Stable non-singular cosmologies in beyond Horndeski theory and disformal transformations

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Abstract

In this note we collect, systemise and generalise the existing results for relations between general Horndeski theories and beyond Horndeski theories via disformal transformations of metric. We derive additional disformal transformation rules relating Lagrangian functions of beyond Horndeski theory and corresponding Horndeski theory and demonstrate that some of them become singular at some moments(s) once one constructs a non-singular cosmological solution in beyond Horndeski theory that is free from ghost, gradient instabilities and strong gravity regime during the entire evolution of the system. The key issue here is that such solutions are banned in Horndeski theory due to existing no-go theorem. The proof of singular behaviour of disformal relations in this case resolves the apparent contradiction between the fact that Horndeski and beyond Horndeski theories appear related by field redefinition but describe different physics in the context of non-singular cosmologies.

1 Introduction

Scalar-tensor theories of modified gravity have become a go-to framework for addressing various cosmological issues ranging from initial singularity problem to explaining late time accelerated expansion. Today Degenerate Higher order scalar-tensor (DHOST) theories are the most general class of theories providing a systemized approach to modifying gravity by adding a scalar field \cite{1,2,3} (see e.g. Ref.\cite{4} for review). Despite involving the second derivatives in the Lagrangian, and, hence, having corresponding equations of motion higher than second order, DHOST theories are protected from naively expected Ostrogradsky ghost by a set of degeneracy conditions imposed
on the Lagrangian, which ensure that the number of propagating degrees of freedom (DOF) is no more than 1 scalar and 2 tensor modes.

In this note we will focus on two renowned special cases of DHOST theories, namely, Horndeski theories [6, 7, 8, 5] and beyond Horndeski or GLPV theories [9, 10, 11] (see e.g. Ref. [12] for review). The former represent the most general subclass, where the equations of motion are manifestly second order in derivatives, while the latter are historically the first successful generalization of Horndeski theories, which gave up the restriction of having second-order equations of motion but still gave rise to $2 + 1$ DOFs thanks to degeneracies among the equations.

The equal number of DOFs in Horndeski theories and their extensions is not entirely surprising since there is a non-trivial relation between these theories via the invertible field redefinition, namely, disformal transformation of metric [13]

$$\bar{g}_{\mu\nu} = \Omega^2(\pi, X)g_{\mu\nu} + \Gamma(\pi, X)\partial_\mu\pi\partial_\nu\pi,$$

where $\pi$ is a scalar field, $X = g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$, while $\Omega^2(\pi, X)$ and $\Gamma(\pi, X)$ are arbitrary functions. Disformal relations between Horndeski and beyond Horndeski subclasses and phenomenologically viable DHOST theories were discussed in details e.g. in Refs. [2, 11] and [14, 15, 16]. In particular, in Ref. [14] it was shown that Horndeski theories are stable under transformations (1) with both $\Omega$ and $\Gamma$ depending on $\pi$ but not $X$. Later it was shown that allowing $\Gamma$ to be function of $X$ transforms Horndeski theory into beyond Horndeski class, and, interestingly, this is exactly how the first example of beyond Horndeski theories was derived [9]. Finally, further generalisation with both $\Omega(\pi, X)$ and $\Gamma(\pi, X)$ allows to generate DHOST family still starting off with Horndeski theories. Naturally, as long as no other matter is involved and disformal transformation (1) is invertible all these disformally related theories describe the same physics.

Another exceptional property of DHOST theories and their subclasses that will be important for us in this note is their ability to violate the Null Energy Condition (NEC) without fatal consequences for the stability of a linearized theory (see Refs. [12, 18] for details). In fact, insofar as gravity is modified NEC is replaced by the Null Convergence Condition [19]. This NEC/NCC-violating feature made Horndeski theories and their extensions particularly attractive for non-standard early Universe cosmology, since violating NEC/NCC is crucial for constructing cosmological scenarios without initial singularity like the Universe with a bounce or Universe starting off with Genesis (see Ref. [20] for a mini review).

However, only beyond Horndeski and DHOST theories became truly successful in pursuit of constructing completely stable non-singular cosmological solutions. "Complete" stability here means that no pathological DOFs like ghosts and gradient instabilities are present in the system during entire evolution followed from $t \to -\infty$ to $t \to +\infty$. In particular, unextended Horndeski theories were ruled out as suitable frameworks for such solutions by a no-go theorem [21, 22], which states that any non-singular cosmological solution in Horndeski theory runs into gradient instabilities at some moment provided one considers the system from $t \to -\infty$ to $t \to +\infty$. The

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1. Recent studies have shows that coupling additional matter to DHOST theories might be somewhat subtle when preserving degeneracy is concerned [17].

2. There is a loophole, though: it is possible to evade the no-go theorem by allowing strong gravity in the
no-go breaks down as soon as one goes beyond Horndeski: this fact was first shown within the effective field theory approach (EFT) in Refs. [25, 26] and later supported by numerous explicit examples of completely healthy solutions, see e.g. Refs. [27, 28, 29, 30].

It might seem contradictory, however, that while healthy Horndeski and beyond Horndeski theories are related via a disformal transformation, the former do not admit completely stable non-singular cosmologies but the latter do. Indeed, disformal transformation (1) is a field redefinition, and while it is invertible it cannot change the number of DOFs, hence, it cannot affect stability of the solution. So the question is how is it possible that there are non-singular solutions without instabilities in beyond Horndeski subclass? This puzzle was resolved in Ref. [26], where it was found that the disformal transformation, which turns beyond Horndeski Lagrangian admitting stable solution into general Horndeski form, becomes singular at some moment of time. This result was obtained within the effective field theory (EFT) approach and later confirmed in the covariant formalism for a quartic subclass of beyond Horndeski theory [20], but not in the most general case of beyond Horndeski theory including the quintic subclass.

One of the main purposes of this note is to generalize the result of Ref. [20] to include both quartic and quintic beyond Horndeski subclasses, i.e. we show explicitly that completely stable non-singular cosmological solutions exist only in those beyond Horndeski theories which are related to Horndeski subclass by disformal transformations that are singular at some point. To be precise, singularities appear in the transformation rules relating the Lagrangian functions, which makes it impossible to map beyond Horndeski Lagrangian into the Horndeski one in our case. Another aim of this note is to gather and systemise the disformal relations between the Lagrangian functions in Horndeski and beyond Horndeski theories in a covariant formalism, which were obtained earlier e.g. in Refs. [11, 16]. This is done in Sec. 2, where we also derive some additional relations necessary for our argument. Then in Sec. 3 we demonstrate that disformal relations for $X$-derivatives of Lagrangian functions are indeed divergent if beyond Horndeski theory admits a stable bouncing or Genesis solution. We conclude in Sec. 4.

2 Disformal relations for Horndeski and GLPV theories

Let us first briefly recall the explicit form of (beyond) Horndeski Lagrangian and introduce our set of notations:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + B^H \mathcal{L}_4 + B^H \mathcal{L}_5,$$

past [22, 23, 24].
Let us note in passing that both eqs. (3a) and (3b) aim to beyond Horndeski type. Here and below we denote metric corresponding to Horndeski theory as \( \bar{g} \), while metric \( g \) is ascribed to beyond Horndeski theory. For our purposes it sufficient to consider disformal transformation (1) where \( \Omega(\pi, X) = 1 \). Then upon transformation of the metric \( \bar{g} \), the Lagrangians \( \mathcal{L}_4[\bar{g}] \) and \( \mathcal{L}_5[\bar{g}] \) in eq. (2) describe general Horndeski theory, while adding \( BH \mathcal{L}_4 \) and \( BH \mathcal{L}_5 \) extend the theory to beyond Horndeski type.

In what follows we focus on disformal transformation of the sum of quartic (3b) and quintic (3c) subclasses of Horndeski theory

\[
\mathcal{L}_H = \mathcal{L}_4[\bar{g}] + \mathcal{L}_5[\bar{g}].
\]

Here and below we denote metric corresponding to Horndeski theory as \( \bar{g} \), while metric \( g \) is ascribed to beyond Horndeski theory. For our purposes it sufficient to consider disformal transformation (1) where \( \Omega(\pi, X) = 1 \). Then upon transformation of the metric \( \bar{g} \), the Lagrangians \( \mathcal{L}_4[\bar{g}] \) and \( \mathcal{L}_5[\bar{g}] \) in eq. (2) describe general Horndeski theory, while adding \( BH \mathcal{L}_4 \) and \( BH \mathcal{L}_5 \) extend the theory to beyond Horndeski type.

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Now let us revisit the existing results for disformal relations between the Lagrangian functions and derive the missing ones. The relations between the original functions $\bar{G}_4$ and $\bar{G}_5$ in Horndeski theory and the new $\hat{G}_4$ and $G_5$ in beyond Horndeski theory in the covariant approach were already found in Refs. [11, 16] and read as follows:

$$\bar{G}_4(\pi, \bar{X}) = \frac{\hat{G}_4(\pi, X)}{\sqrt{1 + X\Gamma}}, \quad \bar{G}_5(\pi, \bar{X}) = \int G_{5X}(\pi, X) \sqrt{1 + \Gamma X} dX, \quad \bar{X} = \frac{X}{1 + X\Gamma}. \quad (9)$$

After disformal transformation the following combinations of $\hat{G}_4$, $G_5$ and $\Gamma$ in the transformed Lagrangians (5)–(6) comprise the beyond Horndeski functions $\hat{F}_4$ and $F_5$:

$$\hat{F}_4 = \frac{\Gamma_X \left( \hat{G}_4 - 2X\hat{G}_{4X} \right)}{1 - \Gamma_X X^2}, \quad (10a)$$
$$F_5 = -\frac{\Gamma_X G_{5X}X}{3(1 - \Gamma_X X^2)}. \quad (10b)$$

At this point modulo $F$ and $K$ the only new functions left in eq. (8) to be related with the original ones are $\tilde{G}_4$ and $F_4$. As we mentioned above disformal transformation of $\mathcal{L}_5[\bar{G}_5]$ generally involves all lower subclasses, and according to eq. (6) $\tilde{G}_4$ and $\tilde{F}_4$ belong to such generated $\mathcal{L}_4$ and $\mathcal{BH}\mathcal{L}_4$, respectively. Hence, both $\tilde{G}_4$ and $\tilde{F}_4$ are given by some combinations in terms of $\Gamma$ and the new $G_5$:

$$\tilde{G}_4(\pi, X) = \frac{X}{4} \int \left[ \int G_{5\pi X} X^{1/2} dX \right] X^{-3/2} dX - \hat{G}_4(\pi, X), \quad (11a)$$
$$\tilde{F}_4(\pi, X) = -\frac{\Gamma_X \left( \tilde{G}_4 - 2X\tilde{G}_{4X} \right)}{1 - \Gamma_X X^2}, \quad (11b)$$

where

$$\tilde{G}_4(\pi, X) = \frac{X}{4\sqrt{1 + X\Gamma}} \int \left[ \int G_{5\pi X} X^{1/2} dX \right] \frac{1 - \Gamma_X X^2}{\sqrt{1 + X\Gamma}} X^{-3/2} dX,$$

was introduced to highlight the structure of the transformation rules (11). Eq. (11a) together with $\tilde{G}_4$ in eq. (9) define the relation between the original $\tilde{G}_4$ and the new $G_4 = \hat{G}_4 + \hat{G}_4$. Then the explicit form of $F_4 = \tilde{F}_4 + \tilde{F}_4$ in eq. (7) follows from eqs. (10a) and (11b) with integrating by parts where it is necessary:

$$F_4 = \frac{\Gamma_X}{1 - \Gamma_X X^2} \left( G_4 - 2XG_{4X} + \frac{1}{2} G_{5\pi X} \right). \quad (12)$$

So in result eqs. (9) together with explicitly derived in this note eqs. (10b), (11a) and (12) completely define the disformal relation between Lagrangian functions in $\mathcal{L}_H$ (4) and $\mathcal{L}_{BH}$ (7). Let us note that it follows immediately from eqs. (10b) and (12) that both $F_4$ and $F_5$ vanish for $\Gamma = \Gamma(\pi)$ in full agreement with the fact that Horndeski theories are stable under disformal transformations with $\Gamma_X = 0$ [14].

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Another important by-product of revisiting the disformal relations above is the ability to explicitly derive a specific constraint on $F_4$ and $F_5$ in the resulting beyond Horndeski theory:\footnote{The aforementioned constraint has been discussed earlier in literature and given without derivation e.g. in Ref. [31].}:

$$F_4 \, G_{5X} X = -3F_5 \left[ G_4 - 2X G_{4X} + \frac{1}{2} G_{5n} X \right], \quad (13)$$

which immediately follows from eqs. (10b) and (12). In particular, the relation (13) is in line with the fact that beyond Horndeski with arbitrary $F_4$ and $F_5$ cannot be disformally transformed into Horndeski theory [11, 15, 16].

Finally, we derive the relations between the original functions $\bar{G}_{4X}$ and $\bar{G}_{5X}$ and the new ones, which immediately follow from eqs. (9):

$$\bar{G}_{4X} = \frac{\partial \bar{G}_4}{\partial X} = \left( \hat{G}_4 (1 + \Gamma X) - \frac{1}{2} \hat{G}_4 (\Gamma + X \Gamma X) \right) \frac{\sqrt{1 + \Gamma X}}{1 - \Gamma \Gamma X X}, \quad (14)$$

$$\bar{G}_{5X} = \frac{\partial \bar{G}_5}{\partial X} = G_{5X} \frac{(1 + \Gamma X)^{5/2}}{1 - \Gamma \Gamma X X}. \quad (15)$$

In the next section based on these relations we will demonstrate that both $\bar{G}_{4X}$ and $\bar{G}_{5X}$ which enter the Lagrangian $\mathcal{L}_H (4)$ inevitably become singular at some moment provided the corresponding beyond Horndeski theory (7) admits stable non-singular cosmological solutions.

### 3 No-go theorem and disformal transformations

Let us start with a quick revision of a linearized theory for Horndeski theories and beyond Horndeski theories (2). The corresponding quadratic action for perturbations about a cosmological FLRW background has the same structure for both theories. In the unitary gauge where the propagating DOFs are curvature perturbation $\zeta$ and two tensor modes $h_{ij}^T$, the action reads (see e.g. Ref. [12] for details):

$$S = \int dt d^3x \ a^3 \left[ \frac{G_T}{8} \left( \dot{h}_{ij}^T \right)^2 - \frac{F_T}{8a^2} \left( \partial_k h_{ij}^T \right)^2 + G_S \dot{\zeta}^2 - \mathcal{F}_S \frac{\left( \partial_k \zeta \right)^2}{a^2} \right], \quad (15)$$

with an overdot standing for the derivative w.r.t. time $t$, scale factor $a$ and the following coefficients involved

$$G_S = \frac{\Sigma G_T^2}{\Theta^2} + 3G_T, \quad (16)$$

$$\mathcal{F}_S = \frac{1}{a} \frac{d}{dt} \left[ a \frac{G_T + \mathcal{D} \pi}{\Theta} G_T \right] - \mathcal{F}_T, \quad (17)$$
where $G_T$, $F_T$, $D$, $\Theta$ and $\Sigma$ are given in terms of Lagrangian functions

$$G_T = 2G_4 - 4G_{4X}X + G_{5\pi}X - 2HG_{5X}X\dot{\pi} + 2F_4X^2 + 6HF_5X^2\dot{\pi}, \quad (18a)$$

$$F_T = 2G_4 - 2G_{5X}X\dot{\pi} - G_{5\pi}X, \quad (18b)$$

$$D = -2F_4X\dot{\pi} - 6HF_5X^2, \quad (18c)$$

$$\Theta = -K_{X}X\dot{\pi} + 2G_4 H - 8HG_{4X}X - 8HG_{4XX}X^2 + G_{4\pi}\dot{\pi} + 2G_{4\pi}X\dot{\pi} - 5H^2G_{5X}X\dot{\pi} - 2H^2G_{5XX}X^2\dot{\pi} + 3HG_{5\pi}X + 2HG_{5\pi X}X^2 + 10HF_4X^2 + 4HF_4X^3 + 21H^2F_5X^2\dot{\pi} + 6H^2F_{5X}X^3\dot{\pi}, \quad (18d)$$

$$\Sigma = F_XX + 2F_{XX}X^2 + 12HK_XX\dot{\pi} + 6HK_{XX}X^2\dot{\pi} - K_{\pi}X - K_{\pi X}X^2 - 6H^2G_4 + 42H^2G_{4X}X + 96H^2G_{4XX}X^2 + 24H^2G_{4XXX}X^3 - 6HG_{4\pi}\dot{\pi} - 30HG_{4\pi X}X\dot{\pi} - 12HG_{4\pi XX}X^2\dot{\pi} + 30H^3G_{5X}X\dot{\pi} + 26H^3G_{5XX}X^2\dot{\pi} + 4H^3G_{5XXX}X^3\dot{\pi} - 18H^2G_{5\pi}X - 27H^2G_{5\pi X}X^2 - 6H^2G_{5\pi XX}X^3 - 90H^2F_4X^2 - 78H^2F_4X^3 - 12H^2F_{4XX}X^4 - 168H^3F_5X^2\dot{\pi} - 102H^3F_{5XX}X^3\dot{\pi} - 12H^3F_{5XXX}X^4\dot{\pi}. \quad (18e)$$

The case of Horndeski theory is recovered immediately once $F_4 = F_5 = 0$ in eqs. (18).

To have a cosmological solution that is free from ghost and gradient instabilities one has to satisfy the following inequalities:

$$G_T, F_T > \epsilon > 0, \quad G_S, F_S > \epsilon > 0, \quad (19)$$

where $\epsilon$ is a positive constant which ensures that there is no naive strong coupling, i.e. $G_{S,T} \neq 0$ and/or $F_{S,T} \neq 0$. This is a strong version of stability constraints adopted here to avoid considering the case of $G_{S,T} \to 0$ and/or $F_{S,T} \to 0$ which might help evade the no-go without encountering the strongly coupled regime, see Refs. [22, 23] for details.

The no-go theorem [21, 22] in Horndeski theory is based on the gradient stability constraint (17) in cosmologies with a scale factor $a \neq 0$:

$$\frac{d}{dt} \left[ aG_T(G_T + D\dot{\pi}) \right] = a(F_S + F_T) > \epsilon > 0, \quad (20)$$

where $D = 0$ for Horndeski theory, see eq. (18c). To have complete stability one requires that the constraint (20) holds at all times. Then according to eq. (20), the coefficient on the left-hand side

$$\xi = \frac{aG_T^2}{\Theta} \quad (21)$$

has to be a monotonously growing function with slope bounded from below by $\epsilon$, which means that $\xi \to -\infty$ as $t \to -\infty$ and $\xi \to +\infty$ as $t \to +\infty$, i.e. $\xi$ has to cross zero at some moment(s) of time. However, for $a > 0$ and $G_T > \epsilon > 0$ (due to no ghost constraint in the tensor sector, see eq. (19)) the only option for $\xi$ to cross zero is when $\Theta \to \infty$, which corresponds to a singularity in the classical solution. Thus, non-singular cosmological solutions in Horndeski theory cannot satisfy the stability conditions (19) at all times.
The situation changes as soon as $\mathcal{D} \neq 0$ in eq. (20), which is the case for beyond Horndeski theory: while $\mathcal{G}_T$ still has to be positive due to the no-ghost condition (19), the combination $(\mathcal{G}_T + \mathcal{D})$ is unconstrained and may take any values including zero, so that

$$\dot{\xi} = \frac{a\mathcal{G}_T(\mathcal{G}_T + \mathcal{D}\dot{\pi})}{\Theta}$$

(22)
on the left-hand side in eq. (20) can safely cross zero and grow monotonously during entire evolution. Therefore, in beyond Horndeski theories it is possible to construct a completely stable non-singular cosmological solution. Let us now demonstrate that in fact right at the moment when $\dot{\xi} \sim (\mathcal{G}_T + \mathcal{D}\dot{\pi})$ crosses zero in beyond Horndeski theory, the original functions $\mathcal{G}_{4\mathcal{X}}$ and $\mathcal{G}_{5\mathcal{X}}$ in a disformally related Horndeski Lagrangian $\mathcal{L}_H$ (4) hit singularity.

First, we express $\Gamma_X$ using a linear combination of eqs. (10b) and (12) $^6$ and cast it in terms of coefficients (18) from the quadratic action:

$$\Gamma_X = -\frac{\mathcal{D}\dot{\pi}}{X^2\mathcal{G}_T}.$$  

(23)

Both transformation rules (14) for $\mathcal{G}_{4\mathcal{X}}$ and $\mathcal{G}_{5\mathcal{X}}$ involve the same denominator, which in terms of $\mathcal{D}$ and $\mathcal{G}_T$ reads:

$$\frac{1}{1 - \Gamma_X X^2} = \frac{\mathcal{G}_T}{\mathcal{G}_T + \mathcal{D}\dot{\pi}}.$$  

(24)

It follows from eq. (24) that the denominator of the transformation rules (14) goes through zero at the same moment as $\dot{\xi}$ in eq. (22) does. This means that both Lagrangian functions $\mathcal{G}_{4\mathcal{X}}$ and $\mathcal{G}_{5\mathcal{X}}$ diverge at that moment of time.

To sum up, once one goes beyond Horndeski to evade the no-go theorem by having $\mathcal{D} \neq 0$, it is possible to comply with the requirement for $\dot{\xi}$ in eq. (20). The latter implies vanishing $(\mathcal{G}_T + \mathcal{D}\dot{\pi})$ at some moment(s) of time. And, thus, in this case beyond Horndeski and Horndeski theories are related by singular transformation rules (14). So in fact there is no contradiction between the no-go theorem and existence of completely stable cosmologies in seemingly disformally related theories.

### 4 Conclusion

Even though disformal transformations within scalar-tensor theories of modified gravity are a highly-developed topic, in this note we aimed to collect, somewhat systemise and generalize the existing results in the covariant formalism for beyond Horndeski theories. In this way, we have addressed a specific issue of disformal relation between Horndeski and beyond Horndeski theories in the context of constructing healthy non-singular cosmological solutions like bouncing Universe or the Universe with Genesis. In particular, disformal relation between the theories implies similar physics behind both of them, but at the same time there is the no-go theorem which states that $^6$Here the linear combination is explicitly the sum of eqs. (10b) and (12) multiplied by $2HX^2\dot{\pi}$ and $2X^2$, respectively.
completely stable non-singular cosmologies exist only in beyond Horndeski theory but not in Horndeski subclass. The resolution of this apparent contradiction is similar to that suggested in Ref. [20] for the quartic subclass of beyond Horndeski theories: the price of evading the no-go theorem is singularity in disformal relations between the Lagrangian functions of corresponding Horndeski and beyond Horndeski theories. In this note we have proved the latter statement for the case of quintic beyond Horndeski subclass, which generalizes the existing result and completes the argument.

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