Infinitely many solutions for three classes of self-similar equations, with the $p$-Laplace operator

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025
kormanp@ucmail.uc.edu

Abstract

We study the global solution curves, and prove the existence of infinitely many positive solutions for three classes of self-similar equations, with $p$-Laplace operator. In case $p = 2$, these are well-known problems involving the Gelfand equation, the equation modeling electrostatic micro-electromechanical systems (MEMS), and a polynomial nonlinearity. We extend the classical results of D.D. Joseph and T.S. Lundgren \cite{Lundgren} to the case $p \neq 2$, and we generalize the main result of Z. Guo and J. Wei \cite{Wei} on the equation modeling MEMS.

Key words: Parameterization of the global solution curves, infinitely many solutions.

AMS subject classification: 35J60, 35B40.

1 Introduction

We consider radial solutions on a ball in $\mathbb{R}^n$ for three special classes of equations, involving the $p$-Laplace operator, the ones self-similar under scaling. We now explain our approach for one of the classes, involving the $p$-Laplace version of the equation which arises in modeling of electrostatic micro-electromechanical systems (MEMS), see \cite{Joseph}, \cite{Lundgren}, \cite{Wei} (with $p > 1$, $\alpha > 0$, $q > 0$, $u = u(x)$, $x \in \mathbb{R}^n$, $n \geq 1$)

\begin{equation}
(1.1) \ \text{div} \left( |\nabla u|^{p-2}\nabla u \right) + \lambda \frac{|x|^{\alpha}}{(1-u)^q} = 0, \text{ for } |x| < 1 \ u = 0, \text{ when } |x| = 1.
\end{equation}
Here $\lambda$ is a positive parameter. We are looking for solutions satisfying $0 < u < 1$. Radial solutions of this equation satisfy

$$\varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda \frac{r^\alpha}{(1-u)^q} = 0 \quad \text{for } 0 < r < 1,$$

$$u'(0) = u(1) = 0, \quad 0 < u(r) < 1,$$

with $\varphi(v) = v|v|^{p-2}$. It is easy to see that $u'(r) < 0$ for all $0 < r < 1$, which implies that the value of $u(0)$ gives the maximum value (or the $L^\infty$ norm) of our solution. Moreover, $u(0)$ is a global parameter, i.e., it uniquely identifies the solution pair $(\lambda, u(0))$, see e.g., P. Korman [10]. It follows that a two-dimensional curve in the $(\lambda, u(0))$ plane completely describes the solution set of (1.2). The self-similarity of this equation allows one to parameterize the global solution curve, using the solution of a single initial value problem:

$$\varphi(w')' + \frac{n-1}{t} \varphi(w') = \frac{\alpha^\alpha}{w^{q}}, \quad w(0) = 1, \quad w'(0) = 0. \quad (1.3)$$

Its solution $w(t)$ is a positive and increasing function, which can be easily computed numerically. Following J.A. Pelesko [13], we show that the global solution curve of (1.2) is given by

$$(\lambda, u(0)) = \left( \frac{t^{\alpha+p}}{w^{p+q-1}(t)}, 1 - \frac{1}{w(t)} \right),$$

parameterized by $t \in (0, \infty)$. In particular, $\lambda = \lambda(t) = \frac{t^{\alpha+p}}{w^{p+q-1}(t)}$, and

$$\lambda'(t) = t^{\alpha+p-1} w^{-p+q} \left[ (\alpha + p)w - t(p + q - 1)w' \right],$$

so that the solution curve travels to the right (left) in the $(\lambda, u(0))$ plane if $(\alpha + p)w - t(p + q - 1)w' > 0$ ($< 0$). This makes us interested in the roots of the function $(\alpha + p)w - t(p + q - 1)w'$. If we set this function to zero

$$(\alpha + p)w - t(p + q - 1)w' = 0,$$

then the general solution of this equation is

$$w(t) = ct^\beta, \quad \beta = \frac{\alpha + p}{p + q - 1}.$$

Quite remarkably, if we choose the constant $c = c_0 = \left[ \frac{1}{[p^{p-1}((p-1)(\beta-1)+n-1)]^{1/p+q-1}} \right]^{-1}$ then

$$w_0(t) = c_0 t^\beta.$$
also solves the equation in (1.3), along with \( w(t) \). We show that \( w(t) \) tends to \( w_0(t) \) as \( t \to \infty \), and the solution curve of (1.2) makes infinitely many turns if and only if \( w(t) \) and \( w_0(t) \) intersect infinitely many times. We give a sharp condition for that to happen, thus generalizing the main result in Z. Guo and J. Wei [6] to the case of \( p \neq 2 \) (with a simpler proof). In [9] we called \( w(t) \) the generating solution, and \( w_0(t) \) the guiding solution.

We apply a similar approach to a class of equations with polynomial \( f(r, u) \) generalizing the well-known results of D.D. Joseph and T.S. Lundgren [8], and to the \( p \)-Laplace version of the generalized Gelfand equation, where we easily recover the corresponding result of J. Jacobsen and K. Schmitt [7].

For each of the three classes of equations we show that along the solution curves (as \( u(0) \to \infty \)), the solutions tend to a singular solution (for which \( u(r) \to \infty \), or \( u'(r) \to \infty \), as \( r \to 0 \)). Moreover, one can calculate the singular solutions explicitly, which is truly a remarkable feature of self-similar equations. Singular solutions were studied previously by many authors, including C. Budd and J. Norbury [2], F. Merle and L. A. Peletier [12], and I. Flores [3].

2 Parameterization of the solution curves

We begin with the \( p \)-Laplace version of the generalized Gelfand equation

\[
(2.1) \quad \varphi(u')' + \frac{n - 1}{r} \varphi(u') + \lambda r^\alpha e^u = 0 \quad \text{for } 0 < r < 1, \quad u'(0) = 0, \quad u(1) = 0,
\]

where \( \varphi(v) = v|v|^{p-2} \), \( p > 1 \). Observe that \( \varphi(sv) = s^{p-1} \varphi(v) \) for any constant \( s > 0 \). Assume that \( u(0) = a > 0 \). We set \( u = w + a, \quad t = br \). The constants \( a \) and \( b \) are assumed to satisfy

\[
\lambda = t^{\alpha+p} e^{-a}.
\]

Then (2.1) becomes

\[
(2.2) \quad \varphi(w')' + \frac{n - 1}{t} \varphi(w') + t^\alpha e^w = 0, \quad w(0) = 0, \quad w'(0) = 0.
\]

The solution of this problem \( w(t) \), which is a negative and decreasing function, is defined for all \( t > 0 \), and it may be easily computed numerically. (Write this equation as \( [t^{n-1} \varphi(w')]' = -t^{n+\alpha-1} e^w < 0 \), and conclude that \( t^{n-1} \varphi(w') < 0 \), and then \( w'(t) < 0 \) for all \( t \).) We have

\[
0 = u(1) = a + w(b),
\]
so that \( a = -w(b) \), and then \( \lambda = b^{\alpha + p}e^{w(b)} \). The solution curve for (2.1) is

\[
(\lambda, u(0)) = \left( b^{\alpha + p}e^{w(b)}, -w(b) \right),
\]

parameterized by \( b \in (0, \infty) \). The solution of (2.1) at \( b \) is \( u(r) = w(br) - w(b) \). It will be convenient to write the solution curve as

(2.3)  
\[
(\lambda, u(0)) = \left( t^{\alpha + p}e^{w(t)}, -w(t) \right),
\]

parameterized by \( t \in (0, \infty) \), and \( w(t) \) is the solution of (2.2). The solution of (2.1) at the parameter value \( t \) is \( u(r) = w(tr) - w(t) \).

We consider next the problem

(2.4)  
\[
\varphi(u')' + \frac{n - 1}{r} \varphi(u') + \lambda \frac{r^\alpha}{(1 - u)q} = 0 \quad \text{for } 0 < r < 1,
\]

\[
u'(0) = u(1) = 0, \quad 0 < u(r) < 1,
\]

which arises in modeling of electrostatic micro-electrome chanical systems (MEMS), see [13], [5], [6]. Here \( \lambda \) is a positive parameter, \( q > 0 \) and \( \alpha > 0 \) are constants, and as before \( \varphi(v) = v|v|^{p-2} \), \( p > 1 \). Any solution \( u(r) \) of (2.4) is a positive and decreasing function (by the maximum principle), so that \( u(0) \) gives its maximum value. Our goal is to compute the solution curve \((\lambda, u(0))\). Let \( 1 - u = v \). Then \( v(r) \) satisfies

(2.5)  
\[
\varphi(v')' + \frac{n - 1}{r} \varphi(v') = \lambda \frac{r^\alpha}{v^q} \quad \text{for } 0 < r < 1, \quad v'(0) = 0, \quad v(1) = 1.
\]

Assume that \( v(0) = a \). We scale \( v(r) = aw(r) \), and \( t = br \). The constants \( a \) and \( b \) are assumed to satisfy

(2.6)  
\[
\lambda = a^{p+q-1}b^{\alpha+p}.
\]

Then (2.5) becomes

(2.7)  
\[
\varphi(w')' + \frac{n - 1}{t} \varphi(w') = \frac{t^\alpha}{w^q}, \quad w(0) = 1, \quad w'(0) = 0.
\]

The solution of this problem is a positive increasing function, which is defined for all \( t > 0 \). We have

\[
1 = v(1) = aw(b),
\]
and so \( a = \frac{1}{w(b)} \), and then \( \lambda = \frac{b^{\alpha+p}}{w^{p+q-1}(b)} \). The solution curve \((\lambda, u(0))\) is 
\( \left( \frac{b^{\alpha+p}}{w^{p+q-1}(b)}, 1 - \frac{1}{w(b)} \right) \), parameterized by \( b \in (0, \infty) \). It will be convenient to write the solution curve in the form
\[
(\lambda, u(0)) = \left( \frac{t^{\alpha+p}}{w^{p+q-1}(t)} \right), 1 - \frac{1}{w(t)} \right),
\] (2.8)
parameterized by \( t \in (0, \infty) \). In case \( p = 2 \), this parameterization was first derived by J.A. Pelesko [13], and was then used in [5]. The solution of (2.4) at \( t \) is \( u(r) = 1 - \frac{w(b)}{w(t)} \).

Finally, we consider the problem (with the constants \( p > 1, q > 1, \alpha \geq 0 \))
\[
\varphi(u')' + \frac{n-1}{t} \varphi(u') + \lambda r^\alpha (1 + u)^q = 0 \quad \text{for } 0 < r < 1,
\]
(2.9)
which was analyzed in case \( p = 2 \) and \( \alpha = 0 \) by D.D. Joseph and T.S. Lundgren [8]. If we set \( 1 + u = v \), then \( v(r) \) satisfies
\[
\varphi(v')' + \frac{n-1}{r} \varphi(v') + \lambda r^\alpha v^q = 0, \quad v'(0) = 0, \quad v(1) = 1.
\]
(2.10)
Assuming that \( v(0) = a \), we scale \( v(r) = aw(r) \), and \( t = br \). The constants \( a \) and \( b \) are assumed to satisfy
\[
\lambda = \frac{b^{\rho+\alpha}}{a^{\gamma-p+1}}.
\]
(2.11)
Then (2.10) becomes
\[
\varphi(w')' + \frac{n-1}{t} \varphi(w') + t^\alpha w^q = 0, \quad w(0) = 1, \quad w'(0) = 0.
\]
(2.12)
The solution of (2.12) satisfies \( w'(t) < 0 \), so long as \( w(t) > 0 \) (the function \( t^{n-1} \varphi(w(t)) \) is zero at \( t = 0 \), and its derivative is negative). It follows that either there is a \( t_0 \), so that \( w(t_0) = 0 \) and \( w(t) > 0 \) on \((0, t_0)\), or \( w(t) > 0 \) on \((0, \infty)\) and \( \lim_{t \to \infty} w(t) = a \geq 0 \). It is easy to see that \( a = 0 \) in the second case. Indeed, assuming that \( a > 0 \), we have \([t^{n-1} \varphi(w')]' \leq -a \varphi',\) and integrating we conclude that \( w(t) \leq 1 - ct^\gamma \), with some \( c > 0 \), and \( \gamma = \frac{\alpha+p}{p+1} > 0 \), contradicting that \( w(t) > 0 \) on \((0, \infty)\).

**Lemma 2.1** Assume that
\[
q > \frac{np - n + p + p\alpha}{n - p}.
\]
(2.13)
Then \( w(t) > 0 \), and \( w'(t) < 0 \) on \((0, \infty)\), with \( \lim_{t \to \infty} w(t) = 0 \).
Proof: In view of the above remarks, we need to exclude the possibility that \( w(t_0) = 0 \) and \( w(t) > 0 \) on \((0, t_0)\). Recall that for the equation

\[
\varphi(w')' + \frac{n-1}{t} \varphi(w') + f(t, w) = 0,
\]

the Pohozaev function

\[
P(t) = t^n \left[ (p-1) \varphi(w')w' + pF(t, w) \right] + (n - p) t^{n-1} \varphi(w')w
\]

is easily seen to satisfy

\[
P'(t) = t^{n-1} \left[ npF(t, w) - (n - p)wf(t, w) + ptF'(t, w) \right],
\]

where \( F(t, w) = \int_0^w f(t, z) \, dz \), see e.g., [10], p. 136. Here

\[
P'(t) = t^{n-1+\alpha} \left[ \frac{np}{q + 1} - (n - p) + \frac{p\alpha}{q + 1} \right] w^{q + 1} < 0.
\]

Since \( P(0) = 0 \), and \( P(t_0) > 0 \), we have a contradiction. ♦

As before, we have

\[
1 = v(1) = aw(b),
\]

and so \( a = \frac{1}{w(b)} \), and then \( \lambda = b^{p+\alpha} w^{q-p+1}(b) \). Under the condition (2.13), the solution curve \((\lambda, u(0))\) is \( \left( b^{p+\alpha} w^{q-p+1}(b), \frac{1}{w(b)} - 1 \right) \), parameterized by \( b \in (0, \infty) \). The solution at \( b \) is \( u(r) = \frac{w(br)}{w(b)} - 1 \). It will be convenient to write the solution curve in the form

\[
(\lambda, u(0)) = \left( t^{p+\alpha} w^{q-p+1}(t), \frac{1}{w(t)} - 1 \right),
\]

parameterized by \( t \in (0, \infty) \). The solution of (2.9) at \( t \) is \( u(r) = \frac{w(tr)}{w(t)} - 1 \).

3 The equation modeling MEMS

We consider the problem (2.4), whose solution curve is given by (2.8), where \( w(t) \) is the solution of (2.7). We have \( \lambda(t) = \frac{w^{p+\alpha}}{w^{p+1}}(t) \), where \( w(t) \) is the solution of (2.7), and so

\[
\lambda'(t) = t^{p+\alpha} w^{-p-1} w^{-p-q} \left[ (\alpha + p)w - t(p + q - 1)w' \right].
\]

We are interested in the roots of the function \((\alpha + p)w - t(p + q - 1)w'\). If we set this function to zero

\[
(\alpha + p)w - t(p + q - 1)w' = 0,
\]

6
then the general solution of this equation is

\[ w(t) = ct^\beta, \quad \beta = \frac{\alpha + p}{p + q - 1}. \]

Quite remarkably, if we choose the constant \( c = c_0 = \left[ \frac{1}{p^{p-1}(p-1)((\beta-1)n-1)} \right]^{1/p-1}, \)
under the condition that

\[ (p-1)(\beta-1) + n - 1 > 0, \]

then

\[ w_0(t) = c_0t^\beta \]

also solves the equation in (2.7), along with \( w(t). \) We shall show that \( w(t), \)
the solution of the initial value problem (2.7), tends to \( w_0(t) \) as \( t \to \infty, \) and
the issue turns out to be whether \( w(t) \) and \( w_0(t) \) cross infinitely many times
as \( t \to \infty. \)

**Lemma 3.1** Assume that \( w(t) \) and \( w_0(t) \) intersect infinitely many times. Then the solution curve of (2.4) makes infinitely many turns.

**Proof:** Assuming that \( w(t) \) and \( w_0(t) \) intersect infinitely many times, let \( \{t_n\} \) denote the points of intersection. At \( \{t_n\}'s, w(t) \) and \( w_0(t) \) have different slopes (by uniqueness for initial value problems). Since \( (\alpha + p)w(t_n) - t_n(p+q-1)w_0'(t_n) = 0, \) it follows that \( (\alpha + p)w(t_n) - t_n(p+q-1)w'(t_n) < 0 \)
(> 0) if \( w(t) \) intersects \( w_0(t) \) from below (above) at \( t_n. \) Hence, on any interval \( (t_n, t_{n+1}) \) there is a point \( t_0, \) where \( (\alpha + p)w(t_0) - t_0(p+q-1)w'(t_0) = 0, \)
i.e., \( \lambda'(t_0) = 0, \) and \( t_0 \) gives a critical point. Since \( \lambda'(t_n) \) and \( \lambda'(t_{n+1}) \) have different signs, the solution curve changes its direction over \( (t_n, t_{n+1}). \)

We shall need the following well-known Sturm-Picone's comparison theorem, see e.g., p. 5 in [11].

**Lemma 3.2** Let \( u(t) \) and \( v(t) \) be respectively classical solutions of

\[ (a(t)u')' + b(t)u = 0, \]

\[ (a_1(t)v')' + b_1(t)v = 0. \]

Assume that the given differentiable functions \( a(t), a_1(t), \) and continuous functions \( b(t) \) and \( b_1(t), \) satisfy

\[ b_1(t) \geq b(t), \text{ and } 0 < a_1(t) \leq a(t) \text{ for } t \geq t_0 > 0. \]

In case \( a_1(t) = a(t) \) and \( b_1(t) = b(t) \) for all \( t, \) assume additionally that \( u(t) \)
and \( v(t) \) are not constant multiples of one another. Then, for \( t \geq t_0, v(t) \)
has a root between any two consecutive roots of \( u(t). \)
Lemma 3.3 Consider the equation

\[(a_0(t)(1 + f(t)) v')' + \frac{n-1}{t}a_0(t)(1 + f(t)) v' + b_0(t)(1 + g(t)) v = 0,\]

with given differentiable functions \(a_0(t) > 0\) and \(f(t)\), and continuous functions \(b_0(t) > 0\) and \(g(t)\). Assume that \(\lim_{t \to \infty} f(t) = \lim_{t \to \infty} g(t) = 0\), and there is an \(\epsilon > 0\) such that any solution of

\[(a_0(t)(1 + \epsilon) v')' + \frac{n-1}{t}a_0(t)(1 + \epsilon) v' + b_0(t)(1 - \epsilon) v = 0\]

has infinitely many roots. Then any solution of (3.5) has infinitely many roots.

Proof: We rewrite (3.5) in the form (3.2), with

\[a(t) = \frac{t^{n-1}a_0(t)(1 + f(t))}{1 + \epsilon},\]

and \(b(t) = \frac{t^{n-1}b_0(t)(1 + g(t))}{1 - \epsilon}\), and we rewrite (3.6) in the form (3.3), with

\[a_1(t) = \frac{t^{n-1}a_0(t)(1 + \epsilon)}{1 + \epsilon},\]

and \(b_1(t) = \frac{t^{n-1}b_0(t)(1 - \epsilon)}{1 - \epsilon}\). For large \(t\), the inequalities in (3.4) hold, and the Lemma 3.2 applies. 

The linearized equation for (2.7) is

\[(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' = -qt^\alpha w^{-q-1}z.\]

At the solution \(w = w_0(t)\), this becomes

\[(a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0,\]

with \(a_0(t) = \varphi'(w_0'(t)) = (p-1)c_0^{-2}\beta^{p-2}2^{(p-2)(p-1)}\), and \(b_0(t) = qt^\alpha w^{-q-1} = qe_0^{-q-1}\beta^{q+1}\). One simplifies (3.7) to read

\[z'' + \frac{[(p-2)(\beta-1) + n-1]}{t}z' + \frac{q\beta [(p-1)(\beta-1) + n-1]}{(p-1)t^2}z = 0,\]

which is an Euler equation! The roots of its characteristic equation,

\[r(r-1) + [(p-2)(\beta-1) + n-1] r + \frac{q\beta [(p-1)(\beta-1) + n-1]}{(p-1)} = 0,\]

are complex valued, provided that

\[[(p-2)(\beta-1) + n-2]^2 < \frac{4q\beta [(p-1)(\beta-1) + n-1]}{p-1}.\]
We write this inequality in the form

(3.8) \[ A\beta^2 + B\beta - C > 0, \]

with \( A = 4(p-1)q - (p-1)(p-2)^2 \), \( B = 4q(n-p) - 2(p-1)(p-2)(n-p) \), and \( C = (p-1)(n-p)^2 \). We shall have \( A > 0 \), provided that

(3.9) \[ 4q - (p-2)^2 > 0. \]

For (3.8) to hold, we need \( \beta = \frac{\alpha + p}{p + q - 1} \) to be greater than the larger root of this quadratic, i.e., \( \beta > -\frac{B + \sqrt{B^2 + 4AC}}{2A} \) (assuming (3.9)), which gives

(3.10) \[ \frac{\alpha + p}{p + q - 1} > \frac{(p-n)(2q - p^2 + 3p - 2) + 2|n-p|\sqrt{q(p+q-1)}}{(p-1)[4q - (p-2)^2]}. \]

**Theorem 3.1** Assume that \( q > 0 \), \( p > 1 \), with \( (p-1)(\beta - 1) + n - 1 > \beta \), and the conditions (3.9) and (3.10) hold. Then the solution curve of (2.4) makes infinitely many turns. Moreover, along this curve (as \( u(0) \to \infty \)), \( \lambda \to \lambda_0 = \frac{1}{c_{\beta 0}} = \beta_{p-1} [(p-1)(\beta - 1) + n - 1] \), and \( u(r) \) tends to \( 1 - r^\beta \) for \( r \neq 0 \), which is a solution of the equation in (2.4).

**Proof:** In view of Lemma 3.1 we need to show that \( w(t) \) and \( w_0(t) \) intersect infinitely many times. Let \( P(t) = w(t) - w_0(t) \). Then \( P(t) \) satisfies

(3.12) \[ (a(t)P')' + \frac{n-1}{t}a(t)P' + b(t)P = 0, \]

where

(3.13) \[ a(t) = \int_0^1 \varphi'(sw(t) + (1-s)w_0'(t)) \, ds, \]

(3.14) \[ b(t) = qt^\alpha \int_0^1 \frac{1}{[sw(t) + (1-s)w_0(t)]^{q+1}} \, ds. \]

We claim that it is impossible for \( P(t) \) to keep the same sign over some infinite interval \((t_0, \infty)\), and tend to a constant as \( t \to \infty \). Assuming the contrary, write

\[ a(t) = (p-1)(w_0')^{p-2} \int_0^1 \left| s \frac{w'(t)}{w_0'(t)} + (1-s) \right|^{p-2} ds = a_0(t) (1 + o(1)), \]
\[ b(t) = qt^{\alpha - 1} \int_0^1 \frac{1}{s^{q+1} \left[ \frac{w(t)}{w_0(t)} + (1-s) \right]^{q+1}} ds = b_0(t) (1 + o(1)). \]

as \( t \to \infty \). (Observe that \( \frac{w(t)}{w_0(t)} \to 1 \), since \( P(t) \) tends to a constant, and \( \frac{w'(t)}{w_0(t)} \to 1 \), by L'Hospital's rule, as \( t \to \infty \).) Since Euler's equation (3.7) has infinitely many roots on \((t_0, \infty)\), we conclude by Lemma 3.3 that \( P(t) \) must vanish on that interval too, a contradiction.

Next we show that if \( P(t_0) = 0 \), then \( P(t) \) remains bounded for all \( t > t_0 \). Assume that \( P'(t_0) < 0 \), and the case when \( P'(t_0) > 0 \) is similar. Then \( P(t) < 0 \) for \( t > t_0 \), with \( t - t_0 \) small. From (3.12), \( t^{n-1}a(t)P'(t) \) is increasing for \( t > t_0 \), so that

\[ P'(t) > -\frac{a_0}{a(t)t^{n-1}}, \quad \text{for } t > t_0 \quad \text{ (with } a_0 = -t_0^{n-1}a(t_0)P'(t_0) > 0). \]

Since solutions of the linear equation (3.12) cannot go to infinity over a bounded interval, we may assume that \( t_0 \) is large, and then by the above \( a(t) \sim a_0(t) \sim a_1 t^{(p-2)(\beta-1)} \) for \( t > t_0 \), and some \( a_1 > 0 \). It follows that for some \( a_2 > 0 \)

\[ P'(t) > -\frac{a_2}{t^{n-1 + (p-2)(\beta-1)}} = -\frac{a_2}{t_{1+\epsilon}}, \quad \text{for } t > t_0, \]

with \( \epsilon = n - 2 + (p-2)(\beta - 1) > 0 \), in view of (3.11). Integrating over \((t_0, t)\), and using that \( n \geq 3 \), we conclude the boundness of \( P(t) \), so long as \( P(t) < 0 \). If another root of \( P(t) \) is encountered, we repeat the argument. Hence, \( P(t) \) remains bounded for all \( t > t_0 \).

From the equation (3.12), we see that \( P(t) \) cannot have points of positive minimum or points of negative maximum. We claim that if \( P(t) \) has one root, it has infinitely many roots. Indeed, assume that \( P(t_1) = 0 \), and say \( P'(t_1) > 0 \). For \( t > t_1 \), \( P(t) \) remains bounded, but cannot tend to a constant. Hence, \( P(t) \) will have to turn back and become decreasing, but it cannot have a positive local minimum, or tend to a constant. Hence, \( P(t_2) = 0 \) at some \( t_2 > t_1 \), and so on.

We have \( P(0) = 1 \), so that \( (t^{n-1}a(t)P'(t))' < 0 \) for small \( t > 0 \). The function \( q(t) \equiv t^{n-1}a(t)P'(t) \) satisfies \( q(0) = 0 \) and \( q'(t) < 0 \), and so \( q(t) < 0 \). It follows that \( P'(t) < 0 \) for small \( t > 0 \). Since \( P(t) \) cannot turn around, or tend to a constant, we conclude the existence of the first root \( t_1 \) of \( P(t) \), implying the existence of infinitely many roots.
We show next that \( w(t) \to w_0(t) \) as \( t \to \infty \). Let \( t_k \) and \( t_{k+1} \) be two consecutive roots of \( P(t) \), and \( P'(t_k) < 0 \), so that \( P(t) < 0 \) on \((t_k, t_{k+1})\). Let \( \tau_k \) be the unique point of minimum of \( P(t) \) on \((t_k, t_{k+1})\). For negative \( P(t) \) we have the inequality (3.15), with \( t_k \) in place of \( t_0 \). Integrating this inequality over \((t_k, \tau_k)\), we get

\[
P(\tau_k) > \bar{c} \left( \tau_k - t_k \right) \quad \text{(with some} \quad \bar{c} > 0
\]

which implies that \(|P(\tau_k)| \to 0\), as \( k \to \infty \). The case when \( P'(t_k) > 0 \) is similar, so that \( w(t) \to w_0(t) \) along the solution curve. Since \( u(r) = 1 - \frac{w(t_r)}{w(t)} \), it follows that along the solution curve \( u(r) \) tends to \( 1 - \frac{w_0(t_r)}{w_0(t)} = 1 - r^\beta \), while \( \lambda(t) \) tends to \( \frac{1}{c^0} \).

Observe that in case \( \beta \in (0,1) \), the limiting solution \( 1 - r^\beta \) is singular, because \( u'(0) \) is not defined. Notice also that the condition (3.11) implies (3.1). Finally, observe that in case \( \beta \in (0,1) \) the condition (3.11) implies that \( n \geq 2 \). Indeed, we can rewrite (3.11) as \( n > 2\beta + p(1 - \beta) \), which is a point between \( p > 1 \), and 2.

One special case when this theorem applies is the following. Assume that \( n \geq p \), so that (3.10) becomes

\[
\alpha + p > \frac{2\sqrt{q(p+q-1)} + p^2 - 3p + 2 - 2q}{(p-1) [4q - (p-2)^2]}
\]

Then (3.10) holds, provided that

\[
2\sqrt{q(p+q-1)} + p^2 - 3p + 2 - 2q > 0,
\]

\[
4q > (p-2)^2,
\]

\[
p \leq n < p + \frac{(\alpha + p)(p-1) [4q - (p-2)^2]}{(p+q-1) \left( 2\sqrt{q(p+q-1)} + p^2 - 3p + 2 - 2q \right)}.
\]

Observe that the third inequality \( n \geq p \) implies that the condition (3.1) holds, and the second inequality is just (3.9). Hence, the three inequalities in (3.16) imply the theorem. In case \( p = 2 \), the first and the second inequalities hold automatically, while the third one gives the condition in Z. Guo and J. Wei [6].
4 The generalized Joseph-Lundgren problem

We now study the problem (2.9). Its solution curve is represented by (2.14), under the condition (2.13), where \( w(t) \) is the solution of (2.12). In particular, \( \lambda(t) = t^{p+\alpha}w^{q-p+1}(t) \), and we wish to know how many times this function changes the direction of monotonicity for \( t \in (0, \infty) \). (Here \( w(t) \) is the solution of (2.12), the generating solution.) Compute

\[
\lambda'(t) = t^{p+\alpha}w^{q-p+1}(t)((q-p+1)tw'(t))',
\]

so that we are interested in the roots of the function \((p+\alpha)w + (q-p+1)tw'\).

If we set this function to zero

\[
(p+\alpha)w + (q-p+1)tw' = 0,
\]

then the general solution of this equation is \( w(t) = at^{-\beta} \), with \( \beta = \frac{p+\alpha}{q-p+1} \).

If we choose the constant \( a \) as

\[
a = a_0 = \left[(n-p)\beta^{p-1} - (p-1)\beta^p\right]^{\frac{1}{q-p+1}},
\]

then \( w_0(t) = a_0t^{-\beta} \) is a solution of (2.12), the guiding solution (we have \((n-p)\beta^{p-1} - (p-1)\beta^p > 0 \), under the condition (2.13), if \( n > p \).

Lemma 4.1 Assume that \( w(t) \) and \( w_0(t) \) intersect infinitely many times. Then the solution curve of (2.9) makes infinitely many turns.

Proof: Indeed, assuming that \( w(t) \) and \( w_0(t) \) intersect infinitely many times, let \( \{t_n\} \) denote their points of intersection. At \( \{t_n\}'s, w(t) \) and \( w_0(t) \) have different slopes (by uniqueness for initial value problems). Since \((p+\alpha)w_0(t_n) + (q-p+1)t_nw'(t_n) = 0\), it follows that \((p+\alpha)w(t_n) + (q-p+1)t_nw'(t_n) > 0 \) if \( w(t) \) intersects \( w_0(t) \) from below (above) at \( t_n \). Hence, on any interval \((t_n, t_{n+1})\) there is a point \( t_0 \), where \((p+\alpha)w(t_0) + (q-p+1)t_0w'(t_0) = 0 \), i.e., \( \lambda'(t_0) = 0 \), and \( t_0 \) is a critical point. Since \( \lambda'(t_n) \) and \( \lambda'(t_{n+1}) \) have different signs, the solution curve changes its direction over \((t_n, t_{n+1})\).

\[\Box\]

The linearized equation for (2.12) is

\[
(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' + qt^\alpha w^{q-1}z = 0.
\]

At the solution \( w = w_0(t) \), this becomes

\[
(4.1) \quad (a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0.
\]
with $a_0(t) = \phi'(w'_0)$, and $b_0(t) = qt^\alpha w'_0$. One simplifies (4.1) to Euler’s equation

$$z'' + \left[ -\frac{(\beta + 1)(p - 2) + n - 1}{t} \right] z' + \frac{qa_0^{q-p+1}}{(p-1)\beta^{p-2}t^2} z = 0. \tag{4.2}$$

Let us consider first the case when $p = 2$ and $\alpha = 0$, and $n > 2$. Then $\beta = \frac{2}{q-1}$, $a_0 = \left[ \beta(n - \beta - 2) \right]^{\frac{1}{q-1}}$, and the equation (4.2) becomes

$$t^2 z'' + (n - 1) t z' + q\beta(n - \beta - 2) z = 0.$$  

Its characteristic equation

$$r(r - 1) + (n - 1)r + q\beta(n - \beta - 2) = 0$$

has the roots

$$r = \frac{-(n - 2) \pm \sqrt{(n - 2)^2 - 4q\beta(n - \beta - 2)}}{2}.$$  

These roots are complex if

$$(n - 2)^2 - 4q\beta(n - 2) + 4q\beta^2 < 0.$$  

On the left we have a quadratic in $n - 2$, with two positive roots. The largest value of $n - 2$, for which this inequality holds, corresponds to the larger root of this quadratic, i.e.,

$$n - 2 < \frac{4q}{q-1} + 4\sqrt{\frac{q}{q-1}}.$$  

We shall show that infinitely many solutions occur if (4.3) holds, and

$$q > \frac{n + 2}{n - 2}. \tag{4.4}$$

(The last condition ensures that the generating solution $w(t)$ is defined for all $t > 0$, by Lemma 2.1.) In terms of $n$, the conditions (4.3) and (4.4) imply

$$\frac{2 + 2q}{q - 1} < n < 2 + \frac{4q}{q - 1} + 4\sqrt{\frac{q}{q - 1}}, \tag{4.5}$$

which is the condition from [8] (it implies that $n > 2$). Thus we shall recover the following classical theorem of D.D. Joseph and T.S. Lundgren [8].
Theorem 4.1 Assume that the conditions (4.3) and (4.4) hold (or (4.5) holds). Then the solution curve of (2.9) makes infinitely many turns. Moreover, along this curve (as \( u(0) \to \infty \)), \( \lambda \to \lambda_0 = a_0^{q-1} \), and \( u(r) \) tends to \( r^{-\beta} - 1 \) for \( r \neq 0 \), which is a singular solution of the equation in (2.9).

We shall give a proof of more general result below.

For general \( p \) and \( \alpha \), the characteristic equation for (4.2) is

\[
(4.6) \quad r(r-1) + Ar + B = 0,
\]

with \( A = -\beta(p-2) + n - p + 1 \), and \( B = \frac{q(n-p)}{p-1} \beta - q\beta^2 \). The roots of (4.6)

\[
r = \frac{-(A-1) \pm \sqrt{(A-1)^2 - 4B}}{2}
\]

are complex, provided that

\[
(A-1)^2 - 4B < 0,
\]

which simplifies to

\[
(4.7) \quad (n-p)^2 - \theta(n-p) + \gamma < 0,
\]

with

\[
(4.8) \quad \theta = 2\beta(p-2) + \frac{4q\beta}{p-1}, \quad \gamma = (p-2)^2 \beta^2 + 4q\beta^2.
\]

On the left in (4.7) we have a quadratic in \( n-p \), with two positive roots. The largest value of \( n-p \), for which the inequality (4.7) holds, corresponds to the larger root of this quadratic, i.e.,

\[
(4.9) \quad n - p < \frac{\theta + \sqrt{\theta^2 - 4\gamma}}{2}.
\]

We shall show that infinitely many solutions occur if the conditions (2.13) and (4.9) hold. In terms of \( n \), the conditions (2.13) and (4.9) imply that

\[
(4.10) \quad \frac{pq + p + p\alpha}{q - p + 1} < n < p + \frac{\theta + \sqrt{\theta^2 - 4\gamma}}{2}.
\]

The first inequality in (4.10) implies that

\[
(4.11) \quad (\beta + 1)(p-2) < n - 2,
\]

which in turn gives that \( n > p \).
Theorem 4.2 Assume that the conditions (2.13) and (4.9) hold (or (4.10) holds). Then the solution curve of (2.9) makes infinitely many turns. Moreover, along this curve (as \( u(0) \to \infty \)), \( \lambda \to \lambda_0 = a_0^{q-1} \), and \( u(r) \) tends to \( r^{-\beta} - 1 \) for \( r \neq 0 \), which is a singular solution of the equation in (2.9).

Proof: In view of Lemma 4.1, we need to show that \( w(t) \) and \( w_0(t) \) intersect infinitely many times, and they tend to each other as \( t \to \infty \). Let \( P(t) = w(t) - w_0(t) \). Then \( P(t) \) satisfies

\[
(a(t)P')' + \frac{n-1}{t} a(t)P' + b(t)P = 0, 
\]

where

\[
a(t) = \int_0^1 \varphi' (sw'(t) + (1 - s)w_0'(t)) \, ds, 
\]

\[
b(t) = qt^\alpha \int_0^1 [sw(t) + (1 - s)w_0(t)]^{q-1} \, ds. 
\]

Since both \( w(t) \) and \( w_0(t) \) tend to zero as \( t \to \infty \) (see Lemma 2.1), we conclude that \( P(t) \to 0 \) as \( t \to \infty \). This simplifies the proof considerably, compared to Theorem 3.1. We claim that it is impossible for \( P(t) \) to keep the same sign over some infinite interval \((t_0, \infty)\). Assuming the contrary, write \((a_0(t) \text{ and } b_0(t) \text{ were defined in (4.1)})

\[
a(t) = (p-1) (-w_0')^{p-2} \int_0^1 \left| \frac{w'(t)}{w_0(t)} + (1 - s) \right|^{p-2} ds = a_0(t) (1 + o(1)), 
\]

\[
b(t) = qt^\alpha w_0^{q-1} \int_0^1 \left[ s \frac{w(t)}{w_0(t)} + (1 - s) \right]^{q-1} ds = b_0(t) (1 + o(1)). 
\]

as \( t \to \infty \). (Observe that \( \frac{w(t)}{w_0(t)} \to 1 \), since \( P(t) \) tends to a constant, and \( \frac{w'(t)}{w_0'(t)} \to 1 \), by L'Hopital's rule, as \( t \to \infty \).) Since Euler's equation (3.7) has infinitely many solutions on \((t_k, \infty)\), we conclude by Lemma 3.3 that \( P(t) \) must vanish on that interval too, a contradiction. It follows that \( P(t) \) has infinitely many roots, which implies that \( w(t) \) and \( w_0(t) \) have infinitely many points of intersection, and hence the solution curve makes infinitely many turns.

Since \( u(r) = \frac{w(r)}{w(t)} - 1 \), it follows that along the solution curve \( u(r) \) tends to \( \frac{w_0(r)}{w_0(t)} - 1 = r^{-\beta} - 1 \) for \( r \neq 0 \).  

\[\diamond\]
5 The generalized Gelfand problem

We now use the representation \((2.3)\) for the solution curve of \((2.1)\). In particular, \(\lambda(t) = t^{\alpha+p}e^{w(t)}\), where \(w(t)\) is the solution of \((2.2)\), and the issue is how many times this function changes its direction of monotonicity for \(t \in (0, \infty)\). Compute

\[
\lambda'(t) = te^{w} (\alpha + p + tw'),
\]

so that we are interested in the roots of the function \(\alpha + p + tw'\). If we set this function to zero

\[
\alpha + p + tw' = 0,
\]

then the solution of this equation is of course \(w(t) = a - (\alpha + p) \ln t\). Quite surprisingly, if we choose the constant \(a = a_0 = \ln \left[ (n-p)(\alpha + p)^p - 1 \right]\), assuming that \(n > p\), then

\[
w_0(t) = \ln \left[ (n-p)(\alpha + p)^p - 1 \right] - (\alpha + p) \ln t
\]

is a solution of the equation in \((2.2)\)! We shall show that \(w(t)\) (the solution of the initial value problem \((2.2)\)) tends to \(w_0(t)\) as \(t \to \infty\), and give a condition for \(w(t)\) and \(w_0(t)\) to cross infinitely many times as \(t \to \infty\).

**Lemma 5.1** Assume that \(w(t)\) and \(w_0(t)\) intersect infinitely many times. Then the solution curve of \((2.1)\) makes infinitely many turns.

**Proof:** Indeed, assuming that \(w(t)\) and \(w_0(t)\) intersect infinitely many times, let \(\{t_n\}\) denote the points of intersection. At \(\{t_n\}\)'s, \(w(t)\) and \(w_0(t)\) have different slopes (by uniqueness for initial value problems). Since \(\alpha + p + t_nw'(t_n) = 0\), it follows that \(\alpha + p + t_nw'(t_n) > 0\) \((< 0)\) if \(w(t)\) intersects \(w_0(t)\) from below (above) at \(t_n\). Hence, on any interval \((t_n, t_{n+1})\) there is a point \(t_0\), where \(\alpha + p + t_0w'(t_0) = 0\), i.e., \(\lambda'(t_0) = 0\), and \(t_0\) is a critical point. Since \(\lambda'(t_n)\) and \(\lambda'(t_{n+1})\) have different signs, the solution curve changes its direction over \((t_n, t_{n+1})\).

The linearized equation for \((2.2)\) is

\[
(\varphi'(w')z')' + \frac{n-1}{t} \varphi'(w')z' + t^\alpha e^w z = 0.
\]

At the solution \(w = w_0(t)\), this becomes

\[
(5.1) \quad (a_0(t)z')' + \frac{n-1}{t} a_0(t)z' + b_0(t)z = 0,
\]

16
with $a_0(t) = \varphi'(w_0') = \frac{(p-1)(p+\alpha)^{p-2}}{t^{p-2}}$, and $b_0(t) = t^\alpha e^{w_0} = \frac{(n-p)(p+\alpha)^{p-1}}{t^p}$.

Simplifying (5.1) gives

$$(p-1)t^2 z'' + (p-1)(n-p+1)tz' + (n-p)(p+\alpha)z = 0,$$

which is Euler’s equation! Its characteristic equation

$$(p-1)r(r-1) + (n-p+1)rt + (n-p)(p+\alpha) = 0,$$

has the roots

$$r = \frac{-(p-1)(n-p) \pm \sqrt{((p-1)(n-p) [p-1](n-p-4(p+\alpha))}}{2(p-1)}.$$

The roots are complex if $n-p > 0$, and the quantity in the square brackets is negative (the opposite inequalities lead to a vacuous condition), i.e., when

$$p < n < \frac{p^2 + 3p + 4\alpha}{p-1}.$$  

We now easily recover the following result of J. Jacobsen and K. Schmitt [7], which was a generalization of the famous theorem of D.D. Joseph and T.S. Lundgren [8].

**Theorem 5.1** Assume that the condition (5.2) holds. Then the solution curve of (2.1) makes infinitely many turns. Moreover, along this curve (as $u(0) \to \infty$), $\lambda \to e^{w_0} = (n-p)(p+\alpha)^{p-1}$, and $u(r)$ tends to $-(p+\alpha)\ln r$ for $r \neq 0$, which is a singular solution of the equation in (2.1).

**Proof:** We follow the proof of the Theorem 3.1. In view of Lemma 5.1 we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(a(t)P')' + \frac{n-1}{t}a(t)P' + b(t)P = 0,$$

where

$$a(t) = \int_0^1 \varphi'(sw'(t) + (1-s)w_0'(t)) \, ds,$$

$$b(t) = t^\alpha \int_0^1 e^{sw(t)+(1-s)w_0(t)} \, ds.$$  

Compared with the proof of the Theorem 3.1 we have a complication here: in case $P(t)$ tends to a constant $p_0$ as $t \to \infty$, we cannot conclude that $b(t) = b_0(t)(1 + o(1))$, unless $p_0 = 0$. 

---

17
We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval $(t_0, \infty)$, and tend to a constant $p_0 \neq 0$ as $t \to \infty$. Assume, on the contrary, that $P(t) > 0$ on $(t_0, \infty)$, and $\lim_{t \to \infty} P(t) = p_0 > 0$. We may assume that

\begin{equation}
(5.6) \quad P(t) > \frac{1}{2} p_0 > 0 \quad \text{on} \quad (t_1, \infty), \quad \text{with some} \quad t_1 > t_0.
\end{equation}

Write (5.3) as

\begin{equation}
(5.7) \quad \left( t^{n-1} a(t) P' \right)' = -t^{n-1} b(t) P.
\end{equation}

As before,

\begin{equation}
(5.8) \quad a(t) = a_0(t) \left( 1 + f(t) \right), \quad \text{with} \quad f(t) \to 0 \quad \text{as} \quad t \to \infty.
\end{equation}

Writing $b(t) = t^\alpha e^{\omega_0(t) \int_0^1 e^{sP(t)} ds}$, we see that

\begin{equation}
(5.9) \quad b(t) = b_0(t) \left( p_1 + g(t) \right),
\end{equation}

for some constant $c_1 > 0$. Integrating this inequality over $(t_1, t)$, we get

\begin{equation}
(5.10) \quad t^{n-1} a(t) P' < c_2 - c_3 t^{n-p} \quad \text{on} \quad (t_1, \infty),
\end{equation}

for some constants $c_2 > 0$, and $c_3 > 0$ (using that $n > p$). By (5.8)

$$a(t) > c_4 t^{-p+2} \quad \text{on} \quad (t_2, \infty),$$

for some constants $c_4 > 0$, and $t_2 > t_1$. Using this in (5.10), we have

$$P' < \frac{c_2}{c_4} t^{-n+p-1} - \frac{c_3}{c_4} t^{-1} \quad \text{on} \quad (t_2, \infty).$$

Integrating this over $(t_2, t)$, and using that $n > p$

$$P(t) < c_5 + \frac{c_2}{c_4 (-n + p)} t^{-n+p} - \frac{c_3}{c_4} \ln t < c_5 - \frac{c_3}{c_4} \ln t,$$

for some constant $c_5 > 0$. Hence, $P(t)$ has to vanish at some $t > t_2$, contradicting the assumption that $P(t) > 0$ on $(t_0, \infty)$. This proves that $p_0 = 0$. We conclude that $p_1 = 1$ in (5.9), and the rest of the proof is similar to that of Theorem 3.1. \qed

If $p = 2$ and $\alpha = 0$, the condition (5.2) becomes $2 < n < 10$, the classical condition of D.D. Joseph and T.S. Lundgren [8].
References

[1] J. Bebernes and D. Eberly, Mathematical Problems from Combustion Theory, Springer-Verlag, New York (1989).

[2] C. Budd and J. Norbury, Semilinear elliptic equations and supercritical growth, *J. Differential Equations* **68** no. 2, 169-197 (1987).

[3] I. Flores, A resonance phenomenon for ground states of an elliptic equation of Emden-Fowler type, *J. Differential Equations* **198**, no. 1, 1-15 (2004).

[4] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.* **68**, 209-243 (1979).

[5] N. Ghoussoub and Y. Guo, On the partial differential equations of electrostatic MEMS devices: stationary case, *SIAM J. Math. Anal.* **38**, no. 5, 1423-1449 (2007).

[6] Z. Guo and J. Wei, Infinitely many turning points for an elliptic problem with a singular non-linearity, *J. Lond. Math. Soc. (2)* **78**, no. 1, 21-35 (2008).

[7] J. Jacobsen and K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, *J. Differential Equations* **184**, no. 1, 283-298 (2002).

[8] D.D. Joseph and T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.* **49**, 241-269 (1972/73).

[9] P. Korman, Global solution curves for self-similar equations, *J. Differential Equations* **257**, no. 7, 2543-2564 (2014).

[10] P. Korman, Global Solution Curves for Semilinear Elliptic Equations, World Scientific, Hackensack, NJ (2012).

[11] K. Kreith, Oscillation Theory, Lecture Notes in Mathematics, Vol. 324. Springer-Verlag, Berlin-New York (1973).

[12] F. Merle and L. A. Peletier, Positive solutions of elliptic equations involving supercritical growth, *Proc. Roy. Soc. Edinburgh Sect. A* **118**, no. 1-2, 49-62 (1991).

[13] J.A. Pelesko, Mathematical modeling of electrostatic MEMS with tailored dielectric properties, *SIAM J. Appl. Math.* **62**, no. 3, 888-908 (2002).

[14] L.A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **81**, no. 2, 181-197 (1983).