THIRD ORDER HANKEL DETERMINANT FOR CERTAIN UNIVALENT FUNCTIONS

Deepak Bansal, Sudhananda Maharan, and Jugal Kishore Prajapat

Abstract. The estimate of third Hankel determinant

\[ H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \]

of the analytic function \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \), for which \( \Re(1 + zf''(z)/f'(z)) > -1/2 \) are investigated. The corrected version of a known results [2, Theorem 3.1 and Theorem 3.3] are also obtained.

1. Introduction

Let \( \mathcal{H}(\mathbb{D}) \) denote the class of analytic functions in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{A} \) be the subclass of \( \mathcal{H}(\mathbb{D}) \) normalized by the condition \( f(0) = 0 = f'(0) - 1 \) and having the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \]

Let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of functions which are also univalent in \( \mathbb{D} \). We denote by \( \mathcal{R} \) a subclass of \( \mathcal{A} \) consisting of functions \( f \) which satisfy \( \Re(f'(z)) > 0 \), \( z \in \mathbb{D} \). Functions in \( \mathcal{R} \) are known to be close-to-convex (and hence univalent) in \( \mathbb{D} \). Further, a function \( f \in \mathcal{A} \) is called starlike (with respect to the origin 0), if \( tw \in f(D) \) whenever \( w \in f(D) \) and \( t \in [0,1] \). We denote by \( \mathcal{S}^* \) the subclass of \( \mathcal{A} \) whose members are starlike in \( \mathbb{D} \). It is well known that \( f \in \mathcal{S}^* \) satisfy the inequality

\[ \Re \left( \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}. \]
Further, let $\mathcal{F}$ be the class of functions $f \in \mathcal{A}$ that are locally univalent and satisfying the inequality

$$
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \quad z \in \mathbb{D}.
$$

It is well known that functions in the class $\mathcal{F}$ are close-to-convex (and hence univalent) in the unit disk. The class $\mathcal{F}$ plays an important role in the discussion on certain extremal problems for the classes of complex-valued and sense-preserving harmonic convex functions and some other related problems in determining univalence criteria for sense-preserving harmonic mappings (see [26]).

For $f \in \mathcal{A}$ of the form (1), the classical Fekete-Szegö functional $\Phi_\lambda(f) = a_3 - \lambda a_2^2$ plays an important role in the function theory. A classical problem settled by Fekete and Szegö [9] is to find for each $\lambda \in [0, 1]$, the maximum value of $|\Phi_\lambda(f)|$ over the function $f \in \mathcal{S}$. By applying the Léwner method they proved that

$$
\max_{f \in \mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 
1 + 2 \exp\{-2\lambda/(1 - \lambda)\}, & \lambda \in [0, 1) \\
1, & \lambda = 1.
\end{cases}
$$

The problem of calculating $\max_{f \in \mathcal{F}} |\Phi_\lambda(f)|$ for various compact subfamilies $\mathcal{F}$ of $\mathcal{A}$, as well as $\lambda$ being an arbitrary real or complex number, was also considered by many authors (see e.g. [1, 5, 12, 13, 14, 20]).

The Hankel determinants $H_{q,n}(f)$ of Taylor’s coefficients of functions $f \in \mathcal{A}$ of the form (1), is defined by

$$
H_{q,n}(f) = \begin{vmatrix}
   a_n & a_{n+1} & \cdots & a_{n+q-1} \\
   a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)}
\end{vmatrix},
$$

where $a_1 = 1$ and $n, q \in \mathbb{N} = \{1, 2, \ldots\}$. The Hankel determinants $H_{q,n}(f)$ are useful, for example, in showing that a function of bounded characteristic in $\mathbb{D}$, i.e., a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [6]. Noonan and Thomas [22] studied the growth rate of the second Hankel determinant of an areally mean $p$-valent function. Pommerenke [25] proved that the Hankel determinants of univalent functions satisfy $|H_{q,n}(f)| < Kn^{-\frac{1}{2} + \beta}q^{\frac{1}{2}}$, where $\beta > 1/4000$ and $K$ depends only on $q$. Later, Hayman [10] proved that $|H_{2,n}(f)| < A n^{1/2}$ ($A$ is an absolute constant) for areally mean univalent functions. Ehrenborg studied Hankel determinant of the exponential polynomials [8] and Noor studied Hankel determinant for the close-to-convex functions [23].

Note that, $H_{2,1}(f) = \Phi_1(f)$ is the Fekete-Szegö functional. Recently many authors have studied the problem of calculating $\max_{f \in \mathcal{F}} |H_{2,2}(f)|$ for various subfamilies $\mathcal{F} \subset \mathcal{A}$ (see e.g. [4, 11, 15, 16]).
$H_{3,1}(f)$ is given by

\begin{equation}
H_{3,1}(f) = \begin{vmatrix}
a_1 & a_2 & a_3 \\
a_2 & a_3 & a_4 \\
a_3 & a_4 & a_5 \\
\end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).
\end{equation}

Recently, Babalola [2] has studied $\max_{f \in F} |H_{3,1}(f)|$ when $F$ are the classes $\mathcal{R}, \mathcal{S}^*$. Also, Raza and Malik [27] have obtained the upper bound on $|H_{3,1}(f)|$ for a subclass of $\mathcal{A}$ associated with right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.

The class of Carathéodory functions $\mathcal{P}$, is the class of functions $p \in \mathcal{H}(\mathbb{D})$ of the form

\begin{equation}
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D},
\end{equation}

having a positive real part in $\mathbb{D}$. Following are the well known results for the functions belonging to the class $\mathcal{P}$:

**Lemma 1.1** ([7]). If $p \in \mathcal{P}$ is of the form (6), then

\begin{equation}
|c_n| \leq 2, \quad n \in \mathbb{N}.
\end{equation}

The inequality (7) is sharp and the equality holds for the function

$$
\phi(z) = \frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n.
$$

**Lemma 1.2** ([18, 19]). If $p \in \mathcal{P}$ is of the form (6), then

\begin{equation}
2c_2 = c_1^2 + x(4 - c_1^2),
\end{equation}

and

\begin{equation}
4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z
\end{equation}

for some $x, z$ with $|x| \leq 1$ and $|z| \leq 1$.

2. Main results

We first provide the corrected form of the results in [2, Theorem 3.1 and Theorem 3.2], given in Theorem 2.1 and Theorem 2.2 below.

**Theorem 2.1.** Let the function $f \in \mathcal{R}$ of the form (1). Then

\begin{equation}
|a_2a_3 - a_4| \leq \frac{1}{2}.
\end{equation}

The inequality (10) is sharp and the equality is attended by the function

\begin{equation}
f(z) = \int_{0}^{z} \frac{1 + \zeta^3}{1 - \zeta^3} d\zeta.
\end{equation}
Proof. If \( f \in \mathcal{R} \) of the form (1), then \( f'(z) = p(z) \), where \( p \in \mathcal{P} \) of the form (6). Equating the coefficients of the series expansion of \( f' \) and \( p \), we get
\[ a_2 = \frac{1}{2} c_1, \quad a_3 = \frac{1}{3} c_2 \quad \text{and} \quad a_4 = \frac{1}{4} c_3. \]
Hence
\[ |a_2a_3 - a_4| = \left| \frac{1}{16} c_1 c_2 - \frac{1}{4} c_3 \right|. \]
Using Lemma 1.2 in (13) for some \( x \) and \( z \) such that \( |x| \leq 1 \) and \( |z| \leq 1 \), we get
\[ |a_2a_3 - a_4| = \frac{1}{48} \left| 4c_1 \left( c_1^2 + x(4 - c_1^2) \right) - 3 \left( c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) \right) \right| + 2(1 - |x|^2)(4 - c_1^2)z \]
\[ = \frac{1}{48} \left| c_1^3 + (4 - c_1^2)(-2c_1x + 3c_1x^2 - 6(1 - |x|^2)z) \right|. \]
By Lemma 1.1, we have \( |c_1| \leq 2 \). Therefore, letting \( c_1 = c \), we may assume without restriction that \( c \in [0, 2] \). Thus applying the triangle inequality with \( \mu = |x| \), we obtain
\[ |a_2a_3 - a_4| \leq \frac{1}{48} \left[ (4 - c^2)(6 + 2c + 3\mu^2(c - 2)) \right] \]
\[ = F(c, \mu). \]
Let \( \Omega = \{(c, \mu) : 0 \leq c \leq 2, 0 \leq \mu \leq 1 \} \). To find the maximum value of \( F \) over the region \( \Omega \) we use the Hessian matrix method. For this, differentiate \( F \) with respect to \( \mu \) and \( c \) and set them equal to zero;
\[ \frac{\partial F}{\partial \mu} = \frac{1}{24} \left[ (4 - c^2)(c + 3\mu(c - 2)) \right] = 0, \]
\[ \frac{\partial F}{\partial c} = \frac{1}{48} \left[ 8\mu + 12\mu^2 + 12(\mu^2 - 1)c + 3(1 - 2\mu - 3\mu^2)c \right] = 0. \]
Solving (15) and (16) with the help of the mathematica software, we get the critical points
\((-2, -(1 + 2\sqrt{7})/6), \quad (2, -3/4) \) and \((8/3, -4/3)\).
Observe that, the only critical point lying in \( \Omega \) is \( (0, 0) \). At this critical point \( (0, 0) \), we find that
\[ \frac{\partial^2 F}{\partial \mu^2} = -1 < 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial c^2} = \left( \frac{\partial^2 F}{\partial \mu \partial c} \right)^2 = \frac{2}{9} > 0. \]
Therefore \( F(c, \mu) \) has a local maximum at \( (0, 0) \).
We now look the critical points on the boundary of \( \Omega \). At \( L_1 = \{(2, \mu) : 0 \leq \mu \leq 1 \} \), we have \( F(2, \mu) = 1/6 \), which is a constant. At \( L_2 = \{(0, \mu) : 0 \leq \mu \leq 1 \} \), we have \( F(0, \mu) = (1 - \mu^2)/2 \), which gives the same critical point \( (0, 0) \). At \( L_3 = \{(c, 1) : 0 \leq c \leq 2 \} \), we have \( F(c, 1) = (5c - c^3)/12 \), which
gives another critical point \((\sqrt{5/3}, 1)\). At \(L_4 = \{(c, 0) : 0 \leq c \leq 2\}\), we have 
\[F(c, 0) = (c^3 - 6c^2 + 24)/48,\]
giving the same critical point \((0, 0)\). Observe that 
\[F(2, \mu) < F(\sqrt{5/3}, 1) < F(0, 0).\]
Thus the local maximum at \((0, 0)\) is also the global maximum on \(\Omega\). Hence 
\[\max_{\Omega} F(c, \mu) = F(0, 0) = 1/2.\]

To show the sharpness, set \(c_1 = x = 0, z = 1\) in (8) and (9), to get \(c_2 = 0\) and \(c_3 = 2\). Using these values in (13), we find that the inequality (10) is sharp and it can be seen easily that the equality in (10) is attended by the function \(f\) given in (11). This completes the proof. \(\square\)

It is well known that, if \(f \in \mathcal{R}\) is of the form (1), then 
\[|a_n| \leq 2/n, n = 2, 3, \ldots, [21], |a_3 - a_2^2| \leq 2/3 \quad [3],\]
and \[|a_2a_4 - a_3^2| \leq 4/9 \quad [11].\]
Using these coefficient bounds and Theorem 2.1, we get 
\[|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \leq 2 \frac{4}{3} + 2 \frac{1}{4} + 2 \frac{2}{5} = \frac{439}{540}.\]
Thus, we state that:

**Theorem 2.2.** Let the function \(f \in \mathcal{R}\) of the form (1). Then 
\[|H_{3,1}(f)| \leq \frac{439}{540}.\]

**Remark 2.3.** Babalola in [2, Theorem 3.3] proved that, if \(f \in \mathcal{S}^*\) is of the form (1), then \([a_2a_3 - a_4] \leq 2. This inequality is sharp and the equality is attended for the Koebe function \(k(z) = z/(1 - z)^2\) and its rotation. While observing its proof, we see, that the author’s claim about \(F'(<0\) is not correct. From the method used in Theorem 2.1, we can easily see that the result in [2, Theorem 3.3] is correct and its proof is similar to that of Theorem 2.1 above. This can easily be worked out, and therefore, we skip giving details in this regard.

**Theorem 2.4.** Let the function \(f \in \mathcal{F}\) of the form (1). Then 
\[|a_3 - a_2^2| \leq \frac{1}{2}.\]
The inequality (17) is sharp.

**Proof.** If \(f \in \mathcal{F}\) of the form (1), then we may write 
\[1 + zf''(z) f'(z) = 3 \frac{p(z)}{2} - \frac{1}{2}.\]
Substituting the series expansion of \(f''(z), f'(z)\) and \(p(z)\) and equating the coefficients, we get 
\[a_2 = \frac{3}{4} c_1, \quad a_3 = \frac{1}{8} (3c_1^2 + 2c_2), \quad a_4 = \frac{1}{64} (9c_1^2 + 18c_1c_2 + 8c_3).\]
Using these values of coefficients and Lemma 1.2 for some \( x \) and \( z \) such that \( |x| \leq 1 \) and \( |z| \leq 1 \), we get

\[
|a_3 - a_2^2| = \frac{1}{16} \left| -c_1^2 + 2x(4 - c_1^2) \right|.
\]

By Lemma 1.1, we may assume \( c_1 = c \in [0, 2] \). Applying the triangle inequality in (19) with \( \mu = |x| \), we obtain

\[
|a_3 - a_2^2| \leq \frac{1}{16} \left[ c_1^2 + 2\mu(4 - c_1^2) \right] = H_1(c, \mu).
\]

Differentiating \( H_1 \) with respect to \( \mu \), we get

\[
\frac{\partial H_1}{\partial \mu} = \frac{1}{8} (4 - c_1^2) \geq 0 \text{ for } 0 \leq \mu \leq 1.
\]

Hence, \( H_1 \) is an increasing function of \( \mu \) on \([0, 1]\). Therefore

\[
\max_{0 \leq \mu \leq 1} H_1(c, \mu) = H_1(c, 1) = \frac{1}{16}(8 - c_1^2) = H(c).
\]

It is clear that \( H(c) \) is a decreasing function of \( c \) \((0 \leq c \leq 2)\), hence the maximum value of \( H_1(c, \mu) \) is attained at the point \((0, 1)\), that is,

\[
\max \Omega H_1(c, \mu) = H_1(0, 1) = \frac{1}{2}.
\]

To show the sharpness of (17), choose \( c_1 = 0 \) and \( x = 1 \) in (8) and (9), we get \( c_2 = 2 \) and \( c_3 = 0 \). Using these values in (19) we find that inequality (17) is sharp. This completes the proof. \( \square \)

**Theorem 2.5.** Let the function \( f \in \mathcal{F} \) of the form (1). Then

\[
|a_2a_3 - a_4| \leq \frac{9}{4\sqrt{15}}.
\]

**Proof.** Using the values of \( a_2, a_3 \) and \( a_4 \) from (18) and using (8) and (9) for some \( x \) and \( z \) such that \( |x| \leq 1 \) and \( |z| \leq 1 \), we get

\[
|a_2a_3 - a_4| = \frac{1}{64} \left| 4c_3^3 + (4 - c_1^2) \{-7c_1x + 2c_1x^2 - 4(1 - |x|^2)z \} \right|.
\]

By Lemma 1.1, we have \( |c_1| \leq 2 \). Letting \( c_1 = c \), we may assume without restriction that \( c \in [0, 2] \). Thus applying the triangle inequality in (20) with \( \mu = |x| \), we obtain

\[
|a_2a_3 - a_4| \leq \frac{1}{64} \left[ 4c^3 + (4 - c^2)(7c\mu + 2c\mu^2 + 4 - 4\mu^2) \right] = H_2(c, \mu).
\]

Differentiating \( H_2 \) with respect to \( \mu \) and \( c \), we get

\[
\frac{\partial H_2}{\partial \mu} = \frac{1}{64} \left[ (4 - c^2)(7c + 4c\mu - 8\mu) \right],
\]

\[
\frac{\partial H_2}{\partial c} = \frac{1}{64} \left[ 12c^2 + 28\mu + 8\mu^2 - 21c^2\mu + 6c^2\mu^2 - 8c + 8c\mu^2 \right].
\]
Solving $\frac{\partial H_2}{\partial \mu} = 0$ and $\frac{\partial H_2}{\partial c} = 0$, we find that the critical points of $H_2$ are 
\(-2, -(7 + \sqrt{177})/8),\quad (-2, -(7 + \sqrt{177})/8),\n(-44/81, -77/206),\quad (0, 0)\quad \text{and}\quad (2, 4/7).

Observe that $H_2(c, \mu)$ does not attain extremum at $(0, 0)$ and $(2, 4/7)$. 
Next, we examine the critical points at the boundary of $\Omega$. We find that, 
along $L_1 = \{(2, \mu) : 0 \leq \mu \leq 1\}$, we have $H_2(2, \mu) = 1/2$, which is a constant and another critical points at the boundary are only $(2/3, 0)$ and $(6/\sqrt{15}, 1)$. 
Since $H_2(2/3, 0) < H_2(2, \mu) < H_2(6/\sqrt{15}, 1)$, we get 
\[\max_{\Omega} H_2(c, \mu) = H_2(6/\sqrt{15}, 1) = \frac{9}{4\sqrt{15}}.\]

This completes the proof. \(\square\)

**Theorem 2.6.** Let the function $f \in \mathcal{F}$ of the form (1). Then 
\[|a_2a_4 - a_3^2| \leq \frac{21}{64}.\]

**Proof.** Using the values of $a_2$, $a_3$ and $a_4$ from (18) and using (8) and (9) for some $x$ and $z$ such that $|x| \leq 1$ and $|z| \leq 1$, we get 
\[|a_2a_4 - a_3^2| = \frac{1}{256} \left| -4c_1^4 + (4 - c_1^2)(7c_1^2x - 6c_1^2x^2 + 12c_1(1 - |x|^2)z - 4x^2(4 - c_1^2)) \right|.\]

By Lemma 1.1, we assume $c_1 = c \in [0, 2]$. Applying the triangle inequality in above equation with $\mu = |x|$, we obtain 
\[|a_2a_4 - a_3^2| \leq \frac{1}{256} \left[ 4c_1^4 + (4 - c_1^2)(7c_1^2\mu + 2c_1^2\mu^2 + 12c_1 + 12c\mu^2 + 16\mu^2) \right] = H_3(c, \mu)\]

Differentiating $H_3$ with respect to $\mu$ and $c$, we get 
\[\frac{\partial H_3}{\partial \mu} = \frac{1}{256} \left[ (4 - c^2)(7c^2 + 4c^2\mu - 24c\mu + 32\mu) \right],\]
\[\frac{\partial H_3}{\partial c} = \frac{1}{256} \left[ 16c^3 + 56c\mu - 16c\mu^2 + 48 - 48\mu^2 - 28c^3\mu - 8c\mu^2 - 36c^2 + 36c^2\mu^2 \right].\]

Solving $\frac{\partial H_3}{\partial \mu} = 0$ and $\frac{\partial H_3}{\partial c} = 0$, we get the critical points are 
\((-2, -(7 + \sqrt{721})/24),\quad (-2, -(7 + \sqrt{721})/24),\quad \text{and}\quad (2, 2/7).\)
We observe that, \((2, 2/7)\) is the only critical point laying inside \(\Omega\), but at this point
\[
\frac{\partial^2 H_3}{\partial \mu^2} \frac{\partial^2 H_3}{\partial c^2} - \left(\frac{\partial^2 H_3}{\partial \mu \partial c}\right)^2 < 0.
\]
Hence \(H_3\) does not attain extremum at \((2, 2/7)\).

Next, we examine the critical points at the boundary of \(\Omega\). We find that, along \(L_1 = \{(2, \mu) : 0 \leq \mu \leq 1\},\) \(H_3(2, \mu) = 1/4\), which is a constant and other critical points at the boundary are only \((0, 1)\) and \((\sqrt{2}, 1)\). Hence \(H_3(0,0) < H_3(2, \mu) = H_3(0, 1) < H_3(\sqrt{2}, 1)\). Therefore
\[
\max_{\Omega} H_3(c, \mu) = H_3(\sqrt{2}, 1) = \frac{21}{64}.
\]
This completes the proof.

It is known that, if \(f \in \mathcal{F}\) of the form \((1)\), then \(|a_n| \leq \frac{n+1}{2}\) for \(n \geq 2\) [26]. Using this bound and Theorem 2.4, Theorem 2.5 and Theorem 2.6, we get:

**Theorem 2.7.** Let the function \(f \in \mathcal{F}\) of the form \((1)\). Then
\[
|H_{3,1}(f)| \leq \frac{180 + 69\sqrt{15}}{32\sqrt{15}}.
\]

**Remark 2.8.** For \(f \in \mathcal{S}\), Thomas [24, p. 166] conjectured that
\[
|H_{2,n}(f)| = |a_n a_{n+2} - a_n^2| \leq 1, \quad n = 2, 3, \ldots,
\]
Subsequently, Li and Srivastava [17, p. 1040] showed that this conjecture is not valid for \(n \geq 4\), i.e., conjecture is valid only for \(n = 2, 3\). From the known result \(|a_2 a_4 - a_3^2| \leq 4/9\) (see [11]) and Theorem 2.6, we found that, if the function \(f\) is a member of the class \(\mathcal{R}\) and \(\mathcal{F}\), respectively and each having form \((1)\), then
\[
|H_{2,2}(f)| \leq \frac{4}{9} \quad \text{and} \quad |H_{2,2}(f)| \leq \frac{21}{64}.
\]
Since all functions in \(\mathcal{R}\) and \(\mathcal{F}\) are close-to-convex and hence also univalent in \(\mathbb{D}\). Therefore, the result in [11] and Theorem 2.6 validate the Thomas conjecture when \(n = 2\) for the function belonging to the classes \(\mathcal{R}\) and \(\mathcal{F}\).

**Acknowledgment.** The authors are grateful to the referees for their valuable suggestions. Also, the present investigation of first author is supported by Department of Science and Technology, New Delhi, Government of India under Sanction Letter No. SR/FTP/MS-015/2010.

**References**

[1] H. R. Abdel-Gawad and D. K. Thomas, *The Fekete-Szegő problem for strongly close-to-convex functions*, Proc. Amer. Math. Soc. 114 (1992), no. 2, 345-349.
[2] K. O. Babalola, *On third order Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl. 6 (2010), 1-7.
THIRD ORDER HANKEL DETERMINANT 1147

[3] K. O. Babalola and T. O. Opoola, On the coefficients of certain analytic and univalent functions, Advances in Inequalities for Series, (Edited by S. S. Dragomir and A. Sofo) Nova Science Publishers (2008), 5–17.

[4] D. Bansal, Upper bound of second Hankel determinant for a new class of analytic functions, Appl. Math. Lett. 26 (2013), no. 1, 103–107.

[5] B. Bhowmik, S. Ponnusamy, and K. J. Wirths, On the Fekete-Szegő problem for concave univalent functions, J. Math. Anal. Appl. 373 (2011), no. 2, 432–438.

[6] D. G. Cantor, Power series with integral coefficients, Bull. Amer. Math. Soc. 69 (1963), 362–366.

[7] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[8] R. Ehrenborg, The Hankel determinant of exponential polynomials, Amer. Math. Monthly 107 (2000), no. 6, 557–560.

[9] M. Fekete and G. Szegő, Eine Bermerkung über ungerade Schlichte funktionen, J. Lond. Math. Soc. 8 (1933), 85–89.

[10] W. K. Hayman, On the second Hankel determinant of mean univalent functions, Proc. Lond. Math. Soc. (3) 18 (1968), 77–94.

[11] A. Janteng, S. Halim, and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math. 7 (2006), no. 2, Article 50, 5 pp.

[12] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20 (1969), 8–12.

[13] Y. C. Kim, J. H. Choi, and T. Sugawa, Coefficient bounds and convolution properties for certain classes of close-to-convex functions, Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), no. 6, 95–98.

[14] W. Koepl, On the Fekete-Szegő problem for close-to-convex functions, Proc. Amer. Math. Soc. 101 (1987), no. 1, 89–95.

[15] B. Kowalczyk and A. Lecko, The Fekete-Szegő inequality for close-to-convex functions with respect to a certain starlike function dependent on a real parameter, J. Inequal. Appl. 2014 (2014), Article 65, 16 pp.

[16] S. K. Lee, V. Ravichandran, and S. Subramaniam, Bounds for the second Hankel determinant of certain univalent functions, J. Inequal. Appl. 2013 (2013), Article 281, 17 pp.

[17] J. L. Li and H. M. Srivastava, some questions and conjectures in the theory of univalent functions, Rocky Mountain J. Math. 28 (1998), no. 3, 1035–1041.

[18] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85 (1982), no. 2, 225–230.

[19] Coefficient bounds for the inverse of a function with derivatives in P, Proc. Amer. Math. Soc. 87 (1983), no. 2, 251–257.

[20] R. R. London, Fekete-Szegő inequalities for close-to-convex functions, Proc. Amer. Math. Soc. 117 (1993), no. 4, 947–950.

[21] T. H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. 104 (1962), 532–537.

[22] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a real mean p-valent functions, Trans. Amer. Math. Soc. 223 (1976), 337–346.

[23] K. I. Noor, Higher order close-to-convex functions, Math. Japon. 37 (1992), no. 1, 1–8.

[24] R. Parvatham and S. Ponnusamy (Editors), New Trends in Geometric Function Theory and Application, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1981.

[25] C. Pommerenke, On the coefficients and Hankel determinant of univalent functions, J. Lond. Math. Soc. 41 (1966), 111–122.
[26] S. Ponnusamy, S. K. Sahoo, and H. Yanagihara, Radius of convexity of partial sums of functions in the close-to-convex family. Nonlinear Anal. 95 (2014), 219–228.

[27] M. Raza and S. N. Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli, J. Inequal. Appl. 2013 (2013), 412, 8 pp.

[28] T. V. Sudharsana, S. P. Vijayakshmi, and B. Adolf Stephen, Third Hankel determinant for a subclass of analytic univalent functions, Malaya J. Mat. 2 (2014), no. 4, 438–444.

Deepak Bansal  
Department of Mathematics  
Govt. College of Engineering and Technology  
Bikaner-334004, Rajasthan, India  
E-mail address: deepakbansal79@yahoo.com

Sudhananda Maharana  
Department of Mathematics  
Central University of Rajasthan  
NH-8, Bandarsindri, Kishangarh-305801  
Distt.-Ajmer, Rajasthan, India  
E-mail address: snmmath@gmail.com

Jugal Kishore Prajapat  
Department of Mathematics  
Central University of Rajasthan  
NH-8, Bandarsindri, Kishangarh-305801  
Distt.-Ajmer, Rajasthan, India  
E-mail address: jkprajapat@curaj.ac.in