Kinetic solutions
for nonlocal scalar conservation laws

Jinlong Wei\textsuperscript{a}, Jinqiao Duan\textsuperscript{b} and Guangying Lv\textsuperscript{c}\textsuperscript{*}

\textsuperscript{a} School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan, Hubei 430073, China
\textsuperscript{b} Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616
\textsuperscript{c} School of Mathematics and Statistics, Henan University, Kaifeng, Henan 475001, China

Abstract This work is devoted to examine the uniqueness and existence of kinetic solutions for a class of scalar conservation laws involving a nonlocal super-critical diffusion operator. Our proof for uniqueness is based upon the analysis on a microscopic contraction functional and the existence is enabled by a parabolic approximation. As an illustration, we obtain the existence and uniqueness of kinetic solutions for the generalized fractional Burgers-Fisher type equations. Moreover, we demonstrate the kinetic solutions’ Lipschitz continuity in time, and continuous dependence on nonlinearities and Lévy measures.

Keywords: Kinetic solution; Nonlocal conservation laws; Uniqueness; Existence; Anomalous diffusion

MSC (2010): 35L03; 35L65; 35R11

1 Introduction

The present paper is concerned with the anomalous diffusion related to the Lévy flights \cite{1,2,3}. At the macroscopic modeling level, this means the Laplacian for normal diffusion is replaced by a fractional power of the (negative) Laplacian. We consider the following partial differential equation, coupling a conservation law with an anomalous diffusion:

\[
\frac{\partial}{\partial t} \rho(t, x) + \text{div}_x F(\rho) + \nu \mathcal{L} \rho = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d,
\]

fulfilling the initial data

\[
\rho(t = 0, x) = \rho_0(x), \quad x \in \mathbb{R}^d,
\]

where \(\nu\) is a nonnegative parameter and

\[
\rho_0 \in L^1 \cap BV(\mathbb{R}^d), \quad F \in W^{1, \infty}_\text{loc}(\mathbb{R}; \mathbb{R}^d), \quad \mathcal{L} = (-\Delta)^{\frac{\alpha}{2}} \text{ or } \sum_{i=1}^d (-\partial_{x_i,x_i})^{\frac{\alpha}{2}}, \quad \alpha \in (0, 1).
\]

*Corresponding author Email: gylvmaths@henu.edu.cn
Moreover, $(−Δ_x)^α$ is the nonlocal or fractional Laplacian in $\mathbb{R}^d$ (see [4]), defined, for any $φ ∈ \mathcal{D}(\mathbb{R}^d)$, $x ∈ \mathbb{R}^d$, by

$$(-Δ_x)^α φ(x) = c(d, α) \text{P.V.} \int_{\mathbb{R}^d} \frac{φ(x) - φ(z + x)}{|z|^{d+α}} dz = c(d, α) \int_{\mathbb{R}^d} \frac{φ(x) - φ(z + x)}{|z|^{d+α}} dz,$$  \hspace{1cm} (1.4)

with $c(d, α) = α 2^{α-1} π^{-d/2} Γ(\frac{d+α}{2})/Γ(\frac{2-α}{2})$. We also denote $(-∂^2_{x,i,x,j})^α$ for the 1-dimensional fractional Laplacian.

The nonlocal Cauchy problem (1.1)-(1.2) has attracted a lot of attention for the past few years due to its broad applications in mathematical finance [4], hydrodynamics [5], acoustics [6], trapping effects in surface diffusion [7], statistical mechanics [8, 9], relaxation phenomena [10], physiology [11, 12] and molecular biology [13, 14], and its relation with stochastic analysis [15, 16, 17].

We briefly mention some recent works on well-posedness of (1.1)-(1.2), which are relevant for the present paper. We first recall a remarkable result on the scalar conservation law without diffusion ($ν = 0$):

$$\frac{∂}{∂t}ρ(t, x) + \text{div}_x F(ρ) = 0, \hspace{1cm} (t, x) ∈ (0, T) × \mathbb{R}^d. \hspace{1cm} (1.5)$$

Since (1.5) is hyperbolic, classical solutions, starting out from smooth initial values, spontaneously develop discontinuities. Hence, in general, only weak solutions may exist. But weak solutions may fail to be unique in general. By introducing an entropy formulation

$$\frac{∂}{∂t}η(ρ) + \text{div}_x Q(ρ) ≤ 0,$$  \hspace{1cm} (1.6)

Kružkov [18] showed the uniqueness results for entropy solutions in $L^∞$ space.

The general Kružkov type theory on well-posedness for nonlocal version of (1.1), i.e. (1.1), with $α ∈ (1, 2)$ (called sub-critical) was initiated by [11] for the fractional Burgers equation ($ν > 0$ and $F(ρ) = aρ^α$) in Bessel potential and/or Morrey spaces. This result was then strengthened by Droniou, Gallouët and Vovelle [12]; using a splitting method, they proved the global existence and uniqueness of regular solutions. A general result in this direction was obtained by Droniou and Imbert [4], by means of the “reverse maximum principle” and Duhamel’s formula; they proved the existence and uniqueness for regular solutions to the Hamilton-Jacobi equation.

The critical ($α = 1$) and super-critical ($α ∈ (0, 1)$) cases are more difficult. Alibaud [14] obtained well-posedness results for $L^∞$-solutions of fractional conservation laws.

Recently, Lions, Perthame and Tadmor [19] proved that, if $ρ$ is an entropy solution and belongs to $L^1$ space, then for any $v ∈ \mathbb{R}$, $u(t, x, v)$ defined by

$$u(t, x, v) = χ_ρ(v) = 1_{(0, ρ(t,x))}(v) − 1_{(ρ(t,x), 0)}(v) \hspace{1cm} (1.7)$$

satisfies

$$\frac{∂}{∂t}u(t, x, v) + f(v) · \nabla_x u(t, x, v) = \frac{∂}{∂v}m(t, x, v), \hspace{1cm} (t, x, v) ∈ (0, T) × \mathbb{R}^d × \mathbb{R}, \hspace{1cm} (1.8)$$

in $\mathcal{D}'((0, T) × \mathbb{R}^{d+1})$ and initial data

$$u(t = 0) = χ_{ρ_0}(v), \hspace{1cm} (x, v) ∈ \mathbb{R}^d × \mathbb{R}, \hspace{1cm} (1.9)$$
where \( f = F' \) and \( m \) is a nonnegative measure. But when discussing \((1.8)-(1.9)\), \( L^1 \) is a natural space for the solutions. Based upon this observation, Perthame extended Kružkov \( L^\infty \) theory for entropy solutions and developed an \( L^1 \) theory for kinetic solutions \([20, 21]\). How to generalize this \( L^1 \) theory to the Cauchy problem \((1.1)-(1.2)\) is an interesting issue.

As claimed in \([14]\), one can define ”intermediate” (for ”classical \( \Rightarrow \) entropy \( \Rightarrow \) intermediate \( \Rightarrow \) weak”) solutions for \((1.1)\) by

\[
\frac{\partial}{\partial t} \eta(t, \rho) + \text{div}_x Q(t) + \nu(-\Delta_x)^{\frac{\alpha}{2}} \eta(t, \rho) \leq 0. \tag{1.10}
\]

Previously mentioned works did not use this entropy formulation, since the doubling variable technique is not appropriate to this solution, and to a very great degree, intermediate solution is non-unique. Furthermore, as inspired by \([20, 21]\), we note that \((1.10)\) may be suitable for us to establish a relationship between \((1.1)\) and the following nonlocal linear convection-diffusion equation

\[
\frac{\partial}{\partial t} u(t, x, v) + f(v) \cdot \nabla_x u + \nu \mathcal{L} u = \frac{\partial}{\partial v} (m + n), \quad (t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}, \tag{1.11}
\]

via a kinetic formulation, with certain nonnegative measures \( m \) and \( n \). When we deal with \((1.11)\), some technical difficulties may be overcome in order to show the uniqueness for kinetic solutions. On account of this fact, in the present paper we introduce a notion of kinetic solution (analogue of \([20]\) and will prove that under the assumption \((1.3)\), the Cauchy problem \((1.1)-(1.2)\) is well-posed. It is non-trivial to get the uniqueness of the kinetic solution to \((1.1)-(1.2)\) because of the nonlocal term \( \mathcal{L}\rho \), see Section 3. Moreover, we revisit the continuous dependence on nonlinearities and Lévy measures. Comparing with the results in \([22, 23]\), we delete the assumption \( \rho_0 \in L^\infty \).

This paper is organized as follows. In Section 2, we introduce some notions on solutions for \((1.1)-(1.2)\), and then prove the uniqueness and existence of kinetic solutions in Section 3. We further discuss the regularity properties and continuous dependence (on nonlinearities and Lévy measures) for kinetic solutions in Section 4.

2 Entropy solutions and kinetic solutions

We take \( \nu > 0 \) and the analysis on \( \nu \mathcal{L} \) is the same as \( \nu(-\Delta_x)^{\frac{\alpha}{2}} \), for writing simplicity, we choose \( \mathcal{L} = (-\Delta_x)^{\frac{\alpha}{2}} \) in the present paper, and we take \( \nu = 1 \) in Section 2 and Section 3. Now we introduce some notions.

**Definition 2.1** (Entropy solution) Let \((1.3)\) hold and \( \rho_0(x) \in L^\infty(\mathbb{R}^d) \). A function \( \rho \in L^\infty([0, T) \times \mathbb{R}^d) \cap C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T); \text{BV}(\mathbb{R}^d)) \) is said to be an entropy solution of \((1.1)-(1.2)\), if for every smooth convex function \( \eta \), there are two non-negative bounded measures \( m^{\eta'}(t, x), n^{\eta'}(t, x) \) satisfying that

\[
m^{\eta'}(t, x) = \int_{\mathbb{R}} \eta''(v)m(t, x, v)dv, \quad \text{with } m(t, x, v) \text{ a nonnegative measure}, \tag{2.1}
\]

and

\[
n^{\eta'}(t, x) = \frac{\eta'(\rho(t, x))(-\Delta_x)^{\frac{\alpha}{2}} \rho(t, x)}{2} - (-\Delta_x)^{\frac{\alpha}{2}} \eta(\rho(t, x)), \tag{2.2}
\]
such that the following identity holds
\[
\frac{\partial}{\partial t}\eta(\rho) + \text{div}_x Q(\rho) + (-\Delta_x)\frac{\partial}{\partial t}\eta(\rho) = -2m_\varepsilon''(t,x) - 2n_\varepsilon(t,x),
\]
(2.3)
in \mathcal{D}'((0,T) \times \mathbb{R}^d) with \(\eta(\rho(t=0)) = \eta(\rho_0)\), where \(Q(\rho) = \int_0^\rho \eta'(v)f(v)dv\).

**Remark 2.1**

(i) We define an entropy solution by the identity (2.3), and the source or motivation for this definition comes from the \(\varepsilon \to 0\) limit of the following equation directly,
\[
\frac{\partial}{\partial t}\rho_\varepsilon(t,x) + \text{div}_x F(\rho_\varepsilon) + (-\Delta_x)\frac{\partial}{\partial t}\rho_\varepsilon - \varepsilon \Delta \rho_\varepsilon = 0.
\]
(2.4)
Indeed, if one multiplies equation (2.4) by \(\eta'(\rho_\varepsilon)\), it yields
\[
\frac{\partial}{\partial t}\eta(\rho_\varepsilon) + \text{div}_x Q(\rho_\varepsilon) + \eta'(\rho_\varepsilon)(-\Delta_x)\frac{\partial}{\partial t}\rho_\varepsilon = \varepsilon \eta''(\rho_\varepsilon) \Delta \rho_\varepsilon.
\]
(2.5)
With the help of the chain rule,
\[
\varepsilon \eta'(\rho_\varepsilon) \Delta \rho_\varepsilon = \varepsilon \Delta \eta(\rho_\varepsilon) - \eta''(\rho_\varepsilon)|\nabla \rho_\varepsilon|^2 =: \varepsilon \Delta \eta(\rho_\varepsilon) - 2m_\varepsilon''.
\]
(2.6)
Moreover, since \(\eta\) is convex, by (1.4),
\[
\eta'(\rho_\varepsilon)(-\Delta_x)\frac{\partial}{\partial t}\rho_\varepsilon(t,x) \geq c_0 \int_{\mathbb{R}^d} \frac{\eta(\rho_\varepsilon(t,x)) - \eta(\rho_\varepsilon(t,z+x))}{|z|^{d+\alpha}} dz = (-\Delta_x)\frac{\partial}{\partial t}\eta(\rho_\varepsilon(t,x)).
\]
(2.7)
Combining (2.6) and (2.7), we conclude from (2.3) that
\[
\frac{\partial}{\partial t}\eta(\rho_\varepsilon) + \text{div}_x Q(\rho_\varepsilon) + (-\Delta_x)\frac{\partial}{\partial t}\eta(\rho_\varepsilon(t,x)) = \varepsilon \Delta \eta(\rho_\varepsilon) - 2m_\varepsilon'' - 2n_\varepsilon',
\]
(2.8)
with non-negative measures \(m_\varepsilon''\) and \(n_\varepsilon'\). So the vanishing viscosity limit in the preceding identity leads to (2.3).

(ii) Another motivation to define entropy solutions is from [24] Definition 2.2 and Lemma 2.4. Since
\[
\|(-\Delta_x)\frac{\partial}{\partial t}\rho(t,x)\|_{L^1(\mathbb{R}^d)} \leq C\|\rho(t)\|_{L^1(\mathbb{R}^d)}^{1-\alpha}\|\rho(t)\|_{BV(\mathbb{R}^d)}^{\alpha},
\]
(2.9)
and
\[
\|(-\Delta_x)\frac{\partial}{\partial t}\eta(\rho(t,x))\|_{L^1(\mathbb{R}^d)} \leq C\|\rho(t)\|_{L^1(\mathbb{R}^d)}^{1-\alpha}\|\rho(t)\|_{BV(\mathbb{R}^d)}^{\alpha},
\]
(2.10)
we have
\[
(-\Delta_x)\frac{\partial}{\partial t}\eta(\rho) + 2n_\varepsilon'(t,x) = \eta'(\rho(t,x))(-\Delta_x)\frac{\partial}{\partial t}\rho(t,x).
\]

The present definition is the same as Definition 2.2 in [24]. The only difference is that, here we define entropy solutions by an identity but not an inequality. As mentioned in introduction, the intermediate solution may fail to be unique, so we give an explicit formula for dissipation measure \(n\), and it comes from [25] Definition 2.1 for non-isotropic degenerate parabolic-hyperbolic equation:
\[
\frac{\partial}{\partial t}\rho(t,x) + \text{div}_x F(\rho) + \nabla \cdot (A(\rho)\nabla \rho) = 0, \quad A(\rho) = \sigma(\rho)\sigma(\rho)\top, \quad \sigma \in \mathbb{R}^{d\times J}.
\]
(2.11)
(iii) The main ingredient in Definition 2.1 of [25] is the chain rule for \( \rho \), i.e.
\[
\psi(t) \sigma_{i,j}(\rho) \partial_{x_i} \rho = \partial_{x_i} \beta_{i,j}^\psi(\rho), \quad \forall \psi \in D_+(\mathbb{R}),
\]
where \( \beta_{i,j}^\psi \) is a special function, see [25].

Even though \( \rho \in L^1 \cap L^\infty \) does not make the left hand side meaningful, the chain rule ensures that all manipulations legitimate in (2.12). When the degenerate parabolic operator is replaced by a fractional operator, this chain rule may no longer hold. However, if \( \rho \in L^\infty([0,T]; BV(\mathbb{R}^d)) \) and \( \alpha \in (0,1) \), with the help of (2.9)–(2.10), for any convex smooth function \( \eta \), we have
\[
0 \leq \frac{\eta'(\rho(t,x))(-\Delta_x)^{\frac{\alpha}{2}} \rho(t,x) - (-\Delta_x)^{\frac{\alpha}{2}} \eta(\rho(t,x))}{2} \in L^1([0,T) \times \mathbb{R}^d).
\]

Note that (2.9) and (2.10) is meaningful if and only if \( \eta' \) is bounded, then the microscopic equation (1.11) is legitimate for \( \rho \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T]; BV(\mathbb{R}^d)) \). By this observation, we introduce the following definition.

**Definition 2.2 (Kinetic solution)** Let (1.3) hold. A function \( \rho \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T]; BV(\mathbb{R}^d)) \) fulfills the condition (1.11), if \( \psi \) is given by (1.7), satisfies (1.11), (1.4) in \( D'(0,T) \times \mathbb{R}^{d+1} \) and

(i) the non-negative measure \( n(t,x,v) \) is given by
\[
n(t,x,v) = \frac{\text{sgn}(\rho(t,x) - v)(-\Delta_x)^{\frac{\alpha}{2}} \rho(t,x) - (-\Delta_x)^{\frac{\alpha}{2}} |\rho(t,x) - v|}{2};
\]

(ii) the nonnegative measure \( m + n \) fulfills the condition
\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (m + n)(dt,dx,dv) \in L^\infty_0(\mathbb{R}).
\]

**Remark 2.2** Note that \( \rho \in C([0,T]; L^1(\mathbb{R}^d)) \), \( n(t,x,v) \in L^\infty_0(\mathbb{R}; L^1_{x,v}([0,T] \times \mathbb{R})), \) Hence the nonnegative measure \( m \) in Definition 2.2 is continuous in \( t \) in the sense that
\[
\lim_{s \to t} \int_0^s \varphi(x) \psi(v)m(ds,dx,dv) = \int_0^t \varphi(x) \psi(v)m(ds,dx,dv),
\]
for \( \varphi \in D(\mathbb{R}^d) \) and \( \psi \in D(\mathbb{R}) \), which imply that the preceding definition is equivalent to
\[
\int_{\mathbb{R}^{d+1}} u(t,x,v)\varphi(x) \psi(v)dx dv - \int_{\mathbb{R}^{d+1}} \chi_{\rho_0}(v)\varphi(x) \psi(v)dx dv
\]
\[
= \int_0^t \int_{\mathbb{R}^{d+1}} u(s,x,v)f(v) \cdot \nabla \varphi(x) \psi(v)dx dv ds - \int_0^t \int_{\mathbb{R}^{d+1}} u(s,x,v)L \varphi(x) \psi(v)dx dv ds
\]
\[
- \int_0^t \int_{\mathbb{R}^{d+1}} \varphi(x) \frac{\partial}{\partial v} \psi(v)(m + n)(ds,dx,dv),
\]
for \( t \in (0,T), \varphi \in D(\mathbb{R}^d) \) and \( \psi \in D(\mathbb{R}) \).

Now, we are in a position to show the relationship between entropy solutions and kinetic solutions for (1.1)–(1.2).
Theorem 2.1  **(Kinetic formulation)** Let (1.3) be valid, $\rho_0 \in L^\infty(\mathbb{R}^d)$ and $u(t, x, v) = \chi_\rho(v)$.

(i) If $\rho$ is an entropy solution of (1.1)-(1.2), then it is also a kinetic solution. Besides, the nonnegative measures $m$ and $n$ are bounded and supported in $[0, T] \times \mathbb{R}^d \times [-M, M]$ for $M = \|\rho\|_{L^\infty([0, T] \times \mathbb{R}^d)}$, and further satisfy (2.15).

(ii) If $\rho$ is a kinetic solution of (1.1)-(1.2), then it is an entropy solution as well.

**Proof.** By the following relationship:

$$\int_\mathbb{R} S'(v)u(t, x, v)dv = S(\rho(t, x)), \quad \forall S \in C^1(\mathbb{R}),$$

we clearly get the conclusion (ii). It remains to verify (i).

Indeed, if $\rho$ is an entropy solution, then from (2.3), by an approximation, we deduce that

$$\frac{\partial}{\partial t}\eta(\rho, v) + \text{div}_x Q(\rho, v) + (-\Delta_x)^{\frac{\alpha}{2}}|\rho - v| = -2m - 2n,$$

where $\eta(\rho, v) = |\rho - v|, Q(\rho, v) = \text{sgn}(\rho - v)[F(\rho) - F(v)]$.

By differentiating (2.16) in $v$ in the distributions sense, we obtain the equation (1.11). Besides, from (2.16), $m$ and $n$ are nonnegative and supported in $[0, T] \times \mathbb{R}^d \times [-M, M]$.

Furthermore, if one integrates the identity (2.16) in $x$ on $\mathbb{R}^d$, then

$$\int_0^T \int_{\mathbb{R}^d} (m + n)(dt, dx, v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\rho_0(x) - v| - |\rho(T, x) - v|dx \leq \sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |\rho(t, x)|dx.$$

Therefore $m + n$ is bounded and (2.15) holds. We complete the proof.

**Remark 2.3** The proof here is analogue to that for

$$\frac{\partial}{\partial t}\rho(t, x) + \text{div}_x F(\rho) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d,$$

in [19]; so we omit some details.

(ii) Observe that

$$(\rho - v)_+ = \frac{|\rho - v| + (\rho - v)}{2}, \quad (\rho - v)_- = \frac{|\rho - v| - (\rho - v)}{2},$$

Thus if $\rho$ is an entropy solution, we can take entropy-entropy flux pairs by $(\eta(\rho, v) = (\rho - v)_+, Q(\rho, v) = \text{sgn}(\rho - v)_+[F(\rho) - F(v)])$ and $(\eta(\rho, v) = (\rho - v)_-, Q(\rho, v) = \text{sgn}(\rho - v)_-[F(\rho) - F(v)])$ respectively, and we can estimate that

$$\int_0^T \int_{\mathbb{R}^d} (m + n)(dt, dx, v) \leq \frac{1}{2} \int_{\mathbb{R}^d} |(\rho_0(x) - v)_+ - (\rho(T, x) - v)_+|dx \quad (2.17)$$

and

$$\int_0^T \int_{\mathbb{R}^d} (m + n)(dt, dx, v) \leq \frac{1}{2} \int_{\mathbb{R}^d} |(\rho_0(x) - v)_- - (\rho(T, x) - v)_-|dx. \quad (2.18)$$

From (2.17) and (2.18), we derive

$$\lim_{v \to \pm \infty} \int_0^T \int_{\mathbb{R}^d} (m + n)(dt, dx, v) = 0,$$

i.e. (2.15) is true. Observing that the right hand sides in (2.17) and (2.18) are meaningful if $\rho$ is a kinetic solution. Thus in Definition 2.2, we add the condition (2.15).
where $b$ is a given function.

**Theorem 3.1.** Our main result. In this section, we are interested in the Cauchy problem (1.1)-(1.2) and it is ready for us to state the result.

### Remark 2.4
The preceding result holds as well for the non-homogeneous fractional convection-diffusion problem

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t}\rho(t,x) + \text{div}_x F(\rho) + (-\Delta_x)^{\frac{\alpha}{2}} B(\rho) = A(\rho), \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\
\rho(t=0,x) = \rho_0(x), \quad x \in \mathbb{R}^d,
\end{array} \right.
$$

(2.19)

if

$$
A(0) = 0, \quad A \in W^{1,1}_0(\mathbb{R}), \quad B(0) = 0, \quad B \in W^{1,\infty}_0(\mathbb{R}), \quad B' \geq 0.
$$

(2.20)

But now the Cauchy problem ([1.3]) with ([1.1]) should be replaced by

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} u + f(v) \cdot \nabla_x u + A(v) \frac{\partial}{\partial t} u + b(v)(-\Delta_x)^{\frac{\alpha}{2}} u = \frac{\partial}{\partial t} (m + n), \quad (t,x,v) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}, \\
u(t=0) = \chi_{\rho_0}(v), \quad (x,v) \in \mathbb{R}^d \times \mathbb{R},
\end{array} \right.
$$

(2.21)

where $b(v) = B' (v)$,

$$
n(t,x,v) = \frac{\text{sgn}(\rho(t,x) - v) (-\Delta_x)^{\frac{\alpha}{2}} B(\rho(t,x)) - (-\Delta_x)^{\frac{\alpha}{2}} |B(\rho(t,x)) - B(v)|}{2}.
$$

(2.22)

### 3 Uniqueness and existence of kinetic solutions

In this section, we are interested in the Cauchy problem ([1.1]-[1.2]) and it is ready for us to state our main result.

**Theorem 3.1.** Let ([1.3]) hold. Then there is a unique kinetic solution of the nonlocal Cauchy problem ([1.1])-([1.2]).

**Proof.** (Uniqueness) Let $\rho_i$ ($i = 1, 2$) be kinetic solutions of ([1.1])-([1.2]). Then for both $u_i = \chi_{\rho_i}$ ($i = 1, 2$)

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} u_i(t,x,v) + f(v) \cdot \nabla_x u_i + (-\Delta_x)^{\frac{\alpha}{2}} u_i = \frac{\partial}{\partial t} m_i(t,x,v) + \frac{\partial}{\partial v} n_i(t,x,v), \\
u_i(t=0) = \chi_{\rho_0}(v) \in L^1(\mathbb{R}^{d+1}),
\end{array} \right.
$$

(3.1)

with the nonnegative measures $m_i, n_i$ satisfying (2.14) and (2.15).

We set

$$
n_1^1(t,x,v) = \text{sgn}(\rho_1(t,x) - v) (-\Delta_x)^{\frac{\alpha}{2}} \rho_1(t,x), \quad n_1^2(t,x,v) = -(-\Delta_x)^{\frac{\alpha}{2}} |\rho_1(t,x) - v|.
$$

(3.2)

Then

$$
n_1^1, n_1^2 \in L^\infty_0(\mathbb{R}_v; L^1([0,T) \times \mathbb{R}^d_x)) \quad \text{and} \quad 2n_i = n_i^1 + n_i^2.
$$

(3.3)

For $\varepsilon_1, \varepsilon_2, \sigma > 0$, define

$$
\varrho_{1,\varepsilon_1}(t) = \frac{1}{\varepsilon_1} \varrho_1 \left( \frac{t}{\varepsilon_1} \right), \quad \varrho_{2,\varepsilon_2}(x) = \frac{1}{\varepsilon_2} \varrho_2 \left( \frac{x}{\varepsilon_2} \right), \quad \varrho_{3,\sigma}(v) = \frac{1}{\sigma} \varrho_3 \left( \frac{v}{\sigma} \right),
$$

here $\varrho_1$, $\varrho_2$ and $\varrho_3$ are three nonnegative normalized regularizing kernels, satisfying

$$
\text{supp} \varrho_1 \subset (-1, 0), \quad \text{supp} \varrho_2 \subset B_1(0), \quad \text{supp} \varrho_3 \subset (-1, 1).
$$
Then \( u_{i,\varepsilon} := u_i * \varrho_{1,\varepsilon} * \varrho_{2,\varepsilon} \) \((i = 1, 2)\) yield
\[
\begin{cases}
\quad \frac{\partial}{\partial t} u_{i,\varepsilon} + f(v) \cdot \nabla x u_{i,\varepsilon} + (-\Delta x)^{\alpha} u_{i,\varepsilon} = \frac{\partial}{\partial v} m_{i,\varepsilon}(t, x, v) + \frac{\partial}{\partial v} n_{i,\varepsilon}(t, x, v), \\
\quad u_{i,\varepsilon}(t = 0) = \chi_{\varrho_0} * \varrho_{2,\varepsilon}(x),
\end{cases}
\tag{3.4}
\]
and \( u_{i,\varepsilon}^\sigma := u_i * \varrho_{1,\varepsilon} * \varrho_{2,\varepsilon} \) \((i = 1, 2)\) fulfill
\[
\begin{cases}
\quad \frac{\partial}{\partial t} u_{i,\varepsilon}^\sigma + f(v) \cdot \nabla x u_{i,\varepsilon}^\sigma + (-\Delta x)^{\alpha} u_{i,\varepsilon}^\sigma = \frac{\partial}{\partial v} m_{i,\varepsilon}^\sigma + \frac{\partial}{\partial v} n_{i,\varepsilon}^\sigma + R_{i,\varepsilon}^\sigma, \\
\quad u_{i,\varepsilon}^\sigma(t = 0) = \chi_{\varrho_0} * \varrho_{2,\varepsilon}(x) * \varrho_{3,\sigma}(v),
\end{cases}
\tag{3.5}
\]
with
\[
R_{i,\varepsilon}^\sigma = f(v) \cdot \nabla x u_{i,\varepsilon}^\sigma - (f(v) \cdot \nabla x u_{i,\varepsilon}) * \varrho_{3,\sigma},
\tag{3.6}
\]
here we define \( u_i(t, x, v) := 0 \), when \( t \in [0, T) \).

In view of \(|u_{i,\varepsilon}| = \text{sgn}(v) u_{i,\varepsilon}\), we get from \((3.4)-(3.5)\) that
\[
\frac{\partial}{\partial t}|u_{i,\varepsilon}| + f(v) \cdot \nabla x |u_{i,\varepsilon}| + (-\Delta x)^{\alpha} |u_{i,\varepsilon}| = \text{sgn}(v) \left[ \frac{\partial}{\partial v} m_{i,\varepsilon} + \frac{\partial}{\partial v} n_{i,\varepsilon} \right],
\]
and
\[
\frac{\partial}{\partial t}(u_{i,\varepsilon}^\sigma u_{i,\varepsilon}^\sigma) + f \cdot \nabla x (u_{i,\varepsilon}^\sigma u_{i,\varepsilon}^\sigma) + [u_{i,\varepsilon}^\sigma (-\Delta x)^{\alpha} u_{i,\varepsilon}^\sigma + u_{i,\varepsilon}^\sigma (-\Delta x)^{\alpha} u_{i,\varepsilon}^\sigma]
\]
\[
= u_{i,\varepsilon}^\sigma \frac{\partial}{\partial v}[m_{i,\varepsilon}^\sigma + n_{i,\varepsilon}^\sigma] + u_{i,\varepsilon}^\sigma \frac{\partial}{\partial v}[m_{i,\varepsilon}^\sigma + n_{i,\varepsilon}^\sigma] + u_{i,\varepsilon}^\sigma R_{i,\varepsilon}^\sigma + u_{i,\varepsilon}^\sigma R_{i,\varepsilon}^\sigma.
\]
Hence
\[
\frac{\partial}{\partial t}|u_{i,\varepsilon}| + |u_{i,\varepsilon}| - 2u_{i,\varepsilon}^\sigma u_{i,\varepsilon}^\sigma + f(v) \cdot \nabla x |u_{i,\varepsilon}| + |u_{i,\varepsilon}| - 2u_{i,\varepsilon}^\sigma u_{i,\varepsilon}^\sigma + (-\Delta x)^{\alpha} |u_{i,\varepsilon}| + |u_{i,\varepsilon}|
\]
\[
= I_1 + I_2 - 2u_{i,\varepsilon}^\sigma R_{i,\varepsilon}^\sigma - 2u_{i,\varepsilon}^\sigma R_{i,\varepsilon}^\sigma,
\tag{3.7}
\]
where
\[
I_1(t, x, v) = \text{sgn}(v) \left[ \frac{\partial}{\partial v}[m_{i,\varepsilon}^\sigma + m_{i,\varepsilon}^\sigma] - 2[u_{i,\varepsilon}^\sigma \frac{\partial}{\partial v} m_{i,\varepsilon}^\sigma + u_{i,\varepsilon}^\sigma \frac{\partial}{\partial v} m_{i,\varepsilon}^\sigma] \right]
\tag{3.8}
\]
and
\[
I_2(t, x, v) = \text{sgn}(v) \left[ \frac{\partial}{\partial v}[n_{i,\varepsilon}^\sigma + n_{i,\varepsilon}^\sigma] - 2[u_{i,\varepsilon}^\sigma \frac{\partial}{\partial v} n_{i,\varepsilon}^\sigma + u_{i,\varepsilon}^\sigma \frac{\partial}{\partial v} n_{i,\varepsilon}^\sigma] \right].
\tag{3.9}
\]

Let \( \theta \) and \( \xi \) be two cut-off functions, with variables \( x \) and \( v \) respectively, i.e. \( \theta \in \mathcal{D}(\mathbb{R}^d), \xi \in \mathcal{D}(\mathbb{R}) \),
\[
0 \leq \theta, \xi \leq 1, \quad \theta = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2 \end{cases}, \quad \xi = \begin{cases} 1, & |v| \leq 1, \\ 0, & |v| \geq 2, \end{cases}
\]
and for \( p, k \in \mathbb{N} \), we denote by \( \theta_p(x) = \theta(\frac{x}{p}) \) and \( \xi_k(v) = \xi(\frac{v}{k}) \).

Now let us estimate the right hand sides in \((3.7)\). Initially, we have the following estimate for the last two error terms,
\[
\lim_{\sigma \to 0} \int_{\mathbb{R}^{d+1}} [u_{i,\varepsilon}^\sigma R_{i,\varepsilon}^\sigma + u_{i,\varepsilon}^\sigma R_{i,\varepsilon}^\sigma] \xi_k(v) \theta_p(x) dx dv = 0
\tag{3.11}
\]

for fixed $\varepsilon_1, \varepsilon_2, k$ and $p$. It remains to reckon the others.

Note that

$$\lim_{\sigma \to 0} \int \partial_v u_{1,\varepsilon}^\sigma(t, x, v)m_{2,\varepsilon}^\sigma(t, x, v)\xi_k(v)\theta_p(x)dx dv = \int \partial_v u_{1,\varepsilon}(t, x, v)m_{2,\varepsilon}(t, x, v)\xi_k(v)\theta_p(x)dx dv$$

Moreover, since $m_{2,\varepsilon} = 0, \varepsilon_2, m_{1,\varepsilon} \geq 0$, from (3.14), it leads to

$$\int I_1(t, x, v)\xi_k(v)\theta_p(x)dx dv \leq \int \xi_k(v)\theta_p(x)dx.$$  \hspace{1cm} (3.13)

Therefore

$$\lim_{\sigma \to 0} \int I_1(t, x, v)\xi_k(v)\theta_p(x)dx dv \leq 2 \int [u_{1,\varepsilon}m_{2,\varepsilon} + u_{2,\varepsilon}m_{1,\varepsilon}]\xi_k\partial_v\xi_k\theta_p dx dv - 2 \int [m_{1,\varepsilon} + m_{2,\varepsilon}]\theta_p\xi_kdxdv. \hspace{1cm} (3.14)$$

From (3.10), one can deduce that $\text{sgn}(v)\xi'(v) \leq 0$. By virtue of (1.7) and (3.10), it follows that

$$u_{1,\varepsilon}(t, x, v)\partial_v\xi_k(v) = \frac{1}{k} \int_0^t \int \xi_k(v)\theta_p(x)dx dv \leq 0$$

and

$$u_{2,\varepsilon}(t, x, v)\partial_v\xi_k(v) = \frac{1}{k} \int_0^t \int \xi_k(v)\theta_p(x)dx dv \leq 0.$$

Moreover, since $m_{1,\varepsilon}, m_{2,\varepsilon} \geq 0$, from (3.14), it leads to

$$\lim_{\sigma \to 0} \int I_1(t, x, v)\xi_k(v)\theta_p(x)dx dv \leq -2 \int [m_{1,\varepsilon} + m_{2,\varepsilon}]\theta_p(\text{sgn}(v))\partial_v\xi_k dx dv.$$

In view of (2.15), we have

$$\lim_{k \to \infty} \lim_{p \to \infty} \lim_{\sigma \to 0} \int I_1(t, x, v)\xi_k(v)\theta_p(x)dx dv \leq 0. \hspace{1cm} (3.15)$$

Now let us estimate the term $I_2$ and firstly, via integration by parts,

$$-2 \int u_{1,\varepsilon}^\sigma(t, x, v)\partial_v m_{2,\varepsilon}^\sigma(t, x, v)\xi_k(v)\theta_p(x)dx dv$$
\[ \begin{align*}
&= 2 \int_{\mathbb{R}^{d+1}} \partial_v u_{1,\varepsilon}^1 n_{2,\varepsilon}^1 \vartheta_p(x) \xi_k(v) dx dv + 2 \int_{\mathbb{R}^{d+1}} u_{1,\varepsilon}^1(t, x, v) n_{2,\varepsilon}^1(t, x, v) \partial_v \xi_k(v) \vartheta_p(x) dx dv \\
&= 2 \int_{\mathbb{R}^{d+1}} \partial_v \int_0^t \int_{\mathbb{R}^{d+1}} \chi_{\rho_1(s, y)}(y - z) \varrho_{1,\varepsilon_1}(t - s) \varrho_{2,\varepsilon_2}(x - y) \varrho_{3,\varepsilon}(z) dy ds dz \\
&\times n_{2,\varepsilon}^1(t, x, v) \vartheta_p(x) \xi_k(v) dx dv + 2 \int_{\mathbb{R}^{d+1}} u_{1,\varepsilon}^1(t, x, v) n_{2,\varepsilon}^1(t, x, v) \partial_v \xi_k(v) \vartheta_p(x) dx dv \\
&= 2 \int_{\mathbb{R}^{d+1}} \int_0^t \int_{\mathbb{R}^{d+1}} [\delta(v - z) - \delta(v - z - \rho_1(s, y))] \varrho_{1,\varepsilon_1}(t - s) \varrho_{2,\varepsilon_2}(x - y) \varrho_{3,\varepsilon}(z) dy ds dz \\
&\times n_{2,\varepsilon}^1 \vartheta_p(x) \xi_k(v) dx dv + 2 \int_{\mathbb{R}^{d+1}} u_{1,\varepsilon}^1(t, x, v) n_{2,\varepsilon}^1(t, x, v) \partial_v \xi_k(v) \vartheta_p(x) dx dv \\
&= 2 \int_{\mathbb{R}^{d+1}} n_{2,\varepsilon}^1(t, x, z) \varrho_{3,\varepsilon}(z) \vartheta_p(x) dx dz \\
&- 2 \int_{\mathbb{R}^{d+1}} [n_{2,\varepsilon}^1(t, x, \rho_1(\cdot, \cdot) + z) \xi_k(\rho_1(\cdot, \cdot) + z)] \varrho_{3,\varepsilon}(z) * \varrho_{1,\varepsilon_1} * \varrho_{2,\varepsilon_2}(t, x) \vartheta_p(x) dx dz \\
&+ 2 \int_{\mathbb{R}^{d+1}} u_{1,\varepsilon}^1(t, x, v) n_{2,\varepsilon}^1(t, x, v) \partial_v \xi_k(v) \vartheta_p(x) dx dv. \quad (3.16)
\end{align*} \]

An analogue calculation also implies that

\[ \begin{align*}
&= 2 \int_{\mathbb{R}^{d+1}} n_{1,\varepsilon}^1(t, x, z) \varrho_{3,\varepsilon}(z) \vartheta_p(x) dx dz \\
&- 2 \int_{\mathbb{R}^{d+1}} [n_{1,\varepsilon}^1(t, x, \rho_2(\cdot, \cdot) + z) \xi_k(\rho_2(\cdot, \cdot) + z)] \varrho_{3,\varepsilon}(z) * \varrho_{1,\varepsilon_1} * \varrho_{2,\varepsilon_2}(t, x) \vartheta_p(x) dx dz \\
&+ 2 \int_{\mathbb{R}^{d+1}} u_{1,\varepsilon}^1(t, x, v) n_{1,\varepsilon}^1(t, x, v) \partial_v \xi_k(v) \vartheta_p(x) dx dv. \quad (3.17)
\end{align*} \]

By (3.3), (3.16), (3.17), we get

\[ \begin{align*}
\lim_{\varepsilon \to 0} &\int_{\mathbb{R}^{d+1}} I_2(t, x, v) \xi_k(v) \vartheta_p(x) dx dv \\
&= -2 \int_{\mathbb{R}^d} [n_2^1(t, x, 0) + n_1^2(t, x, 0)] \vartheta_p dx - 2 \int_{\mathbb{R}^{d+1}} [n_{1,\varepsilon}^1(t, x, v) + n_{2,\varepsilon}^1(t, x, v)] \text{sgn}(v) \partial_v \xi_k \vartheta_p dx dv \\
&\quad - 2 \int_{\mathbb{R}^d} [n_1^1(t, x, \rho_1(\cdot, \cdot)) \xi_k(\rho_1(\cdot, \cdot)) + n_1^1(t, x, \rho_2(\cdot, \cdot)) \xi_k(\rho_2(\cdot, \cdot))] \varrho_{1,\varepsilon_1} * \varrho_{2,\varepsilon_2}(t, x) \vartheta_p(x) dx \\
&\quad + 2 \int_{\mathbb{R}^{d+1}} [u_{1,\varepsilon}^1(t, x, v) n_{2,\varepsilon}^1(t, x, v) + u_{2,\varepsilon}^1(t, x, v) n_{1,\varepsilon}^1(t, x, v)] \partial_v \xi_k(v) \vartheta_p(x) dx dv \\
&= -2 \int_{\mathbb{R}^d} [\rho_1(t, x)]_\varepsilon + [\rho_2(t, x)]_\varepsilon (-\Delta_x)^{\frac{\mu}{2}} \vartheta_p(x) dx \\
&\quad - 2 \int_{\mathbb{R}^{d+1}} [n_{1,\varepsilon}^1(t, x, v) + n_{2,\varepsilon}^1(t, x, v)] \text{sgn}(v) \partial_v \xi_k(v) \vartheta_p(x) dx dv \\
&\quad - 2 \int_{\mathbb{R}^d} [n_1^1(t, x, \rho_1(\cdot, \cdot)) \xi_k(\rho_1(\cdot, \cdot)) + n_1^1(t, x, \rho_2(\cdot, \cdot)) \xi_k(\rho_2(\cdot, \cdot))] \varrho_{1,\varepsilon_1} * \varrho_{2,\varepsilon_2}(t, x) \vartheta_p(x) dx \\
&\quad + 2 \int_{\mathbb{R}^{d+1}} [u_{1,\varepsilon}^1(t, x, v) n_{2,\varepsilon}^1(t, x, v) + u_{2,\varepsilon}^1(t, x, v) n_{1,\varepsilon}^1(t, x, v)] \partial_v \xi_k(v) \vartheta_p(x) dx dv. \quad (3.18)
\end{align*} \]
For $\varepsilon_1, \varepsilon_2$ fixed,

$$\left[ u_{1,\varepsilon}(t, x, v)n_{1,\varepsilon}^1(t, x, v) + u_{2,\varepsilon}(t, x, v)n_{1,\varepsilon}^1(t, x, v) \right] \in C([0, T]; L^1(\mathbb{R}^{d+1})).$$

So

$$\lim_{k \to \infty} \lim_{p \to \infty} \int_{\mathbb{R}^{d+1}} \left[ u_{1,\varepsilon}(t, x, v)n_{1,\varepsilon}^1(t, x, v) + u_{2,\varepsilon}(t, x, v)n_{1,\varepsilon}^1(t, x, v) \right] \partial_v \xi_k(v) \theta_p(x) dx dv = 0. \quad (3.19)$$

By (3.3),

$$\lim_{k \to \infty} \lim_{p \to \infty} \int_{\mathbb{R}^{d+1}} \left[ n_{1,\varepsilon}(t, x, v) + n_{2,\varepsilon}(t, x, v) \right] \text{sgn}(v) \partial_v \xi_k(v) \theta_p(x) dx dv = 0. \quad (3.20)$$

Combining (3.19) and (3.20), from (3.18), we assert that

$$\lim_{k \to \infty} \lim_{p \to \infty} \lim_{\sigma \to 0} \int_{\mathbb{R}^{d+1}} I_2(t, x, v) \xi_k(v) \theta_p(x) dx dv$$

$$= -2 \int_{\mathbb{R}^d} n_{1,\varepsilon}^1(t, x, \rho_1(\cdot, \cdot)) + n_{1,\varepsilon}^1(t, x, \rho_2(\cdot, \cdot)) \ast \varrho_{1, \varepsilon} \ast \varrho_{2, \varepsilon}(t, x) dx. \quad (3.21)$$

From (3.21), if we take $\varepsilon_1 \downarrow 0$, $\varepsilon_2 \downarrow 0$ in turn, then

$$\lim_{\varepsilon_2 \to 0} \lim_{\varepsilon_1 \to 0} \lim_{k \to \infty} \lim_{p \to \infty} \lim_{\sigma \to 0} \int_{\mathbb{R}^{d+1}} I_2(t, x, v) \xi_k(v) \theta_p(x) dx dv$$

$$= -2 \int_{\mathbb{R}^d} \text{sgn}(\rho_2(t, x) - \rho_1(t, x))( -\Delta_x )^\frac{\alpha}{2} \rho_2(t, x) - (-\Delta_x )^\frac{\alpha}{2} \rho_1(t, x) dx$$

$$\leq -2 \int_{\mathbb{R}^d} (-\Delta_x )^\frac{\alpha}{2} |\rho_2(t, x) - \rho_1(t, x)| dx$$

$$= 0. \quad (3.22)$$

By Remark 2.3, then (3.7) implies

$$\frac{d}{dt} \int_{\mathbb{R}^{d+1}} |u_{1,\varepsilon}| + |u_{2,\varepsilon}| - 2u_{1,\varepsilon}^0 u_{2,\varepsilon}^0 |\xi_k(v)\theta_p(x)| dx dv$$

$$= \int_{\mathbb{R}^{d+1}} |u_{1,\varepsilon}| + |u_{2,\varepsilon}| - 2u_{1,\varepsilon}^0 u_{2,\varepsilon}^0 |\xi_k(v) f(v) \cdot \nabla x \theta_p(x)| dx dv$$

$$- \int_{\mathbb{R}^{d+1}} (-\Delta_x )^\frac{\alpha}{2} \theta_p(x) |u_{1,\varepsilon}| + |u_{2,\varepsilon}| |\xi_k(v)| dx dv - 2 \int_{\mathbb{R}^{d+1}} [u_{1,\varepsilon} R_{2,\varepsilon}^\sigma + u_{2,\varepsilon} R_{1,\varepsilon}^\sigma] \xi_k \theta_p dx dv$$

$$+ \int_{\mathbb{R}^{d+1}} I_1(t, x, v) \xi_k(v) \theta_p(x) dx dv + \int_{\mathbb{R}^{d+1}} I_2(t, x, v) \xi_k(v) \theta_p(x) dx dv. \quad (3.23)$$

According to (3.11), (3.15) and (3.22), if we let $\sigma \downarrow 0$ first, $p \uparrow \infty$ second, $k \uparrow \infty$ third, $\varepsilon_1 \downarrow 0$ fourth and $\varepsilon_2 \downarrow 0$ last, we conclude from (3.23) that

$$\frac{d}{dt} \int_{\mathbb{R}^{d+1}} |u_1| + |u_2| - 2u_1 u_2 |dx dv \leq 0. \quad (3.24)$$

Since $|u_1| = |u_1|^2, |u_2| = |u_2|^2, |u_1 - u_2| = |u_1 - u_2|^2$, from (3.24), we end up with

$$\frac{d}{dt} \int_{\mathbb{R}^{d+1}} |u_1(t) - u_2(t)| dx dv \leq 0,$$
which indicates
\[
\int_{\mathbb{R}^d} |\rho_1(t) - \rho_2(t)|dx = \int_{\mathbb{R}^{d+1}} |u_1(t) - u_2(t)|dxdv = 0. \tag{3.25}
\]
From this, we finish the proof for the uniqueness.

**Existence** We prove the existence by a vanishing viscosity method. Assume that \(\rho_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d)\).

Consider the Cauchy problem:
\[
\begin{cases}
\frac{\partial}{\partial t} \rho_\varepsilon(t, x) + \text{div}_x F(\rho_\varepsilon) + (-\Delta_x)^{\frac{3}{2}} \rho_\varepsilon - \varepsilon \Delta \rho_\varepsilon = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho_\varepsilon(t = 0) = \rho_0, & x \in \mathbb{R}^d. \tag{3.26}
\end{cases}
\]

With the classical parabolic theory (see [26]), there is a unique strong solution \(\rho_\varepsilon\) of [3.26] and for any smooth convex function \(\eta\), [2.8] holds (see Remark 2.1). Moreover, the following inequalities hold
\[
\begin{aligned}
\|\rho_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} &\leq \|\rho_0\|_{L^1(\mathbb{R}^d)}, \\
\|\partial_t \rho_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} &\leq C(\|\rho_0\|_{BV(\mathbb{R}^d)}), \\
\|\rho_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} &\leq C(\|\rho_0\|_{BV(\mathbb{R}^d)} + \|\rho_0\|_{L^1(\mathbb{R}^d)}). 
\end{aligned} \tag{3.27}
\]

Indeed, if we choose \(\eta(\rho) = |\rho|\), with the help of entropy inequality [2.3] (since a classical solution is also an entropy solution), it follows that
\[
\frac{\partial}{\partial t} |\rho_\varepsilon| + \text{div}_x [\text{sgn}(\rho_\varepsilon) F(\rho_\varepsilon)] \leq \varepsilon \Delta |\rho_\varepsilon| - (-\Delta_x)^{\frac{3}{2}} |\rho_\varepsilon|. \tag{3.28}
\]

Integrating both hand sides of [3.28] on \(\mathbb{R}^d\), we obtain
\[
\int_{\mathbb{R}^d} |\rho_\varepsilon(t, x)|dx \leq \int_{\mathbb{R}^d} |\rho_0(x)|dx,
\]
which reveals that the first inequality in [3.27] is valid.

If we set \(\rho_\varepsilon^h(t, x) = \rho_\varepsilon(t, x + h)\) and \(\rho_0^h(x) = \rho_0(x + h)\), then
\[
\begin{cases}
\frac{\partial}{\partial t} \rho_\varepsilon^h(t, x) + \text{div}_x F(\rho_\varepsilon^h) + (-\Delta_x)^{\frac{3}{2}} \rho_\varepsilon^h - \varepsilon \Delta \rho_\varepsilon^h = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho_\varepsilon^h(t = 0) = \rho_0^h, & x \in \mathbb{R}^d. \tag{3.29}
\end{cases}
\]

An analogue discussion (as used from [3.2] to [3.24]) leads to
\[
\int_{\mathbb{R}^d} |\rho_\varepsilon^h(t, x) - \rho_\varepsilon(t, x)|dx \leq \int_{\mathbb{R}^d} |\rho_0^h(x) - \rho_0(x)|dx. \tag{3.29}
\]
So the second inequality in [3.27] satisfies if taking \(h\) to zero.

The third inequality in [3.27] is from the following estimate:
\[
\|\partial_t \rho_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leq \|\text{div}_x F(\rho_0) + (-\Delta_x)^{\frac{3}{2}} \rho_0\|_{L^1(\mathbb{R}^d)} \leq C(\|\rho_0\|_{L^1(\mathbb{R}^d)} + \|\rho_0\|_{BV(\mathbb{R}^d)}). \tag{3.30}
\]

By [3.27], using the Helly theorem (see [27 p17]), the Fréchet-Kolmogorov compactness theorem (see [28 p85]) and the Arzela-Ascoli compactness criterion (see [28 p85]), after a standard control of decay at infinity, there is a subsequence (denoted by itself), such that
\[
\rho_\varepsilon \to \rho \text{ in } C([0, T]; L^1(\mathbb{R}^d)), \text{ as } \varepsilon \to 0. \tag{3.31}
\]
By \((3.31)\), from \((3.27)\), if we let \(\varepsilon \to 0\), then
\[
\int_{\mathbb{R}^d} |\rho^h(t, x) - \rho(t, x)| \, dx \leq \int_{\mathbb{R}^d} |\rho^0_h(x) - \rho_0(x)| \, dx,
\]
which implies
\[
\rho \in L^\infty([0, T); BV(\mathbb{R}^d)).
\] (3.32)

Besides, \(\rho\) satisfy \((2.1)-(2.3)\) for any smooth convex function \(\eta\). So \(\rho\) is an entropy solution of \((1.1)-(1.2)\). Then Theorem 2.1 applies and thus \(\rho\) is a kinetic solution.

For \(\rho_0 \in L^1 \cap BV(\mathbb{R}^d)\), we approximate it by \(\rho^0_\varepsilon \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d)\), such that
\[
\rho^0_\varepsilon \to \rho_0 \quad \text{in} \quad \mathbb{R}^1 \cap BV(\mathbb{R}^d), \quad \|\rho^0_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq C\|\rho_0\|_{L^1(\mathbb{R}^d)}, \quad \|\rho^0_\varepsilon\|_{BV(\mathbb{R}^d)} \leq C\|\rho_0\|_{BV(\mathbb{R}^d)}.
\] (3.33)
Then there is a kinetic solution \(m_\sigma\) of \((1.1)-(1.2)\), and for any \(h \in \mathbb{R}\),
\[
\int_{\mathbb{R}^d} |\rho^h_\sigma(t, x) - \sigma(t, x)| \, dx \leq \int_{\mathbb{R}^d} |\rho^0_\varepsilon(x + h) - \rho^0_\varepsilon(x)| \, dx.
\] (3.34)
Correspondingly, the nonnegative measures \(m_\sigma\) and \(n_\sigma\) meet \((2.1)-(2.5)\) and \((3.34)-(3.35)\).

Moreover for any \(\sigma_1, \sigma_2 > 0\),
\[
\|\rho_{\sigma_1}(t) - \rho_{\sigma_2}(t)\|_{L^1(\mathbb{R}^d)} \leq \|\rho^0_{\sigma_1} - \rho^0_{\sigma_2}\|_{L^1(\mathbb{R}^d)}.
\] (3.35)

Denote \(\mathcal{M}_b([0, T] \times \mathbb{R}^d)\) for the space of bounded Borel measures over \([0, T] \times \mathbb{R}^d\), with norm given by the total variation of measures), \(n \in L^\infty(\mathbb{R}^v; L^1([0, T] \times \mathbb{R}^d))\). With the aid of \((2.14)-(2.15)\) and \((3.34)-(3.35)\), by choosing a subsequence (not labeled), there are \(\rho \in \mathcal{C}(\mathbb{R}^v; L^1(\mathbb{R}^d)) \cap L^\infty([0, T]; BV(\mathbb{R}^d)), m \in L^\infty(\mathbb{R}^v; \mathcal{M}_b([0, T] \times \mathbb{R}^d))\) such that
\[
\rho_{\sigma} \to \rho, \quad \text{in} \quad \mathcal{C}([0, T]; L^1(\mathbb{R}^d)),
\]
\[
m_{\sigma} \to m, \quad \text{in} \quad L^\infty(\mathbb{R}^v; \mathcal{M}_b([0, T] \times \mathbb{R}^d)),
\]
\[
n_{\sigma} \to n, \quad \text{in} \quad L^\infty(\mathbb{R}^v; L^1([0, T] \times \mathbb{R}^d)),
\] (3.36)
and \(n\) fulfills \((2.1)-(2.5)\), \(m + n \in L^\infty(\mathbb{R}^v; L^1([0, T] \times \mathbb{R}^d))\).

Moreover, by Remark 2.3, if one takes Kružkov entropy \((\rho_{\sigma} - v)_+\) and \((\rho_{\sigma} - v)_-\), respectively, then \(m_{\sigma} + n_{\sigma}\) satisfies \((2.17)-(2.18)\), respectively. Therefore the nonnegative measures \(m\) and \(n\) fulfilling \((2.15)\), and \(\rho\) is a kinetic solution of \((1.1)-(1.2)\).

**Remark 3.1**

(i) \(m_{i, \varepsilon}(t, x, v)\) \((i = 1, 2)\) meet \((3.1)\), and the left hand side in \((3.1)\) belongs to \(L^1_{loc}\) in variable \(v\), so \(m_{i, \varepsilon}(t, x, v) + n_{i, \varepsilon}(t, x, v)\) are continuous in \(v\). Clearly, \(n_{1, \varepsilon}(t, x, v)\) and \(n_{2, \varepsilon}(t, x, v)\) are continuous in \(v\). Thus \(m_{i, \varepsilon}(t, x, v)(i = 1, 2)\) are continuous in \(v\), which suggests that \(m_{2, \varepsilon}(t, x, 0)\) in \((3.12)\) and \(m_{1, \varepsilon}(t, x, 0)\) in \((3.13)\) are legitimate.

(ii) In our proof, we used the functional \((3.7)\)
\[
G(t, x, v) = |\chi_{\rho_1(t, x)}(v)| + |\chi_{\rho_2(t, x)}(v)| - 2\chi_{\rho_1(t, x)}(v)\chi_{\rho_2(t, x)}(v)
\]
\[
= |u_1(t, x, v)| + |u_2(t, x, v)| - 2u_1(t, x, v)u_2(t, x, v),
\] (3.37)
which was introduced by Perthame (consult to \([20, 21]\)) for first order hyperbolic equations. Then this method was extended to the hyperbolic-parabolic equations by Chen and Perthame \([22]\), to derive the uniqueness for kinetic solutions. Here our proof follows Chen and Perthame’s work, by applying the contraction mapping principle to get the uniqueness of kinetic solutions.
Remark 3.2 Our existence and uniqueness result can be extended in a routine way to the non-homogeneous problem (2.19), if we suppose (2.20) and

\[ B' \in L^\infty(\mathbb{R}), \text{ and } \exists M_1, M_2 \in \mathbb{R}, 1_{v>0}A'(v) \leq M_1, -M_2 \leq 1_{v<0}A'(v) \leq M_1. \quad (3.38) \]

Indeed, if one takes functional \( G \) as in (3.37), by repeating the manipulations from (3.2) to (3.28), we end up with

\[ \frac{d}{dt} \int_{\mathbb{R}^d+1} |u_1(t) - u_2(t)|dx dv \leq \int_{\mathbb{R}^d+1} A'(v)|u_1(t) - u_2(t)|dx dv, \]

which demonstrates the uniqueness.

For existence part, we choose \( \rho_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d) \) first, and consider the approximating problem:

\[ \begin{cases} \frac{\partial}{\partial t} \rho_\varepsilon(t, x) + \text{div}_x F(\rho_\varepsilon) + (-\Delta_x)^{\frac{3}{2}} B(\rho_\varepsilon) - \varepsilon \Delta \rho_\varepsilon = A(\rho_\varepsilon), & (t, x) \in (0, T) \times \mathbb{R}^d, \\ \rho_\varepsilon(t = 0) = \rho_0, & x \in \mathbb{R}^d. \end{cases} \quad (3.39) \]

We can derive an analogue of (3.27)

\[ \begin{cases} \|\rho_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leq \exp(M_1 t)\|\rho_0\|_{L^1(\mathbb{R}^d)}, & \|\rho_\varepsilon(t)\|_{BV(\mathbb{R}^d)} \leq \exp(M_1 t)\|\rho_0\|_{BV(\mathbb{R}^d)}, \\ \|\partial_t \rho_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leq C(t, A, \|\rho_0\|_{L^\infty})(\|\rho_0\|_{BV(\mathbb{R}^d)} + \|\rho_0\|_{L^1(\mathbb{R}^d)}), \end{cases} \quad (3.40) \]

and in view of entropy formulation (2.3) (also see Remark 2.3 (ii)), it yields

\[ \begin{aligned} \int_0^T \int_{\mathbb{R}^d} (m + n)(dt, dx, v) & \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \text{sgn}(\rho - v_+)A(\rho)dx dt + \frac{1}{2} \int_{\mathbb{R}^d} |(\rho_0(x) - v_+) - (\rho(T, x) - v_+)|dx \\
& \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \text{sgn}(\rho - v_-)A(\rho)dx dt + \frac{1}{2} \int_{\mathbb{R}^d} |(\rho_0(x) - v_-) - (\rho(T, x) - v_-)|dx. \end{aligned} \]

Therefore

\[ \int_0^T \int_{\mathbb{R}^d} (m + n)(dt, dx, v) \leq \frac{M_1}{2} \int_0^T \int_{\mathbb{R}^d} \text{sgn}(\rho - v_+)|\rho|dx dt + \frac{1}{2} \int_{\mathbb{R}^d} |(\rho_0(x) - v_+) - (\rho(T, x) - v_+)|_{L^1(\mathbb{R}^d)} \\
+ \frac{M_2}{2} \int_0^T \int_{\mathbb{R}^d} \text{sgn}(\rho - v_-)|\rho|dx dt + \frac{1}{2} \int_{\mathbb{R}^d} |(\rho_0(x) - v_-) - (\rho(T, x) - v_-)|_{L^1(\mathbb{R}^d)}. \]

Combining a compactness argument, we complete the proof for regular initial data.

Secondly, for \( \rho_0 \in L^1 \cap BV(\mathbb{R}^d) \), by an approximate discussion, we gain an analogue conclusion of (3.30).
With the same verification as in Theorem 3.1, we achieve the following result.

**Corollary 3.1 (Comparison Principle)** Let \((1.3)\) \((2.20)\) and \((3.38)\) hold and \(\rho_{0,1}, \rho_{0,2} \in L^1 \cap BV(\mathbb{R}^d)\). Assume that \(\rho_1\) and \(\rho_2\) are two kinetic solutions of \((2.19)_1\), to initial values \(\rho_{0,1}\) and \(\rho_{0,2}\) respectively. Then

\[
\|\rho_1(t) - \rho_2(t)\|_{L^1(\mathbb{R}^d)} \leq \exp(M_1 t)\|\rho_{0,1} - \rho_{0,2}\|_{L^1(\mathbb{R}^d)}.
\]

(3.41)

Besides, if \(\rho_{0,1} \leq \rho_{0,2}\), then \(\rho_1 \leq \rho_2\) and in particular, if the initial value is nonnegative, the unique kinetic solution is nonnegative as well.

**Remark 3.3** From above comparison principle \((3.41)\), if \(M_1 < 0\) (for example \(A(\rho) = M_1 \rho\)), for any initial data \(\rho_0 \in L^1 \cap BV(\mathbb{R}^d)\), then the unique kinetic solution \(\rho(t,x)\) of

\[
\frac{\partial}{\partial t}\rho(t,x) + \text{div}_xF(\rho) + (-\Delta_x)^{1/2}B(\rho) = M_1 \rho, \quad (t,x) \in (0,T) \times \mathbb{R}^d,
\]

(3.42)

converges to zero as \(t \to \infty\), i.e. \(\{0\}\) is the unique global attractor for the solution semigroup.

The restriction conditions on \(A\) seem to be strict, but there are models, in population dynamics, chemical wave propagation and fluid mechanics, satisfying this assumption. We now illustrate it by an example.

**Example 3.1** Consider the following multidimensional fractional Burgers-Fisher type equation

\[
\left\{ \begin{array}{l}
 \frac{\partial}{\partial t}\rho(t,x) + \text{div}_xF(\rho) + \nu(-\Delta_x)^{1/2}\rho = A(\rho), \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\
 \rho(t=0,x) = \rho_0(x), \quad x \in \mathbb{R}^d,
\end{array} \right.
\]

(3.43)

where \(a \in \mathbb{R}^d\) is a vector, \(\beta \geq 0\) \(\iota \in \mathbb{N}\) and

\[
A(\rho) = \begin{cases} 
\beta \rho(1 - \rho^k), & \text{when } \rho \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

(3.44)

When \(\alpha = 2\), \(a = 0\) and \(d = k = 1\), it is well known as Fisher equation, proposed by [29] in population dynamics, where \(\nu > 0\) is a diffusion constant, \(\beta > 0\) is the linear growth rate. When \(\alpha = 2\), \(d = 1\) and \(\iota = k\), it is well known as generalized Burgers-Fisher equation, which is modeled for describing the interaction between reaction mechanisms, convection effects and diffusion transports [30]. And when \(\alpha \in (0,2), \beta = 0\), it is the generalized fractal/fractional Burgers equation appeared in continuum mechanics and discussed by [11]. The aim of this work is to argue the more general form of the Burger-Fisher and fractal/fractional Burgers equations called generalized fractional Burgers-Fisher type equation in order to show the effectiveness of the current method.

Clearly, \(F \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^d)\) and when \(k\) is even, \((3.38)\) holds with \(M_1 = \beta, M_2 = 0\). By Remark 3.2 and Corollary 3.1, we have the following result.

**Corollary 3.2** Let \(0 \leq \rho_0 \in L^1 \cap BV(\mathbb{R}^d)\), \(\alpha \in (0,1)\) and \(k\) be an even number. Then there is a unique kinetic solution to \((3.43)\). Besides, the unique kinetic solution is nonnegative as well.
4 Continuous dependence on nonlinearities and Lévy measures

This section is devoted to discuss the regularity on $t$ and the continuous dependence on $f$, $\nu$ and $\alpha$. Since the argument for nonhomogeneous problem is similar, we only concentrate our attention on homogeneous case and our main result is given by:

**Theorem 4.1** Consider the following Cauchy problems

\[
\begin{aligned}
\frac{\partial}{\partial t} \rho_1^1(t, x) + \text{div}_x F_1(\rho_1^1) + \nu_1(\Delta x)\frac{\partial}{\partial t} \rho_1^1 = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho_1^1(t, x) = \rho_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial}{\partial t} \rho_2^2(t, x) + \text{div}_x F_2(\rho_2^2) + \nu_2(\Delta x)\frac{\partial}{\partial t} \rho_2^2 = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho_2^2(t, x) = \rho_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

where

\[
\rho_0 \in L^1 \cap BV(\mathbb{R}^d), \quad F_1, F_2 \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^d) \text{ and } F_1' - F_2' \in L^\infty(\mathbb{R}; \mathbb{R}^d), \alpha, \beta \in (0, 1). \tag{4.3}
\]

Let $\rho_1^1$, respectively $\rho_2^2$, be the unique kinetic solution to (4.1), respectively to (4.2). Then the following claims hold:

(i) $\rho_1^1$ and $\rho_2^2$ are Lipschitz continuous in $t$ in the following sense: For every $t, s \in [0, T]$,

\[
\|\rho_1^1(t) - \rho_1^1(s)\|_{L^1(\mathbb{R}^d)} \leq \|\rho_0\|_{BV(\mathbb{R}^d)}\|F_1'\|_{L^\infty(\mathbb{R})} |t - s| + C\|\rho_0\|_{L^1(\mathbb{R}^d)}\|\rho_0\|_{BV(\mathbb{R}^d)} |t - s|, \tag{4.4}
\]

and

\[
\|\rho_2^2(t) - \rho_2^2(s)\|_{L^1(\mathbb{R}^d)} \leq \|\rho_0\|_{BV(\mathbb{R}^d)}\|F_2'\|_{L^\infty(\mathbb{R})} |t - s| + C\|\rho_0\|_{L^1(\mathbb{R}^d)}\|\rho_0\|_{BV(\mathbb{R}^d)} |t - s|, \tag{4.5}
\]

if $F_1', F_2' \in L^\infty(\mathbb{R}; \mathbb{R}^d)$;

(ii) Continuous in the nonlinearities and viscosity coefficients: If $\alpha = \beta$, then

\[
\|\rho_1^1 - \rho_2^2\|_{C([0,T]; L^1(\mathbb{R}^d))} \leq T\|\rho_0\|_{BV(\mathbb{R}^d)}\|F_1' - F_2'\|_{L^\infty(\mathbb{R})} + T|\nu_1 - \nu_2|\|\rho_0\|_{L^1(\mathbb{R}^d)}\|\rho_0\|_{BV(\mathbb{R}^d)}, \tag{4.6}
\]

(iii) Lipschitz continuous in Lévy measure: If $F_1 = F_2$, then for every $\lambda \in (0, 1)$

\[
\limsup_{\alpha, \beta \to \lambda} \frac{\|\rho_1^1 - \rho_2^2\|_{C([0,T]; L^1(\mathbb{R}^d))}}{\|\alpha - \beta\|} \leq CT\|\rho_0\|_{L^1(\mathbb{R}^d)}\|\rho_0\|_{BV(\mathbb{R}^d)}(1 + |\log \|\rho_0\|_{BV}|). \tag{4.7}
\]

Before proving the main result, we introduce another notion of solutions and present a useful lemma.

**Definition 4.1** Let $\rho_0 \in L^\infty(\mathbb{R}^d)$, $F_1 \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^d)$ and $\alpha \in (0, 1)$. We call $\rho \in L^\infty([0,T] \times \mathbb{R}^d)$ an entropy solution of (4.2), if for every $v \in \mathbb{R}, r > 0$ and every nonnegative function $\psi \in \mathcal{D}([0,T] \times \mathbb{R}^d)$

\[
\int_0^T \int_{\mathbb{R}^d} \left[ \rho_1^1 - v \frac{\partial}{\partial t} \psi(t, x) + \text{sgn}(\rho_1^1 - v)(F_1(\rho_1^1) - F_1(v)) \cdot \nabla \psi \right] dt dx + \int_{\mathbb{R}^d} |\rho_0 - v| \psi(0, x) dx = 0.
\]
\[ + \int_0^T \int_{\mathbb{R}^d} \left[ \rho_\alpha^1 - v |\mathcal{L}_r^\varphi(\psi(t,x)) + \text{sgn}(\rho_\alpha^1 - v)\mathcal{L}_r^{x} \psi(t,x) \right] dt \, dx \geq 0, \]  
where \( \mathcal{L}_r^\varphi \) and \( \mathcal{L}_r^{x} \) are defined, for \( \varphi \in \mathcal{D}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), by
\[
\mathcal{L}_r^\varphi(x) = c(d, \alpha) \int_{|z| < r} \frac{\varphi(x + z) - \varphi(x) - \nabla \varphi(x) \cdot z}{|z|^{d+\alpha}} \, dz, \\
\mathcal{L}_r^{x} \varphi(x) = c(d, \alpha) \int_{|z| < r} \frac{\varphi(x + z) - \varphi(x)}{|z|^{d+\alpha}} \, dz.
\]

**Lemma 4.1**

(i) \([14] \text{Theorem 2.5}\) If \( \rho_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d) \), then Definition 2.1 and Definition 4.1 are equivalent. Thus by the kinetic formulation in Theorem 2.1, both Definition 2.2 and Definition 4.1 are equivalent.

(ii) \([14] \text{Theorem 2.3 or 3.4 or 14 \text{Theorem 2, Theorem 4}}\) If \( \rho_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d) \), then Theorem 4.1 holds.

**Remark 4.1** Notice that the right hand sides in (4.4)-(4.7) are dependent only on \( \|\rho_0\|_{L^1} \) and \( \|\rho_0\|_{BV} \). So (4.4)-(4.7) may be true if the entropy solutions take values in \( \mathcal{C}([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T]; BV(\mathbb{R}^d)) \). However, when \( \rho \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T]; BV(\mathbb{R}^d)) \), the term \( \int_0^T \int_{\mathbb{R}^d} \text{sgn}(\rho_\alpha^1 - v)(F_1(\rho_\alpha^1) - F_1(v)) \cdot \nabla \psi(t,x) \, dt \, dx \) in the first line in (4.7) is not legitimate. To overcome this obstacle, we introduce the notion of kinetic solutions, and extend Lemma 4.1 to the class of \( L^1 \cap BV(\mathbb{R}^d) \) solutions.

**Proof of Theorem 4.1.** We approximate \( \rho_0 \) by \( \rho_0^\sigma \) such that (3.30) holds. By Remark 2.3, (2.9) and (3.30), we end up with
\[ \|\rho_0^1 \|_{L^1(\mathbb{R}^d)} \leq \|\rho_0^\sigma\|_{BV(\mathbb{R}^d)} \|F_1 \|_{L^\infty(\mathbb{R})} |t - s| + C \|\rho_0^\alpha\|_{L^1(\mathbb{R}^d)} \|\rho_0^\alpha\|_{BV(\mathbb{R}^d)} |t - s|, \]  
and
\[ \|\rho_0^2 \|_{L^1(\mathbb{R}^d)} \leq \|\rho_0^\beta\|_{BV(\mathbb{R}^d)} \|F_2 \|_{L^\infty(\mathbb{R})} |t - s| + C \|\rho_0^\beta\|_{L^1(\mathbb{R}^d)} \|\rho_0^\beta\|_{BV(\mathbb{R}^d)} |t - s|, \]
where the constant \( C \) is dependent only on \( \|F_1 \|_{L^\infty(\mathbb{R})} \), \( \|F_2 \|_{L^\infty(\mathbb{R})} \), \( d \) and \( \alpha, \beta \).

By Lemma 4.1 (i) and (ii), \( \rho_0^1 \) and \( \rho_0^2 \) fulfill
\[ \|\rho_0^1 \|_{L^1(\mathbb{R}^d)} \leq T \|\rho_0^\sigma\|_{BV(\mathbb{R}^d)} \|F_1 \|_{L^\infty(\mathbb{R})} + T|\nu_1 - \nu_2| \|\rho_0^\alpha\|_{L^1(\mathbb{R}^d)} \|\rho_0^\alpha\|_{BV(\mathbb{R}^d)} \]  
and
\[ \|\rho_0^1 \|_{C([0,T]; L^1(\mathbb{R}^d))} \leq T \int_{\mathbb{R}^d} \|\rho_0^\sigma\|_{L^1(\mathbb{R}^d)} d[\mu_\alpha - \mu_\beta], \]  
where
\[ d[\mu_\alpha] = \frac{c(d, \alpha)}{|z|^{d+\alpha}} dz, \quad d[\mu_\beta] = \frac{c(d, \beta)}{|z|^{d+\beta}} dz. \]

Observing that
\[ \rho_0^1, \rho_0^2 \rightarrow \rho^1, \rho^2 \text{ in } C([0,T]; L^1(\mathbb{R}^d)), \text{ as } \sigma \rightarrow 0, \]
as $\sigma \to 0$, and noting (4.9)-(4.12), we arrive at inequalities (4.1)-(4.6). Therefore, the claims (i) and (ii) in Theorem 4.1 hold.

From (4.9)-(4.12), we also have

$$\|\rho_1^1 - \rho_1^2\|_{C([0,T];L^1(\mathbb{R}^d))} \leq T \int_{\mathbb{R}^d} \|\rho_0(\cdot + z) - \rho_0(\cdot)\|_{L^1(\mathbb{R}^d)} d[\mu_\alpha - \mu_{\beta}],$$

and to prove claim (iii), let $0 < r_1 \in \mathbb{R}$, we split the integral in the right hand side in (4.13) into two parts

$$\int_{|z| > r_1} \|\rho_0(\cdot + z) - \rho_0(\cdot)\|_{L^1(\mathbb{R}^d)} d[\mu_\alpha - \mu_{\beta}] + \int_{|z| < r_1} \|\rho_0(\cdot + z) - \rho_0(\cdot)\|_{L^1(\mathbb{R}^d)} d[\mu_\alpha - \mu_{\beta}].$$

Then the proof for Theorem 4 (23) applies, and we obtain (4.6). This completes the proof.

Remark 4.2 We can prove the continuous dependence of solutions on nonlinearities by introducing the functional $G$ (given in (3.37)). Indeed, we write (4.1) and (4.2) in microscopic types by using kinetic formulation first, then we regularize solutions in $t,x,v$ and repeat the calculations from (3.11) to (3.23), to get

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u_\alpha^1(t,x,v) - v_\alpha^2(t,x,v)| dx dv \leq 2\|F_1 - F_2\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{BV(\mathbb{R}^d)} + 2|\nu_1 - \nu_2| \|\rho_0\|_{L^1(\mathbb{R}^d)}^\alpha \|\rho_0\|_{BV(\mathbb{R}^d)}^\alpha,$$

where $u_\alpha^1 = \chi_{\rho_\alpha^1}(v)$ and $u_\alpha^2 = \chi_{\rho_\alpha^2}(v)$.

From (4.13), it follows that

$$\|\rho_1^1 - \rho_1^2\|_{C([0,T];L^1(\mathbb{R}^d))} \leq 2T\|\rho_0\|_{BV(\mathbb{R}^d)} \|F_1 - F_2\|_{L^\infty(\mathbb{R})} + |\nu_1 - \nu_2| \|\rho_0\|_{L^1(\mathbb{R}^d)}^\alpha \|\rho_0\|_{BV(\mathbb{R}^d)}^\alpha.$$ (4.16)

If we define the right hand side of (4.10) by $I(T,\rho_0, F_1, F_2, \nu_1, \nu_2)$, then from (4.6),

$$\|\rho_1^1 - \rho_1^2\|_{C([0,T];L^1(\mathbb{R}^d))} \leq \frac{1}{2}I(T,\rho_0, F_1, F_2, \nu_1, \nu_2).$$

So (4.6) implies (4.16), and in this sense, we say the estimate (4.6) is better than (4.16). Hence in the proof of Theorem 4.1, we adapt the method developed in [22, 23].

Besides the continuous dependence, we also have obtained the limiting equations as $\alpha \downarrow 0$ and $\nu \downarrow 0$. Firstly, we give a useful lemma for fixed $\nu$, which will serve us well for the limiting problem as $\alpha \downarrow 0$, and for simplicity we take $\nu = 1$.

Lemma 4.2 (23) Theorem 3) Let $\rho_0 \in L^\infty(\mathbb{R}^d)$ and for $\alpha \in (0,1)$, let $\rho_\alpha$ be the unique entropy solution (defined by Definition 4.1) of (1.1)-(1.2). If $\rho_0 \in L^1(\mathbb{R}^d)$, then as $\alpha \downarrow 0$, $\rho_\alpha$ converges in $C([0,T];L^1_{\text{loc}}(\mathbb{R}^d))$ to the unique entropy solution (defined by Definition 4.1) $\rho \in L^\infty([0,T] \times \mathbb{R}^d) \cap C([0,T];L^1(\mathbb{R}^d))$ of the Cauchy problem

$$\begin{align*}
\frac{\partial}{\partial t}\rho(t,x) + \text{div}_x F(\rho) + \rho = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\
\rho(t = 0,x) = \rho_0(x), \quad x \in \mathbb{R}^d.
\end{align*}$$

Our main result is given by:
Theorem 4.2 Let $\rho_0 \in L^1 \cap BV(\mathbb{R}^d)$, and for $\alpha \in (0, 1)$, let $\rho^\nu_\alpha$ be the unique kinetic solution of (4.1)–(4.2).

(i) As $\nu \downarrow 0$, $\rho^\nu_\alpha$ converges in $C([0, T]; L^1(\mathbb{R}^d))$ to the unique kinetic solution $\rho$ of the following Cauchy problem

$$\begin{align*}
\frac{\partial}{\partial t} \rho(t, x) + \text{div}_x F(\rho) &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho(t = 0, x) &= \rho_0(x), \quad x \in \mathbb{R}^d.
\end{align*}$$

(4.18)

Moreover, we have the following error estimate: for all $T > 0$,

$$\|\rho^\nu_\alpha - \rho\|_{C([0, T]; L^1(\mathbb{R}^d))} = O(\nu), \quad \text{as } \nu \to 0.$$  

(4.19)

(ii) If $\alpha \downarrow 0$, then $\rho^\nu_\alpha$ converges in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$ to the unique kinetic solution $\rho^\nu$ of the following Cauchy problem

$$\begin{align*}
\frac{\partial}{\partial t} \rho^\nu(t, x) + \text{div}_x F(\rho^\nu) + \nu \rho^\nu &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho^\nu(t = 0, x) &= \rho_0(x), \quad x \in \mathbb{R}^d.
\end{align*}$$

(4.20)

Proof. For every pair of $\nu_1, \nu_2 > 0$, by virtue of Theorem 4.1 (ii), we have

$$\|\rho^\nu_\alpha - \rho^\nu_{\alpha}^\nu\|_{C([0, T]; L^1(\mathbb{R}^d))} \leq T|\nu_1 - \nu_2|\|\rho_0\|_{L^1(\mathbb{R}^d)}^{1-\alpha}\|\rho_0\|_{BV(\mathbb{R}^d)}^\alpha.$$  

(4.21)

which implies that $\{\rho^\nu_{\alpha}\}_\nu$ is a Cauchy sequence in $C([0, T]; L^1(\mathbb{R}^d))$. So $\{u^\nu_{\alpha} = \chi_{\rho^\nu_{\alpha}}\}_\nu$ is a Cauchy sequence in $C([0, T]; L^1(\mathbb{R}^{d+1}))$.

Observe that $u^\nu_{\alpha}$ yields

$$\begin{align*}
\frac{\partial}{\partial t} u^\nu_{\alpha}(t, x, v) + f(v) \cdot \nabla_x u^\nu_{\alpha} + \nu(-\Delta_x) u^\nu_{\alpha} &= \frac{\partial}{\partial \nu} (m^\nu_{\alpha} + n^\nu_{\alpha}), \quad (t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}, \\
u^\nu_{\alpha}(t = 0, x, v) &= \chi_{\rho_0(x)}(v), \quad x \in \mathbb{R}^d.
\end{align*}$$

(4.22)

Combining (2.14) and (2.17), we conclude that

$$n^\nu_{\alpha} \to 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^d; L^1([0, T] \times \mathbb{R}^d)), \quad \text{as } \nu \downarrow 0.$$  

(4.23)

In view of (2.15), (2.17) and (2.18), there is a nonnegative measure $m \in L^\infty(\mathbb{R}^d; \mathcal{M}_b([0, T] \times \mathbb{R}^d))$, so that

$$m^\nu_{\alpha} \to m, \text{ in } L^\infty(\mathbb{R}^d; \mathcal{M}_b([0, T] \times \mathbb{R}^d)), \quad \text{as } \nu \downarrow 0.$$  

(4.24)

By (4.21), (4.23), (4.24) and the following estimate

$$\|\rho^\nu_{\alpha}(t)\|_{BV(\mathbb{R}^d)} \leq \|\rho_0\|_{BV(\mathbb{R}^d)},$$

(4.25)

and take $\nu \downarrow 0$ in (4.22) in the distributions sense, we know that there is $\rho \in C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T]; BV(\mathbb{R}^d))$, satisfying

$$\begin{align*}
\frac{\partial}{\partial t} \chi_{\rho(t, x)}(v) + f(v) \cdot \nabla_x \chi_{\rho(t, x)}(v) &= \frac{\partial}{\partial \nu} m(t, x, v), \quad (t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}, \\
u(t = 0, x, v) &= \chi_{\rho_0(x)}(v), \quad x \in \mathbb{R}^d.
\end{align*}$$

(4.26)

Clearly the kinetic solution for (4.18) is unique, and thus $\rho$ is the unique kinetic solution of (4.18).
The error estimate (4.19) follows from (4.21) by letting \( \nu_2 \downarrow 0 \) and replacing \( \nu_1 \) by \( \nu \), and this finishes the proof for (i).

It remains to show (ii) and without loss of generality, we suppose \( \nu = 1 \).

Let \( \rho_0^\alpha \) and \( \rho_1^\alpha \) be described in (4.9). Then, by Lemma 4.2, as \( \alpha \downarrow 0 \), \( \rho_0^\alpha \) converges in \( C([0, T]; L^1_{loc}(\mathbb{R}^d)) \) to the unique entropy solution (defined by Definition 4.1) \( \rho^\sigma \in L^\infty([0, T] \times \mathbb{R}^d) \cap C([0, T]; L^1(\mathbb{R}^d)) \) of

\[
\begin{aligned}
\frac{\partial}{\partial t} \rho^\sigma (t, x) + \nabla_x F(\rho^\sigma) + \rho^\sigma &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho^\sigma (t = 0, x) &= \rho_0^\alpha(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\tag{4.27}
\]

With the aid of classical kinetic formulation (see [21]), \( \rho^\sigma \in L^\infty([0, T] \times \mathbb{R}^d) \cap C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T]; BV(\mathbb{R}^d)) \) and it is the unique kinetic solution of (4.17), i.e. \( u^\sigma(t, x, \nu) = \chi_{\rho^\sigma}(\nu) \) meets

\[
\begin{aligned}
\frac{\partial}{\partial t} u^\sigma + f(v) \cdot \nabla_x u^\sigma - v \frac{\partial}{\partial \nu} u^\sigma &= \frac{\partial}{\partial \nu} m^\sigma, \quad (t, x, \nu) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}, \\
\nu^\sigma (t = 0) &= \chi_{\rho_0^\alpha}(\nu), \quad (x, \nu) \in \mathbb{R}^d \times \mathbb{R},
\end{aligned}
\tag{4.28}
\]

for some nonnegative measure \( m^\sigma \), which satisfies

\[
\int_0^T \int_{\mathbb{R}^d} m^\sigma(dt, dx, \nu) \in L_0^\infty(\mathbb{R}).
\tag{4.29}
\]

In view of (2.15), (2.17) and (2.18), there is a nonnegative measure \( m \in L_0^\infty(\mathbb{R}_v; \mathcal{M}_b([0, T] \times \mathbb{R}^d)) \), so that

\[
m^\sigma \to m, \quad in \quad L_w^\infty(\mathbb{R}_v; \mathcal{M}_b([0, T] \times \mathbb{R}^d)), \quad as \quad \sigma \downarrow 0.
\tag{4.30}
\]

By (4.24), (4.29), (4.30), if we take \( \sigma \downarrow 0 \) in (4.28) in the distributions sense, then there is \( \rho \in C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T]; BV(\mathbb{R}^d)) \), satisfying

\[
\begin{aligned}
\frac{\partial}{\partial t} \chi_{\rho(t, x)}(\nu) + f(\nu) \cdot \nabla_x \chi_{\rho(t, x)}(\nu) - \nu \frac{\partial}{\partial \nu} \chi_{\rho(t, x)}(\nu) &= \frac{\partial}{\partial \nu} m(t, x, \nu), \quad (t, x, \nu) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}, \\
\nu(t = 0, x, \nu) &= \chi_{\rho_0(x)}(\nu), \quad x \in \mathbb{R}^d.
\end{aligned}
\tag{4.31}
\]

Thus \( \rho \) is the unique kinetic solution of (4.26) and we complete the proof.

**Remark 4.3** The calculations for Corollary 3.1 used here, we gain: if \( \rho_0 \geq 0 \), then the unique kinetic solution \( \rho \) for (4.18), and the unique kinetic solution \( \rho^\nu \) for (4.20) are nonnegative. Besides, we have the following identities

\[
\int_{\mathbb{R}^d} \rho(t, x) dx = \int_{\mathbb{R}^d} \rho_0(x) dx, \quad \int_{\mathbb{R}^d} \rho^\nu(t, x) dx + \nu \int_0^t \int_{\mathbb{R}^d} \rho^\nu(s, x) dx ds = \int_{\mathbb{R}^d} \rho_0(x) dx.
\tag{4.32}
\]

On the other hand, if we let \( \rho_0^\alpha \) be the unique kinetic solution of

\[
\begin{aligned}
\frac{\partial}{\partial t} \rho_0^\alpha(t, x) + \nabla_x F(\rho_0^\alpha) + \nu(-\Delta_x)\rho_0^\alpha &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho_0^\alpha(t = 0, x) &= \rho_0(x) \geq 0, \quad x \in \mathbb{R}^d,
\end{aligned}
\tag{4.33}
\]

then

\[
\int_{\mathbb{R}^d} \rho_0^\alpha(t, x) dx = \int_{\mathbb{R}^d} \rho_0(x) dx.
\tag{4.34}
\]
Therefore, the mass preserving property still holds at the $\nu \downarrow 0$ limit, but will be lost at the $\alpha \downarrow 0$ limit. So, in general speaking, as $\alpha \downarrow 0$, $\rho_{\alpha}^\nu$ does not converges in $\mathcal{C}([0,T];L^1(\mathbb{R}^d))$ to the unique kinetic solution $\rho^\nu$ of (4.20). From this point, the convergence here is sharp. But when discussing (i), the mass preserving property still holds at the limit, so one can expect $L^1$ convergence for (i) as $\nu \downarrow 0$. Moreover, the preceding convergence is in $L^1$ spaces, but $L^1 \cap BV$ is a proper space to ensure this discussion. Based upon this point, we derive analogue results of Theorem 3.3 [14] and Theorem 3 [23] for kinetic solutions, without assuming $\rho_0 \in L^\infty$.

Acknowledgements

This research was partly supported by the NSF of China grants 11501577, 11301146, 11531006, 11371367 and 11271290.

References

[1] D.W. Stroock, Diffusion processes associated with Lévy generators. Z. Wahr. Verw. Geb. 32 (1975) 209-244.
[2] M.F. Shlesinger, G.M. Zaslavsky, U. Frisch, Lévy Flights and Related Topics in Physics. Lecture Notes in Phys. 450, Springer-Verlag, Berlin, 1995.
[3] J. Duan, An Introduction to Stochastic Dynamics. Cambridge University Press, New York, 2015.
[4] J. Droniou, C. Imbert, Fractal first-order partial differential equations. Arch. Ration. Mech. An. 182(2) (2006) 299-331.
[5] N. Sugimoto, T. Kakutani, Generalized Burgers equation for nonlinear viscoelastic waves. Wave Motion 7 (1985) 447-458.
[6] N. Sugimoto, Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves. J. Fluid Mech. 225 (1991) 631-653.
[7] G.M. Zaslavsky, S.S. Abdullaev, Scaling properties and anomalous transport of particles inside the stochastic layer. Phys. Rev. E 51 (1995) 3901-3910.
[8] G.M. Zaslavsky, Fractional kinetic equations for Hamiltonian chaos. Phys. D 76 (1994) 110-122.
[9] P. Biler, W.A. Woyczynski, Global and exploding solutions for nonlocal quadratic evolution problems. SIAM J. Appl. Math. 59(3) (1998) 845-869.
[10] A.S. Saichev, W.A. Woyczynski, Advection of passive and reactive tracers in multidimensional Burgers velocity field. Phys. D 100 (1997) 119-141.
[11] P. Biler, T. Funaki, W.A. Woyczynski, Fractal Burgers equations. J. Differ. Equations 148 (1998) 9-46.
[12] J. Droniou, T. Gallouët, J. Vovelle, Global solution and smoothing effect for a non-local regularization of an hyperbolic equation. J. Evol. Equ. 3 (2003) 499-521.
[13] J. Droniou, Vanishing non-local regularization of a scalar conservation law. Electron. J. Differ. Eq. 117 (2003) 1-20.

[14] N. Alibaud, Entropy formulation for fractal conservation laws. J. Evol. Equ. 7(1) (2007) 145-175.

[15] P. Biler, G. Karch, W.A. Woyczynski, Multifractal and Lévy conservation laws. C. R. Acad. Sci. Paris 330 (2000) 343-348.

[16] P. Biler, G. Karch, W.A. Woyczynski, Critical nonlinear exponent and self-similar asymptotics for Lévy conservation laws. Ann. I. H. Poincaré-AN 18 (2001) 613-637.

[17] X. Zhang, $L^p$-maximum regularity of nonlocal parabolic equations and applications. Ann. I. H. Poincaré-AN 30(4) (2013) 573-614.

[18] N. Kružkov, First order quasilinear equations in several independent variables. Math. USSR Sbornik 10 (1970) 217-243.

[19] P.L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations. J. Am. Math. Soc. 7(1) (1994) 169-191.

[20] B. Perthame, Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure. J. Math. Pure Appl. 77 (1998) 1055-1064.

[21] B. Perthame, Kinetic formulation of conservation laws. Oxford University Press, New York, 2002.

[22] N. Alibaud, S. Cifani, E.R. Jakobsen, Continuous dependence estimates for nonlinear fractional convection-diffusion equations. SIAM J. Math. Anal. 44(2) (2012) 603-632.

[23] N. Alibaud, S. Cifani, E.R. Jakobsen, Optimal continuous dependence estimates for fractional degenerate parabolic equations. Arch. Ration. Mech. An. 213(3) (2014) 705-762.

[24] S. Cifani, E.R. Jakobsen, Entropy solution theory for fractional degenerate convection-diffusion equations. Ann. I. H. Poincaré-AN 28(3) (2011) 413-441.

[25] G.Q. Chen, B. Perthame, Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. Ann. I. H. Poincaré-AN 20(4) (2003) 645-668.

[26] A.I. Volpert, S.I. Hudjaev, Cauchy’s problem for degenerate second order quasilinear parabolic equations. Mat. Sbornik 78 (120) (1969) 374-396; Engl. Transl.: Math. USSR Sb. 7 (3) (1969) 365-387.

[27] C.M. Dafermos, Hyperbolic conservation laws in continuum physics. Springer, New York, 2010.

[28] K. Yosida, Functional analysis. Springer, Berlin, 1968.

[29] R.A. Fisher, The wave of advance of advantageous genes. Ann. Eugenics 7 (1937) 353-369.

[30] H.N.A. Ismail, K. Raslan, A.A.A. Rabboh, Adomian decomposition method for Burger’s-Huxley and Burger’s-Fisher equations. Appl. Math. Comput. 159(1) (2004) 291-301.