Moonshines for $M_{12}$ and $L_2(11)$

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Abstract

In this paper, we revisit an earlier conjecture by one of us that related conjugacy classes of $M_{12}$ to Jacobi forms of weight one and index zero. We construct Jacobi forms for all conjugacy classes of $M_{12}$ that are consistent with constraints from group theory as well as modularity. However, we obtain 1427 solutions that satisfy these constraints (to the order that we checked) and are unable to provide a unique Jacobi form. Nevertheless, as a consequence, we are able to provide a group theoretic proof of the evenness of the coefficients of all EOT Jacobi forms associated with conjugacy classes of $M_{12} : 2 \subset M_{24}$. In the absence of a unique answer for $M_{12}$, we show that there exist moonshines for two distinct $L_2(11)$ sub-groups of the $M_{12}$. We also show that BKM Lie superalgebras are associated with one of the $L_2(11)$ sub-groups.
1 Introduction

Following the discovery of monstrous moonshine, came a moonshine for the largest sporadic Mathieu group, $M_{24}$. This related $\rho$, a conjugacy class of $M_{24}$, to a multiplicative eta product that we denote by $\eta_\rho$ via the map \[(1.1)\]

$$\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N} \rightarrow \eta_\rho(\tau) := \prod_{m=1}^{N} \eta(m\tau)^{a_m}.$$ 

The same multiplicative eta products appeared as the generating function of $\frac{1}{2}$-BPS states twisted by a symmetry element (in the conjugacy class $\rho$) in type II string theory compactified on $K3 \times T^2$. This was further extended to the generating function of $\frac{1}{4}$-BPS states in the same theory. The generating function in this case was a genus-two Siegel modular form that we denote by $\Phi_\rho(Z)$ [3].

Renewed interest in this moonshine (now called Mathieu moonshine) appeared following the work of Eguchi-Ooguri-Tachikawa (EOT) who observed the appearance of the dimensions of irreps of $M_{24}$ in the elliptic genus of $K3$ when expanded in terms of characters of the $\mathcal{N} = 4$ superconformal algebra [4]. The Siegel modular form $\Phi_\rho(Z)$, when it exists, unifies the two Mathieu moonshines, the one related to multiplicative eta products as well as the one related to the elliptic genus [5, 6].

It has now been established that there is a moonshine that relates conjugacy classes of $M_{24}$ to (the EOT) Jacobi forms of weight zero and index one that we denote by $Z_\rho(\tau, z)$ [7–9]. These Jacobi forms arise as twistings (also called ‘twinings’) of the elliptic genus of $K3$. Given a conjugacy class $\rho$ of $M_{24}$, the associated Jacobi form expressed in terms of $\mathcal{N} = 4$ characters: the massless character $C(\tau, z)$ and the massive character $B(\tau, z)$ takes the form \[(1.2)\]

$$Z_\rho(\tau, z) = \alpha_\rho C(\tau, z) + q^{-\frac{1}{8}} \Sigma_\rho(\tau) B(\tau, z),$$

where $\alpha_\rho = 1 + \chi_{23}(\rho)$ and the character expansion of the function $\Sigma_\rho(\tau)$ is as follows:

$$\Sigma_\rho(\tau) = -2 + [\chi_{45}(\rho) + \chi_{\overline{45}}(\rho)] q + [\chi_{231}(\rho) + \chi_{\overline{231}}(\rho)] q^2 + [\chi_{770}(\rho) + \chi_{\overline{770}}(\rho)] q^3 + 2\chi_{2277}(\rho) q^4 + 2\chi_{5796}(\rho) q^5 + \cdots \quad (1.3)$$

where the subscript denotes the dimension of the irrep of $M_{24}$ and $q = e^{2\pi i \tau}$. An all-orders proof of the existence of such an expansion has been given by Gannon [12]. The left-hand side of Figure 1 captures the various connections. In particular, $\Phi_\rho(Z)$ can be constructed in two ways: an additive lift which uses the eta product $\eta_\rho(\tau)$ as input and a multiplicative/Borcherds lift where $Z_\rho(\tau, z)$ is the input. However, the additive lift is not known for all conjugacy classes.

In some cases, the square-root of $\Phi_\rho(Z)$ is related to a Borcherds-Kac-Moody (BKM) Lie superalgebra with the additive and multiplicative lifts providing the
sum and product side of the Weyl denominator formula. An attempt at understanding this square-root was done in [13] where it was argued that there might be a moonshine involving the Mathieu group $M_{12}$ relating its conjugacy classes to BKM Lie superalgebras. In this paper, we revisit that proposal from several viewpoints. Our results may be summarised as follows:\footnote{See the right side of Figure 1 to see the various connections mentioned below.}:

1. We show the existence of a moonshine that relates conjugacy classes of $M_{12}$ to Jacobi forms of weight zero and index one. However, we do not provide a unique solution but find 1427 families of Jacobi forms that are consistent with the constraints that we impose.

2. We show that the very existence albeit non-unique is sufficient to show that all the Fourier-Jacobi coefficients of the EOT Jacobi forms for $M_{24}$ conjugacy classes that reduce to conjugacy classes of $M_{12}:2$ (a maximal sub-group of $M_{24}$) are even. This provides an alternate proof of a theorem of Creutzig et al.

3. We address the non-uniqueness of the $M_{12}$ moonshine by providing two moonshines for $L_2(11)$ that arise as two distinct sub-groups of $M_{12}$.

4. We show the existence of BKM Lie superalgebras for conjugacy classes of one of the $L_2(11)$ subgroups. The sum side and the product side of Weyl denominator formula for all the cases arise as additive and multiplicative lifts of Jacobi forms that lead to genus two Siegel modular forms with positive weight.

\begin{center}
\begin{tikzpicture}

\node (a) at (0,0) {$\Phi^{\rho}(Z)$};
\node (b) at (-3,-2) {$\eta^{\rho}(\tau)$};
\node (c) at (3,-2) {$\eta^{\tilde{\rho}}(\tau)$};
\node (d) at (0,-4) {$Z^{\rho}(\tau,z)$};
\node (e) at (0,-6) {$Z^{\tilde{\rho}}(\tau,z)$};
\node (f) at (3,0) {$\psi^{\rho}(\tau,z)$};
\node (g) at (-3,0) {$\psi^{\tilde{\rho}}(\tau,z)$};
\node (h) at (6,0) {BKM algebra};
\node (i) at (0,-3) {$\Delta^{\tilde{\rho}}(Z)$};

\draw[->] (a) -- (b) node[pos=0.5,above] {Add. Lift};
\draw[->] (a) -- (c) node[pos=0.5,above] {Add. Lift};
\draw[->] (a) -- (d) node[pos=0.5,right] {Mult. Lift};
\draw[->] (a) -- (e) node[pos=0.5,left] {Mult. Lift};
\draw[->] (i) -- (h);
\draw[->] (h) -- (i);\end{tikzpicture}
\end{center}

Figure 1: Moonshines for $M_{24}$ and $M_{12}$: $\rho$ ($\tilde{\rho}$) is an $M_{24}$ ($M_{12}$) conjugacy class.

The plan of the manuscript is as follows. Following the introductory section, we state the main conjecture and study its implications in section 2. In section 3, we discuss evidence for the conjecture and find 1427 solutions that satisfy all the constraints that were imposed. In section 4, we show that there are unique solutions for two $L_2(11)$ subgroups of $M_{12}$ and one of them is related to a family of
BKM Lie superalgebras. We conclude with brief remarks in section 5. Appendices contain some of the relevant results in modular forms and (finite) group theory that are relevant for this paper.

2 The $M_{12}$ conjecture

In some cases, the square-root of the Siegel modular form that unifies the additive and multiplicative moonshines for $M_{24}$ is related to the denominator formula of a Borcherds-Kac-Moody (BKM) Lie superalgebra. A necessary condition is that the multiplicative seed have even coefficients. A group-theoretic answer to this question comes via $M_{12}:2$, a maximal subgroup of $M_{24}$ that can be constructed from $M_{12}$ and its outer automorphism. We denote characters and conjugacy classes of $M_{12}:2$ by $\tilde{\chi}_i$ and $\tilde{\rho}$ throughout this paper. Similarly, we will use a hat for $M_{12}$ characters and conjugacy classes.

$M_{12}:2$ is a maximal subgroup of $M_{24}$. The construction of $M_{12}:2$ has its origin in the order two outer automorphism of $M_{12}$ that is generated by an element that we denote by $\varphi$. There are two classes of elements, $(\hat{g}, e)$ and $(\hat{g}, \varphi)$, where $\hat{g} \in M_{12}$. The composition rule is given by

$$(\hat{g}_1, e) \cdot (\hat{g}_2, h) = (\hat{g}_1 \cdot \hat{g}_2, h), \quad (\hat{g}_1, \varphi) \cdot (\hat{g}_2, h) = (\hat{g}_1 \cdot \varphi(\hat{g}_2), \varphi \cdot h)$$

The existence of the decomposition of $Z^\rho(\tau, z)$ in terms of characters of $M_{24}$ as in Eq. (1.2) immediately implies the following decomposition for all the 21 conjugacy classes of $M_{12}:2$.

$$Z^\tilde{\rho}_{0,1}(\tau, z) = \alpha^\tilde{\rho} \mathcal{C}(\tau, z) + q^{-\frac{1}{8}} \Sigma^\tilde{\rho}(\tau) \mathcal{B}(\tau, z), \quad (2.1)$$

where $\alpha^\tilde{\rho} = 2 + \tilde{\chi}_2(\tilde{\rho}) + \tilde{\chi}_3(\tilde{\rho})$ and the function $\Sigma^\tilde{\rho}(\tau)$ can be expanded in terms of characters of $M_{12}:2$ as follows\(^2\):

$$\Sigma^\tilde{\rho}(\tau) = -2 + \sum_{n=1}^{\infty} \left( \sum_{a=1}^{21} \tilde{N}_a(n) \tilde{\chi}_a(\tilde{\rho}) \right) q^n. \quad (2.2)$$

where the multiplicities $\tilde{N}_a(n)$ are non-negative integers.

Characters of representations of $M_{12}:2$ arise in two ways from characters of $M_{12}$. Those of splitting type where a pair of $M_{12}:2$ characters are determined by a single $M_{12}$ character and of fusion type where a $M_{12}:2$ character is determined by a couple of $M_{12}$ characters.\(^2\)

\(^2\)We indicate conjugacy classes and other objects related to $M_{12}:2$ with a tilde and a hat for $M_{12}$. In addition, characters of $M_{12}:2$ are labelled with the beginning letters of the alphabet while letters beginning from $m$ are used for characters of $M_{12}$.
Let \( \hat{\rho} \) denote a conjugacy class of \( M_{12} \) associated with \( \hat{g} \in M_{12} \). Then, the pair of conjugacy classes \((\hat{\rho}, \varphi(\hat{\rho}))\) become a conjugacy class \( \hat{\rho} \) associated with the element \((\hat{g}, e)\) of \( M_{12}:2 \) and hence of \( M_{24} \) as well. 12 of the 21 conjugacy classes of \( M_{12}:2 \) arise in this fashion.

### Conjecture 2.1 (Govindarajan [13]).

There exists a moonshine for \( M_{12} \) that associates a unique weight zero, index one Jacobi form \( \hat{\psi}^\hat{\rho}_{0,1} \) to every conjugacy class \( \hat{\rho} \) of \( M_{12} \) such that

1. \( Z^\hat{\rho}(\tau, z) = \hat{\psi}^\hat{\rho}_{0,1}(\tau, z) + \hat{\psi}^{\varphi(\hat{\rho})}_{0,1}(\tau, z) \), where \( \hat{\rho} = (\hat{\rho}, \varphi(\hat{\rho})) \) and \( Z^\hat{\rho}_{0,1}(\tau, z) \) is the Jacobi form that appears in the moonshine for \( M_{24} \).

2. The Jacobi form written in terms of \( N = 4 \) characters

\[
\hat{\psi}^\hat{\rho}_{0,1}(\tau, z) = \alpha^\hat{\rho} C(\tau, z) + q^{-\frac{1}{8}} \hat{\Sigma}^\hat{\rho}(\tau) B(\tau, z),
\]

where \( \alpha^\hat{\rho} \) and \( \hat{\Sigma}^\hat{\rho}(\tau) \) can be expressed in terms of \( M_{12} \) characters. One has \( \alpha^\hat{\rho} = 1 + \hat{\chi}_2(\hat{\rho}) \) and

\[
\hat{\Sigma}^\hat{\rho}(\tau) = -1 + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{15} \hat{N}_m(n) \hat{\chi}_m(\hat{\rho}) \right) q^n. \tag{2.3}
\]

with \( \hat{N}_m(n) \) being non-negative integers for all \( m \geq 1 \) and \( n \geq 1 \).

### Implications of the conjecture

**Proposition 2.2.** For \( M_{12} \) conjugacy classes \( \hat{\rho} \neq 4a/4b/8a/8b \)

\[
\hat{\psi}^\hat{\rho}_{0,1}(\tau, z) = \frac{1}{2} Z^\hat{\rho}(\tau, z), \quad .
\]

**Remark:** It is easy to verify that \( \alpha^{\hat{\rho}} = \alpha^{\varphi(\hat{\rho})} \) for these conjugacy classes. Thus it suffices to show that \( \hat{\Sigma}^{\hat{\rho}} = \hat{\Sigma}^{\varphi(\hat{\rho})} \). A complete list of EOT Jacobi forms that are related to conjugacy classes of \( M_{12}:2 \) is given in Table A.2.

**Proof.** The \( M_{12} \) characters \( \hat{\chi}_4 \) and \( \hat{\chi}_5 \) take complex values for the conjugacy classes 11a/b. Reality of the Jacobi forms for these two conjugacy classes implies that the multiplicities that appear in Eq. (2.3) must be such that

\[
\hat{N}_4(n) = \hat{N}_5(n) \text{ for all } n \geq 1. \]

### Table

| Rep. Type | Conjugacy Class |
|-----------|-----------------|
| Splitting | \( \hat{x}_a = \hat{x}_{a'} = \hat{x}_m \) | \( \hat{x}_a + \hat{x}_{a'} = 0 \) |
| Fusion | \( \hat{x}_a = \hat{x}_m + \hat{x}_{m'} \) | \( \hat{x}_a = 0 \) |
It is useful to rewrite the above character decomposition taking into account the outer automorphism $\varphi$ of $M_{12}$. With this in mind, we define

$$
\hat{N}_2^+(n) = (\hat{N}_2(n) \pm \hat{N}_9(n)) \quad , \quad \hat{N}_9^+(n) = (\hat{N}_9(n) \pm \hat{N}_{10}(n)) \quad , \\
\hat{\chi}_2^+(\hat{\rho}) = (\hat{\chi}_2(\hat{\rho}) \pm \hat{\chi}_3(\hat{\rho})) \quad , \quad \hat{\chi}_9^+(\hat{\rho}) = (\hat{\chi}_9(\hat{\rho}) \pm \hat{\chi}_{10}(\hat{\rho})) .
$$

Note that $\hat{\chi}_2^+(\varphi(\hat{\rho})) = \pm \hat{\chi}_2^+(\hat{\rho})$ and $\hat{\chi}_9^+(\varphi(\hat{\rho})) = \pm \hat{\chi}_9^+(\hat{\rho})$ by construction. We can then write the character decomposition as follows

$$
\hat{\Sigma}^\hat{\rho} = -1 + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{15} \hat{N}_m(n) \hat{\chi}_m(\hat{\rho}) + \frac{1}{2} \hat{N}_2^+(n) \hat{\chi}_2^+(\hat{\rho}) + \frac{1}{7} \hat{N}_9^+(n) \hat{\chi}_9^+(\hat{\rho}) \\
+ \frac{1}{2} \hat{N}_2^-(n) \hat{\chi}_2^-(\hat{\rho}) + \frac{1}{2} \hat{N}_9^-(n) \hat{\chi}_9^-(\hat{\rho}) \right) q^n . \quad (2.4)
$$

Then, for conjugacy classes $\tilde{\rho} = (\hat{\rho}, \varphi(\hat{\rho}))$ of $M_{12} : 2$, one has

$$
\hat{\Sigma}^{\hat{\rho}} - \hat{\Sigma}^{\varphi(\hat{\rho})} = \sum_{n=1}^{\infty} \left( \hat{N}_2^-(n) \hat{\chi}_2^-(\hat{\rho}) + \hat{N}_9^-(n) \hat{\chi}_9^-(\hat{\rho}) \right) q^n . \quad (2.5)
$$

$\hat{\chi}_2^-(\hat{\rho}) = 0$ and $\hat{\chi}_9^-(\hat{\rho}) = 0$ for $M_{12}$ conjugacy classes $\hat{\rho} \neq 4a/4b/8a/8b$ since $\varphi(\hat{\rho}) = \hat{\rho}$. Thus, for $M_{12}$ conjugacy classes $\hat{\rho} \neq 4a/4b/8a/8b$, $\hat{\Sigma}^{\hat{\rho}} = \hat{\Sigma}^{\varphi(\hat{\rho})}$, thereby proving the proposition. \hfill \Box

**Remark:** There is a proposal due to Eguchi and Hikami where they propose a moonshine called ‘Enriques’ moonshine [14]. For the four classes that are undetermined, they propose to take one half of the $M_{24}$ Jacobi form as the required Jacobi form. That is not consistent with the decomposition in the expression for $\alpha \hat{\rho}$ as it implies that $\alpha \hat{\rho} = 1 + \frac{1}{2}(\hat{\chi}_2 + \hat{\chi}_3)$ - the coefficients are half integral and do not make sense from group theory. The outer automorphism of $M_{12}$ that we make extensive use of also does not make an appearance in their considerations. We believe that their proposal in not related to our work.

**Proposition 2.3.** For all conjugacy classes of $M_{12} : 2$, the Fourier-Jacobi coefficients of the associated Jacobi form are even i.e.,

$$
Z^{\hat{\rho}}(\tau) = 0 \mod 2 .
$$

**Remark:** This follows from a theorem of Creutzig, Höhn and Miezaki where they show that the $Z^{\rho}(\tau, z)$ for $M_{24}$ conjugacy classes $(7a/b, 14a/b, 15a/b, 23a/b)$ have odd coefficients. These are precisely the conjugacy classes of $M_{24}$ that do not reduce to conjugacy classes of $M_{12} : 2$. We provide an alternate proof assuming that the $M_{12}$ conjecture holds.

**Proof.** Since the $\mathcal{N} = 4$ characters have integer coefficients, it suffices to show that $\alpha \hat{\rho} = 0 \mod 2$ and $\hat{\Sigma}^{\hat{\rho}} = 0 \mod 2$ for all conjugacy classes.
Part 1: $\alpha^{\hat{\bar{\rho}}} = 1 + \chi_2(\bar{\rho}) + \chi_3(\bar{\rho}) = 0 \mod 2$ for all conjugacy classes as can be explicitly checked.

Part 2: Consider the $M_{12} : 2$ conjugacy classes that arise from elements of type $(g, e)$. For such cases,

$$\Sigma^{\hat{\bar{\rho}}} = \Sigma^{\hat{\bar{\rho}}(\bar{\rho})} = -2 + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{15} \tilde{N}_m(n)\hat{x}_m(\bar{\rho}) \right) q^n,$$

Comparing the above equation with Eq. (2.2), we obtain

$$2\tilde{N}_m(n) = \tilde{N}_a(n) + \tilde{N}_{a'}(n)$$

and for the fusion type,

$$\hat{N}_2^+(n) = \hat{N}_3(n), \quad \hat{N}_9^+(n) = \hat{N}_{11}(n), \quad 2\hat{N}_4(n) = 2\hat{N}_5(n) = \hat{N}_4(n).$$

Next, considering the expression for $\Sigma^{\hat{\bar{\rho}}(\tau)}$ above modulo 2, we obtain

$$\Sigma^{\hat{\bar{\rho}}(\tau)} = \hat{N}_2^+(n)\hat{x}_2^+(\hat{\bar{\rho}}) + \hat{N}_9^+(n)\hat{x}_9^+(\hat{\bar{\rho}}) \mod 2.$$

Further, one has $\hat{x}_2^+(\hat{\bar{\rho}}) = \hat{x}_9^+(\hat{\bar{\rho}}) = 0 \mod 2$ as one can explicitly check. Thus, for conjugacy classes for elements of type $(g, e)$, one has $\Sigma^{\hat{\bar{\rho}}(\tau)} = 0 \mod 2$.

Part 3: Now consider conjugacy classes of $M_{12} : 2$ associated with elements of type $(g, \varphi)$. For these conjugacy classes, the character for all fusion representations vanish. Thus,

$$\Sigma^{\hat{\bar{\rho}}(\tau)} = -2 + \sum_{n=1}^{\infty} \left( \sum_{a \in \text{splitting}} \tilde{N}_a(n)\tilde{x}_a(\bar{\rho}) \right) q^n = -2 + \sum_{n=1}^{\infty} \left( \sum_{\text{pairs } (a, a') \in \text{splitting}} (\tilde{N}_a(n)\tilde{x}_a(\bar{\rho}) + \tilde{N}_{a'}(n)\tilde{x}_{a'}(\bar{\rho})) \right) q^n,$$

where we have used the relation $\tilde{x}_a + \tilde{x}_{a'} = 0$ for all pairs of splitting representations. The characters for the pairs $(7, 8)$ and $(16, 17)$ are irrational for these conjugacy classes. For such pairs, rationality (and hence integrality) of the Fourier-Jacobi coefficients of the EOT Jacobi forms implies...
that \((\tilde{N}_a(n) - \tilde{N}_{a'}(n)) = 0\) for \((a, a') = (7, 8), (16, 17)\). For all other pairs, equation (2.6) implies the weaker condition, \((\tilde{N}_a(n) - \tilde{N}_{a'}(n)) = 0 \mod 2\). Thus, for conjugacy classes for elements of type \((g, \varphi)\), one has \(\Sigma^{\tilde{\rho}}(\tau) = 0 \mod 2\).

\[
\square
\]

3 Towards proving for the \(M_{12}\) conjecture

We have seen in Proposition 2.2 that for \(M_{12}\) conjugacy classes \(\hat{\rho} \neq 4a/4b/8a/8b\), one has

\[
\hat{\psi}^{\hat{\rho}}(\tau, z) = \frac{1}{2} Z^\rho(\tau, z),
\]

where \(Z^\rho(\tau, z)\) is the EOT Jacobi form for the \(M_{24}\) conjugacy class \(\rho = (\hat{\rho})^2\) i.e., \(\rho = 1^{2a_1}2^{2a_2} \ldots N^{2aN}\) if \(\hat{\rho} = 1^{a_1}2^{a_2} \ldots N^{aN}\). That leaves four undetermined Jacobi forms associated with the classes \(4a/4b\) and \(8a/8b\). In these cases, we have the relation that relates the sums of Jacobi form of the two \(M_{12}\) conjugacy classes to EOT Jacobi forms.

\[
Z^{4b}(\tau, z) = \hat{\psi}^{4a}(\tau, z) + \hat{\psi}^{4b}(\tau, z)
\]

(3.2)

\[
Z^{8b}(\tau, z) = \hat{\psi}^{8a}(\tau, z) + \hat{\psi}^{8b}(\tau, z).
\]

(3.3)

Thus it suffices to determine \(\hat{\psi}^{4a}(\tau, z)\) and \(\hat{\psi}^{8a}(\tau, z)\) to obtain Jacobi forms for all conjugacy classes. These two examples have vanishing twisted elliptic genus as the corresponding cycle shapes have no one-cycles. Thus, for \(\hat{\rho} = 4a\) and \(8a\), one has

\[
\hat{\psi}^{\hat{\rho}}(\tau, z) = \check{\gamma}^{\hat{\rho}}(\tau) \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6},
\]

(3.4)

where \(\check{\gamma}^{\hat{\rho}}(\tau)\) is a weight two modular form of suitable subgroup of \(SL(2, \mathbb{Z})\).

Constraints

There are two kinds of constraints that we impose on the weight two modular forms.

1. **Non-negativity of coefficients in the character expansion:** We anticipate that all the Jacobi forms associated with the fifteen \(M_{12}\)-conjugacy classes admit a decomposition in terms of the fifteen \(M_{12}\) characters. The \(\mathcal{N} = 4\) decomposition (as carried out by Eguchi-Hikami) implies

\[
\hat{\psi}^{\check{\rho}}_{0,1}(\tau, z) = \check{\alpha}^{\hat{\rho}} \mathcal{C}(\tau, z) + q^{1/8} \check{\Sigma}^{\check{\rho}}(\tau) \mathcal{B}(\tau, z),
\]

(3.5)
where \( \hat{\alpha} \hat{\beta} = 1 + \hat{\chi}_2(\hat{\rho}) \) and
\[
\hat{\Sigma}^\hat{\beta}(\tau) = -1 + \hat{\chi}_6 \; q + [\hat{\chi}_8 + \hat{\chi}_{15}] \; q^2 + [\hat{\chi}_{11} + 2 \; \hat{\chi}_{13} + 2 \; \hat{\chi}_{14} + \hat{\chi}_{15}] \; q^3 + \cdots
\]
We have written out the first four terms in the character expansion of \( \hat{\Sigma}^\hat{\beta} \) as there is no ambiguity arising from the undetermined conjugacy classes. The ambiguity arises from irreps that get exchanged by the outer automorphism of \( M_{12} \) – these correspond to the four characters \( \hat{\chi}_2 \leftrightarrow \hat{\chi}_3 \) and \( \hat{\chi}_9 \leftrightarrow \hat{\chi}_{10} \). In particular, this implies that we know the first four terms in the \( q \)-series for \( \gamma^{4A} \) and \( \gamma^{8A} \). One has
\[
\gamma^{4A} = 1 - 4q + 4q^2 + 4q^3 + \cdots, \quad \gamma^{8A} = 1 - 2q - 2q^2 + 2q^3 + \cdots \quad (3.6)
\]
The constraint from group theory is that coefficients in the character expansion of \( \hat{\Sigma}^\hat{\beta} \) are all non-negative integers.

2. Modularity: The multiplicative eta products for the two classes \( 4a/8a \) are modular forms at level 16 and 64 respectively. From our experience with determining such examples for the \( M_{24} \) Jacobi forms for type II conjugacy classes, these do provide a rough guide to determining the levels for our Jacobi forms. For the class \( 4a \), the first four terms already determined imply that the level must be larger than 8.

We find solutions where both conjugacy classes are determined by weight two modular forms at level 32. Table 3 lists out the multiplicities of the various \( M_{12} \) characters, i.e., \( \hat{N}_m(n) \), in the character decomposition for the \( M_{12} \) Jacobi form for \( n \in [0, 32] \) when \( d_4 = d_5 = d_6 = d_8 = 0 \).

3.1 The conjugacy classes \( 4a/8a \)

Our non-unique proposal for the Jacobi forms for the conjugacy classes \( 4a/8a \) are as follows:
\[
\hat{\psi}^{4a}(\tau, z) = \left[ \gamma^{4a}(\tau) + 4\alpha(\tau) \right] \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6},
\]
\[
\hat{\psi}^{8a}(\tau, z) = \left[ \gamma^{8a}(\tau) - 2\alpha(\tau) \right] \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6},
\]
where it is specified in terms of the following weight two modular forms of \( \Gamma_0(32) \):
\[
\gamma^{4a}(\tau) := -\frac{31}{192} E_2^{(2)}(\tau) + \frac{3}{64} E_2^{(4)}(\tau) - \frac{7}{48} E_2^{(8)}(\tau) + \frac{35}{192} E_2^{(16)}(\tau) - \frac{9}{4} f(\tau)
\]
\[
= 1 - 4q + 4q^2 + 4q^3 - 4q^4 - 20q^5 + 16q^6 + 8q^7 + 24q^8 + \cdots \quad (3.8)
\]
\[
\gamma^{8a}(\tau) := \frac{41}{384} E_2^{(2)}(\tau) - \frac{45}{128} E_2^{(4)}(\tau) + \frac{301}{192} E_2^{(8)}(\tau) - \frac{155}{32} E_2^{(16)}(\tau)
\]
\[
+ \frac{217}{48} E_2^{(32)}(\tau) + \frac{3}{8} f(\tau) + \frac{5}{2} f(2\tau) - \frac{13}{4} \eta_4^{8s}(\tau)
\]
\[
= 1 - 2q - 2q^2 + 2q^3 + 4q^4 + 14q^5 - 12q^6 + 4q^7 - 32q^8 + \cdots,
\]

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with the ambiguity given by a modular form \( \alpha(\tau) \) with integral coefficients and is parametrised by four integers \((d_4, d_5, d_6, d_8)\).

\[
\alpha(\tau) = d_4(-\frac{1}{32}E_2^{(4)}(\tau) + \frac{7}{64}E_2^{(8)}(\tau) - \frac{5}{64}E_2^{(16)}(\tau)) \\
+ d_5(-\frac{1}{256}E_2^{(2)}(\tau) + \frac{1}{256}E_2^{(4)}(\tau) - \frac{1}{16}f(\tau) + \frac{1}{8}\eta_{4281}^2(\tau)) \\
+ d_6(-\frac{1}{384}E_2^{(2)}(\tau) + \frac{3}{256}E_2^{(4)}(\tau) - \frac{2}{768}E_2^{(8)}(\tau) - \frac{1}{8}f(2\tau)) \\
+ d_8(-\frac{7}{192}E_2^{(8)}(\tau) + \frac{15}{128}E_2^{(16)}(\tau) - \frac{31}{384}E_2^{(32)}(\tau)) \\
= d_4 q^4 - d_5 q^5 + d_6 q^6 + d_8 q^8 + \cdots.
\]

We can see that these four integers would be determined if we could fix either \( \gamma^4 A \) or \( \gamma^8 A \) to order \( q^8 \). Positivity for the first few terms give the following inequalities:

\[-4 \leq d_4 \leq 0 \quad , \quad -3 \leq (d_5 - 3d_4) \leq 3 \, , \]

\[-10 \leq d_6 + 3(3d_4 - d_5) \leq 8 \quad , \quad -32 \leq d_8 + 9d_6 - 22d_5 + 51d_4 \leq 46 \, . \]

There exists no solution with \( d_5 = -1 \) and \( d_6 = d_8 = 0 \). If true, this would imply that there is a solution with \( \gamma^4 A \) as a modular form of \( \Gamma_0(16) \). To order \( q^{128} \), we find 1427 solutions that include \( d_4 = d_5 = d_6 = d_8 = 0 \). We have checked that all these solutions continue to satisfy the positivity constraints to order \( q^{512} \) and possibly to all orders\(^3\).

The result can also be understood in terms of the expansion of \( \Sigma^4 \) in terms of representations of \( M_{12} \). The above solution completely determines the multiplicities \( \hat{N}_m(n) \) defined in Eq. (2.3) for all representations except \( m = 9, 10 \). In particular, it uniquely fixes \( \hat{N}_2(n) \) and \( \hat{N}_3(n) \).

It appears that we might need additional inputs beyond those considered in this paper to come up with a unique answer.

4 Moonshines for \( L_2(11) \)

We have seen that we do not have a unique representative of the Jacobi form for all conjugacy classes of \( M_{12} \). With this in mind, we looked for subgroups of \( M_{12} \) for which the characters \( \chi^L_2(\hat{\rho}) \) and \( \chi^L_5(\hat{\rho}) \) vanish on restriction to the sub-group. There are two such sub-groups, both isomorphic to \( L_2(11) \). The first is as a maximal subgroup of \( M_{12} \) and the second is as a maximal subgroup of \( M_{11} \subset M_{12} \). As sub-groups of \( M_{12} \), these two groups are not conjugate to each other and thus lead to distinct moonshines [15].

\(^3\)All 1427 solutions have been listed in the LaTeX source file after the \texttt{\end{document}} command. They can be accessed by downloading the source file.
4.1 \( L_2(11) \)

\( L_2(11) \) is Artin’s notation for the finite simple group \( PSL(2, \mathbb{F}_{11}) = SL(2, \mathbb{F}_{11})/\mathbb{F}^\times_{11} \), where \( \mathbb{F}_{11} \) is the prime field of integers modulo 11. It has a natural action on the projective line, \( PL(11) \), via projective linear transformations:

\[
    x \rightarrow \frac{ax + b}{cx + d}, \quad x \in PL(11).
\]

The projective line \( PL(11) \) consists of 12 points whose inhomogeneous coordinates are given by the set \( \Omega = (0, 1, 2, 3, \ldots, 9, X = 10, \infty) \). This provides a 12-dimensional permutation representation of \( L_2(11) \). In this representation, \( L_2(11) \) is generated as \( \langle \alpha, \beta, \gamma \rangle =: L_2(11)_A \), where

\[
    \alpha : x \rightarrow x + 1, \quad \beta : x \rightarrow 3 \cdot x, \quad \gamma : x \rightarrow -1/x.
\]

Explicitly, one has

\[
    \alpha = (\infty)(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X)
    \beta = (\infty)(0)(1, 3, 9, 5, 4)(2, 6, 7, X, 8)
    \gamma = (\infty, 0)(1, X), (2, 5), (3, 7)(4, 8)(6, 9)
\]

One has \( \alpha^{11} = \beta^5 = \gamma^2 = 1 \). The eight conjugacy classes of \( L_2(11) \) are given by the following cycle shapes in \( L_2(11)_A \):

| \( \rho \) | cycle shape | element |
|---|---|---|
| 1a | 1\^2 | 1 |
| 2a | 2\^6 | \( \gamma \) |
| 3a | 3\^4 | \( \alpha \gamma \) |
| 5a | 5\^2 | \( \beta \) |
| 5b | 5\^2 | \( \beta^{-1} \) |
| 6a | 6\^2 | \( \alpha \gamma \beta \) |
| 11a | 1\^11\^1 | \( \alpha \) |
| 11b | 1\^11\^1 | \( \alpha^{-1} \) |

Let \( \delta \) represent the permutation (with cycle shape 1\^42\^4) acting on \( PL(11) \):

\[
    \delta = (\infty)(0)(1)(2, X)(3, 4)(5, 9)(6, 7)(8).
\]

A second construction of \( L_2(11) \), that we call \( L_2(11)_B \), is generated by \( \langle \alpha, \beta, \delta \rangle \). All three generators fix \( \infty \) and thus \( L_2(11)_B \) permutes points in \( \Omega \setminus \infty \). The cycle shapes for the conjugacy classes for \( L_2(11)_B \) are

| \( \rho \) | cycle shape | element |
|---|---|---|
| 1a | 1\^2 | 1 |
| 2a | 2\^4 | \( \delta \) |
| 3a | 3\^3 | \( \alpha \delta \) |
| 5a | 5\^2 | \( \beta \) |
| 5b | 5\^2 | \( \beta^{-1} \) |
| 6a | 1\^23\^16\^1 | \( \alpha \delta \beta \) |
| 11a | 1\^11\^1 | \( \alpha \) |
| 11b | 1\^11\^1 | \( \alpha^{-1} \) |

The important observation here is that both \( L_2(11)_A \) and \( L_2(11)_B \) do not have any elements of order 4 and 8. Thus the conjugacy classes 4\( a/4b \) and 8\( a/8b \) do not reduce to conjugacy classes of these sub-groups. The conjugacy class 10\( a \) of \( M_{12} \), for which we do know the Jacobi form, also does not appear.
Table 1: $M_{12}$ conjugacy classes and the corresponding cycle shapes. The associated eta products are modular forms of weight $k$ of $\Gamma_0(2M,2)$ with Dirichlet character $\chi$ with $\Gamma_1(2M,2) \subseteq \ker(\chi)$. The values are given in columns 3-5.

| $M_{12}$ Conj. Class | Cycle Shape       | $M$ | $k$ | $\hat{\chi}(d)$ |
|----------------------|------------------|-----|-----|-----------------|
| 1a                   | $1^{12}$         | 1   | 6   |                 |
| 2a                   | $2^6$            | 4   | 3   | $(-\frac{1}{d})$|
| 2b                   | $1^42^4$         | 2   | 2   |                 |
| 3a                   | $1^33^3$         | 3   | 3   | $(-\frac{2}{d})$|
| 3b                   | $3^4$            | 9   | 2   |                 |
| 4a                   | $2^24^2$         | 8   | 2   |                 |
| 4b                   | $1^42^4/2^24^2$  | 8   | 3   | $(-\frac{1}{d})$|
| 5a                   | $1^25^2$         | 5   | 2   |                 |
| 6a                   | $6^2$            | 36  | 1   | $(-\frac{1}{d})$|
| 6b                   | $1^12^13^16^1$   | 6   | 2   |                 |
| 8a                   | $4^18^1$         | 32  | 1   | $(-\frac{2}{d})$|
| 8b                   | $1^22^14^18^1/4^18^1$ | 32 | 2 | |
| 10a                  | $2^110^1$        | 20  | 1   | $(-\frac{20}{d})$|
| 11a/b                | $1^111^1$        | 11  | 1   | $(-\frac{11}{d})$|

Below we provide first few terms that appear in the character expansion which is the analog of Eq. (2.3) for the two $L_2(11)$ subgroups. For $L_2(11)_A \subset M_{12}$, one has$^4$

$$\Sigma = -\chi_1 + (\chi_1 + 2\chi_5 + \chi_7 + \chi_8)q + (\chi_2 + \chi_3 + 5\chi_4 + 2\chi_5 + 5\chi_6 + 4\chi_7 + 4\chi_8)q^2 + (\chi_1 + 8\chi_2 + 8\chi_3 + 9\chi_4 + 12\chi_5 + 13\chi_6 + 14\chi_7 + 14\chi_8)q^3 + (2\chi_1 + 15\chi_2 + 15\chi_3 + 39\chi_4 + 32\chi_5 + 37\chi_6 + 42\chi_7 + 42\chi_8)q^4 + O(q^5) \quad (4.2)$$

For $L_1(11)_B \subset M_{11} \subset M_{12}$

$$\Sigma = -\chi_1 + (\chi_4 + \chi_6 + \chi_7 + \chi_8)q + (\chi_1 + 3\chi_2 + 3\chi_3 + 2\chi_4 + 4\chi_5 + 4\chi_6 + 4\chi_7 + 4\chi_8)q^2 + (\chi_1 + 4\chi_2 + 4\chi_3 + 15\chi_4 + 10\chi_5 + 13\chi_6 + 14\chi_7 + 14\chi_8)q^3 + (4\chi_1 + 19\chi_2 + 19\chi_3 + 31\chi_4 + 38\chi_5 + 35\chi_6 + 42\chi_7 + 42\chi_8)q^4 + O(q^5) \quad (4.3)$$

### 4.2 Towards BKM Lie superalgebras

The construction of Borcherds-Kac-Moody Lie algebras is intimately connected to modular forms that appear as the Weyl denominator formula for the BKM Lie algebra. The sum side of Weyl denominator formula arises from an additive lift while the product side is given by a Borcherds product formula.

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$^4$For the two equations that follow, the characters that appear are those for $L_2(11)$. 

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The additive ‘seed’ for the Siegel modular form is generated by a Jacobi form of weight $k$ and index $1/2$

$$\hat{\phi}^\rho_{k,1/2}(\tau, z) = \frac{\vartheta_1(\tau, z)}{\eta(\tau)^3} \times \eta(\tau).$$

where $\eta(\tau)$ is an eta product given in Table 1. For $k > 0$, the additive lift is given by

$$\Delta^\rho_k(Z) = \sum_{m=1}^{\infty} s^{(2m-1)/2} \hat{\phi}^\rho_{k,1/2} |_{k} T^{-}_M(2m-1)(\tau, z), \tag{4.4}$$

where $T^{-}_M(m)$ is the Hecke operator defined by Clery-Gritsenko [16]. Let $\phi(\tau, z)$ be a Jacobi form of weight $k$ of $\Gamma_0(M)$ with character $\chi$ and index which is integral or half-integral. Then

$$\phi^\rho \big|_{k} T^{-}_M(m)(\tau, z) = \frac{1}{m^{k-1}} \sum_{\text{ad} = m \atop b \mod d} d^{-k} \chi(a) \phi^\rho \left( \frac{at+qb}{d}, az \right).$$

where $q$ is chosen such that $\Gamma_1(Mq, q) \subset \ker(\chi)$. For all the cases of interest, one has $q = 2$ and $M$ as given in Table 1.

The Borcherds or multiplicative lift is given by

$$\Delta^\rho_k(Z) = s^{1/2} \hat{\phi}^\rho_{k,1/2}(\tau, z) \times \exp \left( \sum_{m=1}^{\infty} s^{m} \hat{\psi}^\rho_0 T(m)(\tau, z) \right) \tag{4.5}$$

where the twisted Hecke operator (as defined in [17]) is given by

$$\hat{\psi}^\rho_0 T(m)(\tau, z) = \frac{1}{m} \sum_{\text{ad} = m \atop b \mod d} \hat{\psi}^\rho_m \left( \frac{at+b}{d}, az \right),$$

where $\rho_m$ is the conjugacy class of the $m$-th power of an element in the conjugacy class $\rho$.

A necessary condition for the compatibility of the additive lift with the multiplicative lift is:

$$\left[ \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right] \eta^\rho_0(T^{-}_M(3)(\tau, z)) = \hat{\psi}^\rho(\tau, z) \left[ \frac{\vartheta_1(\tau, z)}{\eta(\tau)^3} \right] \eta^\rho_0(\tau).$$

This is the coefficient of $s^{3/2}$ in both the lifts i.e., the ones given in Eq. (4.4) and Eq. (4.5). We will carry out this compatibility test for the various sub-groups of $M_{12}$. We first consider the cases for which the Jacobi forms are given by half of the EOT Jacobi forms.

5The group $\Gamma_1(Mq, q)$ is defined as follows.

$$\left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2,\mathbb{Z}) \ \middle| \ c = 0 \mod Mq, \ b = 0 \mod q, \ a = 1 \mod Mq, \ d = 1 \mod Mq \right\}.$$
4.2.1  \( L_2(11)_B \)

For situations such as 1\textsuperscript{1}11\textsuperscript{1}, the additive seed has weight zero. In such cases, the additive lift is usually not defined. We shall discuss the weight zero cases separately. The compatibility condition holds for all other cycle shapes. Except for the cycle shape, 1\textsuperscript{1}2\textsuperscript{1}3\textsuperscript{1}6\textsuperscript{1}, it is known \([3, 18]\) that the Siegel modular forms appear as the Weyl denominator formula for rank three BKM Lie superalgebras whose real simple roots have the Cartan matrix

\[
A = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix}.
\]

Given the identification of the walls of the Weyl chambers with walls of marginal stability in the physical string theory \([19, 20]\), the same expectation to hold for the cycle shape 1\textsuperscript{1}2\textsuperscript{1}3\textsuperscript{1}6\textsuperscript{1}. One can verify that this holds \([21]\).

In conclusion, there exists a BKM Lie superalgebra for all conjugacy classes of \( L_2(11)_B \) other than 1\textsuperscript{1}11\textsuperscript{1}.

4.2.2  \( L_2(11)_A \)

There are three new cycles shapes that did not appear in \( L_2(11)_B \): 2\textsuperscript{6}, 3\textsuperscript{4} and 6\textsuperscript{2}. The cycle shape 6\textsuperscript{2} is associated with \( k = 0 \) and we shall discuss it later. The compatibility condition holds for 2\textsuperscript{6} but does not hold for 3\textsuperscript{4}. It suggests that one might look for a BKM Lie superalgebra to be associated with the cycle shape 2\textsuperscript{6}. For 3\textsuperscript{4}, we find

\[
T_{3} \hat{\phi}^{34} - \hat{\psi}^{34} = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \left[ 9 \eta_{13-29^3}(\tau) \right]
\]

where \( T_{3} \hat{\phi}^{3b} \) is short form for \( \hat{\phi}^{3b} \mid T^M(3) \). The extra piece is a weight two multiplicative eta quotient (at level 9) that appears in the list of Martin \([22]\).

4.2.3  The weight \( k = 0 \) examples

In all the \( k = 0 \) examples, we carry out the naive additive lift and find that it does not agree with the multiplicative lift. We find weight two modular forms at the suitable level that give the mismatch between the two lifts.

\[
\begin{align*}
T_{3} \phi^{210^1} - \hat{\psi}^{210^1} &= \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \left[ \frac{20}{3} \eta_{210^2}(\tau) \right] \\
T_{3} \phi^{1111^1} - \hat{\psi}^{1111^1} &= \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \left[ \frac{11}{3} \eta_{1112}(\tau) \right] \\
T_{3} \phi^{62} - \hat{\psi}^{6^2} &= \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \left[ \frac{2}{3} \eta_{6^2}(\tau) + \frac{1}{3} \eta_{13-3g^3}(\tau) + 2\eta_{13-3g^3}(2\tau) + \frac{8}{3} \eta_{13-3g^3}(4\tau) \right]
\end{align*}
\]
We do not expect any BKM Lie superalgebra to be associated with these modular forms.

5 Concluding Remarks

The non-uniqueness of our solution for $M_{12}$ moonshine suggests we look for further constraints beyond the ones that we have imposed. It is known that like $M_{24}$, $M_{12}$ has a non-trivial 3-cocycle $[23]$. One has $H_3(M_{12}, \mathbb{Z}) = \mathbb{Z}_8 \oplus \mathbb{Z}_6$. With this in mind we looked to see if whether suitable powers of the 1427 solutions for $\gamma^4$ and $\gamma^8$ are modular forms of $\Gamma_0(4)$ and $\Gamma_0(8)$ respectively. We did not find any solution.

We also found BKM Lie superalgebras associated with almost all conjugacy classes of $L_2(11)_B$ while that was not true for $L_2(11)_A$. This observation might prove useful in better classifying the BKM Lie algebras that are connected with Mathieu and more generally umbral moonshine. We discuss some issues related to this in our forthcoming paper $[21]$.

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A Modular Forms

The genus-one theta functions are defined by

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (\tau, z) = \sum_{l \in \mathbb{Z}} q^{\frac{1}{2} (l+a)^2} r^{(l+a)^2} e^{i\pi lb},$$

(A.1)

where $a, b \in (0, 1) \mod 2$. We define $\theta_1 (\tau, z) \equiv \theta \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] (\tau, z)$, $\theta_2 (\tau, z) \equiv \theta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] (z_1, z)$, $\theta_3 (\tau, z) \equiv \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\tau, z)$ and $\theta_4 (\tau, z) \equiv \theta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] (\tau, z)$.

The characters of the level 1 $\mathcal{N} = 4$ superconformal algebra that appear in our decomposition of the Jacobi forms of weight zero index 1 are defined by

$$\mathcal{C}(\tau, z) = -\frac{\theta_1 (\tau, z)^2}{\eta(\tau)^3} \frac{i}{\theta_1 (\tau, 2z)} \sum_{n \in \mathbb{Z}} q^{2n^2} r^{4n} \frac{1 + q^n r}{1 - q^n r},$$

(A.2)

$$\mathcal{B}(\tau, z) = -\frac{\theta_1 (\tau, z)^2}{\eta(\tau)^3}.$$ 

(A.3)

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A.1 Weight two modular forms

The Eisenstein series at level \(N > 1\) and weight 2 is defined as follows:

\[ E_2^{(N)}(\tau) := \frac{12i}{\pi(N-1)} \partial_\tau \left[ \ln \eta(\tau) - \ln \eta(N\tau) \right] = 1 + \frac{24}{N-1} q + \cdots. \]

Note that \(\frac{N-1}{24} E_2^{(N)}(\tau)\) has integral coefficients except for the constant term. Let \(f(\tau)\) denote the following weight two modular form of \(\Gamma_0(16)\):

\[ f(\tau) := \frac{1}{4} (\eta_{488-4}(\tau) - \eta_{1224-2}(\tau)) = q - 4q^3 + 6q^5 - 8q^7 + \cdots \quad (A.4) \]

An alternate formula for \(f(\tau)\) is as a generalised Eisenstein series

\[ f(\tau) = E_{2,\chi,\chi}(\tau) := \sum_{m=0}^{\infty} \left[ \sum_{n|m} \chi(n) \chi(\frac{m}{n}) m \right] q^m, \]

where \(\chi(m) = (\frac{-4}{m})\) is a real Dirichlet character modulo 4. A basis for five-dimensional space of weight two modular forms of \(\Gamma_0(16)\) is given by\(^6\)

\[ E_2^{(2)}(\tau), E_2^{(4)}(\tau), E_2^{(8)}(\tau), E_2^{(16)}(\tau) \text{ and } f(\tau). \quad (A.5) \]

The first five Fourier coefficients of any weight two modular form of \(\Gamma_0(16)\) uniquely determine the modular form. A basis for weight two modular forms of \(\Gamma_0(32)\) is obtained by adding three more weight two modular forms:

\[ E_2^{(32)}(\tau), f(2\tau) \text{ and the cusp form } \eta_{488^2}(\tau), \]

to the \(\Gamma_0(16)\) basis. Eight of the first nine Fourier coefficients of any weight two modular form of \(\Gamma_0(32)\) uniquely determines the modular form.

A.2 The EOT Jacobi Forms for \(M_{12}:2\)

Let \(\phi_{0,1}(\tau, z)\) and \(\phi_{-2,1}(\tau, z)\) denote the following Jacobi forms:

\[ \phi_{0,1}(\tau, z) = 4 \left[ \frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2} \right], \]

\[ \phi_{-2,1}(\tau, z) = -\frac{\theta_1(\tau, z)^2}{\eta(\tau)^6}. \]

These are the unique weak Jacobi forms of index 1 and weight \(\leq 0\). They generate the ring of weak Jacobi forms freely over the space of modular forms [26]. Thus, all weak Jacobi forms of index 1 and weight zero can be given in terms of a constant and a modular form of weight 2 at suitable level.

\(^6\)We have used SAGE to obtain the dimension of the spaces of modular forms [25]. SAGE also provides a basis for the modular forms and we have verified that our choices agree with the choices given there.
Table 2: The EOT Jacobi forms as given in [7–9] for all conjugacy classes of $M_{24}$ that reduce to conjugacy classes of $M_{12}:2$.

### B Basic Group Theory

#### B.1 $M_{12}$ and $M_{12}:2$

In the 12-dimensional permutation representation, $M_{12}$ is generated as $\langle \alpha, \beta, \gamma, \delta \rangle$. $L_2(11)_A$ is a maximal subgroup of $M_{12}$ while one has $L_2(11)_B \subset M_{11} \subset M_{12}$. The four conjugacy classes associated with elements of order 4, 8 and 10 do not reduce to conjugacy classes of either $L_2(11)_A$ or $L_2(11)_B$.

Let $g$ denote an element of $M_{12}$ in the 12-dimensional permutation representation. Let $\varphi$ denote the outer automorphism of $M_{12}$. It acts on the generators of $M_{12}$ as

$$
\alpha^\varphi = \varphi \alpha \varphi^{-1} = \alpha^{-1} , \quad \beta^\varphi = \beta , \quad \gamma^\varphi = \gamma^{-1} , \quad \delta^\varphi = \delta .
$$

The 24-dimensional permutation representation of $M_{12}:2$ consists of two classes
of elements given in block-diagonal form below:
\[
(g, e) := \begin{pmatrix}
g & 0 \\
0 & \varphi(g)
\end{pmatrix}
\quad \text{and} \quad
(g, \varphi) := \begin{pmatrix}
0 & g \\
\varphi(g) & 0
\end{pmatrix} .
\]

Conjugacy classes, \(\tilde{\rho}\) of type \((g, e)\) of \(M_{12}:2\) descend to pairs of conjugacy classes of \(M_{12}\). Explicitly, one has \(\tilde{\rho} = (\tilde{\rho}, \varphi(\tilde{\rho}))\), where \(\tilde{\rho}\) is the conjugacy class of \(g\) in \(M_{12}\). One has the sequence of groups
\[
L_2(11)_{A/B} \subset M_{12} \xrightarrow{\varphi} M_{12}:2 \subset M_{24} ,
\]

**B.1.1 \(M_{12}:2\) characters from \(M_{12}\) characters**

Given a group \(G\) and a \(\mathbb{Z}_2\) automorphism \(\varphi\), the characters of the group \(G\) are related to those of the group \(G.2\) in two possible ways [27]:

1. **The splitting case:** A character \(\hat{\chi}_m\) of \(G\) may give rise to two characters of \(G.2\) – call them \(\hat{\chi}_a\) and \(\hat{\chi}_{a'}\). For elements of type \((g, e)\), they are given by \(\hat{\chi}_a = \hat{\chi}_{a'} = \hat{\chi}_m\). For elements of type \((g, \varphi)\), one has \(\hat{\chi}_a + \hat{\chi}_{a'} = 0\). Thus we have a natural pairing \((a, a')\) of representations of \(G.2\) and they are mapped to the \(m\)-th representation of \(G\). For \(M_{12}:2\), the splitting representations are (in the notation \(m \leftrightarrow (a, a')\))
\[
1 \leftrightarrow (1,2) , 6 \leftrightarrow (5,6) , 7 \leftrightarrow (7,8) , 8 \leftrightarrow (9,10) ,
11 + a \leftrightarrow (12 + 2a, 13 + 2a) \text{ for } a = 0, 1, \ldots, 4 . \tag{B.1}
\]

2. **The fusion case:** Two characters \(\hat{\chi}_m\) and \(\hat{\chi}_n\) fuse to give a single character, call it \(\tilde{\chi}_{m,n}\). For elements of \(G.2\) of type \((g, e)\), one has \(\hat{\chi}_a = \hat{\chi}_m + \hat{\chi}_n\) and for elements of type \((g, \varphi)\), one has \(\hat{\chi}_a = 0\). The fusion characters of \(M_{12}:2\) are (in the notation \((m, n) \leftrightarrow a\))
\[
(2, 3) \leftrightarrow 3 , (4, 5) \leftrightarrow 4 , (9, 10) \leftrightarrow 11 . \tag{B.2}
\]

**B.2 Character Tables**

Character table for \(L_2(11)\) obtained from the GAP database [28]

|   | 1a | 2a | 3a | 5a | 5b | 6a | 11a | 11b |
|---|----|----|----|----|----|----|-----|-----|
| \(\chi_1\) | 1   | 1  | 1  | 1  | 1  | 1  | 1   | 1   |
| \(\chi_2\) | 5   | 1  | -1 | 0  | 0  | 1  | -1/2 + \(\sqrt{5}/2\) | -1/2 - \(\sqrt{5}/2\) |
| \(\chi_3\) | 5   | 1  | -1 | 0  | 0  | 1  | -1/2 - \(\sqrt{5}/2\) | -1/2 + \(\sqrt{5}/2\) |
| \(\chi_4\) | 10  | -2 | 1  | 0  | 0  | 1  | -1   | -1   |
| \(\chi_5\) | 10  | 2  | 1  | 0  | 0  | -1 | -1   | -1   |
| \(\chi_6\) | 11  | -1 | -1 | 1  | 1  | -1 | 0   | 0   |
| \(\chi_7\) | 12  | 0  | 0  | -1/2 + \(\sqrt{5}/2\) | -1/2 - \(\sqrt{5}/2\) | 0  | 1   | 1   |
| \(\chi_8\) | 12  | 0  | 0  | -1/2 - \(\sqrt{5}/2\) | -1/2 + \(\sqrt{5}/2\) | 0  | 1   | 1   |

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where \( \alpha = -\frac{1}{2} + i \frac{\sqrt{3}}{2} \). Under the outer automorphism, \( \varphi \), of \( M_{12} \) one has
\[
\varphi : \quad \hat{\chi}_2 \leftrightarrow \hat{\chi}_3 \quad , \quad \hat{\chi}_4 \leftrightarrow \hat{\chi}_5 \quad , \quad \hat{\chi}_9 \leftrightarrow \hat{\chi}_{10} .
\]

The character table for \( M_{12} \): 2 (obtained from the GAP character table database)

\[
\begin{pmatrix}
\text{Label} & 1a & 2a & 3a & 3b & 4a & 4b & 5a & 6a & 6b & 8a & 8b & 10a & 11a & 11b \\
\hat{\chi}_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hat{\chi}_2 & 11 & -1 & 3 & 2 & -1 & -1 & 3 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 \\
\hat{\chi}_3 & 11' & -1 & 3 & 2 & -1 & -1 & 3 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 \\
\hat{\chi}_4 & 16 & 4 & 0 & -2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -\alpha & \alpha^* \\
\hat{\chi}_5 & 16' & 4 & 0 & -2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -\alpha^* & \alpha \\
\hat{\chi}_6 & 45 & 5 & -3 & 0 & 3 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \ \\
\hat{\chi}_7 & 54 & 6 & 6 & 0 & 0 & 2 & 2 & -1 & 0 & 0 & 0 & 1 & -1 & -1 \\
\hat{\chi}_8 & 55' & 5 & -5 & 1 & 1 & 3 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
\hat{\chi}_9 & 55' & -5 & -1 & 1 & 1 & -1 & 3 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
\hat{\chi}_{10} & 66 & 6 & 2 & 3 & 0 & -2 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\
\hat{\chi}_{11} & 66 & 6 & 2 & 3 & 0 & -2 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\
\hat{\chi}_{12} & 99 & -1 & 3 & 0 & 3 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
\hat{\chi}_{13} & 120 & 0 & -8 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\
\hat{\chi}_{14} & 144 & 4 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 \\
\hat{\chi}_{15} & 176 & -4 & 0 & -4 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

The character table for \( M_{12} \): 2 (obtained from the GAP character database)

\[
\begin{pmatrix}
\text{Label} & 1a & 2a & 3a & 3b & 4a & 4b & 5a & 6a & 6b & 8a & 8b & 10a & 11a & 11b \\
\hat{\chi}_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hat{\chi}_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hat{\chi}_3 & 22 & -2 & 6 & 4 & -2 & 2 & 2 & -2 & 0 & -2 & 0 & 0 & 0 & 0 \\
\hat{\chi}_4 & 32 & 8 & 0 & -4 & 2 & 0 & 2 & 2 & 0 & 0 & 2 & -2 & 0 & 0 \\
\hat{\chi}_5 & 45 & 5 & -3 & 0 & 3 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -5 & 3 \\
\hat{\chi}_6 & 45 & 5 & -3 & 0 & 3 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -5 & 3 \\
\hat{\chi}_7 & 54 & 6 & 6 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\hat{\chi}_8 & 54 & 6 & 6 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\hat{\chi}_9 & 55 & -5 & 7 & 1 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 5 & 1 \\
\hat{\chi}_{10} & 55 & -5 & 7 & 1 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 5 & 1 \\
\hat{\chi}_{11} & 110 & -10 & -2 & 2 & 2 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{\chi}_{12} & 66 & 6 & 2 & 3 & 0 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & 2 & 0 \\
\hat{\chi}_{13} & 66 & 6 & 2 & 3 & 0 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & 2 & 0 \\
\hat{\chi}_{14} & 99 & -1 & 3 & 0 & 3 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\
\hat{\chi}_{15} & 99 & -1 & 3 & 0 & 3 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\
\hat{\chi}_{16} & 120 & 0 & -8 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hat{\chi}_{17} & 120 & 0 & -8 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hat{\chi}_{18} & 144 & 4 & 0 & 0 & 0 & -3 & 0 & -1 & 1 & 0 & 0 & 0 & 4 & 2 \\
\hat{\chi}_{19} & 144 & 4 & 0 & 0 & 0 & -3 & 0 & -1 & 1 & 0 & 0 & 0 & 4 & 2 \\
\hat{\chi}_{20} & 176 & -4 & 0 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
\hat{\chi}_{21} & 176 & -4 & 0 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
\end{pmatrix}
\]

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