The $SU(n)$ invariant massive Thirring model with boundary reflection

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Abstract

We study the $SU(n)$ invariant massive Thirring model with boundary reflection. Our approach is based on the free field approach. We construct the free field realizations of the boundary state and its dual. For an application of these realizations, we present integral representations for the form factors of the local operators.

1 Introduction

For without-boundary massive theories, integrability is ensured by the factorized scattering condition or the Yang-Baxter equation. Cherdnik [1] showed that the integrability in the presence of reflecting boundary is ensured by the boundary Yang-Baxter equation and the Yang-Baxter equation for without-boundary theory. A systematic treatment of
integrable model with boundary reflection was initiated by Sklyanin in the framework of the Bethe Ansatz. The boundary state condition was proposed in on the basis of the boundary bootstrap approach, in order to obtain the boundary state. Jimbo et al. developed this idea to obtain the correlation functions of the XXZ chain with a boundary, using the free field approach. The generalization of the XXZ chain with a boundary, was done in .

In this paper we study the $SU(n)$ invariant massive Thirring model with boundary reflection, by means of the free field approach. For $SU(2)$ symmetry case, three men studied the invariant massive Thirring model with boundary reflection, using the free field approach, and derived integral representations for the form factors. In reference , their construction of the boundary state starts with Lukyanov’s u-v cut-off realization of the Zamolodchikov-Faddeev operators, and the form factors are derived after removing the cut-off parameter at the final stage. In this paper we prefer to work directly with operators with cut-off parameter removed. We construct the free field realizations of the boundary state and it’s dual, and give integral representations for form factors of the $SU(n)$ invariant massive Thirring model with a boundary.

Now a few words about the organization of this paper. In section 2 we set up problem, and give the free field realization of the Zamolodchikov-Faddeev operators. In section 3 we construct the free field realizations of the boundary state and it’s dual. In section 4 we give the free field realization of the local operator, and give integral representations of the form factors of the local operators. In Appendix we summarize useful facts about the multi-Gamma functions.

2 Formulation

The purpose of this section is to set up the problem, thereby fixing the notation. We give the free field realization of the Zamolodchikov-Faddeev operators.

2.1 Model

In this section we set up problem, and present briefly necessary tools concerning the completely integral models of quantum field theory with massive spectra. The $SU(n)$ invariant massive Thirring model with boundary reflection is described by the bulk $S$-
matrix and the boundary $S$-matrix ($K$-matrix). Let $V$ be $n$-dimensional vector space $V = \oplus_{j=0}^{n-1} C v_j$. The bulk $S$-matrix $S(\beta) \in \text{End}(V \otimes V)$ of the present model is given by

$$S(\beta) = \frac{\Gamma \left( \frac{n-1}{n} + \frac{\beta}{2\pi i} \right) \Gamma \left( \frac{n-1}{n} - \frac{\beta}{2\pi i} \right)}{\Gamma \left( \frac{\beta}{2\pi i} \right) \Gamma \left( \frac{\beta}{2\pi i} + \frac{1}{2} \right)} \times \frac{\beta - \frac{2\pi i}{n} P}{\beta - \frac{2\pi i}{n}},$$

where the operator $P \in \text{End}(V \otimes V)$ represents the permutation.

$$P(a \otimes b) = b \otimes a.$$ (2.2)

This $S$-matrix $S(\beta)$ satisfies the Yang-Baxter equation.

$$S_{12}(\beta_1 - \beta_2)S_{13}(\beta_1 - \beta_3)S_{23}(\beta_2 - \beta_3) = S_{23}(\beta_2 - \beta_3)S_{13}(\beta_1 - \beta_3)S_{12}(\beta_1 - \beta_2).$$ (2.3)

Let us fix two numbers $L, M$ such that $0 \leq L < M \leq n$, ($L, M \in \mathbb{N}$). The boundary $K$-matrix $K(\beta) \in \text{End}(V)$ of the present model is a diagonal matrix given by

$$K(\beta)v_j = \sum_{j=0}^{n-1} v_k \delta_{j,k} K(\beta)^j_k.$$ (2.4)

Here the diagonal elements are defined by

$$K(\beta)^j_k = k_{L,M}(\beta) \begin{cases} 1, & 0 \leq j \leq L - 1, \\ \frac{\mu - \beta}{\mu + \beta}, & L \leq j \leq M - 1, \\ 1, & M \leq j \leq n - 1. \end{cases}$$ (2.5)

Here the scalar function $k_{L,M}(\beta)$ is given by

$$k_{L,M}(\beta) = \frac{\Gamma \left( \frac{-\beta + \mu}{2\pi i} \right) \Gamma \left( \frac{\beta + \mu}{2\pi i} + 1 - \frac{L}{n} \right)}{\Gamma \left( \frac{\beta + \mu}{2\pi i} \right) \Gamma \left( \frac{-\beta + \mu}{2\pi i} + 1 - \frac{L}{n} \right)} \times \frac{\Gamma \left( \frac{-\beta - \mu}{2\pi i} + \frac{L}{n} \right) \Gamma \left( \frac{\beta - \mu}{2\pi i} + 1 + \frac{L-M}{n} \right)}{\Gamma \left( \frac{\beta + \mu}{2\pi i} + \frac{L}{n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + 1 + \frac{L-M}{n} \right)} \times$$

$$\times \frac{\Gamma \left( \frac{-\beta + \mu}{2\pi i} + \frac{1}{2} - \frac{1}{2n} \right) \Gamma \left( \frac{\beta + \mu}{2\pi i} + \frac{1}{2} - \frac{1}{2n} \right)}{\Gamma \left( \frac{\beta - \mu}{2\pi i} + \frac{1}{2} - \frac{1}{2n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + \frac{1}{2} - \frac{1}{2n} \right)}.$$ (2.6)

The $S$-matrix and the boundary $K$-matrix satisfy the boundary Yang-Baxter equation.

$$K_2(\beta_2)S_{21}(\beta_1 + \beta_2)K_1(\beta_1)S_{12}(\beta_1 - \beta_2) = S_{21}(\beta_1 - \beta_2)K_1(\beta_1)S_{12}(\beta_1 + \beta_2)K_2(\beta_2).$$ (2.7)
For the description of the space of physical states we use the Zamolodchikov-Faddeev operators. The Zamolodchikov-Faddeev operators $Z^*_j(\beta)$, $Z_j(\beta)$, $(j = 1, \cdots, n - 1)$ of the present model satisfy the following commutation relations.

\[
Z^*_j(\beta_1)Z^*_k(\beta_2) = \sum_{k_1,k_2=0}^{n-1} S(\beta_1-\beta_2)^{k_1k_2}_{j_1j_2} Z^*_{k_1}(\beta_2)Z^*_k(\beta_1), \quad (2.8)
\]

\[
Z_j(\beta_1)Z_k(\beta_2) = \sum_{k_1,k_2=0}^{n-1} S(\beta_1-\beta_2)^{j_1j_2}_{k_1k_2} Z_k(\beta_2)Z^*_j(\beta_1). \quad (2.9)
\]

Here $S(\beta)^{cd}_{ab}$ are matrix elements of the $S$-matrix,

\[
S(\beta)v_{k_1} \otimes v_{k_2} = \sum_{j_1,j_2=0}^{n-1} v_{j_1} \otimes v_{j_2} S(\beta)^{k_1k_2}_{j_1j_2}. \quad (2.10)
\]

The Zamolodchikov-Faddeev operators $Z^*_j(\beta)$ and its dual $Z_j(\beta)$ satisfy the inversion relation.

\[
Z^*_j(\beta_1)Z_k(\beta_2 + \pi i) = \frac{\delta_{j,k}}{\beta_1 - \beta_2} + \cdots, \quad (\beta_1 \to \beta_2). \quad (2.11)
\]

where “$\cdots$” means regular term.

For the description of the space of physical state we use the boundary state $|B\rangle$. The boundary state $|B\rangle$ and its dual $\langle B|$ are characterized by the following conditions.

\[
K(\beta)^j_Z|B\rangle = Z^*_j(-\beta)|B\rangle, \quad (j = 0, \cdots, n - 1), \quad (2.12)
\]

\[
K(\beta)^j_Z\langle B|Z_j(-\beta + \pi i) = \langle B|Z_j(\beta + \pi i), \quad (j = 0, \cdots, n - 1). \quad (2.13)
\]

In this paper we shall construct the free field realizations of the boundary state and its dual.

The space of states is generated by the vectors,

\[
|\beta_1, \cdots, \beta_N\rangle_{j_1,\cdots,j_N} = Z^*_j(\beta_1) \cdots Z^*_j(\beta_N)|B\rangle, \quad (2.14)
\]

and the dual space,

\[
\langle j_1,\cdots,j_N|\beta_1, \cdots, \beta_N| = \langle B|Z_j(\beta_1) \cdots Z_j(\beta_N). \quad (2.15)
\]

Consider local operators $\mathcal{O}$ and construct the matrix element,

\[
j_1,\cdots,j_M \langle \beta'_{1}, \cdots, \beta'_{M}|\mathcal{O}|\beta_1 \cdots \beta_N\rangle_{k_1 \cdots k_N}. \quad (2.16)
\]
We call the above matrix elements “form factor”.

In this paper we shall give integral representations of the form factors.

In this subsection we have introduced the basic tools of the bootstrap approach, i.e. the $S$-matrix $S(\beta)$, the boundary $K$-matrix $K(\beta)$, the Zamolodchikov-Faddeev operators $Z_j^*(\beta)$, $Z_j(\beta)$, the boundary state $|B\rangle$ and its dual state $\langle B|$; the basis of the space of the states $|\beta_1, \cdots, \beta_N\rangle_j^{\pm\cdots\pm N}$, it’s dual $\langle \beta_1, \cdots, \beta_N|$; and the form factors.

### 2.2 Zamolodchikov-Faddeev operators

The purpose of this section is to give the free field realization of the Zamolodchikov-Faddeev operators.

Let $a_j(t)$ $(1 \leq j \leq n-1, t \in \mathbb{R})$ be the free bose field satisfying the following commutation relation,

$$[a_j(t), a_k(t')] = -\frac{1}{t} \frac{\sinh((a_j|a_k)\pi t)}{\sinh(\frac{\pi}{n}t)} \delta(t + t').$$

(2.17)

Here $((a_j|a_k)_{1 \leq j, k \leq n-1}$ is the Cartan matrix of type $A_{n-1}$.

$$((a_j|a_k)_{1 \leq j, k \leq n-1} = \begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots \\
0 & -1 & 2 & -1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2 \\
\end{pmatrix}. (2.18)$$

Let us introduce the Fock space $\mathcal{H}$ generated by the vacuum vector $|vac\rangle$ satisfying

$$a_j(t)|vac\rangle = 0 \text{ for } t > 0. \quad (2.19)$$

A normal ordering : $A$ : of an element $A$ is defined as usual : anihilation operators are replaced on the right of the creation operators, for example,

$$a(t_1)a(-t_2) := a(-t_2)a(t_1), \quad (t_1, t_2 > 0). \quad (2.20)$$
Let us introduce the basic operators,

\[ V_j(\beta) = : \exp \left( \int_{-\infty}^{\infty} a_j(t) e^{i \beta t} dt \right) : \quad (1 \leq j \leq n - 1), \quad (2.21) \]

\[ V_0(\beta) = : \exp \left( \int_{-\infty}^{\infty} a_1^*(t) e^{i \beta t} dt \right) :, \quad (2.22) \]

\[ V_n(\beta) = : \exp \left( \int_{-\infty}^{\infty} a_{n-1}^*(t) e^{i \beta t} dt \right) :, \quad (2.23) \]

where we have set

\[ a_j^*(t) = - \sum_{j=1}^{n-1} a_j(t) \frac{\sin((a-j) \pi t)}{\sin \pi t}, \quad (2.24) \]

\[ a_{n-1}^*(t) = - \sum_{j=1}^{n-1} a_j(t) \frac{\tan \pi t}{\sin \pi t}, \quad (2.25) \]

We have

\[ [a_j^*(t), a_j(t')] = \delta_{1,j} \frac{e^{\pi |t|}}{t} \delta(t + t'), \quad (2.26) \]

\[ [a_j(t), a_{n-1}^*(t')] = \delta_{j,n-1} \frac{e^{\pi |t|}}{t} \delta(t + t'). \quad (2.27) \]

The basic operators satisfy the following contraction relations.

\[ V_j(\beta_1)V_{j-1}(\beta_2) = \frac{-ie^{-\gamma}}{\beta_2 - \beta_1 + \frac{\pi i}{n}} : V_j(\beta_1)V_{j-1}(\beta_2) :, \quad (j = 1, \cdots, n), \quad (2.28) \]

\[ V_{j-1}(\beta_1)V_j(\beta_2) = \frac{-ie^{-\gamma}}{\beta_2 - \beta_1 + \frac{\pi i}{n}} : V_{j-1}(\beta_1)V_j(\beta_2) :, \quad (j = 1, \cdots, n), \quad (2.29) \]

\[ V_j(\beta_1)V_j(\beta_2) = e^{2\gamma(\beta_1 - \beta_2)} \left( \beta_2 - \beta_1 + \frac{2\pi i}{n} \right) : V_j(\beta_1)V_j(\beta_2) :, \quad (j = 1, \cdots, n - 1) \quad (2.30) \]

\[ V_0(\beta_1)V_0(\beta_2) = e^{\frac{-2n}{n}(\gamma + \log 2\pi)} \frac{\Gamma \left( \frac{\beta_1 - \beta_2}{2\pi i} + 1 - \frac{1}{n} \right)}{\Gamma \left( \frac{\beta_1 - \beta_2}{2\pi i} \right)} : V_0(\beta_1)V_0(\beta_2) : \quad (2.31) \]

\[ V_n(\beta_1)V_n(\beta_2) = e^{\frac{-2n}{n}(\gamma + \log 2\pi)} \frac{\Gamma \left( \frac{\beta_1 - \beta_2}{2\pi i} + 1 - \frac{1}{n} \right)}{\Gamma \left( \frac{\beta_1 - \beta_2}{2\pi i} \right)} : V_n(\beta_1)V_n(\beta_2) : \quad (2.32) \]

The basic operators satisfy the following commutation relations.

\[ V_j(\beta_1)V_{j-1}(\beta_2) = \frac{\beta_1 - \beta_2 + \frac{n \pi i}{n}}{\beta_2 - \beta_1 + \frac{n \pi i}{n}} V_{j-1}(\beta_2)V_j(\beta_1), \quad (j = 1, \cdots, n), \quad (2.33) \]

\[ V_j(\beta_1)V_j(\beta_2) = -\frac{\beta_1 - \beta_2 - 2\pi i}{\beta_2 - \beta_1 - 2\pi i} V_j(\beta_2)V_j(\beta_1), \quad (j = 1, \cdots, n - 1), \quad (2.34) \]
and

\[
V_0(\beta_1)V_0(\beta_2) = s(\beta_1 - \beta_2)V_0(\beta_2)V_0(\beta_1),
\]

\[
V_n(\beta_1)V_n(\beta_2) = s(\beta_1 - \beta_2)V_n(\beta_2)V_n(\beta_1).
\]

Here we have set

\[
s(\beta) = \frac{\Gamma\left(\frac{\beta}{2\pi i} + 1 - \frac{1}{n}\right) \Gamma\left(-\frac{\beta}{2\pi i}\right)}{\Gamma\left(-\frac{\beta}{2\pi i} + 1 - \frac{1}{n}\right) \Gamma\left(\frac{\beta}{2\pi i}\right)}.
\]

Next we give the free field realization of the Zamolodchikov-Faddeev operators. Let us set the operators \(Z^*_j(\beta), (j = 0, \cdots, n - 1)\) by

\[
Z^*_j(\beta) = \int_{-\infty}^{\infty} d\alpha_1 \cdots \int_{-\infty}^{\infty} d\alpha_j \frac{V_0(\beta)V_1(\alpha_1)\cdots V_j(\alpha_j)}{\prod_{l=1}^{j}(\alpha_l-\frac{\pi i}{n})}\times :V_0(\beta)V_1(\alpha_1)\cdots V_j(\alpha_j): \prod_{l=1}^{j} \frac{1}{(\alpha_l-\alpha_{l-1}+\frac{\pi i}{n})(\alpha_{l-1}-\alpha_l+\frac{\pi i}{n})},
\]

where we set \(\alpha_0 = \beta\).

Let us set the operators \(Z_j(\beta), (j = 0, \cdots, n - 1)\) by

\[
Z_j(\beta) = c_j \int_{-\infty}^{\infty} d\alpha_{j+1} \cdots \int_{-\infty}^{\infty} d\alpha_{n-1} \frac{V_{j+1}(\alpha_{j+1})\cdots V_{n-1}(\alpha_{n-1})V_n(\beta)}{\prod_{l=j+1}^{n-1}(\alpha_l-\alpha_{l+1}+\frac{\pi i}{n})}\times :V_{j+1}(\alpha_{j+1})\cdots V_{n-1}(\alpha_{n-1})V_n(\beta): \prod_{l=j+1}^{n-1} \frac{1}{(\alpha_l-\alpha_{l+1}+\frac{\pi i}{n})(\alpha_{l+1}-\alpha_l+\frac{\pi i}{n})},
\]

where we set \(\alpha_n = \beta\), and \(c_j\) are proper constants chosen to satisfy the relation (2.11). The operators \(Z^*_j(\beta), Z_j(\beta), (j = 0, \cdots, n - 1, \beta \in \mathbb{R})\) generate the Zamolodchikov-Faddeev algebras. The above free field realizations of the Zamolodchikov-Faddeev operators \(Z^*_j(\beta), Z_j(\beta)\) satisfy the commutation relations (2.8), (2.9), and the inversion relation (2.11). The proof is as the same as those in the reference [15].
Note. The free field realization of the Zamolodchikov-Faddeev operators $Z_j^*(\beta)$ of the $A_{n-1}^{(1)}$ Toda fields theory are given in the reference [14]. Naively the Zamolodchikov-Faddeev operators (the type-II vertex operators) of the present model is a limiting case of those of the $A_{n-1}^{(1)}$ Toda field theory. The construction of the dual Zamolodchikov-Faddeev operators $Z_j(\beta)$ is sketched in the reference [15]. In the reference [15] they constructed the type-I vertex operators and it’s dual vertex operators for the $A_{n-1}^{(1)}$ critical chain.

3 Boundary State

In this section we give the free field realization of the boundary state $|B\rangle$, which satisfies

$$K(\beta_j^*)Z_j^*(\beta)|B\rangle = Z_j^*(-\beta)|B\rangle, \quad (j = 0, \ldots, n-1).$$

(3.1)

Here $K(\beta_j^*)$ are diagonal elements of the boundary $K$-matrix.

We make the ansatz that the boundary state has the following form.

$$|B\rangle = e^F|\text{vac}\rangle.$$

(3.2)

Here $F$ is a quadratic form of the free bosons,

$$F = \sum_{j,k=1}^{n-1} \int_0^\infty \alpha_{j,k}(t)a_j(-t)a_k(-t)dt + \sum_{j=1}^{n-1} \int_0^\infty \beta_j(t)a_j(-t)dt,$$

(3.3)

where $\alpha_{j,k}(t)$ and $\beta_j(t)$ are scalar functions.

In this section we show that we can choose the coefficients $\alpha_{j,k}(t)$ and $\beta_j(t)$ such that the state $|B\rangle$ satisfies the characterizing relation (3.1).

The operator $e^F$ has the effect of a Bogoliubov transformation,

$$e^{-F}a_i(-t)e^F = 0, \quad (t > 0),$$

(3.4)

$$e^{-F}a_i(t)e^F = a_i(t) + \sum_{j,k=1}^{n-1} \alpha_{j,k}(t) \left( -\frac{e^{\pi t}}{t} \text{sh} \frac{(a_i|a_j)\pi t}{n} a_k(-t) - \frac{e^{\pi t}}{t} \text{sh} \frac{(a_i|a_k)\pi t}{n} a_j(-t) \right)$$

$$- \sum_{l=1}^{n-1} \beta_l(t) \frac{e^{\pi t}}{t} \text{sh} \frac{(a_i|a_l)\pi t}{n} \frac{a_l}{\text{sh} \frac{n\pi}{n}}, \quad (t > 0).$$

(3.5)
Let us set the coefficient \( \alpha_{j,k}(t) \) by

\[
\alpha_{j,k}(t) = -\frac{te^{-\frac{\pi t}{n}}}{2} \times I_{j,k}(t).
\]

(3.6)

Here \( (I_{j,k}(t))_{1 \leq j,k \leq n-1} \) is the inverse matrix of the quantum Cartan matrix of type \( A_{n-1} \). The quantum Cartan matrix of type \( A_{n-1} \) is defined by

\[
\begin{pmatrix}
\frac{\text{sh}(a_j|a_k)\pi t}{n} \\
\frac{\pi t}{n} \\
\frac{\text{sh}(a_j|a_k)\pi t}{n}
\end{pmatrix},
\]

(3.7)

where \( ((a_j|a_k))_{1 \leq j,k \leq n-1} \) is the Cartan matrix of type \( A_{n-1} \).

Explicitly matrix elements \( I_{j,k}(t) \) of the inverse matrix of the quantum Cartan matrix are given by

\[
I_{j,k}(t) = \frac{\text{sh} j\pi t}{\pi t} \frac{(n-k)\pi t}{n} \text{sh} \frac{\pi t}{n} = I_{k,j}(t), \quad (1 \leq j \leq k \leq n-1).
\]

(3.8)

After specifying \( \alpha_{j,k}(t) \), we have

\[
e^{-F} a_t(t)e^F = a_t(t) + a_t(-t) - \frac{e^{\frac{\pi t}{n}}}{t} \sum_{j=1}^{n-1} \beta_j(t) \times \frac{\text{sh}(a_j|a_l)\pi t}{\text{sh} \frac{\pi t}{n}}, \quad (t > 0).
\]

(3.9)

In what follows we use the abbreviations,

\[
V_j^a(\beta) = \exp \left( \int_0^\infty a_j(t)e^{i\beta t} dt \right),
\]

(3.10)

\[
V_j^c(\beta) = \exp \left( \int_0^\infty a_j(-t)e^{-i\beta t} dt \right),
\]

(3.11)

\[
V_0^a(\beta) = \exp \left( \int_0^\infty a_1(t)e^{i\beta t} dt \right),
\]

(3.12)

\[
V_0^c(\beta) = \exp \left( \int_0^\infty a_1(-t)e^{-i\beta t} dt \right),
\]

(3.13)

\[
V_n^a(\beta) = \exp \left( \int_0^\infty a_{n-1}(t)e^{i\beta t} dt \right),
\]

(3.14)

\[
V_n^c(\beta) = \exp \left( \int_0^\infty a_{n-1}(-t)e^{-i\beta t} dt \right),
\]

(3.15)
The actions of the basic operators on the vector \(|B\rangle\) are given by

\[
V_j^\alpha(\alpha)|B\rangle = G_j(\alpha)V_j^\gamma(-\alpha)|B\rangle, \quad (j = 1, \cdots, n - 1),
\]
\[
V_0^\alpha(\alpha)|B\rangle = H_0(\beta)V_0^\gamma(-\beta)|B\rangle.
\]  
(3.16)  
(3.17)

Here the functions \(G_j(\alpha), \quad (j = 1, \cdots, n - 1)\) and \(H_0(\beta)\) are given by

\[
G_j(\alpha) = \exp\left(-\frac{1}{2} \int_0^\infty \frac{e^{2it}}{t} (1 + e^{\frac{\pi}{n}it})dt - \int_0^\infty \frac{e^{(i\alpha + \frac{\pi}{n})t}}{t} \sum_{l=1}^{n-1} \beta_l(t) \frac{\sh\left(a_l\alpha \pi t \right)}{\sh\left(\frac{\pi}{n} \right) dt},
\]
\[
H_0(\beta) = \exp\left(-\frac{1}{2} \int_0^\infty \frac{e^{(2i\beta + \frac{\pi}{n})t}}{t} \frac{\sh\left(\frac{n-1}{n} \pi t \right)}{\sh\left(\frac{\pi}{n} \right) dt} + \int_0^\infty \frac{e^{(i\beta + \frac{\pi}{n})t}}{t} \beta_1(t)dt\right).
\]
(3.18)  
(3.19)

Let us set the coefficients \(\beta_l(t), \quad (l = 1, \cdots, n - 1)\) by

\[
\beta_l(t) = \frac{1}{2} (e^{\frac{\pi}{n}t} - 1) \sum_{k=1}^{n-1} I_{l,k}(t) + e^{i\mu t + \frac{\pi}{n}(L-1)t} I_{L,l}(t) + e^{-i\mu t + \frac{\pi}{n}(M-2L-1)t} I_{M,l}(t).
\]
(3.20)

Here we should understand \(I_{0,l}(t) = 0\) and \(I_{n,l}(t) = 0\).

After specifying the coefficients \(\beta_l(t)\), we have

\[
e^{-F} a(t) e^F = a(t) + a(-t) + \frac{1}{2t} (e^{\frac{\pi}{n}t} - 1) - \delta_{t,L} \frac{e^{i\mu t + \frac{\pi}{n}Lt}}{t} - \delta_{t,M} \frac{e^{-i\mu t + \frac{\pi}{n}(M-2L)t}}{t},
\]
(3.21)

and

\[
G_j(\alpha) = \begin{cases} 
-i e^\gamma \times \alpha, & j \neq L, M, \\
e^{2\gamma} \times \alpha \left(-\mu + \frac{\pi}{n}L - \alpha\right), & j = L, \\
e^{2\gamma} \times \alpha \left(\mu + \frac{\pi}{n}(M - 2L) - \alpha\right), & j = M.
\end{cases}
\]
(3.22)

\[
H_0(\beta) = \text{Const.} (\mu + \beta)^{\delta_{t,0}} \frac{\Gamma \left(\frac{\beta+\mu}{2\pi i} + \frac{L}{2}\right) \Gamma \left(\frac{\beta-\mu}{2\pi i} + \frac{L}{2} - \frac{1}{2}\right)}{\Gamma \left(\frac{\beta+\mu}{2\pi i} + 1 - \frac{L}{2}\right) \Gamma \left(\frac{\beta-\mu}{2\pi i} + 1 + \frac{M-L}{2}\right) \Gamma \left(\frac{\beta}{2\pi i}\right)}.
\]
(3.23)

We shall prove the characterizing relations (3.1), under the specification of the coefficients \(\alpha_{j,k}(t)\) and \(\beta_j(t)\). In what follows we use the auxiliary function,

\[
\Delta(\alpha_1, \alpha_2) = \left(\alpha_1 + \alpha_2 + \frac{\pi i}{n}\right) \left(-\alpha_1 + \alpha_2 + \frac{\pi i}{n}\right) \left(\alpha_1 - \alpha_2 + \frac{\pi i}{n}\right) \left(-\alpha_1 - \alpha_2 + \frac{\pi i}{n}\right),
\]
(3.24)
which is invariant under the change of variables $\alpha_j \rightarrow -\alpha_j, (j = 1, 2)$,

\[ \Delta(\alpha_1, \alpha_2) = \Delta(-\alpha_1, \alpha_2) = \Delta(\alpha_1, -\alpha_2) = \Delta(-\alpha_1, -\alpha_2). \] (3.25)

The actions of the operators $Z^*_j(\beta)$ on the boundary state $|B\rangle$ are given by

\[
Z^*_j(\beta)|B\rangle = e^{-2j\gamma}H_0(\beta)\prod_{l=1}^j \int_{-\infty}^\infty d\alpha_l \prod_{l=1}^j G_l(\alpha_l) \prod_{l=2}^j \left( \alpha_l + \alpha_{l-1} + \frac{\pi i}{n} \right) \prod_{l=1}^j \frac{1}{\Delta(\alpha_l, \alpha_{l-1})} \prod_{l=0}^j V^c_0(\alpha_l)V^c_0(-\alpha_l)|B\rangle.
\] (3.26)

The 0-th components of the characterizing relation (3.1) become

\[
(LHS) = K(\beta)_0^0H_0(\beta)\prod_{l=1}^j \int_{-\infty}^\infty d\alpha_l \prod_{l=1}^j G_l(\alpha_l) \prod_{l=2}^j \left( \alpha_l + \alpha_{l-1} + \frac{\pi i}{n} \right) K(\beta)_0^0V^c_0(\beta)V^c_0(-\beta)|B\rangle,
\] (3.27)

\[
(RHS) = H_0(-\beta)V^c_0(\beta)V^c_0(-\beta)|B\rangle.
\] (3.28)

The 0-th component of the equation (3.1) follows from

\[
K(\beta)_0^0 = \frac{H_0(-\beta)}{H_0(\beta)}.
\] (3.29)

Next we consider the $0 < j < L$ th component of the equation (3.1). The difference of the both hand side is given by

\[
K(\beta)_0^j Z^*_j(\beta)|B\rangle - Z^*_j(-\beta)|B\rangle
= 2e^{-2j\gamma}\beta H_0(-\beta)\prod_{l=1}^j \int_{-\infty}^\infty d\alpha_l \prod_{l=1}^j G_l(\alpha_l) \prod_{l=2}^j \left( \alpha_l + \alpha_{l-1} + \frac{\pi i}{n} \right) \prod_{l=1}^j \frac{1}{\Delta(\alpha_l, \alpha_{l-1})} \prod_{l=0}^j V^c_0(\alpha_l)V^c_0(-\alpha_l)|B\rangle.
\] (3.30)

The following part :

\[
\prod_{l=1}^j \frac{1}{\Delta(\alpha_l, \alpha_{l-1})} \prod_{l=0}^j V^c_0(\alpha_l)V^c_0(-\alpha_l)|B\rangle,
\] (3.31)

is invariant under the change of variable $\alpha_l \leftrightarrow -\alpha_l$. Therefore a sufficient condition of the equation,

\[
K(\beta)_0^j Z^*_j(\beta)|B\rangle = Z^*_j(-\beta)|B\rangle, \quad (0 < j < L),
\] (3.32)
is a polynomial identity,
\[
\sum_{\epsilon_1 \cdots \epsilon_j = \pm} j \prod_{l=1}^{j} \epsilon_l \prod_{l=2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}) = 0, \quad (0 < j < L).
\] (3.33)

The above equation (3.33) follows from following inductive relation.
\[
\sum_{\epsilon_1 \cdots \epsilon_j = \pm} j \prod_{l=1}^{j} \epsilon_l \prod_{l=2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}) = 2 \alpha_1 \sum_{\epsilon_{L+1} \cdots \epsilon_j = \pm} \prod_{l=L+1}^{j} \epsilon_l \prod_{l=L+2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}).
\] (3.34)

As the same arguments as the above the characterizing relation (3.1) for \( L \leq j < M \), is reduced to the following polynomial identity,
\[
\sum_{\epsilon_1 \cdots \epsilon_j = \pm} \left( \mu - \frac{\pi i}{n} - \epsilon_1 \alpha_1 \right) \left( \mu - \frac{\pi i}{n} L + \epsilon_L \alpha_L \right) \prod_{l=1}^{j} \epsilon_l \prod_{l=2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}) = 0,
\] \((L \leq j < M)\. (3.35)

The above equation (3.35) is deformed to the following.
\[
2^L \prod_{l=1}^{L} \alpha_l \left( \mu - \frac{\pi i}{n} L - \alpha_L \right) \left( \mu - \frac{\pi i}{n} L + \alpha_L \right) \times \sum_{\epsilon_{L+1} \cdots \epsilon_j = \pm} \prod_{l=L+1}^{j} \epsilon_l \prod_{l=L+2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}).
\] (3.36)

Therefore the equation (3.35) is reduced to the equation (3.33).

As the same arguments as the above the characterizing relation (3.1) for \( M \leq j \leq n-1 \), is reduced to the following polynomial identity.
\[
\sum_{\epsilon_1 \cdots \epsilon_j = \pm} \left( \mu - \frac{\pi i}{n} L + \epsilon_L \alpha_L \right) \left( \mu - \frac{\pi i}{n} (2L - M) - \epsilon_M \alpha_M \right) \prod_{l=1}^{j} \epsilon_l \prod_{l=2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}) = 0, \quad (M \leq j \leq n-1).
\] (3.37)

The above equation (3.37) is reduced to the following.
\[
2^M \prod_{l=1}^{M} \alpha_l \left( \mu - \frac{\pi i}{n} (2L - M) - \alpha_M \right) \left( \mu - \frac{\pi i}{n} (2L - M) + \alpha_M \right) \times \sum_{\epsilon_{M+1} \cdots \epsilon_j = \pm} \prod_{l=M+1}^{j} \epsilon_l \prod_{l=M+2}^{j} (\epsilon_l \alpha_l + \epsilon_{l-1} \alpha_{l-1} + \frac{\pi i}{n}).
\] (3.38)
Therefore the equation (3.37) is reduced to the equation (3.33). Now we have proved the characterizing relation (3.1).

Next we consider the dual boundary state \(|B|\), which satisfies

\[ K(\beta)_j^j \langle B| Z_j(-\beta + \pi i) = \langle B| Z_j(\beta + \pi i), \ (j = 0, \cdots, n - 1). \]  

(3.39)

Here \(K(\beta)_j^j\) are diagonal elements of the boundary \(K\)-matrix.

We make the ansatz that the dual boundary state has the following form.

\[ \langle B| = \langle \text{vac}| e^G. \]  

(3.40)

Here \(G\) is a quadratic form of the free bosons,

\[ G = \sum_{j,k=1}^{n-1} \int_0^\infty \gamma_{j,k}(t) a_j(t) a_k(t) dt + \sum_{j=1}^{n-1} \int_0^\infty \delta_j(t) a_j(t) dt, \]  

(3.41)

where \(\gamma_{j,k}(t)\) and \(\delta_j(t)\) are scalar functions.

As the same arguments as the above the following coefficients functions \(\gamma_{j,k}(t)\) and \(\delta_j(t)\) of the dual boundary state are given by

\[ \gamma_{j,k}(t) = -te^{-\frac{(\pi n - 2\pi t)^2}{2}} \times I_{j,k}(t), \]  

(3.42)

and

\[ \delta_j(t) = \frac{1}{2}(e^{-\frac{\pi t}{2}} - 1)e^{-\pi t} \sum_{l=1}^{n-1} I_{l,j}(t) \]

\[ - e^{i(\mu + \frac{\pi}{n}(2M + 1 - L))t} I_{L,j}(t) - e^{i(-\mu + \frac{\pi}{n}(2n - M + 1))t} I_{M,j}(t). \]  

(3.43)

The effect of the operator \(e^G\) is given by

\[ e^G a_l(t) e^{-G} = 0, \ (t > 0), \]  

(3.44)

\[ e^G a_l(-t) e^{-G} = a_l(-t) + e^{-2\pi t} a_l(t) + \frac{e^{-\pi t}}{t} (e^{\frac{\pi t}{2}} - 1) \]

\[ + \delta_{L,l} \frac{e^{i\mu t + \frac{\pi}{n}(L - 2M)t}}{t} + \delta_{M,l} \frac{e^{-i\mu t + \frac{\pi}{n}Mt - 2\pi t}}{t}, \ (t > 0). \]  

(3.45)

The following properties are useful.

\[ \langle B| V_j^c(\alpha + \pi i) = G_j^*(\alpha) \langle B| V_j^a(-\alpha + \pi i), \ (j = 1, \cdots, n - 1), \]  

(3.46)

\[ \langle B| V_n^c(\beta + \pi i) = H_n^*(\beta) \langle B| V_n^a(-\beta + \pi i). \]  

(3.47)
Here we have set

\[ G_j^* (\alpha) = \begin{cases} 
  ie^\gamma \times \alpha, & j \neq L, M, \\
  \alpha \times (-\mu + \frac{\pi}{n} (n + L - 2M) + \alpha)^{-1}, & j = L, \\
  \alpha \times (\mu - \frac{\pi}{n} (n - M) + \alpha)^{-1}, & j = M,
\end{cases} \]  

and

\[ H_n^* (\beta) = \text{Const.} (\mu - \beta)^{\delta_{M,n}} \frac{\Gamma \left( \frac{-\beta + \mu}{2\pi i} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + \frac{L}{n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + 1 - \frac{L}{2n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + 1 + \frac{L-M}{2n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + 1 + \frac{L-M}{2n} \right)}{\Gamma \left( \frac{-\beta + \mu}{2\pi i} + 1 - \frac{L}{2n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + 1 + \frac{L-M}{2n} \right) \Gamma \left( \frac{-\beta - \mu}{2\pi i} + 1 + \frac{L-M}{2n} \right)}. \]  

The function \( H_n^* (\beta) \) satisfies

\[ k_{L,M} (\beta) = \frac{H_n^* (\beta)}{H_n^* (-\beta)} \left( \frac{\mu + \beta}{\mu - \beta} \right)^{\delta_{M,n}}. \]  

Let us summarize the results of this section. We have constructed the free field realization of the boundary state \(|B\rangle\) and the dual boundary state \langle B|.

\[ \langle B| = \langle \text{vac} | e^G, e^F | \text{vac} \rangle = |B\rangle. \]  

Here \( F \) and \( G \) are quadratic forms of free bosons. The explicit formulae of \( F \) and \( G \) are given by (3.3), (3.41). The coefficients \( \alpha_{j,k}(t) \), \( \beta_j(t) \), \( \delta_{j,k}(t) \), \( \gamma_j(t) \), and \( I_{j,k}(t) \) are given by (3.6), (3.20), (3.42), (3.43) and (3.8), respectively.

### 4 Form Factors

The purpose of this section is to give integral representations of the form factors of the local fields, defined by

\[ f(\delta_1, \cdots, \delta_M | \beta_1, \cdots, \beta_N)_{j_1, \cdots, j_N}^{k_1, \cdots, k_M} \]

\[ = \frac{1}{\langle B|B \rangle} \times \langle B|Z'_{k_1} (\delta_1) \cdots Z'_{k_M} (\delta_M) | \beta_1, \cdots, \beta_N \rangle_{j_1, \cdots, j_N}. \]  

Here \( \langle B \rangle \) and \( |B \rangle \) are the boundary state and its dual. The state \(|\beta_1, \cdots, \beta_N\rangle_{j_1, \cdots, j_N}\) is the basis of the space of the physical states defined in the section 2. The operators \( Z'_k (\delta), (k = 0, \cdots, n - 1) \) are the local operators given in the next section.
4.1 Local Operators

In terminology of the Quantum Field Theory, the local operator is the one which commutes with the Zamolodchikov-Faddeev operators, up to scalar function multiplicity. In this subsection we give the free field realization of a class of the local operators $Z_j' (\beta)$ of the present model. Naively the local operators of the present model are a limiting case of the type-I vertex operators $\Phi_j (\beta)$ of the $A_{n-1}^{(1)}$ critical chain [15].

In this section, we give the free field realization of the local operators $Z_j' (\delta)$ which satisfy the following commutation relations.

\[ Z_j' (\beta_1) Z_k' (\beta_2) = \text{Const.} \mathcal{L}(\beta_1 - \beta_2) Z_k' (\beta_2) Z_j' (\beta_1), \]

(4.2)

where we set

\[ \mathcal{L}(\beta) = \frac{\Gamma \left( \frac{\beta}{2\pi i} + 1 - \frac{1}{2n} \right) \Gamma \left( \frac{-\beta}{2\pi i} + 1 - \frac{1}{2n} \right)}{\Gamma \left( \frac{-\beta}{2\pi i} + 1 - \frac{1}{2n} \right) \Gamma \left( \frac{\beta}{2\pi i} + 1 - \frac{1}{2n} \right)}. \]

(4.3)

Let us set the auxiliary fields $b_j(t), (1 \leq j \leq n-1, t \in \mathbb{R})$ by

\[ b_j(t) = e^{-\frac{\pi}{n} |t|} \times a_j(t). \]

(4.4)

The boson field $b(t)$ satisfies the following commutation relation.

\[ [b_j(t), b_k(t')] = -\frac{1}{t} \frac{\text{sh} \left( \frac{\pi a_j a_k \pi t}{n} \right)}{\text{sh} \frac{\pi t}{n}} e^{-\frac{\pi}{n} |t|} \delta(t + t'). \]

(4.5)

Here $((a_j|a_k))_{1 \leq j, k \leq n-1}$ is the Cartan matrix of type $A_{n-1}$.

\[ b_1^*(t) = -\sum_{j=1}^{n-1} b_j(t) \frac{\text{sh} \left( \frac{(n-j) \pi t}{n} \right)}{\text{sh} \frac{\pi t}{n}}. \]

(4.6)

We have

\[ [b_1^*(t), b_j(t')] = \delta_{j,1} \frac{e^{-\frac{\pi}{n} |t|}}{t} \delta(t + t'). \]

(4.7)

Let us introduce the basic operators,

\[ U_j(\delta) = : \exp \left( -\int_{-\infty}^{\infty} b_j(t) e^{it} dt \right) : (1 \leq j \leq n-1), \]

(4.8)

\[ U_0(\delta) = : \exp \left( -\int_{-\infty}^{\infty} b_1^*(t) e^{it} dt \right) :. \]

(4.9)
The basic operators satisfy the following contraction relations.

\[ U_j(\delta_1)U_{j-1}(\delta_2) = \frac{-ie^{-\gamma}}{\delta_2 - \delta_1 - \frac{2\pi i}{n}} : U_j(\delta_1)U_{j-1}(\delta_2) : , \quad (j = 1, \cdots, n-1) , \quad (4.10) \]

\[ U_{j-1}(\delta_1)U_j(\delta_2) = \frac{-ie^{-\gamma}}{\delta_2 - \delta_1 - \frac{2\pi i}{n}} : U_{j-1}(\delta_1)U_j(\delta_2) : , \quad (j = 1, \cdots, n-1) , \quad (4.11) \]

\[ U_j(\delta_1)U_j(\delta_2) = e^{2\gamma(\delta_1 - \delta_2)} \left( \delta_2 - \delta_1 - \frac{2\pi i}{n} \right) : U_j(\delta_1)U_j(\delta_2) : , \quad (j = 1, \cdots, n-1) , \quad (4.12) \]

\[ U_0(\delta_1)U_0(\delta_2) = e^{(\gamma + \log 2\pi)\frac{2\pi i}{n}} \frac{\Gamma \left( \frac{\delta_1 - \delta_2}{2\pi i} + 1 \right)}{\Gamma \left( \frac{\delta_1 - \delta_2}{2\pi i} + \frac{1}{n} \right)} : U_0(\delta_1)U_0(\delta_2) : \quad (4.13) \]

We give the free field realization of the local operators \( Z'_j(\delta) \), \((j = 1, \cdots, n-1)\).

\[ Z'_j(\delta) = \int_{-\infty}^{\infty} d\gamma_1 \cdots \int_{-\infty}^{\infty} d\gamma_j U_0(\delta)U_1(\gamma_1) \cdots U_j(\gamma_j) \prod_{k=1}^{j} (\gamma_{k-1} - \gamma_k - \frac{\pi i}{n}) , \]

\[ = (-ie^{-\gamma})^j \int_{-\infty}^{\infty} d\gamma_1 \cdots \int_{-\infty}^{\infty} d\gamma_j \]

\[ \times : U_0(\delta)U_1(\gamma_1) \cdots U_j(\gamma_j) : \prod_{k=1}^{j} \frac{1}{(\gamma_{k-1} - \gamma_k - \frac{\pi i}{n}) (\gamma_k - \gamma_{k-1} - \frac{\pi i}{n})} , \quad (4.14) \]

where we set \( \delta = \gamma_0 \). The local operators \( Z'_j(\delta) \) satisfy the commutation relations similar to (2.8). See the reference [14]. The two type basic operators \( U_j(\delta) \) and \( V_k(\beta) \) satisfy the following interaction relations.

\[ V_j(\beta_1)U_{j-1}(\beta_2) = \frac{ie^{-\gamma}}{\beta_1 - \beta_2} : V_j(\beta_1)U_{j-1}(\beta_2) : , \quad (j = 1, \cdots, n-1) , \quad (4.15) \]

\[ V_{j-1}(\beta_1)U_j(\beta_2) = \frac{ie^{-\gamma}}{\beta_1 - \beta_2} : V_{j-1}(\beta_1)U_j(\beta_2) : , \quad (j = 1, \cdots, n-1) , \quad (4.16) \]

\[ V_j(\beta_1)U_j(\beta_2) = -e^{2\gamma} \left( \beta_1 - \beta_2 + \frac{\pi i}{n} \right) \left( \beta_2 - \beta_1 + \frac{\pi i}{n} \right) : V_j(\beta_1)U_j(\beta_2) : , \quad (4.17) \]

\[ U_j(\beta_1)V_j(\beta_2) = -e^{2\gamma} \left( \beta_1 - \beta_2 + \frac{\pi i}{n} \right) \left( \beta_2 - \beta_1 + \frac{\pi i}{n} \right) : U_j(\beta_1)V_j(\beta_2) : , \quad (4.18) \]

\[(j = 1, \cdots, n-1), \]

\[ V_0(\beta_1)U_0(\beta_2) = e^{(\gamma + \log 2\pi)\frac{2\pi i}{n}} \frac{\Gamma \left( \frac{\beta_1 - \beta_2}{2\pi i} + 1 - \frac{1}{2n} \right)}{\Gamma \left( \frac{\beta_1 - \beta_2}{2\pi i} + \frac{1}{2n} \right)} : V_0(\beta_1)U_0(\beta_2) : . \quad (4.19) \]

The commutation relation (1.2) is proved as the same manner as the reference [13].
4.2 Integral Representations

In this section we calculate explicit formulae of the matrix elements,

\[ f(\delta_1, \cdots, \delta_M | \beta_1 \cdots \beta_N)_{j_1 \cdots j_N}^{k_1 \cdots k_M}. \]  

(4.20)

In order to evaluate the above expectation value, we invoke the free field realization of the Zamolodchikov-Faddeev operaors, the local operators, and the boundary state and it’s dual state.

Fix the indexes \( \{j_1, \cdots, j_N\} \), where \( j_1, \cdots, j_N \in \{0, 1, \cdots, n-1\} \), and \( \{k_1, \cdots, k_M\} \), where \( k_1, \cdots, k_M \in \{0, 1, \cdots, n-1\} \). We associate the integration variables \( \alpha_{j,r} \), (1 \( \leq r \leq N, 1 \leq j \leq j_r \)) to the basic operator \( V_j(\alpha_{j,r}) \) contained in the Zamolodchikov-Faddeev operator \( Z^*_j(\beta_r) \). We also use the notation \( \alpha_{0,r} = \beta_r \). We associate the integration variables \( \gamma_{k,s} \), (1 \( \leq s \leq N, 1 \leq k \leq k_s \)) to the basic operator \( U_j(\gamma_{k,s}) \) contained in the local operator \( Z'_k(\delta_s) \). We also use the notation \( \gamma_{0,s} = \delta_s \). Let us set the index set \( A_j \) and \( G_k \) by

\[ A_j = \{r | j_r \geq j\}, \quad G_k = \{s | k_s \geq k\}. \]  

(4.21)

By normal-ordering the product of the Zamolodchikov-Faddeev operators and the local operators, we have the following formula.

\[ f(\delta_1, \cdots, \delta_M | \beta_1 \cdots \beta_N)_{j_1 \cdots j_N}^{k_1 \cdots k_M} = E(\{\beta\} | \{\delta\}) \prod_{r=1}^{N} \prod_{j=1}^{j_r} \int_{-\infty}^{\infty} d\alpha_{j,r} \prod_{s=1}^{M} \prod_{k=1}^{k_s} \int_{-\infty}^{\infty} d\gamma_{k,s} I(\{\alpha\} | \{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M}. \]  

(4.22)

Here we set \( E(\{\beta\} | \{\delta\}) \) by

\[ E(\{\beta\} | \{\delta\}) = \prod_{r=1}^{N} \prod_{s=1}^{M} \frac{\Gamma(\frac{\beta_r - \delta_s}{2\pi} + 1 - \frac{1}{n})}{\Gamma(\frac{\beta_r - \delta_s}{2\pi} + \frac{1}{2})} \times \prod_{1 \leq r_1 < r_2 \leq N} \frac{\Gamma(\frac{\beta_{r_1} - \beta_{r_2}}{2\pi} + 1 - \frac{1}{n})}{\Gamma(\frac{\beta_{r_1} - \beta_{r_2}}{2\pi} + \frac{1}{n})} \prod_{1 \leq s_1 < s_2 \leq M} \frac{\Gamma(\frac{\delta_{s_1} - \delta_{s_2}}{2\pi} + 1)}{\Gamma(\frac{\delta_{s_1} - \delta_{s_2}}{2\pi} + \frac{1}{n})}. \]  

(4.23)
Here we set the integrand function by

\[
I(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M} = \prod_{j=1}^{n-1} \left\{ \prod_{r_1, r_2 \in A_{j_1}}^{r_1 \neq r_2} (\alpha_{j,r_1} - \alpha_{j,r_2})(\alpha_{j,r_1} - \alpha_{j,r_2} - \frac{2\pi i}{n}) \right\} \\
\times \prod_{j=1}^{n-1} \left\{ \prod_{r_1 \in A_{j}, r_2 \in A_{j-1}}^{r_1 \neq r_2} (\alpha_{j-1,r_1} - \alpha_{j,r_2} - \frac{\pi i}{n})^{-1} \prod_{r_1 \in A_{j-1}, r_2 \in A_j}^{r_1 \neq r_2} (\alpha_{j,r_1} - \alpha_{j-1,r_2} - \frac{\pi i}{n})^{-1} \right\} \\
\times \prod_{k=1}^{n-1} \left\{ \prod_{s_1, s_2 \in G_{j_k}}^{s_1 \neq s_2} (\gamma_{k,s_1} - \gamma_{k,s_2})(\gamma_{k,s_1} - \gamma_{k,s_2} + \frac{2\pi i}{n}) \right\} \\
\times \prod_{j=1}^{n-1} \left\{ \prod_{s \in G_{j}, r \in A_j} (\alpha_{j,r} - \gamma_{j,s} + \frac{\pi i}{n})(\gamma_{j,s} - \alpha_{j,r} + \frac{\pi i}{n}) \right\} \\
\times \prod_{j=1}^{n-1} \left\{ \prod_{s \in G_{j_k}, r \in A_{j-1}}^{s_1 \neq s_2} (\gamma_{j,s} - \alpha_{j-1,r})^{-1} \prod_{s \in G_{j}, r \in A_j} (\gamma_{j-1,s} - \alpha_{j,r})^{-1} \right\} \times J(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M}.
\]

(4.24)

Here we have set

\[
J(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M} = \frac{1}{\langle B|B \rangle} \times \langle B| \exp \left( \int_0^\infty X_j(t)a_j(-t)dt \right) \exp \left( \int_0^\infty Y_j(t)a_j(t)dt \right) |B \rangle,
\]

(4.25)

where

\[
X_j(t) = \frac{\text{sh}(n-j)t}{n} \left( -\sum_{r=1}^{N} e^{-\frac{\pi i}{n}t} \sum_{s=1}^{M} e^{-\frac{\pi i}{n}t} + \sum_{r \in A_j} e^{-\frac{\pi i}{n}t} \sum_{s \in \bar{G}_j} e^{-\frac{\pi i}{n}t} \right),
\]

(4.26)

\[
Y_j(t) = \frac{\text{sh}(n-j)t}{n} \left( -\sum_{r=1}^{N} e^{\frac{\pi i}{n}t} \sum_{s=1}^{M} e^{\frac{\pi i}{n}t} + \sum_{r \in A_j} e^{\frac{\pi i}{n}t} \sum_{s \in \bar{G}_j} e^{\frac{\pi i}{n}t} \right) + \sum_{r \in A_j} e^{\frac{\pi i}{n}t} \sum_{s \in \bar{G}_j} e^{\frac{\pi i}{n}t} \right),
\]

(4.27)

Next we evaluate the vacuum expectation value, \( J(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M} \). In what follows we use the abbreviation,

\[
A_{j,k}(t) = -\frac{1}{t} \frac{\text{sh}(n-j)t}{\text{sh}(n)t} e^{\frac{\pi i}{n}t} |k|, \quad (j, k = 0, \cdots, n - 1).
\]

(4.28)
For our purpose we use the coherent states, $|\xi_1, \cdots, \xi_{n-1}\rangle$, defined by

$$|\xi_1, \cdots, \xi_{n-1}\rangle = \exp \left( \sum_{k=1}^{n-1} \int_0^\infty \xi_k(s)a_k(-s)ds \right) |\text{vac}\rangle,$$  
(4.29)

and its dual states, $\langle \bar{\xi}_1, \cdots, \bar{\xi}_{n-1}|$,

$$\langle \bar{\xi}_1, \cdots, \bar{\xi}_{n-1}| = \langle \text{vac}| \exp \left( \sum_{k=1}^{n-1} \int_0^\infty \bar{\xi}_k(s)a_k(s)ds \right).$$  
(4.30)

The coherent states enjoy

$$a_j(t)|\xi_1, \cdots, \xi_{n-1}\rangle = \sum_{k=1}^{n-1} A_{j,k}(t)\xi_k(t)|\xi_1, \cdots, \xi_{n-1}\rangle, \quad (t > 0),$$  
(4.31)

$$\langle \bar{\xi}_1, \cdots, \bar{\xi}_{n-1}|a_j(-t) = \sum_{k=1}^{n-1} A_{j,k}(t)\bar{\xi}_k(t)\langle \bar{\xi}_1, \cdots, \bar{\xi}_{n-1}|, \quad (t > 0).$$  
(4.32)

For our purpose the following completeness relation by means of Feynmann path integral is useful.

$$id = \text{Const.} \times \int \prod_{k=1}^{n-1} \prod_{s>0} d\xi_k(s)d\bar{\xi}_k(s)$$

$$\times \exp \left( -\sum_{k_1,k_2=1}^{n-1} \int_0^\infty \sum_{k=1}^{n-1} A_{k_1,k_2}(s)\xi_{k_1}(s)\bar{\xi}_{k_2}(s)ds \right) |\xi_1, \cdots, \xi_{n-1}\rangle \langle \bar{\xi}_1, \cdots, \bar{\xi}_{n-1}|.$$

(4.33)

Here the integration $\int d\xi d\bar{\xi}$ is taken over the entire complex plane with the measure $d\xi d\bar{\xi} = -2i dxdy$ for $\xi = x + iy$.

In what follows we use the following abbreviations.

$$\tilde{\beta}_j(t) = \sum_{k=1}^{n-1} A_{j,k}(t)\beta_k(t) = \frac{1}{2t}(e^{\pi t} - 1) - \frac{e^{i\mu + \frac{\pi}{n}(M-2L)t} + e^{-i\mu + \frac{\pi}{n}(M-2L)t}}{t}\delta_{j,L} - \frac{e^{i\mu + \frac{\pi}{n}(M-2L)t} + e^{-i\mu + \frac{\pi}{n}(M-2n)t}}{t}\delta_{j,M},$$  
(4.34)

$$\tilde{\delta}_j(t) = \sum_{k=1}^{n-1} A_{j,k}(t)\delta_k(t) = \frac{e^{-\pi t}}{2t}(e^{\pi t} - 1) + \frac{e^{i\mu + \frac{\pi}{n}(L-2M)t} + e^{-i\mu + \frac{\pi}{n}(M-2n)t}}{t}\delta_{j,L} + \frac{e^{i\mu + \frac{\pi}{n}(L-2M)t} + e^{-i\mu + \frac{\pi}{n}(M-2n)t}}{t}\delta_{j,M},$$  
(4.35)
\[ \tilde{X}_j(t) = \sum_{k=1}^{n-1} A_{j,k}(t) X_k(t) \]  
\[ = \frac{e^{\pi t}}{t} \left( \sum_{r \in \mathcal{A}_{j-1}} e^{-i\alpha_{j-1,r} t} + \sum_{r \in \mathcal{A}_{j+1}} e^{-i\alpha_{j+1,r} t} \right) - \frac{1}{t} (1 + e^{\frac{2\pi}{n}}) \sum_{r \in \mathcal{A}_j} e^{-i\alpha_{j,r} t} \]
\[ - \frac{1}{t} \left( \sum_{s \in \mathcal{G}_{j-1}} e^{-i\gamma_{j-1,s} t} + \sum_{s \in \mathcal{G}_{j+1}} e^{-i\gamma_{j+1,s} t} \right) + \frac{e^{-\pi t}}{t} (1 + e^{\frac{2\pi}{n}}) \sum_{s \in \mathcal{G}_j} e^{-i\gamma_{j+1,s} t}, \]
\[ \tilde{Y}_j(t) = \sum_{k=1}^{n-1} A_{j,k}(t) Y_k(t) = \tilde{X}_j(t)^*. \]  

Here we understand \( \mathcal{A}_n = \mathcal{G}_n = \{\emptyset\} \).

Using the Bogoliubov transformation of \( e^F \), we have

\[ J(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M} \]
\[ = \frac{1}{\langle vac|e^G e^F|vac\rangle} \times \langle vac|e^G e^F \exp \left( \int_0^\infty \sum_{j=1}^{n-1} (X_j(t) + Y_j(t)) a_j(-t) dt \right) |vac\rangle \]
\[ \times \exp \left( \int_0^\infty \sum_{j=1}^{n-1} \bar{\beta}_j(t) Y_j(t) dt \right) \exp \left( \frac{1}{2} \int_0^\infty \sum_{j_1,j_2=0}^{n-1} \bar{Y}_{j_1}(t) Y_{j_2}(t) dt \right). \]

We then insert the completeness relation of the coherent states between \( e^G \) and \( e^F \), we have without-operator formula,

\[ \langle vac|e^G e^F \exp \left( \int_0^\infty \sum_{j=1}^{n-1} (X_j(t) + Y_j(t)) a_j(-t) dt \right) |vac\rangle \]
\[ = \int \prod_{j=1}^{n-1} \prod_{t>0} \xi_j(t) d\xi_j(t) \exp \left( \int_0^\infty \sum_{j=1}^{n-1} (\Delta_j(t) \xi_j(t) + (\bar{\beta}_j(t) + \bar{X}_j(t) + \bar{Y}_j(t)) \bar{\xi}_j(t) dt \right)
\[ + \int_0^\infty \sum_{j_1,j_2=0}^{n-1} A_{j_1,j_2}(t) \left( \frac{e^{-2\pi t}}{2} \xi_{j_1}(t) \xi_{j_2}(t) - \xi_{j_1}(t) \bar{\xi}_{j_2}(t) + \frac{1}{2} \xi_{j_1}(t) \bar{\xi}_{j_2}(t) \right) dt \). \]

Performing the Gaussian integral (quadratic integral) calculations,

\[ \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \]  

(4.40)
we get the following formula.

\[
J(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N} = \exp \left( \int_0^\infty \frac{-te^{-\frac{t}{2}}}{1 - e^{-2\pi t}} \left( \sum_{l=1}^{n-1} I_{l,l}(t) \left( e^{-2\pi t} \tilde{X}_l(t)\tilde{Y}_l(t) + e^{-2\pi t} \tilde{X}_{l_1}(t)\tilde{X}_{l_2}(t) + e^{-2\pi t} \tilde{X}_l(t)\tilde{Y}_{l_1}(t)\tilde{Y}_{l_2}(t) \right) \right) dt \right),
\]

Therefore we can write \(J(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N}\) by using multi-Gamma functions. We arrive at the following formula.

\[
J(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N} = J_B(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N} \times J_F(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N},
\]

where the factor \(J_B(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N}\) depends on the boundary condition, and the factor \(J_F(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N}\) is free from the boundary condition.

\[
J_F(\{\alpha\}|\{\gamma\})^{k_1\ldots k_M}_{j_1\ldots j_N} = \prod_{l=1}^{n-1} F_{l,l}^{(1)}(\{\alpha + 2\pi\}, \{\gamma + 2\pi\}|\{-\alpha\}, \{-\gamma\})
\]

\[
\times \prod_{l=1}^{n-1} \sqrt{F_{l,l}^{(1)}(\{\alpha - 2\pi i\}, \{\gamma - 2\pi i\}|\{\alpha\}, \{\gamma\}) \cdot F_{l,l}^{(1)}(\{-\alpha\}, \{-\gamma\}|\{-\alpha\}, \{-\gamma\})}
\]

\[
\times \prod_{l_1, l_2} F_{l_1, l_2}^{(1)}(\{\alpha - 2\pi i\}, \{\gamma - 2\pi i\}|\{-\alpha\}, \{-\gamma\}) \cdot F_{l_1, l_2}^{(1)}(\{-\alpha - 2\pi i\}, \{-\gamma - 2\pi i\}|\{\alpha\}, \{\gamma\})
\]

\[
\times \prod_{l_1, l_2} F_{l_1, l_2}^{(1)}(\{\alpha - 2\pi i\}, \{\gamma - 2\pi i\}|\{\alpha\}, \{\gamma\}) \cdot F_{l_1, l_2}^{(1)}(\{-\alpha\}, \{-\gamma\}|\{-\alpha\}, \{-\gamma\})
\]

\[
\times \prod_{l_1, l_2} F_{l_1, l_2}^{(1)}(\{\alpha - \pi i\}, \{\gamma - \pi i\}) \cdot F_{l_1, l_2}^{(1)}(-\alpha, \{-\gamma\})
\]

\[
\times \prod_{l_1, l_2} F_{l_1, l_2}^{(1)}(\{\alpha - \pi i\}, \{\gamma - \pi i\}) \cdot F_{l_1, l_2}^{(1)}(\{\alpha - \pi i\}, \{\gamma - \pi i\})
\]

\[
\times \prod_{l_1, l_2} F_{l_1, l_2}^{(1)}(\{-\alpha\}, \{-\gamma\}) \cdot F_{l_1, l_2}^{(1)}(\{-\alpha\}, \{-\gamma\}).
\]
Here we have set

\[
F^{(1)}_{l_1,l_2}(\{\alpha\}, \{\gamma\}|\{\alpha'\}, \{\gamma'\}) = \prod_{r_1 \in A_{l_1-1}} \frac{\Omega^A_{l_1,l_2}(\alpha_{l_1-1,r_1} + \frac{\pi i}{n} \{\alpha'\})}{\Omega^G_{l_1,l_2}(\alpha_{l_1-1,r_1} + \frac{\pi i}{n} \{\alpha'\})} \prod_{r_1 \in A_{l_1+1}} \frac{\Omega^A_{l_1,l_2}(\alpha_{l_1+1,r_1} + \frac{\pi i}{n} \{\gamma' - \frac{\pi i}{n}\})}{\Omega^G_{l_1,l_2}(\alpha_{l_1+1,r_1} + \frac{\pi i}{n} \{\gamma' - \frac{\pi i}{n}\})} \\
\times \prod_{r_1 \in A_{l_1}} \frac{\Omega^G_{l_1,l_2}(\gamma_{l_1-1,s_1} \{\gamma' - \frac{\pi i}{n}\})}{\Omega^A_{l_1,l_2}(\gamma_{l_1-1,s_1} \{\gamma' - \frac{\pi i}{n}\})} \prod_{r_1 \in A_{l_1+1}} \frac{\Omega^G_{l_1,l_2}(\gamma_{l_1+1,s_1} \{\gamma' - \frac{\pi i}{n}\})}{\Omega^A_{l_1,l_2}(\gamma_{l_1+1,s_1} \{\gamma' - \frac{\pi i}{n}\})} \\
\times \prod_{s_1 \in \mathcal{G}_{l_1}} \frac{\Omega^A_{l_1,l_2}(\gamma_{l_1,s_1} + \frac{\pi i}{n} \{\alpha'\})}{\Omega^G_{l_1,l_2}(\gamma_{l_1,s_1} + \frac{\pi i}{n} \{\alpha'\})} \Omega^A_{l_1,l_2}(\gamma_{l_1,s_1} - \frac{\pi i}{n} \{\gamma' - \frac{\pi i}{n}\}), \tag{4.44}
\]

where

\[
\Omega^A_{l_1,l_2}(\beta|\{\alpha\}) = \frac{\prod_{r_2 \in A_{l_2-1}} e^{(1)}_{l_1,l_2}(\beta + \alpha_{l_2-1,r_2} - \frac{\pi i}{n}) \prod_{r_2 \in A_{l_2+1}} e^{(1)}_{l_1,l_2}(\beta + \alpha_{l_2+1,r_2} - \frac{\pi i}{n}) \prod_{r_2 \in A_{l_2}} e^{(1)}_{l_1,l_2}(\beta + \alpha_{l_2,r_2}) e^{(1)}_{l_1,l_2}(\beta + \alpha_{l_2,r_2} - 2\frac{\pi i}{n})}{\prod_{r_2 \in A_{l_2}} e^{(1)}_{l_1,l_2}(\beta + \alpha_{l_2,r_2} - 2\frac{\pi i}{n})}. \tag{4.45}
\]

Here we have set

\[
e^{(1)}_{l_1,l_2}(\alpha) = \frac{\Gamma_3\left(i\alpha + \frac{\pi}{n}(l_1 + l_2 + 2)\right) 2\pi, 2\pi, 2\pi}{\Gamma_3\left(i\alpha + \frac{\pi}{n}(l_2 - l_1 + 2)\right) 2\pi, 2\pi, 2\pi} \tag{4.46}
\]

We have set

\[
F^{(2)}_{l_1,l_2,l_3}(\{\alpha\}, \{\gamma\}) = \prod_{r \in A_{l_3-1}} \frac{e^{(2)}_{l_1,l_2}(\alpha_{l_3-1,r} - \frac{\pi i}{n}) \prod_{r \in A_{l_3+1}} e^{(2)}_{l_1,l_2}(\alpha_{l_3+1,r} - \frac{\pi i}{n})}{\prod_{r \in A_{l_3}} e^{(2)}_{l_1,l_2}(\alpha_{l_3,r}) e^{(2)}_{l_1,l_2}(\alpha_{l_3,r} - 2\frac{\pi i}{n})} \\
\times \prod_{s \in \mathcal{G}_{l_3}} \frac{e^{(2)}_{l_1,l_2}(\gamma_{l_3,s} + \frac{\pi i}{n}) e^{(2)}_{l_1,l_2}(\gamma_{l_3,s} - \frac{\pi i}{n})}{\prod_{s \in \mathcal{G}_{l_3-1}} e^{(2)}_{l_1,l_2}(\gamma_{l_3-1,s}) \prod_{s \in \mathcal{G}_{l_3+1}} e^{(2)}_{l_1,l_2}(\gamma_{l_3+1,s})}, \tag{4.47}
\]

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where we have set

\[
\begin{aligned}
& e_{l_1,l_2}^{(2)}(\alpha) = \\
& \times \frac{\Gamma_3(i\alpha + \frac{\pi}{n}(l_1 + l_2 + 1)|\pi, 2\pi, \frac{2\pi}{n})}{\Gamma_3(i\alpha + \frac{\pi}{n}(l_1 + l_2 + 2)|\pi, 2\pi, \frac{2\pi}{n})}.
\end{aligned}
\]

We have set

\[
\begin{aligned}
J_B(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N}^{k_1 \cdots k_M} &= \prod_{l=1}^{L} \frac{F_{L,l,l}^{(3)}(\{\alpha\}|\{\gamma\} | \mu + \frac{n\pi}{n}(2M - L))}{F_{L,l,l}^{(3)}(\{\alpha\}|\{\gamma\} | \mu - \frac{n\pi}{n}(2M - L))} \\
& \times \prod_{m=1}^{M} \frac{F_{M,m,m}^{(3)}(\{\alpha\}|\{\gamma\} | \mu - \frac{n\pi}{n}(2M - L))}{F_{M,m,m}^{(3)}(\{\alpha\}|\{\gamma\} | \mu - \frac{n\pi}{n}(2M - L))}.
\end{aligned}
\]

where we have set

\[
\begin{aligned}
& F_{l_1,l_2,l_3}^{(3)}(\{\alpha\}|\{\gamma\} | \nu) = \\
& = \prod_{r \in A_{l_1}} e_{l_1,l_2}^{(3)}(\alpha_{l_1,r} + \frac{\pi}{n} - \nu) \prod_{s \in A_{l_1}} e_{l_1,l_2}^{(3)}(\alpha_{l_1,s} + \frac{\pi}{n} - \nu) \\
& \times \prod_{r \in A_{l_2}} e_{l_1,l_2}^{(3)}(\alpha_{l_2,r} + \frac{\pi}{n} - \nu) \prod_{s \in A_{l_3}} e_{l_1,l_2}^{(3)}(\gamma_{l_3,s} + \frac{\pi}{n} - \nu).
\end{aligned}
\]

Here we have set

\[
\begin{aligned}
e_{l_1,l_2}^{(3)}(\alpha) &= \frac{\Gamma_3(i\alpha + \frac{\pi}{n}(l_1 + l_2 + 2)|\pi, 2\pi, \frac{2\pi}{n})}{\Gamma_3(i\alpha + \frac{\pi}{n}(l_1 + l_2 + 2)|\pi, 2\pi, \frac{2\pi}{n})}.
\end{aligned}
\]
Let us summarize the result of this section. We present integral representations (4.22) for the form factors of the local operators (4.1),

\[ f(\delta_1, \cdots, \delta_M, \beta_1, \cdots, \beta_N)_{j_1 \cdots j_N}. \]

Here the factor \( E(\{\beta\}|\{\delta\}) \) is given in (4.23). The integrand \( I(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N} \) is given in (4.24), where the factor \( J_r(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N} \), \( J_F(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N} \), and \( J_B(\{\alpha\}|\{\gamma\})_{j_1 \cdots j_N} \), are given in (4.42), (4.43), and (4.49). Here the auxiliary functions \( f_{l_1,l_2}^{(1)}, f_{l_1,l_2,l_3}^{(2)}, f_{l_1,l_2,l_3}^{(3)} \) are given by (4.44), (4.47), and (4.50), respectively.

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A Multi-Gamma functions

Here we summarize the multiple gamma and the multiple sine functions. Let us set the functions \( \Gamma_1(x|\omega) \), \( \Gamma_2(x|\omega_1, \omega_2) \) and \( \Gamma_3(x|\omega_1, \omega_2, \omega_3) \) by

\[
\log \Gamma_1(x|\omega) + \gamma_{B_{11}}(x|\omega) = \int_C \frac{dt}{2\pi it} \frac{e^{-xt}}{1-e^{-\omega t}}, \quad (A.1)
\]

\[
\log \Gamma_2(x|\omega_1, \omega_2) - \frac{\gamma}{2} B_{22}(x|\omega_1, \omega_2) = \int_C \frac{dt}{2\pi it} \frac{e^{-xt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}, \quad (A.2)
\]

\[
\log \Gamma_3(x|\omega_1, \omega_2, \omega_3) + \frac{\gamma}{3!} B_{33}(x|\omega_1, \omega_2, \omega_3) = \int_C \frac{dt}{2\pi it} \frac{e^{-xt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})}, \quad (A.3)
\]

where the functions \( B_{jj}(x) \) are the multiple Bernoulli polynomials defined by

\[
\frac{t^r e^{xt}}{\prod_{j=1}^r (e^{\omega_j t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\omega_1 \cdots \omega_r), \quad (A.4)
\]

more explicitly

\[
B_{11}(x|\omega) = \frac{x}{\omega} - \frac{1}{2}, \quad (A.5)
\]

\[
B_{22}(x|\omega_1, \omega_2) = \frac{x^2}{\omega_1 \omega_2} - \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right) x + \frac{1}{2} + \frac{1}{6} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right). \quad (A.6)
\]

Here \( \gamma \) is Euler's constant, \( \gamma = \lim_{n \to \infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n) \).

Here the contour of integral is given by

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0
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Contour C
Let us set

\[ S_1(x|\omega) = \frac{1}{\Gamma_1(\omega - x|\omega)\Gamma_1(x|\omega)}, \quad (A.7) \]

\[ S_2(x|\omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)}, \quad (A.8) \]

\[ S_3(x|\omega_1, \omega_2, \omega_3) = \frac{1}{\Gamma_3(\omega_1 + \omega_2 + \omega_3 - x|\omega_1, \omega_2, \omega_3)\Gamma_3(x|\omega_1, \omega_2, \omega_3)} \quad (A.9) \]

We have

\[ \Gamma_1(x|\omega) = e^{\left(\frac{x}{\omega} - \frac{1}{2}\right)\log\omega} \frac{\Gamma(x/\omega)}{\sqrt{2\pi}} \quad S_1(x|\omega) = 2\sin(\pi x/\omega), \quad (A.10) \]

\[ \frac{\Gamma_2(x + \omega_1|\omega_2)}{\Gamma_2(x|\omega_1, \omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)}, \quad \frac{S_2(x + \omega_1|\omega_1, \omega_2)}{S_2(x|\omega_1, \omega_2)} = \frac{1}{S_1(x|\omega_2)}, \quad \frac{\Gamma_1(x + \omega|\omega)}{\Gamma_1(x|\omega)} = x. \quad (A.11) \]

\[ \frac{\Gamma_3(x + \omega_1|\omega_1, \omega_2, \omega_3)}{\Gamma_3(x|\omega_1, \omega_2, \omega_3)} = \frac{1}{\Gamma_2(x|\omega_2, \omega_3)}, \quad \frac{S_3(x + \omega_1|\omega_1, \omega_2, \omega_3)}{S_3(x|\omega_1, \omega_2, \omega_3)} = \frac{1}{S_2(x|\omega_2, \omega_3)}. \quad (A.12) \]

\[ \log S_2(x|\omega_1 \omega_2) = \int_C \frac{\sinh(x - \omega_1 \omega_2/2)t}{2\sinh(\omega_1 t/2)\sinh(\omega_2 t/2)} \log(-t) \frac{dt}{2\pi it}, \quad (0 < \text{Re} x < \omega_1 + \omega_2). \quad (A.13) \]

\[ S_2(x|\omega_1 \omega_2) = \frac{2\pi}{\sqrt{\omega_1 \omega_2}} x + O(x^2), \quad (x \to 0). \quad (A.14) \]

\[ S_2(x|\omega_1 \omega_2)S_2(-x|\omega_1 \omega_2) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2}. \quad (A.15) \]