Bounding the expectation of the supremum of empirical processes indexed by Hölder classes

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Abstract

We obtain upper bounds on the expectation of the supremum of empirical processes indexed by Hölder classes of any smoothness and for any distribution supported on a bounded set. Another way to see it is from the point of view of integral probability metrics (IPM), a class of metrics on the space of probability measures: our rates quantify how quickly the empirical measure obtained from \( n \) independent samples from a probability measure \( P \) approaches \( P \) with respect to the IPM indexed by Hölder classes. As an extremal case we recover the known rates for the Wasserstein-1 distance.

1 Introduction

Organization of the note In Section 1 we introduce our notations before presenting and formally defining Hölder classes and Integral Probability Metrics (IPM). In Section 2 we expose some important concepts and results from empirical process theory needed for our proofs. We end this note by stating our main theorem in Section 3. The proofs are postponed to the appendix.

Notations A multi-index \( \mathbf{k} \) is a vector with integer coordinates \((k_1, \ldots, k_d)\). We write \(|\mathbf{k}| = \sum_{i=1}^d k_i\). For a given multi-index \( \mathbf{k} = (k_1, \ldots, k_d) \), we define the differential operator

\[
D^\mathbf{k} = \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}}.
\]

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For any positive real number \( x \), \( \lfloor x \rfloor \) denotes the largest integer strictly smaller than \( x \). We let \( \mathcal{X} \) be a convex bounded set in \( \mathbb{R}^d \) with non-empty interior. We assume that all the functions and function classes considered in this note are supported on the bounded set \( \mathcal{X} \).

For any integer \( k \), we denote by \( C^k(\mathcal{X}, \mathbb{R}) \) the class of real-valued functions with domain \( \mathcal{X} \) which are \( k \)-times differentiable with continuous \( k \)-th differentials. For any real-valued bounded function \( f \) on \( \mathcal{X} \), we let \( \| f \|_\infty := \sup_{x \in \mathcal{X}} |f(x)| \in [0, +\infty) \). We let \( \| \cdot \| \) denote some norm on \( \mathbb{R}^d \). We denote by \( \sigma_1, \ldots, \sigma_n \) i.i.d. Rademacher random variables, i.e., discrete random variables such that \( \mathbb{P}(\sigma_1 = 1) = \mathbb{P}(\sigma_1 = -1) = 1/2 \) which are independent of any other source of randomness.

1.1 H"{o}lder classes

A central problem in nonparametric statistics is to estimate a function belonging to an infinite-dimensional space (e.g., density estimation, regression function estimation, hazard function estimation), see Tsybakov (2008) for an introduction to the topic of nonparametric estimation. To obtain nontrivial rates of convergence, some kind of regularity is assumed on the function of interest. It can be expressed as conditions on the function itself, on its derivatives, on the coefficients of the function in a given basis, etc. H"{o}lder classes are one of the most common classes considered in the nonparametric estimation literature, they form a natural extension of Lipschitz-continuous functions and can be formalized with the following simple conditions. For any real number \( \alpha > 0 \), we define the H"{o}lder norm of smoothness \( \alpha \) of a \( \lfloor \alpha \rfloor \)-times differentiable function \( f \) as

\[
\| f \|_{H^\alpha} := \max_{|k| \leq \lfloor \alpha \rfloor} \| D^k f \|_\infty + \max_{|k| = \lfloor \alpha \rfloor} \sup_{x \neq y} \frac{|D^k f(x) - D^k f(y)|}{\|x - y\|^{\alpha - \lfloor \alpha \rfloor}}.
\]

The H"{o}lder ball of smoothness \( \alpha \) and radius \( L > 0 \), denoted by \( \mathcal{H}^\alpha(L) \), is then defined as the class of \( \lfloor \alpha \rfloor \)-times continuously differentiable functions with H"{o}lder norm bounded by the radius \( L \):

\[
\mathcal{H}^\alpha(L) = \{ f \in C^{\lfloor \alpha \rfloor}(\mathcal{X}, \mathbb{R}) \mid \| f \|_{H^\alpha} \leq L \}.
\]

To get a grasp of why H"{o}lder classes are convenient, let us consider the case \( d = 1 \). In this setting, one can easily derive an upper bound on the remainder of the best polynomial approximation of any given H"{o}lder function. Indeed, for any positive \( \alpha > 0 \) with \( \lfloor \alpha \rfloor = \ell \), for any function \( f \in \mathcal{H}^\alpha(L) \), Taylor’s theorem yields that for any points \( x, y \in \mathcal{X} \),

\[
\left| f(y) - \sum_{k=0}^\ell \frac{f^{(k)}(x)}{k!}(y - x)^k \right| \leq \frac{|y - x|^\ell}{(\ell - 1)!} \int_0^1 |f^{(\ell)}(x + t(y - x)) - f^{(\ell)}(x)|(1 - t)^{\ell-1}dt
\]

\[
\leq L \frac{|y - x|^{\alpha}}{(\ell - 1)!} \int_0^1 t^{\alpha - \ell}(1 - t)^{\ell-1}dt
\]

\[
\leq L \frac{|y - x|^{\alpha}}{\ell!}.
\]

Note that this bound holds uniformly over the H"{o}lder ball \( \mathcal{H}^\alpha(L) \).
1.2 Integral probability metrics

The class $\mathcal{H}^1(1)$ of 1-Lipschitz functions has received a lot of attention in the optimal transport literature (see Santambrogio (2015) for an overview of the topic of mathematical optimal transport). This interest comes from the Kantorovitch duality, which implies that the Wasserstein-1 distance (also known as the earth mover’s distance) can be expressed, for any probability measures $P, Q$, as a supremum of some functional over 1-Lipschitz functions:

$$W_1(P, Q) = \sup_{f \in \mathcal{H}^1(1)} |\mathbb{E}_{X \sim P} f(X) - \mathbb{E}_{Y \sim Q} f(Y)|.$$ 

More generally, for a given class $\mathcal{F}$ of bounded functions, one can define a pseudo-metric on the space of probability measures, the integral probability metric (IPM) induced by the class $\mathcal{F}$, as

$$d_F(P, Q) = \sup_{f \in \mathcal{F}} |\mathbb{E}_{X \sim P} f(X) - \mathbb{E}_{Y \sim Q} f(Y)|.$$ 

The literature on IPM has recently been boosted by the advent of adversarial generative models (Arjovsky et al., 2017; Goodfellow et al., 2014). A reason for this is that an IPM can be seen as an adversarial loss: to compare two probability distributions, it seeks for the function which discriminates the most the two distributions in expectation. Initially studied by the deep learning community, impressive empirical results obtained by adversarial generative models on several tasks such as image generation led statisticians to study it theoretically (Briol et al., 2019; Chen et al., 2020; Liang, 2018) (see also Sriperumbudur et al. (2012) for statistical results on IPM in a general framework). Since, as pointed out earlier, Lipschitz functions are also Hölder, one can wonder what happens for IPM indexed by general Hölder classes.

To further motivate this, let us consider the abstract problem of minimum distance estimation: for a given probability measure $P$, find a distribution $Q$ in a given set of probability measures $\mathcal{Q}$ such that $Q$ is close to $P$ under the metric $d_F$:

$$\min_{Q \in \mathcal{Q}} d_F(Q, P).$$

(1)

For example, when $\mathcal{F}$ is taken to be the class of 1-Lipschitz function, this problem is known as minimum Kantorovitch estimation (Bassetti et al., 2006). In statistics, the probability $P$ is usually unknown and one is only given i.i.d. samples $X_1, \ldots, X_n$ from the probability distribution $P$. A natural strategy is then to employ the empirical distribution $P_n = 1/n \sum_{i=1}^n \delta_{X_i}$, as a proxy for the theoretical distribution and instead of (1) solve the problem:

$$\min_{Q \in \mathcal{Q}} d_F(Q, P_n).$$

(2)
Since the triangle inequality yields
\[ |d_\mathcal{F}(Q, P) - d_\mathcal{F}(Q, P_n)| \leq d_\mathcal{F}(P, P_n) = \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E} f(X) \right|, \]
one question of interest is to measure how fast the empirical measure approximates the true measure under the IPM $d_\mathcal{F}$. If the rates are fast, we do not lose much by considering the empirical problem (2) instead of the theoretical one of (1). However, if the rates are slow, one cannot expect the distances of the solutions to the measure $P$ to be close. We will see in the next section that the latter expression corresponds to the supremum of the empirical process indexed by the class $\mathcal{F}$, it will enable us to leverage the rich literature on empirical processes to obtain rates of convergence for $d_\mathcal{F}(P, P_n)$.

2 Empirical processes, metric entropy and Dudley’s bounds

This section provides a short account of the notions and tools from the theory of empirical processes which are necessary for stating and establishing the main result.

2.1 Empirical processes

Empirical process are ubiquitous in statistical learning theory, we refer the reader to Giné and Nickl (2016); Koltchinskii (2011) for a general presentation of results on empirical processes and their link with statistics and learning theory. For clarity, we begin by recalling the definition of an empirical process.

**Definition 1.** Let $\mathcal{F}$ be a class of real-valued functions $f : \mathcal{X} \to \mathbb{R}$, where $(\mathcal{X}, \mathcal{A}, P)$ is a probability space. Let $X$ be a random point in $\mathcal{X}$ distributed according to the law $P$ and let $X_1, \ldots, X_n$ be independent copies of $X$. The random process $(\mathbb{X}_n(f))_{f \in \mathcal{F}}$ defined by
\[ \mathbb{X}_n(f) := \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E} f(X) \]
is called an empirical process indexed by $\mathcal{F}$.

In our case, we are interested in controlling the (expectation of the) supremum of an empirical process, a common case in the literature. Most of the time, the first step to apply for achieving this goal is to “symmetrize” the empirical process as allowed by the following lemma. Let $\hat{R}_n(\mathcal{F})$ be the empirical Rademacher complexity of function class $\mathcal{F}$, defined as
\[ \hat{R}_n(\mathcal{F}) = \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i f(X_i) \middle| X_1, \ldots, X_n \right]. \]
Lemma 1 (Symmetrization). For any class $\mathcal{F}$ of $P$-integrable functions,

$$
\mathbb{E}\left[ \sup_{f \in \mathcal{F}} |X_n(f)| \right] \leq 2 \mathbb{E}\left[ \hat{R}_n(\mathcal{F}) \right].
$$

The advantage of Rademacher processes is that, regardless of the distribution of the random variable $X$ and the function class $\mathcal{F}$, for a fixed sample $X_1, \ldots, X_n$, the random variable $\sum_{i=1}^n \sigma_i f(X_i)$ has a sub-Gaussian behavior, in the following sense.

Definition 2 (Sub-Gaussian behavior). A centered random variable $Y$ has a sub-Gaussian behavior if there exists a positive constant $\sigma$ such that

$$
\mathbb{E}e^{\lambda Y} \leq e^{\lambda^2 \sigma^2 / 2}, \quad \forall \lambda \in \mathbb{R}.
$$

In that case, we define the sub-Gaussian norm$^1$ of $Y$ as

$$
\|Y\|_{\psi_2} = \inf \left\{ t > 0 : \mathbb{E}e^{Y^2/t^2} \leq 2 \right\}.
$$

Having a sub-Gaussian behavior essentially means to be at least as concentrated as a Gaussian random variable around its mean. Our definition is equivalent to the tail inequalities

$$
\mathbb{P}(|Y| > t) \leq 2e^{-t^2/(2\sigma^2)}, \quad \forall t > 0.
$$

This type of behavior will be crucial to obtain the main result of this note. Indeed, as we will see, the behavior of the supremum of an empirical process (and more generally a stochastic process) which has sub-Gaussian increments exclusively depends on the topology of the space by which the process is indexed.

2.2 Metric entropy

Let $(T, d)$ be a totally bounded metric space, i.e., for every real number $\varepsilon > 0$, there exists a finite collection of open balls of radius $\varepsilon$ whose union contains $M$. We give a formal definition of such finite collections, see also Figure 1 for an illustration.

Definition 3. Given $\varepsilon > 0$, a subset $T_\varepsilon \subset T$ is called an $\varepsilon$-cover of $T$ if for every $t \in T$, there exists $s \in T_\varepsilon$ such that $d(s, t) \leq \varepsilon$.

Note that adding any point to an $\varepsilon$-cover still yields an $\varepsilon$-cover. Thus we can look for $\varepsilon$-covers of a set with smallest cardinality, which we call covering number.

$^1$See (Vershynin, 2018, Section 2.5) for the link between definitions of sub-Gaussian random variables (bound on moment-generating function, tail inequalities...) and the Orlicz norm $\psi_2$. 

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Definition 4. The $\varepsilon$-covering number of $T$, denoted by $\mathcal{N}(T,d,\varepsilon)$, is the cardinality of the smallest $\varepsilon$-cover of $T$, that is
$$\mathcal{N}(T,d,\varepsilon) := \min \{|T^\varepsilon| : T^\varepsilon \text{ is an } \varepsilon\text{-cover of } T\}.$$  

The metric entropy of $T$ is given by the logarithm of the $\varepsilon$-covering number.

Remark 1. A totally bounded metric space $(T,d)$ is pre-compact in the sense that its closure is compact. The metric entropy (or entropic numbers) of $(T,d)$ can then be seen as some measure of compactness of the space. Indeed, $\mathcal{N}(T,d,\varepsilon)$ quantifies precisely how many balls of radius $\varepsilon$ are needed to cover the whole space $T$.

Entropic numbers for Hölder classes are known and can be found in e.g. (Shiryayev, 1993; Van Der Vaart and Wellner, 1996).

Theorem 1 (Theorem 2.7.3 in Van Der Vaart and Wellner (1996)). Let $\mathcal{X}$ be a bounded, convex subset of $\mathbb{R}^d$ with nonempty interior. There exists a constant $K_{\alpha,d}$ depending only on $\alpha$ and $d$ such that, for every $\varepsilon > 0$,
$$\log \mathcal{N}(\mathcal{H}^\alpha(1), \|\cdot\|_\infty, \varepsilon) \leq K_{\alpha,d} \lambda_d(\mathcal{X}^1) \varepsilon^{-d/\alpha},$$
where $\lambda_d$ is the $d$-dimensional Lebesgue measure and $\mathcal{X}^1$ is the 1-blowup of $\mathcal{X}$: $\mathcal{X}^1 = \{y : \inf_{x \in \mathcal{X}}\|y - x\| < 1\}$.

2.3 Dudley’s bound and its refined version

We now present classic results which show the link between the topology of the indexing set and the behavior of the supremum of the corresponding empirical process. Following
(Vershynin, 2018, Definition 8.1.1), for $K \geq 0$, we say that a random process $(X_t)_{t \in T}$ on a metric space $(T, d)$ has $K$-sub-Gaussian increments if

$$\|X_t - X_s\|_{\psi_2} \leq K d(t, s), \quad \text{for all } t, s \in T.$$ 

**Theorem 2** (Dudley’s inequality). Let $(X_t)_{t \in T}$ be a mean-zero random process on a metric space $(T, d)$ with $K$-sub-Gaussian increments. Then

$$\mathbb{E}\left[\sup_{t \in T} X_t\right] \leq CK \int_{0}^{+\infty} \sqrt{\log \mathcal{N}(T, d, \varepsilon)} \, d\varepsilon,$$

for some universal constant $C > 0$.

One drawback of Dudley’s bound is that the integral on the right hand side may diverge if the metric entropy of $T$ tends to infinity at a very fast rate when $\varepsilon \to 0$. For example, when the metric entropy is upper bounded by $\varepsilon^{-\gamma}$, as it was seen to be the case with $\gamma = d/\alpha$ for $\alpha$-Hölder-smooth $d$-variate functions, the integral converges if and only if $\gamma < 2$.

An improvement of Dudley’s bound in the case where the process $X_t$ is a Rademacher average indexed by a class of functions $\mathcal{F}$—circumventing the problem of divergence of the integral—was proposed by (Srebro et al., 2010, Lemma A.3) (see also (Srebro and Sridharan, 2010)). Before stating the theorem, let us recall the definition of the $L_2(P_n)$ norm of a function $f$:

$$\|f\|^2_{L_2(P_n)} = \int_{\mathcal{X}} f^2 \, dP_n = \frac{1}{n} \sum_{i=1}^{n} f(X_i)^2.$$

**Theorem 3.** Let $\mathcal{F} \subset \{ f : \mathcal{X} \to \mathbb{R} \}$ be any class of measurable functions containing the uniformly zero function and let $S_n(\mathcal{F}) = \sup_{f \in \mathcal{F}} \|f\|_{L_2(P_n)}$. We have

$$\hat{R}_n(\mathcal{F}) \leq \inf_{\tau > 0} \left\{ 4\tau + \frac{12}{\sqrt{n}} \int_{\tau}^{S_n(\mathcal{F})} \sqrt{\log \mathcal{N}(\mathcal{F}, L_2(P_n), \varepsilon)} \, d\varepsilon \right\}.$$

Note that the refined Dudley bound gives an upper bound on the empirical Rademacher process and depends on the metric entropy with respect to the empirical norm $L_2(P_n)$. The following simple lemma shows that the $L_2(P_n)$-norm can be replaced by the supremum-norm in the refined Dudley bound.

**Lemma 2.** Let $\mathcal{F}$ be any class of bounded functions defined on $\mathcal{X}$. For any sample $X_1, \ldots, X_n$, let $\mathcal{F}|_{X_1, \ldots, X_n}$ be the subset of $\mathbb{R}^n$ defined by

$$\mathcal{F}|_{X_1, \ldots, X_n} = \{ u \in \mathbb{R}^n : \exists f \in \mathcal{F} \text{ such that } u_i = f(X_i) \text{ for all } i = 1, \ldots, n \}.$$

For any $\varepsilon > 0$, we have

$$\mathcal{N}(\mathcal{F}, L_2(P_n), \varepsilon) \leq \mathcal{N}(\mathcal{F}|_{X_1, \ldots, X_n}, \|\cdot\|_{\infty}, \varepsilon) \leq \mathcal{N}(\mathcal{F}, \|\cdot\|_{\infty}, \varepsilon).$$
Proof. Let \( \{u_1, \ldots, u_M\} \) be a minimal \( \varepsilon \)-net for \( \mathcal{F}|_{X_1, \ldots, X_n} \) with respect to the supremum norm. Let \( f_1, \ldots, f_M \in \mathcal{F} \) be such that \((f_j(X_1), \ldots, f_j(X_n)) = u_j\) for every \( j = 1, \ldots, M \). Then, for any \( f \in \mathcal{F} \), there exists an index \( j \in [M] \) such that \( \max_i |f(X_i) - (u_j)_i| = \max_i |f(X_i) - f_j(X_i)| \leq \varepsilon \). Since for any function \( f \) in \( \mathcal{F} \),

\[
\|f - f_j\|_{L_2(P_n)}^2 = \frac{1}{n} \sum_{i=1}^n (f(X_i) - f_j(X_i))^2 \leq \|f - f_j\|_{\infty}^2,
\]

\( \{f_1, \ldots, f_M\} \) is an \( \varepsilon \)-net for \( \mathcal{F} \) with respect to the empirical \( L_2 \) norm. This proves the first inequality. Let now \( f_1, \ldots, f_M \) be an \( \varepsilon \)-net of \( (\mathcal{F}, \|\cdot\|_{\infty}) \). One readily checks that \( u_1, \ldots, u_M \) defined by \( u_j = (f_j(X_1), \ldots, f_j(X_n)) \) is an \( \varepsilon \)-net of \( \mathcal{F}|_{X_1, \ldots, X_n} \). This completes the proof. \( \square \)

3 Main theorem

We are now in a position to state the main theorem which gives, for an IPM defined by a Hölder class, the rate of convergence of the empirical measure towards its theoretical counterpart.

**Theorem 4.** Let \( \mathcal{X} \subset \mathbb{R}^d \) be a convex bounded set with non-empty interior. Let \( \mathcal{H}^\alpha(L) \) be the Hölder class of \( \alpha \)-smooth functions supported on the set \( \mathcal{X} \) and with Hölder norm bounded by \( L \). For any probability distribution \( P \) supported on \( \mathcal{X} \), denoting by \( P_n \) the empirical measure associated to i.i.d. samples \( X_1, \ldots, X_n \sim P \), we have,

\[
\mathbb{E}[d_{\mathcal{H}^\alpha(L)}(P_n, P)] = \mathbb{E}\left[ \sup_{h \in \mathcal{H}^\alpha(L)} |X_n(h)| \right] \leq cL \begin{cases} n^{-\alpha/d} & \text{if } \alpha < d/2, \\ n^{-1/2} \ln(n) & \text{if } \alpha = d/2, \\ n^{-1/2} & \text{if } \alpha > d/2, \end{cases}
\]

where \( c \) is a constant depending only on \( d, \lambda_d(\mathcal{X}^1) \) and \( \alpha \).

We notice two different regimes: for highly smooth functions (\( \alpha > d/2 \)), the rate of convergence does not depend on the smoothness \( \alpha \) nor on the dimension \( d \) and corresponds to the usual parametric rate of convergence (note that it also matches the rate known for the Maximum Mean Discrepancy metric, which is an IPM indexed by the unit ball of a RKHS with bounded kernel (Briol et al., 2019)). For less regular Hölder functions (\( \alpha < d/2 \)), the rate of convergence depends both on the smoothness and on the dimension in a typical curse of dimensionality behavior. These two regimes coincide, up to a logarithmic factor, at their smoothness boundary \( \alpha = d/2 \): we have a continuous transition in terms of the exponent of the sample size. Interestingly the rates we obtain interpolate between the \( n^{-1/d} \) rate known for Wasserstein-1 distance (Weed et al., 2019) when considering \( \mathcal{H}^1(1) \) and the \( n^{-1/2} \) rate for Maximum Mean Discrepancy when consider Hölder classes with enough smoothness. Those observations are summarized in Figure 2.
Figure 2: Exponent $p$ appearing in the rates of convergence $n^{-p}$ in Theorem 4 as a function of the smoothness $\alpha$.

Finally, let us be more precise about the constant $c$ appearing in Theorem 4, while keeping implicit the constant $K = K_{\alpha,d}$ taken from Theorem 1 (which only depends on $\alpha$ and $d$). From the proof of Theorem 4, we obtain

$$c = 12\min\{d,2\alpha\}
\frac{1}{(d-2\alpha)_+} (K_{\alpha,d} \lambda_d(\mathcal{X}^1))^{(\alpha/d)\wedge(1/2)}.$$

In the case $\alpha = d/2$ a more precise and explicit upper bound on the expected distance is given by

$$\mathbb{E}[d_{h^p(L)}(P_n,P)] \leq 12 \sqrt{\frac{K \lambda_d(\mathcal{X}^1)}{n}} \left\{ 1 + 0.5 \ln \left( \frac{n}{9K \lambda_d(\mathcal{X}^1)} \right) \right\}, \quad \alpha = d/2.$$

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4 Appendix: proofs

This section contains the proofs of the main results, Theorems 3 and 4, stated in the main body of the note.

4.1 Proof of Theorem 3

The proof of Theorem 3 can be found in Srebro and Sridharan (2010). We add it here for completeness.

Let \( \gamma_0 = S_n(\mathcal{F}) = \sup_{f \in \mathcal{F}} \| f \|_{L_2(P_n)} \). Define \( \gamma_j = 2^{-j} \gamma_0 \), for every integer \( j \in \mathbb{N} \), and let \( T_j \) be a minimal \( \gamma_j \)-cover of \( \mathcal{F} \) with respect to \( L_2(P_n) \). For any function \( f \in \mathcal{F} \), we denote by \( \hat{f}_j \) an element of \( T_j \) which is a \( \gamma_j \) approximation of \( f \). For any positive integer \( N \) we can decompose the function \( f \) as

\[
 f = f - \hat{f}_N + \sum_{j=1}^{N} (\hat{f}_j - \hat{f}_{j-1})
\]

where \( \hat{f}_0 = 0 \in \mathcal{F} \). Hence, for any positive integer \( N \), we have

\[
 \hat{R}_n(\mathcal{F}) = \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i \left( f(X_i) - \hat{f}_N(X_i) + \sum_{j=1}^{N} (\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i)) \right) \right]
\]

\[
 \leq \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i (f(X_i) - \hat{f}_N(X_i)) \right] + \sum_{j=1}^{N} \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i (\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i)) \right]
\]

\[
 \leq \frac{1}{n} \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} |(f(X_i) - \hat{f}_N(X_i))| + \sum_{j=1}^{N} \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i (\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i)) \right]
\]

\[
 = \sup_{f \in \mathcal{F}} \| f - \hat{f}_N \|_{L_2(P_n)} + \sum_{j=1}^{N} \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i (\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i)) \right]
\]

\[
 \leq \gamma_N + \sum_{j=1}^{N} \frac{1}{n} \mathbb{E}_\sigma \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i (\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i)) \right].
\]

For any positive integer \( j \), the triangle inequality gives

\[
 \| \hat{f}_j - \hat{f}_{j-1} \|_{L_2(P_n)} \leq \| \hat{f}_j - f \|_{L_2(P_n)} + \| f - \hat{f}_{j-1} \|_{L_2(P_n)} \leq \gamma_j + \gamma_{j-1} = 3\gamma_j. \tag{3}
\]

We need the following classic lemma which controls the expectation of a Rademacher average over a finite set\(^2\).

\(^2\)We refer the reader to https://ttic.uchicago.edu/~tewari/lectures/lecture10.pdf for a simple proof of this lemma.
Lemma 3 (Massart’s finite class lemma). Let $\mathcal{X}$ be a finite subset of $\mathbb{R}^n$ and let $\sigma_1, \ldots, \sigma_n$ be independent Rademacher random variables. Denote the radius of $\mathcal{X}$ by $R = \sup_{x \in \mathcal{X}} \|x\|$. Then, we have,

$$\mathbb{E} \left[ \sup_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i x_i \right] \leq R \frac{\sqrt{2 \log |\mathcal{X}|}}{n}.$$ 

Applying this lemma to $\mathcal{X}_j = \{(\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i))_{i=1}^{n} \in \mathbb{R}^n : f \in \mathcal{F}\}$ for any $j = 1, \ldots, n$ and using (3), we get

$$\sum_{j=1}^{N} \frac{1}{n} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \sigma_i (\hat{f}_j(X_i) - \hat{f}_{j-1}(X_i)) \right] \leq \sum_{j=1}^{N} 3 \gamma_j \frac{\sqrt{2 \log(|T_j| \cdot |T_{j-1}|)}}{n}$$

Therefore we have

$$\hat{R}_n(\mathcal{F}) \leq \gamma_N + \sum_{j=1}^{N} 3 \gamma_j \frac{\sqrt{2 \log(|T_j| \cdot |T_{j-1}|)}}{n} \leq \gamma_N + \frac{6}{n} \sum_{j=1}^{N} \gamma_j \sqrt{\log|T_j|} = \gamma_N + \frac{12}{n} \sum_{j=1}^{N} (\gamma_j - \gamma_{j+1}) \sqrt{\log|T_j|} \leq \gamma_N + \frac{12}{n} \int_{\gamma_{N+1}}^{70} \sqrt{\log \mathcal{N}(\mathcal{F}, L_2(P_n), \varepsilon)} \, d\varepsilon.$$ 

For any $\tau > 0$, pick $N = \sup \{ j : \gamma_j > 2\tau \}$. Then $\gamma_N = 2\gamma_{N+1} \leq 4\tau$ and $\gamma_{N+1} = \gamma_N / 2 \geq \tau$. Hence, we conclude that

$$\hat{R}_n(\mathcal{F}) \leq 4\tau + \frac{12}{\sqrt{n}} \int_{\tau}^{70} \sqrt{\log \mathcal{N}(\mathcal{F}, L_2(P_n), \varepsilon)} \, d\varepsilon.$$ 

Since $\tau$ can take any positive value we can take the infimum over all positive $\tau$ and this concludes the proof.

4.2 Proof of Theorem 4

Without loss of generality, we prove the theorem in the case $L = 1$. The general case will follow by homogeneity. For simplicity we write $\mathcal{H}^\alpha = \mathcal{H}^\alpha(1)$, $P_h = \int_{\mathcal{X}} h \, dP$ and
\[ P_n h = \int_X h \, dP_n. \] A symmetrization argument (Lemma 1) gives

\[ \mathbb{E} \left[ \sup_{h \in \mathcal{H}_\alpha} |P_h - P_n h| \right] \leq 2 \mathbb{E} \left[ \hat{R}_n(\mathcal{H}_\alpha) \right], \]

where the empirical Rademacher process \( \hat{R}_n(\mathcal{H}_\alpha) \) is given by

\[ \hat{R}_n(\mathcal{H}_\alpha) = \frac{1}{n} \mathbb{E} \left[ \sup_{h \in \mathcal{H}_\alpha} \sum_{i=1}^n \sigma_i h(X_i) \right]. \]

Noting that, for any \( h \in \mathcal{H}_\alpha \),

\[ P_n h^2 := \frac{1}{n} \sum_{i=1}^n h^2(X_i) \leq \|h^2\|_\infty \leq 1, \]

the improved Dudley bound (Theorem 3) coupled with Lemma 2 yields, for \( \alpha \neq d/2 \),

\[ \mathbb{E} \left[ \sup_{h \in \mathcal{H}_\alpha} |P_n h - Ph| \right] \leq \inf_{\tau > 0} \left( 4 \tau + \frac{12}{\sqrt{n}} \int_{\tau}^{1} \sqrt{\log N(\mathcal{H}_\alpha, \|\cdot\|_\infty, \varepsilon)} d\varepsilon \right) \]

\[ \leq \inf_{\tau > 0} \left( 4 \tau + \frac{12 \sqrt{K \lambda_d(\mathcal{X}^1)}}{\sqrt{n}} \int_{\tau}^{1} \varepsilon^{-d/2\alpha} d\varepsilon \right) \]

\[ \leq \inf_{\tau > 0} \left( 4 \tau + \frac{24 \alpha \sqrt{K \lambda_d(\mathcal{X}^1)}}{2\alpha - d} (1 - \tau^{1-d/2\alpha}) \right) \]

\[ \leq \inf_{\tau > 0} \left( 4 \tau + \frac{24 \alpha \sqrt{K \lambda_d(\mathcal{X}^1)}}{d - 2\alpha} \tau^{-(d-2\alpha)/2\alpha} \right) \]

where \( K = K_{\alpha,d} \) is the constant depending only on \( \alpha \) and \( d \) borrowed from Theorem 1.

**Case** \( \alpha < d/2 \). The minimum is attained for \( \tau_\star = \left( 9K \lambda_d(\mathcal{X}^1)/n \right)^{\alpha/d} \) and it yields the upper bound

\[ 4\tau_\star + \frac{24 \alpha \sqrt{K \lambda_d(\mathcal{X}^1)}}{d - 2\alpha} \frac{1-d/2\alpha}{\tau_\star^{1-d/2\alpha}} = 4\tau_\star + \frac{4\tau_\star}{(d/2\alpha) - 1} = \frac{4\tau_\star d}{d - 2\alpha} = \frac{4d}{d - 2\alpha} \left( 9K \lambda_d(\mathcal{X}^1) \right)^{\alpha/d} n^{-\alpha/d} \]

\[ \leq \frac{12d}{d - 2\alpha} \left( \frac{K \lambda_d(\mathcal{X}^1)}{n} \right)^{\alpha/d}. \]

**Case** \( \alpha > d/2 \). Letting \( \tau \) go to zero, we get an upper bound equal to \( \frac{24 \alpha \sqrt{K \lambda_d(\mathcal{X}^1)}}{2\alpha - d} \).

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Case $\alpha = d/2$. The refined Dudley bound (3) gives

$$
\mathbb{E} \sup_{h \in H^\alpha} |Ph - P_n h| \leq \inf_{\tau > 0} \left\{ 4\tau + \frac{12 \sqrt{K \lambda_d(\mathcal{X}_1)}}{\sqrt{n}} \int_{\tau}^{1} \varepsilon^{-1} d\varepsilon \right\}
$$

$$
= \inf_{\tau > 0} \left\{ 4\tau - \frac{12 \sqrt{K \lambda_d(\mathcal{X}_1)}}{\sqrt{n}} \ln \tau \right\}.
$$

The minimum is attained for \(\tau^* = 3 \sqrt{K \lambda_d(\mathcal{X}_1) n^{-1/2}}\) and it yields an upper bound of order \(C \sqrt{\frac{K \lambda_d(\mathcal{X}_1)}{n}} \left\{ 1 + 0.5 \ln \left( \frac{n}{9 K \lambda_d(\mathcal{X}_1)} \right) \right\}\) where \(C\) is a positive absolute constant.