A Research on Robust Safety Stock Allocation Model to Deal with Uncertain Leadtime In an Acyclic Supply Network

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**Abstract**: Safety stock allocation problems including uncertainty in the supply-side have forced to assume a specific distribution of the lead time yet this is usually not reliable. From a guaranteed-service approach, a new model based on robust optimization in which no distribution is assumed is proposed. The robust model is a mixed-integer linear program. Then, the model is validated and applied to several complex real-case supply chains. Preliminary results are promising and show how the model could be a powerful tool for practitioners to manage risks from the supply-side in a robust yet efficient way.

**Key Words**: Lead time, Robust analysis, Guaranteed-service, Inventory control, Safety stock

1. Introduction

Supply Chain Management (SCM) comprises the management of supply chain assets and product, information, and fund flows to maximize total supply chain profitability (Chopra and Meindl, 2001). Inventory control is then one of the activities that involves SCM and it has been taking a priority role for both practitioners and academia. It is not only because of its significant share in the total assets of the companies (from 20 to 60 per cent (Willems, 2008)) but because of its intrinsic complexity. In fact, a relevant report (CSCO, 2011) stated that majority of managers considers inventory optimization as the main area to improve for real-world supply chains. It should be noted that these supply chains are usually general acyclic systems, conformed by several stages with one or more specific functions such as distribution, manufacturing, procurement, retailing or transportation.

Uncertainty in supply and demand side is, to a greater or lesser extent, inherent to every system and the difficulty of modelling the network efficiently. In the worst case, it could lead to operational disruption when the risks and their consequences are not sufficiently modelled. In 2012, more than 8 out of 10 companies recognized to have suffered from supply-side and demand-side disruptions in the previous two years (Lee et al., 2012). To face these uncertainties, companies rely on safety stock, that serves as a buffer to respond to demand and supply variations in order to mitigate stock-out risk. Therefore, a critical issue is to determine the amount of safety stock that each stage has to keep to holistically minimize the risk of disruption along the network while maximizing the profit.

The model that first had a wider extension was the Stochastic Service (SS) model. In its initial version, the demand followed a probabilistic distribution while the lead time was deterministic (Clark and Scarf, 1960). In the SS model, safety stock stands as the only means to deal with uncertainty (Klosterhafen et al., 2013). The other significant branch that has recently grown in attention is the Guaranteed Service (GS) model, approach that is taken in this research, in which safety stock is employed to meet demand up to a certain bound. Other measures such as expediting should be taken to meet demand above this bound and are not considered inside the model. Operational flexibility is the main difference with the SS model. The model was first proposed by Simpson (1958), being finally extended to general supply chains by Humair and Willems (2011). It was not until the last decade when the model was successfully implemented. In consequence, 80% of the research papers have been launched since then.

In the vast literature, stochasticity in demand due to customer’s behavior has been intensely covered. Nevertheless, supply uncertainties have received less attention despite practitioners have often claimed this weakness in their models. Stochastic lead time has been a common extension in the SS literature while only a few have dealt with this issue in the GS model (Humair et al., 2013; Rambau and Schade, 2012). Moreover, when stochastic lead time was considered, it was either normally distributed or characterized by historic data. Normality assumption is unwarranted in general (Eppen and Martin, 1988) and historic data is often not available or unreliable.

The aim of this research is to propose a new model that extends uncertain lead time to the GS Model without assuming any probabilistic distribution. For that purpose, a robust optimization approach is used. The solution is meant to be reliable and immunized against uncertainty of lead time (Ben-Tal et al., 2009). To the best of our knowledge, this is new. Real-world supply chains (Willems, 2008) are used to validate the model and analyze its performance. The article is structured as follows: Section...
2 covers the state of art of the three big issues covered of the article (GS model, uncertain lead time and robust optimization). In Section 3, the model is proposed and explained in detail. Chapter 4 starts with some considerations about the computational analysis and expounds the main results. Chapter 5 finishes with the conclusions and future lines.

2. Theoretical Framework and Literature Review

An introduction to the core issues of this research is conducted in this chapter. Starting from the more general approach (stochastic service vs. guaranteed service), the GS model is explained as well as its most relevant extensions and current successes. The second subsection is focused on uncertain lead time and how it has been covered by researchers, especially in regard to the GS model. In the third subsection, the robust approach is presented and relevant works related to operations management are mentioned.

2.1 Guaranteed Service Model

There are mainly two competing research approaches to safety stock models for multi-echelon networks: the stochastic-service and the guaranteed-service model. The SS does not allow any flexibility measure for dealing with demand, which has a stochastic nature. Therefore, safety stock is the only mean to cope with uncertainty. On the other hand, the GS model assumes that some extraordinary measures (i.e., expediting, external subsupplier, overtime) could be taken, apart from the safety stock, to satisfy demand and ensure that the demand is always met (guaranteed) within a given service time.

From these definitions, each of the approaches have to make strong assumptions. The SS states that the system behaves the same under all demand conditions while the GS assumes that only a share of demand is met by safety stock, the demand that is comprised within a given boundary of demand. Comparisons of performance using one or other approach has been conducted in different research (Graves and Willems, 2003; Klosterhalfen and Minner, 2010). For an extensive survey on SS, the reader could study van Houtum (2006). Analogously, Eruguz et al. (2016) carried a thorough literature review on the GS model. In this research, the GS approach is adopted.

There are some other assumptions on the initial model proposed by Simpson (1958) and lately extended by Graves and Willems (2000). The lead time is supposed to be deterministic, there are no capacity constraints, service times are unique for each stage, the replenishment policy is periodic-review base stock policy and the extraordinary measures are not modeled explicitly (Eruguz et al., 2016). Later research has been focused on relaxing these assumptions in order to make it easier for practitioners to adopt it. A list of these relaxations with their associated references can be found in Eruguz et al. (2012).

Regarding the nature of the model, Eruguz et al. (2012) argue that the SS has had limited deployment because of its complexity to be implemented in real-word networks, where it is not usual to find serial, assembly or distribution topologies. Dynamic optimization is usually required in GS, since the model is a nonlinear, separable concave minimization problem subject to linear constraints. Magnanti et al. (2006) use successive piecewise linear approximation to reformulate the problem into a linear concave model. This idea is also integrated in the model proposed in section 3. Therefore, general acyclic networks, regardless of its complexity, could be assessed.

Another promising feature of the GS model is the performance in companies that have adopted it. Major companies such as HP, Procter and Gamble, Kraft Foods or Black & Decker have proved its efficiency in industrial environments with significant savings in every case (Humair et al., 2013; Eruguz et al., 2012).

2.2 Uncertain Lead Time

The majority of multi-echelon models for both, the GS and the SS approach, assumes that the lead time is deterministic. However, practitioners are usually reluctant to model their system without capturing this key issue. To a greater or lesser extent, all supply chains experiment lead-time variability (Humair et al., 2013). The variation is typically different depending on the part of the supply chain. In general terms, raw material locations face longer delays while retailers can be more precise. Even when the processing times are deterministic, occasional stock-outs due to demand peaks show its stochastic nature.

Researchers, aware of this reality, have extended models and proposed heuristics in which lead time is stochastic and its probabilistic distribution is known. Since this approach could be adequate for many stages, it is not trivial to determine this distribution. A common assumption in the literature is the normal approximation. However, Eppen and Martin (1988) already showed that this hypothesis is unwarranted in general and could lead to flawed results. When available, a distribution based on historical data is preferable. Real-world considerations for determining lead time variability could be found in Humair et al. (2013).

However, historical data might not be available or can even be not representative of the future behavior of the suppliers. Occurrence of abnormal risks could produce significant delays in their expected lead times. Thus, a model that takes into account lead time variability without assuming any distribution seems beneficial for dealing with supply-side variability.

In concern with the GS model, there is few research that considers lead time variability. Eruguz et al. (2012) already pointed random lead time as a useful but complex relaxation that the GS framework should develop. Later on, Humair et al. (2013) achieved this landmark, assuming a certain distribution for the lead time. Similarly, Rambau and Schade (2012) proposed an stochastic GS model with recourse to deal with uncertainty and keep control of the usage of operational flexibility at the time, based on different lead time/demand scenario.

2.3 Robust Optimization approach

For the following explanation, we rely on Ben-tal (2009). A robust optimization is a methodology for coping with optimization problems in which there is uncertain data.
The robust solution will be feasible for every instance and optimal for the worst instances of the uncertain parameters. This is why it is generally stated that robust optimization gives a solution that is immunized against uncertainty. Formally, an uncertain linear programming (LP) problem is defined as

$$\min \sum_{i \in N(G)} h_i y_i$$

(2)

with the data \((c, A, B)\) varying in a certain perturbation set \(U\).

The tractability of the program will be characterized depending on the perturbation set chosen. Common sets are box, elliptical and conic sets. The uncertain data in our current formulation is the lead time \(L_i\). The lead time will be characterized by its nominal value \(L_i\) and its half-length interval \(\bar{L}_i\).

Robust optimization has been applied to many different fields taking an important role in control, statistics or machine learning. It has also been extensively used in operations management research, with applications in inventory theory (Bertsimas and Thiele, 2006), supply chain management (Li and Liu, 2013) or closed-loop networks (Pishvaee et al., 2011), among others. To the best of our knowledge, robust analysis has not been applied to cope with supply uncertainty, neither to the GS model.

3. Model Formulation

In the first section, the classic GS Model will be presented for the sake of clarity and coherence with the former extensions. Then, the successive piecewise linear approximation and the robust approach is integrated in the model, using this later version for the computational results. Notation is based on Rambau and Schade (2012).

3.1 GS Deterministic Model

The SC graph \(G\), which is acyclic and can take any topology, is formed by \(N\) stages. \(N(G), A(G)\) and \(D(G)\) represent, respectively, the set of nodes, arcs and leaves or demand nodes in \(G\). Inventory holding costs per stage, \(h_i\), are the only costs assessed in the model. The given service time is the service time guaranteed by the demand nodes to external clients. These are the parameters of the model, as well as the lead time \(L_i\) and \(\phi_i(x_i)\), which expresses the level of safety stock in terms of the net replenishment lead time. The decision variables are the service times, \(s_{i}^{in}\) and \(s_{i}^{out}\), the net replenishment lead time, \(x_i\), and the order-point \(y_i\). The inbound service time \(s_{i}^{in}\) is the maximum time that a stage is quoted by its predecessors. The outbound service time, \(s_{i}^{out}\), is the delivery time that a stage guarantees to its successors. The mathematical formulation of the linear optimization is then:

$$\min \; \sum_{i \in N(G)} h_i y_i$$

(2)

$$s_{i}^{out} \geq s_{i}^{out} \forall i \in D(G)$$

(3)

$$s_{i}^{in} \geq s_{j}^{out} \forall j \in A(G)$$

(4)

$$x_i \geq s_{i}^{in} - s_{i}^{out} + L_i \forall i \in N(G)$$

(5)

$$y_i \geq \phi_i(x_i) \forall i \in N(G)$$

(6)

$$s_{i}^{in}, s_{i}^{out}, x_i, y_i \geq 0 \forall i \in N(G)$$

(7)

$$y_i \in \mathbb{Z} \forall i \in N(G)$$

(8)

In this initial formulation, lead time is deterministic. It should be noted that demand is bounded by \(\phi_i(x_i)\), which is assumed to be concave, separable and non-decreasing (6). The first constraint (3), guarantees that external demand is met within the service time required. (4) ensures that the inbound service time is equal to the maximum outbound service time of their immediate successors. (5) defines the net replenishment time (more in Humair et al., 2013). The following constraints establish non-negativity of the variables (7) and the integer nature of the order-point variable (8).

Thus, the model is a nonlinear, separable concave minimization problem subject to linear constraints (Magnanti et al., 2006). This problem is NP-hard in general (Lesnaia, 2004). In the next subsection the approximation taken to solve the tractability is shown and uncertain lead time is incorporated.

3.2 Linear Approximation and Robust Approach

To convert the model into a mixed-integer programming (MIP) problem, a piecewise linear concave approximation is used for \(\phi_i(x_i)\). Figure 1 shows the structure and the notation used for this approximation. The function is integrated in the optimization problem by a common “multiple choice” approach (Magnanti et al., 2006). The problem could then be solved by commercial software, which can cope with large MIP problems.

![Fig. 1 Concave linear piecewise function (Magnanti et al., 2006)](image-url)
In the deterministic GS model, the lead time is assumed to be fixed. For our formulation, it is assumed that the lead time is uncertain within a certain interval, defined by $\tilde{L}_i$ and $\hat{L}_i$. The only uncertain data appears in constraint (3c). Note that there is only one uncertain parameter for each $i$th constraint. Thus, the only possible uncertain set is a box:

$$L_i = \tilde{L}_i + \zeta \hat{L}_i \forall i \in N(G), \zeta \in [-1, 1] \quad (9)$$

Inspired by the budgeted uncertain set presented in Bental et al. (2009) and deployed in Bertsimas and Thiele (2006) in which the uncertainty is bounded, $|\zeta|$ is a parameter set by the user between 0 and 1. Note that a null (2006) in which the uncertainty is bounded, $L_i$ and $\hat{L}_i$ are parameters set by the user between 0 and 1. Note that a null $\zeta$.

The model is evaluated for two different types of real world supply chains. In the database used (Willems, 2008) there are some supply chains that considers lead time as a random variable and some others in which the lead time is deterministic. Therefore, for the first case, a comparison between the robust model and the deterministic model has been done. For the second type of supply chains, the effect of the variability in the lead time has been analyzed, by assuming a certain interval of uncertainty in the deterministic value. Then, the robust model has been applied. In total, the model has been applied to the first twenty supply chains (out of 38), being six of them lead time deterministic supply chains. The number of stages varies from 8 to 159. The lead time variability is expressed either by a discrete distribution based allegedly on historical data or by a normal distribution. In the later situation, the standard deviation for each stage. The service level is only provided on the demand stages so the mean value of the service level is taken for internal nodes. This expression is separable, concave and non-decreasing, having been the most common expression for applications of the GS model (i.e.: Graves & Willems, 2003; Klosterhalfen and Minner, 2007) but some other expressions have been considered, for example, in Humair and Willems (2011), in which they take risk pooling into account.

4. Computational Results

4.1 Initial Considerations

The model is evaluated for two different types of real world supply chains. In the database used (Willems, 2008) there are some supply chains that considers lead time as a random variable and some others in which the lead time is deterministic. Therefore, for the first case, a comparison between the robust model and the deterministic model has been done. For the second type of supply chains, the effect of the variability in the lead time has been analyzed, by assuming a certain interval of uncertainty in the deterministic value. Then, the robust model has been applied. In total, the model has been applied to the first twenty supply chains (out of 38), being six of them lead time deterministic supply chains. The number of stages varies from 8 to 159. The lead time variability is expressed either by a discrete distribution based allegedly on historical data or by a normal distribution. In the later situation, the maximum value assumed is the mean value plus three times its standard deviation. The uncertain interval when a distribution is given is defined by its minimum and maximum value.

The size of the problem is related to the $R$ pieces of the linear approximation. For this purpose, the algorithm proposed in Magnanti et al. (2006), is deployed. The software used for the program is Gurobi, coded in Python. Computation times are not long in any of the networks used (less than a minute).

To model the level of inventory needed, the expression assumed is $\Phi_i(x_i) = k_i \sigma_i \hat{\gamma}(x_i)$, where $k_i$ is determined by the service level of each stage and $\sigma_i$ is the demand standard deviation for each stage. The service level is only provided on the demand stages so the mean value of the service level is taken for internal nodes. This expression is separable, concave and non-decreasing, having been the most common expression for applications of the GS model (i.e.: Graves & Willems, 2003; Klosterhalfen and Minner, 2007) but some other expressions have been considered, for example, in Humair and Willems (2011), in which they take risk pooling into account.

4.2 Results and analysis

Table 1 shows the main results for the supply chains that treats lead time as a random variable. The first column, SC, correspond to the number of supply chain in the public-available supply chain. Then the pipeline stock and safety stock, in absolute values, are displayed. Mid represents the solution of the GS model when the lead time has no variability. The term has to do with the fact that the
lead time value taken is not the mean but the middle value between the minimum and the maximum. Therefore, no distribution is being assumed. Max corresponds to the solution of the robust model when the lead time takes its worst instance value, inside the uncertain interval. $\Delta ss$ is the safety stock variation (in percentage) between the two cases. The next column reflects the variation of the safety stock cost. $S_{ss}$ is the outbound service time in average, that means, the mean of the outbound service time considering all the stages of the supply chain. The next two columns, Stages $w$ stock, shows how many stages hold stock after the solution is applied. Finally, stages stock lead-time expresses the number of stages with stochastic lead time and $N$ is the total number of stages. The last column is the relative variation of lead time between the worst case and the deterministic case.

According to this, it can be stated that generally the number of stages holding stock does not vary or not significantly due to lead time variability. Aggregating data, 481 stages hold safety stock for the mid case and 488, seven stages more, for the worst-case scenario. Nevertheless, this should not be taken as a rule. The normal trend for the robust solution is to have the same or more number of stages with safety stock (six remains the same, five increases the stock) but two of them reduce the number of stages holding stock. For example, SC10 goes from 52 stages to 37 stages, even though the lead time considered is 35% bigger in average. The holding cost structure of its particular supply chain is decisive in this.

Regarding not the number of stages but the safety stock amount, the trend is similar. Most of the supply chains react to uncertainty by increasing the total amount of safety stock. This is the case for all the supply chains except for two (SC7 and SC10). In this case, the safety stock allocation policy changes from more stock in stages with less holding cost to less stock in stages with more costly holding cost. However, the cost obviously increases when the supply chain faces uncertainty from suppliers. The next column, $\Delta ss$ cost shows this in relative terms. We can see that the cost related to the safety stock goes from a 3% difference to a maximum of 40%. To effectively compare this variation, it should be considered the number of stages with random lead time and the relative lead time variation. For example, in SC16, 106 out of 145 stages are characterized with random lead times, varying in average 92%, and the increase of safety stock cost is 31%. The company should then consider if this extra cost is worth to take to avoid unmet demand, especially if it is taken into account that the safety stock costs represents, for this SC, just one fifth of the total stock cost.

Another important issue are the outbound service times, because the longer they are, the less flexible the chain becomes. Usually, uncertainty in supply chain increase it, so the time that each stage each stage quotes to its successor is longer. There are no exceptions for the sample assessed, anyhow, the relative variation is significantly different from one chain to another. The companies could easily impose restrictions on the maximum service time for internal nodes or capacity constraints in case some optimal solutions do not seem realistic or feasible for them.

The other analysis is done in supply chains 2, 4, 13, 17, 18 and 19. Theses supply chains do not consider (in the database) lead time variability. Therefore, robust approach cannot be directly implemented as stated before.
The aim with these chains was to measure how much does uncertainty affect in terms of cost, service time and stock to the optimal solution when the lead time admits some random variability around its mean value. In this case, an aggregate analysis was less meaningful so in Fig. 2 is shown the results for SC 4 and the former will be described as an example. The model has been ran for these six supply chains ten times with uncertainty going from 0 to 100% in intervals of 10%. The trend lines show the variation of safety stock (squares), the variation of safety stock cost (circles) and the variation, in average, of outbound service times (triangles).

For this particular chain, the safety cost increases in a bigger proportion than the safety stock amount, which means that few stock is being allocated in stages with higher holding cost. The trend for both is similar. On the other side, the service times experiment an uncommon trend, having a significant growth for levels of 60% of uncertainty, then staggering and, finally, dramatically decreasing. For a better understanding of this phenomenon, it should be noted that the number of stages holding safety stock remains in 12 until 50% of uncertainty and then it scales to 19 stages for 60%, where it keeps constant. This analysis could be very useful for companies. The model allows the user to set its own level of uncertainty for different stages in the SC.

5. Conclusion
In the present paper, a new model is proposed based on robust analysis to face safety stock allocation problem with uncertain lead time. Thus, the main advantage is that none lead time distribution is needed, which determination is usually a key problem for practitioners. Then, the model is tested in twenty complex real-world supply chain networks with promising results. It has been proven that the model could work as a very useful tool for practitioners to deal with supply-side risks thanks to its operational flexibility adding constraints and setting parameters as well as its relatively small computational times and different risk analyses.
Future work could compare the performance of this model to models that assume a stochastic distribution for the lead time, even when data is incomplete, unreliable or uncertain.

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