Avoiding Monochromatic Rectangles Using Shift Patterns

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Introduction

Ramsey Theory (Graham and Rothschild 1990) deals with patterns that cannot be avoided indefinitely. In this paper we focus on a pattern of coloring a $n \times m$ grid with $k$ colors: Consider all possible rectangles within the grid whose length and width are at least 2. Try to color the grid using $k$ colors so that no such rectangle has the same color for its four corners. When this is possible, we say that the $n \times m$ grid is $k$-colorable while avoiding monochromatic rectangles.

Many results regarding this problem have been derived by pure combinatorial approach: for example, a generalization of Van der Waerden’s Theorem can give an upper bound; it was shown (Fenner et al. 2010) that for each prime power $k$, a $k^2 + k$ by $k^2$ grid is $k$-colorable but adding a row makes it not $k$-colorable. However, these results are unable to decide many grid sizes; whether an 18 by 18 grid is 4-colorable is an example. This grid had been the last missing piece of the question of 4-colorability, and a challenge prize was raised to close the gap (Hayes 2009). Three years later, a valid 4-coloring of that grid was found by encoding the problem into propositional logic and applying SAT-solving techniques (Steinbach and Posthoff 2012).

Thus, to find a 5-coloring of 26 by 26, or rather, to find a valid coloring for any number of colors $k$ in general, an internal symmetry that is applicable to all $k$ is very desirable. We found a novel internal symmetry that is unrestricted by the number of colors $k$. Further analysis on this symmetry gives further constraints on the number of occurrences of each color. Factoring in these constraints, the search time for $G_{24,24}$ and $G_{25,25}$ can be reduced to a few minutes. We also attempted to solve the 26 by 26 grid; many attempts came down to only 2 or 3 unsatisfied clauses, but none succeeded.

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Classifying colorings of smaller grid examples

We consider classifying colorings of simpler grids to be a good starting point that enables us to gain insight into how many “essentially different” solutions there are. As an example, we will analyze $G_{4,4}$ and $G_{10,10}$, the maximal 2-colorable and 3-colorable squares.

For square grids $G_{n,n}$ and number of colors $k$, there are 3 kinds of symmetries that transform a valid coloring to another: (i) permutation of colors (ii) permutation of rows or columns (iii) transposition, i.e. a flip along the diagonal. We define symmetries by sequences of these operations and define a natural equivalence relation between colorings.

Let $Grid(n, n, k)$ be the set of all valid $k$-colorings of $G_{n,n}$. We are interested in counting the equivalence classes of $Grid(10, 10, 3)$. Our approach is to convert grid colorings to graph colorings, and use Bliss (Junttila and Kaski 2007) graph isomorphism tool to assign to each graph a canonical labeling. In particular, we identify each member of $Grid(n, n, k)$ with a graph on $n^2$ vertices, where each vertex corresponds to a cell in the grid and has the same la-
bel and color. Two distinct vertices \((a, b), (c, d)\) are joined by an edge if and only if \(a = c\) or \(b = d\) (but not both).

For Grid\(4, 4, 2\), we generated all 840 solutions and compared the canonical labelings of their graphs up to color permutations. There are three non-isomorphic solutions as shown in Figure 1 (upper-left).

For Grid\(10, 10, 3\), the CNF formula yielded 35 solutions after adding symmetry breaking clauses by Shatter (Aloul, Sakallah, and Markov 2006). Comparison of graph representatives output by Bliss showed that all 35 solutions are isomorphic. That is, there is only one equivalence class in Grid\(10, 10, 3\) shown in Figure 1 (lower-left).

Generalizable pattern and new results

The representative of Grid\(10, 10, 3\) in Figure 1 (left) admits a shift pattern: within each 4 by 4 subgrid, the second row is a copy of the first row except shifted right (or left) by 1; the third row shifted by 2; the last row shifted by 3. All the shifts wrap around on the edge of the subgrid.

The shift pattern greatly reduces search space, as only the first row in each subgrid needs to be chosen. This pattern can be iterated for one more layer: by adding “midgrids” that are made up of smaller subgrids and enforcing that subgrids on subsequent rows are shifted copies of those on the first row in a similar fashion, the search space can be further reduced. This is extremely helpful to solving larger grids, and significantly reduces the running time. Figure 1 shows a 4-coloring of \(G_{18,18}\) with 3-subgrids and 9-midgrids. CaDiCal found this solution in under 1 second; previously (Steinbach and Posthoff 2012), it took the SAT solver clasp roughly 7 hours to find a cyclic-reusable assignment of \(G_{18,18}\). For a fair comparison, we ran the same formula on CaDiCal and it terminated in 4 hours and 40 minutes.

In addition, the 4-coloring of \(G_{18,18}\) in Figure 1 is a “new” solution, in the sense that it is not isomorphic to the previous solution (Steinbach and Posthoff 2012).

Infeasible case: Shift pattern on 26 by 26

If there are no remaining columns and rows, the only choices for subgrid size are 2, 13 and 26. The subgrid size cannot be 26 by the Pigeonhole Principle. If a 5-coloring of subgrid size 2 or 13 exists, then it must satisfy certain necessary conditions. For the case of size 2 and 13, integer constraints can be placed on number of each color in each subgrid. 23 Theorem Prover (De Moura and Bjørner 2008) reported unsatisfiable for both cases. Therefore, no 5-colorings of such shift pattern exist.

Thus, we turn our attention to finding solutions that have shift pattern for the upper-left 25 by 25 or 24 by 24 part, and constrain the remaining column(s) and row(s) appropriately.

Shift pattern on 25 by 25

In the case of 25 by 25, the possible subgrid sizes are 5 and 25. For subgrid size 5, an obvious color distribution is 5 cells of each color in each subgrid. We attempted to find a coloring of \(G_{26,26}\) which has this pattern on the 25 by 25 part and left the last row and column without additional constraints. PalSAT was unable to find a satisfying assignment in 24 hours, with only one unsatisfied clause. This instance is unlikely to be satisfiable because of the cell at the bottom-right corner.

Shift pattern on 24 by 24, under further constraints

This brings us to 24 by 24, which has more possible subgrid sizes. Directly solving \(G_{24,24}\) for subgrid sizes ranging from 3 to 10 shows that subgrid sizes \(\{3, 4, 5, 6, 8, 10\}\) are satisfiable. However, PalSAT was unable to solve \(G_{26,26}\) with the similar shift pattern and diagonal patterns in 24 hours.

Therefore, we sought to constrain them with a “partial shift pattern”: e.g., take the upper-left 26 by 26 part of a 32 by 32 grid, for subgrid size 8. Now, the 25th row under each subgrid is a shifted copy of the 24th row, and the columns are also constrained to be alternating colors, as observed in the solution of \(G_{25,25}\) in Figure 1 (right). However, this was not enough: PalSAT could not get under 2 unsatisfied clauses.
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