INTRINSIC ULTRACONTRACTIVITY OF NON-LOCAL DIRICHLET FORMS ON UNBOUNDED OPEN SETS

XIN CHEN   PANKI KIM   JIAN WANG

Abstract. In this paper we consider a large class of symmetric Markov processes $X = (X_t)_{t \geq 0}$ on $\mathbb{R}^d$ generated by non-local Dirichlet forms, which include jump processes with small jumps of $\alpha$-stable-like type and with large jumps of super-exponential decay. Let $D \subset \mathbb{R}^d$ be an open (not necessarily bounded and connected) set, and $X^D = (X^D_t)_{t \geq 0}$ be the killed process of $X$ on exiting $D$. We obtain explicit criterion for the compactness and the intrinsic ultracontractivity of the Dirichlet Markov semigroup $(P^D_t)_{t \geq 0}$ of $X^D$. When $D$ is a horn-shaped region, we further obtain two-sided estimates of ground state in terms of jumping kernel of $X$ and the reference function of the horn-shaped region $D$.

Keywords: symmetric jump process; non-local Dirichlet form; intrinsic ultracontractivity; ground state; intrinsic super Poincaré inequality.

MSC 2010: 60G51; 60G52; 60J25; 60J75.

1. INTRODUCTION AND MAIN RESULTS

Suppose that $(P^D_t)_{t \geq 0}$ is a strongly continuous Markov semigroup in $L^2(D; dx)$ generated by a symmetric Markov process in an open subset $D$ of $\mathbb{R}^d$, and that $(P^D_t)_{t \geq 0}$ has the transition density $p^D(t, x, y)$ with respect to the Lebesgue measure. If $(P^D_t)_{t \geq 0}$ is a compact semigroup, then it is known that there exists a complete orthonormal set of eigenfunctions $\{\phi_n^D\}_{n \geq 1}$ such that $P^D_t \phi_n^D(x) = e^{-\lambda_n^D t} \phi_n^D(x)$ for all $n \geq 1$, $t > 0$ and $x \in D$, where $\{\lambda_n^D\}_{n \geq 1}$ are eigenvalues of $(P^D_t)_{t \geq 0}$ such that $0 < \lambda_1^D \leq \lambda_2^D \leq \lambda_3^D \leq \cdots \to \infty$ as $n \to \infty$. In the literature, the first eigenfunction $\phi_1^D$ is called ground state. Suppose furthermore that $\phi_1$ (from now we write $\phi_1^D$ as $\phi_1$ for simplicity) can be chosen to be bounded, continuous and strictly positive on $D$. The semigroup $(P^D_t)_{t \geq 0}$ is said to be intrinsically ultracontractive, if for every $t > 0$ there is a constant $C_{t, D} > 0$ such that

$$p^D(t, x, y) \leq C_{t, D} \phi_1(x) \phi_1(y), \quad x, y \in D.$$ 

The notion of intrinsic ultracontractivity for symmetric semigroups was first introduced by Davies and Simon in [24]. It has wide applications in the area of analysis and probability. Recently, the intrinsic ultracontractivity of Markov semigroups (including Dirichlet semigroups and Feynman-Kac semigroups) has been intensively

X. Chen: Department of Mathematics, Shanghai Jiao Tong University, 200240 Shanghai, P.R. China; & Fujian Key Laboratory of Mathematical Analysis and Applications (FJKLMAA), Fujian Normal University, 350007 Fuzhou, P.R. China. chenxin217@sjtu.edu.cn.

P. Kim: Department of Mathematics, Seoul National University, Seoul 151-742, South Korea. pkim@snu.ac.kr.

J. Wang: School of Mathematics and Computer Science & Fujian Key Laboratory of Mathematical Analysis and Applications (FJKLMAA), Fujian Normal University, 350007 Fuzhou, P.R. China. jianwang@fjnu.edu.cn.
established for various Lévy type processes, see e.g. [10, 11, 12, 20, 21, 26, 28, 29, 32, 34, 35, 36].

The aim of this paper is to study the intrinsic ultracontractivity of Dirichlet semigroup $(P_t^D)_{t \geq 0}$ for a large class of symmetric jump processes on an unbounded open set $D$. It is twofold. Firstly, under quite general setting we obtain sufficient conditions and necessary conditions for the intrinsic ultracontractivity of $(P_t^D)_{t \geq 0}$. We emphasize that these results are illustrated to be optimal for symmetric jump processes with small jumps of $\alpha$-stable-like type and with large jumps of super-exponential decay on horn-shaped regions, even the knowledge of such interesting processes is still far from completeness. Secondly, for horn-shaped regions we establish sharp two-sided estimates of ground state explicitly in terms of jumping kernel and the reference function of horn-shaped regions. This is the most sophisticated part of this paper, since usually the ground state is very sensitive with respect to the behavior of the process and the shape of the open set.

1.1. Basic setting. Let $J(x, y)$ be a non-negative symmetric Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ satisfying that

$$
\sup_{x \in \mathbb{R}^d} \int_{y \neq x} \min\{1, |x - y|^2\} J(x, y) \, dy < \infty.
$$

(1.1)

Here $\text{diag}$ denotes the diagonal set, i.e., $\text{diag} = \{(x, x) : x \in \mathbb{R}^d\}$. Consider the following non-local quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; dx)$:

$$
\mathcal{E}(f, f) = \iint_{x \neq y} (f(x) - f(y))^2 J(x, y) \, dx \, dy,
$$

(1.2)

$$
\mathcal{F} = C_c^\infty(\mathbb{R}^d)^\delta.
$$

Here, $C_c^\infty(\mathbb{R}^d)$ denotes the space of $C^\infty$ functions on $\mathbb{R}^d$ with compact support, $\mathcal{E}_1(f, f) := \mathcal{E}(f, f) + \int_{\mathbb{R}^d} f^2(x) \, dx$ and $\mathcal{F}$ is the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the metric $\mathcal{E}_1(f, f)^{1/2}$. Under (1.1), it is known that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$, see e.g. [25, Example 1.2.4]. Hence there exist a subset $N \subset \mathbb{R}^d$ having zero capacity with respect to the Dirichlet form $(\mathcal{E}, \mathcal{F})$, and a symmetric Hunt process $X = (X_t)_{t \geq 0}$ with state space $\mathbb{R}^d \setminus N$. See [25, Chapter 7]. Throughout this paper, we always assume that $N = \emptyset$ (i.e., the associated Hunt process $X$ can start from all $x \in \mathbb{R}^d$), and that there exists a transition density function $p(\cdot, \cdot, \cdot) : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ so that

$$
P_t f(x) = E^x f(X_t) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy, \quad f \in L^2(\mathbb{R}^d; dx), x \in \mathbb{R}^d, t > 0,
$$

where $(P_t)_{t \geq 0}$ is the $L^2(\mathbb{R}^d; dx)$ semigroup associated with $(\mathcal{E}, \mathcal{F})$.

Let $D \subseteq \mathbb{R}^d$ be an open (not necessarily bounded and connected) set. We define a subprocess $X^D$ of $X$ as follows

$$
X^D_t := \begin{cases} 
X_t, & \text{if } t < \tau_D, \\
\partial, & \text{if } t \geq \tau_D,
\end{cases}
$$

where $\tau_D := \inf\{t > 0 : X_t \notin D\}$ and $\partial$ denotes the cemetery point. The process $X^D := (X^D_t)_{t \geq 0}$ is called the killed process of $X$ upon exiting $D$. By the strong
Markov property, it is easy to see that the process $X^D$ has a transition density (or Dirichlet heat kernel) $p^D(t, x, y)$, which enjoys the following relation with $p(t, x, y)$:

\[
p^D(t, x, y) = p(t, x, y) - \mathbb{E}^x \left[ p(t - \tau_D, X_{\tau_D}, y) \mathbb{I}_{\{\tau_D \geq t\}} \right], \quad x, y \in D;
p^D(t, x, y) = 0, \quad x \notin D \text{ or } y \notin D.
\]

Define

\[
P_t^D f(x) = \mathbb{E}^x f(X^D_t) = \int_D p^D(t, x, y) f(y) \, dy, \quad t > 0, x \in D, f \in L^2(D; dx).
\]

It is well known that $(P_t^D)_{t \geq 0}$ is a strongly continuous contraction semigroup on $L^2(D; dx)$, which is called Dirichlet semigroup associated with the process $X^D$. Furthermore, in this paper except Appendix, we assume that for every $t > 0$ the function $p^D(t, \cdot, \cdot)$ is bounded, continuous and strictly positive on $D \times D$. This assumption is also mild in a number of applications. See Propositions 7.1 and 7.2 in Appendix.

1.2. Main results for horn-shaped regions. In the following, denote by $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, and $a^+ = a \vee 0$. For any non-negative function $f$ and $g$, $f \asymp g$ means that there is a constant $c \geq 1$ such that $c^{-1} f(r) \leq g(r) \leq c f(r)$, and $f \asymp g$ means that there exist positive constants $c_i$ ($i = 1, 2, 3, 4$) such that $c_1 f(c_2 r) \leq g(r) \leq c_3 f(c_4 r)$. For any open subset $D \subset \mathbb{R}^d$ and $x \in D$, $\delta_D(x)$ stands for the Euclidean distance between $x$ and $D^c$. We call an open set $D \subset \mathbb{R}^d$ a domain, when it is connected.

The contribution of this paper is to obtain efficient criterion for the intrinsic ultracontractivity of Dirichlet semigroup $(P_t^D)_{t \geq 0}$ generated by non-local Dirichlet forms on general unbounded open sets, and to establish two-sided estimates for the corresponding ground state (i.e., the first eigenfunction). To illustrate how powerful our approach is and to show how precise and sharp our estimates are, here we summarize our results on horn-shaped regions with specific reference functions for several classes of jump processes, including symmetric jump processes with small jumps of stable-like type and large jumps of super-exponential decay.

Let us first recall the definition of horn-shaped region (domain). For any $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, let $\bar{x} = (x_2, x_3, \ldots, x_d)$. Suppose that $f : (0, \infty) \to (0, \infty)$ is a bounded and continuous function such that $\lim_{u \to \infty} f(u) = 0$. Then, the open set $D_f = \{x \in \mathbb{R}^d : x_1 > 0, |\bar{x}| < f(x_1)\}$ is called the horn-shaped region. We call $f$ the reference function of $D_f$. The study of Dirichlet semigroups for Brownian motions on horn-shaped regions has a long history. For (both mathematical and physical) backgrounds and motivations on such subject, see [1, 2, 3, 6, 23, 24, 37] and the references therein.

Let $\Phi$ be a strictly increasing function on $(0, \infty)$ satisfying that there exist constants $0 < \underline{\alpha} \leq \overline{\alpha} < 2$ and $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$ such that

\[
\underline{\alpha} \left( \frac{R}{r} \right)^{\overline{\alpha}} \leq \frac{\Phi(R)}{\Phi(r)} \leq \overline{\alpha} \left( \frac{R}{r} \right)^{\overline{\alpha}}, \quad 0 < r \leq R.
\]

Obviously (1.4) implies that $\Phi(0) := \lim_{r \to 0} \Phi(r) = 0$.

Let $\chi$ be a nondecreasing function on $(0, \infty)$ with $\chi(r) \equiv \chi(0)$ for $r \in (0, 1)$, and there exist constants $c_1, c_2, L_1, L_2 > 0$ and $\gamma \in [0, \infty]$ such that

\[
L_1 \exp(c_1 r^\gamma) \leq \chi(r) \leq L_2 \exp(c_2 r^\gamma), \quad r > 1.
\]
we denote by $L$ and there exists a constant $\kappa(0, \infty)$ is a measurable function satisfying that $\kappa(x, y) = \kappa(y, x)$ and there exists a constant $L_0 > 1$ such that
$$L_0^{-1} \leq \kappa(x, y) \leq L_0, \quad x, y \in \mathbb{R}^d.$$ 

According to [17], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ with jumping kernel $J(x, y)$ above generates an symmetric Hunt process $X = (X_t)_{t \geq 0}$, which starts from all $x \in \mathbb{R}^d$ and has a transition density function $p(t, x, y)$ with respect to the Lebesgue measure, such that $p(\cdot, \cdot, \cdot) : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ is bounded, continuous and strictly positive.

Let $D_f$ be a horn-shaped region with respect to the reference function $f$, and denote by $(P_t^D)_{t \geq 0}$ the associated Dirichlet semigroup. Then, the Dirichlet heat kernel $p_t^D(t, x, y)$ exists, and $(x, y) \mapsto p_t^D(t, x, y)$ is bounded, continuous and strictly positive on $D_f \times D_f$ for every $t > 0$. When $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive, we denote by $\Phi_1$ the corresponding ground state. To obtain two-sided estimates for the intrinsic ultracontractivity of $(P_t^D)_{t \geq 0}$.

(K$\eta$) There are constants $L > 0$ and $\eta > \bar{\alpha}/2$ such that
$$|\kappa(x, x + h) - \kappa(x, x)| \leq L|h|^\eta$$
for every $x, h \in \mathbb{R}^d$ with $|h| \leq 1$, where $\bar{\alpha}$ is the constant in (1.4).

(SD) The function $\Phi \in C^1((0, \infty))$ such that $r \mapsto -(\Phi(r)^{-1}r^{-\bar{\alpha}})\gamma/\gamma$ is decreasing on $(0, \infty)$.

Note that condition (SD) above holds for pure jump isotropic unimodal Lévy processes including all subordinated Brownian motions, whose characteristic exponents satisfy the weak scaling conditions in (1.4). See e.g. [27, Remark 1.4].

Theorem 1.1. With all notations and assumptions above, we have the following two statements.

(1) Suppose that $\gamma = 0$ in (1.5) and that
$$f(s) \simeq \Phi^{-1}(\log^{-\theta} s)$$
for some $\theta > 0$. Then,
(a) $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive if and only if $\theta > 1$;  
(b) if $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive, then for all $x \in D_f$ with $|x|$ large enough,

$$\phi_1(x) \simeq \Phi(\delta_{D_f}(x))^{1/2}\Phi(f(|x|))^{1/2} \frac{1}{|x|^{d\Phi(|x|)}}.$$

(2) Suppose that $\gamma \in (0, \infty)$ in (1.5) and that
$$f(s) \simeq \Phi^{-1}(s^{-\theta})$$

We next consider the jumping kernel $J(x, y)$ in the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ given by (1.2) with the following expression

$$J(x, y) = \frac{\kappa(x, y)}{|x - y|^d \Phi(|x - y|) \chi(|x - y|)},$$

where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ is a measurable function satisfying that $\kappa(x, y) = \kappa(y, x)$ and there exists a constant $L_0 > 1$ such that
$$L_0^{-1} \leq \kappa(x, y) \leq L_0, \quad x, y \in \mathbb{R}^d.$$
for some $\theta > 0$. Then,

(a) $(P^D_{t})_{t \geq 0}$ is intrinsically ultracontractive if and only if $\theta > \gamma \wedge 1$;

(b) if $(P^D_{t})_{t \geq 0}$ is intrinsically ultracontractive, then for all $x \in D_f$ with $|x|$ large enough,

$$
\phi_1(x) \asymp \Phi(\delta_{D_f}(x))^{1/2} \Phi(f(|x|))^{1/2} \exp \left(-|x|^\gamma / 2 \log (\gamma - 1) / \gamma |x| \right).$$

(1.10)

For the class of jump processes considered in this subsection, our method works for $D^f$ with not only specific reference functions in (1.7) and (1.9), but also general reference functions $f$. Here we restrict ourselves on (1.7) and (1.9) to light up the structure of $\phi_1$. In the setting of Theorem 1.1, two-sided estimates of $\phi_1$ can be decomposed into two terms. Roughly speaking, for $x \in D_f$ with $|x|$ large enough, the term $\Phi(\delta_{D_f}(x))^{1/2} \Phi(f(|x|))^{1/2}$ in (1.8) and (1.10) represent the probability (called exiting probability later) of the process $X^{D_f}$ exiting from $B(x, \delta_{D_f}(x))$, and both of the other term in (1.8) and (1.10) describe the probability (called returning probability later) of the process $X^{D_f}$ from $x \in D_f$ to the origin, which is independent of $f$ and correlates with $p(1, x, 0)$. By [14, Theorem 1.2] and [17, Theorem 1.2 and Theorem 1.4], we know that two-sided heat kernel estimates of $p(1, x, 0)$ are comparable to the jumping kernel $J(x, 0)$ for $|x|$ large enough, when $J(x, y)$ enjoys the form (1.6) with $\gamma \in [0, 1]$.

**Theorem 1.2.** Suppose that $\gamma \in (1, \infty]$ in (1.5) and that

$$
f(s) \asymp \exp(-s^\theta)
$$

for some constant $\theta > 0$. Then,

(a) the Dirichlet semigroup $(P^D_{t})_{t \geq 0}$ is intrinsically ultracontractive;

(b) for any $x \in D_f$ with $|x|$ large enough,

$$
\phi_1(x) \asymp \Phi(\delta_{D_f}(x))^{1/2} \Phi(f(|x|))^{1/2} \exp \left(-|x|^{1+\gamma(\gamma - 1) / \gamma} \right).$$

(1.12)

In particular, when $\gamma = \infty$,

$$
\phi_1(x) \asymp \Phi(\delta_{D_f}(x))^{1/2} \Phi(f(|x|))^{1/2} \exp \left(-|x|^{1+\theta} \right).
$$

The result above indicates the term that describes the returning probability of $X^{D_f}$ from $x \in D_f$ to the origin should depend on the reference function $f$, if $f$ is decay faster than polynomials, see e.g. (1.11). In particular, (1.12) implies that, when $1 < \gamma \leq \theta$, the returning probability dominates the exit probability since the exponential term can be absorbed into $\Phi(f(|x|))^{1/2}$; while, when $1 < \theta < \gamma$ the returning probability reveals some delicate interactions between the reference function $f$ and the jumping kernel $J(x, y)$.

To the best of our knowledge, both results above concerning the intrinsic ultracontractivity and two-sided estimates of ground state for general symmetric (non-Lévy) jump processes on unbounded open sets are new. In fact, previously the intrinsic ultracontractivity of symmetric jump processes on unbounded open sets is considered only for symmetric $\alpha$-stable Lévy processes, e.g., see [36, Example 2] for related conclusions on horn-shaped regions. The argument of [36] is heavily based on uniform boundary Harnack inequalities in [8]. Even though the uniform boundary Harnack inequalities for a quite general discontinuous Feller process in metric measure space
have been proved in [9, 33], it is still not available for symmetric jump processes whose jumping kernel \( J(x, y) \) given by (1.6) with \( \gamma \in (1, \infty) \) and it is not true when \( \gamma = \infty \), see [31, Section 6]. Thus the approach of [36] can not yield Theorem 1.1 when \( \gamma \in (1, \infty) \) and Theorem 1.2. Moreover, our criterion could also be applied to a class of jump processes whose scaling orders depend on their position, see e.g. Example 5.4 below.

We would like to mention that, in both theorems above we only present two-sided estimates of ground state \( \phi_1 \) for \( x \in D_f \) with \( |x| \) large enough. The reason is that the horn-shaped region \( D_f \) may not be a \( C^{1,1} \) domain even if \( f \in C^{1,1} \), because the boundary of \( D_f \) at corner near the point \( y_0 = (0, f(0), 0 \cdots 0) \in \partial D_f \) is only Lipschitz. (Note that for Lipschitz domain \( D \) no explicit estimates for ground state are available even when \( D \) is bounded, see e.g. [7].) If we assume additionally that \( D_f \) is a \( C^{1,1} \) domain, then it is not difficult to see that Theorem 1.1 and Theorem 1.2 hold true for all \( x \in D_f \).

1.3. Further comments. We will make further comments on the setting and the approach of our paper.

(1) Brownian motions on horn-shaped regions. As mentioned before, the intrinsic ultracontractivity of Dirichlet semigroups for Brownian motions on horn-shaped regions has been studied by many authors. Our assumptions on horn-shaped regions are more general than those in previous literatures. For example, in [24, Section 7, p. 366] the reference function \( f \) is required to satisfy that \( f'(r)/f(r) \to 0 \) as \( r \to \infty \). In [36, Proposition 1] the function \( f \) fulfills that \( f'/f \) is bounded. Clearly, the function \( f \) given by (1.11) does not necessarily satisfy such assumption. Besides, our estimates in Theorem 1.1 and Theorem 1.2 are quite precise, since they consist both the exit probability term (i.e., involving the behavior of ground state near the boundary) and the returning probability term (i.e., involving the behavior of ground state far from the boundary).

(2) Symmetric pure-jump processes on bounded open sets. When \( D \) is bounded, there are already a lot of works on the intrinsic ultracontractivity of symmetric jump processes, see [10] and the references therein. Several differences and difficulties occur when \( D \) is unbounded. For instance, firstly the Dirichlet semigroup \( (P^D_t)_{t \geq 0} \) is always compact when \( D \) is bounded; however, it is not true when \( D \) is unbounded. Secondly the (uniform) \( C^{1,1} \) property of open set \( D \) has a crucial effect on explicit estimates of ground state in the bounded open set \( D \), see e.g. [18, 27, 30]; however, as mentioned above, even if we assume that the reference function of an unbounded horn-shaped region \( D \) is a \( C^{1,1} \) function, \( D \) only enjoys \( C^{1,1} \) characteristics locally.

We believe that our approach, to yield the intrinsic ultracontractivity of Dirichlet semigroups \( (P^D_t)_{t \geq 0} \) via intrinsic super Poincaré inequalities of non-local Dirichlet forms, is interesting of its own. Such idea has been efficiently used to consider the corresponding problem for Feynman-Kac semigroups of general symmetric jump processes in [11, 12]. Also our methods could be used to study related topics for non-local Dirichlet forms on general metric measure spaces. On the other hand, in order to obtain two-sided estimates for ground state \( \phi_1 \) some new ideas and techniques are required. In particular, the approach via Harnack inequalities or harmonic measure for Brownian motions (see [24] and [2] respectively) and the idea by using uniform
boundary Harnack inequality for symmetric $\alpha$-stable Lévy processes are not feasible in this paper. Instead, we make full use of the formula for the Lévy system, two-sided estimates for Dirichlet heat kernels and sharp estimates for distributions of the first exit time and the return probability.

The remainder of the paper is arranged as follows. In the next section, we study the compactness of Dirichlet semigroup $(P^D_t)_{t \geq 0}$. Our criterion on the compactness of $(P^D_t)_{t \geq 0}$ is quite general and works, in particular, for symmetric jump processes with small jumps of high intensity in Example 2.4. In Section 3 we derive (rough) lower bound estimates for ground state, which is one of key ingredients to establish sufficient conditions for the intrinsic ultracontractivity of $(P^D_t)_{t \geq 0}$. General results about sufficient conditions and necessary conditions for the intrinsic ultracontractivity of $(P^D_t)_{t \geq 0}$ are presented in Section 4. In Section 5, we will apply the results in Section 4 to study the intrinsic ultracontractivity of $(P^D_t)_{t \geq 0}$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ with jumping kernel given by (1.6) on two specific unbounded regions, including horn-shaped regions and an unbounded and disconnected open set with locally $\alpha$-fat property. Proofs of Theorems 1.1(1)(a), 1.1(2)(a) and 1.2(a) are given in the end of subsection 5.1. Finally, by using the characterization of horn-shaped regions and estimates for (Dirichlet) heat kernel, we will obtain two-sided estimates of ground state corresponding to $(P^D_t)_{t \geq 0}$ in Section 6. Proofs of Theorems 1.1(1)(b), 1.1(2)(b) and 1.2(b) are given after the statement of Theorem 6.1.

**Notations** Throughout the paper, we use $c$, with or without subscripts, to denote strictly positive finite constants whose values are insignificant and may change from line to line. We will use “$\ldots =$” to denote a definition, which is read as “is defined to be”. Denote by $B(x, r)$ the ball with center at $x \in \mathbb{R}^d$ and radius $r > 0$. For $A, B \subset \mathbb{R}^d$, $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. For any Borel subset $A \subset \mathbb{R}^d$, we denote $|A|$ the volume of $A$, $A^c$ the complementary set corresponding to $A$, and $\bar{A}$ the closure of the set $A$. For $a \in \mathbb{R}$, $[a]$ is the smallest integer greater than or equal to $a$, and $\lfloor a \rfloor$ is the largest integer smaller than or equal to $a$. For a measurable function $f : (0, \infty) \to (0, \infty)$, we use notations $f^*(r) = \sup_{s \geq r} f(s)$ for all $r > 0$, and $f_*(r) = \inf_{1 \leq s \leq r} f(s)$ for all $r \geq 1$. For a decreasing function $g : (0, \infty) \to (0, \infty)$, we denote $g^{-1}(r) := \inf\{s > 0 : g(s) \leq r\}$ for any $r > 0$, where $\inf \emptyset := \infty$. Similarly, for an increasing function $g : (0, \infty) \to (0, \infty)$, we denote $g^{-1}(r) := \inf\{s > 0 : g(s) \geq r\}$ for $r > 0$. For open set $D \subset \mathbb{R}^d$, denote by $C^\infty_c(D)$ the set of $C^\infty$ functions on $D$ with compact supports.

2. **Compactness of the Dirichlet semigroup**

Consider the symmetric Hunt process $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d\}$ as in Subsection 1.1. The associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by (1.2), and denote by $p(t, x, y)$ the corresponding transition density function with respect to the Lebesgue measure.

For an open (not necessarily bounded and connected) subset $D \subset \mathbb{R}^d$, let $X^D$ be the killed process of $X$ upon exiting $D$. Denote by $(P^D_t)_{t \geq 0}$ its associated semigroup on $L^2(D; dx)$, and by $p^D(t, x, y)$ its transition density (or Dirichlet heat kernel). Recall that we always assume that $p^D(t, \cdot, \cdot)$ is bounded, continuous and strictly positive on $D \times D$ for every $t > 0$. 


According to [25, Theorem 4.4.3 (i)], the Dirichlet form \((\mathcal{E}_D, \mathcal{F}_D)\) associated with 
\((P_t^D)_{t \geq 0}\) is given by 
\[
\mathcal{E}_D(f, f) = \int \int_{D \times D} (f(x) - f(y))^2 J(x, y) \, dx \, dy + 2 \int_D f^2(x) V_D(x) \, dx,
\]
\[
\mathcal{F}_D = C_c^{\infty}(D)^{d_D},
\]
where 
\[
V_D(x) := \int_D J(x, y) \, dy, \quad x \in D.
\]

**Definition 2.1.** Let \(\Theta\) be a decreasing function on \((0, \infty)\) such that 
\[
\int_s^{\infty} \frac{\Theta^{-1}(r)}{r} \, dr < \infty \quad \text{for large } s > 0.
\]
We say that the on-diagonal upper bound estimate \(\text{DHK}_{\Theta, <}\) holds for the Dirichlet heat kernel \(p^D(t, x, y)\), if 
\[
p^D(t, x, y) \leq \Theta(t) \quad \text{for all } t > 0 \text{ and } x, y \in D.
\]

In particular, according to the relation (1.3) between \(p(t, x, y)\) and \(p^D(t, x, y)\), if 
\(\text{DHK}_{\Theta, <}\) holds for \(p(t, x, y)\), then \(\text{DHK}_{\Theta, <}\) holds for \(p^D(t, x, y)\) for all open subsets \(D\). We next present two examples to show that \(\text{DHK}_{\Theta, <}\) is satisfied for a large class of symmetric jump processes.

**Definition 2.2.** Let \(\Phi\) be a strictly increasing function on \((0, \infty)\) satisfying (1.4). We say that \(J_{\Phi, >}\) holds for the jumping kernel \(J(x, y)\), if there are constants \(C_0, r_0 > 0\) such that 
\[
J(x, y) \geq \frac{C_0}{|x - y|^d + 2 \log \frac{2}{|x - y|}}, \quad 0 < |x - y| \leq r_0.
\]

Without loss of generality, whenever the assumption \(J_{\Phi, >}\) is considered, we will assume that the constant \(r_0\) in (2.3) is 1.

**Example 2.3.** According to [17, Proposition 3.1], if \(J_{\Phi, >}\) is satisfied, then \(\text{DHK}_{\Theta, <}\) holds with 
\[
\Theta(t) = c_0 \left( \Phi^{-1}(t)^{-d} \vee t^{-d/2} \right)
\]
for some constant \(c_0 > 0\). In particular, it follows that \(\text{DHK}_{\Theta, <}\) holds for the heat kernel of symmetric jump processes with small jumps of \(\alpha\)-stable-like type with \(\alpha \in (0, 2)\).

**Example 2.4.** Suppose that there is a constant \(\beta \in (0, 1]\) such that 
\[
J(x, y) \simeq \frac{1}{|x - y|^{d+2} \log^{1+\beta} \left( \frac{2}{|x - y|} \right)}, \quad 0 < |x - y| \leq 1.
\]

Then, according to [38, Theorem 1.1], \(\text{DHK}_{\Theta, <}\) holds with 
\[
\Theta(t) = c_0 (t \wedge 1)^{-d/2} \left( \log \frac{2}{t \wedge 1} \right)^{\beta d/2}
\]
for some constant \(c_0 > 0\). This indicates that \(\text{DHK}_{\Theta, <}\) is satisfied for the heat kernel of symmetric jump processes with small jumps of high intensity.
The main contribution of this section is the following result.

**Theorem 2.5.** Suppose that $D$ is an open subset of $\mathbb{R}^d$ and $\text{DHK}_{\Theta, \leq}$ holds for Dirichlet heat kernel $p^D(t, x, y)$. If

$$
|\{x \in D : V_D(x) < r\}| < \infty \quad \text{for every } r > 0,
$$

then the semigroup $(P^D_t)_{t \geq 0}$ is compact.

To prove Theorem 2.5, we first cite [45, Corollary 1.2] in our setting. Let $(\mathcal{E}^D, \mathcal{F}^D)$ be the Dirichlet form associated with $(P^D_t)_{t \geq 0}$, which is given by (2.1). Define

$$
\mathcal{E}^D_0(f, f) = \iint_{D \times D} (f(x) - f(y))^2 J(x, y) \, dx \, dy + \int_D f^2(x) V_D(x) \, dx, \quad f \in \mathcal{F}^D.
$$

Since

$$
\mathcal{E}^D(f, f) = \mathcal{E}^D_0(f, f) + \int_D f^2(x) V_D(x) \, dx, \quad f \in \mathcal{F}^D,
$$

$(\mathcal{E}^D_0, \mathcal{F}^D)$ is also a regular Dirichlet form on $L^2(D; dx)$, and $(\mathcal{E}^D, \mathcal{F}^D)$ can be seen as the perturbation of $(\mathcal{E}^D_0, \mathcal{F}^D)$ with potential $V_D$. Note that, because $V_D \geq 0$, [45, (1.3)] holds trivially. Then, according to [45, Corollary 1.2], we have the following statement.

**Proposition 2.6.** Suppose that condition (2.4) is satisfied and that the following super Poincaré inequality holds

$$
\int_D f^2(x) \, dx \leq s \mathcal{E}^D_0(f, f) + \beta_0(s) \left( \int_D |f|(x) \, dx \right)^2, \quad s > 0, f \in \mathcal{F}^D
$$

with a decreasing function $\beta_0 : (0, \infty) \to (0, \infty)$ satisfying

$$
\int_s^\infty \frac{\beta_0^{-1}(r)}{r} \, dr < \infty \quad \text{for large } s > 0.
$$

Then the semigroup $(P^D_t)_{t \geq 0}$ is compact.

Next, we present the proof of Theorem 2.5. Let $\mathcal{E}^{D,*}_0(f, f) = \frac{1}{t} \mathcal{E}^D(t, f, f)$. Then, the process associated with $(\mathcal{E}^{D,*}_0, \mathcal{F}^D)$ is a time-change of the process associated with $(\mathcal{E}^D, \mathcal{F}^D)$. Let $p^{D,*}(t, x, y)$ be transition density of the process corresponding to $(\mathcal{E}^{D,*}_0, \mathcal{F}^D)$. Then, by DHK$_{\Theta, \leq}$,

$$
p^{D,*}(t, x, y) = p^D(t/2, x, y) \leq \Theta(t/2), \quad t > 0, x, y \in D.
$$

Denote by $(P^{D,*}_t)_{t \geq 0}$ the transition semigroup of $(\mathcal{E}^{D,*}_0, \mathcal{F}^D)$. In particular, according to [42, Theorem 4.5] (or see [44, Theorem 3.3.15]), the following super-Poincaré inequality holds

$$
\int_D f^2(x) \, dx \leq s \mathcal{E}^{D,*}_0(f, f) + \beta_0(s) \left( \int_D |f|(x) \, dx \right)^2, \quad s > 0, f \in \mathcal{F}^D,
$$

where

$$
\beta_0(s) = \inf_{s \geq t, r \geq 0} \frac{l_{P^{D,*}_t}}{r} \left\| \frac{L^1(\mathbb{R}^d dx)}{L^\infty(\mathbb{R}^d dx)} \right\| \exp \left( \frac{r}{t} - 1 \right).
$$
Since
\[
\beta_0(s) \leq \|p_{s,D}^*(\cdot)\|_{L^1(D, dx) \to L^\infty(D, dx)} = \sup_{x, y \in D} p_{s,D}^*(s, x, y) \leq \Theta(s/2)
\]
and \(\mathcal{E}_{0,D}^*(f, f) \leq \mathcal{E}_0^D(f, f)\), we arrive at
\[
\int_D f^2(x) \, dx \leq s\mathcal{E}_0^D(f, f) + \Theta(s/2) \left(\int_D |f(x)| \, dx\right)^2, \quad s > 0, f \in \mathcal{F}^D.
\]
In particular, (2.5) holds with \(\beta_0(s) = \Theta(s/2)\), and (2.6) also holds due to (2.2). Combining these with (2.4), we know that all the assumptions of Proposition 2.6 are satisfied, so \((P^D_t)_{t \geq 0}\) is compact by Proposition 2.6.

As a direct consequence of Theorem 2.5, we immediately have

**Corollary 2.7.** Suppose that \(\text{DHK}_{\Theta, <}\) holds. Then, the semigroup \((P^D_t)_{t \geq 0}\) is compact, if either \(D\) has finite Lebesgue measure or \(\Phi(0+) = 0\).

The result of above corollary under assumption (2.8) extends [36, Lemma 2], where symmetric \(\alpha\)-stable Lévy processes is considered.

In the remainder of this paper, we are concerned with the case that (2.8) holds true. In order to verify (2.8), it is necessary to consider lower bounds of \(V_D(x)\). Thus we will study it under the assumption \(J_{\Phi, >}\). In fact, in the proof below we can relax the assumption in (1.4) to \(0 \leq \alpha \leq \sigma \leq 2\) and \(\Phi(0+) = 0\) (instead of \(0 < \alpha \leq \sigma < 2\)).

Recall that a Borel subset \(U\) is called \((\kappa, r)\)-fat at \(x \in \overline{U}\), if there exists a point \(\xi_x \in U\) such that \(B(\xi_x, \kappa r) \subseteq U \cap B(x, r)\). We say that \(U\) is \(\kappa\)-fat, if there exists a constant \(R_0 > 0\) such that \(U\) is \((\kappa, r)\)-fat at every \(x \in \overline{U}\) for each \(r \in (0, R_0]\).

**Proposition 2.8.** Suppose that \(D\) is an open subset of \(\mathbb{R}^d\) and that \(J_{\Phi, >}\) holds. If there is a constant \(\kappa \in (0, 1)\) such that for any \(x \in D\), there exist a constant \(R_x \in (0, 1)\) and \(z_x \in \partial D\) with \(|x - z_x| = \delta_D(x)\) satisfying that \(D^c\) is \((\kappa, r)\)-fat at \(z_x \in \partial D\) for each \(r \in (0, R_x]\), then there is a constant \(c_1 > 0\) such that
\[
V_D(x) \geq c_1 \frac{(\delta_D(x) \wedge R_x)^d}{\delta_D(x)^d \Phi(\delta_D(x))} \quad \text{for all } x \in D \text{ with } \delta_D(x) \leq \frac{1}{2}.
\]

In particular, if \(D^c\) is \(\kappa\)-fat, then there exists a constant \(c_2 > 0\) such that
\[
V_D(x) \geq \frac{c_2}{\Phi(\delta_D(x))} \quad \text{for all } x \in D \text{ with } \delta_D(x) \leq \frac{1}{2}.
\]

**Proof.** Noticing that \(D^c\) is \((\kappa, \delta_D(x) \wedge R_x)\)-fat at \(z_x \in \partial D\), we can find a point \(\xi_{z_x} \in D^c\) such that
\[
B(\xi_{z_x}, \kappa (\delta_D(x) \wedge R_x)) \subseteq D^c \cap B(z_x, \delta_D(x) \wedge R_x).
\]
Since for \(x \in D\) with \(\delta_D(x) < 1/2\) and \(y \in B(\xi_{z_x}, \kappa (\delta_D(x) \wedge R_x)) \subseteq D^c \cap B(z_x, \delta_D(x) \wedge R_x)\),
\[
|y - x| \leq |y - z_x| + |z_x - x| < (\delta_D(x) \wedge R_x) + \delta_D(x) \leq 2\delta_D(x) \leq 1,
\]
we have, by $J_{\Phi, \geq}$ (i.e., (2.3)),

$$J(x, y) \geq c_1|x - y|^{-d}\Phi(|x - y|)^{-1} \geq c_2(\delta_D(x))^{-d}\Phi(\delta_D(x))^{-1}.$$ 

Therefore,

$$V_D(x) = \int_{D^c} J(x, y) dy \geq \int_{B(\xi_x, \kappa(\delta_D(x) \wedge R_x))} J(x, y) dy$$

$$\geq \frac{c_2}{\delta_D(x)^d\Phi(\delta_D(x))} \int_{B(\xi_x, \kappa(\delta_D(x) \wedge R_x))} dy$$

$$\geq \frac{c_3(\delta_D(x) \wedge R_x)^d}{\delta_D(x)^d\Phi(\delta_D(x))},$$

which proves (2.9).

Furthermore, if $D^c$ is $\kappa$-fat, then we can find some constant $R_0 \in (0, 1)$ such that $R_x = R_0$ for all $x \in D$ in the argument above, and so the second assertion follows. \(\square\)

Combining Example 2.3 and Corollary 2.7 with Proposition 2.8, we have the following simple sufficient condition for the compactness of $(P_t^D)_{t \geq 0}$.

**Corollary 2.9.** Suppose $J_{\Phi, \geq}$ holds. If $D^c$ is $\kappa$-fat and

$$\lim_{x \in D \text{ and } |x| \to \infty} \delta_D(x) = 0,$$

then the semigroup $(P_t^D)_{t \geq 0}$ is compact.

3. Lower bound estimates of ground state

In Section 2, we considered the symmetric Hunt process $X = \{X_t, t \geq 0; P^x, x \in \mathbb{R}^d\}$ as in Subsection 1.1. Throughout this section, we continue considering the symmetric Hunt process $X = \{X_t, t \geq 0; P^x, x \in \mathbb{R}^d\}$ as in Subsection 1.1. Suppose that $D$ is a fixed open subset of $\mathbb{R}^d$, and we assume that the Dirichlet semigroup $(P_t^D)_{t \geq 0}$ is compact. Then, by assumption that $p^D(t, \cdot, \cdot)$ is bounded, continuous and strictly positive on $D \times D$ for every $t > 0$, and the standard theory for symmetric compact semigroups, see e.g. [39, Theorem VI. 16] and [40, Theorem XIII. 43], there exists a complete orthonormal set of eigenfunctions $\{\phi_n^D\}$ such that $P_t^D \phi_n^D(x) = e^{-\lambda_n^D t} \phi_n^D(x)$ for all $t > 0$ and $x \in D$, where $\{\lambda_n^D\}_{n \geq 1}$ are eigenvalues of $(P_t^D)_{t \geq 0}$ such that $0 < \lambda_1^D < \lambda_2^D \leq \lambda_3^D \leq \cdots \to \infty$ as $n \to \infty$; moreover, the first eigenfunction $\phi_1^D$ can be chosen to be bounded, continuous and strictly positive on $D$ (see Proposition 7.3 in the Appendix for this fact). In the literature, $\phi_1^D$ is called ground state. In what follows, we write $\phi_1^D$ as $\phi_1$ for simplicity.

This section is devoted to driving lower bound estimates for $\phi_1$, which is one of key ingredients to establish sufficient conditions for the intrinsic ultracontractivity of $(P_t^D)_{t \geq 0}$.

We begin with the following simple lemma.

**Lemma 3.1.** For any relatively compact open set $D_0 \subset D$ and $t_0 > 0$,

$$\phi_1(x) \geq c_{t_0, D_0} P_{t_0}^D 1_{D_0}(x), \quad x \in D,$$

where $c_{t_0, D_0} := e^{\lambda_1 t_0} (\inf_{y \in D_0} \phi_1(y)) > 0$. 

Proof. For any \( t_0 > 0 \) and \( x \in D \),
\[
P_{t_0} \mathbb{1}_{D_0}(x) = \int_{D_0} p^D(t_0, x, y) \, dy \leq \frac{1}{\inf_{y \in D_0} \phi_1(y)} \int_{D_0} p^D(t_0, x, y) \phi_1(y) \, dy
\]
\[
\leq \frac{1}{\inf_{y \in D_0} \phi_1(y)} \int_D p^D(t_0, x, y) \phi_1(y) \, dy = \frac{P_{t_0} \phi_1(x)}{\inf_{y \in D_0} \phi_1(y)} = e^{-\lambda t_0} \phi_1(x),
\]
where we have used the property that \( \inf_{y \in D_0} \phi_1(y) > 0 \) for any relatively compact open set \( D_0 \subset D \), thanks to the fact that \( \phi_1 \) is continuous and strictly positive. The desired assertion follows from the inequality above. \( \square \)

Recall that for any \( x \in \mathbb{R}^d \), stopping time \( \tau \) (with respect to the natural filtration of the process \( X \)), and non-negative measurable function \( f \) on \([0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) with \( f(s, z, z) = 0 \) for all \( z \in \mathbb{R}^d \) and \( s \geq 0 \), we have the following \( \text{Lévy system} \):
\[
(3.1) \quad \mathbb{E}^x \left[ \sum_{s \leq \tau} f(s, X_s, X_s) \right] = \mathbb{E}^x \left[ \int_0^\tau \left( \int_{\mathbb{R}^d} f(s, X_s, z) J(X_s, z) \, dz \right) \, ds \right],
\]
see e.g. [13, Lemma 4.7] and [14, Appendix A].

Let \( \Psi \) be an increasing function on \((0, \infty)\) satisfying that there exist constants \( 0 < \beta_1 \leq \beta_2 < \infty \) and \( 0 < c_1 \leq c_2 < \infty \) such that
\[
(3.2) \quad c_1 \left( \frac{R}{r} \right)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq c_2 \left( \frac{R}{r} \right)^{\beta_2}, \quad 0 < r \leq R.
\]
For any \( A \subset \mathbb{R}^d \), the first exit time from \( A \) of the process \( X \) is defined by
\[
\tau_A = \inf\{ t > 0 : X_t \in A \}.
\]

**Definition 3.2.** Let \( \Psi \) be an increasing function on \((0, \infty)\) such that (3.2) holds. We say that \( \text{Exit}_{\Psi, \geq} \) holds on \( D \), if there exist constants \( C_0, \tilde{r}_0 > 0 \) and \( C_1 \in (0, 1) \) such that for all \( r \in (0, \tilde{r}_0] \),
\[
(3.3) \quad \inf_{x \in D, B(x, r) \subset D} \mathbb{P}^x (\tau_{B(x, r)} > C_0 \Psi(r)) \geq C_1.
\]
For simplicity, we say \( \text{Exit}_{\Psi, \geq} \) holds if \( \text{Exit}_{\Psi, \geq} \) holds on \( \mathbb{R}^d \).

Obviously, for any open subset \( D_1 \subseteq D_2 \subseteq \mathbb{R}^d \), if \( \text{Exit}_{\Psi, \geq} \) holds on \( D_2 \), then it also holds on \( D_1 \).

**Example 3.3.** Suppose that there exist constants \( 0 < \alpha_1 \leq \alpha_2 < 2 \) and \( c_1, c_2 \in (0, \infty) \) such that
\[
\frac{c_1}{|x - y|^{d + \alpha_1}} \leq J(x, y) \leq \frac{c_2}{|x - y|^{d + \alpha_2}}, \quad 0 < |x - y| \leq 1.
\]
Then, by [12, Lemma 3.1] (or [5, Theorem 2.1]), \( \text{Exit}_{\Psi, \geq} \) holds on \( \mathbb{R}^d \) with
\[
\Psi(r) = r^{\alpha_2 + (\alpha_2 - \alpha_1) d / \alpha_1}.
\]

**Proposition 3.4.** Suppose that \( \text{Exit}_{\Psi, \geq} \) holds on \( D \). For any fixed \( x_0 \in D \), set \( r_{x_0} = (\delta_D(x_0)/4) \wedge \tilde{r}_0 \), where \( \tilde{r}_0 > 0 \) is the constant in (3.3). Then, there exists a constant \( c := c(x_0, r_{x_0}, D) > 0 \) such that for all \( x \in D \),
\[
(3.4) \quad \phi_1(x) \geq c \Psi(\delta_D(x) \wedge r_{x_0}) \left( \inf_{|y - z| \leq |x - x_0| + 2r_{x_0}} J(y, z) \right).
\]
Proof. Fix $x_0 \in D$. For any $x \in D$ such that $|x - x_0| \geq 3r_{x_0}$ or $\delta_D(x) \leq r_{x_0}$, set $D_0 = B(x_0, 2r_{x_0})$, $\tilde{D}_0 = B(x_0, r_{x_0})$, $D_1 = B(x, \delta_D(x) \wedge r_{x_0})$ and $t_0 = C_0 \Psi(r_{x_0})$, where $C_0 > 0$ is the constant in (3.3). Then, it holds that dist($D_1, \tilde{D}_0$) > 0. Indeed, for any $y \in \tilde{D}_0$ and $z \in D_1$, if $|x - x_0| \geq 3r_{x_0}$, then

$$|y - z| \geq |x_0 - x| - |x - z| - |y - x_0| \geq 3r_{x_0} - r_{x_0} - r_{x_0} = r_{x_0} > 0;$$

if $\delta_D(x) \leq r_{x_0}$, then,

$$|y - z| \geq |x_0 - x| - |x - z| - |y - x_0| \geq \delta_D(x_0) - \delta_D(x) - \delta_D(x) - r_{x_0} \geq \delta_D(x_0) - 3r_{x_0} \geq \delta_D(x_0)/4 > 0.$$

Hence, by strong Markov property,

$$P^D_{t_0} 1_{D_0}(x) \geq \mathbb{P}^x \left( 0 < \tau_{D_1} \leq \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}), X_{\tau_{D_1}} \in \tilde{D}_0; \forall s \in [\tau_{D_1}, t_0], X_s \in D_0 \right)$$

$$\geq \mathbb{P}^x \left( \mathbb{P}^{X_{\tau_{D_1}}} \left( \tau_{\tilde{D}_0} > t_0 \right); 0 < \tau_{D_1} \leq \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}), X_{\tau_{D_1}} \in \tilde{D}_0 \right)$$

$$\geq \mathbb{P}^x \left( \mathbb{P}^{X_{\tau_{D_1}}} \left( \tau_{B(X_{\tau_{D_1}}, r_{x_0})} > t_0 \right); 0 < \tau_{D_1} \leq \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}), X_{\tau_{D_1}} \in \tilde{D}_0 \right)$$

$$\geq C_1 \mathbb{P}^x \left( 0 < \tau_{D_1} \leq \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}), X_{\tau_{D_1}} \in \tilde{D}_0 \right),$$

where in the third and fourth inequalities we have used the fact that for all $y \in \tilde{D}_0$,

$$\mathbb{P}^y \left( \tau_{B(y, r_{x_0})} > t_0 \right) \geq \mathbb{P}^y (\tau_{B(x, r_{x_0})} > t_0) \geq C_1 > 0,$$

thanks to Exit$_{\Psi, \geq}$.

Furthermore, by the Lévy system in (3.1),

$$\mathbb{P}^x \left( 0 < \tau_{D_1} \leq \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}), X_{\tau_{D_1}} \in \tilde{D}_0 \right)$$

$$= \mathbb{E}^x \int_0^{\frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0})} \int_{\tilde{D}_0} J(X_s, z) \, dz \, ds$$

$$\geq \left( \inf_{y \in D_1, z \in \tilde{D}_0} J(y, z) \right) |\tilde{D}_0| \int_0^{\frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0})} \mathbb{P}^x (\tau_{D_1} > s) \, ds$$

$$\geq \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}) \left( \inf_{y \in D_1, z \in \tilde{D}_0} J(y, z) \right) |\tilde{D}_0| \mathbb{P}^x \left( \tau_{D_1} > \frac{C_0}{2} \Psi(\delta_D(x) \wedge r_{x_0}) \right)$$

$$\geq c_1 \Psi(\delta_D(x) \wedge r_{x_0}) \left( \inf_{y \in D_1, z \in \tilde{D}_0} J(y, z) \right),$$

where in the equality above we have used the fact that dist($D_1, \tilde{D}_0$) > 0, and in the last inequality we used Exit$_{\Psi, \geq}$ again.

Combining both estimates above, we arrive at

$$P^D_{t_0} 1_{D_0}(x) \geq c_2 \Psi(\delta_D(x) \wedge r_{x_0}) \left( \inf_{y \in D_1, z \in \tilde{D}_0} J(y, z) \right).$$
This along with Lemma 3.1 yields the desired assertion for any \( x \in D \) with \(|x-x_0| \geq 3r_{x_0} \) or \( \delta_D(x) \leq r_{x_0} \).

Next, we set \( D_{x_0} := \{ x \in D : |x-x_0| < 3r_{x_0} \text{ and } \delta_D(x) > r_{x_0} \} \). Then, \( D_{x_0} \) is a precompact open subset of \( D \). Since by assumption \( \phi_1 \) is continuous and strictly positive on \( D \), we have \( \inf_{z \in D_{x_0}} \phi_1(z) \geq c_3 \) for some constant \( c_3 > 0 \). While, since by (1.1)

\[
\infty > c_4 := \sup_{y \in \mathbb{R}^d, z \neq y} (1 \wedge |y-z|^2) J(y, z) dz
\]

\[
\geq (1 \wedge r_{x_0}^2) \sup_{y \in \mathbb{R}^d} \int_{|r_{x_0} < |y-z| < 2r_{x_0}|} J(y, z) dz
\]

\[
\geq (1 \wedge r_{x_0}^2) |\{ r_{x_0} \leq |w| \leq 2r_{x_0} \}| \sup_{y \in \mathbb{R}^d, z \in \mathbb{R}^d} \inf_{|y-z| \leq 2r_{x_0}} J(y, z),
\]

we have

\[
\inf_{|y-z| \leq |x-x_0|+2r_{x_0}} J(y, z) \leq \sup_{y \in \mathbb{R}^d, z \in \mathbb{R}^d} \inf_{|y-z| \leq 2r_{x_0}} J(y, z)
\]

\[
\leq (1 \wedge r_{x_0}^2) |\{ r_{x_0} \leq |w| \leq 2r_{x_0} \}| c_4.
\]

Hence, by changing the constant \( c \) properly, we know that (3.4) also holds on \( D_{x_0} \).

Therefore, the desired assertion follows from both estimates above. \( \square \)

Note that the right hand side of (3.4) is zero if the process \( X \) has finite range jumps and \(|x-x_0|\) is large enough. Thus, Proposition 3.4 mainly concerns with lower bounds of \( \phi_1 \) for processes with infinite range jumps. We next consider another type of lower bounds of \( \phi_1 \), which is suitable for processes with finite range jumps or with super-exponentially decaying jumps, e.g., the jumping kernel given by (1.6) with \( \gamma \in (1, \infty] \).

**Definition 3.5.** For fixed \( y \in D \), we call \( x \in D \) is connected with \( y \) in a reasonable way with respect to constants \( 0 < a_1 \leq a_2 < 1 \), if there exist \( n := n(y,x) \in \mathbb{N} \) and \( \{x^{(i)}\}_{i=0}^n \) such that \( x^{(0)} = y, x^{(n)} = x, x^{(i)} \in D \) and \( a_1 \leq |x^{(i)} - x^{(i-1)}| \leq a_2 \) for \( 1 \leq i \leq n \). For simplicity, we write \( x \sim_{(n,a_1,a_2)} y \) if \( x \) is connected with \( y \) in a reasonable way with respect to constants \( 0 < a_1 \leq a_2 < 1 \).

**Proposition 3.6.** Suppose that \( \text{Exit}_{\Psi, \geq} \) holds on \( D \) and that

\[
(3.6) \quad \inf_{0 \leq |x-y| < 1} J(x, y) > 0.
\]

For any fixed \( x_0 \in D \), there exist constants \( c_1, c_2 > 0 \) (which may depend on \( x_0, a_1, a_2, r_0, C_0, C_1 \) but are independent of \( n \)) such that for all \( x \in D \) with \( x \sim_{(n,a_1,a_2)} x_0 \) it holds that

\[
(3.7) \quad \phi_1(x) \geq c_1 \Psi(\delta_D(x) \wedge r_0) \exp \left[ -c_2 \left( n \log n + \sum_{i=0}^{n-1} \log \frac{1}{r_i} \right) \right],
\]

where \( r_i := (\delta_D(x^{(i)})/3) \wedge r_0 \) for all \( 1 \leq i \leq n \), \( r_0 := (\delta_D(x_0)/3) \wedge (((1-a_2) \wedge a_1)/4) \wedge \tilde{r}_0 \), and \( \tilde{r}_0 \) is the constant in (3.3).
Proof. In the following, we fix $x_0, x \in D$ such that $x \sim_{(n; a_1, a_2)} x_0$. For $0 \leq i \leq n$, set $D_i = B(x^{(i)}, 2r_i)$ and $\tilde{D}_i = B(x^{(i)}, r_i)$. Then, for any $z \in D_i$ and $u \in \tilde{D}_{i-1}$ with $1 \leq i \leq n$,

$$|z - u| \leq |x^{(i)} - x^{(i-1)}| + |x^{(i)} - z| + |x^{(i-1)} - u| \leq a_2 + \frac{(1 - a_2)}{2} + \frac{(1 - a_2)}{4} < 1,$$

and

$$|z - u| \geq |x^{(i)} - x^{(i-1)}| - |x^{(i)} - z| - |x^{(i-1)} - u| > a_1 - \frac{a_1}{2} - \frac{a_1}{4} = \frac{a_1}{4} > 0.$$

In particular, $D_i \cap \tilde{D}_{i-1} = \emptyset$ for $1 \leq i \leq n$.

Define $\tilde{\tau}_{D_i} = \inf\{t \geq \tilde{\tau}_{D_{i+1}} \mid X_t \notin D_i\}$ for $1 \leq i \leq n - 1$, and $\tilde{\tau}_{D_n} = \tau_{D_n}$. By the convention, we also set $\tilde{\tau}_{D_{n+1}} = 0$ (and so $X_{\tilde{\tau}_{D_{n+1}}} = x$ under $P^x$). Let $t_0 = C_0 \Psi(r_0)$, where $C_0 > 0$ is the constant in (3.3). Then,

$$P^0_{t_0, D_0}(x) = \mathbb{P}^x \left( 1_{D_0}(X_{t_0}) 1_{\{ \tau_D > t_0 \}} \right)
\geq \mathbb{P}^x \left( 0 < \tilde{\tau}_{D_i} - \tilde{\tau}_{D_{i+1}} \leq \frac{C_0 \Psi(r_i)}{2n}, X_{\tilde{\tau}_{D_i}} \in \tilde{D}_{i-1} \text{ for each } 1 \leq i \leq n; \right.
X_s \in D_0 \text{ for all } s \in [\tilde{\tau}_{D_i}, t_0] \bigg)$$

$$= \mathbb{P}^x \left( 0 < \tau_{D_n} \leq \frac{C_0 \Psi(r_n)}{2n}, X_{\tau_{D_n}} \in \tilde{D}_{n-1} \right)
\times \mathbb{P}^{X_{\tau_{D_{n}}} \mid \tilde{\tau}_{D_{n+1}}} \left( 0 < \tau_{D_{n-1}} \leq \frac{C_0 \Psi(r_{n-1})}{2n}, X_{\tau_{D_{n-1}}} \in \tilde{D}_{n-2} \right)
\times \mathbb{P}^{X_{\tau_{D_{n-1}}} \mid \cdots \mid \tilde{\tau}_{D_{i}}} \left( \cdots \mathbb{P}^{X_{\tau_{D_{2}}} \mid \cdots \mid \tilde{\tau}_{D_{1}}} \left( 0 < \tau_{D_{1}} \leq \frac{C_0 \Psi(r_1)}{2n}, X_{\tau_{D_{1}}} \in \tilde{D}_0 \right) \right)\left. \times \mathbb{P}^{X_{\tau_{D_{1}}} \mid X_s \in D_0 \text{ for all } s \in [0, t_0 - \tilde{\tau}_{D_1}]} \right)\left. \cdots \right)\right),$$

where in the last equality we have used the strong Markov property.

According to the Lévy system in (3.1), for any $1 \leq i \leq n$, if $X_{\tilde{\tau}_{D_{i+1}}} \in \tilde{D}_{i}$, then,

$$\mathbb{P}^{X_{\tau_{D_{i+1}}} \mid \tilde{\tau}_{D_{i+1}}} \left( 0 < \tau_{D_{i}} \leq \frac{C_0 \Psi(r_i)}{2n}, X_{\tau_{D_{i}}} \in \tilde{D}_{i-1} \right)
\geq \inf_{y \in \tilde{D}_i} \mathbb{P}^y \left( 0 < \tau_{D_{i}} \leq \frac{C_0 \Psi(r_i)}{2n}, X_{\tau_{D_{i}}} \in \tilde{D}_{i-1} \right)
\geq \inf_{y \in \tilde{D}_i} \mathbb{E}^y \left( \int_0^{\mathbb{C}_0 \Psi(r_i)} J(X_s^{D_{i}}, u) \, du \, ds \right)
\geq \left( \inf_{z \in \tilde{D}_i, u \in \tilde{D}_{i-1}} J(z, u) \right) |\tilde{D}_{i-1}| \inf_{y \in \tilde{D}_i} \mathbb{P}^y \left( \tau_{D_{i}} > s \right) ds
\geq C_1 \frac{\Psi(r_i)}{n} |\tilde{D}_{i-1}| \inf_{y \in \tilde{D}_i} \mathbb{P}^y \left( \tau_{B(y, r_i)} > \frac{C_0 \Psi(r_i)}{2} \right)
where in the equality above we used the fact that \( D_i \cap \tilde{D}_{i-1} = \emptyset \), and in the last inequality we used Exit_{\Psi,\geq} and (3.6).

On the other hand, due to Exit_{\Psi,\geq} again, if \( X_{\tau_{D_1}} \in \tilde{D}_0 \), then
\[
P^{X_{\tau_{D_1}}} \left( \forall s \in [0, t_0 - \tau_{D_1}) X_s \in D_0 \right) \geq \inf_{y \in D_0} P^y \left( \tau_{D_0} > t_0 \right)
\]
(3.10)
\[
\geq \inf_{y \in D_0} P^y \left( \tau_{B(y, r_0)} > t_0 \right) \geq C_1 > 0.
\]

Therefore, combining (3.8) with (3.9) and (3.10) and using (3.2), we find that
\[
P^0_1 \mathbb{1}_{D_0}(x) \geq (1 - C_1)c_1 n \prod_{i=1}^{n} \left( \frac{\Psi(r_i)}{n^{d-1}} \right)
\]
\[
\geq c_2 \frac{\Psi(\delta_D(x) \wedge r_0)}{n^n} c_1 n \prod_{i=1}^{n-1} \Psi(r_i) r_i^d
\]
\[
= c_2 \Psi(\delta_D(x) \wedge r_0)
\]
\[
\times \exp \left[ - \left( n \log n - \log c_1 + \sum_{i=1}^{n-1} \log \frac{1}{\Psi(r_i)} + d \sum_{i=1}^{n-1} \log \frac{1}{r_i} \right) \right]
\]
\[
\geq c_3 \Psi(\delta_D(x) \wedge r_0) \exp \left[ -c_4 \left( n \log n + \sum_{i=0}^{n-1} \log \frac{1}{r_i} \right) \right],
\]
where in the last inequality we have used the property that
\[
\log \frac{1}{\Psi(r_i)} \leq \beta_2 \log \frac{1}{r_i} + c_5,
\]
due to (3.2). This along with Lemma 3.1 proves the desired assertion. \( \square \)

4. INTRINSIC ULTRACONTRACTIVITY OF DIRICHLET SEMIGROUPS: GENERAL RESULTS

The main purpose of this section is to present sufficient conditions and necessary conditions for intrinsic ultracontractivity of \((P^D_t)_{t \geq 0}\).

4.1. Sufficient conditions for intrinsic ultracontractivity of \((P^D_t)_{t \geq 0}\). In this part, we consider the Dirichlet form \((\mathcal{E}, \mathcal{F})\) given by (1.2) such that \(J_{\Phi,\geq}\) is satisfied. Recall that, without loss of generality, we have assumed that \(r_0 = 1\) in (2.3), i.e.,
\[
J(x, y) \geq \frac{C_0}{|x - y| d\Phi(|x - y|)}, \quad 0 < |x - y| \leq 1.
\]

Let \(D \subset \mathbb{R}^d\) be an open set such that (2.8) holds; that is,
\[
\lim_{x \in D \text{ and } |x| \to \infty} V_D(x) = \infty.
\]

Then, according to Example 2.3 and Corollary 2.7, we know that \((P^D_t)_{t \geq 0}\) is compact. As stated before, we always assume that the Dirichlet heat kernel \(p^D(t, x, y)\) exists, and \(p^D(t, \cdot, \cdot)\) is bounded, continuous and strictly positive on \(D \times D\) for every \(t > 0\). Let \(\phi_1\) be the associated ground state, which can be chosen to be bounded, continuous and strictly positive, see e.g. Proposition 7.3 below.
Recall that the Dirichlet form \((\mathcal{E}^D, \mathcal{F}^D)\) is given in (2.1). Our approach to prove the intrinsic ultracontractivity of \((P^D_t)_{t \geq 0}\) is based on the intrinsic super Poincaré inequality for \((\mathcal{E}^D, \mathcal{F}^D)\). For any open set \(D\) and constants \(R, r > 0\), let 
\[
D_{R,r} = \{x \in D : |x| < R \text{ and } \delta_D(x) > r\}.
\]
We first present the following form of local intrinsic super Poincaré inequality for \((\mathcal{E}^D, \mathcal{F}^D)\).

**Proposition 4.1.** Suppose that \(J_{\Phi, \geq}\) holds and \(D\) satisfies (2.8). Then there exists a constant \(c > 0\) such that for any \(R, r, s > 0\), and \(f \in C^\infty_c(D)\),
\[
\int_{D_{R,r}} f^2(x) \, dx \leq s \mathcal{E}^D(f, f) + \alpha(R, r, s; \phi_1) \left( \int_{D} |f(x)| \phi_1(x) \, dx \right)^2,
\]
where 
\[
\alpha(R, r, s; \phi_1) := \frac{c}{\inf_{z \in D_{R+1, r/2}} \phi_1^2(z)} \left( \Phi^{-1}(s) \wedge r \wedge 1 \right)^{-d}.
\]

**Proof.** For \(f \in C^\infty_c(D)\), define 
\[
f_s(x) = \frac{1}{|B(x, s)|} \int_{B(x, s)} f(z) \, dz,
\]
and
\[
s > 0, x \in D.
\]
Note that for any \(x \in D_{R,r}\) and \(0 < s < r/2\), by the definition of \(D_{R,r}\), \(B(x, s) \subset D_{R+s, r-s}\). Thus, we obtain
\[
\sup_{x \in D_{R,r}} |f_s(x)| \leq \frac{1}{|B(0, s)|} \int_{D_{R+s, r-s}} |f(z)| \, dz
\]
and
\[
\int_{D_{R,r}} |f_s(x)| \, dx \leq \int_{D_{R,r}} \frac{1}{|B(0, s)|} \int_{B(x, s)} |f(z)| \, dz \, dx
\]
\[
\leq \int_{D_{R+s, r-s}} \left( \frac{1}{|B(0, s)|} \int_{B(z, s)} \, dx \right) |f(z)| \, dz
\]
\[
\leq \int_{D_{R+s, r-s}} |f(z)| \, dz.
\]
Hence,
\[
\int_{D_{R,r}} f_s^2(x) \, dx \leq \left( \sup_{x \in D_{R,r}} |f_s(x)| \right) \int_{D_{R,r}} |f_s(z)| \, dz = \frac{1}{|B(0, s)|} \left( \int_{D_{R+s, r-s}} |f(z)| \, dz \right)^2.
\]
Therefore, we have for any \(f \in C^\infty_c(D)\) and \(0 < s \leq (r/2) \wedge 1\),
\[
\int_{D_{R,r}} f^2(x) \, dx \leq 2 \int_{D_{R,r}} (f(x) - f_s(x))^2 \, dx + 2 \int_{D_{R,r}} f_s^2(x) \, dx
\]
\[
\leq 2 \int_{D_{R,r}} \frac{1}{|B(0, s)|} \int_{B(x, s)} (f(x) - f(y))^2 \, dy \, dx
\]
\[
+ \frac{2}{|B(0, s)|} \left( \int_{D_{R+s, r-s}} |f(z)| \, dz \right)^2
\]
\[
\leq c_1 \Phi(s) \int_{\{(x, y) \in D \times D : |x - y| \leq s\}} (f(x) - f(y))^2 J(x, y) \, dx \, dy
\]
where in the third inequality we have used (2.3) and the fact that for any \( x \in D_{R,r} \) and \( 0 < s \leq r/2, B(x,s) \subset D \).

Setting \( c_1 \Phi(s) \) as \( s \) in the inequality above and using (1.4), we arrive at that there exist constants \( c_3, c_4 > 0 \) such that for any \( 0 < s \leq c_3(\Phi(r) \wedge 1) \) and \( f \in C_c^\infty(D) \),

\[
\int_{D_{R,r}} f^2(x) \, dx \leq s \mathcal{E}^D(f,f) + \frac{c_4(\Phi^{-1}(s))^{-d}}{\inf_{x \in D_{R+1/2, r/2}} \phi_0^2(x)} \left( \int_D |f(x)| \phi_1(x) \, dx \right)^2,
\]

where in the last inequality we used (1.4) again. Therefore, the desired assertion follows from both conclusions above.

The following theorem is the main result in this subsection.

**Theorem 4.2.** Suppose that \( J_{\Phi,\geq} \) holds and \( D \) satisfies (2.8). Then, for any \( f \in C_c^\infty(D) \) and \( s > 0 \),

\[
\int_D f^2(x) \, dx \leq s \mathcal{E}^D(f,f) + \beta(s) \left( \int_D |f(x)| \phi_1(x) \, dx \right)^2,
\]

where

\[
\beta(s) := \inf \left\{ \alpha(R,r,s/2; \phi_1) : 0 < r < 1/2 \text{ and } \inf_{x \in D \setminus D_{R,r}} V_D(x) \geq 1/s \right\},
\]

and \( \alpha(R,r,s; \phi_1) \) is defined in Proposition 4.1. By convention, \( \inf \emptyset = \infty \) here.

Consequently, the semigroup \( (P_t^D)_{t \geq 0} \) is intrinsically ultracontractive, if

\[
\int_t^\infty \frac{\beta^{-1}(s)}{s} \, ds < \infty, \quad t > \inf \beta.
\]

**Proof.** Due to (2.1), it holds that

\[
\int_D f^2(x) V_D(x) \, dx \leq \frac{1}{2} \mathcal{E}^D(f,f), \quad f \in C_c^\infty(D).
\]

Then, for any \( R, r > 0 \),

\[
\int_{D \setminus D_{R,r}} f^2(x) \, dx \leq \frac{1}{\inf_{x \in D \setminus D_{R,r}} V_D(x)} \int_D f^2(x) V_D(x) \, dx
\]
This along with (4.1) yields that for any $t > s > 0$,
\[
\int_D f^2(x) \, dx \leq \left( t + \frac{1}{2 \inf_{x \in D \setminus D_{R,r}} V_D(x)} \right) \mathcal{E}^D(f, f) + \alpha(R,r,t; \phi_1) \left( \int_D |f(x)| \phi_1(x) \, dx \right)^2.
\]

For any $s > 0$, letting $t = s/2$ and choosing $0 < r < 1/2$ and $R > 0$ such that $\inf_{x \in D \setminus D_{R,r}} V_D(x) \geq 1/s$, we obtain the first desired assertion.

The second assertion is a consequence of the first one and [43, Theorem 3.3] or [44, Theorem 3.3.14]. See e.g. [12, Theorem 2.1] for the proof. \hfill \Box

**Remark 4.3.** Since $\inf_{x \in D \setminus D_{R,r}} V_D(x) \leq \inf_{x \in D} V_D(x)$ for any $r > 0$, (2.8) does not guarantee the finiteness of the rate function $\beta(s)$. According to Proposition 2.8, we know that if $D^c$ is $\kappa$-fat such that (2.11) is satisfied, then the rate function $\beta(s)$ defined by (4.4) is finite.

### 4.2. Necessary conditions for intrinsic ultracontractivity of $(P^D_t)_{t \geq 0}$

We first introduce two definitions.

**Definition 4.4.** Let $\Psi$ be an increasing function on $(0, \infty)$ satisfying (3.2). We say that the lower bound near diagonal estimate $\text{NDHK}_{\Psi,\geq}$ holds in $D$, if there are $c_0 \in (0, 1)$ and $r_0, c_1, c_2 > 0$ such that for all $0 < r \leq r_0$ and $B(x, r) \subset D$,
\[
p^{B(x,r)}(c_1 \Psi(r), y, z) \geq c_2 r^{-d}, \quad y, z \in B(x, cr).
\]

**Definition 4.5.** Let $\Gamma$ be a non-increasing positive function on $(1, \infty)$ such that $\lim_{s \to \infty} \Gamma(s) = 0$. We say that the off-diagonal upper bounded estimate $\text{ODHK}_{\Gamma,\leq}$ holds for the Dirichlet heat kernel $p^D(t, x, y)$ if there are $t_0, c_0 > 0$ such that for all $t \in (0, t_0]$ and $x, y \in D$ with $|x - y| \geq c_0$,
\[
p^D(t, x, y) \leq C(t) \Gamma(|x - y|),
\]
where $C(t)$ is a positive constant depending on $t$.

If $\text{NDHK}_{\Psi,\geq}$ holds in $D$, then there exists a positive constant $c > 0$ such that for all $r \in (0, r_0]$ and $B(x, r) \subset D$,
\[
P^\mathbb{P} \left( \tau_{\partial B(x,r)} > c_1 \Psi(r) \right) \geq \int_{B(x,cr)} p^{B(x,r)} \left( c_1 \Psi(r), x, y \right) \, dy \geq c,
\]
and so $\text{Exit}_{\Psi,\geq}$ holds in $D$ too. Next, we give the other consequence of $\text{NDHK}_{\Psi,\geq}$.

**Lemma 4.6.** Suppose that $\text{NDHK}_{\Psi,\geq}$ holds in $D$. Then, there exist positive constants $c_1, c_2$ such that for all $t > 0$ and $x \in D$ with $\delta_D(x) \leq r_0$,
\[
P^D_t \mathbb{1}_{B(x,\delta_D(x))}(x) \geq c_1 e^{-c_2 t^\Psi(\delta_D(x))},
\]
where $r_0$ the positive constant in Definition 4.4.

**Proof.** Throughout the proof, let $r_0$ and $c_1$ be positive constants in the definition $\text{NDHK}_{\Psi,\geq}$. Fix $x \in D$ with $\delta_D(x) \leq r_0$. As mentioned above, $\text{NDHK}_{\Psi,\geq}$ implies
Exit $\Psi_{\geq}$. Hence, if $t > 0$ such that $c_1\Psi(\delta_D(x)) \geq t$, then
\[
P_t^D \mathbf{1}_{B(x, \delta_D(x))}(x) \geq P^x(\tau_{B(x, \delta_D(x))} > t) \geq P^x(\tau_{B(x, \delta_D(x))} > c_1\Psi(\delta_D(x))) \geq C_1 \geq C_1e^{-t/(c_1\Psi(\delta_D(x)))}.
\]

Next, we consider $t > 0$ such that $c_1\Psi(\delta_D(x)) < t$. Set $n = \left\lfloor \frac{t}{c_1\Psi(\delta_D(x))} \right\rfloor$. Then, by NDHK$_{\Psi, \geq}$,
\[
P_t^D \mathbf{1}_{B(x, \delta_D(x))}(x) \geq P^x(\tau_{B(x, \delta_D(x))} > t) \geq P^x(\tau_{B(x, \delta_D(x))} > nc_1\Psi(\delta_D(x))) \geq \int_{B(x, c_0\delta_D(x))} \cdots \int_{B(x, c_0\delta_D(x))} p^{B(x, \delta_D(x))}(c_1\Psi(\delta_D(x)), x, z_1) \cdots \times p^{B(x, \delta_D(x))}(c_1\Psi(\delta_D(x)), z_{n-1}, z_n) dz_n \cdots dz_1 \geq C_2^n \geq C_3e^{-C_4t/\Psi(\delta_D(x))}.
\]

Therefore, combining with both estimates above, we prove the desired assertion. \qed

Now, we are in the position to present necessary conditions for the intrinsic ultracontractivity of $(P_t^D)_{t \geq 0}$.

**Proposition 4.7.** Suppose that $D$ is an open subset satisfying (2.11), and that NDHK$_{\Psi, \geq}$ holds on $D$ and ODHK$_{\Gamma, \leq}$ hold for the Dirichlet heat kernel $p^D(t, x, y)$. If $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive, then
\[
\lim_{x \in D \text{ and } |x| \to \infty} \Psi(\delta_D(x)) \log \Gamma(|x|) = 0.
\]

**Proof.** Since $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive, then, for any $t > 0$ there is a constant $C_{t, D} \geq 1$ such that for all $x, y \in D$,
\[
C_{t, D}^{-1} \phi_1(x)\phi_1(y) \leq p^D(t, x, y) \leq C_{t, D}\phi_1(x)\phi_1(y),
\]
see e.g. [24, Theorem 3.2]. Thus, for any $x \in D$ it holds that
\[
P_t^D \mathbf{1}_{B(x, \delta_D(x))}(x) = \int_{B(x, \delta_D(x))} p^D(t, x, y) dy \leq C_{t, D}\phi_1(x)\delta_D(x)^d.
\]
For any compact subset $D_0$ of $D$ with $|D_0| > 0$ and for any $x \in D$, we also have
\[
P_t^D \mathbf{1}_{D_0}(x) = \int_{D_0} p^D(t, x, y) dy \geq C_{t, D}^{-1}\phi_1(x) \int_{D_0} \phi_1(y) dy \geq C_{t, D_0, D}\phi_1(x).
\]
Combining with both inequalities above, we know that for every $t > 0$ and any compact subset $D_0$ of $D$ with $|D_0| > 0$, there exists a constant $c_{t, D_0, D} > 0$ such that
\[
P_t^D \mathbf{1}_{B(x, \delta_D(x))}(x) \leq c_{t, D_0, D}P_t^D \mathbf{1}_{D_0}(x)\delta_D(x)^d, \quad x \in D.
\]
Furthermore, by NDHK$_{\Psi, \geq}$, (2.11) and Lemma 4.6, there exist constants $c_2, c_3 > 0$ such that for every $t > 0$ and $x \in D$ with $|x|$ large enough,
\[
P_t^D \mathbf{1}_{B(x, \delta_D(x))}(x) \geq c_2e^{-c_3t/\Psi(\delta_D(x))},
\]
On the other hand, according to (2.11) and ODHK$_{Γ,<}$, there is a constant $t_0 ∈ (0,1)$ such that for any compact set $D_0 ⊂ D$, $t ∈ (0,t_0]$ and $x ∈ D$ with $|x| ≥ 2R_0 := 2\sup_{z ∈ D_0} |z|$ large enough,

$$P^D_t \mathbb{1}_{D_0}(x)δ_D(x)^d ≤ \int_{D_0} p_D(t,x,y) dy ≤ c_{4,t,D}|D_0| \sup_{y ∈ D_0} Γ(|x−y|) ≤ c_{5,t,D_0,D}Γ(|x|−R_0) ≤ c_{5,t,D_0,D}Γ(|x|/2).$$

(4.9)

Below, we fix a compact subset $D_0 ⊂ D$ with $|D_0| > 0$. Therefore, combining (4.7) with (4.8) and (4.9), we arrive at that there is $t_0 > 0$ such that for all $t ∈ (0,t_0]$ and $x ∈ D$ with $|x|$ large enough so that $δ(x) ≤ 1$,

$$e^{−c_7t/Ψ(δ_D(x))} ≤ c_{6,t}Γ(|x|/2).$$

(4.10)

Next, we assume that (4.6) does not hold. Then there exist a constant $c_7 > 0$ and a sequence $\{x_n\}^\infty_{n=1} ⊂ D$ such that $\lim_{n→∞} x_n = ∞$ and $|log Γ(|x_n|/2)|Ψ(δ_D(x_n)) ≥ c_7$ for all $n ∈ \mathbb{N}$. Thus, for any $t > 0$,

$$\exp \left(−c_7t/Ψ(δ_D(x_n)) \right) ≥ \exp \left(−t/\log Γ(|x_n|/2)\right) = Γ(|x_n|/2)^t.$$ 

Taking $t = t_1 := ((c_3t_0)/(2c_7)) ∧ (1/2)$ in the inequality above and using (4.10), we get

$$0 < e^{−c_7t_1/(1−t_1)} ≤ Γ(|x_n|/2).$$

Since $\lim_{s→∞} Γ(s) = 0$, letting $n → ∞$ we get a contradiction. That is, if $(P^D_t)_{t≥0}$ is intrinsically ultracontractive, then (4.6) does hold. The proof is complete. □

5. INTRINSIC ULTRACONTRACTIVITY OF DIRICHLET SEMIGROUPS: EXPLICIT RESULTS

We continue considering the symmetric Hunt process $X = \{X_t, t ≥ 0; \mathbb{P}^x, x ∈ \mathbb{R}^d\}$ as in Subsection 1.1. The associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by (1.2). Let $D$ be an open set of $\mathbb{R}^d$. Denote by $(P^D_t)_{t≥0}$ and $p^D(t,x,y)$ the Dirichlet semigroup and the Dirichlet heat kernel associated with the killed process $X^D$ of the process $X$ upon exiting $D$, respectively. Throughout this section, we suppose that both $J_{Φ,≥}$ and Exit$_{Φ,≥}$ hold. See Definitions 2.2 and 3.2. Recall that we also always assume that the Dirichlet heat kernel $p^D(t,x,y)$ exists, and $p(t,\cdot,\cdot)$ is bounded, continuous and strictly positive on $D × D$ for every $t > 0$.

In this section, we will apply results in previous sections to establish criteria for the intrinsically ultracontractivity of $(P^D_t)_{t≥0}$ on two specific types of open sets. One is horn-shaped regions, and the other one is unbounded and disconnected open sets with locally $κ$-fat property.

5.1. Horn-shaped regions. Let $D_f = \{x ∈ \mathbb{R}^d : x_1 > 0, |\tilde{x}| < f(x_1)\}$ be a horn-shaped region with the reference function $f$, where $f : (0,∞) → (0,∞)$ is bounded, continuous and satisfies that $\lim_{u→∞} f(u) = 0$. As mentioned above, we assume that $J_{Φ,≥}$ holds. Then, according to Corollary 2.9, it is easy to see that the semigroup $(P^D_f)_{t≥0}$ is compact if $D_f^c$ is $κ$-fat. Furthermore, by Proposition 7.3 and the assumption that the Dirichlet heat kernel $p^D_f(t,x,y)$ exists and $p^D_f(t,\cdot,\cdot)$ is bounded, continuous and strictly positive on $D_f × D_f$ for every $t > 0$, the corresponding ground state $ϕ_1$ can be chosen to be bounded, continuous and strictly positive on $D_f$. 

Non-local Dirichlet forms on unbounded open sets
To illustrate how powerful Theorem 4.2 is, we begin with horn-shaped regions with general reference functions. For non-negative measurable function $f$, let $f^*(r) = \sup_{s \geq r} f(s)$ and $f^{*-1}(r) = \inf\{ s > 0 : f^*(s) \leq r \}$ for $r > 0$.

**Proposition 5.1.** Assume that $J_{\Phi, r}$ and Exit$_{\Phi, r}$ hold. Let $D_f$ be a horn-shaped region such that $D_f^c$ is $\kappa$-fat. Then, the following two statements hold true.

1. There are positive constants $c_1, c_2$ such that for all $x \in D_f$,
   \[
   \phi_1(x) \geq c_1 \Phi(\delta_{D_f}(x) \wedge c_2) \left( \inf_{|y-z| \leq |x| + c_1} J(y, z) \right).
   \]

2. The super Poincaré inequality (4.3) holds with
   \[
   (5.1) \quad \beta(s) = C_1(\Phi^{-1}(s) \wedge 1)^{-d} \Psi(\Phi^{-1}(s) \wedge 1)^{-2} \sup_{|y-z| \leq C_1 f^{*-1}(C_2(\Phi^{-1}(s) \wedge 1))} \frac{1}{J(y, z)^2}.
   \]

Here $C_1, C_2$ are positive constants. Consequently, the semigroup $(P^D_t)_{t \geq 0}$ is intrinsically ultracontractive, if $\beta(s)$ given above satisfies (4.5).

*Proof.* (1) The first assertion is a consequence of (3.4).

(2) Denote by $D = D_f$ for simplicity. Since $D^c$ is $\kappa$-fat, we can use (2.10) in Proposition 2.8. Let $c_0 > 0$ be the constant $c_2$ in (2.10). Applying (2.10) and (4.4), we find that (4.3) holds with the rate function $\beta(s)$ satisfying that

\[
\beta(s) \leq c_1 \inf \left\{ \frac{(\Phi^{-1}(s) \wedge r)^{-d}}{\inf_{x \in D \setminus D, r} \Phi^2_1(x)} : \inf_{x \in D \setminus D, r} V_D(x) \geq 1/s \text{ and } r \in (0, 1/2) \right\}
\]

\[
\leq c_1 \inf \left\{ \frac{(\Phi^{-1}(s) \wedge r)^{-d}}{\inf_{x \in D \setminus D, r} \Phi^2_1(x)} : \sup_{x \in D \setminus D, r} \Phi(\delta_D(x)) \leq c_0 s \text{ and } r \in (0, 1/2) \right\}.
\]

According to (1.4), we can take a constant $c_* \in (0, r_{0}/2)$ small enough so that $c_* \Phi^{-1}(s) \leq \Phi^{-1}(c_0 s)$ for all $s > 0$. Next, we choose $s_0 > 0$ small enough such that for any $s \in (0, s_0]$, $r = c_* (\Phi^{-1}(s) \wedge 1) < 1/2$ and $R = f^{*-1}(c_* (\Phi^{-1}(s) \wedge 1)) < \infty$. Taking this $r$ and $R$ in the infimum of the last term in the display above, and using assertion (1) for lower bound estimates of $\phi_1$, we find that $\beta(s)$ is not bigger than

\[
c_2(\Phi^{-1}(s) \wedge 1)^{-d} \Psi(\Phi^{-1}(s) \wedge 1)^{-2} \sup_{|y-z| \leq f^{*-1}(c_*(\Phi^{-1}(s) \wedge 1))} \frac{1}{J(y, z)^2}.
\]

This proves the second assertion. \(\square\)

Instead of Proposition 3.4, we can use Proposition 3.6 to obtain the following result.

**Proposition 5.2.** Assume that $J_{\Phi, r}$ and Exit$_{\Phi, r}$ hold. Let $D_f$ be a horn-shaped region such that $D_f^c$ is $\kappa$-fat. Then the following statements hold true.

1. There are positive constants $c_1, c_2$ such that for all $x \in D_f$,
   \[
   \phi_1(x) \geq c_1 \Psi(\delta_{D_f}(x) \wedge c_1) \exp \left( -c_2 (1 + |x|) \log \left( e + \frac{|x|}{\inf_{c_1 \leq s \leq |x| + 1} f(s)} \right) \right).
   \]
The super Poincaré inequality (4.3) holds with
\[ \beta(s) = C_i(\Phi^{-1}(s) \wedge 1)^{-d} \Psi(\Phi^{-1}(s) \wedge 1)^{-2} \]
\[ \times \exp \left\{ C_1 \left[ (1 + F(s)) \log \left( e + \frac{F(s)}{\inf_{C_2 \leq r \leq F(s)} f(r)} \right) \right] \right\}, \]
where
\[ F(s) = f^s - 1(C_3(\Phi^{-1}(s) \wedge 1) \]
and \( C_i \ (i = 1, 2, 3) \) are positive constants. Consequently, the semigroup \((P_t^D)_{t \geq 0}\) is intrinsically ultracontractive, if \( \beta(s) \) defined above satisfies (4.5).

**Proof.** We denote \( D = D_f \) throughout the proof.

1. Choose \( r_* \geq 4 \) large enough such that \( f(r) \leq 2^{-5} \) for any \( r \geq r_* - 1 \). We first consider \( x \in D \) with \( x_1 \geq 2r_* \). Take \( x^{(i)} = (r_* + i(x_1 - r_*)/n, \tilde{0}) =: (x^{(i)}, \tilde{0}) \) for \( 0 \leq i \leq n - 1 \) with \( n := \lceil 4(x_1 - r_*) \rceil \), and \( x^{(n)} = (x_1, \tilde{x}) = x \). Then,
\[ 2^{-3} \leq |x^{(i-1)} - x^{(i)}| = \frac{x_1 - r_*}{n} \leq 2^{-2}, \quad 1 \leq i \leq n - 1, \]
and
\[ 2^{-3} \leq \frac{x_1 - r_*}{n} \leq |x^{(n-1)} - x^{(n)}| \leq \frac{x_1 - r_*}{n} + f(x_1) \leq 2^{-1}. \]

Let
\[ r_i = \frac{\delta_D(x^{(i)})}{3} \wedge 2^{-5} \wedge \tilde{r}_0, \quad 0 \leq i \leq n - 1 \]
\[ r_n = \frac{\delta_D(x)}{3} \wedge 2^{-5} \wedge \tilde{r}_0, \]
denote the constant in (3.3). By the definition of horn-shaped region, for all \( 0 \leq i \leq n - 1 \),
\[ r_i \geq c_1 \inf_{|s - x^{(i)}| \leq f(x^{(i)})} f(s) \geq c_2 \inf_{r_* - 1 \leq s \leq |x| + 1} f(s). \]

Indeed, let \( y = (y_1, \tilde{y}) \in \partial D \) such that \( |x^{(i)} - y| = \delta_D(x^{(i)}) \). Since \( |\tilde{y}| = f(y_1) \) and \( |\tilde{x}| = 0 \), we have
\[ (5.3) \quad \delta_D(x^{(i)}) \geq |\tilde{y} - \tilde{x}| = f(y_1) \geq \inf_{s \in B(x^{(i)}, \delta_D(x^{(i)}))} f(s) \geq \inf_{s \in B(x^{(i)}, f(x^{(i)}))} f(s). \]

Therefore, combining all the estimates above with (3.7), we obtain that for every \( x_1 \geq 2r_* \),
\[ \phi_1(x) \geq c_3 \Psi(\delta_D(x) \wedge c_3) \exp \left[ - c_4 \left( n \log n + \sum_{i=0}^{n-1} \log \frac{1}{r_i} \right) \right] \]
\[ \geq c_3 \Psi(\delta_D(x) \wedge c_3) \exp \left[ - c_5 |x| \log \left( \frac{|x|}{\inf_{r_* - 1 \leq s \leq |x| + 1} f(s)} \right) \right], \]
where in the last inequality we have used the fact that \( c_6 |x| \leq n \leq c_7 |x| \) if \( x_1 \geq 2r_* \).

This proves that (5.2) holds for every \( x \in D \) such that \( x_1 \geq 2r_* \).

Next, we consider \( x \in D \) such that \( x_1 < 2r_* \). Since \( D_{r_*} := \{ x \in D : x_1 < 2r_* \} \) is bounded, we can find \( x_0 \in D \) and positive constants \( s_0, s_1, a_1, a_2, N_0 \) such that

1. \( \overline{B(x_0, s_0)} \subseteq D_{r_*} \).
(ii) For every \( x \in D_r \setminus B(x_0, s_0) \), \( x \sim_{n(x); a_1, a_2} x_0 \) for some positive integer \( n(x) \leq N_0 \), and the connected points \( \{ x^{(i)} : 1 \leq i \leq n(x) \} \) satisfies that \( \delta_D(x^{(i)}) \geq s_1 > 0 \) for all \( 1 \leq i \leq n(x) \).

Therefore, by (3.7) we know that (5.2) holds for every \( x \in D_r \setminus B(x_0, s_0) \). On the other hand, since \( \overline{B}(x_0, s_0) \subseteq D \) and \( \phi_1 \) is continuous, strictly positive on \( D \), \( \inf_{z \in B(x_0, s_0)} \phi_1(z) \geq c_8 \) for some constant \( c_8 > 0 \). So, by changing the constants properly, (5.2) still holds for any \( x \in D_r \).

Combining all the estimates above, we have shown that (5.2) holds for all \( x \in D \).

(2) With (1) at hand, the argument for the proof of (2) is the same as that for Proposition 5.1 (2). So we skip the details. \( \square \)

As a consequence of Propositions 5.1 and 5.2, we have the following corollary.

**Corollary 5.3.** Assume that \( J_{\Phi, \gamma} \) and Exit_{\Phi, \gamma} hold. Let \( D_f \) be a horn-shaped region such that \( D_f \) is \( \kappa \)-fat. Then, the following statements hold.

1. Suppose that there exist constants \( \alpha, \theta, s_0, c_1, c_2 > 0 \) such that
   \[
   J(x, y) \geq c_1|x - y|^{-\alpha}, \quad |x - y| > 1,
   \]
   and
   \[
   f(s) \leq c_2 \Phi^{-1}(\log^{-\theta} s), \quad s \geq s_0.
   \]
   Then \( (P_t^{D_f})_{t \geq 0} \) is intrinsically ultracontractive if \( \theta > 1 \).

2. Suppose that there exist constants \( \theta, s_0, c_1, c_2, c_3 > 0 \) and \( \gamma \in (0, \infty] \) such that
   \[
   J(x, y) \geq c_1 e^{-c_2|x - y|^\gamma}, \quad |x - y| > 1,
   \]
   and
   \[
   f(s) \leq c_3 \Phi^{-1}(s^{-\theta}), \quad s \geq s_0.
   \]
   Then \( (P_t^{D_f})_{t \geq 0} \) is intrinsically ultracontractive if \( \theta > \gamma \).

3. Suppose that \( \gamma \in (1, \infty] \) in (5.6) and that
   \[
   c_3 s^{-\eta} \leq f(s) \leq c_4 \Phi^{-1}(s^{-\theta}), \quad s \geq s_0
   \]
   for some constants \( c_3, c_4, s_0, \theta, \eta > 0 \). Then \( (P_t^{D_f})_{t \geq 0} \) is intrinsically ultracontractive if \( \theta > 1 \).

4. Suppose that \( \gamma \in (1, \infty] \) in (5.6) and that
   \[
   c_3 \exp(-c_4 s^{\theta_1}) \leq f(s) \leq c_3 \exp(-c_5 s^{\theta_2}), \quad s \geq s_0
   \]
   for some constants \( c_3, c_4, c_5, c_6, s_0 > 0 \) and \( \theta_1 \geq \theta_2 > 0 \). Then \( (P_t^{D_f})_{t \geq 0} \) is intrinsically ultracontractive.

**Proof.** (1) By (5.5), for \( s > 0 \) small enough, it holds that
   \[
   f^{*-1} \circ \Phi^{-1}(s) \leq c_1 \exp\left(c_2 s^{-1/\theta}\right).
   \]
   From the above estimate, (1.4), (3.2), (5.1) and the assumption (5.4), we know that (4.3) holds with
   \[
   \beta(s) \leq c_3 \exp\left(c_1 s^{-1/\theta}\right), \quad 0 < s < s_1
   \]
   for some \( s_1 > 0 \) small enough. Hence, it is easy to see that when \( \theta > 1 \), (4.5) is satisfied, which shows that \( (P_t^{D_f})_{t \geq 0} \) is intrinsically ultracontractive.
(2) By (5.7), it is easy to verify that for \( s > 0 \) small enough
\[
(5.9) \quad f_s^{-1} \circ \Phi^{-1}(s) \leq c_5s^{-1/\theta}.
\]
Combining this with (5.1), (1.4), (3.2) and the assumption (5.6) with \( \gamma \in (0, 1] \), we get that (4.3) holds
\[
\beta(s) \leq c_6 \exp \left( c_\gamma s^{-\gamma/\theta} \right), \quad 0 < s < s_1
\]
for some \( s_1 > 0 \) small enough. Obviously \( \theta > \gamma \) implies (4.5), which shows that \((P_t^{D_i})_{t \geq 0}\) is intrinsically ultracontractive if \( \theta > \gamma \).

(3) Note that the inequality (5.9) is still true. Then, according to (5.2), we have the following estimate for the rate function \( \beta(s) \) in (4.3) — there exist constants \( s_1 > 0 \) and \( c_i > 0 \) \((i = 8, 9)\) such that
\[
\beta(s) \leq c_8 \exp \left[ c_9 s^{-1/\theta} \left( \log \frac{1}{s} \right) \right], \quad 0 < s < s_1,
\]
where the first inequality in (5.8) was used. Hence, (4.5) holds if \( \theta > 1 \), which proves that \((P_t^{D_i})_{t \geq 0}\) is intrinsically ultracontractive when \( \theta > 1 \).

(4) The proof is based on Proposition 5.2 as that of (3), and we can see that there exist constants \( s_1 > 0 \) and \( c_i > 0 \) \((i = 10, \ldots, 17)\) such that for all \( 0 < s < s_1 \)
\[
\beta(s) \leq c_{10} (\Phi^{-1}(s) \land 1)^{-d} \Phi(\Phi^{-1}(s) \land 1)^{-2}
\]
\[
\times \exp \left[ c_{11} \left( \log \frac{1}{\Phi^{-1}(s)} \right)^{1/\theta_2} \log \left( \frac{c_{12} \log(1/\Phi^{-1}(s))^{1/\theta_2}}{\exp(-c_{13} (\log(1/\Phi^{-1}(s)))^{\theta_1/\theta_2})} \right) \right]
\]
\[
\leq c_{14} \exp \left[ c_{15} \left( \log \frac{1}{\Phi^{-1}(s)} \right)^{(1+\theta_1)/\theta_2} \right]
\]
\[
\leq c_{16} \exp \left[ c_{17} \left( \log \frac{1}{s} \right)^{(1+\theta_1)/\theta_2} \right],
\]
where in the second inequality we used (1.4) and (3.2), and the last inequality follows from (1.4) again. The rate function \( \beta \) above satisfies (4.5), which yields that \((P_t^{D_i})_{t \geq 0}\) is intrinsically ultracontractive. \(\square\)

Thanks to the milder assumptions in Corollary 5.3, we can obtain sufficient conditions for intrinsic ultracontractivity of \((P_t^{D_i})_{t \geq 0}\) for a class of jumping processes with variable orders as follows.

\textbf{Example 5.4.} Suppose that a function \( \alpha : \mathbb{R}^d \to (0, 2) \) satisfies \( 0 < \underline{\alpha} \leq \alpha(x) \leq \overline{\alpha} < 2 \) and
\[
|\alpha(x) - \alpha(y)| \leq c_1 \log^{-1} \left( \frac{2}{|x - y|} \right), \quad |x - y| < 1
\]
for some positive constant \( c_1 \). We consider the non-local symmetric Dirichlet form \((\mathcal{E}', \mathcal{F})\) given by (1.2), and suppose that the jumping kernel \( J(x, y) \) satisfies
\[
c_2 \left( \frac{1}{|x - y|^{d+\alpha(x)}} \wedge \frac{1}{|x - y|^{d+\alpha(y)}} \right) \leq J(x, y)
\]
\[
\leq c_3 \left( \frac{1}{|x - y|^{d+\alpha(x)}} \vee \frac{1}{|x - y|^{d+\alpha(y)}} \right)
\]
for all \(x, y \in \mathbb{R}^d\) and some positive constants \(c_2, c_3\). Then, according to [5, Example 2.3 and Theorem 3.5], there exists a symmetric Hunt process \(X = (X_t, t \geq 0; \mathbb{P}^x, x \in \mathbb{R}^d)\) on \(\mathbb{R}^d\) associated with \((\mathcal{E}', \mathcal{F})\), and the process \(X\) possesses the transition density function (i.e., heat kernel) \(p(t, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+\) so that \(p(t, x, y)\) is jointly continuous on \((t, x, y)\). Following the argument of [10, Proposition 8], one can prove that \(p(t, x, y)\) is strictly positive for every \(t > 0\) and \(x, y \in \mathbb{R}^d\).

Let \(D_f\) be a horn-shaped region such that \(D_f^\delta\) is \(\kappa\)-fat. Since \(D_f\) is connected, it is easy to verify that the associated Dirichlet heat kernel \(p^{D_f}(t, \cdot, \cdot)\) is bounded, continuous and strictly positive on \(D_f \times D_f\) for every \(t > 0\), see e.g. Proposition 7.1 and [10, Corollary 7 and Remark 8 (2)]. Therefore, all the assumptions in Subsection 1.1 are fulfilled in this setting.

It is clear that \(J_{\Phi, >}\) holds with \(\Phi(r) = r^\alpha\). On the other hand, according to Example 3.3, Exit\(\Phi, >\) holds with \(\Psi(r) = r^{1/(1-\alpha)}\). In fact, according to [5, Theorem 2.1 and Example 2.3] and the continuity assumption on \(\alpha(x)\), we can obtain that Exit\(\Phi, >\) holds with \(\Psi(r) = r^\alpha\). Now, according to Corollary 5.3 (1), we know that \((P_t^{D_f})_{t \geq 0}\) is intrinsically ultracontractive, if there are constants \(\theta > 1/\alpha\) and \(c_0, s_0 > 0\) such that for all \(s \geq s_0\),

\[
\Phi(s) \leq c_0 \log^{-\theta} s.
\]

For the remaining part of this section, we consider the regular Dirichlet form \((\mathcal{E}', \mathcal{F})\) whose jumping kernel \(J(x, y)\) given by (1.6). (Note that we do not assume that Assumptions (\(\mathcal{K}_n\)) and (SD) hold here.) As mentioned in Subsection 1.2, associated with \((\mathcal{E}', \mathcal{F})\) there is a Hunt process \(X\) on \(\mathbb{R}^d\), who has a transition density function \(p(t, x, y)\) with respect to the Lebesgue measure satisfying that for every \(t > 0\), \(p(t, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)\) is bounded, continuous and strictly positive. According to [16, Lemma 2.5] and [16, Theorem 2.4 (ii)], we know that for every open set \(D\) both Exit\(\Phi, >\) and NDHK\(\Phi, >\) hold \(D\) with \(r_0 = 1\). Furthermore, if \(\gamma = 0\), then, by [14, Theorem 1.2] (for the case \(\gamma_1 = \gamma_2 = 0\) in [14, (1.12)]), ODHK\(\Gamma, \leq\) holds with \(t_0 = 1\) and

\[
(5.10) \quad \Gamma(s) = s^{-d} \Phi(s)^{-1};
\]

if \(\gamma > 0\), then, by [17, (1.13) and (1.16) in Theorem 1.2 and (1.20) in Theorem 1.4], ODHK\(\Gamma, \leq\) holds with \(t_0 = 1\) and

\[
(5.11) \quad \Gamma(s) = \exp \left( -c(1 + s)^{\gamma \wedge 1} \log^{(\gamma-1)/\gamma}(1+s) \right)
\]

for some constant \(c > 0\).

Now, we can prove the assertions for the intrinsic ultracontractivity of \((P_t^{D_f})_{t \geq 0}\) in Theorems 1.1 and Theorem 1.2.

**Proofs of Theorems 1.1(1)(a) and 1.1(2)(a).** The sufficiency of the intrinsic ultracontractivity of \((P_t^{D_f})_{t \geq 0}\) can be easily seen from Corollary 5.3(1)–(3). So, one only need to verify the necessity of the corresponding assertions.

(1) Suppose that \(\gamma = 0\) and that (1.7) holds with \(\theta \leq 1\). Then, by the definition of horn-shaped region,

\[
\Phi(\delta_{D_f}(x_n)) \geq c_1 \log^{-\theta} n
\]
for \( x_n := (n, 0) \) and \( n \geq 1 \) large enough. This along with (5.10) implies that (4.6) does not hold. Thus, by Proposition 4.7, \( (P_t^D)_{t \geq 0} \) is not intrinsically ultracontractive.

(2) Suppose that \( \gamma \in (0, \infty] \) and that (1.9) holds with \( \theta \leq \gamma \wedge 1 \). Then,

\[
\Phi(\delta_D(x_n)) \geq c_1 n^{-\theta}
\]

for \( x_n := (n, 0) \) and \( n \geq 1 \) large enough. Combining this with (5.11) and (4.6), we can see from Proposition 4.7 that \( (P_t^D)_{t \geq 0} \) is not intrinsically ultracontractive. \( \square \)

Proof of Theorem 1.2(a). This immediately follows from Corollary 5.3 (4). \( \square \)

5.2. Unbounded and disconnected open set with locally \( \kappa \)-fat property. Recall that we consider the regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) whose jumping kernel \( J(x, y) \) given by (1.6). In this part, let \( h : [0, \infty) \rightarrow [12, \infty) \) be a measurable function such that \( \lim_{s \rightarrow \infty} h(s) = \infty \). Define

\[
D_0 = \bigcup_{n=1}^{\infty} \bigcup_{m=0}^{[h(n)]-1} \left( n + \frac{m}{[h(n)]}, n + \frac{m+1/2}{[h(n)]} \right),
\]

\[
D = \{ x \in \mathbb{R}^d : |x| \in D_0 \text{ or } |x| \leq 1 \}.
\]

The construction of the open set \( D \) above is partially inspired by [36, Example 4].

It is easy to see that for each \( n \geq 1 \), the set \( \overline{D^c} \cap \{ x \in \mathbb{R}^d : n < |x| < n + 1 \} \) is \( (\kappa_{\frac{1}{4}[h(n)]}) \)-fat at every point of \( \overline{D^c} \cap \{ x \in \mathbb{R}^d : n < |x| < n + 1 \} \), and \( \delta_D(x) \leq \frac{1}{[h(n)]} \) for all \( x \in D \) with \( n < |x| < n + 1 \). This along with (2.9) yields that for all \( n \geq 0 \),

\[
V_D(x) \geq c_1 \Phi(\delta_D(x))^{-1} \geq c_2 \Phi(h(n)^{-1})^{-1}, \quad x \in D \text{ and } n < |x| < n + 1.
\]

In particular, due to the fact that \( \lim_{n \rightarrow \infty} h(n) = \infty \), (2.8) holds, and so \( (P_t^D)_{t \geq 0} \) is compact. On the other hand, by Propositions 7.1, 7.2 and 7.3 in the Appendix, we know that the associated ground state \( \phi_1 \) can be chosen to be bounded, continuous and strictly positive on \( D \).

The next result illustrates again that our results for the intrinsic ultracontractivity of Dirichlet semigroups are optimal in some sense.

Theorem 5.5. Consider the regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) whose jumping kernel \( J(x, y) \) given by (1.6). Let \( D \) be the open set defined by (5.12). Then, we have the following statements.

1. Suppose that \( \gamma = 0 \) in (1.5) and

\[
h(s) \simeq \frac{1}{\Phi^{-1}(\log^{-\gamma}s)}, \quad s \geq 2
\]

for some \( \theta > 0 \). Then \( (P_t^D)_{t \geq 0} \) is intrinsically ultracontractive if and only if \( \theta > 1 \).

2. Suppose that \( \gamma \in (0, \infty] \) in (1.5) and

\[
h(s) \simeq \frac{1}{\Phi^{-1}(s^{-\theta})}, \quad s \geq 1
\]

for some \( \theta > 0 \). Then \( (P_t^D)_{t \geq 0} \) is intrinsically ultracontractive if and only if \( \theta > \gamma \wedge 1 \).
Proof. (1) Suppose that \( \gamma = 0 \) and (5.14) holds. Since \( \lim_{x \in D} \delta_D(x) = 0 \) and \( \text{Exit}_{\Phi,\gamma} \) holds, it follows from (3.4) that there is a constant \( c_1 > 0 \) such that for all \( x \in D \)

\[
\phi_1(x) \geq c_1 \Phi(\delta_D(x)) \frac{1}{|x|^d \Phi(|x|)}.
\]

Applying this estimate and (5.13) into (4.4), we get that the intrinsic super Poincaré inequality (4.3) holds with the rate function \( \beta(s) \) as follows

\[
\beta(s) \leq c_2 \inf \left\{ \frac{1}{|x|^d \Phi(|x|)} \left( \Phi^{-1}(s) \wedge r \right)^{-d} \inf_{x \in D_{R+1}} \Phi_2'(x) : V_D(x) \geq 1/s \right\} \leq c_4 \inf \left\{ \frac{1}{|x|^d \Phi(|x|)} \left( \Phi^{-1}(s) \wedge r \right)^{-d} |\Phi'_2(R)|^2 \phi_2(R)^2 : c_5 \Phi(R) \leq s \right\}.
\]

In the infimum above taking \( r = c_6 \Phi^{-1}(s) \) and \( R = \exp(c_7 s^{-1/\theta}) \) for \( s > 0 \) small enough and with some suitable positive constants \( c_6, c_7 \) (thanks to (1.4) and (5.14)), we arrive at

\[
\beta(s) \leq c_8 \exp(c_9 s^{-1/\theta}), \quad 0 < s \leq s_0
\]

for some constant \( s_0 > 0 \). This implies that when \( \theta > 1 \), the rate function above satisfies (4.5), hence \( (P^D_t)_{t \geq 0} \) is intrinsically ultracontractive.

If (5.14) holds with \( \theta = 1 \), then, by the definition of \( D \), we can find a sequence \( \{x_n\}_{n=2}^\infty \subset D \) such that \( n < |x_n| < n + 1 \) and

\[
\delta_D(x_n) \geq \frac{1}{4|h(n)|} \geq c_9 \Phi^{-1}\left( \frac{1}{\log \theta n} \right).
\]

This together with (1.4), (5.10) and the fact that \( \theta \leq 1 \) shows that (4.6) does not hold. Thus, by Proposition 4.7, we know \( (P^D_t)_{t \geq 0} \) is not intrinsically ultracontractive.

(2) We first consider the case \( \gamma > 1 \). Assume that (5.15) holds. For every \( x \in D \) with \( |x| \geq 4 \), let \( n = \lfloor 2|x| \rfloor \), \( x^{(n)} = x \), \( x^{(0)} = 0 \), \( x^{(1)} = \frac{1}{2} \cdot \frac{x}{|x|} \) and

\[
x^{(i)} := \left( \left\lfloor \frac{i}{2} \right\rfloor + \frac{1+2(i-2)}{4} \cdot h\left( \left\lfloor \frac{i}{2} \right\rfloor \right) \right) \cdot \frac{x}{|x|} \in D, \quad 2 \leq i \leq n - 1.
\]

Since

\[
2 \leq \frac{\left\lfloor \frac{i}{2} \right\rfloor}{|a|} \leq \frac{9}{11}, \quad \frac{1}{6} \leq \frac{\left\lfloor \frac{i}{2} \right\rfloor}{|a|} \leq \frac{3}{11}
\]

for all \( a \geq 12 \), using the fact \( h \geq 12 \), we see that \( \frac{1}{6} \leq |x^{(i-1)} - x^{(i)}| \leq \frac{7}{8} \) for all \( 1 \leq i \leq n \) and \( \delta_D(x^{(0)}) \geq \frac{1}{4|h(\lfloor 1/2 \rfloor)|} \) for all \( 0 \leq i \leq n - 1 \). Then, taking such \( \{x^{(i)}\}_{0 \leq i \leq n} \) into (3.7), we obtain that for all \( x \in D \) with \( |x| \) large enough

\[
\phi_1(x) \geq c_{10} \Phi(\delta_D(x)) \exp \left( -c_{11} \left( n \log n + \sum_{i=0}^{n-1} \log \left( \left\lfloor \frac{i}{2} \right\rfloor \right) \right) \right)
\]

\[
\geq c_{12} \Phi(\delta_D(x)) \exp \left( -c_{13} |x| \log |x| \right),
\]

where in the last inequality we used the following fact deduced from (5.15) and (1.4) that

\[
\log \left[ h\left( \left\lfloor \frac{i}{2} \right\rfloor \right) \right] \leq c_{14} \log \frac{1}{\Phi^{-1}\left( (i+1)^{-\theta} \right)} \leq c_{15} \log (e + i), \quad 0 \leq i \leq n.
\]
Furthermore, according to the lower bound estimate for $\phi_1$ above and (5.13), we know the intrinsic super Poincaré inequality (4.3) holds with

$$\beta(s) \leq c_{16} \inf \left\{ \frac{1}{\phi_1^2(x)} \left( \inf_{x \in D_{R+1, r/2}} \Phi^{-1}(s) \land \frac{1}{2} \right)^{-d} : x \in D_{1}, \phi_1(x) \geq 1/s \text{ and } r \in (0, r_0/2) \right\}$$

$$\leq c_{16} \inf \left\{ \frac{1}{\phi^2(r)} \left( \Phi^{-1}(s) \land \frac{1}{2} e^{-c_{17} R \log R} \right)^{-d} : c_{18} \Phi(r \vee h(R)^{-1}) \leq s \text{ and } r \in (0, r_0/2) \right\}.$$ 

Therefore, in the infimum above choosing $r = c_{19} \Phi^{-1}(s)$ and $R = c_{20} s^{-1/\theta}$ for $s > 0$ small enough and with some suitable positive constants $c_{19}, c_{20} > 0$ (thanks to (1.4) and (5.15)), we will arrive at

$$\beta(s) \leq c_{21} \exp \left( c_{22} s^{-1/\theta} \log |s| \right), \quad 0 < s \leq s_0$$

for some $s_0 \in (0, 1)$. In particular, when $\theta > 1$, (4.5) holds, and so $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive.

Assume (5.15) holds with $\theta \leq 1$. Then, we can find a sequence $\{x_n\}_{n=1}^\infty \subset D$ such that $n < |x_n| < n + 1$ and

$$\delta_D(x_n) \geq \frac{1}{4|\log(n)|} \geq c_{23} \Phi^{-1}(n^{-\theta}).$$

This combined with (5.11) shows that (4.6) does not hold, thanks to $\theta \leq 1$. Hence, according to Proposition 4.7, we know that $(P_t^D)_{t \geq 0}$ is not intrinsically ultracontractive.

Next we consider the situation that $\gamma \leq 1$. In this case, we can directly apply (3.4) to derive that

$$\phi_1(x) \geq c_{24} \Phi(\delta_D(x)) \exp \left( -c_{25} |x|^{\gamma} \right)$$

holds for all $x \in D$ with $|x|$ large enough.

Using (5.13), (5.15) and (5.16), and following the same argument as above, we can obtain that the intrinsic super Poincaré inequality (4.3) holds with

$$\beta(s) \leq c_{26} \exp \left( c_{27} s^{-\gamma/\theta} \right), \quad 0 < s \leq s_0.$$ 

If $\theta > \gamma$, then (4.5) holds, and so $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive.

If (5.15) holds with $\theta \leq \gamma$, then, as the same procedure as above, we can show that

$$\limsup_{x \in D \text{ and } |x| \to \infty} \Phi(\delta_D(x)) |x|^\gamma > 0.$$ 

So (4.6) does not hold, and, by Proposition 4.7, $(P_t^D)_{t \geq 0}$ is not intrinsically ultracontractive.

\begin{remark}
We close this section with some comments on our approaches on the compactness and the intrinsic ultracontractivity of $(P_t^D)_{t \geq 0}$. Throughout the arguments up to this section, we make use of abstract assumptions like $\text{J}_{\Phi, \gamma}$, Exit$_{\Phi, \gamma}$, NDHK of $\text{ODHK}_F$, and $\text{ODHK}_P$. Such assumptions have been used in [19] to study heat kernel estimates for non-local Dirichlet forms on general metric measure spaces. In fact, the arguments above do not heavily depend on the characteristics of Euclidean space. We believe that our methods above can be used to study the related topics for non-local Dirichlet forms on general metric measure spaces.
\end{remark}
6. TWO-SIDED ESTIMATES FOR GROUND STATE ON HORN-SHAPED REGIONS

This section, as a continuation of Subsection 5.1, is devoted to establishing two-sided estimates for ground state on horn-shaped regions. We concentrate on the regular Dirichlet form \((E', \mathcal{F})\) with jumping kernel \(J(x, y)\) given by (1.6). Let \(D_f\) be a horn-shaped region with the reference function \(f\). The associated Dirichlet semigroup is denote by \((P_t^{D_f})_{t \geq 0}\).

In order to obtain explicit estimates for \(\phi_1\), we need additionally assumptions on the coefficient function \(\kappa(x, y)\) and the scaling function \(\Phi(r)\) of jumping kernel \(J(x, y)\) and the reference function \(f\) of horn-shaped region \(D_f\). Recall that a function \(g \in C^1(0, \infty)\) is a \(C^{1,1}\) function, if there is a constant \(c_g > 0\) satisfying \(\|g'\|_\infty \leq c_g\) and \(|g'(s) - g'(t)| \leq c_g|s - t|\) for all \(s, t > 0\). Throughout this section, the jumping kernel \(J(x, y)\) is given by (1.6) and we further assume

1. Assumptions \((K_\eta)\) and \((SD)\) hold;
2. The reference function \(f\) of \(D_f\) is a \(C^{1,1}\) function;
3. The semigroup \((P_t^{D_f})_{t \geq 0}\) is intrinsically ultracontractive.

Note that, \(f \in C^{1,1}\) implies that \(D_f\) is a \(\kappa\)-fat set. Thus the semigroup \((P_t^{D_f})_{t \geq 0}\) is compact. Furthermore, by Propositions 7.1, 7.2 and 7.3, the ground state \(\phi_1\) is bounded, continuous and strictly positive on \(D_f\). We also emphasize here that we do not assume neither \(f\) is non-decreasing nor \(\lim_{r \to \infty} f'(r) = 0\). Such assumptions \(f\) were used in [37, (A1) and (A2) in p. 382] to study lower bound estimates of ground state of killed Brownian motion on horn-shaped region.

The following is the main result in this section. Recall that \(f^*(r) = \sup_{s \geq r} f(s)\) and \(f_*(r) = \inf_{1 \leq s \leq r} f(s)\).

**Theorem 6.1.** Under the setting and all the assumptions above, we have the following statements.

1. If \(\gamma = 0\) in (1.5), then there are constants \(c_1, c_2 > 0\) such that for all \(x \in D_f\) with \(|x|\) large enough,
   \[
   c_1 \Phi(\delta_{D_f}(x))^{1/2} \Phi(f_*(x_1 + 1))^{1/2} |x|^{-d} \Phi(|x|)^{-1} \leq \phi_1(x) \leq c_2 \Phi(\delta_{D_f}(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} |x|^{-d} \Phi(|x|)^{-1}.
   \]

2. If \(\gamma \in (0, 1]\) in (1.5), then there are constants \(c_i > 0\) \((i = 1, \ldots, 4)\) such that for all \(x \in D_f\) with \(|x|\) large enough,
   \[
   c_1 \Phi(\delta_{D_f}(x))^{1/2} \Phi(f_*(x_1 + 1))^{1/2} \exp\left(-c_2 |x|^\gamma\right) \leq \phi_1(x) \leq c_3 \Phi(\delta_{D_f}(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \exp\left(-c_4 |x|^\gamma\right).
   \]

3. If \(\gamma \in (1, \infty]\) in (1.5), then, for any increasing function \(g(r)\) satisfying that \(c_0 \log^{1/\gamma} r \leq g(r) \leq r/4\) for all \(r > 0\) large enough and some \(c_0 > 0\), there are constants \(c_i > 0\) \((i = 1, \ldots, 4)\) such that for all \(x \in D_f\) with \(|x|\) large enough,
   \[
   \phi_1(x) \leq c_3 \Phi(\delta_{D_f}(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \times \exp\left[-c_4 \frac{|x|}{g(|x|)} \left(g(|x|)^\gamma \log \frac{1}{f^*(|x|/4)}\right)\right]
   \]

\[(6.1)\]
and
\[
\phi_1(x) \geq c_1 \Phi(\delta_{D_f}(x))^{1/2} \Phi(f_*(x_1 + 1))^{1/2} \\
\times \exp \left\{ -c_2 \left[ |x| \wedge \left( \frac{|x|}{g(|x|)} \left( g(|x|)^\gamma \vee \log \frac{1}{f_*(2|x|)} \right) \right) \right] \right\},
\]
(6.2)

Theorem 6.1 follows from Proposition 6.5 and Proposition 6.7 below, which are concerned with upper bound estimates and lower bound estimates of \( \phi_1 \), respectively. As an application of Theorem 6.1, here we present the second part of proofs of Theorem 1.1 and Theorem 1.2 about two-sided estimates for \( \phi_1 \).

**Proofs of Theorem 1.1(1)(b) and 1.1(2)(b).** Theorem 1.1(1)(b) and (2)(b) with \( \gamma \in (0, 1] \) follow from Theorem 6.1 (1) and (2), respectively. Concerning Theorem 1.1 (2)(b) with \( \gamma \in (1, \infty) \), we take \( g(r) = 4 \log^{1/\gamma}(e + r) \) in (6.1) and (6.2). Then, we can get the desired assertion. Note that, in two-sided estimates for \( \phi_1 \) in the statement of Theorem 1.1(2)(b), the factor \( \Phi(f(|x|))^{1/2} \) can be absorbed into the exponential term, since for \( x \in D_f \) such that \( |x| \) large enough
\[
c_1 \exp \left( -c_2 |x| \log^{(\gamma-1)/\gamma} |x| \right) \leq \Phi(f(|x|))^{1/2} \exp \left( -c_0 |x| \log^{(\gamma-1)/\gamma} |x| \right) \leq c_3 \exp \left( -c_4 |x| \log^{(\gamma-1)/\gamma} |x| \right).
\]
\[
\square
\]

**Proof of Theorem 1.2(b).** When \( \theta \geq \gamma \), taking \( g(r) = r/4 \) in (6.1) and (6.2), we can get (1.12). When \( \theta < 0, \gamma \), we choose \( g(r) = 4r^{\theta/\gamma} \) in (6.1) and (6.2) to obtain (1.12).

\[
\square
\]

6.1. **Upper bound estimates for ground state.** Let \( U \subset \mathbb{R}^d \) be an open set and let \( x \in \partial U \). We say that \( U \) is \( C^{1,1} \) near \( x \) if there exist a localization radius \( r_x > 0 \), a constant \( \lambda_x > 0 \), a \( C^{1,1} \)-function \( \varphi_x : \mathbb{R}^{d-1} \to \mathbb{R} \) satisfying \( \varphi_x(0) = 0 \), \( \nabla \varphi_x(0) = (0, \ldots, 0) \), \( \| \nabla \varphi_x \|_\infty \leq \lambda_x \), \( |\nabla \varphi_x(y) - \nabla \varphi_x(z)| \leq \lambda_x |y - z| \), and an orthonormal coordinate system \( C_{S_x} \) with its origin at \( x \) such that
\[
B(x, r_x) \cap U = \{ y = (y_1, \tilde{y}) \in B(0, r) \in C_{S_x} : y_1 > \varphi_x(\tilde{y}) \},
\]
where \( \tilde{y} := (y_2, \ldots, y_d) \). The pair \((r_x, \lambda_x)\) is called the \( C^{1,1} \) characteristics of \( U \) at \( x \). An open set \( U \subset \mathbb{R}^d \) is said to be a (uniform) \( C^{1,1} \) open set with characteristics \((R, \Lambda)\), if it is \( C^{1,1} \) near every boundary point of \( \partial U \) with the same characteristics \((R, \Lambda)\).

In order to obtain upper bound estimates for ground state \( \phi_1 \), we first present the following upper bound estimate for the expectation of exit time. In what follows, we denote by \( D = D_f \) for simplicity.

**Lemma 6.2.** There exists a constant \( c_1 > 0 \) such that for every \( x \in D \) with \( |x| \) large enough,
\[
\mathbb{E}_x \left[ \tau_{B(z_x, 1) \cap D} \right] \leq c_1 \Phi(\delta_D(x))^{1/2} \Phi(f_*(x_1 - 2))^{1/2},
\]
where \( z_x \in \partial D \) such that \( |x - z_x| = \delta_D(x) \).

**Proof.** Throughout this proof we will consider \( x \in D \) such that \( f_*(x_1 - 2) = 1/2 \) and \( \delta_D(x) \leq 1/2 \). Suppose that \( z_x = (z_1, \tilde{z}) \) with \( |\tilde{z}| = f(z_1) \). By a proper orthonormal
transformation of $\hat{x}$ in $\mathbb{R}^{d-1}$, we can assume that $z_x = (z_1, f(z_1), 0, \ldots, 0)$. Let $y = (z_1, f(z_1) + \delta_D(x), \ldots, 0)$. Then, $y \in D^c$ and $\delta_D(y) \leq |y - z_x| = \delta_D(x)$.

Suppose $y_x \in \partial D$ such that $|y - y_x| = \delta_D(y)$. Since $f$ is a $C^{1,1}$ function, due to the uniform exterior ball condition (see e.g. [18, p. 1309]), there is a constant $0 < \kappa < 1/2$ (which is independent of $x \in D$) such that for every $x \in D$ with $|x|$ large enough, $B(u_x, \kappa) \subset D^c$, where $u_x := y_x + \kappa \cdot (y - y_x)/|y - y_x|$. For every $z \in B(z_x, 1) \cap D$ with $|x|$ large enough,

$$|z - u_x| \leq |z - z_x| + |y - z_x| + |u_x - y| \leq 1 + \delta_D(x) + \kappa \leq 2,$$

and so $B(z_x, 1) \cap D \subseteq B(u_x, 2) \setminus B(u_x, \kappa) =: D_x$. Clearly $D_x$ is a $C^{1,1}$ domain with characteristics $(R_0, \Lambda)$, which are independent of $x$ when $|x|$ is large enough.

According to [27, Theorem 1.6], there exists a constant $c_1 > 0$ such that for every $x \in D$ with $|x|$ large enough and $z \in D_x$,

$$G_{D_x}(x, z) \leq c_1 \Phi(|z - x|) \left(1 + \frac{\Phi(\delta_D(x))}{\Phi(|z - x|)} \right)^{1/2}, \quad (6.4)$$

where $G_{D_x} : D_x \times D_x \to [0, \infty)$ denotes the Green function of the process $X$ on $D_x$. We want to emphasize that the constant $c_1$ above only depends on the characteristics $(R_0, \Lambda)$, and it is independent of $x$.

Using $B(z_x, 1) \cap D \subseteq D_x$ and applying (6.4), we find that

$$E_x[\tau_{B(z_x, 1) \cap D}] = \int_{B(z_x, 1) \cap D} G_{B(z_x, 1) \cap D}(x, z) \, dz \leq \int_{B(z_x, 1) \cap D} G_{D_x}(x, z) \, dz$$

(6.5)

$$\leq c_1 \int_{B(z_x, 1) \cap D} \frac{\Phi(|z - x|)}{|z - x|^d} \left(1 + \frac{\Phi(\delta_D(x))}{\Phi(|z - x|)} \right)^{1/2} \, dz \leq c_1 \Phi(\delta_D(x))^{1/2} \int_{B(z_x, 1) \cap D} \frac{\Phi(|z - x|)^{1/2}}{|z - x|^d} \, dz.$$

Since $y_x \in \partial B(u_x, \kappa) \subseteq \partial D_x$, by (1.4) we have

$$\Phi(\delta_D(x)) \leq \Phi(|z - x|) + |z_x - y| + |y - y_x| \leq \Phi(3\delta_D(x)) \leq c_2 \Phi(\delta_D(x)).$$

On the other hand, since $\delta_D(x) \leq 1/2$, we have $B(z_x, 1) \cap D \subseteq U_1^x \cup U_2^x$, where $U_1^x = B(x, f^*(x_1 - 2))$ and $U_2^x = \{z \in \mathbb{R}^d : f^*(x_1 - 2) \leq |z_x| \leq 2\} \cap D$. Thus,

$$\int_{B(z_x, 1) \cap D} \frac{\Phi(|z - x|)^{1/2}}{|z - x|^d} \, dz \leq \int_{U_1^x} \frac{\Phi(|z - x|)^{1/2}}{|z - x|^d} \, dz + \int_{U_2^x} \frac{\Phi(|z - x|)^{1/2}}{|z - x|^d} \, dz =: I_1 + I_2.$$

Using (1.4), we have

$$I_1 \leq c_3 \int_0^{f^*(x_1 - 2)} \frac{\Phi(r)^{1/2}}{r} \, dr$$

$$= c_3 \Phi(f^*(x_1 - 2))^{1/2} \int_0^{f^*(x_1 - 2)} \frac{\Phi(r)^{1/2}}{\Phi(f^*(x_1 - 2))^{1/2} r} \, dr \leq c_4 \Phi(f^*(x_1 - 2))^{1/2} \int_0^{f^*(x_1 - 2)} \frac{r^{\alpha/2 - 1}}{f^*(x_1 - 2) r^{\alpha/2}} \, dr$$

$$\leq c_6 \Phi(f^*(x_1 - 2))^{1/2}.$$
Moreover, note that the Lebesgue surface measure of $U^2_x \cap \{ z \in \mathbb{R}^d : |z - x| = r \}$ is not bigger than
\[
c_6 \max_{|z - x| \leq 2} f(s)^{d-1} \leq c_6 f^*(x_1 - 2)^{d-1}
\]
for any $r \leq 2$. Using (6.5) and the fact that $1 + \pi/2 < 2 \leq d$, we get
\[
I_2 \leq c_7 f^*(x_1 - 2)^{d-1} \int_{f^*(x_1 - 2)}^2 \Phi(r)^{1/2} dr
\]
\[
= c_8 \Phi(f^*(x_1 - 2))^{1/2} f^*(x_1 - 2)^{d-1} \int_{f^*(x_1 - 2)}^2 \Phi(r)^{1/2} \frac{\Phi(r)^{1/2}}{r^{\pi/2-d}} dr
\]
\[
\leq c_9 \Phi(f^*(x_1 - 2))^{1/2}.
\]
Combining all the estimates with (6.5) yields the desired conclusion (6.3).

The following two lemmas are needed, see [18, Lemma 1.10] or [27, Lemma 5.1] for the first one, and [27, Corollary 2.4] for the second one.

**Lemma 6.3.** Let $U_1$ and $U_3$ be two open subsets of an open set $U \subset \mathbb{R}^d$ such that $\text{dist}(U_1, U_3) > 0$, and let $U_2 = U \setminus (U_1 \cup U_3)$. Then, for every $t > 0$, $x \in U_1$ and $y \in U_3$,
\[
p^U(t, x, y) \leq \mathbb{P}(X_{B_1} \in U_2) \sup_{0 < s < t, z \in U_2} p^{U}(s, z, y)
\]
\[
+ (t \wedge E^{U}[\tau_{U_1}]) \sup_{u \in U_1, z \in U_3} J(u, z).
\]

**Lemma 6.4.** There exists a constant $c > 0$ such that for every open set $U$, $r \in (0, 1]$ and $y \in \mathbb{R}^d$,
\[
\mathbb{P}(X_{B(y, r)} \cap U) \leq \frac{c E^{U}[\tau_{B(y, r)} \cap U]}{\Phi(r)}, \quad x \in B(y, r/2) \cap U.
\]

Now, we can give upper bound estimates for the ground state $\phi_1$.

**Proposition 6.5.** The following three statements hold.

1. Suppose that $\gamma = 0$ in (1.5). Then there exists a constant $c_1 > 0$ such that for every $x \in D$ with $|x|$ large enough,
\[
\phi_1(x) \leq c_1 \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} |x|^{-d} \Phi(|x|)^{-1}.
\]

2. Suppose that $\gamma \in (0, 1]$ in (1.5). Then there exist constants $c_i > 0$ ($i = 1, 2$) such that for every $x \in D$ with $|x|$ large enough,
\[
\phi_1(x) \leq c_i \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \exp \left( -c_2 |x|^{\gamma} \right).
\]

3. Suppose that $\gamma \in (1, \infty]$ in (1.5). Then, for any increasing function $g(r)$ satisfying that $4 \leq r/g(r) \leq r/4$ for $r > 0$ large enough, there exist constants $c_i > 0$ ($i = 1, 2$) such that for every $x \in D$ with $|x|$ large enough,
\[
\phi_1(x) \leq c_i \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \times \exp \left\{ -\frac{c_2 |x|}{g(|x|)} \left[ g(|x|)^{\gamma} \wedge \log \left( 1 + \frac{1}{f^*(|x|/4)} \right) \right] \right\}.
\]
In particular, for $\gamma = \infty$, there exist constants $c_i > 0$ ($i = 3, 4$) such that for every $x \in D$ with $|x|$ large enough,

\begin{equation}
(6.10) \quad \phi_1(x) \leq c_3 \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \exp \left[ -c_4 |x| \log \left( 1 + \frac{1}{f^*(|x|/4)} \right) \right].
\end{equation}

**Proof.** Note that the intrinsic ultracontractivity of $(P^D_t)_{t \geq 0}$ implies that

\begin{equation}
\phi_1(x) \leq \frac{1}{c\phi_1(y_0)} p^D(1, x, y_0) \leq c' p^D(1, x, y_0), \quad x \in D,
\end{equation}

where $y_0 = (1, 0)$ and $c' > 0$ is a constant independent of $x \in D$. Therefore, in order to get the desired assertions it suffices to consider upper bound estimates of $p^D(1, x, y_0)$. Throughout the proof, we consider $x \in D$ with $|x|$ large enough such that $f^*(|x|/3) \vee f^*(x_1 - 2) \vee \delta_D(x) \leq 1/2$. Let $z_x \in \partial D$ such that $|z_x - x| = \delta_D(x)$, and set $U_x = B(z_x, 1) \cap D$.

(1) By [14, Theorem 1.2], it holds that for $t > 0$ and $x, y \in D$,

\begin{equation}
(6.11) \quad p^D(t, x, y) \leq p(t, x, y) \leq c_0 \left( \Phi^{-1}(t) \right)^{-d} \wedge \frac{t}{|x - y|^d \Phi(|x - y|)}.
\end{equation}

Recall $U_x = B(z_x, 1) \cap D$. Let $U_1 = U_x$, $U_2 = \{z \in D : |z - x| \leq |x - y_0|/2\} \setminus U_1$ and $U_3 = D \setminus (U_1 \cup U_2)$ in (6.6). Then, by (6.11), we have

\begin{equation}
(6.12) \quad \sup_{0 < s < 1, z \in U_2} p(s, z, y_0) \leq \sup_{0 < s < 1, |z - y_0| \geq |x - y_0|/2} p(s, z, y_0)
\end{equation}

and

\begin{equation}
(6.13) \quad \sup_{u \in U_1, z \in U_3} J(u, z) \leq \sup_{|u - z| \geq |x - z| - |x - u| \geq |x - y_0|/2 - 3|y - y_0|/3} J(u, z)
\end{equation}

Thus

\begin{equation}
\begin{aligned}
p^D(1, x, y_0) &\leq P^x(X_{\tau_{U_2}} \in U_2) \sup_{0 < s < 1, z \in U_2} p^D(s, z, y_0) \\
&\quad + \left( 1 \wedge E^x[\tau_{U_1}] \right) \sup_{u \in U_1, z \in U_3} J(u, z)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\leq c_2 E^x[\tau_{B(z_x, 1) \cap D}] \left( \sup_{0 < s < 1, z \in U_2} p(s, z, y_0) + \sup_{u \in U_1, z \in U_3} J(u, z) \right)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\leq c_3 \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} |x|^{-d} \Phi(|x|)^{-1},
\end{aligned}
\end{equation}

where the second inequality follows from (6.7) and in the third inequality we have used (6.3), (6.12) and (6.13).

(2) Suppose $\gamma \in (0, 1]$. Then, by [17, Theorem 1.2], for any $t \in (0, 1]$ and $x, y \in D$ with $|x - y| \geq 1$,

\begin{equation}
\begin{aligned}
p^D(t, x, y) &\leq p(t, x, y) \leq c_1 \exp \left( -c_2 |x - y|^{\gamma} \right).
\end{aligned}
\end{equation}

Using this instead of (6.11) and following the same argument as (1), One can get (6.8) immediately.

(3) Suppose $\gamma \in (1, \infty]$. In this part we need much more delicate arguments instead of applying (6.6) directly. The proof is a little long, and it is split into four steps.
(i) For any $x \in D$ with $|x|$ large enough such that $g(|x|) \wedge (|x|/g(|x|)) \geq 4$, define $V_k = \{ z \in D : |z - x| \leq k g(|x|) \}$, 
$$h_k = \sup_{0 < s \leq 1, z \in V_k} p^D(s, z, y_0), \quad 1 \leq k \leq N = \left\lfloor \frac{|x|}{4g(|x|)} \right\rfloor,$$
$h_{N+1} = 1$.

Recall that $U_x = B(z_x, 1) \cap D$. By the strong Markov property, we have
$$p^D(1, x, y_0) = \mathbb{E}^x[p^D(1 - \tau_{U_x}, X_{\tau_{U_x}}, y_0) 1_{\{\tau_{U_x} < 1\}}]$$
(6.14)
$$= \mathbb{E}^x[p^D(1 - \tau_{U_x}, X_{\tau_{U_x}}, y_0) 1_{\{\tau_{U_x} < 1\}} 1_{\{X_{\tau_{U_x}} \in V_1 \setminus U_x\}}] + \mathbb{E}^x[p^D(1 - \tau_{U_x}, X_{\tau_{U_x}}, y_0) 1_{\{\tau_{U_x} < 1\}} 1_{\{X_{\tau_{U_x}} \in D \setminus V_1\}}]$$
$$=: I_1 + I_2.$$

Note that, by (6.7),
$$I_1 \leq \mathbb{P}^x(X_{\tau_{U_x}} \in V_1 \setminus U_x) \sup_{0 < s \leq 1, z \in V_1} p^D(s, z, y_0) \leq c_1 \mathbb{E}^x[\tau_{U_x}] h_1.$$  (6.15)

On the other hand, since $D \setminus V_1 = \left( \bigcup_{k=2}^{N} V_k \setminus V_{k-1} \right) \cup V_{N+1}$, where $V_{N+1} := D \setminus V_N$, by the Lévy system (3.1) of the process $X$,
$$I_2 = \int_0^1 \int_{U_x} \int_{D \setminus V_1} p^{U_x}(s, x, y) J(y, z) p^D(1 - s, z, y_0) \, dz \, dy \, ds$$
$$= \sum_{k=2}^{N} \int_0^1 \int_{U_x} \int_{V_k \setminus V_{k-1}} p^{U_x}(s, x, y) J(y, z) p^D(1 - s, z, y_0) \, dz \, dy \, ds$$
$$+ \int_0^1 \int_{U_x} \int_{V_{N+1}} p^{U_x}(s, x, y) J(y, z) p^D(1 - s, z, y_0) \, dz \, dy \, ds$$
$$\leq \sum_{k=2}^{N} \left[ \left( \int_0^1 \int_{V_k \setminus V_{k-1}} p^{U_x}(s, x, y) \, dy \, ds \right) |V_k \setminus V_{k-1}| \sup_{y \in U_x, z \in V_k \setminus V_{k-1}} J(y, z) \right]$$
$$\times \sup_{0 < s \leq 1, z \in V_k \setminus V_{k-1}} p^D(s, z, y_0)$$
$$+ \left[ \int_0^1 \left( \int_{U_x} p^{U_x}(s, x, y) \, dy \right) \left( \int_D p^D(1 - s, z, y_0) \, dz \right) \, ds \right]$$
$$\times \sup_{y \in U_x, z \in V_{N+1}} J(y, z)$$
$$\leq \mathbb{E}^x[\tau_{U_x}] \sum_{k=2}^{N} \left( \sup_{y \in U_x, z \in V_k \setminus V_{k-1}} J(y, z) \right) |V_k \setminus V_{k-1}| h_k$$
$$+ \mathbb{E}^x[\tau_{U_x}] \sup_{y \in U_x, z \in V_{N+1}} J(y, z).$$

For any $z \in V_N$ and any $x \in D$ with $|x|$ large enough,
$$z_1 \geq x_1 - Ng(|x|) \geq x_1 - (|x|/4g(|x|)) + g(|x|)$$
$$= x_1 - |x|/4 - g(|x|) \geq x_1 - |x|/2 \geq 3|x|/3,$$
and so, by the definition of horn-shaped region,
\[
\max_{2 \leq k \leq N} |V_k \setminus V_{k-1}| \leq c_1 f^*(|x|/3)^{d-1}g(|x|) \leq c_1 2^{1-d}g(|x|).
\]
Since for every \(2 \leq k \leq N\), \(y \in U_x\) and \(z \in D \setminus V_k\),
\[
|y - z| \geq |x - z| - |x - y| \geq kg(|x|) - 2 \geq (k - 1)g(|x|),
\]
which, along with the assumption that \(\gamma > 1\), implies that for every \(2 \leq k \leq N\)
\[
\sup_{y \in U_x, z \in V_{k-1}} J(y, z) \leq c_2 \sup \left\{ e^{-c_0|y-z|^\gamma} : |y - z| \geq (k - 2)g(|x|) \right\}
\[
\leq c_2 \exp \left( -C_0(k - 2)g(|x|)^\gamma \right)
\]
and
\[
\sup_{y \in U_x, z \in V_{N+1}} J(y, z) \leq c_2 \sup \left\{ e^{-c_0|y-z|^\gamma} : |y - z| \geq (N - 1)g(|x|) \right\}
\[
\leq c_2 \exp \left( -C_0(N - 1)g(|x|)^\gamma \right).
\]
Applying all the estimates above into (6.16), we get that
\[
I_2 \leq c_3 E^x[\tau_{U_x}] \sum_{k=2}^{N+1} \exp \left( -C_1(k - 1)g(|x|)^\gamma \right) h_k,
\]
which together with (6.14) and (6.15) in turn yields that
\[
p^D(1, x, y_0)
\leq c_4 E^x[\tau_{U_x}] \sum_{k=1}^{N+1} \exp \left( -C_1(k - 1)g(|x|)^\gamma \right) h_k
\leq c_5 \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \sum_{k=1}^{N+1} \exp \left( -C_1(k - 1)g(|x|)^\gamma \right) h_k,
\]
where we have used (6.3) in the last inequality above.

(ii) For any \(1 \leq k \leq N\), let \(x_1^{(k)} = x_1 - 2 - kg(|x|)\). For any fixed \(u \in V_k\), let \(z_u\) be
a point on \(\partial D\) satisfying \(|u - z_u| = \delta_D(u) \leq 1/2\), and let \(U_u = B(z_u, 1) \cap D\). We first
observe that \(U_u \subset V_{k+1}\), and that for every \(k + 1 \leq l \leq N\), \(y \in U_u\) and \(z \in D \setminus V_l\)
\[
|y - z| \geq |x - z| - |x - u| - \delta_D(u) - |z_u - y|
\geq (l - k)g(|x|) - 2 \geq 2^{-1}(l - k)g(|x|).
\]
Hence, for \(k + 2 \leq l \leq N\),
\[
\sup_{y \in U_u, z \in V_l \setminus V_{l-1}} J(y, z) \leq c_2 \sup \left\{ e^{-c_0|y-z|^\gamma} : |y - z| \geq 2^{-1}(l - k - 1)g(|x|) \right\}
\leq c_2 \exp \left( -C_2(l - k - 1)g(|x|)^\gamma \right)
\]
and
\[
\sup_{y \in U_u, z \in V_{N+1}} J(y, z) \leq c_2 \sup \left\{ e^{-c_0|y-z|^\gamma} : |y - z| \geq 2^{-1}(N - k)g(|x|) \right\}
\leq c_2 \exp \left( -C_2(N - k)g(|x|)^\gamma \right).
Thus, following the same arguments to derive (6.17), we have that for any \( t \in (0, 1] \) and \( u \in V_k \),
\[
p^D(t, u, y_0) \leq P^u(X_{\tau_{u}} \in V_{k+1}) \sup_{0 < s \leq 1, z \in V_{k+1}} p^D(s, z, y_0)
\]
\[+ \sum_{l=k+2}^{N} \int_{0}^{t} \int_{U_l} \int_{V_l \setminus V_{l-1}} p^D(s, u, y) J(y, z) p^D(t - s, z, y_0) \, dz \, dy \, ds \]
\[+ \int_{0}^{t} \int_{U_l} \int_{V_{N+1}} p^D(s, u, y) J(y, z) p^D(t - s, z, y_0) \, dz \, dy \, ds \]
\[\leq c_6 E^u[\tau_{u}] \sum_{l=k+1}^{N+1} \left( \sup_{y \in U_l, z \in V_{l-1}} J(y, z) \right) |V_l \setminus V_{l-1}| h_l + \left( \sup_{y \in U_l, z \in V_{N+1}} J(y, z) \right) \]
\[\leq c_7 E^u[\tau_{u}] \sum_{l=k+1}^{N+1} \exp \left( - C_3(l - (k + 1)) g(|x|^{\gamma}) \right) h_l \]
\[\leq c_8 \Phi(f^*(x_1^{(k)})) \sum_{l=k+1}^{N+1} \exp \left( - C_3(l - (k + 1)) g(|x|^{\gamma}) \right) h_l.\]

Here in the last inequality above we have used (6.3), (1.4) and the facts that \( \delta_D(u) \leq f(u_1) \leq f^*(u_1) \) and \( u_1 \geq x_1 - k g(|x|) \) to get that
\[
E^u[\tau_{u}] \leq c_6' \Phi(\delta_D(u))^{1/2} \Phi(f^*(u_1 - 2))^{1/2}
\leq c_6' \Phi(f^*(u_1))^{1/2} \Phi(f^*(u_1 - 2))^{1/2} \leq c_7 \Phi(f^*(x_1^{(k)})).
\]

Therefore, for \( 1 \leq k \leq N \),
\[
h_k = \sup_{0 < s \leq 1, z \in V_k} p(s, z, y_0)
\leq c_8 \Phi(f^*(x_1^{(k)})) \sum_{l=k+1}^{N+1} \exp \left( - C_3(l - (k + 1)) g(|x|^{\gamma}) \right) h_l.
\]

(iii) Below, we will apply (6.18) into (6.17) to estimate the remaining \( h_k \) term for \( 2 \leq k \leq N - 1 \). Note that if we continue the procedure for \( l \) times, then it only remains \( h_i \) with index \( l + 1 \leq i \leq N \). For simplicity, we relabel and use again constants \( C, C_i > 0 \) \((i \geq 1)\) in the argument below without confusion. Thus
\[
\sum_{k=1}^{N+1} \exp \left( - C(k - 1) g(|x|^{\gamma}) \right) h_k
\leq C_1 \Phi(f^*(x_1^{(1)})) \sum_{l=2}^{N+1} \exp \left( - C(l - 2) g(|x|^{\gamma}) \right) h_l + \sum_{k=2}^{N+1} \exp \left( - C(k - 1) g(|x|^{\gamma}) \right) h_l
\]
\[= C_1 \Phi(f^*(x_1^{(1)})) h_2 + C_1 \Phi(f^*(x_1^{(1)})) \sum_{l=3}^{N+1} \exp \left( - C(l - 2) g(|x|^{\gamma}) \right) h_l
\]
\[+ \exp \left( - C g(|x|^{\gamma}) \right) h_2 + \sum_{k=3}^{N+1} \exp \left( - C(k - 1) g(|x|^{\gamma}) \right) h_k
\]
\[
\begin{align*}
&C_1^2 \Phi(f^*(x_1^{(1)})) \Phi(f^*(x_1^{(2)})) \sum_{l=3}^{N+1} \exp \left( -C(l-3)g(|x|)^\gamma \right) h_l \\
&+ C_1 \Phi(f^*(x_1^{(1)})) \sum_{l=3}^{N+1} \exp \left( -C(l-2)g(|x|)^\gamma \right) h_l \\
&+ C_1 \Phi(f^*(x_1^{(2)})) \exp \left( -Cg(|x|)^\gamma \right) \sum_{l=3}^{N+1} \exp \left( -C(l-3)g(|x|)^\gamma \right) h_l \\
&+ \sum_{k=3}^{N+1} \exp \left( -C(k-1)g(|x|)^\gamma \right) h_k \\
&\leq C_1^3 \Phi(f^*(x_1^{(1)})) \Phi(f^*(x_1^{(2)})) \Phi(f^*(x_1^{(3)})) h_4 \\
&+ C_1^2 \Phi(f^*(x_1^{(1)})) \Phi(f^*(x_1^{(2)})) \exp \left( -Cg(|x|)^\gamma \right) h_4 \\
&+ C_1^2 \Phi(f^*(x_1^{(2)})) \Phi(f^*(x_1^{(3)})) \exp \left( -Cg(|x|)^\gamma \right) h_4 \\
&+ C_1 \Phi(f^*(x_1^{(1)})) \exp \left( -2Cg(|x|)^\gamma \right) h_4 \\
&+ C_1 \Phi(f^*(x_1^{(2)})) \exp \left( -2Cg(|x|)^\gamma \right) h_4 \\
&+ C_1 \Phi(f^*(x_1^{(3)})) \exp \left( -2Cg(|x|)^\gamma \right) h_4 \\
&+ \cdots.
\end{align*}
\]

In the argument above and below we assume that $C_1 > 1$.

Note also that, by [17, (1.16) in Theorem 1.2],

\[
h_N = \sup_{0 < s \leq 1, z \in V_N} p(s, z, y_0) \\
\leq \sup_{0 < s < 1, |z - y_0| \geq |x - y_0| - |z - x| \geq |x - y_0|/2} p(s, z, y_0) \leq c_9 \exp(-c_{10}|x|) \leq c_9.
\]

and $h_{N+1} = 1$.

Therefore, repeating the procedure $N - 1$ times and using the notational convention that $\prod_{l=1}^{N} \Phi(f^*(x_1^{(k_l)})) = 1$, we can conclude from (6.17) that

\[
p^D(1, x, y_0) \\
\leq C_2 \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \\
\times \sum_{l=0}^{N} C_1^l \left( \sum_{1 \leq k_1 < k_2 \cdots < k_l \leq N} \prod_{i=1}^{l} \Phi(f^*(x_1^{(k_i)})) \right) \exp \left( -C_3(N - l)g(|x|)^\gamma \right).
\]

(iv) Since $f^*$ is non-increasing, we have for every $1 \leq k_1 < k_2 \cdots < k_l \leq N - 1$,

\[
\sum_{i=1}^{l} \log \left( 1 + \frac{1}{f^*(x_1^{(k_i)})} \right) \geq \sum_{i=N-l+1}^{N} \log \left( 1 + \frac{1}{f^*(x_1^{(i)})} \right) \geq c_{11} l \log \left( 1 + \frac{1}{f^*(|x|/4)} \right),
\]
where the last inequality follows from the fact that \( x_k^{(k)} \geq |x|/4 \) for every \( 1 \leq k \leq N \). The inequality above finally leads to the following estimate

\[
p^D(1, x, y_0) \leq C_4 C_1^N \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2}
\]

\[
\times \sup_{0 \leq l \leq N} \exp \left\{ - C_5 \left[ l \log \left( 1 + \frac{1}{f^*(|x|/4)} \right) + (N - l) g(|x|)^\gamma \right] \right\} \leq C_6 \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2}
\]

\[
\times \exp \left\{ - C_7 |x| \left[ g(|x|)^\gamma \wedge \log \left( e + \frac{1}{f^*(|x|/4)} \right) \right] \right\},
\]

where in the first inequality we used (1.4), and in the last inequality we have used the fact that \( N = \left\lfloor \frac{|x|}{4g(|x|)} \right\rfloor \). Therefore, we have now obtained (6.9). In particular, when \( \gamma = \infty \),

\[
\phi_1(x) \leq c' \Phi(\delta_D(x))^{1/2} \Phi(f^*(x_1 - 2))^{1/2} \exp \left\{ - \frac{c' |x|}{g(|x|)} \log \left( 1 + \frac{1}{f^*(|x|/4)} \right) \right\}.
\]

Thus, by taking \( g(r) = 4 \) for \( r > 0 \), we obtain (6.10).

6.2. Lower bound estimates for ground state. In this subsection, we turn to lower bound estimates of ground state \( \phi_1 \). We begin with the following lower bound estimate for the survival probability. Recall that \( D = D_f \) is a horn-shaped region.

**Lemma 6.6.** There exists a constant \( c > 0 \) such that for every \( x \in D \) with \( |x| \) large enough

\[
\mathbb{P}^x \left( \tau_{B(z, 1) \cap D} > \Phi(f_*(x_1 + 1)) \right) \geq \frac{c \Phi(\delta_D(x))^{1/2}}{\Phi(f_*(x_1 + 1))^{1/2}},
\]

where \( z \in \partial D \) such that \( |x - z| = \delta_D(x) \).

**Proof.** We assume that \( x \in D \) with \( |x| \) large enough such that \( |x| \geq 1 \) and \( f(x_1) \leq 1/4 \). Since \( f \) is a \( C^{1,1} \) function, due to the local interior ball condition (see [18, p. 1039]), there exist constants \( \kappa \in (0, 1) \) and \( c_0 > 0 \) such that, for all \( x \in D \) with \( |x| \) large enough one can choose a ball \( U := B(\xi_x, \kappa f_*(x_1 + 1)) \) satisfying that \( x \in U \subset B(z, 1) \cap D \) and \( c_0 \delta_D(x) < \delta_U(x) \leq \delta_D(x) \).

Let \( X^0 := (X^0_t)_{t \geq 0} \) (on the same probability space) be a symmetric jump process on \( \mathbb{R}^d \), whose jumping kernel \( J^0(x, y) \) is given by (1.6) with \( \gamma = 0 \), and \( \kappa \) and \( \Phi \) are same as those for jumping kernel \( J(x, y) \). One can regard \( X^0 \) as the process obtained by the Meyer’s construction (see [4, Remark 3.5]) through increasing the intensity of jumps for the process \( X \) larger than 1, so that \( X_t = X^0_t \) for any \( t \in (0, N_0^0) \), where

\[
N_0^0 := \inf \left\{ t > 0 : |X^0_t - X^0_{t-}| > 1 \right\}.
\]

In the following, for any subset \( A \subset \mathbb{R}^d \), let

\[
\tau_A^0 = \inf \left\{ t > 0 : X^0_t \notin A \right\}
\]

be the first exit time from \( A \) of the process \( X^0 \). Note that, under \( \mathbb{P}^x \) the event \( \{X^0_t \in U \text{ for any } t \in [0, \Phi(f_*(x_1 + 1))]\} \) implies that the process \( (X^0_t)_{t \geq 0} \) does not have any jump bigger than 1 in time interval \([0, \Phi(f_*(x_1 + 1))]\). That is, under \( \mathbb{P}^x \),

\[
\{\tau^0 > \Phi(f_*(x_1 + 1))\} \subseteq \{\tau_U > \Phi(f_*(x_1 + 1))\}.
\]
Therefore, for any $x \in D$ with $|x|$ large enough,
\begin{equation}
\mathbb{P}^x(\tau_{B(x,1)} \cap D > \Phi(f_*(x_1 + 1))) \geq \mathbb{P}^x(\tau_U > \Phi(f_*(x_1 + 1))) \geq \mathbb{P}^x(\tau_U^0 > \Phi(f_*(x_1 + 1))).
\end{equation}

Next, we choose an orthonormal coordinate system (CS) with origin at $z_e$ and, for $\delta > 0$, let $X^{(\delta)} := \{\delta^{-1}X^0_{\Phi(\delta)t} : t \geq 0\}$ be the scaled process of $X^0$. Define

$$
\kappa_{(\delta)}(x, y) = \kappa(\delta x, \delta y) \quad \text{and} \quad \Phi(\delta)(r) = \Phi(\delta r)/\Phi(\delta).
$$

Then the jumping kernel $J^{(\delta)}(x, y)$ of the process $X^{(\delta)}$ with respect to the Lebesgue measure is related to that of $X^0$ by the following formula

$$
J^{(\delta)}(x, y) = \delta^d \Phi(\delta)J^0(\delta x, \delta y) = \frac{\kappa_{(\delta)}(x, y)}{\Phi(\delta)(|x - y|)|x - y|^d}.
$$

Clearly, by (SD), $\Phi(\delta) \in C^1(0, \infty)$ and

$$
\kappa_{(\delta)}(x, y) = \kappa(\delta x, \delta y)
$$

and

$$
\Phi(\delta)(r) = \Phi(\delta r)/\Phi(\delta)
$$

is decreasing on $(0, \infty)$. It is also clear that, by (1.4), we have

$$
\left(\frac{R}{r}\right)^{\alpha} \leq \frac{\Phi(\delta)(R)}{\Phi(\delta)(r)} \leq \left(\frac{R}{r}\right)^{\alpha}
$$

for every $\delta > 0$, $0 < r \leq R$.

Finally, if $\delta \leq 1$ and $\eta > \pi/2$, then by $(K_\eta)$, for every $x, h \in \mathbb{R}^d$ with $|h| \leq 1$

$$
|\kappa_{(\delta)}(x, x + h) - \kappa_{(\delta)}(x)| = |\kappa(\delta x, \delta x + \delta h) - \kappa(\delta x, \delta x)| \leq L|\delta h|^\eta \leq L|h|^\eta.
$$

Therefore, for $\Phi(\delta)(r)$ and $\kappa_{(\delta)}(x, y)$ defined above, (1.4), $(K_\eta)$ and (SD) hold uniformly for all $\delta \leq 1$.

Since $f_*(x_1 + 1)^{-1}U = B\left(\frac{x}{f_*(x_1 + 1)}, \kappa\right)$, by using [27, Lemma 7.2] to $X^{(\delta_1)}$ with $\delta_1 = f_*(x_1 + 1)$ and (1.4), we have

$$
\mathbb{P}^x(\tau_U^0 > \Phi(f_*(x_1 + 1))) = \mathbb{P}^{x_1} \left(\tau_{\delta_1^{-1}U} > 1\right) \geq c_1 \Phi(\delta_1)(\delta_1^{-1}U(\delta^{-1}x))
$$

$$
= c_1 \Phi(\delta_1)^{-1/2} \Phi(\delta_1 \delta_1^{-1}U(\delta_1^{-1}x))^{1/2}
$$

$$
= c_1 \Phi(\delta_U(x))^{1/2} \Phi(\delta_1 \delta_1^{-1}U(\delta_1^{-1}x))^{1/2}
$$

$$
\geq c_2 \Phi(\delta_D(x))^{1/2} \Phi(f_*(x_1 + 1))^{1/2},
$$

where $\tau_{A}^{(\delta_1)}$ denotes the first exit time from $A$ for the process $X^{(\delta_1)}$. This together with (6.20) yields (6.19).

**Proposition 6.7.** The following two statements hold.

1. Suppose that $\gamma = 0$ in (1.5). Then there exists a constant $c_1 > 0$ such that for every $x \in D$ with $|x|$ large enough,

$$
\phi_1(x) \geq c_1 \Phi(\delta_D(x))^{1/2} \Phi(f_*(x_1 + 1))^{1/2} |x|^{-d} \Phi(|x|)^{-1}
$$

2. Suppose that $\gamma \in (0, \infty]$ in (1.5). Then, for any increasing function $g(r)$ satisfying that $4 \leq r/g(r) \leq r/4$ for $r > 0$ large enough, there exist constants $c_i > 0$ ($i = 1, 2, 3, 4$) such that for every $x \in D$ with $|x|$ large enough,

$$
\phi_1(x) \geq c_1 \Phi(\delta_D(x))^{1/2} \Phi(f_*(x_1 + 1))^{1/2} \exp \left( -c_2 |x|^\gamma \right)
$$
and
\[
\phi_1(x) \geq c_3 \Phi(\delta_D(x))^{1/2} \Phi(f_s(x_1 + 1))^{1/2} \times \exp \left[ -c_4 \frac{|x|}{g(|x|)} \left( g^\gamma(|x|) + \log \frac{|x|}{g(|x|)} + \log \left( 1 + \frac{1}{f_s(|x| + 1)} \right) \right) \right],
\]

(6.23)

Proof. Recall that Exit_{\Phi, r_0} holds with the constant \( r_0 = 1 \) in (3.3) in this setting (see e.g. [16, Lemma 2.5]). Let \( x_0 = (1, 0) \), \( r_0 = (\delta_D(x_0)/4) \wedge 1 \) and \( t_0 = c_0 \Phi(1) \), where \( c_0 \) is a positive constant \( C_0 \) in (3.3). We will always consider \( x \in D \) such that \( |x| \) is large enough and \( \Phi(f_s(x_1 + 1)) \leq t_0/4 \).

(1) The proof of (6.21) is based on that of Proposition 3.4 with some modifications. Set \( D_0 = B(x_0, 2r_0) \). According to Lemma 3.1, in order to prove the desired lower bound (6.21) of \( \phi_1 \), it suffices to verify the same lower bound (possibly with different constants) for \( P^D_{t_0} 1_{D_0}(x) \).

In the following, let \( D_0 = B(x_0, r_0) \). For any fixed \( x \in D \) with \( |x| \) large enough such that \( \Phi(f_s(x_1 + 1)) \leq t_0/4 \), let \( D_1 = B(z_x, 1) \cap D \), where \( z_x \in \partial D \) such that \( |x - z_x| = \delta_D(x) \). Then, using the strong Markov property and Exit_{\Phi, r_0}, and following the same argument of (3.5), we can get that

\[
P^D_{t_0} 1_{D_0}(x) \geq P^x \left( 0 < \tau_{D_1} \leq \Phi(f_s(x_1 + 1)), X_{\tau_{D_1}} \in \tilde{D}_0; \forall s \in [\tau_{D_1}, t_0], X_s \in D_0 \right)
\]

\[
\geq P^x \left( P^{X_{\tau_{D_1}}} (\tau_{D_0} > t_0); 0 < \tau_{D_1} \leq \Phi(f_s(x_1 + 1)), X_{\tau_{D_1}} \in \tilde{D}_0 \right)
\]

\[
\geq P^x \left( P^{X_{\tau_{D_1}}} (\tau_{B(X_{\tau_{D_1}} - r_0)} > t_0); 0 < \tau_{D_1} \leq \Phi(f_s(x_1 + 1)), X_{\tau_{D_1}} \in \tilde{D}_0 \right)
\]

\[
\geq c_1 P^x \left( 0 < \tau_{D_1} \leq \Phi(f_s(x_1 + 1)), X_{\tau_{D_1}} \in \tilde{D}_0 \right).
\]

(6.24)

According to the \( \text{Lévy} \) system (3.1), we have

\[
P^x \left( 0 < \tau_{D_1} \leq \Phi(f_s(x_1 + 1)), X_{\tau_{D_1}} \in \tilde{D}_0 \right)
\]

\[
= \int_0^{\Phi(f_s(x_1 + 1))} ds \int_{D_1} p^{D_1}(s, x, y) dy \int_{\tilde{D}_0} J(y, z) dz
\]

\[
\geq \left( \inf_{y \in D_1, z \in \tilde{D}_0} J(y, z) \right) \left| \tilde{D}_0 \right| \int_0^{\Phi(f_s(x_1 + 1))} ds \int_{D_1} p^{D_1}(s, x, y) dy \int_{\tilde{D}_0} J(y, z) dz
\]

\[
\geq c_2 |x|^{-d} \Phi(|x|)^{-1} \Phi(f_s(x_1 + 1)) P^x \left( \tau_{D_1} > \Phi(f_s(x_1 + 1)) \right)
\]

\[
\geq c_3 \Phi(\delta_D(x))^{1/2} \Phi(f_s(x_1 + 1))^{1/2} |x|^{-d} \Phi(|x|)^{-1},
\]

where in the second inequality we have used that for \( x \in D \) with \( |x| \) large enough

\[
\inf_{y \in D_1, z \in \tilde{D}_0} J(y, z) \geq \inf_{|y - z| \leq |x|} J(y, z) \geq c_4 |x|^{-d} \Phi(|x|)^{-1},
\]

and the last inequality follows from (6.19).

Combining (6.24) with (6.25), we prove (6.21).

(2) Since the proof of (6.22) is almost the same as (6.21), we omit it here. The proof of (6.23) is partly motivated by those of Propositions 3.6 and 5.2. We consider
$x \in D$ with $|x| \geq 2r_*$ for some $r_* > 0$ large enough. Set $n := \left\lceil \frac{x_1 - r_*}{g(|x|)} \right \rceil$, $x^{(i)} = (r_* + i(x_1 - r_*)/n, \tilde{0}) := (x_1^{(i)}, \tilde{0})$ and $r_i = (\delta_D(x^{(0)})/4) \wedge (\delta_D(x^{(i)})/4) \wedge 1$ for any $0 \leq i \leq n - 1$.

Define

$$D_n = B(z_{x_1}, 1) \cap D, \quad D_i = B(x^{(i)}, 2r_i), \quad \tilde{D}_i = B(x^{(i)}, r_i), \quad 0 \leq i \leq n - 1;$$

$$\tilde{\tau}_{D_n} = \tau_{D_n}, \quad \tilde{\tau}_{D_i} = \inf\{t > \tilde{\tau}_{D_{i+1}} : X_1 \notin D_i\}, \quad 0 \leq i \leq n - 1.$$  

Noting that for $0 \leq i \leq n - 2$,

$$\frac{g(|x|)}{2} \leq |x^{(i+1)} - x^{(i)}| = \frac{x_1 - r_*}{n} \leq g(|x|),$$

we can easily check that $0 < \operatorname{dist}(D_i, D_{i+1}) < g(|x|)$ for $0 \leq i \leq n - 1$, which in turn implies that

\begin{equation}
\inf_{u \in D_i, z \in D_{i+1}} J(u, z) \geq c_1 \exp \left( -c_2 g^7(|x|) \right), \quad 0 \leq i \leq n - 1.
\end{equation}

By using the strong Markov property and following the argument of (3.8), we have

$$P_{D_0}^D 1_{D_0}(x) \geq P^x\left(0 < \tilde{\tau}_{D_n} \leq \Phi(f_s(x_1 + 1)), 0 < \tilde{\tau}_{D_i} - \tilde{\tau}_{D_{i+1}} \leq \frac{c_0 \Phi(r_i)}{2n}, X_{\tilde{\tau}_{D_n}} \notin \tilde{D}_{n-1} \text{ for each } 1 \leq i \leq n - 1; \forall s \in [\tilde{\tau}_{D_0}, t_0] X_s \in D_0 \right)$$

$$= P^x\left(0 < \tau_{D_n} \leq \Phi(f_s(x_1 + 1)), X_{\tau_{D_n}} \notin \tilde{D}_{n-1} \right)$$

$$\times P^{X_{\tau_{D_n}}}(0 < \tau_{D_{n-1}} \leq \frac{c_0 \Phi(r_{n-1})}{2n}, X_{\tau_{D_{n-1}}} \notin \tilde{D}_{n-2})$$

$$\times P^{X_{\tau_{D_{n-1}}}}(\cdots \times P^{X_{\tilde{\tau}_{D_1}}}(0 < \tau_{D_1} \leq \frac{c_0 \Phi(r_1)}{2n}, X_{\tau_{D_1}} \in \tilde{D}_0)$$

$$\times P^{X_{\tilde{\tau}_{D_1}}(\forall s \in [0, t_0 \wedge \tau_{D_1}] X_s \in D_0)) \left(\cdots \right)).$$

For any $r > 0$, set $\check{f}(r) = \inf_{|s - r| \leq \check{f}(r)} f(s)$. It is clear that for $r > 0$ large enough, $\check{f}(r) \geq f_s(r)$. For every $2 \leq i \leq n - 1$, if $X_{\tilde{\tau}_{D_i}} \notin \tilde{D}_{i-1}$, then, by the Lévy system (3.1), we have

$$P^{X_{\tilde{\tau}_{D_i}}}(0 < \tau_{D_{i-1}} \leq \frac{c_0 \Phi(r_{i-1})}{2n}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2})$$

$$\geq \inf_{y \in \tilde{D}_{i-1}} P^y\left(0 < \tau_{D_{i-1}} \leq \frac{c_0 \Phi(r_{i-1})}{2n}, X_{\tau_{D_{i-1}}} \in \tilde{D}_{i-2}\right)$$

$$= \inf_{y \in \tilde{D}_{i-1}} \int_{D_{i-1}} dz \int_0^{\frac{c_0 \Phi(r_{i-1})}{2n}} p^{D_{i-1}}(s, y, z) ds \int_{\tilde{D}_{i-2}} J(z, u) du$$

$$\geq \left(\inf_{z \in \tilde{D}_{i-1}, a \in \tilde{D}_{i-2}} J(z, u) \right) |\tilde{D}_{i-2}| \inf_{y \in \tilde{D}_{i-1}} \int_0^{\frac{c_0 \Phi(r_{i-1})}{2n}} P^y(\tau_{D_{i-1}} > s) ds$$

$$\geq c_3 \frac{\Phi(r_{i-1})}{2n} \exp \left( -c_2 g^7(|x|) \right) \inf_{y \in \tilde{D}_{i-1}} P^y\left(\tau_{D_{i-1}} \geq \frac{c_0 \Phi(r_{i-1})}{2n}\right)$$

$$\geq c_4 \left(\check{f}(x^{(i-2)}) \wedge r_0\right)^d \Phi(\check{f}(x^{(i-1)}) \wedge r_0) g(|x|) / |x|$$
\[ \times \exp\left( -c_2 g^7(|x|) \right) \inf_{y \in \tilde{D}_{n-1}} \mathbb{P}^y \left( \tau_{B(y, r_{i-1})} > \frac{c_3 \Phi(r_{i-1})}{2} \right) \]
\[ \geq c_5 \left( \tilde{f}(x_1^{(i-1)} \cap r_0) d \Phi(\tilde{f}(x_1^{(i-1)}) \cap r_0) \exp \left( -c_6 g^7(|x|) + \log \frac{|x|}{g(|x|)} \right) \right), \]

where the third inequality is due to (6.26), in the forth inequality we have used the facts that \( n \leq c' |x|/g(|x|) \) and, by the definition of horn-shaped region, see e.g. (5.3),
\[ r_{i-2} = \delta_D(x_1^{(i-2)} \cap r_0) \geq c_7 \left( \tilde{f}(x_1^{(i-2)}) \cap r_0 \right), \]

and in the last one we used Exit \( \Phi_{MAX} \).

Furthermore, according to the Lévy system (3.1), (6.26) and (6.19), we know immediately that
\[ \mathbb{P}^z \left( 0 < \tilde{\tau}_{D_n} \leq \Phi(f_n(x_1 + 1)), X_{\tilde{\tau}_{D_n}} \in \tilde{D}_{n-1} \right) \]
\[ = \int_{D_n} dz \int_0^{\Phi(f_n(x_1+1))} p^{D_n}(s, x, z) ds \int_{\tilde{D}_{n-1}} J(z, u) du \]
\[ \geq c_8 r_{n-1}^2 \Phi(f_n(x_1 + 1)) \exp \left( -c_2 g^7(|x|) \right) \mathbb{P}^z \left( \tau_{B(z_1, r_{D})} > \Phi(f_n(x_1 + 1)) \right) \]
\[ \geq c_9 \left( \tilde{f}(x_1^{(n-1)}) \cap r_0 \right) d \Phi(\delta_D(x)) \left( \Phi(f(x_1 + 1)) \right)^{1/2} \exp(-c_2 g^7(|x|)). \]

On the other hand, using Exit \( \Phi_{MAX} \) again, if \( X_{\tilde{\tau}_{D_1}} \in \tilde{D}_0 \), then
\[ \mathbb{P}^{X_{\tilde{\tau}_{D_1}}} \left( \forall s \in [0, t_0 - \tilde{\tau}_{D_1}] X_s \in D_0 \right) \geq \inf_{y \in D_0} \mathbb{P}^y \left( \tau_{D_0} > t_0 \right) \]
\[ \geq \inf_{y \in D_0} \mathbb{P}^y \left( \tau_{B(y, r_0)} > t_0 \right) \geq c_{10}. \]

Combining all the estimates above, we arrive at
\[ P_{t_0}^{D_1} \mathbb{1}_{D_0}(x) \]
\[ \geq c_{11} \Phi(\delta_D(x))^{1/2} \Phi(f_n(x_1 + 1))^{1/2} \]
\[ \times \exp \left( -c_{12} n \left( g^7(|x|) + \log \frac{|x|}{g(|x|)} \right) \right) \prod_{i=1}^{n-1} \left( \tilde{f}(x_1^{(i-1)} \cap r_0) d \Phi(\tilde{f}(x_1^{(i)}) \cap r_0) \right) \]
\[ \geq c_{13} \Phi(\delta_D(x))^{1/2} \Phi(f_n(x_1 + 1))^{1/2} \]
\[ \times \exp \left[ -c_{14} \frac{|x|}{g(|x|)} \left( g^7(|x|) + \log \frac{|x|}{g(|x|)} + \log \frac{1}{\inf_{r_s \leq s \leq |x|+1} f(s)} \right) \right], \]

where the last inequality follows from (1.4) and the fact that
\[ \sum_{i=1}^{n-1} \log \frac{1}{\tilde{f}(x_1^{(i-1)})} \leq c_{16} \sum_{i=1}^n \log \left( e + \frac{1}{\inf_{r_s \leq s \leq |x|+1} f(s)} \right) \]
\[ \leq c_{17} \frac{|x|}{g(|x|)} \log \left( 1 + \frac{1}{\inf_{r_s \leq s \leq |x|+1} f(s)} \right). \]

Hence we have proved (6.23). \[ \square \]
7. Appendix: Boundedness, Continuity and Strict Positivity of Dirichlet Heat Kernel

In this appendix, we make some comments on assumptions of Dirichlet heat kernel in Subsection 1.1. Let \((\mathcal{E}, \mathcal{F})\) be the Dirichlet form given by (1.2) such that its jumping kernel \(J(x, y)\) satisfies (1.1). Since \((\mathcal{E}, \mathcal{F})\) is regular on \(L^2(\mathbb{R}^d; dx)\), it associates a symmetric Hunt process \(X = (X_t)_{t \geq 0}\) starting from quasi-everywhere on \(\mathbb{R}^d\). Suppose that there exist \(0 < \alpha_1 \leq \alpha_2 < 2\) and \(c_1, c_2 \in (0, \infty)\) such that

\[
\frac{c_1}{|x - y|^{d + \alpha_1}} \leq J(x, y) \leq \frac{c_2}{|x - y|^{d + \alpha_2}}, \quad 0 < |x - y| \leq 1.
\]

Then, according to [17, Proposition 3.1], there exist a properly \(\mathcal{E}\)-exceptional set \(\mathcal{N} \subset \mathbb{R}^d\) and a transition density (also called heat kernel) \(p(\cdot, \cdot) : (0, \infty) \times (\mathbb{R}^d \setminus \mathcal{N}) \times (\mathbb{R}^d \setminus \mathcal{N}) \rightarrow [0, \infty)\) such that

\[
p(t, x, y) \leq c\left(t^{-d/\alpha_1} \lor t^{-d/2}\right), \quad x, y \in \mathbb{R}^d \setminus \mathcal{N}, t > 0.
\]

In the following, we further suppose that

\[
C_* := \sup_{|x - y| > 1} J(x, y) < \infty,
\]

and that

\[
\mathcal{N} = \emptyset \quad \text{and} \quad p(\cdot, \cdot, \cdot) : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow (0, \infty) \quad \text{is continuous and strictly positive.}
\]

See [5, 13, 14, 15, 17] and the references therein for sufficient conditions on such assumptions. In particular, the associated semigroup \((P_t)_{t \geq 0}\) enjoys the strong Feller property.

**Proposition 7.1.** Under assumptions above, for every open set \(D\) and \(t > 0\), the transition density for the process \(X^D\), \(p^D(t, \cdot, \cdot) : D \times D \rightarrow (0, \infty)\), is continuous.

**Proof.** We first construct the process \(Z = (Z_t)_{t \geq 0}\) from \(X\) by removing jumps of size larger than 1 via Meyer’s construction (see [4, Remark 3.4]). Let \(p_Z(t, x, y)\) be the transition density of \(Z\). Note that, from [4, (3.30)] one can check that there exist constants \(c_i > 0\) \((1 \leq i \leq 4)\) such that for all \(|x - y| \leq c_1\)

\[
p_Z(t, x, y) \leq c_2 t|x - y|^{-c_3}, \quad t < c_4|x - y|^{\alpha_2}.
\]

By [4, Lemma 3.7(2)], we have

\[
p(t, x, y) \leq p_Z(t, x, y) + t\|J(x, y)\|_{\{\|x - y\| > 1\}} \leq \infty, \quad x, y \in \mathbb{R}^d, t > 0,
\]

We will claim that for all \(\beta > 0\),

\[
\sup_{|x - y| \geq \beta, t > 0} p(t, x, y) + \sup_{|x - y| \geq \beta, t > 1} p(t, x, y) =: M_{1, \beta} + M_{2, \beta} < \infty.
\]

In fact, \(M_{2, \beta} < \infty\) for all \(\beta > 0\) by (7.2). On the other hand, by (7.2), (7.3), (7.5) and (7.6), for all \(\beta \leq c_1\),

\[
M_{1, \beta} \leq \sup_{|x - y| \geq \beta, t < 1} p(t, x, y) + \sup_{|x - y| \geq \beta, 1 \leq t < c_4|x - y|^{\alpha_2}} p(t, x, y)
\]

\[
\leq c_5 \sup_{|x - y| \geq \beta, t < 1} (t|x - y|^{-c_3} + C_* t) + c_5 \sup_{1 \leq t \leq c_4|x - y|^{\alpha_2}} t^{-d/\alpha_1}
\]

\[
\leq c_6 (\beta^{-c_3} + C_* + \beta^{-d\alpha_2/\alpha_1}) < \infty.
\]
On the other hand, by Meyer’s construction and [4, Lemma 3.8, (3.29) and (3.20)], one can obtain that \( \mathbb{P}^x(\tau_D \leq t) \to 0 \) uniformly in any compact \( K \subset D \) as \( t \to 0 \). Using the continuity of \( p(\cdot, \cdot, \cdot) \), the strong Feller property of \((P_t)_{t \geq 0}\) and (7.7), one can follow the proof of [22, Theorem 2.4] line by line and show that, for each \( t > 0 \), \((x, y) \mapsto \mathbb{E}^x[p(t - \tau_D, X_{t\tau_D}, y)1_{\{\tau_D = 0\}}]\) is continuous on \( D \times D \). Thus, by (1.3), for every \( t > 0 \) the function \( p_D(t, \cdot, \cdot) : D \times D \to (0, \infty) \) is continuous. \( \square \)

Under assumptions above, if \( D \) is a domain of \( \mathbb{R}^d \) (i.e., \( D \) is a connected open set), then it is easy to verify that Dirichlet heat kernel \( p_D(t, x, y) \) is strictly positive for any \((t, x, y) \in (0, \infty) \times D \times D \). See [10, Corollary 7 and Remark 8 (2)] or [26, Proposition 2.2 (i)]. For disconnected open set, we need some additional assumption. Following [10, Condition (RC), p. 1120] (or see [32, Definition 4.3]), we call an open set \( D \) is roughly connected by the process \( X \), if for any \( x, y \in D \), there exist \( m \geq 1 \) and distinct connected components \( \{D_i\}_{i=1}^m \) of \( D \), such that \( x \in D_1 \), \( y \in D_m \) and for every \( 1 \leq i \leq m - 1 \), \( \text{dist}(D_i, D_{i+1}) < r_j \), where

\[
r_j := \inf \left\{ r > 0 : \inf_{|x-y| \leq r} J(x, y) > 0 \right\}.
\]

The following proposition essentially has been proved in [10, Proposition 6 and Remark 8].

**Proposition 7.2.** Under assumptions above, if the open set \( D \subset \mathbb{R}^d \) is roughly connected by the process \( X \), then the Dirichlet heat kernel \( p_D(t, x, y) \) is strictly positive for any \((t, x, y) \in (0, \infty) \times D \times D \).

In particular, if there are some constants \( 0 < a_1 \leq a_2 < 1 \) such that for all \( x, y \in D \), \( x \sim_{(t; a_1, a_2)} y \) (that is, \( x \) is connected with \( y \) in a reasonable way with respect to constants \( 0 < a_1 \leq a_2 < 1 \), see Definition 3.5), then the open set \( D \) is roughly connected by the process \( X \) under assumption (3.6).

The result below should be known, see e.g. [41, Chapter V, Theorem 6.6] or [40, Theorem XIII. 43]. The reader can see [12, Proposition 1.2] for the proof.

**Proposition 7.3.** Suppose that \((P_t^D)_{t \geq 0}\) is compact, and let \( \phi_1 \) be its first eigenfunction (called ground state). If \( p_D(t, x, y) \) is bounded, continuous and strictly positive on \((0, \infty) \times D \times D \), then \( \phi_1 \) also has a version which is bounded, continuous and strictly positive on \( D \).

**Acknowledgements.** The research of Xin Chen is supported by National Natural Science Foundation of China (No. 11501361), “Yang Fan Project” of Science and Technology Commission of Shanghai Municipality (No. 15YF1405900), and Fujian Provincial Key Laboratory of Mathematical Analysis and its Applications (FJKLMAA). The research of Panki Kim is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2016R1E1A1A01941893). The research of Jian Wang is supported by National Natural Science Foundation of China (No. 11522106), the Fok Ying Tung Education Foundation (No. 151002), the JSPS postdoctoral fellowship (26-04021), National Science Foundation of Fujian Province (No. 2015J01003), the Program for Nonlinear Analysis and Its Applications (No. IRTL1206), and Fujian Provincial Key Laboratory of Mathematical Analysis and its Applications (FJKLMAA).
References

[1] Bañuelos, R.: Sharp estimates for Dirichlet eigenfunctions in simply connected domains, *J. Differential Equations* **125** (1996), 282–298.

[2] Bañuelos, R. and van den Berg, M.: Dirichlet eigenfunctions for horn-shaped regions and Laplacians on cross sections, *J. London Math. Soc.* **53** (1996), 503–511.

[3] Bañuelos, R. and Davis, B.: Sharp estimates for Dirichlet eigenfunctions in horn-shaped regions, *Comm. Math. Phys.* **150** (1992), 209–215. Erratum: *Comm. Math. Phys.* **162** (1994), 215–216.

[4] Barlow, M.T., Bass, R.F., Chen, Z.-Q. and Kassmann, M.: Non-local Dirichlet forms and symmetric jump processes, *Trans. Amer. Math. Soc.* **361** (2009), 1963–1999.

[5] Bass, R.F., Kassmann, M. and Kumagai, T.: Symmetric jump processes: localization, heat kernels, and convergence, *Ann. Inst. Henri Poincaré Probab. Statist.* **46** (2010), 59–71.

[6] van den Berg, M: On the spectrum of the Dirichlet Laplacian for horn-shaped regions in $\mathbb{R}^n$ with infinite volume, *J. Funct. Anal.* **58** (1984), 150–156.

[7] Bogdan, K., Grzywny, T. and Ryznar, M.: Heat kernel estimates for the fractional Laplacian with Dirichlet conditions, *Ann. Probab.* **38** (2010), 1901–1923.

[8] Bogdan, K., Kulczycki, T. and Kwaśnicki, M.: Estimates and structure of $\alpha$-harmonic functions, *Probab. Theory Relat. Fields* **140** (2008), 345–381.

[9] Bogdan, K., Kumagai, T. and Kwaśnicki, M.: Boundary Harnack inequality for Markov processes with jumps, *Trans. Amer. Math. Soc.* **367** (2015), 477–517.

[10] Chen, X. and Wang, J.: Intrinsic ultracontractivity of general Lévy processes on bounded open sets, *Illinois J. Math.* **58** (2014), 1117–1144.

[11] Chen, X. and Wang, J.: Intrinsic contractivity of Feynman-Kac semigroups for symmetric jump processes with infinite range jumps, *Front. Math. China* **10** (2015), 753–776.

[12] Chen, X. and Wang, J.: Intrinsic ultracontractivity of Feynman-Kac semigroups for symmetric jump processes, *J. Funct. Anal.* **270** (2016), 4152–4195.

[13] Chen, Z.-Q. and Kumagai, T.: Heat kernel estimates for stable-like processes on $d$-sets, *Stoch. Proc. Appl.* **108** (2003), 27–62.

[14] Chen, Z.-Q. and Kumagai, T.: Heat kernel estimates for jump processes of mixed types on metric measure spaces, *Probab. Theory Relat. Fields* **140** (2008), 277–317.

[15] Chen, Z.-Q., Kim, P. and Kumagai, T.: Weighted Poincaré inequality and heat kernel estimates for finite range jump processes, *Math. Ann.* **342** (2008), 833–883.

[16] Chen, Z.-Q., Kim, P. and Kumagai, T.: On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces, *Acta Math. Sin.* **25** (2009), 1067–1086.

[17] Chen, Z.-Q., Kim, P. and Kumagai, T.: Global heat kernel estimates for symmetric jump processes, *Trans. Amer. Math. Soc.* **363** (2011), 5021–5055.

[18] Chen, Z.-Q., Kim, P. and Song, R.: Heat kernel estimates for Dirichlet fractional Laplacian, *J. Euro. Math. Soc.* **12** (2010), 1307–1329.

[19] Chen, Z.-Q., Kumagai, T. and Wang, J.: Stability of heat kernel estimates for symmetric jump processes on metric measure space, arXiv:1604.04035

[20] Chen, Z.-Q. and Song, R.: Intrinsic ultracontractivity and conditional gauge for symmetric stable processes, *J. Funct. Anal.* **150** (1997), 204–239.

[21] Chen, Z.-Q. and Song, R.: Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains, *Illinois J. Math.* **44** (2000), 138–160.

[22] Chung, K.L. and Zhao, Z.: *From Brownian Motion to Schrödinger’s equation*, Springer, New York, 1995.

[23] Cranston, M. and Li, Y.: Eigenfunction and harmonic function estimates in domains with horns and cusps, *Comm. Partial Differential Equations* **22** (1997), 1805–1836.

[24] Davies, E.B. and Simon, B.: Ultracontractivity and heat kernels for Schrödinger operators and Dirichlet Laplacians, *J. Funct. Anal.* **59** (1984), 335–395.

[25] Fukushima, M., Oshima, Y. and Takeda, M.: *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter, Berlin 2011, 2nd.

[26] Grzywny, T.: Intrinsic ultracontractivity for Lévy processes, *Probab. Math. Statist.* **28** (2008), 91–106.
[27] Grzywny, T., Kim, K. and Kim, P.: Estimates of Dirichlet heat kernel for symmetric Markov processes, arXiv:1512.02717.
[28] Kaleta, K. and Kulczycki, T.: Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians, Potential Anal. 33 (2010), 313–339.
[29] Kaleta, K. and Łorinczi, J.: Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes, Ann. Probab. 43 (2015), 1350–1398.
[30] Kim, K. and Kim, P.: Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like process in $C^{1,\eta}$ open sets, Stoch. Proc. Appl. 124 (2014), 3055–3083.
[31] Kim, P. and Song, R.: Potential theory of truncated stable processes, Math. Z. 256 (2007), 139–173.
[32] Kim, P. and Song, R.: Intrinsic ultracontractivity for non-symmetric Lévy processes, Forum Math. 21 (2009), 43–66; Erratum, Forum Math. 21 (2009), 1137–1139.
[33] Kim, P., Song, R. and Vondraček, Z.: Scale invariant boundary Harnack principle at infinity for Feller processes, arXiv:1510.04569v2. To appear in Potential Anal.
[34] Kulczycki, T.: Intrinsic ultracontractivity for symmetric stable processes, Bull. Polish Acad. Sci. Math. 46 (1998), 325–334.
[35] Kulczycki, T. and Siddeja, B.: Intrinsic ultracontractivity of the Feynman-Kac semigroup for relativistic stable processes, Trans. Amer. Math. Soc. 358 (2006), 5025–5057.
[36] Kwaśnicki, M.: Intrinsic ultracontractivity for stable semigroups on unbounded open sets, Potential Anal. 31 (2009), 57–77.
[37] Lindeman, A., Pang, M.H. and Zhao, Z.: Sharp bounds for ground state eigenfunctions on domains with horns and cusps, J. Math. Anal. Appl. 212 (1997), 381–416.
[38] Mimica, A.: Heat kernel estimates for jump processes with small jumps of high intensity, Potential Anal. 36 (2012), 203–222.
[39] Reed, M. and Simon, B.: Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, 1978.
[40] Reed, M. and Simon, B.: Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, 1978.
[41] Schaefer, H.H.: Banach lattices and positive operators, Springer, New York 1974.
[42] Wang, F.-Y.: Functional inequalities, semigroup properties and spectrum estimates, Infinite Dimens. Anal. Quant. Probab. Related Topics 3 (2000), 263–295.
[43] Wang, F.-Y.: Functional inequalities and spectrum estimates: The infinite measure case, J. Funct. Anal. 194 (2002), 288–310.
[44] Wang, F.-Y.: Functional Inequalities, Markov Processes and Spectral Theory, Science Press, Beijing 2005.
[45] Wang, F.-Y. and Wu, J.-L.: Compactness of Schrödinger semigroups with unbounded below potentials, Bull. Sci. Math. 132 (2008), 679–689.