Finite groups whose prime graph on class sizes is a block square

V. Sotomayor

Departmento de Matemáticas y Estadística, Centro Universitario EDEM, Valencia, Spain

ABSTRACT
Let $G$ be a finite group, and let $\Delta(G)$ be the prime graph built on its set of conjugacy class sizes: this is the (simple undirected) graph whose vertices are the prime numbers dividing some conjugacy class size of $G$, and two distinct vertices $p, q$ are adjacent if and only if $pq$ divides some class size of $G$. In this paper, we characterize the structure of those groups $G$ whose prime graph $\Delta(G)$ is a block square.

1. Introduction
Throughout this paper, all groups considered are finite. Within finite group theory, the influence of the arithmetical properties of the conjugacy class sizes of a group on its algebraic structure is a research area that has attracted the interest of several authors over the last decades. The prime graph built on the set of class sizes of a group $G$, which we denote by $\Delta(G)$, is a useful tool that is gaining an increasing interest for analyzing the arithmetical properties of this set. This (simple undirected) graph has as vertex set $V(G)$ the prime divisors of the conjugacy class sizes of $G$, and its edge set $E(G)$ contains pairs $\{p, q\} \subseteq V(G)$ such that $pq$ divides some class size of $G$. In this framework, two relevant questions that arise are: which graphs can occur as $\Delta(G)$ for some finite group $G$, and how is the structure of $G$ affected by the graph-theoretical properties of $\Delta(G)$?

Interestingly, non-adjacency between vertices of $\Delta(G)$ highly restricts the structure of $G$, which suggests that $\Delta(G)$ tends to have “many” edges. In fact, the extreme case when $\Delta(G)$ is disconnected happens if and only if $G$ is a $D$-group, that is, $G = AB$ where $A \leq G$ and $B$ are abelian subgroups of coprime orders, $Z(G) \leq B$, and the factor group $G/Z(G)$ is a Frobenius group with kernel $AZ(G)/Z(G)$ (see Theorem 4 of [3]). In this situation, $G$ has three class sizes, which are $\{1, |A|, |B/Z(G)|\}$. So the vertex sets of the (two) connected components of $\Delta(G)$ turn out to be the sets of prime divisors of the orders of $A$ and $B/Z(G)$, respectively, and both sets are cliques (i.e. they induce complete subgraphs) of $\Delta(G)$.

Casolo et al. [1] studied the structure of those finite groups $G$ such that $\Delta(G)$ has no complete vertices. Moreover, they characterized those groups whose prime graph on class sizes is non-complete and regular, and they basically direct products of certain $D$-groups. In particular, if $\Delta(G)$ is a square with $V(G) = \{p, q, r, s\}$ and $E(G) = \{\{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}\}$, then from their
result it follows that (up to abelian direct factors) \( G = A \times B \) where \( A \) and \( B \) are \( D \)-groups of orders divisible by \( \{p, q\} \) and \( \{r, s\} \), respectively.

A natural way to generalize a square graph is to replace each vertex by a set of vertices. In this spirit, a graph is called a block square if its vertex set can be written as a union of four disjoint, non-empty subsets \( \pi_1, \pi_2, \pi_3, \pi_4 \), where no prime in \( \pi_1 \) is adjacent to any prime in \( \pi_4 \) and no prime in \( \pi_2 \) is adjacent to any prime in \( \pi_3 \), and there exist vertices in both \( \pi_1 \) and \( \pi_4 \) that are adjacent to vertices in \( \pi_2 \) and in \( \pi_3 \). Certainly, any direct product \( G = A \times B \) of two coprime \( D \)-groups yields a block square \( \Delta(G) \). So the question that naturally arises is whether there exist other types of groups whose prime graph on class sizes is a block square. The main result of this paper shows that in fact this is the unique way of obtaining groups with such class-size prime graph.

**Theorem A.** Let \( G \) be a finite group. Then \( \Delta(G) \) is a block square if and only if, up to an abelian direct factor, \( G = A \times B \) where \( A \) and \( B \) are \( D \)-groups of coprime orders.

As a consequence, we have attained a characterization of the block square graphs that can occur as \( \Delta(G) \) for some finite group \( G \).

**Corollary B.** Let \( \Delta \) be a block square graph. Then there exists a finite group \( G \) such that \( \Delta(G) = \Delta \) if and only if \( \pi_i \) is a clique for each \( 1 \leq i \leq 4 \), and all the primes in \( \pi_1 \cup \pi_4 \) are adjacent to all the primes in \( \pi_2 \cup \pi_3 \).

Frequently, the results on the class-size context have a dual version in the context of degrees of irreducible characters. It is worth mentioning that M.L. Lewis and Q. Meng introduced in [4] the concept of block square graphs, and in that paper they carried out an analysis of block squares for the prime graph built on the character degrees of soluble groups. Among other things, they proved an analogous version of Theorem A for the character-degree prime graph in the particular case that the group possesses two normal non-abelian Sylow subgroups.

### 2. Preliminaries

In the sequel, if \( x \) is an element of a group \( G \), then we denote by \( x^G \) the conjugacy class of \( x \) in \( G \), and its size is \( |x^G| = |G : C_G(x)| \). For a positive integer \( n \), we write \( \pi(n) \) for the set of prime divisors of \( n \), and in particular \( \pi(G) \) is the set of prime divisors of \( |G| \). As usual, given a prime \( p \), the set of all Sylow \( p \)-subgroups of \( G \) is denoted by \( \text{Syl}_p(G) \), and \( \text{Hall}_p(G) \) is the set of all Hall \( p \)-subgroups of \( G \) for a set of primes \( p \). The remaining notation and terminology used is standard in the framework of finite group theory.

The following elementary properties will be used without further reference.

**Lemma 2.1.** Let \( G \) be a group. Then the following conclusions hold.

(a) If either \( x, y \in G \) have coprime orders and they commute, or \( x \in M \) and \( y \in N \) with \( M \) and \( N \) normal subgroups of \( G \) such that \( M \cap N = 1 \), then \( \pi(|x^G|) \cup \pi(|y^G|) \subseteq \pi(|(xy)^G|) \).

(b) A given prime \( p \) does not lie in \( V(G) \) if and only if \( G \) has a central Sylow \( p \)-subgroup.

As it was mentioned in the Introduction, non-adjacency between vertices significantly constrains the structure of the group. The next result also illustrates this fact. It is Theorem C of [2].

**Proposition 2.2.** Let \( G \) be a group. If \( \pi \) is a set of vertices which are all non-adjacent in \( \Delta(G) \) to a vertex \( p \), then \( G \) is \( \pi \)-soluble with abelian Hall \( \pi \)-subgroups, and the vertices in \( \pi \) are pairwise adjacent.

Observe that if \( \Delta(G) \) is a block square, then it certainly has no complete vertices. Therefore the result below, which is Theorem C of [1], yields a reduction on the structure of such a group \( G \).
Proposition 2.3. Let G be group. Assume that no vertex of $\Delta(G)$ is complete. Then, up to an abelian direct factor, $G = KL$ with $K \trianglelefteq G$ and $L$ abelian subgroups of coprime orders. Moreover, $K = G'$, $K \cap Z(G) = 1$, and both $\pi(K)$ and $\pi(L)$ are cliques of $\Delta(G)$.

We close this section with the next key fact, which is partially Proposition 3.1 of [2].

Proposition 2.4. Let $G$ be a group, and $p$, $q$ non-adjacent vertices of $\Delta(G)$. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$, and let $M$ be a non-trivial abelian normal subgroup of $G$ such that $|M|$ is a power of a suitable prime $r$. Assume that $C_M(P) = 1$, and that $M$ has a complement in $G$. Then $O_q(G) = Q \cap C_G(M) \leq Z(C_G(M))$.

3. Proof of main results

Proof of Theorem A. First, recall that the class sizes of $G$ are the same that those of $G \times A$, where $A$ is an abelian group. Therefore, we may assume that $G$ has no abelian direct factors, and in particular $\pi(G) = V(G)$.

If $G = A \times B$ where $A$ and $B$ are $D$-groups of coprime orders, then certainly $\Delta(G)$ is a block square, where $\pi_1$ and $\pi_4$ are respectively the set of prime divisors of the Frobenius kernel and complement of $A/Z(A)$, and $\pi_2$ and $\pi_3$ are the set of prime divisors of the Frobenius kernel and complement of $B/Z(B)$.

Therefore, the remainder of the proof is devoted to show that if $\Delta(G)$ is a block square, then $G = A \times B$ where $A$ and $B$ are $D$-groups of coprime orders. In virtue of Proposition 2.2, we have that $\pi_i$ is a clique of $\Delta(G)$ and there exists an abelian Hall $\pi_i$-subgroup $H_i$ of $G$ for every $i \in \{1, 2, 3, 4\}$. In particular, all the Sylow subgroups of $G$ are abelian.

Since there is no complete vertex in $\Delta(G)$, then by Proposition 2.3 it follows that $G = KL$ where $K \trianglelefteq G$ and $L$ are abelian subgroups of coprime orders, $K = G'$, $K \cap Z(G) = 1$, and both $\pi(K)$ and $\pi(L)$ induce complete subgraphs in $\Delta(G)$. In particular, $V(G) = \pi(K) \cup \pi(L)$.

Without loss of generality, we may assume that there exists a prime $p \in \pi_1 \cap \pi_4$, so $G$ has a normal (abelian) Sylow $p$-subgroup. As there is no edge in $\Delta(G)$ between $\pi_1$ and $\pi_4$, and $\pi_1$ is a clique of $\Delta(G)$, then $\pi_4 \subseteq \pi(L)$. Further, as $\pi(L)$ is also a clique of $\Delta(G)$, then necessarily it holds that $\pi_1 \subseteq \pi(K)$, so $H_1 = O_{\pi_1}(G) \leq K$. Arguing analogously, we may suppose that $\pi_3 \subseteq \pi(K)$ and $\pi_2 \subseteq \pi(L)$. It follows that $K = H_1 \times H_3$ and, up to conjugation, $L = H_2 \times H_4$.

Next we proceed in three steps.

Step 1: For each $s \in \pi_4$, there exists $p \in \pi_1$ such that $[P, S] \neq 1$, where $P \in Syl_p(H_1)$ and $S \in Syl_s(H_4)$. Besides, for each $r \in \pi_2$, there exists $q \in \pi_3$ such that $[Q, R] \neq 1$, where $Q \in Syl_q(H_3)$ and $R \in Syl_r(H_2)$.

In order to prove the first assertion, and arguing by contradiction, let us assume that $[S, H_1] = 1$. Since $L$ is abelian and $s \in V(G)$, then there necessarily exist $q \in \pi_3$ and $Q \in Syl_q(H_3)$ such that $[Q, S] \neq 1$. As $Q \trianglelefteq G$, there must exist some $y \in S$ with $q \in \pi([y^G])$.

Let $r \in \pi_2$ and $R \in Syl_r(H_2)$. We claim that $[R, H_1] = 1$. If not, then $R$ does not centralize some $P \in Syl_p(H_1)$ for some $p \in \pi_1$. If $r$ does not divide $|x^G|$ for each element $x \in P$, then $x \in C_G(R^{G*})$ for some $g \in G$, and we may suppose that $g \in K$. But then $x = x^{g_s^{-1}} \in C_G(R)$ because $x \in P \trianglelefteq Z(K)$. Since this is valid for all the elements $x \in P$, we get that $P \trianglelefteq C_G(R)$, a contradiction. Hence we may take an element $x \in P \trianglelefteq H_1$ with $r \in \pi([x^G])$. As $[S, H_1] = 1$, then $qr$ divides $|\langle xy \rangle^G|$, which is a contradiction because $q \in \pi_3$ and $r \in \pi_2$.

Since the previous argument holds for each prime $r \in \pi_2$, we deduce that $H_2$ centralizes $H_1$. But $H_1 \trianglelefteq Z(K)$, so there exists $t \in \pi_4$ and $T \in Syl_t(H_4)$ with $[H_1, T] \neq 1$ (in particular $t \neq s$). So
we can take a suitable prime $p \in \pi_1$ such that $[P, T] \neq 1$ for $P \in \text{Syl}_p(H_1)$. In particular, $p$ divides $|w^G|$ for some $w \in T$.

Next we claim $C_0(T) = 1$. Let us suppose that there exists a non-trivial element $x \in C_0(T)$. Certainly $K \leq C_G(x)$, and since $K \cap Z(G) = 1$, then there exists a prime $u \in \pi_2 \cup \pi_4$ such that $u$ divides $|x^G|$. Recall that $[Q, S] \neq 1$ by the first paragraph, so we can pick an element $z \in Q$ with $s \in \pi(|x^G|)$. If $Q$ centralizes $T$, then $|(wz)^G|$ is divisible by both $p \in \pi_1$ and $s \in \pi_4$, a contradiction. Thus $Q$ does not centralize $T$, and therefore there exists an element $w_2 \in T$ with $q \in \pi(|(uw)^G|)$. Now we distinguish two cases: if $u \in \pi_2$, then the class size of $xw_2$ is divisible by $u$ and $q \in \pi_3$, a contradiction; if $u \in \pi_4$, then $\{u, p\} \subseteq \pi(|(xw)^G|)$, which is also a contradiction. Hence $C_0(T) = 1$.

Recall that $Q$ is an abelian normal Sylow $q$-subgroup of $G$, so it is complemented in $G$. Since $\{p, t\} \not\subseteq E(G)$ for every $p \in \pi_3$, then Proposition 2.4 leads to $P \leq Z(C_G(Q))$ for $P \in \text{Syl}_p(G)$, and this is valid for every prime $p \in \pi_4$. It follows $C_G(Q) \leq C_G(H_1)$.

Note that $\pi_1$ and $\pi_2$ are adjacent in $\Delta(G)$ by hypothesis, so there exist $v_1 \in \pi_1$ and $v_2 \in \pi_2$ such that $v_1v_2 \in \pi(|g^G|)$ for some $g \in G$. We can decompose $g = gg_1$ in such way that $g_1 \in K$, $g_1 \in L$ up to conjugation, and $gg_1 = gg_k$. In particular, since additionally $g_k^G$ and $g_l^G$ have coprime sizes, then $|g^G|$ is the product of $|g_k^G|$ and $|g_l^G|$. Therefore $v_1 \in \pi(|g_k^G|)$ and $v_2 \in \pi(|g_l^G|)$. As $C_G(Q) \leq C_G(H_1)$ by the previous paragraph, then certainly $g_1 \notin C_G(Q)$. This means that $q$ divides the class size of $g_k$, so $|g^G|$ is divisible by both $v_2 \in \pi_2$ and $q \in \pi_3$, a contradiction.

The first assertion of Step 1 is already proved. Observe that the second part analogously follows, since the roles of $\pi_1$ and $\pi_4$ are symmetric with respect to $\pi_3$ and $\pi_2$.

**Step 2:** $H_1$ centralizes $H_2$, and $H_2$ centralizes $H_4$.

For proving that $[H_1, H_2] = 1$, let us suppose that there exist $P \in \text{Syl}_p(H_1)$ and $R \in \text{Syl}_q(H_2)$ such that they do not commute, and we aim to reach a contradiction. Note that we can then take an element $z_1 \in R$ such that $p$ divides $|z_1^G|$.

We claim $C_p(R) = 1$. Otherwise, there exists a non-trivial element $x \in C_p(R)$, and since $K \cap Z(G) = 1$, then $\pi(|x^G|)$ contains a suitable prime $u \in \pi_2 \cup \pi_4$. Using Step 1, there is a prime $q \in \pi_3$ and $Q \in \text{Syl}_q(G)$ such that $Q$ does not centralize $R$. Hence $q$ divides the class size of certain element $z_2 \in R$. It follows that the class sizes of $xz_2$ and $x_2$ are divisible by $pu$ and $qu$, respectively. As $u \in \pi_2 \cup \pi_4$, then this contradicts our assumptions. Therefore $C_p(R) = 1$.

Since $\langle q, r \rangle \not\subseteq E(G)$ for every $q \in \pi_3$, then by Proposition 2.4 we get that $Q \leq Z(C_G(P))$, for $Q \in \text{Syl}_q(G)$, and for all primes $q \in \pi_3$. It follows that $C_G(P) \leq C_G(H_3)$.

By assumptions, we can take an element $g \in G$ such that $v_3v_4 \in \pi(|g^G|)$, where $v_3 \in \pi_3$ and $v_4 \in \pi_4$. This element can be written as a product of two suitable commutative elements $g_k \in K$ and $g_l \in L$. Since $|g^G|$ is the product of the class sizes of $g_k$ and $g_l$, then necessarily we obtain that $v_3 \in \pi(|g_k|)$ and $v_4 \in \pi(|g_l|)$. Thus, in virtue of the above paragraph, $g_l$ cannot centralize $P$, and therefore $p \in \pi(|g_k^G|)$. It follows that $pv_3v_4$ divides $|g^G|$, which is a contradiction because $p \in \pi_1$.

Hence $[H_1, H_2] = 1$, and analogously it can be proved $[H_3, H_4] = 1$. These two facts together with Step 1 yield $G = A \times B$, where $A := H_1H_4$ with $H_4 \leq A$ and $B := H_2H_4$ with $H_2 \leq B$. Note that $A$ and $B$ have coprime orders.

**Step 3:** $A = H_1H_4$ and $B = H_3H_2$ are $D$-groups.

We will show that $A = H_1H_4$ is a $D$-group, and the same arguments are analogously valid for $B$. Note that $H_1$ and $H_4$ are abelian groups of coprime orders. Moreover, $Z(A) \leq H_4$ since the Hall $\pi_1$-subgroup of $Z(A)$ is contained in $K \cap Z(G) = 1$.

Let $Z := Z(A)$. We claim that $A/Z$ is a Frobenius group with Frobenius kernel $H_1Z/Z$. By coprime action, it is enough to prove that $C_{H_1}(x) \leq Z$ for every non-trivial element $x \in H_1$. If this does not hold, then we can take an element $y \in C_{H_1}(x) \setminus Z$, so $|y^A|$ is divisible by some prime
$p \in \pi_1$. Since $x \not\in \mathbb{Z}$, then its class size in $A$ is divisible by certain prime $s \in \pi_4$. Therefore $ps \in \pi((xy)^A) = \pi((xy)^G)$, which contradicts our assumptions.

**Proof of Corollary B.** In virtue on Theorem A, the necessity of the condition is clear. Hence, let us suppose that $\Delta$ is a block square graph where $\pi_i$ is a clique for every $1 \leq i \leq 4$, and all the primes in $\pi_1 \cup \pi_4$ are adjacent to all the primes in $\pi_2 \cup \pi_3$. We aim to show that there exists a suitable group $G$ such that $\Delta(G) = \Delta$.

Let $m_i$ denote the size of each $\pi_i$, for $i \in \{1,2,3,4\}$. Let $n_1 := p_1p_2 \cdots p_{m_1}$ where the $p_j$ are pairwise distinct prime numbers. Let $s_1, s_2, \ldots, s_{m_4}$ be distinct primes such that $n_4 := s_1s_2 \cdots s_{m_4}$ is congruent to 1 modulo $n_1$; we point out that they exist by Dirichlet’s theorem on primes in an arithmetic progression. Let $K_1$ and $L_1$ be cyclic groups of orders $n_1$ and $n_4$, respectively. Consider the semidirect product $A = K_1 \rtimes L_1$ with respect to a Frobenius action of $L_1$ on $K_1$. Certainly $A$ is a $D$-group.

Now let $n_2 := q_1q_2 \cdots q_{m_3}$ where $q_k \not\in \pi(A)$ for each $k \in \{1,\ldots,m_3\}$ and they are pairwise distinct primes. Consider a set $\{r_1, r_2, \ldots, r_{m_2}\}$ of pairwise distinct primes such that none of them lies in $\pi(A)$ and $n_2 := r_1r_2 \cdots r_{m_2}$ is congruent to 1 modulo $n_3$; again they exist by the aforementioned theorem due to Dirichlet. Let $K_2$ and $L_2$ be cyclic groups of orders $n_3$ and $n_2$, respectively. Consider the semidirect product $B = K_2 \rtimes L_2$ with respect to a Frobenius action of $L_2$ on $K_2$, so $B$ is a $D$-group.

Certainly $|A|, |B| = 1$. Let $G = A \times B$. Hence it easily follows that $\Delta(G)$ is a block square graph where all the vertices in $\pi(K_1) \cup \pi(L_1)$ are adjacent to all the vertices in $\pi(K_2) \cup \pi(L_2)$, and $\pi(K_i), \pi(L_i)$ are cliques for $1 \leq i \leq 2$. Thus $\Delta(G) = \Delta$.

**Acknowledgements**

This research has been carried out during a stay of the author at the Dipartimento di Matematica e Informatica "Ulisse Dini" (DIMAI) of Università degli Studi di Firenze. He wishes to thank the members of the DIMAI for their hospitality, and Centro Universitario EDEM for its support.

**Funding**

This research is supported by Proyecto PGC2018-096872-B-I00 from the Ministerio de Ciencia, Innovación y Universidades (Spain), and Proyecto AICO/2020/298 from the Generalitat Valenciana (Spain).

**ORCID**

V. Sotomayor 🌐 http://orcid.org/0000-0001-8649-5742

**References**

[1] Casolo, C., Dolfi, S., Pacifici, E., Sanus, L. (2012). Groups whose prime graph on conjugacy class sizes has few complete vertices. *J. Algebra.* 364:1–12. DOI: 10.1016/j.jalgebra.2012.04.013.

[2] Casolo, C., Dolfi, S., Pacifici, E., Sanus, L. (2013). Incomplete vertices in the prime graph on conjugacy class sizes of finite groups. *J. Algebra.* 376:46–57. DOI: 10.1016/j.jalgebra.2012.10.012.

[3] Dolfi, S. (1995). Arithmetical conditions on the length of the conjugacy classes of a finite group. *J. Algebra.* 174(3):753–771. DOI: 10.1006/jabr.1995.1151.

[4] Lewis, M. L., Meng, Q. (2020). Solvable groups whose character degree graphs generalize squares. *J. Group Theory.* 23(2):217–234. DOI: 10.1515/jgth-2019-0027.