Constructing Monotone Homotopies and Sweepouts

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Abstract. This article investigates when homotopies can be converted to monotone homotopies without increasing the lengths of curves. A monotone homotopy is one which consists of curves which are simple or constant, and in which curves are pairwise disjoint. We show that, if the boundary of a Riemannian disc can be contracted through curves of length less than $L$, then it can also be contracted monotonously through curves of length less than $L$. This proves a conjecture of Chambers and Rotman. Additionally, any sweepout of a Riemannian 2-sphere through curves of length less than $L$ can be replaced with a monotone sweepout through curves of length less than $L$. Applications of these results are also discussed.

1. Introduction

The primary objects of study in this article are monotone homotopies, which we define below. Throughout the article, if $\alpha$ is a simple closed contractible curve, then $D(\alpha)$ denotes the closed disc that $\alpha$ bounds. If the underlying surface has at least one boundary component, then this disc is unique. If it is an oriented sphere, then the orientation of the sphere and the orientation of $\alpha$ determines $D(\alpha)$; it is the unique disc for which, given the orientation of the sphere, the induced orientation of the boundary agrees with that of $\alpha$. If $\alpha$ and $\beta$ are two simple closed contractible curves with $D(\beta) \subset D(\alpha)$, then let $A(\alpha, \beta) = A(\beta, \alpha)$ denote the annulus between $\alpha$ and $\beta$, that is, $D(\alpha)$ with the interior of $D(\beta)$ removed. If $\beta$ is a constant curve, then we extend the definition of $A(\alpha, \beta)$ to denote $D(\alpha)$.

Definition 1.1. Let $(M, g)$ be a Riemannian annulus with boundaries $\gamma_0$ and $\gamma_1$, and let $H : S^1 \times [0, 1] \to M$ be a homotopy between $\gamma_0$ and $\gamma_1$, that is, a smooth map such that $H(t, 0) = \gamma_0$ and $H(t, 1) = \gamma_1$. We will say that $H$ is monotone if every intermediate curve $\gamma_t := H(t, \tau)$ is a simple closed curve parameterized by $t$, and if the closed 2-annuli $A(\gamma_\tau, \gamma_1) \subseteq M$ satisfy the inclusion $A(\gamma_{\tau_2}, \gamma_1) \subseteq A(\gamma_{\tau_1}, \gamma_1)$ for every $\tau_1 < \tau_2$. In this definition, $\gamma_0$ and $\gamma_1$ can be constant curves or simple closed curves.

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A monotone contraction of a Riemannian 2-disc is a monotone homotopy from its boundary to a constant curve. We say that such a monotone homotopy is outward if \( D(\gamma_0) \subset D(\gamma_1) \), and is called inward if \( D(\gamma_1) \subset D(\gamma_0) \).

We prove the following two theorems, the first of which was a conjecture by Chambers and Rotman \[9, \text{Conjecture 0.2}\].

**Theorem 1.2.** Suppose that \((D, g)\) is a Riemannian disc, and suppose that there is a contraction of \(\partial D\) through curves of length less than \(L\). Then there is a monotone contraction of \(\partial D\) through curves of length less than \(L\).

The techniques involved in the proof of this theorem also apply in the setting of a Riemannian annulus and a homotopy between its two boundaries through curves of length less than \(L\), yielding a monotone homotopy through curves of length less than \(L\).

The second theorem concerns a similar monotonicity result for sweepouts of 2-spheres. A sweepout of a Riemannian 2-sphere is a map \(f : S^1 \times S^1 \to S^2\) of degree 1. We can regard a sweepout as a 1-parameter family of connected closed curves \(f(t, *)\) parametrized by \(t \in S^1\). These curves might have self-intersections as well as pairwise intersections.

**Theorem 1.3.** Suppose that \((S^2, g)\) is a Riemannian 2-sphere, and suppose that \(f\) is a sweepout of it composed of curves of length less than \(L\). Then there exists a diffeomorphism from the round sphere \((S^2, \text{round}) = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}\) to \((S^2, g)\) such that the length of the image of each parallel \(\{(x, y, z) : z = \text{constant}\}\) \(\cap (S^2, \text{round})\) is less than \(L\).

The proof of this result holds also if we assume only that there exists such a map of odd degree (which is not necessarily equal to 1).

**Background and related work.** These theorems have numerous applications to metric geometry, and to applied topology. In terms of metric geometry, it improves known estimates of the lengths of the shortest geodesics between pairs of points on Riemannian 2-spheres from \[11\] and \[13\]. It also greatly decreases the complexity of these proofs. Furthermore, it allows the results from \[12\] to be generalized to the free loop space of a Riemannian 2-sphere. For more details, we refer to Chambers and Rotman \[9, \text{Section 0.1}\].

Of special note is that Theorem 1.2 directly implies that, if the boundary of a Riemannian disc is contractible through curves of length less than \(L\), then for any point \(q\) on the boundary of the disc, the boundary is contractible to the point \(q\) through loops based at \(q\) of length less than \(L + 2d\). Here, \(d\) is the diameter of the Riemannian 2-disc.

The sweepouts described in Theorem 1.3 appear in minimal surface and min-max literature. In \[7\], Chambers and Liokumovich show that if there is

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1The proof is even simpler in that case, since case \(b\) of Proposition 2.9 never occurs.
a sweepout of a Riemannian 2-sphere through curves of length less than \( L \), then there is a sweepout of the same Riemannian 2-sphere through simple closed curves and constant curves of lengths less than \( L \). They then use this result to answer a question of Freedman about the minmax levels with respect to different classes of sweepouts. Our theorem is an improvement on this result, proving that such a sweepout can be simplified to not only consist of curves which do not have self-intersections (other than constant curves), but to consist of such curves which are (mostly) pairwise disjoint as well.

From the computational topology literature, much recent work has focused on computing a “best” homotopy between two curves as a means of measuring similarity of the curves or determining optimal morphs between them [4, 6, 10]. The main goal in this setting is to determine the computational complexity of such a problem in the most common settings, generally where the two curves are in the plane (possibly with obstacles) or on a meshed surface, as typically produced by surface reconstruction algorithms.

The type of optimality we study in this work has been investigated in a combinatorial setting, where it was called the “height” of the homotopy [2, 3, 10], and in the graph theoretic setting, where it was called a “b-northward migration” [1]. However, the exact complexity of this problem remains open, and both papers include a conjecture that the best such morphings will proceed monotonically. The monotonicity result we present in this paper is a key ingredient in showing that this problem lies in the complexity class \( \mathcal{NP} \) [5].

Finally, Chambers and Rotman formulated another conjecture [9, Conjecture 0.3] on monotonicity, where the initial curve is not the boundary of the disc. We say that a monotone contraction covers a simple closed curve \( \gamma \) if \( \gamma \) is contained in the disc which is the image of that contraction. They conjectured that if \( M \) is a Riemannian surface and \( \gamma \) a simple closed curve...
contractible through curves of length less than \( L \), then there is a monotone contraction covering \( \gamma \) through curves of length less than \( L \). We observe that this conjecture is false by exhibiting the counter-example in Figure 1.

In that example, the underlying surface is an annulus, and the metric is the Euclidean one, except for two mountains, one taller than the other one. The initial closed curve \( \gamma \) lies as shown in the first picture, half-way up the tall mountain from both sides. An optimal contraction is pictured in the following two pictures; it first climbs down the tall mountain for both sides of the curve in order to reduce the length, before climbing over the smaller one. On the other hand, any monotone contraction covering \( \gamma \) must start at a closed curve \( \alpha \) that also lies half-way up the tall mountain from both sides. Then, by monotonicity, only one side of the curve can climb down the tall mountain. Therefore, the maximum length of the curves in such a monotone contraction will need to be strictly larger than for a non-monotone one.

2. Preliminaries

Throughout the article, a closed curve \( \gamma \) in a Riemannian annulus \( A \) is called a minimizing geodesic if it is essential (homotopic to one of the boundaries), and its length is minimal among the essential curves.

**Definition 2.1.** A zigzag \( Z \) is a collection of homotopies \( H_1, \ldots, H_n \) with the following properties:

1. \( H_i(1) = H_{i+1}(0) \)
2. \( H_i \) alternates between outward and inward monotone homotopies, i.e., each of the \( H_i \) is a monotone homotopy, but for any \( i \in \{1, \ldots, n-1\} \), the concatenation of \( H_i \) and \( H_{i+1} \) is not.

We define \( \gamma_0 = H_1(0) \) and \( \gamma_i = H_i(1) \) for \( 1 \leq i \leq n \).

Each \( H_i \) goes from \( \gamma_{i-1} \) to \( \gamma_i \). We define the order of \( Z \), \( \text{ord}(Z) \), to be \( n \).
We will also need the following definitions and a theorem from the article of Chambers and Rotman [9].

**Definition 2.2.** ([9, Definition 0.6]) Let $\alpha : [0, 1] \to M$ and $\beta : [0, 1] \to M$ be two simple closed curves in a Riemannian manifold $M$. If every two points of intersection between $\alpha$ and $\beta$ are consecutive on $\alpha$ if and only if they are consecutive on $\beta$, then $\alpha$ and $\beta$ are said to satisfy the simple intersection property.

When $\alpha$, $\beta$ defined in 2.2 do not satisfy the simple intersection property, we will say that they are meandering with respect to each other.

**Definition 2.3.** Let $\alpha$ and $\beta$ be two simple closed curves in a closed topological 2-disc $D$. Let $\alpha_i = \alpha_{|[t_i, t_{i+1}]}$ be an arc of $\alpha$, such that the interior of the arc does not intersect $\beta$, while its endpoints $\alpha(t_i), \alpha(t_{i+1}) \in \beta$. Then these points subdivide $\beta$ into two arcs. Let $\lambda$ be an arc that together with $\alpha_i$ bounds a disc in the closed annulus $A(\partial D, \beta(t))$ between $\partial D$ and $\beta$. Then we will call $\lambda$ a corresponding arc. We will refer to the disc $D_i$ with the boundary $\alpha_i \cup \beta$ as a corresponding disc, (see fig. 2 (b). The disc that corresponds to arc $\alpha_i$ is shaded).

**Definition 2.4.** Let $\alpha$, $\beta$ be two simple closed curves in a closed topological 2-disc $D$. Suppose $\alpha$ is meandering with respect to $\beta$. We will call an arc $\alpha_i$ of $\alpha$ that intersects $\beta$ only at its endpoints maximal if it is adjacent to the outer face in the planar graph obtained by superimposing $\alpha$ onto $\beta$.

**Proposition 2.5.** Suppose that there is a contraction of $\partial D$ through curves of length less than $L$ then there exists a zigzag of order $n$ such that $\gamma_0 = \partial D$ and $\gamma_n$ is a constant curve, and such that all curves of all homotopies have length less than $L$.

**Proof.** First, by a result of Chambers and Liokumovich [8, Theorem 1.1], we know that there exists a contraction of $\partial D$ through simple closed curves of length less than $L$.

We say that a corresponding disc between two arcs $\alpha$ and $\alpha'$ is $\delta$-thin if there is a reparameterization of $\alpha$ such that $\alpha(t)$ and $\alpha'(t)$ are at distance at most $\delta$ for any $t \in [0, 1]$. Similarly, an annulus $A(\alpha, \beta)$ is $\delta$-thin if there is a reparameterization of $\alpha$ such that $\alpha(t)$ and $\beta(t)$ are at distance at most $\delta$. Now, we consider a discretized version of the contraction $H$, i.e., we consider an increasing sequence of $n$ times $t_i \in [0, 1]$ so that

- $t_0 = 0$,
- $t_n = 1$, and
- for $0 \leq i \leq n - 1$, if $H(t_i)$ and $H(t_{i+1})$ intersect, they have the simple intersection property and the corresponding discs are $\delta$-thin,
for δ to be determined later. If they do not intersect, the annulus 
\( A(H(t_i), H(t_{i+1})) \) is δ-thin.

The existence of this sequence follows directly by compactness. Now, if 
\( H(t_i) \) and \( H(t_{i-1}) \) intersect, for each \( 0 < i < n \), we define an auxiliary 
curve \( H(t_i^f) \) from \( H(t_i) \): \( H(t_i^f) \) is obtained from \( H(t_i) \) by considering 
all of the arcs of \( H(t_i) \) in \( D(H(t_{i-1})) \) and replacing the other ones by the 
arcs they correspond to in \( H(t_{i-1}) \). Then we claim that there are monotone 
homotopies between \( H(t_i) \) and \( H(t_i^f) \), and between \( H(t_i^f) \) and \( H(t_{i+1}) \) 
such that the intermediate curves have length less than \( L \). Indeed, one can 
go from one to the other using monotone homotopies that interpolate within 
the corresponding discs, and if \( \delta \) is chosen small enough, this interpolation 
can be done with an arbitrarily small overhead on the lengths of the curves.

If \( H(t_i) \) and \( H(t_{i-1}) \) do not intersect, for \( \delta \) small enough, the 
\( \delta \)-thin assumption implies that there exists a monotone homotopy between 
\( H(t_{i-1}) \) and \( H(t_i) \), such that the intermediate closed curves have length less than \( L \).

Gluing together all of these monotone homotopies, we obtain a zigzag 
with curves of length at most \( L \).

One of our main technical tools is the following theorem.

**Theorem 2.6.** Let \( H \) be a monotone homotopy between simple closed 
curves \( \gamma_0 \) and \( \gamma_1 \) such that the intermediate curves have length less than 
\( L \), and let \( \gamma \) be another simple closed curve in \( A(\gamma_0, \gamma_1) \) such that \( \gamma \) is a 
minimizing geodesic in \( A(\gamma, \gamma_1) \). Then there exists a monotone homotopy 
between \( \gamma_0 \) and \( \gamma \) where the intermediate curves have length less than \( L \).

Although being not explicitly stated in Chambers and Rotman [9], this 
theorem is implicit in the proof of their Theorem 0.7. More precisely, their 
proof is divided in two steps, and this is the result obtained in Step 1.

The proof of Theorem 2.6 relies on the following two propositions al-
lowing us to modify small portions of zigzags. The first one follows rather 
directly from Theorem 2.6, but the second one requires more work.

**Proposition 2.7.** Suppose that \( Z \) is an order 2 zigzag through curves of 
length less than \( L \). If \( \gamma_1 \) is not a minimizing geodesic in \( A(\gamma_1, \gamma_2) \), and if a 
minimizing geodesic \( \gamma \) in this annulus also lies in the interior of \( A(\gamma_0, \gamma_1) \) 
and is essential in it, then there is a zigzag \( Z' \) where the intermediate curves 
have length less than \( L \) and such that

1. \( \text{ord}(Z') = 2 \).
2. \( \gamma_1' \) minimizes in \( A(\gamma_1', \gamma_2') \).
3. \( \gamma_0' = \gamma_0, \gamma_1' = \gamma, \text{and} \gamma_2' = \gamma_2 \).

Suppose that \( Z \) is an order 2 zigzag where the intermediate curves have 
length less than \( L \), and that \( \gamma_0 \) is a minimizing geodesic in \( A(\gamma_1, \gamma_2) \), or that
where $\gamma_2$ is a minimizing geodesic in $A(\gamma_0, \gamma_1)$. Then there is an order 1 zigzag $Z'$ (i.e., a monotone homotopy) through curves of length less than $L$ and such that $\gamma_0' = \gamma_0$, and $\gamma_1' = \gamma_2$.

**Proof.** The first part of the proposition follows from two applications of Theorem 2.6. We first apply it to the homotopy $H_1$ and the curve $\gamma$, and then to the reversal of the homotopy $H_2$ and the curve $\gamma$. This yields two new homotopies $H'_0$ and $H'_1$, going respectively from $\gamma_0$ to $\gamma$ and from $\gamma$ to $\gamma_2$ and their concatenation satisfies the needed properties.

For the second part of the proposition, let us first deal with the first case where $\gamma_0$ is a minimizing geodesic in $A(\gamma_1, \gamma_2)$. Then one application of Theorem 2.6 to the homotopy $H_2$ and $\gamma_0$ yields the homotopy from $\gamma_0$ to $\gamma_2$. The other case is obtained by applying the theorem to $H_1$ and $\gamma_2$ instead.

**Lemma 2.8.** Suppose that $Z$ is a zigzag of order 2, and that $\gamma_0$ is a minimizer in $A(\gamma_0, \gamma_1)$. Then there exists an essential curve $\gamma$ which is a minimizer in $A(\gamma_1, \gamma_2)$, and which has the simple intersection property with $\gamma_0$.

**Proof.** We begin by choosing an essential minimizing curve $\alpha$ in $A(\gamma_1, \gamma_2)$. Let $\varrho$ be a segment of $\gamma_0$ whose endpoints are intersections between $\alpha$ and $\gamma_0$, and whose interior is contained in the interior of $A(\gamma_1, \alpha)$. Let the endpoints of $\varrho$ be $\varrho(0)$ and $\varrho(1)$.

From $\alpha$ and $\varrho$ we can define two auxiliary curves: one which goes from $\varrho(0)$ to $\varrho(1)$ following $\alpha$ and then back to $\varrho(0)$ along $\varrho$, and one which goes from $\varrho(1)$ to $\varrho(0)$ following $\alpha$ and then back to $\varrho(1)$ along $\varrho$. Let the first curve be $\beta_1$, and let the other one be $\beta_2$. Note that both $\beta_1$ and $\beta_2$ are contained in $A(\gamma_1, \alpha)$. In this annulus, exactly one of $\beta_1$ or $\beta_2$ is essential, without loss of generality, we may assume that it is $\beta_1$. Since $\beta_2$ is not essential, it bounds a disc which we call by a slight abuse of language its interior. If $\varrho$ lies in the boundary of a portion of the interior of $\beta_2$ outside of $A(\gamma_0, \gamma_1)$, then we do nothing.

If $\varrho$ lies on the boundary of a portion of the interior of $\beta_2$ inside of $A(\gamma_0, \gamma_1)$, we claim that $\beta_1$ has total length not greater than that of $\alpha$. Indeed, let us first build an auxiliary closed curve in the following way: take all the maximal segments $\Sigma$ of $\beta_2$ in the interior of $A(\gamma_0, \gamma_1)$. The segments in $\Sigma$ are also segments of $\alpha$, and since $\gamma_0$ is a minimizing geodesic in $A(\gamma_0, \gamma_1)$, any $\sigma \in \Sigma$ is at least as long as its corresponding arc on $\gamma_0$. The closed curve $\alpha'$ is obtained by replacing all the segments in $\Sigma$ from $\alpha$ by their corresponding arcs on $\gamma_0$, it is not longer than $\alpha$. Now, $\alpha'$ may not be simple, in particular there may be double points on $\varrho(0)$ or $\varrho(1)$. These double points allow us to shortcut $\alpha'$ even more, by just removing the portions of $\alpha'$ between them. After these removals, we obtain the curve $\beta_1$, which is by construction not longer than $\alpha'$ and thus not longer than $\alpha$. 
Figure 3. The different curves in the proof of Lemma 2.8.
For the $\varrho$ on the right, we do nothing, while for the $\varrho$ in the middle we can shortcut $\alpha$ by replacing it with $\beta_1$.

We now build $\gamma$ as follows. For every segment $\varrho$ as above, we replace $\gamma$ by the corresponding $\beta_1$ if in the second case. Since there are only finitely many such segments, and since the number of such segments decreases as we apply this procedure, this process terminates. The final curve has length bounded by the length of $\alpha$, is contained in $A(\gamma_1, \gamma_2)$, and is essential in this annulus (since each $\beta_1$ as above has these properties). Since $\alpha$ is minimizing, $\gamma$ is minimizing.

What remains is to show that $\gamma$ has the simple intersection property with $\gamma_0$; if this not true, then there exists a segment $\varrho$ as above in the second case. However, no such segment can exist, which is a contradiction.

$$\square$$

Proposition 2.9. Suppose that $Z$ is a zigzag of order 3 where the intermediate curves have length at most $L$ and such that $\gamma_1$ is a minimizing geodesic in $A(\gamma_1, \gamma_2)$, but is not a constant curve. Then one of the following two cases is true:

**Case a.** There is a zigzag $Z'$ of order 3 such that

1. $\gamma_0' = \gamma_0$, $\gamma_3' = \gamma_3$, and $\gamma_2' = \gamma_2$.
2. There exists a minimizing geodesic $\gamma \in A(\gamma_2', \gamma_3')$ which is fully contained in the interior of $A(\gamma_1', \gamma_2')$.
3. $\gamma_1'$ is a minimizing geodesic in $A(\gamma_1', \gamma_2')$.
4. All curves in $Z'$ have length less than $L$.

**Case b.** There exists a zigzag $Z'$ of order 1 such that

1. $\gamma_0' = \gamma_0$.
2. $\gamma_1'$ is a constant curve.
3. All curves in $Z'$ have length less than $L$. 
Proof. We begin by applying Lemma 2.8 to the order 2 zigzag form $\gamma_1$ to $\gamma_2$ to $\gamma_3$ to obtain an essential minimizing geodesic $\gamma$ in $A(\gamma_2, \gamma_3)$ which has the simple intersection property with $\gamma_1$.

If $\gamma$ lies in $A(\gamma_1, \gamma_2)$, and is essential in this annulus, then we are done as Case a is satisfied.

We now break the proof into two cases as follows.

Case a. Suppose that $\gamma$ is not entirely contained in $A(\gamma_1, \gamma_2)$. Note that one can always modify the homotopy $H_3$ to obtain a new homotopy $H_3'$ between $\gamma_2$ and $\gamma_3$, such that the lengths of the curves in $H_3'$ is less than $L$ and $\gamma$ is one of the curves of $H_3'$. One achieves this by applying Theorem 2.6 to the homotopy $H_3$ and $\gamma$, and then to the reversal of $H_3$ and $\gamma$ and concatenating the two resulting homotopies. Thus, without loss of generality assume that $\gamma = (H_3)_t$ for some $t \in [0, 1]$.

Since $\gamma$ is entirely contained in $A(\gamma_1, \gamma_2)$, the disc bounded by $\gamma$ intersects the disc bounded by $\gamma_1$. Then the new zigzag $Z'$ is obtained in the following manner. First, we form an auxiliary monotone homotopy $\tilde{H}$ from $\gamma_1$ to a new curve $\tilde{\gamma}$ (see fig. 4(a)). This curve $\tilde{\gamma}$ is defined by replacing the segments of $\gamma$ in $A(\gamma_1, \gamma_2)$ with the corresponding segments of $\gamma_1$.

To form the desired homotopy, we replace segments of the curves in $H_3$ which lie inside $A(\gamma_1, \gamma_2)$ with segments of $\gamma_1$, using the fact that the homotopy $H_3$ starts at $\gamma_2$, and $\gamma_1$ is minimizing in $A(\gamma_1, \gamma_2)$. This procedure is completely analogous to the proof of Theorem 0.7 in [9], and will be summarized at the end of the proof. To form our new homotopy, we append the homotopy $\tilde{H}$ to the end of $H_1$, and we append $\tilde{H}$ in reverse direction to the beginning of $H_2$. Clearly, properties 1 and 4 are satisfied.

Property 2 follows from the fact that $\gamma$ and $\gamma_1$ have the simple intersection property.

To prove property 3, we fix any essential curve $\gamma'$ in $A(\gamma'_1, \gamma'_2)$. If we replace segments of $\gamma'$ which lie in $A(\gamma_1, \gamma_2)$ with segments of $\gamma_1$, then replace segments of the resulting curve which lie in $A(\gamma'_2, \gamma)$ with segments of $\gamma$, we obtain $\tilde{\gamma} = \gamma'_1$. This procedure does not increase the length, so
Figure 5. Discontinuities might occur...

this shows that the length of $\gamma'$ is greater than or equal to the length of $\gamma'_1$, completing the proof.

Case b. Suppose that $\gamma$ is contained in $A(\gamma_1, \gamma_2)$, and is non-essential in it. In this case, the disc bounded by $\gamma$ does not intersect the disc bounded by $\gamma_1$. Thus, by monotonicity, the homotopy $H_3$ “sweeps” $D(\gamma_1)$ completely. Denote by $p$ one of the last points of $D(\gamma_1)$ swept by $H_3$, that is, a point on $\gamma_1$ such that $p \in D(H_3(t))$ for some $t$ but $D(H_3(t') \cap D(\gamma_1)) = \emptyset$ for any $t' > t$. We will construct a new homotopy $H'_1$ between $\gamma_0$ and the constant curve $p$, which will complete the proof. Let us denote by $H'_3$ the restriction of $H_3$ to the interval $[0, t]$. The homotopy $H'_1$ is built almost identically to the ones in Case 1. We begin by replacing segments of curves from $H_3$ which lie outside of $D(\gamma_1)$ by the corresponding arcs of $\gamma_1$. Once again, details of this construction are deferred to the end of the proof. Notice that by definition of $p$, the curve $\tilde{\gamma}$ we obtain at the end of this modified homotopy is contained in the boundary of $D(\gamma_1)$, and is homotopic to $p$ within this boundary. Since this homotopy can be performed without increasing the lengths of the curve, we define $\tilde{H}$ by the concatenation of these two homotopies (one formed by replacing curves from $H_3$, and one formed by contacting the end curve which lies in $D(\gamma_1)$). We then concatenate $\tilde{H}$ to the end of $H_1$ to form $H'_1$ and append $\tilde{H}$ in the reverse order to the beginning of $H_2$ to obtain the new zigzag. Clearly, both properties are satisfied.

Therefore, the proof of the lemma will follow if we can show that there exists a monotone homotopy $H'_1$ between the curves $\gamma_1$ and $\tilde{\gamma}$ where the length of the intermediate curves is less than $L$. To keep the proof simple, we focus on Case a; case b is completely analogous. The existence of such a homotopy follows from the construction given in the proof of Theorem 0.7 in [9].

$H'_1$ will be a concatenation of two homotopies: $H_1$ and a monotone homotopy $G$ between $\gamma_1$ and $\tilde{\gamma}$ obtained from $H_3$ by replacing the segments
of curves of the homotopy $S$ that lie outside the closed disc bounded by $\gamma_1$ by segments of $\gamma_1$; these segments are not longer than the corresponding segments that they are replacing via the procedure described in the previous paragraph. The main difficulty lies in implementing this procedure continuously with respect to the curves in the homotopy. In fact, stated as it is the procedure can result in discontinuities, which appear when the replacement algorithm is not unique. Let $\alpha_s$ denote the curves of homotopy $S$. If for some $s \in [0, 1]$ the intersection between $\alpha_s$ and $\gamma_1$ is not transversal, the procedure can be discontinuous at $\alpha_s$. Fig. 5 depicts such a situation. Here $\alpha_2$ touches $\gamma_1$ at point $Q$. There are two ways to exchange the segments of $\gamma_2$ in the neighborhood of $Q$, (see fig. 6 (a) that depicts this situation locally). One way is to replace the segment of $\alpha_2$ that connects the points $Q_1$ and $Q_2$ that lies outside of the open disc bounded by $\gamma_1$ by path $P_1$, (see fig. 6 (b)). Let us call this replacement the type 1 replacement. Another way is depicted in fig. 6 (c). Here we replace the segment of $\alpha_2$ that connects $Q_1$ and $Q_2$ by $P_2$. $P_2$ is a path that consists of two paths: the first one replaces the segment of $\alpha_2$ that connects $Q$ and $Q_2$, while the second one, $\beta$, replaces the segment of $\alpha_2$ that connects $Q_1$ and $Q$. Let us call this replacement the type 2 replacement. However, while there are two ways of replacing this segment of $\alpha_2$, our procedure gives one canonical way to replace the relevant part of $\alpha_1$, a curve that is close to $\alpha_2$ and is outside of the disc bounded by $\alpha_2$ (fig. 5). If we want the procedure to result in a homotopy, that forces us to choose the type 2 replacement on $\alpha_2$. On the other hand, there is also, one type of replacement that can be performed on $\alpha_3$, the curve that is close to $\alpha_2$ and lies inside the disc that is bounded by $\alpha_2$. Again, if we want that our procedure to result in a homotopy, it forces us to choose the type 1 replacement for $\alpha_2$. Hence, we have a discontinuity at $\alpha_2$. To avoid this discontinuity, note that $P_2 = \beta \ast \bar{\beta} \ast P_1$, (see fig. 6 (c)). Here $\bar{\beta}$ denotes path $\beta$ traversed in the opposite direction. Therefore, $P_1$ and $P_2$ can be connected by the obvious length non-increasing path homotopy, which amounts to contracting $\beta \ast \bar{\beta}$ to $Q_1$. This path homotopy extends to
the homotopy between the two curves derived from \( \alpha_2 \). Allowing both the type 1 and type 2 replacements for the segment of \( \alpha_2 \) and including the homotopy between the two different resulting curves solves the discontinuity problem.

Note that this new zigzag \( Z' \) still satisfies the first property of the hypothesis of the proposition.

\[ \square \]

**Proposition 2.10.** Suppose that \( Z \) is a zigzag of order 2 on a Riemannian sphere such that all curves have length less than \( L \), and such that \( \gamma_0 \) is a constant curve, but \( \gamma_1 \) and \( \gamma_2 \) are not constant curves. Furthermore, assume that the orientation of the sphere is such that the discs bounded by curves close to \( \gamma_0 \) in the first monotone homotopy are close to the image of \( \gamma_0 \). Then one of the following two statements are true. First, there exists an order 2 zigzag \( Z' \) with \( \gamma'_0 = \gamma_0 \), \( \gamma'_2 = \gamma_2 \), and \( \gamma'_1 \) is a minimizing geodesic in \( A(\gamma'_1, \gamma'_2) \). Second, there exists an order 1 zigzag \( Z' \) such that \( \gamma'_0 \) is a constant curve, and \( \gamma'_1 = \gamma_1 = \gamma_2 \).

**Proof.** Let \( \gamma \) be a minimizing geodesic in \( A(\gamma_1, \gamma_2) \). There are two possibilities:

1. \( \gamma_0 \) is contained in both \( D(\gamma_1) \) and \( D(\gamma) \).
2. \( \gamma_0 \) is contained in exactly one of \( D(\gamma_1) \) and \( D(\gamma) \).

If the first condition is true, then we proceed as in the proof of Proposition 2.7. If the second condition is true, then \( H_1 \) “sweepouts out” \( \gamma \), as in the proof of Case (b) of Proposition 2.9. Let \( p \) be the first point on \( \gamma \) which curves of \( H_1 \) intersect at point \( t^* \), that is, \( \gamma \cap H_3(t^*) = \{p\} \), but the intersection is empty for all \( t < t^* \). Note that we can perturb our original zigzag so that the first intersection is a single point while keeping the lengths of curves less than \( L \). If we replace all segments of curves in \( H_1 \) which lie in \( A(\gamma, \gamma_1) \) with the corresponding segments of \( \gamma \), we obtain a monotone homotopy from \( p \) to \( \gamma \). As in the start of the proof of Proposition 2.9, we may assume that \( \gamma \) occurs as a curve in \( H_2 \). Removing the first part of \( H_2 \) from \( \gamma_2 \) to \( \gamma \), and replacing it with the monotone homotopy that we have produced from \( p \) to \( \gamma \), we obtain an order 1 zigzag that starts at a constant curve, passes through \( \gamma \), and ends at \( \gamma_2 \).

\[ \square \]

### 3. Proof of Theorems 1.2 and 1.3

We first find a zigzag which starts at the boundary of the Riemannian disc, ends at a constant curve, and traverses curves of length less than \( L \) which minimizes the order of the zigzag.
Proposition 3.1. Suppose that there exists a contraction of $\partial D$ through curves of length less than $L$. Then there is a zigzag $Z$ of finite order, which consists of curves of length less than $L$, and which begins on $\partial D$, and ends at a point. Furthermore, for every zigzag $\tilde{Z}$ with these properties, the order of $\tilde{Z}$ is greater than or equal to the order of $Z$.

The proof follows directly from Proposition 2.5, and from the fact that the order of a zigzag is a positive integer. We will need one more lemma before we can prove our two theorems.

Lemma 3.2. Suppose that $Z$ is a zigzag of order $n \geq 2$ through curves of length less than $L$, and suppose that at most the initial and final curves are constant. Suppose further that $\gamma_1$ is a minimizing geodesic in $A(\gamma_1, \gamma_2)$. Then one of the following is true. First, there exists a zigzag $Z'$ of order $n$ with $\gamma'_0 = \gamma_0$ and $\gamma'_n = \gamma_n$, every $\gamma_i$ is a minimizing geodesic in $A(\gamma_i, \gamma_{i+1})$ for all $i \in \{1, \ldots, n-1\}$, and some minimizing geodesic in $A(\gamma_{i+1}, \gamma_{i+2})$ is contained and essential in $A(\gamma_i, \gamma_{i+1})$ for all $i \in \{1, \ldots, n-2\}$. Second, there exist two zigzags, $Z'_1$ and $Z'_2$ of orders $m_1 > 0$ and $m_2 > 0$ with $m_1 + m_2 = n$ through curves of length less than $L$, and such that the first curve of $Z'_1$ is equal to the first curve of $Z$, the last curve of $Z'_2$ is equal to the last curve of $Z$, the last curve of $Z'_1$ is a constant curve, and the first curve of $Z'_2$ is equal to the same constant curve.

Proof. We will prove this lemma by induction on $n$, and by using Proposition 2.9 and Proposition 2.7. If $n = 2$, there is nothing to prove. If $n = 3$, then we apply Proposition 2.9 to $Z$, followed by applying Proposition 2.7 to the final order 2 zigzag. If, during the process, we produce a zigzag of smaller order which ends at a constant curve, then we terminate this procedure.

For the inductive step, we first apply the induction hypothesis to the order $n-1$ zigzag at the beginning of $Z$. If we obtain a zigzag of smaller order which ends at a constant curve, then we are in the second case of the lemma and we are done. Otherwise, we obtain a new zigzag $Z'$ where $\gamma_{n-2}$ is a minimizing geodesic in $A(\gamma_{n-2}, \gamma_{n-1})$. Thus we are in a position to apply Proposition 2.9 to the order 3 zigzag at the end of $Z'$, from $\gamma_{n-3}$ to $\gamma_n$. If we are in case $b$ of that proposition, we are done since we obtain a zigzag of smaller order ending at a constant curve.

Otherwise, we obtain a new zigzag, which we replace into $Z'$ to yield $Z''$. Since $\gamma_{n-2}$ may have moved in the last step, it may be the case that $\gamma_{n-3}$ is not a minimizing geodesic in $A(\gamma_{n-3}, \gamma_{n-2})$ anymore. In order to fix this, we apply the induction hypothesis once again, this time to the $n-2$ zigzag at the beginning of $Z''$, yielding yet another zigzag $Z'''$ (once again, we are done if we are in the second case of the lemma). Since $\gamma_{n-2}$ and $\gamma_{n-1}$ have
not been changed in this last step, we still have that $\gamma_{n-2}$ is a minimizing geodesic in $A(\gamma_{n-2}, \gamma_{n-1})$, and some minimizing geodesic in $A(\gamma_{n-1}, \gamma_n)$ is contained and essential in $A(\gamma_{n-2}, \gamma_{n-1})$. Now, either $\gamma_{n-1}$ is a minimizing geodesic in $A(\gamma_{n-1}, \gamma_n)$ and we are done, or we can apply Proposition 2.7 to the final order 2 zigzag of $Z'''$. The resulting zigzag fulfills all the properties of the lemma.

We now have all the tools to prove our main theorems.

**Proof of Theorem 1.2.** Let $Z$ be a zigzag which satisfies the conclusion of Proposition 3.1. If the order of $Z$ is equal to 1, then the proof is finished. As such, assume that $ord(Z) > 1$. Since the zigzag must end at a constant curve, $ord(Z) \geq 3$. We may further assume that no other curve in the zigzag is a constant curve. Additionally, since $\gamma_0 = \partial D$, $A(\gamma_1, \gamma_2)$ is contained in $A(\gamma_0, \gamma_1)$. As a result, we can apply Proposition 2.7 to replace $Z$ with a zigzag with the property that $\gamma_1$ is minimizing in $A(\gamma_1, \gamma_2)$ (this also uses the fact that we cannot produce a zigzag of shorter order from $\partial D$ to a constant curve). As a result, we may assume that $Z$ has this property.

We now apply Lemma 3.2 to $Z$. Since we cannot find a zigzag of smaller order which begins at $\partial D$ and ends at a constant curve, the result is an order $n$ zigzag satisfying the conclusions of the lemma. In particular, $\gamma_{n-1}$ must be a minimizing geodesic in $A(\gamma_{n-1}, \gamma_n)$, but must not be a constant curve. However, $\gamma_n$ is a constant curve, and so $\gamma_{n-1}$ must also be constant, having length 0. This is a contradiction, completing the proof.

We can use a very similar technique to prove Theorem 1.3:

**Proof of Theorem 1.3.** To prove Theorem 1.3, we proceed in a similar way. We first apply Theorem 1.2 from [7], which tells us that we can replace our sweepout $f$ of our Riemannian sphere $(S^2, g)$ of curves of length less than $L$ by a sweepout which contains only simple closed curves and constant curves. Let this sweepout be parametrized by $[0, 1]$, where 0 and 1 are mapped to the same constant curve. Since it is smooth, we can find a finite number of subintervals $I_1, \ldots, I_k$ of $[0, 1]$ such that the boundaries of $I_i$ are mapped to constant curves, the interior of $I_i$ is mapped to simple closed curves, and the degree of the map of $f$ restricted to $I_i$ is $d_i \neq 0$. Furthermore, the sum of all of the degrees of these maps is equal to $1$, the degree of the map. As a result, there is at least one such map that has odd degree.

We now apply Proposition 3.1 to this map to produce a zigzag which starts and ends at constant curves, and contains no other constant curves. Since this zigzag is homotopic to the original map, it also has odd degree. Let $Z$ be a minimal zigzag of odd degree on the sphere which begins and
ends at a constant curve, and only passes through simple closed curves, and which has minimal order.

If $Z$ has order one, then we are done. Otherwise, $Z$ has order at least 3, and we first apply Proposition 2.10 and then Lemma 3.2 to $Z$; if we can divide it into two zigzags (the second possibility of the lemma), then the concatenation is homotopic to $Z$, and so has odd degree as a map, and so one of the zigzags must have odd degree as a map but order smaller than $n$, which contradicts the minimality of the order of $Z$. If we are in the first conclusion, then the result is homotopic to $Z$, and so has odd degree as a map. As in the proof of Theorem 1.2, $\gamma_{n-1}$ must be a constant curve, as it must be a minimizing geodesic in $A(\gamma_{n-1}, \gamma_n)$, and $\gamma_n$ is a constant curve with length 0. Thus, the degree of the last segment of $Z$ is 1, which contradicts the minimality of the order of $Z$. This completes the proof.

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