Landau dynamics of a grey soliton in a trapped condensate

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Following the experimental observation of grey solitons (or more precisely, entities associated with quasi-one-dimensional grey solitons) in Bose-Einstein condensates (BEC’s) [1], a great deal of attention has been paid to the theory of the phenomenon (see e.g. [2, 3] and references therein). It was found that a grey soliton in a parabolic trap displays a number of peculiarities. Among them, we mention the frequency of oscillations, which is \( \sqrt{2} \) times less than the frequency of oscillations of the condensate as a whole [2], and the consequent beatings, which can be observed in the long-time dynamics of a soliton, an internal mode accompanying soliton dynamics, nontrivial phase changes in soliton evolution [3], etc. It turns out, however, that the mathematical treatment of the problem, based on the application of perturbation theory to dark solitons [4] and covering all the main effects, is rather involved, and more important, requires a small soliton velocity. The major problem is that a grey soliton even in the simplest one-dimensional (1D) parabolic trap is dramatically different from the standard grey soliton known from soliton theory. Distinctions emerge from different boundary conditions, which are zero in the case of a trap potential and nonzero in the case of a mathematical grey soliton (see e.g. [3]).

The present paper aims to describe the motion of a grey soliton over a wide range of velocities in a trapped condensate, whose longitudinal size is sufficiently large. The phenomenon is described by the mean-field 1D Gross-Pitaevskii (GP) equation

\[
\frac{i \hbar}{2m} \dot{\Psi} = \frac{\hbar^2}{2m} \nabla^2 \Psi + U(x) \Psi + g_1 |\Psi|^2 \Psi. \tag{1}
\]

We consider a cigar-shaped condensate with the following relations among the transverse \( a_\perp \), the longitudinal \( 2L \) dimensions of the condensate and a healing length \( \xi = \frac{\hbar}{\sqrt{2m \mu}} (c \) hereafter being the sound velocity): \( a_\perp \ll \xi \ll 2L \). The 1D coupling constant \( g_1 \), comes from the “averaging” over the transverse cross section of the condensate [2, 3].

We developed a full description of the dynamics of the soliton, considering it as a quasiparticle and using the Landau theory of superfluidity. Subsequently we show that this approach is consistent with the GP equation and derive the main result by use of the proper perturbation procedure. In conclusion, on the basis of the results obtained we discuss qualitatively the dynamics of a vortex ring.

Soliton in a homogeneous condensate. We start with a grey soliton in a uniform 1D condensate, i.e. when \( U(x) \equiv 0 \). According to Tsuzuki [2] the respective condensate wave function can be presented in the form (see also [3], §5.5):

\[
\Psi (x,t) = \sqrt{n} \left( \frac{v}{c} + \frac{u}{c} \tanh \left[ \frac{1}{\ell} (x - X(t)) \right] \right) e^{-i\mu t/\hbar}, \tag{2}
\]

where \( X(t) = vt \), \( v \) is the velocity of the soliton, \( n \) is the unperturbed linear density, \( \mu = g_1 n = mc^2 \) is the chemical potential, \( u = \sqrt{c^2 - v^2} \), and \( \ell = \hbar/\mu n \) is the width of the soliton. The energy of the soliton is computed to be

\[
\mathcal{E} = \frac{4\hbar m}{3g_1} (c^2 - v^2)^{3/2} = \frac{4\hbar m}{3g_1} u^3. \tag{3}
\]

Landau dynamics of a soliton in an inhomogeneous condensate. Let us now suppose that the condensate length, \( 2L \), is large compared to the width of the soliton:

\[
L \gg \ell. \tag{4}
\]

Then, for such large condensates one can use semiclassical Landau dynamics of the soliton, where the quantity [3] plays role of the Hamiltonian of quasiparticle. We notice that condition (4) is stronger than the condition of applicability of the Thomas-Fermi (TF) approximation to the condensate \( L \gg \xi \) [3, 5]. Requirement (4) ensures, also, that the velocity of motion of the condensate as a whole due to the soliton oscillations is small and one can regard the condensate as being in the rest.

Subject to condition (4), one can use a local density approximation, assuming that equation (5) for the soliton energy is valid in the inhomogeneous condensate, i.e., that \( c \) can be changed to its local value \( c(X) \), where \( X \)
is the position of the center of the soliton. In the first
approximation the soliton wave function has the same
form \([2]\), where \(X\) and \(v\) are functions of time related by
differential equations \(dX/dt = v(t)\). The soliton motion is then defined by the
energy conservation equation
\[
\frac{4\hbar m}{3g_1} \left[ c(X)^2 - v^2 \right]^{3/2} = E = \text{const}
\]
which obviously implies that \(u\) is also a constant. Ex-
pressing \(E\) in terms of \(u\), we finally find the equation
\[
\left( \frac{dX}{dt} \right)^2 = c(X)^2 - u^2
\]
which can be solved by a simple integration.

In order to find the distribution of the sound speed, we
approximate the density by the TF law
\[
n_{TF}(x) = (\mu - U(x))/g_1 .
\]
Assuming, without loss of generality, that \(U(0) = 0\), we find
\[
c(X)^2 = g_1 n_{TF} / m = c_0^2 - U(X) / m
\]
where \(c_0 = \sqrt{\mu/m}\) is the sound speed at \(x = 0\). Substi-
tution into \((6)\) gives
\[
m \left( \frac{dX}{dt} \right)^2 + U(X) = m(c_0^2 - u^2) .
\]
This equation describes classical motion of a particle having
mass \(2m\) and energy \(m(c_0^2 - u^2)\) in the potential
\(U(X)\). It is remarkable that the soliton propagates
through the condensate without change of its density
profile. Indeed, simple calculation gives for the density pertur-
bation in the vicinity of the soliton:
\[
\delta n(x) \equiv |\Psi(x)|^2 - n = -\frac{mu^2}{g_1} \frac{1}{\cosh^2 \left[ \sqrt{\frac{\mu}{m}} (x - X) \right]} .
\]
This quantity does not depend of time (at a given \(x - X\)),
because \(u\) is an integral of motion.

Let us apply these results to a harmonic trap \(U(x) = m\omega_x^2 x^2/2\). Notice that \(\omega_x\) is the frequency of oscillations of the center of mass of the condensate. Now the distribution of the sound velocity is \(c_0^2(X) = c_0^2(1 - X^2/L^2)\),
where \(2L\) is the condensate length defined by \(L^2 = 2\mu/m\omega_x^2\) and \(\omega_x\) takes the form \(\omega_x^2 = \sqrt{\frac{\mu}{m}} \omega_x^2 / 2\). This equation describes pure harmonic oscillations with frequency \(\omega_x = \omega_x / \sqrt{2}\). This frequency coincides with the one obtained in Refs. \([2, 3]\) in a different way for a slow soliton with \(v \ll c_0\). The amplitude of oscillations is equal to the coordinate \(X_1\) of the turning point:
\[
X_1^2 = \frac{2}{\omega_x^2} \left( \frac{\mu}{m} - u^2 \right) = L^2 - \frac{2}{\omega_x^2} u^2 .
\]
The soliton oscillates between points \(\pm X_1\), where the
soliton velocity and central density become zero. Our
approach requires, in addition to the general condition
\([4]\), that the distance between a turning point and the
condensate boundary is larger than the soliton width:
\(L - X_1 \gg \ell\). This condition can be transformed to
\(u/c_0 \gg (\xi/L)^{1/3}\), which indicates that velocity of the
soliton must not be too close to speed of sound.

To conclude this section, we must make an important
remark. Equation \((8)\) was derived for a uniform back-
ground in the absence of an external potential. It is not
obvious that the energy has the same form in the pres-
ence of trapping potential. The point is that a soliton is a
region of decreased density and, thus contains a negative
"number of atoms"
\[
N_s[\Psi] = \int_{-\infty}^{\infty} \delta n(x) dx = -\frac{2\hbar}{g_1} u .
\]
One may think that to the energy \(E\) one must add the po-
tential energy \(N_s U(X)\) that corresponds to these atoms.
Such conclusion would be wrong. Actually, this decrease
of energy is compensated by increasing the energy of
atoms outside of the soliton. We will prove this in the
next paragraph by direct calculation of the energy.

It is worthwhile to notice that the depletion \((12)\) of the
number of atoms in the soliton depends only on its
energy \(E\). This implies that \(N_s\) is also an integral of motion.
A soliton is a quasiparticle of constant (negative) mass \(2mN_s\). However, this "physical" mass of the soliton
depends on energy and is not equivalent to the "effective mass" \(2m\).

**Energy of a soliton in an external field.** In this para-
graph we will show that equation \((8)\) for the energy of a
soliton, obtained for an uniform condensate, is actually
valid also for a trapped condensate in the local density
approximation. Thus the trapping potential does not en-
ter explicitly in the expression for the energy of a soliton.

To take properly into account the expelling of atoms from
the soliton, it is more convenient to work not at a given \(x - X\),
because \(u\) is an integral of motion.

Let us apply these results to a harmonic trap \(U(x) = m\omega_x^2 x^2/2\). Notice that \(\omega_x\) is the frequency of oscillations of the center of mass of the condensate. Now the distribution of the sound velocity is \(c_0^2(X) = c_0^2(1 - X^2/L^2)\),
where \(2L\) is the condensate length defined by \(L^2 = 2\mu/m\omega_x^2\) and \(\omega_x\) takes the form \(\omega_x^2 = \sqrt{\frac{\mu}{m}} \omega_x^2 / 2\). This equation describes pure harmonic oscillations with frequency \(\omega_x = \omega_x / \sqrt{2}\). This frequency coincides with the one obtained in Refs. \([2, 3]\) in a different way for a slow soliton with \(v \ll c_0\). The amplitude of oscillations is equal to the coordinate \(X_1\) of the turning point:
\[
X_1^2 = \frac{2}{\omega_x^2} \left( \frac{\mu}{m} - u^2 \right) = L^2 - \frac{2}{\omega_x^2} u^2 .
\]
an integral \( \int_{|x-x'|<\delta} e^r_{TF} dx \) and, correspondingly, deduct it from the second term in \( E' \). Such an addition transforms the first term in the total energy of the condensate in the absence of the soliton, which will be designated \( E_0' \). In the second term one can safely change the smooth function \( e^r_{TF}(x) \) to \( e^r_{TF}(X) \). Finally we arrive at the result

\[
E' = E_0' + \int_{|x-x'|<\delta} \left[ \frac{\hbar^2}{2m} |\Psi_x|^2 + \frac{1}{2} g_1(n(x)^2 - n_{TF}(X)^2) + (U(X) - \mu)(n(x) - n_{TF}(X)) \right] dx . \tag{14}
\]

Now we can use the theorem about small increments (see [10], §16). According to this theorem small corrections to \( E \) and \( E' \), are equal, being expressed, correspondingly, in terms of \( N \) and \( \mu \). Thus to obtain a correction to the energy, we must express the second term in (14) in terms of the density. One may eliminate \( \mu \) from (14) by taking into account that, according to (7), \( (\mu - U(X)) = g_1 n_{TF}(X) \). Owing to the fast convergence of the integral, it is also possible to change formally the integration limits from \( X \pm \delta \) to \( \pm \infty \) respectively. Finally we get \( E = E_0 + \mathcal{E} \) where

\[
\mathcal{E} = \int_\infty^{\infty} \left[ \frac{\hbar^2}{2m} |\Psi_x|^2 + \frac{1}{2} g_1(n(x) - n_{TF}(X))^2 \right] dx . \tag{15}
\]

Equation (15) does not contain the trapping potential explicitly. Substituting \( \Psi(x) \) from (2) and integrating, we obtain equation (3), where \( n \) is changed to \( n_{TF}(X) \). Equation (1) ensures now conservation of \( E \).

**Perturbative approach to grey-soliton dynamics.** In the previous paragraphs investigating the soliton dynamics we postulated, in the spirit of the Landau theory, that when the soliton moves in a weakly-inhomogeneous background its local energy stays constant. Let us now investigate the relation between the above phenomenological approach and the evolution of a grey soliton emerging from the dynamical approach, based on the mean-field GP equation (1).

A grey soliton propagates against a background, and thus the first step to obtain the soliton dynamics is to determine the background (which above was approximated by the TF density distribution). One can do this by means of the ansatz (3) \( \Psi = \exp(-i\tilde{\Psi} t) F(x) \psi(x,t) \) where \( F(x) \) is a real-value solution of the nonlinear eigenvalue problem

\[
\mu F = -\frac{\hbar^2}{2m} F_{xx} + U(x) F + g_1 n F^3 \tag{16}
\]

with \( F(0) = 1, F_x(0) = 0 \) and \( \lim_{x \to \pm \infty} F(x) = 0 \). As above we stipulate that \( U(0) = 0 \) and without loss of generality impose the normalization condition on \( F(x) \) at \( x = 0 \) assuming that the soliton is placed in the interval \( x \in (-X_1, X_1) \), where \( X_1 \ll L \). The function \( \psi(x,t) \) then solves the equation

\[
\imath \hbar \psi_t + \frac{\hbar^2}{2m} \psi_{xx} - g_1(|\psi|^2 - n) \psi = R[\psi,F] \tag{17}
\]

which is subject to the boundary conditions \( R[\psi,F] \) is given by

\[
R[\psi,F] = -\frac{\hbar^2}{2m} \frac{F_x^2}{F^2} \psi_x + g_1 n (F^2 - 1)(|\psi|^2 - n) \psi \tag{18}
\]

An explicit form of Eq. (17) appears to be useful if \( R[\psi,F] \) is small compared with the left hand side of that equation. In order to estimate it we recall condition (4), which can be viewed as a definition of the small parameter of the problem \( \epsilon = \ell/L \). We restrict our discussion to a soliton near the center of the trap, where \( |U(x)|/g_1 n = O(\epsilon) \) and the potential is smooth enough, which enables one to neglect third order in the expansion around \( x = 0 \). One can ensure then that \( \mu \approx g_1 n \) and \( R[\psi,F] = O(\epsilon g_1 n) \) in the domain specified above.

This naturally leads one to use perturbation theory, the first step of which would be an adiabatic approximation, i.e. an approximation in which the soliton shape is given by (2) with slowly varying parameters. This straightforward approach, however meets difficulties in the case at hand because of the first term in the right hand side of Eq. (13). Mathematical reasons for this are discussed in [3, 4]. Here we will use a more physical approach. Namely, we will prove that it is possible to define an adiabatic approximation for a dark soliton, such that in the leading order the number of particles associated to the soliton is constant. In other words, one has to prove that there exist a function \( \phi(x,t) \), satisfying (4) in the leading order and preserving the quantity \( N_\phi[\phi] = \int |\phi|^2 dx \). Because both \( N_\psi \) and \( \mathcal{E} \) are expressed in terms of the same quantity \( u \), the energy \( \mathcal{E} \) is also an integral of motion in accordance with the Landau theory. By direct algebra, one may verify that the first term in the r.h.s. of (13) results in change of the number of particles in the case of a nonzero current \( \int \psi_x \psi^* (\psi^* - \psi_x^*) dx \). In order to apply our arguments, we again assume that in the adiabatic approximation the soliton is given by (2). One can then ensure
that the imaginary part of all terms in \([17]\) becomes precisely zero in the vicinity of the point \(x = X(t)\) except for the first term in the r.h.s., which even grows with \(X(t)\). This behavior originates from non-adiabatic effects, which must be taken into account before one passes to the Landau quasiparticle description of the soliton.

In order to avoid this difficulty, let us take into account that the solution we are interested in is a function of \(t\) and \(\zeta = x - X(t)\), where \(X(t)\) is the coordinate of the soliton center and can be associated with a coordinate in the Lagrangian description of the condensate flow. Then, we make the substitution

\[
\psi(x, t) = \phi(x, t) - if(t)\phi_x(x, t), \tag{19}
\]

Now one can verify that \(dN_s[\phi]/dt = \mathcal{O}(\epsilon^2)\), i.e \(\phi(x, t)\) describes a soliton with a constant mass. Indeed, while the last two terms in the r.h.s. of \([21]\) obviously give zero contribution to the change of the number of particles, in order to treat the first one we recall the above requirement of the smoothness of the potential \(U(x)\). One then easily estimates

\[
\frac{F_x(x)}{F(x)} - \frac{F_x(X(t'))}{F(X(t'))} = \frac{(x - X)U_{xx}(0)}{2gn} + \mathcal{O}(\epsilon^2).
\]

It is significant that the r.h.s. is an odd function of \(x - X\). On the other hand \(\phi^*\phi_x - \phi_x^*\phi\), with \(\phi\) given by the r.h.s. of \([2]\), is an even function. Thus, in computing \(dN_s[\phi]/dt\), the first term contains an odd function of \(x - X\), which yields zero to the leading order.

According to the previous reasoning, the result obtained proves the conservation of the soliton energy.

Vortex ring. The method developed in this letter can be applied to other localized excitations. The most interesting example is a vortex ring. Let a small ring moves along the axis of the trapped condensate. The energy of the ring in an uniform condensate can be written as \(E = \sqrt{n}\chi(v/c)\), where function \(\chi\) is a decreasing function of its argument. (It was calculated in \([12]\), see also \([8]\), § 5.4.). Then it follows from conservation of \(E\) that when the ring moves in the direction of decreased density, the ratio \(v/c\) (and \(v\) itself) decreases and its radius increases. On the contrary, moving in the direction of increasing \(n\), the ring accelerates. If the central density is big enough, the ring velocity can reach its maximal value \(v = 0.93c\). The behavior of the system near this point cannot be described in the local density approximation. Probably the excitation collapses beyond this point.

where

\[
f(t) = \frac{\hbar}{m} \int_0^t \frac{F_x(X(t'))}{F(X(t'))} dt'. \tag{20}
\]

Here \(\phi\) can be associated with the soliton wave function while the second term on the r.h.s. of Eq. \([19]\) is an internal mode excited when a soliton moves in a potential. Neglecting the terms of orders higher than \(\epsilon\) one obtains the equation for \(\phi(x, t)\)

\[
i\hbar \phi_t + \frac{\hbar^2}{2m} \phi_{xx} - g_1(|\phi|^2 - n)\phi = -\hbar \int \phi_x g_1 n(F^2 - 1)(|\phi|^2 - n)\phi - 2if(t)g_1\phi^2\phi_x^* \tag{21}
\]

In conclusion, we have shown that considering a grey soliton as a quasiparticle in the spirit of the Landau theory of superfluidity, one can obtain a simple solution of the problem of soliton motion in a trapped 1D condensate. The energy and the shape of the soliton are preserved during its motion and soliton moves in a trapping potential as a particle of mass \(2m\).

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