Perfect State Transfer on NEPS of the path $P_3$

Hiranmoy Pal  
Department of Mathematics  
Indian Institute of Technology Guwahati  
Guwahati, Assam, India - 781039  
hiranmoy@iitg.ernet.in

Bikash Bhattacharjya  
Department of Mathematics  
Indian Institute of Technology Guwahati  
Guwahati, Assam, India - 781039  
b.bikash@iitg.ernet.in  

January 6, 2015

Abstract

We consider perfect state transfer on Kronecker product and NEPS of the path $P_3$. We show that Kronecker product on $n$ copies of the path $P_3$ i.e. $P_3^{\times n}$ admits perfect state transfer at a distance 2 and the transfer time is $\frac{\pi}{\sqrt{2}} n$. So the Cartesian product on $k$ copies of $P_3^{\times n}$ admits perfect state transfer at time $\frac{\pi}{\sqrt{2}} n$ and the distance between the vertices having perfect state transfer is $2k$. Thus long distance perfect state transfer is achievable at significantly less time. Also we discuss perfect state transfer on complete bipartite graphs and NEPS($P_3, \ldots, P_3, \Omega$).

Keywords: Perfect state transfer, Cartesian product, Kronecker product, NEPS of graphs.

1 Introduction

Perfect state transfer is highly desirable in quantum-communication networks modelled by a graph with adjacency matrix as the Hamiltonian of the system. The property of perfect state
transfer on quantum networks was originally introduced by S. Bose [8]. The main goal is to find graphs having perfect state transfer. Christandl et al. [11, 12] shown that Cartesian products of the paths $P_2$ and $P_3$ exhibit perfect state transfer at long distances. Again Bernasconi et al. [7] generalized the result of Christandl et al. [11, 12] for the graph $P_2$ and shown that the cubelike Cayley graphs $X(Z_n^2, \Omega)$, which are actually NEPS($P_2, \ldots P_2, \Omega$), admit perfect state transfer whenever the sum $\oplus_{w \in \Omega} w \neq 0$ in $Z_n^2$. So the natural question is to find whether NEPS($P_3, \ldots P_3, \Omega$) admits perfect state transfer or not. The initial step in this direction is to examine perfect state transfer on Kronecker products of $P_3$. We show that the only complete bipartite graphs that exhibit perfect state transfer is $K_{2,n}$ at time $\frac{\pi}{\sqrt{2}}$. With the help of this result, we find precisely which pair of vertices in $P_3^{\times n}$ admit perfect state transfer. Finally we show that some restrictions on the basis set $\Omega$ yields perfect state transfer on NEPS($P_3, \ldots P_3, \Omega$). This generalizes the result of Christandl et al. [11, 12] for the graph $P_3$.

2 Preliminaries

2.1 Perfect state transfer on graphs

Throughout the paper we only consider simple graphs. The transition matrix $H_A(t)$ for a graph $G$ with adjacency matrix $A$ is defined by

$$H_A(t) := \exp(-itA) := \sum_{k \geq 0} (-i)^k A^{\frac{tk}{k!}}.$$ 

The graph $G$ is said to admit perfect state transfer from a vertex $u$ to another vertex $v$ at time $\tau \in \mathbb{R}$ if $|e_u^T H_A(\tau) e_v| = 1$, i.e., if the $uv$-th entry of $H_A(\tau)$ has unit modulus. Note that $e_u$ denotes the unit vector of appropriate size with $u$-th entry 1. If $u = v$ then we say that the graph is periodic at the vertex $u$. A graph is said to be periodic if it is periodic at each of its vertices. We illustrate this by the following example.

Consider the graph $G = P_2$, the path on two vertices. Then

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $A^2 = I$, the identity matrix. Therefore $H_A(t) = \cos(t)I - i \sin(t)A$. Thus $G$ has perfect
state transfer at $t = \frac{\pi}{2}$ and it is periodic at $t = \pi$.

Spectral decomposition can be used to find the transition matrix efficiently. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the distinct eigenvalues of $A$ and the projections (idempotents) onto the corresponding eigenspaces be $E_1, E_2, \ldots, E_p$. Thus the spectral decomposition of $A$ is given by $A = \sum_{r=1}^{p} \lambda_r E_r$. Note that $E_1 + E_2 + \ldots + E_p = I$, $E_r^2 = E_r$ and $E_r E_s = 0$ for $r \neq s$. As the exponential function is defined on the eigenvalues of $A$, the transition matrix becomes $H_A(t) = \sum_{r=1}^{p} \exp(-it\lambda_r)E_r$. By Lagrange interpolation, there is a polynomial $q(x)$ of degree $p-1$ such that $q(\lambda_r) = \exp(-it\lambda_r)$, $1 \leq r \leq p$. Therefore $q(A) = \sum_{r=1}^{p} q(\lambda_r)E_r = \sum_{r=1}^{p} \exp(-it\lambda_r)E_r = H_A(t)$. So $H_A(t)$ is a polynomial in $A$ and hence $H_A(t)$ is a symmetric matrix. Thus we have

$$H_A(t) (H_A(t))^* = H_A(t)H_A(t) = \exp(-itA) \exp(-i(-t)A) = I.$$

Therefore $H_A(t)$ is also a unitary matrix. We illustrate this by the following example.

Let $G = P_3$, a path of length two with the vertices 1, 2 and 3, where both the vertices 1 and 3 are adjacent to the vertex 2. Therefore $P_3$ has the adjacency matrix

$$A = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},$$

with respect to the ordering 1,2,3 of the vertices. Eigenvalues of the adjacency matrix $A$ are $\lambda_1 = -\sqrt{2}, \lambda_2 = 0, \lambda_3 = \sqrt{2}$ and the projections onto the corresponding eigenspaces are

$$E_1 = \frac{1}{4} \begin{pmatrix}
1 & -\sqrt{2} & 1 \\
-\sqrt{2} & 2 & -\sqrt{2} \\
1 & -\sqrt{2} & 1
\end{pmatrix}, \quad E_2 = \frac{1}{2} \begin{pmatrix}
1 & 0 & -1 \\
0 & 0 & 0 \\
-1 & 0 & 1
\end{pmatrix}, \quad E_3 = \frac{1}{4} \begin{pmatrix}
1 & \sqrt{2} & 1 \\
\sqrt{2} & 2 & \sqrt{2} \\
1 & \sqrt{2} & 1
\end{pmatrix}.$$

Therefore the transition matrix for $P_3$ is $H_A(t) = \exp(-i(-\sqrt{2})t)E_1 + E_2 + \exp(-i\sqrt{2}t)E_3$. Hence at $t = \frac{\pi}{\sqrt{2}}$,

$$H_A \left( \frac{\pi}{\sqrt{2}} \right) = -E_1 + E_2 - E_3 = \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{pmatrix}.$$

Therefore $P_3$ admits perfect state transfer from the vertex 1 to 3 at time $t = \frac{\pi}{\sqrt{2}}$ and it is periodic

Therefore $P_3$ admits perfect state transfer from the vertex 1 to 3 at time $t = \frac{\pi}{\sqrt{2}}$ and it is periodic
at the vertex 2 at the same time. More information on perfect state transfer and periodicity can be found in [14, 20].

2.2 Graph products and transition matrix

Kronecker product\[17\] of two matrices \( S = (s_{ij}) \) and \( T \) is defined by the block matrix \( S \otimes T := (s_{ij}T) \). Let \( \lambda \) and \( \mu \) be the eigenvalues of \( S \) and \( T \), respectively, corresponding to the eigenvectors \( x \) and \( y \). Then \( \lambda \mu \) is an eigenvalue of \( S \otimes T \) corresponding to the eigenvector \( x \otimes y \). If \( E_\lambda \) and \( F_\mu \) are the idempotents of \( S \) and \( T \), respectively, corresponding to the eigenvalues \( \lambda \) and \( \mu \), then \( E_\lambda \otimes F_\mu \) is the idempotent of \( S \otimes T \) corresponding to the eigenvalue \( \lambda \mu \). Thus one can find the spectral decomposition of \( S \otimes T \).

**Cartesian Product:** The Cartesian product on two graphs \( G_1 \) and \( G_2 \) with vertex set \( V_1 \) and \( V_2 \) is the graph \( G_1 \Box G_2 \) whose vertex set is \( V_1 \times V_2 \). Two vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent if either \( u_1 = u_2 \) and \( v_1 \) is adjacent to \( v_2 \) in \( G_2 \) or \( u_1 \) is adjacent to \( u_2 \) in \( G_1 \) and \( v_1 = v_2 \). If \( A \) and \( B \) are the adjacency matrices of \( G_1 \) and \( G_2 \), respectively, then the Cartesian product \( G_1 \Box G_2 \) has the adjacency matrix \( C := A \otimes I + I \otimes B \). Here the identity matrices \( I \) have suitable orders to define the sum. As the matrices \( A \otimes I \) and \( I \otimes B \) commutes, the transition matrix for the Cartesian product \( G_1 \Box G_2 \) becomes \( H_C(t) = H_A(t) \otimes H_B(t) \). More details on perfect state transfer of Cartesian product can be found in [11, 12].

**Kronecker Product:** The Kronecker product on two graphs \( G_1 \) and \( G_2 \) with vertex set \( V_1 \) and \( V_2 \) is the graph \( G_1 \times G_2 \), whose vertex set is \( V_1 \times V_2 \). Two vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent whenever \( u_1 \) is adjacent to \( u_2 \) in \( G_1 \) and \( v_1 \) is adjacent to \( v_2 \) in \( G_2 \). Let \( G_1 \) and \( G_2 \) have the adjacency matrices \( A \) and \( B \), respectively, with respect to some ordering of vertices in \( G_1 \) and \( G_2 \). Then the Kronecker product has the adjacency matrix \( C := A \otimes B \) with respect to the dictionary ordering of vertices in \( G_1 \times G_2 \) induced by the ordering of vertices in \( G_1 \) and \( G_2 \). Let the spectral decomposition of \( A \) and \( B \) be \( A = \sum_{r=1}^{p} \lambda_r E_r \) and \( B = \sum_{s=1}^{q} \mu_s F_s \) respectively. Therefore the spectral decomposition of \( C \) is given by \( C = \sum_{r=1}^{p} \sum_{s=1}^{q} \lambda_r \mu_s (E_r \otimes F_s) \). Thus the
transition matrix for Kronecker product becomes

$$H(t) = \sum_{r=1}^{p} \sum_{s=1}^{q} \exp(-it\lambda_{r}\mu_{s}) (E_{r} \otimes F_{s}) = \sum_{s=1}^{q} \left( \sum_{r=1}^{p} \exp(-it\lambda_{r}\mu_{s}) E_{r} \right) \otimes F_{s} = \sum_{s=1}^{q} H_{A}(\mu_{s} t) \otimes F_{s}. \quad (1)$$

More information on perfect state transfer of Kronecker product can be found in [14, 13].

**NEPS of graphs:** Let $\Omega$ be a set of $n$-tuples $\beta = (\beta_{1}, \beta_{2}, \ldots, \beta_{n})$ of symbols 0 and 1, which does not contain the $n$-tuple $(0, 0, \ldots, 0)$. The NEPS [10] on the graphs $G_{1}, G_{2}, \ldots, G_{n}$ with basis $\Omega$ is the graph $NEPS(G_{1}, G_{2}, \ldots, G_{n}; \Omega)$ whose vertex set is $V(G_{1}) \times V(G_{2}) \times \ldots \times V(G_{n})$. Two vertices $(x_{1}, x_{2}, \ldots, x_{n})$ and $(y_{1}, y_{2}, \ldots, y_{n})$ are adjacent in $NEPS(G_{1}, G_{2}, \ldots, G_{n}; \Omega)$ if and only if there is an $n$-tuple $(\beta_{1}, \beta_{2}, \ldots, \beta_{n})$ in $\Omega$ such that $x_{i} = y_{i}$ in $G_{i}$ exactly when $\beta_{i} = 0$ and $x_{i}$ is adjacent to $y_{i}$ in $G_{i}$ exactly when $\beta_{i} = 1$. Let the graphs $G_{1}, G_{2}, \ldots, G_{n}$ have adjacency matrices $A_{1}, A_{2}, \ldots, A_{n}$, respectively, with respect to some ordering of vertices. Then the graph $NEPS(G_{1}, G_{2}, \ldots, G_{n}; \Omega)$ has the adjacency matrix $A_{\Omega} = \sum_{\beta \in \Omega} A_{1}^{\beta_{1}} \otimes A_{2}^{\beta_{2}} \otimes \ldots \otimes A_{n}^{\beta_{n}}$ with respect to the dictionary ordering of vertices induced by the ordering of vertices in each $G_{k}$’s. See [10] for details. For $\beta \in \Omega$, let $A_{\beta} = A_{1}^{\beta_{1}} \otimes A_{2}^{\beta_{2}} \otimes \ldots \otimes A_{n}^{\beta_{n}}$. Here $A_{\beta}$ can be considered as the adjacency matrix for the graph $NEPS(G_{1}, G_{2}, \ldots, G_{n}; \{\beta\})$. Notice that if $\beta, \delta \in \Omega$, then $A_{\beta}^{\delta_{1}}$ and $A_{\delta}^{\beta_{1}}$ are either $A_{1}$ or $I$ and so $A_{\beta}^{\delta_{1}}, A_{\delta}^{\beta_{1}}$ commutes. Using the property that $(S_{1} \otimes T_{1})(S_{2} \otimes T_{2}) = (S_{1}S_{2}) \otimes (T_{1}T_{2})$, one can see that if $S_{1}$ commutes with $S_{2}$ and $T_{1}$ commutes with $T_{2}$ then $S_{1} \otimes T_{1}$ commutes with $S_{2} \otimes T_{2}$. Thus $A_{\beta}A_{\delta} = A_{\delta}A_{\beta}$ where $\beta, \delta \in \Omega$.

While discussing NEPS we write $H_{\beta}(t)$ to denote $H_{A_{\beta}}(t)$ for $\beta \in \Omega$ and $H_{\Omega}(t)$ to denote $H_{A_{\Omega}}(t)$.

The transition matrix for $NEPS(G_{1}, G_{2}, \ldots, G_{n}; \Omega)$ can be calculated as

$$H_{\Omega}(t) = \exp\left(-it \sum_{\beta \in \Omega} A_{\beta}\right) = \prod_{\beta \in \Omega} \exp(-itA_{\beta}), \text{ as } e^{C+D} = e^{C}e^{D} \text{ if } CD = DC$$

$$= \prod_{\beta \in \Omega} H_{\beta}(t).$$

### 3 Perfect state transfer on complete bipartite graphs

A graph is said to be a complete bipartite graph if its vertex set can be partitioned into two partite sets such that each vertex in one partite set is adjacent to every vertex in the other
partite set. A complete bipartite graph is denoted by $K_{m,n}$, where $m, n$ are the sizes of the partite sets.

R. J. Angeles-Canul et al. [1] shown that for $n \geq 2$, the complete bipartite graph $K_{n,n}$ does not exhibit perfect state transfer. Here we discuss perfect state transfer on general complete bipartite graph $K_{m,n}$.

If $u$ be a vertex of a graph $G$ then $\text{Aut}(G)_u$ denotes the group of automorphisms of $G$ that fix $u$. The next lemma shows that if perfect state transfer occurs from a vertex $u$ to $v$ then any automorphism of $G$ that fixes $u$ must also fix $v$ and vice versa.

**Lemma 3.1.** [16] If a graph $G$ admits perfect state transfer from $u$ to $v$ then $\text{Aut}(G)_u = \text{Aut}(G)_v$.

**Proposition 3.2.** The graph $K_{2,n}$ exhibit perfect state transfer at time $\frac{n}{\sqrt{2n}}$ only between the vertices of the first partite set and the graph is periodic at time $\sqrt{\frac{2}{n}}\pi$.

**Proof.** Let $G = K_{2,n}$ and let the partite sets be $X = \{a, b\}$ and $Y = \{1, 2, \ldots, n\}$. Then $G$ has the adjacency matrix

$$
A = \begin{pmatrix}
0 & 0 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 1 \\
1 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 0 & \ldots & 0
\end{pmatrix}
$$

with respect to the ordering of vertices $a, b, 1, 2, \ldots, n$. The eigenvalues of $A$ are $-\sqrt{2n}, \sqrt{2n}$ and $0$. Here the eigenvalues $-\sqrt{2n}, \sqrt{2n}$ are simple and the eigenvalue $0$ has multiplicity $n$. The eigenvectors corresponding to $-\sqrt{2n}$ and $\sqrt{2n}$ are $(-\sqrt{\frac{2}{n}}, -\sqrt{\frac{2}{n}}, 1, \ldots, 1)^T$ and $(\sqrt{\frac{2}{n}}, \sqrt{\frac{2}{n}}, 1, \ldots, 1)^T$ respectively. Therefore the idempotents corresponding to the eigenvalues $-\sqrt{2n}, \sqrt{2n}$ and $0$ are
given by

\[
F_1 = \frac{1}{2n} \begin{pmatrix}
\frac{\sqrt{n}}{2} & \frac{\sqrt{n}}{2} & -\sqrt{n} & \cdots & -\sqrt{n} \\
\frac{\sqrt{n}}{2} & \frac{\sqrt{n}}{2} & -\sqrt{n} & \cdots & -\sqrt{n} \\
-\sqrt{n} & -\sqrt{n} & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\sqrt{n} & -\sqrt{n} & 1 & \cdots & 1
\end{pmatrix}, \quad F_2 = \frac{1}{2n} \begin{pmatrix}
\frac{\sqrt{n}}{2} & \frac{\sqrt{n}}{2} & \sqrt{n} & \cdots & \sqrt{n} \\
\frac{\sqrt{n}}{2} & \frac{\sqrt{n}}{2} & \sqrt{n} & \cdots & \sqrt{n} \\
\sqrt{n} & \sqrt{n} & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sqrt{n} & \sqrt{n} & 1 & \cdots & 1
\end{pmatrix}
\]

and \( F_3 = I - F_1 - F_2 \). Therefore the transition matrix is given by

\[
H_A(t) = \exp \left( \sqrt{2n}it \right) F_1 + \exp \left( -\sqrt{2n}it \right) F_2 + F_3.
\]

Thus at \( t = \frac{\pi}{\sqrt{2n}} \), \((a,b)\)-th entry of \( H_A \left( \frac{\pi}{\sqrt{2n}} \right) \) is

\[
H_A \left( \frac{\pi}{\sqrt{2n}} \right)_{a,b} = \exp \left( i\pi \right) (F_1)_{a,b} + \exp \left( -i\pi \right) (F_2)_{a,b} + (F_3)_{a,b}
\]

\[
= \exp \left( i\pi \right) \frac{1}{4} + \exp \left( -i\pi \right) \frac{1}{4} - \frac{1}{2}
\]

\[
= -1.
\]

Therefore the graph \( G \) admit perfect state transfer at time \( \frac{\pi}{\sqrt{2n}} \) from the vertex \( a \) to \( b \). Note that there is an automorphism of \( G \) fixing only the vertex \( j \) in the set \( Y \), where \( 1 \leq j \leq n \). Thus by Lemma 3.1, the graph \( G \) do not exhibit perfect state transfer between any other pair of vertices in \( G \). Again one can calculate to see that the diagonal entries of \( H_A \left( \frac{\sqrt{2n} \pi}{n} \right) \) are 1. Therefore \( G \) is periodic at each of its vertices at time \( \sqrt{\frac{2}{n}} \pi \) i.e, \( G \) is periodic at time \( \sqrt{\frac{2}{n}} \pi \).

We now show that no other complete bipartite graph except for the graph \( K_{1,1} \) exhibit perfect state transfer. Note that the graph \( K_{1,1} \) is exactly the path \( P_2 \) and we have seen that \( P_2 \) admit perfect state transfer at time \( \frac{\pi}{2} \).

**Proposition 3.3.** The complete bipartite graphs \( K_{1,n} \) and \( K_{m,n} \), where \( m, n \geq 3 \) do not exhibit perfect state transfer between any pair of vertices.

**Proof.** The graph \( K_{1,n} \) is a star graph and there exist automorphisms fixing only the central vertex of a star. Therefore by Lemma 3.1 perfect state transfer cannot occur from the central vertex of the star graph. Thus if the star admits perfect state transfer then it must occur
between two leaves. But that is also not possible because for any two leaves of the star, there is
an automorphism fixing one leaf but not the other.

The same argument also applies to $K_{m,n}$ for $m, n \geq 3$. Hence the result follows. \hfill $\square$

In this section we have characterized perfect state transfer on complete bipartite graphs.
Similarly, one can see that the $k$-partite graph $K_{n_1,n_2,\ldots,n_k}$ does not exhibit perfect state transfer
whenever the number of partite sets having exactly one vertex is not equal to 2 and the other
remaining partite sets have more than 3 vertices.

4 Perfect state transfer on Kronecker product of $P_3$

Before discussing perfect state transfer on Kronecker product of $P_3$, we introduce a few definitions
and lemmas.

**Definition 4.1** (Center of a matrix). We define the center of a square matrix $A = (a_{i,j})$ of odd
order $n$ by $C(A) := a_{\frac{n+1}{2}, \frac{n+1}{2}}$.

Let us consider $G = P_3$ so that the transition matrix for $G$ as in Section 2.1 is given by
$H_A(t) = \exp(i\sqrt{2}t)E_1 + E_2 + \exp(-i\sqrt{2}t)E_3$. Therefore

$$C\left( H_A \left( \frac{\pi}{\sqrt{2}} \right) \right) = C\left( \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right) = -1$$

Note that $C$ is a linear function on the set of all matrices of odd order $n$.

**Lemma 4.1.** Let $B_1, B_2, \ldots, B_n$ be square matrices of odd orders and let $A = B_1 \otimes B_2 \otimes \ldots B_n$.
Then $C(A) = \prod_{i=1}^{n} C(B_i)$.

**Proof.** For $n = 2$ the lemma follows directly from the definition of Kronecker product. Let us
consider $C = B_1 \otimes B_2 \otimes \ldots B_{n-1}$ so that $A = C \otimes B_n$. Now the lemma follows by induction. \hfill $\square$

**Definition 4.2.** Let $A = (a_{i,j})$ be a square matrix of odd order $n \geq 3$. Then we define $M_3(A)$
to be the $3 \times 3$ principal sub-matrix of $A$ for which $C(M_3(A)) = C(A)$.
Let us consider $G = P_3 \Box P_3$, the Cartesian product of two $P_3$'s. If $H_A(t)$ is the transition matrix for $P_3$ then the transition matrix for $G$ is $H_A(t) \otimes H_A(t)$. Therefore

$$
\mathcal{M}_3 \left( H_A \left( \frac{\pi}{\sqrt{2}} \right) \otimes H_A \left( \frac{\pi}{\sqrt{2}} \right) \right) = \mathcal{M}_3 \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{pmatrix} \otimes \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{pmatrix} \\
= (-1) \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
$$

Note that $\mathcal{M}_3$ is also a linear function on the set of all matrices of odd order $n \geq 3$.

**Lemma 4.2.** Let $B_1, B_2, \ldots, B_n$ be square matrices of odd order and let $A = B_1 \otimes B_2 \otimes \ldots B_n$. Then

$$
\mathcal{M}_3(A) = \left( \prod_{i=1}^{n-1} \mathcal{C}(B_i) \right) \cdot \mathcal{M}_3(B_n).
$$

**Proof.** For $n = 2$ the lemma follows directly from the definition of Kronecker product. Let us consider $C = B_1 \otimes B_2 \otimes \ldots B_{n-1}$ so that $A = C \otimes B_n$. By Lemma 4.1, $\mathcal{C}(C) = \prod_{i=1}^{n-1} \mathcal{C}(B_i)$. Therefore we have

$$
\mathcal{M}_3(A) = \mathcal{C}(C) \cdot \mathcal{M}_3(B_n) = \left( \prod_{i=1}^{n-1} \mathcal{C}(B_i) \right) \cdot \mathcal{M}_3(B_n).
$$

Hence the result follows by induction. $\blacksquare$

Let us consider the matrix

$$
P = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
$$

Then we have $E_1 - E_2 + E_3 = P$, where $E_1, E_2$ and $E_3$ are the projections onto the eigenspaces of the adjacency matrix for the path $P_3$ corresponding to the eigenvalues $\lambda_1, \lambda_2$ and $\lambda_3$, respectively. Let $H_n(t)$ be the transition matrix for $P_3^{\otimes n}$, the Kronecker product of $n$ copies of $P_3$. We use the convention that $P_3^{\otimes 1} = P_3$. Let $\tau_n = \frac{\pi}{\sqrt{2}}$ so that $\sqrt{2} \tau_n = \tau_{n-1}$. Therefore $H_1(\tau_1) = -P$. 
Lemma 4.3. If $H_n(t)$ is the transition matrix for $P_3^{\times n}$ then $H_n(-\tau_n) = H_n(\tau_n)$ for all $n \in \mathbb{N}$.

Proof. We have $H_1(-\tau_1) = (-P)^{-1} = -P = H_1(\tau_1)$. Suppose $H_{k-1}(-\tau_{k-1}) = H_{k-1}(\tau_{k-1})$ for $k \geq 2$. Clearly, $P_3^{\times k} = P_3^{\times (k-1)} \times P_3$. Therefore using (1), we have

$$H_k(\tau_k) = \sum_{i=1}^{3} H_{k-1}(\lambda_i \tau_k) \otimes E_i$$

$$= H_{k-1}(-\sqrt{2} \tau_k) \otimes E_1 + H_{k-1}(0) \otimes E_2 + H_{k-1}(-2\sqrt{2} \tau_k) \otimes E_3$$

$$= H_{k-1}(-\tau_{k-1}) \otimes E_1 + I \otimes E_2 + H_{k-1}(\tau_{k-1}) \otimes E_3$$

$$= H_{k-1}(\tau_{k-1}) \otimes E_1 + I \otimes E_2 + H_{k-1}(\tau_{k-1}) \otimes E_3.$$

From the above we find that $H_k(-\tau_k) = H_k(\tau_k)$. Hence the lemma follows by induction. □

Now we evaluate the transition matrix for Kronecker products of $P_3$ recursively. The following result will enable us to find perfect state transfer in Kronecker product and in general NEPS of the path $P_3$.

Proposition 4.4. The transition matrix $H_n(t)$ for $n \geq 2$ at time $\tau_n$ is given by

$$H_n(\tau_n) = (H_{n-1}(\tau_{n-1}) + I) \otimes E_2 + H_{n-1}(\tau_{n-1}) \otimes P.$$

Proof. From the proof of Lemma 4.3 we have

$$H_n(\tau_n) = H_{n-1}(\tau_{n-1}) \otimes E_1 + I \otimes E_2 + H_{n-1}(\tau_{n-1}) \otimes E_3$$

$$= H_{n-1}(\tau_{n-1}) \otimes (E_1 + E_3) + I \otimes E_2$$

$$= H_{n-1}(\tau_{n-1}) \otimes (E_2 + P) + I \otimes E_2$$

$$= (H_{n-1}(\tau_{n-1}) + I) \otimes E_2 + H_{n-1}(\tau_{n-1}) \otimes P.$$

Hence the result. □

Corollary 4.5. Let $H_n(\tau_n)$ be the transition matrix for $P_3^{\times n}$. Then $\mathcal{M}_3(H_n(\tau_n)) = -P$ and $\mathcal{C}(H_n(\tau_n)) = -1$.

Proof. Clearly the result holds for $H_1(\tau_1)$. Suppose we have $\mathcal{M}_3(H_{k-1}(\tau_{k-1})) = -P$ and so $\mathcal{C}(H_{k-1}(\tau_{k-1})) = -1$. We know that both $\mathcal{C}$ and $\mathcal{M}_3$ are linear. Now by Lemma 4.2 and
Proposition 4.4 we have
\[
\mathcal{M}_3(H_k(\tau_k)) = \mathcal{M}_3((H_{k-1}(\tau_{k-1}) + I) \otimes E_2) + \mathcal{M}_3((H_{k-1}(\tau_{k-1})) \otimes P)
\]
\[
= \mathcal{C}(H_{k-1}(\tau_{k-1}) + I) E_2 + \mathcal{C}(H_{k-1}(\tau_{k-1})) P,
\]
as \mathcal{M}_3(E_2) = E_2, \mathcal{M}_3(P) = P
\[
= (\mathcal{C}(H_{k-1}(\tau_{k-1})) + 1) E_2 + \mathcal{C}(H_{k-1}(\tau_{k-1})) P
\]
\[
= (-1 + 1) E_2 + (-1)P
\]
\[
= -P.
\]
Thus by induction, \( \mathcal{M}_3(H_n(\tau_n)) = -P \). Now \( \mathcal{C}(H_n(\tau_n)) = \mathcal{C}(\mathcal{M}_3(H_n(\tau_n))) = \mathcal{C}(-P) = -1 \).
Hence the lemma follows. \( \square \)

Let \( G \) be a graph with adjacency matrix \( A \). The set of all automorphisms of \( G \) is denoted by \( Aut(G) \). If \( G \) admits perfect state transfer from the vertex \( u \) to \( v \) at time \( \tau \) then \( H_A(\tau)e_u = \gamma e_v \), where \( \gamma \) is a complex number of unit modulus. Let \( f \in Aut(G) \) and \( Q \) be the permutation matrix of \( f \). Then \( Q \) commutes with \( A \) and hence \( Q \) commutes with \( H_A(\tau) \) as \( H_A(\tau) \) is a polynomial in \( A \). Therefore we have \( H_A(\tau)Qe_u = QH_A(\tau)e_u = \gamma Qe_v \). Note that \( Qe_u = e_{f(u)} \) and \( Qe_v = e_{f(v)} \).
Thus we have the following result deduced from the proof of the Lemma 4.1 in [16].

**Lemma 4.6.** Let \( f \in Aut(G) \). If perfect state transfer occurs between the vertices \( u \) and \( v \) then perfect state transfer occurs between the vertices \( f(u) \) and \( f(v) \).

Using Lemma 4.6 we find that if \( f : G \to H \) is an isomorphism and if there is perfect state transfer from the vertex \( u \) to \( v \) in \( G \) then there is perfect state transfer from the vertex \( f(u) \) to \( f(v) \) in \( H \) as well.

Now we show that \( P_3^{\times n} \) admits perfect state transfer from the vertex \( u = (2, \ldots, 2, 1) \) to \( v = (2, \ldots, 2, 3) \). Note that the vertices less than \( (2, 1, \ldots, 1) \) in dictionary ordering are those vertices with first entry 1 so that there are \( 3^{n-1} \) vertices less than \( (2, 1, \ldots, 1) \) in dictionary ordering. Similarly one can see that there are \( 3^{n-1} + 3^{n-2} \) vertices less than \( (2, 2, 1, \ldots, 1) \) in dictionary ordering. Therefore the position of the row of \( H_n(\tau_n) \) corresponding to the vertex \( (2, \ldots, 2, 1) \) is \( 3^{n-1} + 3^{n-2} + \ldots + 3^1 + 1 = \frac{3^n-1}{2} \). Similarly, the position of the column of \( H_n(\tau_n) \) corresponding to the vertex \( (2, \ldots, 2, 3) \) is \( \frac{3^n-1}{2} + 2 \), as the vertices \( (2, \ldots, 2, 1), (2, \ldots, 2, 2) \) and \( (2, \ldots, 2, 3) \) are consecutive vertices in dictionary ordering. Again one can see that 1st row and 3rd column of \( \mathcal{M}_3(H_n(\tau_n)) \) correspond to \( \frac{(3^n-1)}{2} \)-th row and \( \frac{(3^n-1)}{2} + 2 \)-th column of \( H_n(\tau_n) \),
respectively, as $H_n(\tau_n)$ has order $3^n$ and $\mathcal{M}_3(H_n(\tau_n))$ is the central $3 \times 3$ block. So that $(1,3)$-th entry of $\mathcal{M}_3(H_n(\tau_n))$ corresponds to the uv-th entry of $(H_n(\tau_n))$ in dictionary ordering. Thus we have the following theorem.

**Theorem 4.7.** Let $u$ and $v$ be two vertices of $P_3^{x_n}$. If the $k$-th entry of $u$ and $v$ are 1 and 3, respectively, and the remaining entries of both $u$ and $v$ are 2 then $P_3^{x_n}$ exhibits perfect state transfer from the vertex $u$ to the vertex $v$ at time $\frac{2\pi}{(\sqrt{2})^k}$. Also the graph is periodic at the vertex $(2,2,\ldots,2)$ at time $\frac{2\pi}{(\sqrt{2})^n}$.

**Proof.** By Corollary 4.5, $\mathcal{M}_3(H_n(\tau_n)) = -P$. Clearly, $(1,3)$-th entry of $\mathcal{M}_3(H_n(\tau_n))$ is $-1$. Thus perfect state transfer occurs from the vertex $(2,2,\ldots,2,1)$ to $(2,2,\ldots,2,3)$ at time $\tau_n$. Similarly $P_3^{x_n}$ is periodic at the vertex $(2,2,\ldots,2)$ at time $\tau_n$. Let $f_k$, $1 \leq k \leq n$ be defined on the vertices of $P_3^{x_n}$ by $f_k(u_1,\ldots,u_{k-1},u_k,u_{k+1},\ldots,u_n) = (u_1,\ldots,u_{k-1},u_n,u_{k+1},\ldots,u_k)$. That is $f_k$ swaps only the $k$-th and $n$-th entry of the vertices. Clearly, $f_k$ preserves the adjacency relation in $P_3^{x_n}$. Therefore $f_k \in \text{Aut}(P_3^{x_n})$. By Lemma 4.6, perfect state transfer occurs between the images of $(2,2,\ldots,2,1)$ and $(2,2,\ldots,2,3)$ under $f_k$ for each $1 \leq k \leq n$. Hence the result. \hfill \boxed

Thus we have atleast $n$ pair of vertices in $P_3^{x_n}$ between which perfect state transfer occurs. Hereafter we show that these are the only pair of vertices in $P_3^{x_n}$ exhibiting perfect state transfer. Note that $(2,2,\ldots,2,1)$ and $(2,2,\ldots,2,3)$ are not adjacent but both of them are adjacent to $(1,1,\ldots,1,2)$. Therefore perfect state transfer occurs in $P_3^{x_n}$ at a distance 2.

Let $G$ and $H$ be two graphs. The map $g \times h$ is defined on $G \times H$ by $(g \times h)(u,v) = (g(u),h(v))$. If $g \in \text{Aut}(G)$, $h \in \text{Aut}(H)$ then $(u_1,v_1)$ is adjacent to $(u_2,v_2)$ iff $(g(u_1),h(v_1))$ is adjacent to $(g(u_2),h(v_2))$ in $G \times H$. Therefore $g \times h \in \text{Aut}(G \times H)$.

**Theorem 4.8.** Let $u$ and $v$ be two vertices of $P_3^{x_n}$. If the $k$-th entry of $u$ is 1 or 3 and the $k$-th entry of $v$ is 2 then $P_3^{x_n}$ does not admit perfect state transfer from $u$ to $v$ for $1 \leq k \leq n$.

**Proof.** Let $\text{Aut}(P_3) = \{e,f\}$, where $e$ is the identity automorphism and $f$ swaps 1 and 3 fixing the vertex 2. Consider the automorphism $F_k = e \times \ldots \times f \times \ldots \times e$ of $P_3^{x_n}$ where only the $k$-th component is $f$. Clearly, $F_k$ fixes $v$ but it does not fix $u$. Hence there is no perfect state transfer from $u$ to $v$. \hfill \boxed

**Theorem 4.9.** Kronecker product of $n \geq 2$ copies of the path $P_3$ does not admit perfect state transfer from $u$ to $v$ where entries of $u$ and $v$ are either 1 or 3.
Proof. Here the vertices with entries 1 or 3 are adjacent to no other than \((2, 2, \ldots, 2)\) and vice-versa. Therefore these vertices together with the vertex \((2, 2, \ldots, 2)\) form the graph \(K_{1, 2^n}, n \geq 2\). Thus by Proposition 3.3, perfect state transfer cannot occur on those vertices.

**Theorem 4.10.** Kronecker product of \(n\) copies of the path \(P_3\) does not admit perfect state transfer from \(u\) to any other vertex, where \(k\) \((1 \leq k \leq n - 2)\) entries of \(u\) are 2 and the other remaining entries in \(u\) are 1 or 3.

Proof. Let \(u = (u_1, u_2, \ldots, u_n)\), where \(k\) \((1 \leq k \leq n - 2)\) entries of \(u\) be 2 and the other remaining entries be either 1 or 3. Let us consider \(X := \{v = (v_1, v_2, \ldots, v_n) \in V(P_3^{\times n}) | v_j = 2 \text{ iff } u_j = 2\}\) and \(Y := \{w = (w_1, w_2, \ldots, w_n) \in V(P_3^{\times n}) | w_j = 2 \text{ iff } u_j \neq 2\}\). By definition of the Kronecker product, each vertex in \(X\) is adjacent to all the vertices in \(Y\) only and vice-verse. Thus \(X\) and \(Y\) together forms the complete bipartite graph \(K_{2^k, 2^{n-k}}\), which appears as a component of \(P_3^{\times n}\). Clearly, the set \(X\) contains more than two vertices. Thus by Proposition 3.2, perfect state transfer cannot occur on vertices in \(X\).

In this section we have shown precisely which pair of vertices in \(P_3^{\times n}\) exhibits perfect state transfer. Note that The components of \(P_3^{\times n}\) are the graphs \(K_{2^{n-j}, 2^j}, 0 \leq j \leq n\) and the pair of vertices between which perfect state transfer occurs belongs to the component \(K_{2, 2^{n-1}}\).

In the following section we discuss perfect state transfer in NEPS of \(P_3\) with the help of some results developed in this section.

## 5 Perfect state transfer on NEPS of \(P_3\)

Let \(\beta = (\beta_1, \beta_2, \ldots, \beta_n)\) be a non-zero \(n\)-tuple of symbols 0 and 1. Let \(G\) be the graph \(\text{NEPS}(P_3, P_3, \ldots, P_3; \{\beta\})\) with adjacency matrix \(A_\beta\). Let the matrix \(P\) be as defined in the previous section. If the transition matrix for \(G\) is assumed to be \(H_\beta(t)\) then we have the following lemma.

**Lemma 5.1.** If \(\beta = (\beta_1, \beta_2 \ldots, \beta_n)\) where \(\sum_{i=1}^{n} \beta_i = k \neq 0\) then \(\mathcal{M}_3(H_\beta(\tau_k))\) is \(-I\) or \(-P\) according as \(\beta_n\) is equal to 0 or 1 with \(H_\beta(-\tau_k) = H_\beta(\tau_k)\).

Proof. We prove this by induction on \(n\), the length of \(\beta\). For \(n = 1\), \(\sum_{i=1}^{n} \beta_i = 1\) so that \(\mathcal{M}_3(H_\beta(\tau_k)) = \mathcal{M}_3(H_1(\tau_1)) = -P\), by Corollary 4.5 and \(H_1(-\tau_1) = H_1(\tau_1)\) by Proposition...
4.4 Let us assume that the result is true for \( n = l \), i.e., for any \( \beta \) of length \( l \) with \( \sum_{i=1}^{l} \beta_i = k' \), \( \mathcal{M}_3(H_{\beta}(\tau_{k'})) \) is \(-I\) or \(-P\) according as \( \beta_i \) is equal to 0 or 1 with \( H_{\beta}(-\tau_{k'}) = H_{\beta}(\tau_{k'}) \). Consider \( \beta = (\beta_1, \beta_2 \ldots \beta_{l+1}) \) and let \( \beta^* = (\beta_1, \beta_2 \ldots \beta_1) \). Then \( \mathcal{M}_3(H_{\beta^*}(\tau_{k'})) \) is \(-I\) or \(-P\) according as \( \beta_i \) is equal to 0 or 1 with \( H_{\beta^*}(-\tau_{k'}) = H_{\beta^*}(\tau_{k'}) \), where \( \sum_{i=1}^{l} \beta_i = k' \). We now consider two cases according as \( \beta_{l+1} \) is 1 or 0.

**Case I:** Let \( \beta_{l+1} = 1 \) so that \( \sum_{i=1}^{l+1} \beta_i = k' + 1 \). Notice that the graph \( \text{NEPS}(P_3, P_3, \ldots, P_3; \{\beta\}) \) is actually the Kronecker product \( \text{NEPS}(P_3, P_3, \ldots, P_3; \{\beta^*\}) \times P_3 \) as \( \beta_{l+1} = 1 \). Thus using (1), we have

\[
H_{\beta}(\tau_{k'+1}) = H_{\beta^*}(-\sqrt{2} \tau_{k'+1}) \otimes E_1 + H_{\beta^*}(0) \otimes E_2 + H_{\beta^*}(\sqrt{2} \tau_{k'+1}) \otimes E_3 \\
= H_{\beta^*}(-\tau_{k'}) \otimes E_1 + I \otimes E_2 + H_{\beta^*}(\tau_{k'}) \otimes E_3 \\
= H_{\beta^*}(\tau_{k'}) \otimes (E_1 + E_3) + I \otimes E_2 \\
= H_{\beta^*}(\tau_{k'}) \otimes (E_2 + P) + I \otimes E_2 \\
= (H_{\beta^*}(\tau_{k'}) + I) \otimes E_2 + H_{\beta^*}(\tau_{k'}) \otimes P.
\]

Therefore \( \mathcal{M}_3(H_{\beta}(\tau_{k'+1})) = -P \) with \( H_{\beta}(-\tau_{k'+1}) = H_{\beta}(\tau_{k'+1}) \).

**Case II:** Let \( \beta_{l+1} = 0 \) so that \( \sum_{i=1}^{l+1} \beta_i = k' \). Observe that as \( \beta_{l+1} = 0 \), the adjacency matrix for \( \text{NEPS}(P_3, P_3, \ldots, P_3; \{\beta\}) \) is \( A_\beta = A_{\beta^*} \otimes I \), where \( I \) is the identity matrix of order 3. Therefore we have

\[
H_{\beta}(\tau_{k'}) = \exp(-i \tau_{k'} (A_{\beta^*} \otimes I)) = H_{\beta^*}(\tau_{k'}) \otimes I.
\]

Thus \( \mathcal{M}_3(H_{\beta}(\tau_{k'})) = \mathcal{G}(H_{\beta}(\tau_{k'}))I = -I \) with \( H_{\beta}(-\tau_{k'}) = H_{\beta}(\tau_{k'}) \). \( \square \)

Let \( U_1, U_2 \) be two unitary matrices of odd order \( n \geq 3 \) and let \( \mathcal{M}_3(U_1), \mathcal{M}_3(U_2) \) also be unitary. Then we have

\[
U_1 U_2 = \begin{pmatrix}
* & O & * \\
O & \mathcal{M}_3(U_1) & O \\
* & O & *
\end{pmatrix}
\begin{pmatrix}
* & O & * \\
O & \mathcal{M}_3(U_2) & O \\
* & O & *
\end{pmatrix}
= \begin{pmatrix}
* & O & * \\
O & \mathcal{M}_3(U_1) \mathcal{M}_3(U_2) & O \\
* & O & *
\end{pmatrix}.
\]

14
Notice that $\mathcal{M}_3(U_1).\mathcal{M}_3(U_2)$ is also a unitary matrix. Thus we have the following lemma.

**Lemma 5.2.** If $U_1, U_2, \ldots, U_k$ are unitary matrices of odd order $n \geq 3$ and $\mathcal{M}_3(U_i)$ is unitary for each $i$ then $\mathcal{M}_3 \left( \prod_{i=1}^{k} U_i \right) = \prod_{i=1}^{k} \mathcal{M}_3(U_i)$.

**Lemma 5.3.** If $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ where $\sum_{i=1}^{n} \beta_i = k \neq 0$ then $\mathcal{M}_3(H_\beta(2l\tau_k)) = I$ for each $l \in \mathbb{N}$.

**Proof.** By Lemma 5.1, $\mathcal{M}_3(H_\beta(\tau_k))$ is equal to $-I$ or $-P$, i.e., $\mathcal{M}_3(H_\beta(\tau_k))$ is a unitary matrix. Again we have $H_\beta(2l\tau_k) = \exp((-i\tau_k A_\beta)2l) = (H_\beta(\tau_k))^{2l}$, where $A_\beta$ is the adjacency matrix for $NEPS(P_3, P_3, \ldots, P_3; \{\beta\})$. Thus by Lemma 5.2 we have

$$
\mathcal{M}_3(H_\beta(2l\tau_k)) = (\mathcal{M}_3(H_\beta(\tau_k)))^{2l} = (-I)^{2l} \text{ or } (-P)^{2l}
$$

$$
= I, \text{ as } P^2 = I.
$$

Hence the lemma follows.

Let $\Omega = \{\beta^1, \ldots, \beta^m\}$, where each $\beta^i$ is a binary sequence $\left(\beta_1^i, \beta_2^i, \ldots, \beta_n^i\right)$ of length $n$. Now consider $\Omega_j := \{\delta^j = \left(\beta_1^j, \beta_2^j, \ldots, \beta_n^j\right) \mid \text{ for } \beta^i \in \Omega\}$ (i.e., $\delta^j$'s are defined by interchanging the $j$-th entry of each $\beta^i$ with the $n$-th entry). We now show that $G := NEPS(P_3, \ldots, P_3; \Omega)$ is isomorphic with $G' := NEPS(P_3, \ldots, P_3; \Omega_j)$. Here both the graph have the same vertex set $V(P_3) \times V(P_3) \times \cdots \times V(P_3)$. Consider the map $f : V(G) \to V(G')$ defined by $f(v_1, \ldots, v_j, \ldots, v_n) = (v_1, \ldots, v_n, \ldots, v_j)$, i.e., $f$ interchanges the $j$-th and $n$-th entry of a vertex and other entries remain fixed. It is easy to see that $f$ is an isomorphism. Thus we have the following theorem.

**Theorem 5.4.** Let $\Omega = \{\beta^1, \beta^2, \ldots, \beta^m\}$, where each $\beta^i$ is a binary sequence $\left(\beta_1^i, \beta_2^i, \ldots, \beta_n^i\right)$ of length $n$. Let $\sum_{j=1}^{n} \beta_j^i = k$ be fixed for each $i$. Then $NEPS(P_3, \ldots, P_3; \Omega)$ admits perfect state transfer at time $\tau_k$ if $\oplus_{i=1}^{m} \beta^i \neq 0$ in $\mathbb{Z}_2^n$.

**Proof.** Since $\oplus_{i=1}^{m} \beta^i \neq 0$ in $\mathbb{Z}_2^n$, there is at least one coordinate $1 \leq j \leq n$ such that $\sum_{i=1}^{m} \beta_j^i \neq 0$ in $\mathbb{Z}_2$. Without loss of generality let $j = n$, as there is an isomorphism of $NEPS(P_3, \ldots, P_3; \Omega)$ with $NEPS(P_3, \ldots, P_3; \Omega_j)$. Let the transition matrix for $NEPS(P_3, \ldots, P_3; \Omega)$ be $H_\Omega(t)$. Recall that $\mathcal{M}_3(H_\beta(\tau_k))$ is equal to $-I$ or $-P$ according as $\beta_n^i$ is 0 or 1. That is, $\mathcal{M}_3(H_\beta(\tau_k))$
is unitary for all \(1 \leq i \leq m\). Therefore by Lemma \ref{thm:unitary} we get

\[
\mathcal{M}_3 (H_{\Omega}(\tau_k)) = \mathcal{M}_3 \left( \prod_{i=1}^{m} H_{\beta_i}(\tau_k) \right) = \prod_{i=1}^{m} \mathcal{M}_3 (H_{\beta_i}(\tau_k)).
\]

Again by the assumption, there are odd numbers of \(i's\) for which \(\beta_i = 1\). Therefore we have

\[
\prod_{i=1}^{m} \mathcal{M}_3 (H_{\beta_i}(\tau_k)) = (−1)^m P, \text{ as } P \text{ has self inverse.}
\]

Now from \(\mathcal{M}_3 (H_{\Omega}(\tau_k)) = (−1)^m P\) we find that \((-1)^m\) is an off diagonal entry of \(H_{\Omega}(\tau_k)\). Hence \(NEPS(P_3, \ldots, P_3; \Omega)\) exhibits perfect state transfer at time \(\tau_k\). Hence the result follows. 

Observe that by Theorem \ref{thm:unitary} if \(\sum_{i=1}^{m} \beta_i^j \neq 0\) in \(\mathbb{Z}_2\) then \((1,3)\)-th entry of \(\mathcal{M}_3 (H_{\Omega}(\tau_k))\) is \((-1)^m\). Again from the discussions in the previous section, we see that \((1,3)\)-th entry of \(\mathcal{M}_3 (H_{\Omega}(\tau_k))\) is actually the \(uv\)-th entry of \(H_{\Omega}(\tau_k)\) where \(u = (2,2,\ldots,2,1)\) and \(v = (2,2,\ldots,2,3)\). Thus perfect state transfer occurs from the vertex \((2,2,\ldots,2,1)\) to \((2,2,\ldots,2,3)\) in \(NEPS(P_3, \ldots, P_3; \Omega)\) whenever \(\sum_{i=1}^{m} \beta_i^j \neq 0\) in \(\mathbb{Z}_2\).

Now if \(\sum_{i=1}^{m} \beta_i^j \neq 0\) in \(\mathbb{Z}_2\), for some \(j\), then \(\sum_{i=1}^{m} \delta_i^j \neq 0\) in \(\mathbb{Z}_2\) where \(\delta_i \in \Omega_j\). Therefore perfect state transfer occurs between the vertices \(u = (2,2,\ldots,2,1)\) and \(v = (2,2,\ldots,2,3)\) in \(NEPS(P_3, \ldots, P_3; \Omega_j)\). Notice that there is an isomorphism of \(NEPS(P_3, \ldots, P_3; \Omega_j)\) with \(NEPS(P_3, \ldots, P_3; \Omega)\), changing the \(j\)-th entry of a vertex with the \(n\)-th entry as already discussed. Thus perfect state transfer occurs between the vertices \(u_j\) and \(v_j\) of \(NEPS(P_3, \ldots, P_3; \Omega)\), where \(j\)-th entry of \(u_j\) and \(v_j\) are \(1\) and \(3\), respectively, and the other remaining entries are \(2\). Thus we have the following corollary.

**Corollary 5.5.** If the conditions of Theorem \ref{thm:unitary} are satisfied then \(NEPS(P_3, \ldots, P_3; \Omega)\) exhibits perfect state transfer between the pair of vertices \(u_j\) and \(v_j\) whenever \(\sum_{i=1}^{m} \beta_i^j \neq 0\) in \(\mathbb{Z}_2\).

If \(\sum_{i=1}^{m} \beta_i^j = 0\) in \(\mathbb{Z}_2\) then by the proof of Theorem \ref{thm:unitary} we have

\[
\mathcal{M}_3 (H_{\Omega}(\tau_k)) = \prod_{i=1}^{m} \mathcal{M}_3 (H_{\beta_i}(\tau_k)) = (-1)^m I.
\]

Now \((1,1)\), \((2,2)\) and \((3,3)\)-th entry of \(\mathcal{M}_3 (H_{\Omega}(\tau_k))\) correspond to \(uu\), \(vv\) and \(ww\)-th entry of \(H_{\Omega}(\tau_k)\) where \(u = (2,2,\ldots,2,1)\), \(v = (2,2,\ldots,2,2)\) and \(w = (2,2,\ldots,2,3)\), respectively. Hence the graph \(NEPS(P_3, \ldots, P_3; \Omega)\) is periodic at the vertices \((2,2,\ldots,2,1),(2,2,\ldots,2,2)\) and \(2,2,\ldots,2,3)\), respectively.
and \((2, 2, \ldots, 2, 3)\) at time \(\tau_k\). Again if \(\sum_{i=1}^{m} \beta_j^i = 0\) in \(\mathbb{Z}_2\), for some \(j\), then \(\text{NEPS}(P_3, \ldots, P_3; \Omega)\) is periodic at the vertices \(u_j, v_j\) and \((2, 2, \ldots, 2, 2)\) by similar arguments discussed already. Thus we have the following corollary.

**Corollary 5.6.** If \(\sum_{j=1}^{n} \beta_j^i = k\) is fixed for each \(i\), then \(\text{NEPS}(P_3, \ldots, P_3; \Omega)\) is periodic at the vertices \(u_j, v_j\) and \((2, 2, \ldots, 2, 2)\) at time \(\tau_k\) whenever \(\sum_{i=1}^{m} \beta_j^i = 0\) in \(\mathbb{Z}_2\).

The following result, which is indeed proved in \([11, 12]\), can also be obtained as a corollary of Theorem 5.4.

**Corollary 5.7.** \([11, 12]\) Cartesian product of \(n\) copies of the path \(P_3\) exhibits perfect state transfer at time \(\frac{\pi}{\sqrt{2}}\) between the vertices \(u_j\) and \(v_j\) \((1 \leq j \leq n)\).

**Proof.** If we consider \(\Omega = \{\beta^1, \beta^2, \ldots, \beta^n\}\), where \(\beta^i\) has the \(i\)-th entry 1 and the other remaining entries 0 then \(\text{NEPS}(P_3, \ldots, P_3; \Omega) = P_3^{\otimes n}\), the Cartesian product of \(n\) copies of the path \(P_3\). Therefore by Theorem 5.4 and Corollary 5.5, \(P_3^{\otimes n}\) admits perfect state transfer at time \(\tau_1 = \frac{\pi}{\sqrt{2}}\) between \(u_j\) and \(v_j\) for each \(j \in \{1, 2, \ldots, n\}\).

Note that there are other pair of vertices in \(P_3^{\otimes n}\) exhibiting perfect state transfer. Namely, the pair of vertices \((1, 1, \ldots, 1)\) and \((3, 3, \ldots, 3)\) also exhibit perfect state transfer in \(P_3^{\otimes n}\) \([12]\). But the method used here does not provide this information.

Now we extend Theorem 5.4 to have more general NEPS of the path \(P_3\) exhibiting perfect state transfer.

**Theorem 5.8.** Let \(\Omega = \{\beta^1, \beta^2, \ldots, \beta^m\}\) where each \(\beta^i\) is a binary sequence \((\beta_j^1, \beta_j^2, \ldots, \beta_j^n)\) of length \(n\). Let \(k_i = \sum_{j=1}^{n} \beta_j^i\) be even (or odd) for all \(\beta^i \in \Omega\) and let \(k = \min_i k_i\). If \(\Omega^* = \left\{\beta^i : n \sum_{j=1}^{n} \beta_j^i = k\right\}\), then \(\mathcal{M}_3(H_\Omega(\tau_k)) = \mathcal{M}_3(H_{\Omega^*}(\tau_k))\).

**Proof.** We have \(k = \min_i k_i\). If for some \(i, k_i \neq k\) then \(k_i\) and \(k\) differ by an even number. Therefore \(\tau_k = \frac{\pi}{\sqrt{2}} = (\sqrt{2})^{k_i - k} \frac{\pi}{\sqrt{2}^{l_i}} = 2l_i\tau_{k_i}\) for some \(l_i \in \mathbb{N}\). Thus by Lemma 5.3, we have
\[ M_3 \left( H_\beta(\tau_k) \right) = M_3 \left( H_\beta(2l_i \tau_k) \right) = I. \] This implies,

\[ M_3 \left( H_\Omega(\tau_k) \right) = M_3 \left( \prod_{i \in \Omega} H_\beta(\tau_k) \right) = \prod_{i \in \Omega} M_3 \left( H_\beta(\tau_k) \right) = M_3 \left( H_\Omega(\tau_k) \right). \]

Hence the theorem. \(\square\)

If \(\oplus_{\beta \in \Omega} \beta \neq 0\) in \(\mathbb{Z}_2^n\) then by Theorem 5.4, \(NEPS(P_3, \ldots, P_3; \Omega^*)\) admits perfect state transfer between some pair of vertices. Thus by Theorem 5.8, \(NEPS(P_3, \ldots, P_3; \Omega)\) also admits perfect state transfer at the same time between the same pair of vertices which are obtained by using Theorem 5.4 in \(NEPS(P_3, \ldots, P_3; \Omega^*)\). We illustrate this by the following example.

Let us consider \(G := NEPS(P_3, P_3, P_3; \Omega)\) where \(\Omega = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}\). Note that the coordinate sum of each tuple in \(\Omega\) is odd and the minimum of those coordinate sum is 1. Therefore \(\Omega^* = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}\). Now \(G' := NEPS(P_3, P_3, P_3; \Omega^*)\) is the Cartesian product of 3 copies of the path \(P_3\). Hence by Corollary 5.7, \(G'\) admits perfect state transfer at time \(\frac{\pi}{\sqrt{2}}\) between the vertices \(u\) and \(v\) where \(j\)-th entry of \(u\) and \(v\) are 1 and 3, respectively, and the remaining entries are 2. Thus by Theorem 5.8, \(G\) admits perfect state transfer at time \(\frac{\pi}{\sqrt{2}}\) between the same pair of vertices as in \(G'\).

Thus with the help of Theorems 5.4 and 5.8, one can construct several NEPS of the path \(P_3\) to exhibit perfect state transfer and periodicity at time \(\frac{\pi}{(\sqrt{2})^n}\) for each \(n\).

6 Conclusion

The problem of finding perfect state transfer in NEPS of the path \(P_3\) has been mentioned by Dragan Stevanović in [20]. We have developed a method to characterize perfect state transfer in NEPS of \(P_3\). The idea was to find a pair of vertices exhibiting perfect state transfer and then with the help of isomorphisms one can get several pairs of vertices exhibiting perfect state transfer. With this technique we provide full characterization of perfect state transfer in Kronecker product of the path \(P_3\). Finally we have shown that several NEPS of \(P_3\) can be constructed to exhibit perfect state transfer at time \(\frac{\pi}{(\sqrt{2})^n}\) for each \(n\). One problem with the
technique is that we may not get all the pairs of vertices exhibiting perfect state transfer. Thus one can try to find whether there are other pairs of vertices in NEPS of $P_3$ exhibiting perfect state transfer. In Theorem 5.8 we have considered $\Omega$ to be a set of non-zero $n$-tuples with their co-ordinate sums either even or odd. Thus one can try to find whether $NEPS (P_3, \ldots, P_3; \Omega)$ exhibit perfect state transfer whenever $\Omega$ has some other restriction.

We also have discussed perfect state transfer on complete bipartite graphs and shown that only $K_{2,n}$ exhibits perfect state transfer. We have mentioned some complete $k$-partite graphs which do not exhibit perfect state transfer. Thus one can try to find whether perfect state transfer occurs on the remaining possible complete $k$-partite graphs.

References

[1] R. J. Angeles-Canul, R. Norton, M. Opperman, C. Paribello, M. Russel, C. Tamon, Quantum perfect state transfer on weighted join graphs, Int. J. Quantum Inform. Vol. 7, No. 8 (2009) 1429-1445.

[2] R. J. Angeles-Canul, R. Norton, M. Opperman, C. Paribello, M. Russel, C. Tamon, Perfect state transfer, integral circulants and join of graphs, Quantum computation and Information 10 (2010) 325-342. arXiv:0907.2148

[3] M. Bašić, M. D. Petković, D. Stevanović, Perfect state transfer in integral circulant graphs, Applied Mathematics Letters 22 (2009) 1609-1615.

[4] M. Bašić, M. D. Petković, Some class of integral circulant graphs either allowing or not allowing perfect state transfer, Applied Mathematics Letters 22 (2009) 1117-1121.

[5] M. Bašić, M. D. Petković, Perfect state transfer in integral circulant graphs of non-square-free order, Linear Algebra and its Applications, 433 (2010) 149-163.

[6] M. Bašić, Characterization of circulant networks having perfect state transfer, Quantum inf. process, 12:345-364 (2011). DOI:10.1007/s11128-012-0381-z

[7] A. Bernasconi, C. Godsil, and S. Severini, Quantum networks on cubelike graphs, Physical Review A, 78:052320, 2008.
[8] S. Bose, *Quantum communication through an unmodulated spin chain*, Physical Review Letters, *91*(20):207901, 2003.

[9] S. Bose, A. Casaccino, S. Mancini, S. Severini, *Communication in XYZ all-to-all quantum networks with a missing link*, Int. J. Quantum Inform. *07*, 713 (2009). DOI: 10.1142/S0219749909005389

[10] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, 1980.

[11] M. Christandl, N. Dutta, A. Ekert, and Landahl, *Perfect state transfer in quantum spin networks*, Physical Review Letters, *92*:187902, 2004.

[12] M. Christandl, N. Dutta, T. Dorlas, A. Ekert, A. Kay, and Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Physical Review A, *71*:032312, 2005.

[13] Y. Ge, B. Greenberg, O. Perez, C. Tamon, *Perfect state transfer, graph products and equitable partitions*, Int. J. Quantum Inform. *09*, 823 (2011). DOI: 10.1142/S0219749911007472

[14] C. Godsil, *State transfer on graphs*, Discrete Mathematics (2011), doi:10.1016/j.disc.2011.06.032.

[15] C. Godsil, *Periodic graphs*, Electron. J. Combin., 18(1): Paper 23, 15, 2011.

[16] C. Godsil, *When can perfect state transfer occur?* Electronic Journal of Linear Algebra, 23:877-890, 2012.

[17] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[18] A. Kay, *Perfect, efficient, state transfer and its application as a constructive tool*, Int. J. Quantum Inform., *08*, 641 (2010). DOI: 10.1142/S0219749910006514

[19] N. Saxena, S. Severini, and I. E. Shparlinski, *Parameters of integral circulant graphs and periodic quantum dynamics*, Int. J. Quantum Inform. *05*, 417 (2007). DOI: 10.1142/S0219749907002918

[20] D. Stevanović, *Application of graph spectra in quantum physics*, in: D. Cvetković, I. Gutman (Eds.), *Selected Topics on Applications of Graph Spectra*, Zbornik radova 14(22), Mathematical Institute SANU, Belgrade, 2011, pp. 85-111.