Braided products for quantum spaces

Hartmut Wachter*

Max-Planck-Institute
for Mathematics in the Sciences
Inselstr. 22, D-04103 Leipzig, Germany

Arnold-Sommerfeld-Center
Ludwig-Maximilians-Universität
Theresienstr. 37, D-80333 München, Germany

Abstract

Attention is focused on quantum spaces of physical importance, i.e. Manin plane, q-deformed Euclidean space in three or four dimensions as well as q-deformed Minkowski space. There are algebra isomorphisms that allow to identify quantum spaces with commutative coordinate algebras. This observation enables the realization of braiding mappings on commutative algebras. The aim of the article is to perform this task explicitly, leading us to formulae for so-called braided products.

*e-mail: Hartmut.Wachter@physik.uni-muenchen.de
1 Introduction

Relativistic quantum field theory is not a fundamental theory, since its formalism leads to divergencies. In some cases like that of quantum electrodynamics one is able to overcome the difficulties with the divergencies by applying the so-called renormalization procedure due to Richard Feynman. Unfortunately, this procedure is not successful if we want to deal with quantum gravity. Despite the fact that gravitation is a rather weak interaction we are not able to treat it perturbatively. The reason for this lies in the fact, that transition amplitudes of nth order to the gravitation constant diverge like a momentum integral of the general form [1]

\[ \int p^{2n-1} dp, \quad n \in \mathbb{N}. \quad (1) \]

This leaves us with an infinite number of ultraviolet divergent Feynman diagrams that cannot be removed by redefining finitely many physical parameters.

It is surely legitimate to ask for the reason for these fundamental difficulties. It is commonplace that the problems with the divergences in relativistic quantum field theory result from an incomplete description of spacetime at very small distances. Niels Bohr and Werner Heisenberg have been the first who suggested that quantum field theories should be formulated on a spacetime lattice [2, 3]. Such a spacetime lattice would imply the existence of a smallest distance \( a \) with the consequence that plane-waves of wave-length smaller than twice the lattice spacing could not propagate. In accordance with the relationship between wave-length \( \lambda \) and momentum \( p \) of a plane-wave, i.e.

\[ \lambda \geq \lambda_{\text{min}} = 2a \quad \Rightarrow \quad \frac{1}{\lambda} \sim p \leq p_{\text{max}} \sim \frac{1}{2a}, \quad (2) \]

it then follows that physical momentum space would be bounded. Hence, the domain of all momentum integrals in Eq. (1) would be bounded as well. Consequently, all of these integrals should take on finite values.

However, discrete spacetime structures in general do not respect classical Poincaré symmetry. A possible way out of this difficulty is to modify not only spacetime but also its corresponding symmetries. How are we to accomplish this? First of all let us recall that classical spacetime symmetries are usually described by Lie groups. Realizing that Lie groups are manifolds the Gelfand-
Naimark theorem tells us that Lie groups can be naturally embedded in the category of algebras [4]. The utility of this interrelation lies in formulating the geometrical structure of Lie groups in terms of a Hopf structure [5]. The point is that during the last two decades generic methods have been discovered for continuously deforming matrix groups and Lie algebras within the category of Hopf algebras. It is this development which finally led to the arrival of quantum groups and quantum spaces [6–12].

From a physical point of view the most realistic and interesting deformations are given by q-deformed versions of Minkowski space and Euclidean spaces together with their corresponding symmetries, i.e. respectively Lorentz symmetry and rotational symmetry [13–17]. Further studies even allowed to establish differential calculi on these q-deformed quantum spaces [18–20] representing nothing other than q-analogs of classical translational symmetry. In this sense we can say that q-deformations of the complete Euclidean and Poincaré symmetries are now available [21, 22]. Finally, Julius Wess and his coworkers were able to show that q-deformation of spaces and symmetries can indeed lead to the wanted discretizations of the spectra of spacetime observables [23–25]. This observation nourishes the hope that q-deformation might give a new method to regularize quantum field theories [26–29].

In order to formulate quantum field theories on q-deformed quantum spaces it is necessary to provide us with some essential tools of a q-deformed analysis. The main question is how to define these new tools, which should be q-analogs of classical notions. Towards this end the considerations of Shahn Majid have proven very useful [30–32]. The key idea of his approach is that all the quantum spaces to a given quantum symmetry form a braided tensor category. Consequently, operations and objects concerning quantum spaces must rely on this framework of a braided tensor category, in order to guarantee their well-defined behavior under quantum group transformations. This so-called principle of covariance can be seen as the essential guideline for constructing a consistent theory.

In our previous work we have applied these rather general considerations (as they are exposed in Refs. [32–35]) to quantum spaces of physical importance, i.e. q-deformed quantum plane, q-deformed Euclidean space with three or four dimensions and q-deformed Minkowski space, resulting in explicit expressions for star products [36], operator representations [37], q-integrals [38], q-exponentials [39], and q-translations [40]. In this article we would like to complete our toolbox of essential elements of q-analysis by
calculating formulae for braided products.

For this to achieve we intend to proceed as follows. In Sec. 2 we cover the ideas our considerations about q-analysis are based on. Important for us is the fact that there is an algebra isomorphism which allows to identify quantum space algebras with algebras of commutative coordinates. In the subsequent sections this observation will be used to calculate formulae for braided products, i.e. realizations of braiding mappings on commutative algebras. In doing so, we restrict attention on braidings referring to quantum spaces we are interested in for physical reasons, i.e. Manin plane, q-deformed Euclidean space in three or four dimensions as well as q-deformed Minkowski space. In Sec. 7 we close our considerations by a short conclusion. Appendix A shall serve as a review of the quantum algebras describing the symmetries of the quantum spaces under consideration, while Appendix B is devoted to some special calculations.

2 Basic ideas on q-analysis

Let us recall that q-analysis can be regarded as a noncommutative analysis formulated within the framework of quantum spaces [41]. Quantum spaces are defined as comodule algebras of quantum groups and can be interpreted as deformations of ordinary coordinate algebras. For our purposes, it is at first sufficient to consider a quantum space as an algebra $A_q$ of formal power series in non-commuting coordinates $X_1, X_2, \ldots, X_n$, i.e.

$$A_q = \mathbb{C}[[X_1, \ldots, X_n]]/I,$$

where $I$ denotes the ideal generated by the relations of the non-commuting coordinates. The three-dimensional Euclidean quantum space shall serve as an example. It consists of all the power series in three coordinates $X^+, X^3$, and $X^-$ being subject to the commutation relations

$$X^3 X^\pm - q^{\pm 2} X^\pm X^3 = 0,$$

$$X^- X^+ - X^+ X^- = \lambda X^3 X^3.$$

We can think of $q > 1$ as a deformation parameter measuring the coupling among different spatial degrees of freedom, whereas $\lambda \equiv q - q^{-1}$ should here be understood as some kind of a relative lattice spacing. In the classical case, i.e. if $q$ tends to 1, it is rather obvious from Eq. 4 that we regain
Next, we would like to focus our attention on the question how to perform calculations on our noncommutative coordinate algebras. As will become evident, this can be accomplished by a kind of pullback, which translates operations on noncommutative coordinate algebras to those on commutative ones. Towards this end, we have to realize that the noncommutative algebra $A_q$ satisfies the Poincaré-Birkhoff-Witt property, i.e. the dimension of a subspace of homogenous polynomials should be the same as for commuting coordinates. This property is the deeper reason why the monomials of normal ordering $X_1 X_2 \ldots X_n$ constitute a basis of $A_q$. In particular, we can establish a vector space isomorphism between $A_q$ and the commutative algebra $A$ generated by ordinary coordinates $x_1, x_2, \ldots, x_n$:

$$W : A \rightarrow A_q,$$

$$W(x_1^{i_1} \ldots x_n^{i_n}) = X_1^{i_1} \ldots X_n^{i_n}. \quad (5)$$

This vector space isomorphism can even be extended to an algebra isomorphism by introducing a noncommutative product in $A$, the so-called star product $[42–44]$. This product is defined via the relation

$$W(f \ast g) = W(f) W(g), \quad (6)$$

being tantamount to

$$f \ast g = W^{-1} \left( W(f) \ W(g) \right), \quad (7)$$

where $f$ and $g$ are formal power series in $A$. In the case of the three-dimensional q-deformed Euclidean space, for instance, the star product takes the form

$$f(x) \ast g(x) = \sum_{i=0}^{\infty} \lambda^i \frac{(x^3)^{2i}}{[[i]]_{q^4}!} q^{2(\hat{n}_3 \hat{n}_3' + \hat{n}_- \hat{n}_-')} \left[ (D_{q^3})^i f(x) \right] \left[ (D_{q^3}^{-1})^i g(x') \right] \bigg|_{x' \rightarrow x}, \quad (8)$$

where

$$h \equiv \ln q \quad \text{and} \quad \hat{n}_A \equiv \frac{\partial}{\partial x^A}, \quad A \in \{ +, 3, - \}. \quad (9)$$
In addition to this, we have introduced in (8) as some sort of discretised version of classical derivatives the so-called *Jackson derivative* \(^{(a)}\)

\[
D_a^A f(x^A) = \frac{f(q^a x^A) - f(x^A)}{q^a x^A - x^A}, \quad a \in \mathbb{C}.
\]

Furthermore, the *antisymmetric q-numbers* are given by

\[
[[n]]_q^a \equiv \sum_{k=0}^{n-1} q^{ak} = \frac{1 - q^{an}}{1 - q^a},
\]

showing the property

\[
[[n]]_q^a \to n \quad \text{for} \quad q \to 1.
\]

Factorials of q-numbers are defined in complete analogy to the classical case, i.e.

\[
[[m]]_q^a! \equiv [[1]]_q^a [[2]]_q^a \ldots [[m]]_q^a, \quad [[0]]_q^a! \equiv 1.
\]

From these expressions it should be evident that star products on q-deformed quantum spaces tend to the commutative product in the limit \(q \to 1\), i.e. our star products can be seen as modifications of commutative products.

Now, we want to deal on with symmetry generators. To this end let us recall that in quantum theory it is necessary to know how symmetry generators act on function spaces. Such actions can be derived from the commutation relations between symmetry generators and space coordinates. As remarked earlier, there are q-deformed analogs of classical symmetry algebras and we also know the commutation relations of the corresponding generators with quantum space coordinates. In the case of three-dimensional q-deformed Euclidean space, for example, quantum space coordinates now commute with angular momentum component \(L^+\) according to

\[
L^+ X^+ - X^+ L^+ = 0, \quad L^+ X^3 - X^3 L^+ = q X^+ \tau^{-1/2}, \quad L^+ X^- - X^- L^+ = X^3 \tau^{-1/2},
\]

and the partial derivative \(\partial_-\) has to obey the q-deformed Heisenberg relations

\[
\partial_- X^+ - X_- \partial^+ = 0, \quad \partial_- X^3 - q^2 X^3 \partial_- = -q^2 \lambda \lambda_+ X^+ \partial_3,
\]
\[
\partial_- X^- - q^4 X^- \partial_+ = q^2 \lambda \lambda_+ X_3 \partial_3 + q \lambda^2 \lambda_+ X^+ \partial_+ ,
\]
with \( \lambda_+ \equiv q + q^{-1} \).

From such identities one can calculate actions of q-deformed symmetry generators on normal ordered monomials of quantum space coordinates. By means of the relation
\[
W(h \triangleright f) = h \triangleright W(f), \quad h \in \mathcal{H}, \ f \in \mathcal{A},
\]
(16)
or
\[
h \triangleright f = W^{-1} (h \triangleright W(f)) ,
\]
(17)
the action of the symmetry algebra \( \mathcal{H} \) on the quantum space algebra \( \mathcal{A}_q \) carries over to a corresponding one on a commutative coordinate algebra \( \mathcal{A} \). We applied these ideas in the work of Ref. [37]. This way we got operator representations of q-deformed symmetry generators. As an example we give the representations that result from the relations in (14) and (15):

\[
L^+ \triangleright f(\underline{x}) = -q^2 x^3 (D_q^- f)(q^{-2} x^-) - qx^+(D_q^3 f)(q^{-2} x^-) ,
\]
(18)
\[
\partial_- \triangleright f(\underline{x}) = D_q^- f(q^2 x^3) + \lambda x^+(D_q^3)^2 f.
\]

In this manner, we obtained discretised versions of classical expressions. Furthermore, let us note that the non-classical terms proportional to \( \lambda \) can again be interpreted as result of the coupling between the different degrees of freedom. So to speak, the situation in quantum spaces is in some sense like that in solids.

As a next step we can introduce q-deformed integrals as inverse operations to partial derivatives [38]. This requires to find solutions of the difference equations
\[
\partial \triangleright F = f
\]
for given \( f \). As an example we present the integral corresponding to the second identity in (18):

\[
F = (\partial_-)^{-1} \triangleright f
\]

\[
= \sum_{k=0}^{\infty} (-\lambda)^k q^{2(k+1)} x^+ (D_q^-)^{-1} (D_q^3)^2 \left[ (D_q^-)^{-1} f(q^{-2(k+1)} x^3) \right]^k
\]
The so-called *Jackson integral* in here is defined by

\[
(D_q^A)^{-1} f\big|_0^{x^A} = -(1 - q^a) \sum_{k=1}^{\infty} (q^{-ak} x^A) f(q^{-ak} x^A).
\] (21)

Again, it can be seen as discretised version of classical integrals, into which it passes for \( q \to 1 \).

Now, we come to another very important ingredient of q-analysis. As we know from quantum field theory, it is convenient to express solutions to wave equations in terms of plane-waves, i.e. exponential functions. In q-analysis there are q-deformed analogs of such exponentials, which are completely determined by their property of being eigenfunctions of partial derivatives, i.e.

\[
\exp(x \mid \partial) \in \mathcal{A}_x \otimes \mathcal{A}_\partial,
\] (22)

with

\[
\partial_A \triangleright \exp(x \mid \partial) = \exp(x \mid \partial) \ast \partial_A \quad \text{for all} \ A.
\] (23)

In some cases like that of three-dimensional q-deformed Euclidean space these exponentials take on a form that is in complete analogy to their classical counterpart [39]:

\[
\exp(x_R \mid \partial_L) = \sum_{n=0}^{\infty} \frac{(x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-} \otimes (\partial_-)^{n_-} (\partial_3)^{n_3} (\partial_+)^{n_+}}{[[n_+]]_q![[n_3]]_q![[n_-]]_q!}.
\] (24)

The q-deformed exponentials can be used to perform translations on quantum spaces, as it was explained in Ref. [34]. In complete analogy to the classical case, translations on quantum spaces are obtained by the formula

\[
f(y \oplus_L x) \equiv \exp(y_R \mid \partial_L) \triangleright f(x),
\] (25)

with the understanding that the partial derivatives act on the function far right in the above equation. By using the last identity in combination with the explicit form for q-exponential and actions of partial derivatives we can obtain q-deformed versions of Taylor rules. As an example we give the result for three-dimensional q-deformed Euclidean space [40]:

\[
f(y \oplus_L x)
\] (26)

8
\[
\sum_{k_+ = 0}^{n_+} \sum_{k_3 = 0}^{n_3} \sum_{k_- = 0}^{n_-} \sum_{i = 0}^{k_3} \frac{(q \lambda \lambda_i)^i}{[2i]_q!!} \frac{(y^+)^{k_+} (y^3)^{k_3-i} (y^-)^{k_-+i}}{[[k_+]]_q^i [[k_3-i]]_q^i [1-q^2]^i} \times \left( x^+ \right)^i \left( (D_{q^4}^+)^{k_+} (D_{q^2}^3)^{k_3+i} (D_{q^4}^-)^{k_-} f \right) (q^{2(k_3-i)+3} x_+, q^{2k_-3}).
\]

Now, we come to the main incentive of this paper, i.e. braided products. To this end, let us recall that in physical applications it is often necessary to deal with tensor products. Such a situation arises, for example, when one wants to describe multiparticle states or the phenomenon of spin. The tensor product of commutative algebras is usually characterized by a componentwise multiplication. For tensor products of q-deformed quantum spaces, however, this is not the case.

For this to become more clear, it is useful to consider quantum spaces from a point of view provided by category theory. A category for our purposes is just a collection of objects \(X, Y, Z, \ldots\) and a set \(\text{Mor}(X, Y)\) of morphisms between any two objects \(X, Y\). The composition of morphisms has similar properties as the composition of maps. We are interested in tensor categories. These categories have a product, denoted \(\otimes\) and called the tensor product. It admits several 'natural' properties such as associativity and existence of a unit object. For a more formal treatment we refer to Refs. [30], [31], [35] or [46]. If the action of a quasitriangular Hopf algebra \(H\) on the tensor product of two quantum spaces \(X\) and \(Y\) is defined by

\[
h \triangleright (v \otimes w) = (h(1) \triangleright v) \otimes (h(2) \triangleright w) \in X \otimes Y, \quad h \in H,
\]

(27)

(the coproduct is written in the so-called Sweedler notation, i.e. \(\Delta(h) = h(1) \otimes h(2)\)) then the representations (quantum spaces) of the given Hopf algebra (quantum algebra) are the objects of a tensor category.

In this tensor category exists a morphism of particular importance. To be more precise, for any pair of objects \(X, Y\) there is an isomorphism \(\Psi_{X,Y} : X \otimes Y \to Y \otimes X\) such that \((g \otimes f) \circ \Psi_{X,Y} = \Psi_{X',Y} \circ (f \otimes g)\) for arbitrary morphisms \(f \in \text{Mor}(X, X')\) and \(g \in \text{Mor}(Y, Y')\). In addition to this one can require the hexagon axiom to hold. The hexagon axiom is the validity of the two conditions

\[
\Psi_{X,Z} \circ \Psi_{Y,Z} = \Psi_{X \otimes Y,Z}, \quad \Psi_{X,Z} \circ \Psi_{X,Y} = \Psi_{X,Y \otimes Z}.
\]

(28)

A tensor category equipped with such mappings \(\Psi_{X,Y}\) for each pair of objects
$X, Y$ is called a braided tensor category. The mappings $\Psi_{X,Y}$ as a whole are often referred to as the braiding of the tensor category. Their property of being morphisms implies that it makes no difference whether we interchange the two tensor factors by the braiding before or after applying the action of a given Hopf algebra. More formally, we have

$$\Psi_{X,Y}(h \triangleright (v \otimes w)) = h \triangleright (\Psi_{X,Y}(v \otimes w)) \in Y \otimes X, \quad h \in \mathcal{H}. \quad (29)$$

To state this another way, the braiding represents a solution to the problem how to commute elements of different quantum spaces in a covariant way, i.e. consistent with the underlying quantum symmetry.

One should notice that the above considerations also hold for the set of inverse mappings $\Psi^{-1}_{X,Y}$. This fact gives rise to a second braiding, which in our cases is indeed different from the first one. In the sequel of this article, it is our aim to derive explicit formulae for the mappings $\Psi_{X,Y}$ and $\Psi^{-1}_{X,Y}$, where $X$ and $Y$ stand for quantum space algebras of physical importance, i.e. Manin plane, q-deformed Euclidean space in three or four dimensions as well as q-deformed Minkowski space. In doing so, it can be useful to realize that the symmetry algebras for these quantum spaces are described by quasitriangular Hopf algebras. Thus, the braiding in the corresponding tensor category is completely determined by a universal $R$-matrix $R = R^{(1)} \otimes R^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ (or its inverse $R^{-1} = (R^{-1})^{(1)} \otimes (R^{-1})^{(2)}$), since we have

$$\Psi_{X,Y}(v \otimes w) = (R^{(2)} \triangleright w) \otimes (R^{(1)} \triangleright v), \quad v \in X, \ w \in Y, \quad (30)$$

$$\Psi^{-1}_{X,Y}(v \otimes w) = ((R^{-1})^{(2)} \triangleright w) \otimes ((R^{-1})^{(1)} \triangleright v).$$

By means of the algebra isomorphism in Eq. (5) we are able to introduce a new product between two commutative algebras $A$ and $A'$, the so-called braided product, which for the two different braidings is given by

$$\mathcal{D}_L, \mathcal{D}_L : A \otimes A' \rightarrow A' \otimes A, \quad (31)$$

$$f(x) \mathcal{D}_L g(y) \equiv (W^{-1} \otimes W^{-1}) \circ \Psi^{-1}_{X,Y}(W(f) \otimes W(g)),$$

$$f(x) \mathcal{D}_L g(y) \equiv (W^{-1} \otimes W^{-1}) \circ \Psi_{X,Y}(W(f) \otimes W(g)).$$

Finally, it should be mentioned that braided products are part of the calculation of q-deformed tensor products. For this to become more clear, let us recall the definition for a tensor product of two quantum spaces. In
the case of the braiding \( \Psi_{X,Y} \) it reads

\[
(X \otimes Y) \otimes (X \otimes Y) \rightarrow X \otimes Y
\]

\[
(w_1 \otimes v_1)(w_2 \otimes v_2) \equiv (m \otimes m) \circ (w_1 \otimes \Psi_{X,Y}(v_1 \otimes w_2) \otimes v_2),
\]

and likewise for the inverse braiding \( \Psi^{-1}_{X,Y} \), where \( m \) denotes multiplication in the quantum space algebras \( X \) and \( Y \). With this identity at hand, it is now straightforward to extend the definition of braided products to that of a q-deformed tensor product between the algebras \( A \) and \( A' \):

\[
\odot_L : (A \otimes A') \otimes (A \otimes A') \rightarrow A \otimes A'
\]

\[
(f_1(x) \otimes g_1(y)) \odot_L (f_2(x) \otimes g_2(y))
\]

\[
\equiv (W^{-1} \otimes W^{-1}) \circ (m \otimes m) \circ [W(f_1) \otimes \Psi(W(g_1) \otimes W(f_2)) \otimes W(g_2)]
\]

\[
= [f_1 \ast W^{-1}(R^{(2)} \triangleright W(f_2))] \otimes [W^{-1}(R^{(1)} \triangleright W(g_1)) \ast g_2]
\]

\[
= f_1(x) \ast_x (g_1(y) \odot_L f_2(x)) \ast_y g_2(y).
\]

It is not very hard to adapt this formula to the inverse braiding. Thus, the details are left to the reader.

3 Quantum plane in two dimensions

In this section we would like to consider the simplest example for a q-deformed quantum space, the so-called Manin plane [47]. It is generated by coordinates \( X^\alpha, \alpha = 1, 2 \), subjected to the relation

\[
X^1 X^2 = q X^2 X^1.
\]

(34)

The corresponding symmetry algebra is given by \( U_q(su_2) \) [48] (for its definition see also Appendix A). If we substitute for the universal R-matrix that of \( U_q(su_2) \) and take into account the action of \( U_q(su_2) \) on the quantum plane, the relations in Eq. (30) enable us to calculate the braiding between two copies of quantum plane coordinates. Proceeding in this way one can obtain [49]

\[
\Psi_{X,Y} (X^\alpha \otimes Y^\beta) = q R^\alpha{}_{\gamma}{}^\beta Y^\gamma \otimes X^\delta,
\]

(35)

\[
\Psi^{-1}_{X,Y} (X^\alpha \otimes Y^\beta) = q^{-1} (R^{-1})^\alpha{}_{\gamma}{}^\beta Y^\gamma \otimes X^\delta,
\]
where $R_{\alpha \beta}^{\gamma \delta}$ and $(R^{-1})_{\alpha \beta}^{\gamma \delta}$ stand for the vector representation of the universal R-matrix and its inverse, respectively. More explicitly, we have the relations

\[
\Psi_{X,Y}(X^1 \otimes Y^1) = q^2 Y^1 \otimes X^1,
\]
\[
\Psi_{X,Y}(X^1 \otimes Y^2) = q Y^2 \otimes X^1 + q \lambda Y^1 \otimes X^2,
\]
\[
\Psi_{X,Y}(X^2 \otimes Y^1) = q Y^1 \otimes X^2,
\]
\[
\Psi_{X,Y}(X^2 \otimes Y^2) = q^2 Y^2 \otimes X^2.
\]

The relations arising from the inverse braiding are derived from the above ones most easily by applying the substitutions

\[
X^\alpha \leftrightarrow X^{\alpha'}, \quad Y^\alpha \leftrightarrow Y^{\alpha'}, \quad q \leftrightarrow q^{-1}, \quad \alpha' \equiv 3 - \alpha, \quad \alpha = 1, 2. \quad (37)
\]

Now, we come to the calculation of braided products for two normal ordered monomials. Towards this end, it suffices to focus our attention on one version of the braiding, since the expressions for the two braidings follow from each other quite easily by using simple transition rules [these transition rules are based on the correspondence in Eq. (37), as we will see later on].

Using Eq. (36) together with Eq. (37), it is a simple exercise to check that

\[
\Psi_{X,Y}^{-1}((X^1)^{n_1} \otimes (Y^1)^{m_1}) = q^{-2n_1m_1}(Y^1)^{m_1} \otimes (X^1)^{n_1},
\]
\[
\Psi_{X,Y}^{-1}((X^2)^{n_2} \otimes (Y^2)^{m_2}) = q^{-2n_2m_2}(Y^2)^{m_2} \otimes (X^2)^{n_2},
\]
\[
\Psi_{X,Y}^{-1}((X^1)^{n_1} \otimes (Y^2)^{m_2}) = q^{-n_1m_2}(Y^2)^{m_2} \otimes (X^1)^{n_1}.
\]

To derive a formula for the braiding between powers of $X^2$ and $Y^1$ is a little bit more complicated. A glance at the second relation in Eq. (36) shows that we can make as ansatz

\[
\Psi_{X,Y}^{-1}((X^2)^{n_2} \otimes (Y^1)^{m_1}) = \sum_{i=0}^{\min(n_2, m_1)} C_{i}^{n_2, m_1} (Y^1)^{m_1-i}(Y^2)^i \otimes (X^1)^{i}(X^2)^{n_2-i},
\]

with unknown coefficients $C_{i}^{n,m} \in \mathbb{C}$. Exploiting Eq. (36) together with Eq. (37) should enable us to find the recursion relation

\[
C_{i}^{n,m} = q^{-n-2i}C_{i}^{n,m-1} - q^{-n}\lambda [n-i+1]q^{-2}C_{i-1}^{n,m-1}, \quad (40)
\]
\[ C_{n,m}^0 = q^{-nm}. \]

As one can prove by insertion this recursion relation has the solution
\[ C_{n,m}^i = q^{-nm}(-\lambda)^i [[i]]_{q^{-2}}! \left[ \begin{array}{c} n \\ i \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} m \\ i \end{array} \right]_{q^{-2}}, \] (41)

where the \textit{q-deformed binomial coefficients} are defined by
\[ \left[ \begin{array}{c} \alpha \\ m \end{array} \right]_{q^a} = \frac{[[\alpha]]_{q^a} [[\alpha - 1]]_{q^a} \cdots [[\alpha - m + 1]]_{q^a}}{[[m]]_{q^a}!}, \] (42)

with \( \alpha \in \mathbb{C}, m \in \mathbb{N} \). Applying the relations of Eqs. (38) and (39) in succession finally yields
\[
\Psi_{X,Y}^{-1} ((X_1)^{n_1} (X_2)^{n_2} \otimes (Y_1)^{m_1} (Y_2)^{m_2})
\]
\[ = \sum_{i=0}^{\min(n_2,m_1)} (-\lambda)^i [[i]]_{q^{-2}}! \left[ \begin{array}{c} n_2 \\ i \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} m_1 \\ i \end{array} \right]_{q^{-2}} q^{-(n_2-i)(m_1-i)} \]
\[ \times q^{2(n_2-i)(m_2+i)-2(m_1-i)(n_1+i)-2(n_1+i)(m_2+i)+i(m_1+n_2-i)} \]
\[ \times (Y_1)^{m_1-i} (Y_2)^{m_2+i} \otimes (X_1)^{n_1+i} (X_2)^{n_2-i}. \] (43)

In order to get a formula for the braiding between arbitrary elements one has to realize that the monomials \((X_1)^{n_1} (X_2)^{n_2}\) form a basis of two-dimensional quantum plane. Due to this fact and the linearity of the mapping \(\Psi_{X,Y}^{-1}\), we are nearly done. What remains is to find a corresponding expression on commutative algebras. As a first step to this aim, we fix the isomorphism of Eq. (5) by
\[ \mathcal{W} ( (x_1)^{n_1} (x_2)^{n_2} ) \equiv (X_1)^{n_1} (X_2)^{n_2}, \quad n_1, n_2 \in \mathbb{N}. \] (44)

Recalling the definition of the braided product, which implies
\[ (x_1)^{n_1} (x_2)^{n_2} \otimes_L (y_1)^{m_1} (y_2)^{m_2} \]
\[ = (\mathcal{W}^{-1} \otimes \mathcal{W}^{-1}) \circ \Psi_{X,Y}^{-1} (\mathcal{W} ((x_1)^{n_1} (x_2)^{n_2}) \otimes \mathcal{W} ((y_1)^{m_1} (y_2)^{m_2})), \] (45)

we can immediately read off from Eq. (43) that
\[ (x_1)^{n_1} (x_2)^{n_2} \otimes_L (y_1)^{m_1} (y_2)^{m_2} \]
\[ = (\mathcal{W}^{-1} \otimes \mathcal{W}^{-1}) \circ \Psi_{X,Y}^{-1} (\mathcal{W} ((x_1)^{n_1} (x_2)^{n_2}) \otimes (y_1)^{m_1} (y_2)^{m_2})). \] (46)
\[
\begin{align*}
q\text{-binomial coefficients by making use of the braiding with respect to } \Psi_{X,Y} \text{ (to be more precise, we should mention that we have } (D_{q})^i (x_1)^{n_1} (x_2)^{n_2} = 0 \text{ for } i > n_\alpha) \text{ and }
q^\lambda \left( (y_1)^{m_1} (y_2)^{m_2} \otimes (x_1)^{n_1} (x_2)^{n_2} \right)
&= q^{\lambda (m_\alpha n_\beta)} \left( (y_1)^{m_1} (y_2)^{m_2} \otimes (x_1)^{n_1} (x_2)^{n_2} \right). \tag{48}
\end{align*}
\]

Next, we would like to move on to the corresponding formula concerning the braiding with respect to } \Psi_{X,Y}. \text{ Starting from the relations in Eq. (36) we have:}

\[
\begin{align*}
f(x_1, x_2) \sqcup_L g(y_1, y_2) = 
&= \sum_{i=0}^{\infty} q^i (-\lambda)^i \left( y_2 \right)^i \otimes (x_1)^i q^{-n_1 \otimes n_2 - 2n_2 \otimes n_1 - 2n_1 \otimes n_1 - n_2 \otimes n_1} \\
&\quad \times (D_{q})^i g(q^{-i}y_1, q^{-2i}y_2) \otimes (D_{q})^i f(q^{-2i}x_1, q^{-i}x_2).
\end{align*}
\]
and repeating the same steps as before, one can obtain

\[
\Psi_{X,Y} \left( (X^2)^{n_2} (X^1)^{n_1} \otimes (Y^2)^{m_2} (Y^1)^{m_1} \right)
\]

\[
= \min(n_1, m_2) \sum_{i=0}^{\min(n_1, m_2)} \lambda^i [i]_{q^2!} \left[ \begin{array}{c} n_1 \\ i \end{array} \right] \left[ \begin{array}{c} m_2 \\ i \end{array} \right] q^{(n_1-i)(m_2-i)}
\]

\[
\times q^{2(n_1-i)+(m_2-i)+2(n_2-i)+(n_1+i)-(i)(m_2+n_1-i)}
\]

\[
\times (Y^2)^{m_2-i} (Y^1)^{m_1+i} \otimes (X^2)^{n_2+i} (X^1)^{n_1-i}.
\]

Specifying the algebra isomorphism by

\[
\mathcal{W} \left( (x^1)^{n_1} (x^2)^{n_2} \right) = (X^2)^{n_2} (X^1)^{n_1}, \quad n_1, n_2 \in \mathbb{N},
\]

leads us to

\[
f(x^2, x^1) \bar{\circ}_L g(y^2, y^1)
\]

\[
= \sum_{i=0}^{\infty} q^{-i^2} \lambda^i \left( \frac{y^1}{x^1} \right)^i \otimes \left( \frac{x^2}{y^2} \right)^i
\]

\[
\times (D_{q^2})^i g(q^i y^2, q^i y^1) \otimes (D_{q^1})^i f(q^i x^2, q^i x^1),
\]

showing, in turn, that the following connection between the two braided products holds:

\[
f(x^2, x^1) \bar{\circ}_L g(y^2, y^1) \xrightarrow{\alpha \to \alpha'} q^{-1/q} \bar{\circ}_R f(x^1, x^2) \bar{\circ}_L g(y^1, y^2),
\]

where the symbol \( \xrightarrow{\alpha \to \alpha'} \) indicates a transition via the substitutions

\[D_{q^\alpha} \to D_{q^{-\alpha}}, \quad \hat{n}^\alpha \to -\hat{n}^{\alpha'}, \quad x^\alpha \to x^{\alpha'}, \quad q \to q^{-1}.
\]

It should be obvious that this correspondence is a direct consequence of the crossing symmetry in Eq. \(54\). One should also notice that in Eq. \(52\) the tilde on the multiplication symbol shall remind us of the fact that the explicit form of this braided product refers to a reversed normal ordering.
4 q-Deformed Euclidean space in three dimensions

In this section we would like to deal with three-dimensional q-deformed Euclidean space. Its algebra is spanned by three coordinates $X^A$, $A \in \{+, 3, -\}$, which fulfill as commutation relations [17]

\[
X^3X^\pm = q^{\pm 2}X^+X^3,
\]
\[
X^-X^+ = X^+X^- + \lambda X^3X^3.
\] (55)

Let us recall that the entries $L^A_B$ of the so-called L-matrix determine the braiding of the quantum space coordinates $X^A$, $A \in \{+, 3, -\}$, since we have (if not stated otherwise summation over repeated indices is to be understood)

\[
\Psi_{X,Y}(X^A \otimes w) = (L^A_B \triangleright w) \otimes X^B.
\] (56)

From the results in Ref. [37] we can read off their explicit form in terms of $U_q(su_2)$-generators, leaving us with

\[
\begin{align*}
\Psi_{X,Y}(X^+ \otimes w) &= (\Lambda^{1/2}(\tau^3)^{-1/2} \triangleright w) \otimes X^+,
\Psi_{X,Y}(X^3 \otimes w) &= (\Lambda^{1/2} \triangleright w) \otimes X^3 + \lambda \lambda_+ (\Lambda^{1/2}(L^- \triangleright w) \otimes X^+,
\Psi_{X,Y}(X^- \otimes w) &= (\Lambda^{1/2}(\tau^3)^{1/2} \triangleright w) \otimes X^-
+ q^{-1}\lambda \lambda_+ (\Lambda^{1/2}(\tau^3)^{1/2}L^- \triangleright w) \otimes X^3
+ q^{-2}\lambda^2 \lambda_+ (\Lambda^{1/2}(\tau^3)^{1/2}(L^-)^2 \triangleright w) \otimes X^+,
\end{align*}
\] (57)

where $w$ denotes an arbitrary element of a quantum space the algebra $U_q(su_2)$ acts upon. One should notice that $\Lambda$ stands for a scaling operator satisfying the relations

\[
\Lambda X^A = q^4 X^A \Lambda, \quad \Lambda \triangleright 1 = 1.
\] (59)

As a next step let us try to derive formulae for the expressions

\[
\Psi_{X,Y}((X^A)^n \otimes w) = (L^A_{B_1}L^A_{B_2}\ldots L^A_{B_n} \triangleright w) \otimes X^{B_1}X^{B_2}\ldots X^{B_n},
\] (60)

where $A \in \{+, 3, -\}$ and $n \in \mathbb{N}$. In order to do so, we perform the following
calculation:
\[ L_{B_1}^+ L_{B_2}^+ \cdots L_{B_n}^+ \otimes X^{B_1} X^{B_2} \cdots X^{B_n} = (L_B^+ \otimes X^B)^n \]
\[ = (\Lambda^{1/2} (r^3)^{-1/2} \otimes X^+)^n = (\Lambda^{1/2} (r^3)^{-1/2})^n \otimes (X^+)^n. \]  

Proceeding in the same way for \( X^3 \) and recalling the q-binomial theorem \([50, 51]\) for q-commuting variables, i.e.
\[ (A + B)^n = \sum_{k=0}^{n} \binom{n}{k} q^k A^k B^{n-k}, \quad \text{if } BA = q^a AB, \]  
we obtain
\[ L_{B_1}^3 L_{B_2}^3 \cdots L_{B_n}^3 \otimes X^{B_1} X^{B_2} \cdots X^{B_n} = (L_B^3 \otimes X^B)^n \]
\[ = (\Lambda^{1/2} \otimes X^3 + \lambda \Lambda^{1/2} L^- \otimes X^+)^n \]
\[ = \sum_{k=0}^{n} \binom{n}{k} q^2 (\lambda \Lambda)^k (\Lambda^{1/2})^n (L^-)^k \otimes (X^+)^k (X^3)^{n-k}. \]  

Notice that application of the q-binomial theorem in the third identity requires to hold
\[ (\Lambda^{1/2} \otimes X^3)(\Lambda^{1/2} L^- \otimes X^+) = q^2 \Lambda L^- \otimes X^+ X^3, \]
which follows directly from the commutation relations for coordinates and symmetry generators (see for example Ref. [37]).

To derive a relation describing the braiding of powers of \( X^- \) is a little bit more involved. It turns out to be convenient to formulate the braiding in terms of the coproduct. On quantum space coordinates the coproduct takes on the form \([40], [49]\)
\[ \Delta_L (X^A) = (X^A)_{(1)} \otimes (X^A)_{(2)} \]
\[ = X^A \otimes 1 + L^A_B \otimes X^B. \]

Now, we are in a position to introduce the identity
\[ \Psi_{X,Y} ((X^A)^n \otimes w) = \left( [(X^A)^n]_{(1)} \triangleright w \right) \otimes [(X^A)^n]_{(2)}, \]
which is valid, if we demand for quantum space coordinates to act on elements of another quantum space by

\[ X^A \triangleright w = 0. \]  

Next, we would like to derive an explicit formula for the coproduct of powers of \( X^\tau - \). To achieve this, we take into account the module coalgebra property of our quantum spaces [35], which implies

\[
\Delta \bar{L} \left( \left[ (L^\tau)^n \right] \mathcal{X}^3 \right) = \Delta \bar{L} \left( \left[ (L^\tau)^n \right] \mathcal{X}^3 \right) \Delta((L^\tau)^n) \Delta((\tau^3)^{n-k}/2).
\]

In this derivation, the first step uses the explicit form for the adjoint action and the second equality makes use of

\[
\Delta(L^n) = (\Delta(L^n))^n = (L^n \otimes (\tau^3)^{-1/2} + 1 \otimes L^n)^n = \sum_k q^{-2k(n-k)} \left[ \binom{n}{k} q^2 \right] (L^n)^{n-k} \otimes (L^n)^k (\tau^3)^{-k}/2,
\]

together with

\[
S((L^n)^n) = ((L^n)^n)^n = (-1)^n q^{n(n-1)}(L^n)^n (\tau^3)^{-1/2}.
\]

On the other hand, using the operator representation of \( L^- \), as it was presented in Ref. [37], we get

\[
\Delta \bar{L} \left( (L^n) \mathcal{X}^3 \right) = q^{-n^2} \left[ [n] q^n \right]! \Delta \bar{L} \left( (X^n) \right).
\]

A short glance at Eqs. (69) and (72) shows us how to reduce coproducts for powers of \( X^- \) to those for powers of \( L^- \), \( \tau^3 \), and \( X^3 \). Thus, it remains to find an explicit formula for coproducts of powers of \( X^3 (\tau^3 \) is a grouplike
element). In Ref. [40] this missing link was found to be

$$\Delta_L(X^3)^n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] q^2 (\lambda \lambda_+)^k (\Lambda^{1/2})^n (L^-)^k \otimes (X^+)^k (X^3)^{n-k} + \ldots, \quad (73)$$

where the insertion points indicate terms that in the end will not contribute in Eq. (67) due to Eq. (68).

From what we have done so far, it is now rather straightforward but laborious to obtain

$$\Psi_{X,Y}((X^-)^n \otimes w) = (\mathcal{L}_{B_1}^- \mathcal{L}_{B_2}^- \ldots \mathcal{L}_{B_n}^- \triangleright w) \otimes X^{B_1} X^{B_2} \ldots X^{B_n} \quad (74)$$

= \sum_{i=0}^{n} \sum_{k=0}^{n-k} (-1)^{k} \binom{\lambda \lambda_+}{i} q^{n^2 + k(k-1)-2l(n-i-k-l)} \times \left[ \begin{array}{c} n-k \\ l \\ q^2 \\ \frac{i}{q^2} \end{array} \right] \left[ \frac{3}{Q(k_3)} \right] (\Lambda^{1/2})^n (L^-)^{i+k+l}(\tau^3)^{n/2} \triangleright w \times (L^-)^{n-k-l} \triangleright (X^+)^i (X^3)^{n-i}. \quad (75)$$

Applying Eqs. (61), (63), and (74) in succession finally yields

$$\Psi_{X,Y}((X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-} \otimes w) \quad (75)$$

= \sum_{i=0}^{n_3} \sum_{k=0}^{n_-} \sum_{l=0}^{n_- - k} (-1)^{k} (\lambda \lambda_+)^{i+j} \binom{\lambda \lambda_+}{i} q^{k(k-1)+2l(k+l+j)-2n_+(i+j+k+l)+n_-(n_- - 2l)} \times \left[ \begin{array}{c} k+l \\ k \\ q^2 \\ \frac{i}{q^2} \end{array} \right] \left[ \begin{array}{c} n_3 \\ n_- \\ j \\ q^2 \end{array} \right] k! \times \left( (\Lambda^{1/2})^{n_3-n_-} (L^-)^{i+j+k+l}(\tau^3)^{(n_- - n_+ - n_+)} \triangleright w \right) \times (X^+)^{n_+ - i} (X^3)^{n_3 - i} \times (L^-)^{n_- - k-l} \triangleright (X^+)^l (X^3)^{n_- - j}. \quad (75)$$

There remains to evaluate the action in the last expression of Eq. (75). This can be achieved by formula (157) in Appendix B. Now, we have everything together to write down an expression for the braiding of two normal
ordered monomials. Having finished that task, we can follow the same line of arguments as in the previous section. More concretely, we determine the algebra isomorphism in Eq. (5) by

\[ \mathcal{W} \left( (x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} \right) \equiv (X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-}, \quad n_+, n_3, n_- \in \mathbb{N}, \quad (76) \]

and read off from (75) an expression for the braided product in very much the same way as was done for the two-dimensional case. In this manner, we finally arrive at (if not stated otherwise, all summation variables in this article have to be integer and non-negative)

\[
\begin{align*}
& f(x^+, x^3, x^-) \circ_{L^3} g(y^+, y^3, y^-) \\
& = \sum_{i,s=0}^{\infty} \sum_{j=0}^{s} \sum_{k+l+t=s} \sum_{0 \leq v \leq u \leq i} \sum_{u+v=t} (-1)^k (\lambda \lambda_+)^{i+j} t_{u,v} \\
& \quad \times q^{k-v+2(k+l+i+j)^2+2s(s-v)+2u(i-v)-2j(i-l)+l^2+lt+t^2} \\
& \quad \times [i]_{q^4}! [k]_{q^4}! [t]_{q^4}! [s]_{q^4}! [u]_{q^4}! [j]_{q^4}! [s]_{q^4}! \] \\
& \quad \times \left[ (L^-)^{i+j+k+l} \triangleright g(q^{2(k+l+i+j)} y^+, q^{2(k+l+i+j)} y^-) \right] \\
& \quad \times \left[ (x^+)^{i+j-u} (x^3)^{s-j+u-v} (x^-)^v \times (D_{q^3}^{-1}) (D_{q^3}^{-1})^s f(q^{2(j-u)} x^3, q^{2(i+j+l)} x^-) \right],
\end{align*}
\]

where the explicit form of \( t_{u,v} \) and that for the action of powers of \( L^- \) can again be looked up in Appendix [3] [see Eqs. (156) and (158)].

Let us end with the comment that we could also have started our considerations from the other braiding determined by

\[
\begin{align*}
& \Psi_{XY}^{-1} (X^- \otimes w) = (\Lambda^{-1/2}(r^3)^{-1/2} \triangleright w) \otimes X^-, \\
& \Psi_{XY}^{-1} (X^3 \otimes w) = (\Lambda^{-1/2} \triangleright w) \otimes X^3 + \lambda \lambda_+ (\Lambda^{-1/2} L^+ \triangleright w) \otimes X^-, \\
& \Psi_{XY}^{-1} (X^+ \otimes w) = (\Lambda^{-1/2}(r^3)^{1/2} \triangleright w) \otimes X \\
& \quad + q \lambda \lambda_+ (\Lambda^{-1/2}(r^3)^{1/2} L^+ \triangleright w) \otimes X^3 \\
& \quad + q^2 \lambda^2 \lambda_+ (\Lambda^{-1/2}(r^3)^{1/2}(L^+)^2 \triangleright w) \otimes X^-. 
\end{align*}
\]

(79)
This way we would get a second braided product denoted by \( \tilde{\odot}_L \), which is linked to the first one via the transformation rule

\[
f(x^+, x^3, x^-) \odot_L g(y^+, y^3, y^-) \xrightarrow{\pm \to \mp} f(x^-, x^3, x^+) \odot_L g(y^-, y^3, y^+),
\]

where \( \xi \xrightarrow{\pm \to \mp} \pm 1/q \) indicates that we can make a transition between the two expressions by applying the substitutions

\[
D_\pm \to D_\mp, \quad \hat{n}_\pm \to \mp \hat{n}_\mp, \quad x_\pm \to x_\mp, \quad q_\pm \to q_\mp.
\]

5 q-Deformed Euclidean space in four dimensions

The four-dimensional Euclidean space [52] can be treated in very much the same way as the three-dimensional one. Therefore we will restrict ourselves to stating the results, only. Again, we begin by considering its defining relation, which in terms of the coordinates \( X^i, i \in \{1, \ldots, 4\} \), explicitly read

\[
X^1 X^j = q X^j X^1, \\
X^j X^4 = q X^4 X^j, \quad j = 2, 3, \\
X^2 X^3 = X^3 X^2, \\
X^4 X^1 = X^1 X^4 + \lambda X^2 X^3.
\]

The results in Ref. [37] tell us that the inverse braiding of a single quantum space coordinate with an element \( w \) of another quantum space is now determined by

\[
\Psi^{-1}_{X,Y}(X^1 \otimes w) = (\Lambda^{-1/2} K_1^{-1/2} K_2^{1/2} \triangleright w) \otimes X^1, \\
\Psi^{-1}_{X,Y}(X^2 \otimes w) = (\Lambda^{-1/2} K_1^{-1/2} K_2^{1/2} \triangleright w) \otimes X^2 \\
+ q \lambda (\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_1^\dagger \triangleright w) \otimes X^1, \\
\Psi^{-1}_{X,Y}(X^3 \otimes w) = (\Lambda^{-1/2} K_1^{1/2} K_2^{-1/2} \triangleright w) \otimes X^3 \\
+ q \lambda (\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_2^+ \triangleright w) \otimes X^1, \\
\Psi^{-1}_{X,Y}(X^4 \otimes w) = (\Lambda^{-1/2} K_1^{-1/2} K_2^{-1/2} \triangleright w) \otimes X^4.
\]
the identities in Eq. (83) now imply

\[ q\lambda(\Lambda^{-1/2}K_1^{-1/2}K_2^{-1/2}L_2^+ \triangleright w) \otimes X^2, \]

\[ q\lambda(\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2}L_1^+ \triangleright w) \otimes X^3 \]

\[ q^2\lambda^2(\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}L_1^+L_2^+ \triangleright w) \otimes X^1, \]

where the symmetry generators are part of \( U_q(so_4) \) (for its defining relations see again Appendix A). With the same reasonings as in the previous section the identities in Eq. (83) now imply

\[
\Psi^{-1}_{X,Y}((X^1)^{n_1} \otimes w) = \left((\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2})^{n_1} \triangleright w\right) \otimes (X^1)^{n_1},
\]

\[
\Psi^{-1}_{X,Y}((X^2)^{n_2} \otimes w) = \sum_{k=0}^{n_2} \binom{n_2}{k} q^{i\lambda} \left((\Lambda^{-1/2})^{n_2}(K_1^{1/2}K_2^{-1/2}L_2^+)^k(K_1^{-1/2}K_2^{1/2})^{n_2-i} \triangleright w\right) \otimes (X^1)^k(X^2)^{n_2-k},
\]

\[
\Psi^{-1}_{X,Y}((X^3)^{n_3} \otimes w) = \sum_{k=0}^{n_3} \binom{n_3}{k} q^{i\lambda} \left((\Lambda^{-1/2})^{n_3}(K_1^{1/2}K_2^{1/2}L_2^+)^k(K_1^{-1/2}K_2^{-1/2})^{n_3-i} \triangleright w\right) \otimes (X^1)^k(X^3)^{n_3-k}.
\]

The corresponding formula for the quantum space coordinate \( X^4 \) takes on a much more complicated form, i.e.

\[
\Psi^{-1}_{X,Y}((X^4)^{n_1} \otimes w) = \sum_{k=0}^{n_4} \sum_{u=0}^{n_4-k} \sum_{v=0}^{n_4-u} (-1)^k \frac{(q\lambda)^u}{[n_4]_q!} q^{n_4(n_4+1)/2+k(k+1)/2-1/2-n_4(k+v)}
\times q^{2u(n_4-k-v)+u(n_4-u)} \left[\begin{array}{c} n_4 \\ k \end{array}\right]_q \left[\begin{array}{c} n_4-k \\ u \end{array}\right]_q \left[\begin{array}{c} n_4 \\ u \end{array}\right]_q
\times \left((\Lambda^{-1/2})^{n_4}(K_1^{1/2})^{-n_4+2(k+v)}(K_2^{1/2})^{n_4+2u}(L_1^+)^{k+v}(L_2^+)^u \triangleright w\right) \otimes (L_1^+)^{n_4-k-v} \otimes (X^1)^u(X^3)^{n_4-u}.
\]
The derivation of this relation is mainly based on

\[
\Delta_L \left( (L_1^+)^n \triangleright (X^3)^n \right) = \Delta_L \left( \left[ (L_1^+)^n \right]_{(1)} \left[ (X^3)^n \right] \mathcal{S} \left( \left[ (L_1^+)^n \right]_{(2)} \right) \right) \tag{88}
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} q^{(k+1)(n-k)} \left[ \frac{n}{k} \right]_{q^{-2}} \Delta((L_1^+)^k) \Delta_L((X^3)^n) \Delta((L_1^+)^{n-k})
\]

and

\[
\Delta_L \left( (L_1^+)^n \triangleright (X^3)^n \right) = q^{-n(n+1)/2}[[n]]q^2! \Delta_L(X^4)^n, \tag{89}
\]

since both identities allow one to proceed in a similar fashion as for the coordinate \(X^c\) of three-dimensional q-deformed Euclidean space. A short glance at Eq. (87) shows us that we also need to know the action of powers of \(L_i^+\), \(i = 1, 2\), on normal ordered monomials (the action of the \(K_i\), \(i = 1, 2\), is very simple). Explicit formulae for this task have been derived in Appendix B [see Eqs. (159) and (160)].

Now, we have everything together for calculating the braiding between a normal ordered monomial and an arbitrary element of another quantum space. Before doing so, we fix the algebra isomorphism of Eq. (5) by

\[
\mathcal{W} \left( (x^1)^{n_1} (x^2)^{n_2} (x^3)^{n_3} (x^4)^{n_4} \right) \equiv (X^1)^{n_1} (X^2)^{n_2} (X^3)^{n_3} (X^4)^{n_4}, \quad n_i \in \mathbb{N}. \tag{90}
\]

With this convention we finally arrive at

\[
f(x^1, x^2, x^3, x^4) \mathcal{L} g(y^1, y^2, y^3, y^4) \tag{91}
\]

\[
= \sum_{i,j=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k,l=s+2}^{\infty} (-1)^{k+v} (q\lambda)^{i+j+u} q^{(i+1)/2+j(j+1)/2+k(k+1)}
\]

\[
\times q^{u(u+1)/2+v(v+1)/2+s(s+1)/2+(u-v)(i+j-2s)-u(v+2j)+i(j-2k-2l)}
\]

\[
\times \frac{1}{[[i]]q^2![[j]]q^2![[k]]q^2![[\ell]]q^2!} \left[ \begin{array}{c} u \\ v \\ s \end{array} \right] q^2 \left[ \begin{array}{c} u \\ s \end{array} \right] q^2
\]

\[
\times q^{(n_1+n_2+n_3+n_4)\otimes(n_1+n_2+n_3+n_4)-(n_1-n_4)-(n_2-n_3)-(n_2-n_3)}
\]

\[
\times \left[ (L_1^+)^{i+l+k} (L_2^+)^{j+l+u} \triangleright g(y^1, y^2, y^3, y^4)
\]

\[
\otimes (x^1)^{i+j+u-v} (x^2)^v (x^3)^s-u
\]

\[
\times (D_{q^2})^{i}(D_{q^2})^{j}(D_{q^2})^{s} f \left( q^{-i+j+u-v} x^2, q^{-(i+j+u-v)} x^2, q^{i+j+u-v-s} x^4 \right),
\]

23
where formulae for the action of powers of $L_i^\pm$, $i = 1, 2$, on commutative functions are again given in Appendix B [see Eqs. (161) and (162)].

In complete analogy to the three-dimensional case we can also assign a braided product to the braiding determined by

$$
\Psi_{X,Y}(X^4 \otimes w) = (\Lambda^{1/2} K_1^{1/2} K_2^{1/2} \triangleright w) \otimes X^4,
$$

$$
\Psi_{X,Y}(X^3 \otimes w) = (\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} \triangleright w) \otimes X^3
+ q^{-1} \lambda (\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_1^- \triangleright w) \otimes X^4,
$$

$$
\Psi_{X,Y}(X^2 \otimes w) = (\Lambda^{1/2} K_1^{1/2} K_2^{-1/2} \triangleright w) \otimes X^2
+ q^{-1} \lambda (\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_2^- \triangleright w) \otimes X^4,
$$

$$
\Psi_{X,Y}(X^1 \otimes w) = (\Lambda^{1/2} K_1^{-1/2} K_2^{-1/2} \triangleright w) \otimes X^1
- q^{-1} \lambda (\Lambda^{1/2} K_1^{1/2} K_2^{-1/2} L_1^- \triangleright w) \otimes X^2
- q^{-1} \lambda (\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} L_2^- \triangleright w) \otimes X^3
- q^{-2} \lambda^2 (\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_1^- L_2^- \triangleright w) \otimes X^4.
$$

The explicit form of that second braided product is related to the first one by the correspondence

$$
f(x^1, x^2, x^3, x^4) \otimes_L g(y^1, y^2, y^3, y^4)
\overset{q \rightarrow 1/q}{\sim} f(x^4, x^3, x^2, x^1) \otimes_L g(y^4, y^3, y^2, y^1),
$$

symbolizing a transition via the substitutions

$$
D^i_{q^a} \rightarrow D'^i_{q^{-a}}, \quad \hat{n}_i \rightarrow -\hat{n}_{i'}, \quad x^i \rightarrow x^{i'}, \quad q^a \rightarrow q^{-1},
$$

where $i' \equiv 5 - i$.

### 6 q-Deformed Minkowski space

In this section we would like to deal with q-deformed Minkowski space [13, 15–17], which from a physical point of view is the most interesting one (for other versions of deformed spacetime see also Refs. [53–58]). First of all, let us make contact with its defining relations. In terms of the coordinates $X^\mu$,
\( \mu \in \{0, +, -, 3/0\} \), they take the form
\[
X^{\mu}X^0 = X^0X^{\mu}, \quad \mu \in \{0, +, -, 3/0\}, \tag{95}
\]
\[
X^{\pm}X^{3/0} = q^{\mp 2}X^{3/0}X^{\pm},
\]
\[
X^-X^+ - X^+X^- = \lambda(X^{3/0}X^{3/0} + X^0X^{3/0}).
\]

Notice that \( X^0 \) can be interpreted as a time coordinate, while the \( X^\mu \) with \( \mu = \{+, 3/0, -\} \) give some sort of light cone coordinates.

As in the previous sections we start our further considerations with the braiding of the Minkowski space coordinates [21]:
\[
\Psi_{X,Y}^{-1}(X^{3/0} \otimes w) = (\Lambda^{-1/2}T^1 \triangleright w) \otimes X^{3/0} \tag{96}
\]
\[ - q^{1/2}\lambda^1_+^{1/2}\lambda(\Lambda^{-1/2}(\tau^3)^{-1/2}S^1 \triangleright w) \otimes X^+,
\]
\[
\Psi_{X,Y}^{-1}(X^+ \otimes w) = (\Lambda^{-1/2}(\tau^3)^{-1/2}\sigma^2 \triangleright w) \otimes X^+ \tag{97}
\]
\[ - q^2\lambda^1_+^{-1/2}\lambda(\Lambda^{-1/2}T^2 \triangleright w) \otimes X^{3/0},
\]
\[
\Psi_{X,Y}^{-1}(X^- \otimes w) = (\Lambda^{-1/2}(\tau^3)^{1/2}\tau^1 \triangleright w) \otimes X^- \tag{98}
\]
\[ - q^{-1/2}\lambda^1_+^{1/2}\lambda(\Lambda^{-1/2}S^1 \triangleright w) \otimes X^0
\]
\[ - \lambda^2(\Lambda^{-1/2}(\tau^3)^{-1/2}T^-S^1 \triangleright w) \otimes X^+
\]
\[ + q^{-1/2}\lambda^1_+^{-1/2}\lambda(\Lambda^{-1/2}(\tau^1T^- - q^{-1}S^1) \triangleright w) \otimes X^{3/0},
\]
\[
\Psi_{X,Y}^{-1}(X^0 \otimes w) = (\Lambda^{-1/2}\sigma^2 \triangleright w) \otimes X^0 \tag{99}
\]
\[ - q^{1/2}\lambda^1_+^{-1/2}\lambda(\Lambda^{-1/2}T^2(\tau^3)^{1/2} \triangleright w) \otimes X^-
\]
\[ + q^{1/2}\lambda^1_+^{-1/2}\lambda(\Lambda^{-1/2}(\tau^3)^{-1/2}(T^-\sigma^2 + qS^1) \triangleright w) \otimes X^+
\]
\[ - \lambda^1_+^{-1}(\Lambda^{-1/2}(\lambda^2T^-T^2 + q(\tau^1 - \sigma^2)) \triangleright w) \otimes X^{3/0},
\]
where \( T^-, T^2, S^1, \sigma^2, \tau^1, \) and \( \tau^3 \) are generators of the q-deformed Lorentz algebra (for its algebraic structure see Appendix \( \Delta \)).

Applying considerations very similar to those for the Euclidean cases we can derive the identities
\[
\Psi_{X,Y}^{-1}((X^+)^n \otimes w) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{(-q^{3/2}\lambda^1_+^{-1/2}\lambda)^k}{q^2}
\]
\[ \times \left[ (\Lambda^{-1/2})^n(T^2)^k(\sigma^2)^{n-k}(\tau^3)^{-(n-k)/2} \triangleright w \right] \tag{100}
\]

25
\( \otimes (X^\dagger)^{n-k} (X^{3/0})^k \)

\[
\Psi_{X,Y}^{-1}((X^{3/0})^n \otimes w) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{(-q^{1/2} \lambda_{+}^{1/2})^k}{q^2} q^{-k(k+1)}
\]

\[
\times \left[ (\Lambda^{-1/2})^n (S^1)^k (\tau^3)^n (T^3)^{-k/2} \triangleright w \right]
\]

\[
\otimes (X^\dagger)^k (X^{3/0})^{n-k}.
\]

To get a corresponding expression for powers of \( X^- \) we have to exploit the relations

\[
\Delta_L ((T^-)^n \triangleright (X^{3/0})^n)
\]

\[
= \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] (-1)^k q^{k(k-1)} \Delta((T^-)^{n-k}) \Delta_L((X^{3/0})^n) \Delta((T^-)^k),
\]

and

\[
\Delta_L ((T^-)^n \triangleright (X^{3/0})^n) = (q^{3/2} \lambda_{+}^{1/2})^n [n]_q^2 ! \Delta_L(X^-)^n,
\]

leading us to

\[
\Psi_{X,Y}^{-1}((X^-)^n \otimes w) =
\]

\[
= \sum_{i=0}^{n} \sum_{k=0}^{n-k} \sum_{l=0}^{n-k} (-1)^{k+i} \chi^i \frac{(q^{1/2} \lambda_{+}^{1/2})^{i-n}}{[n]_q^2 !} q^{k(k+1)+i(i-1)+2(n-k-l-i)(i+k)-n}
\]

\[
\times \left[ \begin{array}{c} n \\ i \end{array} \right] q^2 \left[ \begin{array}{c} n \\ k+l \end{array} \right] \left[ \begin{array}{c} k+l \\ k \end{array} \right]_q^2 q^2
\]

\[
\times \left[ (\Lambda^{-1/2})^n (T^-)^i (S^1)^i (T^-)^k (T^3)^{n-l-i-k/2} \triangleright w \right]
\]

\[
\otimes (T^-)^{n-k-l} \triangleright (X^\dagger)^i (X^{3/0})^{n-i}.
\]

There remains to deal with the coordinate \( X^0 \) in the same manner. Unfortunately, if we try to do so we would get expressions that reveal a rather complicated structure. Hence, we wish to proceed differently. Towards this end, we have to recall that in Ref. [40] it was proven that the monomials

\[
(i^2)^{n_\mu} (X^+)^{n_\mu} (X^{3/0})^{n_\mu/0-n_\mu} (X^-)^{n_-}, \quad n_\mu \in \mathbb{N}_0,
\]

also constitute a basis of q-deformed Minkowski space, where the new central
coordinate \( \hat{r}^2 \) stands for the square of the Minkowski length given by

\[
\hat{r}^2 = -X^0 X^0 + X^3 X^3 - q X^+ X^- - q^{-1} X^- X^+.
\]  

(106)

Let us notice that in Ref. [40] one can also find the relationship between this new basis and a more familiar one with \( \hat{r}^2 \) being replaced by the time coordinate \( X^0 \). The reason for introducing this special basis lies in the fact that the Minkowski length has a rather simple braiding, since it satisfies

\[
\Psi^{-1}_{X,Y}((\hat{r}^2)^n \otimes w) = (\Lambda^{-n} \triangleright w) \otimes \hat{r}^{2n},
\]  

(107)

which follows from the very definition of \( \hat{r}^2 \). Putting everything together we get in a straightforward manner

\[
\Psi^{-1}_{X,Y}((\hat{r}^2)^n (X^+)^{n_+} (X^{3/0})^{n_{3/0}} (X^-)^{n_-} \otimes w) = (108)
\]

\[
(q^{3/2} \lambda_+^{1/2})^{-n_-} \sum_{i=0}^{n_+} \sum_{j=0}^{n_{3/0}} (-q^{3/2} \lambda_+^{1/2} \lambda)^i (-q^{1/2} \lambda_+^{1/2} \lambda)^j q^{-j(j+1)-2(j(n_+ - 2i))}
\]

\[
\times \left[ \begin{array}{c} n_+ \\ i \\ \end{array} \right] \left[ \begin{array}{c} n_{3/0} \\ j \\ \end{array} \right] q^2 \sum_{l=0}^{n_-} \sum_{v=0}^{n_-} (-1)^{l(l-1)} [l+v] q^2 [l+v] q^2
\]

\[
\times \sum_{t=0}^{n_-} (-q^{1/2} \lambda_+^{1/2} \lambda)^t q^{-t(t+1)} \left[ \begin{array}{c} n_- \\ t \\ \end{array} \right] q^2
\]

\[
\times \left[ (T^2)^t (S^1)^j (\sigma^2)^{n_- - i} (\tau^1)^{n_{3/0} - j} (\tau^3)^-(n_+ - n_- - i + j + l + v)/2
\]

\[
\times (S^1)^t (\tau^3)^{-t/2} (T^-)^t (\Lambda^{-1/2})^{2n_+ + n_{3/0} + n_-} \triangleright w
\]

\[
\otimes (\hat{r}^2)^n_r (X^+)^{n_+ - i + j} (X^{3/0})^{n_{3/0} - j} (T^-)^{n_- - l - v} \triangleright ((X^+)^t (X^{3/0})^{n_- - t}).
\]

Our next job is to rearrange the symmetry generators within the square bracket in a way that generators of the same type are brought together. With the help of the commutation relations in Appendix A see Eqs. (144)-(147) this task can be done rather easily in the case of the symmetry generator \( S^1 \). For computational convenience it is also necessary to commute powers of \( \sigma^2 \) and \( \tau^2 \) to the far left of the product of symmetry generators. To achieve this,
we apply the two identities

\[(\tau^1)^n(T^-)^m = \sum_{k=0}^{\min(n,m)} (-\lambda)^k q^{-2m(n-k)} [[k]]_{q=2}! \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} m \\ k \end{array} \right]_q + (S^1)^k (T^-)^{m-k} (\tau^1)^{n-k}, \]

\[(\sigma^2)^n(T^-)^m = \sum_{k=0}^{\min(n,m)} (q^2\lambda)^k q^{2m(n-k)} [[k]]_{q=2}! \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} m \\ k \end{array} \right]_q + (S^1)^k (T^-)^{m-k} (\sigma^2)^{n-k}, \]

which can be proven by the method of induction in combination with the relations of (144) and (145) in Appendix A. With these considerations it follows that

\begin{align*} 
(T^2)^i(S^1)^j(\sigma^2)^{n_i - i}(\tau^1)^{n_{3/0} - j}(\tau^-)^{(n_+ - n_- - i + j + l + v)/2} \times (S^1)^t(\tau^1)^{n_- - t}(\tau^-)^t/(n_+ - n_- + n_{3/0} + n_-) \\
= q^{-2((n_3/0 + t) - 2t(n_+ - n_- + v + i + j))} \\
\times \sum_{u_1=0}^{l} \sum_{u_2=0}^{l-u_1} \sum_{u_3=0}^{l-u_1-u_2} (-\lambda)^{u_1 + u_2} \lambda^{u_3} \\
\times q^{-2(u_1 + u_2 + u_3 + l + 2l(u_1 + u_2))} q^{-2(n_+ - i)(u_1 + u_2 + u_3) + 2(n_{3/0} - j)u_1} \\
\times [[u_1 + u_2 + u_3]]_{q=2}! \left[ \begin{array}{c} l \\ u_1 + u_2 + u_3 \end{array} \right]_q \\
\times \left[ \begin{array}{c} n_+ - t \\ u_1 \end{array} \right]_q \left[ \begin{array}{c} n_{3/0} - j \\ u_2 \end{array} \right]_q \left[ \begin{array}{c} n_+ - i \\ u_3 \end{array} \right]_q \\
\times (T^2)^i(S^1)^j+l+u_1+u_2+u_3 (T^-)^{l-u_1-u_2-u_3}(\sigma^2)^{n_+ - i - u_3} \\
\times (\tau^1)^{n_+ + n_{3/0} - j - l - u_1 - u_2}(\tau^-)^{l/2(n_+ - n_- + i - j - l - v - t)} (\Lambda^{-1/2})^{2n_+ + n_- + n_{3/0} + n_-}.
\end{align*}

A short glance at Eq. (108) makes it obvious that we cannot write down a closed formula for braided products on a commutative algebra until we have not evaluated the action of powers of $T^-$ as it appears in the second tensor factor of the last expression of (108). Using the results of Appendix B [see Eq. (173)], one can show by direct calculation that

\[(r^2)^{n_r}(X^+)^{n_+ - i + j}(X^{3/0})^{n_{3/0} - j + i}(T^-)^{n_- - l - v} \times ((X^+)^t(X^{3/0})^{n_- - t}) \]

(112)
functions. If we choose for the algebra isomorphism of Eq. (5)
directly translate into a formula for the braided product of two com-
mutative
Appendix A. Proceeding this way provides us with an expression we can
be read off from the relations in (180) and (181) in Appendix B [the ac-
tion
B [see Eqs. (166) and (173)]. As a next step we insert Eqs. (111) an-
and (112)
into Eq. (108) and evaluate the actions for
τ

where the explicit forms of (d_q)^{ij}_k l and S^{0}_{i,j} have been introduced in Appendix
E [see Eqs. (166) and (173)]. As a next step we insert Eqs. (111) and (112)
into Eq. (108) and evaluate the actions for τ^1, σ^2, and τ^3 which can easily
be read off from the relations in (180) and (181) in Appendix B the ac-
tion
of the scaling operator Λ is very simple, as one can see from Eq. (111) in
Appendix A. Proceeding this way provides us with an expression we can
directly translate into a formula for the braided product of two commutative
functions. If we choose for the algebra isomorphism of Eq. (5)
\[ W((r^2)^{n_+}(x^+)^{n_+}(x^{3/0})^{n_3/0}(x^-)^{n_-}) \]
\equiv (\hat{r}^2)^{n_+}(X^+)^{n_+}(X^{3/0})^{n_3/0}(X^-)^{n_-}, \quad n_\mu \in \mathbb{N},
we obtain for the braided product after some tedious steps
\[ f(r^2, x^+, x^{3/0}, x^-) \otimes_L g(r^2, x^+, x^{3/0}, x^-) \]
\[ = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-q^{3/2} \lambda_+^{1/2} \lambda)^i (-q^{1/2} \lambda_+^{1/2} \lambda)^j q^{-ij(j+1)+4ij} \\
\times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} (-1)^k (q^{-3/2} \lambda_+^{1/2})^{k+l} (-q^{1/2} \lambda_+^{1/2})^s \\
\times q^{k(k-1)-s(s-1)-2ks+2(i-j)(k+2s)} q^{-[l]q^2} [s]q^2! \\
\times \left[ q^{-[l]q^2} [s]q^2! \right]^{1/2} \\
\times \left[ q^{-[l]q^2} [s]q^2! \right]^{1/2} \\
\times \left[ q^{-[l]q^2} [s]q^2! \right]^{1/2} \]
29
\[ \sum_{t_1=0}^{k} \sum_{t_2=0}^{k-t_1} \sum_{t_3=0}^{k-t_1-t_2} (-\lambda)^{t_1+t_2} (q^2 \lambda)^{t_3} q^{-t_3(t_3-1)} \]
\[ \times \sum_{t_1} \frac{q^{2(t-k+1)(t_1+t_2+t_3)} - 2k(i-s-t_1) + 2(k-t_1)(j+t_2)}{[[t_1]] q^2 ! [[t_2]] q^2 ! [[t_3]] q^2 ! [k - t_1 - t_2 - t_3] q^2 !} \]
\[ \times \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} q^{-u_1(u_1-1) - u_2(u_2-1) - 4u_1u_2} q^{-2u_1(j+t_2) - 2(u_1+u_2)(i+t_3)} \]
\[ \times \sum_{v=0}^{\infty} \sum_{0 \leq w_1 + w_2 \leq \min(v, s)} (q^{-3/2} \lambda_+^{1/2}) v (d_q)_{w_1,w_2}^{v,s} \]
\[ \times \frac{q^{2v+2(v-w_1-w_2)(k+l+v-s)-2(u_1+u_2)(i-j)}}{[[v]] q^2 ! [k + l + v - s] q^2 !} \]
\[ \times \sum_{p=0}^{w_1} q^{-p+2p(i-j+k+l-s+w_1+w_2)} \]
\[ \times \left[ (T^2)^i (S^1)^{j+s+t_1+t_2+t_3} (T^{-})^{k-t_1-t_2-t_3} \left( (y^{3/0})^2 D_q^2 D_q^{-} \right)^{u_1+u_2} \]
\[ \times (x^+)^{\alpha_+} (x^{-3/0})^{\alpha_3 / 0} S^{0}_{w_1,p} (x^2, x^{-3/0}) (x^-)^{\alpha_-} \]
\[ \times (D_q^+)^{i+t_3} (D_q^{-})^{j+t_2+u_2} (D_q^2)^{k+l+v-s} (x^-)^{t_1+u_1} (D_q^{-})^{s+t_1+u_1} \]
\[ \times q^{(n_+ + n_3 / 0 + n_- + 2n_+) \otimes (n_+ + n_3 / 0 + n_- + 2n_+) + 2(n_- - n_+) \otimes (n_- - n_+)} \]
\[ \times g(q^{\beta_+ y^+}, q^{\beta_3 / 0 y^{3 / 0}}, q^{\beta_- y^-}) \otimes f(q^{\gamma_+ x^+}, q^{\gamma_3 / 0 x^{3 / 0}}) \]
Let us mention that in Eq. (114) the tensor product of operators has to act componentwise on the tensor product of commutative functions. Again, the explicit form of the required actions can be looked up in Appendix B [see Eqs. (176)-(181)].

For the sake of completeness, we wish to write down the rule making a connection to the second braiding which is determined by the relations

\[
\Psi_{X,Y}(X^{3/0} \otimes w) = (\Lambda^{1/2}(\tau^3)_{-1/2}\sigma^2 \triangleright w) \otimes X^{3/0}
\]

\[- q^{1/2}\lambda^{1/2}\lambda(\Lambda^{1/2}T^2 \triangleright w) \otimes X^- , \]

\[
\Psi_{X,Y}(X^- \otimes w) = (\Lambda^{1/2}(\tau^1 \triangleright w) \otimes X^-
\]

\[- q^{1/2}\lambda^{1/2}\lambda(\Lambda^{1/2}(\tau^3)_{-1/2}S^1 \triangleright w) \otimes X^{3/0} ,\]

\[
\Psi_{X,Y}(X^+ \otimes w) = (\Lambda^{1/2}(\sigma^2 \triangleright w) \otimes X^+
\]

\[- q^{1/2}\lambda^{1/2}\lambda(\Lambda^{1/2}T^2(\tau^3)_{1/2} \triangleright w) \otimes X^0
\]

\[- q^{1/2}\lambda^{1/2}\lambda(\Lambda^{1/2}(\tau^3)_{1/2}(T^+ \sigma^2 + q\tau^3 T^2) \triangleright w) \otimes X^{3/0}
\]

\[+ q^{2}\lambda^2(\Lambda^{1/2}T^2T^+ \triangleright w) \otimes X^- ,\]

\[
\Psi_{X,Y}(X^0 \otimes w) = (\Lambda^{1/2}(\tau^3)_{1/2}(\tau^3 \triangleright w) \otimes X^0
\]

\[- q^{1/2}\lambda^{1/2}\lambda^{1/2}\lambda(\Lambda^{1/2}S^1 \triangleright w) \otimes X^{
\]

\[- q^{1/2}\lambda^{1/2}\lambda^{1/2}\lambda(\Lambda^{1/2}(qT^+(\tau^3 - T^2)) \triangleright w) \otimes X^-
\]

\[+ \lambda^1(\Lambda^{1/2}(\tau^3)_{1/2}(\lambda^2 T^+ S^1 + q^{-1}(\tau^3 \tau^1 - \sigma^2)) \triangleright w) \otimes X^{3/0}.\]

In very much the same way as for the Euclidean spaces we can write

\[
f(r^2, x^+, x^{3/0}, x^-) \otimes_L g(r^2, y^+, y^{3/0}, y^-) \]

\[\underset{q \rightarrow q}{\overset{\pm}{\longleftrightarrow}} f(x^-, x^{3/0}, x^+, r^2) \otimes_L g(y^-, y^{3/0}, y^+, r^2),\]

where the transition symbol has the same meaning as in Sec. 4.

7 Conclusion

Let us end with a few comments on our results. In our previous work we started a program to provide us with explicit formulae for elements of q-deformed analysis that can be important in formulating physical theories on quantum spaces. By calculating formulae for braided products we have now
finished this program. In this manner we have constructed nothing else than multidimensional extensions of the well-known q-calculus (see for example the presentation in Ref. [59]). In our future work we will show how to use this new formalism in formulating physical theories. One should also notice that the expressions we have obtained so far can serve as starting point for implementing q-analysis on a computer algebra system. This way we will lay the foundations for evaluating physical theories based on quantum symmetries.

However, when we apply the results of this article, we have to take into account the following observations. First of all, let us note that the formulae in this article always refer to the braiding of two coordinate spaces, but they are also valid, if we consider the braiding of two momentum spaces, since momentum generators show the same algebraic properties as coordinate generators. Things become slightly different, if we take our attention to the braiding between a momentum space and a coordinate space. The reason for this lies in the fact that the scaling operators Λ act on coordinate and momentum generators differently [see Eqs. (136), (137), (138), and (151) in Appendix A]. Due to this fact our formulae have to be adapted in the following way:

a) (quantum plane)

\[
\begin{align*}
  f(x) \circ L g(\partial) &= f(x) \circ L g(y) |_{y^A \rightarrow q^{-3} \partial^A}, \\
  f(\partial) \circ L g(y) &= f(x) \circ L g(y) |_{x^A \rightarrow q^{-3} \partial^A}, \\
  f(x) \circ L g(\partial) &= f(x) \circ L g(y) |_{y^A \rightarrow q^3 \partial^A}, \\
  f(\partial) \circ L g(y) &= f(x) \circ L g(y) |_{x^A \rightarrow q^3 \partial^A}.
\end{align*}
\] (123)

b) (three-dimensional Euclidean space)

\[
\begin{align*}
  f(x) \circ L g(\partial) &= f(x) \circ L g(y) |_{y^A \rightarrow q^{-4} \partial^A}, \\
  f(\partial) \circ L g(y) &= f(x) \circ L g(y) |_{x^A \rightarrow q^{-4} \partial^A}, \\
  f(x) \circ L g(\partial) &= f(x) \circ L g(y) |_{y^A \rightarrow q^4 \partial^A}, \\
  f(\partial) \circ L g(y) &= f(x) \circ L g(y) |_{x^A \rightarrow q^4 \partial^A}.
\end{align*}
\] (124)
c) (four-dimensional Euclidean space)

\[ f(\partial) \widetilde{L}_+ g(\partial) = f(\partial) \widetilde{L}_+ g(\partial) \big|_{y^A \rightarrow q^{-2}y^A}, \quad (125) \]
\[ f(\partial) \widetilde{L}_- g(\partial) = f(\partial) \widetilde{L}_- g(\partial) \big|_{x^A \rightarrow q^{-2}x^A}, \]
\[ f(x) \widetilde{L}_+ g(\partial) = f(x) \widetilde{L}_+ g(y) \big|_{y^A \rightarrow q^2 y^A}, \]
\[ f(x) \widetilde{L}_- g(\partial) = f(x) \widetilde{L}_- g(y) \big|_{x^A \rightarrow q^2 x^A}. \]

d) (q-deformed Minkowski space)

\[ f(\partial) \widetilde{L}_+ g(\partial) = f(\partial) \widetilde{L}_+ g(\partial) \big|_{y^A \rightarrow q^{-2}y^A}, \quad (126) \]
\[ f(\partial) \widetilde{L}_- g(\partial) = f(\partial) \widetilde{L}_- g(\partial) \big|_{x^A \rightarrow q^{-2}x^A}, \]
\[ f(x) \widetilde{L}_+ g(\partial) = f(x) \widetilde{L}_+ g(y) \big|_{y^A \rightarrow q^2 y^A}, \]
\[ f(x) \widetilde{L}_- g(\partial) = f(x) \widetilde{L}_- g(y) \big|_{x^A \rightarrow q^2 x^A}. \]

Acknowledgements
First of all I am very grateful to Eberhard Zeidler for interesting and useful discussions, his special interest in my work and his financial support. Also I wish to express my gratitude to Julius Wess for his efforts, suggestions and discussions. Furthermore, I would like to thank Alexander Schmidt, Fabian Bachmaier, Michael Wohlgenannt, and Florian Koch for useful discussions and their steady support. Finally, I thank Dieter Lüst for kind hospitality.

A Quantum algebras

In this Appendix we list for the quantum algebras we are interested in their defining relations and their Hopf structure.

We denote by \( U_{q}(su_{2}) \) the algebra over \( \mathbb{C} \) generated by three generators \( L^+ \), \( L^- \), and \( \tau^3 \) subject to the relations [17]

\[ \tau^3 L^\pm = q^{\mp 4} \tau^3 L^\pm \tau^3, \quad (127) \]
\[ qL^+ L^- - q^{-1}L^- L^+ = q\lambda - \lambda^{-1}(1 - \tau^{-1}). \]
Sometimes it is more convenient to use the generators
\[
T^\pm = q^{1/2} \lambda_+^{1/2} (\tau^3)^{1/2} L^\pm, \\
T^3 = \lambda^{-1} (1 - \tau^3),
\]
which satisfy the relations
\[
q^{-1} T^+ T^- - q T^- T^+ = T^3, \\
q^2 T^3 T^+ - q^{-2} T^+ T^3 = \lambda_+ T^+, \\
q^2 T^- T^3 - q^{-2} T^3 T^- = \lambda_+ T^-.
\]
The Hopf structure of the generators \(L^+, \ L^-, \ \)and \(\tau^3\) is given by
\[
\Delta(L^\pm) = L^\pm \otimes (\tau^3)^{-1/2} + 1 \otimes L^\pm, \\
\Delta(\tau^3) = \tau^3 \otimes \tau^3,
\]
\[
S(L^\pm) = -L^\pm (\tau^3)^{1/2}, \\
S(\tau^3) = (\tau^3)^{-1},
\]
\[
\varepsilon(L^\pm) = 0, \\
\varepsilon(\tau^3) = 1,
\]
and likewise for \(T^+, \ T^-, \ \)and \(T^3, \)
\[
\Delta(T^3) = T^3 \otimes 1 + \tau^3 \otimes T^3, \\
\Delta(T^\pm) = T^\pm \otimes 1 + (\tau^3)^{1/2} \otimes T^\pm,
\]
\[
S(T^3) = -(\tau^3)^{-1} T^3, \\
S(T^\pm) = - (\tau^3)^{-1/2} T^\pm,
\]
\[
\varepsilon(T^A) = 0, \quad A \in \{+, 3, -\}.
\]
For our purposes, it is necessary to extend \(U_q(su_2)\) by a grouplike scaling operator being central in the algebra. With coordinates and partial derivatives it shows the commutation relations
\[
\Lambda X^\alpha = q^{-2} X^\alpha \Lambda, \quad \alpha = 1, 2,
\]
\[ \Lambda \partial^\alpha = q^2 \partial^\alpha \Lambda, \]
\[ \Lambda X^A = q^4 X^A \Lambda, \quad A \in \{+, 3, -\}, \quad (137) \]
\[ \Lambda \partial^A = q^{-4} \partial^A \Lambda, \]

where \( \alpha \) and \( A \) denote spinor and vector indices, respectively.

Next, we come to the quantum algebra \( U_q(so_4) \), which from an algebraic point of view is isomorphic to the tensor product of two \( U_q(su_2) \)-algebras, i.e.
\[ U_q(so_4) \cong U_q(su_2) \otimes U_q(su_2). \quad (138) \]
Thus \( U_q(so_4) \) is spanned by two commuting sets of \( U_q(su_2) \)-generators, denoted by \( L_i^\pm, K_i, i = 1, 2 \). The commutation relations between generators of the same lower index explicitly read [52]
\[ q^{-1}L_i^+ L_i^- - qL_i^- L_i^+ = \lambda^{-1}(1 - K_i^{-1}), \quad (139) \]
\[ L_i^\pm K_i = q^{\mp 2} K_i L_i^\pm, \quad i = 1, 2. \]
The corresponding Hopf structure is given by
\[ \Delta(K_i) = K_i \otimes K_i, \quad (140) \]
\[ \Delta(L_i^\pm) = L_i^\pm \otimes 1 + K_i^{-1} \otimes L_i^\pm, \]
\[ S(K_i) = K_i^{-1}, \quad (141) \]
\[ S(L_i) = -K_i L_i^\pm, \quad (141) \]
\[ \varepsilon(K_i) = 1, \quad (142) \]
\[ \varepsilon(L_i^\pm) = 0. \quad (142) \]

In this case the central scaling operator \( \Lambda \) has to satisfy
\[ \Lambda X^i = q^2 X^i \Lambda, \quad i = 1, \ldots, 4, \quad (143) \]
\[ \Lambda \partial^i = q^{-2} \partial^i \Lambda. \]

Finally, we would like to consider a q-deformed version of Lorentz algebra, as it is presented in Refs. [15] and [21]. Its seven generators are denoted by \( T^+, T^-, S^1, T^2, \tau^3, \tau^1, \) and \( \sigma^2 \). Let us note that the generators \( T^+, T^- \), and \( \tau^3 \) generate an \( U_q(su_2) \)-subalgebra. In addition to the commutation relations
of the $U_q(su_2)$-subalgebra we have the relations

\begin{align}
\tau^1 T^+ &= T^+ \tau^1 + \lambda T^2, \\
\tau^1 T^- &= q^{-2} T^- \tau^1 - \lambda S^1, \\
\tau^1 T^2 &= q^2 T^2 \tau^1, \\
\tau^1 S^1 &= S^1 \tau^1, \\
\sigma^2 T^+ &= T^+ \sigma^2 - q^2 \lambda \tau^3 T^2, \\
\sigma^2 T^- &= q^2 T^- \sigma^2 + q^2 \lambda S^1, \\
\sigma^2 T^2 &= q^{-2} T^2 \sigma^2, \\
\sigma^2 S^1 &= S^1 \sigma^2, \\
T^+ T^2 &= q^{-2} T^2 T^+, \\
T^- T^2 &= T^2 T^- + \lambda^{-1} (\sigma^2 - \tau^1), \\
T^+ S^1 &= q^2 S^1 T^+ + \lambda^{-1} (\tau^3 T^1 - \sigma^2), \\
T^- S^1 &= S^1 T^-, \\
T^+ T^- &= q^2 T^- T^+ + q\lambda^{-1} (1 - \tau^3), \\
T^2 S^1 &= S^1 T^2, \\
\tau^1 \sigma^2 &= \sigma^2 \tau^1 + q\lambda^3 T^2 S^1, \\
\tau^3 \tau^1 &= \tau^1 \tau^3, \\
\tau^3 \sigma^2 &= \sigma^2 \tau^3, \\
\tau^3 T^\pm &= q^{-4} T^\pm \tau^3, \\
\tau^3 T^2 &= q^{-4} T^2 \tau^3, \\
\tau^3 S^1 &= q^4 S^1 \tau^3.
\end{align}

The Hopf structure of $S^1, T^2, \tau^1$, and $\sigma^2$ is given by

\begin{align}
\Delta(\tau^1) &= \tau^1 \otimes \tau^1 + \lambda^2 S^1 (\tau^3)^{-1/2} \otimes T^2, \\
\Delta(\sigma^2) &= \sigma^2 \otimes \sigma^2 + \lambda^2 T^2 (\tau^3)^{1/2} \otimes S^1, \\
\Delta(T^2) &= T^2 \otimes \tau^1 + (\tau^3)^{-1/2} \sigma^2 \otimes T^2, \\
\Delta(S^1) &= S^1 \otimes \sigma^2 + (\tau^3)^{1/2} \tau^1 \otimes S^1, \\
S(T^2) &= -q^2 (\tau^3)^{1/2} T^2, \quad (149)
\end{align}

36
\[ S(S^1) = - (\tau^3)^{-1/2} S^1, \]
\[ S(\tau^1) = \sigma^2, \]
\[ S(\sigma^2) = \tau^1, \]
\[ \varepsilon(\tau^1) = \varepsilon(\sigma^2) = 1, \quad \varepsilon(T^2) = \varepsilon(S^1) = 0. \]  

For the scaling operator we now have
\[ \Lambda X^\mu = q^{-2} X^\mu \Lambda, \quad \mu \in \{+, 0, 3/0, -\}, \]
\[ \Lambda \partial^\mu = q^2 \partial^\mu \Lambda. \]  

B Actions of symmetry generators

In this appendix we present formulae for calculating actions of powers of symmetry generators on normal ordered monomials. Especially for the computations in Sec. 4 we need to know the action of powers of the \( U_q(su_2) \)-generator \( L^- \). With the help of Eq. (70) we can write at first
\[ (L^-)^n \triangleright (X^+)^m (X^3)^m (X^-)^m \]
\[ = \left[ ((L^-)^n)_{(1)} \triangleright (X^+)^m \right] \left[ ((L^-)^n)_{(2)} \triangleright (X^3)^m(X^-)^m \right] \]
\[ = \sum_{k=0}^{n} q^{-2k(n-k)} \binom{n}{k} [q^2] \left[ (L^-)^k \triangleright (X^+)^m \right] \]
\[ \times \left[ (L^-)^{n-k}(\tau^3)^{-k/2} \triangleright (X^3)^m(X^-)^m \right]. \]  

The action contained in the second square bracket is a rather simple one as can be seen from direct inspection of the commutation relations between \( L^- \) and the quantum space coordinates (for their explicit form see for example Ref. [37]). In the case of the action in the first square bracket the same commutation relations show us that we can make as ansatz
\[ (L^-)^n \triangleright (X^+)^m = \sum_{i+j=m} C_{i,j}^{n,m} (X^+)^{m-i}(X^3)^{i-j}(X^-)^j, \]  

\[ 37 \]
which for the unknown coefficients leads to the recursion relation

\[
C_{i,j}^{n+1,m} = q^{-2j} [[m - i + 1]] q^i C_{i-1,j}^{n,m} + q^{-2(j-1)-1} [[i - j + 1]] q^2 C_{i,j-1}^{n,m}.
\]  

(154)

By the method of inserting one can prove that a solution is given by

\[
C_{i,j}^{n,m} = q^{-j^2} t_{i,j} [[i]] q^i \left[ \begin{array}{c} m \\ i \end{array} \right]_{q^4},
\]

(155)

with

\[
t_{i,0} = 1,
\]

(156)

\[
t_{i,j} = \sum_{0 \leq s_1 + \ldots + s_j \leq i} \prod_{l=1}^{j} q^{-2(j-l+1) s_l} [[i - j + 2l - s_1 - \ldots - s_l]] q^2.
\]

Putting everything together we arrive at the expression

\[
(L^-)^n \triangleright (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-}
\]

\[
= \sum_{i+j+k=n \atop 0 \leq j \leq m_+} q^{-2k(n-k+j) - k^2 - 2m_- n + 2j(m_3 - i - j) - j^2} t_{i,j}
\]

\[
\times [[i]] q^i \left[ \begin{array}{c} n \\ k \end{array} \right]_{q^2} \left[ \begin{array}{c} m_+ \\ i \end{array} \right]_{q^4} \left[ \begin{array}{c} m_3 \\ k \end{array} \right]_{q^2}
\]

\[
\times (X^+)^{m_+ - i} (X^3)^{m_3 - k + i - j} (X^-)^{m_- + k + j},
\]

which, in turn, gives rise to the following action on commutative functions:

\[
(L^-)^n \triangleright f(x^+, x^3, x^-)
\]

\[
= \sum_{i+j+k=n \atop 0 \leq j \leq i} q^{-k(2n-k+2j)-j^2} t_{i,j} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q^2}
\]

\[
\times (x^3)^{i-j} (x^-)^{k+j} (D_q^x)^i (D_{q^3}^x)^k f(q^{2j} x^3, q^{-2n} x^-).
\]

The calculations in Sec. \[\Box\] require explicit formulae for the action of powers of the $U_q(so_4)$-generators $L_i^\pm$, \(i = 1, 2\). Since all of the above considerations carry over to that case we limit ourselves to presenting the results

38
without any proof. Using commutation relations listed in Ref. [37] we found for the action on normal ordered monomials

\[(L_1^+)^n \triangleright (X^1)^{m_1} (X^2)^{m_2} (X^3)^{m_3} (X^4)^{m_4} \]

\[= \sum_{k=0}^{n} (-1)^k q^{m_1(n-2k)-(m_2+m_3)(n-k)+k(k-1)/2+(n-k)(n-k-1)/2} \]

\[\times [[k]]_q! [[n-k]]_q! \left[ {n \atop k} \frac{m_1}{q^2} \frac{m_3}{q} \right] \left[ {n-k \atop k} \frac{m_2}{q^2} \right] \]

\[\times (X^1)^{m_1-k} (X^2)^{m_2+k} (X^3)^{m_3-(n-k)} (X^4)^{m_4+(n-k)}, \]

\[(L_2^+)^n \triangleright (X^1)^{m_1} (X^2)^{m_2} (X^3)^{m_3} (X^4)^{m_4} \]

\[= \sum_{k=0}^{n} (-1)^k q^{m_1(n-2k)-(m_2+m_3)(n-k)+k(k-1)/2+(n-k)(n-k-1)/2} \]

\[\times [[k]]_q! [[n-k]]_q! \left[ {n \atop k} \frac{m_1}{q^2} \frac{m_3}{q} \right] \left[ {n-k \atop k} \frac{m_2}{q^2} \right] \]

\[\times (X^1)^{m_1-k} (X^2)^{m_2-(n-k)} (X^3)^{m_3+k} (X^4)^{m_4+(n-k)}. \]

These results imply the following actions on commutative functions:

\[(L_1^+)^n \triangleright f(x^1, x^2, x^3, x^4) \]

\[= \sum_{k=0}^{n} (-1)^{n-k} q^{k(k-1)/2+(n-k)(n-k-1)/2} (x^2)^n q^{-k} (x^4)^k \]

\[\times (D_q^1)^{n-k} (D_q^3)^k f(q^{-n-2k} x^1, q^{-k} x^2, q^{-k} x^3), \]

\[(L_2^+)^n \triangleright f(x^1, x^2, x^3, x^4) \]

\[= \sum_{k=0}^{n} (-1)^{n-k} q^{k(k-1)/2+(n-k)(n-k-1)/2} (x^3)^n q^{-k} (x^4)^k \]

\[\times (D_q^1)^{n-k} (D_q^2)^k f(q^{-n-2k} x^1, q^{-k} x^2, q^{-k} x^3). \]

Now, we come to the actions being relevant for q-deformed Minkowski space. In very much the same way as was done for \(L^-\) in the case of three-dimensional q-deformed Euclidean space, one can derive the action of powers of \(T^-\) on normal ordered monomials. By making use of the coproduct of \(T^-\)
we get
\[
(T^-)^m \triangleright (\hat{r}^2)^n (X^+)^{n_+} (X^{3/0})^{n_{3/0}} (X^-)^{n_-}
\]
\[
= \sum_{k=0}^m \left[ m \atop k \right] q^2 (\hat{r}^2)^n \left[ (T^-)^{m-k} (\hat{r}^3)^{k/2} \triangleright (X^+)^{n_+} \right]
\times \left[ (T^-)^k \triangleright (X^{3/0})^{n_{3/0}} (X^-)^{n_-} \right],
\]
where we have taken into account that the Minkowski length commutes with all Lorentz generators (for the explicit form of commutation relations between Lorentz generators and Minkowski space coordinates see for example Ref. [37]). The action in the second bracket takes on a rather simple form, since it is given by
\[
(T^-)^k \triangleright (X^{3/0})^{n_{3/0}} (X^-)^{n_-}
\]
\[
= (q^{3/2} \lambda^{1/2})^k \left[ [k] \right] q^2 \left[ \frac{n_{3/0}}{k} \right] (X^{3/0})^{n_{3/0} - k} (X^-)^{n_- + k}.
\]

For the action in the first bracket we can assume the general form
\[
(T^-)^m \triangleright (X^+)^n
\]
\[
= \sum_{0 \leq s + l \leq \min(n, m)} (d_q)_{s,l}^{m,n} (X^+)^{n-s-l} (X^0)^s (X^{3/0})^{s+2l-m} (X^-)^{m-s-l}.
\]

If we introduce new coefficients \( \tilde{d}_q \) being subject to
\[
(d_q)_{s,l}^{m,n} = (q^{m/2} q^{m-s-l}) (\tilde{d}_q)_{s,l,m-s-l}^{m,n},
\]
we find the recursion relations
\[
(\tilde{d}_q)_{s,l,i}^{m+1,n} = [n - s - l - i + 1] q^{-2} (\tilde{d}_q)_{s-1,l,i}^{m,n} + q^{-2(n-s-l-i)} [n - s - l - i + 1] q^4 (\tilde{d}_q)_{s,l-1,i}^{m,n} + q^{-2(n-s-l-i)} [l + 1] q^2 (\tilde{d}_q)_{s,l+1,i-1}^{m,n},
\]
\[
(\tilde{d}_q)_{n,0,0}^{m,n} = [n] q^{-2}!,
\]
\[40\]
which are solved by

\[(\bar{d}_q)_{s,l,i}^{m,n} = q^{-2i^2} (q\lambda_+)^{-l} (T_q)_{s,l,i}^n ([s + l]) q^{-2} \left[ \frac{n}{s + l} \right] q^{-2}, \quad (169)\]

with

\[(T_q)_{s,l,i}^n \equiv \sum_{s_1 + \ldots + s_{i+1} = s} \prod_{k=1}^{i} q^{-2 \sum_{j=1}^{k} (s_j + i_j)} [[l - i + k - l_1 - \ldots - l_k]] q^2 \quad (170)\]

\[\times \sum_{0 \leq \alpha_1 \leq \alpha_2} q^{2 \sum_{u=1}^{i+1} \alpha_u (n-s-l)} \prod_{k=1}^{i+1} \left( q^{2k} \sum_{u=1}^{i+1} (s_u + t_u) \left( P_{q}^{s_k l_k} \right) \right),\]

\[(P_q)^{v,p}_{k} \equiv \sum_{0 \leq j_1 + \ldots + j_p \leq v} q^{2 \sum_{u=1}^{p} (j_1 + \ldots + j_u + t_u)}. \quad (171)\]

There remains to replace the time coordinate \(X^0\) by the square of the Minkowski length \(\hat{r}^2\) in Eq. (165). In Ref. [40] we already addressed this problem and suggested as solution

\[(X^+)^{n_+}(X^{3/0})^{n_{3/0}}(X^0)^{n_0}(X^-)^{n_-} = \sum_{p=0}^{n_0} q^{2(n_{3/0} - 1)p} (X^+)^{n_+ + p}(X^{3/0})^{n_{3/0} - p} S_{n_0, p}^{0} (\hat{r}^2, X^{3/0})(X^-)^{n_+ + p}, \quad (172)\]

where

\[S_{k,p}^{0}(\hat{r}^2, X^{3/0}) \quad (173)\]

\[= \begin{cases} \sum_{j_1=0}^{p} \sum_{j_2=0}^{j_1} \ldots \sum_{j_{k-p-1}=0}^{j_{k-p-2}} a_0(\hat{r}^2, q^{2j_1} X^{3/0}), & \text{if } 0 \leq p < k, \\ 1, & \text{if } p = k, \end{cases}\]

and

\[a_0(\hat{r}^2, X^{3/0}) = -\lambda_+^{-1} (q \hat{r}^2 (X^{3/0})^{-1} + q^{-1} X^{3/0}). \quad (174)\]

From what we have done so far, it is now rather straightforward to show that

\[\mathcal{T}^{m} \triangleright (\hat{r}^2)^{n_+} (X^+)^{n_+} (X^{3/0})^{n_{3/0}} (X^-)^{n_-} \quad (175)\]
where, for brevity, we have introduced the operator

\begin{align*}
&= \sum_{k=0}^{m} \sum_{0 \leq s+t \leq \min(k, n_+)} (q^{3/2} \lambda_+^{1/2})^{m-k} q^{-2(m-k)n_+ + 2(k-s-t)(n_{3/0} - m + k)} \\
&\times \sum_{u=0}^{s} q^{-u+2u(n_{3/0} - m + 2t + s)} (d_q)^{k, n_+}_{s, t} [[m - k]]_q^2 \\
&\times \left[ \sum_{m - k \leq 0} \left[ \frac{m}{k} \right] q^2 \right] (\hat{r}^2)^{n_r} (X^+)^{n_+ - s - t + u} \\
&\times (X^{3/0})^{n_{3/0} - m + 2t - u} S_{s, u}^0 (\hat{r}^2, X^{3/0}) (X^-)^{n_- + m - s - t + u}.
\end{align*}

As usual, this expression enables us to read off the corresponding action on commutative functions, i.e.

\begin{equation}
(T^-)^m f(r^2, x^+, x^{3/0}, x^-) = \sum_{k=0}^{m} \sum_{s=0}^{k} \sum_{0 \leq t \leq k-s} \sum_{u=0}^{s} q^{3m/2 - k - t - u} (\lambda_+^{1/2})^{m-2t} \\
\times q^{(k-s-t)(2(s+t)-k-m+1)-2u(m-2t-s)} \left[ \frac{m}{k} \right] q^2 \\
\times (x^+)^u (x^{3/0})^{k+s+2t-u} S_{s, u}^0 (r^2, x^{3/0}) (x^-)^{m - s - t + u} \\
\times (\hat{T}_q)^{s, t, k-s-t} (D_{q^{-1}})^{s+t} (D_{q^2})^{m-k} f(q^{-2(m-k)} x^+, q^{2(k-s-t+u)} x^{3/0}),
\end{equation}

where, for brevity, we have introduced the operator

\begin{align*}
&\quad (\hat{T}_q)^{s, l, i} f(r^2, x^+, x^{3/0}, x^-) \\
&\equiv \sum_{s_1 + \ldots + s_{i+1} = s} \left( \prod_{k=1}^{i} q^{-2 \sum_{j=1}^{k} (s_j + l_j)} \left[ [l - i + k - l_1 - \ldots - l_k] \right] q^2 \right) \\
&\times \sum_{0 \leq t_0 \leq l_0} \left( \prod_{k=1}^{i+1} q^{2t_k \sum_{u=1}^{k-1} (s_u + t_u)} (P_q)^{s_k, l_k} \right) f(q^2 \sum_{u=1}^{i+1} t_u x^+).
\end{align*}

Following the same line of arguments, we can also derive explicit formulae for the action of powers of some more Lorentz generators. For the
computation of braided products in Sec. we need to know

\[(S^1)^m \triangleright f(r^2, x^+, x^{3/0}, x^-) = (-q^{1/2} \lambda_+^{-1/2})^m (x^{3/0})^m (D_q^+)^m f(q^{-m} x^+, q^{-m} x^{3/0}, q^m x^-),\]  

\[(T^2)^m \triangleright f(r^2, x^+, x^{3/0}, x^-) = (q \lambda_+)^{-m/2} (x^{3/0})^m (D_q^-)^m f(q^m x^+, q^{-m} x^{3/0}, q^{-m} x^-),\]  

\[(\tau^1)^m \triangleright f(r^2, x^+, x^{3/0}, x^-) = \sum_{k=0}^{\infty} (-\lambda_+^{-1} \lambda^2)^k q^{-k(k-1)} \binom{m}{k} (x^{3/0})^{2k} (D_q^+)^k (D_q^-)^k f(q^m x^+, q^{-m} x^{-2k} x^{3/0}, q^{-m} x^-),\]  

\[(\sigma^2)^m \triangleright f(r^2, x^+, x^{3/0}, x^-) = f(q^{-m} x^+, q^{-m} x^{3/0}, q^m x^-),\]  

\[(\tau^3)^m \triangleright f(r^2, x^+, x^{3/0}, x^-) = f(q^{-4m} x^+, q^{4m} x^-).\]

References

[1] S. Weinberg, *Gravitation and Cosmologie*, Wiley, New York, 1972.

[2] D. C. Cassidy, *Uncertainty: The Life and Science of Werner Heisenberg*, Freeman, New York, 1992.

[3] W. Heisenberg, "Über die in der Theorie der Elementarteilchen auftretende universelle Länge," Ann. Phys. 32, 20 (1938).

[4] I. M. Gelfand and M. A. Naimark, *On the embedding of normed linear rings into the ring of operators in Hilbert space*, Math. Sbornik 12, 197 (1943).

[5] H. Hopf, "Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen," Ann. Math. 42, 22 (1941).

[6] P. P. Kulish and N. Y. Reshetikin, *Quantum linear problem for the Sine-Gordon equation and higher representations*, J. Sov. Math. 23, 2345 (1983).
[7] S. L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. 111, 613 (1987).

[8] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Dokl. 32, 254 (1985).

[9] M. Jimbo, *A q-analogue of U(g) and the Yang-Baxter equation*, Lett. Math. Phys. 10, 63 (1985).

[10] V. G. Drinfeld, *Quantum groups*, in Proceedings of the International Congress of Mathematicians (A.M. Gleason, ed.), Amer. Math. Soc., 798 (1986).

[11] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. 1, 193 (1990).

[12] M. Takeuchi, *Matrix Bialgebras and Quantum Groups*, Israel J. Math. 72, 232 (1990).

[13] U. Carow-Watamura, M. Schlieker, M. Scholl, and S. Watamura, *Tensor Representations of the Quantum Group SL_q(2) and Quantum Minkowski Space*, Z. Phys. C 48, 159 (1990).

[14] P. Podleś and S. L. Woronowicz, *Quantum Deformation of Lorentz Group*, Commun. Math. Phys. 130, 381 (1990).

[15] W. B. Schmidke, J. Wess, and B. Zumino, *A q-deformed Lorentz Algebra*, Z. Phys. C 52, 471 (1991).

[16] S. Majid, *Examples of braided groups and braided matrices*, J. Math. Phys. 32, 3246 (1991).

[17] A. Lorek, W. Weich, and J. Wess, *Non-commutative Euclidean and Minkowski Structures*, Z. Phys. C 76, 375 (1997), q-alg/9702025.

[18] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B (Proc. Suppl.) 18, 302 (1991).

[19] U. Carow-Watamura, M. Schlieker, and S. Watamura, *SO_q(N)-covariant differential calculus on quantum space and deformation of Schrödinger equation*, Z. Phys. C 49, 439 (1991).
[20] X. C. Song, Covariant differential calculus on quantum minkowski space and q-analog of Dirac equation, Z. Phys. C 55, 417 (1992).

[21] O. Ogievetsky, W. B. Schmidke, J. Wess, and B. Zumino, q-Deformed Poincaré Algebra, Commun. Math. Phys. 150, 495 (1992).

[22] S. Majid, Braided momentum in the q-Poincaré group, J. Math. Phys. 34, 2045 (1993).

[23] M. Fichtenmüller, A. Lorek, and J. Wess, q-deformed Phase Space and its Lattice Structure, Z. Phys. C 71, 533 (1996), hep-th/9511106.

[24] J. Wess, q-Deformed phase space and its lattice structure, Int. J. Mod. Phys. A 12, 4997 (1997).

[25] B. L. Cerchiai and J. Wess, q-Deformed Minkowski Space based on a q-Lorentz Algebra, Eur. Phys. J. C 5, 553 (1998), math.QA/9801104.

[26] H. Grosse, C. Klimčík, and P. Prešnajder, Towards finite quantum field theory in non-commutative geometry, Int. J. Theor. Phys. 35, 231 (1996). [hep-th/9505175.

[27] S. Majid, On the q-regularisation, Int. J. Mod. Phys. A 5, 4689 (1990).

[28] R. Oeckl, Braided Quantum Field Theory, Commun. Math. Phys. 217, 451 (2001).

[29] C. Blohmann, Free q-deformed relativistic wave equations by representation theory, Eur. Phys. J. C 30, 435 (2003). [hep-th/0111172.

[30] S. Majid, Representations, duals and quantum doubles of monoidal categories, Suppl. Rend. Circ. Mat. Palermo, Ser. II, 26, 197 (1991).

[31] S. Majid, Algebras and Hopf Algebras in Braided Categories, Lec. Notes Pure Appl. Math. 158, 55 (1994).

[32] S. Majid, Braided geometry: A new approach to q-deformations, in First Caribbean Spring School of Mathematics and Theoretical Physics (R. Coquereaux et al., eds.), World Scientific, Guadeloupe (1993).

[33] S. Majid, Introduction to Braided Geometry and q-Minkowski Space, preprint. [hep-th/9410241]
[34] S. Majid, *Free braided differential calculus, braided binomial theorem and the braided exponential map*, J. Math. Phys. **34**, 4843 (1993).

[35] S. Majid, *Foundations of Quantum Group Theory*, University Press, Cambridge, 1995.

[36] H. Wachter and M. Wohlgenannt, **-Products on quantum spaces, Eur. Phys. J. C **23**, 761 (2002), [hep-th/0103120](https://arxiv.org/abs/hep-th/0103120).

[37] C. Bauer and H. Wachter, Operator representations on quantum spaces, Eur. Phys. J. C **31**, 261 (2003), [math-ph/0201023](https://arxiv.org/abs/math-ph/0201023).

[38] H. Wachter, *-Integration on quantum spaces, Eur. Phys. J. C **32**, 281 (2004), [hep-th/0206083](https://arxiv.org/abs/hep-th/0206083).

[39] H. Wachter, *-Exponentials on quantum spaces, Eur. Phys. J. C **37**, 379 (2004), [hep-th/0401113](https://arxiv.org/abs/hep-th/0401113).

[40] H. Wachter, *-Translations on quantum spaces, preprint, hep-th/0410205.

[41] J. Wess, *q*-deformed Heisenberg Algebras, in Proceedings of the 38. Internationale Universitätswochen für Kern- und Teilchenphysik (H. Gausterer, H. Grosse, and L. Pittner, eds.), Springer-Verlag, Schladming (2000), math-phy/9910013.

[42] J. E. Moyal, Quantum mechanics as a statistical theory, Proc. Camb. Phil. Soc. **45**, 99 (1949).

[43] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization. I. Deformations of symplectic structures, Ann. Phys. **111**, 61 (1978).

[44] J. Madore, S. Schraml, P. Schupp, and J. Wess, *Gauge Theory on Noncommutative Spaces*, Eur. Phys. J. C **16**, 161 (2000) [hep-th/0001203](https://arxiv.org/abs/hep-th/0001203).

[45] F. H. Jackson, On *-functions and a certain difference operator, trans. Roy. Edin. **46**, 253 (1908); F. H. Jackson, *-Integration, Proc. Durham Phil. Soc. **7**, 182 (1927).

[46] S. Mac Lane, *Categories for the Working Mathematician*, Springer Verlag, 1974.
[47] Y. I. Manin, *Quantum Groups and Non-Commutative Geometry*, Centre de Recherche Mathématiques, Montreal, 1988.

[48] P. P. Kulish and N. Yu. Reshetikhin, *Quantum linear problem for the Sine-Gordon equation and higher representations*, Zap. Nauchn. Sem. LOMI **101**, 101 (1981).

[49] D. Mikulovic, A. Schmidt, and H. Wachter, *Grassmann variables on quantum spaces*, preprint, hep-th/0407273.

[50] T. H. Koornwinder, *Special functions and \(q\)-commuting variables*, preprint, q-alg/9608008.

[51] A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations*, Springer-Verlag, Berlin, 1997.

[52] H. Ocambo, *\(SO_q(4)\) quantum mechanics*, Z. Phys. C **70**, 525 (1996).

[53] J. Lukierski, A. Nowicki, and H. Ruegg, *New Quantum Poincare Algebra and \(\kappa\)-deformed Field Theory*, Phys. Lett. B **293**, 344 (1992).

[54] L. Castellani, *Differential Calculus on ISO\(_q\)(N), Quantum Poincaré Algebra and \(q\)-Gravity*, preprint, hep-th/9312179.

[55] V. K. Dobrev, *New \(q\)-Minkowski space-time and \(q\)-Maxwell equations hierarchy from \(q\)-conformal invariance*, Phys. Lett. B **341**, 133 (1994).

[56] M. Chaichian and A. P. Demichev, *Quantum Poincaré group without dilatation and twisted classical algebra*, J. Math. Phys. **36**, 398 (1995).

[57] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, *On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and its Implications on Noncommutative QFT*, Phys. Lett. B **604**, 98 (2004), hep-th/0408062.

[58] F. Koch and E. Tsouchnika, *Construction of \(\theta\)-Poincaré Algebras and their invariants on \(\mathcal{M}_\theta\)*, Nucl. Phys. B **717**, 387 (2005), hep-th/0409012.

[59] V. Kac and P. Cheung, *Quantum Calculus*, Springer-Verlag, Berlin, 2000.