Thermodynamics of Two Dimensional Black Holes

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Abstract

Thermodynamic relations for a class of 2D black holes are obtained corresponding to observations made from finite spatial distances. We also study the thermodynamics of the charged version of the Jackiw-Teitelboim black holes found recently by Lowe and Strominger. Our results corroborate, in appropriate limits, to those obtained previously by other methods. We also analyze the stability of these black holes thermodynamically.
It is customary in the study of black holes to write the laws of thermodynamics in the asymptotic space-time\(^\text{[1]}\). For a static uncharged black hole in four dimensions, the Hawking temperature is \(T_c = \frac{1}{8\pi M}\), where \(M\) is the mass of the black hole. This temperature is measured at spatial infinity. Then Hawking's interpretation of the gravitational action as the entropy leads to the following asymptotically valid equation for this black hole:

\[
ST_c = \frac{M}{2}. \tag{1}
\]

Similar laws also exist for other black holes. However, if this equation is true for all \(M\), then the specific heat of the black hole is negative; thus rendering the space-time unstable to radiation. It will lose mass with increasing temperature and is destined to meet with a catastrophic end. It has recently been pointed out \(^\text{[2]}\), that the negativity of the specific heat of this black hole could be tracked down to ignoring the \(O\left(\frac{1}{r}\right)\) term in the action. Retention of this term is tantamount to measuring the thermodynamic quantities at a finite \(r\). Therefore one needs a finite-space formulation of black hole thermodynamics.

Such a formulation in the context of an effective 2D string theory has been proposed in \(^\text{[3]}\). One of the major motivations for analysing these 2D black hole solutions in string theory is to gain some insight into the nature of problems in thoroughly understood 4D theory of gravity with quantum effects. The 2D solutions can be obtained from higher dimensional ones, via some compactification scheme, say, and are simpler to deal with.

In string theory, as is well-known, a scalar field, namely dilaton, plays an important role in obtaining the black hole solutions and their physics. For example, in 2D, pure Einstein gravity does not have any nontrivial solutions. But the introduction of dilaton gives rise to many interesting solutions, such as black holes\(^\text{[4]}\).
cosmological universes[3] etc. Dilaton field also plays an important role in defining the thermodynamics of the black holes, as noted earlier[3]. The root of many of these new features is the non-minimality of the coupling of gravity and other fields to the dilaton. Therefore, the study of other non-minimally coupled scalars showing up in the low energy effective action of string theory is also of interest. Example of one such class of 2D black holes have been found in[3](see also[7]). These come from the compactification of the 4D black holes in presence of non-minimally coupled moduli fields in the extremal limit. These 2D black holes asymptote a space-time of constant negative curvature. This led to a vanishing temperature for the corresponding 4D theory, except for the special case when the 2D black hole corresponds to the one in string theory. However we shall show that an analogue of equation (1) is still valid in the 2D theory.

Introduction of an electric field in the low energy effective field theory ensuing from sting theory has more exotic effects. In[8] the construction of such a black hole is given following the observation in[9] that the 3D rotating BTZ black hole[10] might be interpreted as an electrically charged 2D black hole.

Motivated partly by the results of[2] in showing the connection between the stability of the 4D black holes with the entropy at finite distance, we study in this paper various types of 2D black holes. First, we explicitly show that the thermodynamic relations are well-defined for the observations made from a finite distance. We then investigate the thermodynamics of a charged 2D black hole with asymptotic properties similar to the ones mentioned above. In comparing our results with those from other methods, we find that one has to make appropriate gauge transformations of the gauge potentials in order to obtain a consistent value of the entropy.

We also study the specific heat of these solutions. An analysis taking into account the definition of the observed temperature shows that specific heat is positive for all
the observers.

Let us now start with the discussions of the class of 2D black holes obtained in $[6]$. These emerge as the solutions of the action:

$$\mathcal{I} = -\int_M \sqrt{g} \exp(-2\phi) \left[ R + \frac{8k}{k-1}(\nabla \phi)^2 + \lambda^2 \right] - 2 \int_{\partial M} \exp(-2\phi) K,$$

(2)

where $K$ is the trace of the second fundamental form, $\partial M$ is the boundary of $M$ and $k$ is a parameter taking values $|k| \leq 1$. It reduces to the Jackiw-Teitelboim action $[11]$ for $k = 0$. Equations of motion ensuing from (2) are:

$$R + \frac{8k}{k-1}(\nabla^2 \phi - (\nabla \phi)^2) + \lambda^2 = 0$$

(3)

$$R_{ab} - \frac{1}{2}g_{ab} R - \frac{1}{2}g_{ab} \lambda^2 + 2 \nabla_a \nabla_b \phi - \frac{4}{1-k} \nabla_a \phi \nabla_b \phi = 0.$$

(4)

These possess the following exact solutions:

$$ds^2 = -\sinh^2(\kappa \sigma) \cosh^{2k}(\kappa \sigma) dt^2 + d\sigma^2$$

(5)

$$\exp(-2\phi) = \exp(-2\phi_0) \cosh^{1-k}(\kappa \sigma),$$

(6)

where $\kappa = \frac{\lambda}{\sqrt{2(1-k)}}$. These solutions are everywhere regular for any value of $k$ and have a horizon at $\sigma = 0$. They asymptote to the anti-de Sitter background with a linear dilaton for $\sigma \to \infty$. Also note that the solution (5)–(6) describes, for $k = -1$, the usual asymptotically flat stringy dilatonic black hole $[4]$. Thermodynamics in this special case is dealt with at length in $[3]$. We shall study the thermodynamics of the solutions (5)–(6) for generic $k$.

The discussion of thermodynamics begins with the definition of free energy:

$$\mathcal{F} = \frac{\mathcal{I}}{\beta},$$

(7)
where $\mathcal{I}$ is the Euclideanized action evaluated for the metric of the space-time under consideration and $\beta$ is the inverse temperature. Therefore we will start with the evaluation of the action $\mathcal{I}$. Since we are concerned with the thermodynamics of the black holes at a finite spatial separation, we have to evaluate the action with the boundary contribution on a spacelike slice. This is feasible since the solutions admit a Killing vector $k^a = \left( \frac{\partial}{\partial t} \right)^a$. The proper periodicity of the Euclideanized time at a fixed value of the spatial coordinate is interpreted as the local temperature $T_w = \beta^{-1}$.

The conserved dilaton current $j_a \equiv \epsilon^b_a \nabla_b \exp(-2\phi)$ defines another thermodynamic potential

$$\mathcal{D} = \int_{\Sigma} j$$

where $\Sigma$ is a spacelike hypersurface bounded by the wall of the box. Here we note a direct consequence of the non-minimal coupling of the dilaton in two dimensions. In four dimensions, the dilaton field can be decoupled from the Einstein term by rescaling the metric. As a result it becomes the part of a general matter action and does not affect the black hole thermodynamics. Also, the dilaton charge $\mathcal{D}$ in equation is essentially the value of the dilaton field $\exp(-2\phi)$ on the boundary, which is a scalar. Consequently this quantity is a measurable one, in contrast to the coordinates which parametrize it. We will therefore treat $\mathcal{F}$ as a function of the two thermodynamic quantities $T_w$ and $\mathcal{D}$. Then by the first law of thermodynamics, one can define the entropy $S$ and the dilaton potential $\psi$ as:

$$S = -\left[ \frac{\partial \mathcal{F}}{\partial T_w} \right]_{\mathcal{D}}, \quad \psi = -\left[ \frac{\partial \mathcal{F}}{\partial \mathcal{D}} \right]_{T_w}.$$

But it is not the Helmholtz free energy, rather its Legendre transform, $\mathcal{E} = \mathcal{F} + ST_w$, that defines the non-available energy. This corresponds to the mass of the space-time, provided it exists, as the limiting value of the difference between the energies for the black hole and its asymptotic background solution at spatial infinity.
Using the dilaton equation (3), the action (2) can be evaluated for a generic \( k \) to be

\[
\mathcal{I} = -2 \int_{\partial M} \exp(-2\phi) \left( K - \frac{4k}{k-1} n^a \nabla_a \phi \right),
\]

(10)

where \( n^a \) is the unit outward normal on \( \partial M \). For our choice of boundary, \( n^\mu = (0, \frac{1}{\sqrt{g_{11}}}) \), \( K = \frac{\partial_1 \ln \sqrt{|g_{00}|}}{\sqrt{g_{11}}} \) and the action has the form

\[
\mathcal{I} = -\int_{\partial M} \sqrt{\frac{1}{g_{11}}} \exp(-2\phi) \left( \frac{1}{2} \partial_1 g_{00} - \frac{4k}{k-1} \partial_1 \phi \right)
\]

(11)

Then by defining \( x = \kappa \sigma \), the Helmholtz free energy for the solution (3) becomes

\[
\mathcal{F} = \mathcal{I} T_w
\]

(12)

\[
= -2\kappa \frac{T_w}{T_c} \exp(-2\phi_0) \left[ \cosh^2 x - k \sinh^2 x \right]
\]

(13)

where \( T_c \) is the proper periodicity of the Euclidean time at the horizon and is related to \( T_w \) by

\[
T_w = \frac{T_c}{\sqrt{g_{00}}}
\]

(14)

The Euclideanized metric corresponding to (3) has a conical singularity as \( \sigma \to 0 \), unless \( \tau \equiv it \) has a periodicity \( T_c = \frac{\kappa}{2\pi} \). Thus,

\[
T_w = \frac{T_c}{\sinh x \cosh^k x}
\]

(15)

and the dilaton charge for this black hole is

\[
\mathcal{D} = \exp(-2\phi_0) \cosh^{1-k} x.
\]

(16)

Then using (13) and (14) we find

\[
\mathcal{F} = -2\kappa \mathcal{D} (\coth x - k \tanh x).
\]

(17)
In (17) we have eliminated the constant $\exp(-2\phi_0)$ in favor of the dilaton charge $\mathcal{D}$, the basic principle being, that one should not keep arbitrary parameters in the description of the thermodynamic quantities except those which appear in the action itself. But the coordinate $x$ is kept as an implicit variable defined by (15). Since both $\mathcal{F}$ and $T_w$ depend implicitly on $x$, we can write $S$ as

$$S = -\left[ \frac{\partial \mathcal{F}}{\partial x} \right] \mathcal{D} \left[ \frac{dT_w}{dx} \right]^{-1}. \quad (18)$$

This yields

$$S = 4\pi \mathcal{D} \cosh^{k^{-1}} x = 4\pi \exp(-2\phi_0). \quad (19)$$

The black hole energy as defined by the Legendre transform of $\mathcal{F}$ is given by

$$\mathcal{E}_{BH} = -2(1 - k)\kappa \mathcal{D} \tanh x \quad (20)$$

We observe that all the thermodynamic quantities listed above go over to those for the string black hole by choosing $k = -1$. In fact, the entropy is a constant of $x$ for all values of $k$ including the case of string black hole. We however notice some important differences between $k = -1$ and $k \neq -1$ situations. Unlike the $k = -1$ case the solutions (3)–(6) asymptote to the anti-de Sitter (AdS) linear dilaton vacuum for general $k$. As a result $T_w$ vanishes asymptotically as $\exp[-(k + 1)x]$ while $\mathcal{D}$ goes to infinity as $\exp[(k + 1)x]$. The energy of the black hole is to be computed with reference to the AdS linear dilaton vacuum defined by

$$ds^2 = \exp(2(k + 1)\kappa\sigma) dt^2 + d\sigma^2, \quad (21)$$

$$\phi = \phi_0 + \frac{1}{2}(k - 1)\kappa\sigma. \quad (22)$$

The free energy (12) becomes

$$\mathcal{F}_{AdS} = -2\lambda(1 - k)\mathcal{D}, \quad (23)$$
which implies $S_{AdS} = 0$ and $E_{AdS} = -2\lambda(1 - k)D$. Then defining $M \equiv E_{BH} - E_{AdS}$ we obtain

$$M = 2kD(1 - k)[1 - \tanh x].$$

(24)

An analogue of (1) at finite $x$ was written in [3] for $k = -1$ and has the form

$$S = \frac{M}{T_c} \left(1 - \frac{M}{16\pi DT_c}\right).$$

(25)

It can be verified that the above equation generalizes for general $k$ to,

$$S = 4\pi D \left[\frac{M}{2\pi DT_c(1 - k)} \left(1 - \frac{M}{8\pi DT_c(1 - k)}\right)^{(1-k)/2}\right].$$

(26)

Note that unlike the case of asymptotically flat metric, the quantity $M$ for a general $k$ vanishes in the limit $x \to \infty$ by the Tolman redshift factor as $M \sim M_{ADM} \exp[-(k + 1)x]$, where $M_{ADM} = (1 - k)^{\frac{D}{2}} \exp(-2\phi)$ is the ADM-mass of the black hole. Equation (26) is one of the main results of this paper. Also, one can verify that, in the asymptotic limit,

$$ST_c = \frac{2M_{ADM}}{1 - k},$$

(27)

which is precisely the relation given in [3].

We now investigate the thermodynamics for the charged version of the $k = 0$ black hole. For $k = -1$ the charged black hole solution and its thermodynamics is discussed in [13] and [3] respectively. For $k = 0$, gauge fields were introduced in [8] through the dimensional compactification of a three dimensional string effective action using a suggestion in [9]. The 2D action in this case has the form,

$$I = -\int_{M_2} \sqrt{g} \exp(-2\Phi) [R + 2\lambda^2 - \frac{1}{4} \exp(-4\Phi) F^2] - 2 \int_{\partial M} \exp(-2\Phi) K,$$

(28)
where now $\Phi$ is a scalar field coming from the compactification and plays the role of dilaton for the 2D action. Action (28) describes the Jackiw-Teitelboim theory with a gauge field. The equations of motion ensuing from this action are

$$R_{ab} + 2\nabla_a \nabla_b \Phi - 4\nabla_a \Phi \nabla_b \Phi + \frac{1}{2} \exp(-4\Phi) F_a^c F_{cb} =$$

$$g_{ab} \left[ \frac{1}{2} R + \lambda^2 + 2\nabla^2 \Phi - 4(\nabla \Phi)^2 - \frac{1}{8} \exp(-4\Phi) F^2 \right] = 0, \quad (29)$$

$$R + 2\lambda^2 - \frac{3}{4} \exp(-4\Phi) F^2 = 0, \quad (30)$$

$$\partial_a \left( \sqrt{g} \exp(-6\Phi) F^{ab} \right) = 0. \quad (31)$$

These possess the solution

$$ds^2 = -(M - \lambda^2 r^2 - \frac{J^2}{4r^2}) dt^2 + (M - \lambda^2 r^2 - \frac{J^2}{4r^2})^{-1} dr^2, \quad (32)$$

$$A_0 = -\frac{J}{2r^2}, \quad (33)$$

$$\exp(-2\Phi) = r. \quad (34)$$

The parameter $J$ in this solution gives charge to this black hole. The metric has a curvature singularity at $r = 0$ for nonvanishing $J$ as is seen from the Ricci scalar

$$R = -2\lambda^2 - \frac{3J^2}{2r^4}. \quad (35)$$

It also goes asymptotically, $r \to \infty$, to the anti-de Sitter Space-time. Now we study the thermodynamics of these black hole solutions for observations done from finite distances. In this case, the use of the equations of motion (29)–(31) implies the following value of the classical action:

$$I = -\int_{\partial M} \left[ n^a F_{ab} A^b \exp(-6\Phi) + 2K \exp(-2\Phi) \right]. \quad (36)$$

To discuss the thermodynamics, we rewrite the solution (32)–(34) in the non-extremal case, $M^2 > \lambda^2 J^2$, in the coordinates:

$$r^2 = \frac{M + \sqrt{M^2 - \lambda^2 J^2} \cosh 2\lambda \rho}{2\lambda^2} \quad (37)$$
by exploiting the fact that it admits a timelike Killing vector. Then we find,

$$ds^2 = -G(\rho)dt^2 + d\rho^2$$  \hspace{1cm} (38)$$

where

$$G(\rho) = \frac{1}{2M} \frac{(M^2 - \lambda^2 J^2) \sinh^2 2\lambda \rho}{M + \sqrt{M^2 - \lambda^2 J^2} \cosh 2\lambda \rho}.$$  \hspace{1cm} (39)$$

$A_0$ and $\exp(-2\Phi)$ are still given by (33)-(34) with $r$ replaced from (37). In these coordinates the horizon is at $\rho = 0$. The free energy is obtained by the evaluation of (36). We note however that there is an ambiguity in the evaluation of (36) due to the freedom of a constant shift in the gauge potential: $A_a \to A_a + \text{const.}$, in the equations of motion. Constant shifts have been applied earlier\cite{12, 14} in the evaluation of the classical actions in order to avoid divergence in the gauge potential at the horizon. In our case, on the other hand, $A_a$ is well-defined at $\rho = 0$. But as we will see later, this shift is needed in order to show the consistency of the present method of computations with the Noether’s charge prescription\cite{16}.

Once again the temperature is given by the periodicity of the proper time in a local inertial frame around $\rho$ and satisfies the relation:

$$T_w = \sqrt{2} T_c \frac{[M + \sqrt{M^2 - \lambda^2 J^2} \cosh 2x]^{1/2}}{\sqrt{M^2 - \lambda^2 J^2} \sinh 2x}.$$  \hspace{1cm} (40)$$

where $x = \lambda \rho$ and

$$T_c = \frac{\sqrt{2\lambda}}{2\pi} \frac{\sqrt{M^2 - \lambda^2 J^2}}{(M + \sqrt{M^2 - \lambda^2 J^2})^{1/2}}$$  \hspace{1cm} (41)$$

is the proper periodicity at the horizon. The dilaton charge is now given by

$$\mathcal{D} = \left[ \frac{M}{2\lambda^2} (1 + \xi \cosh 2x) \right]^{1/2},$$  \hspace{1cm} (42)$$

where $\xi = \sqrt{1 - \left( \frac{\lambda}{M} \right)^2}$ with $\xi^2 > 0$. Then one can evaluate (36), with a shift in the gauge potential $A_\mu \to A_\mu(\rho) - A_\mu(\rho = 0)$, and the free energy is

$$\mathcal{F} = -2\lambda \mathcal{D} \coth x.$$  \hspace{1cm} (43)$$
The form of equation (43) needs some qualifications. As in (17), an implicit variable \( x \) has been used in writing them. However, the thermodynamic variables are only the dilaton charge \( D \), temperature \( T_w \) and the electric charge \( Q \), defined as 
\[
Q = -\frac{1}{2} \exp(-6\Phi) \epsilon_{ab} F^{ab} \text{ at the boundary.}
\]
To show that the free energy can be written purely in terms of \( \lambda \) and thermodynamic variables \( T_w, Q \) and \( D \), it suffices to record the following relations:
\[
\frac{Q}{D^3} = 2\lambda^2 \xi \sqrt{1 - \xi^2} \sinh 2x \frac{\sinh 2x}{(1 + \xi \cosh 2x)^2},
\]
and
\[
\xi = \frac{\pi^2 T_w^2 \sinh^2 2x - \lambda^2}{\lambda^2 \cosh 2x - \pi^2 T_w^2 \sinh^2 2x}.
\]
Since \( F \) in (43) does not depend explicitly on \( Q \) and \( \xi \), entropy is once again computed using equation (18) and can be written as
\[
S = -4\lambda D \coth \frac{x}{T_w} \left[ \frac{\xi (1 + \xi \cosh 2x)}{1 - \xi^2 - (1 + \xi \cosh 2x)^2} \right].
\]
The consistency of this procedure is provided by the fact that
\[
\frac{d\xi}{dT_w} \equiv \frac{\partial \xi}{\partial T_w} + \frac{\partial \xi}{\partial x} \left( \frac{dT_w}{dx} \right)^{-1} = 0.
\]
As a result, in differentiating with respect to \( T_w \) and \( x \), \( \xi \) is taken as a constant. In the same manner as above, \( S \) can also be thought to be a function of thermodynamic variables only. In the \( x \rightarrow \infty \) limit we find
\[
\lim_{x \to \infty} S = \frac{2\sqrt{2\pi}}{\lambda} \left[ M + \sqrt{M^2 - \lambda^2 J^2} \right]^{1/2}.
\]
Recently, the black hole entropy for asymptotic observers has been evaluated by different methods, including the Noether’s charge prescription[16, 18]. In the case of 2D black holes, this simply leads to \( S = 4\pi \exp(-2\Phi) \) evaluated on the horizon. This
result matches with the ones derived in (48). We would like to remind the reader of the crucial role of the gauge choice in deriving (48) for this comparison.

We now come to the stability analysis of the black holes through the evaluation of the specific heat. The space-time is thermodynamically stable to radiation provided the specific heat is positive. It is noted that, at least in those cases, where $S$ and $T_w$ are asymptotically constants of $x$, say, $S_0$ and $T_0$, respectively, an equation of the type (1) is satisfied and the specific heat,

$$C = T_0 \frac{dS_0}{dT_0},$$

(49)
is negative, viz, $-S_0$. For the case under consideration, however, it is naive to conclude from this that the black hole is unstable.

We now compute the specific heat in the present formulation and show the stability of the black hole solutions. The specific heat is now given by the formula:

$$C_D \equiv T_w \left[\left(\frac{\partial S}{\partial T_w}\right)_{x,D} + \left(\frac{\partial S}{\partial x}\right)_{T_w,D} \left(\frac{dT_w}{dx}\right)^{-1}\right].$$

(50)

For the uncharged black holes (5)–(6), using the entropy (19), we obtain the specific heat as:

$$C_D = 4\pi(1 - k) \frac{\exp(-2\phi_0)}{k + \coth^2 x},$$

(51)

which is positive for all $|k| < 1$. Therefore one concludes that these black holes are stable. For $k = -1$, on the other hand, $C_D$ is infinite asymptotically. This coforms to the observations made earlier[19].

For the charged black hole, the specific heat for a constant $D$ and $Q$ is found to be:

$$C_{DQ} = \frac{8\pi\xi}{\lambda} \sqrt{\frac{M}{2}} (1 + \xi) \frac{\cosh^2 x(1 + \xi \cosh 2x)}{[(1 - \xi^2) - (1 + \xi \cosh 2x)^2]^2} \left[(1 - \xi^2)\right].$$
\[ \begin{align*}
- (1 + \xi \cosh 2x)^2 + 2\xi(1 + \xi \cosh 2x) \\
- 2\xi^2 \sinh^2 2x \frac{(1 - \xi^2) + (1 + \xi \cosh 2x)^2}{(1 - \xi^2) - (1 + \xi \cosh 2x)^2},
\end{align*} \]  
(52)

where once again we have used the constancy of \( \xi \).

We have plotted \( C_{DQ} \) as a function of \( x \) in Fig.1 for certain values of \( \xi \) and found that it is positive throughout. Its asymptotic value is same as that of entropy, \( S \) in (48). In the other limit, \( x \to 0 \), the specific heat vanishes as \( \sim x^2 \). It is now interesting to note that for \( x \) close to zero we also have \( C_{DQ} \sim T_{w}^2 \). As is known that a power law dependence of specific heat on temperature is a signature of the presence of massless modes in a theory. Its significance in our context, in the light of masslessness of the dilaton, should be interesting to analyze.

To summarize, in this paper we have investigated black hole thermodynamics at finite distance for several types of black holes. We have also argued for their thermodynamic stability by calculating the specific heat. We also point out the role of the gauge freedom in the consistent evaluation of thermodynamic potentials. It will be interesting to further generalize our results in many directions. First, it has been already pointed out that the action \( Z \) has a duality symmetry for general \( k \) [17]. It will be of interest to find out the nature of the thermodynamics for the dual black holes and compare it with the present results. In this regard, the duality invariance of thermodynamic quantitites have been shown for string case earlier [20]. Whether these results are still valid for observations at finite distances is worth addressing. Secondly, the surface terms have been written down earlier for the higher curvature gravity theories. In the context of string theories, since the black hole solution is already known to all-orders, one can also study their thermodynamics to higher orders in this method. Whether there is a way to address the all-order (in \( \alpha' \) ) thermodynamics for string theories is an open question. Moreover, one can possibly generalize our
results to include the dilaton potential. We expect that in that case the nature of specific heat will differ from the results presented here, the reason being the absence of a massless mode. Finally, the results for the charged black hole derived in this paper apply only to non-extremal black holes. Although the expression for entropy (46) has a well-defined limit as $\xi \to 0$, this is not quite the correct value of the asymptotic entropy for the extremal black hole. The transformation (37) is valid only if $\xi > 0$. In fact, as has recently been advocated by Hawking et al in [15], the extremal black hole is thermodynamically a different object than the non-extremal one. The extremal black hole has to be treated separately. It will be interesting to see how the considerations of [15] translate to the cases treated here.

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References

[1] R. M. Wald, *General Relativity*, University of Chicago Press, 1984; G. Gibbons and S. Hawking, *Phys. Rev.* D15, 2738 (1977); C. Lousto, *Nucl. Phys.* B410, 155, (1993).

[2] J. D. Brown and J. York, University of North Carolina Preprint, IFP-UNC-491, [gr-qc/9405024](http://arxiv.org/abs/gr-qc/9405024).

[3] G. Gibbons and M. Perry, [hep-th/9204090](http://arxiv.org/abs/hep-th/9204090), (1992).

[4] G. Mandal, A Sengupta and S. Wadia, *Mod. Phys. Lett.* A6, 1685, (1991); E. Witten, *Phys. Rev.* D44, 314 (1991).

[5] G. Veneziano, *Phys. Lett.* B265, 287 (1991); M. Gasperini, J. Maharana and G. Veneziano, *Phys. Lett.* B272, 167, (1991); S. P. Khastgir and A. Kumar, Bhubaneswar preprint, [hep-th/9311068](http://arxiv.org/abs/hep-th/9311068) (to appear in *Phys. Lett.* B).

[6] M. Cadoni and S. Mignemi, INFN Preprint, INFN-CA-20-93, hep-th/9312171.

[7] Y. Kiem, *Phys. Lett.* B322, 323, (1994); J. Lemos and P. Sá, *Phys. Rev.* D49, 2897, (1994).

[8] D. Lowe and A. Strominger, *Phys. Rev. Lett.* 73, 1468 (1994).

[9] A. Achúcarro and M. Ortiz, *Phys. Rev.* D48, 3600 (1993).

[10] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* 69, 1849, (1992); M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Phys. Rev.* D48, 1506, (1993).
[11] R. Jackiw, *Quantum Theory of Gravity*, ed. S Christensen, Higler, Bristolm 1984; C. Teitelboim *ibid*; R. Jackiw, *Nucl. Phys*. **B252**, 343 (1985).

[12] R. Kallosh, T. Ortin and A. Peet, *Phy. Rev.* **D47**, 5400 (1993).

[13] M.D.Mc.Guian, C.R.Nappi and S.A.Yost, *Nucl. Phys*. **B375**, 421 (1992).

[14] G. Gibbons and S. Hawking, *Phy. Rev.* **D15**, 2752 (1977).

[15] S. Hawking, G. Horowitz and S. Ross, Cambridge University Preprint, DAMTP/R 94-26; [gr-qc/9409013](http://arxiv.org/abs/gr-qc/9409013).

[16] R. M. Wald, *Phy. Rev.* **D48**, 3427 (1993); V. Iyer and R. M. Wald, [gr-qc/9403028](http://arxiv.org/abs/gr-qc/9403028).

[17] M. Cadoni and S. Mignemi, INFN Preprint, INFN-CA-TH-94-4.

[18] R.C. Myers, Mc-Gill University preprint/94-22, [hep-th/9405162](http://arxiv.org/abs/hep-th/9405162); see also V. Frolov, *Phy. Rev.* **D46**, 5383, (1992).

[19] T. Fiola, J. Preskill, A. Strominger and S. Trivedi, CalTech Preprint-68-1918, [hep-th/9403137](http://arxiv.org/abs/hep-th/9403137).

[20] G. Horowitz and D. Welch, Santa Barbara Preprint NSF-ITP-93-89, [hep-th/9308074](http://arxiv.org/abs/hep-th/9308074).
Figure Caption

**Fig 1.** Plot of specific heat ($C_DQ$) vs. coordinate($x$) for the black hole solution (32) for $M = \lambda = 1$. Curves are labelled by different values of $\xi$. The origin on the x-axis is shifted. The curves start at $x = 0$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9410068v1