Approximate predictability for pseudo–metric systems

Giordano Pola
DISIM–DEWS
University of L’Aquila
L’Aquila, Italy
giordano.pola@univaq.it

Elena De Santis
DISIM–DEWS
University of L’Aquila
L’Aquila, Italy
elena.desantis@univaq.it

Maria Domenica Di Benedetto
DISIM–DEWS
University of L’Aquila
L’Aquila, Italy
mariadomenica.dibenedetto@univaq.it

Abstract—In this contribution, we introduce the notion of approximate predictability for the general class of pseudo–metric systems, which are a powerful modeling framework to deal with complex heterogeneous systems such as hybrid systems. Approximate predictability corresponds to the possibility of predicting the occurrence of specific states belonging to a particular subset of interest, in advance with respect to their occurrence, on the basis of observations. We establish a relation between approximate predictability of a given pseudo–metric system and approximate predictability of a pseudo–metric system that approximately simulates the given one.

Index Terms—approximate predictability, pseudo–metric systems, approximate simulation

I. INTRODUCTION

Safety issues in modern control systems are presently one of the most significant challenges, see e.g. [1]. In this regard, it is fundamental to be able to understand if the system’s behavior enters a given critical subset of the state space on the basis of the observations. The safety problem can then be addressed by predicting in a deterministic way the occurrence of specific states belonging to the critical set, in advance with respect to their occurrence (predictability property). Predictability (also called prognosability) has been investigated for discrete–event systems and Petri nets, see e.g. [3, 6, 7, 8, 9, 10] and continuous systems, see e.g. [4]. In this contribution we introduce the notion of approximate predictability which extends to the general class of pseudo–metric systems, the notion of (exact) predictability given in [2], see also [3, 10], for finite state machines. Pseudo–metric systems are a powerful modeling framework to deal with complex heterogeneous systems such as hybrid systems. We then establish the relation between the approximate predictability of a given pseudo–metric system and the approximate predictability of a pseudo–metric system that approximately simulates the given one. Among possible applications of the results presented in this contribution, there are smart factories where it is of key importance to predict future critical behavior for early maintenance, which can avoid plant shutdown or can mitigate the effects of a fault. This will be explored in our future work.

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II. NOTATION AND PRELIMINARY DEFINITIONS

The symbols \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}^+ \) and \( \mathbb{R}^+_0 \) denote the set of nonnegative integer, integer, real, positive real, and nonnegative real numbers, respectively. Given \( a, b \in \mathbb{Z} \), we denote \([a; b] = [a, b] \cap \mathbb{Z}\) and \([a; b] = [a, b) \cap \mathbb{Z}\). Given a set \( X \), a function \( d : X \times X \to \mathbb{R}^+_0 \cup \{\infty\} \) is a pseudo–metric for \( X \) if: (i) for any \( x \in X \), \( d(x, x) = 0 \); (ii) for any \( x, y, z \in X \), \( d(x, y) \leq d(x, z) + d(z, y) \), (iii) for any \( x, y \in X \), \( d(x, y) = d(y, x) \). When function \( d \) is a pseudo metric for \( X \), the pair \((X, d)\) is said a pseudo metric space.

III. APPROXIMATE PREDICTABILITY FOR PSEUDO–METRIC SYSTEMS

In this contribution we consider the class of pseudo–metric systems, defined as follows.

Definition 1: [5] A system \( S \) is a tuple \( S = (X, X_0, U, Y, H) \), where \( X \) is the set of states, \( X_0 \subseteq X \) is the set of initial states, \( U \) is the set of inputs, \( \rightarrow \subseteq X \times U \times X \) is the transition relation, \( Y \) is the set of outputs, \( H : X \to Y \) is the output function.

We follow standard practice and denote a transition \((x, u, x')\) of \( S \) by \( x \xrightarrow{u} x' \). The evolution of a system is captured by the notions of state and output runs. Given a sequence of transitions of \( S \)

\[
x(0) \xrightarrow{u(0)} x(1) \xrightarrow{u(1)} \cdots \xrightarrow{u(l-1)} x(l)
\]

with \( x(0) \in X_0 \), the sequences:

\[
x(\cdot) : x(0) x(1) \ldots x(l),
\]

\[
y(\cdot) : H(x(0)) H(x(1)) \ldots H(x(l)),
\]

are called a state run and an output run of \( S \), respectively. State \( x(l) \) is called the ending state of the state run in (2).

Given state run \( x(\cdot) \) in (2) and \( l' \leq l \), the symbol \( x|_{[0,l']} \) denotes the state run \( x(0) x(1) \ldots x(l') \). Given output run \( y(\cdot) \) in (3) and \( l' \leq l \), the symbol \( y|_{[0,l']} \) denotes the output run \( H(x(0)) H(x(1)) \ldots H(x(l')) \). The accessible part of a system \( S \), denoted \( Ac(S) \), is the collection of states of \( S \) that are reached by state runs of \( S \). System \( S \) is said to be symbolic if \( Ac(S) \) and \( U \) are finite sets, and pseudo–metric if \( X \) is equipped with a pseudo–metric. Pseudo–metric systems are general enough to capture heterogeneous dynamics arising for
example in cyber-physical systems, see e.g. [5]. We assume that the inputs $u$ of pseudo–metric system $S$ are not available; this assumption corresponds to the point of view of an external observer that cannot have access to the inputs of the system $S$. We now introduce the notion of approximate predictability.

**Definition 2:** Consider a pseudo–metric system $S = (X, X_0, U, Y, H)$ with pseudo–metric $d$, and denote by $B_\rho(x)$ the closed ball induced by pseudo–metric $d$ centered at $x$ in $X$, with radius $\rho \in \mathbb{R}_+^*$, i.e. $B_\rho(x) = \{x' \in X | d(x,x') \leq \rho\}$. Given $X' \subseteq X$, define $B_\rho(X') = \bigcup_{x \in X'} B_\rho(x')$. Given a desired accuracy $\rho \in \mathbb{R}_+^*$, system $S$ is $(\rho,F)$–predictable if there exists $\Delta \in \mathbb{N}$, such that for any finite state run $x'()$ of $S$ for which the ending state $x'(t_f) \in F$, and $x'(t) \notin F$ for any $t \in [0,t_f)$, there exists $T \in [t_0,t_f)$ such that the following properties hold:

(i) for any state run $x'$, with $y'\big|_{[t_0,T]} = y\big|_{[t_0,T]} \quad x'\big|_{[t_0,T]}$ does not contain states $F$;

(ii) for any state run $x'$ such that $x' \neq x''$, with $y'\big|_{[t_0,T]} = y''\big|_{[t_0,T]}$ and for any infinite state run $x''()$ such that $x''\big|_{[t_0,T]} = x''\big|_{[t_0,T]} \quad x''(T+\delta) \in B_\rho(F)$, for some $\delta \leq \Delta$.

Approximate predictability corresponds to the possibility of distinguishing, on the basis of the observations collected in a certain time interval $[t_0;T]$ and in at most $\Delta > 0$ time steps (i.e., within $T + \Delta$), state runs that will reach for the first time the set of faulty states $F$ from both state runs that will not reach the set $B_\rho(F)$ and state runs that already reached $F$ at a previous time instant $t < T$. The definition above extends to pseudo–metric systems the notion of (exact) predictability given in [2], see also [3, 10], for FSMs. In particular, when $\rho$, the definition above coincides with the one given in [2]. Checking approximate predictability of symbolic pseudo–metric systems is a decidable problem with polynomial computational complexity. We now establish the relation between approximate predictability and approximate simulation that is recalled hereafter.

**Definition 3:** Consider a pair of pseudo–metric systems $S_i = (X_i, X_{i,0}, U_i, Y_i, H_i), \quad i = 1, 2, (4)$

with $X_1$ and $X_2$ subsets of some pseudo–metric set $X$ equipped with pseudo–metric $d$, and let $\epsilon \in \mathbb{R}_+^*$ be a given accuracy. Consider a relation $R \subseteq X_1 \times X_2$ satisfying the following conditions:

(i) $\forall x_1 \in X_{1,0}, \exists x_2 \in X_{0,2}$ such that $(x_1, x_2) \in R$;

(ii) $d(x_1, x_2) \leq \epsilon, \forall (x_1, x_2) \in R$;

(iii) $H_1(x_1) = H_2(x_2), \forall (x_1, x_2) \in R$.

Relation $R$ is an $\epsilon$–approximate simulation relation from $S_1$ to $S_2$ if it enjoys conditions (i)–(iii) and the following one:

(iv) $\forall (x_1, x_2) \in R$ if $x_1 \xrightarrow{u_1} x'_1$ then there exists $x_2 \xrightarrow{u_2} x'_2$ with $(x'_1, x'_2) \in R$.

System $S_1$ is $\epsilon$–simulated by $S_2$, denoted $S_1 \preceq_{\epsilon} S_2$, if there exists an $\epsilon$–approximate simulation relation from $S_1$ to $S_2$.

We can now give the following

**Theorem 1:** Consider a pair of pseudo–metric systems (4) with $X_1$ and $X_2$ subsets of some pseudo–metric set $X$ equipped with pseudo–metric $d$ and suppose that $S_1 \preceq_{\epsilon} S_2$.

Consider a set $F_1 \subseteq X_1$ of faulty states for $S_1$ and define the set $F_2 = B_\rho(F_1) \cap X_2$ of faulty states for system $S_2$. If $S_2$ is $(\rho_2, F_2)$–predictable for some accuracy $\rho_2 \in \mathbb{R}_+^*$, then $S_1$ is $(\rho_1, F_1)$–predictable for all $\rho_1 \geq \rho_2 + 2\epsilon$.

When $S_2$ has fewer states than $S_1$, Theorem 1 can reduce computational complexity in checking approximate predictability of $S_1$. In particular, provided that one is able to construct a symbolic pseudo–metric system approximating a continuous or hybrid control system $\Sigma$ (with an infinite number of states) in the sense of approximate simulation, Theorem 1 allows one to check approximate predictability of $\Sigma$. The construction of symbolic systems approximating continuous or hybrid control systems can be obtained by leveraging many results available in the literature, see e.g. [5] and references therein.

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