Infrared behavior and fixed-point structure in the compactified Ginzburg–Landau model

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We consider the Euclidean $N$-component Ginzburg–Landau model in $D$ dimensions, of which $d$ ($d \leq D$) of them are compactified. As usual, temperature is introduced through the mass term in the Hamiltonian. This model can be interpreted as describing a system in a region of the $D$-dimensional space, limited by $d$ pairs of parallel planes, orthogonal to the coordinates axis $x_1, x_2, \ldots, x_d$. The planes in each pair are separated by distances $L_1, L_2, \ldots, L_d$. For $D = 3$, from a physical point of view, the system can be supposed to describe, in the cases of $d = 1, d = 2$, and $d = 3$, respectively, a superconducting material in the form of a film, of an infinitely long wire having a rectangular cross-section and of a brick-shaped grain. We investigate in the large-$N$ limit the fixed-point structure of the model, in the absence or presence of an external magnetic field. An infrared-stable fixed point is found, whether of not an external magnetic field is applied, but for different ranges of values of the space dimension $D$.

I. INTRODUCTION

A large amount of work has already been done on the Ginzburg–Landau (GL) model, both in its single component and in the $N$-component versions, using the renormalization group approach [1,4]. In particular, an analysis of the renormalization group in finite-size geometries can be found in [3,11] and a general study of phase transitions in confined systems is in [10]. These studies have been performed to take into account boundary effects on thermodynamical quantities for these systems. The existence of phase transitions are in this case associated to some spatial parameters related to the breaking of translational invariance, for instance, the distance $L$ between planes confining the system. Also, in other contexts, the influence of boundaries in the behavior of systems undergoing transitions have been investigated [11,12].

We shall analyze in the present paper the effects of boundaries on the transition by considering that such confined systems are modeled by compactifying spatial dimensions [10]. Compactification will be engendered as a generalization of the Matsubara (imaginary-time) prescription to account for constraints on the spatial coordinates. In the original Matsubara formalism, time is rotated to the imaginary axis, $t \rightarrow i\tau$, where $\tau$ (the Euclidean time) is limited to the interval $0 \leq \tau \leq \beta$, with $\beta = 1/T$ standing for the inverse temperature. The fields then fulfill periodic (bosons) or antiperiodic (fermions) boundary conditions and are compactified on the $\tau$-axis in an $S^1$ topology, the circumference of length $\beta$. Such a formalism leads to the description of a system in thermal equilibrium at the temperature $\beta^{-1}$. Since in a Euclidean field theory space and time are on the same footing, one can envisage a generalization of the Matsubara approach to any set of spatial coordinates as well [13-16].

The topological conceptual framework for studying simultaneously finite temperature and spatial constraints has been developed by considering a simply or nonsimply connected $D$-dimensional manifold with a topology of the type $\Gamma^{d+1}_D = \mathbb{R}^{D-d-1} \times S^1 \times S^1 \times \ldots \times S^1$, with $S^1$ corresponding to the compactification of the imaginary time and $S^1, \ldots, S^1$ referring to the compactification of $d$ spatial dimensions [16,17]. The topological structure of spacetime does not modify the local field equations. However, topology implies modifications of the boundary conditions on fields and Green functions [18]. Physical manifestations of this type of topology include, for instance, the vacuum-energy fluctuations giving rise to the Casimir effect [11,19,21]; in the study of phase transitions, the dependence of the critical temperature on the compactification parameters is found in several situations of condensed-matter physics [10,22,23]. Also, this kind of formalism has been employed in the investigation of the confining phase transition in effective theories for Quantum Chromodynamics [23,24]. In the $\Gamma^{d+1}_D$ topology, the Feynman rules are modified by introducing a generalized Matsubara prescription, performing the following multiple replacements [compactification of a $(d+1)$-dimensional subspace]:

$$
\int \frac{dk_0}{2\pi} \rightarrow \frac{1}{\beta} \sum_{n_1=-\infty}^{+\infty} , \quad \int \frac{dk_i}{2\pi} \rightarrow \frac{1}{L_i} \sum_{n_i=-\infty}^{+\infty} ; \quad k_1 \rightarrow \frac{2(n_1 + c)\pi}{\beta} \quad k_i \rightarrow \frac{2(n_i + c)\pi}{L_i} ,
$$

(1)

where for each $i = 1, 2, \ldots, d$, $L_i$ is the size of the compactified spatial dimension $i$ and $c = 0$ or $c = 1/2$ for, respectively, bosons and fermions.
The compactification formalism described above has been applied to field-theoretical models in \( D \) dimensions, with a \( d \)-dimensional \((d \leq D)\) set of compactified spatial coordinates \([25, 26, 32]\). This formalism has also been developed from a path-integral approach in \([17]\). This allows to generalize to any subspace previous results in the effective potential framework for finite temperature and spatial boundaries. This mechanism generalizes and unifies results from recent work on the behavior of field theories in the presence of spatial constraints \([12, 16, 32]\), and previous results in the literature for finite-temperature field theory as, for instance, in \([33]\).

When studying the compactification of spatial coordinates, however, it is argued in \([10]\) from topological considerations, that we may have a quite different interpretation of the generalized Matsubara prescription: it provides a general and practical way to account for systems confined in limited regions of space at finite temperatures. Distinctly, we shall be concerned here with stationary field theories and employ the generalized Matsubara prescription to study bounded systems by implementing the compactification of spatial coordinates; no imaginary-time compactification will be done, temperature will be introduced through the mass parameter in the Hamiltonian. We will consider a topology of the type \( \Gamma^D = \mathbf{R}^{D-d} \times S^{1_1} \times S^{1_2} \times \cdots \times S^{1_k}, \) where \( S^{1_1}, \ldots, S^{1_k} \) refer to the compactification of \( d \) spatial dimensions.

We consider in the present article the Euclidean vector \( N \)-component \((\lambda \varphi^4)_D\) theory at leading order in \( 1/N \), the system being submitted to the constraint of being limited by \( d \) pairs of parallel planes. Each pair is orthogonal to the coordinate axes \( x_1, \ldots, x_d \), respectively, and in each one of them the planes are at distances \( L_1, \ldots, L_d \) apart from one another. From a physical point of view, we take in particular \( D = 3 \) and introduce temperature by means of the mass term in the Hamiltonian in the usual Ginzburg–Landau fashion. These models can then describe a superconducting material in the shapes of a film \((d = 1)\), of a wire \((d = 2)\) and of a grain \((d = 3)\). With geometries such as these, some of us have been able to obtain general formulas for the dependence of the transition temperature and other quantities on the parameters delimiting the spatial region within which the system is confined (see for instance \([25, 26]\) and other references therein).

We also consider the critical behavior of the system under the influence of an external magnetic field. Physically, for \( D = 3 \), this corresponds to superconducting films, wires and grains in a magnetic field. In \([2]\), a large-\( N \) theory of a second-order transition for arbitrary dimension \( D \) is presented and the fixed-point effective free energy describing the transition is found. The theory is based on the Ginzburg–Landau model with the coupling of scalar and gauge fields. While ignoring gauge-field fluctuations, the model includes an external magnetic field. The authors in \([2]\) also claim that it is possible that in the physical situation of \( N = 1 \), a mechanism of reduction of the lower critical dimension could allow a continuous transition in \( D = 3 \). In \([7]\), the possibility of the existence of a phase transition for a superconductor film in the presence of an external magnetic field has been investigated. This has been done in the renormalization-group framework by looking for the existence of infrared-stable fixed points for the \( \beta \) function.

In this article, we study, for arbitrary space dimension \( D \) and for any number \( d \leq D \) of compactified dimensions (specially wires and grains), the fixed-point structure of the model, thus generalizing the previously quoted studies for films. In both situations, with or without external magnetic field, we shall neglect the minimal coupling with the vector potential corresponding to the intrinsic gauge fluctuations. Also, as usual in the GL model, no imaginary-time compactification will be done, temperature will be introduced through the mass parameter in the Hamiltonian. Our main concern will be to analyze the model from a field-theoretical point of view. In this sense, the present work may be seen as a further development of previous papers by some of us, as for instance \([6, 16, 17]\). The paper is organized in the following way. In Section II below, we establish in all compactified cases the running coupling constant (and hence the fixed point) for the model in which the external field is omitted, while the analogous study when it is considered is the subject of Section III. In Section IV, we present our conclusions.

## II. THE COMPACTIFIED MODEL IN THE ABSENCE OF AN EXTERNAL FIELD

We first consider the \( N \)-component vector model described by the Ginzburg–Landau Hamiltonian density

\[
\mathcal{H} = \partial_\mu \varphi_a \partial^\mu \varphi_a + m^2 \varphi_a \varphi_a + u (\varphi_a \varphi_a)^2 \tag{2}
\]

in Euclidean \( D \)-dimensional space, where \( u \) is the coupling constant and \( m^2 \) is a mass parameter such that \( m^2 = \alpha (T - T_0) \) and \( T_0 \) the bulk transition temperature. Summation over repeated indices \( \mu \) and \( a \) is assumed. In the following, we will consider the model described by the Hamiltonian \([2]\) and take the large-\( N \) limit, such that \( u \to 0 \), \( N \to \infty \) with \( Nu = \lambda \) fixed.

Let us consider the system in \( D \) dimensions confined to a region of space delimited by \( d \) \((d \leq D)\) pairs of parallel planes. Each plane of a pair \( j \) is at a distance \( L_j \) from the other member of the pair, \( j = 1, 2, \ldots, d \), and is orthogonal to all other planes belonging to distinct pairs \( i \), \( i \neq j \). This may be pictured as a parallelepiped-shaped box embedded in the \( D \)-dimensional space, whose parallel faces are separated by distances \( L_1, L_2, \ldots, L_d \). We use Cartesian coordinates \( r = (x_1, \ldots, x_d, z) \), where \( z \) is a \((D - d)\)-dimensional vector, with corresponding momenta \( k = (k_1, \ldots, k_d, q) \), \( q \) being
a \((D-d)\)-dimensional vector in momentum space. Under these conditions, the generating functional of correlation functions is written in the form

\[
Z = \int D\varphi^* D\varphi \exp \left( -\int_0^{L_1} dx_1 \cdots \int_0^{L_d} dx_d \int d^{D-d} z \, \mathcal{H}(|\varphi|, |\nabla \varphi|) \right),
\]  

(3)

with the field \(\varphi(x_1, \ldots, x_d, z)\) satisfying the condition of confinement inside the box, \(\varphi(x_i \leq 0, z) = \varphi(x \geq L, z) = \text{const}\). Then the field should have a mixed series-integral Fourier expansion of the form

\[
\varphi(x_1, \ldots, x_d, z) = \sum_{i=1}^d \sum_{n_i = -\infty}^{+\infty} c_{n_i} \int d^{D-d} q \, b(q) e^{-i\omega_{n_i} x - iq \cdot \tilde{x}} \varphi(\omega_{n_i}, q),
\]  

(4)

where, for \(i = 1, \ldots, d\), \(\omega_{n_i} = 2\pi n_i / L_i\) and the coefficients \(c_{n_i}\) and \(b(q)\) correspond respectively to the Fourier series representation over the \(x_i\) and to the Fourier integral representation over the \((D-d)\)-dimensional \(z\)-space. As explained in the comments leading to Eq. (1), the above conditions of confinement of the \(x_i\)-dependence of the field to a segment of length \(L_i\) allow us to proceed with respect to the \(x_i\)-coordinates, for all \(i\), in a manner analogous as it is done in the imaginary-time Matsubara formalism in field theory. Accordingly, the multiple Matsubara replacements modify the Feynman rules following the prescription

\[
\int \frac{dk_i}{2\pi} \rightarrow \frac{1}{L_i} \sum_{n_i = -\infty}^{+\infty}, \quad k_i \rightarrow \frac{2\pi n_i}{L_i} \equiv \omega_{n_i}, \quad i = 1, \ldots, d.
\]  

(5)

Compactification can be implemented in different ways as, for instance, through specific conditions on the fields at spatial boundaries. We here choose periodic boundary conditions.

A. The boundary-dependent coupling constant in the large-\(N\) limit

The coupling constant will be defined in terms of the four-point function for small external momenta which, at leading order in \(1/N\), is given by the sum of all chains of one-loop diagrams. It is given in momentum space, before compactification, and at the critical point by [3]

\[
\Gamma^{(4)}_D(p, m = 0) = \frac{u}{1 + Nu\Pi(p, m = 0)},
\]  

(6)

where \(\Pi(p, m = 0)\) is the single one-loop integral at the critical point. It is written as (let us keep in mind that \(p\) is the \(D\)-dimensional external momentum vector)

\[
\Pi(p, m = 0) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (p-k)^2} = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + p^2 x(1-x)]^2},
\]  

(7)

where a Feynman parameter \(x\) was introduced.

Performing the Matsubara replacements [5] for \(d\) dimensions, Eq. (7) becomes

\[
\Pi(p, D, \{L_i\}, m = 0) = \frac{1}{L_1 \cdots L_d} \sum_{i=1}^d \sum_{n_i = -\infty}^{+\infty} \int_0^1 dx \int \frac{d^{D-d} q}{(2\pi)^{D-d}} \frac{1}{[q^2 + \omega_{n_1}^2 + \cdots + \omega_{n_d}^2 + p^2 x(1-x)]^2}
\]  

(8)

and we define the effective \(\{L_i\}\)-dependent coupling constant in the large-\(N\) limit as

\[
\lambda(p, D, \{L_i\}) = \lim_{u \to 0; N \to \infty} N\Gamma^{(4)}_D(p, \{L_i\}, m = 0) = \frac{\lambda}{1 + \lambda\Pi(p, D, \{L_i\}, m = 0)}.
\]  

(9)
with $Nu = \lambda$ fixed.

The sum over the $n_i$ and the integral over $q$ above can be treated using the formalism developed in [16]. It concerns the study of expressions of the form

$$I(s) = \sum_{i=1}^{d} \sum_{n_i=-\infty}^{\infty} \int \frac{d^{D-d}q}{(q^2 + a_1 n_1^2 + \cdots + a_d n_d^2 + c^2)^s}. \quad (10)$$

(In our case, for the computation of $\Pi$, we have $s = 2$, $a_i = 1/L_i^2$, $\omega^2 = (2\pi)^2 a_i n_i^2$ and $c^2 = p^2 x(1-x)/(2\pi)^2$; also, a redefinition of the integration variables, $q \to q/2\pi$, has been performed.) Such integral over the $D-d$ noncompactified momentum variables is performed using the well-known dimensional regularization formula [8]

$$\int \frac{d^d q}{(q^2 + M)^s} = \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} \frac{\pi^{d/2}}{M^{s-d/2}}, \quad (11)$$

which, for $\ell = D - d$, leads to

$$I(s) = f(D, d, s) Z_d^2 \left( s - \frac{D-d}{2}; a_1, \ldots, a_d \right), \quad (12)$$

where

$$f(D, d, s) = \pi^{(D-d)/2} \frac{\Gamma(s - \frac{D-d}{2})}{\Gamma(s)} \quad (13)$$

and $Z_d^2(\nu; a_1, \ldots, a_d)$ are Epstein–Hurwitz zeta functions, for $\nu = s - (D-d)/2$, which are defined by

$$Z_d^2(\nu; a_1, \ldots, a_d) = \sum_{n_1, \ldots, n_d=-\infty}^{\infty} (a_1 n_1^2 + \cdots + a_d n_d^2 + c^2)^{-\nu}. \quad (14)$$

It is valid for $\text{Re}(\nu) > d/2$ (in our case, this implies $\text{Re}(s) > D/2$). The Epstein–Hurwitz zeta function can be extended to the whole complex $s$-plane and we obtain, after some manipulations [16, 34],

$$Z_d^2(\nu; a_1, \ldots, a_d) = \frac{2^{\nu - \frac{D}{2} + 1} \pi^{2\nu - \frac{D}{2}}}{a_1 \cdots a_d \Gamma(\nu)} \left[ 2^{\nu - \frac{D-d}{2}} c^{D-2s} \Gamma \left( \nu - \frac{d}{2} \right) \right]
+ 2^d \sum_{i=1}^{d} \sum_{n_i=1}^{\infty} \left( \frac{c}{L_i n_i} \right)^{\frac{D-d}{2}} K_{\nu - \frac{D}{2}} (c L_i n_i) + \cdots
+ 2^d \sum_{n_1, \ldots, n_d=1}^{\infty} \left( \frac{c}{\sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2}} \right)^{\frac{D-d}{2}} K_{\nu - \frac{D}{2}} \left( c \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right). \quad (15)$$

Putting $\nu = s - (D-d)/2$ in Eq. (15), we get

$$I(s) = \frac{h(D, s)}{a_1 \cdots a_d} \left[ 2^{s-D/2-2} c^{D-2s} \Gamma \left( s - \frac{D}{2} \right) \right]
+ \sum_{i=1}^{d} \sum_{n_i=1}^{\infty} \left( \frac{c}{L_i n_i} \right)^{D/2-s} K_{D/2-s}(c L_i n_i)
+ 2^d \sum_{i<j=1}^{d} \sum_{n_{ij}=1}^{\infty} \left( \frac{c}{\sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}} \right)^{D/2-s} K_{D/2-s} \left( c \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2} \right) + \cdots
+ 2^{d-1} \sum_{n_1, \ldots, n_d=1}^{\infty} \left( \frac{c}{\sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2}} \right)^{D/2-s} K_{D/2-s} \left( c \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right), \quad (16)$$
where

\[ h(D, s) = \frac{2^s - D/2 + 2\pi^{2s-D/2}}{\Gamma(s)} \]  \hspace{1cm} (17)

and the \( K_\nu \) are the modified Bessel functions. Applying formula (16) to Eq. (7) the result is

\[ \Pi(p, D, \{ L_i \}, m = 0) = \frac{\sqrt{a_1 \cdots a_d}}{(2\pi)^d} \int_0^1 dx \, I(2) \]

\[ = \frac{h(D, 2)}{(2\pi)^d} \int_0^1 dx \left[ 2^{-D/2} \left( \frac{1}{(2\pi)^2} \sqrt{p x(1-x)} \right)^{D/2-2} \Gamma \left( 2 - \frac{D}{2} \right) \right. \]

\[ + \sum_{i=1}^d \sum_{n_i=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi L_i n_i} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} L_i n_i \right) \]

\[ + 2 \sum_{i<j=1}^d \sum_{n_i, n_j=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2} \right) + \cdots \]

\[ + 2^{d-1} \sum_{n_1, \ldots, n_d=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right) \], \hspace{1cm} (18)

with \( h(D, 2) = (2\pi)^{4-D/2} \), which, replaced in Eq. (18), gives the effective boundary-dependent coupling constant in the large-\( N \) limit.

**B. Infrared behavior**

We can write Eq. (18) in the form

\[ \Pi(p, D, \{ L_i \}, m = 0) = A(D) |p|^{D-4} + B_d(D, \{ L_i \}), \]  \hspace{1cm} (19)

with the coefficient of the \(|p|\)-term being

\[ A(D) = (2\pi)^{4-3D/2} 2^{-D/2} b(D) \Gamma \left( 2 - \frac{D}{2} \right), \]  \hspace{1cm} (20)

where we have defined

\[ b(D) = \int_0^1 dx \, [x(1-x)]^{D/2-2} = 2^{3-D} \sqrt{\pi} \frac{\Gamma \left( \frac{D}{2} - 1 \right)}{\Gamma \left( \frac{D-1}{2} \right)}, \]  \hspace{1cm} \text{for Re}(D) > 2, \]  \hspace{1cm} (21)

and

\[ B_d(D, \{ L_i \}) = \frac{h(D, 2)}{(2\pi)^d} \int_0^1 dx \left[ \sum_{i=1}^d \sum_{n_i=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi L_i n_i} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} L_i n_i \right) \right. \]

\[ + 2 \sum_{i<j=1}^d \sum_{n_i, n_j=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2} \right) + \cdots \]

\[ + 2^{d-1} \sum_{n_1, \ldots, n_d=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right) \]. \hspace{1cm} (22)
We remark that, for the physically interesting dimension $D = 3$, $b(3) = \pi$. This implies that $A(3) = \pi/4$.

If an infrared-stable fixed point exists for any of the models with $d$ confining dimensions, it would be possible to determine it by a study of the infrared behavior of the Callan–Symanzik $\beta$ function, i.e., in the neighborhood of $|p| = 0$. Therefore, we should investigate the above equations for $|p| \approx 0$.

In this case, we consider a typical term in Eq. (22), which has the form

$$\sum_{n_1,\ldots,n_p = 1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi \sqrt{L_1^2n_1^2 + \cdots + L_p^2n_p^2}} \right)^{D/2-s} K_{D/2-s} \left( \frac{1}{2\pi} \sqrt{p^2x(1-x)} \sqrt{L_1^2n_1^2 + \cdots + L_p^2n_p^2} \right),$$

with $s = 2$ and $p = 1, 2, \ldots, d$. In the $|p| \approx 0$ limit, we may use an asymptotic formula for small values of the argument of the modified Bessel functions [35],

$$K_\nu(z) \approx \frac{1}{2} \Gamma(\nu) \left( \frac{z}{2} \right)^{-\nu} \quad (z \sim 0, \quad \text{Re}(\nu) > 0)$$

and Eq. (23) reduces to

$$\frac{1}{2} \Gamma \left( \frac{D}{2} - s \right) E_p \left( \frac{D}{2} - s; L_1, \ldots, L_p \right).$$

It is expressed in terms of one of the multidimensional Epstein zeta functions $E_p \left( \frac{D}{2} - s; L_1, \ldots, L_p \right)$, for $p = 1, 2, \ldots, d$, which are defined by [36]

$$E_p (\nu; \sigma_1, \ldots, \sigma_p) = \sum_{n_1,\ldots,n_p = 1}^{\infty} \left( \sigma_1^2n_1^2 + \cdots + \sigma_p^2n_p^2 \right)^{-\nu}.$$

Notice that, for $p = 1$, $E_p$ reduces to the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. We then see from (24) that in this limit the $p^2$-dependence of the modified Bessel functions exactly compensates the one coming from the accompanying factors. Thus the remaining $p^2$-dependence is only that of the first term of (15), which is the same for all number of compactified dimensions $d$.

One can also construct analytical continuations and recurrence relations for the multidimensional Epstein functions, which permit to write them in terms of modified Bessel and Riemann zeta functions [16, 36]. One gets

$$E_p (\nu; L_1, \ldots, L_p) = -\frac{1}{2p} \sum_{i=1}^{p} E_{p-1} (\nu; \ldots, \hat{L}_i, \ldots) + \sqrt{\frac{\pi}{2d}} \Gamma \left( \nu - \frac{1}{2} \right) \sum_{i=1}^{p} \frac{1}{L_i} E_{p-1} \left( \nu - \frac{1}{2}; \ldots, \hat{L}_i, \ldots \right)$$

$$+ \frac{2\sqrt{\pi}}{p \Gamma(\nu)} W_p \left( \nu - \frac{1}{2}, L_1, \ldots, L_p \right),$$

where the hat over the parameter $L_i$ in the functions $E_{p-1}$ means that it is excluded from the set $\{L_1, \ldots, L_p\}$ (the others being the $p-1$ parameters of $E_{p-1}$), and

$$W_p (\nu; L_1, \ldots, L_p) = \sum_{i=1}^{p} \frac{1}{L_i} \sum_{n_1,\ldots,n_p = 1}^{\infty} \left( \frac{\pi n_i}{L_i \sqrt{\cdots + \hat{L}_i n_i^2 + \cdots}} \right)^{\nu} K_\nu \left( \frac{2\pi n_i}{L_i} \sqrt{\cdots + \hat{L}_i n_i^2 + \cdots} \right),$$

with $\cdots + \hat{L}_i n_i^2 + \cdots$ representing the sum $\sum_{j=1}^{p} L_j^2 n_j^2 - L_i^2 n_i^2$.

We can derive expressions for each particular value of $d$, from 1 to $D$, but let us restrict ourselves to the most expressive values, $d = 1, 2, 3$. For $D = 3$, these correspond respectively to materials in the form of a film, a wire, or a grain.

1. One compactified dimension (a film)

By taking $d = 1$, the compactification of just one dimension, let us say, along the $x_1$-axis, we are considering that the system is confined between two planes, separated by a distance $L_1 = L$. Physically, for $D = 3$, this corresponds
to a film of thickness $L$. Then we have, from Eqs. (22), (24) and (26), in the $|p| \approx 0$ limit,

$$B_{d=1}(D, L) = (2\pi)^{-D/2} \int_0^1 dx \sum_{n=1}^{\infty} \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi n L_1} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} n L_1 \sqrt{p^2 x(1-x)} \right)$$

$$\sim (2\pi)^{-D/2} 2^{D/2-3} L^{1-D} \Gamma \left( \frac{D}{2} - 2 \right) \zeta(D-4), \quad (29)$$

where $\zeta(z)$ is the Riemann zeta function. The above expression is valid for all odd dimensions $D > 5$, due to the poles of the $\Gamma$ and $\zeta$ functions. We can obtain an expression for smaller values of $D$ by using the recurrence relations, Eq. (27); in the present case, this is equivalent to perform an analytic continuation of the Riemann zeta function $\zeta(D-4)$ by means of its reflexion property [35],

$$\zeta(z) = \frac{\Gamma \left( \frac{1-z}{2} \right)}{\Gamma(z/2)} \pi^{z-1/2} \zeta(1-z), \quad (30)$$

which gives

$$\zeta(D-4) = \frac{\Gamma \left( \frac{D}{2} \right)}{\Gamma \left( \frac{D}{2} - 2 \right)} \pi^{D-9/2} \zeta(5-D). \quad (31)$$

Then Eq. (29) becomes an expression valid for $2 < D < 4$ given by

$$B_{d=1}(D, L) = 2^{-3} \pi^{(D-9)/2} L^{1-D} \Gamma \left( \frac{5-D}{2} \right) \zeta(5-D). \quad (32)$$

For $D = 3$, we have $B_{d=1}(3, L) = L/48\pi$.

2. Two compactified dimensions (a wire)

Let us now take the case $d = 2$, in which the system is confined simultaneously between two parallel planes a distance $L_1$ apart from one another normal to the $x_1$-axis and two other parallel planes, normal to the $x_2$-axis separated by a distance $L_2$. That is, in the physical space the material is bounded within an infinite wire of rectangular cross section $L_1 \times L_2$. We then get for $|p| \approx 0$,

$$B_{d=2}(D; L_1, L_2) = (2\pi)^{-D/2} \left[ \int_0^1 dx \sum_{n=1}^{\infty} \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi n L_1} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} n L_1 \sqrt{p^2 x(1-x)} \right) \right. \right.$$

$$+ \left. \int_0^1 dx \sum_{n=1}^{\infty} \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi n L_2} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} n L_2 \sqrt{p^2 x(1-x)} \right) \right.$$

$$+ 2 \int_0^1 dx \sum_{n_1,n_2=1}^{\infty} \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi \sqrt{L_1 n_1^2 + L_2 n_2^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2} \sqrt{p^2 x(1-x)} \right) \right.$$ 

$$\sim 2^{-3} \pi^{(D-9)/2} \left( L_1^{4-D} + L_2^{4-D} \right) \Gamma \left( \frac{5-D}{2} \right) \zeta(5-D) + 2^{-2} \pi^{-D/2} \Gamma \left( \frac{D}{2} - 2 \right) E_2 \left( \frac{D}{2} - 2; L_1, L_2 \right), \quad (33)$$

with $E_2$ defined in Eq. (26) and valid for $\text{Re}(D) > 3$.

In particular, noticing that $E_1(\nu; L_j) = L_j^{-2\nu} \zeta(2\nu)$, one finds

$$E_2 \left( \frac{D-2}{2}; L_1, L_2 \right) = - \frac{1}{4} \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D-2)$$

$$+ \frac{\sqrt{\pi} \Gamma \left( \frac{D-3}{2} \right)}{4 \Gamma \left( \frac{D-2}{2} \right)} \left( \frac{1}{L_1 L_2^{D-3}} + \frac{1}{L_1^{D-3} L_2} \right) \zeta(D-3) + \frac{\sqrt{\pi}}{\Gamma \left( \frac{D-2}{2} \right)} W_2 \left( \frac{D-3}{2}; L_1, L_2 \right), \quad (34)$$
which is a meromorphic function of $D$, symmetric in the parameters $L_1$ and $L_2$. The function $W_2((D - 3)/2; L_1, L_2)$ in Eq. (34) is the particular case of Eq. (23) for $p = 2$.

This equation presents no problems for $3 < D < 4$ but, for $D = 3$, the first and second terms between brackets of Eq. (34) are divergent due to the $\zeta$ function and the $\Gamma$ function, respectively. However, these two divergences cancel out. No regularization is needed. This can be seen by remembering the property

$$\lim_{z \to 1} \left[ \zeta(z) - \frac{1}{z - 1} \right] = \gamma,$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant and, using the expansion of $\Gamma((D - 3)/2)$ around $D = 3$,

$$\Gamma\left(\frac{D - 3}{2}\right) \approx \frac{2}{D - 3} + \Gamma'(1),$$

$\Gamma'(z)$ standing for the derivative of the $\Gamma$ function with respect to $z$. For $z = 1$, it coincides with the Euler digamma function $\psi(1)$, which has the particular value $\psi(1) = -\gamma$. The two divergent terms generated by the use of formulas (35) and (36) cancel exactly for $D = 3$. Thus, remembering Eq. (21), the domain of existence of $B_{d=2}(D; L_1, L_2)$ can, as in the case of films, be extended to $2 < D < 4$.

3. Three compactified dimensions (a grain)

Finally, we may compactify three of the dimensions, which leaves us in $D = 3$ with a system which is a grain of some material in the form of a parallelepiped. We have, for arbitrary $D$, for $|p| \approx 0$,

$$B_{d=3}(D; L_1, L_2, L_3) = (2\pi)^{-D/2} \left[ \int_0^1 dx \sum_{n=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi n L_1} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi n L_1} \sqrt{p^2x(1-x)} \right) 
+ \int_0^1 dx \sum_{n=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi n L_2} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi n L_2} \sqrt{p^2x(1-x)} \right) 
+ \int_0^1 dx \sum_{n=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi n L_3} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi n L_3} \sqrt{p^2x(1-x)} \right) 
+ 2 \int_0^1 dx \sum_{n_1, n_2=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi \sqrt{L_1 n_1^2 + L_2 n_2^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{L_1 n_1^2 + L_2 n_2^2} \sqrt{p^2x(1-x)} \right) 
+ 2 \int_0^1 dx \sum_{n_1, n_3=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi \sqrt{L_1 n_1^2 + L_3 n_3^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{L_1 n_1^2 + L_3 n_3^2} \sqrt{p^2x(1-x)} \right) 
+ 2 \int_0^1 dx \sum_{n_2, n_3=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi \sqrt{L_2 n_2^2 + L_3 n_3^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{L_2 n_2^2 + L_3 n_3^2} \sqrt{p^2x(1-x)} \right) 
+ 4 \int_0^1 dx \sum_{n_1, n_2, n_3=1}^{\infty} \left( \frac{\sqrt{p^2x(1-x)}}{2\pi \sqrt{L_1 n_1^2 + L_2 n_2^2 + L_3 n_3^2}} \right)^{D/2-2} K_{D/2-2} \left( \frac{1}{2\pi} \sqrt{L_1 n_1^2 + L_2 n_2^2 + L_3 n_3^2} \sqrt{p^2x(1-x)} \right) \right] 
\sim \frac{1}{8} \pi^{(D-9)/2} (L_1^{4-D} + L_2^{4-D} + L_3^{4-D}) \Gamma \left( \frac{5 - D}{2} \right) \zeta(5 - D) 
+ \frac{1}{4\pi D/2} \Gamma \left( \frac{D}{2} - 2 \right) \left[ E_2 \left( \frac{D}{2} - 2; L_1, L_2 \right) + E_2 \left( \frac{D}{2} - 2; L_1, L_3 \right) + E_2 \left( \frac{D}{2} - 2; L_2, L_3 \right) \right] 
+ \frac{1}{2\pi D/2} \Gamma \left( \frac{D}{2} - 2 \right) E_3 \left( \frac{D}{2} - 2; L_1, L_2, L_3 \right) \right].$$

(37)
The analytical structure of the function $E_3((D-2)/2;L_1,L_2,L_3)$ in the equation above can be obtained from the general symmetrized recurrence relation given by Eqs. (27) and (28); explicitly, one has

$$E_3\left(\frac{D-2}{2};L_1,L_2,L_3\right) = -\frac{1}{6} \sum_{i,j=1}^{3} E_2\left(\frac{D-2}{2};L_i,L_j\right) + \frac{\sqrt{\pi}}{6 \Gamma\left(\frac{D-2}{2}\right)} \sum_{i,j,k=1}^{3} \frac{1}{L_i} E_2\left(\frac{D-2}{2};L_j,L_k\right)$$

$$+ \frac{2 \sqrt{D}}{3 \Gamma\left(\frac{D-2}{2}\right)} W_3\left(\frac{D-3}{2};L_1,L_2,L_3\right), \quad (38)$$

where $\varepsilon_{ijk}$ is the totally antisymmetric symbol and the function $W_3$ is a particular case of Eq. (28). The first two terms in the square bracket of Eq. (38) diverge as $D \to 3$ due to the poles of the $\Gamma$ and $\zeta$ functions. However, as it happens in the case of wires, it can be shown that these divergences cancel exactly one another, leaving an extended domain of validity $2 < D < 4$, for $B_{d=3}(D;L_1,L_2,L_3)$.

**C. The $\beta$ function and the fixed points**

For all $d \leq D$, within the domain of validity of $D$, we have, by inserting (11) in Eq. (3), the running coupling constant

$$\lambda(|p| \approx 0, D, \{L_i\}) \approx \frac{\lambda}{1 + \lambda \lambda [A(D)|p|^D + B_d(D,\{L_i\})]|. \quad (39)$$

Let us take $|p|$ as a running scale, and define the dimensionless coupling

$$g = \lambda (p, D, \{L_i\}) |p|^D. \quad (40)$$

We recall that in the previous expressions $p$ is a $D$-dimensional vector.

It is widely known that the $\beta$ function controls the rate of the renormalization-group flow of the running coupling constant and that a (nontrivial) fixed point of this flow is given by a (nontrivial) zero of the $\beta$ function. For $|p| \approx 0$, it is obtained straightforwardly from Eq. (40):

$$\beta(g) = |p| \frac{\partial g}{\partial |p|} \approx (D - 4) [g - A(D)g^2], \quad (41)$$

from which we get the infrared-stable fixed point

$$g_\ast(D) = \frac{1}{A(D)}. \quad (42)$$

We see that the $L_\ast$-dependent $B_d$-part of the subdiagram II does not play any role in this expression and, as remarked before, $A(D)$ is the same for all number of compactified dimensions, so is $g_\ast$ only dependent on the space dimension.

**III. THE SYSTEM WITH AN EXTERNAL MAGNETIC FIELD**

**A. The Landau-level basis**

In this section, we take the same $N$-component Ginzburg–Landau model to describe the behavior of confined systems, now in the presence of an external magnetic field, at leading order in $1/N$. The system is again constrained to a $d$-dimensional subspace of $\mathbf{R}^D$ in the form of a parallelepiped. The Hamiltonian density is then modified to

$$\mathcal{H} = [\partial_{\mu} - ieA_{\mu}^{\text{ext}}] \varphi_a \left(\partial_{\mu} - ieA_{\mu}^{\text{ext}}\right) \varphi_a + m^2 \varphi_a \varphi_a + u (\varphi_a \varphi_a)^2, \quad (43)$$

where summation over repeated indices is assumed and $m^2 = \alpha(T - T_c)$, with $\alpha > 0$. For $D = 3$, from a physical point of view, such Hamiltonian is supposed to describe type-II superconductors. In this case, we assume that the external magnetic field $\mathbf{H}$ is parallel to the $z$-axis and we choose the gauge $A_{\text{ext}} = (0, xH, 0)$. The model with $N$ complex components is taken in the large-$N$ limit with $N\mu = \lambda$ fixed. If we consider the system in unlimited space, the field $\varphi$ should be written in terms of the well-known Landau-level basis,

$$\varphi(r) = \sum_{\ell=0}^{\infty} \int \frac{dp_{\ell}}{2\pi} \int \frac{d^{D-2}p}{(2\pi)^{D-2}} \varphi_{\ell,p_\mu,p_\ell,p_\mu,p}(r), \quad (44)$$
where \( \chi_{\ell,p_y,p}(\mathbf{r}) \) are the Landau-level eigenfunctions given by

\[
\chi_{\ell,p_y,p}(\mathbf{r}) = \frac{1}{\sqrt{2^\ell \ell!}} \left( \frac{\omega}{\pi} \right)^{1/4} e^{i(p_y r + p_y y)} e^{-\omega(x - p_y/\omega)^2/2} H_\ell \left( \sqrt{\omega x - \frac{p_y}{\omega}} \right),
\]

with energy eigenvalues \( E_\ell(p) = |p|^2 + (2\ell + 1) \omega + m^2 \) and \( \omega = eH \) is the so-called cyclotron frequency. In the above equation, \( \mathbf{p} \) and \( \mathbf{r} \) are \((D - 2)\)-dimensional vectors.

Let us consider the system confined as in the previous sections, and use Cartesian coordinates \( \mathbf{r} = (x_1, \ldots, x_d, \mathbf{z}) \), where \( \mathbf{z} \) now is a \((D - 2 - d)\)-dimensional vector, with corresponding momenta \( \mathbf{k} = (k_1, \ldots, k_d, \mathbf{q}) \), \( \mathbf{q} \) being a \((D - 2 - d)\)-dimensional vector in momentum space. That is, the superconducting material is confined to a subspace of the \((D - 2)\)-dimensional Euclidean space in the form of a \(d\)-dimensional parallelepiped. Under these conditions, the generating functional of correlation functions is written as

\[
\mathcal{Z} = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \exp \left( -\int_0^{L_1} dx_1 \cdots \int_0^{L_d} dx_d \int d^{D-2} z \mathcal{H}(|\varphi|, |\nabla\varphi|) \right),
\]

with the field \( \varphi(x_1, \ldots, x_d, \mathbf{z}) \) satisfying the box-confinement condition as in Section II. Then the field representation should be modified and have a mixed series-integral Fourier expansion of the form

\[
\varphi(x_1, \ldots, x_d, \mathbf{z}) = \sum_{\ell=0}^{\infty} \sum_{i=1}^{d} \sum_{n_i=-\infty}^{\infty} c_{n_i} \int \frac{dp_y}{2\pi} \int d^{D-d-2} \mathbf{q} b(\mathbf{q}) e^{-i\omega_{n_i} x - i\mathbf{q} \cdot \mathbf{z}} \tilde{\varphi}_\ell(\omega_{n_i}, \mathbf{q}),
\]

where, for \( i = 1, \ldots, d \), \( \omega_{n_i} = 2\pi n_i/L_i \) and the coefficients \( c_{n_i} \) and \( b(\mathbf{q}) \) correspond respectively to the Fourier series representation over the \( x_i \) and to the Fourier integral representation over the \((D - d - 2)\)-dimensional \( \mathbf{z} \)-space. As was done previously, we now apply the Matsubara-like formalism according to \([10]\).

### B. Infrared behavior

In the following, we consider only the lowest Landau level \( \ell = 0 \). For \( D = 3 \), this assumption usually corresponds to the description of superconductors in the extreme type-II limit. Under this assumption, we obtain that the effective \( |\varphi|^4 \) interaction in momentum space and at the critical point as

\[
\lambda(p, D, \{L_1\}; \omega) = \frac{\lambda}{1 + \lambda e^{-(1/2)\omega(p^2 + p_y^2)\Pi(p, D, \{L_1\}, m = 0; \omega)}},
\]

where the single 1-loop bubble \( \Pi(p, D, \{L_1\}, m = 0; \omega) \) is given by

\[
\Pi(p, D, \{L_1\}, m = 0; \omega) = \frac{1}{L_1 \cdots L_d} \sum_{i=1}^{d} \sum_{n_i=-\infty}^{\infty} \int_0^{1} dx \int \frac{d^{D-d-2} \mathbf{q}}{(2\pi)^{D-d-2}} \left[ \frac{1}{\mathbf{q}^2 + \omega_{n_i}^2 + \cdots + \omega_{n_d}^2 + p^2 x (1 - x)} \right]^{D/2-3}.
\]

This is the same kind of expression that is encountered in the previous section, Eq. \([3]\), with the only modification that \( D \to D - 2 \). Also, one should be reminded that \( p \) is now a \((D - 2)\)-dimensional vector. The analysis is then performed along the same lines and we obtain, analogously,

\[
\Pi(p, D, \{L_1\}, m = 0; \omega) = (2\pi)^{1-D/2} \left[ 2^{1-D/2} \left( \frac{1}{(2\pi)^2} e^{\Gamma(D)\left(3 - \frac{D}{2}\right)} \right) \left( \frac{p^2}{2} \right)^{D/2-3} \right. \\
+ \int_0^{1} dx \sum_{i=1}^{d} \sum_{n_i=1}^{\infty} \left( \frac{\sqrt{p^2 x (1 - x)}}{2\pi L_i n_i} \right)^{D/2-3} K_{D/2-3} \left( \frac{1}{2\pi} \sqrt{p^2 x (1 - x) L_i n_i} \right) \\
+ 2 \int_0^{1} dx \sum_{i < j=1}^{d} \sum_{n_i, n_j=1}^{\infty} \left( \frac{\sqrt{p^2 x (1 - x)}}{2\pi \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}} \right)^{D/2-3} K_{D/2-3} \left( \frac{1}{2\pi} \sqrt{p^2 x (1 - x) \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}} \right) + \cdots.
\]
\[ +2^{d-1} \int_0^1 dx \sum_{n_1, \ldots, n_d=1}^\infty \left( \frac{\sqrt{p^2 x(1-x)}}{2\pi \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2}} \right)^{D/2-3} \times K_{D/2-3} \left( \frac{1}{2\pi} \sqrt{p^2 x(1-x)} \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right), \]

where

\[ c(D) = \int_0^1 dx (x(1-x))^{D/2-3} = 2^{5-D} \sqrt{\pi} \frac{\Gamma(\frac{D}{2}-2)}{\Gamma(\frac{D}{2})}, \quad \text{for } \text{Re}(D) > 4. \]

As for the infrared behavior of the \( \beta \) function, it suffices to study it in the neighborhood of \( |p| = 0 \), so that we can again use the asymptotic formula (24). It turns out that in the \( |p| \to 0 \) limit, the bubble \( \Pi \) is written in the form

\[ \Pi(|p| \approx 0, D, \{L_i\}, m = 0; \omega) = A_1(D) |p|^{D-6} + C_d(D, \{L_i\}), \]

with

\[ A_1(D) = (2\pi)^{-D/2-1} 2^{1-D/2} c(D) \Gamma \left( 3 - \frac{D}{2} \right), \]

and where the quantity \( C_d(D, \{L_i\}) \) is obtained by simply making the change \( D \to D-2 \) in the formula for \( B_d(D, \{L_i\}) \) in the preceding section.

C. Fixed points

Let us define a dimensionless coupling constant by

\[ g = \omega \lambda \langle |p| \approx 0, D, \{L_i\}; \omega \rangle |p|^{D-6}. \]

Then, after performing manipulations entirely analogous to those in Section and recalling Eq. (51), we have the extended domain of validity \( 4 < D < 6 \) for the quantities \( C_{d=1}(D; L_1), C_{d=2}(D; L_1, L_2) \) and \( C_{d=3}(D; L_1, L_2, L_3) \).

As in the preceding section, we take as a running scale \( |p| \), and define the dimensionless coupling

\[ g^{(1)} = \omega \lambda (p_1 = p_2 = 0, D, \{L_i\}) |p|^{D-6}, \]

where we remember that in this context \( p \) is a \( (D-2) \)-dimensional vector. Then, we obtain the \( \beta \) function for \( |p| \approx 0 \):

\[ \beta(g) = |p| \frac{dg^{(1)}}{dp} \approx (D-6) \left[ g^{(1)} - A_1(D) g^{(1)} \right], \]

from which the infrared-stable fixed point

\[ g^{(1)}_{*} = \frac{1}{A_1(D)} \]

is obtained.

IV. CONCLUDING REMARKS

In this article, we have discussed the infrared behavior and the fixed-point structure of the \( N \)-component Ginzburg–Landau model in the large-\( N \) limit, the system being confined in a \( d \)-dimensional box with edges of length \( L_i \), \( i = 1, 2, \ldots, d \) (compactification in a \( d \)-dimensional subspace). For \( D = 3 \) and \( d = 1, 2, 3 \), the system is supposed to describe, respectively, a film of thickness \( L \), an infinitely long wire of cross-section \( L_1 \times L_2 \), and a grain of volume \( L_1 \times L_2 \times L_3 \). We have studied the cases in which the system has no external influence and in which the system is submitted to the action of an applied external magnetic field. In both situations, with or without an external magnetic
field, we get the result that the existence of an infrared-stable fixed point depends only on the space dimension $D$; it does not depend on the number of compactified dimensions.

In the absence of an external magnetic field, we find that, for $2 < D < 4$, our result is the existence of an infrared-stable fixed point, in agreement with previous renormalization-group calculations for materials in bulk form (all $L_i = \infty$) in the literature (see, for instance, [8] and other references therein). Taking $D = 3$, we demonstrate directly that in the absence of a magnetic field, the superconducting transition in films, wires and grains is a second-order one. Moreover, the fixed point is independent of the size of the system or, in other words, the nature of the transition in the absence of a magnetic field is insensitive to the confining geometry.

In the case of the system in the presence of an external magnetic field, it is interesting to compare our results with those obtained for type-II materials in bulk form. For instance, a large-$N$ analysis and a functional renormalization-group study performed in Refs. [5, 57, 38] conclude for a second-order transition in dimensions $4 < D < 6$. The same conclusion is obtained in Ref. [10]. The authors of Ref. [57] claim, moreover, that the inclusion of fluctuations does not alter significantly the main characteristic of the system, that is, the existence of a continuous transition into a spatially homogeneous condensate. For the system under the action of an external magnetic field, the existence of a fixed point for $4 < D < 6$ should be taken as an indication, not as a demonstration, of the existence of a continuous transition. As already discussed in [37, 38], in this case, even if infrared fixed points exist, none of them can be completely attractive. The existence of an infrared fixed point in the presence of a magnetic field, as found in this paper, does not assure the (formal) existence of a second-order transition. Anyway, we conclude that, for materials in the form of films, wires and grains under the action of an external magnetic field, as is also the case for materials in bulk form, if there exists a phase transition for $D < 4$, in particular in $D = 3$, it should not be a second-order one.

Our results for a confined material in absence of an external magnetic field in dimension $2 < D < 4$ and for confined materials submitted to an external magnetic field for dimensions $4 < D < 6$ are in agreement with previous results for the bulk. Notice the shift of 2 in the range of dimensions for which a second-order transition would occur in both cases. As a final remark, from Eqs. (21) and (51), we see that, for $D \leq 2$ and for $D \leq 4$, respectively, in particular for $D = 1$, severe infrared divergences appear under the form of a divergence of the integrals in the quantities $b(D)$ and $c(D)$. Here we recover the well-known Peierls theorem, which forbids the existence of phase transitions in unidimensional spaces.

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