RENORMALIZED VOLUME AND
THE EVOLUTION OF APES

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Abstract. We study the evolution of the renormalized volume functional for asymptotically Poincaré-Einstein metrics \((M, g)\) which are evolving by normalized Ricci flow. In particular, we prove that

\[
\frac{d}{dt} \text{RenV}(M^n, g(t)) = - \int_{M^n} (S(g(t)) + n(n-1)) \, dV_{g(t)},
\]

where \(S(g(t))\) is the scalar curvature for the evolving metric \(g(t)\). This implies that if \(S + n(n-1) \geq 0\) at \(t = 0\), then \(\text{RenV}(M^n, g(t))\) decreases monotonically.

We also discuss how, when \(n = 4\), our results describe the Hawking-Page phase transition. Differences in renormalized volumes give rigorous meaning to the Hawking-Page difference of actions and describe the free energy liberated in the transition.

1. Introduction

There is a well-known connection between the Riemannian geometry of noncompact Poincaré-Einstein (PE) manifolds and the conformal geometry on their asymptotic boundaries, which are compact manifolds of one lower dimension. While relevant to the AdS/CFT correspondence in string theory [23, 30, 21, 25, 10], it traces back to earlier mathematical studies by Fefferman and Graham [12, 13], who were motivated by the program of classifying conformal invariants on the boundary in terms of the ‘interior’ Riemannian geometry of the PE space. PE geometry has appeared independently in various other mathematical guises too. Finally, this has other physical roots, related to the thermodynamics of anti-de Sitter black holes [19].

An interesting quantity in this setting is the renormalized volume of a PE space \((M^n, g)\). This is defined by an Hadamard regularization scheme of the volumes of certain special families of compact subdomains which exhaust \(M\). A remarkable theorem, due to Henningson-Skenderis [21] and Graham-Witten [18], see also [15], asserts that this renormalized volume, \(\text{RenV}(M, g)\), is well defined when \(n\) is even in the sense that is independent of the choice of exhausting sequence of domains (subject to the conditions that the boundaries form equidistant families). When \(n\) is odd, one obtains a quantity no longer independent of choices, but which depends on these choices in a very simple and comprehensible way. All of this is explained in great detail in [15], but is recalled below.

In this paper we study a class of spaces \((M^n, g)\) which are asymptotically Poincaré-Einstein (APE) in a strong asymptotic sense. It is still possible in this setting to define \(\text{RenV}([M, g])\) and show that it has the same invariance properties as for PE metrics. Our goal here is to consider the behavior of these APE spaces and of their renormalized volumes under the Ricci flow. We derive a formula for the time derivative of \(\text{RenV}([M, g(t)])\), where \(g(t)\) is a solution to the Ricci flow equations, and show that under certain circumstances this quantity

**Key words and phrases.** Ricci flow, conformally compact metrics, asymptotically hyperbolic metrics, renormalized volume, black hole thermodynamics.
is monotone. We give some applications and conclude by explaining the relevance of this circle of ideas to black hole thermodynamics. There is a well-known example of a manifold with boundary $M$ where the same conformal structure at infinity is induced by three non-isometric PE metrics in the interior [19]. We argue in §4 that the detailed consideration of phase transitions in black hole thermodynamics leads to consideration of Ricci flow of manifolds with APE metrics and their renormalized volumes.

We shall consider the normalized Ricci flow

$$\begin{align*}
\partial_t g &= -2 \left( \text{Rc}(g) + (n-1)g \right) := -2E(g), \quad t \in [0,T) \\
g(0) &= g^0,
\end{align*}$$

(1.1)

where $(M, g^0)$ is APE. The short-time existence for this flow when the initial data is APE (as well as for general smooth conformally compact initial data) was obtained in [5]. Other relevant papers include [28] and [2].

While the long-time behavior of conformally compact manifolds under Ricci flow is certainly no simpler than that in the compact case, one can obtain strong control at spatial infinity in any finite time slice in this class of spaces. Our first result is that the APE condition is preserved under the flow:

**Theorem A.** Let $(M^n, g^0)$, $n \geq 2$, be APE, and let $g(t)$ be a solution of (1.1). Then $(M, g(t))$ remains APE on some possibly smaller interval $[0, T_0)$.

This theorem is only slightly different than the existence result in [28] in that for those authors the approximately Poincaré-Einstein condition for the initial metric $g^0$ and the resulting flow $g(t)$ is phrased purely in terms of the decay of the Einstein tensor $E(g)$ at $\partial M$, whereas our result also posits that $g^0$ has an expansion and this is preserved under the flow. Although this seems like a minor change, this persistence of the expansion is an important point here. In a sequel to this paper we shall study the evolution under Ricci flow of asymptotically hyperbolic metrics which are asymptotic to Poincaré-Einstein metrics in a much weaker sense, and where the behavior of the expansion under the flow becomes one of central technical issues. We also note that it is highly likely that $T_0 = T$, or in other words, that the solution remains APE in the maximal interval of existence.

It is natural to study how various numerical quantities associated to the family of metrics $g(t)$ evolve with $t$. To that end, we prove

**Theorem B.** For $(M, g(t))$ as in Theorem A and $t \in [0, T_0)$, the renormalized volume $\text{RenV}[g(t)]$ satisfies

$$\begin{align*}
\frac{d}{dt} \text{RenV}((M, g(t))) &= - \int_M (S(g(t)) + n(n-1)) \, dV_{g(t)} \equiv - \int_M \text{tr}^g E(g(t)) \, dV_{g(t)} ,
\end{align*}$$

(1.2)

and

$$\begin{align*}
\frac{d^2}{dt^2} \text{RenV}((M, g(t))) &= - \int_M \left( 2 |E|^2 - 2(n-1) \text{tr}^g E - (\text{tr}^g E)^2 \right) \, dV_{g(t)} \\
&\equiv - \int_M \left[ 2 |Z|^2 - \frac{(n-3)}{(n-1)} (\text{tr}^g E)^2 - 2(n-1) \text{tr}^g E \right] \, dV_{g(t)} .
\end{align*}$$

(1.3)

Here $S(g(t))$ is the scalar curvature of $g(t)$, $\text{tr}^g E(g(t)) = S(g(t)) + n(n-1)$, and $Z(g(t))$ is the trace-free part of the Ricci tensor.
Note that if $S(g(t)) + n(n - 1) = 0$ at any value of $t$, so that RenV is stationary under the Ricci flow there, then this second variation reduces to

\begin{equation}
\left. \frac{d^2}{dt^2} \right|_{t=0} \text{RenV}(M, g(t)) = -2 \int_M |Z(g(0))|^2 dV_{g(0)}.
\end{equation}

There is a theory developed by Graham and Witten [18] of renormalized areas and volumes for complete, properly embedded minimal submanifolds in PE spaces, of arbitrary dimension and codimension, which have regular asymptotic boundaries. The special case of renormalized area for such two-dimensional minimal surfaces $\mathcal{Y} \subset \mathbb{H}^3$, studied at length in [3], bears a marked resemblance to the special case of the considerations here concerning renormalized volume of four-dimensional PE spaces. In particular, there is a local formula for renormalized area in that setting which mirrors the formula (1.1) for renormalized volume in four dimensions below. The formulae in [3] for the first and second variations of renormalized area appear quite different from (1.2) and (1.3), however, in that the variation formulae of [3] are entirely localized to the asymptotic boundary at infinity, whereas the formulae here are integrals over the interior. There are two distinct reasons for this. On the one hand, the metrics we consider here are not exactly Einstein, which accounts for the fact that only interior integrals appear here; on the other hand, our APE condition precludes terms in the expansion of the metric at order $n - 1$ (this is explained in the next section), which is the reason that the formulae above have no boundary terms. We will treat two more-involved cases in forthcoming works, specifically that of a metric with general even expansion, and that of the renormalized area of a hypersurface under the mean curvature flow [6].

Continuing on, the maximum principle applied to $S(g) + n(n - 1)$ gives

**Theorem C.** If $(M, g(t))$ is as in Theorem A and $S(g^0) + n(n - 1) \geq 0$, then $S(g(t)) + n(n - 1) \geq 0$ for all $t$ and RenV($g(t)$) is monotone decreasing. Furthermore, RenV($g(t)$) is constant on an interval $t \in I$ iff $S(g(t)) + n(n - 1) = 0$ for all $0 \leq t < T$.

One application of all of this is a no breathers theorem:

**Corollary D.** Let $(M, g(t))$ be as in Theorem A and suppose that $S(g^0) + n(n - 1) \geq 0$. If there exist times $0 \leq t_1 < t_2$ such that $(M, g(t_1))$ is isometric to $(M, g(t_2))$, then $(M, g(t))$ is a stationary solution and Einstein. If, in addition, $n < 8$ and the conformal class induced on $\partial_\infty M$ by $g^0$ is that of the standard round sphere, then $(M, g(t))$ is isometric to hyperbolic space $\mathbb{H}^n$ for all $t$.

The organization of this paper is as follows. In §2 we describe the various asymptotic conditions for metrics, including conformally compactifiable, asymptotically hyperbolic, and APE, and prove Theorem A. Next, in §3, we recall the alternative Riesz regularization method to define RenV and prove Theorem B. The proofs of Theorem C and Corollary D appear here too. Finally, §4 describes how the APE evolution of renormalized volume appears in black hole thermodynamics, where renormalized volume gives rigorous justification for the Hawking-Page difference of (formally divergent) actions. Our monotonicity theorem in the APE setting correlates well with numerical results of Headrick and Wiseman [19] in a different setting, that of “black holes in a finite box”, and suggests the same kind of novel free energy diagram.

We thank Gerhard Huisken for first posing the question which motivated this work. RM acknowledges support by National Science Foundation grant DMS–1105050. EB was supported in part by a new faculty startup grant at Seattle University. EW was supported by
2. The evolution of expansions

The renormalization scheme defining RenV requires that the metric \( g \) admits a particular type of asymptotic expansion near \( \partial M \). We begin by recalling some facts about such expansions and then show that they are preserved under the normalized Ricci flow.

To fix notation, let \( M^n \) be an open manifold which is the interior of a compact manifold with boundary \( \overline{M} \). A metric \( g \) on \( M \) is called conformally compact if there exists a smooth defining function \( x \) for \( \partial M \) such that \( \overline{g} = x^2 g \) extends to a smooth metric on \( \overline{M} \). The restriction of \( \overline{g} \) to \( T\partial M \) determines a conformal class \( \mathcal{C}(g) \) on the boundary, called the conformal infinity of \( g \). An asymptotically hyperbolic (AH) metric is one which is conformally compact and satisfies \( |dx|^2_{\overline{g}} = 1 \) on \( \partial M \). Observe that neither \( x \) nor \( \overline{g} \) are individually well-defined since \( g = x^{-2} \overline{g} = (ax)^{-2} (a^2 \overline{g}) \) for any positive smooth function \( a \), but the AH condition is independent of this ambiguity. It is also useful to consider conformally compact metrics with various other regularity conditions, for example metrics for which \( g \) and \( x \) are both polyhomogeneous rather than smooth.

An AH metric \( g \) is Poincaré-Einstein (PE) if it is also Einstein, i.e. if the Einstein tensor \( E(g) := \text{Rc}(g) + (n - 1)g \) vanishes identically. Any such metric is a stationary solution of the normalized Ricci flow. If \( g^0 \) is AH, then it is proved in [5] that the solution \( g(t) \) of the normalized Ricci flow with this initial data remains AH, and in particular, smoothly conformally compact, for \( t > 0 \). In the optimal situation, this solution \( g(t) \) should converge to a PE metric, but of course this may fail to happen because of the development of singularities. It is still unclear whether there is any possibility of new and specifically boundary singularities forming, or whether singularity formation is confined entirely to the interior, where it can be studied by the same techniques as in the compact case.

If \( g \) is AH, then a theorem of Graham and Lee [17] states that for each representative \( g_0 \) of \( \mathcal{C}(g) \), there is a uniquely defined special boundary defining function \( x \) such that

\[
g = \frac{dx^2 + g_x}{x^2}
\]

near \( \partial M \). Here \( g_x \) is a smooth family of symmetric 2-tensors on \( \partial M \), which can be expanded in a Taylor series as

\[
g_x = g_0 + x g_1 + \ldots + x^n g_n + \ldots
\]

This choice of special boundary defining function is a sort of coordinate gauge which is very useful for many of the algebraic computations concerning these metrics. For example, it is easy to check from (2.1) and (2.2) that if \( g \) is AH, then it is asymptotically Einstein to order 1 in the sense that \( |E(g)|_g = \mathcal{O}(x) \). On the other hand, if \( g \) is actually PE, then it was shown by Fefferman and Graham [12] that if \( g_0 \) and (perhaps surprisingly) also \( g_n \) are fixed, then all the remaining coefficients \( g_j \) are formally determined by the Einstein condition \( E(g) = 0 \). In particular, the coefficients \( g_1, \ldots, g_{n-1} \) are determined formally by \( g_0 \) alone.
More specifically,
\[
g(x) = \begin{cases} 
  x^{-2}(dx^2 + g_0 + x^2g_2 + \cdots + x^{n-2}g_{n-2} + x^{n-1}g_{n-1} + O(x^n)) & \text{for } n \text{ even}, \\
  x^{-2}(dx^2 + g_0 + x^2g_2 + \cdots + x^{n-3}g_{n-3} + x^{n-1}(g_{n-1} + g_{n-1}\log x) + O(x^n\log x)) & \text{for } n \text{ odd},
\end{cases}
\]
where each \(g_j\), \(0 \leq j \leq n - 2\) is obtained by applying a universal differential expression to \(g_0\), and moreover, \(\text{tr}^g g_{n-1} = 0\). For \(n\) odd, \(\tilde{g}_{n-1}\) is also determined by \(g_0\) and \(\text{tr}^g \tilde{g}_{n-1} = 0\).

This leads to the

**Definition 2.1.** An AH metric \(g\) is called asymptotically Poincaré-Einstein (APE) if \(|E(g)|_g = O(x^n)\).

Note that, by definition, our APE metrics have smooth conformal compactifications, so in particular do not have the terms involving log \(x\) in their expansions.

**Proposition 2.2.** If \(n = \dim M\) is even, and if \(g\) is APE, then the expansion of the term \(\hat{g}_x\) in the Graham-Lee normal form for \(g\), relative to any choice of representative \(g_0 \in \mathfrak{c}(g)\), has the form (2.3).

**Remark 2.3.** The key point here is that the coefficients \(g_2, \ldots, g_{n-1}\) and, for odd \(n\), \(\tilde{g}_{n-1}\) are all determined by \(g_0\), and are the same as the corresponding coefficients of the expansion of a PE metric \(g'\) for which \(g_0 \in \mathfrak{c}(g')\). Note, however, that there is no guarantee that a PE metric with the conformal infinity \(\|g_0\|\) necessarily exists!

**Sketch.** Since this result is only a small modification of a well known one when \(g\) is PE, we merely sketch the proof, following the argument in [16].

First observe that in any smooth coordinate system \((x, y_1, \ldots, y_{n-1})\), the APE condition implies that the components of the Einstein tensor in this coordinate system satisfy
\[
E(g)_{ij} = O(x^{n-2}).
\]

Now calculate \(E(g) = \text{Rc}(g) + (n - 1)g\) using (2.1) and (2.3). We use Greek indices to label components on the boundary and \(0\) for \(\partial_x\); a ‘prime’ denotes the derivative with respect to \(x\), and \(\nabla\) is the Levi-Civita connection for \(g_x\) fixed \(x\). From [16] we obtain
\[
2x E_{\alpha\beta} = -x g_{\alpha\beta}'' + x g^{\mu\nu} g_{\alpha\mu} g_{\beta\nu}' - \frac{x}{2} g^{\mu\nu} g_{\alpha\mu}' g_{\beta\nu}' + (n - 2) g_{\alpha\beta}' + g^{\mu\nu} g_{\alpha\mu} g_{\beta\nu}' + 2x \text{Rc}(g_x)_{\alpha\beta}.
\]
(2.5) \(E_{\alpha\beta} = E_{\beta\alpha} = \frac{1}{2} g^{\mu\nu} (\nabla_{\nu} g_{\alpha\mu}' - \nabla_{\alpha} g_{\nu\mu}'\).\)
\[
E_{00} = -\frac{1}{2} g^{\mu\nu} g_{\mu\nu}' + \frac{1}{4} g^{\mu\nu} g^{\lambda\sigma} g_{\lambda\mu} g_{\sigma\nu}' + \frac{1}{2} x^{-1} g_{\mu\nu}' g_{\mu\nu}'.
\]
Differentiating the first equation \(s - 1\) times with respect to \(x\), \(s \leq n - 2\), and then setting \(x = 0\) gives
\[
(n - 1 - s) \partial_x^s g_{\alpha\beta} + g^{\mu\nu} \partial_x^s g_{\mu\nu} \cdot g_{\alpha\beta}
= \partial_x^{s-1}(2x E_{\alpha\beta}) \bigg|_{x=0} + (\text{terms containing } \partial_x^k g_{\alpha\beta} \text{ with } k < s).
\]
(2.6) This yields the first \(n-2\) terms in the expansion of \(g_x\). For \(s = n - 1\), we obtain \(\partial_x^{n-2}(2x E_{\alpha\beta}) = O(x)\). Together with a parity argument, we see that the trace free part of \(g_{n-1}\) must vanish.
too. Finally, observe that all of these computations are insensitive to whether \( g \) is PE or simply APE.

As noted earlier, it is proved in [28] that if \( g^0 \) is a smooth AH metric, then the solution \( g(t) \) of normalized Ricci flow with initial condition \( g^0 \) remains AH, and moreover, \( \epsilon(g(t)) = \epsilon(g^0) \), for all \( t \) in the maximal interval of existence of the solution. Even though the conformal infinity remains constant, so we can choose the same representative \( g_0 \) for all \( t \), the special boundary defining function \( x \) depends on the interior metric \( g(t) \) as well as \( g_0 \), and hence depends on \( t \). This means that the Graham-Lee normal form for \( g(t) \) evolves in a rather complicated way, and cannot be expressed simply in terms of a single boundary defining function. This is the chief difficulty in understanding the variation of the renormalized volumes \( \text{RenV}(g(t)) \).

As a first step toward that, we recall a result from [28] which asserts that pointwise decay at \( \partial M \) of the Einstein tensor persists under the flow.

**Theorem 2.4** (Lemma 4.3 of [28]). Let \( g(t) \) be a solution to the normalized Ricci flow for \( 0 \leq t < T \), with \( g(0) = g^0 \) an AH metric satisfying

\[
(2.7) \quad |\text{Rm}(g^0)| \leq k_0, \quad \text{and} \quad |\nabla \text{Rm}(g^0)| \leq k_1
\]

for two constants \( k_0, k_1 > 0 \). Suppose too that

\[
(2.8) \quad |E(g^0)|_{g^0} \leq C_0 x^\gamma, \quad \text{and} \quad |\nabla E(g^0)|_{g^0} \leq C_0 x^\gamma
\]

for some \( 0 < \gamma < 1 \). Then there exists \( C = C(k_0, k_1, n, C_0, T) > 0 \) such that

\[
(2.9) \quad |E(g(t))|_{g(t)} \leq C x^\gamma, \quad |\nabla E(g(t))|_{g(t)} \leq C x^\gamma, \quad |\nabla^2 E(g(t))|_{g(t)} \leq \frac{C}{\sqrt{t}} x^\gamma.
\]

**Sketch.** As before, we give only a brief indication, outlining the proof for the estimate of \( E = E(g(t)) \), and referring to [28] Lemma 4.3 for more details.

First compute the evolution equation for \( |E|^2 \) along the flow:

\[
(2.10) \quad \partial_t |E|^2 = \Delta |E|^2 - 2|\nabla E|^2 + 4 \text{Rm}_{ijkl} E^{il} E^{jk}.
\]

Shi’s well-known estimate [29] bounds the curvature \( \text{Rm}(g(t)) \) by a time dependent constant. Rescaling \( |E| \) by setting \( E = x^{-n}E \), then there is a new evolution equation which involves derivatives of \( x \), which must be bounded as well. After some work, we obtain

\[
(2.11) \quad \partial_t |E|^2 \leq \Delta |E|^2 + C(T) |E|^2.
\]

Now apply a modification of the Ecker-Huisken maximum principle [11] to conclude that \( |E| \) is bounded for some short time; this means simply that \( g(t) \) remains APE.

This result makes no use the fact that \( g(t) \) has an expansion for \( t > 0 \). However, coupling this theorem with the results of [5] and Proposition [22] we obtain our first main theorem, which we restate more precisely as

**Theorem 2.5.** Let \( g^0 \) be an APE metric, and let \( g(t) \) be the corresponding solution to the normalized Ricci flow. Then \( g(t) \) remains APE on some time interval \( 0 \leq t < T_0 \). We may thus write

\[
(2.12) \quad g(t) = \frac{dx_t^2 + \hat{g}_{x_t}}{x_t^2},
\]
where \( x_t \) is the special boundary defining function associated to \( g_0 \) and \( g(t) \), and

\[
\dot{g}_{x_t} = g_0 + g_2(t)x_t^2 + \cdots + g_{n-2}(t)x_t^{n-2} + g_{n-1}(t)x_t^{n-1} + \cdots ,
\]

only even powers of \( x \)

where \( \text{tr}^g_0 g_{n-1}(t) = 0 \).

This expansion plays a prominent role in the next section.

As a final comment here, recall that we have been using the result from [5] that if \( g_0 \) has a smooth asymptotic expansion (and in particular if it is APE), then its Ricci evolution \( g(t) \) also has a smooth expansion so long as the flow exists. If \( g(t) \) were to exist for all \( t > 0 \) and converge to a PE metric, then there would have to be a ‘jump’ in the expansion at \( t = \infty \) when \( n \) is odd in order to capture the extra log terms in (2.3). This is an interesting effect, though one which may be difficult to analyze precisely. Examples of jumps in the asymptotic structure of the limit metric have previously been observed in Ricci flows on asymptotically conical surfaces [22] and, for \( n \geq 3 \), rotationally symmetric asymptotically flat Ricci flows [24].

3. Renormalized Volume and the Ricci flow

We now turn to the renormalized volume functional. Although we noted in §1 that RenV is defined using Hadamard regularization, we begin by providing an alternative definition using Riesz regularization. This is entirely equivalent, as explained in [1], and we refer to that paper for more details.

If \( g \) is APE, then

\[
dV_g = \left( \frac{\det(\dot{g}_{x_t})}{\det(g_0)} \right)^{\frac{1}{2}} \frac{dV_{g_0}}{x^n} dx_dV_{g_0}(dx) \frac{x^n}{x^n};
\]

where the \( v_i \), \( 0 \leq i \leq n - 1 \), are locally determined functions on \( \partial M \) which have been studied closely due to their connection with the \( Q \)-curvature function, see [9].

Using this expansion, it is straightforward that

\[
z \mapsto \zeta_x(z) = \int_M x^dV_g
\]

extends meromorphically from \( \text{Re}(z) > n - 1 \) to the whole complex plane with simple poles at \( \zeta_j = n - 1 - j \), \( j = 0, 1, 2, \ldots \). The Riesz regularization of volume is then defined as the finite part of this function at \( z = 0 \):

\[
\text{RenV}(M, g) := \text{FP}_{z=0} \zeta_x(z).
\]

In the following, write

\[
x_t = e^{\omega t} x,
\]

and let an overdot denote \( \frac{\partial}{\partial \tau} \). Note that we are not evaluating this derivative at \( t = 0 \), which is a departure from the calculations in [1].
We now begin the proof of Theorem B. To study the renormalized volume under the Ricci flow we compute

\[
\frac{\partial}{\partial t} \int_M x^2 dV_{g_t} = \int_M x x_t^{-1} x dV_{g_t} + \int_M x \left( \frac{1}{2} \text{tr}^g(\dot{g}) \right) dV_{g_t}.
\]

The second term is easier to understand. Indeed, contracting the flow equation

\[\partial_t g = -2(\text{Rc}(g) + (n-1)g)\]

with \(g\) shows that \(\text{tr}^g \dot{g} = -2(S+n(n-1))\) and by the APE condition, this has pointwise norm decaying like \(O(x^n)\). Since \(dV_g \sim x^{-n}\), this shows that

\[
FP \int_M x^2 \left( \frac{1}{2} \text{tr}^g(\partial_t g) \right) dV_{g_t} = - \int_M (S+n(n-1)) dV_g.
\]

Now turn to the first term. We shall show how to express this as a boundary integral which depends only on the fixed conformal infinity \(c(g(t)) \equiv c(g^0)\). Because several constants appearing here depend on \(t\), this proof only shows that the result only holds for \(t\) in a sufficiently small interval, perhaps much less than the overall time of existence of the solution. For simplicity, we still denote the time of existence by \(T\) but shall later set the value of \(T\) to be small. In any case, all of the compactified metrics \(\bar{g}_t\) are uniformly equivalent for \(0 \leq t \leq T\) and

\[
0 < c_T \leq e^{\omega_t} \leq C_T
\]

for some \(c_T < C_T\).

We require some knowledge about the expansion of \(\omega_t\) with respect to the original boundary defining function \(x\).

**Lemma 3.1.** If \(g(t)\) is APE for \(0 \leq t \leq T\), then

\[
\omega_t(x, y, t) = \omega_t(y, t)x^n + O_T(x^{n+1});
\]

here the notation \(O_T\) indicates that the constants depend on \(T\).

**Proof.** Suppose inductively that

\[
\omega_t = \omega_{(k)}(y, t)x^k + O_T(x^{k+1}).
\]

Since \(\omega_t(0, y, t) = 0\), we may take \(k \geq 1\) initially.

Now \(d\omega_t = \omega_{(k)}(y, t)kx^{k-1}dx + O_T(x^k)\), and hence \(d\dot{\omega}_t = \dot{\omega}_{(k)}(y, t)kx^{k-1}dx + O_T(x^k)\), where the error terms are measured in pointwise norm with respect to any one of the compactified metrics \(\bar{g}(t)\). Following [1] Lemma 5.2, differentiating the relation

\[
1 = |dx_t|_{g_t}^2 = x^{-2}g_t(dx, dx) + 2x^{-1}g_t(dx, d\omega_t) + g_t(d\omega_t, d\omega_t)
\]

yields, after some rearrangement, that

\[
g_t(dx, d\omega_t) + xg_t(d\omega_t, d\omega_t) = x^{-1}E_t(dx, dx) + 2E_t(dx, d\omega_t) + xE_t(d\omega_t, d\omega_t).
\]

We rewrite this further using the compactified metrics as

\[
\bar{g}_t(dx, d\omega_t) = x^{-1}x_t^2E_t(dx, dx) + 2x_t^2E_t(dx, d\omega_t) + xx_t^2E_t(d\omega_t, d\omega_t) - x\bar{g}_t(d\omega_t, d\omega_t).
\]

Recall that \(C_1x \leq x_t \leq C_2x\) and

\[
|E_t(dx, dx)|_{g_t} \leq |E_t|_{g_t}|dx|_{g_t}^2 = O(x^{n-2}).
\]
Hence the terms involving $E_t$ are all $O(x^{n-1})$, while the final term on the right is $O(x^{2k-1})$.

Finally, using the inductive hypothesis for $\omega_t$ on the left, then
\begin{equation} \label{eq:3.15} \dot{\omega}_{(k)} = O(x^{n-k}) + O(x). \end{equation}

Thus for $k < n$ we deduce that $\dot{\omega}_{(k)} = 0$, so \eqref{eq:3.10} holds with $k$ replaced by $k + 1$. \hfill \Box

Now return to the first integral in the main computation. We find that
\begin{equation} \label{eq:3.16} \begin{array}{c}
\text{FP}_{z=0} \int_M z x_i^{2-1} \cdot dV_{g_t} = \text{FP}_{z=0} \int_M z x_i^{2} \cdot \dot{\omega}_t dV_{g_t} = \text{Res}_{z=0} \int_M x_i^{2} \cdot \dot{\omega}_t dV_{g_t} \\
= \text{Res}_{z=0} \left[ \left( \int_0^\varepsilon \int_{\partial M} x_i^{2} \cdot \dot{\omega}_t \left( 1 + v_2 x^2 + \text{even terms} + \cdots n x^n + \cdots \right) \frac{dV_{g_t}}{x^n} \right) + C(\varepsilon) \right] \\
= 0.
\end{array} \end{equation}

The key point in this final equality is that $\dot{\omega}_t$ cancels the singularity in the volume measure, so the residue vanishes. This concludes the proof of formula \eqref{eq:1.2}.

To derive the second variation formula \eqref{eq:1.3}, we use the evolution formula for $\text{tr}^{g(t)} E(g(t))$, which is
\begin{equation} \label{eq:3.17} \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}^{g(t)} E(g(t)) = 2 |E(g(t))|^2 - 2 (n-1) \text{tr}^{g(t)} E(g(t)). \end{equation}

Differentiating \eqref{eq:1.2} along the flow yields
\begin{equation} \label{eq:3.18} \frac{d^2}{dt^2} \text{RenV}(M, g(t)) = - \int_M \left[ \frac{\partial}{\partial t} (\text{tr}^g E) + \frac{1}{2} (\text{tr}^g E) g^{ij} \frac{\partial g_{ij}}{\partial t} \right] dV_g. \end{equation}

Plugging \eqref{eq:3.17} into \eqref{eq:3.18} and simplifying, we obtain that
\begin{equation} \label{eq:3.19} \frac{d^2}{dt^2} \text{RenV}(M, g(t)) = - \int_M \left[ 2 |E|^2 - 2 (n-1) (\text{tr}^g E) - (\text{tr}^g E)^2 \right] dV_g. \end{equation}

There is a term at $\partial_{\infty} M$ in this integration by parts which vanishes because the APE condition implies that $\text{tr}^g E$ and $x \partial_t \text{tr}^g E$ are both $O(x^n)$. This concludes the proof of Theorem C.

Theorem D is proved, as in [7], by noting that if $\text{tr}^{g(0)} E(0) \geq 0$, then the maximum principle yields positivity of $E(t)$ so long as the flow exists.

**Proposition 3.2.** If $g(t)$ is an AH solution to the Ricci flow, and if $\text{tr}^{g(0)} E(0) \geq 0$ then $\text{tr}^{g(t)} E(t) \geq 0$.

**Proof.** Recall that if $g$ is AH, then
\begin{equation} \label{eq:3.20} \text{tr}^g E(g) = S(g) + n(n-1) = O(x), \end{equation}

which evolves according to equation \eqref{eq:3.17}. Note that this is much weaker decay than that of an APE metric.

Suppose by way of contradiction that $\inf_{p \in M} \text{tr}^{g(t)} E(g(t))(p, t_s) < 0$ for some $t_s < T$. There is a constant $C = C(t_s)$ such that
\begin{equation} \label{eq:3.21} |\text{tr}^{g(t)} E(g(t))| \equiv |S(g(t)) + n(n-1)| \leq C x. \end{equation}

The classical parabolic minimum principle implies that $\text{tr}^{g(t)} E(g(t))$ cannot attain a negative minimum in the region $\{ x > 0 \} \times [0, t_s]$. Indeed, at such a point, $\Delta \text{tr}^{g(t)} E(g(t)) \geq 0$, so the right side of equation \eqref{eq:3.17} is strictly positive, while the left side is nonpositive.
This leaves the possibility that a negative minimum for $\text{tr}^g(t) E(g(t))$ occurs at $x = 0$. Choose an exhaustion of $M \times [0, t_1]$ by compact sets of the form $M_k = \{ x \geq \frac{1}{k} \} \times [0, t_1]$. We know that $S(g) + n(n - 1)$ cannot attain a negative minimum in the interior $M_k$, and must be at least nonnegative on its boundary, at $x = 1/k$. Taking a limit, we see that $\text{tr}^g(t) E(g(t)) \geq 0$ everywhere, which is a contradiction. 

Using this in (4.2) gives monotonicity, and hence completes the proof of the main statement of Theorem C. Finally, from Proposition 3.2 since $\text{tr}^g(t) E(g(t))$ is nonnegative for $t \in (t_1, t_2)$, if it is not everywhere zero, then by (4.2) $\text{RenV}(M, (g(t_2)) < \text{RenV}(M, (g(t_1)))$. This proves the final statement of Theorem C.

**Proof of Corollary 17.** If $(M, g(t_1))$ and $(M, g(t_2))$ are isometric, then their renormalized volumes are equal. Since $S(g^0) \geq -n(n - 1)$, Theorem C gives that $S(g(t)) + n(n - 1) \equiv 0$. But the evolution equation (3.17) for $\text{tr}^g(t) E(g(t))$ implies that the full Einstein tensor $E(g(t)) \equiv 0$ for all $t$, so that $g(t)$ is Einstein and stationary.

Finally, by Qing’s rigidity theorem [27], any conformally compactifiable Einstein manifold of dimension $n < 8$ whose conformal infinity is a round sphere must be $\mathbb{R}^n$.

**Proposition 3.3.** When $n$ is odd and $(M, g^0)$ is APE, then the conformal anomaly term in the expansion of $g(t)$ is constant along the flow.

**Proof.** Taking the determinant of both sides of (2.12), differentiating with respect to $t$, and using $\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial t} = - \text{tr}^g(t) E(g(t))$, we obtain

\[
\frac{\sqrt{g(t)}}{x_t} - \text{tr}^g E = \frac{\partial}{\partial t} \log \sqrt{g(t)} .
\]

By the estimates for $\dot{\omega}$ in Lemma 3.1 and the fact that $\text{tr}^g(t) E := S(g) + n(n - 1) = \mathcal{O}(x^n)$ since $g$ is APE, we have

\[
\frac{\partial}{\partial t} \log \sqrt{g(t)} = \mathcal{O}(x^n) .
\]

Substituting $\sqrt{g(t)} = v(0) + v(2)x^2 + \cdots + v(n-1)x_t^{n-1}$ into (3.23), we see that $v_{n-1} = 0$. The conclusion now follows since the conformal anomaly is $\int_{\partial M} v_{n-1} dV_{g^0}$. 

4. Discussion

The Hawking-Page phase transition [19] in black hole thermodynamics is related to the renormalized volume of Poincaré-Einstein manifolds. Consider $S^1 \times S^2$ with metric $\gamma$, the product of the circle of length $\beta$ and the standard (curvature equal to +1) metric on $S^2$. There are three well-known Poincaré-Einstein metrics for which $(S^2 \times S^1, [\gamma])$ is the conformal infinity. Two of these, the so-called large and small black hole metrics $g_{bh}$ and $g_{sbh}$, are warped product metrics on the bulk manifold $M_1 = S^2 \times \mathbb{R}^2$, while the third, $g_b$, called \emph{thermal hyperbolic space} in physics, is the hyperbolic metric on $M_2 = B^3 \times S^1$, realized as the quotient of $\mathbb{H}^4$ by a hyperbolic dilation with translation distance $\beta$. The parameter $1/\beta$ is called the \emph{temperature} (in units of the Boltzmann constant).

Now recall the formula of Anderson [4], see also [1], for renormalized volume of PE spaces, which generalizes immediately to APE spaces:

\[
\text{RenV}(M, g) = \frac{4\pi^2}{3} \chi(M) - \frac{1}{24} \int_M (|\text{Rm}|^2 - 4|Z|^2 - 24) dV(g)
\]
We mention in passing that our formula (1.2) for $\partial_t \text{RenV}$ in 4 dimensions can also be obtained by differentiating (4.1) and using an identity due to Berger [8] along with the APE condition to eliminate the boundary terms arising in subsequent integrations by parts. Here $\chi(M)$ is the Euler characteristic and $Z$ the tracefree Ricci tensor. We note too that [4] uses different conventions so the constants in that paper are different. Using (4.1) we obtain that

\begin{equation}
\text{RenV}(M_2, g_h) = 0
\end{equation}

while

\begin{equation}
\text{RenV}(M_1, g_{\text{dbh}}/g_{\text{shh}}) = \frac{8\pi^2 a^2 (1 - a^2)}{3 (1 + 3a^2)}
\end{equation}

The black hole horizon radii $a_- < a_+$ are the two roots of the quadratic equation

\begin{equation}
3a^2 - \frac{4\pi}{\beta}a + 1 = 0,
\end{equation}

see [19, equation (2.7)]. In fact there are two distinct real roots only when $\beta^2 > \frac{4\pi^2}{3}$, which we always assume. In any case, when $\beta < \pi$, which corresponds to high temperature,

\begin{equation}
\text{RenV}(M_1, g_{\text{dbh}}) < \text{RenV}(M_2, g_h) = 0 < \text{RenV}(M_1, g_{\text{shh}})
\end{equation}

On the other hand, thermal hyperbolic space is the minimizer in the low temperature regime $\beta > \pi$.

Now up to an overall normalization to account for physical units, (4.3) is precisely the formula [19, eqn. (2.9)] for the difference between the gravitational action of a black hole and that of thermal hyperbolic space, as computed using the ad hoc regularization procedure of [19]. The renormalized volume formalism here makes this rigorous.

All of this suggests that the renormalized volume functional is relevant to this physical system. We next discuss the role played by APE flows.

Using (1.4), we see that a PE manifold is marginally linearly stable in the space of APE perturbations. (By contrast, RenV is strictly unstable at any non-Einstein constant scalar curvature metric).

It is well-known [26] that the small black hole is linearly unstable in the sense that its Lichnérivicz Laplacian $\Delta_L$ has a negative eigenvalue, and hence it is also linearly unstable for the Ricci flow, since the linearized Ricci-DeTurck flow is $\partial_t - \Delta_L$, see [20]. Thus, a small perturbation of a PE metric produces a nearby non-Einstein metric with the same renormalized volume to second order, and this perturbed metric then evolves under the full (normalized) Ricci flow (1.1) during which its renormalized volume strictly decreases.

The case of black holes with boundary has been studied numerically [20], and there is an analogous picture there. The boundary, with Dirichlet boundary conditions, can be filled by two different black holes, and also by so-called hot flat space ($\mathbb{R}^3 \times S^1$ with a flat metric. In that case, the Einstein-Hilbert action is largest at the small black hole, and this is (linearly) unstable for Ricci flow. The numerical flows can be followed all the way from a small perturbation of the small black hole to either the large black hole or to hot flat space. In the latter space there must be a surgery to account for the change in topology. Monotonicity of the Einstein-Hilbert action is observed numerically along these flows, and is used to construct what is referred to in [20] as a “novel” free energy diagram for the phase transition in that setting.

Our analytical results lend firm support to these numerical and thermodynamic arguments, albeit it in the setting of APE manifolds rather than manifolds with boundary. More
specifically, the renormalized volume is monotonic and approximates the Einstein-Hilbert action near the fixed points, which is where equilibrium thermodynamics can be invoked. It would be very interesting to have numerical studies of the Ricci flow of APE metrics for initial data near the Poincaré-Einstein small black hole.

Finally, many phase transitions admit a modern description in terms of renormalization groups flows. For example, the ferromagnetic 2-dimensional Ising model has a phase transition described by “block spin renormalization” which interpolates between the macroscopic ordered and disordered states of the ferromagnet. It is certainly suggestive that the dynamics of the Hawking-Page phase transition may have a Ricci flow description. This would fit with the well-known fact that Ricci flow is an approximation to the (one loop) renormalization group flow for a 2-dimensional sigma model, and hence also approximately for closed strings [14], the fundamental excitations thought to give rise to collective states such as black holes.

References

[1] P. Albin, Renormalizing curvature integrals on Poincaré-Einstein manifolds, Adv. Math. 221 (2009) 140–169.
[2] P. Albin, C. Aldana and F. Rochon, Ricci flow and the determinant of the Laplacian on non-compact surfaces, Comm. Par. Diff. Eq. 38 (2013) 711–749.
[3] S. Alexakis and R. Mazzeo, Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds Comm. Math. Phys. 297 (2010) No. 3, pp. 621–651.
[4] M.T. Anderson, \(L^2\) curvature and volume renormalization of AHE metrics on 4-manifolds, Math. Res. Lett. 8 (2001) 171–188.
[5] E. Bahuaud, Ricci flow of conformally compact metrics, Ann. Inst. H. Poincaré: Anal. Non-Lin. 28 (2011) 813–835.
[6] E. Bahuaud, R. Mazzeo, and E. Woolgar, work in progress; E. Bahuaud and R. Mazzeo, work in progress.
[7] T. Balehowsky and E. Woolgar, The Ricci flow of asymptotically hyperbolic mass and applications, J. Math. Phys. 53 (2012) 072501.
[8] M. Berger, Quelques formules de variation pour une structure riemannienne, Ann. Sci. ÉNS 4e série 3 (1970) 285–294.
[9] S-Y.A. Chang, H. Fang and C.R. Graham A note on renormalized volume functionals, [arXiv:1211.6422]
[10] S. de Haro, K. Skenderis, and S.N. Solodukhin, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, Comm. Math. Phys. 217 (2001) 595–622.
[11] K. Ecker and G. Huisken, Interior estimates for hypersurfaces moving by mean curvature, Invent. Math., 105 (1991) 547–569.
[12] C. Fefferman and C.R. Graham, Conformal invariants, in Élie Cartan et les Mathématiques d’Aujourd’hui, Astérisque (numéro hors série, 1985) 95–116.
[13] C. Fefferman and C.R. Graham, The ambient metric Princeton Annals of Mathematical Studies 178, Princeton, NJ (2012)
[14] D.H. Friedan, Nonlinear Models in \(2 + \varepsilon\) Dimensions, PhD thesis, University of California, Berkeley, 1980 (unpublished); Phys. Rev. Lett. 45 (1980) 1057–1060; Ann. Phys. (NY) 163 (1985) 318–419.
[15] C.R. Graham, Volume and area renormalizations for conformally compact Einstein metrics in The Proceedings of the 19th Winter School “Geometry and Physics” (Srni, 1999). Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 31–42.
[16] C.R. Graham and K. Hirachi, The ambient obstruction tensor and Q-curvature, in AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries, IRMA Lect Math Theor Phys 8 (European Mathematical Society Zürich, 2005) pp 59–71.
[17] C.R. Graham and J. Lee, Einstein metrics with prescribed conformal infinity on the ball, Adv. Math. 87 (1991) 186–225.
[18] C.R. Graham and E. Witten, Conformal anomaly of submanifold observables in AdS/CFT correspondence, Nucl. Phys. B546 (1999) 52–64.
[19] S.W. Hawking and D.N. Page, Thermodynamics of black holes in anti-de Sitter space, Comm. Math. Phys. 87 (1983) 577–588.
[20] M. Headrick and T. Wiseman, *Ricci flow and black holes*, Class. Quantum Grav. 23 (2006) 6683–6708.
[21] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, J.H.E.P. 9807 (1998) 023.
[22] J. Isenberg, R. Mazzeo, and N. Sesum, *Ricci flow on asymptotically conical surfaces with nontrivial topology*, J. Reine Angew. Math. (Crelle), 676 (2013) 227-248.
[23] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231–252.
[24] T.A. Oliynyk and E. Woolgar, *Asymptotically flat Ricci flows*, Commun. Anal. Geom. 15 (2007) 535–568.
[25] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, in *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, IRMA Lectures in Mathematics and Theoretical Physics 8, pp 73–101, European Mathematical Society, Zürich (2005).
[26] T. Prestidge, *Dynamic and thermodynamic stability and negative modes in Schwarzschild-anti-de Sitter*, Phys. Rev. D61 (2000) 084002.
[27] J. Qing, *On the rigidity for conformally compact Einstein manifolds*, Int. Math. Res. Not. 21 (2003) 1141–1153.
[28] J. Qing, Y. Shi, and J. Wu, *Normalized Ricci flows and conformally compact Einstein metrics*, arXiv:1106.0372
[29] W.X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Diff. Geom. 30 (1989) 223–301.
[30] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. 2 (1998) 505–532.

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