Distance-regular graphs with valency $k$ having smallest eigenvalue at most $-k/2$

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1 Introduction

In this paper, we study the non-bipartite distance-regular graphs with valency $k$ and having a smallest eigenvalue at most $-k/2$ (For notations and explanation of the graphs, see next section and [2] or [12]). There are seven infinite families known, namely

1. The odd polygons with valency 2;
2. The complete tripartite graphs $K_{t,t,t}$ with valency $2t$ at least 2;
3. The folded $(2D + 1)$-cubes with valency $2D + 1$ and diameter $D \geq 2$;
4. The Odd graphs with valency $k$ at least 3;
5. The Hamming graphs $H(D, 3)$ with valency $2D$ where $D \geq 2$;
6. The dual polar graphs of type $B_D(2)$ with $D \geq 2$;

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7. The dual polar graphs of type $2A_{2D-1}(2)$ with $D \geq 2$.

First we will show a valency bound for distance-regular graphs with a relatively large, in absolute value, smallest eigenvalue.

**Theorem 1.1** For any real number $1 > \alpha > 0$ and any integer $D \geq 2$, the number of coconnected non-bipartite distance-regular graphs with valency $k$ at least two and diameter $D$, having smallest eigenvalue $\theta_{\text{min}}$ not larger than $-\alpha k$, is finite.

**Remarks.** (i) Note that the regular complete $t$-partite graphs $K_{t \times s}$ ($s, t$ positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.

(ii) Note that there are infinitely many bipartite distance-regular graphs with diameter 3, for example the point-block incidence graphs of a projective plane of order $q$, where $q$ is a prime power. For diameter 4 this is also true, for example the Hadamard graphs.

(iii) The second largest eigenvalue for a distance-regular graphs behaves quite differently from its smallest eigenvalue. For example $J(n, t) n \geq 2t \geq 4$, has valency $t(n-t)$ and second largest eigenvalue $(n-t-1)(t-1)-1$. So for fixed $t$ there are infinitely many Johnson graphs $J(n, t)$ with second largest eigenvalue larger then $k/2$.

Then we classify the non-bipartite distance-regular graphs with diameter at most 4 with valency $k$ having smallest eigenvalue at most $-k/2$ where for diameter 4 we have also the condition $a_1 \neq 0$.

**Theorem 1.2** Let $\Gamma$ be a non-bipartite distance-regular graph with diameter $D$ at most 4 where, if $D = 4$, then $a_1 \neq 0$, and valency $k$ at least 2, having smallest eigenvalue at most $-k/2$. Then $\Gamma$ is one of the following graphs:

1. **Diameter equals 1:**
   (a) The triangle with intersection array $\{2; 1\}$

2. **Diameter equals 2:**
   (a) The pentagon with intersection array $\{2, 1; 1, 1\}$;
   (b) The Petersen graph with intersection array $\{3, 2; 1, 1\}$;
   (c) The folded 5-cube with intersection array $\{5, 4; 1, 2\}$;
   (d) The $3 \times 3$-grid with intersection array $\{4, 2; 1, 2\}$;
   (e) The generalized quadrangle $GQ(2, 2)$ with intersection array $\{6, 4; 1, 3\}$;
   (f) The generalized quadrangle $GQ(2, 4)$ with intersection array $\{10, 8; 1, 5\}$;
   (g) A complete tripartite graph $K_{t,t,t}$ with $t \geq 2$, with intersection array $\{2t, t-1; 1, 2t\}$;

3. **Diameter equals 3:**
   (a) The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;
   (b) The Odd graph with valency 4, $O_4$, with intersection array $\{4, 3, 3; 1, 1, 2\}$;
(c) The Sylvester graph with intersection array \( \{5, 4, 2; 1, 1, 4\} \);
(d) The second subconstituent of the Hoffman-Singleton graph with intersection array \( \{6, 5, 1; 1, 1, 6\} \);
(e) The Perkel graph with intersection array \( \{6, 5, 2; 1, 1, 3\} \);
(f) The folded 7-cube with intersection array \( \{7, 6, 5; 1, 2, 3\} \);
(g) A possible distance-regular graph with intersection array \( \{7, 6, 6; 1, 1, 2\} \);
(h) A possible distance-regular graph with intersection array \( \{8, 7, 5; 1, 1, 4\} \);
(i) The truncated Witt graph associated with \( M_{23}\) (see [2, Thm 11.4.2]) with intersection array \( \{15, 14, 12; 1, 1, 9\} \);
(j) The coset graph of the truncated binary Golay code with intersection array \( \{21, 20, 16; 1, 2, 12\} \);
(k) The line graph of the Petersen graph with intersection array \( \{4, 2, 1; 1, 1, 4\} \);
(l) The generalized hexagon \( GH(2, 1) \) with intersection array \( \{4, 2, 2; 1, 1, 2\} \);
(m) The Hamming graph \( H(3, 3) \) with intersection array \( \{6, 4, 2; 1, 2, 3\} \);
(n) One of the two generalized hexagons \( GH(2, 2) \) with intersection array \( \{6, 4, 4; 1, 1, 3\} \);
(o) One of the two distance-regular graphs with intersection array \( \{8, 6, 1; 1, 3, 8\} \) (see [2, p. 386]);
(p) The regular near hexagon \( B_3(2) \) with intersection array \( \{14, 12, 8; 1, 3, 7\} \);
(q) The generalized hexagon \( GH(2, 2) \) with intersection array \( \{18, 16, 16; 1, 1, 9\} \);
(r) The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array \( \{24, 22, 20; 1, 2, 12\} \);
(s) The Witt graph associated to \( M_{24} \) (see [2, Thm 11.4.1]) with intersection array \( \{30, 28, 24; 1, 3, 15\} \);
(t) The regular near hexagon \( ^2A_5(2) \) with intersection array \( \{42, 40, 32; 1, 5, 21\} \).

4. Diameter equals 4 and \( a_1 \neq 0 \);

(a) The generalized octagon \( GO(2, 1) \) with intersection array \( \{4, 2, 2, 2; 1, 1, 1, 2\} \);
(b) The distance-regular graph with intersection array \( \{6, 4, 2, 1; 1, 1, 4, 6\} \) (see [2, Thm 13.2.1]);
(c) The Hamming graph \( H(4, 3) \) with intersection array \( \{8, 6, 4, 2; 1, 2, 3, 4\} \);
(d) A generalized octagon \( GO(2, 4) \) with intersection array \( \{10, 8, 8, 8; 1, 1, 1, 5\} \);
(e) The Cohen-Tits regular near octagon associated with the Hall-Janko group (see [2, Thm 13.6.1]) with intersection array \( \{10, 8, 8, 2; 1, 1, 4, 5\} \).
(f) The regular near hexagon \( B_4(2) \) with intersection array \( \{30, 28, 24, 16; 1, 3, 7, 15\} \);
(g) The regular near hexagon \( ^2A_7(2) \) with intersection array \( \{170, 168, 160, 128; 1, 5, 21, 85\} \).

**Remark.** It is not known whether the generalized octagon \( GO(2, 4) \) with intersection array \( \{10, 8, 8, 8; 1, 1, 1, 5\} \) is unique.

This result is an extension of De Bruyn’s results [4, Sects. 3.5 & 3.6] on regular near hexagons and octagons, with lines with size 3, see also Theorem 6.2.

As a consequence of Theorem 1.2 we also obtain a complete classification of the 3-chromatic distance-regular graphs with diameter 3 and the 3-chromatic distance-regular graphs with diameter 4 and intersection number \( a_1 \neq 0 \).
Theorem 1.3  (i) Let $\Gamma$ be a 3-chromatic distance-regular graph with diameter 3. Then $\Gamma$ is one of the following:

1. The 7-gon, with intersection array $\{2,1,1;1,1,1\}$;
2. The Odd graph with valency 4, $O_4$, with intersection array $\{4,3,3;1,1,2\}$;
3. The Perkel graph with intersection array $\{6,5,2;1,1,3\}$;
4. The generalized hexagon $GH(2,1)$ with intersection array $\{4,2,2;1,1,2\}$;
5. The Hamming graph $H(3,3)$ with intersection array $\{6,4,2;1,2,3\}$;
6. The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array $\{24,22,20;1,2,12\}$.

(ii) Let $\Gamma$ be a 3-chromatic distance-regular graph with diameter 4 and $a_1 \neq 0$. Then $\Gamma$ is the Hamming graph $H(4,3)$ with intersection array $\{8,6,4;2,1,2,3,4\}$, or the generalized hexagon $GO(2,1)$ with intersection array $\{4,2,2,2;1,1,1,2\}$.

This result is an extension of Blokhuis et al. [1]. In that paper, they determined all the 3-chromatic distance-regular graphs among the known examples.

This paper is organised as follows, in Section 2 we give definitions and preliminaries, and in Section 3 we give the proof of the valency bound, Theorem 1.3. In Section 4, we treat the strongly regular graphs. In Section 5 we give a bound on the intersection number $c_2$. In Sections 6 we treat the case $a_1 = 1$ and in Section 7 we treat the case $a_1 = 0$. In Section 8, we give the proofs of Theorems 1.2 and 1.3. In the last section we give some open problems.

2 Preliminaries and definitions

All graphs considered in this paper are finite, undirected and simple (for more background information, see [2] or [12]). For a connected graph $\Gamma = (V(\Gamma), E(\Gamma))$, the distance $d(x,y)$ between any two vertices $x,y$ is the length of a shortest path between $x$ and $y$ in $\Gamma$, and the diameter $D$ is the maximum distance between any two vertices of $\Gamma$. For any vertex $x$, let $\Gamma_i(x)$ be the set of vertices in $\Gamma$ at distance precisely $i$ from $x$, where $0 \leq i \leq D$. For a set of vertices $x_1, \ldots, x_n$, let $\Gamma_1(x_1, \ldots, x_n)$ denote $\cap_{i=1}^n \Gamma_i(x_i)$. For a non-empty subset $S \subseteq V(\Gamma)$, $\langle S \rangle$ denotes the induced subgraph on $S$. A graph is coconnected if its complement is connected. A connected graph $\Gamma$ with diameter $D$ is called a distance-regular graph if there are integers $b_i, c_i$ ($0 \leq i \leq D$) such that for any two vertices $x, y \in V(\Gamma)$ with $i = d(x,y)$, there are exactly $c_i$ neighbors of $y$ in $\Gamma_{i-1}(x)$ and $b_i$ neighbors of $y$ in $\Gamma_{i+1}(x)$. The numbers $b_i, c_i$ and $a_i := b_i - b_i - c_i$ ($0 \leq i \leq D$) are called the intersection numbers of $\Gamma$. Set $c_0 = b_D = 0$. We observe $a_0 = 0$ and $c_1 = 1$. The array $\nu(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the intersection array of $\Gamma$. In particular $\Gamma$ is a
regular graph with valency \( k := b_0 \). We define \( k_i := |\Gamma_i(x)| \) for any vertex \( x \) and \( i = 0, 1, \ldots, D \). Then we have \( k_0 = 1, \ k_1 = k, \ c_{i+1}k_{i+1} = b_i k_i \ (0 \leq i \leq D - 1) \) and thus

\[
k_i = \frac{b_1 \cdots b_{i-1}}{c_2 \cdots c_i} k \quad (1 \leq i \leq D).
\]

(1)

A regular graph \( \Gamma \) on \( n \) vertices with valency \( k \) is called a strongly regular graph with parameters \( (n, k, \lambda, \mu) \) if there are two non-negative integers \( \lambda \) and \( \mu \) such that for any two distinct vertices \( x \) and \( y, |\Gamma_1(x, y)| = \lambda \) if \( d(x, y) = 1 \) and \( \mu \) otherwise. A connected non-complete strongly regular graph is just a distance-regular graph with diameter 2.

The adjacency matrix \( A = A(\Gamma) \) is the \((|V(\Gamma)| \times |V(\Gamma)|)\)-matrix with rows and columns indexed by \( V(\Gamma) \), where the \((x, y)\)-entry of \( A \) is 1 if \( d(x, y) = 1 \) and 0 otherwise. The eigenvalues of \( \Gamma \) are the eigenvalues of \( A = A(\Gamma) \). It is well-known that a distance-regular graph \( \Gamma \) with diameter \( D \) has exactly \( D + 1 \) distinct eigenvalues \( \theta = \theta_0 > \theta_1 > \cdots > \theta_D \) which are the eigenvalues of the following tridiagonal matrix

\[
L_1 := \begin{pmatrix}
0 & k \\
c_1 & a_1 & b_1 \\
c_2 & a_2 & b_2 \\
& \ddots & \ddots & \ddots \\
c_i & a_i & b_i \\
& & c_{D-1} & a_{D-1} & b_{D-1} \\
& & & c_D & a_D \\
\end{pmatrix}
\]

(cf. [2 p.128]). The standard sequence \( \{u_i(\theta) \mid 0 \leq i \leq D\} \) corresponding to an eigenvalue \( \theta \) is a sequence satisfying the following recurrence relation

\[
c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) = \theta u_i(\theta) \quad (1 \leq i \leq D)
\]

where \( u_0(\theta) = 1 \) and \( u_1(\theta) = \frac{\theta}{k} \). Then the multiplicity of eigenvalue \( \theta \) is given by

\[
m(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^{D} k_i u_i^2(\theta)}
\]

(2)

which is known as Biggs’ formula (cf. [2 Theorem 4.1.4]).

Let \( \Gamma \) be a distance-regular graph with valency \( k \), \( n \) vertices and diameter \( D \). For \( i = 0, 1, \ldots, D \), let \( A_i \) be the \( \{0, 1\}\)- matrix with where \( (A_i)_{xy} = 1 \) if and only if \( d(x, y) = i \) for vertices \( x, y \) of \( \Gamma \). Let \( \mathcal{A} \) be the Bose-Mesner algebra of \( \Gamma \), i.e. the matrix algebra over the complex numbers generated by \( A = A_1 \). Then \( \mathcal{A} \) has as basis \( \{A_0 = I, A_1 = A, A_2, \ldots, A_D\} \). The algebra \( \mathcal{A} \) also has a basis of idempotents \( \{E_0 = \frac{1}{n} I, E_1, \ldots, E_D\} \). Define the Krein numbers \( q_{ij}^\ell \) where \( 0 \leq i, j, \ell \leq D \) by \( E_i \circ E_j = \frac{1}{n} \sum_{\ell=0}^{D} q_{ij}^\ell E_\ell \). It is known the Krein numbers are non-negative real numbers, see [2 Prop. 4.1.5]. We will also need the absolute bound. Let \( \theta_0 = k > \theta_1, \ldots, \theta_D \) be the distinct eigenvalues of \( \Gamma \) with respective multiplicities \( m_0 = 1, m_1, \ldots, m_D \). Then for \( 0 \leq \ell, \ell' \leq D \) we have

\[
\sum_{\ell \in \{0, \ldots, D\}} m_\ell \leq \left\{ \begin{array}{ll}
m_i m_j & \text{if } i \neq j \\
m_i (m_i + 1)/2 & \text{if } i = j.
\end{array} \right.
\]

This is called the absolute bound.
For a graph $\Gamma$, a partition $\Pi = \{P_1, P_2, \ldots, P_t\}$ of $V(\Gamma)$ is called equitable if there are constants $\alpha_{ij}$ ($1 \leq i, j \leq t$) such that all vertices $x \in P_i$ have exactly $\alpha_{ij}$ neighbours in $P_j$. The $\alpha_{ij}$'s ($1 \leq i, j \leq t$) are called the parameters of the equitable partition.

Let $\Gamma$ be a distance-regular graph. For a set $S$ of vertices of $\Gamma$, define $S_i := \{x \in V(\Gamma) | d(x, S) := \min\{d(x, y) | y \in S\} = i\}$. The number $\rho = \rho(S) := \max\{i | S_i \neq \emptyset\}$ is called the covering radius of $S$. The set $S$ is called a completely regular code of $\Gamma$ if the distance-partition $\{S = S_0, S_1, \ldots, S_{\rho(S)}\}$ is equitable. The following result was first shown by Delsarte [6] for strongly regular graphs and extended by Godsil to the class of distance-regular graphs.

**Lemma 2.1 (Delsarte-Godsil bound)** Let $\Gamma$ be a distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue $\theta_{\text{min}}$. Let $C \subseteq V(\Gamma)$ be a clique with $c$ vertices. Then
\[ c \leq 1 + \frac{k}{1 - \theta_{\text{min}}}, \]
with equality if and only if $C$ is a completely regular code with covering radius $D - 1$.

A clique $C$ with $|C| = 1 + \frac{k}{1 - \theta_{\text{min}}}$ is called a Delsarte clique of $\Gamma$. It is known that parameters of a Delsarte clique as a completely regular code only depend on the parameters of $\Gamma$.

A distance-regular graph $\Gamma$ is called a geometric distance-regular graph if $\Gamma$ contains a set of Delsarte cliques $C$, such that every edge of $\Gamma$ lies in exactly one member $C$ of $\mathcal{C}$.

Examples of geometric distance-regular graphs are for example the bipartite distance-regular graphs, the Johnson graphs, the Grassmann graphs, the Hamming graphs and the bilinear forms graphs. See [10], for more information on geometric distance-regular graphs.

A geometric distance-regular graph with valency $k$ and diameter $D$ is called a regular near $2D$-gon if $c_ia_i = a_i$ for $i = 1, 2, \ldots, D$. A generalized $2D$-gon of order $(s, t)$, where $s, t \geq 1$ are integers, is a regular near $(2D)$-gon with valency $k = s(t + 1)$ and intersection number $c_{D-1} = 1$. A generalized 4-gon of order $(s, t)$ is called a generalized quadrangle of order $(s, t)$ and is denoted by $\text{GQ}(s, t)$. In similar fashion, a generalized 6-gon (respectively 8-gon) of order $(s, t)$ is called a generalized hexagon (octagon) of order $(s, t)$ and is denoted by $\text{GH}(s, t)$ ($\text{GO}(s, t)$).

### 3 Proof of Theorem 1.1

In this section, we give a proof of the valency bound Theorem [1.1]

**Proof of Theorem 1.1**

As $\Gamma$ is coconnected, $\Gamma$ is not complete multipartite. Let $m$ be the multiplicity of $\theta_{\text{min}}$. As $\Gamma$ is not complete multipartite, we see that $k \leq \frac{(m-1)(m+2)}{2}$ holds, by [2] Thm 5.3.2.

We consider the standard sequence $u_0 = 1, u_1, \ldots, u_D$ of $\theta = \theta_{\text{min}}$. Then $u_1 = \theta/k$ and
\[ u_{i+1} = \frac{(\theta - a_i)u_i - c_iu_i - c_iu_i - 1}{b_i} \quad (i = 1, 2, \ldots, D - 1), \]
and as $\theta$ is the smallest eigenvalue $(-1)^i u_i > 0$ for $i = 0, 1, \ldots, D$, see [2] Cor. 4.1.2] We may assume that $k \geq 4\alpha^{-2}$, and hence $c_1 = 1 \leq (\alpha^2/4)k$. Let $\ell := \max\{i \mid 1 \leq i \leq D \text{ such that } c_i \leq \alpha^{i+1}2^{-i-1}k\}$. and let $p := \min\{\ell + 1, D\}$.

**Claim 1**
The number $u_i$, satisfies $|u_i|\alpha^{-i} \geq 2^{-i}$ for $i = 0, 1, 2, \ldots, p$.

**Proof of Claim 1.**
We will show it by induction on $i$. For $i = 0$, it is obvious, as $u_0 = 1$. For $i = 1$ we have $|u_1| = |\theta/k| \geq \alpha$, so the claim holds for $i = 1$.

Let $i \geq 2$. Then $u_i = \frac{(\theta - a_{i-1}) u_{i-1} - c_{i-1} u_{i-2}}{b_{i-1}}$ holds. As $(-1)^i u_i > 0$, $c_i - 1 \leq \alpha^{i-1}2^{-i}k, b_{i-1} \leq k, a_{i-1} \geq 0, |u_{i-2}| \leq 1, \theta \leq -\alpha k$ we see that

$$|u_i| \geq \frac{k\alpha |u_{i-1}| - \alpha^{i-1}2^{-i}k}{k}.$$ 

By the induction hypothesis we obtain $|u_i|\alpha^{-i} \geq 2^{1-i} - 2^{-i} = 2^{-i}$. This shows the claim by induction.

Let $1 \leq q \leq p$ be such that $k_q$ is maximal among $k_0, k_1, \ldots, k_p$.

**Claim 2**
The number of vertices $n$ of $\Gamma$ satisfies $n \leq (D + 1)2^{(q+1)(D-q)} \alpha^{(q+1)(q-D)} k_q$.

**Proof of Claim 2.**
If $q < p$, or if $q = D$, then $n = \sum_{i=0}^D k_i \leq (D + 1)k_q$, as, then $k_q = \max\{k_i \mid 0 \leq i \leq D\}$ and hence the claim follows in this case.

So we may assume $p = q < D$. As $c_q > \alpha^{q+1}2^{-q+1}k$ and

$$k_{q+j} = \frac{k_q b_{q+1} \ldots b_{q+j-1}}{c_{q+1} c_{q+2} \ldots c_{q+j}} < k_q^{j} c_q^{j} < k_q 2^{(q+1)j} \alpha^{-(q+1)j}$$

for $j = 0, 1, \ldots, D - q$. As $n = \sum_{i=0}^D k_i$, Claim 2 follows.

Let $f(D, \alpha) := \max\{(D + 1)2^{(q+1)(D-q)+2q} \alpha^{(q+1)(q-D)} - 2q \mid q = 1, 2, \ldots, D\}$.

By Biggs’ formula, see [2] Thm 4.1.4, Claim 1 and Claim 2, we have

$$m = \frac{n}{\sum_{i=0}^D u_i^2 k_i} < \frac{n}{u_q^2 k_q} \leq (D + 1)2^{(q+1)(D-q)} \alpha^{(q+1)(q-D)} - 2q 2^q \leq f(D, \alpha).$$

Hence $k \leq (f(D, \alpha) - 1)(f(D, \alpha) + 2)/2$. This shows the theorem with $\kappa(D, \alpha) = (f(D, \alpha) - 1)(f(D, \alpha) + 2)/2$. 

4 Diameter 2

In this section we will determine the connected strongly regular graphs with valency \( k \geq 2 \) and smallest eigenvalue at most \(-k/2\).

**Proposition 4.1** Let \( \Gamma \) be a non-complete non-bipartite connected strongly regular graph, valency \( k \geq 2 \) and smallest eigenvalue \( \theta_{\text{min}} \) satisfying \( \theta_{\text{min}} \leq -k/2 \), then \( \Gamma \) is one of the following:

1. The pentagon with intersection array \( \{2,1;1,1\} \);
2. The Petersen graph with intersection array \( \{3,2;1,1\} \);
3. The folded 5-cube with intersection array \( \{5,4;1,2\} \);
4. The \( 3 \times 3 \)-grid with intersection array \( \{4,2;1,2\} \);
5. The generalized quadrangle \( GQ(2,2) \) with intersection array \( \{6,4;1,3\} \);
6. The generalized quadrangle \( GQ(2,4) \) with intersection array \( \{10,8;1,5\} \);
7. A complete tripartite graph \( K_{t,t,t} \) with \( t \geq 2 \), with intersection array \( \{2t, t-1;1,2t\} \)

Before we show this proposition we recall the following classification of Seidel.

**Theorem 4.2** (Seidel \[11\], see also \[2\] Thm 3.12.4(i)) Let \( \Gamma \) be a strongly regular graph and with second smallest eigenvalue \(-2\). Then \( \Gamma \) is one of the following graphs:

1. A Cocktail Party graph \( K_{n \times 2} \), with \( n \geq 2 \);
2. A \( t \times t \)-grid with \( t \geq 2 \);
3. A triangular graph \( T(n) \) with \( n \geq 4 \);
4. The Petersen graph;
5. The Schl"afli graph;
6. The Shrikhande graph;
7. One of the three Chang graphs;
8. The halved 5-cube

From now on let \( \Gamma \) be a non-bipartite distance-regular graph with valency \( k \geq 2 \), diameter \( D \geq 2 \) and smallest eigenvalue \( \theta_{\text{min}} \leq -k/2 \). If \( a_1 \geq 1 \), then, by Lemma \[2.1\] \( \Gamma \) has no 4-cliques and any triangle is a completely regular code.
Let us first consider the case $a_1 \geq 2$. Then any triangle $T = \{x, y, z\}$ is a completely regular code and any vertex $u$ at distance 1 from $T$ has exactly two neighbours in $T$. Let $A_{ab} := \{u \in V(\Gamma) \mid u \sim a, u \sim b\}$ where $a \neq b$ and $a, b \in \{x, y, z\}$. Then $A_{ab}$ forms a coclique, as there are no 4-cliques by the Delsarte-Godsil bound, and if $\{a, b, c\} = \{x, y, z\}$, then each vertex of $A_{ab}$ is adjacent to each vertex of $A_{ac}$. As the valency of $x$, $y$ and $z$ equals $\#A_{xy} + \#A_{xz} + \#A_{yz}$, respectively, it follows that $\Gamma$ is the complete tripartite graph $K_{t,t,t}$ where $t = \#A_{xy} = \#A_{xz} = \#A_{yz}$. This shows:

**Lemma 4.3** Let $\Gamma$ be a distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue $\theta_{\text{min}}$. If $\theta_{\text{min}} \leq -k/2$ and $a_1 \geq 2$, then $\Gamma$ is a complete tripartite graph $K_{t,t,t}$ for some $t \geq 2$.

This shows that, if the distance-regular graph is coconnected, then $a_1 \leq 1$.

Now we are ready to give the proof of Proposition 4.1

**Proof:** Assume the graph is not bipartite. First let us discuss the case when $\theta_{\text{min}}$ is not an integer. Then $\Gamma$ has intersection array $\{2t, t; 1, t\}$ and smallest eigenvalue $\frac{-1 - \sqrt{4t+1}}{2}$. Hence $\theta_{\text{min}} \leq -k/2 = -t$ implies that $t \leq 2$, and we have that $\Gamma$ is the pentagon as for $t = 2$, $\theta_{\text{min}}$ is an integer. So from now we may assume that $\theta_{\text{min}}$ is an integer. Let $\theta_1$ be the other non-trivial eigenvalue of $\Gamma$. Then $\theta_1$ is a non-negative integer. It follows that $c_2 - k = \theta_1 \theta_{\text{min}} \leq -k \theta_1/2$. This implies that $\theta_1 \leq 1$. For $\theta_1 = 0$, we obtain the complete tripartite graphs, and for $\theta_1 = 1$, the complement of $\Gamma$ has smallest eigenvalue $-2$. These have been classified in Theorem 4.2 and by checking them we obtain the proposition.

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\[\]

**5 A bound on $c_2$**

In this section we will give a bound on $c_2$. We first give the following result. This is a slight generalisation of [2] Prop. 4.4.6 (ii)]. We give a proof for the convenience for the reader, following the proof of [2]. Before we do this we need to introduce the following. Let $\Gamma$ be a distance-regular graph with valency $k$ with an eigenvalue $\theta$, say with multiplicity $m$. Let $1 = u_0, u_1, \ldots, u_D$ be the standard sequence of $\theta$. Then there exists a map $\phi : V(\Gamma) \to \mathbb{R}^m : x \mapsto \overline{x}$, such that the standard inner product between $u$ and $v$ satisfies $\langle u, v \rangle = u_0 \delta_{\phi(u), v}$.

**Proposition 5.1** (Cf. [2] Prop. 4.4.6(ii)] Let $\Gamma$ be a distance-regular graph with diameter $D \geq 2$, and valency $k \geq 2$. Assume $\Gamma$ contains an induced $K_{r,s}$ for some positive integers $r$ and $s$. Let $\theta$ be an eigenvalue of $\Gamma$, distinct from $\pm k$, with standard sequence $1 = u_0, u_1, \ldots, u_D$. Then

\[(u_1 + u_2)((r + s) \frac{1 - u_2}{u_1 + u_2} + 2rs) \geq 0 \quad (3)\]

and

\[(u_1 - u_2)((r + s) \frac{1 - u_2}{u_1 - u_2} - 2rs) \geq 0 \quad (4)\]
hold. In particular, if $\theta$ is the second largest eigenvalue, then

$$1 + \frac{b_1}{\theta + 1} \geq \frac{2rs}{r+s}$$

holds.

Proof: Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the two color classes of the induced $K_{r,s}$. Let $G$ be the Gram matrix with respect to the set $\{\overrightarrow{u} | u \in X \cup Y\}$. Now $\{X, Y\}$ is an equitable partition of $G$ with quotient matrix $Q$ where

$$Q = \begin{pmatrix}
1 + (r-1)u_2 & su_1 \\
u_1 & 1 + (s-1)u_2
\end{pmatrix}.$$  

Multiplying the first column of $Q$ by $s$ and the second column by $r$ we obtain the matrix

$$Q' := \begin{pmatrix}
s + s(r-1)u_2 & rsu_1 \\
rsu_1 & r + r(s-1)u_2
\end{pmatrix}.$$  

As $G$ is positive semi-definite, it follows that $Q$ and $Q'$ are both positive semi-definite and hence $(1\,1)Q'(1\,1)^T \geq 0$ and $(1, -1)Q'(1, -1)^T \geq 0$. Hence we obtain Equations (3) and (4). If $\theta$ is the second largest eigenvalue of $\Gamma$, then $1 > u_1 > u_2 > \cdots > u_D$ (using that the largest eigenvalue of the matrix $T$ of [2, p. 130] equals $\theta_1$), and $1 + \frac{b_1}{\theta_1} = \frac{1 - u_2}{u_1 - u_2}$ both hold. This implies the in particular statement.

This leads us to the following result.

**Lemma 5.2** Let $\Gamma$ be a non-bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 2$. If the smallest eigenvalue of $\Gamma$, $\theta_{\min}$, is at most $-k/2$, then $a_1 \leq 1$ and $c_2 \leq 5 + a_1$.

Proof: Let $\theta := \theta_{\min} \leq -k/2$. We already have established that if $a_1 \geq 2$, then the graph is complete tripartite and hence diameter is equal to 2. So this implies $a_1 \leq 1$. Let $1 = u_0, u_1, \ldots, u_D$ be the standard sequence of $\Gamma$ with respect to $\theta$. Then $u_1 + u_2 = \frac{1}{k\theta_1} (\theta + k)(\theta + 1) < 0$. The induced subgraph of $\Gamma$ consisting of two vertices at distance 2 and their common neighbours is a $K_{2,c_2}$. By Equation (3), we obtain that if $a_1 = 0$, then $3 > 1 - \frac{k-1}{\theta_{\min}-1} \geq \frac{4\sigma_2}{2+c_2}$ and hence $c_2 \leq 5$. If $a_1 = 1$ then $\theta_{\min} = -k/2$ and $\sigma_1 = -1/2$ and $\sigma_2 = 1/4$ and again using Equation (3), we obtain $c_2 \leq 6$. This shows the lemma.

### 6 The case $a_1 = 1$

In this section we will discuss the situation for $a_1 = 1$. We will start with the following easy observation.
Proposition 6.1 Let $\Gamma$ be a distance-regular graph with valency $k \geq 3$, diameter $D \geq 2$, intersection number $a_1 = 1$, and smallest eigenvalue $\theta_{\min} \leq -k/2$. Then $\theta_{\min} = -k/2$, $\Gamma$ is geometric, and there exists an integer $i, 2 \leq i \leq D$, such that $a_j = c_j$ for $1 \leq j < i$, $a_i = k/2$ and $a_j = b_j$ for $i + 1 \leq j \leq D$, with the understanding that $b_D = 0$. Moreover, if $a_D = k/2$, then $\Gamma$ is a regular near 2$D$-gon of order $(2, k/2 - 1)$.

Proof: As each triangle is a Delsarte clique, it follows that $\Gamma$ is a geometric distance-regular graph. This implies the proposition. (For details, we refer to Koolen and Bang [10].)

In the following result, we summarise the known existence results about regular near 2$D$-gons with $a_1 = 1$.

Theorem 6.2 Let $D \geq 2$. Let $\Gamma$ be a regular near 2$D$-gon of order $(2, t)$. Then $c_2 \in \{1, 2, 3, 5\}$. Moreover, the following holds:

1. If $c_2 = 5$, then $\Gamma$ is the dual polar graph of type $^2 A_{2D-1}(2)$.
2. If $c_2 = 3$, then $\Gamma$ is the dual polar graph of type $B_D(2)$, or if $D = 3$, the Witt graph associated to $M_{24}$ (see [2 Thm 11.4.1]) with intersection array $\{30, 28, 24; 1, 3, 15\}$;
3. If $c_2 = 2$, then $\Gamma$ is the Hamming graph $H(D, 3)$, or if $D = 3$, the coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$;
4. If $c_2 = 1$ and $D = 3$, then $\Gamma$ is one of the following:
   (a) The generalized hexagon $GH(2, 1)$ with intersection array $\{4, 2, 2; 1, 1, 2\}$;
   (b) The two generalized hexagons $GH(2, 2)$ with intersection array $\{6, 4, 4; 1, 1, 3\}$;
   (c) The generalized hexagon $GH(2, 8)$ with intersection array $\{18, 16, 16; 1, 1, 9\}$.
5. If $c_2 = 1$ and $D = 4$, then $\Gamma$ is one of the following:
   (a) The generalized octagon $GO(2, 1)$ with intersection array $\{4, 2, 2, 2; 1, 1, 1, 2\}$;
   (b) A generalized octagon $GO(2, 4)$ with intersection array $\{10, 8, 8, 8; 1, 1, 1, 5\}$;
   (c) The Cohen-Tits regular near octagon associated with the Hall-Janko group (see [2 Thm 13.6.1]) with intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$.

Proof: For $D = 2$, see for example [8 Cor. 10.9.5]. For $D = 3$, see [4 Sect. 3.5]. For $D = 4$, this follows from [4 Sect. 3.6] and [5]. This shows the theorem for diameter at most 4. The cases $c_2 = 5$ and $c_2 = 4$ follows from Brouwer and Wilbrink [3] who classified the regular near 2$D$-gons having $D \geq 4$, $c_2 \geq 3$ and $a_1 \geq 1$, see also [12 Thm 9.11]. For $D \geq 3$, $c_2 \geq 2$, $c_3 = 3$ and $a_1 \geq 1$, it was shown by Van Dam et al. [12 Thm 9.11] that $\Gamma$ is the Hamming graph $H(D, 3)$. For $D \geq 4$ and $c_2 = 2$, it is shown by Brouwer and Wilbrink that then also the regular near octagon with intersection array $\{2c_4, 2c_4 - 2, 2c_4 - 2c_2, 2c_4 - 2c_3; 1, c_2, c_3, c_4\}$ must exist and as by the diameter 4 case, we find that it must be the Hamming graph $H(4, 3)$, and hence $c_3 = 3$ holds. This shows
that if \( c_2 = 2 \) and \( D \geq 4 \) we must have the Hamming graph \( H(D, 3) \). This finishes the proof of the theorem.

For \( a_1 = 1 \) and \( \theta_{\min} = -k/2 \), we can improve the valency bound of Theorem 6.1. Note that in Hiraki and Koolen [9] a similar bound was obtained for regular near polygons.

**Proposition 6.3** Let \( \Gamma \) be a distance-regular graph with \( a_1 \), valency \( k \geq 4 \) and diameter \( D \geq 2 \). Then \( k \leq 2^{2D+1} - 2 \). Moreover, if \( c_D = k \), then \( k \leq 2^{2D-2} - 2 \).

**Proof:** Let \( (u_0, u_1, \ldots, u_D) \) be the standard sequence corresponding to \( \theta_{\min} = -k/2 \). Let \( m \) be the multiplicity of \( \theta_{\min} \). By Proposition 6.1 there exists \( 2 \leq i \leq D \) such that \( a_i = k/2 \). It is easy to show by induction, again using Proposition 6.1, that \( u_j = (-2)^{j-i} \) if \( j \leq i \) and \( u_j = (-2)^{j-i} \) if \( j \geq i \).

So, by Biggs’ formula, \( m \leq \frac{1}{\min\{u_i^2|i=1,2,\ldots,D\}} \leq 2^{2D} \). As \( 2m \geq k + 2 \), we find \( k \leq 2^{2D+1} - 2 \).

If \( i \leq D - 1 \), then

\[
  k_{i+1} + k_{i-1} \geq \frac{b_i + c_i}{\max\{c_{i+1}, b_{i-1}\}} k_i \geq \frac{k/2}{k} k_i = k_i/2.
\]

For positive real numbers \( a, b, c, d \), if \( a/c \geq b/d \) holds, then

\[
  \frac{a}{c} \geq \frac{a+b}{c+d} \geq \frac{b}{d}
\]

holds. Using this, we see that

\[
  m = \frac{\sum_{j=0}^D k_j}{\sum_{j=0}^D k_j u_j^2} \leq \frac{k_{i-1} + k_i + k_{i+1}}{2^{-2i+2} k_{i-1} + 2^{-2i} k_i + k_{i+1} 2^{-2i+2}},
\]

as \( |u_j| \geq 2^{-i} \) for \( j \notin \{i-1, i, i+1\} \). Hence \( m \leq \frac{3/2^{3i}}{3/2^{2i}} = 2^{i-1} \leq 2^{2D-3} \). As \( k \leq 2m - 2 \), by [9, Prop. 3], we see that \( k \leq 2^{2D-2} - 2 \) if \( i \leq D - 1 \). This shows the proposition as \( c_D = k \) if and only if \( i \leq D - 1 \).

In view of Proposition 6.1 and Theorem 6.2 we only need to classify the distance-regular graphs with diameter \( D \) equals to 3 or 4, \( a_1 = 1 \) and \( c_D = k \).

**Theorem 6.4** Let \( \Gamma \) be a distance-regular graph with diameter \( D \) equals 3 or 4, \( a_1 = 1 \) and valency \( k \), \( c_D = k \) and smallest eigenvalue \( -k/2 \). Then one of the following hold:

1. \( D = 3 \) and \( \Gamma \) is a distance-regular graph with intersection array \( \{8,6,1;1,3,8\} \) (see [2, p. 224]), or the line graph of the Petersen graph with intersection array \( \{4,2,1;1,1,4\} \);

2. \( D = 4 \) and \( \Gamma \) is the distance-regular graph with intersection array \( \{6,4,2,1;1,1,4,6\} \) (see [2, Thm 13.2.1]).
Proof: As \( a_1 = 1 \), the valency \( k \) is even. By Proposition 6.3, we have for diameter 3 that the valency \( k \) is bounded by \( k \leq 14 \) and for diameter 4 we obtain \( k \leq 62 \). We generated all the possible intersection arrays of diameter 3 and 4 with \( k \leq 14 \) for diameter 3 and \( k \leq 62 \) for diameter 4, such that \( c_2 \leq 6 \), the \( c_i \)'s are increasing, the \( b_i \)'s are decreasing, the valencies \( k_i \) are positive integers, satisfying the conditions of Proposition 6.1, \( c_4 = k \) and the multiplicities of the eigenvalues are positive integers. Besides the intersection arrays in the theorem we obtained only the following two intersection arrays \{10, 8, 3; 1, 2, 10\} and \{12, 10, 3; 1, 3, 12\}. As both have eigenvalue \(-k/2\) with multiplicity 7 and the number of vertices equals 63, we find by the absolute bound (see [2, Prop. 4.1.5]) that if the graph exists, it must have a vanishing Krein parameter, but that is not the case. So there is no distance-regular graph with either of these two intersection arrays. This shows the theorem.

7 Diameter 3 and \( a_1 = 0 \)

Note that there are infinitely many bipartite distance-regular graphs with diameter 3.

In the following result, we show that a non-bipartite distance-regular graph with diameter 3, valency \( k \) and smallest eigenvalue at most \(-k/2\) has \( k \) at most 64.

Proposition 7.1 Let \( \Gamma \) be a non-bipartite triangle-free distance-regular graph with diameter 3, valency \( k \) and smallest eigenvalue \( \theta_{\text{min}} \) at most \(-k/2\). Then \( k \leq 64 \) holds.

Proof: Let \((u_0 = 1, u_1, u_2, u_3)\) be the standard sequence with respect to \( \theta := \theta_{\text{min}} \). Let \( \Gamma \) have distinct eigenvalues \( \theta_0 = k > \theta_1 > \theta_2 > \theta_3 = \theta \). Let

\[
L = \begin{pmatrix} 0 & k & 0 & 0 \\ 1 & 0 & k - 1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{pmatrix}.
\]

The matrix \( L \) has as eigenvalues \( \theta_0, \theta_1, \theta_2, \theta_3 \), and hence \( \text{tr}(L^2) = \theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 \geq k^2 + k^2/4 \). On the other hand we have \( \text{tr}(L^2) = a_3^2 + c_3^2 + 2k + 2c_2(k - 1) + 2c_3b_2 \). Replacing \( b_2 \) by \( k - a_2 - c_2 \) and \( a_3 \) by \( k - c_3 \), we obtain \( \text{tr}(L^2) = k^2 + (c_3 - a_2)^2 + k(2 + 2c_2) - 2c_2(1 + c_3) \leq k^2 + (a_2 - c_3)^2 + 12k \) as \( c_2 \leq 5 \) by Lemma 5.2. This means that \( (a_2 - c_3)^2 + 12k \geq k^2/4 \). Now assume \( k \geq 65 \) to obtain a contradiction. Then \( 12k \leq (12/65)k^2 \). This implies that \( |a_2 - c_3| \geq (0.255)k \). This means that at least one of \( c_3 \) and \( a_2 \) is at least \((0.255)k\). We are going to estimate the multiplicity \( m \) of \( \theta \). On the one hand, \( m \geq 6, \) as \( a_1 = 0 \). On the other hand, by Biggs’ formula, we have

\[
m = \frac{n}{\sum_{i=0}^{3} k_i u_i^2},
\]

where \( n \) is the number of vertices of \( \Gamma \). This means that

\[
m \geq \frac{1}{\min\{u_i^2 \mid i = 0, 1, 2, 3\}}.
\]
We have \( u_0 = 1, u_1 = \theta/k \leq -1/2, \)
\[
u_2 = \frac{\theta u_1 - 1}{k - 1} \geq \frac{k - 4}{4k - 4} \geq \frac{61}{256}.
\]

So in order that \( m \geq k \geq 65 \) holds, we must have \( u_3^2 < 1/64 \), or, in other words, \( u_3 > -1/8 \). We obtain
\[
(k - a_2)/8 > b_2/8 > -b_2u_3 = (-\theta + a_2)u_2 + c_2u_1 \geq (k/2 + a_2)(61/256) - 5,
\]
and this implies \( a_2 < \frac{3k + 2560}{186} \). As \( k \geq 65 \), it follows that \( a_2 < k/4 \). As we have already established that at least one of \( a_2 \) and \( c_3 \) is at least \((0.255)k\), we find \( c_3 \geq (0.255)k \). We find \(-b_2u_3 = u_2(a_2 - \theta) + c_2u_1 \geq \frac{61}{256}(-\theta) + 5\frac{2}{k} = -\theta\left(\frac{61}{256} - \frac{5}{k}\right) \geq \frac{k}{2}\left(\frac{61}{256} - \frac{5}{k}\right) \). As \( b_2 < k \) and \( k \geq 65 \), we obtain \(-u_3 \geq \frac{1}{k}\left(\frac{61}{256} - \frac{5}{65}\right) > 41/512 \).

For positive real numbers \( a, b, c, d \), if \( a/c \geq b/d \) holds, then
\[
\frac{a}{c} \geq \frac{a + b}{c + d} \geq \frac{b}{d}
\]
holds. Using this, we see that
\[
m = \frac{\sum_{i=0}^3 k_i}{\sum_{i=0}^3 k_i u_i^2} \leq \frac{k_2 + k_3}{u_2^2k_2 + u_3^2k_3}
\]
holds, as \( \frac{1+k}{1+ku_i^2} \leq 4 \). Now
\[
\frac{k_2 + k_3}{k_2u_2^2 + k_3u_3^2} = \frac{c_3 + b_2}{c_3u_2^2 + b_2u_3^2} \leq \frac{c_3 + k}{c_3u_2^2 + ku_3^2},
\]
as \( |u_3| < 1/8 < u_2, k_3 = \frac{b_2}{c_3}k_2 \) and \( b_2 < k \) all hold. Using \( c_3 \geq (0.255)k, u_2 \geq 61/256 \) and \( |u_3| \geq 41/512 \), we find that
\[
m \leq \frac{k_2 + k_3}{k_2u_2^2 + k_3u_3^2} \leq 64,
\]
a contradiction. This shows the proposition.

Now we come to the main result of this section.

**Theorem 7.2** Let \( \Gamma \) be a non-bipartite distance-regular graph with diameter 3, \( a_1 = 0 \), valency \( k \geq 2 \) and smallest eigenvalue at most \(-k/2\). Then \( \Gamma \) is one of the following:

1. The 7-gon, with intersection array \( \{2, 1, 1; 1, 1, 1\} \);
2. The Odd graph with valency 4, \( O_4 \), with intersection array \( \{4, 3, 3; 1, 1, 2\} \);
3. The Sylvester graph with intersection array \( \{5, 4, 2; 1, 1, 4\} \);
4. The second subconstituent of the Hoffman-Singleton graph with intersection array \( \{6, 5, 1; 1, 1, 6\} \);
5. The Perkel graph with intersection array \( \{6, 5, 2; 1, 1, 3\} \);
6. The folded 7-cube with intersection array \(\{7, 6, 5; 1, 2, 3\}\);

7. A possible distance-regular graph with intersection array \(\{7, 6, 6; 1, 1, 2\}\);

8. A possible distance-regular graph with intersection array \(\{8, 7, 5; 1, 1, 4\}\);

9. The truncated Witt graph associated with \(M_{23}\) (see [2, Thm 11.4.2]) with intersection array \(\{15, 14, 12; 1, 1, 9\}\);

10. The coset graph of the truncated binary Golay code with intersection array \(\{21, 20, 16; 1, 2, 12\}\);

**Proof:** By Proposition 7.1 we have that the valency \(k\) is bounded by \(k \leq 64\). We generated all the possible intersection arrays of diameter 3 with \(k \leq 64\), such that \(a_1 = 0\), \(c_2 \leq 5\), the \(c_i\)’s are increasing, the \(b_i\)’s are decreasing, the valencies \(k_i\) are positive integers, and the multiplicities of the eigenvalues are positive integers. Besides the intersection arrays listed in the theorem, we only found the intersection arrays \(\{5, 4, 2; 1, 1, 2\}\) and \(\{13, 12, 10; 1, 3, 4\}\). It was by Fon-der-Flaass [7] that there are no distance-regular graphs with intersection array \(\{5, 4, 2; 1, 1, 2\}\). The intersection array \(\{13, 12, 10; 1, 3, 4\}\) is ruled out by [2, Thm. 5.4.1]. This shows the theorem.

---

**8 Proofs of Theorems 1.2 and 1.3**

In this section we give the proof of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2** For diameter 2, it follows from Proposition 4.1. For \(a_1 \neq 0\) and diameter 3 and 4, it follows from Proposition 6.1 and Theorems 6.2 and 6.4. For diameter 3 and \(a_1 = 0\), it follows from Theorem 7.2.

Before we give the proof of Theorem 1.3, let us recall the chromatic number of a graph. A proper coloring with \(t\) colors of a graph \(\Gamma\) is a map \(c : \nu(\Gamma) \rightarrow \{1, 2, \ldots, t\}\) where \(t\) is a positive number such that \(c(x) \neq c(y)\) for any edge \(xy\). The chromatic number of \(\Gamma\) denoted by \(\chi(\Gamma)\) is the minimal \(t\) such that there exists a proper coloring of \(\Gamma\) with \(t\) colors. We also say that such a graph is \(\chi(\Gamma)\)-chromatic. An independent set of \(\Gamma\) is a set \(S\) of vertices such that there are no edges between them.

Hoffman showed the following result for regular graphs.

**Lemma 8.1** (Hoffman bound), cf. [2] Prop. 1.3.2. Let \(G\) be a \(k\)-regular graph with \(n\) vertices and with smallest eigenvalue \(\theta_{\text{min}}\). Let \(S\) be an independent set of \(\Gamma\) with \(s\) vertices. Then

\[
s \leq \frac{n}{1 + \frac{k}{\theta_{\text{min}}}}.
\]

This means that if a \(k\)-regular graph \(\Gamma\) on \(n\) vertices is 3-chromatic, then it must have an independent set of size \(n/3\) and by the Hoffman bound we find that the smallest eigenvalue of \(\Gamma\) is at most
Now we are ready to give the proof for Theorem 1.3.

Proof of Theorem 1.3: (i): By above we only need to check the graphs of Theorem 1.2. The six graphs, we list, are shown to be 3-chromatic in [1, Section 3.4]. For the case \(a_1 > 0\), it was shown that the last three graphs are the only 3-chromatic distance-regular graphs with diameter 3 in [1, Thm. 3.6]. So we only need to check the graphs with \(a_1 = 0\). That the distance-regular graphs with intersection arrays \(\{21, 20, 16; 1, 2, 12\}\) and \(\{7, 6, 5; 1, 2, 3\}\) are not 3-chromatic follows from [1, Sect. 3.6]. That the distance-regular graph with intersection array \(\{15, 14, 12; 1, 1, 9\}\) is not 3-chromatic follows from [1, Sect. 3.7].

That the distance-regular graphs with intersection arrays \(\{5, 4, 2; 1, 1, 4\}\), \(\{6, 5, 1; 1, 1, 6\}\), \(\{7, 6, 6; 1, 1, 2\}\), \(\{8, 7, 5; 1, 1, 4\}\) are not 3-chromatic follows from [1, Sect. 3.9].

(ii): The two graphs we list are shown be 3-chromatic in [1, Sect. 3.4]. In [1, Thm. 3.3 & Prop. 3.8], it is shown that the Hamming graph \(H(D, 3)\) is the only 3-chromatic distance-regular graph with \(c_2 \geq 2\), \(a_D > 0\) and \(D \geq 4\). [1, Thm. 3.3] also shows that the distance-regular graph with intersection array \(\{6, 4, 2; 1, 1, 4, 6\}\) is not 3-chromatic, as it has induced pentagons. That the regular near octagon associated with the Hall-Janko group (see [2, Thm 13.6.1]) with intersection array \(\{10, 8, 8, 2; 1, 1, 4, 5\}\), is not 3-chromatic is shown on [1, p. 299]. That a generalized octagon \(GO(2, 4)\) with intersection array \(\{10, 8, 8, 8; 1, 1, 1, 5\}\) is not 3-chromatic, follows from [1, Thm. 3.2]. This shows Theorem 1.3.

9 Open problems

Now we give some open problems.

1. Classify the geometric distance-regular graphs with intersection number \(a_1 = 1\).

2. Finish the classification of the non-bipartite distance-regular graphs with diameter 4, valency \(k\) and smallest eigenvalue at most \(-k/2\).

3. Classify the non-bipartite distance-regular graphs with diameter 3, valency \(k\) and smallest eigenvalue \(-k/3\).

References

[1] A. Blokhuis, A.E. Brouwer, and W.H. Haemers. On 3-chromatic distance-regular graphs. Des. Codes Cryptogr., 44:293–305, 2007.

[2] A.E. Brouwer, A.M. Cohen, and A. Neumaier. Distance-Regular Graphs. Springer-Verlag, Berlin, 1989.
[3] A.E. Brouwer and H.A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata*, 14:145–176, 1983.

[4] B. De Bruyn. *Near Polygons*. Birkhäuser, Basel, 2006.

[5] B. De Bruyn. The nonexistence of regular near octagons with parameters \((s, t, t_2, t_3) = (2, 24, 0, 8)\). *Electron. J. Combin.*, 17:R149, 2010.

[6] P. Delsarte. *An algebraic approach to the association schemes of coding theory*, volume 10 of *Philips Res. Reports Suppl.* 1973.

[7] D.G. Fon-Der-Flaass. There exists no distance-regular graph with intersection array \((5, 4, 3; 1, 1, 2)\). *European J. Combin.*, 14:409–412, 1993.

[8] C.D. Godsil and G.F. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

[9] A. Hiraki and J.H. Koolen. A Higman-Haemers inequality for thick regular near polygons. *J. Algebraic Combin.*, 20:213–218, 2004.

[10] J.H. Koolen and S. Bang. On distance-regular graphs with smallest eigenvalue at least \(-m\). *J. Combin. Theory Ser. B*, 100:573–584, 2010.

[11] J.J. Seidel. Strongly regular graphs with \((-1,1,0)\) adjacency matrix having eigenvalue 3. *Lin. Alg. Appl.*, 1:281–298, 1968.

[12] E.R. van Dam, J.H. Koolen, and H. Tanaka. Distance-regular graphs. [arXiv:1410.6294v1].