A new model for the X-ray continuum of the magnetized accreting pulsars

Ruben Farinelli¹, Carlo Ferrigno², Enrico Bozzo², and Peter A. Becker³

¹INAF-Osservatorio Astronomico di Padova, Vicolo dell’Osservatorio 5, I-35122 Padova, Italy
e-mail: ruben.farinelli@oapd.inaf.it
²ISDC, chemin d’Écogia 16, 1290 Versoix, Switzerland
³Department of Physics and Astronomy, George Mason University, Fairfax, VA 22030

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ABSTRACT

Context. Accreting highly magnetized pulsars in binary systems are among the brightest X-ray emitters in our Galaxy. Although a number of high statistical quality broad-band (0.1-100 keV) X-ray observations are available, the spectral energy distribution of these sources is usually investigated by adopting pure phenomenological models, rather than models linked to the physics of accretion. Aims. In this paper, a detailed spectral study of the X-ray emission recorded from the high-mass X-ray binary pulsars Cen X-3, 4U 0115+63, and Her X-1 is carried out by using BeppoSAX and joined Suzaku + NuStar data, together with an advanced version of the COMPMAG model. The latter provides a physical description of the high energy emission from accreting pulsars, including the thermal and bulk Comptonization of cyclotron and bremsstrahlung seed photons along the neutron star accretion column.

Methods. The COMPMAG model is based on an iterative method for solving second-order partial differential equations, whose convergence algorithm has been improved and consolidated during the preparation of this paper.

Results. Our analysis shows that the broad-band X-ray continuum of all considered sources can be self-consistently modelled by using Gaussian components. From the fits of the COMPMAG model to the data we inferred the physical properties of the accretion columns in all sources, finding values reasonably close to those theoretically expected according to our current understanding of accretion in highly magnetized neutron stars.

Conclusions. The updated version of the COMPMAG model has been tailored to the physical processes that are known to occur in the columns of highly magnetized accreting neutron stars and it can thus provide a better understanding of the high energy radiation from these sources. The availability of broad-band high statistical quality X-ray data, as those provided by BeppoSAX in the past and NuStar plus other facilities in the present, is crucial to fully exploit the potentialities of the model. The close advent of the Astro-H mission, endowed with an unprecedented combination of high sensitivity and X-ray broad-band coverage, provides good perspectives to improve our understanding of accretion onto highly magnetized neutron stars through physical models like the one adopted here.

Key words. X-rays: binaries, magnetic fields, radiative transfer, pulsars: individual: Cen X-3, 4U 0115+63, Her X-1

1. Introduction

X-ray binary pulsars (XRBPs) were discovered more than forty years ago with the pioneering observations of the bright sources Her X-1 (Giacconi et al. 1971) and Cen X-3 (Tananbaum et al. 1972). The origin of the pulsed emission was soon understood to be related to the accretion of the donor star onto a rotating neutron star (NS). If the latter is endowed with a sufficiently strong magnetic field (B ≳ 10¹¹–¹² G), the infalling material from the donor star is halted at the magneto-spheric boundary of the compact object and funneled toward its magnetic poles, forming one or more accretion columns (Pringle & Rees 1972; Davidson & Ostriker 1973). As the NS magnetic field is known to decay over time, accretion columns are expected to be more extended in relatively young binary systems, as the high mass X-ray binaries (HMXB; see, e.g., Walter et al. 2013 for a recent review). Some binary systems with intermediate properties between HMXBs and low mass X-ray binaries (LMXBs) also show evidence for extended accretion columns (IMXBs; see, e.g., the case of Her X-1; Becker & Wolff 2007 hereafter BW07). In all these cases, the gravitational potential energy of the accreting material is first converted into kinetic energy within the accretion column, and then released in the form of X-rays as the plasma decelerates and settles onto the stellar surface. X-ray pulsations are generated owing to the disalignment between the NS magnetic and rotational axes. A peculiar feature that is observed in the X-ray spectra of many HMXBs and IMXBs is the cyclotron resonant scattering absorption line (CRSFs), which provide a direct measurement of the NS magnetic-field strength. The latter can be estimated by using the relation $E_{\text{cr}} = 11.6 B_{12}\times(1+z)^{-1}$ keV, where $E_{\text{cr}}$ is the centroid energy of the fundamental CRSF, $B_{12}$ is the NS magnetic field strength in units of $10^{12}$ G,
and $z$ is the gravitational redshift in the line-forming region (Wasserman & Shapiro 1983). In some cases, higher order harmonics of the fundamental CSRF are also observed at higher energies (see, e.g. Walter et al. 2013, and references therein).

The continuum broad-band X-ray emission of accreting X-ray pulsars is usually described by using phenomenological models, including an absorbed power-law extending up to $\sim 100$ keV with a roll-over at $\sim 30$–50 keV or a broken power-law (Orlandini 2006). In some cases it has been shown, however, that such simplified models are no able to satisfactorily describe high statistical quality observations (Perrigano et al. 2009, hereafter F09). The development of refined spectral models, linking the emission from X-ray pulsars to the physics of accretion, has been limited so far by the complexity of the radiative and dynamical equations that describe the behaviour of the accreting material under extreme gravity and magnetic field conditions. BW07 made a major step forward in this respect by elaborating an analytical approximation of the theoretical model for accretion onto highly magnetized NSs proposed previously by Davidson & Ostriker (1973).

Assuming the case of constant temperature and magnetic field in the emitting region, together with a simplified profile for the velocity of the radiation dominated flow in the accretion column, BW07 found an analytical solution to solve the Compton reprocessing of seed photons in this region (the latter being emitted either at the base of the accretion column or along the accretion stream). Through the definition of a proper Green function, these authors managed to approximate the complex magnetic bremsstrahlung emission of the electrons in the accretion flow (Riffert et al. 1999, hereafter F09) as a combination of a line emission at the cyclotron energy and an ordinary bremsstrahlung emission at lower energies. The angular-dependent cross section of the magnetic Compton scattering was then computed by using weighted averages. The BW07 model was first revised by Farnelli et al. (2012a, hereafter F12), relaxing the previous assumption on the velocity profile of the accreting material in the accretion column. These authors considered the case of a free-falling plasma settling onto the NS surface, and carried out the integration of the Green function along the accretion column. They limited their analysis to the case of low luminosity X-ray pulsars (e.g., the supergiant fast X-ray transients in quiescence; Walter et al. 2015), such that the bulk of the seed photons to be Comptonized can be assumed to originate only from the blackbody emission at the base of the column.

In this work, we further extended the revised approach of F12 by including in the Comptonization process the seed photons produced by the bremsstrahlung and cyclotron emissions along the accretion column. We also allow the value of the dipolar magnetic field to vary along the cylindrical column, thus exploring the more realistic situation in which the energy and the intensity of the cyclotron emission are directly linked to the height of the accretion column and the local density of the plasma. This improvement is motivated by the results of F09, who applied the BW07 model to the emission of the Be/X-ray binary 4U 0115+63 and found that the cyclotron emission is characterized by a magnetic field lower than the one estimated from the centroid energy of the fundamental CSRF. They suggested that this discrepancy might have been caused by a difference in the height of the regions producing the continuum cyclotron emission and the scattering feature along the accretion column.

In order to test the presently improved model, we used a number of observations collected with {	extit{BeppoSAX}} and the joined {	extit{Suzaku + NuStar}} facilities of the three sources Her X-1, Cen X-3, and 4U 0115+63, which are well known to possess prominent cyclotron lines in their X-ray spectra. The X-ray spectrum of 4U 0115+63 is also known to be characterized by the presence of a very strong and broad emission-like feature at 10 keV, i.e. right below the energy of the fundamental cyclotron absorption line (F09). This “10-keV feature” (Coakley et al. 2002) has been observed in a number of X-ray pulsars, and it is commonly modelled in the literature by using a broad Gaussian line (F09; Suchy et al. 2008 Vasci et al. 2013) which origin is not well understood. In our model, we show that the 10-keV feature can be reasonably well explained as the emission from the collisionally excited first Landau level (cyclotron emission) broadened by the Comptonization along the accretion column.

We present an overview of the improved {	extit{COMPMAG}} model in Sect. 2, summarize in Sect. 3 the properties of the X-ray pulsars Her X-1, Cen X-3, and 4U 0115+63 for which we carry out a detailed spectral analysis to test the model. In Sect. 4, we describe our data-set and provide some details on the data reduction procedures. All results are summarized in Sect. 5 and discussed in Sect. 6. The theoretical considerations for the seed photons emission terms and the accretion geometry are discussed in Sect. 7 and and Sect. 8 respectively. We provide our conclusions in Sect. 9.

2. The updated {	extit{COMPMAG}} model

The first version of the {	extit{COMPMAG}} model was presented by F12, and the main difference with respect to the spectral model developed by BW07 was related to the presence of the second-order bulk-Comptonization term in the energy diffusion space operator, together with the possibility to consider a more general velocity profile for the inflowing matter along the pulsar accretion column. The latter was approximated as cylindrical and permeated by a constant magnetic field. The seed photons for the Comptonization were assumed to be uniquely from the blackbody radiation at the bottom of the column and diffused through its height. The algorithm implemented by F12 to solve the Fokker-Planck approximation of the radiative transfer equation in the accretion column is based on a general finite-difference iterative procedure for the solution of partial differential equations with two variables.

The first version of the {	extit{COMPMAG}} model was applied to the case of the Supergiant Fast X-ray Transients, as these sources host pulsars which display usually a relatively low X-ray luminosity ($\approx 10^{33}$–$34$ erg s$^{-1}$; Farinelli et al. 2012b). As shown by BW07, in brighter accreting X-ray pulsars the contribution of the Comptonized blackbody is virtually negligible compared to that of the Comptonized bremsstrahlung and the cyclotron emission. The latter thus need to be considered if bright X-ray pulsars are studied. This extension is presented in this work and represents the key difference between the previous and the updated version of the {	extit{COMPMAG}} model.

For electrons embedded in a strong magnetic field, the motion in the direction perpendicular to field lines has discrete energy states (the Landau levels), while along the field...
lines the velocity is distributed according to a function $f(v)$. The electron distribution function $f(v)$ can be determined from the solution of the integro-differential Boltzmann equation with a particle diffusion term and a Coulomb scattering kernel (Riffert et al. 1999; hereafter R99). The proton-electron Coulomb interaction without photon emission tends to bring the electrons to a thermodynamical equilibrium situation, for which $f(v)$ is a Maxwellian at the proton temperature $T$. If the electron velocity is such that $v > \sqrt{2E_{\text{cyc}}/m_e}$, where $E_{\text{cyc}}$ is the cyclotron energy, the Coulomb scattering cross-section $\sigma(v \rightarrow v')$ presents some specific features (Neuebauer et al. 1990). The first one consists of two narrow peaks at $v' = \pm v$, where the one corresponding to forward scattering ($v' = v$) is many orders of magnitude higher than that of backward scattering ($v' = -v$); in this case there is no photon emission. The second feature, associated with photon emission, is a broad peak around the energies $E_{\text{el}} = \pm (E_{\text{el}} - E_{\text{cyc}})$, where $E_{\text{el}} = 1/2m_ev^2$ and $E_{\text{el}} = 1/2m_ev'^2$ are the electron energies after and before scattering, respectively. In the latter case the intermediate state of the Coulomb scattering brings the electron from the fundamental to higher Landau levels ($n > 1$), with a decay time which is much shorter than the particle relaxation time. The outcome is a significant loss of the initial electron energy, which causes a depopulation of the tail of the Maxwellian distribution that can be described by using a simple electron-proton Coulomb interaction. The energy threshold of this process implies that the higher is the electron temperature $kT_e$ compared to the magnetic field, the more efficient is the depopulation effect.

Numerical computations performed by R99 showed that the emissivity has a narrow energy peak at the cyclotron energy for photons emitted in the direction perpendicular to the magnetic field, while the peak gets broader for photons progressively emitted in the magnetic field direction. When averaging over all the possible directions, the net result can be qualitatively described as a broad cyclotron emission feature overlapped to a smooth continuum resembling a non-magnetized bremsstrahlung emissivity spectrum. However, a full self-consistent treatment is particularly difficult and hardly feasible in a run-time code (see Neuß, Magli & Lamb 1996, for a solution). A suitable approximation can be found by splitting the emissivity into two different seed photon sources (see also BW07), and this is the approach we follow in our treatment. Instead of modeling the cyclotron line emission with a delta function $\delta(E - E_{\text{cyc}})$, we assume a Gaussian profile with a centroid energy of $E_{\text{cyc}}$ and a variable width $\sigma_{\text{cyc}}$. The latter quantity parameterizes all the different effects leading to the intrinsic cyclotron line broadening, such as thermal broadening, spatial variations of the magnetic field not related to the dipolar structure, and the release of photons at all emission angles (this is of particular importance in case of the phase-averaged spectral analyses, as those we carry out in this paper; see Sect. 4).

The radiative transfer equation (RTE) in the observer frame for sub-relativistic bulk motion and arbitrary geometry has been first derived by Blandford & Payne (1981; hereafter BP81). This particular form of the RTE has then been widely used in the literature (e.g. Lyubarskii & Sunyaev 1982, Titarchuk et al. 1993, Farinelli et al. 2008). On the other hand, Psaltis & Lamb (1997; hereafter PL97) outlined how the result derived by BP81 was based on the wrong assumption that the ratio between the zeroth and the second moment of the specific intensity is equal $1/3$, which is strictly true only in the fluid frame where the bulk velocity is zero. The error introduced in the RTE when deriving it in the observer reference frame is at the first order comparable to the adimensional fluid velocity $\beta$ (see derivation in Appendix B). Under particular assumptions for the matter velocity profile, the RTE can be solved analytically by using the variable separation method for which the solutions are determined by the space boundary conditions for a bounded medium (Sunyaev & Titarchuk 1980, Lyubarskii & Sunyaev 1982, Titarchuk et al. 1997).

It is worth noting that analytical solutions of the RTE with bulk motion, even in the case of ad hoc velocity profiles, can be obtained if the energy diffusion operator depends only on the electron temperature, assumed to be constant through the medium. If second-order bulk Comptonization effects are taken into account, a space-dependent term containing a $\beta^2$ factor appears in the energy operator. In this case, the variable-separation method can not longer be adopted and solutions must be obtained through numerical methods. The contribution of the space dependent term to the solutions of the RTE has been investigated by, e.g., Psaltis & Lamb (1997); comparison between analytical and numerical solutions can be found in Titarchuk et al. 1997. As this term provides in all cases more reliable solutions to the RTE, although with an increasing effect depending on the magnitude of $\beta$, we included it in our model.

In the presence of a strong magnetic field ($B \gtrsim 10^{12}$ G), two polarization radiation modes arise as a consequence of the vacuum polarization (e.g. Ventura 1979). These two modes are commonly labelled as ordinary (O) and extraordinary (E). The approximate electron-scattering cross-section for O-photons is

$$\sigma_{\text{ord}}(E, \varphi) = \sigma_T[\sin^2 \varphi + k(E)\cos^2 \varphi], \quad (1)$$

where

$$k(E) \equiv \begin{cases} 1 & \text{for } E \geq E_{\text{cyc}}, \\ \left(E/E_{\text{cyc}}\right)^2 & \text{for } E < E_{\text{cyc}}. \end{cases} \quad (2)$$

The approximate cross-section of the E-photons is instead

$$\sigma_{\text{ext}}(E, \varphi) = \sigma_T k(E) + \sigma_1 \Phi_1, \quad (3)$$

where $\Phi_1$ is a normalized function of the line profile, and

$$\sigma_1 = 1.9 \times 10^6 \sigma_T B_{12}^{-1}. \quad (4)$$

In equations (1) and (3), $\varphi$ is the angle between the photon propagation direction and the magnetic field (Arons et al. 1987). Both O-photons and E-photons interact with the electrons of the plasma with an efficiency dictated by their relative cross-sections, but mode conversion also occurs at a given scattering rate (Arons et al. 1987). A system of two equations for both polarization modes containing the transition-rate term should thus be solved.

As outlined by BW07, the angular dependence of the cross-sections must be replaced by some angle-average values in order to treat the problem in analytical and numerical codes. Additionally, the explicit dependence on energy in equation (2) is neglected and instead we use energy-averaged cross-sections. On one hand, E-photons mostly escape from the bounded configuration, except for energies close to the cyclotron value and its harmonics where the
scattering cross-section becomes highly resonant (see equation [3]) and photons are substantially trapped in the line. On the other hand, O-photons are more efficiently scattered, particularly at orthogonal propagation angles with respect to the magnetic field (equation [1]). Building on the general approach of BW07, it has been assumed as an underlying assumption in the COMPAG model that the continuum Comptonization spectrum has to be computed only for O-photons, while the contribution of E-photons is phenomenologically taken into account by using Gaussian absorption features in the X-ray spectra.

The RTE for the zeroth-moment photon occupation number \(n(x)\) in the cylindrically symmetric case can be written as

\[
\frac{\partial n}{\partial t} - j_{\text{yc}}(x) - j_{\text{bs}}(x) = -v \frac{d}{dx} \frac{d n}{d z} + \frac{1}{3 v} x^2 \frac{d}{dx} \left[ \left( n + f_n \frac{d n}{d x} \right) \frac{d}{dx} \right] - \alpha_{\text{ff}}(x)n + \beta M \left[ \alpha_{\text{ff}}(x) + x \frac{d}{dx} \alpha_{\text{ff}}(x) \right] n, \tag{5}
\]

where \(x = E/kT_\parallel, \Theta = kT_\parallel/m_e c^2, f_n = 1 + m_e c^2/3kT, \beta = v/c, \sigma = 10^{-3} \sigma_T, \tau_T = 10^{-1} \sigma_T, \) and \(T_\parallel\) is the Thomson cross-section. Equation (3) can be easily derived from equation (18) of BP81 by adopting a few modifications. The first one is the addition of the second-order bulk Comptonization term \(f_n\) in the energy distribution operator, as previously discussed (see equation [33] of PL97 for its derivation). The second modification is the inclusion of the free-free absorption as expressed in the observer reference frame with an accuracy of order of \(v/c\) higher terms of the series expansion are of order of \(v^2/c^2\) and can thus be neglected (see equation [B4] of PL97). The free-free absorption was neither considered in the earlier treatment of F12, nor in the analytical treatment by BW07. The absorption coefficient can be written as

\[
\alpha_{\text{ff}}(x) = 4.2 \times 10^{-8} \frac{n_{19}^2 (\sigma^2 - 1) G(x)}{x^3 \Theta^7/2}, \tag{6}
\]

where

\[
G(x) = e^{-x/2} K_0(x/2) \tag{7}
\]

is the Gaunt factor and \(K_0(x/2)\) is the modified Bessel function of the second kind [Titarchuk 1988].

It is important to note that the expression for \(\alpha_{\text{ff}}\) used in equation (3) is for unmagnetized plasma. In the presence of a strong magnetic field, the behavior is different and depends on the photon polarization mode: for O-photons \(\alpha_{\text{ff}}\) is a smooth function of energy which can be well approximated by that of unmagnetized case. For E-photons, the Gaunt factor exhibits strong resonance at the cyclotron fundamental energy and its harmonics \(E_0 = n E_{\text{cyc}}\) [Pavlov & Panov 1976]. Since equation (3) deals with O-photons, the expression for the absorption coefficient in equation (6) is a good approximation, while the free-free absorption resonances of E-photons are not treated in our continuum model. To take into account their contributions in the source spectra, we use phenomenological Gaussian absorption line components. The E-photons mainly interact with the electrons via the cyclotron resonance, but this is effectively an elastic scattering process that preserves the angle-averaged photon distribution [Nagel 1980]. Hence no term describing this process should be included in the RTE.

The function \(M\) in the last term of equation (6) is the first moment occupation number, defined as (see BP81)

\[
M = \frac{1}{3} \left( \frac{\partial n}{\partial \tau} + \beta \frac{d n}{dx} \right), \tag{8}
\]

The addition of the photon escape term is linked to the solution of the RTE along the vertical direction. Actually, the system has a finite size also along the radial direction, and the rate of photon escaped through the walls of the accretion column must be taken into account. In principle, one should find solutions of the RTE considering both vertical and radial space coordinates [Davidson 1973]. However, together with the addition of the energy operator, the problem would become cumbersome and best suitable to be treated by Monte Carlo codes [Odaka et al. 2013, 2014]. We parametrize this effect using the characteristic photon time escape defined as \(t_{\text{esc}} = r_\tau \tau_\parallel/c\), where \(\tau_\parallel\) is the optical depth perpendicular to the magnetic field defined assuming a cross-section \(\sigma_\parallel = \sigma_T\).

As previously mentioned, we consider the bremsstrahlung and cyclotron emissions as separated sources of seed photons in equation (3) (see Appendix A). It is worth noting that the emission coefficient expressed in the observer frame for subrelativistic flows comprises a zeroth term plus correction factors of the order of \(v^2/c^2\) (see equation [B4] of PL97). As the accuracy of the equation (3) is \(\sim v/c\), with the exception of the Comptonization operator, we only retained the zeroth-order term for the emission coefficient.

Using the definition \(d \sigma = n_e \sigma_\parallel dz\) for the optical depth along the \(z\) axis, performing a logarithmic sampling of the adimensional energy through the change of variable \(x = e^\theta\), and defining \(n = J/x^3\) after some lengthy calculations, we obtain the adimensional equation for the zero moment intensity as

\[
\frac{\partial J}{\partial u} - S_{\text{cyc}}(q, \tau) - S_{\text{BS}}(q, \tau) = \left[ 1 + \frac{n_e v(r)^2}{3kT_e} \right] \frac{\partial^2 J}{\partial q^2} + \left[ e^\theta - 3 + \frac{m_e v(r)^2}{kT_e} + \beta \Phi(q) \right] \frac{\partial J}{\partial q} + \left[ e^\theta - 3 - \frac{\xi^2 \beta(r)^2}{H} - \Psi(q) - 3 \beta \Phi(q) \right] J + \frac{1}{3H} \frac{\partial^2 J}{\partial \tau^2} - \frac{\beta(r)}{H} + \Phi(q) \frac{\partial J}{\partial \tau}, \tag{9}
\]

where \(u = n_e \sigma_T c H, \xi = 15.5 \tau_\parallel/\tilde{m}, \beta = 1/(3H) dv/\partial \sigma, H = (\sigma/\sigma_T) \Theta\).

The parameter \(r_\parallel\) contained in the \(\tau\) term, is the adimensional column radius expressed in units of NS Schwarzschild radius through the relation \(R_0 = R_0^{\text{Sch}} \tilde{m} r_\parallel \) km, while \(\tilde{m} = M/M_{\text{Eddy}}\) is the adimensional accretion rate in Edington units.

The explicit form of the cyclotron seed photon term as a function of the \(x\) variable is

\[
S_{\text{cyc}}(x) = 1.7 \times 10^{-8} n_{19} H(x_{\text{cyc}}) e^{-x_{\text{cyc}} x^{-2}} B_{12}^{-3/2} \Theta^{-4} g(x), \tag{10}
\]
with \( x_{\text{cyc}} = E_{\text{cyc}}/kT \) and \( B_{12} = B/10^{12} \). We also used the normalized Gaussian function
\[
g(x) = \sqrt{\frac{2}{\pi \sigma_{\text{cyc}^2}}} e^{-(x-x_{\text{cyc}})^2/2\sigma_{\text{cyc}}^2},
\]
in place of the \( \delta \)-function corresponding to the monochromatic line emission case. The bremsstrahlung source term is
\[
S_{\text{bs}}(x) = 3.8 \times 10^{-13}g(x) n_{19} \Theta^{-9/2} e^{-x/\Theta},
\]
with \( G(x) \) given in equation (10). The derivation of the source terms in equation (10) and (12) is reported in Appendix A. The functions \( \Psi \) and \( \Phi \) in equation (9) are related to the zeroth and first order term of the free-free absorption coefficient and are defined as
\[
\Psi(x) = c_1 \frac{n_{19}(e^x - 1)}{x^3} x K_0(x/2),
\]
and
\[
\Phi(x) = \frac{n_{19} \beta}{3x^6} \left\{ K_0(x/2) \left[ c_1 x \cosh(x/2) \right] - c_2 \sinh(x/2) - c_1 K_1(x/2) \sinh(x/2) \right\},
\]
where \( c_1 = 2.1 \times 10^{-13} \) and \( c_2 = 4c_1 \).

Equation (9) is solved by using the numerical procedure described in F12. We considered as natural boundary conditions that: (i) the intensity vanishes at the extremes of the energy domain and at the top of the column; (ii) the approximation reported in equation (31) of F12 holds at the base of the accretion column. We assumed that the NS magnetic field within the accretion column is dipolar, and thus \( B_{12}(z) = B_{12}^{\text{ns}}(z/z_0)^{-3} \), where \( B_{12}^{\text{ns}} \) is the magnetic field strength at the base of the accretion column (i.e., the NS surface at \( z_0 \)). In the updated version of the COMPmag model being developed here, the free parameters regulating the cyclotron emissions to be determined through the fit to the X-ray data are thus \( B_{12}^{\text{ns}} \) and \( \sigma_{\text{cyc}} \). The additional improvement with respect to the former COMPmag version that we introduce here is the possibility to treat the height of the accretion column \( z_{\text{max}} \) as a free parameter. We also stress that in the new version of the model the mass accretion rate has been used as a free parameter in Xspec in place of the vertical optical depth adopted before (see also Table 2). The relation between these two quantities depends on the form of the velocity profile of the accreting material. Starting from the continuity equation
\[
\dot{M} = \pi R_0^2 m_p n_e |v_e|,
\]
and defining the first adimensional velocity profile as
\[
\beta = -\beta_0 \left( \frac{z}{z_0} \right)^\eta,
\]
one obtains that the optical depth measured at the top of the column is
\[
\tau_0 = 2.17 \times 10^{-3} \frac{n_{\text{cyc}}(z_{\text{max}}^{1+\eta} - z_0^{1+\eta})}{\mathcal{A} r_0^3(1 + \eta)},
\]
where \( \mathcal{A} = \beta_0 n_e \) while \( \beta_0 \) is the matter velocity in units of \( c \) at the NS surface, and \( z_0 \) is the adimensional NS radius. Given the definition of \( r_0 \), we obtain \( H = R_{15}^\text{cyc} m_20 \) km.

Here and throughout the paper we assumed \( m = 1.4 \) and \( R_{\text{ns}} = 10 \) km, implying \( z_0 = 2.42 \).

If the inflowing matter is assumed to decelerate towards the NS surface according to the law (see also equation [27] of BW07)
\[
\beta = -\frac{10.4 r_0}{m z_0} \gamma,
\]
then, the relations between \( \tau_0 \) and \( \dot{m} \) is given by
\[
\tau_0 = \frac{m z_0^{1/2} (z_{\text{max}} - z_0)^{1/2}}{50 r_0^{3/2}}.
\]

It is worth pointing out that the adimensional velocity of equation (15) is limited to be less than unity at the top of the column. In the above equation, \( \tau_0 \) is the approximated vertical optical depth calculated with the reduced Thomson cross-section for photons propagating along the lines of the magnetic field:
\[
\tau_0 = \int_{z_0}^{z_{\text{max}}} n_e \sigma_\parallel dz'.
\]
The above reported equations are implicitly derived under the assumption that the accreting column has a pure cylindrical shape with constant radius over height. Actually, for a dipolar magnetic field the column has a cone-like shape of half-angle \( \theta_c \) given by [Frank et al. 2002]
\[
\sin^2 \theta_c = \frac{R_e}{R_m} \sin^2 \psi,
\]
where \( R_e \) is the NS radius, \( R_m \) is the radius at which matter starts to be channeled by the magnetic field lines, and \( \psi \) is the inclination angle between the magnetic axis and the NS equatorial rotating plane.

Finally, the last important improvement we introduced in the new version of the COMPmag model is related to the geometry of the X-ray emission. We now differentiate between the pencil beam case, when the X-ray radiation is emitted upwards with respect to the NS surface, and the fan beam geometry, when the radiation is released from the lateral boundaries of the accretion column. Assuming a cylindrical accretion column, the adimensional photon flux in the pencil beam case is given by
\[
F^P(x) = \frac{4 \pi x^3}{3} \left( \frac{\partial n}{\partial \tau} + \beta^3 x \frac{\partial n}{\partial x} \right),
\]
where \( \beta^3 \) is the adimensional accreting matter velocity at the top of the column.

In order to pass to physical units, it is necessary to consider the relation between the intensity and the occupation number
\[
J(E) \approx 3 \times 10^{31} (kT_e/\text{keV})^3 x^3 n(x) \text{keVcm}^{-2} \text{s}^{-1} \text{keV}^{-1}\text{ster}^{-1},
\]
which can be easily derived once defining \( h\nu = xkT_e \) (see equation [43]). The COMPmag normalization \( N_{\text{comp}} \) is defined in this case as the ratio between the energy flux multiplied by the top-column surface \( S = \pi R_0^2 \), and divided by
4πD², where D is the source distance. Using equations (23) and (24), we obtain

\[ \mathcal{F}(x) \approx 0.02(kT_e/\text{keV})^3 \frac{r_0^2}{d_{10}^2} F_0(x) \text{ keV cm}^{-2} \text{ s}^{-1} \text{ keV}^{-1}, \]

(24)

where \( r_0 \) and \( d_{10} \) are the adimensional column radius and source distance in units of 10 kpc, respectively.

In the fan-beam case, the high optical depth allows the photons to diffuse and escape through the lateral boundaries of the accretion columns. As COMP MAG is a one-dimensional model and the variations of the physical quantities can be computed only along the vertical (z) and not the radial direction, we adopted some simplification. More specifically, considering the zeroth-moment specific intensity \( J(x, \tau) = x^3 n(x, \tau) \) at the walls of the column and assuming a uniform brightness \( B(x, \tau) = \pi J(x, \tau) \), the vertically-integrated adimensional photon flux is given by

\[ F^b(x) = \frac{P_{\text{cyl}}}{4\pi D^2} \int_0^{r_0} B(x, \tau) \frac{dz}{d\tau} \, d\tau, \]

(25)

where \( \frac{dz}{d\tau} \) can be easily derived by differentiating either equation (17) or equation (19), and \( P_{\text{cyl}} = 2\pi R_0 \) is the column perimeter.

From equation (23) and (24) we then obtain

\[ \mathcal{F}^b(x) \approx 9 \times 10^{-3}(kT_e/\text{keV})^3 \frac{r_0}{d_{10}} \int_0^{r_0} B(x, \tau) \frac{dz}{d\tau} \, d\tau \text{ keV cm}^{-2} \text{ s}^{-1} \text{ keV}^{-1}. \]

(26)

Note that we expressed the fluxes in equations (23) and (24) in keV because these are the standard units when writing Xspec models. The conversion to cgs units is performed by the fitting package at the end of the runs.

The COMP MAG normalization factor \( N_{\text{comp}} \) as derived in equations (23) and (24), deserves however some considerations. Once the source distance is fixed, the other parameters that determine the observed flux are \( kT_e \) and \( r_0 \), to be inferred from the fit to the X-ray data. This means that, in principle, \( N_{\text{comp}} \) should be kept fixed during the fit to a value of the order of \( \sim 1/d_{10}^2 \), where \( d_{10} \) is the distance of the source in units of 10 kpc. However, in the computation presented above (and also reported in BW07) we considered only the approximated case of a purely cylindrical accretion column and a flat space-time. In a more realistic situation, it is likely that hydrodynamical and General Relativity effects will affect the spectral energy distribution of the X-ray radiation. In particular, the gravitational-lensing effect, due to the curved space-time close to the NS surface, is expected to enhance the emerging X-ray flux at some angles compared to the Newtonian case. For this reason, the normalization \( N_{\text{comp}} \) should be proportional to \( f \times 1/d_{10}^2 \), where \( f > 1 \) is a numerical factor that parametrizes all the effects that can not be taken (yet) into account in the present version of the COMP MAG model.

2.1. New convergence procedure of the algorithm

In the first version of the COMP MAG model proposed by F12, the authors developed an empirical criterion to halt the iteration during the resolution of the radiative transfer equation in the pulsar accretion column. The code first estimated the approximated energy index \( \alpha \) from the fit to the source spectrum in the energy range \( E_{\text{min}} < E < E_{\text{max}} \), where \( E_{\text{min}} > 3kT_{\text{th}} \) and \( E_{\text{max}} \lesssim 10 \text{ keV} \). These boundaries were chosen in order to limit the fit to the region where the energy spectral distribution is better approximated with a power-law, and the iteration stopped when \( 1 - \alpha_{\text{fit}}/\alpha_{\text{mod}} < \varepsilon \). Additionally, the vertical optical depth \( \tau \) was always divided into \( N = 10 \) bins independently of its value to achieve a reasonably fast computing time.

This method was shown to be well suited to perform fits to the relatively low quality data that were available to F12. The data that will be used in the present paper to study some among the brightest X-ray pulsars in our Galaxy (see Sect. 3) require, instead, a significant improvement of the convergence procedure for the resolution of the radiative transfer problem in the accretion column. We verified that the presence of cyclotron emission features in the spectra of these sources makes the estimation of the spectral slope through a two-point linear interpolation largely uncertain, resulting in the non-convergence of the algorithm or an unsustainable high number of numerical iterations. We thus changed the initial differential approach of F12 to the convergence problem into an integral one. Defining \( u_{m}^{i,j} \) and \( u_{m+1}^{i,j} \) as the computed values of the source energy spectral distribution at the \( m \)-th and \( m + 1 \)-th iterations, with \( (i,j) \) identifying the steps in the pre-defined grid of energy and vertical optical depth, the new stopping criterion for the convergence of the algorithm can be expressed as

\[ |u_{m+1}^{i,j} - u_{m}^{i,j}| < \varepsilon \quad \forall i, l. \]

(27)

The maximum allowed value of \( \varepsilon \) is fixed by verifying at each iteration step that the \( \chi^2 \) obtained from the fit to the data is monotonically decreasing. We checked \textit{a posteriori} that \( \varepsilon = 10^{-4} \) allows to achieve the required accuracy in the estimate of the model parameters during all fits performed, maintaining at the same time a reasonably short computational time.

We also checked that for the new dataset used in this paper different values of the number \( N_\tau \) of bins selected for \( \tau \) resulted in remarkably different values of the model parameters obtained from the fits. After a number of trials, we found that fixing \( N_\tau = 50 \) provides an optimal solution, as higher values do not significantly alter the results of the fits and the computational time of each fit is still reasonably short (a few hours on a quad-core Linux machine with a 2.4 GHz processor).

3. The sources

3.1. Cen X-3

The high-mass X-ray binary Cen X-3 was discovered by Uhuru (Giacconi et al. 1971) and is known to host a \( \sim 4.8 \text{ s} \) spinning NS with an estimated mass of 1.34 \( M_\odot \) and its corresponding X-ray emission is predicted to be comparable to the brightest X-ray pulsars in our Galaxy (see Sect. 3). The system is \( \sim 2.1 \text{ days} \), and the NS is on a nearly circular orbit eclipsed by the massive companion for about 20% of the time (Hutchings et al. 1973; Ash et al. 1999).

The distance to the source is poorly constrained. A lower limit of 6.2 kpc was suggested by Krzeminski (1974), while Thompson & Rothschild (2009) derived more recently an
estimate of 5.7±1.5 kpc. For the purpose of our work and to ease comparisons with published works, we adopt hereafter a distance of 8 kpc. The system is known to undergo episodes of both disk and wind accretion, as shown by the positive and negative fluctuating derivative of the NS spin period (see e.g., Bildsten et al. 1997).

Prominent fluorescence Iron lines for different ionization states are present at 6.4, 6.7, and 6.97 keV in the X-ray spectrum of the source and have been reported at all orbital phases with variable intensities. Observations performed with the gratings on-board Chandra also evidenced the presence of Doppler broadened SiXIII, SiXIV, and FeXXV complexes caused by recombination in a photoionized plasma (Wojdowski et al. 2003; Iaria et al. 2005).

It has been argued that the fluorescent lines originate relatively close to the NS (probably in the outer regions of the accretion disk), while the other lines are produced at larger distances from the NS in the wind of the massive companion (Iaria et al. 2005; Naik et al. 2011). The stellar wind velocity measured from these latter lines is significantly smaller than that expected for a O-type star, providing convincing indications for a persistent perturbation of the wind by the NS X-ray radiation.

The source X-ray spectrum is generally well described by a power-law model with exponential cutoff and a Gaussian-like emission feature centred at ~13 keV (Suchy et al. 2008). A cyclotron absorption feature was detected at ~30 keV (Burderi et al. 2000). A partial covering component has also been introduced in some cases to model the variable column density of the source and interpret the hour-long dips in the source lightcurve (see, e.g., Naik et al. 2011). The X-ray variability of Cen X-3 ranges from time-scales comparable to the orbital period, down to fractions of the NS spin period. The spectral energy distribution also shows a remarkable dependence on the spin and orbital phase (see, e.g., Suchy et al. 2008; Raichur & Pandey 2008; Naik et al. 2011, and references therein).

3.2. 4U 0115+63

Hard X-ray radiation from 4U 0115+63 was discovered first in 1969 (Johns et al. 1978), during one of the giant type II outbursts displayed by the source. In that occasion, the peak luminosity of the event reached a few 10^{37} erg/s, two orders of magnitude above the usual quiescent emission level of the source. A periodicity of 3 years for the outbursts from 4U 0115+63 was later suggested by Whitlock et al. (1989). The source orbital period was measured by Rapaport et al. (1978) using SAS data at $P_{\text{orb}} = 24.3\,\text{d}$. These authors also estimated the eccentricity of the orbit as $e = 0.34$, and its semimajor axis $a \sin i = 140.1\,\text{h} - \text{s}$. X-ray pulsations with a period of $P_S = 3.6\,\text{s}$ were discovered by Cominsky et al. (1978).

The source has been widely studied in the optical and IR bands, leading to the identification of its optical counterpart (V 635 Cas; Johns et al. 1978) and the determination of its distance (7-8 kpc; Negueruela & Okazaki 2001). The X-ray spectrum of the source above ~10 keV is usually well described by power-law model with an exponential cut-off (e.g., Coburn et al. 2002). The centroid energy of the fundamental cyclotron line is at ~11 keV, and up to six harmonics have been observed and reported in the literature (Wheaton et al. 1978; White et al. 1983; Heindl et al. 1999; Santangelo et al. 1999, F09). This is a peculiar characteristic of 4U 0115+63, as in all other sources displaying cyclotron lines only the second harmonic is usually detected (the only exception being V 0332+53, which also showed evidences for the third cyclotron line harmonic: Coburn et al. 2002; Orlandini 2004; Tsygankov et al. 2006; Caballero et al. 2007). The variations of the centroid energy of the fundamental cyclotron line with the source luminosity were discussed by Mihara et al. (2004), Nakajima et al. (2006) and Tsygankov et al. (2007), but these results were later criticized by Müller et al. (2013). The latter authors showed that slightly different fits to the continuum emission could produce artificial (but significant) variations of the centroid energy. On the other hand, F09 presented the results of the fit to the broad-band X-ray spectrum of 4U 0115+63 with the BW07 model, aiming at constraining the properties of the X-ray emitting region and the high energy radiation mechanisms. These authors showed that the cyclotron emission has a dominant role in cooling the material that is flowing through the accretion column in this source, and inferred a magnetic field strength from the cyclotron continuum that is approximately a factor of 2 lower than the one estimated from the centroid energy of the fundamental cyclotron line. This suggested that the heights of the continuum and cyclotron line formation along the pulsar accretion column were substantially different.

3.3. Her X-1

Her X-1 hosts a NS with a spin period of 1.24 s and a mass of 1.5±0.3 M\text{\odot}, orbiting around an A/F donor star, HZ Her. The orbital period of the system is 1.7 d and the high inclination angle (∼85 degrees) with respect to the observer line of sight gives rise to extended X-ray eclipses, lasting about 20% of the binary orbit (Scott et al. 2000). The super-orbital modulation of the source X-ray emission is a periodicity of 34 d is usually ascribed to the presence of a warped accretion disc (Staubert et al. 2009, and references therein) or to the free precession of the NS-disc system (Staubert et al. 2013). During the super-orbital period, Her X-1 displays a "main-on" state lasting 10-11 days, during which the source achieves the highest X-ray luminosity, and a short slightly dimmer state which typically last 5 to 7 days. At other super-orbital phases, the source is still visible in the X-ray domain, although at a much lower luminosity level.

Her X-1 is characterized by a low Galactic absorption and is situated at a distance of 6.6±0.4 kpc (Reynolds et al. 1993). The source X-ray spectrum shows emission lines produced from the photoionized accretion disc and on the companion surface, as shown by Chandra and XMM-Newton grating observations (see, e.g., Zane et al. 2004; Jimenez-Garate et al. 2005; Li et al. 2009, and references therein). The fundamental CRSF was initially detected with a centroid energy of 40 keV (Truemper et al. 1977), but it has been shown that such energy decreased in the past 20 years by 4.2 keV (Staubert et al. 2013; Klochkov et al. 2015). The centroid energy is also known to positively correlate with the overall source X-ray flux over a wide range of different timescales (from seconds to days; Staubert et al. 2007; Klochkov et al. 2011). This behaviour has been interpreted in terms of the so-called sub-critical accretion regime, during which the height of the cyclotron scattering forming region decreases at higher mass accretion rates (Becker et al. 2012). This conclusion was criticized
by Mushtukov et al. (2013). These authors showed that, at the luminosity of Her X-1 where the cyclotron line is typically monitored, the source should be in the super-critical accretion regime, and thus an opposite trend of the cyclotron centroid energy with the source luminosity should be observed. An alternative possible solution to this problem is that the magnetic field of Her X-1 is strongly non-dipolar, resulting in multiple accretion sites on the NS surface with non-cylindrical sections (Shakura et al. 1991). It is worth noticing that a the combination of a pencil and a fan-beamed emission from a misaligned dipole would also be able to reproduce the pulse profiles of Her X-1 with a reasonable accuracy (Leahy 1981). The latter conclusion was also independently confirmed by Blum & Kraus (2009), who used a pulse profile deconvolution method to infer the pattern of the X-ray emission close to the NS surface and the geometrical properties of the corresponding emitting region.

More detailed studies of the source pulse profile were presented by Vasco et al. (2013) and Enoto et al. (2008). The first authors analyzed RXTE observations of Her X-1 and found a puzzling double peak in the variation of cyclotron line energy and photon index before and in correspondence of the source pulse maximum. Enoto et al. (2008) analyzed a 60 ks-long Suzaku observation of the source and reported on the possible detection of the second cyclotron harmonic close to the maximum of the pulse profile. A extensive analysis of three combined Suzaku/NuSTAR observations was carried out by Fürst et al. (2013), hereafter F13), who used a phenomenological model to describe the source emission during the main-on state.

4. Observations and data analysis

In order to properly test the improved version of the COMPmag model presented in this paper, we made use of a set of high-statistical quality data, providing a broad and continuous coverage in the energy range 0.5-100 keV.

Our sample exploits archive BeppoSAX (see Boella et al. (1997), for a description of the mission) data of Cen X-3 and 4U 0115+63, together with a combined NuSTAR (Harrison et al. 2013) and Suzaku (Mitsuda et al. 2007) observation of Her X-1 during one of the source main-on state (see Sect. 3). We summarize all characteristics of our dataset in Table 1, together with the references to previously published papers that made use of these data. Note that the two Cen X-3 observations reported in Table 1 were obtained with the source at different luminosities and we removed the time intervals corresponding to the source eclipse before carrying out any further analysis.

Concerning the BeppoSAX observations, we analyzed data from the high energy narrow field instruments LECS (0.7-4.0 keV, Parmar et al. 1997), MECS (1.5-10.5 keV, Boella et al. 1997b), HPGSPC (7-44 keV, Manzo et al. 1997) and PDS (15-100 keV, Frontera et al. 1997) after performing the full processing through the standard pipeline (saxdas v.2.1, running only on heasoft v. 5.2). Background subtraction was performed by using the earth occultation for the HPGSPC, the off-source pointing for the PDS, and the standard calibration files for the MECS and the LECS. The LECS and MECS source extraction radii were set to 8' to optimize the signal to noise ratio.

We adopted standard reduction procedures for the Suzaku and NuSTAR data, devoting a particular attention in evaluating the effect of pile-up during the Suzaku/XIS observation. As in the latter case only the outer portion of the instrument chip was active, we made use of data exclusively collected through the optimally calibrated XIS3 (see Fürst et al. 2013, for further details). Considering the scope of this paper, we limited our analysis to the 3-8.5 keV energy range for the Suzaku/XIS, 16-80 keV for the Suzaku/HXD, and 3.5-65 keV energy range for NuSTAR. Any spectral contribution coming from the accretion disk and the surrounding corona was thus excluded from further analysis. We only considered for each source spectra averaged over the entire spin period and did not perform a phase-resolved spectral analysis.

All spectral analyses were performed with the Xspec version 12.8 (Arnaud 1996), using the COMPmag model described in Sect. 2. Inter-calibration constants were included in the fits and found to be compatible with typically expected values. The fundamental and all detected harmonics of the CRSF features in each of the considered source were accounted for in the fits by using the GABS model in Xspec:

\[
G(E) = \exp \left( -\frac{\tau_n}{\sigma_n \sqrt{2\pi}} \right) \sqrt{\frac{E}{E_{\text{rms}}}} \left( 1 - \frac{E_{\text{rms}}}{E} \right)^{2} \right) .
\] (28)

Here \( \tau_n \), \( E_{\text{rms}} \), and \( \sigma_n \) denote the optical depth, energy, and standard deviation of the absorption for the \( n \)-th harmonic, respectively. The line width is fixed to previously measured values whenever it cannot be robustly determined by the fit. For Cen X-3 and Her X-1, we have linked the intensity of the magnetic field in the COMPmag model used to reproduce the continuum to the centroid energy of the fundamental CRSF; for 4U 0115+63 it was linked to the first harmonic, as this is known to be less affected by biases in the continuum modeling of the source (Müller et al. 2013).

5. Results

The updated version of the COMPmag model that we introduced in the present paper permits to compute the spectra of X-ray pulsars emerging in both a pencil-beam or a fan-beam geometry. The former case can be obtained by using equation (16) for the velocity profile of the infalling material in the pulsar accretion column and equation (24) to describe the emerging X-ray flux. The parameter \( \beta_0 \) defines in this geometry the terminal velocity of the infalling material at the NS surface. The corresponding equations for the fan-beam geometry are (13) and (20).

A preliminary fit to the available spectra considered in this work revealed that reasonably good values of the \( \chi^2 \) could be obtained with all the COMPmag parameters left free to vary, but the uncertainties on these parameters were not very constraining. We verified that this is largely due to the computation in the model of the matter velocity profile inside the accretion column, as the latter determines the optical depth of the column in the vertical direction and thus in turns the total source luminosity. As all sources in this paper are characterized by a luminosity that is far above the value at which the fan-beam emission geometry

\footnote{See https://heasarc.gsfc.nasa.gov/docs/sax/abc/saxabc/saxabc.html.}

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Table 1. Log of the observations of the sources analysed in this paper.

| Source          | Start Time [UT] | Stop Time [UT] | EXPOSURES [ks] |
|-----------------|-----------------|----------------|----------------|
| Cen X-3 (LL)    | 1996-08-14 12:38| 1996-08-14 23:46| 4.3            |
| Cen X-3 (HL)    | 1997-02-27 19:45| 1997-02-28 11:00| -              |
| 4U 0115+63(Obs. I) | 1999-03-19 17:05 | 1999-03-20 08:42 | 3.3            |
| 4U 0115+63(Obs. II) | 1999-03-26 17:31 | 1999-03-27 17:34 | 5.1            |
| Her X-1         | 2012-09-22 04:39 | 2012-09-22 18:33 | 22.6           |

Notes. (a) Unpublished BeppoSAX data.
(b) The same data set was studied by Burderi et al. (2004). As for the present paper, these authors excluded the time intervals corresponding to the source eclipse before performing any analysis. LECS data were not available for this observation due to telemetry problems.
(c) An analysis of these data is reported in Fürst et al. (2013).
(d) A study of the source hard X-ray spectrum extracted from this observation is reported by Santangelo et al. (2009).
(e) A study of the broadband spectrum, as derived from these data, is reported by F09.

is expected to set-in, we first fixed the normalization of the COMPMAK model (Ncomp) and concentrate on the fan-beam case with efficient matter deceleration along the accretion column due to the strong radiation pressure (see equations (15) and (21)). This case the comparison with both the previous theoretical treatment of the model reported in BW07 and its first application discussed by F09. For completeness, however, we also provide in the next sessions a discussion about the results of the fits to all data with the COMPMAK model and the pencil-beam geometry.

In all cases, we modelled the effect of the interstellar photoelectric absorption on the X-ray emission from each of the considered sources with the PHABS spectral component in Xspec, assuming Solar abundances.

5.1. Cen X-3

We first fit the spectra of Cen X-3 obtained from the two available BeppoSAX observations (LL and HL); see Table 1 with the COMPMAK model in the fan-beam geometry case, plus a cyclotron Gaussian absorption line centered at ≃30 keV. The results of the fits were not yet acceptable (χ²/dof=449/318 and χ²/dof=675/241 for the LL and HL case, respectively) due to evident residuals around 6.7 keV. The addition of a Gaussian emission line at this energy significantly improved the results of both the LL (χ²/dof=509/315) and HL (χ²/dof=167/238) observations.

In the case of the LL observation, some residuals were still visible around 5 keV and below 2 keV. The former are most likely due to a known feature of the MECS instrument and could be modelled by adding a Gaussian component with a fixed width of σ = 0.1 keV. This improvement led to χ²/dof=447/311. The residuals below ≃2 keV observed in the LECS data (not available during the HL observation) were interpreted them in terms of a second region of absorption partially covering the source. This possibility has been already investigated by Naik et al. (2011), during the phase-resolved spectroscopy of the source performed on some Suzaku data. The authors found a strongly-variable local absorption component with N_H ranging from ≃ 2 × 10^{22} to ≃ 1.3 × 10^{23} cm⁻², and a covering fraction in the range 0.3 ± 1. Following the above mentioned work, we added a multiplicative partial covering absorption component (PCfabs) to our Xspec model and obtained a significantly improved result (χ²/dof=369/311) with N_H ≃ 5 × 10^{22} cm⁻² and a covering fraction of ≃ 0.4.

In the HL spectrum, this component was not necessary to obtain an acceptable fit (see Fig. 1), possibly due to a higher ionization of the absorbing medium and/or by a weaker constrain of the absorption in the data set, due to the lack of LECS data.

Although the above discussed model provided a reasonably good fit to both the available observations of Cen X-3, we obtained from the COMPMAK best fit parameters a similar accretion rate in both observations. This result does not seem to be consistent with the eight times higher X-ray luminosity of the second observation (see Table 2). We verified that this computational issue is due to the degeneracy in the fits between the COMPMAK normalization Ncomp and the mass accretion rate ˙m, as they both regulate the source luminosity (see Sect. 2). To avoid this degeneracy, we performed a new fit to the LL data with the COMPMAK normalization fixed to the value obtained from the fit to the HL data (Ncomp = 60). The fit with this model provided χ²/dof=449/312, a slightly worse value than that obtained with a free to vary normalization, but with the accretion rate parameters ˙m now tracing reasonably well the source luminosity variation between the two observations (see Table 2). No significant changes of the N_H and the covering fraction parameters in the PCfabs components were altered by fixing the COMPMAK normalization. It is noteworthy that the determined value of the N_H is comparable to that measured by Naik et al. (2011) during their phase-averaged spectral analysis (see their Table 1). The most significant change was recorded for the mass accretion rate estimated from the fit, that is now in rough agreement with that expected from the ratio of the X-ray luminosities measured from the LL and HL observations (see Table 2). For completeness, we also tried to fix the normalization of the COMPMAK model in the fit of the HL data to the value obtained from the original fit to the LL data (with Ncomp left free to vary). However, in this case, we could not obtain an acceptable fit to the HL data. From the values of the different parameters obtained through our best fit models.
To perform a fit to the HL and LL data with the pencil-beam geometry (assuming a free-fall velocity profile for the material within the accretion column), we have to introduce the flow terminal velocity $\beta_0$ at the NS surface as an additional model parameter. We found that the fits to the data from both the HL and LL observations were unable to provide reasonable constraints to all model parameters. The situation did not improve by fixing the COMPmag normalization, as different values of $N_{\text{comp}}$ resulted in virtually equivalent $\chi^2$. By performing fits with values of $N_{\text{comp}}$ comprised between few tens and a few thousands, we found that a tight relation exists between this parameter, the accretion rate $\dot{m}$, and the radius of the column $R_0$, i.e. $\dot{m} = a_1 N_{\text{comp}}^{-b_1}$ and $R_0 = a_2 N_{\text{comp}}^{-b_2}$ (as expected according to the parameter definitions; see Sect. 2). We measured the slopes $b_1 = 1.12 \pm 0.03$ and $b_2 = 0.44 \pm 0.01$ from the fits to the LL observation, $b_1 = 0.94 \pm 0.02$ and $b_2 = 0.49 \pm 0.04$ from the HL one. By assuming for the pencil-beam geometry the same value of $R_0$ measured from the fan-beam geometry, we can use the above relations and coefficients to estimate the mass accretion rate in the former case. We obtained $\dot{m} \sim 1.6$ and $\dot{m} \sim 1.9$ for the LL and HL observation, respectively. In the pencil-beam geometry, the spectral parameters related to the cyclotron feature and the continuum are consistent (to within the uncertainties) between the LL and HL observations. The electron temperature is slightly higher for both observations than that measured 

(Fig. 3, we measured a fraction of about 80% and 30% for the Comptonized cyclotron emission of the LL and the HL observation, respectively (see Fig. 1 and Table 2).

To perform a fit to the HL and LL data with the pencil-beam geometry (assuming a free-fall velocity profile for the material within the accretion column), we have to introduce the flow terminal velocity $\beta_0$ at the NS surface as an additional model parameter. We found that the fits to the data from both the HL and LL observations were unable to provide reasonable constraints to all model parameters. The situation did not improve by fixing the COMPmag normalization, as different values of $N_{\text{comp}}$ resulted in virtually equivalent $\chi^2$. By performing fits with values of $N_{\text{comp}}$ comprised between few tens and a few thousands, we found that a tight relation exists between this parameter, the accretion rate $\dot{m}$, and the radius of the column $R_0$, i.e. $\dot{m} = a_1 N_{\text{comp}}^{-b_1}$ and $R_0 = a_2 N_{\text{comp}}^{-b_2}$ (as expected according to the parameter definitions; see Sect. 2). We measured the slopes $b_1 = 1.12 \pm 0.03$ and $b_2 = 0.44 \pm 0.01$ from the fits to the LL observation, $b_1 = 0.94 \pm 0.02$ and $b_2 = 0.49 \pm 0.04$ from the HL one. By assuming for the pencil-beam geometry the same value of $R_0$ measured from the fan-beam geometry, we can use the above relations and coefficients to estimate the mass accretion rate in the former case. We obtained $\dot{m} \sim 1.6$ and $\dot{m} \sim 1.9$ for the LL and HL observation, respectively. In the pencil-beam geometry, the spectral parameters related to the cyclotron feature and the continuum are consistent (to within the uncertainties) between the LL and HL observations. The electron temperature is slightly higher for both observations than that measured
absorbed COMP Maggie model in the fan-beam geometry, plus two absorption Gaussian lines at energies corresponding to the fundamental and the second harmonic of the CRSF. This gave $\chi^2$/dof=710/422 and $\chi^2$/dof=753/526 for Obs. I and II, respectively. In both cases, we noticed a systematic excess at energies $\lesssim$ 2 keV together with an absorption feature around $\sim$ 35 keV (the latter were more significant in Obs. I). We thus first added a partial covering component to the previous spectral model (PCfabs in Xspec), leading to $\chi^2$/dof=518/420 and $\chi^2$/dof=598/524 in Obs. I and II, and then added to the model a further Gaussian gaussian line to take into account the presence of the third CRSF harmonic; we obtained $\chi^2$/dof=444/418 for Obs. I and $\chi^2$/dof=583/522 for Obs. II, respectively. In both cases, the Gaussian widths of the third CRSF could not be constrained during the fit and thus we fixed this parameter to $\sigma = 2$ keV, in agreement with the measured width of the second CRSF harmonic. In Obs. I, we additionally noticed some residuals from the fit around $\sim$45 keV, where the fourth CRSF harmonic is usually detected (see, e.g. F09). Adding a fourth Gaussian line to the spectral model used for Obs. I led to a further improvement in the fit with $\chi^2$/dof=363/415.

No evidence of the Kα iron emission line was found in the spectra of Obs. I and II. We report all the best determined parameters of the spectral fits described above in Table 2. The unfolded spectra and the residuals from the fits are shown in Fig. 2. We also show in Fig. 4 the separated contributions of the Comptonized cyclotron and bremsstrahlung emission to the total X-ray spectra of 4U 0115+63; the former component is about 60% and 70% in Obs. I and II, respectively.

Concerning the applicability of the COMP Maggie model in the pencil-beam geometry to 4U 0115+63 (assuming a free-fall velocity for the material inside the accretion column), the same considerations of Cen X-3 apply (see Section 5.1). From the fit to the source spectrum in Obs. I, we determined $b_1 = 1.06 \pm 0.02$ and $b_2 = 0.52 \pm 0.01$ for the two relations $n \propto N\text{comp}^{-a}$ and $R_0 \propto N\text{comp}^{-b}$, respectively. The values measured from the fit to the source spectrum of Obs. II are instead $b_1 = 0.93 \pm 0.02$ and $b_2 = 0.48 \pm 0.01$. The corresponding values of the mass accretion rate for a column radius $R_0$ equal to the value obtained in the fan-beam geometry are $\sim 0.76$ and $\sim 0.52$. Interestingly, the accretion rate ratio results equal to the luminosity ratio of the two observations (see Table 2). At odds with the case of Cen X-3, the flow terminal velocity and column height determined for the two 4U 0115+63 observations remain roughly constant, with $b_0 \sim 0.2$ and $z_{\text{max}} \sim 3$ km.

5.3. Her X-1

A fit with an absorbed COMP Maggie model, together with a cyclotron absorption line, to the combined Suzaku and NuStar observation of Her X-1 provided an unacceptable result ($\chi^2$/dof=7387/3732) due to a strong emission line around 6.4-6.7 keV. The addition of an emission line at these energies led instead to a formally acceptable fit ($\chi^2$/dof=3909/3729; see Table 2). It is worth noting that F13 modelled similar residuals by using two narrow lines with centroid energies of 6.4 and 6.5 keV, respectively (the estimated line widths were $\sigma_1 \sim 0.1$ keV). One of the two lines was included mainly to smear-out the sharp drop of the adopted phenomenological cut-off power-law model,
and thus the second line is likely to have a model-dependent rather than a physical origin.

Even though the fit was formally acceptable, some residuals were still visible around $\sim 10$ keV. These residuals were also noticed by F13, who argued that they might have been caused by some instrumental effect and excluded the energy interval 10-12 keV from the spectral analysis. The fact that the same residuals emerged by using a complete different spectral model, convinced us that they could have instead a physical origin. We found that the addition of an iron absorption edge at 9.1 keV correctly accounted for the residuals around 10 keV and slightly improved the fit ($\chi^2$/dof = 3769/3727). The estimated contributions of the cyclotron and bremsstrahlung Comptonized component to the total X-ray emission of the source in the 3-100 keV energy range were 60% and 40%, respectively. We report all values of the best fit model to the data of Her X-1 in Table 2 and show the unfolded spectrum with the residuals from the fit in Fig. 8. The high statistical quality of the Her X-1 observations allowed to constrain reasonably well all parameters of the COMPMAG model, providing the most accurate test of the model available so far.

Concerning the details of the CRSF in Her X-1, it is noteworthy that the width of this feature as measured by F13 was found to be dependent on the specific continuum model adopted by the authors and varied between 6 and 8 keV. The width of the CRSF estimated through the usage of our COMPMAG model is compatible with the value measured by F13 when a high-energy cut-off (HIGHECUT in xspec) is used to characterize the data. The latter is also found by F13 to be the most statistically preferable spectral model. The centroid energy of the CRSF measured by F13 and in the current paper are only marginally different ($E_{\text{cyc}} = 37.7 - 39.1$ keV) and $E_{\text{cyc}} = 37.2 \pm 0.2$ keV, respectively.

Finally, in the case of Her X-1 it was not possible to find a formally acceptable fit to the data by using the COMPMAG model in the pencil-beam geometry and considering a free-fall velocity profile for the material within the accretion column.

### 6. Discussion

In this paper, we introduced an updated version of the COMPMAG model, which permits to physically compute the X-ray luminosity emerging from a highly magnetized accreting pulsar in the fan-beam and pencil-beam geometry.

Two particularly critical parameters of the model are the normalization $N_{\text{comp}}$ and the mass accretion rate $\dot{m}$. In a purely Newtonian case, the mass accretion rate entirely regulates the source X-ray luminosity and it is expected that the normalization of the COMPMAG model can be fixed to a constant proportional to the source distance (see Sect. 5). In a more realistic case, the X-ray emission produced relatively close to the NS surface will be strongly affected by General Relativistic (GR) effects and the normalization of the COMPMAG model is no longer trivially connected to the source distance. As in this work the COMPMAG model was tested against observational data in which GR effects cannot be disentangled from the local X-ray emission, we left $N_{\text{comp}}$ free to vary in the fits and determined its value for the different sources from the data. A better definition of $N_{\text{comp}}$ would require the extension of the COMPMAG model to include a full GR treatment of the X-ray emission from the pulsar, but this is outside the scope of the present paper.

The simplified approach adopted here could be successfully applied to the currently best available broad-band X-ray observations of three bright X-ray pulsars: 4U 0115+63, Cen X-3, and Her X-1. We used publicly available BeppoSAX archival data for Cen X-3 and 4U 0115+63, and a recent combined Suzuki +NuStar observation of Her X-1. In all cases, the broad-band coverage of the data ensured a detailed characterization of the source spectrum and provided the possibility to perform tests of the COMPMAG model. In the case of 4U 0115+63, the fit performed to the two BeppoSAX observations with $N_{\text{comp}}$ and $\dot{m}$ left free to vary provided fully self-consistent physical results. We obtained a ratio of $\sim 1.3$ between the mass accretion rate in Obs. I and Obs. II, and a ratio of $\sim 1.8$ between the corresponding values of $N_{\text{comp}}$. For comparison, the ratio between the X-ray luminosities recorded during these observations is $\sim 1.5 (0.1-100$ keV energy range; see Table 2). On one hand, the difference in the mass accretion rate evaluated though COMPMAG thus fully reflect the variation of the source luminosity across the two data-sets. On the other hand, the difference in the pulse profile of the source that was reported previously among these two observations (Santangelo et al. 1999, F99) also justify the measured variation in $N_{\text{comp}}$, as some change has certainly occurred in the pattern of the X-ray radiation arising from the pulsar accretion column. A similar conclusion applies to the case of Cen X-3, for which a ratio of $\sim 7.3$ was measured from the fits with COMPMAG between the mass accretion rates in the LL and HL observations and the correspondingly known ratio of the X-ray luminosity is $\sim 8$ (see Table 2). Note that in this case the value of $N_{\text{comp}}$ of the LL observation had to be fixed to that of the HL observation in order to avoid degeneracies between this parameter and $\dot{m}$ in the spectral fits (see Section 5). A similar check was not possible on Her X-1, as in this case only a single observation was available. However, the COMPMAG model could provide also for this source a reasonably good fit to the broad-band X-ray data, giving indications about the structure of the accretion column (see Table 2). Generally, we remark that a comparison between the values of $\dot{m}$ determined from COMPMAG for different systems should be carried out with caution, as uncertainties on the distances to these sources and intrinsic differences in the accretion geometry could lead to systematic biases. Values of $\dot{m}$ are more easily comparable between different observations of a single source, as in this case the uncertainty on the distance cancels out and variations in this parameters can be directly linked to the physical processes intervening in the release of the X-ray emission.

There are two major limitations in the current version of the COMPMAG model. The first is related to the crude approximation of the scattering cross-section, which is described by parallel and perpendicular components to the magnetic field direction ($\sigma_\parallel$ and $\sigma_\perp$, respectively). The former appears in the Fokker-Planck treatment of the radiative transfer equation through the vertical spatial photon diffusion term, while the latter is related to the photon escape time across the side walls of the accretion column. The cross-section $\sigma$, related to the photon diffusion in the energy space and representing the average scattering direction of photons parallel and perpendicular to the magnetic field, is also crudely approximated. These issues were al-
Table 2. Best fit models for the sample of sources analysed in this paper. For the COMPMMAG model, the emission geometry defined in equation \(20\) combined with velocity profile in equation \(15\) is assumed.

| Source          | Cen X-3 (LL) | Cen X-3 (HL) | 4U 0115+63 (Obs. 1) | 4U 0115+63 (Obs 2) | Her X-1 |
|-----------------|--------------|--------------|---------------------|--------------------|--------|
| Parameter       |              |              |                     |                    |        |
| \(N_H^a\)       | 0.65±0.13    | 0.71±0.07    | 1.12±0.07           | 1.19±0.09          | [10\(^{-4}\)] |
| \(N_H^b\)       | 6.3±0.9      | –            | 5.3±1.0             | 4.3±0.9            | –      |
| Cov. Drag.      | 0.76±0.02    | –            | 0.54±0.03           | 0.51±0.04          | –      |
| \(kT_e\) (keV)  | 1.83±0.04    | 1.43±0.03    | 0.77±0.05           | 0.89±0.04          | 3.09±0.10 |
| \(m_e\)         | 0.83±0.07    | 1.1±0.2      | 0.92±0.13           | 0.47±0.10          | 4.02±0.30 |
| \(B_{12}^c\)    | 2.62±0.09    | 2.51±0.06    | 0.99±0.01           | 0.97±0.01          | 3.22±0.02 |
| \(\sigma_{syc}\) (keV) | [3]       | [3]          | [3]                 | [3]                | [3]    |
| \(R_0\) (km)    | 1.02±0.05    | 1.72±0.06    | 0.44±0.03           | 0.34±0.03          | 1.70±0.12 |
| \(H\) (km)      | 7.3±0.4      | 3.4±0.2      | 2.5±0.1             | 2.6±0.1            | 4.6±0.3  |
| \(N_{comp}\)    | 65±5         | –            | 4.1\(1.1\)\times10\(^3\) | 4.4\(2.0\)\times10\(^3\) | 13±2 |

**Cyclotron absorption features**

| \(E_f\) (keV)   | 30±1         | 29.1±0.7    | 12.2±0.1            | 11.9±0.1           | 37.2±0.2 |
| \(\sigma_f\) (keV) | [3]       | [3]          | [3]                 | [3]                | [3]    |
| \(N_f^a\)       | 9±2          | 7.4±1.7     | 0.9±0.1             | 0.6±0.1            | 9.5±3.1  |
| \(E_s\) (keV)   | –            | –           | 23.0±0.2            | 22.6±0.1           | –      |
| \(\sigma_s\) (keV) | –            | –           | 2.1±0.4             | 2.0±0.2            | –      |
| \(N_s^a\)       | –            | 1.3±0.3     | 1.3±0.3             | 1.3±0.2            | –      |
| \(E_s\) (keV)   | –            | –           | 35.2±0.6            | 36.0±1.7           | –      |
| \(\sigma_s\) (keV) | –            | –           | 3.6±1.7             | 3.6±1.7            | –      |
| \(N_s^a\)       | –            | –           | 2.0±0.7             | 2.0±0.7            | –      |

**Iron emission line**

| \(E_{Fe}\) (keV) | 6.58±0.01    | 6.60±0.03   | –                   | –                   | 6.61±0.03 |
| \(\sigma_{Fe}\) (keV) | 0.33±0.02    | 0.31±0.05   | –                   | –                   | 0.35±0.10 |
| \(N_{Fe}^a\)     | 5.2±0.2      | 5.3\(3.0\)  | –                   | –                   | 5.7\(1.7\) |
| \(F_{cyc} / F_{cont}\) | 0.78        | 0.28        | 0.58                | 0.67                | 0.62 |
| \(L_X^{b,h}\)    | 1.2          | 11.0        | 11.7                | 7.8                 | 4.6    |

\(\chi^2/\text{dof}\) 449/312 167/238 363/415 583/522 3769/3727

**Notes.** (a) Absorption of the PFRSB model in units of 10\(^{22}\) cm\(^{-2}\). (b) Absorption of the pCFRR model in units of 10\(^{22}\) cm\(^{-2}\). (c) In units of 1.5 \times 10\(^{17}\) m gr s\(^{-1}\), with m = 1.4. (d) In units of 10\(^{12}\) G, and linked to the centroid of the scattering features: the fundamental for Her X-1 and Cen X-3, the first harmonic for 4U 0115+63. (e) Normalization corresponding to the parameter \(\tau_0\) in equation \(25\). (f) In units of 10\(^{-3}\) photons cm\(^{-2}\). (g) Computed in the energy range 0.5-100 keV for Cen X-3 and 4U 0115+63. (h) 3-100 keV for Her X-1, and corrected for absorption. (h) In units of 10\(^{37}\) erg s\(^{-1}\) assuming a distances of 8 kpc, 7.5 kpc, and 6.5 kpc for Cen X-3, 4U 0115+63, and Her X-1, respectively.

Note that BW07 treats the parallel and average cross-sections as implicit model parameters, while in the COMPMMAG model we fix them to a suitable values.
Finally, we discuss the usage of the RTE (equations [3] and [9]). In the present case, the RTE is written in the static reference frame of the observer with an accuracy of order $\beta$, the adimensional bulk velocity of the accreting material. It is important to outline that for a deccelerated velocity profile as in equation (18), the electron density progressively increases towards the NS surface. As the combined continuum bremsstrahlung plus narrow cyclotron emission is proportional to $n_e^2$ (see equations [A.1] and [A.2]), most of the seed photons are produced in the region where $\beta \to 0$. Corrections between the reference frames of the fluid and the observer are thus minimal. The same consideration holds for the escape time of photons and the Comptonization efficiency: both processes are more efficient in a high electron-density environment. This also significantly reduces the uncertainty in equation (4) introduced by the usage of the relation $K/J = 1/3$ between the second and zeroth moment of the intensity field. The latter would strictly hold only in the fluid reference frame (see Appendix [B]).

6.1. Comparison with earlier models

As already reported in Section [2] and with more details in F12, there are a number of differences between the current implementation of the COMMAG model and the former analytical treatment presented by BW07. So far, the only application of the BW07 model to a broad-band spectrum of an X-ray pulsar was presented by F09 using BeppoSAX data of 4U 0115 + 63. We thus compare below the results of the present paper for the source 4U 0115 + 63 with those reported in F09.

A first important difference between the two treatments is that the BW07 model was proved unable to provide a satisfactorily fit over the entire energy range spanned by the X-ray spectrum of 4U 0115 + 63. F09 showed that an additional thermal Comptonization component was needed below ~9 keV to correctly describe the data, while the higher energy part could be well interpreted in terms of Comptonized cyclotron emission. A broad Gaussian line had also to be included in the spectral model to achieve an acceptable result, but the origin of this feature could not be easily explained. F09 also showed that the magnetic field strength derived from the centroid energy of the cyclotron line was significantly different from that inferred from the fit to the continuum, and the values of $n_e$ and $R_0$ had to be fixed in the fit ($6 \times 10^{16}$ g s$^{-1}$ and 600 m, respectively) due to unresolved degeneracies of the different model parameters. As we showed in the previous sections, the COMMAG model is able to consistently fit the broad-band spectra of the different sources without the need of including additional components (besides the cyclotron absorption features).

It should be remarked that the lower electron temperature of the bremsstrahlung seed component measured in the COMMAG and BW07 models ($kT_e \sim 8$ keV in F09 and $kT_e \sim 0.7$ keV in the present work) is known to arise from the different approximations of the Compton scattering: BW07 considered only the first-order bulk Comptonization term in the energy-diffusion operator, while we introduced also the second-order term. The latter component, being space-dependent, can not be included when performing the analytical treatment of Comptonization through the variable-separation method (as adopted by BW07). It is a general feature of the thermal plus bulk Comptonization models that the inclusion of higher-order Compton bulk terms lead to lower electron temperatures for the same spectrum (Titarchuk et al. 1997; Psaltis & Lamb [1997]).

In our analysis of the X-ray emission from 4U 0115 + 63, we did not find any statistically significant evidence of the fifth harmonic of the cyclotron line, that was instead reported by F09. We note, however, that the parameters of the cyclotron lines are known to be strongly affected by the choice of the X-ray continuum and thus the detection of particularly weak wiggles in the high-energy end of an exponentially decaying spectrum can easily turn out to be statistically non-significant in different models.

7. Source term emission

In this section we justify the assumption of a combined thermal bremsstrahlung and cyclotron emission (the latter approximated through a Gaussian function) to generate the seed photons used for the Comptonization in the COMMAG model. We mentioned in Sect. [1] that this approximation was originally proposed by BW07, given the current impossibility of providing a self-consistent analytical description of the magnetic bremsstrahlung emission produced by the electrons in the accretion flow. While implementing the BW07 treatment of the problem in an XSPEC model, F09 adopted for the cyclotron term of the seed photons an exponentially attenuated $\delta$-function. These authors thus assumed that the width of the cyclotron line is intrinsically narrow at the origin and is then broadened due to thermal and bulk Comptonization effects. In the treatment presented in this paper, we used a Gaussian profile instead of a $\delta$-function in order to take into account the natural width of the cyclotron emission feature (the COMMAG model parameter describing this effect is $\sigma_{cyc}$ in equation [10]). The latter is indeed expected to result from both thermal broadening and magnetic field mixing instabilities.

7.1. Bremsstrahlung emission

The fits to the data of the X-ray pulsars considered in this paper with the COMMAG model provided an electron temperature of order of few keV (see Tab. [2]). It is straightforward to verify for each source that the fraction of electron with velocities $|v| > \sqrt{2E_{cyc}/m_e}$ is very small (less than 1%) in both the cases of 1D and 3D Maxwellian distributions. The effect of tail depopulation because of the resonant electron-proton interaction with photon emission, and the consequent deviation of $f(v)$ from a Maxwellian distribution is thus very marginal. Moreover, as previously mentioned in Section [2] the angular integration of the bremsstrahlung emission smears-out narrow features arising from the effect of the viewing angle, resulting in a spectrum which fairly well agrees with the three-dimensional classical case. We thus concluded that splitting the bremsstrahlung emission process in a strong magnetic field into a smooth continuum similar to the unmagnetized case plus a narrow cyclotron feature is a reasonably good assumption.

It shall be noticed that, beside the proton-electron bremsstrahlung, there is a second channel for the
production of cyclotron emission photons due to the photon-electron resonant scattering (RS; see, e.g., Canuto et al. 1971; Herold 1976; Daugherty & Harding 1986; Harding & Daugherty 1991). In principle, for a thermal plasma with a temperature of a few keV, only a small fraction of the electrons fulfill the condition $E_{\text{el}} > E_{\text{cyc}}$. The presence of a significant cyclotron feature in the X-ray data (see Fig. 1) would thus support the idea that the $\gamma$-RS dominates over the e-p RS. In reality, a quantitative estimate of the relative contribution provided by the two processes is very difficult. The bremsstrahlung spectra reported by R99 were computed in the case of an optically thin plasma, where the e-p processes dominate over the e-$\gamma$ interactions. A more realistic solution to the problem would require a self-consistent treatment where both processes are taken simultaneously into account, employing a non-linear iterative solution scheme. The possibility of exchanging the bremsstrahlung and cyclotron terms for the seed photons in equation 5 indicates our inability in developing such a scheme.

7.2. Cyclotron emission

Let us now consider the broadening of the cyclotron emission feature due to magneto-hydrodynamical effects. First, we emphasize that the cyclotron emission term in the COMPmag model (equation 10) is self-consistently weighted over the vertical profile of the accretion column, as we considered $B_z = B_0(z_0/z)^3$. An intrinsic broadening of the cyclotron emission line energy $E_{\text{cyc}}$ is thus possible due to geometrical effects (i.e., the distance from the NS) and the assumption on the vertical emissivity profile of the accretion column. The vertical gradient of the magnetic field does not provide, however, a dominant contribution to the broadening of the cyclotron emission feature, as the total height of the column where the overall emission process takes place is relatively small (see Tab. 2) and cannot lead to $\sigma_{\text{cyc}}$ appreciably larger than zero. A similar conclusion holds for the radial gradient of the magnetic field. The exact expression of a dipolar magnetic field in the vertical direction using cartesian coordinates is

$$B_x = \mu \left[ \frac{2z^2}{(R^2 + z^2)^{5/2}} - \frac{1}{(R^2 + z^2)^{3/2}} \right],$$

where $\mu$ is the magnetic moment, and $R^2 = x^2 + y^2$. The assumption $B_z \propto z^{-3}$ considered so far is thus valid only in the asymptotic limit $R \ll z$. However, it is easy to demonstrate that the variations of both $B_z$ and $|B|$ from the center ($R = 0$) to the walls of the accretion column ($R \ll 1$ km) are negligible and cannot contribute significantly to $\sigma_{\text{cyc}} > 0$.

Magnetohydrodynamical simulations of hyper-critical accretion regimes onto a magnetized star have shown that significant fluctuations of the magnetic field intensity in the vertical direction can be produced as a consequence of field submerging effects by the copious inflowing material in the accretion column (Bernal et al. 2010). Numerical simulations of plasma accreted on the NS surface also show that interchange instabilities contribute to increase the gradient of the magnetic field intensity especially at the base of the accretion column (Mukherjee et al. 2013a, b). We thus argue that all these effects can easily give rise to the complex magnetic field structures that are needed to produce the observed broad cyclotron emission feature close to the NS surface.

It is also interesting to discuss the relative contribution of the Comptonized cyclotron and bremsstrahlung components to the total emission of the analyzed sources, even though the two terms for the production of the seed photons for the Comptonization should be mutually linked in a more realistic treatment (see Sect. 2 and the discussion earlier in this section). As shown by Neugebauer et al. (1993) through detailed calculations, if the energy of an electron moving along a magnetic field line is higher than the cyclotron energy, the proton-electron Coulomb cross-section has four resonances. The first two occur at $\omega = \pm \nu$, where $\nu$ and $\nu'$ are the velocities of the electron before and after the scattering, respectively. In this case, the p-e scattering process does not lead to the emission of a photon and the forward scattering probability is much higher than the backward one due to the high proton mass (making this a nearly pure elastic scattering). The other two resonances occur at $\omega = \pm \nu'$, such that $E_{\gamma} = (E_\text{el} - E_{\text{cyc}})$. Here $E = 1/2m_\text{e}v^2, E' = 1/2m_\text{p}v'^2$, and the $\pm$ sign refers to the forward and backward scattering of the electron, respectively. In this case, the parallel velocity component (and thus the energy) is transferred to the quantized perpendicular electron Landau level, from which the electron decays releasing a cyclotron photon within a characteristic time

$$t_{\text{cyc}} \sim 10^{-19} \left( \frac{B_{\text{crit}}}{B} \right)^2 s,$$

where $B_{\text{crit}} = 4.41 \times 10^{13}$ G. The p-e scattering leading to the emission of a photon is the most interesting process from an observational point of view. In particular, the resonance occurs because the electron in the intermediate state (before the decay) is generally in the excited Landau levels with $n = 1$. A p-e scattering leading to a photon emission and an electron in the excited Landau level with $n = 0$ during the intermediate state is also a possibility, but the photons released after the decay would have in this case a lower energy than the cyclotron value. This case is usually considered similar to a zero-magnetic-field bremsstrahlung emission. From the represented Comptonized cyclotron and bremsstrahlung emissions in Fig. 4 we conclude that for all sources considered in this work the bulk of the bremsstrahlung emission is produced at energies lower than the cyclotron emission, in line with the above discussion. The contributions of the two components to the total emission of each source are similar, as expected in case of p-e Coulomb scatterings where the electrons are populating (for the cyclotron component) or not (for the bremsstrahlung component) the Landau levels $n \geq 1$ during their intermediate excited state.

It is also worth noting the strong correlation between the electron temperature and the magnetic field intensity derived from the fits with the COMPmag model to the data of all sources (Fig. 5). The Spearman correlation coefficient of the dataset is $R = 0.9$, and the relation between the two parameters can be analytically described by the function $kT_e = a + bT_c^2$, with $a = 0.81 \pm 0.10, b = 0.01 \pm 0.008$, and $c = 4.61 \pm 0.70$. As the electron temperature in the COMPmag model determines the exponential roll-over of the spectra, the correlation we found is most likely related to the scaling law between the cut-off energy and the centroid energy of the cyclotron emission lines that was dis-
covered thanks to Ginga and RXTE data through fits with phenomenological models (see, e.g., Makishima et al. 1999; Coburn et al. 2002). The advantage of the COMPmag model in this case is that it can provide a physical interpretation of this correlation. In particular, we note that the electron temperature is about an order of magnitude lower than the cyclotron energy, which provides the maximum temperature at which electrons can be heated. As discussed earlier in this section, this is due to the collisional excitation of the Landau levels, which limits the electron Maxwellian distribution below the cyclotron energy. As reported by Arons et al. (1987), the cyclotron emission is an effective cooling channel for the material in the accretion column (together with the Compton cooling). This is also the reason why there is an important contribution of the cyclotron emission term in the spectra of all X-ray pulsars analyzed in this paper. Arons et al. (1987) also showed that the electron temperature is expected to be generally lower than the average photon energy, because photons have the largest heat capacity. This can be easily verified for 4U 0115+63, Her X-1, and Cen X-3 by using the results reported in Table 2.

While the relation between the magnetic field intensity and the electron temperature is thus well understood in terms of the physical processes regulating the accretion in highly magnetized X-ray pulsars, the possible correlations we identified in Sect. 5 between the different geometrical parameters of the COMPmag model (i.e., $R_0$ and $z_{\max}$) should instead be taken with caution. These relations were derived by assuming a purely Newtonian treatment of the X-ray emission process close to the NS surface, where GR effects (e.g., light bending and lensing) can significantly modify the values of geometrical quantities between the emission and the observer frame. In order to be properly understood and interpreted, the same correlations should be investigated by introducing a ray-tracing code providing solutions for the equations describing the geodesics of the light emerging from the accretion column. This is, however, outside the scope of the present paper.

**Fig. 5.** Electron temperature as a function of the magnetic field at the base of the column for the sample of accreting X-ray pulsars analyzed in this paper. The data values are reported in Table 2 and can be fit by a powerlaw (see text for more details).

8. **Geometry of the emission region**

In modelling the source spectra, we have considered both a pencil-beam geometry with an accelerating velocity profile in the accretion column, and a fan-beam geometry, where matter decelerates towards the NS surface. The first case is expected to occur in less luminous sources ($L_X < 10^{37}$ erg s$^{-1}$), where the radiation pressure generated by the soft photons from the NS surface is too low to halt the infalling matter in the accretion column and the radiation is preferentially emitted from the top of the column (fan-beam). At higher luminosities ($L_X > 10^{37}$ erg s$^{-1}$), the strong radiation pressure from the NS surface contrasts the pressure of bulk inflow, leading to the formation of a radiative shock above the NS surface. In this case, the energy dependency of the electron-photon scattering cross section leads to substantially different behaviours below and above the cyclotron line energy. Below the cyclotron energy, the optical depth perpendicular to the magnetic field is higher than in the parallel direction. The Comptonized radiation thus emerges preferentially through the lateral walls of the accretion column (fan-beam), while part of the weakly reprocessed radiation can still be emitted along the column. Above the cyclotron energy, the cross sections in both directions become comparable, increasing the emission through the lateral walls of the column. The COMPmag model is best suited to fit phase-averaged spectra, as in this case the effects described above are averaged out and the treatment of the cross sections in the RTE approximation is justified.

All the sources considered in this paper have X-ray luminosities $L_X > 10^{37}$ erg s$^{-1}$, and thus the usage of the fan-beam geometry in the COMPmag model was considered more appropriate to fit their X-ray spectra. We have shown that in principle fits with the COMPmag model and the pencil-beam geometry are possible for Cen X-3 and 4U 0115+63, while no solution could be found in the case of Her X-1. In the pencil-beam case, the empirical dependence $R_0 \propto N_{\text{comp}}$ discussed in Sect. 5.1 and 5.2 is derived from the definition of the COMPmag normalization for the top-column emission (see equation (22)). Indeed, the $N_{\text{comp}}$ parameter explicitly depends on the emitting area at the top of the accretion column as well as on the magnification factor $f_{\text{geom}}$. The latter parametrizes all the (unknown) geometrical and GR effects that are not taken into account in the present version of the COMPmag model. The inverse proportionality $\dot{m} \propto N_{\text{comp}}^{-1}$ shows that the COMPmag model correctly takes into account a linear dependence between the source flux and the estimated mass accretion rate. For a given value of the source X-ray flux, a decrease of $N_{\text{comp}}$ has to be compensated by a corresponding linear increase in $\dot{m}$. We verified that the changes in $\dot{m}$ and $R_0$ derived as a consequence of a variation in the model normalization, $N_{\text{comp}}$, are sufficient to keep value of the $X^2$ unchanged. As we reported during the analysis of Cen X-3 and 4U 0115+63, both $R_0$ and $z_{\max}$ are insensitive to variations of $N_{\text{comp}}$. This can be explained considering that the emerging flux (and thus the source X-ray luminosity) in the Eddington approximation (see equation (24)) is mostly regulated by the vertical space gradient of the photon field at the boundary of the accretion column and not by the height of the column $z_{\max}$. Although the value of $\partial \dot{m} / \partial t$ at the top of the accretion column is tightly related to the overall space photon distribution, the effect of the value of $z_{\max}$ is marginal in the pencil-beam case compared
to the fan-beam case. In the latter, this parameter explicitly appears in the integration of the source flux performed along the column height (see equation \ref{eq:te}). In this case, \( z_{\text{max}} \) influences more the vertical gradient of the magnetic field and thus also the cyclotron emission term.

On one hand, it is thus evident that we should generally expect some degeneracy in the compmag model parameters and it might not always be possible to verify \textit{a priori} if the fan-beam or the pencil-beam geometry is better suited to describe the X-ray spectrum of an X-ray pulsar. On the other hand, the fact that the pencil-beam geometry could not be used to satisfactorily fit the X-ray spectrum of the source terms in the RTE. In the latter, this parameter explicitly appears in the integration of the source flux performed along the column height (see equation \ref{eq:te})

The units of the radiative transfer equation \ref{eq:te} are \( s^{-1} \), while the occupation number is related to the specific intensity by

\[
J(E) = \frac{2h\nu^3}{c^2} n(E),
\]

where \( J(E) \) is given in units of energy \( \text{cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1} \). Note that in equation \ref{eq:te} both \( J(E) \) and \( n(E) \) are averaged over the angle. To obtain the correct units for the equation \ref{eq:te}, the quantity \( J(E) \) must be multiplied by the extra factor \( c \).

Using the definitions \( E = h\nu \) and \( x = E/kT_e \), together with the equation \ref{eq:te}, the cyclotron and bremsstrahlung source terms in the forms used within the equation \ref{eq:te} can be easily obtained as

\[
j(x) = \frac{c^3 h^2}{2(xkT_e)^3} \Psi(E),\]

Note however that the \textit{the extra factor} \( c \) which is obtained dividing equation \ref{eq:te} by \( n_e \sigma_T c H \). By additionally defining \( \Theta = kT_e/m_e c^2 \), we finally obtain the expression

\[
\delta(E - E_{\text{cyc}}) = \frac{\delta(x - x_{\text{cyc}})}{kT_e},
\]

and then substituted \( \delta(x - x_{\text{cyc}}) \) with a normalized Gaussian.

### Appendix B: Moments of the intensity radiation field in the observer reference frame

Let us consider a fluid moving with bulk velocity \( \mathbf{V} \) with respect to a static system of reference, i.e., the observer reference frame. In the diffusion approximation, the specific intensity in the fluid reference frame can be written as (e.g., Pomraning 1973)

\[
I_f = \frac{I_{0,f}}{4\pi} + \frac{3}{4\pi} \mathbf{k}_f \cdot \mathbf{M}_f,
\]

where \( \mathbf{k}_f \) is the cosine-direction versor, while \( I_{0,f} \) and \( \mathbf{M}_f \) are the zeroth and first moments of the specific intensity, respectively. The latter are defined as

\[
I_{0,f} = \int I_f d\Omega_f,
\]

and

\[
\mathbf{M}_f = \int I_f \mathbf{k}_f d\Omega_f.
\]

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From equations (B.10) and (B.12) we finally obtain

$$K_1^{ij} = \int I_i k_i^j k_j^i d\Omega_i.$$  \hspace{1cm} (B.4)

For the sake of clarity, we consider here the case of a cartesian coordinate system where only the $M^z$ component is different from zero (slab approximation) so that equation (B.1) can be written as

$$I_t = \frac{I_{0t}}{4\pi} + \frac{3}{4\pi} u t M^z,$$  \hspace{1cm} (B.5)

where $k_i^j = u_i$ is the cosine of the photon direction along the $z$ coordinate. Substituting $I_t$ from equation (B.5) into the integrals (B.2) to (B.4) it is straightforward to see that $K_t/I_{0t} = 1/3$, where $K_t = K_{zz}^x$.

We now consider the relation between the zeroth and second moment in the observer reference frame. The transformation law of the photon energy between the fluid and observer reference frame is

$$E_i = E_i \Gamma(1 + \beta u_t),$$  \hspace{1cm} (B.6)

where $\beta = V/c$ is the fluid velocity, and the relation between the angles $u_i$ and $u_1$ is given by the aberration formula

$$u_i = \frac{u_i - \beta}{1 - \beta u_i}.$$

Using now the relativistic invariant

$$\frac{I_t}{E_i^3} = \frac{I_t}{E_i^3},$$

together with the relation (B.6), the specific intensity in the laboratory reference frame can be written as

$$I_t = \Gamma(1 + \beta u_t)^3 I_t,$$  \hspace{1cm} (B.9)

where $I_t$ is given in equation (B.1). Computing the zeroth, first, and second moment of the specific intensity in the observer reference frame we obtain

$$J_t = \Gamma(1 + M_t^z),$$  \hspace{1cm} (B.10)

$$M_t^z = \Gamma(M_t^z + I_{0t}),$$  \hspace{1cm} (B.11)

and

$$K_t^{zz} = \frac{\Gamma}{3^{3/2}} \left[ I_{0t} \beta (-1 + 2 \beta^2) + M_t^z (3 - 5 \beta^2 + 3 \beta^4) \right] + \frac{(I_{0t} \beta - 3 M_t^z) \operatorname{ArcTanh}[\beta]}{3^{1/2} \Gamma^3}.$$  \hspace{1cm} (B.12)

We also made use above of the equations (B.9) and (B.7). From equations (B.10) and (B.12) we finally obtain

$$\frac{K_t^{zz}}{J_t} \approx \frac{1}{3} + \frac{16 M_t^z \beta}{15 I_{0t}} + O[\beta^2]$$  \hspace{1cm} (B.13)

respectively. Note that $M^z_t$ is the spectral flux along the $i^{th}$-coordinate. Together with the zeroth and first moment, one can also define the second moment of the radiation field $K_{t}^{zz}$.

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