INVARIANT EFFECTIVE ACTIONS,
COHOMOLOGY OF HOMOGENEOUS SPACES
AND ANOMALIES

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Abstract

We construct the most general local effective actions for Goldstone boson fields associated with spontaneous symmetry breakdown from a group $G$ to a subgroup $H$. In a preceding paper, it was shown that any $G$-invariant term in the action, which results from a non-invariant Lagrangian density, corresponds to a non-trivial generator of the de Rham cohomology classes of $G/H$. Here, we present an explicit construction of all the generators of this cohomology for any coset space $G/H$ and compact, connected group $G$. Generators contributing to actions in 4-dimensional space-time arise either as products of generators of lower degree such as the Goldstone-Wilczek current, or are of the Wess-Zumino-Witten type. The latter arise if and only if $G$ has a non-zero $G$-invariant symmetric $d$-symbol, which vanishes when restricted to the subgroup $H$, i.e. when $G$ has anomalous representations in which $H$ is embedded in an anomaly free way. Coupling of additional gauge fields leads to actions whose gauge variation coincides with the chiral anomaly, which is carried here by Goldstone boson fields at tree level. Generators contributing to actions in 3-dimensional space-time arise as Chern-Simons terms evaluated on connections that are composites of the Goldstone field.

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1. Introduction

Local effective field theories are widely used to capture the dynamics of Goldstone bosons resulting from the spontaneous breakdown of continuous symmetries. The power of local effective field theories lies in the fact that their structure is largely determined by symmetry considerations alone. If the symmetry of the action is a (compact) group $G$, and the symmetry of the vacuum is a subgroup $H$, then the Goldstone fields $\pi^a(x)$ parametrize the coset space $G/H$ with $a = 1, \ldots, \dim G/H$. The corresponding effective field theory depends on only a finite number of couplings, up to any given order in an expansion in powers of derivatives.

General methods for constructing invariant actions were given in [1] for $SU(2)_L \times SU(2)_R$ and extended to the case of arbitrary $G$ and $H$ in [2]. These methods are based on producing the most general invariant Lagrangian density, but do not consider exceptions where the action is invariant under $G$ while the Lagrangian density transforms with a total derivative term. The Wess-Zumino-Witten (WZW) term, which was originally considered as an effective action for chiral anomalies, is an example of such an exception [3,4].

The success of the effective action approach relies upon the assumption that we can enumerate all possible invariant contributions to the action. In a recent paper [5], we addressed the issue of constructing all invariant contributions to the effective action for arbitrary $G$ and $H$. We exhibited a one to one correspondence between local terms in the effective Lagrangian density, that although not $G$-invariant, yield $G$-invariant contributions to the effective action, and generators of the de Rham cohomology classes of the coset space $G/H$. The invariant effective action $S[\pi]$ in space-time dimension $n - 1$ is given in terms of a de Rham cohomology generator $\Omega$ of degree $n$ as follows

$$S[\pi] = \int_{B_n} \Omega(\tilde{\pi})$$  \hspace{1cm} (1.1)

Here, $n - 1$ dimensional space-time $M_{n-1}$ is extended to an $n$-ball $B_n$ with boundary $M_{n-1}$, and the field $\tilde{\pi}$ interpolates between the original field $\pi$ on $M_{n-1}$ and the field whose value is zero at all space-time points. Each independent cohomology generator produces a $G$-invariant action given by (1.1) which arises from a non-invariant Lagrangian density and escapes the construction of $G$-invariant actions given in [2]. In four dimensions, we are interested in de Rham cohomology generators of degree 5. For the simplest case of
\( G/H = SU(N), \, N \geq 3 \) for example, there is just a single generator which corresponds to the original WZW term.

In the present paper, we derive the general structure of the de Rham cohomology classes for arbitrary compact, connected \( G \) and arbitrary subgroup \( H \). We emphasise the classes of degrees 4 and 5, which are the ones that yield invariant actions in 3- and 4-dimensional space-times respectively. We exhibit a correspondence between cohomology generators and invariant symmetric tensors, which is closely related to the transgression map introduced by A. Weil, H. Cartan and A. Borel \([6,7]\) and to the descent equations that arose in the study of chiral anomalies \([8,9,10]\). More specifically, when a cohomology generator on \( G/H \) contains linear combinations of products of cohomology generators on \( G/H \) of lower degree, the corresponding generator is said to be \textit{decomposable}; in the contrary case, the generator is said to be \textit{primitive}. The full cohomology of degree \( n \) can be obtained simply from the primitive cohomology generators of all degrees up to and including \( n \). For the case of most interest, cohomology of degree 5, we need the primitive generators of degree 5, and also all primitive generators of degrees 1, 2, 3 and 4, which we explicitly construct. We show that primitive cohomology generators are in one to one correspondence with \( G \)-invariant rank 3 symmetric tensors which vanish when restricted to the generators of the subalgebra of \( H \). * The primitive cohomology generators then always arise from non-trivial cohomology generators on \( G \) that can be consistently projected down to \( G/H \). While the analysis of cohomology classes of homogeneous spaces has been the subject of intense study in mathematics, the number and the form of the generators does not seem to be available in explicit form for general \( G/H \).

The elements in the above construction have direct physical counterparts. Cohomology generators of degree 2 arise in connection with the dynamics of electrically charged particles moving in the presence of a magnetic monopole field. (See e.g. ref. \([4]\)) Cohomology generators of degree 3 are associated with the Goldstone-Wilczek current which plays an important role in the study of quantum numbers carried by solitons \([12]\). The same generators are also familiar as the WZW term in 2 space-time dimensions. Cohomology generators of degree 4 are given by the second Chern class, evaluated on a composite \( H \)-valued gauge field on \( G/H \). The resulting invariant action may be recast directly in

* This general result is the one obtained in \([7,11]\).
three dimensions and turns out to be given by the Chern-Simons form [8,13] evaluated on the same $H$-valued gauge field. This construction of 3-dimensional invariant actions generalizes the Hopf type invariants.

The primitive cohomology generators of degree 5 are related to the WZW term. The $G$-invariant symmetric tensors of rank 3 are identical to the $d$-symbols encountered in the study of chiral anomalies in quantum field theories with chiral fermions coupled to non-Abelian gauge fields [13,14]. Thus, primitive cohomology generators of degree 5 occur when the group $G$ has at least some representations in which the $d$-symbol is non-vanishing; in this case the group $G$ by itself admits a WZW term. The corresponding $d$-symbol must vanish upon restriction to the Lie subalgebra of $H$, so that the WZW term for $G$ can be gauged with respect to the anomaly free subgroup $H$, and projected down to a well-defined action where the field takes values in $G/H$. The fact that the WZW term can be constructed this way has been known for some time [10,15]. What we show here is that this construction produces all Lagrangian densities that, although not $G$-invariant, yield $G$-invariant actions.

As is well-known, WZW terms cannot in general be coupled to gauge fields in a locally $G$-invariant way. This failure to maintain gauge invariance in a theory with chiral fermions is the chiral anomaly, and the WZW term was introduced precisely as an effective action reproducing this anomaly [3]. It is remarkable that we have here identified a source of anomalies in a completely classical theory with Goldstone bosons only, without ever mentioning chiral fermions or short distance divergences that underly chiral anomalies. This result seems to emphasize again the fundamental connection between chiral anomalies and spontaneous symmetry breakdown [16].

The remainder of this paper is organized as follows. In §2, we review standard results on the differential calculus of homogeneous spaces and outline the construction of cohomology generators carried out in the paper. In §3, we reduce the cohomology problem to the classification of constant symmetric tensors on $G$ with certain invariance and vanishing properties with respect to $G$ and $H$. In §4, we integrate the equations for the cohomology of degrees 2, 3 and 4 with simply connected $G/H$. In §5, we analyze the cohomology of degree 5 for simply connected $G/H$, and relate primitive cohomology generators of $H^5(G/H; R)$ to $G$-invariant symmetric tensor of rank 3 on $G$, that vanish upon restriction to the Lie
algebra of $H$. We show that generators either factorize into generators of lower degree or are obtained from the projection onto $G/H$ of primitive generators of $G$. In §6, we extend our results to include the case of non-simply connected $G/H$. Finally, in §7, we discuss gauging the $G$-invariant actions constructed here.

2. Differential Calculus on Homogeneous Spaces $G/H$

We assume that $G$ is a compact connected Lie group with $H$ a subgroup and we denote the corresponding Lie algebras by $\mathcal{G}$ and $\mathcal{H}$ respectively. On a homogeneous space, the Lie algebra $\mathcal{G}$ may be decomposed as follows $\mathcal{G} = \mathcal{H} + \mathcal{M}$, with

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H}, \mathcal{M}] \subset \mathcal{M} \quad (2.1)$$

We shall use a notation in which the indices for the generators of $\mathcal{G}$ are Latin capital letters $A = 1, \cdots, \dim G$ and are decomposed into a set of Latin lower case letters $a$ running over the generators of $\mathcal{M}$ and a set of Greek lower case letters $\alpha$ running over the generators of $\mathcal{H}$.

It is a fundamental result * that the cohomology of homogeneous spaces $G/H$ is given by the classes of closed $G$-invariant differential forms on $G/H$ modulo forms that are the exterior derivative of $G$-invariant forms on $G/H$. To construct $G$-invariant differential forms, we begin by introducing a basis of left-invariant differential forms of degree 1 on $G$, denoted by $\theta^A$, $A = 1, \cdots, \dim G$ and expressed in terms of the Goldstone field $U$ as follows :

$$\theta = \theta^A T^A = U^{-1} dU, \quad [T^A, T^B] = f^{ABC} T^C \quad (2.2)$$

Here $T^A$ are matrices in the representation under which $U$ transforms and $f$ are the totally anti-symmetric structure constants of $G$. Throughout, repeated indices are to be summed over and we shall not distinguish between upper and lower indices. From the definition of $\theta$, we see that its components satisfy the Maurer-Cartan equations

$$d\theta^C + \frac{1}{2} f^{ABC} \theta^A \theta^B = 0 \quad (2.3)$$

* Introductions to differential geometry and cohomology of Lie groups and homogeneous spaces may be found in [11,17] and especially in [18]. Following standard physics notation, wedge products will be understood for the multiplication of differential forms.
We may use the basis of left-invariant differentials $\theta^A$ to exhibit general differential forms of degree $n$ on $G$.

$$\Omega = \frac{1}{n!} \omega_{A_1\ldots A_n} \theta^{A_1} \cdots \theta^{A_n} \quad (2.4)$$

For arbitrary differential forms $\Omega$, the coefficients $\omega_{A_1\ldots A_n}$ are arbitrary functions of $G$. However, if we restrict to $G$-invariant differential forms that are well-defined on $G/H$, we have the following Theorem [17] which puts strong conditions on the coefficients $\omega_{A_1\ldots A_n}$.

**Theorem 1**

The ring of invariant forms $\Omega$ on $G/H$ is given by the exterior algebra of multilinear anti-symmetric maps on $G$ which vanish on $H$ and which are invariant under the adjoint action of $H$.

This means that the coefficients $\omega_{A_1\ldots A_n}$ must be constant as functions on $G$, vanish whenever one of the indices $A_i$ corresponds to a generator in $H$ and be invariant under the adjoint action of the group $H$ (or $H$-invariant for short). As a result, the ranges of the summation indices in (2.4) may be restricted to $M$ and we have

$$\Omega = \frac{1}{n!} \omega_{a_1\ldots a_n} \theta^{a_1} \cdots \theta^{a_n} \quad \omega_c^{[a_2\ldots a_n f a_1]} c^\beta = 0 \quad (2.5)$$

Here the symbol $[\ ]$ denotes anti-symmetrization of the indices inside the brackets and the index $\beta$ corresponds to any generator of $H$.

This theorem is easily understood in view of the $G$-transformation properties of the basic differential forms of (2.2). Under left $G$ transformations, the Goldstone fields transform as follows [2]: $U(\pi) \rightarrow g U(\pi) = U(\pi') h(g, \pi)$ where $h(g, \pi) \in H$. As a result, the forms $\theta^a$ transform in the adjoint representation of $H$, whereas $\theta^A$ transform as a $H$-valued connection :

$$\theta^a T^a \mapsto h^{-1} \theta^a T^a h$$

$$\theta^A T^\alpha \mapsto h^{-1} \theta^\alpha T^\alpha h + h^{-1} dh \quad (2.6)$$

Thus, $G$ transformations on $U$ induce $H$-valued gauge transformations on $\theta$, and a $G$-invariant form on $G/H$ should be invariant under $H$-gauge transformations; this is precisely what Theorem 1 guarantees.

We also need differential forms of degree $n$ which are tensors of rank $m$ on the Lie algebra $\mathcal{G}$ (called tensor forms), defined as follows

$$\Omega_{B_1\ldots B_m} = \frac{1}{n!} \omega_{B_1\ldots B_m; A_1\ldots A_n} \theta^{A_1} \cdots \theta^{A_n} \quad (2.7)$$
Such a tensor form is $G$-covariant and well-defined on $G/H$ if the coefficients $\omega_{B_1 \cdots B_m;A_1 \cdots A_n}$ are constant, vanish whenever one of the $A_i$ indices corresponds to a generator of $H$, and when viewed as a tensor of rank $m + n$ are invariant under the action of $H$ in the adjoint representation. We shall not at present assume any particular symmetry amongst $B_i$ indices or between $B_i$ indices and $A_j$ indices.

An operation $O$ on differential forms $\Omega_1$ and $\Omega_2$ of degrees $d_1$ and $d_2$ respectively, satisfying the following rule

$$O(\Omega_1 \Omega_2) = (O\Omega_1)\Omega_2 + (-)^{w_1} \Omega_1 (O\Omega_2)$$

is said to act as a derivative if $w_1 = 0$ and as an anti-derivative if $w_1 = d_1$.

The exterior derivative $d$ on differential forms with constant coefficients is defined by the anti-derivative rule in (2.8) and the Maurer-Cartan equations of (2.3). For example on tensor forms of (2.7) we have

$$d\Omega_{B_1 \cdots B_m} = -\frac{1}{2(n-1)!} \omega_{B_1 \cdots B_m;A_1 \cdots A_n} f^{A_1 \cdots A_n}_{B_1 \cdots B_m} \theta_{B}^{C} \theta_{A_2} \cdots \theta_{A_n}$$

On $G$-invariant forms on $G/H$, as defined by Theorem 1 and (2.5), the action of the exterior derivative takes on a simplified form:

$$d\Omega = -\frac{1}{2(n-1)!} \omega_{a_1 \cdots a_n} f^{a_1 \cdots a_n}_{a_1 \cdots a_n} \theta_{b} \cdots \theta_{a_n}$$

On tensor forms of rank $m$, it is natural to act with the covariant derivative $D$, defined by the anti-derivative rule of (2.8) and its expression on tensor forms of rank 1. On tensor forms of rank $m$ we have

$$D\Omega_{B_1 \cdots B_m} = d\Omega_{B_1 \cdots B_m} + f_{B_1 BC} \theta_{B} \Omega_{C B_2 \cdots B_m} + \cdots + f_{B_m BC} \theta_{B} \Omega_{B_1 \cdots B_{m-1} C}$$

Since the connection $\theta$ has zero curvature, we have $D^2 = 0$ on tensor forms of all ranks. The exterior derivative increases the degree by one unit and leaves the rank of the tensor form unchanged.

We shall also make use of the fundamental operation $i_A$, which lowers the degree by one and increases the rank by one:

$$i_A \Omega_{B_1 \cdots B_m} = \frac{1}{(n-1)!} \omega_{B_1 \cdots B_m;A_1 \cdots A_n} \theta^{A_2} \cdots \theta^{A_n}$$
It is easy to see that $i_A$ acts with the anti-derivative rule of (2.8) and satisfies $i_A i_B + i_B i_A = 0$ on all forms.

Rotation under the action of $G$ obeys the derivative rule of (2.8) and acts on tensor forms of rank $m$ by

$$L_A \Omega_{B_1 \cdots B_m} = f_{AB_1B} \Omega_{B B_2 \cdots B_m} + \cdots + f_{AB_mB} \Omega_{B_1 \cdots B_{m-1} B} \tag{2.12}$$

$L_A$ obeys the commutation relations of $G$ of (2.2), vanishes on forms of rank 0, and its square $L^2 = L_A L_A$ is the quadratic Casimir operator acting on the representation of $G$ under which the tensor form $\Omega_{B_1 \cdots B_m}$ transforms.

There are fundamental relation [18] between the actions of $D$, $i_A$ and $L_A$ valid on tensor forms of any rank:

$$L_A i_B - i_B L_A = f_{ABC} i_C$$
$$L_A D - DL_A = 0 \tag{2.13}$$
$$i_A D + Di_A = - L_A$$

In particular, on tensor forms $\Omega$ of rank 0, the last identity reduces to $i_A d\Omega + Di_A \Omega = 0$.

Finally, we introduce a composite operation $\Delta$, which is neither a derivative nor an anti-derivative, and is defined by

$$\Delta \Omega_{B_1 \cdots B_m} = L_A (i_A \Omega_{B_1 \cdots B_m}) \tag{2.14}$$

An analogous operation was introduced in [10] for the study of Lie algebra cohomology. It obeys the following relations, valid on forms of any rank

$$L_A \Delta - \Delta L_A = 0$$
$$D \Delta + \Delta D = - L^2 \tag{2.15}$$

The last equation will be a crucial ingredient in our analysis of cohomology. It basically implies that any closed form which transforms under a non-trivial representation of $G$ is an exact form on $G$.

Outline of the Construction of Cohomology Generators of $G/H$

First, we use the result – which was already mentioned [17,18] – that the de Rham cohomology of $G/H$ can be identified with the coset of the space of all closed $G$-invariant forms on $G/H$ by the space of exterior derivatives of $G$-invariant forms on $G/H$. 

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Second, using Theorem 1, $G$-invariant forms on $G/H$ are given by (2.4) where the coefficients $\omega_{A_1 \cdots A_n}$ are

(1) constant as functions on $G$,
(2) invariant under the adjoint action of $H$, and
(3) zero whenever any of the indices $A_i$ corresponds to a generator of $H$.

Thus, all cohomology generators on $G/H$ are obtained as forms of the type (2.4) that are closed and satisfy properties (1 - 3) modulo differentials of forms that satisfy properties (1 - 3). The space of forms obeying these properties is finite dimensional.

Third, we construct all closed forms obeying (1), including all exact forms of the type (2.4), but ignoring properties (2) and (3) temporarily. To do this, we make use in §3 of the operation $\Delta$ of (2.14) to map closed forms onto closed forms of lower degree but higher rank. This map is closely related to the transgression map [6,7] and produces a hierarchy of equations which is closely related to the descent equations. Ultimately, one is led to analyzing forms of degree 0 (i.e. constants) which are $G$-invariant completely symmetric tensors. This problem can be studied using group theoretical methods alone. For example the analysis of cohomology of degree 5 reduces to the study of constant symmetric tensors of rank 3, also known as the $d$-symbols of $G$.

Fourth, property (3) will in general put restrictions on the allowed symmetrical tensors discussed in the third point above. In particular, the $d$-symbols must vanish whenever all of its indices correspond to generators of $H$.

Fifth, one may integrate the hierarchy of equations and obtain all closed forms of the type (2.4) that obey properties (1) and (2). Property (3) is imposed explicitly on these forms and the resulting equations can be solved rather easily. This gives all closed forms of the type (2.4) obeying all three properties (1 - 3).

Finally, we discard exact generators by simply enumerating the differentials of forms of the type (2.4) that obey properties (1 - 3) themselves. The remaining forms are precisely the de Rham cohomology generators of $G/H$.

3. Cohomology and Constant Invariant Tensors

In this section as well as in §4 and §5, we assume that $G$ is compact, semi-simple, simply connected, and that $H$ is connected. The Lie groups $G$ and $H$ admit the following
factorization in terms of simple factors $G_i$ and $H_j$ (with corresponding Lie algebras $\mathcal{G}_i$ and $\mathcal{H}_j$), and $U(1)$ components:

$$G = G_1 \times \cdots \times G_p, \quad H = H_1 \times \cdots \times H_q \times U(1)^r \quad (3.1)$$

As a result, $G/H$ is simply connected and has vanishing first cohomology: $H^1(G/H; R) = 0$, which considerably simplifies the discussion. These assumptions will be relaxed in §6, where the general case will be considered.

We begin by formulating the correspondence that maps forms into forms of lower degree but increased rank. Let $\Omega$ be a closed $G$-invariant form of rank 0 and degree $n$. Using the third equation in (2.13) for forms of rank 0, we see that

$$d\Omega = 0 \implies D(i_A \Omega) = 0 \quad (3.2)$$

We have thus constructed from $\Omega$ a new form $i_A \Omega$ which is again closed. We show that $i_A \Omega$ is exact as a form on $G$ and we have

$$i_A \Omega = D\Omega_A \quad (3.3)$$

The form $\Omega_A$ may be constructed explicitly with the help of (2.15), and the fact that the quadratic Casimir operator is invertible on $i_A \Omega$, since the group $G$ is semi-simple. Equation (3.3) determines $\Omega_A$ up to a closed form $\Omega'_A$. Using (2.15) and the invertibility of the quadratic Casimir operator on $\Omega'_A$, we find that $\Omega'_A$ is exact on $G$, so that $\Omega'_A = DM_A$ for some form $M_A$ of degree 2 and rank 1. The addition of $\Omega'_A$ does not modify the original forms $i_A \Omega$ or $\Omega$. Denoting by $C_2(\mathcal{G}_j) \neq 0$ the eigenvalue of the quadratic Casimir operator on each simple component in the adjoint representation, we find

$$\Omega_A = DM_A - \frac{1}{L^2} \Delta (i_A \Omega)$$

$$= DM_A - \sum_{j=1}^p \frac{1}{C_2(\mathcal{G}_j)} \delta^j_{AA'} f_{A'B'C} i_B i_C \Omega \quad (3.4)$$

Here, $\delta^j$ stands for the Kronecker symbol restricted to the the simple component $\mathcal{G}_j$ of $\mathcal{G}$. From $\Omega_A$, we construct a new closed form, again using (2.13)

$$i_A \Omega = D\Omega_A \quad \Rightarrow \quad D(i_A \Omega_B + i_B \Omega_A) = 0 \quad (3.5)$$
The correspondence which maps the form $i_A \Omega$ onto the form $i_A \Omega_B + i_B \Omega_A$ is the simplest case of a transgression map, in which a closed tensor form of rank 1 and degree $n - 1$ is mapped onto a closed tensor form of rank 2 and of degree $n - 3$.

To generalize the above construction to tensor forms of arbitrary rank, we have to overcome two complications. First, when an arbitrary form $\Omega_{B_1 \ldots B_m}$ is closed, the form $i_B \Omega_{B_1 \ldots B_m}$ does not, in general, obey any simple closure relation. Instead, the symmetrized form $i\{B \Omega_{B_1 \ldots B_m}\}$, constructed from a tensor form $\Omega_{B_1 \ldots B_m}$ which is itself symmetrical in all its $B_i$ indices, is automatically closed in view of (2.13). Thus, for totally symmetrical tensor forms $\Omega_{B_1 \ldots B_m}$ we have the general formula

$$D \Omega_{B_1 \ldots B_m} = 0 \implies D(i\{B \Omega_{B_1 \ldots B_m}\}) = 0 \quad (3.6)$$

Second, the tensor form $i\{B \Omega_{B_1 \ldots B_m}\}$ no longer transforms under a single irreducible representation of $G$, which complicates the evaluation of the quadratic Casimir operator $L^2$ in (2.15). Thus, we decompose the tensor into irreducible components $i\{B_{m+1} \Omega_{B_1 \ldots B_m}\}(R_k)$ which transform under the irreducible representation $R_k$ of $G$

$$i\{B_{m+1} \Omega_{B_1 \ldots B_m}\} = \sum_{\{R_k\}} i\{B_{m+1} \Omega_{B_1 \ldots B_m}\}(R_k) \quad (3.7)$$

The quadratic Casimir operator within each irreducible representation $R_k$ is just a numerical constant $C_2(R_k)$.

Using these ingredients, the above map may be generalized to act on tensor forms of any rank and degree, which is the content of the following Theorem. Let $\Omega_{B_1 \ldots B_m}$ be a closed tensor form of rank $m$ and of degree $n \geq 2$, which is completely symmetric in its indices $B_i$.

**Theorem 2**

The forms $i\{B_{m+1} \Omega_{B_1 \ldots B_m}\}$, and every irreducible component $i\{B_{m+1} \Omega_{B_1 \ldots B_m}\}(R_k)$ defined in (3.7), are automatically closed

$$D \Omega_{B_1 \ldots B_m} = 0 \implies D(i\{B_{m+1} \Omega_{B_1 \ldots B_m}\}(R_k)) = 0 \quad (3.8)$$

For non-trivial irreducible representations $R_k$, the form $i\{B_{m+1} \Omega_{B_1 \ldots B_m}\}(R_k)$ is exact on
and determines a form $\Omega_{B_1\ldots B_{m+1}}(R_k)$ up to an exact form $DM_{B_1\ldots B_{m+1}}(R_k)$ on $G$

$$i_{\{B_{m+1}\Omega_{B_1\ldots B_{m}}\}}(R_k) = D\Omega_{B_1\ldots B_{m+1}}(R_k)$$

$$\Omega_{B_1\ldots B_{m+1}}(R_k) = DM_{B_1\ldots B_{m+1}}(R_k) - \frac{m + 1}{C_2(R_k)} f^{AB} \{B_{m+1}, i^A i_{\{B\Omega_{B_1\ldots B_{m}}\}}\}(R_k)$$

(3.9)

The form $i_{\{B_{m+2}\Omega_{B_1\ldots B_{m+1}}\}}(R_k)$ is automatically closed in view of (2.13). For trivial representations $R_0$, the form $i_{\{B_{m+1}\Omega_{B_1\ldots B_{m}}\}}(R_0)$ is a linear combination of products of $G$-invariant tensors and forms of rank 0 and degree $n - 1$.

Theorem 2 is proved by repeated use of (2.13) and (2.15), and by exploiting the fact that the quadratic Casimir operator is simply evaluated in any irreducible representation of $G$.

The map that sends a closed form $i_{\{B_{m+1}\Omega_{B_1\ldots B_{m}}\}}(R_k)$ with non-trivial representation $R_k$ into another closed form $i_{\{B_{m+2}\Omega_{B_1\ldots B_{m+1}}\}}(R_k)$, defined through Theorem 2 is essentially the transgression map of [6,7]. It reduces the degree of a form by two units, while increasing the rank by one unit, and may be iterated until forms or degree 0 (or 1) are encountered. Since we assumed in this section that $G/H$ is simply-connected, cohomology of degree 1 vanishes. The study of invariant forms of degree $n$ and rank 0 is thus reduced to the analysis of constant invariant tensors. This is indeed a fundamental result on cohomology of groups and homogeneous spaces [7]. The algebra of tensors $\omega_{B_1\ldots B_m; A_1\ldots A_n}$, completely symmetrized in $B_i$ and completely anti-symmetrized in $A_i$ is usually referred to as the Weil algebra [6,7,11]. Theorem 2 implies that the cohomology of the Weil algebra is located entirely in forms of degree 0, i.e. constants. We shall now provide a more detailed analysis of how the cohomologies of $G$ and $G/H$ are constructed.

The components in the trivial representation in the decomposition of (3.7) are invariant under the adjoint action of $G$, and automatically closed. They are linear combinations of products of $G$-invariant tensors $d_{B_1\ldots B_{m+1}}^{(k)}$ of degree 0 and closed $G$-invariant forms $\Sigma_{n-2m-1}^{(k)}$ of degree $n - 2m - 1$ and rank 0. Using the procedure of Theorem 2, we may write down explicitly a hierarchy of equations that result from the successive application
of the above map:

\[ D(i_B \Omega) = 0 \Rightarrow i_B \Omega = D \Omega_B + \left( \sum_k d_B^{(k)} \Sigma_{n-1}^{(k)} \right) \]

\[ D(i_{B_2} \Omega_{B_1}) = 0 \Rightarrow i_{B_2} \Omega_{B_1} = D \Omega_{B_1 B_2} + \left( \sum_k d_{B_1 B_2}^{(k)} \Sigma_{n-3}^{(k)} \right) \]

\[ \vdots \]

\[ D(i_{B_{m+1}} \Omega_{B_1 \ldots B_m}) = 0 \Rightarrow i_{B_{m+1}} \Omega_{B_1 \ldots B_m} = D \Omega_{B_1 \ldots B_m} + \left( \sum_k d_{B_1 B_2 \ldots B_{m+1}}^{(k)} \Sigma_{n-2m-1}^{(k)} \right) \]

(3.10)

The hierarchy of (3.10) forms the basis for the analysis of the structure of the cohomologies of \( G \) and of \( G/H \) when combined with the expression of the forms \( \Omega_{B_1 \ldots B_{m+1}} \) given in (3.9). Its structure is related to that of the descent equations [8,9], which proceed in opposite direction by increasing the degrees in \( \theta \) instead.

The sum terms on the right hand side of each of the above lines correspond to the contribution \(^\dagger\) in Theorem 2 of the trivial representations \( R_0 \). They have been put in parentheses for the following reason. If a term \( \sum_k d_{B_1 B_2 \ldots B_m}^{(k)} \Sigma_{n-2m+1}^{(k)} \) arises in a given line \( m \), then a contribution from the hierarchy for the closed forms \( \Sigma_{n-2m+1} \) must be subtracted from the left hand side of the equation of the next line \( m+1 \). This equation becomes instead

\[ D(i_{B_{m+1}} \Omega_{B_1 \ldots B_m}) - \sum_k d_{B_1 B_2 \ldots B_m}^{(k)} i_{B_{m+1}} \Omega_{B_1 \ldots B_m} = 0 \]  \( (3.11) \)

The modifications required by (3.11) are understood in (3.10), but have not been written explicitly to keep the notation as transparent as possible.

The origin of the sum terms in (3.10) may be understood as follows. Given cohomology generators \( \Omega_1 \) and \( \Omega_2 \) of degrees \( n_1 > 0 \) and \( n_2 > 0 \), their (wedge) product \( \Omega = \Omega_1 \Omega_2 \) of degree \( n = n_1 + n_2 \) is a closed form and is a cohomology generator of degree \( n \). The hierarchy equations (3.10) for \( \Omega \) may be simply deduced from those for \( \Omega_1 \) and \( \Omega_2 \) and will contain sum terms on the right hand side of (3.10) with \( \Sigma \) of degree \( n_1 \) or \( n_2 \). More generally, sum terms on the right hand side of (3.10) occur with a degree of \( \Sigma \) greater or

\(^\dagger\) With the assumption that \( G \) is semi-simple, no sum term appears on the right hand side of the first line, since there can be no invariant tensors transforming under the adjoint representation of \( G \).
equal to 1 if and only if the form \( \Omega \) contains linear combinations of products of cohomology generators of strictly lower degree.

It is standard to introduce the notion of \emph{primitive generators}, which do not contain any linear combinations of products of generators of strictly lower degrees. Generators that do contain linear combinations of products of cohomology generators of strictly lower degree are said to be \emph{decomposable generators}. Clearly, the cohomology is completely determined by its primitive generators, since decomposable generators may always be obtained as linear combinations of products of primitive generators including those of lower degree.

Henceforth, without loss of generality, we limit ourselves to the analysis of the primitive cohomology generators.

If \( \Omega \) is a \emph{primitive generator} of degree \( n \), the associated hierarchy contains no \( \Sigma^{(k)} \) sum terms on the right hand side of (3.10), except possibly when their degree vanishes. The hierarchy of (3.10) then considerably simplifies, and terminates at \( m = [(n - 1)/2] \) where forms of degree 0 are encountered. No subtractions of the type encountered in (3.11) are required in this case and we have

\[
D(i_B \Omega) = 0 \quad \Rightarrow \quad i_B \Omega = D\Omega_B \\
D(i_{B_2} \Omega_{B_1}) = 0 \quad \Rightarrow \quad i_{B_2} \Omega_{B_1} = D\Omega_{B_1 B_2} \\
D(i_{B_3} \Omega_{B_1 B_2}) = 0 \quad \Rightarrow \quad i_{B_3} \Omega_{B_1 B_2} = D\Omega_{B_1 B_2 B_3} \\
\vdots \\
D(i_{B_{m+1}} \Omega_{B_1 \cdots B_m}) = 0 \quad \Rightarrow \quad i_{B_{m+1}} \Omega_{B_1 \cdots B_m} = D\Omega_{B_1 \cdots B_{m+1}} + d_{B_1 \cdots B_{m+1}}
\]\n
Here, \( d_{B_1 \cdots B_{m+1}} \) is a constant \( G \)-invariant tensor, completely symmetric in its indices; when \( i_{B_{m+1}} \Omega_{B_1 \cdots B_m} \) above is of degree 0, we set \( \Omega_{B_1 \cdots B_{m+1}} = 0 \). The hierarchy (3.12) associates to a primitive generator \( \Omega \) a sequence of forms that terminates with a constant symmetric \( G \)-invariant tensor. The analysis of cohomology is thus reduced to the analysis of constant tensors with certain invariance properties, a problem that can be solved by group theoretical methods alone.

The cases for \( n \) odd or even differ substantially, and are best analyzed separately. When \( n = 2m + 1 \) is odd, the last equation in (3.12) simplifies, as \( \Omega_{B_1 \cdots B_{m+1}} = 0 \) and we find that to each primitive generator of odd degree there corresponds a constant \( G \)-invariant completely symmetric tensor \( d_{B_1 \cdots B_{m+1}} \) of rank \( m + 1 = (n + 1)/2 \). Conversely, a tensor \( d_{B_1 \cdots B_{m+1}} \) will correspond to a primitive generator on \( G/H \) only provided it vanishes on \( \mathcal{H} \).
When $H = 1$, we have the cohomology of Lie groups, and no extra conditions are needed. In this case, the invariant tensor $d_{B_1 \cdots B_{m+1}}$ produces a unique cohomology generator of degree $2m + 1$ on $G$, given by a well-known expression [8]:

$$\Omega^{(d)} \sim d_{B_1 \cdots B_{m+1}} \theta^{B_1} d\theta^{B_2} \cdots d\theta^{B_{m+1}} \quad (3.13)$$

$G$-invariance of $d$ guarantees closure of $\Omega^{(d)}$.

When $n = 2m + 2$ is even, the tensor $\Omega_{B_1 \cdots B_{m+1}}$ is of degree 0 and thus constant. In contrast with the case of $n$ odd, it does not have to be $G$-invariant, but should be invariant only under the action of $H$ in the adjoint representation. Thus, even dimensional cohomology of $G/H$ is related to constant $H$-invariant tensors, again with certain vanishing conditions on $H$.

4. Low Dimensional Cohomology of degrees 2, 3 and 4

In this section, we compute explicitly all generators of low dimensional cohomology using the methods of §3. We make the same assumptions: $G$ and $H$ are compact and connected, $G$ is semi-simple and simply connected and $H$ is semi-simple times $U(1)$ factors as in (3.1), so that $H^1(G/H; \mathbb{R}) = 0$. We shall also re-express our results directly in terms of the Goldstone field $U$, the composite $H$-valued gauge field $V$ with components $\theta^\alpha$, its field strength $W$ and associated covariant derivative $D_H$, which are defined as follows

$$V = (U^{-1} dU)_H \quad W = dV + V^2 \quad U^{-1} D_H U = (U^{-1} dU)_M \quad (4.1)$$

In particular, this notation will allow us to make contact with the results of [5].

Cohomology of degree 2

We consider a closed form $\Omega$ of rank 0 and degree 2.

$$\Omega = \frac{1}{2} \omega_{a_1 a_2} \theta^{a_1} \theta^{a_2} \quad (4.2)$$

where $\omega_{a_1 a_2}$ is $H$-invariant and the notation of (4.2) shows that the form $\Omega$ vanishes on $H$ as required by Theorem 1. We follow the procedure of Theorem 2, and associate to $\Omega$ a constant form $\Omega_A$ of degree 0 and of rank 1 defined by (3.3–4). The contribution from the
exact form $DM_A$ in (3.4) is absent and $\mathcal{H}$-invariance of $\omega_{ab}$ implies $\mathcal{H}$-invariance of $\Omega_A$. There are two possible cases for the constant $\Omega_A$

1. $A = \alpha$, corresponding to a generator of $\mathcal{H}$. If $A = \alpha$ corresponds to a generator in one of the simple components of $H$ in (3.1), then the tensor $\Omega_\alpha$ cannot be $\mathcal{H}$-invariant. If $\alpha$ corresponds to a generator of one of the $U(1)$ factors in (3.1), then $\Omega_\alpha$ is invariant under $\mathcal{H}$ and may just be set to a constant. Thus, for every $U(1)$ factor of $H$, there is a generator of $H^2(G/H; R)$ given by

$$\Omega^{(l)} = \frac{1}{2} \Omega_{\alpha l} f_{\alpha l ab} \theta^a \theta^b \quad l = 1, \ldots, r \quad (4.3)$$

The forms $\Omega^{(l)}$ are curvature forms of composite Abelian connections $V$ of (4.1) associated with each of the commuting $U(1)$ factors in $H$

$$\Omega^{(l)} = -d(\Omega_{\alpha l} \theta^\alpha) = -\Omega_{\alpha l} W_{\alpha l} \quad (4.4)$$

These forms are the generators of the first Chern class and the associated invariant effective action [5] is that of a charged particle moving in the presence of a magnetic monopole [4].

2. $A = a$, corresponding to a generator of $\mathcal{M}$. This can occur if $\mathcal{M}$ contains a generator that commutes with all of $\mathcal{H}$, so that $f_{\beta a C} = 0$ for all $\beta \in \mathcal{H}$. From $\Omega_a$, we obtain $i_B \Omega = D \Omega_B$, and from $i_B \Omega$, we have

$$\Omega = \frac{1}{2} f_{abc} \Omega_a \theta^b \theta^c = -d(\Omega_a \theta^a) \quad (4.5)$$

The forms $\Omega$ are well-defined on $G/H$, vanish on $\mathcal{H}$, and are the exterior derivatives of well-defined $\mathcal{H}$-invariant forms on $G/H$. These forms are exact on $G/H$, and do not contribute to $H^2(G/H; R)$.

**Cohomology of degree 3**

We consider a closed form $\Omega$ of rank 0 and of degree 3,

$$\Omega = \frac{1}{3!} \omega_{a_1 a_2 a_3} \theta^{a_1} \theta^{a_2} \theta^{a_3} \quad (4.6)$$

where $\omega_{a_1 a_2 a_3}$ is $\mathcal{H}$-invariant and vanishes on $\mathcal{H}$. Following the procedure of Theorem 2, and using (3.3-4) we associate to $\Omega$ a form $\Omega_A$ of degree 1 and an arbitrary form $M_A$ of degree 0. Furthermore, $d_{BC} = i_B \Omega_C$ is closed and of degree 0, which implies that it is
a constant symmetric $G$-invariant tensor. Within each simple component of $G$, $d_{AB}$ must be proportional to the $G$-invariant metric $\delta^{ij}_{AB}$ on $G_j$, so that we have

$$d_{BC} = \sum_{j=1}^{p} d^j_{2} \delta^{ij}_{BC}$$

(4.7)

where $d^j_{2}$ are constants. From (3.4), it is clear that up to an exact form, $\Omega_A$ must be independent of $\theta^{\beta}$, and hence $d_{BC}$ must vanish on $H$: $d_{\beta\gamma} = 0$.

Conversely, to find all primitive closed forms $\Omega$ on $G/H$, we start from the most general invariant tensor $d_{AB}$ of (4.7) satisfying $d_{\beta\gamma} = 0$ for all $\beta$, $\gamma$ corresponding to generators in $H$, integrate the hierarchy equations (3.12) and enforce $H$-invariance and vanishing on $H$, as required by Theorem 1. The most general solution is obtained from $i_B \Omega_C$ by adding an arbitrary constant anti-symmetric tensor:

$$i_B \Omega_C = d_{BC} + K_{BC} \quad K_{BC} = -K_{CB}$$

(4.8)

The contribution of $M_A$ in (3.4) can always be absorbed into $K_{AB}$. Independence of $\Omega_C$ on $\theta^{\beta}$ requires $d_{\beta c} = -K_{\beta c}$ and $K_{\beta\gamma} = 0$. The contribution to $\Omega$ of the $K_{ab}$ components of $K$ is always exact, and will be omitted. The form $\Omega$ is easily reconstructed and we find

$$\Omega = \frac{1}{6} d_{ab} f_{bcd} \theta^a \theta^b \theta^c + \frac{2}{3} d_{a\beta} f_{\beta cd} \theta^a \theta^b \theta^c$$

(4.9)

Notice that the components $d_{ab}$ and $d_{a\beta}$ are related by $G$-invariance of $d_{AB}$.

To construct the form $\Omega$ from $d_{AB}$, one may also proceed directly from the familiar Goldstone-Wilczek current, (which is in the same as the WZW term in 1+1 dimensions) in which the subgroup $H$ has been gauged [12,15,19]. The precise correspondence is achieved by identifying the invariant metrics on $G_j$ with the Cartan-Killing form $\delta^{ij}_{BC} = \text{tr} T^i_j T^j_c = \text{tr}_{G_j} T_B T_C$ and making use of the notation of (4.1). Here $\text{tr}_{G_j}$ stands for the trace in the simple component $G_j$ only. It is easy to see that $\Omega$ of (4.9) is then given by a sum of gauged Goldstone-Wilczek forms

$$\Omega = \frac{1}{3} \sum_j d^j_{2} \text{tr}_{G_j} \left\{ (U^{-1} D_H U)^3 - 3 W U^{-1} D_H U \right\}$$

(4.10)

The exterior derivative of $\Omega$ vanishes in view of the fact that $d_{\alpha\beta} = 0$ for $\alpha$ and $\beta$ corresponding to any generators of $H$. This form is of course nothing but the gauged WZW term in 2 dimensions for the coset model $G/H$. 

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Since we have assumed that $H^1(G/H; R) = 0$, there are no decomposable generators of degree 3, and third cohomology is completely characterized by symmetric, $G$-invariant, rank 2 tensors on $G$ that vanish on $H$.

Cohomology of degree 4

Finally, we consider a form $\Omega$ of degree 4,

$$\Omega = \frac{1}{4!} \omega_{a_1 \ldots a_4} \theta^{a_1} \ldots \theta^{a_4}$$

(4.11)

where $\omega_{A_1 \ldots A_4}$ is $H$-invariant and vanishes on $H$. From the closure of $\Omega$ and (3.3-4), we associate to $\Omega$ the form $\Omega_A$ of degree 2, and an arbitrary form $M_A$ of degree 1. Using (3.5), we see that the form $i_{\{A}\Omega_B}$ is closed, and we again apply Theorem 2 to it. Since $i_{\{A}\Omega_B}$ is a one form it must be a linear combination of the left-invariant forms $\theta^A$. Thus, upon decomposing $i_{\{A}\Omega_B}$ into irreducible representations of $G$, we only encounter the adjoint representation. (If $G$ had not been semi-simple, we would also have encountered trivial representations at this point, associated with possible $U(1)$ components of $G$.) As a result, $i_{\{A}\Omega_B}$ is an exact form according to Theorem 2, and we have

$$i_{\{A}\Omega_B} = D\Omega_{AB}$$

(4.12)

The form $\Omega_{AB}$ is of degree 0 and of rank 2, symmetric in $A$ and $B$ and must be invariant under the adjoint action of $H$.

To find all primitive closed forms $\Omega$, we proceed in analogy with the case of degree 3: we integrate the hierarchy equations (3.12) and impose the conditions that the forms should vanish on $H$ and be $H$-invariant. The most general solution to (4.12) is obtained by adding to $D\Omega_{BC}$ an arbitrary anti-symmetric form $K$ of degree 1 and rank 2

$$i_B \Omega_C = D\Omega_{BC} + K_{BC} \quad K_{BC} = -K_{CB} = k_{BC;A} \theta^A$$

(4.13)

Here, $K_{BC}$ and $\Omega_{BC}$ are constrained only by the requirement that (4.12) be integrable as an equation in $\theta^A$. The most general solution is easily found and we have

$$\Omega_A = -D(\Omega_{AB}\theta^B) - \frac{1}{2} k_{ABC} \theta^B \theta^C$$

(4.14)
Here, $\tilde{k}_{ABC}$ is an arbitrary $\mathcal{H}$-invariant tensor of rank 3 which is completely anti-symmetric in its three indices. Because the contribution to $\Omega_A$ involving $\Omega_{BC}$ in (4.14) is an exact differential, it drops out in the evaluation of $i_A\Omega$, and we find that

$$\Omega = dQ, \quad Q = \frac{1}{3!} \tilde{k}_{ABC} \theta^A \theta^B \theta^C \quad (4.15)$$

To render this form well-defined on $G/H$, we require that $\Omega$ be independent of any of the $\theta^\lambda$, where $\lambda$ corresponds to a generator in $\mathcal{H}$. Putting this requirement directly on $\Omega$ turns out to be rather cumbersome; it is more convenient to require that the combination $\Omega_A - D_M A$ of (3.4) and (3.9) be independent of $\theta^\lambda$.

$$i_\lambda \left( \frac{1}{2} \tilde{k}_{ABC} \theta^B \theta^C + D M_A \right) = 0 \quad (4.16)$$

This equation is easy to solve in terms of a constant tensor $m_{AB}$ with

$$\tilde{k}_{\lambda AB} = m_{C\lambda} f_{ABC} \quad D M_A = M_A - \Omega_{AB} \theta^B = m_{AB} \theta^B \quad (4.17)$$

Addition of an arbitrary exact form to $Q$ leaves $\Omega$ unchanged, and is equivalent to shifting $m_{AB}$ by an arbitrary anti-symmetric rank 2 tensor. Using this property, we may choose $m_{\alpha\beta}$ symmetric and $m_{a\beta} = 0$. Neglecting the contribution to $Q$ of well-defined forms on $G/H$, which lead to exact contributions to $\Omega$ only, we find

$$Q = \frac{1}{2} m_{\beta\lambda} f_{a\beta c} \theta^a \theta^c \theta^\lambda + \frac{1}{6} m_{\beta\lambda} f_{a\beta \gamma} \theta^a \theta^\gamma \theta^\lambda \quad (4.18)$$

which gives the following expression for $\Omega$

$$\Omega = m_{\alpha\beta} W_\alpha W_\beta \quad W_\alpha = d\theta_\alpha + \frac{1}{2} f_{\alpha\gamma\delta} \theta_\gamma \theta_\delta \quad (4.19)$$

Here, $W_\alpha$ are the components of the curvature form $W$ of (4.1); and $m_{\alpha\beta}$ is any constant symmetric tensor invariant under the adjoint action of $\mathcal{H}$. All such tensors may be parametrized in terms of the Cartan-Killing forms on the simple components of $\mathcal{H}$, and arbitrary coefficients on the $U(1)$ components of $\mathcal{H}$ of (3.1). As a result, we have

$$\Omega = \sum_{k=1}^q m_{2}^{(k)} W_\alpha^{(k)} W_\alpha^{(k)} + \sum_{l,m=1}^r m_{1}^{(l,m)} W_\alpha W_{\alpha_m} \quad (4.20)$$
The generators of $H^4(G/H; R)$ in the first sum belong to the second Chern class (or first Pontryagin class) evaluated on the $\mathcal{H}$-valued connection $V$ of (4.10) with components $\theta^\alpha$, while the generators in the second sum arise from products of generators belonging to the first Chern class. The latter were already identified completely in (4.4) in the subsection on cohomology of degree 2. Since $H^1(G/H; R) = 0$, there are no further decomposable generators. The form $Q$ in (4.18) may be viewed as a linear combination of Chern-Simons actions in 3 dimensions [8,13], evaluated on the composite gauge field $V$ of (4.1). The resulting invariant effective action [5] thus coincides with the Chern-Simons action evaluated on composite connections and provides generalizations of the Hopf invariant. (see for example [20] for general references)

5. Cohomology of Degree 5 and the G-Invariant $d$-Symbol

Let $\Omega$ be a closed form of rank 0 and degree 5 on $G/H$ given by

$$\Omega = \frac{1}{5!} \omega_{a_1 \ldots a_5} \theta^{a_1} \ldots \theta^{a_5}.$$  \hspace{1cm} (5.1)

Following the procedure of Theorem 2, and equations (3.12) for primitive cohomology generators, we associate to $\Omega$ the hierarchy

\begin{align*}
  i_A \Omega &= D \Omega_A \quad \Rightarrow \quad \Omega_A = DM_A - (L^2)^{-1} \delta(i_A \Omega) \\
  i_{\{A} \Omega_{B\}} &= D \Omega_{AB} \quad \Rightarrow \quad \Omega_{AB} = DM_{AB} - (L^2)^{-1} \delta(i_{\{A} \Omega_{B\}}) \\
  i_{\{A} \Omega_{BC\}} &= d_{ABC}
\end{align*}  \hspace{1cm} (5.2)

Here, $d_{ABC}$ is a constant totally symmetric and $G$-invariant tensor on $G$, referred to as the $d$-symbol of $G$ in the physics literature on chiral anomalies [14]. It is a simple group theoretical problem to find all such invariant tensors. Assuming that $G$ is semi-simple, the most general $d$-symbol on $G$ is parametrized completely by the $d$-symbols on the simple components $G_j$.

$$d_{ABC} = \sum_{j=1}^{p} d^j_3 \ d^j_{ABC}$$  \hspace{1cm} (5.3)

On each simple component $G_j$, there is at most one such invariant tensor. For all simple compact Lie groups, we have $d_{ABC} = 0$, except for $SU(N)$, $N \geq 3$, where $d_{ABC} \neq 0$; this includes the case of $SO(6) = SU(4)/Z_2$ [14].
In the simplest case where $H = 1$, and $G/H$ is a Lie group, there are no further conditions on the $d$-symbols and we shall see that each non-zero $d$-symbol produces a non-trivial cohomology generator of $G$, given by (3.13).

For more general subgroups $H$, the $d$-symbols corresponding to cohomology generators on $G/H$ must satisfy further conditions. A complete characterization of $H^5(G/H; R)$ is given by the following Theorem:

**Theorem 3**

The $d_{ABC}$ tensor associated with a closed 5-form $\Omega$ on $G/H$ must vanish on the subalgebra $\mathcal{H}$ (i.e. $H$ must be embedded into $G$ in an anomaly free way). Conversely, any non-zero $G$-invariant tensor $d_{ABC}$, which vanishes on the subalgebra $\mathcal{H}$ produces a unique primitive generator of $H^5(G/H; R)$. All other generators of $H^5(G/H; R)$ are decomposable into linear combinations of products of generators of degrees 2 and 3.

To show that a $d$-symbol corresponding to a cohomology generator on $G/H$ must vanish on $\mathcal{H}$, we produce an explicit formula for $d_{ABC}$ in terms of $\Omega$ of (5.1) by solving (5.2). The exact forms $DM_A$ and $DM_{AB}$ of (5.2) do not contribute to $d_{ABC}$: $DM_{AB}$ would add to $d_{ABC}$ a quantity $-Di\{A M_{BC}\}$ which vanishes since $M_{BC}$ is of degree 0; $DM_A$ contributes $-i\{A M_B\}$ to $\Omega_{AB}$ which would add to $d_{ABC}$ a quantity $-i\{A i\{B M_C\}\} = 0$. It follows that the $d$-symbol is given in terms of $\Omega$ by the following expression.

$$d_{ABC} = i\{A \frac{1}{L^2} \delta \left(i\{B \frac{1}{L^2} \delta(i\{C\})\} \Omega\right) \quad (5.4)$$

Both Casimir operators effectively act on the adjoint representation of $G$, and (5.4) may be evaluated in terms of the symbols $f_{ABC}^j$ which are the structure constants of $G$, restricted to the simple component $G_j$, and we find

$$d_{ABC} = \sum_j \frac{1}{3! C_2(G_j)^2} f_{BA_2A_3}^j f_{CA_4A_5}^j \omega_{AA_2A_3A_4A_5} \quad (5.5)$$

The expression on the right hand side is completely symmetrized in $ABC$. Also, one of the three indices $A, B, C$ is always attached directly to $\omega$. Since $\omega$ vanishes whenever one of its indices corresponds to a generator in $\mathcal{H}$, the $d$-symbol restricted to the subalgebra $\mathcal{H}$ must vanish: $d_{\alpha\beta\gamma} = 0$, for all indices $\alpha, \beta, \gamma$ corresponding to generators of $\mathcal{H}$. This is precisely the content of the first part of Theorem 3.
To show that a non-zero $G$-invariant tensor $d_{ABC}$, which vanishes on the subalgebra $\mathcal{H}$, produces a unique primitive generator of $H^5(G/H; R)$, we integrate the hierarchy equations of (5.2). The first step is to obtain the one form $\Omega_{BC}$ from (5.2); it is determined up to tensors $l_{BC:A} = l_{CB:A}$ as follows

$$\Omega_{BC} = (d_{ABC} + l_{BC:A})\theta^A$$

Applying the differential $D$ to the above expression for $\Omega_{BC}$, we find $i_B\Omega_C + i_C\Omega_B$, which determines $\Omega_A$ up to an anti-symmetric form $K$:

$$i_A\Omega_B = D\Omega_{AB} + K_{AB} \quad K_{AB} = -K_{BA} = \frac{1}{2}k_{AB;CD}\theta^C\theta^D$$

From $\Omega_A$, we have $i_A\Omega = D\Omega_A$, and from $i_A\Omega$, we obtain the general solution for $\Omega$.

$$\Omega = \Omega^{(d)} + \Omega^{(k)} + \Omega^{(l)}$$

The three contributions arise from the $d$ tensor and from the $k$ and $l$ forms respectively, and are given by

$$\Omega^{(d)} = \frac{1}{40} d_{A_1B_2C_3} f_{B_2A_3} f_{C_3A_4A_5} \theta^{A_1} \ldots \theta^{A_5}$$

$$\Omega^{(k)} = dQ^{(k)} \quad Q^{(k)} = \frac{1}{24} k_{A_1A_2A_3A_4} \theta^{A_1} \ldots \theta^{A_4}$$

$$\Omega^{(l)} = dQ^{(l)} \quad Q^{(l)} = \frac{1}{12} l_{A_1M_2} f_{M_2A_3A_4} \theta^{A_1} \ldots \theta^{A_4}$$

In the case where $H = 1$, there are no further conditions on the forms $\Omega^{(d)}$, $\Omega^{(k)}$, and $\Omega^{(l)}$, which are well-defined on $G$. The forms $\Omega^{(k)}$ and $\Omega^{(l)}$ are exact on $G$, while for $d_{ABC} \neq 0$, the form $\Omega^{(d)}$ produces a non-trivial generator of $H^5(G; R)$. For simple compact groups, this cohomology has a single generator for $SU(N)$, $N \geq 3$ and $SO(6) = SU(4)/Z_2$, while it is trivial for all other cases [21]. For products of groups, one makes use of the Künneth formula, as in [5].

A more familiar expression for the form $\Omega^{(d)}$ is obtained by casting $d_{ABC}$ in terms of a trace over representation matrices $T^j_A$ of the simple components $\mathcal{G}_j$ of $\mathcal{G}$: $d^j_{ABC} = \text{tr} T^j_AT^j_BT^j_C = \text{tr}_{\mathcal{G}_j} T_AT_BT_C$. Substituting into $\Omega^{(d)}$, and using the fact that the structure constants in (5.9) yield commutators, we obtain

$$\Omega^{(d)} = \sum_j \frac{1}{10} d^j_5 \text{tr}_{\mathcal{G}_j} (U^{-1}dU)^5$$

(5.10)
The forms $\text{tr}_{G_j}(U^{-1}dU)^5$ are the generators of $H^5(G; R)$ for each simple component and correspond to the standard WZW terms on $G_j$.

For more general groups $H$, it is still the case that each non-zero $d$-symbol (which vanishes on $\mathcal{H}$, according to the first part of Theorem 3) gives rise to a non-trivial cohomology generator in $H^5(G/H; R)$. To see this, we must add to $\Omega^{(d)}$ contributions of the form $\Omega^{(k)}$ and $\Omega^{(l)}$ to guarantee that the resulting form will properly vanish on $\mathcal{H}$, and can thus be projected down to $G/H$. The final result is a form $\tilde{\Omega}^{(d)}$ which vanishes on $\mathcal{H}$ and is given by

$$
\tilde{\Omega}^{(d)} = \frac{1}{40} \left\{ d_{a_1 b c} f_{b a_2 a_3} f_{c a_4 a_5} + 7d_{a_1 b \gamma} f_{b a_2 a_3} f_{\gamma a_4 a_5} + 16d_{a_1 b \gamma} f_{b a_2 a_3} f_{\gamma a_4 a_5} \right\} \theta^{a_1} \theta^{a_2} \theta^{a_3} \theta^{a_4} \theta^{a_5}
$$

Notice that $d_{\alpha \beta \gamma}$ does not enter, and that for $H = 1$, we recover $\Omega^{(d)}$ of (5.9).

An alternative procedure for obtaining the same result is already quite familiar and was used in [5]. In terms of the $\mathcal{H}$-valued gauge field $V$, the $\mathcal{H}$-covariant derivative $D_\mathcal{H}$ of (4.1), and the trace expression for the form $\Omega^{(d)}$ of (5.10), we obtain a form [9,22] that vanishes on $\mathcal{H}$

$$
\tilde{\Omega}^{(d)} = \sum_j \frac{1}{10} d^3 \text{tr}_{G_j} \left\{ (U^{-1}D_\mathcal{H}U)^5 - 5W(U^{-1}D_\mathcal{H}U)^3 + 10W^2(U^{-1}D_\mathcal{H}U) \right\}
$$

As is well-known [5,14], closure of this form is guaranteed by the fact that $d_{ABC}$ vanishes on $\mathcal{H}$. The generator obtained in this way cannot be decomposed into a sum of products of generators of lower degree that are well-defined on $G/H$, and thus $\tilde{\Omega}^{(d)}$ is primitive.

The last issue we must address is whether the remaining forms $\Omega^{(k)}$ and $\Omega^{(l)}$ – which are exact on $G$ – by themselves produce any non-trivial cohomology generators on $G/H$. Actually, inspection of (5.9) reveals that the forms $\Omega^{(l)}$ are a special case of the forms $\Omega^{(k)}$, so that we restrict to the latter. We now impose $\mathcal{H}$-invariance, vanishing on $\mathcal{H}$ and discard generators that are exact on $G/H$.

From (3.4), we know that, up to an exact form on $G$, the form $\Omega_B^{(k)}$ must vanish on $\mathcal{H}$. Furthermore, because $\Omega^{(k)}$ in (5.9) is exact on $G$, the form $\Omega_B^{(k)}$ equals $i_B Q^{(k)}$, up to an exact form on $G$. Combining both, the condition for the vanishing of $\Omega^{(k)}$ on $\mathcal{H}$ is that

$$
i_\alpha (i_B Q^{(k)} - DM_B) = 0
$$
Using (5.13) and $\mathcal{H}$-invariance of the tensor $k_{A_1 \ldots A_4}$, certain components of this tensor can be expressed in terms of the form $M_A = \frac{1}{2} m_{A;BC} \theta^B \theta^C$

$$k_{\alpha B;CD} = m_{B;E\alpha} \ f_{ECD} + m_{E;C\alpha} \ f_{EBD} + m_{E;D\alpha} \ f_{ECB}$$ (5.14)

Using (5.14), anti-symmetry of $k$ and $\mathcal{H}$-invariance of $m_{A;BC}$, it is easy to establish that $m$ must satisfy the following condition

$$(m_{A;\epsilon C} + m_{C;\epsilon A}) \ f_{\beta \delta \epsilon} = 0$$ (5.15)

for all indices $A, C$ and $\delta, \beta$. Finally, the form $Q^{(k)}$ is defined only up to an exact form on $G$. The addition of an arbitrary exact form $N = \frac{1}{6} n_{ABC} \theta^A \theta^B \theta^C$ to $Q^{(k)}$ is equivalent to shifting the form $M_B$ by $i_B N$, and amounts to shifting the tensor $m_{A;BC}$ by a totally anti-symmetric tensor $n_{ABC}$. This freedom to shift $m$ may be used to render it symmetric in its indices $A$ and $B$ for example.

Condition (5.15) implies that for all indices corresponding to generators of $\mathcal{H}$, we must have $k_{\alpha \beta;\gamma \delta} = 0$. Condition (5.14) and invariance of $k$ under the adjoint action of $\mathcal{H}$ furthermore imply that the form $\Omega^{(k)}$ of (5.9) is given by

$$\Omega^{(k)} = dQ^{(k)} \quad Q^{(k)} = \frac{1}{6} k_{\alpha b;cd} \theta^a \theta^b \theta^c \theta^d$$ (5.16)

Here, the indices $\alpha$ only run over generators of $\mathcal{H}$ that commute with all of $\mathcal{H}$. These generators are in general non-trivial, because $\theta^\alpha$ is not a well-defined differential on $G/H$. Only its exterior derivative $d\theta^\alpha = -\frac{1}{2} f_{\alpha bc} \theta^b \theta^c$ is well-defined, and is a closed 2 form on $G/H$ as shown in (4.4). The remaining form of degree 3 given by $\frac{1}{6} k_{\alpha bc;\delta} \theta^\alpha \theta^b \theta^c \theta^\delta$ is closed on $G/H$ in view of (5.14) and corresponds to a non-trivial cohomology generator of $H^3(G/H;R)$. Thus, the contributions to $\Omega^{(k)}$ are linear combinations of forms that factorize into products of cohomology generators of degrees 2 and 3. This concludes the proof of Theorem 3, and the complete description of cohomology of degree 5.

6. Cohomology of non-simply Connected $G/H$

We now generalize our results to the case where $G/H$ is not necessarily simply-connected. This may occur because $G$ is not simply connected or because $H$ is not connected. $H^1(G/H;R)$ may then contain non-trivial generators, which are closed forms of
degree 1, invariant under the adjoint action of $\mathcal{H}$. Let us denote a set of independent generators by $\theta^{s_i}$ with $i = 1, \cdots, t = \dim H^1(G/H; R)$. The analysis of the cohomology of degree $n$ proceeds directly from Theorem 1. Let $\Omega$ be a closed $\mathcal{H}$-invariant form on $G/H$.

$$\Omega = \frac{1}{n!} \omega_{a_1 \cdots a_n} \theta^{a_1} \cdots \theta^{a_n} \quad (6.1)$$

Since the generators $\theta^{s_i}$ are $\mathcal{H}$-invariant by themselves, $\Omega$ may be written as a linear combination of forms of degree $n$ that involve $k$ factors of generators $\theta^{s_i}$ with $k \leq t$

$$\Omega = \Omega^{(n)} + \sum_{k=1}^{t} \theta^{s_{i_1}} \cdots \theta^{s_{i_k}} \Omega^{(n-k)}_{s_{i_1} \cdots s_{i_k}} \quad (6.2)$$

Each of the forms $\Omega^{(n-k)}_{s_{i_1} \cdots s_{i_k}}$ is closed, invariant under the adjoint action of $\mathcal{H}$ and well-defined on $G/H$. As a result, each $\Omega^{(n-k)}$ is just a generator of the cohomology group $H^{n-k}(G/H; R)$ for $k \geq 1$, which does not involve any generators of the first cohomology group. Thus, the analysis of the forms $\Omega^{(n-k)}$ is just that of the cohomology for which no generators of degree 1 occur and has already been carried out previously.

To summarize, when $H^1(G/H; R) \neq 0$, the cohomology involving those generators is formed out of the product of cohomology generators of lower degree with products of cohomology generators of degree 1.

7. Gauging Invariant Effective Actions

So far we have dealt with effective actions for Goldstone boson fields only. To couple the Goldstone fields to gauge fields, we follow the procedure of minimal coupling, and gauge the global symmetry group $G$. Manifestly invariant actions, corresponding to $G$-invariant Lagrangian densities, are obtained by the construction of [2]. The Lagrangian density is a local function of derivatives of the Goldstone field which are invariant under global $G$-transformations. Upon gauging $G$, these derivatives are replaced by $G$-covariant derivatives, and the new Lagrangian density obtained in this way is invariant under $G$-valued gauge transformations [2]. (An alternative method for obtaining general effective actions has been advanced in [23], where the nature of locally gauge invariant effective actions is investigated. This work is also related to an approach that starts from equivariant cohomology [24].)
It remains to couple to $G$-valued gauge fields the invariant actions that do not correspond to invariant Lagrangian densities. As we have seen for the case of $SU(N)$, $N \geq 3$ in [5], and proven in this paper, these invariant actions are essentially of the gauged Wess-Zumino-Witten type. It has been known since the time of their construction [3] that WZW actions cannot, in general, be made invariant under the complete gauge group $G$. Instead their gauge variation reproduces the chiral anomaly [13,14]; in fact, the WZW term was conceived as a generating functional for these chiral anomalies [3,4,9].

From our point of view, it is the invariant actions associated with non-invariant Lagrangian densities that are of central importance. Following [4], we wish to produce an action that couples the Goldstone fields to $G$-valued gauge fields, though we foresee the fact that it will not be possible to obtain an action invariant under all of the $G$-valued gauge transformations. Instead, we shall show that it is always possible to construct an action whose variation under gauge transformations only involves the gauge field and not the Goldstone fields.

Remarkably, we also show that invariant actions associated with non-invariant Lagrangian densities – appearing for non-trivial de Rham cohomology – are equivalently governed by the appearance of anomalies. We show that the fifth de Rham cohomology of $G/H$ is non-zero only if the group $G$ has anomalies in at least some of its representations. Furthermore, while the full group $G$ cannot be gauged, the subgroup $H$ always can be completely gauged.

*Gauging a general element of $H(G/H; R)$.*

The forms $\theta^A$ of (2.2) are minimally coupled to a $G$-valued gauge field $A = dx^\mu A_\mu$ and promoted to forms $\tilde{\theta}^A$ which are invariant under $G$-valued gauge transformations acting to the left on $U$.

$$\tilde{\theta} = U^{-1}(d + A)U = \tilde{\theta}^A T^A$$

(7.1)

The 1-forms $\tilde{\theta}$ no longer obey the Maurer-Cartan equations; instead, the connection $\tilde{\theta}$ has curvature, which is related to the curvature of the $G$-valued gauge field $A$.

$$dA^A + \frac{1}{2} f^{ABC} A^B A^C = F^A$$

$$d\tilde{\theta}^A + \frac{1}{2} f^{ABC} \tilde{\theta}^B \tilde{\theta}^C = \tilde{F}^A$$

$$\tilde{F}^A T^A = U^{-1} F^A T^A U$$

(7.2)
The covariant derivative with respect to $\bar{\theta}$ is denoted by $\bar{D}$, and $\bar{F}$ obeys the Bianchi identity $\bar{D}\bar{F} = 0$. The inner product with $\bar{\theta}^A$ is still denoted by $i_A$. To every differential form $\Omega$, we associate a form $\bar{\Omega}$ as follows

$$\Omega_{B_1 \cdots B_m} = \frac{1}{n!} \Omega_{B_1 \cdots B_m; A_1 \cdots A_n} \theta^{A_1} \cdots \theta^{A_n}$$

$$\bar{\Omega}_{B_1 \cdots B_m} = \frac{1}{n!} \Omega_{B_1 \cdots B_m; A_1 \cdots A_n} \bar{\theta}^{A_1} \cdots \bar{\theta}^{A_n}$$

There is a general relation between the covariant derivatives of both forms, given by

$$\bar{D}\Omega_{B_1 \cdots B_m} = \Sigma_{B_1 \cdots B_m}$$

$$\bar{D}\bar{\Omega}_{B_1 \cdots B_m} = \bar{\Sigma}_{B_1 \cdots B_m} + \bar{F}^A i_A \bar{\Omega}_{B_1 \cdots B_m}$$

Making use of this result, we obtain the gauged form of the hierarchy of equations of (3.12), which were the basis for the analysis of cohomology. We start with a closed (globally) $G$-invariant form $\Omega$ of rank 0 and degree $n$. According to the notation of (7.3), we associate a new form $\bar{\Omega}$. This form is no longer closed in general, and we have the following hierarchy of equations

$$d\bar{\Omega} = \bar{F}^A i_A \bar{\Omega}$$

$$\bar{D}(i_B \bar{\Omega}) = \bar{F}^A i_A i_B \bar{\Omega} \quad \Rightarrow \quad i_B \bar{\Omega} = \bar{D} \bar{\Omega}_B - \bar{F}^A i_A \bar{\Omega}_B$$

$$\bar{D}(i_{\{B_2 \bar{\Omega}_{B_1}\}}) = \bar{F}^A i_A i_{\{B_1 \bar{\Omega}_{B_2}\}} \quad \Rightarrow \quad i_{\{B_2 \bar{\Omega}_{B_1}\}} = \bar{D} \bar{\Omega}_{B_1 B_2} - \bar{F}^A i_A \bar{\Omega}_{B_1 B_2}$$

$$\bar{D}(i_{\{B_3 \bar{\Omega}_{B_1 B_2}\}}) = \bar{F}^A i_A i_{\{B_1 \bar{\Omega}_{B_2 B_3}\}} \quad \Rightarrow \quad i_{\{B_3 \bar{\Omega}_{B_1 B_2}\}} = \bar{D} \bar{\Omega}_{B_1 B_2 B_3} - \bar{F}^A i_A \bar{\Omega}_{B_1 B_2 B_3}$$

$$\cdots$$

(7.5)

Here it is understood that a $G$-invariant tensor $d_{B_1 \cdots B_m}$ is added on the right hand side of the second set of equations whenever the corresponding degree vanishes.

To obtain a $G$-gauge invariant form out of $\bar{\theta}$ and $\bar{F}$, we first construct a form whose exterior derivative is independent of the Goldstone field and only depends upon the gauge field. To do so, we introduce a sequence of forms of rank 0 and degree $n$ as follows

$$\bar{\Omega}_{(m)} = \bar{\Omega} + \sum_{k=1}^{m} (-1)^k \bar{F}^{B_1} \cdots \bar{F}^{B_k} \bar{\Omega}_{B_1 \cdots B_k}$$

(7.6)

The exterior derivatives of these forms are given by

$$d\bar{\Omega}_{(m)} = (-1)^m \bar{F}^{B_1} \cdots \bar{F}^{B_{m+1}} i_{\{B_{m+1} \bar{\Omega}_{B_1 \cdots B_m}\}}$$

(7.7)
Of special interest is the case where $\Omega$ and $\bar{\Omega}$ are of degree 5, so that the sequence of $\Omega_m$ terminates at $m = 2$. The form $i_A \bar{\Omega}_{BC}$ is then of degree 0 and we have shown in §3 that it must be constructed out of the $d$-symbols of $\mathcal{G}$. In view of $G$-invariance of the $d$-symbol and the expression for $\bar{F}^A$ in terms of $F^A$ given by (7.2), we find that

$$d\bar{\Omega}_{(2)} = d^{ABC} \bar{F}^A \bar{F}^B \bar{F}^C$$

Thus, the right hand side is a function only of the gauge field $A$.

Regarding the form $\hat{\Omega}_{(2)}$ as a function of the Goldstone field $U$ and of the gauge field $A$, we can easily obtain a closed form by making use of the fact that the right hand side of (7.8) no longer depends upon $U$ as in [9,14,22].

$$\hat{\Omega}(U, A) = \bar{\Omega}_{(2)}(U, A) - \bar{\Omega}_{(2)}(1, A) \quad \quad d\hat{\Omega}(U, A) = 0 \quad (7.9)$$

The form $\hat{\Omega}$ is no longer gauge invariant, and its gauge variation precisely coincides with the chiral anomaly, which can be seen as follows. Since the form $\bar{\Omega}_{(2)}(U, A)$ is gauge invariant, the gauge variation $\delta \hat{\Omega}$ of $\hat{\Omega}$ is given by $\delta \hat{\Omega}(U, A) = -\delta \bar{\Omega}_{(2)}(1, A)$ which may be evaluated using the explicit expression for $\bar{\Omega}_{(2)}$ in (7.6) for $U = 1$. A $\mathcal{G}$-valued gauge transformation acts by $\delta A^B = \delta \theta^B = -\bar{D}\epsilon^B$, and $\delta \bar{F}^B = \bar{D}\delta \theta^B = -\bar{D}^2 \epsilon^B$, where $\epsilon^B$ is an infinitesimal $\mathcal{G}$-valued function of degree 0. With the help of the hierarchy (7.5), we find

$$\delta \hat{\Omega}(U, A) = d\left\{ \epsilon^A \left( i_A \bar{\Omega} - \bar{F}^B i_A \bar{\Omega}_B + \bar{F}^B \bar{F}^C i_A \bar{\Omega}_{BC} \right) \right\} \quad (7.10)$$

As a result, the gauge variation of the gauged effective action is given by a four dimensional integral, which coincides with the chiral anomaly: each term in (7.10) is proportional to some component of $d_{ABC}$. The definition of $\hat{\Omega}(U, A)$ is not unique and can be modified by adding the exterior derivative of any function of $A$ only [14].

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