LYAPUNOV TYPE INEQUALITIES FOR \textit{n}TH ORDER FORCED DIFFERENTIAL EQUATIONS WITH MIXED NONLINEARITIES

RAVI P. AGARWAL
Department of Mathematics, Texas A&M University-Kingsville,
700 University Blvd., Kingsville, TX 78363-8202, USA

ABDULLAH ÖZBEKLER*
Department of Mathematics, Atilim University
06836, Incek, Ankara, Turkey

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Abstract. In the case of oscillatory potentials, we present Lyapunov type inequalities for \textit{n}th order forced differential equations of the form

\[ x^{(n)}(t) + \sum_{j=1}^{m} q_j(t)|x(t)|^{\alpha_j-1}x(t) = f(t) \]

satisfying the boundary conditions

\[ x(a_i) = x'(a_i) = x''(a_i) = \cdots = x^{(k_i)}(a_i) = 0; \quad i = 1, 2, \ldots, r, \]

where \( a_1 < a_2 < \cdots < a_r, 0 \leq k_i \) and

\[ \sum_{j=1}^{r} k_j + r = n; \quad r \geq 2. \]

No sign restriction is imposed on the forcing term and the nonlinearities satisfy

\[ 0 < \alpha_1 < \cdots < \alpha_j < 1 < \alpha_{j+1} < \cdots < \alpha_m < 2. \]

The obtained inequalities generalize and compliment the existing results in the literature.

1. Introduction. Consider the Hill’s equation

\[ x''(t) + q(t)x(t) = 0; \quad a \leq t \leq b, \]

where \( q(t) \in L^1[a,b] \) is a real-valued function. If there exists a nontrivial solution \( x(t) \) of Eq. (1) satisfying the Dirichlet boundary conditions

\[ x(a) = x(b) = 0, \]

where \( a, b \in \mathbb{R} \) with \( a < b \) and \( x(t) \neq 0 \) for \( t \in (a,b) \), then the inequality

\[ \int_{a}^{b} |q(t)|dt > \frac{4}{b-a} \]

holds. This striking inequality was first proved by Lyapunov \cite{Lyapunov} and it is known as “Lyapunov inequality”. Later Wintner \cite{Wintner} and thereafter some authors achieved...
to replace the function $|q(t)|$ in Ineq. (3) by the function $q^+(t)$ i.e. they obtained the following inequality:

$$\int_a^b q^+(t)dt > \frac{4}{b-a}, \quad (4)$$

where $q^+(t) = \max\{q(t), 0\}$. The constant “4” in the right hand side of inequalities (3) and (4) is the best possible largest number (see [37] and [29, Thm. 5.1]).

In [29], Hartman obtained an inequality sharper than both (3) and (4):

$$\int_a^b (b-t)(t-a)q^+(t)dt > b-a. \quad (5)$$

Clearly, Ineq. (5) implies Ineq. (4), since

$$\frac{(b-t)(t-a)}{4(b-a)^2} \leq 1$$

for all $t \in (a,b)$, and equality holds when $t = (a+b)/2$.

It appears that the first generalization of Lyapunov’s result for the equation

$$x^{(n)}(t) + q(t)x(t) = 0; \quad a \leq t \leq b, \quad (7)$$

satisfying the $n$-point boundary conditions

$$x(a_k) = 0; \quad k = 1, 2, \ldots, n, \quad (8)$$

was obtained by Yang [55, Thm. 2], where $n \in \{2, 3, \ldots\}$ and $q(t)$ is a locally Lebesgue integrable real-valued function defined on $\mathbb{R}$.

**Theorem 1.1** (Lyapunov type inequality). Let Eq. (7) has a nontrivial solution satisfying the $n$-point boundary conditions (8) where $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, with $a_1 < a_2 < \cdots < a_{n-1} < a_n$ are consecutive zeros. Then the inequality

$$\int_a^b |q(t)|dt > \frac{(n-2)!a^{n-1}}{(n-1)^{n-2}(b-a)^{n-1}}; \quad n = 2, 3, \ldots \quad (9)$$

holds, where $a = a_1$ and $b = a_n$.

We remark that when $n = 3$, Ineq. (9) reduces to

$$\int_a^b |q(t)|dt > \frac{9}{2(b-a)^2} \quad (10)$$

which is sharper than

$$\int_a^b |q(t)|dt > \frac{4}{(b-a)^2} \quad (11)$$

given by Parhi and Panigrahi [46]. However, when $n = 2$, Ineq. (9) becomes

$$\int_a^b |q(t)|dt > \frac{2}{b-a}$$

which is weaker than the classical Lyapunov inequality, i.e., Ineq. (4) (and also Ineq. (3)).

Thereafter, using analogous techniques as in the proof of Thm. 1.1, Çakmak [15, Thm. 1] improved the Yang’s result.
Theorem 1.2 (Lyapunov type inequality). Let Eq. (7) has a nontrivial solution satisfying the n-point boundary conditions (8) where \(a_k \in \mathbb{R}, k = 1, 2, \ldots n\), with \(a_1 < a_2 < \cdots < a_{n-1} < a_n\) are consecutive zeros. Then the inequality

\[
\int_a^b |q(t)| dt > \frac{(n-2)!n^n}{(n-1)^{n-1}(b-a)^{n-1}}; \quad n = 2, 3, \ldots
\]

(12)

holds, where \(a = a_1\) and \(b = a_n\).

We note that when \(n = 3\), Ineq. (12) reduces to

\[
\int_a^b |q(t)| dt > \frac{27}{4(b-a)^2}
\]

(13)

which is sharper than the results of Parhi and Panigrahi [46], and Yang [55]. Moreover, when \(n = 2\), Ineq. (12) turns into the classical Lyapunov inequality, i.e., Ineq. (3). The proofs of Thm. 1.1 (and Thm. 1.2) are based on Green’s function of the equation

\[-x^{(n)}(t) = 0; \quad n = 2, 3, \ldots \]

(14)

satisfying the n-point boundary conditions (8).

It appears that an explicit form of Green’s function \(g_n(t,s)\) for the n-point boundary value Prb. (14)-(8) was first given by Das and Vatsala [19] in 1973. One of the aims of this paper is to extend and improve the results given in Thm.’s 1.1 and 1.2 to the forced differential equations with mixed nonlinearities under the general boundary conditions than those in (8). Our proofs are based on Green’s function \(G_n(t,s)\) of Eq. (14) satisfying the boundary conditions

\[x(a_i) = x'(a_i) = x''(a_i) = \cdots = x^{(k_i)}(a_i) = 0; \quad i = 1, 2, \ldots r,\]

(15)

where \(a_1 < a_2 < \cdots < a_r\) and

\[\sum_{j=1}^{r} k_j + r = n; \quad k_i \geq 0 \quad \text{for all} \quad i = 1, 2, \ldots r.\]

In 1962, Beesack [11] proved that Green’s function \(G_n(t,s)\) of Prb. (14)-(15) satisfies the inequality

\[|G_n(t,s)| \leq \frac{1}{(n-1)!(a_r-a_1)} \prod_{i=1}^{r} |t-a_i|^{k_i+1}\]

(16)

for \(a_1 < s < a_r\) and \(-\infty < t < \infty\). He also showed that

\[|G_n(t,s)| \leq \left(1 - \frac{1}{n}\right)^{n-1} \frac{(a_r-a_1)^{n-1}}{n!}\]

(17)

for \(a_1 < t, s < a_r\) by using the inequality

\[\prod_{i=1}^{r} |t-a_i|^{k_i+1} \leq \left(1 - \frac{1}{n}\right)^{n-1} \frac{(a_r-a_1)^{n}}{n}\]

(18)

for \(a_1 < t < a_r\).

In 1983, Agarwal [2] extended Ineq. (18) to the inequality

\[\prod_{i=1}^{r} |t-a_i|^{k_i+1} \leq \frac{(n-\sigma-1)^{n-\sigma-1}}{n^n} (\sigma+1)^{\sigma+1} (a_r-a_1)^n,\]

(19)
where \( \sigma = \min\{k_1, k_r\} \). Now, using Ineq.’s (16) and (19), it follows that Green’s function \( G_n(t, s) \) of Prb. (14)-(15) satisfies the inequality
\[
|G_n(t, s)| \leq \frac{(n - \sigma - 1)^{\alpha-1}}{n^{\alpha-1} n!} (\sigma + 1)^{\alpha+1} (a_r - a_1)^{n-1}.
\] (20)

We note that when \( \sigma = 0 \), then Ineq. (20) reduces to Ineq. (17).

The Lyapunov inequality and its generalizations have been used successfully in connection with oscillation and Sturmian theory, asymptotic theory, disconjugacy, eigenvalue problems and various properties of the solutions of (1) and related equations, see for instance [57, 29, 5, 12, 13, 14, 17, 18, 25, 31, 35, 36, 38, 39, 40, 41, 44, 48, 49, 50] and the references cited therein. For some of its extensions to Hamiltonian systems, higher order differential equations, nonlinear and half-linear differential equations, difference and dynamic equations, functional and impulsive differential equations, we refer in particular to [17, 18, 15, 16, 21, 22, 23, 24, 27, 28, 20, 32, 34, 42, 43, 45, 46, 47, 51, 53, 54, 55, 56, 58]. Further, no more Lyapunov and Hartman type inequalities are known for higher-order nonlinear differential equations.

Motivated by the above works for linear equations, in this paper we will find analogs of well-known Lyapunov type inequalities for more general equations of the form
\[
x^{(n)}(t) + \sum_{i=1}^{m} q_i(t)|x(t)|^\alpha_i x(t) = f(t),
\] (21)

satisfying the \( r \)-point boundary conditions (15), where \( n, m \in \mathbb{N} \), the potentials \( q_i(t), i = 1, \ldots, m \), the forcing term \( f(t) \) are real-valued functions, and no sign restrictions are imposed on them. Further, the exponents in Eq. (21) satisfy
\[
0 < \alpha_1 < \cdots < \alpha_j < 1 < \alpha_{j+1} < \cdots < \alpha_m < 2.
\]

It is clear that the two special cases of Eq. (21) are the \( n \)th order forced sub-linear equation
\[
x^{(n)}(t) + q(t)|x(t)|^{\gamma-1} x(t) = f(t); \quad 0 < \gamma < 1
\] (22)

and the \( n \)th order forced super-linear equation
\[
x^{(n)}(t) + p(t)|x(t)|^{\beta-1} x(t) = f(t); \quad 1 < \beta < 2.
\] (23)

Further, we note that letting \( \alpha_i \to 1^- \), \( i = 1, \ldots, j \) and \( \alpha_i \to 1^+ \), \( i = j + 1, \ldots, m \) in (21) results in
\[
x^{(n)}(t) + \nu(t)x(t) = f(t),
\] (24)

where
\[
\nu(t) = \sum_{i=1}^{m} q_i(t).
\] (25)

Since when \( r = n \) the boundary conditions (15) reduce to (8), our results extent and improve the main results of Yang [55, Thm. 2] and Çakmak [15, Thm 1], i.e. Thm’s. 1.1 and 1.2 even when \( f(t) = 0 \), and in particular the classical Lyapunov [37] inequality, see Cor. 5 and Remark 2.

We further remark that the Lyapunov type inequalities have been studied by many authors, see for instance the survey paper [52] and the references therein, but to the best of our knowledge there are no results in the literature for Eq. (21), and in particular for Eq.’s (22), (23) and (24).
2. **Main results.** Throughout this paper we shall assume that \( q_i(t) \in L^1[a, b] \), \( i = 1, \ldots, m \).

In what follows we will need the following lemma, see [8, 10, Lemma 2.1] (we include its proof here for completeness).

**Lemma 2.1.** If \( A \) is positive, and \( B, z \) are nonnegative, then
\[
A z^2 - B z^\mu + (2 - \mu)z^{\mu/(2-\mu)}B^{2/(2-\mu)} \geq 0
\]
for any \( \mu \in (0, 2) \) with equality holding if and only if \( B = z = 0 \).

**Proof.** Let
\[
\mathcal{H}(z) = Az^2 - B z^\mu; \quad z \geq 0
\]
where \( A > 0 \) and \( B \geq 0 \). Clearly, when \( z = 0 \) or \( B = 0 \), (26) is obvious. On the other hand, if \( B > 0 \), then it is easy to see that \( \mathcal{H} \) attains its minimum at \( z_0 = (\mu A^{-1}B/2)^{1/(2-\mu)} \) and
\[
\mathcal{H}_{\text{min}} = -(2 - \mu)z_0^\mu B^{2/(2-\mu)} A^{-\mu/(2-\mu)} B^{2/(2-\mu)}.
\]
Thus, (26) holds. Note that if \( B > 0 \), then Ineq. (26) is strict. \( \square \)

Now we state and prove our first result.

**Theorem 2.2** (Lyapunov type inequality). Let \( x(t) \) be a nontrivial solution of Eq. (21) satisfying the \( n \)-point boundary conditions (15). If \( x(t) \neq 0 \) in \((a_j, a_{j+1})\), \( j = 1, 2, \ldots, r - 1 \), then the inequality
\[
\left( \int_{a_1}^{a_r} \tilde{Q}_m(t) \, dt \right) \left( \int_{a_1}^{a_r} \{ \tilde{Q}_m(t) + |f(t)| \} \, dt \right) > \frac{n^{2n-2}(\sigma + 1)^{-2(\sigma+1)}}{4(n-\sigma-1)^{2(n-\sigma-1)}} \times \frac{n!^2}{(a_r - a_1)^{2n-2}}
\]
holds, where \( \sigma = \min\{k_1, k_r\} \), and
\[
\tilde{Q}_m(t) = \sum_{i=1}^{m} |q_i(t)| \quad \text{and} \quad \tilde{Q}_m(t) = \sum_{i=1}^{m} \rho_i |q_i(t)|
\]
with
\[
\rho_i = (2 - \alpha_i) \alpha_i^{\alpha_i/(2-\alpha_i)} 2^{2/(\alpha_i-2)}.
\]

**Proof.** Let \( x(t) \) be a nontrivial solution of Eq. (21) satisfying the \( n \)-point boundary conditions (15). If \( x(t) \neq 0 \) in \((a_j, a_{j+1})\), \( j = 1, 2, \ldots, r - 1 \), then by using Green’s function of the Prb. (14)-(15), \( x(t) \) can be expressed as
\[
x(t) = \int_{a_1}^{a_r} G_n(t, s) \sum_{i=1}^{m} q_i(s)|x(s)|^{\alpha_i - 1} x(s) \, ds - \int_{a_1}^{a_r} G_n(t, s) f(s) \, ds.
\]
(31)

Let \( |x| = \max_{t \in (a, b)} |x(t)| \). Using this in (31), we obtain
\[
|x(c)| = \left| \int_{a_1}^{a_r} G_n(c, s) \sum_{i=1}^{m} q_i(s)|x(s)|^{\alpha_i - 1} x(s) \, ds - \int_{a_1}^{a_r} G_n(c, s) f(s) \, ds \right|
\]
\[
\leq \int_{a_1}^{a_r} |G_n(c, s)| \sum_{i=1}^{m} |q_i(s)||x(s)|^{\alpha_i} \, ds + \int_{a_1}^{a_r} |G_n(c, s)||f(s)| \, ds.
\]
(32)

Then by Ineq. (26) in Lemma 2.1 with \( A = B = 1 \), we have
\[
x^{\alpha_i}(c) < x^2(c) + \rho_i
\]
(33)
which implies together with Ineq. (32) the quadratic inequality

$$R_1 x^2(c) - x(c) + R_2 > 0,$$

(34)

where

$$R_1 = \int_{a_1}^{a_r} |G_n(c,s)\hat{Q}_m(s)ds$$

and

$$R_2 = \int_{a_1}^{a_r} |G_n(c,s)| \left\{ \hat{Q}_m(s) + |f(s)| \right\} ds. $$

But Ineq. (34) is possible if and only if $R_1R_2 > 1/4$. Finally, using Ineq. (20) with $t = c$, we complete the proof of Thm. 2.2.

Since

$$\lim_{\alpha_i \to 1^+} \rho_i = \lim_{\alpha_i \to 1^-} \rho_i = 1/4, $$

(35)

where $\rho_i$ is defined in (30), we have following result.

**Corollary 1** (Lyapunov type inequality). Let $x(t)$ be a nontrivial solution of Eq. (24) satisfying the n-point boundary conditions (15). If $x(t) \neq 0$ in $(a_j, a_{j+1})$, $j = 1, 2, \ldots, r - 1$, then the inequality

$$\left( \int_{a_1}^{a_r} \hat{Q}_m(t)dt \right) \left( \int_{a_1}^{a_r} \left\{ \hat{Q}_m(t) + 4|f(t)| \right\} dt \right) > \frac{n^{2n-2}(\sigma + 1)^{-2(\sigma+1)}}{(n - \sigma - 1)^{2(n-\sigma-1)}} \times \frac{n!^2}{(a_r - a_1)^{2n-2}}$$

holds, where $\sigma = \min\{k_1, k_r\}$, and the function $\hat{Q}_m(t)$ is defined in Thm. 2.2.

When $q_i(t) = 0$, for all $i = 2, 3, \ldots, m$, then Eq. (21) reduces to the forced sub-linear equation, i.e., Eq. (22) with $q(t) = q_1(t)$ and $\gamma = \alpha_1 \in (0, 1)$. Similarly, when $q_i(t) = 0$, for all $i = 2, 3, \ldots, m - 1$, then Eq. (21) reduces to the forced super-linear equation, i.e., Eq. (23) with $p(t) = q_m(t)$ and $\beta = \alpha_m \in (1, 2)$. For these equations we have the following corollaries.

**Corollary 2** (Lyapunov type inequality). Let $x(t)$ be a nontrivial solution of Eq. (22) satisfying the n-point boundary conditions (15). If $x(t) \neq 0$ in $(a_j, a_{j+1})$, $j = 1, 2, \ldots, r - 1$, then the inequality

$$\left( \int_{a_1}^{a_r} |q(t)|dt \right) \left( \int_{a_1}^{a_r} \{\gamma_0 |q(t)| + |f(t)| \} dt \right) > \frac{n^{2n-2}(\sigma + 1)^{-2(\sigma+1)}}{4(n - \sigma - 1)^{2(n-\sigma-1)}} \times \frac{n!^2}{(a_r - a_1)^{2n-2}}$$

holds, where $\sigma = \min\{k_1, k_r\}$, and

$$\gamma_0 = (2 - \gamma)^\gamma/(2 - \gamma)2^{2/(\gamma-2)}.$$

(36)

**Corollary 3** (Lyapunov type inequality). Let $x(t)$ be a nontrivial solution of Eq. (23) satisfying the n-point boundary conditions (15). If $x(t) \neq 0$ in $(a_j, a_{j+1})$,
\[ j = 1, 2, \ldots, r - 1, \text{ then the inequality} \]
\[
\left( \int_{a_1}^{a_r} |p(t)| dt \right) \left( \int_{a_1}^{a_r} \{ \beta_0 |p(t)| + |f(t)| \} dt \right) > \frac{n^{2n-2}(\sigma + 1)^{-2(\sigma + 1)}}{4(n - \sigma - 1)^{2(n - \sigma - 1)}} \times \frac{n!^2}{(a_r - a_1)^{2n-2}}
\]
holds, where \( \sigma = \min\{k_1, k_r\} \), and
\[
\beta_0 = (2 - \beta)^{\beta/(2 - \beta)} 2^{2/(\beta - 2)}.
\]

Remark 1. It is of interest to find analogues of Cor.'s 2 and 3 for Eq.'s (22) and (23) dropping the forcing term \( f(t) \), i.e., for the \( n \)-th order sub-linear
\[
x^{(n)}(t) + q(t)|x(t)|^{\gamma - 1}x(t) = 0; \quad 0 < \gamma < 1
\]
and the \( n \)-th order super-linear equation
\[
x^{(n)}(t) + p(t)|x(t)|^{\beta - 1}x(t) = 0; \quad 1 < \beta < 2,
\]
respectively. We state these results in the following:

Proposition 1 (Lyapunov type inequality). Let \( x_1(t) \) and \( x_2(t) \) be nontrivial solutions of Eq.'s (38) and (39) satisfying the \( n \)-point boundary conditions (15), respectively. If \( x(t) \neq 0 \) in \( (a_j, a_{j+1}) \), \( j = 1, 2, \ldots, r - 1 \), then the inequalities
\[
\int_{a_1}^{a_r} |q(t)| dt > \frac{n^{n-1}(\sigma + 1)^{-(\sigma + 1)}}{2\sqrt{\gamma_0}(n - \sigma - 1)^{n-\sigma-1}} \times \frac{n!}{(a_r - a_1)^{n-1}}
\]
and
\[
\int_{a_1}^{a_r} |p(t)| dt > \frac{n^{n-1}(\sigma + 1)^{-(\sigma + 1)}}{2\sqrt{\beta_0}(n - \sigma - 1)^{n-\sigma-1}} \times \frac{n!}{(a_r - a_1)^{n-1}}
\]
hold, where \( \sigma = \min\{k_1, k_r\} \), and the constants \( \gamma_0 \) and \( \beta_0 \) are defined in (36) and (37).

When \( \gamma \to 1^- \) (or \( \beta \to 1^+ \)), Eq. (38) (or Eq. (39)) reduces to \( n \)-th order linear equation
\[
x^{(n)}(t) + \mu(t)x(t) = 0,
\]
where \( \mu(t) = q(t) \) (or \( \mu(t) = p(t) \)). Since
\[
\lim_{\beta \to 1^+} \beta_0 = \lim_{\gamma \to 1^-} \gamma_0 = 1/4,
\]
we have the following result.

Corollary 4 (Lyapunov type inequality). Let \( x(t) \) be a nontrivial solution of Eq. (40) satisfying the \( n \)-point boundary conditions (15). If \( x(t) \neq 0 \) in \( (a_j, a_{j+1}) \), \( j = 1, 2, \ldots, r - 1 \), then the inequality
\[
\int_{a_1}^{a_r} |\mu(t)| dt > \frac{n^{n-1}(\sigma + 1)^{-(\sigma + 1)}}{(n - \sigma - 1)^{n-\sigma-1}} \times \frac{n!}{(a_r - a_1)^{n-1}}
\]
holds, where \( \sigma = \min\{k_1, k_r\} \).

When \( r = n \), Prb. (40)-(15) reduces to the Prb. (40)-(8) with \( a_1 = a \) and \( a_r = a_n = b \). Then Cor. 4 yields the following result.
Corollary 5 (Lyapunov type inequality). Let $x(t)$ be a nontrivial solution of Eq. (40) satisfying the $n$-point boundary conditions (8) where $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, with $a = a_1 < a_2 < \cdots < a_{n-1} < a_n = b$ are consecutive zeros. Then the inequality

$$
\int_a^b |\mu(t)| \, dt > \left( \frac{n}{n-1} \right)^{n-1} \frac{n!}{(b-a)^{n-1}}; \quad n = 2, 3, \ldots
$$

holds.

Proof. Since $\sigma = 0$, when $r = n$, Ineq. (41) immediately implies Ineq. (42) with $a_1 = a$ and $a_r = b$. \qed

Remark 2. Cor. 5 is remarkable since Ineq. (42) is sharper than both those obtained in Yang [55, Thm. 2] and Çakmak [15, Thm 1], i.e. Ineq.’s (9) and (12) for all $n = 2, 3, \ldots$.

Remark 3. When $n = 3$, Ineq. (42) turns into

$$
\int_a^b |q(t)| \, dt > \frac{27}{2(b-a)^2}
$$

which is sharper than the results of Parhi and Panigrahi [46], Yang [55] and Çakmak [15], i.e. Ineq.’s (11), (10) and (13). Moreover, when $n = 2$, Ineq. (42) turns into the classical Lyapunov inequality, i.e., Ineq. (3).

Note that, when $r = 2$, $k_1 = k - 1$ and $k_2 = n - k - 1$, (15) turns to $(k, n-k)$ conjugate boundary conditions

$$
x^{(i)}(a) = 0; \quad i = 0, 1, \ldots, k-1,
$$

$$
x^{(i)}(b) = 0; \quad j = 0, 1, \ldots, n-k-1,
$$

(43)

where $a = a_1$ and $b = a_2$.

The following result is analogous to that of our first main result, i.e., Thm. 2.2.

Theorem 2.3 (Lyapunov type inequality). Let $x(t)$ be a nontrivial solution of Eq. (21) satisfying the $(k, n-k)$ conjugate boundary conditions (43) with $a < b$. If $x(t) \neq 0$ in $(a, b)$, then the inequality

$$
\left( \int_a^b \tilde{Q}_m(t) \, dt \right) \left( \int_a^b \{ \tilde{Q}_m(t) + |f(t)| \} \, dt \right) > \frac{n^{2n-2}}{4k^{2k(n-k)^2(n-k)}} \times \frac{n!^2}{(b-a)^{2n-2}}
$$

(44)

holds, where the functions $\tilde{Q}_m(t)$ and $\tilde{Q}_m(t)$ are defined in Thm. 2.2.

Proof. Since $\sigma = \min\{k-1, n-k-1\}$, there are two cases; one is $\sigma = k-1$ and the other is $\sigma = n-k-1$. In the both situations, we have

$$
(\sigma + 1)^{-\sigma+1} = \frac{1}{k^k(n-k)^{n-k}}.
$$

Using this in Ineq. (28) we obtain Ineq. (44) immediately. \qed
3. \((k, n - k)\) Conjugate boundary value problem. In this section, we further consider the \(n\)th order forced mixed nonlinear differential equation of the form
\[
x^{(n)}(t) + (-1)^{n-k-1} \sum_{i=1}^{m} q_i(t)|x(t)|^{(a_i-1)}x(t) = f(t); \quad n = 2, 3, \ldots
\] (45)
satisfying the \((k, n - k)\) conjugate boundary conditions (43).

In 1976, Gustafson [26] obtained Green’s function \(G_n(t, s)\) for the \((k, n - k)\) conjugate boundary value Prb. (14)-(43) which is given by
\[
G_n(t, s) = \sum_{j=0}^{k-1} \left\{ \sum_{i=0}^{k-j-1} \binom{n-k+i-1}{i} \left( \frac{t-a}{b-a} \right)^i \frac{(t-a)^j(a-s)^{n-j-1}}{j!(n-j-1)!} \right\} \times \left( \frac{b-t}{b-a} \right)^k;
\]
\(a \leq s \leq t \leq b\) (46)

and
\[
G_n(t, s) = -\sum_{j=0}^{n-k-1} \left\{ \sum_{i=0}^{n-k-j-1} \binom{k+i-1}{i} \left( \frac{b-t}{b-a} \right)^i \frac{(t-b)^j(b-s)^{n-j-1}}{j!(n-j-1)!} \right\} \times \left( \frac{t-a}{b-a} \right)^k;
\] \(a \leq t \leq s \leq b\). (47)

It was shown simply by Yang [55] that for \(a \leq s, t \leq b\)
\[
|G_n(t, s)| \leq \begin{cases} 
\psi_1(s) & \text{if } a \leq s \leq t \leq b; \\
\psi_2(s) & \text{if } a \leq t \leq s \leq b,
\end{cases}
\] (48)

where
\[
\psi_1(s) = \frac{(b-s)^{n-k}(s-a)^{n-k}}{(k-1)!(n-k)!(b-a)^{n-k}} \{(b-a) + (s-a)\}^{k-1},
\] (49)
\[
\psi_2(s) = \frac{(b-s)^k(s-a)^k}{k!(n-k-1)!(b-a)^{n-k-1}} \{(b-a) + (b-s)\}^{n-k-1}.
\] (50)

In 2003, Yang [55] established Hartman and Lyapunov type inequalities for the equation
\[
x^{(n)}(t) + (-1)^{n-k-1} q(t)x(t) = 0; \quad n = 2, 3, \ldots
\] (51)
satisfying the \((k, n - k)\) conjugate boundary conditions (43) by employing the above Green’s function of Prb. (14)-(43).

**Theorem 3.1** (Hartman type inequality). Let Eq. (51) has a nontrivial solution \(x(t)\) satisfying the \((k, n - k)\) conjugate boundary conditions (43) with \(a < b\). If \(x(t) \neq 0\) in \((a, b)\), then the inequality
\[
\int_a^b \Psi(t)|q(t)|dt > 1
\] (52)
holds, where \(\Psi(t) = \max\{\psi_1(t), \psi_2(t)\}\).

On account of Ineq. (6), Ineq. (52) immediately implies the following Lyapunov type inequality.
Let Eq. (56) hold, then the inequality
\[ \int_a^b |q(t)|dt > \max\{a_n^k, b_n^k\} \] (53)
holds, where
\[ a_n^k = \frac{(k - 1)!(n - k)!2^{2n-3k+1}}{(b-a)^{2n-1}} \quad \text{and} \quad b_n^k = \frac{k!(n-k-1)!2^{3k-n+1}}{(b-a)^{n-1}}. \]

However the function \( \psi_1(s) \) given in (49) is not correct, we will give the modified form of \( \psi_1(s) \). For this, we need the following lemma.

**Lemma 3.3.** Let the function \( G_n(t, s) \) be defined as in (46)-(47). Then
\[ |G_n(t, s)| \leq \begin{cases} 
\theta_1(t, s) & \text{if } a \leq s \leq t \leq b; \\
\theta_2(t, s) & \text{if } a \leq t \leq s \leq b,
\end{cases} \] (54)
where
\[ \theta_1(t, s) = \frac{(b-t)^k(s-a)^{n-k}}{(k-1)!(n-k)!(b-a)^k} \{t + s - 2a\}^{k-1}, \] (55)
\[ \theta_2(t, s) = \frac{(b-s)^k(t-a)^k}{k!(n-k-1)!(b-a)^k} \{2b - t - s\}^{n-k-1}. \] (56)

**Proof.** For \( a \leq s \leq t \leq b \), we have
\[ |G_n(t, s)| \leq \frac{(b-t)^k(s-a)^{n-k}}{(b-a)^k} \sum_{j=0}^{k-1} \left\{ \sum_{i=0}^{k-j-1} \binom{n-k+i-1}{i} \left( \frac{t-a}{b-a} \right)^i \right\} \times \frac{(t-a)^j(s-a)^{k-j-1}}{j!(n-j-1)!}. \]
Since \( (t-a)/(b-a) \leq 1 \), we have
\[ \sum_{i=0}^{k-j-1} \binom{n-k+i-1}{i} \left( \frac{t-a}{b-a} \right)^i \leq \sum_{i=0}^{k-j-1} \binom{n-k+i-1}{i} = \binom{n-j-1}{k-j-1}. \]
Last two inequalities imply that
\[ |G_n(t, s)| \leq \frac{(b-t)^k(s-a)^{n-k}}{(b-a)^k} \sum_{j=0}^{k-1} \binom{n-j-1}{k-j-1} \frac{1}{j!(n-j-1)!} \bigg( \frac{t-a)^j(s-a)^{k-j-1}}{j!(n-j-1)!} \bigg) \]
\[ = \frac{(b-t)^k(s-a)^{n-k}}{(b-a)^k(n-k)!(k-1)!} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{1}{j!(n-j-1)!} \bigg( \frac{t-a)^j(s-a)^{k-1-j}}{j!(n-j-1)!} \bigg) \]
\[ = \frac{(b-t)^k(s-a)^{n-k}}{(b-a)^k(n-k)!(k-1)!} (t + s - 2a)^{k-1} \]
\[ = \theta_1(t, s) \]
for \( a \leq s \leq t \leq b. \)

In a similar way, it can be shown that
\[ |G_n(t, s)| \leq \theta_2(t, s) \]
for \( a \leq t \leq s \leq b. \) \( \square \)
We note that
\[ \theta_1(t, s) \leq \tilde{\theta}_1(s) := \frac{k}{n!} \binom{n}{k} \frac{(b-s)^k(s-a)^{n-k}}{(b-a)^k} (s + b - 2a)^{k-1} \]  
(57)

and
\[ \theta_2(t, s) \leq \tilde{\theta}_2(s) := \frac{n-k}{n!} \binom{n}{k} \frac{(b-s)^k(s-a)^{k}}{(b-a)^k} (2b - a - s)^{n-k-1}, \]  
(58)

for \( a \leq t \leq b \).

Employing the handy inequalities (57) and (58), we can improve and extend the results of Yang [55] for \((k, n-k)\) conjugate boundary value Prb. (45)-(43).

**Theorem 3.4** (Hartman type inequality). Let \( x(t) \) be a nontrivial solution of Eq. (45) satisfying the \((k, n-k)\) conjugate boundary conditions (43) with \( a < b \). If \( x(t) \neq 0 \) in \((a, b)\), then the inequality
\[
\left( \int_a^b (b-t)^k \Phi(t) \Phi_m(t) dt \right) \left( \int_a^b (b-t)^k \Phi(t) \left\{ \tilde{Q}_m(t) + |f(t)| \right\} dt \right)
\geq \frac{1}{4} k^2 (n-k)t^2 (b-a)^{2k}
\]  
(59)

holds, where the functions \( \tilde{Q}_m(t) \) and \( Q_m(t) \) are defined in (29) and
\[
\Phi(t) = \max_{t \in [a, b]} \left\{ (k-t)(a-t)^{n-k}(t+b-2a)^{k-1}, (n-k)(t-a)^k(2b-a-t)^{n-k-1} \right\}
\]  
(60)

for \( k = 1, \ldots, n-1 \).

**Proof.** Let \( x(t) \) be a nontrivial solution of Eq. (45) satisfying the \((k, n-k)\) conjugate boundary conditions (43) with \( a < b \). Then by using Green’s function \( G_n(t, s) \) for Prb. (14)-(43), \( x(t) \) can be expressed as
\[ x(t) = \int_a^b (-1)^{n-k} G_n(t, s) \sum_{i=1}^m q_i(s) |x(s)|^{\alpha_i-1} x(s) ds - \int_a^b (-1)^{n-k} G_n(t, s) f(s) ds. \]  
(61)

Let \( |x(c)| = \max_{t \in (a, b)} |x(t)| \). Using this and Ineq.’s (57)-(58) in (61), we obtain
\[
|x(c)| = \left| \int_a^b (-1)^{n-k} G_n(c, s) \sum_{i=1}^m q_i(s) |x(s)|^{\alpha_i-1} x(s) ds - \int_a^b (-1)^{n-k} G_n(c, s) f(s) ds \right|
\leq \int_a^b |G_n(c, s)| \sum_{i=1}^m |q_i(s)||x(c)|^{\alpha_i} ds + \int_a^b |G_n(c, s)||f(s)| ds
\leq \frac{1}{n! (b-a)^k} \binom{n}{k} \int_a^b (b-s)^k \Phi(s) \sum_{i=1}^m |q_i(s)||x(c)|^{\alpha_i} ds
+ \frac{1}{n! (b-a)^k} \binom{n}{k} \int_a^b (b-s)^k \Phi(s) |f(s)| ds.
\]  
(62)

Then using Ineq. (33) as in the proof of Thm 2.2 in Ineq. (62), we obtain the quadratic inequality
\[ S_1 |x(c)|^2 - |x(c)| + S_2 > 0, \]  
(63)
where
\[ S_1 = \frac{1}{n!(b-a)^k} \binom{n}{k} \int_a^b (b-s)^k \Phi(s) \tilde{Q}_m(s) ds \]
and
\[ S_2 = \frac{1}{n!(b-a)^k} \binom{n}{k} \int_a^b (b-s)^k \Phi(s) \left\{ \tilde{Q}_m(s) + |f(s)| \right\} ds. \]
But Ineq. (63) is possible if and only if \( S_1 S_2 > 1/4 \). The proof of Thm. 3.4 is completed.

We will need to prove the following lemma for the next result.

**Lemma 3.5.** let \( k \) and \( n \) be positive constants. If \( n > k \), then
\[ (b-t)^k (t-a)^{n-k} \leq k^k (n-k)^{n-k} \frac{n^k}{n^n} (b-a)^n \]
for all \( t \in [a,b] \) with equality holding if and only if \( t = (k/n)a + (1-k/n)b \).

**Proof.** Clearly, when \( t = a \) or \( t = b \), (64) is obvious. On the other hand, for \( t \in (a,b) \), using Ineq. (19) with \( r = 2, a_1 = a, a_r = a_2 = b, k_1 = k - 1 \) and \( k_2 = n - k - 1 \) we obtain
\[ (b-t)^k (t-a)^{n-k} \leq \frac{(n-\sigma-1)^{n-\sigma-1}}{n^n} (\sigma + 1)^{\sigma+1} (b-a)^n, \]
where \( \sigma = \min\{k-1, n-k-1\} \). In both cases e.g. \( \sigma = k - 1 \) or \( \sigma = n - k - 1 \), Ineq. (65) yields Ineq. (64). Moreover, if we define the function
\[ \mathcal{F}(t) := (b-t)^k (t-a)^{n-k}; \quad n > k > 0, t \in [a,b], \]
then it is easy to see that \( \mathcal{F} \) attains its maximum at \( t_0 = (k/n)a + (1-k/n)b \), and hence equality in (64) holds if and only if \( \mathcal{F}(t) = \mathcal{F}(t_0) \) for all \( t \in [a,b] \).

Note that if \( k = 0 \) or \( k = n \), then Ineq. (64) is still true for all \( t \in [a,b] \) with the convention that \( 0^0 = 1 \). Moreover, when \( n = 2k = 2 \), Ineq. (64) turns to Ineq. (6).

Note also that the inequality
\[ (b-t)^{n-k} (t-a)^{k} \leq k^k (n-k)^{n-k} \frac{n^k}{n^n} (b-a)^n; \quad t \in [a,b] \]
is true for \( n > k \).

Ineq. (64) and Ineq. (66) allow the following Lyapunov type inequality.

**Theorem 3.6** (Lyapunov type inequality). Let \( x(t) \) be a nontrivial solution of Eq. (45) satisfying the \((k,n-k)\) conjugate boundary conditions (43) with \( a < b \). If \( x(t) \neq 0 \) in \((a,b)\), then the inequality
\[ \left( \int_a^b \tilde{Q}_m(t) dt \right) \left( \int_a^b \left\{ \tilde{Q}_m(t) + |f(t)| \right\} dt \right) > \frac{n^2 k^2 (n-k)^2 (b-a)^{2n-2}}{\Phi_{nk}^2 k^{2k(n-k)} (n-k)^{2(n-k)} (b-a)^{2n-2}} \]
holds, where the functions \( \tilde{Q}_m(t) \) and \( \tilde{Q}_m(t) \) are defined in (29) and
\[ \Phi_{nk} = \max \left\{ k^{2k}, (n-k)^{2n-k} \right\}; \quad k = 1, \ldots, n-1. \]
Proof. In view of (55) and (56), we have

\[
\theta_1(t, s) \leq \frac{(b-t)^k(t-a)^{n-k}}{(k-1)!((n-k)!(b-a)^k)} (2t-2a)^{k-1}
\]

\[
= \frac{2^{k-1}k}{n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(b-t)^k(t-a)^{n-k}}{(b-a)^k} (t-a)^{k-1}
\]

\[
\leq \frac{2^k}{2n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(b-t)^k(t-a)^{n-k}}{b-a}
\]

(69)

and

\[
\theta_2(t, s) \leq \frac{(b-t)^k(t-a)^k}{k!(n-k-1)!(b-a)^k} (2b-2t)^{n-k-1}
\]

\[
= \frac{2^{n-k-1}(n-k)}{n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(b-t)^k(t-a)^k}{(b-a)^k} (b-t)^{k-1}
\]

\[
\leq \frac{2^{n-k}(n-k)}{2n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(b-t)^k(t-a)^k}{b-a}
\]

(70)

for \( a \leq t \leq b \). Using Ineq.'s (64) and (66), Ineq.'s (69) and (70) turn out to

\[
\theta_1(t, s) \leq \frac{2^k}{2n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{k^k(n-k)^{n-k}}{n^n} (b-a)^{n-1}
\]

(71)

and

\[
\theta_2(t, s) \leq \frac{2^{n-k}(n-k)}{2n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{k^k(n-k)^{n-k}}{n^n} (b-a)^{n-1},
\]

(72)

respectively. Then using Ineq.'s (71) and (72) as in the proof of Thm 3.4 in Ineq. (62), we obtain the quadratic inequality

\[
\tilde{S}_1|x(c)|^2 - |x(c)| + \tilde{S}_2 > 0,
\]

(73)

where

\[
\tilde{S}_1 = \frac{\Phi_{nk}}{2n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{k^k(n-k)^{n-k}}{n^n} (b-a)^{n-1} \int_a^b \tilde{Q}_m(s)ds
\]

and

\[
\tilde{S}_2 = \frac{\Phi_{nk}}{2n!} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{k^k(n-k)^{n-k}}{n^n} (b-a)^{n-1} \int_a^b \left\{ \tilde{Q}_m(s) + |f(s)| \right\} ds.
\]

But Ineq. (73) is possible if and only if \( \tilde{S}_1 \tilde{S}_2 > 1/4 \). The proof of Thm. 3.6 is completed.

On account of the limit process (35), as a direct consequence of Thm.'s 3.4 and 3.6, we have following result for the equation

\[
x^{(n)}(t) + (-1)^{n-k-1}\nu(t)x(t) = f(t); \quad n = 2, 3, \ldots,
\]

(74)

where \( \nu(t) \) is defined in (25).

Corollary 6. Let \( x(t) \) be a nontrivial solution of Eq. (74) satisfying the \((k, n-k)\) conjugate boundary conditions (43) with \( a < b \). If \( x(t) \neq 0 \) in \((a, b)\), then the following hold:
Hartman type inequality;

\[
\left( \int_a^b (b - t)^k \Phi(t) \tilde{Q}_m(t) dt \right) \left( \int_a^b (b - t)^k \Phi(t) \left\{ \tilde{Q}_m(t) + 4|f(t)| \right\} dt \right) > k!^2 (n - k)!^2 (b - a)^{2k}
\]  

(75)

Lyapunov type inequality;

\[
\left( \int_a^b \tilde{Q}_m(t) dt \right) \left( \int_a^b \left\{ \tilde{Q}_m(t) + 4|f(t)| \right\} dt \right) > \frac{4n^2k!^2(n - k)!^2}{\Phi_{nk}^2 k^{2k} (n - k)^{2(n - k)} (b - a)^{2n - 2}}
\]  

(76)

where \( \tilde{Q}_m(t) \), \( \Phi(t) \) and \( \Phi_{nk} \) are defined in (29), (60) and (68) respectively.

When \( m = 1 \) and \( f(t) \equiv 0 \), Eq. (74) reduces to the equation

\[
x^{(n)}(t) + (-1)^{n-k-1} q(t)x(t) = 0; \quad n = 2, 3, \ldots
\]  

(77)

with \( q(t) = q_1(t) \). Then Cor. 6 yields the following result.

**Corollary 7.** Let \( x(t) \) be a nontrivial solution of Eq. (77) satisfying the \((k, n - k)\) conjugate boundary conditions (43) with \( a < b \). If \( x(t) \neq 0 \) in \((a, b)\), then the following hold:

(i) Hartman type inequality:

\[
\int_a^b (b - t)^k \Phi(t)|q(t)|dt > k!(n - k)!(b - a)^k
\]  

(78)

(ii) Lyapunov type inequality:

\[
\int_a^b |q(t)|dt > \frac{2n^2k!(n - k)!}{\Phi_{nk}^2 k^k (n - k)^{(n - k)}(b - a)^{n-1}},
\]  

(79)

where \( \Phi(t) \) and \( \Phi_{nk} \) are defined in (60) and (68) respectively.

**Remark 4.** Cor. 7 is remarkable since when \( n = 2 \) and \( k = 1 \), Eq. (77) reduces to Hill’s Eq. (1) and Ineq.’s (78) and (79) reduce to the classical inequalities

\[
\int_a^b (b - t)(t - a)|q(t)|dt > b - a
\]

and

\[
\int_a^b |q(t)|dt > \frac{4}{b - a}.
\]

When \( q_i(t) = 0 \), for all \( i = 2, 3, \ldots, m \), then Eq. (45) reduces to the forced sub-linear equation

\[
x^{(n)}(t) + (-1)^{n-k-1} q(t)|x(t)|^{\gamma - 1}x(t) = f(t); \quad n = 2, 3, \ldots
\]  

(80)

where \( q(t) = q_1(t) \) and \( \gamma = \alpha_1 \in (0, 1) \). Similarly, when \( q_i(t) = 0 \), for all \( i = 1, 2, 3, \ldots, m - 1 \), then Eq. (45) reduces to the forced super-linear equation

\[
x^{(n)}(t) + (-1)^{n-k-1} p(t)|x(t)|^{\beta - 1}x(t) = f(t); \quad n = 2, 3, \ldots
\]  

(81)

where \( p(t) = q_m(t) \) and \( \beta = \alpha_m \in (1, 2) \). For these equations we have the following corollaries.
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Corollary 8. Let \( x(t) \) be a nontrivial solution of Eq. (80) satisfying the \((k, n - k)\) conjugate boundary conditions (43) with \( a < b \). If \( x(t) \neq 0 \) in \((a, b)\), then the following hold:

(i) Hartman type inequality:
\[
\left( \int_a^b (b - t)^k \Phi(t) |q(t)| dt \right) \left( \int_a^b (b - t)^k \Phi(t) \{ \gamma_0 |q(t)| + |f(t)| \} dt \right) > \frac{1}{4} k!^2 (n - k)!^2 (b - a)^{2k} \tag{82}
\]

(ii) Lyapunov type inequality:
\[
\left( \int_a^b |q(t)| dt \right) \left( \int_a^b \{ \gamma_0 |q(t)| + |f(t)| \} dt \right) > \frac{n^{2n} k!^2 (n - k)!^2}{\Phi_{nk}^2 k^{2k} (n - k)^{2(n - k)} (b - a)^{2n - 2}} \tag{83}
\]

where \( \Phi(t), \Phi_{nk} \) and the constant \( \gamma_0 \) are defined in (60), (68) and (36) respectively.

Corollary 9. Let \( x(t) \) be a nontrivial solution of Eq. (81) satisfying the \((k, n - k)\) conjugate boundary conditions (43) with \( a < b \). If \( x(t) \neq 0 \) in \((a, b)\), then the following hold:

(i) Hartman type inequality:
\[
\left( \int_a^b (b - t)^k \Phi(t) |p(t)| dt \right) \left( \int_a^b (b - t)^k \Phi(t) \{ \beta_0 |p(t)| + |f(t)| \} dt \right) > \frac{1}{4} k!^2 (n - k)!^2 (b - a)^{2k} \tag{84}
\]

(ii) Lyapunov type inequality:
\[
\left( \int_a^b |p(t)| dt \right) \left( \int_a^b \{ \beta_0 |p(t)| + |f(t)| \} dt \right) > \frac{n^{2n} k!^2 (n - k)!^2}{\Phi_{nk}^2 k^{2k} (n - k)^{2(n - k)} (b - a)^{2n - 2}} \tag{85}
\]

where \( \Phi(t), \Phi_{nk} \) and the constant \( \beta_0 \) are defined in (60), (68) and (37) respectively.

Remark 5. When \( n = 2 \) or (and) \( f(t) = 0 \) the results given in Cor.’s 8 and 9 are still new.

When \( \gamma \to 1^- \) (or \( \beta \to 1^+ \)), Eq. (80) (or Eq. (81)) reduces to Eq. (74) with \( \nu(t) = q(t) \) (or \( \nu(t) = p(t) \)). Since
\[
\lim_{\beta \to 1^+} \beta_0 = \lim_{\gamma \to 1^-} \gamma_0 = 1/4,
\]
Ineq.’s (82) and (83) (or Ineq.’s (84) and (85)) reduce to Ineq.’s (75) and (76) with \( \bar{Q}_m(t) = q(t) \) (\( \tilde{Q}_m(t) = p(t) \)) given in Cor. 6.

3.1. Further estimations for \( n = 2\ell \). In this section, we consider Prb. (45)-(43) with \( n = 2k = 2\ell \), i.e., the equation
\[
x^{(2\ell)}(t) + (-1)^{\ell - 1} \sum_{i=1}^{m} q_i(t) |x(t)|^{\alpha_i - 1} x(t) = f(t); \quad \ell = 1, 2, \ldots \tag{86}
\]
satisfying the 2-point boundary conditions
\[
x^{(j)}(a) = x^{(j)}(b) = 0; \quad j = 0, 1, \ldots, \ell - 1, \tag{87}
\]
where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros.

It appears that the first generalization of Hartman and Lyapunov results for the linear equation

$$x^{(2\ell)}(t) + (-1)^{\ell-1}q(t)x(t) = 0; \quad a \leq t \leq b,$$

was obtained by Das and Vatsala [20, Thm. 3.1].

**Theorem 3.7.** Let $x(t)$ be a nontrivial solution of Prb. (88)-(87) with $a < b$. If $x(t) \neq 0$ in $(a, b)$, then the following hold:

(i) **Hartman type inequality:**

$$\int_a^b (b - t)^{2\ell-1}(t - a)^{2\ell-1}q^+(t)dt > (2\ell - 1)(\ell - 1)^2(b - a)^{2\ell-1}$$

(ii) **Lyapunov type inequality:**

$$\int_a^b q^+(t)dt > \frac{4^{2\ell-1}(2\ell - 1)(\ell - 1)^2}{(b - a)^{2\ell-1}}.$$  

The proof of Thm. 3.7 is based on Green’s function $G_\ell(t, s)$ of the 2-point boundary value problem

$$-x^{(2\ell)}(t) = 0; \quad a \leq t \leq b,$$

satisfying (87), obtained in the same paper [20] as follows:

$$G_\ell(t, s) = \frac{(-1)^{\ell-1}}{(2\ell - 1)!} \left(\frac{(t - a)(b - s)}{(b - a)}\right)^{\ell-1} \sum_{j=0}^{\ell-1} \left(\frac{\ell - 1 + j}{j}\right) (s - t)^{\ell-1-j} \times \left(\frac{(b - t)(s - a)}{(b - a)}\right)^j; \quad t \leq s \leq b$$

and

$$G_\ell(t, s) = \frac{(-1)^{\ell-1}}{(2\ell - 1)!} \left(\frac{(s - a)(b - t)}{(b - a)}\right)^{\ell-1} \sum_{j=0}^{\ell-1} \left(\frac{\ell - 1 + j}{j}\right) (t - s)^{\ell-1-j} \times \left(\frac{(t - a)(b - s)}{(b - a)}\right)^j; \quad a \leq s \leq t.$$

Recently, present authors [9, Thm. 2.1, Thm. 2.2] extended and improved the main results of Das and Vatsala [20], i.e. Thm. 3.7 for Prb. (86)-(87) with $f(t) = 0$, and in particular the classical Lyapunov and Hartman’s results.

**Theorem 3.8.** Let $x(t)$ be a nontrivial solution of Prb. (86)-(87) with $f(t) = 0$, where $a < b$. If $x(t) \neq 0$ in $(a, b)$, then the following hold:

(i) **Hartman type inequality:**

$$\left(\int_a^b (b - t)^{2\ell-1}(t - a)^{2\ell-1}\tilde{P}_m(t)dt\right) \left(\int_a^b (b - t)^{2\ell-1}(t - a)^{2\ell-1}\tilde{P}_m(t)dt\right) > \frac{1}{4}(2\ell - 1)^2(\ell - 1)^4(b - a)^{4\ell-2}$$

(ii) **Lyapunov type inequality:**

$$\left(\int_a^b \tilde{P}_m(t)dt\right) \left(\int_a^b \tilde{P}_m(t)dt\right) > \frac{4^{4\ell-3}(2\ell - 1)^2(\ell - 1)^4}{(b - a)^{4\ell-2}},$$
where
\[ P_m(t) = \sum_{i=1}^{m} q_i(t) \quad \text{and} \quad \tilde{P}_m(t) = \sum_{i=1}^{m} \rho_i q_i(t). \]

Here the constant \( \rho_i \) is defined in (30).

The results those given in Sec. 3 can be extended to even-order forced differential equations of the form (86) satisfying 2-point boundary conditions (87) and now we present some new Hartman and Lyapunov type inequalities for Prb. (86)-(87).

**Theorem 3.9.** Let \( x(t) \) be a nontrivial solution of Prb. (86)-(87) with \( a < b \). If \( x(t) \neq 0 \) in \( (a, b) \), then the following hold:

(i) **Hartman type inequality:**
\[
\left( \int_{a}^{b} (b-t)^{\ell}(t-a)^{\ell} \Phi^*(t) \tilde{Q}_m(t) \, dt \right) \left( \int_{a}^{b} (b-t)^{\ell}(t-a)^{\ell} \Phi^*(t) \{ \tilde{Q}_m(t) + |f(t)| \} \, dt \right) > \frac{1}{4\ell^2 b-a} 2^{2\ell},
\]

(ii) **Lyapunov type inequality:**
\[
\left( \int_{a}^{b} \tilde{Q}_m(t) \, dt \right) \left( \int_{a}^{b} \{ \tilde{Q}_m(t) + |f(t)| \} \, dt \right) > \frac{4^{2\ell-1} \ell!^4}{\ell^2 (b-a)^{4\ell-2}},
\]

where the functions \( \tilde{Q}_m(t) \) and \( Q_m(t) \) are defined in (29) and
\[ \Phi^*(t) = \max_{t \in [a, b]} \left\{ (t+b-2a)^{\ell-1}, (2b-a-t)^{\ell-1} \right\} \]
for \( \ell = 1, \ldots, n \).

**Proof.** When we take \( n = 2k = 2\ell \), Ineq.’s (59) and (67) immediately imply Ineq.’s (90) and (91). The proof of Thm. 3.9 is completed.

**Remark 6.** Ineq.’s (90) and (91) obtained in Thm. 3.9 are so handy that they can be trivially adapted all the special cases of Prb. (86)-(87) such as (un)forced sub-linear, super-linear, linear differential equations satisfying 2-point boundary conditions (87). We leave these results to the reader.

It will be of interest to find similar results for the \( n \)th order mixed nonlinear equations of the form Eq. (21) and Eq. (45) for some \( \alpha_k \geq 2 \), or super-linear Eq. (23) and Eq. (81) for \( \beta \in [2, \infty) \) even though \( f(t) = 0 \). In fact, the case when \( n = 2 \) (Emden-Fowler super-linear), is of immense interest. Finally, we note that the inequalities obtained in this paper for nonlinear equations may not be the best ones. Therefore, it will be of immense interest to develop new Lyapunov and Hartman type inequalities better than the inequalities presented in this paper. For this, we suggest to the readers to prove an new inequality better than inequality (26) given in Lemma 2.1.

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Received February 2016; revised June 2016.
E-mail address: agarwal@tamuk.edu
E-mail address: aozbekler@gmail.com, abdullah.ozbekler@atilim.edu.tr