Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data

O. Yu. Imanuvilov* and M. Yamamoto†

Abstract

For a semilinear elliptic equation, we prove uniqueness results in determining potentials and semilinear terms from partial Cauchy data on an arbitrary subboundary.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\Gamma$ be a relatively open subset on $\partial \Omega$ and $\Gamma_0 = \partial \Omega \setminus \Gamma$. Consider the following boundary value problem:

$$ P(x,D)u := \Delta u + q(x)u - f(x,u) = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0, $$

(1)

Henceforth we set $L(x,D)u = \Delta u + qu$.

Consider the following partial Cauchy data:

$$ C_{q,f} = \left\{ (u, \frac{\partial u}{\partial \nu}) \left| \Gamma : P(x,D)u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0, \quad u \in H^1(\Omega) \right. \right\}. $$

Here $\nu$ is the unit outward normal vector to $\partial \Omega$.

The paper is concerned with the following inverse problem: Using the partial Cauchy data $C_q$, determine the coefficient $q$.

Assume that

$$ f, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2} \in C^0(\Omega \times \mathbb{R}^1), \quad f(x,0) \equiv \frac{\partial f(x,0)}{\partial y} \equiv 0, $$

(2)

and for some positive constants $p > 1, C_1, C_2$, the following holds true:

$$ f(x,y)y \geq C_1|y|^{p+1} - C_2, \quad \forall (x,y) \in \Omega \times \mathbb{R}^1. $$

(3)

Moreover for some $p_1 > 0, p_2 > 0, C_3 > 0$ and $C_4 > 0$, the following inequalities holds true:

$$ |\frac{\partial f}{\partial y}(x,y)| \leq C_3(1 + |y|^{p_1}), \quad \forall (x,y) \in \Omega \times \mathbb{R}^1, \quad \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right| \leq C_4(1 + |y|^{p_2}), \quad \forall (x,y) \in \Omega \times \mathbb{R}^1. $$

(4)

Our first main result is concerned with the uniqueness in determining a linear part, that is, a potential $q$.

**Theorem 1** Let functions $f_1, f_2$ satisfy (3), (4) and $q_j \in C^{2+\alpha}(\overline{\Omega}), \quad j = 1, 2$, with some $\alpha \in (0,1)$. Suppose that $C_{q_1, f_1} = C_{q_2, f_2}$. Then $q_1 = q_2$ in $\Omega$.

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*Department of Mathematics, Colorado State University, 101 Weber Building, Fort Collins, CO 80523-1874, USA, E-mail: oleg@math.colostate.edu

†contacting author: Department of Mathematical Sciences, The University of Tokyo, Komaba Meguro Tokyo 153-8914 Japan, E-mail:myama@ms.u-tokyo.ac.jp, tel: +81-3-5465-7011, fax: +81-3-5465-7011
Remark 1. Since our assumptions on the potential \( q \) and nonlinear term \( f \) in general do not imply the uniqueness of a solution for the boundary value problem for the elliptic operator \( P(x, D) \), by the equality \( C_q = C_{q_2} \), we mean the following: for any element \((v_1, v_2)\) from \( C_{q_1, f_1} \), there exists a function \( w \in H^1(\Omega) \) such that \( P_2(x, D)w = \Delta w + q_2w - f_2(x, w) = 0 \) in \( \Omega \), \( w|_{\Gamma} = 0 \), \( w|_{\tilde{\Gamma}} = v_1 \) and \( \frac{\partial w}{\partial n}|_{\Gamma} = v_2 \).

Remark 2. Theorem 1 is still true if condition (3) is replaced by following: there exists a continuous function \( G \) such that a solution to the boundary value problem

\[
P(x, D)u = 0 \quad \text{in } \Omega, \quad u|_{\partial \Omega} = g
\]

satisfies the estimate

\[
\|u\|_{H^1(\Omega)} \leq G(\|g\|_{H^{\frac{1}{2}}(\partial \Omega)}).
\]

Condition (3) is used in deriving the inequality (64).

For any \( F(t) \in C([0, 1]; C^{2+\alpha}(\overline{\Omega})) \) with \( \alpha \in (0, 1) \), we introduce the set

\[
O_F = \bigcup_{0 \leq t \leq 1, x \in \Omega} \{(x, (F(t))(x))\}.
\]

Let

\[
U_j = \{F \in C([0, 1]; C^{2+\alpha}(\overline{\Omega})); F(0) = 0, u(\cdot, t) := F(t) \text{ satisfies } \Delta u(x, t) + q_j u(x, t) - f_j(x, u(x, t)) = 0, x \in \Omega, \ u(\cdot, t)|_{\Gamma} = 0, \ j = 1, 2\}.
\]

The second main result asserts the uniqueness for semilinear terms \( f_k, k = 1, 2 \) in some range provided that the potential \( q \) is known:

**Theorem 2** Let \( q_1 = q_2 = q \in C^{2+\alpha}(\overline{\Omega}) \) be arbitrarily fixed. Let functions \( f_1, f_2 \in C^{3+\alpha}(\overline{\Omega} \times \mathbb{R}^1) \) for some \( \alpha \in (0, 1) \), satisfy (3), (4) and \( f_1(\cdot, 0) = f_2(\cdot, 0) = 0 \). Suppose that \( C_{q, f_1} = C_{q, f_2} \). Then

\[
f_1 - f_2 = 0 \quad \text{in } \bigcup_{j \in \{1, 2\}} \bigcup_{F \in U_j} O_F.
\]

Combining Theorems 1 and 2, under stronger conditions on \( f_k, k = 1, 2 \), we can prove the uniqueness in determining both \( q \) and \( f(x, u) \).

**Corollary** Let \( q_1, q_2 \in C^{2+\alpha}(\overline{\Omega}) \) and let functions \( f_1, f_2 \in C^{3+\alpha}(\overline{\Omega} \times \mathbb{R}^1) \) with some \( \alpha \in (0, 1) \), satisfy (2), (3) and (4). Suppose that \( C_{q_1, f_1} = C_{q_2, f_2} \). Then \( q_1 = q_2 \) in \( \Omega \) and

\[
f_1 - f_2 = 0 \quad \text{in } \bigcup_{j \in \{1, 2\}} \bigcup_{F \in U_j} O_F.
\]

**Remark 3.** Under the condition of Theorem 1, we can not completely recover the nonlinear term. Indeed, if \( \rho \in C^2(\overline{\Omega}), \rho|_{\partial \Omega} = 0, \frac{\partial \rho}{\partial n} < 0 \) on \( \partial \Omega \) and \( \rho > 0 \) in \( \Omega \), under assumptions (2) and (3), we have the following a priori estimate proved in (10):

\[
\int_{\Omega} \rho^\kappa (|\nabla u|^2 + |u|^{p+1}) \, dx \leq C
\]

for \( u \in H^1(\Omega) \) satisfying \( P(x, D)u = 0 \) in \( \Omega \). Here a constant \( C \) is independent of \( u \) and \( \kappa \) depends on \( p \). Such a estimate immediately implies that for any \( \Omega_1 \subset \subset \Omega \), there exists a constant \( C(\Omega_1) > 0 \) such that

\[
\|u\|_{C^0(\overline{\Omega_1})} \leq C(\Omega_1).
\]
This estimate and (3) imply that for any \( x \in \Omega \), a nonlinear term \( f(x,y) \) can not be recovered for all sufficiently large \( y \).

Restricted to the linear elliptic equation in two dimensions, there are quite rich references and here we give a very partial list. In the case \( \Gamma = \partial \Omega \) of the full Cauchy data, the uniqueness in determining a potential \( q \) in the two dimensional case was proved for the conductivity equation by Nachman in [21] within \( C^4 \) conductivities, and later in [2] within \( L^\infty \) conductivities. For a convection equation, see [5]. The case of the Schrödinger equation was solved by Bukhgeim [4]. In the case of the partial Cauchy data on arbitrary subboundary, the uniqueness was obtained in [9] for potential \( q \in C^{2+\alpha}(\overline{\Omega}) \), and in [13], the regularity assumption was improved to \( C^{\alpha}(\overline{\Omega}) \) in the case of the full Cauchy data and up to \( W_p^1(\Omega) \) with \( p > 2 \) in the case of partial Cauchy data on arbitrary subboundary. The case of general second-order elliptic equation was studied in the papers [12] and [10]. The results of [9] were extended to a Riemannian surface in [7] and system of linear equations in [14]. The case where voltages are applied and currents are measured on disjoint subboundaries was discussed and the uniqueness is proved in [11]. Conditional stability estimates in determining a potential are obtained in [22]. As for the cases of the dimensions \( \geq 3 \), we refer to [20] and the references therein.

Our main results establish the uniqueness in determining semilinear terms by partial Cauchy data on arbitrary subboundary, and to our best knowledge, there are no publications in this case. On the other hand, we can refer to several works on nonlinear elliptic equations by not arbitrary subboundary as follows. The uniqueness results for recovery of the nonlinear term in the semilinear elliptic equation were first obtained for the full Cauchy data in three dimensional case by Isakov and Sylvester in [18] and in two dimensional case by Isakov and Nachman in [17]. It should be mentioned that the proof of the analog of Theorem 1 in those papers requires the uniqueness of solution for the Dirichlet boundary problem for the operator \( P(x,D) \). Later this result was expanded to the case of a system of semilinear elliptic equations by Isakov in [16]. Also see Kang and Nakamura [19] for determination of coefficients of the linear and the quadratic nonlinear terms in the principal part of a quasilinear elliptic equation. In a special case where a nonlinear term is independent of \( x \), the uniqueness was proved in determining such a nonlinear term from partial Cauchy data [15]. Moreover we note that in [16] and [18], the monotonicity of \( f(x,u) \) with respect to \( u \) is assumed. In general, if a nonlinear term depends on \( x \), \( u \) and the gradient of \( u \), then it is impossible to prove the uniqueness even for the linear case. This can be seen by [13] if we consider the term \(-f(x,u,\nabla u) = A(x) \cdot \nabla u + q(x)u\).

The paper is composed of four sections. In section 2, we prove Theorem 2 provided that Theorem 1 is proved. Sections 3 and 4 are devoted to the proof of Theorem 1.

## 2 Proof of Theorem 2

Henceforth let \( \partial_\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \), \( \beta = (\beta_1, \beta_2) \in (\mathbb{N} \cup \{0\})^2 \) and \( |\beta| = \beta_1 + \beta_2 \). We set \( P_k(x,D)u = \Delta u + q(x)u - f_k(x,u), \ k = 1,2, \) and \( u_{1,t}(x) = u(x,t) \). Let \( u_{2,t} \in H^1(\Omega), \ t \in [0,1] \) satisfy

\[
P_2(x,D)u_{2,t} = 0 \quad \text{in } \Omega, \quad u_{2,t} = u_{1,t} \quad \text{on } \partial \Omega, \quad \forall t \in [0,1].
\]

Then \( C_{q,f_1} = C_{q,f_2} \) yields

\[
\left( \frac{\partial u_{1,t}}{\partial \nu} - \frac{\partial u_{2,t}}{\partial \nu} \right) |_{\Gamma} = 0, \quad \forall t \in [0,1].
\]

By [4] and the Sobolev embedding theorem, \( f_2(\cdot, u_{2,t}(\cdot)) \in L^\kappa(\Omega) \) for any \( \kappa > 1 \). The standard solvability theory for the Dirichlet boundary value problem for the Laplace operator in Sobolev spaces implies \( u_{2,t} \in H^2(\Omega) \). Hence \( f_2(\cdot, u_{2,t}(\cdot)) \in C^{\kappa}(\Omega) \) for any \( \alpha \in (0,1) \). Then, since \( u_{2,t} \in C^{2+\alpha}(\partial \Omega) \), the solvability theory for the Dirichlet boundary value problem for the Laplace operator in Hölder spaces implies \( u_{2,t} \in C^{2+\alpha}(\overline{\Omega}) \).

By the assumption, there exists a constant \( K > 0 \) such that

\[
\sup_{t \in [0,1]} ||u_{1,t}||_{C^0(\overline{\Omega})} \leq K.
\]

We claim that

\[
u_{1,t} \equiv u_{2,t}, \quad \forall t \in [0,1].
\]
Our proof is by contradiction. Suppose that for some \( t_0 \in (0,1] \), this equality fails. Let \( t_* \) be the infimum over such \( t_0 \).

Setting \( u_t = u_{2,t} - u_{1,t} \), we have

\[
\Delta u_t - q(t,x)u_t = f_1(x,u_{1,t}) - f_2(x,u_{1,t}) \quad \text{in } \Omega, \quad u_t|_{\partial\Omega} = 0, \quad \frac{\partial u_t}{\partial \nu}|_{\tilde{\Gamma}} = 0, \tag{7}
\]

where \( q(t,x) = -q(x) + \int_0^1 \frac{\partial f_2}{\partial y}(x,(1-s)u_{2,t}(x)+su_{1,t}(x))ds \). To this equation, applying a Carleman estimate with boundary term (see e.g., [8]), we can choose some function \( \phi \in C^2(\Omega) \) such that

\[
\|e^{\tau \phi}u_t\|_{H^2(\Omega)} \leq C \tau^{-2} \|e^{\tau \phi}(f_1(\cdot,u_1) - f_2(\cdot,u_1))\|_{L^2(\Omega)}, \quad \forall \tau \geq \tau_0.
\]

Here \( \|v\|_{H^2(\Omega)} = \left( \sum_{|\beta| \leq 2} \tau^{-2-2|\beta|}\|v\|^2_{H^1(|\beta|)}(\Omega) \right)^{\frac{1}{2}} \). That is, fixing a large \( \tau > 0 \) arbitrarily,

\[
\|u_t\|_{H^2(\Omega)} \leq C\|f_1(\cdot,u_1) - f_2(\cdot,u_1)\|_{L^2(\Omega)}, \quad \forall t \in [0,1],
\]

where a constant \( C > 0 \) depends on fixed \( \tau \).

Consider the boundary value problem

\[
\Delta v_{k,t} + q(x)v_{k,t} - \frac{\partial f_k}{\partial y}(x,u_{k,t})v_{k,t} - \tilde{f}_k(x,v_{k,t}) = \Delta v_{k,t} + q(x)v_{k,t} - f_k(x,v_{k,t}) + f_k(x,u_{k,t}) = 0 \quad \text{in } \Omega, \quad v_{k,t}|_{\partial\Omega} = 0,
\]

where \( \tilde{f}_k(x,w) = f_k(x,w+u_{k,t}) - f_k(x,u_{k,t}) - \frac{\partial f_k}{\partial y}(x,u_{k,t})w \). Obviously the functions \( \tilde{f}_k \) satisfy [2], [3] and [4]. Moreover

\[
C_{q-\frac{\partial f_2}{\partial y}(x,u_{1,t}),\tilde{f}_1} = C_{q-\frac{\partial f_2}{\partial y}(x,u_{2,t}),\tilde{f}_2}. \quad \text{Indeed let } (w_1,w_2) \in C_{q-\frac{\partial f_2}{\partial y}(x,u_{1,t}),\tilde{f}_1}.
\]

On the one hand, the function \( w + u_{1,t} \) solves the boundary value problem

\[
\Delta w + qw - \frac{\partial f_1}{\partial y}(x,u_{1,t})w - \tilde{f}_1(x,w) = 0 \quad \text{in } \Omega, \quad w|_{\tilde{\Gamma}} = 0, \quad w|_{\tilde{\Gamma}} = w_1.
\]

such that \( \frac{\partial w}{\partial \nu}|_{\tilde{\Gamma}} = w_2 \).

Let \( \tilde{u} \) satisfy

\[
\Delta \tilde{u} + q\tilde{u} - f_2(x,\tilde{u}) = 0 \quad \text{in } \Omega, \quad \tilde{u}|_{\partial\Omega} = 0
\]

and

\[
\tilde{u} = w + u_{1,t} \quad \text{on } \tilde{\Gamma}.
\]

Then, by assumption \( C_{q,f_1} = C_{q,f_2} \), we have

\[
\frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial (w + u_{1,t})}{\partial \nu} \quad \text{on } \tilde{\Gamma}.
\]

Setting \( \tilde{w} = \tilde{u} - u_{2,t} \), we obtain

\[
\Delta \tilde{w} + q\tilde{w} - \frac{\partial f_2}{\partial y}(x,u_{2,t})\tilde{w} - \tilde{f}_2(x,\tilde{w}) = 0 \quad \text{in } \Omega, \quad \tilde{w}|_{\partial\Omega} = 0.
\]

Then on \( \tilde{\Gamma} \) we have

\[
\tilde{w} - w = (\tilde{u} - u_{2,t}) - (\tilde{u} - u_{1,t}) = u_{1,t} - u_{2,t} = 0.
\]
and
\[
\frac{\partial \bar{w}}{\partial \nu} - \frac{\partial w}{\partial \nu} = \frac{\partial \bar{u}}{\partial \nu} - \frac{\partial u_{2,t}}{\partial \nu} = 0.
\]
Therefore \( \bar{w} = w_1 \) and \( \frac{\partial \bar{w}}{\partial \nu} = w_2 \) on \( \tilde{\Gamma} \). Hence \( (w_1, w_2) \in C_q - \frac{\partial f_2}{\partial y}(x, u_{2,t}), f_2 \). Since the reverse inclusion can be proved similarly, we have proved
\[
C_q - \frac{\partial f_1}{\partial y}(x, u_{1,t}), f_1 = C_q - \frac{\partial f_2}{\partial y}(x, u_{2,t}), f_2.
\]

Therefore we can apply Theorem 1 to this equation. Hence we have the uniqueness for the potential, that is,
\[
\frac{\partial f_1}{\partial y}(x, u_{1,t}) = \frac{\partial f_2}{\partial y}(x, u_{2,t}) \quad \text{in} \quad \Omega, \quad \forall t \in [0, 1].
\] (9)

Denote \( \gamma(t) = \|u_{1,t} - u_{1,t}\|_{C^0(\Omega)} + \|u_{2,t} - u_{2,t}\|_{C^0(\Omega)} \). Since \( u_{1,t} = u_{2,t} \) in \( \Omega \), we have \( f_1(x, u_{1,t}) = \Delta u_{1,t} = \Delta u_{2,t} = f_2(x, u_{1,t}) \) in \( \Omega \). Therefore
\[
f_1(x, u_{1,t}(x)) - f_2(x, u_{1,t}(x)) = \int_{u_{1,t}(x)}^{u_{1,t}(x)} \left( \frac{\partial f_1}{\partial y}(x, s) - \frac{\partial f_2}{\partial y}(x, s) \right) ds.
\]
If \( s \in (u_{1,t}(x), u_{1,t}(x)) \), then, by the continuity of \( u_{1,t}(x) \) with respect to \( t \) and the intermediate value theorem, there exists \( t_0(s, x) \in [0, t] \) such that \( s = u_{1,t_0(s,x)}(x) \). Hence
\[
f_1(x, u_{1,t}(x)) - f_2(x, u_{1,t}(x)) = \int_{u_{1,t_0(s,x)}(x)}^{u_{1,t}(x)} \left( \frac{\partial f_1}{\partial y}(x, u_{1,t_0(s,x)}(x)) - \frac{\partial f_2}{\partial y}(x, u_{1,t_0(s,x)}(x)) \right) ds.
\]
Applying (10) and (5), we have
\[
f_1(x, u_{1,t}(x)) - f_2(x, u_{1,t}(x)) = \int_{u_{1,t_0(s,x)}(x)}^{u_{1,t}(x)} \left( \frac{\partial f_2}{\partial y}(x, u_{2,t_0(s,x)}(x)) - \frac{\partial f_2}{\partial y}(x, u_{1,t_0(s,x)}(x)) \right) ds
\]
\leq \left\| \frac{\partial^2 f_2}{\partial y^2} \right\|_{C^0(\Omega \times [-K, K])} \sup_{t \in (0, t)} |(u_{1,\tilde{t}} - u_{2,\tilde{t}})(x)| \gamma(t)
\leq \left\| \frac{\partial^2 f_2}{\partial y^2} \right\|_{C^0(\Omega \times [-K, K])} \sup_{t \in (0, t)} |(u_{1,\tilde{t}} - u_{2,\tilde{t}})(x)| \gamma(t). \] (11)

In order to obtain the last inequality, we used the fact that \( u_{1,\tilde{t}} - u_{2,\tilde{t}} = 0 \) for all \( \tilde{t} \) from \( [0, t_*] \). Therefore inequality (11) implies
\[
\sup_{t \in (t_*, t)} \| f_1(x, u_{1,t}) - f_2(x, u_{1,t}) \|_{L^2(\Omega)} \leq C \gamma(t) \sup_{t \in (t_*, t)} \| u_{1,\tilde{t}} - u_{2,\tilde{t}} \|_{L^2(\Omega)}.
\] (11)

From (9) and (11), we obtain
\[
\| u_t \|_{H^2(\Omega)} \leq C \gamma(t) \sup_{t \in (t_*, t)} \| u_{1,\tilde{t}} - u_{2,\tilde{t}} \|_{L^2(\Omega)}, \quad \tilde{t} \in (t_*, t).
\]
This implies that
\[
\sup_{t \in (t_*, t)} \| u_{\tilde{t}} \|_{H^2(\Omega)} \leq C \gamma(t) \sup_{t \in (t_*, t)} \| u_{\tilde{t}} \|_{L^2(\Omega)}.
\] (12)

From (12) and the fact that \( \gamma(t) \) goes to zero as \( t \to t_* \), we obtain that there exists \( \dot{t} > t_* \) such that \( u_{1,\dot{t}} = u_{2,\dot{t}} \) for all \( t \) from \( (t_*, \dot{t}) \). We arrive at the contradiction. Equality (9) is proved and the statement of the theorem follows from it and (11). ■
3 Preliminaries for the proof of Theorem 1

Henceforth we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$, $\partial z = \frac{1}{2} (\partial x_1 - i \partial x_2)$, $\partial \overline{z} = \frac{1}{2} (\partial x_1 + i \partial x_2)$, $D = (\frac{1}{i} \frac{\partial}{\partial x_1}, \frac{1}{i} \frac{\partial}{\partial x_2})$. The tangential derivative on the boundary is given by $\partial \tau \gamma = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial \Omega$. We set $(u, v)_{L^2(\Omega)} = \int_{\Omega} u dx$ for functions $u, v$, while by $(a, b)$ we denote the scalar product in $\mathbb{R}^2$ if there is no fear of confusion. For $f : \mathbb{R}^2 \to \mathbb{R}^1$, the symbol $f''$ denotes the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space $X$ to another Banach space $Y$. Let $\| \cdot \|_X$ be the norm in a Banach space $X$. We set $\|u\|_{H^{k, \tau}(\Omega)} = (\|u\|^2_{H^k(\Omega)} + |\tau|^{2k} \|\psi\|^2_{L^2(\Omega)})^{1/2}$. By $o_X(\frac{1}{\tau})$ we denote a function $f(\tau, \cdot)$ such that $\|f(\tau, \cdot)\|_X = o(\frac{1}{\tau})$ as $|\tau| \to +\infty$.

Let $\Omega_*$ be a bounded domain in $\mathbb{R}^2$ such that $\Omega \subset \Omega_*$, $\Gamma_0 \subset \partial \Omega_*$ and $\Gamma \cap \partial \Omega_* = \emptyset$. For some $\alpha \in (0, 1)$, we consider a function $\Phi(z) = \varphi(x_1, x_2) + i \psi(x_1, x_2) \in C^{0+\alpha}(\Omega_*)$ with real-valued $\varphi$ and $\psi$ such that

$$\partial_z \Phi(z) = 0 \quad \text{in } \Omega_*, \quad \text{Im } \Phi|_{\Gamma_0} = 0. \tag{13}$$

Denote by $\mathcal{H}$ the set of all the critical points of the function $\Phi$:

$$\mathcal{H} = \{z \in \overline{\Omega}_* : \frac{\partial \Phi}{\partial z}(z) = 0\}.$$ 

Assume that $\Phi$ has no critical points on $\overline{\Gamma}$, and that all critical points are nondegenerate:

$$\mathcal{H} \cap \partial \Omega \subset \Gamma_0, \quad \partial^2 \Phi(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{14}$$

Then $\Phi$ has only a finite number of critical points and we can set:

$$\mathcal{H} \setminus \Gamma_0 = \{\bar{x}_1, \ldots, \bar{x}_\ell\}, \quad \mathcal{H} \cap \Gamma_0 = \{\bar{x}_{\ell+1}, \ldots, \bar{x}_{\ell+\nu}\}. \tag{15}$$

Let $\partial \Omega = \cup_{j=1}^N \gamma_j$, where $\gamma_j$ is a closed contour. The following proposition was proved in [9].

Proposition 1 Let $\bar{x}$ be an arbitrary point in $\Omega$. There exists a sequence of functions $\{\Phi_\epsilon\}_{\epsilon \in (0, 1)}$ satisfying (13), (14) and there exists a sequence $\{\bar{x}_\epsilon\}, \epsilon \in (0, 1)$ such that

$$\bar{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \overline{\Omega_\epsilon} : \frac{\partial \Phi_\epsilon}{\partial z}(z) = 0\}, \quad \bar{x}_\epsilon \to \bar{x} \quad \text{as } \epsilon \to +0.$$ 

Moreover for any $j$ from $\{1, \ldots, N\}$, we have

$$\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \text{if } \gamma_j \cap \overline{\Gamma} \neq \emptyset,$$

$$\mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \text{if } \gamma_j \cap \overline{\Gamma} = \emptyset$$

and

$$\text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \notin \{\text{Im } \Phi_\epsilon(x) : x \in \mathcal{H}_\epsilon \setminus \{\bar{x}_\epsilon\}\} \text{ and } \text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \neq 0.$$ 

Later we use the following proposition (see [9]):

Proposition 2 Let $\Phi$ satisfy (13) and (14). For every $g \in L^1(\Omega)$, we have

$$\int_{\Omega} g e^{\tau(\Phi - \overline{\Phi})} dx \to 0 \quad \text{as } \tau \to +\infty.$$
Consider the boundary value problem
\[ L(x, D)u = f \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0. \]

The following proposition is proved in [12].

**Proposition 3** Suppose that \( \Phi \) satisfies (13) and (14), \( u \in H^1_0(\Omega) \) and \( \|q\|_{L^\infty(\Omega)} \leq K \). Then there exist \( \tau_0 = \tau_0(K, \Phi) \) and \( C = C(K, \Phi) \), independent of \( u \) and \( \tau \), such that
\[
|\tau|\|e^{r\Phi}\|_{L^2(\Omega)}^2 + \|e^{r\Phi}\|_{H^1_0(\Omega)}^2 + \|\frac{\partial u}{\partial \nu}e^{r\Phi}\|_{L^2(\Gamma_0)}^2 + \tau^2\|\frac{\partial e^{r\Phi}}{\partial z}\|_{L^2(\Omega)}^2 \\
\leq C(\|L(x, D)u\|_{L^2(\Omega)} + \|\tau\|_{L^2(\Omega)} + \|\int_\Gamma |\frac{\partial u}{\partial \nu}|^2e^{2r\Phi}d\sigma\|) \quad \forall \tau > \tau_0. \tag{16}
\]

Using estimate (15), we obtain

**Proposition 4** Let \( \Phi \) satisfy (13) and (14). There exists a constant \( \tau_0 \) such that for \( |\tau| \geq \tau_0 \) and any \( f \in L^2(\Omega) \) and \( g \in H^{\frac{1}{2}}(\partial \Omega) \), there exists a solution to the boundary value problem:
\[
L(x, D + i\tau \nabla \varphi)u = f \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = g \tag{17}
\]
such that
\[
\|u\|_{H^{2,\tau}(\Omega)} \leq C(|\tau|^\frac{1}{4}\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_0)}). \tag{18}
\]

The constant \( C \) in (18) is independent of \( \tau \).

**Proof.** First we reduce the problem (17) to the case \( g = 0 \). Let \( r(z) \) be a holomorphic function and \( \bar{r}(\overline{\tau}) \) be an antiholomorphic function such that \( (r + \overline{r})|_{\Gamma_0} = g \). These functions can be chosen such that
\[
|\tau|\|e^{r\varphi}\|_{L^2(\Omega)} + \|\overline{e^{r\varphi}}\|_{L^2(\Omega)} \leq C_24\|g\|_{H^{\frac{1}{2}}(\Gamma_0)}.
\]

We look for a solution \( u \) in the form
\[
u = (e^{r\varphi} + e^{-r\varphi}) + \bar{u},
\]
where
\[
L(x, D + i\tau \nabla \varphi)\bar{u} = \bar{f} \quad \text{in} \quad \Omega_*, \quad \bar{u}|_{\Gamma_0} = 0 \tag{19}
\]
and \( \bar{f} = f - qre^{r\varphi} - q\overline{e}^{-r\varphi} \) is extended by zero on \( \Omega_* \setminus \Omega \). By Proposition 2.1 of [9] there exists a solution to the problem (19) such that
\[
|\tau|^\frac{1}{4}\|\bar{u}\|_{L^2(\Omega_*)} \leq C\|\bar{f}\|_{L^2(\Omega)}. \tag{20}
\]
Obviously the restriction of the function \( \bar{u} \) on \( \Omega \) satisfies the estimate
\[
\|\bar{u}\|_{H^{2,\tau}(\Omega)} \leq C|\tau|^\frac{1}{4}\|\bar{f}\|_{L^2(\Omega)}. \tag{21}
\]

The proof of the proposition is finished. \( \blacksquare \)

Let us introduce the operators:
\[
\partial_z^{-1}g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\zeta - z}d\xi_1d\xi_2, \quad \partial_{\bar{z}}^{-1}g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\zeta - \overline{z}}d\xi_1d\xi_2
\]
and
\[
\operatorname{Re}_\tau = \frac{1}{2}e^{\tau \varphi} \partial_z^{-1}e^{\varphi} - \varphi, \quad \operatorname{Re}_\tau = \frac{1}{2}e^{\tau \varphi} \partial_z^{-1}e^{\tau \varphi}. \tag{22}
\]

Then we have (e.g., p.47, 56, 72 in [23]):

**Proposition 5** A) Let \( m \geq 0 \) be an integer number and \( \alpha \in (0, 1) \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in L(C^{m+\alpha}(\Omega), C^{m+\alpha+1}(\Omega)) \).

B) Let \( 1 \leq p \leq 2 \) and \( 1 < \gamma < \frac{2p}{2p-1} \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in L(L^p(\Omega), L^\gamma(\Omega)) \).

C) Let \( 1 < p < \infty \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in L(L^p(\Omega), W_p^1(\Omega)) \).
4 Proof of Theorem 1

Let the function $\Phi$ satisfy (13) and (14), and $\bar{x}$ be some point from $\mathcal{H}\setminus\Gamma_0$. Without loss of generality, adding to the function $\Phi$ a suitable negative constant, we can always assume that

$$\max_{x \in \Omega} \varphi(x) < 0. \quad (22)$$

Let $a \in C^{\beta + \alpha}(\overline{\Omega})$ be a holomorphic function such that

$$\frac{\partial a}{\partial \tau} = 0 \quad \text{in} \quad \Omega, \quad \text{Re} \ a|_{\Gamma_0} = 0. \quad (23)$$

By Proposition 4.2 of [12], there exists a holomorphic function $a_0(z) \in C^\infty(\overline{\Omega})$ such that $Im \ a_0|_{\Gamma_0} = 0$, $a_0(\bar{x}) = 1$ and $a_0$ vanishes at each point of the set $\mathcal{H}\setminus\{\bar{x}\}$. Then, choosing $\ell_0 \in \mathbb{N}$ large, we see that $a = a_0^\ell$ is holomorphic with the following properties:

$$a(\bar{x}) = 1, \quad \partial_{x\bar{x}}^2 a(x) = 0 \quad \forall x \in \mathcal{H}\setminus\{\bar{x}\} \quad \text{and} \quad \forall \alpha_1 + \alpha_2 \leq 6. \quad (24)$$

For example, we can choose $\ell_0 = 100$ and fix. Short computations yield

$$L_1(x, D)(ae^{\tau \Phi}) = q_1 ae^{\tau \Phi}, \quad L_1(x, D)(\overline{ae^{\tau \Phi}}) = q_1 \overline{ae^{\tau \Phi}}. \quad (25)$$

Let $e_1, e_2$ be smooth functions such that

$$e_1 + e_2 = 1 \quad \text{on} \quad \Omega, \quad (26)$$

and $e_1$ vanishes in a neighborhood of $\partial \Omega$ and $e_2$ vanishes in a neighborhood of the set $\mathcal{H}\setminus\Gamma_0$.

We have

**Proposition 6** Let $q \in C^{2 + \alpha}(\overline{\Omega})$ for some positive $\alpha$ and $\bar{q} \in W^1_p(\overline{\Omega})$ for some $p > 2$. Suppose that $q|_{\mathcal{H}} = \overline{q}|_{\mathcal{H}} = 0$. There exists smooth function $m_+ \in C^2(\partial \Omega)$, which is independent of $\tau$, such that the asymptotic formulae hold true:

$$\overline{\mathcal{R}_+}(e_1(q + \frac{\bar{q}}{\tau}))|_{\partial \Omega} = e^{\tau(\overline{\Phi} - \Phi)} \left( \frac{m_+ e^{2i\tau \psi(x)}}{\tau^2} + o_{C^2(\partial \Omega)}(\frac{1}{\tau^2}) \right) \quad \text{as} \quad |\tau| \to +\infty, \quad (27)$$

$$\mathcal{R}_+(e_1(q + \frac{\bar{q}}{\tau}))|_{\partial \Omega} = e^{\tau(\Phi - \overline{\Phi})} \left( \frac{m_+ e^{-2i\tau \psi(x)}}{\tau^2} + o_{C^2(\partial \Omega)}(\frac{1}{\tau^2}) \right) \quad \text{as} \quad |\tau| \to +\infty. \quad (28)$$

and

$$\|\overline{\mathcal{R}_+}(e_1(q + \frac{\bar{q}}{\tau})) - \frac{e_1 q}{2\tau \partial_\Phi} \|_{L^2(\Omega)} + \|\mathcal{R}_+(e_1(q + \frac{\bar{q}}{\tau})) - \frac{e_1 q}{2\tau \partial_{\overline{\Phi}}} \|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad |\tau| \to \infty. \quad (29)$$

**Proof.** Since $\text{supp} \ e_1 \bar{q} \subset \subset \Omega$ by (26), the functions $\tau e^{\tau(\Phi - \overline{\Phi})}\overline{\mathcal{R}_+}(e_1 \bar{q})$ are uniformly bounded in $C^k(\partial \Omega)$ for any positive integer $k$. By Proposition 4 of [13], the functions $\tau e^{\tau(\Phi - \overline{\Phi})}\overline{\mathcal{R}_+}(e_1 \bar{q})$ converges to zero pointwise on $\partial \Omega$ as $\tau$ approaches to infinity. Therefore

$$\overline{\mathcal{R}_+}(e_1(q + \frac{\bar{q}}{\tau}))|_{\partial \Omega} = e^{\tau(\Phi - \overline{\Phi})} o_{C^2(\partial \Omega)}(\frac{1}{\tau^2}) \quad \text{as} \quad |\tau| \to +\infty. \quad (30)$$

We set $q_* = \sum_{k=1}^\ell e(x - \bar{x}_k)((\nabla q(\bar{x}_k), x - \bar{x}_k) + \frac{1}{2}(q''(\bar{x}_k)(x - \bar{x}_k), (x - \bar{x}_k)))$, where $e$ is a smooth function such that the support is located in a small ball centered at the origin and $e$ is equal to one in some neighborhood of the origin. Integrating by parts, we have

$$\partial_{x_1}^\beta_1 \partial_{x_2}^\beta_2 \overline{\mathcal{R}_+}(e_1(q - q_*))|_{\partial \Omega} = \frac{1}{\pi \tau} \int_\Omega \text{div}(\partial_{x_1}^\beta_1 \partial_{x_2}^\beta_2 \frac{e_1(q - q_*)}{(\zeta - z)^2} \nabla \psi) e^{2i\tau \psi} d\xi|_{\partial \Omega}, \quad \forall \beta_1 + \beta_2 \leq 5.$$
Since \( \text{div}(\partial_{x_1}^2 \partial_{x_2}^2 e^{i(\frac{q}{\tau} + \frac{\alpha}{2})} \nabla \psi) \) belongs to the space \( W^1_p(\Omega) \) with \( p(\alpha) > 2 \), by Proposition \( 4 \) of [14] this integral converges to zero. Then from the stationary phase argument, we obtain

\[
\mathcal{R}_\tau(e_1q)|_{\partial \Omega} = e^{\tau(\Phi - \varphi)} \left( \frac{m_+ e^{2i \tau \psi(\bar{z})}}{\tau^2} + o_{C^2(\partial \Omega)}(\frac{1}{\tau^2}) \right) \quad \text{as } |\tau| \to +\infty.
\]

Therefore the equality (27) is proved. The asymptotic formula (28) follows from (27) and the equality \( \mathcal{R}_\tau(e_1(\bar{q} + \frac{x}{\tau})) = \mathcal{R}_\tau(e_1(q + \frac{\bar{x}}{\tau})) \). In order to prove (29), observe that by Proposition \( 2 \) the functions \( e^{2i \tau \psi} \mathcal{R}_\tau(e_1 \bar{z}) \) and \( e^{-2i \tau \psi} \mathcal{R}_\tau(e_1 \bar{z}) \) converge pointwise to zero and by Proposition \( 5 \) they are bounded in \( H^\alpha(\Omega) \) with some positive \( \alpha \). Thus they converge to zero in \( L^2(\Omega) \) as \( \tau \) goes to infinity. Applying Proposition 3.4 from [12], we finish the proof of (29). □

Denote \( p_1 = \frac{1}{2} \partial_{\bar{z}} \frac{1}{2}(q_1 a) - M(z) \in C^{3+\alpha}(\bar{\Omega}) \), where the function \( M \) is the polynomial such that

\[
p_1(\bar{x}) = 0, \quad \partial_{x_1} \partial_{x_2} p_1(x) = 0 \quad \text{for all } x \in \mathcal{H} \setminus \{\bar{x}\} \text{ and } \forall \alpha_1 + \alpha_2 \leq 3. \quad (31)
\]

Next we introduce holomorphic functions \( a_{-1}, a_+ \in C^2(\bar{\Omega}) \) as follows:

\[
(a_{-1} + \bar{\alpha}_{-1})|_{\partial \Omega} = \text{Re}\{\frac{p_1}{\partial_{\bar{z}} \Phi}\},
\]

\[
\partial_{x_1} \partial_{x_2} a_{-1}(x) = 0 \quad \text{for all } x \in \mathcal{H} \text{ and } \forall \alpha_1 + \alpha_2 \leq 2,
\]

\[
(a_+ + \bar{\alpha}_+)|_{\partial \Omega} = m_+ .
\]

(32)

(33)

We set \( \bar{\rho}_1 = -q_1(\frac{\bar{e}_1 p_1}{2\partial_{\bar{z}} \Phi} + a_{-1}) + L_1(x, D)(\frac{\bar{e}_1 p_1}{2\partial_{\bar{z}} \Phi}) \) and \( \bar{\rho}_1 = \frac{1}{2} \partial_{\bar{z}} \partial_{\bar{z}} \bar{\rho}_1 - \bar{M}(z) \), where \( \bar{M} \) is a polynomial such that

\[
\bar{\rho}_1(\bar{x}) = 0, \quad \partial_{x_1} \partial_{x_2} \bar{\rho}_1(x) = 0 \quad \text{for all } x \in \mathcal{H} \setminus \{\bar{x}\} \text{ and } \forall \alpha_1 + \alpha_2 \leq 3. \quad (34)
\]

Since \( \frac{\bar{e}_1 p_1}{2\partial_{\bar{z}} \Phi} \in H^1(\partial \Omega) \) by (26) and (31), there exists a holomorphic function \( a_{-2} \in H^\frac{3}{2}(\bar{\Omega}) \) such that

\[
(a_{-2} + \bar{\alpha}_{-2})|_{\partial \Omega} = \text{Re}\{\frac{\bar{\rho}_1}{\partial_{\bar{z}} \Phi}\} .
\]

(35)

By Proposition \( 6 \) there exists a function \( m_+ \in C^2(\partial \Omega) \) such that

\[
\mathcal{R}_\tau(e_1(\bar{p}_1 + \bar{\bar{\rho}}_1)) = e^{\tau(\Phi - \varphi)} \left( \frac{m_+ e^{2i \tau \psi(\bar{x})}}{\tau^2} + o_{C^2(\partial \Omega)}(\frac{1}{\tau^2}) \right) \quad \text{as } |\tau| \to +\infty .
\]

(36)

and

\[
\mathcal{R}_\tau(e_1(\bar{p}_1 + \bar{\bar{\rho}}_1)) = e^{\tau(\Phi - \varphi)} \left( \frac{m_+ e^{2i \tau \psi(\bar{x})}}{\tau^2} + o_{C^2(\partial \Omega)}(\frac{1}{\tau^2}) \right) \quad \text{as } |\tau| \to +\infty .
\]

(37)

We introduce the function \( a_\tau \in H^1(\Omega) \) by

\[
a_\tau = a_{-1} - e_2 p_1/2 \partial_{\bar{z}} \Phi + \frac{1}{\tau^2} \left( e^{2i \tau \psi(\bar{x})} a_+ + e^{-2i \tau \psi(\bar{x})} a_{-2} - \bar{\rho}_1 e_2/2 \partial_{\bar{z}} \Phi \right) .
\]

(38)

Using this formula, we prove the following proposition.

**Proposition 7** The asymptotic formulae are true:

\[
L_1(x, D)(a_\tau e^{\Phi} + \bar{a}_\tau e^{\bar{\Phi}} - e^{\Phi} \mathcal{R}_\tau(e_1(\bar{p}_1 + \bar{\bar{\rho}}_1)) - e^{\bar{\Phi}} \mathcal{R}_\tau(e_1(\bar{p}_1 + \bar{\bar{\rho}}_1))) = e^{\tau\phi} o_{L^2(\Omega)}(\frac{1}{\tau^2}),
\]

(39)

\[
(a_\tau e^{\Phi} + \bar{a}_\tau e^{\bar{\Phi}} - e^{\Phi} \mathcal{R}_\tau(e_1(\bar{p}_1 + \bar{\bar{\rho}}_1)) - e^{\bar{\Phi}} \mathcal{R}_\tau(e_1(\bar{p}_1 + \bar{\bar{\rho}}_1)))|_{\partial \Omega} = e^{\tau\phi} o_{H^1(\Omega)}(\frac{1}{\tau^2}).
\]

(40)
Proof. By (13) and (32)-(35), we have
\[
( \alpha \tau e^{r \Phi} + \overline{\alpha} e^{\overline{r} \Phi} - e^{r \Phi} \overline{R}_\tau c_1(q_1 + \overline{p}_1/\tau) - e^{\overline{r} \Phi} R_\tau c_1(\overline{p}_1 + \overline{p}_1/\tau) ) |_{\Gamma_0}
\]
\[
= ( \alpha \tau e^{r \Phi} + \overline{\alpha} e^{\overline{r} \Phi} - e^{r \Phi} \overline{R}_\tau c_1(p_1 + \overline{p}_1/\tau) - e^{\overline{r} \Phi} R_\tau c_1(\overline{p}_1 + \overline{p}_1/\tau) ) |_{\Gamma_0}
\]
\[
= e^{r \Phi} \left( a + \frac{a_1 - e^{2p_1} \overline{2}\partial \Phi}{\tau} + \frac{1}{\tau} (2 \overline{r} e \overline{2} \tau \overline{2} \Phi) + e^{2 \overline{2} i \Phi} (2 \overline{2} \tau \overline{2} \Phi) a_1 \right) + \frac{1}{\tau} \left( e^{2 \overline{2} i \Phi} a_1 + e^{-2 \overline{2} i \Phi} \overline{a}_1 + a - \overline{p}_1 e_2 \overline{2}\partial \Phi \right)
\]
\[
+ \overline{a} + \frac{1}{\tau} \left( e^{2 \overline{2} i \Phi} a_1 + e^{-2 \overline{2} i \Phi} \overline{a}_1 + a - \overline{p}_1 e_2 \overline{2}\partial \Phi \right)
\]
Here in order to obtain the final equality, we used (54) and (55). Proposition 6 and simple computations imply the asymptotic formula:
\[
L_1(x, D)(-e^{r \Phi} \overline{R}_\tau c_1(p_1 + \overline{p}_1/\tau) - \frac{e_2 (p_1 + \overline{p}_1/\tau) e^{r \Phi}}{2 \tau \partial \Phi} - e^{r \Phi} \overline{R}_\tau c_1(\overline{p}_1 + \overline{p}_1/\tau))
\]
\[
- \frac{e_2 (\overline{p}_1 + \overline{p}_1/\tau) e^{r \Phi}}{2 \tau \partial \Phi}
\]
\[
- \frac{e_2 (\overline{p}_1 + \overline{p}_1/\tau) e^{r \Phi}}{2 \tau \partial \Phi}
\]
\[
- e^{r \Phi} L_1(x, D)(\frac{e_2 (p_1 + \overline{p}_1/\tau)}{2 \tau \partial \Phi}) - e^{\overline{r} \Phi} L_1(x, D)(\frac{e_2 (p_1 + \overline{p}_1/\tau)}{2 \tau \partial \Phi})
\]
\[
- e^{r \Phi} \left( \frac{e_1 p_1}{2 \tau \partial \Phi} + \frac{L_1(x, D)(\overline{e_2 p_1}) e^{r \Phi}}{2 \tau \partial \Phi} \right) + \frac{1}{\tau} \left( \frac{e_1 p_1}{2 \tau \partial \Phi} + \frac{L_1(x, D)(\overline{e_2 p_1}) e^{r \Phi}}{2 \tau \partial \Phi} \right)
\]
\[
= - \frac{1}{\tau} q_1 a x e^{r \Phi} - \frac{1}{\tau} q_1 a e^{r \Phi} - q_1 e^{r \Phi} \frac{e_2 \overline{p}_1}{2 \tau \partial \Phi} - q_1 e^{r \Phi} \frac{e_1 p_1}{2 \tau \partial \Phi}
\]
Similarly to (26), we obtain
\[
L_1(x, D)(a_1 e^{r \Phi} + \overline{a}_1 e^{\overline{r} \Phi} - \frac{e_2 (p_1 + \overline{p}_1/\tau) e^{r \Phi}}{2 \tau \partial \Phi} - \frac{e_2 (\overline{p}_1 + \overline{p}_1/\tau) e^{r \Phi}}{2 \tau \partial \Phi})
\]
\[
= q_1 (a - \frac{e_2 (p_1 + \overline{p}_1/\tau)}{2 \tau \partial \Phi} e^{r \Phi} + q_1 (\overline{a}_1 - \frac{e_2 (\overline{p}_1 + \overline{p}_1/\tau)}{2 \tau \partial \Phi} e^{r \Phi})
\]
By (42) and (41), we obtain (39). ■

Using Propositions 4 and 7, we construct the last term $u_{-1}$ in the complex geometric optics solution which satisfies
\[
\| u_{-1} \|_{H^{2+\gamma}(\Omega)}/|\tau|^{\frac{3}{2}} = o(\frac{1}{|\tau|}) \quad \text{as } |\tau| \to +\infty.
\]
Finally we obtain a complex geometric optics solution for the linear operator $L_1(x, D)$ in the form:
\[
u_{1, \gamma}(x) = a_1 e^{r \Phi} + \overline{a}_1 e^{r \Phi} - e^{r \Phi} \overline{R}_\tau (p_1 + \overline{p}_1/\tau) - e^{\overline{r} \Phi} \overline{R}_\tau (\overline{p}_1 + \overline{p}_1/\tau) + e^{r \Phi} u_{-1}.
\]
Obviously
\[ L_1(x, D)u_{1,*} = 0 \quad \text{in } \Omega, \quad u_{1,*}\mid_{\Gamma_0} = 0. \] (45)

Thanks to (3), there exist positive constants \( C \) and \( \kappa \), both independent of \( \tau \), such that
\[ \| e^{-\tau \varphi} P_1(x, D) u_{1,*} \|_{L^2(\Omega)} \leq C e^{-\kappa \tau}. \] (46)

We finish the construction of the complex geometric optics solution for the semilinear elliptic equation \( P_1(x, D) = L_1(x, D) - f_1 \) using the Newton-Kantorovich iteration scheme. More precisely we use the Theorem 6 (1.XVIII) from \[ \text{[I]} \] p.708.

Following the notations of \( \text{[I]} \), we set \( x_0 = 0, X = \{ u \in H^2 \tau(\Omega), u\mid_{\Gamma_0} = 0 \}, \| \cdot \|_X = \| \cdot \|_{H^2 \tau(\Omega)} \) and \( P = I + L_1(x, D + i\tau \nabla \varphi)^{-1}e^{-\tau \varphi}f(x, e^{\tau \varphi} \circ) \). Here by \( L_1(x, D + i\tau \nabla \varphi)^{-1} \) we mean the operator from \( L^2(\Omega) \) into the orthogonal complement of \( \text{Ker} L_1(x, D + i\tau \nabla \varphi) \) in \( X \), and \( I \) is the identity operator. The mapping \( P \) is twice continuously differentiable as the mapping from \( X \) into \( X \). By Proposition \( \text{[I]} \) we have
\[ \| \Gamma_0 \|_{L(X; X)} \leq C \tau^2. \]

From this inequality and (16), we have
\[ \| \Gamma_0 P(x_0) \| \leq C \tau^2 e^{-\kappa \tau} = \eta(\tau). \]

We set \( \Omega_0 = \{ x \mid \| x - x_0 \| \leq r_0 \} \). By (1), we have
\[ \| \Gamma_0 P(x) \| \leq C \tau^2 = K(\tau). \]

Then \( h = K(\tau) \eta \leq \tau^4 e^{-\kappa \tau} \) and \( r_0 = \frac{1 - \sqrt{1 - \frac{4}{\kappa}}}{4} \eta \leq 2\tau^2 e^{-\kappa \tau} < \frac{1}{2} \) for all sufficiently large \( \tau \). Then there exists a solution \( x_* \) to the equation \( P(x) = 0 \) such that \( \| x_* \| \leq r_0 \).

Let \( u_1 \) be a complex geometrical optics solution to the semilinear equation \( P_1(x, D) \) of the form:
\[ u = u_{1,*} + e^{\tau \varphi} u_{\text{cor}}, \quad \| u_{\text{cor}} \|_{H^2 \tau(\Omega)} = O\left( \frac{1}{\tau} \right) \quad \text{as } \tau \to +\infty. \] (47)

Similarly we construct the complex geometric optics solutions to the operator \( L_2(x, D) \):
\[ v(x) = \bar{a}_\tau e^{-\tau \varphi} + \bar{a}_\tau e^{-\tau \varphi} - e^{-\tau \varphi} \bar{R}_{-\tau}(p_2 + \bar{p_2}/\tau) - e^{-\tau \varphi} \bar{R}_{-\tau}(\bar{p_2} + \bar{p_2}/\tau) + e^{-\tau \varphi} v_{-1}. \] (48)

Here the function \( \bar{a}_\tau \) is given by
\[ \bar{a}_\tau = a_\tau e^{-\tau \varphi} \frac{\partial_a \bar{q}_2}{\partial_\tau} - e^{-\tau \varphi} \bar{R}_{-\tau}(p_2 + \bar{p_2}/\tau) - e^{-\tau \varphi} \bar{R}_{-\tau}(\bar{p_2} + \bar{p_2}/\tau) + e^{-\tau \varphi} v_{-1}. \] (49)

where \( p_2 = \frac{1}{\tau} \partial_\tau a_\tau(q_2 a) - M_1(z) \in C^{3+\alpha}(\Omega) \) and the function \( M_1 \) is the polynomial such that
\[ p_2(\bar{x}) = 0, \quad \partial_\alpha \partial_{\bar{x}} \partial_{x_2} p_2(x) = 0 \quad \text{for } \forall x \in \mathcal{H} \setminus \{ \bar{x} \} \quad \text{and } \forall \alpha_1 + \alpha_2 \leq 3. \] (50)

The function \( \bar{a}_{-1} \in C^2(\Omega) \) is the holomorphic functions such that :
\[ (\bar{a}_{-1} + \bar{a}_{-1})\mid_{\Gamma_0} = \text{Re}\{ \frac{p_2}{\partial_\tau \Phi} \}, \] (51)
\[ \frac{\partial_{\bar{x}}}{\partial_\tau} \partial_{\bar{x}} \partial_{x_2} \bar{a}_{-1}(x) = 0 \quad \text{for } \forall x \in \mathcal{H} \quad \text{and } \forall \alpha_1 + \alpha_2 \leq 2. \]

We set \( \bar{p}_2 = \frac{1}{\tau} \partial_\tau \bar{p}_2 - \bar{M}_1(z) \) and \( \bar{p}_2 = -q_2(a \bar{p}_2 + a_{-1}) + L_2(x, D)(\frac{\partial_{\bar{x}}}{\partial_\tau \Phi} \bar{M}_1) \), where \( \bar{M}_1 \) is a polynomial such that
\[ \bar{p}_2(\bar{x}) = 0, \quad \partial_\alpha \partial_{\bar{x}} \partial_{x_2} \bar{p}_2(x) = 0 \quad \text{for } \forall x \in \mathcal{H} \setminus \{ \bar{x} \} \quad \text{and } \forall \alpha_1 + \alpha_2 \leq 2. \] (52)

Since \( \frac{\bar{p}_2}{\partial_\tau \Phi} \in H^1(\Omega) \) by (52), there exists a holomorphic function \( \bar{a}_{-2} \in H^1(\Omega) \) such that
\[ (\bar{a}_{-2} + \bar{a}_{-2})\mid_{\Gamma_0} = \text{Re}\{ \frac{\bar{p}_2}{\partial_\tau \Phi} \}. \] (53)
By Proposition 6 there exists a function \( m_\tau \in C^2(\partial \Omega) \) such that
\[
\tilde{\mathcal{R}}_\tau(e_1(p_2 + \frac{\bar{p}_2}{\tau})) = e^{\tau(\Phi - \Phi)} \left( \frac{m_\tau e^{2\tau\psi(\bar{x})}}{\tau^2} + o_{C^2(\partial \Omega)} \left( \frac{1}{\tau^2} \right) \right) \text{ as } |\tau| \to +\infty
\] (54)
and
\[
\mathcal{R}_\tau(e_1(p_2 + \frac{\bar{p}_2}{\tau})) = e^{\tau(\Phi - \Phi)} \left( \frac{m_\tau e^{2\tau\psi(\bar{x})}}{\tau^2} + o_{C^2(\partial \Omega)} \left( \frac{1}{\tau^2} \right) \right) \text{ as } |\tau| \to +\infty.
\] (55)

Next we introduce a holomorphic function \( a_- \in C^2(\overline{\Omega}) \) such that:
\[
(a_- + \bar{a}_-)|_{\Gamma_0} = m_-.
\] (56)

Obviously the function \( \tilde{a}_\tau \) belongs to \( H^1(\Omega) \). Using Proposition 4 we have
\[
L_2(x, D) \left( -e^{-\tau\Phi} \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau})) + \frac{e^{-\tau\Phi} e_2(p_2 + \frac{\bar{p}_2}{\tau})}{2\tau \partial_2 \Phi} \right)
- e^{-\tau\Phi} \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau})) + \frac{e^{-\tau\Phi} e_2(p_2 + \frac{\bar{p}_2}{\tau})}{2\tau \partial_2 \Phi}
= -L_2(x, D) \left( e^{-\tau\Phi} \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau})) - \frac{e^{-\tau\Phi} e_2(p_2 + \frac{\bar{p}_2}{\tau})}{2\tau \partial_2 \Phi} \right)

= -e^{-\tau\Phi} q_2 \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau})) - e^{-\tau\Phi} q_2 \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau}))

= -q_2(a_+ + \frac{\bar{a}_-}{\tau}) e^{-\tau\Phi} - q_2(\bar{a}_+ + \frac{\bar{a}_-}{\tau}) e^{-\tau\Phi} + e^{-\tau\Phi} o_{L^2(\Omega)} \left( \frac{1}{\tau} \right) \text{ as } |\tau| \to +\infty.
\] (57)

Setting \( v^* = \tilde{a}_\tau e^{-\tau\Phi} + \bar{a}_\tau e^{-\tau\Phi} - e^{-\tau\Phi} \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau})) - e^{-\tau\Phi} \tilde{\mathcal{R}}_{\tau-}(e_1(p_2 + \frac{\bar{p}_2}{\tau})) \), we obtain that
\[
L_2(x, D)v^* = e^{-\tau\Phi} o_{L^2(\Omega)} \left( \frac{1}{\tau} \right) \text{ in } \Omega, \quad v^*|_{\Gamma_0} = e^{-\tau\Phi} o_{H^1(\Gamma_0)} \left( \frac{1}{\tau} \right) \text{ as } |\tau| \to +\infty.
\]

Using (58) and Proposition 4 and 7 we construct the last term \( v_- \in H^2(\overline{\Omega}) \) in the complex geometric optics solution which solves the boundary value problem
\[
L_2(x, D)v_- = L_2(x, D)v^* \text{ in } \Omega, \quad v_-|_{\Gamma_0} = v^*,
\] (59)
and we obtain
\[
\sqrt{|\tau|} ||v_-||_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} ||(\nabla v_-)||_{L^2(\Omega)} = o\left( \frac{1}{\tau} \right) \text{ as } |\tau| \to +\infty.
\] (60)

Finally we have a complex geometric optics solution for the Schrödinger operator \( L_2(x, D) \) in a form:
\[
v = v^* + v_- e^{-\tau\Phi}.
\] (61)

By (51), (58) and (59), we have
\[
L_2(x, D)v = 0 \text{ in } \Omega, \quad \tau|_{\Gamma_0} = 0.
\] (62)

Let \( u_2 \) be a solution to the following boundary value problem:
\[
L_2(x, D)u_2 - f(x, u_2) = 0 \text{ in } \Omega, \quad u_2|_{\partial \Omega} = u_1|_{\partial \Omega}, \quad \frac{\partial u_2}{\partial \nu}|_{\Gamma} = \frac{\partial u_1}{\partial \nu}|_{\Gamma}.
\] (63)
Taking the scalar products of equation (63) with the function \( u_2 \) and integrating by parts, we have
\[
\int_\Omega (|\nabla u_2|^2 + C_2 |u_2|^{p+1}) \, dx \leq \int_\Omega u_2 \frac{\partial u_2}{\partial \nu} \, d\sigma + \int_\Omega q_2 u_2^2 \, dx + C \text{Vol}(\Omega). \tag{64}
\]

From (61), using (15), we have
\[
\|u_2\|_{H^1(\Omega)} \leq C. \tag{65}
\]

Then by (2) and (4), there exists \( q \)
\[
\text{By (49), (43) and (47), we have}
\]
\[
\|u_1 e^{-\tau \varphi}\|_{H^1(\Omega)} \leq C|\tau|^{\frac{2}{p}} \quad \forall \tau \geq \tau_0. \tag{66}
\]

Similarly
\[
\|u_1 e^{-\tau \varphi}\|_{H^1(\Omega)} \leq C|\tau|^{\frac{2}{p}} \quad \forall \tau \geq \tau_0. \tag{67}
\]

By (49), (43) and (47), we have
\[
\|u_1 e^{-\tau \varphi}\|_{H^2(\partial \Omega)} \leq C \tau^2. \tag{68}
\]

Hence, by (66), (67) and (68) we obtain
\[
\|u_1 e^{-\tau \varphi}\|_{H^1(\Omega)} + \|u_2 e^{-\tau \varphi}\|_{H^1(\Omega)} \leq C|\tau|^2 \quad \forall \tau \geq \tau_0. \tag{69}
\]

Therefore, by (69) and (72), there exists \( \tau_1 > 0 \) such that
\[
\|u_1\|_{C^0(\overline{\Omega})} + \|u_2\|_{C^0(\overline{\Omega})} \leq \delta \quad \forall \tau \geq \tau_1. \tag{70}
\]

Setting \( u = u_1 - u_2 \), we have
\[
L_2(x, D)u + (q_1 - q_2)u_1 + f_1(x, u_1) - f_2(x, u_2) = 0 \quad \text{in } \Omega \tag{71}
\]
and
\[
u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\Gamma} = 0. \tag{72}
\]

Let \( v \) be a function given by (61). Taking the scalar products of (71) with \( v \) in \( L^2(\Omega) \) and using (62) and (72), we obtain
\[
0 = \mathfrak{G}(u_1, v) = \int_\Omega (q_1 - q_2)u_1 v \, dx + \int_\Omega (f_1(x, u_1) - f_2(x, u_2))v \, dx. \tag{73}
\]

Our goal is to obtain the asymptotic formula for the right-hand side of (73).

By (2) and (3), there exist positive constants \( C \) and \( \delta \) such that
\[
|f(x, y)| \leq C|y|^p, \quad \forall (x, y) \in \Omega \times [-\delta, \delta]. \tag{74}
\]

Using (74), (70) and (72), we obtain
\[
|\int_\Omega (f_1(x, u_1) - f_2(x, u_2))v \, dx| \leq \int_\Omega (|f_1(x, u_1)| + |f_2(x, u_2)|)|v| \, dx \leq C \int_\Omega (|u_1|^p + |u_2|^p)|v| \, dx \leq C \int_\Omega e^{p-1}\tau \varphi (|e^{-\tau \varphi} u_1|^p + |e^{-\tau \varphi} u_2|^p)|e^{-\tau \varphi} v| \, dx \leq C e^{\tau \max_{x \in \Omega} \varphi} = o\left(\frac{1}{\tau}\right). \tag{75}
\]

By (41), (44), (43) and Proposition 6, we have
\[
u_1(x) = 2Re \left\{ (a + \frac{a^{-1}}{\tau})e^{-\tau \varphi} - \frac{p}{2\tau} \partial_2 \Phi \right\} + e^{-\tau \varphi} o_{L^2(\Omega)}(\frac{1}{\tau}) \text{ as } \tau \to +\infty. \tag{76}
\]
Using (48), (60) and Proposition 6 we obtain
\[
v(x) = 2 \text{Re} \left\{ (a + \frac{\bar{a} - 1}{\tau}) e^{-\tau \phi} + \frac{p_2 e^{-\tau \Phi}}{2 \tau \partial_2 \Phi} \right\} + e^{-\tau \phi} o_{L^2(\Omega)}(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty.
\]

By (76) and (77), we obtain the following asymptotic formula:
\[
\mathcal{O}(u_1, v) = ((q_1 - q_2)u_1, v)_{L^2(\Omega)} = ((q_1 - q_2)((a + \frac{\bar{a} - 1}{\tau}) e^{-\tau \phi} + (\pi + \frac{\bar{p} - 1}{\tau}) e^{-\tau \Phi} - \frac{\bar{p} e^{-\tau \Phi}}{2 \tau \partial_2 \Phi} - \frac{p_1 e^{-\tau \Phi}}{2 \tau \partial_2 \Phi} + e^{-\tau \phi} o_{L^2(\Omega)}(\frac{1}{\tau})),
\]
\[
(a + \frac{\bar{a} - 1}{\tau}) e^{-\tau \phi} + (\pi + \frac{\bar{p} - 1}{\tau}) e^{-\tau \Phi} + \frac{\bar{p} e^{-\tau \Phi}}{2 \tau \partial_2 \Phi} + \frac{p_2 e^{-\tau \Phi}}{2 \tau \partial_2 \Phi} + e^{-\tau \phi} o_{L^2(\Omega)}(\frac{1}{\tau}))_{L^2(\Omega)}
\]
\[
= \int_\Omega (2(q_1 - q_2)\text{Re} \{ (a + \frac{\bar{a} - 1}{\tau} - \frac{p_1}{2 \tau \partial_2 \Phi})(a + \frac{\bar{a} - 1}{\tau} + \frac{p_2}{2 \tau \partial_2 \Phi}) \}) + 2(q_1 - q_2)\text{Re} \{ |a|^2 e^{2\tau \psi} \} dx + o(\frac{1}{\tau}).
\]

Applying the stationary phase argument (see e.g., [3]) to the last integral on the right-hand side of this formula and using (24), we have
\[
\mathcal{O}(u_1, v) = \int_\Omega 2(q_1 - q_2)\text{Re} \{ (a + \frac{\bar{a} - 1}{\tau} - \frac{p_1}{2 \tau \partial_2 \Phi})(a + \frac{\bar{a} - 1}{\tau} + \frac{p_2}{2 \tau \partial_2 \Phi}) \} dx
\]
\[
+ 2\pi (q_1 - q_2)(\bar{x} e^{2\tau \psi(\bar{x})} + (q_1 - q_2)(\bar{x}) e^{-2\tau \psi(\bar{x})})
\]
\[
+ \frac{1}{2\tau i} \int_{\partial\Omega} (q_1 - q_2) |a|^2 e^{2\tau \psi(\bar{x})} \frac{\nu \cdot \nabla \psi}{|\nabla \psi|^2} d\sigma - \frac{1}{2\tau i} \int_{\partial\Omega} (q_1 - q_2) |a|^2 e^{-2\tau \psi(\bar{x})} \frac{\nu \cdot \nabla \psi}{|\nabla \psi|^2} d\sigma + o(\frac{1}{\tau}).
\]

By Proposition 11 we have
\[
\frac{1}{2\tau i} \int_{\partial\Omega} (q_1 - q_2) |a|^2 e^{2\tau \psi(\bar{x})} \frac{\nu \cdot \nabla \psi}{|\nabla \psi|^2} d\sigma - \frac{1}{2\tau i} \int_{\partial\Omega} (q_1 - q_2) |a|^2 e^{-2\tau \psi(\bar{x})} \frac{\nu \cdot \nabla \psi}{|\nabla \psi|^2} d\sigma = o(\frac{1}{\tau}).
\]

Since \( \psi(\bar{x}) \neq 0 \), we obtain from (76) and (77) that \( q_1(\bar{x}) = q_2(\bar{x}) \). Since \( \bar{x} \) can be chosen an arbitrary close to any point in the domain \( \Omega \), we finish the proof.

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