THE CALOGERO-MOSER PARTITION AND ROUQUIER FAMILIES FOR COMPLEX REFLECTION GROUPS

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Abstract. Let $W$ be a complex reflection group. We formulate a conjecture relating blocks of the corresponding restricted rational Cherednik algebras and Rouquier families for cyclotomic Hecke algebras. We verify the conjecture in the case that $W$ is a wreath product of a symmetric group with a cyclic group of order $l$.

1. Introduction

1.1. Let $W$ be a complex reflection group. The aim of this note is to state, and for an infinite family of complex reflection groups, prove a conjecture relating restricted rational Cherednik algebras and cyclotomic Hecke algebras for complex reflection groups. The former algebra is finite dimensional factor algebra of a rational Cherednik algebra with interesting properties, which has been used, for example, to study the existence of symplectic resolutions of quotient singularities. Its simple modules are labeled naturally by the set, $\text{Irr}_W$, of simple $CW$-modules. We can partition $\text{Irr}_W$ according to the blocks of the restricted rational Cherednik algebra; we call this partition the Calogero-Moser partition (the spectra of the centres of rational Cherednik algebras are called Calogero-Moser spaces).

1.2. Cyclotomic Hecke algebras for complex reflection groups are objects of current interest, which are expected to provide insight into the representation theory of finite reductive groups and to display behaviour analogous to Hecke algebras associated to series of reductive algebraic groups. In this latter direction, Rouquier has defined a partition of the set $\text{Irr}_W$, which generalises the notion of Lusztig’s families for Weyl groups, [Rou1]. This is the partition of $\text{Irr}_W$ into Rouquier families. Conjecture 2.7 states that the partition into Rouquier families refines the Calogero-Moser partition and proposes further numerical connections between them. We expect that this conjecture is a natural extension of [GM, Conjecture 1.3] which relates the Calogero-Moser partition to a partition arising (conjecturally) from cells at unequal parameters for Weyl groups. In the classical situation this latter partition equals the partition into families, but since there exists (at present) no cell theory for complex reflection groups which are not Weyl groups, our conjecture seems to be the most suitable generalisation of [GM]. We prove Conjecture 2.7(i) when $W$ is the wreath product $G(l, 1, n) = \mathbb{Z}/l\mathbb{Z} \wr S_n$ by comparing known combinatorial descriptions of the two partitions. It would be very interesting to have a more conceptual understanding of this result.

1.3. In section 2 we introduce the main protagonists and state the precise conjecture, which includes a geometric interpretation of the size of each Rouquier block. In section 3 we prove the conjecture for wreath products and conclude with an interpretation of the combinatorics via higher level Fock spaces.
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2. Calogero-Moser partition, Rouquier families and the conjecture

2.1. Notation. Let $W$ be a complex reflection group and $\mathfrak{h}$ its reflection representation over $\mathbb{C}$. Let $A$ be the set of reflecting hyperplanes in $\mathfrak{h}$. Let $\mathfrak{h}^{\operatorname{reg}} = \mathfrak{h} \setminus (\bigcup_{H \in A} H)$. Given a hyperplane $H \in A$ we define $W_H$ to be the subgroup of $W$ of elements that fix $H$ pointwise. Let $e_H = |W_H|$, and for $C \in A/W$ let $e_C$ be the common value $e_H$ for $H \in C$. For every $H \in A$ we choose $v_H \in \mathfrak{h}$ such that $\mathbb{C}v_H$ is a $W_H$-stable complement to $H$, and choose also a linear form $\alpha_H \in \mathfrak{h}^*$ with kernel $H$. Let $< , >$ denote the natural pairing of $\mathfrak{h}^*$ with $\mathfrak{h}$. Let $A$ be a finite dimensional algebra, then we denote by $\operatorname{Irr} A$ the set of irreducible $A$-modules. We will write $\operatorname{Irr} W$ for the set $\operatorname{Irr} CW$.

2.2. Rational Cherednik algebras. We introduce parameters $k := (k_{C,i})_{C \in A/W, 0 \leq i \leq e_c - 1}$ where $k_{C,i} \in \mathbb{C}$ for all $C$ and $i$, and $k_{C,0} = 0$ for any $C$. We use the convention here and throughout this paper that the subscript $j$ in $k_{C,j}$ is considered modulo $e_C$.

The rational Cherednik algebra (at $t = 0$), $H_k$, is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$, the smash product of the free $\mathbb{C}$-algebra on $\mathfrak{h} \oplus \mathfrak{h}^*$ with $W$, by the relations:

$$[x, x'] = 0,$$  
$$[y, y'] = 0$$  
$$[y, x] = \sum_{H \in A} \frac{< \alpha_H, y > < x, v_H >}{< \alpha_H, v_H >} \gamma_H$$

for $y, y' \in \mathfrak{h}$ and $x, x' \in \mathfrak{h}^*$. Here we set

$$\gamma_H = \sum_{w \in W_H \setminus \{1\}} \left( \sum_{j=0}^{e_H - 1} \det(w)^j (k_{C,j+1} - k_{C,j}) \right) w$$

for all $H \in A$.

This is the definition given in [GGOR] and, as noted there, is equivalent to that given in [EG]. It follows easily from the definition that

$$H_k \cong H_{\lambda k} \quad \text{(1)}$$

for any $\lambda \in \mathbb{C}^*$.

2.3. The Calogero-Moser partition. By [EG] Proposition 4.15, there is an algebra embedding of $A := \mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]^W$ into the centre of $H_k$. Let $A_+ \subset A$ denote the maximal ideal of polynomials with positive degree. The restricted rational Cherednik algebra, $\overline{H}_k$, is the finite dimensional factor algebra $H_k/A_+ H_k$. For more details on the structure of $\overline{H}_k$, see [Gor2].

The Calogero-Moser partition, or $CM_k$-partition, is defined by the equivalence relation on $\operatorname{Irr} \overline{H}_k$ given by: $M \sim_{CM_k} N$ if and only if $M$ and $N$ lie in the same block of $\overline{H}_k$. The set of irreducible $\overline{H}_k$-modules, $\operatorname{Irr} \overline{H}_k$, can be identified with the set $\operatorname{Irr} W$, [Gor2] Proposition 4.3], and so we think of the $CM_k$-partition as a partition of $\operatorname{Irr} W$. The proof of the next lemma follows directly from (1)
Lemma. Let \( k \) be a parameter as in $2.2$ and let \( \lambda \in \mathbb{C} \). Then the \( CM_k \)-partition and \( CM_{\lambda k} \)-partition are equal.

2.4. Geometric interpretation. Let \( Z_k \) be the centre of \( H_k \). The embedding
\[
A \hookrightarrow Z_k
\]
induces a morphism of schemes
\[
\Psi : \text{Spec } Z_k \rightarrow \mathfrak{h}/W \times \mathfrak{h}^*/W.
\]
We write \( \Psi^*(0) \) for the scheme theoretic fibre of 0 and \( \Psi^{-1}(0) \) for the closed points in \( \Psi^*(0) \). By [Gor2] Corollary 5.8 the \( CM_k \)-partition is trivial (that is, each equivalence class is a singleton set) if and only if \( \Psi^{-1}(0) \) consists of smooth points, and this occurs if and only if all irreducible \( H_k \)-modules have dimension \( |W| \).

2.5. Generic Hecke algebras. We retain the notation of $2.4$. For every \( d > 1 \), set \( \eta_d = e^{\frac{2\pi i}{d}} \) and denote by \( \mu_d \) the group of all \( d \)th roots of unity. Let \( \mu_{\infty} \) be the group of all roots of unity in \( \mathbb{C} \) and let \( K \) be a number field contained in \( \mathbb{Q}(\mu_{\infty}) \) such that \( K \) contains \( \mu_{c\mathbb{C}} \), for all \( C \in \mathcal{A}/W \). We denote by \( \mu(K) \) the group of roots of unity in \( K \) and by \( \mathbb{Z}_K \) the ring of integers in \( K \).

Let \( x_0 \in \mathfrak{h}^{\text{reg}} \) and denote by \( \mathfrak{f}_0 \) its image in \( \mathfrak{h}^{\text{reg}}/W \). Let \( B \) be the fundamental group \( \Pi_1(\mathfrak{h}^{\text{reg}}/W, \mathfrak{f}_0) \). Let \( u = (uc_{j,C})_{C \in \mathcal{A}/W, 0 \leq j \leq c_{C} - 1} \) be a set of indeterminates, and let \( \mathbb{Z}[u, u^{-1}] := \mathbb{Z}[u_{C, j}^{-1}; C \in \mathcal{A}/W, 0 \leq j \leq c_{C} - 1] \). The generic Hecke algebra, \( \mathcal{H}_W \), is the quotient of \( \mathbb{Z}[u, u^{-1}]B \) by relations of the form
\[
(s - u_{C,0})(s - u_{C,1}) \cdots (s - u_{C, c_{e} - 1}),
\]
where \( C \in \mathcal{A}/W \) and \( s \) runs over the set of monodromy generators around the images in \( \mathfrak{h}^{\text{reg}}/W \) of the hyperplane orbit \( C \), see [BMR] §4.C. We shall from now on assume the following.

Hypothesis. \( \mathcal{H}_W \) is a free \( \mathbb{Z}[u, u^{-1}] \)-module of rank \( |W| \) and \( \mathcal{H}_W \) has a symmetrising form \( t : \mathcal{H}_W \rightarrow \mathbb{Z}[u, u^{-1}] \) that reduces to the standard symmetrising form on \( \mathbb{Z}_K W \) upon specialising \( u_{C, j} \) to \( \eta_{c_{C}}^{j} \cdot \mu(K)^{j} \). Furthermore, let \( v = (vc_{j,C})_{C \in \mathcal{A}/W, 0 \leq j \leq c_{C} - 1} \) be indeterminates such that \( u_{C, j} = \eta_{c_{C}}^{j} \cdot \mu(K)^{j} \) for all \( C, j \): then the \( K(v) \)-algebra \( K(v)\mathcal{H}_W \) is split semisimple.

It is known that all but a finite number of complex reflection groups satisfy this hypothesis, see [AK], [BM] and [Ari], and it is conjectured to hold for all complex reflection groups. By Tits' deformation theorem (see, for example, [GP] Theorem 7.2]) the specialisation \( v_{C, j} \rightarrow 1 \) induces a bijection \( \text{Irr} W \rightarrow \text{Irr} K(v)\mathcal{H}_W \); \( S \rightarrow S^{K(v)} \) such that \( \text{Dim}_C S = \text{Dim}_K(S^{K(v)}) \).

2.6. Cyclotomic Hecke algebras.

Definition. A cyclotomic Hecke algebra is the \( \mathbb{Z}_K[y, y^{-1}] \)-algebra induced from \( \mathbb{Z}_K[v, v^{-1}]\mathcal{H}_W \) by an algebra homomorphism of the form
\[
\mathbb{Z}_K[v, v^{-1}] \rightarrow \mathbb{Z}_K[y, y^{-1}]
\]
\[
v_{C, j} \mapsto y^{mc_\lambda j}
\]
such that

(i) \( m_{C,j} \in \mathbb{Z} \) for all \( C \in \mathcal{A}/W \) and \( 0 \leq j \leq e_C - 1 \);

(ii) Set \( x := y^{\mu(K)} \). If \( z \) is an indeterminate then the element of \( \mathbb{Z}_K[y,z] \) defined by

\[
\Gamma_C(y,z) = \prod_{j=0}^{e_C-1} (z - \eta_{e_c}^j y^{m_{C,j}})
\]

is invariant under \( \text{Gal}(K(y)/K(x)) \) for all \( C \in \mathcal{A}/W \). In other words, \( \Gamma_C(y,z) \) is contained in \( \mathbb{Z}_K[x^{\pm 1}, z] \).

We write \( m = (m_{C,j})_{C \in \mathcal{A}/W, \ 0 \leq j \leq e_C - 1} \) and denote this algebra by \( \mathcal{H}_W(m) \).

The algebra \( K(y)\mathcal{H}_W(m) \) has a symmetric form induced by \( t \) and is split semisimple by [ChlI §4.3]. Thus by Tits’ deformation theorem we have bijections

\[
\text{Irr}W \cong \text{Irr}K(y)\mathcal{H}_W(m) \cong \text{Irr}K(v)\mathcal{H}_W.
\]

2.7. Rouquier families. We define the Rouquier ring to be \( \mathcal{R}(y) = \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}] \). By Hypothesis 2.6, \( \mathcal{R}(y)\mathcal{H}_W(m) \subset K(y)\mathcal{H}_W(m) \) is free of rank \( |W| \). We define an equivalence relation on \( \text{Irr}K(y)\mathcal{H}_W(m) = \text{Irr}W \) by: \( S \sim_{Rm} T \) if and only if \( S \) and \( T \) belong to the same block of \( \mathcal{R}(y)\mathcal{H}_W(m) \). We call the equivalence classes of this relation Rouquier families, and we call the subalgebras \( K(y)B \subset K(y)\mathcal{H}_W(m) \), where \( B \) is a block of \( \mathcal{R}(y)\mathcal{H}_W(m) \), Rouquier blocks.

Conjecture. Let \( W \) be a complex reflection group satisfying Hypothesis 2.5.

Let \( k = (k_{C,j})_{C \in \mathcal{A}/W, \ 0 \leq j \leq e_C - 1} \) be a parameter as in 2.4 such that \( k_{C,j} \in \mathbb{Z} \) for all \( C \) and \( j \). Let \( m = (m_{C,j})_{C \in \mathcal{A}/W, \ 0 \leq j \leq e_C - 1} \) where \( m_{C,j} = -k_{C, e_c - j} \) for all \( C \) and \( j \). Then:

(i) The partition of \( \text{Irr}W \) into Rouquier families associated to \( \mathcal{H}_W(m) \) refines the \( CM_k \)-partition. For generic values of \( k \) the partitions are equal;

(ii) Let \( k \) be a parameter such that the \( CM_k \)-partition and partition into Rouquier families are equal.

Let \( p \in \Psi^{-1}(0) \) and let \( K(y)B \) be the corresponding Rouquier block. Then \( \text{Dim}_C(C[\Psi^*(0)_p]) = \dim K(y)B \).

A priori there is no reason why the specialisation \( m_{C,j} = -k_{C, e_c - j} \) satisfies condition (ii) of Definition 2.6. By Lemma 2.3 however, we can assume without loss of generality that \( |\mu(K)| \) divides all of the \( k_{C,j} \). It seems necessary to include the condition that \( k \) has integer entries since the Hecke algebra is only defined at integer parameters. If \( k \) has rational entries then \( \lambda \) has integer entries for some \( \lambda \in \mathbb{Z} \), so we can state a version of the conjecture for \( k \) by using Lemma 2.3.

2.8. Evidence for the conjecture. By the Shepherd-Todd classification of complex reflection groups, [ST], \( W \) is either a member of the infinite family \( G(l,p,n) \) where \( l, n \in \mathbb{N} \) and \( p \) divides \( l \), or one of 34 exceptional groups \( G_4, \ldots, G_{37} \).

- Combining [EG] Proposition 16.4(ii), [Gor2] Proposition 7.3 and [Bel] Proposition 3.2 we have that when \( W \) is not \( G(l,1,n) \) or \( G_4 \) there exist irreducible \( \overline{\mathcal{H}}_k \)-modules of dimension \( < |W| \) for all values
of \( k \). By [2,3] this implies that in these cases the \( CM_k \)-partition is nontrivial. The work of [ChH] shows that the Rouquier families of \( H_W(m) \) are never trivial in these cases.

- Suppose now that \( W = G(l,1,n) \): this is the wreath product \( \mathbb{Z}/l\mathbb{Z} \wr S_n \). We prove part (i) of the conjecture for these groups in the next section, Corollary 3.13.

- Let \( W = G(l,1,n) \). Then, for generic rational values of \( k \), \( X_k \) is smooth so that the \( CM_k \)-partition is trivial. In this case if we take \( p \in \Psi^{-1}(0) \) then by [Gor 2, Corollary 5.8], \( \dim \mathbb{C}[\Psi^*(0)_p] = (\dim S)^2 \), where \( S \in \text{Irr} W \) is the module corresponding to \( p \). On the other hand we know from Corollary 3.13 that the Rouquier partition is trivial and so the Rouquier blocks of \( K(y)H_W(m) \) are simply the blocks of this algebra. Now by Tits’ deformation theorem we have that the dimension of the block corresponding to \( S \) has \( K(y) \)-dimension \( (\dim S)^2 \).

3. Proof of the conjecture for \( G(l,1,n) \)

3.1. Parameters for \( G(l,1,n) \). We fix through positive integers \( l \) and \( n \). The wreath product \( W = \mathbb{Z}/l\mathbb{Z} \wr S_n \) has two orbits of reflecting hyperplanes, \( C_1 \) and \( C_2 \), with \( e_{C_1} = 2 \) and \( e_{C_2} = l \). We will use the parameter set \( h = (h; H_1,\ldots,H_{l-1}) \) where \( h = k_{C_1,1} \) and \( H_1 = k_{C_2,l-i+1} - k_{C_2,l-i} \) for each \( i \). We set also \( H_0 = -H_1 - \cdots - H_{l-1} \).

We will assume throughout that \( h \) has rational entries. We also assume that \( h = -1 \) and we fix a positive integer \( d \) such that \( dh = (-d; dH_1,\ldots,dH_{l-1}) \) has integer entries. Our parameter \( m \), calculated with respect to \( dh \), is given by \( m_{C_1,0} = 0, m_{C_1,1} = d \) and \( m_{C_2,j} = d \sum_{i=1}^{j} H_i \) for all \( 0 \leq j \leq l-1 \).

3.2. Sequences. Let \( r \in \mathbb{Z} \). A strictly decreasing sequence of integers \( C = \{C_1, C_2, \ldots\} \) will be said to stabilise with respect to \( r \) if there exists an \( K \geq 1 \) such that \( C_i = r + 1 - i \) for all \( i \geq K \). Given any strictly decreasing set of integers, \( C \), we define its power series

\[
\pi(C) := \sum_{i \geq 1} x^{C_i}.
\]

For any positive integer \( K \) we define the truncated power series \( \pi(C)_{\leq K} := \sum_{i:C_i \leq K} x^{C_i} \). Given any sequence of integers, \( C \), and any \( i \in \mathbb{Z} \) we define \( S^iC \) to be the sequence \( \{C_1 + i, C_2 + i, \ldots\} \).

The next notion will be useful to us.

Definition. Let \( r = (r_0,\ldots,r_{l-1}) \in \mathbb{Z}^l \). Let \( C = (C^q)_{q=0}^{l-1} \) be an \( l \)-tuple where each \( C^q \) is a strictly decreasing sequence of integers which stabilises with respect to \( r_q \). We define \( \chi(C) \) to be the set

\[
\bigcup_{i=0}^{l-1}\{(l(C^q_i) - 1) + i + 1 : 0 \leq q \leq l-1, i \geq 1\}.
\]

We note that \( \chi \) is one-to-one: for \( l \)-tuples of strictly decreasing integers, \( C \) and \( D \), \( \chi(C) = \chi(D) \) implies \( C = D \). We will need the following elementary result.

Lemma. Let \( C \) be as above and let \( r = \sum_{i=0}^{l-1} r_i \). Then the set \( \chi(C) \) can be rearranged into a strictly decreasing sequence of integers which stabilises with respect to \( r \).
Proof. Let $d$ be the smallest integer such each $C^n$ stabilises after $d$ steps. Let $r_{\text{min}} = \min\{r_0, \ldots, r_{l-1}\}$. For each $q$ let $D^q$ be the sequence obtained by removing the first $d + (r_q - r_{\text{min}})$ terms from $C^q$. Thus $D^q = \{r_{\text{min}} - d, r_{\text{min}} - d - 1, \ldots\}$. Let $D = (D^0, \ldots, D^{l-1})$. Then $\chi(D) = \{l(r_{\text{min}} - d), l(r_{\text{min}} - d) - 1, \ldots\}$ and $|\chi(C) \setminus \chi(D)| = \sum_{q=0}^{l-1} d + (r_q - r_{\text{min}}) = r + l(d - r_{\text{min}}).$ \hfill $\square$

3.3. Partitions. A partition of $n$ is a sequence of natural numbers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ such that $|\lambda| := \sum_{i=1}^{k} \lambda_i = n$. The integer $k$ is called the length of $\lambda$, and we will denote this by $L(\lambda)$. We use the convention that $\lambda_i = 0$ for $i > k$, and we denote the set of partitions of $n$ by $P(n)$. The Young diagram of $\lambda$ is $Y(\lambda) := \{(a, b) \in \mathbb{Z}^2 : 1 \leq a \leq L(\lambda), 1 \leq b \leq \lambda_a\}$. The elements of the Young diagram are called nodes and we define the content of a node to be $\text{cont}(a, b) = b - a$. For example, the Young diagram of the partition $(5, 3, 2)$ with nodes labeled with their content is:

```
0 1 2 3 4
-1 0 1
-2 -1
```

3.4. $\beta$-numbers. Let $r \in \mathbb{Z}$. We define the $r$-shifted $\beta$-number of $\lambda$ to be the sequence of decreasing integers

$$\beta^r(\lambda) = \{\lambda_1 + r, \lambda_2 + r - 1, \ldots, \lambda_j + r + 1 - j, \ldots\}.$$  

We define $\beta^r(\lambda)_i = \lambda_i + r + 1 - i$ for each $i \geq 1$. The decreasing sequence $\beta^r(\lambda)$ stabilises with respect to $r$; conversely, any decreasing sequence of integers which stabilises with respect to some $r \in \mathbb{Z}$ equals the $r$-shifted $\beta$-number of a unique partition. When $r = 0$ we shall simply write $\beta(\lambda)$ instead of $\beta^r(\lambda)$.

The residue of a partition $\lambda$, $\text{Res}_\lambda(x)$, is the element of $\mathbb{Z}[x^{\pm 1}]$ given by

$$\sum_{(a,b) \in Y(\lambda)} x^{\text{cont}(a,b)}.$$ 

For $r \in \mathbb{Q}$ we define the $r$-shifted residue of $\lambda$ to be $\text{Res}^r_\lambda(x) := x^r \text{Res}_\lambda(x)$. We have the following relationship between residues and $\beta$-numbers:

$$(x - 1)\text{Res}^r_\lambda(x) = \pi(\beta^r(\lambda)) \leq K - \frac{x^{r+1-K} - x^{r+1}}{1-x}$$

(2)

for any $K \geq L(\lambda)$.

3.5. $J$-hearts. Given $j \in \{0, 1, \ldots, l - 1\}$ we say a node $(a, b) \in Y(\lambda)$ is $j$-removable if $\text{cont}(a, b)$ is equal to $j$ modulo $l$ and if $Y(\lambda) \setminus (a, b)$ is the Young diagram of some partition. Given a subset $J \subseteq \{0, \ldots, l - 1\}$ we define the $J$-heart of $\lambda$ to be the partition obtained by removing as often as possible $j$-removable boxes, where $j \in J$. Denote this partition $\lambda_J$.

The notion of $j$-removability is related to $\beta(\lambda)$ as follows. If a node, $(a, b)$, is removable then it lies at the right hand edge of row $a$ and so $b = \lambda_a$. Now $(a, \lambda_a)$ is $j$-removable for some $j \in J$ if and only if its content is congruent to $j$ modulo $l$ and $(a + 1, \lambda_a)$ does not lie in $Y(\lambda)$. This is equivalent to: $\beta(\lambda)_a = 1 \equiv j \mod l$ and $\beta(\lambda)_{a+1} < \beta(\lambda)_a - 1$. Furthermore, the $\beta$-number of $\lambda \setminus (a, \lambda_a)$ is $(\beta(\lambda) \setminus \beta(\lambda)_a) \cup \{\beta(\lambda)_a - 1\}$.
3.6. Residues and Rouquier families. An $l$-multipartition of $n$ is an $l$-tuple $(\lambda^{(0)}, \ldots, \lambda^{(l-1)})$ of partitions such that $\sum_{i=0}^{l-1} |\lambda^{(i)}| = n$. We denote the set of $l$-multipartitions of $n$ by $\mathcal{P}(l, n)$. There is a natural bijection between $\mathcal{P}(l, n)$ and $\text{Irr} W$, see [Rou2] for example. Thus we think of the $CM_k$-partition and the partition into Rouquier families as partitions of the set $\mathcal{P}(l, n)$.

Given $r = (r_0, \ldots, r_{l-1}) \in \mathbb{Z}^l$ we define the $r$-shifted residue of $\lambda$ to be

$$\text{Res}_r(x) := \sum_{i=0}^{l-1} \text{Res}_{\lambda^{(i)}}(x).$$

Theorem. [BK, Théorème 3.13], [Ch2, Theorem 3.11] Let $\lambda, \mu \in \mathcal{P}(l, n)$ and let $m$ be as in 3.4. Define $m = (0, dH_1, dH_1 + dH_2, \ldots, dH_1 + \cdots + dH_{l-1})$. If $\lambda \sim_R m \mu$ then

$$\text{Res}_\lambda(x^d) = \text{Res}_m(x^d).$$

For generic values of $m$ the converse is also true, since for generic $m$ the partition into Rouquier families is trivial, [Ch2, Proposition 3.12]; in fact, if $l$ is a power of a prime number then the converse statement is true for all $m$.

3.7. Let $\lambda \in \mathcal{P}(l, n)$ and $r \in \mathbb{Z}^l$. We obtain a partition from $\lambda$ and $r$ as follows. Set $r = \sum_{i=0}^{l-1} r_i$. Let $\beta^r = (\beta^0(\lambda^{(0)}), \ldots, \beta^{r_{l-1}}(\lambda^{(l-1)}))$. The set $\chi(\beta^r)$ can be arranged into a decreasing sequence of integers stabilising to $r$, Lemma 3.2. Therefore $\chi(\beta^r)$ equals $\beta^r(\tau_r(\lambda))$ for some partition $\tau_r(\lambda)$,

and we obtain a map

$$\mathbb{Z}^l \times \prod_n \mathcal{P}(l, n) \rightarrow \prod_n \mathcal{P}(n); \ (r, \lambda) \mapsto \tau_r(\lambda).$$

3.8. Affine symmetric group. Throughout this subsection subscripts are considered modulo $l$. Let $S_l$ denote the symmetric group on $l$ letters. We identify $S_l$ with permutations of the set $\{0, \ldots, l-1\}$, which is generated by elements $s_i$ for $1 \leq i \leq l-1$, where $s_i$ is the simple transposition swapping $i-1$ and $i$. There is an action of $S_l$ on $\mathbb{Z}^l$ via:

$$s_i \cdot (\theta_0, \ldots, \theta_{l-1}) = (\theta_0, \ldots, \theta_{i-1} + \theta_i, -\theta_i, \theta_i + \theta_{i+1}, \ldots, \theta_{l-1}) \quad \text{for all } 1 \leq i \leq l-1. \quad (3)$$

Let $e_0, \ldots, e_{l-1}$ denote the standard basis of the lattice $\mathbb{Z}^l$. Let $R$ denote the root lattice of type $\tilde{A}_l$, which is the sublattice of $\mathbb{Z}^l$ generated by the simple roots $\alpha_i = -e_{i-1} + 2e_i - e_{i+1}$ for all $0 \leq i \leq l-1$. The action of $S_l$ preserves $R$ and we define the affine symmetric group, $\tilde{S}_l$, to be the semidirect product $R \rtimes S_l$.

The equations in (3) extend naturally to define an action of $S_l$ on $Q_1^l := \{(\theta_0, \ldots, \theta_{l-1}) \in Q^l : \theta_0 + \cdots + \theta_{l-1} = 1\}$. The group $R$ acts on $Q_1^l$ by translations. These two actions combine to give an action of $\tilde{S}_l$ on $Q_1^l$. Let

$$A = \{(\theta_0, \ldots, \theta_{l-1}) \in Q_1^l : 0 \leq \theta_i \leq 1 \text{ for } 0 \leq i \leq l-1\}$$

By [Hum, Proposition 4.3], for every $\theta \in Q_1^l$ there is a (not necessarily unique) $w_\theta \in \tilde{S}_l$ such that $w_\theta \cdot \theta \in A$. Let $w_\theta \cdot \theta = (\varepsilon_0, \ldots, \varepsilon_{l-1})$. We define the type, $J$, of an element $\theta \in Q_1^l$ to be the set $\{j \in \{0, \ldots, l-1\} : \varepsilon_j = 0\}$. 7
3.9. $CM_h$-partition. We describe in detail the combinatorial algorithm which yields the $CM_h$-partition of $P(l,n)$, This is based on [Gor1, § 7-8] and is stated explicitly in [GM]. Let $\mathbb{Z}_l^0 = \{(\theta_0, \ldots, \theta_{l-1}) \in \mathbb{Z}^l : \theta_0 + \cdots + \theta_{l-1} = 0\}$, an $S_l$ stable sublattice of $\mathbb{Z}^l$. There is an $S_l$-equivariant isomorphism of lattices

$$
\phi : \mathbb{Z}_l^0 \to R; (r_0, \ldots, r_{l-1}) \mapsto (r_{l-1} - r_0)e_0 + (r_0 - r_1)e_1 + \cdots + (r_{l-2} - r_{l-1})e_{l-1}.
$$

Let $S_l$ act on $P(l,n)$ by $s_i(\lambda^{(0)}, \ldots, \lambda^{(l-1)}) = (\lambda^{s_i(0)}, \ldots, \lambda^{s_i(l-1)})$ for all $i$.

**Theorem.** [GM, Theorem 2.5] Let $h = (-1, H_0, H_1, \ldots, H_{l-1}) \in \mathbb{Q}^{l+1}$ be a parameter for $H_h$ as in [74]. Let $\theta = (1 + H_0, H_1, \ldots, H_{l-1}) \in \mathbb{Q}_l^l$. Let $w_0 \in \tilde{S}_i$ be such that $w_0 \cdot \theta \in A$ and suppose that $\theta$ has type $J$. Write $w_0 = \phi(r)w$ with $w \in S_l$ and $r \in \mathbb{Z}_l^0$. Let $\lambda, \mu \in P(l,n)$. Then

$$
\lambda \sim_{CM_h} \mu \text{ if and only if } \tau_r(w(\lambda))_J = \tau_r(w(\mu))_J.
$$

We will refer to $\theta$ as the stability parameter associated to $h$. Following this theorem we define the $J$-heart of a multipartition $\lambda$ to be the partition $\tau_r(w(\lambda))_J$. Note that this definition depends on the parameter $h$.

3.10. Retain the hypotheses of Theorem [33]. Let $\epsilon := \phi(r)w \cdot \theta = (\epsilon_0, \ldots, \epsilon_{l-1})$, so that $0 \leq \epsilon_i \leq 1$ for all $i$ and $\sum \epsilon_i = 1$. Recall the integer $d$ from [3.1] A straightforward calculation shows that $d \epsilon \in \mathbb{Z}_l^l$.

**Definition.** Let $c = (c_0, \ldots, c_{l-1}) \in \mathbb{Z}_l^l$. Then for each $0 \leq i \leq l-1$ define the partial sum $m_i(c) = c_0 + \cdots + c_i$.

We define an equivalence relation on the set $\{0, \ldots, l-1\}$ via:

$$
p \sim q \text{ if and only if } m_p(d \epsilon) = m_q(d \epsilon).
$$

For any $0 \leq t \leq d$, let $I_t$ denote the equivalence class of $p \in \{0, \ldots, l-1\}$ such that $m_p(d \epsilon) = t$. Given $0 \leq a \leq b \leq l-1$, denote the corresponding interval by $[a,b]$. We define $[a,b] = \emptyset$ if $a > b$.

**Lemma.** Let $0 \leq t \leq d$. Then the set $I_t$ is empty or equal to $[a,b]$ for some $0 \leq a \leq b \leq l-1$. Furthermore, if $I_t \neq \emptyset$ then

$$
J \cap I_t = \begin{cases} 
[a + 1, b] & \text{if } t \neq 0 \\
[a,b] & \text{if } t = 0.
\end{cases}
$$

In particular, if $0 \neq p \in J \cap I_t$ for some $t$ then $p - 1 \in I_t$.

**Proof.** This follows immediately from the inequalities

$$
0 \leq m_0(d \epsilon) \leq m_1(d \epsilon) \leq \cdots \leq m_{l-1}(d \epsilon) = d
$$

and the fact that, for all $0 \leq p \leq l-2$, $m_p(d \epsilon) = m_{p+1}(d \epsilon)$ if and only if $\epsilon_{p+1} = 0$. \hfill \Box
3.11. In order to calculate \( \tau_r(w(\lambda)) \) one first considers, for each \( 0 \leq p \leq l - 1 \), the \( \beta \)-numbers

\[
\beta^r p(\lambda^{(w^{-1}(p))}) = \{ \lambda_1^{(w^{-1}(p))} + r_p, \lambda_2^{(w^{-1}(p))} + r_p, \ldots \}.
\]

Let \( C^p = C^p(\lambda) = \beta^r p(\lambda^{(w^{-1}(p))}) \) and let \( C = (C^0, \ldots, C^{l-1}) \). Let \( 0 \leq t \leq d \) and let \( I_t \) be the corresponding equivalence class defined in 3.10. For \( 1 \leq t \leq d - 1 \), let \( C_{[t]} \) be the multiset \( \bigcup_{p \in I_t} C^p \). We define also the multiset \( C_{[0]} = C_{[d]} := \bigcup_{p \in I_0} S^{-1}C^p \). We define \( \pi(C_{[t]}) := \sum_{p \in I_t} \pi(C^p) \) for \( 1 \leq t \leq d - 1 \) and \( \pi(C_{[0]}) = \pi(C_{[d]}) := \sum_{p \in I_0} x^{-1}\pi(C^p) + \sum_{p \in I_d} \pi(C^p) \).

We define \( C = C(\lambda) = (C^0, \ldots, C^{l-1}) \) to be the unique tuple of sets such that

(i) \( \tilde{C}^p \) is a strictly decreasing set of integers for all \( p \),

(ii) \( \tilde{C}_{[t]} = C_{[t]} \) for all \( 0 \leq t \leq d \),

(iii) if \( 0 \neq p \in J \) then \( C^p \subseteq C^{p-1} \) and

(iv) if \( 0 \in I_0 \) (so that \( 0 \in J \)) then \( S^{-1}C^0 \subseteq C^{l-1} \).

Such a tuple, \( \tilde{C} \), exists and is unique. Indeed, if \( C \) does not satisfy (iii) then there is some \( p \) and an element \( x \in C^p \) with \( x \notin C^{p-1} \). We form a new tuple of decreasing sequences by removing \( x \) from \( C^p \) and adding it to \( C^{p-1} \) - this new tuple still satisfies (ii) by Lemma 3.10. We now repeat this process (and the analogous one for (iv)) until we have a tuple with the desired properties. This shows that there exists a \( \tilde{C} \) satisfying (i)-(iv). Suppose that there exists another tuple \( D \) satisfying these conditions and that \( D \neq \tilde{C} \). Thus there is some \( p \) and some \( i \) such that \( D^p \notin \tilde{C}^p \). Suppose that \( p \in I_t \) for some \( 1 \leq t \leq d - 1 \), and let us set \( x = D^p \). By Lemma 3.10, \( I_t \) is of the form \([a, b] \) for some \( a \) and \( b \). By (iii), \( x \in D^p \subseteq D^{p-1} \subseteq \cdots \subseteq D^a \). By properties (ii) and (iii), there exists a \( k \) such that \( x \in \tilde{C}^k \subseteq \tilde{C}^{k-1} \subseteq \cdots \subseteq \tilde{C}^a \) and \( x \notin \tilde{C}^{k+1}, \tilde{C}^{k+2}, \ldots, \tilde{C}^b \). By our assumption on \( x, k < p \). This is a contradiction, since both \( D \) and \( \tilde{C} \) satisfy (ii). A similar argument when \( p \in I_0 \cup I_d \) (using both conditions (iii) and (iv)) also yields a contradiction. Therefore \( \tilde{C} \) is uniquely defined.

**Theorem.** The \( J \)-heart of \( \tau_r(w(\lambda)) \) is the partition with \( \beta \)-number equal to \( \chi(\tilde{C}) \).

**Proof.** Let \( \mu \) denote the partition with \( \beta \)-number \( \chi(\tilde{C}) \). We first show that \( \mu_\tau = \mu \). Let \( y \in Y(\mu) \). Suppose that \( y \) is \( J \)-removable. By 3.10, \( y = (k, \mu_k) \) for some \( k \). The element \( \beta(\mu)_k \in \chi(\tilde{C}) \) is equal to \( l\tilde{C}^p + p - l + 1 \) for some \( 0 \leq p \leq l - 1 \) and \( i \geq 1 \). By 3.10, the \( J \)-removability of \( y \) is equivalent to the fact that there is a \( j \in J \) such that \( l\tilde{C}^p + p - l + 1 - 1 \equiv p \equiv j \mod l \) and \( \beta(\mu)_k + 1 < l\tilde{C}^p + p - l \).

Let us first suppose that \( 0 \neq p \in J \). Consider the equality \( \beta(\mu)_{k+1} = l\tilde{C}^p + p - l \). Since \( \beta(\mu)_{k+1} \equiv p \mod l \), this is equivalent to: \( \beta(\mu)_{k+1} = l\tilde{C}^{p-1} + (p - 1) - l + 1 \) for some \( i' \) and \( \tilde{C}^{p-1} = \tilde{C}^i \). Thus the \( J \)-removability of \( y \) is equivalent to \( \tilde{C}^p \notin \tilde{C}^{p-1} \). By the construction of \( \tilde{C} \), \( \tilde{C}^p \subseteq \tilde{C}^{p-1} \) for all \( 0 \neq p \in J \), which is a contradiction.

Suppose now that \( p = 0 \in J \) and so \( I_0 \neq \emptyset \). Then an analogous argument to that given in the previous paragraph shows that the \( J \)-removability of \( y \) is equivalent to \( \tilde{C}^0 - 1 \notin \tilde{C}^{l-1} \). Again we obtain a contradiction to our construction of \( \tilde{C} \).

Let \( \nu = \tau_r(w(\lambda)) \). We now show that we can obtain \( \mu \) from \( \nu \) by removing \( J \)-removable nodes. If \( \nu \) has no \( J \)-removable nodes then, as follows from the arguments in the previous two paragraphs, this implies
Part (1) follows from the easy fact that

\[ \nu_J = \nu = \mu. \]

If \( \nu \) has a \( J \)-removable node then this means there is some \( 0 \neq p \in J \) (respectively \( 0 = p \in J \)) and some \( i \) such that \( C^p \neq C^{p-1} \) (respectively \( C^i_1 \neq C^{i-1}_1 \)). By 3.5 the partition obtained by removing this node has \( \beta \)-number \( \chi(C') \) where \( C' = (C^0, \ldots, C^{l-1}) \) with \( C^p = C^p \setminus \{C^p_i\} \), \( C^{p-1} = C^{p-1} \setminus \{C^p_i\} \) (respectively \( C^{l-1} = C^{l-1} \setminus \{C^0_i \} \)) and \( C^q = C^q \) otherwise. Now \( \chi(C') = \beta(\nu_J) \) if and only if \( C' \) satisfies the defining properties of \( \bar{C} \). If this is not the case then we can repeat this process: after removing \( k \) \( J \)-removable nodes we have a partition with \( \beta \)-number \( \chi(C^{(k)}) \) for an \( l \)-tuple \( C^{(k)} \) satisfying (i) and (ii) above. This partition has no \( J \)-removable boxes precisely when \( C^{(k)} \) also satisfies conditions (iii) and (iv), that is, \( C^{(k)} = \bar{C} \). Therefore \( \beta(\nu_J) = \chi(\bar{C}) = \beta(\mu) \) and so \( \nu_J = \mu \).

**Corollary.** Two multipartions \( \lambda, \mu \in \mathcal{P}(l, n) \) have the same \( J \)-heart if and only if \( C(\lambda)_{[l]} = C(\mu)_{[l]} \) for all \( 0 \leq t \leq d \).

**Proof.** If \( \lambda \) and \( \mu \) have the same \( J \)-heart then \( \chi(\bar{C}(\lambda)) = \chi(\bar{C}(\mu)) \). Thus \( \bar{C}(\lambda) \) and \( \bar{C}(\mu) \) and in particular, \( C(\lambda)_{[l]} = C(\mu)_{[l]} \) for all \( 0 \leq t \leq d \). On the other hand, if \( C(\lambda)_{[t]} = C(\mu)_{[t]} \) for all \( 0 \leq t \leq d \), then by the uniqueness of \( \bar{C}(\lambda) \) and \( \bar{C}(\mu) \) we have \( \bar{C}(\lambda) = \bar{C}(\mu) \). Therefore \( \lambda, \mu \) have the same \( J \)-heart by the theorem. \( \Box \)

### 3.12. Recall the action of \( S_\ell \) on \( \mathbb{Z}^l \) defined in [3].

**Lemma.** Let \( d \) be as in [3.10] and let \( \epsilon \) be as in [3.10]. Let \( \theta \) be the stability parameter associated to \( h \), and let \( u_\theta = \phi(r)w \) with \( w \in S_\ell \) and \( r \in \mathbb{Z}^l_\ell \). Then

1. for all \( 0 \leq p \leq l-1 \),
   \[ m_p(w \cdot \theta) = m_{w^{-1}(p)}(\theta); \]
2. let \( 0 \leq t \leq d \), then for all \( p \in I_t \),
   \[ m_{w^{-1}(p)}(d\theta) = dr_p - dr_{l-1} + t. \]

**Proof.** Part (1) follows from the easy fact that \( m_p(s_k \cdot \theta) = m_{sk(p)}(\theta) \) for all \( 1 \leq k \leq l-1 \). Part (2) follows from

\[ m_{w^{-1}(p)}(d\theta) = m_{w^{-1}(p)}(w^{-1}(\phi(-dr) + d\epsilon)) = dm_p(\phi(-r)) + m_p(d\epsilon). \]

and the equalities \( dm_p(\phi(-r)) = dr_p - dr_{l-1} \) and \( m_p(d\epsilon) = t \). \( \Box \)

### 3.13. We can now prove our main theorem.

**Theorem.** Let \( \lambda, \mu \in \mathcal{P}(l, n) \). Then \( \lambda \) and \( \mu \) have the same \( J \)-heart if and only if \( \text{Res}_\lambda^\mu(x^d) = \text{Res}_\mu^\lambda(x^d) \).

**Proof.** Let \( K \) be a positive integer such that \( C(\lambda)^p \) stabilises with respect to \( r_p \) after \( K \) steps, for all \( p \) and all \( \lambda \in \mathcal{P}(l, n) \) (we could, for instance, take any \( K > n \)). Let \( \lambda, \mu \in \mathcal{P}(l, n) \). By Corollary 3.11 \( \lambda \) and \( \mu \) have the same \( J \)-heart if and only if \( C(\lambda)_{[t]} = C(\mu)_{[t]} \), or equivalently, \( \pi(C(\lambda)_{[t]}) = \pi(C(\mu)_{[t]}) \) for all \( 0 \leq t \leq d \).

Suppose that \( 1 \leq t \leq d-1 \). By our choice of \( K \), \( C(\lambda)_{[t]} = C(\mu)_{[t]} \) if and only if \( \sum_{p \in I_t} \pi(C(\lambda)^p)_{\leq K} = \sum_{p \in I_t} \pi(C(\mu)^p)_{\leq K} \), and this latter equality is equivalent to

\[ \sum_{p \in I_t} \pi(C(\lambda)^p)_{\leq K}(x^d) = \sum_{p \in I_t} \pi(C(\mu)^p)_{\leq K}(x^d). \]

(5)
By \(2\),
\[
\sum_{p \in I_t} \pi(C(\lambda)^p)_{\leq K}(x^d) = (x^d - 1) \sum_{p \in I_t} \text{Res}_{\lambda^{(w-1)(p)}}^{dr_p}(x^d) + \sum_{p \in I_t} \frac{x^{dr_p+d-dK} - x^{dr_p+d}}{1-x^d}.
\]
Thus (3) is equivalent to
\[
\sum_{p \in I_t} \text{Res}_{\lambda^{(w-1)(p)}}^{dr_p}(x^d) = \sum_{p \in I_t} \text{Res}_{\mu^{(w-1)(p)}}^{dr_p}(x^d).
\]
(6)

By Lemma 3.12(2), (6) is equivalent to
\[
x^{dr_1-1-t} \sum_{p \in I_t} \text{Res}_{\lambda^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d) = x^{dr_1-1-t} \sum_{p \in I_t} \text{Res}_{\mu^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d),
\]
which is equivalent to
\[
\sum_{p \in I_t} \text{Res}_{\lambda^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d) = \sum_{p \in I_t} \text{Res}_{\mu^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d).
\]
(7)

An analogous argument for \(t = 0\) yields \(C(\lambda)[0] = C(\mu)[0]\) if and only if
\[
\sum_{p \in I_0 \cup I_d} \text{Res}_{\lambda^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d) = \sum_{p \in I_0 \cup I_d} \text{Res}_{\mu^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d).
\]
(8)

Let \(m(d\theta) = (m_0(d\theta), \ldots, m_{d-1}(d\theta))\). We claim that equalities (7) for all \(1 \leq t \leq d - 1\) and (8) together are equivalent to \(\text{Res}_{\lambda}^{m(d\theta)}(x^d) = \text{Res}_{\mu}^{m(d\theta)}(x^d)\). Consider the decomposition of vector spaces \(Z[x, x^{-1}] = \bigoplus_{0 \leq k \leq d-1} x^k Z[x^d, x^{-d}]\), and let \(P_k\) denote the projection onto the \(k\)th component. By Lemma 3.12(2),
\[
P_k(\text{Res}_{\lambda}^{m(d\theta)}(x^d)) = \begin{cases} \sum_{p \in I_k} \text{Res}_{\lambda^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d) & \text{if } 1 \leq k \leq d-1 \\ \sum_{p \in I_0 \cup I_d} \text{Res}_{\lambda^{(w-1)(p)}}^{m_{w-1}(p)(d\theta)}(x^d) & \text{if } k = 0. \end{cases}
\]

Since \(\text{Res}_{\lambda}^{m(d\theta)}(x^d) = \text{Res}_{\mu}^{m(d\theta)}(x^d)\) if and only if \(P_k(\text{Res}_{\lambda}^{m(d\theta)}(x^d)) = P_k(\text{Res}_{\mu}^{m(d\theta)}(x^d))\) for all \(k\), this proves the claim.

To complete the proof of the theorem, we note that for all \(\lambda \in \mathcal{P}(l, n)\), \(\text{Res}_{\lambda}^{m(d\theta)}(x^d) = x^{d+dH_0} \text{Res}_{\lambda}^{\mu}(x^d)\).
\(\square\)

We now verify Conjecture 2.7(1) for the groups \(G(l, 1, n)\).

**Corollary.** Let \(\lambda, \mu \in \mathcal{P}(l, n)\). Then \(\lambda \sim_{R_m} \mu\) implies \(\lambda \sim_{CM_{bh}} \mu\).

**Proof.** By Theorem 3.6 if \(\lambda \sim_{R_m} \mu\) then \(\text{Res}_{\lambda}^{\mu}(x^d) = \text{Res}_{\lambda}^{\mu}(x^d)\). The above theorem then implies that \(\lambda\) and \(\mu\) have the same \(J\)-heart and therefore, by Theorem 3.9 \(\lambda \sim_{CM_{bh}} \mu\). By Lemma 2.3 \(\lambda \sim_{CM_{bh}} \mu\) if and only if \(\lambda \sim_{CM_{bh}} \mu\).
\(\square\)

We conclude with a mention of another interpretation of the \(CM_{bh}\)-partition for \(G(l, 1, n)\). Let us suppose that the parameter \(h\) has integer entries. For the corresponding tuple, \(\mathbf{m}\), one can define a representation, \(F(\Lambda_{\mathbf{m}})\), of the quantum algebra \(U_v(\mathfrak{gl}_\infty)\), see [LM, Section 6] for further details. This module is called the higher level Fock space, and it has a basis of weight vectors, \(s_{\lambda}\), labelled naturally by multipartitions of \(n\). As a consequence of Theorem 4.13 and [LM, Section 6.2] we have

**Corollary.** Let \(\lambda, \mu \in \mathcal{P}(l, n)\). Then \(\lambda \sim_{CM_{bh}} \mu\) if and only if \(s_{\lambda}\) and \(s_{\mu}\) have the same weight.
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12