Shortcuts for Graviton Propagation in a Six Dimensional Brane World Model

Elcio Abdalla, Adenauer Casali and Bertha Cuadros-Melgar

Instituto de Física, Universidade de São Paulo
C.P.66.318, CEP 05315-970, São Paulo, Brazil

Abstract

We consider a six dimensional brane world model with asymmetric warp factors for time and both extra spatial coordinates, \( y \) and \( z \). We derive the set of differential equations governing the shortest graviton path and numerically solve it for AdS-Schwarzschild and AdS-Reissner-Nordström bulks. In both cases we derive a set of conditions for the existence of shortcuts in bulks with shielded singularities and show some examples of shortcuts obtained under these conditions. Consequences are discussed.

PACS numbers: 98.80.Hw 97.60.Lf

Keywords: Extra dimensions, Brane world cosmology, Shortcuts, Graviton propagation.
1 Introduction

The ideas of Kaluza and Klein [1], advocating the physical possibility of extra dimensions in order to achieve the unification of different field theories can be considered as a landmark in Quantum Field Theory. Such an importance grew specially half a century after the original works, in the framework of supergravity and string theories. In the latter, the existence of extra dimensions is actually enforced by consistency.

Furthermore, new possibilities to realize the extra dimensions permitted to explore new mechanisms of explaining unified field theories. The possibility of explaining hierarchies in such a context is specially appealing and has been confirmed in the works of Arkani-Hamed and others [2, 3]. The hierarchy between the electro-weak (∼ 100 GeV) and the Planck (10^{19} GeV) scales has been focused by means of the consideration of extra dimensions at a submilimeter size, which shows up in a theory of two extra dimensions connecting both scales. Such an idea replaces the usual one where extra dimensions should only show up at the Planck scale, and the tower of massive particles thus generated is above that level and has a wider validity, including cosmology [4].

Such size constraints on the size of the extra dimensions constitute drawbacks in the formulation of the theory.

More recently, Randall and Sundrum [5, 6] proposed a model -or rather a class of models- where there is a warp factor in the metric, such that even infinitely large extra dimensions are allowed.

The existence of large extra dimensions with a warp factor naturally raises the question of whether information can follow a shorter path outside the brane riding on gravitons [7, 8, 9, 10]. We proposed a simple calculation to establish the shortest path followed by a graviton [11], which propagating in all dimensions in the so called bulk, could in principle follow a path which decouples from the brane, that is, from our universe, returning later to another point, advanced in time with respect to a photon, which by construction must follow a path in the brane, thus being delayed. In our previous paper we considered a model constructed in [13] which was basically a generalized Friedmann-Robertson-Walker universe with cosmological constant, with different scenarios in the brane (where are living the Standard Model fields) and in the bulk. The result was actually a negative one, that is, the shortest path followed by the graviton was the same as the one followed by the photon,
namely, inside the brane.

In a model introduced by Csáki et. al. [14] the speed of light along flat four dimensional sections varies over the extra dimensions due to different warp factors for the space and time coordinates, a construction similar to the one of Randall and Sundrum. Thus the authors proposed that gravitational waves might travel faster than photons, which remain in the brane. The delay between electromagnetic and gravity waves may be experimentally detected with the gravitational waves detectors under way [14, 16].

The models are basically AdS-Schwarzschild or AdS-Reissner-Nordström black holes in the bulk. Brane models in AdS space with Schwarzschild singularities have been used to understand the AdS/CFT correspondence and looks like a promising theoretical model [16]. They are based on the Randall-Sundrum scheme [5, 6], where a large mass hierarchy is obtained with uncompactified dimensions from solutions of Einstein equations in higher dimensions (i.e., in the bulk) with two separated branes. The four dimensional part of the metric is multiplied by a “warp” factor which is a rapidly changing function of the additional dimension.

In this paper we consider a six-dimensional model and look for possible shortcuts for AdS-Schwarzschild and AdS-Reissner-Nordström bulk configurations. The paper is organized as follows. In section 2 we describe a general six-dimensional model, derive Einstein equations, and find the Israel conditions the metric has to satisfy due to the brane embedding. At this point we choose a metric describing a six-dimensional black hole and add a $Z_2$ symmetry. In section 3 we find the Euler-Lagrange equations which define the graviton path in this model. Section 4 is devoted to study the numerical solutions of these equations in the context of AdS-Schwarzschild bulk finding certain analytical requirements for the existence of shortcuts. The AdS-Reissner-Nordström bulk is studied in section 5, where we perform an analytical discussion to impose a set of conditions under which shortcuts can coexist with shielded singularities. Finally, consequences are discussed in section 6.

2 A Six-Dimensional Model

We consider a six-dimensional model, such as the one constructed by Kanti et. al. [17]. We also search for a solution of six-dimensional Einstein equation
in AdS space of the form

\[ ds^2 = -n^2(t, y, z)dt^2 + a^2(t, y, z)d\Sigma_k^2 + b^2(t, y, z) \left\{ dy^2 + c^2(t, y, z)dz^2 \right\} \]  \tag{1} \]

where \( d\Sigma_k^2 \) represents the metric of the three dimensional spatial sections with \( k = -1, 0, 1 \) corresponding to a hyperbolic, a flat and an elliptic space, respectively.

The components of the Einstein tensor read

\[ G_{00} = \frac{3a}{n} \left( \frac{3b}{a} + 2\frac{b}{c} + \frac{c}{n} \right) + \frac{b}{c} \left( \frac{2b}{a} + \frac{c}{n} \right) - n^2 \left\{ \left( \frac{3b^2}{a} + \frac{2\partial y}{b} + 2\frac{\partial y}{bc} \right) \left( \frac{\partial y}{bc} \right)^2 + \frac{2\partial y}{b} + \frac{\partial y}{bc} \right\} \delta_{ij} + \frac{1}{c^2} \left( \frac{3b^2}{a} + 3\frac{\partial y}{a} \right)^2 + \frac{\partial y}{b} \left( \frac{\partial y}{b} \right)^2 + \frac{\partial y}{bc} \left( \frac{\partial y}{bc} \right)^2 \right\} \delta_{ij} \]  \tag{2} \]

\[ G_{ij} = \frac{2a}{n} \left( \frac{2b}{a} + 2\frac{b}{n} \right) + 2\frac{\partial y}{b} + 2\frac{\partial y}{bc} \right\} \delta_{ij} + \frac{1}{c^2} \left( \frac{2b^2}{a} + \frac{3\partial y}{a} \right)^2 + \frac{\partial y}{b} \left( \frac{\partial y}{b} \right)^2 + \frac{\partial y}{bc} \left( \frac{\partial y}{bc} \right)^2 \right\} \delta_{ij} \]  \tag{3} \]

\[ G_{55} = -\frac{k^2}{n^2} \left( \frac{3a}{b} + \frac{2b}{c} + \frac{c}{n} \right) + \frac{2\partial y}{b} + 2\frac{\partial y}{bc} \right\} \delta_{ij} + \frac{1}{c^2} \left( \frac{3b^2}{a} + 3\frac{\partial y}{a} \right)^2 + \frac{\partial y}{b} \left( \frac{\partial y}{b} \right)^2 + \frac{\partial y}{bc} \left( \frac{\partial y}{bc} \right)^2 \right\} \delta_{ij} \]  \tag{4} \]

\[ G_{66} = -\frac{k^2}{n^2} \left( \frac{3a}{b} + \frac{2b}{c} + \frac{c}{n} \right) + \frac{2\partial y}{b} + 2\frac{\partial y}{bc} \right\} \delta_{ij} + \frac{1}{c^2} \left( \frac{3b^2}{a} + 3\frac{\partial y}{a} \right)^2 + \frac{\partial y}{b} \left( \frac{\partial y}{b} \right)^2 + \frac{\partial y}{bc} \left( \frac{\partial y}{bc} \right)^2 \right\} \delta_{ij} \]  \tag{5} \]

\[ G_{05} = -\frac{3\partial y}{bc} - \frac{\partial y}{b} + 3\frac{\partial y}{a} + \frac{2\partial y}{b} + \frac{2\partial y}{b} \right\} \delta_{ij} + \frac{1}{c^2} \left( \frac{3b^2}{a} + 3\frac{\partial y}{a} + \frac{\partial y}{b} \right) + \frac{1}{c^2} \left( \frac{3b^2}{a} + \frac{\partial y}{b} \right) \]  \tag{6} \]

\[ G_{06} = -\frac{3\partial y}{bc} - \frac{\partial y}{b} + 3\frac{\partial y}{a} + \frac{2\partial y}{b} + \frac{2\partial y}{b} \right\} \delta_{ij} + \frac{1}{c^2} \left( \frac{3b^2}{a} + 3\frac{\partial y}{a} + \frac{\partial y}{b} \right) + \frac{1}{c^2} \left( \frac{3b^2}{a} + \frac{\partial y}{b} \right) \]  \tag{7} \]
The total energy-momentum tensor can be decomposed in two parts corresponding to the bulk and the brane as

\[ \tilde{T}^M_N = \hat{T}^M_N(B) + T^M_N(b), \]  

where the brane contribution can be written as

\[ T^M_N(b) = \frac{\delta (z - z_0)}{bc} \text{diag} \left( -\rho, p, p, p, \hat{p}, 0 \right). \]  

In order to have a well-defined geometry, the metric must be continuous across the brane; however, its derivatives with respect to \( z \) can be discontinuous at the position of the brane, generating a Dirac \( \delta \)-function in the second derivatives of the metric with respect to \( z \) [12]. These \( \delta \) function terms must be matched with the components of the brane energy-momentum tensor (10) in order to satisfy Einstein equations. Thus, using (2), (3) and (4) we obtain the following Israel conditions,

\[ \left[ \frac{\partial_z a}{a_0 b_0 c_0} \right] = -\frac{k^2}{6} \left( p - \hat{p} + \rho \right), \]
\[ \left[ \frac{\partial_z b}{b_0^2 c_0} \right] = -\frac{k^2}{4} \left\{ \rho - 3(p - \hat{p}) \right\}, \]
\[ \left[ \frac{\partial_z n}{b_0 c_0 n_0} \right] = \frac{k^2}{4} \left\{ \hat{p} + 3(p + \rho) \right\}. \]  

A metric of the form (11) satisfying six dimensional Einstein equations is given by

\[ ds^2 = -h(z)dt^2 + \frac{z^2}{l^2}d\Sigma_k^2 + h^{-1}(z)dz^2, \]  

where

\[ d\Sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2d\Omega_{(2)}^2 + (1 - kr^2)dy^2, \]

and

\[ h(z) = k + \frac{z^2}{l^2} - \frac{M}{z^3}, \quad \text{for AdS-Schwarzschild bulk,} \]
\[ h(z) = k + \frac{z^2}{l^2} - \frac{M}{z^3} + \frac{Q^2}{z^6}, \quad \text{for AdS-Reissner-Nordström bulk}, \]
with \( l^{-2} \propto -\Lambda \) (\( \Lambda \) being the cosmological constant), which describes a black hole in the bulk, located at \( z = 0 \).

Following [14], we find a further solution by means of a \( Z_2 \) symmetry inverting the space with respect to the brane position. That is, considering a metric of the form

\[
ds^2 = -A^2(z)dt^2 + B^2(z)\Sigma^2_{(4)} + C^2(z)dz^2
\]

(16)

and the brane to be defined at \( z = z_0 \), there is a solution given by

\[
A(z), \ B(z), \ C(z) \quad \text{for} \quad z \leq z_0,
A(z_0^2/z), \ B(z_0^2/z), \ C(z_0^2/z)z_0^2 \quad \text{for} \quad z \geq z_0.
\]

(17)

The \( Z_2 \)-symmetry corresponds to \( z \rightarrow z_0^2/z \).

The static brane still has to obey the Israel conditions (11), which for the metric (12) are written as

\[
\left[ \partial_z a \right] a_0 c_0 = -\frac{\kappa^2}{4} \rho,
\]

\[
\left[ \partial_z n \right] a_0 c_0 n_0 = \frac{\kappa^2}{4} (4p + 3\rho),
\]

(18)

where here

\[
\left[ \partial_z a \right] = -\frac{2}{l},
\]

\[
\left[ \partial_z n \right] = -\frac{h'(z_0)}{\sqrt{h(z_0)}}.
\]

(19)

### 3 The Shortest Cut Equation

We consider the metric (12) with \( k = 0 \)

\[
ds^2 = -n^2(z)dt^2 + a^2(z) f^2(r)dr^2 + b^2(z)dy^2 + d^2(z)dz^2,
\]

(20)

where the graviton path is defined equating (20) to zero. Therefore,

\[
\int_{r_0}^r f(r')dr' = \int_{t_0}^t \sqrt{\n^2(z) - b^2(z)\dot{y}^2 - d^2(z)\dot{z}^2} \frac{a(z)}{a(z)}dt \equiv \int_{t_0}^t \mathcal{L} \left[ y(t), \dot{y}(t), z(t), \dot{z}(t); t \right] dt
\]

(21)
which naturally defines a lagrangian density. The Euler-Lagrange equations of $\mathcal{L}$ define the graviton path. We first choose to work at a constant $y$ to check on the very possibility of (20) allowing shortcuts. In this case the resulting equation is simple but far from trivial,

$$\ddot{z} + \left( \frac{a'}{a} - 2 \frac{n'}{n} + \frac{d'}{d} \right) \dot{z}^2 + \left( \frac{nn'}{d^2} - \frac{a' n^2}{a d^2} \right) = 0. \quad (22)$$

For $z \leq z_0$, $a = z/l$, $n = \sqrt{h(z)}$, and $d = 1/\sqrt{h(z)}$. For $z \geq z_0$ we have to use the $Z_2$ symmetry showed up in (17).

Notice that this case is equivalent to consider the problem in five dimensions with the metric shown in [14].

The most general case includes a $y$ dependence on the graviton path and the two Euler-Lagrange equations are then given by

$$(n^2 - d^2 \dot{z}^2) \ddot{y} + \dot{z} \left\{ \left( - \frac{a'}{a} + 2 \frac{b'}{b} \right) (n^2 - d^2 \dot{z}^2) - mn' + dd' \dot{z}^2 + d^2 \dot{z} \right\} \dot{y} + b^2 \left( \frac{a'}{a} - \frac{b'}{b} \right) \dot{z} \dot{y}^2 = 0 \quad (23)$$

and

$$(n^2 - b^2 \dot{y}^2) \ddot{z} + \left\{ \left( \frac{a'}{a} + \frac{d'}{d} \right) (n^2 - b^2 \dot{y}^2) - 2mn' + \right.$$  

$$+ 2bb' \dot{y}^2 \right\} \dot{z}^2 + (b^2 \dot{y} \dot{z}) \dot{z} +$$  

$$+ \left\{ - \frac{a'}{ad^2} (n^2 - b^2 \dot{y}^2) + \frac{mm' - bb' \dot{y}^2}{d^2} \right\} (n^2 - b^2 \dot{y}^2) = 0. \quad (24)$$

It is clear that the case $\dot{y} = 0$ is a solution of this set of equations when at the same time $z$ obeys (22).

This set of equations can be handled leading to

$$\frac{\ddot{z} \dot{y}}{h(z)} F_z + \left( h(z) - \frac{\dot{z}^2}{h(z)} \right) F_y = 0, \quad (25)$$

$$\left( 1 - \frac{\dot{z}^2 \dot{y}^2}{h(z)} \right) F_z + \frac{\ddot{z} \dot{y}}{h(z)} F_y = 0,$$
where

\[ F_y = \dot{y} + \dot{z} \left( \frac{2}{z} - \frac{h'(z)}{h(z)} \right) \dot{y}, \]
\[ F_z = \frac{\ddot{z}}{z} + \frac{z^2}{z^2} \left( 1 - \frac{3}{2} \frac{h'(z)}{h(z)} \right) + \frac{h(z)}{z} \left( \frac{h'(z)}{2} - \frac{h(z)}{z} \right). \]  

(26) Since the determinant of the set (25) is non-zero, the solutions of (23) and (24) must satisfy \( F_y = 0 \) and \( F_z = 0 \) independently. Furthermore, let us notice that \( F_y = 0 \) and \( F_z = 0 \) are the null geodesic equations for \( y \) and \( z \) respectively obtained from

\[ \ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \lambda \dot{x}^\alpha. \]  

(27) Thus, a null curve is extreme if and only if it is a null geodesic.

Then, our problem is reduced to the previous case with constant \( y \) described by (22).

For \( k \neq 0 \) cases we can also consider (22) as the shortcut equation if we assume the existence of a \( y \)-symmetry in our problem.

### 4 AdS-Schwarzschild Bulk

From the Israel conditions (18) together with (19) we have

\[ \frac{h}{z_0^2} = \frac{\kappa_{(6)}^4 \rho^2}{64}, \]  

(28) \[ \frac{h'}{2z_0} = -\frac{\kappa_{(6)}^4 \rho^2}{64}(4\omega + 3), \]  

(29) and we can obtain the black hole mass \( M \) as a function of the brane energy density \( \rho \), while \( \rho \) is fixed by a fine-tuning,

\[ \frac{M}{z_0^5} = \frac{2}{5} \frac{k}{z_0^2} - (\omega + 1) \frac{\kappa_{(6)}^2 \rho^2}{40}, \]  

(30) \[ \frac{\kappa_{(6)}^2 \rho^2}{64} = -\frac{3k}{z_0^2 (8\omega + 3)} - \frac{5}{(8\omega + 3)l^2}, \]  

(31) where \( \omega = p/\rho \).
As we saw in the previous section, the shortcuts in six dimensions are determined from (22). We should also remember that the brane is static at $z = z_0$.

If a shortcut exists, there must be a time $t = v$ in the graviton path when $\dot{z}(v) = 0$ and $\ddot{z}(v) \geq 0$. Thus, (22) evaluated at this point will give

$$\ddot{z}(v) + h(z_v) \left( \frac{h'(z_v)}{2} - \frac{h(z_v)}{z_v} \right) = 0.$$  

(32)

It is obvious that this minimum must be between the brane and the event horizon $z_h$, if a horizon exists. Otherwise, there is no turning point in the path since the graviton can not return after it goes through the event horizon. Hence, $h(z_v) > 0$. 

Figure 1: $h(z)$ in six-dimensional AdS-Schwarzschild bulk with the brane located at $z = 1/3$. Notice that the singularity is shielded by a horizon.
Thus, from (32) we require
\[
F(z_v) = \frac{h'(z_v)}{2} - \frac{h(z_v)}{z_v} \leq 0 \quad \text{for} \quad z_h < z_v < z_0.
\] (33)

Using (14) this implies
\[
F(z) = \frac{5}{2} \frac{M}{z^4} - \frac{k}{z}.
\] (34)

This equation has a zero in \( z = z_f \neq 0 \) for \( k \neq 0 \)
\[
z_f^3 = \frac{5}{2k} M.
\]

Thus, for the \( k = 0 \) or \( k = -1 \) cases there is no positive root. Since the mass, \( M \), is positive, \( F(z) > 0 \) everywhere preventing the coexistence of shortcuts and horizons.
Figure 3: \( h(z) \) in five-dimensional AdS-Schwarzschild bulk with the brane located at \( z = 1/2 \). Notice that the singularity is shielded by a horizon.

On the other hand, for \( k = 1 \) there is one real and positive root, which must satisfy \( z_f < z_0 \) in order to have shortcuts. This is

\[
\frac{5M}{2z_0^3} - 1 < 0.
\]

Taking into account (30) and the fact that \( \varepsilon^2 \) must be positive in (31) \footnote{From now on, we will denote \( \varepsilon^2 = \kappa^4 \rho^2 / 64 \) in six dimensions.}

\[
- 4(\omega + 1)\varepsilon^2 z_0^2 < 0,
\]

then

\[
\omega + 1 > 0.
\]
Figure 4: Shortcuts for several initial velocities in five-dimensional AdS-Schwarzschild bulk. As in the six dimensional case, there is a threshold initial velocity for which the graviton can not return to the brane and falls into the event horizon.

at the zeros of

\[ z^5 + z^3 l^2 - l^2 M. \]

In the meantime, the non-vanishing zeros of \( h'(z) \) occur when

\[ 2z^5 + 3l^2 M = 0. \]

Since the derivative has no positive zeros with \( M > 0 \), there is just one event horizon. Then as \( h(z) \) goes to \(-\infty\) at the origin, the conditions

\[ M > 0 \]

(37)

and

\[ h(z_0) > 0 \]

(38)
are necessary and, in fact, enough to have a horizon and assure that the brane lies after it.

The condition (38) is automatically satisfied due to equation (28).

To fulfill (37) let us substitute (31) into (30) to have

\[(\omega + \frac{3}{4}) + (\omega + 1)\frac{z_0^2}{l^2} < 0.\]

If \(\omega + 1 \leq 0\), this condition is always satisfied, but this configuration does not produce shortcuts as we would like. However, the condition is also satisfied with \(\omega + 1 > 0\) if we require

\[-1 < \omega < -\frac{3}{4},\]

(39)

and

\[\frac{z_0^2}{l^2} < -\frac{\omega + 3/4}{\omega + 1}.\]

(40)

If we follow both (39) and (40) together with the fine-tunning for the energy (31), we will have several shortcuts in AdS-Schwarzschild bulks with shielded singularity. In figures 1 and 2 we illustrate an example with \(\omega = -4/5\), \(z_0 = 1/3\), and \(l = 1\). Notice in figure 1 that the horizon appears before the brane.

Since this case is equivalent to consider the problem in five dimensions with \(h(z)\), \(M\) and \(\rho\) given in [14], analogous results are obtained. In this case, the fine-tunning in the energy is given by

\[\varepsilon_{(5)}^2 = -\frac{1}{3\omega + 1} \left(\frac{1}{z_0^2} + \frac{2}{l^2}\right),\]

(41)

and \(\omega\) is confined to

\[-1 < \omega < -\frac{2}{3},\]

(42)

while the brane position is given by

\[\frac{z_0^2}{l^2} < -\frac{\omega + 2/3}{\omega + 1}.\]

(43)

An example is shown in figures 3 and 4 for \(\omega = -3/4\), \(z_0 = 1/2\), and \(l = 1\).

\[\text{In this case } \varepsilon_{(5)}^2 = \kappa_{(5)}^4 \rho^2 / 36.\]
5 AdS-Reissner-Nordström Bulk

From the Israel conditions (18) we will have for the black hole mass and charge,

\[ M = \frac{2k}{z^6_0} + \frac{8}{3l^2} + \frac{\kappa_6^4}{24} \rho^2 \omega, \]
\[ Q^2 = \frac{k}{z^6_0} + \frac{5}{3l^2} + \frac{8\omega + 3 \kappa_6^4 \rho^2}{64}. \]  (44)

At this stage it is convenient to carefully study the possibility of existence of shortcuts for every value of \( k \).

5.1 \( k = 0 \) and \( k = -1 \) Cases

As it was found in the AdS-Schwarzschild case, (13) determines the existence of shortcuts. Using (15) we see that (33) has a zero in \( z = z_f \neq 0 \) when

\[ \frac{5}{2} M z_f^3 - 4Q^2 - kz_f^6 = 0. \]  (45)

If \( k = 0 \), we have a real root in

\[ z_f^3 = \frac{8Q}{5M}. \]  (46)

If \( k = 1 \), we have two roots in

\[ z_f^3 = \frac{5}{4} M \pm \frac{1}{4} \sqrt{25M^2 - 64Q^2}. \]  (47)

Finally, if \( k = -1 \), we have

\[ z_f^3 = -\frac{5}{4} M \pm \frac{1}{4} \sqrt{25M^2 + 64Q^2}. \]  (48)

Notice that \( F(z) \) has at most one real and positive zero if \( k = 0, -1 \) and at most two positive zeros if \( k = 1 \).

Analyzing \( h(z) \) and its derivative we see that \( h(z) \) tends to \( +\infty \) both at the singularity and at infinity, while \( h'(z) \) tends to \( -\infty \) at the singularity and to \( +\infty \) at infinity.
The horizons occur at the zeros of \( h(z) \), or equivalently, at the zeros of
\[
z^8 + l^2 k z^6 - l^2 M z^3 + l^2 Q^2 = 0. \tag{49}
\]

On the other hand, the non-vanishing zeros of \( h'(z) \) occur when
\[
2z^8 + 3l^2 M z^3 - 6l^2 Q^2 = 0. \tag{50}
\]
This polynom grows at infinity being negative at the origin. Its derivative has non-vanishing roots when
\[
16z^5 + 9l^2 M = 0. \tag{51}
\]
For \( M > 0 \) this equation is never satisfied. Thus, as the derivative of (50) does not vanish and is positive outside the origin, the polynom (50) grows monotonically and has just one root. The zeros of this polynom are all non-vanishing zeros of \( h'(z) \). Therefore, we conclude that for positive mass there is just one zero for \( h'(z) \), and hence, at most two horizons for \( h(z) \).

When there is one horizon, \( h'(z) \) is negative before it and positive after, crossing \( h(z) \) at the very horizon. If there are two horizons, \( h'(z) \) vanishes at a point between Cauchy and event horizons, being negative before this point and positive after, while \( h(z) \) is positive at all points except between both horizons. Taking into account both the sign and zeros of these functions, \( h'(z) \) crosses \( h(z) \) between the Cauchy horizon and the point at which \( h'(z) \) vanishes.

Since \( h'(z)/2 \) has the same sign as \( h'(z) \) and vanishes at the same point, and in the same way \( h(z)/z \) has the same sign of \( h(z) \) and vanishes at the same points, we conclude that, existing horizons, \( F(z) \) necessarily vanishes at some point \( z = z_c \) such that \( 0 < z_c < z_h \). However, as we pointed out before, for \( k = 0 \) or \( k = -1 \) there is only one positive root of \( F(z) \). As \( F(z) < 0 \) for \( z < z_c \), then \( F(z) > 0 \) for \( z > z_c \). Thus, because \( z_c \leq z_h \), \( F(z) > 0 \) for \( z > z_h \) contrary to what was required in (33). This implies that there are no shortcuts with \( k = 0 \) or \( k = -1 \) when horizons exist.

In five dimensions the proof is very similar and we arrive to the same conclusion.
5.2 $k = 1$ Case

As we saw in the previous section, $F(z)$ has two real, positive and distinct roots for $k = 1$,

$$r_1^3 = \frac{5}{4} M - \frac{1}{4} \sqrt{25M^2 - 64Q^2},$$  \hspace{1cm} (52)

$$r_2^3 = \frac{5}{4} M + \frac{1}{4} \sqrt{25M^2 - 64Q^2}.$$

This is the only situation where the shortcuts can coexist with a shielded singularity. In fact, this situation necessarily requires the second root of $F(z)$ being at some point before the brane position $z_0$. This also implies $F(z_0) < 0$.

In addition, we must have both $Q^2$ and $M$ positive.

Given the fact that we have horizons, if the brane is not between them or at a horizon position, then $h(z_0) > 0$. Furthermore, in order to guarantee that the brane is located after the event horizon, we also need $h'(z_0) > 0$.

From the discussion in the previous section we will have one or two horizons if and only if $h(r_1) \leq 0$.

In summary, shortcuts in bulks with shielded singularities can occur only if $k = 1$ and also if the following conditions are supplied,

1. $h(z_0) > 0$ and $h'(z_0) > 0$ to have both horizons before the brane.
2. $F(z_0) < 0$ and $r_2 < z_0$ to have shortcuts with shielded singularity.
3. $Q^2 > 0$ and $M > 0$, which assures the positivity of the black hole mass and charge.
4. $h(r_1) \leq 0$ in order to have horizons.

We will analyze each condition and impose certain restrictions on $\omega$, $\rho^2$, and $z_0$.

5.2.1 Existence of Both Horizons Before the Brane

These conditions are the simplest to analyze since they restrict $\omega$ directly from the Israel conditions [18] together with [19]

$$\frac{h(z_0)}{a_0^3} = \varepsilon^2,$$  \hspace{1cm} (54)
\[ \frac{h'(z_0)}{2z_0} = -(4\omega + 3)\varepsilon^2. \] (55)

The condition (54) is automatically satisfied since \( \varepsilon^2 > 0 \).

From the condition (53)

\[-(4\omega + 3)\varepsilon^2 > 0, \] (56)

we have our first restriction

\[ \omega < -\frac{3}{4}. \] (57)

### 5.2.2 Existence of Shortcuts with Shielded Singularity

From the definition of \( F(z) \), (33), and using (54) and (55) we see that

\[ 0 > F(z_0) = -4(\omega + 1)\varepsilon^2 z_0. \] (58)

Thus we find another condition on \( \omega \)

\[ \omega + 1 > 0. \] (59)

Besides, from \( r_2 < z_0 \)

\[ \frac{5M}{4} - z_0^3 < -\frac{1}{4}\sqrt{25M^2 - 64Q^2}. \] (60)

This equation will be satisfied if \( \frac{5M}{4} < z_0^3 \)

\[ \frac{5M}{4} - z_0^3 < 0, \] (61)

or using (54)

\[ \frac{3}{2}z_0^3 + \frac{10}{3}z_0^5 + \frac{10}{3}z_0^5\omega\varepsilon^2 < 0, \] (62)

and as \( \omega < -3/4 \)

\[ z_0^2\varepsilon^2 > \frac{1}{\omega} \left( -\frac{9}{20} - \frac{z_0^2}{l^2} \right). \] (63)

\footnote{We assume that \( 25M^2 - 64Q^2 > 0 \). We will return to this condition when we discuss the existence of horizons, where we will impose a stronger restriction, \( M^2 - 4Q^2 > 0 \).}
5.2.3 Positivity of the Black Hole Mass and Charge

Because we require the positivity of the black hole mass, from (44) we have

\[ \frac{M}{z_0^3} > 0 \quad \Rightarrow \quad \frac{z_0^2}{l^2} + \omega \varepsilon^2 z_0^2 > -\frac{3}{4}, \]  

(64)

thus,

\[ z_0^2 \varepsilon^2 < \frac{1}{\omega} \left( -\frac{3}{4} - \frac{z_0^2}{l^2} \right). \]  

(65)

Since \(3/4 > 9/20\) this condition is certainly compatible with (63).

On the other hand, the positivity of the squared black hole charge requires

\[ \frac{Q^2}{z_0^6} > 0 \quad \Rightarrow \quad 1 + \frac{5z_0^2}{3l^2} + \left( \frac{8}{3} \omega + 1 \right) z_0^2 \varepsilon^2 > 0, \]  

(66)

so that

\[ z_0^2 \varepsilon^2 < \frac{1}{8\omega + 3} \left( -3 - \frac{5z_0^2}{l^2} \right). \]  

(67)

In spite of not being trivial, this equation is also compatible with (63). This requires

\[ \frac{1}{\omega} \left( -\frac{9}{20} - \frac{z_0^2}{l^2} \right) < \frac{1}{8\omega + 3} \left( -3 - \frac{5z_0^2}{l^2} \right), \]  

(68)

or

\[ -\frac{1}{5} \left( \omega + \frac{9}{4} \right) - \frac{z_0^2}{l^2} (\omega + 1) < 0, \]  

(69)

what is always true for \(-1 < \omega < -3/4\).

5.2.4 Existence of Horizons

This is the last and the more complicated of our conditions. We must have \(h(r_1) \leq 0\). Let \(x\) be \(r_1^3\),

\[ \frac{x^{8/3}}{l^2} + x^2 - Mx + Q^2 \leq 0. \]  

(70)

We do not need to do a complete study of this equation. For our purposes it will be enough to require

\[ x^2 - Mx + Q^2 < 0. \]  

(71)
Using (52) this implies
\[ M^2 - 4Q^2 > 0. \] (72)

This condition is necessary but not enough to have horizons. However, this restriction added to the others developed in this section will be enough to construct shortcuts with horizons as we will see. Notice that this condition is stronger than that one assumed before, \( M > 8/5Q \).

Using (44) in (72),
\[(1 - \varepsilon^2 l^2) + \frac{16 z_0^2}{9 l^2} \left(1 + \omega \varepsilon^2 l^2\right)^2 > 0.\] (73)

We know from (63) that \( 1 - \varepsilon^2 l^2 \) must be negative. Therefore, we must very carefully analyze (73). We can interpret (73) as a quadratic equation in the energy
\[ \left(1 + \frac{16 z_0^2}{9 l^2}\right) + \left(\frac{32}{9} \omega z_0^2 - l^2\right) \varepsilon^2 + \left(\frac{16}{9} l^2 z_0^2 \omega^2\right) \varepsilon^4 > 0, \] (74)
what implies
\[ z_0^2 \varepsilon^2 > \frac{1}{32} \left(-32 \omega z_0^2 + 9 l^2 + 3 \sqrt{-64 \omega z_0^2 l^2 + 9 l^4 - 64 l^2 z_0^2 \omega^2}\right) / (\omega^2 l^2), \] (75)
or
\[ z_0^2 \varepsilon^2 < \frac{1}{32} \left(-32 \omega z_0^2 + 9 l^2 - 3 \sqrt{-64 \omega z_0^2 l^2 + 9 l^4 - 64 l^2 z_0^2 \omega^2}\right) / (\omega^2 l^2). \] (76)

Notice that because \(-1 < \omega < -3/4,\)
\[-64 \omega z_0^2 l^2 - 64 l^2 z_0^2 \omega^2 = -64 \omega (\omega + 1) z_0^2 l^2 > 0\]
and all the previous roots are real and positive.

Summarizing, from the considerations in the previous sections, we must have for \( \omega \)
\[-1 < \omega < -3/4. \] (77)

For the energy,
\[ \frac{1}{\omega} \left(-\frac{9}{20} - \frac{z_0^2}{l^2}\right) < z_0^2 \varepsilon^2 < \frac{1}{\omega} \left(-\frac{3}{4} - \frac{z_0^2}{l^2}\right), \] or \( (78) \)
\[ \frac{1}{\omega} \left(-\frac{9}{20} - \frac{z_0^2}{l^2}\right) < z_0^2 \varepsilon^2 < \frac{1}{8 \omega + 3} \left(-3 - \frac{5 z_0^2}{l^2}\right), \] (79)
depending on which condition is more restrictive.

In addition,

\[ z_0^2 \epsilon^2 > \frac{1}{32} \left( -32 \omega z_0^2 + 9 l^2 + 3 \sqrt{-64 \omega z_0^2 l^2 + 9 l^4 - 64 l^2 z_0^2 \omega^2} \right) / (\omega^2 l^2) \] (80)

or

\[ z_0^2 \epsilon^2 < \frac{1}{32} \left( -32 \omega z_0^2 + 9 l^2 - 3 \sqrt{-64 \omega z_0^2 l^2 + 9 l^4 - 64 l^2 z_0^2 \omega^2} \right) / (\omega^2 l^2) \] (81)

Now we are going to analyze the situations in which all these conditions are compatible.

Let us begin our analysis with equation (81). To be compatible with (78) and (79), we just need

\[ \frac{1}{\omega} \left( -\frac{9}{20} - \frac{z_0^2}{l^2} \right) < \frac{1}{32} \left( -32 \omega z_0^2 + 9 l^2 - 3 \sqrt{-64 \omega z_0^2 l^2 + 9 l^4 - 64 l^2 z_0^2 \omega^2} \right) / (\omega^2 l^2) , \] (82)

that is,

\[ -\frac{9}{20} \omega - \frac{9}{32} + \frac{3}{32} \sqrt{-64 \omega z_0^2 l^2 + 9 - 64 z_0^2 l^2 \omega^2} < 0 . \] (83)

Since \( 3/4 < |\omega| < 1 \),

\[ -\frac{9}{20} \omega - \frac{9}{32} > 0 \]

will always be positive and thus, (83) will never be satisfied. Then, we conclude that (81) is not compatible either with (78) or (79). This implies that \( z_0^2 \epsilon^2 \) must satisfy (80) together with (78) or (79).

Let us initially compare (78) with (80). We must have

\[ \frac{1}{\omega} \left( -\frac{3}{4} - \frac{z_0^2}{l^2} \right) > \frac{1}{32} \left( -32 \omega z_0^2 + 9 l^2 + 3 \sqrt{-64 \omega z_0^2 l^2 + 9 l^4 - 64 l^2 z_0^2 \omega^2} \right) / (\omega^2 l^2) , \] (84)

that is,

\[ -\frac{3}{4} \omega - \frac{9}{32} - \frac{3}{32} \sqrt{-64 \omega z_0^2 l^2 + 9 - 64 z_0^2 l^2 \omega^2} > 0 . \] (85)

In this case, since \( \omega \) is negative and \( 3/4 < |\omega| < 1 \),

\[ -\frac{3}{4} \omega - \frac{9}{32} > 0 \]
and (83) can be satisfied if

\[
\left( -\frac{3}{4} \omega - \frac{9}{32} \right)^2 > \frac{9}{1024} \left( -64\omega \frac{z_0^2}{l^2} + 9 - 64\frac{z_0^2}{l^2} \omega^2 \right), \tag{86}
\]

or

\[
\frac{9}{16} \omega (\omega + 1) \frac{z_0^2}{l^2} + \frac{9}{64} (3 + 4\omega) \omega > 0. \tag{87}
\]

Since \(\omega + 1 > 0\) and \(3 + 4\omega < 0\), for positive \(z_0\) the inequality will be only fulfilled if

\[
z_0 < \frac{1}{2} \sqrt{\frac{3 + 4\omega}{1 + \omega}}. \tag{88}
\]

So that (83) and (88) can be compatible.

Now we are going to analyze the compatibility between (80) and (79). We must have

\[
\frac{1}{8\omega + 3} \left( -3 - \frac{5z_0^2}{l^2} \right) + \frac{z_0^2}{\omega l^2} - \frac{9}{32\omega^2} > \frac{3}{32\omega^2} \sqrt{-64\omega z_0^2 l^2 + 9l^4 - 64l^2 z_0^2 \omega^2}. \tag{89}
\]

or using that \(8\omega + 3 < 0\), simplifying, and squaring both sides we can write (89) as

\[
\frac{1}{16} \omega^2 (3 + 4\omega)^2 + \omega^2 (\omega + 1)^2 \left( \frac{z_0^2}{l^2} \right)^2 + \frac{\omega}{2} (3 + 4\omega)(\omega + 1) \frac{z_0^2}{l^2} > 0. \tag{90}
\]

This polynom has just one root for \(z_0^2/l^2\)

\[
\frac{z_0^2}{l^2} = -\frac{1}{4} \left( \frac{3 + 4\omega}{1 + \omega} \right). \tag{91}
\]

Since the coefficient of \(z_0^4/l^4\) is positive, the inequality is satisfied with the same condition (88), so we verify that both (79) and (78) are compatible with (80) under the same restrictions.

Furthermore, let us compare the upper limits of (78) and (79). Suppose

\[
\frac{1}{\omega} \left( -\frac{3}{4} - \frac{z_0^2}{l^2} \right) > \frac{1}{8\omega + 3} \left( -3 - \frac{5z_0^2}{l^2} \right), \tag{91}
\]
what can also be written as

\[- \frac{3}{4} (4\omega + 3) - \frac{3z_0^2}{l^2}(\omega + 1) > 0. \tag{92}\]

Thus, (91) is satisfied if and only if

\[\frac{z_0^2}{l^2} < -\frac{1}{4} \left( \frac{3 + 4\omega}{1 + \omega} \right), \tag{93}\]

which is just the same inequality (88), that \(z_0\) must satisfy. Hence, between (78) and (79), it is enough to take into account the latter. Nevertheless, from (90) notice that (73) would be also compatible with (80) if

\[\frac{z_0^2}{l^2} > -\frac{1}{4} \left( \frac{3 + 4\omega}{1 + \omega} \right),\]

and (91) would be satisfied with a change of sign implying that we should consider (78) instead of (79); however, as stated before, the compatibility of (78) and (80) requires

\[\frac{z_0^2}{l^2} < -\frac{1}{4} \left( \frac{3 + 4\omega}{1 + \omega} \right),\]

which contradicts our hypothesis. Therefore, the only possible configuration is (93).

At last, we compare the lower limits of (79) and (80). Suppose

\[\frac{1}{\omega} \left( -\frac{9}{20} - \frac{z_0^2}{l^2} \right) < \frac{1}{32\omega^2l^2} \left( -32\omega z_0^2 + 9l^2 + 3\sqrt{-64\omega z_0^2 l^2 + 9l^4 - 64l^2 z_0^2 \omega^2} \right), \tag{94}\]

or

\[-\frac{9}{20}\omega - \frac{9}{32} < \frac{3}{32} \sqrt{-64\omega \frac{z_0^2}{l^2} + 9 - 64 \frac{z_0^2}{l^2} \omega^2}.\]

Squaring and simplifying we obtain

\[\frac{z_0^2}{l^2} > -\frac{9}{20(\omega + 1)} \left( \frac{4}{5}\omega + 1 \right).\]

For \(-1 < \omega < -3/4\) this inequality is always satisfied since the right hand side is negative. Hence, we conclude that (94) is valid and between the lower limits for the energy in (78) and (80), we just need to choose the latter.
Figure 5: $h(z)$ in six-dimensional AdS-Reissner-Nordström bulk with the brane located at $z = 1$. Notice that the singularity is shielded by two horizons.

In short, by purely analytic considerations we conclude that shortcuts in bulks having no naked singularities and a static brane embedded in can only appear if $k = 1$ and if the following conditions are satisfied,

1. We must choose $\omega$ such that $-1 < \omega < -3/4$;

2. Given $\omega$, the brane must be located at a position such that

$$
\frac{z_0}{l} < \frac{1}{2} \sqrt{-\frac{3 + 4\omega}{1 + \omega}},
$$

which is the same condition as AdS-Schwarzschild case (40);

3. Given (95), the energy $\varepsilon$ must satisfy
In this way, it turns out to be simple to find shortcuts in bulks with shielded singularities.

As an example, let us choose $\omega = -9/10$. From (95) we must have

$$\frac{z_0}{l} < \frac{\sqrt{6}}{2},$$

then we choose $l = 1$ and $z_0 = 1$.

From (96) we have

$$\frac{35}{24} + \frac{5}{72} \sqrt{41} < \varepsilon^2 < \frac{40}{21},$$

**Figure 6:** Shortcuts for several initial velocities in six-dimensional AdS-Reissner-Nordström bulk. Notice that there is threshold initial velocity for which the graviton cannot return to the brane and falls into the event horizon.
Figure 7: $h(z)$ in five-dimensional AdS-Reissner-Nordström bulk with the brane located at $z = 1$. We see that the singularity is protected by two horizons.

so we choose $\varepsilon = \sqrt{238/125}$.

In figure (5) we plot $h(z)$ with these conditions. Notice that the singularity is protected by an event horizon and the brane is at $z = z_0 = 1$.

In figure (6) we plot the graviton paths obtained from (22) under the previous conditions for a variety of initial velocities showing that, in fact, shortcuts appear when we choose the parameters following the complete analysis shown in this section.

The analysis in five dimensions can be performed analogously to six-dimensional one with

$$h(z) = 1 + \frac{z^2}{l^2} - \frac{M}{z^2} + \frac{Q^2}{z^4},$$

(97)
Figure 8: Shortcuts for several initial velocities in five-dimensional AdS-Reissner-Nordström bulk. We see that there is threshold initial velocity for which the graviton can not return to the brane and falls into the event horizon.

and

\[
\frac{M}{z_0^4} = \frac{2}{z_0^2} + \frac{3}{l^2} + 3\omega \varepsilon_{(5)}^2, \quad (98)
\]

\[
\frac{Q^2}{z_0^6} = \frac{1}{z_0^2} + \frac{2}{l^2} + 3\omega \varepsilon_{(5)}^2 + \varepsilon_{(5)}^2, \quad (99)
\]

arriving to the following restrictions:

1. We must choose \( \omega \) such that \(-1 < \omega < -2/3\);

2. Given \( \omega \), the brane must be located at a position such that

\[
\frac{z_0}{l} < \sqrt{-\omega + 2/3 \over 1 + \omega}; \quad (100)
\]
3. Given \((100)\), the energy \(\varepsilon_{(5)}^2 = \kappa_{(5)}^4 \rho^2 / 36\) must satisfy

\[
\frac{1}{9\omega^2} \left( 2 - 9 \frac{z_0^2}{l^2} \omega + 2 \sqrt{1 - 9 \frac{z_0^2}{l^2} \omega (\omega + 1)} \right) < z_0 \varepsilon_{(5)}^2 < \frac{1}{3\omega + 1} \left( -1 - \frac{2z_0^2}{l^2} \right).
\]

As an example, we choose \(\omega = -7/8\). From \((100)\) we must have

\[
\frac{z_0}{l} < \frac{\sqrt{15}}{3},
\]

so we choose \(l = 1\) and \(z_0 = 1\).

From \((101)\) the energy must fulfill

\[
\frac{632}{441} + \frac{16}{441} \sqrt{127} < \varepsilon_{(5)}^2 < \frac{24}{13},
\]

then we choose \(\varepsilon_{(5)} = \sqrt{461/250}\).

In figure (7) we can see \(h(z)\) according to the previous conditions. As in the six dimensional case, the singularity is protected by an event horizon, and the brane is at \(z = z_0 = 1\).

In figure (8) we show several graviton paths obtained under the previous conditions for several initial velocities showing that shortcuts appear when we choose the parameters following the analysis shown in this section analogously to the six dimensional case.

6 Conclusions

In this paper we have shown that the shortest graviton path is governed by just one equation involving the “radial” extra coordinate. We have also seen that a symmetry in the “angular” extra coordinate has permitted us to consider curved spatial sections.

The AdS-Schwarzschild and AdS-Reissner-Nordström bulks open up the possibility of having shortcuts provided both the spatial section has positive curvature and a set of strong restrictions on the brane intrinsic tension must be satisfied. Moreover, its location in the bulk has to be respected. We should also notice that the energy is already fine-tunned from the Israel conditions in the case of the AdS-Schwarzschild bulk.
Hence, it is interesting to notice that despite the fact that the charge contributes to have a negative $F(z_0)$ and thus facilitates the existence of shortcuts, there are more restrictive conditions for the energy coming from $Q^2 > 0$ and from the horizons equation (70) which do not appear in the uncharged case. In this way, the results favor the existence of shortcuts in bulks with shielded singularities with the same conditions for $\omega$ and $z_0$ as the AdS-Schwarzschild case and also impose what is basically a fine-tuning in the energy. Thus, both cases seem to be equivalent for the study of shortcuts in static universes with protected singularities. We should realize that the restrictions to obtain these shortcuts make lose the main advantage we should have when we study the AdS-Reissner-Nordström case, i.e., the absence of fine-tunnings for the intrinsic tension.

In spite of the existence of fine-tunnings, the fact is that shortcuts appear and consequences are manifold. As mentioned in [7, 8], the existence of shortcuts could partially solve the horizon problem. We should notice that our set of conditions to obtain shortcuts in AdS-Schwarzschild and AdS-Reissner-Nordström bulks with protected singularities impose a restriction on the size of the universe, namely $z_0 \sim l$, which corresponds to a primeval universe. So the results shown in the present paper could contribute to the solution of this important problem.

We can also point out experimental consequences. In particular, gravitational waves advanced with respect to photons might be found in the proposed gravitational antennas under way, in case we find a model for the Universe with a physical size.

**Acknowledgements:** This work has been supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

**References**

[1] T. Kaluza, *Sitzungsberichte Preussische Akademie der Wissenschaften* K1 (1921) 966; O. Klein, *Z. F. Physik* 37 (1926) 895; O. Klein, *Nature* 118 (1926) 516.

[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* B429 (1998) 263.
[3] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436 (1998) 257.

[4] E. Abdalla and L. A. Correa-Borbonet, Phys. Lett. B489 (2000) 383; N. Kaloper and A. Linde, Phys. Rev. D60 (1999) 103509.

[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.

[7] H. Ishihara, Phys. Rev. Lett. 86 (2001) 381.

[8] R. Caldwell, D. Langlois, Phys.Lett. B511 (2001) 129; gr-qc/0103070.

[9] D. J. Chung and K. Freese, Phys. Rev. D62 (2000) 063513.

[10] D. J. Chung, E. W. Kolb and A. Riotto, Phys. Rev. D65 (2002) 083516; hep-ph/0008126; D. J. Chung, talk at Santa Fe 2000 Summer Workshop “Supersymmetry, branes and extra dimensions”.

[11] E. Abdalla, B. Cuadros-Melgar, S. Feng and B. Wang, Phys. Rev. D65 (2002) 083512; hep-th/0109024; B. Cuadros-Melgar, talk at Spanish Relativity Meeting E.R.E.2001.

[12] P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. B565 (2000) 269; hep-th/9905012.

[13] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B477 (2000) 285.

[14] C. Csáki, J. Erlich and C. Grojean, Nucl.Phys. B604 (2001) 312; hep-th/0012143.

[15] M. A. Clayton and J. W. Moffat, Phys. Lett. B460 (1999) 263. I. T. Drummond gr-qc/9908058; M. A. Clayton and J. W. Moffat, Phys. Lett. B477 (2000) 269.

[16] Bin Wang, Elcio Abdalla and Ru Keng Su, Phys. Lett. B503 (2001) 394, Mod. Phys. Lett. A17 (2002) 23.

[17] P. Kanti, R. Madden and K. A. Olive, Phys. Rev. D64 (2001) 044021; hep-th/0104177.