THE END OF THE CURVE COMPLEX

SAUL SCHLEIMER

Abstract. Suppose that $S$ is a surface of genus two or more, with exactly one boundary component. Then the curve complex of $S$ has one end.

1. Introduction

We denote the compact, connected, orientable surface of genus $g$ with $b$ boundary components by $S_{g,b}$. The complexity of $S = S_{g,b}$ is $\zeta(S) := 3g - 3 + b$. A simple closed curve $\alpha$ in $S$ is essential if $\alpha$ does not cut a disk out of $S$. Also, $\alpha$ is non-peripheral if it does not cut an annulus out of $S$.

When $\zeta(S) \geq 2$ the complex of curves, $C(S)$, is the simplicial complex where vertices are isotopy classes of essential non-peripheral curves. The $k$–simplices are collections of $k+1$ distinct vertices having disjoint representatives. We regard every simplex as a Euclidean simplex of side-length one. If $\alpha$ and $\beta$ are vertices of $C(S)$ let $d_C(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$ in the one-skeleton $C^1(S)$. It is a pleasant exercise to prove that $C(S)$ is connected. It is an important theorem of H. Masur and Y. Minsky [6] that $C(S)$ is Gromov hyperbolic.

Let $B(\omega, r) := \{ \alpha \in C^0(S) \mid d_C(\alpha, \omega) \leq r \}$ be the ball of radius $r$ about the vertex $\omega$. We will prove:

**Theorem 5.1.** Fix $S := S_{g,1}$ for some $g \geq 2$. For any vertex $\omega \in C(S)$ and for any $r \in \mathbb{N}$: the complex spanned by $C^0(S) \backslash B(\omega, r)$ is connected.

For such surfaces, Theorem 5.1 directly answers a question of Masur’s. It also answers a question of G. Bell and K. Fujiwara [1] in the negative: the complex of curves need not be quasi-isometric to a tree. Theorem 5.1 is also evidence for a positive answer to a question of P. Storm:

**Question 1.1.** Is the Gromov boundary of $C(S)$ connected?

Note that Theorem 5.1 is only evidence for, and not an answer to, Storm’s question: for example, there is a one-ended hyperbolic space...
where the Gromov boundary is a pair of points. Finally, as we shall see in Remark 4.2, it is not obvious how to generalize Theorem 5.1 to surfaces with more (or fewer) boundary components.

Acknowledgments. I thank both Jason Behrstock and Christopher Leininger; their observations, here recorded as Proposition 4.3, were the origin of my thoughts leading to Theorem 5.1. I also thank Kenneth Bromberg for showing me a simplification, given below, of my original proof of Theorem 5.1. Finally, I am grateful to Howard Masur for both posing the main question and for many enlightening conversations.

2. Definitions and necessary results

An important point elided above is how to define \( C(S) \) when \( \zeta(S) = 1 \). The complex as defined is disconnected in these cases. Instead we allow a \( k \)-simplex to be a collection of \( k + 1 \) distinct vertices which have representatives with small intersection. For \( S_{1,1} \) exactly one intersection point is allowed while \( S_{0,4} \) requires two. In both cases \( C(S) \) is the famous Farey tessellation. Note that \( C(S_{0,3}) \) is empty. We will not need to consider the other low complexity surfaces: the sphere, the disk, the annulus, and the torus.

A subsurface \( X \subset S \) is essential if every component of \( \partial X \) is essential in \( S \). We will generally assume that \( \zeta(X) \geq 1 \). A pair of curves, or a curve and a subsurface, are tight if they cannot be isotoped to reduce intersection. We will generally assume that all curves and subsurfaces discussed are tight with respect to each other. We say a curve \( \alpha \) cuts \( X \) if \( \alpha \cap X \neq \emptyset \). If \( \alpha \cap X = \emptyset \) then we say \( \alpha \) misses \( X \).

Following Masur and Minsky [7], we define the subsurface projection map \( \pi_X \): this maps vertices of \( C(S) \) to collections of vertices of \( C(X) \). Fix a vertex \( \alpha \in C(S) \) and, for every component \( \delta \subset \alpha \cap X \), form \( N_\delta := \text{neigh}(\delta \cup \partial X) \), a closed regular neighborhood of \( \delta \cup \partial X \). Take \( \pi_X(\alpha) \) to be the set of all vertices of \( C(X) \) which appear as a boundary component of some \( N_\delta \). If \( \alpha \) misses \( X \) then \( \pi_X(\alpha) = \emptyset \). Note if \( \alpha \subset S \) is contained in \( X \) after tightening then \( \pi_X(\alpha) = \{ \alpha \} \).

As a useful bit of notation, if \( \alpha \) and \( \beta \) both cut \( X \), we set

\[
d_X(\alpha, \beta) := \text{diam}_X(\pi_X(\alpha), \pi_X(\beta))
\]

with diameter computed in \( C^1(X) \). Masur and Minsky give an combinatorial proof [7, Lemma 2.2] that:

Lemma 2.1. If \( \alpha \) and \( \beta \) both cut \( X \) and \( d_S(\alpha, \beta) \leq 1 \) then \( d_X(\alpha, \beta) \leq 2 \). \( \square \)
By geodesic in $\mathcal{C}(S)$ we will always be referring to a geodesic in the one-skeleton. Since $\mathcal{C}(S)$ is Gromov hyperbolic the exact position of the geodesic is irrelevant; we often use the notation $[\alpha, \beta]$ as if the geodesic was determined by its endpoints. We immediately deduce from Lemma 2.1:

**Lemma 2.2.** Suppose that $\alpha, \beta$ are vertices of $\mathcal{C}(S)$, both cutting $X$. Suppose that $d_X(\alpha, \beta) > 2 \cdot d_S(\alpha, \beta)$. Then every geodesic $[\alpha, \beta] \subset \mathcal{C}(S)$ has a vertex which misses $X$. □

This is essentially Lemma 2.3 of [7].

**Remark 2.3.** There is a useful special case of Lemma 2.2: assume all the hypotheses and in addition that $\gamma$ is the unique vertex of $\mathcal{C}(S)$ missing $X$. Then every geodesic connecting $\alpha$ to $\beta$ contains $\gamma$.

In fact, $\gamma$ is the unique vertex missing $X$ exactly when $S \setminus \text{neigh}(\gamma) = X$ or $S \setminus \text{neigh}(\gamma) = X \cup P$ with $P \cong S_{0,3}$: a pants.

**Remark 2.4.** Note that Lemma 2.2 is a weak form of the Bounded Geodesic Image Theorem [7, Theorem 3.1]. The proof of their stronger result appears to require techniques from Teichmüller theory.

We now turn to the mapping class group $\mathcal{MCG}(S)$: the group of isotopy classes of homeomorphisms of $S$. Note that the natural action of $\mathcal{MCG}(S)$ on $\mathcal{C}(S)$ is via isometries. We have an important fact:

**Lemma 2.5.** If $\psi: S \to S$ is a pseudo-Anosov and $\alpha$ is a vertex of $\mathcal{C}^0(S)$ then $\text{diam}_S(\psi^n(\alpha) \mid n \in \mathbb{Z})$ is infinite. □

It follows that the diameter of $\mathcal{C}(S)$ is infinite whenever $\zeta(S) \geq 1$. A proof of Lemma 2.5, relying on Kobayashi’s paper [4], may be found in the remarks following Lemma 4.6 of [7]. As a matter of fact, Masur and Minsky there prove more using train track machinery: any orbit of a pseudo-Anosov map is a quasi-geodesic. We will not need this sharper version.

Note that if $\psi: S \to S$ is a homeomorphism then we may restrict $\psi$ to the curve complex of a subsurface $\psi|X: \mathcal{C}(X) \to \mathcal{C}(\psi(X))$. This restriction behaves well with respect to subsurface projection: that is, $\pi_\psi(X) \circ \psi = \psi|X \circ \pi_X$.

We conclude this discussion by examining partial maps. Suppose that $X \subset S$ is an essential surface, not homeomorphic to $S$. If $\psi: S \to S$ has the property that $\psi|S \setminus X = \text{Id}|S \setminus X$ then we call $\psi$ a partial map supported on $X$. Note that if $\psi$ is supported on $X$ then the orbits of $\psi$ do not have infinite diameter in $\mathcal{C}(S)$. Since $\psi$ fixes $\partial X$ and acts on $\mathcal{C}(S)$ via isometry, every point of an orbit has the same distance to $\partial X$ in $\mathcal{C}(S)$. Nonetheless, Lemmas 2.2 and 2.5 imply:
Lemma 2.6. Suppose $\psi: S \to S$ is supported on $X$ and $\psi|X$ is pseudo-Anosov. Fix a vertex $\sigma \in \mathcal{C}(S)$ and define $\sigma_n := \psi^n(\sigma)$. Then for any $K \in \mathbb{N}$ there is a power $n \in \mathbb{Z}$ so that $d_X(\sigma, \sigma_n) \geq K$. In particular, if $K > 4 \cdot d_S(\sigma, \partial X)$ then every geodesic $[\sigma, \sigma_n] \subset \mathcal{C}(S)$ contains a vertex which misses $X$. \hfill $\square$

3. No dead ends

We require a pair of tools in order to prove Theorem 5.1. The first is:

Proposition 3.1. Fix $S = S_{g,b}$. For any vertex $\omega \in \mathcal{C}(S)$ and for any $r \in \mathbb{N}$: every component of the subcomplex spanned by $\mathcal{C}^0(S) \setminus B(\omega, r)$ has infinite diameter.

A more pithy phrasing might be: the complex of curves has no dead ends. Proposition 3.1 allows us to push vertices away from $\omega$ while remaining inside the same component of $\mathcal{C}(S) \setminus B(\omega, r)$. The proof is a bit subtle due to the behavior of $\mathcal{C}(S)$ near a non-separating curve.

Proof of Proposition 3.1. If $S = S_{0,3}$ is a pants then the curve complex is empty and there is noting to prove. If $\mathcal{C}(S)$ is a copy of the Farey graph then the claim is an easy exercise. So we may suppose that $\zeta(S) \geq 2$.

Now fix a vertex $\alpha \in \mathcal{C}(S) \setminus B(\omega, r)$. Set $n := d_S(\alpha, \omega)$. Thus $n > r$. Our goal is to find a curve $\delta$, connected to $\alpha$ in the complement of $B(\omega, n - 1)$, with $d_S(\delta, \omega) = n + 1$. Doing this repeatedly proves the proposition. Note that finding such a vertex $\delta$ is straight-forward if $r = 0$ and $n = 1$. This is because $\mathcal{C}(S) \setminus \omega$ is connected and because, following Lemma 2.5, we know that the diameter of $\mathcal{C}(S)$ is infinite. Henceforth we will assume that $n \geq 2$; that is, $\omega$ cuts $\alpha$.

Fix attention on a component $X$ of $S \setminus \text{neigh} (\alpha)$ which is not a pair of pants. So $\zeta(X) \geq 1$ and, by the comments following Lemma 2.5, $\mathcal{C}(X)$ has infinite diameter. Since $\omega$ cuts $\alpha$ we find that $\omega$ also cuts $X$. Choose a curve $\beta$ contained in $X$ with $d_X(\beta, \omega) \geq 2n + 1$. Note that $d_S(\alpha, \beta) = 1$. We may assume that $\beta$ is either non-separating or cuts a pants off of $S$. (To see this: if $\beta$ cannot be chosen to be non-separating then $X$ is planar. As $\zeta(X) \geq 1$ we deduce that $X$ has at least four boundary components. At most two of these are parallel to $\alpha$.) It follows from Lemma 2.2 that any geodesic from $\beta$ to $\omega$ in $\mathcal{C}(S)$ has a vertex $\gamma$ which misses $X$.

By the triangle inequality $d_S(\gamma, \omega)$ equals $n$ or $n - 1$. In the former case we are done: simply take $\delta = \beta$ and notice that $d_S(\beta, \omega) = n + 1$. In the latter case $d_S(\beta, \omega) = n$ and we proceed as follows: replace $\alpha$
by $\beta$ and replace $X$ by $Z := S \setminus \text{neigh}(\beta)$. We may now choose $\delta$ to be a vertex of $C(Z)$ with $d_Z(\delta, \omega) \geq 2n + 1$. As above, any geodesic $[\delta, \omega] \subset C(S)$ has a vertex which misses $Z$. Since $\beta$ is the unique vertex not cutting $Z$ Remark 2.3 implies that $\beta \in [\delta, \omega]$. Thus $d_S(\delta, \omega) = n + 1$ and we are done.

4. THE BIRMAN SHORT EXACT SEQUENCE

We now discuss the second tool needed in the proof of Theorem 5.1. Following Kra’s notation in [5] let $\hat{S} = S_{g,1}$ and $S = S_g$ for a fixed $g \geq 2$. Let $\rho: \hat{S} \to S$ be the quotient map crushing $\partial \hat{S}$ to a point, say $x \in S$. This leads to the Birman short exact sequence:

$$\pi_1(S, x) \to \text{MCG}(\hat{S}) \to \text{MCG}(S)$$

for $g \geq 2$. The map $\rho$ gives the second arrow. The first arrow is defined by sending $\gamma \in \pi_1(S, x_0)$ to a mapping class $\psi_\gamma$. There is a representative of this class which is isotopic to the identity, in $S$, via an isotopy dragging $x$ along the path $\gamma$. See Birman’s book [2] or Kra’s paper [5] for further details.

Fix an essential subsurface $\hat{X} \subset \hat{S}$ and let $X = \rho(\hat{X})$. If $\gamma \in \pi_1(S, x)$ is contained in $X$ then $\psi_\gamma$ is a partial map, supported in $\hat{X}$. We say that $\gamma$ fills $X$ if $\gamma \subset X$ and, in addition, every representative of the free homotopy class of $\gamma$ cuts $X$ into a collection of disks and peripheral annuli. For future use we record a well-known theorem of I. Kra [5]:

**Theorem 4.1.** Suppose that $\zeta(\hat{X}) \geq 1$. If $\gamma$ fills $X$ then $\psi_\gamma|\hat{X}$ is pseudo-Anosov.

Now note that, corresponding to the Birman short exact sequence, there is a “fibre bundle” of curve complexes:

$$\mathcal{F}_\tau \to C(\hat{S}) \to C(S).$$

Here $\tau$ is an arbitrary vertex of $C(S)$ and $\mathcal{F}_\tau := \rho^{-1}(\tau)$. The second arrow is given by $\rho$. The first is the inclusion of $\mathcal{F}_\tau$ into $C(\hat{S})$.

**Remark 4.2.** If $|\partial S| \geq 2$ then collapsing one boundary component does not induce a map on the associated curve complexes. Thus, it is not clear how to generalize Theorem 5.1 to such surfaces. If $\partial S$ is empty then I do not know of any interesting quotients or electrifications of $C(S)$.

Using the Birman short exact sequence we obtain an action of $\pi_1(S, x)$ on the curve complex $C(\hat{S})$. Behrstock and Leininger observe that:
Proposition 4.3. The map $\rho: C(\hat{S}) \to C(S)$ has the following properties:

- It is 1–Lipschitz.
- For any $\alpha \in C(\hat{S})$, $\gamma \in \pi_1(S, x)$ we have $\rho(\alpha) = \rho(\psi_\gamma(\alpha))$.
- Every fibre $F_\tau$ is connected.

Remark 4.4. Behrstock and Leininger’s interest in the fibre $F_\tau$ was to give a “natural” subcomplex of $C(S)$ which is not quasi-convex: this is implied by the first pair of properties.

Remark 4.5. More of the structure of $F_\tau$ is known. For example, since $S$ is closed, the fibre $F_\tau$ is either a single $\pi_1(S, x)$–orbit or the union of a pair of orbits depending on whether $\tau$ is non-separating or separating. Furthermore, $F_\tau$ is a tree. See [3] for a detailed discussion.

Proof of Proposition 4.3. Fix an essential non-peripheral curve $\alpha$ in $\hat{S}$. Note that $\rho(\alpha)$ is essential in $S$ and so the induced map $\rho: C(\hat{S}) \to C(S)$ is well-defined. If $\alpha$ and $\beta$ are disjoint in $\hat{S}$ then so are their images in $S$. Thus $\rho$ does not increase distance between vertices and the first conclusion holds.

Now fix a curve $\alpha \subset \hat{S}$ and $\gamma \in \pi_1(S, x)$. Note that $\psi_\gamma$ is isotopic to the identity in $S$. Thus the images $\rho(\psi_\gamma(\alpha))$ and $\rho(\alpha)$ are isotopic in $S$. It follows that $\rho(\alpha) = \rho(\psi_\gamma(\alpha))$ as vertices of $C(S)$, as desired.

Finally, fix $\tau \in C(S)$. Let $F_\tau$ be the fibre over $\tau$. Pick $\alpha, \beta \in F_\tau$. It follows that $a := \rho(\alpha)$ and $b := \rho(\beta)$ are both isotopic to $\tau$ and so to each other. We induct on the intersection number $\iota(\alpha, \beta)$. Suppose the intersection number is zero. Then $\alpha$ and $\beta$ are disjoint and we are done. Suppose that the intersection number is non-zero. Since $a$ and $b$ are isotopic, yet intersect, they are not tight with respect to each other. It follows that there is a bigon $B \subset S \setminus (a \cup b)$. Since $\alpha$ and $\beta$ are tight in $\hat{S}$ the point $x$ must lie in $B$. Let $\hat{b} := \rho^{-1}(\hat{B})$. Now construct a curve $\beta' \subset \hat{S}$ by starting with $\beta$, deleting the arc $\beta \cap \hat{B}$, and adding the arc $\alpha \cap \hat{B}$. Isotope $\beta'$ to be tight with respect to $\alpha$. Now $\beta' \in F_\tau$ because $\rho(\beta')$ is isotopic to $\rho(\beta)$ in $S$. Finally, $\iota(\alpha, \beta') \leq \iota(\alpha, \beta) - 2$. □

5. Proving the theorem

We are now equipped to prove:

Theorem 5.1. Fix $\hat{S} := S_{g,1}$ for some $g \geq 2$. For any vertex $\omega \in C(\hat{S})$ and for any $r \in \mathbb{N}$: the complex spanned by $C^0(\hat{S}) \setminus B(\omega, r)$ is connected.

As above we use the notation $\hat{S} = S_{g,1}$ and $S = S_g$ for some fixed $g \geq 2$. Also, we have defined a map $\rho: C(\hat{S}) \to C(S)$ induced by
collapsing $\partial \hat{S}$ to a point, $x$. As above we use $F_\tau = \rho^{-1}(\tau)$ to denote the fibre over $\tau$.

**Proof of Theorem 5.1.** Choose $\alpha'$ and $\beta'$ vertices of $\mathcal{C}(\hat{S}) \setminus B(\omega,r)$. By Proposition 3.1 we may connect $\alpha'$ and $\beta'$, by paths disjoint from $B(\omega,r)$, to vertices outside of $B(\omega,3r)$. Call these new vertices $\alpha$ and $\beta$. We may assume that both $\alpha$ and $\beta$ are non-separating because such vertices are 1–dense in $\mathcal{C}(\hat{S})$.

Choose any vertex $\tau \in \mathcal{C}(S)$ so that $d_S(\tau,\rho(\omega)) \geq 4r$. This is always possible because $\mathcal{C}(S)$ has infinite diameter. (See the remarks after Lemma 2.5.) It follows from Proposition 4.3 that $F_\tau \cap B(\omega,r) = \emptyset$.

We will now connect each of $\alpha$ and $\beta$ to some point of $F_\tau$ via a geodesic disjoint from $B(\omega,r)$. Since $F_\tau$ is connected, by Proposition 4.3, this will complete the proof of Theorem 5.1.

Let $X := \hat{S} \setminus \alpha$ and take $\hat{X} := \rho(X)$. Fix any point $\sigma$ in $F_\tau$. If $\sigma = \alpha$ then $\alpha$ is trivially connected to the fibre. So suppose that $\sigma \neq \alpha$. Since $\alpha$ is non-separating deduce that $\sigma$ cuts $\hat{X}$. Now, since $\zeta(\hat{S}) \geq 4$ we have $\zeta(\hat{X}) \geq 3$. Let $\gamma \in \pi_1(S,x)$ be any homotopy class so that $\psi_\gamma$ is supported in $\hat{X}$ and so that $\gamma$ fills $X$. By Kra’s Theorem (4.1) $\psi_\gamma | \hat{X}$ is pseudo-Anosov.

Since $F_\tau$ is left setwise invariant by $\pi_1(S,x)$ (Proposition 4.3) the curves $\sigma_n := \psi_\gamma^n(\sigma)$ all lie in $F_\tau$. Since $\psi_\gamma | \hat{X}$ is pseudo-Anosov, Lemma 2.6 gives an $n \in \mathbb{Z}$ so that every geodesic $g = [\sigma, \sigma_n] \subset C(\hat{S})$ has a vertex which misses $\hat{X}$. Since $\alpha$ is non-separating, as in Remark 2.3, it follows that $\alpha$ is actually a vertex of $g$.

We now claim that at least one of the two segments $[\sigma, \alpha] \subset g$ or $[\alpha, \sigma_n] \subset g$ avoids the ball $B(\omega,r)$. For suppose not: then there are vertices $\mu, \mu' \in g$ on opposite sides of $\alpha$ which both lie in $B(\omega,r)$. Thus $d_S(\mu, \mu') \leq 2r$. Since $\rho$ is a geodesic the length along $g$ between $\mu$ and $\mu'$ is at most $2r$. Thus $d_S(\omega, \alpha) \leq 2r$. This is a contradiction.

Thus we can connect $\alpha$ to a vertex of $F_\tau$ (namely, $\sigma$ or $\sigma_n$) avoiding $B(\omega,r)$. Identically, we can connect $\beta$ to a vertex of $F_\tau$ while avoiding $B(\omega,r)$. As noted above, this completes the proof.

**References**

[1] Gregory Bell and Koji Fujiwara. The asymptotic dimension of a curve graph is finite. arXiv:math.GT/0509216.

[2] Joan S. Birman. *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Studies, No. 82.

[3] Richard Kent, Chris Leininger, and Saul Schleimer. Trees and convex co-compactness. In preparation.
Tsuyoshi Kobayashi. Heights of simple loops and pseudo-Anosov homeomorphisms. In *Braids (Santa Cruz, CA, 1986)*, pages 327–338. Amer. Math. Soc., Providence, RI, 1988.

Irwin Kra. On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces. *Acta Math.*, 146(3-4):231–270, 1981.

Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999. arXiv:math.GT/9804098.

Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000. arXiv:math.GT/9807150.

Department of Mathematics, Rutgers University, Piscataway, New Jersey 08854

E-mail address: saulsch@math.rutgers.edu

URL: http://www.math.rutgers.edu/~saulsch