RULED CR–SUBMANIFOLDS OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

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ABSTRACT. The purpose of this paper is to study the canonical totally real foliations of CR–submanifolds in a locally conformal Kähler manifold.

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1. Introduction

The concept of CR–submanifold, first introduced in Kähler geometry by A. Bejancu [3], was later considered and studied in locally conformal Kähler ambient by many authors (see e.g. [1, 7, 8, 10, 11, 13, 14, 19, 20, 21, 23]). Such a submanifold comes naturally equipped with some canonical foliations, which were first investigated by B.Y. Chen and P. Piccinni [9] (see also Chapter 12 from the monograph [13]). One of these foliations, denoted by $\mathfrak{F}^\perp$ and called the totally real foliation, is given by the totally real distribution involved in the definition of the CR–submanifold, proven to be always completely integrable by D.E. Blair and B.Y. Chen [5]. On the other hand, A. Bejancu and H.R. Farran [4] Chapter 5 investigated the relationship between the geometry of the totally real foliation on a CR–submanifold of a Kähler manifold and the geometry of the CR–submanifold itself, stressing on the links between the foliation and the complex structure on the embedding manifold (see also the monograph [2] for an excellent survey concerning foliations in CR geometry). Moreover, they also used the theory of ruled submanifolds (see [22] for a detailed survey on the topic) to characterize some classes of CR–submanifolds in Kähler manifolds. At the end of the chapter, the authors have proposed, as an interesting and useful research, the extension of this study to CR–submanifolds of manifolds endowed with various geometric structures. This was done recently for quaternionic and paraquaternionic Kähler ambient [15, 16, 25].

In this paper, following the same techniques, we study the CR–submanifolds in a locally conformal Kähler manifold. In particular, we obtain necessary and sufficient conditions for a CR–submanifold of a locally conformal Kähler manifold to be ruled with respect to the totally real foliation $\mathfrak{F}^\perp$. In the last part of the paper characterizations are provided for this foliation to become Riemannian, i.e. with bundle–like metric.

2. Preliminaries

Let $(\overline{M}, J, \overline{g})$ be an almost Hermitian manifold of dimension $2n$, where $J$ denotes the almost complex structure and $\overline{g}$ the Hermitian metric. Then $(\overline{M}, J, \overline{g})$ is called a locally conformal Kähler (briefly l.c.K.) manifold if for each point $p$ of $\overline{M}$ there
exists an open neighbourhood $U$ of $p$ and a positive function $f_U$ on $U$ so that the local metric
\[ g_{U} = \exp(-f_U)\overline{g}_{U} \]
is Kählerian (see [17, 24]). If $U = \overline{M}$, then the manifold $(\overline{M}, J, \overline{g})$ is said to be a globally conformal Kähler (briefly g.c.K.) manifold. Equivalently (see [13]), $(\overline{M}, J, \overline{g})$ is l.c.K. if and only if there exists a closed 1–form $\omega$, globally defined on $\overline{M}$, such that
\[ d\Omega = \omega \wedge \Omega, \]
where $\Omega$ is the Kähler 2–form associated with $(J, g)$, i.e.
\[ \Omega(X, Y) = g(X, JY), \]
for $X, Y \in \Gamma(T\overline{M})$, is called Lee vector field. It is known that $(\overline{M}, J, \overline{g})$ is globally conformal Kähler (respectively Kähler) if the Lee–form $\omega$ is exact (respectively $\omega = 0$). It is also known that Levi–Civita connections $\nabla^U$ of the local metrics $\overline{g}_{U}$ glue up to a globally defined torsion free linear connection $\nabla$ on $\overline{M}$, called the Weyl connection of the l.c.K. manifold $\overline{M}$, given by
\[ D_X Y = \nabla_X Y - \frac{1}{2}(\omega(X)Y + \omega(Y)X - \overline{g}(X, Y)B) \]
for any $X, Y \in \Gamma(T\overline{M})$, where $\nabla$ is the Levi–Civita connection of $\overline{g}$. Moreover, Weyl connection $\nabla$ satisfies $\nabla\overline{g} = \omega \otimes \overline{g}$ and $DJ = 0$. As a consequence, considering the anti–Lee form $\theta = \omega \circ J$ and the anti–Lee vector field $A = -JB$, one can obtain a third equivalent definition in terms of the Levi–Civita connection $\nabla$ of the metric $\overline{g}$ (see [13]). Namely, $(\overline{M}, J, \overline{g})$ is l.c.K. if and only if the following equation is satisfied for any $X, Y \in \Gamma(T\overline{M})$:
\[ (\nabla_X J)Y = \frac{1}{2} [\theta(Y)X - \omega(Y)JX - \overline{g}(X, Y)A - \Omega(X, Y)B]. \tag{1} \]

A submanifold $M$ of a l.c.K. manifold $(\overline{M}, J, \overline{g})$ is called a CR–submanifold if there exists a differentiable distribution $D : p \to D_p \subset T_p M$ on $M$ satisfying the following conditions:

i. $D$ is holomorphic, i.e. $JD_p = D_p$, for each $p \in M$;
ii. the complementary orthogonal distribution $D^\perp : p \to D^\perp_p \subset T_p M$ is totally real, i.e. $JD^\perp_p \subset T^\perp_p M$ for each $p \in M$.

If $\dim D^\perp_p = 0$ (resp. $\dim D_p = 0$), then the CR–submanifold is said to be a holomorphic (resp. a totally real) submanifold. A CR–submanifold is called a proper CR–submanifold if it is neither holomorphic nor totally real.

**Remark 2.1.** Let $M$ be a CR–submanifold of a l.c.K. manifold $(\overline{M}, J, \overline{g})$. By the definition of a CR–submanifold we have the orthogonal decomposition
\[ TM = D \oplus D^\perp. \]

Also, the normal bundle has the orthogonal decomposition
\[ TM^\perp = JD^\perp \oplus \mu, \]
where $\mu$ is the subbundle of the normal bundle $TM^\perp$ which is the orthogonal complement of $J D^\perp$. Corresponding to the last decomposition, any normal vector field $N$ can be written as $N = N_{JD^\perp} + N_\mu$, where $N_{JD^\perp}$ (resp. $N_\mu$) is the $JD^\perp$– (resp. $\mu$–) component of $N$. It is easy to see that the subbundle $\mu$ is invariant under the action of $J$. We note that if $\mu = 0$, then the CR-submanifold is said to be an anti-holomorphic submanifold or a generic submanifold.

If we denote by $\nabla$ the Levi–Civita connection on $(M, g)$, where $g$ is the induced Riemannian metric by $g$ on $M$, then the Gauss and Weingarten formulas are given by:

$$\nabla_X Y = \nabla_X Y + h(X, Y), \ \forall X, Y \in \Gamma(TM)$$

and

$$\nabla_X N = -a_N X + \nabla_X^\perp N, \ \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp)$$

where $h$ is the second fundamental form of $M$, $\nabla^\perp$ is the connection on the normal bundle and $a_N$ is the shape operator of $M$ with respect to $N$. It is well-known that $h$ is a symmetric $F(M)$–bilinear form and $a_N$ is a self-adjoint operator, related by:

$$g(a_N X, Y) = g(h(X, Y), N)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. We say (see [4]) that the distribution $D$ (resp. $D^\perp$) is $a_N$–invariant, if $a_N X \in \Gamma(D)$ (resp. $a_N Z \in \Gamma(D^\perp)$) for any $X \in \Gamma(D)$ (resp. $Z \in \Gamma(D^\perp)$).

A CR–submanifold $M$ of a l.c.K. manifold $(\overline{M}, J, \overline{g})$ is called:

i. $D$-geodesic if $h(X, Y) = 0$, $\forall X, Y \in \Gamma(D)$.

ii. $D^\perp$-geodesic if $h(X, Y) = 0$, $\forall X, Y \in \Gamma(D^\perp)$.

iii. mixed geodesic if $h(X, Y) = 0$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$.

We recall now the following result which we shall need in the sequel.

**Theorem 2.2.** Let $M$ be a CR–submanifold of a l.c.K. manifold $(\overline{M}, J, \overline{g})$. Then:

i. The totally real distribution $D^\perp$ is integrable [5].

ii. The holomorphic distribution $D$ is integrable if and only if

$$\overline{g}(h(X, JY), JZ) = \overline{g}(h(JX, Y), JZ) - \overline{\Omega}(X, Y)\theta(Z)$$

for all $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$ [6].

3. Totally real foliation of a CR–submanifold in a locally conformal Kähler manifold

Let $M$ be a CR–submanifold of a l.c.K. manifold $(\overline{M}, J, \overline{g})$. From Theorem 2.2 we have that the distribution $D^\perp$ is always integrable and gives rise to a foliation of $M$ by totally-real submanifolds of $\overline{M}$. So any CR–submanifold of a l.c.K. manifold comes naturally equipped with a foliation denoted by $\mathcal{F}^\perp$ and called the totally real foliation. We note that if the holomorphic distribution $D$ is also integrable, then $M$ carries a foliation by holomorphic submanifolds of $\overline{M}$, called the Levi foliation (see [4, 12]).

We recall that if each leaf of a foliation $\mathcal{F}$ on $M$ is a totally geodesic submanifold of $M$, then we say that $\mathcal{F}$ is a totally geodesic foliation. Next we state some characterizations of totally geodesic totally real foliations on CR–submanifolds.
Proposition 3.1. The canonical totally real foliation $\mathfrak{F}^\perp$ on a CR–submanifold $M$ of a l.c.K. manifold $(\overline{M}, J, \overline{F})$ is a totally geodesic foliation if and only if
\[ \theta(Y)X = 2h_{JD^\perp}(X, Y), \forall X \in \Gamma(D^\perp), Y \in \Gamma(D). \] (5)

Proof. For $X, Z \in \Gamma(D^\perp)$ and $Y \in \Gamma(D)$, using (1)–(4), we derive:
\[
\overline{F}(J\nabla_X Z, Y) = -\overline{F}(\nabla_X Z, JY) \\
= -\overline{F}(\nabla_X Z - h(X, Z), JY) \\
= \overline{F}(-\nabla_X JZ + \nabla_Z JX, Y) \\
= -\frac{1}{2}\overline{F}(\theta(Z)X - \omega(Z)JX - g(X, Z)A - \Omega(X, Y)B - 2\nabla_X JZ, Y) \\
= -\frac{1}{2}\overline{F}(h(X, Z)B - 2\nabla_X JZ, Y) \\
= \frac{1}{2}g(X, Z)\overline{F}(B, JY) + \overline{F}(-a_{JZ}X + \nabla_Z JX, Y) \\
= \frac{1}{2}g(X, Z)\omega(JY) - \overline{F}(h(X, Y), JZ) \\
= \frac{1}{2}\overline{F}(\theta(Y)JX, JZ) - \overline{F}(h(X, Y), JZ).
\]

Therefore, we obtain
\[
\overline{F}(J\nabla_X Z, Y) = \frac{1}{2}\overline{F}(\theta(Y)JX - 2h_{JD^\perp}(X, Y), JZ), \forall X, Z \in \Gamma(D^\perp), Y \in \Gamma(D). \tag{6}
\]

If we suppose now $\mathfrak{F}^\perp$ is a totally geodesic foliation, then $\nabla_X Z \in \Gamma(D^\perp)$, for all $X, Z \in \Gamma(D^\perp)$, and from (6) we deduce:
\[
\overline{F}(\theta(Y)JX - 2h_{JD^\perp}(X, Y), JZ) = 0, \forall Z \in \Gamma(D^\perp)
\]
and the implication follows.

Conversely, if we suppose $\theta(Y)JX = 2h_{JD^\perp}(X, Y)$, for all $X \in \Gamma(D^\perp), Y \in \Gamma(D)$, then from (6) we derive:
\[
\overline{F}(J\nabla_X Z, Y) = 0
\]
and we conclude $\nabla_X Z \in \Gamma(D^\perp)$. Thus $\mathfrak{F}^\perp$ is a totally geodesic foliation. \hfill \Box

Remark 3.2. An alternative proof of the above Proposition can be obtained using [6, Lemma 1, p. 343].

Theorem 3.3. Let $M$ be a CR–submanifold of a l.c.K. manifold $(\overline{M}, J, \overline{F})$ such that the Lee vector field $B$ is normal to $M$. Then the next assertions are equivalent:

i. The canonical totally real foliation $\mathfrak{F}^\perp$ on $M$ is totally geodesic.
ii. $h(X, Y) \in \Gamma(\mu), \forall X \in \Gamma(D^\perp), Y \in \Gamma(D)$.
iii. The totally real distribution $D^\perp$ is $a_N$– invariant for any $N \in \Gamma(JD^\perp)$.
iv. The holomorphic distribution $D$ is $a_N$– invariant for any $N \in \Gamma(JD^\perp)$.

Proof. Since $B$ is normal to $M$, we deduce
\[ \theta(Y) = \omega(JY) = \overline{F}(JY, B) = 0 \]
for any $Y \in \Gamma(D)$. Therefore, from the above Proposition we obtain (i) $\leftrightarrow$ (ii).

The equivalence of (ii) and (iii) follows easily from (1), while the equivalence of (iii) and (iv) holds because $a_N$ is a self–adjoint operator. \hfill \Box
Remark 3.4. We note that Theorem 3.3 extends Theorem 4.1 in [4, p. 247] from the case of an ambient Kählerian manifold to the case of an ambient l.c.K. manifold.

Corollary 3.5. Let $M$ be a CR–submanifold of a l.c.K. manifold $(\overline{M}, J, \overline{g})$ such that the Lee vector field $B$ is normal to $M$. Then:

i. If $M$ is mixed geodesic, then the totally real foliation $\mathfrak{F}^\perp$ on $M$ is totally geodesic.

ii. If $M$ is an anti–holomorphic submanifold, then $M$ is mixed geodesic if and only if the totally real foliation $\mathfrak{F}^\perp$ is totally geodesic.

Proof. The proof is clear from Theorem 3.3.

Remark 3.6. We note that the Corollary 3.5(i.) has been also obtained using a different proof by Dragomir [10] (see also [13, Theorem 12.6, p. 168]). On the other hand, Corollary 3.5(ii.) gives us an interesting geometric characterization of mixed geodesic anti–holomorphic submanifolds in a l.c.K. manifold normal to the Lee vector field. Thus, $M$ is mixed geodesic if and only if any geodesic of a leaf of $D^\perp$ is a geodesic of $M$. On another hand, according to Corollary 3.5(i.), if $M$ is totally geodesic, then $M$ is mixed geodesic and any geodesic of a leaf of $\mathfrak{F}^\perp$ is a geodesic of $M$ which in turn is a geodesic of $\overline{M}$. Therefore any leaf of $\mathfrak{F}^\perp$ is totally geodesic immersed in $(\overline{M}, J, \overline{g})$. It is important to note that this property is also true in Kähler ambient (see [4, Corollary 4.4, p. 148]).

A submanifold $M$ of a Riemannian manifold $(\overline{M}, \overline{g})$ is said to be a ruled submanifold if it admits a foliation whose leaves are totally geodesic immersed in $(\overline{M}, \overline{g})$. A CR–submanifold which is a ruled submanifold with respect to the canonical foliation $\mathfrak{F}^\perp$ is called a totally real ruled CR–submanifold. We are able now to state the following characterization of totally real ruled CR–submanifolds in l.c.K. manifolds.

Theorem 3.7. Let $M$ be a CR–submanifold of a l.c.K. manifold $(\overline{M}, J, \overline{g})$. Then the next assertions are equivalent:

i. $M$ is a totally real ruled CR–submanifold.

ii. $M$ is $D^\perp$–geodesic and the anti–Lee form $\theta$ and the second fundamental form $h$ of the submanifold are related by (5).

iii. The second fundamental form $h$, the anti–Lee form $\theta$ and the anti–Lee vector field $A$ are related by (5) and satisfy:

\[ h(X, Z) \in \Gamma(\mu), \forall X, Z \in \Gamma(D^\perp) \]  

\[ (\nabla_X^\perp JZ)_\mu = -\frac{1}{2}g(X, Z)A_\mu, \forall X, Z \in \Gamma(D^\perp), \]  

where the index $\mu$ denotes the $\mu$–component of the vector field.

Proof. i. $\iff$ ii. For any $X, Z \in \Gamma(D^\perp)$ we have:

\[ \nabla_X Z = \nabla_X Z + h(X, Z) \]

\[ = \nabla_X^{D^\perp} Z + h^{D^\perp}(X, Z) + h(X, Z) \]

and thus we conclude that the leafs of $D^\perp$ are totally geodesic immersed in $\overline{M}$ if and only if $h^{D^\perp} = 0$ and $M$ is $D^\perp$–geodesic. The equivalence follows now easily from Proposition 3.1.
i. **⇔ iii.** For $X, Z \in \Gamma(D^\perp)$, and $U \in \Gamma(D)$ we obtain similarly as in the proof of Proposition 3.1

\[
\mathfrak{g}(\nabla_X Z, U) = \mathfrak{g}(J\nabla_X Z, JU)
\]
\[
= \mathfrak{g}(-\nabla_X JZ + \nabla_X JZ, JU)
\]
\[
= \frac{1}{2}\mathfrak{g}(\theta(JU)JX - 2h_{JD^\perp}(X, JU), JZ).
\]

On the other hand, if $X, Z, W \in \Gamma(D^\perp)$, then taking account of \ref{eq:2} we deduce:

\[
\mathfrak{g}(\nabla_X Z, JW) = \mathfrak{g}(\nabla_X Z + h(X, Z), JW)
\]
\[
= \mathfrak{g}(h(X, Z), JW).
\]

If we consider now $X, Z \in \Gamma(D^\perp)$ and $N \in \Gamma(\mu)$, then making use of \ref{eq:11} and \ref{eq:10} we derive:

\[
\mathfrak{g}(\nabla_X Z, N) = \mathfrak{g}(J\nabla_X Z, JN)
\]
\[
= \mathfrak{g}(-\nabla_X JZ + \nabla_X JZ, JN)
\]
\[
= \frac{1}{2}\mathfrak{g}((\theta(Z)X - \omega(\mu)Z)X + g(X, Z)A - \Omega(X, Z)B - 2\nabla_X JZ, JN)
\]
\[
= \frac{1}{2}\mathfrak{g}(g(X, Z)A + 2\nabla_X JZ, JN)
\]
\[
= \frac{1}{2}\mathfrak{g}(g(X, Z)A + 2\nabla_X JZ, JN)
\]

and thus we obtain:

\[
\mathfrak{g}(\nabla_X Z, N) = \frac{1}{2}\mathfrak{g}(g(X, Z)A_\mu + 2(\nabla_X J)_{\mu}, JN).
\]

Finally, $M$ is a totally real ruled CR–submanifold of $(M, J, \mathfrak{g})$ if and only if $\nabla_X Z \in \Gamma(D^\perp)$, $\forall X, Z \in \Gamma(D^\perp)$ and by using \ref{eq:9}, \ref{eq:10} and \ref{eq:11} we deduce the equivalence. \hfill \Box

**Corollary 3.8.** If $M$ is a CR–submanifold of a l.c.K. manifold $(M, J, \mathfrak{g})$ such that $B \in \Gamma(JD^\perp)$, then the next assertions are equivalent:

i. $M$ is a totally real ruled CR–submanifold.

ii. $M$ is $D^\perp$–geodesic and the second fundamental form satisfies

\[
h(X, Y) \in \Gamma(\mu), \ \forall X \in \Gamma(D^\perp), \ Y \in \Gamma(D).
\]

iii. The subbundle $JD^\perp$ is $D^\perp$–parallel, i.e:

\[
\nabla_X JZ \in \Gamma(JD^\perp), \ \forall X, Z \in \Gamma(D^\perp)
\]

and the second fundamental form satisfies

\[
h(X, Y) \in \Gamma(\mu), \ \forall X \in \Gamma(D^\perp), \ Y \in \Gamma(TM).
\]

iv. The shape operator satisfies

\[
a_{JZ}X = 0, \ \forall X, Z \in \Gamma(D^\perp)
\]

and

\[
a_{N}X \in \Gamma(D), \ \forall X \in \Gamma(D^\perp), \ N \in \Gamma(\mu).
\]
Proof. The equivalence of (i.), (ii.) and (iii.) is clear from the above theorem, since for any \( Y \in \Gamma(D) \) we have

\[
\theta(Y) = g(JY, B) = 0.
\]

The equivalence of (ii) and (iii.) follows from (4).

\[ \square \]

Corollary 3.9. Let \( M \) be a CR–submanifold of a l.c.K. manifold \((\overline{M}, J, g)\) such that Lee vector field \( B \) is normal to \( M \). If \( M \) is totally geodesic, then \( M \) is a totally real ruled CR–submanifold.

Proof. The assertion is clear from Theorem 3.7.

\[ \square \]

4. Foliations with bundle–like metric on CR–submanifolds of locally conformal Kähler manifolds

Let \((M, g)\) be a Riemannian manifold and \( \mathcal{F} \) a foliation on \( M \). The metric \( g \) is said to be bundle–like for the foliation \( \mathcal{F} \) if the induced metric on the transversal distribution \( D^\perp \) is parallel with respect to the intrinsic connection on \( D^\perp \). This is true if and only if the Levi–Civita connection \( \nabla \) of \((M, g)\) satisfies (see [4]):

\[
g(Q^\perp X QY, Q^\perp Z) + g(Q^\perp Z QX, Q^\perp Y) = 0, \quad \forall X, Y, Z \in \Gamma(TM),
\]

(12)

where \( Q^\perp \) (resp. \( Q \)) is the projection morphism of \( TM \) on \( D^\perp \) (resp \( D \)).

If for a given foliation \( \mathcal{F} \) there exists a Riemannian metric \( g \) on \( M \) which is bundle-like for \( \mathcal{F} \), then we say that \( \mathcal{F} \) is a Riemannian foliation on \((M, g)\).

In what follows we provide necessary and sufficient conditions for the induced metric on a CR–submanifold of a l.c.K. manifold to be bundle–like for the totally real foliation \( \mathcal{F}^\perp \).

**Theorem 4.1.** If \( M \) is a CR–submanifold of a l.c.K. manifold \((\overline{M}, J, g)\), then the next assertions are equivalent:

i. The induced metric \( g \) on \( M \) is bundle–like for the canonical totally real foliation \( \mathcal{F}^\perp \).

ii. The second fundamental form \( h \) of the submanifold and anti–Lee vector field \( A \) satisfy:

\[
g(U, V)A + h(U, JV) + h(V, JU) \in \Gamma(TM) \oplus \Gamma(\mu),
\]

for any \( U, V \in \Gamma(D) \).

Proof. From (12) we deduce that \( g \) is bundle-like for the canonical totally real foliation \( \mathcal{F}^\perp \) if and only if:

\[
g(\nabla_U X, V) + g(\nabla_V X, U) = 0, \quad \forall X \in \Gamma(D^\perp), \ U, V \in \Gamma(D).
\]

(13)
On the other hand, using (11)-(14), we obtain for any \( X \in \Gamma(D^\perp), U, V \in \Gamma(D) \):
\[
g(\nabla_U X, V) + g(\nabla_V X, U) = \bar{g}(\nabla_U X - h(U, X), V) + \bar{g}(\nabla_V X - h(V, X), U)
\]
\[
= \bar{g}(\nabla_U X, V) + \bar{g}(\nabla_V X, U)
\]
\[
= \bar{g}(- (\nabla_U J)X + \nabla_U JX, JV) + \bar{g}(- (\nabla_V J)X + \nabla_V JX, JU)
\]
\[
= - \frac{1}{2} \bar{g}(\theta(X)U - \omega(X)JU - g(U, X)A - \Omega(U, X)B - 2\nabla_U JX, JU)
- \frac{1}{2} \bar{g}(\theta(X)V - \omega(X)JV - g(V, X)A - \Omega(V, X)B - 2\nabla_V JX, JU)
\]
\[
= - \frac{1}{2} \bar{g}(\theta(X)U - \omega(X)JU - 2\nabla_U JX, JU)
- \frac{1}{2} \bar{g}(\theta(X)V - \omega(X)JV - 2\nabla_V JX, JU)
\]
\[
= \omega(X)g(U, V) + \bar{g}(\nabla_U JX, JU) + \bar{g}(\nabla_V JX, JU)
\]
\[
= \omega(X)g(U, V) - g(A_{JX} U, JV) - g(A_{JX} V, JU)
\]
\[
= \omega(X)g(U, V) - \bar{g}(h(U, JV), JX) - \bar{g}(h(V, JU), JX).
\]
and taking into account that \( B = \omega^g \) and \( A = -JB \) we derive:
\[
g(\nabla_U X, V) + g(\nabla_V X, U) = -\bar{g}(\theta(U, V), A + h(U, JV) + h(V, JU), JX). \tag{14}
\]

The proof is now complete from (13) and (14). \( \square \)

**Corollary 4.2.** Let \( M \) be a CR–submanifold of a l.c.K. manifold \((\overline{M}, J, \overline{g})\).

i. If \( B \in \Gamma(D) \oplus \Gamma(TM^\perp) \), then the induced metric \( g \) on \( M \) is bundle–like for the canonical totally real foliation \( \mathcal{F}^\perp \) if and only if
\[
h(U, JV) + h(V, JU) \in \Gamma(\mu), \forall U, V \in \Gamma(D).
\]

ii. If \( B \) has a non–vanishing component in \( \Gamma(D^\perp) \), then the induced metric \( g \) on \( M \) is not bundle–like for the canonical totally real foliation \( \mathcal{F}^\perp \).

**Proof.** The proof follows from Theorem 4.1. \( \square \)

**Corollary 4.3.** If \( M \) is an anti-holomorphic submanifold of a l.c.K. manifold \((\overline{M}, J, \overline{g})\), normal to the Lee field of \( M \), then the induced metric \( g \) on \( M \) is bundle–like for the canonical totally real foliation \( \mathcal{F}^\perp \) if and only if
\[
h(U, JV) + h(V, JU) = 0, \forall U, V \in \Gamma(D).
\]

**Proof.** The assertion is clear from the above Corollary. \( \square \)

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**References**

[1] E. Barletta, CR submanifolds of maximal CR dimension in a complex Hopf manifold, Ann. Global Anal. Geom. 22 (2002), No. 2, 99-118.

[2] E. Barletta, S. Dragomir, K.L. Duggal, Foliations in Cauchy-Riemann geometry, Mathematical Surveys and Monographs, Vol. 140, American Mathematical Society, 2007.

[3] A. Bejancu, CR submanifolds of a Kaehler manifold. I, Proc. Am. Math. Soc. 69 (1978), 135–142.
[4] A. Bejancu, H.R. Farran, *Foliations and geometric structures*, Mathematics and Its Applications, Springer, 2006.
[5] D.E. Blair, B.Y. Chen, *On CR submanifolds of Hermitian manifolds*, Israel J. Math. 34 (1979), 353-369.
[6] D.E. Blair, S. Dragomir, *CR products in locally conformal Kähler manifolds*, Kyushu J. Math. 56 (2002), No. 2, 337–362.
[7] V. Bonanzinga, K. Matsumoto, *Warped product CR–submanifolds in locally conformal Kähler manifolds*, Period. Math. Hungar. 48 (2004), No. 1–2, 207–221.
[8] J.L. Cabrerizo, M. Fernandez Andres, *CR–submanifolds of a locally conformal Kähler manifold*, Differential geometry (Santiago de Compostela, 1984), Res. Notes in Math. 131 (1985), Pitman, Boston, MA, 17–32.
[9] B.Y. Chen, P. Piccinni, *The canonical foliations of a locally conformal Kähler manifold*, Ann. Mat. Pura Appl. (4) 141 (1985), 289–305.
[10] S. Dragomir, *Cauchy–Riemann submanifolds of locally conformal Kahler manifolds*. I–II, Geom. Dedicata 28 (1988), No. 2, 181–197; Sem. Mat. Fis. Univ. Modena XXXVII (1989), 1-11.
[11] S. Dragomir, R. Grimaldi, *Cauchy–Riemann submanifolds of locally conformal Kahler manifolds*. III, Serdica 17 (1991), No. 1, 3-14.
[12] S. Dragomir, S. Nishikawa, *Foliated CR manifolds*, J. Math. Soc. Japan 56 (2004), No. 4, 1031–1068.
[13] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Progress in Math. 155, Birkhäuser, Boston, Basel, 1998.
[14] K.L. Duggal, L. Sharma, *Totally umbilical CR–submanifolds of locally conformal Kahler manifolds*, Math. Chronicle 16 (1987), 79–83.
[15] S. Ianuş, A.M. Ionescu, G.E. Vilcu, *Foliations on quaternion CR-submanifolds*, Houston J. Math. 34 (2008), No. 3, 739–751.
[16] S. Ianuş, S. Marchiafava, G.E. Vilcu, *Paraquaternionic CR-submanifolds of paraquaternionic Kähler manifolds and semi-Riemannian submersions*, Cent. Eur. J. Math. 8 (2010), No. 4, 735–753.
[17] P. Libermann, *Sur les structures presque complexes et autres structures infinitésimales régulières*, Bull. Soc. Math. France 83 (1955), 195–224.
[18] K. Matsumoto, *On CR-submanifolds of locally conformal Kahler manifold*. I-II, J. Korean Math. Soc. 21(1) (1984), 49–61; Tensor, N.S., 45 (1987), 144–150.
[19] M. Munteanu, *Doubly warped product CR–submanifolds in locally conformal Kahler manifolds*, Monatsh. Math. 150 (2007), No. 4, 333–342.
[20] L. Ornea, *On CR submanifolds of locally conformal Kahler manifolds*, Demonstratio Math. 19 (1986), No. 4, 863–869.
[21] N. Papaghiuc, *Some remarks on CR-submanifolds of a locally conformal Kahler manifold with parallel Lee form*, Publ. Math. Debrecen 43 (1993), No. 3–4, 337–341.
[22] V. Rovenskii, *Foliations on Riemannian manifolds and submanifolds*, Birkhäuser Boston, Inc., Boston, MA, 1998.
[23] B. Şahin, R. Güneş, *CR-submanifolds of a locally conformal Kahler manifold and almost contact structure*, Math. J. Toyama Univ. 25 (2002), 13–23.
[24] I. Vaisman, *On locally conformal almost Kahler manifolds*, Israel J. Math. 24 (1976), 338–351.
[25] G.E. Vilcu, *Riemannian foliations on quaternion CR-submanifolds of an almost quaternion Kähler product manifold*, Proc. Indian Acad. Sci., Math. Sci. 119 (2009), No. 5, 611–618.

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