Abstract

Aharonov-Kaufherr model of quantum space-time which accounts Reference Frames (RF) quantum effects is considered in Relativistic Quantum Mechanics framework. For RF connected with some macroscopic object its free quantum motion - wave packet smearing results in additional uncertainty of test particle coordinate. Due to the same effects the use of Galilean or Lorentz transformations for this RFs becomes incorrect and the special quantum space-time transformations are introduced. In particular for any RF the proper time becomes the operator in other RF. This time operator calculated solving relativistic Heisenberg equations for some quantum clocks models. Generalized Klein-Gordon equation proposed which depends on both the particle and RF masses.

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* E-mail Mayburov@sgi.lpi.msk.su
1 Introduction

Some years ago Aharonov and Kaufherr have shown that in nonrelativistic Quantum Mechanics (QM) the correct definition of physical reference frame (RF) must differ from commonly accepted one, which in fact was transferred copiously from Classical Physics. The main reason is that to perform exact quantum description one should account the quantum properties not only of studied object, but also RF, despite the possible practical smallness. The most simple of this RF properties is the existence of Schroedinger wave packet of free macroscopic object with which RF is usually associated. If this is the case it inevitably introduces additional uncertainty in the measurement of object space coordinate. Furthermore this effect account results in the coordinate transformations between two such RFs called quantum RFs, principally different from the Galilean ones.

In their work Aharonov and Kaufherr formulated Quantum Equivalence principle - all the laws of Physics are invariant under transformations between both classic and quantum RFs. In their paper its applicability for nonrelativistic QM was proved. The importance of RF quantum properties account was shown already in Quantum Gravity and Cosmology studies. Further studies of quantum RF effects can help also to understand some features of quantum space-time at small distances. The aim of our study is the development of relativistically covariant quantum RF description; our first results were reported in . It will be shown that the transformations of the test particle state vector between two quantum RF obeys to relativistic invariance principles, but due to dependence on RF state vector differs from Poincare Group transformations. The time ascribed to such RF becomes the operator, corresponding to proper time of Classical Relativity. As will be shown this operator introduces the quantum fluctuations in the classical Lorentz time boost in moving RF time measurements. Our paper is organized as follows: in the rest of this chapter our model of quantum RF will be formulated and its compatibility with Quantum Measurement Theory discussed. In a chapter 2 the new canonical formalism of quantum RF states and their transformations described, which is quite simple and more suitable to our purposes. The relativistic equations for quantum RF and the resulting quantum space-time transformations are regarded in chapter 3. In a final chapter the obtained results and their interpretation are discussed.

In QM framework the system regarded as RF presumably should be able to measure the observables of studied quantum states and due to it to include measuring devices - detectors. As the realistic example of such RF we can regard the photoemulsion plate or the diamond crystal which can measure microparticle position relative to its c.m. and simultaneously record it. At first sight it seems that due to it quantum RF problem must use as its basis some model of the state vector collapse. Yet despite the multiple proposals up to now well established theory of collapse doesn’t exist. Alternatively we’ll show that our problem premises doesn’t connected directly with the state vector collapse mechanism and and in place of it the two simple assumptions about RF and detector states properties can be used. The first one is that RF consists of finite number of atoms (usually rigidly connected) and have the finite mass. Our second assumption needs some prelimi-
nary comments. It’s well known that the solution of Schroedinger equation for any free quantum system consisting of $N$ constituents can be presented as:

$$\Psi(\vec{r}_1, \ldots, \vec{r}_n, t) = \sum c_l \Phi_c^l(\vec{R}_c, t) \ast \phi_l(\vec{r}_{ij}, t)$$

(1)

where center of mass coordinate $\vec{R}_c = \sum m_i \ast \vec{r}_i / M$. $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ are the relative or 'internal' coordinates of constituents [4]. Here $\Phi_c^l$ describes the c.m. motion of the system. It means that the evolution of the system is separated into the external evolution of pointlike particle $M$ and the internal evolution completely defined by $\phi_l(\vec{r}_{ij}, t)$. So the internal evolution is independent of whether the system is localized in the macroscopic 'absolute' reference frame (ARF) or not. Relativistic QM and Field Theory evidences that the factorization of c.m. and relative motion holds true even for nonpotential forces and variable $N$ in the secondarily quantized systems [10]. Moreover this factorization expected to be correct for nonrelativistic systems where binding energy is much less then its mass $M$, which is characteristic for the real detectors and clocks. Consequently it’s reasonable to extend this result on the detector states despite we don’t know their exact structure. We’ll use it quite restrictively and assume that the factorization of the c.m. motion holds for RF only in the time interval $T$ from RF preparation moment , until the act of measurement starts ,i.e. when the measured particle collides with it. Formally our second assumption about RF properties is that during period $T$ its state is described by wave function generalizing (1):

$$\Psi(\vec{R}_c, u_i, t) = \sum c_l \Phi_c^l(\vec{R}_c, t) \ast \phi_l(u_i, t)$$

(2)

where $u_i$ denote all internal detector degrees of freedom, and this state evolves during $T$ according to Schroedinger equation (or some field equation). Its possible violation at later time when the particle state collapse occurs is unimportant for our model. Due to it we’ll assume always in our model that all measurements are performed on the quantum ensemble of observers $F^\dagger$. It means that each event is resulted from the interaction between the 'fresh' RF and particle, prepared both in the specified quantum states, alike the particle alone in the standard experiment. To simplify our calculations normally we’ll take below all $c_l = 0$ except $c_1$ which wouldn’t influence our final results.

The common opinion is that to observe experimentally measurable smearing of macroscopic object demands too large time , but for some mesoscopic experiments it can be reasonably small to be tested in the laboratory conditions [7]. We don’t consider in our study the influence of RF recoil effects on the measurements results which can be made arbitrarily small [1].

2 Quantum Coordinates Transformations

To illustrate the meaning of Quantum RF consider gedankenexperiment in two dimensions $x, y$, where in ARF the wave packet of RF $F^\dagger$ described by $\psi_1(x)\xi_1(y)$ at some time moment $T$. The test particle $n$ with mass $m_2$ belongs to narrow
beam which average velocity is orthogonal to \( x \) axe and its wave function at \( T \) is \( \psi_n(x)\xi_n(y) \). Before they start to interact this system wave function is the product of \( F^1 \) and \( n \) packets. We want to find \( n \) wave function for the observer in \( F^1 \) rest frame. In general it can be done by means of the canonical transformations described below, but in the simplest case when \( n \) beam is localized and \( \psi_n(x) \) can be approximated by delta-function \( \delta(x-x_b) \) this wave function in \( F^1 \) as easy to find \( \psi_n(x) = \psi_1(x-x_b) \). It shows that if \( F^1 \) wave packet have average width \( \sigma_x \) then from the 'point of view' of observer in \( F^1 \) each object localized in ARF acquires wave packet of the same width \( \sigma_x \) and any measurement both in \( F^1 \) and ARF, as will be shown below confirms this conclusion.

Now we’ll regard the nonrelativistic formalism alternative to Quantum Potentials one used in \( \mathbb{U} \). Consider the system \( S_N \) of \( N \) objects \( B^i \) which include \( N_g \) pointlike 'particles' \( G^i \) and \( N_f \) frames \( F^i \), which in principle can have also some internal degrees of freedom described by (1). For the start we’ll take that particles and RF coordinates \( \vec{r}_i \) are given in absolute (classical) ARF having very large mass \( m_A \). We should find two transformation operators - from ARF to quantum RF, and between two quantum RF, but it’ll be shown that in general approach they coincide. We’ll use Jacoby canonical coordinates \( \vec{q}_{lj} \), which for \( F^1 \) rest frame are equal :

\[
\vec{q}_i = \frac{\sum_{j=i+1}^{N} m_j r^l_j}{M^{l+1}_N} - \vec{r}_i, j < N; \quad \vec{q}_N = \vec{R}_{cm} \tag{3}
\]

Here \( \vec{r}_j = \vec{r}_j, m_j = m_j \), and for \( l > 1 \):

\[
\vec{r}_1 = \vec{r}_i; \quad \vec{r}_j = \vec{r}_{j-1}, 1 < j < l; \quad \vec{r}_l = \vec{r}_j, j > l \tag{4}
\]

The same relations connect \( m_j \) and \( m_j \), \( M^{l,i}_N = \sum_{j=i}^{N} m_{lj} \) (if upper indexes \( i, l \) are omitted, then \( i, l = 1 \)). Conjugated to \( \vec{q}_i \) canonical momentums can be easily found, for example :

\[
\vec{p}_1 = \mu_i \left( \frac{\vec{p}_{i+1}}{M^{l+1}_N} - \frac{\vec{p}_l}{m_i} \right), \quad \vec{p}_N = \vec{p}_l \tag{5}
\]

where \( \vec{p}_l = \sum_{j=1}^{N} \vec{p}_j \) and reduced mass \( \mu_i^{-1} = (M^{l+1}_N)^{-1} + m_i^{-1} \).

The relative coordinates \( \vec{r}_j - \vec{r}_1 \) can be represented as the linear sum of several coordinates \( \vec{q}_i \); they don’t constitute canonical set due to the quantum motion of \( F^1 \).

We consider first the transformation between two quantum RF and start from the simplest case \( N_f = 2, N_g = 0 \). This is just the space reflection of \( F^1 \) coordinate \( \vec{q}_1 = -\vec{q}_1 \) performed by the parity operator \( \hat{P}_1 \). The next case \( N_f = 2, N_g = 1 \) is \( \vec{q}_1 \) coordinates bilinear transformation exchanging \( \vec{r}_2, \vec{r}_1 \) :

\[
\vec{q}_{1,2}^{B} = \tilde{U}_{2,1} \vec{q}_1^{A} \tilde{U}_{2,1}^{-1} = a_{1,2} \vec{q}_1 + b_{1,2} \vec{q}_2 \tag{6}
\]

Corresponding unitary operator can be decomposed as \( \tilde{U} = \mathcal{C}_2 \mathcal{R} \mathcal{C}_1 \), where \( \mathcal{C}_{1,2} \) are the dilatation operators, which action changes the coordinate scale. For example \( \mathcal{C}_1 \) results in \( \vec{q}_i^B = c_i^1 \vec{q}_i^A \), where \( c_i^1 \) proportional to \( \mu_i \).
$\hat{R}$ is the rotation on $\vec{q}_{1,2}$ intermediate coordinates hypersurface on the angle:

$$\beta = -\arccos\left[\frac{m_2m_1}{(m_3 + m_2)(m_1 + m_3)}\right]$$

(7)

For the general case $N > 3$ it’s possible nevertheless to decompose the transformation from $F^j$ to $F^k$ as the product of analogous bilinear operators. Really if to denote as $\hat{S}_{i+1,j}$ the operator exchanging $F^i, F^{i+1}$ in $\vec{q}_i$ set, as follows from (3) it changes in fact only $\vec{q}_i, \vec{q}_{i+1}$ pair. $\hat{U}_{2,1} = \hat{S}_{2,1}$ and all $\hat{S}_{j,j-1}$ have the analogous form changing only parameters $\beta, c_k$. Then the transformation operator from $F^1$ to $F^k$ is:

$$\hat{U}_{k,1} = \hat{S}_{2,1}\hat{S}_{3,2}...\hat{S}_{k,k-1}$$

(8)

It follows immediately that the transformation from $F^j$ to $F^k$ is $\hat{U}_{j,k} = \hat{U}_{k,1}\hat{U}_{j,1}^{-1}$.

To find the transformation operator from the classical ARF to $F^1$ we’ll regard ARF as the quantum object $B^{N+1}$ with infinite $m_{N+1}$ belonging to extended system $S_{N+1}$. ARF ‘classical’ Jacoby set is $\vec{q}_i^A = \vec{r}_i - \vec{r}_A^c$, but acting by parity operators we’ll transform it to $\vec{q}_i^A = -\vec{q}_i^A$. Then it’s easy to see that for $S_{N+1} \vec{q}_i^A = \vec{q}_i^A$ as follows from (3). Note that formally we can regard also each particle $G^i$ as RF and perform for them the transformations $\hat{S}_{j,j-1}$ described above. Then omitting simple calculations we obtain that operator performing transformations from ARF to $F^1$ is equal to $\hat{U}_{A,1} = \hat{U}_{N+1,1}$ for infinite $m_{N+1}$. In this case new $\vec{q}_i^A$ set for $S_{N+1}$ can be rewritten as the function of $\vec{r}_i^A, \vec{r}_{N+1}^A = \vec{r}_A^c$ of (3) to which formally must be added $\vec{q}_{N+1}^A = \vec{r}_A^c$.

The free Hamiltonian of the system objects motion in ARF is:

$$\hat{H} = \hat{H}_s + \hat{H}_c = \frac{(\vec{p}_N^1)^2}{2M_N} + \sum_{j=1}^{N-1} \frac{(\vec{p}_i^1)^2}{2\mu_i}$$

(9)

Hamiltonian of $S_N$ in $F^1$ should depend on relative $B^i$ momentums only, so we can regard $\hat{H}_c$ as the real candidate for its role. It results into modified Schroedinger equation, in which objects evolution depends in fact on observer mass $m_1$. Yet relativistic analysis given below introduces corrections to $\hat{H}_c$, which normally are small but essential for the interpretation.

In general this quantum transformations in 2 or 3 dimensions should also take into account the possible rotation of quantum RF relative to ARF, which introduce additional angular uncertainty into objects coordinates. Thus after performing coordinate transformation $\hat{U}_{A,1}$ from ARF to $F^1$ c.m. we must rotate all the objects (including ARF) around it on the uncertain polar and azimuthal angles, $\phi, \theta$, so the complete transformation is $\hat{U}_{A,1} = \hat{U}_{A,1}^R \hat{U}_{A,1}$. Such rotation transformation operator commutes with $\hat{H}_c$ and due to it can’t change the evolution of the transformed states (3).

Now we’ll discuss the measurements in quantum RF and for this purpose return to the gedankenexperiment regarded above. Let’s assume in addition that $F^1$ includes detector $D_0$ which can measure the distance between $n$ and $F^1$ c.m. $\Delta = x_n - x_1$. $F^1$ and ARF observers will treat the same event unambiguously as $n$
measurement by $D_0$ (or its flight through $D_0$). In $F^1$ $n$ state vector collapse reveals itself by the detection process in $D_0$ initiated by $n$ absorption. For ARF the collapse results from the $n$ nonobservation in a due time in ARF detectors - so called negative result experiment. This measurement means not only the reduction of $\psi'_n$ in $F^1$ to some $|\Delta_j\rangle$ eigenstate, but also the reduction of $\psi_1(x)$ in ARF to $\delta(x - x_b)$ in our example. Now let’s consider for arbitrary $\psi_n$ the reduction in ARF for $\Delta$ measurement of the state $\psi_s = \psi_n(x_n)\psi_1(x_1)\varphi(u)$ corresponding to the density matrix $\hat{\rho}_m$, where $\varphi$ is $F^1$ internal (detector) state. If the measurement of $|\Delta_j\rangle$ eigenstate results in $\varphi_j$ detector state, described correspondingly by the projection operators $\hat{P}_{\Delta j}, \hat{P}_{\varphi j}$, then the density matrix after the collapse is given by [10]:

$$\hat{\rho}_f = \sum_j \hat{P}_{\Delta j}\hat{P}_{\varphi j}\hat{\rho}_m \hat{P}_{\Delta j}\hat{P}_{\varphi j}$$

In $S_2$ Jacoby coordinates $\Delta = q_{1x}, x_{cm} = q_{2x}$. Then from the relation $|\Delta\rangle|x_{cm}\rangle = |x_1\rangle|x\rangle$, it follows that $\hat{\rho}_f$ is a mixture of the states nonlocalized in ARF of the form $\psi_n(x)\psi_1(x - \Delta_j)$ for the given $\Delta_j$ value. Their effective width is of the order $\sigma_n$, if $\sigma_n \ll \sigma_1$. This $F^1$ state reduction takes place for any $\psi_n$, but without $F^1$ localization in ARF. So in general $n$ coordinate $\Delta$ measurement in $F^1$ transforms initial $F^1$ state into another nonlocalized state. This results demonstrates that nonlocalized RF conserves this delocalization after the measurements of the other nonlocalized states. The Decoherence model calculations of the particle coordinate measurements by $F^1$ detector support this conclusion [7].

### 3 Relativistic Equations

Now we’ll consider possible generalization of Quantum Equivalence principle for relativistic QM. The relativistic covariant formalism will be studied here with the model of relativistic wave packets of macroscopic objects regarded as quantum RF [11]. We’ll take that all RF constituents spins and orbital momentums are compensated so that its total orbital momentum is zero.

In nonrelativistic mechanics time $t$ is universal and is independent of observer, while in relativistic case each observer in principle has its own proper time $\tau$. We don’t know yet the origin of the physical time, but phenomenologically we can associate it with the clock hands motion or some other relative motion of the system parts. Meanwhile it will be shown that like in the case of the position measurement this internal processes can be disentangled from the system c.m. motion. Then RF $F^2$ wave packet evolution can be described by the relativistic equation for their c.m. motion relative to other RF and $F^2$ internal degrees of freedom evolution which define its clocks motion and consequently its proper time $\tau_2$ are factorized from it.

We’ll start the proper time study with the simple models of quantum RFs with clocks, yet we expect its main results to be true also for the more sophisticated models. Consider the evolution of some system $F^2$ where the internal interactions are nonrelativistic, which as was discussed in chap.1 is a reasonable approximation for the measuring devices or clocks. More precisely we consider the complex sys-
tem which Hamiltonian in ARF is analogous to Klein-Gordon square root (KGR) Hamiltonian of pointlike particle \[14\] :

\[ H_T = \left[ (m'_2 + H_c)^2 + \vec{p}_2^2 \right]^\frac{1}{2} \]

, where \( H_c = V + T \) is clocks Hamiltonian , \( m'_2 \) is \( F^2 \) constituents rest mass. We'll use below the parameter \( \alpha_I = \frac{\hbar c}{m'_2} \), describing the relative strength of the internal \( F^2 \) forces , which for the realistic clocks can be as small as \( 10^{-10} \). In \( F^2 \) c.m. it can be defined as \( \alpha_I = m'_2^{-1} \langle \varphi_2 | \hat{H}_c | \varphi_2 \rangle \) where \( \varphi_2 \) is \( F^2 \) internal state of \([13]\) , yet our general formalism doesn’t demand factorizable \( \varphi_2 \) existence at all.

As the illustrative example we’ll consider the model of quantum clocks - rotator \( C_q \) proposed by Peres \([12]\) with the Hamiltonian \( \hat{H}_c = -2\pi\omega i\frac{\bar{\theta}}{m'_2} \),where \( \theta \) is the rotator’s polar angle (\(-\pi < \theta < \pi\)). Preparing the special \( C_q \) initial state at \( t = 0 \), which is analog of Gaussian wave packet :

\[ |\psi_0\rangle = \sum_m e^{im\hat{\theta}} \frac{\hat{H}_c}{\sqrt{N}} \]

where \( m = -J_z, ..., J_z, N = 2J_z + 1 \) one obtains the close resemblance of the classical clocks evolution. Resulting \( C_q \) state \( \varphi_c(\theta - 2\pi t) \) for large \( N \) has the sharp peak at \( \theta = 2\pi t \) with the uncertainty \( \Delta_\theta = \pm \frac{\pi}{N} \) and can be visualized as the constant hand motion on the clocks circle. As was shown in \([13]\) the observable \( \hat{\tau} \) performs the nonshifted measurement of time parameter \( t \) if at any \( t \bar{\tau} = t \) and its dispersion \( D(\tau) \) is finite. Then the operator \( \hat{\theta}_R = \frac{\theta}{2\pi\omega} \) describes nonshifted \( t \) measurement in the interval \( 0 < t < T \), where \( T = \omega^{-1} \) where its dispersion \( D(\hat{\theta}_R) = c\frac{T^2}{4N^2} \) , (for \( t > T \) it describes \( t' = \text{mod}(t, T) \)). Suppose that both \( F^2 \) and ARF carry the clocks \( C_q, C'_q \) performing the proper time measurements for the observables \( \hat{\tau}^p_2 = \hat{\theta}_R, \hat{\tau}'_0 = \hat{\theta}'_R \) in their rest frames. Yet \( C_q \) angle \( \theta \) in principle can be measured also in ARF so that its proper time \( \hat{\tau}_2 \) is the observable in ARF proportional to \( C_q \) angle \( \theta \) and differ from \( \hat{\tau}'_2 \). For the simplicity we’ll suppose below that \( D(\hat{\theta}_R) \gg D(\hat{\theta}'_R) \) and \( T' = T \).

This relativistic time operator \( \hat{\tau}_2 \) can be found solving Heisenberg equation for the \( C_q \) relativistic Hamiltonian \( \hat{H}_T \). In particular to obtain \( C_q \) angle operator \( \theta(\tau_0) \) for Hamiltonian \( H_T \) in ARF the commutation relations for operator functions \([Q, F(P)] = -iF'(P)\) can be used \([13]\). After the simple algebra one obtains the evolution equation for \( \theta(\tau_0) \) :

\[ \dot{\theta} = -i[\theta, H_T] = \frac{2\pi\omega m_2}{(m_2^2 + \vec{p}_2^2)^{\frac{3}{2}}} = 2\pi\omega B_2 \ (11) \]

where \( m_2 = m'_2 + H_c \) is \( F^2 \) mass operator. The solution of this equation is :

\( \theta(\tau_0) = 2\pi\omega B_2 \tau_0 + \theta(0) \) where \( B_2 \) can be called time boost operator. From the relation \( \hat{\tau}_2 = \frac{\theta}{2\pi\omega} \) it follows that the proper time operator is equal to :

\[ \hat{\tau}_2 = B_2 \tau_0 + \frac{\theta(0)}{2\pi\omega} \ (12) \]

where constant operator \( \theta(0) \) can be defined from \( C_q \) nonrelativistic Hamiltonian \( H_c \). Despite \( \bar{\theta}(0) = 0 \) it produces the additional quantum fluctuations. Due to the
cilical $C_q$ evolution equation \( \text{(12) is formally fulfilled only for } \tau_0 < T \), but below it will be shown that it has fundamental origin independent of the particular clocks model and holds for any $\tau_0 > 0$.

Then $\hat{\tau}_2$ expectation value and dispersion are:

\[
\bar{\tau}_2 = \bar{B}_2(\bar{p}_2) \tau_0 \quad \text{(13)}
\]
\[
D(\tau_2) = D(B_2)\tau^2_0 + \bar{G}_2 \tau_0 + D_0
\]

where $D_0 = D(\theta_R)$ and $D(B_2) = \bar{B}^2_2 - (\bar{B}_2)^2$. Operator $\bar{G}_2$ in the second $\alpha_I$ order is equal to:

\[
\bar{G}_2 = \frac{2B_2\theta(0)}{2\pi\omega} + \frac{\theta^2_0}{4\pi^2\omega^2}(m_2^2 + \bar{p}_2^2)^\frac{3}{2}
\]

$G_2$ is connected in fact with the interference between $B_2$ and $\theta(0)$ operators. It seems to be some analogy between this term and nonexponential and momentum dependent corrections to the relativistic decay amplitudes \[14\].

To make the interpretation of the time operator $\hat{\tau}_2$ more clear let’s consider first $C_q$ evolution in its rest frame. Then from Heisenberg equation for $\theta, H_c$ it follows $\hat{\tau}_2 = I\tau_2 + \theta_R(0)$, where $I$ is unit operator. $\hat{\tau}_2$ dispersion is equal to $D_0$ which means that its real difference from $I\tau_2$ can be done very small. $I\tau_2$ represents the parameter $\tau_2$ and $\hat{\tau}_2$ in fact is pseudotime observable which approximates $\tau_2$ with the arbitrary accuracy. In distinction $\hat{\tau}_2$ dispersion grows unrestrictedly, which means that it can’t approximates any time parameter. Consider now the evolution equation for $F^2$ state in the first order of $\alpha_I$, where it’s possible to factorize total $F^2$ Hamiltonian $\hat{H}_T$:

\[
-i\frac{d\Psi_2}{d\tau_0} = (m_2^2 + \bar{p}_2^2)^\frac{3}{2}\Psi_2 \simeq \left[ \frac{m_2^2 \hat{H}_c}{(m_2^2 + \bar{p}_2^2)^\frac{3}{2}} + (m_2^2 + \bar{p}_2^2)^\frac{3}{2} \right]\Psi_2 \quad \text{(14)}
\]

For the simplicity we’ll choose the special initial $F^2$ state $\Psi_2(0) = \Phi_2(\bar{p}_2)\varphi_2$, where $\varphi_2 = |v_0\rangle$, such that $C_q$ orbital momentum $J_z$ and $F^2$ momentum are parallel. So $\Phi_2 = \sum c_i |\bar{p}_2i\rangle$, and $\bar{p}_2i = (0, 0, p_{ci})$. This state describes $F^2$ clocks sinchronized with ARF clocks at $\tau_0 = 0$. It evolves into

\[
\Psi_2(\tau_0) = \sum c_i \varphi_2i(u_i, \tau_0) |\bar{p}_2i\rangle e^{-iE(\bar{p}_2i)\tau_0} \quad \text{(15)}
\]

where $\varphi_2i(u_i, 0) = \varphi_2(u_i, 0)$, $E(\bar{p}) = (m_2^2 + \bar{p}^2)^\frac{3}{2}$. In this case $\varphi_2$ evolution in ARF is described by the boosted Schrödinger equation:

\[
-i\frac{d\varphi_2i}{d\tau_0} = \frac{m_2^2 \hat{H}_c}{(m_2^2 + \bar{p}_2^2)^\frac{3}{2}} \varphi_2i = \hat{B}i0(\bar{p}_2i)\hat{H}_c \varphi_2i \quad \text{(16)}
\]

This equation describes the time dilatation in ARF in comparison with $F^2$ c.m. for any processes in which $F^2$ constituents interact. Due to this factorization it eq. \[16\] is easily solved for $F^2$ clocks:

\[
\Psi_2(\tau_0) = \sum_l c_l \varphi_c(\theta - 2\pi\omega B_0 \tau_0)|\bar{p}_2l\rangle e^{-iE(\bar{p}_2l)\tau_0} \quad \text{(17)}
\]
where \( B_t = B_\theta(\vec{p}_\theta) \). It shows that at any \( \tau_0 > 0 \) \( \Psi_2 \) is the entangled superposition of the states which \( F^2 \) clocks acquires at the consequent \( \tau_2 \) moments. It means for example that in ARF \( F^2 \) clocks can show 3,4 and 5 o'clocks simultaneously which can be checked by its hand angle measurement. Note that \( \hat{B}_2 \) approximates the classical Lorentz factor inverse value - \( \hat{B}_2 = \gamma^{-1}(\vec{v}) \), if the \( \Phi(\vec{p}_2) \) packet width \( \sigma_p \to 0 \). In this case one obtains \( \hat{\tau}_2 = \gamma^{-1}I\tau_0 + \theta_R(0) \) which gives just classical time boost in moving RF for \( \tau_2 \) value.

Obtained results suppose that the proper time of any quantum RF (\( F^2 \)) being the parameter in it simultaneously will be the operator from the ‘point of view’ of other quantum RF. This operator measurement shows how much time passed in \( F^2 \) in this particular event and can give quite different value for another event. It means that the time moments in different RFs corresponds only statistically with the dispersion of \( \tau_2 \) point in ARF given by (13). It differs from Classical Relativity where one - to - one correspondence between \( \tau_2, \tau_0 \) time moments always exists.

In general case \( C_q \) state is quite complicated due to Lorentz transformation of the large orbital momentum components of \( |v_0\rangle \). But as follows from (12) \( \hat{\tau}_2 \) expectation value and the dispersion leading term are independent on it and this state can influence only \( D_0 \) and \( G_2 \) enlarging so the clocks dispersion.

The more appropriate \( C_q \) model of the time measurement considers the free particle \( m \) motion for the time observable \( \hat{t} = \frac{\hbar}{p_x} \) proportional to the particle’s path length . For the Gaussian packet \( \varphi_x = Aexp[-(\vec{p}_x a_x - p_x a_x)^2] \) the operator \( \hat{t} \) in \( F^2 \) c.m. performs the nonshifted \( t \) measurement with the finite dispersion for \( 0 < t < \infty \). In the relativistic case we’ll start with the Hamiltonian of two objects \( a, b \) relative motion ( see eq. (21) below ) in their c.m.s.:

\[
\hat{H}_s = (m_a^2 + q_{ab}^2)^{\frac{1}{2}} + (m_b^2 + q_{ab}^2)^{\frac{1}{2}}
\]

where \( q_{ab} \) is \( m_b \) relative invariant momentum \[17\]. If \( |q_{ab}| \) is small we can choose as \( p_x q_{ab} \) projection along any suitable direction and \( x = i\frac{\partial}{\partial p_x} \). Then \( \hat{H}_T \) mass operator \[ m_2 = m_a + m_b + \frac{v_2^2}{2\mu_{ab}} + E_k(p_y, p_z) \), where \( E_k \) can be neglected in the calculations. So the time in this model can be defined measuring the distance between \( F^2 = a \) and some particle \( b \) emitted by \( F^2 \).

Analogous to (11) evolution equation results into the proper time operator:

\[
\hat{\tau}_2 = \frac{p_x B_2(\vec{p}_2)}{\bar{p}_x} \tau_0 + \frac{\mu_{ab} x(0)}{\bar{p}_x}
\]

(18)

Its expectation value and dispersion are given by (13), but \( G_2 \) and dispersion parameters are different:

\[
G_2 = x(0) \frac{p_x m_2}{(m_2^2 + \bar{p}_2^2)^{\frac{1}{2}}} + \frac{p_x m_2}{(m_2^2 + \bar{p}_2^2)^{\frac{1}{2}}} x(0)
\]

\[
D(B_2) = \frac{\bar{p}_x^2 \bar{B}_2^2 - (\bar{B}_2)^2}{(\bar{p}_x)^2} \bar{B}_2^2; \quad D_0 = \frac{\mu_{ab} a_2^2}{\bar{p}_x}
\]

(19)

The factor \( \frac{\bar{p}_x}{\bar{p}_x} \) produces additional \( \hat{\tau}_2 \) fluctuations resulting from the particle velocity spread. without changing its expectation value. Due to this effect absent in \( C_q \).
rotator model the part of \( D(\tau_2) \):
\[
D_x = D_0 + \frac{\bar{p}_x^2 - (\bar{p}_x)^2}{(\bar{p}_x)^2} (\bar{B}_2)^2 \tau_2^2
\]
can be related to the packet smearing along \( x \) coordinate, regarded as the clocks mechanism uncertainty. The realization of \( x \) measurement in ARF can be the intricated procedure, which scheme we don’t intend to discuss here. Some examples of the analogous nonlocal observables measurements are described in [10].

To calculate the time operator between two quantum RFs it’s necessary first to find the evolution equation for the free motion in quantum RF. For the beginning we’ll consider the evolution of system \( S_2 \) of RF \( F_1 \) and the neutral spinless particle \( G^2 \) which momentums \( \vec{p}_i \) and energies \( E_i \) are defined in classical ARF. If to regard initially prepared states including only positive energy components, then their joint state vector evolution in ARF is defined by the sum of two (KGR) Hamiltonians [10]:
\[
-\frac{i}{\hbar} \frac{d\Psi_s}{d\tau_0} = \left[ (m_1^2 + \bar{p}_1^2)^{\frac{1}{2}} + (m_2^2 + \bar{p}_2^2)^{\frac{1}{2}} \right] \Psi_s
\]  
(20)

From it one should extract Hamiltonian \( \hat{H}^1 \) of \( S_2 \) in \( F_1 \) rest frame which velocity relative to ARF is formally equal to \( \vec{\beta}_1 = \vec{p}_1E^{-1}_1 \). Analogously to the calculations of \( G^2 \) energy and momentum \( s_{12}, \vec{q}_{12} \) in c.m.s. by means of Lorentz transformation with the parameter \( \vec{\beta}_1 \) written here in vector form we define \( S_2 \) energy and \( G^2 \) momentum in \( F_1 \) rest frame:
\[
\vec{p}_{12} = \frac{s_{12}}{m_1} \vec{p}_2 + \frac{(\vec{n}_1 \vec{p}_2)(E_1 - m_1)\vec{n}_1 - E_2 \vec{p}_1}{m_1}
\]
\[
E^1_s = (s^2_{12} + \bar{p}_{12}^2)^{\frac{1}{2}} = m_1 + (m_2^2 + \bar{p}_{12}^2)^{\frac{1}{2}}
\]  
(21)
where \( \vec{n}_1 = \vec{p}_1|\vec{p}_1|^{-1} \). Yet \( E_i, \vec{p}_i \) are the operators and their transformations formally must be performed by the action of Poincare generators. Really it’s easy to show that the transformation (21) can be described as the generalization of Poincare group transformations when its parameters \( \vec{a}, \vec{\beta} \) becomes the operators. The corresponding transformation operator is equal to:
\[
\hat{U}'_{A,1} = e^{i\vec{N}_2 \vec{\beta}_1}
\]  
(22)
where \( \vec{N}_2 = \frac{1}{2}(E_2\vec{r}_2 + \vec{r}_2 E_2) \) is Lorentz generator, \( \vec{r}_2 = i\frac{\partial}{\partial \vec{p}_2} \), \( \vec{\beta}_1 \) is \( F_1 \) velocity operator defined above. Under this transformation \( \vec{p}_2 \rightarrow \vec{p}_{12}, E_2 \rightarrow E_{12} = E^1_1 - m_1 \). To obtain the evolution equation for \( F_1 \) proper time \( \tau_1 \) we’ll assume that the operator relation \( -i\frac{\partial}{\partial \tau_1} = \hat{H} \) is applicable also for quantum RFs. The resulting evolution equation for \( G_2 \) for \( F_1 \) proper time \( \tau_1 \) is:
\[
-\frac{i}{\hbar} \frac{d\psi^1_1}{d\tau_1} = \hat{H}^1 \psi^1_1 = [m_1 + (m_2^2 + \bar{p}_{12}^2)^{\frac{1}{2}}] \psi^1_1
\]  
(23)
It’s easy to note that \( \hat{H}^1 \) depends only on relative motion of \( F_1, G^2 \) and can be rewritten as function of \( \vec{q}_{12} \). \( \hat{H}^1 \) coincides with KGR Hamiltonian, if \( m_1 \) regarded as
the arbitrary constant added to $G^2$ energy. Consequently we can use in $F^1$ the same momentum eigenstates spectral decomposition and the states scalar product [10]. This spectra can be used also as the basis of $G^2$ field secondary quantization in $F^1$. To perform it we can introduce now antiparticles of $G^2$, to which negative energy $E_{12}$ is attributed. Then taking the square of eq. (23), where $m_1$ is subtracted we can take as the field equation in $F^1$ coinciding with Klein-Gordon equation:

$$\frac{\partial^2 \psi^1}{\partial \tau_1^2} = (m_1^2 + \vec{p}_{12}^2) \psi^1$$

(24)

If the number of particles $N_q > 1$ analogous to $\vec{p}_{21}$ of (21) canonical momentums can be defined for each particle separately. Alternatively for their states transformations from ARF to $F^1$ the clasterization formalism can be used described here for $N_q = 2$ [17]. According to previous arguments Hamiltonian in $F^1$ of two free particles $G^2, G^3$ rewritten through the system observables acquires the form:

$$\hat{H}^1 = m_1 + (s_{23}^2 + \vec{p}_{23}^2)^{\frac{1}{2}}$$

(25)

where $s_{23}$ is $G^2, G^3$ invariant mass. In clasterization formalism at the first level the relative motion of $G^2, G^3$ defined by $\vec{q}_{23}$ their relative momentum is considered. At the second level we regard them as the single quasiparticle - cluster $C_{23}$ with mass $s_{23}$ and $\vec{p}_{23}$ momentum in $F^1$ which evolution is studied. So at any level we regard the relative motion of two objects only and this procedure can be extended in the obvious inductive way to arbitrary $N$.

As the space coordinate operator in $F^1$ the generalization of Newton-Wigner ansatz [15] is natural to consider:

$$\hat{x}_{12} = i \frac{d}{dp_{12,x}} - i \frac{p_{12,x}}{2E_{12}^2}$$

(26)

To obtain eq.(23) only equivalence principle was used assuming that any quantum RF has its proper time without use of any relation between $\tau_0$ and $\tau_1$ which will be studied now. In this framework for $F^2$ with $C_q$ clocks its Hamiltonian in $F^1$ can be found substituting in $\hat{H}^1$ $m_2 = m_2' + \hat{H}_c$ and so we can find $\hat{\tau}_2$ solving Heisenberg equation for $\hat{H}^1$ analogously to (11). The similar calculations results in $F^2$ proper time operator $\hat{\tau}_2$ in $F^1$:

$$\hat{\tau}_2 = B_2(\vec{p}_{12})\tau_1 + \frac{\theta_{12}(0)}{2\pi \omega}$$

(27)

Note that this approach is completely symmetrical and the operator obtained from (27) exchanging indexes 1 and 2 relates the time $\hat{\tau}_1$ in $F^1$ and $F^2$ proper time - parameter $\tau_2$. Obtained relation between two finite mass RFs shows that Quantum Equivalence principle can be correct also in relativistic QM. Analogously to Classical Relativity average time boost depends on whether $F^1$ measures $F^2$ clocks observables, as we considered or vice versa, and this measurement makes $F^1$ and $F^2$ nonequivalent. The new effect will be found only when $F^1$ and $F^2$ will compare their initially synchronized clocks. Formally this synchronization means that at the moment $\tau_1^0$ the prepared $F^2$ state factorized as $\Phi^1_2(\vec{p}_{12})\varphi_2(u_{in})$, where $\varphi_2$ is clock
wave function, describing some initial time value ($|\psi_0\rangle$ for $C_q$ state). If this experiment repeated several times (to perform quantum ensemble) it’ll reveal not only classical Lorentz time boost, but also the statistical spread having quantum origin with the dispersion given in (13).

Due to appearance of the time operators the transformation operator between two quantum RFs $\hat{U}_{2,1}(\tau_2, \tau_1)$ is quite intricate, and to obtain it general form demands further studies, here only the simplest situations are regarded. Consider first the transformation of $F^2$ relative state in $F^1$ $\psi^1$ to $F^1$ state in $F^2$. The solution of eq. (28) is $\psi^1 = \Phi_2(\vec{p}_{12}) \exp[-i E^1_s(\vec{p}_{12}) \tau_1]$ and if $F^1, F^2$ are synchronized at $\tau_1 = \tau_2 = 0$ then at this moment one have $\Phi_1(\vec{p}_{21}) = \hat{U}_{2,1}(0, 0) \Phi_2(\vec{p}_{12})$. It corresponds to RF Lorentz transformation $\vec{\beta}_2' = -\vec{\beta}_2$ which up to the quantum phase gives $\Phi_1(\vec{p}_{21}) = \Phi_2(-\frac{m_1 \vec{p}_{12}}{m_2})$. Then it’s easy to find that $\hat{U}_{2,1}(0, 0) = \hat{C}_2 \hat{P}_2$ the product of dilatation and parity operators as was shown obtaining eq. (6). Then the transformation operator for any $\tau_1, \tau_2$ is:

$$\hat{U}_{21}(\tau_1, \tau_2) = \hat{W}_2(\tau_2) \hat{U}_{21}(0, 0) \hat{W}_1^{-1}(\tau_1)$$

(28)

, where $\hat{W}_{i,2}(\tau_{i,2}) = \exp(-i \tau_{i,2} \hat{H}_{1,2})$ are the evolution operators in $F^1, F^2$. Analogously can be described the transformation of the single particle $G^3$ state between $F^1$ and $F^2$. To apply the clusterization formalism we’ll take that in $F^1$ at time $\tau_1 = 0$ the joint state vector of $F^2$ and $m_3$ - is $\psi^{1}_{in}(\vec{p}_{23}, \vec{q}_{23}) = \sum_{j,k} c_{jk} |\vec{p}_{23,j}, \vec{q}_{23,k}\rangle$ , where $\vec{p}_{23}$ is $F^2, G^3$ total momentum in $F^1$. Due to unambiguous correspondence between the $\vec{p}_{13}, \vec{q}_{13}$ and $\vec{p}_{23}, \vec{q}_{23}$ phase space points the state vector $\psi^{1}_{in}(\vec{p}_{13}, \vec{q}_{13})$ in $F^2$ is obtained acting on $\psi^{1}_{in}$ by $\hat{U}_{2,1}(0, 0)$. Analytical relations connecting $\vec{p}_{13}, \vec{q}_{13}$ and $\vec{p}_{23}, \vec{q}_{23}$ are quite complicated and omitted here [13]. Then the joint $G^3, F^1$ state in $F^2$ at any $\tau_2$ can be obtained by the action of the operator $\hat{U}_{2,1}(\tau_1, \tau_2)$ of (28) on $G^3, F^2$ state in $F^1$. It means that despite $\tau_2$ and $\tau_1$ are correlated only statistically through $\tau_2, G^3$ state vectors in $F^2, F^1$ at this moments are related unambiguously.

As was shown above $G^2$ state transformation (21) from ARF to $F^1$ can be described as the generalization of Poincare group transformations. The operator (22) is equal to $\hat{U}_{A,1}(0, 0)$ and the calculation of $\hat{U}_{A,1}(\tau_0, \tau_1)$ becomes straightforward in this case. This approach can be extended also on $F^2, G^3$ system analogously to transformation (22). Combining our previous considerations we’ll define in $F^1$ the transformation operator to $F^2$:

$$\hat{U}_{21}(0, 0) = \hat{C}_2 \hat{P}_2 e^{i \vec{p}_{12} N'_3}$$

(29)

where $\vec{N}_3 = \frac{1}{2} (E_{13} \vec{r}_{13} + \vec{r}_{13} E_{13}), \quad \vec{r}_{13} = i \frac{\partial}{\partial p_{13}}, \quad \vec{\beta}_2' = \vec{p}_{12} E_{12}^{-1}$. Here $\vec{p}_{13}, E_{13}$ are defined for $G^3$ analogous to (21). The first two members act on $F^2$ transforming it to $F^1$ state, and the last part transforms $G^3$ state.

Now we’ll consider obtained results in nonrelativistic limit. It’s easy to see that in the limit $\vec{p}_{12} \rightarrow 0$ Hamiltonian (21) after the masses subtraction differs from $\hat{H}_c$ of (9) by the factor $k_m = m_1 + m_2$, resulting from Lorentz transformation from c.m.s. to $F^1$ rest frame. The space coordinate operator in $F^1 x_{12}$ of (26) in nonrelativistic limit is equal to $\hat{x}_{12} = k_m^{-1}(\hat{x}_2 - \hat{x}_1)$, where $x_1, x_2$ are coordinates in ARF. This result doesn’t broke transformation invariance, because nonrelativistic QM has no fundamental length scale.
4 Concluding Remarks

We’ve shown that the extrapolation of QM laws on free macroscopic objects with which RF are associated demands to change the approach to the space-time which was taken copiously from Classical Physics. It seems that QM admits the existence of RF manifold each element of which is the state vector and the transformations between which principally can’t be reduced to Galilean or Lorentz transformations.

Historically QM formulation started from defining the wave functions on Euclidean 3-space $\mathbb{R}^3$ which constitute Hilbert space $H_s$. In the alternative approach accepted here we can regard $H_s$ as primordial states manifold. Introducing particular Hamiltonian defines $\vec{r}$, $\vec{p}$ axes in $H_s$ and results in the asymmetry of $H_s$ vectors which permit to define $R^3$ as a spectrum of the continuous observable $\hat{\vec{r}}$ which eigenstates are $|\vec{r}_i\rangle$. But as we’ve shown here for several quantum objects one of which is RF this definition become ambiguous and have many alternative solutions defining $R^3$ on $H_s$. In the relativistic case the situation is more complicated, yet as we’ve shown it results in ambiguous Minkovsky space-time definition. Meanwhile each quantum $RF_i$ has its own proper time - parameter $\tau_i$ and the phase space and all this RFs are physically equivalent. We have shown that this parameteres can be related by the proper time operators , which introduces the quantum fluctuations in the time relations. It means that to any time moment $\tau_i$ the time moment $\tau_j$ in $RF_j$ can corresponds only with the uncertainty $\pm D^{\frac{1}{2}}(\tau_i)$. So in this model each observer has its proper space-time which can’t be related unambiguously with the another observers space-time and in this sense is local. As the result we’ve got in any quantum RF Hamiltonian Mechanics of free particles on fundumental 3-dimensional momentum space with time parameter. Due to the invariance of the obtained evolution equation (23) proposed Quantum Equivalence principle was demonstrated is applicable also in the relativistic case.

In our work we demanded strictly that each RF must be quantum observer i.e. to be able to measure state vector parameters. But it isn’t clear whether this ability is the main property characterizing RF. In classical Physics this ability doesn’t influence the system principal dynamical properties. In QM at first sight we can’t claim it true or false finally because we don’t have the established theory of collapse. But it can be seen from our analysis that collapse is needed in any RF only to measure the wave functions parameters at some $t$. Alternatively this parameters at any RF can be calculated given the initial experimental conditions without performing the additional measurements. It’s quite reasonable to take that quantum states have objective meaning and exist independently of their measurability by the particular observer,so this ability probably can’t be decisive for this problem. It means that we can connect RF with the system which doesn’t include detectors ,which can weaken and simplify our assumptions about RF. We can assume that primordial for RF is the ability, which complex solid states have, to reproduce and record the space and time points ordering with which objects wave functions are related.

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