Representation Theory and the Quantum Inverse Scattering Method:
The Open Toda Chain and the Hyperbolic Sutherland Model

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Abstract

Using the representation theory of $\mathfrak{gl}(N,\mathbb{R})$, we express the wave function of the $GL(N,\mathbb{R})$ Toda chain, which two of us recently obtained by the Quantum Inverse Scattering Method, in terms of multiple integrals. The main tool is our generalization of the Gelfand-Zetlin method to the case of infinite-dimensional representations of $\mathfrak{gl}(N,\mathbb{R})$. The interpretation of this generalized construction in terms of the coadjoint orbits is given and the connection with the Yangian $Y(\mathfrak{gl}(N))$ is discussed. We also give the hyperbolic Sutherland model eigenfunctions expressed in terms of integrals in the Gelfand-Zetlin representation. Using the example of the open Toda chain, we discuss the connection between the Quantum Inverse Scattering Method and Representation Theory.

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1 Introduction

In the late seventies two basic approaches to quantum integrable systems were formulated. Olshanetsky and Perelomov discovered a direct relation between the representation theory of non-compact semi-simple Lie groups and Sutherland-Calogero-Moser (SCM) systems [1]-[3]. The methods of Representation Theory provide a complete solution to the quantum SCM system. Soon after, Kostant and Kazhdan realized that more general classes of integrable systems, including open Toda chains, can be solved by the same method [4], [5]. The wave functions of the SCM systems and of Toda chains appear as zonal spherical functions [6], [7], and Whittaker functions [8]-[11], respectively. There is a generalization of the above method to a still more general class of integrable systems, formally corresponding to affine Lie algebras and affine quantum groups (see [12], [13] and references therein). However, up to now this approach has not produced explicit representations of the wave functions in terms of integrals because of analytic difficulties in the corresponding representation theory.

At the same time, another approach to quantum integrability emerged. In the works of Faddeev, Sklyanin, Kulish and Takhtajan [14]-[17], the Quantum Inverse Scattering Method (QISM) was formulated and successively applied to numerous quantum integrable systems [19]. A related recent development is connected with the Separation of Variables (SoV) method [20], [21]. Nevertheless, the interrelations of QISM with Representation Theory have remained unclear. The main objective of this paper is to outline an approach towards establishing such a connection.

1 See the summary of the early period of the development in [18].
Our starting point was a new method of deriving the wave function for the $GL(N, \mathbb{R})$ Toda chain given in [22]. This had no obvious relationship with the Representation Theory approach, but instead QISM was used to construct the wave function. The understanding of this construction in terms of Representation Theory, and the comparison of these two basic approaches to quantum integrability for this simple integrable system, should provide additional insight into the nature of QISM. In this paper we give an interpretation of the integral expression of the wave function of the open Toda chain, obtained in [22], in terms of the Representation Theory approach to quantum integrability. The basic tool is a generalization of the Gelfand-Zetlin method [23], [24] to the case of infinite-dimensional representations of $\mathfrak{gl}(N, \mathbb{R})$. This gives a construction of a representation of $\mathcal{U}(\mathfrak{gl}(N, \mathbb{R}))$ in terms of difference operators $^2$, which one can use for finding explicit expressions for Whittaker vectors. Thus, we reproduce the integral formula for Whittaker functions obtained previously in [22] by QISM.

Comparing the methods of [22] with those proposed in this paper reveals a deep connection between the SoV [20] and Gelfand-Zetlin type representations. In [26], [20] the separated variables for the classical Toda chain were found. They could be successfully quantized to give an explicit solution to the corresponding quantum theory. In this paper we use the generalization of the Gelfand-Zetlin representation for $\mathfrak{gl}(N, \mathbb{R})$, which gives rise to a much larger set of separated variables ($\frac{1}{2}N(N-1)$ instead of $(N-1)$). The idea of introducing the enlarged set of separated variables can be traced back to the work of Gelfand and Kirillov on the field of fractions of the universal enveloping algebra [27]. On the other hand, the representation which we have constructed appears to be closely connected with the representation theory of the Yangian $Y(\mathfrak{gl}(N))$. The enlargement of the set of variables for solving the quantum problem is a crucial part of the Representation Theory approach to quantum integrability. One may hope that understanding the correct notion of separated variables in Representation Theory will turn out to be fruitful for both the theory of Quantum Integrable Systems and Representation Theory.

The plan of the paper is as follows. In section 2 we construct the main tool we will be using throughout the paper: we give a direct generalization of the Gelfand-Zetlin method to the case of infinite-dimensional representations of $\mathcal{U}(\mathfrak{gl}(N))$. The crucial point is to use continuous unconstrained variables instead of integer-valued Gelfand-Zetlin parameters. Another type of analytical continuation was considered in [24] (see also [28]). The main results of this section are the analytic construction of a representation of $\mathcal{U}(\mathfrak{gl}(N))$ and explicit description of mutually dual Whittaker modules.

In section 3 we give the interpretation of the representation described in section 2 in terms of the Kirillov-Kostant orbit method [29]. We give an explicit parameterization of the finite cover of the open part of the coadjoint orbit of $GL(N, \mathbb{R})$ with stabilizer $H_N = GL(1, \mathbb{R})^\otimes N$. We show that this parameterization yields Darboux coordinates with respect to the canonical Kirillov-Kostant symplectic structure on the orbit. The above construction is a version of a construction by Alekseev, Faddeev and Shatashvili [30], [31] adopted for non-compact groups. Our parameterization yields the classical counterparts for the generators of $\mathfrak{gl}(N)$

$^2$Incidentally, a version of this construction for the affine algebra $\widehat{\mathfrak{sl}}(2)$ has been discovered previously in [25]. The generalization to the case $\widehat{\mathfrak{sl}}(N)$ will be given elsewhere.
given in section 2. A subtle point of the orbit method interpretation is that we use only
the finite cover of the open part of the orbit. This leads to a representation of \( \mathfrak{gl}(N, \mathbb{R}) \)
which cannot be integrated to a representation of the group. However, it is obvious that
the action of the Cartan subalgebra integrates to the action of a Cartan torus which is sufficient
for constructing solutions of the open Toda chain. We end this section by giving contour
integral formulas for the generators of \( \mathfrak{gl}(N, \mathbb{R}) \).

In section 4 we relate the Gelfand-Zetlin parameterization to the representation theory
of the Yangian \(^32\). (Closely connected ideas for the classical Gelfand-Zetlin construction
were developed in \(^33\)-\(^35\)). In this section we derive explicit formulas in the Gelfand-
Zetlin parameterization for the images of the Drinfeld generators \( A_n(\lambda), B_n(\lambda), C_n(\lambda) \)
under the natural homomorphism: \( Y(\mathfrak{gl}(N)) \to U(\mathfrak{gl}(N)) \). This result will be used in section
6 to establish the connection of the Gelfand-Zetlin parameterization with the QISM-based
approach to the integrability of the quantum open Toda chain.

In section 5 we prove two of the main results of this paper, Theorems 5.1 and 5.2,
by using the methods developed in section 2. We apply the representation constructed in
section 2 to find the explicit solutions of the quantum open Toda chain and the quantum
hyperbolic Sutherland model. For the Toda chain, we reproduce the results of \(^22\). This
bridges the gap between the Representation Theory and QISM approaches for these theories.
The corresponding integral representation for the wave functions of the quantum hyperbolic
Sutherland model appears to be new.

In the last section, we compare the QISM-based approach to the quantum integrability
of the open Toda chain to the Representation Theory approach.

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2 Whittaker modules in the Gelfand-Zetlin representation

2.1 Generalization of the Gelfand-Zetlin construction to infinite-
dimensional representations

We start by recalling the original Gelfand-Zetlin construction for the canonical basis in the
space of an irreducible finite-dimensional representation of \( GL(N, \mathbb{C}) \) \(^{23}, \(^{24}\). It is based
on the two well-known facts.

Any irreducible finite-dimensional representation is uniquely determined by an \( n \)-dimen-
sional vector $\mathbf{m}_N = (m_{N1}, \ldots, m_{NN})$ with integer entries such that $m_{N1} \geq \ldots \geq m_{NN}$. The vector $\mathbf{m}_N$ is called the label or the highest weight of the representation.

Consider $GL(N - 1, \mathbb{C})$ as the subgroup of $GL(N, \mathbb{C})$ embedded in standard way. Then, each of the irreducible representations of the subgroup $GL(N - 1, \mathbb{C})$ enters into the decomposition of a finite-dimensional irreducible representation of $GL(N, \mathbb{C})$ with multiplicity not greater than one. If we denote by $\mathbf{m}_{N-1}$ the labels of the representations involved in the decomposition, then their entries satisfy the additional conditions: $m_{Nj} \geq m_{N-1,j} \geq m_{N,j+1}$, for all $j = 1, \ldots, N - 1$.

If we continue the decomposition step by step, we obtain a decomposition of an irreducible finite-dimensional representation of $GL(N, \mathbb{C})$ into a direct sum of one-dimensional subspaces. Fixing an element in each one-dimensional summand yields a Gelfand-Zetlin basis. The vectors in this basis are labeled by triangular arrays, consisting of all subspaces. Fixing an element in each one-dimensional summand yields a Gelfand-Zetlin basis. The vectors in this basis are labeled by triangular arrays, consisting of all $\mathbf{m}_n$, $n = 1, \ldots, N$, with entries satisfying the above mentioned constraints.

Now, we wish to generalize the Gelfand-Zetlin (GZ) construction to infinite-dimensional representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(N))$.

We start with the following generalization of the Gelfand-Zetlin representation of $\mathfrak{gl}(N)$. Namely, let $(\gamma_1 \ldots \gamma_N)$ be a triangular array consisting of $\frac{1}{2}N(N + 1)$ variables $\gamma_n = (\gamma_{n1}, \ldots, \gamma_{nn}) \in \mathbb{C}^n$; $n = 1, \ldots, N$.

**Proposition 2.1** Let $M$ be the space of meromorphic functions in $\frac{1}{2}N(N - 1)$ variables $\gamma_1, \ldots, \gamma_{N-1}$. Then the operators \(^3\)

\[
E_{nn} = \frac{1}{i\hbar} \left( \sum_{j=1}^{n} \gamma_{nj} - \sum_{j=1}^{n-1} \gamma_{n-1,j} \right) ; \quad (n = 1, \ldots, N), \tag{2.1a}
\]

\[
E_{m,n+1} = -\frac{1}{i\hbar} \sum_{j=1}^{n} \prod_{r=1}^{n+1} \left( \gamma_{nj} - \gamma_{n+1,r} - \frac{i\hbar}{2} \right) \prod_{s \neq j} \left( \gamma_{nj} - \gamma_{ns} \right) e^{-i\hbar\partial_{nj}} ; \quad (n = 1, \ldots, N - 1), \tag{2.1b}
\]

\[
E_{n+1,n} = \frac{1}{i\hbar} \sum_{j=1}^{n} \prod_{r=1}^{n-1} \left( \gamma_{nj} - \gamma_{n-1,r} + \frac{i\hbar}{2} \right) \prod_{s \neq j} \left( \gamma_{nj} - \gamma_{ns} \right) e^{i\hbar\partial_{nj}} ; \quad (n = 1, \ldots, N - 1) \tag{2.1c}
\]

form a representation of $\mathfrak{gl}(N)$ in $M$.

**Proof.** It is sufficient to show that the generators (2.1) satisfy the standard commutation relations

\[
[E_{nn}, E_{m,m+1}] = (\delta_{nm} - \delta_{n,m+1}) E_{m,m+1},
\]

\[
[E_{nn}, E_{m+1,m}] = -(\delta_{nm} - \delta_{n,m+1}) E_{m+1,m}, \tag{2.2}
\]

\[
[E_{n,n+1}, E_{m+1,m}] = (E_{nn} - E_{n+1,n+1}) \delta_{nm}
\]

\(^3\)Actually, we construct the family of representations depending on the auxiliary parameter $\hbar$. One could rescale the variables $\gamma_{nj} \rightarrow \hbar \gamma_{nj}$ to get rid of this parameter. However, we will retain $\hbar$ in the formulas to make the connection with standard notations in the theory of quantum integrable systems more explicit.
and obey the Serre relations

\[
\begin{align*}
[E_{n,n+1}, [E_{n,n+1}, E_{n+1,n+2}]] &= 0, \\
[E_{n+1,n+2}, [E_{n+1,n}, E_{n+1,n+2}]] &= 0, \\
[E_{n+1,n}, [E_{n+1,n}, E_{n+2,n+1}]] &= 0, \\
[E_{n+2,n+1}, [E_{n+1,n}, E_{n+2,n+1}]] &= 0.
\end{align*}
\]

These follow by direct verification. By defining the composite root generators recursively as \( E_{j,k} = [E_{j,m}, E_{m,k}] \) for \( j < m < k \) and \( j > m > k \), one obtains the complete set of commutation relations

\[
[E_{j,k}, E_{l,m}] = E_{j,m}\delta_{lk} - E_{l,k}\delta_{jm}.
\] (2.4)

The representation described above is parameterized by the label \( \gamma_N = (\gamma_{N1}, \ldots, \gamma_{NN}) \) and extends to a representation of the universal enveloping algebra \( U(gl(N)) \). Obviously, there is a natural representation of any subalgebra \( U(gl(n)) \subset U(gl(N)), n = 1, \ldots, N - 1 \) on the same space \( M \) with the label \( \gamma_n = (\gamma_{n1}, \ldots, \gamma_{nn}) \). Let \( Z(gl(n)) \) be the centre of \( U(gl(n)) \).

**Definition 2.1** We say that a \( U(gl(n)) \)-module \( V \) admits an infinitesimal character \( \xi \) if there is a homomorphism \( \xi : Z(gl(n)) \to \mathbb{C} \) such that \( zv = \xi(z)v \) for all \( z \in Z(gl(n)), v \in V \).

It is possible to show that the \( U(gl(N)) \)-module \( M \) defined above admits an infinitesimal character. Actually, a more general statement holds:

**Proposition 2.2** Each central element of \( U(gl(n)) \) acts on \( M \) via multiplication by a symmetric polynomial in the variables \( \gamma_{nj} \).

**Proof.** Let us use the induction over \( n \). For \( n = 1 \) the statement is obvious. Notice that the operators \( u \in U(gl(m)) \) acting on \( M \) are invariant under permutations in each subset of variables \( \gamma_1, \ldots, \gamma_m \). Now suppose that the action of the centres \( Z(gl(m)), m = 1, \ldots, n - 1 \) are given by the functionally independent symmetric polynomials of \( \gamma_m, m = 1, \ldots, n - 1 \). Let us prove that the centres of \( Z(gl(n)) \) act on \( M \) analogously. The elements of the centre \( Z(gl(n)) \) commute with \( Z(gl(m)) \) and thus do not depend on the shifts over \( \gamma_m \). On the other hand, the elements of \( Z(gl(n)) \) commute with all root generators \( E_{ij}, i,j = 1, \ldots, n \) and, hence, are \( \hbar \)-periodic with respect to any variable \( \gamma_{mj}, m = 1, \ldots, n - 1 \). The polynomial generators of the centre of \( U(gl(n)) \) may act only as rational functions in this representation and thus we conclude that they do not depend on the variables \( \gamma_{mj}, m = 1, \ldots, n - 1 \). Taking into account that the generators of \( U(gl(n)) \) depend polynomially on \( \gamma_n \) we conclude that the generators of the centre \( Z(gl(n)) \) act on \( M \) via symmetric polynomials in \( \gamma_n \). The functional independence may be directly verified on a particular subset of \( M \).
Hence, the elements \( z \in Z(\mathfrak{gl}(n)) \) act on the \( \mathcal{U}(\mathfrak{gl}(n)) \)-module \( M \) as operators of scalar multiplication, and the \( \mathcal{U}(\mathfrak{gl}(n)) \)-module \( M, \ n = 1, \ldots, N, \) admits an infinitesimal character.

In the next subsection we calculate the explicit action of the central elements on \( M \) using the notion of Whittaker vectors.

### 2.2 Whittaker modules

Let us now give a construction for Whittaker modules using the representation of \( \mathcal{U}(\mathfrak{gl}(N)) \) described above. We first recall some facts from [5].

Let \( n_+ \) and \( n_- \) be the subalgebras of \( \mathfrak{gl}(N) \) generated, respectively, by positive and negative root generators. The homomorphisms (characters) \( \chi_+: n_+ \to \mathbb{C}, \chi_-: n_- \to \mathbb{C} \) are uniquely determined by their values on the simple root generators, and are called non-singular if the (complex) numbers \( \chi_+(E_{n,n+1}) \) and \( \chi_-(E_{n+1,n}) \) are non-zero for all \( n = 1, \ldots, N - 1 \).

Let \( V \) be any \( \mathcal{U} = \mathcal{U}(\mathfrak{gl}(N)) \)-module. Denote the action of \( u \in \mathcal{U} \) on \( v \in V \) by \( uv \). A vector \( w \in V \) is called a Whittaker vector with respect to the character \( \chi_+ \) if

\[
E_{n,n+1}w = \chi_+(E_{n,n+1})w, \quad (n = 1, \ldots, N - 1), \tag{2.5}
\]

and an element \( w' \in V' \) is called a Whittaker vector with respect to the character \( \chi_- \) if

\[
E_{n+1,n}w' = \chi_-(E_{n+1,n})w', \quad (n = 1, \ldots, N - 1). \tag{2.6}
\]

A Whittaker vector is cyclic for \( V \) if \( \mathcal{U}w = V \), and a \( \mathcal{U} \)-module is a Whittaker module if it contains a cyclic Whittaker vector. The \( \mathcal{U} \)-modules \( V \) and \( V' \) are called dual if there exists a non-degenerate pairing \( \langle \ldots, \ldots \rangle: V' \times V \to \mathbb{C} \) such that \( \langle Xv', v \rangle = -\langle v', Xv \rangle \) for all \( v \in V, \ v' \in V' \) and \( X \in \mathfrak{gl}(N) \).

We proceed with explicit formulas for Whittaker vectors corresponding to the representation given by (2.1).

**Proposition 2.3** The equations

\[
E_{n+1,n}w' = -i\hbar^{-1}w'_N, \tag{2.7}
\]

\[
E_{n,n+1}w = -i\hbar^{-1}w_N \tag{2.8}
\]

for all \( n = 1, \ldots, N - 1, \) admit the solutions

\[
w'_N = 1, \quad w_N = e^{-\frac{\pi}{\hbar} \sum_{n=1}^{N-1} (n-1) \sum_{j=1}^{n} \gamma_{nj}} \prod_{n=1}^{N-1} s_n(\gamma_n, \gamma_{n+1}), \tag{2.9}
\]
where
\[
s_n(\gamma_n, \gamma_{n+1}) = \prod_{k=1}^{n} \prod_{m=1}^{n+1} \hbar^{\gamma_{nk}-\gamma_{n+1,m}} + \frac{1}{\hbar^{\gamma_{nk}-\gamma_{n+1,m}}} \Gamma\left(\frac{\gamma_{nk} - \gamma_{n+1,m}}{i\hbar} + \frac{1}{2}\right).
\]

(2.10)

**Proof.** The following equality holds:
\[
E_{n,n+1} w_N = -i\hbar^{-1} w_N \sum_{j=1}^{n} \prod_{r=1}^{n-1} \frac{\gamma_{nj} - \gamma_{n-1,r} - \frac{i\hbar}{2}}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})}.
\]

(2.11)

Using the identity
\[
\sum_{j=1}^{n} \prod_{s \neq j} \frac{x_j^m}{(x_j - x_s)} = \sum_{k_r \geq 0}^{k_n} x_1^{k_1} \ldots x_n^{k_n}
\]

(2.12)

one arrives at (2.8). Equation (2.7) is proved similarly.

Evidently, the solutions (2.9), (2.10) are not unique. The set of Whittaker vectors is closed under multiplication by an arbitrary \(i\hbar\)-periodic function in the variables \(\gamma_{nj}\). Hence, there are infinitely many invariant subspaces in \(M\) corresponding to infinitely many Whittaker vectors.

To construct irreducible submodules, let us introduce the Whittaker modules \(W\) and \(W'\), generated cyclically by the Whittaker vectors \(w_N\) and \(w'_N\), respectively.

**Theorem 2.1** Let \(m_n = (m_{n1}, \ldots, m_{nn})\) be the set of non-negative integers. The Whittaker module \(W = U w_N\) is spanned by the elements
\[
w_{m_1, \ldots, m_{N-1}} = \prod_{n=1}^{N-1} \prod_{k=1}^{n} \sigma_k^{m_{nk}}(\gamma_n) w_N,
\]

(2.13)

where \(\sigma_k(\gamma_n)\) is the elementary symmetric function of the variables \(\gamma_{n1}, \ldots, \gamma_{nn}\) of order \(k\):
\[
\sigma_k(\gamma_n) = \sum_{j_1 < \ldots < j_k} \gamma_{nj_1} \cdots \gamma_{nj_k}.
\]

(2.14)

Similarly, the Whittaker module \(W' = U w'_N\) is spanned by the polynomials
\[
w'_{m_1, \ldots, m_{N-1}} = \prod_{n=1}^{N-1} \prod_{k=1}^{n} \sigma_k^{m_{nk}}(\gamma_n).
\]

(2.15)

The Whittaker modules \(W\) and \(W'\) are irreducible.
**Proof.** Let us prove the statement for the module $W$. Using the identity $e^{i\hbar\delta_{nj}}\sigma_k(\gamma_n) = \sigma_k(\gamma_n) + i\hbar \sum_{r=1}^{k-1} (-1)^r \sigma_{k-r-1}(\gamma_n) \gamma_n^r$, and the explicit formula for the action of $E_{n+1,n}$ on the Whittaker vector:

$$E_{n+1,n}w_N = i\hbar^{-1}w_N \sum_{j=1}^{n} \prod_{j \neq r}^{n}(\gamma_n - \gamma_{n+1,r} + \hbar) \sum_{j=1}^{n} \prod_{j \neq r}^{n}(\gamma_n - \gamma_{ns}),$$  \hspace{1cm} (2.16)

one shows, due to (2.12), that the Whittaker module is spanned by elements of the form (2.13).

To prove the irreducibility of Whittaker modules we need the following fact (5, Theorem 3.6.2). Let $M$ be any $\mathcal{U}$-module which admits an infinitesimal character. Assume $w \in M$ is a Whittaker vector. Then the submodule $\mathcal{U}w \subset M$ is irreducible.

Proposition 2.2 and the above theorem implies that the modules $W$ and $W'$ are irreducible. $\blacksquare$

Let us note that for any subalgebra $\mathcal{U}(\mathfrak{gl}(n)) \subset \mathcal{U}(\mathfrak{gl}(N))$, $2 \leq n < N$, the module over the ring of the polynomials in $\gamma_n$ with the basis $\prod_{s=1}^{n-1} \prod_{k=1}^{s} \sigma_{m,s}^n(\gamma_s)w_N$ is a $\mathcal{U}(\mathfrak{gl}(n))$ Whittaker module.

Now one can calculate the explicit form of the action of the central elements of $\mathcal{U}(\mathfrak{gl}(n))$ on the space $M$. It is well known that the generating function $A_n(\lambda)$ of the central elements of $\mathcal{U}(\mathfrak{gl}(n))$ (the Casimir operators) can be represented as follows [38]:

$$A_n(\lambda) = \sum_{p \in P_n} \text{sign} p \left[ (\lambda - i\hbar \rho_1^{(n)})\delta_{p(1),1} - i\hbar E_{p(1),1} \right] \cdots \left[ (\lambda - i\hbar \rho_n^{(n)})\delta_{p(n),n} - i\hbar E_{p(n),n} \right], \hspace{1cm} (2.17)$$

where $\rho_k^{(n)} = \frac{1}{2}(n - 2k + 1)$, $k = 1, \ldots, n$ and the summation is over elements of the permutation group $P_n$.

**Proposition 2.4** The operators (2.17) have the following form on $M$:

$$A_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj}), \hspace{1cm} (n = 1, \ldots, N). \hspace{1cm} (2.18)$$

**Proof.** By Proposition 2.2 it is sufficient to calculate the action of $A_n(\lambda)$ on the Whittaker vector $w_N$ (or $w'_N$). Due to (2.8), one has

$$A_n(\lambda)w_N = A_{n-1}(\lambda - \frac{i\hbar}{2})[\lambda - i\hbar \rho_n^{(n)} - i\hbar E_{nn}]w_N$$

$$- i\hbar \sum_{k=2}^{n} A_{n-k}(\lambda - \frac{i\hbar}{2})E_{n,n-k+1}w_N. \hspace{1cm} (2.19)$$

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Furthermore, using the relation \((2.13)\), it is easy to calculate the action of the composite generators:

\[
E_{n,n-k+1}w_N = i\hbar^{-1}w_N \sum_{j_1=1}^{n-1} \cdots \sum_{j_k=1}^{n-k+1} \frac{\prod_{p \neq j_{p-1}} (\gamma_{n-p,j_p} - \gamma_{n-p+1,r_p} + \frac{i\hbar}{2})}{\prod_{s_p \neq j_p} (\gamma_{n-p,j_p} - \gamma_{n-p,s_p})}.
\]

(2.20)

By substituting \((2.20)\) into \((2.19)\), one arrives, after some combinatorial, at \((2.18)\). Obviously, we can also do the analogous calculation for the Whittaker module \(W'\).

It remains to construct a pairing between \(W\) and \(W'\), and to prove that the Whittaker modules \(W\) and \(W'\) are dual with respect to this pairing.

**Definition 2.2** Let \(\phi \in W'\) and \(\psi \in W\). Define the pairing \(\langle \cdot, \cdot \rangle: W' \otimes W \to \mathbb{C}\) by

\[
\langle \phi, \psi \rangle = \int_{\mathbb{R}} \prod_{n=1}^{N(N-1)/2} \mu_0(\gamma) \bar{\phi}(\gamma) \psi(\gamma) \prod_{j \leq n} d\gamma_{nj},
\]

where

\[
\mu_0(\gamma) = \prod_{n=2}^{N-1} \prod_{s < p} (\gamma_{ns} - \gamma_{np})(e^{\frac{2\pi n p}{\hbar}} - e^{\frac{2\pi n s}{\hbar}}).
\]

(2.21)

The integral \((2.21)\) converges absolutely. Actually, this is a simple corollary of a more general statement. Denote the integrand in the right hand side of \((2.21)\) by \(I_N(\gamma)\).

**Lemma 2.1** For \(\gamma_{nj}\) such that \(\min_j \{\text{Im} \, \gamma_{nj}\} > \max_k \{\text{Im} \, \gamma_{n+1,k}\} - \frac{\hbar}{2}\) , the following inequality holds:

\[
|I_N(\gamma)| \leq |P(\gamma)| \exp \left\{ - \frac{1}{(2N-3)!!} \sum_{n=1}^{N-1} \sum_{j=1}^{n} |\text{Re} \, \gamma_{nj}| \right\}
\]

for some polynomial \(P(\gamma)\) in \(\gamma_1, \ldots, \gamma_{N-1}\).

**Proof.** For any \(x > 0\) the inequality \(|\Gamma(x + iy)| \leq \Gamma(x)|p_x(y)| \cosh^{-\frac{1}{2}}(\pi y)\) is valid for some polynomial \(p_x(y)\) with degree depending on \(x\). Hence, the integrand satisfies:

\[
|I_N(\gamma)| \leq |P(\gamma)| e^\frac{\pi}{\hbar} S_N(\gamma),
\]

(2.24)

where

\[
S_N(\gamma) = \sum_{n=2}^{N-1} \sum_{s < p} |\text{Re} \, \gamma_{ns} - \text{Re} \, \gamma_{sp}|
\]

\[
- \frac{1}{2} \sum_{n=1}^{N-2} \sum_{j,k} |\text{Re} \, \gamma_{nj} - \text{Re} \, \gamma_{n+1,k}| - \frac{N}{2} \sum_{j=1}^{N-1} |\text{Re} \, \gamma_{N-1,j}|.
\]

(2.25)
Using the inequality
\[ \sum_{j,k=1}^{m} |a_j - b_k| - \sum_{j<k}^{m} |a_j - a_k| - \sum_{j<k}^{m} |b_j - b_k| \geq 0, \] (2.26)
which holds for any set of real parameters \( \{a_j, b_j\}; j = 1, \ldots, m \) (see (11.118) in [39]), one proves the recursive relation
\[ S_N \leq \frac{1}{2N-3} (S_{N-1} - \sum_j (\text{Re} \gamma_{N-1,j})). \] This implies (2.23).

**Proposition 2.5** Let \( \gamma_N \in \mathbb{R}^N \). The Whittaker modules \( W \) and \( W' \) are dual with respect to the pairing defined by (2.21). I.e. for any \( \phi \in W' \) and \( \psi \in W \), the generators \( X \in \mathfrak{gl}(N, \mathbb{R}) \) possess the property
\[ \langle \phi, X \psi \rangle = - \langle X \phi, \psi \rangle. \] (2.27)

**Proof.** For example, consider \( \langle \phi, E_{n,n+1} \psi \rangle \), where \( E_{n,n+1} \) is defined by (2.1b). It is easy to see that the expression \( E_{n,n+1} \phi \psi \) does not have poles in the upper hyper-plane and it is therefore possible to deform the integration contour. Consider the shifts \( \gamma_{nj} \rightarrow \gamma_{nj} + i \hbar \) and note the difference equation
\[ e^{i \hbar \partial_{\gamma_{nj}}} \mu_0(\gamma) = \mu_0(\gamma) \prod_{s \neq j} \frac{\gamma_{nj} - \gamma_{ns} + i \hbar}{\gamma_{nj} - \gamma_{ns}}. \] (2.28)
Then, using a deformation of the contour and the estimates (2.23), one obtains (2.27).

3 The orbit method for \( GL(N, \mathbb{R}) \) and the Gelfand-Zetlin representation

In this section we discuss the orbit method interpretation of the explicit algebraic construction of the infinite-dimensional representation of \( U(\mathfrak{gl}(N)) \) given in the previous section. Similar results for the case of a compact group were obtained in [30], [31].

Let \( G \) be a Lie group, \( \mathfrak{g} \) the corresponding Lie algebra, and let \( \mathfrak{g}^* \) be its dual. The coadjoint orbits \( O \) of \( G \) are equipped with a canonical symplectic structure given by the Kirillov-Kostant two-form [29]. The space of linear functions on \( \mathfrak{g}^* \) is closed under the Kirillov-Kostant Poisson bracket and coincides with \( \mathfrak{g} \) as a Lie algebra. Consider the coadjoint orbits of \( GL(N, \mathbb{R}) \) with stabilizer \( H_N = GL(1, \mathbb{R})^\otimes N \). Below, we construct Darboux coordinates on the finite cover \( \tilde{O}^{(0)} \) of the open part \( O^{(0)} \) of such an orbit \( O \). We also give expressions in these coordinates for the restriction onto \( O^{(0)} \) of the linear functions on \( \mathfrak{g}^* \). The corresponding Hamiltonian vector fields act on \( O^{(0)} \). The explicit realization of \( U(\mathfrak{gl}(N)) \) by difference operators given in the previous section may be considered as a “quantization” of this Poisson algebra. Let us stress that this construction gives a representation of \( U(\mathfrak{gl}(N)) \) which cannot be integrated to a representation of the group. Our
discussion below shows that this is quite natural because part of the orbit $O^{(0)}$ is not stable with respect to the action of the group on the orbit $O$.

Let $G = GL(N, \mathbb{R})$ and $g = gl(N, \mathbb{R})$. We identify $g$ and its dual $g^*$ via the Killing form. Consider the coadjoint orbit $O$ of the diagonal element $u_N \in g^*$ with pairwise different diagonal entries: $\gamma_N = (\gamma_{N1}, \gamma_{N2}, \ldots, \gamma_{NN})$:

$$u_N = \begin{pmatrix} \gamma_{N1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{NN} \end{pmatrix}.$$ (3.1)

The stabilizer of $u_N$ is $H_N = GL(1, \mathbb{R})^{\otimes N}$ and the general element of the orbit may be parameterized as $u = g^{-1} u_N g$, where $g \in GL(N, \mathbb{R})$ is defined up to the left action of $H_N$. There is a canonical symplectic two-form on $O$, given in this parameterization by

$$\Omega = Tr u_N (dg g^{-1})^2.$$ (3.2)

Consider the subspace in $GL(N, \mathbb{R})$ which consists of the elements:

$$g = gNg_{N-1} \cdots g_2.$$ (3.3)

The matrices $g_n$ differ from the unit matrix only in the upper-left $n \times n$ corner and are defined recursively as follows. Denote the upper-left $(n \times n)$ sub-matrix of $g_n$ by $f_n$. There should exist certain $\gamma_{n-1,j}$, $j = 1, \ldots, n - 1$ so that the following condition holds:

$$f_n^{-1} \gamma_n f_n = \begin{pmatrix} \gamma_{n1} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \gamma_{n,n-1} & 0 \\ 0 & \cdots & 0 & \gamma_{nn} \end{pmatrix},$$ (3.4)

This means that the diagonal matrix conjugated by $f_n$ can have non-zero, non-diagonal entries only in the last column and on the last row. The matrix $f_n$ is defined uniquely by $\gamma_{nj}, \gamma_{n-1,k}$ up to the left action by the diagonal matrix and the right action by the diagonal matrix with the last diagonal element being equal to one. Let us choose the following representatives for $f_n$ in the coset of $G$ by the left action by diagonal matrices:

$$(f_n)_{jk} = \frac{Q_{n-1,k}}{\gamma_{nj} - \gamma_{n-1,k}} \prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r}) \prod_{s \neq j} (\gamma_{nj} - \gamma_{ns}), \quad (k = 1, \ldots, n - 1),$$ (3.5)

where $Q_{n-1,k}$, $k = 1, \ldots, n - 1$ are additional coordinates on the orbit. This gives a map from $\mathbb{R}^{N(N-1)} \setminus \Delta$ to the open part $O^{(0)}$ of the orbit. Here, $\Delta$ is a union of three
subspaces of $\mathbb{R}^{N(N-1)}$: $\Delta = D_1 \cup D_2 \cup D_3$. The subspace $D_1$ is a subspace where at least two $\gamma$-coordinates of the same level coincide: $\gamma_{nj} = \gamma_{nk}$. The subspace $D_2$ is defined as a subspace where at least two of the $\gamma$-coordinates from consecutive levels coincide $\gamma_{nj} = \gamma_{n-1,k}$. Finally, the subspace $D_3$ is a subspace where at least one of $Q_{nj}$ is zero.

Now, let us construct the inverse map. Consider the element of the orbit $u = g^{-1}u_{\gamma}g$ shifted by the $N \times N$ unit matrix $I$ multiplied by a formal variable $\lambda$:

$$T(\lambda) = \lambda I - u. \quad (3.6)$$

The $n$-th principal minor $a_n(\lambda)$ of $T(\lambda)$ does not depend on $g_m$, $m < n+1$. For the general point of the orbit, the roots of the minors may be complex. Consider the open part $O^{(0)}$ of the orbit $\mathcal{O}$ such that all upper-left sub-matrices are diagonalizable and their determinants have distinct real roots. We also impose the condition that there are no identical roots of minors with consecutive ranks. Then, define the $\gamma$-variables to be the roots of the minors:

$$a_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj}). \quad (3.7)$$

Note that, by this condition the roots are defined only up to the action of the symmetric group $S_n$. As a result we obtain the coordinates on the finite cover of an open part of the orbit.

Now, we give the explicit expressions for the coordinates $Q_{nj}$. Let $b_n(\lambda)$ be the determinant of the $n \times n$ corner sub-matrix of the matrix obtained from $T(\lambda)$ by interchanging the $n$-th and the $(n+1)$-th columns. I.e. $b_n(\lambda) = \det \langle T(\lambda)S_{n,n+1} \rangle$, where the $N \times N$ matrix $S_{n,n+1}$ is a unit matrix with the diagonal $2 \times 2$ sub-matrix in the $n$, $n+1$ rows and the $n$, $n+1$ rows replaced by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Similarly, we define $c_n(\lambda)$ as the determinant of the $n \times n$ corner sub-matrix obtained from $T(\lambda)$ by interchanging the $n$-th and the $(n+1)$-th rows; i.e. $c_n(\lambda) = \det \langle S_{n,n+1}T(\lambda) \rangle$. Then, the rest of the variables may be introduced as follows:

$$Q_{nj} = b_n^{-1}(\gamma_{nj})a_{n+1}(\gamma_{nj}) = -c_n(\gamma_{nj})a_{n-1}^{-1}(\gamma_{nj}), \quad (3.8)$$

where the last equality can be proved by a straightforward check. It is not difficult to verify that this definition is compatible with the explicit parameterization in terms of $(\gamma_{nj}, Q_{nj})$ given above. Due to the condition (3.1), the $n$-th left-upper corner sub-matrix of the matrix $u_n = g_{n+1}^{-1}g_1^{-1}u_{\gamma}g_N \cdots g_{n-1}$ is diagonal matrix; $a_n(\lambda)$ is given by (3.7). The determinant $b_n(\lambda)$ is invariant under conjugation by the matrix which differs from the unit matrix only in the upper-left $(n-1) \times (n-1)$ corner. Thus, $b_n(\lambda)$ may be represented as:

$$b_n(\lambda) = \det(g_n^{-1}) \det \langle (\lambda - u_n)g_nS_{n,n+1} \rangle. \quad (3.9)$$

A straightforward calculation gives

$$b_n(\lambda) = \sum_{j=1}^{n} Q_{nj}^{-1} \prod_{r=1}^{n+1}(\gamma_{nj} - \gamma_{nj+1,r}) \prod_{s \neq j}^{\lambda - \gamma_{ns}} \frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}}. \quad (3.10)$$
Similarly, for \( c_n(\lambda) \) one has
\[
c_n(\lambda) = - \sum_{j=1}^{n} Q_{nj} \prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r}) \prod_{s \neq j} \frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}}.
\] (3.11)

This gives the inverse transformation from the finite cover of the open part of the orbit \( \tilde{O}^{(0)} \) to \( \mathbb{R}^{N(N-1)/2} \setminus \Delta \).

A simple calculation of the Kirillov-Kostant symplectic form in the coordinates \((\gamma_{nj}, Q_{nj})\) gives the following:

**Proposition 3.1** 1. The coordinates \((\gamma_{nj}, Q_{nj})\) parameterize the finite cover \( \tilde{O}^{(0)} \) of the open part of the orbit \( O^{(0)} \). The product of the symmetric groups \( S = \prod_{n=2}^{N} S_n \) acts freely on the fibres of the projection: \( \pi : \tilde{O}^{(0)} \to O^{(0)} \) and the factor is \( O^{(0)} \).

2. The lift of the 2-form \( \Omega \) on \( \tilde{O}^{(0)} \) has the canonical form
\[
\pi^* \Omega = \sum_{n=1}^{N-1} \sum_{j=1}^{n} d\gamma_{nj} \wedge Q_{nj}^{-1} dQ_{nj}.
\] (3.12)

The coordinates \((\gamma_{nj}, Q_{nj})\) give Darboux coordinates on the finite cover of the open part \( O^{(0)} \) of the coadjoint orbit of \( \text{GL}(N, \mathbb{R}) \) with the stabilizer conjugated to \( H_N = \text{GL}(1, \mathbb{R}) \otimes \mathbb{R}^N \).

Now let us give explicit expressions for the elements on the super-diagonal, diagonal and over-diagonal of the matrix \( u = g^{-1}u_N g \) in the GZ-coordinates. These give the classical counterparts of the generators (2.1).

**Proposition 3.2** The linear functions \( u_{nn}, u_{n,n+1}, u_{n+1,n} \) on \( \mathfrak{gl}(N, \mathbb{R})^* \), being restricted to \( O^{(0)} \) and lifted by the projection map \( \pi : \tilde{O}^{(0)} \to O^{(0)} \) to \( \tilde{O}^{(0)} \), have the following form in GZ-coordinates 4:
\[
\begin{align*}
u_{nn} &= \sum_{j=1}^{n} \gamma_{nj} - \sum_{j=1}^{n-1} \gamma_{n-1,j}, \\
u_{n,n+1} &= - \sum_{j=1}^{n} \prod_{r=1}^{n+1} (\gamma_{nj} - \gamma_{n+1,r}) \prod_{s \neq j} \frac{Q_{nj}}{(\gamma_{nj} - \gamma_{ns})}, \\
u_{n+1,n} &= \sum_{j=1}^{n} \prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r}) \prod_{s \neq j} \frac{Q_{nj}}{(\gamma_{nj} - \gamma_{ns})}.
\end{align*}
\] (3.13)

4There is another representation with “square roots” which is closer to that given in [23]. It may be obtained by a different choice of parameterization for the solution of [3.4].
Proof. Let us start with the diagonal elements. The coadjoint action of the diagonal matrix $\Phi = \text{diag}(e^{\phi_1}, \cdots, e^{\phi_N})$ on $u$ may be represented as the right action on the element $g$, (3.13). It is easy to see that this is equivalent to a shift of the variables:

$$Q_{nj} \rightarrow Q_{nj} e^{\phi_{n-\phi_{n+1}}},$$

$$Q_{Nj} \rightarrow Q_{Nj} e^{\phi_N}.$$  

Thus, the corresponding Hamiltonian functions $u_{nn}$ are:

$$u_{nn} = \sum_{j=1}^{n} \gamma_{nj} - \sum_{j=1}^{n-1} \gamma_{n-1,j}.$$  

(3.15)

This result could also be derived from the explicit representation of the matrix $u$. Moreover, from the structure of the matrices $u_n, f_n$ one could easily obtain expressions for the other generators. For example,

$$u_{n+1,n} = \sum_{j=1}^{n} (u_n)_{n+1,j} (f_n)_{jn} = \sum_{j=1}^{n} \prod_{r=1}^{n-1} \frac{(\gamma_{nj} - \gamma_{n-1,r})}{\prod_{s \neq j}(\gamma_{nj} - \gamma_{ns})} Q_{nj}.$$  

(3.16)

Quite similarly,

$$u_{n,n+1} = -\sum_{j=1}^{n} \prod_{r=1}^{n+1} \frac{(\gamma_{nj} - \gamma_{n+1,r})}{\prod_{s \neq j}(\gamma_{nj} - \gamma_{ns})} Q_{nj}^{-1}.$$  

(3.17)

There is a representation for $u_{n,n}$, $u_{n,n+1}$, and $u_{n+1,n}$ in terms of contour integrals using the polynomials $a_n(\lambda), b_n(\lambda)$, and $c_n(\lambda)$:

$$u_{n,n} = -\frac{1}{2\pi i} \oint \frac{a_n(\lambda)}{a_{n-1}(\lambda)} d\lambda,$$

$$u_{n,n+1} = -\frac{1}{2\pi i} \oint a_n^{-1}(\lambda) b_n(\lambda) d\lambda,$$

$$u_{n+1,n} = -\frac{1}{2\pi i} \oint c_n(\lambda) a_n^{-1}(\lambda) d\lambda.$$  

(3.18)

In the next section we discuss the connection of this representation with Drinfeld’s “new realization” of the Yangian [32].

4 The Yangian $Y(\mathfrak{gl}(N))$ and the Gelfand-Zetlin representation

In the previous section we outlined the construction of the Gelfand-Zetlin parameterization of the coadjoint orbit of $GL(N, \mathbb{R})$. The polynomials $a_n(\lambda), b_n(\lambda)$ and $c_n(\lambda)$, which appear
in the invariant formulation of this parameterization, bear an obvious similarity to the basic ingredients of Drinfeld’s “new realization” of the Yangian. In this section, we derive the explicit expressions for the Drinfeld generators of the Yangian $Y(\mathfrak{gl}(N))$ in the Gelfand-Zetlin realization\(^5\). These explicit expressions will be important for a new interpretation of the QISM approach to the solution of the open Toda chain [22].

We recall some well-known facts about the Yangian $Y(\mathfrak{gl}(N))$, [32] (see also the recent review [34]). The Yangian $Y(\mathfrak{gl}(N))$ is an associative Hopf algebra generated by the elements $T_{ij}^{(r)}$, where $i, j = 1, \ldots, N$ and $r = 0, \ldots, \infty$, subject the following relations. Consider the $N \times N$ matrix $T(\lambda) = ||T_{ij}(\lambda)||_{i,j=1}^N$ with operator valued entries

$$T_{ij}(\lambda) = \lambda \delta_{ij} + \sum_{r=0}^{\infty} T_{ij}^{(r)} \lambda^{-r}. \quad (4.1)$$

Let

$$R_N(\lambda) = I \otimes I + i\hbar P/\lambda; \quad P_{ik,jl} = \delta_{il}\delta_{kj} \quad (4.2)$$

be an $N^2 \times N^2$ numerical matrix (the Yang $R$-matrix). Then the relations between the generators $T_{ij}^{(r)}$ can be written in the standard form

$$R_N(\lambda - \mu)(T(\lambda) \otimes I)(I \otimes T(\mu)) = (I \otimes T(\mu))(T(\lambda) \otimes I)R_N(\lambda - \mu). \quad (4.3)$$

The centre of the Yangian is generated by the coefficients of the following formal Laurent series (the quantum determinant of $T(\lambda)$ in the sense of [19]):

$$\det_q T(\lambda) = \sum_{p \in P_N} \text{sign} \ p \ T_{p(1),1}(\lambda - i\hbar \rho_1^{(N)}) \cdots T_{p(k),k}(\lambda - i\hbar \rho_k^{(N)}) \cdots T_{p(N),N}(\lambda - i\hbar \rho_N^{(N)}), \quad (4.4)$$

where $\rho_p^{(N)} = \frac{1}{2}(N - 2n + 1), \ (n = 1, \ldots, N)$ and the summation is over elements of the permutation group $P_N$. Let $X(\lambda) = ||X_{ij}(\lambda)||_{i,j=1}^N$ be an $n \times n$ sub-matrix of the matrix $||T_{ij}(\lambda)||_{i,j=1}^N$. It is obvious from the explicit form of $R_N(\lambda)$ that this sub-matrix satisfies an analogue of the relations [13]. The quantum determinant $\det_q X(\lambda)$ is defined similarly to [14] (with the evident change $N \to n$).

The following way to describe the Yangian $Y(\mathfrak{gl}(N))$ was introduced in [32]. Let $A_n(\lambda), \ n = 1, \ldots, N,$ be the quantum determinants of the sub-matrices, determined by the first $n$ rows and columns, and let the operators $B_n(\lambda), C_n(\lambda), \ n = 1, \ldots, N - 1,$ be the quantum determinants of the sub-matrices with elements $T_{ij}(\lambda)$, where $i = 1, \ldots, n; j = 1, \ldots, n-1, n+1$ and $i = 1, \ldots, n-1, n+1; j = 1, \ldots, n$, respectively. The expansion coefficients of $A_n(\lambda), B_n(\lambda), C_n(\lambda), \ n = 1, \ldots, N - 1,$ with respect to $\lambda$, together with those of $A_N(\lambda)$, generate the algebra $Y(\mathfrak{gl}(N))$. The parts of the relations between the Drinfeld generators,
Proposition 4.1

We arrive at the following definition of the Yangian.

\[ [A_n(\lambda), A_m(\mu)] = 0; \quad (n, m = 1, \ldots, N), \]
\[ [B_n(\lambda), B_m(\mu)] = 0; \quad [C_n(\lambda), C_m(\mu)] = 0; \quad (m \neq n \pm 1), \]
\[ (\lambda - \mu + ih)A_n(\lambda)B_n(\mu) = (\lambda - \mu)B_n(\mu)A_n(\lambda) + ihA_n(\mu)B_n(\lambda), \]  
\[ (\lambda - \mu + ih)A_n(\mu)C_n(\lambda) = (\lambda - \mu)C_n(\lambda)A_n(\mu) + ihA_n(\lambda)C_n(\mu). \]  

(4.5)

Let \( A(\mathfrak{gl}(N)) \) be the commutative subalgebra of \( Y(\mathfrak{gl}(N)) \) generated by \( A_n(\lambda), n = 1, \ldots, N \). It was proved in [37] that \( A(\mathfrak{gl}(N)) \) is the maximal commutative subalgebra of \( Y(\mathfrak{gl}(N)) \).

There is a natural epimorphism \( \pi_N : Y(\mathfrak{gl}(N)) \to \mathcal{U}(\mathfrak{gl}(N)) \),

\[ \pi_N(T_{jk}(\lambda)) = \lambda \delta_{jk} - ih E_{jk}, \quad (j, k = 1, \ldots, N). \]  

(4.6)

Denote the images under \( \pi_N \) of the generators \( A_n(\lambda), B_n(\lambda), \) and \( C_n(\lambda) \) by \( A_n(\lambda), B_n(\lambda) \) and \( C_n(\lambda) \), respectively. Let us describe the images of the Drinfeld generators. The images of the generators of \( A(\mathfrak{gl}(N)) \) under the homomorphism (4.6) have the form (2.17). In particular, the image of \( A_n(\lambda) \), which is a polynomial in \( \lambda \) of order \( N \), gives a generating function for the generators of the centre \( \mathcal{Z} \subset \mathcal{U}(\mathfrak{gl}(N)) \). For the other generators \( B_n(\lambda), C_n(\lambda) \) we have

**Lemma 4.1**

\[ B_n(\lambda) = [A_n(\lambda), E_{n,n+1}], \]  

(4.7)

\[ C_n(\lambda) = [E_{n+1,n}, A_n(\lambda)], \]  

(4.8)

where \( n = 1, \ldots, N - 1 \).

**Proof.** A direct computation using (4.4) and the explicit expressions for the images of the generators \( A_n(\lambda), B_n(\lambda), C_n(\lambda) \) under the homomorphism \( \pi_N \). □

Using (4.7), (4.8) and the explicit expressions for the generators \( E_{jk} \) introduced in Proposition 2.1 we arrive at the following

**Proposition 4.1** The operators

\[ A_n(\lambda) = \prod_{j=1}^{n} (\lambda - \gamma_{nj}), \]
\[ B_n(\lambda) = \sum_{j=1}^{n} \prod_{s \neq j}^{n} \frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}} \prod_{r=1}^{n+1} (\gamma_{nj} - \gamma_{n+1,r} - \frac{ih}{2}) e^{-ih \partial_{nj}}, \]  
\[ C_n(\lambda) = -\sum_{j=1}^{n} \prod_{s \neq j}^{n} \frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}} \prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r} + \frac{ih}{2}) e^{ih \partial_{nj}}. \]  

Define a representation of the Yangian.
We notice that the explicit expressions for \( A_n(\lambda) \), \( B_n(\lambda) \), \( C_n(\lambda) \) may be obtained directly from the defining relations of the Yangian. One may start with the following representation of the maximal commutative subalgebra \( A(\mathfrak{gl}(N)) \) by polynomials with real roots: 
\[
A_n(\lambda) = \prod_{j=1}^{n} (\lambda - \gamma_{nj}), \quad n = 1, \ldots, N.
\]
Then, we should resolve the rest of the Yangian relations to find the explicit expressions for the generators \( B_n(\lambda) \) and \( C_n(\lambda) \) in terms of some operators acting on the space of functions depending on the variables \( \gamma_{nj}, j = 1, \ldots, n; n = 1, \ldots, N-1 \). The full set of Yangian relations in terms of \( A_n(\lambda), B_n(\lambda), C_n(\lambda) \) is known implicitly through the “new realization” by Drinfeld \[32\] (example after Theorem 1). This may be used to fix \( B_n(\lambda), C_n(\lambda) \) up to conjugation by an arbitrary function of \( \gamma_{nj} \). The expressions in (4.9) are obtained by choosing the appropriate representative.

The following integral formulas (which may be considered as a “quantization” of the classical relations (3.18)) express the generators of \( \mathcal{U}(\mathfrak{gl}(N)) \) in terms of \( A_n(\lambda), B_n(\lambda), C_n(\lambda) \):
\[
\begin{align*}
E_{n,n+1} &= \frac{1}{2\pi\hbar} \oint A^{-1}(\lambda)B_n(\lambda)d\lambda, \\
E_{n+1,n} &= \frac{1}{2\pi\hbar} \oint C_n(\lambda)A^{-1}(\lambda)d\lambda, \\
E_{nn} &= \frac{1}{2\pi\hbar} \oint \frac{A_n(\lambda)}{A_{n-1}(\lambda - \frac{i\hbar}{2})} \frac{d\lambda}{\lambda} - \frac{1}{2}(n-1).
\end{align*}
\]
Here, the integrands are understood as Laurent series and the contours of integrations are taken around \( \infty \). This representation is similar to the expressions of the generators in the “new realization” of the Yangian through \( A_n(\lambda), B_n(\lambda), C_n(\lambda) \) \[32\] (example after Theorem 1).

5 Application to quantum integrable systems

5.1 The open Toda chain

The open Toda chain corresponding to \( \mathfrak{gl}(N,\mathbb{R}) \) is one of the simplest examples of an integrable quantum mechanical system with \( N \) degrees of freedom (see \[40\] and references therein). It has \( N \) mutually commuting Hamiltonians, the first two of which are given by
\[
\begin{align*}
h_1 &= \sum_{j=1}^{N} p_j, \\
h_2 &= \sum_{j<k} p_j p_k - \sum_{j=1}^{N-1} e^{x_j-x_{j+1}},
\end{align*}
\]
where \([x_n, p_m] = i\hbar\delta_{nm}\).

Recall the general idea behind the Representation Theory approach to quantum integrability in the case of the open Toda chain. Let \( V \) and \( V' \) be any dual irreducible Whittaker
can be expressed in the form that the action of the Cartan subalgebra is integrated to the action of the Cartan torus, so that the following function is well defined

\[ \psi_{\gamma_1,\ldots,\gamma_N} = e^{-x^{\bullet} \rho(N)} \langle w'_N, e^{-\sum_{k=1}^N x_k E_{kk}} w_N \rangle. \] (5.2)

This is nothing but the \( GL(N, \mathbb{R}) \) Whittaker function \[8\] written in terms of the Gauss decomposition \[11\]. In the case of irreducible Whittaker modules, the action of the elements of the centre \( Z \) of the universal enveloping algebra \( U(\mathfrak{gl}(N, \mathbb{R})) \) is proportional to the action of the unit operator \[5\] (Theorem 3.6.1). Consider the function

\[ \tilde{\psi}_{\gamma_1,\ldots,\gamma_N} = e^{-x^{\bullet} \rho(N)} \langle w'_N, e^{-\sum_{k=1}^N x_k E_{kk}} z w_N \rangle, \] (5.3)

where \( z \) belongs to the centre \( Z \). Using the properties of the Whittaker vectors one could show that there is a differential operator \( D_z \) in the variables \( x_k \) such that \( \tilde{\psi}_{\gamma_1,\ldots,\gamma_N} = D_z \psi_{\gamma_1,\ldots,\gamma_N} \). Thus, taking the first two Casimir operators, one gets the first two Hamiltonians \[5.1\]. On the other hand, \( z \) is an element of the centre \( Z \) and \( \tilde{\psi}_{\gamma_1,\ldots,\gamma_N} \) is proportional to \( \psi_{\gamma_1,\ldots,\gamma_N} \) with some numerical coefficient. Hence, we get the common eigenfunction for the set of differential operators corresponding to the elements of \( Z \).

Let us give an explicit realization of the representation of \( U(\mathfrak{gl}(N, \mathbb{R})) \) in terms of some difference/differential operators, which integrates to the representation of the Cartan group. This will lead to the integral formula for the wave function of the open Toda chain. To find such a formula for the GZ-representation, we substitute the expressions \( (2.9), (2.10), \) and \( (2.1a) \) into \( (5.2) \), thus obtaining

\[ \psi_{\gamma_1,\ldots,\gamma_N} = e^{-x^{\bullet} \rho(N)} \times \int \prod_{n=1}^{N-1} \prod_{k=1}^{n} \prod_{m=1}^{n+1} \frac{\gamma_{nk} - \gamma_{n+1,m}}{i\hbar} + \frac{1}{\hbar} \frac{1}{\prod_{s<p} \Gamma(\gamma_{nk} - \gamma_{np})^2} e^{\pm \sum_{n,j=1}^N (\gamma_{nj} - \gamma_{n-1,j}) x_n \prod_{n=1}^{N-1} d\gamma_{nj}}, \] (5.4)

where by definition \( \gamma_{nj} = 0 \) for \( j > n \).

In the study of the analytic properties of this solution with respect to \( \gamma_N \), the following reformulation is very useful:

**Theorem 5.1** An analytical continuation of the eigenfunction \( (5.4) \) as a function of \( \gamma_N \) can be expressed in the form

\[ \psi_{\gamma_N}(x_1, \ldots, x_N) = \int \prod_{n=1}^{N-1} \prod_{k=1}^{n} \prod_{m=1}^{n+1} \frac{\gamma_{nk} - \gamma_{n+1,m}}{i\hbar} \frac{\Gamma(\gamma_{nk} - \gamma_{n+1,m})}{\prod_{s<p} \Gamma(\gamma_{ns} - \gamma_{np})^2} e^{\pm \sum_{n,j=1}^N (\gamma_{nj} - \gamma_{n-1,j}) x_n \prod_{n=1}^{N-1} d\gamma_{nj}}, \] (5.5)
where the domain of integration $S$ is defined by the conditions $\min_j \{\text{Im} \gamma_{kj}\} > \max_m \{\text{Im} \gamma_{k+1,m}\}$ for all $k = 1, \ldots, N - 1$. The integral (5.5) converges absolutely.

**Proof.** Let us change the variables of integration in (5.4):

$$
\gamma_{nj} \rightarrow \gamma_{nj} - \frac{i\hbar}{n} \sum_{s=1}^{n} \rho_{s}^{(n)}, \quad (n = 1, \ldots, N - 1),
$$

(5.6)

and recall that $\rho_{s}^{(n)} = \frac{1}{n} (N - 2s + 1)$. After elementary calculations, the integral in (5.4) acquires the form (5.5). It is worth mentioning that after the change of variables (5.6) we shift the domain of integration $\mathbb{R}^{N(N-1)/2}$ to the complex plane in such a way that the domain of integration over the variables $\gamma_{n-1,j}$ lies above the domain of integration over the variables $\gamma_{nj}$. Thus, we arrive at the analytic continuation of the wave function described in the theorem.

**Example 5.1** Let $N = 2$. In this case the general formula (5.5) acquires the form

$$
\psi_{\gamma_{21},\gamma_{22}}(x_1, x_2) = \frac{e^{i(x_1 + x_2)} e^{\psi_{\gamma_{21},\gamma_{22}}}}{\pi \hbar} e^{\psi_{\gamma_{21}}(x_1)} e^{\psi_{\gamma_{22}}(x_2)} d\gamma_{11},
$$

(5.7)

where $\sigma > \max \{\text{Im} \gamma_{21}, \text{Im} \gamma_{22}\}$. Using the formulas 7.2(15) and 7.12(34) from [47], one obtains the solution for the $\text{GL}(2, \mathbb{R})$ Whittaker function in terms of the Macdonald function $K_{\nu}(z)$:

$$
\psi_{\gamma_{21},\gamma_{22}}(x_1, x_2) = 4\pi \hbar e^{\psi_{\gamma_{21}}(x_1 + x_2) / \hbar} K_{\gamma_{21} - \gamma_{22}} \left( \frac{2}{\hbar} e^{(x_1 - x_2)/2} \right).
$$

(5.8)

The explicit form of the eigenfunction and the absolute convergence of the integral lead to a recursive relation between the wave functions corresponding to $\mathfrak{gl}(n)$ and $\mathfrak{gl}(n - 1)$; this relation is a direct consequence of the Gelfand-Zetlin inductive procedure:

**Corollary 5.1** Let us fix the solution $\psi_{\gamma_{11}}(x_1) = e^{\pi \gamma_{11} x_1}$ of the $\text{GL}(1, \mathbb{R})$ Toda chain. For any $n = 2, \ldots, N$ there is a recursive relation between the wave functions $\psi_{\gamma_n}$ and $\psi_{\gamma_{n-1}}$:

$$
\psi_{\gamma_n}(x_1, \ldots, x_n)
= \int_{S_n} \mu^{(n-1)}(\gamma_{n-1}) f_n(\gamma_n, \gamma_{n-1}) e^{\psi_{\gamma_{n-1}}(x_1, \ldots, x_{n-1})} \prod_{j=1}^{n-1} d\gamma_{n-1,j},
$$

(5.9)

where

$$
\mu^{(n-1)}(\gamma_{n-1}) = \prod_{s \neq p} \left[ \frac{\Gamma(\frac{\gamma_{n-1,s} - \gamma_{n-1,p}}{\hbar})}{\hbar} \right]^{-1},
$$

(5.10)
and

\[ f_n(\gamma_n, \gamma_{n-1}) = \prod_{k=1}^{n-1} \prod_{m=1}^{n} \hbar^{\gamma_{n-1,k} - \gamma_{nm}} \Gamma\left(\frac{\gamma_{n-1,k} - \gamma_{nm}}{i\hbar}\right). \] (5.11)

The domain of integration \( S_n \) is defined by the conditions \( \min_j \{\text{Im} \gamma_{n-1,j}\} > \max_m \{\text{Im} \gamma_{nm}\} \).

The above results are in complete agreement with the recursive relations between the wave functions and the explicit solution in terms of the Mellin-Barnes representation obtained previously in [22] by the QISM-based approach.

### 5.2 The hyperbolic Sutherland model

In this section we give a short derivation of the integral representation for a wave function of the hyperbolic Sutherland model.

The quantum hyperbolic Sutherland model corresponding to \( \mathfrak{g}l(N) \) is an integrable quantum mechanical system with \( N \) degrees of freedom (see [2] and references therein). It has \( N \) mutually commuting Hamiltonians, the first two of which are given by

\[
h_1 = \sum_{n=1}^{N} p_n, \]
\[
h_2 = \sum_{m<n} \left\{ p_n p_m + \frac{\hbar^2/4}{\sinh^2(x_m - x_n)} \right\}. \] (5.12)

The eigenfunction of the Hamiltonians of the quantum hyperbolic Sutherland model is defined by the equations:

\[
\sum_{n=1}^{N} p_n \Psi = \sigma_1(\gamma_N) \Psi, \\
\sum_{m<n} \left\{ p_n p_m + \frac{\hbar^2/4}{\sinh^2(x_m - x_n)} \right\} \Psi = \sigma_2(\gamma_N) \Psi. \] (5.13)

In a similar fashion to the open Toda chain, the hyperbolic Sutherland model can be explicitly solved using the Representation Theory approach. Let \( V \) be a representation of \( \mathcal{U}(\mathfrak{g}l(N, \mathbb{R})) \) and \( v_N \) be a vector in \( V \) such that

\[
(E_{n,n+1} - E_{n+1,n}) v_N = 0, \quad (n = 1, \ldots, N - 1). \] (5.14)

Consider the matrix element \( \Phi(x_1, \ldots, x_N) \) of the form

\[
\Phi = \langle v_N, e^{-\sum_{k=1}^{N} x_k E_{kk}} v_N \rangle. \] (5.15)
One can show that (5.15) gives (up to a simple factor) the eigenfunction of the Hamiltonian of the hyperbolic Sutherland model. To derive this, consider the more general matrix elements:

\[ \Phi_r = \langle v_N, e^{-\sum_{k=1}^{N} x_k E_{kk}} a_{N+r} v_N \rangle, \]  

where \( a_{N+r} \) are the coefficients of the expansion of \( A_N(\lambda) \) (see (2.17)): \( A_N(\lambda) = \lambda^N - \lambda^{N-1} a_{N1} + \lambda^{N-2} a_{N2} + \ldots \). The first two have the form:

\[ a_{N1} = i \hbar \sum_{n=1}^{N} E_{nn}, \]
\[ a_{N2} = -\hbar^2 \left\{ \sum_{m<n} E_{mm} E_{nn} - \sum_{n=1}^{N} \rho_n^{(N)} E_{nn} - \sum_{m<n} E_{nm} E_{mn} + \sigma_2(\rho^{(N)}) \right\}. \]  

This should be compared with the discussion of the open Toda chain (5.3). The same line of reasoning shows that the matrix element (5.15) satisfies the differential equations

\[ -i \hbar \sum_{n=1}^{N} \partial_{x_n} \Phi = \sigma_1(\gamma_N) \Phi, \]
\[ -\hbar^2 \sum_{m<n} \left\{ \partial_{x_m} \partial_{x_n} - \frac{1}{2} \coth(x_m - x_n)(\partial_{x_m} - \partial_{x_n}) \right\} \Phi = \left( \sigma_2(\gamma_N) + h^2 \sigma_2(\rho^{(N)}) \right) \Phi. \]  

Actually, \( \Phi \) is the \( GL(N, \mathbb{R}) \) zonal spherical function \([6]-[7]\).

Finally, the wave function \( \Psi \) defined by

\[ \Psi = \Phi \prod_{j<k} \sinh^{1/2}(x_j - x_k) \]  

satisfies the equations (5.13).

Now, we give a new explicit integral formula for the eigenfunction of the hyperbolic Sutherland model using the GZ representation discussed in section 2. First, let us find the vector \( v_N \) satisfying (5.14).

**Lemma 5.1** Let

\[ \phi_n(\gamma_n, \gamma_{n+1}) = \prod_{k=1}^{n} \prod_{m=1}^{n+1} h^{\frac{\gamma_{nk} - \gamma_{n+1,m}}{2m}} \Gamma \left( \frac{\gamma_{nk} - \gamma_{n+1,m} + 1}{2i \hbar} \right). \]  

Then the vector

\[ v_N = e^{-\sum_{n=1}^{N-1} (n-1) \sum_{j=1}^{n} \gamma_{nj} \prod_{n=1}^{N-1} \phi_n(\gamma_n, \gamma_{n+1})} \]  

satisfies the equations (5.14).
Proof. Straightforward check.

Now we have the following

**Theorem 5.2** The solution to (5.13) admits the integral representation:

\[
\Psi_{\gamma N_1, \ldots, \gamma N_N}(x_1, \ldots, x_N) = \prod_{j<k} \sinh^{1/2}(x_j - x_k)
\]

\[
\times \int_{\mathbb{R}^N / 2} \prod_{n=1}^{N-1} \prod_{k=1}^{n} \prod_{m=1}^{n+1} \left| \Gamma\left(\frac{\gamma_{nk} - \gamma_{n+1,m}}{2\hbar} + \frac{1}{2}\right) \right|^2 e^{\pi \sum_{n,j=1}^{N} (\gamma_{nj} - \gamma_{n-1,j})x_n} \prod_{n=1}^{N-1} d\gamma_{nj}.
\]

Proof. The only non-obvious thing is the convergence of the integral. However, the integral converges absolutely due to the inequality (2.23).

**Example.** Let \( N = 2 \). Then

\[
\Psi_{\gamma_{21}, \gamma_{22}}(x_1, x_2) = \sinh^{1/2}(x_1 - x_2)
\]

\[
\times e^{\frac{h}{2}(\gamma_{21} + \gamma_{22})x_2} \int_{\mathbb{R}} \prod_{m=1}^{2} \left| \Gamma\left(\frac{\gamma_{11} - \gamma_{2m} + 1/4}{2\hbar}\right) \right|^2 e^{\pi \gamma_{11}(x_1 - x_2)} d\gamma_{11}.
\]

Assuming \( x_1 \geq x_2 \), one can close the contour in the upper half-plane thus calculating the integral by residues. The answer is given in terms of hypergeometric functions as follows:

\[
\Psi_{\gamma_{21}, \gamma_{22}}(x_1, x_2) = \sinh^{1/2}(x_1 - x_2) \frac{4\pi^2 \hbar}{\cosh \frac{\pi}{2\hbar} (\gamma_{21} - \gamma_{22})} e^{\frac{h}{2}(\gamma_{21} + \gamma_{22})(x_1 + x_2)}
\]

\[
\times e^{-\frac{1}{2}(x_1 - x_2)} \left\{ e^{\frac{h}{2\hbar}(\gamma_{21} - \gamma_{22})(x_1 - x_2)} \frac{\Gamma\left(-\frac{\gamma_{21} - \gamma_{22}}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\gamma_{21} - \gamma_{22}}{2\hbar}\right)} F\left(\frac{1}{2}, 1 + \frac{\gamma_{21} - \gamma_{22}}{2\hbar}, 1 + \frac{\gamma_{21} - \gamma_{22}}{2\hbar}; e^{-2(x_1 - x_2)}\right) \right\}.
\]

Up to a trivial factor this solution coincides with the Legendre function \( P_{\frac{\gamma_{21} - \gamma_{22}}{2\hbar} - \frac{1}{2}}(\cosh(x_1 - x_2)) \) (see [11], formulas 3.2(9), 3.2(27)), namely,

\[
\Psi_{\gamma_{21}, \gamma_{22}}(x_1, x_2) = \sinh^{1/2}(x_1 - x_2)
\]

\[
\times e^{\frac{h}{2\hbar}(\gamma_{21} + \gamma_{22})(x_1 + x_2)} P_{\frac{\gamma_{21} - \gamma_{22}}{2\hbar} - \frac{1}{2}}(\cosh(x_1 - x_2)).
\]

Note, in the limit \( x_1 - x_2 \to \infty \) the wave function (5.23) has the asymptotic behaviour

\[
\Psi_{\gamma_{21}, \gamma_{22}}(x_1, x_2) \sim c(\gamma_{21}, \gamma_{22}) e^{\frac{h}{2}(\gamma_{21}x_1 + \gamma_{22}x_2)} + c(\gamma_{22}, \gamma_{21}) e^{\frac{h}{2}(\gamma_{22}x_1 + \gamma_{21}x_2)},
\]

(5.26)
Remark 5.1 Since the integrand in (5.22) is written in the factorized form, it is clear that there exists a recursive relation, similar to (5.19), between the $n$ and $n-1$ particle eigenfunctions of the Sutherland model. This fact is the direct consequence of the Gelfand-Zetlin inductive procedure.

6 Connection with the QISM-based approach

In section 5 we have outlined the derivation of the integral representation of the wave function of an open Toda chain based on the GZ representation of $\mathcal{U}(\mathfrak{gl}(N, \mathbb{R}))$. As was stressed in section 4, this representation bears a direct relationship to $Y(\mathfrak{gl}(N, \mathbb{R}))$. This provides us with another point of view on the way to obtain the wave functions for the open Toda chain. In particular, the algebra generated by the images of the polynomials $A_n(\lambda)$, $B_n(\lambda)$, $C_n(\lambda)$ under the homomorphism $\pi_N : Y(\mathfrak{gl}(N)) \to \mathcal{U}(\mathfrak{gl}(N))$ can successfully be used to derive the recursive relations for the wave functions.

We begin with a short summary of the QISM-based approach to the $GL(N, \mathbb{R})$ Toda chain due to Sklyanin [20]. The model is described with the help of the auxiliary $2 \times 2$ matrix Lax operator with spectral parameter:

$$L_n(\lambda) = \begin{pmatrix} \lambda - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix},$$  

where $p_n = -i\hbar \partial_{x_n}$.

The operator (6.1) satisfies the usual quadratic relation

$$R_2(\lambda - \mu)(L_n(\lambda) \otimes I)(I \otimes L_n(\mu)) = (I \otimes L_n(\mu))(L_n(\lambda) \otimes I)R_2(\lambda - \mu)$$  

with the rational $4 \times 4$ $R$-matrix (see (4.2)). The monodromy matrix

$$T_n(\lambda) = L_n(\lambda) \ldots L_1(\lambda) \equiv \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ C_n(\lambda) & D_n(\lambda) \end{pmatrix}$$  

also satisfies the relation:

$$R_2(\lambda - \mu)(T_n(\lambda) \otimes I)(I \otimes T_n(\mu)) = (I \otimes T_n(\mu))(T_n(\lambda) \otimes I)R_2(\lambda - \mu).$$  

In particular, the following commutation relations hold:

$$[A_n(\lambda), A_n(\mu)] = [B_n(\lambda), B_n(\mu)] = [C_n(\lambda), C_n(\mu)] = 0,$$

$$(\lambda - \mu + i\hbar)A_n(\lambda)B_n(\mu) = (\lambda - \mu)B_n(\mu)A_n(\lambda) + i\hbar A_n(\mu)B_n(\lambda),$$

$$(\lambda - \mu + i\hbar)A_n(\mu)C_n(\lambda) = (\lambda - \mu)C_n(\lambda)A_n(\mu) + i\hbar A_n(\lambda)C_n(\mu).$$
The polynomial $A_n(\lambda)$ is a generating function for the quantum Hamiltonians of the $GL(n, \mathbb{R})$ Toda chain, and the wave functions $\psi_{\gamma_n}(x_1, \ldots, x_n)$, $n = 1, \ldots, N$ are defined as solutions of the differential equations

$$A_n(\lambda) \psi_{\gamma_n1,\ldots,\gamma_nn} = \prod_{j=1}^{n}(\lambda - \gamma_{nj}) \psi_{\gamma_n1,\ldots,\gamma_nn}. \quad (6.6)$$

The idea of the QISM approach is to solve iteratively these equations for $\psi_{\gamma_n}$, $n = 1, \ldots, N$ using the additional operators $B_n(\lambda)$ and $C_n(\lambda)$. From the results of [22] (eq. (2.7)), the following expressions for the operators $A_n(\lambda)$, $B_n(\lambda)$, and $C_n(\lambda)$ can be obtained:

$$A_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj}),$$

$$B_n(\lambda) = i^{1-n} \sum_{j=1}^{n} \prod_{s \neq j} \frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}} e^{-i\hbar\partial_{\gamma_{nj}}}, \quad (6.7)$$

$$C_n(\lambda) = i^{1-n} \sum_{j=1}^{n} \prod_{s \neq j} \frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}} e^{i\hbar\partial_{\gamma_{nj}}}. $$

A simple check shows that, thus defined, the operators (6.7) satisfy the relations (6.5). Hence, one establishes the connection between the generators $A_n(\lambda)$, $B_n(\lambda)$ and $C_n(\lambda)$ entering the description of the GZ representation (4.9), and the corresponding QISM generators (6.7):

$$A_n(\lambda) = A_n(\lambda),$$

$$B_n(\lambda) = s_n^{-1} \circ B_n(\lambda) \circ s_n, \quad (6.8)$$

$$C_n(\lambda) = s_{n-1}^{-1} \circ C_n(\lambda) \circ s_{n-1}, $$

where $s_n$ is defined by (2.10). Therefore, the expressions (6.5) should be compared with (4.5).

At the end of this section, we illustrate the connection between the differential operator $A_n(\lambda)$ entering in (6.3) and the generator $A_n(\lambda)$ defined by (2.17). First, note that (6.1), (6.3) define the relations between the operators $A_n(\lambda)$ for different levels:

$$A_n(\lambda) = (\lambda - p_n)A_{n-1}(\lambda) - e^{x_{n-1} - x_n}A_{n-2}(\lambda), \quad (6.9)$$

where $n = 1, \ldots, N$ and $A_{-1} = 0, A_0 = 1$.

**Proposition 6.1** Let $\psi_{\gamma_{N1},\ldots,\gamma_{NN}}$ be defined by (5.2). Then the operators $A_n(\lambda)$ and $A_n(\lambda)$ are related by

$$A_n(\lambda)\psi_{\gamma_{N1},\ldots,\gamma_{NN}} = e^{-\mathbf{x}\cdot\mathbf{p}^{(N)}}(w_N, e^{-\sum_{k=1}^{N}x_k E_{kk}}A_n(\lambda - \frac{i(N-n)\hbar}{2})w_N). \quad (6.10)$$
Proof. We prove the statement by showing that the operators $A_n(\lambda)$ defined by (6.10) satisfy the recursive relations (6.9). Consider the relation (2.19). The generators $E_{nn}$ and $E_{n,n-k+1}$ commute with $A_{n-1}(\lambda)$ and $A_{n-k}(\lambda)$, respectively. Therefore, due to (2.27) and (2.7),

$$
\langle w'_N, e^{-\sum_{k=1}^N x_k E_{kk}} A_n(\lambda - \frac{i(N-n)\hbar}{2}) w_N \rangle
$$

$$
= (\lambda - i\hbar \rho^{(N)}_n + i\hbar \partial_{x_n}) \langle w'_N, e^{-\sum_{k=1}^N x_k E_{kk}} A_{n-1}(\lambda - \frac{i(N-n+1)\hbar}{2}) w_N \rangle
$$

$$
- e^{x_{n-1-x_n}} \langle w'_N, e^{-\sum_{k=1}^N x_k E_{kk}} A_{n-2}(\lambda - \frac{i(N-n+2)\hbar}{2}) w_N \rangle.
$$

(6.11)

We leave a more systematic discussion of the interrelation between the two sets of operators $A_n(\lambda), B_n(\lambda), C_n(\lambda)$ and $A_n(\lambda), B_n(\lambda), C_n(\lambda)$ for a future publication.
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