Charges and fluxes in Maxwell theory on compact manifolds with boundary

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ABSTRACT

We investigate the charges and fluxes that can occur in higher-order Abelian gauge theories defined on compact space-time manifolds with boundary. The boundary is necessary to supply a destination to the electric lines of force emanating from brane sources, thus allowing non-zero net electric charges, but it also introduces new types of electric and magnetic flux. The resulting structure of currents, charges, and fluxes is studied and expressed in the language of relative homology and de Rham cohomology and the corresponding abelian groups. These can be organised in terms of a pair of exact sequences related by the Poincaré-Lefschetz isomorphism and by a weaker flip symmetry exchanging the ends of the sequences. It is shown how all this structure is brought into play by the imposition of the appropriately generalised Maxwell’s equations. The requirement that these equations be integrable restricts the world-volume of a permitted brane (assumed closed) to be homologous to a cycle on the boundary of space-time. All electric charges and magnetic fluxes are quantised and satisfy the Dirac quantisation condition. But through some boundary cycles there may be unquantised electric fluxes associated with quantised magnetic fluxes and so dyonic in nature.

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1. Introduction

In the search for a unified theory of particle interactions encompassing both the standard model and Einstein’s theory of gravity the most promising candidate seems to be the superstring and M-theories which require space-time to have dimensions 10 and 11 respectively for internal consistency.

A common feature of these is the presence of states known as “p-branes”, objects which, classically at least, can be pictured as extended objects resembling p-dimensional surfaces (or volumes) in space. As time evolves these sweep out surfaces (or volumes) of one dimension higher, \( p + 1 \).

When \( p = 0 \), the object is simply a point particle tracing out a world-line, \( w \), in space-time. It has a geometrically natural interaction with Maxwell’s electromagnetic field specified by the addition of a term in the action taking the schematic form

\[
q \int_w A.
\]

The same can be done for any positive value of \( p \) less than \( m \), the dimension of space-time, with the proviso that the gauge potential \( A \) now has degree \( (p + 1) \), matching the fact that the world-volume, \( w \), has \( (p + 1) \)-dimensions. When \( p \) equals one, so that the brane is a string, \( A \) is the familiar Kalb-Ramond [1974] gauge potential.

Naively Stokes’ theorem implies that expression (1.1) is unchanged when \( A \) is altered according to

\[
A \rightarrow A + d\chi,
\]

where \( \chi \) is of degree \( p \) and arbitrary, if it assumed that \( w \) is closed, i.e. a \((p+1)\)-cycle. This generalised gauge invariance suggests that an important physical role would be played by the following quantity which is invariant with respect to (1.2):

\[
F = dA.
\]

This is the \((p+2)\)-form field strength, reducing to the familiar one of Maxwell when \( p = 0 \).

The most natural equations of motion for \( F \) take the form:

\[
dF = 0,
\]

\[
d\ast(hF) = \ast j,
\]

in exterior calculus notation, although there are more elaborate possibilities. In order to include a common feature of supergravity/superstring theories we have admitted the presence of a positive scalar function of scalar fields, \( h(\phi) \), in (1.5), that equals unity in vacuo. Apart from this feature, these are Maxwell’s equations generalised in the form envisaged by Hodge, and \( \ast \) denotes his duality operation, constructed by means of a metric on space-time, here assumed to be a fixed background [Hodge 1952, Flanders 1963]. We shall henceforth refer to them as Maxwell’s equations.

The inhomogeneous Maxwell equation (1.5) will play an important role in what follows irrespective of the detailed form of the quantity \( h \) as long as it reduces to unity in vacuo.
and we shall not have to consider equations of motion for the scalar fields. The quantity $j$ is the “electric current” and is a $(p + 1)$-form, so possessing the same degree as $A$. By virtue of (1.5) and the nilpotency of the exterior derivative, $d$, it has to be “conserved” so that

$$d^* j = 0,$$

(1.6)

and we shall always suppose this.

Such field theories are indeed part of modern superstring/M-theory and it is therefore important to understand their properties by answering the questions listed below, particularly when the background space-time is taken to be topologically complicated. But these equations are only a part of the larger theory and not the whole. In this subtheory, no account need be taken of supersymmetry and the values of $p$ and $m$, the dimension of space-time $\mathcal{M}$, can treated as arbitrary. Many special features of these subtheories have become familiar, [Nepomechie 1985, Teitelboim 1986, Henneaux and Teitelboim 1986, Deser, Gomberoff, Henneaux and Teitelboim 1997] but our aim is to uncover yet more general structure as will be seen.

When $p$ vanishes and $m$ equals 4 there are three notions familiar since the times of Faraday, electric charge, electric flux and magnetic flux. (Magnetic charge is excluded by (1.4) since magnetic current is). These three quantities are all conserved, that is unchanged by various sorts of evolution, including that in time. It will be seen to be important to distinguish the notions before examining the possibility of any relations between them.

Allowing $p$ and $m$ to be arbitrary, the physical questions considered in this paper concern:

1. the classification and enumeration of the independent charges and fluxes,
2. the understanding of how the Maxwell equations (1.4) and (1.5) relate the notions of electric flux and charge,
3. the determination of the possible numerical values of these charges and fluxes,
4. the understanding of how quantum theory can relate the values of electric and magnetic conserved quantities (yielding the generalised Dirac quantisation condition).

The answers turn out to be more subtle and interesting than we had expected and it is this that motivates this presentation. It is found important to resist the common temptation to simplify by taking the space-time manifold, $\mathcal{M}$, to be closed as this results in oversimplification. When account is taken of the generalised Maxwell equations, (1.4) and (1.5), all electric charges and fluxes then vanish, leaving magnetic fluxes as the only available conserved quantities, as Henneaux and Teitelboim [1986] have pointed out.

Thus it is essential to allow space-time, $\mathcal{M}$, to possess a boundary, $\mathcal{B}$, a manifold of dimension one less, interpreted as corresponding to the “points at spatial infinity” through which “electric field lines” may escape, thereby furnishing a potentially non-trivial flux. When this is done, the answers to the physical questions above are provided by a set of results in pure mathematics whose physical relevance is, we believe, hitherto unappreciated by physicists. Once we have established the appropriate definitions we find the connections, made more precise in the text:

- electric charges $\leftrightarrow$ relative homology of space-time,
- electric fluxes $\leftrightarrow$ absolute homology of the boundary of space-time,
- magnetic fluxes $\leftrightarrow$ absolute homology of space time.
All these charges and fluxes are expressed as integrals over some sort of cycle in space-time and homology deals with the classification of these cycles in the way that is appropriate to the physics. There are precisely three types of homology in the situation just described and all three play a physical role according to the connections just listed. Moreover the relationships between the different sorts of conserved quantity correspond to relationships between these different sorts of homology.

All of the aforementioned types of homology class form elements of an abelian group, the appropriate homology group, \( H_* \), say. Taking into account all values of \( p \) that are possible in the given fixed background space-time, \( \mathcal{M} \), these abelian groups can be arranged in a certain order such that there is a natural homomorphism acting between successive members. This provides a sequence with the property of being exact, that is, at each stage, the homology group, \( H_* \), possesses a subgroup, \( K_* \), say, that is at the same time the kernel of the succeeding homomorphism and the image of the preceding one. This is the exact sequence of relative homology (of space-time). A more refined classification of the physical notions of charge and flux will depend on the distinction between the subgroup \( K_* \) and the coset group \( H_*/K_* \) within each homology group \( H_* \). This structure is explained in more detail in the text as it becomes relevant to the development of the physical arguments and particularly in Sections 7 and 10, as well as the Appendix. Relevant mathematical background together with more detail can be found in [Schwarz 1994 and Massey 1991].

Each homology group, \( H_* \), is abelian, and usually of infinite order. For reasons explained they are essentially discrete lattices of finite dimension, \( b_* \) which is known as the Betti number. The number of linearly independent charges and fluxes will be expressible in terms of Betti numbers in a surprisingly complicated way that we shall determine. These results will answer the first two of the physical questions listed above.

An important subtlety is that although the definition of the conserved electric charge as an integral over the conserved current, \( j \) works irrespective of whether or not the generalised Maxwell equations (1.4) and (1.5) are assumed to hold, the counting of the charges does depend on this choice, being more complicated when they do hold, as they should when account is taken of physical relevance. For example, when space-time is closed, all possible non-trivial electric charges are forced to vanish by the equations (1.4) and (1.5). The point is that there exist conserved currents on space-time for which it is impossible to integrate (1.5) to obtain a field strength, \( F \). Consequently these currents will be forbidden on the physical grounds that the field strengths must exist. It is the aforementioned exact sequence of relative homology that clarifies the occurrence of this phenomenon as explained in section 4 and amplified later.

Answering the third of the physical questions listed above requires an explicit form of the conserved current, \( j(w) \), due to a \( p \)-brane with world-volume \( w \) as implied by (1.1) and (1.5) together. This is provided by a singular differential form involving Dirac \( \delta \)-functions whose support is \( w \). Then the electric charge associated with integrating over a relative cycle \( S \) is \( q \) times the intersection number of \( S \) and the absolute cycle, \( w \). The coefficient \( q \) is defined by (1.1) and the intersection number is well defined as \( w \) and \( S \) have dimensions summing to \( m \), that of space-time. As this intersection number is unchanged by homologies of both \( w \) and \( S \), it is defined on their homology classes. Since the groups formed by these classes are essentially lattices whose dimension is the relevant Betti number, it follows that
the intersection data is encoded in the intersection matrix, $I$, formed of the intersection numbers between elements of bases of the two lattices. This matrix, $I$, has integer entries, is square and unimodular (that is, has determinant equal to $\pm 1$), the latter two properties being consequences of “Poincaré-Lefschetz duality”, another feature of the exact sequence of relative homology.

So far this analysis does not use the “Maxwell equations”, (1.4) and (1.5), and hence applies whether or not they are chosen to hold. If not, the electric charges take values equal to an integer times $q$. Conversely the unimodularity of the intersection matrix means that there exist brane configurations realising all possible values of these sets of values.

If Maxwell’s equations are chosen to hold, as they should, the situation is more complicated as many potential electric charges are forced to vanish, apparently contradicting the unimodular property of the intersection matrix. The resolution of this paradox depends on the recognition that some brane configurations are forbidden as they yield conserved electric currents for which the Maxwell’s equations (1.4) and (1.5) cannot be integrated to yield a field strength. This is explained in more detail in section 8 and requires the intersection matrix to have a more detailed structure than so far apparent. This is revealed by writing it in block form according to the kernel subgroup, $K_*$, of each $H_*$, and the coset $H_*/K_*$. One block has to vanish identically and this is verified explicitly in section 10 and the Appendix. This leaves square matrices on the block diagonal each of which have to be unimodular.

The upshot is that the only brane configurations that are allowed by the integrability requirement are those that are homologous to cycles in the boundary, $\mathcal{B}$, of space-time, $\mathcal{M}$. The surviving electric charges again take values that are integral multiples of $q$. Conversely there are allowed brane configurations that realise all possible sets of these values.

Another phenomenon quantified by the exact sequence of homology is the existence of electric fluxes which are not equal to electric charges and hence not quantised. Through the same cycles there may flow quantised magnetic fluxes of the field coupling to the brane dual to that coupling to the electric field so that the overall effect is suggestive of something dyonic.

All results so far are “classical”, invoking no quantum theory. Taking the latter into account requires that the schematic term (1.1) in the action be unambiguous when suitably exponentiated. This constrains the values of the magnetic fluxes to satisfy a generalisation of Dirac’s celebrated quantisation when compared with any of the electric charges [Dirac 1931, Wu and Yang 1975, Alvarez and Olive 2000].

The resulting picture is beautifully consistent yet unexpectedly rich. Nevertheless our analysis made a number of implicit simplifications compared with the full superstring theory that are so far unavoidable. Some of these are listed in the conclusion, section 11, and it is hoped that a subsequent elaboration of our present methods will lead to answers removing these assumptions.

A technical Appendix extends the idea of a distribution valued form associated with a bulk cycle (such as the brane world-volume) to chains both in the bulk and on the boundary. These constructions are used to derive the weak form of Poincaré-Lefschetz duality used in establishing the vanishing of an off-diagonal block of the previously mentioned intersection matrix.
Relative topology has been used previously to discuss certain aspects of branes in M-theory. A partial description of the role of relative cohomology in the classification of charges in generalised Maxwell theory was sketched in section 2 of [Moore and Witten 2000]. In [Kalkkinen and Stelle 2003] relative cohomology is used to present a geometric description of certain brane intersections in M-theory. A similar analysis of D2-branes in Wess-Zumino-Witten theory can be found in [Figueroa-O’Farrill and Stanciu 2001].

2. First notions

Taken as given is a fixed background space-time $\mathcal{M}$, assumed oriented and compact, but possibly of complicated topology. It has dimension $m$ and initially it is assumed to be closed. On it is defined a field strength $F$ that is a $(p+2)$-form satisfying the generalised Maxwell equations (1.4) and (1.5). According to the first of these $F$ is closed so that locally there is defined a $(p+1)$-form gauge potential $A$, (1.3), with a gauge ambiguity with respect to the gauge transformations (1.2) where $\chi$ is a $p$-form also defined locally. The quantity $j$ is a $p+1$-form denoting the electric current due to the matter degrees of freedom. For the time being it does not have to be assumed that it has the form that (1.1) would imply.

Electric current conservation is the statement that $\ast j$ is a closed form on $\mathcal{M}$, (1.6). This follows from the above Maxwell equation (1.5) as $d^2$ vanishes, but we shall assume its validity even when Maxwell’s equations are disregarded.

The first notion of an electric charge is associated with the current $j$ without any reference to the field strength, $F$. Hence Maxwell’s equations can be temporarily disregarded. It is formulated by considering an oriented region $S$ that is a $(m-p-1)$-chain over which it is possible to integrate the matching form $\ast j$:

$$Q(S) = \int_S \ast j$$

(2.1)

Conventionally the region $S$ would be thought of as “space-like” but this is not essential. The virtue of the definition (2.1) is that it is insensitive to alterations of $S$ by homologies that preserve its boundary, $\partial S$. Thus, if $S' = S + \partial C$, so $\partial S' = \partial S$,

$$Q(S') = Q(S)$$

as $Q(\partial C) = \int_{\partial C} \ast j = \int_C d \ast j = 0$, using Stokes’ theorem and current conservation (1.6). This establishes a good sense in which the charge $Q$ is conserved. The disadvantage of this definition is that the regions $S$ for which the charge is defined lack any real homological significance unless it is assumed that $S$ is closed, $\partial S = 0$. Now the result means that each electric charge, $Q(S)$, is preserved by homologies of $S$, that is, unchanged by the kinds of evolution associated with these homologies. Homologous surfaces form absolute homology classes which themselves form an abelian group under addition of surfaces, in this case the absolute homology group of $\mathcal{M}$, denoted $H_{m-p-1}(\mathcal{M}; \mathbb{Z})$. Without any field strengths satisfying the Maxwell equations this would be the end of the story as there would be no fluxes to consider.
Since field strengths are included, it is necessary to consider the effect of applying Maxwell’s equation (1.5):

$$Q(S) = \int_S * j = \int_S d * (hF) = \int_{\partial S} * (hF) = 0$$

as $\partial S$ vanishes. Thus all electric charges vanish when Maxwell’s equations hold on a closed space-time, $\mathcal{M}$. In physical terms, the problem is that the Maxwell equation (1.5) attaches electric lines of force to the electric charge distribution and these lines have nowhere to go. Mathematically the point is that when the conserved electric current, $j$, is such that any $Q(S)$ fails to vanish, it is impossible to integrate (1.5) to obtain the field strength $F$ on $\mathcal{M}$. This is unacceptable on physical grounds.

An obvious remedy is to provide a destination for the lines of force by allowing space-time, $\mathcal{M}$, to be non compact, and this will be considered next. But it will remain necessary to check the integrability of Maxwell’s equations in the sense just described.

3. Electric Charges and Relative Homology

Instead of allowing space-time, $\mathcal{M}$, to be non-compact, as just suggested, we shall do something slightly different and keep it compact but allow it to have a non-trivial boundary, $\mathcal{B} = \partial \mathcal{M}$, of one dimension less. This can be thought of as comprising those points at spatial infinity through which electric lines of force may escape. On the other hand, electric current, $j$, is not allowed to escape, that is its Hodge dual, $*j$, is assumed to be localised and this is expressed by the boundary condition:

$$*j\big|_\mathcal{B} = 0. \quad (3.1)$$

More precisely this means that the restriction of the differential form $*j$ to $\mathcal{B}$ vanishes. Thus the normal components of $j$ vanish on $\mathcal{B}$. In addition, it is assumed that the scalar function, $h$, occurring in (1.5), takes its vacuum value on $\mathcal{B}$:

$$h\big|_\mathcal{B} = 1 \quad (3.2)$$

Of course Maxwell’s equations, (1.4) and (1.5) and also current conservation, (1.6) still hold on $\mathcal{M}$, or as we shall say, in the bulk.

The same expression (2.1) for the electric charge holds good except that now, instead of assuming $\partial S$ vanishes, we assume that it lies in $\mathcal{B}$, and so $S$ has become what is called a relative cycle. Suppose that $S$ is altered by a relative homology:

$$S \rightarrow S' = S + \partial C + \beta, \quad C \in \mathcal{M}, \quad \beta \in \mathcal{B}.$$ 

Then $Q(\partial C) = \int_{\partial C} * j = \int_C d * j = 0$, by Stokes’ theorem and (1.6), while $Q(\beta) = \int_{\beta} * j = 0$ by (3.1). So

$$Q(S) = Q(S') \quad \text{if } S \sim S' \quad (3.3)$$

in relative homology $\mathcal{M}\mod\mathcal{B}$. In particular $Q(S)$ vanishes if $S \sim 0$. Thus the electric charge is well defined as an integral over relative homology classes, denoted $\mathbf{[S]}$ and forming.
an abelian group, $H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$. This is one sense in which the electric charges are conserved. The abelian group structure arises because two like relative cycles can be added to form a third. According to (2.1) this addition law is respected by the electric charges:

$$Q([S]) + Q([S']) = Q([S + S']) = Q([S] + [S'])$$

and this furnishes another sense in which they are conserved.

Some elements of this homology group have finite order and are called torsion elements. Thus if $S$ is not trivial, that is not relatively homologous to 0, yet has the property that there exists an integer $n$ such that $n[S] = [nS]$ is trivial, then, by the above, $Q([S]) = Q([nS])/n = 0$. Altogether such elements form a finite abelian subgroup $T$, (the torsion group), which can be divided out of $H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$ to form a free group

$$F_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) = H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})/T$$

which can be regarded as a lattice of finite dimension $b_{m-p-1}(\mathcal{M}, \mathcal{B})$. This dimension is the corresponding Betti number. Because there are no contributions from torsion elements, electric charges are only defined on $F_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$. Hence the integer $b_{m-p-1}(\mathcal{M}, \mathcal{B})$ counts what appears to be the number of linearly independent electric charges that can be defined on the space-time $\mathcal{M}$. This conclusion is an overestimate for reasons to be explained in the next section.

From now on, the conventions of this section will be adopted, and absolute chains will be denoted by lower case Roman letters (a,b,c ... s,t,u,v,w...), relative chains by upper case Roman letters (A,B,C ... S,T,U,V,W..) and chains in the boundary by Greek letters ($\alpha, \beta, \gamma... \phi, \chi, \psi..$). The letters later in the alphabet will denote the corresponding cycles.

4. Electric fluxes and electric charges

The above derivation of the topological classification of electric charges by relative homology $F_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$, used only the properties (1.6) and (3.1) of the current $j$, and not the Maxwell equations $dF = 0$ and (1.5). Current conservation (1.6) can be regarded as a necessary local condition for the integrability of the field strength $F$, given the current, $j$, but it is not sufficient, as already has been seen when space-time, $\mathcal{M}$, has no boundary, nor will it be so when it does have a boundary.

Assuming the Maxwell equation (1.5) does hold, the electric charge $Q(S)$ can be rewritten as an electric flux:

$$Q(S) = \int_S *j = \int_S d*(hF) = \int_{\partial S} *(hF) = \int_{\partial S} *F,$$

by Stokes’ theorem and (3.2). Of course $\partial S$ is in the space-time boundary, $\mathcal{B}$, and is a cycle though not necessarily a boundary of a chain there, even though it is in the bulk, $\mathcal{M}$.

But it is possible to provide a more general definition of electric flux than this by considering any cycle in $\mathcal{B}$, not just one that is a boundary of a relative cycle:

$$\Phi_E(\phi) = \int_{\phi} *F, \quad \phi \in \mathcal{B}, \quad \partial \phi = 0. \quad (4.2)$$
This extended definition works on all the absolute homology classes of the boundary, $H_{m-p-2}(B; \mathbb{Z})$, or, more precisely, the free parts, $F_{m-p-2}(B; \mathbb{Z})$, defined as before. To check, consider the absolute homology in the boundary, $\phi' \to \phi + \partial \gamma$, $\gamma \in \mathcal{B}$. Then $\Phi_E(\partial \gamma) = \int_{\partial \gamma} *F = \int_{\gamma} d \ast(hF) = \int_{\gamma} \ast j = 0$, using Stokes’ theorem and equations (1.5) and (3.1). So indeed $\Phi_E(\phi) = \Phi_E(\phi')$ if $\phi \sim \phi'$ in absolute homology in $\mathcal{B}$, in distinction to the electric charges that appeared to correspond to relative homology.

So, since their classifications differ, electric charges and electric fluxes must be distinguished. This distinction manifests itself in two different physical ways.

First, not all electric fluxes are expressible as electric charges because not all cycles on the boundary, $\mathcal{B}$, are boundaries of chains on $\mathcal{M}$. The electric fluxes that are equal to charges are associated with cycles on the boundary, $\mathcal{B}$, that are also boundaries of chains on $\mathcal{M}$, as in (4.1). These classes of cycles form a subgroup of the absolute homology group of the boundary, $\mathcal{B}$, $H_{m-p-2}(B; \mathbb{Z})$, that we shall denote as follows:

$$K_{m-p-2}(B; \mathbb{Z}) = \text{classes of boundary cycle } \phi \text{ satisfying } \phi = \partial S \text{ for some bulk chain } S.$$  \hfill (4.3)

Secondly there are apparently non-trivial electric charges, $Q(S)$, which must vanish if they are expressible as fluxes. This happens precisely when $\partial S$ is a boundary in $\mathcal{B}$, as well as in $\mathcal{M}$, according to Stoke’s theorem applied to (4.1). The classes of these cycles form a subgroup of the relative homology group that we shall denote as follows:

$$K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) = \{ \text{classes of relative cycle, } R, \text{ satisfying } \partial R = \partial \alpha, \alpha \in \mathcal{B} \}. \hfill (4.4)$$

It follows that it is the vanishing of the charges associated with these cycles that is the extra integrability condition on the Maxwell’s equation (1.5) in order to obtain a field strength, $F$, given a conserved current, $j$.

To recap, electric charges defined on $K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$ all vanish, leaving non-trivial charges associated with each coset element of this subgroup. Furthermore only those electric fluxes defined on $K_{m-p-2}(B; \mathbb{Z})$ are expressible as electric charges. What is happening mathematically is that the boundary operation $\partial$ mapping relative cycles to boundary cycles induces a map

$$\partial_* : \ H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \longrightarrow H_{m-p-2}(B; \mathbb{Z})$$  \hfill (4.5)

In fact this is a group homomorphism with kernel $K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$, (4.4), and image $K_{m-p-2}(B; \mathbb{Z})$, (4.3). So, by Lagrange’s theorem,

$$H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})/K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \equiv K_{m-p-2}(B; \mathbb{Z}).$$

and it is this group (or more precisely the free part) that classifies the non-trivial electric charges. Applying this to the free parts that are lattices classifying the corresponding charges and fluxes, the number of linearly independent electric charges is given by

$$b_{m-p-1}(\mathcal{M}, \mathcal{B}) - s_{m-p-1}(\mathcal{M}, \mathcal{B}) = s_{m-p-2}(\mathcal{B}), \hfill (4.6)$$
explaining the overestimate mentioned previously. The integers \( s_*(X) \) are the dimensions of the lattices specifying the free part of \( K_*(X; \mathbb{Z}) \).

As there are \( b_{m-p-2}(\mathcal{B}) \) linearly independent fluxes, \( s_{m-p-2}(\mathcal{B}) \) of which are expressible as electric charges the difference, the number \( b_{m-p-2}(\mathcal{B}) - s_{m-p-2}(\mathcal{B}) \), specifies the number of linearly independent electric fluxes that cannot be equated to electric charges of the form (2.1).

5. Relation between the preliminary and final versions of electric charge

For reasons that become clear later, it is worth asking a question that seems rather ridiculous from a physical point of view, namely how to relate the class of electric charge obtained by integrating \( \ast j \) over a bulk cycle to the class obtained by integrating over a relative cycle. The first class, considered in our preliminary discussion still makes sense when space-time has a boundary since a bulk cycle can be considered as a special case of a relative cycle. The reason the question is apparently ridiculous from a physical point of view is that these preliminary charges do all vanish when account is taken of Maxwell’s equations as already seen.

Consider an absolute bulk \((m - p - 1)\)-cycle, \( r \), and decompose it into the sum of a \((m - p - 1)\)-chain in \( \mathcal{B} \) and a remainder that contains no such chain:

\[
r = R + \alpha.
\]

As \( \partial r = 0 \), \( \partial R = -\partial \alpha \in \mathcal{B} \), so that \( R \) is a relative cycle. Furthermore, if \( r \) is trivial as a bulk cycle, so \( r = \partial a \), then \( R = \partial a - \alpha \) and so is trivial as a relative cycle. Hence the projection map \( j : r \to R \) induces a map, \( j_* \), of absolute bulk homology classes to relative homology classes:

\[
j_* : \quad H_{m-p-1}(\mathcal{M}; \mathbb{Z}) \to H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}).
\] (5.1)

This is actually a homomorphism. Its kernel consists of bulk cycles, \( r \) for which \( R \) is relatively trivial, so \( r = (\partial C + \beta) + \alpha = \partial C + \gamma \), where \( \gamma \in \mathcal{B} \). Since \( r \) is closed, so is \( \gamma \). Thus \( \gamma \) is a cycle in the boundary and the kernel can be denoted

\[
K_{m-p-1}(\mathcal{M}, \mathbb{Z}) = \{ \text{classes of bulk cycle homologous to cycles in } \mathcal{B} \}. \quad (5.2)
\]

On the other hand the image of \( j_* \) consists of classes of relative cycle whose boundary is the boundary of a chain within \( \mathcal{B} \). This coincides with the subgroup \( K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \) already defined as being the kernel of \( \partial_* \) in the previous section, (4.3).

Putting together \( j_* \) and \( \partial_* \) as two successive homomorphisms:

\[
H_{m-p-1}(\mathcal{M}; \mathbb{Z}) \xrightarrow{j_*} H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \xrightarrow{\partial_*} H_{m-p-2}(\mathcal{B}; \mathbb{Z}),
\]

we see that this sequence is exact at \( H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \) as \( K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \) is both the the image of \( j_* \) and the kernel of \( \partial_* \). This is a short segment of the exact sequence of relative homology alluded to in the introduction and more segments will be seen when magnetic fluxes are considered next. The complete exact sequence will be presented in later sections. A textbook presentation can be found in [Massey 1991].
Of course, as we saw at the start, all electric charges vanish that are integrals over cycles in $H_{m-p-1}(\mathcal{M}; \mathbb{Z})$. This agrees with the fact already found above that they also vanish on $K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$, which is the image of the former group under the action of $j_\ast$.

6. Action principle for p-branes and magnetic flux quantisation

The standard (naive) expression for the term in the action describing the interaction of the field strength, $F$, with the current, $j$, that is its source, according to (1.4) and (1.5), is

$$\int_\mathcal{M} A \wedge * j.$$  \hspace{1cm} (6.1)

Naively, this term is gauge invariant on its own with respect to the transformation (1.2) given that the electric current, $j$, is conserved, (1.6), and localised, (3.1). Ideally the current should be expressible in terms of quantum mechanical wave functions for the matter but it is only really understood how to do this when $p = 0$ so that the branes are point particles.

By default, the only accepted way to proceed is to adopt the classical geometric picture described in the introduction. The evolution of the $p$-brane in space-time is specified by its world-volume, $w$, an absolute bulk $(p+1)$-cycle on $\mathcal{M}$. Then the action term (6.1) takes the form (1.1) mentioned at the start.

Because we already know the classical equations of motion in the Maxwell form (1.4) and (1.5), the detailed form of the action is only really relevant in the quantum theory. In that context, the expressions (1.1) and (6.1) are equally problematical (which explains the use of the words “schematic or naive”) as they involve the gauge potential, $A$, which is only defined locally, whilst the integration extends globally over all of space-time, $\mathcal{M}$. Consequently, in a topologically complicated space-time such as the one being imagined, there are problems in patching together this expression in overlapping neighbourhoods of space-time. Fortunately it is the exponentiated action $e^{i q \int_w A/\hbar}$ that enters the Feynman action principle and this is more amenable. One needs to know how this phase alters when $w$ is altered by a boundary. That is tantamount to requiring that the phase has a meaning when $w$ is a boundary of a bulk chain. This can be done provided the background field strength $F$ satisfies the Dirac quantisation conditions for all magnetic fluxes through bulk $(p+2)$-cycles:

$$\Phi_M(v) = \int_v F \in \frac{2\pi \hbar}{q} \mathbb{Z}, \quad \partial v = 0.$$  \hspace{1cm} (6.2)

As $dF = 0$ these fluxes are defined on the classes of the absolute homology of space-time $\mathcal{M}$, forming the group $H_{p+2}(\mathcal{M}, \mathbb{Z})$, or more precisely the free part of this, $F_{p+2}(\mathcal{M}; \mathbb{Z})$, a lattice of dimension $b_{p+2}(\mathcal{M})$.

A parenthetic remark concerning this quantisation condition (6.2) is that it is known not really to be correct when wave functions are considered, as is, so far, only possible when $p$ vanishes and the brane is therefore a point particle. Then there is a possibility of fractional quantisation conditions when the wave function is of a spinor nature (involving half-integers instead of integers). The precise rule is easy to state when $m = 4$ [Alvarez and Olive 2000].

By this stage of the argument it has become established that, as claimed in the introduction, there is a connection between the physical notions of electric charge, electric flux
and magnetic flux of a $p$-brane and mathematical notions of relative homology, absolute boundary homology and absolute bulk homology and more precisely, with the free parts of the abelian homology groups, $H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$, $H_{m-p-2}(\mathcal{B}; \mathbb{Z})$ and $H_{p+2}(\mathcal{M}; \mathbb{Z})$, respectively. But there is a more detailed structure connected to the subgroups $K_*$ of $H_*$, for short, that plays a role in the exact sequence of relative homology and moreover possesses a physical relevance. Let us illustrate this last point by investigating magnetic fluxes through $\mathcal{B}$ cycles with a view to comparing electric and magnetic fluxes. Later on we shall see how this comparison will indicate a generalised dyonic phenomenon that is possibly related to the Zwanziger-Schwinger quantisation condition [Zwanziger 1968, Schwinger 1969].

Magnetic fluxes can already be defined for cycles in the boundary, $\mathcal{B}$, rather than in the bulk, $\mathcal{M}$, but nothing appears to be gained by this as cycles in $\mathcal{B}$ are automatically cycles in $\mathcal{M}$ but may become boundaries of bulk chains when regarded as $\mathcal{M}$-cycles and hence homologically trivial in the bulk. Associated with this idea is the inclusion map, $i$, which induces the homomorphism:

$$i_* : H_{p+2}(\mathcal{B}; \mathbb{Z}) \rightarrow H_{p+2}(\mathcal{M}; \mathbb{Z}),$$

with kernel consisting of the classes of cycle just mentioned that become boundaries. This is precisely the subgroup $K_{p+2}(\mathcal{B}; \mathbb{Z})$ of the type met before, (4.3), (with $p + 2$ replaced by $m - p - 2$), as the image of the homomorphism, $\partial_*$, (4.5), induced by the boundary operator and met before in the comparison of electric charges and fluxes. The image of this homomorphism is clearly given by classes of bulk cycle homologous to a cycle in the boundary and these precisely form the subgroup $K_{p+2}(\mathcal{M}; \mathbb{Z})$, (5.2), already met as the kernel of the homomorphism $j_*$, (5.1), (again with $p + 2$ replaced by $m - p - 2$).

All magnetic fluxes on cycles of $K_{p+2}(\mathcal{B}; \mathbb{Z})$ vanish, as

$$\int_{\phi} F = \int_{\partial S} F = \int_{S} dF = 0,$$

by (4.3), Stokes’ theorem and (1.4), corresponding to the fact that these cycles are trivial as bulk cycles. So the only non-trivial magnetic fluxes through boundary cycles correspond to the $b_{p+2}(\mathcal{B})-s_{p+2}(\mathcal{B})$ cosets of $K_{p+2}(\mathcal{B}; \mathbb{Z})$ in $H_{p+2}(\mathcal{B}; \mathbb{Z})$. These observations will become more interesting when we are able to compare them with the corresponding properties of electric fluxes through boundary cycles later on.

Thus we have two more examples of a coincidence between images and kernels of different homomorphisms. This phenomenon is part of the exact sequence of relative homology mentioned in the introduction, an important pattern that has been emerging gradually and will be elaborated now.

7. The exact sequence of relative homology of space-time

Our study within a general setting of the physical concepts of electric charge, electric flux and magnetic flux has revealed how these are described as integrals over cycles in space-time that are respectively relative, boundary and bulk type and unchanged by the appropriate homologies. So they are certainly classified by the corresponding homology groups $H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$, $H_{m-p-2}(\mathcal{B}; \mathbb{Z})$ and $H_{p+2}(\mathcal{M}; \mathbb{Z})$, when $p$-branes are considered.
We have also met three different types of homomorphism between the three types of homology group, denoted $i_*$, $j_*$ and $\partial_*$, and illustrated by (6.3), (5.1) and (4.5). Associated with all of these is an image and kernel which is always a very specific subgroup of the relevant homology group, illustrated by (4.4), (4.3) and (5.2).

If $p$ is allowed to run over all the values compatible with possible $p$-branes in the given background space-time $\mathcal{M}$, the set of all possible homology groups can be arranged as an ordered sequence with homomorphisms of one or other of the above three types relating each successive pair:

$$
\ldots \quad \partial_* \to H_{m-p-1}(\mathcal{B}) \quad i_* \to H_{m-p-1}(\mathcal{M}) \quad j_* \to H_{m-p-1}(\mathcal{M}, \mathcal{B}) \quad \partial_* \to H_{m-p-2}(\mathcal{B}) \quad i_* \to \ldots \quad (7.1)
$$

This is the exact sequence of relative homology well known to pure mathematicians in the context of algebraic topology, and more careful and detailed treatments can be found in various textbooks. The notation has been compressed by omitting reference to the integers $\mathbb{Z}$.

Assuming space-time, $\mathcal{M}$, is connected, this exact sequence of abelian groups starts and finishes with the trivial group, written as 1 in multiplicative notation:

$$
1 \to H_m(\mathcal{M}, \mathcal{B}) \to H_{m-1}(\mathcal{B}) \to H_{m-1}(\mathcal{M}) \to H_{m-1}(\mathcal{M}, \mathcal{B}) \to \ldots
$$

and

$$
\ldots H_1(\mathcal{B}) \to H_1(\mathcal{M}) \to H_1(\mathcal{M}, \mathcal{B}) \to H_0(\mathcal{B}) \to H_0(\mathcal{M}) \to 1.
$$

Thus, besides the two trivial terms terminating the exact sequence, there are $3m$ terms. From the sequence it is now possible to evaluate in terms of the Betti numbers the numbers $s_q(\mathcal{B})$, $s_q(\mathcal{M})$ and $s_q(\mathcal{M}, \mathcal{B})$ that are the dimensions of the free parts of the kernels (4.3), (5.2) and (4.4) and entered the counts of the various charges and fluxes.

The exact sequence (7.1) implies a similar but simpler exact sequence for the free parts of the homology groups (obtained by dividing out the torsion subgroup). Working over real coefficients rather than integers yields an exact sequence of vector spaces with dimensions given by the Betti numbers and linked by linear maps replacing the group homomorphisms. To understand what happens consider such a sequence in simplified notation:

$$
1 \to V_0 \to V_1 \to V_2 \to V_3 \to \ldots V_N \to 1 \quad (7.2)
$$

If $K_n \subset V_n$ is the kernel/image, then, by exactness $K_n \equiv V_{n-1}/K_{n-1}$ (retaining multiplicative notation). So repeating

$$
K_n = V_{n-1}/V_{n-2}/V_{n-3}/\ldots/V_1/V_0/1, \quad (7.3)
$$

and taking dimensions,

$$
s_n = \dim K_n = b_{n-1} - b_{n-2} \ldots (-1)^{n+1}b_0 = b_n - b_{n+1} \ldots (-1)^{N-n}b_N, \quad (7.4)
$$

using the fact that $s_{N+1}$, which equals the alternating sum of all the Betti numbers, vanishes.
Applying these formulae to the exact sequence of relative homology, (7.1), yields

\[ s_n(M) = b_n(M) - b_n(M, B) + b_{n-1}(B) - b_{n-1}(M) \ldots, \]

\[ s_n(M, B) = b_n(M, B) - b_{n-1}(B) + b_{n-1}(M) - b_{n-1}(M, B) \ldots, \]

\[ s_n(B) = b_n(B) - b_n(M) + b_n(M, B) - b_{n-1}(B) \ldots, \]

showing how the count of electric charges, (4.6), depends on the topology of space-time, \( M \).

It is familiar that in the understanding of electromagnetic duality on closed space-time manifolds, \( M \), a property known as Poincaré duality is important. There is an analogous property for manifolds with boundary that will play an important role in the present context. This is known as Poincaré-Lefschetz duality and a short explanation follows.

Corresponding to the integer homology groups already defined it is possible to define integer cohomology groups denoted \( H^q(M; \mathbb{Z}) \) and so on. There is also an exact sequence of homomorphisms linking these in the sense of ascending superscript:

\[ \ldots \rightarrow H^p(B) \rightarrow H^{p+1}(M, B) \rightarrow H^{p+1}(M) \rightarrow H^{p+1}(B) \rightarrow \ldots \]  

(7.5)

The statement of Poincaré-Lefschetz duality is that the corresponding terms in the two exact sequences (7.1) and (7.5) are isomorphic as groups. So

\[ H^{p+1}(M, B; \mathbb{Z}) \equiv H_{m-p-1}(M; \mathbb{Z}), \quad H^{p+1}(M; \mathbb{Z}) \equiv H_{m-p-1}(M, B; \mathbb{Z}), \]  

(7.6)

and

\[ H^p(B; \mathbb{Z}) \equiv H_{m-p-1}(B; \mathbb{Z}). \]  

(7.7)

The last isomorphism is simply Poincaré duality for the boundary, \( B \), which is automatically a closed manifold of dimension \( m - 1 \). Notice how the superscripts and subscripts in an isomorphism are always complementary in the sense of summing to the dimension of the relevant manifold and how (7.6) relates relative topology to absolute topology in the bulk.

There is yet another relation between homology and cohomology that results from the universal coefficient theorem by considering the coefficients to be real numbers rather than integers. The resultant groups are simply the vector spaces, with dimension equal to the Betti number, spanned by the lattices given by the free parts of the integer groups as previously mentioned. Then a homology group of given suffix and type is the dual of the cohomology group of corresponding superscript and type:

\[ H^q(M; \mathbb{R}) = H_q(M; \mathbb{R})^*, \quad H^q(M, B; \mathbb{R}) = H_q(M, B; \mathbb{R})^*, \quad H^q(B; \mathbb{R}) = H_q(B; \mathbb{R})^*. \]  

(7.8)

By means of these and the Poincaré-Lefschetz duality relations (7.6) and (7.7), the cohomology groups can be eliminated to yield the following relations between homology groups:

\[ H_q(M; \mathbb{R}) = H_{m-q}(M, B; \mathbb{R})^* \quad \text{and} \quad H_q(B; \mathbb{R}) = H_{m-q-1}(B; \mathbb{R})^*. \]  

(7.9)
Because the Betti numbers are the dimensions of these real vector spaces, particular consequences are the following equalities:

\[ b_q(\mathcal{M}) = b_{m-q}(\mathcal{M}, \mathcal{B}) \quad \text{and} \quad b_q(\mathcal{B}) = b_{m-q-1}(\mathcal{B}). \quad (7.10) \]

The corresponding duality relations for the dimensions on the image/kernels of the exact sequence, the numbers \( s_q(\mathcal{M}) \), \( s_q(\mathcal{B}) \) and \( s_q(\mathcal{M}, \mathcal{B}) \), will be important and are easily obtained by recognising that in the simplified notation for the exact sequence, (7.2), \( V_n = V_{N-n}^* \).

So \( b_n = b_{N-n} \) and hence by (7.4),

\[ s_n = s_{N+1-n}. \]

As a consequence

\[ \dim V_n = b_n = s_n + s_{n+1} = \dim K_n + \dim(V/K)_n \]

and

\[ b_{N-n} = \dim V_{N-n} = s_{N+1-n} + s_{N-n} = \dim(V/K)_{N-n} + \dim K_{N-n}. \]

This means the dimensions of the two complementary subspaces of \( V \), namely \( K \) and \( V/K \) interchange under duality, \( N \leftrightarrow N-n \).

In particular

\[ s_{m-p-1}(\mathcal{M}, \mathcal{B}) = b_{p+1}(\mathcal{M}) - s_{p+1}(\mathcal{M}) \quad \text{and} \quad s_{p+1}(\mathcal{M}) = b_{m-p-1}(\mathcal{M}, \mathcal{B}) - s_{m-p-1}(\mathcal{M}, \mathcal{B}). \quad (7.11) \]

In fact, by (7.10) these two equations are the same as each other.

8. Electric charges as intersection numbers

With this information we are now well prepared to consider the physical question as to the possible numerical values of the generalised electric charges (2.1). Given a suitable expression for the conserved, localised electric current, \( j \), the charges are evidently determined without recourse to Maxwell’s equations (1.4) and (1.5). Hence in this calculation these equations can be temporarily renounced, provided it is remembered that their reinstatement will reduce the number of independent electric charges, as explained in section 3. We shall defer this reinstatement and the detailed understanding of the issues it raises until the following section.

Just as in the discussion of magnetic fluxes and their quantisation in section 6, we shall have to resort to the geometrical picture of a brane world-volume, as this will give us tractable form for the current. This is found by equating (1.1) and (6.1), the two versions of the term in the action responsible for the brane coupling to the gauge potential:

\[ q \int_w A = \int_\mathcal{M} A \wedge *j. \quad (8.1) \]

Now \( A \) is taken to be an arbitrary \((p+1)\)-form on \( \mathcal{M} \), so it follows that

\[ *j = q\mu(w), \quad (8.2) \]
where $\mu(w)$ is a singular $(m-p-1)$-form involving a product of the same number of Dirac $\delta$-functions with support on the absolute $(p+1)$-cycle $w$ and differentials in the variables transverse to it. It follows that its restriction to $\mathcal{B}$ vanishes, as it should (3.1). In the Appendix it will be shown to be closed as well.

Inserting the $p$-brane current (8.2) into the electric charge (2.1) yields

$$Q(w;S) = q \int_S \mu(w).$$

This is invariant under relative homologies of $S$ according to the work of section 3. Now consider a bulk homology of the world volume, $w$, $w \rightarrow w' = w + \partial a$. By linearity $\mu(w') = \mu(w) + \mu(\partial a)$. As discussed in the Appendix, $\mu(a)$ exists for a bulk chain, $a$ (and now involves step functions as well as Dirac-delta functions) and, moreover, obeys $d\mu(a) = \mu(\partial a)$, up to a sign. Hence the change in the electric charge, $Q(w;S)$, due to this homology is

$$Q(w' - w;S) = q \int_S \mu(\partial a) = q \int_S d\mu(a) = q \int_{\partial S} \mu(a) = \int_{\partial S} \mu(a)|_{\mathcal{B}} = 0$$

So $Q(w;S)$ is defined on the homology classes $H_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \times H_{p+1}(\mathcal{M}; \mathbb{Z})$, or rather on the corresponding product of free parts. So it can be assumed that the relative cycle $S$ intersects the absolute bulk cycle of complementary dimension, $w$, at discrete points. Then the integral for the electric charge is recognised as [Henneaux and Teitelboim 86]

$$Q(w;S) = qI(w,S)$$

where $I(w,S)$ denotes the intersection number of the absolute bulk cycle $w$ with the relative cycle $S$, being the algebraic sum of the number of these points, taking into account signs due to relative orientation. This intersection number possesses a number of mathematical properties that are important for the physical interpretation of this result that we shall now describe.

Choose bases $S_j$ and $w_i$ in the lattices $F_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$ and $F_{p+1}(\mathcal{M}; \mathbb{Z})$ that are the free parts of the two relevant homology groups. Then all intersection numbers are specified by knowledge of the matrix

$$I(w_i, S_j) = I_{ij} \in \mathbb{Z}$$

This intersection matrix, $I$, has $b_{p+1}(\mathcal{M})$ rows and $b_{m-p-1}(\mathcal{M}, \mathcal{B})$ columns and hence is square, by (7.10). Yet another consequence of Poincaré-Lefschetz duality is that this matrix $I$ is unimodular:

$$\det I = \pm 1.$$  \hspace{1cm} (8.4)

Putting these results together it follows that all electric charges are quantised:

$$Q(S) \in q\mathbb{Z}$$  \hspace{1cm} (8.5)

as integral multiples of the coupling constant, $q$, that enters the action. Thus any electric charge paired with any magnetic flux satisfies the Dirac quantisation condition:

$$Q(S)\Phi_M(v) \in 2\pi\hbar \mathbb{Z}.$$  \hspace{1cm} (8.6)
If Maxwell’s equations are permitted, then it follows that certain of the electric fluxes are indeed quantised, namely those obtained by integrating over boundary cycles within $K_{m-p-2}(B; \mathbb{Z})$, as these fluxes are equal to electric charges.

On the other hand, there is no reason to believe that the remaining electric fluxes are quantised, and we shall return to some comments on this later. All that can be said as a result of (8.5) is that, for the latter fluxes, the quantities $\exp(2\pi i \Phi_E(v)/q)$ are well defined on the cosets $(H/K)_{m-p-2}(B; \mathbb{Z})$.

The physical consequence of $I$ being unimodular is that a configuration of the brane-world-volume can be found that realises any assignment of charges satisfying (8.5).

Finally let us comment on the connection between the physical arguments of this section and the mathematical arguments of the preceding one, outlining Poincaré-Lefschetz duality. By current conservation, (1.6), the dual current, $\ast j$, is a closed $(m-p-1)$-form, that, in addition, has vanishing restriction on the boundary of space-time, (3.1). This means that it is a relative $(m-p-1)$-cocycle in the sense of de Rham cohomology, and so defines a class of $H_{\text{de Rham}}^{m-p-1}(\mathcal{M}, B; \mathbb{R})$. The same is therefore true of $\mu(w)$ by (8.2), which therefore provides a map from the homology class of the world-volume, $w$, $H_{p+1}(\mathcal{M}; \mathbb{Z})$ to $H_{\text{de Rham}}^{m-p-1}(\mathcal{M}, B; \mathbb{R})$. This is part of the Poincaré-Lefschetz isomorphism, (7.6). It is possible to develop this line of thought and this is done in the Appendix.

9. Maxwell’s equations and the intersection matrix

Two (correct) arguments have been developed in this paper that apparently lead to a contradiction. We shall now explain what this is, and how it is resolved by finding that the intersection matrix, $I$, (8.3), has further detailed properties, hitherto unexpected.

In section 4 it was shown that the effect of Maxwell’s equations is to force all electric charges, $Q(S)$, to vanish when the relative cycle over which they are integrated, $S$, belongs to $K_{m-p-1}(\mathcal{M}, B; \mathbb{Z})$. Yet according to the preceding section, irrespective of Maxwell’s equations, the electric charge due to a brane configuration with world-volume, $w$ was seen to be proportional to the intersection number of $w$ with $S$. The apparent contradiction arises from the fact that the intersection matrix is non-singular, as a consequence of its being unimodular (8.4).

To make this clearer, it is natural to partition the intersection matrix in a way that distinguishes each kernel $K$ within each $H$ from the cosets $H/K$. This is done by choosing the basis $\{S_j\}$ so that the first $s_{m-p-1}(\mathcal{M}, B)$ elements form a basis of $K_{m-p-1}(\mathcal{M}, B)$ while the remainder refer to the cosets $(H/K)_{m-p-1}(\mathcal{M}, B)$. The basis $\{w_i\}$ is chosen so that the last $s_{p+1}(\mathcal{M})$ elements form a basis for $K_{p+1}(\mathcal{M})$ while the remainder refer to the cosets. Corresponding to this, the intersection matrix, (8.3), is written in the block form

$$I(w, S) = \begin{pmatrix}
K(\mathcal{M}, B) & (H/K)(\mathcal{M}, B) \\
K(\mathcal{M}) & (H/K)(\mathcal{M}, B)
\end{pmatrix}.
$$

That electric charges associated with $K_{m-p-1}(\mathcal{M}, B)$ all vanish seems to imply that the submatrices $A$ and $X$ vanish, apparently contradicting the fact that the overall matrix has determinant equal to $\pm 1$.  

17
But this is not a correct interpretation of what has been shown. The correct interpretation is that the absolute bulk homology classes of the brane world-volume, \( w \), that yield non-zero charges associated with relative cycles of homology belonging to the subgroup \( K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \) are all forbidden because Maxwell’s equations cannot then be integrated to yield field strengths, given the corresponding currents (8.2). Thus the only permitted homology classes of world-volume are those whose intersection number with all elements of \( K_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \) vanish. These classes should form a subgroup and it natural to anticipate that this be provided by the kernel \( K_{p+1}(\mathcal{M}; \mathbb{Z}) \). The condition for this is that the submatrix \( X \) in (9.1) vanish. This is perfectly consistent with the unimodularity of \( I \), (8.4), since, by (7.11), the consequences of Poincaré-Lefschetz duality for the kernels, the block diagonal submatrices \( A \) and \( B \) are both square. Consequently

\[
\pm 1 = \det I = \det A \det B,
\]

implying that the block diagonal submatrices \( A \) and \( B \), possessing integer entries, are both unimodular too.

Thus it is the submatrix \( B \) that gives the physical charges, \( Q(S) \), for \( S \) in the coset \((H/K)_{m-p-1}(\mathcal{M}, \mathcal{B}; \mathbb{Z})\) as it determines the intersection numbers between these relative classes and the permitted homology classes of brane world-volume. According to (5.2), (with \( m-p-1 \) replaced by \( p+1 \)) these permitted world-volumes are those homologous to cycles in the boundary of space-time, \( \mathcal{B} \). It is remarkable that such a selection rule on brane configurations can be derived without recourse to any equations of motion for the brane degrees of freedom.

Since the submatrix, \( B \), that determines the physical charges, is unimodular, the previous conclusion there exist brane configurations realising any assignment of quantised charges, (8.5), holds good even when the selection rule is taken into account. What remains is to provide an independent check that the block submatrix \( X \) in (9.1) vanishes.

This is a geometrical condition that should hold for any background space-time, \( \mathcal{M} \), with boundary \( \mathcal{B} \), and it can be rewritten as:

\[
I(K_{p+1}(\mathcal{M}), K_{m-p-1}(\mathcal{M}, \mathcal{B})) = 0. \quad (9.2)
\]

This vanishing theorem will be demonstrated in the next section using some results developed in the Appendix.

**10. De Rham cohomology, field strengths and currents**

The argument will be interesting as it brings into play further parallels between physical and mathematical concepts and sheds light on the more abstract ideas involving the two related exact sequences mentioned previously and to be elaborated below.

We can no longer avoid describing de Rham cohomology which deals with the exterior derivative, \( d \), of differential forms (such as the field strengths and currents we have been talking about). We have to explain the three types of cohomology group that arise, given a manifold with boundary; how they can arranged in an exact sequence, and how that exact sequence is related to the one for homology groups already explained.
A real $q$-form, $\omega$, on $\mathcal{M}$ is an absolute bulk cocycle if it is coclosed, $d\omega = 0$. Two such cocycles are absolutely cohomologous in the bulk if

$$H^q(\mathcal{M}; \mathbb{R}) : \quad \omega' \sim \omega \iff \omega' = \omega + d\alpha. \quad (10.1)$$

As indicated, these form equivalences classes which constitute elements of the absolute bulk homology group $H^q(\mathcal{M})$ (in the sense of de Rham). The groups are abelian since composition is by addition. Actually these groups are real vector spaces since they are closed under multiplication by real numbers.

Essentially the same concepts can be applied to $q$-forms, $\phi$, on the boundary, $\mathcal{B}$. $\phi$ is a boundary cocycle if it exists on $\mathcal{B}$ and is coclosed there. Two such cocycles are absolutely cohomologous on the boundary if

$$H^q(\mathcal{B}; \mathbb{R}) : \quad \phi' \sim \phi \iff \phi' = \phi + d\beta. \quad (10.2)$$

Again these are equivalence relations whose classes form the group indicated.

The third and last concept is that of a relative cocycle, $\eta$, which is defined in the bulk, on $\mathcal{M}$, is coclosed there, $d\eta = 0$ and has vanishing restriction to the boundary, $\eta|_{\mathcal{B}} = 0$. Two such cocycles are relatively cohomologous if they differ by a coexact form $d\alpha$ with the property that the restriction of $\alpha$ to the boundary is coexact there.

$$H^q(\mathcal{M}, \mathcal{B}; \mathbb{R}) : \quad \eta' \sim \eta \iff \eta' = \eta + d\alpha, \quad \alpha|_{\mathcal{B}} = d\beta. \quad (10.3)$$

Again these are equivalence relations whose classes form the group indicated. In each case the cohomology relation preserves the appropriate coclosure property. Physical examples are provided by the field strength, $F$, which is an absolute bulk $(p + 2)$-cocycle, $\ast F|_{\mathcal{B}}$, which is an absolute boundary $(m - p - 2)$-cocycle and the dual current, $\ast j$, which is a relative $(m - p - 1)$-cocycle.

Thus there are three types of cohomology matching the three types of homology already explained. Furthermore they both exist for a range of values of the integer $q$ specifying the dimension of the cycle or the degree of the form as appropriate. When taken over real numbers, homology and cohomology groups of matching type and integer $q$ are related in a nice way, as dual vector spaces, see (7.8). To understand the first example of these relations, let $\omega$ be a $q$-cocycle and $v$ a $q$-cycle, both in the absolute bulk sense, and consider

$$\int_v \omega \in \mathbb{R}.$$ 

This integral enjoys a number of properties:

1) It is invariant under the appropriate homologies of $v$, $v \rightarrow v' = v + \partial a$, and cohomologies of $\omega$, (10.1).

2) It is linear in $v$ and $\omega$ separately and hence provides a real bilinear form.

3) It is nonsingular; that is there is no nontrivial class of either type such that the integral vanishes for all classes of the other type.

The first two properties are easy to check but the third, nonsingularity, is quoted as a known theorem (of de Rham). Of course the magnetic flux (6.2) already defined is an
example of such an integral. Precisely analogous constructions work for the other two
types of homology/cohomology and yield the remaining duality relations (7.8). Physical
elements of these integrals are provided by electric charge, (2.1), and electric flux, (4.2),
involving relative and boundary homology/cohomology respectively.

Space-time, $\mathcal{M}$, is itself a relative $m$-cycle and hence it is appropriate to integrate relative
$m$-cocycles over it. The wedge product $\eta \wedge \omega$ is such a cycle if $\eta$ and $\omega$ are respectively
relative and absolute bulk cocycles of complementary degree (summing to $m$). So it is
natural to consider

$$\int_{\mathcal{M}} \eta \wedge \omega \in \mathbb{R}.$$ 

This integral is
(1) invariant under the appropriate cohomologies of $\omega$ and $\eta$, (10.1) and (10.3),
(2) bilinear in $\omega$ and $\eta$
(3) nonsingular.

As a consequence there results the duality relation

$$H^q(\mathcal{M}; \mathbb{R}) = H^{m-q}(\mathcal{M}, \mathcal{B}; \mathbb{R})^*$$

which, when combined with the previous duality relations (7.8), implies

$$H^q(\mathcal{M}; \mathbb{R}) = H_{m-q}(\mathcal{M}, \mathcal{B}; \mathbb{R}) \quad \text{and} \quad H^q(\mathcal{M}, \mathcal{B}; \mathbb{R}) = H_{m-q}(\mathcal{M}; \mathbb{R}),$$

a weak version of Poincaré-Lefschetz duality, (7.6) (weak because it is over the reals rather
than the integers). A weak version (over the reals) of the similar relation for the boundary,
(7.7), can likewise be checked.

These results are sufficient to show that there exists an exact sequence of de Rham
cohomology groups but it is worth demonstrating this explicitly in order to find precise
definitions of the common kernel/image subgroups of these groups.

Relative cocycles are automatically absolute cocycles in the bulk too and this leads to
the homomorphism

$$j^* : H^q(\mathcal{M}, \mathcal{B}) \to H^q(\mathcal{M}). \quad (10.4)$$

The kernel of $j^*$ is made up of the elements that are trivial in $H^q(\mathcal{M})$:

$$K^q(\mathcal{M}, \mathcal{B}) = \{ \text{classes of } H^q(\mathcal{M}, \mathcal{B}) \text{ satisfying } \eta = d\alpha, \quad d\alpha|_{\mathcal{B}} = 0 \}, \quad (10.5)$$

while the image appears to consist of elements of $H^q(\mathcal{M})$ with $\omega|_{\mathcal{B}}$ vanishing.

Absolute cocycles in the bulk automatically yield cocycles in the boundary when re-
stricted to it. So $\omega \to \omega|_{\mathcal{B}}$ yields the homomorphism

$$i^* : H^q(\mathcal{M}) \to H^q(\mathcal{B}) \quad (10.6)$$

with kernel

$$K^q(\mathcal{M}) = \{ \text{classes of } H^q(\mathcal{M}) \text{ with } \omega|_{\mathcal{B}} \text{ coexact} \} \quad (10.7)$$

This obviously includes the image of $j^*$ and tallies after applying bulk cohomologies (10.1).
The image of $i^*$ will be specified below.
Given a coclosed form, \( \beta_0 \), on the boundary, \( d\beta_0 = 0 \) on \( B \), there is a way to find a closed form \( \eta_\beta \), of one degree higher on the bulk whose restriction to the boundary automatically vanishes so that it is relatively coclosed. Although the procedure is not unique, the degree of ambiguity lies in a single relative cohomology class and so the procedure leads to a homomorphism, known as the Bockstein homomorphism:

\[
d^* : H^q(B) \to H^{q+1}(M, B)
\]  

(10.8)

Let \( \beta \) denote an extension of \( \beta_0 \) from the boundary, that is, a form on \( M \), not necessarily closed, satisfying \( \beta|_B = \beta_0 \). Then, if \( \eta_\beta = d\beta \), \( d\eta_\beta = 0 \) and \( \eta_\beta|_B = d\beta|_B = d\beta_0 = 0 \) and so \( \eta_\beta \) is a relative cocycle. Consider now \( \beta_0 \) and \( \beta_0' \), forms which are cohomologous in \( H^q(B) \), (10.2), so \( \beta_0' = \beta_0 = d\alpha \), (on \( B \)). If they have extensions \( \beta \) and \( \beta' \), respectively to the bulk

\[
\eta_{\beta'} - \eta_\beta = d(\beta' - \beta) \quad \text{and} \quad (\beta' - \beta)|_B = d\alpha,
\]

which means \( \eta_{\beta'} \) and \( \eta_\beta \) are relatively cohomologous, (10.3), as desired. In particular, this applies to the ambiguity arising when \( \beta' \) and \( \beta \) are different extensions of the same \( \beta_0 \).

The image of \( d^* \) is obviously given by increasing \( q \) by unity in (10.5), originally the kernel of \( j^* \) but the kernel of \( d^* \) is trickier. Obviously \( \eta_\beta \) is trivial in relative cohomology (10.3) whenever \( \beta_0 \) is coexact but this means it is trivial in \( H^q(B) \). But \( \eta_\beta \) is also trivial if it vanishes, that is if \( \beta_0 \) extends to a form \( \beta \) in the bulk which is still coclosed. Thus

\[
K^q(B) = \{ \text{classes of } H^q(B) \text{ extending to coclosed forms on } M \}.
\]  

(10.9)

This is also the image of \( i^* \). Thus we have a series of identifications of images and kernels and the results can be all assembled in the following grand diagram:

\[
\begin{array}{ccccccccc}
.. & \overset{i^*}{\to} & H^p(B) & \overset{d^*}{\to} & H^{p+1}(M, B) & \overset{j^*}{\to} & H^{p+1}(M) & \overset{i^*}{\to} & H^{p+1}(B) & \overset{d^*}{\to} & .. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
.. & \overset{\partial}{\to} & H_{m-p-1}(B) & \overset{i}{\to} & H_{m-p-1}(M) & \overset{j}{\to} & H_{m-p-1}(M, B) & \overset{\partial}{\to} & H_{m-p-2}(B) & \overset{i}{\to} & .. 
\end{array}
\]  

(10.10)

The upper sequence is composed of the homomorphisms of the de Rham cohomology groups just described. It is exact because at each stage the kernels and images coincide as was just explained. The lower sequence is the exact sequence of homology (7.1) explained in previous sections whilst the vertical arrows indicate the Poincaré-Lefschetz isomorphisms (7.6) and (7.7). The most powerful version of this diagram refers to groups taken over the integers, \( \mathbb{Z} \), but for some parts of the diagram we have only given arguments establishing a weaker version, over the reals, \( \mathbb{R} \).

By (7.3), a consequence of exactness, the kernel subgroups of the pairs of groups related by the Poincaré-Lefschetz isomorphism are themselves isomorphic. This suggests that the result we wish to prove, (9.2), is equivalent to its cohomological counterpart:

\[
\int_\mathcal{M} \eta \wedge \omega = 0 \quad \text{if} \quad \eta \in K^{m-q}(\mathcal{M}, B; \mathbb{R}) \quad \text{and} \quad \omega \in K^q(\mathcal{M}; \mathbb{R}).
\]  

(10.11)
But this is quite easy to prove using the results above, as we now see. By (10.5) the integral equals
\[ \int_M d\alpha \wedge \omega = \int_M d(\alpha \wedge \omega) \] as \( d\omega \) vanishes by (10.7). By Stokes’ theorem on \( M \) the integral equals
\[ \int_B \alpha \wedge \omega = \int_B \alpha \wedge d\gamma = \int_B \alpha \wedge \omega \] as \( \omega \) is coexact by (10.7). But, by (10.5), \( \alpha \) is coclosed so the integral equals
\[ \int_B d(\alpha \wedge \gamma) = \int_{\partial B} \alpha \wedge \gamma = 0, \] by Stokes’ for the boundary, \( B \), and the fact that the latter is automatically closed.

Vanishing theorems analogous to (10.11) also apply to the integrals like \( \int_w \omega \) coupling a pair of like homology and cohomology groups. For example, the electric charge \( Q(S) = \int_S \ast j \) couples the relative homology of the integration domain, \( S \), \( H_{m-p-1}(M,B) \) to the relative de Rham cohomology of the dual current, \( \ast j \), \( H^{m-p-1}(M,B) \) and vanishes when \( S \in K_{m-p-1}(M,B) \), (4.4) and \( \ast j \in K^{m-p-1}(M,B) \), (10.5). The latter condition certainly hold when Maxwell’s equation, (1.5), for the field strength, \( F \), holds. This vanishing theorem is then precisely what was proven in our earlier discussion of electric charges, and that is now seen to be part of a more general pattern.

The last step is the derivation of the vanishing theorem (9.2) for the intersection matrix from the vanishing theorem for cohomology, (10.11), proven above, using the upward arrow in the Poincaré-Lefschetz isomorphism, (10.10). A convenient concrete version of this map is provided by the quantity \( \mu(w) \) that enters the expression (8.2) for the dual current \( \ast j \) due to a brane whose world-volume is the absolute cycle \( w \), and generalisations of this to be explained in the Appendix. These maps will provide homomorphisms between the groups indicated in (10.10) mapping the appropriate kernel subgroups into each other. The desired result follows by combining these results with the fact that the intersection number can be written
\[ I(w,S) = \int_M \mu(w) \wedge \mu(S) \]

11. Discussion

Motivated by the physical questions of elucidating and counting the types of conservation law occurring in the sorts of generalised Maxwell theories that arise naturally in string/superstring theories formulated on a fixed background space-time of possibly complicated topology, we have been led to a well established area of pure mathematics. This is the theory of relative homology/cohomology associated with the space-time, assumed to have a boundary, and it seems not to be so familiar to physicists despite its evident physical relevance. Accordingly we have tried to build it up systematically, as guided by physics, and in particular, the generalised Maxwell’s equations, and included reasonably self-contained proofs.

Given an understanding of the overall grand mathematical structure, comprising the two exact sequences of homology and cohomology and the Poincaré-Lefschetz isomorphism relating them, as depicted by (10.10), and the duality relations, (7.8), indicating a horizontal reflection symmetry of the exact sequences, it is relatively easy to explain the relevance to physics. This is what we now do because of the value of the new perspectives afforded.

The first step is the recognition that there are precisely three types of conserved quantity, electric charge, (2.1), electric flux, (4.2) and magnetic flux, (6.2), and that these are associated with the three possible types of homology/cohomology, namely relative, boundary and absolute bulk, respectively. In fact these conserved quantities are invariant under the
appropriate homologies/cohomologies and, indeed, constitute non-singular bilinear forms on the free parts of these groups, thereby being responsible for the duality relations, (7.8), of the exact sequences (10.10).

However this argument makes only partial use of the generalised Maxwell’s equations, (1.4) and (1.5), and the associated boundary conditions (3.1) and (3.2). What is used for each conserved quantity in turn is:

- Electric charge, (2.1): \( d \ast j = 0 \) and \( j|_B = 0 \)
- Electric flux, (4.2): \( d\{\ast F\}|_B = 0 \)
- Magnetic Flux, (6.2): \( dF = 0 \).

With this limited information these three conserved quantities appear unrelated to each other and counted by the relevant Betti numbers, \( b_{m-p-1}(\mathcal{M}, \mathcal{B}) \), \( b_{m-p-2}(\mathcal{B}) \) and \( b_{p+2}(\mathcal{M}) \) as explained above. The content in Maxwell’s equations that has not so far been exploited is the inhomogeneous Maxwell in the bulk, (1.5), and it has many extra consequences, as we have seen in the text. From the point of view of de Rham cohomology the most immediate is that the dual current, \( \ast j \), is not just coclosed (current conservation, (1.6)) but coexact, and hence an element of the subgroup \( K^{m-p-1}(\mathcal{M}, \mathcal{B}) \), (10.5), of the relative de Rham cohomology group. This is the subgroup that plays the role of kernel/image at this stage of the exact sequence of cohomology. Thus the exact sequence is now brought into play by means of the bulk Maxwell’s equations.

When this current, \( j \), is determined by the geometrical picture in terms of the \( p \)-brane world-volume, \( w \), by (8.2), the fact that \( \mu \) realises the Poincaré-Lefschetz isomorphism as explained in the Appendix, means that the world-volume \( w \) must belong to a class of \( K_{p+1}(\mathcal{M}) \), (4.3), and hence be homologous to a cycle on the boundary, \( \mathcal{B} \), of space-time. This was one of our main results, obtained by a more roundabout, though more self-contained, method, when we were not taking the complete mathematical structure for granted.

This conclusion is contrary to what would have seemed intuitively likely, that any configuration of brane world-volumes in space-time is possible. The reason unsuitable configurations are forbidden is that they provide topological obstructions to the integration of the generalised Maxwell equations for which they provide sources, as argued in the text. Notice that in obtaining this selection rule it was not necessary to take into account any equations of motion for the brane degrees of freedom.

We saw that a another, related, consequence of Maxwell’s inhomogeneous equations in the bulk was the reduction of the count of linearly independent electric charges from the Betti number \( b_{m-p-1}(\mathcal{M}, \mathcal{B}) \) to \( s_{p+1}(\mathcal{M}) = b_{m-p-1}(\mathcal{M}, \mathcal{B}) - s_{m-p-1}(\mathcal{M}, \mathcal{B}) \), corresponding to the number of linearly independent homology classes permitted for the \( p \)-brane world-volume.

In section 4 we saw that \( p \)-brane electric fluxes are classified by the boundary homology group, \( H_{m-p-2}(\mathcal{B}; \mathbb{Z}) \), or, more precisely by the free part of this abelian group obtained by dividing out the torsion subgroup, namely a lattice of dimension given by the Betti number \( b_{m-p-2}(\mathcal{B}) \). The effect of the inhomogeneous bulk Maxwell equations is to equate to electric charges those electric fluxes on the sublattice of dimension \( s_{m-p-2}(\mathcal{B}) \), corresponding to \( K_{m-p-2}(\mathcal{B}; \mathbb{Z}) \). As a result these electric fluxes are quantised, as integer multiples of \( q \), but this result does not apply the remaining \( b_{m-p-2}(\mathcal{B}) - s_{m-p-2}(\mathcal{B}) \) electric fluxes. There
is no reason for them to be quantised.

Through these boundary cycles there may also be magnetic fluxes, this time associated with $\tilde{p}$-branes, dual to the $p$-branes (so $p + \tilde{p} + 4 = m$). As seen in section 6, these vanish on the afore-mentioned sublattice of dimension $s_{m-p-2}(B)$ whilst the remaining fluxes are quantised as integer multiples of $2\pi \hbar / q$. Thus there is evidence of states carrying just quantised electric charge and no magnetic charge, and these must be the input $p$-brane states. But the quantised magnetic flux and non-quantised electric flux through the $(H/K)_{m-p-2}(B; \mathbb{Z})$ cycles is rather reminiscent of known solutions [Witten 1979] to the Zwanziger-Schwinger quantisation condition [Zwanziger 1968, Schwinger 1969] applying to particles in four dimensional space-time and so provides evidence of mysterious and intriguing dyonic objects that are not situated on the space-time, $\mathcal{M}$, according to (1.4). Maybe a better understanding of this phenomenon is important in connection with electromagnetic duality.

At this stage, we should explain that one motivation for the present work was to gain a better understanding of electromagnetic duality [Montonen and Olive 1977]. It has been understood that in a closed space-time of four dimensions, certain partition functions exhibit a beautiful covariance under the action of the modular group implementing electro-magnetic duality transformations [Verlinde 1995, Witten 1995] and this is further enhanced when spin is taken into account [Alvarez and Olive 2001]. It is also possible to include Wilson loops [Zucchini 02]. But, in closed space-times there are neither non-vanishing electric charges nor electric fluxes, only magnetic fluxes. Yet in supersymmetric gauge theories on flat space-time it is familiar that electromagnetic duality transformations permute electric and magnetic charges [Sen 1994].

So it might be important to consider space-times with boundary, as we have. As just explained there is a beautiful topological classification of conservation laws involving electric charge, electric flux and magnetic flux but leaving no room for the classification of magnetic charge. As a result we are left with a dilemma to be resolved by future work.

There are many other questions left open for future work and many of them concern undesirable simplifications that have been made relative to the full complication of super-string theory. We shall conclude by listing some of these. Some of these oversimplifications are routine practice in the subject but should none-the-less be removed when possible.

1) Branes have been treated as geometrical objects, cycles in space-time, and not assigned any sort of generalised quantum mechanical wave function as ideally they should. In the absence of this there is lacking the concept of intrinsic spin which is familiar for particle ($p = 0$) wave-functions on four dimensional space-time, and known to play a role in the understanding of electromagnetic duality [Alvarez and Olive 2001].

2) No account is taken of any internal brane structure, such as gauge theories confined to the brane as sometimes required by supersymmetry. If so presumably a $K$-theory classification of this internal structure would be relevant, [Witten 2001].

3) Brane world-volumes have been treated as cycles. It might be more reasonable to allow them to have boundaries in the infinite past or future but we do not know how to do this.

4) No equations of motion for $p$-brane degrees of freedom have been considered. Partly this is because these equations ought to involve the wave-functions, not yet formulated
properly anyway when \( p > 0 \).

5) No account is taken of any supersymmetry. This usually requires a spectrum of
values of \( p \) and the fact that some branes may possess boundaries situated on other branes
[Strominger 1996]. It would be interesting to know how the charges and fluxes we have
discussed could be related to the tensor charges occurring in the supersymmetry algebra.

6) Branes have been treated as carrying only electric charge but maybe an additional
magnetic charge should be allowed as an input in (1.4).

7) No account of Chern-Simons type terms has been taken in the generalised Maxwell
equations. This could only occur when \( p + 2 \) is even and divides \( m + 1 \), as for the familiar
case of \( p = 2 \) and \( m = 11 \), [Cremmer, Julia and Scherk 1978].

8) No special consideration has been made of the self-dual case when \( m \) equals twice
\( p + 2 \) (so \( p = \bar{p} \)).

12. Appendix

The basic idea stems from the way the term in the action, (1.1), describing the geomet-
rical coupling of the \( p \)-brane to the gauge potential \( A \), defines the dual electric current,
\( *j \), via (6.1) to be proportional to a distribution valued differential form, \( \mu(w) \), depending
on the world-volume, \( w \). Clearly this idea is motivated by physical considerations. A
mathematical version had earlier been proposed by de Rham [1955,1984]. So far the idea
applies to absolute cycles and it has to be extended to relative cycles and to chains both
in the bulk and on the boundary and this is now done.

If \( C \) is a \( q \)-chain containing no sub \( q \)-chain lying in the boundary, \( B \), its dual current,
\( \mu(C) \), is defined by

\[
\int_C f = \int_M f \wedge \mu(C)
\]

where \( f \) is an arbitrary \( q \)-form. On the other hand if \( \gamma \) is a \( q \)-chain lying on the boundary,
\( B \), the dual surface current, \( \nu(\gamma) \) is defined by

\[
\int_B g = \int_B g \wedge \nu(\gamma),
\]

where \( g \) is an arbitrary \( q \)-form on the boundary. Notice that even though \( C \) and \( \gamma \) are
chains of the same dimension, \( q \), \( \mu(C) \) and \( \nu(\gamma) \) are forms of different degree, \( m - q \) and
\( m - q - 1 \) respectively.

The boundary of \( C \), \( \partial C \), can be decomposed into two terms of the type just described,
each of dimension one less:

\[
\partial C = U + \alpha,
\]

so that \( \mu(C) \), \( \mu(U) \) and \( \nu(\alpha) \) are all well-defined.

The integral \( \int_{\partial C} h \) can be evaluated in two ways, first as
\( \int_M h \wedge \mu(U) + \int_B h|_B \wedge \nu(\alpha) \),
and secondly as \( \int_M dh \wedge \mu(C) \), using Stokes’ theorem. On integrating by parts this equals
\( \int_B h|_B \wedge \mu(C)|_B + (-1)^{d(C)} \int_M h \wedge d\mu(C) \), where \( d(C) \) is the dimension of \( C \). Equating the
bulk and boundary terms separately yields the identities

\[
d\mu(C) = (-1)^{d(C)} \mu(U) \quad \text{and} \quad \mu(C)|_B = \nu(\alpha).
\]
These are precisely what is needed to check the properties of the upwards Poincaré-Lefschetz homomorphism from homology to de Rham cohomology. This will be done by exploiting the different ways of interpreting these relations and special cases of them.

$C$ is a relative cycle if $U$ vanishes. Then $d\mu(C) = 0$ and so $\mu(U)$ is an absolute bulk de Rham cocycle as $\mu(C)|_B$ need not vanish.

$C$ is an absolute bulk cycle if $U$ and $\alpha$ both vanish. Then $d\mu(C)$ and $\mu(C)|_B$ both vanish, implying that $\mu(C)$ is a relative de Rham cocycle.

$U$ is a relative boundary and $\mu(U)$ is coexact and so trivial in absolute bulk cohomology.

$U$ is an absolute boundary if $\alpha$ vanishes. Again $\mu(U)$ is coexact but in addition $\mu(U)|_B = \nu(\alpha) = 0$ so that now $\mu(U)$ is trivial in relative cohomology.

These four observations are sufficient to show that $\mu$ maps absolute or relative homology classes into relative or absolute cohomology classes respectively. By linearity these maps are homomorphisms and it is easy to see that their kernels include the torsion subgroups so more properly $\mu$ acts on the free parts, $F$, of the homology groups $H$ (obtained by dividing out the torsion). So

$$\mu : F_q(\mathcal{M}; \mathbb{Z}) \to F^{m-q}(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \quad \text{and} \quad F_q(\mathcal{M}, \mathcal{B}; \mathbb{Z}) \to F^{m-q}(\mathcal{M}; \mathbb{Z}).$$

The last step is to check that $\mu$ maps the appropriate kernel subgroups into each other.

$U$ is an absolute bulk cycle homologous to a boundary cycle, $-\alpha$ if $\partial U$ vanishes and so in a class of $K_q(\mathcal{M})$ by (5.2). But then $\mu(U) = d( (-1)^{d(C)} \mu(C) )$ and $\mu(C)|_B = \nu(\alpha)$ where $d\nu(\alpha) = (-1)^{d(C)} \nu(\partial \alpha) = 0$. So, by (10.5), $\mu$ maps from a class of $K_q(\mathcal{M})$ to a class of $K^{m-q}(\mathcal{M}, \mathcal{B})$.

Finally if $U$ vanishes and $\alpha = \partial \gamma$, then $\partial C = \partial \gamma$ meaning that $C$ is a relative cycle in a class of $K_q(\mathcal{M}, \mathcal{B})$ by (4.4). Hence $d\mu(C)$ vanishes and $\mu(C)|_B = \nu(\alpha) = \nu(\partial \gamma) = (-1)^{d(C)} d\nu(\gamma)$. Thus, by (10.7), $\mu$ maps from a class of $K_q(\mathcal{M}, \mathcal{B})$ to a class of $K^{m-q}(\mathcal{M})$.

By the work of this Appendix, the intersection number

$$I(w, S) = \int_S \mu(w) = \int_{\mathcal{M}} \mu(w) \wedge \mu(S).$$

Furthermore if $w \in K_q(\mathcal{M}; \mathbb{Z})$ and $S \in K_{m-q}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$ then $\mu(w) \in K^{m-q}(\mathcal{M}, \mathcal{B}; \mathbb{Z})$ and $\mu(S) \in K^q(\mathcal{M}; \mathbb{Z})$ so that, finally, $I(w, S)$ vanishes by (10.11), as desired.

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