Block-avoiding point sequencings

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Abstract

Recent papers by Kreher, Stinson and Veitch have explored variants of the problem of ordering the points in a triple system (such as a Steiner triple system, directed triple system or Mendelsohn triple system) so that no block occurs in a short segment of consecutive entries (so the ordering is locally block-avoiding). The paper describes a greedy algorithm which shows that such an ordering exists, provided the number of points is sufficiently large. This algorithm leads to improved bounds on the number of points in cases where this was known, but also extends the results to a significantly more general setting (for example, orderings that avoid the blocks of a block design). Similar results for a cyclic variant of this situation are also established.

The results above were originally inspired by results of Alspach, Kreher and Pastine, who (motivated by zero-sum avoiding sequences in abelian groups) were interested in orderings of points in a partial Steiner triple system where no segment is a union of disjoint blocks. Alspach et al. show that, when the system contains at most $k$ pairwise disjoint blocks, an ordering exists when the number of points is more than $15k - 5$. By making use of a greedy approach, the paper improves this bound to $9k + O(k^{2/3})$. 

1
1 Introduction

Let $V$ be a finite set of cardinality $v$. A sequence $x_1, x_2, \ldots, x_v$ over $V$ is a **sequencing** if the elements $x_i$ are a permutation of the elements of $V$. A **segment** (or window) of a sequence is a subsequence of consecutive entries.

Suppose that $V$ is the set of points of a Steiner triple system (STS). Recall that a STS is a set of 3-subsets, *blocks*, of $V$ such that every pair of points is contained in a unique block. Several recent papers have explored variations on the idea of sequencings that are block-avoiding. For example, Stinson and Kreher [4] define a sequencing to be $\ell$-good (for some integer $\ell$) if no segment of length $\ell$ contains a block. An STS is $\ell$-good if it possesses an $\ell$-good sequencing. Stinson and Kreher show that any STS with $v > 3$ is 3-good, and any STS with $v > 71$ is 4-good. More generally, Stinson and Veitch [9] show that an STS with $v > \ell^6/16 + \mathcal{O}(\ell^5)$ is $\ell$-good. They use a greedy algorithm to establish this result. The naive greedy algorithm chooses the elements $x_i$ of the sequencing in order, making sure the elements are distinct and avoiding the points that would create a block when combined with some of the $\ell - 1$ most recently chosen elements of the sequence. But this naive algorithm might fail towards the end of the sequencing, because the elements that have not yet been used all form a block with recent elements. So Stinson and Veitch carefully design the beginning of the sequence so that any problematic elements can be swapped with elements there, allowing the algorithm to complete.

Kreher, Stinson and Veitch [6, 7, 8] have explored other variants of this problem, including cyclic versions of the $\ell$-good property (where segments can ‘wrap around’ from the end to the start of the sequence) and variants where the triples are ordered in some way (Mendelsohn triple systems, and directed triple systems). Using greedy algorithms, all these variants of the $\ell$-good property can be shown to be feasible when $v$ is sufficiently large compared to $\ell$.

All of the results above also hold in the more general situation of partial Steiner triple systems: a PSTS is a set of 3-subsets, *blocks*, of $V$ that pairwise intersect in at most one element.

The main aim of this paper is to provide a greedy algorithm which provides sequencings for all these variants of the $\ell$-good property and more, and which improves on the greedy algorithms in the papers above (in the sense that it succeeds for smaller values of $v$). To aid understanding, in Section 2 we first describe the algorithm in the special case of partial Steiner.
triple systems. We show that any PSTS with \( v > 3\ell^4/4 + \mathcal{O}(\ell^3) \) is \( \ell \)-good. (A slightly more careful argument given in the section which follows shows that the bound may be improved to \( v > \ell^4/2 + \mathcal{O}(\ell^3) \).) In Section 4 we will describe the general problem and give provide an algorithm to solve it. Some consequences of this general algorithm, including results on \( \ell \)-good sequencings of Mendelsohn triple systems and directed triple systems, are also provided there. We consider the cyclic case in Section 4.

The paper also includes an improvement on a result due to Alspach, Kreher and Pastine [2] which inspired the results of Kreher, Stinson and Veitch above. Alspach et al. considered the following problem, motivated by the properties of zero-sum free sequences in abelian groups [1, 3]. A triple system is sequenceable if there exists a sequencing with no non-empty segment having elements equal to a union of disjoint blocks. Kreher and Stinson [5] provide infinitely many examples of PSTS that are not sequenceable. However, Aspach et al. show that a PSTS with at most 3 pairwise disjoint blocks is sequenceable, and that if a PSTS has at most \( k \) disjoint blocks then it is sequenceable whenever \( v \geq 18k - 5 \). We modify their method, treating large and small segments differently, and show (see Section 5) that this bound may be improved to \( v \geq 9k - \mathcal{O}(k^{2/3}) \).

2 Sequenceability of partial Steiner triple systems

We begin by defining and recapping notation. Let \( V \) be a finite set of cardinality \( v \). We say that a sequence \( x_1, x_2, \ldots, x_n \) is a partial sequencing (of \( V \)) if the elements \( x_i \in V \) are distinct. If \( n = v \), we say that \( x_1, x_2, \ldots, x_n \) is a sequencing. For a partial Steiner triple system (PSTS) with point set \( V \), we say that a sequence \( x_1, x_2, \ldots, x_n \) of points is good if \( \{x_1, x_2, \ldots, x_n\} \) does not contain a block. We say that a (partial) sequencing is \( \ell \)-good if all segments of length \( \ell \) or less are good.

**Theorem 1.** Let \( \ell \) be a positive integer. Whenever

\[
v \geq \left( 2\ell + 3\binom{\ell-1}{2} \right) \left( \frac{\ell-1}{2} \right) + \ell = 3\ell^4/4 + \mathcal{O}(\ell^3),
\]

all partial Steiner Triple Systems of order \( v \) have \( \ell \)-good sequencings.
Proof. Since every sequencing is 1-good and 2-good, we may assume that \( \ell \geq 3 \). Let \( v \) be an integer such that (1) holds. Fix a PSTS with point set \( V \), where \(|V| = v\). Set \( L = \binom{\ell-1}{2} \), so \( v \geq (2\ell + 3L)L + \ell \). To prove the theorem, it suffices to construct a sequencing for our PSTS. The following algorithm constructs such a sequencing.

**Stage 1:** Greedily construct an \( \ell \)-good partial sequencing \( x_1, x_2, \ldots, x_{(\ell-1)L} \).

So at each stage we choose the point \( x_i \in V \) so that

- \( x_i \) is distinct from \( x_1, x_2, \ldots, x_{i-1} \), and
- there is no block in the PSTS contained in the set \( \{x_r, x_{r+1}, \ldots, x_i\} \), where \( r = \max\{1, i - \ell + 1\} \).

Note that at most \( i - 1 \) points are ruled out by the first condition. We claim that there are at most \( L \) points ruled out by the second condition. To see this, first note that any block contained in \( \{x_r, x_{r+1}, \ldots, x_i\} \) must involve \( x_i \), since we may assume by induction that \( x_1, x_2, \ldots, x_{i-1} \) is \( \ell \)-good. So the block must intersect \( \{x_r, x_{r+1}, \ldots, x_{i-1}\} \) in a set of size 2. There are at most \( L \) such blocks, as there are \( L \) choices for the two points that lie in \( \{x_r, x_{r+1}, \ldots, x_{i-1}\} \), and once these points are chosen, the third point in the block is determined. Each block rules out at most once choice for \( x_i \), and so our claim follows. Since \( v \geq (\ell - 1)L + L > i - 1 + L \), the greedy algorithm will always succeed.

We divide the resulting sequence into \( L \) segments \( x^1, x^2, \ldots, x^L \), each of length \( \ell - 1 \). So \( x^j = x_{(j-1)(\ell-1)+1}, x_{(j-1)(\ell-1)+2}, \ldots, x_{(j-1)(\ell-1)+\ell-1} \).

Let \( V' = V \setminus \{x_1, x_2, \ldots, x_{(\ell-1)L}\} \), so

\[ |V'| = v - (\ell - 1)L. \]

The initial segment of the sequencing we will construct will have the form \( y_1x^1y_2x^2 \cdots y_Lx^L \), for distinct points \( y_1, y_2, \ldots, y_L \in V' \) that we have not yet specified. Of course, not all choices of points \( y_j \) will preserve the \( \ell \)-good property of the sequence. We say that a point \( y \in V' \) is unfortunate if the \( \ell \)-good property of the sequencing fails in this way, and define \( U \) to be the set of unfortunate points. Since being \( \ell \)-good is a property of segments of length at most \( \ell \), and the segments \( x_i \) are all of length \( \ell - 1 \), we see that \( U \) is the set of points \( y \in V' \) such that for some \( i \in \{1, 2, \ldots, L - 1\} \) the sequence \( x_iyx_{i+1} \) is not \( \ell \)-good. (Note that we have covered the case that \( yx_1 \) fails to
be \(\ell\)-good, since this case implies that \(x_1y\), and so \(x_1yx_2\), fails to be \(\ell\)-good.) There are \(L - 1\) choices for \(i\). There are at most \((\ell - 1)(\ell - 2) + \left(\frac{\ell - 1}{2}\right)\) pairs of positions in \(x_i\) or \(x_{i+1}\) that are contained in a segment of \(x_1yx_{i+1}\) of length at most \(\ell\). Each pair of positions gives rise to at most one unfortunate element (the third point in the block containing the points in these positions, if such a block exists). So

\[
|U| \leq \frac{3}{2}(\ell - 1)(\ell - 2)(L - 1) \leq 3L^2. \tag{2}
\]

Note that the partial sequencing \(y_1x_1y_2x_2\cdots y_Lx_L\) is \(\ell\)-good whenever the points \(y_i\) are distinct, and none of the points \(y_i\) are unfortunate.

Our strategy in stages 2 to 4 will be to construct a partial sequencing \(z\) over \(V'\) such that: all but \(L\) points of \(V'\) are used; all unfortunate points are used; and \(x_\ell z\) is \(\ell\)-good. Once we have accomplished this, we set \(y_1, y_2, \ldots, y_L\) to be the \(L\) points that do not appear in \(z\). The sequence \(y_1x_1y_2x_2\cdots y_Lx_\ell z\) is an \(\ell\)-good sequencing, as required.

**Stage 2:** If \(|U| > L\), greedily find a partial sequencing \(u\) of \(|U| - L\) unfortunate points, so that the partial sequencing \(x_\ell u\) is \(\ell\)-good. Note that this is possible, since at each stage maintaining the \(\ell\)-good condition rules out at most \(L\) points, and there are always more than \(L\) unfortunate points available.

**Stage 3:** Add each of the remaining unfortunate points to our partial sequencing \(u\) (maintaining the \(\ell\)-good property) as follows. For each remaining \(u \in U\), extend the partial sequencing by \(z_1z_2\cdots z_{\ell-1}u\), where \(z_1, z_2, \ldots, z_{\ell-1} \in V'\) are chosen greedily so that the \(\ell\)-good property is maintained and also so that \(\{z_1, z_2, \ldots, z_{\ell-1}, u\}\) does not contain a block. Note that when we choose an element \(z_j\), we have previously used fewer than \(|U| + (\ell - 1)L\) elements of \(V'\). Moreover, at most \(L\) elements are ruled out to maintain the \(\ell\)-good property, and at most \(\ell - 2\) elements by the fact that \(z_j\) cannot be in a block with \(u\) and one of \(z_1, z_2, \ldots, z_{j-1}\). Now (1) and (2) imply that

\[
|V' - (|U| + L + \ell - 1)| \geq v - ((\ell - 1)L + 3L^2 + L + \ell - 1) > 0,
\]

and so the greedy algorithm always succeeds.

After Stage 3, we have produced a partial sequencing \(v\) which extends \(u\). All unfortunate elements appear in \(v\), and \(x_L v\) is \(\ell\)-good. Moreover, since we
have used at most $|U| + L\ell$ points from $V'$, we see that (2) implies at least $L$ points in $V'$ have not yet been used:

$$|V'| - (|U| + L\ell) = v - ((\ell - 1)L + 3L^2 + L\ell) > L.$$ 

**Stage 4:** Use the greedy algorithm to extend our partial $\ell$-good sequencing $v$ until just $L$ unused elements of $V'$ remain. Again, we require that the resulting sequencing $z$ has the property that $x_Lz$ is $\ell$-good. Since there are always more than $L$ points from $V'$ that are not yet been used, the greedy algorithm always succeeds in constructing $z$.

At the end of Stage 4, the partial sequencing $z$ contains $v - L$ points of $V'$. Let $y_1, y_2, \ldots, y_L$ be the $L$ remaining points in $V'$. Then the sequence $y_1x^1y_2x^2\cdots y_Lx^Ly$ is an $\ell$-good sequencing, as desired. 

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**Comment 1.** The lower bound on $v$ can be improved a little. First, slightly better estimates could be used. For example, the bound on the number of unfortunate points can be improved. Indeed, Theorem 2 below does provide a slightly improved bound when specialised to the PSTS case. Secondly, the algorithm itself could be improved, by taking unfortunate elements arising from the start of the sequence $x_1, x_2, \ldots, x_{(\ell - 1)L}$ and including them towards the end of this sequence. But even with these changes the lower bound on $v$ is still of the order of $\ell^4$: new methods are needed to significantly improve the bound (which is dominated by the size of the set of unfortunate elements).

### 3 The general (non-cyclic) case

When we are interested in $\ell$-good sequencings of partial Steiner triple systems, we first choose a positive integer $v$ and a set $V$ of cardinality $v$. We then choose a PSTS over $V$, and declare a segment to be good if the set of its elements does not contain a block. To rephrase, we define a set $F$ of ('forbidden') sequences over $V$, namely the sequences of length 3 whose elements form a block, and we declare a sequence over $V$ to be good if none of its subsequences lie in $F$. We then define a sequence $x_1, x_2, \ldots, x_v$ of length $v$ to be an $\ell$-good sequencing if the elements $x_i$ of the sequence are distinct and if the segments $x_i, x_{i+1}, \ldots, x_{i+\ell-1}$ for $1 \leq i \leq v - \ell + 1$ are all good.
We first generalise this process, and extract the properties we need for our greedy algorithm to work.

Let $V$ be a finite set of cardinality $v$. As before, we say that a sequence $x_1, x_2, \ldots, x_n$ over $V$ is a partial sequencing if the elements $x_i$ are distinct, and a sequencing if in addition $n = v$.

Let $F$ be a set of finite sequences over $V$. We say that a sequence is $F$-good if none of its subsequences lie in $F$. We say that a (partial) sequencing is an $(\ell, F)$-good (partial) sequencing if the elements $x_i$ of the sequence are distinct and if all segments of the sequence of length $\ell$ or less are $F$-good.

We note that when $F$ is the set of sequences of length 3 whose elements form blocks of a partial Steiner triple system the notions of $(\ell, F)$-good sequencing and $\ell$-good sequencing are identical.

Let $\ell$ be a positive integer. For an integer $L$ (possibly depending on $\ell$), we say that $F$ has the $L$-suffix property if the following statement holds. For any non-negative integer $n$ with $n \leq \ell - 1$ and any $F$-good sequence $x_1, x_2, \ldots, x_n$ over $V$, there are at most $L$ choices for $x \in V$ such that the sequence $x_1, x_2, \ldots, x_n, x$ fails to be $F$-good. Similarly, for an integer $L'$, we define $F$ to have the $L'$-prefix property if there are at most $L'$ choices for $x \in V$ such that the sequence $x, x_1, x_2, \ldots, x_n$ fails to be $F$-good. We note that in the PSTS case, the choices for $x$ where $x_1, x_2, \ldots, x_n, x$ fails to be $F$-good are precisely the third points in the blocks intersecting $\{x_1, x_2, \ldots, x_n\}$ in two distinct points; since there are at most $\binom{\ell - 1}{2}$ such blocks we see that $F$ has the $\binom{\ell - 1}{2}$-suffix property. Similarly, $F$ has the $\binom{\ell - 1}{2}$-prefix property in this case.

For some sets of sequences $F$ (for example, those sets of sequences that are closed under permuting their elements) for any sequence $x_1, x_2, \ldots, x_n$ the sets of elements $x$ that are counted by the $L$-suffix and $L$-prefix property are equal. In this situation, we say that $F$ is symmetric. So in the PSTS case, $F$ is symmetric.

For an integer $K$, possibly depending on $\ell$, we say that $F$ has the $K$-insertion property if the following statement holds. For any $(\ell, F)$-good sequence $x_1, x_2, \ldots, x_{2\ell - 2}$ over $V$, there are at most $K$ choices for $x \in V$ such that the sequences $x_1, x_2, \ldots, x_{\ell - 1}, x$ and $x, x_\ell, x_{\ell + 1}, \ldots, x_{2\ell - 2}$ are $(\ell, F)$-good but the sequence $x_1, x_2, \ldots, x_{\ell - 1}, x, x_\ell, x_{\ell + 1}, \ldots, x_{2\ell - 2}$ fails to be $(\ell, F)$-good. In the partial STS case, $F$ has the $\binom{\ell - 1}{2}$-insertion property. To see this, note that there are $\binom{\ell - 1}{2}$ ways of choosing positions $i \in \{1, 2, \ldots, \ell - 1\}$ and $j \in \{\ell, \ell + 1, \ldots, 2\ell - 2\}$ with $j - i < \ell - 1$. There is at most one element
such that \( \{x_i, x_j, x\} \) is a block, and all the elements \( x \) we are counting arise in this way.

For integers \( J \) and \( s \), possibly depending on \( \ell \), we say that \( \mathcal{F} \) has the \((J, s)\)-reachability property if the following statement holds. Let \( W \subseteq V \) with \( |W| \geq J \). Let \( x_1, x_2, \ldots, x_{\ell-1} \) be a partial \((\ell, \mathcal{F})\)-good sequencing whose elements \( x_i \) do not lie in \( W \). Let \( w \in W \). Then there exist elements \( w_1, w_2, \ldots, w_s \in W \) such that \( x_1, x_2, \ldots, x_{\ell-1}, w_1, w_2, \ldots, w_s, w \) is a partial \((\ell, \mathcal{F})\)-good sequencing. In the PSTS case, we claim that \( \mathcal{F} \) has the \((\binom{\ell-1}{2} + 2\ell, \ell - 1)\)-reachability property. For we may greedily choose distinct elements \( w_i \in W \setminus \{w\} \) such that

- The partial sequencing \( x_1, x_2, \ldots, x_{\ell-1}, w_1, w_2, \ldots, w_i \) is \( \ell \)-good, and
- There is no block of the form \( \{w_j, w_i, w\} \) for \( 1 \leq j < i \).

At the point when the greedy algorithm chooses \( w_i \), maintaining the first condition rules out at most \( \left( \frac{\ell - 1}{2} \right) + \ell - 2 \) points (as we must not pick points that form a block with two of the previous \( \ell - 1 \) points in the partial sequencing, and we must also avoid the points that lie in \( \{w_1, w_2, \ldots, w_{i-1}\} \cup \{w\} \)). Maintaining the second condition rules out up to \( \ell - 2 \) further points. So when \( |W| \geq \binom{\ell-1}{2} + 2\ell \), the greedy algorithm always succeeds. This establishes our claim.

**Theorem 2.** Let \( \ell \) be a positive integer, and let \( V \) be a finite set of order \( v \). Using the notation above, suppose that \( \mathcal{F} \) has the \( L \)-suffix, \( L' \)-prefix, \( K \)-insertion and \((J, s)\)-reachability properties. Then \( \mathcal{F} \) has an \((\ell, \mathcal{F})\)-good sequencing when

\[
v \geq \ell L + \left( L + L' + K \right)L + sL + J.
\]

(3)

If \( \mathcal{F} \) is symmetric, then \( \mathcal{F} \) has an \((\ell, \mathcal{F})\)-good sequencing when

\[
v \geq \ell L + \left( L + K \right)L + sL + J.
\]

(4)

Before proving the theorem, we illustrate its usefulness by providing some corollaries.

**Corollary 3.** There exists a function \( f_{\text{STS}} : \mathbb{N} \rightarrow \mathbb{N} \) such that any partial Steiner triple system of order \( v \) with \( v \geq f_{\text{STS}}(\ell) \) has an \( \ell \)-good sequencing. Moreover,

\[
f_{\text{STS}}(\ell) \leq \frac{1}{2} \ell^4 + O(\ell^3).
\]
Proof. Let $\mathcal{F}$ be the set of sequences $x_1, x_2, x_3$ such that $\{x_1, x_2, x_3\}$ is a block in our partial Steiner Triple System. Then $\mathcal{F}$ is symmetric, and has the $L$-suffix, $K$-insertion and $(J, s)$-reachability properties where $L = K = \frac{1}{2} \ell^2 + O(\ell)$, where $J = \frac{1}{2} \ell^2 + O(\ell)$ and where $s = \ell + O(1)$. The corollary follows by Theorem 2.

Two ordered variations of partial Steiner triple systems have been considered in the context of sequencings: directed triple systems [6] and Mendelsohn triple systems [8]. We take each in turn, and derive a corollary of Theorem 2 in this context.

Let $V$ be a finite set. A transitive triple is a sequence $x, y, z$ over $V$ where $x$, $y$ and $z$ are distinct. (The terminology comes from the transitive ordering $x < y < z$ determined by the sequence.) We may depict a transitive triple $x, y, z$ as a directed triangle of the following form:

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 x -----> y -----> z
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So a transitive triple can be thought of as a triangle of three directed edges in the complete directed graph on $V$. We define a partial directed triple system (DTS) to be a set $\mathcal{F}$ of transitive triples such that each directed edge in the complete directed graph on $V$ is contained in at most one transitive triple in $\mathcal{F}$. Following [6], we define, for an integer $\ell$, a sequencing of a partial directed triple system $\mathcal{F}$ to be $\ell$-good if it is $(\ell, \mathcal{F})$-good. So no segment of the sequencing of length at most $\ell$ contains a transitive triple $x, y, z$ from the system as a subsequence.

**Corollary 4.** There exists a function $f_{DTS} : \mathbb{N} \to \mathbb{N}$ such that any partial directed triple system of order $v$ with $v \geq f_{DTS}(\ell)$ has an $\ell$-good sequencing. Moreover,

$$f_{DTS}(\ell) \leq \frac{3}{4} \ell^4 + O(\ell^3).$$

Proof. A partial directed triple system is not necessarily symmetric. But nevertheless it is not difficult to show that $\mathcal{F}$ has the $(\ell^2)$-suffix, $(\ell^2)$-prefix and $(\ell^2)$-insertion properties, and the $((\ell^2-1) + 2\ell, \ell-1)$-reachability property. The corollary follows by Theorem 2. □
A 3-cycle $(x, y, z)$ over $V$, where $x$, $y$ and $z$ are distinct, may be thought of as a directed triangle of the following form:

![Diagram of a 3-cycle](https://via.placeholder.com/150)

Note that $(x, y, z)$, $(y, z, x)$ and $(z, x, y)$ are the same cycle. We may think of a 3-cycle as a triangle of three directed edges in the complete directed graph on $V$. We define a partial Mendelsohn triple system (MTS) to be a set of 3-cycles over $V$ such that each directed edge in the complete directed graph on $V$ is contained in at most one of these 3-cycles. A partial Mendelsohn triple system over $V$ gives rise to a set $F$ of sequences over $V$ by identifying each 3-cycle $(x, y, z)$ with three sequences: $x, y, z$, and $y, z, x$ and $z, x, y$. We define, for an integer $\ell$, a sequencing of a partial Mendelsohn triple system $F$ to be $\ell$-good if it is $(\ell, F)$-good. (Note that this definition differs from [8], as there the sequencing itself is regarded as a $v$-cycle. So all segments of length $\ell$ in the $v$-cycle, including those that overlap the ends of the sequencing, need to be $F$-good according to the definition in [8]. We consider this cyclic version of the $\ell$-good property in Section 4.)

**Corollary 5.** There exists a function $f_{MTS} : \mathbb{N} \to \mathbb{N}$ such that any partial Mendelsohn Triple System of order $v$ with $v \geq f_{MTS}(\ell)$ has an $\ell$-good (non-cyclic) sequencing. Moreover,

$$f_{MTS}(\ell) \leq \ell^4/2 + O(\ell).$$

**Proof.** Let $F$ be the set of sequences $x_1, x_2, x_3$ such that $(x_1, x_2, x_3)$ is a 3-cycle in our partial Mendelsohn Triple System. Then $F$ is symmetric. Moreover, it is not difficult to show that $F$ has the $L$-suffix, $K$-insertion and $(J, s)$-reachability properties where $L = K = \frac{1}{2}\ell^2 + O(\ell)$, where $J = \frac{1}{2}\ell^2 + O(\ell)$, and where $s = \ell + O(1)$. The corollary follows by Theorem 2.

Before proving Theorem 2, we give one more corollary. Given a $t$-$(v, k, \lambda)$ design with point set $V$, we may define a set $F$ of sequences by setting $x_1, x_2, \ldots, x_k$ to lie in $F$ if and only if $\{x_1, x_2, \ldots, x_k\}$ is a block in the design. We say that a sequencing is $\ell$-good if it is $(\ell, F)$-good. When $\ell < k$, all sequencings are $\ell$-good. So we may assume that $\ell \geq k$. 

10
Corollary 6. Let $t$, $k$ and $\lambda$ be fixed integers, with $2 \leq t < k$ and $\lambda \geq 1$. There exists a function $f_{BD} : \mathbb{N} \to \mathbb{N}$ such that any $t$-$(v,k,\lambda)$ design with $v \geq f_{MTS}(\ell)$ has an $\ell$-good sequencing. Moreover,

$$f_{BD}(\ell) \leq O(\ell^{2t}).$$

Proof. The set $F$ of sequences associated with the design is clearly symmetric. We see that $F$ has the $L$-suffix property where $L$ is any upper bound on the number of blocks that intersect $\ell - 1$ points in a set of size $k - 1$, since the remaining points in these blocks are exactly the points we must avoid to extend our sequencing. Counting the number of $t$-subsets of these blocks (in two ways) shows that we may take $L = \lambda \binom{\ell - 1}{t}/\binom{k - 1}{t} = O(\ell^t)$.

Let $K$ be the maximum number of blocks that intersect a partial $(\ell,F)$-good sequencing $x_1, x_2, \ldots, x_{2(\ell-1)}$ in $k - 1$ points, that intersect both the subsets $\{x_1, x_2, \ldots, x_{\ell-1}\}$ and $\{x_\ell, x_{\ell+1}, \ldots, x_{2(\ell-1)}\}$ non-trivially, and are contained in a segment of length $\ell - 1$ of $x_1, x_2, \ldots, x_{2(\ell-1)}$. We have that $F$ has the $K$-insertion property, since the remaining points of each of these blocks include all those those we wish to count. But clearly

$$K \leq \lambda \binom{2(\ell - 1)}{t} = O(\ell^t),$$

since every block we are counting contains a $t$-subset of $\{x_1, x_2, \ldots, x_{2(\ell-1)}\}$, and each such $t$-subset is contained at most $\lambda$ blocks.

It is not hard to see that $F$ satisfies the $(J, \ell - 1)$-insertion property with $J = L + \ell - 1 + \lambda \binom{\ell - 1}{t} = O(\ell^t)$, using a straightforward generalisation of the argument for Steiner triple systems. The corollary now follows, by Theorem 2.

Proof of Theorem 2. Suppose that (3) holds. When $L = 0$, the $L$-suffix property implies that the naive greedy algorithm constructs an $(\ell,F)$-good sequencing. So we may assume that $L \geq 1$. Similarly, we may assume that $L' \geq 1$.

We construct a $(\ell,F)$-good sequencing as follows.

Stage 1: We begin by constructing a sequence $x_1, x_2, \ldots, x_{(\ell-1)L}$ that is a partial $(\ell,F)$-good sequencing. We do this in a greedy fashion, by choosing each element $x_i \in V$ so that:

(a) the element $x_i$ is distinct from $x_1, x_2, \ldots, x_{i-1}$, and
(b) the sequence $x_r, x_{r+1}, \ldots, x_i$, where $r = \max \{1, i - (\ell - 1)\}$, is $\mathcal{F}$-good.

When choosing $x_i$, at most $i - 1$ elements from $V$ are ruled out because of condition (a) and, since $\mathcal{F}$ has the $L$-suffix property, at most $L$ choices for $x_i$ are ruled out by condition (b). Since $v \geq (\ell - 1)L + L > i - 1 + L$, there are always choices for the elements $x_i$ and so the greedy algorithm will always succeed in constructing this partial sequencing.

We divide the partial sequencing into $L$ segments $x^1, x^2, \ldots, x^L$, each of length $\ell - 1$. Define $V' = V \setminus \{x_1, x_2, \ldots, x_{(\ell - 1)L}\}$. The initial segment of the sequencing we construct will have the form $y_1x^1y_2x^2\cdots y_Lx^L$ for elements $y_1, y_2, \ldots, y_L \in V'$ that are yet to be specified. Not all choices of elements $y_j \in V'$ will preserve the $(\ell, \mathcal{F})$-good property of the sequence. We say that $y \in V'$ is unfortunate if the $\ell$-good property of the sequencing fails in this way, and let $U \subseteq V'$ be the set of unfortunate elements. So an element $y \in V'$ lies in $U$ exactly when

- $yx^i$ fails to be $(\ell, \mathcal{F})$-good for some $i \in \{1, 2, \ldots, L\}$, or
- $x^iy$ fails to be $(\ell, \mathcal{F})$-good for some $i \in \{1, 2, \ldots, L - 1\}$, or
- the above two cases do not hold, but $x^iyx^{i+1}$ fails to be $(\ell, \mathcal{F})$-good for some $i \in \{1, 2, \ldots, L - 1\}$.

Since $\mathcal{F}$ has the $L$-suffix, $L'$-prefix and $K$-insertion properties, and since there are at most $L$ choices for $i$ in the conditions above, we see that $|U| \leq (L + L' + K)L$.

Our strategy will be to extend $x^L$ to an $(\ell, \mathcal{F})$-good partial sequencing $x^Lz$ whose elements consist of all elements of $U$, and all but $L$ elements $y_1, y_2, \ldots, y_L$ of $V'$. Since $y_1, y_2, \ldots, y_L \notin U$, we find that $y_1x^1y_2x^2\cdots y_Lx^L$ is a partial $(\ell, \mathcal{F})$-good sequencing. Since the sets of elements occurring in the partial sequencings $y_1x^1y_2x^2\cdots y_Lx^L$ and $z$ are pairwise disjoint, we see that $y_1x^1y_2x^2\cdots y_Lx^Lz$ is a sequencing. Moreover, this sequencing is $(\ell, \mathcal{F})$-good, since $x^L$ has length $\ell - 1$ and so every segment of length $\ell$ or less is a segment of one of the $(\ell, \mathcal{F})$-good partial sequencings $x^0y_1x^1y_2x^2\cdots y_Lx^L$ and $x^Lz$.

So to establish the theorem, it remains to construct the sequence $z$ with the properties we require. We continue as follows.

Stage 2: If $|U| > L$, we greedily extend the $(\ell, \mathcal{F})$-good partial sequencing $x^L$ using $|U| - L$ unfortunate elements (to produce another, longer, $(\ell, \mathcal{F})$-good partial sequencing). This is possible since at each stage at most $L$
elements are ruled out by the \((\ell, F)\)-good condition, by the \(L\)-suffix property, and because there are always more than \(L\) unfortunate elements available.

**Stage 3:** Next we use the \((J, s)\) reachability property to add each of the remaining unfortunate elements to our partial sequencing (maintaining the \((\ell, F)\)-good property). So for each remaining element \(u \in U\), defining \(W \subseteq V'\) to be the set of elements we have not yet used, we extend the partial sequencing by \(w_1w_2\cdots w_su\), where \(w_1, w_2, \ldots, w_s \in W\) are chosen so that they maintain the \((\ell, F)\)-good property. At any stage of this process, we have used at most \((\ell - 1)L + |U| + sL\) elements, and so (3) implies that

\[
|W| \geq v - ((\ell - 1)L + (L + L' + K)L + sL) \geq J.
\]

So the \((J, s)\)-reachability property guarantees the existence of the elements \(w_i\).

After Stage 3, all unfortunate elements have been used in our partial sequencing. Moreover, since \(v \geq ((\ell - 1)L + (L + L' + K)L + sL + J) + L\) at least \(L\) elements of \(V'\) have not yet been used.

**Stage 4:** We now greedily extend our partial \((\ell, F)\)-good sequencing by elements of \(V'\) until just \(L\) unused elements remain. The greedy algorithm always succeeds, by the \(L\)-suffix property. At the end of this process, we have obtained a partial \((\ell, F)\)-good sequencing \(x_Lz\) with the properties we require: \(z\) contains all but \(L\) of the elements of \(V'\), and all the elements of \(U\). The first statement of the theorem now follows by the remarks at the end of Stage 1.

Now suppose in addition that \(F\) is symmetric, and (4) holds. The strategy above constructs a \((\ell, F)\)-good sequencing once we note that the number of unfortunate elements is at most \((L + K)L\). This follows since the set of elements \(y \in V'\) where \(yx_i\) is not \((\ell, F)\)-good is equal to the set of elements \(y \in V'\) where \(x_iy\) is not \((\ell, F)\)-good.

\[\square\]

4 **Cyclic results**

This section considers a cyclic variant of the problems in Section 3. This approach has been considered for Mendelsohn Triple Systems by Kreher,
Stinson and Veitch [6], but (to our knowledge) has not been considered previously for Steiner triple systems, Directed Triple Systems or block designs. We formalise this as follows.

The cyclic segments of a sequence $x_1, x_2, \ldots, x_n$ are the segments of the sequence, together with the sequences $x_r, x_{r+1}, \ldots, x_n, x_1, x_2, \ldots, x_s$ for $1 \leq s < r \leq n$.

Let $V$ be a set of cardinality $v$, and let $F$ be a set of sequences over $V$. We say that a sequencing $x_1, x_2, \ldots, x_v$ is a cyclic $(\ell, F)$-good sequencing if all its cyclic segments of length at most $\ell$ are $F$-good. (When we consider Mendelsohn triple systems, we remark that a cyclic $(\ell, F)$-good sequencing is exactly what Kreher et al. [6] call an $\ell$-good sequencing.)

Roughly speaking, our strategy for proving a cyclic variation of Theorem 2 is to produce a long $(\ell, F)$-good partial sequencing with ‘gaps’ as before, then to add elements to produce a cyclic $(\ell, F)$-good sequencing with gaps, before finally filling the gaps. We need one more definition, closely related to the $K$-insertion property, which is associated with the penultimate step in this process. Let $K'$ and $s'$ be positive integers. We say that $F$ has the $(K', s')$-completion property if, for any partial $(\ell, F)$-good sequencings $x$ and $x'$ of length $\ell - 1$ over $V$, and any subset $X \subseteq V$ of cardinality at least $K'$ and disjoint from the elements in $x$ and $x'$, there exists a sequence $y$ of length $s'$ over $X$ such that $xyx'$ is a partial $(\ell, F)$-good sequencing.

Theorem 7. Let $\ell$ be a positive integer, and let $V$ be a finite set of order $v$. Using the notation of Section 3 and the notation above, suppose that $F$ has the $L$-suffix, $L'$-prefix, $K$-insertion, $(J, s)$-reachability and $(K', s')$-completion properties. Then $F$ has a cyclic $(\ell, F)$-good sequencing when

$$v \geq (K' - s')\ell + \ell + (K' - s')(L + L' + K) + (s - 1)L + J + K'.$$  

If $F$ is symmetric, then $F$ has a cyclic $(\ell, F)$-good sequencing when

$$v \geq (K' - s')\ell + \ell + (K' - s')(L + K) + (s - 1)L + J + K'.$$  

Proof. The algorithm in Theorem 2 can be modified to construct an $(\ell, F)$-good cycle, as follows.

At Stage 1, we greedily choose an $(\ell, F)$-good sequencing of the form $x^0x^1 \cdots x^{K'-s'}$, where each sequence $x^i$ has length $\ell - 1$. The cycle we construct will start with the sequence $x^0y_1x^1y_2x^2 \cdots y_{K'-s'}x^{K'-s'}$ where the elements $y_i$ are yet to be specified. We define $V'$ to be the set of elements...
of $V$ that do not occur in any of the sequences $x^i$, and define the set $U$ of unfortunate elements to be those elements $u \in U$ such that $x^iux^{i+1}$ fails to be $(\ell, \mathcal{F})$-good for some $i$. Counting as before, we see that

$$|U| \leq (K' - s')(L + L' + K),$$

since $\mathcal{F}$ satisfies the $L$-suffix, $L'$-prefix and $K$-insertion properties. When $\mathcal{F}$ is symmetric, we have the following slightly stronger upper bound for $|U|$:

$$|U| \leq (K' - s')(L + K).$$

We now extend $x^{K' - s'}$ to a longer partial $(s, \mathcal{F})$-good sequencing including all unfortunate elements, just as before. So in Stage 2, we greedily extend $x^{K' - s'}$ using elements of $U$ until at most $L$ unfortunate elements remain, and in Stage 3 we use the $(J, s)$-reachability property to add the remaining $L$ unfortunate elements to our partial $(s, \mathcal{F})$-good sequencing. At the end of this process, we have used at most $|U| + (s - 1)L$ elements from $V'$.

We claim that, provided $L$ is taken to be as small as possible subject to $\mathcal{F}$ having the $L$-suffix property, we must have that $K' \geq L$. To see this, we note that the minimality of $L$ implies that there must exist a partial $(\ell, \mathcal{F})$-good sequencing $x$ of length at most $\ell - 1$ and a disjoint subset $W \subseteq V$ with $|W| = L - 1$ such that none of the sequences $xz$ with $z \in W$ are $(\ell, \mathcal{F})$-good. Then $K' > L - 1$, since otherwise $x$ and $W$ (together with another sequencing $x'$) would form a counterexample to the $(K', s)$-completion property. This establishes our claim.

In Stage 4, we continue to greedily add elements to our partial $(s, \mathcal{F})$-good sequencing until just $K'$ elements of $V'$ remain unused. Note that this makes sense, since at least $K'$ elements of $V'$ remain unused at the start of this process:

$$|V'| - (|U| + (s - 1)L) \geq v - (K' - s')L - (K' - s')(L + L' + K) - (s - 1)L \geq K'$$

in general, and

$$|V'| - (|U| + (s - 1)L) \geq v - (K' - s')L - (K' - s')(L + K) - (s - 1)L \geq K'$$

when $\mathcal{F}$ is symmetric. Also note that the greedy algorithm succeeds, since we may assume that $K' \geq L$. At the end of this process we have a sequence $z$ of length $|V'|-K'$ over $V'$ such that $x^{K' - s'}z$ is a partial $(s, \mathcal{F})$-good sequencing. Let $Y$ be the set of elements of $V'$ that do not occur in $z$. 

15
Define $x$ to be the sequence consisting of the final $\ell - 1$ entries of $z$. Since $|Y| = K'$, the $(K', s')$ completion property implies there exists a sequence $y$ over $Y$ of length $s'$ such that $xyx_0$ is a partial $(\ell, F)$-good sequencing. Let $y_1, y_2, \ldots, y_{K'-s'}$ be the elements of $Y$ that do not occur in $y$. In Stage 5 we output the cycle

$$x^0y_1x^1y_2x^2 \cdots y_{K'-s'}x^{K'-s'}yz.$$ 

Note that $x^0y_1x^1y_2x^2 \cdots y_{K'-s'}x^{K'-s'}$ is $(\ell, F)$-good, as none of the elements $y_i$ are unfortunate. Moreover, the sequences $x^{K'-s'}z$ and $xyx_0$ are also $(\ell, F)$-good by our construction of $z$ and by the $(K', s')$-completion property. So we have a cyclic $(\ell, F)$-sequencing, as required.

For a partial STS, MTS, DTS or block design, a cyclic $\ell$-good sequencing is a cyclic $(\ell, F)$-good sequencing, where $F$ is the set of sequences defined in Section 3.

**Corollary 8.** There exists a function $g_{\text{STS}} : \mathbb{N} \to \mathbb{N}$ such that any partial Steiner Triple System of order $v$ with $v \geq g_{\text{STS}}(\ell)$ has a cyclic $\ell$-good sequencing. Moreover, 

$$g_{\text{STS}}(\ell) \leq \frac{3}{2} \ell^4 + \mathcal{O}(\ell^3).$$

**Proof.** We fix a partial Steiner triple system, and define $F$ to be the set of sequences of length 3 whose elements form the points in some block. Recall that $F$ is symmetric, and has the $L$-suffix, $L'$-prefix, $K$-insertion and $(J, \ell-1)$-reachability properties, where $L, L', K$ and $J$ are each of the form $1, 2, 3, \ldots, 2\ell$. The corollary follows by Theorem 7 provided we can show that $F$ has the $(K', \ell-1)$ completion property where $K' = \frac{3}{2} \ell^2 + \mathcal{O}(\ell)$. In fact, we will show that $F$ has the $(K', \ell - 1)$-completion property where $K' = 3(\ell - 1)^2 + \ell - 1$.

Fix $F$-good sequences $x$ and $x'$, and a set $X$ with $|X| \geq K'$ whose elements do not occur in $x$ or $x'$. We have $x = x_1, x_2, \ldots, x_{\ell-1}$ and $x' = x'_1, x'_2, \ldots, x'_{\ell-1}$, say, where for any $i$ we have $x_i \notin X$ and $x'_i \notin X$. We may greedily choose elements $y_1, y_2, \ldots, y_{\ell-1} \in X$, choosing $y_i$ so that the following three conditions all hold:

(a) $y_i \in X \setminus \{y_1, y_2, \ldots, y_{i-1}\}$;

(b) There is no block of the form $\{y_i, a, b\}$ where $a, b \in \{x_1, x_{i+1}, \ldots, x_{\ell-1}, y_1, y_2, \ldots, y_{i-1}\}$;
(c) For \( j \in \{1, 2, \ldots, i - 1\} \), there is no block of the form \( \{y_j, y_i, b\} \) where \( b \in \{x'_1, x'_2, \ldots, x'_j\} \).

(d) There is no block of the form \( \{y_i, a, b\} \) where \( a, b \in \{x'_1, x'_2, \ldots, x'_i\} \).

Note that condition (a) rules out at most \( \ell - 2 \) elements \( y_i \in X \), and condition (b) rules out at most \( \binom{\ell - 1}{2} \) elements. Condition (c) rules out at most \( \sum_{j=1}^{i-1} j = \binom{i}{2} \leq \binom{\ell - 1}{2} \) elements, and condition (d) rules out at most \( \binom{i}{2} \leq \binom{\ell - 1}{2} \) elements. So the greedy algorithm succeeds in finding suitable elements \( y_i \), since \( |X| \geq K' = 3\binom{\ell - 1}{2} + \ell - 1 \). Now consider the sequence

\[ x_1, x_2, \ldots, x_{\ell - 1}, y_1, y_2, \ldots, y_{\ell - 1}, x'_1, x'_2, \ldots, x'_{\ell - 1}. \]

We have a partial sequencing, since condition (a) guarantees that the elements \( y_i \) are distinct. To prove our claim, it suffices to show that this sequence is a partial \((\ell, F)\)-good sequencing. Suppose for a contradiction that there are is a segment of length \( \ell \) containing a block \( B \). Since \( x \) and \( x' \) are separated by \( \ell - 1 \) elements, \( B \) cannot contain both elements of \( x \) and \( x' \). Since \( x \) and \( x' \) are \((\ell, F)\)-good, \( B \) must contain at least one element of \( \{y_1, y_2, \ldots, y_{\ell - 1}\} \). Let \( i \) be the largest integer so that \( y_i \in B \). Condition (b) shows that \( B \) must contain an element from \( x' \). If \( B \) contains just one element from \( x' \), condition (c) shows we have a contradiction; if \( B \) contains two elements from \( x' \), this contradicts condition (d). So the sequence is \((\ell, F)\)-good, as required.

The methods of Corollary 3 apply equally to the situation when we are interested in Directed or Mendelsohn Triple Systems to show that we have the \((3\binom{\ell - 1}{2} + \ell - 1, \ell - 1)\)-completion property. Theorem 7, together with properties from the previous section, then imply the following two corollaries.

**Corollary 9.** There exists a function \( g_{DTS} : \mathbb{N} \to \mathbb{N} \) such that any partial Directed Triple System of order \( v \) with \( v \geq g_{DTS}(\ell) \) has a cyclic \( \ell \)-good sequencing. Moreover,

\[
g_{DTS}(\ell) \leq \frac{9}{4} \ell^4 + O(\ell^3).
\]
Corollary 10. There exists a function $g_{MTS} : \mathbb{N} \to \mathbb{N}$ such that any partial Mendelsohn Triple System of order $v$ with $v \geq g_{MTS}(\ell)$ has a cyclic $\ell$-good sequencing. Moreover,

$$g_{MTS}(\ell) \leq \frac{3}{2} \ell^4 + O(\ell^3).$$

Finally, the following corollary holds for block designs:

Corollary 11. Let $t$, $k$ and $\lambda$ be fixed integers, with $2 \leq t < k$ and $\lambda \geq 1$. There exists a function $g_{BD} : \mathbb{N} \to \mathbb{N}$ such that any $t$-$(v, k, \lambda)$ design with $v \geq f_{MTS}(\ell)$ has a cyclic $\ell$-good sequencing. Moreover,

$$g_{BD}(\ell) \leq O(\ell^{2t}).$$

Proof. Let $\mathcal{F}$ be the set of all sequences of length $k$ whose points form a block in our design. Just as in the proof of Corollary 8, the corollary follows from Theorem 7 and the results of Section 3 provided we can show that $\mathcal{F}$ has the $(K', \ell - 1)$-completion property where $K' = O(\ell^t)$. In particular, it suffices to prove that $\mathcal{F}$ has the $(K, \ell - 1)$-completion property with

$$K = \lambda \left(2\ell - 1\right) + \ell - 1.$$

As in the proof of Corollary 8, we fix $\mathcal{F}$-good sequences $\mathbf{x}$ and $\mathbf{x}'$ of length $\ell - 1$ and a set $X$ with $|X| \geq K'$ whose elements do not occur in $\mathbf{x}$ or $\mathbf{x}'$. We have $\mathbf{x} = x_1, x_2, \ldots, x_{\ell-1}$ and $\mathbf{x}' = x_1', x_2', \ldots, x_{\ell-1}'$, say, where for any $i$ we have $x_i \notin X$ and $x_i' \notin X$. We greedily choose elements $y_1, y_2, \ldots, y_{\ell-1} \in X$, choosing $y_i$ so that the following two conditions hold:

(a) $y_i \in X \setminus \{y_1, y_2, \ldots, y_{i-1}\}$;

(b) There is no block that contains $y_i$ and intersects the set $W_i$ in a set of size $k - 1$, where

$$W_i = \{x_i, x_{i+1}, \ldots, x_{\ell-1}\} \cup \{y_1, y_2, \ldots, y_{i-1}\} \cup \{x'_1, x'_2, \ldots, x'_{i}\}.$$

Note that enforcing (a) rules out at most $\ell - 2$ choices for $y_i$. Since the set $W_i$ in condition (b) has cardinality at most $2(\ell - 1)$, there are at most $\lambda \binom{2(\ell - 1)}{t}$ blocks that intersect $W_i$ in $t$ or more points, and so there are at most $\lambda \binom{3(\ell - 1)}{t}$ blocks that intersect $W_i$ in $k - 1$ points. Each such block rules out at most one choice for $y_i$, and so enforcing (b) rules out at most $\lambda \binom{2(\ell - 1)}{t}$ choices for $y_i$. Since $K' > (\ell - 2) + \lambda \binom{2(\ell - 1)}{t}$, the greedy algorithm always succeeds.
We note that condition (a) guarantees that
\[ x_1, x_2, \ldots, x_{\ell-1}, y_1, y_2, \ldots, y_{\ell-1}, x'_1, x'_2, \ldots, x'_{\ell-1} \]
is a partial sequencing. Moreover, we claim that condition (b) guarantees
that this partial sequencing is \((\ell, \mathcal{F})\)-good. For suppose that \(B\) is a block
that is contained in a segment of length \(\ell\) in the partial sequencing. Since
\(x\) and \(x'\) are \(\mathcal{F}\)-good, and are separated by \(\ell - 1\) elements in the sequence,
\(B\) must intersect \(\{y_1, y_2, \ldots, y_{\ell-1}\}\) non-trivially. Let \(i\) be the largest integer
such that \(y_i \in B\). Then \(B \subseteq W_i \cup \{y_i\}\), violating condition (b). This
contradiction establishes our claim. So \(\mathcal{F}\) has the \((K', \ell - 1)\)-completion
property with \(K' = \lambda(\binom{2(\ell-1)}{t}) + \ell - 1\), as required. \(\square\)

5 Sequenceable partial Steiner triple systems

Recall [1, 2] that a partial Steiner triple system is sequenceable if there exists
a sequencing so that for all \(r\), no segment of length \(3r\) consists of the points
of \(r\) pairwise disjoint blocks.

**Theorem 12.** Fix a partial Steiner triple system on a point set \(V\) of cardinality \(v\). Suppose that the triple system has \(k\) disjoint blocks, but does not have \(k + 1\) disjoint blocks. Then the triple system is sequenceable provided that
\[ v > 9k + 22k^{2/3} + 10. \] (7)

Let \(X\) be the set of points in some union of \(k\) disjoint blocks. A key
observation due to Alspach et al. [2] is that any segment of length \(3r\) that
contains fewer than \(r\) elements of \(X\) cannot be the union of disjoint blocks.
For then one of the \(r\) blocks will be disjoint from \(X\), which would mean
that the system contains \(k + 1\) disjoint blocks, contradicting the definition
of \(k\). Using this observation, Alspach et al. showed that a triple system is
sequenceable provided \(v \geq 15k - 5\). We use an extra trick to improve the
bound on \(v\): we use the key observation for long segments, but use a greedy
algorithm to make sure that short segments are also not unions of disjoint
blocks.

**Proof of Theorem 12** Let \(X\) be the set of points in the union of a set of \(k\)
disjoint blocks, so \(|X| = 3k\). Define \(Y = V \setminus X\), so \(Y\) is the set of points not
in the union of these \(k\) disjoint blocks. Suppose that the lower bound (7) on
v holds. To prove the theorem, it suffices to construct a sequencing of the form we want.

We begin by determining where the elements of X and Y should be placed in our sequencing. We construct a binary sequence s, where the positions of zeroes in s will determine positions of elements of X in our sequencing, as follows. Let \( \ell = \lfloor k^{1/3} \rfloor \). We begin by concatenating \( \lfloor 3k/\ell \rfloor \) copies of sequence \((011)^{\ell - 1}0111\) of length \( 3\ell + 1 \). If necessary, we append copies of the sequence 011 until the symbol zero occurs \( 3k \) times in the sequence. We now have a sequence of length \( 9k + \lfloor 3k/\ell \rfloor \) with zero occurring \( 3k \) times; these occurrences are separated by runs of ones of length 2 or 3. Every \( \ell \)-th run of ones has length 3, and the remainder have length 2. Finally, we append copies of the symbol one until the sequence has length \( v \). We claim that the sequence s that results has the following three properties:

(a) Every segment of s of length \( 3r \) contains at most \( r \) zero entries.

(b) If a segment of s of length \( 3r \) ends at position \( 9k + \lfloor 3k/\ell \rfloor + 1 \) or later, the segment contains at most \( r - 1 \) zero entries.

(c) When \( r \geq 3\ell + 1 \), a segment of s of length \( 3r \) contains at most \( r - 1 \) zero entries.

Claim (a) follows since zero entries are separated by 11 or 111, so no three consecutive entries can contain more than one zero. Claim (b) follows since the first \( 3(r - 1) \) entries of the segment contain at most \( r - 1 \) zero entries, by Claim (a), and the final 3 entries of the segment are 111. To see why Claim (c) is true, suppose for a contradiction that the segment contains \( r \) zeroes. The segment must therefore contain at least \( r - 1 \) runs of elements equal to one. All these runs have at least length 2, and since \( r - 1 \geq 3\ell \) at least three of these runs have length 3 (since every \( \ell \)-th run of ones has length 3). The segment will therefore contain \( r \) occurrences of zero and at least \( 2(r - 1) + 3 \) occurrences of one. But this is impossible as the segment has length \( 3r \).

We now greedily construct our sequencing \( z_1, z_2, \ldots, z_v \) as follows. Let \( X = \{x_1, x_2, \ldots, x_{3k}\} \). For each \( i \in \{1, 2, \ldots, v\} \) in turn, let \( j \) be the number of zeroes in the initial segment \( s_1, s_2, \ldots, s_i \) of s. If the \( i \)-th entry \( s_i \) of s is equal to 0, we set \( z_i = x_j \in X \). If \( s_i = 1 \) and \( i \geq 9k + \lfloor 3k/\ell \rfloor \) we set \( z_i \) to be any unused element of Y. Otherwise we set \( z_i = y \in Y \), where \( y \) is chosen as follows. We define the set \( X' \subseteq X \) by

\[
X' = \{x_m : \min(1, j - 3\ell + 1) \leq m \leq \min(3k, j + 1)\},
\]

20
so $X'$ is made up of the last $3\ell$ elements of $X$ we have used, together with the next element of $X$ (if any) we will use. We set $Y'$ to be the last $6\ell$ elements of $Y'$ we have used (or all used elements of $Y$ if we have not yet used $6\ell$ elements). We choose $y \in Y$ so that there is no block of the form \{ $x', y', y$\} for $x \in X'$ and $y' \in Y'$. Note that this last condition rules out at most $|X'||Y'| \leq 18\ell^2 + 6\ell$ elements. Since $i \leq 9k + \lfloor 3k/\ell \rfloor$, the number of choices for $Y$ is at least

$$v - (9k + \lfloor 3k/\ell \rfloor) - 18\ell^2 - 6\ell > 0$$

by (7). Thus the algorithm always succeeds.

Finally, we check that the sequencing $z_1, z_2, \ldots, z_v$ has the property we want. Let $r$ be a positive integer, and consider a segment of the sequencing of length $3r$. Suppose for a contradiction that the segment can be expressed as the union of $r$ disjoint blocks $B_1, B_2, \ldots, B_r$. The segment contains at most $r$ elements of $X$, by condition (a) above, and no block $B_i$ can be disjoint from $X$, since $X$ is the set of points in a maximal set of disjoint blocks. So the segment must contain exactly $r$ elements of $X$, and each block $B_i$ must contain exactly one element of $X$ and two elements of $Y$. Condition (b) shows that the segment must be contained in the initial segment of length $9k + \lfloor 3k/\ell \rfloor$ of the sequencing. Condition (c) shows that $r \leq 3\ell$. So the segment contains at most $6\ell$ elements of $Y$ and at most $3\ell$ elements of $X$. Now consider the last element $y \in Y$ in our segment. Our choice of $y$ implies there is no block of the form \{ $x', y', y$\} with $x' \in X'$ and $y' \in Y'$, where $X \subseteq X$ and $Y' \subseteq Y$ are defined above. But the definition of the set $X'$ implies that the set of elements from $X$ in the segment are all contained in the set $X'$. Similarly, the set of elements $Y$ in the segment are all contained in the set $Y' \cup \{y\}$. So $y$ cannot be contained in a block in our segment, which gives us our required contradiction.

\[ \square \]

References

[1] Brian Alspach, ‘Variations on the sequenceable theme’, in Fan Chung, Ronald L. Graham, Frederick Hoffman, Leslie Hogben, Ronald C. Mullin, Douglas B. West (eds) 50 Years of Combinatorics, Graph Theory, and Computing, CRC Press, Boca Raton FL, to appear.
[2] Brian Alspach, Donald L. Kreher and Adrián Pastine, ‘Sequencing partial Steiner triple systems’, *J. Combin. Des.*, to appear. See *arXiv* 1907.10760v1, 24 July 2019.

[3] Brian Alspach and Georgina Liversidge, ‘On strongly sequenceable abelian groups’, *Art Disc. Appl. Math*, to appear.

[4] Donald L. Kreher and Douglas R. Stinson, ‘Block-avoiding sequencings of points in Steiner triple systems’, *Australas. J. Comb.* 74 (2019), 498–500.

[5] Donald L. Kreher and Douglas R. Stinson, ‘Nonsequenceable Steiner triple systems’, *Bull. Inst. Combin. Appl.* 86 (2019), 64–68.

[6] Donald L. Kreher, Douglas R. Stinson and Shannon Veitch, ‘Block-avoiding point sequencings of directed triple systems’, *Discrete Math.* 343 (2020), article 111773.

[7] Donald L. Kreher, Douglas R. Stinson and Shannon Veitch, ‘Good sequencings for small directed triple systems’, *arXiv* 1907.11144v1, 26 July 2019.

[8] Donald L. Kreher, Douglas R. Stinson and Shannon Veitch, ‘Block-avoiding point sequencings of Mendelsohn triple systems’, *Discrete Math.*, to appear. See *arXiv* 1909.09101v1, 19 September 2019.

[9] Douglas R. Stinson and Shannon Veitch, ‘Block-avoiding point sequencings of arbitrary length in Steiner triple systems’, *arXiv* 1907.04416v1, 9 July 2019.