VARIATIONAL PRINCIPLES FOR MULTISYMPLECTIC SECOND-ORDER CLASSICAL FIELD THEORIES

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December 4, 2014

Abstract

We state a unified geometrical version of the variational principles for second-order classical field theories. The standard Lagrangian and Hamiltonian formulations of these principles and the corresponding field equations are recovered from this unified framework.

Key words: Second-order classical field theories; Variational principles; Unified, Lagrangian and Hamiltonian formalisms

AMS s. c. (2010): 70S05, 49S05, 70H50
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(All the manifolds are real, second countable and $C^\infty$. The maps and the structures are assumed to be $C^\infty$. Usual multi-index notation introduced in [4] is used).

1 Higher-order jet bundles

(See [4] for details). Let $M$ be an orientable $m$-dimensional smooth manifold, and let $\eta \in \Omega^m(M)$ be a volume form for $M$. Let $E \xrightarrow{\pi} M$ be a bundle with $\dim E = m + n$. If $k \in \mathbb{N}$, the $k$th-order jet bundle of the projection $\pi$, $J^k\pi$, is the manifold of the $k$-jets of local sections $\phi \in \Gamma(\pi)$; that is, equivalence classes of local sections of $\pi$ by the relation of equality on every partial derivative up to order $k$. A point in $J^k\pi$ is denoted by $j^k_x\phi$, where $x \in M$ and $\phi \in \Gamma(\pi)$ is a representative of the equivalence class. We have the following natural projections: if $r \leq k$,

\[
\begin{align*}
\pi^k_r : J^k\pi & \longrightarrow J^r\pi \\
j^k_r : J^k\pi & \longrightarrow J^r\pi \\
\pi^k_r : J^k\pi & \longrightarrow E \\
j^k_r : J^k\pi & \longrightarrow \phi(x) \\
\pi^k_r : J^k\pi & \longrightarrow x
\end{align*}
\]

Observe that $\pi^k_r \circ \pi^k_s = \pi^k_{s \circ r}$, $\pi^k_0 = \pi^k$, $\pi^k_{k} = \text{Id}_{J^k\pi}$, and $\pi^k = \pi \circ \pi^k$.

If local coordinates in $E$ adapted to the bundle structure are $(x^i, u^\alpha)$, $1 \leq i \leq m$, $1 \leq \alpha \leq n$, then local coordinates in $J^k\pi$ are denoted $(x^i, u_\alpha^I)$, with $0 \leq |I| \leq k$.

If $\psi \in \Gamma(\pi)$, we denote the $k$th prolongation of $\phi$ to $J^k\pi$ by $j^k\phi \in \Gamma(\pi^k)$.

Definition 1 A section $\psi \in \Gamma(\pi^k)$ is holonomic if $j^k(\pi^k \circ \psi) = \psi$; that is, $\psi$ is the $k$th prolongation of a section $\phi = \pi^k \circ \psi \in \Gamma(\pi)$.

In the following we restrict ourselves to the case $k = 2$. According to [5], consider the $\pi_{J^1\pi}$-transverse submanifold $J^2\pi^\dagger$ of $\Lambda^2_2(J^1\pi)$ defined locally by the constraints $p^i_{\alpha\beta} = p^\alpha_{i\beta}$, which fibers over $J^1\pi$ and $M$ with projections $\pi^\dagger_{J^1\pi} : J^2\pi^\dagger \to J^1\pi$ and $\pi^\dagger : J^2\pi^\dagger \to M$, respectively. The submanifold $j_s : J^2\pi^\dagger \hookrightarrow \Lambda^2_2(J^1\pi)$ is the extended 2-symmetric multimomentum bundle.

All the canonical geometric structures in $\Lambda^m_2(J^1\pi)$ restrict to $J^2\pi^\dagger$. Denote $\Theta^\dagger_1 = j^*_s\Theta_1 \in \Omega^m(J^2\pi^\dagger)$ and $\Omega^\dagger_1 = j^*_s\Omega_1 \in \Omega^{m+1}(J^2\pi^\dagger)$ the pull-back of the Liouville forms in $\Lambda^m_2(J^1\pi)$, which we call the symmetrized Liouville forms.

Finally, let us consider the quotient bundle $J^2\pi^\dagger = J^2\pi^\dagger/\Lambda^m_1(J^1\pi)$, which is called the restricted 2-symmetric multimomentum bundle. This bundle is endowed with a natural quotient map, $\mu : J^2\pi^\dagger \to J^2\pi^\dagger$, and the natural projections $\pi^\dagger_{J^1\pi} : J^2\pi^\dagger \to J^1\pi$ and $\pi^\dagger : J^2\pi^\dagger \to M$. Observe that $\dim J^2\pi^\dagger = \dim J^2\pi^\dagger - 1$. 

2 Lagrangian-Hamiltonian unified formalism

(See [3] for details). Let \( \pi: E \rightarrow M \) be the configuration bundle of a second-order field theory, where \( M \) is an orientable \( m \)-dimensional manifold with volume form \( \eta \in \Omega^m(M) \), and \( \dim E = m + n \). Let \( \mathcal{L} \in \Omega^m(J^2\pi) \) be a second-order Lagrangian density for this field theory. The 2-symmetric jet-multimomentum bundles are

\[
\mathcal{W} = J^3\pi \times_{J^1\pi} J^2\pi^1; \quad \mathcal{W}_r = J^3\pi \times_{J^1\pi} J^2\pi^1.
\]

These bundles are endowed with the canonical projections \( \rho_1^r: \mathcal{W}_r \rightarrow J^3\pi, \rho_2: \mathcal{W} \rightarrow J^2\pi^1, \rho_3^r: \mathcal{W}_r \rightarrow J^2\pi^1, \) and \( \rho_3^M: \mathcal{W}_r \rightarrow M \). In addition, the natural quotient map \( \mu: J^2\pi^1 \rightarrow J^2\pi^1 \) induces a natural submersion \( \mu_\mathcal{W}: \mathcal{W} \rightarrow \mathcal{W}_r \).

Using the canonical structures in \( \mathcal{W} \) and \( \mathcal{W}_r \), we define a Hamiltonian section \( \hat{h} \in \Gamma(\mu_\mathcal{W}) \), which is specified by giving a local Hamiltonian function \( \hat{H} \in C^\infty(\mathcal{W}_r) \). Then we define the forms \( \Theta_r = (\rho_2 \circ \hat{h})^*\Theta \in \Omega^m(\mathcal{W}_r) \) and \( \Omega_r = -d\Theta_r \in \Omega^{m+1}(\mathcal{W}_r) \). Finally, \( \psi \in \Gamma(\rho_3^M) \) is holonomic in \( \mathcal{W}_r \) if \( \rho_1^r \circ \psi \in \Gamma(\pi^3) \) is holonomic in \( J^3\pi \).

The Lagrangian-Hamiltonian problem for sections associated with the system \( (\mathcal{W}_r, \Omega_r) \) consists in finding holonomic sections \( \psi \in \Gamma(\rho_3^M) \) satisfying

\[
\psi^* i(X)\Omega_r = 0, \quad \text{for every } X \in \mathfrak{X}(\mathcal{W}_r). \quad (1)
\]

**Proposition 1** A section \( \psi \in \Gamma(\rho_3^M) \) solution to the equation (1) takes values in a \( n(m+m(m+1)/2) \)-codimensional submanifold \( J^L: \mathcal{W}_r \rightarrow \mathcal{W}_r \) which is identified with the graph of a bundle map \( \mathcal{F}\mathcal{L}: J^3\pi \rightarrow J^2\pi^1 \) over \( J^1\pi \) defined locally by

\[
\mathcal{F}\mathcal{L}^* p^\alpha_\alpha = \frac{\partial \hat{L}}{\partial u^\alpha_\alpha} - \sum_{j=1}^m \frac{1}{n(ij)} \frac{d}{dx^j} \left( \frac{\partial \hat{L}}{\partial u^i_{\alpha+1j}} \right); \quad \mathcal{F}\mathcal{L}^* p^I_\alpha = \frac{\partial \hat{L}}{\partial u^i_\alpha}.
\]

The map \( \mathcal{F}\mathcal{L} \) is the restricted Legendre map associated with \( \mathcal{L} \), and it can be extended to a map \( \mathcal{F}\mathcal{L}: J^3\pi \rightarrow J^2\pi^1 \), which is called the extended Legendre map.

3 Variational Principle for the unified formalism

If \( \Gamma(\rho_3^M) \) is the set of sections of \( \rho_3^M \), we consider the following functional (where the convergence of the integral is assumed)

\[
\mathcal{L}\mathcal{H}: \Gamma(\rho_3^M) \rightarrow \mathbb{R}, \quad \psi \mapsto \int_M \psi^*\Theta_r.
\]

**Definition 2** (Generalized Variational Principle) The Lagrangian-Hamiltonian variational problem for the field theory \( (\mathcal{W}_r, \Omega_r) \) is the search for the critical holonomic sections of the functional \( \mathcal{L}\mathcal{H} \) with respect to the variations of \( \psi \) given by \( \psi_t = \sigma_t \circ \psi, \) where \( \{ \sigma_t \} \) is a local one-parameter group of any compact-supported \( \rho_3^M \)-vertical vector field \( Z \) in \( \mathcal{W}_r \), that is,

\[
\left. \frac{d}{dt} \right|_{t=0} \int_M \psi_t^*\Theta_r = 0.
\]

**Theorem 1** A holonomic section \( \psi \in \Gamma(\rho_3^M) \) is a solution to the Lagrangian-Hamiltonian variational problem if, and only if, it is a solution to equation (1).
(Proof) This proof follows the patterns in [1] (see also [2]). Let \( Z \in \mathcal{X}^{V(r_M^\rho)}(W_r) \) be a compact-supported vector field, and \( V \subset M \) an open set such that \( \partial V \) is a \((m - 1)\)-dimensional manifold and \( \rho_M^\rho(\text{supp}(Z)) \subset V \). Then,

\[
\frac{d}{dt}\Big|_{t=0} \int_M \psi_t^* \Theta_r = \frac{d}{dt}\Big|_{t=0} \int_V \psi_t^* \Theta_r = \frac{d}{dt}\Big|_{t=0} \int_V \psi^* \sigma_t^* \Theta_r = \int_V \psi^* \left( \lim_{t \to 0} \frac{\sigma_t^* \Theta_r - \Theta_r}{t} \right)
\]

\[
= \int_V \psi^* L(Z) \Theta_r = \int_V \psi^* (i(Z) d \Theta_r + d i(Z) \Theta_r) = \int_V \psi^* (-i(Z) \Omega_r + d i(Z) \Theta_r)
\]

\[
= - \int_V \psi^* i(Z) \Omega_r + \int_V d(\psi^* i(Z) \Theta_r) = - \int_V \psi^* i(Z) \Omega_r + \int_{\partial V} \psi^* i(Z) \Theta_r
\]

as a consequence of Stoke’s theorem and the assumptions made on the supports of the vertical vector fields. Thus, by the fundamental theorem of the variational calculus, we conclude

\[
\frac{d}{dt}\Big|_{t=0} \int_M \psi_t^* \Theta_r = 0 \iff \psi^* i(Z) \Omega_r = 0,
\]

for every compact-supported \( Z \in \mathcal{X}^{V(r_M^\rho)}(W_r) \). However, since the compact-supported vector fields generate locally the \( C^\infty(W_r) \)-module of vector fields in \( W_r \), it follows that the last equality holds for every \( \rho_M^\rho \)-vertical vector field \( Z \) in \( W_r \). Now, for every \( w \in \text{Im} \psi \), we have a canonical splitting of the tangent space of \( W_r \) at \( w \) in a \( \rho_M^\rho \)-vertical subspace and a \( \rho_M^\rho \)-horizontal subspace,

\[
T_w W_r = V_w(\rho_M^\rho) \oplus T_w(\text{Im} \psi).
\]

Thus, if \( Y \in \mathcal{X}(W_r) \), then

\[
Y_w = (Y_w - T_w(\psi \circ \rho_M^\rho)(Y_w)) + T_w(\psi \circ \rho_M^\rho)(Y_w) \equiv Y_w^V + Y_w^\psi,
\]

with \( Y_w^V \in V_w(\rho_M^\rho) \) and \( Y_w^\psi \in T_w(\text{Im} \psi) \). Therefore,

\[
\psi^* i(Y) \Omega_r = \psi^* i(Y^V) \Omega_r + \psi^* i(Y^\psi) \Omega_r = \psi^* i(Y^\psi) \Omega_r,
\]

since \( \psi^* i(Y^V) \Omega_r = 0 \), by the conclusion in the above paragraph. Now, as \( Y_w^\psi \in T_w(\text{Im} \psi) \) for every \( w \in \text{Im} \psi \), then the vector field \( Y^\psi \) is tangent to \( \text{Im} \psi \), and hence there exists a vector field \( X \in \mathcal{X}(M) \) such that \( X \) is \( \psi \)-related with \( Y^\psi \); that is, \( \psi_* X = Y^\psi \big|_{\text{Im} \psi} \). Then \( \psi^* i(Y^\psi) \Omega_r = i(X) \psi^* \Omega_r \). However, as \( \dim \text{Im} \psi = \dim M = m \) and \( \Omega_r \) is a \((m + 1)\)-form, we obtain that \( \psi^* i(Y^\psi) \Omega_r = 0 \). Hence, we conclude that \( \psi^* i(Y) \Omega_r = 0 \) for every \( Y \in \mathcal{X}(W_r) \).

Taking into account the reasoning of the first paragraph, the converse is obvious since the condition \( \psi^* i(Y) \Omega_r = 0 \), for every \( Y \in \mathcal{X}(W_r) \), holds, in particular, for every \( Z \in \mathcal{X}^{V(r_M^\rho)}(W_r) \).

4 Lagrangian variational problem

Consider the submanifold \( j_L : W_L \hookrightarrow W_r \). Since \( W_L \) is the graph of the restricted Legendre map, the map \( \rho_L^\rho \circ j_L : W_L \to J^3 \pi \) is a diffeomorphism. Then we can define the Poincaré-Cartan \( m \)-form as \( \Theta_L = (j_L \circ (\rho_L^\rho)^{-1})^* \Theta_r \in \Omega^m(J^3 \pi) \).

Given the Lagrangian field theory \((J^3 \pi, \Omega_L)\), consider the following functional

\[
L : \Gamma(\pi) \longrightarrow \mathbb{R} \quad \phi \longmapsto \int_M (j^3_\phi)^* \Theta_L.
\]
Definition 3 (Generalized Hamilton Variational Principle) The Lagrangian variational problem (or Hamilton variational problem) for the second-order Lagrangian field theory \((J^3 \pi, \Omega_L)\) is the search for the critical sections of the functional \(L\) with respect to the variations of \(\phi\) given by \(\phi_t = \sigma_t \circ \phi\), where \(\{\sigma_t\}\) is a local one-parameter group of any compact-supported \(Z \in \mathcal{X}^{V(\pi)}(E)\); that is,

\[
\frac{d}{dt} \bigg|_{t=0} \int_M (\psi_{\phi_t})^* \Theta_L = 0.
\]

Theorem 2 Let \(\psi \in \Gamma(\rho_M^* )\) be a holonomic section which is critical for the unified variational problem given by the functional \(LH\). Then, the section \(\phi = \pi^3 \circ \rho_1^* \circ \psi \in \Gamma(\pi)\) is a critical section for the variational problem given by the functional \(L\).

(Proof) The proof follows the same patterns as in Theorem[1].

5 Hamiltonian variational problem

Let \(\tilde{\mathcal{P}} = \text{Im}(\mathcal{F}L) \overset{i}{\rightarrow} J^2 \pi^1\) and \(\mathcal{P} = \text{Im}(\mathcal{F}L) \overset{j}{\rightarrow} J^2 \pi^1\) the image of the extended and restricted Legendre maps, respectively; \(\tilde{\pi}_P : \mathcal{P} \rightarrow M\) the natural projection, and \(\mathcal{F}L_o : J^3 \pi \rightarrow \mathcal{P}\) the map defined by \(\mathcal{F}L = j \circ \mathcal{F}L_o\).

A Lagrangian density \(L \in \Omega^m(J^2 \pi)\) is almost-regular if (i) \(\mathcal{P}\) is a closed submanifold of \(J^2 \pi^1\), (ii) \(\mathcal{F}L\) is a submersion onto its image, and (iii) for every \(j \in J^3 \pi\), the fibers \(\mathcal{F}L^{-1}(\mathcal{F}L(j \phi))\) are connected submanifolds of \(J^3 \pi\).

The Hamiltonian section \(\hat{h} \in \Gamma(\mu\mathcal{W})\) induces a Hamiltonian section \(h \in \Gamma(\mu)\) defined by \(\rho_2 \circ \hat{h} = h \circ \rho_2^\mu\). Then, we define the Hamilton-Cartan m-form in \(\mathcal{P}\) as \(\Theta_h = (h \circ j)^* \Theta_1^* \in \Omega^m(\mathcal{P})\). Observe that \(\mathcal{F}L_o^* \Theta_h = \Theta_L\).

In what follows, we consider that the Lagrangian density \(L \in \Omega^m(J^2 \pi)\) is, at least, almost-regular. Given the Hamiltonian field theory \((\mathcal{P}, \Omega_h)\), let \(\Gamma(\tilde{\pi}_P)\) be the set of sections of \(\tilde{\pi}_P\). Consider the following functional

\[
\mathbf{H} : \Gamma(\tilde{\pi}_P) \rightarrow \mathbb{R},
\]

\[
\psi_h \mapsto \int_M \psi_h^* \Theta_P.
\]

Definition 4 (Generalized Hamilton-Jacobi Variational Principle) The Hamiltonian variational problem (or Hamilton-Jacobi variational problem) for the second-order Hamiltonian field theory \((\mathcal{P}, \Omega_h)\) is the search for the critical sections of the functional \(H\) with respect to the variations of \(\psi_h\) given by \(\psi_{\phi_t} = \sigma_t \circ \psi_h\), where \(\{\sigma_t\}\) is a local one-parameter group of any compact-supported \(Z \in \mathcal{X}^{V(\pi)}(\mathcal{P})\),

\[
\frac{d}{dt} \bigg|_{t=0} \int_M (\psi_{\phi_t})^* \Theta_h = 0.
\]

Theorem 3 Let \(\psi \in \Gamma(\rho_M^* )\) be a solution to the variational problem given by the functional \(LH\). Then, the section \(\psi_h = \mathcal{F}L_o \circ \rho_1^* \circ \psi \in \Gamma(\tilde{\pi}_P)\) is a solution to the variational problem given by the functional \(H\).

(Proof) The proof follows the same patterns as in Theorem[1].
Acknowledgments

We acknowledge the financial support of the MICINN, projects MTM2011-22585 and MTM2011-15725-E. P.D. Prieto-Martínez thanks the UPC for a Ph.D grant.

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