Wave equations for the perturbations of a charged black hole

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A pair of simple wave equations is presented for the symmetric gravitational and electromagnetic perturbations of a charged black hole. One of the equations is uncoupled, and the other has a source term given by the solution of the first equation. The derivation is presented in full detail for either axisymmetric or stationary perturbations, and is quite straightforward. This result is expected to have important applications in astrophysical models.

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I. INTRODUCTION

Recent discoveries of astrophysical X-ray sources of extraordinary intensity have instigated research on the nature of these objects. Current speculation involving mechanisms such as the Blandford-Znajek effect in the vicinity of black holes has been hampered by a lack of a viable theory of perturbations of charged black holes.

In the three decades following the discovery of the master equation of the perturbative treatment of gravitational, electromagnetic and Weyl neutrino fields in the presence of an uncharged black hole, much effort has been spent on finding a corresponding description of the fields for charged Kerr-Newman black holes. It has been shown, for example, that a decoupled equation exists for either the electromagnetic or the gravitational component.

The purpose of this paper is to show for the first time that master equations do exist for both the electromagnetic and the gravitational perturbation components of the charged black hole. Symmetry will be assumed to hold in order to keep the discussion simple. The required symmetry can be any one-parameter group of isometries with a time- or spacelike Killing vector.

It is hoped that the perturbation formalism presented in this paper will find important applications in creating models of relativistic sources of electromagnetic and gravitational radiation, both in analytical and in numerical approaches. Axisymmetric perturbations, characterized by a vanishing angular frequency of the normal modes, for instance, may already suffice to construct models of astrophysical X-ray sources.

In the next section, we shall briefly review the relevant theory.

II. EINSTEIN-MAXWELL FIELDS WITH A SYMMETRY

An Einstein-Maxwell system with one Killing vector, spacelike or timelike, may be fully characterized by the complex 3-covectors:

\[ \mathbf{G} = \frac{\nabla \mathcal{E} + 2 \bar{\Phi} \nabla \Phi}{2(\text{Re} \mathcal{E} + \Phi \bar{\Phi})}, \quad \mathbf{H} = \frac{\nabla \Phi}{(\text{Re} \mathcal{E} + \Phi \bar{\Phi})^{1/2}} \]  \hspace{1cm} (2.1)

where \( \mathcal{E} \) and \( \Phi \) are the complex gravitational and electromagnetic Ernst potentials, respectively. In the notation referring to the metric of the three-space of Killing trajectories, the field equations can be written

\[ R_{\mu\nu} = -G_{\mu} \bar{G}_{\nu} - \bar{G}_{\mu} G_{\nu} + H_{\mu} \bar{H}_{\nu} + \bar{H}_{\mu} H_{\nu} \]  \hspace{1cm} (2.2)

\[ (\nabla - \mathbf{G}) \cdot \mathbf{G} = \mathbf{H} \cdot \mathbf{H} - \bar{\mathbf{G}} \cdot \mathbf{G} \]  \hspace{1cm} (2.3)

\[ (\nabla - \mathbf{G}) \times \mathbf{G} = \mathbf{H} \times \mathbf{H} - \bar{\mathbf{G}} \times \mathbf{G} \]  \hspace{1cm} (2.4)

\[ (\nabla - \mathbf{G}) \cdot \mathbf{H} = \frac{1}{2} (\mathbf{G} - \bar{\mathbf{G}}) \cdot \mathbf{H} \]  \hspace{1cm} (2.5)

\[ \nabla \times \mathbf{H} = -\frac{1}{2} (\mathbf{G} + \bar{\mathbf{G}}) \times \mathbf{H}. \]  \hspace{1cm} (2.6)

We introduce a complex triad of basis vectors \( z_0 = \ell, z_+ = m \) and \( z_- = \bar{m} \). In close analogy with the null tetrad in space-time, the normalization is chosen

\[ \ell \cdot \ell = m \cdot \bar{m} = 1, \]
while all other independent products of the triad vectors vanish. In their role as linear operators, the triad vectors are denoted \( \{ \ell, m, \bar{m} \} \) and have the commutators

\[
D\delta - \delta D = \kappa D + (\bar{\sigma} + \epsilon)\delta + \sigma\bar{\delta}
\]

(2.7a)

\[
\delta\bar{\delta} - \bar{\delta}\delta = (\bar{\sigma} - \rho)D + \bar{\sigma}\bar{\delta} - \tau\delta.
\]

(2.7b)

When the triad components of the field equations are taken, there result five complex Ricci equations from (2.2), of the following form,

\[
D\sigma - \delta\kappa = (\rho + \bar{\sigma} + 2\epsilon)\sigma + \bar{\sigma}\kappa + \kappa^2 + 2G_+G_- - 2\bar{H}_+H_+
\]

(2.8a)

\[
D\rho - \delta\kappa = \rho^2 + \sigma\sigma + (\bar{\sigma} - \tau)\kappa + G_o\bar{G}_o - H_o\bar{H}_o
\]

(2.8b)

\[
D\tau - \delta\epsilon = (\bar{\sigma} + \tau)\rho + (\bar{\sigma} - \tau)\epsilon - (\kappa + \bar{\sigma})\sigma + G_o\bar{G}_o - H_o\bar{H}_o - H_o\bar{H}_o
\]

(2.8c)

\[
\delta\sigma - \delta\rho = 2\tau\sigma + \kappa(\rho - \bar{\rho}) - \bar{G}_oG_o - G_o\bar{G}_o + \bar{H}_oH_o + H_o\bar{H}_o
\]

(2.8d)

\[
\delta\tau + \delta\bar{\epsilon} = \rho\bar{\sigma} - \sigma\bar{\sigma} + 2\tau\epsilon(\rho - \bar{\rho}) - G_o\bar{G}_o + G_o\bar{G}_o + G_-\bar{G}_+ + \bar{H}_oH_o + H_o\bar{H}_o + \bar{H}_+H_- - \bar{H}_-H_+
\]

(2.8e)

one equation each from (2.3) and (2.5), furthermore three equations each from (2.4) and (2.6):

\[
(D - \rho - \bar{\rho})G_o + (\bar{\delta} + \bar{\sigma} - \tau)G_- + (\delta + \kappa - \bar{\sigma})G_* = (G_o - \bar{G}_o)G_o + (G_+ - \bar{G}_+)(G_- + (G_- - G_+)G_o + \bar{H}_oH_o + \bar{H}_+H_- + H_oH_+ + H_+H_-)
\]

(2.9)

\[
(\delta + \bar{\sigma})G_o - (\bar{D} - \bar{\rho} + \epsilon)G_- + \bar{\sigma}G_+ = \bar{G}_oG_+ - \bar{G}_-G_o - H_o\bar{H}_o + H_+H_- + H_-H_+
\]

(2.10a)

\[
(\delta + \bar{\sigma})G_o - (\bar{D} - \bar{\rho} + \epsilon)G_- + \bar{\sigma}G_+ = \bar{G}_oG_+ - \bar{G}_-G_o - H_o\bar{H}_o + H_+H_- + H_-H_+
\]

(2.10b)

\[
(\bar{\delta} - \tau)G_+ - (\delta - \bar{\tau})G_- + (\bar{\rho} - \rho)G_o = \bar{G}_+G_- - \bar{G}_+G_- + H_+H_- + \bar{H}_+H_- + H_-H_+
\]

(2.10c)

\[
(D - \rho - \bar{\rho})H_o + (\bar{\delta} + \bar{\sigma} - \tau)H_- + (\delta + \kappa - \bar{\sigma})H_* = \frac{1}{2} [(3G_o - \bar{G}_o)H_o + (3G_+ - \bar{G}_+)H_- + (3G_- - \bar{G}_-)H_+]
\]

(2.11)

\[
(\delta + \bar{\sigma})H_o - (\bar{D} - \bar{\rho} + \epsilon)H_- + \bar{\sigma}H_+ = \frac{1}{2} [(G_o + \bar{G}_o)H_+ - (G_- + \bar{G}_-)H_o]
\]

(2.12a)

\[
(\delta + \bar{\sigma})H_o - (\bar{D} - \bar{\rho} + \epsilon)H_- + \bar{\sigma}H_+ = \frac{1}{2} [(G_o + \bar{G}_o)H_+ - (G_+ + \bar{G}_+)H_o]
\]

(2.12b)

\[
(\bar{\delta} - \tau)H_+ - (\delta - \bar{\tau})H_- + (\bar{\rho} - \rho)H_o = \frac{1}{2} [(G_+ + \bar{G}_+)H_- - (G_- + \bar{G}_-)H_+]
\]

(2.12c)

These relations contain the four complex Ricci rotation coefficients \( \kappa = m_{\mu\nu}^{\ell}\ell\nu, \rho = m_{\mu\nu}^{\ell}\ell\mu\nu, \sigma = m_{\mu\nu}^{\ell\mu}m^{\ell\nu}, \) and \( \tau = m_{\mu\nu}^{\ell\mu\nu}m^{\ell\nu}, \) and the imaginary rotation coefficient \( \epsilon = m_{\mu\nu}^{\ell\mu\ell\nu}. \) (Note that the complex conjugate of, say, \( G_+ \) is \( \bar{G}_- \).)

A solution of particular relevance for us is the Kerr-Newman metric with mass \( m \), rotation parameter \( a \), electric charge \( e \) and Ernst potentials

\[
\mathcal{E} = 1 - \frac{2m}{\xi}, \quad \Phi = \frac{\xi}{\zeta}
\]

where

\[
\zeta = r - ia \cos \theta.
\]

Given the two Killing vectors \( \partial/\partial t \) and \( \partial/\partial \varphi \) of the space-time, the three-space may be defined with respect to any of these two (or their linear combinations). In what follows, we choose the three-space to be positive-definite.

The orientation of the triad vectors can be chosen at will. Here we adopt a triad for which the eigenray condition holds,

\[
G_+ = 0,
\]

(2.13)

thereby fixing the direction of the vector \( \ell \). We then have

\[
D = \frac{\partial}{\partial \varphi}
\]

(2.14)

\[
\delta = \frac{1}{2D} \left( -ia \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \varphi} + \frac{i}{\sin \varphi} \frac{\partial}{\partial \varphi} \right).
\]

(2.15)

Here \( f = \text{Re}\mathcal{E} + \Phi \Phi \). For the charged Kerr solution, the following quantities vanish,

\[
\kappa = \sigma = H_+ = 0.
\]
III. THE PERTURBED ELECTROVACUUM

When the perturbed space-time is either axisymmetric or stationary, the description in Sec. 2 holds once again. In particular, the three-space and the complex fields $G$ and $H$ do exist. The perturbed triad is chosen to satisfy the eigenray condition (2.13) once again. Three of the fields, $\kappa, \sigma$ and $H_+$ are small since they vanish for the unperturbed space-time.

In this gauge, it is a straightforward matter to derive an uncoupled wave equation. Taking the sum of Eqs. (2.11) and (2.12a), we get

$$(D - 2\rho)H_o + 2|\delta - \tau|H_+ - G_- H_+ + \kappa H_- + \frac{1}{2}(G_o - 3G_o)H_o = 0. \quad (3.1)$$

The commutator (2.7a) is applied on the complex function $H_o$ and the derivatives $DH_o$ and $\delta H_o$ are eliminated by use of the field equations (3.1) and (2.12b), respectively. The derivative $DG_o$ is expressed from the sum of Eqs. (2.9) and (2.10d), and the derivatives $DG_-, DH_-, \delta G_+, \delta H_+, \delta G_-, \delta H_-, \delta \rho$ and $D\tau$ from the respective equations (2.10a), (2.12a), (2.10b), (2.12b), (2.10c), (2.12d), (2.8c) and (2.8d). In this way, all purely unperturbed terms can be removed by the remaining field equations, with the result

$$\begin{align*}
\{G_o \left[DD + 2\delta - D(\bar{\rho} + \epsilon) + (G_o + 2\rho)\epsilon \right]
- (G_o - \bar{G}_o + 2\rho + 2\bar{\rho} + 2\epsilon)G_o - 2\bar{H}_o H_o \} D,
-(G_o + 2\tau)G_o \delta + (\bar{G}_o + 2H_+ H_o)\delta
-G_o (\delta G_o - 2\bar{\rho}G_o - H_+ \bar{\rho} + (\bar{\rho} + \epsilon)^2 + 2\tau G_+ - 2\bar{\rho} \bar{H}_+ - (\bar{\rho} + \epsilon)^2 + 2\bar{\tau} \bar{H}_+ - 2\bar{\rho} \bar{H}_+)
+G_o (\delta G_o - 2\bar{\rho}G_o + 3\bar{H}_+ \bar{H}_+ - 3H_+ G_o + 2(\rho + \epsilon)\bar{H}_o H_o) H_+
-2\tau \bar{\rho}_o + 3G_- \bar{H}_+ - 3H_+ G_o + 2(\rho + \epsilon)\bar{H}_o H_o)
H_+
= [(G_- H_o - 2G_o H_o + 2\delta H_o)G_o - 2H_o \delta G_o] \sigma.
\end{align*}$$

All terms on the left contain a factor $H_+$ and those on the right contain the rotation coefficient $\sigma$. Both of these quantities are of first order, thus the operators acting on them and the factors can be taken to have their values in the charged Kerr metric. When this is done for the perturbation function

$$\phi = H_+ (\epsilon^2 \zeta^{-1} + \text{const}) \quad (3.3)$$

the terms on the right-hand side cancel and an uncoupled separable wave equation results:

$$\Box_1 \phi = 0 \quad (3.4)$$

where the wave operator is defined

$$\Box_s = \Delta^{-s} \frac{\partial}{\partial r} \Delta^{-s+1} \frac{\partial}{\partial \varphi} + \sin^{-1} \frac{\partial}{\partial r} \sin \frac{\partial}{\partial \varphi} + s \left( s + 1 - \frac{s}{\sin^2 \varphi} \right)$$

$$-2a \frac{\partial^2}{\partial r \partial \varphi} + \frac{1}{\sin^2 \varphi} \frac{\partial^2}{\partial \varphi^2} + 2i \frac{\cos \varphi}{\sin \varphi} \frac{\partial}{\partial \varphi} + \frac{1}{\sin \varphi} \frac{\partial}{\partial \psi}$$

(3.5)

and $\Delta = r^2 - 2mr + a^2 + e^2$ is the horizon function.

Next, applying the commutator (2.7a) on the function $G_o$ , we get the relation

$$D\kappa = \left( \epsilon - \rho \right) \kappa + (4\tau + 2G_- - 2\bar{\tau}) \sigma$$

$$-\frac{\partial}{\partial r} (\bar{H}_o D H_+ + \bar{H}_o \delta H_+ + \sigma \bar{G}_o)$$

$$+ \frac{\bar{H}_+}{G_o} (5G_o - 3G_o + 2\rho + 2\epsilon) \bar{H}_o + (3G_- + 2\tau) \bar{H}_+).$$

(3.6)

Now we act with the commutator (2.7a) on $\kappa$ and eliminate the derivatives $D\kappa$ and $\delta \kappa$ by use of (3.6) and (2.8a) respectively. The pure first-order relation (2.8a) is one of the five Ricci equations [7]. We obtain another second-order differential equation

$$G_o \left[ G_o \left[ (2G_o - \rho) + 4\epsilon^2 + \rho^2 - 2\bar{H}_+ H_- + 2\bar{G}_+ G_- - 4\delta \tau \right]
+ 2\delta \delta G_o + G_o (2\delta \delta + DD + 2\tau \delta - D\rho - 2D \kappa - D \bar{\rho})
+ 2G_o (2\epsilon - G_o) \bar{\rho} - 2(G_+ + \tau) \bar{\rho} - (G_- + 2\tau) \delta - (\bar{\rho} + 2\epsilon) \bar{D} \right] \sigma$$

$$- 2\delta G_o \left[ H_+ H_o - \tau G_o - \bar{G}_o - G_o \delta \right] \sigma.$$
J paper. relies on a gauge adapted to the Killing bivector of the black-hole background. The details will be given in a follow-up to consider either axisymmetric or stationary waves. The procedure for general perturbations is more involved and already provides a framework for devising models of astrophysical sources of radiation in situations where it is sufficient.

\[ \begin{align*}
& \{ G_o \} \hat{H}_o (2\delta \tau - 2\delta D - 3\delta \bar{G}_o) + 2\hat{H}_+ (\delta \tau - \delta \bar{\delta}) + (3G_- + 2\tau)\delta \hat{H}_+ \\
& + (3G_o \hat{H}_- - 2H_o \tau - 3G_- \hat{H}_o)\hat{H}_+^2 + (3G_o + 2\rho + \delta \bar{G}_o) G_o \delta \hat{H}_o \\
& - \left[ 3\hat{G}_+ \hat{H}_o - \hat{H}_+ \hat{G}_o \right] G_o - 3(\hat{H}_+ H_o - \tau G_o) \hat{H}_o \hat{G}_o G_o \\
& + \left[ (G_o + 2\rho + 2\delta) \hat{H}_o + 2\hat{H}_+ \tau \right] \hat{G}_+ G_o \\
& + \left[ (5G_o + 2\rho + \delta) \bar{G}_o - 2(G_o + \rho + \delta) \hat{H}_+ H_o \right] \hat{H}_o \\
& + \left[ (3G_- + \tau) \hat{H}_+ - 3\bar{G}_o \hat{H}_o + (5G_o + 2\rho + 2\delta) \hat{H}_o \right] G_o \delta \\
& + 2 \left[ (\hat{H}_+ H_o - \tau G_o - \hat{G}_+ G_o) \hat{H}_+ - \delta \hat{H}_+ \right] \delta \\
& - 2 \left[ (\hat{G}_+ \hat{H}_o + \hat{H}_+ G_o + \delta \hat{H}_o) G_o - (\hat{H}_+ H_o - \tau G_o) \hat{H}_o \right] D \\
& + \left[ (G_o - 3\rho + 2\delta) \bar{G}_o + 2(3G_- + \tau) \bar{\tau} \hat{H}_+ G_o \right] H_+. \end{align*} \] (3.7)

Each term on the left contains the small function

\[ \psi = \sigma G_o (e^{2\xi} + m) \] (3.8)

and each term on the right contains an \( H_+ \), to be expressed in terms of \( \phi \). When inserting the unperturbed values of the operators and factors, neither the \( \psi \) terms, nor the \( \phi \) terms cancel, and what we get is the wave equation

\[ \Box \psi = J(\phi). \] (3.9)

Thus the terms containing a solution \( \phi \) of Eq. (3.4) will provide the source function \( J(\phi) \) for the equation for \( \psi \). The source term is a functional of the field \( \phi \) containing up to second derivatives. By considering the differential structure of \( J(\phi) \) in the same gauge, it is possible to derive a decoupled equation also for \( \psi \), which, however, is quite lengthy.

Given a solution of Eqs. (3.4) and (3.9), the perturbation functions \( \sigma \) and \( H_+ \) are available from the simple relations (3.8) and (3.9), respectively. One can next compute the first-order function \( \kappa \) by integrating Eq. (3.4). Continuing the step-by-step integration procedure, we have an algorithm for systematically getting the full description of the perturbed space-time.

Choosing a different gauge with the roles of the fields \( G_+ \) and \( H_+ \) interchanged, one gets an uncoupled wave equation for the gravitational perturbation and one with a source term for the electromagnetic perturbation.

IV. CONCLUSIONS

The present description of the simultaneous gravitational and electromagnetic excitations of a charged black hole already provides a framework for devising models of astrophysical sources of radiation in situations where it is sufficient to consider either axisymmetric or stationary waves. The procedure for general perturbations is more involved and relies on a gauge adapted to the Killing bivector of the black-hole background. The details will be given in a follow-up paper.

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