ON COMPUTATIONAL COMPLEXITY OF SIEGEL JULIA SETS

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Abstract. It has been previously shown by two of the authors that some polynomial Julia sets are algorithmically impossible to draw with arbitrary magnification. On the other hand, for a large class of examples the problem of drawing a picture has polynomial complexity. In this paper we demonstrate the existence of computable quadratic Julia sets whose computational complexity is arbitrarily high.

1. Foreword

Let us informally say that a compact set in the plane is computable if one can program a computer to draw a picture of this set on the screen, with an arbitrary desired magnification. It was recently shown by the second and third authors, that some Julia sets are not computable [BY]. This in itself is quite surprising to dynamicists – Julia sets are among the “most drawn” objects in contemporary mathematics, and numerous algorithms exist to produce their pictures. In the cases when one has not been able to produce informative pictures (the dynamically pathological cases, like maps with a Cremer or a highly Liouville Siegel point) the feeling had been that this was due to the immense computational resources required by the known algorithms.

The next surprise came with the discovery by the authors of this paper in [BBY] that all Cremer quadratics (or more generally, rational maps without rotation domains) have computable Julia sets. The non-computable examples constructed in [BY] were Siegel quadratic polynomials, and one would expect the Cremer case to be at least as bad if not worse computationally.

The natural question to ask is then whether in those cases in which we know the Julia set is computable, but no good pictures exist, the computational complexity of such a set is indeed high. Here at least, our original intuition seems to be correct: it is shown in the present paper that there exist computable Siegel quadratic Julia sets with arbitrarily high computational complexity. An irritating possibility still remains that some Cremer Julia sets are computationally easy (and we just do not go about trying to draw them in the right way). This, however, seems unlikely. We note that the examples constructed in this paper are the first known cases of Julia sets which are not poly-time computable. The second author [Brv1] and independently Rettinger [Ret] have previously shown that hyperbolic Julia sets are poly-time computable. More recently the second author has shown [Brv2]
that some Julia sets with parabolics are poly-time computable as well. The last result was yet another surprise, as the time complexity of all previously known algorithms for these Julia sets was exponential.

The structure of the paper is as follows. In §2.2 of the Introduction, having stated the principal definitions, we formulate the main result of the paper. In §2.4 we give a sketch of the argument. In §4 we prove several technical lemmas. The final §5 contains the proof of the Main Theorem.

Acknowledgement. We would like to thank Giovanni Gallavotti for very helpful suggestions on the exposition.

2. Introduction

2.1. Computability of real sets. The reader is directed to [BY] for a more detailed discussion of the notion of computability of subsets of \( \mathbb{R}^n \) as applied, in particular, to Julia sets. We recall the principal definitions here. The exposition below uses the concept of a Turing Machine. This is a standard model for a computer program employed by computer scientists. Readers unfamiliar with this concept should think instead of an algorithm written in their favorite programming language. These concepts are known to be equivalent.

Denote by \( \mathbb{D} \) the set of the dyadic rationals, that is, rationals of the form \( \frac{p}{2^m} \). We say that \( \phi : \mathbb{N} \to \mathbb{D} \) is an oracle for a real number \( x \), if \( |x - \phi(n)| < 2^{-m} \) for all \( n \in \mathbb{N} \). In other words, \( \phi \) provides a good dyadic approximation for \( x \). We say that a Turing Machine (further abbreviated as TM) \( M^\phi \) is an oracle machine, if at every step of the computation \( M \) is allowed to query the value \( \phi(n) \) for any \( n \). This definition allows us to define the computability of real functions on compact sets.

Definition 2.1. We say that a function \( f : [a, b] \to [c, d] \) is computable, if there exists an oracle TM \( M^\phi(m) \) such that if \( \phi \) is an oracle for \( x \in [a, b] \), then on input \( m \), \( M^\phi \) outputs a \( y \in \mathbb{D} \) such that \( |y - f(x)| < 2^{-m} \).

To understand this definition better, the reader without a Computer Science background should think of a computer program with an instruction

\[
\text{READ real number } x \text{ WITH PRECISION } n(m).
\]

On the execution of this command, a dyadic rational \( d \) is input from the keyboard. This number must not differ from \( x \) by more than \( 2^{-n(m)} \) (but otherwise can be arbitrary). The algorithm then outputs \( f(x) \) to precision \( 2^{-n} \).

It is worthwhile to note why the oracle mechanism is introduced. There are only countably many possible algorithms, and consequently only countably many computable real numbers which such algorithms can encode. Therefore, one wants to separate the hardness of encoding the real number \( x \) from the hardness of computing the value of the function \( f(x) \), having the access to the value of \( x \).

Let \( K \subset \mathbb{R}^k \) be a compact set. We say that a TM \( M \) computes the set \( K \) if it approximates \( K \) in the Hausdorff metric. Recall that the Hausdorff metric is a metric on compact subsets
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of \( \mathbb{R}^n \) defined by

\[
d_H(X, Y) = \inf \{ \epsilon > 0 \mid X \subset U_\epsilon(Y) \text{ and } Y \subset U_\epsilon(X) \},
\]

where \( U_\epsilon(S) \) is defined as the union of the set of \( \epsilon \)-balls with centers in \( S \).

We introduce a class \( \mathcal{C} \) of sets which is dense in metric \( d_H \) among the compact sets and which has a natural correspondence to binary strings. Namely \( \mathcal{C} \) is the set of finite unions of dyadic balls:

\[
\mathcal{C} = \left\{ \bigcup_{i=1}^{n} B(d_i, r_i) \mid \text{where } d_i, r_i \in \mathbb{D} \right\}.
\]

Members of \( \mathcal{C} \) can be encoded as binary strings in a natural way.

We now define the notion of computability of subsets of \( \mathbb{R}^n \) (see \[Wei\], and also \[RW\]).

**Definition 2.2.** We say that a compact set \( K \subset \mathbb{R}^k \) is computable, if there exists a TM \( M(d, n) \), where \( d \in \mathbb{D}, n \in \mathbb{N} \) which outputs a value 1 if \( \text{dist}(d, K) < 2^{-n} \), the value 0 if \( \text{dist}(d, K) > 2 \cdot 2^{-n} \), and in the “in-between” case it halts and outputs either 0 or 1.

In other words, it computes, in the classical sense, a function from the family \( F_K \) of functions of the form

\[
f(d, n) = \begin{cases} 
0, & \text{if } \text{dist}(d, K) > 2 \cdot 2^{-n} \\
1, & \text{if } \text{dist}(d, K) < 2^{-n} \\
0 \text{ or } 1, & \text{otherwise}
\end{cases}
\]

**Theorem 2.1.** For a compact \( K \subset \mathbb{R}^k \) the following are equivalent:

1. \( K \) is computable as per definition 2.2.
2. there exists a TM \( M(m) \), such that on input \( m \), \( M(m) \) outputs an encoding of \( C_m \in \mathcal{C} \) such that \( d_H(K, C_m) < 2^{-m} \) (global computability),
3. the distance function \( d_K(x) = \inf \{|x - y| \mid y \in K\} \) is computable as per definition 2.1.

Note that in the case \( k = 2 \) computability means that \( K \) can be drawn on a computer screen with arbitrarily good precision (if we imagine the screen as a lattice of pixels).

In the present paper we are interested in questions concerning the computability of the Julia set \( J_c = J(f_c) = J(z^2 + c) \). Since there are uncountably many possible parameter values for \( c \), we cannot expect for each \( c \) to have a machine \( M \) such that \( M \) computes \( J_c \) (recall that there are countably many TMs). On the other hand, it is reasonable to want \( M \) to compute \( J_c \) with an oracle access to \( c \). Define the function \( J : \mathbb{C} \to K^* \) (\( K^* \) is the set of all compact subsets of \( \mathbb{C} \)) by \( J(c) = J(f_c) \). In a complete analogy to Definition 2.1 we can define

**Definition 2.3.** We say that a function \( \kappa : S \to K^* \) for some bounded set \( S \) is computable, if there exits an oracle TM \( M^\phi(d, n) \), where \( \phi \) is an oracle for \( x \in S \), which computes a function \( 2.2 \) of the family \( \mathcal{F}_{\kappa(x)} \).

Equivalently, there exists an oracle TM \( M^\phi(m) \) with \( \phi \) again representing \( x \in S \) such that on input \( m \), \( M^\phi \) outputs a \( C \in \mathcal{C} \) such that \( d_H(C, \kappa(x)) < 2^{-m} \).
In the case of Julia sets:

**Definition 2.4.** We say that $J_c$ is computable if the function $J : d \mapsto J_d$ is computable on the set $\{c\}$.

We have the following (see [Brv1]):

**Theorem 2.2.** Suppose that a TM $M^\phi$ computes the function $J$ on a set $S \subset \mathbb{C}$. Then $J$ is continuous on $S$ in Hausdorff sense.

**Proof.** Let $c$ be any point in $S$, and let $\varepsilon = 2^{-k}$ be given. Let $\phi$ be an oracle for $c$ such that $|\phi(n) - c| < 2^{-(n+1)}$ for all $k$. We run $M^\phi(k+1)$ with this oracle $\phi$. By the definition of $J$, it outputs a set $L$ which is a $2^{-k}$ approximation of $J_c$ in the Hausdorff metric.

The computation is performed in a finite amount of time. Hence there is an $m$ such that $|x-c| < 2^{-(m+1)}$, $\phi$ is a valid oracle for $x$ up to parameter value of $m$. In particular, we can create an oracle $\psi$ for $x$ that agrees with $\phi$ on $1, 2, \ldots, m$. If $x \in S$, then the execution of $M^\psi(k+1)$ will be identical to the execution of $M^\phi(k+1)$, and it will output $L$ which has to be an approximation of $J_x$. Thus we have

$$d_H(J_c, J_x) \leq d_H(J_c, L) + d_H(J_x, L) < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$ 

This is true for any $x \in B(c, 2^{-(m+1)}) \cap S$. Hence $J$ is continuous on $S$. \qed

The second and third authors have demonstrated in [BY]:

**Theorem 2.3.** There exists a parameter value $c \in \mathbb{C}$ such that the Julia set of the quadratic polynomial $f_c(z) = z^2 + c$ is not computable.

The quadratic polynomials in Theorem 2.3 possess Siegel disks (see §2.3 below for the definitions of Siegel and Cremer points). It was further shown by the authors of the present paper in [BBY] that the absence of rotation domains, that is either Siegel disks or Herman rings, guarantees computability of the rational Julia set. This implies, in particular, that all Cremer quadratic Julia sets are computable – this despite the fact that no informative high resolution images of such sets have ever been produced. One expects, however, that such “bad” but still computable examples have high algorithmic complexity, which makes the computational cost of producing such a picture prohibitively high. We note, that the second author [Brv1] and independently Rettinger [Ret] have shown:

**Theorem 2.4.** Hyperbolic Julia sets are computable in polynomial time. That is, if $J$ is the Julia set of a hyperbolic rational mapping $R$, then there exists a TM $M(d,n)$ which computes a function of the family (2.3) in time polynomial in the bit size of the input $(d,n)$. It is worth noting that the same oracle TM $M^\phi(d,n)$ with the oracle representing the parameters of the rational mapping $R$, can be selected for all hyperbolic Julia sets of the same degree. Moreover, the asymptotics of the polynomial time bound depends only on $R$ but not on the input $(d,n)$. 
2.2. **Statement of the Main Theorem.** On the other end of the complexity spectrum we expect to find “bad” but computable Siegel Julia sets and Cremer Julia sets. Indeed, it the present paper we show:

**Theorem 2.5.** There exist quadratic Siegel Julia sets of arbitrarily high computational complexity. More precisely, for any computable increasing function \( h : \mathbb{N} \rightarrow \mathbb{N} \) there exists a computable Siegel parameter value \( c \in \mathbb{C} \) such that:

- the Julia set \( J_c \) is computable by an oracle TM;
- for any oracle TM \( M^\phi(m) \) which computes the \( 2^{-m} \)-approximations to \( J_c \), there exists a sequence \( \{m_i\}_{i=1}^{\infty} \) such that \( M^\phi \) requires the time of at least \( h(m_i) \) to compute the approximation \( C_{m_i} \in \mathbb{C} \).

From this statement for global computational complexity immediately follows the corresponding local statement:

**Corollary 2.6.** There exist computable parameter values \( c \) for which the Julia set \( J_c \) is computable, and the complexity of the problem of computing a function \([2.3]\) in the family \( \mathcal{F}_{J_c} \) is arbitrarily high.

2.3. **Siegel disks of quadratic maps.** Let \( R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) be a rational map of the Riemann sphere. For a periodic point \( z_0 = R^p(z_0) \) of period \( p \) its multiplier is the quantity \( \lambda = \lambda(z_0) = DR^p(z_0) \). We may speak of the multiplier of a periodic cycle, as it is the same for all points in the cycle by the Chain Rule. In the case when \( |\lambda| \neq 1 \), the dynamics in a sufficiently small neighborhood of the cycle is governed by the Intermediate Value Theorem: when \( 0 < |\lambda| < 1 \), the cycle is attracting (super-attracting if \( \lambda = 0 \)), if \( |\lambda| > 1 \) it is repelling. Both in the attracting and repelling cases, the dynamics can be locally linearized:

\[
\psi(R^p(z)) = \lambda \cdot \psi(z)
\]

where \( \psi \) is a conformal mapping of a small neighborhood of \( z_0 \) to a disk around 0. By a classical result of Fatou, a rational mapping has at most finitely many non-repelling periodic orbits.

In the case when \( \lambda = e^{2\pi i \theta} \), \( \theta \in \mathbb{R} \), the simplest to study is the parabolic case when \( \theta = n/m \in \mathbb{Q} \), so \( \lambda \) is a root of unity. In this case \( R^p \) is not locally linearizable; it is not hard to see that \( z_0 \in J(R) \). In the complementary situation, two non-vacuous possibilities are considered: Cremer case, when \( R^p \) is not linearizable, and Siegel case, when it is. In the latter case, the linearizing map \( \psi \) from \([2.3]\) conjugates the dynamics of \( R^p \) on a neighborhood \( U(z_0) \) to the irrational rotation by angle \( \theta \) (the rotation angle) on a disk around the origin. The maximal such neighborhood of \( z_0 \) is called a Siegel disk.

Let us discuss in more detail the occurrence of Siegel disks in the quadratic family. For a number \( \theta \in [0, 1) \) denote \([r_0, r_1, \ldots, r_n, \ldots] \), \( r_i \in \mathbb{N} \cup \{\infty\} \) its possibly finite continued
fraction expansion:

\[
[r_0, r_1, \ldots, r_n, \ldots] \equiv \frac{1}{r_0} + \frac{1}{\frac{1}{r_1} + \frac{1}{\ldots + \frac{1}{r_n + \ldots}}}
\]

Such an expansion is defined uniquely if and only if \( \theta \notin \mathbb{Q} \). In this case, the rational convergents \( \frac{p_n}{q_n} = [r_0, \ldots, r_{n-1}] \) are the closest rational approximants of \( \theta \) among the numbers with denominators not exceeding \( q_n \). In fact, setting \( \lambda = e^{2\pi i \theta} \), we have

\[
|\lambda^h - 1| > |\lambda^{q_n} - 1| \quad \text{for all } 0 < h < q_{n+1}, \ h \neq q_n.
\]

The difference \( |\lambda^{q_n} - 1| \) lies between \( \frac{2}{q_{n+1}} \) and \( \frac{2\pi}{q_{n+1}} \), therefore the rate of growth of the denominators \( q_n \) describes how well \( \theta \) may be approximated with rationals.

We recall a theorem due to Brjuno (1972):

**Theorem 2.7 (Brj).** Let \( R \) be an analytic map with a periodic point \( z_0 \in \hat{\mathbb{C}} \). Suppose that the multiplier of \( z_0 \) is \( \lambda = e^{2\pi i \theta} \), and

\[
B(\theta) = \sum \log(q_n+1) < \infty.
\]

Then \( z_0 \) is a Siegel point.

Note that a quadratic polynomial with a fixed Siegel disk with rotation angle \( \theta \) after an affine change of coordinates can be written as

\[
P_\theta(z) = z^2 + e^{2\pi i \theta} z.
\]

In 1987 Yoccoz proved the following converse to Brjuno’s Theorem:

**Theorem 2.8 (Yoc).** Suppose that for \( \theta \in [0, 1) \) the polynomial \( P_\theta \) has a Siegel point at the origin. Then \( B(\theta) < \infty \).

The numbers satisfying (2.5) are called Brjuno numbers; the set of all Brjuno numbers will be denoted \( \mathcal{B} \). It is a full measure set which contains all Diophantine rotation numbers. In particular, the rotation numbers \( [r_0, r_1, \ldots] \) of bounded type, that is with \( \sup r_i < \infty \) are in \( \mathcal{B} \). The sum of the series (2.5) is called the Brjuno function. For us a different characterization of \( \mathcal{B} \) will be more useful. Inductively define \( \theta_1 = \theta \) and \( \theta_{n+1} = \{1/\theta_n\} \). In this way,

\[
\theta_n = [r_{n-1}, r_n, r_{n+1}, \ldots].
\]

We define the Yoccoz’s Brjuno function as

\[
\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1 \theta_2 \cdots \theta_{n-1} \log \frac{1}{\theta_n}.
\]
One can verify that
\[ B(\theta) < \infty \iff \Phi(\theta) < \infty. \]
The value of the function \( \Phi \) is related to the size of the Siegel disk in the following way.

**Definition 2.5.** Let \((U, u)\) be a simply-connected subdomain of \( \mathbb{C} \) with a marked interior point. Consider the unique conformal isomorphism \( \phi : \mathbb{D} \rightarrow U \) with \( \phi(0) = u \), and \( \phi'(0) > 0 \). The **conformal radius** of \((U, u)\) is the value of the derivative \( r(U, u) = \phi'(0) \).

Let \( P(\theta) \) be a quadratic polynomial with a Siegel disk \( \Delta_\theta \not\ni 0 \). The **conformal radius** of the Siegel disk \( \Delta_\theta \) is \( r(\theta) = r(\Delta_\theta, 0) \). For all other \( \theta \in [0, \infty) \) we set \( r(\theta) = 0 \), and \( \Delta_\theta = \{0\} \).

By the Koebe 1/4 Theorem of classical complex analysis (see e.g. [Ahl]), the radius of the largest Euclidean disk around \( u \) which can be inscribed in \( U \) is at least \( r(U, u) / 4 \).

We note that one has the following direct consequence of the Carathéodory Kernel Theorem (see e.g. [Pom]):

**Proposition 2.9.** The conformal radius of a quadratic Siegel disk varies continuously with respect to the Hausdorff distance on Julia sets.

Yoccoz [Yoc] has shown that the sum
\[ \Phi(\theta) + \log r(\theta) \]
is bounded below independently of \( \theta \in \mathcal{B} \). Recently, Buff and Chéritat have greatly improved this result by showing that:

**Theorem 2.10 ([BC]).** The function \( \theta \mapsto \Phi(\theta) + \log r(\theta) \) extends to \( \mathbb{R} \) as a 1-periodic continuous function.

In [BBY] we obtain the following result on computability of quadratic Siegel disks:

**Theorem 2.11.** The following statements are equivalent:

1. The Julia set \( J(P_\theta) \) is computable;
2. The conformal radius \( r(\theta) \) is computable;
3. The inner radius \( \inf_{z \in \partial \Delta_\theta} |z| \) is computable.

We note that when \( \theta \) is not a Brjuno number, the quantities in (II) and (III) are each equal to zero, and the claim is simply that \( J(P_\theta) \) is computable in this case.

We will make use of the following Lemma which bounds the variation of the conformal radius under a perturbation of the domain. It is a direct consequence of the Koebe Theorem (see e.g. [RZ] for a proof).

**Lemma 2.12.** Let \( U \) be a simply-connected subdomain of \( \mathbb{C} \) containing the point 0 in the interior. Let \( V \subseteq U \) be a subdomain of \( U \). Assume that \( \partial V \subseteq B_\epsilon(\partial U) \). Then
\[ 0 < r(U, 0) - r(V, 0) \leq 4 \sqrt{r(U, 0) \sqrt{\epsilon}}. \]
2.4. **Outline of the construction.** We can now describe the idea of our construction. This outline is rather sketchy and suffers from obvious logical deficiencies, however, it presents the construction in a simple to understand form. Consider the oracle Turing machines $M^\phi$ with $\phi$ representing the parameter $\theta$ in $P_\theta$. Since there are only countably many Turing machines, we may order these machines in a sequence $M_1^\phi, M_2^\phi, \ldots$ We denote $S_i$ the domain on which $M_i^\phi$ computes $J(P_\theta)$ properly. We thus have that for each $i$, the function $J : \theta \mapsto J(P_\theta)$ is continuous on $S_i$.

Let us start with a machine $M_{n_1}^\phi$ which computes $J(P_{\theta_*})$ for $\theta_* = [1,1,1,\ldots]$. If any of the digits $r_i$ in this infinite continued fraction is changed to a sufficiently large $N \in \mathbb{N}$, the conformal radius of the Siegel disk will become small. For $N \to \infty$ the Siegel disk will implode and its center will become a parabolic fixed point in the Julia set (see [Do2]).

If we are careful, we may select $i_1 > 1$ and $N_1 >> 1$ in such a way that for $\theta_1$ given by the continued fraction where all digits are ones except $r_{i_1} = N_1$ we have

\begin{equation}
(2.7) \quad r(\theta_*)(1 - 1/4) < r(\theta_1) < r(\theta_*)(1 - 1/8).
\end{equation}

By the Koebe 1/4-Theorem, there exists $\ell_1 > 0$ such that the distance between the two Julia sets

\[ d_H(J(P_{\theta_*}), J(P_{\theta_1})) > 2^{-\ell_1}. \]

To ensure that the machine $M_{n_1}^\phi$ will not be able to produce an accurate $2^{-\ell_1}$-approximation of $J(P_{\theta_1})$ faster than in the time $h(\ell_1)$ we simply select $i_1 > h(\ell_1)$. This guarantees that the TM will have to read at least $h(\ell_1)$ digits of the oracle $\phi$ to distinguish the two Julia sets, which takes the time $h(\ell_1)$.

To “fool” the machine $M_{n_2}^\phi$ we then change a digit $r_{i_2}$ for $i_2 > i_1$ sufficiently far in the continued fraction of $\theta_1$ to a large $N_2$. In this way, we will obtain a Brjuno number $\theta_2$ for which

\begin{equation}
(2.8) \quad r(\theta_*)(1 - 1/4 - 1/8) < r(\theta_2) < r(\theta_*)(1 - 1/4).
\end{equation}

Again, there exists $\ell_2$ such that for any such Brjuno number, we have

\[ d_H(J(P_{\theta_1}), J(P_{\theta_2})) > 2^{-\ell_2}, \]

and we choose $i_2 > h(\ell_2)$. Continuing inductively, we arrive at the desired limiting Brjuno number $\theta_\infty$.

To convince the reader that this construction is not artificial, and not due to the peculiarities of the selected computation model let us recast it somewhat informally as follows. It is possible by an arbitrarily small perturbation of the parameter $\theta$ to cause a detectable disturbance in the picture of $J(P_\theta)$. To distinguish the picture of the new Julia set from the old one, in practice one needs to draw it with *arbitrary precision arithmetic*. That is, not only the input of the parameter (reading the oracle) will take a long time due to the number of significant digits, but also all the arithmetic manipulations with this parameter. Of course, the former consideration is already sufficient to prove the theorem.
3. Computing Noble Siegel Disks

The primary goal of the present paper is to show that there are computationally hard yet computable Julia sets with Siegel disks. To establish this computability we need a computability result for noble Siegel disks. The term “noble” is applied in the literature to rotation numbers of the form \([a_0, a_1, \ldots, a_k, 1, 1, 1, \ldots]\). The noblest of all is the golden mean \(\gamma^* = [1, 1, 1, \ldots]\).

Lemma 3.1. There is a Turing Machine \(M\), which given a finite sequence of numbers \([a_0, a_1, \ldots, a_k]\) computes the conformal radius \(r_\gamma\) for the noble number \(\gamma = [a_0, \ldots, a_k, 1, \ldots]\).

The idea is to approximate the boundary of \(\Delta_\gamma\) with the iterates of the critical point \(c_\gamma = -e^{2\pi i \gamma}/2\). It is known that in this case the critical point itself is contained in the boundary. The renormalization theory for golden-mean Siegel disks (constructed in \([McM]\)) implies that the boundary \(\Delta_\gamma^*\) is self-similar up to an exponentially small error. In particular, there exist constants \(C > 0\) and \(\lambda > 1\) such that

\[
d_{H}(\{P^i_\gamma(c_\gamma), i = 0, \ldots, q_n\}, \partial \Delta_\gamma^*) < C\lambda^{-n}
\]

Below we derive a similar estimate for all noble Siegel disks with constructive constants \(C\) and \(\lambda\). For this, we do not need to invoke the whole power of renormalization theory. Rather, we will use a theorem of Douady, Ghys, Herman, and Shishikura \([Do1]\) which specifically applies to quadratic noble Siegel disks.

Noble (or more generally, bounded type) Siegel quadratic Julia sets may be constructed by means of quasiconformal surgery on a Blaschke product

\[
f_\gamma(z) = e^{2\pi i \tau(\gamma)}z^2 \frac{z^3 - 3}{1 - 3z}.
\]

This map homeomorphically maps the unit circle \(\mathbb{T}\) onto itself with a single (cubic) critical point at 1. The angle \(\tau(\gamma)\) can be uniquely selected in such a way that the rotation number of the restriction \(\rho(f_\gamma|_\mathbb{T}) = \gamma\).

For each \(n\), the points

\[\{1, f_\gamma(1), f_\gamma^2(1), \ldots, f_\gamma^{2n+1}(1)\}\]

form the \(n\)-th dynamical partition of the unit circle. We have (cf. Theorem 3.1 of \([dFdM]\)) the following:

**Theorem 3.2 (Universal real a priori bound).** There exists an explicit constant \(B > 1\) independent of \(\gamma\) and \(n\) such that the following holds. Any two adjacent intervals \(I\) and \(J\) of the \(n\)-th dynamical partition of \(f_\gamma\) are \(B\)-commensurable:

\[
B^{-1}|I| \leq |J| \leq B|I|
\]

Let us now consider the mapping \(\Psi\) which identifies the critical orbits of \(f_\gamma\) and \(P_\gamma\) by

\[
\Psi : f_\gamma^i(1) \mapsto P_\gamma^i(c_\gamma).
\]

We have the following (Theorem 3.10 of \([YZ]\)):
Theorem 3.3 (Douady, Ghys, Herman, Shishikura). The mapping $\Psi$ extends to a $K$-quasiconformal homeomorphism of the plane $\mathbb{C}$ which maps the unit disk $\mathbb{D}$ onto the Siegel disk $\Delta_\gamma$. The constant $K$ depends on $B$ and $a_0, \ldots, a_k$ in a constructive fashion.

Elementary combinatorics implies that each interval of the $n$-th dynamical partition contains at least two intervals of the $(n + 2)$-nd dynamical partition. This in conjunction with Theorem 3.2 implies that the size of an interval of the $(n + 2)$-nd dynamical partition of $f_\gamma$ is at most $\tau^n$ where

$$\tau = \sqrt{\frac{B}{B + 1}}.$$ 

We now complete the proof of Lemma 3.1. Denote $W_n$ the connected component containing 0 of the domain obtained by removing from the plane a closed disk of radius $2K\tau^n$ around each point of

$$\Omega_n = \{P_\gamma^i(c_\gamma), i = 0, \ldots, q_{n+2}\}.$$ 

By Theorem 3.3

$$\text{dist}_H(\Omega_n, \partial\Delta_\gamma) < K\tau^n,$$

and we have

$$W_n \subset \Delta_\gamma \text{ and } \text{dist}_H(\partial\Delta_\gamma, \partial W_n) \leq \epsilon_n = 2K\tau^n.$$ 

Any constructive algorithm for producing the Riemann mapping of a planar region (e.g. that of [BB]) can be used to estimate the conformal radius $r(W_n, 0)$ with precision $\epsilon_n$. Denote this estimate $r_n$.

Elementary estimates imply that the Julia set $J(P_\gamma) \subset \overline{B(0, 2)}$. By Schwarz Lemma this implies $r(\Delta_\gamma, 0) < 2$. By Lemma 2.12 we have

$$|r(\Delta_\gamma, 0) - r_n| \leq |r(\Delta_\gamma, 0) - r(W_n, 0)| + |r(W_n, 0) - r_n| < 4\sqrt{\epsilon_n} + \epsilon_n \xrightarrow{n \to \infty} 0,$$

and the proof is complete.

4. Making Small Changes to $\Phi$ and $r$

For a number $\gamma = [a_1, a_2, \ldots] \in \mathbb{R} \setminus \mathbb{Q}$ we denote

$$\alpha_i(\gamma) = \frac{1}{a_i + \frac{1}{a_{i+1} + \frac{1}{a_{i+2} + \ldots}}}$$

so that

$$\Phi(\gamma) = \sum_{n \geq 1} \alpha_1(\gamma)\alpha_2(\gamma) \cdots \alpha_{n-1}(\gamma) \log \frac{1}{\alpha_n(\gamma)}.$$ 

We will show the following two lemmas.
Lemma 4.1. For any initial segment \( I = (a_0, a_1, \ldots, a_n) \), write \( \omega = [a_0, a_1, \ldots, a_n, 1, 1, 1, \ldots] \). Then for any \( \varepsilon > 0 \), there is an \( m > 0 \) and an integer \( N \) such that if we write \( \beta = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots] \), where the \( N \) is located in the \( n + m \)-th position, then
\[
\Phi(\omega) + \varepsilon < \Phi(\beta) < \Phi(\omega) + 2\varepsilon.
\]

Lemma 4.2. For \( \omega \) as above, for any \( \varepsilon > 0 \) there is an \( m_0 > 0 \), which can be computed from \( (a_0, a_1, \ldots, a_n) \) and \( \varepsilon \), such that for any \( m \geq m_0 \), and for any tail \( I = [a_{n+m}, a_{n+m+1}, \ldots] \)
if we denote
\[
\beta^I = [a_1, a_2, \ldots, a_n, 1, 1, \ldots, 1, a_{n+m}, a_{n+m+1}, \ldots],
\]
then
\[
\Phi(\beta^I) > \Phi(\omega) - \varepsilon.
\]

We first prove lemma 4.1. Denote
\[
\Phi^{-}(\omega) = \Phi(\omega) - \alpha_0(\omega) \alpha_1(\omega) \ldots \alpha_{n+m-1}(\omega) \log \frac{1}{\alpha_{m+n}(\omega)}.
\]

The value of the integer \( m > 0 \) is yet to be determined. Denote
\[
\beta^N = (a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots).
\]

We will need the following estimates, which are proven by induction

Lemma 4.3. For any \( N \), the following holds:
\( (1) \) For \( i \leq n + m \) we have
\[
\left| \log \frac{\alpha_i(\beta^N)}{\alpha_i(\beta^{N+1})} \right| < 2^{i-(n+m)}/N;
\]
\( (2) \) for \( i < n + m \),
\[
\left| \log \frac{\alpha_i(\beta^N)}{\alpha_i(\beta^I)} \right| < 2^{i-(n+m)},
\]
\( (3) \) for \( i < n + m \),
\[
\left| \log \frac{\frac{1}{\alpha_i(\beta^N)}}{\frac{1}{\alpha_i(\beta^I)}} \right| < 2^{i-(n+m)+1},
\]
\( (4) \) for \( i < n + m - 1 \),
\[
\left| \log \frac{\frac{1}{\alpha_i(\beta^N)}}{\frac{1}{\alpha_i(\beta^I)}} \right| < 2^{i-(n+m)+1}.
\]

The estimates yield the following.

Lemma 4.4. For any \( \omega \) of the form as in lemma 4.1 and for any \( \varepsilon > 0 \), there is an \( m_0 > 0 \) such that for any \( N \) and any \( m \geq m_0 \),
\[
|\Phi^{-}(\beta^N) - \Phi^{-}(\beta^I)| < \frac{\varepsilon}{4}.
\]
Proof. The $\sum$ in the expression for $\Phi(\beta^1)$ converges, hence there is an $m_1 > 1$ such that the tail of the sum $\sum_{i \geq n + m_1} \alpha_1 \alpha_2 \ldots \alpha_i \log \frac{1}{\alpha_i} < \frac{\varepsilon}{16}$. We will show how to choose $m_0 \geq m_1$ to satisfy the conclusion of the lemma.

We bound the influence of the change from $\beta^1$ to $\beta^N$ using lemma 1.3 parts 2 and 3.

The influence on each of the “head elements” ($i < n + m_1 < n + m_1 - 1$) is bounded by

$$\log \frac{\alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)}}{\alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)}} < \sum_{j=1}^{i-1} 2^{j-(n+m)} + 2^{i-(n+m)+1} < 2^{i-(n+m)+2} < 2^{m_1+2-m}.$$ 

By making $m$ sufficiently large (i.e. by choosing a sufficiently large $m_0$ we can ensure that

$$1 - \frac{\varepsilon}{16 \Phi(\beta^1)} < \frac{\alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)}}{\alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)}} < 1 + \frac{\varepsilon}{16 \Phi(\beta^1)},$$

hence

$$\left| \alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)} - \alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)} \right| < \frac{\varepsilon}{16 \Phi(\beta^1)} \alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)}.$$ 

Adding the inequality for $i = 1, 2, \ldots, n + m_1 - 1$ we obtain

$$\left| \sum_{i=1}^{n+m_1-1} \alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)} - \sum_{i=1}^{n+m_1-1} \alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)} \right| < \frac{\varepsilon}{16 \Phi(\beta^1)} \sum_{i=1}^{n+m_1-1} \alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)} < \frac{\varepsilon}{16 \Phi(\beta^1)} \Phi(\beta^1) = \frac{\varepsilon}{16}.$$ 

Hence the influence on the “head” of $\Phi^-$ is bounded by $\frac{\varepsilon}{16}$.

To bound the influence on the “tail” we consider three kinds of terms $\alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)}$: $n + m_1 \leq i \leq n + m - 2$, $i = m + n - 1$ and $i \geq m + n + 1$ (recall that $i = n + m$ is not in $\Phi^-$).

For $n + m_1 \leq i \leq n + m - 2$:

$$\log \frac{\alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)}}{\alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)}} < \sum_{j=1}^{i-1} 2^{j-(n+m)} + 2^{i-(n+m)+1} < 2^{i-(n+m)+2} \leq 1.$$ 

Hence in this case each term can increase at most by a factor of $e$.

For $i = n + m - 1$ Note that the change decreases $\log \frac{1}{\alpha_{n+m-1}(\beta^N)}$ so that $\log \frac{1}{\alpha_{n+m-1}(\beta^N)} \leq \log \frac{1}{\alpha_{n+m-1}(\beta^1)}$, hence we have

$$\log \frac{\alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N) \log \frac{1}{\alpha_i(\beta^N)}}{\alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)}} \leq \log \frac{\alpha_1(\beta^N) \ldots \alpha_{i-1}(\beta^N)}{\alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1)} <$$
Hence this term could increase by a factor of \( \sqrt{e} \) at most.

For \( i \geq n + m + 1 \): Note that \( \alpha_j \) for \( j > n + m \) are not affected by the change, and the change decreases \( \alpha_{n+m} \), so that \( \alpha_{n+m}(\beta^N) \leq \alpha_{n+m}(\beta^1) \). Hence

\[
\log \frac{\alpha_1(\beta^N) \ldots \alpha_i(\beta^N) \log \frac{1}{\alpha_1(\beta^N)} \ldots \alpha_i(\beta^1) \log \frac{1}{\alpha_1(\beta^1)} = \log \frac{\alpha_1(\beta^N) \ldots \alpha_{n+m}(\beta^N)}{\alpha_1(\beta^1) \ldots \alpha_{n+m}(\beta^1)} \leq \\
\log \frac{\alpha_1(\beta^N) \ldots \alpha_{n+m-1}(\beta^N)}{\alpha_1(\beta^1) \ldots \alpha_{n+m-1}(\beta^1)} < \sum_{j=1}^{n+m-1} 2^{-i(n+m)} < 1
\]

So in this case each term could increase by a factor of \( e \) at most.

We see that after the change each term of the tail could increase by a factor of \( e \) at most. The value of the tail remains positive in the interval \((0, \frac{\varepsilon}{16}]\), hence the change in the tail is bounded by \( \frac{\varepsilon \varepsilon}{16} < \frac{3 \varepsilon}{16} \).

So the total change in \( \Phi^- \) is bounded by

\[
\text{change in the “head” + change in the “tail”} < \frac{\varepsilon}{16} + \frac{3 \varepsilon}{16} = \frac{\varepsilon}{4}.
\]

\[\square\]

Lemma 4.4 immediately yields:

**Lemma 4.5.** For any \( \varepsilon \) and for the same \( m_0(\varepsilon) \) as in lemma 4.4, for any \( m \geq m_0 \) and \( N \),

\[
|\Phi^- (\beta^N) - \Phi^- (\beta^{N+1})| < \frac{\varepsilon}{2}.
\]

Denote \( \Phi^1(\omega) = \alpha_0(\omega) \alpha_1(\omega) \ldots \alpha_{n+m-1}(\omega) \log \frac{1}{\alpha_{n+m}(\omega)} = \Phi(\omega) - \Phi^-(\omega) \). We are now ready to prove the following.

**Lemma 4.6.** For sufficiently large \( m \), for any \( N \),

\[
\Phi^1(\beta^{N+1}) - \Phi^1(\beta^N) < \frac{\varepsilon}{2}.
\]

**Proof.** According to lemma 4.4 part 4 we have

\[
\left| \log \frac{\alpha_1(\beta^{N+1}) \ldots \alpha_{n+m-1}(\beta^{N+1})}{\alpha_1(\beta^N) \ldots \alpha_{n+m-1}(\beta^N)} \right| < \sum_{i=1}^{n+m-1} 2^{-i(n+m)} / N < \frac{1}{N}.
\]

Hence \( \alpha_1(\beta^{N+1}) \ldots \alpha_{n+m-1}(\beta^{N+1}) < \alpha_1(\beta^N) \ldots \alpha_{n+m-1}(\beta^N) e^{1/N} \), and

\[
\Phi^1(\beta^{N+1}) < \Phi^1(\beta^N) e^{1/N} \log \frac{1}{\alpha_{n+m}(\beta^{N+1})} = \Phi^1(\beta^N) e^{1/N} \frac{\log(N + 1 + \phi)}{\log(N + \phi)}.
\]
Hence
\[
\Phi^1(\beta^{N+1}) - \Phi^1(\beta^N) < \Phi^1(\beta^N) \left( e^{1/N} \log(N+1/\phi) - 1 \right) < 
\Phi^1(\beta^N) \left( 1 + \frac{e}{N} \log(N+1/\phi) - 1 \right).
\]

We make the following calculations. Denote \( x = \log\left( \frac{N+1}{\phi} \right) \), then \( (N+1/\phi)^x = N+1+1/\phi \).
\( x = \log\left( \frac{N+1}{\phi} \right) \), then \( (N+1/\phi)^x < e^{N+1/\phi} \).

It is not hard to see that \( \alpha_k - 1/2 \) for all \( k > 1 \), and we have
\[
\Phi^1(\beta^N) = \alpha_1(\beta^N) \cdots \alpha_{n+m-1}(\beta^N) \log \frac{1}{\alpha_{n+m}(\beta^N)} < \left( \frac{1}{2} \right)^{(n+m-1)/2} \log(N+1/\phi).
\]

Thus
\[
\Phi^1(\beta^{N+1}) - \Phi^1(\beta^N) < \Phi^1(\beta^N) \left( 1 + \frac{e}{N} \log(N+1/\phi) - 1 \right) < 
\left( \frac{1}{2} \right)^{(n+m-1)/2} \log(N+1/\phi) \left( (1 + e/N)(1 + 3/N) - 1 \right) < 
\left( \frac{1}{2} \right)^{(n+m-1)/2} \log(N+1/\phi) \frac{14}{N}.
\]
Since \( \frac{14}{N} \in o(1/\log(N+1/\phi)) \), this expression can be always made less than \( \frac{3}{2} \) by choosing \( m \) large enough. □

Since \( \Phi = \Phi^- + \Phi^1 \), summing the inequalities in Lemmas 4.5 and 4.6 yields the following.

**Lemma 4.7.** For sufficiently large \( m \), for any \( N \),
\[
\Phi(\beta^{N+1}) - \Phi(\beta^N) < \varepsilon.
\]

It is immediate from the formula of \( \Phi(\beta^N) \) that:

**Lemma 4.8.**
\[
\lim_{N \to \infty} \Phi(\beta^N) = \infty.
\]

We are now ready to prove lemma 4.1.

**Proof.** (of lemma 4.1). Choose \( m \) large enough for lemma 4.7 to hold. Increase \( N \) by one at a time starting with \( N = 1 \). We know that \( \Phi(\beta^1) = \Phi(\omega) < \Phi(\omega) + \varepsilon \), and by lemma 4.8, there exists an \( M \) with \( \Phi(\beta^M) > \Phi(\omega) + \varepsilon \). Let \( N \) be the smallest such \( M \). Then \( \Phi(\beta^{N-1}) \leq \Phi(\omega) + \varepsilon \), and by lemma 4.7
\[
\Phi(\beta^N) < \Phi(\beta^{N-1}) + \varepsilon \leq \Phi(\omega) + 2\varepsilon.
\]

Hence
\[
\Phi(\omega) + \varepsilon < \Phi(\beta^N) < \Phi(\omega) + 2\varepsilon.
\]
Choosing \( \beta = \beta^N \) completes the proof. □
The second part of the following Lemma follows by the same argument as Lemma 4.4 by taking \( N \geq 1 \) to be an arbitrary real number, not necessarily an integer. The first part is obvious, since the tail of \( \omega \) has only 1’s.

**Lemma 4.9.** For an \( \omega = \beta^1 \) as above, for any \( \varepsilon > 0 \) there is an \( m_0 > 0 \), such that for any \( m \geq m_0 \), and for any tail \( I = [a_{n+m}, a_{n+m+1}, \ldots] \) if we denote \( \beta^I = [a_1, a_2, \ldots, a_{n+m}, 1, 1, 1, \ldots] \),
then
\[
\sum_{i \geq n+m} \alpha_1(\beta^1) \alpha_2(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)} < \varepsilon,
\]
and
\[
\sum_{i=1}^{n+m-1} \left| \alpha_1(\beta^I) \ldots \alpha_{i-1}(\beta^I) \log \frac{1}{\alpha_i(\beta^I)} - \alpha_1(\beta^1) \ldots \alpha_{i-1}(\beta^1) \log \frac{1}{\alpha_i(\beta^1)} \right| < \varepsilon.
\]

We can now prove lemma 4.2.

**Proof.** (of lemma 4.2). Applying lemma 4.9 with \( \varepsilon/2 \) instead of \( \varepsilon \), we get
\[
\Phi(\beta^I) - \Phi(\omega) = \sum \{ \text{“head”}(\beta^I) - \text{“head”}(\omega) \} + \sum \{ \text{“tail”}(\beta^I) - \text{“tail”}(\omega) \} > -\varepsilon/2 - \sum \{ \text{“tail”}(\omega) \} > -\varepsilon/2 - \varepsilon/2 = -\varepsilon.
\]

We will need a *computable* version of Lemma 4.1 for modifying the conformal radius of the corresponding Julia set.

**Lemma 4.10.** For any given initial segment \( I = (a_0, a_1, \ldots, a_n) \) and \( m_0 > 0 \), write \( \omega = [a_0, a_1, \ldots, a_n, 1, 1, 1, \ldots] \). Then for any \( \varepsilon > 0 \), we can uniformly compute \( m > m_0 \) and an integer \( N \) such that if we write \( \beta = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots] \), where the \( N \) is located in the \( n + m \)-th position, we have
\[
(4.1) \quad r(\omega) - 2\varepsilon < r(\beta) < r(\omega) - \varepsilon,
\]
and
\[
(4.2) \quad \Phi(\beta) > \Phi(\omega).
\]

**Proof.** We first show that such \( m \) and \( N \) exist, and then give an algorithm to compute them. By Lemma 4.1 we can increase \( \Phi(\omega) \) by any controlled amount by modifying one term arbitrarily far in the expansion.

By Theorem 2.10 \( f : \theta \mapsto \Phi(\theta) + \log r(\theta) \) extends to a continuous function. Hence for any \( \varepsilon_0 \) there is a \( \delta \) such that \( |f(x) - f(y)| < \varepsilon_0 \) whenever \( |x - y| < \delta \). In particular, there is an \( m_1 \) such that \( |f(\beta) - f(\omega)| < \varepsilon_0 \) whenever \( m \geq m_1 \).

This means that if we choose \( m \) large enough, a controlled increase of \( \Phi \) closely corresponds to a controlled drop of \( r \) by a corresponding amount, hence there are \( m > m_0 \) and
N such that (4.1) holds. (4.2) is satisfied almost automatically. The only problem is to computably find such m and N.

To this end, we apply Lemma 3.1. It implies that for any specific m and N we can compute r(β). This means that we can find the suitable m and N, by enumerating all the pairs (m, N) and exhaustively checking (4.1) and (4.2) for all of them. We know that eventually we will find a pair for which (4.1) and (4.2) hold. □

5. Proving the Main Theorem

There are countably many oracle Turing Machines. Let us enumerate them in some arbitrary computable fashion M_1^φ, M_2^φ, . . . so that every machine appears infinitely many times in the enumeration. Recall that r(θ) is the conformal radius of the Siegel disk associated with the polynomial P_θ(z) = z^2 + e^{2πiθ}z, or zero, if θ is not a Brjuno number.

We will argue by induction. On each iteration i of the argument we shall maintain an initial segment I_i = [a_0, a_1, . . . , a_N] an interval H_i = [l_i, r_i], and ℓ_i = ℓ(H_i) = r_i − l_i such that the following properties are maintained:

(5.1) 
\[ r_i = r(γ_i), \text{ where } γ_i = [I_i, 1, 1, . . .], \]

and

(5.2) 
For any \( β = [I_i, t_{N, i+1}, t_{N, i+2}, . . .] \) with \( r(β) ∈ [l_i, r_i] \),

the machine \( M_β^φ \) requires at least the time \( h(2 \lceil − \log ℓ_i \rceil + 1) \) to compute the \( \frac{ℓ_i}{2} \)-approximation to \( J(P_β) \). And

(5.3) 
for \( i ≥ 1 \), \( Φ(β) > Φ(γ_{i−1}) − 2^{−(i−1)} \), for any \( β = [I_i, t_{N, i+1}, t_{N, i+2}, . . .] \).

Moreover, the intervals we construct are nested: \([l_i, r_i] ⊂ [l_{i−1}, r_{i−1}] \), and the sequence \( I_i \) contains \( I_{i−1} \) as the initial segment. The numbers \( 2 \lceil − \log ℓ_i \rceil \) form a strictly increasing sequence.

For the basis of induction, set \( I_0 = [1], r_0 = r(γ_0) < 2 \) (by the Schwarz Lemma) and \( l_0 = r_0/2 \), where \( γ_0 = [1, 1, 1, . . .] \). Then for \( i = 0 \) condition (5.1) holds by definition and conditions (5.2) and (5.3) hold because they are empty.

The induction step. We now have the conditions (5.1), (5.2) and (5.3) for some \( i \) and would like to extend them to \( i + 1 \).

Consider the machine \( M^φ_{i+1} \). Set \( ℓ_{i+1} = ℓ_i/20 \). Simulate \( M^φ_{i+1} \) on \( γ_i \) for at most \( h(2 \lceil − \log ℓ_{i+1} \rceil + 1) \) steps to compute \( J_{γ_i} \) with precision \( ℓ_{i+1}/2 \). The machine reads at most \( h(2 \lceil − \log ℓ_{i+1} \rceil + 1) \) bits of the input, and we can compute \( m_0 \) such that this run does not distinguish between \( γ_i \) and \( γ = [I_i, 1, 1, . . ., 1, N_{m_0+1}, N_{m_0+2}, . . .] \). There are two cases:

Case 1: \( M^φ_{i+1} \) does not terminate in the assigned time, or does not output a proper set. In this case, we proceed by setting \( I_{i+1} = [I_i, 1, . . ., 1] \) (with 1’s up to position \( m_0 \)), \( γ_{i+1} = γ_i \),
\( r_{i+1} = r_i \) and \( l_{i+1} = r_{i+1} - \ell_{i+1} \). By Lemma 4.2, we can choose sufficiently many 1’s in \( I_{i+1} \), so that for any \( \beta \) beginning with \( I_{i+1} \), we have \( \Phi(\beta) > \Phi(\gamma_i) - 2^{-i} \).

**Case 2:** \( M^\gamma_{i+1} \) outputs a set \( S \). Compute the conformal radius \( r(S) \). Considerations of Schwarz Lemma imply that for any quadratic Siegel disk, \( r(\Delta) < 2 \). Using the above consideration to bound the constant in Lemma 2.12, we know that for any Julia set \( J(P_\omega) \) which is \( \ell^2_i \)-accurately described by \( S \), we have \( |r(\omega) - r(S)| < 4\sqrt{2}\ell_{i+1} < 6\ell_{i+1} \). Again, there are two cases (if both hold, it doesn’t matter which way to proceed):

**Subcase 2a:** \( r_i - \ell_{i+1} > r(S) + 8\ell_{i+1} \). In this case we proceed by setting \( I_{i+1} = [I_i, 1, \ldots, 1] \) (with 1’s up to position \( m_0 \)), \( \gamma_{i+1} = \gamma_i \), \( r_{i+1} = r_i \), and \( l_{i+1} = r_{i+1} - \ell_{i+1} \).

**Subcase 2b:** \( l_i + 2\ell_{i+1} < r(S) - 8\ell_{i+1} \). By Lemma 4.10, we can select \( \gamma_{i+1} = [I_{i+1}, 1, 1, \ldots] \) by modifying \( \gamma_i \) at an arbitrarily far position, and set \( r_{i+1} = r(\gamma_{i+1}) \) so that \( \Phi(\gamma_{i+1}) > \Phi(\gamma_i) \), \( r_{i+1} \geq l_i + \ell_{i+1} \) and \( [r_{i+1} - \ell_{i+1}, r_{i+1}] \cap [r(S) - 8\ell_{i+1}, r(S) + 8\ell_{i+1}] = \emptyset \). The number \( r_{i+1} \) is computable since it is the conformal radius of a noble Siegel disk. Set \( l_{i+1} = r_{i+1} - \ell_{i+1} \).

We see that the induction is maintained for these parameters.

In either subcase, by Lemma 4.2, we can add sufficiently many 1’s to \( I_{i+1} \), so that for any \( \beta \) beginning with \( I_{i+1} \), we have \( \Phi(\beta) > \Phi(\gamma_i) - 2^{-i} \), and condition (5.3) is satisfied.

**Lemma 5.1.** Denote \( \gamma = \lim_{i \to \infty} \gamma_i \). Then the following equalities hold:

\[
\Phi(\gamma) = \lim_{i \to \infty} \Phi(\gamma_i) \quad \text{and} \quad r(\gamma) = \lim_{i \to \infty} r(\gamma_i).
\]

**Proof.** By the construction, the limit \( \gamma = \lim \gamma_i \) exists. We also know that the sequence \( r(\gamma_i) = r_i \) converges uniformly to some number \( r \), and that the sequence \( \Phi(\gamma_i) \) is monotone non-decreasing, and hence converges to a value \( \psi \) (a priori we could have \( \psi = \infty \)). By the Carathéodory Kernel Theorem (see e.g. [Pom]), we have \( r(\gamma) \geq r > 0 \), so \( \psi < \infty \). On the other hand, by the property we have maintained through the construction, we know that \( \Phi(\gamma) > \Phi(\gamma_i) - 2^{-i} \) for all \( i \). Hence \( \Phi(\gamma) \geq \psi \).

From [BC], we know that

\[
\psi + \log r = \lim_{i \to \infty} (\Phi(\gamma_i) + \log r(\gamma_i)) = \Phi(\gamma) + \log r(\gamma).
\]

Hence we must have \( \psi = \Phi(\gamma) \), and \( r = r(\gamma) \), which completes the proof. \( \square \)

The conformal radius \( r(\gamma) \) is computable, since the convergence \( r(\gamma_i) \to r(\gamma) \) is uniform. Thus \( J_{P_\gamma} \) is also computable by Theorem 2.11. By construction, it satisfies all of the required properties. Note that the value \( \gamma \) itself is also computable.
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