Products and Relations
in Symplectic Floer Homology

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§1. Introduction

This paper gives a detailed and functorial treatment of products, operations and relations in Floer homology and Floer cohomology of monotone symplectic manifolds. Floer (co)homology groups were introduced by A. Floer in a series of papers [F1], [F2], [F3] and [F4]. Basic material on Floer (co)homology can also be found in [HS], [HZ], [M], [MS1], [S] and [SZ]; see also [Sch1]. Let \( M \) be a monotone symplectic manifold of dimension \( 2n \). The Floer homology groups \( HF_*(M) \) are given by a chain complex \( CF_*(M) \) with modules generated by the contractible periodic orbits of a Hamiltonian flow on \( M \). The differential is defined by counting points in moduli spaces of perturbed pseudoholomorphic curves parametrized by the cylinder \( \mathbb{R} \times S^1 \). The Floer cohomology groups \( HF^*(M) \) are given by the dual complex \( CF^*(M) \). The Floer (co)homology groups are isomorphic to the ordinary (co)homology groups \( H_*(M) \) and \( H^*(M) \), except that the grading may be reduced modulo an even integer; see [F3] and [F4] Thm. 5.

The cylinder \( \mathbb{R} \times S^1 \) can be viewed as a twice punctured sphere. In this paper we consider moduli spaces of perturbed pseudoholomorphic curves parametrized by Riemann surfaces \( \Sigma_{g,k^-,k^+} \) of genus \( g \) with \( k = k^- + k^+ \) punctures, of which \( k^- \) punctures are labeled “negative” and \( k^+ \) punctures are labeled “positive”. These moduli spaces are determined by a choice of conformal structure on \( \Sigma_{g,k^-,k^+} \) and some additional perturbation data. These choices are parametrized by a space \( \hat{\mathcal{J}}_{g,k^-,k^+} \). The construction is invariant under a framed diffeomorphism group \( \text{Diff}_{g,k} \), and descends to \( \mathcal{J}_{g,k^-} = \hat{\mathcal{J}}_{g,k^-} / \text{Diff}_{g,k} \). The space \( \hat{\mathcal{J}}_{g,k^-} \) is contractible. The group \( \text{Diff}_{g,k} \) acts freely on a subset \( \mathcal{J}_{g,k^-}^* \) of \( \hat{\mathcal{J}}_{g,k^-} \) whose complement has infinite codimension. We let \( \mathcal{J}_{g,k^-}^* = \mathcal{J}_{g,k^-}^* / \text{Diff}_{g,k} \). This is essentially a classifying space for \( \text{Diff}_{g,k} \).

The singular homology of \( \mathcal{J}_{g,k^-}^* \) is given by a chain complex \( C_*(\mathcal{J}_{g,k^-}^*) \) where the modules are generated by maps \( \sigma \) from a standard simplex to \( \mathcal{J}_{g,k^-}^* \). Such a
equivariant homomorphism points in compactifications of 1-dimensional moduli spaces. Hence there is an induced equivariant chain map of degree 2.

The product $S_{k^-} \times S_{k^+}$ of the permutation groups on $k^-$ and $k^+$ letters acts on the left hand side by permuting the punctures, and on the right hand side by permuting the factors. Theorem 4.3.2 and Theorem 4.3.3 essentially state that there is an induced equivariant homomorphism

$$Q : H_*(\mathcal{J}_{g,k^-+k^+}) \to HF^*(M)^{\otimes k^-} \otimes CF_*(M)^{\otimes k^+}.$$ 

Theorem 4.3.4 states that the diagram

$$\begin{array}{ccc}
H_*(\mathcal{J}_{g_1,k^-+k^+}) \otimes H_*(\mathcal{J}_{g_2,k^-+k^+}) & \xrightarrow{Q \otimes Q} & H((CF^*(M)^{\otimes k^-} \otimes CF_*(M)^{\otimes k^+}) \\
\downarrow \diamond_{ij} & \downarrow & \downarrow \diamond_{ij} \\
H_*(\mathcal{J}_{g_1+g_2,k^-+k^+}) & \xrightarrow{Q} & H((CF^*(M)^{\otimes (k^-+k^+)} \otimes CF_*(M)^{\otimes (k^-+k^+)})
\end{array}$$

commutes. The homomorphism on the right, also denoted $\diamond_{ij}$, is induced by the pairing $CF^*(M) \otimes CF_*(M) \to \mathbb{Z}$. In §4.4 we show that as a consequence, symplectic Poincaré duality is an isomorphism, the symplectic cup and intersection products are associative, they are intertwined by symplectic Poincaré duality, the symplectic unit class is an identity element for the symplectic cup product, and the symplectic top class
is an identity element for the symplectic intersection product. As another consequence, in §4.5 we show that the usual scheme for identifying Floer (co)homology groups defined using different Hamiltonians and almost complex structures is consistent, and that the homomorphism \(Q\) is independent of these choices.

One obtains the punctured surface \(\Sigma_{g+1,k^-,k^+}\) by gluing the punctured surface \(\Sigma_{g,k^-,k^+}^{+1}\) to itself. This gives a map

\[
\mathcal{C}_{ij}^{\ell} : \mathfrak{J}_{g,k^-,k^+}^{+1} \rightarrow \mathfrak{J}_{g+1,k^-,k^+}^{+1}.
\]

Theorem 4.3.5 states that the diagram

\[
\begin{array}{ccc}
H_*(\mathfrak{J}_{g,k^-,k^+}^{+1}) & \xrightarrow{Q} & H(CF^*(M)^{\otimes(k^-+1)} \otimes CF_*(M)^{\otimes(k^++1)}) \\
\downarrow \mathcal{C}_{ij} & & \downarrow \mathcal{C}_{ij} \\
H_*(\mathfrak{J}_{g+1,k^-,k^+}) & \xrightarrow{Q} & H(CF^*(M)^{\otimes k^-} \otimes CF_*(M)^{\otimes k^+})
\end{array}
\]

commutes. The homomorphism on the right, also denoted \(\mathcal{C}_{ij}\), is induced by the pairing \(CF^*(M) \otimes CF_*(M) \rightarrow \mathbb{Z}\). As a consequence of this theorem, the number of isolated perturbed pseudoholomorphic tori in \(M\), with a given conformal structure, equals the Euler characteristic of \(M\).

By Theorem 4.3.6 the homomorphism \(Q\) factors through a homomorphism

\[
Q^0 : H_*(\mathfrak{J}_{g,k^-,k^+}^{0}) \rightarrow H(CF^*(M)^{\otimes k^-} \otimes CF_*(M)^{\otimes k^+})
\]

where \(\mathfrak{J}_{g,k^-,k^+}^{0}\) is the unframed analogue of \(\mathfrak{J}_{g,k^-,k^+}^{*}\). In particular, \(Q\) vanishes on the classes given by twists of the framings and on the classes corresponding to Dehn twists.

These theorems form an attempt to make rigorous the conjectural Gromov-Witten classes \([W],[V],[Ru],[KM]\); see also \([Fu]\). These are homomorphisms

\[
Q : H_*(J_{g,k^-,k^+}) \rightarrow H(C^*(M)^{\otimes k^-} \otimes C_*(M)^{\otimes k^+})
\]

with the same gluing properties as our homomorphisms \(Q\), where \(J_{g,k^-,k^+}\) is the Mumford-Deligne compactification of the moduli space of conformal structures on \(\Sigma_{g,k^-,k^+}\).

Another rigorous approach to Gromov-Witten classes, in the genus 0 case, is the quantum cohomology of Y. Ruan and G. Tian \([RT1]\); see also \([MS1]\). The symplectic cup product and its relation to quantum cohomology is discussed in \([MS],[P],[RT2]\) and \([Sch2]\).

In a second paper we will extend our results to weakly monotone symplectic manifolds, in the sense of Hofer and Salamon \([HS]\), by imposing additional transversality conditions on the simplices spanning \(C_*(\mathfrak{J}_{g,k^-,k^+}^{*})\).

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§2. Moduli Spaces of Perturbed Pseudoholomorphic Curves

2.1. Floer’s moduli spaces $\mathcal{M}^d(\alpha^-, \alpha^+)$. Let $M$ be a compact smooth symplectic manifold of dimension $2n$ with symplectic form $\omega$. Let $H$ be a smooth function $M \times S^1 \to \mathbb{R}$. The function $H$ will play the role of a periodic time-dependent Hamiltonian on $M$. The periodic orbits for $H$ are the solutions $\alpha : S^1 \to M$ to Hamilton’s equations

$$\frac{d\alpha}{d\theta} = X_H(\alpha(\theta), \theta),$$

where $X_H(\cdot, \theta)$ denotes the symplectic gradient of $H(\cdot, \theta)$. Let $\mathcal{C}_H$ denote the set of periodic orbits. Let $\mathcal{C}_0^H$ denote the set of contractible periodic orbits.

The Hamiltonian induces an exact symplectomorphism $\phi_H : M \to M$ by $\phi_H(p) = \alpha(2\pi)$, where $\alpha : [0, 2\pi] \to M$ is the unique solution to (2.1) with initial data $\alpha(0) = p$. The periodic orbits correspond to the fixed points of $\phi_H$.

Definition 2.1.1. A periodic orbit $\alpha$ is regular if and only if $1$ is not an eigenvalue of $d_{\alpha(0)}\phi_H$. Equivalently, $\alpha$ is regular if and only if the graph of $\phi_H$ intersects the diagonal transversely in $M \times M$ at $(\alpha(0),\alpha(0))$.

A Hamiltonian $H$ is regular if and only if all periodic orbits for $H$ are regular.

Let $\mathcal{H}_M$ denote the space of smooth time-dependent Hamiltonians on $M$.

Proposition 2.1.2. The regular time-dependent Hamiltonians form a dense open subset of $\mathcal{H}_M$.

This is proven in [F4] Prop. 2c.1 and [SZ] Thm. 8.1.

An almost complex structure on $M$ is a smooth bundle map $J : TM \to TM$ such that $J^2 = -1$. An almost complex structure $J$ on $M$ is said to be compatible with $\omega$ if $\langle \xi, \zeta \rangle = \omega(\xi, J\zeta)$ is a Riemannian metric on $M$. We denote the space of compatible almost complex structures on $M$ by $\mathcal{J}_M$. If $J$ is compatible with $\omega$, then $X_H = J\nabla H$. Hamilton’s equations (2.1) then take the form

$$J \frac{d\alpha}{d\theta} + \nabla H = 0.$$

Proposition 2.1.3. The space $\mathcal{J}_M$ is contractible.

This is well known; see for instance [HZ] p. 15. It follows from Prop. 2.1.3 that $TM$ has well defined Chern classes, independently of the choice of $J$. 
Assumption 2.1.4. Throughout the paper we assume that $M$ is monotone. This means that there exists $k \geq 0$ such that $\langle \omega, a \rangle = k \langle c_1(TM), a \rangle$ for all $a \in \pi_2(M)$.

In a second paper we will extend the results of this paper to weakly monotone symplectic manifolds in the sense of [HS].

Given a regular $A = (H, J)$, Floer considered smooth maps

$$u : \mathbb{R} \times S^1 \to M$$

that satisfy the equation

$$(2.2) \quad \frac{\partial u}{\partial t} + J \frac{\partial u}{\partial \theta} + \nabla H(u, \theta) = 0,$$

where $t$ and $\theta$ are the $\mathbb{R}$ and $S^1$ coordinates, and have finite energy,

$$\int_{-\infty}^{\infty} \int_{S^1} \left( |\frac{\partial u}{\partial t}|^2 + |J \frac{\partial u}{\partial \theta} + \nabla H|^2 \right) dt d\theta < \infty.$$

Floer proved that if the Hamiltonian $H$ is regular, then for any finite energy solution $u$ to (2.2), there exist periodic orbits $\alpha^-$ and $\alpha^+$ such that

$$(2.3) \quad u(t, \theta) \to \alpha^-(\theta) \quad \text{as } t \to -\infty$$

$$u(t, \theta) \to \alpha^+(\theta) \quad \text{as } t \to \infty.$$  

Here $u$ and all its derivatives converge exponentially fast. For more details, see [F4] and [R] §3. He then defined the moduli space

$$\mathcal{M}(\alpha^-, \alpha^+)$$

as the set of finite energy solutions to (2.2) that satisfy (2.3), and he set up a deformation theory for these moduli spaces. He showed that

$$\mathcal{M}(\alpha^-, \alpha^+) = \bigcup_{d \in \mathbb{Z}} \mathcal{M}^d(\alpha^-, \alpha^+)$$

where $\mathcal{M}^d(\alpha^-, \alpha^+)$ is the zero set of a Fredholm section $\Psi^d(\alpha^-, \alpha^+)$ of index $d$ of a Hilbert space bundle $F^d(\alpha^-, \alpha^+)$ over a Hilbert manifold $\mathcal{P}^d(\alpha^-, \alpha^+)$. For more details, see [F4] and [R] §2.
**Definition 2.1.5.** The moduli space $M^d(\alpha^-, \alpha^+)$ is regular if and only if $H$ is regular and the section $\Psi^d(\alpha^-, \alpha^+)$ is transverse to the zero section of $F^d(\alpha^-, \alpha^+)$. The pair $A = (H, J)$ is regular if and only if $H$ is regular, the moduli spaces $M^d(\alpha^-, \alpha^+)$ are regular for all $d \in \mathbb{Z}$ and all $\alpha^-, \alpha^+ \in \mathbb{C} H$, and there are no pseudo-holomorphic spheres in $M$ with $c_1 = 1$ that intersect periodic orbits of $H$.

A regular moduli space $M^d(\alpha^-, \alpha^+)$ is an embedded submanifold of $P^d(\alpha^-, \alpha^+)$ of dimension $d$.

**Proposition 2.1.6.** The regular pairs $A = (H, J)$ form a dense subset of $\mathcal{H}_M \times J_M$. In fact, it is a subset of the second category in the sense of Baire. An outline of a proof was given in [F4] Prop. 2c.2. More details can be found in [SZ] Thm. 8.4, [HS] and [FHS].

The moduli spaces $M^d(\alpha^-, \alpha^+)$ are translation invariant. The only translation invariant finite energy solutions to (2.2) are the trivial ones,

$$\overline{\sigma}_0(t, \theta) = \alpha_0(\theta),$$

in $M^0(\alpha_0, \alpha_0)$. This has the following consequence.

**Proposition 2.1.7.** If $A = (H, J)$ is regular, then the moduli space $M^0(\alpha_0, \alpha_0)$ contains a single point, the trivial solution $\overline{\sigma}_0$, and if $\alpha^- \neq \alpha^+$, then the moduli space $M^0(\alpha^-, \alpha^+)$ is empty.

### 2.2. The parameter space $\mathcal{J}_{g,k^-}^{g,k^+}(A)$

Floer considered maps from the cylinder $\mathbb{R} \times S^1$ to $M$. The cylinder $\mathbb{R} \times S^1$ can also be viewed as a twice punctured sphere. We will consider maps from a Riemann surfaces of any genus with any number of punctures. We need to take into account that such a surface in general has a nontrivial moduli spaces of conformal structures.

Fix a regular pair $A = (H, J)$. For each triple $(g, k^-, k^+)$ of non-negative integers, we fix a compact oriented surface $\Sigma_{g,k^-}^{g,k^+}$ of genus $g$, $k = k^- + k^+$ distinct points $p_1^-, \ldots, p_{k^-}^-, p_1^+, \ldots, p_{k^+}^+$ on $\Sigma_{g,k^-}^{g,k^+}$, and rays $X_i^\pm \in (T_{p_i^\pm} M \setminus \{0\})/\mathbb{R}$. Let

$$\Sigma_{g,k^-}^{g,k^+} = \Sigma_{g,k^-}^{g,k^+} \setminus \{p_1^-, \ldots, p_{k^-}^-, p_1^+, \ldots, p_{k^+}^+\}.$$

Let $\mathcal{J}_g$ denote the space of smooth conformal structures on $\Sigma_{g,k^-}^{g,k^+}$. Let $\mathcal{D}_{g,k}$ denote the space of $k$-tuples

$$\Delta = (\Delta_1^-, \ldots, \Delta_{k^-}^-, \Delta_1^+, \ldots, \Delta_{k^+}^+)$$
of smooth closed pairwise disjoint disks on $\Sigma_{g, k- \cdot k+}$ such that $p_{i}^{\pm}$ lies in the interior of $\Delta_{i}^{\pm}$.

It follows from Riemann’s mapping theorem, that for any $j \in J_{g}$ and $\Delta \in D_{g,k}$, there exist unique $j$-holomorphic coordinate functions $z_{i}^{\pm} = x_{i}^{\pm} + iy_{i}^{\pm}: \Delta_{i}^{\pm} \to \{ z \in \mathbb{C} : |z| \leq 1 \}$ such that $x_{i}^{\pm}(p_{i}^{\pm}) = 0$ and $X_{i}^{\pm}$ is the ray spanned by the tangent vector $\partial/\partial x_{i}^{\pm}$. On $\Delta_{i}^{\pm} \setminus \{ p_{i}^{\pm} \}$ we use cylindrical coordinates $(t_{i}^{\pm}, \theta_{i}^{\pm})$, with $t_{i}^{\pm} + i \theta_{i}^{\pm} = \log z_{i}^{\pm}$ and $t_{i}^{\pm} + i \theta_{i}^{\pm} = \log z_{i}^{\mp}$. Then $(t_{i}^{\pm}, \theta_{i}^{\pm})$ maps $\Delta_{i}^{\pm} \setminus \{ p_{i}^{\pm} \}$ to $[0, \infty) \times S^{1}$, and $(t_{i}^{\mp}, \theta_{i}^{\mp})$ maps $\Delta_{i}^{\mp} \setminus \{ p_{i}^{\mp} \}$ to $(-\infty, 0] \times S^{1}$.

Let $\mathcal{J}_{g, k- \cdot k+}(A)$ be the smooth Fréchet space bundle over $\mathcal{J}_{g} \times D_{g,k}$ whose fiber over the pair $(j, \Delta)$ consists of all triples $(j, \Delta, R)$ where $R$ is a section of the bundle $(T^{0,1} \Sigma_{g, k- \cdot k+})^{\ast} \boxtimes TM$ over $\Sigma_{g, k- \cdot k+} \times M$ such that

$$R = (dt_{i}^{\pm} - i d\theta_{i}^{\pm}) \otimes \nabla H$$

on each $\Delta_{i}^{\pm} \times M$.

We define the framed diffeomorphism group $\text{Diff}_{g,k}$ as the group of diffeomorphisms of $\Sigma_{g, k- \cdot k+}$ that fix the points $p_{i}^{\pm}$ and the rays $X_{i}^{\pm}$. The group $\text{Diff}_{g,k}$ acts on the spaces $\mathcal{J}_{g} \times D_{g,k}$ and $\mathcal{J}_{g, k- \cdot k+}(A)$ by push-forward. We let $\mathcal{J}_{g, k- \cdot k+}^{\ast}(A)$ denote the largest subset of $\mathcal{J}_{g, k- \cdot k+}(A)$ on which $\text{Diff}_{g,k}$ acts freely. We let

$$\mathcal{J}_{g, k- \cdot k+}(A) = \mathcal{J}_{g, k- \cdot k+}^{\ast}(A)/ \text{Diff}_{g,k}$$

and

$$\mathcal{J}_{g, k- \cdot k+}^{\ast}(A) = \mathcal{J}_{g, k- \cdot k+}^{\ast}(A)/ \text{Diff}_{g,k}^{\ast}.$$

There is an alternative description of $\mathcal{J}_{g, k- \cdot k+}^{\ast}(A)$. Instead of fixing the rays $X_{i}^{\pm}$, we can make them part of the data. Let $\mathcal{J}_{g, k- \cdot k+}^{\ast}(A)$ denote the set of quadruples $(j, \Delta, X, R)$ where $X = ((X_{1}^{\pm}, \ldots, X_{k}^{\pm}) , (X_{1}^{\mp}, \ldots, X_{k}^{\mp}))$ and $X_{j}^{\pm} \in (T_{p_{j}^{\pm}} M \setminus \{ 0 \})/\mathbb{R}$. Let $\text{Diff}_{g,k}^{\ast}$ be the group of diffeomorphisms of $\Sigma_{g, k- \cdot k+}$ that fix the points $p_{i}^{\pm}$. Then

$$\mathcal{J}_{g, k- \cdot k+}(A) = \mathcal{J}_{g, k- \cdot k+}^{\ast}(A)/ \text{Diff}_{g,k}^{\ast}.$$

Let $\mathcal{J}_{g, k- \cdot k+}^{\ast}(A)$ be the largest subset of $\mathcal{J}_{g, k- \cdot k+}^{\ast}(A)$ on which $\text{Diff}_{g,k}^{\ast}$ acts freely. Then

$$\mathcal{J}_{g, k- \cdot k+}(A) = \mathcal{J}_{g, k- \cdot k+}^{\ast}(A)/ \text{Diff}_{g,k}^{\ast}.$$

If the Hamiltonian $H$ is time independent, then the section $R$ can be chosen independently of the rays $X_{i}^{\pm}$. Then the group $\text{Diff}_{g,k}^{\ast}$ acts naturally on $\mathcal{J}_{g, k- \cdot k+}$, and we define

$$\mathcal{J}_{g, k- \cdot k+}(A) = \mathcal{J}_{g, k- \cdot k+}^{\ast}(A)/ \text{Diff}_{g,k}^{\ast}.$$
Let $\widehat{\mathcal{J}}_{g,k_-}^{0*}(A)$ denote the largest subset of $\widehat{\mathcal{J}}_{g,k_-}^{0}(A)$ on which $\text{Diff}^+_g,k$ acts freely. Then we define
\[
\widehat{\mathcal{J}}_{g,k_-}^{0*}(A) = \widehat{\mathcal{J}}_{g,k_-}^{0}(A)/\text{Diff}^+_g,k.
\]

**Proposition 2.2.1.** If $k = k^- + k^+ > 0$, then $\widehat{\mathcal{J}}_{g,k_-}^{0}(A) = \widehat{\mathcal{J}}_{g,k_-}^{0*}(A)$ and $\mathcal{J}_{g,k_-}^{0*}(A) = \mathcal{J}_{g,k_-}^{0}(A)$. The space $\mathcal{J}_{g,k_-}^{0*}(A)$ is a weakly contractible smooth Fréchet manifold. The space $\mathcal{J}_{g,k_-}^{0*}(A)$ is a smooth Fréchet manifold. The manifold $\mathcal{J}_{g,k_-}^{0*}(A)$ is a smooth principal $\text{Diff}^+_g,k$ bundle over the manifold $\mathcal{J}_{g,k_-}^{0*}(A)$.

If $H$ is time-independent, then similar statements hold for $\widehat{\mathcal{J}}_{g,k_-}^{0*}(A)$, $\text{Diff}^+_g,k$, and $\mathcal{J}_{g,k_-}^{0*}(A)$.

It is tempting to say that $\mathcal{J}_{g,k_-}^{0*}(A)$ and $\mathcal{J}_{g,k_-}^{0}(A)$ are classifying spaces for the groups $\text{Diff}^+_g,k$ and $\text{Diff}^+_g,k$. However, our arguments fall slightly short of this statement, as we do not show that the spaces and group actions are simplicial.

**Proof.** The space $\mathcal{J}_{g,k_-}^{0*}(A)$ is clearly a smooth Fréchet manifold by definition. The space $\mathcal{J}_{g,k_-}^{0*}(A)$ is an open subset of $\mathcal{J}_{g,k_-}^{0}(A)$ and is hence a smooth Fréchet manifold.

The space $\mathcal{J}_{g}$ is contractible by Prop. 2.1.3. It is not hard to show, using Riemann’s mapping theorem, that the space $\mathcal{D}_{g,k}$ is contractible. The space of smooth sections $R$ is a vector space, and is hence contractible. It follows that $\mathcal{J}_{g,k_-}^{0*}(A)$ is contractible. The complement of $\mathcal{J}_{g,k_-}^{0*}(A)$ essentially has infinite codimension. A simple perturbation argument shows that $\mathcal{J}_{g,k_-}^{0*}(A)$ is weakly contractible.

Assume that $k > 0$. If a diffeomorphism $\varphi \in \text{Diff}^+_g,k$ fixes a conformal structure $J$, a disk $\Delta_i^\pm$ and the ray $X_i^\pm$, it then follows from Riemann’s mapping theorem that the restriction of $\varphi$ to $\Delta_i^\pm$ is the identity map. By analytic continuation $\varphi$ is then identity map. Hence $\mathcal{J}_{g,k_-}^{0*}(A) = \mathcal{J}_{g,k_-}^{0}(A)$.

We now turn to the main part of the theorem. The action of $\text{Diff}^+_g,k$ on $\mathcal{J}_{g,k_-}^{0*}(A)$ is free by definition. To show that the action defines a principal bundle over a smooth Fréchet manifold, we need to construct smooth local sections. The tools for doing this are provided by Teichmüller theory. For a global analysis approach to Teichmüller theory, see [EE], [J] and [T]. Let $\text{Diff}^0_{g,0}$ denote the identity component of $\text{Diff}^+_{g,0}$. The Teichmüller space $T_{g,0}$ is defined as the quotient $\mathcal{J}_g/\text{Diff}^0_{g,0}$. By Teichmüller’s theorem, $T_{g,0}$ is a finite dimensional complex manifold, diffeomorphic to an open ball. The action of $\text{Diff}^0_{g,0}$ on $\mathcal{J}_g$ is free, except for $g = 1$, in which case the stabilizer of each point is $S^1 \times S^1$, and $g = 0$, in which case the stabilizer is the Möbius group. By the Earle-Eells theorem the action of $\text{Diff}^0_{g,0}$ on $\mathcal{J}_g$ defines a smooth fiber bundle.
The mapping class group $\Gamma_g = \text{Diff}_{g,0}/\text{Diff}^0_{g,0}$ is a discrete group that acts on $T_{g,0}$. The quotient $T_{g,0}/\text{Diff}_{g,0} = T_{g,0}/\Gamma_g$ is the moduli space of conformal structures on $\Sigma_{g,0,0}$. This action of $\Gamma_g$ on $T_{g,0}$ is free except at a discrete set of points that have finite stabilizers, and by Kravetz’ theorem, the action is properly discontinuous.

The case $g \geq 1$: Let $\text{Diff}^0_{g,k} = \text{Diff}_{g,k} \cap \text{Diff}^0_{g,0}$. Then $\text{Diff}_{g,k}/\text{Diff}^0_{g,0} = \Gamma_g$. Let $\mathcal{T}_{g,k} = \mathcal{J}_{g}/\text{Diff}^0_{g,k}$. It follows from Kravetz’ theorem that the action of $\Gamma_k$ on $\mathcal{T}_{g,k}$ is properly discontinuous. In particular it admits local slices. It follows from the Earle-Eells theorem that the action of $\text{Diff}^0_{g,k}$ on $\mathcal{J}_g$ defines a trivial smooth fiber bundle. In particular the action admits local slices. It follows that the action of $\text{Diff}_{g,k}$ on $\mathcal{J}_g$ admits local slices. Each point has a compact stabilizer. It follows that the action of $\text{Diff}_{g,k}$ on $\mathcal{J}_{g,k}^{-,k+}(A)$ admits local slices. As this action is free, it gives a principal fiber bundle.

The case $g = 0$, $k \geq 3$: Select three of the points $p_{i}^{\pm}$. Let $\text{Diff}^*_{0,3}$ be the group of diffeomorphisms that fix these three points, but not necessarily the corresponding rays $X_i^{\pm}$. The action of $\text{Diff}^*_{0,3}$ on $\mathcal{J}_0$ is free and transitive. By the Earle-Eells theorem it gives a diffeomorphism between $\text{Diff}^*_{0,3}$ and $\mathcal{J}_0$. We can then argue as in the case $g \geq 1$, using the group $\text{Diff}^*_{0,3}$ in place of $\text{Diff}_{g,0}$.

The case $g = 0$, $k = 1, 2$: If $k = 2$, then the quotient of $\mathcal{J}_0$ by $\text{Diff}_{0,2}$ is $S^1$, and the stabilizer of any $j \in \mathcal{J}_0$ is $\mathbb{R}$. If $k = 1$, then the group $\text{Diff}_{0,1}$ acts transitively on $\mathcal{J}_0$, and the stabilizer of any $j \in \mathcal{J}_0$ is the group of affine transformations of the form $z \mapsto az + b$ with $a \in \mathbb{R}$ and $b \in \mathbb{C}$. In either case we need to verify that the action of the stabilizer of any $j \in \mathcal{J}_0$ on $\mathcal{D}_{0,k}$ defines a principal fiber bundle. To show that a free action by a noncompact finite dimensional group defines a principal bundle one has to show that the orbits are closed. It is not hard to see that the orbit of any $\Delta \in \mathcal{D}_{0,k}$ under these two groups is closed.

The case $g = k = 0$: This case is handled the same way as the case $g = 0$, $k = 1, 2$.

Theorem follows from the observation that the orbit of any nonzero section $R$ of $(T^{0,1}\Sigma_{0,0,0})^* \mathcal{E}_C TM$ under the action of the Möbius group is closed.

2.3. The parametrized moduli spaces $\mathcal{M}_g^d((\alpha_1^-, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))$. Let $\eta$ be a smooth cut-off function with $\eta = 0$ on $(-\infty, \frac{1}{3})$ and $\eta = 1$ on $[\frac{2}{3}, \infty)$. We then define the energy of a map $u : \Sigma_{g,k}^{-,k+} \rightarrow M$ as

\begin{equation}
E[u] = \int_{\Sigma_{g,k}^{-,k+}} |du - \sum \eta(\pm t_i^+) \frac{\partial}{\partial t_i} \otimes \nabla H(u, \theta_i^+)|^2 dA.
\end{equation}
**Definition 2.3.1.** A smooth map $u : \Sigma_{g,k-} \to M$ is a perturbed pseudoholomorphic curve with data $(j, \Delta, R) \in \mathcal{J}_{g,k-}(A)$, if and only if $E[u] < \infty$, and, in terms of local complex coordinates $z = x + iy$, compatible with $j$,

\[
(2.5) \quad (dx - idy) \otimes \left( \frac{\partial u}{\partial x} + J(u(x,y)) \frac{\partial u}{\partial y} \right) + R((x,y),u(x,y)) = 0.
\]

We often write this equation $\overline{\partial} u + R(\cdot, u) = 0$. The left hand side is a section of $(T^{0,1}\Sigma_{g,k-})^* \otimes C TM$. On the coordinate charts $(t^\pm_i, \theta^\pm_i)$ the equation takes the simpler form

\[
\frac{\partial u}{\partial t_i^\pm} + J(u(t_i^\pm, \theta_i^\pm)) \frac{\partial u}{\partial \theta_i^\pm} + \nabla H(u(t_i^\pm, \theta_i^\pm), \theta_i^\pm) = 0.
\]

We define the moduli space $\mathcal{M}_{g,k-}$ as the set of equivalence classes $[u, c]$, under the action of the group $\text{Diff}_{g,k}$ of pairs $(u,c)$, where $u$ is a perturbed pseudoholomorphic curve with data $c = (j, \Delta, R)$. It follows from the results of [F4], see also [R] Prop. 3.1, that if $[u, c] \in \mathcal{M}_{g,k-}$, then $u$ converges exponentially fast to a periodic orbit $\alpha_i^\pm$ at each puncture $p_i^\pm$. This gives a decomposition

\[
\mathcal{M}_{g,k-} = \bigcup_{\alpha_i^- \ldots, \alpha_i^- \in \mathcal{U}} \mathcal{M}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+)\).
\]

Let

\[
\mathcal{P}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+))
\]

denote the space of maps $u : \Sigma_{g,k-} \to M$ such that $u$ is of Sobolev class $H^3$ locally, and on each $\Delta_i^\pm$, the map $u$ satisfies the same asymptotic conditions as in the definition of $\mathcal{P}(\alpha^-, \alpha^+)$, see [F4] and [R] §2, but with limit $\alpha_i^\pm$. Thus defined, $\mathcal{P}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+))$ is a Hilbert manifold.

The tangent bundle $T\mathcal{P}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+))$ is the Hilbert space bundle with fibers

\[
T_u \mathcal{P}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+)) = H^3(\Sigma_{g,k-}, u^* TM).
\]

By Prop. 2.2.1,

\[
\mathcal{P}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+)) = \mathcal{P}_{g,k-}((\alpha_i^+, \ldots, \alpha_i^-), (\alpha_i^+, \ldots, \alpha_i^+)) \times_{\text{Diff}_{g,k}} \mathcal{J}_{g,k-}(A)
\]
Finally, let \( F_{g,k} \) be a smooth Hilbert space bundle over \( P \). The fiber with fiber \( P \) is a smooth Hilbert space bundle over \( (2.6) \). The fiber \( \Psi_{g,k} \) contained in \( P \) is a smooth Hilbert space bundle over \( (2.6) \). Arguing as in [F4], see also [R] Prop. 3.1, we see that the moduli space \( F_{g,k} \) of \( (2.6) \) is contained in \( \Psi_{g,k} \). Furthermore, by Prop. 2.2.1,

\[
T^{\text{fib}} \Psi_{g,k} = T^g_{\text{fib}} \Psi_{g,k} + (\alpha_1^-, \ldots, \alpha_{k_1}^-), (\alpha_1^+ \ldots, \alpha_{k_1}^+) \times \text{Diff}_{g,k} \hat{\mathcal{J}}_{g,k}^* \rightarrow F_{g,k}(A).
\]

is a smooth Hilbert space bundle over \( \Psi_{g,k} \). The fiber \( T^{\text{fib}} \Psi_{g,k} \) of \( T^{\text{fib}} \Psi_{g,k}^* \) at \( [u,c] \) is

\[
H^2(\Sigma_{g,k}^*, u^*TM).
\]

Finally, let \( F_{g,k} \) be the smooth Hilbert space bundle over \( P \) with fibers \( H^2((T^{0.1} \Sigma_{g,k}^*)^* \otimes u^*TM) \). By Prop. 2.2.1,

\[
F_{g,k} = F_{g,k} \times \text{Diff}_{g,k} \hat{\mathcal{J}}_{g,k}^* \rightarrow F_{g,k}(A)
\]

is a smooth Hilbert space bundle over \( \Psi_{g,k} \). The fiber \( (F_{g,k})_{[u,c]} \) of \( F_{g,k} \) at \( [u,c] \) is

\[
H^2((T^{0.1} \Sigma_{g,k}^*)^* \otimes u^*TM).
\]

The left hand side of (2.5) defines a section

\[
\Psi_{g,k} : \Psi_{g,k} \rightarrow F_{g,k}.
\]

Arguing as in [F4], see also [R] Prop. 3.1, we see that the moduli space \( \mathcal{M}_{g,k} \) is contained in \( \Psi_{g,k} \) and is the zero locus of \( \Psi_{g,k} \).

For \( [u,c] \in \mathcal{M}_{g,k} \), we let

\[
D_{[u,c]} \Psi_{g,k} : T_{[u,c]} \Psi_{g,k} \rightarrow \Psi_{g,k} \rightarrow (F_{g,k})_{[u,c]}
\]

de note the intrinsic derivative of \( \Psi_{g,k} \) at \( [u,c] \). We let

\[
D_{[u,c]}^{\text{fib}} \Psi_{g,k} : T^{\text{fib}}_{[u,c]} \Psi_{g,k} \rightarrow (F_{g,k})_{[u,c]}
\]
denote the restriction of $D_{[u,c]} u \rightarrow \Psi_{g,k} - k^+ + T_{[u,c]}^{\text{fib}} \Psi_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))$.

As in [R] Prop. 2.1,

\[
D_{[u,c]}^{\text{fib}} u \rightarrow \Psi_{g,k} - k^+ + \xi = (dx - i dy) \otimes \left( \nabla_x \xi + J \nabla_y \xi + T \left( \xi, \frac{\partial u}{\partial x} \right) + J T \left( \xi, \frac{\partial u}{\partial y} \right) \right) + \nabla \xi \nabla R.
\]

Arguing as in [F4], see also [R] Thm. 2.2, we see that $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi_{g,k} - k^+ + \xi$ is Fredholm. This gives a decomposition

\[
\mathcal{M}_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+)) = \bigcup_{d \in \mathbb{Z}} \mathcal{M}^d_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))
\]

according to index of $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi_{g,k} - k^+ + \xi$. By taking (2.7) as a definition, we get a family $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi_{g,k} - k^+ + \xi$ of Fredholm operators $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi_{g,k} - k^+ + \xi$ parametrized by $\mathcal{P}_{g,k} - k^+ + \xi$. This gives a decomposition

\[
\mathcal{P}_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+)) = \bigcup_{d \in \mathbb{Z}} \mathcal{P}^d_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))
\]

according to the index of $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi_{g,k} - k^+ + \xi$. We write $\mathcal{F}^d_{g,k} - k^+ + \xi$ and $\mathcal{P}^d_{g,k} - k^+ + \xi$ for the restrictions of $\mathcal{F}_{g,k} - k^+ + \xi$ and $\mathcal{P}_{g,k} - k^+ + \xi$ to $\mathcal{P}^d_{g,k} - k^+ + \xi$.

**Proposition 2.3.2.** The moduli space $\mathcal{M}^d_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))$ is an embedded smooth Fréchet submanifold of $\mathcal{P}^d_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))$. The projection

\[
\pi : \mathcal{M}^d_{g,k} - k^+ ((\alpha_1, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+)) \rightarrow \mathcal{F}^d_{g,k} - k^+ + \xi (A)
\]

is a $\sigma$-proper Fredholm map of index $d$.

**Proof.** Let $[u,c] \in \mathcal{M}^d_{g,k} - k^+ + \xi$. The annihilator of the range of $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi^d_{g,k} - k^+ + \xi$ is the kernel of the adjoint operator. This is an elliptic differential operator. By the Aronzajn unique continuation theorem, a section in the annihilator of the range can not vanish on an open subset of $\Sigma_{g,k} - k^+$. On the other hand, by varying $R$ we see that the range of $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi^d_{g,k} - k^+ + \xi$ contains all elements of $(\mathcal{F}^d_{g,k} - k^+ + \xi)_{[u,c]}$ that are smooth and supported on the complement of the disks $\Delta_i^\pm$. Hence the annihilator of the range consists of sections supported on the disks $\Delta_i^\pm$. It follows that the annihilator of the range of $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi^d_{g,k} - k^+ + \xi$ is trivial, and that the range is dense. As $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi^d_{g,k} - k^+ + \xi$ is Fredholm, it is then surjective.

Let $U \subset T_{[u,c]}^{\text{fib}} \mathcal{M}^d_{g,k} - k^+ + \xi$ be the kernel of $D_{[u,c]}^{\text{fib}} u \rightarrow \Psi^d_{g,k} - k^+ + \xi$. Let $U^\perp$ be a linear complement to $U$ in $T_{[u,c]}^{\text{fib}} \mathcal{M}^d_{g,k} - k^+ + \xi$. Let $W$ be a linear complement to $T_{[u,c]}^{\text{fib}} \mathcal{M}^d_{g,k} - k^+ + \xi$ in
$T_{[u,c]} \mathcal{M}^d_{g,k-}$. As $D_{[u,c]} \mathcal{P}^d_{g,k-}$ is surjective, there exists a finite dimensional subspace $V$ of $W$ such that $D_{[u,c]} \mathcal{P}^d_{g,k-} : U \perp V \to (\mathcal{P}^d_{g,k-})_{[u,c]}$ is invertible. Let $V^\perp$ be a linear complement to $V$ in $W$. We can then parametrize a neighborhood of $[u,c]$ in $\mathcal{P}^d_{g,k-}$ by a neighborhood of $0$ in $T_{[u,c]} \mathcal{P}^d_{g,k-} = U \oplus U \perp V \oplus V^\perp$. By the implicit function theorem, applied to the Hilbert space $U \perp V$ with the Fréchet space $U \oplus V^\perp$ as parameter space, there exist smooth maps $S_1 : U \oplus V^\perp \to U$ and $S_2 : U \oplus V^\perp \to V$ such that the map $u \oplus v \perp \mapsto u + v^\perp + S_1(u \oplus v^\perp) + S_2(u \oplus V^\perp)$ gives a local parametrization of a neighborhood of $[u,c]$ in $\mathcal{M}^d_{g,k-}$. Hence $\mathcal{M}^d_{g,k-}$ is an embedded Fréchet submanifold of $\mathcal{P}^d_{g,k-}$.

We can parametrize a neighborhood of $[c]$ in $\mathcal{J}^*_g,k-,(A)$ by a neighborhood of $0$ in $W = V \oplus V^\perp$. We may assume that the projection $\mathcal{P}^d_{g,k-} \to \mathcal{J}^*_g,k-,(A)$ is given, in terms of the parametrizations, by the projection $U \oplus U \perp \oplus V \oplus V^\perp \to V \oplus V^\perp$. In terms of these parametrizations $\pi$ is the map

$$U \oplus V^\perp \to V \oplus V^\perp$$

given by

$$u \oplus v^\perp \mapsto S_2(u \oplus v^\perp) + v^\perp.$$ 

Hence

$$\text{index } D\pi = \dim U - \dim V = \text{index } D_{[u,c]} \mathcal{P}^d_{g,k-} = d.$$ 

Finally, that the map $\pi$ is $\sigma$-proper follows by Uhlenbeck-Gromov compactness.

\[\square\]

\textbf{Definition 2.3.3.} For any smooth $\sigma : [0,1]^q \to \mathcal{J}^*_g,k-,(A)$ the parametrized moduli space

$$\mathcal{M}^d_\sigma((\alpha^-_1, \ldots, \alpha^-_k), (\alpha^+_1, \ldots, \alpha^+_k))$$

is defined as the set of pairs

$$(u,x) \in \mathcal{M}^{d-q}_{g,k-}((\alpha^-_1, \ldots, \alpha^-_k), (\alpha^+_1, \ldots, \alpha^+_k)) \times [0,1]^q$$

such that $\pi(u) = \sigma(x)$.

\textbf{Definition 2.3.4.} A map $\sigma : [0,1]^q \to \mathcal{J}^*_g,k-,(A)$ is regular if and only if $\sigma$, and the restrictions of $\sigma$ to the boundary faces of $[0,1]^q$ of any dimension, are transverse to $\pi : \mathcal{M}^d_{g,k-}((\alpha^-_1, \ldots, \alpha^-_k), (\alpha^+_1, \ldots, \alpha^+_k)) \to \mathcal{J}^*_g,k-,(A)$ for all integers $d$ and all $\alpha^\pm \in \mathcal{C}_H$. 

\[\square\]
It is semi-regular if and only if \( \sigma \) is transverse to \( \pi \) for \( d \leq -q \), and its restriction to any boundary face of \([0, 1]^q\) is transverse to \( \pi \) for \( d \leq -q' \) where \( q' \) is the dimension of the boundary face.

If \( \sigma : [0, 1]^q \to J^*_{g,k-}+ (A) \) is regular, then \( \mathcal{M}_\sigma^d \) is empty for \( d < 0 \), and for \( d \geq 0 \) it is an embedded submanifold of \( \Psi_{g,k-,k+}^d \) of dimension \( d \) with corners. If \( \sigma : [0, 1]^q \to J^*_{g,k-}+ (A) \) is semi-regular, then \( \mathcal{M}_\sigma^d \) is empty for \( d < 0 \), and it is discrete set of points for \( d = 0 \).

Remark 2.3.5. There is an alternative description of \( \mathcal{M}_\sigma^d \). Let \( \Psi_\sigma^d = \sigma^* \Psi_{g,k-,k+}^{d-q}. \)

The pull-backs are \( \Psi_\sigma^d \) is a fiber bundle over \([0, 1]^q\) with fiber \( \Psi_{g,k-,k+}^{d-q} \). Let \( F_\sigma^d = \sigma^* F^{d-q}. \) Then \( F_\sigma^d \) is a Hilbert space bundle over \( \Psi_\sigma^d \). Let \( \Psi_\sigma^d = \sigma^* \Psi_{g,k-,k+}^{d-q}. \) Then \( \Psi_\sigma^d \) is the zero locus of \( \Psi_\sigma^d \).

Proposition 2.3.6. A map \( \sigma : [0, 1]^q \to J^*_{g,k-}+ (A) \) is transverse to \( \pi : \mathcal{M}_{g,k-,k+}^{d-q} \to J^*_{g,k-}+ (A) \) if and only if the section \( \Psi_\sigma^d \) is transverse to the zero section of \( F_\sigma^d \).

Proof. The map \( \sigma : [0, 1]^q \to J^*_{g,k-}+ (A) \) is transverse to the map \( \pi : \mathcal{M}_{g,k-,k+}^{d-q} \to J^*_{g,k-}+ (A) \) if and only if the lifted map \( \sigma : \Psi_\sigma^d \to \Psi_{g,k-,k+}^{d-q} \) is transverse to \( \mathcal{M}_{g,k-,k+}^{d-q} \). This is the case if and only if the lifted map \( \sigma : F_\sigma^d \to F_{g,k-,k+}^{d-q} \) is transverse to the intersection of the section \( \Psi_{g,k-,k+}^{d-q} \) and the zero section of \( F_{g,k-,k+}^{d-q} \). This map and these two manifolds are pairwise transverse. Hence the above condition says that the triple is transverse. But the triple is transverse if and only if the pull-backs of the section \( \Psi_{g,k-,k+}^{d-q} \) and the zero section of \( F_{g,k-,k+}^{d-q} \) under \( \sigma \) are transverse sections of \( \Psi_\sigma^d \). The pull-backs are \( \Psi_\sigma^d \) and the zero section of \( F_\sigma^d \). \( \square \)

Remark 2.3.7. Note that

\[ M^d(\alpha^-, \alpha^+) = \mathcal{M}_\sigma^d((\alpha^-), (\alpha^+)) \]

and

\[ \mathcal{P}^d(\alpha^-, \alpha^+) = \mathcal{P}_\sigma^d((\alpha^-), (\alpha^+)) \]

where \( \sigma_0 : [0, 1]^0 \to J_{0,1,1}^0 \) maps the standard 0-simplex to the standard conformal structure on \( \Sigma_{0,1,1} = \mathbb{R} \times S^1 \) and the standard perturbation \( R = (dt - i d\theta) \otimes \nabla H \). For the sake of definiteness, we can take \( \Delta^- \) and \( \Delta^+ \) to be the closures of \((-\infty, -1] \times S^1 \) and \([1, \infty) \times S^1 \) in \( \Sigma_{0,1,1} \). Thus the results about \( M_\sigma^d \) and \( \mathcal{P}_\sigma^d \) that we will show in §3 apply, mutatis mutandis, to \( M^d(\alpha^-, \alpha^+) \) and \( \mathcal{P}^d(\alpha^-, \alpha^+) \).
§3. Excision

3.1. The gluing maps $\diamondsuit_{ij}^\ell$ and $\mathcal{C}^\ell_{ij}$. For any $\ell \geq 0$ we define a smooth map

$$\diamondsuit_{ij}^\ell : \mathcal{J}^*_{g_1,k_1^-,k_1^+ + 1}(A) \times \mathcal{J}^*_{g_2,k_2^-,k_2^+ + 1}(A) \to \mathcal{J}^*_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}(A)$$

as follows. The definition is illustrated in fig. 1. We denote the punctures, disks, and coordinates on $\Sigma_{g_1,k_1^-,k_1^+ + 1}$ by $p^i_{1,i}$, $\Delta^i_{1,i}$, and $(t^i_{1,i}, \theta^i_{1,i})$. We denote the punctures, disks, and coordinates on $\Sigma_{g_2,k_2^-,k_2^+ + 1}$ by $p^i_{2,i}$, $\Delta^i_{2,i}$, and $(t^i_{2,i}, \theta^i_{2,i})$. We can then form a surface $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ by gluing $p^i_{1,i}$ to $p^i_{2,i}$ as follows. We remove the disk $t^i_{1,i} < \ell$ from $\Sigma_{g_1,k_1^-,k_1^+ + 1}$ and the disk $t^i_{2,j} < -\ell$ from $\Sigma_{g_2,k_2^-,k_2^+ + 1}$. We then identify the annulus $0 \leq t^i_{1,i} \leq \ell$ on $\Sigma_{g_1,k_1^-,k_1^+ + 1}$ with the annulus $-\ell \leq t^i_{2,j} \leq 0$ on $\Sigma_{g_2,k_2^-,k_2^+ + 1}$ by identifying $(t^i_{1,i}, \theta^i_{1,i})$ with $(t^i_{2,j}, \theta^i_{2,j})$ if $t^i_{2,j} = t^i_{1,i} - \ell$ and $\theta^i_{2,j} = \theta^i_{1,i}$. On the corresponding annulus on $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ we use coordinates $(t_0, \theta_0)$ defined by $t_0 = t^i_{1,i} = t^i_{2,j} + \ell$ and $\theta_0 = \theta^i_{1,i} = \theta^i_{2,j}$. We refer to these annuli on $\Sigma_{g_1,k_1^-,k_1^+ + 1}$, $\Sigma_{g_2,k_2^-,k_2^+ + 1}$ and $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ as the necks. We order the punctures on $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ as follows:

$$((p^i_{2,j-1}, \ldots, p^i_{1,1}, \ldots, p^i_{1,i}, \ldots, p^i_{2,j+1}, \ldots, p^i_{2,j+1}),$$

$$(p^i_{1,1}, \ldots, p^i_{1,i-1}, p^i_{1,i}, \ldots, p^i_{2,j}, \ldots, p^i_{2,j}, p^i_{1,1}, \ldots, p^i_{1,i+1})).$$

Then $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ is diffeomorphic to the fixed surface $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$. Choose a diffeomorphism $\varphi^\ell : \Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+} \to \Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$.

Let $(j_1, \Delta_1, R_1) \in \mathcal{J}^*_{g_1,k_1^-,k_1^+ + 1}(A)$ and $(j_2, \Delta_2, R_2) \in \mathcal{J}^*_{g_2,k_2^-,k_2^+ + 1}(A)$. Identify the complement of the neck in $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ with appropriate parts of the complement of the necks in $\Sigma_{g_1,k_1^-,k_1^+ + 1} \cup \Sigma_{g_2,k_2^-,k_2^+ + 1}$. Let $j_3$ be the conformal structure on $\Sigma_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}$ that equals $j_1$ and $j_2$ off the neck, and equals the conformal structure given by the coordinates $(t_0, \theta_0)$ on the neck.

Let $\Delta_3 = ((\Delta_{2,1}^-, \ldots, \Delta_{2,j-1}^-, \Delta_{1,1}^-, \ldots, \Delta_{1,k_1}^-, \Delta_{2,j+1}^-, \ldots, \Delta_{2,k_2}^-),$

$$(\Delta_{1,1}^+, \ldots, \Delta_{1,i-1}^+, \Delta_{2,1}^+, \ldots, \Delta_{2,k_2}^+, \Delta_{1,j+1}^+, \ldots, \Delta_{1,k_1+1}^+))$$

Let $R_3$ be equal to $R_1$ and $R_2$ off the neck, and equal to $(dt_0 - i d\theta_0) \otimes \nabla H$ on the neck. Then $(\varphi^\ell_{*} j_3, \varphi^\ell_{*} \Delta_3, \varphi^\ell_{*} R_3) \in \mathcal{J}^*_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}(A)$. The corresponding equivalence class $[\varphi^\ell_{*} j_3, \varphi^\ell_{*} \Delta_3, \varphi^\ell_{*} R_3] \in \mathcal{J}^*_{g_1+g_2,k_1^-,k_2^+,k_1^+ + k_2^+}(A)$ only depends on $[j_1, \Delta_1, R_1] \in \mathcal{J}^*_{g_1,k_1^-,k_1^+ + 1}(A)$ and $[j_2, \Delta_2, R_2] \in \mathcal{J}^*_{g_2,k_2^-,k_2^+ + 1}(A)$. We define (3.1) by

$$[j_1, \Delta_1, R_1] \circ \diamondsuit_{ij}^\ell [j_2, \Delta_2, R_2] = [\varphi^\ell_{*} j_3, \varphi^\ell_{*} \Delta_3, \varphi^\ell_{*} R_3].$$
In particular, given smooth maps
\[ \sigma_1 : [0, 1]^{q_1} \to J^*_{g_1, k^-_1, k^+_1 + 1}(A) \]
and
\[ \sigma_2 : [0, 1]^{q_2} \to J^*_{g_2, k^-_2 + 1, k^+_2}(A), \]
we can form the map
\[ \sigma_1 \diamond_{ij}^\ell \sigma_2 : [0, 1]^{q_1 + q_2} \to J_{g_1 + g_2, k^-_1 + k^-_2, k^+_1 + k^+_2}(A). \]

defined as
\[ (\sigma_1 \diamond_{ij}^\ell \sigma_2)(x_1, \ldots, x_{q_1 + q_2}) = \sigma_1(x_1, \ldots, x_{q_1}) \diamond_{ij}^\ell \sigma_2(x_{q_1 + 1}, \ldots, x_{q_1 + q_2}). \]

We obtain a similar map
\[ (3.2) \quad C_{ij}^\ell : J^*_{g, k^- + 1, k^+ + 1}(A) \to J_{g + 1, k^-, k^+}(A) \]
by gluing \( p^+_i \) to \( p^+_j \) on \( \Sigma_{g, k^- + 1, k^+ + 1} \) to form a surface \( \Sigma_{g, k^-, k^+} \). The details are the same as in the definition of \( \diamond_{ij}^\ell \). We order the punctures on \( \Sigma_{g, k^-, k^+} \) as follows:
\[ ((p^-_1, \ldots, p^-_{j-1}, p^-_{j+1}, \ldots, p^-_{k-1}), (p^+_1, \ldots, p^+_{i-1}, p^+_{i+1}, \ldots, p^+_{k+1})). \]

In particular, given a smooth map
\[ \sigma : [0, 1]^q \to J^*_{g, k^- + 1, k^+ + 1}(A), \]
we can form the map
\[ C_{ij}^\ell \sigma : [0, 1]^q \to J_{g + 1, k^-, k^+}(A). \]

3.2. The excision maps \( g_{ij}^\ell \) and \( c_{ij}^\ell \). In this section we will construct a lift
\[ g_{ij}^\ell : \mathfrak{P}^+_{g_1, k^-_1, k^+_1 + 1; i} [\alpha_0] \times \mathfrak{P}^-_{g_2, k^-_2 + 1, k^+_2; j} [\alpha_0] \to \mathfrak{P}^*_{g_1 + g_2, k^-_1 + k^-_2, k^+_1 + k^+_2} \times \mathfrak{P}(\alpha_0, \alpha_0) \]
of the map (3.1), where
\[ \mathfrak{P}^+_{g_1, k^-_1, k^+_1 + 1; i} [\alpha_0] = \mathfrak{P}_{g_1, k^-_1, k^+_1 + 1}((\alpha^+_1, 1, \ldots, \alpha^+_1, k^-_1), (\alpha^+_1, i, \ldots, \alpha^+_1, i-1, \alpha_0, \alpha^+_1, i+1, \ldots \alpha^+_1, k^+_1 + 1)), \]
\[ \mathfrak{P}^-_{g_2, k^-_2 + 1, k^+_2; j} [\alpha_0] = \mathfrak{P}_{g_2, k^-_2 + 1, k^+_2}((\alpha^-_{2, 1}, \ldots, \alpha^-_{2, j-1}, \alpha_0, \alpha^-_{2, j+1}, \ldots \alpha^-_{2, k^+_2 - 1}), (\alpha^+_{2, 1}, \ldots, \alpha^+_{2, k^+_2})). \]
and
\[
\mathcal{P}_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2} = \mathcal{P}_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2}((\alpha_{2,1}, \ldots, \alpha_{2,j-1}, \alpha_{1,1}, \ldots, \alpha_{1,k^-_1}, \alpha_{2,j+1}, \ldots, \alpha_{2,k^-_2+1}), \\
(\alpha^+_1, \ldots, \alpha^+_{1,i-1}, \alpha^+_2, \ldots, \alpha^+_{2,k^+_2}, \alpha^+_{1,i+1}, \ldots, \alpha^+_{1,k^+_1+1}).
\]

We use similar notation for the bundles \( T^n \mathcal{P} \) and \( \mathbb{R} \) over these spaces. The map \( g^\ell_{ij} \) will only be defined on an open subset of \( \mathcal{P}_{g_1,k^-_1+k^-_2,1;i}[\alpha_0] \times \mathcal{P}^*_{g_2,k^-_2,1,j}[\alpha_0] \).

Let \( \Sigma_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2} \) be the surface obtained by gluing \( p_i^* \) on \( \Sigma_{g_1,k^-_1,k^+_1+1} \) to \( p_j^{-1} \) on \( \Sigma_{g_2,k^-_2,1,k^+_2} \) as in §3.1. We also get a copy of \( \mathbb{R} \times S^1 \) by forming the union of the punctured disk \( t^+_1 \) \( \theta^+_{1,i} \) and \( t^-_2 \) \( \theta^-_{2,j} \) if \( t^+_2 = t^-_1 + \ell \) and \( \theta^-_{2,j} = \theta^+_{1,i} \). On \( \mathbb{R} \times S^1 \) we have global coordinates \((t, \theta)\) defined by \( t = t^+_1 \) and \( \theta = \theta^+_{1,i} \), or \( t = t^-_2 + \ell \) and \( \theta = \theta^-_{2,j} \). Let \( \eta \) be a smooth function \( \mathbb{R} \to [0,1] \) with \( \eta = 0 \) on \( (-\infty, 1/3) \) and \( \eta = 1 \) on \( [2/3, \infty) \).

The domain of \( g^\ell_{ij} \) consists of all
\[
([u_1, c_1], [u_2, c_2]) \in \mathcal{P}_{g_1,k^-_1,k^+_1+1;i}[\alpha_0] \times \mathcal{P}^-_{g_2,k^-_2,1,j}[\alpha_0],
\]
such that
\[
[c_1] \triangleright^\ell_{ij} [c_2] \in \mathcal{J}^*_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2}(A),
\]
dist\((u_1(t^+_1, \theta^+_{1,i}), \alpha_0(\theta^+_{1,i})), \alpha_0(\theta^+_{1,i})) \leq r_0/2 \) for \( t^+_1 \geq \ell/3 \), and dist\((u_2(t^-_2, \theta^-_{2,j}), \alpha_0(\theta^-_{2,j})), \alpha_0(\theta^-_{2,j})) \leq r_0/2 \) for \( t^-_2 \leq -\ell/3 \). Then
\[
u_1(t^+_1, \theta^+_{1,i}) = \exp \xi_1(t^+_1, \theta^+_{1,i})
\]
for \( t^+_1 \geq \ell/3 \) and
\[
u_2(t^-_2, \theta^-_{2,j}) = \exp \xi_2(t^-_2, \theta^-_{2,j})
\]
for \( t^-_2 \leq -\ell/3 \) as in [R] §2. We define
\[
g^\ell_{ij}([u_1, c_1], [u_2, c_2]) = ([u_3, c_1 \triangleright^\ell_{ij} c_2], u_4)
\]
where \( u_3 \) and \( u_4 \) are as follows. Identify the complement of the necks in \( \Sigma_{g_1,k^-_1,k^+_1+1} \cup \Sigma_{g_2,k^-_2+1,k^+_2} \) with the complement of the necks in \( \Sigma_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2} \cup (\mathbb{R} \times S^1) \). Then we let \( u_3 \) and \( u_4 \) be equal to to \( u_1 \) and \( u_2 \) off the necks. Identify the necks of \( \Sigma_{g_1,k^-_1,k^+_1+1} \), \( \Sigma_{g_2,k^-_2+1,k^+_2} \), \( \Sigma_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2} \), and \( (\mathbb{R} \times S^1) \). Then we let \( u_3 = \exp u_3 \) and \( u_4 = \exp \xi_4 \) on the necks, where
\[
\xi_3 = \left( \cos(\pi \eta(t_0/\ell)) + J \sin(\pi \eta(t_0/\ell)) \right) \cos(\pi \eta(t_0/\ell)/2) \xi_1 + \sin(\pi \eta(t_0/\ell)/2) \xi_2
\]
\[ \xi_4 = - \left( \cos(\pi \eta(t_0/\ell)) + J \sin(\pi \eta(t_0/\ell)) \right) \cos(\pi \eta(t_0/\ell)/2) \xi_1 + \cos(\pi \eta(t_0/\ell)/2) \xi_2. \]

The point of this definition is that
\[
\begin{pmatrix}
\cos(\pi s) + i \sin(\pi s) & \cos(\pi s/2) \\
- (\cos(\pi s) + i \sin(\pi s)) & \cos(\pi s/2)
\end{pmatrix}
0 \leq s \leq 1
\]
is a path from \((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\) and \((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\) in the Lie group \(U(2)\).

The map \(g_{ij}^\ell\) lifts to bundle maps
\[
(g_{ij}^\ell)_*: T^\text{fib} \mathcal{P}^+_{g_1,k^+_1,k^-_1+1,;i} [\alpha_0] \boxplus T^\text{fib} \mathcal{P}^-_{g_2,k^-_2+1,;j} [\alpha_0] \rightarrow T^\text{fib} \mathcal{P}_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2} \boxplus T \mathcal{P}(\alpha_0,\alpha_0)
\]
and
\[
(g_{ij}^\ell)_*: F^+_{g_1,k^-_1,k^+_1+1,;i} [\alpha_0] \boxplus F^-_{g_2,k^-_2+1,;j} [\alpha_0] \rightarrow F_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2} \boxplus F(\alpha_0,\alpha_0)
\]
defined as follows. Let \([(u_1,c_1),[u_2,c_2)]\) be in the domain of \(g_{ij}^\ell\). Let \(s_1 \oplus s_2\) be an element of
\[
T^\text{fib}_{[u_1,c_1],;i} \mathcal{P}^+_{g_1,k^+_1,k^-_1+1,;i}[\alpha_0] \boxplus T^\text{fib}_{[u_2,c_2],;j} \mathcal{P}^-_{g_2,k^-_2+1,;j}[\alpha_0]
\]
or
\[
(F^+_{g_1,k^-_1,k^+_1+1,;i} [\alpha_0])_{[u_1,c_1]} \boxplus (F^-_{g_2,k^-_2+1,;j} [\alpha_0])_{[u_2,c_2]}.
\]
Then \(s_1 = (\exp \xi_1)_* \xi_1\) and \(s_2 = (\exp \xi_2)_* \xi_2\), as in \([R]\) \S2. We define \((g_{ij}^\ell)_*(s_1 \oplus s_2) = s_3 \oplus s_4\) where \(s_3\) and \(s_4\) are as follows. Identify the complement of the necks in \(\Sigma_{g_1,k^-_1,k^+_1+1} \cup \Sigma_{g_2,k^-_2+1,k^+_2}\) with the complement of the necks in \(\Sigma_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2}\) \(\cup \mathbb{R} \times S^1\). We then let \(s_3 \oplus s_4\) be equal to \(s_1 \oplus s_2\) off the necks. Identify the necks of \(\Sigma_{g_1,k^-_1,k^+_1+1,} \Sigma_{g_2,k^-_2+1,k^+_2,} \Sigma_{g_1+g_2,k^-_1+k^-_2,k^+_1+k^+_2}\) and \(\mathbb{R} \times S^1\). We then let \(s_3 = (\exp \xi_3)_* \xi_3\) and \(s_4 = (\exp \xi_4)_* \xi_4\) on the necks, where
\[ \zeta_3 = \left( \cos(\pi \eta(t_0/\ell)) + J \sin(\pi \eta(t_0/\ell)) \right) \cos(\pi \eta(t_0/\ell)/2) \xi_1 + \sin(\pi \eta(t_0/\ell)/2) \xi_2 \]
and
\[ \zeta_4 = - \left( \cos(\pi \eta(t_0/\ell)) + J \sin(\pi \eta(t_0/\ell)) \right) \sin(\pi \eta(t_0/\ell)/2) \xi_1 + \cos(\pi \eta(t_0/\ell)/2) \xi_2. \]

The map \(g_{ij}^\ell\) induces a map
\[ g_{ij}^\ell: \mathcal{P}_{\sigma_1,i}^+[\alpha_0] \times \mathcal{P}_{\sigma_2,j}^- [\alpha_0] \rightarrow \mathcal{P}_{\sigma_1 \circ \sigma_2,j} \times \mathcal{P}(\alpha_0,\alpha_0) \]
the obvious way, where

\[ \mathcal{P}_{\sigma_1; i}^+ [\alpha_0] = \mathcal{P}_{\sigma_1}((\alpha_{1, 1}, \ldots, \alpha_{1, k_1^+}), (\alpha_{1, 1}, \ldots, \alpha_{i, i-1}, \alpha_0, \alpha_{i, i+1}, \ldots, \alpha_{1, k_1^+})), \]

\[ \mathcal{P}_{\sigma_2; j}^- [\alpha_0] = \mathcal{P}_{\sigma_2}((\alpha_{2, 1}, \ldots, \alpha_{2, j-1}, \alpha_0, \alpha_{2, j+1}, \ldots, \alpha_{2, k_2^-}+1), (\alpha_{2, 1}, \ldots, \alpha_{2, k_2^-}+1)) \]

and

\[ \mathcal{P}_{\sigma_1 \circ j_{\sigma_2}}^- = \mathcal{P}_{\sigma_1 \circ j_{\sigma_2}}^+((\alpha_{2, 1}, \ldots, \alpha_{2, j-1}, \alpha_{1, 1}, \ldots, \alpha_{1, k_1^+}, \alpha_{2, j+1}, \ldots, \alpha_{2, k_2^-}+1), \]

\[ (\alpha_{1, 1}, \ldots, \alpha_{i, i-1}, \alpha_{2, 1}, \ldots, \alpha_{2, k_2^-}, \alpha_{1, i+1}, \ldots, \alpha_{i, k_i^+}+1) \]

This map is invertible. The inverse of the matrices (3.3) form a path

\[
\begin{pmatrix}
\cos(\pi s) - i \sin(\pi s) & \cos(\pi s/2) \\
\sin(\pi s/2) & \cos(\pi s/2)
\end{pmatrix}
\]

\[ 0 \leq s \leq 1. \]

Hence the inverse of \( g_{ij}^\ell \) is given by \((g_{ij}^\ell)^{-1}(u_3, u_4) = (u_1, u_2)\) where \( u_1 \) and \( u_2 \) are equal \( u_3 \) or \( u_4 \) off the necks, and \( u_1 = \exp \xi_1 \) with

\[ \xi_1 = \left( \cos \left( \pi \eta(t_0/\ell) \right) - J \sin \left( \pi \eta(t_0/\ell) \right) \right) \cos \left( \pi \eta(t_0/\ell)/2 \right) \xi_3 \]

\[ - \left( \cos \left( \pi \eta(t_0/\ell) \right) - J \sin \left( \pi \eta(t_0/\ell) \right) \right) \sin \left( \pi \eta(t_0/\ell)/2 \right) \xi_4 \]

and \( u_2 = \exp \xi_2 \) with

\[ \xi_2 = \sin \left( \pi \eta(t_0/\ell)/2 \right) \xi_3 + \cos \left( \pi \eta(t_0/\ell)/2 \right) \xi_4 \]

on the necks.

This map lifts to bundle maps

\[ (g_{ij}^\ell)_*: T^{\text{fib}} \mathcal{P}_{\sigma_1; i}^+ [\alpha_0] \oplus T^{\text{fib}} \mathcal{P}_{\sigma_2; j}^- [\alpha_0] \to T^{\text{fib}} \mathcal{P}_{\sigma_1 \circ j_{\sigma_2}}^+ \oplus T\mathcal{P}([\alpha_0, \alpha_0]) \]

and

\[ (g_{ij}^\ell)_*: \mathcal{P}_{\sigma_1; i}^+ [\alpha_0] \oplus \mathcal{P}_{\sigma_2; j}^- [\alpha_0] \to \mathcal{P}_{\sigma_1 \circ j_{\sigma_2}}^+ \oplus \mathcal{P}([\alpha_0, \alpha_0]) \]

defined the same way as before. These bundle maps are invertible. We denote the inverse \((g_{ij}^\ell)^*\).

Let

\[ \mathcal{P}_{g_{i, k+1}^+, k+1, k+1; i, j}^+ [\alpha_0, \alpha_0] = \mathcal{P}_{g, k+1, k+1}^+((\alpha_{1, i}, \ldots, \alpha_{1, i-1}, \alpha_0, \alpha_{1, i+1}, \ldots, \alpha_{k, k+1}), \]

\[ (\alpha_{1, i}, \ldots, \alpha_{1, i-1}, \alpha_0, \alpha_{1, i+1}, \ldots, \alpha_{k, k+1}) \]
and
\[ \Psi_{g+1,k-,k^+} = \Psi_{g+1,k-,k^+}(\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{k^+}), \]
\[ \left(\alpha_1^+, \ldots, \alpha_{i-1}^+, \alpha_{i+1}^+, \ldots, \alpha_{k^+}^+\right). \]

Then the map (3.2) lifts to a map
\[ c_{ij}^\ell: \Psi_{g,k-,k^+}^{+,\ell}[\alpha_0, \alpha_0] \to \Psi_{g+1,k-,k^+} \times \mathcal{P}(\alpha_0, \alpha_0), \]
defined the same way as \( g_{ij}^\ell \). The map \( c_{ij}^\ell \) lifts to bundle maps
\[ (c_{ij}^\ell)_*: T_{g,k-,k^+}^{+,\ell}[\alpha_0, \alpha_0] \to T_{g+1,k-,k^+}^{+,\ell} \bigoplus T\mathcal{P}(\alpha_0, \alpha_0) \]
and
\[ (c_{ij}^\ell)_*: \mathbb{F}_{g,k-,k^+}^{+,\ell}[\alpha_0, \alpha_0] \to \mathbb{F}_{g+1,k-,k^+} \bigoplus F(\alpha_0, \alpha_0). \]
defined the same way as \( (g_{ij}^\ell)_* \). Similarly, for any smooth \( \sigma: [0,1]^q \to \mathcal{J}_{g,k-,k^+}^{\ast} \) there are local diffeomorphisms
\[ c_{ij}^\ell: \Psi_{g,k-,k^+}^{\ast}[\alpha_0, \alpha_0] \to \Psi_{g,k-,k^+}^{\ast}[\alpha_0, \alpha_0] \]
with lifts
\[ (c_{ij}^\ell)_*: T_{g,k-,k^+}^{\ast}[\alpha_0, \alpha_0] \to T_{g+1,k-,k^+}^{\ast}[\alpha_0, \alpha_0] \bigoplus T\mathcal{P}(\alpha_0, \alpha_0) \]
and
\[ (c_{ij}^\ell)_*: \mathbb{F}_{g,k-,k^+}^{\ast}[\alpha_0, \alpha_0] \to \mathbb{F}_{g+1,k-,k^+} \bigoplus F(\alpha_0, \alpha_0). \]

We denote the inverse of \( (c_{ij}^\ell)_* \) by \( (c_{ij}^\ell)_*^{-1} \).

3.3. Index calculations. The following is an excision principle for the index of \( D_{\Psi_{g,k-,k^+}}^{\ast} \). Recall that \( \Psi_{g,k-,k^+} \) denotes the subset of \( \Psi_{g,-k^+} \) where index \( D_{\Psi_{g,k-,k^+}}^{\ast} \) is \( d \). Similarly we add a superscript \( d \) to the notation introduced in §3.2.

**Lemma 3.3.1.**
\[ g_{ij}^\ell(\Psi_{g_1,k_1-,k_1^+}^{d_1;+}[\alpha_0] \times \Psi_{g_2,k_2-,k_2^+}^{d_2;-}[\alpha_0]) \subseteq \Psi_{g_1+g_2,k_1^+,k_2^+}[\alpha_0] \times \mathcal{P}(\alpha_0, \alpha_0) \]
\[ c_{ij}^\ell(\Psi_{g,k-,k^+}^{d;+,-}[\alpha_0]) \subseteq \mathcal{P}(\alpha_0, \alpha_0) \]
\[ g_{ij}^\ell(\Psi_{g,k-,k^+}^{d;+,-}[\alpha_0] \times \mathcal{P}(\alpha_0, \alpha_0)) \subseteq \Psi_{g+1,k-,k^+}[\alpha_0] \times \mathcal{P}(\alpha_0, \alpha_0) \]
\[ c_{ij}^\ell(\Psi_{g,k-,k^+}^{d;+,-}[\alpha_0] \times \mathcal{P}(\alpha_0, \alpha_0)) \subseteq \Psi_{g+1,k-,k^+}[\alpha_0] \times \mathcal{P}(\alpha_0, \alpha_0). \]

**Proof.** By Remark 2.3.5, it suffices to prove the first two statements. We will prove the first one. The second is proven the same way. Let \( ([\alpha_3, c_1 \odot_{ij} c_2, \alpha_4] = \)
$g^j_{ij}(\{u_1, c_1\}, \{u_2, c_2\})$. To keep the notation simple, we write $u_1$, $u_2$, and $u_3$ for $[u_1, c_1]$, $[u_2, c_2]$, and $[u_3, c_1 \triangleleft_j c_2]$. We need to show that

$$\text{index } D_{u_3}^{fib} \Psi_{g_1+g_2, k^-_1+k^-_2, k^+_1+k^+_2} = \text{index } D_{u_1}^{fib} \Psi_{g_1, k^-_1, k^+_1+1} + \text{index } D_{u_2}^{fib} \Psi_{g_2, k^-_2+1, k^+_2}$$

and

$$\text{index } D_{u_4} \Psi(\alpha_0, \alpha_0) = 0.$$ 

The map $u_4$ is a perturbation of $\overline{\sigma}_0$. Hence $\text{index } D_{u_4} \Psi(\alpha_0, \alpha_0) = 0$. Thus it suffices to show that

$$\text{index } D_{u_1}^{fib} \Psi_{g_1, k^-_1, k^+_1+1} + \text{index } D_{u_2}^{fib} \Psi_{g_2, k^-_2+1, k^+_2} = \text{index } D_{u_3}^{fib} \Psi_{g_1+g_2, k^-_1+k^-_2, k^+_1+k^+_2} + \text{index } D_{u_4} \Psi(\alpha_0, \alpha_0).$$

This is simply invariance of the index under excision. However, as we are working on non-compact surfaces, we cannot appeal to the standard excision principle of Atiyah and Singer. Instead we argue as follows. It follows from [R] Prop. 2.1 that

$$(D_{u_1}^{fib} \Psi_{g_1, k^-_1, k^+_1+1})(\exp \xi_1)_* \zeta_1 \oplus (D_{u_2}^{fib} \Psi_{g_2, k^-_2+1, k^+_2})(\exp \xi_2)_* \zeta_2$$

$$= (\exp \xi_1)_* \left( \frac{\partial \zeta_1}{\partial \xi} + J(D_{\alpha_0} \Phi_H) \zeta_1 + (\xi_1 \otimes \zeta_1) \cdot f \right)$$
$$\oplus (\exp \xi_2)_* \left( \frac{\partial \zeta_2}{\partial \xi} + J(D_{\alpha_0} \Phi_H) \zeta_2 + (\xi_2 \otimes \zeta_2) \cdot f \right)$$

on the necks. Similarly,

$$(D_{u_3}^{fib} \Psi_{g_1+g_2, k^-_1+k^-_2, k^+_1+k^+_2})(\exp \xi_3)_* \zeta_3 \oplus (D_{u_4} \Psi(\alpha_0, \alpha_0))(\exp \xi_4)_* \zeta_4$$

$$= (\exp \xi_3)_* \left( \frac{\partial \zeta_3}{\partial \xi} + J(D_{\alpha_0} \Phi_H) \zeta_3 + (\xi_3 \otimes \zeta_3) \cdot f \right)$$
$$\oplus (\exp \xi_4)_* \left( \frac{\partial \zeta_4}{\partial \xi} + J(D_{\alpha_0} \Phi_H) \zeta_4 + (\xi_4 \otimes \zeta_4) \cdot f \right)$$

on the necks. It follows that the excision map $g^j_{ij}$ intertwines $D_{u_1}^{fib} \Psi_{g_1, k^-_1, k^+_1+1} \oplus D_{u_2}^{fib} \Psi_{g_2, k^-_2+1, k^+_2}$ and $D_{u_3}^{fib} \Psi_{g_1+g_2, k^-_1+k^-_2, k^+_1+k^+_2} \oplus D_{u_4} \Psi(\alpha_0, \alpha_0)$ modulo lower order terms that are supported on the necks. These terms define a compact operator. Hence

$$\text{index } (D_{u_1}^{fib} \Psi_{g_1, k^-_1, k^+_1+1} \oplus D_{u_2}^{fib} \Psi_{g_2, k^-_2+1, k^+_2})$$
$$= \text{index } (g^j_{ij})^* (D_{u_3}^{fib} \Psi_{g_1+g_2, k^-_1+k^-_2, k^+_1+k^+_2} \oplus D_{u_4} \Psi(\alpha_0, \alpha_0))$$
$$= \text{index } (D_{u_3}^{fib} \Psi_{g_1+g_2, k^-_1+k^-_2, k^+_1+k^+_2} \oplus D_{u_4} \Psi(\alpha_0, \alpha_0)).$$

$\square$
In our setup two different minimal Chern numbers arise. The minimal Chern number of Floer is $N_0$.

**Definition 3.3.2.** We define the homotopical minimal Chern number $N_0$ by

$$\langle c_1(TM) | \pi_2(M) \rangle = N_0 \mathbb{Z}$$

and the homological minimal Chern number $N_1$ by

$$\langle c_1(TM) | H_2(M, \mathbb{Z}) \rangle = N_1 \mathbb{Z}.$$ 

The following is the main index formula.

**Proposition 3.3.3.** There exist a unique function

$$\mu_H : C^0_H \to \mathbb{Z}/2N_0 \mathbb{Z}$$

such that if $\alpha^+_i \in C^0_H$ and $\Psi^d_{g, k^+, k^-}((\alpha^-_1, \ldots, \alpha^-_k), (\alpha^+_1, \ldots, \alpha^+_k))$ is nonempty, then

$$d \equiv 2n(1 - g - k^-) + \sum_{i=1}^{k^-} \mu_H(\alpha^-_i) - \sum_{i=1}^{k^+} \mu_H(\alpha^+_i) \mod 2N_1,$$

and, for $g = 0$,

$$d \equiv 2n(1 - k^-) + \sum_{i=1}^{k^-} \mu_H(\alpha^-_i) - \sum_{i=1}^{k^+} \mu_H(\alpha^+_i) \mod 2N_0.$$ 

The function $\mu_H$ is independent of $J$.

If $\sigma$ is a smooth map $[0, 1]^q \to \mathcal{J}_{g, k^-}, k^+ \in C^0_H$, and $\Psi^d_{\sigma}((\alpha^-_1, \ldots, \alpha^-_k), (\alpha^+_1, \ldots, \alpha^+_k))$ is nonempty, then

$$d \equiv q + 2n(1 - g - k^-) + \sum_{i=1}^{k^-} \mu_H(\alpha^-_i) - \sum_{i=1}^{k^+} \mu_H(\alpha^+_i) \mod 2N_1,$$

and, for $g = 0$,

$$d \equiv q + 2n(1 - k^-) + \sum_{i=1}^{k^-} \mu_H(\alpha^-_i) - \sum_{i=1}^{k^+} \mu_H(\alpha^+_i) \mod 2N_0.$$
If \( \alpha^+ \in \mathcal{C}_H^0 \) and \( \mathcal{P}^d(\alpha^-, \alpha^+) \) is non-empty, then

\[
d \equiv \mu(\alpha^-) - \mu(\alpha^+) \mod 2N_0.
\]

**Proof.** For each \( \alpha \in \mathcal{C}_H^0 \), we choose \([u_\alpha^+, c_\alpha^+] \in \mathcal{P}_{0,0,1}(\emptyset, (\alpha))\). To keep the notation simple we write \( u \) for \([u, c]\). In particular we write \( u_\alpha^+ \) for \([u_\alpha^+, c_\alpha^+]\). If there exists a function \( \mu_H \) as in the Proposition, then

\[
(3.4) \quad \text{index } \mathcal{D}^{\text{fib}}_{u_\alpha^+} \Psi_{0,0,1} \equiv 2n - \mu_H(\alpha) \mod 2N_0.
\]

This shows uniqueness.

To show existence, we take (3.4) as the definition of \( \mu_H(\alpha) \), and show that the first index formula in the Proposition holds. We first note that if \( u^0 \in \mathcal{P}_{g,0,0} \), then \( \mathcal{D}^{\text{fib}}_{u^0} \Psi_{g,0,0} \) is a perturbed \( \mathcal{D} \)-complex on the compact Riemann surface \( \Sigma_{g,0,0} \) coupled to \((u^0)^*TM\). By the Riemann-Roch theorem,

\[
\text{index } \mathcal{D}^{\text{fib}}_{u^0} \Psi_{g,0,0} = 2n(1 - g) + 2\langle c_1(TM) | (u^0)_* |[\Sigma] \rangle.
\]

In particular,

\[
(3.5) \quad \text{index } \mathcal{D}^{\text{fib}}_{u_\alpha^+} \Psi_{g,0,0} \equiv 2n(1 - g) \mod 2N_1.
\]

If \( g = 0 \), then this and the following congruences hold modulo \( 2N_0 \). We also note that by Prop. 2.1.7,

\[
(3.6) \quad \text{index } \mathcal{D}_{\overline{\alpha}} \Psi(\alpha, \alpha) = 0.
\]

For each \( \alpha \in \mathcal{C}_H^0 \), we also choose \( u_\alpha^- \in \mathcal{P}_{0,1,0}(\emptyset, (\alpha)) \). Let \((u^0, \overline{\alpha}) = g_{11}^*(u_\alpha^+, u_\alpha^-)\). With a slight abuse of notation we here write “\( \overline{\alpha} \)” for a a map that is only approximately equal to \( \overline{\alpha} \). By Lemma 3.3.1,

\[
\text{index } \mathcal{D}^{\text{fib}}_{u_\alpha^-} \Psi_{0,0,1} + \text{index } \mathcal{D}^{\text{fib}}_{u_\alpha^+_0} \Psi_{0,1,0} = \text{index } \mathcal{D}^{\text{fib}}_{u_\alpha^-} \Psi_{0,0,0} + \text{index } \mathcal{D}_{\overline{\alpha}} \Psi(\alpha, \alpha).
\]

It then follows from (3.4), (3.5) and (3.6) that

\[
(3.7) \quad \text{index } \mathcal{D}^{\text{fib}}_{u_\alpha^+_0} \Psi_{0,1,0} \equiv \mu_H(\alpha) \mod 2N_1.
\]
Let $u \in \Psi_{g,k-}^{k+}(\alpha^-_1, \ldots, \alpha^-_k, (\alpha^+_1, \ldots, \alpha^+_k))$. By repeated use of Lemma 3.3.1,

$$\begin{align*}
\text{index } D^\text{fib}_u \Psi_{g,k-}^{k+} &+ \text{index } D^\text{fib}_{-u^\alpha_{k^-}} \Psi_{0,1} + \cdots + \text{index } D^\text{fib}_{u^\alpha_{k^-}} \Psi_{0,0,1} \\
&+ \text{index } D^\text{fib}_{u^\alpha_{k^+}} \Psi_{0,1,0} + \cdots + \text{index } D^\text{fib}_{u^\alpha_{k^+}} \Psi_{0,1,0} \\
&= \text{index } D^\text{fib}_u \Psi_{g,0,0} + \text{index } D_{\alpha^-_1} \Psi(\alpha^-_1, \alpha^-_1) + \cdots + \text{index } D_{\alpha^-_k} \Psi(\alpha^-_k, \alpha^-_k) \\
&+ \text{index } D_{\alpha^+_1} \Psi(\alpha^+_1, \alpha^+_1) + \cdots + \text{index } D_{\alpha^+_k} \Psi(\alpha^+_k, \alpha^+_k).
\end{align*}$$

The first index formula now follows from (3.4), (3.5), (3.6), and (3.7). The other index formulas are easy consequences of the first index formula; see Remark 2.3.5 and 2.3.7. It follows from Prop. 2.1.3 by continuity that $\mu_H$ does not depend on $J$. \hfill \square

A different approach to index formulas for perturbed pseudoholomorphic curves uses the Maslov index, see [F1], [SZ] and [MS2] Sect. 9.2.

3.4. Coherent orientations. We will orient the moduli space by performing excisions on the orientation sheaves. Our construction is inspired by [D] Sect. 3, [DK] Sect. 7.1.6, and [FH]. However, the details of our construction are different, as we do not localize the kernels and cokernels.

Let $T$ be a Fredholm operator. We then let $\mathcal{O}_T$ denote the set of orientations of the vector space $\ker T \oplus (\text{coker } T)^*$, or, equivalently, the set of orientations of the line $\Lambda^\text{max} \ker T \otimes (\Lambda^\text{max} \text{coker } T)^*$. Then $\mathcal{O}_T$ is a 1-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. The following properties of $\mathcal{O}_T$ will be used extensively.

- If $T$ is invertible, then $\mathcal{O}_T = \mathbb{Z}/2\mathbb{Z}$ canonically.
- If $T - T'$ is a compact operator, then $\mathcal{O}_T = \mathcal{O}_{T'}$ canonically.
- $\mathcal{O}_{T_1 \circ T_2} = \mathcal{O}_{T_1} \otimes \mathcal{O}_{T_2}$ canonically.
- Hence, if $S_1 T - U S_2$ is compact, and $S_1$ and $S_2$ are invertible, then we can identify

$$\begin{align*}
\mathcal{O}_T &= (\mathbb{Z}/2\mathbb{Z}) \otimes \mathcal{O}_T = \mathcal{O}_{S_1} \otimes \mathcal{O}_T = \mathcal{O}_{S_1 \circ T} \\
&= \mathcal{O}_{U \circ S_2} = \mathcal{O}_U \otimes \mathcal{O}_{S_2} = \mathcal{O}_U \otimes (\mathbb{Z}/2\mathbb{Z}) = \mathcal{O}_U.
\end{align*}$$

We denote the resulting isomorphism

$$S_* : \mathcal{O}_T \to \mathcal{O}_U.$$

- We can identify $\mathcal{O}_{T_1 \oplus T_2}$ with $\mathcal{O}_{T_1} \otimes \mathcal{O}_{T_2}$. In other words, elements of $\mathcal{O}_{T_1}$ and $\mathcal{O}_{T_2}$ determine an element of $\mathcal{O}_{T_1 \oplus T_2}$. We use the following convention.
Choose orientations of \( \ker T_1 \), \( \coker T_1 \), \( \ker T_2 \) and \( \coker T_2 \) compatible with the orientations of \( \ker T_1 \oplus (\coker T_1)^* \) and \( \ker T_1 \oplus (\coker T_2)^* \). We then use the induced orientation of \( (\ker T_1) \oplus (\coker T_1)^* \oplus (\ker T_2) \oplus (\coker T_2)^* \).

- The spaces \( \mathcal{O}_{T_1} \otimes \mathcal{O}_{T_2} \) and \( \mathcal{O}_{T_2} \otimes \mathcal{O}_{T_1} \) are identical. However, if we choose elements of \( \mathcal{O}_{T_1} \otimes \mathcal{O}_{T_2} \) and \( \mathcal{O}_{T_2} \otimes \mathcal{O}_{T_1} \), and make the identifications \( \mathcal{O}_{T_1} \otimes \mathcal{O}_{T_2} = \mathcal{O}_{T_1} \oplus \mathcal{O}_{T_2} \) and \( \mathcal{O}_{T_2} \otimes \mathcal{O}_{T_1} = \mathcal{O}_{T_2} \oplus \mathcal{O}_{T_1} \) as above, then the induced elements of \( \mathcal{O}_{T_1} \otimes \mathcal{O}_{T_2} \) and \( \mathcal{O}_{T_2} \otimes \mathcal{O}_{T_1} \) differ by \((-1)^{\text{index } T_1 \text{ index } T_2} \).

- More generally, if \( \rho \) is a permutation on \( k \) letters, and we choose elements of \( \mathcal{O}_{T_1}, \ldots, \mathcal{O}_{T_k} \), then the induced elements of \( \mathcal{O}_{T_1} \otimes \cdots \otimes \mathcal{O}_{T_k} \) and \( \mathcal{O}_{T_{\rho(1)}} \otimes \cdots \otimes \mathcal{O}_{T_{\rho(k)}} \) differ by the graded sign of the permutation \( T_1, \ldots, T_k \rightarrow (T_{\rho(1)}, \ldots, T_{\rho(k)}) \), where \( T_i \) is assigning the degree index \( T_i \).

The graded sign of a permutation is defined as the sign of the permutation obtained by removing all elements of even degree.

- If \( T = \{T_x \}_{x \in X} \) is a continuous family of Fredholm operators parametrized by a topological space \( X \), then \( \mathcal{O}_T = \bigcup_{x \in X} \mathcal{O}_{T_x} \) forms a continuous double cover of \( X \). We call \( \mathcal{O}_T \) the orientation sheaf for the family \( T \).

Let \( \mathfrak{D}_{g,k^-,k^+} \) denote the orientation sheaf for the family \( D^\text{fib} \Psi_{g,k^-,k^+} \) parametrized by \( \Psi_{g,k^-,k^+}((\alpha_1^-, \ldots, \alpha_{k^-}^-), (\alpha_1^+, \ldots, \alpha_{k^+}^+)) \). Let \( \mathfrak{D}_\sigma \) denote the orientation sheaf for the family \( D \Psi_\sigma \) parametrized by \( \Psi_\sigma((\alpha_1^-, \ldots, \alpha_{k^-}^-), (\alpha_1^+, \ldots, \alpha_{k^+}^+)) \). Let \( \mathcal{O}(\alpha^-, \alpha^+) \) denote the orientation sheaf for the family \( D \Psi(\alpha^-, \alpha^+) \) parametrized by \( \Psi(\alpha^-, \alpha^+) \). Let \( \mathcal{O}_{[0,1]^g} \) denote the orientation sheaf for the manifold \([0,1]^g\).

A global section of \( \mathfrak{D}_{g,k^-,k^+} \) determines a global section of \( \mathfrak{D}_\sigma = \mathcal{O}_{[0,1]^g} \otimes \sigma^* \mathfrak{D}_{g,k^-,k^+} \), see Remark 2.3.5, and a global section of \( \mathcal{O}(\alpha^-, \alpha^+) = \mathfrak{D}_{\sigma_0} \), see Remark 2.3.7. If the moduli space \( \mathcal{M}_\sigma \) is regular, then a global section of \( \mathfrak{D}_\sigma \) determines an orientation of \( \mathcal{M}_\sigma \); see for instance [DK] Sect. 5.4.1. Hence it suffices to choose global sections of the orientation sheaves \( \mathfrak{D}_{g,k^-,k^+} \).
We first note that in some cases $\mathcal{D}_{g,k^{-},k^{+}}$ has a canonical section. If $[u,c] \in \Psi_{g,0,0}(\emptyset, \emptyset)$, then $D_{[u,c]}\Psi_{g,0,0}$ is a perturbed $\overline{D}$-operator coupled to a complex vector bundle over the compact surface $\Sigma_{g,0,0}$. Hence the operator has a complex index line bundle. The real index line bundle is the determinant of the complex index line bundle. Hence the real index line bundle has a canonical orientation. This gives a canonical section of $\mathcal{D}_{g,0,0}$.

By Prop. 2.1.7, the operator $D_{\overline{\sigma}_{0}}\Psi(\alpha_{0},\alpha_{0})$ is invertible. Thus $\Theta_{\overline{\sigma}_{0}}(\alpha_{0},\alpha_{0})$ has a canonical element. By continuity, we also get canonical elements of $\Theta_{u}(\alpha_{0},\alpha_{0})$ for $u$ close to $\overline{\sigma}_{0}$.

In the proof of Lemma 3.3.1 we saw that the maps $(g_{ij}^{\ell})_{*}$ intertwine the families

$$D^{\text{fib}}\Psi_{g_{1},k_{1}^{-},k_{1}^{+}+1} \sqcup D^{\text{fib}}\Psi_{g_{2},k_{2}^{-},k_{2}^{+}}$$

and

$$D^{\text{fib}}\Psi_{g_{1}+g_{2},k_{1}^{-}+k_{2}^{-},k_{1}^{+}+k_{2}^{+}} \sqcup D\Psi(\alpha_{0},\alpha_{0})$$

up to compact operators. Hence there are induced excision maps

$$(g_{ij}^{\ell})_{*} : \mathcal{D}_{g_{1},k_{1}^{-},k_{1}^{+}+1} \boxtimes \mathcal{D}_{g_{2},k_{2}^{-},k_{2}^{+}} \rightarrow \mathcal{D}_{g_{1}+g_{2},k_{1}^{-}+k_{2}^{-},k_{1}^{+}+k_{2}^{+}} \otimes \Theta_{\overline{\sigma}_{0}}(\alpha_{0},\alpha_{0}).$$

Here we write $\Theta_{\overline{\sigma}_{0}}(\alpha_{0},\alpha_{0})$ for $\Theta_{u}(\alpha_{0},\alpha_{0})$ with $u$ close to $\overline{\sigma}_{0}$.

Similarly there are excision maps

$$(c_{ij}^{\ell})_{*} : \mathcal{D}_{g,k^{-},k^{+}+1} \sqcup \mathcal{D}_{g+1,k^{-},k^{+}} \rightarrow \mathcal{D}_{g+1,k^{-}+1,k^{+}} \otimes \Theta_{\overline{\sigma}_{0}}(\alpha_{0},\alpha_{0}).$$

Definition 3.4.1. A coherent system of orientations is a choice of a global section of the double cover $\mathcal{D}_{g,k^{-},k^{+}}$ of each $\Psi_{g,k^{-},k^{+}}(\alpha_{1}^{-},\ldots,\alpha_{k}^{-},\alpha_{1}^{+},\ldots,\alpha_{k}^{+})$, with $\alpha_{i}^{\pm} \in \mathcal{C}_{H}^{0}$, such that the following four conditions are satisfied:

- For any $g \geq 0$, the coherent section of $\mathcal{D}_{g,0,0}$ is given by the canonical orientations.
- For each $\alpha_{0} \in \mathcal{C}_{H}$, the coherent element of $\Theta_{\overline{\sigma}_{0}}(\alpha_{0},\alpha_{0})$ is equal to the canonical orientation.
- The excision map (3.8) preserves or reverses the coherent orientations according to the graded sign of the permutation

$$\begin{align*}
(\alpha_{1}^{-},\ldots,\alpha_{k}^{-},\alpha_{1}^{+},\ldots,\alpha_{k}^{+}) & \mapsto (\alpha_{1}^{-},\ldots,\alpha_{k}^{-},\alpha_{1}^{+},\ldots,\alpha_{k}^{+}).
\end{align*}$$

(3.10)
where $\alpha$ has degree $\mu_H(\alpha)$.

- The excision map (3.9) preserves or reverses the coherent orientations according to the graded sign of the permutation

$$
\left(\alpha_1^-, \ldots, \alpha_{j-1}^-, \alpha_0, \alpha_{j+1}^-, \ldots, \alpha_k^-, \alpha_1^+, \ldots, \alpha_{i-1}^+, \alpha_0^+, \alpha_{i+1}^+, \alpha_k^+ight)
\rightarrow
\left(\alpha_1^-, \ldots, \alpha_{j-1}^-, \alpha_{j+1}^-, \ldots, \alpha_k^-, \alpha_1^+, \ldots, \alpha_{i-1}^+, \alpha_{i+1}^+, \alpha_k^+, \alpha_0^+, \alpha_0^-ight),
\tag{3.11}
$$

where $\alpha$ has degree $\mu_H(\alpha)$.

We also refer to the induced sections of $O_\sigma$ and $O(\alpha^-, \alpha^+)$ as coherent orientations.

Recall that graded sign means the sign of the permutation obtained by removing all elements of even degree.

**Proposition 3.4.2.** For any regular Hamiltonian $H$, there exist coherent systems of orientations. Given one coherent system of orientations, any other coherent system of orientations can be obtained by choosing a function $\tau : C^0_H \rightarrow \{-1, 1\}$ and changing the section of $O_{g,k^-}^{-,k^+}((\alpha_1^-, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))$ by

$$
\prod_{i=1}^{k^-} \tau(\alpha_i^-) \prod_{i=1}^{k^+} \tau(\alpha_i^+).
\tag{3.12}
$$

**Proof.** To keep the notation simple, we write $u$ for $[u, c] \in \Psi_{g,k^-}^{-,k^+}$, and $O_u$ for $(O_{g,k^-}^{-,k^+})_{[u, c]}$. For each $\alpha \in C^0_H$, we choose elements $u_{\alpha}^+ \in \Psi_{0,0,1}(\emptyset, (\alpha))$ and $u_{\alpha}^- \in \Psi_{0,1,0}((\alpha), \emptyset)$. We then arbitrarily choose an element of each $O_{g,k^-}^{-,k^+}$ that will serve as a coherent orientation. For $\ell$ large enough, $(u_{\alpha}^0, \pi) = g_{11}(u_{\alpha}^+, u_{\alpha}^-)$ is defined. (Recall that we write “$\pi$” for maps that are only approximately equal to $\pi$.) The excision map

$$(g_{11})_* : O_{u_{\alpha}^+} \otimes O_{u_{\alpha}^-} \rightarrow O_{u_{\alpha}^0} \otimes O_{\pi}$$

has to preserve coherent orientations. As we have chosen the coherent element of $O_{u_{\alpha}^+}$, and $O_{u_{\alpha}^-}$ and $O_{\pi}$ have canonical elements, this determines the coherent element of $O_{u_{\alpha}^-}$.

Let

$$u \in \Psi_{g,k^-}^{-,k^+}((\alpha_1^-, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+)).$$

By repeated use of the excision maps we get a map

$$O_{u_{\alpha}^+} \otimes \cdots \otimes O_{u_{\alpha}^-} \otimes O_{u} \otimes O_{u_{\alpha}^+} \otimes \cdots \otimes O_{u_{\alpha}^-} \rightarrow O_{\pi} \otimes \cdots \otimes O_{\pi} \otimes O_{u} \otimes O_{\pi} \otimes \cdots \otimes O_{\pi}$$

where $\alpha$ has degree $\mu_H(\alpha)$. 

where \( u^0 \in \Psi_{g,0,0}(\emptyset, \emptyset) \). This excision map has to preserve the coherent orientations. This determines the coherent element of \( \Omega_\alpha \).

These orientations are unique, once we have chosen elements of \( \Omega_{u^\pm_\alpha} \) for all \( \alpha \in \mathcal{C}_H^0 \). A different choice of elements of \( \Omega_{u^\pm_\alpha} \) changes the coherent orientation by a factor (3.12).

We have chosen elements of \( \Omega_{u^+_\alpha}, \Omega_{u^-_\alpha} \) and \( \Omega_\pi \) twice; first in the initial step of the construction, and then when we defined the coherent element of an arbitrary \( \Omega_u \). Both elements of \( \Omega_{u^+_\alpha} \) and \( \Omega_{u^-_\alpha} \) are determined by requiring the map \( (g^\ell_{11})_* \) to preserve the orientations. Thus the two elements of \( \Omega_{u^+_\alpha} \) and \( \Omega_{u^-_\alpha} \) are identical.

We need to verify that these orientations satisfy the four conditions of Def. 3.4.1. The first condition is satisfied trivially. The third condition is verified as follows. We have chosen elements of \( \Omega_{u^\pm_\alpha} \), but with a designated element that is \((-1)^{\mu(\alpha)}\) times the coherent element. It follows from Prop. 3.3.3, or rather the simplified formula

\[
d \equiv \sum_{i=1}^{k^-} \mu(\alpha^-_i) + \sum_{i=1}^{k^+} \mu(\alpha^+_i) \mod 2,
\]

and the discussion of permutations earlier in this section that the excision maps

\[
\tilde{\Omega}_{u^-_\alpha} \otimes \tilde{\Omega}_{u^-_\alpha} \rightarrow \Omega_{u^0} \otimes \Omega_\pi
\]

and

\[
\Omega_u \otimes \Omega_{u^-_{k^+}} \otimes \cdots \otimes \Omega_{u^-_{1}} \otimes \tilde{\Omega}_{u^-_{k^-}} \otimes \cdots \otimes \tilde{\Omega}_{u^-_{1}} \rightarrow \Omega_{u^0} \otimes \Omega_\pi^+ \otimes \cdots \otimes \Omega_\pi^+ \otimes \Omega_\pi^- \otimes \cdots \otimes \Omega_\pi^-
\]

preserve the orientations induced by the coherent orientations and the designated elements of the spaces \( \tilde{\Omega}_{u^-_{i}} \). That the third condition is satisfied now follows from the commutative diagram on the next page. The two maps on the left and the map on the lower right preserve the orientations. The bottom map is defined to make the diagram commute. It is induced by a fairly complicated composition of excision operators on the bundles \( T^{fib}_u \mathcal{P} \) and \( \mathcal{E} \). However, a moment’s thought shows that if we choose the parameter \( \ell \) for the gluing of the \( u_\alpha^\pm \)'s at least three times larger than the parameter \( \ell \) for the gluing of \( u_1 \) and \( u_2 \), then this operator splits as a direct sum of operators

\[
T^{fib}_{u_1^+} \mathcal{P}_{g_1,0,0} \oplus T^{fib}_{u_2^+} \mathcal{P}_{g_2,0,0} \rightarrow T^{fib}_{u_3^+} \mathcal{P}_{g_1+g_2,0,0} \oplus T^{fib}_{u_\alpha^0} \mathcal{P}_{0,0,0} \text{ and } \mathcal{E}_{g_1,0,0} (u_1^0 \oplus (\mathcal{E}_{g_2,0,0})u_2^0 \rightarrow \mathcal{E}_{g_1+g_2,0,0})u_\alpha^0 \oplus (\mathcal{E}_{0,0,0})u_\alpha^0 \),
\]

and the identity operators on \( T^{\mathcal{P}_{\pi^+,1}} \mathcal{P} \cdots, T^{\mathcal{P}_{\pi^+,n}} \mathcal{P} \) and \( T^{\mathcal{P}_{\pi^-,n}} \mathcal{P} \cdots, T^{\mathcal{P}_{\pi^-,1}} \mathcal{E} \mathcal{E}_{\alpha^0} \). The operators \( T^{fib}_{u_1^+} \mathcal{P}_{g_1,0,0} \oplus T^{fib}_{u_2^+} \mathcal{P}_{g_2,0,0} \rightarrow T^{fib}_{u_3^+} \mathcal{P}_{g_1+g_2,0,0} \oplus T^{fib}_{u_\alpha^0} \mathcal{P}_{0,0,0} \text{ and } \mathcal{E}_{g_1,0,0} (u_1^0 \oplus (\mathcal{E}_{g_1+g_2,0,0})u_2^0 \rightarrow \mathcal{E}_{g_1+g_2,0,0})u_\alpha^0 \oplus (\mathcal{E}_{0,0,0})u_\alpha^0 \) commute with \( \mathcal{J} \), up to lower order terms. In this case the coherent orientations are induced by the complex structures. It follows that the induced map \( \Omega_{u_1^0} \otimes \Omega_{u_2^0} \rightarrow \Omega_{u_3^0} \otimes \Omega_{u_\alpha^0} \) preserves the coherent orientations. Hence the bottom map is orientation preserving.
The top map is a permutation, and changes the orientation according to the graded sign of the permutation (3.10). As the diagram commutes, it follows that the upper right map also changes the orientation according to the graded sign of the permutation (3.10). But this map is given by the excision map \( \mathcal{D}_{u_1} \otimes \mathcal{D}_{u_2} \to \mathcal{D}_{u_3} \otimes \mathcal{O}_{\mathcal{D}_0} \). It follows that the third condition is satisfied. The fourth condition is verified the same way.

Finally we verify the second condition. By definition, the excision map \( g^{\ell}_{11} \) above preserves the orientations given by the coherent element of \( \mathcal{O}_{\mathcal{D}_0} \). By the third condition \( g^{\ell}_{11} \) preserves the orientations given by the canonical element of \( \mathcal{O}_{\mathcal{D}_0} \). Hence the two elements of \( \mathcal{O}_{\mathcal{D}_0} \) are identical.

Given \( \sigma_1 : [0, 1]^{q_1} \to \mathcal{T}_{g_1, k_1^{-}, k_1^{+} + 1}(A) \) and \( \sigma_2 : [0, 1]^{q_2} \to \mathcal{T}_{g_2, k_2^{-} + 1, k_2^{+}}(A) \) we get two families \( D\Psi_{\sigma_1} \oplus D\Psi_{\sigma_2} \) and \( D\Psi_{\sigma_1^{\circ} \sigma_2} \oplus D\Psi_{\sigma_1^{\circ} \sigma_2} \) of Fredholm operators parametrized by \([0, 1]^{q_1 + q_2}\). These families are intertwined by the excision operators \( (g^{\ell}_{ij})_* \), up to compact operators. Thus there is an induced excision map

\[
(g^{\ell}_{ij})_* : \mathcal{D}_{\sigma_1} \otimes \mathcal{D}_{\sigma_2} \to \mathcal{D}_{\sigma_1^{\circ} \sigma_2} \otimes \mathcal{O}_{\mathcal{D}_0}.
\]

Similarly there is an excision map

\[
(c^{\ell}_{ij})_* : \mathcal{D}_{\sigma} \to \mathcal{D}_{\sigma_{ij}^{\circ} \sigma} \otimes \mathcal{O}_{\mathcal{D}_0}.
\]

**Proposition 3.4.3.** Any coherent system of orientations of the moduli spaces \( \mathcal{M}^d_g \) has the following three properties.

- The moduli space \( \mathcal{M}^0(\alpha_0, \alpha_0) \) consists of a single point \( \mathcal{D}_0 \), and this point is positively oriented.
- The map (3.13) preserves or reverses the orientation according to the graded sign of the permutation

\[
(\sigma_1, \alpha_{1,1}, \ldots, \alpha_{1,k_1^{-}}, \alpha_{1,1}^{+}, \ldots, \alpha_{1,i_1^{-} - 1}, \alpha_{0,1,i_1+1}, \ldots, \alpha_{1,k_1^{+} + 1},
\]

\[
\sigma_2, \alpha_{2,1}, \ldots, \alpha_{2,j_2^{-} - 1}, \alpha_{0,1,j_2+1}, \ldots, \alpha_{2,k_2^{-} + 1}, \alpha_{2,1}^{+}, \ldots, \alpha_{2,k_2^{+} + 1}
\]

\[
\mapsto (\sigma_1, \sigma_2, \alpha_{2,1}, \ldots, \alpha_{2,j_2^{-} - 1}, \alpha_{1,1}, \ldots, \alpha_{1,k_1^{-}}, \alpha_{2,j_2+1}, \ldots, \alpha_{2,k_2^{-} + 1},
\]

\[
\alpha_{1,1}^{+}, \ldots, \alpha_{1,i_1^{-} - 1}, \alpha_{2,1}^{+}, \ldots, \alpha_{2,k_2^{+} + 1}, \alpha_{1,1}^{+}, \ldots, \alpha_{1,i_1+1}, \alpha_{2,1}^{+}, \ldots, \alpha_{2,k_2^{+} + 1}
\]

where \( \alpha \) has degree \( \mu_\mu(\alpha) \), \( \sigma_1 \) has degree \( q_1 \), and \( \sigma_2 \) has degree \( q_2 \).
- The map (3.14) preserves or reverses the orientations according to the graded sign of the permutation (3.11).
To show this one argues the same way as when we verified the third condition of Def. 3.4.1 in the proof of Prop. 3.4.2, keeping track of the additional factors $O_{[0,1]}^\alpha$, $O_{[0,1]}^{\rho_1}$, and $O_{[0,1]}^\rho$. Let $S_k$ denote the symmetric group on $k$ letters. Let $\rho = (\rho^-, \rho^+)$ with $\rho^- : \{1, \ldots, k^-\} \to \{1, \ldots, k^-\}$ and $\rho^+ : \{1, \ldots, k^+\} \to \{1, \ldots, k^+\}$ be an element of $S_{k^-} \times S_{k^+}$. Let $\varphi$ be any diffeomorphism with $\varphi(p_i^\pm) = p_{\rho^\pm (i)}^\pm$ and $\varphi_* F_i^\pm = F_{\rho^\pm (i)}^\pm$. Then for any $c \in J^*_g,k^-,k^+(A)$, $[\varphi_* c] \in J^*_g,k^-,k^+(A)$ is uniquely determined by $[c] \in J^*_g,k^-,k^+(A)$ and $(\rho^-, \rho^+) \in S_{k^-} \times S_{k^+}$. This defines a natural group action

$$(S_{k^-} \times S_{k^+}) \times J^*_g,k^-,k^+(A) \to J^*_g,k^-,k^+(A).$$

For each $\rho \in S_{k^-} \times S_{k^+}$ a similar construction gives a map

$$\mathcal{P}_\rho((\alpha_1^-, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+)) \to \mathcal{P}_\rho.d((\alpha^-_{\rho^- (1)}, \ldots, \alpha^-_{\rho^- (k^-)}), (\alpha^+_{\rho^+ (1)}, \ldots, \alpha^+_{\rho^+ (k^+)})�.\nu

This map has natural lifts to bundle maps $T^\text{fib}\mathcal{P}_\rho \to T^\text{fib}\mathcal{P}_\rho.d$ and $F_\rho \to F_\rho.d$. These bundle maps intertwine $D\mathcal{P}_\rho$ and $D\mathcal{P}_\rho.d$. Hence there is an induced map

$$(3.16) \quad \mathcal{O}_\rho \to \mathcal{O}_\rho.d. \nu

\textbf{Proposition 3.4.4.} \quad \text{The map (3.16) preserves or reverses the coherent orientation according to the graded sign of the permutation}

$$(3.17) \quad (\alpha_1^-, \ldots, \alpha_k^-, \alpha_1^+, \ldots, \alpha_k^+) \to (\alpha^-_{\rho^- (1)}, \ldots, \alpha^-_{\rho^- (k^-)}, \alpha^+_{\rho^+ (1)}, \ldots, \alpha^+_{\rho^+ (k^+)})�.\nu

This is shown the same way as Prop. 3.4.3.

3.5. Uhlenbeck-Gromov compactness. The usual integration by parts argument, shows that if

$$u \in \mathcal{P}^d((\alpha_1^-, \ldots, \alpha_k^-), (\alpha_1^+, \ldots, \alpha_k^+))$$

then

$$E[u] \leq C + \int_{\Sigma_g,k^-,k^+} u^* \omega,$$

where $C$ only depends on $H$ and $R$. The integral of $u^* \omega$ is uniquely determined by the periodic orbits $\alpha_i^\pm$ and $d$. In fact, if $u_1 \in \mathcal{P}^{d_1}$ and $u_2 \in \mathcal{P}^{d_2}$, then $\int_{\Sigma_g} (u_1^* \omega - u_2^* \omega)$ equals $k$ times the relative first Chern class of $u_1^* TM$ and $u_2^* TM$, which, by excision and the Atiyah-Singer theorem, equals $k(d_1 - d_2)/2$. With a uniform estimate of $E[u]$ at hand, the following compactness result follows by the standard Uhlenbeck-Gromov argument; see for instance [PW].
Proposition 3.5.1. If $\sigma : [0,1]^q \to \mathcal{J}^*_{g, k^- + k^+}(A)$ is semi-regular, then the moduli space $\mathcal{M}^d_a((\alpha_i^- \ldots \alpha_k^-), (\alpha_i^+ \ldots \alpha_k^+))$ is empty for all $d < 0$, and any sequence in $\mathcal{M}^0_a((\alpha_i^- \ldots \alpha_k^-), (\alpha_i^+ \ldots \alpha_k^+))$ has a strongly convergent subsequence.

There is a similar compactness result for 1-dimensional moduli spaces.

Proposition 3.5.2. If $\sigma : [0,1]^q \to \mathcal{J}^*_{g, k^- + k^+}(A)$ is regular, then any sequence $[u_n, c_n]$ in $\mathcal{M}^1_a((\alpha_i^- \ldots \alpha_k^-), (\alpha_i^+ \ldots \alpha_k^+))$ has a subsequence, that we also denote $[u_n, c_n]$, such that one of the following conditions holds:

- The subsequence converges strongly to an element of $\mathcal{M}^1_a((\alpha_i^- \ldots \alpha_k^-), (\alpha_i^+ \ldots \alpha_k^+))$.

- There exists $i \in \{1, \ldots, k^\circ\}$, $\alpha_0 \in C_H$, and a sequence $\ell_n$, such that $\ell_n \to \infty$ as $n \to \infty$, $(\ell_n, [u_n, c_n])$ is in the range of $g_{11}^{\ell_n}$, and $(g_{11}^{\ell_n})^{-1}(\alpha_0, [u_n, c_n])$ converges strongly to a pair in $M^1(\alpha_i^-, \alpha_0) \times M^0_a((\alpha_i^-, \ldots, \alpha_i^{-1}, \alpha_0, \alpha_{i+1}^-, \ldots, \alpha_{k^-}^-), (\alpha_i^+, \ldots, \alpha_{k^+}^+))$. If $\alpha_i^- \in C^0_H$, then $\alpha_0 \in C^0_H$.

- There exists $i \in \{1, \ldots, k^\circ\}$, $\alpha_0 \in C_H$, and a sequence $\ell_n$, such that $\ell_n \to \infty$ as $n \to \infty$, and $(\ell_n, [u_n, c_n], \alpha_0)$ is in the range of $(g_{11}^{\ell_n})^{-1}$ and $(g_{11}^{\ell_n})^{-1}(u_n, c_n, \alpha_0)$ converges strongly to a pair in $\mathcal{M}^0((\alpha_i^-, \ldots, \alpha_k^-), (\alpha_i^+, \ldots, \alpha_{k^-}^+, 0, \alpha_{i+1}^+, \ldots, \alpha_{k^+}^+)) \times M^1(\alpha_0, \alpha_i^+)$. If $\alpha_i^- \in C^0_H$, then $\alpha_0 \in C^0_H$.

There are similar estimates for $E[u]$ with $u \in \mathcal{M}^d_{\sigma_1 \circ \sigma_2}$ that are uniform in $\ell$, if the energy integrand $\mathcal{J}^*_{g, k^- + k^+}(A)$ is redefined as $|\nabla u - \eta(t_0) \eta(t - t_0) d\theta_0 \otimes \nabla H|^2$ on the neck. Then the standard Uhlenbeck-Gromov argument gives the following compactness result.

Proposition 3.5.3. If $\sigma_1 : [0,1]^n \to \mathcal{J}^*_{g_1, k_{1^-} + k_{1^+}}(A)$ and $\sigma_2 : [0,1]^{n_2} \to \mathcal{J}^*_{g_2, k_{2^-} + k_{2^+}}(A)$ are semi-regular, and $\sigma_1 \circ \sigma_2$ takes values in $\mathcal{J}^*_{g_1 + g_2, k_{1^-} + k_{1^+} + k_{2^-} + k_{2^+}}(A)$ for all $\ell \geq 0$, then, for $\ell$ large enough,

$$\mathcal{M}^d_{\sigma_1 \circ \sigma_2}(\alpha_1^+, \ldots, \alpha_{i+1}^+, \alpha_{i+2}^+, \ldots, \alpha_k^+, \alpha_{i+1}^-, \alpha_{i+2}^-, \ldots, \alpha_k^-)$$
is empty for all \(d < 0\), and for any sequence \(\ell_n\) of positive numbers such that \(\ell_n \to \infty\) as \(n \to \infty\) and for any sequence

\[
[u_n, c_n] \in \mathcal{M}_d^0 \times \mathcal{M}_d^0((\alpha_{1,1}, \ldots, \alpha_{1,j-1}, \alpha_1, \ldots, \alpha_{1,k^-}, \alpha_{2,j+1}, \ldots, \alpha_{2,k^+} + 1), (\alpha_{1,1}', \ldots, \alpha_{1,j-1}', \alpha_1', \alpha_{1,k^-}', \alpha_{2,j+1}', \ldots, \alpha_{2,k^+}'))
\]

there exists \(\alpha_0 \in \mathcal{C}_H\) and a subsequence, that we also denote \(\ell_n\) and \([u_n, c_n]\), such that \(((u_n, c_n), \overline{\sigma}_0)\) lies in the range of \(g_{ij}^\ell\) and \((g_{ij}^\ell)^{-1}([u_n, c_n], \overline{\sigma}_0)\) converges strongly to a pair in

\[
\mathcal{M}^0_{\sigma_1}((\alpha_{1,1}, \ldots, \alpha_{1,k^-}'), (\alpha_{1,1}', \ldots, \alpha_{1,k^-}', \alpha_0', \alpha_{1,i+1}', \ldots, \alpha_{1,k^-}'))
\]

\[
\times \mathcal{M}^0_{\sigma_2}((\alpha_{2,1}', \ldots, \alpha_{2,j-1}', \alpha_0, \alpha_{2,j+1}', \ldots, \alpha_{2,k^+}'), (\alpha_{2,1}', \ldots, \alpha_{2,k^+}')).
\]

If \(g_1 = 0\) and \(\alpha_{1,\nu}^\pm \in \mathcal{C}_H^0\) for \(\nu = 1, \ldots, k^+_i\), or if \(g_2 = 0\) and \(\alpha_{2,\nu}^\pm \in \mathcal{C}_H^0\) for \(\nu = 1, \ldots, k^+_2\), then \(\alpha_0 \in \mathcal{C}_H^0\).

There is a similar compactness theorem for \(\mathcal{M}^d_{\mathcal{C}_{ij}^\sigma}\) with \(d \leq 0\).

### 3.6. Gluing of moduli spaces.

We now change our notation slightly, and let \(A\) denote a triple \((H, J, \sigma)\) where \((H, J)\) is a regular pair of a time-dependent Hamiltonian and almost complex structure on \(M\) and \(\sigma\) is a coherent system of orientations. It follows from Prop. 3.5.1 that if \(H\) is regular and \(\sigma : [0, 1]^q \to \mathcal{L}^{*}_{g,k^-, k^+}(A)\) is transverse to

\[
\pi : \mathcal{M}^{-q}((\alpha_{1,1}, \ldots, \alpha_{k^-_1}), (\alpha_{1,1}', \ldots, \alpha_{k^-_1}')) \to \mathcal{L}^{*}_{g,k^-, k^+}(A),
\]

then the moduli space \(\mathcal{M}^0_\sigma((\alpha_{1,1}, \ldots, \alpha_{k^-_1}), (\alpha_{1,1}', \ldots, \alpha_{k^-_1}'))\) consists of a finite number of points. If all \(\alpha_{i}^\pm \in \mathcal{C}_H^0\), then the moduli space becomes a finite set of points, with signs given by \(\sigma\). We denote the number of points, counted with signs, by

\[
\#\mathcal{M}^0_\sigma((\alpha_{1,1}, \ldots, \alpha_{k^-_1}), (\alpha_{1,1}', \ldots, \alpha_{k^-_1}')).
\]

Similarly, \(\mathcal{M}(\alpha^-, \alpha^+)\) is a finite union of affine lines. The affine structures give the lines canonical orientations. The coherent system \(\sigma\) also induces orientations. We count a line as positive if the coherent and the canonical orientations agree, and negative otherwise. We denote the number of lines, counted with signs, by

\[
\#\mathcal{M}(\alpha^-, \alpha^+).
\]
Proposition 3.6.1. For any contractible periodic orbit $\alpha$,

$$\#M^0(\alpha, \alpha) = 1.$$ 

This is an immediate consequence of Prop. 3.4.3.

Proposition 3.6.2. If $\rho = (\rho^-, \rho^+) \in S_{k^-} \times S_{k^+}$, then

$$\#M^0_{\rho, \sigma}((\alpha^-_{\rho^-(1)}, \ldots, \alpha^-_{\rho^-(k^-)}), (\alpha^+_{\rho^+(1)}, \ldots, \alpha^+_{\rho^+(k^+)})$$

$$= \pm \#M^0_{\rho, \sigma}((\alpha^-_1, \ldots, \alpha^-_{k^-}), (\alpha^+_1, \ldots, \alpha^+_{k^+}))$$

where the sign is the graded sign of the permutation (3.17).

This is an immediate consequence of Prop. 3.4.4.

Proposition 3.6.3. If $\sigma_1 : [0, 1]^{q_1} \to J_{g_1, k^-_{1}, k^+_{1} + 1}(A)$ and $\sigma_2 : [0, 1]^{q_2} \to J_{g_2, k^-_{2}, k^+_{2} + 1}(A)$ are semi-regular and

$$\sigma_1 \circ_{ij} \sigma_2 : [0, 1]^{q_1 + q_2} \to J_{g_1 + g_2, k^-_{1} + k^-_{2} + 1, k^+_{1} + k^+_{2} + 1}(A)$$

takes values in $J_{g_1 + g_2, k^-_{1} + k^-_{2}, k^+_{1} + k^+_{2}}(A)$ for all $\ell \geq 0$, then, for sufficiently large $\ell$, $\sigma_1 \circ_{ij} \sigma_2$ is semi-regular, and if all periodic orbits of $H$ are contractible, then

$$\#M^0_{\sigma_1 \circ_{ij} \sigma_2}((\alpha^-_1, \ldots, \alpha^-_{2,j-1}, \alpha^-_{1,k^-_{1}}, \alpha^-_{2,j+1}, \ldots, \alpha^-_{2,k^-_{1} + 1}),$$

$$(\alpha^-_{1,1}, \ldots, \alpha^-_{1,i-1}, \alpha^-_{2,1}, \ldots, \alpha^-_{2,k^-_{2}}, \alpha^+_{1,i+1}, \ldots, \alpha^+_{1,k^+_{1} + 1}))$$

$$= (-1)^{q_1 q_2} \sum_{\alpha_0 \in c_H^0} \pm \#M^0_{\sigma_1}((\alpha^-_{1,1}, \ldots, \alpha^-_{1,k^-_{1}}),$$

$$(\alpha^-_{1,1}, \ldots, \alpha^-_{1,i-1}, \alpha^0, \alpha^+_{1,i+1}, \ldots, \alpha^+_{1,k^+_{1} + 1}))$$

$$\pm \#M^0_{\sigma_2}((\alpha^-_{2,1}, \ldots, \alpha^-_{2,j-1}, \alpha^-_{2,j+1}, \ldots, \alpha^-_{2,k^-_{2} + 1}),$$

$$(\alpha^+_{2,1}, \ldots, \alpha^+_{2,k^+_{2}})),$$

where the signs equal the graded signs of the permutations (3.10).

The assumption that all periodic orbits of $H$ are contractible can be replaced by $g_1 = 0$ and $\alpha^+_{1,\nu} \in c_H^0$ for $\nu = 1, \ldots, k^+_{1}$, or by $g_2 = 0$ and $\alpha^+_{2,\nu} \in c_H^0$ for $\nu = 1, \ldots, k^+_{2}$,
Proposition 3.6.4. If \( \sigma : [0, 1]^q \rightarrow \mathcal{J}^*_g, k-1, k+1(A) \) is semi-regular and
\[
\mathcal{C}'_\ell \sigma : [0, 1]^q \rightarrow \mathcal{J}^*_{g+1, k-, k+}(A)
\]
takes values in \( \mathcal{J}^*_{g+1, k-, k+}(A) \) for all \( \ell \geq 0 \), then, for sufficiently large \( \ell \), \( \mathcal{C}'_\ell \sigma \) is semi-regular, and if all periodic orbits of \( H \) are contractible, then
\[
\#M^0_{\mathcal{C}'_\ell \sigma}((\alpha_1^-, \ldots, \alpha_{j-1}^-, \alpha_{j+1}^-, \ldots, \alpha_{k+1}^-), (\alpha_1^+, \ldots, \alpha_{i-1}^+, \alpha_{i+1}^+, \ldots, \alpha_{k+1}^+))
\]
\[
= \sum_{\alpha_0 \in \mathcal{C}^0_H} \pm \#M^0_\sigma((\alpha_1^-, \ldots, \alpha_{j-1}^-, \alpha_0, \alpha_{j+1}^-, \ldots, \alpha_{k+1}^-), (\alpha_1^+, \ldots, \alpha_{i-1}^+, \alpha_0, \alpha_{i+1}^+, \ldots, \alpha_{k+1}^+)),
\]
where the signs equal the graded signs of the permutation (3.11).

Given a smooth map
\( \sigma : [0, 1]^q \rightarrow \mathcal{J}^*_g, k-, k+ (A) \),
we can form the boundary maps
\[
\partial^0 \sigma, \partial^1 \sigma : [0, 1]^{q-1} \rightarrow \mathcal{J}^*_g, k-, k+ (A)
\]
given by
\[
\partial^0 \sigma(x_1, \ldots, x_{q-1}) = \sigma(x_1, \ldots, x_{\nu-1}, 0, x_\nu, \ldots, x_{q-1})
\]
and
\[
\partial^1 \sigma(x_1, \ldots, x_{q-1}) = \sigma(x_1, \ldots, x_{\nu-1}, 1, x_\nu, \ldots, x_{q-1}).
\]

We use the following convention for orienting boundaries of manifolds. Let \( X \) be a manifold with boundary \( \partial X \). We orient the normal bundle \( N \partial X \) of \( \partial X \) by taking the outward direction to be positive. Given an orientation of \( X \) this determines an orientation of \( \partial X \) by the requirement that \( N \partial X \oplus T \partial X = TX \) as oriented vector spaces.

Proposition 3.6.5. If \( \sigma : [0, 1]^q \rightarrow \mathcal{J}^*_g, k-, k+ (A) \) is semi-regular and \( \alpha_i^+ \in \mathcal{C}^0_H \) for
\( i = 1, \ldots, k^+ \), then
\[
\sum_{j=1}^{k^-} \sum_{\alpha_0 \in \mathcal{C}^0_H} (-1)^{q+\mu(\alpha_1^-)+\ldots+\mu(\alpha_{j-1}^-)} \#M^1_\sigma((\alpha_1^-, \ldots, \alpha_{j-1}^-), (\alpha_1^+, \ldots, \alpha_{k}^+)), \#M^0_\sigma((\alpha_1^-, \ldots, \alpha_{j-1}^-, \alpha_0, \alpha_{j+1}^-, \ldots, \alpha_{k}^-), (\alpha_1^+, \ldots, \alpha_{k}^+))
\]
\[
- \sum_{i=1}^{k^+} \sum_{\alpha_0 \in \mathcal{C}^0_H} (-1)^{q+\mu(\alpha_1^+)\ldots+\mu(\alpha_{i-1}^+)+\mu(\alpha_{i}^+)+\ldots+\mu(\alpha_{k}^+)+\mu(\alpha_0)} \#M^0_\sigma((\alpha_1^-, \ldots, \alpha_{k}^-), (\alpha_1^+, \ldots, \alpha_{i-1}^+, \alpha_0, \alpha_{i+1}^+, \ldots, \alpha_{k}^+)) \#M^1_\sigma(\alpha_0, \alpha_i^+)
\]
\[
+ \sum_{\nu=1}^q (-1)^{\nu+1} \left( \#M^0_{\partial^0 \sigma}((\alpha_1^-, \ldots, \alpha_{k}^-), (\alpha_1^+, \ldots, \alpha_{k}^+)) - \#M^0_{\partial^0 \sigma}((\alpha_1^-, \ldots, \alpha_{k}^-), (\alpha_1^+, \ldots, \alpha_{k}^+)) \right) = 0.
\]
Proposition 3.6.6. ([F4] Thm. 4.) For $\alpha^-, \alpha^+ \in C_H^0$,

$$\sum_{\alpha_0 \in C_H} \#M^1(\alpha^-, \alpha_0) \#M^1(\alpha_0, \alpha^+) = 0.$$ 

Proof of Thm. 3.6.3. We denote the moduli spaces $M_{\sigma_1, i}^{0, +}[\alpha_0]$, $M_{\sigma_2, j}^{0, -}[\alpha_0]$ and $M_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0]$. In this notation, we have to show that for $\ell$ large enough, $M_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0]$ is semi-regular, and

$$\#M_{\sigma_1, i}^{0, \ell}[\alpha_0] = \sum_{\alpha_0 \in C_H} \pm \#M_{\sigma_1, i}^{0, +}[\alpha_0] \#M_{\sigma_2, j}^{0, -}[\alpha_0].$$

The moduli space $M_{\sigma_1, i}^{0, +}[\alpha_0] \times M_{\sigma_2, j}^{0, -}[\alpha_0]$ is the zero locus of the section $\Psi_{\sigma_1, i}^{0, +}[\alpha_0] \oplus \Psi_{\sigma_2, j}^{0, -}[\alpha_0]$ of the bundle $F_{\sigma_1, i}^{0, +}[\alpha_0] \oplus F_{\sigma_2, j}^{0, -}[\alpha_0]$ over $M_{\sigma_1, i}^{0, +}[\alpha_0] \times M_{\sigma_2, j}^{0, -}[\alpha_0]$. By Prop. 2.3.6 this section is transverse to the zero section. Let $\mathcal{U}$ be a neighborhood of $M_{\sigma_1, i}^{0, +}[\alpha_0] \times M_{\sigma_2, j}^{0, -}[\alpha_0]$ in $F_{\sigma_1, i}^{0, +}[\alpha_0] \times F_{\sigma_2, j}^{0, -}[\alpha_0]$. It follows from the implicit function theorem that if we choose $\mathcal{U}$ small enough, then any sufficiently small perturbation of the restriction of $\Psi_{\sigma_1, i}^{0, +}[\alpha_0] \oplus \Psi_{\sigma_2, j}^{0, -}[\alpha_0]$ to $\mathcal{U}$ is also transverse to the zero section and its zero locus is a small perturbation of $M_{\sigma_1, i}^{0, +}[\alpha_0] \times M_{\sigma_2, j}^{0, -}[\alpha_0]$.

By Prop. 3.4.3, we can count the number of points in $M_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0]$ instead of $M_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0]$. It follows from Prop. 3.5.1 that for $\ell$ large enough, $M_{\sigma_1, i}^{0, +}[\alpha_0] \times M_{\sigma_2, j}^{0, -}[\alpha_0]$ is contained in the domain of $g^j_{ij}$. It follows from Prop. 3.5.3 that for $\ell$ large enough, $M_{\sigma_1, i}^{0, +}[\alpha_0] \times M^{0}(\alpha_0, \alpha)$ lies in the range of $g^j_{ij}$. It also follows from Prop. 3.5.3 that $(g^j_{ij})^{-1} \circ M_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0, \alpha]) \subseteq \mathcal{U}$ for $\ell$ large enough.

Now $\Psi_{\sigma_1, i}^{0, +}[\alpha_0] \oplus \Psi_{\sigma_2, j}^{0, -}[\alpha_0]$ and $(g^j_{ij})^{*}(\Psi_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0, \alpha])$ are equal off the necks. By [R] Prop. 2.1 that on the necks, with $u_1 = \exp \xi_1$ and $u_2 = \exp \xi_2$,

$$\Psi_{\sigma_1, i}^{0, +}[\alpha_0](u_1) \oplus \Psi_{\sigma_2, j}^{0, -}[\alpha_0](u_2) = (\exp \xi_1)(d\xi_1/\xi_2 + J(D_{\alpha}(\Phi_H))\xi_1 + (\xi_1 \otimes \xi_1) \cdot f_1)$$

$$\oplus (\exp \xi_2)(d\xi_2/\xi_2 + J(D_{\alpha}(\Phi_H))\xi_2 + (\xi_2 \otimes \xi_2) \cdot f_2)$$

where $f_1$ and $f_2$ are smooth functions of $\xi$, $\nabla \xi$, $\eta$, and $\nabla \eta$. Similarly, on the necks, with $u_3 = \exp \xi_3$ and $u_4 = \exp \xi_4$,

$$\Psi_{\sigma_1 \circ \sigma_2}^{0, \ell}[\alpha_0, \alpha]$$

$$= (\exp \xi_3)(d\xi_3/\xi_3 + J(D_{\alpha}(\Phi_H))\xi_3 + (\xi_3 \otimes \xi_3) \cdot f)$$

$$\oplus (\exp \xi_4)(d\xi_4/\xi_4 + J(D_{\alpha}(\Phi_H))\xi_4 + (\xi_4 \otimes \xi_4) \cdot f).$$
It follows that
\[(\Psi_{\sigma_1,1}^0[a_0] \boxplus \Psi_{\sigma_2,j}^0[a_0]) - (g_{ij})^*(\Psi_{\sigma_1 \circ f_j \sigma_2}^0(a_0, a_0))\]
vansishes off the necks, and on the necks it is given by terms of the form \((\xi_\mu \otimes \xi_\nu) \cdot f\) and \((d\eta/dt) \xi_\mu \cdot f\). We can make these terms, and their first order Fréchet derivatives, arbitrarily small by choosing \(\Omega\) small and \(\ell\) large. It then follows from the implicit function theorem that \(M_{\sigma_1,1}^{0,1}[a_0] \times M_{\sigma_2,j}^{0,1}[a_0]\) is a small perturbation of \((g_{ij})^{-1}(M_{\sigma_1 \circ f_j \sigma_2}^0 \times M^0(a_0, a_0))\). The Proposition follows, if we disregard signs.

Finally, the excision map \(g_{ij}\) induces a map (3.13). By Prop. 3.4.3 this map changes the coherent orientations by the graded sign of the permutation (3.15). By Prop. 3.3.3, if the moduli spaces are non-empty, then
\[q_1 \equiv \mu_\nu(a_{1,1}) + \cdots + \mu_\nu(a_{1,k_1}) + \mu_\nu(a_{0,1}) + \cdots + \mu_\nu(a_{0,2}) + \mu_\nu(a_{0,1}) + \cdots + \mu_\nu(a_{0,1})\]
modulo 2. It follows that the signs of the permutations (3.15) and (3.10) differ by \((-1)^{q_1} q_2\). This accounts for the signs in the Proposition.

\[\square\]

**Proof of Thm. 3.6.4.** This is proven the same way as Thm. 3.6.3.

\[\square\]

**Proof of Thm. 3.6.5.** For simplicity we denote the moduli spaces \(M_{\sigma_1}^{0,1}[a_0], M_{\partial g, \sigma}^0, M_{\partial g, \sigma}^0\). In this notation we have to show that
\[
\sum_{j, a_0} \pm \#M^1(\alpha_j, a_0) \#M_{\sigma, j}^{0,1}[a_0] - \sum_{i, a_0} \pm \#M_{\sigma, i}^{0,1}[a_0] \#M^1(a_0, \alpha_i^+) \\
+ \sum_{\nu} (-1)^{\nu+1} \left(\#M_{\partial g, \sigma}^0 - \#M_{\partial g, \sigma}^0\right) = 0.
\]
(3.18)

We also let
\[M_{\partial g, \sigma}^1 = M_{\partial g, \sigma}^1((\alpha_1^+, \ldots, \alpha_{k_+}^+), (\alpha_1^-, \ldots, \alpha_{k_-}^-)).\]

At first we assume that \(\mu\) is transverse to \(\pi: M_{g, k_-, k_+}^{1-q} \to \mathcal{J}_{g, k_-, k_+}^*(A)\). At the end of the proof we will show how to eliminate this assumption. The idea behind the proof is to identify the boundary of \(M_{\partial g, \sigma}^1\), and use the fact that the number of boundary points counted with signs is zero. We also have to consider that \(M_{\partial g, \sigma}^1\) may be non-compact, and include the number of ends, also counted with signs. This accounts for the first two sums in (3.18).

The map \(\sigma\), and the boundary maps \(\partial^0 g\) and \(\partial^1 g\), are transverse to \(\pi: M_{g, k_-, k_+}^{1-q} \to \mathcal{J}_{g, k_-, k_+}^*(A)\). It follows that the moduli space \(M_{\sigma}^1\) is a smooth 1-manifold with boundary,
and its boundary points are given by the moduli spaces $\mathcal{M}_{\partial^0 \sigma}^0$ and $\mathcal{M}_{\partial^1 \sigma}^0$. This accounts for the third sum in (3.18).

The moduli spaces $\mathcal{M}_{\partial^0 \sigma}^0$ and $\mathcal{M}_{\partial^1 \sigma}^0$ have two natural orientations, the coherent orientation and the orientation as boundary of $\mathcal{M}_{\sigma}^1$. We have $\partial^0 \sigma = \partial [0,1]^{-1} \otimes \sigma^* \Omega_{g,k}$. The coherent orientation of $\mathcal{M}_{\partial^0 \sigma}^0$ is given by the canonical orientation of $[0,1]^{-1}$ and the coherent orientation of $\Omega_{g,k}$. The boundary orientation of $\mathcal{M}_{\partial^0 \sigma}^0$ is given by the boundary orientation of $[0,1]^{-1}$ and the coherent orientation of $\Omega_{g,k}$. These orientations differ by $\alpha^0$ for $\partial^0 \sigma$ and by $\alpha^1$ for $\partial^1 \sigma$. This accounts for the signs in the third sum in (3.18).

Define $\mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm}$ by

$$\mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm} \times \mathcal{M}^0(\alpha_0, \alpha_0) = (\mathcal{M}_{\sigma}^1 \times \mathcal{M}^0(\alpha_0, \alpha_0)) \cap \text{range } g^{\ell}_{11}.$$ 

where $g^{\ell}_{11}$ is the excision map

$$(g^{\ell}_{11}) : \Psi_{\sigma,i}^0(\alpha_0) \times F^1(\alpha_0, \alpha_i^+) \rightarrow \Psi_{\sigma,i}^0 \times F^0(\alpha_0, \alpha_0).$$

Similarly, define $\mathcal{M}_{\sigma,i}^{1,\ell,\pm}(\alpha_0, \alpha_i^+)$ by

$$\mathcal{M}_{\sigma,i}^{0,\ell,\pm}(\alpha_0) \times \mathcal{M}_{\sigma,i}^{1,\ell,\pm}(\alpha_0, \alpha_i^+) = (\mathcal{M}_{\sigma,i}^{0,\ell,\pm}(\alpha_0) \times \mathcal{M}_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)) \cap \text{domain } g^{\ell}_{11}.$$

It follows from Prop. 3.5.2 that for $\ell$ large enough,

$$\mathcal{M}_{\sigma}^1 \setminus \bigcup_{\pm,i,\alpha_0} \mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm}$$

is a compact subset of $\mathcal{M}_{\sigma}^1$. Thus it suffices to count the number of ends of each $\mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm}$.

First we count the number of ends of $\mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm}$. The moduli space $\mathcal{M}_{\sigma,i}^{0,\ell,\pm}[\alpha_0] \times \mathcal{M}_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$ is a union of affine lines. It is the zero locus of a translation invariant section $\Psi_{\sigma,i}^{0,\ell,\pm}[\alpha_0] \oplus \Psi_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$. By Prop. 2.3.6, this section is transverse to the zero section. Hence there exists a translation invariant neighborhood $U$ of $\mathcal{M}_{\sigma,i}^{0,\ell,\pm}[\alpha_0] \times \mathcal{M}_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$ in $\Psi_{\sigma,i}^{0,\ell,\pm}[\alpha_0] \oplus \Psi_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$ such that the zero locus of any sufficiently small perturbation of $\Psi_{\sigma,i}^{0,\ell,\pm}[\alpha_0] \times \Psi_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$ is a small perturbation of $\mathcal{M}_{\sigma,i}^{0,\ell,\pm} \times \mathcal{M}_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$. It follows from Prop. 3.5.2 that $(g^{\ell}_{11})^{-1}(\mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm} \times \mathcal{M}_{\sigma,i}^{0}(\alpha_0, \alpha_0)) \subset U$ for $\ell$ large enough. Now $(g^{\ell}_{11})^{-1}(\mathcal{M}_{\sigma,i}^{1,\alpha_0,\ell,\pm} \times \mathcal{M}_{\sigma,i}^{0}(\alpha_0, \alpha_0))$ is the zero locus of $(g^{\ell}_{11})^* (\Psi_{\sigma,i}^{1,\alpha_0,\ell,\pm} \oplus \Psi_{\sigma,i}^{0}(\alpha_0, \alpha_0))$. As in the proof of Prop. 3.6.3, these two sections of $\mathcal{F}_{\sigma,i}^{0,\alpha_0,\ell,\pm} \oplus \mathcal{F}_{\sigma,i}^{1}(\alpha_0, \alpha_i^+)$ differ by terms of the form $(df/ds) \xi_\mu \cdot f$ and $(\xi_\mu \otimes \xi_\nu) \cdot f$ supported on the neck. These terms, and their first order Fréchet derivatives, can be made arbitrarily small.
by choosing $\Delta$ small and $\ell$ large. It then follows from the implicit function theorem that $(g_{i_1}^\ell)^{-1}(M_{i_1}^{\alpha_0,\ell,\mu} \times M^0(\alpha_0,\alpha_0))$ is a small perturbation of $M_{i_1}^{0,\alpha_0} \times M^1(\alpha_0,\alpha_0)$. Thus the ends of $(M_{i_1}^{1,\alpha_0,\ell,\mu} \times M^0(\alpha_0,\alpha_0))$ can be identified with the ends of $M_{i_1}^{1,\alpha_0} \times M^1(\alpha_0,\alpha_0)$. This accounts for the first sum in (3.18).

It follows from Prop. 3.4.3 that the map $g_{i_1}$ changes the orientations by the graded sign of the permutation

$$(\sigma, \alpha_1^+, \ldots, \alpha_k^-, \sigma, \ldots, \alpha_i^+, \ldots, \alpha_{i-1}^-, \alpha_0, \alpha_0, \alpha_0)$$

$$\mapsto (\sigma, \alpha_1^-, \ldots, \alpha_k^-, \sigma, \ldots, \alpha_i^-, \ldots, \alpha_{i-1}^+, \alpha_0, \alpha_0),$$

which is

$$(-1)^{\mu + (\alpha_{i-1}^+)^{\mu} + \cdots + \mu + (\alpha_+^+)} = (-1)^{q + \mu(\alpha_1^-) + \cdots + \mu(\alpha_k^-) + \mu(\alpha_i^+) + \cdots + \mu(\alpha_{i-1}^-) + \mu(\alpha_0)}.$$

This accounts for the signs in the first sum in (3.18). We get an additional minus sign as the number of ends of $M^{1,\ell,\mu}(\alpha_0,\alpha_0)$, counted with signs, is $-(\# M^1(\alpha_0,\alpha_0))$.

Similarly, the second sum in (3.18) gives the number of ends of $M_{i_1}^{1,\alpha_0,\ell,\mu}$. The sign of this term is given by the graded sign of the permutation

$$(\alpha_j^-, \alpha_0, \sigma, \alpha_1^-, \ldots, \alpha_{j-1}^-, \alpha_0, \alpha_{j+1}^-, \ldots, \alpha_{k-1}^-, \alpha_0, \alpha_k^+, \ldots, \alpha_0)$$

$$\mapsto (\sigma, \alpha_1^-, \ldots, \alpha_k^-, \sigma, \ldots, \alpha_{j-1}^+, \ldots, \alpha_{k+}^+, \alpha_0, \alpha_0),$$

which is

$$(-1)^{q + \mu(\alpha_1^-) + \cdots + \mu(\alpha_{j-1}^-)}.$$

Finally, assume that $\sigma$ is not transverse to $\pi : \mathcal{M}_{g,k^-}^{1-q} \to \mathcal{J}_{g,k^-}^{*,\mu}(A)$. By assumption $\partial_{\mu}^g\sigma$ and $\partial_{\mu}^b\sigma$ are semi-regular and hence transverse to $\pi : \mathcal{M}_{g,k^-}^{1-q} \to \mathcal{J}_{g,k^-}^{*,\mu}(A)$. Then there exists a sequence of maps $\sigma_n : [0,1]^q \to \mathcal{J}_{g,k^-}^{*,\mu}(A)$ such that $\sigma_n$ is transverse to $\pi : \mathcal{M}_{g,k^-}^{1-q} \to \mathcal{J}_{g,k^-}^{*,\mu}(A)$, $\sigma_n \to \sigma$ as $n \to \infty$, $\partial_{\mu}^g\sigma_n = \partial_{\mu}^g\sigma$, and $\partial_{\mu}^b\sigma_n = \partial_{\mu}^b\sigma$. The Proposition holds with $\sigma$ replaced by $\sigma_n$. Thus we only have to verify that $\# \mathcal{M}_{\sigma_n,\iota}^{0,\pm}[\alpha_0] = \# \mathcal{M}_{\sigma,\iota}^{0,\pm}[\alpha_0]$ for $n$ large enough.

Let $\mathcal{W}$ be a neighborhood of $\mathcal{M}_{\sigma_n,\iota}^{0,\pm}[\alpha_0]$ in $\mathcal{M}_{\sigma_n,\iota}[\alpha_0]$. The moduli spaces $\mathcal{M}_{\sigma_n,\iota}^{0,\pm}[\alpha_0]$ and $\mathcal{M}_{\sigma,\iota}[\alpha_0]$ are the zero loci of sections $\Psi_{\sigma_n,\iota}^{0,\pm}[\alpha_0]$ and $\Psi_{\sigma,\iota}^{0,\pm}[\alpha_0]$. It follows from the implicit function theorem that if we choose $\mathcal{W}$ small enough and $n$ large enough, then $\mathcal{M}_{\sigma_n,\iota}^{0,\pm}[\alpha_0] \cap \mathcal{W}$ is a small perturbation of $\mathcal{M}_{\sigma,\iota}^{0,\pm}[\alpha_0] \cap \mathcal{W}$. On the other hand, in Prop. 3.5.1 we can let $v_n \in \mathcal{M}_{\alpha_n}^{0,\pm}$, where $\sigma_n \to \sigma$ as $n \to \infty$. It follows that for any $\mathcal{W}$, $\mathcal{M}_{\sigma_n,\iota}[\alpha_0] \subset \mathcal{W}$ for $n$ large enough. □
Remarks on Prop. 3.6.6. A proof of this is outlined in [F4]. No complete proof has appeared in print. A gap in [F4], the possibility that, when \( N_0 = 1 \), a sequence in \( \mathcal{M}^2(\alpha, \alpha) \) degenerates to an element of \( \mathcal{M}^0(\alpha, \alpha) \) and a holomorphic sphere with \( c_1 = 1 \), was bridged in [HS] by adding an additional transversality condition; see Definition 2.1.5.

Our scheme for orienting the moduli spaces differs from the method used by Floer, so we need to check the signs in Prop. 3.6.6. We get the same signs as Floer, for by Prop. 3.4.3, the excision maps

\[(g_{11}^f)_* : \mathcal{P}^1(\alpha^-, \alpha_0) \times \mathcal{P}^1(\alpha_0, \alpha^+) \to \mathcal{P}^2(\alpha^-, \alpha^+) \times \mathcal{P}^0(\alpha_0, \alpha_0)\]

preserve the coherent orientations.

\[\square\]

§4. Products and Relations

4.1. The homological gluing maps \( \diamond_{ij} \) and \( \mathfrak{c}_{ij} \). The singular homology groups \( H_*(\mathfrak{M}_{g,k-},(A), \mathbb{Z}) \) can be defined as the homology of the complex \( C_*(\mathfrak{M}_{g,k-},(A), \mathbb{Z}) \) of smooth cubical simplicial chains, modulo degenerate chains; see [HW] Sect. 8.3. These chains are linear combinations with integer coefficients of smooth cubical simplices, \( \sigma : [0,1]^q \to \mathfrak{M}_{g,k-}^*(A) \). The boundary operator is defined by

\[d\sigma = \sum_{\nu=1}^q (-1)^{\nu+1} (\partial_\nu^1 \sigma - \partial_\nu^0 \sigma).\]

If \( k^+_1 + k^-_1 + k^+_2 + k^-_2 \geq 1 \), then \( \mathfrak{M}^g_{g_1 + g_2, k^-_1 + k^-_2, k^+_1 + k^+_2}^*(A) = \mathfrak{M}^g_{g_1 + g_2, k^-_1 + k^-_2, k^+_1 + k^+_2}^*(A) \), and the gluing map (3.1) induces a chain map

\[\diamond_{ij} : C_*(\mathfrak{M}^g_{g_1, k^-_1, k^+_1}^*(A)) \otimes C_*(\mathfrak{M}^g_{g_2, k^-_2, k^+_2}^*(A)) \to C_*(\mathfrak{M}^g_{g_1 + g_2, k^-_1 + k^-_2, k^+_1 + k^+_2}^*(A)),\]

and hence a homomorphism

\[\diamond_{ij} : H_*(\mathfrak{M}^g_{g_1, k^-_1, k^+_1}^*(A)) \otimes H_*(\mathfrak{M}^g_{g_2, k^-_2, k^+_2}^*(A)) \to H_*(\mathfrak{M}^g_{g_1 + g_2, k^-_1 + k^-_2, k^+_1 + k^+_2}^*(A)).\]

Similarly, if \( k^- + k^+ \geq 1 \) then the gluing map (3.2) induces a chain map

\[\mathfrak{c}_{ij} : C_*(\mathfrak{M}^g_{g,k^-+1,k^++1}(A)) \to C_*(\mathfrak{M}^g_{g+1,k^-+1,k^++1}(A)),\]

and hence a homomorphism

\[\mathfrak{c}_{ij} : H_*(\mathfrak{M}^g_{g,k^-+1,k^++1}(A)) \to H_*(\mathfrak{M}^g_{g+1,k^-+1,k^++1}(A)).\]
The homology maps are independent of \( \ell \).

There are natural maps
\[
\diamond_{11} : H_*(\mathcal{J}_{g_1,0,1}(A)) \otimes H_*(\mathcal{J}_{g_2,1,0}(A)) \to H_*(\mathcal{J}_{g_1+g_2,0,0}(A))
\]
and
\[
\mathbb{C}_{11} : H_*(\mathcal{J}_{g,1,1}(A)) \to H_*(\mathcal{J}_{g+1,0,0}(A))
\]
as well. These are defined using the following Lemma.

**Lemma 4.1.1.** For any \( \sigma \in C_q(\mathcal{J}_{g,1,1}(A)) \) there exists \( \sigma' \in C_q(\mathcal{J}_{g,1,1}(A)) \) and \( \tau \in C_{q+1}(\mathcal{J}_{g,1,1}(A)) \) such that \( \sigma - \sigma' = d\tau \) and \( \mathbb{C}_{11}^\ell \sigma' \in C_q(\mathcal{J}_{g+1,0,0}(A)) \) for all \( \ell \geq 0 \).

Similarly, for any \( \sigma_1 \in C_q(\mathcal{J}_{g_1,0,1}(A)) \) and \( \sigma_2 \in C_{q_2}(\mathcal{J}_{g_2,1,0}(A)) \) there exist \( \sigma_1' \in C_{q_1}(\mathcal{J}_{g_1,0,1}(A)) \) and \( \tau \in C_{q_1+1}(\mathcal{J}_{g_1,1,1}(A)) \) such that \( \sigma_1 - \sigma_1' = d\tau \) and \( \sigma_1' \diamond_{11} \sigma_2 \in C_{q_1+q_2}(\mathcal{J}_{g_1+g_2,0,0}(A)) \) for all \( \ell \geq 0 \).

**Proof of Lemma 4.1.1.** We say that \( c = [j, R, \Delta] \in \mathcal{J}_{g,1,1}(A) \) is somewhere injective if it has the following property. There exists an open subset \( U \) of \( \Sigma_{g,1,1} \) such that for any \( p \in U \) there does not exist \( q \in \Sigma_{g,1,1} \) and a complex linear map \( L : (T^0_\Sigma)^* \to (T^0_\Sigma)^* \) such that for all \( m \in M \), \( R(q,m) = (L \otimes \text{id})R(p,m) \). If \( c \) is somewhere injective, \( \varphi_*j = j \) and \( \varphi_*R = R \), then \( \varphi \) is the identity on \( U \). By analytic continuation, \( \varphi \) is the identity on all of \( \Sigma_{g,1,1} \). In other words, if \( c \) is somewhere injective, then \( \mathbb{C}_{11}^\ell c \) is somewhere injective. The first part of the lemma follows.

The complement of the somewhere injective elements has infinite codimension in the following sense. Any map from a finite dimensional manifold into \( \mathcal{J}_{g,1,1}(A) \), such that the boundary values are somewhere injective, can be perturbed in the interior so that all its values are somewhere injective. If \( c \) is somewhere injective, then \( \mathbb{C}_{ij}^\ell c \) is somewhere injective. The first part of the lemma follows.

The second part is proven in a similar way.

### 4.2. Floer (co)homology and the contraction maps \( \diamond_{ij} \) and \( \mathbb{C}_{ij} \)

We first recall a few facts from homological algebra. The tensor product \( C_* \otimes C'_* \) of two chain complexes \( C_* \) and \( C'_* \) is the chain complex defined by \( (C \otimes C')_k = \bigoplus_{i+j=k} C_i \otimes C'_j \), and \( d(x \otimes x') = dx \otimes x' + (-1)^i x \otimes dx' \) for \( x \otimes x' \in C_i \otimes C'_j \). With this choice of signs, \( (C_* \otimes C'_*) \otimes C''_* = C_* \otimes (C'_* \otimes C''_*) \) and \( \mathbb{Z} \otimes C_* = C_* \otimes \mathbb{Z} = C_* \), where \( \mathbb{Z} \) is the chain complex given by \( \mathbb{Z}_0 = \mathbb{Z} \) and \( \mathbb{Z}_i = (0) \) for \( i \neq 0 \).

The dual \( C^* \) of a complex \( C_* \) is defined by \( C^i = \text{Hom}(C_i, \mathbb{Z}) \) and \( d^*x = (-1)^i d^1x \) for \( x \in C^i \) where \( d^1 \) is the adjoint of \( d \). With this choice of signs, the contraction \( C_* \otimes C^* \to \mathbb{Z} \) is a chain map.
If \( \rho \) is a permutation on \( k \) letters, then there is a chain map \( C^*_\ast \otimes \cdots \otimes C^*_{\rho(k)} \) defined by \( \alpha_1 \otimes \cdots \otimes \alpha_k \mapsto \pm \alpha_{\rho(1)} \otimes \cdots \otimes \alpha_{\rho(k)} \) where the sign is the graded sign of the permutation \((\alpha_1, \ldots, \alpha_k) \mapsto (\alpha_{\rho(1)}, \ldots, \alpha_{\rho(k)})\). (Recall that the graded sign of a permutation is defined as the sign of the permutation obtained by removing all elements of even degree.) To show this, it suffices to check that the homomorphism \( C^*_\ast \otimes C'_\ast \to C'_\ast \otimes C^*_\ast \) given by \( \alpha \otimes \beta \mapsto (-1)^{ij} \beta \otimes \alpha \) for \( \alpha \in C_i \) and \( \beta \in C'_j \) is a chain map. In particular, the symmetric group \( S_k \) acts on \( C^*_\ast \otimes \cdots \otimes C^*_\ast \) by chain maps.

Fix a triple \( A = (J, H, \sigma) \) where \( J \) is an almost complex structure on \( M \) compatible with \( \omega \), \( H \) is a time-dependent Hamiltonian on \( M \) and \( \sigma \) is a coherent system of orientations. We assume that the pair \((J, H)\) is regular.

**Definition 4.2.1.** The Floer chain complex \( CF_\ast(M, A) \) and the Floer cochain complex \( CF^\ast(M, A) \) are \( \mathbb{Z}/2N_0\mathbb{Z} \)-graded and defined as follows. For \( p \in \mathbb{Z}/2N_0\mathbb{Z} \), \( CF_p(M, A) \) and \( CF^p(M, A) \) are both the free \( \mathbb{Z} \)-module generated by periodic orbits \( \alpha \in \mathcal{C}_H^0 \) with \( \mu_H(\alpha) = p \). The differential
\[
d : CF_i(M, A) \to CF_{i-1}(M, A)
\]
is defined by
\[
d\alpha = \sum_{\beta \in \mathcal{C}_H^0} (-1)^{\mu(\beta)} \#M^1(\alpha, \beta) \beta.
\]
The differential
\[
d^* : CF^i(M, A) \to CF^{i+1}(M, A)
\]
is defined by
\[
d^* \alpha = \sum_{\beta \in \mathcal{C}_H^0} \#M^1(\beta, \alpha) \beta.
\]
The pairing
\[
\langle \alpha, \beta \rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}
\]
for \( \alpha, \beta \in \mathcal{C}_H^0 \). The Floer homology groups \( HF_\ast(M, A) \) are the homology groups of the chain complex \( CF_\ast(M, A) \). The Floer cohomology groups \( HF^\ast(M, A) \) are the cohomology groups of \( CF^\ast(M, A) \).

This definition is consistent for it follows from Prop. 3.3.3 that if \( M^1(\alpha, \beta) \) is non-empty, then \( \mu(\beta) = \mu(\alpha) - 1 \), and it follows from Prop. 3.6.6 that \( d^2 = 0 \). Note that \( CF^\ast(M, A) \) is the dual of \( CF_\ast(M, A) \).

In the following, we will view \( CF^\ast(M, A) \) as a chain complex, where \( CF^q(M, A) \) has degree \(-q\), rather than as a cochain complex.
Definition 4.2.4. The homomorphism
\[
\diamond_{ij} : CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+} \to CF^*(M, A)^{\otimes (k^- + k^+)}
\]
is defined by
\[
\alpha_{1,1} \otimes \ldots \otimes \alpha_{1,k^-} \otimes \alpha_{2,1}^+ \otimes \ldots \otimes \alpha_{2,k^+} \diamond_{ij} \alpha_{2,1}^+ \otimes \ldots \otimes \alpha_{2,k^+} + \alpha_{2,2}^+ \otimes \ldots \otimes \alpha_{2,k^+} \\
= \pm (\alpha_{1,1}^+, \alpha_{2,j}) \alpha_{2,1} \otimes \ldots \otimes \alpha_{2,j-1} \otimes \alpha_{1,1}^+ \otimes \ldots \otimes \alpha_{1,k^-} \otimes \alpha_{2,j+1} \otimes \ldots \otimes \alpha_{2,k^-} + \alpha_{2,k^+} \\
\otimes \alpha_{1,1}^+ \otimes \ldots \otimes \alpha_{1,j-1}^+ \otimes \alpha_{2,1}^+ \otimes \ldots \otimes \alpha_{2,k^+} \otimes \alpha_{1,j+1}^+ \otimes \ldots \otimes \alpha_{1,k^+}
\]
where the sign is the graded sign of the permutation obtained by comparing the ordering of the \(\alpha\)'s on the left and the right hand sides.

The homomorphism
\[
\mathcal{C}_{ij} : CF^*(M, A)^{\otimes (k^- + 1)} \otimes CF_*(M, A)^{\otimes (k^+ + 1)} \to CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+}
\]
is defined by
\[
\mathcal{C}_{ij} (\alpha_{1,-} \otimes \ldots \otimes \alpha_{k,-} \otimes \alpha_{1,+} \otimes \ldots \otimes \alpha_{k,+}) \\
= \pm (\alpha_{1,1}^+, \alpha_{2,j}) \alpha_{1,-} \otimes \ldots \otimes \alpha_{j-1,-} \otimes \alpha_{j,+} \otimes \ldots \otimes \alpha_{k,-} \\
\otimes \alpha_{1,1}^+ \otimes \ldots \otimes \alpha_{j-1,+} \otimes \alpha_{j,+} \otimes \ldots \otimes \alpha_{k,+}
\]
where the sign again is the graded sign of the permutation obtained by comparing the ordering of the \(\alpha\)'s on the left and the right hand sides.

Lemma 4.2.5. The homomorphisms \(\diamond_{ij}\) and \(\mathcal{C}_{ij}\) are chain maps.

Proof. The homomorphisms \(\diamond_{ij}\) and \(\mathcal{C}_{ij}\) are compositions of the permutation maps and contraction maps discussed at the beginning of this section. \(\square\)

4.3. The chain map \(Q\). Let \(C^\pi_s(\mathbb{J}_{g,k-,k+}(A))\) be the sub-complex of \(C_s(\mathbb{J}_{g,k-,k+}(A))\) generated by semi-regular cubical simplices \(\sigma\) as in Def. 2.3.4. If the boundary faces of a simplex are semi-regular, then the simplex can be perturbed, keeping the boundary fixed, to be semi-regular. It follows that the inclusion
\[
C^\pi_s(\mathbb{J}_{g,k-,k+}(A)) \to C_s(\mathbb{J}_{g,k-,k+}(A))
\]
is a chain homotopy equivalence. In the following we will work with the chain complex \(C^\pi_s(\mathbb{J}_{g,k-,k+}(A))\) rather than \(C_s(\mathbb{J}_{g,k-,k+}(A))\). Recall that for semi-regular simplices \(\sigma\), the numbers \#\(\mathcal{M}_\sigma((\alpha_{1,-}, \ldots, \alpha_{k,-}), (\alpha_{1,+}, \ldots, \alpha_{k,+}))\) are well defined.
Definition 4.3.1. The homomorphism

\[ Q : C^*_*(3_{g,k-1}^*(A)) \rightarrow CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+} \]

is defined by

\[ Q\sigma = (-1)^{q(q-1)} \sum_{\alpha_i^+ \in e^0_H} \#M^0_\sigma((\alpha_1^+, \ldots, \alpha_{k^-}, \alpha_{k^-}, \alpha_{k^+}, \alpha_{k^+})) \]

If \( \sigma \) is degenerate and semi-regular, then the moduli spaces \( M^0_\sigma \) are empty. Thus \( Q \) is well defined.

Theorem 4.3.2. If \( g = 0 \), then \( Q \) is a chain map of degree \( 2n(1 - k^-) \). In general, \( Q \) is a chain map of mixed degree \( 2n(1 - g - k^-) + 2\nu N_1, \nu \in \mathbb{Z} \). In particular, there is an induced homomorphism,

\[ Q : H_*(3_{g,k-1}^*(A)) \rightarrow H(CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+}). \]

Proof. It follows from Prop. 3.6.5 that if \( \sigma : [0,1]^q \rightarrow 3_{g,k-1}^*(A) \) is smooth and semi-regular, then

\[ dQ\sigma = \sum_{j=1}^{k^-} \sum_{\alpha_i^+, \alpha_j \in C^0_H \text{ omit } \alpha_j^+} (-1)^{q(q-1)+\mu(\alpha_j^-)+\cdots+\mu(\alpha_j^-)} \]

\[ \#M^0_\sigma((\alpha_1^+, \ldots, \alpha_{k^-}, \alpha_0, \alpha_{j-1}, \alpha_{j-1}, \alpha_{k^-}, \alpha_{k^-}, \alpha_{k^+}, \alpha_{k^+})) \]

\[ \cdot \alpha_1^+ \otimes \cdots \otimes \alpha_{k^-}^+ \otimes d^\sigma \alpha_0 \otimes \alpha_{j+1}^+ \otimes \cdots \otimes \alpha_{k^-}^+ \otimes \alpha_{k^+}^+ \]

\[ + \sum_{i=1}^{k^+} \sum_{\alpha_i^+ \in C^0_H \text{ omit } \alpha_i^+} (-1)^{q(q-1)+\mu(\alpha_i^-)+\cdots+\mu(\alpha_i^-)+\mu(\alpha_i^-)+\cdots+\mu(\alpha_i^-)} \]

\[ \#M^0_\sigma((\alpha_1^+, \ldots, \alpha_{k^-}, \alpha_0, \alpha_{i-1}, \alpha_{i-1}, \alpha_{k^-}, \alpha_{k^-}, \alpha_{k^+}, \alpha_{k^+})) \]

\[ \cdot \alpha_1^+ \otimes \cdots \otimes \alpha_{k^-}^+ \otimes \alpha_i^+ \otimes \cdots \otimes \alpha_{i-1}^+ \otimes \alpha_{i+1}^+ \cdots \otimes \alpha_{k^+}^+ \]

\[ - \sum_{i=1}^{k^+} \sum_{\alpha_i^+, \alpha_j \in C^0_H \text{ omit } \alpha_j^+} (-1)^{q(q-1)+\mu(\alpha_j^-)+\cdots+\mu(\alpha_j^-)+\mu(\alpha_j^-)+\cdots+\mu(\alpha_j^-)+\mu(\alpha_j^-)} \]

\[ \#M^0_\sigma((\alpha_1^+, \ldots, \alpha_{k^-}, \alpha_0, \alpha_{i-1}, \alpha_{i-1}, \alpha_{k^-}, \alpha_{k^-}, \alpha_{k^+}, \alpha_{k^+})) \]

\[ \cdot \alpha_1^+ \otimes \cdots \otimes \alpha_{k^-}^+ \otimes \alpha_i^+ \otimes \cdots \otimes \alpha_{i+1}^+ \]
\[(S_k^- \times S_k^+) \times H_*(\mathcal{J}_{g,k^-,-k^+}(A)) \xrightarrow{id \times Q} (S_k^- \times S_k^+) \times H(CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+}) \]

\[H_*(\mathcal{J}_{g,k^-,-k^+}(A)) \xrightarrow{Q} H(CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+}) \]

commutes.

**Proof.** This is an immediate consequence of Proposition 3.6.2. \(\square\)

Next we state the gluing relations.

**Theorem 4.3.4.** If the simplices \(\sigma_1 : [0, 1]^{q_1} \to \mathcal{J}_{g_1,k_1^-,-k_1^+}(A)\) and \(\sigma_2 : [0, 1]^{q_2} \to \mathcal{J}_{g_2,k_2^-,-k_2^+}(A)\) are semi-regular, \(\sigma_1 \diamond_{ij}^\ell \sigma_2\) takes values in \(\mathcal{J}_{g_1+g_2,k_1^-+k_2^-,-k_1^++k_2^+}(A)\) for all \(\ell \geq 0\), and either all periodic orbits of \(H\) are contractible, \(g_1 = 0\), or \(g_2 = 0\), then \(Q(\sigma_1 \diamond_{ij}^\ell \sigma_2) = (Q\sigma_1) \diamond_{ij} (Q\sigma_2)\) for all sufficiently large \(\ell\). In particular, the diagram

\[
\begin{array}{ccc}
H_*(\mathcal{J}_{g_1,k_1^-,-k_1^+}(A)) \otimes H_*(\mathcal{J}_{g_2,k_2^-,-k_2^+}(A)) & \xrightarrow{\otimes Q} & H(CF^*(M, A)^{\otimes k_1^-} \otimes CF_*(M, A)^{\otimes (k_1^+ + 1)}) \\
\downarrow \diamond_{ij} & & \downarrow \diamond_{ij} \\
H_*(\mathcal{J}_{g_1+g_2,k_1^-+k_2^-,-k_1^++k_2^+}(A)) & \xrightarrow{Q} & H(CF^*(M, A)^{\otimes (k_1^- + k_2^-)} \otimes CF_*(M, A)^{\otimes (k_1^+ + k_2^+)})
\end{array}
\]

commutes.
Proof. By Prop. 3.6.3,

\[(Q\sigma_1) \diamond_{ij} (Q\sigma_2)\]

\[= (-1)^{\frac{1}{2}q_1(n-1)} \sum_{\alpha_{i,j}^+ \in e_H^0} #\mathcal{M}_{\sigma_1}^0((\alpha_{i,j}^+, \alpha_{1,k_1}^+, \ldots, \alpha_{i,k_1}^+), (\alpha_{i,j}^+, \alpha_{1,k_1}^+, \ldots, \alpha_{i,k_1}^+))\]

\[\diamond_{ij} (-1)^{\frac{1}{2}q_2(q-1)} \sum_{\alpha_{i,j}^- \in e_H^0} #\mathcal{M}_{\sigma_2}^0((\alpha_{i,j}^-, \alpha_{2,k_2}^+, \ldots, \alpha_{i,k_2}^+), (\alpha_{i,j}^-, \alpha_{2,k_2}^+, \ldots, \alpha_{i,k_2}^+))\]

\[= \sum_{\alpha_{i,j}^+, \alpha_{i,j}^- \in e_H^0} \pm (-1)^{\frac{1}{2}q_1(n-1) + \frac{1}{2}q_2(q-1)} (\alpha_{i,j}^+, \alpha_{2,j}^-)\]

\[\#\mathcal{M}_{\sigma_1}^0((\alpha_{1,k_1}^+, \ldots, \alpha_{1,k_1}^+), (\alpha_{1,k_1}^+, \ldots, \alpha_{1,k_1}^+))\]

\[\#\mathcal{M}_{\sigma_2}^0((\alpha_{2,k_2}^+, \ldots, \alpha_{2,k_2}^+), (\alpha_{2,k_2}^+, \ldots, \alpha_{2,k_2}^+))\]

\[\alpha_{2,1}^+ \otimes \ldots \otimes \alpha_{2,j-1}^+ \otimes \alpha_{1,k_1}^+ \otimes \alpha_{2,j+1}^+ \otimes \ldots \otimes \alpha_{2,k_2}^+ \otimes \alpha_{i,j+1}^+ \otimes \ldots \otimes \alpha_{i,k_1}^+\]

\[\otimes \alpha_{2,1}^+ \otimes \ldots \otimes \alpha_{2,j-1}^+ \otimes \alpha_{1,k_1}^+ \otimes \alpha_{2,j+1}^+ \otimes \ldots \otimes \alpha_{2,k_2}^+ \otimes \alpha_{i,j+1}^+ \otimes \ldots \otimes \alpha_{i,k_1}^+\]

\[= Q(\sigma_1 \diamond_{ij} \sigma_2)\]

for sufficiently large \(\ell\), where the sign is the graded sign of the permutation (3.10). The Proposition follows. \(\square\)

**Theorem 4.3.5.** If \(\sigma : [0, 1]^q \to J_{g,k-1,k+1}(A)\) is semi-regular, \(C_{ij}^\ell\) takes values in \(J_{g+1,k-1,k+1}(A)\) for all \(\ell \geq 0\), and all periodic orbits of \(H\) are contractible, then \(Q\mathcal{C}_{ij}^\ell \sigma = C_{ij} Q\sigma\) for all sufficiently large \(\ell\). In particular, the diagram

\[
\begin{array}{ccc}
H_* (J_{g,k-1,k+1}(A)) & \overset{Q}{\longrightarrow} & H(CF^* (M, A)^{\otimes (k+1)} \otimes CF_*(M, A)^{\otimes (k+1)}) \\
\downarrow C_{ij} & & \downarrow C_{ij} \\
H_* (J_{g+1,k-1,k+1}(A)) & \overset{Q}{\longrightarrow} & H(CF^* (M, A)^{\otimes k} \otimes CF_*(M, A)^{\otimes k})
\end{array}
\]

commutes.
Proof. By Prop. 3.6.4, 

$$C_{ij}Q\sigma = (-1)^{\frac{1}{2}q(g-1)} \sum_{\alpha^\pm \in C_{ij}} \#M^0_{\sigma}(\alpha^-_1, \ldots, \alpha^-_{k-1}, \alpha^+_1, \ldots, \alpha^+_{k+1})$$

$$= (-1)^{\frac{1}{2}q(g-1)} \sum_{\alpha^\pm \in C_{ij}} \pm \langle \alpha^+_i, \alpha^-_j \rangle \#M^0_{\sigma}(\alpha^-_1, \ldots, \alpha^-_{k-1}, \alpha^+_1, \ldots, \alpha^+_{k+1})$$

$$= (-1)^{\frac{1}{2}q(g-1)} \sum_{\alpha^\pm \in C_{ij}} \#M^0_{\sigma}(\alpha^-_1, \ldots, \alpha^-_{k-1}, \alpha^+_1, \ldots, \alpha^+_{k+1})$$

$$= Q^0C_{ij}$$

for sufficiently large $\ell$, where the sign is the graded sign of the permutation (3.11). The Proposition follows. 

Every compact symplectic manifold admits a regular Hamiltonian for which all periodic orbits are contractible, and for which Thm. 4.3.4 and Thm. 4.3.5 therefore apply. In §4.5 we prove that the map $Q$ is essentially independent of $A$, and thus that the diagrams in Thm. 4.3.4 and Thm. 4.3.5 will commute for any $A$. 

The rays $X^\pm_i$ are needed to define the gluing maps. However, if $H$ is time-independent, then they play no role in the construction of the moduli spaces $M^0_{\sigma}$. Thus there is a chain map $Q^0 : C_*(\mathcal{J}_{g,k-}^{0})(A) \to CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+}$ defined the same way as $Q$. 

There is a natural projection map $\mathcal{J}_{g,k-}^{0}(A) \to \mathcal{J}_{g,k-}^{0}(A)$. The induced map $\mathcal{J}_{g,k-}^{0}(A) \to \mathcal{J}_{g,k-}^{0}(A)$ is in general only defined on a large subset of $\mathcal{J}_{g,k-}^{0}(A)$. A simple perturbation argument as in §4.1 still gives a homomorphism $H_*(\mathcal{J}_{g,k-}^{0}(A)) \to H_*(\mathcal{J}_{g,k-}^{0}(A))$. This proves the following Theorem.

**Theorem 4.3.6.** If the Hamiltonian $H$ is time-independent, then $Q$ factors through a homomorphism 

$$Q^0 : H_*(\mathcal{J}_{g,k-}^{0}(A)) \to H(CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+}).$$
4.4. Products and relations. In this section we show how the homomorphisms $Q$ give products and other operations in Floer (co)homology, and how Thm. 4.3.4 and 4.3.5 give relations for these products. The definitions and proofs are illustrated in fig. 2 and 3.

Let $\Theta_{g,k^-,-k^+}$ denote the canonical generator of $H_0(\mathcal{J}_g^{*}, k^-,-k^+)$.

**Lemma 4.4.1.** The elements $Q(\Theta_{g,k^-,-k^+})$ have degree $2n(1-g-k^-)$ modulo $2N_0$.

**Proof.** For $g = 0$, this follows directly from Prop. 4.3.2. For $g \geq 1$ Prop. 4.3.2 only gives the degree modulo $2N_1$. If all periodic orbits of $H$ are contractible, then it follows from Thm. 4.3.5 that

$$Q(\Theta_{g,k^-,-k^+}) = Q(\mathcal{C}^g_{11} \Theta_{0,k^-+g,k^+-+g}) = \mathcal{C}^g_{11} Q(\Theta_{0,k^-+g,k^+-+g}).$$

By Prop. 4.3.2 the element $Q(\Theta_{0,k^-+g,k^+-+g})$ has degree $2n(1-g-k^-)$ modulo $2N_0$. The chain map $\mathcal{C}^g_{11}$ has degree 0 modulo $2N_0$. Hence $Q(\Theta_{g,k^-,-k^+})$ has degree $2n(1-g-k^-)$ modulo $2N_0$. By the results of §4.5, this holds even if $H$ has non-contractible periodic orbits. □

To simplify the notation, for $x_1 \in H(CF^*_*(M,A)^{\otimes k^-} \otimes CF^*_*(M,A)^{\otimes k^+})$ and $x_2 \in H(CF^*_*(M,A)^{\otimes k^-_1} \otimes CF^*_*(M,A)^{\otimes k^+_1})$ we write $x_1 \Diamond x_2$ for $x_1 \Diamond_{k^+_1} x_2$. In this case the sign in the definition of $\Diamond$ is positive.

**4.4.1. The identity map.** The class $\Theta_{0,1,1}(A,A)$ can be represented by the simplex $\sigma_0$ as in Remark 2.3.7. By Prop. 3.6.1, $Q(\sigma_0) = \sum_{\alpha \in c^0} \alpha \otimes \alpha$. In other words, the class $Q(\Theta_{0,1,1})$ can be represented by the Floer cycle

$$\sum_{\alpha \in c^0} \alpha \otimes \alpha.$$

It follows that for $a \in HF^*(M,A)$,

$$a = Q(\Theta_{0,1,1}) \Diamond a.$$

Similarly, for $x \in HF^*(M,A)$,

$$x = x \Diamond Q(\Theta_{0,1,1}).$$

Thus the class $Q(\Theta_{0,1,1})$ induces the identity maps $HF^*(M,A) \to HF^*(M,A)$ and $HF_*(M,A) \to HF_*(M,A)$. Other homomorphisms are given by the classes $Q(\Theta_{g,1,1})$. By Lemma 4.4.1, these have degree $2ng$ and $-2ng$ respectively.
4.4.2. The symplectic cup product. We define the symplectic cup product of $a, b \in HF^*(M, A)$ as

$$a \cup b = (Q(\Theta_{0,1,2}) \diamond a) \diamond b.$$ 

By Lemma 4.4.1 the cup product has degree 0.

It follows from Thm. 4.3.4 that the symplectic cup product is associative,

$$((a \cup b) \cup c = (Q(\Theta_{0,1,2}) \diamond ((Q(\Theta_{0,1,2}) \diamond a) \diamond b)) \diamond c$$

$$= ((Q(\Theta_{0,1,2}) \diamond a) \diamond b) \diamond c$$

$$= (Q(\Theta_{0,1,2}) \diamond (Q(\Theta_{0,1,2}) \diamond a) \diamond b) \diamond c$$

$$= (Q(\Theta_{0,1,2}) \diamond (Q(\Theta_{0,1,2}) \diamond b) \diamond c)$$

$$= a \cup (b \cup c).$$

Even though we have permuted $a$ and $Q(\Theta_{0,1,2})$, the signs are positive, for $Q(\Theta_{0,1,2})$ has degree 0.

It follows from Thm. 4.3.4 and 4.3.3 that the cup product is graded commutative. In fact, let $\rho_0$ denote the permutation of order two in the symmetric group $S_2$ on two letters. Then

$$a \cup b = (Q(\Theta_{0,1,2}) \diamond a) \diamond b$$

$$= (-1)^{ij}((1,\rho_0).Q(\Theta_{0,1,2}) \diamond b) \diamond a$$

$$= (-1)^{ij}(Q((1,\rho_0).\Theta_{0,1,2}) \diamond b) \diamond a$$

$$= (-1)^{ij}(Q(\Theta_{0,1,2}) \diamond b) \diamond a$$

$$= (-1)^{ij}b \cup a.$$

for $a \in HF^i(M, A)$ and $b \in HF^j(M, A)$. Finally, define the unit class 1 as

$$1 = Q(\Theta_{0,1,0}) \in HF^0(M).$$

It follows from Prop 4.3.4 and §4.4.1 that 1 is an identity element for the symplectic cup product,

$$1 \cup a = (Q(\Theta_{0,1,2}) \diamond Q(\Theta_{0,1,0})) \diamond a = Q(\Theta_{0,1,1}) \diamond a = a.$$ 

The classes $Q(\Theta_{g,1,2})$ give higher genus cup products. However, by Prop. 3.4.3, $Q(\Theta_{g,1,2}) = Q(\Theta_{g,1,1}) \diamond Q(\Theta_{0,1,2})$. Hence the higher genus cup products are simply the composition of the cup product and the homomorphisms $HF^*(M, A) \to HF^*(M, A)$ given by $Q(\Theta_{g,1,1})$. 


4.4.3. The symplectic intersection product. We define the symplectic intersection product of \(x, y \in HF_\ast(M, A)\) as

\[
x \cdot y = x \diamond (y \diamond Q(\Theta_{0,2,1})).
\]

The same argument as in §4.4.2 shows that the intersection product is graded commutative, associative, and has degree \(-2n\). The symplectic top class \([M]_\omega = Q(\Theta_{0,0,1}) \in HF_{2n}(M, A)\) is an indentity element for the symplectic intersection product.

4.4.4. The symplectic cap product. We define the symplectic cap product of \(x \in HF_\ast(M)\) and \(a \in HF^\ast(M)\) as

\[
x \cap a = x \star Q(\Theta_{0,1,2}) \star a.
\]

Then

\[
(x \cap a) \cap b = (x \star (Q(\Theta_{0,1,2}) \cap a) \star Q(\Theta_{0,1,2})) \diamond b
\]

\[
= x \diamond ((Q(\Theta_{0,1,2}) \cap_{11} Q(\Theta_{0,1,2})) \cap a) \diamond b
\]

\[
= x \diamond (Q(\Theta_{0,1,3}) \cap a) \diamond b
\]

\[
= x \diamond ((Q(\Theta_{0,1,2}) \cap_{21} Q(\Theta_{0,1,2})) \cap a) \diamond b
\]

\[
= x \diamond Q(\Theta_{0,1,2}) \diamond ((Q(\Theta_{0,1,2}) \cap a) \diamond b)
\]

\[
= x \cap (a \cup b).
\]

4.4.5. Symplectic Poincaré duality. We define the symplectic Poincaré duals of \(x \in HF_\ast(M, A)\) and \(a \in HF^\ast(M, A)\) as

\[
x^\sharp = x \diamond Q(\Theta_{0,2,0})
\]

\[
a^\flat = Q(\Theta_{0,0,2}) \cap a
\]

By Thm. 4.3.4 and §4.4.1, these maps are inverses,

\[
(x^\sharp)^\flat = Q(\Theta_{0,0,2}) \cap (x \diamond Q(\Theta_{0,2,0}))
\]

\[
= x \diamond (Q(\Theta_{0,0,2}) \cap_{22} Q(\Theta_{0,2,0}))
\]

\[
= x \diamond Q(\Theta_{0,1,1}) = x,
\]

and similarly

\[
(a^\flat)^\sharp = a.
\]

It also follows from Thm. 4.3.4 that

\[
(a \cup b)^\flat = a^\flat \cdot b^\flat
\]

\[
x \cap a = x \cdot a^\flat
\]

\[
1^\flat = [M]_\omega.
\]
4.4.6. The Euler characteristic and the number of tori. The Floer cohomology groups are the ordinary cohomology groups with the grading reduced modulo $2N_0$; see [F3] and [F4] Thm. 5. In particular, the Euler characteristic of the Floer cohomology is the same as the ordinary Euler characteristic $\chi(M)$. Choose $A$ such that all periodic orbits for $H$ are contractible. It then follows from Thm. 4.3.5 and §4.4.1 that the number of perturbed pseudo-holomorphic tori with a given conformal structure is

$$Q(\Theta_{1,0,0}) = \mathcal{C}_1 Q(\Theta_{0,1,1}) = \mathcal{C}_1 \sum_{\alpha \in C_0^\ast} \alpha \otimes \alpha = \sum_{\alpha \in C_0^\ast} (-1)^{\mu(\alpha)} = \chi(M).$$

4.4.7. Twists. If the Hamiltonian $H$ is time-independent, then we get a loop in $\tilde{\Theta}_{g,k,-k+}(A)$ by a full counterclockwise twist of the ray $X_i^\pm$. The image of this loop in $\mathcal{H}_{g,k,-k+}(A)$ defines an element $\Xi_i^\pm \in H_1(\mathcal{H}_{g,k,-k+}(A))$. This class exists even if $H$ is time-dependent. If $H$ is time-independent, then the image of $\Xi_i^\pm$ in $H_1(\mathcal{H}_{g,k,-k+}(A))$ is zero. It follows from Thm. 4.3.6 that if $H$ is time-independent, then $Q(\Xi_i^\pm) = 0$. The class $\mathcal{C}_i \Xi_i^\pm = -\mathcal{C}_i \Xi_i^\pm$ corresponds to a Dehn twist of $\Xi_{g+1,k-1,k+1}$. It follows from Thm. 4.3.5 that $Q(\mathcal{C}_i \Xi_i^\pm) = 0$ as well. By the results of §4.5, these identities also hold for time-dependent Hamiltonians.

4.4.8. The symplectic Massey product. The associativity relation $a \cup (b \cup c) = (a \cup b) \cup c$ gives rise to a secondary operation, the Massey product. The Massey product $M(a,b,c)$ is an element of $HF^\ast(M,A)/(a \cup HF^\ast(M,A) + HF^\ast(M,A) \cup c)$, which is defined for $a, b, c \in HF^\ast(M,A)$ with $a \cup b = b \cup c = 0$.

Let $\theta_{0,1,2} \in C_0(\mathcal{H}_{0,1,2}(A))$ be a cycle that represents the class $\Theta_{0,1,2}$. Now $\Theta_{0,1,2} \otimes_{11} \Theta_{0,1,2} = \Theta_{0,1,2} \otimes_{21} \Theta_{0,1,2}$. Hence $\theta_{0,1,2} \otimes_{11} \theta_{0,1,2} - \theta_{0,1,2} \otimes_{21} \theta_{0,1,2} = d\lambda$ for some $\lambda \in C_1(\mathcal{H}_{0,1,3}(A))$. Let $\alpha, \beta, \gamma$ be Floer cocycles that represent $a, b$ and $c$. Then there exist cochains $\zeta$ and $\xi$ such that $\alpha \cup \beta = d\zeta$ and $\beta \cup \gamma = d\xi$. Then the Massey product $M(a,b,c)$ is represented by the cochain

$$((Q(\lambda) \otimes \alpha) \otimes \beta) \otimes \gamma + (Q(\theta_{0,1,2}) \otimes \zeta) \otimes \gamma - (-1)^{\mu(\alpha)}(Q(\theta_{0,1,2}) \otimes \alpha) \otimes \xi.$$

A similar construction is discussed in [Fu].

4.4.9 Other operations. Using analogous constructions, the higher dimensional homology classes in $H_a(\mathcal{H}_{g,k,-k+}(A))$ also give products. Thms. 4.3.4 and 4.3.5 give relations between these products, and corresponding secondary operations can be defined.
4.5. Independence of the choice of Hamiltonian, almost complex structure,
and coherent system of orientations. In this section we put Floer’s proof, [F4]
Thm. 4, that the Floer (co)homology groups $HF_*(M, A)$ and $HF^*(M, A)$ are inde-
pendent of the choice of $A = (H, J, \sigma)$ into our framework. We then show that the chain
map $Q$ is also independent of the choice of $A$.

That $HF_*(M, A)$ is independent of $A$ means that there exists a canonical system
of homomorphisms $L_{AA'} : HF_*(M, A) \rightarrow HF_*(M, A')$ which has the following two
properties:

Reflexivity: $L_{AA}$ is the identity

Transitivity: $L_{AA'} \circ L_{AA''} = L_{AA''}$

To define the chain maps $L_{AA'}$ we need a generalization of the homomorphism
$Q : H_*(\mathcal{J}_{g, k^-, k^+}(A)) \rightarrow H(CF^*(M, A)^{\otimes k^-} \otimes CF_*(M, A)^{\otimes k^+})$ of §4.3. Let

$$\mathcal{J}_{g, k^-, k^+}((A_1^-, \ldots, A_k^-), (A_1^+, \ldots, A_k^+)),$$

with $A_i^\pm = (H_i^\pm, J_i^\pm, \sigma_i^\pm)$, be a space defined the same way as $\mathcal{J}_{g, k^-, k^+}(A)$ in §2.2, but
we now let $J$ be a family of conformal structures on $M$ parametrized by $\Sigma_{g, k^-, k^+}$. We
require that $J = J_i^\pm$ and $R = (dt_i^\pm - i\theta_i^\pm) \otimes \nabla H_i^\pm$ on $\Delta_i^\pm$. There then exists a gluing
map $\diamond_{ij}$ as in §4.1, which is defined if $A_{1,i}^+ = A_{2,j}^-$. Similarly there exists a gluing map
$\mathcal{C}_{ij}$ which is defined if $A_{2,j}^- = A_{1,i}^+$.

If $A_{1,i}^+ = A_{2,j}^-$, there is a chain map

$$\diamond_{ij} : \left( \bigotimes_{\nu=1}^{k_-} CF^*(M, A_{1,\nu}^-) \otimes \bigotimes_{\nu=1}^{k_+} CF_*(M, A_{1,\nu}^+) \right)$$

$$\otimes \left( \bigotimes_{\nu=1}^{k_-} CF^*(M, A_{2,\nu}^-) \otimes \bigotimes_{\nu=1}^{k_+} CF_*(M, A_{2,\nu}^+) \right)$$

$$\rightarrow \bigotimes_{\nu=1}^{j-1} CF^*(M, A_{2,\nu}) \otimes \bigotimes_{\nu=1}^{i-1} CF_*(M, A_{1,\nu}) \otimes \bigotimes_{\nu=j+1}^{k_-} CF^*(M, A_{2,\nu})$$

$$\otimes \bigotimes_{\nu=1}^{k_-} CF_*(M, A_{1,\nu}) \otimes \bigotimes_{\nu=1}^{k_+} CF_*(M, A_{2,\nu}) \otimes \bigotimes_{\nu=i+1}^{k_+} CF_*(M, A_{1,\nu})$$

defined the same way as the chain map $\diamond_{ij}$ of Def. 4.2.4. Similarly, if $A_{2,j}^- = A_{1,i}^+$, there
is a chain map

$$\mathcal{C}_{ij} : \bigotimes_{\nu=1}^{k_-} CF_*(M, A_{\nu}^-) \otimes \bigotimes_{\nu=1}^{k_+} CF_*(M, A_{\nu}^+) \rightarrow \bigotimes_{\nu=1}^{k_-} CF^*(M, A_{\nu}^-) \otimes \bigotimes_{\nu=1}^{k_+} CF_*(M, A_{\nu}^+).$$
It is straightforward to generalize the construction of $Q$ to get a chain map

$$Q : C^*_\ast (\mathfrak{z}_{g,k^-}^{-k^+} (A_1^{-k}, \ldots, A_{k^-}^{-k}), (A_1^{k^+}, \ldots, A_{k^+}^{k^-}))$$

$$\rightarrow \bigotimes_{\nu = 1}^{k^-} CF^\ast (M, A_{\nu}^-) \otimes \bigotimes_{\nu = 1}^{k^+} CF^\ast (M, A_{\nu}^+) .$$

The main modification is that we replace $J(u(x, y))$ in (2.5) by $J((x, y), u(x, y))$. This chain map satisfies the appropriate analogues of the theorems of §4.3.

Let $\Theta_{0,1,1} (A^-, A^+)$ be the canonical generator of $H_0(\mathfrak{z}_{0,1,1}^\ast (A^-, A^+))$. We define the homomorphism

$$L_{AA'} : HF^\ast (M, A) \rightarrow HF^\ast (M, A')$$

as

$$x \mapsto x \diamond Q(\Theta_{0,1,1} (A, A')).$$

Reflexivity follows from §4.4.1. By Thm. 4.3.4,

$$L_{A''A'}L_{AA'}x = (x \diamond Q(\Theta_{0,1,1} (A, A'))) \diamond Q(\Theta_{0,1,1} (A', A''))$$

$$= x \diamond (Q(\Theta_{0,1,1} (A, A')) \diamond_{11} Q(\Theta_{0,1,1} (A', A'')))$$

$$= x \diamond Q(\Theta_{0,1,1} (A, A') \diamond_{11} \Theta_{0,1,1} (A', A''))$$

$$= x \diamond Q(\Theta_{0,1,1} (A, A'')) = L_{AA''}x$$

and we have established transitivity. Therefore we can identify the groups $HF^\ast (M, A)$ for different $A$ using the maps $L_{AA'}$. This gives the abstract Floer homology groups $HF^\ast (M)$.

Similarly we define a map $HF^\ast (M, A) \rightarrow HF^\ast (M, A')$ by

$$L^\ast_{A'A}x = Q(\Theta_{0,1,1} (A', A)) \diamond x .$$

More generally we define homomorphisms

$$H \left( \bigotimes_{\nu = 1}^{k^-} CF^\ast (M, A_{\nu}^-) \otimes \bigotimes_{\nu = 1}^{k^+} CF^\ast (M, A_{\nu}^+) \right) \rightarrow H \left( \bigotimes_{\nu = 1}^{k^-} CF^\ast (M, A_{\nu}^-') \otimes \bigotimes_{\nu = 1}^{k^+} CF^\ast (M, A_{\nu}^{'+}) \right)$$

by

$$x \mapsto Q(\Theta_{0,1,1} (A_{\nu}^-', A_{\nu}^-)) \diamond_{k^-} \cdots \diamond_{12} Q(\Theta_{0,1,1} (A_{1}^{-}, A_{1}^{-})) \diamond_{11} x$$

$$\diamond_{11} Q(\Theta_{0,1,1} (A_{1}^{+}, A_{1}^{+})) \diamond_{21} \cdots \diamond_{k+1} Q(\Theta_{0,1,1} (A_{k^+}^{+}, A_{k^+}^{+})).$$

These homomorphisms are also reflexive and transitive.
The spaces $\mathcal{J}_{g,k-}^*((A_1^-, \ldots, A_k^-), (A_1^+, \ldots, A_k^+))$ with different $A_i^\pm$ are canonically homotopy equivalent. Thus we can simply write $H_*(\mathcal{J}_{g,k-}^*).$ The canonical homotopy equivalence

$$\mathcal{J}_{g,k-}^*((A_1^-, \ldots, A_k^-), (A_1^+, \ldots, A_k^+)) \rightarrow \mathcal{J}_{g,k-}^*((A_1'^-, \ldots, A_k'^-), (A_1'^+, \ldots, A_k'^+))$$

can be realized explicitly by the map

$$c \mapsto c_0(A_{k-}'^-, A_{k-}^-) \bowtie_{1k-} \cdots \bowtie_{12} c_0(A_{1-}'^-, A_{1}^-) \bowtie_{11} c$$

$$\bowtie_{11} c_0(A_{1+}'^+, A_{1}^+) \bowtie_{21} \cdots \bowtie_{k+1} c_0(A_{k-}'^+, A_{k}^+)$$

where $c_0(A, A')$ is any element of $\mathcal{J}_{0,1,1}(A, A')$ and $\ell \geq 0.$ It follows that the induced isomorphisms

$$H_*(\mathcal{J}_{g,k-}^*((A_1^-, \ldots, A_k^-), (A_1^+, \ldots, A_k^+)))$$

$$\rightarrow H_*(\mathcal{J}_{g,k-}^*((A_1'^-, \ldots, A_k'^-), (A_1'^+, \ldots, A_k'^+)))$$

are given by

$$x \mapsto \Theta_{0,1,1}(A_{k-}'^-, A_{k-}^-) \bowtie_{1k-} \cdots \bowtie_{12} \Theta_{0,1,1}(A_{1-}'^-, A_{1}^-) \bowtie_{11} x$$

$$\bowtie_{11} \Theta_{0,1,1}(A_{1+}'^+, A_{1}^+) \bowtie_{21} \cdots \bowtie_{k+1} \Theta_{0,1,1}(A_{k-}'^+, A_{k}^+)$$

It follows, by repeated use of Thm. 4.3.4 that the diagram

$$H_*(\mathcal{J}_{g,k-}^*((A_1^-, \ldots, A_k^-), (A_1^+, \ldots, A_k^+))) \xrightarrow{Q} \bigotimes_{j=1}^{k-} CF^*(M, A_j^-) \otimes \bigotimes_{i=1}^{k+} CF_*(M, A_i^+)$$

$$\downarrow$$

$$H_*(\mathcal{J}_{g,k-}^*((A_1'^-, \ldots, A_k'^-), (A_1'^+, \ldots, A_k'^+))) \xrightarrow{Q} \bigotimes_{j=1}^{k-} CF^*(M, A_j'^-) \otimes \bigotimes_{i=1}^{k+} CF_*(M, A_i'^+)$$

commutes. Thus we can view $Q$ as a homomorphism

$$Q : H_*(\mathcal{J}_{g,k-}^*) \rightarrow H(CF^*(M)^{\otimes k^-} \otimes CF_*(M)^{\otimes k^+}).$$
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(a \cup b) \cup c = a \cup (b \cup c)

1 \cup a = a

[M]_\omega \cdot x = x

(a^x)^x = a

(x^y)^b = x

(a \cup b)^b = a^b \cdot b^b

Figure 3