Covariance of WDVV Equations

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The (generalized) WDVV equations for the prepotentials in \(2d\) topological and \(4,5d\) Seiberg-Witten models are covariant with respect to non-linear transformations, described in terms of solutions of associated linear problem. Both time-variables and the prepotential change non-trivially, but period matrix (prepotential’s second derivatives) remains intact.

1 Summary

The WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations \cite{1,2,3} form an overdefined set of non-linear equations for a function (prepotential) of \(r\) variables (times), \(F(t^i),\ i=1,\ldots,r\). According to \cite{4}, they can be written in the form:

\[
F_i G^{-1} F_j = F_j G^{-1} F_i,
\]

\[
G = \sum_{k=1}^{r} \eta^k F_k, \quad \forall i, j = 1, \ldots, r \quad \text{and} \quad \forall \eta^k(t)
\]  

where \(F_i\) are \(r \times r\) matrices \((F_i)_{jk} = F_{ij} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}\) and the ”metric” matrix \(G\) is an arbitrary linear combination of \(F_k\)’s, with coefficients \(\eta^k(t)\) that can be time-dependent\cite{4}.

\[\text{Note that homogeneity of } F \implies \text{that } t^0\text{-derivatives are expressed through those w.r.t. } t^i, \ e.g.\]

\[t^0 F_{0ij} = -F_{ij} k^k, \quad t^0 F_{00i} = F_{ikl} t^k t^l, \quad t^0 F_{000} = -F_{klm} t^k t^l t^m \quad \text{etc.}\]

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1 The prepotential, as it arises in applications, is naturally a homogeneous function of degree 2, but it depends on one extra variable \(t^0\):

\[F(t^0, t^1, \ldots, t^r) = (t^0)^2 F(t^i/t^0),\]

see \cite{5} for emerging the general theory. As explained in \cite{6}, the WDVV equations \cite{7} for \(F(t^i)\) can be also rewritten in terms of \(F(t^I)\):

\[
F_I \hat{G}^{-1} F_J = F_J \hat{G}^{-1} F_I, \quad \forall I, J = 0, 1, \ldots, r; \{\eta^K(t)\}
\]

where this time \(F_I\) are \((r + 1) \times (r + 1)\) matrices of the third derivatives of \(F\) and

\[
\hat{G} = \sum_{k=0}^{r} \eta^k F_K, \quad \hat{G}^{-1} = (\det \hat{G}) \hat{G}^{-1}
\]
The WDVV equations imply consistency of the following system of differential equations (2):

\[
\left( F_{ijk} \frac{\partial}{\partial t^i} - F_{ijl} \frac{\partial}{\partial t^k} \right) \psi^j(t) = 0, \quad \forall i, j, k
\]

Contracting with the vector \( \eta^l(t) \), one can also rewrite it as

\[
\frac{\partial \psi^i}{\partial t^k} = C^i_{jk} D \psi^j, \quad \forall i, j
\]

where

\[
C_k = G^{-1} F_k, \quad G = \eta^l F_l, \quad D = \eta^l \frac{\partial}{\partial t^l}
\]

(note that the matrices \( C_k \) and the differential \( D \) depend on choice of \( \{ \eta^l(t) \} \), i.e. on choice of metric \( G \)) and (3) can be rewritten as

\[
[C_i, C_j] = 0, \quad \forall i, j
\]

The set of the WDVV equations (1) is invariant under linear change of time variables with the prepotential unchanged (4). According to (2) and especially to (8), there can exist also non-linear transformations which preserve the WDVV structure, but they can change the prepotential. We shall show that such transformations are naturally induced by solutions of the linear system (2):

\[
t^i \longrightarrow \tilde{t}^i = \psi^i(t),
\]

\[
F(t) \longrightarrow \tilde{F}(\tilde{t}),
\]

so that the period matrix remains intact:

\[
F_{ij} = \frac{\partial^2 F}{\partial t^i \partial t^j} = \frac{\partial^2 \tilde{F}}{\partial \tilde{t}^i \partial \tilde{t}^j} \equiv \tilde{F}_{\tilde{i}\tilde{j}}
\]

2 Comments

2.1. As explained in (7), the linear system (2) has infinitely many solutions. "Original" time-variables are among them: \( \psi^i(t) = t^i \).

2.2. Condition (7) guarantees that the transformation (6) changes linear system (2) only by (matrix) multiplicative factor, i.e. the set of solutions \( \{ \psi^i(t) \} \) is invariant of (6). Among other things this implies that repeated application of (6) does not provide new sets of time-variables.

2.3. In the case of quantum cohomologies (2d topological models) (1) (2) (3) (8) there is a distinguished time-variable, say, \( t^r \), such that all \( F_{ijk} \) are independent of \( t^r \):

\[
\frac{\partial}{\partial t^r} F_{ijk} = 0 \quad \forall i, j, k = 1, \ldots, r
\]

Thus, all "metrics" \( G \) are degenerate, but \( \tilde{G}^{-1} \) are non-degenerate. Entire discussion of the present paper allows one to put forward such reformulation in terms of \( \tilde{F} \), e.g. the Baker-Akhiezer vector-function \( \psi(t) \) should be just substituted by the explicitly homogeneous (of degree 0) function \( \psi(t^i/t^0) \). The extra variable \( t^0 \) should not be mixed with the distinguished "zero-time" associated with the constant metric in quantum cohomology theory. Generically such a variable does not exist (when it does, see comment 2.3 below, we will identify it with \( t^r \)).
where we introduced the new matrix \( A \). The r.h.s. of this formula can be understood as matrix elements of the matrix 

\[
\frac{\partial}{\partial t^j} \hat{\psi}^i = z C_{jk}^i \hat{\psi}^k, \quad \forall i, j
\]

where \( \hat{\psi}^k(t^1, \ldots, t^{r-1}) = \int \psi^k_z(t^1, \ldots, t^{r-1}, t^r) e^{zt^r} dt^r \). In this case the set of transformations (5) can be substituted by a family, labeled by a single variable \( z \):

\[
t^i \rightarrow \tilde{t}^i_z = \hat{\psi}^i_z(t)
\]

In the limit \( z \to 0 \) and for the particular choice of the metric, \( \bar{G} = F_r \), one obtains the particular transformation

\[
\frac{\partial \tilde{t}^i}{\partial t^j} = \bar{C}^i_{jk} h^k, \quad h^k = \text{const},
\]

discovered in [8]. (Since \( \bar{C}^i_j = \partial_j \bar{C}_i \), one can also write \( \bar{C}^i_j = \bar{C}^i_k h^k, \bar{C}^i_k = (F_r^{-1})^i_k F_r^k \).)

2.4. Parametrization like (11) can be used in generic situation (3) as well (i.e. without distinguished \( t^r \)-variable and for the whole family (3)), only \( h^k \) is not a constant, but solution to

\[
(\partial_j - DC_j)^i_k h^k = 0
\]

\((h^k = D\psi^k)\) is always a solution, provided \( \psi^k \) satisfies (3)).

### 3 Infinitesimal variation of the WDVV equations

In this section we look for the infinitesimal variation of the WDVV equations, which preserve their shape. To this end, consider the small variation of time-variables and the prepotential

\[
\tilde{F}(t) = F(t) + \epsilon f(t), \quad t^i = \tilde{t} + \epsilon \xi^i(t)
\]

This variation induces the variation of the third derivatives of the prepotential

\[
\tilde{F}_{,ijk} = F_{,ijk} + \epsilon \left[ f + F_{,ik} \xi^l \right]_{,ijk} - F_{,ijkl} \xi^l
\]

The r.h.s. of this formula can be understood as matrix elements of the matrix \( F_j (I + \epsilon A_j) \)

where we introduced the new matrix \( A \) with matrix elements defined by

\[
F_{,ijn} A_{jk}^n \equiv \left[ f + F_{,ik} \xi^l \right]_{,ijn} - F_{,ijkl} \xi^l
\]

The form of the WDVV equations (14) is preserved by the transformation (3) provided

\[
F_i (A_i - A_j) F_i^{-1} = F_k (A_k - A_j) F_k^{-1}
\]

where the matrix \((A_i)_k^l \equiv A_{ik}^l\). Solution to this equation is any constant (\( i \)-independent) matrix \( A: A_i = A, \forall i \). Therefore,

\[
F_{,ijn} A_{jk}^n = \left\{ f + F_{,ik} \xi^l \right\}_{,ijn} - F_{,ijkl} \xi^l = f_{,ijn} + \left( F_{,it} \xi_{j}^l + F_{,ij} \xi_{l}^i + F_{,ij} \xi_{ij} \right)_{,k} + F_{,ij} \xi_{ij}^l
\]

The last term in the r.h.s. of this equation is of the desired form \( F_j \times A \), while it is hard to represent the remaining terms in this form. Therefore, it is natural to request them vanish

\[
F_{,it} \xi_{j}^l + F_{,ij} \xi_{l}^i + F_{,ij} \xi_{ij} = -f_{,ij}
\]
This is nothing but the infinitesimal version of formula (7)! Therefore, we obtain the invariance of the period matrix, eq.(7), as the condition of the WDVV system covariance.

Formula (18) allows one to find $f$ once the change of time variables, i.e. functions $\xi^l(t)$, is known. However, one still needs to find $\xi^l(t)$. In fact, they are restricted by the condition of self-consistency of (18). Namely, the l.h.s. of this formula is the second derivative of something w.r.t. $t^i$ and $t^j$ iff its third derivative w.r.t. $t^k$ is symmetric over all the three indices. This is equivalent to the condition

$$F_{,ijkl}^l\xi^l_{,k} = F_{,kjl}^l\xi^l_{,i}$$

i.e. symmetricity under the permutations of $i$ and $k$. This symmetry occurs if $\xi^l$ have form

$$\xi^l_{,i} = C^l_{ij}h^j$$

where $h^j$ are some new functions. Then, the symmetry is a simple corollary of (1) and (3).

Now, however, one should consider also restrictions on the functions $h^i$, which can be derived from (20). Quite similarly to the previous step, the r.h.s. of (20) is the derivative of something w.r.t. $t^i$ iff its derivative w.r.t. to $t^j$ is symmetric under the permutation $i \leftrightarrow j$. This condition is solved explicitly by the formula

$$h^l_{,i} = D \left(C^l_{ij}h^j_{(1)}\right)$$

with some new functions $h^j_{(1)}$ that are also subject to the consistency conditions. Iteratively repeating this procedure, one can express $h^i_{(1)}$ through $h^j_{(2)}$ using formula (21) etc.

In fact, what we are doing in this way is the iterative procedure ($P$-exponential) for the following (matrix) equation (the sign $\circ$ means the composition of operators)

$$h^l_{,i} = D \left(C^l_{ij}h^j\right), \text{ i.e. } \left(\partial_i - D \circ C_i\right) h = 0$$

Therefore, we come to formula (12). In the next section we check that this equation provides also the non-infinitesimal transformation of time-variables. This is not surprising, since it is linear.

4 Proof of covariance

Now we investigate the non-infinitesimal transformations of time-variables $t^i \rightarrow \psi^i$ given by the formula

$$\frac{\partial \psi^i}{\partial t^i} = \left(C^l_{ij}h^j\right)$$

where functions $h^i$ satisfy the equation (12). The proof is a combination of reasoning from [8, 7].

First of all, let us check that the differential operators $D_i = \partial_i - D \circ C_i$ in eq.(12) induce a self-consistent system, i.e. they are commuting. Indeed, one can use the identity

$$\partial_j C_i - \partial_i C_j = C_j (DC_i) - C_i (DC_j),$$

i.e.

$$D \left(\partial_j C_i - \partial_i C_j\right) = [DC_j, DC_i] + C_j D^2 C_i - C_i D^2 C_j$$
In order to get (24), we used the WDVV equations and the definitions (3) and the identity $\partial_l F_{ijk} = \partial_i F_{ljk}$. Then, one can easily see that
\[
[D_i, D_j] = D (\partial_i C_j - \partial_j C_i) [DC_i, DC_j] + C_i D^2 C_j - C_j D^2 C_i + \]
\[
(C_i DC_j - DC_j C_i + DC_j C_i - C_j DC_i) D =
\]
\[
= D (\partial_i C_j - \partial_j C_i) [DC_i, DC_j] + C_i D^2 C_j - C_j D^2 C_i + D ([C_i, C_j]) D = 0
\]
because of (25) and (3).

Now we should check the consistency of the change of time-variables (23), i.e. that the r.h.s. of formula (23) can be presented as the derivative of something w.r.t. $t_i$. To this end, as usual, we should check the symmetricity of the derivative of (23) w.r.t. $i \leftrightarrow k$. This is equivalent to the condition that $\psi_{i,j,k}$ is symmetric under the permutation of indices $j$ and $k$. Indeed,
\[
\psi_{i,j,k} = C_j \partial_k h + (\partial_l C_j) h = C_j C_k D h + (C_j DC_k + \partial_l C_j) h
\]
This expression is really symmetric because of (5) and (24).

The next step of our proof is to check the invariance of the period matrix (7), i.e. existence of a function $\tilde{F}$ such that (7) is fulfilled. In other words, one needs to check that $F_{ij}$ can be presented as a second derivative w.r.t. the time-variables $\{\psi^i\}$. Therefore, one should again take the derivative of $F_{ij}$ w.r.t. $\psi^k$ and check its symmetricity:
\[
\frac{\partial F_{ij}}{\partial \psi^k} = F_{ijl} \frac{\partial t^l}{\partial \psi^k} \equiv (F_j U)_{ik}
\]
where $U$ is the matrix with elements $(U)^i_k = \frac{\partial t^i}{\partial \psi^k}$. Now, instead of checking the symmetricity of the matrix $F_j U$, we check the symmetricity of the inverse matrix $U^{-1} F^{-1}_j$. Its matrix elements are (see (23))
\[
C^{i}_{lm} \left( F^{-1}_j \right)^{lk} h^m = \left( G^{-1} F^{-1}_m F^{-1}_j \right)^{ik} h^m
\]
This matrix is indeed symmetric, since every one of matrices $F_i$ is symmetric and because of the WDVV equations.

Thus, we proved that our construction is consistent. It remains to check that the WDVV equations are covariant under the change $F_i \to \tilde{F}_i$, $t^i \to \psi^i$. Indeed,
\[
\tilde{F}_i \tilde{G}^{-1} \tilde{F}_j = F_i U U^{-1} G^{-1} F_j U = F_j G^{-1} F_i U = \tilde{F}_j \tilde{G}^{-1} \tilde{F}_i
\]
This completes the proof.

Note that, from (23) and (12), it follows that the new time-variables $\psi^k$ are solutions to the equation (cf. (4))
\[
(\partial_i - C_i D) \psi = 0
\]
provided $h^k = D \psi^k$. Therefore, they solve the linear problem for the WDVV equations.

5 Conclusion and acknowledgments

To conclude, we described a set of non-trivial non-linear transformations which preserve the structure of the WDVV equations (1). The consideration above does not prove that all such
transformations are of the form \((6), (7)\), and even in this sense the story is incomplete. Still \((6)\) is already unexpectedly\((?)\) large, because \((1)\) is an *overdefined* system and it could seem to be *rigid*, if has any solutions at all.

Even more obscure are the *origins and implications* of this covariance. Two remarks deserve to be made.

First, we see that the ”period matrix” \(F_{,ij}\) appears ”more rigid” than the prepotential itself: according to \((7)\) it does not change under \((6)\). Note that, in the framework of Seiberg-Witten theory \([9]\), \(F_{,ij}\), not the prepotential \(F\) itself, has the direct physical meaning (it describes coupling constants of the low-energy abelian effective theory). Thus, in certain sense, the transformations \((6), (7)\) leave ”physics” invariant, as one could wish.

Second, an essential ingredient of the Seiberg-Witten theory is its hidden integrable structure \([10, 11]\) which is also relevant for consistency with the brane theory \([12]\). In this context, the role of time-variables \(t^i\) is played by the periods \(a^i\) of the presymplectic 1-form (of some 0 + 1-dimensional quantum-mechanical system), \(t^i = a^i = \oint A_i dS\) on the family of spectral curves. It is unclear if the transformations \((6)\) can be lifted to some deformation of this structure. Note that ref.\([13]\) suggests that the problem of integral representation for \(\psi^i(t)\) (defined as a solution of the linear system \((2)\)) is related to that of the mirror-like maps.

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2In terms of footnote \([6]\) \(f_{,ij}\) is a homogeneous function of degree 0, which one can naturally expect to be more ”stable” than \(F\), which is a homogeneous function of degree 2.
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