REMAINDER PADÉ APPROXIMANTS FOR THE HURWITZ ZETA FUNCTION

MARC PRÉVOST

Abstract. Following our earlier research, we use the method introduced by the author in [8] named Remainder Padé Approximant in [9], to construct approximations of the Hurwitz zeta function. We prove that these approximations are convergent on the positive real line. Applications to new rational approximations of \( \zeta(2) \) and \( \zeta(3) \) are given.

1. Introduction

In [8], we gave a new proof of the irrationality of \( \zeta(2) = \sum_{k=1}^{\infty} 1/k^2 \) (and also of \( \zeta(3) = \sum_{k=1}^{\infty} 1/k^3 \)) based on an explicit construction of some Padé approximants of the remainder term \( R_2(1/n) = \sum_{k=n}^{\infty} 1/k^2 \). Precisely, we have that

\[
\zeta(2) = \sum_{k=1}^{n-1} 1/k^2 + R_2(1/n) + E_n(z)
\]

with

\[
R_2(z) = \sum_{k=0}^{\infty} \frac{z^2}{(zk+1)^2}.
\]

The function \( R_2 \) is meromorphic in \( \mathbb{C} \setminus \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \cdots \} \) and so is not holomorphic at 0.

However, it is \( C^\infty \) at \( z = 0 \) and admits a Taylor series \( \hat{R}_2(z) = \sum_{k=0}^{\infty} B_k z^{k+1} \) which is an asymptotic development convergent only at \( z = 0 \), (the radius of convergence is 0), where \( B_k \) is the \( k \)th Bernoulli number.

The idea was to compute an explicit representation of diagonal Padé approximant \( P_n(z)/Q_n(z) = [n/n]_{\hat{R}_2} (z) \in \mathbb{Q}(z) \) of the series \( \hat{R}_2(z) \) with a good estimation of the error term \( E_n(z) = R_2(z) - [n/n]_{\hat{R}_2} (z) \).

At last, we find that

\[
Q_n(1/n) E_n(1/n) = Q_n(1/n) \zeta(2) - Q_n(1/n) \sum_{k=1}^{n-1} 1/k^2 - P_n(1/n) = q_n \zeta(2) - p_n
\]

provides a sequence of rational approximation \( P_n/q_n \) which proves the irrationality of \( \zeta(2) \). The surprise was to find that it is exactly the sequences used by Apéry [1] for the same purpose.

The same method applied to the series \( \zeta(3) = \sum_{k=1}^{\infty} 1/k^3 \) provides also the Apéry numbers given in [1].

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The name Remainder Padé Approximant (RPA) has been used later in a paper written with T. Rivoal [9] for the exponential function.

In this paper, we apply this method and prove that some sequence of RPA for Hurwitz zeta function is convergent on the real line.

For \( a \in \mathbb{C}, \mathcal{R}(a) > 0 \), the Hurwitz zeta function is defined as

\[
\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k + a)^s}
\]  

(1.1)

where the Dirichlet series on the right hand side of (1.1) is convergent for \( \mathcal{R}(s) > 1 \) and uniformly convergent in any finite region where \( \mathcal{R}(s) \geq 1 + \delta \), with \( \delta > 0 \). It defines an analytic function for \( \mathcal{R}(s) > 1 \). The Riemann \( \zeta \) function is \( \zeta(s, 1) \).

An integral representation of \( \zeta(s, a) \) is

\[
\zeta(s, a) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} \, dx \quad \mathcal{R}(s) > 1, \mathcal{R}(a) > 0,
\]

(1.2)

where \( \Gamma(s) = \int_{0}^{\infty} y^{s-1} e^{-y} dy \) is the gamma function.

If \( \mathcal{R}(s) \leq 1 \) the Hurwitz zeta function can be defined by the equation:

\[
\zeta(s, a) = \Gamma(1 - s) \frac{1}{2\pi i} \int_{C} \frac{z^{s-1} e^{az}}{1 - e^{z}} \, dz
\]

where \( C \) is some path in \( \mathbb{C} \), which provides the analytic continuation of \( \zeta(s, a) \) over the whole \( s \)-plane.

If we write the formula (1.1) as:

\[
\zeta(s, a) = \sum_{k=0}^{n-1} \frac{1}{(k + a)^s} + \sum_{k=0}^{\infty} \frac{1}{(n + a + k)^s}
\]  

(1.3)

and set

\[
\Psi(s, x) := t^{1-s} \sum_{k=0}^{\infty} \left( \frac{x}{1+kt} \right)^s
\]

then

\[
\zeta(s, a) = \sum_{k=0}^{n-1} \frac{1}{(k + a)^s} + (n + a)^{1-s} \Psi\left(s, \frac{1}{n + a}\right).
\]

(1.4)

Another expression of \( \Psi(s, t) \) is [12, chapt. XIII]

\[
\Psi(s, t) = \frac{t^{1-s}}{\Gamma(s)} \int_{0}^{\infty} u^{s-1} \frac{e^{-u/t}}{1 - e^{-u}} \, du
\]

(1.5)

whose Taylor series is
\[ \Psi(s, t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} (s)_{k-1} (-1)^k t^k, \] (1.6)

convergent only at \( t = 0 \), where \( B_k \) is the \( k \)th Bernoulli number and \( (s)_k \) is the Pochammer symbol \( ((s)_p := s(s+1)(s+2) \cdots (s+p-1), (s)_{-1} = 1/(s-1)) \).

We replace in the remainder term in (1.4), \( \Psi \left( s, \frac{1}{n+a} \right) \) by some Padé approximant \([m_1/m_2]_{\Psi(s,t)} = \tilde{Q}_{m_1}(t)/\tilde{P}_{m_2}(t)\) to the function \( \Psi(s, t) \) computed at \( t = 1/(n+a) \). We obtain the RPA approximant

\[ \text{RPA}(n, m_1, m_2) := \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + (n+a)^{1-s} \frac{\tilde{Q}_{m_1}(1/(n+a))}{\tilde{P}_{m_2}(1/(n+a))}. \] (1.7)

The crucial point is the convergence of this sequence \( \text{RPA}(n, m_1, m_2) \) when one or more of the parameter \( n, m_1, m_2 \) tends to infinity.

The convergence of Padé approximants is proved for many functions as meromorphic function, Stieltjes function, etc...[3]. In this paper, we prove that the function \( \Psi(s, t) \) is a Stieltjes function in the variable \( t \) when \( s \) is a real positive number. First we have to recall the definition and some properties of Padé approximants.

2. Padé approximants

Let \( f \) be a function whose Taylor expansion about \( t = 0 \) is

\[ f(t) = \sum_{i=0}^{\infty} c_i t^i. \]

The Padé approximant \([m_1/m_2]_f\) to \( f \) is a rational fraction \( \tilde{Q}_{m_1}/\tilde{P}_{m_2} \) whose denominator has degree \( m_2 \) and whose numerator has degree \( m_1 \) so that its expansion in ascending powers of \( t \) coincides with the expansion of \( f \) up to the degree \( m_1 + m_2 \), i.e.

\[ \tilde{P}_{m_2}(t)f(t) - \tilde{Q}_{m_1}(t) = 0(t^{m_1+m_2+1}), \quad t \to 0 \] (2.1)

The theory of Padé approximation is linked with the theory of orthogonal polynomials as following [6]: let us define the linear functional \( c \) acting on \( \mathcal{P} \), space of polynomials by

\[ c : \mathcal{P} \to \mathbb{R} \text{ (or } \mathbb{C} \text{)} \] (2.2)

\[ x^i \to < c, x^i > = c_i \quad i = 0, 1, 2, \ldots \] (2.3)

and

\[ \text{if } p \in \mathbb{Z} \quad < c^{(p)}, x^i > := < c, x^{i+p} > = c_{i+p} \quad i = 0, 1, 2, \ldots (c_i = 0, i < 0) \] (2.4)
then the denominators of the Padé approximants $[m_1/m_2]_f$ satisfy the following orthogonality property:

$$< c^{(m_1-m_2+1)}, x^i P_{m_2}(x) > = 0 \quad i = 0, 1, 2, \ldots, n - 1$$

where $P_{m_2}(x) = x^{m_2} \tilde{P}_{m_2}(x^{-1})$.

Let us define the associated polynomials:

$$R_{m_2-1}(t) := < c^{(m_1-m_2+1)}, \frac{P_{m_2}(x) - P_{m_2}(t)}{x - t} >, R_{m_2-1} \in \mathcal{P}_{m_2-1}$$

(2.5)

where $c^{(m_1-m_2+1)}$ acts on the variable $x$. Then

$$\tilde{Q}_{m_1}(t) = \left( \sum_{i=0}^{m_2-m_1} c_i t^i \right) \tilde{P}_{m_2}(t) + t^{m_1-m_2+1} \tilde{R}_{m_2-1}(t)$$

(2.6)

where $\tilde{R}_{m_2-1}(t) = t^{m_2-1} \tilde{R}_{m_2-1}(t^{-1})$ and $c_j = 0$ for $j < 0$.

If $c$ admits an integral representation with a function $\alpha$ non decreasing, with bounded variation,

$$c_i = \int_L x^i d\alpha(x),$$

(2.7)

then the function $f$ is a Stieltjes function and the theory of Gaussian quadrature shows that the polynomials $P_n$ orthogonal with respect to $c$, have all their roots in the support of the function $\alpha$. Moreover, the convergence is proved if the coefficients $c_i$ satisfy the Carleman condition [3, p.240].

The aim of the paper is to find, in the case of Hurwitz zeta function, the weight function $d\alpha$ depending on $s$ and prove that it is a positive function.

The error is defined by

$$\text{error} : f(t) - [m_1/m_2]_f(t) = \frac{t^{m_1+m_2+1}}{P_{m_2}^2(t)} c^{(m_1-m_2+1)} \left( \frac{P_{m_2}^2(x)}{1-xt} \right).$$

(2.8)

The above expression of the error is understood as a formal one if $c$ is only a formal linear functional [6, chapt. 3], but if $c$ admits the integral representation (2.7) then the error becomes:

$$f(t) - [m_1/m_2]_f(t) = f(t) - \tilde{Q}_{m_1}(t) = \frac{t^{m_1+m_2+1}}{P_{m_2}^2(t)} \int_L x^{m_1-m_2+1} \frac{P_{m_2}^2(x)}{1-xt} d\alpha(x)$$

(2.9)

In the particular case $m_1 = m_2 - 1$,

$$f(t) - [m_2-1/m_2]_f(t) = \frac{t^{2m_2}}{P_{m_2}^2(t)} \int_L \frac{P_{m_2}^2(x)}{1-xt} d\alpha(x).$$

(2.10)
Note that

\[
[m_2/m_2]_f(t) = c_0 + t[m_2 - 1/m_2]_f,
\]

and

\[
[m_2 + p/m_2]_f(t) = c_0 + c_1 t + \cdots + c_p t^p + t^{p+1}[m_2 - 1/m_2]_{f_p}(t)
\]

where

\[
f_p(t) = \sum_{i=0}^{\infty} c_{i+p+1} t^i.
\]

The orthogonal polynomial can be expressed with determinant:

Let \((e_k)_{k\in\mathbb{N}}\) a basis of the space of polynomials. If

\[
\langle c, e_i e_j \rangle = \alpha_{i,j}
\]

then

\[
P_n(x) = k_n \begin{vmatrix}
\alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{n,0} \\
\alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{n,1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{0,n-1} & \alpha_{1,n-1} & \cdots & \alpha_{n,n-1} \\
e_0(x) & e_1(x) & \cdots & e_n(x)
\end{vmatrix}
\]

where \(k_n\) is some constant.

This property will be used in section 5.

3. Statements of the results

In this section, we will prove that it is possible to compute the weight function underlying the coefficients \(B_k\) of the function \(\Psi(s, t) = \sum_{k=0}^{\infty} B_k (s)^k t^k\). The weight function \(w_s\) depends on \(s\). Thus, for particular values of \(s\), \(\Psi(s, t)\) is a Stieltjes function in the variable \(t\) and the convergence of RPA will be proved for \(t = 1/(n + a)\).

**Theorem 1.** For any complex numbers \(s, a, \Re(s) > -1, \Re(a) > 0\) and any positive integer \(m, m > \Re(s) - 1\), then

\[
\zeta(s, a) = \frac{1}{a^{s-1}} \left( \frac{1}{s-1} + \frac{1}{2a} + \int_0^\infty \frac{1}{a^2 + x^2} w_s(x) dx \right)
\]

where the weight \(w_s\) is defined by:

\[
w_s(x) := \frac{2(-1)^m x^s}{\Gamma(s) \Gamma(m + 1 - s)} \int_x^\infty (t-x)^{m-s} \frac{d^m}{dt^m} \left( \frac{1}{e^{2\pi t} - 1} \right) dt.
\]
We will prove in the section 4 that this formula (3.1) is a consequence of Hermite’s formula for the function \( \zeta(s, a) \).

As found by Touchard [11], Bernoulli numbers satisfy

\[
B_k = -\frac{i\pi}{2} \int_L a^k \frac{dx}{\sin^2(\pi x)},
\]

where \( L \) is the line \( L := -\frac{1}{2} + i\mathbb{R} \) and thus can be viewed as moments of order \( k \geq 0 \) of the positive weight function \( 1/\sin^2(\pi x) \) on the line \( L \). So for \( s = 2 \), the coefficients \( B_k(s)_{k-1} = B_k \) of (1.6) are moments of a positive weight. For \( s = 3 \), these coefficients appear as their derivative \((k + 1)B_k\). The derivative of \( 1/\sin^2(\pi x) \) equals is a weight function symmetric around \(-1/2\) whose support is also the line \( L \). But for integer value of \( s \) greater than 4, it is no more true for all \( k \geq 0 \). A much more difficult case is the case \( s \) is real.

From the previous Theorem, we obtain an integral representation of \( \frac{B_k}{k!}(s)_{k-1} \) but for \( k \geq 2 \).

**Theorem 2.** Integral representation of Bernoulli numbers.

If \( s \) a complex number such that \( \Re(s) > -1 \), then

\[
B_{k+2} \left( \frac{s+1}{k+2} \right) = (-1)^{k/2} \int_0^{\infty} x^k w_s(x) dx \quad (k \text{ even}) \tag{3.2}
\]

\[
= 0 \quad (k \text{ odd}) \tag{3.3}
\]

with

\[
w_s(x) := \frac{2(-1)^m x^s}{\Gamma(s) \Gamma(m+1-s)} \int_x^{\infty} (t-x)^{m-s} \frac{dm}{dt} \left( \frac{1}{e^{2\pi t} - 1} \right) dt \tag{3.4}
\]

where \( m \) is any integer satisfying \( m > \Re(s) - 1 \).

**Proof** We use the expansion of \( \zeta(s, a) \) in terms of Bernoulli numbers [10, p.160].

\[
\zeta(s, a) = a^{-s} + \sum_{k=0}^{n} (s)_{k-1} \frac{B_k}{k!} a^{-k-s+1} + \frac{1}{\Gamma(s)} \int_0^{\infty} \left( \frac{1}{e^t - 1} - \sum_{k=0}^{n} \frac{B_k}{k!} t^{k-1} \right) e^{-a t} t^{s-1} dt
\]

valid for \( \Re(s) > -2n + 1, \Re(a) > 0 \) and \( n \) a positive integer.

By identification with formula (3.1), we get the formula (3.2).

**Remark:** The positivity of the weight function is important because it will imply the convergence of Padé approximants. In the case where \( s \) is a positive real number then \( w_s \) is positive on its support. This gives the following main theorem.

**Theorem 3.** For all positive real number \( s \), for all complex number \( a, \Re(a) > 0 \), for all integers \( n \geq 0, p \geq 1 \), the following sequence
\[ RPA(n, m + p, m) := \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + (n+a)^{1-s}[m+p/m] \psi(s,t) \big|_{t=1/(n+a)}. \]  

(3.5)

converges to \( \zeta(s,a) \) when \( m \) tend to infinity.

These Theorems will be proved in Section 4.

Remark

For \( n = 0, p = -1 \), this result is proved only for \( s = 2, 3 \) in [8]. If \( n = 0 \), then formula (3.5) reduces to

\[ \forall s > 0, \forall a \in \mathbb{C}, \Re(a) > 0, \forall p \geq 1, \lim_{m \to \infty} a^{1-s}[m+p/m] \psi(s,t) \big|_{t=1/a} = \zeta(s,a). \]

Remark

For \( n = 0, a = 1 \), then \( RPA(0, m + p, m) \) is a rational function depending on the variable \( s \) and which converges to \( \zeta(s,1) = \zeta(s) \) when \( m \) tends to \( \infty \) (by Theorem [3]).

For example, for \( p = 1 \),

\[ RPA(0, 2, 1) = \frac{(s+2)(s+3)}{12(s-1)}, \]

\[ RPA(0, 3, 2) = \frac{(s+2)(s^2+12s+31)}{2(s-1)(s^2+3s+62)}, \]

\[ RPA(0, 4, 3) = \frac{(s+2)(s^3+48s^2+88s+405)}{120(s-1)(s^3+7s+54)}, \]

\[ RPA(0, 5, 4) = \frac{(s+2)(s^4+124s^3+2644s^2+23730s^2+92939s+122130)}{2(s-1)(s^5+101s^4+1755s^3+21415s^2+153884s+657564s+977040)}, \]

\[ RPA(0, 6, 5) = \frac{(s+2)(s+4)(s+6)(s^5+746s^4+28162s^3+498112s^2+3612925s+8457750)}{84(s-1)(s^6+429s^5+10387s^4+134511s^3+1044772s^2+4891020s+9666000)}. \]

If \( n \geq 1 \) and \( a = 1 \), then \( RPA(n, m + p, m) \) is no more a rational fraction since it contains powers of \( (n+a) \) with \( s \) as exponent:

\[ RPA(1, 2, 1) = \frac{2^{1-s}(s^2+11s^2+36)}{48(s-1)} + 1, \]

\[ RPA(1, 3, 2) = \frac{2^{-s-1}(s^3+26s^2+231s+726)}{(s-1)(s^2+3s+242)} + 1, \]

\[ RPA(1, 4, 3) = \frac{2^{-s-4}(s^3+166s^2+2937s^2+22334s+64800)}{15(s-1)(s^2+7s+180)} + 1, \]

\[ RPA(2, 2, 1) = \frac{3^{1-s}(s^2+17s+90)}{108(s-1)} + 2^{-s} + 1, \]

\[ RPA(2, 3, 2) = \frac{3^{-s}(s^3+38s^2+527s+2710)}{2(s-1)(s^2+3s+542)} + 2^{-s} + 1. \]

Nevertheless, if \( s \) is an integer, if \( a = 1 \) and \( p \geq 1 \), we obtain a sequence of rational approximations of \( \zeta(s) \) by computing \( RPA(n, m + p, m) \), depending on the two integers parameters \( n \) and \( m \).
4. Proof of Theorems

4.1. Proof of Theorem \[1\] We start from Hermite’s formula for \( \zeta(s, a), \Re(a) > 0 \), which is a consequence of Plana’s summation formula:

\[
\zeta(s, a) = \frac{1}{2} a^{-s} + \frac{a^{1-s}}{s-1} + 2 \int_{0}^{\infty} (a^2 + y^2)^{-s/2} \sin \left( s \arctan \frac{y}{a} \right) \frac{dy}{e^{2\pi y} - 1}. \quad (4.1)
\]

Note that this integral converges for all complex number \( s \neq 1 \).

Let us define

\[
I_s := \int_{0}^{t} \frac{x^s(t-x)^{-s}}{1+x^2} dx \quad (t \geq 0, -1 < \Re(s) < 1).
\]

Using the identity

\[
\int_{0}^{\infty} \frac{v^{s-1}}{1+\alpha v} = \alpha^{-s} \Gamma(s) \Gamma(1-s)
\]

we get, with the change of variable \( x = \frac{t v}{1+v} \),

\[
I_s = t \int_{0}^{\infty} \frac{v^s}{(1+v)^2 + t^2 v^2} dv
\]

\[
= \frac{1}{2i} \int_{0}^{\infty} v^{s-1} \left( \frac{1}{1 + v(1-it)} - \frac{1}{1 + v(1+it)} \right) dv
\]

\[
= \Gamma(s) \Gamma(1-s) \frac{1}{2i} \left( (1-it)^{-s} - (1+it)^{-s} \right).
\]

Setting \( t = \tan \theta \), we obtain

\[
I_s = \Gamma(s) \Gamma(1-s) \cos^s(\theta) \sin(s \theta) = \Gamma(s) \Gamma(1-s)(1+i^2)^{-s/2} \sin(s \arctan t).
\]

Thus, the following identity

\[
\frac{1}{\Gamma(s) \Gamma(1-s)} \int_{0}^{t} \frac{x^s(t-x)^{-s}}{1+x^2} dx = (1+i^2)^{-s/2} \sin(s \arctan t)
\]

holds for all real \( t \geq 0 \) and complex \( s \), \(-1 < \Re(s) < 1\).

Now, for \( s \) complex satisfying \( \Re(s) > -1 \) and \( m \) an integer such that \( m > \Re(s) - 1 \), by recurrence on \( m \), it is easy to prove the following formula,

\[
\sin(s \arctan t) \frac{(1+t^2)^{s/2}}{(1+t^2)^{s/2}} = \frac{1}{\Gamma(s) \Gamma(m+1-s)} \frac{d^m}{dt^m} \left( \int_{0}^{t} \frac{x^s(t-x)^{m-s}}{1+x^2} dx \right).
\]

To prove Theorem \[1\] we replace \( \sin(s \arctan(t/a)) \) in Hermite’s formula (4.1), then apply \( m \) integration by parts and permutation of the integrals.
\[
\zeta(s, a) = \frac{a^{1-s}}{2} \frac{1}{s-1} \frac{\Gamma(s)\Gamma(m+1-s)}{\Gamma(s)\Gamma(m+1-s)} \int_0^\infty \frac{d^m}{dm} \left( \int_0^t x^s(t-x)^{m-s} \frac{dx}{a^2+x^2} \right) \frac{dt}{e^{2\pi t} - 1}
\]

\[
= \frac{a^{1-s}}{2} \frac{1}{s-1} \frac{\Gamma(s)\Gamma(m+1-s)}{\Gamma(s)\Gamma(m+1-s)} \int_0^\infty \left( \int_0^t x^s(t-x)^{m-s} \frac{dx}{a^2+x^2} \right) \frac{d^m}{dm} \left( \frac{1}{e^{2\pi t} - 1} \right) dt.
\]

The permutation in the last integral is valid since the function \(\frac{x^s}{a^2+x^2} (t-x)^{m-s} \frac{d^m}{dm} \left( \frac{1}{e^{2\pi t} - 1} \right)\) is less than \(\frac{\text{mer}}{a^2} \frac{d^m}{dm} \left( \frac{1}{e^{2\pi t} - 1} \right)\) which integrable on \([0, \infty[\).

**Remark**

If we consider \(a = 1\), we get an integral representation of the Riemann zeta function:

\[
\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \int_0^\infty \frac{1}{1+x^2} w_s(x) dx
\]

with

\[
w_s(x) := \frac{2(-1)^m x^s}{\Gamma(s)\Gamma(m+1-s)} \int_x^\infty (t-x)^{m-s} \frac{d^m}{dm} \left( \frac{1}{e^{2\pi t} - 1} \right) dt
\]

where \(m\) is an integer satisfying \(m > \Re(s) - 1 > 0\).

So, if \(s\) is an integer, we can take \(m = s\), and

\[
\zeta(s) = \frac{1}{2} + \frac{1}{(s-1)} + \frac{2(-1)^{s-1}}{\Gamma(s)} \int_0^\infty \frac{x^s}{1+x^2} \frac{d^{s-1}}{ds-1} \left( \frac{1}{e^{2\pi x} - 1} \right) dx
\]

\[
= \frac{1}{2} + \frac{1}{(s-1)} + \frac{2}{\Gamma(s)} \int_0^\infty \frac{1}{e^{2\pi x} - 1} \frac{d^{s-1}}{ds-1} \left( \frac{x^s}{1+x^2} \right) dx
\]

\[
\zeta(2) = \frac{3}{2} + \pi \int_0^\infty \frac{x^2}{1+x^2} \sinh^2 \pi x dx = \frac{3}{2} + 4 \int_0^\infty \frac{x}{e^{2\pi x} - 1} \frac{1}{(1+x^2)^2} dx
\]

\[
\zeta(3) = 1 + \pi^2 \int_0^\infty \frac{x^3}{1+x^2} \cosh \pi x dx = 1 + 2 \int_0^\infty \frac{1}{e^{2\pi y} - 1} \frac{x(x^2-3)}{(1+x^2)^3} dx
\]

\[
\zeta(4) = \frac{5}{6} + \frac{\pi^3}{3} \int_0^\infty \frac{x^4}{1+x^2} \frac{2 + \cosh(2\pi x)}{\sinh^4 \pi x} dx = \frac{5}{6} + 8 \int_0^\infty \frac{1}{e^{2\pi x} - 1} \frac{x(x^2-1)}{(1+x^2)^4} dx
\]

4.2. **Proof of Theorem ??**: We consider the function \(\Psi(s,t)\) given in \([L.6]\) written as

\[
\Psi(s,t) = \frac{B_0}{s-1} - B_1 t + t^2 \Psi_2(s,t)
\]
where
\[ \Psi_2(s, t) := \sum_{k=0}^{\infty} \frac{B_{k+2}}{(k+2)!} (s)_{k+1} (-1)^k t^k. \]

Thus the Padé approximant \([m + p/m]_\Psi(s, t)\) satisfies
\[ [m + p/m]_\Psi(s, t) = \frac{B_0}{s-1} - B_1 t + t^2 [m + p - 2, m]_\Psi_2(s, t). \]

For \(s\) positive real number, the weight \(w_s\) is positive (Theorem 2) and so \(\Psi_2(s, t)\) is a Stieltjes function. Its coefficients \(c_k := \frac{B_{k+2}}{(k+1)!} (s)_{k+1} (-1)^k\) are positive and satisfy the Carleman condition, i.e. the series
\[ \sum_{k=0}^{\infty} c_k^{-1/2k} \] diverges.

Actually, the Bernoulli numbers of even index satisfy (Bernoulli of odd index greater than 2 are zero)
\[
|B_{2k}| \sim 4\sqrt{\pi k} \left( \frac{k}{\pi e} \right)^{2k}
\]
and so
\[
c_{2k}^{-1/4k} \sim \left( \frac{k}{\pi e} \right)^{-1/2}
\]
which is the general term of a divergent series.

We now apply the Theorem of [4, p.240] to conclude that for all integer \(p, p \geq 1\),
\[
\lim_{m \to \infty} t^{s-1} [m + p/m]_\Psi(s, t) = t^{s-1} \Psi(s, t) = \zeta(s, 1/t) \text{ for all complex number } t, \Re(t) > 0
\]

5. PARTICULAR CASE: \(s\) IS AN INTEGER

If \(s\) is an integer, we can improve the approximation of \(\zeta(s, a)\). In (1.3), we replace \(\Psi\left(s, \frac{1}{n+a}\right)\) by some Padé approximant \([m + p/m]_\Psi(s, t)\) to obtain an approximation of \(\zeta(s, a)\):
\[
\zeta(s, a) = \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} + (n+a)^{1-s} \lim_{m \to \infty} [m + p/m]_\Psi(s, t)\bigg|_{t=1/(n+a)}.
\]

For \(s = 2\) and \(s = 3\), we will find in this section the formal expression of these approximants and the expression of the error.

The case for \(s \geq 4\) remains an open problem.
5.1. Case $s = 2$. For $s = 2$, the weight in the expression (3.1) is

$$w_2(x) := 2x^2 \int_x^\infty \frac{d^2}{d^2t} \left( \frac{1}{e^{2\pi t} - 1} \right) dt = \frac{\pi x^2}{\sinh^2(\pi x)}. \quad (5.1)$$

Another expression of $w_2(x)$ is

$$w_2(x) = \frac{1}{\pi} |\Gamma(1 + ix)\Gamma(1 - ix)|^2.$$

Generalizing a result by Carlitz [7], Askey and Wilson [2] gave an explicit expression: the orthogonal polynomial $P_n$ with respect to the weight function $w_2$ satisfies:

$$P_m(x) = (m + 1)(m + 2)\, {}_3F_2\left( \begin{array}{c} -m, m + 3, 1 - x \\ 2, 2 \end{array} ; 1 \right) \quad (5.2)$$

$$= \sum_{k=0}^m \binom{m + 1}{k + 1} \binom{m + k + 2}{k + 1} \binom{x - 1}{k}, \quad (5.3)$$

and

$$\int_{i\mathbb{R}} P_n(x)P_m(x)\frac{\pi x^2}{\sin^2(\pi x)} dx = 0 \quad (n \neq m), \quad (5.4)$$

$$= \frac{(-1)^n 2(n + 1)(n + 2)}{2n + 3} \quad (n = m). \quad (5.5)$$

The roots of the polynomials $P_n$ are located on the imaginary axis since the weight $\frac{\pi x^2}{\sin^2(\pi x)}$ is positive on this line (see 2.7).

The three terms recurrence relation is

$$P_{m+1}(x) = \frac{2(2m + 3)}{(m + 1)(m + 2)} x P_m(x) + P_{m-1}(x) \quad (5.6)$$

with initial conditions: $P_{-1} = 0, P_0(x) = 2$.  

**Associated polynomials** (see 2.4)

Before computing the associated polynomials, we need the modified moments, i.e., the moment of the binomial $\binom{x - 1}{k}$.

Let us define the linear functional $c^{(s)}$ acting on the space of polynomials as

$$\langle c^{(s)}, x^j \rangle := \frac{B_j}{j!} (s)_{j-1}(-1)^j, \quad j \in \mathbb{N},$$

and $x^2 c^{(s)}$ by

$$\langle x^2 c^{(s)}, x^j \rangle := \langle c^{(s)}, x^{j+2} \rangle.$$
By recurrence, it is not difficult to prove that
\[ \left\langle x^2 c^{(2)}, \binom{x - 1}{k} \right\rangle = (-1)^k \frac{(k + 1)(k + 1)!}{(k + 3)!} \] (5.7)
and
\[ \left\langle x^2 c^{(2)}, \binom{x - 1}{k} \left( \binom{x - 1}{j} \right) \right\rangle = (-1)^{k+j} \frac{(k + 1)(k + 1)!(j + 1)(j + 1)!}{(k + j + 3)!}. \] (5.8)

If we define the following polynomial basis
\[ e_k(x) := \frac{1}{(k + 1)(k + 1)!} \left( \binom{x - 1}{k} \right), k \in \mathbb{N}, \]
then
\[ \left\langle x^2 c^{(2)}, e_k e_j \right\rangle = \frac{(-1)^{k+j}}{(k + j + 3)!}. \]

These moments are related with the coefficients of the exponential function
\[ g(x) := \frac{e^{-x} - (1 - x + x^2/2)}{x^3}. \]

So we can recover the expression (5.3) of the orthogonal polynomials for the functional \( x^2 c^{(2)} \) by substituting in the orthogonal polynomials for the function \( g \) which are
\[ \sum_{k=0}^{m} \binom{m + 1}{k + 1} \binom{m + k + 2}{k + 1} x^k \]
the monomials \( x^k \) by the \( e'_k \)'s (see 2.11).

The associated polynomials \( R_{m-1} \) (2.5) are defined as
\[ R_{m-1}(t) := < x^2 c^{(2)}, \frac{P_m(x) - P_m(t)}{x - t} > \]
where the variable is \( x \). From the expression (5.3) for \( P_m \), we get the following formula for \( R_{m-1} \)
\[ R_{m-1} = \sum_{k=0}^{m} \binom{m + 1}{k + 1} \binom{m + k + 2}{k + 1} \left\langle x^2 c^{(2)}, \frac{\left( \binom{x - 1}{k} \right) - \left( \binom{t - 1}{k} \right)}{x - t} \right\rangle. \] (5.9)

Using the expression of the polynomial \( \frac{\left( \binom{x - 1}{k} \right) - \left( \binom{t - 1}{k} \right)}{x - t} \) on the Newton basis on \( 0, 1, 2, \cdots, k - 1 \)
\[ \frac{\left( \binom{x}{k} \right) - \left( \binom{t}{k} \right)}{x - t} = \left( \binom{t - 1}{k} \right) \sum_{i=1}^{k} \frac{\left( \binom{x - 1}{i - 1} \right)}{i} , \]
we can write a compact formula for $R_{n-1}$:

$$R_{m-1}(t) = \sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{t-1}{k} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i} (i+1)(i+2).$$

So, we get the $[m+1/m]$ Padé approximant to the function $\Psi(2, t)$:

$$[m + 1/m]_{\Psi(2, t)} = B_0 - B_1 t + t^2 \frac{R_{m-1}(t)}{P_m(t)} = B_0 - B_1 t + t \frac{R_{m-1}(1/t)}{P_m(1/t)}$$

and an approximation of $\zeta(2, a)$

$$\zeta(2, a) \approx \sum_{k=0}^{n-1} \frac{1}{(k+a)^2} + \frac{1}{n+a} [m + 1/m]_{\Psi(2, t)} |_{t=1/(n+a)}$$

where

$$\varepsilon_m(a) = \frac{\sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k+1} \sum_{i=1}^{k} \frac{(n+a-i-1)}{k-i} (-1)^{i-1}}{\sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k+1} \binom{n+a-1}{k}}.$$

**Irrationality of $\zeta(2)$**

A consequence of the previous formula is another proof of the well known irrationality of $\zeta(2)$ since it equals to $\pi^2/6$.

Actually, if $d_k := \text{LCM}[1, 2, \cdots, k]$, then (8)

$$\frac{d_k}{i \binom{k}{i}} \in \mathbb{N}.$$ 

Then, the numerator of $\varepsilon_m(1)$ multiplied by $d_{m+2}^2$ is an integer and for all integers $n, m$, $d_{m+2}^2 v_{n,m}(1) \in \mathbb{N}, d_{m+2}^2 u_{n,m}(1) \in \mathbb{N}$.

The error (formula 2.10) applied to the function $\Psi(2, t)$ becomes

$$\Psi(2, t) - [m + 1/m]_{\Psi(2, t)}(t) = \frac{t^{2m}}{P_m(t)} \int_{1}^{t} \frac{P_m^2(x) x^2}{1 - xt \sin^2(\pi x)} dx$$
and the error term satisfies
\[
\left| \zeta(2) - \frac{v_{n,m}(1)}{u_{n,m}(1)} \right| \leq \frac{1}{P_m(n+1)P_m(n+1)} \int_{\mathbb{R}} \left| \frac{P_m^2(x)}{1-x/(n+1)} \right| \frac{x^2}{\sin^2(\pi x)} dx
\]
(5.10)
\[
\leq \frac{1}{P_m^2(n+1)} \int_{\mathbb{R}} \left| P_m^2(x) \right| \frac{x^2}{\sin^2(\pi x)} dx
\]
(5.11)
\[
\leq \frac{1}{P_m^2(n+1)} \frac{2(m+1)(m+2)}{\pi(2m+3)}
\]
(5.12)

Now, we consider \( r \in \mathbb{Q} \) such that \( m = rn \in \mathbb{N} \). Applying the Stirling formula to the expression (5.3) for orthogonal polynomials \( P_n \), we get
\[
\limsup_n (P_m(r+1)/n) = \max_{t \in [0,1]} \frac{(r + t)^r + t}{t^r(r - t)^r - t(1 - t)^{1 - t}}
\]
\[
= \frac{(r + \sigma(r))^r}{(1 - \sigma(r))(r - \sigma(r))^r} = \rho(r)
\]
where \( \sigma(r) = \frac{-r^2 + \sqrt{r^4 + 4r^2}}{2} \) is a zero of \( t^2 + r^2t - r^2 = 0 \).

So
\[
\limsup_n \left| \zeta(2)d_{rn,2}^2u_{n,rn}(1) - d_{rn+2}^2v_{n,rn}(1) \right|^{1/n} \leq \limsup_n (d_{rn,2}^2u_{n,rn}(1))^{1/n} \limsup_n \frac{1}{(u_{n,rn}(1))^{2/n}}
\]
\[
= e^{2\max(r,1)} / \rho(r)
\]
The inequality \( e^{2\max(r,1)} / \rho(r) < 1 \) is satisfied for \( r \in [0.74, 1.53] \).

The irrationality of \( \zeta(2) \) follows from the following limit
\[
\lim_n \left( \zeta(2)d_{rn,2}^2u_{n,rn}(1) - d_{rn+2}^2v_{n,rn}(1) \right) = 0.
\]

5.2. Case \( s = 3 \). For \( s = 3 \), the weight in the expression (3.1) is
\[
w_3(x) := \frac{x^3}{3} \int_x^\infty \frac{d^3}{dt^3} \left( \frac{1}{e^{2\pi t} - 1} \right) dt = \frac{x^2}{3} \frac{x^3 \cosh \pi x}{\sinh^3(\pi x)}.
\]
(5.13)

Another expression of \( x^2w_3(x) \) is
\[
x^2w_3(x) = \frac{1}{12\pi} \frac{\Gamma(1 + ix)\Gamma(2ix)}{\Gamma^2(2ix)}.
\]

This weight has been investigated by Wilson [13]. The orthogonal polynomial \( P_n \) satisfies:
\[
P_m(x^2) = (m + 1)(m + 2)_4 F_3 \left( \begin{array}{ccc} -m & m + 3 & 1 - x & 1 + x \\ 2 & 2 & 2 \end{array} ; 1 \right)
\]
(5.14)
\[
= \sum_{k=0}^m \binom{m + 1}{k + 1} \binom{m + k + 2}{k + 1} \binom{x - 1}{k} \binom{x + k}{k} / (k + 1)
\]
(5.15)
and
\[
\int_{\mathbb{R}} P_n(x^2) P_m(x^2) \frac{x^2}{3} \frac{x^5 \cos \pi x}{\sin^3 \pi x} \, dx = 0 \quad (n \neq m) \quad (5.16)
\]
\[
= \frac{1}{3} \frac{(-1)^n(n+1)(n+2)}{2n+3} \quad (n = m) \quad (5.17)
\]

The roots of the polynomials \(P_n\) are located on the imaginary axis because the weight \(x^5 \cos \pi x \sin^3 \pi x\) is positive on this line.

We set \(\Pi_m(x) := P_n(x^2)\).

The three terms recurrence relation is
\[
\Pi_{m+1}(x) = \left( \frac{2(2m+3)}{(m+1)(m+2)^2} x^2 + \frac{2m+3}{m+2} \right) \Pi_m(x) - \frac{m+1}{m+2} \Pi_{m-1}(x) \quad (5.18)
\]
with initial conditions: \(\Pi_{-1} = 0, \Pi_0 = 2\).

**Associated polynomials**

Before computing the associated polynomials, we need the modified moments, i.e., the moment of the product of binomials \(\left( \frac{x-1}{k} \right) \left( \frac{x+k}{k} \right)\).

In the previous subsection, we have define the linear functional \(c^{(s)}\) acting on the space of polynomials as
\[
\langle c^{(s)}, x^j \rangle := \frac{B_j}{j!} (s)_{j-1} (-1)^j, \quad j \in \mathbb{N},
\]
and \(x^4 c^{(s)}\) by
\[
\langle x^4 c^{(s)}, x^j \rangle := \langle c^{(s)}, x^{j+4} \rangle.
\]
By recurrence, we can prove
\[
\langle x^4 c^{(3)}, \left( \frac{x-1}{k} \right) \left( \frac{x+k}{k} \right) \rangle = (-1)^{k+1} \frac{(k+1)^2}{2(k+3)(k+2)}
\]
and
\[
\langle x^4 c^{(3)}, \left( \frac{x-1}{k} \right) \left( \frac{x+k}{k} \right) \left( \frac{x-1}{j} \right) \left( \frac{x+j}{j} \right) \rangle = (-1)^{k+j+1} \frac{(k+1)^2(j+1)^2}{2(k+j+3)(k+j+2)}.
\]
If we define the following polynomial basis
\[
\theta_k(x) := \frac{(-1)^k}{(k+1)^2} \left( \frac{x-1}{k} \right) \left( \frac{x+k}{k} \right)
\]
then
\[
\langle x^4 c^{(3)}, \theta_k \theta_j \rangle = \frac{(-1)}{2(k+j+3)(k+j+2)} = \frac{1}{2(k+j+3)} - \frac{1}{2(k+j+2)}.
\]
These moments are those of the weight function \(x(1-x)\) on the interval \([0,1]\).

So we can recover the expression of orthogonal polynomials for the functional \(x^4 c^{(3)}\) by substituting in the Jacobi orthogonal polynomials with parameters \(\alpha = 1, \beta = 1\).
\[
P_m^{(1,1)}(2x - 1) = \sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k} (-1)^k x^k
\]

the monomials \(x^k\) by the \(\theta'_k\)s (see 2.11).

The associated polynomials \(\Theta_{m-1}\) (2.5) are defined as

\[
\Theta_{m-1}(t) := \langle x^4 c^{(3)}, \frac{\Pi_m(x) - \Pi_m(t)}{x - t} \rangle
\]

where the variable is \(x\). From the expression (5.15) for \(\Pi_m\), we get the following formula for \(\Theta_{m-1} \in P_{2m-2}\)

\[
\Theta_{m-1} = \sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k+1} \frac{1}{k+1} \left\langle x^4 c^{(3)}, \frac{(x - 1)}{k} \frac{(x+k)}{k} - \frac{(t - 1)}{k} \frac{(t+k)}{k} \right\rangle
\]

Using the expression of the polynomial \(\frac{(x - 1)}{k} \frac{(x+k)}{k} - \frac{(t - 1)}{k} \frac{(t+k)}{k}\) on the Newton basis on \(0, 1, -1, 2, -2 \cdots, k, -k\)

\[
\frac{(x - 1)}{k} \frac{(x+k)}{k} - \frac{(t - 1)}{k} \frac{(t+k)}{k} = \sum_{j=0}^{2k-1} \nu_{2j}(t) \nu_{j+1}(x)
\]

where

\[
\nu_{2j+1}(x) = \binom{x - 1}{j+1} \binom{x+j}{j}
\]

(5.20)

\[
\nu_{2j}(x) = \binom{x - 1}{j} \binom{x+j}{j}.
\]

(5.21)

Using

\[
\langle x^4 c^{(3)}, \nu_{2j+1}(x) \rangle = -\langle x^4 c^{(3)}, \nu_{2j}(x) \rangle = (-1)^j \frac{(j+1)^2}{2(j+3)(j+2)},
\]

we can write a compact formula for \(\Theta_{m-1}\):

\[
\Theta_{m-1}(t) = \sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k+1} \frac{1}{k+1} \sum_{j=0}^{2k-1} \frac{\nu_{2j}(t)}{\nu_{j+1}(t)} \langle x^4 c^{(3)}, \nu_{j}(x) \rangle
\]

\[
= \sum_{k=0}^{m} \binom{m+1}{k+1} \binom{m+k+2}{k+1} \nu_{2j}(t) \frac{1}{k+1} \sum_{p=1}^{k} \frac{t}{t-p} \frac{(t+p)}{p} \frac{(-1)^p}{2(p+1)(p+2)}
\]
So, for $m \geq 2$, we get the $[2m + 2/2m]$ Padé approximant to the function $\Psi(3, t)$,

$$[2m + 2/2m] \psi_{3,t} = \frac{B_0}{2} - B_1 t + \frac{3}{2} t^2 B_2 + t^3 \frac{\Theta_{m-1}(t)}{\Pi_m(t)} = \frac{B_0}{2} - B_1 t + \frac{3}{2} t^2 B_2 + t^4 \frac{\Theta_{m-1}(1/t)}{\Pi_m(1/t)}$$

and an approximation of $\zeta(3, a)$

$$\zeta(3, a) \approx \sum_{k=0}^{n-1} \frac{1}{(k+a)^2} + \frac{1}{2(n+a)^2} + \frac{1}{2(n+a)^3} + \frac{1}{4(n+a)^4} + \frac{1}{(n+a)^5} \varepsilon_m(a) =: \frac{f_{n,m}(a)}{g_{n,m}(a)}$$

where

$$\varepsilon_m(a) = \frac{\Theta_{m-1}(n+a)}{\Pi_m(n+a)} - \sum_{k=0}^{m} \frac{(m+1)(m+k+2)}{k+1} \sum_{p=1}^{n+a} \frac{(n+a-p-1)(n+a+k)}{k-p} (-1)^p \frac{k^2}{p^2(p+1)(p+2)} \sum_{k=0}^{m} \frac{(m+1)(m+k+2)}{k+1} \frac{(n+a-1)}{k} \frac{(n+a+k)}{k} / (k+1).$$

**Irrationality of $\zeta(3)$**

The irrationality of $\zeta(3)$ has been proved by Apery in a celebrated paper [1]. A little later, a particular straightforward and elegant proof of this irrationality was given by Beukers [5]. The author gave another proof in [8] using the RPA. Actually, $RPA(m, 2m-1, 2m)$ provides exactly the Apery’s numbers. In the following, we show that the previous approximation also proves irrationality of $\zeta(3)$ with RPA’s of various degree.

Actually,

$$\forall n, a \in \mathbb{N}, d_{m+1}^3 \Theta_{m-1}(n+a) \in \mathbb{Z}$$

and $d_{m+1}^3 f_{n,m}(n+a) \in \mathbb{Z}, d_{m+1}^3 g_{n,m}(n+a) \in \mathbb{Z}.$

The proof is similar than in previous subsection.

We take $a = 1.$

The error (formula 2.10) applied to the function $\Psi(3, t)$ becomes

$$\Psi(3, t) - [2m + 2/2m] \psi_{3,t} = \frac{t^{2m}}{\Pi_m^2(t)} \int_{iR} \frac{\Pi_m^2(x)}{1 - xt} \frac{\pi^2 x^5 \cos \pi x}{3 \sin^3(\pi x)} dx$$

and the error term

$$\zeta(3) - \frac{f_{n,m}(1)}{g_{n,m}(1)}$$

satisfies
\begin{align*}
\left| \zeta(3) - \frac{f_{n,m}(1)}{g_{n,m}(1)} \right| & \leq \frac{1}{\Pi^2_m(n+1)} \int_{\mathbb{R}} \frac{\Pi^2_m(x) \pi^2 x^5 \cos \pi x}{|1 - x/(n+1)| 3 \sin^3(\pi x)} \, dx \quad (5.22) \\
& \leq \frac{1}{\Pi^2_m(n+1)} \int_{\mathbb{R}} \Pi^2_m(x) \frac{\pi^2 x^5 \cos \pi x}{3 \sin^3(\pi x)} \, dx \quad (5.23) \\
& \leq \frac{1}{\Pi^2_m(n+1)} \frac{(m+1)(m+2)}{2m+3} \quad (5.24)
\end{align*}

Now, we consider \( r \in \mathbb{Q} \) such that \( m = rn \in \mathbb{N} \). Using the Stirling formula in the expression of the orthogonal polynomials \( (5.15) \),

\[
\limsup_n (\Pi(n+1))^{1/n} = \max_{t \in [0,1]} \frac{(r+t)^{r+t}(1+t)^{1+t}}{r^t(1-t)^{1-t}} = \frac{(1+\mu(r))(r+\mu(r))^r}{(1-\mu(r))(r-\mu(r))^r} = \eta(r)
\]

where \( \mu(r) = \frac{r}{\sqrt{1+r^2}} \) is a zero of \( (1-t^2)(r^2-t^2) = t^4 \).

So

\[
\limsup_n \left| \zeta(3)d_{r+1}^3 g_{n,rn}(1) - d_{r+1}^3 f_{n,rn}(1) \right|^{1/n} \leq \limsup_n (d_{r+1}^3 g_{n,rn}(1))^{1/n} \limsup_n \frac{1}{(g_{n,rn}(1))^{2/n}} = c^{\max(r,1)} / \eta(r)
\]

The inequality \( c^{\max(r,1)} / \eta(r) < 1 \) is satisfied for \( r \in [0.74, 1.36] \).

The irrationality of \( \zeta(3) \) then follows from the following limit

\[
\lim_n \left( \zeta(3)d_{r+1}^3 g_{n,rn}(1) - d_{r+1}^3 f_{n,rn}(1) \right) = 0.
\]

**References**

[1] Apéry, R. Irrationality of \( \zeta(2) \) and \( \zeta(3) \). *Astérisque* 61 (1979), 11–13.

[2] Askey, R., and Wilson, J. A set of hypergeometric orthogonal polynomials. *SIAM J. Math. Anal.* 13, 4 (1982), 651–655.

[3] Baker, Jr., G. A., and Graves-Morris, P. Padé approximants, second ed., vol. 59 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[4] Baker, Jr., G. A., and Graves-Morris, P. Padé approximants, second ed., vol. 59 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[5] Beukers, F. A note on the irrationality of \( \zeta(2) \) and \( \zeta(3) \). *Bull. London Math. Soc.* 11, 3 (1979), 268–272.

[6] Brezinski, C. Padé-type approximation and general orthogonal polynomials, vol. 50 of *International Series of Numerical Mathematics*. Birkhäuser Verlag, Basel-Boston, Mass., 1980.

[7] Carlitz, L. Bernoulli and Euler numbers and orthogonal polynomials. *Duke Math. J* 26 (1959), 1–15.

[8] Prévost, M. A new proof of the irrationality of \( \zeta(2) \) and \( \zeta(3) \) using Padé approximants. *J. Comput. Appl. Math.* 67, 2 (1996), 219–235.

[9] Prévost, M., and Rivoal, T. Remainder Padé approximants for the exponential function. *Constr. Approx.* 25, 1 (2007), 109–123.
[10] Srivastava, H. M., and Choi, J. *Zeta and q-Zeta functions and associated series and integrals*. Elsevier, Inc., Amsterdam, 2012.

[11] Touchard, J. Nombres exponentiels et nombres de Bernoulli. *Canad. J. Math. 8* (1956), 305–320.

[12] Whittaker, E. T., and Watson, G. N. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[13] Wilson, J. A. Some hypergeometric orthogonal polynomials. *SIAM J. Math. Anal. 11*, 4 (1980), 690–701.

LMPA Joseph Liouville, CENTRE UNIVERSITAIRE DE LA MI-VOIX, BAT H. POINCARÉ, 50 RUE F. BUISSON, BP 699, 62228 CALAIS CEDEX

E-mail address: marc.prevost@univ-littoral.fr