Quenching Dynamics of a quantum XY spin-1/2 chain in presence of a transverse field

Victor Mukherjee$^{1}$, Uma Divakaran$^{2}$, Amit Dutta$^{3}$, and Diptiman Sen$^{4}$

$^{1,2,3}$Department of Physics, Indian Institute of Technology Kanpur 208016, India
$^{4}$Center for High Energy Physics, Indian Institute of Science, Bangalore 560012, India

We study the quantum dynamics of a one-dimensional spin-1/2 anisotropic XY model in a transverse field when the transverse field or the anisotropic interaction is quenched at a slow but uniform rate. The two quenching schemes are called transverse and anisotropic quenching respectively. Our emphasis in this paper is on the anisotropic quenching scheme and we compare the results with those of the other scheme. In the process of anisotropic quenching, the system crosses all the quantum critical lines of the phase diagram where the relaxation time diverges. The evolution is non-adiabatic in the time interval when the parameters are close to their critical values, and is adiabatic otherwise. The density of defects produced due to non-adiabatic transitions is calculated by mapping the many-particle system to an equivalent Landau-Zener problem and is generally found to vary as $1/\sqrt{\tau}$, where $\tau$ is the characteristic time scale of quenching, a scenario that supports the Kibble-Zurek mechanism. Interestingly, in the case of anisotropic quenching, there exists an additional non-adiabatic transition, in comparison to the transverse quenching case, with the corresponding probability peaking at an incommensurate value of the wave vector. In the special case in which the system passes through a multi-critical point, the defect density is found to vary as $1/\tau^{1/5}$. The von Neumann entropy of the final state is shown to maximize at a quenching rate around which the ordering of the final state changes from antiferromagnetic to ferromagnetic.

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I. INTRODUCTION

A quantum phase transition corresponds to a fundamental change in the symmetry of the ground state of a quantum system when the strength of the quantum fluctuations is appropriately tuned at zero temperature$^{1,2}$. In a quantum system, statics and dynamics are intermingled, and a quantum critical point is therefore associated with a diverging correlation length as well as a diverging relaxation time. The diverging time scale plays a non-trivial role when the system is driven through the quantum critical point at a uniform rate$^{3,4,5,6,7}$. This implies that no matter how slow the quenching may be, the dynamics of the system fails to be completely adiabatic when a quantum critical point is crossed. The possibility of experimental studies of non-equilibrium strongly correlated quantum systems$^{8}$ has paved the way for rigorous theoretical investigations$^{9,10,11,12,13}$ of the dynamics of various model Hamiltonians when swept through their quantum critical points. Recently a general analysis has been carried out of the effects of quenching in gapless systems$^{14}$.

The above mentioned non-adiabaticity arising due to the diverging relaxation time near the critical point leads to the production of topological defects. The spatial distribution of the spins in the final state is quite complex in comparison to the situation in which the dynamics is adiabatic for the entire range of time. The rate of production of defects can be quantified using the prediction of the Kibble-Zurek (KZ) theory extended to quantum spin chains$^{15,16}$. If the gap of a one-dimensional quantum Hamiltonian is changed linearly as $t/\tau$, where $\tau$ is the characteristic time scale of quenching, the dynamics is adiabatic for almost the entire span of time, except for a region in the vicinity of the quantum critical point called the “impulse” region where non-adiabatic transitions dominate$^{13}$. The system enters the impulse region at a time $t$ when the rate of change of the gap is of the order of the relaxation time of the system. One can show that $t \sim \sqrt{\tau}$ which eventually results in the density of defects decreasing as $1/\sqrt{\tau}$. In other words, there is, on average, a single defect in a region of length $\xi$ which also scales as $\sqrt{\tau}$. A smaller quenching rate (larger value of $\tau$) yields a larger $\xi$ and the defect production is less. A detailed analysis of the above mechanism for exactly solvable quantum spin chains is presented in Refs.$^{15,16,17}$ and$^{18}$. Recently, the above studies have been extended to explore the dynamics of disordered transverse Ising chains$^{15,18}$. The quenching behavior of Bose-Hubbard models has also been explored in recent years$^{20,21}$.

There is a recent upsurge in the study of quantum dynamics from the point of view of an optimization problem where the strength of the quantum fluctuations is quenched from a very high value to zero in order to arrive at the true ground state of a frustrated classical system. This approach of adiabatic quantum computation is popularly known as “quantum annealing”$^{22,23,24}$. The measure of non-adiabaticity in this approach is given by the residual energy $E_{\text{res}} = E_{\text{fin}} - E_{\text{cl}}$, where $E_{\text{fin}}$ is the energy of the final state, and $E_{\text{cl}}$ is the energy of the true classical ground state. The two measures of the degree of non-adiabaticity, the density of defects in the previous approach and the residual energy in the quantum annealing approach, are proportional to each other for a disorder free system. In the present literature on quantum dynamics, the terms “quenching” and “annealing” are often used synonymously.

In this work, we study the dynamics of an anisotropic
transverse XY spin-1/2 chain, which is driven across various quantum critical lines at a steady and finite rate. In the transverse quenching scheme, the transverse field is varied from $-\infty$ to $\infty$, whereas in the “anisotropic quenching”, which constitutes the main theme of this paper, the interaction term is quenched from $-\infty$ to $\infty$ keeping the transverse field unchanged. We study the “anisotropic” quenching scheme in detail, and compare the results with those of the transverse quenching scheme. In both cases, the dynamics is exactly solved via a mapping to an equivalent Landau-Zener problem through a Jordan-Wigner transformation from spins to fermions.

The paper is organized in the following way. In Section II, we discuss the Hamiltonian and the corresponding Zener problem exactly solved via a mapping to an equivalent Landau-Wilczek transformation from spins to fermions.

The Hamiltonian in Eq. (1) can be exactly diagonalized using the standard Bogoliubov transformation; we then arrive at an expression for the gap in the excitation spectrum given by

$$
H = -\sum_{k>0} \left\{ (J_x + J_y) \cos k + h \right\} (c_k^+ c_{-k} + c_{-k}^+ c_k) + i(J_x - J_y) \sin k (c_k^+ c_{-k}^+ - c_{-k} c_k).
$$

This Hamiltonian is quadratic in the $c$ operators and can therefore be diagonalized using the standard Bogoliubov transformation; we then arrive at an expression for the gap in the excitation spectrum given by

$$
\epsilon_k = |h^2 + J_x^2 + J_y^2 + 2h(J_x + J_y) \cos k + 2J_x J_y \cos 2k|^{1/2}.
$$

The gap given in Eq. (4) vanishes at $h = \mp (J_x + J_y)$ for wave vectors $k = 0$ and $\pi$ respectively; it turns out that this signals a quantum phase transition from a quantum paramagnetic phase to a ferromagnetically ordered phase in which a discrete $Z_2$ symmetry of the Hamiltonian in Eq. (1) is spontaneously broken. This transition belongs to the universality class of the transverse Ising model and is therefore referred to as the “Ising” transition. The spectrum is also gapless in the limit when the anisotropy vanishes, $J_x \rightarrow J_y$, provided that $|h| \leq 1$. The line $J_x = J_y$ marks the phase boundary between two ferromagnetically ordered phases denoted by $FM_x$ and $FM_y$, and the corresponding phase transition is called the “anisotropic transition”. In the $FM_x$ phase, $J_x > J_y$ and hence the ferromagnetic ordering is in the $x$ direction, while it is the other way around in the $FM_y$ phase. One can also check that the long-range order in the $FM_x$ or $FM_y$ phase only exists for a relatively weak transverse field, in the range $-J_x < h < J_x + J_y$. Each ferromagnetically ordered phase is further divided into a commensurate and an incommensurate region with the incommensurate wave vector $k_0$ given by

$$
\cos k_0 = -\frac{h(J_x + J_y)}{4J_x J_y}.
$$

On the anisotropic phase boundary $J_x = J_y$, the incommensurate wave vector therefore has a value $k_0 = \cos^{-1}(-h/2J_x)$. The boundary between these two regions inside a ferromagnetic phase is given by the relation $h/(J_x + J_y) = \pm(1 - \gamma^2)$, with $\gamma \equiv (J_x - J_y)/(J_x + J_y)$, as shown by the thick dashed lines in Fig. 1. The exponents associated with the anisotropic transition are different from the Ising case and are identical to the exponents of a pair of decoupled Ising models.

We note that the points corresponding to $J_x = J_y$ and $h = \pm 2J_y$ are multi-critical points since more than one phase boundary passes through them. We will see later that the quenching dynamics shows a different behavior when the system passes through those points.
III. QUENCHING SCHEME AND RESULTS

In this section, we will discuss the results obtained for the quenching dynamics of the Hamiltonian in Eq. (1) for two different schemes. In the first scheme, the time-dependent transverse field is of the form $h(t) = t/\tau$, where $t$ is varied from $-\infty$ to $\infty$, and $\tau$ is the characteristic time scale of quenching, often referred to as the “quenching time”. This “transverse quenching” scheme has been studied extensively for the XY chain by Cherubini and Levitov, and for the transverse Ising case by Dziarmaga and by Polkovnikov. We shall briefly present the results for the transverse quenching which will be helpful in comparing it with the results obtained in the other scheme, namely, the “anisotropic quenching”.

As mentioned in the Introduction, in the “anisotropic quenching” scheme, the interaction term $J_x (= t/\tau)$ is quenched from a very large positive initial value to a very large positive final value, with $J_y$ and $h$ held fixed at some positive values. For our numerical studies, we will set $2J_y > h$ for the reason explained below. In the initial state, all the spins are antiferromagnetically ordered in the $x$ direction. On the other hand, the final state should correspond to a state with a perfect ferromagnetic order in the $x$ direction had the dynamical evolution been adiabatic for the entire span of time. However, due to the non-adiabatic transitions near the quantum critical regions, the final state in the limit $t \to \infty$ will not be perfectly ordered and will include a finite fraction of kinks or spins aligned anti-parallel to the direction of ordering.

The Hamiltonian in Eq. (3) decouples into a sum of independent terms, $H(t) = \sum_{k>0} H_k(t)$, where each $H_k(t)$ operates on a four-dimensional Hilbert space spanned by the basis vectors $|0\rangle$, $|k\rangle = c_{k}^{\dagger}|0\rangle$, $|-k\rangle = c_{-k}^{\dagger}|0\rangle$, and $|k, -k\rangle = c_{k}^{\dagger}c_{-k}^{\dagger}|0\rangle$. The vacuum state where no $c$-particle is present is denoted by $|0\rangle$ which corresponds to a spin configuration with all spins pointing in the $-z$ direction. The occurrence of only bilinear terms like $c_{k}^{\dagger}c_{-k}^{\dagger}$ in the Hamiltonian in (3) ensures that the parity (even or odd) of the total number of fermions given by $n_k = c_k^{\dagger}c_k + c_{-k}^{\dagger}c_{-k}$ is conserved for each value of $k > 0$. Thus the states $|0\rangle$ and $|k, -k\rangle$ are coupled to each other by the Hamiltonian, while the states $|k\rangle$ and $|-k\rangle$ remain invariant.

To study the dynamics of transverse quenching, it is sufficient to project the Hamiltonian $H_k(t)$ to the two-dimensional subspace spanned by $|0\rangle$ and $|k, -k\rangle$ since the ground state of Eq. (3) for each value of $k$ lies within this subspace. In this subspace, the Hamiltonian takes the form

$$H_k(t) = -\left[ h + (J_x + J_y) \cos k \right] I_2 + \left[ -i(J_x - J_y) \sin k \right] + \left[ -i(J_x - J_y) \sin k \right] -h - (J_x + J_y) \cos k,$$

where $I_2$ denotes the 2 $\times$ 2 identity matrix. A state in this subspace can be represented as a linear superposition $\psi_k(t) = u_k(t)|0\rangle + v_k(t)|k, -k\rangle$, where the amplitudes $u_k(t)$ and $v_k(t)$ are time-dependent. The initial condition in the transverse quenching scheme is given by $u_k(-\infty) = 1$ and $v_k(-\infty) = 0$. The time evolution of a generic state is governed by the Schrödinger equation

$$i\partial_t \psi_k(t) = H_k(t) \psi_k(t).$$

The projection of the Hamiltonian to the 2 $\times$ 2 Hilbert space has effectively reduced the many-body problem to the problem of a two-level system. The off-diagonal term of the projected Hamiltonian, $\Delta = (J_x - J_y) \sin k$, represents the interaction between the two time-dependent levels $E_{1, 2} = \pm [h(t) + (J_x + J_y) \cos k]$. The Schrödinger equation given above is identical to the Landau-Zener problem of a two-level system, where the off-diagonal terms of the Hamiltonian determines the non-adiabatic transition probability $p_k$ for the wave vector $k$. Using the Landau-Zener transition formula, this transition probability of excitations at the final time is given by

$$p_k = e^{-2\pi \tilde{\gamma}},$$

where $\tilde{\gamma} = \Delta^2 / \left| \frac{4}{\hbar}(E_1 - E_2) \right|$. Equivalently, $p_k$ determines the probability that the system remains in the initial state $|0\rangle$ at the final time. For the transverse quench-
ing case, one can use the relation in Eq. (7) to obtain \( p_k \) as a function of the anisotropy term \( J_x - J_y \) and \( \tau \) as
\[
p_k = e^{-\pi \tau (J_x - J_y)^2} \sin^2 k.
\] (8)

This leads to an expression for the density of kinks \( n \) generated due to non-adiabatic transitions,
\[
n = \int_0^\pi \frac{dk}{\pi} p_k \approx \frac{1}{\pi \sqrt{\tau} |J_x - J_y|}
\] (9)

Eq. (9) shows that the kink density decreases as \( 1/\sqrt{\tau} \) for large \( \tau \) as predicted by the Kibble-Zurek theory and proved by Cherng and Levitov in the case of transverse quenching. We shall now focus on the anisotropic quenching scheme where the interaction term \( J_x(t) = t/\tau \) is changed from \(-\infty\) to \( \infty \), with \( J_y \) and \( h \) held fixed at some positive values. Let us first point out the range in time in which the system fails to follow the instantaneous ground state (namely, the region where the relaxation time is large or divergent) as \( J_x \) is varied. If \( J_x(t_1) + J_y = -h \), i.e., for \( t_1 = -\tau(h + J_y) \), the system undergoes an Ising transition from the initial antiferromagnetic phase to a paramagnetic phase. When \( J_x \) is further increased so that \( J_x(t_2) + J_y = h \), i.e., for \( t_2 = (h - J_y)\tau \), there is a phase transition from the paramagnetic to FM\(_x\) phase. It is to be noted however, that for \( J_x = -J_y \), there is no further anisotropic transition since the magnitude of \( h \) is greater than \( J_x + J_y = 0 \), and the system stays paramagnetic. Eventually, at a time \( t_3 \) given by \( J_x(t_3) = J_y \), with \( h < 2J_x \), the system evolves to the FM\(_x\) phase. The non-adiabaticity dominates in the vicinity of these three quantum critical points, i.e., for \( t \) close to \( t_1, t_2 \) and \( t_3 \). One should also note that if we fix \( h > 2J_x \) at the outset, there cannot be any anisotropic transition, and the system directly evolves from the paramagnetic phase to the FM\(_x\) phase through the Ising transition only; hence there are two, rather than three, regions of non-adiabaticity.

All the transition points are depicted in the static phase diagram of the model shown in Fig. 1. The adiabatic and non-adiabatic regions of time evolution are shown schematically in Fig. 2.

As mentioned already, in the anisotropic case, the reduced Hamiltonian in the \(|0\rangle\), \(|k, -k\rangle\) subspace includes a time-dependent \( J_x \) term with static positive values of \( J_x \) and \( h \). The eigenstates of the Hamiltonian in the limit \( t \to \pm \infty \) are
\[
|e_{1k}\rangle = \sin(k/2)|0\rangle + i \cos(k/2)|k, -k\rangle
\]
and
\[
|e_{2k}\rangle = \cos(k/2)|0\rangle - i \sin(k/2)|k, -k\rangle,
\]
with eigenvalues \( \lambda_1 = t/\tau \) and \( \lambda_2 = -t/\tau \) respectively;
the system is in the state \( |e_{1k}\rangle \) initially. A general state vector can be expressed as a linear combination of \( |e_{1k}\rangle \) and \( |e_{2k}\rangle \),
\[
|\psi_k(t)\rangle = C_{1k}(t)|e_{1k}\rangle + C_{2k}(t)|e_{2k}\rangle.
\] (10)

The Landau-Zener transition probability can be obtained by numerically solving the Schrödinger equation in Eq. (6) using the basis vectors \(|0\rangle\) and \(|k, -k\rangle\), with the appropriate initial conditions for \( u_k \) and \( v_k \). The non-adiabatic transition probability \( p_k \) at the final time is simply given by \( p_k = |C_{1k}(\infty)|^2 \). We have shown the variation of \( p_k \) with \( k \) obtained through numerical integration in Fig. 3.

The situation is apparently complicated in the anisotropic quenching case because the off-diagonal terms of the \( 2 \times 2 \) Hamiltonian matrix in the \(|0\rangle\) and \(|k, -k\rangle\) basis are time-dependent. However, by an unitary transformation to the new set of basis vectors \(|e_{1k}\rangle \) and \(|e_{2k}\rangle \), one can get rid of the time-dependence of the off-diagonal terms and map the anisotropic quenching problem to an equivalent Landau-Zener problem. The unitary transformation is given by \( H'_k(t) = U^\dagger H_k(t)U \), where
\[
U = \begin{bmatrix}
\cos(k/2) & \sin(k/2) \\
-i \sin(k/2) & i \cos(k/2)
\end{bmatrix},
\]
and the new Hamiltonian is
\[
H'_k(t) = -[h + (J_x + J_y) \cos k] I_2 + [J_x + J_y \cos 2k + h \cos k] \sin 2k + h \sin k J_y \sin 2k + h \sin k -J_x - J_y \cos 2k - h \cos k
\]

Therefore, the anisotropic quenching is now placed on the same footing as the transverse quenching case, with
off-diagonal terms in the Hamiltonian matrix which are time-independent. The time evolution of the amplitudes $C_{1k}(t)$ and $C_{2k}(t)$ is dictated by the Schrödinger equation,

\[
\frac{dC_{1k}}{dt} = (J_x(t) + J_y \cos 2k + h \sin k) C_{1k}(t) \\
+ (J_y \sin 2k + h \sin k) C_{2k}(t) \\
\frac{dC_{2k}}{dt} = (J_y \sin 2k + h \sin k) C_{1k}(t) \\
- (J_x(t) + J_y \cos 2k + h \cos k) C_{2k}(t),
\]

where we have dropped the effect of the term involving the identity matrix in the Hamiltonian since this affects $C_{1k}$ and $C_{2k}$ by the same time-dependent phase factor, and will therefore have no effect on the density matrix. The non-adiabatic transition probability now depends on $J_y$ and $h$, and is given by

\[
p_k = |C_{1k}(\infty)|^2 = e^{-\pi \tau(J_y \sin 2k + h \sin k)^2}.
\]

In Fig. 3, we plot $p_k$ as a function of the wave vector $k$ along with the numerical results. We see that in contrast to the transverse case, there is an additional region of non-adiabaticity peaked at the incommensurate wave vector $k_0 = \cos^{-1}(-h/2J_y)$. This non-adiabaticity arises due to the existence of the anisotropic transition of the underlying static XY model at $J_x(t) = J_y$. The situation with zero transverse field is a special case where the regions of non-adiabaticity peak at $k = 0, \pi/2$ and $\pi$, respectively.

As in the transverse quenching case, we now measure the density of kinks given by

\[
n = \int_0^\pi \frac{dk}{\pi} p_k = \int_0^\pi \frac{dk}{\pi} e^{-\pi \tau(J_y \sin 2k + h \sin k)^2}.
\]

In Fig. 4, the plot of the kink density as a function of the quenching time $\tau$ is shown, which clearly shows that $n \propto 1/\sqrt{\tau}$ for large values of $\tau$. This finding supports the prediction of the Kibble-Zurek mechanism even in the anisotropic quenching case. One can also find an approximate analytical form of $n$ in the following way: for large $\tau$, only the modes very close to the critical modes contribute. For $k \rightarrow 0$ and $\pi$, $p_k$ can be approximated as $\exp[-\pi \tau(2J_y + h)^2k^2]$ and $\exp[-\pi \tau(2J_y - h)^2(\pi - k)^2]$ respectively. With this $p_k$, the density of kinks produced by the modes near $k = 0$ and $\pi$ is given by

\[
n_1 \simeq \frac{1}{2\pi \sqrt{\tau}} \left[ \frac{1}{2J_y + h} + \frac{1}{2J_y - h} \right] = \frac{2J_y}{\pi \sqrt{\tau}(4J_y^2 - h^2)}.
\]

By expanding around $k_0$, we find that the contribution to $n$ from modes with $h \approx k_0$ is equal to the contribution from $k = 0$ and $k = \pi$ taken together,

\[
n_2 \simeq \int_0^{\pi} \frac{dk}{\pi} e^{-\pi \tau(2J_y \cos 2k_0 + h \cos k_0)^2(k-k_0)^2} \\
\simeq \frac{1}{\pi \sqrt{\tau}} \left[ \frac{1}{2J_y \cos 2k_0 + h \cos k_0} \right].
\]

Therefore the total kink density is given by

\[
n = n_1 + n_2 \simeq \frac{4J_y}{\pi \sqrt{\tau}(4J_y^2 - h^2)}.
\]

Fig. 4 shows that this approximate form embraces the exact result perfectly in the limit of large $\tau$. 

FIG. 3: $p_k$ vs $k$ as obtained numerically and analytically for $\tau$ equal to 0.5 and 5. We have fixed $J_y = 1$ and $h = 0.2$. In the region marked $\tau = 0.5$, the solid line is numerical and dashed line is analytical. In the region marked $\tau = 5$, the dashed line with smaller spacing is numerical and dashed line with larger spacing is analytical.

FIG. 4: Variation of kink density $n$ with $\tau$ as obtained by numerical integration of Eq. (14) for $h = 0.2, 0.4, 0.6, 0.8$ (from bottom to top in the large $\tau$ region), with $J_y = 1$. For large $\tau$, $n$ increases with increasing $h$, whereas for small $\tau$, it decreases with increasing $h$ as shown in the inset.
Using Eq. (15), one can find the variation of the kink density with $h$, for large values of $\tau$, as
\[
\frac{\partial n}{\partial h} \approx \frac{8hJ_y}{\pi \sqrt{\tau} \left(4J_y^2 - h^2\right)^2}.
\]

The slope is positive for all values $h$ indicating an increment in kink density with increasing $h$ as shown in Fig. 4. On the other hand, for small $\tau$, the density $n$ is found to decreases with increasing $h$. In the limit $\pi \tau (J_y \sin 2k + h \sin k)^2 \ll 1$, one can expand the exponential in $p_k$ in Eq. (14), retaining only terms up to first order in $\tau$. One can then show that $\partial n/\partial h$ is negative. The crossover from the small $\tau$ to the large $\tau$ behavior occurs around a typical quenching time $\tau = \ln 2/[\pi (J_y \sin 2k + h \sin k)^2]$. The significance of the crossover time is explained below.

To derive a characteristic time scale of the anisotropic quenching, we first note that the minima of the probability $p_k$ occurs at a wave vector value $\tilde{k}$, where
\[
\cos \tilde{k} = -h \pm \sqrt{h^2 + 32J_y^2}/8J_y. \tag{16}
\]

The presence of a non-zero transverse field $h$ leads to an asymmetry in $p_k$ on the either side of the maxima at $k_0$ as is evident at smaller values of $\tau$ (Fig. 3). We now define a time scale $\tau_0$ so that at $\tau = \tau_0$, $p_k = 1/2$ at the minimum $k = \tilde{k}$. This implies
\[
\tau_0 = \frac{2}{\pi (J_y \sin 2k + h \sin k)^2}. \tag{17}
\]

The two conjugate values of $\tilde{k}$ given in Eq. (16) lead to a pair of $\tau_0$’s for the anisotropic quenching, in contrast to the transverse quenching case. For $h = 0$, these two values coalesce into one. The crossover of density shown in Fig. 4, takes place roughly around the smaller value of $\tau_0$ denoted by $\tau_{02}$. For $\tau \gg \tau_{01}, \tau_{02}$, the dynamics is nearly adiabatic except for the modes very close to the critical modes (Fig. 5).

Eq. (15) is valid for $2J_y > h$. On the other hand, if we take $h > 2J_y$, the system evolves directly from the paramagnetic to the FM phase; there is no anisotropic transition. In this case, Eq. (14) gives the kink density to be
\[
n \approx \frac{1}{2\pi \sqrt{\tau}} \left[ \frac{1}{h + 2J_y} + \frac{1}{h - 2J_y} \right] \frac{h}{\pi \sqrt{\tau} \left(h^2 - 4J_y^2\right)}. \tag{18}
\]
(Note that $\partial n/\partial h$ is negative in this case). The transverse Ising model result given in Ref. [3] can be obtained by setting $J_y = 0$ in the above expression.

Finally, we consider what happens if $1 - 2J_y/h$ is close to zero. If $2J_y = h$, the system passes through a multicritical point when $J_y(t)$ equals $J_y$; we therefore expect an unusual behavior of the kink density $n$ in this case. If $2J_y = h$ and $\tau$ is large, we find that Eq. (14) is dominated by the region near $k = \pi$; since $p_k$ can be approximated there by $\exp[-\pi \tau h^2(\pi - k)^2/4]$, the contribution of this region to $n$ goes as $1/(h^{1/3}\tau^{1/6})$. [The contribution from the region near $k = 0$ to $n$ goes as $1/(h\sqrt{T})$ which is much smaller than $1/(h^{1/3}\tau^{1/6})$ if $h^2\tau \gg 1$.] Now, if $1 - 2J_y/h$ is non-zero but small, $p_k$ can be approximated near $k = \pi$ by $\exp[-\pi \tau \left(h - 2J_y \right)(\pi - k)/2\pi h^2(\pi - k)^2/4]$. We then see that the term $\tau (h - 2J_y)^2(\pi - k)^2$ will become more important than the term $\tau h^2(\pi - k)^2$ and the kink density will show a crossover from a $1/\sqrt{T}$ behavior to a $1/\tau^{1/6}$ behavior when $\tau$ increases beyond a value which is of order $h/h - 2J_y$.

IV. ENTROPY AND MAGNETIZATION OF THE FINAL STATE

The quenching dynamics of the spin model is dictated by the Schrödinger equation and is obviously unitary. Therefore, the final state must be a pure state described by a density matrix of the product form $\rho = \bigotimes \rho_k$, where $\rho_k$ is given by
\[
\begin{bmatrix}
p_k & q_k \\
q_k & 1 - p_k
\end{bmatrix}
\tag{19}
\]
and $q_k = C_{1k}(\infty)C_{2k}(\infty)$. The diagonal elements of the reduced $2 \times 2$ density matrix are smooth functions of $k$ and are independent of the total time of quenching (which we now set equal to $T$ for convenience, with $T \gg 1$), whereas the off-diagonal terms are rapidly oscillating functions of both $k$ and $T$.

The final state, even though a pure state, has a fairly complicated local structure; its local properties in the real space is identical to that of a mixed state with a finite entropy. In other words, the final state can be
viewed as a superposition of different configurations of magnetically ordered domains. The off-diagonal terms of the density matrix $\rho_k$ can be made to vanish at the final time $T$ upon coarse-graining in the wave vector $k$. The final state therefore may be viewed locally as a mixed state described by a decohered reduced density matrix $\rho_D$ given by

$$
\begin{bmatrix}
p_k & 0 \\
0 & 1 - p_k
\end{bmatrix}.
$$

To quantify the amount of information lost in the decoherence process, we consider the von Neumann entropy density of the system, $s = -\text{tr}(\rho_D \ln \rho_D)$, namely,

$$
s = -\int_0^{\tau} \frac{dk}{\pi} \left[ p_k \ln(p_k) + (1 - p_k) \ln(1 - p_k) \right].
$$

In the case of the transverse quenching, the maximum of the entropy density occurs near the value of $\tau = 2\tau_0$ where the minimum value of $p_k = 1/2$. In the anisotropic case, as mentioned already, $\tau_0$ is non-unique (Fig. 5). The entropy density $s$ when plotted against $\tau/\tau_02$ shows a maxima at a quenching rate $\tau \approx 2\tau_02$ (see Fig. 6), while no special behavior is noted near $\tau_01$; this establishes $\tau_02$ as the characteristic time scale of the anisotropic quenching. For sufficiently small $\tau$, non-adiabaticity dominates and the system stays in its initial state with high probability, namely, the system more or less retains its initial antiferromagnetic ordering even at the final time. The system looks like a pure state even at small length scales. In the large $\tau$ limit, on the other hand, the system evolves essentially in an adiabatic way to the final ground ground state, and the non-adiabatic transition probability is small. The final state is ferromagnetically ordered. Therefore, the entropy density vanishes asymptotically for both large and small values of $\tau$, and the maximum occurs at an intermediate value of $\tau \approx 2\tau_02$ where the defect in the local ordering of the final state is maximum.

The above arguments can be justified further by exploring the magnetization of the system in the final state. Let us recall that the initial state is antiferromagnetically oriented in the $x$ direction. The average energy of the mode $k$ at the final time is related to $p_k$ as $(2p_k - 1)J_x$; hence, integrating over all the modes gives the total energy per site to be $(2n - 1)J_x$, where $n$ is the density of kinks defined above. Above, our emphasis lies on a new scheme, namely, the anisotropic quenching, where the interaction strength in the $x$ direction is quenched from $-\infty$ to $\infty$ at a uniform rate dictated by a time scale $\tau$. At first sight, the Landau-Zener theory does not seem to be applicable in this case due to the presence of time-dependent off-diagonal terms in the reduced $2 \times 2$ Hamiltonian. However, through a unitary transformation to an appropriate basis, the time dependence can be entirely shifted to the diagonal terms, and Landau-Zener theory turns out to be applicable once again.

In the process of the anisotropic quenching, the system is swept across all the three quantum critical lines in the phase diagram if $h < 2J_y$, and across two quantum critical lines if $h > 2J_y$. In both these cases, the density of defects is found to vary as $1/\sqrt{\tau}$. In the special case of $h = 2J_y$, the system is swept across a multi-critical point, and the density of defects is found to vary as $1/\tau^{1/6}$.

V. CONCLUSIONS

We have studied the adiabatic quantum dynamics of an anisotropic $XY$ spin-$1/2$ chain in a transverse field when the parameters of the Hamiltonian are quenched at a steady and finite rate. Our emphasis lies on a new scheme, namely, the anisotropic quenching, where the interaction strength in the $x$ direction is quenched from $-\infty$ to $\infty$ at a uniform rate dictated by a time scale $\tau$. At first sight, the Landau-Zener theory does not seem to be applicable in this case due to the presence of time-dependent off-diagonal terms in the reduced $2 \times 2$ Hamiltonian. However, through a unitary transformation to an appropriate basis, the time dependence can be entirely shifted to the diagonal terms, and Landau-Zener theory turns out to be applicable once again.
Finally, we find a characteristic quenching time $\tau_{02}$ around which the von Neumann entropy of the final state maximizes and ferromagnetic ordering starts to set in. The residual energy of the final state is proportional to the defect density and hence scales as $1/\sqrt{\tau}$.

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E-mail: 1victor@iitk.ac.in
2udiva@iitk.ac.in
3dutta@iitk.ac.in,
4diptiman@cts.iisc.ernet.in

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