Two-dimensional supersolidity in a circular trap

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Dipolar condensates have recently been coaxed into supersolid phases supporting both superfluid and crystal excitations. While one-dimensional (1D) supersolids may be prepared via a roton instability, we find that such a procedure in two dimensions (2D) leads to greater heating. To go on to show that 2D roton modes have little in common with the supersolid configuration: instead, unstable centralised rotons trigger a process of nonlinear crystal growth. By evaporatively cooling directly into the supersolid phase—hence bypassing the first-order roton instability—we experimentally produce a 2D supersolid in a near-circular trap. We develop a stochastic Gross-Pitaevskii theory that includes beyond-meanfield effects to further explore the formation process. We calculate the static structure factor for a 2D supersolid, and compare to a 1D array. These results provide insight into the process of supersolid formation in 2D, and define a realistic path to the formation of large two-dimensional supersolid arrays.

The supersolid phase was predicted to simultaneously exhibit crystalline order and superfluidity \cite{10, 11}. Whereas it remains elusive in helium, recent developments in ultracold quantum gases have finally made supersolidity a reality, providing an excellent platform for the control and observation of these states. Important early advances were made in systems with spin-orbit coupling \cite{13, 14} and cavity-mediated interactions \cite{15}, where supersolid properties were observed in rigid crystal configurations. Bose-Einstein condensates (BECs) with dipole-dipole interactions have now been observed in a supersolid state with deformable crystals \cite{10, 12}, with their lattices genuinely arising from the atom-atom interactions \cite{16, 17}.

In the first dipolar supersolid experiments translational symmetry was broken only along one axis, giving rise to a one-dimensional (1D) density wave, commonly referred to as 1D droplet arrays \cite{10, 12}. A more recent experiment has created the first states with two-dimensional (2D) supersolidity in anisotropic traps of variable aspect ratio \cite{19}. This opens the door to study vortices and persistent currents \cite{17, 20}, as well as exotic ground state phases predicted for large atom numbers \cite{21, 22}.

It is still an open question whether 2D arrays provide as favorable conditions for supersolidity as 1D arrays do. In 1D, with the right parameters, the density pattern induced by roton instability smoothly connects with the final supersolid array. The BEC-to-supersolid transition is hence weakly first-order, or even continuous \cite{22}, and quenches through the transition cause small excitations of the resulting supersolid \cite{10, 12}, leading to minimal heating. While it has been predicted that a similar procedure in 2D may lead to coherence between three droplets in a triangular configuration \cite{24}, it remains unknown whether this procedure can be scaled up for large supersolid crystals.

In this work, we explore the formation of large 2D supersolids in circular traps. We investigate the role of roton modes in seeding crystal formation during an interaction quench, and find this initiates a process of non-linear crystal growth beginning at the trap center and growing radially outwards, leaving the resulting crystal substantially excited. Interestingly, the final crystal configuration is unrelated to any combination of roton modes of the original condensate. As a result, in contrast to the case of a 1D array, quenching from an unmodulated BEC into a 2D supersolid has a strong first-order nature that results in substantial heating. To avoid crossing this first-order phase transition we theoretically explore an alternative approach based on evaporative cooling directly into the supersolid state \cite{12, 16, 27}. We develop a finite temperature theory—that includes quantum fluctuations—which can model the entire evaporative cooling process. We then test this experimentally and observe a 7-droplet hexagonal supersolid crystal in a near-circular trap. We theoretically delve deeper by calculating the static structure factor for a 2D supersolid, and compare to the 1D case.

We utilize dynamic and ground state calculations of the extended Gross-Pitaevskii equation (eGPE), given by

\begin{equation}
\hat{H}_{\text{GP}}[\psi] = -\frac{\hbar^2 \nabla^2}{2m} + \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) + \int d^3 \mathbf{x}' U(\mathbf{x} - \mathbf{x}') |\psi(\mathbf{x}', t)|^2 + \gamma_{\text{QF}} |\psi(\mathbf{x}, t)|^3,
\end{equation}

where $m$ is the mass and \( \omega_{x,y,z} = 2\pi f_{x,y,z} \) are the external trapping frequencies. Two-body contact interactions and the long-ranged, anisotropic dipole-dipole interactions are described by the pseudo-potential, respectively, \( U(\mathbf{r}) = 4\pi \epsilon_0 \hbar^2 \frac{\Delta(\mathbf{r})}{m} + \frac{3\hbar^2}{m} \frac{1 - 3 \cos^2 \theta}{r^3} \), with \( \theta \) being the angle between the polarization axis (z-axis) and...
the vector joining two particles. This is characterized by s-wave scattering length $a_s$ and dipole length $a_{dd} = \mu_0 \mu^2 n/m/12 \hbar^2$, with magnetic moment $\mu_m$. To find the ground state we employ a conjugate-gradients technique minimizing the corresponding energy functional $\mathcal{E}_s$. The last term appearing in Eq. (1) represents quantum fluctuations in the form of a dipolar Lee-Huang-Yang correction $\Delta \mathcal{E}$. $\gamma_{QF} \approx \frac{\hbar^2 k^2}{3m} \sqrt{\pi a_s^2} \Re \{ Q_5(\varepsilon_{dd}) \}$, where $Q_5(\varepsilon_{dd}) = \int d\nu (1 - u + 3a_5 \varepsilon_{dd})^{5/2}$ is the auxiliary function, which can be solved analytically, and $\varepsilon_{dd} = a_{dd}/a_s$. We always consider $^{164}$Dy, such that $a_{dd} = 130.8a_0$.

It is pertinent to review the nature of the elementary excitations obtained in the Bogoliubov-de Gennes (BdG) framework for experimentally relevant parameters, which requires linearizing the eGPE around a stationary state $\varphi_0$. We present the results of this analysis in the form of the dynamic structure factor $S(k, \omega)$, which predicts the system response to perturbations of momentum $\hbar k$ and energy $\hbar \omega$.

Taking a dipolar BEC ground state close to the roton instability, in Fig. 1 we see the roton minimum at finite momentum $k_x l_z \approx 1.1$ (c.f. quasi-2D prediction in the Thomas-Fermi regime $k p l_z = \sqrt{2}$, where $k_p = \sqrt{k_x^2 + k_y^2}$). Roton modes have been observed experimentally in 1D and more recently in 2D. We also plot the 5 lowest energy roton modes corresponding to $m = 0, 1, 2, 3, 4$, with $m$ being the angular quantum number in the $z$ direction. Plotted on the bottom-right corner of Fig. 1 is the density obtained by adding the $m = 2$ roton mode to the condensate wavefunction, revealing the localized nature of roton modes near harmonic trap centers. When the scattering length is quenched across the BEC to supersolid transition multiple roton modes become nearly equally unstable, resulting in an intricate modulational instability at the center of the condensate, variable from shot-to-shot.

We perform dynamic interaction quench simulations, constructing the initial state by adding $T = 30\mu K$ of noise to a BEC ground state with parameters similar to Fig. 1, but for $a_s = 95a_0$, i.e. further above the roton instability. Our interaction quench protocol is: after 20ms for thermalization the interaction strength is linearly ramped to $a_s = 88a_0$ over 30ms–crossing the roton phase transition to the supersolid regime–and then held constant again for the remainder of the simulation. The ground state for the final parameters is the pancake crystal shown under Fig. 2(c).

The density isosurfaces in Figs. 2(a1-a4) represent snapshots at different times for a single simulation run. These images reveal intriguing formation dynamics, initially with two droplets developing near the trap center, but in an asymmetric manner [Fig. 2(a2)]. In fact, one droplet forms nearly at the origin, but generally the positions are influenced by the random thermal fluctuations. Only an $m = 0$ roton instability [Fig. 1] could directly form a droplet at the origin, but such a mode cannot produce off-center droplets.

Although the initial droplets are seeded through unstable roton modes, subsequent droplet formation reveals a process of nonlinear crystal growth, as highlighted from the column densities shown as insets to Figs. 2(a). In Fig. 2(a2), two central droplets have already attained their final peak density, while a secondary ring of droplets is beginning to form. Then in Fig. 2(a3), eight droplets have fully matured and the process continues radially outwards. This sequential formation supports that no combination of roton modes [Fig. 1] is directly related to the final 19 droplet ground state, such a relationship would conversely imply that all droplets emerge simultaneously. Interestingly, the resulting crystals are excited, with some outer droplets dissolving and reemerging from the halo.

The colors on the density isosurfaces in Figs. 2(a1-a4) represent the wavefunction phase distributions. The colors are re-centered at the origin in each subplot, so that an ideal phase coherent solution would be blue everywhere. Importantly, the crystal growth process disrupts the global phase coherence, as is evidenced by the various colors in Fig. 2(a4). We have verified that even for a lower initial temperature ($T = 15\mu K$), and alternative interaction ramp protocols, phase coherence is still disrupted by crossing this first-order phase transition. Previous works in 1D have maximized phase coherence by increasing the final $a_s$, and hence increasing the superfluid connection between droplets. However, the droplet number is strongly dependent on $a_s$, and we find that such a strategy significantly decreases the number of droplets.

We quantify the phase excitations after the quench, taking the phase incoherence $\alpha^I$ as presented in Ref. 11. $\alpha^I = 0$ (1) implies global phase coherence (incoherence).
The dynamics are governed by all highly-populated modes up to a fixed energy cut-off. The SeGPE describes the ‘classical’ field, $\Psi(r)$, to test the supersolid formation through evaporative cooling much longer times–around $t=63N$ ms after crystal formation. We note, however, that after $a_s=88a_0$, coherence is restored but the crystal remains highly excited. Simulation videos are provided in supplemental material.

Undeterred by these findings, we perform an alternative dynamic simulation to create a robust 19 droplet, pancake supersolid. We phenomenologically introduce a simple growth Stochastic eGPE (SeGPE) in order to test the supersolid formation through evaporative cooling. The SeGPE describes the ‘classical’ field, $\Psi(r, t)$, of all highly-populated modes up to a fixed energy cut-off. The dynamics are governed by

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{P} \left\{ (1 - i\gamma) (\hat{H}_{\text{GP}}[\Psi] - \mu\Psi) + \eta \right\}. \quad (2)$$

Here, $\gamma$ arises from the coupling of the classical field modes $\Psi$ to the high-lying modes. We find that $\gamma = 7.5 \times 10^{-3}$ gives good agreement to the atom number growth rate of a recent experiment Ref. \[27\]. The dynamical noise term $\eta$, subject to noise correlations given by $\langle \eta^* (r, t) \eta(r', t') \rangle = 2\hbar \gamma k_B T \delta(t - t') \delta(r - r')$, means that each simulation run is unique. Finally, $\hat{P}$ is a projector which constrains the dynamics of the system up to an energy cutoff $\epsilon_{\text{cut}}(\mu) = 2\mu$–consistent with previous treatments \[51, 52\]–for input chemical potential $\mu$.

Each simulation of Eq. (2) begins with 200ms equilibration time at fixed high temperature $T=150nK$, to generate an initial condition emulating a thermal cloud.

To approximate the experimental cooling process, the chemical potential and temperature are linearly changed for a ramp duration of 100ms, from $(\mu, T) = (-95k_B, 150)nK$ to $(95k_B, 30)nK$, mimicking both the changing temperature and condensate number observed in experiments \[53\], whilst the scattering length is always held fixed at $a_s=88a_0$. Figure 2(b) shows a series of snapshots showing the condensate growth. Remarkably, both the crystal structure and the global phase coherence–evidenced by the uniformly blue color [Fig. 2(b4)]–remain robust against thermal fluctuations. More quantitatively, global phase coherence is predicted by the low value of $\alpha_t \approx 0.19$ at late times.

While experiments have evaporatively cooled directly into the supersolid phase for linear and zigzag configurations \[12, 16, 27\], this could prove an optimal method in circular traps for avoiding the excitations associated with crossing the roton instability. We confirm this by producing a 7 droplet hexagon supersolid in a near-circular trap, as shown in Fig. 3. The experimental apparatus and procedure is similar to that described previously \[18\], but new modifications in the optical dipole trap setup have enabled us to tune between anisotropic and round traps. The current optical trap consists of three 1064 nm wavelength trapping beams, each propagating in the plane perpendicular to gravity. Two of the beams, which cross perpendicularly, have approximately 60 $\mu$m waists, and define the horizontal trapping frequencies. The third, crossing at a roughly 45 degree angle from the others, has a waist of approximately 18 $\mu$m and is rapidly scanned to create a time-averaged light sheet that defines the vertical confinement.
We perform the same BdG linearization as above, but focusing here on 2D supersolid excitations. Helping to observe the reproducible interference pattern, it is estimated by a Fourier transform of the wavefunction.

In a harmonic trap with frequencies $f_{x,y,z} = (47(1), 43(1), 133(5))$Hz, we observe in-trap a 7-droplet state consisting of a hexagon with a central droplet, with a condensate atom number of $N \approx 4 \times 10^4$ [Fig. 3(a)]. To confirm that this state is phase-coherent, we release the atoms from the trap and image the interference pattern after 36 ms time-of-flight [Fig. 3(b)]. The presence of clear modulation in the interference pattern after 36 ms time-of-flight (TOF) expansion, averaged over 68 trials of the experiment. Hexagonal modulation structure is clearly present in the averaged image. Note the rotation of the hexagon between in-situ and TOF images. (c,d) Corresponding eGPE result for same trap, and with $a_s = 90a_0$ and $\approx 4.4 \times 10^5$ atoms within the droplets. In (d), TOF expansion is estimated by a Fourier transform of the wavefunction.

In summary, we predict that large, highly-excited, 2D supersolids can be obtained from the unmodulated BEC phase via an interaction quench. Droplets appear sequentially, rather than simultaneously, with the final crystal structure being unrelated to the roton modes that seed the instability. As a result, in contrast to 1D arrays, the quench has a strong first-order nature, resulting in a significantly excited crystal structure and in the disruption of interdroplet phase coherence in the period following crystal formation. We developed a stochastic Gross-Pitaevskii theory to explore evaporative cooling directly into the crystal phase, showing that it provides a more robust path for creating a 2D supersolid. We experimentally employ this method to create a reproducible 7-droplet supersolid in a near-circular trap.

Future studies will explore further the role the thermal cloud has on supersolid formation, improved quench protocols into large 2D supersolids that may result in less

FIG. 4. Static structure factors for (a) linear and (b) 7-droplet hexagon supersolids. Convergence is achieved within the dashed ellipses (see text). Parameters for linear chain: $a_s = 90a_0$, $f_{x,y,z} = (52.83, 130, 167)$Hz, $N = 4 \times 10^4$. For 2D crystal: $a_s = 90a_0$, $f_{x,y,z} = (52.83, 52.83, 167)$Hz, $N = 9.5 \times 10^4$. For 2D, $k_x l_z \approx 1.43$ every 60°, azimuthally. These peaks reflect the hexagonal structure of the ground state, however this value does not directly reflect the inter-droplet spacing (3.05μm, which would correspond to $k_x l_z \approx 1.25$), but rather the spacing of Bragg lattice planes between the droplets. Crucially, the six inner momentum peaks are rotated compared to the droplet crystal, analogous to what we observed experimentally in the TOF images. Similar to the 1D chain, we find that the out-of-phase Goldstone modes—manifestation of superfluidity—contribute to the majority of the peak signal.
heating, as well as the physics of quantum vortices in 2D dipolar supersolids.\cite{11,12,13}

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