Unbottling the Gini: New Tools from an Old Concept.

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Abstract

The Gini index signals only the dispersion of the distribution and is not very sensitive to income differences at the tails of the distribution, where it would matter most. In the current work, novel inequality measures are proposed that address these limitations. In addition, two related measures of skewness are established. They all are based on a pair of functions that is obtained by attaching simple weights to the distances between the Lorenz curve and the 45-degree line, both in ascending and descending order, resulting in a pair of alternative inequality curves. The inequality measures derived from these two alternative inequality curves either complement the information about distributional dispersion measured by the Gini coefficient with information about distributional asymmetry, or are more sensitive to income differences at both tails of the distribution. The novel tools measure inequality appropriately and their Lorenz-based graphical representations give them intuitive appeal. Beyond socioeconomics, they can be applied in other disciplines of science.

Keywords: Inequality, Lorenz curve, Gini-coefficient, skewness

JEL-classification: D31, D63
1. Introduction

Two limitations of the widely-used Gini coefficient are well established, the index neither is sensitive to the asymmetry of a given distribution (Bowden, 2016), nor particularly sensitive to income differences that relate to values further away from the middle of the distribution (Jenkins, 2009). Figure 1 depicts two Lorenz curves with opposite asymmetry and the same Gini coefficient. A perfectly symmetric income distribution that leads to the same area between the Lorenz curve and the 45-degree line necessarily would result in the same Gini coefficient of 0.33. This is inconsistent with the notion of an inequality index that is sensitive to those distances that are of more social concern since distributional asymmetry is regularly driven by remarkably low or high incomes at the tails of the distribution. Several authors have suggested the use of indices that are sensitive to the asymmetry of the distribution (Bowden (2016), Clementi et al. (2019)). I argue that measures of statistical dispersion that attach a higher weight to income differences at the tails of the distribution compared to the Gini can be just as useful to measure inequality more appropriately.

![Figure 1: Two Lorenz curves with the same Gini coefficient (G=0.33) and opposite asymmetry.](image)

The paper proceeds as follows. In Section 2 inequality measures that address the limitations of the Gini coefficient are introduced, Section 3 establishes two related measures of skewness.
2. Inequality Measures with different Focus

The Gini coefficient can simply be calculated using the information from the Lorenz curve. Let \( L_{E_i} \) denote the points on the 45-degree line of perfect equality and let \( L_{F_i} \) denote the corresponding points on the Lorenz curve, then the Gini coefficient is obtained by comparing the observed distances between the Lorenz curve and the 45-degree line to the sum of distances we would observe in the case of absolute inequality:

\[
G = \frac{n - 1}{n} \frac{\sum_{i=1}^{n-1} (L_{E_i} - L_{F_i})}{\sum_{i=1}^{n} L_{E_i}} = \frac{2n}{n^2} \sum_{i=1}^{n-1} (L_{E_i} - L_{F_i})
\]  

(1)

Interest in the social evaluation function behind inequality measures led to the finding that the Gini coefficient can also be expressed as (Weymark(1981)):

\[
G = 1 - \frac{1}{n^2} \sum_{i=1}^{n} \frac{(2i - 1) y_i}{\hat{\mu}(y)}
\]

(2)

The weights put on the incomes \( y_i \) that result in the Gini coefficient are the first \( n \) odd numbers. Several authors have suggested indices that extend the Gini coefficient by attaching weights to ranked incomes, two families of indices can be distinguished that belong to the wider class of generalized Gini indices. Single-series Ginis are defined by weights that are a nondecreasing sequence of numbers starting with 1. The Gini coefficient is part of this family with weights given by \( (2i-1) \). Single-series Ginis, in general, don’t satisfy the population principle. But the extended Gini measures do, for this family of indices a single parameter is sufficient to determine the weights for all the ranked incomes (Donaldson and Weymark(1980), Weymark(1981)). The extended Gini is given by (Xu, 2000):

\[
EG_\delta(y) = 1 - \frac{1}{n^2} \sum_{i=1}^{n} \frac{(n - i + 1)\delta - (n - i)\delta}{\hat{\mu}} y_i
\]

(3)

where \( \hat{\mu} \) is the mean and \( \delta > 1 \) is a parameter that expresses inequality aversion. For \( \delta = 2 \) we obtain the regular Gini coefficient. With a higher \( \delta \) the inequality index becomes more sensitive to low values in the distribution. A lower \( \delta \) makes it more sensitive to high values. These previous attempts are similar to the approach presented in this paper, but for the calculation of the novel measures the weights are attached directly to the \( n-1 \) distances between \( L_F \) and the 45-degree line, not to individual incomes or their ranks. It has been noted before, that the most useful part of the Lorenz curve is its distance from the 45-degree line of perfect equality(Clementi et al., 2019).

We derive two functions of the Lorenz curve that are sensitive to deviations from the 45-degree line at opposite sides of the cumulative distribution. Without any loss of generality, the two alternative inequality curves are deduced by
attaching weights to distances along the y-axis between the points that define \( L_F \) and the corresponding points on the 45-degree line. The weights are the first \( n-1 \) natural numbers from the infinite sequence \( \mathbb{N} \) starting with 1, each divided by the mean of the number sequence which is \( n/2 \). This results in weights that reflect the ranks of the cumulative distribution and that range from 0\(^+\) to 2, with a mean weight of 1.

To accentuate distances between \( L_F \) and the 45-degree line at the upper end of the cumulative distribution the weights increase with the rank. To accentuate deviations at the lower end of the cumulative distribution the same weights are used, but they decrease with the rank. Correlation analysis exposes that the resulting measures are closely linked to extended Gini measures calculated with a \( \delta \) of about 1.5 and precisely 3. For these extended Gini measures the weights attached to each income does also either increase or decrease with the rank(Yitzhaki, 1983). But it is the slight change to the old concept of attaching weights that delivers the novel measures of inequality and skewness. Unlike the Lorenz curve, the resulting curves \( L_{sk^-} \) and \( L_{sk^+} \) aren’t characterized by strict increasing monotonicity and convexity. They nonetheless are useful, since novel tools for the measurement of distributional dispersion and skewness can be derived from them.

Two properties of the novel inequality curves can be deduced from Figure 2. First, if the original distribution is perfectly symmetric deviations of the alternative curves from the Lorenz curve that occur below and above intersection point \( T \) due to the attached weights cancel each other out. In that case, the area between each of the two derived functions and the 45-degree line is the same as the area between \( L_F \) and the 45-degree line. The corresponding inequality measures \((G_{sk^-}, G_{sk^+})\) then take on the same value as the Gini coefficient. Second, both functions cross the Lorenz curve at the same point \( T \). If \( n \) is an even number, \( T \) coincides with the median of the first \( n-1 \) values of the cumulative distribution, since both alternative functions attach a weight of 1 to this value.

\( G_{sk^-} \) and \( G_{sk^+} \) are given by:

\[
G_{sk^-} = \frac{n-1}{n} \sum_{i=1}^{n-1} (L_{E_i} - L_{F_i}) \left( \frac{(2n - 2i)}{n} \right) \\
G_{sk^+} = \frac{n-1}{n} \sum_{i=1}^{n-1} (L_{E_i} - L_{F_i}) \left( \frac{2i}{n} \right)
\]

Just like the Gini coefficient, they are geometrically defined as the areas between the line segments of the respective function and the 45-degree line, over the total area below the 45-degree line. When \( G_{sk^-} \) and \( G_{sk^+} \) take on the same value for different distributions, these distributions are very similar. But, both \( G_{sk^-} \) and \( G_{sk^+} \) are sensitive to only one type of skewness. If the
distribution is skewed one of them always indicates a lower level of inequality than the Gini coefficient, the other one a higher level. Taken each by itself the measures are very similar to existing generalizations of the Gini coefficient, which are only sensitive to inequality at one end of the distribution. To solve this problem and create an index that is sensitive to both types of skewness, I propose a skewness-adjusted Gini coefficient:

$$G_{sk} = G + \frac{|G_{sk+} - G_{sk-}|}{2}$$ (6)

Here $G$ is the usual Gini coefficient and equation (6) evaluates to the value of $G_{sk-}$ if the original distribution is skewed to the left, the value of $G_{sk+}$ if it is skewed to the right, and $G$ if it has zero skewness. Therefore the index value of $G_{sk}$ is always at least as high as the value of the Gini coefficient. Like the Gini coefficient $G_{sk}$ can take on the same value for distributions with opposite asymmetry, but in contrast to the Gini coefficient, the skewness component of inequality is taken into account. Put in other words, $G_{sk}$ complements the information about statistical dispersion expressed in the Gini with information about distributional asymmetry. This measure geometrically $G_{sk}$ corresponds to the area between the $L_E$ and $L_F$ plus 1.5 times the difference between $U$ and $L$, see Figure 3. As noted before, there exists a complete linear correlation

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Figure 2: Symmetric Lorenz curve, $G_{sk-}$, $G$, and $G_{sk+}$ have the same value 0.33, and $L = U = L' = U'$. )
between an $G_{sk-}$ and an extended Gini with $\delta = 3$, the measure $G_{sk-}$ can also be calculated by multiplying the result of equation (3) with $2/3$. Clearly, together with the value of $G$ the calculation of $G_{sk+}$ and $G_{sk}$ then also is straightforward. For these measures, any choice between the two calculation methods has to be based on computational convenience only. For $n$ large enough, the upper bound of $G_{sk-}$ and $G_{sk+}$ are $2/3$ and $4/3$ respectively, the upper bounds for a specific $n$ are given by:

\[
\max G_{sk-} = \frac{2}{3} \left(\frac{2n^2 - 2}{2n^2}\right) \\
\max G_{sk+} = \frac{(2n - 2)}{n} - \max G_{sk-}
\]

Figure 3: Lorenz curve for a negatively skewed original distribution ($G_{sk-} = 0.3456$, $G = 0.33$, $G_{sk+} = 0.3144$).

It's also possible to derive a measure of statistical dispersion and inequality from two alternative inequality curves. Index $S$ is defined as the area between the two functions $L_{sk-}$ and $L_{sk+}$, over the total area below the 45-degree line. Bounded between 0 and 1, index $S$ is more sensitive to inequality at both tails of the distribution, compared to the Gini coefficient. Index $S$ is also more sensitive to a small number of extreme values at the tails of the distribution, compared to $G_{sk}$ that is more sensitive if more values contribute to the asymmetry of the
distribution. If \( n \) is an even number, we get the upper bound of index \( S \) for a given \( n \) by:

\[
\max S = 1 - \frac{2}{n}
\]  

(9)

And if \( n \) is an odd number:

\[
\max S = \frac{n - 1}{n} (1 - \frac{1}{n})
\]  

(10)

A final index can be derived that can be viewed as an extension of both \( S \) and \( G \). Index \( W \) is defined as the area between the 45-degree line and the linear segments connecting the points of the functions \( L_{sk-} \) and \( L_{sk+} \) that are below \( L_F \) or coincide with it at point \( T \). I want to note that point \( T \) is only part of the function if \( n \) is an even number. If \( n \) is an odd number, the points before and after point \( T \) are joined (see Figure 4). Index \( W \) can also be expressed as \( G + \frac{S}{2} \), since the area relevant for the calculation equals the area of the Gini coefficient plus half the area of inequality index \( S \). Index \( W \) takes values between 0 and 1.5.

Figure 4: Inequality index \( S \) is based on absolute distances between the points that define functions \( L_{sk-} \) and \( L_{sk+} \), and geometrically defined as \( \frac{A}{A+B} \).
The proposed inequality indices possess a set of desirable qualities. Their computation is simple, they can accommodate nonpositive values, and their geometrical representation gives them intuitive appeal. They also satisfy the following axioms:

1. $G_{sk}$, $W$, and $S$ are scale invariant, their index values do not change if all incomes are changed proportionally.

2. $G_{sk}$, $W$, and $S$ satisfy the population principle, their index values do not change if the original population is replicated.

3. $G_{sk}$ and $W$ decrease if a progressive transfers occur among individuals in the income distribution. A special case is index $S$, only if $n$ is an even number and for a transfer from the individual right above the median of the income distribution to the individual right below the median does the index value not change.

4. $G_{sk}, W$ and $S$ are normalized, the index value is zero if all individuals have the same income.

Figure 5: Lorenz curve for a positively skewed original distribution ($G=0.33, b=0.469, d=0.245$)
3. Two Lorenz-based Measures of Skewness

The functions $L_{sk-}$ and $L_{sk+}$ can also be used to derive two novel Lorenz-based measures of skewness that possess simple geometrical representations. A measure of skewness that is based on percentage differences between $G_{sk-}$ and $G_{sk+}$ is:

$$b = \frac{(1 - [1/G_{sk+}]G_{sk-}) + ([1/G_{sk-}]G_{sk+}) - 1)}{1.5}$$

Whenever the distribution is symmetric both $G_{sk+}$ and $G_{sk-}$ take on the same value and the numerator in equation(11) will be zero. Skewness measure $b$ takes on values between -1 and 1. Geometrically the measure corresponds to percent differences between the respective areas of $G_{sk+}$ and $G_{sk-}$ in the unit square. Measure $b$ is more robust than conventional measures of skewness that depend on the third moment, a single extreme value at the left or right side of the distribution has a smaller effect on the value of $b$ than on moment-based measures of skewness like the one given by equation(12).

$$Skew = \frac{1}{n} \sum \frac{(x_i - \bar{x})^3}{s^3}$$

A second skewness measure that can be derived from $L_{sk-}$ and $L_{sk+}$ is the $d$-metric. It can be viewed as the minimum percentage of total income that needs to be redistributed to transform the original income distribution into one with net envy equal to zero. This measure can be written as:

$$d = \frac{n}{n-1} [1.5(G_{sk+} - G_{sk-})]$$

If $T$ is total income, the corresponding monetary amount is given by $D = dT$. For $n$ large enough, $d$ is defined as three times the difference between areas $U$ and $L$, over the total area below the 45-degree line (see Figure 5). It follows that inequality index $G_{sk}$ given in equation (6) equals $G$ plus half the percentage expressed with the $d$-metric. The measure is similar in spirit to an earlier attempt by Bowden(2016), called the $v$-metric.

The novel tools $b$ and $d$ posses a set of natural and desirable features for skewness measures (see also Brys et al.(2003)):

1. Both $b$ and $d$ are scale-invariant. Their values don’t change if all incomes are multiplied by the same factor. Measure $b$ is also location invariant, its value doesn’t change when the same amount is added to all incomes.

2. If the distribution is inverted such that distances to the mean that previously were above the mean are now below and vice versa, then the index values of $b$ and $d$ are inverted as well. Only the sign before the index value does change.
3. If the parent distribution $F$ is symmetric, and as a consequence the distances between $L_F$ and $L_E$ are symmetric, than $b=0$ and $d=0$.

4. Measure $b$ satisfies the population principle, this ensures that if the population is replicated and $n$ increases to $2n$, the index value stays the same.

4. Conclusion

Novel measures of inequality and skewness are introduced that are based on attaching rank-related weights to the distances between the Lorenz curve and the 45-degree line. The two functions derived by attaching the weights in ascending and descending order are closely related to the well-known family of extended Gini measures. The measures that are based on these two functions address major limitations of the Gini coefficient. A skewness-adjusted Gini coefficient is derived, and other measures that are more sensitive to income differences at both tails of the distribution than the Gini coefficient. The proposed measures are simple to compute, have intuitive appeal thanks to graphical representations and satisfy most axioms that are useful in inequality measures. This makes them good alternatives for researchers and practitioners. Any choice of a particular measure has to be informed by the underlying purpose. The decomposition by subgroup and income source for the proposed measures still has to be investigated.

I also introduce two novel measures of skewness, that are based on the same methodology used to derive the inequality measures and their graphical representations. One of them is more robust than conventional skewness measures that depend on the third moment. The other one is linked to inequality measurement since it has a concrete meaning as the percentage of total income that needs to be redistributed to transform the original distribution into a perfectly symmetric distribution, i.e. one in which incomes are evenly distributed around the mean.

Although the proposed measures are discussed in the context of income inequality, in socioeconomics they can also be applied to study wealth inequality. Beyond that, they can be used to study distributional characteristics of variables in other disciplines of science.
References

Bowden, R. J. (2016). Giving Gini direction: An asymmetry metric for economic disadvantage. *Economics Letters, 138*, 96-99.

Brys, G., Hubert, M., & Struyf, A. (2003). A comparison of some new measures of skewness. In *Developments in robust statistics* (pp. 98-113). Physica, Heidelberg.

Clementi, F., Gallegati, M., Gianmoena, L., Landini, S., & Stiglitz, J. E. (2019). Mis-measurement of inequality: a critical reflection and new insights. *Journal of Economic Interaction and Coordination, 14*(4), 891-921.

Donaldson, D., & Weymark, J. A. (1980). A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory, 22*(1), 67-86.

Jenkins, S. P. (2009). Distributionally-sensitive inequality indices and the GB2 income distribution. *Review of Income and Wealth, 55*(2), 392-398.

Weymark, J. A. (1981). Generalized Gini inequality indices. *Mathematical Social Sciences, 1*(4), 409-430.

Xu, K. (2000). Inference for generalized Gini indices using the iterated-bootstrap method. *Journal of Business & Economic Statistics, 18*(2), 223-227.

Yitzhaki, S. (1983). On an extension of the Gini inequality index. *International economic review, 617-628.*