Numerical radius inequalities of sectorial matrices

Pintu Bhunia1 · Kallol Paul2 · Anirban Sen2

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Abstract
We obtain several upper and lower bounds for the numerical radius of sectorial matrices. We also develop several numerical radius inequalities of the sum, product and commutator of sectorial matrices. The inequalities obtained here are sharper than the existing related inequalities for general matrices. Among many other results we prove that if $A$ is an $n \times n$ complex matrix with the numerical range $W(A)$ satisfying $W(A) \subseteq \{r e^{\pm i \theta} : \theta_1 \leq \theta \leq \theta_2\}$, where $r > 0$ and $\theta_1, \theta_2 \in [0, \pi/2]$, then

(i) $w(A) \geq \frac{\csc \gamma}{2} \|A\| + \frac{\csc \gamma}{2} \|\Re(A)\| - \|\Im(A)\|$, and

(ii) $w^2(A) \geq \frac{\csc^2 \gamma}{4} \|AA^* + A^*A\| + \frac{\csc^2 \gamma}{2} \left(\|\Im(A)\|^2 - \|\Re(A)\|^2\right)$,

where $\gamma = \max\{\theta_2, \pi/2 - \theta_1\}$. We also prove that if $A, B$ are sectorial matrices with sectorial index $\gamma \in [0, \pi/2)$ and they are double commuting, then $w(AB) \leq (1 + \sin^2 \gamma) w(A)w(B)$.

Keywords Numerical radius · Numerical range · Accretive matrix · Sectorial matrix

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✉ Kallol Paul
kallol.paul@jadavpuruniversity.in; kallooldada@gmail.com
Pintu Bhunia
pintubhunia5206@gmail.com
Anirban Sen
anirbansenfulia@gmail.com

1 Department of Mathematics, Indian Institute of Science, Bengaluru, Karnataka 560012, India
2 Department of Mathematics, Jadavpur University, Kolkata, West Bengal 700032, India
1 Introduction

Let $\mathcal{M}_n$ denote the algebra of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$, the operator norm of $A$, denoted by $\|A\|$, is defined as

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}.$$  

The numerical range and the numerical radius of $A$, denoted by $W(A)$ and $w(A)$ respectively, are defined as

$$W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

and

$$w(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.$$  

It is well known that the numerical radius $w(\cdot)$ defines a norm on $\mathcal{M}_n$, and $w(\cdot)$ is equivalent to the operator norm $\| \cdot \|$. In fact, for each $A \in \mathcal{M}_n$,

$$\frac{1}{2} \|A\| \leq w(A) \leq \|A\|. \quad (1.1)$$

Note that $\frac{1}{2} \|A\| = w(A)$ if $A^2 = 0$ (see [18]), and $w(A) = \|A\|$ if $AA^* = A^*A$ (see [15]), where $A^*$ denotes the adjoint of $A$. Computation of the exact value of the numerical radius $w(A)$ for an arbitrary matrix $A$ is not an easy task, except for some special class of matrices. Therefore, for arbitrary matrices $A \in \mathcal{M}_n$, the researchers tried to develop upper and lower bounds of $w(A)$, which are sharper than the bounds in (1.1). For recent developments of upper and lower bounds of the numerical radius we refer the readers to see [2, 5, 7–11, 18, 23] and the references therein. In [1, 3], the authors studied the numerical radius inequalities of a particular class of matrices, known as sectorial matrices. They developed several sharper upper and lower bounds of the numerical radius of sectorial matrices. Let us first mention the definition of a sectorial matrix. A matrix $A \in \mathcal{M}_n$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$. The positive matrix $A$ is said to be positive semi-definite (positive definite) if $\langle Ax, x \rangle \geq 0$ for all non-zero $x \in \mathbb{C}^n$ ($\langle Ax, x \rangle > 0$ for all non-zero $x \in \mathbb{C}^n$), and it is denoted by $A \geq 0$ ($A > 0$). The Cartesian decomposition of $A \in \mathcal{M}_n$ is $A = \Re(A) + i \Im(A)$, where $\Re(A) = \frac{A + A^*}{2}$ and $\Im(A) = \frac{A - A^*}{2i}$. A matrix $A \in \mathcal{M}_n$ is said to be accretive if $\Re(A) > 0$. Clearly, all positive definite matrices are accretive. A matrix $A \in \mathcal{M}_n$ is said to be accretive-dissipative if $\Re(A) > 0$ and $\Im(A) > 0$. For a complex number $z$, let $\Re z$ and $\Im z$ denote the real and imaginary part of the complex number $z$. Geometrically, a matrix $A \in \mathcal{M}_n$ is accretive if and only if $W(A) \subseteq \{z \in \mathbb{C} : \Re z > 0\}$.

**Definition 1.1** A matrix $A \in \mathcal{M}_n$ is said to be sectorial if $W(A) \subseteq S_\gamma$ for some $\gamma \in [0, \frac{\pi}{2})$, where

$$S_\gamma = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\tan \gamma)\Re z\}.$$
Geometrically, the numerical range of a sectorial matrix lies entirely in a cone in the right half complex plane (i.e., \( \{ z \in \mathbb{C} : \Re z > 0 \} \)), with vertex at the origin and half-angle \( \gamma \). Clearly, a positive definite matrix is a sectorial matrix with \( \gamma = 0 \). Henceforth, for simplicity, we write \( A \in \Pi_\gamma \) when \( W(A) \subseteq S_\gamma \) for some \( \gamma \in [0, \pi/2) \).

Note that for a sectorial matrix, 0 does not belong to the numerical range. On the other hand if 0 does not belong to the numerical range then there exists \( \theta \) such that \( e^{i\theta} A \) is a sectorial matrix. Such a matrix is known as rotational sectorial matrix. Next we observe that any sectorial matrix is an accretive matrix. Moreover, the converse is also true, i.e., if \( A \in \mathcal{M}_n \) is an accretive matrix, then \( A \in \Pi_\gamma \) for some \( \gamma \in [0, \pi/2) \). For \( A \in \Pi_\gamma \), the following inequality

\[
\cos \gamma \| A \| \leq w(A)
\]  

holds (see [3, Prop. 3.1]). We would like to note that the inequality (1.2) gives better bound than the first bound in (1.1), when \( \gamma \in [0, \pi/3) \).

In this article we present several numerical radius inequalities of the sectorial matrices, which improve on the existing inequalities. Further, we develop numerical radius inequalities of the commutator and anti-commutator of matrices in \( \mathcal{M}_n \).

Before we end this section we introduce the fractional powers of matrices via Dunford-Taylor integral. Let \( f : \mathcal{D} \to \mathbb{C} \) be an analytic function defined on a domain.
\(D\). Let \(A \in \mathcal{M}_n\) and let the spectrum of \(A\), \(\sigma(A)\) (i.e., the eigenvalues of \(A\)), lies in \(D\). Then by using Dunford–Taylor integral, \(f(A)\) can be written as

\[
f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz,
\]

(1.3)

where \(\Gamma \subset D\) be a simple closed smooth curve with positive direction enclosing all eigenvalues of \(A\) in its interior (see [17, p. 44] and [24, p. 287]). If \(A\) is an accretive matrix then \(A\) has no eigenvalues in \((-\infty, 0]\) and so

\[
A^t = \frac{1}{2\pi i} \int_{\Gamma} z^t(zI - A)^{-1} dz,
\]

(1.4)

where \(t \in (0, 1)\) and \(z^t\) is the principal branch of the multi-valued function \(z \to z^t\), and \(\Gamma\) is a suitable curve. Thus for \(0 < t < 1\), \(A^t\) is sectorial whenever \(A\) is sectorial. Note that for \(t > 1\), it is not guaranteed that \(A^t\) is sectorial when \(A\) is sectorial. Moreover (see [12]), if \(A \in \prod_{\gamma}\), then

\[
A^t \in \prod_{\pi_{t\gamma}}, \quad 0 < t < 1.
\]

(1.5)

For further related discussion we refer to [4, 20].

2 Main results

We begin with the following lemma (see [1]), proof follows from the Cartesian decomposition of \(A \in \Pi_{\gamma}\) and the inequality

\[
|\langle \Re(A)x, x \rangle| \leq (\tan \gamma)|\langle \Im(A)x, x \rangle| \quad \text{for all} \quad x \in \mathbb{C}^n \quad \text{with} \quad \|x\| = 1.
\]

Lemma 2.1 [1] Let \(A \in \Pi_{\gamma}\). Then

\[
\|\Im(A)\| \leq (\sin \gamma)w(A).
\]

Now, we are in a position to present our first inequality.

Theorem 2.2 Let \(A \in \Pi_{\gamma}\) with \(\gamma \neq 0\). Then

\[
w^2(A) \geq \frac{csc^2 \gamma}{4} \|AA^* + A^*A\| + \frac{csc^2 \gamma}{2} \left(\|\Im(A)\|^2 - \|\Re(A)\|^2\right).
\]

(2.1)

Proof Let \(x \in \mathbb{C}^n\) be such that \(\|x\| = 1\). From the Cartesian decomposition of \(A\), we have,

\[
|\langle Ax, x \rangle|^2 = (\Re(A)x, x)^2 + \langle \Im(A)x, x \rangle^2.
\]

This gives that

\[
w^2(A) \geq \|\Re(A)\|^2.
\]

(2.2)
Also, Lemma 2.1 gives that

\[ w^2(A) \geq \csc^2 \gamma \| \Im(A) \|^2. \]  

(2.3)

Therefore, from the inequalities (2.2) and (2.3) we have,

\[
\begin{align*}
   w^2(A) & \geq \max \{\| \Re(A) \|^2, \csc^2 \gamma \| \Im(A) \|^2 \} \\
   & = \frac{\| \Re(A) \|^2 + \csc^2 \gamma \| \Im(A) \|^2}{2} + \frac{\| \Re(A) \|^2 - \csc^2 \gamma \| \Im(A) \|^2}{2} \\
   & = \frac{\csc^2 \gamma}{2} \left( \| \Re^2(A) + \Im^2(A) \| + \frac{1}{2} \| \Re(A) \|^2 \right) + \frac{\| \Re(A) \|^2 - \csc^2 \gamma \| \Im(A) \|^2}{2} \\
   & \geq \frac{\csc^2 \gamma}{4} \| AA^* + A^*A \| - \frac{\csc^2 \gamma}{2} \| \Re(A) \|^2 \\
   & + \frac{\| \Re(A) \|^2 - \| \Re^2(A) + \Im^2(A) \|}{2} \\
   & = \frac{\csc^2 \gamma}{4} \| AA^* + A^*A \| + \frac{\csc^2 \gamma}{2} \left( \| \Im(A) \|^2 - \| \Re(A) \|^2 \right) \\
   & \geq \frac{\csc^2 \gamma}{4} \| AA^* + A^*A \|. \\
\end{align*}
\]

This completes the proof. \[\square\]

**Remark 2.3**  
(i) Let \( A \in \Pi_\gamma \) (\( \gamma \neq 0 \)). If \( \| \Im(A) \| \geq \| \Re(A) \| \) then from Theorem 2.2 we get

\[
w^2(A) \geq \frac{\csc^2 \gamma}{4} \| AA^* + A^*A \| + \frac{\csc^2 \gamma}{2} \left( \| \Im(A) \|^2 - \| \Re(A) \|^2 \right) \\
\geq \frac{1}{4} \| AA^* + A^*A \| + \frac{1}{2} \left( \| \Im(A) \|^2 - \| \Re(A) \|^2 \right) \quad (\csc \gamma > 1 \forall \gamma \in [0, \pi/2])
\]

Thus in this case Theorem 2.2 gives sharper bound than the well known bound [19, Th. 1], namely,

\[(2.4)\]

\[ w^2(A) \geq \frac{1}{4} \| A^*A + AA^* \|. \]
(ii) Let $A \in \Pi_\gamma$ $(\gamma \neq 0)$ with $\|\Im(A)\| < \|\Re(A)\|$. Then Theorem 2.2 gives sharper bound than the bound (2.4) if $\sin \gamma < \sqrt{1 - \|\Re(A)\|^2 - \|\Im(A)\|^2 /\|\Re(A)\|^2 + \|\Im(A)\|^2}$. For example, let $A = \begin{pmatrix} 3 + 2i & 0 \\ 0 & 1 \end{pmatrix}$. Then, $A \in \Pi_\gamma$ where $\sin \gamma = 2/\sqrt{13} < \sqrt{1 - \|\Re(A)\|^2 - \|\Im(A)\|^2 /\|\Re(A)\|^2 + \|\Im(A)\|^2} = 2\sqrt{2}/\sqrt{13}$. Theorem 2.2 gives $w^2(A) \geq 13$ whereas (2.4) gives $w^2(A) \geq 13/2$.

(iii) For any $A \in M_n$, it was proved in [6, Th. 2.9] that

$$w^2(A) \geq \frac{1}{4} \|A^*A + AA^*\| + \frac{1}{2} \|\Re(A)\|^2 - \|\Im(A)\|^2. \tag{2.5}$$

We would like to remark that when $A \in \Pi_\gamma$ with $\|\Im(A)\| \geq \|\Re(A)\|$, Theorem 2.2 gives sharper bound than the bound (2.5).

Now, by applying Theorem 2.2 we develop upper bounds for the numerical radius of generalized commutator and anti-commutator matrices. Recall that for $A, B \in M_n$, the matrix $AB - BA$ is called commutator matrix and $AB + BA$ is called anti-commutator matrix.

**Theorem 2.4** Let $A, B, X, Y \in M_n$ with $A \in \Pi_\gamma (\gamma \neq 0)$. Then

$$w(AXB \pm BYA) \leq 2\sqrt{2}(\sin \gamma)\|B\| \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{\csc^2\gamma}{2} \left(\|\Im(A)\|^2 - \|\Re(A)\|^2\right)}. \tag{2.6}$$

**Proof** Let $x \in \mathbb{C}^n$ with $\|x\| = 1$. Suppose $\|X\| \leq 1$ and $\|Y\| \leq 1$. Then, by Cauchy–Schwarz inequality we have,

$$|\langle (AX \pm YA)x, x \rangle| \leq |\langle AXx, x \rangle| + |\langle YAx, x \rangle|$$

$$= |\langle Xx, A^*x \rangle| + |\langle Ax, Y^*x \rangle| \leq \|A^*x\| + \|Ax\| \leq \sqrt{2(\|A^*x\|^2 + \|Ax\|^2)} \quad \text{(by convexity of } f(x) = x^2)$$

$$\leq \sqrt{2\|AA^* + A^*A\|}. \tag{2.6}$$

Now, it follows from Theorem 2.2 that

$$\|AA^* + A^*A\| \leq 4\sin^2 \gamma \left(w^2(A) - \frac{\csc^2\gamma}{2} \left(\|\Im(A)\|^2 - \|\Re(A)\|^2\right)\right). \tag{2.7}$$

Therefore, by using the inequality (2.7) in (2.6) we have,

$$|\langle (AX \pm YA)x, x \rangle| \leq 2\sqrt{2}(\sin \gamma) \sqrt{w^2(A) - \frac{\csc^2\gamma}{2} \left(\|\Im(A)\|^2 - \|\Re(A)\|^2\right)}. \tag{2.7}$$
Taking supremum over \( x \in \mathbb{C}^n, \|x\| = 1 \) we have,

\[
\|x\| = 1 \quad \text{have}, \quad w(AX \pm YA) \leq 2\sqrt{2}(\sin \gamma) \sqrt{w^2(A) - \frac{csc^2\gamma}{2} \left( \|\Im(A)\|^2 - \|\Re(A)\|^2 \right)}. \tag{2.8}
\]

Now, Theorem 2.4 holds trivially when \( X = Y = 0 \). Let \( \max\{\|X\|, \|Y\|\} \neq 0 \). Then \( \|X\|_{\text{max}} \leq 1 \) and \( \|Y\|_{\text{max}} \leq 1 \). Therefore, from (2.8) we have,

\[
w(AX \pm YA) \leq 2\sqrt{2}(\sin \gamma) \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{csc^2\gamma}{2} \left( \|\Im(A)\|^2 - \|\Re(A)\|^2 \right)}.
\]

Replacing \( X \) and \( Y \) by \( XB \) and \( BY \), respectively, we get,

\[
w(AXB \pm BYA) \leq 2\sqrt{2}(\sin \gamma) \max\{\|XB\|, \|BY\|\} \sqrt{w^2(A) - \frac{csc^2\gamma}{2} \left( \|\Im(A)\|^2 - \|\Re(A)\|^2 \right)}.
\]

This completes the proof. \( \square \)

The following corollary follows from Theorem 2.4 by taking \( X = Y = I \).

**Corollary 2.5** Let \( A, B \in \mathcal{M}_n \) with \( A \in \prod_{\gamma} (\gamma \neq 0) \). Then

\[
w(AB \pm BA) \leq 2\sqrt{2}(\sin \gamma) \|B\| \sqrt{w^2(A) - \frac{csc^2\gamma}{2} \left( \|\Im(A)\|^2 - \|\Re(A)\|^2 \right)}.
\]

**Remark 2.6** (i) Let \( A, B \in \mathcal{M}_n \) with \( A \in \prod_{\gamma} (\gamma \neq 0) \) and \( \|\Im(A)\| \geq \|\Re(A)\| \). Then from Corollary 2.5 we have,

\[
w(AB \pm BA) \leq 2\sqrt{2}(\sin \gamma) \|B\| \sqrt{w^2(A) - \frac{csc^2\gamma}{2} \left( \|\Im(A)\|^2 - \|\Re(A)\|^2 \right)}
\]

\[
\leq 2\sqrt{2}\|B\| \left\|w(A) \right\| - \frac{1}{2} \left\|w(A) \right\|^2 - \|\Re(A)\|^2
\]

\[
< 2\sqrt{2}\|B\| w(A),
\]

\( \square \)
as \( \sin \gamma < 1 \) for all \( \gamma \in (0, \pi/2) \). In this case, Corollary 2.5 provides sharper bound than the well known bound \([13, \text{Th. 11}]\), namely,

\[
 w(AB \pm BA) \leq 2\sqrt{2} \|B\| w(A). \tag{2.9}
\]

(ii) Let \( A \in \prod_\gamma (\gamma \neq 0)\) with \( \|\Im(A)\| < \|\Re(A)\| \). Then Corollary 2.5 gives better bound than the bound (2.9) when \( \cos^2 \gamma > \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2w^2(A)} \).

(iii) For any \( A, B \in \mathcal{M}_n \), it was proved in \([16, \text{Cor. 3.4}]\) that

\[
 w(AB \pm BA) \leq 2\sqrt{2} \|B\| \sqrt{w^2(A)} - \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2}. \tag{2.10}
\]

We would like to remark that Corollary 2.5 gives sharper bound than the bound (2.10) when \( A \in \prod_\gamma \) with \( \|\Im(A)\| \geq \|\Re(A)\| \).

On the basis of Corollary 2.5 we obtain the following inequality.

**Corollary 2.7** Let \( A, B \in \prod_\gamma (\gamma \neq 0) \). Then

\[
 w(AB \pm BA) \leq \min\{\beta_1, \beta_2\}, \tag{2.11}
\]

where

\[
 \beta_1 = 2\sqrt{2}(\sin \gamma) \|B\| \sqrt{w^2(A)} - \frac{\csc^2 \gamma}{2} \left(\|\Im(A)\|^2 - \|\Re(A)\|^2\right)
\]

and

\[
 \beta_2 = 2\sqrt{2}(\sin \gamma) \|A\| \sqrt{w^2(B)} - \frac{\csc^2 \gamma}{2} \left(\|\Im(B)\|^2 - \|\Re(B)\|^2\right).
\]

**Proof** The proof follows from Corollary 2.5 by interchanging \( A \) and \( B \). \( \square \)

Next, we need the following proposition (which gives a norm inequality for the sum of \( n \) matrices), the proof follows from \([1, \text{Lemma 3.2}]\) by using induction.

**Proposition 2.8** Let \( A_i \in \mathcal{M}_n \) with no eigenvalues in \((-\infty, 0]\) for \( i = 1, \ldots, n \). Then

\[
 \left\| \sum_{i=1}^n A_i \right\|^{1/2} \leq \sum_{i=1}^n \|A_i^{1/2}\|.
\]

Now, by using Proposition 2.8 we obtain the following corollary, and for this first we note that the following lemma which follows from the norm inequalities \( \|A\|^2 \leq \|\Re(A)\|^2 + 2\|\Im(A)\|^2 \) when \( A \in \mathcal{M}_n \) is accretive and \( \|A\|^2 \leq \|\Re(A)\|^2 + \|\Im(A)\|^2 \) when \( A \in \mathcal{M}_n \) is accretive-dissipative (see \([14, 21])\).
Lemma 2.9 [1] Let $A \in \Pi_{\gamma}$. Then

$$\|A\| \leq \sqrt{1 + 2 \sin^2 \gamma w(A)}.$$

Moreover, if $A \in \Pi_{\gamma}$ is accretive-dissipative, then

$$\|A\| \leq \sqrt{1 + \sin^2 \gamma w(A)}.$$

Corollary 2.10 Let $A_i \in \Pi_{\gamma}$ for $i = 1, \ldots, n$. Then

$$w^{1/2} \left( \sum_{i=1}^{n} A_i \right) \leq \sqrt{1 + 2 \sin^2 \gamma / 2 \sum_{i=1}^{n} w(A_i^{1/2})}.$$

Moreover, if $A_i$ is accretive-dissipative for $i = 1, \ldots, n$, then

$$w^{1/2} \left( \sum_{i=1}^{n} A_i \right) \leq \sqrt{1 + \sin^2 \gamma / 2 \sum_{i=1}^{n} w(A_i^{1/2})}.$$

Proof Using Proposition 2.8, Lemma 2.9 and (1.5) we have,

$$w^{1/2} \left( \sum_{i=1}^{n} A_i \right) \leq \left\| \sum_{i=1}^{n} A_i \right\|^{1/2} \leq \sum_{i=1}^{n} \|A_i^{1/2}\| \leq \sqrt{1 + 2 \sin^2 \gamma / 2 \sum_{i=1}^{n} w(A_i^{1/2})}.$$

Similarly, for accretive-dissipative matrices $A_i$ for $i = 1, \ldots, n$, we have,

$$w^{1/2} \left( \sum_{i=1}^{n} A_i \right) \leq \left\| \sum_{i=1}^{n} A_i \right\|^{1/2} \leq \sum_{i=1}^{n} \|A_i^{1/2}\| \leq \sqrt{1 + \sin^2 \gamma / 2 \sum_{i=1}^{n} w(A_i^{1/2})}.$$

In particular, for $n = 1$ in Corollary 2.10, we get the following result, which is also given in [1, Corollary 3.6].

Corollary 2.11 Let $A \in \Pi_{\gamma}$ Then

$$w^{1/2}(A) \leq \sqrt{1 + 2 \sin^2 \gamma / 2 w(A^{1/2})}.$$

Further, if $A$ is accretive–dissipative, then

$$w^{1/2}(A) \leq \sqrt{1 + \sin^2 \gamma / 2 w(A^{1/2})}.$$

Here we note that Corollary 2.10 also follows directly from [1, Corollary 3.6] by induction. From the above corollary we prove the following power inequality.

\[\Box\]
Theorem 2.12 Let $A \in \prod_{\gamma}$. Then

$$w^{1/2^n}(A) \leq \prod_{i=1}^{n} \left(1 + 2 \sin^2 \gamma/2^i\right)^{\frac{1}{2n+1-i}} w(A^{1/2^n}),$$

(2.12)

for all $n \in \mathbb{N}$. Further, if $A$ is accretive–dissipative, then

$$w^{1/2^n}(A) \leq \prod_{i=1}^{n} \left(1 + \sin^2 \gamma/2^i\right)^{\frac{1}{2n+1-i}} w(A^{1/2^n}),$$

(2.13)

for all $n \in \mathbb{N}$.

Proof We prove the inequality (2.12) by mathematical induction. It is clear form Corollary 2.11 that the inequality (2.12) holds for $n = 1$. Now, we assume that the inequality (2.12) holds for $n = k$ ($k \geq 1$), i.e.,

$$w^{1/2^k}(A) \leq \prod_{i=1}^{k} \left(1 + 2 \sin^2 \gamma/2^i\right)^{\frac{1}{2k+1-i}} w(A^{1/2^k}).$$

(2.14)

By (1.5) we have, $A^{1/2^k} \in \prod_{\gamma/2^k}$. So, it follows from Corollary 2.11 that

$$w(A^{1/2^k}) \leq (1 + 2 \sin^2 \gamma/2^{k+1}) w^2(A^{1/2^{k+1}}).$$

(2.15)

From (2.14) and (2.15), we have

$$w^{1/2^k}(A) \leq \prod_{i=1}^{k} \left(1 + 2 \sin^2 \gamma/2^i\right)^{\frac{1}{2k+1-i}} (1 + 2 \sin^2 \gamma/2^{k+1}) w^2(A^{1/2^{k+1}})$$

$$= \prod_{i=1}^{k+1} \left(1 + 2 \sin^2 \gamma/2^i\right)^{\frac{1}{2k+1-i}} w^2(A^{1/2^{k+1}}).$$

Therefore,

$$w^{1/2^{k+1}}(A) \leq \prod_{i=1}^{k+1} \left(1 + 2 \sin^2 \gamma/2^i\right)^{\frac{1}{2k+2-i}} w(A^{1/2^{k+1}}).$$

Therefore, the inequality (2.12) holds for $n = k + 1$. Hence the inequality (2.12) holds for all $n \in \mathbb{N}$. Proceeding similarly as above, and using Corollary 2.11 for accretive-dissipative matrices, we get the desired inequality (2.13).

Next proposition reads as follows:
Proposition 2.13 Let $A, B \in \mathcal{M}_n$ with no eigenvalues in $(-\infty, 0]$. Then

(i) $\|A + B\| \leq (\|A^\alpha\| + \|B^\alpha\|)(\|A^{1-\alpha}\| + \|B^{1-\alpha}\|)$,

(ii) $\|A + B\| \leq (\|A^\alpha\| + \|B^{1-\alpha}\|)(\|A^{1-\alpha}\| + \|B^\alpha\|),$

for all $\alpha \in (0, 1)$.

Proof (i) We have,

$$\|A + B\| = \left\| \begin{pmatrix} A + B & 0 \\ 0 & 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} A^\alpha B^\alpha & A^{1-\alpha} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{1-\alpha} & 0 \\ B^{1-\alpha} & 0 \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} A^\alpha B^\alpha & A^{1-\alpha} \\ 0 & 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} A^{1-\alpha} & 0 \\ B^{1-\alpha} & 0 \end{pmatrix} \right\|$$

$$\leq \left( \|A^\alpha\| + \|B^\alpha\| \right) \left( \|A^{1-\alpha}\| + \|B^{1-\alpha}\| \right).$$

(ii) We have,

$$\|A + B\| = \left\| \begin{pmatrix} A + B & 0 \\ 0 & 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} A^\alpha B^{1-\alpha} & A^{1-\alpha} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{1-\alpha} & 0 \\ B^\alpha & 0 \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} A^\alpha B^{1-\alpha} & A^{1-\alpha} \\ 0 & 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} A^{1-\alpha} & 0 \\ B^\alpha & 0 \end{pmatrix} \right\|$$

$$\leq \left( \|A^\alpha\| + \|B^{1-\alpha}\| \right) \left( \|A^{1-\alpha}\| + \|B^\alpha\| \right).$$

Based on Proposition 2.13 we obtain the following corollary.

Corollary 2.14 Let $A, B \in \Pi_\gamma$. Then for all $\alpha \in (0, 1),$

$$w(A + B) \leq \sqrt{1 + 2 \sin^2 \alpha \gamma} \sqrt{1 + 2 \sin^2 (1 - \alpha) \gamma} \times \left( w(A^\alpha) + w(B^\alpha) \right) \left( w(A^{1-\alpha}) + w(B^{1-\alpha}) \right)$$
and
\[
\begin{align*}
    w(A + B) & \leq \left( \sqrt{1 + 2 \sin^2 \alpha \gamma \ w(A^\alpha) + 1 + 2 \sin^2 (1 - \alpha) \gamma \ w(B^{1 - \alpha})} \right) \\
    & \times \left( \sqrt{1 + 2 \sin^2 (1 - \alpha) \gamma \ w(A^{1 - \alpha}) + 1 + 2 \sin^2 \alpha \gamma \ w(B^\alpha)} \right).
\end{align*}
\]

Next, we obtain the numerical radius inequalities of the sum and the product of double commuting sectorial matrices. Recall that a set of matrices \( \{ A_i : i = 1, 2, \ldots, n \} \subseteq \mathcal{M}_n \) is called double commuting if \( A_i A_j = A_j A_i \) and \( A_i A_j^* = A_j^* A_i \) for all \( i \neq j \).

First, we note the following lemma, proved in \([22]\).

**Lemma 2.15** Let \( \{ A_i, B_i : i = 1, 2, \ldots, n \} \subseteq \mathcal{M}_n \) be a set of double commuting matrices. Then
\[
    w\left( \sum_{i=1}^{n} A_i B_i \right) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} A_i^* A_i + A_i A_i^* \right\|^{1/2} \left\| \sum_{i=1}^{n} B_i^* B_i + B_i B_i^* \right\|^{1/2}.
\]

Applying the above lemma we prove the following theorem.

**Theorem 2.16** Let \( \{ A_i, B_i : i = 1, 2, \ldots, n \} \subseteq \mathcal{M}_n \) be a set of double commuting sectorial matrices. Then
\[
    w\left( \sum_{i=1}^{n} A_i B_i \right) \leq \left( 1 + \sin^2 \gamma \right) \left( \sum_{i=1}^{n} w^2(A_i) \right)^{1/2} \left( \sum_{i=1}^{n} w^2(B_i) \right)^{1/2}.
\]

**Proof** From Lemma 2.15 we get,
\[
    w^2\left( \sum_{i=1}^{n} A_i B_i \right) \leq \left\| \sum_{i=1}^{n} (\Re^2(A_i) + \Im^2(A_i)) \right\| \left\| \sum_{i=1}^{n} (\Re^2(B_i) + \Im^2(B_i)) \right\| \leq \sum_{i=1}^{n} \left( \| \Re(A_i) \|^2 + \| \Im(A_i) \|^2 \right) \sum_{i=1}^{n} \left( \| \Re(B_i) \|^2 + \| \Im(B_i) \|^2 \right) \leq (1 + \sin^2 \gamma)^2 \left( \sum_{i=1}^{n} w^2(A_i) \right) \left( \sum_{i=1}^{n} w^2(B_i) \right) \quad \text{(by Lemma 2.1)}.
\]

Therefore, we get the required result. \(\Box\)

From Theorem 2.16 (for \( n = 1 \)) we obtain the following corollary.

**Corollary 2.17** Let \( A, B \in \Pi_\gamma \) be such that \( AB = BA \) and \( AB^* = B^* A \). Then
\[
    w(AB) \leq (1 + \sin^2 \gamma) w(A) w(B).
\]
Remark 2.18 Let $A, B \in \Pi_\gamma$ be such that $AB = BA$ and $AB^* = B^* A$. Then it is clear from Corollary 2.17 that the upper bound of $w(AB)$ depends on $\gamma$, and interpolates between $w(A)w(B)$ and $2w(A)w(B)$. If $A, B$ are positive definite (i.e., $A, B > 0$), then $w(AB) \leq w(A)w(B)$ and if $\gamma \to \frac{\pi}{2}$, then $w(AB) \leq 2w(A)w(B)$.

3 On special sectorial matrices

The accretive-dissipative matrices are a particular class of sectorial matrices that satisfy refined norm and numerical radius inequalities (see [1]). Motivated by these special sectorial matrices, we define new class of sectorial matrices whose numerical range lie in the forth quadrant and we obtain stronger results on the numerical radius inequalities for this new class of sectorial matrices. Let $A \in \mathcal{M}_n$ be a sectorial matrix with $\Im(A) < 0$. Then, it is clear from the Cartesian decomposition of $A$ that $iA$ is also a sectorial matrix. Moreover, $iA$ is accretive-dissipative. Conversely, if $A$ and $iA$ are both sectorial matrices, then $\Re(A) > 0$ and $\Im(A) < 0$. Let $A \in \mathcal{M}_n$ be such that

$$W(A) \subseteq \{re^{-i\theta} : \theta_1 \leq \theta \leq \theta_2\},$$

(3.1)

where $r > 0$ and $\theta_1, \theta_2 \in (0, \pi/2)$. Then, $A$ and $iA$ are both sectorial matrices with sectorial index $\theta_2$ and $\pi/2 - \theta_1$, respectively. Conversely, it is easy to observe that when $A$ and $iA$ are both sectorial then there exists $r > 0$ and $\theta_1, \theta_2 \in (0, \pi/2)$ such that $W(A) \subseteq \{re^{-i\theta} : \theta_1 \leq \theta \leq \theta_2\}$. Considering $A = e^{-i\pi/4}\begin{pmatrix} 2 & 4 \sqrt{3} \\ 0 & 2 \end{pmatrix}$, we see that the numerical range of $A$ satisfies the relation (3.1) with $\theta_1 = \pi/12$ and $\theta_2 = 5\pi/12$.

Theorem 3.1 Let $A \in \mathcal{M}_n$ satisfies the property (3.1). Then

$$\|A\| \leq \sqrt{1 + \cos^2 \theta_1} \ w(A) \leq \sqrt{2} w(A).$$

Proof We only prove the first inequality, as second inequality follows trivially. Since $iA$ is accretive-dissipative, and $iA \in \Pi_{\pi/2 - \theta_1}$, by using Lemma 2.9 we have,

$$\|A\| \leq \sqrt{1 + \cos^2 \theta_1} \ w(A).$$

(3.2)

We note that when $A \in \mathcal{M}_n$ satisfies the property (3.1), then there exists $\theta \in [0, \pi/2]$ such that $e^{i\theta}A \in \Pi(\theta_2 - \theta_1)/2$, and by Lemma 2.9 we infer that

$$\|A\| \leq \sqrt{1 + 2 \sin^2 \left(\frac{\theta_2 - \theta_1}{2}\right)} \ w(A).$$

(3.3)

To prove the next theorem we need the following lemma, proved in [1].
Lemma 3.2  Let $A, B \in \Pi_{\gamma}$. Then

$$w(AB) \leq (1 + 2 \sin^2 \gamma)w(A)w(B).$$

Moreover, if $A, B \in \Pi_{\gamma}$ be accretive-dissipative, then

$$w(AB) \leq (1 + \sin^2 \gamma)w(A)w(B).$$

Theorem 3.3  Let $A, B \in \mathcal{M}_n$ which satisfy the property (3.1). Then

$$w(AB) \leq (1 + \cos^2 \theta_1)w(A)w(B) \leq 2w(A)w(B).$$

Proof  Clearly, $iA$ and $iB$ are accretive-dissipative, and $iA, iB \in \prod_{\pi/2 - \theta_1}$. Then from Lemma 3.2 we get,

$$w(AB) \leq (1 + \cos^2 \theta_1)w(A)w(B). \quad (3.4)$$

The second inequality follows easily. \hfill \Box

We also note that when $A, B \in \mathcal{M}_n$ satisfy the property (3.1), then there exists $\theta \in [0, \pi/2]$ such that $e^{i\theta} A, e^{i\theta} B \in \Pi(\theta_2 - \theta_1)/2$. Then by Lemma 3.2, we have

$$w(AB) \leq \left(1 + 2 \sin^2 \left(\frac{\theta_2 - \theta_1}{2}\right)\right)w(A)w(B). \quad (3.5)$$

Now, we observe that when $A \in \mathcal{M}_n$ satisfies the property (3.1), then both $A, iA \in \Pi_{\gamma_1}$, where $\gamma_1 = \max\{\theta_2, \pi/2 - \theta_1\}$. From this observation we prove the following theorem. First we need the following lemma, see [1, Th. 3.5].

Lemma 3.4  Let $A \in \Pi_{\gamma} (\gamma \neq 0)$. Then

$$w(A) \geq \frac{\csc \gamma}{2} \|A\| + \frac{\csc \gamma}{2} \left(\|\Im(A)\| - \|\Re(A)\|\right).$$

Theorem 3.5  Let $A \in \mathcal{M}_n$ be satisfies the property (3.1). Then

$$w(A) \geq \frac{\csc \gamma_1}{2} \|A\| + \frac{\csc \gamma_1}{2} \left(\|\Im(A)\| - \|\Re(A)\|\right), \quad (3.6)$$

where $\gamma_1 = \max\{\theta_2, \pi/2 - \theta_1\}$.

Proof  Since $A$ and $iA$ are in $\Pi_{\gamma_1}$, by using Lemma 3.4 we get,

$$w(A) \geq \frac{\csc \gamma_1}{2} \|A\| + \frac{\csc \gamma_1}{2} \left(\|\Im(A)\| - \|\Re(A)\|\right) \quad (3.7)$$
\[ w(A) \geq \frac{\csc \gamma_1}{2} \| A \| + \frac{\csc \gamma_1}{2} (\| \Re(A) \| - \| \Im(A) \|). \]  

(3.8)

Thus, combining (3.7) and (3.8) we get the desired inequality. \( \square \)

**Remark 3.6** Since \( \gamma_1 \in (0, \pi/2) \), \( \csc \gamma_1 > 1 \). Therefore, from Theorem 3.5 we get,

\[ w(A) \geq \frac{\csc \gamma_1}{2} \| A \| + \frac{\csc \gamma_1}{2} (\| \Re(A) \| - \| \Im(A) \|). \]

Thus, we would like to remark that Theorem 3.5 gives sharper bound than the existing bound [6, Th. 2.1], namely,

\[ w(A) \geq \frac{\| A \|}{2} + \frac{\| \Im(A) \| - \| \Re(A) \|}{2}. \]  

(3.9)

Next refinement reads as:

**Theorem 3.7** Let \( A \in \mathcal{M}_n \) satisfies the property (3.1). Then

\[ w^2(A) \geq \frac{\csc^2 \gamma_1}{4} \| AA^* + A^*A \| + \frac{\csc^2 \gamma_1}{2} (\| \Im(A) \|^2 - \| \Re(A) \|^2), \]  

(3.10)

where \( \gamma_1 = \max\{\theta_2, \pi/2 - \theta_1\} \).

**Proof** Proceeding as Theorem 3.5 and using the inequality in Theorem 2.2 we obtain the desired inequality. \( \square \)

**Remark 3.8** Since \( \gamma_1 \in (0, \pi/2) \), \( \csc^2 \gamma_1 > 1 \). Hence, from Theorem 3.7 we get,

\[ w^2(A) \geq \frac{\csc^2 \gamma_1}{4} \| AA^* + A^*A \| + \frac{\csc^2 \gamma_1}{2} (\| \Im(A) \|^2 - \| \Re(A) \|^2) \]

Thus, we would like to remark that Theorem 3.7 gives sharper bound than that in [6, Th. 2.9], namely,

\[ w^2(A) \geq \frac{\| AA^* + A^*A \|}{4} + \frac{\| \Im(A) \|^2 - \| \Re(A) \|^2}{2}. \]  

(3.11)

Finally we note that when \( A \in \mathcal{M}_n \) with the numerical range \( W(A) \) satisfying the property

\[ W(A) \subseteq \{ re^{i\theta} : \theta_1 \leq \theta \leq \theta_2 \}, \]  

(3.12)
where \( r > 0 \) and \( \theta_1, \theta_2 \in (0, \pi/2) \), then \( A \) and \(-iA\) are both in \( \Pi_{\gamma_1} \), where
\[
\gamma_1 = \max\{\theta_2, \pi/2 - \theta_1\}.
\] Therefore, Theorems 3.5 and 3.7 also hold when \( A \in \mathcal{M}_n \) satisfies (3.12). Further, we see that Theorems 3.5 and 3.7 hold when \( \theta_1, \theta_2 \in (0, \pi/2) \), see (3.9) and (3.11).

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