TORSION IN KERNELS OF INDUCED MAPS ON DIVISOR CLASS GROUPS

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ABSTRACT. We investigate torsion elements in the kernel of the map on divisor class groups of excellent local normal domains $A$ and $A/I$, for an ideal $I$ of finite projective dimension. The motivation for this work is a result of Griffith-Weston which applies when $I$ is principal.

INTRODUCTION

In this section, let $A \to B$ be a homomorphism of noetherian normal integral domains. Under certain circumstances, one can show that such a map induces a group homomorphism $\text{Cl}(A) \to \text{Cl}(B)$ on divisor class groups given by $[a] \mapsto [(a \otimes_A B)_{BB}]$. (See Section 1 for definitions and background material.) For instance, if $A$ is excellent and local, $t$ is a prime element of $A$ such that $A/tA$ satisfies the regularity condition $(R_1)$, and $B$ is the integral closure of $A/tA$, then it is well-known that the natural map $A \to B$ induces such a homomorphism. In this setting, the kernel of this induced map has been studied by V.I. Danilov, P. Griffith, J. Lipman, C. Miller, S. Spiroff, D. Weston, and others. For instance, Griffith and Weston prove the following:

**Theorem A.** [9, Theorem 1.2] Let $(A, m)$ be an excellent, local, normal domain and let $t$ be a principal prime in $m$ such that $A/tA$ satisfies the condition $(R_1)$. Let $B$ denote the integral closure of $A/tA$ and let $e > 1$ be an integer which represents a unit in $A$. Then the kernel of the homomorphism $\text{Cl}(A) \to \text{Cl}(B)$ contains no element of order $e$.

A careful analysis of their intricate argument reveals an overlooked detail. Specifically, they suppose that $[a]$ is an element of order $e$ in $\text{Ker}(\text{Cl}(A) \to \text{Cl}(B))$, and let $a \in A$ be such that $a^{(e)} = aA$. They describe an $A$-algebra structure on the direct sum $S = \oplus_{i=0}^{e-1} a^{(i)}$ that makes $S$ isomorphic to the integral closure of $A[T]/(T^e - a)$. They then show that the integral closure of $S/tS$ is étale over $B$ and conclude that $e = 1$. However, in their proof of the last step, they assume that the integral closure of $S/tS$ is local. While this may be true in general, we are only certain of it when $A/tA$ is normal, by a result of M. Hochster and C. Huneke [13, (3.9) Proposition c)], once one interprets the integral closure of $S/tS$ as the $S_2$-ification of $S/tS$.

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Using the point of view of $S_2$-ifications, we obtain the theorem below. Its proof is the content of Section 3. This result allows us to obtain the special case of Theorem A where $A/tA$ is normal; see Corollary 4.1.

**Theorem B.** Let $(A, m)$ be an excellent, local, normal domain and $I$ a prime ideal of $A$ with finite projective dimension such that:

(i) $\bar{A} = A/I$ is normal;

(ii) $I$ is a complete intersection on the punctured spectrum of $A$, i.e., for each prime ideal $p \neq m$, the localization $I_p$ is either equal to $A_p$ or generated by a regular sequence in $A_p$; and

(iii) $\mu(I) \leq \dim A - 2$, where $\mu(I)$ is the minimal number of generators of $I$.

Then for any positive integer $e > 1$ which represents a unit in $A$, the kernel of the homomorphism $Cl(A) \to Cl(\bar{A})$ contains no elements of order $e$.

A significant tool in the proof of Theorem B is the next result, the proof of which is the content of Section 2.

**Theorem C.** Let $(A, m)$ be a complete, local, normal domain such that $A/m$ is separably closed and let $I$ be a prime ideal of $A$ with finite projective dimension such that $\bar{A} = A/I$ is normal. Let $e > 1$ be an integer which represents a unit in $A$ and assume that $A$ contains a primitive $e$-th root of unity. Let $[a] \in Cl(A)$ with order $e$. Set $R = A \oplus a \oplus a^{(2)} \oplus \cdots \oplus a^{(e-1)}$. If any of the equivalent conditions of Fact 1.14 hold for $R/IR$, then $[a] \notin \text{Ker}(Cl(A) \to Cl(\bar{A}))$.

Note that the $A$-algebra structure on $R$ is described explicitly in paragraph 2.1 and that Lemma 2.4 shows that $R/IR$ satisfies the hypotheses of Fact 1.14. Also, the proofs of Theorems B and C use ideas from [9, Theorem 1.2], but with some key changes. For completeness and clarification, we provide the details. Because of their length and technicality, the proofs are presented in a series of lemmas. We conclude the paper with a few corollaries and an example in Section 4.

Finally, we mention that a thorough discussion of the background material and list of terminology is given Section 1 for the reader’s convenience. We provide useful definitions and results on divisor class groups, $S_2$-ifications, and unramified/étale extensions. Undefined terms can be found in [16].

1. Background

*Throughout this paper, we assume that all rings are commutative and noetherian.*

**Divisor Class Groups.**

We begin with our working definition of the divisor class group of a normal domain. It can be found in J. Lipman [15, §0] and is equivalent to the classical additive definition of the divisor class group appearing in N. Bourbaki [4, VII §1] and R. Fossum [7, §6]. A discussion of this equivalence appears in [19, 2.10].

**Definition 1.1.** Let $A$ be a normal domain and $M$ a finitely generated $A$-module. The dual of $M$ is $M^A = \text{Hom}_A(M, A)$ and the double dual of $M$ is $M^{AA} = (M^A)^A$. The natural biduality map $\sigma^A_M : M \to M^{AA}$ is the $A$-module homomorphism given
by $\sigma^A_M(m)(g) = g(m)$ for all $m \in M$ and all $g \in M^A$. We say that $M$ is reflexive if $\sigma^A_M$ is an isomorphism.

**Remark 1.2.** As in our previous work [20], we use the notation $M^A$ for the dual of an $A$-module $M$ in order to avoid ambiguity when we work with two or more rings simultaneously. This replaces the notation $M^*$ from the classical literature.

**Definition 1.3.** The divisor class group of a normal domain $A$, denoted $\text{Cl}(A)$, is the group of isomorphism classes of reflexive $A$-modules of rank one, or equivalently, non-zero reflexive ideals of $A$. An element $[a] \in \text{Cl}(A)$ is called a divisor class, and multiplication is defined by $[a] \cdot [b] = [(a \otimes_A b)^A]$. The identity element is $[A]$, and the inverse of $[a]$ is $[a^A]$.

As $A$ is normal, it satisfies the Serre conditions $(R_1)$ and $(S_2)$. Thus, each reflexive ideal $a$ can be written uniquely as the primary decomposition $a = \bigcap_{j=1}^{s} p_j^{(e_j)}$, where the $p_j$ are the height one prime ideals containing $a$. For each positive integer $d$, we set $a^{(d)} = \bigcap_{j=1}^{s} p_j^{(e_j d)}$. In $\text{Cl}(A)$, this definition implies that $d[a] = [a^{(d)}]$.

**Fact 1.4.** Let $A \to B$ be a homomorphism of finite flat dimension between normal domains. (For instance, if $I$ is a prime ideal of $A$ with finite projective dimension such that $A$ and $A/I$ are normal domains, then we can consider the natural surjection $A \to A/I$.) Then [19] Theorem A provides a well-defined group homomorphism $\text{Cl}(A) \to \text{Cl}(B)$ given by $[a] \mapsto [(a \otimes_A B)^B]$.

We complement this with the following.

**Lemma 1.5.** Let $A$ be a normal domain, and let $I$ be a prime ideal of $A$ such that $\bar{A} = A/I$ is excellent and satisfies $(R_1)$. If $B$ is the integral closure of $\bar{A}$, then the natural ring homomorphism $\varphi : A \to B$ induces a well-defined group homomorphism $\text{Cl}(A) \to \text{Cl}(B)$ given by $[a] \mapsto [(a \otimes_A B)^B]$.

**Proof.** Note that the fact that $\bar{A}$ is excellent implies that $B$ is noetherian. Let $Q$ be a height one prime ideal of $B$. Then $\bar{P} := Q \cap A = P/I$ for some prime ideal $P$ of $A$. By [20] Theorem 1.10, it suffices to show that $A_P$ is a unique factorization domain, e.g., that it is regular. By [14] Theorems 4.8.6 and B.5.1, we know that $\text{ht} \bar{P} = \text{ht} Q$; this also uses the fact that $\bar{A}$ is excellent. Since $\bar{A}$ satisfies $(R_1)$, it follows that $\bar{A}_P$ is regular. Since the induced map $A_P \to \bar{A}_P$ has finite projective dimension, we conclude from [3] Theorem 6.1(1) that $A_P$ is regular. \hfill \Box

**$S_2$-ifications.**

We present here an exposition on $S_2$-ifications, taking much of our content from a paper by M. Hochster and C. Huneke [14]. This topic comes to the fore in the guise of the integral closure when one assumes that the domain in question satisfies the regularity condition $(R_1)$, but is not necessarily normal. This allows one to consider a broader class of rings when studying divisor class groups.

**Definition 1.6.** Let $(A, m)$ be a local ring, and let $E$ be the injective hull of $A/m$ over $A$. A canonical module for $A$ is a finitely generated $A$-module $\omega$ such that $\text{Hom}_A(\omega, E) \cong \text{Prim}(A)(A)$.

**Fact 1.7.** Let $A$ be a local ring. If $A$ is a homomorphic image of a Gorenstein local ring, e.g., $A$ is complete, then it has a canonical module.
Definition 1.8. Let $A$ be a local ring. Denote by $j(A)$ the largest ideal which is a submodule of $A$ of dimension smaller than $\dim A$. Specifically, 

$$j(A) = \{a \in A : \dim(A/\text{ann}_A(a)) < \dim A\}.$$ 

Fact 1.9. The following items are from [13, (2.2f), (2.1)].

(a) If $A$ is local with canonical module $\omega$, then $j(A) = \text{Ker}(A \to \text{Hom}_A(\omega,\omega))$.

(b) A local ring $A$ is equidimensional and $(0)$ has no embedded primes if and only if $j(A) = (0)$. In particular, if $A$ is a local domain, then $j(A) = 0$.

Definition 1.10. [13, (2.3)] Let $A$ be a local ring.

(a) If $j(A) = 0$, then a subring $S$ of the total ring of quotients of $A$ is an $S_2$-ification of $A$ if:

- $S$ is module finite over $A$;
- $S$ satisfies the Serre condition $(S_2)$ over $A$; and
- $\text{Coker}(A \to S)$ has no prime ideal of $A$ of height less than two in its support.

(b) When $A$ is equidimensional but possibly $j(A) \neq 0$, then by an $S_2$-ification of $A$, we mean an $S_2$-ification of $A/j(A)$.

Fact 1.11. [13, (2.2), (2.7)] A local ring $A$ has an $S_2$-ification if it has a canonical module $\omega$. Specifically, $\text{Hom}_A(\omega,\omega)$ is a commutative $A$-algebra that satisfies $(S_2)$ both as a ring and an $A$-module, and it is an $S_2$-ification of $A$.

Lemma 1.12. Assume that $A$ is a local ring with a canonical module that satisfies $(R_1)$, such that $j(A) = 0$. Then the $S_2$-ification $S$ of $A$ coincides with the integral closure of $A$ in its total ring of quotients.

Proof. Since the $S_2$-ification of $A$ is unique up to isomorphism, by [13, (2.5)], it suffices to show that $S$ is integrally closed in its total ring of fractions. It suffices to show that $S$ satisfies $(R_1)$ and $(S_2)$ as a ring. Let $P$ be a prime ideal of $S$, and set $p = P \cap A$. From [13 Proposition 3.5(a)], we know that $\text{ht} p = \text{ht} P$. Hence, the fact that $S$ satisfies $(S_2)$ as a ring follows from the fact that it satisfies $(S_2)$ as an $A$-module, by [12 Corollaire (5.7.11)].

To verify that $S$ satisfies $(R_1)$, assume that $\text{ht} P \leq 1$. Since $A$ satisfies $(R_1)$, the ring $A_p$ is regular. Since $\text{Coker}(A \to S)$ has no prime ideal of $A$ of height less than two in its support, we conclude that $A_p \cong S_p$, hence $A_p \cong S_P$ is regular. \qed

Fact 1.13. When $(A, \mathfrak{m})$ is an excellent local domain and $t$ is an element of $\mathfrak{m}$ such that $A/ta$ satisfies $(R_1)$, then the integral closure of $A/ta$ in its total ring of quotients is local. (See A. Grothendieck [10 §XIII], or [13 Proposition 3.9].) It follows from [14 Corollary 37.6] that $t$ is a prime element.

The fact above, that the $S_2$-ification of $A/ta$ is local, does not easily generalize to $A/I$ for a non-principal ideal $I$ of $A$, even under the assumptions that $A/I$ is complete, local, and equidimensional. One would need to know, for example, that the canonical module of $A/I$ is indecomposable; or more importantly, that any of the equivalent conditions below hold for the ring $A/I$.

Fact 1.14. [13 Theorem 3.6] For $(T, \mathfrak{n})$ a complete local equidimensional ring, the following conditions are equivalent:

Proof. Since the $S_2$-ification of $A$ is unique up to isomorphism, by [13, (2.5)], it suffices to show that $S$ is integrally closed in its total ring of fractions. It suffices to show that $S$ satisfies $(R_1)$ and $(S_2)$ as a ring. Let $P$ be a prime ideal of $S$, and set $p = P \cap A$. From [13 Proposition 3.5(a)], we know that $\text{ht} p = \text{ht} P$. Hence, the fact that $S$ satisfies $(S_2)$ as a ring follows from the fact that it satisfies $(S_2)$ as an $A$-module, by [12 Corollaire (5.7.11)].

To verify that $S$ satisfies $(R_1)$, assume that $\text{ht} P \leq 1$. Since $A$ satisfies $(R_1)$, the ring $A_p$ is regular. Since $\text{Coker}(A \to S)$ has no prime ideal of $A$ of height less than two in its support, we conclude that $A_p \cong S_p$, hence $A_p \cong S_P$ is regular. \qed

Fact 1.13. When $(A, \mathfrak{m})$ is an excellent local domain and $t$ is an element of $\mathfrak{m}$ such that $A/ta$ satisfies $(R_1)$, then the integral closure of $A/ta$ in its total ring of quotients is local. (See A. Grothendieck [10 §XIII], or [13 Proposition 3.9].) It follows from [14 Corollary 37.6] that $t$ is a prime element.

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Fact 1.14. [13 Theorem 3.6] For $(T, \mathfrak{n})$ a complete local equidimensional ring, the following conditions are equivalent:
(i) $H^{\dim T}_{n}(T)$ is indecomposable;
(ii) The canonical module of $T$ is indecomposable;
(iii) The $S_2$-ification of $T$ is local;
(iv) For every ideal $J$ of height at least two, $\text{Spec}(T) - V(J)$ is connected;
(v) Given any two distinct minimal primes $p, q$ of $T$, there is a sequence of minimal primes $p = p_0, \ldots, p_r = q$ such that for $0 \leq i < r$, $\text{ht}(p_i + p_{i+1}) = 1$.

We apply this result in Section 4. (See Lemmas 3.5, 3.6, and 3.7.)

Unramified and étale extensions, and isomorphisms in codimension one.

For the remainder of this section, let $R$ be a ring, and let $S$ be an $R$-algebra.

**Definition 1.15.** For a prime ideal $P$ of $S$ and $p = P \cap R$, we say that $P$ is **unramified over** $R$ if $pSP = PSP$ and $S_p/pSP$ is a separable field extension of $R_p/pR_p$.

**Definition 1.16.** A prime ideal $p$ of $R$ is **unramified in** $S$ if (1) every prime lying over $p$ is unramified over $R$, and (2) there are only finitely many prime ideals of $S$ lying over $p$. (Note that if $R \hookrightarrow S$ is module finite, then condition (2) is automatic.)

**Definition 1.17.** For $S$ to be **unramified over** $R$ means that (1) every prime ideal of $S$ is unramified over $R$, and (2) for every $p \in \text{Spec}(R)$ there are only finitely many prime ideals of $S$ lying over $p$. (Note that if $R \hookrightarrow S$ is module finite, then condition (2) is automatic.)

**Definition 1.18.** We say that $S$ is **unramified in codimension $i$** (or more accurately, in codimension less than or equal to $i$) over $R$ if every prime ideal of $S$ with height less than or equal to $i$ is unramified over $R$.

**Definition 1.19.** For $S$ to be **étale over** $R$ means that $S$ is unramified and flat over $R$.

**Definition 1.20.** We say that $S$ is **étale in codimension one** (or more accurately, in codimension less than or equal to one) over $R$ if, for every prime ideal $P$ of $S$ with height less than or equal to one and $p = P \cap R$, the ring $S_p$ is étale over $R_p$.

The next fact is a version of “purity of branch locus” from Auslander and Buchsbaum [2, Corollary 3.7] that is very useful for the proof of Theorem $B$.

**Fact 1.21.** Assume that $R$ is local and that $R \hookrightarrow S$ be a module finite extension of normal domains which is unramified in codimension one. If $S$ is free as a $R$-module, then $S$ is unramified over $R$.

**Definition 1.22.** Let $M, N$ be finitely generated modules over a ring $R$. The map $\varphi : M \rightarrow N$ is said to be an **isomorphism in codimension one** if for each prime ideal $p$ in $R$ of height less than or equal to one, the induced homomorphism $\varphi_p : M_p \rightarrow N_p$ is an isomorphism.

The next fact follows with a bit of work from a result of Auslander and Buchsbaum [2, Proposition 3.4]. It is a useful tool in establishing many of the ring isomorphisms in our main argument.

**Fact 1.23.** Let $A$ be a normal domain. If $M$ is a reflexive $A$-module, $N$ is a torsion-free $A$-module, and $\psi : M \rightarrow N$ is an $A$-module homomorphism, then $\psi$ is an isomorphism if and only if it is an isomorphism in codimension one.
Let \( (A, \mathfrak{m}) \) be a complete, local, normal domain such that \( A/\mathfrak{m} \) is separably closed; let \( I \) be a prime ideal of \( A \) with finite projective dimension such that \( A = A/I \) is normal. Let \( e > 1 \) be an integer which represents a unit in \( A \) and assume that \( A \) contains a primitive \( e \)-th root of unity. Let \( \mathfrak{a} \) be a non-zero reflexive ideal of \( A \) such that \( [(a \otimes_A \overline{A})^{A}] \) is trivial in \( \text{Cl}(A) \), and suppose that the order of \( [\mathfrak{a}] \) in \( \text{Cl}(A) \) is \( e \). Thus, \( \mathfrak{a}^{(e)} = aA \) for some \( a \in \mathfrak{a} \). Set \( K \) to be the fraction field of \( A \).

Since \( A \) is a normal ring, it is the intersection over all height one primes \( \mathfrak{p} \) of the family of discrete valuation rings \( \{ \mathfrak{R}_\mathfrak{p} \} \), each of which is contained in \( K \), with corresponding valuations \( v_\mathfrak{p} \). There are height one primes \( \mathfrak{p}_i \) of \( A \) and positive integers \( n_i \) such that \( \mathfrak{a} = \mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_r^{(n_r)} \). Then \( \mathfrak{a}^{(e)} = \mathfrak{p}_1^{(en_1)} \cap \cdots \cap \mathfrak{p}_r^{(en_r)} = aA \) and \( v_\mathfrak{p}(a) = en_i \). By the Approximation Theorem [16, Theorem 12.6], there is an element \( b \in A \) such that \( v_\mathfrak{p}(b) = n_i \) for \( i = 1, \ldots, r \) and \( v_\mathfrak{q}(b) \geq 0 \) for all other height one primes \( \mathfrak{q} \) of \( A \). Set \( u = b^{e} / a \in K \). Then \( v_\mathfrak{p}(u) = en_i - en_i = 0 \) for \( i = 1, \ldots, r \) and \( v_\mathfrak{q}(u) = v_\mathfrak{q}(b^{e}) \geq 0 \) for all other \( \mathfrak{q} \).

Let \( \sqrt{u} \) be a fixed \( e \)-th root of \( u \) in an algebraic closure of \( K \). Then \( \sqrt{a} := b / \sqrt{u} \) is an \( e \)-th root of \( a \). Since \( au = b^{e} \), we have \( K[\sqrt{a}] = K[\sqrt{u}] \). Moreover, the ring

\[
R = A \oplus \left[ a \cdot \frac{\sqrt{u}}{b} \right] \oplus \left[ a^{(2)} \cdot \frac{\sqrt{u^{2}}}{b^{2}} \right] \oplus \cdots \oplus \left[ a^{(e-1)} \cdot \frac{\sqrt{u^{e-1}}}{b^{e-1}} \right]
\]

is the integral closure of \( A \) in the field extension \( K[\sqrt{a}] \), as per [8, Theorem 2.4]. In particular, \( R \) is a domain. It is worth describing the ring structure on \( R \): for \( a_i \in a^{(i)} \) and \( a_j \in a^{(j)} \), we have

\[
\left( a_i \cdot \frac{\sqrt{u^{i}}}{b^{i}} \right) \left( a_j \cdot \frac{\sqrt{u^{j}}}{b^{j}} \right) = \begin{cases} a_i a_j \cdot \frac{\sqrt{u^{i+j}}}{b^{i+j}} & \text{if } i + j < e \\ a_i a_j \cdot \frac{\sqrt{u^{i+j}}}{a^{b^{i+j}}} & \text{if } i + j \geq e. \end{cases}
\]

**Lemma 2.2.** Let \( \mathfrak{p} \in \text{Spec}(R) \) and set \( \mathfrak{p} = P \cap A \). Then \( \text{ht} \mathfrak{p} = \text{ht} P \). Furthermore, if \( A_\mathfrak{p} \) is regular, then the extensions \( A_\mathfrak{p} \to R_\mathfrak{p} \) and \( A_\mathfrak{p} \to R_P \) are étale. In particular, the extension \( A \to R \) is étale in codimension one.

**Proof.** Since \( A \) is normal and \( R \) is a domain that is module finite over \( A \), the theorems of going-up and going-down apply, and hence \( \text{ht} \mathfrak{p} = \text{ht} P \).

Now assume that \( A_\mathfrak{p} \) is regular. There are two cases to consider:

Suppose \( a \notin \mathfrak{p} \). Then \( a \) is a unit in \( A_\mathfrak{p} \). By [5, Theorem III.4.4, p. 113], the polynomial \( T^{e} - a \) is separable in \( A_\mathfrak{p}[T] \). Thus, \( A_\mathfrak{p}[T]/(T^{e} - a)A_\mathfrak{p}[T] \) is a separable \( A_\mathfrak{p} \)-algebra, by definition, cf. [5, p. 109]. Since \( a \notin \mathfrak{p} \), we have \( A_\mathfrak{p} = A_\mathfrak{p} \), so \( R_\mathfrak{p} \cong R \otimes_A A_\mathfrak{p} \cong A_\mathfrak{p} \oplus A_\mathfrak{p} \oplus \cdots \oplus A_\mathfrak{p} \) (\( e \) copies); i.e., \( R_\mathfrak{p} \cong A_\mathfrak{p}[T]/(T^{e} - a)A_\mathfrak{p}[T] \). In other words, \( R \otimes_A A_\mathfrak{p} \) is a separable \( A_\mathfrak{p} \)-algebra. Set \( R_\mathfrak{p} = R \otimes_A A_\mathfrak{p} \). By definition [5 p. 40], \( R_\mathfrak{p} \) is a projective \((R_\mathfrak{p} \otimes A_\mathfrak{p}, R_\mathfrak{p})\)-module. In addition, it is the integral closure of \( A_\mathfrak{p} \) in \( K[\sqrt{a}] = K[\sqrt{u}] \). Thus, by [2, Proposition 2.2], \( R_\mathfrak{p} \) is unramified over \( A_\mathfrak{p} \).

On the other hand, if \( a \in \mathfrak{p} \), then \( \mathfrak{p} = \mathfrak{p}_i \) for some \( i \). Let \( S \) denote the complement in \( A \) of \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \), the height one primes in the decomposition of \( a \) i.e., \( S = A - \)
(p_1 \cup \cdots \cup p_r).\) Note that \(u\) is a unit in \(A_S\) because \(v_{p_i}(u) = 0\) for all \(i = 1, \ldots, r.\) As above, \(T^e - u\) is separable in \(A_S[T]\). Thus, \(A_S[T]/(T^e - u)A_S[T]\) is a separable \(A_S\)-algebra, by definition, \([3, p. 109]\). Since \(u \in S\), we have \(R \otimes_A A_S \cong \oplus_i A_S\) and the proof proceeds as in the case above. In conclusion, for each \(i\), \(A_{p_i} \to R_{p_i}\) is unramified.

The fact that \(A_p\) is regular implies that \(R_p\) is free because \(\mathfrak{a}A_p \cong A_p\). Thus, \(R_p\) is flat over \(A_p\). In particular, the extension \(A_p \to R_p\) is étale. Moreover, it follows that \(R_p\) is also flat over \(A_p\), and hence the extension \(A_p \to R_p\) is étale. Finally, since \(A_p\) is regular for any prime \(p = P \cap A\), where \(ht_R P \leq 1\), it follows that \(A \to R\) is étale in codimension one. \(\square\)

**Lemma 2.3.** The ring \(R\) is a complete local normal domain.

**Proof.** Since \(R\) is the integral closure of \(A\) in \(K[\sqrt{a}]\), it is normal. Moreover, \(R\) is complete since it is a module finite extension of the complete ring \(A\). It remains to show that \(R\) is local.

Since \(R\) is the integral closure of \(A\) in \(K[\sqrt{a}]\), it follows that \(R\) is the integral closure of \(A[\sqrt{a}]\) in \(K[\sqrt{a}]\); that is, in the field of fractions of \(A[\sqrt{a}]\). Note that \(A[\sqrt{a}]\) is an integral domain and finitely generated as an \(A\)-module. Therefore, since \(A\) is complete, it follows that \(A[\sqrt{a}]\) is also complete, both in the \(\mathfrak{m}\)-adic topology and in the \(J\)-adic topology, where \(J\) is the Jacobson radical of \(A[\sqrt{a}]\). We will show that \(A[\sqrt{a}]\) is local and then apply \([17, Corollary 37.6]\), which states that the number of maximal ideals in \(R\), the integral closure of \(A[\sqrt{a}]\), is equal to the number of minimal prime ideals in the completion of \(A[\sqrt{a}]\). Since \(A[\sqrt{a}]\) is complete, \(R\) must be local.

To show that \(A[\sqrt{a}]\) is local, it suffices to show that \(J\) is maximal. Since \(A[\sqrt{a}]\) is module finite over \(A\), each of its maximal ideals of contracts to \(\mathfrak{m}\) and \(A[\sqrt{a}]\) has only finitely many maximal ideals. It follows that \(J \cap A = \mathfrak{m}\). Consider the induced homomorphism \(\psi: A/\mathfrak{m} \to A[\sqrt{a}]/J\), which we claim is bijective. Injectivity is satisfied because \(J \cap A = \mathfrak{m}\). For surjectivity, consider an element \(\xi = \sum_{i=0}^{e-1} a_i (\sqrt{a})^i\) in \(A[\sqrt{a}]\). Since \((\sqrt{a})^e = a \in \mathfrak{m} \subseteq J\), we have \(\sqrt{a} \in J\), and hence \(a_i (\sqrt{a})^i \in J\), for \(1 \leq i \leq e - 1\). In \(A[\sqrt{a}]/J\), we then have

\[
\xi = \sum_{i=0}^{e-1} a_i (\sqrt{a})^i = a_0 = \psi(a_0).
\]

Thus, \(\psi\) is surjective, hence an isomorphism, and so \(A[\sqrt{a}]\) is local. \(\square\)

**Lemma 2.4.** Set \(\bar{R} = R \otimes_A \bar{A} = R/I\bar{R}\). Let \(\bar{P} \subset \bar{R}\) be a prime ideal and set \(\bar{p} = \bar{P} \cap \bar{A}\). Then \(ht \bar{p} = ht \bar{P}\). Furthermore, the ring \(R\) is equidimensional, complete, and local. In particular, Fact \([1, 14]\) applies with \(T = \bar{R}\).

**Proof.** By Lemma 2.3, the ring \(\bar{R}\) is local and complete. Since \(A\) embeds in \(\bar{R}\) as the degree zero direct summand, we see that \(\bar{A}\) embeds in \(\bar{R}\) as the degree zero direct summand. It follows that the induced map \(\bar{A} \to \bar{R}\) is a module finite monomorphism. Hence, going-up holds, and as a result \(ht \bar{p} \geq ht \bar{P}\).
We consider the following commutative diagram of local ring homomorphisms,

\[
\begin{array}{ccc}
A & \longrightarrow & R \\
\tau & \downarrow & \pi \\
\bar{A} & \longrightarrow & \bar{R}
\end{array}
\]

where the vertical maps are the natural surjections, and the horizontal maps are the inclusions. Set \( P = \pi^{-1}(\bar{P}) \) and \( p = \tau^{-1}(\bar{p}) = P \cap A. \)

Note that the map \( A \rightarrow R \) satisfies going-down because \( A \) is integrally closed and \( R \) is an integral domain. Let \( 0 = p_0 \subset p_1 \subset \cdots \subset p_h = p \) be a chain of prime ideals of \( A \) such that \( h = \text{ht } \bar{p} \). It follows that \( I = p_0 \subset p_1 \subset \cdots \subset p_h = p \) is a chain of prime ideals of \( A \). By going-down, there are prime ideals \( P_0 \subset P_1 \subset \cdots \subset P_h = P \) of \( R \) such that \( P_i \cap \bar{A} = p_i \) for each \( i \). In particular, we have \( I = p_0 = P_0 \cap A \) and so \( IR \subseteq P_0 \). It follows that the ideals \( \bar{P}_1 = P_1 \bar{R} \subseteq \bar{R} \) are prime and form a chain \( P_0 \subset \bar{P}_1 \subset \cdots \subset \bar{P}_h = \bar{P} \) and thus \( \text{ht } \bar{p} = h \leq \text{ht } \bar{P}, \) as desired.

To show that \( \bar{R} \) is equidimensional, assume that \( \bar{P} \) is a minimal prime of \( \bar{R} \). Then \( \text{ht } \bar{p} = \text{ht } \bar{P} = 0 \) and so \( p = 0 \). Thus, the induced map \( \bar{A} \rightarrow \bar{R}/\bar{P} \) is a module finite monomorphism. It follows that \( \dim(\bar{R}/\bar{P}) = \dim \bar{A} = \dim \bar{R}. \) \( \square \)

**Lemma 2.5.** For any prime ideal \( \bar{P} \subset \bar{R} \) of height less than or equal to one and \( \bar{p} = \bar{P} \cap \bar{A} \), the extensions \( \bar{A}_\bar{p} \rightarrow \bar{R}_\bar{p} \) and \( \bar{A}_\bar{p} \rightarrow \bar{R}_\bar{P} \) are étale. In particular, the extension \( \bar{A} \rightarrow \bar{R} \) is étale in codimension one. Moreover, the ring \( \bar{R} \) satisfies the Serre condition \((R_1)\).

**Proof.** Let \( \bar{P} \subset \bar{R} \) be a prime of height less than or equal to one. Let \( P, \bar{p} \) and \( p \) be as in the proof of Lemma 2.4, hence \( \text{ht } \bar{p} = \text{ht } \bar{P} \). Since \( \bar{A} \) is a normal domain, it follows that \( \bar{A}_\bar{p} \) is regular. The mapping \( A_\bar{p} \rightarrow \bar{A}_\bar{p} = \bar{A}_\bar{p} \) has finite projective dimension since, by hypothesis, \( A \rightarrow \bar{A} \) has finite projective dimension. It now follows from [3, Theorem 6.1(1)] that \( A_\bar{p} \) is regular. Next, by Lemma 2.2, the extensions \( A_\bar{p} \rightarrow R_\bar{p} \) and \( A_\bar{p} \rightarrow R_\bar{P} \) are étale. It follows by base-change that the extensions \( \bar{A}_\bar{p} \rightarrow \bar{R}_\bar{p} \) and \( \bar{A}_\bar{p} \rightarrow \bar{R}_\bar{P} \) are étale. Thus, the extension \( \bar{A} \rightarrow \bar{R} \) is étale in codimension one. Moreover, the extension \( \bar{A}_\bar{p} \rightarrow \bar{R}_\bar{p} \) has a regular closed fibre (in fact, the closed fibre is a field because the extension is étale). Since the extension is also flat, as it is étale, and \( \bar{A}_\bar{p} \) is regular, it follows that \( \bar{R}_\bar{p} \) is regular. Therefore, \( \bar{R} \) satisfies \((R_1)\). \( \square \)

**Lemma 2.6.** For the biduality map \( \sigma: \bar{R} \rightarrow \bar{R}^{\bar{A} \bar{A}}, \) we have\( \text{Ker } \sigma = \text{nil}(\bar{R}) = j(\bar{R}); \) see Definition 1.8.

**Proof.** Recall that \( j(\bar{R}) = \{ \bar{r} \in \bar{R} : \dim(\bar{R}/\text{ann}_{\bar{R}}(\bar{r})) < \dim \bar{R} \} \) is an ideal of \( \bar{R} \).

First of all, to see that \( \text{nil}(\bar{R}) \subseteq j(\bar{R}) \), let \( \bar{r} \in \text{nil}(\bar{R}) \). By the previous lemma, the extension \( \bar{A}(0)_{\bar{A}} \rightarrow \bar{R}(0)_{\bar{A}} \) is étale. It is also module finite. Since \( \bar{A}(0)_{\bar{A}} \) is a field, the ring \( \bar{R}(0)_{\bar{A}} \) is a finite product of fields; hence, it is reduced. Thus, the image of \( \bar{r} \) in \( \bar{R}(0)_{\bar{A}} \) is zero. In other words, we have \( (\bar{r}\bar{R})(0)_{\bar{A}} = \bar{r}\bar{R}(0)_{\bar{A}} = (0) \) and so \( (0)_{\bar{A}} \notin \text{Supp}_{\bar{A}}(\bar{r}\bar{R}) \). It follows that \( \dim_{\bar{A}}(\bar{r}\bar{R}) < \dim(\bar{A}) \). Now \( \dim_{\bar{R}}(\bar{r}\bar{R}) = \dim_{\bar{A}}(\bar{r}\bar{R}) \) and \( \dim \bar{R} = \dim \bar{A} \), hence \( \bar{r} \in j(\bar{R}). \)
We now show that \( \text{Ker}\; P \psi \ideal \bar{\text{b}} \) is an \( \bar{\text{b}} \) ideal. In particular, we have \( 0 \neq \bar{r}/1 \in \bar{R}_\bar{P} \) and so \( \bar{P}_0 \in \text{Supp}_\bar{R}(\bar{r}\bar{R}) \). It follows that \( \dim_{\bar{R}}(\bar{r}\bar{R}) = \dim \bar{R} \) since \( \bar{R} \) is equidimensional. This contradicts the fact that \( \bar{r} \in j(\bar{R}) \). Therefore, \( j(\bar{R}) \subseteq \text{nil}(\bar{R}) \).

We now show that \( \text{Ker}\; \sigma = j(\bar{R}) \). The module \( \bar{R}_{(0)}\bar{A} \) is free over \( \bar{A}_{(0)}\bar{A} \), hence the localized map \( \sigma_{(0)}\bar{A} : \bar{R}_{(0)}\bar{A} \to (\bar{R}\bar{A})_{(0)}\bar{A} \) is an isomorphism. Thus, \( (\text{Ker}\; \sigma)_{(0)}\bar{A} \cong \text{Ker}(\sigma_{(0)}\bar{A}) = (0) \), so \( \dim_{\bar{A}}(\text{Ker}\; \sigma) < \dim \bar{A} \). Now \( \dim_{\bar{R}}(\text{Ker}\; \sigma) = \dim_{\bar{A}}(\text{Ker}\; \sigma) \) and \( \dim \bar{R} = \dim \bar{A} \), hence \( \text{Ker}\; \sigma \subseteq j(\bar{R}) \).

For the reverse containment, let \( \bar{r} \in j(\bar{R}) \). Then \( \dim_{\bar{A}}(\bar{r}\bar{R}) < \dim \bar{A} \) as above, which means that \( \bar{r}\bar{R}_{(0)}\bar{A} = (0) \). It follows that there exists a non-zero element \( \bar{b} \in \bar{A} \) such that \( \bar{b}\bar{r} = 0 \), and so \( \dim_{\bar{A}}(\bar{r}\bar{A}) < \dim \bar{A} \). For each psi \( \in \text{Hom}_{\bar{A}}(\bar{R}, \bar{A}) \), \( \dim_{\bar{A}}(\psi(\bar{r}\bar{A})) \leq \dim_{\bar{A}}(\bar{r}\bar{A}) < \dim \bar{A} \), since \( \bar{r}\bar{A} \to \psi(\bar{r}\bar{A}) \) via psi. However, \( \psi(\bar{r}\bar{A}) \) is an \( \bar{A} \)-submodule of \( \bar{A} \), that is, an ideal of \( \bar{A} \). If \( \psi(\bar{r}\bar{A}) \neq (0) \), then \( \psi(\bar{r}\bar{A}) \leftarrow \psi(\bar{r}\bar{A})_{(0)}\bar{A} \) because \( \bar{A} \) is a domain; but this would mean that \( \psi(\bar{r}\bar{A})_{(0)}\bar{A} \neq (0) \), hence \( \dim_{\bar{A}}(\psi(\bar{r}\bar{A})) = \dim \bar{A} \). This is a contradiction. Therefore, \( \psi(\bar{r}\bar{A}) = (0) \). This says that \( \psi(\bar{r}) = 0 \) for all psi \( \in \text{Hom}_{\bar{A}}(\bar{R}, \bar{A}) \). By definition, \( \sigma(\bar{r}) : \text{Hom}_{\bar{A}}(\bar{R}, \bar{A}) \to \bar{A} \) is the map \( \psi(\bar{r}) \) which is \( \psi(\bar{r}) = 0 \). Thus, \( \sigma(\bar{r}) = 0 \), which means that \( \bar{r} \in \text{Ker}\; \sigma \), as desired.

**Lemma 2.7.** The ring \( \bar{R}/j(\bar{R}) \) satisfies \((R_1)\), so its integral closure \((\bar{R}/j(\bar{R}))'\) in its total ring of quotients is its \( S_2 \)-ification.

**Proof.** Take \( \bar{P} = \bar{P}/j(\bar{R}) \) a height one prime of \( \bar{R}/j(\bar{R}) \). Then \( \bar{P} \) is a height one prime of \( \bar{R} \) since \( j(\bar{R}) = \text{nil}(\bar{R}) \), and \( (\bar{R}/j(\bar{R}))_{\bar{P}} = \bar{R}_{\bar{P}}/(j(\bar{R}))_{\bar{P}} \). Since \( \bar{R} \) is complete and local, it has a canonical module \( \omega_{\bar{R}} \). Recalling Fact 1.9, \( j(\bar{R}) = \text{Ker}(\bar{R} \to \text{Hom}_{\bar{A}}(\omega_{\bar{R}}, \omega_{\bar{R}})) \), hence \( j(\bar{R})_{\bar{Q}} \cong j(\bar{R}_\bar{Q}) \) for any prime ideal \( \bar{Q} \) of \( \bar{R} \).

Note that this uses the fact that \( \bar{R} \) is equidimensional; see Lemma 2.4 and 13 (2.2i)]. In particular, since \( \bar{R} \) satisfies the \((R_1)\) condition as per Lemma 2.6, we can obtain \( j(\bar{R}_\bar{P}) = 0 \) and hence \( (\bar{R}/j(\bar{R}))_{\bar{P}} \cong \bar{R}_{\bar{P}} \) is regular.

Thus, \( \bar{R}/j(\bar{R}) \) satisfies \((R_1)\). Lemma 2.7 implies that the integral closure of \( \bar{R}/j(\bar{R}) \) in its total ring of quotients is the \( S_2 \)-ification of \( \bar{R}/j(\bar{R}) \), that is, of \( \bar{R} \).

**Lemma 2.8.** The \( \bar{A} \)-module \( \bar{R}\bar{A} \) is free of rank \( e \) and thus satisfies \((S_2)\) over \( \bar{A} \).

**Proof.** Since \( [a] \in \text{Ker}(\text{Cl}(A) \to \text{Cl}(\bar{A})) \), we have \( (a \otimes_{\bar{A}} \bar{A})^{\bar{A}} \cong (\bar{a}) \), for some \( \bar{a} \in \bar{A} \). Thus, using the fact that \( \gamma \) is a group homomorphism, \( (a^{(t)} \otimes_{\bar{A}} \bar{A})^{\bar{A}} = \gamma((a^{(t)})) \cong (\bar{a}^{(t)}) \), for all \( t \); i.e., each component of \( \bar{R}\bar{A} \) is a free \( \bar{A} \)-module. Consequently, \( \bar{R}\bar{A} \) is free of rank \( e \) and satisfies the \((S_2)\) condition, since \( \bar{A} \) is \((S_2)\).

**Lemma 2.9.** For each prime \( \bar{P} \) in \( (\bar{R}/j(\bar{R}))' \), we have

\[
\text{ht}(\bar{P} \cap \bar{A}) = \text{ht}(\bar{P} \cap (\bar{R}/j(\bar{R}))) = \text{ht} \bar{P}.
\]

Moreover, \((\bar{R}/j(\bar{R}))'\) satisfies \((S_2)\) as an \( \bar{A} \)-module and as a ring.
Proof. The equality $\text{ht}(\bar{P} \cap (\bar{R}/j(\bar{R}))) = \text{ht} \bar{P}$ is from \cite[Proposition 3.5(a)]{[13]}. Lemma 2.4 implies that $\text{ht}(\bar{P} \cap \bar{A}) = \text{ht} \bar{P}$. Since $(\bar{R}/j(\bar{R}))'$ satisfies (S2) as an $\bar{R}/j(\bar{R})$-module by Lemma 2.7, it follows from \cite[Corollaire (5.7.11)]{[24]} that $(\bar{R}/j(\bar{R}))'$ satisfies (S2) as an $\bar{A}$-module and as a ring. □

Lemma 2.10. There are $\bar{A}$-isomorphisms $(\bar{R}/j(\bar{R}))' \cong (\bar{R}/j(\bar{R}))^{\bar{A}A} \cong \bar{R}^{\bar{A}A}$.

Proof. Let $\bar{p}$ be a prime of height less than or equal to one in $\bar{A}$. Recall that $A_{\bar{p}} \to R_{\bar{p}}$ is étale, by Lemma 2.5, therefore, $R_{\bar{p}}$ is free over $A_{\bar{p}}$. The maximal ideals of $R_{\bar{p}}$ are of the form $\bar{P}R_{\bar{p}}$, where $\bar{P} \cap \bar{A} = \bar{p}$, thus $\text{ht} \bar{p} \bar{P} = 1$ by Lemma 2.4. Since $(\bar{R}_{\bar{p}})\bar{P}_{\bar{p}} \cong \bar{P}$ and $\bar{R}$ satisfies the $(R_1)$ condition, $\bar{R}_{\bar{p}}$ is a regular local ring. Therefore, $R_{\bar{p}}$ is a regular ring. Moreover, $j(\bar{R}_{\bar{p}}) = 0$ since $\bar{R}_{\bar{p}}$ is a domain; hence, $j(\bar{R}_{\bar{p}})P_{\bar{p}_{\bar{p}}} = 0$ for all $\bar{p}R_{\bar{p}}$ (as in the proof of Lemma 2.7). We conclude that $j(\bar{R}_{\bar{p}}) = 0$. Consequently, $(\bar{R}/j(\bar{R}))_{\bar{p}} = \bar{R}_{\bar{p}}$. Since $\bar{R}_{\bar{p}}$ is regular, hence normal, we deduce that $(\bar{R}/j(\bar{R}))'_{\bar{p}} = (R/j(\bar{R}))'_{\bar{p}} = R_{\bar{p}}$. It follows that the $\bar{A}$-homomorphisms $\bar{R}^{\bar{A}A} \to (\bar{R}/j(\bar{R}))^{\bar{A}A} \to ((\bar{R}/j(\bar{R}))')^{\bar{A}A}$ are isomorphisms in codimension one. Note that $\bar{R}/j(\bar{R})$ is complete, local, and reduced, since it was shown in Lemma 2.6 that $j(\bar{R}) = \text{nil}(\bar{R})$. Therefore, $(\bar{R}/j(\bar{R}))'$ is finite over $\bar{R}/j(\bar{R})$, (see, e.g., \cite[p. 263]{[16]}), and hence finitely generated over $\bar{A}$. It follows that the maps $\bar{R}^{\bar{A}A} \to (\bar{R}/j(\bar{R}))^{\bar{A}A} \to ((\bar{R}/j(\bar{R}))')^{\bar{A}A}$ are isomorphisms of reflexive $\bar{A}$-modules and isomorphisms in codimension one. Since reflexive implies torsion free (see e.g., \cite[Corollary 3.7]{[6]}), these maps are isomorphisms, as per Fact 1.23.

Lemma 2.9 says that $(\bar{R}/j(\bar{R}))'$ satisfies the (S2) condition as an $\bar{A}$-module. Hence it is reflexive as an $\bar{A}$-module, by \cite[Theorem 3.6]{[6]}. Thus, we have $(\bar{R}/j(\bar{R}))' \cong ((\bar{R}/j(\bar{R}))')^{\bar{A}A} \cong (\bar{R}/j(\bar{R}))^{\bar{A}A}$ as $\bar{A}$-modules. □

Remark 2.11. The upshot of the previous lemma is that $(\bar{R}/j(\bar{R}))'$ is the $S_2$-ification of $\bar{R}$. (See Definition 1.10.) Therefore, $(\bar{R}/j(\bar{R}))'$ is local, by assumption.

Lemma 2.12. The map $\bar{A} \to (\bar{R}/j(\bar{R}))'$ is étale in codimension one.

Proof. Recall the composition $\bar{A} \to \bar{R} \to (\bar{R}/j(\bar{R})) \to (\bar{R}/j(\bar{R}))'$. Let $\bar{p}$ be a prime ideal of $\bar{A}$ with height less than or equal to one. By Lemma 2.5, $A_{\bar{p}} \to R_{\bar{p}}$ is étale. Next, $(\bar{R}/j(\bar{R}))'_{\bar{p}} = (R_p/j(R_p))' = R_p \equiv (\bar{R}/j(\bar{R}))_{\bar{p}}$ since $j(\bar{R})_{\bar{p}} = j(R_p) = 0$; see the proof of Lemma 2.10. Hence the composition $A_{\bar{p}} \to ((\bar{R}/j(\bar{R}))')_{\bar{p}}$ is étale. Now, let $\bar{P}$ be a prime ideal of $(\bar{R}/j(\bar{R}))'$ with height less than or equal to one, and set $\bar{p} = \bar{P} \cap \bar{A}$. Lemma 2.9 implies that $\text{ht} \bar{p} \leq 1$, so the map $A_{\bar{p}} \to ((\bar{R}/j(\bar{R}))')_{\bar{p}}$ is étale, by the previous paragraph. □

Lemma 2.13. The ring $(\bar{R}/j(\bar{R}))'$ is a normal domain.

Proof. Lemma 2.9 implies that $(\bar{R}/j(\bar{R}))'$ is (S2) as a ring. We claim that it is also $(R_1)$ as a ring. To this end, let $\bar{P}$ be a height one prime of $(\bar{R}/j(\bar{R}))'$, and set $\bar{p} = \bar{P} \cap \bar{A}$. Then we have $\text{ht} \bar{p} = 1$, again by Lemma 2.9, so $A_{\bar{p}}$ is regular. From Lemma 2.12, we know that $((\bar{R}/j(\bar{R}))')_{\bar{p}}$ is étale over $A_{\bar{p}}$. Since $A_{\bar{p}}$ is regular, this implies that $((\bar{R}/j(\bar{R}))')_{\bar{p}}$ is regular.
Using Serre’s criterion for normality, it follows that \((\bar{R}/j(\bar{R}))'\) is a finite direct product of normal domains; see [16, p. 64]. However, since \((\bar{R}/j(\bar{R}))'\) is local, by assumption, it cannot decompose into a non-trivial product of domains. Hence it is a normal domain.

**Lemma 2.14.** The extension \(\bar{A} \hookrightarrow (\bar{R}/j(\bar{R}))'\) is étale.

**Proof.** Lemma 2.13 implies that \((\bar{R}/j(\bar{R}))'\) is a normal domain. The extension \(\bar{A} \hookrightarrow (\bar{R}/j(\bar{R}))'\) between normal domains is module finite, free, and étale in codimension one by Lemmas 2.8, 2.10, and 2.12. Thus, it is étale by purity of branch locus, specifically, Fact 1.21 applies since \((\bar{R}/j(\bar{R}))'\) is a free \(\bar{A}\) module, and hence projective over \(\bar{A}\).

**Conclusion of proof of Theorem C.** Let \(S = (\bar{R}/j(\bar{R}))'\). Since the extension \(\bar{A} \hookrightarrow S\) is étale and local, the induced field extension \(\bar{A}/\bar{m} \rightarrow S/\bar{m}S\) is separable and algebraic. Since \(\bar{A}/\bar{m}\) is separably closed, this induced map is an isomorphism. Recall that \(S\) is free over \(\bar{A}\) of rank \(e\). Consequently, there are isomorphisms \(\bar{A}/\bar{m} \cong S/\bar{m}S \cong (\bar{A}/\bar{m})^e\).

It follows that \(e = 1\), but this contradicts the assumptions in paragraph 2.1.

**3. Proof of Theorem B**

**3.1.** Let \((A, m)\) be an excellent, local, normal domain and \(I\) a prime ideal of \(A\) with finite projective dimension such that:

(i) \(\bar{A} = A/I\) is normal;
(ii) \(I\) is a complete intersection on the punctured spectrum of \(A\); and
(iii) \(\mu(I) \leq \dim A - 2\).

Let \(e > 1\) be an integer which represents a unit in \(A\). Let \(a\) be a non-zero reflexive ideal of \(A\) such that \([\langle a \otimes_A \bar{A} \rangle]^{\bar{A}}\) is trivial in \(\text{Cl}(\bar{A})\), and suppose that the order of \([a]\) in \(\text{Cl}(A)\) is \(e\). Thus, \(a^{(e)} = aA\) for some \(a \in a\). Set \(K\) to be the fraction field of \(A\).

The theorem is trivial if \(I = (0)\), therefore assume \(I\) is non-zero. If \(\dim A \leq 2\), then the \((R_1)\) condition on \(\bar{A}\) forces the factor ring to be regular, and hence \(A\) must itself be regular by [3, Theorem 6.1(1)]. Consequently, \(\text{Cl}(A) = 0\), which again provides a trivial result. Therefore, assume that \(\dim A \geq 3\). We make a series of reductions to reduce to the case of Theorem C and as in the previous proof, ultimately obtain our conclusion by way of contradiction.

The prime ideal \(I\) has finite projective dimension, therefore

\[
\text{pd}_{A_I}(A_I/I A_I) \leq \text{pd}_A \bar{A} < \infty.
\]

It follows that the localization \(A_I\) is regular and hence a unique factorization domain. This implies that \(a A_I = b A_I\) for some \(b \in a\), and hence

\[
b^{(e)} A_I = (b A_I)^{(e)} = (a A_I)^{(e)} = a^{(e)} A_I = a A_I.
\]

**Lemma 3.2.** \(A\) can be assumed to be complete.
Proof. Because $A$ is an excellent local normal domain, the completion $\hat{A}$ is a complete local normal domain by [17] Corollary 37.6. Next, $\text{pd}_A I\hat{A} < \infty$ since that map $A \to \hat{A}$ is flat. Moreover, because the map is also local, $\mu_\hat{A}(I\hat{A}) = \mu_A(I) \leq \dim A - 2 = \dim \hat{A} - 2$. We need to show that $I\hat{A}$ is a complete intersection on the punctured spectrum of $A$, denoted $\text{Spec}^0(\hat{A})$.

Let $P \in \text{Spec}^0(\hat{A})$ and set $p = P \cap A$. Since the closed fibre $\hat{A}/m\hat{A}$ is isomorphic to $A/m$, the fact that $P$ is not maximal implies that $p \neq m$. Thus, the ideal $IA_p$ is either $A_p$ or generated by an $A_p$-regular sequence. If $IA_p = A_p$, then $IA_P = IA_p \cdot A_P = A_P$. If $IA_p$ is generated by an $A_p$-regular sequence, then $IA_P = IA_p \cdot A_P$ is generated over $A_P$ by the same sequence, which is $A_P$-regular since the induced map $A_p \to A_P$ is flat and local.

The left-most diagram below is commutative and each map has finite flat dimension:

\[
\begin{array}{ccc}
A & \longrightarrow & \hat{A} \\
\rho \downarrow & & \downarrow \\
\hat{A} & \longrightarrow & \hat{A}
\end{array}
\quad
\begin{array}{ccc}
\text{Cl}(A) & \longrightarrow & \text{Cl}(\hat{A}) \\
\text{Cl}(\rho) \downarrow & & \downarrow \\
\text{Cl}(\hat{A}) & \longrightarrow & \text{Cl}(\hat{A}).
\end{array}
\]

(3.2.1)

It follows that the composition $A \to \hat{A}$ also has finite flat dimension, so [20] Theorem 1.14 implies that the second diagram above also commutes.

Suppose the result holds on $\text{Cl}(\hat{A}) \to \text{Cl}(\hat{A})$. Since $[a] \in \ker(\text{Cl}(A) \to \text{Cl}(\hat{A}))$, it follows that $\text{Cl}(\rho)([a]) \in \ker(\text{Cl}(\hat{A}) \to \text{Cl}(\hat{A}))$. By properties of group monomorphisms, the order of $\text{Cl}(\rho)([a])$ is $e$, which is a unit in $\text{Cl}(\hat{A})$. The condition $e > 1$ contradicts the complete case. \qed

Lemma 3.3. A can be assumed to contain a primitive $e$-th root of unity.

Proof. Suppose that $A$ does not contain a primitive $e$-th root of unity and let $\zeta$ be a primitive $e$-th root of unity in the algebraic closure of $K$, the fraction field of $A$. Note that $\zeta$ exists because $e$ is a unit in $A$. By assumption, $A$ is a complete local normal domain.

As per [18] Proposition VI.1, the extension $A \to A[T]/(T^e - 1)$ is étale. Let $\Phi_e(T) \in K[T]$ be the minimal polynomial of $\zeta$ over $K$. Since $A$ is integrally closed and the coefficients of $\Phi_e(T)$ are integral over $A$, it follows that $\Phi_e(T) \in A[T]$. Using the Division Algorithm in $A[T]$, we have $A[\zeta] \cong A[T]/\Phi_e(T)$, and $\Phi_e(T)$ divides $(T^e - 1)$ in $A[T]$. Set $d = \deg \Phi_e(T)$. Note that $A[T]/\Phi_e(T) \cong \oplus_{i=0}^{d-1} A$ as $A$-modules, hence the extension $A \to A[T]/\Phi_e(T)$ is faithfully flat. The maps $A \to A[T]/(T^e - 1)$ and $A[T]/(T^e - 1) \to A[T]/\Phi_e(T)$ are both unramified, thus the composition is unramified; i.e., the extension $A \to A[T]/\Phi_e(T)$ is étale.

The ring $A[\zeta]$ is excellent since excellence is preserved under finitely-generated extensions. Moreover, $A[\zeta]$ is a domain since it is a subring of $K[\zeta]$. Since $A$ is a normal domain and the finite extension $A \to A[\zeta]$ is étale, $(A[\zeta])_Q$ is a normal domain for each prime $Q$ of $A[\zeta]$ by [16] Theorem 23.9.
Let $Q$ be a maximal ideal of $A[\zeta]$. Since $A \to A[\zeta]$ is finite, hence $Q \cap A$ is maximal; that is, $Q \cap A = \mathfrak{m}$. We next note that the ring $A[\zeta]_Q$ and the ideal $IA[\zeta]_Q$ satisfy the hypothesis of the theorem. Indeed, since $A[\zeta]_Q$ is an excellent local normal domain, it follows that $\widehat{A[\zeta]_Q}$ is a complete normal local domain.

The maps in the sequence $A \to A[\zeta] \to A[\zeta]_Q \to \widehat{A[\zeta]_Q}$ are all flat, hence the composition $A \to \widehat{A[\zeta]_Q}$ is flat. It follows that the extension $IA[\zeta]_Q$ has finite projective dimension over $\widehat{A[\zeta]_Q}$. It remains to show that $\widehat{A[\zeta]_Q}/IA[\zeta]_Q$ is a normal domain. To this end, note that $IA[\zeta]_Q$ has finite projective dimension over $A[\zeta]_Q$. The map $A \to A[\zeta]_Q$ is flat and unramified because $A \to A[\zeta]$ is flat and unramified. Therefore, $A/I \to A[\zeta]_Q/IA[\zeta]_Q$ is flat and unramified. The fact that $A/I$ is normal implies that $A[\zeta]_Q/IA[\zeta]_Q$ is normal; since $A[\zeta]_Q/IA[\zeta]_Q$ is local, it is also an excellent local normal domain. Hence its completion $A[\zeta]_Q/IA[\zeta]_Q$ is a local normal domain.

As in Lemma 3.2 we have $\mu_{A[\zeta]_Q}(IA[\zeta]_Q) \leq \dim A[\zeta]_Q - 2$ and the fact that the map $A \to \widehat{A[\zeta]_Q}$ is flat and local, with closed fibre a field, implies that $\widehat{A[\zeta]_Q}$ is a complete intersection on $\text{Spec}^o\left(A[\zeta]_Q\right)$. Likewise, as in Lemma 3.2 we have a commutative diagram

$$\begin{align*}
\text{Cl}(A) & \longrightarrow \text{Cl}(\hat{A}) \\
\text{Cl}(A[\zeta]_Q) & \longrightarrow \text{Cl}(\widehat{A[\zeta]_Q}/IA[\zeta]_Q)
\end{align*}$$

(3.3.1)

and the same argument shows that we may replace $A$ with $\widehat{A[\zeta]_Q}$ to assume that $A$ contains a primitive $\epsilon$-th root of unity. 

\begin{lemma}
$A/\mathfrak{m}$ can be assumed to be separably closed.
\end{lemma}

\begin{proof}
Set $k = A/\mathfrak{m}$ and let $k^{\text{sep}}$ be a separable closure of $k$; (i.e., the set of all separable elements in a fixed algebraic closure). This means that $k \subseteq k^{\text{sep}}$ is a separable algebraic extension, and $k^{\text{sep}}$ has no nontrivial separable algebraic extensions. Grothendieck [11] Proposition (0.10.3.1)] shows that there is a flat local ring homomorphism $\rho: (A, \mathfrak{m}) \to (B, \mathfrak{n})$ such that $\mathfrak{n} = \mathfrak{m}B$, the ring $B$ is complete, and $B/\mathfrak{n} \cong k^{\text{sep}}$.

It follows that the extension $\rho$ is regular, in the terminology of Matsumura [10], pp. 255–256]. To see this, observe that the extension $k \to k^{\text{sep}}$ is 0-smooth by [10] Theorem 26.9]. Using [10] Theorem 28.10 it follows that $B$ is $\mathfrak{m}B$-smooth over $A$, that is, that $B$ is $\mathfrak{n}$-smooth over $A$. Now, apply the theorem of M. André from [11] Lemme II.5.7] to conclude that the extension $\rho$ is regular. (See also [10] p. 260.)

From [10] Theorem 32.2] it follows that $B$ is a normal domain. The same reasoning implies that the ring $B/IB$ is a normal domain. The fact that the extension $\rho$ is flat and local implies that $\text{pd}_B(B/IB) = \text{pd}_A(\hat{A}) < \infty$.

\end{proof}
Again as in Lemma 3.2 we have $\mu_B(IB) \leq \dim B - 2$, and the fact that the map $\rho$ is flat and local, with closed fibre a field, implies that $IB$ is a complete intersection on $\text{Spec}^e(B)$.

Now argue as in the proof of Lemma 3.2 there is a commutative diagram of divisor class groups as in 3.2.1 and 3.3.1, but with bottom row $\text{Cl}(B) \to \text{Cl}(B/IB)$. As $[a] \in \ker(Cl(A) \to Cl(\tilde{A}))$ has order $e$ in $Cl(A)$, the element $Cl(\rho)([a]) \in \ker(Cl(B) \to Cl(B/IB))$ has order $e$ as well. If the theorem holds for $B \to B/IB$, then we have a contradiction. □

The three lemmas below give the explicit details, implicit in 13 proof of Proposition 3.9(c)], as to the fact that if the $S_2$-ification of $\tilde{R}$ is not local, then there exists a prime ideal $\tilde{P}$ of $\tilde{R}$ of height at least two such that the punctured spectrum of $\tilde{R}/\tilde{P}$ is disconnected. (It should be noted that all assumptions in these lemmas are as stated and do not depend upon previously set notation in 3.1.)

**Lemma 3.5.** Let $B$ be a complete local equidimensional ring. Assume that the $S_2$-ification of $B$ is not local. Then there exist ideals $K_1, K_2 \subseteq B$ such that

1. $\text{ht}(K_1 + K_2) \geq 2$,
2. $K_1 \cap K_2 \subseteq \text{nil}(B)$, and
3. $K_j \notin \text{nil}(B)$ for $j = 1, 2$.

**Proof.** Since the $S_2$-ification of $B$ is not local, we know from [14, Theorem 3.6] that there is an ideal $J \subseteq B$ such that $\text{ht} J \geq 2$ and $\text{Spec}(B) - V(J)$ is disconnected. Note that we are using the subspace topology on $\text{Spec}(B) - V(J)$ induced from the Zariski topology on $\text{Spec}(B)$. It follows that there are non-empty disjoint open subsets $U_1, U_2 \subseteq \text{Spec}(B) - V(J)$ such that $\text{Spec}(B) - V(J) = U_1 \cup U_2$. Since each $U_j$ is open in $\text{Spec}(B) - V(J)$, it is of the form $U_j = \text{Spec}(B) - V(J_k) \cap W_j$ for some open subset $W_j \subseteq \text{Spec}(B)$. Thus, there are ideals $I_j \subseteq B$ such that $W_j = \text{Spec}(B) - V(I_j)$, and it follows that

$$U_j = [\text{Spec}(B) - V(J)] \cap [\text{Spec}(B) - V(I_j)]$$
$$= \text{Spec}(B) - [V(J) \cup V(I_j)]$$
$$= \text{Spec}(B) - V(JI_j).$$

For $j = 1, 2$, set $K_j = JI_j$. This implies that $U_j = \text{Spec}(B) - V(K_j)$ for $j = 1, 2$.

The condition $\text{Spec}(B) - V(J) = U_1 \cup U_2$ implies that

$$\text{Spec}(B) - V(J) = [\text{Spec}(B) - V(K_1)] \cup [\text{Spec}(B) - V(K_2)]$$
$$= \text{Spec}(B) - [V(K_1) \cap V(K_2)]$$
$$= \text{Spec}(B) - V(K_1 + K_2).$$

So we have

$$\text{ht}(K_1 + K_2) = \min\{\text{ht} Q \mid Q \in V(K_1 + K_2)\}$$
$$= \min\{\text{ht} Q \mid Q \in V(J)\}$$
$$\geq 2.$$

This explains condition (1) from the statement of the lemma.
Next, we use the fact that $U_1$ and $U_2$ are disjoint:

$$
\emptyset = U_1 \cap U_2
= \{ \text{Spec}(B) - V(K_1) \} \cap \{ \text{Spec}(B) - V(K_2) \}
= \text{Spec}(B) - [V(K_1) \cup V(K_2)]
= \text{Spec}(B) - V(K_1 \cap K_2)
$$

so that $V(K_1 \cap K_2) = \text{Spec}(B)$. It follows that, for each $Q \in \text{Spec}(B)$ we have $K_1 \cap K_2 \subseteq Q$. It follows that $K_1 \cap K_2 \subseteq \text{nil}(B)$, i.e., we have condition (2) from the statement of the lemma.

For condition (3), we argue by contradiction. Suppose that $K_j \subseteq \text{nil}(B)$. This implies that $V(K_j) = \text{Spec}(B)$, so $U_j = \text{Spec}(B) - V(K_j) = \emptyset$. This contradicts our choice of $U_j$. Thus, condition (3) is satisfied. □

**Lemma 3.6.** Let $B$ be a complete local equidimensional ring. Assume that the $S_2$-ification of $B$ is not local. Then there exist ideals $L_1, L_2 \subseteq B$ such that

1. $\text{ht}(L_1 + L_2) \geq 2$,
2. $L_1 \cap L_2 = \text{nil}(B)$,
3. $L_j \not\subseteq \text{nil}(B)$ for $j = 1, 2$, and
4. $L_j = P_{j,1} \cap \cdots \cap P_{j,t_j}$ for $j = 1, 2$, such that $t_j \geq 1$ and each $P_{j,k} \in \text{min}(B)$.

**Proof.** Let $K_1, K_2 \subseteq B$ be as in Lemma 3.5. It follows that

1. $\text{ht}(\text{rad } K_1 + \text{rad } K_2) \geq \text{ht}(K_1 + K_2) \geq 2$,
2. $\text{rad } K_1 \cap \text{rad } K_2 = \text{rad}(K_1 \cap K_2) \subseteq \text{rad } \text{nil}(B) = \text{nil}(B)$,
3. $\text{rad } K_j \not\subseteq \text{nil}(B)$ for $j = 1, 2$, and
4. each $\text{rad } K_j$ is an intersection of primes of $B$.

Thus, we may replace $K_j$ with $\text{rad } K_j$ to assume that each $K_j$ is an intersection of primes of $B$.

Claim: each $K_j$ is contained in a minimal prime of $B$. We prove this by contradiction. By symmetry, suppose that $K_1$ is not contained in any minimal prime of $B$. Then for each $Q \in \text{min}(B)$, we have

$$
K_1 K_2 \subseteq K_1 \cap K_2 \subseteq \text{nil}(B) \subseteq Q.
$$

Since $Q$ is prime, we have $K_j \subseteq Q$ for some $j = 1, 2$. But $K_1 \not\subseteq Q$ by assumption, so we must have $K_2 \subseteq Q$. Since $Q$ was chosen arbitrarily from $\text{min}(B)$, we conclude that $K_2$ is contained in the intersection of the minimal primes of $B$, that is, $K_2 \subseteq \text{nil}(B)$. This contradicts condition (3) from Lemma 3.5. Thus, the claim is established.

By condition (4) above, for $j = 1, 2$ we can write $K_j = P_{j,1} \cap \cdots \cap P_{j,n_j}$ where each $P_{j,k}$ is prime. Re-order the $P_{j,k}$ if necessary to assume that $P_{j,1}, \ldots, P_{j,t_j} \in \text{min}(B)$ and $P_{j,t_j+1}, \ldots, P_{j,n_j} \not\in \text{min}(B)$. Note that the claim above implies that $t_j \geq 1$. For $j = 1, 2$ set $L_j = P_{j,1} \cap \cdots \cap P_{j,t_j} \supseteq K_j$. By definition of $L_j$, we have condition (4) from the statement of the lemma. Furthermore, the condition $L_j \supseteq K_j$ implies that

$$
\text{ht}(L_1 + L_2) \geq \text{ht}(K_1 + K_2) \geq 2
$$

and $L_j \not\subseteq \text{nil}(B)$, so conditions (1) and (3) are satisfied.
We conclude the proof by verifying condition (ii). Since each $L_j$ is an intersection of primes of $B$, we have $L_j \supseteq \text{nil}(B)$, and hence $L_1 \cap L_2 \supseteq \text{nil}(B)$. To show the reverse containment, let $Q \in \min(B)$; it suffices to show that $L_1 \cap L_2 \subseteq Q$. We know that

$$P_{1,1} \cap \cdots \cap P_{1,n_1} \cap P_{2,1} \cap \cdots \cap P_{2,n_2} = K_1 \cap K_2 \subseteq \text{nil}(B) \subseteq Q.$$ 

Thus, we have $P_{j,k} \subseteq Q$ for some $j,k$. Since $Q$ is minimal, we must have $P_{j,k} = Q$ so $k \leq t_j$. From this, we have $L_j = P_{j,1} \cap \cdots \cap P_{j,t_j} \subseteq P_{j,k} = Q$, as desired. \hfill \Box

**Lemma 3.7.** Let $B$ be a complete local equidimensional ring. Assume that the $S_2$-ification of $B$ is not local. Then there is a prime ideal $P$ of $B$ such that $\text{ht} \ P \geq 2$ and $\text{Spec}^o(B_P)$ is disconnected.

**Proof.** Let $L_1, L_2$ be as in Lemma 3.6 and let $P$ be minimal in $V(L_1 + L_2)$. It follows that $\text{ht} \ P \geq \text{ht}(L_1 + L_2) \geq 2$. We show that $\text{Spec}^o(B_P)$ is disconnected. For $j = 1, 2$ we set

$$V^o(L_j B_P) = V(L_j B_P) \cap \text{Spec}^o(B_P) = V(L_j B_P) - \{PB_P\}.$$ 

Since we are using the subspace topology on $\text{Spec}^o(B_P)$ induced from the Zariski topology on $\text{Spec}(B_P)$, the sets $V^o(L_j B_P)$ are closed in $\text{Spec}^o(B_P)$. We show that the closed sets $V^o(L_j B_P)$ give a disconnection of $\text{Spec}^o(B_P)$.

Claim 1: $\text{Spec}^o(B_P) = V^o(L_1 B_P) \cup V^o(L_2 B_P)$. The containment $\supseteq$ is by definition of $V^o(L_j B_P)$. For the reverse containment, let $Q B_P \in \text{Spec}^o(B_P)$. Thus, we have $Q \in \text{Spec}(B)$ and thus $L_1 L_2 \subseteq L_1 \cap L_2 = \text{nil}(B) \subseteq Q$, by condition (ii) of Lemma 3.6. It follows that $L_j \subseteq Q$ for some $j$, hence $L_j B_P \subseteq Q B_P$, so

$$Q B_P \in V(L_j B_P) \cap \text{Spec}^o(B_P) = V^o(L_j B_P) \subseteq V^o(L_1 B_P) \cup V^o(L_2 B_P)$$ 

as desired.

Claim 2: $V^o(L_1 B_P) \cap V^o(L_2 B_P) = \emptyset$. By way of contradiction, suppose that $V^o(L_1 B_P) \cap V^o(L_2 B_P) \neq \emptyset$, and let $Q B_P \in V^o(L_1 B_P) \cap V^o(L_2 B_P)$. It follows that $Q \subseteq \text{nil}(B)$ and $L_j B_P \subseteq Q B_P$ for $j = 1, 2$. It follows that

$$L_j \subseteq B \cap L_j B_P \subseteq B \cap Q B_P = Q \subseteq P$$ 

for $j = 1, 2$. (Given an ideal $I \subseteq B$, we use the notation $B \cap IB_P$ to denote the contraction of $IB_P$ along the natural map $B \rightarrow B_P$.) From this, we have $L_1 + L_2 \subseteq Q \subseteq P$, contradicting the fact that $P$ is minimal in $V(L_1 + L_2)$.

To complete the proof, it suffices to show that $V^o(L_j B_P) \neq \emptyset$ for $j = 1, 2$. For this, it suffices to show that there is a prime $P_{j,k}$ in $B$ such that $L_j \subseteq P_{j,k} \subseteq P$. Using condition (iv) of Lemma 3.6, we have

$$P_{j,1} \cap \cdots \cap P_{j,t_j} = L_j \subseteq L_1 + L_2 \subseteq P$$ 

so we find that $P_{j,k} \subseteq P$ for some $k$. By construction, we have

$$L_j = P_{j,1} \cap \cdots \cap P_{j,t_j} \subseteq P_{j,k} \subseteq P.$$ 

Moreover, we have $\text{ht}(P_{j,k}) = 0 < 2 = \text{ht} \ P$, so $P_{j,k} \subseteq P$, as desired. \hfill \Box

**Conclusion of proof of Theorem 2.1.** We have reduced the theorem to the case where $A$ is complete with separably closed residue field and contains a primitive $e$-th root of unity. Set $R$ to be the truncated symbolic Rees algebra as in equation (2.1.1).
It only remains to show that \( \tilde{R} = R/IR \) satisfies any of the equivalent conditions in Fact 1.14. The result will then follow from Theorem 2.

We will show that the \( S_2 \)-ification of \( R \) is local. To this end, note that \( R \) is a local complete domain, hence equidimensional; see Lemma 2.2. Set \( n = \dim A = \dim R \), where the equality is from the fact that \( R \) is finite over \( A \).

By way of contradiction, suppose the \( S_2 \)-ification of \( R \) is not local. Lemma 3.7 provides a prime ideal \( \tilde{P} = P/IR \subset \tilde{R} \) of height at least two such that \( \text{Spec}^\circ(\tilde{R}_P) \) is disconnected. Now \( R_P \) has a canonical module (since it is a homomorphic image of a regular local ring). Moreover, it is \( (S_2) \), therefore \( H_{\text{dim} R_P}^n R_P \) is indecomposable by [13, Theorem 3.7].

**Case 1.** If \( P \) is maximal in \( R \), then \( \tilde{R}_P = \tilde{R} \), and \( \mu(\text{IR}_P) \leq \mu(I) \leq n - 2 = \dim R - 2 = \dim R_P - 2 \), by condition (ii) of the hypotheses. Thus, \( H^n_{\text{dim} R_P} R_P \) is indecomposable. By [13, Theorem 3.3], \( \text{Spec}^\circ(\tilde{R}_P) = \text{Spec}^\circ(\tilde{R}) \) is connected, contradicting our choice of \( \tilde{P} \).

**Case 2.** If \( P \) is not maximal in \( R \), then \( p = P \cap A \) is not maximal in \( A \) since \( R \) is finite over \( A \). Furthermore, we have \( p \supseteq IR \cap A \supseteq I \) since \( P \supseteq IR \). Therefore \( I_p \) is generated by an \( A_p \)-regular sequence \( x = x_1, \ldots, x_c \in I_p \) by condition (iii) of the hypotheses.

We claim that \( x \) is part of a system of parameters for \( R_P \). (Then [13, Theorem 3.9(c)] implies that \( \text{Spec}^\circ(\tilde{R}_P) \) is connected, again contradicting our choice of \( \tilde{P} \).)

Let \( x' = x_1, \ldots, x_c, \ldots, x_d \in p_p \) be a system of parameters for \( A_p \). It follows that \( A_p/(x') \) has finite length. Since \( R_p \) is finitely generated over \( A_p \), it follows that \( R_p/(x') \) has finite length, that is, \( R_p/(x') \) is artinian. The ring \( R_p/(x') \) is a localization of \( R_p/(x') \), so it is also artinian. Thus, the fact that the sequence \( x' \in P_P \) has length \( d = \dim(A_p) = \dim(\tilde{R}_P) \) by Lemma 2.2 implies that \( x' \) is a system of parameters for \( R_P \), as claimed.

Thus, the \( S_2 \)-ification of \( R \) is local. The result now follows from Theorem 2. □

### 4. Corollaries and an Example

We begin this section by partially recovering [9, Theorem 1.2], as described in the discussion before Theorem 1 in the introduction.

**Corollary 4.1.** Let \((A, \mathfrak{m})\) be an excellent, local, normal domain and \( I \) a prime ideal of \( A \) generated by an \( A \)-regular sequence such that \( \bar{A} = A/I \) is normal. For any integer \( e > 1 \) which represents a unit in \( A \), the kernel of the homomorphism \( \text{Cl}(A) \to \text{Cl}(\bar{A}) \) contains no element of order \( e \).

**Proof.** **Case 1.** \( \dim(A/I) \leq 1 \). In this case, the fact that \( A/I \) is normal (hence, it satisfies \((R_1)\)) implies that \( A/I \) is regular. Since \( I \) is generated by an \( A \)-regular sequence, it follows from [3, Theorem 6.1(1)] that \( A \) is regular, so the result follows in this case.

**Case 2.** \( \dim(A/I) \geq 2 \). In this case, we have \( \mu(I) = \dim(A) - \dim(A/I) \leq \dim(A) - 2 \) since \( I \) is generated by an \( A \)-regular sequence. Thus, the desired conclusion follows from Theorem 2 in this case. □
The next two results follow from Theorem B as in [9].

Corollary 4.2. (See [9, Corollary 1.3]) Under the same assumptions as in Theorem B, with the additional hypothesis that \(A\) is \(Q\)-algebra, the kernel of the homomorphism \(\text{Cl}(A) \to \text{Cl}(\bar{A})\) is torsion free.

Corollary 4.3. (See [9, Corollary 1.4]) If the local hypothesis on \(A\) is removed in Theorem B, and instead it is assumed that \(\text{Pic}(A) = 0\) and \(I\) is a prime ideal of the Jacobson radical of \(A\) with finite projective dimension satisfying the three conditions of Theorem B, then for any integer \(e > 1\) representing a unit in \(A\), the kernel of the homomorphism \(\text{Cl}(A) \to \text{Cl}(\bar{A})\) contains no element of order \(e\).

We conclude with an example where the hypotheses of Theorem B are satisfied by an ideal \(I\) that is not generated by an \(A\)-regular sequence.

Example 4.4. Let \((A, \mathfrak{m})\) be an excellent normal local integral domain, and let the positive integer \(e\) represent a unit in \(A\). Assume that \(A\) has an \(A\)-regular sequence \(f_1, \ldots, f_6 \in \mathfrak{m}\) such that \(A/(f_1, \ldots, f_6)A\) is a normal domain. (Examples of such rings are constructed in [20, Example 3.4].) In particular, we have \(6 \leq \text{depth } A \leq \dim A\). Localize at a minimal prime of the ideal \(J = (f_1, \ldots, f_6)A\), if necessary, to assume that \(J\) is \(\mathfrak{m}\)-primary. It follows that \(A\) is Cohen-Macaulay of dimension 6.

Arrange the sequence \(f_1, \ldots, f_6\) in a \(2 \times 3\) matrix \(F = (f_1, f_2, f_3; f_4, f_5, f_6)\), and consider the ideal \(I = I_2(F)\) generated by the \(2 \times 2\) minors \(d_1 = f_1 f_5 - f_2 f_4, d_2 = f_2 f_6 - f_3 f_5, d_3 = f_1 f_6 - f_3 f_4\). We claim that \(A\) and \(I\) satisfy the hypotheses of Theorem B. The proof of [20, Proposition 3.1] shows that \(I\) is a height-2 prime ideal of \(A\) of finite projective dimension generated by 3 elements (so \(\mu(I) < \dim A - 2\)) such that \(A/I\) is a normal domain. Thus, we need only show that \(I\) is a complete intersection on the punctured spectrum. Let \(p \in \text{Spec}^0(A)\). We need to assume that \(I \subseteq p\) and show that \(IA_p\) is generated by an \(A_p\)-regular sequence.

Since \(J\) is \(\mathfrak{m}\)-primary and \(p \neq \mathfrak{m}\), we have \(J \not\subseteq p\), so some generator of \(J\) is not in \(p\). By symmetry, assume that the generator \(f_1\) is not in \(p\). It follows that \(f_1\) is a unit in \(A_p\). Thus, a routine computation shows that in \(A_p\) we have

\[
d_2 = f_2 f_6 - f_3 f_5 = \frac{f_2}{f_1} d_3 - \frac{f_3}{f_1} d_1
\]

so we conclude that \(IA_p = (d_1, d_3)A_p\). Since \(A_p\) is Cohen-Macaulay and \(\text{ht}(IA_p) = 2\), it follows that the sequence \(d_1, d_3\) is \(A_p\)-regular (e.g., by [16, Theorem 17.4]), as desired.

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