EFFECT OF AN EXTERNAL MAGNETIC FIELD ON SOME STATISTICAL PROPERTIES OF THE 2+1 DIRAC–MOSHINSKY OSCILLATOR

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Abstract

We map the 2+1 Dirac–Moshinsky oscillator (2+1 DMO) into the generalized Jaynes–Cummings model (GJCM), where an external magnetic field is coupled to an external isospin field. We solve analytically the basic equations of the model, where the coherent state is considered as an initial state. The results obtained show that the magnetic field strength and the coupling parameter of the isospin field play important roles when some statistical properties such as the entanglement, population inversion, and degree of coherence are considered. We show that these parameters are important for increasing the entanglement and also demonstrate the collapse and revival phenomena.

Keywords: Dirac–Moshinsky oscillator, generalized Jaynes–Cummings model, entanglement.

1. Introduction

In quantum optics, the Jaynes–Cummings model (JCM) is composed of a single two-level particle interacting with a single quantized cavity mode of the electromagnetic field [1]. This model is exactly solvable in the rotating wave approximation and experimentally realized [2]. It has been found that the JCM has some statistical properties not existing in classical fields, such as the degree of coherence, the collapse and revival phenomena, and squeezing [3,4].

The JCM has been used to elucidate strong quantum correlations (entanglement) — an important aspect of quantum systems; it exhibits correlations that cannot be discussed classically [5].

The Dirac oscillator was suggested [6,7] and reinvestigated with a linear term added to the relativistic momentum of the free-particle Dirac equation [8,9]. The 1+1 Dirac–Moshinsky oscillator (1+1 DMO) has been exactly solved, in view of the theory of the nonrelativistic harmonic oscillator [10,11]. The Dirac oscillator has attracted a lot of attention and found many applications in different branches of physics [12–15]. On the other hand, the 2+1 dimensions have been related to quantum optics through the JCM [15,16].

One of the most exciting properties of the DMO is its connection with quantum optics [17,18]. The relation to quantum optics allows one to conceive quantum-optics experiments that emulate this system.
The dynamics of the 2+1 DMO was studied in [19] where an exact mapping of this quantum relativistic system into the JCM is obtained. In [20], the 2+1 DMO in the presence of an external magnetic field has been studied. Also, the connection between anti-JCM with the DMO in a magnetic field was established without imposing any limit on the magnetic-field strength.

The 1+1 and 2+1 DMO have been mapped onto the JCM, and the dynamical features of a Dirac particle under the influence of the external field have been studied only in the vacuum state [21]. The previous attempts [21] concentrated on the number state without using the coherent state, where the external isospin field is included only. In [19], the 2+1 DMO in an external magnetic field has been studied without any study of statistical properties of the system.

In [22], the 2+1 DMO coupled to an external field has been mapped onto the GJCM. The effect of both the detuning parameter and the coherence angle on the entanglement and the population inversion has been studied in two cases of the initial state: the number state and the coherent state. It has been shown that the coherent state provides a fair description for the entanglement and the population inversion.

In this paper, we study the Dirac oscillator coupled to an isospin field in the presence of an external magnetic field by mapping it to the JCM. The wave function is obtained using the coherent state as an initial state. In addition, we study the influence of both the magnetic-field strength and the coupling parameter of the isospin field on some nonclassical properties of the system.

This paper is arranged as follows.

In Sec. 2, we introduce basic equations and relations. In Sec. 3, we explain the mapping of 2+1 DMO in an external magnetic field coupled to an external isospin field into the GJCM. Section 4 is devoted to the analytical solution of the model, which is followed by a discussion of some nonclassical properties in Sec. 5. Finally, in Sec. 6, we conclude this paper with some brief remarks.

2. Basic Equations and Relations

2.1. The 2+1 Dirac–Moshinsky Oscillator

The DMO is introduced by Moshinsky and Szczepaniak [8] by adding the linear term to the Dirac Hamiltonian for a free particle. In the nonrelativistic limit, it corresponds to the harmonic oscillator plus a spin–orbit coupling term. The DMO model in 2+1 dimensions reads [21]

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \left[ \sum_{j=1}^{2} c\alpha_j (p_j + im\omega r_j) + mc^2 \beta \right] |\psi\rangle,$$

where $c$ is the speed of light, $m$ is the rest mass of the particle, and $\alpha_j$ and $\beta$ are the Dirac matrices in the standard representation; they are taken here as $\alpha_1 = -\hat{\sigma}_y$, $\alpha_2 = -\hat{\sigma}_x$, and $\beta = \hat{\sigma}_z$, where the potential for this particular magnetic field is $A = \left(-\frac{B}{2} y, \frac{B}{2} x, 0\right)$, or $A = \frac{1}{2}(B \wedge r)$.

2.2. The Generalized Jaynes–Cummings Model

We briefly introduce the GJCM, in order to study and connect it with more general and complicated systems, besides the Dirac oscillator. It is a theoretical model in quantum optics. It describes the system
of a two-level particle interacting with one mode of the electromagnetic field without using the rotating wave approximation. The Hamiltonian in the interaction picture takes the following form [24]:

$$\hat{H}_{JC} = \Omega (\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) + \delta \hat{\sigma}_z,$$

(2)

where $\Omega$ is the particle–field coupling constant, and the operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are the raising and lowering operators for the two-level system; they satisfy the commutation relations $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$ and $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$. Here, $\hat{a}^\dagger$ and $\hat{a}$ are the boson creation and annihilation operators, respectively, which satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, and $\delta$ stands for the detuning of the atomic transition frequency from the cavity mode frequency.

3. Mapping of the 2+1 DMO Model in an External Magnetic Field Coupled to an External Isospin Field into the GJCM

In view of the spinor $|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$ and $\hat{H}^2|\psi\rangle = E|\psi\rangle$, Eq. (2) converts to a set of coupled equations as follows:

$$\begin{align*}
(E - mc^2)|\psi_1\rangle &= (2cp_z + im\tilde{\omega}\bar{z})|\psi_2\rangle, \\
(E + mc^2)|\psi_2\rangle &= (2cp_z - im\tilde{\omega}z)|\psi_1\rangle,
\end{align*}$$

(3, 4)

where

$$p_z = \frac{p_x - ip_y}{2}, \quad p_{\bar{z}} = \frac{p_x + ip_y}{2}, \quad z = (x + iy), \quad \bar{z} = (x - iy), \quad \tilde{\omega} = \omega + \omega_c/2,$$

(5)

with $\omega_c = -|eB|/mc$, the cyclotron frequency. We can write $\hat{H}^2$ in the following matrix form:

$$\hat{H}^2 = \begin{pmatrix} mc^2 & 2cp_z + me\tilde{\omega}\bar{z} \\ 2cp_z - ime\tilde{\omega}z & -mc^2 \end{pmatrix}.$$  

(6)

We note that the 2+1 DMO with angular frequency $\omega$ in the presence of the magnetic field maps into 2+1 DMO, where the angular frequency $\omega$ changes to $\tilde{\omega} = \omega + \omega_c/2$, which means that the magnetic field decreases the angular frequency by half the cyclotron frequency of this system.

In order to find the solution, we define the creation and annihilation operators

$$\begin{align*}
\hat{a} &= \frac{1}{\sqrt{m\omega h}}p_z - i\frac{1}{2}\sqrt{\frac{m\omega}{h}}z, \\
\hat{a}^\dagger &= \frac{1}{\sqrt{m\omega h}}p_z + i\frac{1}{2}\sqrt{\frac{m\omega}{h}}\bar{z},
\end{align*}$$

(7, 8)

where $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{a}, \hat{a}] = 0 = [\hat{a}^\dagger, \hat{a}^\dagger]$.

Now, we can write the Hamiltonian $\hat{H}^2$ (6) in terms of the creation and annihilation operators as

$$\hat{H}^2 = \eta(\hat{a}^\dagger \hat{\sigma}_+ + \hat{a} \hat{\sigma}_-) + mc^2 \hat{\sigma}_z,$$

(9)
where $\eta = 2\sqrt{mc^2\omega}\hbar$. This equation represents the Hamiltonian of the anti-JCM in quantum optics.

In the presence of an external isospin field $\Phi$, the dynamics of the total system is given by the Hamiltonian

$$\tilde{H} = \tilde{H}^2 + \Phi,$$

(10)

where $\tilde{H}^2$ is given by Eq. (9) and $\Phi$ is the Hermitean operator of the form [25]

$$\Phi = (A + \hat{\sigma}_z B)(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_- + \gamma\hat{\sigma}_z),$$

(11)

with $\hat{\sigma}$'s being the vectors of Pauli matrices; they have the same commutation relations as $\hat{\sigma}$'s, i.e., the corresponding ladder operators are defined by

$$\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2.$$

(12)

We use the simplest form of $\Phi$, i.e., linear one,

$$\Phi = \chi(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_- + \gamma\hat{\sigma}_z).$$

(13)

In quantum optics, $\tilde{H}$ can be described as GJCM, where $\hat{a}$ is the annihilation operator of the cavity field and each isospin with an atom, $\eta$ and $\chi$ are the coupling of each atom to the cavity isospin, and $mc^2$ and $\gamma$ are described as the detuning of each transition level with the cavity mode frequency. This model can be seen as a linear combination of the two JCM. It can be considered as more general than the model discussed in [22]. In our model, we take into account the influence of the magnetic field, which leads us to map this model into the GJCM.

In order to solve the total system described by (10), we use the Heisenberg equation of motion to deduce the constant of motion as follow:

$$I = \hat{n} + (\hat{\sigma}_z - \hat{\sigma}_z)/2,$$

(14)

with $\hat{n} = \hat{a}^\dagger\hat{a}$.

4. The Analytical Solution

In this section, we derive the wave function $|\psi(t)\rangle$ and the reduced density operators. We assume that the two particles (the particle in DMO and the isospin field) and the electromagnetic field are initially prepared in the ground state and coherent state, respectively. In this case, the wave function of this system at $t = 0$ can be written as (we use the notation adopted from [21])

$$|\psi(0)\rangle = |\tilde{\psi}\rangle_{Ds} \otimes |\tilde{\psi}\rangle_{Is} \otimes |\alpha\rangle_F,$$

(15)

where

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle,$$

(16)

with

$$\alpha_n = \exp\left(-\frac{|\alpha|^2}{2}\right) \frac{\alpha^n}{\sqrt{n!}}, \quad \alpha \in \mathbb{C}.$$

(17)
Using the constant of motion (14), we obtain the wave function $|\psi(t)\rangle$ at $t > 0$; it is

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} (B_1(n,t)|-\frac{1}{2}, n + 2\rangle + B_2(n,t)|+\frac{1}{2}, n + 3\rangle + B_3(n,t)|-\frac{1}{2}, n + 1\rangle + B_4(n,t)|+\frac{1}{2}, n + 2\rangle).$$

(18)

We obtain the coefficients $B_j(n,t), j = 1, 2, 3, 4$, by solving the Schrödinger equation; thus, we arrive at the following system of differential equations for the $B_j(n,t)$ coefficients

$$i\hbar \dot{B}_1(n,t) = a(n)B_2(n,t) + d(n)B_3(n,t) - 2\Omega B_1(n,t),$$

(19)

$$i\hbar \dot{B}_2(n,t) = a(n)B_1(n,t) + c(n)B_4(n,t),$$

(20)

$$i\hbar \dot{B}_3(n,t) = b(n)B_2(n,t) + d(n)B_1(n,t),$$

(21)

$$i\hbar \dot{B}_4(n,t) = b(n)B_3(n,t) + c(n)B_2(n,t) + 2\Omega B_1(n,t),$$

(22)

where

$$a(n) = \lambda_1 \sqrt{n + \frac{3}{2}}, \quad b(n) = a(n-1), \quad c(n) = \lambda_2 \sqrt{n + \frac{3}{2}}, \quad d(n) = c(n-1), \quad mc^2 = \gamma = \Omega,$$

(23)

with

$$\lambda_1 = \frac{2\sqrt{1 + \xi}}{\eta}, \quad \lambda_2 = \frac{\chi}{\eta \sqrt{mc^2 \omega}}, \quad \xi = \frac{eB}{2mc\omega}.$$  

Then we take $c = 1 = \hbar$ and obtain the time-dependent coefficients $B_j(n,t), j = 1, 2, 3, 4$, by solving the above differential equations (19)–(22). After calculating the wave function $|\psi(t)\rangle$, one can perform the calculations for any property related to the particles or the field.

The reduced density operator of the isospin field $\hat{\rho}(t)$ can be obtained as follows:

$$\hat{\rho}(t) = \text{Tr}_F \text{Tr}_{D_{\text{S}}} |\psi(t)\rangle \langle \psi(t)| = \hat{\rho}_{ee}(t) |\frac{1}{2}, \frac{1}{2}\rangle \langle \frac{1}{2}, \frac{1}{2}| + \hat{\rho}_{gg}(t) |\frac{1}{2}, -\frac{1}{2}\rangle \langle \frac{1}{2}, -\frac{1}{2}| + \hat{\rho}_{ge}(t) |\frac{1}{2}, \frac{1}{2}\rangle \langle \frac{1}{2}, -\frac{1}{2}| + \hat{\rho}_{ge}(t) |\frac{1}{2}, -\frac{1}{2}\rangle \langle \frac{1}{2}, \frac{1}{2}|,$$

(24)

where

$$\hat{\rho}_{ee}(t) = \sum_{n=0}^{\infty} (|B_3(n,t)|^2 + |B_4(n,t)|^2), \quad \hat{\rho}_{gg}(t) = \sum_{n=0}^{\infty} (|B_1(n,t)|^2 + |B_2(n,t)|^2),$$

(25)

$$\hat{\rho}_{eg}(t) = \sum_{n=0}^{\infty} (B_3(n+1,t)B_1^*(n,t) + B_4(n+1,t)B_2^*(n,t)) = \hat{\rho}_{ge}(t).$$

(26)

Also, to obtain the reduced density matrix of the two particles, we take the trace over the oscillator degree of freedom

$$\rho(t) = \text{Tr}_F |\psi(t)\rangle \langle \psi(t)| = \begin{pmatrix}
\rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t) \\
\rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t) \\
\rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t) \\
\rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(t)
\end{pmatrix},$$

(27)
where
\[
\rho_{11}(t) = \sum_{n=0}^{\infty} |B_1(n, t)|^2, \quad \rho_{22}(t) = \sum_{n=0}^{\infty} |B_2(n, t)|^2, \\
\rho_{33}(t) = \sum_{n=0}^{\infty} |B_3(n, t)|^2, \quad \rho_{44}(t) = \sum_{n=0}^{\infty} |B_4(n, t)|^2, \\
\rho_{12}(t) = \sum_{n=0}^{\infty} B_1(n+1, t) B_2^*(n, t) = \rho_{21}^*(t), \\
\rho_{13}(t) = \sum_{n=0}^{\infty} B_1(n, t) B_3^*(n+1, t) = \rho_{31}^*(t), \\
\rho_{14}(t) = \sum_{n=0}^{\infty} B_1(n, t) B_4^*(n, t) = \rho_{41}^*(t), \\
\rho_{23}(t) = \sum_{n=0}^{\infty} B_2(n, t) B_3^*(n+2, t) = \rho_{32}^*(t), \\
\rho_{34}(t) = \sum_{n=0}^{\infty} B_3(n+1, t) B_4^*(n, t) = \rho_{43}^*(t).
\]

With these operators given, different statistical properties of this system can be studied.

5. Nonclassical Properties

In this section, we discuss some statistical properties of the system under study and concentrate on the influence of the magnetic-field strength and the coupling constant parameter of the isospin field on the behavior of the entanglement, the population inversion, and the correlation function.

5.1. Entanglement

In this section, we study the entanglement between the DMO and the isospin field through the von Neumann entropy. In quantum optics, the von Neumann entropy is used to study the dynamic characteristics of a two-level atom interacting with light [26]. It is noted that this measure is a useful physical quantity for estimating the degree of entanglement in a pure state.

In quantum mechanics, the von Neumann entropy is defined as [22,27]
\[
S(t) = \lambda_- (t) \ln \lambda_- (t) - \lambda_+ (t) \ln \lambda_+ (t),
\]

(29)

where \(\lambda_\pm(t)\) are the eigenvalues of the reduced density matrix \(\rho(t)\) Eqs. (24)–(26). They can be easily evaluated through the following form:
\[
\lambda_\pm (t) = \frac{1}{2} \pm \frac{1}{2} \sqrt{\langle \hat{\sigma}_x(t) \rangle^2 + \langle \hat{\sigma}_y(t) \rangle^2 + \langle \hat{\sigma}_z(t) \rangle^2},
\]

(30)

where
\[
\langle \hat{\sigma}_x(t) \rangle = 2 \text{Re} [\hat{\rho}_{eg}(t)], \quad \langle \hat{\sigma}_y(t) \rangle = 2 \text{Im} [\hat{\rho}_{eg}(t)], \quad \langle \hat{\sigma}_z(t) \rangle = \hat{\rho}_{ee}(t) - \hat{\rho}_{gg}(t).
\]

(31)
In Figs. 1 and 2, we show the effect of the magnetic-field strength and the coupling parameter of the isospin field on the evolution of the von Neumann entropy against the scaled time $\lambda t$, where $\lambda = \eta \sqrt{mc^2 \omega}$. The atoms are initially in the ground state, and the field is prepared in the coherent state. The value of the intensity of the initial coherent parameter has been fixed as $\alpha = 3$, and the detuning parameter has been fixed as $\Omega = 0.2\lambda$.

We note that $S(t)$ starts from zero and is then followed by a sequence of fluctuations in the oscillation. This means that this system begins by a disentangled state (at $\lambda t = 0$), and then it develops to a mixed state (at $\lambda t > 0$) and never reaches the pure state again.

In Fig. 1, the effect of the magnetic-field strength ($\lambda_1$) clearly appears, where there is a sudden decrease in the value of $S(t)$ as $\lambda t$ ranges from 20 to 40; see Fig. 1a. Also an extra minimum (decrease of $S(t)$) occurs as $\lambda_1$ increases at the same period; see Fig. 1b; also the entanglement increases for a longer period ($\lambda t > 40$). With increasing $\lambda_1$, $S(t)$ oscillates near the maximum value ($\ln 2$).

Figure 2 shows the effect of the coupling parameter of the isospin field ($\lambda_2$) on the entanglement. We observe that with increase in the value of this parameter, the value of entanglement and the number of the fluctuation increase.

We conclude that, to obtain strong entanglement between the isospin field and the Dirac oscillator, one needs to increase the value of $\lambda_1$ or $\lambda_2$.

- Fig. 1. The von Neumann entropy as a function of $\lambda t$ with $\Omega = 0.2\lambda$ and $\lambda_2 = 0.3\lambda$ at $\lambda_1 = 0.2\lambda$ and $\lambda_1 = 1.2\lambda$.

- Fig. 2. The von Neumann entropy as a function of $\lambda t$ with $\Omega = 0.2\lambda$ and $\lambda_1 = 0.3\lambda$ at $\lambda_2 = 0.2\lambda$ and $\lambda_2 = 1.2\lambda$. 

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5.2. Concurrence

In this section, we use the concurrence to measure the entanglement between the two particles. It ensures the scale between 0 for a separable (disentangled) state and \( \sqrt{2(N-1)/N} \) for the maximum entangled state. The concurrence may be written as follows \([28,29]\):

\[
C(t) = \sqrt{2 \sum_{i,j=1,2,3,4} \left( \rho_{ii}(t)\rho_{jj}(t) - \rho_{ij}(t)\rho_{ji}(t) \right)}, \quad i \neq j,
\]

where \( \rho_{ii}(t) \), \( \rho_{jj}(t) \), \( \rho_{ji}(t) \), and \( \rho_{ij}(t) \) are given by (28).

In Figs. 3 and 4, we show the evolution of concurrence \( C(t) \) versus the scaled time \( \lambda t \), in order to observe the effect of \( \lambda_1 \) and \( \lambda_2 \) on the degree of entanglement between the two particles (the isospin field and the particle in DMO). We use the same initial parameters as in the previous figures.

We note that \( C(t) \) starts from zero and is then followed by a sequence of fluctuations between zero and 1.2. This means that the entanglement between the two particles cannot be performed before the interaction is switched on. To visualize the effect of \( \lambda_1 \), in Fig. 3 we take different values of \( \lambda_1 \). We can observe that for a large effect of \( \lambda_1 \), the entanglement increases after a short time from the start, and the number of rapid fluctuations increases; see Fig. 3b.

The same behavior is observed in Fig. 4, where we use different values of \( \lambda_2 \). We can say that the effect of \( \lambda_1 \) on the degree of entanglement between the two particles is similar to the effect of \( \lambda_2 \), where with increase in the value of any of these parameters, one can increase the degree of entanglement.

5.3. The Population Inversion

The population inversion gives us information on the behavior of the particle during the interaction period. This determines when the particle reaches its maximum state and enables one to observe when the particle is in its excited or ground state or in a superposition state. From the mathematical point of view, the population inversion is the expectation value of the operator \( \hat{\sigma}_z \); thus we have

\[
W(t) = \rho_{ee}(t) - \rho_{gg}(t).
\]

We display the evolution of the population inversion of the isospin field for different values of \( \lambda_1 \) in Fig. 5, and we use different values of \( \lambda_2 \) in Fig. 6. We use the same initial parameters as in the previous figures. We note that the collapse and revival phenomena are very obvious in all figures. The function \( W(t) \) is symmetric with respect to \( W(t) = 0 \), and the population inversion oscillates between \((-1)\) and \((+1)\).

We observe that the magnetic-field strength does not strongly influence the behavior of the isospin field; see Fig. 5. In contrast, in Fig. 6 we see that, with increase in the values of \( \lambda_2 \), the collapse period decreases, and the oscillation increases during the revival period. Taking large values of \( \lambda_2 \), one can rapidly increase the oscillations; also the collapse and revival phenomena do not appear as clearly as before, due to interference between the patterns; see Fig. 6b.

We can say that, to study the behavior of the isospin field and show the collapse and revival phenomena clearly in this system, one should simply increase the value of \( \lambda_2 \).

5.4. The Second-Order Coherence

No doubt, the examination of the second-order correlation function leads to better understanding of the nonclassical behavior of the system. For this reason, we devote the present section to the discussion
of the behavior of the correlation function for the system under study. The correlation function is usually
used to discuss the sub-Poissonian and super-Poissonian behavior of the photon distribution. In view
of this function, we can distinguish between classical and nonclassical behavior of the system. The
normalized second-order correlation function is defined by [24]

\[ g^{(2)}(t) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}^2 \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}. \] (33)

The light field has a sub-Poissonian distribution if \( g^{(2)}(t) < 1 \), which is the nonclassical effect. This
means that the probability of detecting an incident pair of photons is smaller than it would be for a
coherent field described by the Poissonian distribution. On the other hand, the light field has the super-
Poissonian distribution if \( g^{(2)}(t) > 1 \), which is the classical effect, and the Poissonian distribution of
photon (standard for the coherent state) if \( g^{(2)}(t) = 1 \). In the meantime, the system displays thermal
statistics when \( g^{(2)}(t) = 2 \) and super-thermal for \( g^{(2)}(t) > 2 \). In order to discuss the distribution of this

![Fig. 3. The concurrence as a function of \( \lambda t \) at \( \lambda_1 = 0.2\lambda \) and \( \lambda_1 = 1.2\lambda \). The parameters are similar to Fig. 1.](image)

![Fig. 4. The concurrence as a function of \( \lambda t \) at \( \lambda_2 = 0.2\lambda \) and \( \lambda_2 = 1.2\lambda \). The parameters are similar to Fig. 2.](image)
system, we calculate the expectation value of the quantities $\langle \hat{a}^\dagger \hat{a}^2 \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle^2$; they are

$$\langle \hat{a}^\dagger \hat{a}^2 \rangle = \langle \hat{n}(\hat{n} - 1) \rangle = \sum_{n=0}^{\infty} [(n+1)(n+2)|B_1(n,t)|^2 + (n+2)(n+3)|B_2(n,t)|^2 + n(n+1)|B_3(n,t)|^2 + (n+1)(n+2)|B_4(n,t)|^2], \quad (34)$$

$$\langle \hat{a}^\dagger \hat{a} \rangle^2 = \langle \hat{n} \rangle^2 = \left( \sum_{n=0}^{\infty} [(n+2)|B_1(n,t)|^2 + (n+3)|B_2(n,t)|^2 + (n+1)|B_3(n,t)|^2 + (n+2)|B_4(n,t)|^2] \right)^2. \quad (35)$$

In view of Eqs. (33)–(35), we can easily obtain $g^{(2)}(t)$.

Now, we discuss the numerical calculations of the second-order correlation function $g^{(2)}(t)$ in Figs. 7 and 8. We see that the oscillation fundamental line is oscillating around 0.987 and never reaches 1 after $\lambda t > 0$; this means that the system is exhibiting sub-Poissonian distribution but, at the beginning, the

Fig. 5. $\lambda_1 = 0.2\lambda; \lambda_2 = 1.2\lambda; \lambda_2 = 0.2\lambda$ and $\lambda_1 = 1.2\lambda$. The population inversion of the isospin field as a function of $\lambda t$ at $\lambda_1 = 0.2\lambda$ and $\lambda_2 = 1.2\lambda$. The parameters are similar to Fig. 1.

Fig. 6. The population inversion of the isospin field as a function of $\lambda t$ at $\lambda_2 = 0.2\lambda$ and $\lambda_2 = 1.2\lambda$. The parameters are similar to Fig. 2.
distribution is Poissonian. Also, increase in values of $\lambda_1$ or $\lambda_2$ enables the oscillations to be squeezed and still remain sub-Poissonian.

![Image](image1)

**Fig. 7.** The second-order correlation as a function of $\lambda t \lambda_1 = 0.2\lambda$ and $\lambda_1 = 1.2\lambda$. The parameters are similar to Fig. 1.

![Image](image2)

**Fig. 8.** The second-order correlation as a function of $\lambda t$ at $\lambda_2 = 0.2\lambda$ and $\lambda_2 = 1.2\lambda$. The parameters are similar to Fig. 2.

### 6. Conclusions

We studied how the 2+1 DMO coupled to an external isospin field in an external magnetic field is mapped into the GJCM. Also, we studied the effect of the magnetic-field strength ($\lambda_1$) and the coupling parameter of the isospin field ($\lambda_2$) on the entanglement, the population inversion, and the second-order correlation function. We used the coherent state as the initial state and fixed the value of the initial coherent parameter as $\alpha = 3$. The model considered is more general than the model obtained in [22], since we added to the 2+1 DMO an external magnetic field while it was neglected in [22]. Also it is more general than the model studied in [19], since we used an external isospin field and studied some statistical properties of the system.

We would like to clarify that the magnetic-field strength and the coupling parameter have clear effects on the entanglement between the isospin field and the Dirac oscillator, as well as between the two particles.
The 2+1 DMO and the isospin field are separated at $\lambda t = 0$, and they are in a mixed state and never reach the pure state again for any time $\lambda t > 0$.

By increasing the value of $\lambda_1$ or the value of $\lambda_2$, we can obtain strong entanglement between the isospin field and the Dirac oscillator; see Figs. 1 and 2. The degree of entanglement between the two particles increases in a similar way; see Figs. 3 and 4.

The behavior of the isospin field and the collapse and revival phenomena are shown clearly by increasing the value of $\lambda_2$; see Fig. 6.

The system exhibits the sub-Poissonian distribution for any time $\lambda t > 0$.

Thus, in this paper we show how important is the link between quantum optics and quantum relativistic theory. This link helps to study some statistical properties of the 2+1 DMO coupled to an external isospin field in an external magnetic field. These statistical properties have not been studied without this link.

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