Isogeometric computation reuse method for complex objects with topology-consistent volumetric parameterization

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Abstract

Volumetric spline parameterization and computational efficiency are two main challenges in isogeometric analysis (IGA). To tackle this problem, we propose a framework of computation reuse in IGA on a set of three-dimensional models with similar semantic features. Given a template domain, B-spline based consistent volumetric parameterization is first constructed for a set of models with similar semantic features. From the properties of Bézier polynomials, an efficient quadrature-free method is investigated to compute the entries of stiffness matrix by Bézier extraction and polynomial approximation. In our approach, evaluation on the stiffness matrix and imposition of the boundary conditions can be pre-computed and reused during IGA on a set of CAD models. Examples with complex geometry are presented to show the effectiveness of our methods, and efficiency similar to the computation in linear finite element analysis can be achieved for IGA taken on a set of models.

Keywords: computation reuse; isogeometric analysis; consistent volume parameterization

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Fig. 1. Comparison of results obtained from IGA computation reuse and direct mapping. (a) and (c) are two different planar domains that have the same tensor-product B-spline space. (b) and (d) are the IGA solution of a heat conduction problem solved on these two domains separately. (e) is the result by directly mapping (b) into (c), and (f) shows the error-map of (e) comparing to (d). Meanwhile, (g) shows the error-map of our computation reuse solution – it is much more accurate.

1. Introduction

The isogeometric analysis (IGA) approach, which was proposed by Hughes et al. [8], offers the possibility of seamless integration of computational analysis and geometric design. Two major challenges in the current development of IGA are volumetric parameterization and computational efficiency. In the recent book of Cottrell and Hughes [5], it has been pointed out that the most significant challenges towards isogeometric analysis is how to construct analysis-suitable parameterizations from given CAD models. On the other hand, high-order basis functions are often employed to achieve smooth solution fields with high continuity, which however also increases the computational cost when filling stiffness matrices. In this paper, we propose a method for computation reuse in IGA on three-dimensional models with similar semantic features, by which the computational efficiency can be significantly improved.

It should be mentioned that the result of IGA-solution cannot be directly reused by the mapping of a consistent parameterization between models having similar semantic features [11, 12]. As illustrated in Fig.1(a,b) and (c,d), a heat conduction problem with homogeneous boundary
condition and the following source function

\[ f(x, y) = -2\pi^2 \sin(\pi x) \sin(\pi y) \]

is solved separately over two different planar domains that have the same tensor-product B-spline space. When directly mapping the result from Fig.1(b) into the domain of (c), the result of heat distribution is as shown in (e) which is different from the result of computation (i.e., Fig.1(d)). Fig.1(f) shows the error-map in color between (e) and (d). On the other aspects, when using the computation reuse approach developed in this paper, the approximation errors brought in are trivial – see the error-map shown in Fig.1(g).

The main contribution of our work can be summarized as follows:

- From the properties of Bézier polynomials, an efficient quadrature-free method is proposed to compute the entries of stiffness matrix with the help of Bézier extraction and polynomial approximation techniques applying to trivariate rational Bézier functions.

- For a set of models with consistent volumetric parametrization that can be constructed according to similar semantic features, evaluation to fill the stiffness matrix and imposition of the boundary conditions can be partially pre-computed and reused during isogeometric analysis. This saves lots of computational costs.

The rest of our paper is organized as follows. In Section 2, the related work on volumetric parameterization and computational efficiency of IGA will be reviewed. The method to construct consistent B-spline based volumetric parameterization is presented in Section 3. Section 4 presents the quadrature-free IGA method using the Bézier extraction and polynomial approximation techniques. By combining techniques presented in Sections 3 and 4, the computation reuse framework for a set of models with similar semantic features is presented in Section 5. Lastly, we conclude this paper and discuss possible future works in Section 6.

2. Related work

Volumetric parameterization From the viewpoint of graphics applications, volume parameterization of 3D models has been studied in [14, 26, 27]. On the other aspect, there are also some recent work on volume parameterization in the literature of IGA. Martin et al. [16] proposed a
method to fit a genus-0 triangular mesh by B-spline volume parameterization, based on discrete volumetric harmonic functions. A variational approach for constructing NURBS parameterization of swept volumes is proposed by Aigner et al. [1]. Escobar et al. [6] proposed a method to construct a trivariate T-spline volume of complex genus-zero solids by using an adaptive tetrahedral meshing and mesh untangling technique. Zhang et al. [31] proposed a robust and efficient algorithm to construct injective solid T-splines for genus-zero geometry from a boundary triangulation. Based on the Morse theory, a volumetric parameterization method of mesh model with arbitrary topology is proposed in [24]. In [29, 30], a constraint optimization framework is proposed to obtain analysis-suited planar and volumetric parameterization of computational domain. Pettersen and Skytt proposed the spline volume faring method to obtain high-quality volume parameterization for isogeometric applications [18]. Zhang et al. [32] studied the construction of conformal solid T-spline from boundary T-spline representation by Octree structure and boundary offset. For volume parameterization of three-manifold solid models having homeomorphic topology, Kwok and Wang [13] proposed an algorithm to constructing volumetric domains with consistent topology. The generated volumetric parameterizations share the same set of base domains and are constrained by the corresponding semantic features in the form of anchor points. In this paper, a compactly supported radial basis function method is proposed to construct consistent volumetric B-spline parameterization for models with similar semantic features.

**Efficiency issues of isogeometric analysis** High-order basis functions are used to represent the geometry and the physical field in IGA to achieve high-accuracy simulation results. Hence, computational efficiency is a key issue in the field of isogeometric analysis. In order to improve the efficiency, several methods have been proposed. There is a trend to use graphic possessing units (GPU) to improve the computational efficiency of assembling the stiffness matrix (e.g., [10]). On the other hand, efficiency improvement on integral computing has also been studied. Hughes et al. [9] proposed an efficient quadrature rules for NURBS-based isogeometric analysis. Antolin et al. [2] developed a sum-factorization approach to save the quadrature computational cost significantly based on the tensor-product structure of splines and NURBS shape functions. Mantzaflaris and Jüttler [15] presented a quadrature-free integration method by interpolation and look-up table for Galerkin-based isogeometric analysis. In this paper, we propose the concept of *computation reuse* to improve the efficiency of IGA on a set of CAD models with consistent topology.
Given the template base domain, \( \Psi \), and a target surface model, \( M \), the surface cross-parameterization \cite{11} can be applied to partition \( M \) consistently to the surface of \( \Psi \). After obtaining a surface model \( \Psi_M \) by mapping \( M \) to \( \Psi \), the vertices on \( \Psi_M \) and \( M \) are used as handles to map the sampled points on the interior patches to \( M \) by volumetric parameterization \cite{22}. B-spline solid for \( M \) can be obtained by fitting it to the boundary surface and the sampled points of \( M \).

3. Consistent B-spline volumetric parameterization of complex 3D models

To prepare for the computation reuse in IGA, we need to partition a model into a set of base domains consistent to the pre-defined one, where each base domain will be represented by a trivariate spline. We assume the reuse of IGA is applied to the models with similar semantic features, e.g., a whole sequence of products having different shapes but the same topology. The analysis will respect the semantic features, which are specified as anchor points, such that the boundary conditions can also be reused.

Given a connectivity of volumetric base domains (\( \Psi \)), e.g., the template base domain in Fig. 2, we can partition a target surface model (\( M \)) into a set of volumetric sub-domains consistent to \( \Psi \). The boundary surface is first partitioned consistently according to anchor points \cite{11, 12}, and the boundary surface is used as the constraints to construct volumetric parameterization using CSRBF-based elastic functions. The volumetric parameterization is used to wrap the sample points on the interior surface of \( \Psi \) to \( M \). The trivariate spline solids are constructed in each domain by fitting the boundary surface of \( M \) and the sample points on above determined interior surfaces.
3.1. Consistent boundary surface decomposition

Based on the anchor points specified on $M$, the surface of $M$ is partitioned consistently according to the nodes in $\Psi$ using the method in [11, 12]. The basic idea is to trace shortest paths between anchor points on $M$ for each edge on the boundary surface of $\Psi$. To ensure topological consistency, we need to ensure the paths are free of intersections, blocking and wrong cyclical order when tracing the paths [19]. Surface cross-parameterization [11, 12] can then be computed through this partition, and a surface model $\Psi_M$ is obtained by mapping the boundary surface of $M$ to $\Psi$. Once the boundary surface is partitioned, the next step is to construct the interior patches.

3.2. CSRBF-based volumetric parameterization

Due to the reason that $M$ is a surface model without volumetric information, we need to construct the interior patches for $M$. A sampling is first taken on the interior patches in $\Psi$. By the volumetric parameterization presented in [22], using the vertices on $\Psi_M$ and $M$ as handles, sample points on the interior patches of $\Psi$ can be mapped to the interior of $M$. Specifically, the volumetric parameterization can be expressed by the elastic function:

$$f(x) = \sum_{j=1}^{n} d_j \phi(x - v_j) + P(x),$$

where $v_j$s are the locations of the constraint vertices on $\Psi_M$, $d_j$s are the weights, and $P(x)$ is a linear polynomial that accounts for the linear and constant portions of $f$.

To solve for the set of $d_j$ that will satisfy the constraints, $v'_i = f(v_i)$, on the elastic function, we can substitute the right side of Eq.(1) for $f(v_i)$ and yield to

$$v'_i = \sum_{j=1}^{k} d_j \phi(v_i - v_j) + P(x).$$

Since this equation is linear with respect to the unknowns $d_j$s and the coefficients of $P(x)$, the unknowns can be formulated and solved as a linear system.

We take the Wendland’s compactly supported radial basis function as $\phi(r)$, it is given by

$$\phi(r) = (1 - r^6_+) (3 + 18r + 35r^2), \text{ with } r = \frac{\|x\|}{\lambda}, \quad a_+ = \max\{a, 0\},$$

which has a compact support with the radius $\lambda$, and has $C^4$-continuity. As a compactly supported kernel function is used, the linear equation system will become sparse and can be efficiently solved by Cholesky decomposition or LU decomposition.
Now the volumetric parameterization has been established. The sample points on the interior patches in Ψ are mapped to $M$ by $f(\cdot)$.

3.3. B-spline solid construction

The trivariate spline solid $S(u, v, w)$ (i.e., B-spline in our implementation) for each base domain inside $M$ can be constructed by fitting the boundary surface of $S(u, v, w)$ to its corresponding boundary surface on $M$ and the interior sample points obtained by the nonlinear elastic function $f(\cdot)$. By this way, we can convert the target model $M$ into a set of connected trivariate spline solids with consistent topology as Ψ. When applying this to a sequence of models $\{M_i\}$, all models will have spline solids in the same connectivity but different control points (i.e., different shapes). With the help of this setup, we will show how to reuse the computation of IGA taken on one model in the IGA of other models.

4. Quadrature-free isogeometric analysis with Bézier extraction and polynomial approximation

In this section, an efficient quadrature-free method is proposed to compute the entries of stiffness matrix with the help of Bézier extraction and polynomial approximation techniques of trivariate rational Bézier functions.

4.1. Preliminary on Bernstein polynomials

Some preliminary on Bernstein polynomials will be used in our method. They are reviewed below.

**Lemma 4.1. Product of Bernstein polynomials**

$$B_i^m(t)B_j^n(t) = \binom{m}{i}\binom{n}{j} B_{i+j}^{m+n}(t)$$  (4)

**Lemma 4.2. Integration of Bernstein polynomials**

$$\int_0^1 B_i^m(t)dt = \frac{1}{m+1}$$  (5)

**Lemma 4.3. Degree elevation of Bernstein polynomials**

$$B_i^n(t) = tB_{i1}^{n-1}(t) + (1-t)B_{i}^{n-1}(t)$$  (6)
Proposition 4.4. Let \( R(u,v,w) \) and \( S(u,v,w) \) be parametric function defined by

\[
R(u,v,w) = \sum_{i=0}^{l_1} \sum_{j=0}^{m_1} \sum_{k=0}^{n_1} a_{ijk} B^{l_1}_i(u) B^{m_1}_j(v) B^{n_1}_k(w),
\]

and

\[
S(u,v,w) = \sum_{i=0}^{l_2} \sum_{j=0}^{m_2} \sum_{k=0}^{n_2} b_{ijk} B^{l_2}_i(u) B^{m_2}_j(v) B^{n_2}_k(w),
\]

where \( a_{ijk} \) and \( b_{ijk} \) are scale values. Then the product of \( R(u,v,w) \) and \( S(u,v,w) \) can be defined as

\[
R(u,v,w) \times S(u,v,w) = \sum_{i=0}^{l_1+l_2} \sum_{j=0}^{m_1+m_2} \sum_{k=0}^{n_1+n_2} c_{ijk} B^{l_1+l_2}_i(u) B^{m_1+m_2}_j(v) B^{n_1+n_2}_k(w), \tag{7}
\]

where

\[
c_{ijk} = \sum_{r=\max(0,i-l_1)}^{\min(l_2,i)} \sum_{s=\max(0,j-m_1)}^{\min(m_2,j)} \sum_{t=\max(0,k-n_1)}^{\min(n_2,k)} (\begin{array}{c} l_1 + l_2 \\ i \\ \end{array}) (\begin{array}{c} m_1 + m_2 \\ j \\ \end{array}) (\begin{array}{c} n_1 + n_2 \\ k \\ \end{array}) a_{rst} b_{(i-r),(j-s),(k-t)}
\]

This proposition can be proved directly by Eq.(4).

4.2. Bézier extraction of B-spline volume

In order to achieve an efficient computation, the isogeometric analysis problem is solved with Bézier extraction [3], in which piece-wise B-spline representation is converted into Bézier form.

Since B-spline basis defined on a knot vector can be written as a linear combination of the Bernstein polynomials, that is,

\[
N(\xi) = CB(\xi) \tag{8}
\]

where \( C \) denotes the Bézier extraction operator and \( B(\xi) \) are the Bernstein polynomials which are defined on \([0,1] \). The conversion matrix \( C \) is sparse and its entries can be obtained by knot insertions or recursive computation. Details on Bézier extraction can be found in Borden et al. [3] and Scott et al. [20].

With the conversion matrix \( C \), the Bézier extraction of B-spline volume can be represented as follows

\[
P = C(\xi) \otimes C(\eta) \otimes C(\zeta) Q \tag{9}
\]

where \( Q \) are the control points of the B-spline volume, and \( P \) are the control points of the extracted Bézier volume. An example with cubic B-spline volume is shown in Fig. 3 to illustrate the extraction results and the corresponding control lattice of the extracted four Bézier volumes with different colors.
4.3. **Isogeometric analysis of heat conduction problem**

Given a domain $\Omega$ with boundary $\Gamma = \partial \Omega_D$ and a volumetric parameterization as follows

$$S(\xi, \eta, \zeta) = (x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{l} B^p_i(\xi) B^q_j(\eta) B^r_k(\zeta) P_{i,j,k}.$$ 

We consider the following thermal conduction problem:

$$\nabla (\nabla T(x)) = g(x) \quad \text{in } \Omega$$

$$T(x) = T_0(x) \quad \text{on } \partial \Omega_D$$

where $x$ are the Cartesian coordinates, $T$ represents the temperature field. Dirichlet condition with imposed temperature $T_0$ is applied on $\partial \Omega_D$ . $g$ is a user-defined function as a source term to the classical heat conduction equation.

According to a classical variational approach, we seek for a solution $T \in H^1(\Omega)$, such as $T(x) = T_0(x)$ on $\partial \Omega_D$ and:

$$\int_{\Omega} \nabla (\nabla T(x)) \psi(x) \, d\Omega = \int_{\Omega} g(x) \psi(x) \, d\Omega \quad \forall \psi \in H^1_{\partial \Omega_D}(\Omega),$$

where $\psi(x)$ are trial functions. After integrating by parts and applying the boundary conditions, we can obtain:

$$-\int_{\Omega} \nabla T(x) \nabla \psi(x) \, d\Omega = \int_{\Omega} f(x) \psi(x) \, d\Omega.$$ 

(11)

According to the paradigm of IGA, the temperature field is represented using trivariate spline basis functions as follows:

$$T(\xi, \eta, \zeta) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} \hat{B}^p_i(\xi) \hat{B}^q_j(\eta) \hat{B}^r_k(\zeta) T_{ijk},$$
where $\hat{B}_i$ functions are Bernstein basis functions and $u = (\xi, \eta, \zeta) \in P$ are domain parameters. Then, we define the trial functions $\psi(x)$ in the physical domain as:

$$B_{ijk}(x) = \hat{B}_{ijk} \circ \sigma^{-1}(x, y, z) = \hat{B}_{ijk}(\xi, \eta, \zeta) = \hat{B}_i^p(\xi) \hat{B}_j^p(\eta) \hat{B}_k^p(\zeta).$$

The weak form in Eq. (11) can then be written as:

$$\sum_{r=1}^{n_r} \sum_{s=1}^{n_s} \sum_{t=1}^{n_t} T_{rst} \int_{\Omega} \nabla B_{rst}(x) \cdot \nabla B_{ijk}(x) \, d\Omega = - \int_{\Omega} g(x) \, B_{ijk}(x) \, d\Omega.$$

Finally, we obtain a linear system similar to that resulting from the classical finite-element methods, with a stiffness matrix and a right-hand side vector defined as

$$\sum_{r=1}^{n_r} \sum_{s=1}^{n_s} \sum_{t=1}^{n_t} T_{rst} M_{ijk,rst} = S_{ijk}$$

with

$$M_{ijk,rst} = \int_{\Omega} \nabla B_{rst}(x) \cdot \nabla B_{ijk}(x) \, d\Omega = \int_{P} \nabla u \hat{B}_{rst}^T(u) J(u) K(u) \nabla u \hat{B}_{ijk}(u) \, dP$$

$$S_{ijk} = \int_{\Omega} g(x) \, B_{ijk}(x) \, d\Omega = \int_{P} \hat{B}_{ijk}(u) J(u) g(T(u)) \, dP,$$

where $J$ is the Jacobian of the transformation from physical domain to parametric domain, and $K$ is the inverse of the Jacobian matrix.

4.4. Explicit representation of stiffness matrix entries

Suppose that the entries of the stiffness matrix are denoted by $M_{ijk,rst} = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} F(u) \, du$, we can derive the following proposition.

**Proposition 4.5.** $F(u)$ can be represented as a trivariate rational Bézier function with the following form

$$F(u) = \frac{\sum_{i=0}^{6l-4} \sum_{j=0}^{6m-4} \sum_{k=0}^{6n-4} F_{ijk} B_{i}^{6l-4}(u) B_{j}^{6m-4}(v) B_{k}^{6n-4}(w)}{\sum_{i=0}^{3l-1} \sum_{j=0}^{3m-1} \sum_{k=0}^{3n-1} j_{ijk} B_{i}^{3l-1}(u) B_{j}^{3m-1}(v) B_{k}^{3n-1}(w)}$$

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Proof.

\[
F(u) = \hat{B}_{rst}(u) J(u) K(u) \kappa(u) \nabla_u \hat{B}_{rst}(u)
\]

\[
= J(u) \left( \frac{\partial \hat{B}_{rst}(u)}{\partial \xi} \frac{\partial \hat{B}_{rst}(u)}{\partial \eta} \frac{\partial \hat{B}_{rst}(u)}{\partial \zeta} \right) \left( \begin{array}{ccc}
\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\
\frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\
\frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z}
\end{array} \right) \left( \begin{array}{ccc}
\frac{\partial B_{rst}(u)}{\partial \xi} & \frac{\partial B_{rst}(u)}{\partial \eta} & \frac{\partial B_{rst}(u)}{\partial \zeta} \\
\frac{\partial B_{rst}(u)}{\partial \xi} & \frac{\partial B_{rst}(u)}{\partial \eta} & \frac{\partial B_{rst}(u)}{\partial \zeta} \\
\frac{\partial B_{rst}(u)}{\partial \xi} & \frac{\partial B_{rst}(u)}{\partial \eta} & \frac{\partial B_{rst}(u)}{\partial \zeta}
\end{array} \right)
\]

\[
= J(u) \left( \frac{\partial \hat{B}_{rst}(u)}{\partial \xi} \frac{\partial \hat{B}_{rst}(u)}{\partial \eta} \frac{\partial \hat{B}_{rst}(u)}{\partial \zeta} \right) \left( \begin{array}{ccc}
a & b & c \\
b & d & e \\
c & e & f
\end{array} \right) \left( \begin{array}{ccc}
\frac{\partial \hat{B}_{rst}(u)}{\partial \xi} \\
\frac{\partial \hat{B}_{rst}(u)}{\partial \eta} \\
\frac{\partial \hat{B}_{rst}(u)}{\partial \zeta}
\end{array} \right)
\]

\[
= J(u) \left( a \left( \frac{\partial \hat{B}_{rst}(u)}{\partial \xi} \right)^2 + d \left( \frac{\partial \hat{B}_{rst}(u)}{\partial \eta} \right)^2 + f \left( \frac{\partial \hat{B}_{rst}(u)}{\partial \zeta} \right)^2 + 2b \frac{\partial \hat{B}_{rst}(u)}{\partial \xi} \frac{\partial \hat{B}_{rst}(u)}{\partial \eta} \right)
\]

\[
+ 2c \frac{\partial \hat{B}_{rst}(u)}{\partial \xi} \frac{\partial \hat{B}_{rst}(u)}{\partial \zeta} + 2e \frac{\partial \hat{B}_{rst}(u)}{\partial \eta} \frac{\partial \hat{B}_{rst}(u)}{\partial \zeta} \right)
\]

in which

\[
a = \xi_x^2 + \eta_x^2 + \zeta_x^2 \quad b = \xi_x \xi_y + \eta_x \eta_y + \zeta_x \zeta_y \quad c = \xi_x \xi_z + \eta_x \eta_z + \zeta_x \zeta_z
\]

\[
b = \xi_y^2 + \eta_y^2 + \zeta_y^2 \quad d = \xi_y \xi_z + \eta_y \eta_z + \zeta_y \zeta_z \quad e = \xi_z^2 + \eta_z^2 + \zeta_z^2
\]

\[
\xi_x = \frac{y \eta \zeta - z \eta \zeta}{J}, \quad \xi_y = \frac{x \eta \zeta - x \xi \zeta}{J}, \quad \xi_z = \frac{y \eta \zeta - y \xi \zeta}{J},
\]

\[
\eta_x = \frac{y \xi \zeta - \xi \eta \zeta}{J}, \quad \eta_y = \frac{x \xi \zeta - x \eta \zeta}{J}, \quad \eta_z = \frac{y \xi \zeta - y \eta \zeta}{J},
\]

\[
\zeta_x = \frac{y \eta \zeta - y \xi \zeta}{J}, \quad \zeta_y = \frac{x \eta \zeta - x \xi \zeta}{J}, \quad \zeta_z = \frac{y \eta \zeta - y \xi \zeta}{J}.
\]

\[
J(u) = \begin{vmatrix}
\xi_x & \xi_y & \xi_z \\
\eta_x & \eta_y & \eta_z \\
\zeta_x & \zeta_y & \zeta_z
\end{vmatrix} = \sum_{i=0}^{3l-1} \sum_{j=0}^{3m-1} \sum_{k=0}^{3n-1} J_{ijk} B_i^{3l-1}(u) B_j^{3m-1}(v) B_k^{3n-1}(w),
\]

in which \( J_{ijk} \) has the following form as given in [25]

\[
J_{ijk} = \sum_{i_1+i_2+i_3=i} \sum_{j_1+j_2+j_3=j} \sum_{k_1+k_2+k_3=k} D_{ijk} \cdot \det \begin{pmatrix}
P_{i_1+1,j_1,k_1} - P_{i_1,j_1,k_1} \\
P_{i_2,j_2+1,k_2} - P_{i_2,j_2,k_2} \\
P_{i_3,j_3,k_3+1} - P_{i_3,j_3,k_3}
\end{pmatrix}^T,
\]

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with
\[
D_{ijk} = \ln \frac{\binom{l-1}{i1} \binom{m-1}{j2} \binom{n-1}{k3}}{\binom{3l-1}{i} \binom{3m-1}{j} \binom{3n-1}{k}}.
\] (18)

According to Eq.(8) the product formula of two trivariate Bernstein polynomials in Proposition 4.4, we can rewritten \(F(u)\) as a high-order trivariate rational Bernstein polynomial,
\[
F(u) = \frac{\sum_{i=0}^{6l-4} \sum_{j=0}^{6m-4} \sum_{k=0}^{6n-4} F_{ijk} B_i^{6l-4}(u) B_j^{6m-4}(v) B_k^{6n-4}(w)}{\sum_{i=0}^{3l-1} \sum_{j=0}^{3m-1} \sum_{k=0}^{3n-1} J_{ijk} B_i^{3l-1}(u) B_j^{3m-1}(v) B_k^{3n-1}(w)}
\] (19)
in which \(F_{ijk}\) can be computed according to Eq.(7).

In general cases, the integration of a rational Bézier function over \([0, 1]\) is either very complex or has no analytically solution. Gaussian-quadrature method is usually employed in general IGA to compute the integration of rational function in Eq.(13) approximately. As shown in Eq.(5), the integration of polynomial Bézier functions (non-rational) has an explicit and exact form. As a classical problem in CAGD, approximating rational Bézier curves and surfaces with polynomial Bézier curves and surfaces has been studied in \([7, 21, 23]\). In this paper, we generalize the weighted least-squares approach \([21]\) to trivariate splines and approximate the trivariate rational function \(F(u)\) with trivariate polynomial Bézier function \(G(u) = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \sum_{k=0}^{\gamma} G_{ijk} B_i^{\alpha}(u) B_j^{\beta}(v) B_k^{\gamma}(w)\).

Suppose that \(F(u)\) can be rewritten as \(F(u) = F_2(u, v, w)\), our Bézier approximation problem can be stated as: given the fixed degree \(\alpha, \beta, \gamma\), to find the control variables \(G_{ijk}\) such that the trivariate Bézier representation \(G(u, v, w) = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \sum_{k=0}^{\gamma} G_{ijk} B_i^{\alpha}(u) B_j^{\beta}(v) B_k^{\gamma}(w)\) is the best approximation of \(F(u)\). That is, if \(F_2(u, v, w)\) is set to be the weight function, the following objective function is minimized
\[
D(F, G) = \int_0^1 \int_0^1 \int_0^1 F_2(u, v, w)(\frac{F_1(u, v, w)}{F_2(u, v, w)} - G(u, v, w))^2 dudvdw
\]
Hence, the coefficient \(G_{ijk}\) satisfy
\[
\frac{\partial D(F, G)}{\partial G_{ijk}} = 0
\] (20)
which is
\[
2 \int_0^1 \int_0^1 (F_1(u, v, w) - F_2(u, v, w)G(u, v, w))B_i^{\alpha}(u)B_j^{\beta}(v)B_k^{\gamma}(w)dudvdw = 0
\] (21)
Eq.(21) can be rewritten as
\[
\int_0^1 \int_0^1 \int_0^1 F_2(u, v, w)G(u, v, w)B_i^{\alpha}(u)B_j^{\beta}(v)B_k^{\gamma}(w)dudvdw
\]
\[
= \int_0^1 \int_0^1 \int_0^1 F_1(u, v, w)B_i^{\alpha}(u)B_j^{\beta}(v)B_k^{\gamma}(w)dudvdw
\]
From the product and integral computation properties of Bézier polynomials, we have

\[
\begin{aligned}
&\sum_{a=0}^{\alpha+3l-1} \sum_{b=0}^{\beta+3m-1} \sum_{c=0}^{\gamma+3n-1} \frac{H_{a,b,c}}{a+i} \binom{2\alpha+3l-1}{b+j} \binom{2\beta+3m-1}{c+k} \\
&= \sigma \sum_{p=0}^{6l-4} \sum_{q=0}^{6m-4} \sum_{r=0}^{6n-4} \frac{F_{p,q,r}}{p+i} \binom{\beta+6m-3}{q+j} \binom{\gamma+6n-3}{r+k},
\end{aligned}
\]  

(22)

in which

\[
\sigma = \frac{(2\alpha + 3l)(2\beta + 3m)(2\gamma + 3n)}{(\alpha + 6l - 3)(\beta + 6m - 3)(\gamma + 6n - 3)},
\]  

(24)

Then, from Eq. (22) for all the \(G_{ijk}\), we can obtain a linear system in the following form,

\[
L \cdot E \cdot G = \sigma \cdot Q \cdot F,
\]  

(25)

in which \(G = [G_{ijk}]_{(\alpha + 1) \times (\beta + 1) \times (\gamma + 1)}\) is a vector with unknown variables as entries, \(\sigma\) is defined in Eq. (24), \(L = [L_{ijk}^{abc}]\) is a matrix with

\[
L_{ijk}^{abc} = \frac{1}{\binom{2\alpha+3l-1}{a+i} \binom{2\beta+3m-1}{b+j} \binom{2\gamma+3n-1}{c+k}},
\]  

(26)

\(E = [E_{rst}^{abc}]\) is a matrix with

\[
E_{rst}^{abc} = \binom{3l-1}{r} \binom{\alpha}{a-r} \binom{3m-1}{s} \binom{\beta}{b-s} \binom{3n-1}{t} \binom{\gamma}{c-t} J_{rst},
\]  

(27)

\(Q = [Q_{ijk}^{pqr}]\) is a matrix with

\[
Q_{ijk}^{pqr} = \binom{6l-4}{p} \binom{6m-4}{q} \binom{6n-4}{r} \binom{\alpha+6l-4}{p+i} \binom{\beta+6m-4}{q+j} \binom{\gamma+6n-4}{r+k},
\]  

(28)

\(F = [F_{pqr}]\) is a vector with \(F_{pqr}\) as entries defined in Eq. (19).

By solving this linear system, the Bézier approximation \(G(u, v, w)\) of the rational function \(F(u)\) can be obtained as

\[
G = \sigma \cdot E^{-1} \cdot L^{-1} Q \cdot F.
\]  

(29)

By using Eq. (5) in Lemma 4.2, the entries \(M_{ijk, rst}\) of the stiffness matrix has the following explicit form

\[
M_{ijk, rst} = \int_0^1 \int_0^1 \int_0^1 F(u)du \approx \int_0^1 \int_0^1 \int_0^1 G(u)du = \frac{1}{(\alpha + 1)(\beta + 1)(\gamma + 1)} \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \sum_{k=0}^{\gamma} G_{ijk}.
\]  

(30)
5. Computation-reuse for models with consistent volume parameterization

In this section, we propose a computation reuse framework for a set of models with consistent volume parameterization based on the proposed quadrature-free IGA method. In the first phase, we obtain consistent B-spline volumetric parameterization from given B-rep models and the template based domain. In the second phase, with the help of Bézier extraction and quadrature-free techniques, we perform analysis-reuse on the resulting consistent volumetric parameterization.

5.1. Framework overview

Generally, given a set of CAD models with consistent topology, our computation reuse framework can be described as following:

**Input**: a set of CAD models with consistent topology

**Output**: IGA results and topology-consistent volume parameterization

Step 1: construct topology-consistent volumetric parameterization for the input set of CAD models and the template base domain as described in Section 3.

Step 2: perform Bézier extraction for the B-spline based volume parameterization of one CAD model, then the conversion matrix can be stored and reused for the Bézier extraction of volume parameterization of other CAD models.

Step 3: for a specified PDE problem (i.e., heat conduction problem), impose the boundary condition by boundary collocation method described in subsection 5.2.2.

Step 4: solve the specified PDE problem on the CAD model in Step 3 with the quadrature-free IGA method proposed in Section 4.

Step 5: solve the specified PDE problem for the other CAD models, in which the boundary collocation matrix can be reused and the entry evaluation of stiffness matrix in Step 4 can be partially reused.

Step 6: output the results of IGA.

Based on this framework, a significant improvement on the efficiency can be achieved for a set of models with consistent-topology volume parameterizations. In the following subsection, we will discuss the computation reuse mechanism during the entry evaluation of stiffness matrix and the boundary condition imposition.
5.2. Computational reuse for entries of stiffness matrix

In our computation-reuse framework, there are mainly two reuse parts: the entry evaluation of element stiffness matrix and the boundary condition imposition.

5.2.1. Entry evaluation of element stiffness matrix

After the heat conduction problem is solved on one of the input models, we want to reuse some computation for the IGA solving on the other models. As describe in subsection 4.4, there are mainly three computing parts for the entry evaluation of element stiffness matrix.

The first part is for the computation of $J_{ijk}$ in Proposition 4.5. From the formula of $J_{ijk}$ as shown in Eq.(17), the values of $D_{ijk}$ in Eq.(18) can be stored and reused for the following models. Therefore, we only need to compute the values related the control points for different Bézier solids with the same degree.

The second part is for the computation of $F_{ijk}$ in Proposition 4.5. As shown in the proof of Proposition 4.5, the computational of $F_{ijk}$ can be considered as a repeated process of production on two trivariate Bézier functions. In total, we have to apply the production for 54 times to obtain the value of $F_{ijk}$. As the set of given B-spline models have the same basis functions with the same degree, some coefficients can be stored and reused to compute the $F_{ijk}$ for the other volumetric models in a similar way as the computational reuse for $J_{ijk}$.

The third part is for the Bézier approximation of the rational trivariate Bézier function $F(u)$. As shown in Eq.(29), the Bézier approximation $G(u,v,w)$ of $F(u)$ can be obtained as

$$G = \sigma \cdot E^{-1} \cdot L^{-1} Q \cdot F.$$  \hspace{1cm} (31)

We can find that in Eq.(31), given the degree $(\alpha, \beta, \gamma)$ of Bézier approximation and the degree $(l, m, n)$ of the B-spline model, $\sigma, L^{-1}$ and $Q$ are independent with the control points information. Hence, after solving the IGA problem on one model, $\sigma, L^{-1}$ and $Q$ can be stored and reused for the following IGA solving process on other models.

Overall, given a set of volumetric models with the same B-spline basis function representation (i.e., the same degree and the same knot vectors for each block), the computation can be partially reused after solving the IGA problem on one model. For the new IGA solving process, the parts that we need compute for element stiffness matrix filling are the following formula involved in the
computation of $J_{ijk}$,

$$\text{det} \begin{pmatrix} P_{i_1+1,j_1,k_1} - P_{i_1,j_1,k_1} \\ P_{i_2,j_2+1,k_2} - P_{i_2,j_2,k_2} \\ P_{i_3,j_3,k_3+1} - P_{i_3,j_3,k_3} \end{pmatrix}^T,$$

and the vector $F = [F_{pqr}]$ with $F_{pqr}$ as entries defined in Eq. (14).

After the local stiffness matrix for each element are filled, the global stiffness matrix can be obtained by assembling. In the assembly process, the boundary condition described in the governing equation must be imposed and the corresponding entries related to the boundary condition will also be filled. In the following subsection, we will discuss the reuse for the imposition of boundary conditions.

5.2.2. Imposition of boundary conditions

As known, the Bernstein basis functions do not have interpolating property at control points. Hence, we cannot impose the essential boundary conditions directly onto the boundary control values. Special treatments should be implemented to achieve the specified boundary conditions, such as the least square approach, the penalty function method or the Nitsche method. In this paper, a boundary collocation method is employed to impose boundary conditions. For a set of computational models with topology-consistent volume parameterization, the boundary control variables can be reused for the same boundary conditions, and the collocation matrix can be reused for different boundary conditions.

Suppose that $\{x_i\}_{i=0}^{nb}$ are collocation points on the boundary surfaces, and $\{\xi_i, \eta_i, \zeta_i\}_{i=0}^{nb}$ are corresponding parametric coordinates in the parametric domain. Then, the Dirichlet boundary condition $U|_{\partial \Omega} = h(\{x_i\})$ can be defined as,

$$\sum_{j=1}^{k} N_j(\xi_i, \eta_i, \zeta_i) b_j = h(\{x_i\}), i = 0, \cdots, nb$$

in which $h(\{x_i\})$ are boundary value to be interpolated, $b_j$ are control variables to be solved. This equation can be rewritten into a matrix form as

$$MB = H,$$

in which the entry of matrix $M$ is $M_{ij} = N_j(\xi_i, \eta_i, \zeta_i)$, $H = [h_i] = [h(\{x_i\})], i,j = 0, \cdots, nb$. Then
the boundary control variables $b_j$ can be solved by

$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{H}. \quad (34)$$

In this paper, since the models with topology-consistent volume parameterization have the same basis functions for the corresponding blocks, the boundary control variables which is determined by Eq. (34) can be used for all other models for a PDE problem with the same boundary conditions. Furthermore, for the PDE problem with different boundary conditions, the inverse of collocation matrix $\mathbf{M}^{-1}$ can also be reused for the models with topology-consistent volume parameterization.

5.3. Experimental results

In this subsection, some experimental results will be presented to show the advantage of the proposed computation reuse method.

Three sets of CAD models are tested in this paper. The first set has two hand models (Fig.4), there are four airplane models in the second set (Fig.5), and the third set consists of four human models with consistent topology (Fig.6). For each model in Figs.4-6, from left to right we show the input surface model with patch-partition information, the results of boundary B-spline surface fitting, the results of volume parameterization, and the IGA results for a heat conduction problem respectively.

In order to illustrate the effectiveness of the proposed computation reuse approach, the corresponding average assembling time of the IGA and IGA-reuse approach are shown in Table 1 for the models in Figs.4-6 respectively. All the computations are implemented in C++ and timed on a Macbook Pro with a quad core 2.4 GHz Intel Core i7 processor and 8GB RAM. From the performance statistic shown in the table, we can find that the computational costs in assembling stiffness matrix can be reduced significantly by our method. Furthermore, as shown in Fig.7, the acceleration ratio keeps increasing while increasing the degree of freedom in IGA.

Overall, since the evaluation of high-order basis function for one model can be reused for other models with consistent topology, a similar performance of classical linear finite element method can be achieved for isogeometric analysis on a set of models. This addresses a main shortcoming of isogeometric analysis and makes it cost-efficient in solving large-scale problems in computational mechanics.
Table 1: Quantitative data and average assembling time for computation reuse in IGA

| Example       | $p$ | $h$-refinement | #DOF   | IGA      | IGA-reuse | acceleration ratio |
|---------------|-----|----------------|--------|----------|-----------|-------------------|
| Hand (Fig.4)  | 2   | $h = 1$        | 85,169 | 3.93 sec.| 1.62 sec. | ×2.42             |
|               |     | $h = 2$        | 562,952| 19.83 sec.| 5.43 sec. | ×3.65             |
|               |     | $h = 3$        | 4,240,664| 136.14 sec.| 15.81 sec.| ×8.61             |
| Airplane (Fig.5) | 3   | $h = 1$        | 35,154 | 3.48 sec.| 1.14 sec. | ×3.05             |
|               |     | $h = 2$        | 263,424| 14.38 sec.| 3.53 sec. | ×4.07             |
|               |     | $h = 3$        | 1,843,968| 93.22 sec.| 10.15 sec.| ×9.18             |
| Human (Fig.6) | 4   | $h = 1$        | 153,472| 13.97 sec.| 2.71 sec. | ×5.15             |
|               |     | $h = 2$        | 1,185,408| 87.45 sec.| 9.24 sec. | ×9.46             |
|               |     | $h = 3$        | 7,938,056| 584.82 sec.| 46.23 sec.| ×12.65            |

#DOF: the degree of freedom; $p$: the degree of basis function; $h$: refinement step.

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Fig. 4. Computation reuse in IGA on human body models with consistent volumetric parameterization.
Fig. 5. Computation reuse in IGA on airplane models with consistent volumetric parameterization.
Fig. 6. Computation reuse in IGA on human body models with consistent volumetric parameterization.
6. Conclusion

In this paper, in order to improve the computational efficiency of isogeometric analysis, the concept of computation reuse is proposed for three-dimensional models with similar semantic features. For a set of models with consistent topology, a CSRBF-based method is proposed to construct topology-consistent volumetric B-spline parameterization from given template domains. From the properties of Bézier polynomials, we propose an efficient quadrature-free method to compute the entries of stiffness matrix with the techniques of Bézier extraction and polynomial approximation. With the help of our method, evaluation on the stiffness matrix and imposition of the boundary conditions can be pre-computed and partially reused for models having consistent volumetric parametrization. Effectiveness of the proposed methods has been verified on several examples with complex geometry, and the computation efficiency similar to classical finite element method can be achieved for IGA on these models.

We plan to test the proposed computation reuse approach on other physical simulation problems in the future. Problems such as linear elasticity and flow simulation for a set of models with consistent topology have been widely practiced in the design of a family of products, which will potentially be benefited from our approach.

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