Running Nonlocal Lagrangians

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Abstract
We investigate the renormalization of “nonlocal” interactions in an effective field theory using dimensional regularization with minimal subtraction. In a scalar field theory, we write an integro-differential renormalization group equation for every possible class of graph at one loop order.

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1 Introduction

In its traditional form, an effective field theory calculation goes like this: Start at a very large scale, that is with the renormalization scale, \( \mu \), very large. In a strongly interacting theory or a theory with unknown physics at high energy, this starting scale should be sufficiently large that nonrenormalizable interactions produced at higher scales are too small to be relevant. In a renormalizable, weakly interacting theory, one starts at a scale above the masses of all the particles, where the effective theory is given simply by the renormalizable theory, with no nonrenormalizable terms. The theory is then evolved down to lower scales. As long as no particle masses are encountered, this evolution is described by the renormalization group. However, when \( \mu \) goes below the mass, \( \Lambda \), of one of the particles in the theory, we must change the effective theory to a new theory without that particle. In the process, the parameters of the theory change, and new, nonrenormalizable interactions may be introduced. Both the changes in existing parameters, and the coefficients of the new interactions are computed by “matching” the physics just below the boundary in the two theories. It is this process that determines the relative sizes of the nonrenormalizable terms associated with the heavy particles.

Because matching is done for \( \mu \approx \Lambda \), the rule for the size of the coefficients of the new operators is simple for \( \mu \approx \Lambda \). At this scale, all the new contributions scale with \( \Lambda \) to the appropriate power (set by dimensional analysis) up to factors of coupling constants, group theory or counting factors and loop factors (of \( 16\pi^2 \), etc.) \([1]\). Then when the new effective theory is evolved down to smaller \( \mu \), the renormalization group introduces additional factors into the coefficients. Thus a heavy particle mass appears in the parameters of an effective field theory in two ways. There is power dependence on the mass that arises from matching conditions. There is also logarithmic dependence that arises from the renormalization group.

The matching correction at tree level is simply a difference between a calculation in the full theory and a calculation in the low energy effective theory

\[
\int \delta \mathcal{L}^0(\Phi) = S_{\mathcal{L}_H+\mathcal{L}}^0(\Phi) - S_{\mathcal{L}}^0(\Phi)
\]

(1)

where \( S_{\mathcal{L}_H+\mathcal{L}}^0(\Phi) \) denotes the light particle effective action in the full theory and \( S_{\mathcal{L}}^0(\Phi) \) denotes the same in the low energy theory \([2]\). The matching correction so obtained is nonlocal because
it depends on $p/\Lambda$ through the virtual heavy particle propagators. It is also analytic in $p/\Lambda$ in the region relevant to the low energy theory, i.e. for characteristic momentum $\ll \Lambda$. Thus it can be expanded in powers of $p/\Lambda$ with the higher order terms decreasing in importance: this corresponds to a local operator product expansion in the domain of analyticity, equivalent to a local nonrenormalizable Lagrangian which can be treated as an honest-to-goodness local field theory. However in general an infinite series of terms of increasingly higher dimension are generated by matching at tree level. These cause no problem when the scales are well separated, because their effects quickly become negligible. But if there are two or more scales close together (such as $m_t$ and $M_Z$ may be), then we may not be justified in ignoring terms at higher orders in the expansion. How does one understand how to interpolate smoothly between the well-understood situation in which the scales are very different and the well-understood situation in which the scales are very close together? How can we keep track of all the infinite number of higher derivative operators efficiently? Is it possible to deal with the nonlocal effective Lagrangian directly, without expanding? In the particular context of a scalar field theory, we will attempt to answer questions at the one loop level. The approach will be direct. We will manipulate the nonlocal interactions as if they are expanded in a momentum expansion, and then show that the resulting $\beta$ functions for the terms in the momentum expansion can be collected into integro-differential renormalization group equations for the nonlocal couplings.

The present paper is organized as follows: in section 2 we calculate the $\beta$ function for a nonlocal four point coupling arising from a one-loop graph with two internal lines in nonlocal $\phi^4$ theory. We give this example before the general results of the following sections in order to point out the important features of our method. In section 3 the method is then applied to obtain the contribution to the renormalization group equation for a general class of graphs in an arbitrary massive nonlocal, non-renormalizable parity invariant scalar field theory.

## 2 Basic Example

The matrix elements in a general nonlocal field theory [3] are calculated by writing down a Lagrangian with ‘smeared’ vertices. For instance, for a nonlocal $\phi^4$ interaction in a scalar field theory with the discrete symmetry $\phi \rightarrow -\phi$, the interaction term in the action is

$$S_4 = \int dx_1 dx_2 dx_3 dx_4 F(x_1 - x_4, x_2 - x_4, x_3 - x_4) \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$$

(2)
where $x_i$ are spacetime coordinates and $F$ is a nonlocal ‘form-factor’. Energy-momentum conservation at each vertex of the corresponding Feynman graph is expressed in terms of the Fourier transform $G$ of $F$: $G = G(p_1, p_2, p_3)$ for the basic $\phi^4$ interaction shown in figure 1. Bose symmetry implies that the nonlocal coupling $G$ is symmetric and satisfies

$$G(p_1, p_2, p_3) = G(p_1, p_2, -p_1 - p_2 - p_3).$$

Figure 1: Basic nonlocal $\phi^4$ vertex.

Of course, Lorentz invariance dictates that in the final expression for the matrix element in momentum space, only scalar products of momenta will appear as arguments of $G$. *Implicit in the definition of $G$ is a mass scale $\Lambda$ which sets the limit for the region of analyticity of $G$, and for characteristic momenta $p < \Lambda$, $G$ is analytic. We go to the effective low energy theory by expanding in a Taylor expansion in $p/\Lambda$. One can think of this Taylor expansion as the formal implementation of a local operator product expansion of $G$. At this point, in going to the effective low energy theory, we have actually changed the high energy behavior of the theory so that integrals which were convergent in the full theory are divergent in the effective theory. This trades logs of $\Lambda$ in the full theory for anomalous dimensions in the low energy theory. This trade allows us to calculate the logs more simply and to sum them using the the renormalization group.*

Let us first discuss as an example the renormalization of the nonlocal $\phi^4$ interaction from the graph shown in figure 2. For massless fields, this is the only contribution in one loop. In the massive case, there is also a contribution from a tadpole graph. In addition, in either case, there are renormalization of $\phi^{2k}$ interactions for $k > 2$. We will systematically consider them in the next section.
Figure 2: Feynman graph contributing to the renormalization of $G$.

The Feynman integral for the graph in figure 2 is

$$\int \frac{d^4k}{(2\pi)^4} \frac{G(p_1, p_2, k) G(p_3, p_4, k + p_1 + p_2)}{[k^2 - m^2 + i\epsilon][(k + p_1 + p_2)^2 - m^2 + i\epsilon]}$$

(4)

Combine denominators

$$\int_0^1 d\alpha \frac{1}{[(1 - \alpha)k^2 + \alpha(k + p_1 + p_2)^2 - m^2 + i\epsilon]^2}$$

(5)

and shift momenta

$$k = \ell - \alpha(p_1 + p_2)$$

(6)

to obtain

$$\int_0^1 d\alpha \int \frac{d^4\ell}{(2\pi)^4} \frac{G(p_1, p_2, \ell - \alpha(p_1 + p_2)) G(p_3, p_4, \ell + (1 - \alpha)(p_1 + p_2))}{[\ell^2 + \alpha(1 - \alpha)(p_1 + p_2)^2 - m^2 + i\epsilon]^2}$$

(7)

Now, here is the crucial point. We get the low energy effective theory by expanding the $G$s in a momentum expansion. An equivalent procedure is to treat the $G$s as if they were analytic everywhere in momentum space. Thus we can deal with the $\ell$ dependence of $G$ by writing a symbolic Taylor expansion. Here we are effectively doing the momentum

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1 Analyticity of the nonlocal couplings may alternatively be exploited by formally Laplace transforming in the loop dependent arguments, which yields instead of $e^{\ell \cdot \mathcal{M}} G$ an exponential $e^{-\ell \cdot s \mathcal{G}}$ in terms of the Laplace transform of $G$, along with an extra Laplace inversion integral. The rest of the computation is essentially identical.
expansion. But the key is that we can resum the final result into a finite integral over the original nonlocal couplings.

\[
G(p_1, p_2, \ell - \alpha(p_1 + p_2)) G(p_3, p_4, \ell + (1 - \alpha)(p_1 + p_2))
\]

\[
= e^{\ell \frac{q}{2\pi}} \left[ G(p_1, p_2, q) G(p_3, p_4, q + p_1 + p_2) \right]_{q = -\alpha(p_1 + p_2)}
\]  

(8)

Now all the dependence on the loop momentum is in the denominators and in the exponential, so we just have to do the Feynman integral of the exponential

\[
\int \frac{d^4\ell}{(2\pi)^4} \frac{e^{\ell \frac{q}{2\pi}}}{[\ell^2 + \alpha(1 - \alpha)(p_1 + p_2)^2 - m^2 + i\epsilon]^2}
\]  

(9)

or to be more precise, the dimensionally regularized integral

\[
\int \frac{d^{4-\epsilon}\ell}{(2\pi)^4} \frac{e^{\ell \frac{q}{2\pi}}}{[\ell^2 + \alpha(1 - \alpha)(p_1 + p_2)^2 - m^2 + i\epsilon]^2}
\]  

(10)

We will do this by manipulating the exponential like a power series, because the basic assumption is analyticity in the momenta. It is at this point that we have irrevocably changed the high energy behavior and gone over the effective low energy theory. Because of symmetry, only even terms in \(\ell\) contribute,

\[
\sum_{r=0}^{\infty} \frac{1}{(2r)!} \int \frac{d^{4-\epsilon}\ell}{(2\pi)^4} \frac{\left(\ell \frac{q}{2\pi}\right)^{2r}}{[\ell^2 + \alpha(1 - \alpha)(p_1 + p_2)^2 - m^2 + i\epsilon]^2}
\]  

(11)

We calculate this as follows

\[
\int \frac{d\Omega}{\Omega} \ell_{\mu_1} \ell_{\mu_2} \cdots \ell_{\mu_{2r}}^{(2r)!/(2^r r!)} \text{ terms}
\]

(12)

\[
= A_r \left( \ell^2 \right)^r \left[ g_{\mu_1 \mu_2} \cdots g_{\mu_{2r-1} \mu_{2r}} + \text{perms} \right],
\]

where

\[
\Omega \equiv \int d\Omega_\ell.
\]  

(13)

Contracting with \(g^{\mu_1 \mu_2}\) gives

\[
A_{r-1} = (4 + 2(r - 1)) A_r = 2(r + 1) A_r
\]  

(14)

or

\[
A_r = \frac{1}{2^r(r + 1)!}
\]  

(15)
and thus
\[ \frac{1}{(2r)!} \int \frac{d\Omega}{\Omega} \left( \ell \frac{\partial}{\partial q} \right)^{2r} = \frac{1}{4^r r!(r+1)!} \left[ e^2 \right]^r \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r. \]  
(16)

So the integral is
\[ \sum_{r=0}^{\infty} \frac{1}{4^r r!(r+1)!} \int \frac{d^4\ell}{(2\pi)^4} \frac{[\ell^2]^r \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r}{[\ell^2 + \alpha(1-\alpha)(p_1 + p_2)^2 - m^2 + i\epsilon]^2}, \]  
(17)

which we can calculate in the usual way. Wick rotate
\[ i \sum_{r=0}^{\infty} \frac{(-1)^r}{4^r r!(r+1)!} \int \frac{d^4\ell}{(2\pi)^4} \frac{[\ell^2]^r \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r}{[\ell^2 + A(\alpha)^2]^2}, \]  
(18)

where
\[ A(\alpha)^2 = -\alpha(1-\alpha)(p_1 + p_2)^2 + m^2, \]  
(19)

and \( A(\alpha)^2 \) is positive for Euclidean momenta. Now doing the \( \ell \) integral gives
\[ \frac{2i}{(4\pi)^2\epsilon \mu^{-2\epsilon}} \sum_{r=0}^{\infty} \frac{1}{4^r r!^2} A(\alpha)^{2r} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r + \ldots \]  
(20)

where we have dropped everything but the \( 1/\epsilon \) pole and the associated \( \mu \) dependence. We can write the result as
\[ \frac{2i}{(4\pi)^2\epsilon \mu^{-2\epsilon}} \frac{\partial}{\partial x} \left[ \sum_{r=0}^{\infty} \frac{x}{4^r r!(r+1)!} [\sqrt{-xA(\alpha)} e^{\frac{\partial}{\partial q}}]^{2r} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r \right]_{x=1} \]  
(21)

This can then be turned back into the (Euclidean) angular average of an exponential, just inverting (19),
\[ \frac{2i}{(4\pi)^2\epsilon \mu^{-2\epsilon}} \frac{\partial}{\partial x} \left[ \int \frac{d\Omega}{\Omega} x e^{\sqrt{-xA(\alpha)} e^{\frac{\partial}{\partial q}}} \right]_{x=1} \]  
(22)

where \( e \) is a Euclidean unit vector. Now that we have done the \( \ell \) integral and extracted the \( 1/\epsilon \) pole, we can use the Taylor series to put the form (22) in terms of the \( G \)s. The result is
\[ \frac{i}{(8\pi)^2\epsilon \mu^{-2\epsilon}} \frac{\partial}{\partial x} \left[ \int \frac{d\Omega}{\Omega} \int_0^1 d\alpha x \left[ G(p_1, p_2, \sqrt{xA(\alpha)} e - \alpha(p_1 + p_2)) \right. \right. \]  
\[ G(p_3, -p_1 - p_2 - p_3, \sqrt{xA(\alpha)} e + (1-\alpha)(p_1 + p_2)) \left. \right]_{x=1} \]  
(23)

with the derivative evaluated at \( x = 1 \). Thus we have reduced the divergent part of the original Feynman graph to an integral of the \( G \)s and their derivatives over finite ranges (for Euclidean momenta). These last integrals are well-behaved and could be done numerically. Hence, with
Figure 3: Feynman graph contributing to the running of $\phi^6$ coupling.

$\Omega = 2\pi^2$ in for Euclidean four space, we obtain the one-loop $\beta$ function for the nonlocal interaction, $G$:

$$
\beta_{G(p_1,p_2,p_3)} = \frac{1}{16\pi^4} \frac{\partial}{\partial x} \left[ \int d\Omega_e \int_0^1 d\alpha \, x \left( G(p_1,p_2,\sqrt{x}A(\alpha)e - \alpha(p_1+p_2)) - G(p_3,-p_1-p_2-p_3,\sqrt{x}A(\alpha)e + (1-\alpha)(p_1+p_2)) \right) \right]_{x=1} + \text{cross terms}
$$

(24)

Now in the massless case, where there are no other contributions to the renormalization of the $\phi^4$ coupling, the “running” nonlocal coupling satisfies the integrodifferential renormalization group equation

$$
\mu \frac{\partial}{\partial \mu} G(p_1,p_2,p_3) = \beta_{G(p_1,p_2,p_3)}.
$$

(25)

A useful check on this result is obtained by going to the ‘local limit’, i.e. $G \to g$ where $g$ is the usual coupling constant of local $\phi^4$ theory. Indeed, we know that the contribution to the $\beta$ function in local $\phi^4$ theory at one loop is

$$
\beta(g) = \frac{3g^2}{16\pi^2} > 0.
$$

(26)

Substituting $G = g$, inserting an explicit symmetry factor of 1/2, and summing over the crossed graphs, (24) indeed yields

$$
\beta_{G(p_1,p_2,p_3)} \to \beta(g) \quad \text{as} \quad G \to g.
$$

(27)

Now the nonlocal quartic coupling $G$ can induce changes in the coupling terms with more fields. For instance, the one-loop diagram relevant to the renormalization of the $\phi^6$ coupling is shown in figure 3.
For massless fields, this does not mix back into $G$ in a mass independent renormalization scheme, because the relevant Feynman graph of figure 4 vanishes. However, in the massive case, the graph of figure 4 gives a non-vanishing contribution. We will now systematically compute the $\beta$ functions of the general nonlocal couplings in a massive nonrenormalizable theory.

## 3 General Results

Working within the formalism of a massive nonlocal effective theory induced by some unknown full theory, we are forced to consider an effective Lagrangian with operators with an arbitrary number of low energy fields. In this section, we will compute the one-loop running of a general non-renormalizable, nonlocal scalar effective theory with $\phi \rightarrow -\phi$ symmetry. Specifically, our purpose is to isolate the dimensional regularization pole of a $2m$ point function of type $n$ (i.e. with $n$ vertices or propagators) at one loop. The plan is as follows: we first construct the expressions corresponding to the Feynman integral of a general graph with a special choice of momentum labelling conventions. We then describe the application of the method of the last section to two relevant examples: the tadpole renormalization of the nonlocal $\phi^4$ coupling with a $\phi^6$ operator insertion, and the renormalization of a nonlocal $\phi^6$ coupling with three $\phi^4$ operator insertions. The last example is useful in highlighting the special features of renormalizing graphs with more than two internal lines. Finally, we attack the general case, and give the complete expression for the $1/\epsilon$ pole of the $2m$ point function of type $n$ at one loop as a surprisingly compact multi-Feynman-parameter, multi-dimensional angular integral.
3.1 Preliminaries

We assume that the dimensionally continued \( d = 4 - \epsilon \) dimensional non-local Lagrangian has a \( \phi \rightarrow -\phi \) symmetry and hence has an interaction term proportional to

\[
\mathcal{L}_{\text{int.}} = \sum_{r=1}^{\infty} \mu^{\epsilon(r-1)} G_{2r} \phi^{2r}. \tag{28}
\]

The mass scale \( \mu \) is introduced to keep the dimensions of the nonlocal couplings \( G_{2r} \) fixed under dimensional continuation. The interaction is not normal ordered\(^2\), and each \( G_{2r} \) is some nonlocal function which is analytic in the region under consideration, depends as a consequence of momentum conservation on \( 2r - 1 \) linearly independent momenta, and may have dimensions proportional to some power of an implicit scale of nonlocality \( \Lambda \).

A diagram with \( n \) internal lines will be called type-\( n \). At one loop, a type-\( n \) graph has \( n \) vertices connected to external lines. Let

\[
N = \sum_{i=1}^{n} v_i, \tag{29}
\]

where \( v_i \) is the number of external lines emanating from the \( i \)th vertex. The loop integral for the type-\( n \) one-loop renormalization of the \( N \) point function (shown in figure 5) is

\[
I = \int \frac{d^4k}{(2\pi)^4} \prod_{i=1}^{n} G_{v_i+2}^i (\ldots, k + Q_{i-1}) \tag{30}
\]

where the upper index on the \( G' \)s is used to number the vertices as one goes along the loop.

\(^2\)Hence fields at the same point can be contracted, and tadpoles will occur explicitly.
Here all external momenta are taken to be incoming (signified by \ldots, in the numerator, different for different $G^i$) and we have defined

$$
Q_0 = 0 \quad (31)
$$

$$
Q_1 = P_1
$$

$$
Q_2 = P_2 + Q_1
$$

$$
Q_3 = P_3 + Q_2
$$

\[ : : : \]

$$
Q_i = P_i + Q_{i-1} = \sum_{j=1}^{j=i} P_j
$$

$$
Q_n = Q_{n-1} + P_n = 0,
$$

and $P_i$ is the sum of the external momenta flowing into the $i$th vertex. Energy momentum conservation is simply $Q_n = 0$. Symmetry factors are suppressed.

Using this notation, the loop integral for the for the $\phi^4$ coupling that we computed in the last section is obtained by setting $n = 2, v_1 = 2, v_2 = 2$:

$$
I = \int \frac{d^4k}{(2\pi)^4} \frac{G_1^4(p_1, p_2, k)G_2^2(p_3, p_4, k + Q_1)}{(k^2 + i\epsilon)[(k + Q_1)^2 + i\epsilon]}. \quad (32)
$$

Using the generalized Feynman identity

$$
\frac{1}{A_1^{\rho_1} A_2^{\rho_2} \cdots A_n^{\rho_n}} = \frac{\Gamma(\rho_1 + \rho_2 + \ldots + \rho_n)}{\Gamma(\rho_1)\Gamma(\rho_2)\ldots\Gamma(\rho_n)} \times \int_0^1 d\alpha_1 \ldots d\alpha_n \frac{\alpha_1^{\rho_1-1}\alpha_2^{\rho_2-1} \ldots \alpha_n^{\rho_n-1}\delta(1 - \alpha_1 - \alpha_2 - \ldots - \alpha_n)}{(\alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_n A_n)^{\rho_1 + \rho_2 + \ldots + \rho_n}}, \quad (33)
$$

combine the denominators

$$
\frac{1}{(k^2)(k + Q_1)^2 \ldots (k + Q_{n-1})^2} = (n - 1)! \int \prod_i d\alpha_i \frac{1}{[k^2(1 - \sum_{i=1}^{n-1} \alpha_i) + \sum_{r=1}^{n-1} (k + Q_r)^2 \alpha_r]^n}, \quad (34)
$$

and shift the loop momentum

$$
k = \ell - \sum_{s=1}^{n-1} Q_s \alpha_s \quad (35)
$$

so that

$$
k^2 = \ell^2 + (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 - 2\ell \cdot \sum_{s=1}^{n-1} Q_s \alpha_s, \quad (36)
$$
which, when put into the denominator gives

\[
D = [\ell^2 + \left(\sum_{s=1}^{n-1} Q_s \alpha_s\right)^2 - 2\ell \cdot \sum_{s=1}^{n-1} Q_s \alpha_s - \ell^2 \sum_{t=1}^{n-1} \alpha_t - \left(\sum_{s=1}^{n-1} Q_s \alpha_s\right)^2 \sum_{t=1}^{n-1} \alpha_t + (2\ell \cdot \sum_{s=1}^{n-1} Q_s \alpha_s) (\sum_{t=1}^{n-1} \alpha_t) + \sum_{r=1}^{n-1} \ell^2 \alpha_r + \sum_{r=1}^{n-1} (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 \alpha_r - 2\sum_{r=1}^{n-1} \ell \cdot \sum_{s=1}^{n-1} Q_s \alpha_s \alpha_r + \sum_{r=1}^{n-1} Q_r^2 \alpha_r + 2\sum_{r=1}^{n-1} \ell \cdot Q_r \alpha_r - 2\sum_{r=1}^{n-1} Q_r (\sum_{s=1}^{n-1} Q_s \alpha_s) \alpha_r]^{n}.
\]

After cancellation (37) yields the denominator

\[
D = [\ell^2 - A^2]^{n}
\]

where

\[
A^2 \equiv (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 - \sum_{r=1}^{n-1} Q_r^2 \alpha_r.
\]

For the massive case, we simply substitute in (34)\(^3\)

\[
k^2 \rightarrow k^2 - m^2
\]

\[
(k + Q_r)^2 \rightarrow (k + Q_r)^2 - m^2
\]

so that we just get an additional term in (37)\(^4\) when combining denominators after the shift (33):

\[
-m^2 (1 - \sum_{s=1}^{n-1} \alpha_s) + \sum_{s=1}^{n-1} (-m^2) \alpha_s = -m^2.
\]

So the general form for the denominator of the right hand side of (34) with massive fields is

\[
D = [k^2 - (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 + (\sum_{r=1}^{n-1} Q_r^2 \alpha_r) - m^2]^{n}.
\]

Now the shift (35) changes the argument of the numerator factors in (30) also, to yield the final expression for the Feynman integral\(^3\)

\[
(n - 1)! \int \prod_{j=1}^{n-1} d\alpha_j \int \frac{d^d \ell}{(2\pi)^d} \frac{\prod_{i=1}^{n} G_{n+2}^{i}(\ldots, \ell - \sum_{s=1}^{n-1} Q_s \alpha_s + Q_{i-1})}{[\ell^2 - (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 + (\sum_{r=1}^{n-1} Q_r^2 \alpha_r) - m^2 + i\epsilon]^{n}}
\]

\(^3\)The reader may check that with \(n = 2, v_1 = v_2 = 2\), the above relations give the correct integral for the example of the first section:

\[
\int d\alpha \int \frac{d^d \ell}{(2\pi)^d} \frac{G_4(p_1, p_2, \ell - \alpha(p_1 + p_2))G_4(p_3, p_4, \ell + (1 - \alpha)(p_1 + p_2))}{[\ell^2 + \alpha(1 - \alpha)Q_1^2 - m^2]^{2}}
\]

with \(Q_1 = p_1 + p_2\).
**Important remarks on notation** : (1) In intermediate steps of the computations, we denote the dependence of each nonlocal function on the (distinct!) external momenta by ellipsis. This is useful since the external momenta play a trivial part in the loop integration, and may be reinstated by examination in the final expressions. (2) Also, since at one loop any vertex shares two and only two lines with the loop, by energy momentum conservation the loop momentum appears only once as an argument of any $G^i$ and will be put in its last slot. (3) The first vertex will have only the loop momentum as its last entry.

**Crossing** : The external momenta can be exchanged amongst themselves. The final result, however, depends only on the number of distinct Lorentz invariants of the four momenta $p_i$ ($i = 2m$ for renormalization of the $2m$ point function) under the condition $\sum_{i=1}^{2m} p_i = 0$ and $p_i^2 = m^2$. This equals the total number of graphs related by ‘crossing’ where the crossed graphs can be obtained by exchanging external momenta. We will give explicit expressions only for one member of each crossed set.

**Counting $i$’s** : Ignoring for the moment the $i$’s appearing due to Wick rotation and integration (see below), the integrand itself gives no powers of $i$. This is seen as follows: each propagator is $\frac{i}{p^2 - m^2 + ic}$, and each nonlocal vertex has a Feynman rule $-iG_2$. Since the number of vertices equals the number of internal lines at one loop, for a type-$n$ graph we obtain $(-i)^n i^n = (-1)^n (i)^{2n} = (-1)^{2n} = 1$. So the only source of $i$’s and minus signs is the integration formula.

**Computational Tools** :

In order to do the dimensionally regularized loop integrals we will need the well-known integration formula (for Euclidean momenta):

$$
\int d^d q \frac{(q^2)^r}{(q^2 - A^2)^n} = i \pi^{d/2} (\pi^2)^{r-n} \frac{\Gamma(r + \frac{d}{2}) \Gamma(n - r - \frac{d}{2}) \Gamma(\frac{d}{2}) \Gamma(n)}{\Gamma(\frac{d}{2}) \Gamma(\frac{d}{2})},
$$

and the expansion

$$
\Gamma(-n + \epsilon) = \left(\frac{-1}{n!}\right)^n \left[ 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma \right] + \mathcal{O}(\epsilon),
$$

where $\gamma = 0.5772157$ is Euler’s constant.

Also note that in $2\kappa - 2$ dimensions ($\kappa$ a positive integer)

$$
\int \frac{d\Omega_{2\kappa-2}^\epsilon}{\Omega^{2\kappa-2}} \ell_{\mu_1} \ell_{\mu_2} \cdots \ell_{\mu_{2r}},
$$

$(2r)!/(2^r r!)$ terms,

$$
= A_{r}^{2\kappa-2} \left( \ell^2 \right)^r \left[ g_{\mu_1 \mu_2} \cdots g_{\mu_{2r-1} \mu_{2r}} + \text{perms} \right]
$$

13
where we have defined
\[ \Omega^d \equiv \int d\Omega^d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \] (49)
Contracting with \( g^{\mu_1 \mu_2} \) gives
\[ A_{r-1}^{2\kappa-2} = (2\kappa - 2 + 2(r - 1)) A_r^{2\kappa-2} \]
(50)
So
\[ A_r^{2\kappa-2} = \frac{1}{2^r(r + \kappa - 2)!} \] (51)
and thus in \( 2\kappa - 2 \) dimensions
\[
\frac{1}{(2r)!} \frac{d\Omega_r^{2\kappa-2}}{\Omega^{2\kappa-2}} \left( \ell \frac{\partial}{\partial q} \right)^{2r} = \frac{1}{4^r r!(r + \kappa - 2)!} \left[ (\ell^2) \left( \frac{\partial}{\partial q} \right)^2 \right]^r. \] (52)
The importance of the result of (52) will be seen when we isolate the pole pieces appearing from graphs with more than two internal lines. The point is that the leading terms in the Taylor expansion then give Feynman integrals which are manifestly convergent by power counting, and divergences appear only at some higher order in the Taylor expansion. This implies that the sum of dimensional regularization poles does not begin at zero, and we cannot immediately write the result as an angular integral over a Euclidean unit vector in four dimensions, as we did in the last section. One choice is to redefine the infinite sum to start at zero and compensate for the redefinition by subtracting off a finite sum (which vanishes under the action of differential operators in the dummy parameters). Alternatively, using (52), we can re-sum the poles for a graph with \( n > 2 \) internal lines, \textbf{without reference to any dummy variables}, in terms of a \( 2n - 2 \) dimensional angular integral! The example of the \( 2 - 2 - 2 \) graph below will give the explicit details of how this is done.

The \( \phi^6 \) operator can be renormalized at one loop by a type-1 (tadpole) with a \( G_8 \) coupling insertion, a type-2 graph with \( G_6, G_4 \) insertions, or the most convergent graph, of type-3, with three \( G_4 \) insertions. For the last one, the relevant graph with incoming external momenta is given in figure 6.

As mentioned in the last section, for massless fields this will not mix back into \( G_4 \) in a mass independent renormalization scheme, because the Feynman graph in figure 7 vanishes. However, for massive light fields the tadpole graph does not vanish, and we will therefore evaluate this type-1 ‘tadpole’ graph first.
Figure 6: Type-3 Feynman graph contributing to $\beta_{G_6}$.

Figure 7: Tadpole graph that vanishes for massless $\phi$s.
3.2 Tadpole diagram contribution to $\beta_{G_4}$

We need to do the integral (see figure 7), where $r = 1$ in (28), and $n = 1, v_1 = 4, p_4 = -(p_1 + p_2 + p_3)$,

$$
\mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{G_6(p_1, p_2, p_3, p_4, k)}{(k^2 - m^2 + i\epsilon)}.
$$

(53)

No momentum shift is needed since the denominator is purely quadratic in the loop momentum. Now because $G_6$ is analytic, we can deal with the $k$ dependence by writing a symbolic Taylor expansion

$$
G_6(p_1, p_2, p_3, p_4, k) = e^{k \frac{\partial}{\partial q}} [G_6(p_1, p_2, p_3, p_4, q)]_{q=0}
$$

(54)

Then we just have to do the dimensionally regularized integral

$$
\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon} \mu^{-2\epsilon}} \left[ k^2 - m^2 + i\epsilon \right].
$$

(55)

We again do this by manipulating the exponential like a power series, because the basic assumption is analyticity in the momenta. Because of symmetry, only even terms in $k$ contribute, so the integral to be done equals

$$
\sum_{r=0}^{\infty} \frac{1}{(2r)!} \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon} \mu^{-2\epsilon}} \left[ k^2 - m^2 + i\epsilon \right]^{2r}.
$$

(56)

which finally yields the integral (just as in example of last section)

$$
\sum_{r=0}^{\infty} \frac{1}{4^r r! (r+1)!} \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon} \mu^{-2\epsilon}} \left[ k^2 - m^2 + i\epsilon \right]^{r}.
$$

(57)

After performing a Wick rotation, and doing the integral using (16) and (17) we obtain for the $1/\epsilon$ pole and the associated $\mu$ dependence:

$$
\frac{im^2}{(4\pi)^2 \epsilon \mu^{-2\epsilon}} \sum_{r=0}^{\infty} \frac{1}{4^r r! (r+1)!} [m^2]^r \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r + \cdots
$$

(58)

This can then be turned back into the (Euclidean) angular average of an exponential, by just inverting (18),

$$
\frac{im^2}{(4\pi)^2 \epsilon \mu^{-2\epsilon}} \int \frac{d\Omega^4}{\Omega^4} e^{m e^{m \frac{\partial}{\partial \eta}}}
$$

(59)

where $e$ is a Euclidean unit vector, and $\Omega^4 = 2\pi^2$. Now that we have done the loop integral and extracted the $1/\epsilon$ pole, we can use the Taylor series to put the form (55) in terms of $G_6$. The result for the pole part then equals

$$
\frac{im^2}{32\pi^4 \epsilon \mu^{-2\epsilon}} \int d\Omega^4 \left[ G_6(p_1, p_2, p_3, -p_1 - p_2 - p_3, mc) \right].
$$

(60)
Thus we have reduced the divergent part of the original Feynman graph to an integral of $G_6$ over a finite region (for Euclidean momenta). Hence we have obtained the tadpole contribution to the one-loop $\beta$ function for the nonlocal interaction, $G_4$:

$$\beta_{G_4}(p_1,p_2,p_3) \equiv \mu \frac{d}{d\mu} G_4[4](p_1,p_2,p_3)$$

$$= \frac{m^2}{32\pi^2} \int d\Omega_4^4 G_6(p_1,p_2,p_3,-p_1-p_2-p_3,\mu e),$$

where the subscript on $G$ on the LHS denotes the coupling that is renormalized, with the number of external lines at each vertex for the graph under consideration in square brackets (i.e. type of graph considered).

### 3.3 2-2-2 diagram contribution to $\beta_{G_6}$

The Feynman integral from figure 6, with $Q_1 = p_1 + p_2, Q_2 = p_1 + p_2 + p_3 + p_4, \sum_{i=1}^{i=6} p_i = 0; \text{ and } v_1 = v_2 = v_3 = 2$, equals

$$= \mu^2 \int \frac{d^4k}{(2\pi)^4} \frac{G_4(p_1,p_2,k) G_4(p_3,p_4,k+Q_1) G_4(p_5,p_6,k+Q_2)}{(k^2-m^2+i\epsilon)(k+Q_1)^2-m^2+i\epsilon][(k+Q_2)^2-m^2+i\epsilon]}. \quad (62)$$

Combining denominators and making the shift $k = \ell - (Q_1\alpha_1 + Q_2\alpha_2)$, we get the denominator

$$D = \left[ \ell^2 - Q_1^2\alpha_1(\alpha_1 - 1) - Q_2^2\alpha_2(\alpha_2 - 1) - 2Q_1 \cdot Q_2\alpha_1\alpha_2 - m^2 \right]^3 \quad (63)$$

so that the integral becomes (with a factor of 2 from the Feynman trick)

$$= 2 \int \prod_i d\alpha_i \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{D} \left[ G_4(p_1,p_2,\ell - (Q_1\alpha_1 + Q_2\alpha_2)) G_4(p_3,p_4,\ell - (Q_1(\alpha_1 - 1) + Q_2\alpha_2)) G_4(p_5,p_6,\ell - Q_1\alpha_1 - Q_2(\alpha_2 - 1)) \right] \quad (64)$$

Define the numerator

$$N(\ell,p,\alpha) = [G_4(p_1,p_2,\ell - (Q_1\alpha_1 + Q_2\alpha_2)) G_4(p_3,p_4,\ell - (Q_1(\alpha_1 - 1) + Q_2\alpha_2)) G_4(p_5,p_6,\ell - Q_1\alpha_1 - Q_2(\alpha_2 - 1))]. \quad (65)$$

---

4. Also note than in the local limit, replacing $G_6$ by a ‘constant’ $g_6$, we get $\beta_{g_4[4]} = \frac{m^2 g_6}{16\pi^2}$ which may be directly obtained in the local limit as a check.
Now, because the $G$s are analytic, we can deal with the $\ell$ dependence of $G$ by writing a symbolic Taylor expansion

$$N(\ell, p, \alpha) = e^{\ell \frac{\partial}{\partial q}} \left[ G_4(p_1, p_2, q) \right. \right.$$  

$$\left. G_4(p_3, p_4, q + Q_1) \right]_{q=-(Q_1 \alpha_1 + Q_2 \alpha_2)}$$

and we just have to do the dimensionally regularized integral

$$\int \frac{d^{4-\epsilon} \ell}{(2\pi)^{4-\epsilon} \mu^{-3\epsilon}} e^{\ell \frac{\partial}{\partial q}} \left[ \ell^2 - A^2 + i\epsilon \right]^{3(66)}$$

where $A^2 = Q_1^2 \alpha_1 (\alpha_1 - 1) + Q_2^2 \alpha_2 (\alpha_2 - 1) + 2Q_1 \cdot Q_2 \alpha_1 \alpha_2 + m^2$.

Again, we do this by manipulating the exponential like a power series, because the basic assumption is analyticity in the momenta. Because of symmetry, only even terms in $\ell$ contribute, and doing the $\ell$ integral yields a sum for the pole pieces:

$$\sum_{r=1}^{\infty} \frac{1}{4^r r!(r-1)!} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{r-1} (A^2)^r$$

where we have retained only the $1/\epsilon$ pole and the associated $\mu$ dependence. We cannot yet convert this sum to an angular integral, because it starts at $r=1$, which just reflects the fact that the leading term in the Taylor expansion gives a convergent Feynman integral, with no $1/\epsilon$ pole piece.

To put the contribution of the sum back into the $G$’s, we attempt to massage it further:

$$\sum_{r=0}^{\infty} \frac{1}{4^r r!(r+1)!} \left( \frac{x}{A^2} \right)^r \frac{\partial^2}{\partial x^2} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{x=1}$$

Note that the finite sum subtracted off to redefine the sum to start at zero gets annihilated by the differential operator.

---

$^5$Note also that the factor of $2/\epsilon$ gets compensated by a factor of $1/2$ from a $\Gamma$ function.
The terms in the last sum (69) are exactly of the form seen before in (16), and it can be written in terms of a four dimensional Euclidean angular integral over a finite range:

\[
\frac{\partial^2}{\partial x^2} \left[ \frac{x}{A^2} \int \frac{d\Omega^4}{\Omega^4} e^\sqrt{x} A e^{\frac{\partial}{\partial q}} \right]_{x=1} \tag{70}
\]

which can be put back into the \( G' \)'s by inverting the Taylor expansion. The contribution to the \( \beta \) function for the nonlocal \( \phi^6 \) vertex arising from a graph with three \( G_4 \) vertices is thus

\[
\beta_{G_{6[2,2,2]}(p_1, \ldots, p_5)} = \frac{1}{8\pi^2} \frac{\partial^2}{\partial x^2} \left[ \int \frac{d\Omega^4}{\Omega^4} \int d\alpha_1 d\alpha_2 \frac{x}{A^2} \left( G_4(p_1, p_2, \sqrt{x} A e - (Q_1 \alpha_1 + Q_2 \alpha_2)) + G_4(p_1, p_4, \sqrt{x} A e - (Q_1 \alpha_1 + Q_2 \alpha_2) + Q_1) + G_4(p_5, p_6, \sqrt{x} A e - Q_1 (\alpha_1 - 1) - Q_2 (\alpha_2 - 1)) \right) \right]_{x=1}
\]

+ cross terms

where \( e \) is a Euclidean unit vector and the square root of \( A^2 \) is real for Euclidean momenta, so we have written it as \( A \).

Alternatively, we can eliminate reference to the extraneous parameter \( x \) by the following trick. Consider the sum of (68) again:

\[
\sum_{r=1}^{\infty} \frac{1}{4^r r! (r-1)!} (A^2)^{r-1} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r, \tag{72}
\]

With

\[
p \equiv r - 1 \tag{73}
\]

we obtain

\[
\sum_{p=0}^{\infty} \frac{1}{4^{p+1} p! (p+1)!} (A^2)^p \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{p+1} \tag{74}
\]

\[
= \frac{1}{4} \left( \frac{\partial}{\partial q} \right)^2 \left[ \sum_{p=0}^{\infty} \frac{1}{4^p p! (p+1)!} (A^2)^p \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^p \right] \tag{75}
\]

\[
= \frac{1}{4} \left( \frac{\partial}{\partial q} \right)^2 \left[ \int \frac{d\Omega^4}{\Omega^4} e^{\sqrt{x} A e^{\frac{\partial}{\partial q}}} \right], \tag{76}
\]
and using (49) for four dimensions, \( \int d\Omega^4 = 2\pi^2 \), we get a more compact form for the nonlocal \( \beta \) function:

\[
\beta_{G_{[2,2,2]}(p_1,\ldots,p_6)} = \frac{1}{64\pi^4} \left( \frac{\partial}{\partial q} \right)^2 \int d\Omega^4 \int d\alpha_1 d\alpha_2 \left[ G_4(p_1, p_2, Ae + q) G_4(p_3, p_4, Ae + q + Q_1) G_4(p_5, p_6, Ae + q + Q_2) \right]_{q=-(Q_1\alpha_1 + Q_2\alpha_2)} + \text{cross terms},
\]

(77)

The last form is free of extraneous parameters.

We reiterate that the evaluation of the \( 2-2-2 \) diagram above highlights two important features which will be crucial for the application of the method to the general case below. First, for graphs with more than two internal lines, the integral of the leading (zeroth) term in the formal Taylor expansion does not give a divergence (it is power-counting convergent), so the infinite sum for the poles starts only at some higher order: for a type \( n \) graph \( (n > 1) \) the first divergent contribution comes from the \( (n-2) \)th term in the Taylor expansion, so the sum of poles starts at \( n-2 \). To rewrite this sum in terms of the angular integral over a Euclidean unit vector, we can subtract off a finite sum which vanishes under the action of the differential operator of order \( n-1 \) in the dummy parameter \( x \). In fact, we can do much better - the parameter \( x \) can be eliminated by writing the pole sum a power of \( \left( \frac{\partial}{\partial q} \right)^2 \) acting on a \( 2n-2 \) dimensional Euclidean unit angular integral of \( G \)'s. Let us explain why this makes sense: For one-loop graphs with one or two internal lines, the four dimensional Feynman integral has a divergence which begins with the zeroth order term in the Taylor expansion (and higher terms in the Taylor expansion are even more divergent), so the sum of poles starts at zero. However, for graphs with more than two internal lines, enough powers of momentum in the numerator are required to cancel the denominator powers - thus the pole sum starts at a higher order. **This sum can be made to start at the zeroth order by defining the loop integral in the appropriate number of higher dimensions.** Hence the resummation of the pole terms for a sum starting at zero gives a higher dimensional angular integral.
3.4 2m-point function of type n at one loop

For the non-trivial general graph with \( n \geq 2 \), after combining denominators and shifting the loop momenta, we get\footnote{The overall factor of \((n - 1)!\) comes from the Feynman trick.} the integral

\[
(n - 1)! \int_0^1 \prod_{j=1}^{n-1} d\alpha_j \int \frac{d\ell}{(2\pi)^d} \frac{1}{D} \prod_{i=1}^{n} \mu^{\Delta(i)} G_{v_i+2}(\ldots, \ell - \sum_{s=1}^{n-1} Q_s \alpha_s + Q_{i-1})
\]  \hfill (78)

where \( v_i \) is the number of external lines emanating from the \( i \)th vertex, \( \alpha_j \) are the Feynman parameters,

\[
D = [\ell^2 - (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 + (\sum_{r=1}^{n-1} Q^2_r \alpha_r) - m^2]^n
\]  \hfill (79)

is the combined denominator, and

\[
\Delta(i) = \frac{\epsilon}{2} v_i
\]  \hfill (80)

carries the renormalization scale dependence. The ellipsis denote the dependence of each \( G^i \) on the external momenta and the momenta \( Q_i \) are defined as in (31).

For the renormalization of a 2m point function with \( n \) internal lines

\[
\sum_{i=1}^{i=n} v_i = 2m
\]  \hfill (82)

so summing up the \( \mu \) dependence from all the vertices gives the overall power of \( \mu \) appearing as \( \mu^{\Delta_m} \) where

\[
\Delta_m = \sum_{i=1}^{i=n} v_i \left( \frac{\epsilon}{2} \right) = m\epsilon.
\]  \hfill (83)

Now, using again the fact that all the \( G^i \)'s are analytic, we can write

\[
\prod_{i=1}^{n} G^i_{v_i+2}(\ldots, \ell - \sum_{s=1}^{n-1} Q_s \alpha_s + Q_{i-1})
\]  \hfill (84)

\[
= \epsilon^{\frac{d}{2}} \left[ \prod_{i=1}^{n} G^i_{v_i+2}(\ldots, q + Q_{i-1}) \right]_{q=\sum_{s=1}^{n-1} Q_s \alpha_s}.
\]

\footnote{By momentum conservation at the last vertex, we may make the structure of the last vertex simpler by making the replacement}

\[
G^m_{v_n+2}(\ldots, \ell - \sum_{s=1}^{n-1} Q_s \alpha_s + Q_{n-1}) \rightarrow G^m_{v_n+2}(\ldots, -\ell + \sum_{s=1}^{n-1} Q_s \alpha_s).
\]  \hfill (81)
Now we just have to do the dimensionally regularized integral
\[
\int \frac{d^{4-\epsilon} \ell}{(2\pi)^{4-\epsilon} \mu^{-\Delta m \epsilon}} \frac{\epsilon \ell \delta^4}{(\ell^2 - A^2 + i\epsilon)^n}
\]  
where
\[
A^2 = (\sum_{s=1}^{n-1} Q_s \alpha_s)^2 - (\sum_{r=1}^{n-1} Q_r \alpha_r) + m^2.
\]

Remembering the analyticity in momenta, we do this as usual by manipulating the exponential like a power series, with only even terms in \(\ell\) contributing. Doing the integral as before using the key formula (46), and expanding the \(\Gamma\) function using (47), with
\[
\Gamma(n - r - 2 + \frac{\epsilon}{2}) = 2^\epsilon \left(\frac{-1}{2 + r - n}\right)^n \frac{n!}{(2 + r - n)!}
\]
for \(n \leq r + 2\), we get the pole piece
\[
(n - 1)! \frac{i}{8\pi^2 \mu^{-\Delta m \epsilon}} \sum_{r=n-2}^{\infty} \frac{(A^2)^{2+r-n}(2+r-n)!}{4^r r!(2 + r - n)!} \left[ \frac{\partial}{\partial q} \right]^2.
\]

Quite nicely, the \((n - 1)!\) coming from the integration formula cancels with the \((n - 1)!\) appearing from the Feynman trick. The sum (88) can then be written as
\[
\frac{i}{8\pi^2 \mu^{-\Delta m \epsilon}} \sum_{r=n-2}^{\infty} \frac{(A^2)^{2+r-n} x r^{r+1}}{4^r r!(r + 1)!} \left[ \frac{\partial}{\partial q} \right]^2_{x=1} - T
\]
where we have defined the finite sum
\[
T = \frac{x}{(A^2)^{n-2}} \sum_{r=0}^{n-3} \frac{(A^2)^r x^r}{4^r r!(r + 1)!} \left[ \frac{\partial}{\partial q} \right]^2_{x=1}.
\]
taken to be vanishing for \(n < 3\). As in the last example, this finite sum is subtracted off to make the first sum start at \(r = 0\). Now the differential operator is of order \(n - 1\), and the finite sum is of order \(n - 2\), so the finite sum gets annihilated by the differential operator, giving finally the pole piece:
\[
\frac{i}{8\pi^2 \mu^{-\Delta m \epsilon}} \frac{\partial^{n-1}}{\partial x^{n-1}} \left[ \frac{x}{(A^2)^{n-2}} \sum_{r=0}^{n-3} \frac{(A^2)^r x^r}{4^r r!(r + 1)!} \left[ \frac{\partial}{\partial q} \right]^2 \right]_{x=1}.
\]
Now, inverting as usual using (16) for a Euclidean unit vector, we may cast this pole contribution as an integral over a finite four-dimensional Euclidean angular region:

\[
Pole = \frac{i}{8\pi^2\epsilon\mu^{-me}} \frac{\partial^{n-1}}{\partial x^{n-1}} \left[ \int d\Omega^4_\epsilon \int_0^1 \prod_{j=1}^{j=n-1} d\alpha_j \frac{x}{(A^2)^{n-2}} \cdot \prod_{i=1}^{i=n} G_{v_i+2}^{i} \left( \ldots, \sqrt{x}Ae - (\sum_{s=1}^{n-1} Q_s\alpha_s) + Q_{i-1} \right) \right]_{x=1}
\]

which gives the \( \beta \) function in the form:

\[
\beta_{G_{2n[v_1,\ldots,v_n]}(p_1,\ldots,p_{2m-1})} = \frac{1}{16\pi^4} \frac{\partial^{n-1}}{\partial x^{n-1}} \left[ \int d\Omega^4_\epsilon \int_0^1 \prod_{j=1}^{j=n-1} d\alpha_j \frac{x}{(A^2)^{n-2}} \cdot \prod_{i=1}^{i=n} G_{v_i+2}^{i} \left( \ldots, \sqrt{x}Ae - (\sum_{s=1}^{n-1} Q_s\alpha_s) + Q_{i-1} \right) \right]_{x=1} + \text{cross terms},
\]

However, we can again do much better, and obtain a more compact expression in terms of a higher dimensional angular integral. Referring back to (88), we can write

\[
\sum_{r=n-2}^{\infty} \frac{(A^2)^{2+r-n}}{4^r r!(2+r-n)!} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^r
\]

in terms of

\[
p \equiv r - n + 2,
\]

for

\[
n \geq 3,
\]

\[8\]With the property that at the last vertex in the product we may make the replacement, by momentum conservation,

\[
G_{v_n+2}^{n}(\ldots, \sqrt{x}Ae - (\sum_{s=1}^{n-1} Q_s\alpha_s) + Q_{n-1}) \\
\equiv G_{v_n+2}^{n}(\ldots, \sqrt{x}Ae + (\sum_{s=1}^{n-1} Q_s\alpha_s)).
\]
as
\[
\sum_{p=0}^{\infty} \frac{(A^2)^p}{4^{p+n-2} p!(p+n-2)!} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{p+n-2} \tag{98}
\]
\[
= \frac{1}{4^{n-2}} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{n-2} \sum_{p=0}^{\infty} \frac{(A^2)^p}{4^{p}(p+n-2)!} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{p}
\]
\[
= \frac{1}{4^{n-2}} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{n-2} \int \frac{d\Omega_{e}^{2n-2}}{\Omega_{e}^{2n-2}} e^{Ae^2/\alpha_q} \tag{99}
\]
so that we finally get the pole
\[
\text{Pole} = \frac{i}{2^{2n-1} \pi^2 \epsilon \mu^m} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{n-2} \left[ \int \frac{d\Omega_{e}^{2n-2}}{\Omega_{e}^{2n-2}} \int_{0}^{1} \prod_{j=1}^{j=n-1} d\alpha_j \right. \\
\left. \prod_{i=1}^{i=n} G_{v_{i}+2}^{i}(\ldots, Ae + q + Q_{i-1}) \right]_{q=-\left(\sum_{s=1}^{n-1} Q_{s} \alpha_{s}\right)} 
\]
which, using (49) yields the contribution to the \( \beta \) function in terms of a \( 2n-2 \) dimensional angular integral over a finite Euclidean region:
\[
\beta_{G_{2m[n_1,\ldots,n_n]}}(p_{1},\ldots,p_{2m-1}) = \frac{(n-2)!}{4^{n} \pi^{n+1}} \left[ \left( \frac{\partial}{\partial q} \right)^2 \right]^{n-2} \left[ \int \frac{d\Omega_{e}^{2n-2}}{\Omega_{e}^{2n-2}} \int_{0}^{1} \prod_{j=1}^{j=n-1} d\alpha_j \right. \\
\left. \prod_{i=1}^{i=n} G_{v_{i}+2}^{i}(\ldots, Ae + q + Q_{i-1}) \right]_{q=-\left(\sum_{s=1}^{n-1} Q_{s} \alpha_{s}\right)} + \text{cross terms} \tag{100}
\]
Expressions (24), (93) and (101) for the \( \beta \) function of an arbitrary nonlocal coupling are the most general results of this paper.

\( \text{It is verified easily that with } n = 2, m = 2 \text{ the } \beta \text{ function calculation for the nonlocal } \phi^4 \text{ coupling, as in section 1, is reproduced. For } n = 3, m = 3, \text{ and } n = 1, m = 2, \text{ the previous results for the } \phi^6 \text{ maximally convergent graph } (2-2-2) \text{ and } \phi^4 \text{ tadpole are also verified from the general expression.} \)
4 Concluding Remarks

We have illustrated explicitly how to obtain the renormalization group coefficients in a nonlocal scalar field theory. Note that our general results were computed with massive fields, and for this reason techniques such as the Gegenbauer polynomial method of [5], valid only for massless propagators, are not useful here.

We regard our results as suggestive, but not final. We have obtained integro-differential renormalization group equations by the brute-force technique of expanding, renormalizing and then resumming. We would like to find the rule that allows us to do this in one step, rather than three. It would be interesting if, by modifying the analytic structure of the nonlocal theory, we could actually give a one-step prescription to isolate the divergent part of the Feynman integral, without ever expanding in terms of a formal Taylor expansion. Work along these lines is in progress [6].

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