Duality and Stability in Complex Multiagent State-Dependent Network Dynamics

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Abstract

Many of the current challenges in science and engineering are related to complex networks and distributed multiagent network systems are currently the focal point of many new applications. Such applications relate to the growing popularity of social networks, the analysis of large network data sets, and the problems that arise from interactions among agents in complex political, economic, and biological systems. Despite extensive progress for stability analysis of conventional multiagent networked systems with weakly coupled state-network dynamics, most of the existing results have shortcomings to address multiagent systems with highly coupled state-network dynamics. Motivated by numerous applications of such dynamics, in our previous work [1], we initiated a new direction for stability analysis of such systems using a sequential optimization framework. Building upon that, in this paper we complete our results by providing another angle to multiagent network dynamics from a duality perspective which allows us to view the network structure as dual variables of a constrained convex program. Leveraging this idea, we show that the evolution of the coupled state-network multiagent dynamics can be viewed as iterates of a primal-dual algorithm to a static constrained optimization/saddle-point problem. This bridges the Lyapunov stability of state-dependent network dynamics and frequently used optimization techniques such as block coordinate descent, mirror descent, Newton method, and subgradient method. As a result, we develop a systematic framework to analyze the Lyapunov stability of state-dependent network dynamics using well-known techniques from nonlinear optimization.

Index Terms

Lyapunov stability; multiagent systems; state-dependent network dynamics; saddle-point dynamics; block coordinate descent; Newton method; subgradient method; convex optimization.

I. INTRODUCTION

Many of the current challenges in science and engineering are related to complex networks. These challenges may involve modeling the interactions of agents in complex networks, the establishment of stability in the agents’ interaction dynamics, and the design of efficient algorithms to obtain or approximate the equilibrium points. We can offer many motivating examples of relationships in political, social, and engineering applications that are governed by complex networks of heterogeneous agents. Agents may be strategic or the networks can be dynamic in the sense that they can vary over time depending on the agents’ states or decisions. The following are just a few examples that one can consider.

– Network security: A basic task in network security is that of providing a mechanism for securing the operation of a set of networked heterogeneous agents (e.g., service providers, computers, or data centers) despite external malicious attacks (Figure 1). One way of doing that is to incentivize the agents to invest in their security (e.g., by installing antivirus software) [2], [3]. However, since the agents are interconnected, the compromise of one agent may affect its neighbors, and such a failure can cascade over the entire network. As a result, the decision made by each agent on how much to invest in its own security level will indirectly affect all the others, and hence the connectivity structure of the network. Thus we face a highly dynamic network of heterogeneous agents where the agents’ states/decisions and the network structure are highly influenced by each other.
Fig. 1. Compromising an agent changes the network structure and hence the security states/decisions of all others.

Fig. 2. Aircraft must keep a certain formation while the communication network among them is subject to change.

Fig. 3. The social network affects the opinions which in turn creates new friendships, and hence a new social network.

– **Formation control**: A goal in formation control is to design a distributed protocol such that a set of agents (e.g., the aircraft in Figure 2) collectively form a certain structure and eventually accomplish a task [4]. Agents may have different communication capabilities and can only communicate with those in their local neighborhoods. Consequently, depending on the agents’ states (e.g., remaining power or relative positions), the communication network they share is subject to change. As a result, the agents’ states and the communication network are highly coupled and dynamically evolve based on each other.

– **Social networks**: In social networks, there are often clear affinities among people based on heterogeneous political or cultural beliefs that define an interaction network among them. However, on specific issues, alliances form among people from different groups. Almost every congressional vote provides an example of this phenomenon, wherein some representatives break away from their respective parties to vote with the other party [5] (Figure 3).

– **Stability of smart grids**: In the emerging smart grid, a significant amount of energy stems from renewable sources, electric vehicles, and storage units, many of which may be owned by consumers rather than utility companies. That phenomenon is turning every grid component into a prosumer: a joint producer and consumer of energy. Prosumers (agents) in the smart grid strategically interact with each other subject to power network constraints [6]. In particular, depending on their states (e.g., energy consumption/production decisions) they may decide to buy/sell energy to different agents. Thus, a major challenge is that of providing decentralized algorithms to stabilize the demand and response given that the structure of the agents’ interactions is a function of their own and their neighbors’ states/decisions.

Motivated by the above, and many other real applications, our objective in this paper is to provide a systematic approach to analyze stability and convergence of agents interacting over a rich dynamic network which may evolve or vary based on agents’ states. To this end, we provide new connections between analysis of multiagent network systems and developed techniques in the mature field of nonlinear programming. Utilizing such connections, we show how Lyapunov stability of seemingly complex multiagent network dynamics can be analyzed using iterative optimization algorithms for finding a minimum or saddle-point of nonlinear functions.

A. Related Works

A general multiagent network problem involves a set of $[n] = \{1, 2, \ldots, n\}$ agents (social individuals, grid prosumers, unmanned vehicles, etc). At each time instance $k = 0, 1, 2, \ldots$, there is an underlying network $\mathcal{G}_k = ([n], \mathcal{E}_k)$ that determines the communication network shared by the agents. Here, $\mathcal{E}_k$ denotes the set of edges of the network at time $k$ which can be undirected or directed. The state of each agent $i \in [n]$ at time $k$ is given by a vector $x^k_i$, which evolves based on the interaction of agent $i$ with its neighbors. In particular, the overall state of the system at time $k + 1$, denoted by $x^{k+1}$, can be
obtained by using a general update rule \( x^{k+1} = f_k(x^k, G_k) \), \( k = 0, 1, 2, \ldots \), where \( f_k(\cdot) \) can be a general time-varying function, depending on the problem setup, and captures the interaction laws among the agents. Therefore, the main goal here is to understand whether the generated sequence of states \( \{x^k\}_{k=0}^\infty \) will converge (stabilize) to any equilibrium. That has been the subject of much research effort, including work in distributed control and computation [7]. Unfortunately, despite enormous efforts in the area, the stability problem for such dynamics in its full generality is still far from having been solved. However, partial solutions to this problem under certain simplifying assumptions are known. For instance, there has been a rich body of literature on the analysis of multiagent network systems, mainly from the static point of view, in which a set of agents iteratively interact over a fixed network to achieve a certain goal, such as consensus or optimization of an objective function. The classical models of DeGroot [8] and Friedkin-Johnsen [9] in social sciences are two special types of such systems [7], [10]. Below is a sample result in this area [11], [12]:

**Theorem 1:** Given a fixed and undirected connected network \( G = ([n], E) \), at any time \( k = 0, 1, \ldots \), let every agent \( i \) take the average of its own state and those of its neighbors, \( x_i^{k+1} = \sum_{j \in N_i} a_{ij} x_j^k, \forall i \in [n] \), where \( a_{ij} > 0 \) are positive constant weights such that \( \sum_{j \in N_i} a_{ij} = 1, \forall i \). Then agents’ states will converge to a consensus point, i.e., \( \lim_{k \to \infty} x_i^k = x^*, \forall i \). Further, the convergence rate is exponential, i.e., \( \|x^k - x^*\| \leq \lambda^k \|x^0 - x^*\| \), where \( 1 \) is the vector of all ones and \( \lambda \in (0, 1) \).

By comparing this result with the aforementioned general dynamics, one can identify several simplifying assumptions that have been made in most of the existing results on multiagent network systems. Particularly: i) the underlying networks are fixed as \( G_k = G, \forall k \); ii) the underlying networks are connected and undirected, and iii) the underlying networks \( G_k \) do not depend on the agents’ states \( x^k \). To relax these simplifying assumptions, a large body of literature has thus been developed to establish the stability of the above general dynamics under less restrictive situations. The results in the static case can often be generalized to time-varying networks by assuming a certain “independency” between the network process and the state dynamics. For instance, one of the commonly used assumptions is that the network dynamics are governed by an exogenous process that is uncoupled from the state dynamics [13]–[20]. Here is an extension which is given in [21]:

**Theorem 2:** Consider a sequence of time-varying directed graphs \( G_k = ([n], E_k) \) with the weight of edge \((i, j)\) at time \( k \) being \( a_{ij}(k) \). Assume that the sequence of graphs is \( B \)-strongly connected, meaning that for any \( k \geq 0 \), the graph \( G = ([n], \bigcup_{k=0}^\infty E_k) \) is strongly connected. Moreover, given a positive constant \( \alpha \in (0, 1) \), assume \( a_{ij}(k) \in [\alpha, 1] \cup \{0\}, \forall i, j, k, \) and \( a_{ii}(k) \geq \alpha, \sum_j a_{ij}(k) = 1, \forall i, k \). Then the dynamics \( x_i^{k+1} = \sum_{j \in N_i} a_{ij}(k) x_j^k, i \in [n] \), will asymptotically converge to a consensus point.

While Theorem 2 relaxes the static communication network to time-varying networks, it still has shortcomings in addressing many realistic multiagent systems. For instance, the network connectivity must be preserved over any time window of length \( B \), and it is hard to check whether that is happening (especially if the networks are generated endogenously based on the agents’ states). Secondly, the assumptions on the weight matrices are somewhat restrictive, as in real situations the weights can approach 0 and then increase again to 1. Besides, the theorem uses an implicit assumption on the symmetry of the networks by imposing strong connectivity. Finally, in realistic situations the evolution of the network itself depends on the evolution of agents’ states, while in the above theorem, the network dynamics are driven by an exogenous process that is independent of how the states evolve. A generalization of Theorem 2 is to allow weak coupling between the state and network dynamics, with certain network connectivity/symmetry assumptions [11], [22], [23]. We refer to [24], [25] for other extensions of such results using backward product of stochastic matrices.

While the existing results can properly address a large class of multiagent network systems, there are still many examples that do not fit into any of the aforementioned categories or for which the application of the above techniques provides poor results on the behavior of the agents. Our work is fundamentally
different from the earlier literature in a sense that departing from conventional methods for stability analysis of multiagent averaging dynamics (e.g., Markov chains or product of stochastic matrices), we provide a deeper analysis than the existing model-free results by capturing the internal co-evolution of state and network dynamics. This approach allows us to relax some of the common assumptions such as global knowledge on the network connectivity over the course of the dynamics. It is worth mentioning that our work is also related to dynamic clustering where the goal is to provide a theoretical justification for cluster synchronization in multiagent systems using saddle-point dynamics [26], [27]. However, the network structure in that application is fixed and captured by a set of linear constraints, while in our work the network dynamically evolves as a complex function of the state variables.

B. Contributions and Organization

Inspired by the above shortcomings and building upon our previous work [1], in this paper we provide a principled framework from an optimization perspective to study Lyapunov stability of multiagent state-dependent network dynamics. We show that despite the challenges due to state-network coupling, it is still possible to capture the co-evolution of network and state dynamics for a broad class of multiagent systems, even under an asymmetric or nonlinear environment. More precisely, we show that often the network structure among the agents can be viewed as dual variables of a constrained optimization problem where the existence of an edge is related to the tightness of the corresponding constraint. As a result, we can view multiagent network dynamics as an iterative primal-dual algorithm to a static constrained optimization problem where the primal updates correspond to state updates of the dynamics and the dual updates correspond to the network evolution. The KKT optimality conditions also guide the coupling between the network and state dynamics. This allows us to view the constrained Lagrangian of the underlying static problem as a Lyapunov or “semi-Lyapunov” function for the multiagent dynamics. Therefore, we obtain a principled way to establish the stability of multiagent network dynamics in terms of asymptotic convergence of an iterative optimization algorithm. This makes a variety of iterative optimization methods amenable to study the stability of multiagent state-dependent network dynamics.

In Section II, we first provide our problem formulation, modeling a large class of state-dependent network dynamics. In Section III we apply a sequential optimization framework based on block coordinate descent method to establish Lyapunov stability for a large class of state-dependent network dynamics. We consider this method under both symmetric and asymmetric network structure and use the change of variables to generate other classes of state-dependent network dynamics. In Section IV we use a saddle-point model to extend our results to a case where there is a conflict between the network structure and the state evolution. In Section V we consider continuous-time dynamics where the edge emergence between agents is no longer a binary event, but rather a continuous weight process. We conclude the paper by identifying some future directions of research in Section VI.

II. Problem Formulation

Let us consider a multiagent network system consisting of \([n] := \{1, 2, \ldots, n\}\) agents. At any given time \(k = 0, 1, 2, \ldots\), we denote the state of agent \(i\) by \(x_i^k \in \mathbb{R}\), and the state of the entire system at that time by \(x^k = (x_1^k, \ldots, x_n^k)^T\), where the superscript \(T\) refers to the transpose of a vector\(^1\). Moreover, we assume that each agent \(i \in [n]\) has \(n - 1\) measurement functions \(g_{ij}(x_i, x_j) : \mathbb{R}^2 \to \mathbb{R}\), one for every other agent \(j \in [n] \setminus \{i\}\), which is assumed to be a convex function of its own state \(x_i\) and agent \(j\)’s state \(x_j\). For most parts in this paper we assume that the measurement functions \(g_{ij}(x_i, x_j), i \neq j\) are twice continuously differentiable, denoted by \(g_{ij} \in C^2\).

Given any state \(x\), we assume that the set of neighbors of an agent \(i \in [n]\) is determined by the logic constraints \(g_{ij}(x_i, x_j) \leq 0, j \in [n] \setminus \{i\}\). In other words, for a given state \(x\), agent \(i\) is influenced

\(^1\)For simplicity of presentation, we assume that the agents’ states are scalar real numbers. However, all the results can be naturally extended to the case where agents’ states are vectors in \(\mathbb{R}^d\).
by agent \( j \) (or \( j \) is a neighbor of \( i \)) if and only if \( g_{ij}(x_i, x_j) \leq 0 \). In particular, we denote the set of neighbors of agent \( i \) at a state \( x \) by \( N_i(x) := \{ j : g_{ij}(x_i, x_j) \leq 0 \} \). At any time instance \( k \), each agent \( i \in [n] \) interacts with its neighbors and updates its state at the next time step to

\[
x_i^{k+1} = \phi_i(x^k, N_i(x^k)), \quad i \in [n],
\]

where \( \phi_i(\cdot) \) is an agent-specific update rule which is a function of states of agent \( i \)'s neighbors at time \( k \). Note that the above discrete-time update dynamics contain a fairly large class of state-dependent network dynamics where here the network at time \( k \) is given by \( G_k = ([n], \{(i, j) : j \in N_i(x^k)\}) \). It is evident that the network structure at time \( k \) depends on the agents’ states at that time, and the state at the next time step \( k+1 \) is a function of the network structure at the current time \( k \). Therefore, our main objective in this paper is to provide a general class of update rules \( \phi_i(\cdot) \) such that the state-dependent network dynamics \( \Pi \) converge to some equilibrium point or are Lyapunov stable in the following sense:

**Definition 1:** A function \( V : \mathbb{R}^n \to \mathbb{R} \) is called a Lyapunov function for the discrete time dynamical system \( z^{k+1} = h_k(z^k), \ k = 0, 1, 2, \ldots \), if it is decreasing along the trajectories of the dynamics, i.e., \( V(z^{k+1}) < V(z^k), \forall k \). We refer to a dynamical system which admits a Lyapunov function as Lyapunov stable.

To illustrate the generality of the above model, let us consider the well-known Hegselmann-Krause (HK) model from social science [5]. In the HK model, there is a set of \([n]\) agents, and it is assumed that at each time instance \( k = 0, 1, 2, \ldots \), the opinion (state) of agent \( i \in [n] \) can be represented by a scalar \( x_i^k \in \mathbb{R} \). Each agent \( i \) updates its state at time \( k+1 \) by taking the arithmetic average of its own state and those of all the others that are in its \( \epsilon \)-neighborhood at time \( k \), i.e.,

\[
x_i^{k+1} = \frac{x_i^k + \sum_{j \in N_i(x^k)} x_j^k}{1 + |N_i(x^k)|}, \quad i \in [n].
\]

Here \( \epsilon > 0 \) is a constant parameter, and \( N_i(x^k) = \{ j \in [n] \setminus \{ i \} : |x_i^k - x_j^k| \leq \epsilon \} \) denotes the set of neighbors of agent \( i \) at time \( k \). In fact, it is known that such dynamics are Lyapunov stable and converge to an equilibrium point [5], [28]. Now it is easy to see that HK dynamics are a very special case of the state-dependent network dynamics \( \Pi \), in which the measurement functions are given by \( g_{ij}(x_i, x_j) = (x_i - x_j)^2 - \epsilon^2 \), and the update rule is given by \( \phi_i(x, N_i(x)) := \frac{x_i + \sum_{j \in N_i(x)} x_j}{1 + |N_i(x)|} \).

### III. Lyapunov Stability Using Block Coordinate Descent

A popular approach to solving optimization problems is the so-called block coordinate descent (BCD) method, which is also known as the Gauss-Seidel method. At each iteration of this method, the objective function is minimized with respect to a single block of variables while the rest of the blocks are held fixed. More specifically, consider the optimization problem: \( \min \{ F(y_1, \ldots, y_n) : y_i \in Y_i, \forall i \} \), where \( Y_i \subseteq \mathbb{R}^{n_i} \) is a closed convex set, and \( F : \prod_{i=1}^n Y_i \to \mathbb{R} \) is a continuous function. At iteration \( t = 0, 1, \ldots \) of the BCD method, the block variable \( y_i \) is updated by solving the subproblem: \( y_i^t = \arg \min_{z_i \in Y_i} F(y_1^t, \ldots, y_{i-1}^t, z_i, y_{i+1}^t, \ldots, y_n^t), \ i \in [n] \). Since in practice finding the exact minimum in each iteration might be difficult, one can consider an inexact BCD method, where a smooth regularizer is added to the objective function or it is approximated by a simpler convex function. In either case, and under some mild assumptions, it can be shown that the inexact BCD method will converge to a stationary point of the objective function \( F(\cdot) \).

Now let us consider the following constrained convex program:

\[
\min f(x) := \sum_{i=1}^n f_i(x_i)
\]

s.t. \( g_{ij}(x_i, x_j) \leq 0, \ \forall i \neq j, \ x \in \mathbb{R}^n \),
where \( f_i(x_i), i \in [n] \) are continuous convex functions and \( g_{ij}(x_i, x_j), i \neq j \) are the measurement convex functions between each pair of the agents. In other words, each agent \( i \in [n] \) has a private convex function \( f_i(x_i) \), and the agents collectively want to choose their states to minimize the global objective function

\[
L(x) := \sum_{i=1}^{n} f_i(x_i),
\]

which is a function consisting of two block variables, namely a state block variable \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \), and a nonnegative network block variable \( \lambda := (\lambda_{ij}, i \neq j) \). Now assume that we want to minimize the Lagrangian function using BCD method subject to the box constraints \( \lambda_{ij} \in [0, 1], \forall i, j \). As \( L(x, \lambda) \) is a linear function of \( \lambda \), fixing the state block variable and minimizing \( L(x, \lambda) \) with respect to \( \lambda \in [0, 1]^{n(n-1)} \), we get\( \lambda_{ij} = 1 \) if \( g_{ij}(x_i, x_j) \leq 0 \) (i.e., there is a directed edge from agent \( i \) to agent \( j \)), and \( \lambda_{ij} = 0 \) if \( g_{ij}(x_i, x_j) > 0 \) (i.e., no such an edge exists). In other words, fixing the state variable and minimizing the Lagrangian with respect to \( \lambda \in [0, 1]^{n(n-1)} \), the dual variables precisely capture the network structure among the agents for that state. Motivated by this observation, we have the following theorem.

**Definition 2:** The measurement functions \( g_{ij}(\cdot) \) are called symmetric if for all \( i \neq j \) we have \( g_{ij}(x_i, x_j) = g_{ji}(x_j, x_i) \). Note that for symmetric measurement functions, the communication network among the agents is always an undirected graph.

**Theorem 3:** Let \( g_{ij} \in \mathbb{C}^2 \) be symmetric convex and \( f_i(x_i) \in \mathbb{C}^2 \) be strictly convex functions such that their second order partial derivatives are bounded above by \( m \). Assume that two agents \( i \) and \( j \) become each others’ neighbors if \( g_{ij}(x_i, x_j) \leq 0 \). Then the following state-dependent network dynamics

\[
x^{k+1}_i = x^k_i - \frac{\partial f_i(x^k_i)}{\partial x_i} + \sum_{j \in N_i(x^k_i)} \frac{\partial g_{ij}(x^k_i, x^k_j)}{\partial x_i}, \quad i \in [n]
\]

will converge. In particular, \( V(x) := \sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \min\{g_{ij}(x_i, x_j), 0\} \) serves as a Lyapunov function for dynamics (2) such that \( V(x^{k+1}) \leq V(x^k) - m\|x^k - x^{k+1}\|^2 \).

**Proof:** Let us consider the following Lagrangian function

\[
L(x, \lambda) = \sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \lambda_{ij} g_{ij}(x_i, x_j),
\]

and consider the BCD method applied to this function when \( \lambda \in [0, 1]^{n(n-1)} \) and \( x \in \mathbb{R}^n \). Fixing the state variable to \( x^k \), and setting \( \lambda^k := \arg\min_{\lambda \in [0, 1]^{n(n-1)}} L(x^k, \lambda) \), it is easy to see that \( \lambda^k \) precisely captures the network structure among the agents at the current state \( x^k \). Next let us fix the network variable to \( \lambda^k \), and consider

\[
L_k(x) := L(x, \lambda^k) = \sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \lambda_{ij}^k g_{ij}(x_i, x_j) = \sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \sum_{j \in N_i(x^k)} g_{ij}(x_i, x_j),
\]

which is a strictly convex function. Ideally, we want to set the state at the next time step \( x^{k+1} \) to the unique minimizer of \( L_k(x) \). However, since solving the minimization problem \( \min_{x \in \mathbb{R}^n} L_k(x) \) exactly might be difficult, we use an inexact BCD method, where instead a quadratic upper approximation of this function is minimized. More precisely, consider the quadratic approximation of \( L_k(x) \) at the current point \( x^k \):

\[
L_k(x) \sim L(x^k) + (x - x^k)^T \nabla L_k(x^k) + \frac{1}{2}(x - x^k)^T \nabla^2 L_k(x^k)(x - x^k),
\]

\(^2\)For instance, any \( m \)-smooth function (a function with \( m \)-Lipschitz gradient) has this property.
where $\nabla L_k(x^k)$ is the gradient of $L_k(x)$ at $x^k$, whose $i$th component is given by $[\nabla L_k(x^k)]_i = \frac{\partial}{\partial x_i} f_i(x_i) + \sum_{j \in N_i(x^k)} \frac{\partial^2 g_{ij}(x^k, x^k)}{\partial x_i \partial x_j}$. Moreover, $\nabla^2 L_k(x^k)$ is the Hessian of $L_k(x)$ at $x^k$, where the Hessian matrix function of $L_k(x)$ is given by

$$[\nabla^2 L_k(x)]_{ij} = \begin{cases} \frac{\partial^2 g_{ij}(x^k, x^k)}{\partial x_i \partial x_j} & \text{if } j = i \\ \frac{\partial g_{ij}(x^k, x^k)}{\partial x_i} & \text{if } j \in N_i(x^k) \\ 0 & \text{if } j \notin N_i(x^k). \end{cases}$$

Now by the assumption $|\frac{\partial g_{ij}(x^k, x^k)}{\partial x_i} | \leq m$, $|\frac{\partial^2 f_i(x^k)}{\partial x_i^2} | \leq m$, $\forall i, j$, and using Gershgorin Circle Theorem one can see that for any $x$, the Hessian $\nabla^2 L_k(x)$ is dominated by the diagonal matrix $Q_k := 2m \cdot \text{diag}([|N_i(x^k)| + 1, \ldots, |N_n(x^k)| + 1])$. Using the Tailor expansion, there exists an $\zeta \in \mathbb{R}^n$ such that,

$$L_k(x) = L(x^k) + (x - x^k)^T \nabla L_k(x^k) + \frac{1}{2} (x - x^k)^T \nabla^2 L_k(\zeta)(x - x^k) \leq L(x^k) + (x - x^k)^T \nabla L_k(x^k) + \frac{1}{2} (x - x^k)^T Q_k(x - x^k) := u_k(x).$$

Therefore, $u_k(x)$ is a quadratic upper approximation for $L_k(x)$ for any $x$. Letting

$$x^{k+1} = \arg\min_{x \in \mathbb{R}^n} u_k(x) = x^k - Q_k^{-1} \nabla L_k(x^k), \quad (3)$$

we can write,

$$L_k(x^{k+1}) \leq u_k(x^{k+1}) \leq u_k(x^k) = L_k(x^k).$$

This shows that the state-dependent network dynamics:

$$x_i^{k+1} = x_i^k - \frac{1}{2m} \cdot \frac{\partial^2 f_i(x^k) + \sum_{j \in N_i(x^k)} \frac{\partial g_{ij}(x^k, x^k)}{\partial x_i} |N_i(x^k)| + 1}{\min_{i,j} g_{ij}(x^k, x^k)},$$

are Lyapunov stable ($L(\cdot)$ decreases regardless of the state or network updates), and

$$V(x) := \min_{\lambda \in [0,1]^{n(n-1)}} L(x, \lambda) = \sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \min\{g_{ij}(x^k), 0\}$$

serves as a Lyapunov function for them. In particular, denoting the $Q_k$-norm of a vector $v$ by $\|v\|_{Q_k}^2 = v^T Q_k v$, the drift of this Lyapunov is lower bounded by,

$$V(x^k) - V(x^{k+1}) = L(x^k, \lambda^k) - L(x^{k+1}, \lambda^{k+1}) \geq L(x^k, \lambda^k) - L(x^{k+1}, \lambda^k)$$

$$= L_k(x^k) - L_k(x^{k+1}) = u_k(x^k) - L_k(x^{k+1})$$

$$\geq u_k(x^k) - u_k(x^{k+1}) = \frac{1}{2} (\nabla L_k(x^k))^T Q_k^{-1} \nabla L_k(x^k)$$

$$= \frac{1}{2} ||Q_k^{-1} \nabla L_k(x^k)||_{Q_k}^2 = \frac{1}{2} ||x^k - x^{k+1}||_{Q_k}^2 \geq m ||x^k - x^{k+1}||^2,$$

where the last equality follows from (3), and the last inequality holds because all the diagonal entries of $Q_k$ are greater than $2m$. Therefore, $V(x^{k+1}) \leq V(x^k) - m ||x^k - x^{k+1}||^2$. As $V(\cdot)$ is lower bounded by a finite value, we get $\lim_{k \to \infty} ||x^{k+1} - x^k|| = 0$. This in view of (3) and the fact that diagonal entries of $Q_k^{-1}$ are lower bounded by $\frac{1}{2m}$ also implies $\lim_{k \to \infty} \nabla L_k(x^k) = 0$.

Finally, to show the convergence of the dynamics (2) to an equilibrium point, we note that for every $k$, $L_k(x)$ belongs to the finite family of strictly convex functions $\mathcal{H} := \{\sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \lambda_{ij} g_{ij}(x_i, x_j) : \lambda_{ij} \in \{0, 1\}, \forall i, j\}$, containing at most $2^{O(n^2)}$ functions. This is because $L_k(x) = L(x, \lambda^k)$, where $\lambda^k$ is the solution of the linear program $\min_{\lambda \in [0,1]^{n(n-1)}} L(x^k, \lambda)$, and must be an extreme point of
Now given any $h(x) \in \mathcal{H}$, let $h_1 < h_2 < \ldots$, be all the indices $k$ for which $L_k(x) = h(x)$. Then we can partition the sequence $\{x^k\}$ into at most $|\mathcal{H}|$ subsequences $\{x^h\}_{h \in \mathcal{H}}$. Since $\lim_{k \to \infty} \nabla L_k(x^k) = 0$, this means that for any subsequence $\{x^{h^r}\}_{r \geq 1}$ we have, $\lim_{k \to \infty} \nabla h(x^{h^r}) = \lim_{k \to \infty} \nabla L_{h^r}(x^{h^r}) = 0$. As $h(\cdot)$ is a strictly convex function, this means that the subsequence $\{x^{h^r}\}_{r \geq 1}$ must converge to the unique minimizer of $h(\cdot)$, denoted by $x_h$. Since there are a finite number of such subsequences, for any $\epsilon > 0$, there exists $K_\epsilon$ such that $\|x^{h^r} - x_h\| < \epsilon$, $\forall h \in \mathcal{H}$, $\ell > K_\epsilon$. Let $\mathcal{X} = \{x_h = \text{argmin} \, h(x) : h \in \mathcal{H}\}$ be the finite set of minimizers of all the functions in $\mathcal{H}$, and choose $\epsilon := \frac{1}{3} \min_{x_p \neq x_q} \|x_p - x_q\|$. Then for $\ell > K_\epsilon$, each subsequence $\{x^{h^r}\}_{r \geq 1}$ lies in an $\epsilon$-neighborhood of its limit point $x_h$, and moreover, there is no jump of the iterates between two distinct $\epsilon$-neighborhood (otherwise, $\|x^{k+1} - x^k\| > \frac{\epsilon}{2}$ for some $k$, contradicting the fact that $\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0$). This shows that for $\ell > K_\epsilon$, all the subsequences $\{x^{h^r}\}_{r \geq 1}$ must lie in the same $\epsilon$-neighborhood, and hence the sequence $\{x^k\}$ converge to a limit point in $x^* \in \mathcal{X}$. \hfill \blacksquare

Example 1: Consider a special case where $g_{ij}(x_i, x_j) = \frac{(x_i - x_j)^2}{2} - \epsilon_{ij}$ with $\epsilon_{ij} = \epsilon_{ji}$ (thus agents $i$ and $j$ communicate if and only if their Euclidean distance is at most $\sqrt{\epsilon_{ij}}$), and $f_i(x_i) = \frac{x_i^2}{2}$. It is easy to see that $\frac{\partial g_{ij}(x_i, x_j)}{\partial x_i, x_j} = \frac{\partial^2 f_i(x_i)}{\partial x_i^2} = 1$, $\forall i, j$, so choosing $m = 1$ will satisfy the assumption of Theorem 3. This shows that the dynamics:

$$x_{i}^{k+1} = x_{i}^{k} - \frac{\nabla_x \psi(x^k)}{2} + \frac{\sum_{j \in N_i(x^k)} (x_{i}^{k} - x_{j}^{k})}{|N_i(x^k)| + 1} = \frac{1}{2} x_{i}^{k} + \frac{1}{2} \sum_{j \in N_i(x^k)} x_{j}^{k}$$

will converge to an equilibrium point. Note that these dynamics are the lazy version of the HK dynamics where each agent is more suborn about its own opinion. Moreover, we know that the equilibrium point $x^*$ must be the minimizer of a strictly convex function of the form $h^*(x) := \sum_i \frac{x_i^2}{2} + \sum_{i,j} \lambda^*_{ij} (\frac{(x_i - x_j)^2}{2} - \frac{\epsilon_{ij}}{2})$ for some fixed undirected network adjacency matrix $\Lambda^* \in \{0, 1\}^{n^2}$. Thus, $x^*$ is a solution to $\nabla h^*(x) = 0$, or equivalently $(I + \mathcal{L}^*) x = 0$, where $\mathcal{L}^*$ is the Laplacian matrix associated with $\Lambda^*$. As $I + \mathcal{L}^*$ is positive definite, we must have $x^* = 0$.

In the proof of Theorem 3 we restricted our attention to quadratic upper approximations. However, motivated by the mirror descent algorithm in convex optimization [29], we can use any smooth convex mirror map $\Psi : \mathbb{R}^n \to \mathbb{R}$ to construct an upper approximation for $L_k(x)$ at the point $x^k$. Doing this, we obtain alternative state-dependent network dynamics whose Lyapunov stability and convergence can be established using a similar fashion as in Theorem 3. More precisely, given a smooth and strictly convex function $\Psi$, let $D_\Psi(x, x^k) := \psi(x) - \psi(x^k) - \nabla \psi(x^k)^T (x - x^k)$ be the Bregman divergence with respect to $\Psi$. Then as long as $\nabla^2 \psi(x) \geq 2m \mathcal{I}$, $u_k(x) := L_k(x^k) + \frac{x - x^k}{2} \nabla L_k(x^k) + D_\Psi(x, x^k)$ serves as a convex upper approximation for $L_k(x)$. Therefore, updating the state at the next time step to $x_{i}^{k+1} = \argmin_{x \in \mathbb{R}^n} u_k(x)$, or equivalently to the solution of

$$\nabla \psi(x^{k+1}) = \nabla \psi(x^k) - \nabla L_k(x^k),$$

will guarantee the decrease of the Lyapunov $V(x) = \sum_i f_i(x_i) + \frac{1}{2} \sum_{i,j} \min\{g_{ij}(x), 0\}$. For instance, choosing the mirror map to be the negative entropy function, i.e., $\psi(x) := \sum\ln x_i$, and using (4), we obtain the following Lyapunov stable multiplicative dynamics:

$$x_{i}^{k+1} = x_{i}^{k} \cdot \exp\left(-\frac{\partial f_i(x_i)}{\partial x_i} - \sum_{j \in N_i(x^k)} \frac{\partial g_{ij}(x_i^k, x_j^k)}{\partial x_i}\right).$$

A. Asymmetric State-Dependent Network Dynamics

Asymmetric (directed) interconnections among the agents often introduce a major challenge in the analysis of the multiagent network dynamics. Unfortunately, the gradient operator is a “symmetric” operator in a sense that fixing the network variable in the BCD method and updating the state variable
in the negative direction of the gradient will always generate a symmetric class of dynamics (i.e., an agent’s state not only is influenced by its out-neighbors but also by its in-neighbors). However, one way of tackling this issue using sequential optimization is to introduce an independent copy of the state variable while making sure that these two copies remain close to each other. In other words, we capture the asymmetry between the agents by introducing an extra block variable into the BCD method and adding an extra (possibly asymmetric) penalty term to the objective function to enforce the two copies of the state variables remain close to each other. Here, the choice of the penalty function can be very problem-specific, resulting in different asymmetric state-dependent network dynamics. However, one natural choice for the penalty function is the Bregman divergence between the two copies of the state variables, as it is shown in the following theorem.

**Theorem 4:** Let \( g_{ij}(x_i, x_j) \in \mathbb{C}^2 \) be \( L \)-Lipschitz convex and \( f(x) = \sum_{i=1}^n f_i(x_i) \) be a smooth convex function whose second order partial derivatives are all bounded above by \( m \). Let \( \|y - x\|_1 \leq \frac{1}{nL} D_f(y, x), \forall x, y, \) where \( \| \cdot \|_1 \) is the \( l_1 \)-norm and \( D_f(\cdot) \) denotes the Bregman divergence with respect to \( f(\cdot) \), then \( V(x) = \sum_{i,j} \min \{g_{ij}(x_i, x_j), 0\} \) serves as a Lyapunov function for the asymmetric state-dependent network dynamics:

\[
x_{i}^{k+1} = x_{i}^{k} - \frac{\sum_{j \in N_{i}(x^{k})} \frac{\partial}{\partial x_{j}} g_{ij}(x_{i}^{k}, x_{j}^{k})}{m(|N_{i}(x^{k})| + 1)} , \quad i \in [n].
\]

**Proof:** Let \( c_i(x, \lambda_i) := \sum_{j \neq i} \lambda_{ij} g_{ij}(x_i, x_j) \) denote the cost of agent \( i \) with respect to its neighbors, and let \( y \) be an independent copy of the state variable \( x \). Consider the following function with three independent block variables \( \lambda \in [0, 1]^{n(n-1)}, x, y \in \mathbb{R}^n \):

\[
L(y, x, \lambda) := \sum_{i=1}^n c_i(y_i, x_i - \lambda_i, \lambda_i) + D_f(y, x) = \sum_{i=1}^n \sum_{j \neq i} \lambda_{ij} g_{ij}(y_i, x_j) + D_f(y, x),
\]

where \( D_f(y, x) := f(y) - f(x) - (y - x)^T \nabla f(x) \). Note that here we no longer require the symmetry assumption \( g_{ij}(x_i, x_j) = g_{ji}(x_j, x_i) \). The reason for introducing the Bregman distance \( D_f(y, x) \) into the objective function is that ideally we want the two copies of state variables coincide. But instead of adding the hard constraint \( y = x \) into our optimization problem, we relax it by adding a soft penalty term to the objective function. Now let us apply the BCD method to the following minimization:

\[
\min_{\lambda \in [0, 1]^{n(n-1)}} \min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^n} \left \{ \sum_{i=1}^n \sum_{j \neq i} \lambda_{ij} g_{ij}(y_i, x_j) + D_f(x, y) \right \}.
\]

First, assume that both state variables are fixed to \( y = x = x_k \). Then minimizing the objective function \( (6) \) with respect to \( \lambda \in [0, 1]^{n(n-1)} \), we precisely capture the asymmetric network structure \( \lambda^k \) associated with the state \( x_k \) (i.e., \( \lambda_{ij}^k = 1 \) if and only if \( g_{ij}(x_i^k, x_j^k) \leq 0 \)). Next, let us fix \( \lambda = \lambda^k \) and \( x = x^k \), and consider minimizing \( (6) \) with respect to the \( y \) variable. However, to obtain a closed-form for the optimal solution, instead of solving this minimization exactly, we minimize its quadratic upper approximation at the current state \( x^k \), given by

\[
z_k(y) := L(x^k, x^k, \lambda^k) + (y - x^k)^T \nabla_y L(x^k, x^k, \lambda^k) + \frac{1}{2} (y - x^k)^T P_k (y - x^k),
\]

where \( P_k \) is a diagonal matrix whose \( i \)th diagonal entry is \( m(|N_{i}(x^k)| + 1) \). To see why \( L(y, x^k, \lambda^k) \leq z_k(y), \forall y \), we note that,

\[
[\nabla_y L(y, x^k, \lambda^k)]_i = \sum_{j \in N_i(x^k)} \frac{\partial}{\partial y_i} g_{ij}(y_i, x_j^k) + (\frac{\partial}{\partial y_i} f_i(y_i) - \frac{\partial}{\partial x_i} f_i(x_i^k)),
\]
which implies that $\nabla^2 L(y, x^k, \lambda^k)$ is a diagonal matrix with diagonal entries being

$$[\nabla^2 L(y, x^k, \lambda^k)]_{ii} = \frac{\partial^2}{\partial y_i^2} f_i(y_i) + \sum_{j \in N_i(x^k)} \frac{\partial g_{ij}(y_i, x_j^k)}{\partial y_i^2}.$$  

As $|\frac{\partial^2 f_i(y_i)}{\partial y_i^2}| \leq m$ and $|\frac{\partial g_{ij}(y_i, x_j^k)}{\partial y_i^2}| \leq m$, $\forall i$, the Hessian matrix must be dominated by $P_k$, and the result follows from the Taylor expansion. Thus, the optimal solution to $\min_{y \in \mathbb{R}^n} z_k(y)$ is given by $x^k + P_k^{-1} \nabla_y L(x^k, x^k, \lambda^k)$, which is precisely the next state of the dynamics (5). Therefore, updating the block variable $y$ to $x^k$, while fixing the other variables to $\lambda = \lambda^k, x = x^k$, will decrease the objective function as,

$$L(x^{k+1}, x^k, \lambda^k) \leq z_k(x^{k+1}) = \min_{y \in \mathbb{R}^n} z_k(y) \leq z_k(x^k) = L(x^k, x^k, \lambda^k).$$

Finally, let us fix $y = x^k, \lambda = \lambda^k$ (which are the solutions to their corresponding sub-optimizations in the BCD method), and consider $\min_{x \in \mathbb{R}^n} L(x^{k+1}, x, \lambda^k)$. In particular, showing that $L(x^{k+1}, x^{k+1}, \lambda^k) \leq L(x^{k+1}, x^k, \lambda^k)$ will complete the BCD loop and imply that $L(y, x, \lambda)$ is decreasing along the trajectory of the asymmetric dynamics (5). Here is where the role of the penalty term in the objective function comes into play. More precisely, using Lipschitz property and the fact that $D_f(x^{k+1}, x^{k+1}) = 0$, we can write,

$$L(x^{k+1}, x^{k+1}, \lambda^k) - L(x^{k+1}, x^k, \lambda^k) = -D_f(x^{k+1}, x^k) + \sum_{i=1}^{n} \sum_{j \in N_i(x^k)} \left( g_{ij}(x_i^{k+1}, x_j^{k+1}) - g_{ij}(x_i^{k+1}, x_j^k) \right)$$

$$\leq -D_f(x^{k+1}, x^k) + \sum_{i,j} |g_{ij}(x_i^{k+1}, x_j^{k+1}) - g_{ij}(x_i^{k+1}, x_j^k)|$$

$$\leq -D_f(x^{k+1}, x^k) + nL \sum_{j} |x_j^{k+1} - x_j^k|$$

$$= -D_f(x^{k+1}, x^k) + nL||x^{k+1} - x^k|| < 0,$$

where the last inequality is by the assumption on the choice of the Bregman map $f(\cdot)$. This shows that $V(x) := \min_{\lambda \in [0,1]^{n-1}} L(x, x, \lambda) = \sum_{i,j} \min\{g_{ij}(x_i, x_j), 0\}$ is a decreasing function along the trajectories of the asymmetric dynamics (5).

**B. BCD Method with Change of Variables**

In this part, we show how a suitable change of block variables in the BCD method can generate new state-dependent network dynamics, whose Lyapunov stability can be established using the same approach as before. The change of variable can be applied to either the state or the network variable (or a combination of both). However, in this section, we focus on a more interesting case where the change of variable is applied on the network variable, and we only illustrate the idea of a change of variable for the state through the following simple example.

**Example 2:** Let us recall the HK model where the state of agent $i$ at the next time step is updated to

$$x_i^{k+1} = \frac{x_i^k + \sum_{j \in N_i(x^k)} x_j^k}{1 + |N_i(x^k)|}, \quad i \in [n],$$

where the neighborhood set of agent $i$ is given by $N_i(x^k) = \{ j \in [n] \setminus \{ i \} : |x_i^k - x_j^k| \leq \epsilon \}$. It is easy to see that HK dynamics are invariant with respect to translation of all the states by a constant value. Therefore, without loss of generality we may assume that $x_i^k > 0, \forall i, k$. Now let us define a new state variable by setting $y_i := \ln(x_i), i \in [n]$. Applying the HK dynamics on this new logarithmic states, we obtain

$$y_i^{k+1} = \frac{y_i^k + \sum_{j \in N_i(y^k)} y_j^k}{1 + |N_i(y^k)|}, \quad i \in [n] \setminus \{ i \} : |y_i^k - y_j^k| \leq \epsilon,$$

that are Lyapunov stable and converge to an equilibrium. Rewriting these dynamics in terms of $x$ variables,
we get a new class of state-dependent geometric averaging dynamics \( x_i^{k+1} = (\prod_{j \in N_i(x^k)} x_j^k)^{1/|N_i(x^k)|} \),
where \( N_i(x^k) = \{ j \in [n] : e^{-\epsilon} \leq x_i^k / x_j^k \leq e^\epsilon \} \), that are also Lyapunov stable and converge.

Now we turn our attention to the more interesting case of changing the network variables. So far, the dual variable \( \lambda_{ij} \) in the Lagrangian function \( L(x, \lambda) \) was used to capture the existence of an edge from agent \( i \) to agent \( j \). In particular, we saw that restricting \( \lambda_{ij} \) to the unit interval \([0, 1]\), and minimizing \( L(x, \lambda) \) for the network variable in the BCD method would automatically enforce \( \lambda_{ij} \) to take binary values in \([1, 0]\) (hence capturing the switching behavior of the existence of an edge from \( i \) to \( j \)). In particular, fixing the state variable to \( x \), we get a new class of state-dependent geometric averaging dynamics

\[
x_i^{k+1} = x_i^k - \frac{\sum_j f_{ij}(x_i^k, x_j^k) \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k)}{2m \sum_j f_{ij}(x_i^k, x_j^k)}, \quad i \in [n]
\]

are Lyapunov stable with the Lyapunov function \( V(x) = \sum_{i \neq j} \int_0^{g_{ij}(x_i, x_j)} f_{ij}(\lambda) d\lambda \).

**Proof:** Let us consider the BCD method applied to the transformed Lagrangian

\[
\hat{L}(x, \lambda) = \min_{\lambda \in \mathbb{R}^{n(n-1)}} \hat{L}(x, \lambda) = \min_{\lambda \in [0, 1]^{n(n-1)}} L(x, \lambda) = L(x, \lambda^*)
\]

Such a transformation on the network variables has three advantages: i) it removes the box constraints (and hence switching behavior) on the network variables and absorbs them into the structure of the transformation function, ii) the optimal network variable in the BCD method after the change of variable has a simpler form, and iii) by choosing different transfer functions one can obtain different classes of state-dependent network dynamics. The following theorem provides a sample result using the idea of a change of network variables.

**Theorem 5:** Let \( g_{ij}(x_i, x_j) \in \mathbb{C}^2 \) be symmetric convex functions such that their second order partial derivatives are bounded above by \( m \). Moreover, let \( f_{ij}(\lambda) \in \mathbb{C} \) be symmetric and nonnegative decreasing functions. Then the following state-dependent network dynamics

\[
x_i^{k+1} = x_i^k - \sum_j f_{ij}(g_{ij}(x_i^k, x_j^k)) \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k)
\]

are Lyapunov stable with the Lyapunov function \( V(x) = \sum_{i \neq j} \int_0^{g_{ij}(x_i, x_j)} f_{ij}(\lambda) d\lambda \).

**Proof:** Let us consider the BCD method applied to the transformed Lagrangian

\[
\hat{L}(x, \lambda) := \sum_{i \neq j} \left( f_{ij}(\lambda_{ij}) g_{ij}(x_i, x_j) - h_{ij}(\lambda_{ij}) \right),
\]

with \( x \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^{n(n-1)} \), for some real valued functions \( h_{ij}(\lambda_{ij}) \) to be determined later. If for any fixed state \( x \), the function \( \hat{L}(x, \lambda) \) has a unique minimum with respect to \( \lambda \in \mathbb{R}^{n(n-1)} \), we can apply the BCD method and assure that this function decreases due to the network updates. Now let us first fix the state variable to \( x^k \). Assuming differentiability of the functions \( f_{ij}, h_{ij} \), to find \( \arg\min_{\lambda \in \mathbb{R}^{n(n-1)}} \hat{L}(x^k, \lambda) \), we set

\[
\frac{\partial}{\partial \lambda_{ij}} \hat{L}(x^k, \lambda) = f_{ij}'(\lambda_{ij}) g_{ij}(x_i^k, x_j^k) - h_{ij}'(\lambda_{ij}) = 0,
\]

which implies \( h_{ij}''(\lambda_{ij}) f_{ij}'(\lambda_{ij}) = g_{ij}(x_i^k, x_j^k) \). Therefore, if we define \( h_{ij}'(\lambda) := \lambda f_{ij}'(\lambda) \), or equivalently, \( h_{ij}(\lambda) := \int_0^\lambda s f_{ij}(s) ds \), the equation \( (8) \) has a unique solution \( \lambda_{ij}^* = g_{ij}(x_i^k, x_j^k) \). To show that this solution is the
minimizer of $\hat{L}(x^k, \lambda)$, we note that
\[
\frac{\partial}{\partial \lambda_{ij}} \hat{L}(x^k, \lambda) = f'_{ij}(\lambda_{ij})[g_{ij}(x_i^k, x_j^k) - \lambda_{ij}].
\]
Since $f_{ij}(\cdot)$ is a decreasing function, $f'_{ij}(\lambda_{ij}) < 0$. Thus for $\lambda_{ij} \leq g_{ij}(x_i^k, x_j^k)$ the function $\hat{L}(x^k, \lambda)$ is decreasing with respect to $\lambda_{ij}$, and for $\lambda_{ij} \geq g_{ij}(x_i^k, x_j^k)$, it is increasing (note that $\hat{L}(x^k, \lambda)$ is splittable over its $\lambda$-components so we can analyze each of its summands separately). Thus given a fixed state $x^k$, the unique global minimum of $\Phi(x^k, \lambda)$ is obtained at $\lambda^* = g_{ij}(x_i^k, x_j^k)$.

The rest of the proof follows along the same analysis as in Theorem 3 and we only sketch it here. Let us fix the network variable to $\lambda^*$, and consider $\min_{x \in \mathbb{R}^n} \hat{L}(x, \lambda^*)$. To find a minimizer we use an inexact method by using its quadratic upper approximation at $x^k$. The $i$th component of the gradient of $\hat{L}(x, \lambda^*)$ at $x^k$ is given by
\[
[\nabla_x \hat{L}(x^k, \lambda^*)]_i = \sum_j \left( f_{ij}(\lambda^*_i) \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k) + f_{ji}(\lambda^*_i) \frac{\partial}{\partial x_i} g_{ji}(x_j^k, x_i^k) \right) = 2 \sum_j f_{ij}(g_{ij}(x_i^k, x_j^k)) \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k),
\]
where the second equality is by the symmetry of the functions $f_{ij}, g_{ij}$. Similarly, the Hessian matrix is given by
\[
[\nabla^2 \hat{L}(x, \lambda^*)]_{ij} = \begin{cases} 
2 \sum_j f_{ij}(g_{ij}(x_i^k, x_j^k)) \frac{\partial^2}{\partial x_i \partial x_j} g_{ij}(x_i, x_j) & \text{if } j = i \\
2f_{ij}(g_{ij}(x_i^k, x_j^k)) \frac{\partial}{\partial x_i} g_{ij}(x_i, x_j) & \text{if } j \neq i,
\end{cases}
\]
which is dominated by a diagonal matrix with $i$th diagonal entry $4m \sum_j f_{ij}(g_{ij}(x_i^k, x_j^k))$. Therefore, the optimal solution to the quadratic upper approximation of $\hat{L}(x, \lambda^*)$ is given by (7), and we have $\hat{L}(x^{k+1}, \lambda^*) < \hat{L}(x^k, \lambda^*)$. As a result
\[
V(x) = \min_{\lambda \in \mathbb{R}^n} \hat{L}(x, \lambda) = \sum_{i \neq j} \left( f_{ij}(g_{ij}(x_i, x_j))g_{ij}(x_i, x_j) - \int_0^{\lambda f_{ij}(\lambda)} g_{ij}(x_i, x_j) d\lambda \right) = \sum_{i \neq j} \int_0^{g_{ij}(x_i, x_j)} f_{ij}(\lambda)d\lambda,
\]
serves as a Lyapunov function for the dynamics (7), where the last equality is by integration by parts.

Remark 1: In fact, the differentiability of the transfer functions $f_{ij}$ in the above theorem can be further relaxed to any nonnegative decreasing symmetric functions $f_{ij}$. In particular, Theorem 5 is a substantial extension of [30, Corollary 1] (see, also [31]) when $f_{ij}(\lambda) = f(\sqrt{\lambda}), \forall i, j$ and $g_{ij}(x_i, x_j) = (x_i - x_j)^2$.}

IV. Stability Using Discrete-Time Saddle-Point Dynamics

In the previous section, we considered the stability of state-dependent network dynamics when the network structure and agents’ states are aligned with each other. More precisely, in the application of the BCD method on the Lagrangian function $L(x, \lambda)$, we considered a double minimization problem $\min_x \min_\lambda L(x, \lambda)$, which essentially means that the network coordinator (viewed as a network player), breaks/adds the links in favor of the agents’ states (viewed as a state player). This essentially means that there is no conflict between the network and state players as they are both minimizing the same Lagrangian function. But what if the network and state players have conflicting objectives? In that case, we have a 2-player zero-sum game between the network and the state with the payoff function $L(x, \lambda)$, so that the network player aims to maximize it while the state player aims to minimize it, i.e., $\min_x \max_\lambda L(x, \lambda)$. 

To model such a conflicting behavior, as before we assume that each agent $i \in [n]$ holds $n - 1$ convex measurement functions $g_{ij}(x_i, x_j), j \in [n] \setminus \{i\}$. In this section, we only restrict our attention to symmetric measurement functions, however, for asymmetric measurement functions, similar results as in Theorem 4 can be obtained. For a given state $x$, two agents $i$ and $j$ become each others’ neighbors if $g_{ij}(x_i, x_j) \geq 0$ (note that as opposed to the previous section, the side of this logic constraint is now reversed). Intuitively, an edge is formed between two agents $i$ and $j$ if and only if their states are far from each other. Now let us consider the same convex program:

$$
\begin{align*}
\min & f(x) := \sum_{i=1}^{n} f_i(x_i) \\
\text{s.t.} & \quad \frac{1}{2} g_{ij}(x_i, x_j) \leq 0, \ \forall i \neq j, \ x \in \mathbb{R}^n, \\
\end{align*}
$$

where $f_i(x_i), i \in [n]$ are agents’ private convex functions. To solve this problem, one can form the Lagrangian function $L(x, \lambda) = f(x) + \frac{1}{2} \sum_{i,j} \lambda_{ij} g_{ij}(x_i, x_j)$, and solve the saddle-point problem:

$$
\min_{x \in \mathbb{R}^n} \max_{\lambda \geq 0} L(x, \lambda).
$$

Now using KKT optimality conditions, we know that if the constraint $g_{ij}(x_i, x_j) \leq 0$ is satisfied but not tight (i.e., $g_{ij}(x_i, x_j) < 0$), then the corresponding optimal dual variable must be zero, i.e., $\lambda_{ij} = 0$. Viewing the dual variables as network variables, this means that there is no edge between the agents $i$ and $j$. This is consistent with the logical condition of not having an edge between $i$ and $j$ as $g_{ij}(x_i, x_j) < 0$. On the other hand, if the constraint $g_{ij}(x_i, x_j) \leq 0$ is not satisfied (i.e., $g_{ij}(x_i, x_j) > 0$), then one must set the corresponding dual variable to $\lambda_{ij} = \infty$ to maximize $\max_{\lambda \geq 0} L(x, \lambda)$. But if the dual variables are upper bounded by 1, to achieve the maximum value in $\max_{\lambda \in [0,1]} L(x, \lambda)$, we must set $\lambda_{ij}$ to its upper bound, i.e., $\lambda_{ij} = 1$. This is again consistent with the logical condition of having an edge between $i$ and $j$ as $g_{ij}(\bar{x}_i, \bar{x}_j) > 0$. These facts together suggest that the network switches that may occur during the update process of state-dependent network dynamics are merely the KKT optimality conditions that guide the iterates to the optimal solution of (9), assuming that there is a budget constraint on the dual variables. In other words, if the dual constraints were free to be chosen from $[0, \infty)$, then the iterates of the dynamics would converge to the optimal solution of (9).

However, the budget constraints on the dual variables do not allow us to penalize the violated constraints arbitrarily large and enforce them to be feasible. Therefore, the solutions that are obtained from state-network updates may not necessarily generate a feasible solution to (9). Nevertheless, this allows us to view the state-network dynamics as an iterative primal-dual algorithm guided by KKT optimality conditions for solving a saddle-point problem with box constraints on the dual variables. Alternatively, the state-dependent network dynamics can be viewed as Nash dynamics in a zero-sum game between a network player and a state player with budget constraints on the action set of the network player. In the following, we use these observations to develop Lyapunov stable and convergent state-dependent network dynamics using discrete-time saddle-point dynamics.

**Theorem 6:** Let $g_{ij}(x_i, x_j) \in \mathbb{C}$ be symmetric convex and $f_i(x_i) \in \mathbb{C}$ be convex functions. Consider the following dynamics in which agent $i$ updates its state as

$$
\begin{align*}
x_i^{k+1} &= x_i^k - \alpha^k \left[ \frac{\partial}{\partial x_i} f_i(x_i^k) + \sum_{j \in N_i(x_i^k)} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k) \right],
\end{align*}
$$

where $N_i(x_i^k) := \{ j : g_{ij}(x_i^k, x_j^k) > 0 \}$ denotes the set of neighbors of agent $i$ at time $k$. Then for any positive sequence $\alpha^k = \gamma k \left[ \sum_{i} \left( \frac{\partial}{\partial x_i} f_i(x_i^k) + \sum_{j \in N_i(x_i^k)} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k) \right)^2 \right]^{-\frac{1}{2}}$, with $\lim k \gamma_k = 0$ and

3Here each constrained is scaled by $\frac{1}{2}$ without changing the actual feasible set.
4It means finding a solution $(\bar{x}, \bar{\lambda})$ such that $L(\bar{x}, \bar{\lambda}) \leq L(x, \lambda) \leq L(\bar{x}, \bar{\lambda})$, $\forall x \in \mathbb{R}^n, \lambda \geq 0$. 
\[ \sum_k \gamma_k = \infty, \] the dynamics (10) converge to an equilibrium \( x^* \). Moreover, for sufficiently small \( \alpha^k \), \( V(x) = \| x - x^* \|^2 \) serves as a Lyapunov function.

**Proof:** Let us consider the following Lagrangian function

\[
L(x, \lambda) = f(x) + \frac{1}{2} \sum_{i,j} \lambda_{ij} g_{ij}(x_i, x_j),
\]

where \( f(x) = \sum_{i=1}^n f_i(x_i) \), and suppose that we want to solve the following saddle-point problem with box constraints on the dual variables:

\[
\min_{x \in \mathbb{R}^n} \max_{\lambda \in [0,1]^{n(n-1)}} L(x, \lambda) = \min_{x \in \mathbb{R}^n} \max_{\lambda \in [0,1]^{n(n-1)}} \{ f(x) + \frac{1}{2} \sum_{i,j} \lambda_{ij} g_{ij}(x_i, x_j) \} = \min_{x \in \mathbb{R}^n} \{ f(x) + \frac{1}{2} \sum_{i,j} \max\{ g_{ij}(x_i, x_j), 0 \} \}.
\]

Defining \( \Phi(x) := f(x) + \frac{1}{2} \sum_{i,j} \max\{ g_{ij}(x_i, x_j), 0 \} \), and noting that for any \( i, j \), \( \max\{ g_{ij}(x_i, x_j), 0 \} \) is a convex function, one can easily see that \( \Phi(x) \) is a convex function of \( x \). Therefore, applying a subgradient algorithm to the unconstrained convex problem \( \min_{x \in \mathbb{R}^n} \Phi(x) \) with appropriate choice of step sizes \( \alpha^k, k = 1, 2, \ldots \), will converge to a minimizer of \( \Phi(x) \), denoted by \( x^* \). More precisely, let us denote the subgradient of \( \Phi(x) \) at \( x^k \) by \( g^k \). Then, it is known that the discrete time dynamics

\[
x^{k+1} = x^k - \alpha_k g^k, \tag{11}
\]

with diminishing step length \( \alpha^k = \frac{\gamma_k}{\| g^k \|} \) with \( \lim k \gamma_k = 0 \) and \( \sum k \gamma_k = \infty \) will converge to \( x^* \) [32]. Now let \( J = \{(r, s) : g_{rs}(x_r^k, x_s^k) > 0 \} \) and \( \bar{J} = \{(r, s) : g_{rs}(x_r^k, x_s^k) \leq 0 \} \). Then for every \((r, s) \in J \) the function \( \max\{ g_{rs}(x_r, x_s), 0 \} \) has a unique subgradient at \( x^k \), which is \( \nabla g_{rs}(x_r^k, x_s^k) \). Moreover, for every \((r, s) \in \bar{J} \) the minimum of the convex function \( \max\{ g_{rs}(x_r, x_s), 0 \} \) equals 0 which is achieved at \( x^k \). Thus 0 is a subgradient of \( \max\{ g_{rs}(x_r, x_s), 0 \} \) at \( x^k \) for every \((i, j) \in \bar{J} \). Using additivity rule of the subgradient, we conclude that \( g^k = \nabla f(x) + \frac{1}{2} \sum_{(r, s) \in J} \nabla g_{rs}(x_r^k, x_s^k) \) is a subgradient for \( \Phi(\cdot) \) at \( x^k \). In particular, the \( ith \) component of \( g^k \) is given by

\[
g^k_i = \frac{\partial}{\partial x_i} f(x^k) + \frac{1}{2} \sum_j \left( 1_{\{ g_{ij}(x_i^k, x_j^k) > 0 \}} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k) + 1_{\{ g_{ij}(x_i^k, x_j^k) > 0 \}} \frac{\partial}{\partial x_i} g_{ij}(x_j^k, x_i^k) \right)
= \frac{\partial}{\partial x_i} f_i(x^k) + \sum_j \left( 1_{\{ g_{ij}(x_i^k, x_j^k) > 0 \}} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k) \right)
= \frac{\partial}{\partial x_i} f_i(x^k) + \sum_{j \in N_i(x^k)} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k), \tag{12}
\]

where \( 1_{\{ \cdot \}} \) is the indicator function, the second equality holds by symmetry of the functions \( g_{ij}(x_i, x_j) = g_{ji}(x_j, x_i) \), and the last equality is due to the definition of an edge emergence between two nodes \( i \) and \( j \). Substituting (12) into (11) we obtain the desired dynamics (10).

Finally, using the definition of the subgradient we can write,

\[
\| x^{k+1} - x^* \|^2 = \| x^k - x^* - \alpha^k g^k \|^2
= \| x^k - x^* \|^2 + (\alpha^k)^2 \| g^k \|^2 - 2\alpha^k \Phi(\Phi^k - x^*)
\leq \| x^k - x^* \|^2 + (\alpha^k)^2 \| g^k \|^2 - 2\alpha^k (\Phi(\Phi^k) - \Phi(x^*)).
\]

Therefore, for any \( \alpha^k \in [0, \frac{2(\Phi(\Phi^k) - \Phi(x^*))}{\| g^k \|^2}] \), we have \( \| x^{k+1} - x^* \|^2 \leq \| x^k - x^* \|^2 \).
Remark 2: Let \( X \) be the set of minimizers of \( \min_{x \in \mathbb{R}^n} \Phi(x) \), which is a nonempty closed convex set. Moreover, let \( d(x, X) = \|x - \Pi_X(x)\| \) be the minimum distance of the point \( x \) from the set \( X \), where \( \Pi_X(x) \) is the projection of \( x \) on the set \( X \). As for any \( x^* \in X \), \( \|x_k - x^*\|^2 \leq \|x_k - x^*\|^2 \), by choosing \( x^* = \Pi_X(x_k) \) we have,
\[
\begin{align*}
    d(x^{k+1}, X) &= \|x^{k+1} - \Pi_X(x^{k+1})\| \leq \|x^{k+1} - \Pi_X(x^k)\| \leq \|x^k - \Pi_X(x^k)\| = d(x^k, X).
\end{align*}
\]
Thus for sufficiently small step size \( \alpha \in [0, \frac{2(\Phi(x^k) - \Phi(x^*))}{\|g^k\|^2}, \|g^k\|^2] \), the distance of the iterates (10) to the optimal set \( X \) also serves as a Lyapunov function.

Example 3: An interesting special case of Theorem [6] is when \( f_i(x_i) = 0, \forall i \in [n] \), and the set of constraints \( \{g_{ij}(x_i, x_j) \leq 0, \forall i, j \} \) is feasible. In this case the set of minimizers of the function \( \Phi(x) = \frac{1}{2} \sum_{i,j} \max \{g_{ij}(x_i, x_j), 0\} \) is precisely the feasible set \( \{x \in \mathbb{R}^n : g_{ij}(x_i, x_j) \leq 0, \forall i, j \} \). In particular, the minimum value of \( \Phi(x) \) is zero which is obtained at any feasible point \( x^* \in \{x \in \mathbb{R}^n : g_{ij}(x_i, x_j) \leq 0, \forall i, j \} \). Now if the norm of the gradient of each measurement function \( g_{ij} \) is bounded above by a constant \( G \), we can write,
\[
\begin{align*}
    \frac{2(\Phi(x^k) - \Phi(x^*))}{\|g^k\|^2} &= \frac{2\Phi(x^k)}{\|g^k\|^2} \leq \frac{\sum_i \sum_{j \in \mathcal{N}_i(x^k)} g_{ij}(x^k, x_j^k)}{\|g^k\|^2} \leq \frac{\sum_i \sum_{j \in \mathcal{N}_i(x^k)} g_{ij}(x^k, x_j^k)^2}{n \sum_{i,j} \left( \frac{\partial}{\partial x_{ij}} g_{ij}(x_i^k, x_j^k) \right)^2} \geq \max_{i,j} \left\{ g_{ij}(x_i^k, x_j^k) \right\} \\
    &\geq \frac{\sum_i \sum_{j \in \mathcal{N}_i(x^k)} g_{ij}(x^k, x_j^k)^2}{n^2 G^2} \geq \frac{\max_{i,j} \left\{ g_{ij}(x_i^k, x_j^k) \right\}}{n^2 G^2}.
\end{align*}
\]
Let us define the \( \epsilon \)-equilibrium set as the set of all the points where each constraint \( g_{ij}(x_i, x_j) \leq 0 \) is violated by at most \( \epsilon \), i.e., \( X_\epsilon = \{x \in \mathbb{R}^n : g_{ij}(x_i, x_j) \leq \epsilon, \forall i, j \} \), and consider the dynamics (10) with the constant step size \( \alpha_k = \frac{\epsilon}{n^2 G^2} \). Then if \( x^k \notin X_\epsilon \), we have \( \max_{i,j} \left\{ g_{ij}(x_i^k, x_j^k) \right\} > \epsilon \), which in view of (13) implies that \( \alpha_k \in [0, \frac{2(\Phi(x^k) - \Phi(x^*))}{\|g^k\|^2}, \|g^k\|^2] \). This shows that as long as \( x^k \notin X_\epsilon \), \( d(x, X) \) serves as a Lyapunov function and we can write,
\[
\begin{align*}
    d(x^{k+1}, X) &\leq d(x^k, X) + (\alpha_k)^2 \|g^k\|^2 - 2\alpha_k \Phi(x^k) \\
    &= d(x^k, X) + \frac{\epsilon^2}{n^4 G^4} \|g^k\|^2 - 2\frac{\epsilon}{n^2 G^2} \Phi(x^k) \\
    &\leq d(x^k, X) + \frac{\epsilon^2}{n^4 G^4} n^2 G^2 - 2\frac{\epsilon}{n^2 G^2} \epsilon = d(x^k, X) - \frac{\epsilon^2}{n^2 G^2},
\end{align*}
\]
where in the last inequality we have used the fact that \( \|g^k\|^2 \leq n^2 G^2 \) and \( \Phi(x^k) \geq \epsilon \) (as \( x^k \notin X_\epsilon \)). Since \( d(x^k, X) \geq 0, \forall k \), we conclude that after at most \( \frac{d(x^k, X) n^2 G^2}{\epsilon^2} \) iterations the state-dependent network dynamics \( x_i^{k+1} = x_i^k - \frac{\epsilon}{n^2 G^2} \sum_{j \in \mathcal{N}_i(x^k)} \frac{\partial}{\partial x_{ij}} g_{ij}(x_i^k, x_j^k) \) will reach to an \( \epsilon \)-neighborhood of its equilibrium set \( X_\epsilon \).

A. Saddle-Point Dynamics with Heterogeneous Step Size

The subgradient method is not the only algorithm for minimizing a convex function and one can consider other alternative algorithms that can result in different state-dependent network dynamics. The following theorem provides another multiagent network dynamics motivated by the fact that often different agents have different scaling parameters in their update rules. These dynamics can be viewed as the quasi-Newton method [32] in the context of multiagent network dynamics.

Definition 3: A function \( V : \mathbb{R}^n \to \mathbb{R} \) is called a semi-Lyapunov function for the discrete-time dynamics \( z^{k+1} = h(z^k) \), \( k = 0, 1, 2, \ldots \), if \( V(z^{k+1}) < V(z^k) \) for any \( z^k \in \mathbb{R}^m \setminus D \), where \( D \) is a measure-zero subset of \( \mathbb{R}^n \).

Theorem 7: Let \( g_{ij} \in \mathbb{C}^2 \) be symmetric convex, and \( f \in \mathbb{C} \) be a convex function. Define \( \Phi(x) := f(x) + \frac{1}{2} \sum_{i,j} \max \{g_{ij}(x_i, x_j), 0\} \), and let \( D := \{x : g_{ij}(x_i, x_j) = 0 \text{ for some } i, j \} \) be the measure-zero
set of nondifferentiability points of $\Phi(x)$. If for any $x \notin D$, there exists a positive-definite diagonal matrix $G^k$ such that $\Phi(y) \leq \Phi(x) + (y - x)^T g^k + \frac{1}{2} (y - x)^T G^k (y - x)$, $\forall y \in L_x$, where $g^k$ and $L_x := \{y : \Phi(y) \leq \Phi(x)\}$ are the subgradient and the level set of $\Phi(\cdot)$ at $x$, then the dynamics
\begin{equation}
\begin{aligned}
x_i^{k+1} &= x_i^k - \frac{1}{G_{ii}^k} \frac{\partial}{\partial x_i} f(x^k) + \sum_{j \in N_i(x^k)} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k), \quad i \in [n],
\end{aligned}
\end{equation}

admit the semi-Lyapunov function $\Phi(x)$. In particular, if there exists a constant $m$ such that $G^k \leq m I$, $\forall k$, and $x^k \in D$ for at most finitely many iterates $k$, then the dynamics (14) will converge to the set of minimizers of $\Phi(x)$.

**Proof:** Consider the convex function $\Phi(x) := f(x) + \frac{1}{2} \sum_{i,j} \max\{g_{ij}(x_i, x_j), 0\}$, which is differentiable at any point except at $D := \{x : g_{ij}(x_i, x_j) = 0 \text{ for some } i, j\}$. As before, we know that the $i$th component of the gradient of $\Phi(x)$ at $x^k$ (subgradient if $x^k \in D$) is given by $g_i^k = \frac{\partial}{\partial x_i} f(x^k) + \sum_{j \in N_i(x^k)} \frac{\partial}{\partial x_i} g_{ij}(x_i^k, x_j^k)$. By the assumption, there is a positive-definite diagonal matrix $G^k$ such that
\begin{equation}
\begin{aligned}
u_k(y) &= \Phi(x^k) + (y - x^k)^T g^k + \frac{1}{2} (y - x^k)^T G^k (y - x^k),
\end{aligned}
\end{equation}
forms a quadratic approximation upper bound for $\Phi(y)$, $\forall y \in L_{x^k}$. Clearly, we have $\nu_k(x^k) = \Phi(x^k)$.

On the other hand, it is easy to see that
\begin{equation}
x^{k+1} = x^k - (G^k)^{-1} g^k = \arg\min_{y \in \mathbb{R}^n} \nu_k(y).
\end{equation}

Now let us consider an arbitrary $x^k \notin D$. Then $g^k = \nabla \Phi(x^k)$, and thus $-(G^k)^{-1} g^k$ is a descent direction for any positive definite matrix $(G^k)^{-1}$. This means that for sufficiently small $\delta > 0$, $\Phi(x^k - \delta(G^k)^{-1} g^k) \leq \Phi(x^k)$, and hence $x^k - \delta(G^k)^{-1} g^k \in L_{x^k}$. Therefore, the line segment $\{(1 - \alpha)x^k + \alpha x^k \mid \alpha \in [0, 1]\}$ intersects $L_{x^k}$ in at least two different points (for $\alpha = 0, \delta$). Now if $x^{k+1} \notin L_{x^k}$, that line segment must intersect the boundary of $L_{x^k}$ at another point $\bar{x} := \bar{x}^k + (1 - \bar{\alpha}) x^{k+1}$, for some $\bar{\alpha} \in (0, 1)$ (note that the level set $L_{x^k}$ is a closed convex set). Using continuity of $\Phi(\cdot)$, $\Phi(\bar{x}) = \Phi(x^k)$, and we can write
\begin{equation}
\begin{aligned}
u_k(x^k) = \Phi(x^k) = \Phi(\bar{x}) \leq \nu_k(\bar{x}) \leq \bar{\alpha} u_k(x^k) + (1 - \bar{\alpha}) u_k(x^{k+1}) < u_k(x^k),
\end{aligned}
\end{equation}
where the first inequality is because $\Phi(y) \leq u_k(y)$, $\forall y \in L_{x^k}$, and the second inequality is by convexity of $u_k(\cdot)$. This contradiction shows that $x^{k+1} \in L_{x^k}$, which implies $\Phi(x^{k+1}) \leq \Phi(x^k)$. Therefore, $\Phi(\cdot)$ serves as a semi-Lyapunov function for the dynamics (14). In particular, the drift of this Lyapunov function at $x^k \notin D$ equals to
\begin{equation}
\begin{aligned}
\Phi(x^k) - \Phi(x^{k+1}) = \Phi(x^k) - [\Phi(x^k) + (x^{k+1} - x^k)^T g^k + \frac{1}{2} (x^{k+1} - x^k)^T G^k (x^{k+1} - x^k)]
&= \frac{1}{2} (g^k)^T (G^k)^{-1} g^k = \frac{1}{2} \|g^k\|_2^2 (G^k)^{-1}.\end{aligned}
\end{equation}

Summing the above inequality for $k = 0, \ldots, K - 1$, we obtain
\begin{equation}
\begin{aligned}
\Phi(x^K) + \sum_{k : x^k \notin D} (\Phi(x^{k+1}) - \Phi(x^k)) \leq \Phi(x^0) - \frac{1}{2} \sum_{k : x^k \notin D} \|g^k\|_2^2 (G^k)^{-1}.\end{aligned}
\end{equation}

As this relation holds for any $K$, and $|\{k : x^k \in D\}| < \infty$ by the assumption, we must have $\sum_{k : x^k \notin D} \|g^k\|_2^2 (G^k)^{-1} < \infty$, and hence $\lim_{K \to \infty} \|g^k\|_2^2 (G^k)^{-1} = 0$. Thus, if there exists $m > 0$ such that $G^k \leq m I$, $\forall k$, we get $\lim_{k \to \infty} \|g^k\|_2^2 = 0$. Since $\Phi(\cdot)$ is a convex function, we know that the set of
minimizers of $\Phi(\cdot)$ are exactly the set of points having 0 as their subgradient. This shows that $\{x^k\}_{k=0}^{\infty}$ must converge to the set of minimizers of $\Phi(\cdot)$.

A natural choice for the matrices $C^k$ in Theorem 7 is the Hessian matrix $\nabla^2 \Phi(x^k)$ which is used in the Newton method for minimizing a smooth convex function. However, in practice it is often easier to work with a sparse modification of $\nabla^2 \Phi(x^k)$ given by a diagonal matrix containing only the diagonal entries of $\nabla^2 \Phi(x^k)$. In particular, to assure positive definiteness, an identity matrix is added to this diagonal matrix to form the quasi-Newton update rule. Using such a quasi-Newton method for minimizing $\Phi(\cdot)$, one obtains the following state-dependent network dynamics

$$
\begin{align*}
  x_i^{k+1} &= x_i^k - t_k \frac{\partial}{\partial x_i} f(x^k) + \sum_{j \in N_i(x^k)} \frac{\partial^2}{\partial x_i \partial x_j} g_{ij}(x_i^k, x_j^k), \\
  \quad &\quad \frac{1}{1 + \frac{\partial^2}{\partial x_i^2} f(x^k)} + \sum_{j \in N_i(x^k)} \frac{\partial^2}{\partial x_i \partial x_j} g_{ij}(x_i^k, x_j^k),
\end{align*}
$$

(15)

where $t_k$ is an appropriately chosen step size obtained using a line search or diminishing rule. In fact, it is known that for sufficiently small neighborhood of the minimizers of $\Phi(\cdot)$, the newton method with step size $t_k = 1$ will converge quadratically fast to the set of optimal points [32]. Therefore, we obtain a simple explanation for the convergence properties and equilibrium points of seemingly complex state-dependent network dynamics (15) using the well-known quasi-Newton method. In particular, this provides a rigorous explanation on why the trajectories of the state-dependent network dynamics of the form (15) (such as HK model) converge exponentially fast as they get close to their equilibrium points.

**Example 4**: Let us consider a special case where $f = 0$ and $g_{ij}(x_i, x_j) = \frac{1}{2}(x_i - x_j)^2 - \frac{\epsilon_{ij}}{2}$, where $\epsilon_{ij} = \epsilon_{ji} > 0$. This means that two agents $i$ and $j$ become each others’ neighbors if their distance is larger than $\epsilon_{ij}$. Note that this is the complement of the original HK model. In this case, $\Phi(x) = \frac{1}{2} \sum_{ij} \max\{\frac{1}{2}(x_i - x_j)^2 - \frac{\epsilon_{ij}^2}{4}, 0\}$, and thus for $x^k \notin D := \{x : |x_i - x_j| = \epsilon_{ij}, \text{for some } i, j\}$, we have,

$$
\nabla_i \Phi(x^k) = \sum_{j \in N_i(x^k)} (x_i^k - x_j^k) = |N_i(x^k)|x_i^k - \sum_{j \in N_i(x^k)} x_j^k,
$$

$$
\nabla_{ij}^2 \Phi(x^k) = \begin{cases} 
|N_i(x^k)| & \text{if } i = j \\
-1 & \text{if } j \in N_i(x^k) \\
0 & \text{else}
\end{cases}.
$$

In other words, the Hessian matrix at $x^k$ is equal to the Laplacian of the connectivity network at state $x^k$. As a result, the quasi-Newton dynamics (15) for minimizing the piecewise quadratic function $\Phi(x)$ becomes,

$$
\begin{align*}
  x_i^{k+1} &= x_i^k - \frac{|N_i(x^k)|x_i^k - \sum_{j \in N_i(x^k)} x_j^k}{|N_i(x^k)| + 1},
\end{align*}
$$

In particular, for sufficiently small choice of step size $t_k$ the function $\Phi(x)$ serves as a semi-Lyapunov function. Note that for unit step size $t_k = 1$, the above dynamics can be explicitly written as

$$
\begin{align*}
  x_i^{k+1} &= \frac{\sum_{j \in N_i(x^k) \cup \{i\}} x_j^k}{|N_i(x^k)| + 1}.
\end{align*}
$$

(16)

As a result the dynamics of the complement-HK model can be viewed as iterates of a quasi-Newton method with unit step size for minimizing $\Phi(x)$. Of course, for $t_k = 1$, there is no reason on why $\Phi(x)$ should serve as a Lyapunov function, unless the initial point of the dynamics is sufficiently close to a minimizer of $\Phi(\cdot)$ (in which case the exponentially fast convergence of the quasi-Newton method with $t_k = 1$ is guaranteed). Nevertheless the function $\Phi(x)$ is still very useful as it globally guides the dynamics based on quasi-Newton iterates. In particular, the set of minimizers of $\Phi(x)$
characterize the equilibrium points of (16). This is because if \( \lim_k x^k = x^* \), we must have \( x^* = \lim_k x^{k+1} = x^* - \lim_k (G_k)^{-1} \nabla \Phi(x^k) \), where here \( (G_k)^{-1} = \text{diag}(\frac{1}{\|x^k_i\|+1}, \ldots, \frac{1}{\|x^k_n\|+1}) \). This implies that \( \lim_{k \to \infty} G_k^{-1} \nabla \Phi(x^k) = 0 \). As the entries of \( G_k^{-1} \) are uniformly bounded below by \( \frac{1}{n+1} \), we must have \( \lim_{k \to \infty} \nabla \Phi(x^k) = 0 \), and the result follows from convexity of \( \Phi(\cdot) \).

V. CONTINUOUS-TIME CONSTRAINED SADDLE-POINT DYNAMICS

In this section, we extend our discrete-time saddle-point dynamics to their continuous-time counterparts and show how they can be leveraged to establish Lyapunov stability of state-dependent network dynamics. Here due to continuity of the time index \( t \in [0, \infty) \), an edge connectivity between a pair of agents \((i,j)\) is no longer a binary event \( \lambda_{ij} \in \{0,1\} \), but rather a continuous weight function of time \( \lambda_{ij}(t) \in [0,1] \). Thus \( \lambda_{ij}(t) \) can be viewed as a connectivity strength between agents \( i \) and \( j \) at time \( t \) such that the maximum influence that two agents can have on each other is \( 1 \) (i.e., fully connected) and the minimum influence is \( 0 \) (i.e., fully disconnected).

Motivated by the method of change of variables for discrete time dynamics in Section III-B, we state our results for continuous-time dynamics in a more general form where the agents’ state are transformed from \( x_t \) to \( p_t(x_i) \), and the network variables are transformed from \( \lambda_{ij} \) to \( q_{ij}(\lambda_{ij}) \). Here we assume that \( p_t(\cdot), q_{ij}(\cdot) \) are continuous and nondecreasing functions such that \( p_t(0) = q_{ij}(0) = 0, \forall i,j \). In particular, we let the Lagrangian function to have a more general form of \( L(p(x),q(\lambda)) \), as long as its partial derivatives exist and is convex with respect to its first argument \( p(x) = (p_1(x_1), \ldots, p_n(x_n))^T \), and concave with respect to its second argument \( q(\lambda) = (q_{ij}(\lambda_{ij}), i \neq j)^T \).

Remark 3: A special case of the above setting is when \( p_t(x_i) = x_i, q_{ij}(\lambda_{ij}) = \lambda_{ij} \) are identity functions, and \( L(x, \lambda) = \sum_i f_i(x_i) + \sum_{i \neq j} \lambda_{ij} g_{ij}(x_i, x_j), \lambda \geq 0, x \in \mathbb{R}^n \). It is clear that for convex measurement functions \( g_{ij}, f_i \), the standard Lagrangian function \( L(x, \lambda) \) is convex with respect to \( x \), and concave (linear) with respect to \( \lambda \).

To introduce a general class of continuous-time state-dependent network dynamics, let us consider the following static constrained saddle-point problem:

\[
\min_{x \in \mathbb{R}^n} \max_{\lambda \in [0,1]^{n(n-1)}} L(p(x), q(\lambda)).
\]  

(17)

To solve the above static saddle-point problem using continuous-time dynamics, we use the idea of gradient flow which was initially introduced in the seminal work of Arrow-Hurwicz-Uzawa [33] and subsequently used in devising primal-dual algorithms for solving constrained optimization problems [34]. However, to adopt these dynamics to our more general setting (17) which has both lower and upper bound constraints on the dual variable \( \lambda \), we introduce the following generalized gradient flow dynamics:

\[
\dot{x}(t) = -\nabla_{p(x)} L(p(x), q(\lambda))
\]

\[
\dot{\lambda}(t) = \left[ \nabla_{q(\lambda)} L(p(x), q(\lambda)) \right]_{\lambda}^{[0,1]},
\]  

(18)

where in the above dynamics \( \nabla_{p(x)} L(p(x), q(\lambda)) := \left( \frac{\partial L(p(x), q(\lambda))}{\partial p_1(x_1)}, \ldots, \frac{\partial L(p(x), q(\lambda))}{\partial p_n(x_n)} \right)^T \) (similarly for \( \nabla_{q(\lambda)} L(p(x), q(\lambda)) \)), and \( [a]_{\lambda}^{[0,1]} \) denotes the projection of the network dynamics to the unit interval,

\[
[a]_{\lambda}^{[0,1]} = \begin{cases} 
\min\{0,a\}, & \text{if } \lambda = 1 \\
\max\{0,a\} & \text{if } 0 < \lambda < 1 \\
a & \text{if } \lambda = 0.
\end{cases}
\]

\(^5\)In fact, using a sorted vector Lyapunov function \( V(x) = \text{sort}(|x_i - x_j|, i \neq j) \), it can be shown that the dynamics (16) do converge where after each iteration \( V(x) \) decreases lexicographically.
When \( a \) is a vector rather than a scalar, the above projection is taken coordinatewise. The reason for introducing such a projection is that if for a pair of agents \((i, j)\) we have \( \lambda_{ij}(t) \in (0, 1) \), the edge variable \( \lambda_{ij}(t) \) has not hit the boundary points \((0, 1)\), and it can freely increase or decrease without violating the box constraint \( \lambda_{ij}(t) \in [0, 1] \). But if \( \lambda_{ij}(t) = 1 \), then this edge variable is only allowed to decrease, and thus \( \dot{\lambda}_{ij}(t) \leq 0 \). Therefore, if \( \frac{\partial L(p(x), q(\lambda))}{\partial q_{ij}(\lambda_{ij})} \geq 0 \), we set \( \dot{\lambda}_{ij}(t) = 0 \) to block further increase of \( \lambda_{ij}(t) \). Similarly, if \( \lambda_{ij}(t) = 0 \) and \( \frac{\partial L(p(x), q(\lambda))}{\partial q_{ij}(\lambda_{ij})} \leq 0 \), we set \( \dot{\lambda}_{ij}(t) = 0 \) to block further decrease of \( \lambda_{ij}(t) \). Therefore, (18) provide a fairly general class of continuous-time state-dependent network dynamics where the strength of the edge connectivity dynamically changes as a function of the state variables.

Remark 4: It is worth noting that in the special setting of Remark 3, the network dynamics in (18) decompose to a simple form of \( \dot{\lambda}_{ij}(t) = [g_{ij}(x_i(t), x_j(t))]_{0,1}^{[0,1]} \), \( \forall i, j \). Thus, the more distant two agents \( i \) and \( j \) are from each other (i.e., larger measurement value \( g_{ij}(x_i, x_j) \)), the faster the edge connectivity between them grows (until it achieve its maximum connectivity at 1). This is consistent with the discrete-time counterpart that an edge emerges between agents \( i \) and \( j \) if \( g_{ij}(x_i, x_j) > 0 \).

In order to establish the Lyapunov stability of the continuous-time state-network dynamics (18), let \((\bar{x}, \bar{\lambda})\) be a saddle-point solution to (17). Note that by continuity and convex-concave property of the Lagrangian function, the existence of a saddle-point in (17) is always guaranteed. Let us define \( P_i(x_i) = \int_{x_i}^{x_{ij}} p_i(s)ds \) and \( Q_{ij}(\lambda_{ij}) = \int_{\lambda_{ij}}^{\bar{\lambda}_{ij}} q_{ij}(s)ds \), where we note that by continuity and monotonicity of \( p_i, q_{ij} \), the functions \( P_i \) and \( Q_{ij} \) are differentiable convex functions. Now we are ready to state the main result of this section.

Theorem 8: Let \( L(p(x), q(\lambda)) \) be a convex function in \( p(x) \) and a concave function in \( q(\lambda) \). Then, the continuous-time state dependent network dynamics (18) are Lyapunov stable. In particular,

\[
V(x, \lambda) := \sum_{i=1}^{n} D_{P_i}(x_i, \bar{x}_i) + \sum_{i \neq j} D_{Q_{ij}}(\lambda_{ij}, \bar{\lambda}_{ij})
\]

serves as a Lyapunov function for the dynamics (18), where \( D_{q}(u, v) = \phi(u) - \phi(v) - \phi'(v)(u - v) \) denotes the Bregman divergence with respect to the convex function \( \phi(\cdot) \).

Proof: Using the definition of the Bregman divergence, for every \( i \) and \( j \) we have:

\[
\dot{D}_{P_i}(x_i, \bar{x}_i) = \frac{\partial D_{P_i}(x_i, \bar{x}_i)}{\partial x_i} \dot{x}_i = -(p_i(x_i) - p_i(\bar{x}_i)) \frac{\partial L(p(x), q(\lambda))}{\partial p_i(x_i)},
\]

\[
\dot{D}_{Q_{ij}}(\lambda_{ij}, \bar{\lambda}_{ij}) = \frac{\partial D_{Q_{ij}}(\lambda_{ij}, \bar{\lambda}_{ij})}{\partial \lambda_{ij}} \dot{\lambda}_{ij} = (q_{ij}(\lambda_{ij}) - q_{ij}(\bar{\lambda}_{ij})) \frac{\partial L(p(x), q(\lambda))}{\partial q_{ij}(\lambda_{ij})},
\]

Now we can write,

\[
\dot{V}(x, \lambda) = -\sum_{i} (p_i(x_i) - p_i(\bar{x}_i)) \frac{\partial L(p(x), q(\lambda))}{\partial p_i(x_i)} \leq -\sum_{i} (p_i(x_i) - p_i(\bar{x}_i)) \frac{\partial L(p(x), q(\lambda))}{\partial p_i(x_i)},
\]

\[
= (\nabla_p L(p(x), q(\lambda)))^T (p(x) - p(\bar{x})) + (\nabla_q L(p(x), q(\lambda)))^T (q(\lambda) - q(\lambda)) \leq L(p(\bar{x}), q(\lambda)) - L(p(x), q(\lambda)) - L(p(x), q(\lambda)) - L(p(\bar{x}), q(\lambda)) \leq 0.
\]

where in the above derivations the last inequality is due to the definition of the saddle-point, and the second inequality follows by convexity/concavity of \( L(\cdot) \) with respect to its first/second argument. Finally, the first inequality is obtained by considering the following three cases:
Thus in either of the above cases we have

\[ (q_{ij}(\lambda_{ij}) - q_{ij}(\bar{\lambda}_{ij}))\left[\frac{\partial L(p(x), q(\lambda))}{\partial q_{ij}(\lambda_{ij})}\right]_{\lambda_{ij}} \leq (q_{ij}(\lambda_{ij}) - q_{ij}(\bar{\lambda}_{ij}))\left[\frac{\partial L(p(x), q(\lambda))}{\partial q_{ij}(\lambda_{ij})}\right], \]

and the result follows.

It is worth noting that Theorem 8 is a continuous-time counterpart of Theorem 6 in a sense that in both of these theorems the Bregman distance of the iterates to a saddle-point serves as a Lyapunov function. However, due to the continuity of the network variables in the continuous-time model, the choice of the step size becomes irrelevant in Theorem 8 while for the discrete-time counterpart the step sizes should be small enough to guarantee the convergence of the dynamics.

VI. CONCLUSIONS

In this paper, we developed a new framework for the stability analysis of multiagent state-dependent network dynamics. We showed that the co-evolution between the network and the state dynamics can be cast as a primal-dual optimization algorithm to a convex program where the primal updates capture the state dynamics while the dual updates capture the network evolution. In particular, the constrained Lagrangian function serves as a Lyapunov function for the state-network dynamics. We considered our framework under two different settings: i) when the network and state dynamics are aligned, and ii) when the network and state dynamics have conflicting objectives. In the first case, we showed that the application of the BCD method with the change of variables can generate a variety of interesting state-dependent network dynamics. In particular, we provided a new technique to handle asymmetry in the network dynamics. In the second case, we reduced the stability of the state-network dynamics to a zero-sum game between the network player and the state player. This allowed us to establish the Lyapunov stability of the multiagent dynamics using saddle-point dynamics and in particular using the subgradient method and the quasi-Newton method. Finally, we extended our results to a continuous-time model and provided a general class of continuous-time state-dependent network dynamics in terms of generalized gradient flow and established their Lyapunov stability.

As a future direction of research, one can use augmented Lagrangian functions or apply alternative optimization techniques to generate a broader class of stable state-dependent network dynamics. Moreover, in our analysis, we mainly used a quadratic upper approximation to derive the state updates. Thus a natural extension is to use other function approximations that include the quadratic approximation as their special case or to use approximations that are suitable to specific applications.

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