb{X}), \right. \\
I_i := (t_i, t_{i+1}], \quad i = 0, 1, \ldots, N, \ x(t_i^+) \text{ and } x(t_i^-) \\
\left. \text{exist for each } i = 1, \ldots, N, \ \text{and satisfy } x(t_i) = x(t_i^-) \right\},
\]

with the norm $\| x \|_{\text{PC}(J; \mathbb{X})} := \sup_{t \in J} \| x(t) \|_{\mathbb{X}}$. Moreover, we also define the set

\[
\text{PC}(J_q; \mathbb{X}) := \left\{ x : J_q \to \mathbb{X} : x|_{t \in [-q, 0)} \in \mathcal{D} \text{ and } x|_{t \in J} \in \text{PC}(J; \mathbb{X}) \right\},
\]

(2.12)
equipped with the norm \(\|x\|_{\text{PC}([-q,T];X)} := \frac{1}{q} \int_{-q}^{0} \|x(s)\|_{X} \, ds + \sup_{t \in J} \|x(t)\|_{X}\), where

\[\mathcal{D} := \{\phi : [-q, 0] \to \mathbb{X} : \phi \text{ is piecewise continuous with jump discontinuity}\},\]

endowed with the norm \(\|\phi\|_{\mathcal{D}} = \frac{1}{q} \int_{-q}^{0} \|\phi(s)\|_{X} \, ds\) (see [18]).

In order to determine the existence and approximate controllability results for the system (1.1a)–(1.1f), we impose the following assumptions:

**Assumption 2.1 (H0)** For each \(i = 0, \ldots, N\) and every \(y \in \mathbb{X}\),

\[z_{\lambda,i}(y) = \lambda(\lambda I + \Psi_{t_i}^{t_{i+1}}J)^{-1}(y) \to 0 \quad \text{as} \quad \lambda \downarrow 0\]

in strong topology, where \(z_{\lambda,i}(y)\) is a solution of the equation

\[
\lambda z_{\lambda,i}(y) + \Psi_{t_i}^{t_{i+1}}J[z_{\lambda,i}(y)] = \lambda y. \tag{2.14}
\]

for each \(i = 0, 1, \ldots, N\).

**(H1)** \(S_0(t), t \in J\) is compact.

**(H2)** (i) Let \(x : J_q \to \mathbb{X}\) be such that \(x_0 = \phi\) and \(x|_{J} \in \text{PC}(J; \mathbb{X})\). The function \(f : J_1 \times \mathcal{D} \to \mathbb{X}\), where \(J_1 = \bigcup_{i=0}^{N} [s_i, t_i+1]\) is strongly measurable in \(t\), for each \(\phi \in \mathcal{D}\) and continuous in \(\phi\), for a.e. \(t \in J_1\).

(ii) There exists a function \(\gamma \in L^1(J_1; \mathbb{R}^+),\) such that

\[\|f(t, \phi)\|_{X} \leq \gamma(t), \quad \text{for a.e.} \ t \in J_1 \text{ and for all } \phi \in \mathcal{D}.\]

**(H3)** The non-instantaneous impulses \(\rho_i : [t_i, s_i] \times \mathbb{X} \to \mathbb{X}\), for \(i = 1, \cdots, N\), are such that

(i) the impulses \(\rho_i(\cdot, \cdot) : [t_i, s_i] \to \mathbb{X}\) are continuously differentiable for each \(x \in \mathbb{X}\),

(ii) for all \(t \in [t_i, s_i]\), the impulses \(\rho_i(t, \cdot) : \mathbb{X} \to \mathbb{X}\) are completely continuous and their derivatives \(\rho'_i(t, \cdot) : \mathbb{X} \to \mathbb{X}\) are continuous,

(iii) \(\|\rho_i(t, x)\|_{X} \leq d_i, \quad \|\rho'_i(t, x)\|_{X} \leq e_i, \quad \text{for all } t \in [t_i, s_i], \ x \in \mathbb{X}, \ i = 1, \cdots, N,\) where \(d_i\)'s and \(e_i\)'s are positive constants.

**(H4)** The functions \(g, h : \text{PC}(J_q; \mathbb{X}) \to \mathbb{X}\) are such that

(i) \(g\) is completely continuous and \(h\) is continuous,

(ii) for all \(x \in \text{PC}(J_q; \mathbb{X})\), there exist a constants \(M_g, M_h\) such that

\[\|g(x)\|_{X} \leq M_g \left(\|x\|_{\text{PC}(J_q; \mathbb{X})} + 1\right), \quad \|h(x)\|_{X} \leq M_h \left(\|x\|_{\text{PC}(J_q; \mathbb{X})} + 1\right).\]
Remark 2.3 Note that for each \( i = 0, 1, \ldots, N \) the equation (2.14) has a unique solution \( z_{\lambda,i}(y) = \lambda(\lambda I + \Psi_{\lambda,i}^{-1}J)^{-1}(y) = \lambda R(\lambda, \Psi_{\lambda,i}^{-1})(y) \) for every \( y \in \mathbb{X} \) and \( \lambda > 0 \), followed by Lemma 2.2 [33]. Moreover

\[
\|z_{\lambda,i}(y)\|_\mathbb{X} = \|\mathcal{J}[z_{\lambda,i}(y)]\|_\mathbb{X} \leq \|y\|_\mathbb{X}, \text{ for } i = 0, \ldots, N. \tag{2.15}
\]

We now introduce the concept of mild solution for the system (1.1a)–(1.1f) (cf. [21]).

Definition 2.3 A function \( x(\cdot; \phi, \eta, u) : J_q \to \mathbb{X} \) is called a mild solution of (1.1a)–(1.1f), if \( x(t) = \phi(t), \ t \in [-q, 0) \) and satisfies the following:

\[
x(t) = \begin{cases} 
C(t, 0)[\phi(0) + g(x)] + S(t, 0)[\eta + h(x)] + \int_0^t S(t, s)[Bu(s) + f(s, x_s)]ds, & t \in [0, t_1], \\
C(t, s_i)[\rho_i(t, x(t_i^-)) + S(t, s_i)[B\rho_i'(s_i, x(t_i^-)))] + \int_{s_i}^t S(t, s)[Bu(s) + f(s, x_s)]ds, & t \in (s_i, t_{i+1}], i = 1, \ldots, N.
\end{cases}
\]

\[x(\cdot; \phi, \eta, u) : J_q \to \mathbb{X} \]

Remark 2.4 Note that a mild solution of the system (1.1a)–(1.1f) satisfies the conditions (1.1b), (1.1d) and (1.1e). Nevertheless, a mild solution need not be differentiable at \( t \in \bigcup_{i=1}^N (t_i, s_i] \cup \{0\} \).

Definition 2.4 (Definition 3.2, [34]) The system (1.1a)–(1.1f) is said to be approximately controllable on \( J \), for any initial function \( \phi \in \mathcal{D} \) and any \( \eta \in \mathbb{X} \), if the closure of the reachable set is the whole space \( \mathbb{X} \), that is, \( \bar{K}(T, \phi, \eta) = \mathbb{X} \), where the reachable set is defined as

\[
K(T, \phi, \eta) = \{x(T; \phi, \eta, u) : u(\cdot) \in L^2(J; \mathbb{U})\}.
\]

3 Approximate Controllability of the Semilinear Non-autonomous System

This section is devoted for investigating the approximate controllability of the system (1.1a)–(1.1f). A set of sufficient conditions will be obtained by studying the approximate controllability of the linear control problem corresponding to the system (1.1a)–(1.1f).
3.1 Linear Control Problem

To establish the approximate controllability of the linear problem, we first determine the existence of an optimal control by minimizing the cost functional given by

$$F(x, u) = \|x(T) - x_T\|_X^2 + \lambda \int_{s_N}^{T} \|u(t)\|_U^2, \quad (3.1)$$

where $x(\cdot)$ is the unique mild solution of the linear control system:

$$\begin{cases} x''(t) = A(t)x(t) + Bu(t), & a.e. t \in I = [s_N, T], \\ x(s_N) = v, \ x'(s_N) = w, \end{cases} \quad (3.2)$$

with the control $u \in L^2(I; U), \ x_T \in X$ and $\lambda > 0$. Since $Bu \in L^1(J; X)$, the existence of a unique mild solution

$$x(t) = C(t, s_N)v + S(t, s_N)w + \int_{s_N}^{t} S(t, s)Bu(s)ds, \ t \in I, \quad (3.3)$$

for any $u \in L^2(I; U) = \mathcal{U}_{ad}$, to the system (3.2) is immediate by an application of Theorem 2.2, [20]. Next, we define the admissible class $\mathcal{A}_{ad}$ for the system (3.2) as

$$\mathcal{A}_{ad} = \{(x, u) : x \text{ is a unique mild solution of (3.2) with control } u \in \mathcal{U}_{ad}\}.$$ 

For a given control $u \in \mathcal{U}_{ad}$, the system (3.2) possesses a unique mild solution, and hence the set $\mathcal{A}_{ad}$ is nonempty.

The existence of an optimal pair minimizing the cost functional (3.1) is discussed in the next theorem:

**Theorem 3.1** For given $v, w \in X$, there exists a unique optimal pair $(x^0, u^0) \in \mathcal{A}_{ad}$ of the problem:

$$\min_{(x,u) \in \mathcal{A}_{ad}} F(x, u). \quad (3.4)$$

**Proof** Let us first define

$$L := \inf_{u \in \mathcal{U}_{ad}} F(x, u).$$

Since, $0 \leq L < +\infty$, there exists a minimizing sequence $\{u^n\}_{n=1}^{\infty} \in \mathcal{U}_{ad}$ such that

$$\lim_{n \to \infty} F(x^n, u^n) = L,$$
where \((x^n, u^n) \in \mathcal{A}_ad\), for each \(n \in \mathbb{N}\). Note that \(x^n(\cdot)\) satisfies
\[
x^n(t) = C(t, s_N) v + S(t, s_N) w + \int_{s_N}^t S(t, s) B u^n(s) ds, \quad t \in I.
\] (3.5)

Since \(0 \in \mathcal{U}_ad\), without loss of generality, we may assume that \(F(x^n, u^n) \leq F(x, 0)\), where \((x, 0) \in \mathcal{A}_ad\). Using the definition of \(F(\cdot, \cdot)\), we easily get
\[
\|x^n(T) - x_T\|_X + \lambda \int_{s_N}^T \|u^n(t)\|_U^2 dt \leq \|x(T) - x_T\|_X^2 + \lambda \int_{s_N}^T \|u^n(t)\|_U^2 dt
\leq 2 \left(\|x(T)\|_X^2 + \|x_T\|_X^2\right) < +\infty.
\] (3.6)

The above estimate implies, there exist a large \(\tilde{C} > 0\) such that
\[
\int_{s_N}^T \|u^n(t)\|_U^2 dt \leq \tilde{C} < +\infty.
\] (3.7)

Moreover, from (3.5), we have
\[
\|x^n(t)\|_X \leq \|C(t, s_N) v\|_X + \|S(t, s_N) w\|_X + \int_{s_N}^t \|S(t, s) B u^n(s)\|_X ds
\leq \|C(t, s_N)\|_{\mathcal{L}(\mathcal{X})} \|v\|_X + \|S(t, s_N)\|_{\mathcal{L}(\mathcal{X})} \|w\|_X
\quad + \int_{s_N}^t \|S(t, s)\|_{\mathcal{L}(\mathcal{X})} \|B\|_{\mathcal{L}(U; \mathcal{X})} \|u^n(s)\|_U ds
\leq M \|v\|_X + \tilde{M} \|w\|_X + \tilde{M} M_B \sqrt{T - s_N} \left(\int_{s_N}^t \|u^n(s)\|_U^2 ds\right)^{1/2}
\leq M \|v\|_X + \tilde{M} \|w\|_X + \tilde{M} M_B \sqrt{T - s_N} \tilde{C}^{1/2} < +\infty.
\] (3.8)

for all \(t \in I\). Since \(L^2(I; \mathcal{X})\) is reflexive, an application of the Banach-Alaoglu theorem yields the existence of a subsequence \(\{x^{n_k}\}_{k=1}^\infty\) of \(\{x^n\}_{n=1}^\infty\) such that
\[
x^{n_k} \overset{w}{\rightharpoonup} x^0 \quad \text{in} \quad L^2(I; \mathcal{X}), \quad \text{as} \quad k \to \infty.
\] (3.9)

The estimate (3.7) ensures that the sequence \(\{u^n\}_{n=1}^\infty\) is uniformly bounded in the space \(L^2(I; \mathcal{U})\). Since \(L^2(I; \mathcal{U})\) is a separable Hilbert space (in fact reflexive), using the Banach-Alaoglu theorem, we can find a subsequence \(\{u^{n_k}\}_{k=1}^\infty\) of \(\{u^n\}_{n=1}^\infty\) such that
\[
u^{n_k} \overset{w}{\rightharpoonup} u^0 \quad \text{in} \quad L^2(I; \mathcal{U}) = \mathcal{U}_ad, \quad \text{as} \quad k \to \infty.
\]

Since \(B\) is a bounded linear operator from \(\mathcal{U}\) to \(\mathcal{X}\), the above convergence also implies
\[
B u^{n_k} \overset{w}{\rightharpoonup} B u^0 \quad \text{in} \quad L^2(I; \mathcal{X}).
\] (3.10)
Note that
\[ \left\| \int_0^t S(t, s)Bu^n(s)ds - \int_0^t S(t, s)Bu^0(s)ds \right\|_\mathcal{X} \to 0, \quad \text{as } k \to \infty, \quad (3.11) \]
for all \( t \in I \). Here, we used the weak convergence given in (3.10) and the compactness of the operator \((Qf)(\cdot) = \int_0^\cdot S(\cdot, s)f(s)ds : L^2(I; \mathbb{X}) \to C(I; \mathbb{X})\). Using the above convergence, we compute
\[ \left\| x^{nk}(t) - x^*(t) \right\|_\mathcal{X} = \left\| \int_0^t S(t, s)Bu^{nk}(s)ds - \int_0^t S(t, s)Bu^0(s)ds \right\|_\mathcal{X} \to 0, \quad \text{as } k \to \infty, \quad \text{for all } t \in I, \quad (3.12) \]
where
\[ x^*(t) = C(t, s_N)v + S(t, s_N)w + \int_{s_N}^t S(t, s)Bu^0(s)ds, \quad t \in I. \]
The above expression guarantees that the function \( x^* \in C(I; \mathbb{X}) \) is the unique mild solution of the Eq. (3.2) with the control \( u^0 \in \mathcal{U}_{ad} \). Using the convergences (3.9) and (3.12), and the fact the weak limit is unique, we obtain \( x^*(t) = x^0(t) \) for all \( t \in I \). Thus, the function \( x^0(\cdot) \) is the unique limit of (3.2) with the control \( u^0 \in \mathcal{U}_{ad} \), therefore the whole sequence \( x^n \to x^0 \) in \( C(J; \mathbb{X}) \). Hence, we have \( (x^0, u^0) \in \mathcal{A}_{ad} \).

Next, we prove that \((x^0, u^0)\) is a minimizer, that is, \( L = \mathcal{F}(x^0, u^0) \). Since the cost functional \( \mathcal{F}(\cdot, \cdot) \) is continuous and convex (see Proposition III.1.6 and III.1.10, [14]) on \( L^2(I; \mathbb{X}) \times L^2(I; \mathbb{U}) \), it follows that \( \mathcal{F}(\cdot, \cdot) \) is sequentially weakly lower semi-continuous (Proposition II.4.5, [14]). That is, for a sequence
\[ (x^n, u^n) \rightharpoonup (x^0, u^0) \text{ in } L^2(I; \mathbb{X}) \times L^2(I; \mathbb{U}), \]
we have
\[ \mathcal{F}(x^0, u^0) \leq \liminf_{n \to \infty} \mathcal{F}(x^n, u^n). \]
Therefore, we obtain
\[ L \leq \mathcal{F}(x^0, u^0) \leq \liminf_{n \to \infty} \mathcal{F}(x^n, u^n) = \lim_{n \to \infty} \mathcal{F}(x^n, u^n) = L. \]
Hence, \((x^0, u^0)\) is a minimizer of the problem (3.2). Note that the cost functional given in (3.1) is convex, the constraint given in (3.2) is linear and the admissible class \( \mathcal{U}_{ad} = L^2(I; \mathbb{U}) \) is convex, then the optimal control obtained above is unique. \( \Box \)

The expression for the optimal control in the feedback form is provided by the following lemma:
Lemma 3.1 Assume that \((x, u)\) is the optimal pair for the problem (3.4). Then the optimal control \(u\) is given by

\[
u(t) = B^*S^*(T, t)J\left[R(\lambda, \Psi^T_{s_N})\ell(x(\cdot))\right], \quad t \in I,
\]

where

\[
\ell(x(\cdot)) = x_T - C(T, s_N)v - S(T, s_N)w.
\]

Proof Let \((x, u)\) be an optimal solution of (3.4) with the control \(u\) and the corresponding trajectory be \(x\). Then \(\varepsilon = 0\) is the critical point of

\[
I(\varepsilon) = F(x_{u+\varepsilon z}, u + \varepsilon z),
\]

with \(z \in L^2(J; U)\), where \(x_{u+\varepsilon z}\) is the unique mild solution of (3.2) with respect to the control \(u + \varepsilon z\) and \(x_{u+\varepsilon z}(\cdot)\) satisfies:

\[
x_{u+\varepsilon z}(t) = C(t, s_N)v + S(t, s_N)w + \int_{s_N}^t S(t, s)B(u + \varepsilon z)(s)ds.
\]

Let us now compute the first variation of the cost functional \(F\) (defined in (3.1)) as

\[
\frac{d}{d\varepsilon} I(\varepsilon) \bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left[ \|x_{u+\varepsilon z}(T) - x_T\|^2_X + \lambda \int_{s_N}^T \|u(t) + \varepsilon z(t)\|^2_U dt \right]_{\varepsilon=0}
\]

\[
= 2 \left( \langle J(x_{u+\varepsilon z}(T) - x_T), \frac{d}{d\varepsilon}(x_{u+\varepsilon z}(T) - x_T) \rangle + 2\lambda \int_{s_N}^T (u(t) + \varepsilon z(t), \frac{d}{d\varepsilon}(u(t) + \varepsilon z(t)))dt \right)_{\varepsilon=0}
\]

\[
= 2 \left( \langle J(x(T) - x_T), \int_{s_N}^T S(t, s)Bz(t)dt \rangle + 2\lambda \int_{s_N}^T (u(t), z(t))dt \right).
\]

(3.15)

By taking the first variation of the cost functional is zero, we obtain

\[
0 = \left( \langle J(x(T) - x_T), \int_{s_N}^T S(t, s)Bz(t)dt \rangle + \lambda \int_{s_N}^T (u(t), z(t))dt \right)
\]

\[
= \int_{s_N}^T \langle J(x(T) - x_T), S(t, s)Bz(t) \rangle dt + \lambda \int_{s_N}^T (u(t), z(t))dt
\]

\[
= \int_{s_N}^T (B^*S^*(T, t)J(x(T) - x_T) + \lambda u(t), z(t)) dt.
\]

(3.16)
Since $z \in L^2(I; \mathbb{U})$ is an arbitrary element (one can choose $z$ to be $B^* S^*(T, t) \mathcal{J}[x(T) - x_T] + \lambda u(t)$), it follows that the optimal control is given by

$$u(t) = -\lambda^{-1} B^* S^*(T, t) \mathcal{J}[x(T) - x_T],$$

(3.17)

for a.e. $t \in I$. It also holds for all $t \in I$, since from the expressions (3.16) and (3.17), it is clear that $u$ is continuous and belongs to $C(I; \mathbb{X})$. Using the above expression of the control, we obtain

$$x(T) = C(T, s_N)v + S(T, s_N)w - \int_{s_N}^{T} \lambda^{-1} S(T, t)BB^* S(T, t)^* \mathcal{J}[x(T) - x_T] \, ds$$

$$= C(T, s_N)v + S(T, s_N)w - \lambda^{-1} \Psi_{s_N}^T \mathcal{J}[x(T) - x_T].$$

(3.18)

Let us now define

$$\ell(x(\cdot)) := x_T - C(T, s_N)v - S(T, s_N)w.$$  

(3.19)

Combining (3.18) and (3.19), we have the following

$$x(T) - x_T = -\ell(x(\cdot)) - \lambda^{-1} \Psi_{s_N}^T \mathcal{J}[x(T) - x_T].$$

(3.20)

From (3.20), one can easily deduce that

$$x(T) - x_T = -\lambda I(\lambda I + \Psi_{s_N}^T \mathcal{J})^{-1} \ell(x(\cdot)) = -\lambda R(\lambda, \Psi_0^T) \ell(x(\cdot)).$$

(3.21)

Finally, from (3.17), we have

$$u(t) = B^* S^*(T, t) \mathcal{J} \left[ R(\lambda, \Psi_{s_N}^T) \ell(x(\cdot)) \right], \quad t \in I,$n

which completes the proof. \hfill \Box

\textbf{Lemma 3.2} (Theorem 3.2, [2]) \textit{The linear control system (3.2) is approximately controllable on $I$ if and only if the Assumption (H0) holds.}

A proof of the above lemma is a straightforward adaptation of the proof of Theorem 3.2, [2].

\textbf{Remark 3.1} From Theorem 2.3 in [33], we know that the Assumption (H0) holds implies that the operator $\Psi_{s_N}^T$ is positive and vice versa. The positivity of $\Psi_{s_N}^T$ is equivalent to say that

$$\langle x^*, \Psi_{s_N}^T x^* \rangle = 0 \Rightarrow x^* = 0.$$  

Since, we have

$$\langle x^*, \Psi_{s_N}^T x^* \rangle = \int_{s_N}^{T} \| B^* S(T, t)^* x^* \| \, dt.$$  

Hence by the above fact and Lemma 3.2, one can ensure that the approximate controllability of the linear system (3.2) is equivalent to the condition

$$B^* S(T, t)^* x^* = 0, \ t \in I \Rightarrow x^* = 0.$$ 

### 3.2 Approximate Controllability of the Semilinear System

In this subsection, we establish the approximate controllability of the system (1.1a)–(1.1f). To achieve this goal, we prove the existence of a mild solution for \(\lambda > 0\), \(x_T \in X\) of the system (1.1a)–(1.1f) with the control (see, [31] for a detailed derivation)

$$u_\lambda(t) = \sum_{i=0}^N u_{i, \lambda}(t) \chi_{[s_i, t_{i+1}]}(t), \ t \in J, \quad (3.22)$$

where

$$u_{i, \lambda}(t) = B^* S^*(t_{i+1}, t) J \left[ R(\lambda, \Psi^{t+1}_{s_i}) g_i(x(\cdot)) \right],$$

for \(t \in [s_i, t_{i+1}], \ i = 0, 1, \ldots, N\) with

$$g_0(x(\cdot)) = x_T - C(t_1, 0)[\phi(0) + g(\tilde{x})] - S(t_1, 0) [\eta + h(\tilde{x})]$$

$$- \int_0^{t_1} S(t_1, s) f(s, \tilde{x}) ds,$$

$$g_i(x(\cdot)) = x_T - C(t_{i+1}, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) - S(t_{i+1}, s_i) \rho_i'(s_i, \tilde{x}(t_i^-))$$

$$- \int_{s_i}^{t_{i+1}} S(t_{i+1}, s) f(s, \tilde{x}_s) ds, \ i = 1, \ldots, N,$$

and \(\tilde{x} : J_q \to X\) such that \(\tilde{x}(t) = \phi(t), \ t \in [-q, 0)\) and \(\tilde{x}(t) = x(t), \ t \in J\).

**Remark 3.2** Since the operator \(\Psi^{t+1}_{s_i}\), for each \(i = 0, 1, \ldots, N\) is non-negative, linear and bounded, Remark 2.3 is also valid for each \(\Psi^{t+1}_{s_i}\), for \(i = 0, 1, \ldots, N\).

**Theorem 3.2** If Assumptions (H1)–(H4) hold true, then for every \(\lambda > 0\) and fixed \(x_T \in X\), the system (1.1a)–(1.1f) with the control (3.22) has at least one mild solution on \(J\), provided

$$K \left[ 1 + \left(\tilde{M} M_B \right)^2 T / \lambda \right] < 1,$$  

where \(K = M M_B + \tilde{M} M_h\).

**Proof** Let us consider a set

$$E_r = \{x \in PC(J : X) : \|x\|_{PC(J : X)} \leq r\},$$
where \( r \) is a positive constant. For \( \lambda > 0 \), let us define an operator \( \Phi_\lambda : PC(J : \mathbb{X}) \to PC(J : \mathbb{X}) \) as

\[
(\Phi_\lambda x)(t) = z(t),
\]

where

\[
z(t) = \begin{cases} 
\mathcal{C}(t, 0)[\phi(0) + g(\tilde{x})] + S(t, 0)[\eta + h(\tilde{x})] + \int_0^t S(t, s)[Bu_\lambda(s) + f(s, \tilde{x}_s)]ds, \ t \in [0, t_1], \\
\mathcal{C}(t, s_i)\rho_i(s_i, \tilde{x}(t_i^-)) + S(t, s_i)\rho_i'(s_i, \tilde{x}(t_i^-)) + \int_{s_i}^t S(t, s)[Bu_\lambda(s) + f(s, \tilde{x}_s)]ds, \ t \in (s_i, t_{i+1}], \ i = 1, \ldots, N,
\end{cases}
\]

with \( u_\lambda \) is defined in (3.22) and \( \tilde{x}(t) = \phi(t), \ t \in [-q, 0) \) and \( x'(t) = x(t), \ t \in J \). By the definition of \( \Phi_\lambda \), one can ensure that the problem of finding a mild solution of the system (1.1a)–(1.1f) is equivalent to finding a fixed point of the operator \( \Phi_\lambda \). A proof that the operator \( \Phi_\lambda \) has a fixed point is divided into the following steps.

**Step (1)** First we prove that for each \( \lambda > 0 \) that satisfies the condition (3.23), there exists a constant \( r = r(\lambda) \) such that \( \Phi_\lambda (E_r) \subset E_r \). In contrast, we assume that our claim is not true. Then for any \( \lambda > 0 \) and for every \( r > 0 \), there exists \( x'(\cdot) \in E_r \) such that \( \| (\Phi_\lambda x')(t) \|_\mathbb{X} > r \), for some \( t \in J \) (\( t \) may depend upon \( r \)). Taking \( t \in [0, t_1] \) and using the estimate (2.15), Assumption 2.1 (H2)–(H4), we calculate

\[
r < \| (\Phi_\lambda x')(t) \|_\mathbb{X}
= \| \mathcal{C}(t, 0)[\phi(0) + g(\tilde{x})] + S(t, 0)[\eta + h(\tilde{x})]
+ \int_0^t S(t, s)[Bu_\lambda(s) + f(s, \tilde{x}_s)]ds \|_\mathbb{X}
\leq \| \mathcal{C}(t, 0)[\phi(0) + g(\tilde{x})] \|_\mathbb{X} + \| S(t, 0)[\eta + h(\tilde{x})] \|_\mathbb{X}
+ \int_0^t \| S(t, s)[Bu_\lambda(s) + f(s, \tilde{x}_s)] \|_\mathbb{X} ds
\leq M[\| \phi(0) \|_\mathbb{X} + M_g(\| \tilde{x} \|_{PC(J_q; \mathbb{X})} + 1)] + \tilde{M}[\| \eta \|_{\mathbb{X}} + M_h(\| x \|_{PC(J_q; \mathbb{X})} + 1)]
+ \tilde{M}M_B \int_0^t \| u_{\lambda}(s) \|_U ds + \tilde{M} \int_0^t \| f(s, \tilde{x}_s) \|_\mathbb{X} ds
\leq M[\| \phi(0) \|_\mathbb{X} + M_g(r_1 + 1)] + \tilde{M}[\| \eta \|_{\mathbb{X}} + M_h(r_1 + 1)]
+ \frac{(\tilde{M}M_B)^2 T}{\lambda} \| g_0(x(\cdot)) \|_\mathbb{X}
+ \tilde{M} \int_0^t \gamma(s) ds
\]
\[ \begin{align*}
&\leq M \|\phi(0)\|_X + \tilde{M} \|\eta\|_X + K(r_1 + 1) \\
&\quad + \frac{(\tilde{M}M_B)^2 T}{\lambda} \left[ \tilde{K} + K(r_1 + 1) + \tilde{M} \|\gamma\|_{L^1(J_1;X)} \right] \\
&\quad + \tilde{M} \|\gamma\|_{L^1(J_1;X)},
\end{align*} \tag{3.25} \]

where \( K = MM_B + \tilde{M}M_h, \quad \tilde{K} = \|x_T\|_X + M \|\phi(0)\|_X + \tilde{M} \|\eta\|_X \) and \( r_1 = r + \|\phi\|_D \).

For \( t \in (t_i, t_{i+1}], \quad i = 1, \ldots, N \), we estimate

\[ r < \| (\Phi_{3\lambda} x^r)(t) \|_X = \| \rho_i(t, \tilde{x}(t_i^-)) \|_X \leq d_i \leq d_i + K r_1 + \frac{(\tilde{M}M_B)^2 K r_1 T}{\lambda}. \tag{3.26} \]

Taking \( t \in (s_i, t_{i+1}], \quad i = 1, \ldots, N \), we evaluate

\[ \begin{align*}
r &< \| (\Phi_{3\lambda} x^r)(t) \|_X \\
&= \| C(t, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) + S(t, s_i) \rho'_i(s_i, \tilde{x}(t_i^-)) \|_X \\
&\quad + \int_{s_i}^t S(t, s) [Bu_{i,\lambda}(s) + f(s, \tilde{x})] ds \\
&\leq \| C(t, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) \|_X + \| S(t, s_i) \rho'_i(s_i, \tilde{x}(t_i^-)) \|_X \\
&\quad + \int_{s_i}^t \| S(t, s) Bu_{i,\lambda}(s) \|_X ds \\
&\quad + \int_{s_i}^t \| S(t, s) f(s, \tilde{x}) \|_X ds \\
&\leq M d_i + \tilde{M} e_i + \tilde{M} M_B \int_{s_i}^t \| u_{i,\lambda}(s) \|_U ds + \tilde{M} \int_{s_i}^t \| f(s, \tilde{x}) \|_X ds \\
&\leq M d_i + \tilde{M} e_i + \frac{(\tilde{M}M_B)^2 T}{\lambda} \| g_i(x(\cdot)) \|_X + \tilde{M} \int_{s_i}^t \| f(s, \tilde{x}) \|_X ds \\
&\leq M d_i + \tilde{M} e_i + \frac{(\tilde{M}M_B)^2 T}{\lambda} \left[ \| x_T \|_X + M d_i + \tilde{M} e_i + \tilde{M} \|\gamma\|_{L^1(J_1;\mathbb{R}^+)} \right] \\
&\quad + \tilde{M} \|\gamma\|_{L^1(J_1;\mathbb{R}^+)} \\
&\leq M d_i + \tilde{M} e_i + \frac{(\tilde{M}M_B)^2 T}{\lambda} \left[ K_i + \tilde{M} \|\gamma\|_{L^1(J_1;\mathbb{R}^+)} \right] \\
&\quad + \tilde{M} \|\gamma\|_{L^1(J_1;\mathbb{R}^+)} \\
&\leq M d_i + \tilde{M} e_i + K r_1 + \frac{(\tilde{M}M_B)^2 T}{\lambda} \left[ K_i + K r_1 + \tilde{M} \|\gamma\|_{L^1(J_1;\mathbb{R}^+)} \right] \\
&\quad + \tilde{M} \|\gamma\|_{L^1(J_1;\mathbb{R}^+)} ,
\end{align*} \tag{3.27} \]

where \( K_i = \|x_T\|_X + M d_i + \tilde{M} e_i \) for \( i = 1, \ldots, N \). Thus, dividing by \( r \) in the expressions (3.25), (3.26) and (3.27), and then passing \( r \to \infty \), we obtain
which is a contradiction to (3.23). Hence, for each \( \lambda > 0 \) for which the condition (3.23) is holds, there is an \( r = r(\lambda) > 0 \) such that \( \varPhi_\lambda(\mathcal{E}_r) \subset \mathcal{E}_r \).

\textbf{Step (2)} The operator \( \varPhi_\lambda \) is continuous. For this, we take a sequence \( \{x^n\}_{n=1}^\infty \subset \mathcal{E}_r \) such that \( x^n \to x \) in \( \mathcal{E}_r \), that is,

\[ \lim_{n \to \infty} \|x^n - x\|_{PC(J;\mathcal{X})} = 0. \]

For \( s \in J \), we estimate

\[ \|\tilde{x}^n_s - \tilde{x}_s\|_\mathcal{D} = \frac{1}{q} \int_{-q}^{0} \|\tilde{x}^n(\theta) - \tilde{x}(\theta)\|_\mathcal{X} \, d\theta = \frac{1}{q} \int_{-q}^{0} \|\tilde{x}^n(s + \theta) - \tilde{x}(s + \theta)\|_\mathcal{X} \, d\theta \]

\[ = \frac{1}{q} \int_{s-q}^{s} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau. \] (3.29)

If \( s < q \), then one can write the above expression as

\[ \|\tilde{x}^n_s - \tilde{x}_s\|_\mathcal{D} \leq \frac{1}{q} \int_{-q}^{0} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau + \frac{1}{q} \int_{0}^{s} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau \]

\[ \leq \frac{1}{q} \int_{-q}^{0} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau + \frac{1}{q} \int_{0}^{T} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau \]

\[ = \frac{1}{q} \int_{0}^{T} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau \leq \frac{T}{q} \|x^n - x\|_{PC(J;\mathcal{X})} \to 0, \text{ as } n \to \infty. \]

If \( s \geq q \), then by the expression (3.29), we obtain

\[ \|\tilde{x}^n_s - \tilde{x}_s\|_\mathcal{D} \leq \frac{1}{q} \int_{0}^{s} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau \leq \frac{1}{q} \int_{0}^{T} \|\tilde{x}^n(\tau) - \tilde{x}(\tau)\|_\mathcal{X} \, d\tau \]

\[ \leq \frac{T}{q} \|x^n - x\|_{PC(J;\mathcal{X})} \to 0, \text{ as } n \to \infty. \]

By using the above convergences along with Assumption 2.1 (H2), we immediately have

\[ \|f(s, \tilde{x}^n_s) - f(s, \tilde{x}_s)\|_\mathcal{X} \to 0 \text{ as } n \to \infty, \text{ uniformly for } s \in J. \] (3.30)

From the convergence (3.30), Assumption 2.1 (H2),(H4) and the dominated convergence theorem, we obtain

\[ \|g_0(x^n(\cdot)) - g_0(x(\cdot))\|_\mathcal{X} \leq \|C(t_1, 0)[g(x^n) - g(\tilde{x})]\|_\mathcal{X} + \|S(t_1, 0)[h(x^n) - h(\tilde{x})]\|_\mathcal{X} \]
\[
+ \int_{0}^{t_{i}} \| S(t_{i}, s) [ f(s, \tilde{x}^{n}_{s}) - f(s, \tilde{x}_{s})] \|_{X} ds \\
\leq M \| g(\tilde{x}^{n}_{\lambda}) - g(\tilde{x}) \|_{X} + \tilde{M} \| h(\tilde{x}^{n}_{\lambda}) - h(\tilde{x}) \|_{X} \\
+ M \int_{0}^{t_{i}} \| f(s, \tilde{x}^{n}_{s}) - f(s, \tilde{x}_{s}) \|_{X} ds \\
\to 0 \text{ as } n \to \infty.
\] (3.31)

Similarly, for \( i = 1, \ldots, N \), we estimate
\[
\| g_{i}(x^{n}(\cdot)) - g_{i}(x(\cdot)) \|_{X} \\
\leq \| C(t_{i+1}, s_{i}) [ \rho_{i}(s_{i}, \tilde{x}^{n}(t_{i}^{-})) - \rho_{i}(s_{i}, \tilde{x}(t_{i}^{-}))] \|_{X} \\
+ \| S(t_{i+1}, s_{i}) [ \rho'_{i}(s_{i}, \tilde{x}^{n}(t_{i}^{-})) - \rho'_{i}(s_{i}, \tilde{x}(t_{i}^{-}))] \|_{X} \\
+ \int_{s_{i}}^{t_{i+1}} \| S(t_{i+1}, s) [ f(s, \tilde{x}^{n}_{s}) - f(s, \tilde{x}_{s})] \|_{X} ds \\
\leq M \| \rho_{i}(s_{i}, \tilde{x}^{n}(t_{i}^{-})) - \rho_{i}(s_{i}, \tilde{x}(t_{i}^{-})) \|_{X} \\
+ \tilde{M} \| \rho'_{i}(s_{i}, \tilde{x}^{n}(t_{i}^{-})) - \rho'_{i}(s_{i}, \tilde{x}(t_{i}^{-})) \|_{X} \\
+ \tilde{M} \int_{s_{i}}^{t_{i+1}} \| f(s, \tilde{x}^{n}_{s}) - f(s, \tilde{x}_{s}) \|_{X} ds \\
\to 0 \text{ as } n \to \infty,
\] (3.32)

where we used the convergence (3.30), Assumption 2.1 (H2)–(H3) and the dominated convergence theorem. From Remark 2.2 (iii), we infer that the operators \( R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) \) for \( i = 0, \ldots, N \) are uniformly continuous on every bounded subset of \( X \). Thus we have
\[
\| R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) g_{i}(x^{n}(\cdot)) - R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) g_{i}(x(\cdot)) \|_{X} \to 0 \text{ as } n \to \infty,
\]
for \( i = 0, 1, \ldots, N \).

Since the mapping \( J : X \to X \) is demicontinuous, we have
\[
J \left[ R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) g_{i}(x^{n}(\cdot)) \right] \rightharpoonup J \left[ R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) g_{i}(x(\cdot)) \right] \text{ as } n \to \infty \text{ in } X^{*},
\] (3.33)

for \( i = 0, 1, \ldots, N \). Since by Theorem 2.1, we know that the operator \( S(t, s) \) is compact for all \( s \leq t \). Therefore, the operator \( S(t, s)^{*} \) is also compact. Hence, by using the compactness of that operator along with the weak convergence (3.33), we obtain
\[
\| u_{i, \lambda}^{n}(t) - u_{i, \lambda}(t) \|_{U} \\
\leq \| B^{*} S(t_{i+1}, t)^{*} J \left[ R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) g_{i}(x^{n}(\cdot)) \right] - J \left[ R(\lambda, \Psi_{s_{i}}^{t_{i+1}}) g_{i}(x(\cdot)) \right] \|_{U}
\]
\[
\begin{align*}
\| S(t_{i+1}, t)^* \left( J \left[ R(\lambda, \Psi_{s_i}^{t+1}) g_i(x^n(t)) \right] - J \left[ R(\lambda, \Psi_{s_i}^{t+1}) g_i(x(t)) \right] \right) \|_X \\
\to 0 \text{ as } n \to \infty, \text{ for } t \in (s_i, t_{i+1}), \ i = 0, 1, \ldots, N.
\end{align*}
\]

(3.34)

Using the convergences (3.30), (3.34) and the dominated convergence theorem, we compute

\[
\begin{align*}
\| (\Phi_\lambda x^n)(t) - (\Phi_\lambda x)(t) \|_X & \\
\leq \int_0^t \| S(t, s) [Bu_{0,\lambda}(s) - Bu_{0,\lambda}(s)] \|_X \, ds \\
+ \int_0^t \| S(t, s) [f(s, \tilde{x}_s^n) - f(s, \tilde{x}_s)] \|_X \, ds \\
\leq \tilde{M} M_B \int_0^t \| u_{0,\lambda}(s) - u_{0,\lambda}(s) \|_U \, ds + \tilde{M} \int_0^t \| f(s, \tilde{x}_s^n) - f(s, \tilde{x}_s) \|_X \, ds \\
\to 0 \text{ as } n \to \infty, \text{ for } t \in [0, t_1].
\end{align*}
\]

Similarly, for \( t \in (s_i, t_{i+1}), \ i = 1, \ldots, N, \) we calculate

\[
\begin{align*}
\| (\Phi_\lambda x^n)(t) - (\Phi_\lambda x)(t) \|_X & \\
\leq \| C(t, s_i) \left( \rho_i(s_i, \tilde{x}_i^n(t_i^\ominus)) - \rho_i(s_i, \tilde{x}(t_i^\ominus)) \right) \|_X \\
+ \int_{s_i}^t \| S(t, s_i) \left( \rho'_{i}(s_i, \tilde{x}_i^n(t_i^\ominus)) - \rho'_{i}(s_i, \tilde{x}(t_i^\ominus)) \right) \|_X \\
\to 0 \text{ as } n \to \infty.
\end{align*}
\]

Moreover, for \( t \in (s_i, t_i], \ i = 1, \ldots, N, \) applying the Assumption 2.1 (H3), we get

\[
\| (\Phi_\lambda x^n)(t) - (\Phi_\lambda x)(t) \|_X = \| \rho_i(t, \tilde{x}_i^n(t_i^\ominus)) - \rho_i(t, \tilde{x}(t_i^\ominus)) \|_X \\
\to 0 \text{ as } n \to \infty.
\]

Hence, it follows that \( \Phi_\lambda \) is continuous.

**Step (3)** \( \Phi_\lambda \) is a compact operator. To achieve this goal, we use the infinite-dimensional version of the generalized Ascoli-Arzelà theorem (see, Theorem 2.1,
To apply the generalized Ascoli-Arzela theorem, in view of Step 1, it is enough to show that

(i) The set \{ (\Phi_\lambda x) : x \in E_r \} is equicontinuous for \( t \in (t_i, t_{i+1}) \), \( i = 0, 1, \ldots, N \),

(ii) the set \( V(t) = \{ (\Phi_\lambda x)(t) : x \in E_r, \ t \in J/\{t_1, \ldots, t_N\} \} \), \( V(t_i^+) = \{ (\Phi_\lambda x)(t_i^+) : x \in E_r \} \) and \( V(t_i^-) = \{ (\Phi_\lambda x)(t_i^-) : x \in E_r \} \) for \( i = 1, \ldots, N \), are relatively compact.

Firstly, we claim that the set \{ (\Phi_\lambda x) : x \in E_r \} is equicontinuous for \( t \in (t_i, t_{i+1}) \), \( i = 0, 1, \ldots, N \). For \( t_1, t_2 \in (0, t_1) \) with \( t_1 < t_2 \) and \( x \in E_r \), we compute the following

\[
\| (\Phi_\lambda x)(t_2) - (\Phi_\lambda x)(t_1) \|_X \\
\leq \| C(\tau_2, 0) \phi(0) - C(\tau_1, 0) \phi(0) \|_X + \| C(\tau_2, 0) g(\tilde{x}) - C(\tau_1, 0) g(\tilde{x}) \|_X \\
+ \| [S(\tau_2, 0) - S(\tau_1, 0)] \eta \|_X + \| [S(\tau_2, 0) - S(\tau_1, 0)] h(\tilde{x}) \|_X \\
+ \int_{t_1}^{t_2} \| S(\tau_2, s) f(s, \tilde{x}_s) \|_X \, ds + \int_{t_1}^{t_2} \| S(\tau_2, s) Bu(\lambda(s)) \|_X \, ds \\
+ \int_{t_1}^{t_2} \| S(\tau_2, s) - S(\tau_1, s) Bu(\lambda(s)) \|_X \, ds \\
\leq \| C(\tau_2, 0) \phi(0) - C(\tau_1, 0) \phi(0) \|_X + \| C(\tau_2, 0) g(\tilde{x}) - C(\tau_1, 0) g(\tilde{x}) \|_X \\
+ \| S(\tau_2, 0) - S(\tau_1, 0) \|_{L(X)} \| \eta \|_X + \| S(\tau_2, 0) - S(\tau_1, 0) \|_{L(X)} (r_1 + 1) \\
+ \tilde{M} \int_{t_1}^{t_2} \gamma(s) \, ds + \left( \frac{\tilde{M} M_B}{\lambda} \right)^2 \left[ \tilde{K} + K(r_1 + 1) + \tilde{M} \| \gamma \|_{L^1(J; X)} \right] (t_2 - t_1) \\
+ \sup_{t \in [0, t_1]} \| S(\tau_2, t) - S(\tau_1, t) \|_{L(X)} M_B \int_{0}^{t_1} \| u_0, \lambda(s) \|_U \, ds \\
+ \sup_{t \in [0, t_1]} \| S(\tau_2, t) - S(\tau_1, t) \|_X \int_{0}^{t_1} \gamma(s) \, ds. \tag{3.35}
\]

Similarly for \( t_1, t_2 \in (s_i, t_{i+1}) \), \( i = 1, \ldots, N \) with \( t_1 < t_2 \) and \( x \in E_r \), we estimate

\[
\| (\Phi_\lambda x)(t_2) - (\Phi_\lambda x)(t_1) \|_X \\
\leq \| C(\tau_2, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) - C(\tau_1, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) \|_X \\
+ \| (S(\tau_2, s_i) - S(\tau_1, s_i)) \rho_i'(s_i, \tilde{x}(t_i^-)) \|_X \\
+ \int_{t_1}^{t_2} \| S(\tau_2, s) Bu_i(\lambda(s)) \|_X \, ds + \int_{t_1}^{t_2} \| S(\tau_2, s) f(s, \tilde{x}_s) \|_X \, ds \\
+ \int_{s_i}^{t_1} \| (S(\tau_2, s) - S(\tau_1, s)) Bu(s, \lambda(s)) \|_X \, ds \\
+ \int_{s_i}^{t_1} \| (S(\tau_2, s) - S(\tau_1, s)) f(s, \tilde{x}_s) \|_X \, ds \\
\leq \| C(\tau_2, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) - C(\tau_1, s_i) \rho_i(s_i, \tilde{x}(t_i^-)) \|_X \\
+ \| S(\tau_2, s_i) - S(\tau_1, s_i) \|_{L(X)} e_i
\]
\[
+ \frac{(\hat{M} MB)^2}{\lambda} \left[ K_i + Kr_1 + \hat{M} \| \gamma \|_{L^1(\mathbb{R}; \mathbb{R}^+)} \right] (\tau_2 - \tau_1) + \hat{M} \int_{\tau_1}^{\tau_2} \gamma(s) ds
\]
\[
+ \sup_{t \in [s_i, \tau_1]} \| S(\tau_2, t) - S(\tau_1, t) \|_{\mathcal{L}(\mathbb{X})} MB \int_{s_i}^{\tau_1} \left\| \mu_i(\lambda, s) \right\|_{\mathcal{U}} ds
\]
\[
+ \sup_{t \in [s_i, \tau_1]} \| S(\tau_2, t) - S(\tau_1, t) \|_{\mathcal{L}(\mathbb{X})} \int_{s_i}^{\tau_1} \gamma(s) ds.
\]
(3.36)

Moreover, for \( \tau_1, \tau_2 \in (t_i, s_i] \), with \( \tau_1 < \tau_2 \) and \( x \in \mathcal{E}_r \), we have
\[
\| (\Phi_{\lambda} x)(\tau_2) - (\Phi_{\lambda} x)(\tau_1) \|_{\mathbb{X}} = \| \rho(t_2, \tilde{x}(t_i^-)) - \rho(t_1, \tilde{x}(t_i^-)) \|_{\mathbb{X}}.
\]
(3.37)

Using the facts, the operator \( C(t, s)x, \ x \in \mathbb{X} \) is uniformly continuous for \( t, s \in J \), the operator \( S(\cdot, s) \) is Lipschitz continuous for all \( s \in J \) in uniform operator topology (see (2.5)) and the continuity of the impulses \( \rho_i(\cdot, x) \) for each \( x \in \mathbb{X} \), we obtain that the right hand side of the expressions (3.35), (3.36) and (3.37), converge to zero as \( |\tau_2 - \tau_1| \to 0 \).

Hence, the set \( \{ (\Phi_{\lambda} x) : x \in \mathcal{E}_r \} \) is equicontinuous for \( t \in (t_i, t_{i+1}), \ i = 0, 1, \ldots, N \).

Next, we show that the sets \( \mathcal{V}(t) = \{ (\Phi_{\lambda} x)(t) : x \in \mathcal{E}_r, \ t \in J \setminus \{t_1, \ldots, t_N\} \} \), \( \mathcal{V}(t^+) = \{ (\Phi_{\lambda} x)(t^+) : x \in \mathcal{E}_r \} \) and \( \mathcal{V}(t^-) = \{ (\Phi_{\lambda} x)(t^-) : x \in \mathcal{E}_r \} \) for \( i = 1, \ldots, \) \( N \) are relatively compact. These sets are relatively compact in \( \mathcal{E}_\mathcal{R} \) follows by the facts that the operator \( S(t, s) \) is compact for \( t \leq s \), the impulses \( \rho_i(t, \cdot) \), \( i = 1, \ldots, N \), \( t \in J \) and the functions \( g(\cdot) \) are completely continuous for each \( x \in PC(J_q; \mathbb{X}) \) and also the compactness of the operator \( (Q)(\cdot) = \int_0^\cdot S(\cdot, s) f(s) ds \) (Lemma 3.2, Corollary 3.3, Chapter 3, [28]).

Hence, the operator \( \Phi_{\lambda} \) is compact in view of generalized Ascoli-Arzela theorem. Then by an application of Schauder’s fixed point theorem yields that the operator \( \Phi_{\lambda} \) has a fixed point \( \tilde{x}(\cdot) \) in \( \mathcal{E}_r \). Thus, by the definition of \( \tilde{x}(\cdot) \), we obtain that \( \tilde{x}(\cdot) \) is a mild solution of the system (1.1a)–(1.1f). \( \square \)

**Remark 3.3** Theorem 3.2 is proved using the well-known Schauder fixed point theorem. In order to do this, one needs the boundedness, continuity and compactness of the operator \( \Phi_{\lambda} \), which is difficult to establish under the Assumption 2.1.

In next theorem, we determine the approximate controllability of the system (1.1a)–(1.1f). For this, we impose the following assumption on \( f(\cdot, \cdot) \).

**Assumption 3.1** *(H5)* The function \( f : J_1 \times \mathcal{D} \to \mathbb{X} \) satisfies the Assumption *(H2)*(i) and there exists a constant \( \mathcal{N}(t) \in L^2(J_1; \mathbb{R}^+) \) such that
\[
\| f(t, \phi) \|_{\mathbb{X}} \leq \mathcal{N}(t), \text{ for a.e. } t \in J_1 \text{ and } \phi \in \mathcal{D}.
\]

*(H6)* Assumption *(H4)*(i) holds true and there exist two constants \( M_g, M_h \) such that
\[
\| g(x) \|_{\mathbb{X}} \leq M_g, \| h(x) \|_{\mathbb{X}} \leq M_h, \text{ for all } x \in PC(J_q; \mathbb{X}).
\]

**Remark 3.4** Note that under the Assumption 3.1, the condition (3.23) of the above Theorem holds automatically.
Theorem 3.3 Suppose that the assumptions (H0), (H1), (H3) (see 2.1) and (H5)–(H6) (see 3.1) are satisfied. Then the system (1.1a)–(1.1f) is approximately controllable.

Proof By using Theorem 3.2, for every $\lambda > 0$ and $x_T \in \mathbb{X}$, there exists a mild solution $x^\lambda(\cdot)$ with the control defined in (3.22) satisfying

$$
x^\lambda(t) = \begin{cases}
C(t, 0) [\phi(0) + g(x)] + S(t, 0) [\eta + h(x)] \\
+ \int_0^t S(t, s) [Bu_\lambda(s) + f(s, x^\lambda_s)] ds, t \in (0, t_1],
\end{cases}
$$

$$
C(t, s_i) \rho_i(t, x^\lambda(t^-_i)), \ t \in \bigcup_{i=1}^N (t_i, s_i],
$$

$$
+ \int_{s_i}^t S(t, s) [Bu_\lambda(s) + f(s, x^\lambda_s)] ds, \ t \in \bigcup_{i=1}^N (s_i, t_{i+1}].
$$

(3.38)

Next, we estimate

$$
x^\lambda(T) = C(T, s_N) \rho_N(s_N, x^\lambda(t^-_N)) + S(T, s_N) \rho'_N(s_N, x^\lambda(t^-_N))
$$

$$
+ \int_{s_N}^T S(T, s) [Bu_{\lambda,s}(s) + f(s, x^\lambda_s)] ds
$$

$$
= C(T, s_N) \rho_N(s_N, x^\lambda(t^-_N)) + S(T, s_N) \rho'_N(s_N, x^\lambda(t^-_N))
$$

$$
+ \int_{s_N}^T S(T, s) f(s, x^\lambda_s) ds
$$

$$
+ \int_{s_N}^T S(T, s) BB^* S^*(T, s) J \left[ R(\lambda, \Psi_{s_N}^T) g_N(x^\lambda(\cdot)) \right] ds
$$

$$
= C(T, s_N) \rho_N(s_N, x^\lambda(t^-_N)) + S(T, s_N) \rho'_N(s_N, x^\lambda(t^-_N))
$$

$$
+ \int_{s_N}^T S(T, s) f(s, x^\lambda_s) ds
$$

$$
+ \Psi_{s_N}^T J \left[ R(\lambda, \Psi_{s_N}^T) g_N(x^\lambda(\cdot)) \right]
$$

$$
x_T = x_T - \lambda R(\lambda, \Psi_{s_N}^T) g_N(x^\lambda(\cdot)).
$$

(3.39)

Moreover, since $x^\lambda \in \mathcal{E}_r$ implies that the set $\{x^\lambda(t) : \lambda > 0\}$, for each $t \in J$ is bounded in $\mathbb{X}$. Then by an application of the Banach-Alaoglu theorem and reflexivity of $\mathbb{X}$, there is a subsequence $\{x^{\lambda_k}(t)\}_{k=1}^\infty$ such that

$$
x^{\lambda_k}(t) \overset{w}{\rightarrow} z(t) \text{ in } \mathbb{X} \text{ as } \lambda_k \to 0^+(k \to \infty), \ t \in J.\]
Using the assumption \((H3)\), we obtain
\[
\rho_N(t, x^{\lambda_k}(t_N^-)) \to \rho_N(t, z(t_N^-)) \quad \text{in } X \text{ as } \lambda_k \to 0^+(k \to \infty),
\]
(3.40)
\[
\rho_N'(t, x^{\lambda_k}(t_N^-)) \rightharpoonup \rho_N'(t, z(t_N^-)) \quad \text{in } X \text{ as } \lambda_k \to 0^+(k \to \infty).
\]
(3.41)
Furthermore, by the assumption \((H5)\), we have
\[
\int_{s_N}^{T} \| f(t, x_s^\lambda) \|_X^2 \, ds \leq \int_{s_N}^{T} N^2(s) \, ds < +\infty.
\]

Therefore, the set \( \{ f(\cdot, x_s^\lambda) : \lambda > 0 \} \) in \( L^2([s_N, T]; X) \) is bounded. Once again by the Banach-Alaoglu theorem, there exists a subsequence \( \{ f(\cdot, x_s^{\lambda_k}) \}_{k=1}^\infty \) such that
\[
f(\cdot, x_s^{\lambda_k}) \rightharpoonup f(\cdot) \quad \text{in } L^2([s_N, T]; X) \text{ as } \lambda_k \to 0^+(k \to \infty).
\]
(3.42)
Next, we evaluate
\[
\| g_N(x^{\lambda_k}(\cdot)) - \omega \|_X \leq \| C(T, s_N)[\rho_N(s_N, x^{\lambda_k}(t_N^-)) - \rho_N(s_N, z(t_N^-))] \|_X
\]
\[
+ \| S(T, s_N)[\rho_N'(s_N, x^{\lambda_k}(t_N^-)) - \rho_N'(s_N, z(t_N^-))] \|_X
\]
\[
+ \int_{s_N}^{T} \| S(T, s)(f(s, x_s^{\lambda_k}) - f(s)) \|_X \, ds
\]
\[
\leq M \| \rho_N(s_N, x^{\lambda_k}(t_N^-)) - \rho_N(s_N, z(t_N^-)) \|_X
\]
\[
+ \| S(T, s_N)[\rho_N'(s_N, x^{\lambda_k}(t_N^-)) - \rho_N'(s_N, z(t_N^-))] \|_X
\]
\[
+ \int_{s_N}^{T} \| S(T, s)(f(s, x_s^{\lambda_k}) - f(s)) \|_X \, ds
\]
\[
\to 0 \text{ as } \lambda_k \to 0^+(k \to \infty).
\]
(3.43)
where
\[
\omega = x_T - C(T, s_N)\rho_N(s_N, z(t_N^-)) - S(T, s_N)\rho_N'(s_N, z(t_N^-)) - \int_{s_N}^{T} S(T, s)f(s) \, ds,
\]
and we used the convergences (3.40), (3.41), (3.42) and the compactness of the operators \( S(\cdot, \cdot) : X \to X \) and \( (Q)(\cdot) = \int_0^T S(\cdot, s)f(s) \, ds : L^2(J; X) \to C(J; X) \) (Lemma 3.2, Corollary 3.3, Chapter 3, [28]).

The equality 3.39 ensures that \( z_{\lambda_k} = x^{\lambda_k}(T) - \omega \) for each \( \lambda_k > 0, \ k \in \mathbb{N} \) is a solution of the equation
\[
\lambda_k z_{\lambda_k} + \Psi_{s_N}^T \mathcal{J}[z_{\lambda_k}] = \lambda_k h_{\lambda_k},
\]
where
\[
    h_{\lambda k} = -g_N(x^{\lambda k}(\cdot)) = C(T, s_N)\rho_N(s_N, \tilde{x}(t_N^-)) + S(T, s_N)\rho_N'(s_N, \tilde{x}(t_N^-)) + \int_{s_N}^{T} S(T, s) f(s, \tilde{x}_s) ds - x_T.
\]

Since the assumption (H0) implies that the operator $\Psi_{s_N}^T$ is positive. Then by Theorem 2.5, [33], we obtain
\[
    \|x^{\lambda k}(T) - x_T\|_X = \|z_{\lambda k}\|_X \to 0 \text{ as } \lambda_k \to 0^+(k \to \infty).
\]

Hence, the system (1.1a)–(1.1f) is approximately controllable on $J$. \qed

**Remark 3.5** It should be noted that Theorem 3.3 is derived under the restricted Assumption 3.1, since we need the existence of mild solution of the system (1.1a)–(1.1f) for all $\lambda > 0$.

### 4 Application

In this section, we provide two examples of wave equation with non-instantaneous impulses and finite delay.

**Example 4.1** Let us consider the following wave equation:

\[
    \begin{aligned}
        \frac{\partial^2 y(t, \xi)}{\partial t^2} & = \frac{\partial^2 y(t, \xi)}{\partial \xi^2} + b(t) \frac{\partial y(t, \xi)}{\partial \xi} + \mu(t, \xi) + k_0 \cos \left(\frac{2\pi t}{T}\right) \sin(y(t - r, \xi)), \\
        t \in \bigcup_{i=0}^{N}(t_i, t_{i+1}] \subset J = [0, T], \ \xi \in [0, 2\pi], \\
        y(t, \xi) & = \rho_i(t, y(t_i^-, \xi)), \ \ t \in (t_i, s_i], \ i = 1, \ldots, N, \ \xi \in [0, 2\pi], \\
        \frac{\partial y(t, \xi)}{\partial t} & = \frac{\partial \rho_i(t, y(t_i^-, \xi))}{\partial t}, \ t \in (t_i, s_i], \ i = 1, \ldots, N, \ \xi \in [0, 2\pi], \\
        y(t, 0) & = y(t, 2\pi), \ t \in J, \\
        y(0, \xi) & = \varphi(0, \xi) + \int_{-r}^{T} g(s) \sin^2(y(s, \xi)) ds, \ \xi \in [0, 2\pi], \\
        \frac{\partial y(0, \xi)}{\partial t} & = \xi_0(\xi) + \sum_{j=1}^{q} c_j \cos(y(\tau_j, \xi)), \ -r < \tau_1 < \tau_2, \ldots, \tau_q < T, \ \xi \in [0, 2\pi], \\
        y(\theta, \xi) & = \varphi(\theta, \xi), \ \xi \in [0, 2\pi], \ \theta \in [-r, 0),
    \end{aligned}
\]

where $\varphi : [-r, 0] \times [0, 2\pi] \to \mathbb{R}$ is a piecewise continuous function, the function $\mu : J \times [0, 2\pi] \to \mathbb{C}$ is continuous in $t$ and satisfies $\mu(t, 0) = \mu(t, 2\pi)$, for each $t \in J$, and the function $b : J \to \mathbb{R}$ is continuously differentiable. The functions $g$ and $\rho_i$ for $i = 1, \ldots, N$ satisfy suitable conditions, which will be described later.
Let $X_p = L^p(T; \mathbb{C})$, for $p \in [2, \infty)$, be the space of $p$-integrable functions (Lebesgue) from $\mathbb{R}$ into $\mathbb{C}$ with period $2\pi$, where $T$ is the quotient group $\mathbb{R}/2\pi \mathbb{Z}$ and $U = L^2(T; \mathbb{C})$. We consider the operator $A_p z(\xi) = z''(\xi)$ with the domain $D(A_p) = W^{2,p}(T; \mathbb{C})$. The operator $A_p$ can be written as

$$A_p z = \sum_{n=1}^{\infty} -n^2 \langle z, w_n \rangle w_n,$$

where $-n^2 (n \in \mathbb{Z})$ and $w_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}$, are the eigenvalues and the corresponding normalized eigenfunctions of the operator $A_p$. Moreover, we can verify in a similar way as in [45] that the operator $A_p$ generate a cosine family $C_{0,p}(t), t \in \mathbb{R}$ on $X_p$ which is strongly continuous and the associated sine family $S_{0,p}(t), t \in \mathbb{R}$ on $X_p$ is compact. Thus, the condition $(H1)$ of Assumption 2.1 holds. Furthermore, the cosine and sine families can explicitly be written as

$$C_{0,p}(t) z = \sum_{n \in \mathbb{Z}} \cos(nt) \langle z, w_n \rangle w_n, \ z \in X_p,$$

$$S_{0,p}(t) z = t \langle z, w_0 \rangle w_0 + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n} \sin(nt) \langle z, w_n \rangle w_n, \ z \in X_p.$$

Let us define

$$F_p(t) z(\xi) = b(t) z'(\xi) \text{ on } W^{1,p}(T; \mathbb{C}).$$

It is easy to verify that the linear operator $A_p(t) = A_p + F_p(t)$ is closed. Next, we show that $A_p + F_p(t)$ generates an evolution family. For this, we consider the following scalar initial value problem:

$$\begin{cases} h''(t) = -n^2 h(t) + i nb(t) h(t), \\ h(s) = 0, \ h'(s) = w. \end{cases}$$

(4.2)

The solution of the above equation satisfies the integral equation

$$h(t) = \frac{w}{n} \sin[n(t-s)] + i \int_s^t \sin[n(t-\tau)] b(\tau) h(\tau) d\tau.$$

By using the Gronwall-Bellman lemma, we obtain

$$|h(t)| \leq \frac{|w|}{n} e^{\delta(t-s)}, \text{ for } s \leq t,$$

(4.3)

where $\delta = \sup_{t \in J} |b(t)|$. Let us denote the solution of the initial value problem (4.2) by $h_n(t, s)$, for each $n \in \mathbb{Z}$ and define
\[ S_p(t, s)z = \sum_{n \in \mathbb{Z}} h_n(t, s) \langle z, w_n \rangle w_n, \quad z \in X_p. \]  

(4.4)

The estimate (4.3) ensures that \( S_p(t, s) : X_p \to X_p \) is well defined and it satisfies the conditions of Definition 2.1.

Let us define

\[ x(t)(\xi) := y(t, \xi), \quad \text{for} \ t \in J \text{ and } \xi \in [0, 2\pi], \]

and the linear operator \( B : U \to X_p \) as

\[ Bu(t)(\xi) := \mu(t, \xi) = \int_0^{2\pi} K(\zeta, \xi)u(t)(\zeta) d\zeta, \quad t \in J, \ \xi \in [0, 2\pi], \]

where \( K \in C([0, 2\pi] \times [0, 2\pi] ; \mathbb{R}) \) and satisfy \( K(\xi, \xi) = K(\xi, \zeta) \). Assume that the operator \( B \) is one-one. Let us estimate

\[ \| Bu(t) \|^p_{X_p} = \int_0^{2\pi} \left| \int_0^{2\pi} K(\zeta, \xi)u(t)(\zeta) d\zeta \right|^p d\xi. \]

Applying the Cauchy-Schwarz inequality, we have

\[
\| Bu(t) \|^p_{X_p} \leq \int_0^{2\pi} \left[ \left( \int_0^{2\pi} |K(\zeta, \xi)|^2 d\zeta \right)^{1/2} \left( \int_0^{2\pi} |u(t)(\zeta)|^2 d\zeta \right)^{1/2} \right]^p d\xi \\
= \left( \int_0^{2\pi} |u(t)(\zeta)|^2 d\zeta \right)^{p/2} \int_0^{2\pi} \left( \int_0^{2\pi} |K(\zeta, \xi)|^2 d\zeta \right)^{p/2} d\xi.
\]

Since the kernel \( K(\cdot, \cdot) \) is continuous, we arrive at

\[ \| Bu(t) \|_{X_p} \leq C \| u(t) \|_U, \]

so that we get \( \| B \|_{L(U; X_p)} \leq C \). Hence, the operator \( B \) is bounded. Moreover, the symmetry of the kernel implies that the operator \( B = B^* \) (self adjoint). In particular, one can take \( K(\xi, \zeta) = 1 + \xi^2 + \zeta^2, \) for all \( \xi, \zeta \in [0, 2\pi] \).

Let us now define \( f : J_1 = \bigcup_{i=0}^N [s_i, t_{i+1}] \times D \to X_p \) as

\[ f(t, x_t)(\xi) := k_0 \cos \left( \frac{2\pi t}{T} \right) \sin(x_t), \quad \xi \in [0, 2\pi], \]  

(4.5)

where \( k_0 \) is some positive constant and \( D \) is given in (2.13). Clearly, \( f \) is continuous and

\[ \| f(t, x_t) \|_{L^p} = k_0 \left( \int_0^{2\pi} \left| \cos \left( \frac{2\pi t}{T} \right) \sin(x_t(\zeta)) \right|^p d\zeta \right)^{1/p} \]
\[ \leq k_0(2\pi)^{\frac{1}{p}} |\cos(2\pi t)| = \gamma(t). \]

It is clear that the function \( \gamma(t) = k_0(2\pi)^{\frac{1}{p}} |\frac{2\pi t}{2\pi}| \in L^1(J_1; \mathbb{R}^+) \cap L^2(J_1; \mathbb{R}^+) \)

Hence, the condition \((H2)\) of Assumption 2.1 and the condition \((H5)\) of Assumption 3.1 is satisfied.

Next, we define the impulse functions \( \rho_{i,p} : [t_i, s_i] \times \mathbb{X}_p \to \mathbb{X}_p \), for each \( i = 1, \ldots, N \) are defined as

\[ \rho_{i,p}(t, x)(\xi) := \int_0^{2\pi} g_{i}(t, \xi, z) \cos^2(x(t_i^-)z)dz, \]

where, \( g_i \in C^1([0, T] \times [0, 2\pi]^2; \mathbb{R}) \). It is not difficult to verify that the impulses \( \rho_{i,p}, \) for \( i = 1, \ldots, N \) satisfy the condition \((H3)\) of Assumption 2.1.

Finally, we consider the nonlocal initial conditions \( g_p, h_p : PC([-r, T]; \mathbb{X}_p) \to \mathbb{X}_p \) as

\[
g_p(x)(t) = g(y(t, \xi)), \quad \xi \in [0, 2\pi],
\]

\[
h_p(x)(t) = h(y(t, \xi)), \quad \xi \in [0, 2\pi].
\]

Let us choose

\[
g_p(y(t, \xi)) = \int_{-r}^{\tau} \varrho(s) \sin^2(y(s, \xi))ds, \quad \xi \in [0, 2\pi],
\]

\[
h_p(y(t, \xi)) = \sum_{j=1}^{q} c_j \cos(y(\tau_j, \xi)), \quad -r < \tau_1 < \tau_2, \ldots, < \tau_q < T, \quad \xi \in [0, 2\pi],
\]

where \( \varrho \in L^1([-r, \tau]; \mathbb{R}) \) and \( c_j \) for \( j = 1, \ldots, q \), are positive constants. Let \( \|\varrho\|_{L^1([-r, \tau]; \mathbb{R})} \leq l \). Hence, the functions \( g_p(\cdot) \) and \( h_p(\cdot) \) satisfy the condition \((H4)\) of Assumption 2.1, whenever the constant \( l \) and \( c_j \)'s are small enough (see [30] for more details) and it is also satisfy the condition \((H6)\) of Assumption 3.1.

Using the above notions, the system (4.1) can be expressed as (1.1a)–(1.1f) and it satisfies Assumption 2.1 \((H1)–(H4)\) and Assumption 3.1 \((H5)\). Finally, we verify that the linear problem corresponding to the system (1.1a)–(1.1f) is approximately controllable. To complete this, we consider (see Remark 3.1)

\[ B^* S_p(T, t)^* x^* = 0, \quad \text{for any } x^* \in \mathbb{X}^*, \quad t \in J. \]

By using definition of \( B^* \), we obtain

\[ \int_0^{2\pi} K(\xi, \xi) S_p(T, t)^* x^*(\xi)d\xi = 0, \quad \text{for any } x^* \in \mathbb{X}^*, t \in J. \]

Since \( K(\cdot, \cdot) \) is one-one, then we have

\[ S_p(T, t)^* x^* = 0 \quad \text{for all } t \in J. \]
The property of $S_p(T,t)^*$ also implies that
\[
\frac{\partial}{\partial t} S_p(T,t)^* x^* = 0 \text{ for all } t \in J.
\]

Using the condition (D1)-(b) in Definition 2.1, we obtain $x^* = 0$. Thus, the linear system is approximately controllable. Finally, Theorem 3.3 yield that the system (4.1) is approximately controllable.

Let us now consider another form of wave equation with non-instantaneous impulses and finite delay.

**Example 4.2** Consider the following system:

\[
\begin{aligned}
\frac{\partial^2 y(t, \xi)}{\partial t^2} &= \frac{\partial^2 y(t, \xi)}{\partial \xi^2} + b(t) y(t, \xi) + \mu(t, \xi) + k_0 \sin(y(t - r, \xi)), \\
& \quad t \in \bigcup_{i=0}^{N} (s_i, t_{i+1}] \subset J = [0, T], \ \xi \in [0, \pi], \\
y(t, \xi) &= \rho_i(t, y(t_i^-, \xi)), \ t \in (t_i, s_i], \ i = 1, \ldots, N, \ \xi \in [0, \pi], \\
\frac{\partial y(t, \xi)}{\partial t} &= \frac{\partial \rho_i(t, y(t_i^-, \xi))}{\partial t}, \ t \in (t_i, s_i], \ i = 1, \ldots, N, \ \xi \in [0, \pi], \\
y(t, 0) &= y(t, \pi) = 0, \ t \in J, \\
y(0, \xi) &= \varphi(0, \xi) + \int_{-r}^{T} \varrho(s)(1 + \cos^2(y(s, \xi))) ds, \ \xi \in [0, \pi], \\
\frac{\partial y(0, \xi)}{\partial t} &= \zeta_0(\xi) + \sum_{j=1}^{q} c_j (1 + \cos(y(\tau_j, \xi))), \ -r < \tau_1 < \tau_2, \ldots, \\
& < \tau_q < T, \ \xi \in [0, \pi], \ y(\theta, \xi) = \varphi(\theta, \xi), \ \xi \in [0, \pi], \ \theta \in [-r, 0),
\end{aligned}
\]

where $\varphi : [-r, 0] \times [0, \pi] \rightarrow \mathbb{R}$ is a piecewise continuous function, the function $\mu : J \times [0, \pi] \rightarrow \mathbb{R}$ is continuous in $t$ and the function $b : J \rightarrow \mathbb{R}$ is continuously differentiable. Also the functions $\varrho$ and $\rho_i$ for $i = 1, \ldots, N$ satisfy the same conditions as in the previous example.

Likewise Example 4.1, we prove the approximate controllability of the above system. Let $\mathbb{X}_p = L^p([0, \pi]; \mathbb{R})$, for $p \in [2, \infty)$ and $\mathbb{U} = L^2([0, \pi]; \mathbb{R})$. We consider the operator $A_p z(\xi) = z''(\xi)$ with the domain $D(A_p) = W^{2,p}([0, \pi]; \mathbb{R}) \cap W^{1,p}_0([0, \pi]; \mathbb{R})$. The operator $A_p$ can be written as

\[
A_p z = \sum_{n=1}^{\infty} -n^2 (z, w_n) w_n, \quad \langle z, w_n \rangle := \int_{0}^{2\pi} z(\xi) w_n(\xi) d\xi,
\]

where $-n^2 (n \in \mathbb{Z})$ and $w_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n \xi)$, are the eigenvalues and the corresponding normalized eigenfunctions of the operator $A_p$. Same as earlier, one can verify that the operator $A_p$ generate a cosine family $C_{0,p}(t), \ t \in \mathbb{R}$ on $\mathbb{X}_p$ which is strongly
continuous and the associated sine family $S_{0,p}(t), t \in \mathbb{R}$ on $\mathbb{X}_p$ is compact. Hence, the condition $(H1)$ of Assumption 2.1 holds. The generated cosine and sine families can be written as

$$C_{0,p}(t)z = \sum_{n=1}^{\infty} \cos(nt)\langle z, w_n \rangle w_n, \ z \in \mathbb{X}_p,$$

$$S_{0,p}(t)z = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nt)\langle z, w_n \rangle w_n, \ z \in \mathbb{X}_p.$$ 

We now define

$$F_p(t)z(\xi) = b(t)z(\xi) \text{ on } \mathbb{X}_p.$$ 

It is easy to verify that the linear operator $A_p(t) = A_p + F_p(t)$ is closed. Next, we show that $A_p + F_p(t)$ generates an evolution family. For this, we consider the following scalar initial value problem:

$$\begin{cases}
  h''(t) = -n^2h(t) + b(t)h(t), \\
  h(s) = 0, \ h'(s) = w.
\end{cases} \quad (4.7)$$

The solution of the above equation satisfies the integral equation

$$h(t) = \frac{w}{n} \sin[n(t - s)] + \frac{1}{n} \int_s^t \sin[n(t - \tau)]b(\tau)h(\tau)d\tau.$$ 

By using the Gronwall-Bellman lemma, we obtain

$$|h(t)| \leq \frac{|w|}{n} \delta e^{\delta(t-s)}, \text{ for } s \leq t, \quad (4.8)$$

where $\delta = \sup_{t \in J}|b(t)|$. Let us denote the solution of the initial value problem (4.7) by $h_n(t, s)$, for each $n \in \mathbb{Z}$ and define

$$S_p(t, s)z = \sum_{n \in \mathbb{Z}} h_n(t, s)\langle z, w_n \rangle w_n, \ z \in \mathbb{X}_p. \quad (4.9)$$

The estimate (4.8) ensures that $S_p(t, s) : \mathbb{X}_p \rightarrow \mathbb{X}_p$ is well defined and it satisfies the conditions of Definition 2.1.

Let us define

$$x(t)(\xi) := y(t, \xi), \text{ for } t \in J \text{ and } \xi \in [0, \pi],$$
and the linear operator \( B : \mathbb{U} \to X_p \) as

\[
Bu(t)(\xi) := \mu(t, \xi) = \int_0^\pi K(\zeta, \xi)u(t)(\zeta)\,d\zeta, \quad t \in J, \quad \xi \in [0, \pi],
\]

where \( K \in C([0, \pi] \times [0, \pi]; \mathbb{R}) \) and satisfy \( K(\zeta, \xi) = K(\xi, \zeta) \). Assume that the operator \( B \) is one-one. Next, we define \( f : J_1 = \bigcup_{i=0}^N [s_i, t_{i+1}] \times \mathcal{D} \to X_p \) as

\[
f(t, x_t)(\xi) := k_0 \sin(x_t), \quad \xi \in [0, \pi], \quad (4.10)
\]

where \( k_0 \) is some positive constant and \( \mathcal{D} \) is given in (2.13). It is not difficult to verify that the function \( f(\cdot, \cdot) \) satisfies the condition \((H2)\) of Assumption 2.1 and the condition \((H5)\) of Assumption 3.1.

Proceeding similar way as Example 4.1, one can obtain by Theorem 3.3 that the system (4.6) is also approximately controllable on \( J \).

5 Conclusions

In this paper, we established the approximate controllability of a non-autonomous second order impulsive evolution system (1.1a)–(1.1f) involving more general impulses referred to as non-instantaneous impulses. We first formulated the linear regulator problem and obtained the optimal control in the feedback form. Then the approximate controllability of the linear control system (3.2) has been obtained by using the optimal control. Further, we proved the existence of a mild solution of the non-autonomous second order semilinear control system (1.1a)–(1.1f) with a suitable control function (3.22). To prove these results, we utilized Schauder’s fixed point theorem. Then, we derived sufficient conditions for the approximate controllability of the system (1.1a)–(1.1f). The major drawback of the method used in this work is that we are not able to relax the assumptions on the nonlinear term \( f(\cdot, \cdot) \) (see Assumption 2.1 (H2)). Therefore, we are not able to include more general type of nonlinearities. Furthermore, the results of this work does not hold in general Banach spaces.

In future, we will try to focus our study on the approximate controllability for a class of nonlocal second order neutral impulsive evolution differential inclusions with non-instantaneous impulses in Banach spaces. One may consider the approximate controllability of a nonlocal autonomous fractional neutral differential inclusion of order \( \alpha \in (1, 2) \) with impulses also.

Acknowledgements  S. Arora would like to thank Council of Scientific and Industrial Research, New Delhi, Government of India (File No. 09/143(0931)/2013 EMR-I), for financial support to carry out his research work and Department of Mathematics, Indian Institute of Technology Roorkee (IIT Roorkee), for providing stimulating scientific environment and resources. J. Dabas would like to thank the Department of Atomic Energy (DAE), Mumbai, Government of India, project (file no-02011/12/2021 NBHM(R.P)/R&D II/7995).
References

1. Ahmed, H.M., El-Borai, M.M., El Bab, A.O., Ramadan, M.E.: Approximate controllability of non-instantaneous impulsive Hilfer fractional integrodifferential equations with fractional Brownian motion. Bound. Value Probl. 2020, 1–25 (2020)

2. Arora, S., Mohan, M.T., Dabas, J.: Approximate controllability of the non-autonomous impulsive evolution equation with state-dependent delay in Banach space. Nonlinear Anal. Hybrid Syst. 39, 100989 (2020)

3. Arora, S., Singh, S., Dabas, J., Mohan, M.T.: Approximate controllability of semilinear impulsive functional differential system with nonlocal conditions. IMA J. Math. Control Inform. 37, 1070–1088 (2020)

4. Arora, S., Mohan, M.T., Dabas, J.: Approximate controllability of a Sobolev type impulsive functional evolution system in Banach spaces. Math. Control. Relat. Fields (2020). https://doi.org/10.3934/mcrf.2020049

5. Arora, U., Sukavanam, N.: Approximate controllability of second order semilinear stochastic system with nonlocal conditions. Appl. Math. Comput. 258, 111–119 (2015)

6. Barbu, V.: Analysis and Control of Nonlinear Infinite Dimensional Systems. Academic Press, New York (1993)

7. Barbu, V.: Controllability and Stabilization of Parabolic Equations. Springer, New York (2018)

8. Benchohra, M., Henderson, J., Ntouyas, S.: Impulsive Differential Equations and Inclusions. Hindawi Publishing Corporation, New York (2006)

9. Bochenek, J.: Existence of the fundamental solution of a second order evolution equation. Ann. Polon. Math. 66, 15–35 (1997)

10. Byszewski, L., Lakshmikantham, V.: Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl. Anal. 40, 11–19 (1991)

11. Chen, F., Sun, D., Shi, J.: Periodicity in a food-limited population model with toxicants and state dependent delays. J. Math. Anal. Appl. 288, 136–146 (2003)

12. Colao, V., Muglia, L., Xu, H.K.: Existence of solutions for a second order differential equation with non-instantaneous impulses and delay. Ann. Mat. 195, 697–716 (2016)

13. Da Prato, G., Zabczyk, J.: Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Notes. Cambridge University Press, Cambridge (1996)

14. Ekeland, I., Turnbull, T.: Infinite-Dimensional Optimization and Convexity. The University of Chicago press, Chicago and London (1983)

15. Feckan, M., Wang, J.: A general class of impulsive evolution equations. Topol. Methods Nonlinear Anal 46, 915–933 (2015)

16. Fu, X.: Approximate controllability of semilinear non-autonomous evolution systems with state-dependent delay. Evol. Equ. Control Theory 6, 517–534 (2017)

17. Fu, X., Rong, H.: Approximate controllability of semilinear non-autonomous evolution systems with nonlocal conditions. Autom. Remote Control 77, 428–442 (2016)

18. Grudzka, A., Rykaczewski, K.: On approximate controllability of functional impulsive evolution inclusions in a Hilbert space. J. Optim. Theory Appl. 166, 414–439 (2015)

19. Guedda, L.: Some remarks in the study of impulsive differential equations and inclusions with delay. Fixed Point Theory 12, 349–354 (2011)

20. Henríquez, H.R.: Existence of solutions of non-autonomous second order functional differential equations with infinite delay. Nonlinear Anal. 74, 3333–3352 (2011)

21. Hernández, E., Henríquez, H.R., McKibben, M.A.: Existence results for abstract impulsive second order neutral functional differential equations. Nonlinear Anal. 70, 2736–2751 (2009)

22. Hernández, E., O’Regan, D.: On a new class of abstract impulsive differential equations. Proc. Am. Math. Soc. 141, 1641–1649 (2013)

23. Kisyński, J.: On cosine operator functions and one parameter group of operators. Stud. Math. 49, 93–105 (1972)

24. Kozak, M.: An abstract second order temporally inhomogeneous linear differential equation II. Univ. lagenl. Acta Math. 32, 263–274 (1995)

25. Kumar, A., Muslim, M., Sakthivel, R.: Controllability of the second order nonlinear differential equations with non-instantaneous impulses. J. Dyn. Control Syst. 24, 325–342 (2018)

26. Kumar, A., Vats, R.K., Kumar, A.: Approximate controllability of second order nonautonomous system with finite delay. J. Dyn. Control Syst. 26, 611–627 (2020)
27. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
28. Li, X.J., Yong, J.M.: Optimal Control Theory for Infinite-Dimensional Systems, Systems & Control: Foundations & Applications. Birkhäuser Boston Inc, Boston (1995)
29. Lin, Y.: Time-dependent perturbation theory for abstract evolution equations of second order. Stud. Math. 130, 263–274 (1998)
30. Liang, J., Liu, J.H., Xiao, T.J.: Nonlocal impulsive problems for nonlinear differential equations in Banach spaces. Math. Comput. Model. 49, 798–804 (2009)
31. Liu, S., Wang, J., O’Regan, D.: Trajectory approximately controllability and optimal control for non-instantaneous impulsive inclusions without compactness Topol. Methods Nonlinear Anal. 58, 19–49 (2021)
32. Lunardi, A.: On the linear heat equation with fading memory. SIAM J. Math. Anal. 21, 1213–1224 (1990)
33. Mahmudov, N.I.: Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. SIAM J. Control. Optim. 42, 1604–1622 (2003)
34. Mahmudov, N.I., Vijayakumar, V., Murugesu, R.: Approximate controllability of second order evolution differential inclusions in Hilbert spaces. Mediterr. J. Math. 13, 3433–3454 (2016)
35. Malik, M., Kumar, A., Feckan, M.: Existence, uniqueness, and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses. J. King Saud Univ. Sci. 30, 204–213 (2018)
36. Ntouyas, S.K.: Nonlocal initial and boundary value problems: a survey. In: Handbook of Differential Equations: Ordinary Differential Equations, vol. 2, pp. 461–557. Elsevier (2006)
37. Nunziato, J.W.: On heat conduction in materials with memory. Q. Appl. Math. 29, 187–204 (1971)
38. Obrecht, E.: Evolution operators for higher order abstract parabolic equations. Czechoslov. Math. J. 36, 210–222 (1986)
39. Obrecht, E.: The Cauchy problem for time-dependent abstract parabolic equations of higher order. J. Math. Anal. Appl. 125, 508–530 (1987)
40. Perelson, A.S., Neumann, A.U., Markowitz, M., Leonard, J.M., Ho, D.D.: HIV-1 dynamics in vivo: virion clearance rate, infected cell life-span, and viral generation time. Science 271, 1582–1586 (1996)
41. Pierri, M., Henríquez, H.R., Prokopczyk, A.: Global solutions for abstract differential equations with non-instantaneous impulses. Mediterr. J. Math. 49, 1685–1708 (2016)
42. Ravi Kumar, K., Mohan, M.T., Anguraj, A.: Approximate controllability of a nonautonomous evolution equation in Banach spaces. Numer. Algebra Control Optim. (2020)
43. Sakthivel, R., Anandhi, E.R., Mahmudov, N.I.: Approximate controllability of second order systems with state-dependent delay. Numer. Funct. Anal. Optim. 29, 1347–1362 (2008)
44. Serizawa, H., Watanabe, M.: Time-dependent perturbation for cosine families in Banach spaces. Houst. J. Math. 2, 579–586 (1986)
45. Singh, S., Arora, S., Mohan, M.T., Dabas, J.: Approximate controllability of second order impulsive systems with state-dependent delay in Banach spaces. Evol. Equ. Control Theory (2020). https://doi.org/10.3934/eect.2020103
46. Shu, L., Shu, X.B., Mao, J.: Approximate controllability and existence of mild solutions for Riemann–Liouville fractional stochastic evolution equations with nonlocal conditions of order $1 < \alpha < 2$. Fract. Calc. Appl. Anal. 22, 1086–1112 (2019)
47. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Hungar. 32, 75–96 (1978)
48. Travis, C.C., Webb, G.F.: Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. Houst. J. Math. 3, 555–567 (1977)
49. Travis, C.C., Webb, G.F.: Second order differential equations in Banach space. In: Nonlinear Equations in Abstract Spaces, pp. 331–361. Academic Press (1978)
50. Triggiani, R.: Addendum: a note on the lack of exact controllability for mild solutions in Banach spaces. SIAM J. Control Optim. 18, 98 (1980)
51. Triggiani, R.: A note on the lack of exact controllability for mild solutions in Banach spaces. SIAM J. Control Optim. 15, 407–411 (1977)
52. Vijaykumar, V., Udhayakumar, R., Dineshkumar, C.: Approximate controllability of second order nonlocal neutral differential evolution inclusions. IMA J. Math. Control Inform. (2020). https://doi.org/10.1093/imamci/dnaa001
53. Wei, W., Xiang, X., Peng, Y.: Nonlinear impulsive integro-differential equations of mixed type and optimal controls. Optimization 55, 141–156 (2006)
54. Yan, Z.: Approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay. Int. J. Control 85, 1051–1062 (2012)
55. Zuazua, E.: Controllability and observability of partial differential equations: some results and open problems. In: Handbook of Differential Equations: Evolutionary Equations, vol. 3, pp. 527–621 (2007)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.