GENERALIZED $q$-GAUSSIAN VON NEUMANN ALGEBRAS WITH COEFFICIENTS, I. RELATIVE STRONG SOLIDITY.

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Abstract. We define $\Gamma_q(B, S \otimes H)$, the generalized $q$-gaussian von Neumann algebras associated to a sequence of symmetric independent copies $(\pi_j, B, A, D)$ and to a subset $1 \in S = S^* \subset A$ and, under certain assumptions, prove their strong solidity relative to $B$. We provide many examples of strongly solid generalized $q$-gaussian von Neumann algebras. We also obtain non-isomorphism and non-embedability results about some of these von Neumann algebras.
1. Introduction

In this article we introduce a new class of von Neumann algebras and prove some structural results about them. Specifically, we introduce the generalized $q$-gaussian von Neumann algebras with coefficients associated to a sequence of symmetric independent copies $(\pi_j, B, A, D)$. A 4-tuple $(\pi_j, B, A, D)$ is called a sequence of symmetric independent copies (of $A$) if $B, A, D$ are finite tracial von Neumann algebras such that $B \subset A \cap D$ and $\pi_j : A \to D$, $j \in \mathbb{N} \setminus \{0\}$ are unital trace-preserving normal $\ast$-homomorphisms satisfying

1. $\pi_j|_B = id_B$, for all $j$, where $E_B : D \to B$ is the conditional expectation;
2. $E_B(\pi_j(a_1) \cdots \pi_j(a_m)) = E_B(\pi_{\sigma(j_1)}(a_1) \cdots \pi_{\sigma(j_m)}(a_m))$, for all finite permutations $\sigma$ on $\mathbb{N} \setminus \{0\}$, all indices $j_1, \ldots, j_m$ in $\mathbb{N} \setminus \{0\}$ and all $a_1, \ldots, a_m$ in $A$, where $E_B : D \to B$ is the canonical trace-preserving conditional expectation;
3. For any subset $I \subset \mathbb{N} \setminus \{0\}$, we denote by $A_I = \bigvee_{i \in I} \pi_i(A) \subset D$. Then, for any finite subsets $I \subset J \subset \mathbb{N} \setminus \{0\}$, $j \not\in J$, $d \in A_I$ and $a, a' \in A$, we have

$$E_{A_I}(\pi_j(a)d\pi_j(a')) = E_{A_J}(\pi_j(a)d\pi_j(a')),$$

where $E_{A_I} : D \to A_I$ is the canonical trace-preserving conditional expectation;
4. For any finite subsets $I, J \subset \mathbb{N} \setminus \{0\}$, we have $E_{A_I}E_{A_J} = E_{A_{I\cap J}}$.
5. $A_{\mathbb{N} \setminus \{0\}} = D$.

Let $-1 < q < 1$ be fixed. For $H$ an infinite dimensional (real) Hilbert space and $S$ a self-adjoint subset of $A$ containing 1, the generalized $q$-gaussian von Neumann algebra $M = \Gamma_q(B, S \otimes H)$ with coefficients in $B$ and associated to the symmetric copies $(\pi_j, B, A, D)$ is defined as the von Neumann subalgebra of the ultraproduct $(\Gamma_q(\ell^2 \otimes H) \otimes D)^\omega$ generated by the elements

$$s_q(a, h) = (n^{-\frac{1}{2}} \sum_{j=1}^{n} s_q(e_j \otimes h) \otimes \pi_j(a))_n, a \in BSB = \{b_1b_2 : b_1, b_2 \in B, a \in S\}, h \in H.$$

Here $\omega$ is a free ultrafilter on the natural numbers and $\Gamma_q(\ell^2 \otimes H)$ is the $q$-gaussian von Neumann algebra. When $H$ is finite dimensional, one needs to further apply a “closure operation” to $\Gamma_q(B, S \otimes H)$, which ensures that the canonical generators - the so-called ”Wick words” - will remain in $M$ (see Def.3.2 and Prop.3.14 for more details). The generators $s_q(a, h)$ satisfy the following moment formula

$$\tau(s_q(a_1, h_1) \cdots s_q(a_m, h_m)) = \delta_{m\in \mathbb{N}} \sum_{\sigma \in P_2(m)} q^{\tau^|(\sigma)} \prod_{\{i,r\} \in \sigma} \langle h_i, h_r \rangle \tau_D(\pi_{j^q_i}(a_1) \cdots \pi_{j^q_m}(a_m)),$$

where $\langle h_i, h_r \rangle$ is the inner product in the real Hilbert space $H$, $P_2(m)$ is the set of all pair partitions on $\{1, \ldots, m\}$, $\tau^|(\sigma)$ is the number of crossings of the pair partition $\sigma \in P_2(m)$ and for every pair partition $\sigma \in P_2(m)$ the $m$-tuple $(j^q_1, \ldots, j^q_m)$ is chosen such that $j^q_k = j^q_l$ if and only if $\{k, l\} \in \sigma$. So in fact the generalized $q$-gaussians von Neumann algebras can roughly be defined as the von Neumann algebras generated by the elements $s_q(a, h)$, $a \in BSB, h \in H$ subject to the moment formula above. The main result we prove about these generalized $q$-gaussian algebras is

**Theorem 1.1.** Let $(\pi_j, B, A, D)$ be a sequence of symmetric independent copies, $H$ be a finite dimensional Hilbert space and $A \subset M = \Gamma_q(B, S \otimes H)$ be a diffuse von Neumann subalgebra which is amenable relative to $B$ inside $M$. For every $k \geq 0$, define $D_k(S)$ to be the following right $B$-submodule of $L^2(D)$:

$$\text{span}\{E_{A_1, \ldots, m}^k(\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)) : m \geq 1, \sigma \in P_{12}(m), |\sigma_s| = k, (j_1, \ldots, j_m) = \sigma, x_i \in BSB\},$$

where $P_{12}(m)$ is the set of pair-singleton partitions on $\{1, \ldots, m\}$, $|\sigma_s|$ is the numbers of singletons of $\sigma$ and the notation $(j_1, \ldots, j_m) = \sigma$ means that $j_h = j_l$ if and only if $\{j, k\} \in \sigma$. Assume
that there exist constants \( d, C > 0 \) such that \( \dim_B(D_k(S)) \leq Cd^k \) for all \( k \geq 1 \). Then at least one of the following statements is true:

1. \( A \prec_M B \), or
2. the von Neumann algebra \( P = N_B(A)' \) generated by the normalizer of \( A \) in \( B \) is amenable relative to \( B \) inside \( M \).

The notation \( A \prec_M B \) means that a corner of \( A \) embeds into \( B \) inside \( M \), in the sense of Popa (see [29], Thm. 2.1). Popa and Vaes coined the phrase "relative strong solidity" to describe the situation above. Namely, a von Neumann algebra \( M \) is strongly solid relative to \( B \), for \( B \subset M \) a subalgebra, if for every von Neumann subalgebra \( A \subset M \) which is amenable relative to \( B \) inside \( M \) (see [29]), it is either the case that \( A \prec_M B \) or that \( N_M(A)' \) is amenable relative to \( B \) inside \( M \). When \( B = \mathbb{C} \), \( M \) is called simply strongly solid. Strong solidity is in turn an enhancement of Ozawa’s concept of solidity (see [28]). Ozawa called a von Neumann algebra \( M \) solid if for every diffuse von Neumann subalgebra \( A \subset M \) one has that \( A' \cap M \) is amenable. It’s easy to see that a non-amenable solid factor \( M \) is automatically prime, i.e. cannot be written as \( M = M_1 \overline{\otimes} M_2 \), with \( M_i \) infinite dimensional. The first strong solidity results have been obtained by Ozawa and Popa in their ground-breaking seminal paper [29], for von Neumann algebras arising from profinite actions of free groups on amenable von Neumann algebras. These results were then extended to cover profinite actions of weakly amenable groups having a proper 1-cocycle into some multiple of their left regular representation. Subsequent generalizations to the case of profinite actions of groups having quasi-cocycles or direct products of such have been obtained in [11] and [11]. Recently, Popa and Vaes generalized these results even further, by completely removing any assumption on the action of the group. Specifically, they proved the following result (see Thm. 1.6 in [10] and Thm. 1.4 in [11]).

**Theorem 1.2.** Let \( \Gamma \) be a weakly amenable group having either a proper 1-cocycle or a proper 1-quasi-cocycle into a (representation which is weakly contained into) a multiple of its left regular representation. Let \( \Gamma \rhd B \) be any trace-preserving action of \( \Gamma \) on the finite von Neumann algebra \( B \), and let \( A \subset M = B \rtimes \Gamma \) be a von Neumann subalgebra which is amenable relative to \( B \) inside \( M \). Then either \( A \prec_M B \), or \( N_M(A)' \) is amenable relative to \( B \) inside \( M \).

In particular, this leads to the conclusion that for \( B \) abelian diffuse and for any p.m.p. free ergodic action \( \Gamma \rhd B \), the von Neumann algebra \( M = B \rtimes \Gamma \) has a unique Cartan subalgebra, up to unitary conjugacy. Let us also mention that Popa and Vaes also obtain the uniqueness of Cartan subalgebra for actions \( \Gamma \rhd B \), where \( B \) is abelian diffuse and \( \Gamma \) admits an unbounded (rather than proper) 1-cocycle into a mixing representation which is weakly contained into a multiple of the left regular representation of \( \Gamma \).

As a consequence of our Theorem 1.1, we find a wide range of examples of generalized q-gaussians which are strongly solid (when \( B = \mathbb{C} \) or finite dimensional) or strongly solid relative to \( B \), for diffuse \( B \). The examples in the corollary below are introduced in Section 4.

**Corollary 1.3.** The following von Neumann algebras are strongly solid relative to \( B \):

1. \( \mathbb{B} \otimes \mathbb{H}_q(H) \), for \( \mathbb{H} \) a finite dimensional Hilbert space;
2. \( \mathbb{H}_q(B, S \otimes H) \) associated to the symmetric independent copies \( (\pi_j, B, A, D) \) constructed in the following way: take a trace preserving action \( \alpha \) of \( \mathbb{Z} \) on a finite von Neumann algebra \( N \). Let \( \mathcal{H} = \langle g_j \mid j \geq 0 \rangle \) be the Heisenberg group, take \( \eta: \mathcal{H} \to \mathbb{Z} \) an onto group homomorphism and define \( \beta: \mathcal{H} \rhd N \) by \( \beta_g(x) = \alpha_{\eta(g)}(x), g \in \mathcal{H}, x \in N \).

Let \( \mathcal{H}_1 = \langle g_0, g_1 \rangle \) and take \( B = N \rtimes \mathbb{Z} = N \overline{\otimes} L(\mathbb{Z}), A = N \rtimes \mathcal{H}_1 \) and \( S = \{1, g_1, g_1^{-1}\} \). Define \( \pi_j: A \to D \) by \( \pi_j(xu_{g_1}) = \alpha_{\eta(g_j)}(x)u_{g_j}, \pi_j(xu_{g_0}) = xu_{g_0}, x \in N, j, k \in \mathbb{N} \).
(3) $\Gamma_q(\mathbb{C}, S \otimes K)$ associated to the symmetric copies $(\pi_j, B = \mathbb{C}, A = \Gamma_q(H), D = \Gamma_q(\ell^2 \otimes H))$, where $\pi_j(s_{\mathbb{C}}(h)) = s_g(e_j \otimes h)$ and $K$ is a finite dimensional Hilbert space;

(4) $\Gamma_q(B_d, S \otimes H)$ associated to the symmetric copies $(\pi_j, B_d, A_d, D_d)$, where $B_d = L(\Sigma_{[-d,0]}), A_d = L(\Sigma_{[-d,1]}), D_d = L(\Sigma_{[-d,\infty]}" and $S = \{1, u_{(01)}\}$ for a fixed $d \in \mathbb{N} \setminus \{0\}$; here $\Sigma_d$ is the group of finite permutations on $\mathbb{Z}$ and for a subset $F \subset \mathbb{Z}$, $\Sigma_F \subset \Sigma_d$ is the group of finite permutations on $F$ naturally embedded into $\Sigma_d$; the copies are defined by $\pi_j(a) = u_{(1j)^i}a u_{(1j)}$, $a \in A_d$;

(5) $\Gamma_q(\mathbb{C}, S \otimes H)$ associated to the symmetric copies $(\pi_j, B = \mathbb{C}, A = L(\mathbb{Z}), D = \bigotimes_{\mathbb{N}} L(\mathbb{Z}))$, where the $j$-th copy of $L(\mathbb{Z})$ is generated by the Haar unitary $u_j$, $A = \{u_i\}^\omega$, the symmetric copies $\pi_j : A \to D$ are defined by the relations $\pi_j(u_1) = u_j$ and $S = \{1, u_1, u_n\}$.

Thus the examples in (3), (4) and (5) are strongly solid and hence solid non-amenable von Neumann algebras. In particular, they are prime von Neumann algebras.

Thus the generalized $q$-gaussian von Neumann algebras with coefficients constitute a wholly new class of von Neumann algebras for which this structural property holds. Using Thm. 1.1 we also deduce the following

Corollary 1.4. Let $M_i = \Gamma_q(B_i; \mathcal{S}_i \otimes H_i)$ be associated with two sequences of symmetric independent copies $(\pi^i_j, B_i, A_i, D_i)$ and two subsets $\mathcal{S}_i \subset A_i$, and $-1 < q_i < 1$, $i = 1, 2$. Assume that $2 \leq \dim(H_i) < \infty$, $\dim_q(D_k(S_i)) \leq C d^k$ for fixed constants $d, C > 0$ and $B_i$ are amenable, for $i = 1, 2$. If $M_1 \subset M_2$, then $B_1 \preceq_M B_2$. Moreover, if $M_1 = M_2 = M$, it follows that $B_1 \preceq_M B_2$ and $B_2 \preceq_M B_1$.

This result can be regarded as an analogue of the "uniqueness of Cartan subalgebra" results in the group measure space construction setting. Note however, that even when $B$ is abelian, it is not a MASA in $M = \Gamma_q(B, S \otimes H)$. Indeed, $B$ always commutes with a copy of $\Gamma_q(H)$ inside $M = \Gamma_q(B, S \otimes H)$, hence it can never be maximally abelian. Thus, even when $B_1$ and $B_2$ are both abelian diffuse, we cannot avail ourselves of Popa’s results about unitary conjugacy of Cartan subalgebras [Popa]. Appendix, Thm. A.1) to conclude that $B_1$ is unitarily conjugate to $B_2$, so this double intertwining result is optimal in our case. Finally, we deduce some non-isomorphism and non-embedability results for generalized $q$-gaussians.

Corollary 1.5. Under the assumption of Cor.1.4, if we moreover assume that

(1) $B_1$ is finite dimensional and $B_2$ is amenable diffuse, or

(2) $B_1$ is abelian and $B_2$ is the hyperfinite $II_1$ factor,

then $M_2 = \Gamma_{q_2}(B_2, S_2 \otimes H_2)$ cannot be realized as a von Neumann subalgebra of $M_1 = \Gamma_{q_1}(B_1, S_1 \otimes H_1)$. In particular $M_1$ and $M_2$ are not *-isomorphic.

Thus, none of the examples in items (1) or (2) in Cor.1.2. above can be *-isomorphic, or even embed into any of the von Neumann algebras in items (3), (4) or (5) of the same corollary.

The $q$-gaussian von Neumann algebras $\Gamma_q(H)$ for $H$ a real Hilbert space were first introduced by Bozejko and Speicher and further studied, among others, by Bozejko, Speicher, Kummerer, Ricard, Sniady, Krolak, Nica, Dykema, Avsec, Dabrowski, and others. Bozejko and Speicher studied the $q$-gaussians from a variety of different perspectives: as concrete implementations of the $q$-anti-commutation relations, as examples of non-commutative brownian motions, as being canonically associated to certain completely positive definite functions on Coxeter groups, in particular the permutations group $S_n$ and also as constituting an interpolating family between the bosonic ($q = 1$) and fermionic ($q = -1$) commutation relations. The study of the $q$-gaussian von Neumann algebras is hence connected to a variety of fields, from non-commutative probability theory, physics, operator spaces, operator algebras, etc. Likewise, the generalized $q$-gaussians can be thought of
in a number of ways. First they can be viewed as natural generalizations of the pure Hilbert space gaussians \( \Gamma_q(H) \), pretty much in the same way the group-measure space construction generalizes group von Neumann algebras. Second, they are a class of von Neumann algebras constructed from some generators satisfying a natural moment formula, i.e. from a certain set of combinatorial data. Third, they can be regarded as probabilistic objects, which additionally constitute an interpolating family between the well-known cases \( q = -1 \) and \( q = 1 \). Fourth, they are von Neumann algebras associated with a system of \( B \)-valued \( q \)-gaussian elements, much in the spirit of Schlyakhtenko’s \( B \)-valued semi-circular systems.

When compared to pure \( q \)-gaussians, the generalized \( q \)-gaussian von Neumann algebras with coefficients can be viewed as an analogue of the cross-product von Neumann algebras \( B \rtimes \Gamma \) as opposed to pure group von Neumann algebras \( L(\Gamma) \). Even the notation \( \Gamma_q(B, S \otimes H) \) is chosen to suggest that the Hilbert space \( H \) (the analogue of the group) “acts” on the von Neumann algebra \( B \). However, it should be stressed that this analogy, though fruitful, is misleading on several counts. First of all, the von Neumann algebra \( \Gamma_q(B, S \otimes H) \) is not, in general, generated by \( B \) and a copy of \( \Gamma_q(H) \), and not even by \( B, \Gamma_q(H) \) and \( S \subset A \). Second, the way \( \Gamma_q(B, S \otimes H) \) is generated by the \( s_q(x,h) \)'s is much more subtle and intricate than in the cross-product case, and the generators remain highly elusive, even after our ”reduction procedure” described in Thm.3.11; consequently, the standard Hilbert space on which \( M \) acts remains highly mysterious itself and we cannot fully and explicitly describe it, though we obtain a partially concrete description of it (see Prop.3.14). Third, insofar as \( H \) can be viewed as acting on \( B \), this ”action” is always trivial, due to our definition of \( \Gamma_q(B, S \otimes H) \) and to axiom (1). It is rather on \( D \), or certain elements of \( D \), that \( H \) “acts”.

This paper is the first one in a series which will systematically investigate the structural properties of the generalized \( q \)-gaussians. In doing so, we employ a variety of high-powered tools and techniques: non-commutative probability theory, specifically central limit theorems ([34], [20]), operator spaces and especially the Haagerup tensor product, orthogonal polynomials ([49], [32]), randomization methods in a non-classical probabilistic setting ([50]), noncommutative \( L^p \)-spaces ([16, 17]) and crucially, Popa’s deformation-rigidity theory. Let us mention here that in 2007 we introduced another class of so-called generalized \( q \)-gaussian von Neumann algebras which were a mix of pure \( q \)-gaussians and the group measure space construction, and we proved some partial classification results about them. These previous \( q \)-gaussians with group action are not, strictly speaking, particular cases of \( q \)-gaussians arising from symmetric copies, although they can be realized as von Neumann subalgebras of them, see Section 4.3 in [20] and 4.3.1. in the present work. We begin by carefully introducing the objects involved, then we go on to prove their basic properties. Theorem 1.1 will be derived from a technical theorem much along the lines of Thm. 3.1 in [20], whose statement and proof can be found in Section 7.

Finally, a couple of words about the proof of the technical theorem, or rather its particular case Thm. 1.1. We follow the approach of Popa and Vaes in [29] and [40] approach which in turn is a development of the original ground-breaking insight in [39, 40]. Let \( A \subset M = \Gamma_q(B, S \otimes H) \) a diffuse von Neumann subalgebra which is amenable over \( B \). The two main ingredients of the proof are, just as in [39, 40]:

1. The fact that the embedding \( A \subset M \) is weakly compact relative to \( B \). This is the existence of a sequence of normal states viewed as unit vectors \( \xi_n \in L^2(\mathcal{N}) \), where \( \mathcal{N} \supset M \) is a suitable (in general non-tracial) von Neumann algebra, which are asymptotically invariant to the action of ("the double of") the normalizer of \( A \) in \( M \); the existence of these states is a consequence of the weak amenability (with Cowling-Haagerup constant 1) of the pure \( q \)-gaussian von Neumann algebras \( \Gamma_q(H) \);

2. The existence of a 1-parameter group of \(*\)-automorphisms \( (\alpha_t) \) of a suitable dilation \( \tilde{\mathcal{N}} \) of \( \mathcal{N} \) having good properties.
The proof proceeds by applying the deformation $\alpha_t$ to the vectors $\xi_n$. Then either the deformation significantly displaces the vectors, or it does not. The first case yields the amenability of $P = N_M(A)''$ relative to $B$, while the second implies that $A \subset_M B$, via the fact that the maps $T_t$ (where $t \to T_t$ is the canonical semi-group of ucp maps on $M$) are compact over $B$, in the terminology of Popa and Ozawa.

While it’s true that conceptually we follow closely the approach of Popa and Vaes in [PoVa1, PoVa2], it has to be strongly emphasized that the technical difficulties of our approach are vastly larger. First of all, since our objects are much more elusive and complicated than cross-product von Neumann algebras, being defined as subalgebras of an ultraproduct to begin with, the proof of Theorem 5.1 (the existence of the invariant states), which is the key ingredient in the proof of the technical theorem, is ridden with daunting challenges. Even constructing the von Neumann algebras involved (e.g. $N$ above) and particularly the spaces on which they act, seemed to be an almost insurmountable task. Also, proving the complete boundedness of certain maps used in the proof turns out to be surprisingly non-trivial and requires the use of delicate operator spaces techniques, particularly the Haagerup tensor product (in the pure Hilbert space setting, somewhat similar techniques have been used in [Avsec, NouI, NouII]). Second, and just as important, we cannot use the reduction to the "trivial action case" (i.e. the tensor product case), as Popa and Vaes do. The reduction step plays a crucial role in their proof, because it is only in the tensor product setting that they are able to prove the relative weak compactness property and subsequently carry out the deformation-rigidity arguments. The reduction is essentially based on the use of the co-multiplication map, we cannot reduce to the tensor product case, and hence everything becomes much more complicated and technically involved, including the standard forms of the von Neumann algebras involved, which are in general infinite (non-tracial). We thus have to prove the relative compactness of the inclusion $A \subset M$ over $B$ in a generic situation, although we need the assumption that the spaces $D_k(S)$ are finitely dimensional over $B$, for every $k$. It is tempting to regard this assumption as an analogue of the action $\Gamma \curvearrowright B$ being profinite in the cross-product case $B \rtimes \Gamma$, but this would be again misleading. That’s because our finite dimensionality assumptions do not imply that $M = \Gamma_q(B, S \otimes H)$ is of the form $(\bigcup_n \Gamma_q(B_n, S \otimes H))''$, for an increasing sequence of finite dimensional subalgebras $B_n \subset B$ such that $B = (\bigcup_n B_n)''$. Instead, our finite dimensionality condition is rather the analogue of the group cocycle being proper in the case of cross-product von Neumann algebras of the form $B \rtimes \Gamma$, where $\Gamma$ is a weakly amenable group having a non-trivial 1-cocycle into some multiple of its left regular representation.

The article contains six sections beside the introduction, and is organized as follows: Section 2 contains some needed technical preliminaries. In Section 3 we introduce the generalized $q$-gaussian von Neumann algebras and prove their basic properties; among other things, we exhibit the canonical generators of $\Gamma_q(B, S \otimes H)$ (the Wick words), prove that they actually belong to the algebra and prove a very useful reduction result about them. Section 4 lists a rather wide range of examples of generalized $q$-gaussian von Neumann algebras constructed from a variety of symmetric independent copies. Some of our examples are interesting even in the case $B = \mathbb{C}$ or finite dimensional, which provide examples of strongly solid generalized $q$-gaussian von Neumann algebras. We devote Section 5 to the proof of the relative weak compactness of the embedding $A \subset M$; the second half of this section contains some technical results about the complete boundedness of certain multipliers used in the proof. In Section 6 we prove that under the assumption of sub-exponential growth of the dimensions of the modules $D_k(S)$ over $B$, the natural deformation bimodules used in the technical theorem are weakly contained in $L^2(M) \otimes_B K$ for some $B - M$ bimodule $K$, fact which will be further used in combination with the technical theorem to derive Thm. 1.1. The proof is based on a novel and "non-deterministic"
approach. Indeed, the calculations of the deformation bimodules in the $q$-gaussian setting is a real challenge even in the case of pure $\Gamma_q(H)$ von Neumann algebras, see [1], and it becomes even more so when we allow $q$-gaussians with coefficients. Section 7 contains the proof of the main technical theorem and its applications. Beside many examples of strongly solid generalized $q$-gaussian von Neumann algebras, we also obtain some non-isomorphism and non-embedability results.
2. Preliminaries

2.1. Popa’s intertwining techniques. We will briefly review the concept of intertwining two subalgebras inside a finite von Neumann algebra, along with the main technical tools developed by Popa in [36, 37]. Let $(M, \tau)$ be a finite von Neumann algebra, let $f \in \mathcal{P}(M)$ and $Q \subset fMf, B \subset M$ be two von Neumann subalgebras. We say that a corner of $Q$ can be intertwined into $B$ inside $M$ and denote it by $Q \prec_M B$ (or simply $Q \prec B$) if there exist two non-zero projections $q \in Q$, $p \in B$, a non-zero partial isometry $v \in qMp$, and a $*$-homomorphism $\psi : qQq \to pBp$ such that $vv\psi(x) = xv$ for all $x \in qQ$. The partial isometry $v$ is called an intertwiner between $Q$ and $B$. Popa proved in [37] the following intertwining criterion:

Theorem 2.1 (Corollary 2.3 in [37]). Let $M$ be a von Neumann algebra and let $Q \subset fMf, B \subset M$ be diffuse subalgebras for some projection $f \in M$. Then the following are equivalent:

1. $Q \prec_M B$.
2. There exists a finite set $F \subset fMf$ and $\delta > 0$ such that for every unitary $u \in U(Q)$ we have
   $$\sum_{x,y \in F} \|E_B(xuv^*)\|_2^2 \geq \delta.$$ 

Let $(M, \tau)$ be a finite von Neumann algebra and $\Phi : M \to M$ a normal, completely positive map. We say that $\Phi$ is sub-tracial if $\tau \circ \Phi \leq \tau$. If $\Phi$ is sub-tracial, then, due to the Schwartz inequality, we automatically have
$$\|\Phi(x)\|_2^2 = \tau(\Phi(x)^*\Phi(x)) \leq \tau(\Phi(x^*x)) \leq \tau(x^*x) = \|x\|_2^2,$$
i.e. $\Phi$ is automatically $\| \cdot \|_2$-contractive, and hence extends to a bounded operator on $L^2(M)$ defined by
$$T_\Phi : L^2(M) \to L^2(M), T_\Phi(x) = \Phi(x), x \in M.$$ 
Let $B \subset (M, \tau)$ be an inclusion of finite von Neumann algebras. The basic construction (of $M$ with $B$) is defined by (see e.g. [101])
$$\langle M, e_B \rangle = (M \cup \{e_B\})'' = (JBJ)' \subset B(L^2(M)),$$
where $L^2(M)$ is the standard form of $M$ and $J : L^2(M) \to L^2(M)$ the associated conjugation. The definition of the compact ideal space of the basic construction (more generally of any semi-finite von Neumann algebra) can be found in [101], 1.3.3.

Definition 2.2. Let $(M, \tau)$ be a finite von Neumann algebra, $B \subset M$ a von Neumann subalgebra and $\Phi : M \to M$ a normal, completely positive, sub-unital, sub-tracial map. We say that $\Phi$ is compact over $B$ if the canonical operator $T_\Phi : L^2(M) \to L^2(M)$ belongs to the compact ideal space of the basic construction $(M, e_B)$.

The following result is Prop.2.7 in [29] (see also [36], 1.3.3.).

Proposition 2.3. Let $(M, \tau)$ be a finite von Neumann algebra and let $B, P \subset M$ be two von Neumann subalgebras. Let $\Phi : M \to M$ be a normal, completely positive, sub-unital, sub-tracial map which is compact over $B$ and assume that
$$\inf_{u \in U(P)} \|\Phi(u)\|_2 > 0.$$ 
Then $P \prec_M B$. 

2.2. Bimodules over von Neumann algebras and weak containment. Let \( M, Q \) be two von Neumann algebras. An \( M \prec Q \) Hilbert bimodule \( K \) is simply a Hilbert space together with a pair of normal *-representations \( \lambda : M \to B(K), \rho : Q^{op} \to B(K) \) with commuting ranges. To these one can associate a *-representation \( \pi : M \otimes_{\text{bin}} Q^{op} \to B(K) \) by
\[
\pi(\sum_k x_k \otimes y_k^{op})\xi = \sum_k \lambda(x_k)\rho(y_k^{op})\xi, \quad x_k \in M, y_k \in Q, \xi \in K.
\]

**Definition 2.4.** Let \( M, Q \) be two von Neumann algebras and \( H, K \) be two \( M \prec Q \) bimodules. We say that \( K \) is weakly contained in \( H \) and denote it by \( K \prec H \) if \( \|\pi_K(x)\| \leq \|\pi_H(x)\| \) for all \( x \in M \otimes_{\text{alg}} Q \), where \( \pi_K, \pi_H \) are the *-representations canonically associated to the left and right actions on \( H, K \) respectively.

Give an \( M \prec Q \) bimodule \( K \) and an \( Q \prec N \) bimodule \( H \) we will denote by \( K \otimes_Q H \) their Connes tensor product, which is an \( M \prec N \) bimodule. For the definition and basic properties of the Connes tensor product, see sections 2.3, 2.4 in PoVaI. The Connes tensor product is well behaved with respect to weak containment (see idem).

**Definition 2.5** (Def. 2.3 in PoVaI). Let \((M, \tau_M)\) and \((Q, \tau_Q)\) be finite tracial von Neumann algebras and \( P \subseteq M \) a von Neumann subalgebra. We say that an \( M \prec Q \) bimodule \( K \) is left \( P \)-amenable if one of the following equivalent conditions holds:

1. There exists a \( P \)-central state \( \Omega \) on \( B(K) \cap (Q^{op})' \) such that \( \Omega|_M = \tau_M \).
2. \( L^2(M) \prec K \otimes_Q K \) as \( M \prec P \) bimodules.

**Definition 2.6.** Let \((M, \tau)\) be a tracial von Neumann algebra, and let \( B, P \subseteq M \) be two von Neumann algebras. We say that \( P \) is amenable relative to \( B \) inside \( M \) if one of the following equivalent conditions holds:

1. The \( M \prec B \) bimodule \( L^2(M) \) is left \( P \)-amenable;
2. \( L^2(M) \prec L^2(M) \otimes_B L^2(M) \) as \( M \prec P \) bimodules.

**Remark 2.7.** Let \((M, \tau)\) be a finite von Neumann algebra and \( B, P \subseteq M \) two von Neumann subalgebras. Let \( K \) be a \( P \)-amenable \( M \prec B \) bimodule such that \( K \prec L^2(M) \otimes_B B \) for some \( B \prec M \) bimodule \( H \). Then \( P \) is amenable relative to \( B \) inside \( M \). Indeed, we have that, as \( M \prec P \) bimodules
\[
L^2(M) \prec K \otimes_M K \prec (L^2(M) \otimes_B H) \otimes_M (\bar{H} \otimes_B L^2(M)) \prec L^2(M) \otimes_B L^2(M).
\]

2.3. Standard forms of non-tracial von Neumann algebras. In some instances we will have to consider non-tracial von Neumann algebras \( M \) and their standard forms. Let us recall that a (hyper) standard form for a von Neumann algebra is given by \((M, H, J, P)\), where \( J : H \to H \) is an antilinear unitary, \( P \subseteq H \) is a self-dual cone such that

i) \( Jx = x \) for \( \xi \in \mathbb{C} \);
ii) \( Jx = x \) for \( \xi \in \mathbb{P} \);
iii) \( xJxJ(P) \subseteq P \) for \( x \in M \).

The standard form of \( M \) is unique up to *-isomorphism, see e.g. HaagerupSF. A particularly useful way of describing the standard form of \( M \) is the abstract Haagerup \( L^2(M) \) space, which we briefly describe below. The reader can find more details in HaagerupSF. Let \((M, \varphi)\) a von Neumann algebra endowed with a normal semi-finite faithful (n.s.f.) weight. Consider \( M = M \rtimes \sigma_t \mathbb{R} \) the cross-product von Neumann algebra of \( M \) with \( \mathbb{R} \) by the modular automorphism group \( \sigma_t \).
Then \( M \) is semi-finite and there exists a n.s.f. trace \( \tau \) on \( M \) such that
\[
(D\varphi : D\tau)_t = \lambda(t), \quad t \in \mathbb{R},
\]
where $\hat{\varphi}$ is the dual weight, $(D\hat{\varphi} : D\tau)_t$ is the Connes cocycle and $\lambda(t)$ is the group of translations on $\mathbb{R}$. Moreover, $\tau$ is the unique n.s.f. trace on $M$ which satisfies
\[
\tau \circ \hat{\varphi}_t = e^{-t}\tau, \quad t \in \mathbb{R}.
\]
Given another n.s.w. $\psi$ on $M$, denote by $h_\psi$ the Radon-Nikodym derivative of $\hat{\psi}$ with respect to $\tau$, i.e. the unique positive self-adjoint operator affiliated to $M$ such that
\[
\hat{\psi}(x) = \tau(h_\psi^{\frac{1}{2}}xh_\psi^{\frac{1}{2}}), \quad x \in M_+.
\]
Then the following condition holds:
\[
\hat{\varphi}_t(h_\psi) = e^{-t}h_\psi, \quad t \in \mathbb{R}.
\]
Moreover, the map $\psi \mapsto h_\psi$ is a bijection from the set of n.s. weights on $M$ to the set of positive self-adjoint operators affiliated to $M$ which satisfy the above condition. Let $L_0(M, \tau)$ be the *-algebra consisting of all the operators on $L^2(\mathbb{R}, H)$ which are measurable with respect to $(M, \tau)$. For $p > 0$, the Haagerup $L^p(M, \varphi)$ is defined by
\[
L^p(M, \varphi) = \{x \in L_0(M, \tau) : \hat{\varphi}_t(x) = e^{-\frac{t}{p}}x, \forall t \in \mathbb{R}\}.
\]
One can define a bi-continuous linear isomorphism from $M_*$ to $L^1(M, \varphi)$ as the linear extension of the map
\[
M_+^* \ni \psi \mapsto h_\psi \in L^1(M, \varphi).
\]
The norm $\| \cdot \|_1$ on $L^1(M, \varphi)$ is defined by requiring that the above isomorphism be isometric. One can define a norm one linear functional $tr$ on $L^1(M, \varphi)$ by $tr(h_\psi) = \psi(1)$, and thus $\|h\|_1 = tr(|h|), h \in L^1(M, \varphi)$. This "trace" is indeed tracial, i.e.
\[
tr(xy) = tr(yx), \quad \text{for} \quad x, y \in L^2(M).
\]
Let $x = u|x|$ be the polar decomposition of an element $x \in L_0(M, \tau)$. Then we have
\[
x \in L^p(M, \varphi) \iff u \in M \quad \text{and} \quad |x| \in L^p(M, \varphi) \iff u \in M \quad \text{and} \quad |x|_p \in L^1(M, \varphi).
\]
This allows one to introduce the $\| \cdot \|_p$-norm on $L^p(M, \varphi)$, by $\|x\|_p = \| |x|_p \|_1^{\frac{1}{p}}$ for $x \in L^p(M, \varphi)$. Let’s also remark that the weight $\varphi$ can be recovered from the trace. Define
\[
N_\varphi = \{x \in M : \varphi(x^*x) < \infty\}, \quad M_\varphi = N_{\varphi}^*N_{\varphi} = \text{span}\{y^*x : x, y \in N_\varphi\}.
\]
The dual weight $\hat{\varphi}$ has a Radon-Nikodym derivative with respect to $\tau$, which will be denoted $d_\varphi$. Then for every $x \in M_\varphi$ the operator $d_\varphi^{\frac{1}{2}}xd_\varphi^{\frac{1}{2}}$ is closable, its closure belongs to $L^1(M, \varphi)$ and we have the following relation
\[
\varphi(x) = tr(d_\varphi^{\frac{1}{2}}xd_\varphi^{\frac{1}{2}}), \quad x \in M_\varphi.
\]
If $\varphi$ is a bounded functional, then $d_\varphi \in L^1(M, \varphi)$ and the above identity becomes
\[
\varphi(x) = tr(d_\varphi^{\frac{1}{2}}xd_\varphi^{\frac{1}{2}}) = tr(xd_\varphi), \quad x \in M.
\]
The Haagerup space $L^p(M, \varphi)$ does not depend on the choice of the n.s.f. weight $\varphi$ up to isomorphism, hence it can simply be denoted by $L^p(M)$. It’s easy to see that $M$ is naturally represented in standard form on the Haagerup space $L^2(M)$ via the obvious left and right actions. When $M$ is finite and $\tau$ is a faithful trace on $M$, the Haagerup space $L^2(M) = L^2(M, \tau)$ coincides with the usual one.
2.4. **W* Hilbert modules.** We also have to recall some facts about (right) Hilbert $W^*$ modules. According to \cite{PaschkeI, PaschkeII} (see also JungeSherman \cite{JungeSherman}) a right Hilbert $C^*$-module $X$ over a von Neumann algebra $M$ is self-dual if and only if admits a module basis, i.e. a family $\{\xi_\alpha\} \subset X$ such that

$$X = \text{span} \sum_\alpha \xi_\alpha M \quad \text{and} \quad \langle \xi_\alpha, \xi_\beta \rangle = \delta_{\alpha\beta} e_\alpha \in \mathcal{P}(M).$$

Here, $\langle \cdot, \cdot \rangle$ denotes the $M$-valued inner product. In this situation, there exists an index set $I$, a projection $e \in B(\ell_2(I) \otimes M)$, and a right module isomorphism $u : X \to e(\ell_2(I) \otimes M)$. Indeed, for a basis $\xi_\alpha$ with $\langle \xi_\alpha, \xi_\alpha \rangle = e_\alpha$ the map $u$ is given by $u(\sum_\alpha \xi_\alpha m_\alpha) = [e_\alpha m_\alpha]$. Here $\ell_2(I) \otimes M$ denotes the space of strongly convergent columns indexed by $I$. Then it is easy to see that the $C^*$-algebra $\mathcal{L}(X)$ of adjointable operators on $X$ is indeed a von Neumann algebra, and isomorphic to $e(B(\ell_2(I)) \otimes M)e$. Moreover, the $M$-compact operators $\mathcal{K}(X)$ spanned by the maps $\Phi_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle$ are weakly dense in $\mathcal{L}(X)$, because $K(\ell_2(I)) \otimes_{\min} M$ is weakly dense in $B(\ell_2(I)) \otimes M$. With the help of a normal faithful state, we can complete $X$ to the Hilbert space $L_2(X, \phi)$ with inner product $(\xi, \eta) = \phi(\langle \xi, \eta \rangle)$. Let $\iota_\phi : X \to L_2(X, \phi)$ the inclusion map. Then

$$\pi : \mathcal{L}(X) \to B(L_2(X, \phi)), \pi(T)(\iota_\phi(x)) = \iota_\phi(Tx)$$

defines a normal faithful $^*$-homomorphism such that

$$\pi(\mathcal{L}(X)) = B(L_2(X, \phi)) \cap (M^{op})'. \quad \text{PaschkeI, PaschkeII, JungeSherman \cite{PaschkeI, PaschkeII, JungeSherman}}$$

This is indeed very easy to check for $\mathcal{L}(X) = e(B(\ell_2(I)) \otimes M)e$. See \cite{PaschkeI, PaschkeII, JungeSherman} for more details and references.
3. The generalized gaussian von Neumann algebras with coefficients -
    definition and basic properties

Throughout this section we will freely use the basic properties of the pure Hilbert space
    gaussian von Neumann algebras \( \Gamma_q(H) \), as they can be found in Section 4 of
    [1].

The following result is due to Dykema, Nica and Voiculescu and can be found in [1].

**Proposition 3.1.** Let \( (M, \varphi) \) and \( (N, \psi) \) be two von Neumann algebras endowed with faithful normal tracial states. Let \( (x_i)_{i=1}^\infty \) and \( (y_j)_{j=1}^\infty \) be countable systems of generators for \( M \) and \( N \), respectively. Assume that for every \( \pi \)

**Definition 3.2.** Let \( A \) and \( D \) be two finite tracial von Neumann algebras and \( B \) a von Neumann subalgebra of \( A \cap D \). Let \( \pi_j : A \to D, j \in \mathbb{N} \) be a countable family of unital, normal, faithful, trace-preserving \(*\)-homomorphisms. The 4-tuple \( (\pi_j, B, A, D) \) is called a sequence of symmetric independent copies of \( A \) if the following properties hold:

1. \( \pi_j|_B = \text{id}_B \), for all \( j \);
2. \( E_B(\pi_j(a_1) \cdots \pi_j(a_m)) = E_B(\pi_{\sigma(j_1)}(a_1) \cdots \pi_{\sigma(j_m)}(a_m)) \), for all finite permutations \( \sigma \) on \( \mathbb{N} \setminus \{0\} \), all indices \( j_1, \ldots, j_m \) in \( \mathbb{N} \setminus \{0\} \) and all \( a_1, \ldots, a_m \) in \( A \), where \( E_B : D \to B \) is the canonical trace-preserving conditional expectation;
3. For \( i \in \mathbb{N} \setminus \{0\} \) denote by \( A_i = \pi_i(A) \subset D \) and for \( I \subset \mathbb{N} \setminus \{0\} \), denote by \( A_I = \bigvee_{i \in I} \pi_i(A) = \bigvee_{i \in I} A_i \subset D \) (by convention, set \( A_\emptyset = B \)); then, for any finite subsets \( I \subset J \subset \mathbb{N} \setminus \{0\} \), \( j \notin J \), \( d \in A_I \) and \( a, a' \in A \), we have

\[
E_{A_I}(\pi_j(d)\pi_j(a')) = E_{A_J}(\pi_j(d)\pi_j(a')),
\]

where \( E_{A_I} : D \to A_I \) is the canonical conditional expectation;
4. for any finite subsets \( I, J \subset \mathbb{N} \setminus \{0\} \), we have \( E_{A_I}E_{A_J} = E_{A_{I\cap J}} \). Note that this automatically implies \( E_{A_I}E_{A_J} = E_{A_{I\cap J}} \) and in particular \( A_I \cap A_J = A_{I\cap J} \).
5. \( A_{\mathbb{N}\setminus\{0\}} = D \).

If the 4-tuple \( (\pi_j, B, A, D) \) only satisfies axioms (1) and (2), we call it a sequence of symmetric copies.

In what follows, the expectations \( E_{A_I} \) will be denoted \( E_I \).

**Proposition 3.3.** Let \( (\pi_j, B, A, D) \) a sequence of symmetric copies. Let \( \Sigma = S(\infty) \) be the group of finite permutations on \( \mathbb{N} \setminus \{0\} \). Then for every \( \sigma \in \Sigma \) there exist a trace preserving automorphism \( \alpha_\sigma \) of \( D_0 = A_{\mathbb{N}\setminus\{0\}} \subset D \) such that \( \alpha_\sigma(\pi_j(x_1) \cdots \pi_j(x_m)) = \pi_{\sigma(j_1)}(x_1) \cdots \pi_{\sigma(j_m)}(x_m) \), for all \( x_1, \ldots, x_m \in A \) and \( j_1, \ldots, j_m \in \mathbb{N} \). Moreover

\[
\Sigma \ni \sigma \mapsto \alpha_\sigma \in \text{Aut}(D_0, \tau)
\]

is an action of \( \Sigma \) on \( D_0 \) by trace-preserving automorphisms. Moreover, if the symmetric copies satisfy axiom 4, then the fixed points algebra of this action is \( B \).

**Proof.** The map \( V_\sigma : L^2(D_0) \to L^2(D_0) \) defined by

\[
\sum \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \mapsto \sum \pi_{\sigma(j_1)}(x_1) \cdots \pi_{\sigma(j_m)}(x_m)
\]

is easily seen to be a well-defined unitary because of axiom 2. Then \( \alpha_\sigma = \text{Ad}(V_\sigma)|_D \) is a trace preserving automorphism of \( D \) which satisfies the required condition. The verification of the second statement is straightforward and we leave it to the reader. \( \blacksquare \)
Remark 3.5. Let \( \sigma : \Sigma \to Aut(D, \tau) \) be a trace preserving action by *-automorphism of the finite von Neumann algebra \( D \), where \( \Sigma \) is now the finite permutation group on \( \mathbb{N} \) instead of \( \mathbb{N} \setminus \{0\} \). Denote by \( B = D^{\Sigma} \) the fixed points algebra of this action. Denote by \( \Sigma_0 = Stab_\Sigma(0) = \{ \sigma \in \Sigma : \sigma(0) = 0 \} \). Set \( A = D^{\Sigma_0} = \{ d \in D : \alpha_\sigma(d) = d, \forall \sigma \in \Sigma_0 \} \). Note that \( \Sigma_0 \subset \Sigma \) is a subgroup isomorphic to \( S(\infty) \) and that \( B \subset A \subset D \). For every \( j \geq 1 \), define \( \pi_j : A \to D \) by the formula \( \pi_j(a) = \alpha_{(0j)}(a), a \in A \), where \( (0j) \in \Sigma \) is the transposition interchanging \( 0 \) and \( j \). Then \( (\pi_j, B, A, D) \) represents a sequence of symmetric copies. Indeed, for any \( j \geq 1 \) and \( b \in B \) we have \( \pi_j(b) = \alpha_{(0j)}(b) = b \) because \( B \) is the fixed points algebra of the action \( \alpha \), so (1) is true. Note that \( \alpha_\sigma(a) = a \) for every \( \sigma \in \Sigma_0 \) and \( a \in A \) and \( E_B \circ \alpha_\sigma = E_B \) for all \( \sigma \in \Sigma \), due to (1) and the facts that \( \alpha \) is trace preserving and the trace preserving conditional expectation \( E_B : D \to B \) is unique. Then for every \( \sigma \in \Sigma_0 \cong S(\infty) \) and for all \( j_1, \ldots, j_m \geq 1 \) and \( a_1, \ldots, a_m \in A \) we have

\[
E_B(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m)) = E_B(\alpha_{(0j_1)}(a_1) \cdots \alpha_{(0j_m)}(a_m)) = E_B(\alpha_{(0j_1)}(a_1) \cdots \alpha_{(0j_m)}(a_m)) = E_B(\alpha_{(0j_1)}(a_1) \cdots \alpha_{(0j_m)}(a_m)) = E_B(\alpha_{(0j_1)}(a_1) \cdots \alpha_{(0j_m)}(a_m)) = E_B(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m)),
\]

so (2) is also true. As noted before, we can also assume without loss of generality that \( D = \bigvee_{j \geq 1} \pi_j(A) = \bigvee_{j \geq 1} A_j \), by simply replacing \( D \) with a von Neumann subalgebra. Moreover, Koestler remarked that, if we define the so-called tail algebra \( A_\infty \subset D \) by

\[
A_\infty = \bigcap_{n \geq 1} \left( \bigvee_{j \geq n} A_j \right),
\]

and if \( A_\infty \subset A \), then the copies \( (\pi_j, B = A_\infty, A, D) \) are automatically symmetric independent. Hence, to every trace-preserving action \( \Sigma \cap D \) we can automatically associate the sequence of independent symmetric \( (\pi_j, B = A_\infty, A, D) \) as described.

**Notation.** In what follows, given an \( m \)-tuple \( (j_1, \ldots, j_m) = \emptyset \), we denote by \( \alpha_{j_1, \ldots, j_m} = \alpha_{\sigma_{j_1, \ldots, j_m}} \), where \( \sigma_{j_1, \ldots, j_m}(i) = j_i, 1 \leq i \leq m \).

**Definition 3.4.** Let \( (\pi_j, B, A, D) \) be a sequence of symmetric independent copies, \( S \) a subset of \( A \) such that \( 1 \in S = S^* \), \( H \) a Hilbert space and \( \omega \) a free ultrafilter on \( \mathbb{N} \). Denote by \( \{e_j\} \) the canonical orthonormal basis of \( \ell^2 = \ell^2(\mathbb{N}) \). Let \(-1 < q < 1\). Define

\[
\Gamma_q^0(B, S \otimes H) = (B \cup \{ s_q(a, h) : a \in S, h \in H \})'' \subset (\Gamma_q(\ell^2 \otimes H) \otimes D)^\omega,
\]

where

\[
s_q(a, h) = (n^{-\frac{1}{2}} \sum_{j=1}^n s_q(e_j \otimes h) \otimes \pi_j(a))_n.
\]

Finally define

\[
\Gamma_q(B, S \otimes H) = (E_{\Gamma_q(\ell^2 \otimes H) \otimes id})_n(\Gamma_q^0(B, S \otimes K)),
\]

where \( K \) is an infinite dimensional Hilbert space containing \( H \), \( \ell^2_n = \text{span}\{e_1, \ldots, e_n\} \) and for each \( n \)

\[
E_{\Gamma_q(\ell^2 \otimes H)} : \Gamma_q(\ell^2 \otimes K) \to \Gamma_q(\ell^2_n \otimes H)
\]

is the canonical conditional expectation.

As \( q \) will be fixed throughout this section, we will simply use the notation \( s(x, h) \) instead of \( s_q(x, h) \) from now on.

**Remark 3.5.** Due to functoriality, the definition of \( \Gamma_q(B, S \otimes H) \) does not depend on the particular choice of \( K \supset H \). When \( H \) is infinite dimensional \( \Gamma_q(B, S \otimes H) = \Gamma_q^0(B, S \otimes H) \).
Lemma 3.8. \( \Gamma_q^0(B, S \otimes H) = \{ (s(a, h) : a \in B \cup S, h \in H) \}'' \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)'' \).

Remark 3.6. \( \Gamma_q(B, S \otimes H) \) is a von Neumann algebra. Indeed, since the map
\[
E = (E_{\Gamma_q(\ell^2 \otimes H)} \otimes \text{id})_n : (\Gamma_q(\ell^2 \otimes K) \bar{\otimes} D)'' \to (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)''
\]
is a normal linear projection (i.e. idempotent map) of norm one, it follows that \( \Gamma_q(B, S \otimes H) \) is an ultraweakly closed, self-adjoint subspace of \( (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)'' \) containing the identity. It’s straightforward to see that the map \( E \) has the following bimodularity property: \( E(x)E(y)E(z) = E(E(x)yE(z)) \), for all \( x, y, z \in \Gamma_q(B, S \otimes K) \). Thus, for \( x, y \in \Gamma_q(B, S \otimes K) \) we have \( E(x)E(y) = E(E(x)y) \in \Gamma_q(B, S \otimes H) \).

The canonical generators \( s_q(a, h) \) are not easy to work with in a variety of situations. The classical \( q \)-gaussians possess a system of generators, the so-called Wick words, whose linear span is an ultraweakly dense \( * \)-subalgebra. Generalized \( q \)-gaussians also have such a well-behaved system of linear generators, which will be called Wick words by analogy with the classical case. In order to find these Wick words let us first define, for every \( n \in \mathbb{N} \), \( x \in A \) and \( h \in H \),
\[
(\sum_{j=1}^{n} s(e_j \otimes h) \otimes \pi_j(x)) \in \Gamma_q(\ell^2 \otimes H) \bar{\otimes} D.
\]
It’s easy to see that \( s(x, h) = (u_n(x, h))_n \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)'' \), for \( x \in A, h \in H \). For \( x_1, \ldots, x_m \in BSB = \{ b_1a_2 : b_1, b_2 \in B, a \in S \} \) and \( h_1, \ldots, h_m \in H \) we will analyze the product
\[
u_n(x_1, h_1) \cdots u_n(x_m, h_m) = \sum_{\sigma \in P(m)} (\sum_{j_1, \ldots, j_m \leq n} s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) \otimes \pi_j(1) \pi_j(2) \cdots \pi_j(n)),
\]
where \( P(m) \) is the set of all partitions of the set \( \{1 \ldots m \} \) and the notation \( (j_1, \ldots, j_m) = \sigma \) means that \( j_i = j_k \) if and only if there exists an \( C \in \sigma \) such that \( i, k \in C \). We will also denote by \( P_{1,2}(m) \) the set of all pair-singleton partitions of \( \{1 \ldots m \} \). For \( \sigma \in P(m) \) let’s define
\[
x_\sigma(x_1, h_1, \ldots, x_m, h_m) = \sum_{1 \leq j_1, \ldots, j_m \leq n, (j_1, \ldots, j_m) = \sigma} s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) \otimes \pi_j(1) \cdots \pi_j(n),
\]
and \( x_\sigma(x_1, h_1, \ldots, x_m, h_m) = (x_\sigma(x_1, h_1, \ldots, x_m, h_m))_n \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)'' \). To keep the notation less cumbersome, we will omit the parameters \( x_k, h_k \) whenever they are clearly understood from the context. Next we see that
\[
u_n(x_1, h_1) \cdots u_n(x_m, h_m) = \sum_{\sigma \in P(m)} x_\sigma,
\]
and also
\[
s(x_1, h_1) \cdots s(x_m, h_m) = (u_n(x_1, h_1) \cdots u_n(x_m, h_m))_n = \sum_{\sigma \in P(m)} x_\sigma.
\]

Lemma 3.8. Let \( (\pi_j, B, A, D) \) be a sequence of symmetric copies. Then
1. \( \sup_n \| x_\sigma^n \|_\infty < \infty \) for all \( m \geq 1 \) and \( \sigma \in P_{1,2}(m) \);
2. If \( \sigma \notin P_{1,2}(m) \) and \( 0 < p < \infty \) then
\[
\lim_n \| x_\sigma^n \|_p = 0.
\]
In particular \( s(x_1, h_1) \cdots s(x_m, h_m) = \sum_{\sigma \in P_{1,2}(m)} x_\sigma \).

Proof. The proof is the same as that of Prop. 4.1. in [20].
Proposition 3.9. We have the following convolution formula for the multiplication of Wick words:

\[ x_\sigma(x_1, h_1, \ldots, x_m, k_m)x_\theta(y_1, k_1, \ldots, y_m', k_m') = \sum_{\gamma \in P_1,2(m+m')} x_\gamma(x_1, h_1, \ldots, y_m', k_m'). \]

Moreover, item i) in the lemma above shows that in the summation we can restrict ourselves to pair-singleton partitions whose only additional pairings are between the singletons of \( \sigma \) and \( \theta \). In particular, the linear span of the Wick words is a \(*\)-algebra.

Proof. We have

\[ x_\sigma(x_1, h_1, \ldots, x_m, k_m)x_\theta(y_1, k_1, \ldots, y_m', k_m') = \]

\[ (n^{-\frac{m+m'}{2}}) \sum_{(j_1, \ldots, j_m) = \sigma, (l_1, \ldots, l_m') = \theta} s(e_{j_1} \otimes h_1) \cdots s(e_{l_m'} \otimes k_m') \otimes \pi_{j_1}(x_1) \cdots \pi_{l_m'}(y_m')) = \]

\[ \sum_{\gamma \in P_1,2(m+m'), \gamma_1(1, \ldots, m) = \sigma, \gamma_1(1, \ldots, m') = \theta} \sum_{(j_1, \ldots, j_m) = \gamma} (n^{-\frac{m+m'}{2}}) \sum_{(l_1, \ldots, l_m') = \gamma} s(e_{j_1} \otimes h_1) \cdots s(e_{l_m'} \otimes k_m') \otimes \pi_{j_1}(x_1) \cdots \pi_{l_m'}(y_m')) = \]

\[ \sum_{\gamma \in P_1,2(m+m'), \gamma_1(1, \ldots, m) = \sigma, \gamma_1(1, \ldots, m') = \theta} x_\gamma(x_1, h_1, \ldots, y_m', k_m'). \]

Now if \( \gamma \in P_1,2(m+m') \) connects a singleton in \( \sigma \) with a leg of a pair in \( \theta \) or the leg of a pair in \( \sigma \) with either a singleton or a leg of a pair in \( \theta \), the resulting \( x_\gamma \) is associated to a partition containing a 3-set or a 4-set and hence vanishes according to Lemma 3.8. So in the above sum we may only allow \( \gamma \)'s which preserve the pair sets of both \( \sigma \) and \( \theta \) and can only additionally pair singletons "on different sides of the marker", which ends the proof.

Our next result provides a reduction method for the Wick words.

Lemma 3.10. Let \( \pi_j : A \to D \) be symmetric independent copies, and \( 1 \in S = S^* \subset A \). Let \( x_1, \ldots, x_m \in \mathbb{B} \mathbb{S} \mathbb{B}, \sigma \in P_{1,2}(m) \) having \( s \) singletons and \( p \) pairs and \( \phi : \{1, \ldots, m\} \to \{1, \ldots, s+p\} \) which encodes \( \sigma \), i.e., \( \phi(k_t) = t \) for every singleton \( \{k_t\} \in \sigma \), \( 1 \leq t \leq s \) and \( \phi(k_t') = \phi(k_t'') = t+s \), for every pair \( \{k_t', k_t''\} \in \gamma, 1 \leq t \leq p \). Consider \( (\varepsilon_k) \) a sequence of Bernoulli independent random variables on a probability space \((X, \mu)\), i.e. \( \varepsilon_k : X \to \{\pm 1\}, \mathbb{E}(\varepsilon_k = 1) = \mathbb{E}(\varepsilon_k = -1) = \frac{1}{2} \). Then

\[ \| \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) - E_{l_1, \ldots, l_s} \pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m) \|_2 \leq C(m, x_j) n^{-\frac{m-1}{2}}. \]

In particular we have

\[ (n^{-\frac{m}{2}}) \sum_{(l_1, \ldots, l_s, l_{s+1}, \ldots, l_{s+p}) = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) \]

\[ = (n^{-\frac{s}{2}}) \sum_{(l_1, \ldots, l_s) = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes E_{l_1, \ldots, l_s} \pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) \]

\[ = (n^{-\frac{s}{2}}) \sum_{(l_1, \ldots, l_s) = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes \alpha_{l_1, \ldots, l_s} (F_\sigma(x_1, \ldots, x_m)), \]

where \( F_\sigma(x_1, \ldots, x_m) = E_{1, \ldots, s} \pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m) \) and the last equality takes place in \( (L^\infty(X) \otimes D)^\omega \).
Proof. Throughout the proof we endow $L^\infty(X) \otimes D$ with the natural trace $\mu \otimes \tau$, where $\tau$ is the faithful trace on $D$. The $\| \cdot \|_2$ in the first statement is the one corresponding to $\mu \otimes \tau$. The approach we take is somewhat similar to the one in [23].

Step 1. Let $x_1, \ldots, x_m \in BSB$ and $n$ be fixed. Consider

$$\Omega_n = \{(C_1, \ldots, C_{s+p}) : C_1 \sqcup \ldots \sqcup C_{s+p} = \{1, \ldots, n\}, C_i \neq \emptyset, \forall i\}.$$ 

Make $\Omega_n$ into a probability space with the normalized counting measure. For every $s + p$-tuple $(l_1, \ldots, l_{s+p}) = \emptyset$, consider the indicator function $\delta_{l_1,\ldots,l_{s+p}} : \Omega_n \to \{0,1\}$ which is 1 if $l_i \in C_1$ for all $1 \leq i \leq s + p$ and 0 otherwise. Then one can easily check that

$$\mathbb{E}(\delta_{l_1,\ldots,l_{s+p}}) = \frac{\# \Omega_{n-1}}{\# \Omega_n} = C(n)$$

only depends on $n$ and not on the $s + p$-tuple and moreover $C(n)$ tends to a finite positive limit as $n \to \infty$. Thus $(C(n)^{-1})$ is a bounded sequence and take $C > 0$ such that $C(n)^{-1} \leq C$ for all $n$. Set

$$F(l_1, \ldots, l_{s+p}) = \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_1}(x_1) \cdots \pi_{l_p}(x_m)) - E l_1, \ldots, l_s (\pi_{l_1}(x_1) \cdots \pi_{l_p}(x_m)).$$

Then we have

$$\| \sum_{(l_1, \ldots, l_{s+p}) = \emptyset} F(l_1, \ldots, l_{s+p}) \|_2 = C(n)^{-1} \| C(n) \sum_{(l_1, \ldots, l_{s+p}) = \emptyset} F(l_1, \ldots, l_{s+p}) \|_2 = C(n)^{-1} \| \mathbb{E}(\delta_{l_1,\ldots,l_{s+p}}) F(l_1, \ldots, l_{s+p}) \|_2 =$$

$$C(n)^{-1} \| \sum_{(l_1, \ldots, l_{s+p}) = \emptyset} \sum_{(C_1, \ldots, C_{s+p}) \in \Omega_n} \delta_{l_1,\ldots,l_{s+p}}((C_1, \ldots, C_{s+p})) F(l_1, \ldots, l_{s+p}) \|_2 =$$

$$C(n)^{-1} \| \sum_{(l_1, \ldots, l_{s+p}) = \emptyset} \sum_{(C_1, \ldots, C_{s+p}) \in \Omega_n} \sum_{l_1 \subseteq C_1, \ldots, l_s \subseteq C_{s+p}} F(l_1, \ldots, l_{s+p}) \|_2 = C(n)^{-1} \| \mathbb{E}(G) \|_2$$

$$\leq C \sup_{(C_1, \ldots, C_{s+p}) \in \Omega_n} \| G((C_1, \ldots, C_{s+p})) \|_2 = C \sup_{(C_1, \ldots, C_{s+p}) \in \Omega_n} \| \sum_{l_1 \subseteq C_1, \ldots, l_s \subseteq C_{s+p}} F(l_1, \ldots, l_{s+p}) \|_2,$$

where we define $G : \Omega_n \to L^\infty(X) \otimes D$ by $G((C_1, \ldots, C_{s+p})) = \sum_{l_1 \subseteq C_1, \ldots, l_s \subseteq C_{s+p}} F(l_1, \ldots, l_{s+p})$.

Step 2. It suffices thus to estimate $\| \sum_{l \subseteq C_1, \ldots, l_s \subseteq C_{s+p}} F(l_1, \ldots, l_{s+p}) \|_2$, for a fixed non-degenerate partition $C_1, \ldots, C_{s+p}$ of $\{1, \ldots, n\}$. Fix such an arbitrary partition. We define the sets $I_l = C_1 \cup \cdots C_{s+p-1} \cup \{1, \ldots, l\} \cap C_{s+p}$ and for $l \in C_{s+p}$

$$d_l = \sum_{l_1 \subseteq C_{l_1}, \ldots, l_s \subseteq C_{l_s}, l_{s+1} \subseteq C_{l_{s+1}}, \ldots, l_{s+p-1} \subseteq C_{l_{s+p-1}}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_1}(x_1) \cdots \pi_{l_l}(x_k) \cdots \pi_{l_s}(x_{k_s}) \cdots \pi_{l_{s+p}}(x_m))$$

$$- E_{l_{s+p-1}}((\pi_{l_{s+1}}(x_1) \cdots \pi_{l_{s+p-1}}(x_{k_{s+p-1}})) \cdots \pi_{l_{l_s}}(x_{k_{l_s}}) \cdots \pi_{l_{l_s}}(x_m))).$$

Note that $D_l = L^\infty(X) \otimes A_l$, $l \in C_{s+p}$, form an increasing finite sequence of von Neumann subalgebras of $L^\infty(X) \otimes D$. Now $d_l \in D_l$ and $E_{D_{l_{s+p-1}}}(d_l) = 0$, for all $l \in C_{s+p}$. The martingale inequality yields

$$\| \sum_{l \in C_{s+p}} d_l \|_2 \leq n^2 \sup_{l \in C_{s+p}} \| d_l \|_2.$$

On the other hand, since the products $\varepsilon_{l_1} \cdots \varepsilon_{l_s}$ are mutually orthogonal for different $s$-tuples $(l_1, \ldots, l_s)$, we see that

$$\| d_l \|_2 = \| \sum_{l_1 \subseteq C_{l_1}, \ldots, l_s \subseteq C_{l_s}, l_{s+1} \subseteq C_{l_{s+1}}, \ldots, l_{s+p-1} \subseteq C_{l_{s+p-1}}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes \pi_{l_1}(x_1) \cdots \pi_{l_l}(x_k) \cdots \pi_{l_s}(x_{k_s}) \cdots \pi_{l_{s+p}}(x_m)$$

$$- E_{l_{s+p-1}}((\pi_{l_{s+1}}(x_1) \cdots \pi_{l_{s+p-1}}(x_{k_{s+p-1}})) \cdots \pi_{l_{l_s}}(x_{k_{l_s}}) \cdots \pi_{l_{l_s}}(x_m))).$$

Note that $D_l = L^\infty(X) \otimes A_l$, $l \in C_{s+p}$, form an increasing finite sequence of von Neumann subalgebras of $L^\infty(X) \otimes D$. Now $d_l \in D_l$ and $E_{D_{l_{s+p-1}}}(d_l) = 0$, for all $l \in C_{s+p}$. The martingale inequality yields

$$\| \sum_{l \in C_{s+p}} d_l \|_2 \leq n^2 \sup_{l \in C_{s+p}} \| d_l \|_2.$$
\((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)) - E_{l_{l-1}}((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)))\|_2^2 \\
= \| \sum_{l_1 \in C_1, \ldots, l_s \in C_x} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes \left( \sum_{l_{s+1} \in C_{s+1}, \ldots, l_{s+p-1} \in C_{s+p-1}} ((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)) - E_{l_{l-1}}((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)))\|_2^2 - \sum_{l_{s+1} \in C_{s+1}, \ldots, l_{s+p-1} \in C_{s+p-1}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes ((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)) - E_{l_{l-1}}((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)))\|_2^2 \leq n^{s \frac{m-1}{2}} \|x_1\|_\infty \cdots \|x_m\|_\infty \leq C' n \frac{m-1}{2}.

Steps 1 and 2 so far imply that
\[
\| \sum_{l_1 \in C_1, \ldots, l_s \in C_x} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes ((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)) - E_{l_{l-1}}((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)))\|_2^2 \leq C' n \frac{m-1}{2}.
\]

Step 3. Now we may proceed inductively. Denote by \(y = \pi_{l\phi(1)}(x_1) \cdots \pi_{l\phi(m)}(x_m)\). Then, using axiom (4) and because the conditional expectations commute, we see that
\[
y - E_{C_1 \cup \cdots \cup C_{s+p-2}}(y) = y - E_{C_1 \cup \cdots \cup C_{s+p-1}}(y) + E_{C_1 \cup \cdots \cup C_{s+p-1}}(y) - E_{C_1 \cup \cdots \cup C_{s+p-2}}(y) = y - E_{C_1 \cup \cdots \cup C_{s+p-1}}(y) + E_{C_1 \cup \cdots \cup C_{s+p-1}}(y) - E_{C_1 \cup \cdots \cup C_{s+p-2} \cup C_{s+p}}(E_{C_1 \cup \cdots \cup C_{s+p-1}}(y)) = y - E_{C_1 \cup \cdots \cup C_{s+p-1}}(y).
\]

Using the previous steps and the fact that the conditional expectations are \(\| \cdot \|_2\)-contractive, we obtain
\[
\| \sum_{l_1 \in C_1, \ldots, l_s \in C_x} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes ((\pi_{l\phi(1)}(x_1) \cdots \pi_l(x_{k_p^l}) \cdots \pi_{l\phi(m)}(x_m)) - E_{C_1 \cup \cdots \cup C_{s+p-2} \cup C_{s+p}}(\pi_{l\phi(1)}(x_1) \cdots \pi_{l\phi(m)}(x_m)))\|_2^2 \leq C' n \frac{m-1}{2}.
\]
After using the triangle inequality \( p \) times, we get
\[
\| \sum_{l_1 \in C_1, \ldots, l_{s+p} \in C_{s+p}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m) - E_{C_1 \cup \cdots \cup C_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))) \|_2 \leq pC'^{n^{\frac{m-1}{2}}} = CN^{n^{\frac{m-1}{2}}}. 
\]

Now we claim that
\[
E_{C_1 \cup \cdots \cup C_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) = E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)).
\]
This can be established using axioms 3 and 4. Indeed, since \( l_{s+p} \notin C_1 \cup \cdots \cup C_{s+p-1} \supset \{l_1, \ldots, l_{s+p-1}\} \), by applying axiom 3 we see that
\[
E_{\{l_1, \ldots, l_{s+p-1}\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) = \pi_{l_{\phi(1)}}(x_1) \cdots E_{\{l_1, \ldots, l_{s+p-1}\}}(\pi_{l_{\phi(1)}}(x_{k_1'} \cdots \pi_{l_{\phi(m)}}(x_{k_p'})) \cdots \pi_{l_{\phi(m)}}(x_m)) = E_{C_1 \cup \cdots \cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))
\]

and then
\[
E_{C_1 \cup \cdots \cup C_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) = E_{C_1 \cup \cdots \cup C_s}(E_{C_1 \cup \cdots \cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))) = E_{C_1 \cup \cdots \cup C_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) = E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))
\]
which proves the claim. Now the claim together with the last inequality entail
\[
\| \sum_{l_1 \in C_1, \ldots, l_{s+p} \in C_{s+p}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m) - E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))) \|_2 \leq CC'^{n^{\frac{m-1}{2}}}.
\]
Step 1 now implies
\[
\| \sum_{\{l_1, \ldots, l_{s+p}\} = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m) - E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))) \|_2 \leq CC'^{n^{\frac{m-1}{2}}},
\]
which proves the first statement in the lemma. For the second statement, we first note that 
\( E_{l, \ldots, l_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) \) only depends on \( l_1, \ldots, l_s \), and not on \( l_{s+1}, \ldots, l_{s+p} \). Indeed, let \( l_1, \ldots, l_s, l'_{s+1}, \ldots, l'_{s+p} = \emptyset \) another \( s+p \)-tuple with the same first \( s \) entries. Take a finite permutation \( \sigma \) such that \( \sigma(l_i) = i, l, i \leq s \) and \( \sigma(l_{s+i}) = l'_{s+i}, i \leq p \). Then \( \alpha_\sigma \) is the identity on \( A_{l_1, \ldots, l_s} \), hence
\[
E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_{k_1'} \cdots \pi_{l_{\phi(m)}}(x_{k_p'}))) = E_{\{l_1, \ldots, l_s\}} \circ \alpha_\sigma(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_{k_1'} \cdots \pi_{l_{\phi(m)}}(x_{k_p'}))) = E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_{k_1'} \cdots \pi_{l_{\phi(m)}}(x_{k_p}))),
\]
which proves the claim. Now the first statement of the lemma together with an easy counting argument shows that
\[
(n^{-\frac{p}{2}} \sum_{\{l_1, \ldots, l_{s+p}\} = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes \pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) =
= (n^{-\frac{p}{2}} \sum_{\{l_1, \ldots, l_{s+p}\} = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes E_{\{l_1, \ldots, l_s\}}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))) =
= (n^{-\frac{p}{2}} \sum_{\{l_1, \ldots, l_s\} = \emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes E_{l_1, \ldots, l_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m))).
\]
Finally, let’s note that
\[
E_{l_1, \ldots, l_s} \circ \alpha_{l_1, \ldots, l_{s+p}} = \alpha_{l_1, \ldots, l_s} \circ E_{l_1, \ldots, l_s},
\]
which implies
\[E_{t_1, \ldots, t_s}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) = (E_{t_1, \ldots, t_s} \circ \alpha_{t_1, \ldots, t_{s+p}})(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) = (\alpha_{t_1, \ldots, t_s} \circ E_{1, \ldots, s})(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) = \alpha_{t_1, \ldots, t_s}(F_\sigma(x_1, \ldots, x_m)).\]

**Theorem 3.11.** Let \((\pi_j, B, A, D)\) be a sequence of symmetric independent copies, \(x_1, \ldots, x_m \in A, \sigma \in P_{1,2}(m)\) having \(s\) singletons and \(p\) pairs and \(\phi : \{1, \ldots, m\} \to \{1, \ldots, s+p\}\) which encodes \(\sigma\). Then
\[x_\sigma(x_1, h_1, \ldots, x_m, h_m) = (n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m)=\sigma} s(e_j \otimes h_1) \cdots s(e_m \otimes h_m) \otimes \pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) = f_\sigma(h_1, \ldots, h_m)(n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s)=0} s(e_l \otimes h_1) \cdots s(e_l \otimes h_s) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m))) = f_\sigma(h_1, \ldots, h_m)W_\sigma(x_1, h_1, \ldots, x_m, h_m),\]
where \(F_\sigma(x_1, \ldots, x_m) = E_{1, \ldots, s}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)), f_\sigma(h_1, \ldots, h_m) = \eta^{s(\sigma)} \prod_{(k, l) \in \sigma}(h_k, h_l)\) and \(\{k_1, \ldots, k_s\}\) are the singletons of \(\sigma\). The elements
\[W_\sigma(x_1, h_1, \ldots, x_m, h_m) = (n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s)=0} s(e_l \otimes h_1) \cdots s(e_l \otimes h_s) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)))\]
will be called reduced Wick words.

**Proof.** We will use the previous lemma. Let \(\hat{B} = B, \hat{A} = \Gamma_q(H) \otimes A, \hat{D} = \Gamma_q(\ell^2 \otimes H) \otimes D\) and \(\hat{\pi}_j : A \to \Gamma_q(\ell^2 \otimes H) \otimes D\) be the \(*\)-homomorphisms given by
\[\hat{\pi}_j(s(h) \otimes x) = s(e_j \otimes h) \otimes \pi_j(x) .\]
Then \((\hat{\pi}_j, B, \hat{A}, \hat{D})\) represents a sequence of independent symmetric copies. Moreover, it is easy to see that \(\hat{A}_I = \Gamma_q(\ell^2(I) \otimes H) \otimes A_I\). Now according to the previous lemma we have
\[(n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m)=\sigma} \epsilon_j \otimes \epsilon_j \otimes s(e_j \otimes h_1) \cdots s(e_j \otimes h_m) \otimes \pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) = (n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_{s+p})=0} \epsilon_l \otimes s(e_l \otimes h_1) \cdots s(e_l \otimes h_s) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(s_1, \ldots, x_m))) = (n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_{s+p})=0} \epsilon_l \otimes \hat{\pi}_{\phi(1)}(s(h_1) \otimes x_1) \cdots \hat{\pi}_{\phi(m)}(s(h_m) \otimes x_m))) = (n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s)=0} \epsilon_l \otimes \hat{\alpha}_{l_1, \ldots, l_s}(E_{\hat{A}_I}(\pi_{\phi(1)}(s(h_1) \otimes x_1) \cdots \pi_{\phi(m)}(s(h_m) \otimes x_m))) = (n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s)=0} \epsilon_l \otimes s(e_l \otimes h_1) \cdots s(e_l \otimes h_s) \otimes \alpha_{l_1, \ldots, l_s}(E_{\hat{A}_I}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m))) = f_\sigma(h_1, \ldots, h_m)(n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s)=0} \epsilon_l \otimes s(e_l \otimes h_1) \cdots s(e_l \otimes h_s) \otimes \alpha_{l_1, \ldots, l_s}(E_{\hat{A}_I}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m))) = f_\sigma(h_1, \ldots, h_m)(n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s)=0} \epsilon_l \otimes s(e_l \otimes h_1) \cdots s(e_l \otimes h_s) \otimes \alpha_{l_1, \ldots, l_s}(E_{\hat{A}_I}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m))) .\]
\[
= f_\sigma(h_1, \ldots, h_m)(n^{-\frac{3}{2}} \sum_{(l_1, \ldots, l_s)=\emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes s(e_{l_1} \otimes h_{k_1}) \cdots s(e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m))).
\]

To see why the equality on line 6 is true, note that
\[
s(e_{\phi(1)} \otimes h_1) \cdots s(e_{\phi(m)} \otimes h_m) = \sum_{\theta \in P_{1,2}(m)} f_\theta(e_{\phi(1)} \otimes h_1 \otimes \cdots \otimes e_{\phi(m)} \otimes h_m) W((e_{\phi(1)} \otimes h_1 \otimes \cdots \otimes e_{\phi(m)} \otimes h_m)_\theta),
\]
where the notation \((\cdot)_\theta\) means that the pair positions of \(\theta\) have been removed. After the application of \(E_{\Gamma_q}(\ell^2 \otimes H)\), we see that the only surviving partition is \(\theta = \sigma\) and
\[
E_{\Gamma_q}(\ell^2 \otimes H)(s(e_{\phi(1)} \otimes h_1) \cdots s(e_{\phi(m)} \otimes h_m)) = f_\sigma(h_1, \ldots, h_m) W(e_1 \otimes h_{k_1} \cdots e_s \otimes h_{k_s}).
\]

Now, let’s define
\[
\tilde{s}(x, h) = (n^{-\frac{1}{2}} \sum_{j=1}^n \varepsilon_j \otimes s(e_j \otimes h) \otimes \pi_j(x) \in (L^\infty(X) \otimes \Gamma_q(\ell^2 \otimes H) \otimes D)^\omega.
\]

We claim that the new Wick words \(\tilde{x}_\sigma\) associated to the variables \(\tilde{s}(x, h)\) have the same moments as \(x_\sigma\) and hence generate an isomorphic von Neumann algebra. Indeed, fix \(\sigma \in P_{1,2}(m)\). Note that for \((l_1, \ldots, l_s) = \emptyset\), we have \(\mu(\varepsilon_{l_1} \cdots \varepsilon_{l_s}) = \mu(\varepsilon_{l_1}) \cdots \mu(\varepsilon_{l_s}) = \delta_{s=0}\) due to the fact that \(\varepsilon_j\) are mean-zero, independent random variables. Then
\[
\tau_\omega(\tilde{x}_\sigma(x_1, h_1, \ldots, x_m, h_m)) = \tau_\omega((n^{-\frac{3}{2}} \sum_{(l_1, \ldots, l_s+p)=\emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes s(e_{l_1} \otimes h_1) \cdots s(e_{l_s} \otimes h_m) \otimes \pi_{l_1(1)}(x_1) \cdots \pi_{l_s(1)}(x_m)) =
\]
\[
\lim_{n \to \infty} \tau_\omega((n^{-\frac{1}{2}} \sum_{(l_1, \ldots, l_s+p)=\emptyset} \mu(\varepsilon_{l_1} \cdots \varepsilon_{l_s}) \tau(s(e_{l_1} \otimes h_1) \cdots s(e_{l_s} \otimes h_m) = \tau_D(\pi_{l_1(1)}(x_1) \cdots \pi_{l_s(1)}(x_m)) =
\]
\[
\delta_{s=0} \lim_{n \to \infty} (n^{-\frac{3}{2}} \sum_{(l_1, \ldots, l_s+p)=\emptyset} \tau(s(e_{l_1} \otimes h_1) \cdots s(e_{l_s} \otimes h_m) \otimes \pi_{l_1(1)}(x_1) \cdots \pi_{l_s(1)}(x_m) =
\]
\[
\tau_\omega(x_\sigma(x_1, h_1, \ldots, x_m, h_m)).
\]

Define \(\mathcal{M} \subset (\Gamma_q(\ell^2 \otimes H) \otimes D)^\omega\) to be the von Neumann algebra generated by all the Wick words \(x_\sigma\). Also define \(\mathcal{M} \subset (L^\infty(X) \otimes \Gamma_q(\ell^2 \otimes H) \otimes D)^\omega\) to be the von Neumann algebra generated by the elements \(\tilde{x}_\sigma\). Using the claim, the convolution formula and Proposition 3.1 we see that the map
\[
\mathcal{M} \ni \sum x_\sigma \mapsto \sum \tilde{x}_\sigma \in \mathcal{M}
\]
is a *-isomorphism. Applying the inverse of this isomorphism to the equality
\[
(n^{-\frac{3}{2}} \sum_{(l_1, \ldots, l_s+p)=\emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes s(e_{l_1} \otimes h_1) \cdots s(e_{l_s} \otimes h_m) \otimes \pi_{l_1(1)}(x_1) \cdots \pi_{l_s(1)}(x_m)) =
\]
\[
= f_\sigma(h_1, \ldots, h_m)(n^{-\frac{3}{2}} \sum_{(l_1, \ldots, l_s)=\emptyset} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes s(e_{l_1} \otimes h_{k_1}) \cdots s(e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m))),
\]
we obtain the desired identity. \(\blacksquare\)

**Proposition 3.12.** Let \(x_1, \ldots, x_m \in A, h_1, \ldots, h_m \in H, \sigma \in P_{1,2}(m)\) and \((j_1, \ldots, j_m) = \sigma\). Then we have the following moment formula:
\[
\tau(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) = \delta_{\sigma \in P_{1,2}(m)} f_\sigma(h_1, \ldots, h_m) \tau(\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)),
\]
which does not depend on the choice of \((j_1, \ldots, j_m) = \sigma\) and in particular

\[
\tau(s(x_1, h_1) \cdots s(x_m, h_m)) = \delta_{m,2} \sum_{\sigma \in P_2(m)} f_{\sigma}(h_1, \ldots, h_m) \tau(\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)),
\]

where \(f_{\sigma}(h_1, \ldots, h_m) = q^{\text{cr}(\sigma)} \prod_{(i, r) \in \sigma} (h_i, h_r)\).

Proof. It’s a straightforward application of the reduction formula.

\[\boxed{\text{Remark 3.13. Proposition 3.13 shows that } M = \Gamma_q^0(B, S \otimes H) \text{ could be introduced abstractly as the tracial von Neumann algebra } (M, \tau) \text{ generated by elements } s(x, h), x \in BSB, h \in H \text{ which satisfy the above moment formula.}}\]

\[\text{Proposition 3.14. Let } K \text{ be infinite dimensional and } x_1, \ldots, x_n \in BSB, h_1, \ldots, h_m \in K, \sigma \in P_{12}(m). \text{ Then } x_\sigma(x_1, h_1, \ldots, x_n, h_m) \in \Gamma_q^0(B, S \otimes K). \text{ For every Hilbert space } K, \text{ all the Wick words } x_\sigma(x_1, h_1, \ldots, x_n, h_m), x_i \in BSB, h_i \in H, \text{ are in } M = \Gamma_q(B, S \otimes H). \text{ In particular, } M \text{ is the ultraweakly closed linear span of the (reduced) Wick words and } L^2(M) \text{ is the } \|\cdot\|_2 \text{-closed span of the (reduced) Wick words.}}\]

Proof. We need a basic fact about infinite dimensional Hilbert spaces.

**Fact.** Let \(K\) be an infinite dimensional Hilbert space and \(\lambda_1, \ldots, \lambda_p \in \mathbb{C}\). Then there exist norm bounded sequences \(\xi_n^k, \eta_n^k \in K\), for \(1 \leq k \leq p\) such that \(\xi_n^k \to 0\), \(\eta_n^k \to 0\) weakly and \((\xi_n^k, \eta_n^k) = \lambda_k\), for all \(1 \leq k \leq p\), and moreover \(\xi_n^k, \eta_n^k \perp \xi_n^j, \eta_n^j\) for \(k \neq j\). Indeed, let \((e_n)\) be an orthonormal infinite sequence in \(K\). Define \(\xi_1^1 = \lambda_1 e_n, \eta_1^1 = e_n, \xi_1^2 = \lambda_2 e_n+1, \eta_1^2 = e_n+1, \ldots, \xi_n^p = \lambda_pe_{n+p-1}, \eta_n^p = e_{n+p-1}\).

To prove the proposition we will use induction on \(s\), the numbers of singletons in \(\sigma\). For \(s = 0\) the statement is trivial. For a given \(\sigma\) with pairs \(B_1, \ldots, B_p \text{ and } B = \{l, r\}\) we use the Fact to find uniformly norm bounded vectors \(h_{l,B}(k), h_{r,B}(k) \in K\) which converge to 0 weakly and such that \((h_{l,B}(k)), h_{r,B}(k)) = (h_l, h_r)\) for all pairs \(B = \{l, r\}\), and such that the \(h_{l,r,B}(k)\)'s are orthogonal for different pairs \(B\). Let us define \(\tilde{h}_l(k) = h_l\) for any singleton \(\{i\} \in \sigma\) and \(\tilde{h}_r(k) = h_{l,r,B}(k)\) if \(i \in B\) and \(i = l\) or \(i = r\). For every other Wick word \(x_{\sigma'}(y_1, f_1, \ldots, y_m', f_m')\), with \(y_j \in BSB, f_j \in K\), we have

\[
\lim_{k \to \infty} \tau(s(x_1, \tilde{h}_1(k)) \cdots s(x_m, \tilde{h}_m(k))x_{\sigma'}) = \lim_{k \to \infty} \sum_{\theta \in P_{12}(m)} \tau(x_\theta(x_1, \tilde{h}_1(k), \ldots, x_m, \tilde{h}_m(k))x_{\sigma'}).
\]

Indeed, for every \(\theta \in P_{12}(m)\) which does not contain all the pairs of \(\sigma\), we use the convolution and the moment formulas to obtain

\[
\tau(x_\theta x_{\sigma'}) = \sum_{\nu \in P_2(m+m')} \tau(x_\nu(x_1, \tilde{h}_1(k), \ldots, y_m', f_m')) = \sum_{\nu \in P_2(m+m')} f_\nu(\tilde{h}_1(k), \ldots, \tilde{h}_m(k), f_1, \ldots, f_m')\tau(W_\nu(x_1, \tilde{h}_1(k), \ldots, y_m', f_m')),
\]

the sum being taken over all \(\nu\) that preserve the pairs of \(\theta\) and \(\sigma'\) and additionally pair all the singletons of \(\theta\) and \(\sigma'\). Now since \(\theta\) does not contain all the pairs of \(\sigma\) there must be a leg \(l\) of a pair \(\{l, r\} = B \in \sigma\) which is connected by \(\theta\) to something else than its other leg in \(\sigma\). There are three possibilities:

1. \(\theta\) connects \(l\) to a leg \(l'\) of another pair \(B' = \{l', r'\} \in \sigma\). Then \((\tilde{h}_l(k), \tilde{h}_r(k)) = 0\), hence for every \(\nu\) in the sum above we have \(f_\nu(h_1, \ldots, f_m') = 0\).
(2) $\theta$ connects $l$ to a singleton \( \{i\} \in \sigma \). Then, since $\tilde{h}_1(k) \to 0$ weakly, we have $\tilde{h}_1(k, h_i) \to 0$, hence for every $\nu$ we also have that $f_\nu(h_1, \ldots, f_m') \to 0$ as $k \to \infty$.

(3) $\{l\}$ is a singleton of $\theta$. In this case, every $\nu \in P_{1,2}(m + m')$ which appears in the sum has to connect $l$ to a singleton $j \in \{1, \ldots, m'\}$. Thus, $(\tilde{h}_1(k, f_j) \to 0$ and again $f_\nu(h_1, \ldots, f_m') \to 0$ as $k \to \infty$.

Summing up, we see that for every $\theta$ such that $\sigma_p \not\subseteq \theta_p$, we have $\tau(x_\theta(x')x'_\sigma) \to 0$ as $k \to \infty$.

Thus, when letting $k \to \infty$, only those $\theta$'s containing the pairs of $\sigma$ make a non-zero contribution. Among them, there is exactly one which has $s$ singletons, namely $\sigma$, all the others have more pairs and hence less than $s$ singletons. We deduce that

$$x_\sigma = w - \lim_{k \to \infty} (s(x_1, \tilde{h}_1(k)) \cdots s(x_m, \tilde{h}_m(k))) - \sum_{\theta_p \supseteq \sigma_p, |\theta_s| < s} x_\theta(x_1, \tilde{h}_1(k), \ldots, x_m, \tilde{h}_m(k)).$$

Since by the induction hypothesis all the $x_\theta$'s, with $|\theta_s| < s$, are in $\Gamma^0_q(B, S \otimes K)$, this proves the statement. For the second statement, let $H$ be any Hilbert space and $K$ an infinite dimensional Hilbert space containing $H$. Let $x_i \in BSB$, $h_i \in H$ and $\sigma \in P_{1,2}(m)$. Then, by the first part, $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in \Gamma^0_q(B, S \otimes K)$. But $x_\sigma = (E_{\Gamma_q(\ell_2^{s} \otimes H)} \otimes id)_n(x_\sigma)$, hence $x_\sigma \in \Gamma_q(B, S \otimes H)$. \hfill \square

**Remark 3.15.** The reader can now better appreciate why we needed the "closure operation" in the definition of $\Gamma_q(B, S \otimes H)$. Indeed, Definition 3.4 ensures that the Wick words belong to $M = \Gamma_q(B, S \otimes H)$ for every Hilbert space $H$, finite or infinite dimensional. Also, Prop. 3.14 shows that $M = \Gamma_q(B, S \otimes H)$ could have been defined as the ultra-weakly closed span of the Wick words.

In the following we use the notation $L^2_k(M)$ for the $\| \cdot \|_2$-closed span of the Wick words of degree $k$ and $W_k(M)$ for the linear span of the Wick words of degree $k$.

**Theorem 3.16.** Let $(\pi_j, B, A, D)$ be a sequence of symmetric independent copies, $1 \in S = S^* \subset A$, $H$ a Hilbert space and $M = \Gamma_q(B, S \otimes H)$. Denote by $\tilde{H} = H \oplus H$. Take an infinite dimensional Hilbert space $K \supset H$ and denote by $\tilde{K} = K \oplus K$.

(1) For every angle $\theta$, let $\alpha_\theta$ be the canonical rotation on $\tilde{K}$. Then

$$\theta \mapsto \alpha_\theta = (\Gamma_q(id \otimes \alpha_\theta) \otimes id)_n \in Aut((\Gamma_q(\ell_2^{s} \otimes \tilde{K}) \otimes D)^\omega)$$

defines by restriction a one parameter group of automorphisms of $\tilde{M} = \Gamma_q(B, S \otimes \tilde{H})$.

Moreover, for every Wick word $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in M$ we have

$$\alpha_\theta(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) = x_\sigma(x_{\sigma}(x_1, \alpha_\theta(h_1), \ldots, x_m, \alpha_\theta(h_m))).$$

(2) For every Wick word $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in M$, the following formula holds

$$(E_M \circ \alpha_\theta)(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) = (\cos(\theta))^s x_\sigma(x_1, h_1, \ldots, x_m, h_m),$$

where $E_M : \tilde{M} \to M$ is the conditional expectation and $s$ is the number of singletons of $\sigma$.

(3) For every $\theta \in [0, \frac{\pi}{2})$, let $t = -\ln(\cos(\theta))$. Then $t \mapsto T_t = E_M \circ \alpha_{\theta_\theta}|_M$ defines a one parameter semi-group of normal, trace preserving, ucp maps on $M$. Moreover, for every Wick word $x_\sigma \in M$ we have $T_t(x_\sigma) = e^{-ts}x_\sigma$, where $s$ is the number of singletons of $\sigma$. Hence, when viewed as a contraction on $L^2_k(M)$, we have $T_t = \sum_{s \geq 0} e^{-ts}P_s$, where $P_s$ is the orthogonal projection of $L^2_k(M)$ on $L^2_s(M)$ and the series is $\| \cdot \|_\infty$-convergent, for every $t > 0$. In particular, if $L^2_k(M)$ is finitely generated as a right $B$ module for every $s$, then $T_t$ is compact over $B$ for every $t > 0$. 

Lemma 3.19. The generator $N$ of $T_t$ is a positive, self-adjoint, densely defined operator in $L^2(M) = \bigoplus_{k=0}^{\infty} L^2_k(M)$, acting by

$$N(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) = k x_\sigma(x_1, h_1, \ldots, x_m, h_m),$$

for every $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in L^2_k(M)$. The spectrum of $N$ is the set of non-negative integers $\mathbb{N}$, all of which are eigenvalues. $N$ is called the number operator.

Proof. It suffices to consider

$$\text{due to (4), we have that for every } h_1, h_2, \ldots, h_m \in \mathcal{H}, \text{ is easily checked, due to entry-wise functoriality, and it shows that } \alpha_\theta \text{ restricts to a one - parameter group of automorphisms on } M = \Gamma_q(B, S \otimes H). \text{ This proves (1). Then, using the reduction formula and the functoriality in each entry, we see that}

$$(E_M \circ \alpha_\theta)(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) =$$

$$(E_M \circ \alpha_\theta)((n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s) = \emptyset} s(e_{l_1} \otimes h_{k_1}) \cdots s(e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)))) =$$

$$(E_M \circ \alpha_\theta)((n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s) = \emptyset} W(e_{l_1} \otimes h_{k_1} \cdots e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)))) =$$

$$(E_M \circ \alpha_\theta)((n^{-\frac{m}{2}} \sum_{(l_1, \ldots, l_s) = \emptyset} W(e_{l_1} \otimes P_H \alpha_\theta(h_{k_1}) \cdots e_{l_s} \otimes P_H \alpha_\theta(h_{k_s})) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)))) =$$

$$(\cos(\theta))^s (E_M \circ \alpha_\theta)(x_\sigma(x_1, h_1, \ldots, x_m, h_m),$$

which establishes (2). (3) is straightforward using (2). To obtain (4), we calculate

$$\lim_{t \to 0} \frac{1}{t} (T_t(x_\sigma) - x_\sigma) = \lim_{t \to 0} \frac{e^{-st} - 1}{t} x_\sigma = -sx_\sigma,$$

for any Wick word $x_\sigma$ of degree $s$. The rest of the statements are straightforward. □

Remark 3.17. Due to (4), we have that for every $x \in M$, the function

$$[0, \frac{\pi}{2}) \ni \theta \mapsto \|\alpha_\theta(x) - x\|_2$$

is increasing.

Definition 3.18. We denote by $D_k(S) \subset L^2(D)$ the $\| \cdot \|_2$-closed linear span of the expressions

$$F_\sigma(x_1, \ldots, x_m) = E_{\ldots, k} (\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)),$$

for all $m \geq 1$, $x_1, \ldots, x_m \in \mathcal{B}$, $\sigma \in P_{1,2}(m)$ having $k$ singletons and $\phi$ which encodes $\sigma$.

Lemma 3.19. Let $y(j_1, \ldots, j_k) \in L^p(D)$ be such that sup $\|y(j_1, \ldots, j_k)\|_p < \infty$ and $h_1, \ldots, h_k \in H$. Then

$$\sup_n \|n^{-k/2} \sum_{(l_1, \ldots, l_k) = \emptyset} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k)\|_p < \infty.$$

Proof. It suffices to consider

$$\| \sum_{l_1, \ldots, l_k \in C_k} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k)\|$$

with $C_1 \cup \cdots C_k = \{1, \ldots, n\}$. Using the martingale decomposition from Lemma 3.10 we deduce

$$\| \sum_{l_1, \ldots, l_k \in C_k} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k)\|_p \leq c(p) \sqrt{n} \sup_{l \in C_k} \| \sum_{l_1, \ldots, l_{k-1}} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k)\|_p.$$
Proposition 3.20. Let $(\pi_j, B, A, D)$ be a sequence of independent symmetric copies, $H$ a finite dimensional Hilbert space and $1 \in S = S^* \subset A$, and assume that $D_\delta(S)$ is finitely generated as a right $B$-module. Then $L^2_\delta(M)$ is finitely generated as a right $B$-module. In particular, when $D_\delta(S)$ is finitely generated over $B$ for every $s$, the maps $T_t$ are compact over $B$, for every $t > 0$.

Proof. Let $N$ be the dimension of $D_\delta$ as a right $B$-module, and let $\{\xi_1, \ldots, \xi_N\}$ be a basis of $D_\delta$ over $B$. Then, for every $\sigma \in P_{1,2}(m)$ having $s$ singletons, and every $x_1, \ldots, x_m \in BSB$, we can find coefficients $b_k(\sigma, x_1, \ldots, x_m) \in B$ such that

$$F_\sigma(x_1, \ldots, x_m) = \sum_{k=1}^N \xi_k b_k(\sigma, x_1, \ldots, x_m).$$

For every $(l_1, \ldots, l_s) = \emptyset$ we have

$$\alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)) = \sum_{k=1}^N \alpha_{l_1, \ldots, l_s}(\xi_k) b_k(\sigma, x_1, \ldots, x_m).$$

Fix a finite basis $B$ of $H$. Then, for every $\sigma$ having $s$ singletons, every $x_1, \ldots, x_m \in BSB$ and every $h_1, \ldots, h_m \in B$ we have, due to the reduction formula

$$x_\sigma(x_1, h_1, \ldots, x_m, h_m) =$$

$$= \left(n^{-s/2} \sum_{(l_1, \ldots, l_s) = \emptyset} s_{l_1}(h_{i_1}) \cdots s_{l_s}(h_{i_s}) \otimes \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)) \right) =$$

$$= \sum_{k=1}^N \left(n^{-s/2} \sum_{(l_1, \ldots, l_s) = \emptyset} s_{l_1}(h_{i_1}) \cdots s_{l_s}(h_{i_s}) \otimes \alpha_{l_1, \ldots, l_s}(\xi_k) \right) b_k(\sigma, x_1, \ldots, x_m).$$

Thus $L^2_\delta(M)$ is spanned over $B$ by at most $N|B|^s = N(dim(H))^s$ elements, namely

$$\left(n^{-s/2} \sum_{(l_1, \ldots, l_s) = \emptyset} s_{l_1}(h_{i_1}) \cdots s_{l_s}(h_{i_s}) \otimes \alpha_{l_1, \ldots, l_s}(\xi_k) \right),$$

with $h_i \in B$ and $1 \leq k \leq N$. These elements belong to $L^2_\delta(M)$ by the previous lemma, and this finishes the proof.

Remark 3.21. Since the dimension of $D_\delta(S)$ over $B$ is finite, the basis elements $\xi_s \in D_\delta \subset L^2(D)$ could be chosen in fact to be bounded, i.e. $\xi_s \in D$, due to \textcolor{blue}{PaschkeI, PaschkeII, \cite{[11, 52]}}. This implies that $L^2_\delta(M)$ admits a basis over $B$ consisting of elements in $M$.

Corollary 3.22. Assume moreover that the dimension $N_s$ of $D_\delta(S)$ over $B$ has sub-exponential growth, i.e. there exist constants $d, C > 0$ such that $N_s \leq Cd^s$ for all $s$. Then the dimension of $L^2_\delta(M)$ over $B$ is less then $C(dim(H)d)^s$ for all $s$, i.e. the dimension of $L^2_\delta(M)$ over $B$ also has sub-exponential growth.

The following argument is essentially due to Sniady (see \textcolor{blue}{SniadyII, [18]}) and Krolak (see \textcolor{blue}{Kro2, [20]})
Proposition 3.23. Let $M = \Gamma_q(B, S \otimes H)$. There exists $d = d(q)$ such that for $\dim(H) \geq d$ we have $\mathcal{Z}(M) \subset \mathcal{Z}(B)$. In particular, $M$ is a factor whenever $B$ is.

Proof. Let $\{e_i\}_{1 \leq i \leq k}$ be an orthonormal set in $H$. We consider the operator $T : L^2(M) \to L^2(M)$ given by

$$T = \sum_{i=1}^{k} (L_{s(1,e_i)} - R_{s(1,e_i)})^2.$$

Here $L_x$ and $R_x$, where $x \in M$ are the canonical left and right multiplication operators, respectively, on $L^2(M)$. We see that

$$T - 2\text{id} = \sum_{i=1}^{k} (L_{s(1,e_i)} - R_{s(1,e_i)})^2 - 1 - 2 \sum_{i=1}^{k} L_{s(1,e_i)}R_{s(1,e_i)}.$$

Since $s(1,e_i)^2 - 1$ is a mean zero element, we deduce from $\|z\|_\infty \leq c_q \sqrt{k}$.

Let us denote by $V = \sum_{i=1}^{k} L_{s(1,e_i)}R_{s(1,e_i)}$ and $\iota : L^2(M) \to \mathcal{F}_q(\ell^2 \otimes H \otimes L^2(D))^\omega$ the natural embedding given by the definition. Then we see that

$$\iota(V \xi) = (V_n(\xi)_n, \xi \in L^2(M),$$

where

$$V_n = \frac{1}{n} \sum_{1 \leq i \leq n, 1 \leq j \leq \beta} L_{s(1,e_i \otimes e_j)}R_{s(1,e_i \otimes e_j)}.$$

Now we can easily modify the argument from 20 to show that

1. $\|k_{j,j'}^+(e_i \otimes e_j)L_{s(1,e_i \otimes e_j)}\| \leq c_q \sqrt{kn^2};$
2. $\|k_{j,j'}^+(e_i \otimes e_j)L_{s(1,e_i \otimes e_j)}\| \leq c_q \sqrt{kn^2};$
3. $\|k_{j,j'}^+(e_i \otimes e_j)L_{s(1,e_i \otimes e_j)}\| \leq c_q \sqrt{kn^2};$
4. $\|k_{j,j'}^+(e_i \otimes e_j)L_{s(1,e_i \otimes e_j)}\| \leq q + c_q \sqrt{kn}.$

Here $l^+, l^-, r^+, r^-$ are the left and right creation operators on the $q$-Fock space coming from the decomposition $L_{s(h)} = l^+(h) + l^-(h), R_{s(h)} = r^+(h) + r^-(h)$. The main estimate is derived from

$$l^-(h)r^+(k)(\xi) = q[k]_\xi + l^-(h)(\xi) \otimes k.$$

The second part can then be estimated via (2). This yields

$$\|(T - 2\text{id})(\text{id} - E_B)(\xi)\| \leq 2qk\|\text{id} - E_B(\xi)\| + 2c_q \sqrt{k}\|\text{id} - E_B(\xi)\|.$$

Now take $z \in \mathcal{Z}(M)$ with $E_B(z) = 0$. Thus $T(z) = 0$ and also

$$0 = \|T(z)\| = \|2zk - (T(z) - 2zk)\| \geq 2k\|z\| - 2qk\|z\| - C_q \sqrt{k}\|z\| = (2k(1 - q) - C_q \sqrt{k})\|z\|.$$

Thus for $2k(1 - q) - C_q \sqrt{k} > 0$, i.e. $k > \frac{C_q}{2(1 - q)}$, we have that $z = 0$. This implies $z = E_B(z)$, for all $z \in \mathcal{Z}(M)$, hence $\mathcal{Z}(M) \subset B$ and also $\mathcal{Z}(M) \subset \mathcal{Z}(B)$.

Finally, let us mention that there is an $H$-less version of the generalized $q$- gaussians, which can be described as follows: let $(\pi_j, B, A, D)$ a sequence of symmetric independent copies. For $1 \in S = S^* \subset A$, define the von Neumann algebra $\Gamma_q(B, S) \subset (\Gamma_q(\ell^2 \otimes D)^\omega$ as being generated by the elements $s_q(x) = (n^{-\frac{1}{2}} \sum_{j=1}^{n} s_q(e_j) \otimes \pi_j(x)), x \in BS$. This is equivalent to taking $H$ to be 1-dimensional in the Def. 3.4 above, hence the $H$-less $q$- gaussians are a particular case of Def. 3.4. Surprisingly, the $H$ generalized $q$- gaussians can also be obtained as a particular
case of this construction. Indeed, let \( H \) be a (real) Hilbert space and \((\pi_j, B, A, D)\) a sequence of symmetric independent copies. Let \((X, \mu)\) be a standard probability measure space and define a new sequence of symmetric independent copies \((\tilde{\pi}_j, \tilde{B}, \tilde{A}, \tilde{D})\) by taking \( \tilde{B} = B, \tilde{A} = A \otimes L^\infty(X), \tilde{D} = D \otimes (\bigotimes_1^\infty L^\infty(X)) \) and \( \tilde{\pi}_j : \tilde{A} \to \tilde{D} \) by

\[
\tilde{\pi}_j(a \otimes f) = \pi_j(a) \otimes (1 \otimes 1 \otimes \cdots \otimes \underbrace{f}_{\text{j-th position}} \otimes \cdots \otimes 1 \otimes \cdots), a \in A, f \in L^\infty(X). \]

Using Rademacher variables, we see that there exists a dense subspace \( H_0 \subset H \) and an isometric embedding \( \iota : H_0 \to L^\infty(X) \subset L^2(X). \) Take \( \tilde{S} = S \otimes \iota(H_0) = \{a \otimes \iota(h) : a \in S, h \in H_0\} \subset \tilde{A}. \) The reader can check that

\[
\Gamma_q(B, S \otimes H) = \Gamma_q(\tilde{B}, \tilde{S}).
\]
4. Examples

We will discuss several type of examples of generalized $q$-gaussian von Neumann algebras.

4.1. Tensor products. Let $B$ and $C$ be finite von Neumann algebras. Define $A = B \hat{\otimes} C$ and $D = B \otimes C^N = B \otimes (\bigotimes N C)$. Define $\pi_j : A \to D$ by the formula

$$\pi_j(b \otimes a) = b \otimes 1 \otimes 1 \otimes \cdots \otimes \underbrace{a}_{\text{j-th position}} \otimes \cdots \otimes 1 \otimes \cdots .$$

Then it’s easy to check that $(\pi_j, B, A, D)$ is a sequence of symmetric independent copies. It’s likewise easy to see that

$$\Gamma_q(B, A \otimes H) = B \otimes \Gamma_q(L^2_{sa}(C) \otimes H).$$

For any finite subset $S \subset L^2_{sa}(C) \otimes H$, the space $D_k(S)$ has finite dimension over $B$.

4.2. Free products with amalgamation. Let $B \subset A$ be an inclusion of finite tracial von Neumann algebras. Take $D = *_BA_j$ the amalgamated free product of a countable number of copies $A_j$, $j \in \mathbb{N}$ of $A$. Define $\pi_j : A \to D$ by the formula

$$\pi_j(a) = 1 \ast 1 \ast \cdots \ast \underbrace{a}_{\text{j-th position}} \ast \cdots \ast 1 \ast \cdots .$$

Then $(\pi_j, B, A, D)$ represents a sequence of independent symmetric copies. To see why this is true it suffices to consider elements $a_i$ such that $E_B(a_i) = 0$. Then we have to calculate

$$\tau_\sigma(a_1, \ldots, a_m) = \tau(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m))$$

such that $(j_1, \ldots, j_m) = \sigma$. If $\sigma$ has no crossings, we can inductively replace neighboring pairs by $E_B(\pi_{j_1}(a_i) \pi_{j_2}(a_{i+1})) = E_B(a_i a_{i+1})$ and finally find an element in $B$. For a non-crossing pair partition we can also join all the pairs, but then we find an expression of the form

$$\tau(b_1 \pi_{j_1}(a_{i_1}) b_2 \pi_{j_2}(a_{i_2}) \cdots \pi_{j_k}(a_{i_k})) = 0 .$$

Thus in the moment formula we only have to expand over non-crossing pair partitions. Let $1 \in S = S^* \subset A$. In this case we find a $B$-valued semicircular system with respect to the matrix-valued completely positive map $\Phi_{s,t}(b) = E_B(a_s^* b a_t)$. The associated $q$-gaussian von Neumann algebras $M = \Gamma_q(B, S \otimes H)$ are not easy to concretely describe. We also have no real control over the dimension of $D_k(S)$ unless we add extra assumptions on the family of functions $\Phi_{s,t}$.

4.3. Group actions.

4.3.1. Second quantization. Let $G \ltimes_\alpha C$ be a trace preserving action of the discrete group $G$ on the finite von Neumann algebra $C$. Also let $\nu : G \to \mathcal{O}(H_{\mathbb{R}})$ be an orthogonal representation of $G$ on a real Hilbert space $H_{\mathbb{R}}$. Let $(\Omega, \mu)$ be the gaussian construction associated to $\nu$ (see e.g. [PedersenSinclair]). We also denote the corresponding action $G \ltimes L^\infty(\Omega)$ by $\nu$. Then define $B = C \ltimes_\alpha G$, $A = (C \hat{\otimes} L^\infty(\Omega)) \ltimes_\rho G$, $D = (C \otimes L^\infty(\Omega^N)) \ltimes_\rho G$ where the action $\rho$ is given by $\rho_\nu(d \otimes f) = \alpha_g(d) \otimes \nu_\nu(f)$. Define the $^*$-homomorphisms $\pi_j : A \to D$ by

$$\pi_j((d \otimes f) u_g) = (d \otimes 1 \otimes 1 \otimes \cdots \otimes \underbrace{f}_{\text{j-th position}} \otimes \cdots \otimes 1 \otimes \cdots) u_g .$$

Then it is easy to see that the fixpoint algebra is $C \ltimes_\alpha G$. Again the moments only depend on the inner product. Moreover, the gaussian functor yields a map $Br : H \to L^2(\Omega)$. Then we find

$$M = \Gamma_q(C \ltimes G, Br(H)) = (C \otimes \Gamma_q(H)) \ltimes G .$$

The spaces $D_k(S)$ are finite dimensional modules over $B = C \ltimes G$ if $L^2_k(H) \ltimes G$ has a finite basis over $G$. For $k = 1$ this means that $H$ is finite dimensional. In a forthcoming paper we will
also analyze the case of profinite actions and / or representations, i.e. when $H$ can be written as $H = \bigcup_i H_i$ such that every $H_i$ is a finite dimensional $G$-invariant Hilbert subspace. However, discrete subgroups of $\mathcal{O}_n = \mathcal{O}(\mathbb{R}^n)$ provide a large class of non-trivial, non-amenable examples. The examples in [28] are subalgebras of $M$.

4.3.2. Symmetric group action. Throughout this subsection $\Sigma$ will denote the group of finite permutations on $\mathbb{N}$. Let us consider a countable discrete group $G$ on which $\Sigma$ acts by automorphisms. Examples for such a symmetric action are given by the natural action of $\Sigma$ on the free group with countably many generators, or by the natural action of $\Sigma$ on the direct product groups $\prod_{n\in\mathbb{N}} G$. More generally, let $R \subset \mathbb{F}_\infty$ be a set of generators which is invariant under the action of $\Sigma$, and assume that $(R) \subset \mathbb{F}_\infty$ is a normal subgroup. Then $G = \mathbb{F}_\infty/(R)$ is a group on which $\Sigma$ acts. A perfect example is given by an amalgamated free product $*_H G_j$ where $G_j = G$. To make things more concrete, we may consider the discrete Heisenberg group

$$H = \langle \mathbb{Z}, \mathbb{Z}^\infty, \mathbb{Z} \rangle$$

with generators $\{g_k\}_{k \geq 1}$ such that $Z = \langle g_0 \rangle$, $Z^\infty = \langle g_k, k \geq 1 \rangle$ and the following relations hold

$$g_k^{-1} g_j g_k = g_0 g_j, \quad k \neq j.$$  

Then $\Sigma$ acts on $H$ by permuting the generators $g_k$ for $k \geq 1$, and leaving $g_0$ fixed. Now we assume that such a $G$, with action $\Sigma \rtimes_\beta G$, acts trace-preservingly on a finite von Neumann algebra $A$ and $B$ is the fixed points algebra of this action $\alpha$. Let $g \in G$ be an arbitrary element and $g_j = \beta_{(1j)}(g)$. We can then construct a sequence of symmetric copies $(\pi_j, B, A, D)$ by defining $\pi_j : A \to A_j$, via $\pi_j(x) = \alpha_{g_j}(x)$. Working in the crossed product $(A \rtimes_\alpha G) \rtimes_\beta \Sigma$ it is easy to see that the $\pi_j$’s are symmetric copies, and that $B$ is the fixed points algebra for these symmetric copies. In fact we may and will always assume that $G$ is generated by the $g_j$’s and then $\pi_j(x) = x$ for all $j$ is exactly the fixed points algebra of the action. In general $\pi_j(A) = A$ and hence we find an example of symmetric, but not necessarily independent copies. In general independent copies are obtained from considering a suitable subalgebra $B \subset A_1 \subset A$. More generally for a subset $S \subset A$ we may however consider the algebras

$$A_j(S) = \{\pi_j(x) | x \in S, j \in A\}.$$  

This is particularly interesting for a single selfadjoint $x$. Then independence depends on the mixing properties of the sequence $\pi_j(x)$, and has to be analyzed on a case by case basis. Similarly, the dimensions over $B$ of the spaces $D_k(S)$ depend on the maps

$$\Phi_{x,y}(b) = E_B(\alpha_g(x) b \alpha_g(y)) = E_B(\alpha_g(xy)) = \lim_{j \to \infty} \alpha_{g_j}(xy)$$

for $x, y \in S$.

A more specific example can be constructed starting from a trace preserving action $\alpha$ of $\mathbb{Z}$ on a finite von Neumann algebra $N$. Take $D = N \rtimes_\beta \mathcal{H}$ where the action $\beta$ is obtained by lifting the action of $\mathbb{Z}$ via the group homomorphism $\pi : \mathcal{H} \to \mathbb{Z}$ given by $\pi(g_0) = 0$ and $\pi(g_j) = 1$ for $j \geq 1$. In other words,

$$\beta_{g_j}(x) = \alpha_{\pi(g_j)}(x), g \in \mathcal{H}, x \in N.$$  

Let $\mathcal{H}_1$ be the group generated by $g_0$ and $g_1$ and take $B = N \rtimes \mathbb{Z} = N \tilde{\otimes} L(\mathbb{Z})$ and $A = N \rtimes \mathcal{H}_1$. Define $\pi_j : A \to D$ by

$$\pi_j(x u_{g_1}) = \alpha_{\pi(g_j)}(x) u_{g_j}, \pi_j(x u_{g_0}) = x u_{g_0}, x \in N, j, k \in \mathbb{N}.$$  

Then $(\pi_j, B, A, D)$ is a sequence of symmetric independent copies. In full generality the dimensions of the spaces $D_k(S)$ or $L_k^2(M)$, where $M = \Gamma_q(B, A \otimes H)$, cannot be controlled. If we restrict ourselves to a small set of generators, e.g. $S = \{1, g_1, g_1^{-1}\}$, then we get a more well-behaved example. In calculating the dimensions over $B$ of the spaces $D_k(S)$ we use the crossed product structure and find expressions of the form

$$\pi_{j_1}(u_{g_1}) \cdots \pi_{j_k}(u_{g_k}) u_{g_0}^{\ell(\sigma)} \alpha_{n(\sigma)}(x)$$
after reduction. Thus \( \dim_B(D_k(S)) \leq (2\dim(H))^k \). For more general group actions and \( S \subset L(G) \), we find coefficients in \( B = L((G, G)) \otimes N \) and finite dimension over \( B \) as long as we have finite generating sets. Note however, that \( L((G, G)) \) is in general not invariant under the action of \( \Sigma \), and hence a more detailed case by case analysis is required. Again a particularly nice class of examples comes from one step nilpotent groups with commutators in the center, such as the Heisenberg groups.

4.4. Colored Brownian motion.

4.4.1. Top up \( q \)-gaussians. Let \( H \) be a Hilbert space and \( q_0 \in [-1, 1] \). Symmetric independent copies can be obtained from second quantization, or simply by defining \( \pi(s_{q_0}(h)) = s_{q_0}(e_j \otimes h) \). This provides symmetric copies of \( A = \Gamma_{q_0}(H) \) into \( D = \Gamma_q(\ell_2(H)) \). By looking at Wick words it is easy that the fixpoint algebra is \( C \). Moreover, independence follows from the moment formula for \( q_0 \)-gaussian random variables. Let \( S \) be a finite, selfadjoint subset given by elements of the form \( x_i = s_{q_0}(h_1(i)) \cdots s_{q_0}(h_{l(i)}(i)) \). Then we see that

\[
\tau(s_q(k_1, x_1) \cdots s_q(k_m, x_m)) = \sum_{\sigma \in \Sigma(m)} q^{cr(\sigma)} f(k_1, \ldots, k_m)_\sigma \tau(\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)),
\]

where \((j_1, \ldots, j_m) = \sigma\). Now we may use the formula for \( q_0 \) gaussian and find for \( L = \sum_l l_i \) that

\[
\tau(\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)) = \sum_{\sigma' \in \Sigma(L) : \sigma' \prec \phi(\sigma)} q^{cr(\sigma')} q_0^{cr(\sigma')} f(h_1(1), \ldots, h_l(1)(1), \ldots, h_1(m), \ldots, h_l(m)(m))_{\sigma'}
\]

Here \( \phi(\sigma) \) is the block partition which gives the same color to the union of two blocks connected via pairs in \( \sigma' \). This means

\[
\tau(s_q(k_1, x_1) \cdots s_q(k_m, x_m)) = \sum_{\sigma \in \Sigma(m), \sigma' \prec \phi(\sigma)} q^{cr(\sigma')} q_0^{cr(\sigma')} f(k_1, \ldots, k_m)_{\sigma} f(h_1, \ldots, h_L)_{\sigma'}
\]

is also obtained by summing over all pair partitions of the total set, using the tensor product \( k \otimes h \). Note that then \( \Gamma_q(C, \Gamma_{q_0}(H) \otimes K) \) contains both \( \Gamma_q(K) \) and \( \Gamma_{q_0}(H) \) in case \( S \) contains \( s_{q_0}(H) \subset S \). In calculating the dimension of the space \( D_k(S) \) we have to understand the weak limit

\[
\lim_{j \to \infty} \tau(W_{k_1}(h_1) \pi_j(W_{k_2}(h_2)) W_{k_3}(h_3) \pi_j(W_{k_4}(h_4)) = \delta_{k_2, k_4} q^{k_2 k_4} \tau(W_{k_1}(h_1) W_{k_3}(h_3)) \tau(W_{k_2}(h_2) W_{k_4}(h_4))
\]

Using a decomposition into minimal links, we deduce that the reduction procedure yields a term

\[
c(\sigma, x_1, \ldots, x_r) \pi_{j_1}(x_{i_1}) \cdots \pi_{j_r}(x_{i_r}),
\]

where \( c(\sigma, x_1, \ldots, x_r) \) is a scalar. This means for a finite set \( S \) of generators, the dimension of \( D_k(S) \) over \( B = C \) is less than \( \langle |S| \dim(K) \rangle^k \).

4.4.2. Actions of \( \Sigma \) by conjugation. Let us consider the finite permutations group \( \Sigma_Z \) acting on \( Z \) instead of \( N \setminus \{0\} \). For every subset \( F \subset Z \) we can identify \( \Sigma_F \), the permutations group on \( F \), with a subgroup of \( \Sigma_Z \) by viewing the elements of \( \Sigma_F \) as acting non-trivially only on \( F \) and acting as the identity on \( Z \setminus F \). For convenience, we use interval notation for the subsets of \( Z \). In particular we have \( \Sigma = \Sigma_{[1, \infty)} \subset \Sigma_Z \) in this way. Let \( \Sigma \) act on \( \Sigma_Z \) by conjugation. This gives rise to an action \( \alpha \) of \( \Sigma \) on the von Neumann algebra \( L(\Sigma_Z) \) (which is in fact isomorphic to the hyperfinite factor). We denote the canonical unitaries generating \( L(\Sigma_Z) \) by \( u_\sigma, \sigma \in \Sigma_Z \). The fixed points algebra of this action is \( B = L(\Sigma_{(-\infty,0)} \cup \{0\}) \). Take \( A = L(\Sigma_{(-\infty,1]} = B \cup \{u_{(01)}\}'' \), \( D = L(\Sigma_Z) \) and define \( \pi_j : A \to D \) by \( \pi_j(a) = \alpha_{(j1)}(a) \) for \( a \in A \) and \( j \geq 2 \), where \( (j1) \) is the transposition interchanging \( j \) and \( 1 \), and \( \pi_1 = id \). Then \( (\pi_j, B, A, D) \) is a sequence of symmetric
independent copies. Indeed, we recall that $A$ is generated by transpositions $(k1), k \leq 0$ and that for $j \geq 2$ we have

$$ (j1)(k1)(j1) = (kj). $$

This means $A_j = B \cup \{u_{(0j)}\}''$ and $A_{i-1} = L(\Sigma_{(\infty,i)}).$ In particular, we have a coset representation $\sigma = \sigma'(j1)$ with $\sigma' \in \Sigma_{(\infty,i)}.$ The algebras $A_j$ are generated by $\Sigma_{i},\Sigma_{(\infty,i)}$ and one generator $(j1)$ for $j \in I$. This easily implies independence. We take $S = \{1,u_{(01)}\} \subset A$ and define $M = \Gamma_{q}(B,S \otimes H).$ Fix $\sigma \in \Pi_{2}(m)$ having $k$ singletons and $p$ pairs, take $\phi : \{1,\ldots,m\} \to \{1,\ldots,k+p\}$ which encodes $\sigma$. This means $\phi(j1) = t,$ where $\{j1\}, 1 \leq t \leq k$ are the singletons of $\sigma,$ and $\phi(j1') = \phi(j1'') = k + t,$ where $\{j_1', j_1''\}, 1 \leq t \leq p$ are the pairs of $\sigma$. Then $D_k(S)$ is the closed span of elements of the form

$$ E_{1,\ldots,k}(u_{(\phi(1)0)}u_{\gamma_1}u_{(\phi(2)0)}u_{\gamma_2} \cdots u_{(\phi(m)0)}u_{\gamma_{m+1}}) $$

$$ = E_{1,\ldots,k}(u_{(\phi(1)0)}\text{ad}(u_{(\phi(2)0)}) \cdots \text{ad}(u_{(\phi(\gamma_{m})0)}(u_{(\phi(m)0)})u_{\gamma_1} \cdots u_{\gamma_{m+1}}) $$

$$ = E_{1,\ldots,k}(u_{(\phi(1)0)}u_{(\phi(2)\gamma_12(0))} \cdots u_{(\phi(m)(\gamma_1 \cdots \gamma_{m})0)}u_{\gamma_1} \cdots u_{\gamma_{m+1}}) $$

where $\gamma_1, \ldots, \gamma_{m+1} \in \Sigma_{(\infty,i)}$ are arbitrary. Here $s_1 = \gamma_1(0), s_2 = \gamma_1 \gamma_2(0), \ldots s_m = \gamma_1 \gamma_2 \cdots \gamma_m(0)$ in $(-\infty,0]$ depend only the $\gamma_i$’s. In full generality the modules $D_k(S)$ do not have finite dimensions over $B$. If we however replace $B$ by $B_d = L(\Sigma_{[-d,0]} \cong L(\Sigma_{d+1}), A$ by $A_d = L(\Sigma_{[-d,1]} = B_d \cup \{u_{(01)}\}''$ and $D$ by $D_d = L(\Sigma_{[-d,\infty]})$ for a fixed $d \in \mathbb{N} \setminus \{0\},$ then we obtain a new sequence of symmetric independent copies $(\pi_j, B_{d,j}, A_{d,j}, D_{d,j})$ and in this case we have at most $(d+1)^k$ different choices for the $s_j$’s. After repeated conjugation with the units on the pair positions, the above expression becomes

$$ u(s_{j_1}') \cdots u(s_{j_k}')E_{1,\ldots,k}(u_{(s_{j_{k+1}}')k+1}) \cdots u(s_{j_{k+p}}'k+p), $$

for some new indices $s_i' \in (-\infty,0] \cap \mathbb{Z}$ which in general depend on the $\gamma_i$’s and $\sigma$. Since for an inclusions of groups $H \subset G$ and $g \in G$ we have $E_{L(H)}(ug) = \delta_{g \in H}ug,$ and the product $(s_{j_{k+1}}'k+1) \cdots (s_{j_{k+p}}'k+p)$ belongs to $\Sigma_{[-d,k]}$ only if it’s equal to 1, we see that a spanning set of $D_k(S)$ over $B_d$ is given by the elements

$$ u(s_{j_1}')u(s_{j_2}') \cdots u(s_{j_k}'), $$

for all choices of $-d \leq s_i' \leq 0, 1 \leq i \leq k,$ which in particular implies that the dimension of $D_k(S)$ over $B_d$ is at most $(d+1)^k$. Note that $B_d$ and $A_d$ are finite dimensional von Neumann algebras. Thus, for the von Neumann algebras $M(d) = \Gamma_{q}(B_d,S \otimes H),$ the spaces $D_k(S)$ have sub-exponential growth of their dimensions over $B_d$. This remains true for any finite subset $1 \in S = S^* \subset A_d.$

4.4.3. Actions by permutations of the generators. Let $D = \bigotimes_{j \geq 1}L(\mathbb{Z}),$ where each copy of $L(\mathbb{Z})$ is generated by a Haar unitary $u_j, j \geq 1.$ The symmetric group $\Sigma = \mathbb{S}(\infty)$ acts naturally on $D$ by permuting the generators. Equivalently, one can define $B = \mathbb{C}, A = \{u_1\}''$ and the copies $\pi_j : A \to D$ by $\pi_j(u_1) = u_j.$ A second example comes from taking $D = L(\mathbb{F}_\infty) = \ast_{\mathbb{N}}L(\mathbb{Z}), B = \mathbb{C}, A = \{u_1\}''$ and $\pi_j(u_1) = u_j,$ where the $j$-th copy of $L(\mathbb{Z})$ is generated by the unitary $u_j.$ More generally, for every finite von Neumann algebra $B,$ one can define $A = B \otimes L(\mathbb{Z}), D = \bigotimes_{\mathbb{N}}(B \otimes L(\mathbb{Z}))$ and $\pi_j : A \to D$ by $\pi_j(b \otimes u_j) = b \otimes u_j.$ In the second case one takes $D = \ast_B(B \otimes L(\mathbb{Z})) = B \otimes L(\mathbb{Z})$ and the symmetric copies as above. For every finite subset $1 \in S = S^* \subset A$ (e.g. $S = \{1,u_1,u_1^2\}$), the dimensions of $D_k(S)$ over $B = \mathbb{C}$ are sub-exponential in both cases. In the case of the free product with amalgamation and $S = \{1,u_1,u_1^2\}$, it’s easy to see that a spanning set for $D_k(S)$ over $B$ is given by the elements $u_1^{e_{ij}}u_1^{e_{ij}} \cdots u_1^{e_k},$ where $k \leq s, i_j \in \{1,\ldots,s\}$ and $e_j \in \{1,*\}$ for all $j \leq k.$ Hence $\dim_B(D_k(S)) \leq 2^{2s}.$
4.5. Operator-valued gaussian. This example is motivated by Shlyahktenko’s A-valued semi-circular algebras and derived from the tensor product construction. Let \( x_k \in N \) be selfadjoint operators and \( X = \sum_k g_kx_k \). We consider \( A_1 = L^\infty(\mathbb{R}) \) and the independent symmetric copies over \( N \) given by

\[
\pi_j(f) = f(\sum_k g_{k,j}x_k),
\]

where \( g_{k,j} \) are i.i.d. gaussians (we could also work with \( q \)-gaussians). The copies are independent over \( N \), and hence independent over every subalgebra of \( N \). We prefer a more internal approach. Let \( D \) be the von Neumann algebra generated by the \( \pi_j(f) \)'s and \( B \) be the tail algebra

\[
B = \bigcap_m (\{ \pi_j(f) : f \in L^\infty(\mathbb{R}), j \geq m \})'.
\]

Koestler showed that the copies \( \pi_j \) are independent symmetric in the sense of our definition 3.2. Note that \( N \) is invariant under the shift from tensor product construction and hence \( B \subset N \). Thus \( M = \Gamma_q(B, S \otimes H) \) is a legitimate example where \( S = \sum_k g_kx_k \) is obtained by approximating \( X \) with bounded functions. Since \( X \in \bigcap_{p<\infty} L_p \) one can actually directly work with one generator \( x \). The dimension of the \( L^2_k(M) \) over \( B \) is in general hard to determine. Indeed, in the reduction we have to work with

\[
\Phi(y) = \sum_k x_k^*yx_k = \lim_j \pi_j(x)y\pi_j(x)^*.
\]

As an example let us consider \( \sigma = \{\{1,3\},\{2,4\}\} \). Then we find the constant coefficient

\[
\sum_{k_1,k_2} x_{k_1}x_{k_2}x_{k_1}x_{k_2} = \lim_{n \to \infty} n^{-2} \sum_{j_1,j_2} \pi_{j_1}(X)\pi_{j_2}(X)\pi_{j_1}(X)\pi_{j_2}(X).
\]

Similarly, we obtain a bracket operator

\[
\Phi_{\{1,5\},\{2,4\},\{3\}}(y) = \sum_{k_1,k_2} x_{k_1}x_{k_2}y_{x_{k_1}x_{k_2}} = \lim_{n \to \infty} n^{-2} \sum_{j_1,j_2} \pi_{j_1}(X)\pi_{j_2}(X)y\pi_{j_1}(X)\pi_{j_2}(X).
\]

This leads to a new reduced Wick word of length 1

\[
n^{-1/2} \sum_{1 \leq j \leq n} s_j \otimes \Phi_\sigma(\pi_j(X)) = \sum_k n^{-1/2} \sum_{1 \leq j \leq n} s_j \otimes g_{jk}\Phi_\sigma(x_k).
\]

Without further assumption, all singleton pair partitions with one singleton might produce new Wick words with some coefficients \( \Phi_\sigma(x_k) \). For results on strong solidity (over \( \mathbb{C} \)) we will have to assume that the space of possible coefficients is finite dimensional. Even when considering the dimension over \( N \) these coefficients play a non-trivial role. For Wick words of higher order we have even larger spaces of ‘reduced coefficients’.

The following example goes back to a conversation of Avsec and Speicher. Let \( N = M_m \) and consider

\[
X = \sum_{r,s} g_{rs}\left(\frac{e_{rs} + e_{sr}}{2}\right).
\]

Let \( y^t \) be the transposed of a matrix \( y \). Then

\[
\Phi(y) = y^t + \text{tr}(y)1
\]

turns out to be a completely positive map. It is easy to calculate

\[
\Phi_{\{1,5\},\{2,4\},\{3\}}(y) = (n+2)y^t + y
\]

and

\[
\Phi_{\{1,3\},\{2,5\},\{4\}}(y) = (n+1)y^t + \text{tr}(y)1 + y.
\]
We note that $e_{rs} + e_{sr}$ is an eigenvalue of the first map, and thus the reduction method does not contribute something new. However, the second map behaves differently for $r \neq s$ and $r = s$ and gives an additional linearly independent Wick word. At the time of this writing the explicit dimensions over $B = \mathbb{C}$ are not known. Note however, that all the additional coefficients are in $M_n$ and hence the dimension of $L_{2}^{(k)}(\Gamma_q(\mathbb{C}, H))$ is at most $n^2 \dim(H)^k$. This class of examples provides an example of a brownian motions interpolating between the classical brownian motion and the free brownian motion.

Remark 4.1. The examples in 4.1., 4.2., 4.4.1, 4.4.2 for $d = 0$ and 4.4.3 are all factors if $B$ is a factor and $\dim(H) \geq d(q)$. 

5. Weak amenability produces approximately invariant states

Let \((\pi, B, A, D)\) a sequence of symmetric independent copies, \(1 \in S = S^* \subset A\) and assume that \(D_x(S)\) is finitely generated over \(B\) for all \(s \geq 1\). Let \(M = \Gamma_q(B, S \otimes H)\) for a finite dimensional space \(H, A \subset M\) be a von Neumann subalgebra which is amenable relative to \(B\) inside \(M\), and let \(P = N_M(A)^\omega\). Define \(M = (\Gamma_q(\ell^2 \otimes H) \otimes D) \cap (\Gamma_q(\ell^2 \otimes H) \otimes D)^\omega\), where \(\Gamma_q(\ell^2 \otimes H) \otimes D\) is embedded as constant sequences. Let

\[
\mathcal{H} \subset ((L^2(M) \otimes_A L^2(P)) \otimes \mathcal{F}_q(\ell^2 \otimes H))\]

be the \(\| \cdot \|\)-closed span of the sequences

\[
(n^{-m \over 2} \sum_{(j_1, \ldots, j_m) = }^{(j_1, \ldots, j_m) = } (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes_A z) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)),
\]

for all \(m \geq 1, \sigma \in \ell_{1,2}(m)\), \(x_i \in BSB, y \in M, z \in P\) and \(h_1, \ldots, h_m \in H\). Define two \(*\)-representations \(\pi : M \to B(H), \theta : P^{op} \to B(H)\) by

\[
\pi(x_{\sigma'}) (n^{-m \over 2} \sum_{(j_1, \ldots, j_m) = }^{(j_1, \ldots, j_m) = } (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes_A z) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)) =
\]

\[
(n^{-m \over 2} \sum_{(i_k) = \sigma', (j_i) = }^{(i_k) = \sigma', (j_i) = } (\pi_{i_1}(y_1) \cdots \pi_{j_m}(x_m) y \otimes_A z) \otimes s(e_{i_1} \otimes k_1) \cdots s(e_{j_m} \otimes h_m))
\]

and

\[
\theta(w^{op}) (n^{-m \over 2} \sum_{(j_1, \ldots, j_m) = }^{(j_1, \ldots, j_m) = } (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes_A z) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)) =
\]

\[
(n^{-m \over 2} \sum_{(j_1, \ldots, j_m) = }^{(j_1, \ldots, j_m) = } (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes_A z w) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m))
\]

where \(x_{\sigma'} = (n^{-m \over 2} \sum_{(i_1, \ldots, i_m) = \sigma'}^{(i_1, \ldots, i_m) = \sigma'} \pi_{i_1}(y_1) \cdots \pi_{i_m}(y_m) \otimes s(e_{i_1} \otimes k_1) \cdots s(e_{i_m} \otimes k_m)) \in M\) is a Wick word in \(M\) and \(w \in P\). Define \(\mathcal{N} = \pi(M) \lor \theta(P^{op}) \subset B(H)\). Note that \(\pi(M)\) and \(\theta(P^{op})\) commute.

**Theorem 5.1.** There exists a sequence of normal states \(\omega_n \in N_s\) satisfying the following properties

1. \(\omega_n(\pi(x)) \to |x|, x \in M\).
2. \(\omega_n(\pi(u) \theta(\bar{u})) \to 1, a \in U(A)\).
3. \(\|\omega_n \circ Ad_\pi(u) \theta(\bar{u}) - \omega_n\| \to 0, u \in N_M(A)\).

**Proof.** Throughout the proof \(m_n\) will be the completely contractive finite rank multipliers on \(\Gamma_q(\ell^2 \otimes H)\) given by multiplication with a positive finitely supported function \(f_n\) constructed by Avsec in [41] and \(\varphi_n := (m_n \otimes id) : M \to M\) the corresponding cb map on \(M\). Take

\[
\mathcal{K} \subset (L^2(M) \otimes_D L^2(M))^\omega
\]

to be the \(\| \cdot \|\)-closed span of the sequences

\[
(n^{-m \over 2} \sum_{(j_1, \ldots, j_m) = }^{(j_1, \ldots, j_m) = } (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) \otimes_D y)) = (x_n \otimes_D y),
\]

where \(x_i \in BSB\) and \(y \in M\). Note that \(\mathcal{K}\) is naturally an \(M - M\) bimodule with the actions

\[
x_{\sigma'} \cdot (n^{-m \over 2} \sum_{(j_1, \ldots, j_m) = }^{(j_1, \ldots, j_m) = } (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) \otimes_D y)) \cdot z =
\]
Lemma 5.2. To our previous remark, we have $B$-expectations. Since $\lambda$ and $\rho$ are the representations of $M$ and $M^\text{op}$ canonically associated to the left and right actions on $K$, respectively.

Step 1. There exists a normal, unital, completely positive map $\mathcal{E} : \mathcal{N} \to \mathcal{S}_A$ such that $\mathcal{E}(\pi(x)\theta(y^\text{op})) = \lambda(x)\rho(E_A(y)^\text{op})$, $x \in M, y \in P$.

Indeed, define an isometry $V : \mathcal{K} \to \mathcal{H}$ by

\[
(n^{-\frac{m+n'}{2}} \sum_{(j_1, \ldots, j_m, \ldots) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_{1}) \cdots s(e_{j_m} \otimes h_m) ) \otimes D y),
\]

where $x_{\sigma'} = (y_{1}, k_{1}, \ldots, y_{m'}, k_{m'}) \in M$ and $z \in M$. Denote by $\mathcal{S}_A = \lambda(M) \vee \rho(A^\text{op}) \subset B(K)$, where $\lambda$ and $\rho$ are the representations of $M$ and $M^\text{op}$ canonically associated to the left and right actions on $K$, respectively.

Step 2. There exist normal functionals $\mu_n^A : \mathcal{S}_A \to \mathbb{C}$ such that $\mu_n^A(\lambda(x)\rho(a^\text{op})) = \tau(\varphi_n(x)a)$, $x \in M, a \in A$.

We need two lemmas.

Lemma 5.2. $L^2(M) \otimes_B L^2(M)$ embeds as an $M - M$ bimodule into $\mathcal{K}$.

Proof. The map $L^2(M) \otimes_B L^2(M) \ni (n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m, \ldots) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)) \otimes_B y) \mapsto$

\[
(n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m, \ldots) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)) \otimes D y) \in \mathcal{K},
\]

or in other words $(x_n^a \otimes_B y) \mapsto (x_n^a \otimes D y)$ is an $M - M$ bimodular isometry. The bimodularity is obvious, so it remains to check that it preserves inner products, in other words that $\langle (x_n) \otimes_B y, (x_n') \otimes_B y' \rangle = \langle (x_n \otimes D y), (x_n' \otimes D y') \rangle$.

Let’s denote by $E_D : \mathcal{M} \to D$ and by $E_{D \otimes 1} : \Gamma_q(\ell^2 \otimes H) \otimes D \to D \otimes 1$ the canonical conditional expectations. Since $D = D \otimes 1 \subset \mathcal{M} \subset (\Gamma_q(\ell^2 \otimes H) \otimes D)^\sigma$ is embedded as constant sequences, for every $(x_n) \in \mathcal{M}$ we have $E_D((x_n)) = w - \lim_{n \to \infty} E_{D \otimes 1}(x_n)$.

We now claim that for any $(x_n) \in M \subset \mathcal{M}$ we have $E_B((x_n)) = E_D((x_n))$. It suffices to prove this for $(x_n) = W_\sigma \in M$ a reduced Wick word. Let $s$ be the number of singletons in $\sigma$. Let

\[
W_\sigma = (n^{-\frac{s}{2}} \sum_{(l_1, \ldots, l_s) = \emptyset} \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)) \otimes s(e_{l_1} \otimes h_{1}) \cdots s(e_{l_s} \otimes h_{k_s})).
\]

We have two possibilities. If $s = 0$, then $W_\sigma = F_\sigma(x_1, \ldots, x_m) = E_B(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m)) \in B$, hence $E_D(W_\sigma) = W_\sigma = E_B(W_\sigma)$. If $s > 0$, then $E_B(W_\sigma) = 0$. On the other hand, according to our previous remark, we have

\[
E_D(W_\sigma) = w - \lim_{n \to \infty} E_{D \otimes 1}(n^{-\frac{s}{2}} \sum_{(l_1, \ldots, l_s) = \emptyset} \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m)) \otimes s(e_{l_1} \otimes h_{1}) \cdots s(e_{l_s} \otimes h_{k_s}))
\]

\[
= w - \lim_{n \to \infty} n^{-\frac{s}{2}} \sum_{(l_1, \ldots, l_s) = \emptyset} \tau(s(e_{l_1} \otimes h_{1}) \cdots s(e_{l_s} \otimes h_{k_s})) \alpha_{l_1, \ldots, l_s}(F_\sigma(x_1, \ldots, x_m))
\]

We can then define
\[ f := \lim_{n \to \infty} \sum_{i_1, \ldots, i_s} \tau(W(e_{i_1} \otimes h_{k_1} \cdots e_{i_s} \otimes h_{k_s})) \alpha_{i_1, \ldots, i_s}(F_\alpha(x_1, \ldots, x_m)) = 0. \]

This proves our claim. Now, for \((x_n), (x'_n) \in M\) we have
\[
\langle (x_n) \otimes_B y, (x'_n) \otimes_B y' \rangle = \tau_M(E_B((x'_n x_n)yy'^*) = \tau_M(E_D((x'_n x_n)yy'^*) = \\
\lim_n \tau_M(E_{1 \otimes_D}(x'_n x_n)yy'^*) = \lim_n \tau_M(\alpha_{x_n \otimes_D y, x'_n \otimes_D y'}) = \langle (x_n) \otimes_D y, (x'_n \otimes_D y') \rangle,
\]
which finishes the proof of the lemma.

**Lemma 5.3.** There exists an orthonormal basis \(Y_\alpha\) of \(L^2(M)\) over \(B\) such that for every \(n\), \(f_n(Y_\alpha) = 0\) for all but finitely many \(\alpha\)'s, where we denote somewhat abusively \(f_n(Y_\alpha) = f_n(s)\), \(s = \text{the degree of } Y_\alpha\).

**Proof.** Since \(D_s\) is finitely generated over \(B\) for all \(s\), according to Lemma 1.14, for every \(s \geq 0\) we can find a finite orthonormal basis \((Y^n_\alpha)\) of \(L^2(M)\) over \(B\). The union \((Y_\alpha)\) of all the \(Y^n_\alpha\)'s is a basis of \(L^2(M)\) over \(B\). For a fixed \(n\), there exists \(s = s(n)\) such that \(f_n(\xi) = 0\) for all \(\xi \in H \otimes k^n\), for \(k > s(n)\). For any \(Y_\alpha \in \bigoplus_{k > s(n)} L^2(M)\) we have \(f_n(Y_\alpha) = 0\), due to the reduction formula.

On the other hand, the set of those \(Y_\alpha \in \bigoplus_{k = 0} L^2(M)\) is finite, which finishes the proof. ■

Denote by \(\iota\) the \(M\)-bimodular embedding in Lemma 2.2 and define
\[
\mu^\alpha_n(T) = \sum_\alpha f_n(Y_\alpha)(\iota(1 \otimes_B 1), \iota(Y_\alpha * \otimes_B Y_\alpha^*)), T \in \mathcal{S}_A.
\]

Then \(\mu^\alpha_n \in (\mathcal{S}_A)_s\) satisfies all the required properties.

**Step 3.** Set \(\gamma_n = \mu^\alpha_n \circ \iota \in N_s\), and \(\omega_n = |\gamma_n|^{-1}|\gamma_n|\). We will prove that the \(\omega_n\)'s satisfy all the required properties. First note that, by construction,
\[
\gamma_n(\pi(x) \delta(y^{op})) = \tau(\varphi_n(x) E_A(y)), x \in M, y \in P.
\]

Toward proving the required properties of the \(\omega_n\)'s, we will first establish the following two claims:

**Claim 1.** \(\limsup_n \|\mu^\alpha_n\| = 1\);

**Claim 2.** \(\lim_n \|\mu^\alpha_n \circ \text{Ad}(\lambda(u) \rho(\bar{u})) - \mu^\alpha_n\| = 0, u \in \mathcal{N}_M(A)\).

**proof of the first claim.** Fix a von Neumann subalgebra \(Q \in P\) which is amenable over \(B\). Just as in Step 2 above one can construct normal functionals \(\mu^\gamma_n\) on \(\mathcal{S}_Q = \lambda(M) \vee \rho(Q^{op}) \subset B(K)\) satisfying \(\mu^\gamma_n(\lambda(x) \rho(y^{op})) = \tau(\varphi_n(x) y), x \in M, y \in Q\). We will show that \(\limsup \|\mu^\gamma_n\| = 1\), and this will help us establish both claims. Since \(\mu^\gamma_n\) is normal, it suffices to estimate its norm on an ultraweakly dense \(C^*\)-subalgebra of \(\mathcal{S}_Q\). Denote by \(S_Q\) the ultraweakly dense \(C^*\)-subalgebra of \(\mathcal{S}_Q\) generated by \(\lambda(x_\sigma)\), for \(x_\sigma \in M\) the Wick words and \(\rho(Q^{op})\). First we note that there exist cb maps \(\varphi_n : S_Q \rightarrow S_Q\) such that
\[
\varphi_n(\lambda(x_\sigma) \rho(y^{op})) = \lambda(\varphi_n(x_\sigma)) \rho(y^{op}), x_\sigma \in M, y \in Q,
\]
and \(\|\varphi_n\|_{cb} = \|\varphi_n\|_{cb}\). To prove this take \(\hat{\mathcal{K}} \subset L^2((M \otimes \Gamma_q(\ell^2 \otimes H))^\omega)\) to be the \(\|\cdot\|_2\)-closed linear span of the sequences
\[
(n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)) = (x_\sigma^n(y \otimes 1))
\]
for all \(x_i \in BSB, h_i \in H, y \in M\). Now define an unitary operator \(U : \mathcal{K} \rightarrow \hat{\mathcal{K}}\) by
\[
(x_\sigma^n(x \otimes D y) \mapsto (x_\sigma^n(y \otimes 1)).
\]
We can then define
\[
\hat{\varphi}_n(z) = U^*(id \otimes m^n)\omega(UzU^*)U, z \in S_Q.
\]
Then the maps \( \tilde{\varphi}_n \) satisfy all the required properties. The complete boundedness of the \( \tilde{\varphi}_n \) is a delicate matter and it will be addressed in the subsection 5.1 below. On the other hand, since \( Q \) is amenable relative to \( B \), we see that the \( M - Q \) bimodule \( L^2(M) \) is weakly contained in \( L^2(M) \otimes_B L^2(M) \), which in turn is contained in \( K \). This produces a \( * \)-homomorphism \( \Theta : S_Q \to B(L^2(M)) \) such that \( \Theta(\lambda(x)\rho(y^{op})) = \lambda_M(x)\rho_M(y^{op}) \), where \( \lambda_M, \rho_M \) are the natural actions of \( M \) on \( L^2(M) \). But then

\[
\mu_n^Q(z) = \langle \Theta(\tilde{\varphi}_n(z))1, 1 \rangle, \quad z \in S_Q,
\]

and this implies that \( \limsup \|\mu_n^Q\| = 1 \). Then by taking \( Q = A \) we get \( \limsup \|\mu_n^A\| = 1 \), which finishes the proof of the first claim.

**Proof of the second claim.** Fix a unitary \( u \in \mathcal{N}_M(A) \). The algebra \( Q = \langle A, u \rangle \subset P \) is amenable relative to \( B \), so by the proof of Claim 1 \( \limsup \|\mu_n^Q\| = 1 \). Now since \( \mu_n^Q(1) = \tau(\phi_n(1)) \to 1 \) and \( \mu_n^Q(\lambda(u)\rho(u^*)) \to 1 \), we see that \( \|\mu_n^Q \circ \text{Ad}(\lambda(u)\rho(u^*)) - \mu_n^Q\| \to 0 \), hence by restricting to \( S_A \) we get \( \|\mu_n^A \circ \text{Ad}(\lambda(u)\rho(u^*)) - \mu_n^A\| \to 0 \). Using the fact that \( \text{Ad}(\lambda(u)\rho(u^*)) \circ \mathcal{E} = \mathcal{E} \circ \text{Ad}(\pi(u)\theta(u)) \) and the fact that \( \gamma_n = \mu_n^A \circ \mathcal{E} \) we see at once that \( \|\gamma_n \circ \text{Ad}(\pi(u)\theta(u)) - \gamma_n\| \to 0 \). But since \( \gamma_n(1) = \tau(\phi_n(1)) \to 1 \) and \( \limsup \|\gamma_n\| = 1 \) we see that \( \|\gamma_n - \omega_n\| \to 0 \). This further implies \( \|\omega_n \circ \text{Ad}(\pi(u)\theta(u)) - \omega_n\| \to 0 \), which establishes the third required property, and the other two follow easily.

### 5.1. CB-estimates for the multipliers

Here we will prove that some multipliers defined on certain \( C^* \)-algebras or von Neumann algebras are completely bounded. The first case is that of the maps \( \tilde{\varphi}_n \) which were used in the proof of Thm. 5.1. above. In the second case we prove the cb boundedness of some normal multipliers on the von Neumann algebra \( N \), introduced above, which are needed to construct a concrete standard form for \( N \). We recall some notation.

**Notation:** \( M = (\Gamma_q(\ell^2 \otimes H) \otimes D) \vee M \subset \Gamma_q(\ell^2 \otimes H) \otimes D) \), where we regard \( \Gamma_q(\ell^2 \otimes H) \) and \( D \) as constant sequences. Let \( K = L^2(M) \) or \( K = L^2(M) \otimes A L^2(P) \). We introduce the subspace

\[
\mathcal{L} \subset (K \otimes \mathcal{F}_q(\ell^2 \otimes H))^{\omega}
\]

as the \( \| \cdot \| \)-closed linear span of the sequences

\[
\sum_{(j_1, \ldots, j_m) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) = (x^n(y \otimes 1)) \in (K \otimes \mathcal{F}_q(\ell^2 \otimes H))^{\omega},
\]

for \( m \geq 1, \sigma \in P_{12}(m), x_i \in BSB, h_i \in H, y \in M \). Let’s define the extended Wick words \( x_\sigma = x_\sigma(x_1, h_1, \ldots, x_m, h_m, y^{op}) \) by

\[
x_\sigma = (n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y^{op} \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)),
\]

where \( m \geq 1, \sigma \in P_{12}(m), x_i \in BSB, h_i \in H, y \in P \), viewed as operators in \( B(K) \), i.e. acting naturally on sequences in \( \mathcal{L} \). The reader can check that

a) \( \mathcal{L} \) is invariant to the natural action of the extended Wick words;

b) \( \mathcal{L} = \text{span}\{\lambda(x_\sigma)\rho(y^{op})(1 \otimes 1), x_\sigma \in M, y \in M\} \) in the first case or \( \text{span}\{\pi(x_\sigma)(1 \otimes y^{op})(1 \otimes \mathcal{A} 1 \otimes 1), x_\sigma \in M, y \in M, z \in \mathcal{P}\} \) in the second case.

c) \( \mathcal{L} \) is invariant to the natural action by orthogonal transformations of \( H \) given by

\[
O(H) \ni o \to \alpha_o = (id \otimes \Gamma_q(id \otimes o)) \in \text{Aut}(\mathcal{M} \otimes \Gamma_q(\ell^2 \otimes H))^{\omega}.
\]

Let \( C(H) \subset B(\mathcal{L}) \) be the \( C^* \)-algebra generated by the elements

\[
(n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m) = \sigma} \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y^{op} \otimes s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m)) = (x^n(y^{op} \otimes 1)),
\]
Lemma 5.6. Let $m_\alpha$ be the multipliers on $\Gamma_q(H)$ associated to the non-negative finite support functions $f_\alpha : \mathbb{N} \to \mathbb{R}$.

(1) One may assume that for every $k$, $f_\alpha(k) = 1$ for $\alpha$ large enough and that $\lim \sup_\alpha \|m_\alpha\|_{cb} = 1$;
(2) $(id \otimes m_\alpha) : \hat{C}(H) \to (B(K) \otimes_{\min} \Gamma_q(\ell^2 \otimes H))^{\omega}$ are completely bounded, and the restriction of a normal map.

Lemma 5.5. Let $\hat{C}(H), C(H)$ and $m_\alpha$ be defined as above.

(1) Let $\rho : (B(K) \otimes_{\min} \Gamma_q(\ell^2 \otimes H))^{\omega} \to B((K \otimes F_q(\ell^2 \otimes H))^{\omega})$ be the *-homomorphism defined by $\rho(T_n)(\xi_a) = (T_n \xi_a)$. Then $\rho(\hat{C}(H))(\mathcal{L}) \subseteq \mathcal{L}$, so $[\rho(\hat{C}(H)), P_\mathcal{L}] = 0$.
(2) The map $\Phi : \hat{C}(H) \to C(H)$ defined by $\Phi(T) = \rho(T)P_\mathcal{L}$ is a surjective *-homomorphism.
(3) If $\gamma \notin P_{1,2}(m)$, then $\Phi((x_\gamma^n(y^\otimes 1))) = 0$. In particular, $C(H) = \Phi(\hat{C}(H))$ is spanned by the elements $\Phi((x_\gamma^n(y^\otimes 1)))$, for $m \geq 1, \sigma \in P_{1,2}(m)$.
(4) If $(x_n) = (x_n^\sigma) \in M$, then $\Phi((x_n(y^\otimes 1))) = \Phi((x_n^\sigma(y^\otimes 1)))$. In particular, $C(H)$ is spanned by the elements $\Phi((W_\sigma(y^\otimes 1)))$, where $W_\sigma \in M, \sigma \in P_{1,2}(m)$ are the reduced Wick words.

Proof. Take $(x_\gamma^n(y^\otimes 1)) \in \hat{C}(H), (x_\gamma^n(z^\otimes 1)) \in \mathcal{L}$. Due to the convolution rule we have

$$\Phi((x_\gamma^n(y^\otimes 1)))(x_\sigma^n(z^\otimes 1)) = \sum_{\gamma \in P(m+m')} (x_\gamma^n(zy^\otimes 1)),$$

the summation being taken over all those $\gamma$’s which preserve the connections of both $\sigma$ and $\sigma'$, i.e. if some indices are connected by $\sigma$ or $\sigma'$, they will remain connected in $\gamma$. Now for all $\gamma \notin P_{1,2}(m + m')$, the corresponding term vanishes, because $\|x_\gamma^n(zy^\otimes 1)\|_2 \leq \|zy\|_\infty \|x_\gamma^n\|_2 \to 0$. Thus

$$\Phi((x_\gamma^n(y^\otimes 1)))(x_\sigma^n(z^\otimes 1)) = \sum_{\gamma \in P_{1,2}(m+m')} (x_\gamma^n(zy^\otimes 1)) \in \mathcal{L},$$

which proves 1. Also, if $\sigma \notin P_{1,2}(m)$ to begin with, every $\gamma$ in the sum will also not be in $P_{1,2}(m + m')$, hence the whole sum vanishes, which proves 3. The second statement is trivial. If $(x_n), (x_n') \in M$ such that $\lim \|x_n - x_n'\|_2 = 0$, then for every $y_n(z^\otimes 1) \in K$, we have $\|x_n y_n(z^\otimes 1) - x_n' y_n(z^\otimes 1)\|_2 \leq \|y_n\|_\infty \|zy\|_\infty \|x_n - x_n'\|_2 \to 0$, i.e. $\Phi((x_n(y^\otimes 1)) = \Phi((x_n'(y^\otimes 1)))$. The last statement then follows from the reduction formula. 

Our goal is to prove that under certain conditions the maps $(id \otimes m_\alpha)$ descend to a multiplier on the quotient algebra, namely $C(H)$. This is done via a careful analysis of $\Phi_\alpha$.

**Lemma 5.6.** There exists a complete contraction

$$\psi : (K \otimes L^2(\Gamma_q(\ell^2 \otimes H)))^c \otimes_h (K \otimes L^2(\Gamma_q(\ell^2 \otimes H))^c \to L^1(B(K) \otimes \Gamma_q(\ell^2 \otimes H))$$

such that

$$\psi((h \otimes a) \otimes (k \otimes b)) = (h \otimes k) \otimes ab$$

and $tr((S \otimes T)(\psi((h \otimes a) \otimes (k \otimes b))^*)) = ((S \otimes T)(k \otimes b), h \otimes a)$. Here $(k \otimes h)$ is the rank one operator with entries $(k_i h_j)$ in a given basis and $\otimes_h$ denotes the Haagerup tensor product of operator spaces.
Corollary 5.7. Let \( L \subset (K \otimes L^2(\Gamma_q(\ell^2 \otimes H)))^\omega \) be defined as above. Then there exists a completely contractive map

\[
\Psi : L^\omega \otimes_h L^\omega \to (L_1(B(K) \otimes \Gamma_q(\ell^2 \otimes H)))^\omega
\]

and a complete contraction \( q : (L_1(B(K) \otimes \Gamma_q(\ell^2 \otimes H)))^\omega \to [(B(K) \otimes \Gamma_q(\ell^2 \otimes H))^\omega]^* \) such that

\[
(q \circ \Psi)(k \otimes \bar{h})(T) = (T(k), \bar{h})
\]

In particular, \( \Psi^*|_{\hat{L}(H)} = \Phi \).

Proof. For \( \xi, \eta \in L \) given by \( \xi = (\xi_n)_n, \eta = (\eta_n)_n \) we may define

\[
\Psi(\xi \otimes \eta) = (\psi(\xi_n \otimes \eta_n))_n,
\]

where \( \psi \) is the map from Lemma 5.6. Now \( \Psi \) obviously extends by linearity, thanks to the definition of the Haagerup tensor product and the well-known fact that \( M_m((X_n)^\omega) = (M_m(X_n))^\omega \) (see [32]). The map \( q \) is given by the limit

\[
q(\overline{(T_n)_n}) = \lim_{n, \omega}(tr \otimes \tau)(T_n \xi_n^*)
\]

Now the assertion follows from Lemma 5.6 and the fact that the duality pairing is given by the limit along the ultraproduct. \( \blacksquare \)
Remark 5.8. Let $H$ be an infinite Hilbert space and $H \subset H'$. Thanks to the definition of the $\mathcal{C}^*$-algebra $\hat{C}(H)$ as a subalgebra of the ultraproduct, we clearly have an isometric inclusion $\hat{C}(H) \subset \hat{C}(H')$. The $\mathcal{C}^*$-algebra $C(H) \subset B(\mathcal{L}(H))$ depends on our minimalistic definition of $\mathcal{L}(H)$. Certainly, $\mathcal{L}(H) \subset \mathcal{L}(H')$ and hence the tautological map $\iota(x) = x_\sigma$, $\iota(y^{op}) = y$ produces a larger norm on $\mathcal{L}(H')$ than on $\mathcal{L}(H)$. Let us consider a noncommutative polynomial $p$ in a finite number of $x_\sigma$’s and $y^{op}$’s, and we may assume that the $x_\sigma$ only contain vectors from a finite dimensional subspace $H_0 \subset H$. Then we can find norm attaining vectors $\xi, \eta \in \mathcal{L}(H')$ for $p$. Then we write $H' = H_0 \oplus H_0^\perp$ and may also assume that the $\xi$ and $\eta$ are linear combination of elements in $\mathcal{L}(H_0)$ and $\mathcal{L}(H_1)$ where $H_1 \subset H_0^\perp$ is a finite dimensional subspace. Using the moment formula, we see that the inner product remains unchanged after applying an orthogonal transformation $o$ which sends $H_1$ to a finite dimensional subspace of $H$ orthogonal to $H_0$ and leaves $H_0$ invariant. This implies that

$$\|p\|_{C(H')} = \sup_{\xi, \eta} |(\xi, p\eta)| = \sup_{\xi, \eta} |(\alpha^o(\xi), p\alpha^o(\eta))| \leq \|p\|_{C(H)}.$$ 

Let us denote by $q_H = \Phi|_{C(H)}: \hat{C}_H \to C(H)$ the quotient map. Then we obtain a commutative diagram

\[
\begin{array}{ccc}
\hat{C}(H) & \xrightarrow{q_H} & C(H) \\
\downarrow & & \downarrow \\
\hat{C}(H') & \xrightarrow{q_H'} & C(H')
\end{array}
\]

where the left hand downarrow is the natural ultraproduct inclusion and the right hand downarrow the tautological inclusion (which is well-defined and injective). This allows us to identify elements in the kernel of $q_H$ by considering $q_H'$. We recall that thanks to Avsec’s result, the orthogonal projection $P_k : \Gamma_q(H) \to \Gamma_q(H)$ onto Wick words of length $k$ is a normal completely bounded map. We use the same notation $P_k : \Gamma_1(\Gamma_q(H)) \to \Gamma_1(\Gamma_q(H))$ and $id \otimes P_k : \Gamma_1(\Gamma_q(H)) \to \Gamma_1(\Gamma_q(H))$. Let us note that $P_k^\ast : \prod \Gamma_1(B(K) \otimes \Gamma_q(H))$ the extension to the ultraproduct on $L^1$ spaces, which satisfies

$$\langle (id \otimes P_k(T_n))_n, (\xi_n) \rangle = \langle (T_n)_n, P_k((\xi_n)_n) \rangle$$

with respect to the anti-linear bracket given by the ultraproduct trace (see also Raynaud).

Lemma 5.9. The kernel of $\Phi P_k$ contains the kernel of $q_H$.

Proof. The map $\Phi P_k^\ast$ is normal. According to Remark 5.8, it therefore suffices to show that for $\xi, \eta \in \mathcal{L}$ we have $id \otimes P_k(\Psi(\xi \otimes \eta)) \in \operatorname{Im}\psi_{H'}$ for some potentially larger Hilbert space $H'$. Let us now consider Wick words $(x_\sigma)_n, \tilde{x}_\sigma^n,$ and $y^{op}, \tilde{y}^{op}$. We have to consider

$$\langle \Psi((\tilde{x}_\sigma^n y^{op})_n \otimes (x_\sigma^n y^{op})_n) \rangle = \langle \Psi(\tilde{x}_\sigma^n y^{op} \otimes (x_\sigma^n y^{op}))_n \rangle.$$ 

For fixed $n \in \mathbb{N}$ we see that

$$\Psi(\langle \tilde{x}_\sigma^n y^{op} \otimes (x_\sigma^n y^{op}) \rangle) = n^{-\frac{m+n}{2}} \sum_{(j) = \tilde{\sigma}, (j) = \sigma} \langle \tilde{\pi}(a) y^{op} \otimes \tilde{\pi}(\tilde{a}) \tilde{y}^{op} \rangle \otimes \tilde{s}_j(\tilde{h}) \tilde{s}_j^*$$

$$= \sum_{\sigma' \in \Gamma(m+n')} \Psi(\tilde{x}_\sigma^{\sigma'} y^{op} \otimes x_{\sigma'} y^{op}) \rangle.$$ 

Here $\Psi^\sigma$ refers to the expression obtained from restricting to $(j_1, ..., j_m, j_n, ..., j_1) = \sigma'$. Note also that $\sigma'$ has to be obtained by joining singletons from $\sigma$ and $\tilde{\sigma}$. In this context we observe
again that is is enough to consider $\sigma' \in P_{1,2}(\tilde{m} + m)$. In the following example we see that

$$\left\| \sum_{j_1} (\pi_{j_1}(a_1)\pi_{j_1}(a_2)y^{op} \otimes \pi_{j_1}(\tilde{a})y^{op}) \otimes s_{j_1}^2 s_{j_1} \right\|_1 \leq \left\| \sum_{j_1} \pi_{j_1}(a_1)\pi_{j_1}(a_2)y^{op} \otimes s_{j_1}^2 \otimes e_{1,j_1} \right\|_1 \sum_{j_1} \pi_{j_1}(\tilde{a})y^{op} \otimes s_{j_1} e_{j_1,1} \leq c_q(a, \tilde{a})n \ll n^{3/2}$$

is much smaller than the predesigned $n^{3/2}$ and hence vanishes in the limit. For more complicated configurations, we may assume that $\sigma$ and $\tilde{\sigma}$ are pair/singleton partitions, and that new links in $\sigma' \in P_{1,2}(m + \tilde{m})$ are obtained from joining pairs or singletons in $\sigma$ with pairs in $\tilde{\sigma}$ (or the other way round). All the joint pairings can be estimated using the definition of the Haagerup tensor product as above which yields the bound

$$\left\| \sum_{j_k = \tilde{\sigma}, (j_k) = \sigma, (j_k, j_1) = \sigma'} (\pi_{j_1}(a)y^{op} \otimes \pi_{j}(\tilde{a})y^{op}) \otimes s_{j_1}(h) \right\| \leq c_q n^f(\sigma, \tilde{\sigma}, \sigma') \sup_j ||a_j|| \sup_j ||\tilde{a}_j|| ||y^{op}|| ||y^{op}||.$$

The function $f$ is obtained as follows. Let $\alpha$ be the number of pairs in $\sigma$ being linked to either a pair or singleton in $\tilde{\sigma}$, and similarly $\beta$ be the number of linked pairs. Then we find

$$f(\sigma, \tilde{\sigma}, \sigma') = \frac{|\sigma_a| + |\sigma_p|}{2} + \alpha + \alpha/2 + \frac{|\tilde{\sigma}_a|}{2} + |\tilde{\sigma}_p| - \beta + \beta/2 = \frac{m + \tilde{m}}{2} - \frac{\alpha + \beta}{2}$$

using row and column vectors $e_1, i_1, \ldots, i_l, e_i, \ldots, i_1$ for the number $l$ of links in $\sigma'$. Thus for $\alpha + \beta > 0$ we obtain $0$ in the limit and therefore only those $\sigma'$ which link singletons to singletons give a contribution in the limit. Now we use Pisier’s version [54, Sublemma 3.3] of the Möbius transform. Let $\sigma'$ be a fixed partition with pairs $\{(l_1, r_1), \ldots, (l_p, r_p)\}$. Then there are unitaries $\lambda_j^{\sigma'}$ in a product of free group factors such that

$$S_j(h) = s_j \otimes \lambda_j^{\sigma'}$$

satisfies

$$a(\sigma') := \sum_{\sigma'' \geq \sigma'} \Psi(\tilde{x}_\sigma y^{op} \otimes x_\sigma y^{op}) = \sum_{j_k} (\pi_{j_k}(a)y^{op} \otimes \pi_{j}(\tilde{a})y^{op}) \otimes (id \otimes E)(\tilde{S}_j(h)^* S(h))$$

$$= (id \otimes E)\psi(\tilde{X}_\sigma \otimes X^{\sigma'}).$$

Here

$$X^{\sigma'} = (n^{-m/2} \sum_{(j_1, \ldots, j_m) \leq \sigma} \pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m) \otimes S_j(h_1) \cdots S_j(h_m))$$

and the corresponding expression for $\tilde{X}^{\sigma'}_\sigma$ depends on $\sigma'$. Moreover, there exists a Möbius function $\mu(\cdot, \cdot)$ such that (see [54, Proposition 1])

$$\Psi(\tilde{x}_\sigma y^{op} \otimes x_\sigma y^{op}) = \sum_{\sigma' \geq \sigma} \mu(\sigma, \pi) a(\pi).$$

The advantage of this representation comes from the fact that we can actually calculate $P_k$ for such a fixed $\sigma'$. Recall that we may assume that $\sigma'$ is a pair/ singleton partition. For fixed $n \in \mathbb{N}$ and an element $\eta_n = S \otimes W_n$, $W_n$ a Wick word of length $k$ we obtain

$$tr(S((\pi_{j_1}(a)y^{op} \otimes \pi_{j}(\tilde{a})y^{op}))\tau(W_n s_{j_1} \cdots s_{j_n} s_{j_1} \cdots s_{j_1})$$
such that \((\tilde{j}_1, \ldots, \tilde{j}_{\tilde{m}}, j_m, \ldots, j_1) = \sigma'\). Then we obtain a non-zero term only if \(|\sigma'_s| = k\) has exactly \(k\) singletons. Hence we find that

\[
(id \otimes P_k)(\psi(\xi \otimes \eta)) = \sum_{|\sigma'_s| = k} \Psi_{\sigma'}(\tilde{x}_s \tilde{y}^{op} \otimes x_s y^{op}) = \sum_{|\sigma'_s| = k} \sum_{\pi \geq \sigma'} \mu(\sigma, \pi)a(\pi) .
\]

Therefore we are left to consider

\[
a(\pi) = (id \otimes E)(\Psi(\tilde{X}_\sigma^s \otimes X_\sigma^s)) .
\]

In order to use Remark \(\text{linf}\) we have to modify the variables \(X_\sigma^s\). Indeed, for every pair \(p = \{l, r\}\) in \(\pi\) we introduce a label \(e_p\) and replace \(s(e_{ji} \otimes h_l)\) by \(s(e_{ji} \otimes e_p \otimes h_l)\), and \(s(e_{jr} \otimes h_r)\) by \(s(e_{jr} \otimes e_p \otimes h_r)\). For the remaining singletons we replace \(s(e_j \otimes h_j)\) by \(S_j = s(e_j \otimes e_0)\) and work in the Hilbert space \(H' = H \otimes \ell_2\). Using the so modified \(X_\sigma^s\)'s we still have

\[
a(\pi) = (id \otimes E_{\Gamma}(\tilde{e}_2 \otimes \tilde{H})_0)(n) \psi(\tilde{X}_\sigma^s \otimes X_\sigma^s) = \lim_{j \to \infty} \psi(\alpha_0(\tilde{X}_\sigma^s) \otimes \alpha_0(X_\sigma^s))
\]

for any sequence \((\alpha_j)\) of orthogonal transformations such that \(\alpha_j(e_0) = e_0\), which converges weakly to \(e_0\). For elements in \(\hat{C}(H)\) the limit for \(j \to \infty\) converges, and hence this remains true for the norm closure. Thus for an element \(x \in \hat{C}(H)\) in the kernel of \(q_H\) we find \(q_H(x) = 0\) and hence

\[
\langle x, a(\pi) \rangle = \lim_{j \to \infty} \langle x, \psi(\alpha_0(\tilde{X}_\sigma^s) \otimes \alpha_0(X_\sigma^s)) \rangle = 0 .
\]

Using linear combinations we deduce indeed that \(\langle P_k(x), \Psi(\tilde{x}_s \tilde{y}^{op} \otimes x_s y^{op}) \rangle = 0\).

**Corollary 5.10.** Let \(m_\alpha\) be multipliers given by the cb-approximation property for \(\Gamma_q(H)\).

i) Then \((id \otimes m_\alpha)_n\) extend to completely bounded maps on \(C(H)\) with \(\limsup \|\cdot\|_cb = 1\) and \(\lim n f_\alpha(k) = 1\), where \(f_\alpha\) are the associated scalar finitely supported functions. In particular, the maps \(\tilde{\varphi}_n\) used in the proof of Thm. 5.1 above are completely bounded with \(\lim sup \|\varphi_n\|_cb = 1\).

ii) Let \(L(H) = \tilde{C}(H)^{\varepsilon_0} \subset B(L)\) and note that \(L(H)\) is spanned by "extended Wick words" (i.e. images of extended Wick words through \(\Phi\)) such that \(L_2^\varepsilon(\Phi(L(H)))\) (i.e. the \(\|\cdot\|_2\)-closed span of the extended Wick words of degree \(k\)) is finitely generated over \(B\). Then there exists a modified family \(f_\alpha(N)^* : L(H)_* \to L(H)_*\) converging in the point norm topology.

**Proof.** Since \((id \otimes m_\alpha)(T) = \sum_k f_\alpha(k)P_k(T),\) we see that \(\|\Phi_H \circ (id \otimes m_\alpha)\|cb \leq 1\) and also \(\ker(q_H) \subset \ker(\Phi_B \circ (id \otimes m_\alpha))\). But that means that there is a unique map \(\tilde{m}_\alpha : \hat{C}(H)/\ker(q_H) \to B(K)\) such that

\[
\|\tilde{m}_\alpha\|cb = \|m_\alpha\|cb \leq 1 + \varepsilon_0 .
\]

However, \(\hat{C}(H)/\ker(q_H) = C(H)\) completely isometrically, and hence \(\tilde{m}_\alpha = (id \otimes m_\alpha)\) coincides with the densely defined map \((id \otimes m_\alpha)W(\sigma, \xi, a, y) = f_\alpha(\|\sigma_s\|)W(\sigma, \xi, a, y)\). Let us now consider a finite dimensional subspace \(H_0 \subset H\). Since \(L_2^\varepsilon(\Phi(L(H)))\) is finitely generated over \(B\), we deduce that the projection \(P_0\) is normal on \(L(H_0)\). Hence the maps \(m_\alpha\) are also normal and restricted to the weakly dense subspace \(C(H_0)\) we know that

\[
\|m_\alpha\|cb \leq (1 + \varepsilon_0) .
\]

Since a weakly dense subspace is norming for \((L(H_0)\), we deduce that \(||(m_\alpha)_*: L(H_0)_* \to L(H_0)_*\|cb \leq (1 + \varepsilon_0)\). Hence the normal map \(m_\alpha\) coincides with the normal map \((m_\alpha)_*\) and satisfies the same cb-norm estimate. Moreover, since we have normal conditional expectations \(\mathcal{E}_{H_0} : L(H) \to L(H_0)\) so that \(\cup H, \mathcal{E}_{H_1}(\mathcal{L}(H))_*\) is norm dense in \(L(H)_*\), we deduce that \((m_\alpha)_*\) extends to a completely bounded map of cb-norm \(\leq (1 + \varepsilon_0\) and hence \(m_\alpha = ((m_\alpha)_*)^*\) is indeed
Lemma 5.11. There exists an action by \(*\)-automorphisms \(\alpha : \mathcal{O}(H) \to \text{Aut}(\mathcal{N})\) such that
\[
\alpha_o(\pi(x)\theta(y^{op})) = \pi(\alpha_o(x))\theta(y^{op}), \quad o \in \mathcal{O}(H), \quad x \in M, \quad y^{op} \in P^{op}.
\]
Moreover, let \(E_0\) be the orthogonal projection of \(L\) onto the closed linear span of the extended Wick words of degree zero. For \(T \in \mathcal{N}\) the condition
\[
\alpha_o(T) = T, \quad \forall o \in \mathcal{O}(H)
\]
implies that \([T, E_0] = 0\).

Proof. Let us recall that \(\mathcal{N}\) acts on
\[
\mathcal{H} = \text{span}(\pi(x_{\sigma})((g \otimes_A 1) \otimes 1), x_{\sigma} \in M, y \in M, z \in P)
\subset ((L^2(\mathcal{M}) \otimes_A L^2(P)) \otimes L_2(\Gamma_q(P^2 \otimes H)))^\omega.
\]
Recall here that \(H\) is infinite dimensional, and thanks to second quantization \(u_o = (id \otimes \alpha_o)_n\) acts on and \(\mathcal{H}\) as a unitary. By normality, we deduce that \(\alpha_o(x) = u_o u_o^*\) extends to a \(*\)-automorphism of \(\mathcal{N}\) and moreover, \(\alpha_o(\theta(y^{op})) = \theta(y^{op})\). Let \(o_i \in \mathcal{O}(H)\) be a family of orthogonal transformations of \(H\) such that \(o_i(h)\) goes to 0 weakly in \(H\). Let \(\xi = \pi(x_{\sigma})(y \otimes_A z \otimes 1)\) and \(\eta = \pi(x_{\sigma'})(y' \otimes_A z' \otimes 1)\). Then we obtain
\[
\lim_{n,\omega} n^{-1/2} \sum_{i} \tau(s_{j_1}(o(h_1)) \cdots o(h_m)) s_{j'_1} \cdots s_{j'_r}(h'_1) = 0
\]
Indeed, we expand the sum into the summation over \(\sigma'' \in P_{1,2}(m + m')\) and execute the limit over \(n\). Then we observe that the coefficients remain uniformly bounded. However, \(o_i(h_k)\) is eventually orthogonal to every \(h'_k\) and then the moment formula for \(q\)-gaussian yields 0 in the limit. We have therefore shown that \(u_o\) converges weakly to \(E_0\), the projection onto words of length 0 in the second component. By taking convex combinations we find a net such that
\[
\text{sot} - \lim_i \sum_i \alpha_i^* u_o i = E_0.
\]
Thus for \(T \in \mathcal{N}\) with \(\alpha_o(T) = T\) for all \(o\), we deduce that \([u_o, T] = 0\) and hence
\[
E_0(T(\xi)) = \lim_i \sum_i \alpha_i^* u_o i T(\xi) = T(\lim_i \sum_i \alpha_i^* u_o i (\xi)) = T(E_0(\xi)).
\]
This means \(E_0 T = TE_0\) as desired. 

Lemma 5.12. Let \(B \vee P^{op} \subset B(L^2(\mathcal{M}) \otimes_A L^2(P))\). Then the natural inclusion map
\[
\pi : B \vee P^{op} \to \mathcal{N}
\]
is normal.
Proposition 5.13. Assume that for every finite dimensional Hilbert space $H$, $L^2(M(H))$ is finitely generated as a right $B$-module (note that in particular this is the case if $\dim B(D_k(S)) < \infty$, for all $k$). Then

i) There exists a faithful normal conditional expectation $\mathcal{E} : \mathcal{N} \to B_P = \pi(B) \vee \theta(P^{op})$;

ii) The action $\alpha$ is implemented by an $\ast$-continuous family of unitary operators $(V_\theta)_{\theta \in \mathcal{O}(H)}$ on $L^2(\mathcal{N})$;

iii) $L^2(\mathcal{N}) = \bigoplus_{k \geq 0} W_k(M)L^2(B_P)$ and $V_\theta(\pi(x_\sigma)\xi) = \pi(\alpha_\sigma(x_\sigma))\xi$ for $x_\sigma \in M, \xi \in L^2(B_P)$.

Moreover, $\mathcal{E}|_{\pi(M)} = E_B$, where $E_B : \pi(M) \to \pi(B)$ is the conditional expectation.

Proof. For a subspace $H' \subset H$ we use the notation

$$\mathcal{H}(H') = \{\pi(x_\sigma)((y \otimes_A z) \otimes 1)|y \in M, z \in P, x_\sigma = x_\sigma(x_1, ..., x_m, h_1, ..., h_m), h_i \in H'\}$$

for the subspace generated by $H'$-Wick words. Let $\iota_{H'} : \mathcal{H}(H') \subset \mathcal{H}$ be the canonical inclusion map and $F_{H'}(T) = \iota_{H'}^* T \iota_{H'}$ the induced completely positive map. Certainly, we have $F_{H'}(\theta(y^{op})) = \theta(y^{op})$ and

$$F_{H'}(x_\sigma) = E_B(x_\sigma).$$

Indeed, if a Wick word $x_\sigma$ contains a singleton $h_i \in (H')^\perp$, then $F_{H'}(x_\sigma) = 0$. Using $h_i \in H' \cup (H')^\perp$ we deduce the assertion by linearity. Thus $F_{H'}(\mathcal{N}(H)) = \mathcal{N}(H_i) \subset B(\mathcal{H}(H'))$ defines a normal surjective conditional expectation $F_{H'}$. Let $e_{H'}$ be the support of $F_{H'}$. We observe that $\pi(M(H'))$ and $\theta(P^{op})$ belong to the multiplicative domain of $F_{H'}$. Let $\tilde{N}(H') \subset \mathcal{N}(H)$ be the von Neumann algebra generated by $\pi(M(H'))$ and $\theta(P^{op})$ inside $N_P(H)$. According to remark 2.10 and Kaplansky’s density theorem, we deduce that $F_{H'}$ induces the same weak$^*$ topology on the unit ball of $\tilde{N}(H')$. This means that the tautological embedding $\sigma_{H'} : \mathcal{N}(H_i) \to \tilde{N}(H)$ given by $\sigma_{H'}(x_\sigma) = x_\sigma$ and $\sigma_{H'}(\theta(y^{op})) = \theta(y^{op})$ satisfies $\sigma_{H'} \sigma = id_{\mathcal{N}(H_i)}$ and $F_{H'}$ is an isomorphism when restricted to $\tilde{N}(H')$. We denote by $\mathcal{E}_{H'} = \sigma_{H'} F_{H'} : \mathcal{N} \to \tilde{N}$ the resulting, not necessarily faithful, conditional expectation. Let $H_i$ be an increasing net of finite dimensional spaces whose union is dense. Since $\bigcup H_i$ is norm dense, we deduce that $\mathcal{E}_{H_i}(x)$ converges weakly to $x$ as $i$ goes to infinity along the net of finite dimensional subspaces. Recall that the
multiplier maps \( m_\alpha \) are normal and commute with every \( \mathcal{E}_{H_i} \). Adding convex combination we may find a new completely contractive net, still denoted by \( m_\alpha \), converging in the strong, strong* operator topology. Thus we may assume that

\[
\lim_{i} \lim_{\alpha} (\hat{E}_{H_i}(m_\alpha x)) = x
\]

converges strongly for all \( x \in \mathcal{N} \). In our next step we consider \( H' = \emptyset \), i.e. the map \( \iota : L^2(M) \otimes_A L^2(P) \to \mathcal{H} \), given by \( \iota(y \otimes_A z) = (y \otimes_A z) \otimes 1 \). This yields a completely positive map \( \Phi(T) = \iota^* T \iota \) such that \( \Phi(\theta(y^{op})) = \theta(y^{op}) \) and \( \Phi(\pi(b)) = \pi(b) \). On the other hand for a Wick word \( x = W_\sigma \), we see that

\[
(\pi(x)\iota(y \otimes_A z), \iota(y \otimes_A z')) = \lim_{n,\omega} \left( \frac{\pi(x)(y \otimes_A z), y' \otimes_A z'}{\pi(s_1(h_1) \cdots s_m(h_m))} \right) = 0. 
\]

By normality, we deduce that \( \Phi(\mathcal{N}) = \mathcal{B} \cap \mathcal{B}(L^2(M) \otimes_A L^2(P)) \). Let us denote by \( B_P = \Phi(\mathcal{N}) \) the resulting von Neumann algebra and by \( e_{B_P} \), the support of \( \mathcal{E} = \Phi|_{\mathcal{N}_P} \). Since the Wick words of order 0 are obviously invariant under \( \alpha_o \) for all \( o \in \mathcal{O}(H) \) and \( \mathcal{E}_{\alpha_o}(x) = \alpha_o(\mathcal{E}(x)) = \mathcal{E}(x) \)

we must have \( \alpha_o(e_{B_P}) = e_{B_P} \) for every \( o \in \mathcal{O}(H) \). More precisely, \( 1 - e_{B_P} \) is the projection of the ideal \( I = \{ x : \mathcal{E}(x^*x) = 0 \} \) and we certainly have \( \alpha_o(I) = I \). This implies \( \alpha_0(1 - e_{B_P}) = 1 - \alpha_0(e_{B_P}) \). We deduce that for all \( o \) we have \( \alpha_o(m_{e_{B_P}}) = m_{e_{B_P}} \) and hence, thanks to Lemma 5.12 we know that \([E_0, m_{e_{B_P}}] = 0\). Now, we fix \( \alpha \) and consider \( x_{i,\alpha} = \mathcal{F}_{H_i}(m_{e_{B_P}}) = m_{e_{B_P}} \mathcal{F}_{H_i}(m_{e_{B_P}}) \). This means

\[
x_{i,\alpha} = \sum_{k \leq k(\alpha)} x_k,
\]

where \( x_k = P_k(x) \). However, we have a finite basis \( \xi_{k,s} \) of \( L^2_k(M(H)) \) over \( B \) made of elements in \( W_k(H_i) \) and hence for all \( z = \pi(x_{\sigma'}) \theta(y^{op}) \) we find

\[
P_k(z) = \sum_s \pi(\xi_{k,s}) E_B(\xi_{s,x_{\sigma'}}) \theta(y^{op}).
\]

Since \( P_k \) is normal we deduce that there are coefficients \( a_s \in \pi(B) \cap \theta(P^{op}) \) such that

\[
x_k = \sum_s \pi(\xi_{k,s}) a_{s,k} \in \mathcal{N}(H_i)
\]

Note here that we have rewritten \( m_{e_{B_P}} \) as normal map, because the maps \( T_k(x) = \pi(\xi_{k,s}) \sigma(\mathcal{E}(\xi_{k,s}^* x)) \) are normal, thanks to Lemma 5.12. Note also that due to Lemma 5.12 \( \sigma(B \cap \mathcal{B}(P^{op})) = \pi(B) \cap \theta(P^{op}) \) contains the range of \( \iota \) and hence

\[
[E_0, \iota_H, m_{e_{B_P}}] = \iota_H^* [E_0, m_{e_{B_P}}]\iota_H = 0.
\]

Thus we have \([E_0, x_{i,\alpha}] = 0\). Let us consider \( \eta = (y \otimes_A z) \otimes 1 \). We deduce that

\[
x_{i,\alpha}(\eta) = \sum_{k \leq k(\alpha)} \sum_s \pi(\xi_{k,s})(a_{s,k}(\eta))
\]

Moreover, we see that

\[
E_B(\xi_{s,k}^* x_{i,\alpha}(\eta)) = E_B(\xi_{s,k}^* \xi_{s,k}) a_{s,k}(\eta).
\]

We may assume that \( f_{k,s} = E_B(\xi_{s,k}^* \xi_{s,k}) \) is a projection in \( B \) and \( a_{s,k} = f_{k,s} a_{s,k} \). Since the conditional expectation can be calculated using vectors in the Hilbert space, we deduce that

\[
a_{s,k}(\eta) = E_B(\xi_{s,k}^* x_{i,\alpha}(\eta)) = E_B(\xi_{s,k}^* E_0(x_{i,\alpha}(\eta))) = 0
\]
for all $k > 0$. Thus only the coefficient for $k = 0$ survives and hence $x_{i,\alpha} \in \sigma(B \vee P^op)$. This remains true for the limit along $\alpha$, i.e. $x_i = F_{H_i}(e_{B}) \in \sigma(B \vee P^op)$. Since $\bigcup_i H_i$ is norm dense we find that

$$e_{B} = w^* - \lim_{i} F_{H_i}(e_{B}) \in \sigma(B \vee P^op).$$

The restriction of the normal map $\sigma \circ \mathcal{E}$ to $\sigma(B \vee P^op)$ is the identity. This implies

$$1 - e_{B} = \sigma \circ \mathcal{E}(1 - e_{B}) = \sigma \circ \mathcal{E}(\sigma_{B}(1 - e_{B})e_{B}) = 0.$$

Thus $e_{B} = 1$ and $\mathcal{E}$ is indeed a faithful normal expectation. Now it is easy to conclude the proof of the crucial assertion iii). Indeed, we may assume that $\pi(B)$ and $\theta(P^op)$ both admit weakly dense separable sub $C^*$-algebras and hence fix a faithful normal state $\phi$ on $B_{P}$. Then $\psi = \phi \circ \mathcal{E}$ satisfies the Connes' commutativity relation for the modular group $\mathcal{E}(\sigma_{i}(x)) = \sigma_{i}(\mathcal{E}(x))$. We refer to [1] for the fact that we have a natural embedding of the Haagerup spaces $L^{2}(B_{P}) \to L^{2}(\mathcal{N})$ given by

$$\iota_{P}(x_{\psi^1}) = x_{\psi}^{1/2}$$

for the densities $d_{\psi} \in L^{1}(B \vee P^op)$, $d_{\psi} \in L^{1}(\mathcal{N})$ associated with the states. Moreover, the support of $d_{\psi}$ is 1. This implies that $L^{2}(\mathcal{N}) = \mathcal{N}L^{2}(B_{P})$. By approximation in the $C^*$-algebra generated by $\pi(M)$ and $\theta(P^op)$ we see that span of elements of the form

$$\pi(x_{\sigma})\theta(y^{op})d_{\psi}^{1/2}$$

are dense in $L^{2}(\mathcal{N})$. However, we have

$$tr((\pi(x_{\sigma})\theta(y^{op})d_{\psi}^{1/2})^{*}\pi(x_{\sigma})\theta(z^{op})d_{\psi}^{1/2}) = tr(\theta(y^{op})^{*}\pi(x_{\sigma})^{*}\pi(x_{\sigma})\theta(z^{op})d_{\psi}^{1/2})$$

$$= tr(\theta(y^{op})^{*}\theta(z^{op})\pi(x_{\sigma})^{*}\pi(x_{\sigma})d_{\psi}^{1/2}) = |\psi(\theta(y^{op})^{*}\theta(z^{op})\pi(x_{\sigma})^{*}\pi(x_{\sigma}))|$$

$$= \phi(\mathcal{E}(\theta(y^{op})^{*}\theta(z^{op})\pi(x_{\sigma})^{*}\pi(x_{\sigma}))) = \phi(\theta(y^{op})^{*}\theta(\mathcal{E}(\pi(x_{\sigma})^{*}\pi(x_{\sigma}))))$$

$$= \phi(\theta(y^{op})^{*}\theta(\mathcal{E}(E_{B}\pi(x_{\sigma})^{*}\pi(x_{\sigma})))) \ .$$

For the proof of the last equality, we may assume that $x_{\sigma}$ and $x_{\nu}$ are reduced Wick words. As in Lemma 5.12, we see that

$$(\pi(x_{\xi})(y \otimes_{A} z), \pi(x_{\nu})(\tilde{y} \otimes_{A} \tilde{z}) = \lim_{n} n^{-|\sigma| - |\nu|/2} \sum_{(j_{k}) = \sigma, (\tilde{j}_{k}) = \tilde{\sigma}}$$

$$\tau(\tilde{z}^{*}E_{A}(\tilde{y}_{j_{k}}^{*}\tilde{n}_{j_{k}}^{*})\tilde{n}_{j_{k}}(x)y)z \tau(s_{j_{m}}^{*}(\tilde{h}_{m})s_{j_{m}}^{*}(h_{m}))$$

$$= \delta_{|\sigma|, |\nu|} \sum_{\gamma \in S_{k}} q^{inv(\sigma)} n^{-|\sigma|} \sum_{(j_{1}, \ldots, j_{k}) = \gamma} \tau((\alpha_{j_{1}}(\tilde{x})(\tilde{x}))^{*}) \alpha_{j_{1}, \ldots, j_{k}}(x)yE_{A}(z^{*}^{\gamma})^{1/2}$$

$$= \delta_{|\sigma|, |\nu|} \sum_{\gamma \in S_{k}} q^{inv(\sigma)} \tau(b(x, \tilde{x}, \gamma)yE_{A}(z^{*}^{\gamma})^{1/2}) \ .$$

The limit $b(x, \tilde{x}, \gamma) \in B$ only depends on $x$ and $\tilde{x}$ and the permutation $\gamma$, see Lemma 5.12. Placing the summation inside we find indeed $E_{B}(x_{\sigma}^{*}x_{\sigma})$. Thus we have shown that $\mathcal{E}_{|\pi(M)} = E_{B}$. Together with 5.3, we deduce that the spaces $W_{k}(M)L_{2}(B_{P})$ are mutually orthogonal. Finally, we have to discuss the action $\alpha : O(H) \to Aut(\mathcal{N})$. For an arbitrary $*$-automorphism $\alpha$ of $\mathcal{N}$, we may define the action on $L^{2}(\mathcal{N})$ via

$$\alpha(x_{\psi^{1/2}}) = \alpha(x)(d_{\psi} \circ \alpha^{-1})^{1/2} \ .$$

It is easy to show that this action is independent of the choice of a normal faithful density $d$ associated with state $\psi$. Here $d \circ \alpha^{-1}$ is the density of $\psi \circ \alpha^{-1}$. Thus we deduce from $\alpha_{\phi}(\theta(y^{op}) = \theta(y^{op})$ and the fact that $\psi \circ \alpha_{\phi} = \psi$, that

$$\alpha_{\phi}(\pi(x_{\sigma})\theta(y^{op})d_{\psi}^{1/2}) = \alpha_{\phi}(\pi(x_{\sigma}))\theta(y^{op})d_{\psi}^{1/2} \ .$$
as expected. 

\textbf{Remark 5.14.} A posteriori, we deduce that under the assumptions above \( F_{H'} \) is faithful for every subspace \( H' \subset H \) because \( \mathcal{E} = \mathcal{E}_{F_{H'}} \).
6. The deformation bimodules are weakly contained in $L^2(M) \otimes_B \mathcal{K}$ for sub-exponential dimensions of $D_k(S)$ over $B$

6.1. Norm estimates for decomposable maps. Let $H$ be an $M$-$N$ bimodule over finite von Neumann algebras $M$ and $N$. We will introduce some norms which will enable us to show that the $M$-$N$ bimodules associated to certain maps $\Phi : M \to L^1(N) = N^\text{op}$ are weakly contained in $H$. To be more precise define

$$\|\Phi\|_H = \inf \{ \sum_j \|\xi_j\|\|\eta_j\| : \tau(\Phi(x)y) = \sum_j \langle x \otimes y^\text{op}\rangle \xi_j, \eta_j \}. $$

The infimum is taken over elements $\xi_j, \eta_j \in H$.

**Lemma 6.1.** Let $K$ be an $M$-$N$ bimodule such that for a total set of vectors $\xi \in K$ the map $\Phi_\xi : M \to L^1(N)$ defined by

$$\tau(\Phi_\xi(x)(y)) = \langle x \otimes y^\text{op}\rangle \xi, \xi \rangle = \langle x_\xi y, \xi \rangle$$

satisfies $\|\Phi_\xi\|_H < \infty$. Then $K$ is weakly contained in $H$.

**Proof.** Let us recall that $K \prec H$ if and only if we have the relation between the kernels

$$\ker(\pi_H) \subset \ker(\pi_K),$$

where $\pi_H : M \otimes_{\text{bin}} N^\text{op} \to B(H)$, respectively $\pi_K : M \otimes_{\text{bin}} N^\text{op} \to B(K)$ are the canonical representations. Let $z = \lim z_j$ be a limit of norm one elementary tensors which converges to an element $z \in \ker(\pi_H)$ with respect to the max norm. Let $\xi \in K$ such that $\|\Phi_\xi\|_H < \infty$. This means we may assume that

$$\tau(\Phi_\xi(x)y) = \sum_l \alpha_l \langle \xi_l, x_\xi y \rangle, \quad \|\xi_j\|\|\eta_j\| \leq 1$$

and $\sum_l |\alpha_l|$ is finite. Using $\|z_j\|_{\text{bin}} \leq 1$ and uniform convergence, we may interchange limits and deduce

$$\langle z_\xi, z_\xi \rangle = \lim \langle \xi_j, z^*_j z_j \xi \rangle = \sum_l \alpha_l \lim \langle \xi_l, z^*_l z_j \eta_l \rangle = \sum_l \alpha_l \langle z_\xi_l, z_\eta_l \rangle = 0.$$

Thus for any linear combination $\xi = \sum_k \xi_k$ of elements such that the $\Phi_{\xi_k}$’s have finite $H$ norm, we still have $\pi_K(z)\xi = 0$. By density this holds for all $\xi \in K$. \hfill \blacksquare

As an illustration for the norm estimates let us prove the following result.

**Lemma 6.2.** Let $H_B = L_2(M) \otimes_B L_2(M)$, and assume that $L^2_k(M)$ has dimension $d_k$ over $B$. Let $P_k : L^2(M) \to L^2_k(M)$ be the orthogonal projection. Then

$$\|P_k\|_{H_B} \leq d_k.$$ 

**Proof.** We recall that

$$\langle x \otimes y^\text{op}(c \otimes d), a \otimes b \rangle = \tau(b^* E_B(a^* x) E_B(d y b^*)) = \tau(E_B(a^* x) E_B(d y b^*)) = \tau(E_B(a^* x) E_B(d y b^*)) = \tau(E_B(a^* x) E_B(d y b^*)) = \tau(E_B(a^* x) E_B(d y b^*)) = \tau(E_B(a^* x) E_B(d y b^*)).$$

Assuming that $\xi_j$ is a basis with $E_B(\xi_j^* \xi_i) = \delta_{ij} e_i, e_i$ a projection, we see that

$$\tau(\Phi_k(x)) = \sum_j \tau(\Phi_k E_B(\xi_j^* x)) = \sum_j \langle x \otimes y^\text{op}(1 \otimes 1), \xi_j \otimes \xi_j^* \rangle.$$

Since $\langle \xi_j \otimes \xi_j^*, \xi_j \otimes \xi_j^* \rangle = \tau(E(\xi_j^* \xi_j) E(\xi_j^* \xi_j)) \leq \tau(e_j)$, we deduce the assertion. \hfill \blacksquare
6.2. Configurations. Our main goal here is to analyze the operators \( \Phi_{\xi,\eta} : M \to L_1(M) \) given by \( \Phi_{\xi,\eta}(x) = E_M(\xi x \eta) \), where \( \xi, \eta \) are elements in \( \Gamma_q(B, A \otimes (H \oplus H)) \). We will start with monomials

\[
\xi = s(x_1, h_1) \cdots s(x_m, h_m), \quad \eta = s(x_1', h_1') \cdots s(x_m', h_m')
\]

where \( h_i, h_i' \in H \times \{0\} \cup \{0\} \times H \). Although our goal is to obtain estimates for arbitrary \( x \), we will first assume that \( x = \zeta \) is a reduced Wick word from \( M \) and only contains singletons from \( H \times \{0\} \). By considering the moment formula we can reorganize the trace using configurations

\[
\tau(\zeta' \xi \zeta \eta) = \sum_{\alpha \text{ configuration}} \tau(\zeta' \Phi_\alpha(\zeta))
\]

whenever \( \zeta' \) is another reduced Wick. Here a configuration \( \alpha = (\sigma_{0 \times H}, \sigma_{H \times 0}, I_{\xi,\zeta}, I_{\zeta,\eta}) \) is given by

i) A pair partition \( \sigma_{0 \times H} \) of \( \{1, \ldots, m\} \cup \{m', \ldots, 1\} \) so that all the pairs \( \{l, r\} \) have indices in \( 0 \times H \);

ii) A pair partition \( \sigma_{H \times 0} \) of \( \{1, \ldots, m\} \cup \{m', \ldots, 1\} \) so that all the pairs \( \{l, r\} \) have indices from \( H \times 0 \);

iii) Subsets \( I_{\xi,\zeta} \subset \{1, \ldots, m\} \), \( I_{\zeta,\eta} \subset \{m', \ldots, 1\} \) disjoint from the support \( \cup \sigma_{0 \times H} \cup \cup \sigma_{H \times 0} \) of the partitions above.

Indeed, using the moment formula for \( \tau(\zeta' \xi \zeta \eta) \) we know that we have to take the sum over all pair partitions of length \( m + m' + k + k' \), \( k = |\zeta|, \quad k' = |\zeta'| \). Every such pair partition has to respect the pairs of \( 0 \times H \) and that defines our \( \sigma_{0 \times H} \). Some pairs can combine elements from \( \xi \) and \( \eta \) with coefficients in \( H \times 0 \). This defines \( \sigma_{H \times 0} \). Some partitions connect \( \xi \) and \( \zeta \) and some \( \zeta \) with \( \eta \). The left hand sides of the pairs between \( \xi \) and \( \eta \) define the set \( I_{\xi,\zeta} \) and the right hand sides of the pairs from \( \zeta \) to \( \eta \) define \( I_{\zeta,\eta} \). All the remaining pairs will connect \( \zeta' \) and \( \zeta \). Since \( \zeta \) and \( \zeta' \) are themselves Wick words, there are no pairs connecting elements from \( \zeta \) (\( \zeta' \)) with itself.

We see that indeed, the sum over all partitions can be regrouped into first summing over all configuration (which only depend on \( \xi \) and \( \eta \)), and then sum over all partitions supported by these configurations. Let us note that once a configuration \( \alpha \) is known we can determine exactly how many crossings will be produced by pairs in \( 0 \times H \). Indeed, we know that \( |I_{\xi,\zeta} + I_{\zeta,\eta}| \) many singletons will be removed from \( \zeta \). According to the position of the left legs in \( \sigma_{0 \times H} \) some extra crossing will be produced from the set \( I_{\xi,\zeta} \). The same applies for \( I_{\xi,\zeta} \). Here is an example

\[
a_1 \quad a_2 \quad b_1 \quad a_3 \quad b_3 \quad | \quad c_1 \quad c_2 \quad a_3 \quad d_1 \quad c_3 \quad c_4 \quad | \quad d_2 \quad b_3 \quad d_1 \quad a_1 \quad b_1
\]

Here \( \sigma_{0 \times H} \) are given by the positions of \( b_1 \) and \( b_3 \). The set \( I_{\xi,\zeta} \) is given by the position of \( a_3 \) and \( \sigma_{H \times 0} \) is given by the positions of \( a_1 \). The \( b \)'s are responsible for \( 8 + 1 + 1 + 1 \) crossings, \( 8 \) crossing with \( c \)'s, one crossing among themselves, one crossings coming from \( a \)'s and \( b \)'s, one crossing from the \( b \) and \( d \)'s. Thus \( k(\alpha) = 2 \times (6 - 2) + 1 + 2 \).

In our next step we replace the monomials \( \xi \) and \( \eta \) by Wick words. This means we only have to sum over those configurations such that \( \sigma_{0 \times H} \) and \( \sigma_{H \times 0} \) connect \( \xi \) and \( \eta \) and no pairs \( \xi \) and \( \eta \) with itself. In addition the reduction procedure produces scalars and new operator valued expression \( \alpha_{j_1 \ldots j_l}(\beta) \) with \( \beta \in D_k(S) \). We have proved the following simple combinatorial fact:

**Lemma 6.3.** Let \( \xi \) and \( \eta \) be Wick words obtained by reduction and \( \zeta \in M \) be a Wick word of length \( k = |\zeta| \). For a fixed configuration \( \alpha \) there is a number \( k(\alpha) \) such that for all \(-1 \leq q \leq 1\)

\[
\Phi_\alpha(\zeta) = q^{k(\alpha)} \tilde{\zeta},
\]

where \( \tilde{\zeta} \) is a linear combination of reduces words with smaller length \( k - |I_{\xi,\zeta}| - |I_{\zeta,\eta}| \). Moreover, if \( k \geq m + m' \) is the length of \( \zeta \), and \( L \) is the cardinality of \( \sigma_{0 \times H} \), then

\[
k(\alpha) \geq (k - m - m')L.
\]
We will give more precise information about $\tilde{\zeta}$ in the next paragraph.

6.3. **Generalized $Q$-gaussians.** As a tool we will use a slight generalization of the von Neumann algebra $\Gamma_q(B, A \otimes H)$. This generalization is based on matrix models of the ordinary $q$-gaussian von Neumann algebras. This approach was invented by Speicher ([49, 50]) and has been applied successively in many situations. Let $Br : H \to \bigcap_p L^p(\Omega, \Sigma, \mu)$ the standard brownian motion so that $Br(h)$ is a normal random variable and and $(Br(h), Br(h')) = (h, h')$. The $\sigma$-algebra is chosen minimal. This construction is well-known as the gaussian measure space construction. Given a selfadjoint matrix $v$ with values $\{-1, 1\}$ there are symmetries $v_j \in M_{2^n}(\mathbb{C})$ such that

$$v_i v_j = \varepsilon_{ij} v_j v_i.$$  

Speicher’s important idea is to choose the matrix $\varepsilon_{ij}$ independently at random for all pairs. We will work with double indices $\varepsilon_{(j,t),(k,s)}$, which are independent as functions of the pairs $\{(j,t), (k,s)\}$ whenever $t \neq s$ or $j \neq k$ and satisfy

$$P(\varepsilon_{(j,t),(k,s)} = 1) = \frac{1 - Q_{s,t}}{2}$$

as long as $(j, t) \neq (ks)$ for a given matrix $Q_{s,t}$. This allows us to construct matrix models

$$u(t, h) = \left( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} v_{j,t} \otimes g_j(h) \right)_n \in \prod_{n, \omega} (M_{2^n}(\mathbb{C}) \otimes L^\infty(\Omega))_n$$

which satisfy

$$\tau(u(t_1, h_1) \cdots u(t_m, h_m)) = \sum_{\sigma \in P_2(n)} \prod_{\{a,b\} \in \sigma} Q_{t_a, t_c} \prod_{\{a,b\} \in \sigma} (h_a, h_b).$$

In other words the constant term $q^{\text{inv}(\sigma)}$ is replaced by the product of the crossing inversions weighted according to $Q$. Indeed, by independence

$$\prod_{\{a,b\} \in \sigma, \{c,d\} \in \sigma} Q_{t_a, t_c} = E\tau(v_{j_1, t_1} \cdots v_{j_m, t_m})$$

for $(j_1, ..., j_m) \leq \sigma$. In particular for a fixed $t$ and $\|h\| = 1$ the random variable $u(t, h)$ is just an ordinary $q$-gaussian. This central limit theorem is well-known and goes back to [49, 50], see also [23].

We may easily generalize this to the $A$-valued situation by considering a sequence of symmetric independent copies $(\pi_j, B, A, D)$ and defining

$$u(t, h, a) = \left( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} v_{j,t} \otimes g_j(h) \otimes \pi_j(a) \right)_n \in \prod_{n, \omega} (M_{2^n}(\mathbb{C}) \otimes L^\infty(\Omega) \otimes D)_n.$$

For a subset $1 \leq S = S^* \subset A$, we denote the von Neumann algebra generated by the elements $u(t, h, a), t \in Q, h \in H, a \in S$ by $\Gamma^S_0(B, S \otimes H)$. Then define the von Neumann algebra $\Gamma_q(B, S \otimes H)$ by the same procedure as in Def. 3.4. A look at the moment formula allows us to state the following fact.

**Lemma 6.4.** Let $T_0 \subset T$ be a non-empty subset such that $Q_{st} = q$ for all $s, t \in T_0$. Then $\Gamma_q(B, S \otimes H)$ embeds into $\Gamma_q(B, S \otimes H)$ in a trace preserving way.

**Remark 6.5.** As observed in [23] the reduction procedure still works in the generalized $Q$-gaussian setting.
Let us return to a configuration $\alpha$ as in 6.2. above. We replace the Wick words $W_q(\vec{a}, \vec{h})$ and $W_q(\vec{a}', \vec{h}')$ by new Wick words $W_Q(\vec{a}, \vec{h})$ $W_Q(\vec{a}', \vec{h}')$ as follows. For a configuration $\alpha$ with a partition $\sigma_{0 \times H}$ of the indices labeled with $0 \times H$, we define a new matrix

$$Q_{\alpha t}(\vec{q}) = \begin{cases} \vec{q} & \text{if } h_s \text{ and } k_s \text{ are both in } 0 \times H \\ q & \text{else.} \end{cases}$$

Note that the matrix only depends on the first component $\sigma_{0 \times H}$ of a configuration. For every pair $p = \{l, r\} \in \sigma_{0 \times H}$ we introduce a label $e_p$ and replace $h_l$ and $h_r'$ by $h_l \otimes e_p$ and $h_r' \otimes e_p$ to avoid over-counting. We denote by $H_s, H'_t$ the modified vectors. Starting from

$$\xi_Q(\vec{q}) = s_{Q(\vec{q})}(H_1, a_1) \cdots s_{Q(\vec{q})}(H_m), \quad \eta_Q(\vec{q}) = s_{Q(\vec{q})}(H_{m'}, a_{m'}) \cdots s_{Q(\vec{q})}(H'_{1}, a'_{1})$$

we apply the same reduction procedure (eliminating all the pairs from the non-reduced words $X_{\sigma}(\vec{h}, \vec{a})$) for the $W_q$’s and obtain the reduced Wick words $W_{Q(\vec{q})}(\vec{H}, \vec{a}), W_{Q(\vec{q})}(\vec{H}', \vec{a}')$.

**Lemma 6.6.** Fix $\sigma_{0 \times H}$. The function

$$F(\vec{q}) = \sum_{\alpha, \alpha' = \sigma_{0 \times H}} E_M(W_{Q(\vec{q})}(\vec{H}, \vec{a})\zeta W_{Q(\vec{q})}(\vec{H}', \vec{a}'))$$

is a polynomial in $\vec{q}$ with lowest degree at least $|\zeta| - m + m'$ and largest degree at most $(|\zeta| + m + m')L$.

**Proof.** Let $\alpha$ be a configuration which contains $\sigma_{0 \times H}$. Comparing the terms in the moment formula for

$$\tau(\zeta' W_q(\vec{h}, \vec{a})\zeta W_q(\vec{h}', \vec{a}')) \quad \text{and} \quad \tau(\zeta' W_{Q(\vec{q})}(\vec{h}, \vec{a})\zeta W_{Q(\vec{q})}(\vec{h}', \vec{a}'))$$

we see that they differ by the factor $(\frac{\vec{q}}{q})^{k(\alpha)}$ number of pairs. Note however, that $k(\alpha)$ only depends on $\alpha$. This implies the assertion. 



### 6.4. Weak containment

We need a simple fact about polynomials:

**Lemma 6.7.** Let $[a, b]$ be an interval, $P_d(a, b)$ the set of polynomials of degree $d$, and $a < t_0 < t_1 < \cdots < t_d < b$ distinct points. Then the map $\Phi : P_d(a, b) \to \mathbb{C}^{d+1}$, $\phi(f) = f(t_j)$ is injective. Moreover, there exists a matrix $a_{i, j}$ such that for every polynomial

$$p(t) = \sum_{0 \leq k \leq d} \alpha_k t^k$$

of degree $\leq d$ we have

$$\alpha_k = \sum_j a_{k,j} f(t_j).$$

**Proof.** For $0 \leq j \leq d$ we define the polynomial $p_j(t) = \left(\prod_{i \neq j} (t - t_i)\right)^{-1} \prod_{i \neq j} (t - t_i)$ which has degree $d$. Then we see that $p_j(t_j) = 1$ and $p_j(t_i) = 0$ for $i \neq j$. In particular, the polynomials $(p_j)_{0 \leq j \leq d}$ are linearly independent and hence $P_d(a, b) = \text{span}\{p_j | 0 \leq j \leq d\}$. This implies

$$p(t) = \sum_{0 \leq j \leq d} p(t_j) p_j(t)$$

and in particular $\Phi$ is injective. Since moreover, the monomials are linearly independent in $C_\infty(a, b)$, we see that the linear map $\Psi(\alpha_0, \ldots, \alpha_d) = \Phi(\sum_k \alpha_k t_j^k)$ is invertible and can be represented by the matrix $C_{j,k} = t_j^k$, the well known Vandermonde matrix. Then $A = C^{-1}$ does the job.
Corollary 6.8. Fix $m, m'$ and $L$. Then there exists a degree $D = D(m, m', L)$ such that for $q \in [a, b]$ and $a \leq t_1 < \cdots < t_D \leq b < 1$ there are coefficients $\gamma_l$ such that

$$E_M(\xi \eta) = \sum_{\sigma} \sum_{l} (\frac{q}{t_l})^{(k-m-m')}L \gamma_l F_\sigma(t_l)$$

holds for $k = |\xi| \geq 2(m + m')$. Moreover, for some possibly different coefficients $\tilde{\gamma}_l$

$$E_M(\xi \eta) = \sum_{\sigma} \sum_{l} \tilde{\gamma}_l F_\sigma(t_l)$$

holds for $|\xi| \leq 2(m + m')$.

Proof. We fix $\sigma$ and $k \geq m + m'$. Let $[a, b] \subset (-1, 1)$ be an interval and $a = q$. The $t_i$’s are all chosen bigger than $a$. We define the polynomial $p_k(t) = t^{-(k-m-m')}L F(t)$ which has degree at most $(k + m' + m - (k - m - m'))L \leq (2m + 2m')L$ and hence

$$p_k(t) = \sum_{0 \leq j \leq (2m + 2m')L} a_j t^j$$

holds for mutually different points $a \leq t_1, \ldots, t_d \leq b$ where $d \leq (2m + 2m')L + 1$ are independent of $k$. Hence we get

$$F_\sigma(q) = q^{(k-m-m')}L p_k(q) = q^{(k-m-m')}L \sum_{j,i} c_{ij} q^i \overline{p}_k(t_i)$$

$$= q^{(k-m-m')}L \sum_{j,i} c_{ij} q^i t_i^{-(k-m-m')}L F_\sigma(t_i) = \sum_{i} (\sum_j c_{ij} q^i)(\frac{q}{t_i})^{(k-m-m')}L F_\sigma(t_i) .$$

This defines the coefficients $\gamma_i$. For $k \leq 2(m + m')$ we work directly with the polynomial $F(t)$ of degree at most $2(2m + 2m')L$.

Let $M = \Gamma_q(B, S \otimes H)$, $\tilde{M} = \Gamma_q(B, S \otimes (H \oplus H))$. Define the $M - M$ bimodule $F_m \subset L^2(\tilde{M})$ as the $\| \cdot \|_2$-closed linear span of the reduced Wick words $W_\sigma(x_1, \ldots, x_t, h_1, \ldots, h_N), N \geq 1$ such that $h_i \in H \times \{0\} \cup \{0\} \times H$ for all $i$ and at least $m$ of the vectors $h_i$ belong to $\{0\} \times H$. This bimodule will play a crucial role in our deformation-rigidity arguments in the next section.

Theorem 6.9. Let $M = \Gamma_q(B, S \otimes H)$ and let $C > 0, d > 0$ be two constants such that the dimension of $L_k^2(M)$ over $B$ is smaller than $Cd^k$ for all $k$. Let $|q| < 1$. Then there exists an $L_0 \in \mathbb{N}$ and a $B$-$M$ bimodule $K$ such that $F_l$ is weakly contained in $L_2(M) \otimes_B K$ for all $l \geq L_0$.

Proof. Let us recall that

$$\langle \zeta \otimes (\zeta')^op(a \otimes_B b), \alpha \otimes_B \beta \rangle = \tau(\beta^* E_B(\alpha^* \zeta a)b\zeta') = \tau(E_B(\alpha^* \zeta a)E_B(b\zeta'/\beta^*)) .$$

Now we may assume that $(\xi_i)_{i \in I_k}$ is a basis of dimension $d_k$ over $B$ so that

$$P_k(\zeta) = \sum_{i \in I_k} \xi_i E_B(\xi_i^* \zeta) \quad \text{and} \quad E_B(\xi_i^* \zeta) \leq 1 .$$

This implies

$$\tau(\zeta' \xi_i \eta) = \sum_{i \in I_k} \tau(\zeta' \xi_i E_B(\xi_i^* \zeta) \eta) = \sum_{i \in I_k} \tau(\xi_i E_B(\xi_i^* \zeta) \eta \zeta')$$
Thus it suffices to know that
\[ \sum_{i \in I_k} (\zeta \otimes (\zeta')^\text{op}) (1 \otimes_B \eta), \xi_i \otimes_B (\xi_i^*)^*). \]

Let \( q_0 < 1 \) so that \( q / q_0 < 1 \). Then we define the \( B - M \) bimodule
\[ \mathcal{K} = \bigoplus_{q/q_0 < 1} L^2(\Gamma_{Q(t)}(B, S \otimes H)) \]
with the natural left and right actions. For fixed \( \xi, \eta \) we choose \( a = \pm q \) and \( |q| / q_0 < t_0 < \cdots t_D < b \) for some \( b < 1 \). This allows us to define \( W_{Q(t_i)}(H, \bar{a}) \) and \( W_{Q(t_i)}(H', \bar{a}') \) in \( K \). With the help of Corollary 6.1, we deduce that the map \( \Phi^+(\zeta) = \sum_k \langle \xi \otimes_B (\xi^*_k) \rangle \) satisfies
\[ \|\Phi^+\|_{L^2(M) \otimes_B \mathcal{K}} \leq \sum_{k \geq 2(m+m')} q_0^{k(m-m')} L \sum_{i \in I_k} \|1 \otimes_B W_{Q(t_i)}(H', \bar{a}')\| \|\xi_i \otimes_B (W_{Q(t_i)}(H, \bar{a}) \xi_i^*)\|. \]

Now we note that
\[ \|1 \otimes_B W_{Q(t_i)}(H', \bar{a}')\| = \|W_{Q(t_i)}(H', \bar{a}')\|_{L^2(\Gamma_{Q(t_i)})} \leq c(t_i) \]
and
\[ \|\xi_i \otimes_B (W_{Q(t_i)}(H, \bar{a}) \xi_i^*)\| = \tau(W_{Q(t_i)}(H', \bar{a}) \xi_i E_B(\xi^*_i \xi_i)(W_{Q(t_i)}(H, \bar{a}) \xi_i^*)^* \leq \tau(\xi_i E^*_i W_{Q(t_i)}(H', \bar{a}) W_{Q(t_i)}(H, \bar{a})) \leq \|W_{Q(t_i)}(H, \bar{a})\|^2_{L^2(\Gamma_{Q(t_i)})} \leq c(t_i). \]

Thus it suffices to know that \( \sum_k q_0^{k(m-m')} L d^k \) is finite. Note here that \( m \) and \( m' \) depend on the Wick word and that we may assume \( l \geq L_0 \). Thus \( q_0^{L_0} d < 1 \) and \( b < 1 \) is enough to achieve summability. Using the second part of Corollary 6.8, we also have summability for \( k \leq 2(m+m') \). Lemma 6.1 then yields the assertion.

**Corollary 6.10.** Let \( M = \Gamma_q(B, S \otimes H) \) and assume that \( H \) is finite dimensional and \( \dim_B(D_k(S)) \leq C d^k \) for some constants \( C, d > 0 \). Then there exists an \( B - M \) bimodule \( \mathcal{K} \) such that for \( m \geq 1 \) large enough we have \( \mathcal{F}_m \prec L^2(M) \otimes_B \mathcal{K} \). In particular, if \( B \) is amenable, then for \( m \) large enough, \( \mathcal{F}_m \) is weakly contained into a multiple of the coarse bimodule \( L^2(M) \otimes L^2(M) \).
7. The main theorem and applications

We first need some preliminaries. Throughout this section we use the notations \( M = M(H) = \Gamma_q(B,S \otimes H), \tilde{M} = \Gamma_q(B,S \otimes (H \oplus H)) = M(H \oplus H) \). Let
\[
\mathcal{M} = (D\Gamma_q(\ell^2 \otimes H)) \vee M \subset (D\Gamma_q(\ell^2 \otimes H))^\omega.
\]
As in Section 5, let
\[
\mathcal{H} \subset ((L^2(\mathcal{M}) \otimes A L^2(P)) \otimes \mathcal{F}_q(\ell^2 \otimes H))^\omega
\]
be the norm closed linear span of the sequences
\[
(n^{-\frac{m}{2}} \sum_{(j_1,\ldots,j_m) = \sigma} (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)y \otimes_A z) \otimes s_{j_1}(h_1) \cdots s_{j_m}(h_m)),
\]
for \( m \geq 1, \sigma \in P_{1,2}(m), x_i \in BSB, h_i \in H, y \in M, z \in P \). Take the representations
\[
\pi : M \to B(\mathcal{H}), \theta : P^{op} \to B(\mathcal{H})
\]
introduced in Section 5 and define \( \mathcal{N} = \pi(M) \vee \theta(P^{op}) \subset B(\mathcal{H}) \). As seen is Section 5, we choose a normal faithful state \( \phi \) on \( B_P = \pi(B) \vee \theta(P^{op}) \subset B(\mathcal{H}) \) and then define a normal faithful state \( \psi \) on \( \mathcal{N} \) by \( \psi = \phi \circ E_{B_P} \), where \( E_{B_P} : \mathcal{N} \to B_P \) is the normal faithful conditional expectation. Let \( d_\psi \in L^1(\mathcal{N}) \) be the density of \( \psi \) and \( \xi_0 = d_\psi^2 \). Then \( L^2(\mathcal{N}) \) is the norm closed span of the elements \( \pi(x_\sigma)\theta(y^{op})\xi_0 \), for \( x_\sigma \in M \) a Wick word and \( y \in P \). Let \( W_k(M) \) be the linear span of the Wick words of degree \( k \) in \( M \) and \( L^2(B_P) = L^2(B_P, \phi) \) be the standard form for \( B_P \subset B(\mathcal{H}) \). Then \( \mathcal{N} \) is standardly represented on
\[
L^2(\mathcal{N}) = \bigoplus_{k \geq 0} W_k(M)L^2(B_P)
\]
by the formulas
\[
\pi(x_\sigma)\theta(y^{op})(\pi(x_\nu)\theta(z^{op})\xi_0) = \pi(x_\sigma x_\nu)\theta((zy)^{op})\xi_0, x_\sigma, x_\nu \in M, y, z \in P.
\]
The conjugation \( \mathcal{J} : L^2(\mathcal{N}) \to L^2(\mathcal{N}) \) associated to the standard representation of \( \mathcal{N} \) is given by
\[
\mathcal{J}(\pi(x_\sigma)\theta(y^{op})\xi_0) = \sigma_\psi^\frac{1}{2} (\pi(x_\sigma^* y^\sigma)\xi_0, x_\sigma \in M, y \in P,
\]
where \( \sigma_\psi^\frac{1}{2} \) is the modular group on \( \mathcal{N} \) associated to \( \psi \). We will also consider \( \tilde{\mathcal{N}} = \mathcal{N}(\tilde{H}) \) constructed in the same way as \( \mathcal{N} \) by using \( \tilde{H} = H \oplus H \) instead of \( H \). Thus take
\[
\tilde{\mathcal{H}} \subset ((L^2(\mathcal{M}) \otimes A L^2(P)) \otimes \mathcal{F}_q(\ell^2 \otimes \tilde{H}))^\omega
\]
to be the norm closed linear span of the sequences
\[
(n^{-\frac{m}{2}} \sum_{(j_1,\ldots,j_m) = \sigma} (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m)y \otimes_A z) \otimes s_{j_1}(\tilde{h}_1) \cdots s_{j_m}(\tilde{h}_m)),
\]
for \( m \geq 1, \sigma \in P_{1,2}(m), x_i \in BSB, \tilde{h}_i \in \tilde{H} \). Exactly as in Section 5, define the \(*\)-representations
\[
\pi : \tilde{M} \to B(\tilde{\mathcal{H}}), \theta : P^{op} \to B(\tilde{\mathcal{H}})
\]
and then define \( \tilde{\mathcal{N}} = \pi(\tilde{M}) \vee \theta(P^{op}) \). Then \( \tilde{\mathcal{N}} \) is standardly represented on
\[
L^2(\tilde{\mathcal{N}}) = \bigoplus_{k \geq 0} W_k(\tilde{M})L^2(B_P),
\]
and the associated conjugation \( \tilde{\mathcal{J}} : L^2(\tilde{\mathcal{N}}) \to L^2(\tilde{\mathcal{N}}) \) is given by the formula
\[
\tilde{\mathcal{J}}(\pi(x_\sigma)\theta(y^{op})\xi_0) = \sigma_\psi^\frac{1}{2} (\pi(x_\sigma^* y^\sigma)\xi_0, x_\sigma \in \tilde{M}, y \in P,
\]
where $\sigma^\psi_t$ is the modular automorphisms group on $\tilde{N}$ associated to $\psi$. For every angle $t$ define the unitary $V_t$ on $L^2(\tilde{N})$ by

$$
\pi(x_\sigma(x_1, h_1, \ldots, x_m, h_m))\theta(y^{op})\xi_0 \mapsto \pi(x_\sigma(x_1, \alpha_t(h_1), \ldots, x_m, \alpha_t(h_m)))\theta(y^{op})\xi_0.
$$

Then the one parameter group of *-automorphisms $\text{Ad}(V_t)$ of $B(L^2(\tilde{N}))$ restricts to a group $\alpha_t$ of *-automorphisms of $\tilde{N}$, acting according to the formula

$$
\alpha_t(\pi(x_\sigma(x_1, h_1, \ldots, x_m, h_m))\theta(y^{op})) = \pi(x_\sigma(x_1, \alpha_t(h_1), \ldots, x_m, \alpha_t(h_m)))\theta(y^{op}).
$$

When further restricted to $\tilde{M} = \Gamma_q(B, S \otimes \tilde{H})$ this group of *-automorphisms coincides with the one introduced in Theorem 3.16 and we have the following identity

$$
T_t(x) = E_M(\alpha_s(x)), x \in M, 0 < s < \frac{\pi}{2},
$$

where $T_t$ is the heat semigroup introduced in Theorem 3.16 and $t = -\ln(\cos(s))$. We finally introduce the bimodules needed in the deformation argument. To do this, recall that $\tilde{M} = M(H \oplus H)$ is the generalized $q$-gaussian algebra generated by $B$, $s_q((a, h, 0))$ and $s_q(a, 0, h)$, where $a \in S$ runs through the generating set and $h \in H$ are unit vectors. Let $F \subset H$ be an orthonormal basis. Then we define an $M - M$ bimodule $F_m \subset L^2(M)$ by

$$
F_m = \overline{\text{span}}\{W_\sigma(k_1, \ldots, k_N, a_1, \ldots, a_N)|k_i \in F \times \{0\} \cup \{0\} \times F, \#\{i|k_i \in \{0\} \times F\} = m\}.
$$

Note that we use reduced Wick words. This means $N = |\sigma_s|$ and the vectors $(k_1, \ldots, k_N)$ are the ones obtained after contracting the pairs. Here $\sigma \in P_{1,2}(N')$ and $a_1, \ldots, a_N'$ are the original coefficients from $S$. One can see that $F_m$ is exactly the eigenspace of vectors $\xi \in L^2(M)$ such that $E_{\tilde{M}(0 \oplus H)}(\alpha_t(\xi)) = e^{-tm}\xi$ for all (some) $t > 0$. Likewise we define the $M - M$ bimodule $F^P_m \subset L^2(\tilde{N})$ as the $\|\cdot\|_2$-closed span of the elements

$$
\pi(W_\sigma(x_1, h_1, \ldots, x_m, h_m))\theta(y^{op})\xi_0, x_i \in BSB, h_i \in F \times \{0\} \cup \{0\} \times F,
$$

such that exactly $m$ of the vectors $h_i$ belong to $\{0\} \times F$. It's easy to see that $F_m$ can be described by

$$
F^P_m = \{\xi \in L^2(\tilde{N})|E_{\tilde{M}(0 \oplus H)}(\alpha_t(\xi)) = e^{-tm}\xi, \forall t > 0\}.
$$

Finally, we set

$$
F_m = \bigoplus_{m' \geq m} F_{m'} \subset L^2(M) \quad , \quad F^P_m = \bigoplus_{m' \geq m} F^P_{m'} \subset L^2(N).
$$

Let's remark that we have the following transversality property, whose proof is virtually the same as that of Prop. 5.1 in [1].

**Lemma 7.1.** There exists a constant $C = C(m) > 0$ such that for $0 < t < 2^{-m-1}$ we have

$$
\|V_{t}^{m+1}(\xi) - \xi\| \leq C\|e^{tV_t}(\xi)\| \quad \text{for all} \quad \xi \in \bigoplus_{k \geq m+1} L^2_k(N) \subset L^2(\tilde{N}).
$$

**Theorem 7.2.** Let $M = \Gamma_q(B, S \otimes \tilde{H})$ associated to a sequence of symmetric independent copies $(\pi_j, B, A, D)$ and assume that the dimension of $D_k(S)$ over $B$ is finite for every $k$ and that $H$ is finite dimensional. Let $A \subset M$ be a von Neumann subalgebra which is amenable relative to $B$ and denote $P = N^*_M(A)^\prime$. Let $m \geq 1$ be fixed. Then at least one of the following statements holds:

1. The $M - M$ bimodule $F_m$ is left $P$-amenable;
2. there exist $t, \delta > 0$ such that $\inf_{a \in H(A)} \|T_t(a)\|_2 \geq \delta$.

**Proof.** The approximately invariant states $\omega_n \in N$ constructed in Thm. 5.1 are implemented by unit vectors $\xi_n \in L^2(N) \subset L^2(\tilde{N})$. Using the Powers-Stormer inequalities we see that the vectors $\xi_n$ have the following properties
\( \langle \pi(x)\xi_n, \xi_n \rangle \to \tau(x), x \in M; \)
\( \| \pi(a)\theta(\bar{a})\xi_n - \xi_n \| \to 0, a \in \mathcal{U}(A); \)
\( \| \pi(u)\theta(\bar{u})\mathcal{J}\pi(u)\theta(\bar{u})\mathcal{J}\xi_n - \xi_n \| \to 0, u \in \mathcal{N}_M(A). \)

Let \( e^\perp : L^2(\mathcal{N}) \to \mathcal{F}_m^P \) be the orthogonal projection. We have the following alternative:

**Case 1.** For every non-zero projection \( p \in \mathcal{Z}(P) \) and for every \( t > 0 \) we have

\[ \limsup_n \| e^\perp V_t \pi(p)\xi_n \| > \frac{\| p \|^2}{8C}. \]

We will prove that in this case the \( M - M \) bimodule \( \mathcal{F}_m \) is left \( P \)-amenable.

**Lemma 7.3.** Let \( X \) be the the strong operator topology completion of \( \mathcal{F}_m \) as a right \( M \)-module with respect to the \( M \)-valued inner product \( \langle x, y \rangle = E_M(x^* y), x, y \in \mathcal{F}_m \). Let \( \mathcal{L}(X) \) be the von Neumann algebra of adjointable operators on \( X \). Then there exists a normal *-homomorphism \( \Psi : \mathcal{L}(X) \to B(L^2(\mathcal{F}_m^P)) \) such that \( \Psi(\mathcal{L}(X)) \subset B(L^2(\mathcal{F}_m^P)) \cap (\mathcal{N}^\prime \cap (\theta(P^\prime))^\prime). \)

**Proof.** The condition \( \text{Lem. (7.3)} \) implies that \( \mathcal{F}_m^P = X \otimes_M L^2(\mathcal{N}) \), where the left action on \( \mathcal{N} \) is that of \( \pi(M) \). Therefore the map \( \Psi : \mathcal{L}(X) \to B(\mathcal{F}_m^P) \) given by

\[ \mathcal{L}(X) \ni T \mapsto T \otimes_M id \in B(\mathcal{F}_m^P) \]

is a well-defined normal *-homomorphism. Let us consider a rank one operator \( \xi \otimes \bar{\eta} \in \mathcal{L}(X) \) with \( \xi, \eta \) Wick words in \( \hat{M} \). Then we calculate

\[ \Psi(\xi \otimes \bar{\eta})(\pi(x_\sigma)\theta(y^\prime)\xi_0) = \pi(\xi)\pi(E_M(\eta^* x_\sigma))\theta(y^\prime)\xi_0. \]

Let \( e_{\mathcal{N}} \) be the orthogonal projection of \( \mathcal{N} \) onto the closure of \( \mathcal{N}\xi_0 \), which exists thanks to the fact that \( E_{\mathcal{N}}^{B_{P'}} \) is faithful, see Remark. Then we note that for \( \bar{x}_\sigma \in \mathcal{M} \) we have

\[ \langle \pi(E_M(\eta^* x_\sigma))\theta(y^\prime)\xi_0, \bar{x}_\sigma \rangle \langle \bar{\eta}^\prime \bar{\eta} \xi_0 \rangle = \psi(\theta(\bar{y}^\prime)^* \theta(y^\prime)E_{\mathcal{N}}^{B_{P'}}(\bar{x}_\sigma^* x_\sigma)) \]

\[ = \psi(\theta(\bar{y}^\prime)^* \theta(y^\prime)(E_{\pi(B)}^{\pi(M)}) \circ E_{\pi(M)}(\bar{x}_\sigma^* x_\sigma)) = \langle \pi(E_M(\eta^* x_\sigma))\theta(y^\prime)\xi_0, \pi(\bar{x}_\sigma)\theta(\bar{y}^\prime)\xi_0 \rangle. \]

This shows that

\[ \pi(E_M(\eta^* x_\sigma))\theta(y^\prime)\xi_0 = e_{\mathcal{N}}(\pi(\eta^* x_\sigma)\theta(y^\prime)\xi_0). \]

Thus we deduce that for all the rank one operators \( \xi \otimes \bar{\eta} \in \mathcal{L}(X) \)

\[ \Psi(\xi \otimes \bar{\eta}) = L_{\pi(\xi)}e_{\mathcal{N}}L_{\pi(\bar{\eta})} \]

is a right \( \mathcal{N} \)-module map, hence belongs to \( B(L^2(\mathcal{F}_m^P)) \cap (\mathcal{N}^\prime \cap (\theta(P^\prime))^\prime). \) It’s also trivial to check that \( \Psi(\xi \otimes \bar{\eta}) \) commutes with the operators \( L_{\theta(y^\prime)p} \), for all \( y \in P \). Since \( \Psi \) is normal and the linear span of the rank one operators is so-dense in \( \mathcal{L}(X) \) we have \( \Psi(\mathcal{L}(X)) \subset B(L^2(\mathcal{F}_m^P)) \cap (\mathcal{N}^\prime \cap (\theta(P^\prime))^\prime), \) as desired.

The lemma provides a normal *-homomorphism

\[ \Psi : B(\mathcal{F}_m) \cap (M^\prime)^\prime \to B(\mathcal{F}_m^P) \cap (\theta(P^\prime)^\prime) \]

such that \( \Psi(\lambda(x)) = \pi(x) \) for \( x \in M \), where \( \lambda \) is the natural left action of \( M \) on \( L^2(\hat{M}) \). From this point on, the proof proceeds verbatim as in the proof of Case 1 in Thm. 3.1.

**Case 2.** There exist a non-zero central projection \( p \in \mathcal{Z}(P) \) and \( t > 0 \) such that

\[ \limsup_n \| e^\perp V_t \pi(p)\xi_n \| \leq \frac{\| p \|^2}{8C}. \]

In this case we prove that there exist \( s, \delta > 0 \) such \( \| T_s(a) \| \geq \delta \) for all \( a \in \mathcal{U}(A) \). Write \( \pi(p)\xi_n = \xi_n + \eta_n \), where \( \xi_n \in \bigoplus_{k \leq m} L^2_k(\mathcal{N}) \), \( \eta_n \in \bigoplus_{k \geq m+1} L^2_k(\mathcal{N}) \). Note that \( \| \xi_n \| \leq 1, \| \eta_n \| \leq 1. \)
Since $V_i$ converges uniformly on $(\bigoplus_{k \leq m} L_k^2(\mathcal{N}))_1$, there exists a $t_0 > 0$ such that for $0 < s < t_0$ we have
\[
\|V_s \xi - \xi\| \leq \min\left\{\frac{\|p\|_2}{8}, \frac{\|p\|_2}{8C}\right\} \quad \text{for} \quad \xi \in (\bigoplus_{k \leq m} L_k^2(\mathcal{N}))_1.
\]
Fix $0 < s < \min\{t^{m+1}, t_0, t_0^{m+1}, 2^{-(m+1)^2}\}$. For every $n \geq 1$ we have the following estimate:
\[
\|V_s \pi(p)\xi_n - \pi(p)\xi_n\| \leq \|V_s\xi_n - \xi_n\| + \|V_s\eta_n - \eta_n\| \leq \frac{\|p\|_2^2}{8} + \|V_s\eta_n - \eta_n\| \\
\leq \frac{\|p\|_2^2}{8} + C\|e^{1/8}V_{m+\sqrt{s}}\eta_n\| \leq \frac{\|p\|_2^2}{8} + C\|e^{1/8}V_{m+\sqrt{s}}\pi(p)\xi_n\| + C\|e^{1/8}V_{m+\sqrt{s}}\xi_n\| \\
\leq \frac{\|p\|_2^2}{8} + C\|e^{1/8}V_{m+\sqrt{s}}\pi(p)\xi_n\| + C\|e^{1/8}V_{m+\sqrt{s}}\xi_n\| - \xi_n\| \leq \frac{\|p\|_2^2}{8} + C\|e^{1/8}V_i\pi(p)\xi_n\|.
\]
Taking $\limsup$ with respect to $n$ we obtain
\[
\limsup_n \|V_s\pi(p)\xi_n - \pi(p)\xi_n\| \leq \frac{3\|p\|_2}{8}.
\]
From this point on, the proof proceeds verbatim as in the proof of Case 2 in Thm. 3.1.

**Corollary 7.4.** Let $M = \Gamma_q(B,S \otimes H)$ and let $A \subset M$ be a von Neumann subalgebra which is amenable relative to $B$. Assume that there are constants $C,d > 0$ such that $\dim_B(D_k(S)) \leq Cd^k$, for all $k \geq 1$ and that $\dim(H) < \infty$. Let $P = \mathcal{N}_M(A)''$. Then at least one of the following statements is true:

1. $A \prec_M B$;
2. $P$ is amenable relative to $B$.

In other words, $M = \Gamma_q(B,S \otimes H)$ is strongly solid relative to $B$.

**Corollary 7.5.** The following von Neumann algebras are strongly solid relative to $B$:

1. $B \otimes \Gamma_q(H)$, for $H$ a finite dimensional Hilbert space;
2. $\Gamma_q(B,S \otimes H)$ associated to the symmetric independent copies $(\pi_j,B,A,D)$, where $B = N \otimes L(\mathbb{Z})$, $A = N \rtimes \mathcal{H}_1$, $D = N \rtimes \mathcal{H}$, $\mathcal{H} = \langle g_j : j \geq 0 \rangle$ is the Heisenberg group, $\mathcal{H}_1 = \langle g_1 \rangle$, $S = \{1,g_1,g_1^{-1}\}$ and $H$ a finite dimensional Hilbert space - the examples constructed in 4.3.2.;
3. $\Gamma_q(C,S \otimes K)$ associated to the symmetric copies $(\pi_j,C,G_0(H),\Gamma_q(\ell^2 \otimes H))$, where $\pi_j(s_0(g_1)) = s_0(g_j \otimes h)$ and $K$ is a finite dimensional Hilbert space - these are the examples in 4.4.1. or "the colored brownian motion";
4. $\Gamma_q(L(\Sigma_{[-d,0]}),S \otimes H)$ associated to the symmetric copies $(\pi_j,B_d,A_d,D_d)$, where $B_d = L(\Sigma_{[-d,0]})$, $A_d = L(\Sigma_{[-d,1]})$, $D = L(\Sigma_{[-d,\infty}})$ and $S = \{1,u_{0(1)}\}$ for a fixed $d \in \mathbb{N}$ - these are the examples in 4.4.2.;
5. $\Gamma_q(B,S \otimes H)$ associated to the symmetric copies $(\pi_j,B,A,D)$, where $D = \bigotimes_{\mathbb{N}}(B \otimes L(\mathbb{Z}))$ or $D = *_B(B \otimes L(\mathbb{Z}))$, the $j$-th copy of $L(\mathbb{Z})$ is generated by the Haar unitary $u_j$, $A = B \otimes \{u_1\}''$, the copies $\pi_j$ are defined by $\pi_j(b \otimes u_1) = b \otimes u_j$ and $S = \{1,u_1,u_1^*\}$ - the examples in 4.4.3.

In particular, the examples in (3), (4) and (5) are strongly solid and a fortiori, solid von Neumann algebras.
**Corollary 7.6.** Let $M_i = \Gamma_q(B_i, S_i \otimes H_i)$ be associated with two sequences of symmetric independent copies $(\pi_i^j, B_i, A_i, D_i)$ and two subsets $S_i \subset A_i$, $i = 1, 2$ and $-1 < q_1, q_2 < 1$. Assume that $\dim B_i(D_i^j(S_i)) \leq C d^k$ for some constants $C, d > 0$, $2 \leq \dim(H_i) < \infty$ and $B_i$ are amenable, for $i = 1, 2$. Then if $M_1 \subset M_2$, it follows that $B_1 \triangleleft M_2 B_2$. If moreover $M_1 = M_2 = M$, then we have $B_1 \triangleleft M B_2$ and $B_2 \triangleleft M B_1$.

**Corollary 7.7.** With the notations and assumptions of Cor. 7.6. above, if we moreover assume that

1. $B_1$ is finite dimensional and $B_2$ is amenable diffuse, or
2. $B_1$ is abelian and $B_2$ is the hyperfinite $I_1$ factor,

then $M_1 = \Gamma_{q_1}(B_1, S_1 \otimes H_1)$ and $M_2 = \Gamma_{q_2}(B_2, S_2 \otimes H_2)$ (for $-1 < q_1, q_2 < 1$) are not $\ast$-isomorphic. In fact, $M_2$ does not even embed into $M_1$, i.e. cannot be realized as a von Neumann subalgebra of $M_1$. Consequently, none of the examples in items (1) (for diffuse $B$) or (2) in Cor. 7.5. above can be $\ast$-isomorphic to, or even embed into, any of the examples in items (3), (4) or (5) of the same corollary.

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