Research Article

Application of Natural Transform Method to Fractional Pantograph Delay Differential Equations

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In this paper, a new method based on combination of the natural transform method (NTM), Adomian decomposition method (ADM), and coefficient perturbation method (CPM) which is called "perturbed decomposition natural transform method" (PDNTM) is implemented for solving fractional pantograph delay differential equations with nonconstant coefficients. The fractional derivative is regarded in Caputo sense. Numerical evaluations are included to demonstrate the validity and applicability of this technique.

1. Introduction

Delay differential equations (DDEs) occur when the rate of change of process is specified by its state at a certain past state known as its history. These types of differential equations occur in traffic models, control systems, population dynamics, and many natural phenomena [1–3]. The expression of delay in mathematical modeling of real-life phenomena leads to a precise and acceptable description of the dynamics of the model. DDEs are complex in nature because the history of the system over the delayed interval is given as an initial condition. Due to this reason, these equations are difficult to solve analytically; hence, a numerical method is required. Pantograph delay equations are one of the well-known functional differential equations with proportional delay and often appear in many scientific models such as mechanics and electrodynamics [4, 5]. In 1851, it was the first time that a device named “pantograph” was used in construction of the electric locomotive after which this name was originated from that time. Pantograph was modeled mathematically in 1971 (see [6] for more details). Thereafter, pantograph delay equation was studied by many authors and solved by several numerical methods. The most important of them are collocation method [7], spline method [8], Runge–Kutta method [9], homotopy perturbation method [10], and Adomian decomposition method [11]. In this paper, we intend to apply the natural transform method to solve fractional delay differential equations with variable coefficients. To solve differential and integral equations, several integral transforms such as Fourier, Laplace, and Sumudu are used [12–16]. The natural transform method is a new integral transform was introduced by Khan and Khan [17], and its properties were described. Belgacem used the natural transform method to solve Bessel’s differential equation, Maxwell’s equation, and nonlinear Klein–Gordon equations (see [18, 19] and its references). A table for some natural transformation properties on certain functions was presented by Khan and Khan in [17]. Moreover, this method is applied to find the solution of diffusion equations [20]. Finally, they proved that the natural transform method converges to the Laplace and Sumudu transforms.

In this paper, a novel technique is applied to find an approximate solution for the fractional pantograph delay differential equation of the following form:

\[
\begin{align*}
D^\alpha_0 x(t) &= f(t) + \sum_{i=1}^{n} a_i(t)x(q_i t), \quad 0 < q_i \leq 1, \\
x(0) &= c,
\end{align*}
\]

where \(0 < \alpha < 1\), \(f(t)\) and \(a_i(t)\) are the given analytic functions, and \(D^\alpha_0\) denotes the Caputo fractional differential
operator. The proposed method is a combination of natural transform method (NTM), Adomian decomposition method (ADM) [21], and coefficient perturbation method (CPM) [22], which is called "perturbed decomposition natural transform method." Unlike existing analytical and numerical methods, this method gives an approximate and/or exact solution for problem (1) by a simple calculation and in an elegant way.

The structure of this paper is as follows: In Section 2, a brief description of the fractional calculus, natural transform method, and Adomian decomposition method are provided. The existence of unique solution for problem (1) is investigated in Section 3. In Section 4, we introduce our new method (called briefly, PDNTM) for obtaining the approximate solution for problem (1). Numerical examples illustrate efficiency and applicability of the proposed method in Section 5, and conclusions are drawn in Section 6.

2. Preliminaries

In this section, we recall some preliminary results which will be needed throughout the paper.

2.1. Fractional Calculus. There are some kinds of definitions for fractional derivatives and integrals, such as Riemann–Liouville integral [23], which are described as follows.

For \( f \in L_1[a, b] \), the Riemann–Liouville fractional integral of order \( \alpha \) is defined as

\[
\int_a^x f(t) \, dt = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) \, dt, \quad \alpha > 0.
\]

For \( \alpha = 0 \), set \( \int_a^x f(t) \, dt = I \), the identity operator. Let \( f(x) = (x - a)^\beta \) for some \( \beta > -1 \) and \( \alpha > 0 \). Then,

\[
\int_a^x f(t) \, dt = \frac{1}{\Gamma(\alpha + \beta + 1)} (x - a)^{\alpha+\beta}.
\]

Let \( m \) be the order of the Caputo differential operator of order \( \alpha \). For \( \alpha = 0 \), set \( D^\alpha_0 = I \), the identity operator. The Caputo differential operator of order \( n \) is defined by

\[
D^m_a f(x) = D^m_0 \left[ f(x) - T_{m-1}[f(x); a] \right],
\]

whenever \( D^m_0 \left[ f(x) - T_{m-1}[f(x); a] \right] \) exists, where \( T_{m-1}[f(x); a] \) denotes the Taylor polynomial of degree \( m - 1 \) for the function \( f \) around the point \( a \). In the case \( m = 0 \), define \( T_{m-1}[f(x); a] = 0 \). Under the above conditions, it is easy to show that

\[
D^m_a f(x) = T^{m-a} D^m_0 f(x).
\]

More details can be known by referring to [23, 24].

2.2. Natural Transform Method (NTM). In this section, we review some properties of the atural transform method that will be used in later sections. For the function \( f(t) \) defined in [0, \( \infty \)), piecewise function continues in every finite interval \( 0 \leq t \leq K \) and \( f \in A \) where

\[
A = \{ f(t) : \exists M, r_1, r_2 > 0, |f(t)| < Me^{r_1t}, t \in (-1)^j \times [0, \infty) \}.
\]

The natural transform of \( f(t) \) is defined as follows:

\[
R(u, s) = N(f) = \int_0^\infty e^{-ut} f(ut) \, dt.
\]

With the above conditions, the right-hand side integral in (39) is convergent. By taking \( u = 1 \) in (39), the Laplace transform of \( f(t) \),

\[
F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) \, dt, \quad s > 0,
\]

and for \( s = 1 \), the Sumudu transform of \( f(t) \),

\[
S(u) = S(f(t)) = \int_0^\infty e^{-t} f(ut) \, dt, \quad t > 0,
\]

will be obtained. The inverse natural transform of \( R(u, s) \) is defined in [19] by

\[
f(t) = N^{-1}[R(u, s)] = \frac{1}{2\pi} \int_{c-\infty}^{c+\infty} e^{stu} R(u, s) \, ds.
\]

We recall some useful properties of the natural transform method from [19] in Table 1.

2.3. Adomian Decomposition Method (ADM). Consider the general form of a differential equation as

\[
Ly(t) + Sy(t) + Ry(t) = g(t),
\]

where \( Ly \) is the linear invertible part, \( Ny \) corresponds to the nonlinear term, and \( Ry \) is the remaining part. Solving \( Ly \) from (12), we have

\[
Ly(t) = g(t) - Ny(t) - Ry(t).
\]

As \( L \) is invertible,

\[
y(t) = y(0) + L^{-1}g(t) - L^{-1}(Ny(t)) - L^{-1}(Ry(t)).
\]

The Adomian decomposition method (ADM) [21] is a powerful tool for solving both linear and nonlinear functional equations. This method decomposes the solution \( y(x) \) into a rapidly convergent series of solution components and then decomposes the analytic nonlinearity \( N(y) \) into the series of the Adomian polynomials:

\[
y(t) = \sum_{i=0}^{\infty} y_i(t),
\]

\[
N(y(t)) = \sum_{i=0}^{\infty} A_i(t),
\]

where \( A_i = A_i(y_0(t), y_1(t), \ldots, y_i(t)) \) are the well-known Adomian polynomials as follows:
The composition method was studied by Cherruault in [25, 26]. The equivalent integral equation representation of (1) is of the following form (see [24]):

\[ f(t) = \sum_{i=0}^{\infty} a_i(t) \phi_i(t) \]

This study is based on the Banach contraction principle. In this section, we prove existence and uniqueness of the solution for the fractional delay differential equation (1).

3. Existence and Uniqueness of Solution

In this section, we prove existence and uniqueness of the solution for the fractional delay differential equation (1). This study is based on the Banach contraction principle.

**Theorem 1.** Assume \( f(t) \) and \( a_i(t) \) are the given analytic functions in (1) and \( M_i = \max_{t \in [0, T]} |a_i(t)| \). If

\[ \sum_{i=0}^{\infty} T^a M_i / \Gamma(a+1) < 1, \]

then problem (1) has a unique solution \( x(t) \in C^1[0, T] \).

**Proof.** The equivalent integral equation representation of problem (1) is of the following form (see [24]):

\[ x(t) = c + \int_0^t f(s) ds + \sum_{i=0}^{n} \int_0^t a_i(t) x(q_i) dq_i. \]

Based on the Banach contraction principle, it is enough to prove that the operator \( A \) defined by

\[ (Ax)(t) = c + \int_0^t f(s) ds + \sum_{i=0}^{n} \int_0^t a_i(t) x(q_i) dq_i, \]

has a unique fixed point. It is clear that the operator \( A \) maps \( C[0, T] \) into itself and that

\[ \|A - I\|_{\infty} = \min_{\|x\|_{\infty} = 1} |Ax - x| \]

which implies, under our assumption, that \( A \) is a contraction mapping. Then, the Banach contraction principle implies that \( A \) has a unique fixed point \( x(t) \).

4. Main Results

Let us approximate the coefficients \( f(t) \) and \( a_i(t), (i = 1, \ldots, n) \), in equation (1) by polynomials of fractional order \( a \) as follows:

\[ f(t) = f_\rho(t) = \sum_{i=0}^{\rho} f_i t^a, \]

\[ a_i(t) = a_{i,\rho}(t) = \sum_{i=0}^{\sigma_i} a_{i,\rho} t^a, \]

Then, problem (1) can be written in perturbed form as

\[ D^a x(t) = \sum_{i=0}^{\rho} f_i t^a + \sum_{i=1}^{n} \sum_{i=0}^{\sigma_i} a_{i,\rho} t^a \bar{x}(q_i), \]

\[ \bar{x}(0) = x(0) = c, \]

which is called “perturbed problem” with the exact solution \( \bar{x}(t) \) that is an approximation for the exact solution \( x(t) \) of the main problem (1). This is a first step of our new method that converts the main problem (1) to new one with polynomial coefficients. This strategy will simplify the calculation in the implementation of the proposed method. Similar ideas have been used in [27, 28] for solving boundary value problem and eigenvalue problems.

**Theorem 2.** Let \( n = 1 \) and

\[ \|f - f_\rho\|_{\infty} \leq \epsilon, \]

\[ \|a_1 - a_{1,\rho}\|_{\infty} \leq \epsilon, \]

Table 1: Natural transforms of some functions.

| Functional form | Natural transform |
|-----------------|-------------------|
| (1) \( af(t) + bg(t) \) | \( aN(f(t)) + bN(g(t)) \) |
| (2) 1 | \( 1/\sigma \) |
| (3) \( t^a \) | \( \Gamma(a+1)t^{a+1} \) |
| (4) \( \sum_{i=0}^{\infty} a_i(t)x \) | \( \sum_{k=0}^{\infty} \Gamma((ia+1)t^{a+1})/t^{a+1} \) |
| (5) \( D^a_0 f(t) \) | \( (s^a/\sigma^a)R(s, u) - \sum_{k=0}^{n-1} (s^{a-(k+1)/t^{a+1}})^{f(k)(0)}, n-1 < a \leq n \) |
for some constant $e$ in (23). Then, we have
\[ \|x(t) - \tilde{x}(t)\|_\infty = O(e), \text{ i.e.,} \]
\[ \|x(t) - \tilde{x}(t)\|_\infty \rightarrow 0, \quad e \rightarrow 0. \]  
(26)

Proof. It can be easily proved that $x(t) \in C[0, T]$ is a solution of equation (1) if and only if $x(t) \in C[0, T]$ is a solution for the integral equation
\[ x(t) = c + \int_0^t f(t) \, dt + J_0^\alpha (a(t)x(qt)), \]  
(27)

where $J_0^\alpha$ is the Riemann–Liouville fractional differential operator of order $\alpha$ (see [24]). Consequently, for perturbed problem (24), we have
\[ \tilde{x}(t) = c + \int_0^t f(t) \, dt + J_0^\alpha (a(t)\tilde{x}(qt)). \]  
(28)

Subtracting equation (28) from equation (27) and some manipulation yield
\[ x(t) - \tilde{x}(t) = \int_0^t (f(t) - f_\rho(t)) \, dt + \int_0^t (a(t)x(qt) - a_\rho(t)\tilde{x}(qt)) \, dt + \int_0^t (a(t)\tilde{x}(qt) - \tilde{x}(qt)) \, dt. \]  
(29)

By taking $\| \cdot \|_\infty$ from both sides of equation (29), we obtain
\[ \|x(t) - \tilde{x}(t)\|_\infty \leq \| \int_0^t (f(t) - f_\rho(t)) \, dt \|_\infty + \| \int_0^t (a(t) - a_\rho(t))\tilde{x}(qt) \|_\infty \|_\infty + \| \int_0^t a(t)(x(qt) - \tilde{x}(qt)) \|_\infty \|_\infty. \]  
(30)

From (25), we have
\[ \| \int_0^t (f(t) - f_\rho(t)) \, dt \|_\infty \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f(s) - f_\rho(s) \|_\infty \, ds \]
\[ \leq \frac{e}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds = \frac{eT^\alpha}{\Gamma(\alpha + 1)}, \]
\[ \| \int_0^t (a(t) - a_\rho(t))\tilde{x}(qt) \|_\infty \|_\infty \leq \frac{\| \tilde{x}(qt) \|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| a(s) - a_\rho(s) \|_\infty \, ds \]
\[ \leq \frac{\| \tilde{x}(qt) \|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds = \frac{\| \tilde{x}(qt) \|_\infty}{\Gamma(\alpha + 1)}, \]
\[ \| \int_0^t a(t)(x(qt) - \tilde{x}(qt)) \|_\infty \|_\infty \leq \frac{\| a(t) \|_\infty}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \| x(s) - \tilde{x}(s) \|_\infty \, ds \]
\[ \leq \frac{\| a(t) \|_\infty}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \, ds \times \frac{T^\alpha}{\alpha}. \]

Then, from (30) and (31), we get
\[ \|x(t) - \tilde{x}(t)\|_\infty \leq \frac{eT^\alpha}{\Gamma(\alpha + 1)} + \frac{\| \tilde{x}(qt) \|_\infty}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \, ds \]
\[ + \frac{T^\alpha}{\alpha} \| a(t) \|_\infty \| x(t) - \tilde{x}(t)\|_\infty. \]  
(32)

Consequently,
\[ \|x(t) - \tilde{x}(t)\|_\infty \left(1 - \frac{T^\alpha}{\Gamma(\alpha + 1)} \right) \leq \frac{\| \tilde{x}(qt) \|_\infty}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \, ds \]
\[ + \frac{T^\alpha}{\alpha} \| a(t) \|_\infty \| x(t) - \tilde{x}(t)\|_\infty. \]  
(33)

After some manipulation on (33), we obtain
\[ \|x(t) - \tilde{x}(t)\|_\infty \leq \frac{eT^\alpha (1 + \tilde{x}(qt))}{\Gamma(\alpha + 1) - T^\alpha a(t)} \| x(t) - \tilde{x}(t)\|_\infty. \]  
(34)

From the upper bound $(eT^\alpha (1 + \tilde{x}(qt)))/\Gamma(\alpha + 1) - T^\alpha a(t)$ in (34), it is seen that if $e \rightarrow 0$, then $\|x(t) - \tilde{x}(t)\|_\infty \rightarrow 0$. \( \square \)

4.1. Method of Solution. Taking the natural transform of both sides of equation (24) yields
\[ N[\mathcal{D}_s^\alpha \tilde{x}(t)] = N \left[ \sum_{i=0}^{\rho} f_i t^\alpha \right] + N \left[ \sum_{i=1}^n \sum_{l=0}^{\sigma_i} a_{il} t^\alpha \tilde{x}(qt) \right]. \]  
(35)

Using the properties of the natural transform method in Table 1, we get
\[ \frac{\partial^\alpha}{\partial t^\alpha} R(u, s) - \frac{\partial^{\alpha - 1}}{\partial t^{\alpha - 1}} \tilde{x}(0) = \sum_{l=0}^{\rho} f^fl \frac{(l + 1) t^{(l+1)\alpha} }{\Gamma(l+1)\alpha + 1} + \sum_{l=1}^{n} \frac{a_l}{s^\alpha} N[t^{l\alpha} \widetilde{x}(q,t)]. \]

Consequently,
\[ R(u, s) = \tilde{x}(0) + \sum_{l=0}^{\rho} f^fl \frac{(l + 1) t^{(l+1)\alpha} }{\Gamma(l+1)\alpha + 1} + \sum_{l=1}^{n} \frac{a_l}{s^\alpha} N[t^{l\alpha} \widetilde{x}(q,t)]. \]

By taking the inverse natural transform of (37), we obtain
\[ \tilde{x}(t) = \tilde{x}(0) + \sum_{l=0}^{\rho} f^fl \frac{(l + 1) t^{(l+1)\alpha} }{\Gamma(l+1)\alpha + 1} + \sum_{l=1}^{n} \frac{a_l}{s^\alpha} N[t^{l\alpha} \widetilde{x}(q,t)]. \]

Remark 1. As is clear, in recurrence relation (41), the terms \( N[t^{l\alpha} \widetilde{x}_k(q,t)] \) and \( N^{-1}[(t^\alpha/s^\alpha) N[t^{l\alpha} \widetilde{x}_k(q,t)]] \) can be calculated easily using properties (3) in Table 1, since the term \( t^{l\alpha} \widetilde{x}_k(q,t) \) is a fractional-order polynomial for \( i = 1, \ldots, n \) and \( k = 0, 1, 2, \ldots \). This is a main advantage in our proposed method for calculating the components \( \tilde{x}_k(t) \) in (43) and making our new method easy to be used and writing a computer program for it.

We summarize the steps of our proposed method (called PDNTM method) as an implementation algorithm as follows (Algorithm 1).

5. Numerical Results

In this section, we present some examples to support the accuracy and simplicity of the proposed method. These results in this section indicate the applicability of the method and verification of the theoretical results in the previous section.

Example 1. We consider the following initial value problem:
\[ D^\alpha_0^x x(t) = 1 + 2x \left( \frac{t}{2} \right) - x(t), \quad x(0) = 0. \] (45)

The exact solution for \( \alpha = 1 \) is \( x(t) = t \). According to the proposed method, by taking natural transform to both sides of equation (45), we obtain
\[ \frac{\partial^\alpha}{\partial t^\alpha} R(s, u) - \frac{\partial^{\alpha - 1}}{\partial t^{\alpha - 1}} x(0) = N[1] + N \left[ 2x \left( \frac{t}{2} \right) - x(t) \right]. \] (46)

From initial condition \( x(0) = 0 \), we obtain
\[ R(s, u) = \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha - 1}}{s^\alpha} N \left[ 2x \left( \frac{t}{2} \right) - x(t) \right]. \] (47)

Applying the inverse of natural transform in (47) yields
\[ x(t) = \frac{t^\alpha}{\Gamma(\alpha + 1)} + N^{-1} \left[ \frac{t^\alpha}{s^\alpha} N \left[ 2 \left( \frac{t}{2} \right) \right] - x(t) \right]. \] (48)

Now, based on (41), we obtain
\[ \sum_{n=0}^{\infty} x_n(t) = \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha - 1}}{s^\alpha} N \left[ \sum_{n=0}^{\infty} \frac{t}{2} - \sum_{n=0}^{\infty} x_n(t) \right], \] (49)

which gives
\[ \begin{aligned}
    x_0(t) &= \frac{t^\alpha}{\Gamma(\alpha + 1)}, \\
    x_{k+1}(t) &= N^{-1} \left[ \frac{t^\alpha}{s^\alpha} N \left[ 2x_k \left( \frac{t}{2} \right) - x_h(t) \right] \right], \quad k = 0, 1, 2, \ldots
\end{aligned} \] (50)

Then, all of the components \( x_k(t) \) can be obtained as follows:
\[ x_1(t) = \frac{1}{\Gamma(2\alpha + 1)} \sum_{i=1}^{\infty} \left( \frac{1}{2^\alpha - 1} - 1 \right) \frac{t^{2\alpha}}{i^{(i+1)\alpha}} \] (51)

Continuing the above recurrence relation yields

\[ x(t) = \frac{t^\alpha}{\Gamma(\alpha + 1)} + \sum_{i=1}^{\infty} \left( \frac{1}{2^\alpha - 1} - 1 \right) \frac{t^{(i+1)\alpha}}{i^{(i+1)\alpha}} \] (52)

For the case \( \alpha = 1 \), the second term in (52) is equal to zero; therefore, \( x(t) = t \), which is the exact solution of problem (45). For \( M = 5 \), the approximate solution is

\[ x(t) = \frac{t^\alpha}{\Gamma(\alpha + 1)} + \sum_{i=1}^{5} \left( \frac{1}{2^\alpha - 1} - 1 \right) \frac{t^{(i+1)\alpha}}{i^{(i+1)\alpha}} \] (53)

In Figure 1, the convergence of approximate solutions to exact solution as \( \alpha \rightarrow 1 \) is shown.

Example 2. Consider the following fractional pantograph problem (see [29]):

\[ D_0^\alpha x(t) = t^2 - 1 - \frac{5}{6} x(t) + 4x\left( \frac{t}{2} \right) + 9x\left( \frac{t}{3} \right), \quad x(0) = 1. \] (54)

In the case \( \alpha = 1 \), the exact solution is \( x(t) = 1 + (67/7)t + (1675/72)t^2 + (12175/1296)t^3 \).

According to the proposed method for \( \alpha = 1 \), we obtain
The exact solution for $\alpha = 1$ is $x(t) = \exp(t)$. Let

$$a_1(t) = \frac{1}{2} \exp\left(\frac{t}{2}\right) = a_{1,\alpha}(t) = \sum_{i=0}^{\alpha} a_{1,i} t^i,$$

$$a_2(t) = a_{2,\alpha}(t) = \frac{1}{2}$$

Then, the perturbed problem is as follows:

$$D_0^\alpha \tilde{x}(t) = a_{1,\alpha}(t)x\left(\frac{t}{2}\right) + a_{2,\alpha}(t)x(t), \quad \tilde{x}(0) = 1. \quad (59)$$

The approximate solutions for $\alpha = 0.75, 0.85,$ and $0.95$ and exact solution for $\alpha_1 = 5, \alpha_2 = 0,$ and $M = 5$ are shown in Figure 3. Table 2 denotes the behavior of approximate solution $\tilde{x}_{10}(t)$ for different values of $\alpha_1 = 5, 10, 15$ with $\alpha = 1$. These results are consistent with Theorem 2 which shows convergence of the solution of perturbed problem (59) to (57).

6. Future Work

In our future work, the PDNTM method will be studied for the general class of problems such as the system of fractional pantograph delay differential equations of the form

$$D_0^\alpha x_i(t) = f_i(t) + \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(t) x_j(q_i t), \quad i = 1, \ldots, n, \quad (60)$$

subject to the initial conditions

$$x_i(0) = \beta_i, \quad i = 1, \ldots, n, \quad (61)$$

where $0 < q_i < 1$ and $0 < q_i < 1$. To this end, we can approximate the coefficients $f_i(t)$ and $a_{ij}(t)$ by polynomials of fractional order $0 < \alpha < 1$ as follows:

![Figure 2: Approximate solution for different values of $\alpha$.](image1)

![Figure 3: Approximate solution for different values of $\alpha$.](image2)

| $t$ | Exact. Sol. | App. Sol. | Abs. Err |
|-----|-------------|-----------|----------|
| 0.0 | 1.0000000000 | 1.0000000000 | 0 |
| 0.1 | 1.1051709282 | 1.1051709163 | 1.783752 × 10^{-9} |
| 0.2 | 1.2214027582 | 1.2214234795 | 1.959785 × 10^{-9} |
| 0.3 | 1.3498580076 | 1.3498580054 | 2.167082 × 10^{-9} |
| 0.4 | 1.4918246976 | 1.4918246952 | 2.394817 × 10^{-9} |
| 0.5 | 1.6487212707 | 1.6487212680 | 2.646726 × 10^{-9} |
| 0.6 | 1.8221188004 | 1.8221187975 | 2.925254 × 10^{-9} |
| 0.7 | 2.0137527075 | 2.0137527042 | 3.233969 × 10^{-9} |
| 0.8 | 2.2255409285 | 2.2255409249 | 3.580051 × 10^{-9} |
| 0.9 | 2.4596031112 | 2.4596031072 | 3.969453 × 10^{-9} |
| 1.0 | 2.7182818285 | 2.7182818285 | 4.431174 × 10^{-9} |

| $t$ | Exact. Sol. | App. Sol. | Abs. Err |
|-----|-------------|-----------|----------|
| 0.0 | 1.0000000000 | 1.0000000000 | 0 |
| 0.1 | 1.1051709181 | 1.1051709163 | 1.783752 × 10^{-9} |
| 0.2 | 1.2214027582 | 1.2214234795 | 1.959785 × 10^{-9} |
| 0.3 | 1.3498580076 | 1.3498580054 | 2.167082 × 10^{-9} |
| 0.4 | 1.4918246976 | 1.4918246952 | 2.394817 × 10^{-9} |
| 0.5 | 1.6487212707 | 1.6487212680 | 2.646726 × 10^{-9} |
| 0.6 | 1.8221188004 | 1.8221187975 | 2.925254 × 10^{-9} |
| 0.7 | 2.0137527075 | 2.0137527042 | 3.233969 × 10^{-9} |
| 0.8 | 2.2255409285 | 2.2255409249 | 3.580051 × 10^{-9} |
| 0.9 | 2.4596031112 | 2.4596031072 | 3.969453 × 10^{-9} |
| 1.0 | 2.7182818285 | 2.7182818285 | 4.431174 × 10^{-9} |
\[ f_i(t) = \sum_{\tau=0}^{\infty} f_i^t \lambda_{\tau}, \]
\[ a_{ij}^\tau(t) = \sum_{\tau=0}^{\infty} a_{ij}^t \tau^{\alpha}, \]

and extend all processes of Section 4.1.

7. Conclusion

In this paper, a new method based on the combination of the natural transform method, Adomian decomposition method, and coefficient perturbation method is introduced to solve the fractional pantograph delay differential equations with nonconstant coefficients. From this method, we obtain an approximate or exact solution for the problem with some simple iterative calculation that is easy in use and computer programming. The numerical results confirm the accuracy of the method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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