RELATIONS OF THE CLASS $U(\lambda)$ TO OTHER FAMILIES OF FUNCTIONS

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ABSTRACT. In this article, we consider the family of functions $f$ analytic in the unit disk $|z| < 1$ with the normalization $f(0) = 0 = f'(0) - 1$ and satisfying the condition $|(z/f(z))^2 f'(z) - 1| < \lambda$ for some $0 < \lambda \leq 1$. We denote this class by $U(\lambda)$ and we are interested in the relations between $U(\lambda)$ and other families of functions holomorphic or harmonic in the unit disk. Our first example in this direction is the family of functions convex in one direction. Then we are concerned with the subordinates to the function $1/((1-z)(1-\lambda z))$. We prove that not all functions $f(z)/z$ ($f \in U(\lambda)$) belong to this family. This disproves an assertion from [14]. Further, we disprove a related coefficient conjecture for $U(\lambda)$. We consider the intersection of the class of the above subordinates and $U(\lambda)$ concerning the boundary behaviour of its functions. At last, with the help of functions from $U(\lambda)$, we construct functions harmonic and close-to-convex in the unit disk.

1. Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $T = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle. Let $A$ be the family of all functions $f$ analytic in $D$ with the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let $S$ denote the subset of $A$ consisting of functions that are univalent in $D$. See [4] for the general theory of univalent functions. For $0 < \lambda \leq 1$, consider the class

$$U(\lambda) = \{f \in A : |U_f(z)| < \lambda \text{ for } z \in D\},$$

where

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = \frac{z}{f(z)} - z \left(\frac{z}{f(z)}\right)' - 1.$$

A well-known fact about this class is that every $f \in U := U(1)$ is univalent in $D$ and hence, $U(\lambda) \subset U \subset S$ for $\lambda \in (0, 1]$ (cf. [12,10]). Several properties of this family are established in [14]. For example, the class $U(\lambda)$ is preserved under a number of elementary operations such as rotation, conjugation, dilation and omitted-value transformations [14, Lemma 1]. These properties do hold for the family $S$. On the other hand, it was also pointed out in [14] that, although $S$ is preserved under the square-root transformation, the family $U$ (and hence $U(\lambda)$) does not. Because $f'(z) (z/f(z))^2$ ($f \in U$) is bounded, we see that $(z/f(z))^2 f'(z) \neq 0$ in $D$ and thus, each $f \in U$ is non-vanishing in $D \setminus \{0\}$. It is well-known (cf [5,11,17,18]) that $U$ neither is included in $S^*$ nor includes $S^*$. Here $S^*$ denotes the

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class of starlike functions, namely, functions \( f \in \mathcal{S} \) such that \( f(\mathbb{D}) \) is starlike with respect to the origin. Typical members of \( \mathcal{U} \) are in the set \( \mathcal{S}_Z \), where

\[
\mathcal{S}_Z = \left\{ z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\}
\]

It is worth pointing out that \( \mathcal{S}_Z \subset \mathcal{U} \cap \mathcal{S}^* \) and the set \( \mathcal{S}_Z \) is precisely the set of functions in \( \mathcal{S} \) having integral coefficients in the power series expansion (cf. \[6\]). Furthermore, each of these functions plays an important role in function theory since they together with rotations are extremal for some well-known subfamilies of \( \mathcal{S} \).

Let \( B \) denote the set of the functions \( \omega \) which are analytic in \( \mathbb{D} \) and satisfy \( |\omega(z)| \leq 1 \) for all \( z \in \mathbb{D} \). Also, we consider the subfamily \( B_0 = \{ \omega \in B : \omega(0) = 0 \} \). It is a simple exercise to see that each \( f \in \mathcal{U}(\lambda) \) has the characterization (cf. \[12\]),

\[
\frac{z}{f(z)} = 1 - a_2z + \lambda z^2 \int_0^z \omega(t) \, dt,
\]

for some \( \omega \in B \), where \( a_2 = f''(0)/2 \). By simplification, we may rewrite (1) as

\[
\frac{z}{f(z)} = 1 - a_2z + \lambda \omega_1(z),
\]

where \( \omega_1 \in B_0 \) and \( \omega_1'(0) = 0 \).

In Section 2, we find some functions from \( \mathcal{U}(\lambda) \) that are convex in one direction and in addition a member of \( \mathcal{U} \) that is convex in no direction.

In \[14\] Theorems 3 and 4 it was stated that \( f \in \mathcal{U}(\lambda) \) satisfies the subordination relations

\[
\frac{f(z)}{z} \prec \frac{1}{1 + (1 + \lambda)z + \lambda z^2} = \frac{1}{(1 + z)(1 + \lambda z)}, \quad z \in \mathbb{D},
\]

and

\[
\frac{z}{f(z)} + a_2z \prec 1 + 2\lambda z + \lambda z^2, \quad z \in \mathbb{D}.
\]

Here \( \prec \) denotes the usual subordination \[3\]. Section 3 is devoted to these two subordination relations and to the study of the boundary behaviour of the Schwarz functions that define subordination. One of the results of this discussion is the fact that (3) is not valid. The formula (3) was indeed the motivation for the following coefficient conjecture of the functions \( f(z) = z + \sum_{n=2}^{\infty} a_n(z)z^k \in \mathcal{U}(\lambda) \).

**Conjecture 1.** If \( f \in \mathcal{U}(\lambda) \), then

\[
|a_n(f)| \leq \sum_{k=0}^{n-1} \lambda^k \quad \text{for } n \geq 2.
\]

The conjecture for \( n = 2 \) has been shown in \[22\] and one of its alternate proofs was given in \[14\]. One of our aims of this paper is to present a counterexample to this conjecture for the case \( n = 3 \).

In Section 4 we use functions from the family

\[
\mathcal{U}_2(\lambda) := \mathcal{U}(\lambda) \cap \{ f \in \mathcal{A} : f''(0) = 0 \}
\]

and construct functions that are harmonic and close-to-convex (univalent) in \( \mathbb{D} \).
2. Convex in some direction

We recall that a univalent mapping \( f \) analytic in \( D \) is called convex if \( f(D) \) is a convex domain. A function \( f \) analytic in \( D \) is said to be close-to-convex if the complement \( D = \mathbb{C} \setminus f(D) \) of \( f(D) \) can be filled with non-intersecting half-lines emanating on the boundary \( \partial D \) and lying completely in \( D \). Every close-to-convex function is known to be univalent in \( D \). However, close-to-convexity of harmonic mappings requires univalence in the definition. This will be considered in Section 4.

A domain \( D \) in \( \mathbb{C} \) is said to be convex in the direction \( \gamma \) \((0 \leq \gamma < \pi)\) if every line parallel to the line through 0 and \( e^{i\gamma} \) has a connected intersection with \( D \). A univalent function \( f \) analytic in \( D \) is said to be convex in the direction \( \gamma \) if \( f(D) \) is convex in the direction \( \gamma \).

Obviously, every function that is convex in the direction \( \gamma \) is necessarily close-to-convex, but the converse is not true. Clearly, a convex function is convex in every direction. The class of functions convex in one direction has been introduced by Robertson [19] and investigated by many others (see, for example, [8, 20]). In particular, convex in the direction \( \gamma = 0 \) is said to be convex along horizontal/real direction.

In order to prove Theorem 1, we need the following results.

**Theorem A.** (Royster and Ziegler [20]) Let \( \varphi \) be a non-constant analytic function in \( D \). The function \( \varphi \) maps \( D \) univalently onto a domain convex in the direction \( \gamma \) if and only if there are numbers \( \mu \) and \( \nu \), \( 0 \leq \mu < 2\pi \) and \( 0 \leq \nu \leq \pi \), such that

\[
\text{Re}\left\{ P_{\mu, \nu, \varphi}(z) \right\} \geq 0, \quad z \in D,
\]

where

\[
P_{\mu, \nu, \varphi}(z) = e^{i(\mu-\gamma)}(1 - 2ze^{-i\mu} \cos \nu + z^2e^{-2\mu})\varphi'(z).
\]

**Theorem B.** ([14][22]) If \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{U}(\lambda) \), then \( |a_2| \leq 1 + \lambda \). Moreover, if \( |a_2| = 1 + \lambda \), then \( f \) must be of the form

\[
f_\theta(z) = \frac{z}{1 - (1 + \lambda)e^{i\theta}z + \lambda e^{2i\theta}z^2}, \quad \theta \in [0, 2\pi).
\]

**Remark 1.** By a standard argument, a consequence of this theorem is that the functions \( f_\theta \) are extreme points of the class \( \mathcal{U}(\lambda) \). It is an open problem, whether there exist other extreme points of \( \mathcal{U}(\lambda) \). See also Remark 2 below.

**Theorem 1.** The function \( f_\theta \) defined in Theorem B is convex in the direction \( 2\pi - \theta \).

**Proof.** Let \( f_\theta(z) \) be defined by [11]. Then

\[
f_\theta'(z) = \frac{1 - \lambda e^{2i\theta}z^2}{(1 - \lambda e^{i\theta}z)^2(1 - e^{i\theta}z)^2}.
\]

We next consider \( P_{\mu, \nu, \gamma, f_\theta}(z) \) for \( \mu = 2\pi - \theta, \nu = 0 \) and \( \gamma = 2\pi - \theta \) to get

\[
P_{2\pi - \theta, 0, 2\pi - \theta, f_\theta}(z) = (1 - e^{i\theta}z)^2 f_\theta'(z) = \frac{(1 - \lambda e^{2i\theta}z^2)(1 - \lambda e^{-i\theta}z)^2}{|1 - \lambda e^{i\theta}z|^4}.
\]
This gives by (5) that
\[ P_{2\pi-\theta,0,2\pi-\theta,f_\theta}(z) = \frac{1 - \lambda e^{2i\theta}z^2}{(1 - \lambda e^{i\theta}z)^2} = \frac{N(e^{i\theta}z)}{D(e^{i\theta}z)}, \]
where \( N(z) = (1 - \lambda z^2) \) and \( D(z) = |1 - \lambda z|^4 > 0 \) in \( \mathbb{D} \). Clearly, \( f_\theta \) is convex in the direction \( 2\pi - \theta \) by Theorem A if \( \text{Re} \{N(z)\} \geq 0 \) for \( |z| = 1 \), since
\[ P_{2\pi-\theta,0,2\pi-\theta,f_\theta}(0) = 1 > 0. \]
Thus, it suffices to show that \( \text{Re} \{N(z)\} > 0 \) for \( |z| = 1 \). It is a simple to see that
\[ \text{Re} \{N(e^{i\theta})\} = 1 - \lambda^3 - 2\lambda(1 - \lambda) \cos \theta - \lambda(1 - \lambda) \cos 2\theta \geq 1 - \lambda^3 - 3\lambda(1 - \lambda) = (1 - \lambda)^3 \]
and hence,
\[ \text{Re} \{P_{2\pi-\theta,0,2\pi-\theta,f_\theta}(z)\} = \text{Re} \{(1 - e^{i\theta}z)^2 f'_\theta(z)\} > 0 \quad \text{for} \quad z \in \mathbb{D}, \]
showing that \( f_\theta \) is convex in the direction \( 2\pi - \theta \) by Theorem A. \( \square \)

Theorem 1 motivates us to consider whether all the functions in \( \mathcal{U}(\lambda) \) are convex in some direction. Next we show that the answer is negative in general.

Example 1. If \( g(z) = \frac{z}{1 + z^2} \), then \( g \) is not convex in any direction.

Proof. First we see that
\[ \left( \frac{z}{g(z)} \right)^2 g'(z) - 1 = -z^2 \quad \text{and} \quad g(e^{\frac{2\pi}{3}}z) = e^{\frac{2\pi}{3}}g(z), \]
and thus, it follows that \( g \in \mathcal{U} \) and \( g \) is 3-fold symmetric. Therefore,
\[ D = g \left( \left\{ z = re^{i\theta} : 0 \leq r < 1, 0 \leq \theta < \frac{2\pi}{3} \right\} \right), \quad g(\mathbb{D}) = D \cup e^{\frac{2\pi}{3}}D \cup e^{\frac{4\pi}{3}}D, \]
where \( e^{i\theta}D = \{e^{i\theta}z : z \in D\} \). Thus, we only need to prove that \( g \) is not convex in the direction \( \gamma \) for \( \gamma \in [0, \frac{\pi}{3}) \). There are three inward pointing (zero-angle) cusps, which are at \( z = 1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}} \). Therefore, \( g \) is not convex in the direction \( \gamma \) for \( \gamma \in [0, \frac{\pi}{3}) \). See Figure 1. \( \square \)

Naturally one may ask the following:

Problem 1. Which functions in \( \mathcal{U}(\lambda) \) are convex in some direction?

The characterization (2) shows that each \( f \in \mathcal{U}(\lambda) \) is completely determined by its second coefficient \( a_2 \), \( \lambda \) and the Schwarz function \( \omega_1 \). One of the simple forms of \( f \) is when \( a_2 = 0 \) and \( \omega_1(z) = az^n \) \((n \geq 2 \) and \( |a| \leq 1 \)) \). So we begin to answer Problem 1 in this case.

Theorem 2. Suppose that \( f \in \mathcal{U}_2(\lambda) \) is in the form (2), where \( \lambda \in (0, 1] \). If \( \omega_1(z) = az^2 \) and \( |a| \leq 1 \), then \( f \) is convex in the direction \( \gamma = 2\pi - \frac{a\pi}{2} \).
Proof. There is nothing to prove if $a = 0$. So, we assume that $a \neq 0$. To prove the assertions, we take in Theorem A $\mu = \gamma$, and $\nu = \pi/2$. We have from (2) with $a_2 = 0$ that

$$f'(z) = \frac{1 - \lambda az^2}{(1 + \lambda az^2)^2} = \frac{1 - \lambda ps^2}{(1 + \lambda ps^2)^2},$$

where $s = ze^{i\arg a/2}$ and $p = |a|$. By Theorem A, we need to show that for all $|s| < 1$,

$$\text{Re} \left\{ f'(z)(1 + s^2) \right\} = \text{Re} \left\{ \frac{(1 - \lambda ps^2)(1 + s^2)}{(1 + \lambda ps^2)^2} \right\} \geq 0. \quad (6)$$

If $s = 0$, then $f'(z)(1 + s^2)$ assumes the value 1 and thus, it suffices to prove (6) for all $|s| = 1$. Now

$$\text{Re} \left\{ f'(z)(1 + s^2) \right\} \big|_{|s|=1}$$

$$= \text{Re} \left\{ \frac{(1 - \lambda^2 p^2 - \lambda ps^2 + \lambda p s^2)(1 + \lambda p + s^2 + \lambda p s^2)}{|1 + \lambda ps|^4} \right\}$$

$$= \frac{(1 - \lambda p)(1 + 4\lambda p + \lambda^2 p^2 + (1 + \lambda p)^2 \text{Re} (s^2) - 2\lambda p (\text{Re} (s^2))^2)}{|1 + \lambda ps|^4} \geq 0$$

for all $|s| = 1$. \qed

We end this section with a picture to demonstrate the image of the function $f(z)$ involved in Theorem 2 with certain special values of $\lambda$ and $a$. See Figure 2.
3. Relations between Schwarz functions and \( f \in \mathcal{U}(\lambda) \)

To prove our next result, we need the following result.

**Theorem C.** ([15, Proposition 4.13]) Let \( \varphi \) be analytic in \( \mathbb{D} \) with an angular limit \( \varphi(\zeta) \) at \( \zeta \in T \). If \( \varphi(\mathbb{D}) \subset \mathbb{D} \), \( \varphi(\zeta) \in T \), then the angular derivative \( \varphi'(\zeta) \) exists and

\[
0 < \frac{\varphi'(\zeta)}{\varphi(\zeta)} = \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \frac{|\varphi(\zeta) - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq +\infty.
\]

**Theorem 3.** Let \( \tilde{\omega}_1(z) = 1 - a_2 z + \lambda \omega_1(z) \in \mathcal{U}(\lambda) \), where \( \omega_1 \in B_0 \) and \( \omega_1'(0) = 0 \). If there exists a point \( \zeta \) of the unit circle \( T \) such that \( |\omega_1(\zeta)| = 1 \), then \( \omega_1 \) is of the form \( \omega_1(z) = e^{i\theta} z^2 \).

**Proof.** Let \( \omega_1(z) = z^2 \psi(z) \). Then \( \psi \) is analytic in \( \mathbb{D} \), \( \psi(\mathbb{D}) \subset \overline{\mathbb{D}} \), and \( |\psi(\zeta)| = 1 \). If \( \psi(\mathbb{D}) \subset \mathbb{D} \), then Theorem C implies that the angular derivative \( \psi'(\zeta) \) exists and satisfies

\[
0 < \frac{\zeta \psi'(\zeta)}{\psi(\zeta)} \leq +\infty.
\]

We now consider

\[
U_f(z) = -\lambda z^2 (\psi(z) + z \psi'(z)).
\]

As \( z \) approaches \( \zeta \) along the segment connecting 0 and \( \zeta \),

\[
|U_f(z)| \to \lambda \left( 1 + \frac{\zeta \psi'(\zeta)}{\psi(\zeta)} \right) > \lambda,
\]

which is a contradiction. Thus \( \psi(z) \equiv e^{i\theta} \) for some \( \theta \in \mathbb{R} \). The proof is completed. \( \square \)

**Example 2.** Let \( \lambda \in (0, 1) \), \( k \geq 2 \) and

\[
\frac{z}{f(z)} = (1 - z) \left( 1 - \frac{\lambda z}{k} \sum_{\nu=0}^{k-1} z^\nu \right) = 1 - z \left( 1 + \frac{\lambda}{k} \right) + \frac{\lambda z^{k+1}}{k}.
\]
Then \( f(z) \in \mathcal{U}(\lambda) \) but \( z/f(z) \) is not subordinate to \((1 - z)(1 - \lambda z)\).

**Proof.** Since \((1 - z) \left( 1 - \frac{z}{k} \sum_{\nu=0}^{k-1} z^\nu \right) \) does not vanish in the unit disk \( \mathbb{D} \), and

\[
\frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' - 1 = -\lambda z^{k+1},
\]

the function \( f \) belongs to \( \mathcal{U}(\lambda) \).

Suppose on the contrary that \( z/f(z) < (1 - z)(1 - \lambda z) \). Then there exists a function \( \psi \in B \) such that

\[
1 - (1 + \lambda)z\psi(z) + \lambda z^2\psi^2(z) = 1 - \left( 1 + \frac{\lambda}{k} \right) z + \frac{\lambda z^{k+1}}{k}.
\]

Since \( k \geq 2 \), Theorem B implies that \( |\psi(z)| < 1 \) for \( z \in \mathbb{D} \). Setting \( z = 1 \) on both sides of the last relation gives

\[
(1 - \psi(1))(1 - \lambda \psi(1)) = 0.
\]

Moreover, because \( \lambda \in (0, 1) \), we obtain that \( \psi(1) = 1 \) and \( \psi'(1) = 0 \), which contradicts Theorem C, namely \( \square \). Therefore, \( z/f(z) \) is not subordinate to \((1 - z)(1 - \lambda z)\).

Based on the erroneous subordination statement that if \( f(z) \in \mathcal{U}(\lambda) \) then \( z/f(z) \) is subordinate to \((1 - z)(1 - \lambda z)\), Conjecture \( 1 \) was proposed in \([14]\). The idea of Example \( 2 \) was the investigation of functions in \( \mathcal{U}(\lambda) \) that have a pole at the point \( z = 1 \). Now such functions will be used to give a counterexample to Conjecture \( 1 \).

**Example 3.** For any \( \lambda \in (0, \delta) \), where \( \delta = \frac{3 + 4 \ln 2}{4 \ln^2 2} \approx 0.29435 \), there exists \( f \in \mathcal{U}(\lambda) \) such that \( a_3(f) > 1 + \lambda + \lambda^2 \).

**Proof.** For \( a \in (0, 1) \), we consider the functions

\[
f_a(z) = \frac{z}{(1 - z) \left( 1 - \frac{z}{1-a} f_z \frac{1}{1+at} dt \right)} = \frac{z}{1 - z \left( 1 + \lambda \frac{1}{1+at} dt \right)}.
\]

Obviously, the functions \( f_a \) are holomorphic in \( \mathbb{D} \), and since

\[
\frac{z}{f_a(z)} - z \left( \frac{z}{f_a(z)} \right)' - 1 = -\lambda z^2 \frac{z + a}{1 + az},
\]

they belong to the class \( \mathcal{U}(\lambda) \). Setting

\[
v(a) := \int_0^1 \frac{t + a}{1 + at} dt = \frac{1}{a} - \frac{1 - a^2}{a^2} \ln(1 + a) = \frac{1}{2} + \sum_{k=1}^\infty \frac{2(-1)^{k-1} a^k}{k(k + 2)}.
\]

The Taylor expansion of \( f_a \) can be derived from

\[
f_a(z) = z \sum_{k=0}^\infty z^k \left( 1 + \lambda v(a) - \lambda \int_0^z \frac{t + a}{1 + at} dt \right)^k
\]

\[
= z + z^2 (1 + \lambda v(a)) + z^3 ((1 + \lambda v(a))^2 - \lambda a) + o(z^3).
\]

In particular, the coefficient of \( z^3 \) of \( f_a \) gives

\[
a_3(f_a) = 1 + \lambda (2v(a) - a) + \lambda^2 v^2(a),
\]
and we prove that there exist real numbers $a \in (0, 1)$, $\lambda \in (0, 1)$ such that

$$1 + \lambda (2v(a) - a) + \lambda^2 v^2(a) > 1 + \lambda + \lambda^2.$$  

Firstly, we will be concerned with the derivatives of $w(a) := 2v(a) - a$, i.e.,

$$w'(a) = 2v'(a) - 1 = \frac{2}{a} - \frac{4}{a^2} + \frac{4}{a^3} \ln(1 + a) - 1,$$

so that $w'(0) = \frac{1}{3}$, $w'(1) = 4 \ln 2 - 3 < 0$. Clearly

$$w''(a) = 4 \sum_{n=4}^{\infty} \frac{(-1)^{n-1}(n-3)}{n} a^{n-4},$$

so that $w''(0) = -1$. Moreover, it can be easily seen that

$$w''(a) = \frac{2}{a^4(1 + a)} (6a + 3a^2 - a^3 - 6(1 + a) \ln(1 + a)) := \frac{2u(a)}{a^4(1 + a)}.$$

As $u(0) = 0$, $u'(a) = 6a - 3a^2 - 6 \ln(1 + a)$, $u'(0) = 0$, and

$$u''(a) = \frac{-6a^2}{1 + a} < 0, \quad a \in (0, 1),$$

the function $u'$ is monotonically decreasing on $[0, 1]$ and negative on $(0, 1]$, and the function $u$ is negative on $(0, 1]$. Hence, $w''(a) < 0$ for $a \in (0, 1]$. This implies that the function $w$ is concave on $[0, 1]$. Since $w(0) = w(1) = 1$, we get the inequality $w(a) > 1$ for $a \in (0, 1)$.

On the other hand,

$$v(a) = \frac{w(a) + a}{2}, \quad 2v'(a) = w'(a) + 1 \quad \text{and} \quad 2v''(a) = w''(a) < 0 \quad \text{for} \quad a \in [0, 1],$$

which implies that $v'(a)$ is monotonically decreasing for $a \in [0, 1]$. So

$$v'(a) > v'(1) = 2 \ln 2 - 1 > 0 \quad \text{for} \quad a \in [0, 1].$$

Therefore, $v(a)$ is monotonically increasing for $a \in [0, 1]$ and $\frac{1}{2} = v(0) \leq v(a) \leq v(1) = 1$.

The above estimates for $w(a)$ and $v(a)$ yield that for any $a \in (0, 1)$ and

$$0 < \lambda < \frac{w(a) - 1}{1 - v^2(a)}$$

inequality (8) holds. We observe that $\frac{w(a) - 1}{1 - v^2(a)}$ is monotonically increasing for $a \in [0, 1]$. To get the more precise about the above assertion on the values of $\lambda$ for which (8) is valid for $a \in (0, 1)$, we calculate by L’Hôpital’s rule

$$\lim_{a \to 1} \frac{w(a) - 1}{1 - v^2(a)} = \lim_{a \to 1} \frac{w'(a)}{-2v(a)v'(a)} = -\frac{w'(1)}{v(1)v'(1) + 1} = \frac{3 - 4 \ln 2}{4 \ln 2 - 2}.$$ 

This proves the assertion of our example. ☐
Remark 2. Example 3 shows that for any \( \lambda \in (0, \delta) \), \( \delta = 0.2943 \cdots \), there are other extreme points in \( U(\lambda) \) besides the functions of the form

\[
f_\theta(z) = \frac{z}{1 - (1 + \lambda)e^{i\theta}z + \lambda e^{2i\theta}z^2}, \quad \theta \in [0, 2\pi).
\]

In the rest of this section, we consider \( f \in U(\lambda) \) of the form

\[
(9) \quad f(z) = \frac{1}{1 - (1 + \lambda)\phi(z) + \lambda \phi^2(z)}, \quad z \in \mathbb{D},
\]

where \( \phi \in B_0 \). That is, \( z/f(z) \) is subordinate to \((1 - z)(1 - \lambda z)\).

The authors in [18] discussed the \( \phi \) such that \( f \) in the form of (9) belongs to \( U(\lambda) \), which is represented as follows.

Theorem D. Let \( f \in U(\lambda) \) be given by (9) with a function \( \phi \in B_0 \) analytic on the closed unit disk \( \mathbb{D} \). If there exists a point \( \zeta \) of the unit circle \( \mathbb{T} \) such that \( \phi(\zeta) = -1 \), then \( \phi \) is of the form \( \phi(z) = e^{i\theta}z \).

Julia’s lemma plays a key role in the proof of Theorem D. We will show that Theorem D still holds even if \( \phi \) is not analytic on the whole closed unit disk \( \mathbb{D} \) by using the following lemma.

Lemma 1. Let \( f \in U(\lambda) \) be given by (9), with a function \( \phi \in B_0 \). Suppose that there exists a point \( \zeta \in \mathbb{T} \) such that the radial limit \( \phi(\zeta) = e^{i\theta_0} \).

(i) If \( \cos \theta_0 \in [-1, \frac{2\lambda}{1 + \lambda}] \), then \( \frac{\zeta \phi'(\zeta)}{\phi(\zeta)} = 1 \).

(ii) If \( \cos \theta_0 \in (\frac{2\lambda}{1 + \lambda}, 1] \), then

\[
1 \leq \frac{\zeta \phi'(\zeta)}{\phi(\zeta)} \leq \frac{(1 + \lambda)(1 + 2\lambda - 2\lambda \cos \theta_0)}{5\lambda^2 + 2\lambda + 1 - 4\lambda(1 + \lambda) \cos \theta_0}.
\]

Proof. The assumption together with Julia-Wolff-Lemma (cf. [21]) gives that the angular derivative \( \phi'(\zeta) \) exists and satisfies that

\[
1 \leq |\phi'(\zeta)| \leq +\infty.
\]

On the other hand, Theorem C implies that

\[
0 < \frac{\zeta \phi'(\zeta)}{\phi(\zeta)} \leq +\infty.
\]

These facts show that

\[
1 \leq \frac{\zeta \phi'(\zeta)}{\phi(\zeta)} \leq +\infty.
\]

Set

\[
t = \frac{\zeta \phi'(\zeta)}{\phi(\zeta)} - 1 \quad \text{and} \quad L(\phi)(z) = \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right|.
\]

Then (9) implies that

\[
L(\phi)(z) = |-(1 + \lambda) (\phi(z) - z\phi'(z)) + \lambda \phi(z) (\phi(z) - 2z\phi'(z))| < \lambda, \quad z \in \mathbb{D}.
\]
Let \( z \) approach \( \zeta \) along the segment connecting 0 and \( \zeta \). We obtain that
\[
L(\phi)(z) \to |(1 + \lambda)t - \lambda e^{i\theta_0}(2t + 1)| =: R(e^{i\theta_0})(t).
\]
A calculation shows that
\[
R^2(e^{i\theta_0})(t) = (1 + \lambda)^2t^2 + \lambda^2(2t + 1)^2 - 2t(2t + 1)\lambda(1 + \lambda)\cos \theta_0.
\]
(i) If \( \cos \theta_0 \in [-1, \frac{2\lambda}{1 + \lambda}] \), then \( R^2(e^{i\theta_0})(t) \) is monotonically increasing for \( t \geq 0 \) and attains the minimum \( \lambda^2 \) at \( t = 0 \).

If \( t > 0 \), then \( R(e^{i\theta_0}) > \lambda \), which contradicts the fact that \( L(\phi)(z) < \lambda \) for all \( z \in \mathbb{D} \).

Therefore, we have \( t = 0 \) and
\[
\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} = 1.
\]
(ii) If \( \cos \theta_0 \in (\frac{2\lambda}{1 + \lambda}, 1] \), then \( R^2(e^{i\theta_0})(t) \) is monotonically decreasing for \( t \in [0, t_0] \) and monotonically increasing for \( t \in [t_0, \infty) \), which implies that \( R^2(e^{i\theta_0})(t) \leq \lambda^2 \) when \( t \in [0, 2t_0] \) and \( R^2(e^{i\theta_0})(t) > \lambda^2 \) when \( t \in (2t_0, \infty) \), where
\[
t_0 = \frac{\lambda((1 + \lambda)\cos \theta_0 - 2\lambda)}{5\lambda^2 + 2\lambda + 1 - 4\lambda(1 + \lambda)\cos \theta_0}.
\]
The above facts and the assumption imply that
\[
1 \leq \frac{\zeta\phi'(\zeta)}{\phi(\zeta)} \leq \frac{(1 + \lambda)(1 + \lambda - 2\lambda\cos \theta_0)}{5\lambda^2 + 2\lambda + 1 - 4\lambda(1 + \lambda)\cos \theta_0}.
\]
This completes the proof of the lemma. \( \square \)

Naturally, it follows that

**Theorem 4.** Let \( f \in \mathcal{U}(\lambda) \) be given by (9) with a function \( \phi \in B_0 \). If there exists a point \( \zeta \in \mathbb{T} \) such that the radial limit \( \phi(\zeta) \in \mathbb{T} \) and \( \cos \phi(\zeta) \in [-1, \frac{2\lambda}{1 + \lambda}] \). Then \( \phi \) must be of the form \( \phi(z) = e^{i\theta}z \).

**Proof.** By assumption, we may write \( \phi(z) = z\psi(z) \), where \( \psi(\mathbb{D}) \subset \mathbb{D} \) or \( |\psi(z)| \equiv 1 \) on \( \mathbb{D} \). Since \( \phi(\zeta) \in \mathbb{T} \), we have that \( \psi(\zeta) \in \mathbb{T} \). If \( \psi(\mathbb{D}) \subset \mathbb{D} \), then Theorem C implies that the angular derivative \( \psi'(\zeta) \) exists and
\[
0 < \frac{\zeta\psi'(\zeta)}{\psi(\zeta)} \leq +\infty,
\]
which yields that
\[
\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} = 1 + \frac{\zeta\psi'(\zeta)}{\psi(\zeta)} > 1.
\]
This contradicts Lemma 1. So \( |\psi(z)| \equiv 1 \) on \( \mathbb{D} \). The proof is completed. \( \square \)

Since each finite Blaschke product maps \( \mathbb{T} \) onto \( \mathbb{T} \), by Theorem 4 we have the following conclusion.

**Corollary 1.** Let \( f \in \mathcal{U}(\lambda) \) be given by (9), with \( \phi \) being a finite Blaschke product and \( \phi(0) = 0 \). Then \( \phi \) must be of the form \( \phi(z) = e^{i\theta}z \).

For \( \lambda = 1 \), then \( \frac{2\lambda}{1 + \lambda} = 1 \). Theorem 4 yields that
Corollary 2. Let \( f \in \mathcal{U} \) be given by (11) with a function \( \phi \in B_0 \). Suppose that there exists a point \( \zeta \in \mathbb{T} \) such that the radial limit \( \phi(\zeta) \in \mathbb{T} \). Then \( \phi \) must be of the form \( \phi(z) = e^{i\theta}z \).

Since each inner function has radial limit of modulus one a.e., we obtain that

Corollary 3. Let \( f \in \mathcal{U} \) be given by (11), with \( \phi \) being an inner function and \( \phi(0) = 0 \). Then \( \phi \) must be of the form \( \phi(z) = e^{i\theta}z \).

Example 4. Let \( r_n = 1 - \frac{1}{2^m}, \theta_n = \frac{1}{2^n} \) and \( a_n = r_n e^{i\theta_n} \), where \( n \geq 1 \). Then

\[
B_1(z) = z \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z}
\]

is a Blaschke product, where

(a) \( B_1(z) \) and all its subproducts have unimodular radial limit at every point of \( \mathbb{T} \);
(b) \( B_1(z) \) has finite angular derivative at every point of \( \mathbb{T} \) and

\[
1 < \frac{\zeta B_1'(\zeta)}{B_1(\zeta)} < \infty, \quad \forall \zeta \in \mathbb{T};
\]

(c) \( B_1 \) has finite phase derivative at every point of \( \mathbb{T} \);
(d)

\[
\frac{z}{(1 - B_1(z))^2} \notin \mathcal{U}.
\]

Proof. It is obvious that

\[
\sum_{n=1}^{\infty} (1 - r_n) = \sum_{n=1}^{\infty} \frac{1}{2^{4n}} < \infty,
\]

and thus \( B_1(z) \) is a Blaschke product.

(a) We can see that \( a_n \to 1 \) as \( n \to \infty \). So for each \( e^{i\theta} \in \mathbb{T} \setminus \{1\}, \{a_n\} \) is bounded away from such \( e^{i\theta} \) and

\[
\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta} - a_n|} = \sum_{n=1}^{\infty} \frac{1}{2^{4n}} \frac{1}{|e^{i\theta} - a_n|} < \infty.
\]

Since \( |1 - a_n|^2 = (1 - r_n)^2 + 4r_n \sin^2 \frac{\theta_n}{2} \) and \( \frac{2}{\pi} |x| \leq \sin |x| \leq |x| \) when \( |x| \leq \frac{\pi}{2} \), we have

\[
|1 - a_n| \geq 2\sqrt{r_n} \sin \frac{\theta_n}{2} \geq \frac{2\sqrt{1 - \frac{1}{2^m}}}{\pi} \theta_n \geq \frac{\sqrt{15}}{2\pi} \theta_n = \frac{\sqrt{15}}{2\pi} \frac{1}{2^n}
\]

which yields that

\[
\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|} \leq \frac{2\pi}{\sqrt{15}} \sum_{n=1}^{\infty} \frac{1}{2^{3n}} < \infty.
\]

The above estimates together with the result in [7] imply that (a) is proved.

(b) We consider

\[
G(B_1, e^{i\theta}) := \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta} - a_n|^2}.
\]
Since \( \{a_n\} \) is bounded away from \( \mathbb{T} \setminus \{1\} \), for each \( e^{i\theta} \in \mathbb{T} \setminus \{1\} \), we have

\[
G(B_1, e^{i\theta}) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{1}{|e^{i\theta} - a_n|^2} < \infty.
\]

For \( \theta = 0 \), (10) gives

\[
G(B_1, 1) \leq \frac{\pi^2}{15} \sum_{n=1}^{\infty} \frac{1}{2^{2n-2}} < \infty.
\]

The above estimates together with the result in [7] imply that for each \( \zeta \in \mathbb{T} \),

\[
1 < \frac{\zeta B_1'(\zeta)}{B_1(\zeta)} = 1 + \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|e^{i\theta} - a_n|^2} < \infty.
\]

Thus the conclusion (b) is proved. The conclusion (c) follows from (b). Finally, (d) follows from (b) and Lemma 1. \( \square \)

**Example 5.** Let \( r_n = 1 - \frac{1}{2^n} \), \( \theta_n = \frac{1}{2^n} \), \( a_n = r_n e^{i\theta_n} \), where \( n \geq 1 \). Then

\[
B_2(z) = z \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{a_n - 1 - a_n z}
\]

is a Blaschke product, where

(a) \( B_2(z) \) and all its subproducts have unimodular radial limit at every point of \( \mathbb{T} \);

(b) \( B_2(z) \) has finite angular derivative at every point of \( \mathbb{T} \setminus \{1\} \) and

\[
\frac{B_2'(1)}{B_2(1)} = \infty;
\]

(c)

\[
\frac{z}{(1 - B_2(z))(1 - \lambda B_2(z))} \notin \mathcal{U}(\lambda).
\]

**Proof.** By similar reasoning as in the proof of Example 4, we obtain that \( B_2(z) \) is a Blaschke product, \( B_2(z) \) and all its subproducts have unimodular radial limit at every point of \( \mathbb{T} \), and \( B_2(z) \) has finite angular derivative at every point of \( \mathbb{T} \setminus \{1\} \).

Since

\[
|1 - a_n|^2 = (1 - r_n)^2 + 4r_n \sin^2 \frac{\theta_n}{2} \leq (1 - r_n)^2 + r_n \theta_n^2 = \frac{1}{2^{2n}},
\]

it follows easily that

\[
G(B_2, 1) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{1}{|1 - a_n|^2} \geq \sum_{n=1}^{\infty} \frac{1}{2^{2n}} 2^{2n} = \infty.
\]

Finally, (c) follows from (b) and Lemma 1. \( \square \)
4. Harmonic mappings generated by functions in $U_2(\lambda)$

A complex-valued function $F$ is called harmonic in $\mathbb{D}$ if $\text{Re} \, F$ and $\text{Im} \, F$ are real harmonic in $\mathbb{D}$. Such function has the canonical representation $F = H + G$, where $H$ and $G$ are analytic in $\mathbb{D}$ with $G(0) = 0$. The Jacobian $J_F$ of $F$ is given by $J_F = |H'|^2 - |G'|^2$. Consequently, $F$ is locally univalent and sense-preserving if and only if $J_F(z) > 0$ in $\mathbb{D}$; or equivalently, $H' \neq 0$ in $\mathbb{D}$ and the dilatation $\omega_F =: \omega = G'/H'$ belongs to $B$. See [3][9].

Lemma E. [16] Let $F = H + G$ be a harmonic mapping normalized by $H(0) = G(0) = H'(0) - 1 = 0$ and $h \in S$ be convex in $\mathbb{D}$. If $F$ satisfies that

$$\text{Re} \left\{ e^{\gamma} \frac{H'(z)}{H(z)} \right\} > \left| \frac{G'(z)}{H'(z)} \right|$$

for all $z \in \mathbb{D}$ and for some $\gamma \in \mathbb{R}$, then $F$ is sense-preserving and close-to-convex (univalent) in $\mathbb{D}$.

Theorem 5. Suppose that $H \in U_2(\lambda)$ and $0 < \lambda \leq \sqrt{2} - 1$. If $F = H + G$ with the dilatation $\omega(z) = \frac{G'(z)}{H'(z)}$, where $\omega(0) = 0$ and

$$|\omega(z)| \leq \frac{1 - 2\lambda - \lambda^2}{(1 + \lambda)^2} \quad \text{for all } z \in \mathbb{D},$$

then $F$ is sense-preserving and close-to-convex (univalent) in $\mathbb{D}$.

Proof. By using Corollary 3.6 in [13], we know that $H \in S^*$ for $0 < \lambda \leq \sqrt{2} - 1 < \sqrt{2}$. Set $h'(z) = \frac{H(z)}{z}$. Then $h \in S$ and is convex in $\mathbb{D}$. By Lemma D, it suffices to prove that for all $z \in \mathbb{D}$,

$$\text{Re} \left\{ \frac{zH'(z)}{H(z)} \right\} > \left| \frac{\omega(z)}{H(z)} \right|$$

for all $z \in \mathbb{D}$.

Since $H \in U_2(\lambda)$, the representation (11) for $H$ shows that

$$H(z) = \frac{z}{1 + \lambda z \int_0^z \omega(t) \, dt},$$

where $\omega \in B$. Then

$$\frac{zH'(z)}{H(z)} = \frac{1 - \lambda z^2 \omega(z)}{1 + \lambda z \int_0^z \omega(t) \, dt},$$

and (11) is equivalent to

$$|\omega(z)| < \frac{\text{Re} \left\{ \left( 1 - \lambda z^2 \omega(z) \right) \left( 1 + \lambda z \int_0^z \omega(t) \, dt \right) \right\}}{|1 - \lambda z^2 \omega(z)| \left| 1 + \lambda z \int_0^z \omega(t) \, dt \right|} =: M.$$

Set $|z| = r < 1$. Then we have

$$M \geq A(r, \lambda) := \frac{1 - 2\lambda r^2 - \lambda^2 r^4}{(1 + \lambda r^2)^2},$$

and since

$$\min_{r \in [0,1]} A(r, \lambda) = \frac{1 - 2\lambda - \lambda^2}{(1 + \lambda)^2}$$
we see that the conclusion holds since $1 - 2\lambda r^2 - \lambda^2 r^4 > 0$ when $0 < \lambda \leq \sqrt{2} - 1$.

**Theorem 6.** Suppose that $H \in U_2(\lambda)$ and $0 < \lambda \leq \frac{1}{2}$. If $F = H + G$ with the dilatation $\omega(z) = \frac{G'(z)}{H'(z)}$, where $\omega(0) = 0$ and

$$|\omega(z)| \leq \sqrt{(1 - \lambda^2)(1 - 4\lambda^2)^2} \quad \text{for all } |z| = r < 1,$$

then $F$ is sense-preserving and close-to-convex (univalent) in $\mathbb{D}$.

**Proof.** Using the method of the proof of Theorem 5, we see that

$$H'(z) = \frac{1 - \lambda z^2 \omega(z)}{(1 + \lambda z \int_0^z \omega(t) \, dt)^2},$$

where $\omega \in B$. Set $|z| = r$ and $F_\varepsilon(z) = H(z) + \varepsilon G(z)$, where $|\varepsilon| = 1$. Then

$$F'_\varepsilon(z) = (1 + \varepsilon \omega(z)) H'(z) \quad \text{and} \quad |\arg F'_\varepsilon(z)| \leq \arcsin(|\omega(z)|) + 3\arcsin(\lambda r^2).$$

Now it comes to prove

$$\arcsin(|\omega(z)|) + 3\arcsin(\lambda r^2) \leq \frac{\pi}{2},$$

which is equivalent to $|\omega(z)| \leq B(r, \lambda)$, where

$$B(r, \lambda) = \cos(3\arcsin(\lambda r^2)) = \sqrt{1 - \sin^2(3\arcsin(\lambda r^2))} = \sqrt{1 - (3\lambda r^2 - 4\lambda^3 r^6)^2},$$

which is simplified to

$$B(r, \lambda) = \sqrt{(1 - \lambda^2 r^4)(1 - 4\lambda^2 r^4)^2}.$$

It is a simple exercise to see that

$$\min_{r \in [0, 1]} B(r, \lambda) = \sqrt{(1 - \lambda^2)(1 - 4\lambda^2)^2}.$$

This completes the proof.

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