Soliton Scattering on Impurities with Modified Exchange Interactions in Anisotropic Ferromagnetic Chains

M T Primatarowa and R S Kamburova
Georgi Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences, 72 Tzarigradsko Chaussee Blvd. BG-1784 Sofia, Bulgaria
E-mail: prima@issp.bas.bg

Abstract. The soliton propagation in an anisotropic ferromagnetic chain where the exchange interactions of an impurity spin with its neighbors are modified is investigated. We considered easy-axis anisotropy of the chain which leads to the formation of bright soliton solutions. The character of the soliton-impurity interaction is different when the exchange coupling of the spin is modified in the x,y-plane or in the z-direction. Our results show that the action of the defect depends on the soliton wave number in a complicated manner and for large velocities becomes significant. A comparison with the soliton dynamics in the presence of point defects is made.

1. Introduction
The existence of spin solitons in magnetic chains including isotropic as well as anisotropic models has been investigated for decades [1-3]. The properties of the solitons as nonlinear excitations which propagate with constant form and velocity were extensively studied both from fundamental point of view and for possible application in real systems. An important problem in this connection is the soliton interaction with defects and inhomogeneities. Widely investigated are linear, nonlinear, bond defects etc. in condensed media [4-7].

In the present paper we investigate the interaction of propagating solitons with an impurity spin with modified exchange interaction in an anisotropic Heisenberg chain.

2. Hamiltonian of the system
We consider a ferromagnetic Heisenberg chain of $N_{ch}$ spins with magnitude $S$ where an impurity spin determined by different exchange interactions with its neighbors is placed at the position $n_0$. The Hamiltonian of such a system in the nearest-neighbor approximation can be written as

$$\hat{H} = -\mu H_0 \sum_n \hat{S}_n^z - J \sum_n [1 + \eta/2 (\delta_{n,n_0} + \delta_{n+1,n_0})] (\hat{S}_n^x \hat{S}_{n+1}^x + \hat{S}_n^y \hat{S}_{n+1}^y) - \tilde{J} \sum_n [1 + \tilde{\eta}/2 (\delta_{n,n_0} + \delta_{n+1,n_0})] \hat{S}_n^z \hat{S}_{n+1}^z,$$

(1)

where $J > 0$ and $\tilde{J} > 0$ are the exchange integrals in the $x,y$-plane and in the $z$-direction, respectively. The parameters $\eta$ and $\tilde{\eta}$ describe the modification for the interactions of the
impurity with its neighbors. They can be of arbitrary sign and strength. For \( J = \tilde{J} \) the model is isotropic. In what follows we study the case of easy-axis anisotropy (\( \tilde{J} > J \)) which unsure us the existence of bright solitons in the chain. \( H_0 \) is the external magnetic field applied along the \( z \)-axis, so that in the ground state of the system all spins are aligned in the \( z \)-direction and \( \mu \) is the magnetic moment per spin.

We have used \( \hat{S}_n^z = \hat{S}_n^+ \pm i\hat{S}_n^- \). Then the Hamiltonian (1) takes the form

\[
\hat{H} = -\mu H_0 \sum_n \hat{S}_n^z - \frac{J}{2} \sum_n \left[ 1 + \frac{\eta}{2} (\delta_{n,n_0} + \delta_{n+1,n_0}) \right] (\hat{S}_n^+ \hat{S}_{n+1}^- + \hat{S}_n^- \hat{S}_{n+1}^+) \\
- \tilde{J} \sum_n \left[ 1 + \frac{\tilde{\eta}}{2} (\delta_{n,n_0} + \delta_{n+1,n_0}) \right] \hat{S}_n^z. \tag{2}
\]

Further, we treat the spins as classical vectors and define \( \alpha_n = \hat{S}_n^z / S \), \( \alpha_n^* = \hat{S}_n^- / S \). Then \( \hat{S}_n^z / S = \sqrt{1 - |\alpha_n|^2} \) and the equations of motion for the complex functions \( \alpha_n \) become (\( \hbar = 1 \))

\[
i \frac{\partial \alpha_n}{\partial t} = \mu H_0 \alpha_n - JS \left\{ [1 + \frac{\eta}{2} (\delta_{n-1,n_0} + \delta_{n,n_0})] \alpha_{n-1} + [1 + \frac{\eta}{2} (\delta_{n,n_0} + \delta_{n+1,n_0})] \alpha_{n+1} \right\} \sqrt{1 - |\alpha_n|^2} \\
+ \tilde{J} S \left\{ [1 + \frac{\tilde{\eta}}{2} (\delta_{n-1,n_0} + \delta_{n,n_0})] \sqrt{1 - |\alpha_n-1|^2} + [1 + \frac{\tilde{\eta}}{2} (\delta_{n,n_0} + \delta_{n+1,n_0})] \sqrt{1 - |\alpha_{n+1}|^2} \right\} \alpha_n. \tag{3}
\]

The set of differential equations (3) describes our system. The case \( \tilde{J} = 0 \) would lead to an equation similar to the perturbed Ablowitz-Ladik one considered in Ref. [8].

3. Soliton solutions

We shall look for solutions in the form of amplitude-modulated waves

\[
\alpha_n(t) = \varphi_n(t) e^{i(kn - \omega t)}, \tag{4}
\]

where \( k \) and \( \omega \) are the wave number and the frequency of the carrier wave (the lattice constant equals unity) and the envelope \( \varphi_n(t) \) is a real slowly varying function of the position and time. Then, in the continuum limit and \( \varphi^2 \ll 1 \), equation (3) transforms into the following perturbed nonlinear Schrödinger equation for the envelope:

\[
i \left( \frac{\partial \varphi}{\partial t} + 2J S [1 + \eta \delta(x - x_0)] \sin k \frac{\partial \varphi}{\partial x} \right) = [\omega_0 - \omega - 2S (J \eta \cos k - \tilde{J} \tilde{\eta}) \delta(x - x_0)] \varphi \\
- JS [1 + \eta \delta(x - x_0)] \cos k \frac{\partial^2 \varphi}{\partial x^2} + [g + S (J \eta \cos k - \tilde{J} \tilde{\eta}) \delta(x - x_0)] |\varphi|^2 \varphi, \tag{5}
\]

\[
\omega_0 = \mu H_0 - 2g, \quad g = (J \cos k - \tilde{J}) S. \tag{6}
\]

We would like to point out that the perturbing terms proportional to the impurity’s parameters \( \eta \) and \( \tilde{\eta} \) have different character. \( \eta \) introduces a linear and a nonlinear \( \delta \)-function perturbing terms, while the impurity \( \eta \) leads to perturbations in all terms of the equation (5) which are furthermore wave number dependent and hence velocity dependent. As a leading perturbing term we can consider the linear proportional to \( \varepsilon \)

\[
\varepsilon = -2S (J \eta \cos k - \tilde{J} \tilde{\eta}). \tag{7}
\]

The interaction of solitons with such linear point defects in condensed media is widely studied and we use it as a basis for our further investigations. The other perturbing terms in (5) are of order \( 1/L \) (first derivative term) or \( 1/L^2 \) (second derivative and nonlinear terms) smaller to
the linear one but under certain conditions they can become significant. To understand their influence on the soliton dynamics is the aim of our work.

For a homogeneous chain $\eta = \tilde{\eta} = 0$ equation (5) possesses a bright-soliton solution

$$\varphi(x,t) = \varphi_0 \text{sech} \left( \frac{x-vt}{L} \right), \quad \varphi_0^2 = - \frac{2JS \cos k}{gL^2}, \quad \omega = \omega_0 - \frac{JS \cos k}{L^2}, \quad v = 2JS \sin k$$

(8)

when $g \cos k < 0$ and $|\varphi(x,t)| \to 0$ at $x \to \pm \infty$. The parameters $L$ and $v$ are the width and the velocity of the soliton.

Figure 1. Scattering of a slow soliton with $k = 0.016$ from an attractive impurity with (a) $\eta = \tilde{\eta} = -0.015$, (b) $\eta = \tilde{\eta} = -0.2$, (c) $\eta = \tilde{\eta} = -0.4$, and (d) $\eta = \tilde{\eta} = -0.75$. $J = S = 1$, $\tilde{J} = 1.2$ and $L = 10$. The time is in units $100/JS$.

4. Soliton scattering on the impurities

We have investigated the propagation of a soliton which at the initial time $t = 0$ is launched at site $n_s$ far enough from the impurity position $n_0$

$$\alpha_n(0) = \frac{1}{L} \sqrt{\frac{2J \cos k}{J - J \cos k}} \text{sech} \left( \frac{n-n_s}{L} \right) e^{ikn}$$

(9)

solving numerically the system (3). For our simulations we have chosen $\tilde{J} = 1.2$, $N_{ch} = 1000$, $n_s = 450$, $n_0 = 500$, and $L = 10$. The soliton evolution depends strongly on the initial velocity $v$ ($v = 2JS \sin k$) and the impurity parameters $\eta$ and $\tilde{\eta}$ which can be equal or different.

To understand the different evolution processes of the soliton we shall consider its kinetic energy $E_k$, nonlinear energy $E_{nl}$ and interaction energy $E_d$ with the linear impurity $\varepsilon$

$$E_k = \frac{v^2}{4JS} N, \quad E_{nl} = \frac{JS}{3L^2} N, \quad E_d = \frac{\varepsilon S}{2L} N,$$

(10)

where $N = 2L \varphi_0^2$ is the norm of the function $\alpha$.

First, we have studied the propagation of slow solitons or the case of strong nonlinearity where the soliton behaves similar to a particle ($E_k < E_{nl}$). The results for an initial value
Figure 2. Scattering of a slow soliton with $k = 0.016$ from a repulsive impurity with (a) $\eta = \tilde{\eta} = 0.012$ and (b) $\eta = \tilde{\eta} = 0.013$. All other parameters are the same as in figure 1.

$k = 0.016$ and different impurity strengths are presented in figures 1 and 2. The values for $\eta$ and $\tilde{\eta}$ are chosen so that the total defect $\varepsilon$ (7) acts as an attractive ($\varepsilon < 0$) or repulsive ($\varepsilon > 0$) one. In the case of attraction the soliton can be completely transmitted when $E_k > |E_d| \ [\varepsilon = -0.006$, figure 1(a)] or completely reflected when $E_k \ll |E_d| \ [\varepsilon = -0.3$, figure 1(d)]. In the intermediate region the soliton can be transmitted + trapped, completely trapped [figures 1(b); $\varepsilon = -0.08$, figure 1(b)] or trapped + reflected [figures 1(c); $\varepsilon = -0.16$, figure 1(c)]. In the case of repulsion the soliton is either transmitted [figures 2(a); $\varepsilon = 0.0048$, figure 2(a)] or reflected [figures 2(b); $\varepsilon = 0.0052$, figure 2(b)].

We have compared our results with the case when instead of the impurity spin with modified exchange interactions $\eta, \tilde{\eta}$ a linear point defect $\varepsilon$ is placed at $n_0$. This can be described by a term $\varepsilon \delta_{n,n_0} a_n$ on the right side of equation (3), where the values of $\varepsilon$ are determined by (7). We have obtained that for slow solitons the scattering pattern for a linear point defect are similar to those shown on figures 1 and 2. The differences in the soliton velocity and amplitude between the two cases are negligible.

Figure 3. Scattering of a soliton with $k = 0.157$ from a defect with modified exchange interactions (a) $\eta = \tilde{\eta} = -0.75$, (a') $\eta = \tilde{\eta} = 0.75$ or from a linear point defect (b) $\varepsilon = -0.318$, (b') $\varepsilon = 0.318$. All other parameters are the same as in figure 1.

To make the difference between an impurity with modified exchange interactions and a linear point defect visible we have increased the initial soliton velocity ($k = 0.157$). We fixed $\eta = \tilde{\eta} = \mp 0.75$ which leads to $\varepsilon = \mp 0.318$ (figure 3). Then, the soliton is split in transmitted and reflected parts and in the case of attractive impurities there is also a small trapped part (figures 3(a),(b)). The transmitted and reflected parts for the two signs of $\varepsilon$ are proportional (figures 3(b),(b')) while for the two signs of $\eta$ ($\tilde{\eta}$) they are different (figures 3(a),(a')). This is due to the circumstance that for defects with modified exchange interactions the contribution from
the second derivative perturbing term \( \sim \eta \cos k \) in (5) has influence on the soliton amplitude and respectively on the ratio between \( E_k \) and \( E_{nl} \). Positive values of \( \eta \) lead to an increase of the soliton amplitude at the moment of collision [figure 3(a')] . For negative values the amplitude and \( E_{nl} \) will decrease and consequently the transmitted part becomes larger [figure 3(a)].

![Figure 4](image-url)

**Figure 4.** Soliton evolution with \( k = 0.785 \) from a defect with modified exchange interactions \( J \eta \cos k = 0.566, \tilde{\eta} = 0 \) (a); \( J \eta \cos k = -0.566, \tilde{\eta} = 0 \) (a'); \( \eta = 0, J\tilde{\eta} = -0.566 \) (b); \( \eta = 0, J\tilde{\eta} = 0.566 \) (b'); or from a linear defect \( \varepsilon \) placed on one site of the chain [(c), (c')] or on three sites of the chain [(d), (d')]. \( \varepsilon = -1.132 \) [(c), (d)] and \( \varepsilon = 1.132 \) [(c'), (d')].

Now, we like to show in more detail the difference between \( \eta \) and \( \tilde{\eta} \) (figure 4). We have increased further the initial soliton velocity \( k = 0.785 \) and chosen the impurity parameters so that the resulting value is \( |\varepsilon| = 1.132 \). For such large soliton velocities we have the case of weak nonlinearity \( (E_k > E_{nl}) \) where the scattering picture for point defects is similar to this of linear waves. The soliton is split only into transmitted \( |T|^2 = v^2/(v^2 + \varepsilon^2) \) and reflected \( |R|^2 = \varepsilon^2/(v^2 + \varepsilon^2) \) parts which do not depend on the sign of \( \varepsilon \). The results for \( \varepsilon < 0 \) and \( \varepsilon > 0 \) are equivalent [figures 4(c),(c')]. Considerably different are the cases for the modified exchange interactions \( \eta \) or \( \tilde{\eta} \) where the process depends also on their sign. For \( \eta \neq 0 \) and \( \tilde{\eta} = 0 \) the first derivative perturbing term \( \sim \eta \sin k \) becomes important. If \( \eta > 0 \) the soliton velocity increases and the transmitted part is larger [figure 4(a)], while for \( \eta < 0 \) it decreases and the reflected part becomes larger [figure 4(a')]. For the impurity \( \tilde{\eta} (\tilde{\eta} = 0) \) we have a linear and a nonlinear perturbing terms in (5) which do not depend on \( k \). For such a model with constant linear and nonlinear point defects we would have a similar behavior as in figures 4(c),(c') independent of the sign of \( \eta \). Our results however [figures 4(b),(b')] show a difference between the two cases for attractive and repulsive impurities. This can be explain with the fact that the modified exchange interactions are between the impurity spin and its neighbors and have some spatial extension. We have performed a simulation with a linear defect \( \varepsilon \) expanded on three chain sites which has confirm our assumption [figures 4(d),(d')].

Finally, we have considered the possibility to compensate the influence of the two parameters \( \eta \) and \( \tilde{\eta} \) (figure 5). We chose \( J\eta \cos k = J\tilde{\eta} = \text{const} \ (\varepsilon = 0) \). This leads to elimination of the
Figure 5. Soliton evolution for $k = 0.157$ [(a), (b)] and $k = 0.785$ [(c), (d)]. $J\eta \cos k = \tilde{J}\tilde{\eta} = -0.48$ [(a), (c)] and $J\eta \cos k = \tilde{J}\tilde{\eta} = 0.48$ [(b), (d)].

linear and nonlinear perturbing terms in equation (5) and we can study the influence of the two derivative perturbing terms proportional to $\eta$. For small velocities ($k = 0.157$) they are negligibly small for both sign of $\eta$ and the soliton propagates through the impurity without any significant changes figures 5(a),(b)]. When the initial soliton velocity increases ($k = 0.785$) the second derivative perturbing term begins to act and a part from the soliton is reflected figures 5(c),(d)]. Depending on the sign of $\eta$ the first derivative perturbing term leads to a decrease or increase of the soliton velocity i.e. the reflected part is larger $[\eta < 0$, figure 5(c)] or smaller $[\eta > 0$, figure 5(d)]. There is no way to eliminate the impact of $\eta$ and $\tilde{\eta}$ due to their asymmetrical contribution to the perturbing terms in equation (5).

5. Conclusion
We have studied the influence of an impurity spin with modified exchange interactions $\eta$ in the $x, y$-plane and $\tilde{\eta}$ in the $z$-direction on the soliton dynamics. We have found that for slow solitons the scattering process is similar to this on point defects with the corresponding strength while for fast solitons it is completely different. Then, in addition to the strength of $\eta$ and $\tilde{\eta}$ the soliton evolution depends considerably on their spatial extension and sign.

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