Distributed Voting in Beep Model
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Abstract—In this paper, we propose two distributed algorithms, named Distributed Voting with Beeps (DVB), for multi-choice voting in beep model where each node can communicate with its neighbors merely by sending beep signals. The two proposed algorithms have simple structures and can be employed in networks having severe constraints on the size of messages. In the first proposed algorithm, the adjacent nodes having the same value form a set called spot. Afterwards, the spots with majority value try to corrode the spots with non-majority values. The second proposed algorithm is based on pairwise interactions between nodes. The proposed algorithms have a termination detection procedure to check whether voting is achieved. We show that the probability of success of the first algorithm goes to one as the percentage of nodes choosing the majority vote as their initial values gets close to one. The second algorithm returns the correct vote with high probability. Our experiments show that the algorithms are fairly invariant to the network topology, initial distribution of values, and network size.

Index Terms—Distributed voting, majority voting, consensus, beep model, communication protocols.

I. INTRODUCTION

C
ommunication protocols can be divided into two main models: shared memory and message-passing [1], [2]. In the shared memory model, agents communicate with each other by writing in or reading from a global memory shared among them. On the other hand, message-passing model provides communication through sending and receiving messages by agents. The message based communication protocols can be categorized into different types, such as population protocol, stone-age protocol, and beep model [2].

In population protocol, pairs of agents, modeled as finite state machines, communicate with each other to update their states. The order of asynchronous pairwise interactions between agents is determined by a fair scheduler [3]. In the stone-age protocol, however, the interactions are not necessarily pairwise. This protocol is based on a network of finite state machines (nFSM) [4] and allows the agents to communicate asynchronously through rich-sized messages but restricts the capacity of their memory to a constant value [2]. Beep protocol, proposed by [5], gives the agents internal memory of logarithmic size but restricts the size of messages to one bit [2]. In beep model, which is synchronous, nodes can merely beep or listen in every time slot [5], [2].

Distributed communication protocols have been studied recently in systems biology [2], [6]. Some of the examples are in Chemical Reaction Networks (CRN) [7], [8], protein-protein interactions [9], and natural animal interactions [10], [11]. The population protocol, which models asynchronous pairwise communication between mobile agents, can be commonly seen in the nature [2] and a good example of studying it on natural life was given in [11]. Furthermore, in biology, memory of a biological cell is highly limited [12]. Hence, the stone-age protocol can be a good model for biological communications [2]. On the other hand, biological interactions carry small effective information [13], encouraging to use beep model for biological communication because of its one-bit messages [2]. For instance, it can be used for distributed voting among cells and DNA molecules [8].

A. Related Work

Although beep model is almost a newly developed communication protocol, it has been received much attention in recent years and various algorithms have been proposed for different distributed problems in this model. As examples of these problems, we can mention interval coloring and graph coloring [5], [14], [15], [16], leader election [17], [18], [19], [20], [21], [22], [23], maximal independent set [24], [25], [26], [27], [28], [29], minimum connected dominating set [30], network size approximation and counting [31], [32], [33], [34], deterministic rendezvous problem [35], naming problem [36], membership problem [37], [38], broadcasting [39], [40], [41], and consensus [42].

In the problem of consensus, each agent chooses an initial value and all agents try to decide on a common value from the set of initial values. In [42], a distributed consensus algorithm has been proposed for the beep model in a fully-connected network. First, for the sake of synchronization, authors in [42] proposed an algorithm for obtaining global synchronization clock out of local clocks of agents. For achieving consensus, they transformed the binary representation of values to a bit stream. After the synchronization, every agent beeps and listens in several defined time slots according to its bit stream. If the agent does not hear any beep, it concludes that all the agents have the same value and thus it outputs its value; otherwise, it outputs the default value. The agents can reach consensus in time \(O(\log l_{\text{min}})\), where \(l_{\text{min}}\) is the smallest initial value among agents [42].

Distributed voting is an extended version of consensus problem in which all the agents should decide on a value that has been selected by majority of agents as their initial values. In this paper, we propose a distributed voting algorithm in beep model. To the best of our knowledge, there is no distributed voting algorithm proposed so far in this model. Here, we review some of the voting algorithms in the other types of communication protocols. Some of these algorithms have been proposed for the binary case. Binary voting using randomized gossip algorithm [43], average consensus [44], two-sample voting [45], and via exponential distribution [46] are the...
examples of binary voting algorithms. There are also some methods for multi-choice voting, such as voting using pairwise asynchronous graph automata [47] and union and intersection operations [48]. It is noteworthy to mention that there exist some voting algorithms in the literature which perform voting using ranking [48], [49] and plurality consensus [50], [51], [52].

B. Our Contribution

In this paper, we propose Distributed Voting with Beeps 1 (DVB1) and Distributed Voting with Beeps 2 (DVB2) algorithms, which work in the beep model [5] for multi-choice voting and any arbitrary network topology. The two DVB algorithms have simple structures and both can be utilized in both wireless networks and biological networks with limited message size. The proposed algorithms have the following characteristics:

- Let $l_m$ and $l_{m'}$ be the levels with the first and second largest supporters in the network which select these levels as their initial values. In fully connected networks with $K$ number of levels, DVB1 algorithm returns the correct result with probability at least $(1 - \exp(-\sqrt{\#l_m(0)/\#l_{m'}(0)})) \times \exp(-(K - 1)\#l_m(0)/(\sqrt{\#l_m(0)}\#l_m(0) - 1))$ where $\#l_m(0)$ and $\#l_{m'}(0)$ are the number of nodes which select levels $l_m$ and $l_{m'}$ as their initial values, respectively. Thus, the probability of success in DVB1 goes to one as the ratio $\#l_m(0)/\#l_{m'}(0)$ increases. The DVB2 returns the correct vote with probability one.
- DVB1 algorithm is empirically shown to be fairly invariant to the network topology, initial distribution of values, and the population of nodes. DVB2 returns the correct result with high probability (w.h.p.) and it is also invariant to initial values and population according to [48].
- The two DVB algorithms include termination detection for voting as well as the distributed consensus.
- The time, message, and space complexities of DVB1 algorithm are respectively $O(KD \log(N))$, $O(ND \log(N) + NDK)$, and $O(\log(KD) + \log^2(N))$ where $K$ is the number of existing value levels, $D$ is the diameter of network, $N$ is the number of nodes, and $\log^2(.) = \log(\log(.))$. The time, message, and space complexities of DVB2 algorithm are $O(D\Delta^2(\log(\Delta))^2 \log(N)) + KD \Delta \log(\Delta) \log(N)$, $O(\log(KD N \log(N))$, and $O(K \log(K) + \log(D\Delta) + \log^2(\Delta))$, respectively, where $\Delta$ is the maximum degree in the network.

The remainder of the paper is organized as follows. Section II introduces the beep model and defines the distributed voting problem. The DVB1 and DVB2 algorithms are explained in Sections III and IV, respectively. The analysis of time, message, and space complexities of the two algorithms is given in Section V. Comparison to alternative approaches is reported in Section VI. The probability of success of DVB1 algorithm is analyzed both in exact and lower bounds in Section VII. The proposed algorithms are evaluated experimentally in Section VIII. Finally, Section IX concludes the paper and mentions the possible future direction.

II. System Model

In this section, we describe the system model and introduce the concepts that are used in the description of the proposed algorithms.

A. Beep Model

We assume that nodes are communicating with each other based on a beep model [5]. In this model, each node can send a beep signal and all its neighbor nodes will receive it. Note that every node can either beep or listen; therefore, listening while beeping is not possible in this model. Every node cannot distinguish the number of beeps it receives if these beep signals are sent at the same time. We assume that the global synchronization has been achieved, as also assumed in the original beep model [5], and the nodes send beep signals at the specified time slots.

B. Problem Definition

Consider a network of $N$ nodes where each node has an initial value. The network topology can be determined by a graph $G = (V,E)$ where $V$ is the set of vertices (nodes) as $V = \{1,2,...,N\}$ and $E$ denotes the set of edges existing at the network. The edge set can be determined as $E \subseteq V \times V$, such that $(i,j) \in E$ if and only if nodes $i$ and $j$ can communicate directly. We say that nodes $i$ and $j$ are neighbors if $(i,j) \in E$. The edges of network are all bidirectional. The diameter of network is denoted by $D$. We assume upper bounds on $N$ and $D$ (or just on $N$ which is also an upper bound on $D$) are already known by nodes.

We denote the value of node $i$ at time slot $t$ by $v_i(t)$. Moreover, we consider a set of $K$ levels for the values of nodes, i.e., $L = \{l_1,l_2,...,l_K\}$. We assume that all nodes know the number of levels, i.e., $K$.

Definition 1. We denote the number of nodes having value $l_j$ at time slot $t$ by $\#l_j(t)$. Thus, we have: $\#l_j(t) \triangleq \{i | i \in V, v_i(t) = l_j\}$.

Our goal is to design distributed algorithms based on the beep model such that all nodes find a level $l_k$ which is chosen by the majority of nodes as their initial values. In other words, we want to determine the level $l_k$ where $\#l_k(0) \geq \#l_i(0), \forall i \neq k$.

III. Distributed Voting with Beeps (DVB 1) Algorithm

The DVB1 algorithm is based on two concepts which we call “spot” and “corrosion of spots”. We first explain these concepts and then present the description of DVB1 algorithm.
A. Spots

Definition 2. A path between nodes $i$ and $j$ is denoted by $i \rightarrow j : i \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_r \rightarrow j$, $\{i_1, \ldots, i_r\} \subseteq V$ where any two node $i_k$ and $i_{k+1}$ on the path, $1 \leq k \leq r - 1$, are neighbors.

Definition 3. We say a set of nodes $V' \subseteq V$ form a spot if:

$$\forall i, j \in V', \exists i \rightarrow j : i \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_r \rightarrow j,$$

such that $v_i(t) = v_{i_1}(t) = v_{i_2}(t) = \cdots = v_{i_k}(t) = v_j(t)$.

Please note that several spots may exist in a network. Furthermore, there might be a node which has different value from all its neighbors. In this case, that node is forming a spot with merely itself. It is worth to mention that some similar concepts to “spot” can be found in literature; for example, “cluster” in [20] is defined as a set of nodes including one candidate node for leadership.

Example 1. As an example, suppose we have three different values $l_0$, $l_1$, and $l_2$ in the network. Figure 1-a depicts an example of initial values in a 2D mesh grid network. As shown in Fig. 1-b, the initial values form spots in the network. This network has spots of nodes $\{1, 2\}$, $\{3, 4, 6, 7, 8\}$, $\{5\}$, $\{9, 13\}$, $\{10\}$, $\{11, 12, 14, 15\}$, $\{16\}$. As it can be seen, some of the spots, e.g., $\{5\}$, are one-node spots.

B. Corrosion Of Spots

The main idea of the DVB1 algorithm is to shrink small spots, i.e., spots with a few number of nodes. We call this procedure “corrosion”. After multiple corrosions, large spots with similar values will merge and create one large spot containing all the nodes in the network. To draw an analogy, suppose that there are two neighbor spots and one of them is smaller than the other one. Imagine that the small spot is like a material which is sensitive to a special type of acid. The larger spot can be considered as the acid which corrodes the small spot until it disappears and solely the acid remains.

We propose DVB1 algorithm to corrode small spots. The DVB1 algorithm consists of two main subroutines “Corrosion()” and “TerminationDetection()” (see Algorithm 1 which is performed by every node $i$). The proposed algorithm executes iteratively until it terminates. We call each iteration of the algorithm a phase. In DVB1 algorithm, each phase consists of $T$ rounds for the corrosion subroutine. Every $D$ phases, the termination detection subroutine is run having at most $(K - 1)$ periods (see Fig. 2). As shown in this figure, a round and a period respectively include $K$ and $(D + 1)$ time slots. The details of timing of algorithm will be explained in next part.

Algorithm 1 DVB1 Algorithm

1: counter ← 0
2: repeat
3: \hspace{1em} counter ← counter + 1
4: \hspace{1em} Corrosion()
5: \hspace{1em} if counter = $D$ then
6: \hspace{2em} \hspace{1em} counter ← 0
7: \hspace{2em} \hspace{1em} TerminationDetection()
8: \hspace{1em} end if
9: until not terminated

Definition 4. A round $R_j$ is a sequence of $K$ time slots $(t_{j,1}, t_{j,2}, \ldots, t_{j,K})$ in which time slot $t_{j,k}$ is allocated to the level $l_k$.

In round $R_j$, at time slot $t_{j,k}$, each node having the value of $l_k$ beeps if it is alive in that round. Then, it decides to either remain alive or die for the next round with probability $p = \frac{1}{2}$ (see lines 5-10 in Algorithm 2). In line 7 in Algorithm 2, $U(0,1)$ denotes the uniform random number in the range $[0, 1]$. Please note that a node cannot beep until the end of corrosion in a phase if it dies after beeping in a round of that phase. In Algorithm 2, we check this condition using “isAllowedToBeep” variable.

Definition 5. In every phase, we call nodes allowed and not allowed to beep “alive” nodes and “dead” nodes, respectively. At the first round of every phase, all nodes are alive. Note that
dead nodes can hear but cannot beep until the end of corrosion of that phase.

If a node is alive, with a chance of $p$, it will remain alive and may beep again in the time slot allocated to its value level in the next round (see line 7 in Algorithm 2). Furthermore, if the majority of neighbor nodes of a node have a same value, let say $l_m$, we can expect that with a good chance, after passing several rounds in a phase, that node will merely hear beep in time slot $t_{j,m}$, and receive no beeps in other time slots of the round. Thus, if a node hears a beep at time slot $t_{j,m}$ and does not hear any other beeps at the other time slots of that round, it concludes that the majority of its neighbors may have the same value $l_m$ corresponded to the time slot $t_{j,m}$ and thus modifies its value to $l_m$ (see lines 15-17 in Algorithm 2).

**Proposition 1.** After $\lceil \log_2(N/\varepsilon) \rceil$ rounds in each phase, all nodes are dead with probability at least $1 - \varepsilon$.

**Proof.** For a node, the probability of being dead after $r$ rounds is $1 - p^r$. Since each node is dying independently, all nodes are dead after $r$ rounds with probability $(1 - p^r)^N$. Moreover, we know $(1 - p^r)^N \geq (1 - Np^r)$. Hence, the probability of all nodes being dead after $r$ rounds is at least $(1 - Np^r)$ which we want to be greater than $1 - \varepsilon$. In other words, we want $Np^r \leq \varepsilon$ where $\varepsilon$ is a small constant. Since $p = \frac{1}{2}$, we have $(1/2)^r \leq \frac{\varepsilon}{N}$. Thus, all nodes will be dead after $r \geq \log_2(N/\varepsilon)$ rounds with probability at least $1 - \varepsilon$. \hfill $\Box$

According to the above proposition, we can set the number of rounds in each phase, i.e. $T$, to $\lceil c_1 \log_2(N) \rceil$ where $c_1$ is a large enough constant, e.g., 20.

**Example 2.** Consider a network with $N = 16$ nodes (see Fig. 1). There are three levels of values ($K = 3$) and thus, every round includes three time slots. Figures 1-c, -d, and -e show possible results of the first, second, and third phases. Several phases are passed to reach consensus in the entire network.

In Fig. 1, the number of nodes having values $l_0$, $l_1$, and $l_2$ are respectively 7, 5, and 4. Thus, it is expected that the voting algorithm finally converges to value $l_0$. As can be seen in Fig. 1-f, all the nodes consensus on value $l_0$ and only one large unique spot has been left.

It is worth mentioning that four cases might happen for a node during a phase: (I) The node remains within its spot and therefore will not change its value because its value is equal to the values of all its neighbors (e.g., node 4 in figures 1-b and 1-c). (II) The node might be at the boundary of its spot and its spot does not have sufficient merit to be expanded at that point and thus the node changes its value to another value level (e.g., node 2 in figures 1-b and 1-c). (III) The node might be at the boundary of its spot and the majority of its neighbors have similar value; therefore, with a good chance, the node will merely hear beep at the time slot of its own value in one of the rounds close to the end of corrosion in that phase. Thus, the node keeps its own value in that phase (e.g., node 7 in figures 1-b and 1-c). (IV) The node is at the boundary of its spot but it hears at least two beeps in different time slots of the last round of the phase. In other words, at least its two different neighbors remain alive in the whole phase. In this case, the node does not change its value in that phase (e.g., node 9 in figures 1-b and 1-c).

**C. Termination Detection**

Every $D$ phases, the termination detection procedure is executed at the end of the phase to check whether voting is reached (line 7 in Algorithm 1). The procedure “Termination Detection($k$)” is described in Algorithm 3. This procedure contains at most $(K - 1)$ periods each of which contains $(D + 1)$ time slots.

**Definition 6.** A period $P_k$, allocated to the value level $l_k$, is a sequence of $(D + 1)$ time slots $(t_{k,1}, t_{k,2}, \ldots, t_{k,D+1})$.

**Algorithm 3 Termination Detection**

1: Terminated $\leftarrow$ 1
2: for $k$ from 1 to $K - 1$ do
3: if $v_i(t_{k,1}) = l_k$ then
4: beep
5: else if hears beep then
6: Terminated $\leftarrow$ 0 \hfill $\triangleright$ Has not terminated
7: end if
8: for $d$ from 1 to $D$ do
9: if Terminated = 0 or hears beep then
10: Terminated $\leftarrow$ 0
11: beep \hfill $\triangleright$ Tell others voting has not reached
12: end if
13: end for
14: if Terminated = 0 then
15: Break procedure
16: end if
17: end for
4 in Algorithm 3). Meanwhile, if a node hears beep while its value is different from the value of that period, it finds out that there are nodes with different value and voting has not reached yet. So, it sets its “Terminated” variable to zero (lines 5 and 6 in Algorithm 3). Every node which finds out the voting has not reached, will beep for the rest of that period (line 11 in Algorithm 3). Please note that $D$ time slots is sufficient to inform the most distant nodes (line 8 in Algorithm 3).

At the end of the period $P_k$, every node checks if the “Terminated” variable has been reset or not. If it is zero, the algorithm has not terminated yet and needs at least another phase of corrosion. Thus, the node will break the TerminationDetection() (line 15 in Algorithm 3) because it knows that there is no more need for checking termination. Otherwise, it starts a new period with another associated value level.

Notice that if the algorithm reaches the voting, i.e., all the nodes have the same majority value, no one hears a beep in any periods of the procedure. Hence, after $(K - 1)$ periods or equally after $(K - 1) \times (D + 1)$ time slot, they all decide to terminate. Please note that we need $(K - 1)$ periods instead of $K$ periods since the $K$-th period is redundant in checking termination.

IV. DISTRIBUTED VOTING WITH BEEPS (DVB) 2

ALGORITHM

The DVB2 algorihm is based on the Distributed Multi-choice Voting/Ranking (DMVR) algorithm [48] and results in correct voting for any network topology and any number of nodes with high probability (w.h.p) because (DMVR) [48] is correct with probability one. In fact, if the nodes already have unique IDs, the DVB2 algorithm eventually returns correct result with probability one. Although, its time complexity might be greater than the one of DVB1 algorithm if the maximum degree of network is large. The advantage of DVB2 over DVB1 is that it guarantees returning the correct result while its time complexity is higher than DVB1. Note that DMVR [48] assumes that one of the levels is in strict majority; here, the same assumption is considered in DVB2. It is noteworthy that with a slight change in DVB2, it can be used for distributed ranking of values in beep model.

A schematic of a phase in the DVB2 algorithm is shown in Fig. 3. Algorithm 4 describes the DVB2 algorithm which is performed by every node $i$. First, each node chooses a random ID from one to $Y$ (line 2 in Algorithm 4). According to URN model [53], if $Y := \lceil c_2 \Delta \log_2(\Delta) \rceil$ where $\Delta$ is the maximum degree in the network and $c_2$ is a large enough constant (e.g., 20), then with high probability (w.h.p) the neighbor nodes will have different IDs. It is noteworthy that the probability of having different IDs for neighbors goes to one by increasing $\Delta$ or $c_2$. This step can be omitted if the nodes already have unique IDs.

Afterwards, in $Y$ time slots, the nodes find out the IDs of their neighbors (and collecting them in a set $\text{ID}_v^i$) by beeping at the slot associated to their ID and listening at the other slots (lines 4-11 in Algorithm 4). Then, the phases (iterations) start where interactions and termination detection (every $D$ phases) are performed repeatedly until the convergence of voting (lines 16-23 in Algorithm 4). The termination detection in Algorithm 4 is the same as in the Algorithm 3.

Algorithm 4 DVB2 Algorithm

1: /* Choosing a random ID: */
2: $\text{ID}_v^i \sim \text{Uniform}\{1, 2, \ldots, Y\}$
3: /* Finding out IDs of neighbors: */
4: $\text{ID}_v^i \leftarrow \emptyset$
5: for $j$ from 1 to $Y$ do
6: if $\text{ID}_v^i = j$ then
7: beep
8: else if it hears beep then
9: $\text{ID}_v^i \leftarrow \text{ID}_v^i \cup \{j\}$
10: end if
11: end for
12: /* DMVR initialization: */
13: $V_i(0) \leftarrow v_i(0)$
14: /* The phases: */
15: counter $\leftarrow 0$
16: repeat $\triangleright$ every iteration is a phase
17: counter $\leftarrow$ counter + 1
18: Interactions($V_i$)
19: if counter $= D$ then
20: counter $\leftarrow 0$ $\triangleright$ reset the counter
21: TerminationDetection()
22: end if
23: until not terminated

The interactions include different steps. First, every node decides to invite or listen for invitation with a probability denoted by $p_{inv}$ (lines 2-6 in Algorithm 5).

Definition 7. The invitation of an inviter node is defined as a collision because an inviter cannot be invitee.

Proposition 2. Suppose that all nodes have the same degree. The probability that an inviter node can interact with one of its neighbor nodes is maximized when $p_{inv}$ is equal to 0.5.

Proof. Each node $i$ becomes inviter with probability $p_{inv}$. The probability that one of its neighbors is inviter, is also $p_{inv}$. Furthermore, the probability that its inviter neighbor selects it, is $1/\Delta$. Thus, the probability that at least one of its neighbors invites it, is $\sum_{j=1}^{\Delta} (p_{inv}/\Delta) = p_{inv}$. Therefore, the probability of being inviter for node $i$ while its neighbors do not invite it, is $p_{inv} (1 - p_{inv})$ which is maximized when $p_{inv} = 0.5$.

After deciding whether to be inviter or not, every inviter node randomly selects which neighbor node to invite, hoping that neighbor is not an inviter in this phase (line 9 in Algorithm 5). We have a loop of $Y$ times each of which contains $Y$ time slots. The indices of outer and inner loops are the IDs of inviter and listener (not inviter) nodes, respectively (lines 12-21 in Algorithm 5). The inviter node beeps at the time slot associated to both its ID ($\text{ID}_v^i$) and its chosen neighbor’s ID ($\text{ID}_n^{\text{inviter}}$). The listener node collects the IDs of nodes which have invited it in a set ($\text{ID}_l^{\text{inviter}}$). After these nested loops, if $\text{ID}_l^{\text{inviter}} = \emptyset$ for a listener node, it understands no one has invited it; otherwise, we call that node an invitee.
The invitee node chooses one of its inviters randomly whose ID is denoted by \( \text{ID}_{i}^{\text{inviter}} \) (line 24 in Algorithm 5). In \( Y \) time slots indexing the ID of invitee, the invitee node beeps at the time slot associated to its chosen inviter’s ID. If the inviter hears a beep in the time slot of its ID, it finds out that its invitation has been accepted (lines 26-35 in Algorithm 5).

The DVB2 algorithm uses DMVR algorithm [48] which guarantees the correct voting using a gossiping mechanism.

**Algorithm 5 Interactions (first part)**

1: /* decide to be inviter or listener: */
2: if \( U(0, 1) < 0.5 \) then
3: Inviter ← 1
4: else
5: Inviter ← 0
6: end if
7: /* invite or listen for invitation: */
8: if Inviter then
9: \( \text{ID}_{i}^{\text{inviter}} \) ← choose randomly from \( \text{ID}_{i} \)
10: end if
11: \( \text{ID}_{1}^{\text{inviter}} \) ← \( \emptyset \) // used if node is not invitee
12: for \( j_{1} \) from 1 to \( Y \) do
13: for \( j_{2} \) from 1 to \( Y \) do
14: if Inviter and \( \text{ID}_{i} = j_{1} \) and \( \text{ID}_{i}^{\text{inviter}} = j_{2} \) then beep
15: end if
16: if not Inviter and hears beep then
17: \( \text{ID}_{i}^{\text{inviter}} \) ← \( \text{ID}_{i}^{\text{inviter}} \) ∪ \( \{j_{1}\} \)
18: end if
19: end for
20: end for
21: /* invitee chooses an inviter: */
22: if not Inviter then
23: \( \text{ID}_{i}^{\text{inviter}} \) ← choose randomly from \( \text{ID}_{i}^{\text{inviter}} \)
24: end if
25: for \( j_{1} \) from 1 to \( Y \) do
26: if Inviter and \( \text{ID}_{i} = j \) and hears beep then
27: accepted ← 1
28: else
29: accepted ← 0
30: end if
31: if not Inviter and \( \text{ID}_{i}^{\text{inviter}} = j \) then beep
32: end if
33: end for
34: end if
35: end for

**Algorithm 6 Interactions (second part)**

1: /* accepted inviter sends its value set and value (memory) to invitee: */
2: \( \mathcal{V}_{i}^{\text{inviter}}(t) \) ← \( \emptyset \) // used if node is invitee
3: for \( j \) from 1 to \( Y \) do
4: /* send/receive value (memory): */
5: for \( k \) from 1 to \( K \) do
6: if Inviter and accepted and \( \text{ID}_{i} = j \) then
7: if \( k \in \mathcal{V}_{i}(t) \) then beep
8: end if
9: end if
10: if not Inviter and \( \text{ID}_{i}^{\text{inviter}} = j \) and hears beeps then
11: \( \mathcal{V}_{i}^{\text{inviter}}(t) \) ← \( \mathcal{V}_{i}^{\text{inviter}}(t) \cup \{k\} \)
12: end if
13: end for
14: /* send/receive value (memory): */
15: for \( k \) from 1 to \( K \) do
16: if Inviter and accepted and \( \text{ID}_{i} = j \) then
17: if \( k = \mathcal{V}_{i}(t) \) then beep
18: end if
19: end if
20: if not Inviter and \( \text{ID}_{i}^{\text{inviter}} = j \) and hears beeps then
21: \( \mathcal{V}_{i}^{\text{inviter}}(t) \) ← \( k \)
22: end if
23: end for
24: end for
25: /* DMVR rules by the invitee: */
26: if not Inviter and \( \text{ID}_{i}^{\text{inviter}} \neq \emptyset \) (i.e., was invited) then
27: \( \mathcal{V}_{i}(t^{+}), \mathcal{V}_{i}^{\text{inviter}}(t^{+}) \) ← \( \text{DMVR}(\mathcal{V}_{i}(t), \mathcal{V}_{i}^{\text{inviter}}(t), \mathcal{V}_{i}(t^{+}), \mathcal{V}_{i}^{\text{inviter}}(t^{+})) \)
29: \( \mathcal{V}_{i}(t^{+}), \mathcal{V}_{i}^{\text{inviter}}(t^{+}) \) ← \( \text{DMVR}(\mathcal{V}_{i}(t), \mathcal{V}_{i}^{\text{inviter}}(t), \mathcal{V}_{i}(t), \mathcal{V}_{i}^{\text{inviter}}(t)) \)
30: end if

In the DMVR, each node has a value set \( \mathcal{V}_{i}(t) \) and a memory at the current time \( t \). In our work, we can consider the value \( \mathcal{V}_{i}(t) \) of a node as its memory in DMVR. The value set of every node is initialized to the initial value of the node (line 13 in Algorithm 4). We have a loop of \( Y \) iterations for indexing the ID of inviter and two back-to-back inner loops of \( K \) time slots for encoding the value set and the value (lines 3-26 in Algorithm 6). The inviter sends its value set and value to the invitee by these nested loops, and the invitee collects them \( (\mathcal{V}_{i}^{\text{inviter}}(t) \text{ and } \mathcal{V}_{i}^{\text{inviter}}(t)) \).

When the invitee node receives the value set and value of
Algorithm 7 Interactions (third part)

1: /* invitee sends the updated value set and value (memory) of inviter back to it */
2: if Inviter and accepted then
3: \( V_i(t) \leftarrow \emptyset \)
4: end if
5: for \( j \) from 1 to \( Y \) do
6: for \( k \) from 1 to \( K \) do
7: if Inviter and accepted and \( ID_i = j \) and hears beep then
8: \( V_i(t^+) \leftarrow V_i(t) \cup \{ k \} \)
9: end if
10: if not Inviter and \( ID_i^{\text{inviter}} = j \) then
11: \( V_i(t^+) \leftarrow V_i(t) \cup V_i(t^+) \)
12: beep
13: end if
14: end if
15: end for
16: for \( k \) from 1 to \( K \) do
17: if Inviter and accepted and \( ID_i = j \) and hears beep then
18: \( V_i(t^+) \leftarrow V_i(t) \cup \{ k \} \)
19: end if
20: if not Inviter and \( ID_i^{\text{inviter}} = j \) then
21: \( V_i(t^+) \leftarrow V_i(t) \cup V_i(t^+) \)
22: beep
23: end if
24: end if
25: end for
26: end for

the inviter, it applies the DMVR rules on its and the inviter’s value set and value (line 29 in Algorithm 6). The DMVR rules [48] are given in Algorithm 8. After applying the DMVR rules, the invitee sends the updated value set and value of the inviter. Afterwards, the inviter collects them using the similar loops (see Algorithm 7). The updated value set and value of a node are denoted by \( V_i(t^+) \) and \( v_i(t^+) \), respectively.

V. COMPLEXITY ANALYSIS OF DVB ALGORITHMS

In this section, we analyze the time, message, and space complexities of the DVB1 and DVB2 algorithms. Note that in DVB1, a phase propagates values by one distance hop; therefore, at least \( D \) phases are required. Here, we assume that DVB1 is executed for \( O(D) \) phases.

A. Complexity Analysis of DVB1

Proposition 3. The time complexity of DVB1 algorithm is in the order of \( O(KD\log(N)) \).

Proof. As can be seen in Fig. 2, the DVB1 algorithm is performed iteratively. In each phase, the corrosion takes \( K \times \lceil c_1 \log(N) \rceil \) time slots because it consists of \( \lceil c_1 \log(N) \rceil \) rounds, each of which includes \( K \) time slots. Assuming that we have \( O(D) \) phases, the complexity of corrosion is \( O(DK\lceil c_1 \log(N) \rceil) = O(KD\log(N)) \). Moreover, at the end of every \( D \) phases, up to \( (K - 1) \) periods are performed, each of which includes \( (D + 1) \) time slots. This results in \( O(KD) \) time slots in worst case for checking termination. Therefore, the time complexity of algorithm is in the order of \( O(KD\log(N)) \).

Corollary 1. As we will see later in Section VIII-A1, the DVB1 algorithm for fully connected network does not require checking termination and can be executed solely for one phase. Therefore, the time complexity of DVB1 algorithm for fully connected network is \( O(K\log N) \).

Proposition 4. The message complexity of DVB1 algorithm is in the order of \( O(ND\log(N) + NDK) \) bits.

Proof. We consider every beep as one bit message. The corrosion consists of \( \lceil c_1 \log(N) \rceil \) rounds, in which every node probably beeps only once when its value is equal to the value of time slot. In the worst case, all nodes beep until the last round of corrosion. Assuming that we have \( O(D) \) phases, the message complexity of corrosion phase is \( O(ND\lceil c_1 \log(N) \rceil) \). In checking termination, there are at most \( (K - 1) \) periods. In each period, in the worst case, each node beeps for \( (1 + D) \) times. As termination is checked every \( D \) phases and we have \( O(D) \) phases, its message complexity is \( O(NKD) \) for \( N \) nodes. Thus, the message complexity of the algorithm is in the order of \( O(ND\lceil c_1 \log(N) \rceil + NKD) = O(ND\log(N) + NDK) \).

Corollary 2. As the DVB1 algorithm for fully connected network can be executed solely for one phase, the message...
The complexity of DVB1 algorithm in fully connected networks is in the order of $O(N \log(N))$.

**Proposition 5.** The space complexity of DVB1 algorithm is in the order of $O(\log(KD) + \log^2(N))$ where $\log^2(\cdot) = \log(\log(\cdot))$.

**Proof.** Every node should have a memory for counting time slots of corrosion and checking termination. The amount of memory needed for encoding the value is $\log(K)$, for corrosion is $\log(Kc)\log(N)$), for checking termination is $\log((K-1)(D+1))$, and for counting every $D$ phases for checking termination is $O(\log(D))$. Thus, the total required space is $O(\log(K) + \log(Kc)\log(N)) + \log((K-1)(D+1)) + O(\log(D)) = O(\log(K\log(N)) + \log(KD)) = O(\log(KD) + \log^2(N))$. □

**Corollary 3.** As the DVB1 algorithm for fully connected network can be executed solely for one phase, the space complexity of DVB1 algorithm in fully connected networks is in the order of $O(\log(K\log(N)))$.

**B. Complexity Analysis of DVB2**

**Proposition 6.** The time complexity of DVB2 algorithm is in the order of $O(D\Delta^2(\log(\Delta))^2 \log(N) + DK\Delta \log(\Delta) \log(N))$.

**Proof.** As can be seen in Fig. 3, a phase excluding termination detection includes $O(Y^2 + Y + 2YK + 2YK) = O(Y^2 + YK)$ time slots. The DMVR algorithm [48] has the time complexity $O(D\log(N))$ for a large class of network topologies when all the nodes randomly interact with one another at every phase. In our work, according to the proof of Proposition 2, the probability that two nodes interact without collision is $p_{inv}(1-p_{inv}) = 1/4$. Therefore, the number of phases is in the order of $O(D\log(N))$ and the total time complexity excluding termination detection is $O(DY^2 \log(N) + DYK \log(N))$. As the termination detection is performed every $D$ phases, its complexity is $O(KD^2 \log(N)/D) = O(KD \log(N))$. □

**Proposition 7.** The message complexity of DVB2 algorithm is in the order of $O(DKN \log(N))$.

**Proof.** In finding out neighbors’ IDs, we have one beep per node resulting in $O(N)$. In inviting, $N/2$ nodes are expected to be inviters (in worst case) each beeping once resulting in $O(N)$. Similarly, when invitee nodes notify inviters, we have $O(N)$ number of messages. Then, in worst case, $N$ inviters send $NK$ and $N$ beeps for their value set and value, respectively. This also holds for returning the updated value sets and values. The number of phases is in the order of $O(D\log(N))$. According to the explanation in proof of Proposition 4, the complexity of termination detection is $O(NKD^2 \log(N)/D) = O(NKD \log(N))$. Thus, the overall complexity is $O(D\log(N)(N+N+N+N(K+1)+N(K+1))+DNK \log(N)) = O(DKN \log(N))$. □

**Proposition 8.** The space complexity of DVB2 algorithm is in the order of $O(K\log(K) + \log(D\Delta) + \log^2(\Delta))$.

**Proof.** The amount of memory needed in every node for steps shown in Fig. 3 is $O(\log(Y)), O(\log(Y^2)), O(\log(Y)), O(\log(KY)), O(\log(KY))$ and $O(\log(KD))$, respectively. Also, counting every $D$ phases for checking termination needs $O(\log(D))$ memory. Moreover, encoding the value (memory) and the value set require $O(\log(K))$ and $O(K \log(K))$, respectively. Thus, the total required space is $O(\log(Y) + 2\log(Y) + \log(KY) + \log(KD) + \log(D) + \log(K) + K \log(K)) = O(\log(K\log(N) + \log(KDY)) = O(K\log(K) + \log(D\Delta) + \log^2(\Delta))$. □

Comparing the complexities of DVB1 and DVB2 shows that DVB1 has better time and message complexities especially when $\Delta$ or $D$ are large. Noticing that $K$ is usually a small integer, if the maximum degree of network is small enough, DVB2 has better space complexity than DVB1; otherwise, DVB1 is better in terms of space complexity.

**VI. COMPARISON TO ALTERNATIVE APPROACHES**

One can consider some alternative approaches for distributed voting in beep model based on distributed primitives such as leader election or broadcasting. However, these approaches may not be applicable in practice for several reasons such as node failures or stringent resource constraints. For instance, in the literature of gossip algorithms, the main works on computing average [43], [54], voting [47], or sum [55], [56] do not utilize such distributed primitives. In fact, functions are computed merely by local interactions among nodes and there is a common belief in the literature that these solutions are much more robust to single point of failures or unreliable network conditions [54]. However, for the sake of completeness, we compare the complexities of three possible alternative approaches with ours and show that our algorithm performs better in terms of time, space, and message complexities. For fair comparison, we assume upper bounds on $N$ and $D$ are already known to these approaches. In the following, we first briefly describe the distributed primitives that are used in these alternative approaches and provide their time, message, and space complexity. Note that the time complexities are based on those reported in the cited papers and the space and message complexities are based on our analysis of those algorithms.

**Leader Election:** In this distributed primitive, exactly one of the nodes is elected as a leader. A time-optimal algorithm for leader election [17], [18] has been recently proposed in beep model. The eventual version of leader election should be used to know the time of termination which is essential when it is used with other algorithms.

- **Time complexity:** $O(D + \log N)$.
- **Space complexity:** $O(N \log N)$.
- **Message complexity:** $O(N(D + \log N))$.

**Broadcasting:** In this distributed primitive, a message with length $\log M$ from a source node is sent to all other nodes in the network. The beep wave [20], [40] algorithm has been proposed to send all bits from the source to all other nodes sequentially in time.

- **Time complexity:** $O(D + \log M)$.
- **Space Complexity:** $O(\log M)$. 


Multi-Broadcasting: In this distributed primitive, a set of source nodes $S$ broadcast their messages, each with length $\log M$ to all nodes in the network. For simplicity in notations, $S$ also denotes the cardinality of this set. The multi-broadcasting with provenance [40] has been proposed to solve this problem where each node in $S$ sends its message as well as its ID so that nodes can distinguish the origin of each message. In our analysis for this solution, we exclude leader election, diameter estimation, and message length calculation (because $\log M$ (or $\log K$) is known here). We also exclude collecting and broadcasting IDs because here all nodes are in $S$ and assuming that nodes are aware of $N$, they can infer the lexicographical order of IDs.

- Time complexity: $O(D + S \log M)$.
- Space complexity: $O(\log N + S \log M)$.
- Message complexity: $O(NS \log M)$.

Message Gathering: In this problem, the messages of length $\log M$ from source nodes in the set $S$ are gathered in the leader node. In order to solve this problem, one can use the multi-broadcasting algorithm in [40] except its last step where the leader broadcasts the messages. Thus, its time and space complexities are equal to the multi-broadcasting algorithm and its message complexity is $O(S^2 \log M)$. Again, in our analysis, we excluded leader election, diameter estimation, message length calculation, and collecting/broadcasting IDs.

Based on the above distributed primitives, we can consider the following two alternative approaches:

1) First Alternative Approach: The first approach consists of the following steps: (i) One of the nodes is elected as a leader. (ii) All nodes send their values to the leader by executing the message gathering algorithm. (iii) Leader counts the votes and broadcasts the majority vote via the broadcasting algorithm.

By considering $M = K$ and $S = N$, the time, space, and message complexities of this approach are $O(D + \log N + N \log K)$, $O(N \log (KN))$, and $O(ND + N \log N + N^2 \log K)$, respectively.

2) Second Alternative Approach: The second approach includes these steps: (i) One of the nodes is elected as a leader. (ii) All nodes broadcast their values to all nodes in the network via multi-broadcasting. (iii) Each node counts the votes and obtains the majority vote itself. For this approach, the time, space, and message complexities are equal to the first approach.

Comparing the time, space, and message complexities of the two alternative approaches with the ones for DVB1 algorithm (which are $O(KD \log(N))$, $O(\log(KD) + \log^2(N))$, and $O(ND \log(N) + NDK)$, respectively) shows the proposed algorithm performs better in terms of time, space, and message complexities. Note that $K$ is most often a small integer and we usually have $D \ll N$.

If we compare the alternative approaches to the time, space, and message complexities of DVB2 algorithm (which are $O(DA^2(\log(\Delta))^2 \log(N)) + KD \Delta \log(\Delta) \log(N)$), $O(K \log(K) + \log(D \Delta) + \log^2(\Delta))$, and $O(KDN \log(N))$, respectively), we see that DVB2 outperforms in space and message complexities. It is also better in time complexity if $\Delta$ is small (e.g., in mesh grid or torus networks). Moreover, DVB2 is not prone to fail because of possible failure of leader and also does not require pre-defined unique IDs.

VII. Probability of Success for DVB1

In this section, we first propose two lower bounds for the probability of success of a phase in DVB1 algorithm when the topology is fully connected. Thereafter, we propose the exact probability of success of a phase, but not in a closed form, for binary voting in fully connected networks using Markov chain.

A. Lower Bound on The Probability of Success

Proposition 9. In DVB1 algorithm, a lower bound on the probability of success in fully connected networks is

$$\max_r \left[ 1 - \left( \left(1 - p^r\right)^{\#l_m(0)} + \left(\#l_{m'}(0)\right)p^r\left(1 - p^r\right)^{\#l_{m'}(0) - 1}\right) \right]$$

where $\#l_{m'}(0) > \#l_k(0)$, for all $k \neq m$.

Proof. Proof in Appendix A.

In order to get more insight about the behavior of DVB1 algorithm, we consider the probability of the event that at least one node remains alive in level $l_m$ and all nodes having other level $l_k \neq l_m$ are dead in a time slot $r$. This probability of success, as another lower bound, is as in the following.

Proposition 10. In DVB1 algorithm, a lower bound on the probability of success in fully connected networks is

$$\left(1 - \exp\left(-\sqrt{\frac{\#l_m(0)}{\#l_{m'}(0)}}\right)\right) \times \exp\left(-\frac{(K-1)\#l_{m'}(0)}{\sqrt{\#l_m(0)\#l_{m'}(0)}} - 1\right),$$

where $l_m$ and $l_{m'}$ are respectively the value levels with the largest and the second largest initial supporters.

Proof. Proof in Appendix B.

It is noteworthy to mention that, as it is expected, the lower bound of Proposition 10 gets close to one as the ratio $\#l_m(0)/\#l_{m'}(0)$ increases. The lower bounds in Propositions 9 and 10 are illustrated in Fig. 4 for one phase execution of binary voting in a fully connected network with 100 nodes.

Corollary 4. For $\#l_m(0)/\#l_{m'}(0) \geq K^2$, the probability of success of DVB1 is greater than $1 - \varepsilon$, if we have

$$\frac{\#l_m(0)}{\#l_{m'}(0)} \geq \frac{1}{4}\left(\ln(1 - \varepsilon) + \sqrt{\ln(1 - \varepsilon)^2 + 4K}\right)^2.$$

Proof. Proof in Appendix C.
B. Analysis Using Markov Chain

We study the probability of success of DVB1 algorithm in fully connected networks using Markov Chain model as explained in the following.

We define \( l_j \) as the number of alive nodes with level \( l_j \) which are still beeping in the current round. We define the state of the algorithm as a vector of remaining alive nodes in each level at the beginning of a round, i.e. \( S = (#l_1, #l_2, \ldots, #l_K) \). The winning states of level \( l_m \) are those states where \( #l_m \geq 1 \), and \( #l_k = 0 \) for any \( k \neq m \) or \( #l_m \geq 2 \) and \( #l_k \leq 1 \) for a \( k \neq m \) and \( #l_k = 0 \) for any \( k_2 \neq m, k_2 \neq k_1 \) (see Appendix A for more details). Furthermore, the draw state is equal to the vector of all zeros. We consider all winning states and the draw state as halting states.

Let \( #l_{j1} \) and \( #l_{j2} \) be the number of alive nodes having value \( l_j \) in two preceding rounds (before and after transition between states), respectively. It can be seen that the probability of transition from state \( S_1 = (#l_{11}, #l_{21}, \ldots, #l_{K1}) \) to another state \( S_2 = (#l_{12}, #l_{22}, \ldots, #l_{K2}) \) can be stated as follows (considering \( S_1 \) is not a halting state),

\[
\prod_{k=1}^{K} \left( \frac{#l_{k1}}{#l_{k1} - #l_{k2}} \right) (1-p)^{#l_{k1} - #l_{k2}} p^{#l_{k2}}.
\]

An execution of the algorithm can be interpreted as a random walk on this Markov chain which is started from the initial state \( S_1 = (#l_{11}(0), #l_{21}(0), \ldots, #l_{K1}(0)) \) and will be terminated in one of the halting states corresponding to the cases that one of the levels wins or we have a draw. We can compute the probability of success by summing over probabilities of being in the halting states which correspond to the cases that we get the correct result.

In Figure 4, the result of Markov chain is shown for simulating one phase of algorithm almost coincides with the result of simulations, as expected.

VIII. Simulations

A. Simulations for DVB1

1) Experiments on the size of majority level: We simulated DVB1 algorithm on fully connected, 2D mesh grid, and Erdős-Rényi (with probability \( \frac{2}{N} \log_2(N) \)) networks with 100 nodes. Figures 5 and 6 show the average results of simulation over 1000 number of experiments for binary and ternary voting, respectively. In experiments, the initial values are assigned randomly to the nodes such that the percentage of initial values for different levels are \( (1-\delta, \delta) \) and \( (\frac{2}{3}-\delta, \frac{1}{3}, \delta) \) for binary and ternary voting, respectively. The range of \( \delta \) is \( (\frac{1}{2}, 1) \) for binary and \( (0, \frac{1}{3}) \) for ternary voting. The \( D \) is taken to be its upper bound
The average number of phases until reaching the voting in the simulations of Figures 5 and 6 are depicted in Figures 7a and 7b, respectively. As can be seen in this Fig. 7a, the average time decreases by increasing $\delta$ because when $\delta$ is close to 0.5, the population of majority value is close to the other value and it takes more time for DVB1 algorithm to reach voting. The same analysis holds for $\delta$ close to 1/3 in Fig. 7b. Moreover, as shown in Figures 7a and 7b, the average number of phases until reaching the voting is almost always one in fully connected topology. We expect with an acceptable chance that the voting will get completed after only one phase since we gave the algorithm sufficient number of rounds for convergence as mentioned before. Therefore, for fully connected networks, DVB1 algorithm might be performed with only one phase excluding the termination detection.

2) Experiments on the number of nodes: We performed another experiment by changing the number of nodes in the network between 20-100 nodes for binary voting with $\delta = 2/3$. Figure 8 shows the average results of 1000 number of simulations for this experiment. The $D$ was set to its upper bound, i.e. $N$, in the simulations. As can be seen in this figure, the DVB1 algorithm is almost invariant with respect to the number of nodes in the network. Notice that this property also exists for ternary voting but for the sake of brevity the binary case is just shown here.

In overall, it was empirically shown that the DVB1 algorithm is fairly invariant to the following network attributes:
- The topology of network
- The initial distribution of values in the network
- The number of nodes

B. Simulations for DVB2

We simulated the DVB2 algorithm on fully connected, 2D mesh grid, and Erdős-Rényi with probability $\frac{2}{N} \log_2(N)$. The number of nodes and the number of simulations per any value of $\delta$ were 100 and 1000, respectively. The experiments and the sweeps over $\delta$ are similar to the set up in Section VIII-A1. According to [48], the DVB2 is invariant to the topology, the initial distribution of values, and the network size, and it has correct voting with probability one. Figures 9a and 9b show the average number of phases until reaching the voting in the simulations. As expected, the average number of phases increases when the $\delta$ is close to 0.5 (for binary voting) and 1/3 (for ternary voting). This is because in these cases, the number of value levels are very close to each other requiring more time for reaching the global voting. Moreover, comparing Figures 7a and 7b with 9a and 9b shows that the DVB2 algorithm requires more phases than DVB1 to reach voting which makes sense because of its time complexity. However,
is invariant to topology, initial values, and the network size. Our experiments verified the fairly invariance of this algorithm to the network topology, initial distribution of values, and the size of network. The second algorithm is based on DMVR algorithm [48] and is invariant to topology, initial values, and the network size. As a possible future work, one can focus on designing an asynchronous distributed voting algorithm using beep signals.

the DVB2 algorithm returns the correct results regardless of initial distribution of values or network topology.

IX. CONCLUSION & FUTURE WORK

In this paper, we proposed two simple-structure algorithm for distributed voting in beep model. The first algorithm is based on forming spots of nodes having the same values, and corroding the small (non-majority) spots against the large (majority) spots to reach the correct result. Our experiments verified the fairly invariance of this algorithm to the network topology, initial distribution of values, and the size of network. The second algorithm is based on DMVR algorithm [48] and is invariant to topology, initial values, and the network size. As a possible future work, one can focus on designing an asynchronous distributed voting algorithm using beep signals.

APPENDIX A

PROOF OF PROPOSITION 9

In fully connected networks, the DVB1 algorithm returns correct result in both of the following events:

- At least two nodes of level \( l_m \) remain alive while there is at most one node alive in one of the other values \( k \neq m \), or
- One alive node remains in level \( l_m \) and all nodes in other values \( l_k \neq l_m \) are dead, where \( \#l_m(0) > \#l_k(0), \forall k \neq m \).

In the first event, the value \( l_m \) wins definitely because the only alive node in another value \( l_k \) (\( k \neq m \)) hears beep from the alive nodes in value \( l_m \) (and not hears itself) in the current round and it changes its value to \( l_m \). However, the case in which there exists one alive node in each level should be deferred waiting to next rounds until all except one of them die (so that level wins) or all die. In the second event, level \( l_m \) wins because in the current round, all the nodes will hear beep from the alive node with value \( l_m \) and thus change their values to \( l_m \).

The probability of having at least two nodes alive in level \( l_m \) at round \( r \) (i.e., not having all dead or all except one dead) is

\[
1 - \left[ (1 - p^r)^{\#l_m(0)} + \binom{\#l_m(0)}{1} p^r (1 - p^r)^{\#l_m(0) - 1} \right].
\]

The probability of having all nodes dead in other levels \( l_k \) (\( k \neq m \)) at round \( r \) is \( \prod_{k=1, k \neq m}^K (1 - p^r)^{\#l_k(0)} \) as the probabilities are independent. Moreover, at round \( r \), the probability of having one alive node in value level \( l_k \) (\( k \neq m \)) while all nodes of other value levels are dead is

\[
\binom{\#l_k(0)}{1} p^r (1 - p^r)^{\#l_k(0) - 1} \prod_{k'=1, k' \neq m, k}^K (1 - p^r)^{\#l_{k'}(0)}. \]

This expression should be summed up over all value levels to cover all possibilities. Finally, the probability of the first event is

\[
\prod_{k=1, k \neq m}^K (1 - p^r)^{\#l_k(0)} + \binom{\#l_m(0)}{1} p^r (1 - p^r)^{\#l_m(0) - 1}. \]

The probability of the second event is

\[
\sum_{k=1}^K \binom{\#l_k(0)}{1} p^r (1 - p^r)^{\#l_k(0) - 1} \prod_{k'=1, k' \neq m, k}^K (1 - p^r)^{\#l_{k'}(0)}. \]

The overall probability of success at round \( r \) is at least

\[
1 - \left[ (1 - p^r)^{\#l_m(0)} + \binom{\#l_m(0)}{1} p^r (1 - p^r)^{\#l_m(0) - 1} \right] \times \prod_{k=1, k \neq m}^K (1 - p^r)^{\#l_k(0)} + \sum_{k=1}^K \binom{\#l_k(0)}{1} p^r (1 - p^r)^{\#l_k(0) - 1} \sum_{k'=1, k' \neq m, k}^K (1 - p^r)^{\#l_{k'}(0)}. \]

APPENDIX B

PROOF OF PROPOSITION 10

If in a time slot, at least one node remains alive in level \( l_m \) (i.e., not having all of them dead) and all nodes having other level \( l_k \neq l_m \) are dead, the voting is done correctly. By
maximizing over $r$, the probability of this event in a time slot $r$ is obtained as

$$
\max_r \left( 1 - (1 - p^r)^{\#m(0)} \right) \times \prod_{l_k \neq m} (1 - p^r)^{\#l_k(0)}.
$$

The above expression is a lower bound on the probability of success. Consider $r = \log_2(\sqrt{\#m(0)/\#l_m(0)})$. Assuming $p = 1/2$ and substituting the value of $r = \log_2(\sqrt{\#m(0)/\#l_m(0)})$ in the term $(1 - p^r)$, we have $(1 - p^r) = 1 - 1/\sqrt{\#m(0)/\#l_m(0)}$. We know that $(1 - p^r)^{\#l_k(0)} = \exp(-\#l_k(0)\ln(1 - p^r)) = \exp(-\#l_k(0)/\sqrt{\#m(0)/\#l_m(0)})$. Hence, $\ln(1 - p^r)$ results in the proposed lower bound.

Now, we consider the second part of expression which is $\prod_{l_k \neq m} (1 - p^r)^{\#l_k(0)}$. Similar to the previous approach, we have $(1 - p^r)^{\#l_k(0)} = \exp(-\#l_k(0)/\sqrt{\#m(0)/\#l_m(0)})$. As we have $x/(1+x) \leq \ln(1+x)$ for $x > -1$, we can say $(1 - p^r)^{\#l_k(0)} \geq \exp(-\#l_k(0)/\sqrt{\#m(0)/\#l_m(0)} - 1)$. We rewrite the second expression as $\max_r \prod_{l_k \neq m} (1 - p^r)^{\#l_k(0)} \geq \exp(-\#m(0)/\sqrt{\#m(0)/\#l_m(0)} - 1)) \geq \exp(-\#m(0)/\sqrt{\#m(0)/\#l_m(0)} - 1))$.

Finally, substituting the two derived expressions in the first and second expressions of $\left( 1 - (1 - p^r)^{\#m(0)} \right) \times \prod_{l_k \neq m} (1 - p^r)^{\#l_k(0)}$ results in the proposed lower bound.

## Appendix C

### Proof of Corollary 4

Assuming that $\#m(0)/\#l_m(0) \geq K^2$, the probability of success, proposed in Proposition 10, becomes greater than $(1 - \exp(-y)) \times \exp(-K/y)$ where $y = \sqrt{\#m(0)/\#l_m(0)}$. Thus, we need to have $\exp(-y) \times \exp(-K/y) \geq 1 - \epsilon$. Taking logarithm from both sides of this inequality, we have $\ln(1 - \exp(-y)) - K/y \geq \ln(1 - \epsilon)$. As we have $y > 0$ and $0 < \exp(-y) < 1$, we can say $\ln(1 - \exp(-y)) \leq -\exp(-y) \leq y$. Therefore, $y - K/y \geq \ln(1 - \exp(-y)) - K/y \geq \ln(1 - \epsilon)$ which results in $y \geq 0.5 \left( \ln(1 - \epsilon) + \sqrt{\ln(1 - \epsilon)^2 + 4K} \right)$.
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