Further simplification of the light deflection formula for solar system objects

Sven Zschocke, Sergei A. Klioner

Lohrmann Observatory, Dresden Technical University, Mommsenstr. 13, 01062 Dresden, Germany

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The transformation $n$ to $k$ in post-post-Newtonian order is simplified. All post-post-Newtonian terms of the order $\mathcal{O}\left(\frac{m^2}{d^2}\right)$ are neglected and we show that the total sum of these terms is smaller than $\frac{15}{4} \pi \frac{m^2}{d^2}$. This simpler transformation will improve the efficiency of Gaia data reduction.
I. INTRODUCTION

The approximative analytical solution of the problem of light deflection has been presented in [1–3]. One of the main results of these investigations is the transformation \( n \) to \( k \) for solar system objects in post-post-Newtonian approximation. A detailed analysis [3, 4] has shown that most of the terms in this transformation can be neglected at the microarcsecond level of accuracy, leading to a simplified formula \( n \) to \( k \) for the data reduction. This simplified formula \( n \) to \( k \) has been given in Eqs. (92) and (93) in [1] and in Eqs. (52) and (53) in [3]. In this report we will show that this transformation can be further simplified. The report is organized as follows. In Section II we will present the transformation \( n \) to \( k \) in post-post-Newtonian order. The estimate of post-post-Newtonian terms and the new simplified transformation \( n \) to \( k \) is given in Section III. A new estimation will be given in Section IV. A summary is given in Section V. Detailed proofs of the estimates used are given in the appendices.

II. TRANSFORMATION \( n \) TO \( k \) IN POST-POST-NEWTONIAN ORDER

The transformation \( n \) to \( k \) in post-post-Newtonian order has been given in Eq. (87) in [1], Eq. (57) in [2], and in Eq. (45) in [3]. We will present this transformation in the following equivalent form:

\[
\begin{align*}
N & \quad n = k \\
pN & \quad - (1 + \gamma) m \frac{k \times (x_0 \times x_1)}{x_1 (x_1 x_0 + x_1 \cdot x_0)} \\
\Delta pN & \quad + (1 + \gamma)^2 m^2 \frac{k \times (x_0 \times x_1)}{(x_1 x_0 + x_1 \cdot x_0)^2} \frac{R}{x_1} \\
\text{scaling} & \quad - \frac{1}{8} (1 + \gamma)^2 \frac{m^2 k}{x_1^2} \frac{((x_1 - x_0)^2 - R^2)^2}{|x_1 \times x_0|^2} \\
ppN & \quad + m^2 k \times (x_0 \times x_1) \left[ \frac{1}{2} (1 + \gamma)^2 \frac{R^2 - (x_1 - x_0)^2}{x_1^2 |x_1 \times x_0|^2} \right] \\
ppN & \quad + \frac{1}{4} \alpha \epsilon \frac{1}{R} \left( \frac{1}{R x_1^2} - \frac{1}{R x_1^2} - 2 \frac{k \cdot x_1}{x_1^4} \right) \\
ppN & \quad - \frac{1}{4} (8(1 + \gamma - \alpha \gamma)(1 + \gamma) - 4 \alpha \beta + 3 \alpha \epsilon) R \frac{k \cdot x_1}{x_1^2 |x_1 \times x_0|^2} \\
ppN & \quad + \frac{1}{8} (8(1 + \gamma - \alpha \gamma)(1 + \gamma) - 4 \alpha \beta + 3 \alpha \epsilon) \frac{x_1^2 - x_0^2 - R^2}{|x_1 \times x_0|^2} \delta(x_1, x_0) \\
ppN & \quad + (1 + \gamma)^2 m^2 \frac{k \times (x_0 \times x_1)}{(x_1 x_0 + x_1 \cdot x_0)^2} \frac{x_1 + x_0 - R}{x_1} \\
ppN & \quad + O(\epsilon^{-6}).
\end{align*}
\]

Here we have classified the nature of the individual terms by labels \( N \) (Newtonian), \( pN \) (post-Newtonian), \( ppN \) (post-post-Newtonian) and \( \Delta pN \) (terms that are formally of post-Newtonian order). The scaling and \( ppN \) terms are expressed in non-dimensional form. The \( ppN \) terms include the leading order values and \( O(\epsilon^{-6}) \) terms are the next order in \( \epsilon \).
post-Newtonian order, but may numerically become significantly larger than other post-post-Newtonian terms, see estimates in (6).

III. SIMPLIFIED TRANSFORMATION n TO k

The effect of all the “ppN” terms in (1) can be estimated as (cf. Eq. (91) in [1] or Eq. (50) in [3])

\[ |\omega'_{ppN}| \leq \frac{15}{4} \pi \frac{m^2}{d^2} \tag{2} \]

The proof of (2) is given in Appendix A. These terms can attain 1 μas only for observations within about 3.3 angular radii from the Sun and can be neglected. Accordingly, we obtain a simplified formula for the transformation from k to n keeping only the post-Newtonian and “enhanced” post-post-Newtonian terms labelled as “pN” and “ΔpN” in (1):

\[ n = k + d P \left(1 + P x_1\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3), \tag{3} \]

\[ P = -(1 + \gamma) \frac{m}{d^2} \left(\frac{x_0 - x_1}{R} + \frac{k \cdot x_1}{x_1}\right). \tag{4} \]

The simplified transformation n to k given in Eq. (3) has now simpler structure than the former expression given in Eq. (92) in [1] or in Eq. (52) in [3]. Therefore, (3) is more efficient for the data reduction. Furthermore, the transformation in Eq. (3) has now similar structure as the simplified transformation n to σ given in Eq. (102) in [1] or in Eq. (62) in [3].

IV. A NEW ESTIMATION

The enhanced post-post-Newtonian term \[ |\omega'_{ΔpN}| \] in Eq. (1) is, for \(\gamma = 1\), given by (cf. Eq. (89) in [1] or Eq. (48) in [3])

\[ |\omega'_{ΔpN}| = 4 m^2 \left| \frac{k \times (x_0 \times x_1)}{(x_1 x_0 + x_1 \cdot x_0)^2 x_1} \right| \tag{5} \]

This term differs from the corresponding term \[ |\omega_{ΔpN}| \] defined in Eq. (89) in [1] or Eq. (48) in [3] only by a factor \(\frac{R}{x_0 + x_1} \leq 1\). Therefore, we conclude that the estimates given in Eqs. (89) and (90) of [1] or in Eqs. (48) and (49) of [3] are also valid for \[ |\omega'_{ΔpN}| \], that means:

\[ |\omega'_{ΔpN}| \leq 16 \frac{m^2}{d^2} \frac{R^2 x_1 x_0^2}{(x_1 + x_0)^2} \leq 16 \frac{m^2}{d^2} \frac{R x_1 x_0^2}{(x_1 + x_0)^3} \leq 16 \frac{m^2}{d^2} \frac{x_1 x_0^2}{(x_1 + x_0)^3} \leq 16 \frac{m^2}{d^2} \frac{x_1}{d}, \tag{6} \]

where the first expression given in (6) represents a new estimation. Another estimation can be given, namely (cf. Eq. (90) in [1] or Eq. (49) in [3]).
\[ |\omega'_{\Delta pN}| \leq \frac{64}{27} \frac{m^2}{d^2} \frac{R}{d}, \tag{7} \]

which cannot be related to the estimations in (6) and reflect different properties of \(|\omega'_{\Delta pN}|\) as function of multiple variables.

V. SUMMARY

In Eq. (57) in [2] the complete transformation \(n\) to \(k\) in post-post-Newtonian order has been given. In [3] we have shown that most of the terms can be neglected because they are of the order \(O\left(\frac{m^2}{d^2}\right)\) and can attain 1 \(\mu\)as only for observations within about 3.3 angular radii from the Sun. These investigations have yielded a simplified transformation, given in Eqs. (92) and (93) in [1] or in Eqs. (52) and (53) in [3], and applicable for an efficient data reduction. In this report we have shown that Eq. (92) in [1] or Eq. (52) in [3] can further be simplified. The main result of this report is Eq. (3), where we give a new simplified transformation \(n\) to \(k\) which will improve the efficiency of Gaia data reduction. We have shown that the total sum of the neglected ppN-terms is smaller than \(\frac{15}{4} \pi \frac{m^2}{d^2}\). Furthermore, estimations of the enhanced post-post-Newtonian term has been given in Eqs. (6) and (7).

[1] S.A. Klioner, S. Zschocke, Numerical versus analytical accuracy of the formulas for light propagation, Class. Quantum Grav. 27 (2001) 075015.
[2] Sergei A. Klioner, Sven Zschocke, GAIA-CA-TN-LO-SK-002-2, Parametrized post-post-Newtonian analytical solution for light propagation, available on arXiv:astro-ph/0902.4206.
[3] Sven Zschocke, Sergei A. Klioner, GAIA-CA-TN-LO-SZ-002-2, Analytical solution for light propagation in Schwarzschild field having an accuracy of 1 micro-arcsecond, available on arXiv:astro-ph/0904.3704.
[4] Sven Zschocke, Sergei A. Klioner, GAIA-CA-TN-LO-SZ-003-1, Formal proof of some inequalities used in the analysis of the post-post-Newtonian light propagation theory, available on arXiv:astro-ph/0907.4281.
Appendix A: Proof of inequality (2)

The sum of all ppN-terms in Eq. (1) can be written as follows (here $\alpha = \beta = \gamma = \epsilon = 1$):

$$|\omega'_{ppN}| = \frac{1}{4} m^2 \frac{d^2 f_{10}'}{d^2},$$

(A1)

where the function is defined by (cf. with $f_{10}$ defined in Eq. (84) in [4])

$$f_{10}' = \left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} + 16 \frac{z (1 - \cos \Phi)^2 (1 + \sqrt{1 + z^2 - 2z \cos \Phi})}{(1 + z^2 - 2z \cos \Phi) \sin \Phi} \right|. \quad (A2)$$

Here we have used the notation $\Phi = \delta(x_0, x_1)$ and $z = \frac{x_0}{x_1}$. By means of the inequalities (note that (A4) improves the inequality given in Eq. (C1) in [4])

$$f_2 = 16 \frac{z (1 - \cos \Phi)^2 (1 + \sqrt{1 + z^2 - 2z \cos \Phi})}{(1 + z^2 - 2z \cos \Phi) \sin \Phi} \leq 8 \sin \Phi,$$

(A3)

$$f_3 = \frac{|z (1 - 3z^2 + 2z^3 \cos \Phi)| \sin^3 \Phi}{(1 + z^2 - 2z \cos \Phi)^2} \leq 3 \sin \Phi,$$

(A4)

(proof of (A3) and (A4) are shown in Appendices B and C respectively) we obtain

$$f_{10}' \leq \left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} + 11 \sin \Phi \right|. \quad (A5)$$

In [4] we have shown $z (16z - z \cos \Phi - 15) \sin \Phi + 15z (\cos \Phi - z) \Phi \leq 0$. Accordingly, due to $\sin \Phi \geq 0$, we obtain

$$f_{10}' \leq \left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} - 15 \sin \Phi \right|, \quad (A6)$$

where, for convenience, we have replaced the term $11 \sin \Phi$ by the larger term $15 \sin \Phi$. Furthermore, in [4] we have shown that

$$\left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} - 15 \sin \Phi \right| \leq 15 \pi.$$

(A7)

Thus, we obtain

$$f_{10}' \leq 15 \pi. \quad (A8)$$

The inequality (A8) in combination with (A1) shows the validity of inequality (2).
Appendix B: Proof of inequalities (A3)

In order to show (A3), we rewrite this inequality as follows:

\[
\frac{z (1 - \cos \Phi)}{1 + z^2 - 2 z \cos \Phi} \frac{1 + z - \sqrt{1 + z^2 - 2 z \cos \Phi}}{1 + \cos \Phi} \leq \frac{1}{2}.
\]  

(B1)

The inequality (B1) can be split into two factors satisfying the following inequalities:

\[
\frac{z (1 - \cos \Phi)}{1 + z^2 - 2 z \cos \Phi} \leq \frac{1}{2},
\]  

(B2)

\[
\frac{1 + z - \sqrt{1 + z^2 - 2 z \cos \Phi}}{1 + \cos \Phi} \leq 1.
\]  

(B3)

The inequality (B2) is obviously valid, because by multiplying (B2) with the denominator we obtain \(- (1 - z)^2 \leq 0\). The inequality (B3) is also straightforward, because it can be rewritten as \(z - \cos \Phi \leq \sqrt{1 + z^2 - 2 z \cos \Phi}\), which is obviously valid due to \(z - \cos \Phi \leq |z - \cos \Phi|\). Thus we have shown the validity of inequality (B1) and (A3), respectively.
Appendix C: Proof of inequality \((A4)\)

Using the notation \(w = \cos \Phi\), the inequality \((A4)\) can be written as follows:

\[
f_3 = \frac{z \left| 1 - 3z^2 + 2z^3w \right| (1 - w^2)}{(1 + z^2 - 2zw)^2} \leq 3. \tag{C1}
\]

Using the inequality (proof see below)

\[
\left| 1 - 3z^2 + 2z^3w \right| \leq 1 - 3wz^2 + 2z^3, \tag{C2}
\]

we obtain

\[
f_3 \leq \frac{z (1 - 3wz^2 + 2z^3) (1 - w^2)}{(1 + z^2 - 2zw)^2} = h_1 + h_2 \leq 3. \tag{C3}
\]

In \((C3)\) the relation \(1 - 3wz^2 + 2z^3 = (1 + z^2 - 2wz) + (-3wz^2 + 2z^3 - z^2 + 2wz)\) has been used. The functions are defined by

\[
h_1 = \frac{z (1 - w^2)}{1 + z^2 - 2wz} \leq \frac{2z (1 - w)}{1 + z^2 - 2wz} \leq 1, \tag{C4}
\]

\[
h_2 = \frac{z^2 \left| -3wz + 2z^2 - z + 2w \right| (1 - w^2)}{(1 + z^2 - 2zw)^2} \leq 2. \tag{C5}
\]

The inequality \((C4)\) has been shown in \([3]\). In order to show \((C5)\), we factorize the function \(h_2\) as follows:

\[
h_2 = h_2^A h_2^B, \tag{C6}
\]

\[
h_2^A = \frac{z^2 (1 - w^2)}{1 + z^2 - 2wz} \leq 1, \tag{C7}
\]

\[
h_2^B = \frac{|-3wz + 2z^2 - z + 2w| (1 - w^2)}{1 + z^2 - 2wz} \leq 2. \tag{C8}
\]

Thus, by means of the inequalities \((C2)\) and \((C4) - (C8)\) we have shown the validity of inequality \((C1)\) and \((A4)\), respectively. We still have to proof of inequalities \((C2), (C7)\) and \((C8)\).

Let us consider \((C2)\). First, we remark that \(1 - 3wz^2 + 2z^3 \geq 0\) because of \(1 - 3z^2 + 2z^3 \geq 0\). Then, squaring both sides of \((C2)\) and subtracting from each other leads to

\[
h_3 = 2z^3 + 2wz^3 - 3z^2 - 3wz^2 + 2 \geq 0. \tag{C9}
\]

The boundaries of \(h_3\) are
\[
\lim_{w \to -1} h_3 = 2 \geq 0, \quad (C10)
\]
\[
\lim_{w \to +1} h_3 = 2 (2z + 1) (z - 1)^2 \geq 0, \quad (C11)
\]
\[
\lim_{z \to 0} h_3 = 2 \geq 0, \quad (C12)
\]
\[
\lim_{z \to \infty} h_3 = 2 (1 + w) \lim_{z \to \infty} z^3 \geq 0. \quad (C13)
\]

The extremal conditions \(h_3, w = 0\) and \(h_3, z = 0\) lead to

\[
z^2 (2z - 3) = 0, \quad (C14)
\]
\[
z (1 + w) (z - 1) = 0. \quad (C15)
\]

The common solutions of (C14) and (C15) are given by

\[
P_1 (w = -1, z = 0), \quad (C16)
\]
\[
P_2 \left(w = -1, z = \frac{3}{2}\right). \quad (C17)
\]

The numerical values of \(h_3\) at these turning points are

\[
h_3 (P_1) = 2 \geq 0, \quad (C18)
\]
\[
h_3 (P_2) = 2 \geq 0. \quad (C19)
\]

Thus, we have shown (C9) and, therefore, the validity of inequality (C2).

Now we consider the inequality (C7). Multiplying both sides of this relation with the denominator leads to the inequality

\[
h_4 = -z^2 w^2 - 1 + 2wz \leq 0. \quad (C20)
\]

The boundaries of \(h_4\) are

\[
\lim_{w \to -1} h_4 = - (1 + z)^2 \leq 0, \quad (C21)
\]
\[
\lim_{w \to +1} h_4 = - (1 - z)^2 \leq 0, \quad (C22)
\]
\[
\lim_{z \to 0} h_4 = -1 \leq 0, \quad (C23)
\]
\[
\lim_{z \to \infty} h_4 = -w^2 \lim_{z \to \infty} z^2 \leq 0. \quad (C24)
\]
The extremal conditions $h_{4,w} = 0$ and $h_{4,z} = 0$ lead to

\[ z (1 - wz) = 0, \quad (C25) \]
\[ w (1 - wz) = 0. \quad (C26) \]

The common solution of $(C25)$ and $(C26)$ is given by

\[ P_3 (w = 0, z = 0), \quad (C27) \]

and the numerical value of $h_4$ at this turning point is

\[ h_4 (P_3) = -1 \leq 0. \quad (C28) \]

Thus, we have shown $(C20)$ and, therefore, the inequality $(C7)$.

Now we consider the inequality $(C8)$. Squaring both sides of $(C8)$ and subtracting from each other leads to the inequality

\[ h_5 = 4 z^2 - z - 7 wz + 2 + 2w \geq 0. \quad (C29) \]

The boundaries of $h_5$ are

\[ \lim_{w \to -1} h_5 = 2 z (3 + 2z) \geq 0, \quad (C30) \]
\[ \lim_{w \to +1} h_5 = 4 (z - 1)^2 \geq 0, \quad (C31) \]
\[ \lim_{z \to 0} h_5 = 2 (1 + w) \geq 0, \quad (C32) \]
\[ \lim_{z \to \infty} h_5 = 4 \lim_{z \to \infty} z^2 \geq 0. \quad (C33) \]

The extremal conditions $h_{5,w} = 0$ and $h_{5,z} = 0$ lead to

\[ -7z + 2 = 0, \quad (C34) \]
\[ 8z - 1 - 7w = 0. \quad (C35) \]

The common solution of $(C34)$ and $(C35)$ is given by

\[ P_4 \left( w = \frac{9}{49}, z = \frac{2}{7} \right), \quad (C36) \]

and the numerical value of $h_5$ at this turning point is

\[ h_5 (P_4) = \frac{100}{49} \geq 0. \quad (C37) \]

Thus, we have shown $(C29)$ and, therefore, the inequality $(C8)$. 
