AKCOGLU’S DILATION THEORY IN $L_1$-SPACES

OLIVIA MAH

Abstract. We provide an outline for the proof of Akcoglu’s dilation theory in $L_1$ [1] which is used by Gustafson et al. [9] to investigate the embedding of a probabilistic process into a larger deterministic dynamical system.

1. Introduction

The motivation for studying dilations in operator theory was to gain insight into the properties of an operator on a smaller space by using the properties of the operator on a larger space.

Over the years, researchers have applied results from dilation theory to mathematical physics. One example is the study conducted by Gustafson et al. in investigating the embedding of a probability process into a larger deterministic dynamical system [8, 9, 1]. The physical significance of their research is related to the work on irreversible processes by Proggigne, a Nobel Prize chemist [18].

In their work [8, 9], Gustafson et al. used two dilation theories: one from Rokhlin [7, 23, 21], the other from Akcoglu [1, 2, 3, 4]. A general result was proved in [2] using Akcoglu’s dilation theorem.

With deep and interesting results, Akcoglu’s dilation theorem was first proved in $L_1$ and subsequently generalized to $L_p$. The original proof in Akcoglu’s dilation theorem was rather complicated and simpler versions of the proof were presented [14, 19]. Nonetheless, it may still be worthwhile for those who are new to the field to go through Akcoglu’s proof. With this in mind, the purpose of this article is to provide an outline for the proof in $L_1$ [1], delineating the interplay of several fundamental ideas.

---

1R.J. Reynolds Tobacco Company requested a reprint of [8] shortly after it was published [11].
The organization of this article is as follows. In Section 2, we present the definition of a dilation, the most general result of Gustafson et al. in [9] and Akcoglu’s dilation theorem in $L_1$. In section 3, we outline the main ideas of Akcoglu’s proof. In the last section, we state the open problem discussed in [10] and present a dilation result from Rota [22].

2. Preliminaries

We first introduce the definition of a dilation. An operator $B$ on a Hilbert space $K$ is said to be a “dilation” of an operator $A$ on a Hilbert space $H$ provided $H$ is a subspace of $K$ and for all positive integers $n$,

$$A^n = PB^n|H,$$

where $P$ is the orthogonal projection of $K$ onto $H$ with range $H$. (The original definition introduced by Paul Halmos was a weaker form than the above as it did not have the power $n$ [12]).

A landmark theorem on dilations was proven by Sz-Nagy, which states that very contraction has a power dilation that is unitary [24]. For a historical development on dilation theory, see [6, 13, 15].

The dilation theories used to determine the types of Markov semigroup which could be embedded in a deterministic dynamical system without approximations in [8, 9] include that of Rokhlin and Akcoglu. Roughly speaking, Sz-Nagy’s dilation theory can be viewed as dilating Hilbert spaces, Rokhlin’s as dilating the underlying dynamical systems and Akcoglu’s the measure spaces.

Using Akcoglu’s dilation theorem, Gustafson et al. proved that an arbitrary Markov semigroup satisfying certain basic properties can be dilated into a deterministic dynamical system.

**Theorem 1.** Every Markov semigroup $M_t$ acting on the Hilbert space $K$ arises as a projection of a dynamical system in a larger Hilbert space $H$.

The Markov semigroup family $\{M_t\}_{t \geq 0}$ is defined in [9] as a semigroup of Markov operators, which are the positive integral-preserving contractions $T$ in Akcoglu’s Theorem (see Theorem 2 below). Under certain assumptions,
a Markov operator is induced by transition probabilities, which in turn, determines a stochastic process called the Markov process (see Proposition V.4.4 in [20] and [5]).

In the next section, we state Akcoglu’s dilation theorem and outline its proof.

3. AKCOGLU’S THEOREM

Akcoglu’s dilation result (see Theorem 2 below) was first proved in $L_1$ [1] and then extended to $L_p$ [2, 3, 4]. Since the ideas in the $L_p$ proof are similar to those in $L_1$, for the rest of the paper, we outline the $L_1$ proof in [1].

Here is the dilation result in $L_1$: [1]

**Theorem 2.** Let $(X, \mathcal{F}, \mu)$ be a Borel Space and let $T$ be a positive contraction on $L_1(X, \mathcal{F}, \mu)$. Then there exists another Borel Space $(Y, \mathcal{G}, \nu)$ and a non-singular invertible transformation $\tau : Y \to Y$ so that the positive isometry $Q$ on $L_1(Y, \mathcal{G}, \nu)$ by $\tau$ is a dilation of $T$.

The setup of the proof is as follows:

Let $J = [0, 1]$, $\mathcal{B}$ be the $\sigma$-algebra of the Borel subsets of $J$ and $\mu$ is the (finite) measure on $(J, \mathcal{B})$. The cartesian product of finitely or countably many copies of $(J, \mathcal{B})$ is denoted by:

$$(J, \mathcal{B})^n = (J^n, \mathcal{B}^n) \quad \text{and} \quad (J, \mathcal{B})^\infty = (J^\infty, \mathcal{B}^\infty).$$

Also, let $(J_i, \mathcal{B}_i)$ be the copies of $(J, \mathcal{B})$.

Let $f \in L_1(J^\infty, \mathcal{B}^\infty, \mu^\infty)$ be a function depending only on $x_0$ and consider $f$ being a member of $L_1(J, \mathcal{B}, \mu)$ also. Then Theorem 2 was proved by showing that $EQ^n f = T^n f$ where

- $T$ is a positive contraction on $L_1(J_0, \mathcal{B}_0, \mu)$,
- $Q$ is the positive contraction on $L_1(J^\infty, \mathcal{B}^\infty, \mu^\infty)$, and
- $E$ is the conditional expectation operator with respect to $\mathcal{B}_0$ i.e. $E : L_1(J_\infty, \mathcal{B}_\infty, \mu_\infty) \to L_1(J, \mathcal{B}_0, \mu)$.

The two main questions that the proof has to address are:

1. From the original measure space $X$, how do we obtain a larger space $Y$?
(2) How do we obtain the dilation \( Q \) from the larger space \( Y \)?

A short answer to the first question is to "dilate" the measure space \( X \) via conditioned measures (see Section 4.1.3). As for the second question, once we obtain a larger space \( Y \), we can then construct a non-singular invertible transformation \( \tau \), which in turn invokes a Frobenius-Perron operator serving as the dilation of \( T \) (see Section 4.5). We will give a more detailed discussion in the next section.

4. Outline of Proof

4.1. Definitions.

4.1.1. Positive Contractions. A linear operator \( T : L_1(S, S, \vartheta) \to L_1(S, S, \vartheta) \) is called a positive operator if \( Tf \geq 0 \) for every \( f \geq 0 \) in \( L_1(S, S, \vartheta) \). It is a contraction if its norm is less than 1, that is, \( ||T|| \leq 1 \).

4.1.2. Transporting Measures. Suppose that \((S, S, \vartheta)\) is a finite measure space. Let \( \rho \) be a measurable map from \((S, S)\) to another measurable space \((S', S')\). If \( \rho \) is invertible, that is, it has a measurable inverse, then the set function \( \varsigma \) on \( S' \) defined by

\[
\varsigma(A') = \vartheta(\rho^{-1}(A')), \quad A' \in S',
\]

is a measure in \((S', S')\) and is called the transport of the measure \( \vartheta \) via \( \rho \) i.e. \( \rho \) transports measure \( \vartheta \) to \( \varsigma \). (If \( \vartheta \) is a probability measure and \((S', S') = (\mathbb{R}^n, \mathbb{R}^n)\), then \( \varsigma \) is called the distribution of \( \rho \).)

4.1.3. Conditioned Measures. Let \((S, S)\) and \((\Xi, M)\) be two measurable spaces. A family of normalized measures \( \{\eta\} \) on \((\Xi, M)\) is said to be conditioned by \((S, S)\) if this family is indexed by \( s \in S \) and denoted by \( \{\eta\}_S \). Let \((Z, \mathcal{Z}) = (S, S) \times (\Xi, M)\). With the conditioned family \( \{\eta\}_S \) on \((\Xi, M)\) and a measure \( \vartheta \) on \((S, S)\), there exists a measure \( \varpi = \vartheta \times \{\eta\} \) on \((Z, \mathcal{Z})\) such that

\[
\varpi(A \times M) = \int_A \eta(M, s) \vartheta(ds),
\]

for each \( A \in S \) and \( M \in M \).
With $\varpi = \varphi \times \{\eta\}$, we can define the conditional expectation operator $E : L_1(Z, \mathcal{S}, \varpi) \to L_1(Z, \mathcal{S}, \varpi)$ with respect to $\mathcal{S}$ as:

$$(E f)(s) = \int_Y f(s, y) \eta(dy, s), \quad f \in L_1(Z, \mathcal{S}, \varpi), \ s \in S.$$

4.1.4. **Equivalence.** A mapping between two measurable spaces is called an equivalence if it is measurable and invertible in both directions.

4.2. **Obtaining a Larger Space.** Conditioned families of measures were investigated by both Rokhlin [21] and Maharam [17]. The Rokhlin-Maharam theorem, stated in the following form in [1] and [3], provides us with tools to dilate spaces.

**Theorem 3** (Rokhlin-Maharam Theorem). Let $(\Omega, \Sigma, \sigma)$ be a Borel space and let $f : \Omega \to J$ be a measurable function transporting $\sigma$ to a measure $\nu$ on $(J, \mathcal{B})$. Then there exists an isomorphism $\Phi : J^2 \to \Omega$ between $(J^2, \mathcal{B}^2, \nu \times \{\eta\})$ and $(\Omega, \Sigma, \sigma)$, for some choice of the conditioned family $\{\eta\}_J$, so that $f \Phi : J^2 \to J$ is the projection of $J^2$ to its first component $J$, $\nu \times \eta$-a.e. (If $(\Omega, \Sigma)$ is equivalent to $(J, \mathcal{B})$, then $\Phi$ can be chosen as an equivalence.)

The Rokhlin-Maharam Theorem shows how we can embed $J$ (the first component) in $J^2$ and then recover it from $J^2$ via an isomorphism $\Phi$ and a conditioned family $\{\eta\}_J$:

\[
\begin{array}{ccc}
\Omega & \xrightarrow{f} & J \\
\downarrow{\Phi} & & \downarrow{f \circ \Phi} \\
J \times J & & \\
\end{array}
\]

We will briefly show how the Rokhlin-Maharam Theorem is used in the following theorem, which is key to proving Akcoglu’s dilation result in Theorem [2].

**Theorem 4.** Let $T$ be an integral-preserving positive contraction on $L_1(J_0, \mathcal{B}_0, \mu)$. Then there exists a conditioned family $\{\alpha\} = \{\alpha\}_{J_0}$ on $(J_{-1}, \mathcal{B}_{-1})$ and an equivalence $\varphi : J_{-1} \times J_0 \to J_0 \times J_1$ so that $\varphi$ transports $\{\alpha\} \times \mu$ on $J_{-1} \times J_0$.
to $\nu \times \lambda$ on $J_0 \times J_{-1}$ where $d\nu = (T1)d\mu$, so that

\[(Tf)(x_0) = (T1)(x_0) \int_{J_1} f(\varphi_0^{-1}(x_0, x_1)) \, dx_1 \quad (4.1)\]

for each $f \in L_1(J_0, \mathcal{B}_0, \mu)$ and for $\mu$-a.a. $x_0 \in J_0$. Here the integration is with respect to the standard Borel measure $\lambda$ on $(J_1, \mathcal{B}_1)$ and $\varphi_0^{-1}(x_0, x_1)$ denotes the $J_0$-coordinate of $\varphi^{-1}(x_0, x_1) \in J_{-1} \times J_0$, where $(x_0, x_1) \in J_0 \times J_1$.

Here is how the Rokhlin-Maharam Theorem is used. We first start with a measurable function $g : J_0 \times J_1 \to J$. Then apply Theorem 3 to $g$ and obtain an equivalence $\varphi : J_{-1} \times J_0 \to J_0 \times J_1$ on $(J_{-1}, \mathcal{B}_{-1})$ so that $g \varphi : J_{-1} \times J_0 \to J$ is the identification of $J_0$-component in $J_{-1} \times J_0$ via the conditioned family $\{\alpha\}_{J_0}$ on $(J_{-1}, \mathcal{B}_{-1})$ as illustrated below:

4.3. Integral Preserving. Note that the positive contraction $T$ in Theorem 4 is integral-preserving, that is, $\int f \, d\mu = \int Tf \, d\mu$ for all $f \in L_1$. With the following theorem [1], in proving Akcoglu’s dilation result, we can reduce the case of a general positive contraction to that of an integral preserving positive contraction.

**Theorem 5.** Every positive contraction on the $L_1$ space of a Borel space has a dilation to an operator of the same type which is also integral preserving.

4.4. Construct $\tau$. Recall our ultimate goal is to obtain a dilation defined on $L_1(J_{-\infty}^\infty, \mathcal{B}_{-\infty}^\infty, \mu_{-\infty}^\infty)$. To do that, we first need to use $\varphi$ in Theorem 4 to construct an equivalence $\tau : J_{-\infty}^\infty \to J_{-\infty}^\infty$ on the measurable space $(J_{-\infty}^\infty, \mathcal{B}_{-\infty}^\infty)$ as follows: Recall that $\varphi$ is defined as $\varphi : J_{-1} \times J_0 \to J_0 \times J_1$. For each $x \in J_{-\infty}^\infty$, we define $\tau$ as:

$\tau_i x = x_{i-1}$, \quad if $i \neq 0, i \neq 1$,

$\tau_0 x = \varphi_0(x_{-1}, x_0)$,

$\tau_1 x = \varphi_1(x_{-1}, x_0)$.
Then via the conditioned measures \( \{\alpha_n\} \in (\mathfrak{B}_{-n}) \), it can be shown that \( \tau \) transports the measure

\[
\mu^\infty_{\infty} = \cdots \times \{\alpha_2\} \times \{\alpha_1\} \times \mu \times \lambda \times \lambda \times \cdots
\]
on \((J_{-\infty}, \mathfrak{B}_{-\infty}, \mu^\infty_{\infty})\) to

\[
\nu^\infty_{\infty} = \cdots \times \{\alpha_2\} \times \{\alpha_1\} \times \nu \times \lambda \times \lambda \times \cdots
\]
on the same space. It follows then that

\[
\frac{d\nu^\infty_{\infty}}{d\mu^\infty_{\infty}}(\cdots, x_{-1}, x_0, x_1, \cdots) = \frac{d\nu}{d\mu}(x_0) = (T1)(x_0),
\] (4.2)

which is a key result to use in obtaining a dilation \( Q \) as shown in the next section.

4.5. **Obtaining a Dilation.** So far, we still have not shown how an integral-preserving positive contraction \( T \) can invoke a dilation operator \( Q \). The key to that issue lies with the transformation \( \tau \): as a nonsingular transformation, \( \tau \) induces a positive contraction called the Frobenius-Perron operator, which becomes the dilation operator \( Q \).

Here is the definition of a Frobenius-Perron operator.

4.5.1. **Frobenius-Perron Operator.** If \( h : X \to X \) is a nonsingular transformation, then \( h \) induces a positive contraction \( Q \) of \( L_1(X, A, \mu) \) defined uniquely by:

\[
\int_{h^{-1}A} f \, d\mu = \int_A Qf \, d\mu
\] (4.3)

for each \( f \in L_1(X, A, \mu) \) and for each \( A \in A \) and we call \( Q \) the Frobenius-Perron operator.

If \( h \) is invertible and we let \( \nu(A) = \mu(h^{-1}A) \), that is, \( h \) transports measure \( \mu \) to \( \nu \), then

\[
(Qf)(x) = \frac{d\nu}{d\mu}(x)f(h^{-1}x).
\] (4.4)

(See Lasota (1994), Ch 1 & 3 in [16].)
4.5.2. $\tau$ induces $Q$. Recall that $\tau$ is defined on $(J_{-\infty}^{\infty}, \mathfrak{B}_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$. Since $\tau$ is invertible and non-singular (because $\nu_{-\infty}^{\infty}$ is absolutely continuous with respect to $\mu_{-\infty}^{\infty}$), it invokes a positive contraction $Q$ of $L_1(J_{-\infty}^{\infty}, \mathfrak{B}_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$ as in (4.3). Then by (4.3) and (4.2), we obtain

$$
(Qf)(\cdots, x_{-1}, x_0, x_1, \cdots) = \frac{d\nu_{-\infty}^{\infty}}{d\mu_{-\infty}^{\infty}}(\cdots, x_{-1}, x_0, x_1, \cdots) f(\tau^{-1}(\cdots, x_{-1}, x_0, x_1, \cdots)) = (T1)(x_0) f(\tau^{-1}(\cdots, x_{-1}, x_0, x_1, \cdots)),
$$

(4.5)

which links the dilation $Q$ with the original operator $T$. Then it can be shown that $EQf = EQEf$ and eventually $EQ^n f = T^n f$.

5. Open Question

An open question was raised in [10] as to whether the semigroup property of $M_t$ in Theorem 1 can be relaxed to some wider stochastic structures which are of more martingale-type or which permit memory effects.

While the open problem still remains unsolved, we state without proof an interesting result from Rota relating dilations and reverse martingales [22].

Here is Rota’s dilation theorem.

Theorem 6. Let $P$ be a linear positive contraction in $L_2(S, \sum, \mu)$ which is self-adjoint and maps the constant function of value 1 to 1 i.e. $P1 = 1$. Then

(1) there is a dilation of the sequence of operators $P^{2n}$ into a (reversed) martingale $E_n$ and

(2) for $f$ in $L_p(S, F, \mu)$, $p > 1$, $\lim_{n \to \infty} P^{2n} f$ exists almost everywhere.

Here, the dilation is in the sense that $P^{2n} = \hat{E} \circ E_n$ where $E_n$ are conditional expectations of a decreasing filtration and $\hat{E}$ is a conditional expectation projecting onto $L_2(S, \sum, \mu)$.

References

[1] M. Akcoglu, Positive Contractions on $L_1$-spaces, Mathematische Zeitschrift, 143 (1975), 5-13.

[2] M. Akcoglu, On Convergence of Iterates of Positive Contractions in $L_p$ Spaces, Journal of Approximation Theory, 13 (1975), 348-362.
[3] M. Akcoglu and L. Sucheston, *On Positive Dilations to Isometries in \( L_p \)-spaces*, Lecture Notes in Mathematics, **541** (1976), Springer, Berlin, 389-401.

[4] M. Akcoglu and P.E. Kopp, *Construction of Dilations of Positive \( L_p \)-Contractions*, Mathematische Zeitschrift, **155** (1977), 119-127.

[5] I. Antoniou and Z. Suchanecki, *Time Operators Associated to Dilations of Markov Processes*, Progress in Nonlinear Differential Equations and Their Applications, **55** (1997), 13-23.

[6] J.S. Byrnes, *Twentieth Century Harmonic Analysis*, Springer (2001).

[7] I. Cornfeld, S. Formin and Ya. B. Sinai, *Ergodic Theory*, Springer (1982).

[8] I. Antoniou and K. Gustafson, *From Probabilistic Description to Deterministic Dynamics*, Physica A, **197** (1993), 153-166.

[9] I. Antoniou and K. Gustafson, *From Irreversible Markov Semigroups to Chaotic Dynamics*, Physica A, **236** (1997), 296-308.

[10] K. Gustafson, *Lectures on Computational Fluid Dynamics, Mathematical Physics, and Linear Algebra*, World Scientific (1997).

[11] K. Gustafson, *Deterministic and Indeterministic Descriptions*, in H. Atmanspacher and R. Bishop (eds.), *Between Chance and Choice: Interdisciplinary Perspectives on Determinism*. Imprint Academic (2002), pp. 115-148.

[12] P. Halmos, *Normal Dilations and Extensions of Operator*, Summa Brasilienis Math, **2** (1950), 125-134.

[13] P.R. Halmos, J. Ewing and F.W. Gehring, *Paul Halmos: Celebrating 50 years of Mathematics*, Springer (1991).

[14] M. Kern, R. Nagel and G. Palm, *Dilations of Positive Operators: Construction and Ergodic Theory*, Mathematische Zeitschrift, **156** (1977), 265-277.

[15] H. Landau, *Moments in Mathematics*, American Mathematical Society (1987).

[16] A. Lasota and M.C. Mackey, *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*, 2nd ed., Springer-Verlag (1994).

[17] D. Maharam, *Decompositions of Measure Algebras and Spaces*, Transactions of the American Mathematica, **69** (1950), 142-160.

[18] B. Misra, I. Prigogine and M. Courbage, *From Deterministic Dynamics to Probabilistic Descriptions*, Physica A, **98** (1979), 1-26.

[19] R. Nagel and G. Palm, *Lattice Dilations of Positive Contractions on \( L^p \)-Spaces*, Canadian Mathematical Bulletin, **25** (1982), 371-374.

[20] J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day (1965).

[21] V. Roklin, *On the Fundamental Ideas of Measure Theory*, American Mathematical Society Translation, **71** (1952), 1-54.
[22] GC. Rota, An “Alternierende Verfahren” for General Positive Operators, Bulletin of the American Mathematical Society, 68 (1962), 95-102.

[23] Ya. B. Sinai, ed., Dynamical Systems II: Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics, Springer (1989).

[24] B. S-Nagy, C. Foias, H. Bercovici, and L. Kérchy Harmonic Analysis of Operators in Hilbert Space, 2nd ed., Springer (2002).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SAN FRANCISCO, SAN FRANCISCO, CA 94117

E-mail address: ommah@usfca.edu