Weighted Approximation theorem for Choldowsky generalization of the q-Favard-Szász operators

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Abstract. We study the convergence of these operators in a weighted space of functions on a positive semi-axis and estimate the approximation by using a new type of weighted modulus of continuity and error estimation.

Keywords: q-Favard- Szász operators, error estimation.

1. Introduction and auxiliary results

The classical Favard–Szász operators are given as follows

\[ S_n(f,x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \]

These operators and the generalizations have been studied by several other researcher (see. [1]-[7]) and references there in. In 1969, Jakimovski and Leviatan [11] introduced the Favard-Szász type operator, by using Appell polynomials \( p_k(x)(k \geq 0) \) defined by

\[ g(u)e^{-ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \]

where \( g(z) = \sum_{n=0}^{\infty} a_n z^n \) is analytic function in the disc \( |z| < R, R > 1 \) and \( g(1) \neq 0 \),

\[ P_{n,t}(f,x) = \frac{e^{nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx)f\left(\frac{k}{n}\right) \]

and they investigated some approximation properties of these operators.

Atakut at el. [10] defined a choldowsky type of Favard–Szász operators as follows:

\[ P^*_n(f,x) = \frac{e^{bx}}{g(1)} \sum_{k=0}^{\infty} p_k(x) \frac{b_n}{b_k} f\left(\frac{k}{b_n}\right), \]

with \( b_n \) a positive increasing sequence with the properties \( \lim_{n \to \infty} b_n = \infty \) and \( \lim_{n \to \infty} \frac{b_n}{n} = 0 \). They also studied some approximation properties of the operators.

Recently, A. Karaisa [12] defined Choldowsky type generalization of the Favard-Szász operators as follows:

\[ P^*_n(f; q; x) = \frac{E_q^{\frac{m}{n}}}{{A(1)}^m} \sum_{k=0}^{\infty} P_k(q; \frac{[m]_{q^n}}{[k]_{q^n}}) \frac{b_n}{[k]_q} f\left(\frac{[k]_q}{n}_{q^n}\right), \]

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Motivated by these results, in this paper we study weighted approximation and error estimation of these operators.

The rapid development of $q$-calculus has led to the discovery of various generalizations of Bernstein polynomials involving $q$-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

To make the article self-content, here we mention certain basic definitions of $q$-calculus, details can be found in [9] and the other recent articles. For each non negative integer $n$, the $q$-integer $[n]_q$ and the $q$-factorial $[n]_q!$ are, respectively, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q...[1]_q, & n = 1,2,..., \\ 1, & n = 0. \end{cases}$$

Then for $q > 0$ and integers $n,k,k \geq n \geq 0$, we have

$$[n+1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n[k-n]_q = [k]_q.$$

The $q$-derivative $D_q f$ of a function $f$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$

The $q$-analogues of the exponential function are given by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q},$$

and

$$E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q}.$$

The exponential functions have the following properties:

$$D_q (e_q^{ax}) = a e_q^{ax}, \quad D_q (E_q^{ax}) = a E_q^{aqx}, \quad e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1.$$

**Lemma 1.** [12] The following hold:

(i) $P_n^*(e_0; q; x) = 1$,
(ii) $P_n^*(e_1; q; x) = x + \frac{D_q(A(1))E_{\frac{[n]_q}{[n]_q}} e_q^{[n]_q x} [n]_q}{A(1)} b_n x$,
(iii) $P_n^*(e_2; q; x) = x^2 + \frac{E_{\frac{[n]_q}{[n]_q}} e_q^{[n]_q x} [n]_q}{A(1)} q D_q(A(q)) b_n x + \frac{D_q^2(A(1))E_{\frac{[n]_q}{[n]_q}} e_q^{[n]_q x} [n]_q}{A(1)} b_n^2 x$.

where $e_i(x) = x^i, \quad i = 0,1,2$.

Now we give an auxiliary lemma for the Korovkin test functions.

**Lemma 2.**

(i) $P_n^*(t-x; q; x) = \frac{D_q(A(1))E_{\frac{[n]_q}{[n]_q}} e_q^{[n]_q x} [n]_q}{A(1)} b_n x$,
(ii) $P_n^*((t-x)^2; q; x) = \frac{E_{\frac{[n]_q}{[n]_q}} e_q^{[n]_q x} [n]_q}{A(1)} q D_q(A(q)) b_n x + \frac{D_q^2(A(1))E_{\frac{[n]_q}{[n]_q}} e_q^{[n]_q x} [n]_q}{A(1)} b_n^2 x$. 

where $D_q f$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$
2. Weighted approximation

Let $B_{x^2}[0, \infty)$ be the set of functions defined on $[0, \infty)$ satisfying the condition $|f(x)| \leq M_f(1 + x^2)$, where $M_f$ is a constant depending on $f$ only. By $C_{x^2}[0, \infty)$, we denote subspace of all continuous functions belonging to $B_{x^2}[0, \infty)$. Also, let $C_{x^2}^+[0, \infty)$ be the subspace of all $f \in C_{x^2}[0, \infty)$ for which $\lim_{x \to \infty} \frac{f(x)}{x}$ is finite. The norm on $C_{x^2}^+[0, \infty)$ if $\|f\|_2^a = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}$. For any positive number $a$, we define

$$\omega_a(f, \delta) = \sup_{|t - x| \leq \delta, x \in [0, a]} |f(t) - f(x)|,$$

and denote the usual modulus of continuity of $f$ on the closed interval $[0, a]$. We know that for a function $f \in C_{x^2}[0, \infty)$, the modulus of continuity $\omega_a(f, \delta)$ tends to zero.

Now, we shall discuss the weighted approximation theorem, when the approximation formula holds true on the interval $[0, \infty)$.

**Theorem 1.** For each $f \in C_{x^2}^+[0, \infty)$, we have

$$\lim_{n \to \infty} P_n^*(f; x) = f(x) = 0.$$

**Proof.** Using the theorem in [8], we see that it is sufficient to verify the following three conditions

\[ (2.1) \lim_{n \to \infty} \|P_n^*(t^r; q; x) - x^r\|_2 = 0, \quad r = 0, 1, 2. \]

Since, $P_n^*(1, x) = 1$, the first condition of (2.1) is satisfied for $r = 0$. Now,

$$\|P_n^*(t; q; x) - x\|_2 = \sup_{x \in [0, \infty)} \frac{|P_n^*(t; q; x) - x|}{1 + x^2} \leq \sup_{x \in [0, \infty)} \left( t + \frac{D_q(A(1))E_q\frac{n!}{n} e_q \frac{n!}{n} b_n}{A(1)} \right) \frac{1}{1 + x^2} \leq \sup_{x \in [0, \infty)} \left( D_q(A(1))E_q\frac{n!}{n} e_q \frac{n!}{n} b_n \right) \frac{1}{1 + x^2}$$

which implies that

$$\|P_n^*(t, x) - x\|_2 = 0.$$

Finally,

$$\|P_n^*(t^2; q; x) - x^2\|_2 = \sup_{x \in [0, \infty)} \frac{|P_n^*(t^2; q; x) - x^2|}{1 + x^2} \leq \sup_{x \in [0, \infty)} \left( t^2 + \frac{E_q\frac{n!}{n} e_q \frac{n!}{n} \{qD_q(A(q)) + D_q(A(1))\} b_n x}{A(1)} + \frac{D_q^2(A(1))E_q\frac{n!}{n} e_q \frac{n!}{n} b_n^2}{A(1)} \right) \frac{1}{1 + x^2}$$

which implies that $\|P_n^*(t^2; q; x) - x^2\|_2 \to 0$ as $|n| \to \infty$. Thus proof is completed.

We give the following theorem to approximate all functions in $C_{x^2}[0, \infty)$.

**Theorem 2.** For each $f \in C_{x^2}[0, \infty)$ and $\alpha > 0$, we have

$$\lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{P_n^*(f; q; x) - f(x)}{(1 + x^2)^{1+\alpha}} = 0.$$

**Proof.** For any fixed $x_0 > 0$,

$$\sup_{x \in [0, \infty)} \frac{|P_n^*(f; q; x) - f(x)|}{(1 + x^2)^{1+\alpha}} \leq \sup_{x \leq x_0} \frac{|P_n^*(f; x) - f(x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|P_n^*(f; q; x) - f(x)|}{(1 + x^2)^{1+\alpha}} \leq \|P_n^*(f; q; x) - f\|_{C[0, x_0]} + \|f\|_2 \sup_{x \geq x_0} \frac{|P_n^*(f; q; x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}}.$$

The first term of the above inequality tends to zero from Theorem 3. By Lemma 3, for any fixed $x_0 > 0$ it is easily seen that $\sup_{x \geq x_0} \frac{|P_n^*(f; q; x)|}{(1 + x^2)^{1+\alpha}}$ tends to zero as $|n| \to \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be made small enough. Thus the proof is completed.

□
3. Error Estimation

The usual modulus of continuity of $f$ on the closed interval $[0, b]$ is defined by

$$\omega_b(f, \delta) = \sup_{|t-x| \leq \delta, x,t \in [0,b]} |f(t) - f(x)|, \ b > 0.$$ 

We first consider the Banach lattice, for a function $f \in E$, $\lim_{\delta \to 0^+} \omega_b(f, \delta) = 0$, where

$$E := \left\{ f \in C[0, \infty) : \lim_{x \to \infty} \frac{f(x)}{1 + x^2} \text{ is finite} \right\}.$$

The next theorem gives the rate of convergence of the operators $P^*_n(f, x)$ to $f(x)$, for all $f \in E$.

**Theorem 3.** Let $f \in E$ and $\omega_{b+1}(f, q; \delta), \ 0 < q < 1$ be its modulus of continuity on the finite interval $[0, b+1] \subset [0, \infty)$, where $a > 0$ then, we have

$$\|P^*_n(f; q; x) - f\|_{C[0,b]} \leq M_f(1 + b^2)\delta_n(b) + 2\omega_{b+1}\left(f, \sqrt{\delta_n(b)}\right).$$

**Proof.** The proof is based on the following inequality

$$\|P^*_n(f; q; x) - f\|_{C[0,b]} \leq M_f(1 + b^2)\omega_{b+1}(f, \delta).$$

For all $(x, t) \in [0, b] \times [0, \infty) := S$. To prove (3.1), we write

$$S = S_1 \cup S_2 := \{(x, t) : 0 \leq x \leq b, 0 \leq t \leq b + 1\} \cup \{(x, t) : 0 \leq x \leq b, t > b + 1\}.$$

If $(x, t) \in S_1$, we can write

$$|f(t) - f(x)| \leq \omega_{b+1}(f, |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right)\omega_{b+1}(f, \delta)$$

where $\delta > 0$. On the other hand, if $(x, t) \in S_2$, using the fact that $t - x > 1$, we have

$$|f(t) - f(x)| \leq M_f(1 + x^2 + t^2) \leq M_f(1 + 3x^2 + 2(t - x)^2) \leq N_f(1 + b^2)(t - x)^2$$

where $N_f = 6M_f$. Combining (3.2) and (3.3), we get (3.1). Now from (3.1) it follows that

$$|P^*_n(f; q; x) - f(x)| \leq N_f(1 + b^2)P^*_n((t - x)^2; q; x) + \left(1 + \frac{P^*_n((t - x)^2, x)}{\delta}\right)\omega_{b+1}(f, \delta)$$

By Lemma 2, we have

$$P^*_n(t - x)^2 \leq \delta_n(b).$$

Choosing $\delta = \sqrt{\delta_n(b)}$, we get the desired estimation. \(\square\)

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