Optimal Chernoff and Hoeffding Bounds for Finite State Markov Chains

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Abstract

This paper develops an optimal Chernoff type bound for the probabilities of large deviations of sums $\sum_{k=1}^{n} f(X_k)$ where $f$ is a real-valued function and $(X_k)_{k \in \mathbb{Z}_{\geq 0}}$ is a finite state Markov chain with an arbitrary initial distribution and an irreducible transition probability matrix satisfying a mild assumption on its positivity pattern, related to the function $f$ being considered. The novelty lies in this being a non-asymptotic finite sample bound. Further, our bound is optimal in the large deviations sense, attaining a constant prefactor and an exponential decay with the optimal large deviations rate. Moreover, through a Pinsker type inequality and a Hoeffding type lemma, we are able to loosen up our Chernoff type bound to a Hoeffding type bound and reveal the sub-Gaussian nature of the sums. Finally, under the same mild assumption on the positivity pattern of the transition probability matrix, we prove a uniform multiplicative ergodic theorem for the exponential family of tilted transition probability matrices corresponding to $f$.

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1 Introduction

Let $S$ be a finite set and $(X_k)_{k \in \mathbb{Z}_{\geq 0}}$ the coordinate process on $S^{\mathbb{Z}_{\geq 0}}$. Given an initial distribution $q$ on $S$, and a stochastic matrix $P$, there exists a unique probability measure $P_q$ on the sequence space such that the coordinate process $(X_k)_{k \in \mathbb{Z}_{\geq 0}}$ is a Markov chain with transition probability matrix $P$, with respect to the filtration of $\sigma$-fields ($\mathcal{F}_n := \sigma(X_0, \ldots, X_n), n \geq 0$). If we assume further that $P$ is irreducible, then there exists a unique stationary distribution $\pi$ for the transition probability matrix $P$, and for any real-valued function $f : S \rightarrow \mathbb{R}$ the empirical mean $n^{-1} \sum_{k=1}^{n} f(X_k)$ converges $P_q$-almost-surely to the stationary mean $\pi(f) := \sum_x f(x) \pi(x)$. The goal of this work is to quantify the rate of this convergence by developing finite sample upper bounds for the large deviations probability

$$P_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \mu \right), \text{ for } \mu \geq \pi(f).$$

The significance of studying finite sample bounds for such tail probabilities is not only theoretical but also practical, since concentration inequalities for Markov dependent random variables have wide applicability in statistics, computer science and learning theory. Just to mention a few applications, first and foremost this convergence forms the backbone behind all Markov chain Monte Carlo (MCMC) integration techniques, see Metropolis et al. (1953). Moreover, tail bounds of this form have been used by Jerrum et al. (2001) to develop an approximation algorithm for the permanent of a non-negative matrix. In addition, in the stochastic multi-armed bandit literature the analysis of learning algorithms is based on tail bounds of this type, see the survey of Bubeck and Cesa-Bianchi (2012). More specifically the work of Moulos (2019) uses such a bound to tackle a Markovian identification problem.

1.1 Chernoff Bound

The classic large deviations theory for Markov chains due to Miller (1961); Donsker and Varadhan (1975); Gärtner (1977); Ellis (1984); Dembo and Zeitouni (1998) suggests that asymptotically the large deviations probability decays exponentially and the rate is given by the convex conjugate $\Lambda^*(\mu)$ of the log-Perron-Frobenius eigenvalue $\Lambda(\theta)$ of the nonnegative irreducible
matrix $\tilde{P}_\theta(x, y) := P(x, y) e^{\theta f(y)}$. In particular
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \mu \right) = -\Lambda^*(\mu), \quad \text{for } \mu \geq \pi(f).
\]

Our objective is to develop a finite sample bound which captures this exponential decay and has a constant prefactor that does not depend on $\mu$, and is thus useful in applications. A counting based approach by Davisson et al. (1981) is able to capture this exponential decay but with a suboptimal prefactor that depends polynomially on $n$. Through the development in the book of Dembo and Zeitouni (1998) (Theorem 3.1.2), which is also presented by Watanabe and Hayashi (2017), one is able to obtain a constant prefactor, which though depends on $\mu$. This is unsatisfactory because exact large deviations for Markov chains, see Miller (1961); Kontoyiannis and Meyn (2003), yield that, at least when the supremum $\sup_{\theta \in \mathbb{R}} \{\theta \mu - \Lambda(\theta)\} = \Lambda^*(\mu)$ is attained at $\theta_\mu$, then
\[
\mathbb{P}_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \mu \right) \sim \frac{\mathbb{E}_{X \sim q}[v_{\theta_\mu}(X)]}{\theta_\mu \sqrt{2\pi n \sigma^2_{\theta_\mu}}} e^{-n\Lambda^*(\mu)}, \quad \text{as } n \to \infty,
\]
where $\sigma^2_{\theta_\mu} = \Lambda''(\theta_\mu)$ and $v_{\theta_\mu}$ is a right Perron-Frobenius eigenvector of $\tilde{P}_{\theta_\mu}$. Here $\sim$ denotes that the ratio of the expressions on the left hand side and the right hand side converges to 1, and $\Lambda''(\theta_\mu)$ denotes the second derivative in $\theta$ of $\Lambda(\theta)$ at $\theta = \theta_\mu$. Thus, if we allow dependence on $\mu$, then the prefactor should be able to capture a decay of the order $1/\sqrt{n}$. If we insist on no dependence on $\mu$ though, the best that we can hope for is a constant prefactor, because otherwise we will contradict the central limit theorem for Markov chains. This is argued formally at the end of Section 3.

In our work we establish a tail bound with the optimal rate of exponential decay and a constant prefactor which depends only on the function $f$ and the stochastic matrix $P$, under the following conditions on $P$. Let $a := \min_{x} f(x)$, and $b := \max_{x} f(x)$. Based on $f$, we define two set of states, $S_b := \{x \in S : f(x) = b\}$ and $S_a := \{x \in S : f(x) = a\}$. We will require that $P$ satisfies some subset of the following structural assumptions on the positivity pattern of $P$. We will enforce A 1-A 2 for upper tail bounds, A 3-A 4 for lower tail bounds, and A 1-A 4 when we want to bound both tails.

A 1. The submatrix of $P$ with rows and columns in $S_b$ is irreducible.
A 2. For every $x \in S - S_b$, there exists $y \in S_b$ such that $P(x, y) > 0$.

A 3. The submatrix of $P$ with rows and columns in $S_a$ is irreducible.

A 4. For every $x \in S - S_a$, there exists $y \in S_a$ such that $P(x, y) > 0$.

As we will see shortly, with these assumptions we are essentially enforcing that after suitable tilts of the transition probability matrix we are able to produce new Markov chains that can realize any stationary mean in $(a, b)$. Our assumptions are general enough to capture all Markov chains, reversible or not, for which all the transitions have a positive probability.

The key technique to derive our Chernoff type bound is the old idea due to Esscher (1932) of an exponential tilt, which lies at the heart of large deviations theory. In the world of statistics those exponential changes of measure go by the name exponential families and the standard reference is the book of Brown (1986). Exponential tilts of stochastic matrices generalize those of finitely supported probability distributions, and were first introduced in the work of Miller (1961). Subsequently they formed one of the main tools in the study of large deviations for Markov chains, see Donsker and Varadhan (1975); Gärtner (1977); Ellis (1984); Dembo and Zeitouni (1998); Balaji and Meyn (2000); Kontoyiannis and Meyn (2003). Naturally they are also the key object when one conditions on the pair empirical distribution of a Markov chain and considers conditional limit theorems, as in Csiszár et al. (1987); Bolthausen and Schmock (1989). A more recent development by Nagaoka (2005) gives an information geometry perspective to this concept, while Hayashi and Watanabe (2016) examine the problem of parameter estimation for exponential families of stochastic matrices.

Here we build on exponential families of stochastic matrices and by studying the analyticity properties of the Perron-Frobenius eigenvalue and its associated eigenvector as we parametrically move the mean of $f$ under exponential tilts, together with conjugate duality, we are able to establish our main Chernoff type bound.

**Theorem 1.** Let $P$ be an irreducible stochastic matrix on the finite state space $S$, with stationary distribution $\pi$, which, combined with a real-valued function $f : S \to \mathbb{R}$, satisfies A 1-A 2. Then, for any initial distribution $q$, we have

$$
\mathbb{P}_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \mu \right) \leq K_u e^{-n\Lambda^*(\mu)}, \text{ for } \mu \geq \pi(f),
$$
where $K_u = K_u(P, f)$ is the constant from Proposition 1, and depends only on the stochastic matrix $P$ and the function $f$.

Remark 1. Since $f$ is arbitrary and our assumptions A 1-A 2 and A 3-A 4 are symmetric, we can substitute $f$ with $-f$, so that Theorem 1 yields a Chernoff type bound for the lower tail as well. In particular, assuming A 3-A 4 we have

$$
P_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \mu \right) \leq K_l e^{-n \Lambda^*(\mu)}, \quad \text{for } \mu \leq \pi(f),$$

where $K_l = K_u(P, -f)$.

Remark 2. Similarly assuming A 1-A 4 we have the following two-sided Chernoff type bound.

$$
P_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \in F \right) \leq 2K e^{-n \inf_{\mu \in F} \Lambda^*(\mu)}, \quad \text{for any } F \text{ closed in } \mathbb{R},$$

where $K = \max\{K_l, K_u\}$.

Remark 3. According to Proposition 1, when $P$ is a positive stochastic matrix, i.e. all the transitions have positive probability, we can replace $K$ with

$$
K \leq \max_{x,y,z} \frac{P(x,z)}{P(y,z)}.
$$

Remark 4. According to Proposition 1, when $P$ induces an IID sequence, i.e. all the rows of $P$ are identical, then $K = 1$. Thus Theorem 1 generalizes the classic bound of Chernoff (1952) for finitely supported IID sequences.

### 1.2 Hoeffding Bound

Although Chernoff type bounds for Markov chains have not been extensively studied in the literature, and that’s exactly the focus of this work, there is a vast literature on Hoeffding type inequalities for Markov chains. Gillman (1993) obtained the first Hoeffding type bound for reversible finite state Markov chains. Reversibility is a key assumption in his work because it allows one to restrict attention to self-adjoint operators and then it is possible to apply the matrix perturbation theory of Kato (1966) in order to derive a bound on the largest eigenvalue of the self-adjoint operator $\tilde{P}_\theta$ defined in (1)
Below. Later on Dinwoodie (1995) obtained an improved prefactor. Using
the same spectral techniques Lezaud (1998) obtained a Bernstein type in-
equality which is also applicable to some nonreversible finite state Markov
chains, and which was later improved in the work of Paulin (2015). Kahale
(1997) introduced the idea of reducing the problem to a two state chain,
which turned out to be very fruitful. León and Perron (2004) employed this
idea and, by performing exact calculations, they obtained a bound which
is optimal for two state chains in the large deviations sense, as well as a
Hoeffding type bound with variance proxy \( \frac{1 + \lambda \sigma_0^2}{4} (b-a)^2 \), where \( \lambda \) is the sec-
ond largest eigenvalue of the reversible stochastic matrix \( P \), as opposed to
the classic variance proxy for IID sequences \( \frac{(b-a)^2}{4} \) due to Hoeffding (1963).
Miasojedow (2014) extended this work to general state spaces without the re-
versibility assumption, Rao (2019) considered stationary finite state Markov
chains but allowed time-varying functions \( f_i \), and finally Jiang et al. (2018)
and Fan et al. (2018) obtained both Bernstein and Hoeffding type bounds
for general state space Markov chains and time-varying functions \( f_i \).

Here we develop a Hoeffding type bound by loosening up our Chernoff
type bound in Theorem 1 using a Pinsker type inequality in Lemma 8. In
the process a Hoeffding type lemma, in Lemma 9, is established as the dual
of our Pinsker type inequality.

**Theorem 2.** Let \( P \) be an irreducible stochastic matrix on the finite state
space \( S \), with stationary distribution \( \pi \), which, combined with a real-valued
function \( f : S \rightarrow [a, b] \), satisfies A 1-A 2. Then, for any initial distribution
\( q \), we have

\[
\mathbb{P}_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \mu \right) \leq K_u e^{-n \left( \frac{(\mu - \pi(f))^2}{2 \sigma_u^2} \right)} \leq K_u e^{-n \left( \frac{2(\mu - \pi(f))^2}{(b-a)^2 + 2K_u L_u} \right)}, \text{ for } \mu \geq \pi(f),
\]

where \( \sigma_u^2 = \sigma_u^2(P, f) := \sup_{\theta \in \mathbb{R}} \Lambda''(\theta) < \infty \), \( \Lambda''(\theta) \) denotes the second
derivative of \( \Lambda(\theta) \) in \( \theta \), and \( K_u = K_u(P, f) \), \( L_u = L_u(P, f) \) are the constants
from Proposition 1.

**Remark 5.** Since \( f \) is arbitrary and our assumptions A 1-A 2 and A 3-A 4
are symmetric, we can substitute \( f \) with \( -f \), so that Theorem 2 yields a
Hoeffding type bound for the lower tail as well. In particular, assuming A 3-
A 4 we have

\[
\mathbb{P}_q \left( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \mu \right) \leq K_f e^{-n \left( \frac{(\mu - \pi(f))^2}{2 \sigma_f^2} \right)} \leq K_f e^{-n \left( \frac{2(\mu - \pi(f))^2}{(b-a)^2 + 2K_f L_f} \right)}, \text{ for } \mu \leq \pi(f),
\]
where $\sigma^2_l := \sigma^2_u(P, -f)$, and $L_l := L_u(P, -f)$.

**Remark 6.** According to Proposition 1, when $P$ induces an IID sequence, i.e. all the rows of $P$ are identical, then $K = \max\{K_u, K_l\} = 1$ and $L = \max\{L_u, L_l\} = 0$. Thus Theorem 2 generalizes the classic bound of Hoeffding (1963) for finitely supported IID sequences.

**Remark 7.** Our variance proxy $\sigma^2_u = \sup_{\theta \in \mathbb{R} \geq a} \Lambda''(\theta) \leq (b-a+2K_uL_u)^2/4$, according to Lemma 2, has an interpretation as a worst case variance among all the tilted Markov chains, and thus parallels the variance proxy from the IID case which is the supremum of the variances among the tilted distributions, and which can be upper bounded by $(b-a)^2/4$.

### 1.3 Organization of Paper

The rest of the paper proceeds as follows. Section 2 contains the classic construction of exponential families of stochastic matrices, the duality between the canonical and mean parametrization, as well as many other useful properties for our bounds. In Section 3 and Section 4 we analyze the limiting behavior of the family under our assumptions A 1-A 2, and we establish our Chernoff (Theorem 1) and Hoeffding (Theorem 2) type bounds. Finally in Section 5 we develop a uniform multiplicative ergodic theorem (Theorem 3).

### 2 Exponential Family of Stochastic Matrices

#### 2.1 Construction

Exponential tilting of stochastic matrices originates in the work of Miller (1961). Following this, we define an exponential family of stochastic matrices which is able to produce Markov chains with shifted stationary means. The generator of the exponential family is an irreducible stochastic matrix $P$, which for this section is not assumed to satisfy A 1-A 4, and $\theta \in \mathbb{R}$ represents the canonical parameter of the family. Then we define

$$\tilde{P}_\theta(x, y) := P(x, y)e^{\theta f(y)},$$

(1)

(or $P_\theta(x, y)$, where $\tilde{(\cdot)}_\theta$ is thought as an operator over matrices). $\tilde{P}_\theta$ has the same nonnegativity structure as $P$, hence it is irreducible and we can
use the Perron-Frobenius theory in order to normalize it and turn it into a stochastic matrix. Let $\rho(\theta)$ (or $\rho(\tilde{P}_\theta)$) be the spectral radius of $\tilde{P}_\theta$, which from the Perron-Frobenius theory is a simple eigenvalue of $\tilde{P}_\theta$, called the Perron-Frobenius eigenvalue, associated with unique left and right eigenvectors $u_\theta$, $v_\theta$ (or $u_{\tilde{P}_\theta}$, $v_{\tilde{P}_\theta}$) such that they both have all entries strictly positive, $\sum_x u_\theta(x) = 1$, and $\sum_x u_\theta(x) v_\theta(x) = 1$, see for instance Theorem 8.4.4 in the book of Horn and Johnson (2013). Using $\tilde{P}_\theta$ we define a family of nonnegative irreducible matrices, parametrized by $\theta$, in the following way

$$(P)_{\theta}(x, y) = P_{\theta}(x, y) := \frac{\tilde{P}_{\theta}(x, y) v_\theta(y)}{\rho(\theta) v_\theta(x)}, \quad (2)$$

which are stochastic, since

$$\sum_y P_{\theta}(x, y) = \frac{1}{\rho(\theta) v_\theta(x)} \cdot \sum_y \tilde{P}_{\theta}(x, y) v_\theta(y) = 1, \quad \text{for } x \in S.$$ 

In addition the stationary distributions of the $P_\theta$ are given by

$$\pi_{\theta}(x) := u_\theta(x) v_\theta(x), \quad \text{for } x \in S,$$

since

$$\sum_x \pi_{\theta}(x) P_\theta(x, y) = \frac{v_\theta(y)}{\rho(\theta)} \cdot \sum_x u_\theta(x) \tilde{P}_{\theta}(x, y) = \pi_\theta(y), \quad \text{for } y \in S.$$ 

Note that the generator stochastic matrix, $P$, is the member of the family that corresponds to $\theta = 0$, i.e. $P_0 = P$, $\rho(0) = 1$, $u_0 = \pi$, $v_0 = 1$, and $\pi_0 = \pi$, where 1 is the all ones vector. In general it is possible that the family is degenerate as the following example suggests.

**Example 1.** Let $S = \{\pm 1\}$, $P(x, y) = 1\{x \neq y\}$, and $f(x) = x$. Then $\rho(\theta) = 1$, $v_\theta(-1) = \frac{1+e^\theta}{2}$, $v_\theta(1) = \frac{1+e^{-\theta}}{2}$, and $P_\theta = P$ for any $\theta \in \mathbb{R}$.

A basic property of the exponential family $P_\theta$ is that the composition of $(\cdot)_{\theta_1}$ with $(\cdot)_{\theta_2}$, is the transform $(\cdot)_{\theta_1+\theta_2}$, and so composition is commutative. Furthermore we can undo the transform $(\cdot)_\theta$ by applying $(\cdot)_{-\theta}$. We state this formally for convenience.

**Lemma 1.** For any irreducible stochastic matrix $P$, and any $\theta_1, \theta_2 \in \mathbb{R}$

$$((P)_{\theta_2})_{\theta_1} = (P)_{\theta_1+\theta_2}.$$ 

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Proof. It suffices to check that \( \left( \frac{v_{\theta_1 + \theta_2}}{v_{\theta_2}}, y \in S \right) \) is a right eigenvector of the matrix with entries \( \left( \frac{P(x,y)e^{\theta_2 f(y)}v_{\theta_2}(y)}{\rho(\theta)v_{\theta_2}(x)} \right)e^{\theta_1 f(y)} \), with the corresponding eigenvalue being \( \frac{\rho(\theta_1 + \theta_2)}{\rho(\theta_2)} \). This is a straightforward calculation. \( \square \)

2.2 Mean Parametrization

The exponential family \( P_\theta \) defined in (2) can be reparametrized using the mean parameters \( \mu = \pi_\theta(f) \). The duality between the canonical parameters \( \theta \) and the mean parameters \( \mu \) is manifested through the log-Perron-Frobenius eigenvalue \( \Lambda(\theta) := \log \rho(\theta) \). More specifically, from Lemma 2 it follows that there are two cases for the mapping \( \theta \mapsto P_\theta \). In the nondegenerate case that this mapping is nonconstant, \( \Lambda'(\theta) \) is a strictly increasing bijection between the set \( \mathbb{R} \) of canonical parameters and the set

\[
\mathcal{M} := \{ \mu \in \mathbb{R} : \pi_\theta(f) = \mu, \text{ for some } \theta \in \mathbb{R} \}
\]

of mean parameters, which is an open interval. Therefore, with some abuse of notation, for any \( \mu \in \mathcal{M} \) we may write \( u_\mu, v_\mu, P_\mu, \pi_\mu \) for \( u_{\Lambda^{-1}(\mu)}, v_{\Lambda^{-1}(\mu)}, P_{\Lambda^{-1}(\mu)}, \pi_{\Lambda^{-1}(\mu)} \).

In the degenerate case that the mapping is constant, \( \Lambda'(\theta) = \pi(f) \), and the set \( \mathcal{M} \) is the singleton \( \{ \pi(f) \} \). An illustration of the degenerate case is Example 1.

Lemma 2. Let \( P \) be an irreducible stochastic matrix, and \( f : S \rightarrow \mathbb{R} \) a real-valued function on the state space \( S \). Then

(a) \( \rho(\theta), \Lambda(\theta), u_\theta \) and \( v_\theta \) are analytic functions of \( \theta \) on \( \mathbb{R} \).

(b) \( \Lambda'(\theta) = \pi_\theta(f) \).

(c) \( \Lambda''(\theta) = \text{var}_{(X,Y) \sim \pi_\theta \circ P_\theta} \left( f(Y) + \frac{v_\theta(X)}{v_\theta(Y)} \frac{d\pi_\theta(Y)}{d\pi_\theta(X)} \right) \), where \( \pi_\theta \circ P_\theta \) denotes the bivariate distribution defined by \( (\pi_\theta \circ P_\theta)(x,y) := \pi_\theta(x)P_\theta(x,y) \).

(d) Either \( P_\theta = P_0 = P \) for all \( \theta \in \mathbb{R} \) (degenerate case), or \( \theta \mapsto P_\theta \) is an injection (nondegenerate case).

Moreover, in the degenerate case \( \Lambda(\theta) = \pi_0(f)\theta \) is linear, while in the nondegenerate case \( \Lambda(\theta) \) is strictly convex.

The proof of Lemma 2 can be found in Appendix B.
2.3 Relative Entropy Rate and Conjugate Duality

For two probability distributions $Q$ and $P$ over the same measurable space we define the relative entropy between $Q$ and $P$ as

$$D(Q \parallel P) := \begin{cases} \mathbb{E}_Q \left[ \log \frac{dQ}{dP} \right], & \text{if } Q \text{ is absolutely continuous with respect to } P, \\ \infty, & \text{otherwise.} \end{cases}$$

Relative entropies of stochastic processes are most of the time trivial, and so we resort to the notion of relative entropy rate. Let $Q, P$ be two stochastic matrices over the same state space $S$. We further assume that $Q$ is irreducible with associated stationary distribution $\pi_Q$. For any initial distribution $q$ on $S$ we define the relative entropy rate between the Markov chain $Q_q$ induced by $Q$ with initial distribution $q$, and the Markov chain $P_q$ induced by $P$ with initial distribution $q$ as

$$D(Q \parallel P) := \lim_{n \to \infty} \frac{1}{n} D(Q_q | F_n \parallel P_q | F_n),$$

where $Q_q | F_n$ and $P_q | F_n$ denote the finite dimensional distributions of the probability measures restricted to the sigma algebra $F_n$. Note that the definition is independent of the initial distribution $q$, since we can easily see using ergodic theory that

$$D(Q \parallel P) = \sum_{x,y} \pi_Q(x)Q(x,y) \log \frac{Q(x,y)}{P(x,y)} = D(\pi_Q \circ Q \parallel \pi_Q \circ P),$$

where $\pi_Q \circ Q$ denotes the bivariate distribution

$$(\pi_Q \circ Q)(x, y) := \pi_Q(x)Q(x, y),$$

and we use the standard notational conventions $\log 0 = \infty$, $\log \frac{\alpha}{0} = \infty$ if $\alpha > 0$, and $0 \log 0 = 0 \log \frac{0}{0} = 0$.

For stochastic matrices which are elements of the exponential family $P_\theta$ defined in (2) we simplify the relative entropy rate notation as follows. For $\theta_1, \theta_2 \in \mathbb{R}$ and $\mu_1 = \Lambda'(\theta_1)$, $\mu_2 = \Lambda'(\theta_2)$ we write

$$D(\theta_1 \parallel \theta_2), D(\mu_1 \parallel \mu_2) := D(\pi_{\theta_1} \circ P_{\theta_1} \parallel \pi_{\theta_1} \circ P_{\theta_2}).$$

For those relative entropy rates Lemma 3 suggests an alternative representation based on the parametrization. Its proof can be found in Appendix B.
Lemma 3. Let $\theta_1, \theta_2 \in \mathbb{R}$ and $\mu_1 = \Lambda'(\theta_1)$, $\mu_2 = \Lambda'(\theta_2)$. Then

$$D(\theta_1 \parallel \theta_2) = \Lambda(\theta_2) - \Lambda(\theta_1) - \mu_1(\theta_2 - \theta_1).$$

We further define the convex conjugate of $\Lambda(\theta)$ as $\Lambda^*(\mu) := \sup_{\theta \in \mathbb{R}} \{\theta \mu - \Lambda(\theta)\}$. Moreover, since we saw in Lemma 2 that $\Lambda(\theta)$ is convex and analytic, we have that the biconjugate of $\Lambda(\theta)$ is $\Lambda(\theta)$ itself, i.e. $\Lambda(\theta) = \sup_{\mu \in \mathbb{R}} \{\mu \theta - \Lambda^*(\mu)\}$. The convex conjugate $\Lambda^*(\mu)$ represents the rate of exponential decay for large deviation events, and in the following Lemma 4, which is established in Appendix B, we derive a closed form expression for it.

Lemma 4.

$$\Lambda^*(\mu) = \begin{cases} 
D(\mu \parallel \pi(f)), & \text{if } \mu \in \mathcal{M}, \\
\lim_{\hat{\mu} \to \mu} D(\hat{\mu} \parallel \pi(f)), & \text{if } \mu \in \partial \mathcal{M}, \\
\infty, & \text{otherwise}, 
\end{cases}$$

where $\mathcal{M}$ is defined in (3).

An inspection of how the supremum was obtained in the previous Lemma 4 yields the following Corollary 1.

Corollary 1.

$$\Lambda^*(\mu) = \begin{cases} 
\sup_{\theta \geq 0} \{\theta \mu - \Lambda(\theta)\}, & \text{if } \mu \geq \pi(f), \\
\sup_{\theta \leq 0} \{\theta \mu - \Lambda(\theta)\}, & \text{if } \mu \leq \pi(f). 
\end{cases}$$

3 Optimal Chernoff Bound

3.1 The Class of Stochastic Matrices of Interest

In order to develop our upper tail bounds we assume that the irreducible stochastic matrix $P$ satisfies A 1-A 2, for the given function $f : S \to \mathbb{R}$. Under those conditions we are able to show in Proposition 1 that the ratio of the entries of the right Perron-Frobenius eigenvector $v_{\theta}(y)/v_{\theta}(x)$ is uniformly bounded for $\theta \in \mathbb{R}_{>0}$. Note that the conditions A 1-A 2 are satisfied by Markov chains where every transitions has a positive probability. For these Markov chains, and in particular for Markov chains that induce IID processes, we provide explicit uniform bounds in Proposition 1.
The following example suggests that we cannot meet the requirement that the ratios of the entries of the right Perron-Frobenius eigenvector is uniformly bounded if we drop assumption A 1.

\textit{Example 2.} Let \( S = \{ \pm 1 \} \), \( P(x, y) = 1\{ x = -1 \}/2 + 1\{ x = 1, y = -1 \} \), and \( f(x) = x \). Then \( \rho(\theta) = \frac{1 + \sqrt{1 + 8e^{2\theta}}}{4} e^{-\theta} \), and \( v_\theta(-1)/v_\theta(1) = \rho(\theta)e^\theta \to \infty \) as \( \theta \to \infty \).

Similarly a birth-death chain illustrates the role of assumption A 2.

\textit{Example 3.} Let \( S = \{-1, 0, 1\} \), \( P(x, y) = 1\{ x + y \neq 0 \}/2 \) and \( f(x) = -x \). Then \( \rho(\theta) = \frac{1}{4} e^\theta \left( 1 + e^{-\theta} + e^{-2\theta} + \sqrt{1 + 2e^{-\theta} - 5e^{-2\theta} + 2e^{-3\theta} + e^{-4\theta}} \right) \), and \( v_\theta(0)/v_\theta(1) = 2\rho(\theta) - e^{-\theta} \to \infty \) as \( \theta \to \infty \).

The natural interpretation of conditions A 1-A 2 is that they allow us to create new Markov chains with any stationary mean in the interval \([\pi(f), b)\), by selecting appropriate tilting levels \( \theta \in \mathbb{R}_{\geq 0} \).

3.2 The Limiting Behavior of the Exponential Family

Define the matrix

\[ \overline{P}_\theta(x, y) := e^{-\theta b} P_\theta(x, y) = e^{-\theta b} P(x, y)e^{\theta f(y)}, \]

and note that \( \rho(\overline{P}_\theta) = e^{-\theta b}\rho(\theta) \), as well as \( u_{\overline{P}_\theta} = u_\theta \), \( v_{\overline{P}_\theta} = v_\theta \). Hence \( \overline{P}_\theta \) will help us study the asymptotic behavior of \( P_\theta \), since

\[ P_\theta(x, y) = \frac{\overline{P}_\theta(x, y)v_\theta(y)}{\rho(\overline{P}_\theta)v_\theta(x)}. \]

Note that

\[ \overline{P}_\infty(x, y) := \lim_{\theta \to \infty} \overline{P}_\theta(x, y) = \begin{cases} P(x, y), & \text{if } y \in S_b, \\ 0, & \text{otherwise.} \end{cases} \]

Due to the structure imposed on \( P \) through A 1-A 2, the following Lemma 5, which constitutes a simple extension of the Perron-Frobenius theory for matrices which are not necessarily irreducible, suggests that \( \rho(\overline{P}_\infty) > 0 \) is a simple eigenvalue of \( \overline{P}_\infty \), which is associated with unique left and right eigenvectors \( u_\infty, v_\infty \) such that \( u_\infty(x) > 0 \) for \( x \in S_b \) and \( u_\infty(x) = 0 \) for \( x \notin S_b \), \( v_\infty \) is positive, \( \sum_x u_\infty(x) = 1 \) and \( \sum_x u_\infty(x)v_\infty(x) = 1 \).
Lemma 5. Let \( M \in \mathbb{R}^{s \times s}_{\geq 0} \) be a nonnegative matrix such that after a consistent renumbering of its rows and columns we can assume, for some \( k \in \{1, \ldots, s\} \), that \( M \) consists of an irreducible square block \( A \in \mathbb{R}^{k \times k}_{\geq 0} \), and a rectangular block \( B \in \mathbb{R}^{(s-k) \times k}_{\geq 0} \) such that none of the rows of \( B \) is zero, assembled together in the following way

\[
M = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}.
\]

Then, \( \rho(M) = \rho(A) > 0 \) is a simple eigenvalue of \( M \), which we call the Perron-Frobenius eigenvalue, and is associated with unique left and right eigenvectors \( u_M, v_M \) such that \( u_M \) has its first \( k \) coordinates positive and its last \( s-k \) coordinates equal to zero, \( v_M \) is positive, \( \sum_{x=1}^{k} u_M(x) = 1 \), and \( \sum_{x=1}^{k} u_M(x)v_M(x) = 1 \).

Proof. Let \( u_A, v_A \) be the unique left and right eigenvectors of \( A \) corresponding to the Perron-Frobenius eigenvalue \( \rho(A) \), such that both of them are positive, \( \sum_{x=1}^{k} u_A(x) = 1 \) and \( \sum_{x=1}^{k} u_A(x)v_A(x) = 1 \). Observe that the vectors

\[
u_M = \begin{bmatrix} u_A \\ 0 \end{bmatrix}, \quad v_M = \begin{bmatrix} v_A \\ Bv_A/\rho(A) \end{bmatrix},
\]

are left and right eigenvectors of \( M \) with associated eigenvalue \( \rho(A) \), and satisfy all the conditions.

In addition, any eigentriple \( \lambda, \begin{bmatrix} u_h^\top & u_l^\top \end{bmatrix}^\top, \begin{bmatrix} v_h^\top & v_l^\top \end{bmatrix}^\top \) of eigenvalue and corresponding left and right eigenvectors of \( M \), will certainly have \( u_l = 0 \), and gives rise to an eigentriple \( \lambda, u_h, v_h \) for \( A \). Therefore, \( \rho(M) = \rho(A) \) and the uniqueness of \( u_M, v_M \) follows from the uniqueness of \( u_A, v_A \). \( \square \)

Note that from Lemma 5 for \( k = s \) we recover the classic Perron-Frobenius theorem (which we have of course used in the proof of Lemma 5).

A continuity argument for simple eigenvalues and their corresponding eigenvectors, enables us to describe the asymptotic behavior of \( P_\theta \) in Lemma 6.

Lemma 6. \((u_\theta, \rho(P_\theta), v_\theta) \to (u_\infty, \rho(P_\infty), v_\infty)\), as \( \theta \to \infty \), and so the following is a well defined stochastic matrix

\[
P_\infty(x, y) := \lim_{\theta \to \infty} P_\theta(x, y) = \frac{P_\infty(x, y)v_\infty(y)}{\rho(P_\infty)v_\infty(x)}.
\]
Proof. Note that $\overline{P}_\infty$ possess the structure of Lemma 5. Consider Lemma 10 in Appendix A, with $M$ taken to be $\overline{P}_\infty$. For $W$ in a sufficiently small neighborhood of $M$ the function $g(W)$ identified in the proof of that lemma is analytic and equals $[u^\top_W \rho(W) v^\top_W]$ for all $W$ in that neighborhood that have the structure in Lemma 5. Now, since $\overline{P}_\theta \to \overline{P}_\infty$ as $\theta \to \infty$, we have $\overline{P}_\theta$ is in this neighborhood for all sufficiently large $\theta$, and $\overline{P}_\theta$, being irreducible, satisfies the conditions of Lemma 5. The conclusion is now immediate. \[\square\]

Remark 8. The combination of the extended Perron-Frobenius theorem in Lemma 5 and the limiting behavior of the exponential family established in Lemma 6 imply that

$$\pi_\theta(f) \to b \quad \text{as} \quad \theta \to \infty,$$

which together with Lemma 2 (b) means that any mean $\mu$ in the interval $[\pi(f), b)$ can be realized by some exponential tilt $\theta \in \mathbb{R}_{\geq 0}$.

A critical ingredient to obtain our tail bounds is the following Proposition 1 which states that under the assumptions A 1-A 2 the ratio of the entries of the right Perron-Frobenius eigenvector stays uniformly bounded.

Proposition 1. Let $P$ be an irreducible stochastic matrix on the finite state space $S$, which, combined with a real-valued function $f : S \to \mathbb{R}$, satisfies A 1-A 2. Then

$$K_u := \sup_{\theta \in \mathbb{R}_{\geq 0}, x, y \in S} \frac{v^{\overline{P}_\theta}(x)}{v^{\overline{P}_\theta}(y)} < \infty, \quad \text{and} \quad L_u := \sup_{\theta \in \mathbb{R}_{\geq 0}, x, y \in S} \left| \frac{d}{d\theta} \frac{v^{\overline{P}_\theta}(x)}{v^{\overline{P}_\theta}(y)} \right| < \infty,$$

where $K_u = K_u(P, f)$ and $L_u = L_u(P, f)$ are constants depending on the stochastic matrix $P$, and the function $f$. In particular

- if $P$ induces an IID process, i.e. $P$ has identical rows, then $K_u = 1$ and $L_u = 0$;

- if $P$ is a positive stochastic matrix, then $K_u \leq \max_{x,y,z} \frac{P(x,z)}{P(y,z)}$.

Proof. Lemma 2 yields that $\theta \mapsto v_\theta(x)/v_\theta(y)$ is continuous, and so in conjunction with Lemma 6 we have that the ratio of the entries of the right Perron-Frobenius eigenvector is uniformly bounded, hence $K_u < \infty$.

Moreover, using the chain rule we see that, for $\theta > 0$, we have

$$\frac{d}{d\theta} \frac{v^{\overline{P}_\theta}(x)}{v^{\overline{P}_\theta}(y)} = \sum_{z,w : P(z,w) > 0} -(b-f(w))e^{-\theta(b-f(w))}P(z,w) \frac{\partial}{\partial W(z,w)} \frac{v_W(x)}{v_W(y)} \bigg|_{W=P_{\theta}}.$$

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To see why this formula holds, first observe that $\bar{P}_\theta$, being irreducible, satisfies the conditions of Lemma 5. Next, observe that the last $s$ coordinates of $g(\bar{P}_\theta)$, in the notation of the proof of Lemma 10, are all strictly positive. With some abuse of notation since we are not really thinking of $S$ as being enumerated, let us write the last $s$ coordinates of $g(W)$ as $g_W(s + 1 + x)$, for $x \in S$. Lemma 10 then implies, that for all $x, y \in S$, the ratio $\frac{g_W(s + 1 + x)}{g_W(s + 1 + y)}$ is analytic in a sufficiently small neighborhood of $\bar{P}_\theta$. Since a small enough variation in $\theta$ centered around the given $\theta$ results in a variation of $W$ centered around $\bar{P}_\theta$ that lies in the set of matrices in this neighborhood that satisfy Lemma 5 (in fact all such matrices are irreducible and, even further, are of the form $\bar{P}_{\theta'}$, for some $\theta'$), we may use the notation $\frac{\partial W}{\partial \theta}$, for the ratio $\frac{g_W(s + 1 + x)}{g_W(s + 1 + y)}$. The point is that what is really intended in the partial derivatives on the right hand side of the preceding equation is $\frac{\partial}{\partial W(z, w)} \frac{g_W(s + 1 + x)}{g_W(s + 1 + y)}|_{W = \bar{P}_\theta}$.

We claim that $\lim_{\theta \to \infty} \frac{\partial}{\partial \theta} \frac{v_W(x)}{v_W(y)} = 0$. This is true because Lemma 10 in Appendix A ensures that $\frac{\partial}{\partial W(z, w)} \frac{v_W(x)}{v_W(y)}$ is continuous at $\bar{P}_\infty$, more precisely

$$
\lim_{\theta \to \infty} \frac{\partial}{\partial W(z, w)} \frac{v_W(x)}{v_W(y)}|_{W = \bar{P}_\theta} = \frac{\partial}{\partial W(z, w)} \frac{g_W(s + 1 + x)}{g_W(s + 1 + y)}|_{W = \bar{P}_\infty} \in \mathbb{R}.
$$

Here, to be able to write the expression on the right hand side of the preceding equation, we first observe that $\bar{P}_\infty$ satisfies the conditions of Lemma 5 and so the last $s$ coordinates of $g(\bar{P}_\infty)$, in the notation of the proof of Lemma 10, are all strictly positive, and so, by Lemma 10, for all $x, y \in S$ the ratio $\frac{g_W(s + 1 + x)}{g_W(s + 1 + y)}$ is analytic in a neighborhood of $\bar{P}_\infty$. Further, the equality in the preceding equation is justified by the fact that, for all $\theta > 0$, $\frac{\partial}{\partial W(z, w)} \frac{v_W(x)}{v_W(y)}|_{W = \bar{P}_\theta}$ is just an alternate notation for $\frac{\partial}{\partial W(z, w)} \frac{g_W(s + 1 + x)}{g_W(s + 1 + y)}|_{W = \bar{P}_\theta}$, and, for all $\theta$ large enough, $\bar{P}_\theta$ lies in the neighborhood around $\bar{P}_\infty$ guaranteed by Lemma 10.

Furthermore for the two cases for which we have a special handle on $K$ we argue as follows.

- Let $\rho$ be the probability distribution driving the IID process, i.e. all the rows of $P$ are identical and equal to $\rho$. Then we can see that $u_\theta(y) = \frac{p(y)e^{\theta f(y)}}{\sum_x p(x)e^{\theta f(x)}}$, $\rho(\bar{P}_\theta) = \sum_x p(x)e^{\theta f(x)}$, and $v_\theta(y) = 1$ for all $y \in S$, since $\bar{P}_\theta$ is the rank one matrix $\bar{P}_\theta(x, y) = \rho(\bar{P}_\theta)v_\theta(x)u_\theta(y)$.

- If $P$ is a positive stochastic matrix then, for any $\theta \in \mathbb{R}$, $x, y \in S$ we
have that
\[
\frac{v_\theta(x)}{v_\theta(y)} = \frac{\sum_z \tilde{P}_\theta(x,z) v_\theta(z)}{\sum_z \tilde{P}_\theta(y,z) v_\theta(z)} \leq \max_{x,y,z} \frac{\tilde{P}_\theta(x,z)}{\tilde{P}_\theta(y,z)} = \max_{x,y,z} \frac{P(x,z)}{P(y,z)}.
\]

Moreover under conditions A 1-A 2 we are able to establish an explicit formula for the limiting relative entropy rate.

**Lemma 7.** For any \(\theta_2 \in \mathbb{R}\), let \(\mu_2 = \Lambda'(\theta_2)\). Then
\[
D(b \parallel \mu_2) := \lim_{\theta_1 \to \infty} D(\theta_1 || \theta_2) = -\log \rho(\overline{P}_\infty) - (\theta_2 b - \Lambda(\theta_2)).
\]

**Proof.** From Remark 8 we have that \(\lim_{\theta \to \infty} \Lambda'(\theta) = b\), so from Lemma 3 it suffices to show that
\[
\theta \Lambda'(\theta) - \Lambda(\theta) \to -\log \rho(\overline{P}_\infty), \text{ as } \theta \to \infty.
\]

Let \(c = \max_{x \not\in S_b} f(x)\). Fix \(x \in S\) and \(y \not\in S_b\). Pick \(y_b \in S_b\) such that \(P(x,y_b) > 0\) and as large as possible. From Proposition 1 we have that for any \(\theta \in \mathbb{R}_{\geq 0}\)
\[
P_\theta(x,y) \leq K_u \frac{P(x,y)}{P(x,y_b)} e^{-\theta(b - f(y))} P_\theta(x,y_b) \leq K_u K_0 e^{-\theta(b-c)},
\]
where
\[
K_0 := \max_{x \in S} \min_{y \in S} \frac{1}{P(x,y_b)} < \infty.
\]

Therefore the stationary probability of any such \(y\) is at most \(\pi_\theta(y) \leq K_u K_0 e^{-\theta(b-c)}\), and so
\[
\pi_\theta(f) \geq (1 - K_u K_0 |S| e^{-\theta(b-c)}) b + K K_0 |S| e^{-\theta(b-c)} a.
\]

From this we obtain that \(\lim_{\theta \to \infty} \theta(b - \Lambda'(\theta)) = 0\), and the conclusion follows since from Lemma 6 we have that \(\lim_{\theta \to \infty} (-\theta b + \Lambda(\theta)) = \log \rho(\overline{P}_\infty)\).

\(\square\)
3.3 Chernoff Bound

*Proof of Theorem 1.* In order to derive our bounds we use a change of measure argument, an idea due to Esscher (1932). We denote by \( P^{(\theta)} \) the probability distribution of the Markov chain with initial distribution \( q \) and stochastic matrix \( P^{(\theta)} \), while for \( \theta = 0 \) we just write \( P_q \) for \( P^{(0)} \). The finite dimensional distributions \( P_q |_{\mathcal{F}_n} \) and \( P^{(\theta)} |_{\mathcal{F}_n} \) are absolutely continuous with each other and their Radon-Nikodym derivative is given by

\[
\frac{dP^{(\theta)} |_{\mathcal{F}_n}}{dP_q |_{\mathcal{F}_n}} = \frac{v_{\theta}(X_0)}{v_q(X_0)} \exp \{-\theta S_n + n \Lambda(\theta)\},
\]

where we denote the sums by \( S_n := \sum_{k=1}^n f(X_k) \).

Fix \( \theta \in \mathbb{R}_{\geq 0} \). Then

\[
P_q(S_n \geq n\mu) = \mathbb{E}_q[1\{S_n \geq n\mu\}] = \mathbb{E}_q^{(\theta)} \left[ \frac{v_{\theta}(X_0)}{v_q(X_0)} e^{-\theta S_n + n \Lambda(\theta)} 1\{S_n \geq n\mu\} \right] \leq K_u \mathbb{E}_q^{(\theta)} \left[ e^{-\theta(S_n - n\mu)} 1\{S_n \geq n\mu\} e^{-n(\theta \mu - \Lambda(\theta))} \right] \leq K_u e^{-n(\theta \mu - \Lambda(\theta))},
\]

where in the first inequality we used Proposition 1.

When \( \mu \in [\pi(f), b] \), we can set \( \theta = \Lambda^{-1}(\mu) \geq \Lambda^{-1}(\pi(f)) = 0 \) and then from Lemma 3 we have that \( D(\mu \parallel \pi(f)) = \theta \mu - \Lambda(\theta) \). When \( \mu = b \), we let \( \theta \) go to \( \infty \) and use Lemma 7. The conclusion follows from Corollary 1. \( \square \)

In this bound we cannot hope for something more than a constant prefactor. First of all, by differentiating twice the formula proved in Lemma 3 we obtain

\[
\lim_{\mu \to \pi(f)} \frac{1}{(\mu - \pi(f))^2} D(\mu \parallel \pi(f)) = \frac{1}{2 \Lambda''(0)}.
\]

In addition, if we fix \( z \geq 0 \) and set \( \mu = \pi(f) + cz/\sqrt{n} \), where \( c^2 = \pi \left( \hat{f}^2 - (P \hat{f})^2 \right) \) and \( \hat{f} \) is a solution of the Poisson equation \((I - P)\hat{f} = f - \pi(f)\), then due to the central limit theorem for Markov chains, see for instance Chung (1960), we have that

\[
\lim_{n \to \infty} P_q(S_n \geq n\mu) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.
\]
Therefore if we want the optimal rate of exponential decay and a prefactor which does not depend on \( \mu \), then the best we can attain is a constant prefactor.

## 4 Hoeffding Bound

In this section we relax the Chernoff type bound to a Hoeffding type bound for finite Markov chains. We achieve this by establishing a Pinsker type inequality in Lemma 8 for the exponential family of stochastic matrices which provides a quadratic lower bound to the relative entropy rate. Through conjugate duality this leads to a Hoeffding type lemma in Lemma 9 which constitutes a quadratic upper bound on the log-Perron-Frobenius eigenvalue.

We first develop a Pinsker type inequality for the exponential family of stochastic matrices under consideration.

**Lemma 8.** Let \( P \) be an irreducible stochastic matrix on the finite state space \( S \), which combined with a real-valued function \( f : S \to [a,b] \) satisfies A1-A2. Then

\[
D(\mu \| \pi(f)) \geq \frac{1}{2} \frac{(\mu - \pi(f))^2}{\sigma_u^2} \geq 2 \frac{(\mu - \pi(f))^2}{(b - a + 2K_uL_u)^2}, \text{ for any } \mu \geq \pi(f),
\]

where \( \sigma_u^2 = \sigma_u^2(P, f) := \sup_{\theta \in \mathbb{R} \geq 0} \Lambda''(\theta) \in (0, \infty) \), and \( K_u, L_u \) are the constants from Proposition 1.

**Proof.** We have that \( \Lambda''(\theta) > 0 \) up to a null set, and so \( \sigma^2 > 0 \). Furthermore, from Lemma 2 we have

\[
\Lambda''(\theta) = \text{var}_{(X,Y) \sim \pi_{\theta} \otimes P_{\theta}} \left( f(Y) + \frac{v_{\theta}(X)}{v_{\theta}(Y)} \frac{d}{d\theta} v_{\theta}(X) \right),
\]

which, according to Proposition 1, is the variance of a random variable living in the interval \([a - K_uL_u, b + K_uL_u]\), and hence \( \sigma^2 \) is finite and upper bounded by \((b - a + 2K_uL_u)^2/4\).

Differentiating twice the formula proved in Lemma 3 we obtain

\[
\frac{\partial^2}{\partial \mu^2} D(\mu \| \pi(f)) = \frac{1}{\Lambda''(\Lambda^{-1}(\mu))} \geq \frac{1}{\sigma_u^2}, \text{ for almost every } \mu \in \mathcal{M} \cap [\pi(f), b].
\]
Using the fact that \( \frac{\partial}{\partial \mu} D (\mu \parallel \pi (f)) |_{\mu=\pi (f)} = 0 \), we conclude that

\[
D (\mu \parallel \pi (f)) = \int_{\pi (f)}^\mu (\mu - \nu) \frac{\partial^2}{\partial \nu^2} D (\nu \parallel \pi (f)) d\nu \geq \frac{1}{2} \frac{(\mu - \pi (f))^2}{\sigma_u^2}.
\]

Combining our Pinsker type inequality in Lemma 8 and the Chernoff type bound in Theorem 1, the Hoeffding type bound in Theorem 2 follows directly.

**Remark 9.** Under assumptions A 3-A 4, due to symmetry, we have that

\[
D (\mu \parallel \pi (f)) \geq \frac{1}{2} \frac{(\mu - \pi (f))^2}{\sigma_l^2} \geq \frac{2}{(b - a + 2K_lL_l)^2}, \text{ for any } \mu \leq \pi (f),
\]

where \( \sigma_l^2 = \sigma_u^2 (P, -f) \), \( K_l = K_u (P, -f) \), and \( L_l = L_u (P, -f) \).

It is also possible to establish this Hoeffding bound directly using the following Hoeffding type lemma for Markov chains which is essentially the dual of our Pinsker type inequality.

**Lemma 9.** Let \( P \) be an irreducible stochastic matrix on the finite state space \( S \), with stationary distribution \( \pi \), which combined with a real-valued function \( f : S \rightarrow [a, b] \) satisfies A 1-A 4. Then

\[
\Lambda (\theta) \leq \pi (f) \theta + \frac{1}{2} \sigma^2 \theta^2 \leq \pi (f) \theta + \frac{(b - a + 2KL)^2}{8} \theta^2, \text{ for any } \theta \in \mathbb{R},
\]

where \( \sigma^2 = \max \{ \sigma_u^2, \sigma_l^2 \} \), \( K = \max \{ K_u, K_l \} \), and \( L = \max \{ L_u, L_l \} \).

**Proof.** Plugging in \( \mu_2 = \pi (f) \) in Lemma 8, and using Lemma 4 it is easy to see that

\[
\Lambda^* (\mu) \geq \frac{1}{2} \frac{(\mu - \pi (f))^2}{\sigma^2}, \text{ for all } \mu \in \mathbb{R}.
\]

Finally, using the fact that \( \Lambda (\theta) \) is the convex conjugate of \( \Lambda^* (\mu) \) we conclude that

\[
\Lambda (\theta) \leq \sup_{\mu \in \mathbb{R}} \left\{ \mu \theta - \frac{1}{2} \frac{(\mu - \pi (f))^2}{\sigma^2} \right\} = \pi (f) \theta + \frac{1}{2} \sigma^2 \theta^2.
\]

We call Lemma 9 a Hoeffding type lemma, since we will establish in Section 5 that \( \frac{1}{n} \log \mathbb{E} [\exp \{ \theta \sum_{k=1}^n f (X_k) \}] \) converges to \( \Lambda (\theta) \), and so at least asymptotically this reveals the sub-Gaussian structure of the sums.
5 A Uniform Multiplicative Ergodic Theorem

The classic linear ergodic theory for Markov chains, Chung (1960) suggests that

$$\frac{1}{n} \mathbb{E}_q \left[ \sum_{k=1}^{n} f(X_k) \right] \to \pi(f), \text{ as } n \to \infty.$$  

Balaji and Meyn (2000) and Kontoyiannis and Meyn (2003) have proved a multiplicative version of this under appropriate assumptions, which states that the scaled log-moment-generating-function $\Lambda_n(\theta)$ converges pointwise to the log-Perron-Frobenius eigenvalue

$$\Lambda_n(\theta) \to \Lambda(\theta), \text{ as } n \to \infty, \text{ for any } \theta \in \mathbb{R},$$  

where

$$\Lambda_n(\theta) := \frac{1}{n} \log \mathbb{E}_q \left[ \exp \left\{ \theta \sum_{k=1}^{n} f(X_k) \right\} \right].$$  

For our class of finite Markov chains we are able to establish a uniform multiplicative ergodic theorem in the terminology of Balaji and Meyn (2000).

**Theorem 3.** Let $P$ be an irreducible stochastic matrix on the finite state space $S$, which combined with a real-valued function $f : S \to \mathbb{R}$ satisfies $A\ 1- A\ 4$. Then

$$\sup_{\theta \in \mathbb{R}} |\Lambda_n(\theta) - \Lambda(\theta)| \leq \frac{\log K}{n}.$$  

where $K$ is the constant from Proposition 1.

Therefore $\Lambda_n(\theta)$ converges uniformly on $\mathbb{R}$ to $\Lambda(\theta)$ as $n \to \infty$.

**Proof.** We start with the calculation

$$e^{n\Lambda_n(\theta)} = \sum_{x_0, x_1, \ldots, x_{n-1}, x_n} q(x_0) P(x_0, x_1)e^{\theta f(x_1)} \cdots P(x_{n-1}, x_n)e^{\theta f(x_n)}$$  

$$= \sum_{x_0, x_n} q(x_0) \tilde{P}_\theta^n(x_0, x_n).$$

From this, using the fact that $v_\theta$ is a right Perron-Frobenius eigenvector of $\tilde{P}_\theta$, we obtain

$$\min_{x,y} \frac{v_\theta(y)}{v_\theta(x)} \leq \exp \left\{ n\Lambda_n(\theta) - n\Lambda(\theta) \right\} \leq \max_{x,y} \frac{v_\theta(y)}{v_\theta(x)}.$$  

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The conclusion now follows by applying Proposition 1.

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Appendix A Analyticity of Perron-Frobenius Eigenvalues and Eigenvectors

Here we use the implicit function theorem in order to deduce in Lemma 10 that the Perron-Frobenius eigenvalue and eigenvectors are analytic functions of the entries of the matrix, at a level of generality adequate for our purposes.

Lemma 10. Let \( M \in \mathbb{R}^{s \times s}_{\geq 0} \) be a nonnegative matrix possessing the structure of Lemma 5, so in particular \( M \) can be a nonnegative irreducible matrix. Let \( W \) range over \( \mathbb{R}^{s \times s} \) in an open neighborhood of \( M \). Then \( u_W, \rho(W) \) and \( v_W \) are analytic as functions of the entries of the matrix \( W \) in an open neighborhood of \( M \) where \( W \) satisfies the conditions of Lemma 5.

Proof. We define the vector-valued function \( F : \mathbb{R}^{(s+1)^2} \rightarrow \mathbb{R}^{2(s+1)} \)

\[
F(W, u, \rho, v) := \begin{bmatrix}
(W^T - \rho I)u \\
\mathbf{1}^T u - 1 \\
(W - \rho I)v \\
u^T v - 1
\end{bmatrix},
\]

where we use column vectors, and \( \mathbf{1} \) denotes the all ones vector. At this point no assumptions are made about the structure of \( W \). Note that each coordinate of the vector \( F(W, u, \rho, v) \) is a multivariate polynomial of degree at most two, and hence each coordinate is an analytic function of \( W, u, \rho \) and \( v \).

In addition \( F(M, u_M, \rho(M), v_M) = 0 \), and the Jacobian of \( F \) with respect to \( u, \rho, v \) evaluated at \( W = M, u = u_M, \rho = \rho(M), v = v_M \) is

\[
J_{F,u,\rho,v}(M, u_M, \rho(M), v_M) = \begin{bmatrix}
M^T - \rho(M)I & -u_M & 0 \\
\mathbf{1}^T & 0 & 0 \\
0 & -v_M & M - \rho(M)I \\
v_M^T & 0 & u_M^T
\end{bmatrix}.
\]
We can easily verify that this Jacobian is left invertible. If \( [u^\top \rho v^\top]^\top \) is in the kernel of \( J_{F,u,v}(M,u_M,\rho(M),v_M) \), then \( M^\top u = \rho(M)u + \rho u_M \), so if we multiply from the left with \( v_M^\top \), we get that \( \rho = 0 \). In the same fashion, using Lemma 5, we can deduce that \( u = v = 0 \), and thus the kernel of the Jacobian is trivial.

Then the analytic implicit function theorem guarantees that there exists a unique vector-valued function \( g : \mathbb{R}^{s^2} \to \mathbb{R}^{2s+1} \) with each coordinate analytic, such that

\[
g(M) = \begin{bmatrix} u_M \\ \rho(M) \\ v_M \end{bmatrix}, \quad \text{and } F(W,g(W)) = 0, \text{ for all } W \text{ in a neighborhood of } M.
\]

Let \( k(M) \) denote the \( 1 \leq k \leq s \) corresponding to \( M \) in the context of the discussion of the structure of \( M \) in the statement of Lemma 5. Then, for \( W \) in a sufficiently small neighborhood of \( M \), the first \( k(M) \) and the last \( s + 1 \) coordinates of \( g(W) \) have to be strictly positive. If we now restrict to those \( W \) in such a neighborhood of \( M \) that satisfy the conditions of Lemma 5, then \( k(W) \geq k(M) \) and, by Lemma 5, \( g(W) \) has to equal \( [u_W^\top \rho(W) v_W^\top]^\top \) for such matrices in this neighborhood of \( M \).

**Appendix B  Proofs from Section 2**

**Proof of Lemma 2.**

(a) Each entry of \( \tilde{P}_\theta \) is an analytic function of \( \theta \), and the conclusion follows from Lemma 10 in Appendix A.

(b) For any \( x,y \in S \) such that \( P(x,y) > 0 \) we have

\[
\log P_\theta(x,y) = \log P(x,y) + \theta f(y) - \Lambda(\theta) + \log \nu_\theta(y) - \log \nu_\theta(x).
\]

Differentiating with respect to \( \theta \), and taking expectations with respect to \( \pi_\theta \odot P_\theta \) we obtain

\[
\mathbb{E}_{(X,Y) \sim \pi_\theta \odot P_\theta} \frac{d}{d\theta} \log P_\theta(X,Y) = \pi_\theta(f) - \Lambda'(\theta).
\]

The conclusion follows because

\[
\mathbb{E}_{(X,Y) \sim \pi_\theta \odot P_\theta} \frac{d}{d\theta} \log P_\theta(X,Y) = \sum_x \pi_\theta(x) \frac{d}{d\theta} \left( \sum_y P_\theta(x,y) \right) = 0.
\]
(c) For any \( x, y \in S \) such that \( P(x, y) > 0 \) we have
\[
\frac{d^2}{d\theta^2} \log P_\theta(x, y) = -\Lambda''(\theta) + \frac{d^2}{d\theta^2} \log v_\theta(y) - \frac{d^2}{d\theta^2} \log v_\theta(x).
\]
Taking expectations with respect to \( \pi_\theta \circ P_\theta \) we obtain
\[
\Lambda''(\theta) = -\mathbb{E}_{(X,Y) \sim \pi_\theta \circ P_\theta} \left( \frac{d}{d\theta} \log P_\theta(X,Y) \right)^2
= \mathbb{E}_{(X,Y) \sim \pi_\theta \circ P_\theta} \left( f(Y) - \pi_\theta(f) + \frac{v_\theta(X)}{v_\theta(Y)} \frac{d}{d\theta} v_\theta(X) \right)^2.
\]

(d) Part (c) already ensures that \( \Lambda(\theta) \) is convex. Moreover we see that
\[\Lambda''(\theta) = 0 \text{ for all } \theta \in (\theta_1, \theta_2), \quad \text{iff} \quad P_\theta = P_{\frac{\theta_1 + \theta_2}{2}} \text{ for all } \theta \in (\theta_1, \theta_2).\]
If such an interval \((\theta_1, \theta_2)\) exists, then we claim that we can enlarge it to the whole real line. To see this fix any \( 0 < \epsilon < \frac{\theta_2 - \theta_1}{2} \). Then using Lemma 1 twice we obtain that for any \( \theta \in (\theta_1, \theta_2) \)
\[
P_{\theta \pm \epsilon} = (P_\theta)_{\pm \epsilon} = \left( P_{\frac{\theta_1 + \theta_2}{2}} \right)_{\pm \epsilon} = P_{\frac{\theta_1 + \theta_2}{2} \pm \epsilon} = P_{\frac{\theta_1 + \theta_2}{2}}.
\]
By repeating this process we see that \( P_\theta = P_0 = P \) for all \( \theta \in \mathbb{R} \).
Alternatively, if no such interval exists, then \( \Lambda'(\theta) \) is strictly increasing and \( \Lambda(\theta) \) is strictly convex. Moreover, for \( \theta_1 < \theta_2 \) we have that \( \pi_{\theta_1}(f) = \Lambda'(\theta_1) < \Lambda'(\theta_2) = \pi_{\theta_2}(f) \), and so \( P_{\theta_1} \neq P_{\theta_2} \), establishing that in this case \( \theta \mapsto P_\theta \) is an injection.

\[\square\]

**Proof of Lemma 3.**
\[
D(\theta_1 \parallel \theta_2) = \mathbb{E}_{(X,Y) \sim \pi_{\theta_1} \circ P_{\theta_1}} \log \frac{P_{\theta_1}(X,Y)}{P_{\theta_2}(X,Y)}
= \Lambda(\theta_2) - \Lambda(\theta_1) - \Lambda'(\theta_2 - \theta_1)
+ \mathbb{E}_{(X,Y) \sim \pi_{\theta_1} \circ P_{\theta_1}} \log \frac{v_{\theta_1}(Y)}{v_{\theta_1}(X)} - \mathbb{E}_{(X,Y) \sim \pi_{\theta_1} \circ P_{\theta_1}} \log \frac{v_{\theta_2}(Y)}{v_{\theta_2}(X)}
= \Lambda(\theta_2) - \Lambda(\theta_1) - \mu_1(\theta_2 - \theta_1),
\]
where the second equality is using the calculations from the proof of Lemma 2 (b).

Proof of Lemma 4. From Lemma 2 we have that $\theta \mapsto \theta \mu - \Lambda(\theta)$ is either the linear function $\theta \mapsto (\mu - \pi(f))\theta$, in which case the conclusion follows right away, or otherwise it is strictly concave.

In the latter case $\mathcal{M} = (\mu_-, \mu_+)$ for some $\mu_- < \mu_+$. If $\mu \in \mathcal{M}$, then $\theta = \Lambda^{-1}(\mu)$ is the unique maximizer and the conclusion follows from Lemma 3. If $\mu = \mu_+$, then the function keeps on growing as $\theta \to \infty$, or equivalently as $\hat{\mu} \to \mu$, which in conjunction with the representation of the relative entropy rate from Lemma 3 establishes this case. If $\mu > \mu_+$, then $\lim_{\theta \to \infty} (\theta \mu - \Lambda(\theta)) = \lim_{\theta \to \infty} \theta(\mu - \mu_+) + \lim_{\hat{\mu} \to \mu_+} D(\hat{\mu} \parallel \pi(f)) = \infty$. The arguments are the same for the other two cases.

□