On the generalised Brézis–Nirenberg problem

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Abstract. For $p \in (1,N)$ and a domain $\Omega$ in $\mathbb{R}^N$, we study the following quasi-linear problem involving the critical growth:

$$-\Delta_p u - \mu g|u|^{p-2}u = |u|^{p^*-2}u \quad \text{in} \quad D_p(\Omega),$$

where $\Delta_p$ is the $p$-Laplace operator defined as $\Delta_p(u) = \text{div}(\nabla |\nabla u|^{p-2}\nabla u)$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent and $D_p(\Omega)$ is the Beppo-Levi space defined as the completion of $C_\infty^\infty(\Omega)$ with respect to the norm $\|u\|_{D_p} := \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$. In this article, we provide various sufficient conditions on $g$ and $\Omega$ so that the above problem admits a positive solution for certain range of $\mu$. As a consequence, for $N \geq p^2$, if $g$ is such that $g^+ \neq 0$ and the map $u \mapsto \int_\Omega |g||u|^p dx$ is compact on $D_p(\Omega)$, we show that the problem under consideration has a positive solution for certain range of $\mu$. Further, for $\Omega = \mathbb{R}^N$, we give a necessary condition for the existence of positive solution.

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1. Introduction

For $p \in (1,N)$ and a domain $\Omega$ in $\mathbb{R}^N$, the Beppo-Levi space $D_p(\Omega)$ is defined as the completion of $C_\infty^\infty(\Omega)$ with respect to the norm $\|u\|_{D_p} := \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$.

In this article, we study the following quasi-linear partial differential equation involving the critical growth:

$$-\Delta_p u - \mu g|u|^{p-2}u = |u|^{p^*-2}u \quad \text{in} \quad D_p(\Omega),$$

(1.1)
where $\Delta_p$ is the $p$-Laplace operator defined as $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. Since the seminal work of Brézis–Nirenberg [11], many different classes of elliptic boundary value problems involving the critical exponent have been explored. For example, see [3,13,17,18,42,47] for Laplacian ($p = 2$) and [12,16,26,27,29,41,43] for $p$-Laplacian ($p \in (1,N)$). It is well known that, if $g \equiv 0$ in (1.1) i.e. the problem

$$-\Delta_p u = |u|^{p^* - 2} u \quad \text{in} \quad \mathcal{D}_p(\Omega),$$

(1.2)
does not admit any positive solution on a bounded, star-shaped domain [11, Remark 1.2]. However, if $g \equiv 1$ on a bounded domain $\Omega$, then (1.1) admits a positive solution for certain range (depending on $p$ and $N$) of $\mu$, see [11, for Laplacian] and [27, for $p$-Laplacian]. This suggests that certain perturbations of (1.2) of the form (1.1) may admit a positive solution. Also, in [52], authors multiplied a weight function $g$ to the right-hand side of (1.2) and studied the existence of positive solutions. In this article, we are interested in identifying a general class of $g$ so that (1.1) admits a positive solution. Many results have been appeared in this direction. For example, for bounded domain $\Omega$, $g$ is positive constant [11,27], $g$ is bounded non-negative function [23,29], sign-changing $g \in C(\Omega)$ [30]. For $\Omega = \mathbb{R}^N$, Charles Swanson and Lao Sen [48] considered nontrivial, non-negative $g \in L^{\frac{N}{p}}(\mathbb{R}^N)$. For $p = 2$, Chabrowski [15] considered sign-changing $g$ which is positive on a positive measure set and $g \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ with \( \lim_{|x| \to \infty} \int_{B_l(x)} |g|^{\frac{1}{r-2}} dx = 0 \) for some $r \in (2,2^*)$, $l > 0$, and for $p \in (1,N)$, Huang [31] has taken $g \in L^{\frac{N}{p^*}}(\mathbb{R}^N)$ for some $r \in (1,p^*)$. In [22], Drábek-Huang considered $g$ such that $g^+ \in L^\infty(\mathbb{R}^N) \cap L^{\frac{N}{p}}(\mathbb{R}^N)$ and $g^- \in L^\infty(\mathbb{R}^N)$. In all the above mentioned results, the assumptions on $g$ ensure that the map

$$G_p(u) := \int_{\Omega} |g| |u|^p dx \quad \text{on} \quad \mathcal{D}_p(\Omega)$$

is compact (i.e. $G_p(u_n) \to G_p(u)$ whenever $u_n \rightharpoonup u$ in $\mathcal{D}_p(\Omega)$). In this case, the non-compactness issues in dealing with (1.1) arise only due to the presence of critical exponent or the unboundedness of the domain. Indeed, to tackle these kinds of non-compactness, one can use the celebrated concentration compactness principles of P. L. Lions [38,39]. Moreover, in [14], authors considered multiple perturbations of (1.2), where the non-compactness of one perturbation is compensated by another perturbation. The main objective of this article is to consider a single non-compact perturbation of (1.2) by a potential $g$ and enlarge the class of $g$ that ensures the existence of a positive solution of (1.1).

We use a concentration compactness principle that depends on $g$ (and in the later part, depends on a closed subgroup of $O(N)$) that can simultaneously address all the aforementioned non-compactness (critical exponent, unboundedness of the domain, non-compact perturbations).

Notice that, if $u \in \mathcal{D}_p(\Omega)$ is a solution to (1.1), then

$$\int_{\Omega} |\nabla u|^p dx = \mu \int_{\Omega} g |u|^p dx + \int_{\Omega} |u|^{p^*} dx \geq \mu \int_{\Omega} g |u|^p dx.$$
We priory assume that \( g \) must satisfy the following Hardy type inequality:

\[
\int_{\Omega} |g||u|^p dx \leq C \int_{\Omega} |
abla u|^p dx, \quad \forall u \in \mathcal{D}_p(\Omega),
\]

for some \( C > 0 \).

**Definition 1.1.** A function \( g \in L^1_{loc}(\Omega) \) satisfying the above inequality is called a Hardy potential and the space of all Hardy potentials is denoted by \( \mathcal{H}_p(\Omega) \).

One can define a norm on \( \mathcal{H}_p(\Omega) \) which makes it a Banach function space (see Sect. 2.5). For \( g \in \mathcal{H}_p(\Omega) \), we define

\[
\mu_1(g, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} |g||u|^p dx = 1, u \in \mathcal{D}_p(\Omega) \right\}.
\]

If the underlying domain is unambiguous, then we simply write \( \mu_1(g) \) instead of \( \mu_1(g, \Omega) \).

Observe that \( \mu_1(g) > 0 \), and for each \( \mu \in (0, \mu_1(g)) \), \( \|u\|_{\mathcal{D}_p, \mu} := \left( \int_{\Omega} [|\nabla u|^p - \mu g|u|^p] dx \right)^{\frac{1}{p}} \) is a quasi-norm on \( \mathcal{D}_p(\Omega) \) and it is equivalent to \( \|u\|_{\mathcal{D}_p} \). We also define

\[
\mu_1(g, x) = \lim_{r \to 0} \left\{ \inf \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} |g||u|^p dx = 1 : u \in \mathcal{D}_p(\Omega \cap B_r(x)) \right\} \right\} ; \quad x \in \overline{\Omega},
\]

and

\[
\Sigma_g = \{ x \in \overline{\Omega} : \mu_1(g, x) < \infty \}.
\]

Now, for \( g \in \mathcal{H}_p(\Omega) \) and \( \mu \in (0, \mu_1(g)) \), we consider the functional

\[
J_{g, \mu}(u) = \int_{\Omega} [|\nabla u|^p - \mu g|u|^p] dx
\]

and define

\[
\mathcal{E}_{g, \mu}(\Omega) = \inf \{ J_{g, \mu}(u) : u \in \mathcal{S}_p(\Omega) \},
\]

where \( \mathcal{S}_p(\Omega) := \{ u \in \mathcal{D}_p(\Omega) : \|u\|_{p^*} = 1 \} \). If \( \mathcal{E}_{g, \mu}(\Omega) \) is attained at \( v \in \mathcal{S}_p(\Omega) \), then the standard variational arguments ensure that \( [J_{g, \mu}(v)]^{\frac{1}{p^*-p}} v \) is a non-trivial solution of (1.1). In order to investigate whether \( \mathcal{E}_{g, \mu}(\Omega) \) is attained or not, consider a minimizing sequence of \( \mathcal{E}_{g, \mu}(\Omega) \), say \( (w_n) \) in \( \mathcal{D}_p(\Omega) \). It is not difficult to see that \( (w_n) \) converges weakly to some \( w \in \mathcal{D}_p(\Omega) \). By zero extension, \( w_n \in \mathcal{D}_p(\Omega) \) can be considered as a \( \mathcal{D}_p(\mathbb{R}^N) \) function whenever convenient. Using this convention, we define the measures \( \nu_n \) as

\[
\nu_n(E) = \int_E |w_n - w|^{p^*} dx, \quad \text{for every Borel set } E \in \mathbb{R}^N,
\]

and the following quantity

\[
\nu_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \int_{B_R^c} |w_n - w|^{p^*} dx.
\]

Since \( (w_n) \) is bounded in \( \mathcal{D}_p(\Omega) \), it follows that \( (\nu_n) \) is bounded in the space of all regular, finite, Borel signed-measures \( \mathcal{M}(\mathbb{R}^N) \) with respect to the norm \( \|\nu_n\| := \nu_n(\mathbb{R}^N) \) (total variation of measures). By the Reisz representation
theorem [1, Theorem 14.14, Chapter 14], \( M(\mathbb{R}^N) \) is the dual of \( C_0(\mathbb{R}^N) := C_c(\mathbb{R}^N) \) in \( L^\infty(\mathbb{R}^N) \). Hence, by the Banach-Aloglu theorem it follows that \( \nu_n \rightharpoonup^* \nu \) in \( M(\mathbb{R}^N) \). It is important to mention that the measure \( \nu \) together with the quantity \( \nu_\infty \) captures the possible failure of strong convergence of \( (w_n) \) in \( L^p(\Omega) \). Indeed, if \( \nu = 0 = \nu_\infty \), then \( w_n \rightharpoonup w \) in \( L^p(\Omega) \) (see Corollary 3.5), and consequently, \( E_{g,\mu}(\Omega) \) is attained at \( w \). If \( \nu \neq 0 \) (or \( \nu_\infty \neq 0 \)), then we say that \( (w_n) \) concentrate on \( \Omega \) (or at \( \infty \)). To study the concentration behaviour of \( \nu \), we consider the following concentration function of \( g \):

\[
C_{g,\mu}(x) := \lim_{r \to 0} E_{g,\mu}(\Omega \cap B_r(x)), \quad C_{g,\mu}(\infty) := \lim_{R \to \infty} E_{g,\mu}(\Omega \cap B_R^c).
\]

We denote \( C_{g,\mu}^*(\Omega) = \inf_{\Omega} C_{g,\mu}(x) \). Notice that

\[
E_{g,\mu}(\Omega) \leq C_{g,\mu}^*(\Omega) \quad \text{and} \quad E_{g,\mu}(\Omega) \leq C_{g,\mu}(\infty). \tag{1.5}
\]

Later we see that the nature of the above inequalities (strict or not) helps us to determine whether \( \nu \) is concentrated or not. For brevity, we make the following definition.

**Definition 1.2.** Let \( g \in H_p(\Omega) \) and \( \mu \in (0, \mu_1(g)) \). We say

(i) \( g \) is sub-critical in \( \Omega \) for level \( \mu \) if \( E_{g,\mu}(\Omega) < C_{g,\mu}^*(\Omega) \), and sub-critical at infinity for level \( \mu \) if \( E_{g,\mu}(\Omega) < C_{g,\mu}(\infty) \),

(ii) \( g \) is critical in \( \Omega \) for level \( \mu \) if \( E_{g,\mu}(\Omega) = C_{g,\mu}^*(\Omega) \), and critical at infinity for level \( \mu \) if \( E_{g,\mu}(\Omega) = C_{g,\mu}(\infty) \).

Let

\[
\mathcal{F}_p(\Omega) := \overline{C_c^\infty(\Omega)} \quad \text{in} \quad H_p(\Omega).
\]

In [6], it is proved that \( \mathcal{F}_p(\Omega) \) is the optimal space for \( g \) such that \( G_p \) is compact in \( D_p(\Omega) \). In this article, we assume that \( g \) satisfies:

**H1**) \( g \in H_p(\Omega), g^- \in \mathcal{F}_p(\Omega), \) and \( |\Sigma g| = 0 \).

Now we state our first result.

**Theorem 1.3.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and \( g \) satisfies 1. If for some \( \lambda \in (0, \mu_1(g)) \), \( g \) is sub-critical in \( \Omega \) and at infinity for level \( \lambda \), then (1.1) admits a positive solution for \( \mu = \lambda \).

Our proof for the above theorem is based on the fact that, if \( g \) is subcritical in \( \Omega \) and at infinity for some level \( \mu \), then \( \nu = 0 = \nu_\infty \) for any minimising sequence of \( E_{g,\mu}(\Omega) \). In other words, the subcriticality (in \( \Omega \) and at \( \infty \)) of \( g \) ensures that the minimising sequence does not concentrate in \( \Omega \) as well as at \( \infty \).

Now, it is natural to ask whether the compact perturbations are subcritical or not. The answer is no, for example, \( g \equiv 0 \) is a compact perturbation (as \( G_p \equiv 0 \) on \( D_p(\Omega) \)) but not a sub-critical potential. On a bounded domain \( \Omega \), \( g \equiv 1 \) is a compact perturbation and it is sub-critical for all \( \mu \in (0, \mu_1(g)) \) if \( N \geq p^2 \), see [11, Theorem 1.2] (for \( p = 2 \)) and [19, Theorem 4.2] (for general \( p \)). Next we show that every compact perturbation (with non-trivial positive part) behaves like \( g \equiv 1 \).
Theorem 1.4. Let $N \geq p^2$ and $g \in \mathcal{F}_p(\Omega)$ be such that $g^+ \neq 0$. Then $g$ is sub-critical in $\Omega$ and at the infinity for each level $\mu \in (0, \mu_1(g))$.

Remark 1.5. Indeed, the condition $N \geq p^2$ is crucial, for instance, if $p = 2$ and $N = 3$ then $g \equiv 1$ is not sub-critical for level $\mu$ near 0. [11, Corollary 1.1].

Next we ask, what happens if $g$ fails to be sub-critical either in $\Omega$ or at infinity? Recall that, for the critical potential $g \equiv 0$, (1.1) does not admit any positive solution when $\Omega$ is a bounded star-shaped domain. However, it admits a positive solution when $\Omega$ is an annular domain [33, Section 4] or entire $\mathbb{R}^N$ [39]. Similarly, if $\Omega$ contains the origin, then (1.1) does not admit a positive solution for any $\mu \in \mathbb{R}$ [28, Theorem 2.1], when $\Omega$ is bounded star-shaped. On the other hand, for the same $g$, (1.1) does admit a positive radial solution on entire $\mathbb{R}^N$ for every $\mu \in (0, \mu_1(g))$ [53, Theorem 1.41]. This indicates that, when the subcriticality fails, the symmetry of the domain plays a vital role in the existence of solutions for (1.1). Here we study the non-subcritical cases under some additional symmetry assumptions on $\Omega$ and $g$.

Consider the action of a closed subgroup $H$ of $\mathcal{O}(N)$ (the group of all orthogonal matrices on $\mathbb{R}^N$) on $\mathbb{R}^N$ given by $\text{x} \rightarrow h \cdot \text{x}$, for $x \in \mathbb{R}^n$, $h \in H$, where $\cdot$ represents the matrix multiplication. We will be writing $h(x)$ instead of $h \cdot x$. For $x \in \mathbb{R}^N$, the orbit of $x$ is denoted by $Hx$ and for $E \subset \mathbb{R}^N$, the orbit of $E$ is denoted by $H(E)$. Thus, $Hx = \{h(x) : h \in H\}$ and $H(E) = \{h(x) : x \in E, h \in H\}$.

Definition 1.6. Let $\Omega$ be a domain in $\mathbb{R}^N$ and $f : \Omega \rightarrow \mathbb{R}$. If $H(\Omega) = \Omega$, then we say $\Omega$ is $H$-invariant. If $\Omega$ is $H$-invariant and $f(h(x)) = f(x), \forall h \in H$ (i.e., $f$ is constant on each $H$ orbit), then we say $f$ is $H$-invariant.

For a $H$-invariant domain $\Omega$, the $H$-action on $\Omega$ naturally induces an action of $H$ on $D_p(\Omega)$ given by

$$\pi(h)(u) = u_h, \text{ where } u_h(x) = u(h^{-1}(x)).$$

Thus, $u \in D_p(\Omega)$ is $H$-invariant if, and only if, $u_h = u$. The set of all $H$-invariant functions in $D_p(\Omega)$ and $S_p(\Omega)$ are denoted by $D_p(\Omega)^H$ and $S_p(\Omega)^H$ respectively, i.e.,

$$D_p(\Omega)^H = \{u \in D_p(\Omega) : u = u_h, \forall h \in H\}$$

$$S_p(\Omega)^H = \{u \in S_p(\Omega) : u = u_h, \forall h \in H\}.$$

Clearly, for $H = \{Id_{\mathbb{R}^n}\}$, $D_p(\Omega)^H = D_p(\Omega)$ and $S_p(\Omega)^H = S_p(\Omega)$, and if $H = \mathcal{O}(N)$ and $\Omega$ is a radial domain, then $D_p(\Omega)^H$ and $S_p(\Omega)^H$ corresponds to the space of all radial functions in $D_p(\Omega)$ and $S_p(\Omega)$ respectively. Next, we consider a $H$-depended minimization problem analogous to (1.3)

$$\mathcal{E}^H_{g,\mu}(\Omega) = \inf \{ J_{g,\mu}(u) : u \in S_p(\Omega)^H \}.$$

It is clear that $\mathcal{E}^H_{g,\mu}(\Omega) \leq \mathcal{E}^H_{g,\mu}(\Omega)$. It may happen that $\mathcal{E}^H_{g,\mu}(\Omega)$ is attained in $D_p(\Omega)^H$ without $\mathcal{E}^H_{g,\mu}(\Omega)$ being attained in $D_p(\Omega)$. Now one may ask, does a minimizer of $\mathcal{E}^H_{g,\mu}(\Omega)$ actually solve (1.1)? i.e., whether a critical point of
$J_{g,\mu}$ over $\mathbb{S}_p(\Omega)^H$ can be a critical point of $J_{g,\mu}$ over $\mathbb{S}_p(\Omega)$? The principle of symmetric criticality theory answers this question affirmatively. We used the following version of the principle of symmetric criticality by Kobayashi and Ótani. [35, Theorem 2.2].

**Theorem.** (Principle of symmetric criticality) Let $V$ be a reflexive and strictly convex Banach space and $H$ be a group that acts on $V$ isometrically i.e.,

$$\|h(v)\|_V = \|v\|_V, \forall h \in H, v \in V.$$  

If $F : V \rightarrow \mathbb{R}$ is a $H$-invariant, $C^1$, then

$$(F|_{V^H})'(v) = 0 \text{ implies } F'(v) = 0 \text{ and } v \in V^H,$$

where $V^H$ is the set of all $H$-invariant elements of $V$.

For the remaining part of this section, we make the following assumptions on $\Omega$ and $g$:

**(H2)** The domain $\Omega$ and the Hardy potential $g$ are $H$-invariant, where $H$ is a closed subgroup of $\mathcal{O}(N)$.

For any closed subgroup $H$ of $\mathcal{O}(N)$, and $\Omega, g$ as in 1, it is easy to verify that

$$\int_{\Omega} [\|\nabla u_h\| - \mu g |u_h|]dx = \int_{\Omega} [\|\nabla u\| - \mu g |u|]dx,$$

i.e., $J_{g,\mu}$ is $H$-invariant, and also $J_{g,\mu}$ is $C^1$ as $g \in \mathcal{H}_p(\Omega)$. Thus, the principle of symmetric criticality holds for $J_{g,\mu}$, and hence a minimizer of $\mathcal{E}_{g,\mu}^H(\Omega)$ (if exists) solves (1.1) (up to a constant multiple).

Now, for $H, \Omega$, and $g$ as given in 1 and 1, we investigate whether $\mathcal{E}_{g,\mu}^H(\Omega)$ is achieved or not. For this purpose, analogous to $C_{g,\mu}$, we introduce a $H$-dependent concentration function of $g$ as follows:

$$C_{g,\mu}^H(x) := \lim_{r \to 0} \mathcal{E}_{g,\mu}^H(\Omega \cap H(B_r(x))) \quad \text{and} \quad C_{g,\mu}^{H,\infty} := \lim_{R \to \infty} \mathcal{E}_{g,\mu}^H(\Omega \cap B_R^c)$$

and $C_{g,\mu}^{H,\infty}(\Omega) := \inf_{\Omega} C_{g,\mu}^{H,\infty}(x)$. Clearly, $C_{g,\mu}^H$ is constant on each $H$-orbits, and if $H = \{1_{2N}\}$, then $C_{g,\mu}^H = C_{g,\mu}$. Now, in a similar fashion as in Definition 1.2, we make the following definition:

**Definition 1.7.** Let $g \in \mathcal{H}_p(\Omega)$ be $H$-invariant. Then for a $\mu$ in $(0, \mu_1(g))$, we say

(i) $g$ is $H$-subcritical in $\Omega$ for level $\mu$ if $\mathcal{E}_{g,\mu}^H(\Omega) < C_{g,\mu}^{H,\infty}(\Omega)$, and $H$-subcritical at infinity for level $\mu$ if $\mathcal{E}_{g,\mu}^H(\Omega) < C_{g,\mu}^H(\infty)$.

(ii) $g$ is $H$-critical in $\Omega$ for level $\mu$ if $\mathcal{E}_{g,\mu}^H(\Omega) = C_{g,\mu}^{H,\infty}(\Omega)$, and $H$-critical at infinity for level $\mu$ if $C_{g,\mu}^{H,\infty}(\infty) = \mathcal{E}_{g,\mu}^H(\Omega)$.

Now analogous to Theorem 1.3, we have the following result:

**Theorem 1.8.** Let $H$, $\Omega$ and $g$ as given in 1 and 1. For $\lambda \in (0, \mu_1(g))$, assume that $g$ is $H$-subcritical in $\Omega$ and at infinity for level $\lambda$. Then (1.1) admits a positive solution for $\mu = \lambda$.

**Remark 1.9.** Indeed, there are Hardy potentials that are not subcritical but $H$-subcritical for some $H$, see Example 5.2-(iii).
Next, to understand the situation when \( g \) fails to be \( H \)-subcritical either in \( \Omega \) or at infinity for some level \( \mu \), we consider a minimizing sequence \((w_n)\) of \( \mathcal{E}_{g,\mu}^H(\Omega) \). In this case, \((w_n)\) can concentrate only on a finite \( H \)-orbit in \( \overline{\Omega} \) and/or at infinity (see Corollary 3.5). In particular, if all the \( H \)-orbits are infinite under the \( H \)-action, then \((w_n)\) can not concentrate anywhere in \( \overline{\Omega} \). These ideas lead to our next result.

**Theorem 1.10.** Let \( H \), \( \Omega \) and \( g \) be as given in 1 and 1 such that the orbits \( Hx \) are infinite for all \( x \in \overline{\Omega} \). Assume that for some \( \lambda \in (0, \mu_1(g)) \), \( g \) is \( H \)-subcritical at infinity for level \( \lambda \). Then (1.1) admits a positive solution for \( \mu = \lambda \).

**Remark 1.11.** For \( 2 \leq k \leq N \), let \( H = \mathcal{O}(k) \times \{ I_{N-k} \} \), where any \( h \in H \) is considered as \( \begin{pmatrix} h_k & 0_{N-k} \\ 0_{N-k} & I_{N-k} \end{pmatrix} \); \( h_k \in \mathcal{O}(k) \). Let \( \Omega = \Omega_k \times \Omega_{N-k} \), where \( \Omega_{N-k} \) is a general domain in \( \mathbb{R}^{N-k} \) and
\[
\Omega_k := \{ x \in \mathbb{R}^k : a \leq |x| \leq b \}^c, \ a, b \geq 0, \tag{1.6}
\]
\( A^c \) denotes the interior of a set \( A \). Depending on the values of \( a, b, \) the above domain \( \Omega \) in (1.6) can be a ball, an annulus, exterior of a ball, unbounded cylinders, or entire \( \mathbb{R}^N \). In particular, if \( k = N \), then \( \Omega \) is all radial domains in \( \mathbb{R}^N \). Observe that \( \Omega \) is \( H \)-invariant, and only the origin has a finite \( H \)-orbit. If \( a > 0 \) then \( 0 \notin \overline{\Omega} \), and hence Theorem 1.10 helps us to identify \( H \)-invariant critical Hardy potentials on \( \Omega \) for which (1.1) admits a positive solution. For instance, for \( k = N \) and \( 0 < a < b < \infty \) (i.e., annular domains in \( \mathbb{R}^N \)), \( g \equiv 0 \) is \( O(N) \)-subcritical at infinity (since the annular domains are bounded). Hence, for \( g \equiv 0 \), (1.1) admits a positive solution. See Example 5.2 for more of such potentials.

Next we address the case when \( g \) is not necessarily \( H \)-subcritical at infinity. In this case, we assume that \( \Omega = \mathbb{R}^N \), and \( g \) satisfies the following homogeneity condition:

\( \text{(H3)} \) \( g(rz) \geq \frac{g(z)}{r^p} ; z \in \mathbb{R}^N, r > 0. \)

**Theorem 1.12.** Let \( H \) be a closed subgroup of \( \mathcal{O}(N) \) and \( \Omega = \mathbb{R}^N \) and \( g \) be as in 1 and 1.

(i) If \( g \) is \( H \)-subcritical in \( \mathbb{R}^N \) for some level \( \lambda \in (0, \mu_1(g)) \) and satisfies 1 for small values of \( r > 0 \), then (1.1) admits a positive solution for \( \mu = \lambda \).

(ii) Let the orbits \( Hx \) are infinite for all \( x(\neq 0) \in \mathbb{R}^N \). If \( g \) satisfies 1 for all \( r > 0 \), then (1.1) admits a positive solution for all \( \mu \in (0, \mu_1(g)) \).

**Remark 1.13.** Consider \( g(x) = 0 \) or \( \frac{1}{|x|^p} \) on \( \mathbb{R}^N \) and \( H = O(N) \). Then \( g \) satisfies 1, and the above theorem assures that (1.1) admits a positive solution on entire \( \mathbb{R}^N \) for all \( \mu \in (0, \mu_1(g)) \).

Next we provide a necessary condition for the existence of a positive solution for (1.1). In this direction, a Pohozaev type identity is available for bounded star-shaped domain [23, Theorem 1.1]. Here we establish such an identity for the entire \( \mathbb{R}^N \).
Theorem 1.14. Let \( g \in C^\alpha_{loc}(\mathbb{R}^N) \), for some \( \alpha \in (0, 1) \) and \( u \in \mathcal{D}_p(\mathbb{R}^N) \) such that \( g(x)|u|^p \in L^1(\mathbb{R}^N) \) and also \( x.\nabla g(x)|u|^p \in L^1(\mathbb{R}^N) \), where \( \nabla g \) denotes the gradient of \( g \) in weak sense. Further, if \( u \) solves 
\[-\Delta_p u - g|u|^{p-2}u = |u|^{p^* - 2}u \quad \text{in} \ \mathbb{R}^N,\]
in weak sense, then the following holds:
\[
\int_{\mathbb{R}^N} [x.\nabla g(x) + pg(x)]|u|^p dx = 0.
\]
A similar identity has been derived when the problem does not involve critical exponent \([49, \text{Proposition 4.5}]\) (for \( p = 2 \)) and \([4, \text{Theorem 6.1.3}]\) (for general \( p \)).

The rest of the paper is organised as follows. In Sect. 2, we briefly discuss about the spaces \( H_p(\Omega) \) and \( F_p(\Omega) \), and recall some important results such as principle of symmetric criticality, a strong maximum principle. In Sect. 3, we derive a \( H \)-depended concentration-compactness lemma. Section 4 is devoted to the discussions on subcritical potentials and it includes our proofs for Theorem 1.3 and Theorem 1.4. In Sect. 5, Theorem 1.10 and Theorem 1.12 are proved. We prove Theorem 1.14 in Sect. 6.

2. Preliminaries

In this section, we briefly discuss many essential results that are required for the development of this article.

2.1. Principle of symmetric criticality

Let \((V, \|\cdot\|_V)\) be a real Banach space and \( BL(V, V) \) be the space of all bounded linear operators on \( V \). Let \( H \) be a group. A representation \( \pi : H \to BL(V, V) \) is a map that satisfies the following:

(i) \( \pi(e)v = v \); \( \forall v \in V \), where \( e \) is the identity element in \( H \).

(ii) \( \pi(h_1h_2)v = \pi(h_1)\pi(h_2)v \); \( \forall h_1, h_2 \in H, v \in V \).

The action of \( H \) on \( V \) is said to be isometric if \( \|\pi(h)v\|_V = \|v\|_V \); \( \forall h \in H, v \in V \). The subspace \( V^H := \{v \in V : \pi(h)v = v, \forall h \in H\} \) is called the \( H \)-invariant subspace of \( V \). A functional \( J : V \to \mathbb{R} \) is called \( H \)-invariant if \( J(\pi(h)v) = J(v), \forall h \in H, \forall v \in V \). A differentiable, \( H \)-invariant functional \( J : V \to \mathbb{R} \) is said to satisfy the principle of symmetric criticality if

\( (P) \quad (J|_{V^H})'(v) = 0 \) implies \( J'(v) = 0 \) and \( v \in V^H \).

In 1979, Palais [44] introduced the notion of the principle of symmetric criticality. Since then many versions of this principle were proved e.g., [35, Theorem 2.2], [35, Theorem 2.7], [36, Theorem 2.1]. In this article, we use the following version due to Kobayashi and Ôtani [35, Theorem 2.2].

Theorem 2.1. [Principle of symmetric criticality] [35, Theorem 2.2] Let \( V \) be reflexive and strictly convex and the action of the \( H \) on \( V \) is isometric. If \( J \) is a \( H \)-invariant, \( C^1 \) functional, then \( (P) \) holds.
Remark 2.2. Let \( \Omega \) be a domain in \( \mathbb{R}^N \). Consider \( V = D_p(\Omega) \) and \( H \) be a closed subgroup of \( O(N) \) which acts on \( V \) as \( \pi(h)v = v_h, \) where \( v_h(x) := v(h^{-1}(x)), \forall x \in \Omega. \) One can verify the following:

(a) \( D_p(\Omega) \) is reflexive and strictly convex,
(b) the action of \( H \) on \( V \) is isometric.

Furthermore, for \( g \in H_p(\Omega), \) the functional \( J_{g,\mu} \) is \( C^1. \) Thus, by Theorem 2.1, (P) holds for \( J_{g,\mu}. \)

2.2. The space of signed measures

We denote the collection of all Borel sets in \( \mathbb{R}^N \) by \( \mathcal{B}(\mathbb{R}^N). \) Let \( \mathcal{M}(\mathbb{R}^N) \) be the space of all regular, finite, Borel signed-measures on \( \mathbb{R}^N. \) It is well known that \( \mathcal{M}(\mathbb{R}^N) \) is a Banach space with respect to the norm \( \| \nu \| = |\nu|(\mathbb{R}^N) \) (total variation of \( \nu \)). By the Riesz Representation theorem [1, Theorem 14.14, Chapter 14], \( \mathcal{M}(\mathbb{R}^N) \) is the dual of \( C_c(\mathbb{R}^N) \) in \( L^\infty(\mathbb{R}^N). \) A sequence \((\nu_n)\) is said to be weak* convergent to \( \nu \) in \( \mathcal{M}(\mathbb{R}^N), \) if

\[
\int_{\mathbb{R}^N} \phi \, d\nu_n \to \int_{\mathbb{R}^N} \phi \, d\nu, \text{ as } n \to \infty, \forall \phi \in C_c(\mathbb{R}^N).
\]

In this case we write \( \nu_n \rightharpoonup^{*} \nu. \)

The following result is a consequence of the Banach-Alaoglu theorem [20, Chapter 5, Section 3] which states that for any normed linear space \( X, \) the closed unit ball in \( X^* \) is weak* compact.

Proposition 2.3. Let \((\nu_n)\) be a bounded sequence in \( \mathcal{M}(\mathbb{R}^N), \) then there exists \( \nu \in \mathcal{M}(\mathbb{R}^N) \) such that \( \nu_n \rightharpoonup^{*} \nu \) up to a subsequence.

The next proposition follows from the uniqueness part of the Riesz representation theorem [1, Theorem 14.14, Chapter 14].

Proposition 2.4. Let \( \nu \in \mathcal{M}(\mathbb{R}^N) \) be a positive measure. Then for an open \( V \subseteq \Omega, \)

\[
\nu(V) = \sup \left\{ \int_{\mathbb{R}^N} \phi \, d\nu : 0 \leq \phi \leq 1, \phi \in C_c(\mathbb{R}^N) \text{ with } \text{Supp}(\phi) \subseteq V \right\},
\]

and for any \( E \in \mathcal{B}(\mathbb{R}^N), \nu(E) := \inf \left\{ \nu(V) : E \subseteq V \text{ and } V \text{ is open} \right\}. \)

The following result plays an important role in proving the concentration compactness principle. For a proof we refer to [39, Lemma 1.2].

Proposition 2.5. Let \( \nu, \Gamma \) be two non-negative, bounded measures on \( \mathbb{R}^N \) such that

\[
\left[ \int_{\mathbb{R}^N} |\phi|^q \, d\nu \right]^\frac{1}{q} \leq C \left[ \int_{\mathbb{R}^N} |\phi|^p \, d\Gamma \right]^\frac{1}{p}, \forall \phi \in C_c(\mathbb{R}^N),
\]

for some \( C > 0 \) and \( 1 \leq p < q < \infty. \) Then there exist a countable set \( \{x_j \in \mathbb{R}^N : j \in \mathbb{J}\} \) and \( \nu_j \in (0, \infty) \) such that

\[
\nu = \sum_{j \in \mathbb{J}} \nu_j \delta_{x_j}.
\]
**Definition 2.6.** A measure $\Upsilon \in \mathcal{M}(\mathbb{R}^N)$ is said to be concentrated on a Borel set $F$ if

$$\Upsilon(E) = \Upsilon(E \cap F), \ \forall \ E \in \mathcal{B}(\mathbb{R}^N).$$

If $\Upsilon$ is concentrated on $F$, then one can observe that $\Upsilon(F) = \|\Upsilon\|$.

For any measure $\Upsilon \in \mathcal{M}(\mathbb{R}^N)$ and a $E \in \mathcal{B}(\mathbb{R}^N)$, we denote the restriction of $\Upsilon$ on $E$ as $\Upsilon_E$. Observe that $\Upsilon_E$ is concentrated on $E$.

### 2.3. Brézis–Lieb lemma

The next lemma is due to Brezis and Lieb (see Theorem 1 of [10]).

**Lemma 2.7.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $(f_n)$ be a sequence of complex-valued measurable functions which are uniformly bounded in $L^p(\Omega, \mu)$ for some $0 < p < \infty$. Moreover, if $(f_n)$ converges to $f$ a.e., then

$$\lim_{n \to \infty} \|f_n\|_{(p, \mu)} - \|f_n - f\|_{(p, \mu)} = \|f\|_{(p, \mu)}.$$ 

We also require the following inequality [39, Lemma I.3] that played an important role in the proof of Brézis–Lieb lemma: for $a, b \in \mathbb{R}^N$,

$$||a + b|^p - |a|^p| \leq \epsilon |a|^p + C(\epsilon, p)|b|^p$$

valid for each $\epsilon > 0$ and $0 < p < \infty$.

### 2.4. A strong maximum principle

We use a version of strong maximum principle that holds for $p$-Laplacian for showing the positivity of the solution of (1.1), see [32, Proposition 3.2].

**Lemma 2.8.** Let $u \in D_p(\Omega)$ and $g \in L^1_{loc}(\Omega)$ be two non-negative functions on $\Omega$ such that $gu^{p-1} \in L^1_{loc}(\Omega)$ and $u$ satisfies the following differential inequality (in the sense of distributions)

$$-\Delta_p u + gu^{p-1} \geq 0 \ \text{in} \ \Omega.$$ 

Then either $u \equiv 0$ or $u > 0$ in $\Omega$.

### 2.5. The spaces $\mathcal{H}_p(\Omega)$ and $\mathcal{F}_p(\Omega)$

We briefly recall the spaces $\mathcal{H}_p(\Omega)$ and $\mathcal{F}_p(\Omega)$ that have been introduced in [6]. The space of Hardy potentials $\mathcal{H}_p(\Omega)$ is defined as the set of all $g \in L^1_{loc}(\Omega)$ such that the following Hardy type inequality holds:

$$\int_{\Omega} |g||u|^p dx \leq C \int_{\Omega} |\nabla u|^p dx, \forall \ u \in D_p(\Omega),$$

for some $C > 0$. In [6], using Mazya’s $p$-capacity, we provide a Banach function space structure on $\mathcal{H}_p(\Omega)$. Recall that for $F \subset \subset \Omega$, the $p$-capacity of $F$ relative to $\Omega$ is defined as,

$$\text{Cap}_p(F, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in N_p(F) \right\},$$
where $\mathcal{N}_p(F) = \{u \in \mathcal{D}_p(\Omega) : u \geq 1 \text{ in a neighbourhood of } F\}$. We define the Banach function space norm on $\mathcal{H}_p(\Omega)$ as follows,

$$\|g\|_{\mathcal{H}_p} = \sup \left\{ \frac{\int_F |g| \, dx}{\operatorname{Cap}_p(F, \Omega)} : F \subset \subset \Omega; |F| \neq 0 \right\}.$$ 

Now we define $\mathcal{F}_p(\Omega) = C^\infty_c(\Omega)$ in $\mathcal{H}_p(\Omega)$. The following result is proved in [6, Theorem 1].

**Proposition 2.9.** Let $g \in \mathcal{H}_p(\Omega)$. Then $G_p : \mathcal{D}_p(\Omega) \rightarrow \mathbb{R}$ is compact if and only if $g \in \mathcal{F}_p(\Omega)$.

### 3. A variant of concentration compactness principle

In this section, we derive a variant of the concentration compactness principle that we use extensively in this article.

**Notations.** We will be following the notations below throughout this article.

- For each $R > 0$, we fix a function $\Phi_R \in C^1_b(\mathbb{R}^N)$ (bounded $C^1$ functions on $\mathbb{R}^N$) satisfying $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $B_R$ and $\Phi_R = 1$ on $B_R^c$.
- For a sequence $(u_n)$ in $\mathcal{D}_p(\mathbb{R}^N)^H$ with $u_n \rightharpoonup u$ in $\mathcal{D}_p(\mathbb{R}^N)$, each of the following sequences corresponds to a bounded sequence in $\mathcal{M}(\mathbb{R}^N)$ and hence weak* converges to a measure in $\mathcal{M}(\mathbb{R}^N)$. For convenience, we denote the sequences and their limits as given below:

$$\nu_n := |u_n - u|^{p^*} \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N),$$

$$\Gamma_n := |\nabla(u_n - u)|^p \rightharpoonup \Gamma \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N),$$

$$\gamma_n := g|u_n - u|^{p^*} \rightharpoonup \gamma \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N),$$

$$\tilde{\Gamma}_n := |\nabla u_n|^p \rightharpoonup \tilde{\Gamma} \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N),$$

$$\tilde{\gamma}_n := g|u_n|^{p^*} \rightharpoonup \tilde{\gamma} \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N).$$

The following limits will be denoted by:

$$\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| \geq R} |u_n - u|^{p^*} \, dx = \nu_\infty$$

$$\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| \geq R} |\nabla(u_n - u)|^p \, dx = \Gamma_\infty$$

$$\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| \geq R} g|u_n - u|^p \, dx = \gamma_\infty.$$

**Proposition 3.1.** The following statements are true:

(i) $\lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \Phi_R \, dx = \nu_\infty$.

(ii) $\lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} g|u_n|^p \Phi_R \, dx = \gamma_\infty$.

(iii) $\lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \Phi_R \, dx = \Gamma_\infty$. 
Proof. (i) Use Brézis-Lieb lemma to obtain
\[
\lim_{n \to \infty} \left| \int_{|x| \geq R} |u_n - u|^p \, dx - \int_{|x| \geq R} |u_n|^p \, dx \right| = \int_{|x| \geq R} |u|^p \, dx. \tag{3.1}
\]
Now, by the sub-additivity property of \( \limsup \), we get
\[
\left| \lim_{n \to \infty} \int_{|x| \geq R} |u_n|^p \, dx - \lim_{n \to \infty} \int_{|x| \geq R} |u_n - u|^p \, dx \right|
\leq \lim_{n \to \infty} \left( \int_{|x| \geq R} |u_n - u|^p \, dx - \int_{|x| \geq R} |u_n|^p \, dx \right).
\]
Thus, using (3.1), we obtain
\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| \geq R} |u_n|^p \, dx = \lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| \geq R} |u_n - u|^p \, dx = \nu_{\infty}. \tag{3.2}
\]
Notice that
\[
\int_{|x| \geq R+1} |u_n|^p \, dx \leq \int_{\mathbb{R}^N} |u_n|^p \Phi_R \, dx \leq \int_{|x| \geq R} |u_n|^p \, dx.
\]
By taking \( n, R \to \infty \) and using (3.2) we get (i).

(ii), (iii) follows by the similar set of calculations as given in (i). \( \square \)

**Proposition 3.2.** Let \( g \in H_p(\mathbb{R}^N) \) be a non-negative Hardy potential, and \( (u_n) \) be a sequence in \( D_p(\mathbb{R}^N) \) such that \( u_n \rightharpoonup u \) in \( D_p(\mathbb{R}^N) \). Then the followings hold:

(i) if \( \Phi \in C^1_0(\Omega) \) is such that \( \nabla \Phi \) has compact support, then
\[
\lim_{n \to \infty} \int_{\Omega} |\nabla ((u_n - u)\Phi)|^p \, dx = \lim_{n \to \infty} \int_{\Omega} |\nabla (u_n - u)|^p \Phi \, dx. \tag{3.3}
\]

(ii) there exists a countable set \( J \) such that \( \nu = \sum_{j \in J} \nu_j \delta_{x_j} \), where \( \nu_j \in (0, \infty) \), \( x_j \in \mathbb{R}^N \). In particular, \( \nu \) is supported on the countable set \( F_j := \{ x_j \in \mathbb{R}^N : j \in J \} \),

(iii) \( \gamma \) is supported on \( \sum_g \).

Proof. (i) Let \( \epsilon > 0 \) be given. Using (2.1),
\[
\left| \int_{\Omega} |\nabla ((u_n - u)\Phi)|^p \, dx - \int_{\Omega} |\nabla (u_n - u)|^p \Phi \, dx \right|
\leq \epsilon \int_{\Omega} |\nabla (u_n - u)|^p \Phi \, dx + C(\epsilon, p) \int_{\Omega} |u_n - u|^p |\nabla \Phi|^p \, dx.
\]
Since \( \nabla \Phi \) is compactly supported, by Rellich-Kondrachov compactness theorem [34, Theorem 2.6.3], the second term in the right-hand side of the above inequality goes to 0 as \( n \to \infty \). Further, as \( (u_n) \) is bounded in \( D_p(\Omega) \) and \( \epsilon > 0 \) is arbitrary, we obtain the desired result.

(ii) Let \( \mu \in (0, \mu_1(g)) \). From the definition of \( E_{g,\mu}(\mathbb{R}^N) \), we have
\[ \mathcal{E}_{g,\mu}(\mathbb{R}^N) \leq \frac{\int_{\mathbb{R}^N} |\nabla \phi(u_n - u)|^p - \mu g|\phi(u_n - u)|^p \, dx}{\left( \int_{\mathbb{R}^N} |\phi(u_n - u)|^p \, dx \right)^{\frac{p}{p^*}}} \leq \frac{\int_{\mathbb{R}^N} |\nabla \phi(u_n - u)|^p \, dx}{\left( \int_{\mathbb{R}^N} |\phi(u_n - u)|^p \, dx \right)^{\frac{p}{p^*}}} \]  

for all \( \phi \in C^\infty_c(\mathbb{R}^N) \). By the above assertion (i) we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla \phi(u_n - u)|^p \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\phi \nabla (u_n - u)|^p \, dx.
\]

Thus, by taking \( n \to \infty \) in (3.4),

\[
\mathcal{E}_{g,\mu}(\mathbb{R}^N) \left[ \int_{\mathbb{R}^N} |\phi|^{p^*} \, d\nu \right]^{\frac{1}{p^*}} \leq \int_{\mathbb{R}^N} |\phi|^p \, d\Gamma.
\]

Now, (ii) follows from Proposition 2.5.

(iii) For \( \phi \in C^\infty_c(\Omega) \), \( (u_n - u)\phi \in D_p(\Omega) \), and since \( g \in H_p(\Omega) \), it follows that

\[
\int_{\mathbb{R}^N} |\phi|^p \, d\gamma_n = \int_{\Omega} g((u_n - u)\phi) \, d\rho \leq \frac{1}{\mu_1(g, \Omega)} \int_{\Omega} |\nabla ((u_n - u)\phi)|^p \, dx
\]

\[
= \frac{1}{\mu_1(g, \Omega)} \int_{\mathbb{R}^N} |\nabla ((u_n - u)\phi)|^p \, dx.
\]

Take \( n \to \infty \) and use assertion (i) to obtain

\[
\int_{\mathbb{R}^N} |\phi|^p \, d\gamma \leq \frac{1}{\mu_1(g, \Omega)} \int_{\mathbb{R}^N} |\phi|^p d\Gamma.
\]

Now by the density of \( C^\infty_c(\mathbb{R}^N) \) in \( C_0(\mathbb{R}^N) \) together with Proposition 2.4, we get

\[
\gamma(E) \leq \frac{\Gamma(E)}{\mu_1(g, \Omega)} \forall E \in \mathcal{B}(\mathbb{R}^N).
\]

In particular, \( \gamma \ll \Gamma \) and hence by Radon-Nikodym theorem,

\[
\gamma(E) = \int_E \frac{d\gamma}{d\Gamma} \, d\Gamma \forall E \in \mathcal{B}(\mathbb{R}^N).
\]

Further, by Lebesgue differentiation theorem (page 152–168 of [25]) we have

\[
\frac{d\gamma}{d\Gamma}(x) = \lim_{r \to 0} \frac{\gamma(B_r(x))}{\Gamma(B_r(x))}.
\]

Now replacing \( g \) by \( g\chi_{B_r(x)} \) and proceeding as before, one can get an analogue of (3.6) as follows:

\[
\gamma(B_r(x)) \leq \frac{\Gamma(B_r(x))}{\mu_1(g, B_r(x))}.
\]

Thus from (3.8) we get

\[
\frac{d\gamma}{d\Gamma}(x) \leq \frac{1}{\mu_1(g, x)}.
\]
Now from (3.7) and (3.9), we conclude that $\gamma$ is supported on $\Sigma_g$. \hfill \Box

As we have seen from the above proposition that $\nu$ is supported on the countable set $F_j = \{x_j \in \mathbb{R}^N : j \in J\}$. Now let us define

$$
\Gamma F_j = \sum_{j \in J} \Gamma_j \delta_{x_j}, \quad \gamma F_j = \sum_{j \in J} \gamma_j \delta_{x_j}, \quad \text{and} \quad \zeta_{\mu} = \Gamma_{F_j} - \mu \gamma_{F_j}
$$

for $\mu \in (0, \mu_1(g))$, where $\Gamma_j = \Gamma(\{x_j\})$ and $\nu_j = \nu(\{x_j\})$ for $j \in J$. Then we have the following proposition.

**Proposition 3.3.** Let $H$ be a closed subgroup of $\mathcal{O}(N)$, $g \in \mathcal{H}_p(\mathbb{R}^N)$ be a non-negative $H$-invariant Hardy potential, and $(u_n)$ be a sequence in $\mathcal{D}_p(\mathbb{R}^N)^H$ such that $u_n \to u$ in $\mathcal{D}_p(\mathbb{R}^N)^H$. If $u = 0$ and $\mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \|\nu\|^\frac{p}{p^*} = \|\zeta_\mu\|$ for some $\mu \in (0, \mu_1(g))$, then $\nu$ is either zero or concentrated on a single finite $H$-orbit in $\mathbb{R}^N$.

**Proof.** Let $u = 0$ and $\mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \|\nu\|^\frac{p}{p^*} = \|\zeta_\mu\|$. First we show that $\nu$ is supported on a single $H$ orbit. By the definition of $\mathcal{E}_{g,\mu}^H(\mathbb{R}^N)$

$$
\left[ \int_{\mathbb{R}^N} |\phi u_n|^{p^*} \, dx \right]^\frac{p}{p^*} \leq \left[ \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \right]^{-1} \left[ \int_{\mathbb{R}^N} |\nabla(\phi u_n)|^p - \mu g|\phi u_n|^p \right] \, dx
$$

for any $H$-invariant function $\phi \in C_c^\infty(\mathbb{R}^N)$. Since $u = 0$, the above inequality together with (3.3) yield

$$
\left[ \int_{\mathbb{R}^N} |\phi|^{p^*} \, d\nu \right]^\frac{p}{p^*} \leq \left[ \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \right]^{-1} \left[ \int_{\mathbb{R}^N} |\phi|^p \, d\Gamma - \mu \int_{\mathbb{R}^N} |\phi|^p \, d\gamma \right]
$$

Consequently, $\nu^{\frac{p}{p^*}} \leq \left[ \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \right]^{-1} [\Gamma - \mu \gamma]$. Since $\nu$ is supported on $F_j$, it follows that

$$
\nu^{\frac{p}{p^*}} \leq \left[ \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \right]^{-1} \zeta_\mu, \quad (3.10)
$$

Further, by applying Holder’s inequality we get

$$
\left[ \int_{\mathbb{R}^N} |\phi|^p \, d\zeta_\mu \right]^\frac{p}{p^*} \leq \left[ \int_{\mathbb{R}^N} |\phi|^p \, d\zeta_\mu \right] \|\zeta_\mu\|^\frac{p^*}{p} \cdot 1. \text{ This gives } \zeta_\mu^{\frac{p}{p^*}} \leq \|\zeta_\mu\|^\frac{p^*}{p} \cdot 1. \text{ Thus, } (3.10) \text{ gives } \nu(E) \leq \left[ \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \right]^{-1} \|\zeta_\mu\| \|\zeta_\mu\| \epsilon_{A_N}(E) \text{ for all } H\text{-invariant } E \subseteq \mathbb{B}(\mathbb{R}^N). \text{ Now, since } \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \|\nu\|^\frac{p}{p^*} = \|\zeta_\mu\|, \text{ it follows that } \nu(E) = \left[ \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \right]^{-1} \|\zeta_\mu\| \|\zeta_\mu\| \epsilon_{A_N}(E) \text{ for all } H\text{-invariant } E \subseteq \mathbb{B}(\mathbb{R}^N). \text{ We use this equality in } (3.10), \text{ and also } \mathcal{E}_{g,\mu}^H(\mathbb{R}^N) \|\nu\|^\frac{p}{p^*} = \|\zeta_\mu\| \text{ to obtain }$

$$
\nu(E)^{\frac{p}{p^*}} \nu(\mathbb{R}^N)^{\frac{1}{p}} \leq \nu(E)^{\frac{1}{p}},
$$

for any $H$-invariant $E \subseteq \mathbb{B}(\mathbb{R}^N)$. Thus, $\nu(E)$ is either 0 or $\|\nu\|$, and hence $\nu$ is concentrated on a single $H$ orbit. Now, let $\nu$ be concentrated on the orbit $H\xi$ for some $\xi \in \mathbb{R}^N$. We show that $H\xi$ is finite. It follows from the inequality (3.10) and Lemma 2.5 that there exist a countable set $F_j = \{x_j \in \mathbb{R}^N : j \in J\}$ and $\nu_j \in (0, \infty)$ such that $\nu = \sum_{j \in J} \nu_j \delta_{x_j}$. Since $\nu$ is concentrated at $H\xi$, it
is clear that \( F_j = H \xi \). Noticing \( \nu \) is invariant under any orthogonal transformations \( h \in H \) (i.e., \( \nu(h(E)) = \nu(E) \) for all \( E \in B(\mathbb{R}^N) \) and \( h \in H \)), we infer that \( \nu_i = \nu_j \) for all \( i, j \in \mathbb{J} \). Thus, \( \mathbb{J} \) has to be finite (as \( \|\nu\| < \infty \)).

Now, we are ready to derive \((g, H)\)-dependent concentration compactness lemma. For \( g \equiv 0 \), a similar result is obtained in [52, Lemma 4.3]. Here we obtain an analogous result for the case \( g \neq 0 \) under the assumption \( |\sum_g| = 0 \).

**Lemma 3.4.** Let \( H \) be a closed subgroup of \( O(N) \), \( g \in H_p(\mathbb{R}^N) \) be non-negative and \( H \)-invariant. Assume that \((u_n)\) is a sequence in \( D_p(\mathbb{R}^N)^H \) such that \( u_n \rightharpoonup u \) in \( D_p(\mathbb{R}^N)^H \). If \( |\sum_g| = 0 \), then for \( \mu \in (0, \mu_1(g)) \), the following holds:

\[
\begin{align*}
& (a) \quad C^{H, \ast}_{g, \mu}(\mathbb{R}^N) \|\nu\| \frac{p^*}{p} + \mu \|\gamma\| \leq \|\Gamma_{\sum_g \cup \mathbb{J}}\|,
& (b) \quad C^{H, \ast}_{g, \mu}(\infty) \nu^\frac{p^*}{p} + \mu \gamma_\infty \leq \Gamma_\infty,
& (c) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} |u|^{p^*} dx + \|\nu\| + \nu_\infty,
& (d) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} g|u_n|^p dx = \int_{\mathbb{R}^N} g|u|^p dx + \|\gamma\| + \gamma_\infty,
& (e) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx \geq \int_{\mathbb{R}^N} |\nabla u|^p dx + \|\Gamma_{\sum_g \cup \mathbb{J}}\| + \Gamma_\infty,
& (f) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} [|\nabla u_n|^p - \mu g|u_n|^p] dx \geq \int_{\mathbb{R}^N} [|\nabla u|^p - \mu g|u|^p] dx + C^{H, \ast}_{g, \mu}(\mathbb{R}^N) \|\nu\| \frac{p^*}{p} + C^{H, \ast}_{g, \mu}(\infty) \nu^\frac{p^*}{p}.
\end{align*}
\]

**Proof.** (a) By the definition of \( \mathcal{E}^H_{g, \mu}(H(B_r(x))) \), we have

\[
\mathcal{E}^H_{g, \mu}(H(B_r(x))) \leq \frac{\int_{\mathbb{R}^N} [|\nabla \phi(u_n - u)|^p - \mu g|\phi(u_n - u)|^p] dx}{\int_{\mathbb{R}^N} |\phi(u_n - u)|^{p^*} dx + \mu \int_{\mathbb{R}^N} |\phi|^{p^*} d\gamma} \leq \int_{\mathbb{R}^N} |\phi|^p d\Gamma.
\]

In particular, for \( x_j \) with \( j \in \mathbb{J} \), by taking \( \phi = 1 \) on \( H(\{x_j\}) \) and then letting \( r \to 0 \), we get

\[
\mathcal{E}^H_{g, \mu}(x_j) \|\nu(Hx_j)\|^{\frac{p^*}{p}} + \mu \gamma(Hx_j) \leq \Gamma(Hx_j) \forall j \in \mathbb{J}.
\]

Taking the sum over \( \mathbb{J} \) and using the concavity of the map \( f(t) := t^{\frac{p^*}{p}} \), we obtain

\[
\mathcal{E}^H_{g, \mu}(\mathbb{R}^N) \|\nu\|^{\frac{p^*}{p}} + \mu \|\gamma_{\mathbb{J}}\| \leq \|\Gamma_{\mathbb{J}}\|.
\]

Next we show that \( \mu \|\sum_{F_j} \gamma\| \leq \|\Gamma_{\mathbb{J}}\| \). Since \( g \in H_p(\Omega) \) and \( \mu \in (0, \mu_1(g)) \), we have

\[
\mu \int_{\mathbb{R}^N} g|u_n - u|^p dx \leq \frac{\mu}{\mu_1(g)} \int_{\mathbb{R}^N} |\nabla (u_n - u)|^p dx \leq \int_{\mathbb{R}^N} |\nabla (u_n - u)|^p dx
\]
for any $\phi \in C^\infty_c(\mathbb{R}^N)$. By taking $n \to \infty$ and using (3.3), one can get
\[
\mu \int_{\mathbb{R}^N} |\phi|^p d\gamma \leq \int_{\mathbb{R}^N} |\nabla \phi|^p d\Gamma
\]
for any $\phi \in C^\infty_c(\mathbb{R}^N)$. Thus, we have $\mu \gamma(\sum_g \setminus F_j) \leq \Gamma(\sum_g \setminus F_j)$. In particular, we get
\[
\mu \|\gamma_{\sum_g \setminus F_j}\| \leq \|\Gamma_{\sum_g \setminus F_j}\|.
\]
(3.12)

Now, by adding (3.12) and (3.11), and using the fact that $\nu$, $\Gamma$ are supported on $F_J$, $\sum g$, we get
\[
\mathcal{C}^{H,*,\mu}_g(\mathbb{R}^N) \|\nu\|^{\frac{p}{p'}} + \mu \|\gamma\| \leq \|\Gamma_{\sum_g \cup F_j}\|.
\]
(3.13)

(b) For $R > 0$, choose $\Phi_R \in C^b_0(\mathbb{R}^N)^H$ satisfying $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $\overline{B_R}$ and $\Phi_R = 1$ on $B_{R+1}^c$. Then $(u_n - u)\Phi_R \in \mathcal{D}_p(B_R^c)^H$. In order to prove (b), one can start with $\mathcal{E}^{H,\mu}_g(H(B_r(x)))$ instead of $\mathcal{E}^{H,\mu}_g(H(B_r(x)))$ and follow the similar arguments as in (a) to get
\[
\mathcal{E}^{H,\mu}_g(H(B_R^c)) \lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} |\Phi_R|^{p'} |u_n - u|^{p'} dx \right]^{\frac{1}{p'}} + \mu \lim_{n \to \infty} \int_{\mathbb{R}^N} g|u_n - u| |\Phi_R|^p dx
\]
\[
\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\Phi_R|^{p'} |\nabla (u_n - u)|^{p'} dx.
\]
(3.14)

Thus, by taking $R \to \infty$ in (3.14) and using Proposition 3.1, we prove the assertion (b).

(c) Using Brézis-Lieb lemma together with Proposition 3.1-(i), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p'} dx
\]
\[
= \lim_{R \to \infty} \lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} |u_n|^{p'} (1 - \Phi_R) dx + \int_{\mathbb{R}^N} |u_n|^{p'} \Phi_R dx \right]
\]
\[
= \lim_{R \to \infty} \lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} |u|^{p'} (1 - \Phi_R) dx
\]
\[
+ \int_{\mathbb{R}^N} |u_n - u|^{p'} (1 - \Phi_R) dx + \int_{\mathbb{R}^N} |u_n|^{p'} \Phi_R dx \right]
\]
\[
= \int_{\mathbb{R}^N} |u|^{p'} dx + \|\nu\| + \nu_{\infty}.
\]
(d) As in (c), using Brezis-Lieb lemma together with Proposition 3.1-(ii), we deduce
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} g|u_n|^p dx = \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} g|u_n|^p (1 - \Phi_R) dx + \int_{\mathbb{R}^N} g|u_n|^p \Phi_R dx \right) \\
= \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} g|u|^p (1 - \Phi_R) dx + \int_{\mathbb{R}^N} g|u|^p \Phi_R dx \right) \\
+ \int_{\mathbb{R}^N} g|u_n - u|^p (1 - \Phi_R) dx + \int_{\mathbb{R}^N} g|u_n|^p \Phi_R dx \\
= \int_{\mathbb{R}^N} g|u|^p dx + \|\gamma\| + \gamma_\infty.
\]

(e) We break this proof into several steps.

Claim 1: \( \tilde{\Gamma}_{F_j} = \Gamma_{F_j} \). Let \( \phi_\varepsilon \in C_c^\infty (B_\varepsilon (\omega)) \) satisfying \( 0 \leq \phi_\varepsilon \leq 1, \phi_\varepsilon (\omega) = 1 \), where \( \omega \in F_j \). Then, using (2.1), we have the following:
\[
\left| \tilde{\Gamma}(\phi_\varepsilon) - \Gamma(\phi_\varepsilon) \right| = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla (u_n)|^p - |\nabla u_n - u|^p | \phi_\varepsilon dx \\
\leq \varepsilon \int_{\mathbb{R}^N} \phi_\varepsilon |\nabla u_n|^p dx + C(\varepsilon, p) \int_{\mathbb{R}^N} \phi_\varepsilon |\nabla u|^p dx.
\]
By taking \( \varepsilon \to 0 \), we get \( \tilde{\Gamma}(\omega) = \Gamma_{F_j}(\omega) \).

Claim 2: \( \tilde{\Gamma} = \Gamma \), on \( \sum_g \). Let \( E \subset \sum_g \) be a Borel set. Thus, for each \( m \in \mathbb{N} \), there exists an open subset \( O_m \) containing \( E \) such that \( |O_m| = |O_m \setminus E| < \frac{1}{m} \). Let \( \varepsilon > 0 \) be given. Then, for any \( \phi \in C_c^\infty (O_m) \) with \( 0 \leq \phi \leq 1 \), using (2.1) we have
\[
\left| \int_{O_m} \phi d\Gamma_n - \int_{O_m} \phi d\tilde{\Gamma}_n \right| = \left| \int_{O_m} \phi |\nabla (u_n - u)|^p dx - \int_{O_m} \phi |\nabla u_n|^p dx \right| \\
\leq \varepsilon \int_{O_m} \phi |\nabla u_n|^p dx + C(\varepsilon, p) \int_{O_m} \phi |\nabla u|^p dx \\
\leq \varepsilon L + C(\varepsilon, p) \int_{O_m} |\nabla u|^p dx,
\]
where \( L = \sup_n \left\{ \int_{O_m} |\nabla u_n|^p dx \right\} \). Now letting \( n \to \infty \), we obtain \( \int_{O_m} \phi d\Gamma - \int_{O_m} \phi d\tilde{\Gamma} \leq \varepsilon L + C(\varepsilon, p) \int_{O_m} |\nabla u|^p dx \). Therefore,
\[
|\Gamma(O_m) - \tilde{\Gamma}(O_m)| = \sup \left\{ \left| \int_{O_m} \phi d\Gamma - \int_{O_m} \phi d\tilde{\Gamma} \right| : \phi \in C_c^\infty (O_m), 0 \leq \phi \leq 1 \right\} \\
\leq \varepsilon L + C(\varepsilon, p) \int_{O_m} |\nabla u|^p dx,
\]
Now as \( m \to \infty \), \( |O_m| \to 0 \) and hence \( |\Gamma(E) - \tilde{\Gamma}(E)| \leq \varepsilon L \). Since \( \varepsilon > 0 \) is arbitrary, we conclude \( \Gamma(E) = \tilde{\Gamma}(E) \).
Claim 3: \( \| \tilde{\Gamma} \| \geq \frac{1}{n} \int_{\mathbb{R}^N} |\nabla u|^p dx + \| \Gamma_{\sum_g \cup F} \| \). Choose an arbitrary \( \phi \in C_c^\infty (\mathbb{R}^N) \) with \( 0 \leq \phi \leq 1 \). Hence, by the lower semicontinuity, we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \phi | \phi dx \geq \int_{\mathbb{R}^N} |\nabla u|^p \phi | \phi dx.
\]

This yields \( \tilde{\Gamma} \geq |\nabla u|^p dx \). On the other hand, the measure \( |\nabla u|^p dx \) is singular to \( \Gamma_{\sum_g \cup F} \) (as \( \sum_g \cup F \) has Lebesgue measure zero) and hence

\[
\| \tilde{\Gamma} \| \geq \int_{\mathbb{R}^N} |\nabla u|^p dx + \| \Gamma_{\sum_g \cup F} \|.
\]

(3.15)

Now we are ready to prove (e). Observe that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p (1 - \Phi_R) dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \Phi_R dx.
\]

Using Proposition 3.1-(iii), (3.15), and (3.11), we infer that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \| \tilde{\Gamma} \| + \Gamma_\infty \geq \int_{\mathbb{R}^N} |\nabla u|^p dx + \| \Gamma_{\sum_g \cup F} \| + \Gamma_\infty.
\]

(f) Since \( \mu \in (0, \mu_1 (g)) \), it follows that \( \int_{\mathbb{R}^N} |\nabla u_n|^p - \mu g |u_n|^p dx \geq 0 \). Thus, using the sub-additivity of limsup, (e), (d) we get

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p - \mu g |u_n|^p dx \geq \int_{\mathbb{R}^N} |\nabla u|^p - \mu g |u|^p dx
\]

\[
+ (\| \Gamma_{\sum_g \cup F} \| - \mu \| \gamma \| ) + (\Gamma_\infty - \mu \gamma_\infty).
\]

Now use (a) and (b) to obtain (f).

The next corollary states the above results for a general domain in place of \( \mathbb{R}^N \).

Corollary 3.5. Let \( H \) be a closed subgroup of \( \mathcal{O} (N) \), and \( g \geq 0, \Omega \) be as in (H2). Assume that \( (u_n) \) be a sequence in \( \mathcal{D}_p (\Omega)^H \) such that \( u_n \rightharpoonup u \) in \( \mathcal{D}_p (\Omega)^H \). If \( |\sum_g| = 0 \), then for \( \mu \in (0, \mu_1 (g)) \), the following holds:

(a) there exists a countable set \( J \) such that \( \nu = \sum_{j \in J} \nu_j \delta_{x_j} \), where \( \nu_j \in (0, \infty), x_j \in \overline{\Omega} \). In particular, \( \nu \) is supported on the countable set \( F_\infty := \{ x_j \in \overline{\Omega} : j \in J \} \),

(b) \( \gamma \) is supported on \( \sum_g \),

(c) \( c_{g, \mu, \infty} (\Omega) |\nu|^p + \mu |\gamma| \leq \| \Gamma_{\sum_g \cup F} \| \),

(d) \( c_{g, \mu, \infty} (\mathbb{R}) |\nu|^p + \mu \gamma_\infty \leq \Gamma_\infty \),

(e) \( \lim_{n \to \infty} \int_{\Omega} |u_n|^p dx = \int_{\Omega} |u|^p dx + \| \nu \| + \nu_\infty \),

(f) \( \lim_{n \to \infty} \int_{\Omega} g |u_n|^p dx = \int_{\Omega} g |u|^p dx + \| \gamma \| + \gamma_\infty \),

(g) \( \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^p dx \geq \int_{\Omega} |\nabla u|^p dx + \| \Gamma_{\sum_g \cup F} \| + \Gamma_\infty \).
\( \lim_{n \to \infty} \int_\Omega \| \nabla u_n \|^p - \mu g |u_n|^p \, dx \geq \int_\Omega \| \nabla u \|^p - \mu g |u|^p \, dx + C_{g,H}(\Omega) \| \nu \|^p + C_{g,H}(\infty) \nu_\infty \)

(i) if \( u = 0 \) and \( E_{g,H}(\Omega) \| \nu \|^p = \| \zeta_\mu \| \), where \( \zeta_\mu = \sum_{j \in \mathbb{J}} (\Gamma_j - \mu \gamma_j) \delta_{x_j} \), then \( \nu \) is either zero or concentrated on a single finite \( H \)-orbit in \( \mathbb{R}^N \).

\[ \text{Proof.} \] Recall that any function \( u \in D_p(\Omega) \) can be considered as a \( D_p(\mathbb{R}^N) \) function whenever convenient. Since \( g, (u_n) \) are supported inside \( \Omega \), it is not difficult to see that (i) \( \nu, \Gamma = 0 \) outside \( \Omega \), (ii) \( C_{g,H}(\Omega) = C_{g,H}(\mathbb{R}^N) \). Hence the corollary follows as a consequence of Proposition 3.2, Proposition 3.3, and Lemma 3.4. \( \square \)

Remark 3.6. (i) It follows from Corollary 3.5-(c) that \( E_{g,H}(\Omega) \| \nu \|^p + \mu \| \gamma \| \leq \| \Gamma_{\sum_{g \in F_1}} \| \). Now, if \( E_{g,H}(\Omega) \| \nu \|^p < \| \zeta_\mu \| \) (\( \zeta_\mu \) as in Proposition 3.3), then the previous inequality has to be strict. Therefore, if \( E_{g,H}(\Omega) \| \nu \|^p + \mu \| \gamma \| = \| \Gamma_{\sum_{g \in F_1}} \| \), then \( E_{g,H}(\Omega) \| \nu \|^p = \| \zeta_\mu \| \).

(ii) Let \( H \) be an infinite closed subgroup on \( O(N) \) and \( 0 \notin \overline{\Omega} \). Assume all the hypothesis of Proposition 3.3 (or Corollary 3.5-(i)). Then, it follows that \( \nu = 0 \).

(iii) Observe that, for a bounded sequence \( (u_n) \) in \( D_p(\Omega) \), the measure \( \nu \) in the above corollary helps us to determine whether \( u_n \to u \) in \( L^{p^*}(\Omega) \) or not. Precisely, if \( \nu = 0 = \nu_\infty \), then \( u_n \to u \) in \( L^{p^*}(\Omega) \).

4. Subcritical potentials

This section is devoted to proving Theorem 1.3 and Theorem 1.4. We bring up several important consequences of these theorems and provide a few examples of subcritical and non-subcritical potentials. Let us commence with the following proposition.

Proposition 4.1. Let \( g \in H_p(\Omega) \) be such that \( g^- \in F_p(\Omega) \). Then \( C_{g,H}(\Omega) = C_{g,H}(\infty) \) and \( C_{g,H}(\infty) = C_{g,H}(\infty) \) for all \( \mu \in (0, \mu_1(g)) \).

\[ \text{Proof.} \] Let us fix \( x \in \overline{\Omega} \) and \( u_n \in S_p(\Omega \cap B_{\frac{1}{n}}(x)) \) be such that

\[ C_{g,H}(x) = \lim_{n \to \infty} \int_\Omega \| \nabla u_n \|^p - \mu g^+ |u_n|^p \, dx. \]

Since \( \mu \in (0, \mu_1(g)) \), the quasi-norm \( \| u \|_{D_{p,H}} := \left( \int_\Omega \| \nabla u \|^p - \mu g^+ |u|^p \, dx \right)^{\frac{1}{p}} \) is equivalent to the norm \( \| u \|_{D_p} \) on \( D_p(\Omega) \). Thus, \( (u_n) \) is bounded in \( D_p(\Omega) \). Furthermore, the supports of \( (u_n) \) converge to the singleton set \( \{ x \} \). Consequently, \( u_n \to 0 \) in \( D_{p,1}(\Omega) \) (up to a subsequence). Now, since \( g^- \in F_p(\Omega) \), we
have $\lim_{n \to \infty} \int_{\Omega} g^-|u_n|^pdx = 0$ (by Proposition 2.9). Hence,

$$C_{g^+,\mu}(x) = \lim_{n \to \infty} \int_{\Omega} \left[|\nabla u_n|^p - \mu g^+|u_n|^p\right]dx + \lim_{n \to \infty} \int_{\Omega} \mu g^-|u_n|^pdx$$

$$= \lim_{n \to \infty} \int_{\Omega} \left[|\nabla u_n|^p - \mu g|u_n|^p\right]dx \geq C_{g,\mu}(x).$$

The other way inequality holds trivially. Hence $C_{g,\mu}(x) = C_{g^+,\mu}(x)$. Since $x \in \Omega$ is arbitrary, we prove that $C_{g,\mu}(\Omega) = C_{g^+,\mu}(\Omega)$. The other assertion follows from a similar set of arguments. \[\square\]

**Proof of Theorem 1.3.** We choose $H = \{Id_{\mathbb{R}^n}\}$ in Corollary 3.5. Then $D_p(\Omega)^H = D_p(\Omega)$, and $S_p(\Omega)^H = S_p(\Omega)$. Let $u_n \in S_p(\Omega)$ be a minimizing sequence of $E_{g,\lambda}(\Omega)$. Since $\lambda \in (0, \mu_1(g))$, the quasi-norm $\|u\|_{D_p,\lambda} := \left(\int_{\Omega} |\nabla u|^p - \lambda g|u|^p\right)^{\frac{1}{p}}$ is equivalent to the norm $\|u\|_{D_p}$ on $D_p(\Omega)$. Consequently, $(u_n)$ is bounded in $D_p(\Omega)$, which implies $u_n \rightharpoonup u$ in $D_p(\Omega)$ (up to a subsequence).

As $g$ satisfies 1, we have $\lim_{n \to \infty} \int_{\Omega} g^-|u_n|^pdx = \int_{\Omega} g^-|u|^pdx$ (Proposition 2.9). Now, we use Corollary 3.5 to $g^+$ and Proposition 4.1 to obtain the following:

$$E_{g,\lambda}(\Omega) = \lim_{n \to \infty} \int_{\Omega} \left[|\nabla u_n|^p - \lambda g|u_n|^p\right]dx$$

$$\geq \int_{\Omega} \left[|\nabla u|^p - \lambda g|u|^p\right]dx + C_{g,\lambda}(\Omega)\|\nu\|_{\mathbb{R}^n}^{\frac{p}{p'}} + C_{g,\lambda}(\infty)\nu_{\infty}^{\frac{p}{p'}}$$

$$\geq E_{g,\lambda}(\Omega) \left[\int_{\Omega} |u|^pdx\right]^{\frac{p}{p'}} + C_{g,\lambda}(\Omega)\|\nu\|_{\mathbb{R}^n}^{\frac{p}{p'}} + C_{g,\lambda}(\infty)\nu_{\infty}^{\frac{p}{p'}}.$$

If one of $\|\nu\|$ or $\nu_{\infty}$ is non-zero, then using the hypothesis that $E_{g,\lambda}(\Omega) < C_{g,\lambda}(\Omega)$ and $E_{g,\lambda}(\Omega) < C_{g,\lambda}(\infty)$ (i.e., $g$ is subcritical in $\Omega$ and at infinity for level $\lambda$), we infer

$$E_{g,\lambda}(\Omega) > E_{g,\lambda}(\Omega) \left(\int_{\Omega} |u|^pdx + \|\nu\| + \nu_{\infty}\right)^{\frac{p}{p'}}.$$

By (e) of Corollary 3.5, $\int_{\Omega} |u|^pdx + \|\nu\| + \nu_{\infty} = \lim_{n \to \infty} \|u_n\|_{p^*}^{p^*} = 1$ (as $u_n \in S_p(\Omega)$). Hence, $E_{g,\lambda}(\Omega) > E_{g,\lambda}(\Omega)$, a contradiction. Therefore, $\|\nu\| = 0 = \nu_{\infty}$. As a consequence $\|u\|_{p^*} = 1$. Hence, $E_{g,\lambda}(\Omega)$ is attained at $u$. Thus, $v = [E_{g,\lambda}(\Omega)]^{\frac{1}{p'-p}} u$ is a non-trivial solution to (1.1). Notice that, if $u$ is a minimizer of $E_{g,\lambda}(\Omega)$, then $|u|$ is also so. Thus, we prove that there exists a non-negative, non-trivial solution $v$ of (1.1). Therefore, we have

$$-\Delta_p v + \lambda g^-v^{p-1} = \lambda g^+v^{p-1} \geq 0,$$

in distribution sense, and Lemma 2.8 ensures that $v$ is positive. \[\square\]

**Remark 4.2.** The above proof gives not only the existence of a positive solution of (1.1) but also it assures that $E_{g,\lambda}(\Omega)$ is attained at some positive $u \in D_p(\Omega)$.
In the next proposition, we prove a particular case of Theorem 1.4.

**Proposition 4.3.** Let \( N \geq p^2 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). Then any positive constant function \( g \) is sub-critical in \( \Omega \) as well as at infinity for each level \( \mu \in (0, \mu_1(g)) \).

**Proof.** Without loss of generality, we consider the constant function \( 1(z) = 1 \); \( z \in \Omega \) and prove that \( 1 \) is sub-critical in \( \Omega \) and at infinity for each level \( \mu \in (0, \mu_1(1)) \). Notice that \( \mu_1(1) = \lambda_1 \), where \( \lambda_1 \) is the first Dirichlet eigenvalue of \( \Delta_p \). Fix \( \mu \in (0, \lambda_1) \). For \( x \in \overline{\Omega} \), there exists \( v_n \in \mathbb{S}_p(\Omega \cap B_{\frac{1}{n}}(x)) \) (by definition of \( \mathcal{E}_{1,\mu}(\Omega \cap B_{\frac{1}{n}}(x)) \)) such that

\[
\int_{\Omega} [\nabla v_n]^p - \mu |v_n|^p \, dx < \mathcal{E}_{1,\mu}(\Omega \cap B_{\frac{1}{n}}(x)) + \frac{1}{n} \leq \mathcal{E}_{0,\mu}(\Omega \cap B_{\frac{1}{n}}(x)) + 1 = \mathcal{E}_{0,\mu}(\Omega) + 1. \tag{4.1}
\]

The last equality follows from the fact that \( \mathcal{E}_{0,\mu}(\cdot) \) does not depend on the domain. From (4.1), we have

\[
\mathcal{E}_{0,\mu}(\Omega) - \mu \int_{\Omega} |v_n|^p \, dx < \mathcal{E}_{1,\mu}(\Omega \cap B_{\frac{1}{n}}(x)) + \frac{1}{n}. \tag{4.2}
\]

Since \( \mu \in (0, \lambda_1) \), it follows from (4.1) that \( (v_n) \) is bounded in \( \mathcal{D}_p(\Omega) \). Further, their supports are shrinking to a null set, namely \( \{x\} \) as \( n \to \infty \). This implies, \( v_n \to 0 \) in \( \mathcal{D}_p(\Omega) \), and the Rellich-Kondrachov compactness theorem assures that \( v_n \to 0 \) in \( L^p(\Omega) \). Now, by taking \( n \to \infty \) in the above inequality (4.2), we obtain \( \mathcal{E}_{0,\mu}(\Omega) \leq C_{1,\mu}(x) \), for each \( x \in \overline{\Omega} \). This implies \( \mathcal{E}_{0,\mu}(\Omega) \leq C_{1,\mu}^*(\Omega) \).

Further, for all \( \mu \in (0, \lambda_1) \), we have \( \mathcal{E}_{1,\mu}(\Omega) < \mathcal{E}_{0,\mu}(\Omega) \) when \( N \geq p^2 \), see [27, Lemma 7.1]. Hence, for all \( \mu \in (0, \lambda_1) \), \( 1 \) is sub-critical in \( \Omega \) when \( N \geq p^2 \). On the other hand, since \( \Omega \) is bounded, \( 1 \) is sub-critical at infinity too. \( \square \)

**Remark 4.4.** By the above proposition and Theorem 1.3, we conclude that (1.1) with \( g \equiv 1 \) on a bounded domain in \( \mathbb{R}^N \) admits a positive solution for all \( \mu \in (0, \lambda_1) \), provided \( N \geq p^2 \). Thus the results of Brézis–Nirenberg [11, Theorem 1.1] (for \( p = 2 \)) and [27, Theorem 7.4] (for general \( p \)) follows as a particular case of Theorem 1.3.

**Example 4.5.** We provide some examples of \( g \) that are not subcritical.

(i) **A potential critical in \( \Omega \).** Let \( \Omega \) be a domain in \( \mathbb{R}^N \). Consider the zero function \( 0(z) \equiv 0 \) on \( \Omega \). Fix \( \mu \in \mathbb{R} \). Recall that

\[
\mathcal{E}_{0,\mu}(\Omega) = \inf_{u \in \mathcal{D}_p(\Omega)} \left\{ \int_{\Omega} |\nabla u|^p \, dx : \|u\|_{p^*} = 1 \right\}.
\]

It is not difficult to see that \( \mathcal{E}_{0,\mu}(\cdot) \) is independent of domain (as \( \mathcal{E}_{0,\mu}(\cdot) \) is invariant under translations and dilations in \( \mathbb{R}^N \)), and hence \( C_{0,\mu}(x) = \mathcal{E}_{0,\mu}(\Omega) \) for any \( x \in \overline{\Omega} \). This implies \( C_{0,\mu}^*(\Omega) = \mathcal{E}_{0,\mu}(\Omega) \). Therefore, \( 0 \) is critical in \( \Omega \) for the level \( \mu \).

(ii) **A potential critical in \( \Omega \).** Let \( \Omega = \Omega_k \times \Omega_{N-k} \) be as in (1.6) with \( 0 \in \Omega \). Let \( g(z) = \frac{1}{|z|^p} \); \( z \in \Omega \). Fix a \( \mu \in \mathbb{R} \). Recall that

\[
\mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega) = \inf_{u \in \mathcal{D}_p(\Omega)} \left\{ \int_{\Omega} [||\nabla u||^p - \frac{\mu}{|z|^p}] \, dz : \|u\|_{p^*} = 1 \right\}.
\]
Since $\mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega)$ is invariant under dilation, using the scaling arguments it can be seen that $\mathcal{C}_{\frac{1}{|z|^p},\mu}(0) = \mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega)$. This implies $\mathcal{C}^{*}_{\frac{1}{|z|^p},\mu}(\Omega) = \mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega)$.

Therefore, $\frac{1}{|z|^p}$ is critical in $\Omega$ for the level $\mu$.

(iii) A potential critical at infinity. Let $\Omega = \Omega_k \times \Omega_{N-k}$ be as in (1.6) with $k = N$, $a > 0$, and $b = \infty$ i.e., the exterior of a ball $B_a(0)$ in $\mathbb{R}^N$. Consider $g(z) = \frac{1}{|z|^p}$; $z \in \Omega$ and fix $\mu \in (0, \mu_1(g))$. For each $\epsilon > 0$, there exists $w \in \mathcal{S}_p(\Omega)$ such that

$$\int_{\Omega} \left[ |\nabla w|^p(z) - \mu \frac{|w|^p(z)}{|z|^p} \right] dz < \mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega) + \epsilon.$$ 

For $w_R(z) = R^{\frac{p-N}{p}} w(\frac{z}{R})$, one can see that $w_R \in \mathcal{S}_p(\Omega \cap B^c_{aR})$ for $R > 1$, and

$$\int_{\Omega} \left[ |\nabla w_R|^p(z) - \mu \frac{|w_R|^p(z)}{|z|^p} \right] dz = \int_{\Omega} \left[ |\nabla w|^p(z) - \mu \frac{|w|^p(z)}{|z|^p} \right] dz.$$ 

This gives $\mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega \cap B^c_{aR}) \leq \mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega)$ for all $R > 1$ and hence $\mathcal{C}_{\frac{1}{|z|^p},\mu}(\infty) \leq \mathcal{E}_{\frac{1}{|z|^p},\mu}(\Omega)$, and the other way inequality always holds. Thus, $g$ is critical at infinity for level $\mu$.

Next, we are going to prove Theorem 1.4.

Remark 4.6. Recall that, for any $\mu \in \mathbb{R}$, $\mathcal{E}_{0,\mu}(\cdot)$ is invariant under dilation and translation in $\mathbb{R}^N$. Hence, $\mathcal{E}_{0,\mu}(\Omega) = \mathcal{C}_{0,\mu}^{*}(\Omega)$ i.e., $g \equiv 0$ is critical in $\Omega$. In addition, if $\Omega$ is bounded and star-shaped, using the Pohozaev identity, we can show that $\mathcal{E}_{0,\mu}(\Omega)$ is not attained in $\mathcal{D}_p(\Omega)$. However, for $\Omega = \mathbb{R}^N$, $\mathcal{E}_{0,\mu}(\mathbb{R}^N)$ is attained by the following radial functions in $\mathcal{D}_p(\mathbb{R}^N)$:

$$\Psi_{\epsilon,x_0}(x) = \epsilon^{-\frac{N}{p}} \left( 1 + C(N,p) \left| \frac{x-x_0}{\epsilon} \right|^{\frac{p}{p-1}} \right)^{\frac{p-N}{p}},$$

for any $\epsilon > 0$ and $x_0 \in \mathbb{R}^N$ [39, Corollary I.1].

Let $g \in \mathcal{F}_p(\Omega)$ be such that $g^+ \neq 0$. Then there exists a compact set $K \subseteq \Omega$ with $|K| > 0$ such that $g$ is positive on $K$. Furthermore, due to Lusin’s theorem, we can assume that $g$ is continuous on $K$. Thus,

$$g_{\text{min}} := \min\{g(x) : x \in K\} > 0. \quad (4.3)$$

Let $\Phi_K \in C^\infty_c(\Omega)$ such that $\Phi_K = 1$ on $K$. For each $y \in \mathbb{R}^N$, we consider

$$u_{\epsilon,y} = \Phi_K U_{\epsilon,y}$$

where $U_{\epsilon,y}(x) = \chi_{\epsilon,y}(x) \left( \epsilon + C(N,p)|x-y|^{\frac{p}{p-1}} \right)^{\frac{p-N}{p}}$ and $\chi_{\epsilon,y}(x) = \chi_0 \left( \frac{x-y}{\epsilon} \right)$ with $\chi_0 \in C^\infty_c(\Omega)$ is such that $0 \leq \chi_0 \leq 1$, $\chi_0 = 1$ on $B_1(0)$ and vanishes outside $B_2(0)$. Next, we list some properties of $u_{\epsilon,y}$.

Lemma 4.7. Let $g \in \mathcal{H}_p(\Omega)$ be such that $g^+ \neq 0$. For each $\epsilon > 0$, $u_{\epsilon,y}$ satisfies the following properties:

(i) $\|u_{\epsilon,y}\|_{p^*} = \|\Psi_{1,y}\|_{p^*} \epsilon^{\frac{p-N}{p}} + O(1)$,
(ii) \( \| \nabla u_{\epsilon,y} \|_p^p = \| \nabla \Psi_{1,y} \|_p^p \leq \frac{p-N}{p} + O(1), \)

(iii) for \( y \in K, \) we have

\[
\int \Omega g|u_{\epsilon,y}|^p dx \geq \begin{cases} A g_{\min} \frac{p^2-N}{p} + O(1) & \text{if } p^2 < N \\ A g_{\min} |\log(\epsilon)| + O(1) & \text{if } p^2 = N, \end{cases}
\]

where \( A > 0 \) depends only on \( N, p, \) and \( g_{\min} \) is defined as (4.3).

**Proof.** For a proof of (i) and (ii), we refer to the assertion (7.7) of [27] (one can also see [11, assertion 1.11, 1.12] for \( p = 2 \)). To prove (iii), we first recall the following estimate [27, (c) of 7.7] (one can also see [11, assertion 1.11, 1.12] for \( p = 2 \)):

\[
\| U_{\epsilon,0}^p \|_1 = \begin{cases} A \epsilon \frac{2-N}{p} + O(1), & \text{if } p^2 < N, \\ A |\log(\epsilon)| + O(1), & \text{if } p^2 = N, \end{cases}
\]

\( A \) is a positive constant independent of \( \epsilon. \) Observe that \( u_{\epsilon,y}(x) = \Phi_K(x)U_{\epsilon,y}(x) = \Phi_K(x)U_{\epsilon,0}(x-y). \) By applying \( \frac{U_{\epsilon,0}^p}{\| U_{\epsilon,0}^p \|_1} \) as an approximate identity, we compute the following:

\[
\int \Omega g(x)|u_{\epsilon,y}(x)|^p dx = \int \Omega g(x)\Phi_K(x)U_{\epsilon,0}(x-y)^p dx
\]

\[
= \| U_{\epsilon,0}^p \|_1 \int \Omega (g\Phi_K^p)(x) \frac{U_{\epsilon,0}^p}{\| U_{\epsilon,0}^p \|_1} (x-y) dx
\]

\[
= \| U_{\epsilon,0}^p \|_1 (g\Phi_K^p)(y) + O(1)
\]

\[
\geq \begin{cases} A \epsilon \frac{2-N}{p} g_{\min} + O(1) & \text{if } p^2 < N \\ A |\log(\epsilon)| g_{\min} + O(1) & \text{if } p^2 = N, \end{cases}
\]

\( \square \)

**Proof of Theorem 1.4:** Let \( q \in \mathcal{F}_p(\Omega). \) Proposition 2.9 assures that \( G_p \) is compact on \( D_p(\Omega). \) Fix \( \mu \in (0, \mu_1(g)) \) and choose \( \epsilon > 0. \) By definition of \( C_{g,\mu}(x), \) there exists \( \delta > 0 \) such that for each \( r \in (0, \delta), \) there exists \( u_r \in D_p(\Omega \cap B_r(x)) \) with \( \| u_r \|_{p^*} = 1 \) satisfying

\[
\int \Omega |\nabla u_r|^p - \mu g|u_r|^p dx < C_{g,\mu}(x) + \epsilon.
\]

This yields,

\[
\mathcal{E}_{0,\mu}(\Omega) - \mu \int \Omega g|u_r|^p dx < C_{g,\mu}(x) + \epsilon.
\]

Since \( \mu \in (0, \mu_1(g)), \) it follows that \( (u_r) \) is bounded in \( D_p(\Omega). \) Further, their supports are decreasing to a singleton set \( \{x\}. \) Hence, \( u_r \to 0 \) in \( D_p(\Omega) \) as \( r \to 0, \) and hence the compactness of \( G_p \) implies \( \int \Omega g|u_r|^p dx \to 0 \) as \( r \to 0. \)

Thus, \( \mathcal{E}_{0,\mu}(\Omega) \leq C_{g,\mu}(x), \) for any \( x \in \overline{\Omega}. \) This shows that \( \mathcal{E}_{0,\mu}(\Omega) \leq C^*_{g,\mu}(\Omega). \)

By the similar reasoning we conclude \( \mathcal{E}_{0,\mu}(\Omega) \leq C_{g,\mu}(\infty). \) Thus, in order to show that \( g \) is subcritical in \( \Omega \) and at infinity for the level \( \mu \in (0, \mu_1(g)), \) it is enough to show that \( \mathcal{E}_{g,\mu}(\Omega) < \mathcal{E}_{0,\mu}(\Omega). \) To establish this strict inequality, we
recall the functions \( u_{\epsilon,y} \in C_c^\infty(\Omega) \) (for \( \epsilon > 0 \) small) in Lemma 4.7 and get the following estimate:

\[
Q_g(u_{\epsilon,y}) := \int_\Omega |\nabla u_{\epsilon,y}|^p dx - \mu \int_\Omega g |u_{\epsilon,y}|^p dx - \mu \int_\Omega \frac{g}{|u_{\epsilon,y}|^{p'} dx} \frac{p}{p'} 
\]

\[
\leq \begin{cases} 
\mathcal{E}_{0,\mu}(\Omega) + O(\epsilon^{\frac{N-p}{p}}) - A g_{\min} \epsilon^{p-1}, & \text{if } p^2 < N \\
\mathcal{E}_{0,\mu}(\Omega) + O(\epsilon^{\frac{N-p}{p}}) - A g_{\min} \epsilon^{\frac{N-p}{p}} |\log(\epsilon)|, & \text{if } p^2 = N,
\end{cases}
\]

This implies \( Q_g(u_{\epsilon,y}) < \mathcal{E}_{0,\mu}(\Omega) \) for sufficiently small \( \epsilon \). By taking \( w_\epsilon = \|u_{\epsilon,y}\|_{p^*} \), we have

\[
\mathcal{E}_{g,\mu}(\Omega) \leq \int_\Omega |\nabla w_\epsilon|^p dx - \mu \int_\Omega g |w_\epsilon|^p dx = Q_g(u_{\epsilon,y}) < \mathcal{E}_{0,\mu}(\Omega).
\]

This completes our proof.

In the following remark, we exhibit certain classical spaces that are contained in \( \mathcal{F}_p(\Omega) \).

**Remark 4.8.** For \( p = 2 \) and \( \Omega \) bounded, \( L^r(\Omega) \subseteq \mathcal{F}_p(\Omega) \) with \( r > \frac{N}{2} \) \([40]\), \( r = \frac{N}{2} \) \([2]\). For \( p \in (1, \infty) \) and for general domain \( \Omega \), \( L^{N,d}(\Omega) \subseteq \mathcal{F}_p(\Omega) \) with \( d < \infty \), in \([51]\). Furthermore, a larger space \( C_c^\infty(\Omega) \) in \( L^{N,\infty}(\Omega) \) is contained in \( \mathcal{F}_p(\Omega) \) \([8, \text{for } p = 2]\) and \([5, \text{for } p \in (1, N)]\). For \( g \in L^1_{loc}(\Omega) \), we consider

\[
\tilde{g}(r) = \text{ess sup}\{|g(y)| : |y| = r\}, \quad r > 0,
\]

where the essential supremum is taken with respect to \((N-1)\) dimensional surface measure. Let \( I_p(\Omega) = \{g \in L^1_{loc}(\Omega) : \tilde{g} \in L^1((0, \infty), r^{p-1}dr)\}\). In \([7]\), authors showed that \( I_p(B_1) \subseteq \mathcal{F}_p(B_1) \) for \( p \in (1, N) \).

5. Critical potentials

In this section, we prove Theorems 1.10 and 1.12. First, we observe that for a Hardy potential \( g \in \mathcal{H}_p(\Omega) \) which is not sub-critical in \( \Omega \) or at infinity, one of the following cases occurs:

(i) \( g \) is \( H \)-subcritical in \( \Omega \) as well as at infinity,
(ii) \( g \) is \( H \)-critical in \( \Omega \) but \( H \)-subcritical at infinity,
(iii) \( g \) is \( H \)-critical at infinity but \( H \)-subcritical in \( \Omega \),
(iv) \( g \) is \( H \)-critical in \( \Omega \) as well as at infinity,

for a closed subgroup \( H \) of \( \mathcal{O}(N) \). Theorem 1.8 deals with the case (i), while Theorems 1.10 and 1.12 address the rest of the three cases.

**Proof of Theorem 1.8.** Let \( H, \Omega, \) and \( g \) satisfy 1 and 1. If \( g \) is \( H \)-subcritical in \( \Omega \) and at infinity for level \( \lambda \), then one can repeat similar arguments as in Theorem 1.3 to show that \( \mathcal{E}_{g,\lambda}(\Omega) \) is attained in \( D_p(\Omega)^H \). Further, the principle of symmetric criticality and strong maximum principle can be applied to ensure that (1.1) admits a positive solution.

Next we prove Theorem 1.10.
Proof of Theorem 1.10: Let $H$, $\Omega$, and $g$ satisfy 1 and 1. Let $(u_n)$ be a minimizing sequence of $E_{g,\lambda}^H(\Omega)$ i.e., $u_n \in \mathcal{D}_p(\Omega)^H$ with $\|u_n\|_{p^*} = 1$ and

$$\int_{\Omega} |\nabla u_n|^p - \lambda g|u_n|^p \, dx \to E_{g,\lambda}^H(\Omega),$$

as $n \to \infty$.

Since $\lambda \in (0, \mu_1(g))$, we have $(u_n)$ is bounded in $\mathcal{D}_p(\Omega)^H$. Hence, $u_n \rightharpoonup u$ in $\mathcal{D}_p(\Omega)^H$ (upto a subsequence). As $g$ satisfies 1, we have $\lim_{n \to \infty} \int_{\Omega} g^-|u_n|^p \, dx = \int_{\Omega} g^-|u|^p \, dx$ (Proposition 2.9). Now, we use Corollary 3.5 to $g^+$ and Proposition 4.1 to obtain

$$E_{g,\lambda}^H(\Omega) = \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^p - \lambda g|u_n|^p \, dx$$

$$\geq \int_{\Omega} |\nabla u|^p - \lambda g|u|^p \, dx + C_{g,\lambda}^H(\Omega)p \nu \|u\|_{p^*}^p + C_{g,\lambda}(\infty)p \nu_{\infty}^p$$

$$\geq E_{g,\lambda}^H(\Omega) \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p^*}} + C_{g,\lambda}^H(\Omega)p \nu \|u\|_{p^*}^p + C_{g,\lambda}(\infty)p \nu_{\infty}^p$$

$$\geq E_{g,\lambda}^H(\Omega) \left( \int_{\Omega} |u|^p \, dx + \|u\|_{p^*} + \nu_{\infty} \right)^{\frac{p}{p^*}} = E_{g,\lambda}^H(\Omega) \quad (5.1)$$

Thus, equality occurs in all the above inequalities. As $g$ is $H$-subcritical at infinity for level $\lambda$, by the same arguments as in the proof of Theorem 1.3 we infer that $\nu_{\infty} = 0$. In the view of Corollary 3.5, observe that the equality in (5.1) implies that $E_{g,\lambda}^H(\Omega)p \nu \|u\|_{p^*}^p + \lambda \|\gamma\| = \|\Gamma_{\sum g \cup \Omega}\|$. Thus, by Remark 3.6-(i), we have $E_{g,\lambda}^H(\Omega)p \nu \|u\|_{p^*}^p = \|\zeta\|$, where $\zeta$ is as in Proposition 3.3. Notice that, the equality in (5.1) and Corollary 3.5-(e) implies that

$$\left[ \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{p}{p^*}} + \|u\|_{p^*} \nu_{\infty}^{\frac{p}{p^*}} \right] = \left[ \left( \int_{\Omega} |u|^p \, dx \right) + \|u\| + \nu_{\infty} \right]^{\frac{p}{p^*}} = 1.$$
$g$ is $H$-invariant, in fact, $\mathcal{O}(N)$-invariant, and also $g$ is $H$-subcritical at infinity (as $\mathcal{A}$ is bounded). Therefore, by Theorem 1.10, $\mathcal{E}_g^H(\mathcal{A})$ must be attained.

(ii) Notice that the solution obtained in Theorem 1.10 is always $H$-invariant. Thus, in particular, if $H = \mathcal{O}(N)$, then the solution (if exists) will be radial.

Example 5.2. (i) A potential critical in $\Omega$: Let $\Omega = \Omega_k \times \Omega_{N-k}$ be as in (1.6) and $g(z) = \frac{1}{|y|^p}$, $z = (x, y) \in \Omega$. It is known that $g \in \mathcal{H}_p(\Omega)$ if $N - k \geq 2$ and $p < N - k$, see [9, Theorem 2.1]. We show that $g$ is critical in $\Omega$ for the level $\mu \in (0, \mu_1(g))$. For a fix $\mu \in (0, \mu_1(g))$ and $\epsilon > 0$, there exists $w \in C_c^\infty(\Omega)$ (by density of $C_c^\infty(\Omega)$ in $\mathcal{D}_p(\Omega)$) with $\|w\|_{p^*} = 1$ such that

$$\int_{\Omega} |\nabla w|^p(z) - \mu \frac{|w|^p(z)}{|y|^p} |dz| < \mathcal{E}_g(\Omega) + \epsilon.$$ 

By taking $(\xi, 0) \in \Omega_k \times \Omega_{N-k}$ and

$$w_r(z) = r^{\frac{p-N}{p}} w \left( \frac{x - \xi}{r}, \frac{y}{r} \right),$$

on $\Omega_r := \{(x, y) \in \mathbb{R}^k \times \Omega_{N-k} : ar \leq |x - \xi| \leq br\}$, where $b > a$ is such that $w(x, y) = 0$, $\forall |x| > b$. It is clear that $\|w_r\|_{p^*} = 1$ and $\Omega_r \subseteq \Omega$ for small $r > 0$. Now, using the change of variable $\frac{x - \xi}{r} = x'$ and $\frac{y}{r} = y'$ we obtain

$$\int_{\Omega_r} |\nabla w_r|^p(z) - \mu \frac{|w_r|^p(z)}{|y|^p} |dz| = \int_{\Omega} |\nabla w|^p(z) - \mu \frac{|w|^p(z)}{|y|^p} |dz|.$$ 

This gives $C_{g, \mu}((\xi, 0)) \leq \mathcal{E}_g(\Omega)$. Consequently, $C_{g, \mu}^*(\Omega) \leq \mathcal{E}_g(\Omega)$ holds for all $\mu \in (0, \mu_1(g))$, and the other way inequality indeed holds. Therefore, $g$ is critical in $\Omega$ for level $\mu$.

(ii) A potential $H$-subcritical at infinity in an unbounded domain: Consider the same example as above where $\Omega = \Omega_k \times \Omega_{N-k}$ as in (1.6) with $0 < a, b = \infty$ and $N - k \geq 2, p < N - k$. Then it follows from the similar arguments used in Example 4.5-(iii) that $g$ is critical at infinity for the level $\mu \in (0, \mu_1(g))$. On the other hand, if $b < \infty$, then $\Omega_k$ is bounded, and on top of that, if $\Omega_{N-k}$ is bounded, then $\Omega$ becomes bounded. In that case $g$ is $H$-subcritical at infinity for any subgroup $H$ of $\mathcal{O}(N)$. Next consider that $b < \infty$ and $\Omega_{N-k} = \mathbb{R}^{N-k}$. In this case, we show that $g$ is $H_{N-k}$-subcritical at infinity, where $H_{N-k} = \{Id_k\} \times \mathcal{O}(N-k)$. Fix $\mu \in (0, \mu_1(g))$. For each $R > 0$, by definition of $C_{g, \mu}^{H_{N-k}}(\infty)$, there exists $v_R \in \mathcal{S}_p(\Omega \cap B_R^c)$ such that

$$\int_{\Omega} |\nabla v_R|^p(x, y) - \mu \frac{|v_R|^p(x, y)}{|y|^p} |dz| < C_{g, \mu}^{H_{N-k}}(\infty) + \frac{1}{R}.$$ 

Therefore,

$$\mathcal{E}_{\Omega_{N-k}}^{H_{N-k}}(\Omega) - \mu \int_{\Omega_R} \frac{|v_R|^p(x, y)}{|y|^p} |dz| < C_{g, \mu}^{H_{N-k}}(\infty) + \frac{1}{R},$$

where $\Omega_R = \Omega \cap B_R^c$. Notice that $\mu \in (0, \mu_1(g))$ implies that $(v_R)$ is bounded in $\mathcal{D}_p(\Omega_R)$ and their supports are decreasing to infinity. Hence, $v_R \to 0$ in $\mathcal{D}_p(\Omega)$. 


Now, since \( g \in L^\infty_0(\Omega_R) \), it follows that \( g \in \mathcal{F}_p(\Omega_R) \) (Remark 4.8) and hence
\[
\int_{\Omega_R} \frac{|v_R|^p(x,y)}{|y|^p} \, dz \to 0, \quad \text{as } R \to \infty.
\]
By taking \( R \to \infty \) in (5.2), we obtain \( \mathcal{E}^{H_{N-k}}_{0,\mu}(\Omega) \leq \mathcal{C}^{H_{N-k}}_{g,\mu}(\infty) \). Now, in order to show that \( g \) is \( H_{N-k} \)-subcritical at infinity for level \( \mu \in (0, \mu_1(g)) \), we require to show that \( \mathcal{E}^{H_{N-k}}_{g,\mu}(\Omega) < \mathcal{E}^{H_{N-k}}_{0,\mu}(\Omega) \). To show this, we recall the well-known compact embedding proved by Lions [37, LEMME III.2] that ensures \( \mathcal{D}_p(\Omega)^{H_{N-k}} \hookrightarrow L^p(\Omega) \) is compact. As a consequence \( \mathcal{E}^{H_{N-k}}_{0,\mu}(\Omega) \) is achieved at some \( u \in \mathbb{S}_p(\Omega)^{H_{N-k}} \) and consequently,
\[
\mathcal{E}^{H_{N-k}}_{0,\mu}(\Omega) = \int_{\Omega} |\nabla u|^p \, dz \geq \int_{\Omega} ||\nabla u|^p - \mu g|u|^p|| \, dz \geq \mathcal{E}^{H_{N-k}}_{g,\mu}(\Omega).
\]

(iii) A potential critical at infinity but \( H \)-subcritical at infinity: Let \( \Omega = \Omega_k \times \mathbb{R}^{N-k} \) as in (1.6) with \( 0 < a, b < \infty \) and \( \Omega_{N-k} = \mathbb{R}^{N-k} \). Also, assume that \( N - k \geq 2, p < N - k \). Consider \( g(z) = \frac{1}{|z|^\alpha}; z = (x, y) \in \Omega \). One can repeat the arguments of Example 5.2-(ii) to show that \( g \) is \( H_{N-k} \)-subcritical at infinity for the level \( \mu \in (0, \mu_1(g)) \). Now, we show that \( g \) is critical at infinity. On the contrary, if \( g \) is subcritical at infinity, then it follows from the proof of Theorem 1.10 that \( \mathcal{E}_{g,\mu}(\Omega) \) is achieved. However, this is possible only if \( \Omega = \mathbb{R}^N \) [46, Theorem 2.2]. Hence, \( g \) must be critical at infinity for level \( \mu \).

(iv) A potential \( H \)-subcritical at infinity: Let \( \Omega = \Omega_k \times \mathbb{R}^{N-k} \) be as in (1.6) with \( b < \infty \) and \( N - k \geq 2, p < N - k \). For \( z = (x, y) \in \Omega \), let
\[
g(z) = \frac{1}{|x|^{\alpha}(1 + |y|^2)^{\frac{p-2}{2}}} \quad \alpha \in (0, \frac{p_0}{N}).
\]
Using the same arguments as in the previous example, one can show that \( g \) is \( H_{N-k} \)-subcritical at infinity for the level \( \mu \in (0, \mu_1(g)) \), where \( H_{N-k} = \{Id_k\} \times \mathcal{O}(N - k) \).

Remark 5.3. Let \( \Omega = \Omega_k \times \Omega_{N-k} \) be as in (1.6) with \( 0 < a, b < \infty \) and \( g \) be as in Example 5.2-(ii), (iii). Then \( g \) is \( H_{N-k} \)-invariant and \( H_{N-k} \)-subcritical at infinity for the level \( \mu \in (0, \mu_1(g)) \) for certain range of \( k \), where \( H_{N-k} = \{Id_k\} \times \mathcal{O}(N - k) \). In these cases, Theorem 1.10 can be applied to show that (1.1) admits a positive solution.

Next we prove Theorem 1.12.

Proof of Theorem 1.12: (i) Let \( \Omega = \mathbb{R}^N \) and \( H \) be a closed subgroup of \( \mathcal{O}(N) \). Let \( (u_n) \) be a minimising sequence of \( \mathcal{E}^{H}_{g,\lambda}(\mathbb{R}^N) \) on \( \mathbb{S}_p(\mathbb{R}^N)^H \) i.e., \( u_n \in \mathcal{D}_p(\mathbb{R}^N)^H \) with \( \|u_n\|_{p^*} = 1 \) and
\[
\int_{\mathbb{R}^N} |\nabla u_n|^p - \lambda g|u_n|^p \, dx \to \mathcal{E}^{H}_{g,\lambda}(\mathbb{R}^N), \quad \text{as } n \to \infty.
\]
For each \( u_n \), there exists \( R_n > 0 \) such that
\[
\int_{B_{R_n}} |u_n|^{p^*} \, dx \geq \frac{1}{2}.
\]
We define \( w_n(z) = R_n^{-p} u_n(R_nz) \) on \( \mathbb{R}^N \). Then, \( w_n \in D_p(\mathbb{R}^N)^{H} \) and \( \|w_n\|_{p^*} = 1 \). Also, since \( g \) satisfies 1 for small \( r > 0 \), we have

\[
\mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) \leq \int_{\mathbb{R}^N} \|\nabla w_n\|^p - \lambda g|w_n|^p \, dz \leq \int_{\mathbb{R}^N} \|\nabla u_n\|^p - \lambda g|u_n|^p \, dx \rightarrow \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N), \quad \text{as} \quad n \rightarrow \infty.
\]

\( \lambda \in (0, \mu_1(g)) \) ensures that \( (w_n) \) is bounded in \( D_p(\mathbb{R}^N)^{H} \), and hence \( w_n \rightharpoonup w \) in \( D_p(\mathbb{R}^N)^{H} \). As \( g \) satisfies 1, we have \( \lim_{n \rightarrow \infty} \int_{\Omega} g^-|w_n|^p \, dx = \int_{\Omega} g^-|w|^p \, dx \) (Proposition 2.9). Now, we use Corollary 3.5 to \( g^+ \) and Proposition 4.1 to obtain

\[
\mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \|\nabla w_n\|^p - \lambda g|w_n|^p \, dz
\geq \int_{\mathbb{R}^N} \|\nabla w\|^p - \lambda g|w|^p \, dz + C^H_{g,\lambda}(\mathbb{R}^N)\|\nu\|_{p^*}^p + C^H_{g,\lambda}(\mathbb{R}^N)\|\nu\|_{p^*}^p
\geq \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) \left( \int_{\mathbb{R}^N} |w|^{p^*} \, dz \right)^{\frac{p}{p^*}} + C^H_{g,\lambda}(\mathbb{R}^N)\|\nu\|_{p^*}^p + C^H_{g,\lambda}(\mathbb{R}^N)\|\nu\|_{p^*}^p
\geq \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) \left[ \int_{\mathbb{R}^N} |w|^{p^*} \, dz + \|\nu\| + \nu_{\infty} \right]^{\frac{p}{p^*}} \geq \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) \quad (5.3)
\]

Thus, equality occurs in

\[
\left[ \left( \int_{\mathbb{R}^N} |w|^{p^*} \, dz \right)^{\frac{p}{p^*}} + \|\nu\|_{p^*}^{\frac{p}{p^*}} + \nu_{\infty} \right] = \left[ \int_{\mathbb{R}^N} |w|^{p^*} \, dz + \|\nu\| + \nu_{\infty} \right]^{\frac{p}{p^*}}
\]

Hence, exactly one of \( \|w\|_{p^*}, \|\nu\| \) or \( \nu_{\infty} \) is 1, others are 0. Since

\[
\int_{B_1(0)} |w_n(z)|^{p^*} \, dz = \int_{B_{R_n}(0)} |u_n(x)|^{p^*} \, dx \geq \frac{1}{2}, \quad \forall n \in \mathbb{N},
\]

it follows that \( \nu_{\infty} = 0 \). Since \( g \) is \( H \)-subcritical in \( \mathbb{R}^N \) for level \( \lambda \), we have \( \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) < C^H_{g,\lambda}(\mathbb{R}^N) \). Hence, one can use the arguments in the proof of Theorem 1.3 to conclude \( \|\nu\| = 0 \). Therefore, \( \|w\|_{p^*} = 1 \), and again using the arguments as in Theorem 1.3, we infer that (1.1) admits a positive solution in \( \mathbb{R}^N \).

\( (ii) \) We consider \( \Omega = \mathbb{R}^N \) and \( H \) is a closed subgroup of \( \mathcal{O}(N) \) as in the hypothesis. Let \( (u_n) \) be a minimising sequence of \( \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N) \) on \( S_{p}(\mathbb{R}^N)^{H} \) i.e., \( u \in D_p(\mathbb{R}^N)^{H} \) with \( \|u_n\|_{p^*} = 1 \) and

\[
\int_{\mathbb{R}^N} \|\nabla u_n\|^p - \lambda g|u_n|^p \, dx \rightarrow \mathcal{E}^H_{g,\lambda}(\mathbb{R}^N), \quad \text{as} \quad n \rightarrow \infty.
\]

Now, for each \( n \in \mathbb{N} \), there exists \( R_n > 0 \) such that

\[
\int_{B_{R_n}(0)} |u_n|^{p^*} \, dx = \frac{1}{2}.
\]
We define \( w_n(z) = R_{n-p}^{-1} u_n(R_n z) \) on \( \mathbb{R}^N \). Then, \( w_n \in D_{p}(\mathbb{R}^N)^H \) with \( \| w_n \|_{p^*} = 1 \) and
\[
\int_{B_1(0)} |w_n|^{p^*} \, dz = \frac{1}{2} \tag{5.4}
\]
Further, since \( g \) satisfies \( 1 \) for all \( r > 0 \), we obtain
\[
\mathcal{E}_{g, \lambda}(\mathbb{R}^N) \leq \int_{\mathbb{R}^N} [\| \nabla w_n \|^p - \lambda g |w_n|^p] \, dz \leq \int_{\mathbb{R}^N} [\| \nabla u_n \|^p - \lambda g |u_n|^p] \, dx \rightarrow \mathcal{E}_{g, \lambda}(\mathbb{R}^N), \quad \text{as } n \rightarrow \infty.
\]
Now, \( \lambda \in (0, \mu_1(g)) \) ensures that \( (w_n) \) is bounded in \( D_{p}(\mathbb{R}^N)^H \), and hence \( w_n \rightarrow w \) in \( D_{p}(\mathbb{R}^N)^H \). As \( g \) satisfies \( 1 \), we have \( \lim_{n \rightarrow \infty} \int_{\Omega} g^- |w_n|^p \, dx = \int_{\Omega} g^- |w|^p \, dx \) (Proposition 2.9). Now, we use Corollary 3.5 to \( g^+ \) and Proposition 4.1 to obtain
\[
\mathcal{E}_{g, \lambda}(\mathbb{R}^N) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [\| \nabla w_n \|^p - \lambda g |w_n|^p] \, dz
\]
\[
\geq \mathcal{E}_{g, \lambda}(\mathbb{R}^N) \left( \int_{\mathbb{R}^N} |w|^{p^*} \, dz \right)^{\frac{p}{p^*}} + C_{g, \lambda}(\mathbb{R}^N)||\nu||^{\frac{p}{p^*}} + C_{g, \lambda}(\infty)\nu_\infty^{\frac{p}{p^*}}
\]
\[
\geq \mathcal{E}_{g, \lambda}(\mathbb{R}^N) \left[ \int_{\mathbb{R}^N} |w|^{p^*} \, dz + ||\nu|| + \nu_\infty \right]^{\frac{p}{p^*}} \geq \mathcal{E}_{g, \lambda}(\mathbb{R}^N) \tag{5.5}
\]
Thus, equality occurs in (5.5). Therefore, as we have seen in the proof of Theorem 1.10 that \( \mathcal{E}_{g, \lambda}(\mathbb{R}^N)||\nu||^{\frac{p}{p^*}} = ||\zeta_\lambda|| \), where \( \zeta_\lambda \) is as in Proposition 3.3, and also
\[
\left[ \left( \int_{\mathbb{R}^N} |w|^{p^*} \, dz \right)^{\frac{p}{p^*}} + ||\nu||^{\frac{p}{p^*}} + \nu_\infty\right]^{\frac{p}{p^*}} = \left[ \int_{\mathbb{R}^N} |w|^{p^*} \, dz + ||\nu|| + \nu_\infty \right]^{\frac{p}{p^*}} = 1.
\]
Hence, exactly one of \( ||w||_{2^*}, ||\nu|| \) or \( \nu_\infty \) is 1, others are 0. By (5.4), \( \nu_\infty \leq \frac{1}{2} \) and hence \( \nu_\infty = 0 \). Now, if \( ||\nu|| = 1 \), then \( ||w||_{p^*} = 0 \), and since \( \mathcal{E}_{g, \lambda}(\Omega)||\nu||^{\frac{p}{p^*}} = ||\zeta_\lambda|| \), it follows from Corollary 3.5-(i) that, either \( \nu = 0 \) or it is concentrated on a finite \( H \) orbit. Since only 0 has a finite \( H \)-orbit, it follows that either \( \nu = 0 \) or \( \nu = \delta_0 \). Let \( \nu = \delta_0 \). Choose \( \phi \in C_c^\infty(\mathbb{R}^N) \) with \( 0 \leq \phi \leq 1, \phi = 1 \) on \( B_\frac{1}{2}(0) \) and \( \phi = 0 \) outside \( B_1(0)^c \). Using (5.4) we obtain,
\[
\frac{1}{2} = \lim_{n \rightarrow \infty} \int_{B_1(0)} |w_n|^{p^*} \, dz \geq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |w_n|^{p^*} \phi \, dz = \delta_0(\phi) = 1,
\]
which is a contradiction. Thus, \( \nu = 0 \) and hence, \( ||w||_{p^*} = 1 \). Now, following the arguments as in Theorem 1.3, we infer that (1.1) admits a positive solution in \( \mathbb{R}^N \).

\[\square\]

6. A necessary condition

In this section, we prove a necessary condition for \( g \) so that (1.1) admits a positive solution in entire \( \mathbb{R}^N \). A similar result has been derived in [4, Theorem
[6.1.3] for the existence of solution to the problem:

$$-\Delta_p u = g(x)|u|^{p - 2}u \text{ in } D_p(\mathbb{R}^N).$$

We adapt their ideas for proving Theorem 1.14. For this we need the following regularity result.

**Proposition 6.1.** Let \( g \in C^{\alpha}_{\text{loc}}(\mathbb{R}^N) \) with \( \alpha \in (0, 1) \) and \( u \) be a solution to

$$-\Delta_p u - g|u|^{p - 2}u = |u|^{p^* - 2}u \text{ in } D_p(\mathbb{R}^N).$$

Then \( u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \).

**Proof.** Let \( \Omega \) be any bounded domain in \( \mathbb{R}^N \). Then, by following the arguments of Proposition A.1 of [23] (we also refer to [45, Theorem E.0.20]) we can show that \( u \in L^\infty(\Omega) \). Subsequently, using Tolksdorf’s regularity results [50] for general quasilinear operator, we have \( u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \). \( \square \)

**Proof of Theorem 1.14:** Using the above regularity (Proposition 6.1) we first show that any solution \( u \) of (1.1) satisfies a pointwise identity outside the set where \( \nabla u \) vanishes. For each \( \eta > 0 \), we set \( \Omega_\eta = \{ x \in \mathbb{R}^N : |\nabla u| > \eta \} \). Now since \( -\Delta_p \) is uniformly elliptic on \( \Omega_\eta \) and \( u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \), it follows from standard elliptic regularity theory [24, Section 8.3, page 482] that \( u \in C^{2,\alpha}_{\text{loc}}(\Omega_\eta) \). Thus we obtain the following point-wise identity on \( \Omega_\eta \):

$$-\Delta_p u - \mu g(x)|u|^{p - 2}u = |u|^{p^* - 2}u \text{ a.e in } \Omega_\eta. \tag{6.1}$$

Now we choose a cut-off function \( \zeta \in C^\infty_c(\mathbb{R}) \) with \( 0 \leq \zeta \leq 1 \) such that \( \zeta(t) = 1 \) for \( t \in [0, 1] \) and \( \zeta = 0 \) for \( t \geq 2 \). For each \( n \in \mathbb{N} \), we consider

$$\psi_n(x) = \zeta\left(\frac{|x|^2}{n^2}\right).$$

Then there exists \( C > 0 \) independent of \( n \) such that

$$|\psi_n(x)|, |x||\nabla\psi_n(x)| \leq C,$n \text{ for all } x \in \mathbb{R}^N \text{ and } n \in \mathbb{N}.$$ Now multiplying (6.1) by \( \{x.\nabla u\}\psi_n \),

$$-\Delta_p u\{x.\nabla u\}\psi_n - g(x)|u|^{p - 2}u\{x.\nabla u\}\psi_n = |u|^{p^* - 2}u\{x.\nabla u\}\psi_n \text{ a.e in } \Omega_\eta. \tag{6.2}$$

For convenience, we denote

$$L_n = |\nabla u|^{p - 2}\nabla u\{x.\nabla u\}\psi_n - (x|\nabla u|^p)\psi_n$$

$$K_n = - \left[g(x)|u|^{p - 2}u + |u|^{p^* - 2}u\right]\{x.\nabla u\}\psi_n$$

$$+ \left(1 - \frac{N}{p}\right)|\nabla u|^p\psi_n + \{x.\nabla u\}|\nabla u|^{p - 2}\nabla u.\nabla \psi_n.$$ Using (6.2) and following the estimates in [4, Theorem 6.1.3] we can show that \( \text{div}L_n = K_n \) a.e. in \( \Omega_\eta \), and furthermore

$$\text{div} L_n = K_n \text{ in } \mathbb{R}^N.$$
in distribution sense. Since we have $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ regularity of $u$ (Proposition 6.1), by using weak divergence theorem [21, Lemma A.1], we obtain

$$\int_{B_N} K_n(x)\,dx = 0.$$ 

Furthermore, following the steps of [4, Theorem 6.1.3] we estimate

$$\text{div} \left\{ \left[ g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right] x\psi_n(x) \right\} = N \left[ g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right] \psi_n(x) + \frac{|u|^p}{p} [x,\nabla g(x)]\psi_n(x) + [g(x)|u|^{p-2}u + |u|^{p^*-2}u] \{x,\nabla u\} \psi_n$$

$$+ \left[ g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right] x.\nabla \psi_n(x)$$
a.e. in $\mathbb{R}^N$. Therefore,

$$K_n = N \left[ g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right] \psi_n(x) + \frac{|u|^p}{p} [x,\nabla g(x)]\psi_n(x) + \left[ g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right] x.\nabla \psi_n(x)$$

$$+ \left( 1 - \frac{N}{p} \right) |\nabla u|^p \psi_n + \{x,\nabla u\}|\nabla u|^{p-2}\nabla u.\nabla \psi_n.$$ 

Hence,

$$\int_{B_N} \left[ N \left( g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right) + \frac{|u|^p}{p} [x,\nabla g(x)]\psi_n(x) + \left( 1 - \frac{N}{p} \right) |\nabla u|^p \right] \psi_n \,dx$$

$$+ \int_{B_N} \left\{ x.\nabla |\nabla u|^{p-2}\nabla u + \left[ g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right] x \right\} \nabla \psi_n(x) \,dx = 0.$$ 

Notice that $\psi_n \to 1$ and $\nabla \psi_n \to 0$ as $n \to \infty$. Since each of the above integrals are integrable in entire $\mathbb{R}^N$, we use dominated convergence theorem to obtain

$$\int_{\mathbb{R}^N} \left[ N \left( g(x) \frac{|u|^p}{p} + \frac{|u|^{p^*}}{p^*} \right) + \frac{|u|^p}{p} [x,\nabla g(x)] + \left( 1 - \frac{N}{p} \right) |\nabla u|^p \right] \,dx = 0. \quad (6.3)$$

As $u$ is a solution,

$$\int_{\mathbb{R}^N} |\nabla u|^p \,dx = \int_{\mathbb{R}^N} \left[ g|u|^p + |u|^{p^*} \right] \,dx.$$ 

Substituting this in (6.3),

$$\int_{\mathbb{R}^N} [x.\nabla g(x) + pg(x)]|u|^p \,dx = 0.$$ 

$\Box$
Remark 6.2. Consider the function
\[ g(z) = \frac{1}{(1 + |y|^2)^{\frac{p}{2}}} \]
for \( z = (x, y) \in \mathbb{R}^N \). Then one can verify that \( z \cdot \nabla g(z) + pg(z) > 0 \) in \( \mathbb{R}^N \). Therefore, (1.1) does not have any solution in entire \( \mathbb{R}^N \). However, if we consider the domain \( \Omega = \Omega_k \times \Omega_{N-k} \) as in (1.6) with \( 0 < a < b < \infty \), then following the arguments of Example 5.2-(ii), it can be shown that \( g \) is sub-critical at infinity for the level \( \mu \in (0, \mu_1(g)) \) if \( N - k \geq 2, p < N - k \). Thus, it follows from Theorem 1.10 that (1.1) admits a positive solution in \( \Omega \).

Remark 6.3. Instead of \( \mu_1(g) \), one may also consider
\[ \tilde{\mu}_1(g) := \inf \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} g|u|^p dx = 1 \right\}, \]
\[ \hat{\mu}_1(g) := -\inf \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} g|u|^p dx = -1 \right\}. \]

It is easy to verify that \( \mu_1(g) \leq \tilde{\mu}_1(g) \) and also \( \mu_1(g) \leq -\hat{\mu}_1(g) \). If \( g \) is not sign-changing, then one can deduce the existence and non-existence of the solutions for (1.1) using the results of this article. For example, for \( g \geq 0 \), \( \tilde{\mu}_1(g) = \mu_1(g) \), and hence, we have the existence of solutions for (1.1) when \( \mu \in (0, \tilde{\mu}_1(g)) \), and the non-existence for \( \mu \leq 0 \) (due to the Pohozaev identity). On the other hand, for \( g = -g_1 \leq 0 \), \( -\hat{\mu}_1(g) = \mu_1(g_1) \). Thus, in this case studying the existence of solution for (1.1) when \( \mu \in (\hat{\mu}_1(g), 0) \) is equivalent to study the same for (1.1) with \( g = g_1 \) and \( \mu \in (0, \mu_1(g_1)) \). Also, we have the non-existence if \( \mu \geq 0 \) (due to the Pohozaev identity).

We conclude this article with the following open question.

**Open problem** For a sign changing \( g \), it will be interesting to study the analogous existence results as in Theorems 1.3, 1.8, 1.10, & 1.12 for \( \mu \in [\mu_1(g), \tilde{\mu}_1(g)) \) and \( \mu \in (\hat{\mu}_1(g), 0) \). Recall that, for \( \mu \in (0, \mu_1(g)) \), the quasi norm \( \| \cdot \|_{D_p,\mu} \) is equivalent to the norm \( \| \cdot \|_{D_p} \) in \( D_p(\Omega) \) and this fact has been used extensively to prove the theorems mentioned above. Unfortunately, we do not have this benefit while dealing with the cases \( \mu \in [\mu_1(g), \tilde{\mu}_1(g)) \) and \( \mu \in (\hat{\mu}_1(g), 0) \).

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