THE HOMOTOPY TYPE OF THE INDEPENDENCE COMPLEX OF GRAPHS
WITH NO INDUCED CYCLES OF LENGTH DIVISIBLE BY 3

JINHA KIM

ABSTRACT. We prove Engström’s conjecture that the independence complex of graphs with
no induced cycle of length divisible by 3 is either contractible or homotopy equivalent to a
sphere. Our result strengthens a result by Zhang and Wu, verifying a conjecture of Kalai and
Meshulam which states that the total Betti number of the independence complex of such a
graph is at most 1. A weaker conjecture was proved earlier by Chudnovsky, Scott, Seymour,
and Spirkl, who showed that in such a graph, the number of independent sets of even size
minus the number of independent sets of odd size has values 0, 1, or −1.

1. INTRODUCTION

We assume all graphs are finite and contain no loops and no multiple edges. A subgraph
of a graph $G$ is an induced subgraph if it can be obtained from $G$ by deleting vertices and all
edges incident with those vertices. An induced cycle is an induced subgraph that is a cycle.
An independent set is a set of pairwise non-adjacent vertices. The independence complex of
a graph $G$ is the abstract simplicial complex $I(G)$ on the vertex set $V(G)$ whose faces are the
independent sets of $G$. A graph is ternary if it contains no induced cycle of length divisible
by 3.

Here is our main theorem.

Theorem 1.1. A graph is ternary if and only if every induced subgraph has the independence
complex that is contractible or homotopy equivalent to a sphere.

We can easily deduce the converse of Theorem 1.1 as follows. Kozlov [6, Proposition 5.2]
showed that the independence complex of a cycle is not homotopy equivalent a sphere if and
only if the cycle has length divisible by 3. More precisely, if $C_\ell$ is a cycle of length $\ell \geq 3$,
then the homotopy type of the independence complex is given by

$$I(C_\ell) \simeq \begin{cases} S^k \lor S^k & \text{if } \ell = 3k + 3, \\ S^k & \text{if } \ell = 3k + 2 \text{ or } 3k + 4, \end{cases}$$

Therefore, if every induced subgraph of a graph $G$ has the independence complex that is
contractible or homotopy equivalent to a sphere, then $G$ does not contain an induced cycle of
length divisible by 3.

Our main result is motivated from a conjecture by Kalai and Meshulam. For a simplicial
complex $K$, let $\tilde{H}_i(K)$ be the $i$-th reduced homology group of $K$ over $\mathbb{Z}$ and $\tilde{\beta}_i(K)$
the $i$-th reduced Betti number of $K$, which is the rank of $\tilde{H}_i(K)$. Let $\beta(K)$ be the total Betti

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number, that is, \( \beta(K) = \sum_{i \geq 0} \tilde{\beta}_i(K) \). Decades ago, Kalai and Meshulam [5] conjectured that the independence complex of every ternary graph has total Betti number at most 1. This conjecture was recently proved by Zhang and Wu [7].

In general, for a graph \( G \), \( \beta(I(G)) \leq 1 \) does not imply that \( I(G) \) is contractible or homotopy equivalent to a sphere. To see why, observe that the barycentric subdivision of any cell complex can be expressed as the independence complex of some graph. For example, considering the real projective plane \( \mathbb{RP}^2 \), one can find a graph whose independence complex is homotopy equivalent to \( \mathbb{RP}^2 \), which has the total Betti number 0, but is neither contractible nor homotopy equivalent to a sphere.

Our result, as well as the result by Zhang and Wu, is a generalization of a result about the reduced Euler characteristic of the independence complex of ternary graphs. Given a simplicial complex \( K \), the reduced Euler characteristic of \( K \) is defined as

\[
\chi(K) = \sum_{i \geq 0} (-1)^i \tilde{\beta}_i(K).
\]

It is a well-known fact in algebraic topology that \( \chi(K) = \sum_{A \subseteq K} (-1)^{|A|} \) (see [4]). Therefore, for a graph \( G \), \( |\chi(I(G))| \) is the difference between the number of independent sets of \( G \) of even size and the number of those of odd size. Kalai and Meshulam [5] also posed a weaker conjecture that a graph \( G \) is ternary if and only if \( |\chi(I(H))| \leq 1 \) for every induced subgraph \( H \), and this conjecture was proved by Chudnovsky, Scott, Seymour and Spirkl [1].

**Theorem 1.1.** [1] A graph \( G \) is ternary if and only if \( |\chi(I(H))| \leq 1 \) for every induced subgraph \( H \).

Earlier, Gauthier [3] proved a special case of the conjecture: if a graph \( G \) contains no (not necessarily induced) cycles of length divisible by 3, then \( |\chi(I(G))| \leq 1 \). By extending the work of Gauthier, Engström [2] showed that the independence complex of such a graph is either contractible or homotopy equivalent to a sphere. Engström conjectured that his result can be extended to ternary graphs. Theorem 1.1 confirms Engström’s conjecture.

Here is an overview of the paper. In Section 2, we discuss some topological background on the homotopy type of simplicial complexes, including useful lemmas about the independence complexes of graphs. The proof of Theorem 1.1 will be presented in Section 3.

## 2. Preliminaries

In this section, we introduce some topological background (for details, see [4]) and useful lemmas to determine the homotopy type of independence complexes.

For a graph \( G \) and \( v \in V(G) \), let \( N(v) := \{ u \in V : uv \in E \} \) and \( N[v] := N(v) \cup \{v\} \). For \( W \subseteq V(G) \), let \( N[W] := \cup_{w \in W} N[w] \).

### 2.1. Mayer-Vietoris Sequences.

Let \( K, A \) and \( B \) be simplicial complexes such that \( K = A \cup B \). The **Mayer-Vietoris sequence** of the triple \((K, A, B)\) is the following long exact sequence of the homology groups of \( K, A \) and \( A \cap B \):

\[
\cdots \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(K) \to \tilde{H}_{i-1}(A \cap B) \to \cdots
\]

The following are basic observations about exact sequences.

- If \( 0 \to A \to B \to 0 \) is exact, then \( A \cong B \).
• If $0 \to A \to B \to C \to 0$ is exact, then $\text{rk}(B) = \text{rk}(A) + \text{rk}(C)$ where $A, B, C$ are finitely generated $\mathbb{Z}$-modules.\footnote{This was left as an exercise in p.146 of [4]. The proof can be found, for example, in [8].}

Now let $G$ be a graph on $V$. For each $v \in V$, observe that every independent set of $G$ containing $v$ is contained in $G - N(v)$. This implies

$$I(G) = I(G - v) \cup I(G - N(v)).$$

Note that $I(G - N(v))$ is a cone with apex $v$, thus it is contractible. Finally, we observe

$$I(G - v) \cap I(G - N(v)) = I(G - N[v]).$$

Then, by applying the Mayer-Vietoris sequence to the triple $(I(G), I(G - v), I(G - N(v)))$, we obtain the following long exact sequence:

(2)

$$\cdots \to \tilde{H}_i(I(G - N[v])) \to \tilde{H}_i(I(G - v)) \to \tilde{H}_i(I(G)) \to \tilde{H}_{i-1}(I(G - N[v])) \to \cdots.$$ 

For disjoint subsets $X$ and $Y$ of $V$ such that $X$ is independent in $G$, let $G(X|Y)$ be the subgraph of $G$ induced by $V - N[X] - Y$. If $v \notin N[X] \cup Y$, then we obtain the following exact sequence by replacing $G$ with $G - N[X] - Y$ in (2):

(3)

$$\cdots \to \tilde{H}_i(I(G(X \cup \{v\}|Y))) \to \tilde{H}_i(I(G(X|Y \cup \{v\}))) \to \tilde{H}_i(I(G(X|Y))) \to \cdots.$$ 

Recalling that $\tilde{H}_i(S^k) = 0$ if $i \neq k$ and $\tilde{H}_k(S^k) \simeq \mathbb{Z}$, we can prove the following lemma.

**Lemma 2.1.** Let $A$, $B$ and $C$ be simplicial complexes such that the following sequence is exact:

$$\cdots \to \tilde{H}_i(A) \to \tilde{H}_i(B) \to \tilde{H}_i(C) \to \tilde{H}_{i-1}(A) \to \cdots.$$ 

Suppose $A \simeq S^k$ and $B \simeq S^\ell$ for some non-negative integers $k$ and $\ell$. Then the following hold.

(i) If $k > \ell$, then $\tilde{\beta}_k+1(C) = \tilde{\beta}_\ell(C) = 1$.

(ii) If $k = \ell$, then either $\tilde{\beta}_i(C) = 0$ for all non-negative integer $i$ or both $\tilde{H}_{k+1}(C) \neq 0$ and $\tilde{H}_k(C) \neq 0$.

**Proof.** To prove (i), assume $k > \ell$. Since $\tilde{H}_\ell(A) = \tilde{H}_{\ell-1}(A) = 0$, we have $\tilde{H}_\ell(C) \simeq \tilde{H}_\ell(B)$. Similarly, since $\tilde{H}_{k+1}(B) = \tilde{H}_k(B) = 0$, we obtain $\tilde{H}_{k+1}(C) \simeq \tilde{H}_k(A)$. Therefore, $\tilde{\beta}_{k+1}(C) = \tilde{\beta}_k(A) = 1$ and $\tilde{\beta}_\ell(C) = \tilde{\beta}_\ell(B) = 1$.

Now to prove (ii), suppose $k = \ell$. Since $\tilde{H}_{i-1}(A) = \tilde{H}_i(B) = 0$ for $i \neq k, k + 1$, we have $\tilde{H}_i(C) = 0$ for $i \neq k, k + 1$. It is sufficient to show that if one of $\tilde{H}_k(C)$ and $\tilde{H}_{k+1}(C)$ is the trivial group, then the other has rank 0. Suppose $\tilde{H}_{k+1}(C) = 0$. Then we have a short exact sequence

$$0 \to \tilde{H}_k(A) \to \tilde{H}_k(B) \to \tilde{H}_k(C) \to 0.$$ 

This implies $\tilde{\beta}_k(B) = \tilde{\beta}_k(A) + \tilde{\beta}_k(C)$. Since $\tilde{\beta}_k(A) = \tilde{\beta}_k(B) = 1$ by the assumption, we have $\tilde{\beta}_k(C) = 0$. Applying a similar argument, one can show that $\tilde{\beta}_{k+1}(C) = 0$ if $\tilde{H}_k(C) = 0$. \hfill $\Box$
2.2. Homotopy type theory. Let $A$ and $B$ be two topological spaces.

- $A$ and $B$ are homotopy equivalent if there are continuous maps $f : A \to B$ and $g : B \to A$ such that $g \circ f \simeq \text{id}_A$ and $f \circ g \simeq \text{id}_B$, where $\text{id}_X$ is the identity map on $X$. We write $A \simeq B$ if $A$ and $B$ are homotopy equivalent. In particular, if $A$ is contractible, i.e. $A$ is homotopy equivalent to a point, then we write $A \simeq \ast$.

- The wedge sum of $A$ and $B$ is the space obtained by taking the disjoint union of $A$ and $B$ and identifying a point of $A$ and a point of $B$. We denote the wedge sum of $A$ and $B$ by $A \vee B$.

- Let $\sim$ be an equivalence relation on $A$. Then we denote the quotient space of $A$ under $\sim$ by $A/\sim$. Let $B \subset A$. Then we define $A/B$ as the quotient space $A/\sim$ where for all $a \neq b$ in $A$, $a \sim b$ if and only if $a, b \in B$.

- The suspension of $A$ is the quotient space
  \[ \Sigma A := A \times [0, 1]/\sim \]
  where for all $(a, s) \neq (b, t)$ in $A \times [0, 1]$, $(a, s) \sim (b, t)$ if and only if either $s = t = 0$ or $s = t = 1$.

Note that if $S^n$ is the $n$-dimensional sphere, then $\Sigma S^n \simeq S^{n+1}$.

Now let $K$, $K_1$ and $K_2$ be simplicial complexes where $K_1 \cap K_2 \neq \emptyset$, and let $L \neq \emptyset$ be a subcomplex of $K$. Then,

(A) If $K = K_1 \cup K_2$, then $K/K_2 \simeq K_1/(K_1 \cap K_2)$.

(B) Suppose the inclusion map $L \hookrightarrow K$ is homotopic to a constant map $c : L \to K$, that is, $L$ is contractible in $K$. Then $K/L \simeq K \vee \Sigma L$. In particular, $K/L \simeq \Sigma L$ when $K$ is contractible, and $K/L \simeq K$ when $L$ is contractible.

By applying (B), we can deduce the following well-known statement:

**Lemma 2.2.** Let $X$ be a simplicial complex, and $Y$ be a subcomplex of $X$. If $X \simeq S^k$ and $Y \simeq S^\ell$ for some non-negative integers $k$ and $\ell$ with $\ell < k$, then $X/Y$ is homotopic to $S^k \vee S^\ell$.

By a similar argument as in Section 2.1, we obtain the following lemma about homotopy equivalence of independence complexes.

**Lemma 2.3.** Let $G$ be a graph and $v$ a vertex of $G$. If $X$ and $Y$ are disjoint subsets of $V(G)$ such that $X$ is independent, $N[X] \cup Y$ does not contain $v$, and $N[X] \cup N[v] \cup Y \neq V(G)$, then
\[ I(G(X|Y)) \simeq I(G(X|Y \cup \{v\}))/I(G(X \cup \{v\}|Y)). \]

**Proof.** Observe that
\[ I(G(X|Y)) = I(G(X|Y \cup \{v\}) \cup I(G(X|Y) - N(v)) \]
and
\[ I(G(X|Y \cup \{v\}) \cap I(G(X|Y) - N(v)) = I(G(X \cup \{v\}|Y)). \]

Applying (A), we obtain
\[ I(G(X|Y))/I(G(X|Y) - N(v)) \simeq I(G(X|Y \cup \{v\}))/I(G(X \cup \{v\}|Y)). \]
Since $I(G(X|Y) - N(v)) \simeq *$, by (B), we have

$$I(G(X|Y)/I(G(X|Y) - N(v)) \simeq I(G(X|Y)).$$

Therefore,

$$I(G(X|Y)) \simeq I(G(X|Y \cup \{v\})/I(G(X \cup \{v\}|Y)),$$

as required.

\[ \square \]

3. Proof of Theorem 1.1

In this section, we prove the main result. By (1), it is sufficient to show the following.

**Theorem 3.1.** Let $G$ be a ternary graph. Then $I(G)$ is either contractible or homotopy equivalent to a sphere.

To prove Theorem 3.1 by contradiction, take a counter-example $G$ on $V$ which is minimal in the following sense: $I(G)$ is neither contractible nor homotopy equivalent to a sphere, but $I(H)$ is either contractible or homotopy equivalent to a sphere for every proper induced subgraph $H$ of $G$.

Let $X$ and $Y$ be disjoint vertex subsets of $G$. We define $d(X|Y)$ as the following:

$$d(X|Y) = \begin{cases} 
    d & \text{if } X \text{ is independent and } I(G(X|Y)) \simeq S^d, \\
    * & \text{otherwise.}
\end{cases}$$

If a graph $G$ has no vertex, then we write $I(G) \simeq S^{-1}$. Note that if $X$ is an independent set, then $d(X|Y) = *$ implies that $I(G(X|Y))$ is contractible.

We need to prove two lemmas. The first lemma describes all possible types of the triples of the form $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\}))$ under certain conditions, and the second lemma shows that there is a universal constant $k$ such that $d(\emptyset|v) = d(v|\emptyset) = k$ for all $v$.

**Lemma 3.2.** Let $X$ and $Y$ be vertex subsets of $G$ such that $X \cup Y \neq \emptyset$ and $X \cap Y = \emptyset$. For every vertex $v \not\in X \cup Y$, the triple $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\}))$ equals to one of the following:

$$(*,*,*), (k,*,*), (*,k,k), (k+1,k,*)$$

for some integer $k \geq -1$.

**Proof.** If $X \cup Y = V$, then there is nothing to prove, so we assume $X \cup Y \neq V$. If $X$ is not an independent set, then we have $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\})) = (*,*,*)$. Thus we assume that $X$ is an independent set.

If $v \in N(X)$, then we have $d(X \cup \{v\}|Y) = *$ since $X \cup \{v\}$ is not independent, and we have $G(X|Y) = G(X|Y \cup \{v\})$. Thus it must be $d(X \cup \{v\}|Y) = *$ and $d(X|Y) = d(X|Y \cup \{v\})$, implying that $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\}))$ is either $(*,*,*)$ or $(k,*,k)$ for some integer $k \geq -1$.

Now assume $v \not\in N(X)$. There are two cases.

\[ \text{Note that if } \beta_d(I(G(X|Y))) \neq 0 \text{ for some } d, \text{ then } I(G(X|Y)) \simeq S^d \text{ by the minimality of } G, \text{ and hence } d(X|Y) = d. \text{ In this sense, Lemma 3.2 is a natural analogue of [7, Lemma 3.1]. We include a proof for the completeness of the paper.} \]
Claim 3.3. If $\beta(I(G)) \geq 2$, then there is an integer $k \geq 0$ such that $\tilde{\beta}_k(I(G)) \geq 2$ and $\tilde{\beta}_i(I(G)) = 0$ for all $i \neq k$. 

Note that Claim 3.3 is a weaker version of [7, Claim 3.3]. The proof can be found in Appendix A.
If \( \beta(I(G)) \geq 2 \), then we have \(|\chi(I(G))| \geq 2 \) by Claim 3.3. Then, Theorem 1.2 implies that \( G \) is not a ternary graph, which is a contradiction. Therefore, we have \( \beta(I(G)) \leq 1 \).

**Lemma 3.4.** There is a non-negative integer \( k \) such that \( d(\emptyset|v) = d(v|\emptyset) = k \) for all \( v \in V \).

**Proof.** Since \( G \) is a ternary graph, if \( N[v] = V \) for some \( v \in V \), we have \( I(G) \simeq S^0 \), which is a contradiction to the assumption on \( G \). Thus, we may assume \( N[v] \neq V \) for all \( v \in V \). Then by Lemma 2.3, for any vertex \( v \in V \), we have

\[
I(G) \simeq I(G(\emptyset|v)) / I(G(v|\emptyset)).
\]

Note that each of \( I(G(\emptyset|v)) \) and \( I(G(v|\emptyset)) \) is either contractible or homotopy equivalent to a sphere. By (B),

- if \( I(G(v|\emptyset)) \simeq * \), then \( I(G) \simeq I(G(\emptyset|v)) \), and
- if \( I(G(\emptyset|v)) \simeq * \), then \( I(G) \simeq \Sigma I(G(v|\emptyset)) \).

In both cases, it is clear that \( I(G) \) is either contractible or homotopy equivalent to a sphere. Thus we may assume both \( I(G(\emptyset|v)) \) and \( I(G(v|\emptyset)) \) are homotopy equivalent to spheres.

Assume \( d(v|\emptyset) = \ell \) and \( d(\emptyset|v) = k \) for some non-negative integers \( k \) and \( \ell \). If \( k > \ell \), then \( I(G) \simeq S^k \lor S^{\ell+1} \) by Lemma 2.2, which is a contradiction to \( \beta(I(G)) \leq 1 \). If \( k < \ell \), then by Lemma 2.1 (i), we have \( \beta_k(I(G)) = \beta_{\ell+1}(I(G)) = 1 \), which implies \( \beta(I(G)) \geq 2 \). This is again a contradiction to \( \beta(I(G)) \leq 1 \). Thus we conclude that \( k = \ell \).

Now suppose there exist \( u, v \in V \) such that \( d(u|\emptyset) = d(\emptyset|u) = p \) and \( d(v|\emptyset) = d(\emptyset|v) = q \) for two non-negative integers \( p, q \) with \( p < q \). By Lemma 3.2, we have

\[
(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) \text{ is either } (q, *, q) \text{ or } (q, q-1, *).
\]

If \( d(u, v|\emptyset) = q - 1 \), then

\[
(d(u|\emptyset), d(u, v|\emptyset), d(u|v)) = (p, q - 1, d(u|v))
\]

which is possible only when \( p = q \) by Lemma 3.2. Thus we obtain

\[
(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (q, *, q).
\]

On the other hand, Lemma 3.2 implies that it must be

\[
(d(u|\emptyset), d(u, v|\emptyset), d(u|v)) = (p, *, p).
\]

Combining the above information, we have

\[
(d(\emptyset|u), d(v|u), d(\emptyset|u, v)) = (p, q, d(\emptyset|u, v)).
\]

However, Lemma 3.2 implies \( q = p - 1 \), which is a contradiction. \( \square \)

Now we are ready to complete the proof of Theorem 3.1. By Lemma 3.4, there is a non-negative integer \( k \) such that \( d(v|\emptyset) = d(\emptyset|v) = k \) for all \( v \in V \).

We claim \( d(u, v|\emptyset) = k - 1 \) for any two distinct vertices \( u, v \) of \( G \). By Lemma 3.2,

\[
(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) \text{ is either } (k, *, k) \text{ or } (k, k - 1, *)
\]

and

\[
(d(\emptyset|u), d(v|u), d(\emptyset|u, v)) \text{ is either } (k, *, k) \text{ or } (k, k - 1, *).
\]

Then \( d(v|u) \) must be *, and hence we obtain

\[
(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (k, k - 1, *).
\]
Since \( d(u, v|\emptyset) = k - 1 \neq \ast \), we obtain that \( \{u, v\} \) is an independent set of \( G \). Since this holds for every pair of two distinct vertices \( u \) and \( v \), we conclude that the whole vertex set \( V \) is an independent set of \( G \). Thus, \( I(G) \) is contractible, which is a contradiction to the assumption on \( G \).

Therefore, if \( G \) is ternary, then \( I(G) \) is either contractible or homotopy equivalent to a sphere.

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APPENDIX A. PROOF OF CLAIM 3.3

As it was mentioned in Section 3, Claim 3.3 follows from [7, Claim 3.3]. Here, we give a proof of Claim 3.3 for the completeness of the paper. We need the following lemma about an exact sequence of homology groups.

Lemma A.1. Let \( A, B \) and \( C \) be simplicial complexes such that the following sequence is exact:
\[
\cdots \to \tilde{H}_i(A) \to \tilde{H}_i(B) \to \tilde{H}_i(C) \to \tilde{H}_{i-1}(A) \to \cdots.
\]
Suppose that each of \( A \) and \( B \) is either contractible or homotopy equivalent to a sphere. If \( \tilde{\beta}_k(C) > 0 \) and \( \tilde{\beta}_\ell(C) > 0 \) for some distinct non-negative integers \( k, \ell \), then either \( A \simeq S^{k-1} \) and \( B \simeq S^\ell \) or \( A \simeq S^\ell \) and \( B \simeq S^k \).

Proof. Since \( \tilde{\beta}_k(C) \neq 0 \), we know that either \( \tilde{H}_{k-1}(A) \neq 0 \) or \( \tilde{H}_k(B) \neq 0 \). Similarly, since \( \tilde{\beta}_\ell(C) \neq 0 \), we have either \( \tilde{H}_{\ell-1}(A) \neq 0 \) or \( \tilde{H}_\ell(B) \neq 0 \). Suppose \( \tilde{H}_{k-1}(A) \neq 0 \). Then we have \( A \simeq S^{k-1} \), and hence we obtain \( \tilde{H}_{\ell-1}(A) = 0 \). This implies \( \tilde{H}_\ell(B) \neq 0 \). Thus we can conclude that \( A \simeq S^{k-1} \) and \( B \simeq S^\ell \). By a similar argument, we can obtain \( A \simeq S^{\ell-1} \) and \( B \simeq S^k \) if \( \tilde{H}_k(B) \neq 0 \). \( \square \)

Now we are ready to prove Claim 3.3.
Claim 3.3. If $\beta(I(G)) \geq 2$, then there is an integer $k \geq 0$ such that $\bar{\beta}_k(I(G)) \geq 2$ and $\bar{\beta}_i(I(G)) = 0$ for all $i \neq k$.

Proof. To show by a contradiction, assume that $\beta_k(I(G)) > 0$ and $\beta_\ell(I(G)) > 0$ for some non-negative integers $k$ and $\ell$ with $k < \ell$. Take a vertex $v \in V(G)$. By applying Lemma A.1 to (2), we have either $d(v|\emptyset) = k - 1$ and $d(\emptyset|v) = \ell$ or $d(v|\emptyset) = \ell - 1$ and $d(\emptyset|v) = k$. Thus we can partition the vertex set $V(G)$ into two parts $V_1$ and $V_2$, where

$$V_1 = \{v \in V(G) : d(v|\emptyset) = k - 1\},$$

$$V_2 = \{v \in V(G) : d(v|\emptyset) = \ell - 1\}.$$

First, we claim that $V_2$ is an independent set. It is sufficient to show that any two vertices in $V_2$ are not adjacent. Take $u, v \in V_2$. By Lemma 3.2,

$$(d(u|\emptyset), d(u, v|\emptyset), d(u|v))$$

is either $(\ell - 1, \ell - 2, *)$ or $(\ell - 1, *, \ell - 1)$ and

$$(d(\emptyset|v), d(u|v), d(u|\emptyset), d(u|v))$$

is either $(k, k - 1, *)$ or $(k, *, k)$.

If $d(u|v) \neq *$, then $k - 1 = d(u|v) = \ell - 1$, which is a contradiction to $k < \ell$. Thus we have

$$(d(u|\emptyset), d(u, v|\emptyset), d(u|v)) = (\ell - 1, \ell - 2, *).$$

Then $d(u, v|\emptyset) = \ell - 2$ implies that $u$ and $v$ are not adjacent, as required.

Now, if $V_1 = \emptyset$, then $I(G)$ is a simplex on $V_2 \neq \emptyset$, which is contractible. This is a contradiction to the assumption. Therefore, we may assume $V_1 \neq \emptyset$.

Next, we claim that for any two disjoint subsets $X$ and $Y$ of $V_1$ such that $X \cup Y \neq \emptyset$,

$$d(X|Y) = \begin{cases} k - |X| & \text{if } |Y| = 0, \\ * & \text{if } |X|, |Y| > 0, \\ \ell & \text{if } |X| = 0. \end{cases} \tag{4}$$

We prove by induction on $|X| + |Y|$. If $|X| + |Y| = 1$, then it is obvious from the definition of $V_1$. Now suppose

(i) (4) holds for any two disjoint subsets $X, Y$ of $V_1$ such that $|X| + |Y| = m$ for some positive integer $m < |V_1|$.

Take $A \subseteq V_1$ with $|A| = m + 1$. By the induction hypothesis (i), we have $d(\emptyset|A \setminus \{a\}) = \ell$ for all $a \in A$. Hence by Lemma 3.2, for all $a \in A$, we know that

$$(d(\emptyset|A \setminus \{a\}), d(\{a\}|A \setminus \{a\}), d(\emptyset|A))$$

is either $(\ell, \ell - 1, *)$ or $(\ell, *, \ell)$.

Now, for any partition $A = A_1 \cup A_2$ such that $|A_1|, |A_2| > 0$, we claim that

$$d(A_1|A_2) = \begin{cases} \ell - 1 & \text{if } d(\emptyset|A) = *, \\ * & \text{if } d(\emptyset|A) = \ell. \end{cases} \tag{5}$$

This can be shown by induction on $|A_1|$. If $|A_1| = 1$, then the statement is true by the above argument. Suppose

(ii) (5) holds for any partition $A = A_1 \cup A_2$ with $|A_1| = n$ for some positive integer $n < m$.
Take a partition $A = A_1 \cup A_2$ with $|A_1| = n + 1 \leq m$. Then for any $a \in A_1$, since $|A_1 \setminus \{a\}| = n > 0$ and $|A_2| = |A| - |A_1| = m - n > 0$, we have $d(A_1 \setminus \{a\}|A_2) = \ast$ by the induction hypothesis (i). By Lemma 3.2, this implies $d(A_1|A_2) = d(A_1 \setminus \{a\}|A_2 \cup \{a\})$. By the induction hypothesis (ii), we obtain

$$d(A_1|A_2) = d(A_1 \setminus \{a\}|A_2 \cup \{a\}) = \begin{cases} \ell - 1 & \text{if } d(\emptyset|A) = \ast, \\ \ast & \text{if } d(\emptyset|A) = \ell. \end{cases}$$

This shows that (5) holds for any partition $A = A_1 \cup A_2$ such that $|A_1|, |A_2| > 0$.

Now take $a \in A$. By the induction hypothesis (i), we have $d(A \setminus \{a\}|\emptyset) = k - |A| + 1$. Then, by Lemma 3.2, $(d(A \setminus \{a\}|\emptyset), d(A|\emptyset), d(A \setminus \{a\}|\{a\}))$ equals to either $(k - |A| + 1, k - |A|, \ast)$ or $(k - |A| + 1, \ast, k - |A| + 1)$. Thus $d(A \setminus \{a\}|\{a\})$ must be either $\ast$ or $k - |A| + 1$.

On the other hand, (5) implies that

$$d(A \setminus \{a\}|\{a\}) = \begin{cases} \ell - 1 & \text{if } d(\emptyset|A) = \ast, \\ \ast & \text{if } d(\emptyset|A) = \ell. \end{cases}$$

If $d(A \setminus \{a\}|\{a\}) \neq \ast$, then $k - |A| + 1 = d(A \setminus \{a\}|\{a\}) = \ell - 1$, which is impossible since $k - |A| + 1 = k - m \leq k - 1 < \ell - 1$. Thus we must have $d(A \setminus \{a\}|\{a\}) = \ast$. This implies $d(A|\emptyset) = k - |A|$ and $d(\emptyset|A) = \ell$. In addition, by (5), $d(\emptyset|A) = \ell$ implies $d(A_1|A_2) = \ast$ for any partition $A = A_1 \cup A_2$ with $|A_1|, |A_2| > 0$. This shows that (4) holds for any two disjoint subsets $X$ and $Y$ of $V_1$ such that $X \cup Y \neq \emptyset$.

Recall that $V_2$ is an independent set. This implies that $I(G(\emptyset|V_1))$ is a simplex on $V_2$. Thus,

$$d(\emptyset|V_1) = \begin{cases} \ast & \text{if } V_2 \neq \emptyset, \\ -1 & \text{if } V_2 = \emptyset. \end{cases}$$

On the other hand, we have $d(\emptyset|V_1) = \ell$ by (4). This is a contradiction, since $\ell$ is a non-negative integer.

Therefore, there exists a non-negative integer $k$ such that $\tilde{\beta}_k(I(G)) = \beta(I(G)) \geq 2$ and $\tilde{\beta}_i(I(G)) = 0$ for all $i \neq k$. This completes the proof. \qed

**Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, Republic of Korea**

*Email address: jinhakim@ibs.re.kr*