THE MYSTERIES OF THE BEST APPROXIMATION AND CHEBYSHEV EXPANSION FOR THE FUNCTION WITH LOGARITHMATIC REGULARITIES

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Abstract. The best polynomial approximation and Chebyshev approximation are both important in numerical analysis. In tradition, the best approximation is regarded as more better than the Chebyshev approximation, because it is usually considered in the uniform norm. However, it not always superior to the latter noticed by Trefethen [11, 12] for the algebraic singularity function. Recently Wang [14] have proved it in theory. In this paper, we find that for the functions with logarithmic regularities, the pointwise errors of Chebyshev approximation are smaller than the ones of the best approximations except only in the very narrow boundaries at the same degree. The pointwise error for Chebyshev series, truncated at the degree $n$ is $O(n^{-\kappa})$ ($\kappa = \min\{2\gamma + 1, 2\beta + 1\}$), but is worse by one power of $n$ in narrow boundary layer near the weak singular endpoints. Theorems are given to explain this effect.

Key words. Chebyshev Projection, Endpoint Singularities, Steepest Descent Method, Best Approximation, Chebyshev Interpolation

AMS subject classifications. 41A10, 41A25, 41A50, 65D05

1. Introduction. The best polynomial approximations and the Chebyshev approximations are dense in numerical analysis and scientific computing with applications in rootfinding, function approximation, integral equations and differential equations. The best approximation is an old idea, which goes back to Chebyshev himself. Before the fast computers, the best approximations received more attentions than the alternatives. Thus in a long history, the best approximation is regarded as the optimal. At the early age of nineteen century, with the computer appearance and the Fast Fourier Transform being proposed [5], researchers have gradually begun to pay attentions to the Chebyshev approximation. In fact, the Chebyshev approximation is more practical and useful, because computing the best approximation is much more expensive. Furthermore, the Chebyshev approximation can easily be used to solve the quadrature and differential problems. During past several decades, it is widely used to solve many mathematical or engineering problems, for more details you can refer [3, 4, 8, 9, 10, 12, 16]. At the same time, connections and differences between best approximation and other orthogonal polynomial projection are discovered [13]. To begin, a bit more notation needs to be given.

In this paper, the interval $\Omega = [-1, 1]$ is considered. $T_k(x) = \cos(k \arccos(x))$ is the Chebyshev polynomial of the first kind. If a function $f(x)$ satisfies the Dini-Lipschitz continuous condition, the following series can uniformly convergence to the function itself

\begin{equation}
(1.1) \quad f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \quad x \in [-1, 1], \quad a_k = \frac{2}{\pi} \int_{-1}^{1} f(x) T_k(x) \frac{1}{\sqrt{1-x^2}} dx,
\end{equation}

where the prime on the sum denotes the first term is divided by 2. Truncate the exact series to a polynomial of degree $n$

\begin{equation}
(1.2) \quad f_n = \sum_{k=0}^{n} a_k T_k(x).
\end{equation}

Let $\mathcal{P}_n$ to be subspace of all real polynomial of degree at most $n$ in $C[-1, 1]$. $p_n^*$ denotes the best polynomial approximation in $\mathcal{P}_n$ to function $f(x)$ in the maximum norm, i.e., $\| f - p_n^* \|_{L^\infty(\Omega)} = \min_{h \in \mathcal{P}_n} \| f - h \|_{L^\infty(\Omega)}$. Just as pointed out in [12, 14], the best approximation is usually more accurate than the same degree Chebyshev truncation in view of maximum norm as is shown in Figure 1. The classical theorem can confirm this.

**Theorem 1.1** (Chebyshev truncation is near-best[12]). Let $f$ be continuous on $[-1, 1]$ with degree $n$ Chebyshev truncation $f_n$ and best approximation $p_n^*$, $n > 1$. Then

\begin{equation}
(1.3) \quad \| f - f_n \| \leq \left( 4 + \frac{4}{\pi^2} \ln(n) \right) \| f - p_n^* \| .
\end{equation}

The proof can refer [12]. Most people focus on the comparisons the two approximations in the uniform norm, but very little attention has been paid to the pointwise errors. Thus, on the face of it we

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pointwise error analysis of Chebyshev truncation is indispensable. In this paper, we will consider the functions with singularities. By comparison the Figure 2 and the Figure 3, we can see that Chebyshev approximation is more accurate than the best approximation for the algebraic singular function. But recently, Trefethen [12] have noticed the surprising phenomenon that the Chebyshev truncation seem to have the best approximation is always accurate more than its younger brother, the Chebyshev approximation. Wang [14] gave an detail explanation in theory why the Chebyshev truncation is better than the best approximation except only in the very narrow layer centered at the singularity point. In fact, the behavior of the two approximations are rather bewildering as is shown in Figure 2. In this figure, all the functions has no singularities on the real interval \([-1, 1]\). Figure 3 shows the pointwise error curves of the functions with singularities. By comparison the Figure 2 and the Figure 3, we can see that

\[ f(x) = |x - 0.25|^{\alpha} g(x), \quad x \in [-1, 1], \quad \alpha > 0, \quad g(x) \text{ is analytic}, \]  

without other singularities in the interval except in the narrow layer centered the algebraic singularity. Wang [14] gave an detail explanation in theory why the Chebyshev truncation is better than the best approximation except only in the very narrow layer centered at the singularity point. In fact, the behavior of the two approximations are rather bewildering as is shown in Figure 2. In this figure, all the functions has no singularities on the real interval \([-1, 1]\). Figure 3 shows the pointwise error curves of the functions with singularities. By comparison the Figure 2 and the Figure 3, we can see that

- For the analytic function \(f(x)\) on the complex plane, the both approximations seem almost equal that the difference hardly matters.
- Although the function \(f(x)\) has no singularities on the interval \([-1, 1]\), but owns singularities on the whole complex plane. The pointwise error of the Chebyshev approximation near the singularities on the whole complex plane might be larger than the counterpart error of the best approximation.
- Just like the algebraic singular function, the logarithm singular function owns a much better Chebyshev approximation rather than the best approximation for almost all values of \(x \in [-1, 1]\). There are interesting phenomena as depicted in the right panel of Figure 3, specifically,
  (i) Why does the pointwise error of Chebyshev truncation blow up at the endpoints for the logarithm singularity function?
  (ii) The both points \(x = -1\) and \(x = 1\) cause the weak singularity for the function \(f(x) = (1 + \frac{x}{2}) (1 - x^3 \ln(1 - x^3))\). Why does the pointwise error of Chebyshev truncation at the point \(x = -1\) is smaller than the counterpart error at the point \(x = 1\)?

Inspired by the idea in the article [14], we make above observations. To answer the questions, the pointwise error analysis of Chebyshev truncation is indispensable. In this paper, we will consider the
Fig. 2. Comparisons of the Chebyshev approximations with the best approximations for the function \( f(x) \), \( x \in [-1, 1] \). Blue line: the pointwise error of the best approximation of degree \( n \), red circle: the pointwise error of the Chebyshev projection of degree \( n \). Upper left: \( f(x) = e^x \), Upper right: \( f(x) = \sin(10x) \), Lower left: \( f(x) = \frac{1}{1 + 25x^2} \), Lower right: \( f(x) = \ln(1.1 - x) \).

Fig. 3. Pointwise Error curves of \( f_n \) and \( p_\ast_n \) for the function \( f(x) \) with \( n = 50 \). Left: \( f(x) = |x - 0.5| \), Right: \( f(x) = (1 + \frac{x}{2})(1 - x^2)\ln(1 - x^2) \).

Following function

\[
f(x) = (1 - x)^\gamma \ln^\mu(1 - x)g_1(x) + (1 + x)^\delta \ln^\mu(1 + x)g_2(x), \quad x \in [-1, 1],
\]

where \( \gamma > -1/2, \delta > -1/2 \), \( g_1(x), g_2(x) \) are analytic on \([-1, 1] \), \( \mu \) is a positive integer. The function plays an important role in function approximations and Flory Huggins theory etc. Similar to the reference [14], to explain the behavior of error for the logarithm singularity function, one needs to analyze the asymptotic behavior of the following functions

\[
\Psi^C(t, n) = \sum_{k=n+1}^{\infty} \frac{\cos(k t)}{k^{\nu+1}}, \quad \Psi^{C,L}(t, n) = \sum_{k=n+1}^{\mu} \frac{\cos(k t)}{k^{\nu+1}} \ln^\mu(k)
\]

where \( t \in [0, \pi], \nu > 0, \mu \in \mathbb{N} \). For the series \( \Psi^C(t, n) \), Wang [14] has given the asymptotic representation
as $n \to +\infty$. In section 3, we will use a more simple technique to estimate the asymptotic behavior of $\Psi_{C}(t, n)$ and $\Psi_{C,L}(t, n)$.

The paper is organized as follows. In section 2, the steepest descent method is used to provide the asymptotic coefficients of the Chebyshev expansion for the function $(1.5)$. In section 3, we discuss the pointwise errors of the Chebyshev truncations for the function $(1.5)$. In section 4, based on the theory analyses in section 3, the asymptotic behavior of pointwise error of Chebyshev interpolation is provided.

2. Asymptotic Coefficients of Chebyshev expansions. In this section, we will use the steepest descent method to give the asymptotic coefficients for the function $(1.5)$. The asymptotic estimate of Chebyshev coefficients for the function $(1.5)$ when $\mu = 1$ was given by Boyd [2]. In this paper, the asymptotic coefficients for the function with the general positive integer $\mu = 1, 2, \ldots$, are extended. Xiang [15] gave the Jacobi coefficients for the similar logarithmic singularity function. Now we consider the function

$$f(x) = (1 - x)^\gamma \ln^\mu (1 - x)g_1(x), \quad x \in [-1, 1],$$

where $\gamma > -\frac{1}{2}, \mu \in \mathbb{N}_+$, and $g_1(x)$ is analytic on $[-1, 1]$. If $\mu = 0, f(x) = (1 - x)^\gamma$, the function may only own branch singularity when $\gamma$ is not an integer. Before getting the asymptotic coefficients of Chebyshev expansion, two lemmas are given.

**Lemma 2.1.** Suppose $\text{Re}(\gamma) > -1, \text{Re}(k) > 0$, then

$$\int_0^\infty s^\gamma \ln(s) e^{-ks} ds = \frac{1}{k^{\gamma+1}} \Gamma(\gamma + 1) [\Psi(\gamma + 1) - \ln(k)],$$

where $\Psi(\gamma + 1)$ is the digamma function.

**Proof.** Set $t = ks$, then the integral can be written as

$$\int_0^\infty s^\gamma \ln(s) e^{-ks} ds = \frac{1}{k^{\gamma+1}} \int_0^\infty t^\gamma [\ln(t) - \ln(k)]e^{-t} dt = \frac{1}{k^{\gamma+1}} \Gamma(\gamma + 1) (\Psi(\gamma + 1) - \ln(k)).$$

The lemma can also be found in identity 4.352 of Gradshteyn and Rhyzik [7], but the proof was not be provided, so the detail proof is given above.

**Lemma 2.2.** Suppose $\text{Re}(\gamma) > -1, \text{Re}(k) > 0$ and $\mu$ is a given positive integer, then

$$\int_0^\infty s^\gamma \ln^\mu(s) e^{-ks} ds = \frac{1}{k^{\gamma+1}} \sum_{\ell=0}^\mu \left( \begin{array}{c} \mu \\ \ell \end{array} \right) \Gamma(\mu - \ell) (\gamma + 1)(-1)^\ell \ln^\ell(k)$$

$$\sim (-1)^k \Gamma(\gamma + 1) \frac{\ln^\mu(k)}{k^{\gamma+1}}, \quad \text{as } n \to +\infty.$$

**Proof.** To proof this lemma, one only needs to take the $(\mu - 1)$th derivative of Eq.(2.2) about $\gamma$.

**Theorem 2.3.** If a function $f(x)$ has a singularity at $x = 1$ of the form (2.1), then the coefficients of Chebyshev series are asymptotically given by

$$a_k \sim \frac{2^{\mu+1-\gamma}}{\pi} g_1(1) \frac{(-1)^{\mu+1} \sin(\gamma \pi)}{\Gamma(2\gamma + 1)} \frac{\ln^\mu(k)}{k^{2\gamma+1}} + (-1)^\mu \frac{\mu}{2^\gamma} g_1(1)$$

$$\times \left( \frac{2}{\pi} \sin(\gamma \pi) \Gamma'(2\gamma + 1) + \left( \cos(\gamma \pi) - \frac{\ln(2)}{\pi} \right) \Gamma(2\gamma + 1) \right) \frac{\ln^{\mu-1}(k)}{k^{2\gamma+1}} + O\left( \frac{\ln^{\mu-2}(k)}{k^{2\gamma+1}} \right).$$

**Proof.** The Chebyshev expansion can be expressed as

$$f(x) = \sum_{k=0}^\infty a_k T_k(x),$$

where

$$a_k = \frac{2}{\pi} \int_{-1}^1 f(x) \frac{1}{\sqrt{1-x^2}} T_k(x) dx, \quad k > 0.$$
Set \( x = \cos(t), t \in [0, \pi] \), the coefficients \( a_k \) are
\[
a_k = \frac{2}{\pi} \int_0^\pi f(\cos(t)) \cos(k t) dt, \quad k \in \mathbb{N}_+.
\]

Since the Chebyshev Coefficients are dominated by the worst singularities, thus we only need to evaluate the worst singularity integral. Consequently, the asymptotic coefficients can be simplified as
\[
a_k \sim \frac{2^\mu + 1 - \gamma}{\pi} g_1(1) \int_0^\pi t^{2\gamma} \ln^\mu(t) \cos(k t) dt
\sim \frac{2^\mu + 1 - \gamma}{\pi} g_1(1) \Re \left\{ \int_0^\pi t^{2\gamma} \ln^\mu(t) e^{ikt} dt \right\}, \quad k \geq 1.
\]

The special case \( \gamma = 0, \mu = 1 \) are used to illustrate the steepest descent method [1]. The key idea is to replace the \( \cos(n t) \) by \( \Re(e^{int}) \) and deform the contour of integration into three line segments:

(a) \( t = 0 + i \cdot 0 \) to \( t = 0 + i \cdot \infty \),

(b) \( t = 0 + i \cdot \infty \) to \( t = \pi + i \cdot \infty \),

(c) \( t = \pi + i \cdot \infty \) to \( t = \pi + i \cdot 0 \)

The second contour integration approximates zero. The third is much less than \( O(k^{-2\gamma - 1}) \). Thus the main contribution comes from the integration in the first segment. The asymptotic expression of the first is
\[
a_k \sim \frac{2^\mu + 1 - \gamma}{\pi} g_1(1) \Re \left\{ \int_{0+is}^{0+i\infty} t^{2\gamma} \ln^\mu(t) e^{ikt} dt \right\}
\]

Set \( t = is \) and
\[
I_1 = \int_{0+is}^{0+i\infty} t^{2\gamma} \ln^\mu(t) e^{ikt} ds,
\]

thus, the integration becomes
\[
I_1 = (i)^{2\gamma + 1} \int_0^\infty s^{2\gamma} \ln^\mu(is) e^{-ks} ds.
\]

Applying the identity \( \ln(is) = \ln(s) + \frac{\pi}{2} i \) and taking the dominant terms, then the integration \( I_1 \) simplifies as
\[
I_1 \sim (i)^{2\gamma + 1} \int_0^\infty s^{2\gamma} \sum_{\ell=0}^\mu \left( \frac{\mu}{\ell} \right) \ln^{\mu-\ell}(s) \left( \frac{\pi}{2} i \right)^\ell
\sim (i)^{2\gamma + 1} \int_0^\infty s^{2\gamma} \ln^\mu(s) e^{-ks} ds + \frac{\pi}{2} i \sum_{\ell=0}^{\mu} i^{\ell+2} \int_0^\infty s^{2\gamma} \ln^{\mu-1}(s) e^{-ks} ds + \cdots.
\]

By Lemma 2.2, the asymptotic coefficients of Chebyshev expansion are finally obtained.

It is easy to see that when \( \gamma \) is an integer in (2.1), the order of \( \ln(n) \) in coefficients will reduce one as is shown in Figure 4. It is also shown in [15]. In [2], Boyd pointed out the special case when \( \gamma \) is an integer and \( \mu = 1 \), the \( \ln(n) \) will disappear, which is consistent with the current result.

Next consider the Chebyshev coefficients for the function
\[
f(x) = (1 + x)^{\mu} \ln^\mu(1 + x) g_2(x), \quad x \in [-1, 1],
\]

where \( \delta > 0, \mu \in \mathbb{N}_+ \) and \( g_2(x) \) is analytic on the interval \([-1, 1] \). Here the asymptotic Chebyshev coefficients are given in the following.

**Corollary 2.4.** If a function \( f(x) \) has a singularity at \( x = -1 \) of the form (2.6), then the coefficients of Chebyshev series are asymptotically proportion to
\[
a_k \sim \frac{2^\mu + 1 - \delta}{\pi} g_2(-1) \left( -1 \right)^{k + \mu + 1} \sin(\delta \pi) \Gamma(2\delta + 1) \ln^\mu(k) \frac{k^{2\delta + 1}}{k^{2\delta + 1}} + (-1)^{k + \mu - \delta} g_2(-1)
\times \left( \frac{2}{\pi} \sin(\delta \pi) \Gamma'(2\delta + 1) + \left( \cos(\delta \pi) - \frac{\ln(2)}{\pi} \right) \Gamma'(2\delta + 1) \right) \ln^{\mu-2}(k) \frac{k^{2\delta + 1}}{k^{2\delta + 1}} + O \left( \frac{\ln^{\mu-2}(k)}{k^{2\delta + 1}} \right).
\]
Proof. By the Theorem 2.3, the asymptotic Chebyshev coefficients $\tilde{a}_k$ for the function $f(-x)$ is

\[
\tilde{a}_k \sim \frac{2^{\mu+1-\delta}}{\pi} g_2(-1) (-1)^{\mu+1} \sin(\delta \pi) \Gamma(2\delta + 1) \frac{\ln^\mu(k)}{k^{2\delta+1}} + (-1)^\mu \frac{2^{\mu-\delta}}{\pi} g_2(-1) \times \left(\frac{2}{\pi} \sin(\delta \pi) \Gamma'(2\delta + 1) + \left(\cos(\delta \pi) - \frac{\ln(2)}{\pi}\right) \Gamma(2\delta + 1) \frac{\ln^{\mu-1}(k)}{k^{2\delta+1}} + O\left(\frac{\ln^{\mu-2}(k)}{k^{2\delta+1}}\right)\right).
\]

(2.8)

Since the symmetries of the Chebyshev polynomials $T_k(-x) = (-1)^k T_k(x)$, it is not difficult to find $a_k = (-1)^k \tilde{a}_k$. $\square$

![Graphs showing Chebyshev coefficients for different values of \( \gamma \) and \( \mu \).](image)

**Fig. 4.** The Chebyshev coefficients for the function $f(x) = (1 + \frac{x}{2})(1 - x)\gamma \ln^\mu(1 - x)$ with different $\gamma$ and $\mu$.

Before this, Boyd [2] have noticed the similar findings for $\mu = 1$. To illustrate the rules of asymptotic coefficient given in Corollary 2.4, the coefficients for the function $f(x) = (1 + \frac{x}{2})(1 + x) \ln^2(1 + x)$ are shown in Figure 5. Then we consider more general weak singularity function (1.5).

**Corollary 2.5.** If the function (1.5) has weak singularities at $x = 1, x = -1$, the asymptotic Chebyshev coefficients are

\[
|a_k| = \begin{cases} O\left(\frac{\ln^\mu(k)}{k^{\min(2\gamma + 2, 2\gamma + 1)}}\right), & \gamma \notin \mathbb{Z}_+, \ \delta \notin \mathbb{Z}_+; \\
O\left(\frac{\ln^{\mu-1}(k)}{k^{\min(2\gamma, 2\gamma + 1)}}\right), & \gamma \in \mathbb{Z}_+, \ \delta \notin \mathbb{Z}_+; \\
\max\left\{O\left(\frac{\ln^{\mu-1}(k)}{k^{2\gamma+1}}\right), O\left(\frac{\ln^{\mu}(k)}{k^{2\gamma+1}}\right)\right\}, & \gamma \in \mathbb{Z}_+, \ \delta \notin \mathbb{Z}_+; \\
\max\left\{O\left(\frac{\ln^{\mu}(k)}{k^{2\gamma+2}}\right), O\left(\frac{\ln^{\mu-1}(k)}{k^{2\gamma+2}}\right)\right\}, & \gamma \notin \mathbb{Z}_+, \ \delta \in \mathbb{Z}_+.
\end{cases}
\]

(2.9)

The Chebyshev coefficients decay rate for the function $f(x) = \sin(x)(1 - x)^\gamma(1 + x)^\delta \ln^\mu(1 - x^2)$ are illustrated in Figure 6. It is shown that a very narrow peak near the right boundary layer appears in the curves of the functions. When the $\mu$ increases larger, the peak becomes more narrow and shaper. That is the reason why the Chebyshev coefficients for the function $f(x)$ with $\mu = 2$ decay slower.

**Remark 2.6.** In Corollary 2.5, the asymptotic behavior also proved by Xiang [15], different with his method, the steepest descent method is used in this article.
Fig. 5. The Chebyshev coefficients for the function $f(x) = (1 + x)(1 + x)^{\gamma} \ln^{\mu}(1 + x)$ with different $\gamma$ and $\mu$.

Fig. 6. The Chebyshev coefficients for the function $f(x) = \sin(x)(1 - x)^{\gamma}(1 + x)^{\delta} \ln(1 - x^2)$ [15]. Upper: Chebyshev coefficients; Lower: the curves of the functions.

3. Pointwise error estimates of Chebyshev truncation. In this section, we will investigate the pointwise error estimates of Chebyshev truncation for the function (1.5). Before this, we give two lemmas.

**Lemma 3.1.** Let $\Psi_C(t, n)$ be the function defined in (1.6). For $n \gg 1$, the following statement holds. 

(i) If $t \bmod 2\pi \neq 0$ and $\nu > 0$, then

$$
\Psi_C(t, n) = -D_n(t) n^{-\nu - 1} + O(n^{-\nu - 2}),
$$

where \( D_n(t) = \frac{\sin\left(\frac{2n+1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)} \) is Dirichlet’s kernel.

(ii) If \( t \) (mod, 2\( \pi \)) = 0 and \( \nu > 0 \), then

\[
\Psi_{\nu}^C(t, n) = \frac{1}{\nu} n^{-\nu} - \frac{1}{2} n^{-\nu-1} + O(n^{-\nu-2}).
\]

**Proof.** The Lemma 3.1 is proved by Wang [14] used the asymptotic presentation of the Hurwitz zeta function.

Here an alternative method is provided to give a sharp upper bound of the \( \Psi_C(t, n) \). For \( x \in [\delta_1, 2\pi - \delta_1], 0 < \delta_1 < 2\pi \) is a small positive parameter, one has

\[
\left| \sum_{k=n+1}^{n+p} \cos(kt) \right| = \left| \frac{\sin\left(\frac{2n+2p+1}{2}t\right) - \sin\left(\frac{2n+1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)} \right| \leq \frac{1}{|\sin(\frac{t}{2})|}, \quad t \in [\delta_1, 2\pi - \delta_1]
\]

and the series \( \left\{ \frac{1}{\nu n} \right\} \) is monotonic and converges to zero as \( k \to \infty \). By the Dirichlet test theorem, one can conclude that the series

\[
\Psi_{\nu}^C(t, n) = \sum_{k=n+1}^{\infty} \frac{\cos(kt)}{k^{\nu+1}}
\]

is uniformly convergent on \( t \in [\delta_1, 2\pi - \delta_1] \). Thus, we have

\[
\left| \Psi_{\nu}^C(t, n) \right| \leq \frac{2}{\sin\left(\frac{t}{2}\right)} \frac{1}{(n+1)^{\nu+1}} + 2 \frac{1}{(n+p)^{\nu+1}}, \quad p \in \mathbb{N}_+.
\]

By taking the limit \( p \to \infty \) in (3.3), it holds, finally

\[
\left| \Psi_{\nu}^C(t, n) \right| \leq \frac{1}{\sin\left(\frac{t}{2}\right)} \frac{1}{(n+1)^{\nu+1}} = O(n^{-\nu-1}), \quad n \gg 1.
\]

**Remark 3.2.** In [14], it is required \( \nu > 0 \), but in current estimate we extend the interval to \( \nu > -1 \).

**Lemma 3.3.** Let the function \( \Psi_{\nu,\mu}^C(t, n) \) is defined in (1.6), and

(i) if \( t \) (mod, 2\( \pi \)) \( \neq 0 \) and \( \nu > 0 \), then there exists the asymptotic expression

\[
\Psi_{\nu,\mu}^C(t, n) = -D_n(t) n^{-\nu-1} \ln^\mu(n) + O\left(n^{-\nu-2} \ln^\mu(n)\right).
\]

(ii) if \( t \) (mod, 2\( \pi \)) = 0 and \( \nu > 0 \), then there exists the asymptotic expression

\[
\Psi_{\nu,\mu}^C(t, n) \sim \frac{1}{\nu} n^{-\nu} \ln^\mu(n) - \frac{1}{2} n^{-\nu-1} \ln^\mu(n) + O\left(n^{-\nu-2} \ln^\mu(n)\right).
\]

**Proof.** We will prove (i),(ii) respectively in the following.

(i) For any \( t \in \mathbb{R} \), if \( t \) (mod, 2\( \pi \)) \( \neq 0 \), there must exist a positive small parameter satisfies \( \delta_1 < 2\pi \) such that the series

\[
\Psi_{\nu,\mu}^C(t, n) = \sum_{k=n+1}^{\infty} \frac{\cos(kt)}{k^{\nu+1}} \ln^\mu(k)
\]

is uniformly convergent on the interval \([\delta_1, 2\pi - \delta_1]\). Thus, taking the derivative about \( \nu \) in both sides of (3.1) \( \mu \) times, gives

\[
\Psi_{\nu,\mu}^C(t, n) = \frac{-\sin\left(\frac{2n+1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)} n^{-\nu-1} \ln^\mu(n) + O\left(n^{-\nu-2} \ln^\mu(n)\right).
\]

(ii) To prove this, we only need compute the \( \mu \)th derivative about \( \nu \) in both sides of (3.2). 

Similar to the analysis above, for (i), we can also use the Dirichlet test theorem to estimate an upper bound of the function \( \Psi_{\nu,\mu}^C(t, n) \), satisfying

\[
\left| \Psi_{\nu,\mu}^C(t, n) \right| \leq \frac{2}{\sin\left(\frac{t}{2}\right)} (n^{-\nu-1} \ln^\mu(n)) = O(n^{-\nu-1} \ln^\mu(n)).
\]

A detailed understanding of Lemma 3.3 will lead insight into the pointwise error estimates. Here we state the main results about pointwise error estimates as Theorem 3.4.
THEOREM 3.4. Let \( f \) be the function defined in (2.1) and \( f_n \) be Chebyshev truncation of degree \( n \). For \( n \to \infty \), the following establishes

(i) if \( x \in [-1, 1] \), then

\[
f - f_n \approx \frac{\alpha_1}{\pi} \sin(\gamma \pi) \Gamma(2\gamma + 1) U_{2n}(\frac{x}{2}) \frac{\ln^\mu(n)}{n^{2\gamma+1}} - \frac{1}{2} \mu \alpha_1 \alpha_2 U_{2n}(\frac{x}{2}) \frac{\ln^{\mu-1}(n)}{n^{2\gamma+1}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\gamma+1}}\right).
\]

where

\[
\alpha_1 = (-1)^\mu 2^{\mu-\gamma} g_1(1)
\]
\[
\alpha_2 = \frac{2}{\pi} \sin(\gamma \pi) \Gamma'(2\gamma + 1) + \left(\cos(\gamma \pi) - \frac{\ln(2)}{\pi}\right) \Gamma(2\gamma + 1).
\]

(ii) if \( x = 1 \), then

\[
f - f_n \approx \frac{1}{\gamma} \frac{\alpha_1}{\pi} \sin(\gamma \pi) \Gamma(2\gamma + 1) \frac{\ln^\mu(n)}{n^{2\gamma}} + \frac{\mu}{2\gamma} \alpha_1 \alpha_2 \frac{\ln^{\mu-1}(n)}{n^{2\gamma}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\gamma}}\right)
\]

with \( \alpha_1, \alpha_2 \) is given in (i) above.

Proof. For the function (2.1), the error of the Chebyshev truncation

\[
f(x) - f_n = \sum_{k=n+1}^{\infty} a_k T_k(x) = a_k \cos(kt), \quad n \gg 1, \quad x \in [-1, 1], \quad x = \cos(t),
\]

is considered. First, we prove the (i), by the Theorem 2.3, the error can be written as

\[
f(x) - f_n \approx \sum_{k=n+1}^{\infty} \left( -\frac{2}{\pi} \frac{\alpha_1}{\pi} \sin(\gamma \pi) \Gamma(2\gamma + 1) \frac{\ln^\mu(k)}{k^{2\gamma+1}} + \alpha_1 \alpha_2 \frac{\ln^{\mu-1}(k)}{k^{2\gamma+1}} \right) T_k(x) + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\gamma}}\right).
\]

Due to \( x \in [-1, 1] \), which corresponds the variable \( t \in (0, \pi] \). Combining this and the identity (3.4) given in Lemma 3.1, establishes the item (i). Next, taking into account that \( x = 1 \), by the identity \( T_k(1) = 1 \), the error gives

\[
f(x) - f_n \approx \sum_{k=n+1}^{\infty} \left( -\frac{2}{\pi} \frac{\alpha_1}{\pi} \sin(\gamma \pi) \Gamma(2\gamma + 1) \frac{\ln^\mu(k)}{k^{2\gamma+1}} + \alpha_1 \alpha_2 \frac{\ln^{\mu-1}(k)}{k^{2\gamma+1}} \right) + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\gamma}}\right), \quad n \gg 1.
\]

Using the asymptotic expression (3.5), the item (ii) is proved. \( \square \)

It turns out that the pointwise error at \( x = 1 \) is \( O(n) \) larger than the error at other point in \( x \in [-1, 1] \), a fact that plays an important role in approximation theory. The main reason is that it exists a weak singularity at \( x = 1 \).

THEOREM 3.5. Let \( f \) be the function defined in (2.6) and \( f_n \) be Chebyshev truncation of degree \( n \). For \( n \to \infty \), the following statements establishes

(i) if \( x \in (-1, 1] \), then

\[
f - f_n \approx \frac{\beta_1}{\pi} \sin(\delta \pi) \Gamma(2\delta + 1) U_{2n}\left(\frac{x + \pi}{2}\right) \frac{\ln^\mu(n)}{n^{2\delta+1}} - \frac{1}{2} \mu \beta_1 \beta_2 U_{2n}\left(\frac{x + \pi}{2}\right) \frac{\ln^{\mu-1}(n)}{n^{2\delta+1}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\delta+1}}\right).
\]

where

\[
\beta_1 = (-1)^\mu 2^{\mu-\delta} g_2(-1)
\]
\[
\beta_2 = \frac{2}{\pi} \sin(\delta \pi) \Gamma'(2\delta + 1) + \left(\cos(\delta \pi) - \frac{\ln(2)}{\pi}\right) \Gamma(2\delta + 1)
\]

(ii) if \( x = -1 \), then

\[
f - f_n \approx \frac{1}{\delta} \frac{\beta_1}{\pi} \Gamma(2\delta + 1) \sin(\delta \pi) \frac{\ln^\mu(n)}{n^{2\delta}} + \frac{\mu}{2\delta} \beta_1 \beta_2 \frac{\ln^{\mu-1}(n)}{n^{2\delta}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\delta}}\right)
\]

with \( \beta_1, \beta_2 \) is given in (i) above.
Proof. To prove this theorem, the identity
\[
\sum_{k=n+1}^{\infty} \frac{\cos(kt)}{k^\nu} (-1)^k = \sum_{k=n+1}^{\infty} \frac{\cos(k(t+\pi))}{k^\nu}.
\]
is used. It follows from the above identity and the Corollary 2.4, the leading term of pointwise error is
\[
f(x) - f_n \sim -\frac{\sin\left(\frac{2n+1}{x}(t+\pi)\right)}{2\sin\left(\frac{1+\pi}{2}\right)} - \frac{2\beta_1}{\pi} \sin(\delta \pi) \Gamma(2\delta + 1) \ln(n) + \mu \beta_1 \beta_2 \frac{\ln^{\mu-1}(n)}{n^{2\delta+1}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\delta+1}}\right).
\]
Next, the item (ii) is to be proved. For the case $x = -1$, the corresponding error of Chebyshev truncation is asymptotic to
\[
f(x) - f_n \sim -\frac{1}{\delta} \frac{\beta_1}{\pi} \Gamma(2\delta + 1) \sin(\delta \pi) \frac{\ln^{\mu}(n)}{n^{2\delta}} + \frac{\mu}{2\delta} \beta_1 \beta_2 \frac{\ln^{\mu-1}(n)}{n^{2\delta}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\delta}}\right).
\]
By the Lemma 3.3, the error can be asymptotically written as
\[
f(x) - f_n \sim -\frac{1}{\delta} \frac{\beta_1}{\pi} \Gamma(2\delta + 1) \sin(\delta \pi) \frac{\ln^{\mu}(n)}{n^{2\delta}} + \frac{\mu}{2\delta} \beta_1 \beta_2 \frac{\ln^{\mu-1}(n)}{n^{2\delta}} + O\left(\frac{\ln^{\mu-2}(n)}{n^{2\delta}}\right).
\]

The theorem implies that the pointwise error at the singularity $x = 1$ or $x = -1$ for the logarithm singularity function is $O(n)$ larger than other points that is not singular, as is shown in Figure 7. Just as the error in $L_\infty$ norm on whole interval is $O(n)$ larger than the interior interval that cut off the logarithm singularity boundary. We can also see that the error of the best approximation and the error of the Chebyshev truncation decay at a same power of $n$, but the former is only a constant multiplier less than the latter. Thus, the Theorem 1.1 is so easy to make readers misunderstand that the Chebyshev truncation is always inferior to the best approximation on the whole interval. Fortunately, Trefethen [12] have noticed that it is not always right for the algebraic singularity function. Wang [14] have provided a firm theory to explain this phenomenon. Based on their great work, we make a further step for analysing the logarithm singularity function. In fact, the Chebyshev approximation is superior to the best approximation on almost whole interval except the narrow, narrow singular boundary layer for the logarithm singularity function. It is also easy to obtain that the leading term of error at the point $x = 1$ for the function (2.1) is same as the one at point $x = -1$ for the function (2.6) if $g_1(1) = g_2(-1)$. Thus, if $g_1(x) = g_2(x)$, the errors of the both Chebyshev truncations in $L_\infty$ norm are same for corresponding functions.

Now we can explain why the pointwise error on the left boundary (at $x = -1$) is smaller than the pointwise error on the right boundary (at $x = 1$) for the function $f(x) = (1 + \frac{x}{2})(1 - x^2)\ln(1 - x^2)$ as is shown in Figure 3. At first glance, the result seems so surprising. Maybe some puzzles that the points $x = -1$ and $x = 1$ are both singularity points. Why their behavior are different. In fact, the result is not surprising, the reason is following. For the function $f(x) = (1 + \frac{x}{2})(1 - x^2)\ln(1 - x^2)$, we can write as
\[
f(x) = \left(1 + \frac{x}{2}\right)(1 + x)(1 - x)\ln(1 - x) + \left(1 + \frac{x}{2}\right)(1 - x)(1 + x)\ln(1 + x)
= g_1(x)(1 - x)\ln(1 - x) + g_2(x)(1 + x)\ln(1 + x)
\]
It is easy to obtain $g_1(1) = 3g_2(-1)$, thus the pointwise error at $x = 1$ is three times as large as the error at $x = -1$. It is caused by the function $g_1(x)$ and $g_2(x)$ rather than the singularities at $x = -1$ and $x = 1$. Just as the description pointed out before, if $g_1(x) = g_2(x)$, the maximum errors at two points $x = -1$ and $x = 1$ are same.

Indeed, before this the authors [17] have noticed the maximum pointwise error ($L_\infty$ norm) of the truncated Chebyshev series is $O(n^{-2\gamma - 1})$ in the interior of the interval $[-1, 1]$, but the maximum pointwise error is $O(n^{-2\gamma})$ in the whole interval $[-1, 1]$, for the function $f(x) = g(x)(1 - x^2)^\gamma\ln(1 - x^2)$.

4. Pointwise error estimates of Chebyshev interpolation. In this section, the pointwise error of Chebyshev interpolations are analyzed. The two most widely used interpolants are considered.
The relationship of the truncation series coefficients and the interpolant coefficients for a finite \( n \) i.e.,

\[
\ell = \sum_{k=0}^{n} b_k T_k(x), \quad b_k = \frac{2}{n+1} \sum_{j=0}^{n} f(x_j) T_k(x_j).
\]

The detail derivation of the relation is given in [6, 14]. The error of Chebyshev interpolant \( p_n^f \) is

\[
f(x) - p_n^f(x) = \sum_{k=n+1}^{\infty} a_k T_k(x) + \sum_{k=0}^{n} \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \left( a_{2\ell(n+1)-k} + a_{2\ell(n+1)+k} \right) T_k(x).
\]

Next the second term of (4.3) need to be estimated

\[
\sum_{k=0}^{n} \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \left( a_{2\ell(n+1)-k} + a_{2\ell(n+1)+k} \right) T_k(x) \sim \sum_{k=0}^{n} \left( a_{2(n+1)-k} + a_{2(n+1)+k} \right) \cos(kt).
\]

Since

\[
1 + \cos(t) + \cos(2t) + \cos(3t) + \cos(4t) + \cdots + \cos(nt) = \frac{\sin\left((2n+1)\frac{t}{2}\right) + \sin\left(\frac{t}{2}\right)}{2 \sin\left(\frac{t}{2}\right)}
\]
is uniformly bounded on $t \in [\delta_1, \pi]$ for any small positive number $\delta_1$. Thus, one can further obtain

$$\left| \sum_{k=0}^{n} (a_{2(n+1)-k} + a_{2(n+1)+k}) \cos(kt) \right| \leq \frac{2}{|\sin(\frac{\pi}{2})|} (|a_{n+1}| + 3|a_{2n+2}| + 2|a_{3n+2}|).$$

Finally, the interpolant error estimates are given in the following.

For the function (2.1), one obtains the pointwise error is

$$|f(x) - p_n^L(x)| = \begin{cases} O\left(\frac{\ln^\gamma(n)}{n^{\gamma+1}}\right), & x \in [-1, 1), \gamma \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma}}\right), & x \in [-1, 1), \gamma \text{ is an integer}, \\ O\left(\frac{\ln^\gamma(n)}{n^{\gamma+2}}\right), & x = 1, \gamma \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma+1}}\right), & x = 1, \gamma \text{ is an integer}. \end{cases}$$

Similarly, for the function (2.6), one can also obtain that

$$|f(x) - p_n^L(x)| = \begin{cases} O\left(\frac{\ln^\gamma(n)}{n^{\gamma+1}}\right), & x \in (-1, 1], \delta \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma}}\right), & x \in (-1, 1], \delta \text{ is an integer}, \\ O\left(\frac{\ln^\gamma(n)}{n^{\gamma+2}}\right), & x = -1, \delta \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma+1}}\right), & x = -1, \delta \text{ is an integer}. \end{cases}$$

### 4.2. Chebyshev points of the second kind.

Let $\{x_j\}_{j=0}^n$ be the set of the Chebyshev points of the second kind, i.e., $x_j = \cos \left(\frac{j}{n} \pi\right), j = 0, 1, \ldots, n$ and let $p_n^{II}$ be the interpolation polynomial of degree $n$ obtained by the Chebyshev points of the second. Based on the discrete orthogonality of the Chebyshev polynomials, the Chebyshev interpolant $p_n^{II}$ can be written as

$$p_n^{II} = \sum_{k=0}^{n} c_k T_k(x), \quad c_k = \sum_{j=0}^{n} \frac{2}{n} f(x_j) T_k(x_j).$$

where the double prime denotes the first term and last term is to be halved. The relations between the coefficients $a_k$ of infinite series and the coefficients $c_k$ of interpolant approximation used the Chebyshev points of the second kind is

$$c_k = a_k + \sum_{\ell=1}^{\infty} (a_{2\ell n-k} + a_{2\ell n+k}).$$

The error is

$$f(x) - p_n^{II} = \sum_{k=-n+1}^{\infty} a_k T_k(x) - \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} (a_{2\ell n-k} + a_{2\ell n+k}) T_k(x).$$

To analyze the error of the interpolant approximation based on the Chebyshev points of second kind, the second term (4.10) need to be estimated. Similar analysis to one for the Chebyshev points of first kind, the errors of interpolant for the functions (2.1), (2.6) are given in the following.

For the function (2.1), one obtains the pointwise error is

$$|f(x) - p_n^{II}(x)| = \begin{cases} O\left(\frac{\ln^\gamma(n)}{n^{\gamma+1}}\right), & x \in [-1, 1), \gamma \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma}}\right), & x \in [-1, 1), \gamma \text{ is an integer}, \\ O\left(\frac{\ln^\gamma(n)}{n^{\gamma+2}}\right), & x = 1, \gamma \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma+1}}\right), & x = 1, \gamma \text{ is an integer}. \end{cases}$$

For the function (2.6), one can also obtain that

$$|f(x) - p_n^{II}(x)| = \begin{cases} O\left(\frac{\ln^\gamma(n)}{n^{\gamma+1}}\right), & x \in (-1, 1], \delta \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma}}\right), & x \in (-1, 1], \delta \text{ is an integer}, \\ O\left(\frac{\ln^\gamma(n)}{n^{\gamma+2}}\right), & x = -1, \delta \text{ is not an integer}, \\ O\left(\frac{\ln^{\gamma-1}(n)}{n^{\gamma+1}}\right), & x = -1, \delta \text{ is an integer}. \end{cases}$$
It deserves to point out the pointwise errors are not equal of the two kind interpolant approximations for a logarithm singularity function. Although we can not give a definite conclusion which is better, the errors at the weak singularity points are considered in the following. Here the function (2.1) is considered, other logarithm singularity functions can be analysed in the similar way. For the function (2.1), the weak singular point is \( x = 1 \). The error of the interpolant \( p_n^I \) at this point is

\[
[f - p_n^I]|_{x=1} = \sum_{k=n+1}^{\infty} a_k - n \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \left( a_{2\ell(n+1)-k} + a_{2\ell(n+1)+k} \right),
\]

and the error the interpolant \( p_n^{II} \) at \( x = 1 \) is

\[
[f - p_n^{II}]|_{x=1} = \sum_{k=n+1}^{\infty} a_k - n \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} (a_{2\ell n-k} + a_{2\ell n+k}).
\]

By the Theorem 2.3, the sign of the coefficients \( a_n \) will not change when \( n \) is more than a definite positive integer. To simplify the analysis, without loss of the generality we assume the coefficients \( a_n \) are positive when \( n \) is more than a definite positive integer. The first term on the right in (4.13) is same as one in (4.14). It is not difficult to see that the second term in (4.14) is much larger than the one in (4.13). Thus the absolute value of error of interpolant \( p_n^I \) is larger than one of interpolant \( p_n^{II} \), as is shown in Figure 8.

5. Conclusions. In [15], the author gave the asymptotic Chebyshev coefficients for the functions with logarithms regularities based on the Hib type formula between Jacobi polynomials and Bessel functions. Different with the proposed methods, the steepest descent method is used to evaluate the asymptotic Chebyshev coefficients for the infinite series of a function with logarithm regularities (1.5). Then the pointwise error of Chebyshev truncations are estimated. For the function (1.5), the pointwise errors at the singularity boundaries are \( O(n) \) larger than ones at the interior points. This provides a more through explanation the error behavior of the Chebyshev truncations, furthermore, it also helps us make a full understand the real relation between the best polynomial approximation and the Chebyshev truncation. In the last section showed that the behaviors of pointwise errors of interpolant approximations based on the two kind interpolation points are analogous to the ones of the Chebyshev truncation errors.

Finally, we echo a similar sentence in [12], best polynomial approximation is optimal in uniform norm, but Chebyshev interpolants are better for the functions with logarithmic regularities.

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