Equations and Integrals of Motion in Discrete Integrable $A_{k-1}$ Algebra Models

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Abstract

We study integrals of motion for Hirota bilinear difference equation that is satisfied by the eigenvalues of the transfer-matrix. The combinations of the eigenvalues of the transfer-matrix are found, which are integrals of motion for integrable discrete models for the $A_{k-1}$ algebra with zero and quasiperiodic boundary conditions. Discrete analogues of the equations of motion for the Bullough-Dodd model and non-Abelian generalization of Liouville model are obtained.

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1. Introduction

Thermodynamical Bethe-anzatz (TBA) \cite{1} is one of the main methods of studying statistical properties of integrable low-dimensional systems. Its formulation yields the functional equations \cite{2, 3, 4} which have the form of multidimensional recurrent relations. The same equations (identical in form but different in the analytical properties of solutions) are known in the theory of nonlinear waves as the Hirota equations. Discrete Hirota equations with specified boundary conditions may yield all the known exactly integrated (1+1)D equations at the continuous limit. The problem of integrability of discrete equations was discussed in recent papers \cite{5, 6, 7, 8, 9, 10, 11}. In Refs. \cite{5, 6, 7, 8, 13} a significant progress was achieved in applying Bethe-ansatz (BA) for integrable discrete (1+1)D systems with the internal degrees of freedom (see also \cite{15}). It was found out thereby that the discrete time variable and the variable corresponding to the internal degrees of freedom, as well as discrete rapidities enter the Hirota equation as the arguments of the searched function on equal footing. The limit of continuous time in this case corresponds to large spin values \cite{16}.

The behavior of the considered system is universal due to several reasons. The formulation of the model on the lattice leads to the solutions \cite{7}, which are expressed via elliptic functions. Complexification of the parameters includes the possibility to consider the quantum-group situation with the deformation parameter being a root of unity \cite{17, 18}. Different values for the shifts of rapidities along the imaginary axis in the searched functions of the discrete equations lead to different functional equations appearing in the TBA approach. In the particular case, when the rapidities are much greater than these shifts, we deal with recurrent relations \cite{19, 20, 21} existing in the theory that takes into account the Haldane’s generalized principle of exclusion statistics \cite{22}.

The eigenvalues $T(u)$ of the transfer matrix, satisfying Eq.(5) with certain boundary conditions (see below), are characterized by analytical properties of solutions specific for the problem under consideration. Thus, for example, the functional recurrent relations of a thermodynamical Bethe-anzatz \cite{4}, being outwardly similar in form with equations for $A_1$—case with zero boundary condition, are satisfied by functions with different analytical properties. The distinction in analytical properties consists in the fact that for recurrent TBA equations the shift of rapidities occurs along the imaginary axis.

In the present paper we consider the case of real shifts along the axis of rapidities. The main goal of the paper is search for integrals of motion following from the analysis of the Hirota classical equation for zero and quasiperiodic boundary conditions in the case of $A_{k-1}$ algebra. The integrals of motion are useful in seeking solutions of BA equations. As the example of the equations that require integrals of motion to be analyzed we consider
equations in the discrete and continuous limits for the $k = 3$ case. The general approach for studying the integrals of motion for discrete sine-Gordon model was used in Ref. [23]. The representation of the zero curvature [24, 25] for the classical discrete sine-Gordon model via the R-matrix was studied in Ref. [26].

The present paper is organized as follows. In the second section for consistency we provide the main expressions from Refs. [13, 14]. Based on these equations we write out the discrete equations of motion for the non-Abelian case $k = 3$ under zero and quasiperiodic boundary conditions. Here we give their form in the continuous limit. The third section is devoted to the analysis of integrals of motion in the case of quasiperiodic and zero boundary conditions. In the discussion we analyze the form of solutions for the $A_1$ algebra with the quasiperiodic boundary conditions.

2. Hirota Equation

The Hirota bilinear difference equation may be written down in different forms. One of the presentation of this equation for the function $\tau(n, l, m)$ depending on the discrete variables $n, l$ and $m$ is

$$\alpha \tau(n, l + 1, m)\tau(n, l, m + 1) + \beta \tau(n, l, m)\tau(n, l + 1, m + 1) +$$

$$\gamma \tau(n + 1, l + 1, m)\tau(n - 1, l, m + 1) = 0. \tag{1}$$

The arbitrary constants $\alpha, \beta, \gamma$ in this equation are restricted by $\alpha + \beta + \gamma = 0$. Further specification of Eq.(1) is performed with the aid of boundary conditions that set up the correspondence with the searched model in the continuous limit. Preserving integrability, various methods of the realization of the continuous limit permit to obtain the whole spectrum of soliton equations in the continuous case. The transform of the function $\tau(n, l, m)$,

$$\tau(n, l, m) = \frac{(-\alpha/\gamma)^{n^2/2}}{(1 + \gamma/\alpha)^{lm}} \tau_n(l, m) \tag{2}$$

eliminates the arbitrary constants $\alpha, \beta, \gamma$ in Eq.(1):

$$\tau_n(l + 1, m)\tau_n(l, m + 1) - \tau_n(l, m)\tau_n(l + 1, m + 1) + \tau_{n+1}(l + 1, m)\tau_{n-1}(l, m - 1) = 0. \tag{3}$$

Substitution of the variables

$$a = n, \quad s = l + m, \quad u = l - m - n,$$
leads to the following bilinear functional relations (expressing the fusion rules) for the eigenvalues $T^a_s(u)$ of the transfer matrix in integrable lattice models for the $A_{k-1}$ algebra:

$$T^a_s(u + 1)T^a_s(u - 1) - T^a_{s+1}(u)T^a_{s-1}(u) = T^{a+1}_s(u)T^{a-1}_s(u).$$  \hspace{1cm} (5)

The discrete index $a = 0, 1, \ldots k$ for the $A_{k-1}$ algebra and the discrete time $s$ in this equation are, respectively, the length and the height of the Young rectangular diagram, to which the eigenvalue $T^a_s(u)$ of the transfer matrix corresponds; $u$ in Eq.(5) is the discrete spectral parameter. Equation (5) preserves its form if the function $T^a_s(u)$ is multiplied by the product of four arbitrary functions $\chi_i$ depending on one variable with the following arguments:

$$T^a_s(u) \to \chi_1(a + u + s)\chi_2(a - u + s)\chi_3(a + u - s)\chi_4(a - u - s)T^a_s(u).$$  \hspace{1cm} (6)

For the function

$$Y^a_s(u) = \frac{T^a_{s+1}(u)T^a_{s-1}(u)}{T^{a+1}_s(u)T^{a-1}_s(u)},$$  \hspace{1cm} (7)

which is invariant with respect to the transform (6), Hirota equation (5) has the form

$$Y^a_s(u + 1)Y^a_s(u - 1) = \frac{(1 + Y^a_{s+1}(u))(1 + Y^a_{s-1}(u))}{(1 + (Y^a_{s+1}(u))^{-1})(1 + (Y^a_{s-1}(u))^{-1})}. $$  \hspace{1cm} (8)

The zero boundary conditions for the function $T^a_s(u)$ are as follows:

$$T^a_s(u) = 0 \text{ for } a < 0 \text{ and } a > k,$$

$$T^k_s(u) = \phi(u - s - k), \quad T^0_s(u) = \phi(u + s),$$

where $\phi(u) = \prod_{i=1}^N \sigma(\eta(u - y_i))$, $\sigma$ is the Weierstrass function, $y_i$ being its roots; $N$ is the size of the system and $\eta$ is a parameter. In the case of $A_1-$ algebra, these boundary conditions lead to the discrete Liouville equation. In terms of the gauge-invariant variables it has the form

$$Y^a_s(u - 1)Y^a_s(u + 1) = (1 + Y^a_{s+1}(u))(1 + Y^a_{s-1}(u))$$  \hspace{1cm} (9)

with the boundary condition $Y_0(u) = 0$. In the continuous limit after replacing $Y^a_s(u) = \delta^{-2}\exp(-\phi(x,t)), n = x/\delta, s = t/\delta$ with $\delta \to 0$ Eq.(10) transforms to Liouville equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 2e^\phi.$$  \hspace{1cm} (11)
The quasiperiodic boundary conditions for the function

$$T_a^s(u) = e^{\alpha} \lambda^{-2a} T_s^{a+2}(u + 2)$$ \hspace{1cm} (12)

in the $A_1$-algebra case lead to a discrete sine-Gordon equation\[13\] for the gauge-invariant function $X_a^s(u) = -\lambda(1 + Y_a^s(u))$\[28, 29, 30\]:

$$X_{s+1}^a(u)X_{s-1}^a(u) = \frac{(\lambda + X_a^s(u + 1))(\lambda + X_a^s(u - 1))}{(1 + \lambda X_a^s(u + 1))(1 + \lambda X_a^s(u - 1))}.$$ \hspace{1cm} (13)

Due to the boundary condition \[\text{(12)},\] the functions $X_{s+1}^a(u)$ with different values of variable "$a$" are related as

$$X_a^s(u)X_{a+1}^s(u + 1) = 1.$$ \hspace{1cm} (14)

We consider discrete equations of motion for the $A_2$ algebra for the zero and quasiperiodic boundary conditions, and their continuous limit. Applying the general approach of the paper \[13\] in the case of zero boundary conditions for gauge-invariant functions $Y_a^s(u)$ with $a = 1, 2$ we obtain the following system of discrete equations:

\begin{align*}
(1 + Y_2^s(u))Y_1^s(u + 1)Y_1^s(u + 1) &= (1 + Y_{s+1}^1(u))(1 + Y_{s-1}^1(u))Y_s^2(u), \hspace{1cm} (15) \\
(1 + Y_1^s(u))Y_2^s(u + 1)Y_2^s(u + 1) &= (1 + Y_{s+1}^2(u))(1 + Y_{s-1}^2(u))Y_s^1(u). \hspace{1cm} (16)
\end{align*}

The Eqs.\((15),(16)\) are the first result of this paper.

Similarly to the consideration of the transition from Eq.\((10)\) to Eq.\((11)\) we now introduce the functions $\phi_l(x, t)$ with $l = 1, 2$ in such a way that $Y_1^s(u) = \delta^{-2} \exp(-\phi_1(x, t))$. Let $u = x/\delta, s = t/\delta$. Then the continuous limit $\delta \to 0$ yields $SU(3)$ Toda equations \[31\]:

\begin{align*}
\frac{\partial^2 \phi_1}{\partial t^2} - \frac{\partial^2 \phi_1}{\partial x^2} &= 2e^{\phi_1} - e^{\phi_2}, \hspace{1cm} (17) \\
\frac{\partial^2 \phi_2}{\partial t^2} - \frac{\partial^2 \phi_2}{\partial x^2} &= 2e^{\phi_2} - e^{\phi_1}. \hspace{1cm} (18)
\end{align*}

In the case of non-Abelian discrete sine-Gordon model the quasiperiodic boundary condition \[\text{(12)},\] should be generalized for the $A_{k-1}$ algebra in the following way:

$$T_a^s(u) = e^{\alpha} \lambda^{-ka} T_s^{a+k}(u + k).$$ \hspace{1cm} (19)

Here the functions $X_a^s(u)$ satisfy the relation

$$X_a^s(u)X_{a+1}^s(u + 1) \ldots X_{a+k-1}^s(u + k - 1) = (-1)^k,$$ \hspace{1cm} (20)
which generalizes Eq.(14).

Consider the case \( k = 3 \). The Hirota equation for gauge-invariant functions \( X^1_s(u) \) for the \( A_2 \)-algebra with boundary conditions \([19]\) can be written as

\[
(\lambda + X^1_s(u))X^1_{s+1}(u)X^1_{s-1}(u) = \frac{(\lambda + X^1_s(u-1))(\lambda + X^1_s(u+1))}{(1 - \lambda X^1_s(u+1)X^1_s(u+2))} X^2_s(u),
\]

\[
(\lambda + X^1_s(u))X^2_{s+1}(u)X^2_{s-1}(u) = \frac{(\lambda + X^2_s(u-1))(\lambda + X^2_s(u+1))}{(1 - \lambda X^2_s(u+2)X^2_s(u-1))} X^1_s(u).
\]

This equations of motion is the second result of our paper.

The continuous limit in these equations can be achieved as follows. Assume that \( \lambda \to 0 \) and \( s \to \infty \) in such a way that \( s = t/\sqrt{\lambda} \) for a fixed \( t \). Similarly, assume that \( \lambda \to 0 \) and \( u \to \infty \) with \( u = x/\sqrt{\lambda} \) for a fixed \( x \). Being parametrized with the aid of the function \( X_l(x, t) = -\exp(-\phi_l(x, t)) \) Eqs. (21) and (22) transform in the continuous limit to the following equations of motion

\[
\frac{\partial^2 \phi_1}{\partial t^2} - \frac{\partial^2 \phi_1}{\partial x^2} = 2e^{\phi_1} - e^{\phi_2} - e^{-(\phi_1 + \phi_2)},
\]

\[
\frac{\partial^2 \phi_2}{\partial t^2} - \frac{\partial^2 \phi_2}{\partial x^2} = 2e^{\phi_2} - e^{\phi_1} - e^{-(\phi_1 + \phi_2)},
\]

which correspond to \( SU(3) \)-case of affine Toda equation.

Let us assume that in Eq.(21) \( X^1_s(u) = X^2_s(u) = X_s(u) \). Then we have

\[
\frac{X_{s+1}(u)X_{s-1}(u)}{X_s(u)} = \frac{(\lambda + X_s(u-1))(\lambda + X_s(u+1))}{(1 - \lambda X_s(u+2)X_s(u-1))(\lambda + X_s(u))},
\]

or from Eq.(22) with the same assumption

\[
\frac{X_{s+1}(u)X_{s-1}(u)}{X_s(u)} = \frac{(\lambda + X_s(u-1))(\lambda + X_s(u+1))}{(1 - \lambda X_s(u+2)X_s(u+1))(\lambda + X_s(u))}.
\]

In this case we have two versions of discrete equation. These equations are symmetrical to each other with respect to changing the variable \( u \to -u \). One can symmetrize these equations adding together Eqs.(25), (27)
\[
\frac{(\lambda + X_s(u))(1 - (\lambda/2)X_s(u - 2)X_s(u - 1) - (\lambda/2)X_s(u + 2)X_s(u + 1))}{(\lambda + X_s(u - 1))(\lambda + X_s(u + 1))}.
\]

The continuous limit of the Eq.(27) yields
\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = e^\phi - e^{-2\phi},
\]
which is the Bullough - Dodd equation.

Another discrete variant of the Bullough-Dodd equation was found from geometrical point of view in Refs. [12]. Generalizing the viewpoint, we may conclude also that Eqs. (21) and (22) are the discrete analog of the equations of motion in SU(3)-case of affine Toda model. To study solutions of the discrete equations of motion [13] and [10] and [21] (22) with zero [7] and quasiperiodic [19] boundary conditions, respectively, one has to find the integrals of motion. We consider this problem in the next section.

### 2. Integrals of motion

Integrals of motion are known to play the key role in the study of types of motion. Under integrals of motion we assume such combination of the functions \(T_s(u)\) which doesn’t depend on argument \(s\) or \(u\). In Ref. [13] integrals of motion for discrete dynamics were obtained for the \(A_1\) algebra and zero boundary conditions. A general determinant representation of integrals of motion was also written out for the \(A_k\) algebra with zero boundary conditions. In this section we shall show (i) how to get integrals of motion in the \(A_2\) case from the known integrals for the \(A_1\) algebra. In fact, we use this example to show how integrals of motion are transformed in the Bäcklund flow. Besides, (ii) we write down some integrals of motion for quasiperiodic boundary conditions.

We shall use the equations of the linear problem, which ensue from the gauge and dual transformations on the lattice [14] of the values for the variables in question. These equations are equivalent to those resulting from the zero curvature condition [24, 25]. Using the notations of Ref. [13], the equations of the linear problem are written as follows
\[
g_n(l, m)\tau_n(l + 1, m) - g_n(l + 1, m)\tau_n(l, m) = c^{-1}g_{n-1}(l, m)\tau_{n+1}(l + 1, m),
\]
\[
g_{n-1}(l, m)\tau_n(l, m + 1) - g_{n-1}(l, m + 1)\tau_n(l, m) = cg_n(l, m)\tau_{n-1}(l, m + 1).
\]

Here \(c\) is the arbitrary constant. When comparing Eqs. (29), (30) with the equations of the linear problem of the paper [14], the transformation [2] and the relation \(c_+c_- = -\gamma/\alpha\), \(c_- = c\) should be taken into account in (29), (30). This relation is the necessary condition
for the function \( g_n(l, m) \) to satisfy also Hirota equation. The inverse equations of the linear problem have the form

\[
g_n(l, m) \tau_n(l - 1, m) - g_n(l - 1, m) \tau_n(l, m) = -c_n^{-1} g_{n-1}(l - 1, m) \tau_{n+1}(l, m), \tag{31}
\]

\[
g_n(l, m) \tau_{n+1}(l, m - 1) - g_n(l, m - 1) \tau_{n+1}(l, m) = -c_n g_n(l, m - 1) \tau_n(l, m). \tag{32}
\]

In the terms of the functions \( T_s^a(u) \) and \( G_s^a(u) \), using the notations from \(^4\), when \( \tau_n(l, m) = T_{n-m}^a(l - m - n) = T_s^a(u) \) and \( g_n(l, m) = G_{l+m}^a(l - m - n) = G_s^a(u) \), one can write Eqs \((29-32)\) in the following way. For the direct equations of the linear problem we have

\[
G_s^a(u) T_s^{a+1}(u + 1) - G_s^a(u + 1) T_s^a(u) = c G_s^{-1}(u + 1) T_s^{a+1}(u), \tag{33}
\]

\[
G_s^{-1}(u + 1) T_s^{a+1}(u - 1) - G_s^{-1}(u) T_s^a(u) = -c G_s^{-1}(u) T_s^{a-1}(u). \tag{34}
\]

Similarly, the inverse equations are written as

\[
G_s^a(u) T_s^{-a+1}(u - 1) - G_s^{-a+1}(u - 1) T_s^a(u) = -c G_s^{-1}(u) T_s^{-a+1}(u - 1), \tag{35}
\]

\[
G_s^{-a+1}(u) T_s^{-a+1}(u + 1) - G_s^{-a+1}(u + 1) T_s^a(u) = -c G_s^{a+1}(u) T_s^{-a}(u). \tag{36}
\]

To make the comparison of the approaches and notations of various papers \(^1\) easier, we write out the equations of the linear problems having expressed the functions included in them via the eigenvalues of the transfer matrix, used in Ref. \(^1\). We note that the complete equivalence is achieved at \( c = -1 \) and the substitution \( s \rightarrow -s \).

Let us focus our attention on the symmetry of pairs of equations \((33), (34)\) and \((35), (36)\). It is seen that after the substitution \( T_s^a(u) \rightarrow G_s^{-a}(u) \) and \( G_s^a(u) \rightarrow T_s^{-a}(u) \) Eqs. \((33), (34)\) transform to \((35), (36)\).

In the case of zero boundary conditions \(^1\) for the function \( T_s^a(u) \), the function \( G_s^a(u) \) satisfies also the same boundary conditions, but with the maximum values a less by a unity, i.e., \( k \) should be replaced by \((k - 1)\). This feature expresses Backlund transform which realizes the transform \( A_k \rightarrow A_{k-1} \). It was used in the paper \(^1\) to get nested Bethe ansatz equation. This transform for \( k = 3 \) case can be schematically presented in the following form:

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & Q_1(u + s) & \bar{Q}_1(u - s) & 0 \\
0 & Q_2(u + s) & G_1(u) & \bar{Q}_2(u - s) & 0 \\
0 & \phi(u + s) & T_1^1(u) & T_1^2(u) & \bar{\phi}(u - s) & 0 \\
\end{array}
\]
Here $Q_\alpha(u)$ are some boundary functions expressed via Weierstrass functions $\sigma(\eta u)$, $\bar{Q}_\alpha(u) = Q_\alpha(u - \alpha)$, $\bar{\phi}(u) = \phi(u - k)$. It follows also from the boundary condition $Q_\alpha(u) = \phi(u)$ (for details see Ref.[13]).

The functions at each level satisfy Hirota equation. There is a relation between two adjacent levels, defined by the equations of the linear problem. The function $G^1_s(u)$ corresponds to the case of $A_1$ algebra. The equations of the linear problem written for it allow to obtain integrals of motion, which for $u$-dynamics are:

\[ I_1(s) = \frac{Q_2(s - 2)G^1_{s-u+1}(u) + Q_2(s)G^1_{s-u-1}(u - 2)}{G^1_{s-u}(u - 1)}, \tag{38} \]
\[ I_2(s) = \frac{\bar{Q}_2(s + 2)G^1_{u-s+1}(u) + \bar{Q}_2(s)G^1_{u-s-1}(u + 2)}{G^1_{u-s}(u + 1)}. \tag{39} \]

For $A_2$ case Eq.[33] of the linear problem for $\sigma = 1$ and $\sigma = 2$ may be written in the following way:

\[ Q_2(u + s)T^1_{s+1}(u) - Q_2(u + s + 2)T^1_s(u - 1) = c^{-1}G^1_{s+1}(u)\phi(u + s), \tag{40} \]
\[ G^1_s(u)T^2_{s+1}(u) - G^1_{s+1}(u + 2)T^2_s(u - 1) = c^{-1}\bar{Q}_2(u - s - 1)T^2_s(u). \tag{41} \]

Similarly Eq.[34] is written down:

\[ G^1_s(u)T^1_{s+1}(u) - G^1_{s+1}(u - 1)T^1_s(u) = cQ_2(u + s + 1)T^2_s(u - 1), \tag{42} \]
\[ \bar{Q}_2(u - s)T^2_{s+1}(u - 1) - \bar{Q}_2(u - s - 2)T^2_s(u) = cG^1_{s+1}(u)\phi(u + s). \tag{43} \]

Having expressed from [13] the necessary values $G^1_s(u)$ and substituting them in [33] we get the first integral of motion

\[ I_1(s) = \phi(s - 2) \left[ \frac{Q_2(s - 2) \left[ T^1_{s-u+1}(u)Q_2(s) - T^1_{s-u}(u - 1)Q_2(s + 2) \right]}{\phi(s) \left[ T^1_{s-u}(u - 1)Q_2(s - 2) - T^1_{s-u-1}(u - 2)Q_2(s) \right]} + \right] + \]
\[ \frac{Q_2(s) \left[ T^1_{s-u-1}(u - 2)Q_2(s - 4) - T^1_{s-u-2}(u - 3)Q_2(s - 2) \right]}{\phi(s - 4) \left[ T^1_{s-u}(u - 1)Q_2(s - 2) - T^1_{s-u-1}(u - 2)Q_2(s) \right]}, \tag{44} \]
Similarly having expressed from (41) the corresponding values $G^1_s(u)$ and substituting them in (39) we get the second integral of motion for $u$-dynamics

$$I_2(s) = \phi(-s-2) \left[ \frac{Q_2(2-s)}{\phi(s)} \left( T_{s+u+1}(u-1)Q_2(-s) - T^2_{s+u}(u)Q_2(-s-2) \right) \right] -$$

$$\frac{Q_2(-s)}{\phi(s-4)} \left( T^2_{s+u-1}(u+1)Q_2(2-s) - T^2_{s+u-2}(u+2)Q_2(s-2) \right)$$

In order to obtain integrals of motion for $s$-dynamics one should use the other pair of equations - Eqs. (11), (13). These integrals of motion are related with the form (8) of boundary conditions. Here one of them is associated with the boundary condition at the left end of the segment $[0,k]$ and the other - at the right end. The chosen example $k = 3$ illustrates the method of obtaining integrals of motion from the integrals of motion for $k$ less by a unity.

Now we consider quasiperiodic boundary conditions. In order Eqs. (13), (14), and (15), are fulfilled, the function $G^a_s(u)$ in the case of the quasiperiodic boundary conditions (19) should also satisfy the same boundary conditions. Therefore there are no Backlund flows in this case and the transition from $A_k$ algebra to $A_{k-1}$ algebra does not take place. It follows from the condition, that two functions $T^a_s(u)$ and $G^a_s(u)$ satisfy one and the same Hirota equation with equal boundary conditions, that they are proportional to each other. The relations between the indices in these functions are defined by the law of the transformation of direct equations of the linear problem (33), (34) into the inverse ones (52), (53). The coefficient of proportionality should retain Hirota equation invariant, i.e. it should coincide with the gauge transformation (3). Thus we may write down

$$G^a_s(u) = \chi_1(a + u + s)\chi_2(a - u + s)\chi_3(a + u - s)\chi_4(a - u - s)T^{-a}_s(-u).$$

We use Eq. (14) in (52), (53). As the result we obtain

$$\frac{\chi_4(a - u - s)}{\chi_4(a - u - s - 2)} T^{-a}_s(-u) T^a_{s+1}(u+1) - \frac{\chi_1(a + u + s + 2)}{\chi_1(a + u + s)} T^{-a}_{s-1}(-u-1) T^a_s(u) =$$

$$e^{-1} \frac{\chi_1(a + u + s - 2)}{\chi_2(a - u + s)} T^{-a+1}_s(-u-1) T^a_{s+1}(u),$$

$$\frac{\chi_2(a - u + s - 2)}{\chi_2(a - u + s)} T^{-a+1}_s(-u-1) T^a_{s+1}(u-1) - \frac{\chi_3(a + u - s - 2)}{\chi_3(a + u - s)} T^{-a+1}_{s-1}(-u) T^a_s(u) =$$
We use this form of equations to get integrals of motion. Here we take into account that quasiperiodic boundary conditions limit the type of the functions $\chi_1$ and $\chi_3$. They should be periodic with the period $2k$, while the functions $\chi_2$ and $\chi_4$ may be arbitrary. We single out the functions $\chi_1$ and $\chi_3$ from Eqs. (17), (19), having constructed the relations

$$
\frac{\chi_1(a + u + s + 2)}{\chi_1(a + u + s)} = \frac{\chi_4(a - u - s)T_{-s}^{a}(-u)T_{s+1}^{a+1}(u)}{\chi_4(a - u - s - 2)T_{-s-1}^{a}(-u - 1)T_{s}^{a}(u)} - c^{-1} \frac{\chi_2(a - u + s)T_{-s}^{a+1}(-u - 1)T_{s+1}^{a+1}(u)}{\chi_2(a - u + s)T_{-s-1}^{a+1}(-u)T_{s}^{a}(u)},
$$

$$
\frac{\chi_3(a + u - s - 2)}{\chi_3(a + u - s)} = \frac{\chi_2(a - u + s - 2)T_{-s}^{a+1}(-u - 1)T_{s+1}^{a+1}(u - 1)}{\chi_2(a - u + s)T_{-s-1}^{a+1}(-u)T_{s}^{a}(u)} - c \frac{\chi_4(a - u - s)T_{-s}^{a}(-u)T_{s+1}^{a-1}(u)}{\chi_4(a - u - s - 2)T_{-s-1}^{a-1}(-u - 1)T_{s}^{a}(u)}.
$$

It follows from the periodicity of the functions $\chi_1$ and $\chi_3$ that the right parts of Eqs. (23), (24) are periodic with the period $2k$ with respect to three indices. After the replacement $u \to u - s$ in Eq. (23) and the replacement $u \to u + s$ in (24), the right part of these equations does not depend on the time $s$. Consequently, after this replacement of variables the right hand size of the equations is the integral of motion for $s$–dynamics. The presence of the arbitrary functions $\chi_1$ and the constant $c$ in the formulas for the integrals of motion shows the degrees of freedom up to which we can consider the restriction imposed by them.

Similarly we may obtain the preserving values of $u$–dynamics, substituting $s \to s - u$ in Eq. (23) and $s \to u + s$ in Eq. (24). For simplicity we assume that $\chi_2 = \chi_4 = 1$ and we consider the case of $A_1$–algebra. For $a = 0$ and $a = 1$ from Eq. (33) we may get two invariant combinations of functions:

$$
\frac{\chi_1(u + s + 2)}{\chi_1(u + s)} = \frac{T_{-s}^{0}(-u)T_{s+1}^{0}(u + 1)}{T_{-s-1}^{0}(-u - 1)T_{s}^{0}(u)} - c^{-1} \frac{T_{-s}^{1}(-u - 1)T_{s+1}^{1}(u)}{T_{-s-1}^{1}(-u - 1)T_{s}^{1}(u)},
$$

(51)
\[
\frac{\chi_1(u+s+3)}{\chi_1(u+s+1)} = \frac{T^1_{-s}(-u+2)T^1_{s+1}(u+1)}{T^1_{-s-1}(-u+1)T^1_s(u)} - e^{-1} \frac{T^0_{-s}(-u-1)T^0_{s+1}(u-2)(\lambda e^\alpha)^2}{T^0_{-s-1}(-u+1)T^0_s(u)}.
\] (52)

The combinations made of the functions \(\chi_3\), which follow from (50), have the form

\[
\frac{\chi_3(u-s-2)}{\chi_3(u-s)} = \frac{T^1_{-s}(-u-1)T^0_{s+1}(u-1)}{T^1_{-s-1}(-u)T^0_s(u)} - e^{-1} \frac{T^0_{-s}(-u)T^1_{s+2}(u+2)}{T^0_{-s-1}(-u)T^0_s(u)},
\] (53)

\[
\frac{\chi_3(u-s-1)}{\chi_3(u-s+1)} = \frac{T^0_{-s}(-u-1)T^1_{s+1}(u-1)}{T^0_{-s-1}(-u)T^0_s(u)} - e^{-1} \frac{T^1_{-s}(-u+2)T^0_{s+1}(u)}{T^0_{-s-1}(-u)T^0_s(u)}.
\] (54)

The Eqs.(49)-(54) are the main result of the paper.

It is seen directly from these equations that after the substitution \(u \to u-s\) \((u \to u+s)\) or \(s \to s-u\) \((s \to s+u)\) in Eqs.(51),(52) ((53),(54)) we obtain in the right hand sizes of these equations the magnitudes which do not depend on \(s\) \((on u)\) after the change of these discrete variables.

4. Discussion

The preserving combinations (51) - (54) of the functions \(T^a_s(u)\) may be used for studying discrete dynamics in sine-Gordon model. To find the solutions of the equation of motion in the discrete sine-Gordon model, Bethe-Ansatz method was used in paper [3]. In the case of \(A_1\)-algebra for the function

\[
T^0_{s+1}(u) = A^{0,1}_s e^{\mu(s)u} \prod_{j=1}^N \sigma \left( \eta \left( u - z^{0,1}_j(s) \right) \right),
\] (55)

where the index \(a\) takes the values 0, 1, the function \(\mu(s) = \mu_0 s + \mu_1\) \((\mu_0, \mu_1\) are the arbitrary constants), two functions \(z^{0,1}_j(s)\) are the zero functions of Weierstrass function and are defined by the solutions of BA equations:

\[
\prod_{j=1}^N \left( \frac{\sigma(\eta(z^{1}_j(s+1) - z^{0}_j(s+1) - 1))}{\sigma(\eta(z^{1}_j(s+1) - z^{0}_j(s+2) - 3))} \right) \left( \frac{\sigma(\eta(z^{1}_j(s) - z^{1}_j(s+1) + 1))}{\sigma(\eta(z^{1}_j(s+1) - z^{1}_j(s+2) + 1))} \right)
\]
expressing the conservation laws in the considered system, this will allow to find multipliers (56), (57). If we substitute the functions by the dependence of the roots see that taking into account (60) the dependence of these functions on change certainly the functions in the left hand size of Eqs.(56),(57) we get that the right hand size of these equations equals −constant. Using the solutions obtained (60), (61) in the right hand size of Eqs.(56), (57) γ Here the coefficients exp(−s).

In the rational limit, when \( \sigma(\eta u) \to u \), the functions \( A_s \) may be found from the system of equations

\[
(A_1^s)^2 - A_{s+1}^1 A_{s-1}^1 = e^{\alpha \lambda^2} (A_0^1)^2 \equiv a(A_0^1)^2 \tag{58}
\]

\[
(A_0^s)^2 - A_{s+1}^0 A_{s-1}^0 = e^{-\alpha \lambda^2} (A_1^0)^2 \equiv b(A_1^0)^2 \tag{59}
\]

The equations (58) - (59) describe the discrete dynamics of the zeroes \( z_j^{0,1}(s) \) as the function of the time \( s \).

In the presence of the restrictions following from (51), (54) to solve Eqs.(56), (57), one should find the solutions of Eqs.(58), (59). We pay attention to the invariance of Eqs.(58), (59) with respect to the transformation of \((A_0^0, A_1^1) \to (A_0^1, A_1^0)e^{\delta s}\), where \( \delta - \) is the constant. This invariance is already displayed in Eq.(55) with the aid of the functions exp(\( \mu(s) u \)) with \( \mu = \mu_0 s + \mu_1 \).

It follows from Eqs.(58), (59) that the functions \( A_s^{0,1} \) have the form

\[
A_0^s = Ae^{-\gamma s^2}, \tag{60}
\]

\[
A_1^s = Ae^{-\gamma s^2 + \alpha/2}, \tag{61}
\]

Here the coefficients \( \gamma \) and \( \lambda \) are connected by the relation \( \lambda^2 = 1 - \exp(-2\gamma) \), \( A \) is the constant. Using the solutions obtained (60), (61) in the right hand size of Eqs.(56), (57) we get that the right hand size of these equations equals −1. In this case, we have to change certainly the functions in the left hand size of Eqs.(56),(57) by \( \sigma(\eta u) \to u \).

Substituting the solutions \( T_s^{0,1}(u) \) of (55) to the gauge-invariant functions \( Y_s^0(u) \) we see that taking into account (60) the dependence of these functions on \( s \) is defined only by the dependence of the roots \( z_s^{0,1} \) on the discrete time \( s \) and does not depend on Gauss multipliers (56), (57). If we substitute the functions \( T_s^{0,1}(u) \) from Eq. (55) to the limitations (51), (54) expressing the conservation laws in the considered system, this will allow to find the limitations on the dependencies of the roots \( z_s^{0,1} \) on the discrete time \( s \). In the case
$N = 2$ this problem may be presented in the form of the finite number of mappings. The thermodynamic limit $N \to \infty$ requires numerical calculations.

In conclusion, we found the integrals of motion for $s$– and $u$–dynamics for $A_k$–algebra with quasiperiodic boundary conditions. The case $k = 2$ corresponds to the discrete sine-Gordon model. It is shown that for $A_2$– algebra the discrete equations of motion at zero boundary conditions convert in the continuous limit to the equations of motion of non-Abelian generalization of Liouville equation and at the quasiperiodic boundary conditions - to the equation of motion of the $SU(3)$ affine Toda equation and to the equation of motion of the Bullough-Dodd model.

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