Theory of generating spaces of convex sets and their applications to solvability of convex programs in Banach spaces

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Abstract When optimization theorists consider optimization problems in infinite dimensional spaces, they need to deal with closed convex subsets (usually cones) which mostly have empty interior. These subsets often prevent optimization theorists from applying powerful techniques to study these optimization problems. In this paper, by nonsupport point, we present generating spaces which are relative to a Banach space and a nonsupport point of its convex closed subset. Then for optimization problems in infinite dimensional spaces, in some general cases, we replace original spaces by generating spaces while containing solutions. Thus this method enable us to apply powerful classical techniques to optimization problems in very general class of infinite dimensional spaces. Based on functional analysis, from classical Banach spaces to separable Banach spaces, from Banach lattice to latticization, we give characterizations of generating spaces and conclude that they are actually linearly isometric to $L_\infty(\ell_\infty)$ or their closed subspaces. Thus continuous linear functional involved in these techniques could be chosen from $L^*_\infty(\ell^*_\infty)$. After that, applications in Penalty principle, Lagrange duality and scalarization function are further studied by this method.

Keywords generating space · nonsupport point · isometric · optimization · infinite dimensional · latticization

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1 Introduction

In infinite dimensional Banach spaces, we always deal with cones. Solidness assumption of these target cones is essential in many cases, including duality theory, variational analysis and so on. A convex subset of a Banach space $X$ is said to be solid if it has nonempty interior. A challenge in these programs is that these cones are not solid in most cases. For example, if the space in question is an infinite dimensional $L_p(\Omega, \mu)$, or, $\ell_p$, then its natural positive cone $L^+_p(\Omega, \mu)$, or, $\ell^+_p$, has empty interior. Indeed, every Banach space admitting a reproducing cone with nonempty interior is "close" to a $C(K)$-space (see, Theorem 13). This means that if $X$ is a Banach space admitting a reproducing cone with nonempty interior, then there is a Banach space $C(K)$ for some compact Hausdorff space $K$ such that $X$ is isomorphic to a subspace $E$ of $C(K)$ satisfying that the "$C(K)$-lattice hull" of $E$ is dense in $C(K)$. In other words, the class of $C(K)$-spaces is almost the only class of Banach spaces admitting a reproducing cone with nonempty interior. Therefore, it has become a significantly important work to find an appropriate substitute for the interior of a reproducing cone in an infinite dimensional Banach space.

The Bishop-Phelps theorem states that for a nonempty closed convex set $C$ in a Banach space $X$, support functionals of $C$ are dense in the cone $C^* \subset X^*$ consisting of all functionals which are bounded above on $C$; and support points of $C$ are always dense in the boundary of $C$. (See, for instance, [46].) We denote by $C_S$, the set of all support points of $C$, and by $C_N = C \setminus C_S$, the set of all non-support points of $C$. Then by the separation theorem of convex sets it is easy to see that $C_N = \text{int} C$ (the interior of $C$) if the latter is nonempty. In 1974, R.R. Phelps [45] further studied topological properties of $C_S$ of $C$, and obtained that $C_S$ is always a $F_\sigma$ set of $C$, and its nonsupport point set $C_N$ is a dense $G_\delta$-subset of $C$ whenever $C_N \neq \emptyset$. Taking $C_N$ as a starting point, many mathematicians used this notion to study differentiability of real-valued convex, or locally Lipschitz functions defined on closed convex sets $C$ with $\text{int} C = \emptyset$ but $C_N \neq \emptyset$ and call such a set "small set" [57, 49, 58], to show some "embedding" theorems [17,21,16], to study "lattice" properties of induced by a cone $C$ with $C_N \neq \emptyset$ [52], and to characterize a cone in a Banach space with a base [20,13]. Coincidentally, [11,38,39] applied nonsupport point subset of a convex set (but they call it "quasi interior") to solvability discussion of convex program, vector optimization and to convex duality theory. For more information in this direction, we refer the reader to [60,13,9,10,34,22] and references therein.

From the facts mentioned above, for a closed convex set $C$ it seems that such a substitute ("$C_N$" for the interior $\text{int} C$) has already exist. Therefore, the question that how to solve the following new problems has moved toward into the central stage.

**Problem 1** Assume that $C$ is a closed convex subset of a Banach space $(X, \| \cdot \|)$ with empty interior but $C_N \neq \emptyset$.

I. How to produce a new Banach space $X_C$ so that
   i) $X_C$ is algebraically contained in $X$;
   ii) $C_0 = X_C \cap C$ is dense in $C$ with respect to the norm of $X$; and
iii) \( \text{int}_C C_0 \) (the interior of \( C_0 \)) is nonempty?

II. How to represent the new space \( X_C \)?

If such a space \( X_C \) exists, then we call it a “\( C \)-generating space”, or simply, a “generating space”.

By a cone \( C \) of a Banach space \( X \) we mean that it is a convex set satisfying \( C + C \subset C \). A cone \( C \) is said to be reproducing (resp., almost reproducing) if \( C - C = X \) (resp., \( C - C = X \), where \( \overline{A} \) denotes the norm closure of \( A \subset X \)). An ordering cone \( C \) of \( X \) is a reproducing or an almost reproducing cone containing no nontrivial subspaces of \( X \), or equivalently, satisfying \( C \cap (-C) = \{0\} \).

This paper is organized as follows. In the first part of this paper (Sections 2-8), we are devoted to solving I of Problem 1. As a result, we show that

1. for every nonempty closed convex \( C \) of a separable Banach space \( X \), the generating spaces \( X_C \) always exist (Theorem 3), and they are not unique in general;
2. if \( C \) is a closed reproducing cone of a separable Banach space \( X \), then two \( C \)-generating spaces \( X_{C,p} \) and \( X_{C,q} \) are isomorphic if and only if the two vectors \( p, q \) are equivalent with respect to the order induced by the “positive cone” \( C \) (Theorem 5);
3. if \( X \) has an unconditional basis, in particular, \( X = \ell_p \) (1 \( \leq \) \( p \) \( < \) \( \infty \)), or, a separable \( L_p \) (1 \( < \) \( p \) \( < \) \( \infty \)), and \( C \) is the positive cone with respect to the basis, then every generating space \( X_C \) is isometric to \( \ell_\infty \) (Theorem 7);
4. if \( X \) is a separable Banach lattice and \( C \) is the positive cone, then every generating space \( X_C \) is isometric to \( L_\infty \).

As applications of the results mentioned above, in the second part of this paper (Sections 9-11), we consider solvability of the following problems.

**Problem 2** Let \( X, Y \) be Banach spaces, \( Y \) be ordered by an ordering cone \( C \subset Y \) and \( f : X \to Y \) be a function. Assume that \( \Omega \subset X \) is a nonempty subset. Consider solvability of the following program.

\[
\min \{ f(x) : x \in \Omega \}.
\]

A vector \( \bar{x} \in \Omega \) is said to be a minimum point of the program (1) provided

\[
(f(\Omega) - f(\bar{x})) \cap (-C \setminus \{0\}) = \emptyset.
\]

If such a vector \( \bar{x} \) exists, then it is said to be a classical solution of the program (1). In order to obtain a classical solution of (1), one usually assume that the ordering cone \( C \) has nonempty interior, that is, \( \text{int} C \neq \emptyset \). However, for an infinite dimensional Banach space \( Y \), it is often impossible to claim that the assumption is true unless that \( Y \) is a \( C(K) \)-like space. On the other hand, the assumption \( \text{int} C \neq \emptyset \) is very important. For example, when \( \text{int} C = \emptyset \), any scalarization function of the program (1) will deny continuity, while just the continuity guarantees that the Moreau-Rockafellar Theorem, i.e., Lemma 2 can be effectively used to solvability discussion of the program (1).
These facts have forced optimization theorists to consider a kind of “weak solution” \( \bar{x} \in \Omega \) of the program (1), namely, generalized approximate solutions. That is, they substitute a subset \( A \) of \( Y \) containing the ordering cone \( C \) but with \( \text{int} A \neq \emptyset \) for \( C \) so that

\[
(f(\Omega) - f(\bar{x})) \cap (-A \setminus \{0\}) = \emptyset. \tag{3}
\]

See, for example, [27, 4, 56, 37]. Nevertheless, a solution \( \bar{x} \) of (3) is often not the classical one which optimization theorists are most concerned.

In Section 9, we will apply results established in Sections 2-4 to generalize the exact penalty principle due to J.J. Ye [59] to all separable Banach spaces. In Section 11, with the help of the results presented in Sections 2-8 and 10, we will overcome the discontinuity difficulty of scalarization functions of the program (1) in box constraints without any additional assumptions.

**Problem 3 (Lagrange model with box constraint)**

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g(x) \in -Y^+, \\
& \quad h(x) = 0_Z, \\
& \quad x \in \Omega = \{x \in X : x_a \leq x \leq x_b\},
\end{align*} \tag{4}
\]

where \( X, Y, Z \) are Banach spaces and \( X, Y \) are ordered by their corresponding ordering cones \( X^+, Y^+ \), and \( f : X \to \mathbb{R} \) is a continuous convex function, \( g : X \to Y \) is continuous and convex-like with respect to \( Y^+ \) and \( h : X \to Z \) is a continuous affine function. \( \Omega = \{x \in X : x_a \leq x \leq x_b\} \) is called “box constraint”. We should mention that box constraint is a very common condition in applications, and that the program (4) is a very general model which contains many concrete models as its special cases. See, for instance, [14, 23, 25, 26, 27, 42, 50] and references therein.

In applications, one often considers the following Lagrange duality program.

**Problem 4 (Lagrange duality model)**

\[
\begin{align*}
\max \quad & \inf_{(y^*, z^*, x_1^*, x_2^*) \in X} \left( f(x) + \langle (y^*, x_1^*, x_2^*), (g(x), x - x_a, x_b - x) \rangle + \langle z^*, h(x) \rangle \right) \\
\text{s.t.} & \quad x \in X, \ y^* \in Y^{**}, \ x_1^* \in X^{**}, \ x_2^* \in X^{**}, \ z^* \in Z^*.
\end{align*} \tag{5}
\]

If extremal values of the programs (4) and (5) are equal, then we say the duality between the two programs holds.

Slater’s condition is an important tool in obtaining the duality between (4) and (5). However, the duality may not hold in the case that \( \text{int}(Y^+) = \emptyset \), because Slater’s condition requires \( \text{int}(Y^+) \neq \emptyset \). It is worth to mention that, Daniele, Giuffrè, and Idone [23] in their remarkable work gave a condition called assumption (S) which assure the Lagrange duality by substituting the nonsupport point set \( (Y^+)_{\text{n}} \) of \( Y^+ \) for the interior \( \text{int}(Y^+) \). Taking that as a starting point, a number of mathematicians presented some reasonable conditions and assumptions in various optimization problems (see, for instance, [25, 26, 7, 42, 43] and references therein). It is worth to mention that in a few special cases, some applications are successful. (See, for example, Donato [26]).
Nevertheless, the mentioned conditions and assumptions are usually hard to verify in practice, although some of these works contain great theoretical elegance [9, 60, 41, 60].

To overcome the difficulty that \( \text{int}(Y^+) = \emptyset \) in Slater’s condition and Lagrange duality theory, in Section 10, we will use the results presented in Sections 2-8 to transform program (4) with box constraints to another equivalent program which Slater’s condition can be applied directly. In Section 11, we consider the convex vector optimization program (1). As a result, we present some subdifferential inclusion theorems of Gerstewitz scalarization functions (Theorems 17 and 18); and as their application, solvability of the vector variational inequality problem (8) is discussed.

In this paper, unless stated explicitly otherwise, we always assume that \( X \) is a Banach space, and \( X^* \) its dual. For a subset \( A \subset X \), we denote successively, by \( \overline{A} \), \( \text{co}A \) and \( \text{co}A \), the (norm) closure, the convex hull and the closed convex hull of \( A \).

2 Preliminaries

In this section, we will recall some concepts and basic properties related to the subset of nonsupport points of a closed convex set in a Banach space \( X \).

**Definition 1** Let \( C \) be a closed convex set of a Banach space \( X \).

i) A point \( x \in C \) is said to be a support point of \( C \) provided there exists a non-zero functional \( x^* \in X^* \) such that

\[
\langle x^*, x \rangle = \max \{ \langle x^*, y \rangle : y \in C \}.
\]

In this case, \( x^* \) is called a support functional of \( C \), which is supporting \( C \) at \( x \). We denote by \( C_S \) the set of all support points of \( C \).

ii) A point of the complement \( C_N \equiv C \setminus C_S \) is called a non-support point of \( C \), and \( C_N \) is said to be the quasi-interior of \( C \).

iii) We say that a point \( x \) of \( C \) is a proper support point if there is a functional \( x^* \in X^* \) such that

\[
\max \{ \langle x^*, y \rangle : y \in C \} = \langle x^*, x \rangle > \inf \{ \langle x^*, y \rangle : y \in C \}.
\]

We denote by \( C_{PS} \) the set of all proper support points of \( C \).

iv) A point \( x \) of \( C \) is said to be a non-proper support point if it is not a proper support point of \( C \). We use \( C_{NP} \) to denote the set of all non-proper support points of \( C \). Clearly,

\[
C_{NP} = C \setminus C_{PS}.
\]

**Definition 2** Let \( C \) be a convex set of a Banach space \( X \).

i) \( C \) is said to be a cone with vertex at the origin 0 provided it is closed under the operations of addition and positive scalar multiplication, i.e.,

\[
x, y \in C, \lambda \geq 0 \text{ imply } x + y \in C \text{ and } \lambda x \in C.
\]
We say that $C$ is a cone with vertex at $x_0 \in X$ if $C - x_0$ is a cone with vertex at the origin 0. When the vertex is origin 0, we also say $C$ is pointed.

A cone $C$ is called a reproducing (resp., an almost reproducing) cone if $C - C = X$ (resp., $C - C = X$).

The next property is easy to observe.

**Proposition 1** Let $C$ be a closed cone of a Banach space $X$ with $C_N \neq \emptyset$. Then $C$ is almost reproducing. But the converse version is not true. (See, Examples 2 and 3.)

**Example 1**

i) Let $(\Omega, \Sigma, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then the positive cone $L_p^+(\mu)$ of $L_p(\mu)$ is reproducing. $(L_p^+(\mu))_N \neq \emptyset$ if and only if $(\Omega, \Sigma, \mu)$ is $\sigma$-finite.

ii) For any nonempty set $\Gamma$ and $1 \leq p \leq \infty$, the positive cone $\ell_p^+(\Gamma)$ of $\ell_p(\Gamma)$ is reproducing.

iii) For every topological space $K$, the positive cone $C^+(K)$ of the real-valued bounded continuous function space $C(K)$ endowed with the sup-norm is reproducing.

iv) For any nonempty set $\Gamma$, the positive cone $c_0^+(\Gamma)$ of $c_0(\Gamma)$ is reproducing.

For a closed convex $C \subset X$, and $x \in C$, let $C_x$ be the cone generated by $C$ with vertex 0 which is defined by

$$C_x = \bigcup_{\lambda > 0} \lambda(C - x).$$ (6)

Each property in the following lemma is either easy to observe, or, to be found in Phelps [45], Cheng and Dong [19], and Holmes [35].

**Lemma 1** Suppose that $C$ is a nonempty closed convex set in a Banach space $X$. Then

(i) $C_N$ is a convex subset of $C$ (maybe empty);

(ii) [45] Ex. 2.18 if $X$ is separable, then $C_N = \emptyset$ if and only if $C$ is contained in a closed hyperplane;

(iii) $C_\emptyset = \text{int} C$ if the latter is nonempty;

(iv) [45] if $C_\emptyset \neq \emptyset$, then $C_N$ is a dense $G_\delta$-subset of $C$, hence, Baire’s category;

(vi) [19] Prop. 2.2 if $C$ is separable, then $C_{NP} \neq \emptyset$.

Now, we give several examples related to the nonsupport point sets of the natural positive cones of some classical Banach spaces as follows.

**Example 2** Let $(\Omega, \Sigma, \mu)$ be a measure space, and $L_p^+(\mu)$ be the positive cone of $L_p(\mu)$ for $1 \leq p \leq \infty$.

i) $(L_p^+(\mu))_N = \{f \in L_p(\mu) : f(\omega) > 0 \text{ for a.e. } \omega \in \Omega\} = \text{int}(L_p^+(\mu))$;

ii) $(L_p^+(\mu))_N = \text{int}(L_p^+(\mu)) \neq \emptyset (1 \leq p < \infty)$ if and only if $L_p(\mu)$ is finite dimensional;

iii) $(L_p^+(\mu))_N = \{f \in L_p(\mu) : f(\omega) > 0 \text{ for a.e. } \omega \in \Omega\} \neq \emptyset (1 \leq p < \infty)$ if and only if $(\Omega, \Sigma, \mu)$ is $\sigma$-finite.
Example 3 Let $\Gamma$ be a nonempty set and $\ell_p^+(\Gamma)$ (resp., $c_0^+(\Gamma)$) be the positive cone of $\ell_p(\Gamma)$ for $1 \leq p \leq \infty$ (resp., $c_0(\Gamma)$). Then

i) $(\ell_p^+(\Gamma))_N = \{x \in \ell_\infty : x(\gamma) > 0 \text{ for all } \gamma \in \Gamma\}$

ii) $(\ell_p^+(\Gamma))_N = \text{int}(\ell_p^+(\Gamma))$ if and only if $\Gamma$ is a finite set;

iii) $(\ell_p^+(\Gamma))_N = \{x \in \ell_p(\Gamma) : x(\omega) > 0 \text{ for all } \gamma \in \Gamma, \gamma \neq \omega\}$ if and only if $\Gamma$ is countable.

iv) $(C_0^+(\Gamma))_N = \{x \in c_0(\Gamma) : x(\gamma) = 0 \text{ for all } \gamma \in \Gamma\}$ if and only if $\Gamma$ is a finite set;

v) $(c_0^+(\Gamma))_N = \{x \in c_0(\Gamma) : x(\gamma) > 0 \text{ for all } \gamma \in \Gamma\}$ if and only if $\Gamma$ is countable.

Example 4 Let $K$ be a compact Hausdorff space, and $C^+(K)$ be the positive cone of $C(K)$. Then

$$(C^+(K))_N = \{f \in C(K) : f(k) > 0 \text{ for all } k \in K\} = \text{int}(C^+(K)).$$

Definition 3 Assume that $C \subset X^*$ is a cone with vertex at the origin $0$. Then

i) $x^* \in X^*$ is said to be a positive functional with respect to $C$ provided $\langle x^*, x \rangle \geq 0$ for all $x \in C$. Without causing confusion, we call it a positive functional for short.

ii) A functional $x^* \in X^*$ is strictly positive if $\langle x^*, x \rangle > 0$ for all $x \in C \setminus \{0\}$.

iii) We denote by $C^{*+}$ the cone of all positive functionals (with respect to $C$).

Definition 4 Let $C \subset X$ be a convex set with $0 \in C$. Then the Minkowski functional $p : X \to \mathbb{R}^+ \cup \{+\infty\}$ generated by $C$ is defined for $x \in X$ by

$$p(x) = \inf\{\lambda > 0 : \lambda^{-1}x \in C\}.$$  

The following property easily follows.

Proposition 2 Let $C \subset X$ be a convex set with $0 \in C$, and $p : X \to \mathbb{R}^+ \cup \{+\infty\}$ be the Minkowski functional generated by $C$. Then

i) $p$ is a nonnegative extended-real-valued sublinear function on $X$;

ii) $p$ is lower semicontinuous if and only if $C$ is closed in $X$; or, equivalently, $C^* = \{x \in X : p(x) \leq 1\}$ is closed;

iii) $p$ is continuous if and only if $0 \in \text{int}C$.

Following theorem is from [20, Lemma 4.1].

Theorem 1 Suppose that $C$ is a closed almost reproducing cone with vertex at the origin of a Banach space $X$. Then $x^* \in X^*$ is a strictly positive functional if and only if $x^*$ is a non-support point of $C^{*+}$.

At the end of this section, we recall some definitions and lemmas in convex analysis.

Definition 5 Let $X$ be a Banach space, $f : X \to \mathbb{R}$ be a convex function. Suppose $f$ is lower semicontinuous at $\bar{x} \in \text{dom}f$. Then the subdifferential of $f$ at $\bar{x}$, denoted by $\partial f(\bar{x})$ is defined as

$$\partial f(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x})\}.$$
The normal cone of a convex set \( \Omega \subset X \) at \( x \in \Omega \) is given by
\[
N(x, \Omega) = \{x^* \in X^* : \langle x^*, y-x \rangle \leq 0, \forall y \in \Omega \}.
\]
If \( x \notin \Omega \), we put \( N(x, \Omega) = \emptyset \). Then we see
\[
\partial f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N((\bar{x}, f(\bar{x})), \text{epi} f)\},
\]
Recall \( \delta(x, \Omega) = 0 \) if \( x \in \Omega \); \( \delta(x, \Omega) = \infty \), otherwise. We have
\[
\partial \delta(x, \Omega) = N(x, \Omega).
\]
Next we introduce the Moreau-Rockafellar Theorem.

**Theorem 2** Suppose that \( f, g : X \to \overline{\mathbb{R}} \) are convex proper lower semicontinuous functions on Banach space \( X \) and that there is a point \( x \in \text{dom} f \cap \text{dom} g \) where \( f \) is continuous at \( x \). Then
\[
\partial(f + g)(x) = \partial f(x) + \partial g(x), \ x \in \text{dom}(f + g).
\]

**Definition 6** Let \( X, Y \) be Banach spaces, \( Y \) be ordered by \( Y^+ \). Suppose \( f : X \to Y \), \( f \) is said to be convex-like with respect to \( Y^+ \) in \( \Omega \subset X \) if the set \( \{ f(x) + y : y \in Y^+, x \in \Omega \} \) is convex.
\( f \) is said to be convex respect to \( Y^+ \) on \( \Omega \) if
\[
\lambda f(x) + (1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y)
\]
(\( \leq \) is in \( Y^+ \) sense) holds for any \( x, y \in \Omega \), \( 0 \leq \lambda \leq 1 \).

3 **\( C \)-generating spaces**

**Definition 7** Assume that \( C \) is a convex subset of a real Banach space \( X \).

i) We denote by \( [C] \) the closure of \( \text{span} C \) in \( X \), and by \( C_{\overline{N}(C)} \) the set of all nonsupport points of \( C \) with respect to \( [C] \).

ii) For any fixed \( x \in C \), we say that the subspace \( C_x \cap (-C_x) \) is the \( (C, x) \)-generating space. If it causes no confusion, we call it a \( C \)-generating space for short, where \( C_x = \bigcup_{\lambda > 0} \lambda (C - x) \).

**Proposition 3** Assume that \( C \) is a nonempty closed convex set of a finite dimensional normed space \( X \). Then
\[
C_N = \text{int} C.
\]

**Proof** Clearly, \( C_N = \text{int} C \) if \( \text{int} C \neq \emptyset \). Assume \( \dim X = n \in \mathbb{N} \). Suppose, to the contrary, that \( \text{int} C = \emptyset \), and that there is \( x \in C_N \). Then by Lemma 1 iv),
\[
C_x = \bigcup_{\lambda > 0} \lambda (C - x) = \bigcup_{n=1}^{\infty} n(C - x)
\]
is a dense convex set in \( X \). Therefore, \( C_x \) contains \( n \) linearly independent vectors \( x_1, x_2, \cdots, x_n \) so that \( \pm x_1, \pm x_2, \cdots, \pm x_n \in C_x \). Since \( \text{co}\{\pm x_1, \pm x_2, \cdots, \pm x_n\} \) is a symmetric convex body containing the origin in its interior, \( C_x = X \). Since each \( n(C - x) \) is closed in \( X \), it follows from completeness of \( X \) and Baire’s category theorem that \( \text{int}[n(C - x)] \neq \emptyset \). Consequently, \( \text{int} C \neq \emptyset \), and this is a contradiction. \( \square \)
Lemma 2 Suppose that \( C \subseteq X \) is a nonempty convex set. Then
\[
C_{\text{NP}} = C_{N,C}.
\]

Proof By definition of \( X_{e} \) and Lemma [iv], \( x \in C_{N,[C]} \) if and only if \( C_{e} = \bigcup_{\lambda > 0} \lambda (C - x) \) is dense in \( [C] \). If \( x \not\in C_{\text{NP}} \), then \( x \in C_{PS} \). Therefore, \( x \in C_{PS} \cap [C_{N,[C]}] = \emptyset \), which is a contradiction.

Lemma 3 Suppose \( C \) is a closed convex set of a Banach space \( X \), and that \( x \in C_{NP} \). Then the \( (C,x) \)-generating space satisfies
\[
C_{e} \cap (-C_{e}) = \bigcup_{\lambda > 0} \lambda (\{(x - C) \cap (-C + x)\}),
\]
and \( C_{e} \cap (-C_{e}) \) is a dense subspace of \([C]\).

Proof Without loss of generality, we assume that \([C] = X\). We first show (9). Clearly,
\[
C_{e} \cap (-C_{e}) = \bigcup_{\lambda > 0} \lambda (\{(x - C) \cap (-C + x)\}) \supset \bigcup_{\lambda > 0} \lambda (\{(x - C) \cap (-C + x)\}).
\]
On the other hand, note that
\[
0 \neq z \in C_{e} \cap (-C_{e}) \iff \exists \lambda_{1}, \lambda_{2} > 0, c_{1}, c_{2} \in C \text{ so that } \lambda_{1}(c_{1} - x) = \lambda_{2}(c_{2} + x).
\]
Equivalently, \( z \) is absorbed by both \( C - x \) and \( x - C \), which is equivalent to that \( z \) is absorbed by \( (C - x) \cap (x - C) \). Consequently, \( z \in \bigcup_{\lambda > 0} \lambda (\{(C - x) \cap (-C + x)\}) \). Therefore, (9) holds.

To show that \( C_{e} \cap (-C_{e}) \) is a dense subspace of \( X \), it suffices to prove that \( C_{e} \cap (-C_{e}) \) is dense. Otherwise, it is contained in a closed hyperplane containing the origin. Let \( 0 \neq x^{\ast} \in X^{\ast} \) be such that
\[
C_{e} \cap (-C_{e}) \subset H(x^{\ast};0) \equiv \{z \in X : \langle x^{\ast}, z \rangle = 0 \}.
\]
Thus, for each \( z \in C_{e} \) with \( \langle x^{\ast}, z \rangle > 0 \), \( -z \notin C_{e} \). Consequently,
\[
\max\{\langle -x^{\ast}, z \rangle : z \in C_{e}\} = -\min\{\langle x^{\ast}, z \rangle : z \in C_{e}\} = 0 = \langle -x^{\ast}, 0 \rangle,
\]
which says that \( 0 \) is a support point of \( C_{e} \). This is a contradiction.

Theorem 3 Suppose that \( C \subseteq X \) is a closed bounded convex set, and \( e \in C_{N} \). Let \( X_{e} \) be the \( (C,e) \)-generating space \( C_{e} \cap (-C_{e}) \) endowed with the norm \( \| \cdot \|_{e} \) defined for \( x \in X_{e} \) as
\[
\|x\|_{e} = \inf\{\lambda > 0 : \lambda^{-1}x \in (C - e) \cap (e - C)\}.
\]
Then
i) Regarding as a subspace of \( X \), \( X_{e} \) is dense in \( X \);
ii) The new norm \( \| \cdot \|_{e} \) topology on \( X_{e} \) is stronger than the original norm \( \| \cdot \| \) topology on \( X_{e} \);
iii) \( X_{e} = (X_{e}, \| \cdot \|_{e}) \) is a Banach space;
iv) \( (X_{e}, \| \cdot \|_{e}) \) is isomorphic to \( (X_{e}, \| \cdot \|) \) if and only if \( e \in \text{int}C \).
Proof
i) This is just Lemma 3.

ii) & iii) Note that closed unit ball \((C - e) \cap (e - C)\) of \((X_e, \| \cdot \|_e)\) is \(\| \cdot \|\)-closed in \(X\). Then it is necessarily \(\| \cdot \|\)-complete. Since \((C - e) \cap (e - C)\) is bounded symmetric convex absorbing set of \(X_e\), and since \(\| \cdot \|_e\) is the Minkowski functional generated by \((C - e) \cap (e - C)\), the new norm \(\| \cdot \|_e\)-topology is not weaker than the original norm topology on \(X_e\). Therefore, ii) is shown. Completeness of \((C - e) \cap (e - C), \| \cdot \|\) entails that \((C - e) \cap (e - C), \| \cdot \|_e\) is complete. Consequently, \(X_e = (X_e, \| \cdot \|_e)\) is a Banach space. Hence, iii) is true.

iv) By ii), \(\| \cdot \|_e\)-topology is stronger than \(\| \cdot \|\)-topology on \(X_e\). This and iii) imply that \((X_e, \| \cdot \|_e)\) is isomorphic to \((X_e, \| \cdot \|)\) if and only if \((X_e, \| \cdot \|)\) is also a Banach space.

\(\Box\)

Theorem 4 Let \(C\) be a closed bounded convex set of a Banach space \(X\) containing at least two points. If \(C\) is separable, then

i) \(C_N[\mathcal{C}] \neq \emptyset\);

ii) for every \(e \in C_N[\mathcal{C}]\), the \((C, e)\)-generating space \((X_e, \| \cdot \|_e)\) is linearly isometric to a closed subspace of \(\ell_\infty\).

Proof
i) This is just Lemma [3].

ii) Without loss of generality, we assume \([\mathcal{C}] = X\). Therefore, \(X\) is separable and every subset \(A\) of \(X^*\) is \(w^*\)-separable, i.e., \(A\) is separable in the weak-star topology \(w^*\) of \(X\). By Theorem [3] \(\| \cdot \|_e\) is stronger than \(\| \cdot \|\) on \(X_e\) and \(X_e\) is \(\| \cdot \|\)-dense in \(X\).

Therefore, \(X^* \subseteq X_e^*\).

Since \((C - e) \cap (e - C)\) is a closed bounded convex subset of \(X\), \(f \equiv \| \cdot \|_e\) acting as an extended real-valued Minkowski functional generated by \((C - e) \cap (e - C)\) defined on \(X\) is lower semicontinuous with its effective domain

\[
\text{dom}(f) = X_e = \bigcup_{n=1}^\infty n((C - e) \cap (e - C)).
\]

Note that

\[
x^* \in \partial f(x) \iff \langle x^*, z \rangle \leq f(z) \text{ for all } z \in X \text{ with } \langle x^*, x \rangle = \|x\|_e,
\]

and that

\[
x^* \in \partial \|x\|_e \iff \langle x^*, z \rangle \leq f(z) \text{ for all } z \in X_e \text{ with } \langle x^*, x \rangle = \|x\|_e.
\]

Then we see

\[
\partial f(x) \subseteq \partial \|x\|_e \subseteq S_{X_e^*}, \text{ for all } x \in X_e \setminus \{0\}, \tag{10}
\]

where \(S_{X_e^*}\) is the unit sphere of \(X_e^*\).

We denote by \(D = \bigcup_{x \in X_e \setminus \{0\}} \partial f(x) \subseteq X^*\), the range of the subdifferential mapping \(\partial f\) of \(F\). Then by the Brondsted-Rockafellar theorem [46, Theorem 3.17], we obtain

\[
f(x) = \sup_{x^* \in D} \langle x^*, x \rangle, \text{ for all } x \in X.
\]
Note the restriction of $f$ to $X_c$ is just $\| \cdot \|_c$, $w^*$-separability of $D$ entails that there is a $w^*$-dense sequence $\{x^*_n\}$ of $D$ so that

$$f(x) = \sup_{n \in \mathbb{N}} \langle x^*_n, x \rangle = \|x\|_c, \text{ for all } x \in X_c.$$ 

Finally, we define $T : X_c \to \ell_\infty$ by

$$Tx = (\langle x^*_1, x \rangle, \langle x^*_2, x \rangle, \cdots, \langle x^*_n, x \rangle, \cdots), \quad x \in X_c.$$ 

Clearly, $T : X_c \to \ell_\infty$ is a linear isometry. \hfill $\square$

Let $P$ be a reproducing cone containing no nontrivial subspaces with vertex at the origin of a real Banach space $X$. Then there is an order on $X$ induced by $P$:

$$x \geq y \iff x - y \in P, \quad \forall x, y \in X. \quad (11)$$

If, in addition, $P$ satisfies that $P \cap -P = \{0\}$, then

$$x \in X, \exists y \in P \text{ such that } -ty \leq x \leq ty, \forall t > 0 \implies x = 0. \quad (12)$$

The following theorem states that in particular, if $C$ is an almost reproducing cone with vertex at the origin containing no nontrivial subspaces, then the generating space can be induced the “cone order”.

**Theorem 5** Suppose that $C \subset X$ is a closed almost reproducing cone with vertex at the origin containing no nontrivial subspaces of $X$, $X$ is ordered by $C$ defined as (11), and that $C_X \neq \emptyset$. Let $u \in C_N$, $X_u = C_u \cap (-C_u)$, and $\| \cdot \|_u$ is defined for $x \in X_u$ by

$$\|x\|_u = \inf \{ \lambda > 0 : -\lambda u \leq x \leq \lambda u \} \equiv \inf \{ \lambda > 0 : x \in \lambda [-u, u] \}. \quad (13)$$

Then

i) Regarding as a subspace of $X$, $X_u$ is dense in $X$;

ii) $\| \cdot \|_u$ is a norm on $X_u$, and the norm topology generated by it is stronger than the topology generated by $\| \cdot \|_c$ on $X_u$;

iii) The closed unit ball $[-u, u]$ of $(X_u, \| \cdot \|_u)$ is just $(C - u) \cap (u - C)$;

iv) The new norm $\| \cdot \|_u$ is a lower semicontinuous function in $X$ with its essential domain $\text{dom} \| \cdot \|_u = X_u$;

v) $(X_u, \| \cdot \|_u)$ is a Banach space.

**Proof** i) It follows immediately from Lemma 5  

ii) We first show that $[-u, u]$ is an absorbing set of $X_u$. Given $x \in X_u$, by (9) of Lemma 5 there exist $\lambda > 0$, $c_j \in C$, $j = 1, 2$ such that

$$-\lambda u \leq \lambda (c_1 - u) = x = \lambda (u - c_2) \leq \lambda u.$$  

Hence, $x \in \lambda [-u, u]$, and this says that $[-u, u]$ is an absorbing set of $X_u$. By (10), it is easy to observe that $[-u, u]$ is bounded closed in $X$. Since $[-u, u]$ is convex and symmetric, $\| \cdot \|_u$ is a norm on $X_u$, and the new norm topology is stronger than the topology generated by $\| \cdot \|_c$ on $X_u$. Therefore, ii) has been shown.
iii) It suffices to note that
\[ x \in (C - u) \cap (u - C) \iff \exists c_1, c_2 \in C \text{ such that } c_1 - u = x = u - c_2 \]
\[ \iff -u \leq x \leq u \iff x \in [-u, u]. \]

iv) Note \( \bigcup_{\lambda \geq 0} \lambda [-u, u] = X_v. \) It follows from that \([-u, u]\) is a closed bounded symmetrically convex set and that \( \| \cdot \|_u \) is just the Minkowski functional generated by \([-u, u]\).

v) It follows from that \([u, u]\) is complete in \((X, \| \cdot \|)\) and ii). \( \square \)

**Theorem 6 (Equivalence theorem)** Suppose that \( C \subset X \) is a closed reproducing cone containing no nontrivial affine subspaces of \( X \), \( X \) is ordered by \( C \) defined as \( \{ \lambda u : u \in X, \lambda > 0 \} \), and that \( C_X \neq \emptyset. \) Let \( u, v \in C_X, X_u = C_u \cap (-C_u), X_v = C_v \cap (-C_v), \| \cdot \|_u \) and \( \| \cdot \|_v \) are defined by

\[ \| x \|_u = \inf \{ \lambda > 0 : x \in \lambda [-u, u] \}, \quad x \in X_u, \]

and

\[ \| x \|_v = \inf \{ \lambda > 0 : x \in \lambda [-v, v] \}, \quad x \in X_v. \]

Then the following statements are equivalent.

i) \( X_u = X_v \) algebraically;

ii) \( u \sim v \), i.e., there is a constant \( c \geq 1 \) such that \( c^{-1} u \leq v \leq c u; \)

iii) \( \| \cdot \|_u \sim \| \cdot \|_v \), i.e., there is a constant \( c \geq 1 \) such that

\[ c^{-1} \| \cdot \|_u \leq \| \cdot \|_v \leq c \| \cdot \|_u. \]

**Proof** i) \( \implies \) iii). Suppose that \( X_u = X_v. \) By Theorem 5, the closed unit ball \([-u, u]\) of \((X_u, \| \cdot \|_u)\) is \( \| \cdot \|_u \)-complete. Indeed, since \([-u, u]\) is \( \| \cdot \|_u \)-complete, and since the norm \( \| \cdot \|_u \)-topology is stronger than the original norm \( \| \cdot \|_v \)-topology on \( X_v (= X_u) \), \([-u, u]\) is necessarily \( \| \cdot \|_v \)-complete. Note that \( X_v = \bigcup \lambda [-u, u] \) endowed with \( \| \cdot \|_v \) is a Banach space. Then by Baire’s category theorem, \( 0 \in \text{int}_{\| \cdot \|_v}[-u, u] \). Therefore, there is a constant \( a > 0 \) such that \([-v, v] \subset a[-u, u] \), or, equivalently, \( \| \cdot \|_v \geq a^{-1} \| \cdot \|_u \).

We can show that there is a constant \( b > 0 \) such that \( \| \cdot \|_u \geq b^{-1} \| \cdot \|_v \) in the same way. Thus, we finish the proof of “i) \( \implies \) iii)” by taking \( c = \max \{ a, b, 1 \} \).

iii) \( \implies \) i). It follows directly from

\[ c^{-1} \| \cdot \|_u \leq \| \cdot \|_v \leq c \| \cdot \|_u \]

that \( X_u = X_v. \)

ii) \( \iff \) iii). It suffices to note that ii) \( c^{-1} u \leq v \leq c u \) if and only if

\[ c^{-1} [-u, u] \subset [-v, v] \subset c [-u, u], \]

which is equivalent to iii)

\[ c^{-1} \| \cdot \|_u \leq \| \cdot \|_v \leq c \| \cdot \|_u. \]
4 Examples of C-generating spaces

For a closed convex set $C$ of a Banach space $X$ with $C_N \neq \emptyset$, and for each $e \in C_N$, by Theorem 3, $(X_e, \| \cdot \|_e)$ is a Banach space but $(X_e, \| \cdot \|)$ is a dense subspace of $X$. This means that $X_e$ is algebraically smaller. The following examples will show that, usually, $(X_e, \| \cdot \|_e)$ is bigger topologically, unless $\text{int}C = \emptyset$.

Example 5 Let $(\Omega, \Sigma, \mu)$ be a probability space, i.e., $\mu(\Omega) = 1$. In particular, $\Omega = [0,1]$, $\Sigma$ is the Borel $\sigma$-algebra of $[0,1]$, and $\mu$ is the Lebesgue measure. Given $1 \leq p < \infty$, we consider $X = L_p(\mu)$. Then it is easy to see that the constant function $u = 1$ a nonsupport point of the positive cone $L^+_p(\mu)$. Since

$$[-u,u] = \{ f \in L_p(\mu) : -1 \leq f(\omega) \leq 1 \text{ for almost all } \omega \in \Omega \},$$

Therefore, $X_u = L_{\infty}(\mu)$. If $p = \infty$, then $u = 1 \in \text{int}(L^+_\infty(\mu))$ is just the “generating unit element” of $L_{\infty}(\mu)$. Therefore, $X_u = X$.

Example 6 We consider $X = \ell_1$. Let $\{e_n\}$ be the standard unit vector basis of $\ell_1$ and $u = (\frac{1}{2^j})_{j=1}^{\infty}$. Then $X_u \cong \ell_\infty$. Indeed, it is clear that $u \in (\ell^+_1)_N$ with $\|u\| = 1$ and

$$[-u,u] = \{ x \in \ell_1 : -\frac{1}{2^n} \leq x(n) \leq \frac{1}{2^n} \text{ for all } n \in \mathbb{N} \}.$$

The set $\text{ext}[-u,u]$ of all extreme points of $[-u,u]$ satisfies

$$\text{ext}[-u,u] = \{ (\pm 2^{-1}, \pm 2^{-2}, \ldots, \pm 2^{-n}, \ldots) \}.$$

Therefore,

$$\|(\pm 2^{-1}, \pm 2^{-2}, \ldots, \pm 2^{-n}, \ldots)\|_u = 1.$$

It also follows that $\| \cdot \|_u$ is monotone non-decreasing, i.e., for every pair of sequences $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty} \subset \mathbb{R}$

$$|a_j| \geq |b_j|, j = 1, 2, \ldots, n \implies \| \sum_{j=1}^{n} a_j e_j \|_u \geq \| \sum_{j=1}^{n} b_j e_j \|_u,$$

and

$$\|e_n\|_u = 2^n, n = 1, 2, \ldots.$$

We defined a linear operator $T : X_u \to \ell_\infty$ for $x = (x(n))_{n=1}^{\infty} \in X_u$ by

$$T(x) = (2x(1), 2^2x(2), \ldots, 2^nx(n), \ldots).$$

Then

$$T(\pm 2^{-1}, \pm 2^{-2}, \ldots, \pm 2^{-n}, \ldots) = (\pm 1, \pm 1, \ldots, \pm 1, \ldots).$$

Therefore,

$$\text{ext}[-u,u] = \text{ext}B_\ell_\infty.$$

Consequently,

$$T(B_{X_u}) = B_{\ell_\infty},$$

This says that $T : X_u \to \ell_\infty$ is a surjective linear isometry.
5 A congruence theorem

In this section, we focus on generating spaces of Banach spaces with unconditional bases.

Lemma 4 Let $X$ be a Banach space with a normalized 1-unconditional basis $\{e_n\}$, and $X^+$ be the positive cone of $X$ related to the basis, i.e.,

$$X^+ = \{ x = \sum_n a_n e_n \in X : a_n \geq 0 \text{ for all } n \in \mathbb{N} \}.$$ 

Then

i) $(X^+)_N = \{ x = \sum_n a_n e_n \in X : a_n > 0 \text{ for all } n \in \mathbb{N} \}$.

ii) For all $u \in (X^+)_N$, $(X_u, \| \cdot \|_u)$ is isometrically isomorphic to $\ell_\infty$.

Proof We write

$$X_{00} = \{ x = (x(n)) \in X, \text{ supp} x = \{ n \in \mathbb{N}, x(n) \neq 0 \} \} \text{ is a finite set}.$$ 

Then it is a dense subspace of $X$.

i) Suppose that $u = \sum_n a_n e_n \in X$ with $a_n > 0$ for all $n \in \mathbb{N}$. Then for any $x \in X_{00}$, let $m = \max \{ n \in \text{supp} x \}$, and let $\alpha = \min \{ a_n : n = 1, 2, \ldots, m \}$. Then there is $M > 0$ such that

$$|x(j)| \leq Ma, \quad j = 1, 2, \ldots, m.$$ 

Therefore, $x \in C_u$. Since $x \in X_{00}$ is arbitrary, it follows $X_{00} \subseteq C_u$. This and Lemma [iv) imply that $u \in (X^+)_N$. Conversely, suppose that $u = \sum_n a_n e_n \in X^+$ with $a_j = 0$ for some $j \in \mathbb{N}$. Let $e_j^*$ be the functional such that $\ker e_j^* = \text{span}\{e_1, e_2, \ldots, e_{j-1}, e_j, \ldots\}$ and $\langle e_j^*, e_j \rangle = -1$. Then

$$0 = \langle e_j^*, u \rangle = \max \{ \langle e_j^*, z \rangle : z \in X^+ \}.$$ 

Therefore, $u$ is a support point of $X^+$.

ii) Let $u = \sum_n a_n e_n \in X^+$ with $a_n > 0$ for all $n \in \mathbb{N}$. Then

$$[-u, u] = \{ x \in X : -a_n \leq x(n) \leq a_n \},$$

and

$${\text{ext}}[-u, u] = \{ (e_1 a_1, e_2 a_2, \ldots, e_n a_n, \ldots) : e_j \in \{-1, 1\}, j = 1, 2, \ldots \}.$$ 

Let $T : X_u \to \ell_\infty$ for $x = (x(n))_{n=1}^\infty \in X_u$ be defined by

$$T(x) = (a_1^{-1}x(1), a_2^{-1}x(2), \ldots, a_n^{-1}x(n), \ldots).$$

Then

$$T(e_1 a_1, e_2 a_2, \ldots, e_n a_n, \ldots) = (e_1, e_2, \ldots, e_n, \ldots),$$

$e_j \in \{-1, 1\}, j = 1, 2, \ldots$. Therefore,

$$T({\text{ext}}[-u, u]) = \text{ext}B_{\ell_\infty}.$$ 

Consequently,

$$T(B_{X_u}) = B_{\ell_\infty}.$$ 

This says that $T : X_u \to \ell_\infty$ is a surjective linear isometry. \qed
As a consequence of Lemma 3, we have the following result.

**Corollary 1** Let $1 \leq p \leq \infty$, $\ell_p^+ = \ell_p$ be the natural positive cone of $\ell_p$ and $u \in (\ell_p^+)_{\text{ext}}$. Then

$$\{X_u, \|u\| \} \cong \ell_\infty.$$

**Proof** Since the natural unit vector basis $\{e_n\}$ of $\ell_p$ ($1 \leq p < \infty$) is an unconditional 1-basis of $\ell_p$, and since the natural positive cone $\ell_p^+$ of $\ell_p$ is just the positive cone of $\ell_p$ related to the basis $\{e_n\}$, it follows from Theorem 4 whenever $1 \leq p < \infty$.

If $p = \infty$, then

$$\ell_p^+ = \text{int} \ell_p = \{x = (x(n)) : 0 < \alpha \equiv \inf_n x(n) \leq \sup_n x(n) \equiv \beta < \infty\}.$$

Therefore, for any fixed $u = (u(n)) \in (\ell_p^+)_{\text{ext}}$ we have

$$[-u, u] = \{x \in \ell_\infty : -u(n) \leq x(n) \leq u(n), \forall n \in \mathbb{N}\},$$

and

$$\text{ext}[-u, u] = \{(e_1 u(1), e_2 u(2), \ldots, e_n u(n), \ldots) : e_j \in \{-1, 1\}\}.$$

Consequently, $T : X_u \rightarrow \ell_\infty$ defined for $x = (x(n))_{n=1}^\infty \in X_u$ by

$$T(x) = (u(1)^{-1} x(1), u(2)^{-1} x(2), \ldots, u(n)^{-1} x(n), \ldots)$$

is a linear surjective isometry. □

**Theorem 7** Let $X$ be a Banach space with an unconditional basis $\{e_n\}$, and $X^+$ be the positive cone of $X$ associated with $\{e_n\}$, i.e.,

$$X^+ = \{x = \sum_n a_n e_n \in X : a_n \geq 0 \text{ for all } n \in \mathbb{N}\}.$$

Then

i) $(X^+)_{\text{ext}} = \{x = \sum_n a_n e_n \in X : a_n > 0 \text{ for all } n \in \mathbb{N}\}$.

ii) For all $u \in (X^+)_{\text{ext}}$, $(X_u, \|u\|)$ is isometrically isomorphic to $\ell_\infty$.

**Proof** Without loss of generality, we can assume that $\{e_n\}$ is a 1-unconditional basis of $X$. Otherwise, let $f_n = e_n / \|e_n\|$, $n = 1, 2, \ldots$, and let $\| \cdot \|$ be defined for $x \in X$ by

$$\|x\| = \sup_{N \in \mathbb{N}, e_n \in \{-1, 1\}} \left\| \sum_{n=1}^N e_n a_n f_n \right\|, \quad x = \sum_{n=1}^\infty a_n f_n.$$

Then $\{f_n\}$ is a normalized 1-unconditional basis of $(X, \| \cdot \|)$.

Note $(X, \| \cdot \|)^+ = (X, (\| \cdot \|)^+)^+$. Then by Lemma 3, for each $u \in (X^+)_{\text{ext}}$, the Banach space $X_u = (X_u, \|u\|)$ generated by the closed convex set $[-u, u]$ is isometric to $\ell_\infty$. □
Recall that the Haar system, i.e. the sequence of functions \( \{ \chi_n(t) \}_{n=1}^{\infty} \) defined on the interval \([0, 1]\) by \( \chi_1(t) \equiv 1 \), and for \( k = 0, 1, 2, \ldots, j = 1, 2, \ldots, 2^k \),

\[
\chi_{2^k+j}(t) = \begin{cases} 
  1, & \text{if } t \in [(2j-2)2^{-k-1}, (2j-1)2^{-k-1}]; \\
-1, & \text{if } t \in ((2j-1)2^{-k-1}, 2j2^{-k-1}]; \\
0, & \text{otherwise}.
\end{cases}
\]

is (in the given order) a monotone (but not normalized) unconditional basis of \( L_p[0, 1] \) \((1 < p < \infty)\) with basis constant at most \( p^* - 1 \), where \( q^* = \max\{p, q\} \), and \( \frac{1}{p} + \frac{1}{q} = 1 \) (See, [1] Theorem 6.1.7). That is,

\[
\| \sum_{n=1}^{N} \epsilon_n a_n \chi_n \| \leq (p^* - 1) \| \sum_{n=1}^{N} a_n \chi_n \|,
\]

for every real number sequence \( \{a_n\}_{n=1}^{\infty} \), \( N \in \mathbb{N} \) and \( \epsilon_n \in \{-1, 1\} \). Then we obtain the following result.

**Corollary 2** Let \( 1 < p < \infty \), and \( C \) be the positive cone of \( L_p[0, 1] \) with respect to the Haar basis \( \{ \chi_n(t) \}_{n=1}^{\infty} \), i.e.

\[
C = \{ x = \sum_{n} a_n \chi_n \in X : a_n \geq 0 \text{ for all } n \in \mathbb{N} \}.
\]

Then for every \( u \in C_N \),

\[
(X_u, \| \cdot \|_u) \cong \ell_\infty.
\]

**Proof** Since \( \{ \chi_n(t) \}_{n=1}^{\infty} \) is an unconditional basis of \( L_p[0, 1] \), it follows from Theorem 4 that for all \( u \in C_N \), \((X_u, \| \cdot \|_u)\) is isometrically isomorphic to \( \ell_\infty \). \( \square \)

**Corollary 3** Let \( c_0^+ \) be the positive cone of \( c_0 \). Then for every \( u \in (c_0^+)_{N} \),

\[
(X_u, \| \cdot \|_u) \cong \ell_\infty.
\]

**Proof** By Theorem 4 it suffices to note that \( c_0^+ \) is just the positive cone of \( c_0 \) with respect to the standard unit vector basis \( \{ e_n \} \) of \( c_0 \). \( \square \)

### 6 More on \( L_p \) spaces

**Lemma 5** For any \( u \in (L^+_1[0, 1])_N \), or, \( (L^+_\infty[0, 1])_N \), we have \( X_u \cong L_\infty[0, 1] \).

**Proof** i) Let \( u \in (L^+_1[0, 1])_N \). Then \( u(t) > 0 \) a.e. for \( t \in [0, 1] \). Let \( T : X_u \to L_\infty[0, 1] \) be defined for \( f \in X_u \) by \( T f = u^{-1} f \). Then we obtain that

\[
T[-u, u] = [-1, 1] = B_{L_\infty[0, 1]}.
\]

Therefore, \( X_u \cong L_\infty[0, 1] \).

ii) Note that

\[
(L^+_\infty[0, 1])_N = \text{int}(L^+_\infty[0, 1]) = \{ f \in L^+_\infty[0, 1] : 0 < \text{essinf} f \}.
\]

For every \( u \in (L^+_\infty[0, 1])_N \), let \( T : X_u \to L_\infty[0, 1] \) be defined for \( f \in X_u \) by \( T f = u^{-1} f \). Then \( T \) is a bounded linear operator from \( L_\infty[0, 1] \) to itself. Clearly,

\[
T[-u, u] = [-1, 1] = B_{L_\infty}.
\]

Therefore, \( T \) is a surjective isometry from \( L_\infty[0, 1] \) to itself. That is, \( X_u \cong L_\infty[0, 1] \). \( \square \)
Theorem 8 Let \((\Omega, \sum, \mu)\) be a \(\sigma\)-finite measure space, and \(1 \leq p \leq \infty\). Then
i) \((L^+_p(\mu))_N \neq \emptyset\);
ii) for each \(u \in (L^+_p(\mu))_N \neq \emptyset\), the generating space \((X_u, \| \cdot \|_u) \cong L_\infty(\mu)\).

Proof
i) Assume \(1 \leq p < \infty\). Then it is easy to observe that
\[
(L^+_p(\mu))_N = \{ f \in L_p(\mu) : f(\omega) > 0 \text{ a.e. } \omega \in \Omega \}.
\]
Therefore, \((L^+_p(\mu))_N \neq \emptyset\). If \(p = \infty\), then \((L^+_\infty(\mu))_N = \text{int}L^+_\infty(\mu) \neq \emptyset\).
ii) Given \(u \in (L^+_p(\mu))_N\), let \(T : X_u \rightarrow L_\infty(\mu)\) be defined by
\[
Tf = u^{-1}f, f \in X_u.
\]
Then we obtain that \(T\) is a bounded linear operator and satisfies
\[
T[-u, u] = [-1, 1].
\]
Consequently, \(T : X_u \rightarrow L_\infty(\mu)\) is a linear surjective isometry.

Theorem 9 Let \((\Omega, \sum, \mu)\) be a \(\sigma\)-finite measure space, and \(1 < p < \infty\). Then there is a closed reproducing cone \(C\) of \(L_p(\mu)\) with \(C_N \neq \emptyset\) such that for each \(u \in C_N\), we have the generating space \(X_u \cong \ell_\infty\).

Proof Let \(\Omega_1, \Omega_2 \in \sum\) satisfy that \(\Omega = \Omega_1 \cup \Omega_2\), \(\Omega_1 \cap \Omega_2 = \emptyset\) and that \((\Omega_1, \sum_1, \mu)\) is atomless and \((\Omega_2, \sum_2, \mu)\) is atomic, where \(\sum_j = \Omega_j \cap \sum, j = 1, 2\). Since \((\Omega, \sum, \mu)\) is \(\sigma\)-finite, \(\Omega_2\) is countable. Therefore,
\[
L_p(\Omega, \sum, \mu) = L_p(\Omega_1, \sum_1, \mu) \oplus_p L_p(\Omega_2, \sum_2, \mu) \\
\cong \begin{cases} L_p(\Omega_1, \sum_1, \mu) \oplus_p \ell_p & \text{if } \Omega_2 \text{ is infinite;} \\
L_p(\Omega_1, \sum_1, \mu) \oplus_p \ell^p & \text{if } \Omega_2 \text{ has } n \text{ elements.}
\end{cases}
\]

Case I. \(\mu(\Omega_1) = 0\). It is trivial.
Case II. \(0 < \mu(\Omega_1) < \infty\). Since \(L_p(\Omega_1, \sum_1, \mu) \cong L_p[0, 1]\), it follows from Corollary 2.
Case III. \(\mu(\Omega_1) = \infty\). Since \((\Omega_1, \sum_1, \mu)\) is \(\sigma\)-finite, there is a \(\sigma\)-partition \(\{E_j\}\) of \(\Omega_1\) such that \(0 < \mu(E_j) < \infty\) for all \(j \in \mathbb{N}\). Therefore, \(L_p(E_j, \mu) \cong L_p[0, 1]\) for all \(j \in \mathbb{N}\). Consequently,
\[
L_p(\Omega_1, \sum_1, \mu) = \bigoplus_j L_p(E_j, \mu) \cong \bigoplus_j L_p([0, 1]) \cong L_p([0, 1]).
\]
It again follows from Corollary 2.
Lemma 6

Let $X$ be a Banach lattice and $X \ni x \lor y \in Z$.

Then the lattice operations are norm continuous.

Proof. It follows from iv) and

$$| - y | = | x \lor z - y \lor z | + | x \land z - y \land z |,$$

for all $x, y, z \in Z$, (15)

that the lattice operations are norm continuous.

By a sublattice of a Banach lattice $Z$ we mean a linear subspace $Y$ of $Z$ so that $x \lor y$ (and also $x \land y = x + y - x \lor y$) belongs to $Y$ whenever $x, y \in Y$. A lattice ideal in $Z$ is a sublattice of $Z$ satisfying that $| z | \leq | y |$ for $z \in Z$ and for some $y \in Y$ implies $z \in Y$.

A Banach lattice $X$ is said to be an abstract $M$-space (AM-space, for short) if

$$x, y \in X, \ | x | \land | y | = 0 \implies \| x + y \| = \max \{ |x|, |y| \}.$$ A mapping $T$ from a partially ordered real Banach space $X$ to a partially ordered real Banach space $Y$ is said to be an order isometry provided it is a fully order preserving isometry, i.e. $\| T x - T y \| = \| x - y \|$ for all $x, y \in X$ and $T x \geq T y$ if and only if $x \geq y$.

Lemma 6 Let $X$ be a Banach lattice and $X^+ = \{ | x | : x \in X \}$ be the positive cone. Then

i) for every $u \in (X^+)_u$, $X_u \equiv (X_u, \| \cdot \|_u)$ is again a Banach lattice and $u$ is its unit.

ii) if, in addition, $X$ is a function space consisting of real-valued functions defined on a set $\Omega$, and the lattice operations are induced by its natural order, i.e., $x \geq y$ if and only if $x(\omega) \geq y(\omega)$ for all $\omega \in \Omega$, then $(X_u, \| \cdot \|_u)$ is an AM-space with the unit $u$.

Proof i) Note that the closed unit ball $[ - u, u ]$ of $X_u$ is a bounded closed symmetric convex subset of $X$ with respect to the norm topology of $X$. Clearly, $a x \geq 0$, for all $x \geq 0$ in $X_u$ and $a \in R^+$. For every pair $x, y \in [ - u, u ]$, by properties of the lattice operations on $X$, we have

$$- u \leq x \land y \leq x \lor y \leq u.$$ This implies that both $x \lor y$ and $x \land y$ exist for all $x, y \in X_u$, and further, $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X_u$. Let $x, y \in X_u$. Then $| y | \leq u$ and $| x | \leq y$ imply that $- u \leq x \leq u$. This is clearly equivalent to that $\| y \|_u \leq 1$ and $| x | \leq y$ entail $\| x \|_u \leq 1$. Thus, for every $x, y \in X_u$, if $| x | \leq | y |$, then $\| x \|_u \leq \| y \|_u$. Thus, $(X_u, \| \cdot \|_u)$ is again a Banach lattice with the unit $u$.
ii) Note that $|x| \wedge |y| = 0$ is equivalent to that $\text{supp}(x) \cap \text{supp}(y) = \emptyset$ if $X$ is a function space. Let $x, y \in X_u$ satisfy $|x| \wedge |y| = 0$. Then

$$|x + y| = |x| + |y| = |x| \vee |y| + |x| \wedge |y| = |x| \vee |y|.$$  \hspace{1cm} (16)

Since $(X_u, \| \cdot \|_u)$ is a Banach lattice, it follows from (16) that

$$\|x + y\|_u \geq \|x\|_u \vee \|y\|_u.$$  \hspace{1cm} (17)

On the other hand, for every pair $x, y \in X_u$ with $|x| \wedge |y| = 0$, $x + y \in [-u, u]$ if and only if $x, y \in [-u, u]$. Consequently, $\|x\|_u = 1$ and $|x| \wedge |y| = 0$ imply that

$$\|x + y\|_u \leq 1 = \|x\|_u = \|x\|_u \vee \|y\|_u.$$  \hspace{1cm} (18)

(17) and (18) together entail that for every pair $x, y \in X_u$ with $|x| \wedge |y| = 0$ we have

$$\|x + y\|_u = \|x\|_u \vee \|y\|_u.$$  

Therefore, $(X_u, \| \cdot \|_u)$ is an AM-space with the unit $u$. \hfill \Box

Remark 1 Lemma 6 i) has been shown in [51, Coro. p.102]. Here the proof is a different but simplified approach.

Let $X$ be a Banach lattice with an order unit $e$, and $X^+$ be the positive cone of $X^*$, i.e. $X^+$ is the closed cone of $X^*$ consisting of all positive functionals with respect to the positive cone $X^+$ of $X$. We denote by $B^+_X = B_X \cap X^+$. Let

$$K = \{x^* \in B^+_X : x^* \in \text{ext}B_X \text{ with } \langle x^*, e \rangle = 1 \}.$$  \hspace{1cm} (19)

Then $K$ is a compact Hausdorff space when it is endowed with the $w^*$-topology of $X^*$.

Keep these notions in mind. Then we state the Kakutani-Bohnenblust-M. Krein-S. Krein Theorem as follows. The proof of “i) $\iff$ iii)” can be seen in Bohnenblust and Kakutani [8] and the proof of “i) $\iff$ ii)” is just [3, Theorem 9.32].

**Theorem 10 (Kakutani-Bohnenblust-M. Krein-S. Krein)** Let $X$ be a Banach lattice. Then the following statements are equivalent.

i) $X$ is a Banach lattice with a unit;

ii) $X$ is an AM-space with a unit;

iii) $X$ is order isometric to $C(K)$. The space $K$ defined as (19) is unique up to homeomorphism.

**Theorem 11** Let $X$ be a Banach lattice so that its positive cone has nonempty interior. Then it is order isomorphic to a Banach space $C(K)$ for some compact Hausdorff space $K$, i.e., there is an equivalent lattice norm on $X$ so that $X$ (with respect to the new norm) is order isometric to $C(K)$.

**Proof** Suppose that $X^+$ has nonempty interior. Then $(X^+)_K = \text{int}X^+$. By Lemma 6, for any $u \in \text{int}X^+$, the generating space $X_u$ is again a Banach lattice with the unit $u$. Therefore, it follows from Theorem 10 that $X_u$ is order isometric to a space $C(K)$ for some compact Hausdorff space $K$. By Theorem 6, $\| \cdot \|_u$ is an equivalent norm on $X$. \hfill \Box
Theorem 12 Let $X$ be a separable Banach lattice. Then
i) the set $(X^+_N)$ of non-support points of the positive cone $X^+$ is nonempty;
ii) for every $u \in (X^+_N)v$, $X_u \equiv (X_u, \|u\|)$ is order isometric to a Banach space $C(K)$ for some compact Hausdorff space $K$.

Proof i) It follows from Lemma 7; ii) It is a consequence of Lemma 6 and Theorem 10.

8 Latticization

Let $X$ be a real Banach space, $\Omega = (B_{X^*}, w^*)$, the closed unit ball $B_{X^*}$ of $X^*$ endowed with the weak-star topology $w^*$ of $X^*$. Let $E_{X^*}$ be the closed subspace of $C(\Omega)$ consisting of all $b$-$w^*$-continuous positive homogenous functions, that is, all bounded weak-star continuous homogenous functions on $X^*$ but restricted to $\Omega$. It is shown in [18] that $E_{X^*}$ is a Banach lattice so that $X$ is order isometric to a subspace of $E_{X^*}$. Therefore, we also call it the latticization of $X$.

We use $\mathcal{K}(X)$ (resp., $\mathcal{K}_0(X)$) to denote the cone of all nonempty convex compact subsets (resp., containing the origin 0) of $X$ endowed with the Hausdorff metric $d_H$, i.e.

$$d_H(A, B) = \inf\{r > 0 : A \subset B + rB_X, B \subset A + rB_X\}, A, \text{ } B \in \mathcal{K}(X).$$

Next, let $J : \mathcal{K}(X) \to E_{X^*}$ be defined for $C \in \mathcal{K}(X)$ by

$$J(C)(x^*) = \sup_{x \in C} \langle x^*, x \rangle, \text{ } x^* \in \Omega.$$  

Let “∨” and “∧” be the usual lattice operations defined on $E_{X^*}$, i.e., for all $u, v \in E_{X^*}$ and $x^* \in \Omega$,

$$(u \lor v)(x^*) = \max\{u(x^*), v(x^*)\}, \text{ } (u \land v)(x^*) = \min\{u(x^*), v(x^*)\}.$$  

Keep these notations in mind. Then we have the following result.

Lemma 7 Let $X$ be a real Banach space. Then
i) $E_{X^*} = J\mathcal{K}(X) - J\mathcal{K}_0(X)$ is a Banach lattice;
ii) $J\mathcal{K}(X) - J\mathcal{K}_0(X) = J\mathcal{K}_0(X) - J\mathcal{K}_0(X)$;
iii) $E_{X^*}$ is separable if $X$ is separable.

Proof i) This is just [18] Theorem 3.2 i).
ii) It is easy to check that $J\mathcal{K}(X) - J\mathcal{K}_0(X) = J\mathcal{K}_0(X) - J\mathcal{K}_0(X)$. Indeed, let $u = JC - JD$ for $C, D \in \mathcal{K}(X)$. Choose any $x_0 \in C$, $y_0 \in D$ and let $A = \text{co}\{\pm x_0, \pm y_0\}$. Then $A + C$, $B + D \in \mathcal{K}_0(X)$ and $u = J(A + C) - J(A + D)$.
iii) Assume that $X$ is separable. Let $\{x_n\} \subset X$ be a dense subsequence of $X$, and for all $m \in \mathbb{N}$, let

$$\mathcal{F}_m = \{\text{co} F : F \text{ is a subset of } m \text{ elements of } \{x_n\}\},$$

and

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n.$$
Then $\mathcal{F}$ is a dense countable subset of $\mathcal{K}(X)$. Since $J : \mathcal{K}(X) \to J\mathcal{K}(X) \subset E_{\mathcal{K}}$ is an order isometry \cite[Theorem 2.3]{18}, $J\mathcal{K}(X)$ is separable in $E_{\mathcal{K}}$. Consequently, $E_{\mathcal{K}}$ is separable.

\begin{lemma}
Let $X$ be a separable Banach space, and $Z \equiv E_{\mathcal{K}}$ be its latticization. For any nonsupport point $u$ of the positive cone $Z^+ = J\mathcal{K}(X)$ of $Z$, we obtain

i) $X \cap Z_u$ is a dense subspace of $X$ with respect to the norm of $X$;

ii) $X \cap Z_u$ is a dual space with respect to the new norm $\| \cdot \|_u$ of $Z_u$.

\end{lemma}

\begin{proof}

i) Let $C$ be a compact convex set of $X$ containing the origin such that $u = JC$. Since $u \in (Z^+)_{\nu}$, $Z_u = \bigcup_{n=1}^{\infty} n[-u,u]$ is a $\| \cdot \|_u$-dense subspace of $Z = E_{\mathcal{K}}$. This entails that $\bigcup_{n=1}^{\infty} n(C \cap -C)$ is a $\| \cdot \|_u$-dense subspace of $X$. Note $x \in X \cap Z_u$ if and only if there exists $\lambda > 0$ such that $-JC = -u \leq \lambda x \leq u = JC$, which is equivalent to $\pm \lambda x \in C$. Therefore, $X \cap Z_u = \bigcup_{n=1}^{\infty} n(C \cap -C)$ is a $\| \cdot \|_u$-dense subspace of $X$.

ii) Since $X$ is a closed subspace of $Z = E_{\mathcal{K}}$, $X \cap Z_u$ is closed in $Z_u$. Note that $x \in X \cap Z_u$ with $\|x\|_u = 1$ if and only if $x \in C \cap -C$, i.e. the closed unit ball of $X \cap Z_u$ is just $C \cap -C$. Since $C \cap -C$ is compact in $X$, by the Dixier-Ng theorem (see, for instance, \cite[Theorem 2.3]{35}), $X \cap Z_u$ is a dual space with respect to the new norm $\| \cdot \|_u$ of $Z_u$.

\end{proof}

\begin{theorem}
Let $X$ be a separable Banach space, and $Z \equiv E_{\mathcal{K}}$ be its latticization. For any nonsupport point $u$ of the positive cone $Z^+ = J\mathcal{K}(X)$ of $Z$,

i) the space $Z_u$ is order isometric to $L_{\infty}(\mu)$ for some probability measure $\mu$;

ii) $X \cap Z_u$ is a closed subspace of $Z_u$ and it is also a dual space.

\end{theorem}

\begin{proof}

i) Since $Z$ is a separable Banach lattice, by Lemma \cite{3} $Z_u$ is again a Banach lattice with its unit $u$. By Theorem \cite{11} $Z_u$ is order isometric to $C(K)$, where the Hausdorff space $K$ is defined as \cite{15}. On the other hand, note that the closed unit ball of $Z_u$ is $[-u,u]$, where $u = JC$ for some compact convex set $C \subset X$ containing the origin. It is easy to see that $[-u,u]$ is a Lipschitz set in $C_b(\Omega)$, where $\Omega = (B_{X^*}, w^*)$. Therefore, $[-u,u]$ is compact in $C_b(\Omega)$. Again by the Dixier-Ng theorem, $Z_u$ is dual space. Consequently, there is an abstract $L$ space $Y$ so that $Y^* = Z_u$ (See, for example, \cite[Theorem 9.27]{3}). By the Kakutani theorem, there is a probability measure $\mu$ such that $Y = L_1(\mu)$. Therefore, $Z_u = L_1(\mu)^* = L_{\infty}(\mu)$.

\end{proof}

\section{Exact penalization}

In this section, we will use properties of $C$-generating spaces established in the third section to generalize Ye’s exact penalty principle \cite{59}. To begin with this section, we recall some definitions.

\begin{definition}
Let $X, Y$ be Banach spaces, $Y$ be ordered by a (reproducing) cone $C$ of $Y, S \subset X$ be a nonempty subset and $f : S \to Y$ be a mapping.

i) $f$ is said to be $C$-lipschitz on $S$ of rank $L_f$ with respect to $e$ if there exist a positive constant $L_f$ and an element $e \in C$ with $\|e\| = 1$ such that

$$f(x) \leq f(y) + L_f \|x - y\|e, \forall x, y \in S.$$  

\end{definition}
ii) $f$ is called locally $C$-Lipschitz near $\bar{x} \in X$ with respect to $e$ if there is a neighborhood $U$ of $\bar{x}$ such that $f$ is $C$-Lipschitz with respect to $e$ on $U$.

The following problem is said to be constrained vector optimization problem.

$$\min \ f(x)$$
$$\text{s.t.} \ x \in \Omega \subset S,$$

where $X, Y$ are Banach spaces, $Y$ is ordered by a (reproducing) cone $C$ of $Y$, $\Omega \subset S \subset X$ are nonempty subsets and $f : S \to Y$ is a mapping.

The exact penalty approach aims at replacing a constrained optimization problem by an equivalent unconstrained optimization problem. Most results in the literature of exact penalization are mainly concerned with finding conditions under which a solution of the constrained optimization problem is a solution of an unconstrained penalized optimization problem, and the reverse property is rarely studied. In [59], Jane J. Ye considered the reverse property. Precisely, she studied the following (unconstrained) penalty program.

$$\min \ f(x) + L_f d(x, \Omega)e$$
$$\text{s.t.} \ x \in S,$$

where $L_f \in \mathbb{R}^+$, $e \in C$ and $d(x, \Omega) = \inf\{\|x - z\| : z \in \Omega\}$ is the distance function.

She obtained the following global exact penalization for distance function [59, Theorem 3.1], which extends Clarke’s Exact Penalty Principle from a scalar function to a vector function, and which states that the constrained optimization problem (22) and the unconstrained exact penalized problem (23) under some conditions are exactly equivalent.

**Theorem 14 (Ye [59])** Let $X$ and $Y$ be Banach spaces, $Y$ be ordered by a convex (reproducing) cone $C$ of $Y$, $S \subset X$ and $\Omega \subset S$ be nonempty subsets. Let $f : S \to Y$ be $C$-Lipschitz on $S$ of rank $L_f$, and $e$ be an element in $C$ given by the $C$-Lipschitz continuity of $f$.

i) Assume that $C \setminus \{0\}$ is an open set. Then any global minimizer of $f$ on $\Omega$ is a global minimizer of the exact penalty function $f(x) + L_f d(x, \Omega)e$ on $S$.

ii) Assume that either $S$ is closed or that $C \setminus \{0\}$ is an open set. Then, for any $L > L_f$, $x$ is a global $C$-minimizer of $f$ on $S$ if and only if it is a global $C$-minimizer of the exact penalty function $f(x) + L d(x, \Omega)e$ on $S$.

As we have mentioned in Section 1, and Theorem[13] a Banach space admitting a cone with nonempty interior is an almost a $C(K)$-space. Therefore, Ye’s Global Exact Penalization Theorem above can be understood as a perfect extension of Clarke’s exact penalty principle from a scalar function to a vector function valued in finite dimensional spaces. On the other hand, in infinite dimensional spaces, it is limited to the class of $C(K)$-spaces. Note that for a closed reproducing cone $K$ of a Banach space $Y$ with the set $K_N$ of all nonsupport points of $K$ being nonempty, in particular, $Y$ is separable, the set $C \equiv K_N \cup \{0\}$ is again an almost reproducing cone. We extend this principle in the following manner.
Theorem 15 (Generalized global exact penalization for distance function) Let $X$ and $Y$ be Banach spaces, $Y$ be ordered by the cone $C = K_Y \cup \{0\}$ of a closed reproducing cone $K$ with $K_Y \neq \emptyset$ of $Y$, $S \subset X$ and $\Omega \subset S$ be two nonempty subsets. Assume that $f : S \to Y$ is $C$-Lipschitz on $S$ of rank $L_f$, and that $e$ is the element in $C$ given by the $C$-Lipschitz continuity of $f$. Then we have the following assertions.

i) Every global $C$-minimizer of $f$ on $\Omega$ is a global $C$-minimizer of the exact penalty function $f(x) + L_f d(x, \Omega) e$ on $S$.

ii) For every $L > L_f$, a global $C$-minimizer of the exact penalty function $f(x) + L d(x, \Omega) e$ on $S$ if and only if it is a global $C$-minimizer of $f$ on $\Omega$.

Proof i) Suppose, to the contrary, that there exists a global minimizer $\bar{x} \in \Omega$ of the program (23) on $\Omega$ but it is not a global minimizer of the program (25) on $S$. Then there exists $\bar{z} \in S$ such that

$$f(\bar{z}) + L_f d(\bar{z}, \Omega) e < f(\bar{x}). \tag{24}$$

Therefore, $f(\bar{x}) - f(\bar{z}) \in C \setminus \{0\} = K_Y$. Since $f : S \to Y$ is $C$-Lipschitz of rank $L_f$ with respect to $e$,

$$-L_f \|\bar{z} - \bar{x}\| e \leq f(\bar{x}) - f(\bar{z}) \leq L_f \|\bar{z} - \bar{x}\| e.$$

This and (24) lead to

$$L_f d(\bar{z}, \Omega) e < f(\bar{x}) - f(\bar{z}) \leq L_f \|\bar{z} - \bar{x}\| e. \tag{25}$$

Denote $q = f(\bar{x}) - f(\bar{z})$. Then $q \in K_Y$. Note $\overline{C} = K$. Then by Theorem 5 the following two $K$-generating spaces

$$Y_e = \bigcup_{\lambda > 0} \lambda \{K_e \cap (-K_e)\}, \quad Y_q = \bigcup_{\lambda > 0} \lambda \{K_q \cap (-K_q)\}$$

are Banach spaces with respect to their new norms $\| \cdot \|_e$ and $\| \cdot \|_q$ defined by

$$\|x\|_e = \inf\{\lambda > 0 : \bar{x} \in \lambda ((C - e) \bigcap (e - C))\}$$

$$= \inf\{\lambda > 0 : \bar{x} \in [-e, e]\}, \quad x \in Y_e,$$

and

$$\|x\|_q = \inf\{\lambda > 0 : \bar{x} \in \lambda ((C - q) \bigcap (q - C))\}$$

$$= \inf\{\lambda > 0 : \bar{x} \in [-q, q]\}, \quad x \in Y_q.$$

It follows from (25) and Theorem 6, $Y_e = Y_q$ algebraically, and that $\| \cdot \|_e, \| \cdot \|_q$ are equivalent. Clearly, $e, q \in \text{int} C_e \cap \text{int} C_q = \text{int} C_e$, where $C_e = \bigcup_{\lambda > 0} \lambda (C - e)$. Consequently, there is $\delta > 0$ so that

$$f(\bar{x}) - f(\bar{z}) - e e = q - e e \in C_e, \quad \text{for all } 0 < e < \delta. \tag{26}$$

It follows from (25) and (26) that for all sufficiently small $0 < e < \delta$,

$$f(\bar{x}) - f(\bar{z}) - e e > L_f d(\bar{z}, \Omega) e. \tag{27}$$
On the other hand, there exists $x_e \in \Omega$ such that
\[
\|\bar{x} - x_e\| < \varepsilon + d(\bar{x}, \Omega).
\]
This and (27) imply
\[
f(x_e) \leq f(\bar{x}) + L_f \|\bar{x} - x_e\| \varepsilon < f(\bar{x}) + L_f (\varepsilon + d(\bar{x}, \Omega)) \varepsilon < f(\bar{x}).
\]
This is a contradiction!

ii) Sufficiency. It follows from i) we have just proven.

Necessity. Suppose, to the contrary, that there exists a global $C$-minimizer $\bar{x} \in S$ of the penalty function $f(x) + Ld(x, \Omega)e$ on $S$ but it is not a global $C$-minimizer of the program (22) on $\Omega$. If $\bar{x} \in \Omega \triangle S$, then there exists $\bar{z} \in \Omega$ such that
\[
f(\bar{z}) > f(\bar{x}).
\]
This and $d(\bar{x}, \Omega) = 0 = d(\bar{z}, \Omega)$ imply that
\[
f(\bar{x}) + Ld(\bar{x}, \Omega) = f(\bar{x}) > f(\bar{z}) = f(\bar{z}) + Ld(\bar{z}, \Omega),
\]
which is a contradiction to that $\bar{x}$ is a global $C$-minimizer of the penalty function $f(x) + Ld(x, \Omega)e$ on $S$. Therefore, $\bar{x} \notin \Omega \cap S$. It follows from (29) that
\[
0 < q \equiv f(\bar{x}) - f(\bar{z}) \in C \setminus \{0\} = K_N.
\]
By Lipschitz continuity of $f$,
\[
q = f(\bar{x}) - f(\bar{z}) \leq L_f \|\bar{x} - \bar{z}\| \varepsilon.
\]
Therefore, $[-q, q] \subset [-\beta \varepsilon, \beta \varepsilon]$, where $\beta = L_f \|\bar{x} - \bar{z}\|$. Consequently, $Y_q \subset Y_e$ algebraically, and $q \in \text{int} C_e \cap K_N$. Hence, there is $\delta > 0$ so that
\[
f(\bar{x}) - f(\bar{z}) - \varepsilon e = q - \varepsilon e \in \text{int} C_e \cap K_N, \text{ for all } 0 < \varepsilon < \delta.
\]
On the other hand, for each such $\varepsilon > 0$, there exists $x_e \in \Omega$ such that
\[
\|\bar{x} - x_e\| < \varepsilon + d(\bar{x}, \Omega).
\]
This, $d(x_e, \Omega) = 0$ and Lipschitz continuity of $f$ imply
\[
f(x_e) + Ld(x_e, \Omega) = f(x_e)
\leq f(\bar{x}) + L_f \|x_e - \bar{x}\| \varepsilon
< f(\bar{x}) + L_f (\varepsilon + d(\bar{x}, \Omega)) \varepsilon
= f(\bar{x}) + L_f d(\bar{x}, \Omega) \varepsilon + L_f \varepsilon e.
\]
Note that $L > L_f$ and $d(\bar{x}, \Omega) > 0$. By (30), we choose $0 < \varepsilon < \delta$ so that
\[
L_f \varepsilon < (L - L_f) d(\bar{x}, \Omega).
\]
Then it follows
\[
f(x_e) + Ld(x_e, \Omega) \varepsilon < f(\bar{x}) + Ld(\bar{x}, \Omega) \varepsilon.
\]
This is a contradiction to that $\bar{x}$ is a global $C$-minimizer of the penalty function $f(x) + Ld(x, \Omega)e$ on $S$. \qed
Problem 5 (Lagrange model with box constraint)

In this section, we consider solvability of the box constraint of Lagrange model (Problem 5 and the Lagrange duality model (Problem 6) mentioned in Section 1. We have already known that every Banach space $X$ is contained in its latticization $E_\infty$, and the density character of $E_\infty$ is the same as that of $X$ (Lemma 6). Therefore, without loss of generality, we can assume that every Banach space $X$ in question is a subspace of a Banach lattice. Thus, $X^+$ and $|x| = x \vee -x$ for every $x \in X$ are meaningful.

Now, we restate Problems 5 and 6 as follows.

**Problem 5 (Lagrange model with box constraint)**

$$
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g(x) \in -Y^+, \\
& \quad h(x) = 0, \\
& \quad x \in \Omega = \{x \in X : x_a \leq x \leq x_b\},
\end{align*}
$$

(31)

where $X, Y, Z$ are Banach spaces and $X, Y$ are ordered by their corresponding ordering cones $X^+, Y^+$ with $(X^+_N) \neq \emptyset \neq (Y^+_N)$, and $f : X \rightarrow R \cup \{+\infty\}$ is a lower semicontinuous convex function with $[\frac{1}{2}x_a, r_a] \subset \text{dom} f$ for some $r > 1$, $g : X \rightarrow Y$ is $Y^+$-Lipschitz on $\Omega$ of rank $\text{lip}(g) = 1$ (Definition 5) and convex-like with respect to $Y^+$ (Definition 5 and 6) and $h : X \rightarrow Z$ is a continuous affine function, and $x_a, x_b \in X$ with $x_b - x_a \in (X^+_N)$. Denote $S = \{x \in \Omega : g(x) \in -Y^+, h(x) = 0\}$.

**Proposition 4** Suppose $g$ is convex-like respect to $Y^+$ and $g$ is $Y^+$-Lipschitz respect to $e \in (Y^+_N)$ in $\Omega$, then $g$ is also convex-like respect to $Y^+_e$ in $\Omega$.

**Proof** Denote $A_1 = \{f(x) + y : y \in Y^+, x \in \Omega\}$, $A_2 = \{f(x) + y : y \in Y^+_e, x \in \Omega\}$. For any $x_1, x_2 \in \Omega$, $y_1, y_2 \in Y^+$, any $\alpha \in [0, 1]$, since $g$ is convex-like with respect to $Y^+$, there exist $x_1, x_2 \in \Omega$ and $y_a \in Y^+$ such that

$$
g(x_1) + y_a = \alpha(g(x_1) + y_1) + (1 - \alpha)(g(x_2) + y_2) \in A_1.
$$

Because $g(x_1), g(x_2), g(x_0), y_1, y_2$ are all in $Y^+$, we have $g(x_0) + y_a \in Y^+$ which implies $y_a \in Y^+_e$ and $g(x_0) + y_a \in A_2$. Thus $g$ is also convex-like respect to $Y^+_e$ in $\Omega$. 

**Problem 6 (Modified Lagrange duality model)**

$$
\begin{align*}
\max & \quad \inf_{(y^*, z^*, x_1^*, x_2^*) \in X} f(x) + \langle y^*, x_1^* \rangle + \langle x_2^*, h(x) \rangle \\
\text{s.t.} & \quad y^* \in Y^+_e, \quad x_1^* \in X^+_e, \quad z^* \in Z^*,
\end{align*}
$$

(32)
where $X_r$ (resp., $Y_r$) is the $(X^+, \pi)$ (resp., $(Y^+, e')$) -generating space for some $\pi \in (X^+)_N$ (resp., $e' \in (Y^+)_X$) and $X^+_r$ (resp., $Y^+_{e'}$) is the positive cone of its dual $X^*_r$ (resp., $Y^*_{e'}$).

**Theorem 16** [Modified Slater’s condition] If the primal program (31) is solvable and satisfying that there is $x \in S$ such that there exists $\lambda > 0$, $-\lambda g(x) > e$ (e is from Lipschitz property of $g$), $h(x) = 0$, and such that $h(\Omega)$ contains a neighborhood of 0, then the dual program (32) is also solvable, that is, it admits an optimal solution, and the extremal values of the two programs are equal.

**Proof** Without loss of generality, we can assume $x_0, x_b \in (X^+_\Omega)$. Otherwise, we substitute successively, $x_c = x_b - x_a$ for $x_a$ and $x_d = 2(x_b - x_a)$ for $x_b$; $\bar{f}(x) = f(x - x_b + 2x_a)$ for $f(x)$; $\bar{g}(x) = g(x - x_b + 2x_a)$ for $g(x)$; $\bar{h}(x) = h(x - x_b + 2x_a)$ for $h(x)$, and $\Omega' = \{x \in X : x_c \leq x \leq x_d\}$ for $\Omega$ in the programs (31) and (32).

Let $\pi = x_b$ and $(X_\pi, \|\cdot\|)$ be the $(X^+, \pi)$-generating space. Then there exists $r > 1$ such that

$$\Omega \subset \frac{1}{r} x_a, rx_b \subset \text{int}_\pi \{X^+ \cap X^*_r\}. \quad (33)$$

Since $f$ is real-valued and lower semicontinuous convex on $[\frac{1}{r} x_a, rx_b]$, $f$ is lower semicontinuous convex on $[\frac{1}{r} x_a, rx_b]$ with respect to the norm $\|\cdot\|_\pi$. (33) implies that $f$ is continuous (hence, locally Lipschitz) on $(\frac{1}{r} x_a, rx_b)$. Therefore, $f$ is locally Lipschitz on $\Omega$.

Let $\bar{x} \in S$ such that $g(\bar{x}) \in -(Y^+_\Omega)$, $h(\bar{x}) = 0$, and such that $h(\Omega)$ contains a neighborhood of 0. Note that $e - g(\bar{x}) > e$. Let $e' = \frac{e - g(\bar{x})}{\|e - g(\bar{x})\|}$, and let $(Y_{e'}, \|\cdot\|_{e'})$ be the $(Y^+, e')$-generating space. Then $-\lambda g(\bar{x}) > e$ for some $\lambda > 0$ implies

$$-g(\bar{x}) \in \text{int}_{\|\cdot\|_{e'}} (Y^+_{e'}).$$

Since $\|e'\| = 1$, we obtain $\|\cdot\|_{e'} \geq \|\cdot\|$ on $Y_{e'}$. Since $g : X \to Y$ is $Y^+$-Lipschitz on $\Omega$ of rank $L_g$ with respect to $e \in (Y^+_\Omega)$, it is also $Y^+$-Lipschitz on $\Omega$ of rank $L_g$ for some $L_g > 0$ with respect to $e' \in (Y^+)_\Omega$, that is,

$$-L_g \|x - y\|_{e'} \leq g(x) - g(y) \leq L_g \|x - y\|_{e'}, \quad \text{for all} \ x, y \in \Omega.$$

Therefore,

$$-L_g \|x - y\|_{e'} e' \leq g(x) - g(y) \leq L_g \|x - y\|_{e'} e', \quad \text{for all} \ x, y \in \Omega.$$

Consequently,

$$\{g(x) - g(y) : x, y \in X\} \subset Y_{e'} \quad (34)$$

Since $g$ is convex-like with respect to $Y^+$, that is, $\{g(x) + y : y \in Y^+, x \in X\}$ is a convex set of $Y$. This and (34) entail that $\{g(x) + y : y \in Y_{e'}^+, x \in X\}$ is a convex set of $Y_{e'}$ by Proposition 3.

Now, the program (31) can be equivalently translated into the following one.

$$\min \quad f(x)$$

$$\begin{array}{l}
\text{s.t.} \quad g(x) \in -Y_{e'}^+, \\
h(x) = 0, \\
x \in \Omega = \{x \in X_\pi : x_a \leq x \leq x_b\}.
\end{array} \quad (35)$$
By the hypothesis of the theorem, the program (35) is solvable. We consider the corresponding dual program (32).

\[
\max_{(y^*, x_1^*, x_2^*, z^*) \in \mathbb{R}} f(x) + \langle (y^*, x_1^*, x_2^*, z^*), (g(x), x - x_a, x_b - x) \rangle + \langle z^*, h(x) \rangle
\]

\[
s.t. \quad y^* \in Y^+_a, \quad x_1^* \in X^+_a, \quad x_2^* \in Z^*.
\]

It satisfies that there exists \( \bar{x} \in S \) such that \( g(\bar{x}) \in -\text{int}(Y^+_a), h(\bar{x}) = 0, \) and \( h(\Omega) \) contains a neighborhood of 0. By Slater’s condition (See, for instance, [14, Theorem 2], also, [36, Theorem 5.3]), the dual program is also solvable.

As an application of Theorem 16, we consider solvability of the following elastic plastic torsion problem. The elastic plastic torsion problem goes back decades (see, for instance, [44, 48, 55]). Later, a number of mathematicians studied this problem (see [12, 29, 24, 42, 32, 33]). This problem could be formulated as follows.

**Example 7** [Elastic plastic torsion problem] Let \( \Omega \subset \mathbb{R}^n \) be a nonempty bounded open Lipschitz domain with its boundary \( \partial \Omega \), \( H^1_0(\Omega) \) be the Sobolev space defined by

\[
H^1_0(\Omega) = \{ u \in L^2(\Omega) : u = 0 \text{ on } \partial \Omega, \quad \nabla u \in L^2(\Omega) \},
\]

endowed with the norm

\[
\|u\| = \|u\|_{H^1_0(\Omega)} = \sqrt{\int_{\Omega} |\nabla u|^2 \, d\omega}.
\]

Let

\[
K \equiv \{ u \in H^1_0(\Omega) : u \geq 0, \quad (\nabla u)^2 = \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 \leq 1 \text{ a.e. in } \Omega \},
\]

and let

\[
Y = L_1(\Omega).
\]

Then for every \( u \in H^1_0(\Omega) \), we have

\[
Y \equiv L_1(\Omega) \ni \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 - 1.
\]

A vector \( \bar{u} \in K \) is said to be a solution of the elastic plastic torsion problem if it is a minimum to following convex program:

\[
\min_{\bar{u} \in K} f(\bar{u}, u) \equiv a(\bar{u}, u) + \int_{\Omega} gudx
\]

\[
s.t. \quad u \in K
\]

where \( a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R} \) is a bilinear function, and \( g \in L^2(\Omega) \).

Let \( X \equiv H^1_0(\Omega) \). Then \( X \) is a separable Hilbert space. Since \( X \) is separable, by Lemma 1 \((X^+)_N \neq \emptyset \). Indeed,

\[
(X^+)_N = \{ u \in X : u > 0 \text{ a.e. } \Omega \}.
\]
Note that $K$ is a bounded absolutely convex set of the (Hilbert) space $X$. It is easy to check that $0 \in K_X$ (the set of nonsupport points of $K$). Let $X_\pi$ be the $(K,0)$-generating space, that is $X_\pi = \bigcup_{n=1}^\infty nK$ endowed with the norm $\| \cdot \|_\pi$ which is the Minkowski functional generated by $K$. Then by Theorem 14, $X_\pi$ is linearly isometric to a closed subspace of $L_\infty(\Omega)$.

Let $Y \equiv L_1(\Omega)$, $e = \frac{1}{\mu(\Omega)} \in Y$, i.e. the constant function on $\Omega$, where $\mu(\Omega)$ is the Lebesgue measure of $\Omega$. Then $e \in (Y^+) \cap Y$ with $\| e \| = \int Y e \, d\mu = 1$. By Example 5, the $(Y^+,e)$-generating space $Y_e = L_\infty(\Omega)$. Therefore, the operator $\sum_{i=1}^n (\frac{\partial}{\partial x_i})^2 - 1$ can be regarded as a continuous operator from $X_\pi$ to $Y_e$. Indeed, $\sum_{i=1}^n (\frac{\partial}{\partial x_i})^2 - 1$ is trivially from $X_\pi$ to $Y_e$. To show continuity of $\sum_{i=1}^n (\frac{\partial}{\partial x_i})^2 - 1$, it suffices to note $\sum_{i=1}^n (\frac{\partial}{\partial x_i})^2 - 1 = (\nabla u)^2 - 1$, and $\nabla : X_\pi \to Y_e = L_\infty(\Omega)$ is a bounded linear operator.

Since $\Omega$ is an open set, for every fixed $x_0 \in \Omega$, there exist $r > 0$ such that $\{ x : \| x - x_0 \| \leq r \} \subset \Omega$. Fix $0 < \lambda < \frac{1}{\theta}$ and let

$$u(t) = \begin{cases} \frac{\theta}{4}(t^2 - \| x - x_0 \|^2), & \| x - x_0 \| \leq r, \\ 0, & \text{otherwise}. \end{cases} \quad (38)$$

Then

$$\sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2 - 1 = \begin{cases} 4\lambda^2 \| x - x_0 \|^2 - 1, & \| t - t_0 \| \leq r, \\ -1, & \text{otherwise}. \end{cases} \quad (39)$$

Note that $v \in \text{int} L^+_{\infty}(\Omega)$ if and only if there is $\alpha > 0$ such that $| v | \geq \alpha$ a.e. $\Omega$. Then $\sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2 - 1 \in -\text{int} L^+_{\infty}(\Omega)$. If program (36) is solvable with a solution $\bar{u}$, then by Theorem 14 we know the following duality program is solvable with same extreme value of the program (36).

$$\max_{y^*} \inf_{u \in X} f(\bar{u},u) + \left( y^*, \left( \sum_{i=1}^n \frac{\partial u}{\partial x_i} \right)^2 - 1 \right) \quad (40)$$

subject to $y^* \in L^+_{\infty}(\Omega)$,

where $X_\pi = \bigcup_{n=1}^\infty nK$.

11 Convex vector optimization

In this section, we will apply the theory of generating spaces to convex vector programs with box constraints in (infinite dimensional) Banach spaces.

Let us come back to the program (14) with the box constraints $\Omega \subset \{ x \in X : x_a \leq x \leq x_b \}$.

$$\min f(x) \quad \text{s.t.} \quad x \in \Omega \subset \{ x \in X : x_a \leq x \leq x_b \}, \quad (41)$$
where $X$, $Y$ are Banach spaces ordered by their ordering cones $X^+$, $Y^+$ respectively with $(X^+)_N \neq \emptyset \neq (Y^+)_N$. $\Omega$ is a closed convex subset with some $x_a < x_b \in X$. $f : X \to Y$ is $Y^+$-Lipschitz on $\Omega$ with respect to some $e \in (Y^+)_N$ (Definition\cite{28}, $\|e\| = 1$, and is convex with respect to $Y^+$.

Usually, the first step to deal with the program \eqref{41} is to consider a scalarized program. There are many scalarization methods (See, for instance, \cite{28, 47, 53}). In this section, we will use the following Gerstewitz scalarization function $\phi_{e, C} : Y \to \mathbb{R} \cup \{+\infty\}$ respect to $C = Y^+$ and $e$ is defined for $y \in Y$ by

$$\phi_{e, C}(y) = \inf \{ t \in \mathbb{R} : te \in y + C \}. \quad \text{(42)}$$

Note that $te \in y + C$ is equivalent to $0 \in y - te + C$. Then

$$\phi_{e, C}(y) = \inf \{ t \in \mathbb{R} : y - te \in -C \}. \quad \text{(43)}$$

Keep these notions in mind. We have following properties \cite{53}.

**Lemma 9** Let $Y$ be a Banach space and $C = Y^+$ be its ordering cone.

1. $\phi_{e, C}$ is lower semicontinuous if and only if $C$ is closed in $Y$.
2. $\phi_{e, C}$ is a monotone nondecreasing sublinear functional with respect to $Y^+$.
3. For all $y \in Y$, $r \in \mathbb{R}$, $\phi_{e, C}(y) \leq r \iff y \in re - C$; in particular, $\phi_{e, C}(y) \leq 0$ if and only if $y \in -C$.
4. $\phi_{e, C}$ is continuous on $Y$ if and only if $e \in \text{int}(C)$.
5. The effective domain of $\phi_{e, C}$ is $\mathbb{R}(e - C) \equiv \bigcup_{r \in \mathbb{R}} re - C$.

We consider following scalarization program. Given $\bar{x} \in \Omega$,

$$\min_{x \in \Omega} \phi_{e, Y^+}(f(x) - f(\bar{x})) \quad \text{(44)}$$

It is obvious that if $\bar{x}$ is a local minimizer of program \eqref{41} if and only if $\phi_{e, Y^+}(f(x) - f(\bar{x})) \geq 0$. In this case, $\bar{x}$ is a minimizer of program \eqref{44}.

The most fundamental and important optimality condition induced by Fermat rule for this program is

$$\theta \in \partial \phi_{e, Y^+}(f(.) - f(\bar{x})) + \delta(\cdot, \Omega)(\bar{x}),$$

where $\delta(\cdot, \Omega)$ is the indicator function of $\Omega$, that is, $\delta(x, \Omega) = 0$, if $x \in \Omega$; $= \infty$, otherwise. Since the Moreau-Rockafellar theorem requires one of $\phi_{e, Y^+}$ and $\delta(\cdot, \Omega)$ is continuous, the following equation

$$\partial (\phi_{e, Y^+}(f(\cdot) - f(\bar{x})) + \delta(\cdot, \Omega))(\bar{x}) = \partial \phi_{e, Y^+}(f(\cdot) - f(\bar{x}))(\bar{x}) + \partial \delta(\cdot, \Omega)(\bar{x}),$$

does not hold if $\text{int}(Y^+) = \emptyset$. This is an essential difficulty and does not depend on the choice of scalarization methods. In order to circumvent this difficulty, the next best thing is to replace $Y^+$ by a larger set $A \supset Y^+$ with $\text{int}(A) \neq \emptyset$. See, for instance, \cite{4, 28, 37, 47, 53, 56} and references therein. Nevertheless, by this procedure, one would obtain \textit{“approximate minimal point”}, instead of minimum point of \eqref{41}. In the
following, we will show that this difficulty can be overcome by the generating space theory whenever $(Y^+)_N \neq \emptyset$.

Assume that the Banach spaces $X$, $Y$, the domain $\Omega$, the function $f$ and the vector $e \in (Y^+)_N$ are the same as in the program (41). For a fixed $\bar{x} \in \Omega$, denote $\bar{f}(x) = f(x) - f(\bar{x})$. Let $Y_e$ be $(Y^+, e)$-generating space with its positive cone $Y_e^+$ and the dual positive cone $\bar{Y}_e^+$. Note that $e \in \text{int}Y_e^+$.

**Theorem 17** With notions and symbols as above, suppose that $\bar{x}$ is a minimum of the program (42). Then

$$\theta \in \partial \phi_{e,Y_e^+} \circ \bar{f}(\bar{x}) + N(\bar{x}, \Omega),$$

where $N(\bar{x}, \Omega)$ is from Definition 5.

**Proof** Since $f$ is $Y^+$-Lipschitz on $\Omega$ with respect to $e \in (Y^+)_N$, we see that

$$\{f(x) - f(\bar{x}) : x \in \Omega\} \subset Y_e,$$

and that $\bar{f}$ is $Y_e^+$-Lipschitz on $\Omega$. Note that $e \in \text{int}(Y_e^+)$. Then by Lemma 9, $\phi_{e,Y_e^+} : (Y_e, \|\cdot\|_e) \to \mathbb{R}$ is a continuous convex function. By the Fermat rule, the Moreau-Rockafellar Theorem (Theorem 2) and the equation (7), we obtain

$$\theta \in \partial \phi_{e,Y_e^+} \circ \bar{f}(\bar{x}) + N(\bar{x}, \Omega),$$

where $N(\bar{x}, \Omega)$ is from Definition 5.

**Lemma 10** (54) Assume that $Y$ is a Banach space, $e \in Y \setminus \{\theta\}$, and $C \subseteq Y$ is a closed convex cone not containing the real line $\mathbb{R}e$ with $C + [0, \infty)e \subseteq C$. Let $C^*$ be the dual positive cone, that is, $C^* \equiv \{y^* \in Y^* : \langle y^*, e \rangle \geq 0, \forall y \in C\}$, and $\phi_{e,C} : Y \to \mathbb{R} \cup \{+\infty\}$ be the corresponding Gerstewitz scalarization function. Then

$$\partial \phi_{e,C}(y) \neq \emptyset, \text{ for all } y \in \text{dom} \phi_{e,C}$$

and

$$\partial \phi_{e,C}(y) = \{y^* \in C^* : \langle y^*, e \rangle = 1, \langle y^*, y \rangle = \phi_{e,C}(y)\}.$$

In particular,

$$\partial \phi_{e,C}(\theta) = \{y^* \in C^* : \langle y^*, e \rangle = 1\}.$$

The next result is a representation theorem of $\partial \phi_{e,Y_e^+} \circ \bar{f}(\bar{x})$.

**Theorem 18** With the notions and conditions as previously mentioned, suppose that $\bar{x}$ is a minimum of the program (42). Then

$$\partial \phi_{e,Y_e^+} \circ \bar{f}(\bar{x}) \subset \bigcup \{\partial \langle y^*, \bar{f} \rangle(\bar{x}) : y^* \in Y_e^{++}, \langle y^*, e \rangle = 1\}$$

$$= \overline{\text{co}}\left(\bigcup \{\partial \langle y^*, \bar{f} \rangle(\bar{x}) : y^* \in K_e\}\right),$$

where $K_e = \text{extr}\{y^* \in Y_e^{++} : \langle y^*, e \rangle = 1\}$, the set of all extreme points of the $w^*$-closed convex set $Y_e^{++} \cap \{y^* \in Y_e^* : \langle y^*, e \rangle = 1\}$, and $\overline{\text{co}}(A)$ denotes the $w^*$-closed convex hull of a set $A \subseteq Y_e^*$. 


Proof Since $f$ is $Y^+$-Lipschitz with respect to $e$, $\{f(x) - f(\bar{x}) : x \in \Omega\} \subset Y_e$. Since $f$ is convex with respect to $Y^+$, $\bar{f}$ is convex with respect to $Y_e^+$. Note that $e \in \text{int} Y_e^+$. Then by Lemma 9, $\phi_{e,Y_e^+}$ is a continuous sublinear and monotone nondecreasing functional with respect to the ordering cone $Y_e^+$ on $Y_e$. Let $\bar{y} = \bar{f}(\bar{x})$. Since $\bar{x}$ is a minimum of the program (44), nondecreasing monotonicity of $\phi_{e,Y_e^+}$ entails

$$\phi_{e,Y_e^+} \circ \bar{f} = \inf_{y \in Y_e^+} \{ \phi_{e,Y_e^+}(y) + \delta((\bar{x}, y); \text{epi} \bar{f}) \} = \phi_{e,Y_e^+}(\bar{y}) + \delta((\bar{x}, \bar{y}); \text{epi} \bar{f}).$$

(50)

Since $\phi_{e,Y_e^+}$ is a continuous sublinear functional and since $\delta((\bar{x}, \cdot); \text{epi} \bar{f})$ is a lower semicontinuous convex function on $Y_e$, by the Moreau-Rockafellar Theorem,

$$\partial \phi_{e,Y_e^+} \circ \bar{f}(\bar{x}) = \partial \phi_{e,Y_e^+}(\bar{y}) + \partial \delta((\bar{x}, \bar{y}); \text{epi} \bar{f}) = \{ x^* \in X^* : y^* \in \partial \phi_{e,Y_e^+}(\bar{y}), (x^*, -y^*) \in N((\bar{x}, \bar{y}), \text{epi} \bar{f}) \}.$$  

(51)

Note that for every $(x^*, -y^*) \in N((\bar{x}, \bar{y}), \text{epi} \bar{f})$, and for every pair $(x, y) \in X \times Y_e$, we have

$$\delta((x, y), \text{epi} \bar{f}) - \delta((\bar{x}, \bar{y}), \text{epi} \bar{f}) \geq (x^*, x - \bar{x}) - (y^*, y - \bar{y}).$$

Then for all $(x, y) \in \text{graph} \bar{f}$,

$$0 \geq (x^*, x - \bar{x}) - (y^*, y - \bar{y}) = (x^*, x - \bar{x}) - (y^*, \bar{f}(x) - \bar{f}(\bar{x})).$$

Therefore,

$$x^* \in \partial (y^*, \bar{f})(\bar{x}).$$

(52)

This and Lemma 10 entail

$$\partial (\phi_{e,Y_e^+} \circ \bar{f}(\bar{x})) \subset \bigcup \{ \partial (y^*, \bar{f})(\bar{x}) : y^* \in \partial \phi_{e,Y_e^+}(\theta) \} = \bigcup \{ \partial (y^*, \bar{f})(\bar{x}) : y^* \in Y_e^+, (y^*, e) = 1 \}.$$  

(53)

Since $\partial \phi_{e,Y_e^+}(\theta) = \{ y^* \in Y_e^+ : (y^*, e) = 1 \}$ is a $w^*$-compact convex set in $Y_e^+$.

By the Krein-Milman theorem,

$$\partial \phi_{e,Y_e^+}(\theta) = \overline{w^*} (\text{extr}(\partial \phi_{e,Y_e^+}(\theta))) = \overline{w^*} (K_e).$$

Therefore, for all $m \in \mathbb{N}$, $y_i^* \in (K_e), \lambda_i \geq 0, i = 1, \ldots m$ with $\sum_{i=1}^m \lambda_i = 1$, and $y^* = \sum_{i=1}^m \lambda_i y_i^*$, we obtain

$$\partial (y^*, \bar{f})(\bar{x}) = \partial \left( \sum_{i=1}^m \lambda_i y_i^*, \bar{f}(\bar{x}) \right) = \sum_{i=1}^m \lambda_i \partial (y_i^*, \bar{f})(\bar{x}).$$

Consequently,

$$\bigcup \{ \partial (y^*, \bar{f})(\bar{x}) : y^* \in \partial \phi_{e,Y_e^+}(\theta) \} = \overline{w^*} (\bigcup \{ \partial (y^*, \bar{f})(\bar{x}) : y^* \in K_e \}).$$

(54)

(53) and (54) together imply (49).
As an application of Theorem 18 we will give a necessary condition for a vector variational inequality in infinite dimensional Banach spaces.

Vector variational inequalities have been widely studied since they were introduced by Giannessi [30] in 1980. See, for example, [5,6,8,15]. To due with vector variational inequalities in infinite dimensional spaces, one would encounter significant difficulties of the non-solidness of ordering cones in question. Especially, scalarization methods in vector variational inequalities are surprisingly restricted. (See, for example, Ansari [6].)

Example 8 Let \((K, \sum, \mu)\) be a probability space. Assume that \(x_a, x_b, x_b - x_a \in (L_2^\mu)_N\), and that \(\Omega \equiv \{x \in L_2^\mu : x_a \leq x \leq x_b\}\) is a closed convex set. Suppose \(T : L_2^\mu \rightarrow B(L_2^\mu)\) is a bounded linear operator, which satisfies that for each fixed \(x \in L_2^\mu\), \(Tx\) is \(L_2^\mu\)-Lipschitz on \(\Omega\) with respect to some \(e \in (L_2^\mu)_N^+\) with \(\|e\| = 1\). We use \(y \notin L_2^\mu(0)\) to denote that \(y \notin L_2^\mu \setminus \{0\}\). Then the variational inequality problem is to find a \(\bar{x} \in \Omega\) such that

\[
\langle T\bar{x}, x - \bar{x} \rangle \notin L_2^\mu(0), \text{ for all } x \in \Omega,
\]

(55)

where \(\langle T\bar{x}, x - \bar{x} \rangle = (T\bar{x})(x - \bar{x})\).

Suppose \(\bar{x} \in \Omega\) is a solution to the program (55). Then \(\bar{x}\) is a minimum of following program.

\[
\min \{ T\bar{x}, x - \bar{x} \} \\
\text{s.t. } x \in \Omega.
\]

Let \(X \equiv L_2^\mu = Y\). Since \(x_a, x_b, x_b - x_a \in (L_2^\mu)_N\), \(\pi \equiv \frac{x_b - x_a}{\|x_b - x_a\|} \in (L_2^\mu)_N^+\). Let \(X_\pi\) be the \((Y^+, \pi)\)-generating space. Then \(\Omega \subset X_\pi\), and (by Example 5) \(X_\pi = L_{\omega}^\mu\). Since \(e \in (L_2^\mu)_N^+ \equiv Y^+, \) the \((Y^+, e)\)-generating space \(Y_e\) is also \(L_{\omega}^\mu\). Since \(T\) is \(L_2^\mu\)-Lipschitz on \(\Omega\) with respect to \(e \in (L_2^\mu)_N^+\), we obtain that \(T(\Omega) \subset Y_e = L_{\omega}^\mu\). Therefore, \(X_\pi = (L_{\omega}^\mu)^+ = Y_e\). Consequently, \(\bar{x}\) is a solution of the following scalarization program.

\[
\min \phi_{L_{\omega}^\mu}(\langle T\bar{x}, x - \bar{x} \rangle) \\
\text{s.t. } x \in \Omega \subset L_{\omega}^\mu.
\]

By Theorems 17 and 18

\[
\theta \in \partial \phi_{L_{\omega}^\mu}(\langle T\bar{x}, \cdot \rangle)(\bar{x}) + N(\bar{x}, \Omega) \\
= \bigcup \{ y^* \circ T(\bar{x}) : \langle y^*, e \rangle = 1, y^* \in L_{\omega}^\mu(\mu)^{++} \} + N(\bar{x}, \Omega).
\]

This is a necessary optimality condition of the program (55).

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