ON EXTENSIONS OF THE LOOMIS-WHITNEY INEQUALITY AND BALL’S INEQUALITY FOR CONCAVE, HOMOGENEOUS MEASURES

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ABSTRACT. The Loomis-Whitney inequality states that the volume of a convex body is bounded by the product of volumes of its projections onto orthogonal hyperplanes. We provide an extension of both this fact and a generalization of this fact due to Ball to the context of $q$–concave, $\frac{1}{q}$–homogeneous measures.

1. Introduction

The Loomis-Whitney inequality [LW49] is a well-known geometric inequality concerning convex bodies, compact and convex sets with nonempty interior. Explicitly, the inequality states that if $u_1, \ldots, u_n$ form an orthonormal basis of $\mathbb{R}^n$ and $K$ is a convex body in $\mathbb{R}^n$, then

$$|K|^{n-1} \leq \prod_{i=1}^{n} |K|u_i^1|,$$

where $K|u_i^1$ denotes the projection of $K$ onto $u_i^1$, the hyperplane orthogonal to $u_i$. Equality occurs only when $K$ is a box with faces parallel to the hyperplanes $u_i^\perp$. This was generalized by Ball [Bal91], who showed that if $u_1, \ldots, u_m$ are vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ positive constants such that

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n,$$

then

$$|K|^{n-1} \leq \prod_{i=1}^{m} |K|u_i|^{c_i}.$$

Here $u_i \otimes u_i$ denotes the rank 1 projection onto the span of $u_i$, so $(u_i \otimes u_i)(x) = \langle x, u_i \rangle u_i$, and $I_n$ is the identity on $\mathbb{R}^n$. What will be useful later is the fact that

$$\sum_{i=1}^{m} c_i = n,$$

which follows by comparing traces in (1.1).

The Loomis-Whitney inequality and Ball’s inequality have been the subject of various generalizations. For instance, Li and Huang [HL17] provided an extension of Ball’s inequality with intrinsic volumes replacing volume and an arbitrary even isotropic measure replacing the discrete measure $\sum_{i=1}^{m} c_i \delta_{u_i}$ in
the condition \( \int_{\mathbb{S}^{n-1}} u \otimes u \, d(\sum_{i=1}^{m} c_i \delta_{u_i}) (u) = I_n \) of (1.1). Li and Huang [LH16] also demonstrated the \( L_p \) Loomis-Whitney inequality for even isotropic measures, while Lv [Lv19] very recently demonstrated the \( L_\infty \) Loomis-Whitney inequality.

In this paper, we will first give a generalization of the original Loomis-Whitney inequality to the context of \( q \)-concave, \( \frac{1}{q} \)-homogeneous measures. Using a different argument, we shall then prove a generalization of Ball’s inequality. Our two theorems are independent in the sense that the first is not recovered when specializing the second to the case of \( u_1, \ldots, u_n \) being an orthonormal basis and \( c_1 = \ldots = c_n = 1 \). Therefore, in fact, two different extensions of the Loomis-Whitney inequality are given.

Let us recall the necessary definitions.

**Definition 1.1.** A function \( f : \mathbb{R}^n \to [0, \infty] \) is \( p \)-concave for some \( p \in \mathbb{R} \setminus \{0\} \) if for all \( \lambda \in [0, 1] \) and \( x, y \in \text{supp}(f) \) we have

\[
 f(\lambda x + (1 - \lambda) y) \geq (\lambda f^p(x) + (1 - \lambda) f^p(y))^{\frac{1}{p}}
\]

**Definition 1.2.** A function \( f : \mathbb{R}^n \to [0, \infty] \) is \( r \)-homogeneous if for all \( a > 0, x \in \mathbb{R}^n \) we have

\[
 f(ax) = a^r f(x).
\]

We will interested in functions \( g \) that are both \( s \)-concave for some \( s > 0 \) and \( \frac{1}{p} \)-homogeneous for some \( p > 0 \). In this case, we get that in fact \( g \) is \( p \)-concave (see e.g. Livshyts [Liv]). Continuity will be assumed throughout. An example of a \( p \)-concave, \( \frac{1}{p} \)-homogeneous function is \( g(x) = 1_{\langle x, \theta \rangle > 0} \langle x, \theta \rangle^{\frac{1}{p}} \).

All such functions \( g \), with the exception of constant functions, will be supported on convex cones. A notation we will use is \( \tilde{g}(x) = g(x) + g(-x) \).

If \( \mu \) is a measure with a \( p \)-concave, \( \frac{1}{p} \)-homogeneous density, then a change of variables will show that \( \mu \) is \( \frac{n + \frac{1}{p}}{n + \frac{1}{p}} \) homogeneous, that is \( \mu(tK) = t^{n + \frac{1}{p}} \mu(K) \). From a result of Borell [Bor75], we also have concavity:

**Lemma 1.3 (Borell).** Let \( p \in (-\frac{1}{n}, \infty) \) and let \( \mu \) be a measure on \( \mathbb{R}^n \) with \( p \)-concave density \( g \). For \( q = \frac{n + \frac{1}{p}}{n + \frac{1}{p}} \), \( \mu \) is a \( q \)-concave measure, that is for measurable \( E, F \) and \( \lambda \in [0, 1] \) we have

\[
 \mu(\lambda E + (1 - \lambda) F) \geq (\lambda \mu(E)^q + (1 - \lambda) \mu(F)^q)^{\frac{1}{q}}.
\]

To now define the generalized notion of projection for measures, one requires the definition of mixed measure.
**Definition 1.4.** Let $A, B$ be measurable sets in $\mathbb{R}^n$. We define

$$
\mu_1(A, B) = \liminf_{\varepsilon \to 0} \frac{\mu(A + \varepsilon B) - \mu(A)}{\varepsilon}
$$

to be the mixed $\mu$–measure of $A$ and $B$.

An important simple fact, which follows from Lemma 3.3 in Livshyts [Liv], is that mixed measure is linear in the second variable, so

$$(1.3) \quad \mu_1(K, E + tF) = \mu_1(K, E) + t\mu_1(K, F)$$

for $t \geq 0$.

For $q$–concave measures, we have the following generalization of Minkowski’s first inequality (see e.g. Milman and Rotem [MR14]):

**Lemma 1.5.** Let $\mu$ be a $q$–concave measure and $A, B$ be measurable sets in $\mathbb{R}^n$. Then,

$$
\mu(A)^{1-q} \mu(B)^q \leq q \mu_1(A, B).
$$

We now turn to discussing the generalized notion of projection. This notion, defined in Livshyts [Liv], is

$$(1.4) \quad P_{\mu,K}(\theta) = \frac{n}{2} \int_0^1 \mu_1(tK, [-\theta, \theta])dt$$

for $\theta \in S^{n-1}$, where $K$ is a convex body and $\mu$ is an absolutely continuous measure. This is a natural extension of the identity $|K|_{\theta^\perp} = \frac{1}{2} \lambda_1(K, [-\theta, \theta])$, with $\lambda$ denoting Lebesgue measure, which can be readily seen for polytopes and follows in the general case by approximation.

In Livshyts [Liv], a version of the Shephard problem for $q$–concave, $\frac{1}{q}$–concave measures was proven with this notion of measure. The author in [Hos] studied related section and projection comparison problems, including for this same class of $q$–concave, $\frac{1}{q}$–homogeneous measures.

With (1.4), we can now state our first theorem:

**Theorem 1.6.** Let $\mu$ be a measure with $p$–concave, $\frac{1}{p}$–homogeneous density $g$ for some $p > 0$. Then, for any convex body $K$ and an orthonormal basis $(u_i)_{i=1}^n$ with $[-u_i, u_i] \cap \text{supp}(g) \neq \emptyset$ for each $1 \leq i \leq n$,

$$
\mu(K)^{n+\frac{1}{p}-1} \leq 2^{n+\frac{1}{p}} \left(1 + \frac{1}{pn}\right)^n \left(\sum_{k=1}^n \tilde{g}^p(u_k)\right)^{-\frac{1}{p}} \prod_{i=1}^n P_{\mu,K}(u_i)^{1+\frac{\tilde{g}^p(u_i)}{p \sum_{k=1}^n \tilde{g}^p(u_k)}}.
$$

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Before we state our generalization of Ball’s inequality, we introduce another definition. Let \( S = \{(u_i)_{i=1}^m\} \) be a set of unit vectors in \( \mathbb{R}^n \). Then we define \( S^{(1)} \) to be the set of \( u_{ij} = \frac{u_i - \langle u_i, u_j \rangle u_j}{|u_i - \langle u_i, u_j \rangle u_j|} \), the normalized projection of \( u_i \) onto the hyperplane \( u_j \), for \( 1 \leq i, j \leq m \). Recursively defining \( S^{(k)} = (S^{(k-1)})^{(1)} \), we set

\[
P = P((u_i)_{i=1}^m) := S \cup S^{(1)} \cup ... \cup S^{(n-1)},
\]

some finite set depending on our initial choice of \( \{(u_i)_{i=1}^m\} \). Our generalization of Ball’s inequality is the following:

**Theorem 1.7.** Let \( \mu \) be a measure with \( p \)-concave, \( \frac{1}{p} \)-homogeneous density \( g \) for some \( p > 0 \). If \( (u_i)_{i=1}^m \) are unit vectors in \( \mathbb{R}^n \) and \( (c_i)_{i=1}^m \) are positive constant such that

\[
\sum_{i=1}^m c_i u_i \otimes u_i = I_n
\]

and moreover \( [-u, u] \cap \text{supp}(g) \neq \emptyset \) for each \( u \in P((u_i)_{i=1}^m) \), then

\[
\mu(K)^{n+\frac{1}{p}-1} \leq 2^{n+\frac{1}{p}} \left( \inf_{u \in P} \tilde{g}(u) \right)^{-1} \prod_{k=1}^n \left( 1 + \frac{1}{kp} \right) \prod_{i=1}^m P_{\mu,K}(u_i)^{c_i \left( 1 + \frac{1}{mp} \right)}
\]

for any convex body \( K \).

Observe that the condition \( [-u, u] \cap \text{supp}(g) \neq \emptyset \) is not particularly restrictive. For instance, if we consider \( g \) whose support is a half space with boundary a half plane \( P \), then the condition simply reduces to the fact that some finite number of points do not lie on \( P \).

**Remark 1.** Note that when \( p \to \infty \), Theorem 1.6 and Theorem 1.7 recover the results for Lebesgue measure up to a dimensional constant of \( 2^n \). The reason for this extra factor of \( 2^n \) comes from the fact that nonconstant \( p \)-concave, \( \frac{1}{p} \)-homogeneous densities are supported on at most a half-space, which therefore restricts us to only being able to get inequalities on ‘half’ of our domain.

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2. **Extension of the Loomis-Whitney Inequality**

We begin with a lemma providing us with a lower bound for the measure of a face of a parallelapiped. With homogeneity, this will give us a lower bound for the measure of a parallelapiped, which will be a key ingredient in the proof of Theorem 1.6.
Lemma 2.1. Let $g, \mu,(u_i)_{i=1}^n$ be as in the statement of Theorem 1.6, let

$$F_i = \{u = \alpha_i u_i + \sum_{j \neq i} \beta_j u_j : |\beta_j| \leq \alpha_j\},$$

where $\alpha_1,...,\alpha_n$ are positive constants, and suppose that $u_i \in \text{supp}(g)$. Then,

$$\mu_{n-1}(F_i) \geq \left(\frac{pm}{pm+1}\right)^n \left(1 + \frac{\tilde{g}^p(u_i)}{p \sum_{k=1}^{m} \tilde{g}^p(u_k)}\right) \left(\sum_{i=1}^{n} \tilde{g}^p(u_i)\right)^{\frac{1}{p}} \prod_{j=1}^{n} \alpha_j^{\frac{1}{p}} \sum_{i=1}^{n} \tilde{g}^p(u_j)^{\frac{1}{p}}.$$

Proof. For simplicity of notation, we deal with the case $i = 1$. We begin by writing $\mu_{n-1}(F_1)$ as an integral of $g$ over $F_1$, subdividing the domain of integration, and using homogeneity:

$$\mu_{n-1}(F_1) = \int_{\sigma = (\pm 1,\ldots,\pm 1)} \int_0^{\alpha_1} \ldots \int_0^{\alpha_2} g(v) dv$$

$$= \sum_{\sigma = (\pm 1,\ldots,\pm 1)} \int_0^{\alpha_1} \ldots \int_0^{\alpha_2} \left(\alpha_1 u_1 + \sum_{j=2}^{n} \beta_j \sigma(j) u_j\right) d\beta_2 \ldots d\beta_n$$

$$= \sum_{\sigma = (\pm 1,\ldots,\pm 1)} I_{\sigma}.$$

If we take $\sigma'$ such that $\sigma'(j) u_j \in \text{supp}(g)$ for each $j$ (which can be done by the hypothesis of Theorem 1.6), then

$$\mu_{n-1}(F_1) \geq I_{\sigma'}.$$

By $p$--concavity and the fact that $g(\sigma'(j) u_j) = \tilde{g}(u_j)$,

$$I_{\sigma'} \geq \int_0^{\alpha_1} \ldots \int_0^{\alpha_2} \left(\alpha_1 + \sum_{j=2}^{n} \beta_j\right)^{\frac{1}{p}} \left(\frac{\alpha_1}{\alpha_1 + \sum_{j=2}^{n} \beta_j} \tilde{g}^p(u_1) + \sum_{j=2}^{n} \beta_j \tilde{g}^p(u_j)\right)^{\frac{1}{p}} d\beta_2 \ldots d\beta_n$$

$$= \int_0^{\alpha_1} \ldots \int_0^{\alpha_2} \left(\alpha_1 \tilde{g}^p(u_1) + \sum_{j=2}^{n} \beta_j \tilde{g}^p(u_j)\right)^{\frac{1}{p}} d\beta_2 \ldots d\beta_n$$

$$= \left(\sum_{i=1}^{n} \tilde{g}^p(u_i)\right)^{\frac{1}{p}} \int_0^{\alpha_1} \ldots \int_0^{\alpha_2} \left(\alpha_1 \sum_{i=1}^{n} \tilde{g}^p(u_i) + \sum_{j=2}^{n} \beta_j \sum_{i=1}^{n} \tilde{g}^p(u_i)\right)^{\frac{1}{p}} d\beta_2 \ldots d\beta_n.$$
Inserting the bound
\[
\alpha \frac{\tilde{g}^P(u_1)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} + \sum_{j=2}^{n} \beta_j \frac{\tilde{g}^P(u_j)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \geq \alpha \frac{\tilde{g}^P(u_1)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \prod_{j=2}^{n} \beta_j \frac{\tilde{g}^P(u_j)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)}
\]
from the Arithmetic Mean-Geometric Mean Inequality under the integral gives
\[
I_{\sigma'} \geq \left( \sum_{i=1}^{n} \frac{\tilde{g}^P(u_i)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \right)^{\frac{1}{p}} \frac{\tilde{g}^P(u_1)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \prod_{j=2}^{n} \left( 1 + \frac{1}{\tilde{g}^P(u_j)} \frac{\tilde{g}^P(u_j)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \right) \frac{1}{\alpha_j} \frac{\tilde{g}^P(u_j)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \alpha_j.
\]
Again by the Arithmetic Mean-Geometric Mean inequality,
\[
\prod_{j=1}^{n} \left( 1 + \frac{\tilde{g}^P(u_j)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \right) \leq \left( 1 + \frac{1}{p} \right)^n
\]
and thus
\[
I_{\sigma'} \geq \left( \frac{pn}{pn + 1} \right)^n \left( 1 + \frac{\tilde{g}^P(u_1)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \right) \left( \sum_{i=1}^{n} \tilde{g}^P(u_i) \right)^{\frac{1}{p}} \alpha_1 \prod_{j=2}^{n} \alpha_j \frac{\tilde{g}^P(u_j)}{\sum_{i=1}^{n} \tilde{g}^P(u_i)} \alpha_j.
\]
By (2.1), our proof is complete.

For the proof of our theorem, we will recall the definition of a zonotope. A zonotope is simply a Minkowski sum of line segments
\[
Z = \sum_{i=1}^{m} [-x_i, x_i].
\]
By linearity (1.3), if \(Z = \sum_{i=1}^{m} \alpha_i [-u_i, u_i]\) for unit vectors \(u_i\) and \(\alpha_i\) positive constants, then
\[
\mu_1(K, Z) = \sum_{i=1}^{m} \alpha_i \mu_1(K, [-u_i, u_i])
\]
for a convex body \(K\). Since our measure \(\mu\) is homogeneous,
\[
P_{\mu, K}(u_i) = \frac{n}{2} \int_{0}^{1} \mu_1(tK, [-u_i, u_i]) dt = \frac{1}{2} \int_{0}^{1} t^{\alpha - 1} \mu_1(K, [-u_i, u_i]) dt = \frac{q}{2} \mu_1(K, [-u_i, u_i])
\]
by (1.4). Therefore,

\begin{equation}
\mu_1(K, Z) = \frac{2}{nq} \sum_{i=1}^{m} \alpha_i P_{\mu, K}(u_i).
\end{equation}

We now prove our theorem:

**Proof of Theorem 1.6.** Let \( Z \) be the zonotope \( \sum_{i=1}^{n} \alpha_i [-u_i, u_i] \) with \( \alpha_i = \frac{1}{P_{\mu, K}(u_i)} \) for \( 1 \leq i \leq n \). By Lemma 1.5, (2.2), and our choice of \( \alpha_i \)

\[ \mu(K)^{1-q} \leq q \mu(Z)^{-q} \mu_1(K, Z) = 2 \mu(Z)^{-q} \]

and so

\begin{equation}
\mu(K)^{\frac{1}{q}-1} \leq 2^{\frac{1}{q}} \mu(Z)^{-1}.
\end{equation}

Without loss of generality, we assume that \( u_i \in \text{supp}(g) \) and \( g(-u_i) = 0 \) for each \( i \). Let \( F_i \) denote the face of \( Z \) orthogonal to and touching \( \alpha_i u_i \), and subdivide \( Z \) into pyramids with bases of \( F_i \), apex at the origin, and height of \( \alpha_i \). By homogeneity,

\[ \mu(Z) = \sum_{i=1}^{n} \int_{0}^{\alpha_i} \mu_{n-1} \left( \frac{t}{\alpha_i} F_i \right) dt \]

\[ = \sum_{i=1}^{n} \left( \int_{0}^{\alpha_i} \frac{1}{t^{\frac{1}{q}-1}} dt \right) \alpha_i \mu_{n-1}(F_i) \]

\[ = q \sum_{i=1}^{n} \alpha_i \mu_{n-1}(F_i). \]

Applying Lemma 2.1, we have

\[ \mu(Z) \geq \frac{1}{n + \frac{1}{p}} \left( \frac{pm}{pm + 1} \right)^n \left( \sum_{i=1}^{n} \tilde{g}^p(u_i) \right)^{\frac{1}{p}} \left( \prod_{j=1}^{n} \alpha_j \right) \left( \frac{1 + \frac{\tilde{g}^p(u_i)}{p \sum_{k=1}^{n} \tilde{g}^p(u_k)}}{\sum_{i=1}^{n} \left( 1 + \frac{\tilde{g}^p(u_i)}{p \sum_{k=1}^{n} \tilde{g}^p(u_k)} \right)} \right) \]

\[ = \left( \frac{pm}{pm + 1} \right)^n \left( \sum_{i=1}^{n} \tilde{g}^p(u_i) \right)^{\frac{1}{p}} \prod_{j=1}^{n} \alpha_j \left( \frac{1 + \frac{\tilde{g}^p(u_i)}{p \sum_{k=1}^{n} \tilde{g}^p(u_k)}}{\sum_{i=1}^{n} \left( 1 + \frac{\tilde{g}^p(u_i)}{p \sum_{k=1}^{n} \tilde{g}^p(u_k)} \right)} \right) \]

Combining this bound with (2.3) and recalling that \( \alpha_i = \frac{1}{P_{\mu, K}(u_i)} \), our desired inequality is proven. \( \square \)
3. Extension of Ball’s Inequality

As in the previous section, we will require an estimate from below for the measure of a zonotope. However, mimicking the approach of Ball [Bal91], rather than estimating the measures of the faces directly, we shall first project them. A main difference from Ball’s proof stems from the lack of translation invariance of our measure, but we will circumvent this obstacle by an appropriate inequality (3.2) coming from concavity.

**Lemma 3.1.** Let $g, \mu, (u_i)_{i=1}^m, (c_i)_{i=1}^m$ be as in the statement of Theorem 1.7. Let $Z = \sum_{i=1}^m \alpha_i[-u_i, u_i]$ be a zonotope. Then

$$\mu(Z) \geq \left( \inf_{u \in P} \tilde{g}(u) \right) \left( \prod_{k=1}^n \frac{k}{k + \frac{1}{p}} \right) \prod_{i=1}^m \left( \frac{\alpha_i}{c_i} \right)^{c_i} \left( 1 + \frac{1}{pn} \right).$$

*Proof.* Following Ball [Bal91], we induct on the dimension $n$. First consider the case $n = 1$. We can then assume $u_1 = \ldots = u_m$ and without loss of generality $g(u_1) = \tilde{g}(u_1) > 0$ and $g(-u_1) = 0$. Then

$$\mu(Z) = \mu \left( \left( \sum_{i=1}^m \alpha_i \right) [-u_1, u_1] \right)$$

$$= \int_0^{\sum_{i=1}^m \alpha_i} g(tu_1) dt$$

$$= \left( \int_0^{\sum_{i=1}^m \alpha_i} t^\frac{1}{p} dt \right) g(u_1)$$

$$= \frac{1}{1 + \frac{1}{p}} \left( \sum_{i=1}^m \alpha_i \right)^{1 + \frac{1}{p}} g(u_1).$$

Since $n = 1$, (1.2) implies $\sum_{i=1}^m c_i = 1$, and therefore by the Arithmetic Mean-Geometric Mean inequality

$$\sum_{i=1}^m \alpha_i \geq \prod_{i=1}^m \left( \frac{\alpha_i}{c_i} \right)^{c_i}.\]$$

This concludes the proof for $n = 1$.

Let us assume we now have our result for dimension $n - 1$, and consider the case of dimension $n$. Firstly, observe that homogeneity implies

$$\mu_1(Z, Z) = \lim_{\varepsilon \to 0} \frac{\mu(Z + \varepsilon Z) - \mu(Z)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \mu(Z) \left( \frac{1}{p} \right)^{\frac{1}{q}} - 1$$

$$= \frac{1}{q} \mu(Z).$$
Therefore, 
\[
\mu(Z) = q\mu_1(Z, Z) \\
= q \sum_{i=1}^{m} \alpha_i \mu_1(Z, [-u_i, u_i]) \\
= qn \sum_{i=1}^{m} \frac{c_i}{c_n} \alpha_i \mu_1(Z, [-u_i, u_i]).
\]

Since \( \sum_{i=1}^{m} \frac{c_i}{c_n} = 1 \), we use the Arithmetic Mean-Geometric Mean inequality once again to get

\[
(3.1) \quad \mu(Z) \geq qn \prod_{i=1}^{m} \left( \frac{\alpha_i}{c_i} \mu_1(Z, [-u_i, u_i]) \right)^{\frac{c_i}{c_n}}.
\]

Let \( P_iZ \) denote the projection of \( Z \) onto the hyperplane \( u_i^\perp \). We wish to show

\[
(3.2) \quad \mu_1(Z, [-u_i, u_i]) \geq \mu_{n-1}(P_iZ),
\]

where here \( \mu_{n-1} \) denotes integration of the density \( g \) over the \((n-1)\)-dimensional set \( P_iZ \). This will compensate for lack of translation invariance of our measure.

By assumption, one of \( u_i \) and \(-u_i\) lies in \( \text{supp}(g) \). Without loss of generality, \( u_i \in \text{supp}(g) \). For \( w \in \mathbb{R}^n \) and \( t > 0 \), concavity and homogeneity gives us

\[
g(w + tu_i) \geq (g^p(w) + t g^p(u_i))^\frac{1}{p} \geq g(w).
\]

To be precise, concavity gives this to us when \( w \in \text{supp}(g) \), but when \( w \not\in \text{supp}(g) \) this is trivial. This inequality is equivalent to the statement that

\[
(3.3) \quad g(w + t_1 u_i) \geq g(w + t_2 u_i)
\]

for any \( w \in \mathbb{R}^n \) and \( t_1 \geq t_2 \).

For each \( w \in P_iZ \), let \( t(w) \geq 0 \) be taken so that \( w + t(w)u_i \in \partial Z \). We now write

\[
\mu_1(Z, [-u_i, u_i]) = \liminf_{\varepsilon \to 0} \frac{\mu((Z + [-u_i, u_i]) \setminus Z)}{\varepsilon} \\
\geq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{P_iZ} \int_{t(h)+\varepsilon}^{t(h)} g(h + su_i)dsdh,
\]

where our integral of the density is taken over the region \((Z + [0, u_i]) \setminus Z \). By (3.3) and continuity,

\[
\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{P_iZ} \int_{t(h)+\varepsilon}^{t(h)} g(h + su_i)dsdh \geq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{P_iZ} \int_{0}^{\varepsilon} g(h + su_i)dsdh
\]
\[= \mu_{n-1}(P_iZ).\]

This proves (3.2).

Denoting the projection of \(u_j\) onto \(u_i^\perp\) by \(P_i(u_j)\), we have that \(P_iZ\) is the zonotope

\[P_iZ = \sum_{j=1}^m \alpha_j[-P_i(u_j), P_i(u_j)]\]

\[= \sum_{i=1}^m \alpha_i \gamma_{ji}[-u_{ji}, u_{ji}],\]

where \(\gamma_{ji} = |u_j - \langle u_i, u_j \rangle u_i|\). A simple computation shows \(\gamma_{ji}^2 = 1 - \langle u_i, u_j \rangle^2\).

We have also

\[P_i = \sum_{j=1}^m c_j P_i u_j \otimes P_i u_j\]

\[= \sum_{j=1}^m \gamma_{ji}^2 c_j u_{ji} \otimes u_{ji},\]

and this is the identity operator on \(u_i^\perp\). By (3.1), (3.2), and our inductive hypothesis,

\[\mu(Z) \geq n + \frac{2}{p} \prod_{i=1}^m \left( \frac{\alpha_i}{c_i} \mu_{n-1}(P_iZ) \right) \frac{c_i}{n}\]

\[\geq \prod_{k=1}^n \left( \frac{\alpha_i}{c_i} \left( \inf_{u \in \mathcal{P}(u_{ji})} \tilde{g}(u) \right) \right) \prod_{j=1}^m \left( \frac{\alpha_j \gamma_{ji}}{c_j \gamma_{ji}^2} \right) \frac{c_j \gamma_{ji}^2 (1 + \frac{1}{p(n-1)})}{\mu_i}.\]

From the inequality \(\frac{1}{\gamma_{ji}} \geq 1\) and the relation

\[\sum_{i=1}^m c_i \gamma_{ji}^2 = \sum_{i=1}^m c_i (1 - \langle u_i, u_j \rangle^2) = n - 1,\]

an appropriate grouping of elements in our product completes the proof. \(\square\)

As before, the proof of Theorem 1.7 now follows:

**Proof of Theorem 1.7.** Let \(Z\) be the zonotope \(\sum_{i=1}^m \alpha_i [u_i, u_i]\) where \(\alpha_i = \frac{c_i}{\mu_i(K)(u_i)}\) for \(1 \leq i \leq m\). By the same argument as in the proof of Theorem 1.6, where we must use (1.2),

\[\mu(K)^{\frac{1}{n}-1} \leq 2^n \mu(Z)^{-1}.\]
By Lemma 3.1, we reach

$$
\mu(K)^{\frac{1}{q}-1} \leq 2^\frac{1}{q} \left( \inf_{u \in P} \tilde{g}(u) \right)^{-1} \prod_{k=1}^{n} \left( 1 + \frac{1}{kp} \right) \prod_{i=1}^{m} P_{\mu, K}(u_i)^{c_i \left(1 + \frac{1}{pn} \right)}
$$

as desired. □

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