A BLOW-UP CRITERIA AND THE EXISTENCE OF 2D GRAVITY WATER WAVES WITH ANGLED CRESTS

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Abstract. We consider the two dimensional gravity water wave equation in the regime that includes free surfaces with angled crests. We assume that the fluid is inviscid, incompressible and irrotational, the air density is zero, and we neglect the surface tension. In [21] it was shown that in this regime, only a degenerate Taylor inequality \(-\frac{\partial P}{\partial n} \geq 0\) holds, with degeneracy at the singularities; an energy functional \(\mathcal{E}\) was constructed and an aprori estimate was proved. In this paper we show that a (generalized) solution of the water wave equation with smooth data will remain smooth so long as \(\mathcal{E}(t)\) remains finite; and for any data satisfying \(\mathcal{E}(0) < \infty\), the equation is solvable locally in time, for a period depending only on \(\mathcal{E}(0)\).

1. Introduction

A class of water wave problems concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in \(n\)-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is \(-k\), where \(k\) is the unit vector pointing in the upward vertical direction, and at time \(t \geq 0\), the free interface is \(\Sigma(t)\), and the fluid occupies region \(\Omega(t)\). When surface tension is zero, the motion of the fluid is described by

\[
\begin{align*}
\nabla v + v \cdot \nabla v &= -k - \nabla P & \text{on } \Omega(t), \; t \geq 0, \\
\text{div } v &= 0, & \text{on } \Omega(t), \; t \geq 0, \\
\text{curl } v &= 0, & \text{on } \Omega(t), \; t \geq 0, \\
P &= 0, & \text{on } \Sigma(t) \\
(1, v) & \text{ is tangent to the free surface } (t, \Sigma(t)),
\end{align*}
\]

(1.1)

where \(v\) is the fluid velocity, \(P\) is the fluid pressure. There is an important condition for these problems:

\[-\frac{\partial P}{\partial n} \geq 0\] (1.2)

point-wise on the interface, where \(n\) is the outward unit normal to the interface \(\Sigma(t)\) [29]. It is well known that when surface tension is neglected and the Taylor sign condition (1.2) fails, the water wave motion can be subject to the Taylor instability [29, 4, 5]. In [30, 31], we showed that for dimensions \(n \geq 2\), the strong Taylor stability criterion

\[-\frac{\partial P}{\partial n} \geq c_0 > 0\] (1.3)

always holds for the infinite depth water wave problem (1.1), as long as the interface is non-self-intersecting and smooth; and the initial value problem of the water wave system (1.1) is uniquely solvable locally in time in Sobolev spaces \(H^s\), \(s \geq 4\) for arbitrary given data. Earlier work include Nalimov [25], Yosihara [36] and Craig [12] on local existence and uniqueness for small and smooth data for the 2d water wave equation (1.1). There have been much work recently, local wellposedness for water waves with additional effects such as surface tension, bottom and vorticity have been proved, c.f. [11, 15, 17, 22, 24, 26, 27, 37].

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local wellposedness of \((1.1)\) in low regularity Sobolev spaces where the interfaces are in \(C^{3/2}\) has been obtained, c.f. \([12,2]\). In all of these work, the strong Taylor stability criterion \((1.3)\) is assumed. In addition, in the last few years, almost global and global wellposedness for the water wave equation \((1.1)\) in both two and three dimensional spaces for small, smooth localized initial data have been proved, c.f. \([32,33,14,18,3]\).

In \([21]\), we studied the 2d water wave equation \((1.1)\) in the regime that includes free interfaces with angled crests. We constructed an energy functional \(E(t)\) in this framework and proved an a priori estimate. In this paper we introduce a notion of generalized solutions of \((1.1)\) — a generalized solution is classical provided the interface is non-self-intersecting\(^4\) and we show that for data satisfying \(E(0) < \infty\), a generalized solution of the 2d water wave equation \((1.1)\) exists for a time period depending only on \(E(0)\); if in addition the initial interface is chord-arc\(^3\) there is a \(T > 0\), depending only on \(E(0)\) and the chord-arc constant, so that the interface remains chord-arc and a classical solution of the 2d water wave equation \((1.1)\) exists for time \(t \in [0,T]\). The (generalized) solution is constructed by mollifying the initial data and by showing that the sequence of (generalized) solutions for the mollified data converges to a (generalized) solution for the given data.

The rest of the paper is organized as follows: in section 2 we state and refine the earlier results this paper is built upon, this includes the local wellposedness result for Sobolev data in \([30]\), and the energy functional \(E\) constructed and the a priori estimate proved in \([21]\), in the context of generalized solutions; the notion of generalized solutions will be introduced in \([2.2]\) and \([2.3]\). In section 3 we present the main results: a blow-up criteria via the energy functional \(E\) and the local existence of water waves with angled crests. We prove the blow-up criteria in sections 4 and the local existence in section 5. The majority of the notation are introduced in \([2.1]\) with the rest throughout the paper. Some basic preparatory results in analysis are given in Appendix A, various identities that are useful for the paper are derived in Appendix B. Finally in Appendix C we list the quantities which have been shown in \([21]\) are controlled by \(E\).

2. Preliminaries

2.1. Notation and convention. We consider solutions of the water wave equation \((1.1)\) in the setting where the fluid domain \(\Omega(t)\) is simply connected, with the free interface \(\Sigma(t) := \partial \Omega(t)\) a Jordan curve\(^2\)

\[\mathbf{v}(z,t) \to 0, \quad \text{as } |z| \to \infty\]

and the interface \(\Sigma(t)\) tending to horizontal lines at infinity\(^1\).

We use the following notations and conventions: \([A,B] := AB - BA\) is the commutator of operators \(A\) and \(B\). \(H^s(\mathbb{R})\) is the Sobolev space with norm \(\|f\|_{H^s} := (\int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi)^{1/2}\), \(H^{1/2}\) is the Sobolev space with norm \(\|f\|_{H^{1/2}} := (\int |\xi|^2 |\hat{f}(\xi)|^2 d\xi)^{1/2}\), \(L^p = L^p(\mathbb{R})\) is the \(L^p\) space with \(\|f\|_{L^p} := (\int |f(x)|^p dx)^{1/p}\) for \(1 \leq p < \infty\) and \(\|f\|_{L^\infty} := \text{ess sup} |f(x)|\). We write \(f(t) := f(\cdot,t)\), with \(\|f(t)\|_{H^s}\) being the Sobolev norm, \(\|f(t)\|_{L^p}\) being the \(L^p\) norm of \(f(t)\) in the spatial variable. When not specified, all the \(H^s\) and \(L^p\) norms are in terms of the spatial variables. Compositions are always in terms of the spatial variables and we write for \(f = f(\cdot,t), \ g = g(\cdot,t), \ f(g(\cdot,t),t) := f \circ g(\cdot,t) := U_g f(\cdot,t).\)

\(^1\)When there is surface tension, or vorticity, or a bottom, \([13]\) doesn’t always hold.
\(^2\)By non-self-intersecting we mean it is a Jordan curve.
\(^3\)A curve is chord-arc if the arc-length and the chord length between any two points on the curve are comparable.
\(^4\)That is, \(\Sigma(t)\) is homeomorphic to the line \(\mathbb{R}\).
\(^5\)The problem with velocity \(\mathbf{v}(z,t) \to (c,0)\) as \(|z| \to \infty\) can be reduced to the one with \(\mathbf{v} \to 0\) at infinity by studying the solutions in a moving frame. \(\Sigma(t)\) may tend to two different lines at \(+\infty\) and \(-\infty\).
identify \((x, y)\) with the complex number \(x + iy\): \(\text{Re} \ z, \text{Im} \ z\) are the real and imaginary parts of \(z; \overline{z} = \text{Re} \ z - i \text{Im} \ z\) is the complex conjugate of \(z\). \(\overline{\Omega}\) is the closure of the domain \(\Omega\), \(\partial \Omega\) is the boundary of \(\Omega\), \(P_- := \{z \in \mathbb{C} : \text{Im} \ z < 0\}\) is the lower half plane. We write
\[
[f, g; h] := \frac{1}{\pi i} \int \frac{(f(x) - f(y))(g(x) - g(y))}{(x - y)^2} h(y) \, dy.
\]

We use \(c, C\) to denote universal constants and \(c(a, b), C(a), M(a)\) etc. to denote constants that depends on \(a, b\) and respectively \(a\) etc. Constants appearing in different contexts need not be the same. We write \(f \lesssim g\) if there is a universal constant \(c\), such that \(f \leq cg\). RHS, LHS are the short codes for the "right hand side" and the "left hand side".

### 2.2. The equation for the free surface in Lagrangian and Riemann mapping variables

Let the free interface \(\Sigma(t) : z = z(\alpha, t)\), \(\alpha \in \mathbb{R}\) be given by Lagrangian parameter \(\alpha\), so \(z_t(\alpha, t) = v(z(\alpha, t); t)\) is the velocity of the fluid particles on the interface, \(z_{tt}(\alpha, t) = v_t + (v \cdot \nabla) v(z(\alpha, t); t)\) is the acceleration; notice that \(P = 0\) on \(\Sigma(t)\) implies that \(\nabla P\) is normal to \(\Sigma(t)\), therefore \(\nabla P = -i a z_\alpha\), where
\[
a = -\frac{1}{|z_\alpha|} \frac{\partial P}{\partial n};
\]
and the first and third equation of (1.1) gives
\[
z_{tt} + i = i a z_\alpha.
\]

The second equation of (1.1): \(\text{div} \ v = \text{curl} \ v = 0\) implies that \(\nabla\) is holomorphic in the fluid domain \(\Omega(t)\), hence \(\overline{\Sigma}_t\) is the boundary value of a holomorphic function in \(\Omega(t)\). By Proposition A.1 the second equation of (1.1) is equivalent to \(\nabla = \mathcal{H}_{\alpha} \nabla_{\alpha}\), where \(\mathcal{H}\) is the Hilbert transform associated with the fluid domain \(\Omega(t)\). So the motion of the fluid interface \(\Sigma(t) : z = z(\alpha, t)\) is given by
\[
\begin{cases}
z_{tt} + i = i a z_\alpha \\ \overline{\Sigma}_t = \mathcal{H}_{\alpha} \nabla_{\alpha}.
\end{cases}
\]

is a fully nonlinear equation. In [30], Riemann mapping was introduced to analyze the quasi-linear structure of (2.4).

Let \(\Phi(\cdot, t) : \Omega(t) \to \overline{\Omega}_-\) be the Riemann mapping taking \(\overline{\Omega}(t)\) to the closure of the lower half plane \(\overline{\Omega}_-\), satisfying \(\lim_{z \to \infty} \Phi_2(z, t) = 1\). Let
\[
h(\alpha, t) := \Phi(z(\alpha, t), t),
\]
so \(h : \mathbb{R} \to \mathbb{R}\) is a homeomorphism. Let \(h^{-1}\) be defined by
\[
h(h^{-1}(\alpha', t), t) = \alpha', \quad \alpha' \in \mathbb{R};
\]
and
\[
Z(\alpha', t) := z \circ h^{-1}(\alpha', t), \quad Z_t(\alpha', t) := z_t \circ h^{-1}(\alpha', t), \quad Z_{tt}(\alpha', t) := z_{tt} \circ h^{-1}(\alpha', t) \quad (2.6)
\]
be the reparametrization of the position, velocity and acceleration of the interface in the Riemann mapping variable \(\alpha'\). Let
\[
Z_{\alpha'}(\alpha', t) := \partial_{\alpha'} Z(\alpha', t), \quad Z_{t, \alpha'}(\alpha', t) := \partial_{\alpha'} Z_t(\alpha', t), \quad Z_{tt, \alpha'}(\alpha', t) := \partial_{\alpha'} Z_{tt}(\alpha', t) \quad (2.7)
\]
etc. \(Z(\alpha', t)\) is the position function, \(Z_t(\alpha', t)\) is the velocity function, \(Z_{tt}(\alpha', t)\) is the acceleration function. We note that \(\Phi^{-1}(\alpha', t) = Z(\alpha', t)\), so \((\Phi^{-1})_t(\alpha', t) = Z_{\alpha'}(\alpha', t)\), and by Proposition A.1
\[
Z_{\alpha'} - 1 = \mathbb{H}(Z, \alpha' - 1), \quad \frac{1}{Z_{\alpha'} - 1} = \mathbb{H}(\frac{1}{Z_{\alpha'} - 1}) \quad (2.8)
\]
Observe that $\nabla \Phi^{-1} : P_- \to \mathbb{C}$ is holomorphic in the lower half plane $P_-$ with $\nabla \Phi^{-1}(\alpha', t) = \overline{Z_t}(\alpha', t)$. Precomposing (2.23) with $h^{-1}$ and applying Proposition A.1 to $\nabla \Phi^{-1}$ on $P_-$ gives the free surface equation in the Riemann mapping variable:

$$
\begin{align*}
Z_{tt} + i &= i\mathcal{A}Z_{,\alpha'} \\
Z_t &= h\mathcal{H}z_t
\end{align*}
$$

(2.9)

where $\mathcal{A} \circ h = ah_\alpha$ and $\mathcal{H}$ is the Hilbert transform associated with the lower half plane $P_-$. In [30], it was shown that systems (1.1), (2.4) and (2.9)-(2.8) with $h, A_1$ given by (2.18), (2.19) are equivalent in the regime of nonself-intersecting interfaces $z = z(\cdot, t)$.

However the system (2.9)-(2.8) is well defined even if $Z = Z(\cdot, t)$ is self-intersecting. In constructing the approximating sequence of solutions from the mollified data, it is convenient to allow self-intersecting solutions of (2.9)-(2.8). In this context, $Z$ and $z, Z_t, z_t$ etc. are related via (2.10) and (2.11) through a homeomorphism $h = h(\cdot, t) : \mathbb{R} \to \mathbb{R}$, and from (2.9)-(2.8) we can show that $h, A_1$ satisfy (2.18)-(2.19), see Appendix B.4. For not necessarily non-self-intersecting solutions $Z$ of (2.9)-(2.8), we will abuse terminologies by continue saying $Z, Z_t$ etc. are in the Riemann mapping variable, $z, z_t$ etc. are in the Lagrangian coordinates, $Z, Z_t, Z_{tt}$ are the interface, velocity and acceleration.

Let’s consider the solution of (2.9)-(2.8) in the ”fluid domain”.\footnote{When $\Sigma(\cdot) : Z = Z(\cdot, t)$ becomes self-intersecting, it is not physical to assume $P \equiv 0$ on $\Sigma(t)$. In general we do not consider beyond the regime of non-self-intersecting interfaces.}

### 2.3. Generalized solutions of the water wave equation.\footnote{It makes sense to talk about fluid domain only when $Z = Z(\cdot, t)$ is non-self-intersecting. Here we just abuse the terminology.}

Let $Z = Z(\cdot, t)$ be a solution of (2.9)-(2.8), let $F(\cdot, t) : P_- \to \mathbb{C}, \Psi(\cdot, t) : P_- \to \mathbb{C}$ be holomorphic functions, continuous on $P_-$, such that

$$
F(\alpha', t) = \overline{Z_t}(\alpha', t), \quad \Psi(\alpha', t) = Z(\alpha', t), \quad \Psi'(\alpha', t) = Z_{,\alpha'}(\alpha', t).
$$

(2.11)

By (B.4) of Appendix B.1 and (2.11),

$$
\frac{Z_t}{Z_{,\alpha'}} \Psi_t = \frac{\overline{F}}{\Psi'} - \frac{\overline{\Psi}}{\Psi'}.
$$

(2.12)

Now $\overline{Z_t}(\alpha, t) = \overline{Z_t}(h(\alpha, t), t) = F(h(\alpha, t), t)$, so

$$
\overline{Z}_{tt} = F_t \circ h + F_{z'} \circ h h_t = U_{h}(F_t - \frac{\Psi_t}{\Psi'} F_{z'} + \overline{F} F_{z'} - i \Psi_{z'})
$$

therefore $\overline{Z}_{tt}$ is the trace of the function $F_t - \frac{\Psi_t}{\Psi'} F_{z'} + \overline{F} F_{z'} - i \Psi_z$ on $\partial P_-; Z_{,\alpha'}(\overline{Z}_{tt} - i)$ is then the trace of the function $\Psi_{z'} F_t - \Psi_t F_{z'} + \overline{F} F_{z'} - i \Psi_{z'}$ on $\partial P_-$. By (2.10),

$$
\Psi_{z'} F_t - \Psi_t F_{z'} + \overline{F} F_{z'} - i \Psi_{z'} = i A_1, \quad \text{on } \partial P_-. \quad (2.13)
$$
On the left hand side of (2.13), $\Psi^t F_t - \Psi_t F_e - i\Psi^e$ is holomorphic on $P_-$, while $\overline{F} F_e = \partial_x (\overline{F} F)$; we recall from complex analysis, $\partial_x = \frac{1}{2}(\partial_x - i\partial_y)$. So there is a real valued function $\mathcal{P} : \mathbb{P}_- \to \mathbb{R}$, such that
\[
\Psi^t F_t - \Psi_t F_e + \overline{F} F_e - i\Psi^e = -(\partial_x - i\partial_y)\mathcal{P}, \quad \text{on } P_-(2.14)
\]
moreover by (2.13), because $iA_1$ is purely imaginary,
\[
\mathcal{P} = 0, \quad \text{on } \partial P_- . (2.15)
\]
We note that by applying $\partial_x + i\partial_y$ to both sides of (2.14), $\mathcal{P}$ satisfies
\[
\Delta \mathcal{P} = -2|F_x|^2 \quad \text{on } P_-. \tag{2.16}
\]
If in addition $\Sigma(t) = \{Z = Z(\alpha', t) : \Psi(\alpha', t) | \alpha' \in \mathbb{R}\}$ is a Jordan curve with
\[
\lim_{|\alpha'| \to \infty} Z_{,\alpha'}(\alpha', t) = 1,
\]
let $\Omega(t)$ be the domain bounded by $Z = Z(\cdot, t)$ from the above, then $Z = Z(\alpha', t)$, $\alpha' \in \mathbb{R}$ winds the boundary of $\Omega(t)$ exactly once. By the argument principle, $\Psi : \overline{P}_- \to \overline{\Omega}(t)$ is one-to-one and onto, $\Psi^{-1} : \Omega(t) \to P_-$ exists and is a holomorphic function. In this case, it is easy to check by the chain rule that equation (2.14) is equivalent to
\[
(F \circ \Psi^{-1})_t + \overline{F} \circ \Psi^{-1}(F \circ \Psi^{-1})_z + (\partial_x - i\partial_y)(\mathcal{P} \circ \Psi^{-1}) = i, \quad \text{on } \Omega(t) \tag{2.17}
\]
This is the Euler equation, i.e. the first equation of (1.1) in complex form. Therefore $\nu = F \circ \Psi^{-1}$, $P = \mathcal{P} \circ \Psi^{-1}$ is a solution of the water wave equation (1.1), with $\Sigma(t) : Z = Z(\cdot, t)$ the boundary of the fluid domain $\Omega(t)$.

In what follows we give the local wellposedness result of [30] and the a priori estimate of [21] for solutions of (2.9)-(2.8).

### 2.4. Local wellposedness in Sobolev spaces.

In [30] we derived a quasi-linearization of (2.9)-(2.8), the system (4.6)-(4.7) of [30] by taking one derivative to $t$ to equation (2.3) and analyzed the quantities $b$ and $A_1$ and via $A_1$, we showed that the strong Taylor inequality (1.3) always holds for smooth non-self-intersecting interfaces. In addition, we proved that the Cauchy problem of the system (4.6)-(4.7) of [30] is locally well-posed in Sobolev spaces.

**Proposition 2.1** (Lemma 3.1 and (4.7) of [30], Proposition 2.2 and (2.18) of [35]). We have
\[
b := h_t \circ h^{-1} = \text{Re} \left( [Z_t, H] \left( \frac{1}{Z_{,\alpha'}} - 1 \right) \right) + 2 \text{Re} Z_t. \tag{2.18}
\]
2.
\[
A_1 = 1 - \text{Im} [Z_t, H] [Z, Z_{,\alpha'}] = 1 + \frac{1}{2\pi} \int |Z_t(\alpha', t) - Z_t(\beta', t)|^2 (\alpha' - \beta')^2 d\beta ' \geq 1. \tag{2.19}
\]
3.
\[
- \frac{\partial P}{\partial n} |_{Z=Z(\cdot, t)} = \frac{A_1}{|Z_{,\alpha'}|}. \tag{2.20}
\]
in particular if the interface $\Sigma(t) \in C^{1,\gamma}$ for some $\gamma > 0$, then the strong Taylor sign condition (1.3) holds.

**Remark 2.2.** By (2.20), the Taylor sign condition (1.2) always holds. Assume $\Sigma(t)$ is non-self-intersecting with angled crests, assume the interior angle at a crest is $\nu$. Around the crest, we know the Riemann mapping $\Phi^{-1}$ (we move the singular point to the origin) behaves like
\[
\Phi^{-1}(z') \approx (z')^\nu, \quad \text{with } \nu = r\pi
\]

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9 [35] has a slightly different and shorter derivation. [21] has the derivation in a periodic setting. The reader may want to consult [21] [35] for the derivations. The identities in Appendix B.1 provide yet another derivation of the quasi-linearization and (2.15), (2.19) from (2.9)-(2.8), without assuming $Z = Z(\cdot, t)$ being non-self-intersecting. We note that (2.20) only makes sense for non-self-intersecting interfaces.
so $Z_{\alpha'} \approx (\alpha')^{r-1}$. From (2.10) and the fact $A_1 \geq 1$, the interior angle at the crest must be $\leq \pi$ if the acceleration $|Z_{tt}| \neq \infty$, and $-\frac{\partial \nu}{\partial n} = 0$ at the singularities where the interior angles are $< \pi$ [cf. 21], §3.

Let $h(\alpha, 0) = \alpha$ for $\alpha \in \mathbb{R}$; let the initial interface $Z(\cdot, 0) := Z(0)$, the initial velocity $Z_t(\cdot, 0) := Z_t(0)$ be given such that $Z(0)$ satisfy (2.8) and $Z_t(0)$ satisfy $Z_t(0) = \mathbb{H} Z_t(0)$; let $A_1$ be given by (2.19), the initial acceleration $Z_{tt}(0)$ satisfy (2.10), and $a_0 = \frac{A_1(\cdot, 0)^{3+1}}{\nu(\cdot, 0)}$. By Theorem 5.11 of [30] and a refinement of the argument in §6 of [30], the following local existence result holds.

**Proposition 2.3** (local existence in Sobolev spaces, cf. Theorem 5.11, §6 of [30]). Let $s \geq 4$. Assume that $Z_t(0) \in H^{s+1/2}(\mathbb{R})$, $Z_{tt}(0) \in H^s(\mathbb{R})$ and $a_0 \geq c_0 > 0$ for some constant $c_0 > 0$. Then there is $T > 0$ such that on $[0, T]$, the initial value problem of (2.9)-(2.8)-(2.13)-(2.19) has a unique solution $Z = Z(\cdot, t)$, satisfying $(Z_t, Z_{t t}) \in C^t([0, T], H^{s-1}(\mathbb{R}) \times H^{s+1/2-1}(\mathbb{R}))$, and $Z_{tt} = -1 \in C^l([0, T], H^{s-1}(\mathbb{R}))$, for $l = 0, 1$.

Moreover if $T^*$ is the supremum over all such times $T$, then either $T^* = \infty$, or $T^* < \infty,$ but

$$\sup_{[0, T^*)} (\|Z_{tt}(t)\|_{H^s} + \|Z_t(t)\|_{H^{s+1/2}}) = \infty. \quad (2.21)$$

**Proof.** Notice that the system (4.6)-(4.7) of [30] is a system for the horizontal velocity $w = \text{Re} Z_t$ and horizontal acceleration $u = \text{Re} Z_{tt}$, the interface doesn’t appear explicitly; it is well-defined even if the interface $Z = Z(\cdot, t)$ is self-intersecting. The first part of Proposition 2.3 follows from Theorem 5.11, and the argument from the second half of page 70 to the first half of page 71 of §6 of [30].

Now assume $T^* < \infty$, and

$$\sup_{[0, T^*)} (\|Z_{tt}(t)\|_{H^s} + \|Z_t(t)\|_{H^{s+1/2}}) := M_0 < \infty. \quad (2.22)$$

We want to show that the solution $Z$ of the system (2.9)-(2.8)-(2.13)-(2.19) can be extended beyond $T^*$ by a time $T' > 0$ that depends only on $M_0$, $c_0$, $\|Z_t(0)\|_{H^s}$ and $\|Z_{tt}(0)\|_{H^s}$, contradicting with the maximality of $T^*$.

Let $T < T^*$ be arbitrary chosen. Let $a = a(\cdot, t)$, $b = b(\cdot, t)$ be given by (4.7) of [30], and let $h = h(\cdot, t)$ satisfy

$$\begin{cases}
\frac{dh}{dt} = b(h, t) \\
h(\alpha, 0) = \alpha.
\end{cases} \quad (2.23)$$

By Theorem 5.11 of [30] and the argument in §6 of [30], we know $b \in C([0, T], H^{s+1/2}(\mathbb{R}))$ with $\|b(t)\|_{H^{s+1/2}} \leq c(\|Z_t(t)\|_{H^{s+1/2}}, \|Z_{tt}(t)\|_{H^s})$, and $h(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism with $h(\alpha, t) - \alpha \in C([0, T], H^{s+1/2})$. Moreover $Z(\alpha', t) := z \circ h^{-1}(\alpha', t)$ satisfies (2.10), and for $t \in [0, T]$,

$$\|Z_{tt}(t)\|_{H^s} + \|Z_t(t)\|_{H^{s+1/2}} \leq d_0 e^{Kt} (\|Z_{tt}(0)\|_{H^s} + \|Z_t(0)\|_{H^{s+1/2}}), \quad (2.24)$$

where $K = K(M(T), a(T), s)$, $d_0 = d(M(T), a(T), s)$ are constants depending on

$$a(T) := \inf_{\mathbb{R} \times [0, T]} a(\alpha', t), \quad M(T) = \sup_{[0, T]} (\|Z_{tt}(t)\|_{H^s} + \|Z_t(t)\|_{H^{s+1/2}}),$$

and $K(M(T), a(T), s) \to \infty$, $d(M(T), a(T), s) \to \infty$ as $M(T) \to \infty$, $a(T) \to 0$. We want to show that $a(T) \geq \frac{1}{c(M_0, c_0)}$ for some constant $C(M_0, c_0) > 0$.

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10 $A := \frac{A_1}{|Z_{tt}(0)|}$ equals zero alongside $-\frac{\partial \nu}{\partial n}$.
11 Let $s \geq 4$. As a consequence of $(Z_t(0), Z_{tt}(0)) \in H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ and $a_0 \geq c_0 > 0$, $Z_{tt}(0) - 1 \in H^s(\mathbb{R})$. In general by (2.10), $(Z_t, Z_{tt}) \in H^{s+1/2} \times H^s$ implies $\frac{1}{Z_{tt} - 1} \in H^s$, and $(Z_t, \frac{1}{Z_{tt} - 1}) \in H^{s+1/2} \times H^s$ implies $Z_{tt} \in H^s$.
12 $T$ depends only on $c_0$, $\|Z_t(0)\|_{H^{s+1/2}}$ and $\|Z_{tt}(0)\|_{H^s}$. 

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By (2.10),

\[ a(\alpha', t) := \frac{|Z_{tt}(t) + i|^2}{A_1(t)} = \frac{A_1(t)}{|Z_{,\alpha'}(t)|^2}, \]

so it suffices to show that there is a constant \( c(M_0, c_0) \), such that \( \|Z_{,\alpha'}(t)\|_{L^\infty} \leq c(M_0, c_0) \) for all \( t \in [0, T] \). From the assumption \( a_0 = \frac{A_1(\cdot, 0)}{|Z_{,\alpha'}(\cdot, 0)|^2} \geq c_0, \ |Z_{,\alpha'}(\cdot, 0)|^2 \leq \frac{A_1(\cdot, 0)}{c_0} \). Applying the Hardy’s inequality Proposition A.3 and Cauchy-Schwarz on (2.19) yields

\[ \|Z_{,\alpha'}(0)\|^2_{L^2} \leq \frac{\|A_1(0)\|_{L^\infty}}{c_0} \leq 1 + \|Z_{,\alpha'}(0)\|^2_{L^2}. \]

We calculate \( \|Z_{,\alpha'}(t)\|_{L^\infty} \) by the fundamental theorem of calculus. Differentiating (2.23) gives

\[
\begin{align*}
\frac{dh}{dt} & = b_{\alpha'}(h, t)h_{\alpha} \\
h_{\alpha}(\alpha, 0) & = 1.
\end{align*}
\]

(2.25)

So on \([0, T]\),

\[ e^{-\int_0^t \|b_{\alpha'}(\tau)\|_{L^\infty(\mathbb{R})} d\tau} \leq h_{\alpha}(\alpha, t) \leq e^{\int_0^t \|b_{\alpha'}(\tau)\|_{L^\infty(\mathbb{R})} d\tau}, \]

and by Sobolev embedding, \( \|b_{\alpha'}(\tau)\|_{L^\infty(\mathbb{R})} \lesssim \|b(\tau)\|_{H^1(\mathbb{R})} \leq c(\|Z_t(\tau)\|_{H^2}, \|Z_{tt}(\tau)\|_{H^2}) \). Because \( h(\alpha, 0) = \alpha, z(\alpha, 0) = Z(\alpha, 0) \) and

\[ z_{\alpha}(\alpha, t) = Z_{,\alpha}(\alpha, 0) + \int_0^t z_{t,\alpha}(\alpha, \tau) d\tau. \]

By the chain rule \( z_{t,\alpha} = Z_{,\alpha'} \circ hh_{\alpha}, z_{\alpha} = Z_{,\alpha'} \circ hh_{\alpha}; \) so for \( t \in [0, T] \),

\[ \|Z_{,\alpha'}(t)\|_{L^\infty} \leq (\|Z_{,\alpha'}(0)\|_{L^\infty} + \int_0^t \|Z_{,t,\alpha'}(\tau)\|_{L^\infty} \|h_{\alpha}(\tau)\|_{L^\infty} d\tau) \frac{1}{h_{\alpha}(t)} \|L^\infty \leq C(M_0, c_0) \]

for some constant \( C(M_0, c_0) \) depending on \( M_0, c_0 \) and \( T^* \), and

\[ a(T) = \inf_{R \times [0, T]} a(\alpha', t) = \inf_{R \times [0, T]} \frac{A_1}{|Z_{,\alpha'}|^2} \geq \frac{1}{C(M_0, c_0)}. \]

(2.26)

Now (2.22), (2.24) and (2.26) gives

\[ \|Z_{tt}(T)\|_{H^s} + \|Z_t(T)\|_{H^{s+1/2}} \leq c(M_0, c_0, \|Z_{tt}(0)\|_{H^s}, \|Z_t(0)\|_{H^{s+1/2}}), \]

and

\[ a(\cdot, T) \geq \frac{1}{C(M_0, c_0)} > 0. \]

So by the first part of Proposition 2.3, the solution \( Z \) can be extended onto \([T, T' + T]\), for some \( T' > 0 \) depending only on \( M_0, c_0 \) and \( \|Z_{tt}(0)\|_{H^s}, \|Z_t(0)\|_{H^{s+1/2}} \). This contradicts with the definition of \( T^* \), so either \( T^* = \infty \) or (2.21) holds.

\[ \square \]

Let \( D_\alpha := \frac{1}{z_{\alpha}} \partial_\alpha \) and \( D_{\alpha'} := \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \). By (2.23) and the basic fact that product of holomorphic functions is holomorphic, if \( g \) is the boundary value of a holomorphic function on \( F_- \), then \( D_{\alpha'} g \) is also the boundary value of a holomorphic function on \( P_- \). Notice that for any function \( f \),

\[ (D_{\alpha} f) \circ h^{-1} = D_{\alpha'}(f \circ h^{-1}). \]
2.5. An a priori estimate for water waves with angled crests. In [21], we studied the water wave equation (1.1) in the regime that includes interfaces with angled crests in a symmetric periodic setting, we constructed an energy functional for this regime and proved an a priori estimate. The same analysis applies to the whole line setting. The main difference is that in the whole line case, we do not need to consider the means of the various quantities; and in the proof of the a priori estimate, the argument in the footnote 21 of [21] works, so we do not need the Peter-Paul trick. Hence the whole line case, that part of the proof is simpler. Additionally, with the minor modifications given in Appendix B.1 the argument in [21] applies more generally to solutions of (2.9)–(2.8)–(2.18)–(2.19), without any non-self-intersecting assumptions, and the characterization of the energy given in §10 of [21] also holds. In this subsection we present the results of [21] in the whole line setting for solutions of (2.9)–(2.8)–(2.18)–(2.19), we will only show how to handle the differences between the symmetric periodic and the whole line cases.

Let

\[ E_a(t) = \| (\partial_t D_\alpha^2 \bar{z}_t) \circ h^{-1}(t) \|^2_{L^2(\mathbb{R})} + \| \frac{1}{Z_{t,\alpha'}} D_{\alpha'}^2 \bar{Z}_t(t) \|^2_{H^{1/2}} + \| D_{\alpha'}^2 \bar{Z}_t(t) \|^2_{L^2} \]  

and

\[ E_b(t) = \| (\partial_t D_\alpha \bar{z}_t(t)) \|^2_{L^2(\mathbb{R})} + \| D_{\alpha'} \bar{Z}_t(t) \|^2_{H^{1/2}} + \| D_{\alpha'}(\bar{Z}_t(t)) \|_{L^2} \]  

Let

\[ \mathcal{E}(t) = E_a(t) + E_b(t) + \| \bar{Z}_t(t) - i \|_{L^\infty} \]  

Notice from the estimate for \( \mathcal{E}(0) < \infty \) that for each given \( \alpha, \) we have

\[ \| \alpha_t \|_{L^\infty} + \| D_\alpha \bar{z}_t \|_{L^\infty} \leq c(\alpha) \]  

where \( c(\alpha) \) is a polynomial with nonnegative universal coefficients. Therefore

\[ \| \alpha_t \|_{L^\infty} + \| D_\alpha \bar{z}_t \|_{L^\infty} \leq c(\alpha) \]  

By Gronwall, \( \| \alpha_t \|_{L^\infty} \) is decreasing with respect to \( \alpha \), and \( c(\alpha) \) is increasing with respect to \( \alpha \).
so $\mathcal{E}(t) \leq \mathcal{E}_1(t)$, and $\mathcal{E}(0) = \mathcal{E}_1(0)$. By the whole line counterpart of Theorem 2 of [21],

$$\frac{d}{dt} \mathcal{E}_1(t) \leq p(\mathcal{E}_1(t)) + C(\mathcal{E}_1(t)).$$

(2.32)

Applying Gronwall again yields the conclusion of Theorem 2.4.

□

Let

$$\mathcal{E}(t) = \|Z_{t,\alpha'}\|_{L^2}^2 + \|D_{\alpha'}Z_t\|_{L^2}^2 + \|\partial_{\alpha'} \left( \frac{1}{Z_{t,\alpha'}} \right) \|_{L^2}^2 + \|D_{\alpha'} \left( \frac{1}{Z_{t,\alpha'}} \right) \|_{L^2}^2$$

$$+ \|\frac{1}{Z_{t,\alpha'}} D_{\alpha'} Z_t\|_{H^{1/2}}^2 + \|D_{\alpha'} Z_t\|_{H^{1/2}}^2 + \|\frac{1}{Z_{t,\alpha'}}\|_{L^\infty}^2.$$  

(2.33)

As was shown in §10 of [21], we have the following characterization of the energy $\mathcal{E}$.

**Proposition 2.5** (A characterization of $\mathcal{E}$ via $\mathcal{E}'$, cf. §10 of [21]). There are polynomials $C_1$ and $C_2$, with nonnegative universal coefficients, such that for solutions $Z$ of (2.9)-(2.8),

$$\mathcal{E}(t) \leq C_1(\mathcal{E}(t)), \quad \text{and} \quad \mathcal{E}(t) \leq C_2(\mathcal{E}(t)).$$

(2.34)

### 2.6. A description of the class $\mathcal{E} < \infty$ in the fluid domain.

We give here an equivalent description of the class $\mathcal{E} < \infty$ for solutions $Z$ of (2.9)-(2.8) in the "fluid domain".

Let $1 < p \leq \infty$, and

$$K_y(x) = \frac{-y}{\pi(x^2 + y^2)}, \quad y < 0$$

(2.35)

be the Poisson kernel. We know for any holomorphic function $G$ on $P_-$,

$$\sup_{y<0} \|G(x + iy)\|_{L^p(\mathbb{R}, dx)} < \infty$$

if and only if there exists $g \in L^p(\mathbb{R})$ such that $G(x + iy) = K_y * g(x)$. In this case, $\sup_{y<0} \|G(x + iy)\|_{L^p(\mathbb{R}, dx)} = \|g\|_{L^p}$. Moreover, if $g \in L^p(\mathbb{R})$, $1 < p < \infty$, $\lim_{y \to 0^-} K_y * g(x) = g(x)$ in $L^p(\mathbb{R})$ and if $g \in L^\infty(\mathbb{R}), \lim_{y \to 0^-} K_y * g(x) = g(x)$ for all $x \in \mathbb{R}$.

Let $Z = Z(\cdot, t)$ be a solution of (2.9)-(2.8), let $\Psi, F$ be holomorphic functions on $P_-$, continuous on $\overline{P}_-$, such that

$$Z(\alpha', t) = \Psi(\alpha', t), \quad Z_t(\alpha', t) = F(\alpha', t).$$

Notice that all the quantities in (2.33) are boundary values of some holomorphic functions on $P_-$. Let $z' = z' + iy'$, where $z', y' \in \mathbb{R}$. $\mathcal{E}(t) < \infty$ is equivalent to

$$\mathcal{E}_1(t) := \sup_{y' < 0} \|F_{z'}(t)\|_{L^2(\mathbb{R}, dz')}^2 + \sup_{y' < 0} \|\partial_{z'} \left( \frac{1}{\Psi_{z'}} \right) F_{z'}(t)\|_{L^2(\mathbb{R}, dz')}^2$$

$$+ \sup_{y' < 0} \|\partial_{z'} \left( \frac{1}{\Psi_{z'}} \right) F_{z'}(t)\|_{L^2(\mathbb{R}, dz')}^2 + \sup_{y' < 0} \|\partial_{z'} \left( \frac{1}{\Psi_{z'}} \right) F_{z'}(t)\|_{L^\infty(\mathbb{R}, dz')}^2$$

$$+ \sup_{y' < 0} \|\partial_{z'} \left( \frac{1}{\Psi_{z'}} \right) F_{z'}(t)\|_{H^{1/2}(\mathbb{R}, dz')}^2 < \infty.$$ 

(2.36)

\[\text{It is clear } \mathcal{E}(t) = \mathcal{E}_1(t) \text{ for smooth } Z = Z(\cdot, t). \text{ Otherwise this equivalence is understood at a formal level, and is made rigorous according to the circumstances.}\]
3. The main results

We are now ready to state the main results of the paper. For simplicity we present and prove the results in the whole line setting. The same results hold for the symmetric periodic setting as studied in [21] and the proofs are similar, except for some minor modifications.

Let \( h(\alpha, 0) = \alpha \) for \( \alpha \in \mathbb{R} \); let the initial interface \( \tilde{Z}(\cdot, 0) := \tilde{Z}(0) \), the initial velocity \( \tilde{Z}_t(\cdot, 0) := \tilde{Z}_t(0) \) be given such that \( \tilde{Z}(0) \) satisfy (2.8) and \( \tilde{Z}_t(0) \) satisfy \( \tilde{Z}_t(0) = \mathbb{H}\tilde{Z}_t(0) \); let \( A_1 \) be given by (2.19), the initial acceleration \( \tilde{Z}_{tt}(0) \) satisfy (2.10).

**Theorem 3.1** (A blow-up criteria via \( E \)). Let \( s \geq 4 \). Assume \( Z, \alpha' \in L^\infty(\mathbb{R}) \), \( Z_t(0) \in H^{s+1/2}(\mathbb{R}) \) and \( Z_{tt}(0) \in H^s(\mathbb{R}) \). Then there is \( T > 0 \), such that on \([0, T] \), the initial value problem of (2.9)–(2.8) has a unique solution \( Z = Z(\cdot, t) \), satisfying \((Z_{tt}, Z_t) \in C^l([0, T], H^{s-l} (\mathbb{R}) \times H^{s+1/2-l} (\mathbb{R})) \) for \( l = 0, 1 \), and \( Z, \alpha' - 1 \in C([0, T], H^s(\mathbb{R})) \).

Moreover if \( T^* \) is the supremum over all such times \( T \), then either \( T^* = \infty \), or \( T^* < \infty \), but

\[
\sup_{[0, T^*)} \mathcal{E}(t) = \infty \tag{3.1}
\]

**Remark 3.2.** 1. Assume \( Z, \alpha'-0 \in L^\infty(\mathbb{R}) \). We note that by the definition \( A := \frac{A_1}{|Z, \alpha'|^2} \),

\[
a_0 = \frac{A_1(0)}{|Z(0, 0)|^2} \geq c_0 > 0 \quad \text{for some constant } c_0 > 0.
\]

So the first part of Theorem 3.1 is the local wellposedness in Sobolev spaces as stated in Proposition 2.3. The novelty of Theorem 3.1 is the new blow up criteria via the energy functional \( \mathcal{E} \).

2. Notice that \( \sup_{[0, T^*)} \mathcal{E}(t) < \infty \) if and only if \( \sup_{[0, T^*]} \mathcal{E}(t) < \infty \), by Proposition 2.5.

By the discussion of (2.3) a solution of (2.9)–(2.8) is a solution of the water wave equation (1.1) if and only if \( \Sigma(t) = \{ Z = Z(\cdot, t) \mid \alpha' \in \mathbb{R} \} \) is Jordan. So we can modify the statement of Theorem 3.1 to give a blow-up criteria for the water wave equation (1.1). For the first half of the statements in Corollary 3.3 see Theorem 6.1 of [30].

**Corollary 3.3** (A blow-up criteria via \( E \)). Let \( s \geq 4 \). Assume in addition \( Z = Z(\cdot, 0) \) is non-self-intersecting. Then there is \( T > 0 \), such that on \([0, T] \), the initial value problem of (1.1) has a unique solution, with the properties that the interface \( Z(\cdot, t) \) is nonself-intersecting and \((Z_{tt}, Z_t) \in C^l([0, T], H^{s-l} (\mathbb{R}) \times H^{s+1/2-l} (\mathbb{R})) \) for \( l = 0, 1 \), and \( Z, \alpha' - 1 \in C([0, T], H^s(\mathbb{R})) \).

Moreover if \( T^* \) is the supremum over all such times \( T \), then either \( T^* = \infty \), or \( T^* < \infty \), but

\[
\sup_{[0, T^*)} \mathcal{E}(t) = \infty, \quad \text{or} \quad Z = Z(\cdot, t) \text{ becomes self-intersecting at } t = T^* \tag{3.2}
\]

**3.1. The initial data.** \(^{14}\) Let \( \Omega(0) \) be the initial fluid domain, with the interface \( \Sigma(0) := \partial \Omega(0) \) being a Jordan curve that tends to horizontal lines at infinity, and let \( \Phi(\cdot, 0) : \Omega(0) \to P_- \) be the Riemann Mapping such that \( \lim_{z \to \infty} \Phi(z, 0) = 0 \). We know \( \Phi(\cdot, 0) : \Omega(0) \to P_- \) is a homeomorphism. Let \( \Psi(\cdot, 0) := \Phi^{-1}(\cdot, 0) \), and \( Z(\cdot, 0) := \Psi(\cdot, 0) \), so \( Z = Z(\cdot, 0) : \mathbb{R} \to \Sigma(0) \) is the parametrization of \( \Sigma(0) \) in the Riemann Mapping variable. Let \( v(\cdot, 0) : \Omega(0) \to \mathbb{C} \) be the initial velocity field, and \( F(z', 0) = \nabla \Phi(z', 0) \). Assume \( \nabla \Phi(\cdot, 0) \) is holomorphic on \( \Omega(0) \), so \( F(\cdot, 0) \) is holomorphic on \( P_- \). Assume \( F(\cdot, 0), \Psi(\cdot, 0) \) satisfy (2.30) at \( t = 0 \). In addition, assume \(^ {15}\)

\[
c_0 := \sup_{y' < 0} \| F(x' + iy', 0) \|_{L^2(\mathbb{R}, dx')} + \sup_{y' < 0} \left| \frac{1}{\Psi_x(x' + iy', 0)} - 1 \right|_{L^2(\mathbb{R}, dx')} < \infty. \tag{3.3}
\]

\(^ {14}\) We only need to assume that \( F(\cdot, 0), \Psi(\cdot, 0) \) are holomorphic on \( P_- \) and continuous on \( P_{-} \), satisfying \( \lim_{z \to \infty} \Psi(z', 0) = 1, \Psi_z(z', 0) \neq 0 \) on \( P_{-} \) at \( t = 0 \) and \( y' < 0 \). We give the initial data as is to put it in the context of the water waves (1.1).

\(^ {15}\) Let \( Z_{tt}(0) \) be given by (2.10). Under the assumption (2.30) at \( t = 0 \), this is equivalent to assuming \( \| Z_t(0) \|_{L^2} + \| Z_{tt}(0) \|_{L^2} < \infty \).
Theorem 3.4 (Local existence in the $\mathcal{E} < \infty$ regime). 1. There exists $T_0 > 0$, depending only on $\mathcal{E}_1(0)$, such that on $[0, T_0]$, the initial value problem of the water wave equation (1.1) has a generalized solution $(F, \Psi, \mathcal{P})$ in the sense of (2.14)–(2.15), with the properties that $F(\cdot, t), \Psi(\cdot, t)$ are holomorphic on $P_\gamma$; for each fixed $t \in [0, T_0], F, \Psi, \mathcal{P}$ are continuous on $P_\gamma \times [0, T_0]$, $F, \Psi$ are continuous differentiable on $P_\gamma \times [0, T_0]$ and $\mathcal{P}$ is continuous differentiable with respect to the spatial variables on $P_\gamma \times [0, T_0]$; during this time, $\mathcal{E}_1(t) < \infty$ and

$$\sup_{y' < 0} \|F(x' + iy', t)\|_{L^2(\mathbb{R}, dx')} + \sup_{y' < 0} \|\Psi(x' + iy', t) - 1\|_{L^2(\mathbb{R}, dx')} < \infty. \tag{3.4}$$

The generalized solution gives rise to a solution $(\nabla, P) = (F \circ \Psi^{-1}, \mathcal{P} \circ \Psi^{-1})$ of the water wave equation (1.1), so long as $\Sigma(t) = \{Z = \Psi(\alpha', t) \mid \alpha' \in \mathbb{R}\}$ is a Jordan curve.

2. If in addition, the initial interface is chord-arc, that is, $Z_{\alpha'}(\cdot, 0) \in L^1_{\text{loc}}(\mathbb{R})$ and there is $0 < \delta < 1$, such that

$$\delta \int_{\alpha'}^{\beta'} |Z_{\alpha'}(\gamma, 0)| d\gamma \leq |Z(\alpha', 0) - Z(\beta', 0)| \leq \int_{\alpha'}^{\beta'} |Z_{\alpha'}(\gamma, 0)| d\gamma, \quad \forall - \infty < \alpha' < \beta' < \infty.$$  

Then there is $T_0 > 0, T_1 > 0, T_0, T_1$ depend only on $\mathcal{E}_1(0)$, such that on $[0, \min\{T_0, T_1\}]$, the initial value problem of the water wave equation (1.1) has a solution, satisfying $\mathcal{E}_1(t) < \infty$ and (3.1), and the interface $Z = Z(\cdot, t)$ is chord-arc.

4. The proof of Theorem 3.4

We only need to prove the second part, the blow-up criteria of Theorem 3.1. We assume $T^* < \infty$, for otherwise we are done.

Let $Z = Z(\cdot, t), t \in [0, T^*)$ be a solution of (2.9)–(2.8):

$$Z_{tt} + i = iAz_{\alpha}, \tag{4.1}$$

with constraint

$$\begin{cases}
Z_t = HZ_t, \\
Z_{\alpha} - 1 = H(Z_{\alpha} - 1), \quad Z_{\alpha} - 1 = H(\frac{1}{Z_{\alpha}} - 1);
\end{cases} \tag{4.2}$$

satisfying $(Z_{tt}, Z_t) \in C^1([0, T^*), H^{s-l}(\mathbb{R}) \times H^{s+1/2-l}(\mathbb{R}))$ for $l = 0, 1$, and $Z_{\alpha} - 1 \in C([0, T^*), H^s(\mathbb{R}))$. Precompose (4.1) with $h$ gives

$$z_{tt} + i = i\zeta z_{\alpha} \tag{4.3}$$

where $ah\zeta := A \circ h$. Differentiating (4.3) with respect to $t$ yields

$$z_{ttt} + i = i\zeta z_{\alpha} = \frac{a_t}{a} (z_{tt} - i) \tag{4.4}$$

Precompose (4.4) with $h^{-1}$. This gives the corresponding equation in the Riemann mapping variable:

$$Z_{ttt} + iAz_{tt,\alpha} = \frac{a_t}{a} \circ h^{-1}(Z_{tt} - i) \tag{4.5}$$

We know $Z_{ttt} = (\partial_t + b\partial_{\alpha})^2Z_t$ and $Z_{tt} = (\partial_t + b\partial_{\alpha})Z_t$, where $b := h_t \circ h^{-1}$. The analysis in Appendix [3.1] shows that $b$ and $A_1 := A|Z_{\alpha}|^2$ are as given in (2.18), (2.19), and

$$\frac{a_t}{a} \circ h^{-1} = -\text{Im}(2[Z_t, H][Z_{tt}, Z_t] + 2[Z_{tt}, H][\partial_t, Z_t] - [Z_t, Z_t; D_{\alpha}Z_t]), \tag{4.6}$$

where

$$[Z_t, Z_t; D_{\alpha}Z_t] := \frac{1}{2\pi i} \int \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{(\alpha' - \beta')^2} D_{\beta'}Z_t(\beta') d\beta'. \tag{4.7}$$

(4.4)–(4.2) or equivalently (4.5)–(4.2) with $b, A_1$ and $\frac{a_t}{a} \circ h^{-1}$ given by (2.18), (2.19) and (4.6) is a quasilinear equation of the hyperbolic type in the regime of smooth interfaces,
with the right hand side consisting of lower order terms.\footnote{4.2}  However in the regime that includes interfaces with angled crests, since $A$ and $-\frac{\partial \psi}{\partial n}$ equal to zero at the crests where the interior angles are $<\pi$, the left hand side of \eqref{equation:1} (or \eqref{equation:1}) is degenerate hyperbolic.

We have the following basic energy inequality.

**Lemma 4.1** (Basic energy inequality). Assume $\theta = \theta(\alpha, t)$, $\alpha \in \mathbb{R}$, $t \in [0, T)$ is smooth, decays fast at the spatial infinity and satisfies $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$ and

$$\partial^2_t \theta + i a \partial_\alpha \theta = G_\theta. \quad (4.8)$$

Let

$$E_\theta(t) := \int \frac{1}{a} |\theta_t|^2 \, d\alpha + i \int \partial_\alpha \theta \bar{\theta} \, d\alpha + \int \frac{1}{a} |\theta|^2 \, d\alpha. \quad (4.9)$$

Then

$$\frac{d}{dt} E_\theta(t) \leq \left( \left\| \frac{\alpha_t}{a} \right\|_{L^\infty} + 1 \right) E_\theta(t) + 2 E_\theta(t)^{1/2} \left( \int \frac{|G_\theta|^2}{a} \, d\alpha \right)^{1/2}. \quad (4.10)$$

**Remark 4.2.** Since $A \circ h := a h_\alpha$, upon changing to the Riemann mapping variable,

$$E_\theta(t) = \int \frac{1}{A} \left( |\theta_t| \circ h^{-1}|^2 + |\theta \circ h^{-1}|^2 \right) \, d\alpha' + i \int \partial_\alpha' (\theta \circ h^{-1}) \bar{\theta} \circ h^{-1} \, d\alpha'$$

By $\theta \circ h^{-1} = \mathbb{H}(\theta \circ h^{-1})$ and $(A.5)$,

$$i \int \partial_\alpha \theta \bar{\theta} \, d\alpha = i \int \partial_\alpha (\theta \circ h^{-1}) \bar{\theta} \circ h^{-1} \, d\alpha' = \| \theta \circ h^{-1} \|^2_{H^{1/2}} \geq 0.$$  

**Proof.** We have\footnote{13}

$$\frac{d}{dt} E_\theta(t) = 2 \text{Re} \int \frac{1}{a} \theta_t \bar{\theta_t} \, d\alpha + \int \frac{\alpha_t}{a^2} |\theta_t|^2 \, d\alpha + i \int \partial_\alpha \theta \bar{\theta_t} \, d\alpha + i \int \bar{\theta_t} \partial_\alpha \bar{\theta} \, d\alpha$$

$$+ 2 \text{Re} \int \frac{1}{a} \theta_h \bar{\theta} \, d\alpha - \int \frac{\alpha_t}{a^2} |\theta|^2 \, d\alpha$$

$$= 2 \text{Re} \int \frac{1}{a} (\theta_{tt} + i a \partial_\alpha \theta) \bar{\theta_t} \, d\alpha - \int \frac{\alpha_t}{a^2} (|\theta_t|^2 + |\theta|^2) \, d\alpha + 2 \text{Re} \int \frac{1}{a} \theta_h \bar{\theta} \, d\alpha$$

Here in the second step we used integration by parts on the third term. \footnote{13} Cauchy-Schwarz and the fact that $i \int \partial_\alpha \theta \bar{\theta} \, d\alpha \geq 0$ gives \eqref{equation:10}.

Apply $D_\alpha (\frac{\partial h}{h_\alpha})^{-k-1}$, $k = 2, 3$ to \eqref{equation:1}, then commute $D_\alpha (\frac{\partial h}{h_\alpha})^{-k-1}$ with $\partial^2_t + i a \partial_\alpha$ yields

$$(\partial^2_t + i a \partial_\alpha) D_\alpha (\frac{\partial h}{h_\alpha})^{-k-1} \bar{\theta_t} = D_\alpha (\frac{\partial h}{h_\alpha})^{-k-1} \left( -i a_\alpha \bar{\theta_t} \right) = \left( D_\alpha (\frac{\partial h}{h_\alpha})^{-k-1} \right) \bar{\theta_t}$$

Let

$$E_k(t) := E_{D_\alpha (\frac{\partial h}{h_\alpha})^{-k-1} \bar{\theta_t}}(t). \quad (4.12)$$

Because $A = \frac{A_1}{|Z_{\alpha'}|^2}$ and $U_{\alpha'}^{-1} D_\alpha U_h = D_{\alpha'} = \frac{1}{|Z_{\alpha'}|^2} \partial_\alpha$,

$$E_k(t) = \int \frac{1}{A_1} \left( |\partial^k_{\alpha'} Z_{\alpha'}|^2 + |Z_{\alpha'} U_{\alpha'}^{-1} \partial_\alpha U_h \frac{1}{|Z_{\alpha'}|^2} \partial^k_{\alpha'} Z_{\alpha'}|^2 \right) \, d\alpha' + \frac{1}{|Z_{\alpha'}|^2} \partial^k_{\alpha'} Z_{\alpha'} \right|^2 \right|_{H^{1/2}} \quad (4.14)$$

We prove Theorem 3.1 via the following two Propositions.

**Proposition 4.3.** There exists a polynomial $p_1 = p_1(x)$ with universal coefficients such that

$$\frac{d}{dt} E_2(t) \leq p_1(\mathcal{E}(t)) E_2(t). \quad (4.15)$$

\footnote{13} \eqref{equation:1} is equivalent to the quasi-linear system \eqref{equation:6}-\eqref{equation:7} of \cite{30}. The only difference is that \eqref{equation:13} is in terms of $Z_t$ and $Z_{\alpha'}$ and \eqref{equation:6}-\eqref{equation:7} of \cite{30} is in terms of the real components $\text{Re} Z_t$ and $\text{Re} Z_{\alpha'}$.

\footnote{18} Some variants of the proof have been given in \cite{32} and \cite{21}. We prove \eqref{equation:10} nevertheless.
Proposition 4.4. There exist polynomials $p_2 = p_2(x, y)$ and $p_3 = p_3(x, y)$ with universal coefficients such that

$$\frac{d}{dt}E_3(t) \leq p_2(E(t), E_2(t))E_3(t) + p_3(E(t), E_2(t)).$$  \hfill (4.16)

Propositions 4.3 and 4.4 give that

$$E_2(t) \leq E_2(0)e^{\int_0^t p_1(E(s)) \, ds}, \quad \text{and} \quad E_3(t) \leq (E_3(0) + \int_0^t p_3(E(s), E_2(s)) \, ds)e^{\int_0^t p_2(E(s), E_2(s)) \, ds},$$  \hfill (4.17)

so for $T^* < \infty$, $E_2(0) + E_3(0) < \infty$ and $\sup_{[0, T^*)} E(t) < \infty$ implies $\sup_{[0, T^*)}(E_2(t) + E_3(t)) < \infty$. In \ref{steinberg} and \ref{steinberg1} we will prove Propositions 4.3 and 4.4. We will complete the proof of Theorem 8.1 in \ref{steinberg} by showing that $\sup_{[0, T^*)}(\|Z_t(t)\|_{H^{3/2}} + \|Z_t(t)\|_{L^1})$ is controlled by $\sup_{[0, T^*)}(E_2(t) + E_3(t))$ and the initial data.

4.1. The proof of Proposition 4.3

Proof. We prove Proposition 4.3 by applying the basic energy inequality, Lemma 4.1 to $D_\alpha(D_\alpha^{-1}Z_t)$ of \ref{steinberg}, notice that $(I - i \Pi)(U_h^{-1}D_\alpha(D_\alpha^{-1}Z_t)) = (I - i \Pi)D_\alpha(Z_{t, \alpha'}) = 0$. Using \ref{steinberg1}, \ref{steinberg2} and \ref{steinberg3}, we expand the right hand side of \ref{steinberg}:

$$G_2 := D_\alpha \frac{\partial}{\partial h}(-i \alpha \bar{\alpha} z_t) + \frac{\partial^2 v}{\partial h} + i\alpha \bar{\alpha} D_\alpha \frac{\partial}{\partial h} \frac{\partial}{\partial h} z_t$$
$$= D_\alpha \frac{\partial}{\partial h}(-i \alpha \bar{\alpha} z_t) - 2(D_\alpha z_t D_\alpha \frac{\partial}{\partial h} z_t + D_\alpha z_t \partial_t D_\alpha \frac{\partial}{\partial h} z_t)
- D_\alpha \partial_t U_h \{(h_t \circ h^{-1})_\alpha \bar{\alpha} Z_{t, \alpha'}\} - D_\alpha U_h \{(h_t \circ h^{-1})_\alpha \bar{\alpha} Z_{t, \alpha'}\} - i D_\alpha U_h \{A_{\alpha'} Z_{i, \alpha'}\}$$

We can control $\|\frac{\partial}{\partial h}\|_{L^\infty}$ by a polynomial of $\bar{\alpha}$, see Appendix C. What remains to be shown is that

$$\int \frac{|G_2|^2}{a} \, d\alpha \leq C(\bar{\alpha})E_2,$$  \hfill (4.19)

for some polynomial $C(\bar{\alpha})$. Changing to the Riemann mapping variables and using $A = \frac{1}{|Z_{\alpha'}|}, \ A_1 \geq 1$,

$$\int \frac{|G_2|^2}{a} \, d\alpha = \int \frac{|G_2|^2}{ah_\alpha} \, d\alpha = \int \frac{|Z_{\alpha'}U_h^{-1}G_2|^2}{A_1} \, d\alpha' \leq \int |Z_{\alpha'}U_h^{-1}G_2|^2 \, d\alpha'. $$  \hfill (4.20)

So it suffices to show that

$$\int |Z_{\alpha'}U_h^{-1}G_2|^2 \, d\alpha' \leq C(\bar{\alpha})E_2.$$  

Let

$$G_{2,0} := D_\alpha \frac{\partial}{\partial h}(-i \alpha \bar{\alpha} z_t);$$  \hfill (4.21)

$$G_{2,1} := -2(D_\alpha z_t D_\alpha \frac{\partial}{\partial h} z_t + D_\alpha z_t \partial_t D_\alpha \frac{\partial}{\partial h} z_t); \quad \text{and} \quad G_{2,2} := -D_\alpha \partial_t U_h \{(h_t \circ h^{-1})_\alpha \bar{\alpha} Z_{t, \alpha'}\} - D_\alpha U_h \{(h_t \circ h^{-1})_\alpha \bar{\alpha} Z_{t, \alpha'}\} - i D_\alpha U_h \{A_{\alpha'} Z_{i, \alpha'}\},$$  \hfill (4.22)

so $G_2 = G_{2,0} + G_{2,1} + G_{2,2}$. We know by $\bar{\alpha}_t - i = -i \alpha \bar{\alpha}$ \ref{steinberg} and $U_h^{-1}D_\alpha U_h = D_\alpha' := \frac{\partial Z_{\alpha}}{\partial z_{\alpha'}}$, \ref{steinberg2}

$$Z_{\alpha'}U_h^{-1}G_{2,0} = \partial_{\alpha'} \partial_\alpha^2 \circ h^{-1}(Z_{t, t} - i);$$  \hfill (4.23)

$$Z_{\alpha'}U_h^{-1}G_{2,1} = -2(D_\alpha' Z_{t, t} \partial_\alpha^2 \bar{\alpha}' Z_t + D_\alpha' Z_{t, t} (Z_{\alpha'}U_h^{-1} \partial_\alpha U_h \frac{1}{Z_{\alpha'}^2} \partial_\alpha^2 \bar{\alpha}' Z_t));$$  \hfill (4.24)
\[ Z_{\alpha'} U_h^{-1} G_{2,2} = -\partial_{\alpha'} U_h^{-1} \partial_t U_h \{(h_t \circ h^{-1})_{\alpha'} Z_{t,\alpha'}\} - \partial_{\alpha'} \{(h_t \circ h^{-1})_{\alpha'} Z_{t,\alpha'}\} \]
\[ - i \partial_{\alpha'} \{A_{\alpha'} Z_{t,\alpha'}\} \] (4.26)

**Step 1.** Quantities controlled by \( E_2 \) and a polynomial of \( \mathcal{E} \). By the definition of \( E_2 \), and the fact that \( \|A_1\|_{L^\infty} \leq C(\mathcal{E}) \) (cf. Appendix [C]), we know

\[ \int \left| \frac{D_a \frac{\partial_{\alpha'} Z_t}{a}}{a} \right|^2 da, \quad \int \left| \frac{\partial_t D_a \frac{\partial_{\alpha'} Z_t}{a}}{a} \right|^2 da \leq E_2 \] (4.27)

\[ \|\partial_{\alpha'} Z_t\|^2_{L^2}, \quad \left\| Z_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t \right\|^2_{L^2}, \quad \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t \right\|^2_{L^{1/2}} \leq C(\mathcal{E}) E_2. \] (4.28)

We commute \( Z_{\alpha'} \) with \( U_h^{-1} \partial_t U_h \) in the second quantity of (4.28)

\[ Z_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t = U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t + [Z_{\alpha'}, U_h^{-1} \partial_t U_h] \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t \] (4.29)

By (4.26) and Appendix [C]

\[ \left\| U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 Z_t \right\|^2_{L^2} \leq (\mathcal{E}) E_2 \] (4.30)

so

\[ \|U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 Z_t\|^2_{L^2} \leq C(\mathcal{E}) E_2 \] (4.31)

**Step 2.** Controlling \( G_{2,1} \). By (4.25), Appendix [C] and (4.28),

\[ \int \left| \frac{\partial_{\alpha'} Z_{t,\alpha'}}{a} \right|^2 da \leq C(\mathcal{E}) E_2. \] (4.32)

**Step 3.** Controlling \( G_{2,2} \). We expand further the terms in \( Z_{\alpha'} U_h^{-1} G_{2,2} \) by the product rule,

\[ \partial_{\alpha'} U_h^{-1} \partial_t U_h \{(h_t \circ h^{-1})_{\alpha'} Z_{t,\alpha'}\} = (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} U_h^{-1} \partial_t U_h Z_{t,\alpha'} \]
\[ + \{U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'}\} \partial_{\alpha'} Z_{t,\alpha'} + \{\partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'}\} U_h^{-1} \partial_t U_h Z_{t,\alpha'} \]
\[ + \{\partial_{\alpha'} U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'}\} Z_{t,\alpha'}; \]
\[ \partial_{\alpha'} \{(h_t \circ h^{-1})_{\alpha'} Z_{t,\alpha'}\} = \{\partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'}\} Z_{t,\alpha'} + (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} Z_{t,\alpha'}; \]
\[ \partial_{\alpha'} \{A_{\alpha'} Z_{t,\alpha'}\} = \{\partial_{\alpha'} A_{\alpha'}\} Z_{t,\alpha'} + A_{\alpha'} \partial_{\alpha'} Z_{t,\alpha'} \] (4.33)

**Step 3.1.** The quantity \( \partial_{\alpha'}^k (h_t \circ h^{-1}) \). By equation (4.33) in Appendix [B.1]

\[ h_t \circ h^{-1}(\alpha',t) = \frac{Z_t(\alpha',t)}{Z_{\alpha'}(\alpha',t)} + \Xi(\alpha',t). \] (4.35)

where \((I - \mathcal{H})\Xi(\cdot,t) = 0\). Differentiating with respect to \( \alpha' \) yields

\[ (h_t \circ h^{-1})_{\alpha'} = \frac{Z_{t,\alpha'}}{Z_{\alpha'}} + Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} + \partial_{\alpha'} \Xi. \] (4.36)

Rewrite \( \frac{Z_{t,\alpha'}}{Z_{\alpha'}} = 2 \text{ Re} \frac{Z_{t,\alpha'}}{Z_{\alpha'}} - \frac{Z_{t,\alpha'}}{Z_{\alpha'}} \) and move \( 2 \text{ Re} \frac{Z_{t,\alpha'}}{Z_{\alpha'}} \) to the left, we obtain

\[ (h_t \circ h^{-1})_{\alpha'} - 2 \text{ Re} \frac{Z_{t,\alpha'}}{Z_{\alpha'}} = - \frac{Z_{t,\alpha'}}{Z_{\alpha'}} + Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} + \partial_{\alpha'} \Xi; \] (4.37)

differentiating (4.36) with respect to \( \alpha' \) and using the fact \( \frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}} = 2 \text{ Re} \frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}} - \frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}} \) gives

\[ \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} - 2 \text{ Re} \frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}} = 2 Z_{t,\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - \frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}} + Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} + \partial_{\alpha'}^2 \Xi. \] (4.38)
Notice that \((I - \mathbb{H})\partial_t^k \mathcal{Z} = 0, k = 1, 2\). Apply \((I - \mathbb{H})\) to both sides of (4.37) and (4.38), then take the real parts. Rewrite the last two terms on the right hand sides as commutators via the fact that \((I - \mathbb{H})\partial_t^k \mathcal{Z}_t = 0\) and \((I - \mathbb{H})\partial_t^k \mathcal{Z}_t = 0, k = 1, 2\). We get

\[
(h_t \circ h^{-1})_{t'} - 2 \text{Re} \frac{\partial^2_t \mathcal{Z}}{\partial \alpha'_{t'}} = \text{Re}\{-\frac{1}{\mathcal{Z}_{t',\alpha'}} \mathbb{H}[\mathcal{Z}_{t',\alpha'} + [\mathcal{Z}_t, \mathbb{H}]\partial_{t'} 1_{\mathcal{Z}_{t',\alpha'}}\} \quad (4.39)
\]

and

\[
\partial_{t'}(h_t \circ h^{-1})_{t'} - 2 \text{Re} \frac{\partial^2_t \mathcal{Z}_t}{\partial \alpha'_{t'}} = \text{Re}\{(I - \mathbb{H})(\mathcal{Z}_{t,\alpha'} - [\mathcal{Z}_t, \mathbb{H}]\partial_{t'} 1_{\mathcal{Z}_{t',\alpha'}} - \frac{1}{\mathcal{Z}_{t,\alpha'}} \mathbb{H}[\partial^2_t \mathcal{Z}_t + [\mathcal{Z}_t, \mathbb{H}]\partial_{t'} 1_{\mathcal{Z}_{t',\alpha'}}\} \quad (4.40)
\]

From (4.30), by Hölder’s inequality, (A.8) and (A.12),

\[
\|\partial_{t'}(h_t \circ h^{-1})_{t'} - 2 \text{Re} \frac{\partial^2_t \mathcal{Z}_t}{\partial \alpha'_{t'}}\|_{L^2} \lesssim \|\mathcal{Z}_{t,\alpha'}\|_{L^\infty} \|\partial_{t'} 1_{\mathcal{Z}_{t,\alpha'}}\|_{L^2} \quad (4.41)
\]

**Step 3.2. The estimates for the quantities involving \(\mathcal{Z}_t\).** Commuting \(\partial_{t'}\) with \(U_h^{-1} \partial_t U_h\) and using (B.19) gives

\[
\partial_{t'} U_h^{-1} \partial_t U_h \mathcal{Z}_{t,\alpha'} = U_h^{-1} \partial_t U_h \partial_{t'} \mathcal{Z}_t + [\partial_{t'}, U_h^{-1} \partial_t U_h] \mathcal{Z}_{t,\alpha'} = U_h^{-1} \partial_t U_h \partial_{t'} \mathcal{Z}_t + (h_t \circ h^{-1})_{t'} \partial_{t'} \mathcal{Z}_t, \quad (4.42)
\]

so by (4.28), (4.31) and Appendix C

\[
\|\partial_{t'} U_h^{-1} \partial_t U_h \mathcal{Z}_{t,\alpha'}\|_{L^2} \leq C(\mathcal{E})E_2. \quad (4.43)
\]

We estimate \(\|\mathcal{Z}_{t,\alpha'}\|_{L^\infty}\) by (A.3), Appendix C and (4.28),

\[
\|\mathcal{Z}_{t,\alpha'}\|_{L^\infty} \lesssim 2\|\mathcal{Z}_{t,\alpha'}\|_{L^2} \leq C(\mathcal{E})E_2^{1/2}. \quad (4.44)
\]

We compute \(\partial_{t'}^2 \mathcal{Z}_t\) by (B.19),

\[
\partial_{t'}^2 \mathcal{Z}_t = U_h^{-1} \partial_t U_h \partial_{t'}^2 \mathcal{Z}_t + [\partial_{t'}, U_h^{-1} \partial_t U_h] \mathcal{Z}_t = 2(h_t \circ h^{-1})_{t'} \partial_{t'}^2 \mathcal{Z}_t + \partial_{t'}(h_t \circ h^{-1})_{t'} \mathcal{Z}_{t,\alpha'}, \quad (4.45)
\]

where by (4.41), (4.28), (4.44) and Appendix C

\[
\|\partial_{t'}(h_t \circ h^{-1})_{t'} \mathcal{Z}_{t,\alpha'}\|_{L^2} \lesssim \|D_t \mathcal{Z}_t\|_{L^\infty} \|\partial_{t'}^2 \mathcal{Z}_t\|_{L^2} + \|\mathcal{Z}_{t,\alpha'}\|_{L^\infty} \|\partial_{t'} 1_{\mathcal{Z}_{t,\alpha'}}\|_{L^2} \lesssim C(\mathcal{E})E_2^{1/2}. \quad (4.46)
\]

Therefore (4.45), (4.46), (4.31), (4.28) and Appendix C gives that

\[
\|\partial_{t'}^2 \mathcal{Z}_t\|_{L^2} \lesssim C(\mathcal{E})E_2. \quad (4.47)
\]

As a consequence of (A.3), (4.47) and Appendix C

\[
\|\partial_{t'} \mathcal{Z}_t\|_{L^\infty} \lesssim 2\|\partial_{t'} \mathcal{Z}_t\|_{L^2} \|\partial_{t'}^2 \mathcal{Z}_t\|_{L^2} \lesssim C(\mathcal{E})E_2^{1/2}. \quad (4.48)
\]

We compute \(U_h^{-1} \partial_t U_h \mathcal{Z}_{t,\alpha'}\) by commuting \(U_h^{-1} \partial_t U_h\) with \(\partial_{t'}\) and using (B.18),

\[
U_h^{-1} \partial_t U_h \mathcal{Z}_{t,\alpha'} = \partial_{t'} \mathcal{Z}_t + [U_h^{-1} \partial_t U_h, \partial_{t'}] \mathcal{Z}_t = \mathcal{Z}_{tt,\alpha'} - (h_t \circ h^{-1})_{t'} \mathcal{Z}_{t,\alpha'}; \quad (4.49)
\]

(4.48), (4.44) and Appendix C imply that

\[
\|U_h^{-1} \partial_t U_h \mathcal{Z}_{t,\alpha'}\|_{L^\infty} \lesssim C(\mathcal{E})E_2^{1/2}. \quad (4.50)
\]

\footnote{If \((I - \mathbb{H})g = 0\), then \((I - \mathbb{H})(fg) = [f, \mathbb{H}]g\).}
Step 3.3. The estimate for the terms involving \( \partial_{\alpha'} (h_t \circ h^{-1}) \) in (4.33) and (4.34). By Steps 3.1 and 3.2, we can give the estimates for some of the terms in (4.33) and (4.34). First, because \( \|(h_t \circ h^{-1})_{\alpha'}\|_{L^\infty} \leq C(\mathcal{E}) \) (cf. Appendix C) and (4.43),
\[
\|(h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} U_h^{-1} \partial_t U_h Z_t,\alpha'\|_{L^2}^2 \leq C(\mathcal{E}) E_2; \tag{4.51}
\]
and from (4.47),
\[
\|(h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} Z_{t,\alpha'}\|_{L^2}^2 \leq C(\mathcal{E}) E_2. \tag{4.52}
\]
From (4.31), (4.28), (4.44), (4.48), and Appendix C,
\[
\|\partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} Z_{t,\alpha'}\|_{L^2} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} \|\partial_{\alpha'} Z_t\|_{L^2} + \|Z_{t,\alpha'}\|_{L^\infty} \|Z_{t,\alpha'}\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} \lesssim C(\mathcal{E}) E_2^{1/2}, \tag{4.53}
\]
additionally from (4.49),
\[
\|\partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} U_h^{-1} \partial_t U_h Z_{t,\alpha'}\|_{L^2} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} \|\partial_{\alpha'} Z_t\|_{L^2} + \|Z_{t,\alpha'}\|_{L^\infty} \|Z_{t,\alpha'}\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} \lesssim C(\mathcal{E}) E_2^{1/2}. \tag{4.54}
\]
Step 3.4. The terms involving \( \partial_{\alpha'} U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} \), \( k = 0, 1 \). We first consider \( U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} \). Applying \( U_h^{-1} \partial_t U_h \) to (4.39) gives
\[
U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} = 2 \text{Re} U_h^{-1} \partial_t U_h \frac{Z_{t,\alpha'}}{Z_{\alpha'}}
\]
\[
+ \text{Re} \{ -U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \mathbb{H} Z_{t,\alpha'} + U_h^{-1} \partial_t U_h [Z_t, \mathbb{H}] \frac{1}{Z_{\alpha'}} \}; \tag{4.55}
\]
we know
\[
U_h^{-1} \partial_t U_h \frac{Z_{t,\alpha'}}{Z_{\alpha'}} = U_h^{-1} \partial_t \frac{Z_t}{Z_{\alpha'}} = \frac{Z_{t,\alpha'}}{Z_{\alpha'}} - \left( \frac{Z_{t,\alpha'}}{Z_{\alpha'}} \right)^2 = D_{\alpha'} Z_{tt} - (D_{\alpha'} Z_t)^2. \tag{4.56}
\]

We compute the last two terms on the RHS of (4.55) by (1.28),
\[
U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \mathbb{H} Z_{t,\alpha'} = [U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}}, \mathbb{H}] Z_{t,\alpha'}
\]
\[
+ \frac{1}{Z_{\alpha'}} \mathbb{H} (\partial_{\alpha'} Z_{tt}) - \frac{1}{Z_{\alpha'}} [h_t \circ h^{-1}, Z_{t,\alpha'}]; \tag{4.57}
\]
and
\[
U_h^{-1} \partial_t U_h [Z_t, \mathbb{H}] \frac{1}{Z_{\alpha'}} = [Z_{tt}, \mathbb{H}] \frac{1}{Z_{\alpha'}}
\]
\[
+ [Z_t, \mathbb{H}] (\partial_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}}) - [Z_t, h_t \circ h^{-1}, \partial_{\alpha'} \frac{1}{Z_{\alpha'}}]. \tag{4.58}
\]
Now by the product rule,
\[
\partial_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} = \partial_{\alpha'} \left\{ \frac{1}{Z_{\alpha'}}((h_t \circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t) \right\}
\]
\[
= (\partial_{\alpha'} \frac{1}{Z_{\alpha'}})((h_t \circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t) + (D_{\alpha'} (h_t \circ h^{-1})_{\alpha'} - D_{\alpha'}^2 Z_t); \tag{4.59}
\]
commuting \( U_h^{-1} \partial_t U_h \) with \( \partial_{\alpha'} \) and using (B.18) gives
\[
U_h^{-1} \partial_t U_h \partial_{\alpha'} \frac{1}{Z_{\alpha'}} = \partial_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} + [U_h^{-1} \partial_t U_h, \partial_{\alpha'}] \frac{1}{Z_{\alpha'}}
\]
\[
= \partial_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} - (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}. \tag{4.60}
\]
Applying Appendix C yields
\[ \| \partial_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \|_{L^2} + \| U_h^{-1} \partial_t U_h \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{L^2} \leq C(\mathcal{E}); \tag{4.61} \]
and from (4.58), (4.59), (4.57), (4.61) and (A.18), (A.17) and Appendix C
\[ \| U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} \|_{L^\infty} \lesssim C(\mathcal{E}). \tag{4.62} \]
We analyze \( \partial_{\alpha'} U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} \) similarly. Commuting \( \partial_{\alpha'} \) with \( U_h^{-1} \partial_t U_h \) and using (B.18) gives
\[ \partial_{\alpha'} U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} = [\partial_{\alpha'}, U_h^{-1} \partial_t U_h] (h_t \circ h^{-1})_{\alpha'} + U_h^{-1} \partial_t U_h \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} \]
\[ = (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} + U_h^{-1} \partial_t U_h \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'}. \tag{4.63} \]
We compute the second term on the RHS of (4.63) via (4.40):
\[ U_h^{-1} \partial_t U_h \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} - 2 \text{Re} U_h^{-1} \partial_t U_h \frac{\partial^2 Z_t}{Z_{\alpha'}} = \text{Re}(2U_h^{-1} \partial_t U_h (I - \mathbb{H})(Z_{t,\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \]
\[ - U_h^{-1} \partial_t U_h \frac{1}{Z_{t,\alpha'}} \mathbb{H} \partial^2_{\alpha'} Z_t + U_h^{-1} \partial_t U_h |Z_t,\mathbb{H}| \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}}); \tag{4.64} \]
commuting \( U_h^{-1} \partial_t U_h \) with \( \frac{\partial_{\alpha'}}{Z_{\alpha'}} := D_{\alpha'} \) and using (B.12) gives
\[ U_h^{-1} \partial_t U_h \frac{\partial^2 Z_t}{Z_{\alpha'}} = D_{\alpha'} U_h^{-1} \partial_t U_h Z_{t,\alpha'} + [U_h^{-1} \partial_t U_h, D_{\alpha'}] Z_{t,\alpha'} \]
\[ = \frac{1}{Z_{\alpha'}} \partial_{\alpha'} U_h^{-1} \partial_t U_h Z_{t,\alpha'} - \frac{1}{Z_{\alpha'}} (D_{\alpha'} Z_{t,\alpha'}) (\partial^2_{\alpha'} Z_t); \tag{4.65} \]
for the first term on the RHS of (4.64), we commute \( U_h^{-1} \partial_t U_h \) with \( (I - \mathbb{H}) \) and use (B.23),
\[ U_h^{-1} \partial_t U_h (I - \mathbb{H})(Z_{t,\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) = (I - \mathbb{H}) U_h^{-1} \partial_t U_h (Z_{t,\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}), \tag{4.66} \]
we use product rule to expand further the terms on the RHS of (4.66), By (4.50), (4.61), (4.44), Appendix C and (A.11),
\[ \left\| U_h^{-1} \partial_t U_h (I - \mathbb{H})(Z_{t,\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \right\|_{L^2}^2 \leq C(\mathcal{E}) E^2_2/2. \tag{4.67} \]
We use (B.25) to compute the last two terms on the RHS of (4.64), then use (A.11), (A.12) and (4.61), (4.50), (4.44), (4.48) and Appendix C to do the estimates, we get
\[ \left\| U_h^{-1} \partial_t U_h \left( \frac{1}{Z_{\alpha'}} \mathbb{H} \partial^2_{\alpha'} Z_t \right) \right\|_{L^2}^2 + \left\| U_h^{-1} \partial_t U_h |Z_t,\mathbb{H}| \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2}^2 \leq C(\mathcal{E}) E^2_2/2, \tag{4.68} \]
therefore
\[ \left\| U_h^{-1} \partial_t U_h \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} - 2 \text{Re} U_h^{-1} \partial_t U_h \frac{\partial^2 Z_t}{Z_{\alpha'}} \right\|_{L^2}^2 \leq C(\mathcal{E}) E^2_2/2. \tag{4.69} \]
We now conclude the estimates for the two terms involving \( U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} \) in (4.33).
By (4.63), (4.41), (4.44), (4.69) and (4.28) and Appendix C
\[ \left\| \{ \partial_{\alpha'} U_h^{-1} \partial_t U_h (h_t \circ h^{-1})_{\alpha'} \} Z_{t,\alpha'} \right\|_{L^2}^2 \leq C(\mathcal{E}) E^2_2; \tag{4.70} \]
by (4.62) and (4.28),
\[
\|\{U_h^{-1}\partial_t U_h(h_t \circ h^{-1})\}_t\partial_{\alpha'} Z_{t,\alpha'}\|^2_{L^2} \leq C(\mathfrak{C}) E_2. \quad (4.71)
\]

Finally we estimate the \(L^2\) norms of the two terms on the RHS of the second equation in (4.34).

**Step 3.5. The \(L^2\) norm of \(\partial_{\alpha'}(A_{\alpha'} Z_{t,\alpha'})\).** We begin with the first equation of (2.9) \(A := \frac{Z_{tt} + i}{Z_{t,\alpha'}}\). Differentiating with respect to \(\alpha'\) gives
\[
\partial_{\alpha'} A = -iD_{\alpha'} Z_{tt} - i(Z_{tt} + i)\partial_{\alpha'} \frac{1}{Z_{\alpha'}}. \quad (4.72)
\]

By Appendix C
\[
\|\partial_{\alpha'} A\|_{L^\infty} \leq C(\mathfrak{C}), \quad (4.73)
\]
therefore by (4.28),
\[
\|A_{\alpha'} \partial_{\alpha'}^2 Z_t\|^2_{L^2} \leq C(\mathfrak{C}) E_2. \quad (4.74)
\]

We now consider the term \((\partial_{\alpha'} A_{\alpha'}) Z_{t,\alpha'}\) in (4.34). We calculate \(\partial_{\alpha'}^2 A\) by differentiating the equation \(iA = \frac{Z_{tt} + i}{Z_{t,\alpha'}}\) twice:
\[
i\partial_{\alpha'}^2 A = \frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}} + 2\partial_{\alpha'} Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} + (Z_{tt} + i)\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}}. \quad (4.75)
\]

Applying \((I - \mathbb{H})\) then taking the imaginary parts gives
\[
\partial_{\alpha'}^2 A = \text{Im}(I - \mathbb{H})((\partial_{\alpha'}^2 Z_t) + 2 \text{Im}(I - \mathbb{H})((\partial_{\alpha'} Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) + \text{Im}(I - \mathbb{H})((Z_{tt} + i)\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}}); \quad (4.76)
\]
we rewrite the first term on the right by commuting out \(\frac{1}{Z_{\alpha'}}\)
\[
(I - \mathbb{H})(\frac{\partial_{\alpha'}^2 Z_t}{Z_{\alpha'}}) = \frac{1}{Z_{\alpha'}}(I - \mathbb{H})(\partial_{\alpha'}^2 Z_t) + [\frac{1}{Z_{\alpha'}}, \mathbb{H}](\partial_{\alpha'}^2 Z_t); \quad (4.77)
\]
using \((I - \mathbb{H})\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} = 0\) we rewrite the third term on the right of (4.70) as a commutator
\[
(I - \mathbb{H})(Z_{tt} + i)\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} = [Z_{tt}, \mathbb{H}]\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \quad (4.78)
\]
so
\[
\partial_{\alpha'}^2 A = \text{Im}\{\frac{1}{Z_{\alpha'}}(I - \mathbb{H})(\partial_{\alpha'}^2 Z_t) + [\frac{1}{Z_{\alpha'}}, \mathbb{H}](\partial_{\alpha'}^2 Z_t) + [Z_{tt}, \mathbb{H}]\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}}\} \quad (4.79)
\]

We apply (A.11), (A.12) and Hölder. This gives
\[
\left\|\partial_{\alpha'}^2 A - \text{Im}\left\{\frac{1}{Z_{\alpha'}}(I - \mathbb{H})(\partial_{\alpha'}^2 Z_t)\right\}\right\|_{L^2} \leq \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_{L^2} \|Z_{tt,\alpha'}\|_{L^\infty}, \quad (4.80)
\]
so
\[
\left\|\partial_{\alpha'}^2 A Z_{t,\alpha'}\right\|_{L^2} \leq \|D_{\alpha'} Z_{t}\|_{L^\infty} \|\partial_{\alpha'}^2 Z_t\|_{L^2} + \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^\infty} \|Z_{tt,\alpha'}\|_{L^\infty}. \quad (4.81)
\]
By (4.28), (4.44), (4.48) and Appendix C
\[
\left\|\partial_{\alpha'}^2 A Z_{t,\alpha'}\right\|^2_{L^2} \leq C(\mathfrak{C}) E_2.
\]
This completes the proof for
\[
\int |Z_{\alpha'} U_h^{-1} G_{2,2}|^2 d\alpha' \leq C(\mathfrak{C}) E_2. \quad (4.82)
\]
**Step 4. Controlling** $\|Z_{,a'}U_{h}^{-1}G_{2,0}\|_{L^{2}}$. We are left with controlling $\|Z_{,a'}U_{h}^{-1}G_{2,0}\|_{L^{2}}$. By (4.24), we must show

$$\int |\partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1}(Z_{,tt} - i))|^{2} \, da' \leq C(\mathcal{E})E_{2}. \quad (4.83)$$

We expand $\partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1}(Z_{,tt} - i))$ by the product rule

$$\partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1}(Z_{,tt} - i)) = \frac{a_{t}}{a} h^{-1} \partial_{t}^{2}_{\alpha'}Z_{,tt} + 2 \partial_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})Z_{,tt,\alpha'} + \partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})(Z_{,tt} - i) \quad (4.84)$$

where we estimate the $L^{2}$ norm of $\partial_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})$ by (4.70), (A.8), (A.11), (A.12) and (A.9)

$$\left\|\partial_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})\right\|_{L^{2}} \lesssim \|Z_{,t,\alpha'}\|_{L^{\infty}} \|Z_{,tt,\alpha'}\|_{L^{2}} + \|Z_{,t,\alpha'}\|_{L^{\infty}} \|Z_{,t,\alpha'}\|_{L^{2}} \|D_{\alpha'}Z_{t}\|_{L^{\infty}} \quad (4.85)$$

so by Appendix C (4.28) and (4.43), (4.48),

$$\int \left|\frac{a_{t}}{a} \circ h^{-1} \partial_{t}^{2}_{\alpha'}Z_{,tt} + 2 \partial_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})(Z_{,tt} - i)\right|^{2} \, da' \leq C(\mathcal{E})E_{2}. \quad (4.86)$$

What remains is the term $\int |\partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})(Z_{,tt} - i)|^{2} \, da'$. We begin with (4.18), together with (4.21), (4.22), (4.23):

$$\left(\partial_{t}^{2} + ia\partial_{\alpha}\right)D_{\alpha}(\frac{a_{t}}{a_{\alpha}})z_{t} = D_{\alpha}(\frac{a_{t}}{a_{\alpha}})(-i\alpha_{t}z_{\alpha}) + G_{2,1} + G_{2,2}. \quad (4.87)$$

Precomposing with $h^{-1}$ then multiply $Z_{,\alpha'}$ gives, using $z_{t} - i = -i\alpha_{t}z_{\alpha}$ (4.3),

$$Z_{,\alpha'}U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha})D_{\alpha}(\frac{a_{t}}{a_{\alpha}})z_{t} = \partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1}(Z_{,tt} - i)) + Z_{,\alpha'}U_{h}^{-1}(G_{2,1} + G_{2,2}). \quad (4.88)$$

By commuting $(\partial_{t}^{2} + ia\partial_{\alpha})$ with $\frac{h_{\alpha}}{z_{\alpha}}$ we rewrite the left hand side as

$$U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha})(\frac{a_{t}}{a_{\alpha}})z_{t} + Z_{,\alpha'}U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha}, \frac{h_{\alpha}}{z_{\alpha}})(\frac{a_{t}}{a_{\alpha}})z_{t}; \quad (4.89)$$

(4.88) now yields

$$U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha})(\frac{a_{t}}{a_{\alpha}})z_{t} = \partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})(Z_{,tt} - i) + \epsilon \quad (4.90)$$

where

$$\epsilon : = -Z_{,\alpha'}U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha}, \frac{h_{\alpha}}{z_{\alpha}})(\frac{a_{t}}{a_{\alpha}})z_{t} + Z_{,\alpha'}U_{h}^{-1}(G_{2,1} + G_{2,2}) \quad (4.91)$$

Observe $(I - \mathcal{F})U_{h}^{-1}(\frac{h_{\alpha}}{z_{\alpha}})^{2}z_{t} = (I - \mathcal{F})\partial_{t}^{2}_{\alpha'}Z_{t} = 0$. We want to use the "almost holomorphicity" of the LHS of (4.90) and the fact that $\partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})$ is real valued to estimate $\int |\partial_{t}^{2}_{\alpha'}(\frac{a_{t}}{a} \circ h^{-1})(Z_{,tt} - i)|^{2} \, da'$. We first show that the error term $\epsilon$ is well behaved. By (B.29),

$$Z_{,\alpha'}U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha}, \frac{h_{\alpha}}{z_{\alpha}})(\frac{a_{t}}{a_{\alpha}})z_{t} = 2(h_{t} \circ h^{-1})_{\alpha'} - D_{\alpha'}Z_{t}U_{h}^{-1}\partial_{t}U_{h}\partial_{t}^{2}_{\alpha'}Z_{t}$$

$$+ ((h_{t} \circ h^{-1})_{\alpha'} - D_{\alpha'}Z_{t})^{2}\partial_{t}^{2}_{\alpha'}Z_{t} + (U_{h}^{-1}\partial_{t}U_{h}(h_{t} \circ h^{-1})_{\alpha'} - U_{h}^{-1}\partial_{t}U_{h}D_{\alpha'}Z_{t})\partial_{t}^{2}_{\alpha'}Z_{t} \quad (4.92)$$

$$+ (Z_{tt} + i)\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\partial_{t}^{2}_{\alpha'}Z_{t};$$

from (B.12), $U_{h}^{-1}\partial_{t}U_{h}D_{\alpha'}Z_{t} = D_{\alpha'}Z_{tt} - (D_{\alpha'}Z_{t})^{2}$, therefore by Appendix C (4.23), (4.33) and (4.62),

$$\int \left|Z_{,\alpha'}U_{h}^{-1}(\partial_{t}^{2} + ia\partial_{\alpha}, \frac{h_{\alpha}}{z_{\alpha}})(\frac{a_{t}}{a_{\alpha}})z_{t}\right|^{2} \, da' \leq C(\mathcal{E})E_{2}. \quad (4.93)$$
The estimates (4.93), (4.86), (4.82) and (4.32) give that

$$\int |e|^2 \, d\omega \leq C(\mathcal{E}) E_2. \quad (4.94)$$

Now we apply \((I - \mathbb{H})\) to both sides of (4.90), then rewrite \((I - \mathbb{H})(\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})) (Z_{tt} - i)\) by commuting out \((Z_{tt} - i)\):

$$(I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a) (\frac{a}{a'} h^{-1})) = (Z_{tt} - i)(I - \mathbb{H})(\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})) + [Z_{tt}, \mathbb{H}](\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})). \quad (4.95)$$

Since \(\mathbb{H}\) is purely imaginary, \(|\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})| \leq |(I - \mathbb{H})(\partial_{\alpha}^2 (\frac{a}{a'} h^{-1}))|\) hence

$$\|(Z_{tt} - i)\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})\|_{L^2} \leq \|(Z_{tt} - i)(I - \mathbb{H})(\partial_{\alpha}^2 (\frac{a}{a'} h^{-1}))\|_{L^2} \leq \|(I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a) (\frac{a}{a'} h^{-1})))\|_{L^2} + \|Z_{tt}, \mathbb{H}[(\partial_{\alpha}^2 (\frac{a}{a'} h^{-1}))]\|_{L^2} + |e|_{L^2}. \quad (4.96)$$

By (A.12) and (4.85), (4.48), (4.44) and Appendix C

$$\|Z_{tt}, \mathbb{H}[\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})]\|_{L^2} \leq \|Z_{tt,\alpha'}\|_{L^\infty} \|\partial_{\alpha'} (\frac{a}{a'} h^{-1})\|_{L^2} \leq C(\mathcal{E}) E_2^{1/2} \quad (4.97)$$

therefore

$$\|(Z_{tt} - i)\partial_{\alpha}^2 (\frac{a}{a'} h^{-1})\|_{L^2} \lesssim \|(I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a) (\frac{a}{a'} h^{-1})))\|_{L^2} + C(\mathcal{E}) E_2^{1/2}. \quad (4.98)$$

In what follows we will show that

$$\|(I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a) (\frac{a}{a'} h^{-1})))\|_{L^2} \leq C(\mathcal{E}) E_2^{1/2}$$

and complete the proof for Proposition 4.3.

**Step 4.1. Controlling \((I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a) (\frac{a}{a'} h^{-1})))\|_{L^2}.** We introduce the following notations. We write \(f_1 \equiv f_2\), if \((I - \mathbb{H})(f_1 - f_2) = 0\). We define \(P_H := \frac{(I + \mathbb{H})}{2}\) and \(P_A := \frac{(I - \mathbb{H})}{2}\), so \(P_H + P_A = I\), and \(P_H - P_A = \mathbb{H}\). By Proposition A.1 \(P_H\) is the projection onto the space of holomorphic functions in the lower half plane \(P^-\), and \(P_A\) is the projection onto the space of anti-holomorphic functions in \(P^-\).

We want to derive an estimate of \(\|(I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a)U o h\|_{L^2}\) for a generic \(U\) satisfying \(U = \mathbb{H} U\), i.e. \(U \equiv 0\). Observe \(D_{\alpha'} U \equiv 0\). By (B.8) of Proposition B.1 \(U\) satisfies

$$U_h^{-1}(\partial_t^2 + i a \partial_a)U o h \equiv \frac{Z_t}{Z_{\alpha'}} \partial_{\alpha'} (U_h^{-1} \partial_t U_h - \frac{Z_t}{Z_{\alpha'}} \partial_{\alpha'}) U + Z_t^2 D_{\alpha'}^2 U + 2(Z_{tt} + i) D_{\alpha'} U. \quad (4.99)$$

What we will do first is to use (4.99) to rewrite \((I - \mathbb{H})(U_h^{-1}(\partial_t^2 + i a \partial_a) U o h)\) into a favorable form so that desired estimate will follow.

We expand on the RHS of (4.99) the term

$$Z_t^2 D_{\alpha'}^2 U = \left(\frac{Z_t}{Z_{\alpha'}}\right)^2 \partial_{\alpha'}^2 U + \frac{Z_t^2}{Z_{\alpha'}} \partial_{\alpha'} (\frac{1}{Z_{\alpha'}}) \partial_{\alpha'} U$$
by the product rule, and decompose $\frac{\partial}{\partial x} = P_A(\frac{\partial}{\partial x}) + P_H(\frac{\partial}{\partial x})$. We have, because $\partial_{\alpha'} (U_h^{-1} \partial_t U_h - \frac{\partial}{\partial \alpha'} \partial_{\alpha'}) U \equiv 0$ by (136),

$$U_h^{-1} (\partial_t^2 + i a \partial_{\alpha'}) U \circ h \equiv 2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'} (U_h^{-1} \partial_t U_h - (P_A + P_H) \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'}) U$$

$$+ \left( (P_A + P_H) \left( \frac{Z_t}{Z_{\alpha'}} \right) \right)^2 \partial_{\alpha'}^2 U$$

$$+ \frac{Z_t^2}{Z_{\alpha'}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \partial_{\alpha'} U + 2 (Z_t + i) D_{\alpha'} U.$$  (4.100)

We expand further the factor $\partial_{\alpha'} (P_H(\frac{Z_t}{Z_{\alpha'}}) \partial_{\alpha'} U)$ on the RHS by the product rule. After cancelation we obtain

$$U_h^{-1} (\partial_t^2 + i a \partial_{\alpha'}) U \circ h \equiv 2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'} (U_h^{-1} \partial_t U_h - P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'}) U$$

$$- 2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \{ P_H \left( \frac{Z_t}{Z_{\alpha'}} \right) + P_H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \} \partial_{\alpha'} U +$$

$$\left( P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \right)^2 \partial_{\alpha'}^2 U + \frac{Z_t^2}{Z_{\alpha'}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \partial_{\alpha'} U + 2 (Z_t + i) D_{\alpha'} U.$$  (4.101)

Now because $P_H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}})$ and $\partial_{\alpha'} U$ are holomorphic,

$$2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) P_H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \partial_{\alpha'} U \equiv 2 \frac{Z_t}{Z_{\alpha'}} P_H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \partial_{\alpha'} U,$$

moreover

$$- 2 \frac{Z_t}{Z_{\alpha'}} P_H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) + \frac{Z_t^2}{Z_{\alpha'}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) = - \frac{Z_t}{Z_{\alpha'}} H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}});$$

and

$$- \frac{Z_t}{Z_{\alpha'}} H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \partial_{\alpha'} U = - Z_t \overline{H} (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \left( \frac{1}{Z_{\alpha'}} \right) U$$

$$+ Z_t \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \overline{H} (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) U;$$

and by straightforward expansion,

$$\left[ Z_t, [Z_t, \overline{H}] \right] \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \equiv - 2 Z_t \overline{H} (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}});$$

and

$$Z_t \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \overline{H} (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) = \left( \left( P_H (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \right)^2 - \left( P_A (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \right)^2 \right).$$

Therefore

$$U_h^{-1} (\partial_t^2 + i a \partial_{\alpha'}) U \circ h \equiv 2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'} (U_h^{-1} \partial_t U_h - P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'}) U$$

$$- 2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) P_H \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'} U + (P_A \left( \frac{Z_t}{Z_{\alpha'}} \right))^2 \partial_{\alpha'}^2 U$$

$$+ \frac{1}{2} \left[ Z_t, [Z_t, \overline{H}] \right] \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) U - \left( P_A (Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}) \right)^2 U$$

$$+ 2 (Z_t + i) D_{\alpha'} U.$$  (4.102)

We further rewrite

$$2 P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) P_H \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'} U \equiv P_H \left( \frac{Z_t}{Z_{\alpha'}} \right) (I - \overline{H})(P_A \left( \frac{Z_t}{Z_{\alpha'}} \right) \partial_{\alpha'} U).$$
Apply \((I - H)\) to both sides of (4.102), and rewrite terms of the form \((I - H)(g_1g_2)\) with 
\[g_2 = Hg_2\] as \([g_1, H]g_2\). We obtain
\[
(I - H)U_{1}^{-1}(\partial_t^2 + i\alpha \partial_a)U \circ h = 2[p_A(Z_t^{\alpha})]H\partial_{a'}(U_{1}^{-1}\partial_t U_h - p_A(Z_t^{\alpha}))\partial_a)U
\]
\[
- (I - H)[\{p_H(Z_t^{\alpha'})[p_A(Z_t^{\alpha})]H\partial_{a'}U\} + ([p_A(Z_t^{\alpha})]H\partial_{a'}^2U
\]
\[
\frac{1}{2}[1; Z_t, [Z_t, H]]]H\partial_{a'}(\frac{1}{Z_t^{\alpha'}})U
\]
\[
- (I - H)[(p_A(Z_t^{\alpha})\frac{1}{Z_t^{\alpha'}})^2U\} + 2\frac{Z_{tt} + i}{Z_t^{\alpha'}}H\partial_{a'}U.\]

We further use the identity\[20\]
\[-2[g_1, H]H\partial_{a'}(g_1g_2) + [g_1^2, H]H\partial_{a'}g_2 = -\frac{1}{\pi i} \int \left(\frac{g_1(\alpha') - g_1(\beta')}{\alpha' - \beta'}\right)g_2(\beta')d\beta' := -[g_1, g_1; g_2]
\]
to rewrite the sum of second part of the first and the third terms on the right:
\[
-2[p_A(Z_t^{\alpha})], H\partial_{a'}\left[p_A(Z_t^{\alpha})\partial_a)U\right] + ([p_A(Z_t^{\alpha})]H\partial_{a'}^2U
\]
\[
- [p_A(Z_t^{\alpha}), p_A(Z_t^{\alpha})]H\partial_{a'}U\].

We are now ready to give the estimate for \((I - H)U_{1}^{-1}(\partial_t^2 + i\alpha \partial_a)U \circ h\). We have, by (A.11), (A.19), and Hölder’s inequality,
\[
\|[(I - H)U_{1}^{-1}(\partial_t^2 + i\alpha \partial_a)U \circ h]\|_{L^2} \lesssim \|\partial_{a'}p_A(Z_t^{\alpha})\|_{L^\infty} \|U_{1}^{-1}\partial_t U_h\|_{L^2} + 
\]
\[
\|[p_H(Z_t^{\alpha'})[p_A(Z_t^{\alpha})]H\partial_{a'}U\|_{L^\infty} \|\partial_{a'}p_A(Z_t^{\alpha})\|_{L^\infty} \|U\|_{L^2} + \|\partial_{a'}p_A(Z_t^{\alpha})\|_{L^\infty} \|U\|_{L^2}
\]
\[
+ \|[Z_t, Z_t, H]]H\partial_{a'}(\frac{1}{Z_t^{\alpha'}})\|_{L^2} \|\partial_{a'}(\frac{Z_{tt} + i}{Z_t^{\alpha'}})\|_{L^\infty} \|U\|_{L^2}.
\]

Now by (A.19),
\[
\|\partial_{a'}(\{[Z_t, [Z_t, H]]H\partial_{a'}(\frac{1}{Z_t^{\alpha'}})\|_{L^2} \lesssim \|Z_t, \partial_{a'}\|_{L^2}^2 \|\partial_{a'}(\frac{1}{Z_t^{\alpha'}})\|_{L^2}
\]
\[
\|\partial_{a'}(\frac{Z_{tt} + i}{Z_t^{\alpha'}})\|_{L^\infty} \lesssim \|D_{\alpha'} Z_{tt}\|_{L^\infty} + \|(Z_{tt} + i)\partial_{a'}(\frac{1}{Z_t^{\alpha'}})\|_{L^\infty}.
\]

We can conclude now by Appendix [C] that for any \(U\) satisfying \(U = HU\),
\[
\|(I - H)U_{1}^{-1}(\partial_t^2 + i\alpha \partial_a)U \circ h\|_{L^2} \leq C(E)(\|U\|_{L^2} + \|U_{1}^{-1}\partial_t U_h\|_{L^2} + \|\frac{1}{Z_t^{\alpha'}}U\|_{H^{1/2}}). \quad (4.106)
\]

As a consequence of (4.106) and (4.28), (4.31),
\[
\|(I - H)U_{1}^{-1}(\partial_t^2 + i\alpha \partial_a)(\frac{\partial_{a}}{\partial_{a'}})^2\|_{L^2} \leq C(E)E_2^{1/2}. \quad (4.107)
\]

\[20\]It is an easy consequence of integration by parts.
We expand
\[ \| (Z_{tt} - i) \partial^2_{\alpha'} \left( \frac{\alpha}{a} \circ h^{-1} \right) \|_{L^2} \leq C(\xi) E_2. \]  
(4.108)

Sum up (4.82), (4.32), (4.89) and (4.108),
\[ \int |Z_{\alpha'} U_h^{-1} G_2|^2 d\alpha' \leq C(\xi) E_2. \]
This finishes the proof of Proposition 4.3.

$\square$

4.2. The proof of Proposition 4.4

Proof. We prove Proposition 4.4 by applying Lemma 4.1 to (4.12) for $k = 3$, notice that $(I - H) U_h^{-1} D_{\alpha}(\frac{\alpha}{a})^2 \xi_t = (I - H) D_{\alpha'} \partial^2_{\alpha'} Z_t = 0.$ For $k = 3$, the right hand side of (4.12) is
\[ G_3 := D_{\alpha}(\frac{\alpha}{h})^2 (-i a \xi_{\alpha} + [\partial^2_t + i a \partial_{\alpha'}, D_{\alpha}(\frac{\alpha}{h})^2] \xi_t) \]  
(4.109)

Similar to the proof for Proposition 4.3 we only need to show that
\[ \int |Z_{\alpha'} U_h^{-1} G_3|^2 d\alpha' \leq C(\xi, E_2) E_3. \]  
(4.110)

We expand $Z_{\alpha'} U_h^{-1} G_3$ by (B.16), (B.15), (B.22). We have
\[ Z_{\alpha'} U_h^{-1} G_3 = \partial^3_{\alpha'} \left( \frac{\alpha}{a} \circ h^{-1} (Z_{tt} - i) \right) + Z_{\alpha'} U_h^{-1} [\partial^2_t + i a \partial_{\alpha'}, D_{\alpha}(\frac{\alpha}{h})^2] \xi_t \]
\[ + \partial_{\alpha'} U_h^{-1} [\partial^2_t + i a \partial_{\alpha'}, D_{\alpha}(\frac{\alpha}{h})] \xi_t + \partial^2_{\alpha'} U_h^{-1} [\partial^2_t + i a \partial_{\alpha'}, D_{\alpha}(\frac{\alpha}{h})] \xi_t \]
\[ := Z_{\alpha'} U_h^{-1} G_{3,0} + Z_{\alpha'} U_h^{-1} G_{3,1} + Z_{\alpha'} U_h^{-1} G_{3,2} + Z_{\alpha'} U_h^{-1} G_{3,3} \]

where
\[ Z_{\alpha'} U_h^{-1} G_{3,0} := \partial^3_{\alpha'} \left( \frac{\alpha}{a} \circ h^{-1} (Z_{tt} - i) \right) \]
\[ = \partial^3_{\alpha'} \left( \frac{\alpha}{a} \circ h^{-1} (Z_{tt} - i) \right) + 3 \partial^2_{\alpha'} \left( \frac{\alpha}{a} \circ h^{-1} \right) \xi_{tt,\alpha'} \]
\[ + 3 \partial_{\alpha'} \left( \frac{\alpha}{a} \circ h^{-1} \right) \partial^2_t \xi_t + \frac{\alpha}{a} \circ h^{-1} \partial^3_t \xi_t \]
\[ Z_{\alpha'} U_h^{-1} G_{3,1} := Z_{\alpha'} U_h^{-1} [\partial^2_t + i a \partial_{\alpha'}, D_{\alpha}(\frac{\alpha}{h})^2] \xi_t \]
\[ = -2 D_{\alpha'} Z_t \partial^3_{\alpha'} \xi_t - 2(D_{\alpha'} Z_t) Z_{\alpha'} U_h^{-1} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \partial^3_t \xi_t \]
\[ Z_{\alpha'} U_h^{-1} G_{3,2} := \partial_{\alpha'} U_h^{-1} [\partial^2_t + i a \partial_{\alpha'}, D_{\alpha}(\frac{\alpha}{h})^2] \xi_t \]
\[ = -\partial_{\alpha'} U_h^{-1} \partial_t U_h \left\{ (h_t \circ h^{-1}) \partial^3_{\alpha'} \xi_t \right\} - \partial_{\alpha'} \left\{ (h_t \circ h^{-1}) \partial_{\alpha'} \partial_t U_h \xi_t \right\} \]
\[ - i \partial_{\alpha'} \{ A_{\alpha'} \partial^2_t \xi_t \} \]
\[ = - (\partial_{\alpha'} U_h^{-1} \partial_t U_h \partial^3_t \xi_t - (U_h^{-1} \partial_t U_h \partial^2_t \xi_t + \partial_{\alpha'} U_h^{-1} \partial_t U_h \partial_{\alpha'} \xi_t) \partial^3_t \xi_t \]
\[ - \partial_{\alpha'} (h_t \circ h^{-1}) \partial_t U_h (h_t \circ h^{-1}) \partial^3_t \xi_t - (h_t \circ h^{-1}) \partial_{\alpha'} U_h^{-1} \partial_t U_h \partial_{\alpha'} \xi_t \]
\[ - \partial_{\alpha'} (h_t \circ h^{-1}) \partial_t U_h \partial^3_t \xi_t - \partial_{\alpha'} U_h^{-1} \partial_t U_h \partial_{\alpha'} \xi_t \]
\[ - i (\partial_{\alpha'} A_{\alpha'}) \partial^2_t \xi_t - i A_{\alpha'} \partial^3_t \xi_t. \]  
(4.114)
and
\[
Z,\alpha U_h^{-1} G_{3,3} := \partial^3_{\alpha} U_h^{-1} [\partial^2_t + i \alpha \partial^3_{\alpha} \mathcal{Z}_t] \\
= - \partial^3_{\alpha} U_h^{-1} \partial_t U_h \{(h_t \circ h^{-1})_\alpha \partial^3_{\alpha} \mathcal{Z}_t\} - \partial^3_{\alpha} \{ (h_t \circ h^{-1})_\alpha \partial^3_{\alpha} \mathcal{Z}_t \} \\
- i \partial^3_{\alpha} \{ A_{\alpha} \partial^3_{\alpha} \mathcal{Z}_t \}.
\] (4.115)

**Step 1. Quantities controlled by \( E_3 \) and a polynomial of \( \mathcal{E} \) and \( E_2 \).** By the definition of \( E_3 \), and the fact that \( \| A_1 \|_{L^\infty} \leq C(\mathcal{E}) \) (cf. Appendix C),
\[
\| \partial^3_{\alpha} \mathcal{Z}_t \|_{L^2}^2, \quad \left\| Z,\alpha U_h^{-1} \partial_t U_h \left( \frac{1}{Z,\alpha} \partial^3_{\alpha} \mathcal{Z}_t \right) \right\|_{L^2}^2, \quad \left\| \frac{1}{Z,\alpha} \partial^3_{\alpha} \mathcal{Z}_t \right\|_{H^{1/2}}^2 \leq C(\mathcal{E})E_3.
\] (4.116)

We commute \( Z,\alpha \) with \( U_h^{-1} \partial_t U_h \) of the second quantity in (4.116):
\[
U_h^{-1} \partial_t U_h \partial^3_{\alpha} \mathcal{Z}_t = Z,\alpha U_h^{-1} \partial_t U_h \left( \frac{1}{Z,\alpha} \partial^3_{\alpha} \mathcal{Z}_t \right) - [Z,\alpha, U_h^{-1} \partial_t U_h] \frac{1}{Z,\alpha} \partial^3_{\alpha} \mathcal{Z}_t
\] (4.117)

By (B.26) and Appendix C we have
\[
\left\| U_h^{-1} \partial_t U_h \partial^3_{\alpha} \mathcal{Z}_t \right\|_{L^2}^2 \leq C(\mathcal{E})E_3 \]
(4.118)

By (B.18),
\[
\partial_\alpha U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t - U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t = \partial_\alpha [ Z,\alpha, U_h^{-1} \partial_t U_h ] \partial^2_{\alpha} \mathcal{Z}_t
\]
so
\[
\left\| \partial_\alpha U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t \right\|_{L^2}^2 \leq C(\mathcal{E})E_3.
\] (4.120)

As a consequence of (A.3), (4.23), (4.116), (4.31) and (4.120),
\[
\| \partial^2_{\alpha} \mathcal{Z}_t \|_{L^\infty} \leq C(\mathcal{E}, E_2) E_3^{1/2}, \quad \| U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t \|_{L^\infty} \leq C(\mathcal{E}, E_2) E_3^{1/2}.
\] (4.121)

By (B.18) again,
\[
\partial^3_{\alpha} U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t - \partial^3_{\alpha} U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t = \partial^3_{\alpha} [ Z,\alpha, U_h^{-1} \partial_t U_h ] \partial^2_{\alpha} \mathcal{Z}_t
\]
\[
= \partial^3_{\alpha} (h_t \circ h^{-1})_\alpha \partial^2_{\alpha} \mathcal{Z}_t + (h_t \circ h^{-1})_\alpha \partial^2_{\alpha} \mathcal{Z}_t,
\]
so by (4.120), (4.116), and (4.41), (4.121), (4.28), (4.44) and Appendix C
\[
\| \partial^3_{\alpha} U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t \|_{L^2}^2 \leq C(\mathcal{E}, E_2) E_3^{1/2} + C(\mathcal{E}, E_2); \]
(4.122)

and consequently by (A.3) and (4.43),
\[
\| \partial_\alpha U_h^{-1} \partial_t U_h \partial^2_{\alpha} \mathcal{Z}_t \|_{L^\infty} \leq C(\mathcal{E}, E_2) E_3^{1/2} + C(\mathcal{E}, E_2).
\] (4.123)

**Step 2. Controlling \( G_{3,1} \).** By (4.113), Appendix C and (4.116),
\[
\int \| Z,\alpha U_h^{-1} G_{3,1} \|_{L^2}^2 \, d\alpha \leq C(\mathcal{E})E_3.
\] (4.124)

**Step 3. Controlling \( G_{3,2} \).** By (4.63), (4.69), (4.41), (4.65), (4.43), (4.28), (4.121), (4.62), (4.110), (4.120), (4.122), (4.78), (4.80), (4.48) and Appendix C we can control each of the terms in (4.114). Sum up, we have
\[
\int \| Z,\alpha U_h^{-1} G_{3,2} \|_{L^2}^2 \, d\alpha \leq C(\mathcal{E}, E_2) E_3 + C(\mathcal{E}, E_2).
\] (4.125)
Step 4. Controlling $G_{3,3}$. Expanding $G_{3,3}$ in (4.115) by the product rule, we find that the additional types of terms that have not already appeared in (4.114) and controlled in the previous step are

\[ (\partial^2_\alpha U^{-1}_h \partial_t U_h (h_t \circ h^{-1})_{\alpha'}) \partial_{\alpha'} Z_t, \]

\[ (\partial^2_\alpha (h_t \circ h^{-1})_{\alpha'}) U^{-1}_h \partial_t U_h \partial_{\alpha'} Z_t, \]

\[ \partial^2_\alpha \{ (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} Z_{tt} \}, \quad \text{and} \quad (\partial^3_\alpha A) \partial_{\alpha'} Z_t. \]

Step 4.1. Controlling $\partial^2_\alpha (h_t \circ h^{-1})_{\alpha'}$ and $\partial^2_\alpha Z_{tt}$. We begin with controlling $\partial_{\alpha'} A_1$, $\partial^2_\alpha A_1$ and $\partial^2_\alpha Z_{\alpha'}$. Differentiating (2.19) gives

\[ \partial_{\alpha'} A_1 = -\Im[Z_{t,\alpha'}, \Im\bar{Z}_{t,\alpha'} - \Im[Z_t, \Im\partial_{\alpha'} Z_{t,\alpha'}] \]  

so by (A.18), Appendix C and (4.28),

\[ \| \partial_{\alpha'} A_1 \|_{L^\infty} \lesssim \| Z_{t,\alpha'} \|_{L^2} \| \partial^2_\alpha Z_t \|_{L^2} \lesssim C(\mathcal{E}) E_2^{1/2}. \]  

(4.127)

Differentiating again with respect to $\alpha'$ then apply (A.11), (A.13) and (A.3) gives

\[ \| \partial^2_\alpha A_1 \|_{L^2} \lesssim \| Z_{t,\alpha'} \|_{L^\infty} \| \partial^2_\alpha Z_t \|_{L^2} \lesssim C(\mathcal{E}, E_2). \]  

(4.128)

To estimate $\partial^2_\alpha \frac{1}{Z_{\alpha'}}$, we begin with (2.10):

\[ -i \frac{1}{Z_{\alpha'}} = \frac{\bar{Z}_{tt} - i}{A_1} \]

Taking two derivatives with respect to $\alpha'$ gives

\[ -i \partial^2_\alpha \frac{1}{Z_{\alpha'}} = \frac{\partial^2_\alpha Z_{tt}}{A_1} - 2 Z_{tt,\alpha'} \frac{\partial_{\alpha'} A_1}{A_1^2} + (\bar{Z}_{tt} - i) (- \frac{\partial^2_\alpha A_1}{A_1^2} + 2 \frac{\partial_{\alpha'} A_1^2}{A_1^3}); \]

(4.129)

therefore

\[ \| \partial^2_\alpha \frac{1}{Z_{\alpha'}} \|_{L^2} \lesssim \| \partial^2_\alpha Z_{tt} \|_{L^2} + \| \partial_{\alpha'} Z_{tt} \|_{L^2} \| \partial_{\alpha'} A_1 \|_{L^\infty} \]

\[ + \| \frac{1}{Z_{\alpha'}} \|_{L^\infty} \left( \| \partial^2_\alpha A_1 \|_{L^2} + \| \partial_{\alpha'} A_1 \|_{L^\infty} \| \partial_{\alpha'} A_1 \|_{L^\infty} \right) \lesssim C(\mathcal{E}, E_2), \]

(4.130)

and consequently by (A.3),

\[ \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{L^\infty} \lesssim C(\mathcal{E}, E_2). \]  

(4.131)

We are now ready to give the estimates for $\| \partial^2_\alpha (h_t \circ h^{-1})_{\alpha'} \|_{L^2}$ and $\| \partial^3_\alpha Z_{tt} \|_{L^2}$. Rewriting the first term on the right of (4.40) as a commutator then differentiating yields,

\[ \partial^2_\alpha (h_t \circ h^{-1})_{\alpha'} - 2 \Re(\frac{\partial^2_\alpha Z_t}{Z_{\alpha'}} + \partial^2_\alpha Z_t \Re(\frac{1}{Z_{\alpha'}})) = \Re(2 \partial_{\alpha'} [Z_{t,\alpha'}, \Im\partial_{\alpha'} \frac{1}{Z_{\alpha'}}]) \]

\[ + \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \Re[\bar{Z}_{tt} + \partial_{\alpha'} \Im\partial^2_\alpha \frac{1}{Z_{\alpha'}}] \right). \]  

(4.132)

Expanding the right hand side of (4.132) by the product rule. By (A.11), (A.12),

\[ \| \partial^2_\alpha (h_t \circ h^{-1})_{\alpha'} \|_{L^2} \lesssim \| \frac{1}{Z_{\alpha'}} \|_{L^\infty} \| \partial^2_\alpha Z_t \|_{L^2} + \| \partial_{\alpha'} Z_t \|_{L^\infty} \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{L^2} \]

\[ + \| Z_{t,\alpha'} \|_{L^\infty} \| \partial^2_\alpha \frac{1}{Z_{\alpha'}} \|_{L^2} \lesssim C(\mathcal{E}, E_2) E_3^{1/2} + C(\mathcal{E}, E_2). \]  

(4.133)

For $\| \partial^3_\alpha Z_{tt} \|_{L^2}$, we differentiate (4.45) with respect to $\alpha'$:

\[ \partial^3_\alpha Z_{tt} - \partial_{\alpha'} U^{-1}_h \partial_t U_h \partial^2_\alpha Z_t = 3 \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} \partial^2_\alpha Z_t + 2 (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} \partial^3_\alpha Z_t \]

\[ + \partial^2_\alpha (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} Z_t, \]  

(4.134)
therefore by (4.11), (4.116), (4.120), (4.121),
\[ \| \partial_{\alpha'}^2 Z_{tt} \|_{L^2} \leq C(\mathcal{E}, E_2) E_3 + C(\mathcal{E}, E_2), \]
(4.135)
and as a consequence of (A.3),
\[ \| \partial_{\alpha'}^2 Z_{tt} \|_{L^\infty} \leq C(\mathcal{E}, E_2) E_3^{1/2} + C(\mathcal{E}, E_2). \]
(4.136)

**Step 4.2. Controlling \( \partial_{\alpha'}^3 \mathcal{A} \).** We differentiate (4.79) with respect to \( \alpha' \) and use the product rule to expand. We have,
\[
\| \partial_{\alpha'}^3 \mathcal{A} \|_{L^2} \leq \left\| \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} \left\| \partial_{\alpha'}^3 Z_{tt} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} \left\| \partial_{\alpha'}^2 Z_{tt} \right\|_{L^2} \\
+ \left\| \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_{L^2} \left\| \partial_{\alpha'} Z_{tt} \right\|_{L^\infty} \leq C(\mathcal{E}, E_2) E_3^{1/2} + C(\mathcal{E}, E_2).
\]
(4.137)

**Step 4.3. Controlling \( \partial_{\alpha'}^2 U_h^{-1} \partial_t U_h(h_t \circ h^{-1})_{\alpha'} \).** By (B.16) and (B.18),
\[
\partial_{\alpha'}^2 U_h^{-1} \partial_t U_h(h_t \circ h^{-1})_{\alpha'} = U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \\
+ (\partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'})^2 + 2(h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'}
\]
(4.138)
where
\[
\left\| (\partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'})^2 + 2(h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} \leq \left\| \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} \left\| \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} \right\|_{L^\infty} + \left\| \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} \left\| (h_t \circ h^{-1})_{\alpha'} \right\|_{L^\infty}
\]
\[
\leq \left\| \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} \left\| \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} + \left\| \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} \left\| (h_t \circ h^{-1})_{\alpha'} \right\|_{L^\infty} \leq C(\mathcal{E}, E_2) E_3^{1/2} + C(\mathcal{E}, E_2).
\]
(4.139)
For \( U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \), we differentiate (4.132) and use the product rule and (B.27) to expand the derivatives,
\[
U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} - 2 \text{Re}(U_h^{-1} \partial_t U_h \partial_{\alpha'}^3 Z_t) + U_h^{-1} \partial_t U_h (\partial_{\alpha'} \partial_{\alpha'} \partial_{\alpha'}^2 Z_t)
\]
\[
= \text{Re} U_h^{-1} \partial_t U_h \left\{ 2 \partial_{\alpha'} [Z_t, L, \partial_{\alpha'}] \frac{1}{Z_{\alpha'}} - \partial_{\alpha'} \left\{ 2 \partial_{\alpha'} [Z_t, \mathcal{H}, \partial_{\alpha'}] \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_t + \partial_{\alpha'} [Z_t, \mathcal{H}, \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}}] \right\} \right\};
\]
(4.140)
we then use (A.11), (A.12), (A.13), (A.16) and Hölder’s inequality to do the estimates. We have
\[
\left\| U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \right\|_{L^2} \leq \left\| U_h^{-1} \partial_t U_h \partial_{\alpha'} \partial_{\alpha'} Z_t \right\|_{L^2} \left\| \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} + \left\| U_h^{-1} \partial_t U_h \partial_{\alpha'} \partial_{\alpha'} Z_t \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} \\
+ \left\| \partial_{\alpha'}^2 Z_t \right\|_{L^\infty} \left\| (h_t \circ h^{-1})_{\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} + \left\| U_h^{-1} \partial_t U_h \partial_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2}
\]
\[
+ \left\| \partial_{\alpha'}^2 Z_t \right\|_{L^\infty} \left\| (h_t \circ h^{-1})_{\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} + U_h^{-1} \partial_t U_h \partial_{\alpha'} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2}
\]
\[
+ \left\| \partial_{\alpha'}^2 Z_t \right\|_{L^\infty} \left\| (h_t \circ h^{-1})_{\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} + \left\| Z_t, \alpha' \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} + \left\| Z_t, \alpha' \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2}
\]
\[
+ \left\| Z_t, \alpha' \right\|_{L^\infty} \left\| U_h^{-1} \partial_t U_h \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2}.
\]
(4.141)
Now by (B.18), (B.16),
\[
U_h^{-1} \partial_t U_h \partial_{\alpha'} \frac{1}{Z_{\alpha'}} = \partial_{\alpha'} U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} - \partial_{\alpha'} (h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - 2(h_t \circ h^{-1})_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}
\]
(4.142)
and
\[ U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \frac{1}{Z_{\alpha'}} ((h_t \circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t); \]  
(4.143)

\[ \partial_{\alpha'}^2 U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} = (\partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}})((h_t \circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t) \]
+ \[2(\partial_{\alpha'} \frac{1}{Z_{\alpha'}})((h_t \circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t)\]
\[+(\partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} - \partial_{\alpha'} D_{\alpha'} Z_t) + \frac{1}{Z_{\alpha'}} (\partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} - \partial_{\alpha'}^2 D_{\alpha'} Z_t); \]
(4.144)

we further expand
\[ \partial_{\alpha'} D_{\alpha'} Z_t = \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t + \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_t; \]
and
\[ \partial_{\alpha'}^2 D_{\alpha'} Z_t = \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^3 Z_t + 2 \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 Z_t + \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} Z_{t,\alpha'} \]

Therefore
\[ \| U_h^{-1} \partial_t U_h \frac{1}{Z_{\alpha'}} \|_{L^\infty} \leq C(\mathcal{E}), \]  
\[ \| U_h^{-1} \partial_t U_h \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{L^2} \leq C(\mathcal{E}, E_2)E_3^{1/2} + C(\mathcal{E}, E_2). \]  
(4.145)

By (4.144),
\[ \| U_h^{-1} \partial_t U_h \partial_{\alpha'}^2 (h_t \circ h^{-1})_{\alpha'} \|_{L^2} \leq C(\mathcal{E}, E_2)E_3^{1/2} + C(\mathcal{E}, E_2). \]  
(4.146)

**Step 4.4. Conclusion for G_{3.3}**. We expand G_{3.3} by product rules. Sum up the estimates in Steps 4.1-4.3, we have
\[ \int |Z_{\alpha'} U_h^{-1} G_{3.3}|^2 \ d\alpha' \leq C(\mathcal{E}, E_2)E_3 \]  
(4.147)

**Step 5. Controlling G_{3.0}**. We estimate \( \| Z_{\alpha'} U_h^{-1} G_{3.0} \|_{L^2} \) using similar ideas as that in Step 4 for Proposition 4.3. By (4.144), we must control \( \| \partial_{\alpha'}^2 (\frac{a_t}{a} \circ h^{-1}) (Z_{tt} - \iota) \|_{L^2}, \| \partial_{\alpha'}^2 (\frac{a_t}{a} \circ h^{-1}) \|_{L^2}, \) and \( \| \partial_{\alpha'} (\frac{a_t}{a} \circ h^{-1}) \|_{L^2} \). First by (4.133)
\[ \| (\frac{a_t}{a} \circ h^{-1}) \partial_{\alpha'}^3 Z_{tt} \|_{L^2} \leq C(\mathcal{E}, E_2)E_3 + C(\mathcal{E}, E_2). \]  
(4.148)

By (4.85) and (A.3),
\[ \| \partial_{\alpha'} (\frac{a_t}{a} \circ h^{-1}) \partial_{\alpha'}^2 Z_{tt} \|_{L^2} \leq C(\mathcal{E}, E_2)E_3 + C(\mathcal{E}, E_2). \]  
(4.149)

By (4.6),
\[ \| \partial_{\alpha'}^2 (\frac{a_t}{a} \circ h^{-1}) \|_{L^2} \leq \| \partial_{\alpha'}^2 Z_{tt} \|_{L^2} (\| Z_{tt,\alpha'} \|_{L^\infty} + \| \Pi Z_{tt,\alpha'} \|_{L^\infty} \| Z_{t,\alpha' \alpha} \|_{L^\infty} + \| \partial_{\alpha'}^2 Z_{t} \|_{L^2} \| Z_{t,\alpha' \alpha} \|_{L^\infty} ) \]
\[+ \| \| \partial_{\alpha'}^2 Z_{tt} \|_{L^2} \| Z_{t,\alpha' \alpha} \|_{L^\infty} \| D_{\alpha'} Z_t \|_{L^\infty} \]
\[+ \| \| \partial_{\alpha'} Z_{t} \|_{L^2} \| \partial_{\alpha'} D_{\alpha'} Z_t \|_{L^2} + \| \partial_{\alpha'} (\frac{a_t}{a} \circ h^{-1}) \|_{L^2} \| \partial_{\alpha'} A_1 \|_{L^\infty} \]
\[+ \| \frac{a_t}{a} \|_{L^\infty} \| \partial_{\alpha'} A_1 \|_{L^2}. \]  
(4.150)

so
\[ \| \partial_{\alpha'}^2 (\frac{a_t}{a} \circ h^{-1}) \|_{L^2} \leq C(\mathcal{E}, E_2)E_3^{1/4} + C(\mathcal{E}, E_2), \]  
(4.151)

therefore
\[ \| \partial_{\alpha'}^2 (\frac{a_t}{a} \circ h^{-1}) \partial_{\alpha'} Z_{tt} \|_{L^2} \leq C(\mathcal{E}, E_2)E_3 + C(\mathcal{E}, E_2). \]  
(4.152)

Now similar to (4.92) and (4.93), we compute \( Z_{\alpha'} U_h^{-1} [(\partial_{\alpha'}^2 + i a \partial_{\alpha'}), \frac{h_{\alpha}}{z_{\alpha}}] (\frac{a_t}{h_{\alpha}})^3 \) by (4.28) and have
\[ \int \left| Z_{\alpha'} U_h^{-1} [(\partial_{\alpha'}^2 + i a \partial_{\alpha'}), \frac{h_{\alpha}}{z_{\alpha}}] (\frac{a_t}{h_{\alpha}})^3 \right|^2 \ d\alpha' \leq C(\mathcal{E})E_3. \]  
(4.153)
Now we begin with (4.12) for $k = 3$. After expansion, commuting and precomposing with $h^{-1}$, and using the above estimates, we arrive at

$$U_h^{-1} (\partial_t^3 + ia \partial_\alpha) U_h \partial_\alpha^3 \overline{Z}_t = (\overline{Z}_{tt} - i) \partial_\alpha^3 (\frac{\alpha_t}{a} \circ h^{-1}) + e_1$$

(4.154)

with

$$\int |e_1|^2 \, d\alpha' \leq C(\mathcal{E}, E_2) E_3 + C(\mathcal{E}, E_2).$$

(4.155)

Going through similar calculations as in (4.135) to (4.138), then applying (4.100) to $U = \partial_\alpha^3 \overline{Z}_t$, we obtain

$$\| (\overline{Z}_{tt} - i) \partial_\alpha^3 (\frac{\alpha_t}{a} \circ h^{-1}) \|^2_{L^2} \leq C(\mathcal{E}, E_2) E_3 + C(\mathcal{E}, E_2).$$

(4.156)

This finishes the proof for Proposition 4.4.

\[\square\]

4.3. Completing the proof for Theorem 3.1

**Proof.** Let $s \geq 4$. Let the initial interface $Z(\cdot, 0) = Z(0)$, the initial velocity $Z_t(\cdot, 0) = Z_t(0)$ be given and satisfy (2.28) and $Z(0) = \mathbb{H} Z(0)$; let $A_1(0)$ satisfy (2.19) and the initial acceleration $Z_{tt}(0)$ satisfy (2.10). Assume $Z_{\alpha'}(0) - 1 \in L^\infty(\mathbb{R})$, $Z_t(0) \in H^{s+1/2}(\mathbb{R})$, and $Z_{tt}(0) \in H^s(\mathbb{R})$. It is clear that $E_2(0) + E_3(0) < \infty$. Assume $\tilde{Z} = Z(\cdot, t)$, for $t \in [0, T^*)$ is a solution of (2.24)-(2.28), such that $(Z_t, Z_{tt}, Z_{\alpha'} - 1) \in C([0, T^*), H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}))$, and $T^*$ is the maximum existence time as defined in Theorem 3.1. Assume $T^* < \infty$, for otherwise we are done; and assume $\sup_{t \in [0, T^*)} \mathcal{E}(t) := M < \infty$. We want to show $\sup_{t \in [0, T^*)} (\| Z_t(t) \|_{H^3} + \| Z_{tt}(t) \|_{H^{3+1/2}}) < \infty$.

**Step 1.** Controlling $\| Z_{tt}(t) \|_{L^2}$ and $\| Z_t(t) \|_{L^2}$ by $\mathcal{E}$ and the initial data. We start with $\| Z_{tt}(t) \|_{L^2}$. By a change of the variables,

$$\frac{d}{dt} \| Z_{tt}(t) \|^2_{L^2} = \frac{d}{dt} \int |z_{tt}|^2 h_\alpha \, d\alpha = 2 \Re \int z_t \overline{z}_{tt} h_\alpha \, d\alpha + 2 \int |z_{tt}|^2 \frac{h_\alpha}{h_\alpha} h_\alpha \, d\alpha;$$

(4.157)

we estimate

$$\int |z_{tt}|^2 \frac{h_\alpha}{h_\alpha} h_\alpha \, d\alpha \leq \left\| \frac{h_\alpha}{h_\alpha} \right\|_{L^\infty} \| Z_{tt}(t) \|^2_{L^2}.$$

(4.158)

Switching back to the Riemann mapping variable and using (4.5) gives

$$\int z_t \overline{z}_{tt} \frac{h_\alpha}{h_\alpha} \, d\alpha = -i \int Z_{tt} \overline{Z}_{tt} \, d\alpha' + \int Z_{tt} \frac{\alpha_t}{\alpha} \circ h^{-1} (Z_{tt} - i) \, d\alpha' = I + II$$

(4.159)

Replacing $A := \frac{A_{0}}{|Z_{\alpha'}(\cdot, t)|}$, we estimate $I$ by

$$|I| \leq \| A_1 \|_{L^\infty} \left\| \frac{1}{|Z_{\alpha'}|} \right\|_{L^\infty}^2 \| Z_{tt} \|_{L^2} \| Z_{\alpha'} \|_{L^2}$$

(4.160)

In $II$ we estimate $\| \frac{\alpha_t}{\alpha} \circ h^{-1} \|_{L^2}$ by (4.16), where we rewrite $D_{\alpha'}^3 Z_{\alpha'} := \frac{1}{Z_{\alpha'}} \overline{Z}_{tt} Z_{\alpha'}$. Using (A.12), (A.13) and (A.16) yields

$$\| \frac{\alpha_t}{\alpha} \circ h^{-1} \|_{L^2} \lesssim \| Z_{\alpha'} \|_{L^2} \| Z_t \|_{L^\infty} + \| Z_{tt} \|_{L^2} \left\| \frac{1}{|Z_{\alpha'}|} \right\|_{L^\infty},$$

(4.161)

so

$$|II| \lesssim \left( \| Z_{\alpha'} \|_{L^2} \| Z_t \|_{L^\infty} + \| Z_{tt} \|_{L^2} \left\| \frac{1}{|Z_{\alpha'}|} \right\|_{L^\infty} \right) \| Z_{tt} \|_{L^2} (\| Z_t \|_{L^\infty} + 1).$$

(4.162)
Sum up the above estimates and apply Appendix C, we arrive at
\[
\frac{d}{dt} \|Z_t(t)\|^2_{L^2} \leq c(\mathcal{E}(t))\|Z_t(t)\|^2_{L^2} + c(\mathcal{E}(t)).
\]
Consequently by Gronwall,
\[
\sup_{[0,T^*]} \|Z_t(t)\|_{L^2} \leq c(\|Z_t(0)\|_{L^2}, M) < \infty. \tag{4.163}
\]

Changing to the Lagrangian coordinate, we have
\[
\int |Z_t(\alpha', t)|^2 \, d\alpha' = \int |z_t(\alpha, t)|^2 h_\alpha(\alpha, t) \, d\alpha,
\]
so
\[
\frac{d}{dt} \int |z_t|^2 h_\alpha \, d\alpha = 2 \Re \int z_t \overline{z_t} h_\alpha \, d\alpha + \int |z_t|^2 h_\alpha \frac{h_{\alpha t}}{h_\alpha} \, d\alpha. \tag{4.164}
\]
Using Cauchy-Schwarz and changing back to the Riemann mapping variable,
\[
\frac{d}{dt} \int |z_t|^2 h_\alpha \, d\alpha \leq 2 \|Z_t(t)\|_{L^2} \|Z_t(t)\|_{L^2} + \| (h_t \circ h^{-1})_{\alpha'} \|_{L^\infty} \|Z_t(t)\|^2_{L^2},
\]
therefore
\[
\frac{d}{dt} \|Z_t(t)\|^2_{L^2} \leq C(\mathcal{E}(t))\|Z_t(t)\|^2_{L^2} + \|Z_t(t)\|^2_{L^2}, \tag{4.166}
\]
by Appendix C. Consequently by Gronwall’s inequality and (4.163),
\[
\sup_{t \in [0,T^*]} \|Z_t(t)\|^2_{L^2} \leq C(\|Z_t(0)\|_{L^2}, \|Z_t(0)\|_{L^2}, M) < \infty. \tag{4.167}
\]

**Step 2. Controlling \(\|Z_{\alpha'}\|_{L^\infty}\).** We know
\[
Z_{\alpha'} \circ h = \frac{z_\alpha}{h_\alpha},
\]
and
\[
\frac{d}{dt} \left| \frac{z_\alpha}{h_\alpha} \right|^2 = 2 \left| \frac{z_\alpha}{h_\alpha} \right|^2 \Re(D_{\alpha'} z_t - \frac{h_{\alpha t}}{h_\alpha}),
\]
so by Appendix C
\[
\sup_{t \in [0,T^*)} \left| \frac{z_\alpha}{h_\alpha} \right|^2 \leq C(\mathcal{E}) \left| \frac{z_\alpha}{h_\alpha} \right|^2
\]
therefore
\[
\sup_{t \in [0,T^*)} \|Z_{\alpha'}(t)\|^2_{L^\infty} \leq \|Z_{\alpha'}(0)\|^2_{L^\infty} e^{C(M)T^*} < \infty. \tag{4.168}
\]

**Step 3. Controlling \(\|Z_t(t)\|_{H^{2+\gamma/2}} + \|Z_t(t)\|_{H^3}\).** Taking sup over \([0,T^*)\) on (4.17) gives
\[
\sup_{t \in [0,T^*)} E_2(t) \leq E_2(0) e^{p_1(M)T^*} := M_2 < \infty; \tag{4.169}
\]
\[
\sup_{t \in [0,T^*)} E_3(t) \leq (E_3(0) + p_3(M, M_2)T^*) e^{p_2(M, M_2)T^*} := M_3 < \infty,
\]
By (4.133), (4.163),
\[
\sup_{[0,T^*)} \|Z_{tt}(t)\|_{H^3} \lesssim \sup_{[0,T^*)} (\|\partial_{\alpha'}^3 Z_t(t)\|_{L^2} + \|Z_t(t)\|_{L^2}) < \infty. \tag{4.170}
\]

Now by (A.6),
\[
\|\partial_{\alpha'}^3 Z_t\|_{H^{1/2}} \lesssim \|Z_{\alpha'}\|_{L^\infty} (\|\frac{1}{Z_{\alpha'}} \partial_{\alpha'}^3 Z_t\|_{H^{1/2}} + \|\partial_{\alpha'}^3 Z_t\|_{L^2} + \|Z_t\|_{L^2} + \|Z_{\alpha'}\|_{L^2} + \|\partial_{\alpha'} \frac{1}{Z_{\alpha'}} Z_t\|_{L^2}).
\]
We know by (4.116) and Appendix C
\[
\|\partial_{\alpha'}^3 Z_t\|_{L^2}, \quad \|\frac{1}{Z_{\alpha'}} \partial_{\alpha'}^3 Z_t\|_{H^{1/2}} \leq C(\mathcal{E}) E_3, \quad \|\partial_{\alpha'} \frac{1}{Z_{\alpha'}} Z_t\|_{L^2} \leq C(\mathcal{E});
\]
so using (4.168) we have
\[
\sup_{[0,T^*]} \| \partial^3_{\alpha^2} \mathcal{Z} \|_{H^{1/2}} \leq \| F_{\alpha'}(0) \|_{L^\infty}^2 \epsilon^{3(M)} C(M) M_3 < \infty
\] (4.171)

Combine with (4.167), we have
\[
\sup_{[0,T^*)} \| Z_1(t) \|_{H^{3+1/2}} < \infty.
\] (4.172)

By Proposition 2.3 this brings us a contradiction. This finishes the proof for Theorem 3.1.

5. THE PROOF OF THEOREM 3.4

We prove Theorem 3.4 by mollifying the initial data by the Poisson Kernel and approximating. We denote \( z' = x' + iy' \), where \( x', y' \in \mathbb{R} \). \( f \ast g \) is the convolution in the spatial variable.

5.1. The initial data. Let \( F'(z',0) \) be the initial fluid velocity in the Riemann mapping coordinate, \( \Psi(z',0) : P_- \to \Omega(0) \) be the Riemann mapping as given in (3.1) with \( Z(\alpha,0) = \Psi(\alpha,0) \) the initial interface. We note that by the assumption

\[
\sup_{y'<0} \| \partial_{z'}(\frac{1}{\Psi_{z'}(z',0)}) \|_{L^2(\mathbb{R},dx')} \leq \mathcal{E}_1(0) < \infty, \quad \sup_{y'<0} \| \frac{1}{\Psi_{z'}(z',0)} - 1 \|_{L^2(\mathbb{R},dx')} \leq c_0 < \infty;
\]

\[
\sup_{y'<0} \| F_{z'}(z',0) \|_{L^2(\mathbb{R},dx')} \leq \mathcal{E}_1(0) < \infty, \quad \sup_{y'<0} \| F(z',0) \|_{L^2(\mathbb{R},dx')} \leq c_0 < \infty,
\]

\( \frac{1}{\Psi_{z'}(\cdot,0)}, F(\cdot,0) \) can be extended continuously onto \( \overline{P_-} \). We denote their boundary values by \( \frac{1}{\Psi_{z'}}(\alpha',0) \) and \( F(\alpha',0) \). So \( Z(\cdot,0) = \Psi(\cdot,0) \) is continuous differentiable on the open set where \( \frac{1}{\Psi_{z'}}(\alpha',0) \neq 0 \) and \( \frac{1}{\Psi_{z'}}(\alpha',0) = \frac{\overline{Z_{\alpha'}}(\alpha,0)}{Z_{\alpha'}}(\alpha,0) \) where \( \frac{1}{\Psi_{z'}}(\alpha',0) \neq 0 \). By \( \frac{1}{\Psi_{z'}}(\cdot,0) - 1 \in H^1(\mathbb{R}) \) and Sobolev embedding, there is \( N > 0 \) sufficiently large, such that for \( |\alpha'| \geq N \), \( |\frac{1}{\Psi_{z'}}(\alpha',0) - 1| \leq 1/2 \), so \( Z = Z(\cdot,0) \) is continuous differentiable on \( (-\infty,-N) \cup (N,\infty) \), with \( |Z_{\alpha'}(\alpha',0)| \leq 2 \), for all \( |\alpha'| \geq N \). Moreover, \( Z_{\alpha'}(\cdot,0) - 1 \in H^1((-\infty,-N) \cup (N,\infty)) \).

5.2. The mollified data and the approximate solutions. Let \( \epsilon > 0 \). We take

\[
Z'(\alpha',0) = \Psi(\alpha' - \epsilon i,0), \quad \overline{Z}_t'(\alpha',0) = F(\alpha' - \epsilon i,0), \quad h'(\alpha,0) = \alpha,
\]

\[
F'(z',0) = F(z' - \epsilon i,0), \quad \Psi'(z',0) = \Psi(z' - \epsilon i,0).
\] (5.1)

Notice that \( F'(\cdot,0), \Psi'(\cdot,0) \) are holomorphic on \( P_- \), \( Z'(\cdot,0) \) satisfies (2.8) and \( \overline{Z}_t'(0) = \overline{H}_t'(0) \). Let \( Z_0 '(0) \) be given by (2.10). It is clear \( Z'(\cdot,0) \), \( Z_t'(0) \) and \( Z_{tt}'(0) \) satisfy the assumption of Theorem 3.1. Let \( Z'(t) \) be the solution as given by Theorem 3.1 with the homeomorphism \( h'(t) \) and \( z'(t) = Z'(h'(\alpha,t),t) \). We know \( z_t'(\alpha,t) = Z_t'(h'(\alpha,t),t) \). Let

\[
F'(x' + iy',t) = K_{y'} \ast \overline{Z}_t'(x',t), \quad \Psi_{z'}'(x' + iy',t) = K_{y'} \ast \overline{Z}_{\alpha'}'(x',t), \quad \Psi'(x',t)
\]

be the holomorphic functions on \( P_- \) with boundary values \( \overline{Z}_t'(t), Z_{\alpha'}'(t) \) and \( Z'(t) \):

\[
\frac{1}{\Psi_{z'}'}(x' + iy',t) = K_{y'} \ast \frac{1}{\overline{Z}_{\alpha'}}(x',t)
\]

by uniqueness. \(^{21}\) We denote the energy functional \( \mathcal{E} \) for \( Z'(t), \overline{Z}_t'(t) \) by \( \mathcal{E}_t'(t) \) and the energy functional \( \mathcal{E}_1 \) for \( F'(t), \Psi'(t) \) by \( \mathcal{E}_1'(t) \). It is clear \( \mathcal{E}_1'(0) = \mathcal{E}_1'(0) \leq \mathcal{E}_1(0) \). By Theorem 3.1 Theorem 2.3 and Proposition 2.3 there exists \( T_0 > 0 \), \( T_0 \) depends only on \( \mathcal{E}_1(0) \), such that

\(^{21}\)By the maximum principle, \( (K_{y'} \ast \frac{1}{\overline{Z}_{\alpha'}}) (K_{y'} \ast \overline{Z}_{\alpha'}) \equiv 1 \) on \( P_- \).
on $[0, T_0]$, the system $2.9 - 2.8 - 2.18 - 2.19$ has a unique solution $Z^r = Z^r(\cdot, t)$, satisfying $(Z^r_t, Z^r_H, \frac{1}{Z^r_{\alpha'}} - 1) \in C([0, T_0], H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ for $s > 4$, and

$$\sup_{[0, T_0]} \mathcal{E}_1^e(t) = \sup_{[0, T_0]} \mathcal{E}_1^e(t) \leq M(\mathcal{E}_1(0)) < \infty. \quad (5.2)$$

Moreover by $2.10$, $4.1.63$ and $4.1.67$,

$$\sup_{[0, T_0]} (\|Z^r_t(t)\|_{L^2} + \|Z^r_H(t)\|_{L^2} + \|\frac{1}{Z^r_{\alpha'}}(t) - 1\|_{L^2}) \leq c(c_0, \mathcal{E}_1(0)),$$

so there is a constant $C_0 := C(c_0, \mathcal{E}_1(0)) > 0$, such that

$$\sup_{[0, T_0]} \{ \sup_{y' < 0} \|F^r(x' + iy', t)\|_{L^2(\mathbb{R}^2)} + \sup_{y' < 0} \|\Psi_{x'}(x' + iy', t) - 1\|_{L^2(\mathbb{R}^2)} \} < C_0 < \infty. \quad (5.4)$$

5.3. Uniformly bounded quantities. We would like to apply some compactness results to pass to the limits of the various quantities for the water waves. It is necessary to understand the boundedness properties of these quantities.

Let $b^r := h^r_t(0)^{-1} = 2 \text{Re} Z^r_t + \text{Re}[Z^r_1, \mathbb{H}](\frac{1}{Z^r_{\alpha'}} - 1)$ be as given by $(2.18)$. By $(A.18)$,

$$\|b^r(t)\|_{L^\infty} = \|h^r_t(t)\|_{L^\infty} \lesssim \|Z^r_t(t)\|_{L^\infty} + \|Z^r_{t, \alpha'}(t)\|_{L^2} \|\frac{1}{Z^r_{\alpha'}}(t) - 1\|_{L^2}. \quad (5.5)$$

Using $(4.2)$ to rewrite $b^r = \text{Re}(I - \mathbb{H})(Z^r_t \frac{1}{Z^r_{\alpha'}})$, differentiating to get

$$\|b^r_{\alpha'}(t)\|_{L^2} \lesssim \|Z^r_{t, \alpha'}(t)\|_{L^2} \|\frac{1}{Z^r_{\alpha'}}(t) - 1\|_{L^\infty} + \|Z^r_t(t)\|_{L^\infty} \|\partial_{\alpha'} \frac{1}{Z^r_{\alpha'}}(t)\|_{L^2}. \quad (5.6)$$

We know $h^r$ satisfies

\[
\begin{cases}
\frac{d}{dt} h^r = b^r(h^r, t)\\
h^r(\alpha, 0) = \alpha.
\end{cases}
\]

Differentiating $(5.7)$ gives

\[
\begin{cases}
\frac{d}{dt} h^r_{\alpha'} = b^r_{\alpha'}(h^r, t) h^r_{\alpha'}\\
h^r_{\alpha'}(\alpha, 0) = 1
\end{cases}
\]

therefore

$$e^{-t \sup_{[0, 1]} \|b^r_{\alpha'}(s)\|_{L^\infty}} \leq h^r_{\alpha'}(\alpha, t) = e^{t \int_0^1 b^r_{\alpha'}(h^r, s) ds} \leq e^{t \sup_{[0, 1]} \|b^r_{\alpha'}(s)\|_{L^\infty}} \quad (5.9)$$

Now by $(2.18)$, $(3.24)$, with an application of $(A.18)$ and $(A.17)$,

$$\|U_{h^r}^{-1} \partial_t U_{h^r} b^r(t)\|_{L^\infty} \lesssim \|Z^r_t(t)\|_{L^\infty} + \|Z^r_{t, \alpha'}(t)\|_{L^2} \|\frac{1}{Z^r_{\alpha'}}(t) - 1\|_{L^2} \quad (5.10)$$

where $U_{h^r}^{-1} \partial_t U_{h^r} \frac{1}{Z^r_{\alpha'}} = \frac{1}{Z^r_{\alpha'}}((h^r_t(0)^{-1})_{\alpha'} - D_{\alpha'} Z^r_t)$ gives

$$\|U_{h^r}^{-1} \partial_t U_{h^r} \frac{1}{Z^r_{\alpha'}}(t)\|_{L^2} \leq \|\frac{1}{Z^r_{\alpha'}}(t)\|_{L^\infty} (\|b^r_{\alpha'}(t)\|_{L^2} + \|\frac{1}{Z^r_{\alpha'}}(t)\|_{L^\infty} \|Z^r_{t, \alpha'}(t)\|_{L^2}) \quad (5.11)$$

and $U_{h^r}^{-1} \partial_t U_{h^r} = \partial_t + b^r \partial_{\alpha'}$ gives

$$\|\partial_t b^r(t)\|_{L^\infty} \leq \|U_{h^r}^{-1} \partial_t U_{h^r} b^r(t)\|_{L^\infty} + \|b^r(t)\|_{L^\infty} \|b^r_{\alpha'}(t)\|_{L^\infty}. \quad (5.12)$$
Finally, differentiating (4.3) gives \( z_{tt} = (z_{tt}^i + i)(D_\alpha z_t^i + \xi_{\alpha}^i) \), so

\[
\| z_{tt}^i(t) \|_{L^\infty} \leq \| z_{tt}^i(t) \|_{L^\infty} + \| (D_\alpha z_t^i)(t) \|_{L^\infty} + \| \xi_{\alpha}^i(t) \|_{L^\infty}.
\]  

(5.13)

Let \( M(\mathcal{E}_1(0)), c_0, \mathcal{E}_1(0), C_0 \) be the bounds in (5.2), (5.3) and (5.4). By Proposition 2.3 Sobolev embedding, Appendix C and (5.11), the following quantities are uniformly bounded with bounds depending only on \( M(\mathcal{E}_1(0)), c_0, \mathcal{E}_1(0), C_0 \):

\[
\sup_{[0,T_0]} \| Z_t^i(t) \|_{L^\infty}, \sup_{[0,T_0]} \| Z_{t,\alpha}^i(t) \|_{L^2}, \sup_{[0,T_0]} \| Z_{tt}^i(t) \|_{L^\infty}, \sup_{[0,T_0]} \| Z_{tt,\alpha}^i(t) \|_{L^2},
\]

(5.14)

and with a change of the variables and (5.9), (5.13) and Appendix C

\[
\sup_{[0,T_0]} \| z_t^i(t) \|_{L^\infty} + \| z_{t,\alpha}^i(t) \|_{L^2} + \sup_{[0,T_0]} \| z_{tt}^i(t) \|_{L^\infty} \leq C(c_0, \mathcal{E}_1(0)),
\]

(5.15)

Furthermore, by the estimates in (5.9)–(5.12), using (5.9) (5.14) and Appendix C the following quantities are uniformly bounded:

\[
\sup_{[0,T_0]} \| h_t^i(t) \|_{L^\infty} + \| h_{t,\alpha}^i(t) \|_{L^2} + \sup_{[0,T_0]} \| h_{tt}^i(t) \|_{L^\infty} \leq C(c_0, \mathcal{E}_1(0))
\]

(5.16)

In particular, by (5.9) and Appendix C there are \( c_1, c_2 > 0 \), depending only on \( c_0 \) and \( \mathcal{E}_1(0) \), such that

\[
0 < c_1 \leq \frac{h_t^\alpha(\alpha, t) - h_t^\beta(\beta, t)}{\alpha - \beta} \leq c_2 < \infty, \quad \forall \alpha, \beta \in \mathbb{R}, \ t \in [0, T_0].
\]

(5.17)

5.4. Some useful compactness results. Here we give two compactness results that we will use to pass to the limits.

**Lemma 5.1.** Let \( \{f_n\} \) be a sequence of smooth functions on \( \mathbb{R} \times [0, T] \). Let \( 1 < p \leq \infty \). Assume that there is a constant \( C \), independent of \( n \), such that

\[
\sup_{[0,T]} \| f_n(t) \|_{L^\infty} + \| \partial_x f_n(t) \|_{L^p} + \sup_{[0,T]} \| \partial_t f_n(t) \|_{L^\infty} \leq C.
\]

(5.18)

Then there is a function \( f \), continuous and bounded on \( \mathbb{R} \times [0, T] \), and a subsequence \( \{f_{n_j}\} \), such that \( f_{n_j} \to f \) uniformly on compact subsets of \( \mathbb{R} \times [0, T] \).

Lemma 5.1 is an easy consequence of Arzela-Ascoli Theorem, we omit the proof.

**Lemma 5.2.** Assume that \( f_n \to f \) uniformly on compact subsets of \( \mathbb{R} \times [0, T] \), and assume there is a constant \( C \), such that \( \sup_n \| f_n \|_{L^\infty(\mathbb{R} \times [0, T])} \leq C \). Then \( K \delta^* f_n \) converges uniformly to \( K \delta^* f \) on compact subsets of \( \mathbb{R} \times [0, T] \).

The proof follows easily by considering the convolution on two sets \( |x'| < N \), and \( |x'| \geq N \). We omit the proof.

**Definition 5.3.** We write

\[
f_n \Rightarrow f \quad \text{on } E
\]

(5.19)

if \( f_n \) converge uniformly to \( f \) on compact subsets of \( E \).
5.5. **Passing to the limit.** Notice that \( h'(\alpha, t) - \alpha = \int_0^t h'(\alpha, s) \, ds \), so

\[
\sup_{\mathbb{R} \times [0, T_0]} |h'(\alpha, t) - \alpha| \leq T_0 \sup_{[0, T_0]} \|h'(t)\|_{L^\infty} \leq T_0 C(\alpha, \mathcal{E}_1(0)) < \infty. \tag{5.20}
\]

By Lemma [5.1](#), there is a subsequence \( \epsilon_j \to 0 \), which we still write as \( \epsilon \) instead of \( \epsilon_j \), and functions \( b, h - \alpha, w, u, q := w_t \), continuous and bounded on \( \mathbb{R} \times [0, T_0] \), such that

\[
b\epsilon \Rightarrow b, \quad h\epsilon \Rightarrow h, \quad z\epsilon \Rightarrow w, \quad \frac{h\epsilon}{\alpha} \Rightarrow u, \quad z_{\epsilon t} \Rightarrow q, \quad \text{on} \ \mathbb{R} \times [0, T_0], \tag{5.21}
\]

as \( \epsilon = \epsilon_j \to 0 \). Moreover by (5.17),

\[
0 < c_1 \leq \frac{h(\alpha, t) - h(\beta, t)}{\alpha - \beta} \leq c_2 < \infty, \quad \forall \alpha, \beta \in \mathbb{R}, \ t \in [0, T_0]; \tag{5.22}
\]

hence \( h(\cdot, t) : \mathbb{R} \to \mathbb{R} \) is a homeomorphism, and

\[
(h\epsilon)^{-1} \Rightarrow h^{-1} \quad \text{on} \ \mathbb{R} \times [0, T_0], \quad \text{as} \ \epsilon = \epsilon_j \to 0. \tag{5.23}
\]

This gives

\[
\begin{align*}
Z_t & \Rightarrow w \circ h^{-1}, \\
\frac{1}{Z_{\epsilon t}'} & \Rightarrow u \circ h^{-1}, \\
Z_{\epsilon t} & \Rightarrow w_t \circ h^{-1}, \quad \text{on} \ \mathbb{R} \times [0, T_0]. \tag{5.24}
\end{align*}
\]

as \( \epsilon = \epsilon_j \to 0 \). Now

\[
F^\epsilon(z', t) = K_{y'} \ast Z_t', \quad \frac{1}{\Psi(z', t)} = F^\epsilon(z', t) \Rightarrow \Lambda(z', t) \quad \text{on} \ \bar{P}_- \times [0, T_0]; \tag{5.25}
\]

as \( \epsilon = \epsilon_j \to 0 \). Moreover \( F(\cdot, t), \Lambda(\cdot, t) \) are holomorphic on \( P_- \) for each \( t \in [0, T_0] \), and continuous on \( \bar{P}_- \times [0, T] \). Furthermore applying the Cauchy integral formula to the first limit in (5.26) yields

\[
F_{z'}^\epsilon(z', t) \Rightarrow F_{z'}(z', t) \quad \text{on} \ P_- \times [0, T_0]. \tag{5.27}
\]

as \( \epsilon = \epsilon_j \to 0 \).

**Step 1. The limit of \( \Psi^\epsilon \).** We consider the limit of \( \Psi^\epsilon \), as \( \epsilon = \epsilon_j \to 0 \). We know

\[
z'_{\alpha}(\alpha, t) = z'_{\alpha}(\alpha, 0) + \int_0^t z'_{\alpha}(\alpha, s) \, ds \tag{5.28}
\]

\[
= \Psi(\alpha - \epsilon i, 0) + \int_0^t z'_{\alpha}(\alpha, s) \, ds,
\]

therefore

\[
Z'(\alpha', t) - Z'(\alpha', 0) = \Psi((h')^{-1}(\alpha', t) - \epsilon i, 0) - \Psi(\alpha' - \epsilon i, 0) \tag{5.29}
\]

\[
+ \int_0^t z'_{\alpha}(h')^{-1}(\alpha', s) \, ds.
\]

Let

\[
W'(\alpha', t) := \Psi((h')^{-1}(\alpha', t) - \epsilon i, 0) - \Psi(\alpha' - \epsilon i, 0) + \int_0^t z'_{\alpha}(h')^{-1}(\alpha', t, s) \, ds. \tag{5.30}
\]

Observe \( Z'(\alpha', t) - Z'(\alpha', 0) \) is the boundary value of the holomorphic function \( \Psi^\epsilon(z', t) - \Psi^\epsilon(z', 0) \). By (5.21) and (5.23), \( \int_0^t z'_{\alpha}(h')^{-1}(\alpha', t, s) \, ds \to \int_0^t w(h^{-1}(\alpha', t), s) \, ds \) uniformly on compact subsets of \( \mathbb{R} \times [0, T_0] \), and by (5.15), \( \int_0^t z'_{\alpha}(h')^{-1}(\alpha', t, s) \, ds \) is continuous and uniformly bounded in \( L^\infty(\mathbb{R} \times [0, T_0]) \). By the assumptions \( \lim_{z' \to 0} \Psi(z', 0) = 1, \Psi(\cdot, 0) = 0 \) is continuous on \( \bar{P}_- \) and (5.20), (5.23),

\[
\Psi((h')^{-1}(\alpha', t) - \epsilon i, 0) - \Psi(\alpha' - \epsilon i, 0)
\]
is continuous and uniformly bounded in $L^\infty(\mathbb{R} \times [0, T_0])$ for $0 < \epsilon < 1$, and converges uniformly on compact subsets of $\mathbb{R} \times [0, T_0]$, as $\epsilon = \epsilon_j \to 0$. This gives

$$\Psi^\epsilon(z', t) - \Psi^\epsilon(z', 0) = K_{y'} \ast W^\epsilon(x', t)$$ (5.31)

and by Lemma 5.2, $\Psi^\epsilon(z', t) - \Psi^\epsilon(z', 0)$ converges uniformly on compact subsets of $\mathcal{P}_- \times [0, T_0]$ to a function that is holomorphic on $\mathcal{P}_-$. For every $t \in [0, T_0]$ and continuous on $\mathcal{P}_- \times [0, T_0]$. Therefore there is a function $\Psi(\cdot, t)$, holomorphic on $\mathcal{P}_-$ for every $t \in [0, T_0]$ and continuous on $\mathcal{P}_- \times [0, T_0]$, such that

$$\Psi^\epsilon(z', t) \Rightarrow \Psi(z', t) \quad \text{on} \quad \mathcal{P}_- \times [0, T_0]$$ (5.32)

as $\epsilon = \epsilon_j \to 0$; as a consequence of the Cauchy integral formula,

$$\Psi^\epsilon(z', t) \Rightarrow \Psi(z', t) \quad \text{on} \quad \mathcal{P}_- \times [0, T_0]$$ (5.33)

as $\epsilon = \epsilon_j \to 0$. Combining with (5.23), we have $\Lambda(z', t) = \frac{1}{\Psi^\epsilon(z', t)}$, so $\Psi^\epsilon(z', t) \neq 0$ for all $(z', t) \in \mathcal{P}_- \times [0, T_0]$ and

$$\frac{1}{\Psi^\epsilon(z', t)} \Rightarrow \frac{1}{\Psi(z', t)} \quad \text{on} \quad \mathcal{P}_- \times [0, T_0]$$ (5.34)

as $\epsilon = \epsilon_j \to 0$. Denote $Z(\alpha', t) := \Psi(\alpha', t)$, $\alpha' \in \mathbb{R}$, and $z(\alpha, t) = Z(h(\alpha, t), t)$. (5.32) gives $Z^\epsilon(\alpha', t) \Rightarrow Z(\alpha', t)$, and with (5.21) it gives $z^\epsilon(\alpha, t) \Rightarrow z(\alpha, t)$ on $\mathbb{R} \times [0, T_0]$, as $\epsilon = \epsilon_j \to 0$. Moreover by (5.28),

$$z(\alpha', t) = z(\alpha', 0) + \int_0^t w(\alpha, s) \, ds,$$

so $w = z_t$. We denote $Z_t = z_t \circ h^{-1}$.

**Step 2. The limits of $\Psi^\epsilon_t$ and $F^\epsilon_t$.** Observe that by (5.30), for fixed $\epsilon > 0$, $\partial_t W^\epsilon(\cdot, t)$ is a bounded function on $\mathbb{R} \times [0, T_0]$, so by (5.31), $\Psi^\epsilon = K_{y'} \ast \partial_t W^\epsilon$ is bounded on $\mathcal{P}_- \times [0, T_0]$. However we will not use this to pass to the limit for $\Psi^\epsilon_t$, instead, we use (5.4).

By (5.4) and the above observation, since $\Psi^\epsilon_{z'}$ is bounded and holomorphic on $\mathcal{P}_-$,

$$\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}} = K_{y'} \ast \left( \frac{Z_t^\epsilon}{Z_{\alpha'}^\epsilon} - b^\epsilon \right).$$ (5.35)

By (5.21), (5.24) and Lemma 5.2, $\Psi^\epsilon_{z'}$ converges uniformly on compact subsets of $\mathcal{P}_- \times [0, T_0]$ to a function that is holomorphic on $\mathcal{P}_-$ for each $t \in [0, T_0]$ and continuous on $\mathcal{P}_- \times [0, T_0]$. By (5.32), (5.33), we can conclude that $\Psi^\epsilon_t \Rightarrow \Psi_t$ on $\mathcal{P}_- \times [0, T_0]$ (5.36)

as $\epsilon = \epsilon_j \to 0$.

Now we consider the limit of $F^\epsilon_t$ as $\epsilon = \epsilon_j \to 0$. Since for fixed $\epsilon > 0$, $\partial_t Z^\epsilon_t = Z^\epsilon_{tt} - b^\epsilon Z^\epsilon_{t, \alpha'}$ is in $L^\infty(\mathbb{R} \times [0, T_0])$, by (5.25),

$$F^\epsilon_t(z', t) = K_{y'} \ast \partial_t Z^\epsilon_t = K_{y'} \ast (\overline{Z}_{tt}^\epsilon - b^\epsilon \overline{Z}_{t, \alpha'}^\epsilon).$$ (5.37)

By Lemma 5.2, $K_{y'} \ast \overline{Z}_{tt}$ converges uniformly on compact subsets of $\mathcal{P}_- \times [0, T_0]$. With a change of variables

$$K_{y'} \ast (b^\epsilon \overline{Z}_{t, \alpha'}^\epsilon) = \frac{-1}{\pi} \int \frac{y'}{(x' - h^\epsilon(\alpha, t))^2 + y'^2} \circ h^\epsilon(\alpha, t) \overline{Z}_{t, \alpha'}(\alpha, t) \, d\alpha.$$ (5.38)

Because (5.21): $z^\epsilon_t \to z_t$, $z_{tt}^\epsilon \to z_{tt}$ uniform on compact subsets of $\mathbb{R} \times [0, T_0]$, and (5.14): $\sup_{[0, T_0]} \|\overline{Z}_{t, \alpha'}(t)\|_{L^2} \leq C(c_0, \mathcal{E}_1(0)), \sup_{[0, T_0]} \|\overline{Z}_{t, \alpha'}(t)\|_{L^2} \leq C(c_0, \mathcal{E}_1(0)), \overline{Z}_{t, \alpha'}, \overline{Z}_{tt} \alpha'$ exist in
$L^2(\mathbb{R})$ for each $t \in [0, T_0]$, with $\sup_{[0,T_0]} \| \overline{\zeta}_{t,\alpha}(t) \|_{L^2} \leq C(c_0, \mathcal{E}_1(0))$, $\sup_{[0,T_0]} \| \overline{\zeta}_{t,\alpha}(t) \|_{L^2} \leq C(c_0, \mathcal{E}_1(0))$, and by (5.21), (5.20) and $K_{y'} * (b' \overline{Z}_{t,\alpha'})$ converges point-wise on $P_- \times [0, T_0]$ to the continuous function

$$\frac{-1}{\pi} \int \frac{y'}{(x' - h(\alpha, t))^2 + y'^2} b \circ h(\alpha, t) \overline{\zeta}_{t,\alpha}(\alpha, t) \, d\alpha$$

as $\epsilon = \epsilon_j \to 0$ and by (5.14), (5.10),

$$\sup_{[0,T_0]} \| F'_{t}(z', t) \|_{L^{\infty}(\mathbb{R}, dx')} \leq (1 + \frac{1}{|y'|^{1/2}})C(c_0, \mathcal{E}_1(0)).$$

Therefore $F$ is continuously differentiable with respect to $t$, with $\sup_{[0,T_0]} \| F_t(z', t) \|_{L^{\infty}(\mathbb{R}, dx')} \leq (1 + \frac{1}{|y'|^{1/2}})C(c_0, \mathcal{E}_1(0))$ and

$$F'_t(z', t) \to F_t(z', t), \quad \text{as } \epsilon = \epsilon_j \to 0 \quad (5.39)$$

point-wise on $P_- \times [0, T_0]$.

**Step 3.** The limit of $\Psi'$. By the calculation in (2.8) we know $Z'_{\alpha}(\overline{Z}_{t,\alpha} - i)$ is the boundary value of the function $\Psi'_{x}F'_{t} - \Psi'_{t}F'_{x} + \overline{F}'_{t} \Psi'_{x} - i\Psi'_{x}$ on $\partial P_-$. Since $\Psi'_{x}F'_{t} - \Psi'_{t}F'_{x} - i\Psi'_{x}$ is holomorphic and $\overline{F}'_{t} \Psi'_{x} = \partial_{x'}(\overline{F}' F')$, where $\partial_{x'} = \frac{1}{2}(\partial_{x'} - i\partial_{y'})$, there is a real valued function $\Psi'$, such that

$$\Psi'_{x}F'_{t} - \Psi'_{t}F'_{x} - i\Psi'_{x} = -(\partial_{x'} - i\partial_{y'})(\Psi' + \frac{1}{2}|F'|^2), \quad \text{in } P_-; \quad (5.40)$$

and by $Z'_{\alpha}(\overline{Z}_{t,\alpha} - i) = iA'_1$, which is pure imaginary, we know

$$\Psi' = \text{constant}, \quad \text{on } \partial P_- \quad (5.41)$$

Without loss of generality we take the constant $= 0$. We now explore a few other properties of $\Psi'$. Moving $\overline{F}'_{t} \Psi'_{x} = \partial_{x'}(\overline{F}' F')$ to the right of (5.40) gives

$$\Psi'_{x}F'_{t} - \Psi'_{t}F'_{x} - i\Psi'_{x} = -(\partial_{x'} - i\partial_{y'})(\Psi' + \frac{1}{2}|F'|^2), \quad \text{in } P_-; \quad (5.42)$$

Applying $(\partial_{x'} + i\partial_{y'}) = 2i\mathcal{L}_{x'}$ to (5.42) yields

$$- \Delta(\Psi' + \frac{1}{2}|F'|^2) = 0, \quad \text{in } P_- . \quad (5.43)$$

So $\Psi' + \frac{1}{2}|F'|^2$ is a harmonic function on $P_-$ with boundary value $\frac{1}{2}|\overline{Z}_{t,\alpha}|^2$. On the other hand, it is easy to check that $\lim_{y' \to -\infty} (\Psi'_{x} F'_{t} - \Psi'_{t} F'_{x} - i\Psi'_x) = -i$. Therefore

$$\Psi'(z', t) = -\frac{1}{2}|F'(z', t)|^2 - y + \frac{1}{2}K_{y'} * (|\mathcal{Z}'_t|^2)(x', t) . \quad (5.44)$$

By (5.20), (5.21) and Lemma 5.2

$$\Psi'(z', t) \Rightarrow -\frac{1}{2}|F'(z', t)|^2 - y + \frac{1}{2}K_{y'} * (|\mathcal{Z}'_t|^2)(x', t), \quad \text{on } P_- \times [0, T_0] \quad (5.45)$$

as $\epsilon = \epsilon_j \to 0$. We write

$$\Psi := -\frac{1}{2}|F'(z', t)|^2 - y + \frac{1}{2}K_{y'} * (|\mathcal{Z}'_t|^2)(x', t).$$

$\Psi$ is continuous on $P_- \times [0, T_0]$ with $\Psi \in C([0, T_0], C^\infty(P_-))$, and

$$\Psi = 0, \quad \text{on } \partial P_- . \quad (5.46)$$

Moreover, since $K_{y'} * (|\mathcal{Z}'_t|^2)(x', t)$ is harmonic on $P_-$, by interior derivative estimate for harmonic functions and by (5.20),

$$(\partial_{x'} - i\partial_{y'}) \Psi' \Rightarrow (\partial_{x'} - i\partial_{y'}) \Psi \quad \text{on } P_- \times [0, T_0] \quad (5.47)$$

as $\epsilon = \epsilon_j \to 0$. 


Step 4. Conclusion. We now sum up Steps 1-3. We have shown that there are functions $\Psi(\cdot, t)$ and $F(\cdot, t)$, holomorphic on $P_-$ for each fixed $t \in [0, T_0]$, continuous on $\overline{P}_- \times [0, T_0]$, and continuous differentiable on $P_- \times [0, T_0]$, with $\frac{1}{|\Psi|}$ continuous on $\overline{P}_- \times [0, T_0]$, such that $\Psi^\epsilon \to \Psi$, $\frac{1}{\Psi^\epsilon} \to \frac{1}{\Psi}$, $F^\epsilon \to F$ uniform on compact subsets of $\overline{P}_- \times [0, T_0]$, $\Psi^\epsilon \to \Psi_t$, $\Psi^\epsilon_z \to \Psi_z$, $F^\epsilon_z \to F_z$ uniform on compact subsets of $P_- \times [0, T_0]$, and $F^\epsilon_t \to F_t$ pointwise on $P_- \times [0, T_0]$, as $\epsilon = \epsilon_j \to 0$. We have also shown there is $\Psi$, continuous on $\overline{P}_- \times [0, T_0]$ with $\Psi = 0$ on $\partial P_-$ and $(\partial_x - i\partial_y)\Psi$ continuous on $P_- \times [0, T_0]$, such that $(\partial_x - i\partial_y)\Psi^\epsilon \to (\partial_x - i\partial_y)\Psi$ uniformly on compact subsets of $P_- \times [0, T_0]$, as $\epsilon = \epsilon_j \to 0$. Let $\epsilon = \epsilon_j \to 0$ in equation (5.40), we have

$$\Psi^\epsilon_z F^\epsilon_t - \Psi_t F^\epsilon_z + \overline{F} F^\epsilon_z - i\Psi^\epsilon_z = -(\partial_x - i\partial_y)\Psi,$$  

on $P_- \times [0, T_0].$ (5.48)

This shows $\Psi$, $F$ is a generalized solution of the water wave equation in the sense given in (2.3). Furthermore because of (5.2), (5.3), letting $\epsilon = \epsilon_j \to 0$ gives

$$\sup_{[0, T_0]} \mathcal{E}_1(t) \leq M(\mathcal{E}_1(0)) < \infty.$$  

(5.49)

and

$$\sup_{[0, T_0]} \{ \sup_{y' < 0} \| F(x' + iy', t) \|_{L^2(\mathbb{R}, dx')} + \sup_{y' < 0} \| \frac{1}{\Psi^\epsilon_z(x' + iy', t)} - 1 \|_{L^2(\mathbb{R}, dx')} \} < C_0 < \infty.$$  

(5.50)

5.6. The invertibility of $\Psi(\cdot, t)$. If in addition $\Sigma(t) = \{ Z = \Psi(\alpha', t) := Z(\alpha', t) \mid \alpha' \in \mathbb{R} \}$ is a Jordan curve, then because $\lim_{|\alpha'| \to \infty} \Psi_{\alpha'}(\alpha', t) = 1$ the domain $\Omega(t)$ bounded above by $\Sigma(t)$ is winded by $\Psi(\cdot, t)$ exactly once. By the argument principle, $\psi(\cdot, t) : \overline{P}_- \to \Omega(t)$ is one-to-one and onto, $\Psi^{-1}(\cdot, t) : \Omega(t) \to P_-$ exists and is holomorphic. By the chain rule, it is easy to check (5.48) is equivalent to

$$(F \circ \Psi^{-1})_t + \overline{F} \circ \Psi^{-1} (F \circ \Psi^{-1})_z - i = - (\partial_x - i\partial_y)(\Psi \circ \Psi^{-1}),$$  

on $\Omega(t).$ (5.51)

This is the Euler equation, the first equation of (1.1) in complex form. Let $\nu = F \circ \Psi^{-1}$, $P = \Psi \circ \Psi^{-1}$. Then $(\nu, P)$ is a solution of the water wave equation (1.1) in $\Omega(t)$, with fluid interface $\Sigma(t) : Z = Z(\alpha', t), \alpha' \in \mathbb{R}$.

5.7. The chord-arc interfaces. Now assume at time $t = 0$, the interface $Z = \Psi(\alpha', 0) := Z(\alpha', 0), \alpha' \in \mathbb{R}$ is chord-arc, that is, there is $0 < \delta < 1$, such that

$$\delta \int_{\alpha'}^{\beta'} |Z_{\alpha'}(\gamma, 0)| d\gamma \leq |Z(\alpha', 0) - Z(\beta', 0)| \leq \int_{\alpha'}^{\beta'} |Z_{\alpha'}(\gamma, 0)| d\gamma, \quad \forall - \infty < \alpha' < \beta' < \infty.$$

We want to show there is $T_1 > 0$, depending only on $\mathcal{E}_1(0)$, such that for $t \in [0, \min\{T_0, T_1\}]$, the interface $Z = Z(\alpha', t) := \Psi(\alpha', t)$ remains chord-arc. We begin with

$$- z^\epsilon(\alpha, t) + z^\epsilon(\beta, t) + z^\epsilon(\alpha, 0) - z^\epsilon(\beta, 0) = \int_0^t \int_{\alpha}^{\beta} z^\epsilon_{\alpha}(\gamma, s) d\gamma ds$$  

(5.52)

for $\alpha < \beta$. Because

$$\frac{d}{dt}|z^\epsilon_{\alpha}(t)|^2 = 2|z^\epsilon_{\alpha}(t)|^2 \text{Re} \, D_{\alpha} z^\epsilon_{\alpha}$$  

by Gronwall, for $t \in [0, T_0]$,

$$|z^\epsilon_{\alpha}(t)|^2 \leq |z^\epsilon_{\alpha}(0)|^2 e^{2 \int_0^t |D_{\alpha} z^\epsilon_{\alpha}(\alpha, \tau)| d\tau};$$  

so

$$|z^\epsilon_{\alpha}(\alpha, t)| \leq |z^\epsilon_{\alpha}(0)| |D_{\alpha} z^\epsilon_{\alpha}(\alpha, t)| e^{\int_0^t |D_{\alpha} z^\epsilon_{\alpha}(\alpha, \tau)| d\tau};$$  

(5.54)

by Appendix [5.2] and Proposition [2.5]

$$\sup_{[0, T_0]} |z^\epsilon_{\alpha}(\alpha, t)| \leq |z^\epsilon_{\alpha}(0)| C(\mathcal{E}_1(0)).$$  

(5.55)

23By a similar argument as in [5.3]
therefore for \( t \in [0, T_0] \),
\[
\int_0^t \int_\alpha^\beta |z'_\alpha(\gamma, s)| \, d\gamma \, ds \leq tC(|\mathcal{E}_1(0)|) \int_\alpha^\beta |z'_\alpha(\gamma, 0)| \, d\gamma
\]  
(5.57)

Now \( z'(\alpha, 0) = Z'(\alpha, 0) = \Psi(\alpha - ei, 0) \). Because \( Z_{,\alpha'}(\cdot, 0) \in L^1_{\text{loc}}(\mathbb{R}) \), and \( Z_{,\alpha'}(\cdot, 0) - 1 \in H^1(\mathbb{R} \setminus [-N, N]) \) for some large \( N \),
\[
\lim_{\varepsilon \to 0} \int_\alpha^\beta |\Psi(z' - ei, 0)| \, d\gamma \leq \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma
\]  
(5.58)

Let \( \varepsilon = \varepsilon_j \to 0 \) in (5.52). We get, for \( t \in [0, T_0] \),
\[
||z(\alpha, t) - z(\beta, t)| - |Z(\alpha, 0) - Z(\beta, 0)|| \leq tC(|\mathcal{E}_1(0)|) \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma
\]  
(5.59)

hence for all \( \alpha < \beta \) and \( 0 \leq t \leq \min\{T_0, \frac{\delta}{2C(|\mathcal{E}_1(0)|)}\} \),
\[
\frac{1}{2} \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma \leq |z(\alpha, t) - z(\beta, t)| \leq 2 \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma
\]  
(5.60)

This show that for \( 0 \leq t \leq \min\{T_0, \frac{\delta}{2C(|\mathcal{E}_1(0)|)}\} \), \( z = z(\cdot, t) \) is absolute continuous on compact intervals of \( \mathbb{R} \), with \( z_{,\alpha}(\cdot, t) \in L^1_{\text{loc}}(\mathbb{R}) \), and is chord-arc. So \( \Sigma(t) = \{z(\alpha, t) \mid \alpha \in \mathbb{R}\} \) is Jordan. This finishes the proof of Theorem 3.4.

**Appendix A. Basic analysis preparations**

Let \( \Omega \subset \mathbb{C} \) be a domain with boundary \( \Sigma : z = z(\alpha), \alpha \in I, \) oriented clockwise. Let \( \mathcal{H} \) be the Hilbert transform associated to \( \Omega \):
\[
\mathcal{H} f(\alpha) = \frac{1}{\pi i} \text{pv.} \int \frac{z_{,\beta}(\beta)}{z(\alpha) - z(\beta)} f(\beta) \, d\beta
\]  
(A.1)

We have the following characterization of the trace of a holomorphic function on \( \Omega \).

**Proposition A.1.** \([19]\)  
\(a\). Let \( g \in L^p \) for some \( 1 < p < \infty \). Then \( g \) is the boundary value of a holomorphic function \( G \) on \( \Omega \) with \( G(z) \to 0 \) at infinity if and only if
\[
(I - \mathcal{H})g = 0.
\]  
(A.2)

\(b\). Let \( f \in L^p \) for some \( 1 < p < \infty \). Then \( \mathcal{P}_H f := \frac{1}{2}(I + \mathcal{H})f \) is the boundary value of a holomorphic function \( \mathcal{G} \) on \( \Omega \), with \( \mathcal{G}(z) \to 0 \) as \( |z| \to \infty \).

\(c\). \( \mathcal{H}1 = 0 \).

Observe that Proposition \([A.1]\) gives \( \mathcal{H}^2 = I \) in \( L^p \).

We next present the basic estimates we will rely on for this paper. We start with the Sobolev inequality.

**Proposition A.2** (Sobolev inequality). Let \( f \in C^1_0(\mathbb{R}) \). Then
\[
\|f\|_{L^\infty}^2 \leq 2\|f\|_{L^2} \|f'\|_{L^2}
\]  
(A.3)

**Proposition A.3** (Hardy’s Inequality). Let \( f \in C^1(\mathbb{R}) \), with \( f' \in L^2(\mathbb{R}) \). Then there exists \( C > 0 \) independent of \( f \) such that for any \( x \in \mathbb{R} \),
\[
\left| \int \frac{(f(x) - f(y))^2}{(x - y)^2} \, dy \right| \leq C \|f'\|_{L^2}^2.
\]  
(A.4)

Let
\[
\mathbb{H} f(x) = \frac{1}{\pi i} \text{pv.} \int \frac{1}{x - y} f(y) \, dy.
\]
be the Hilbert transform associated with $P_\omega$. Let $f : \mathbb{R} \to \mathbb{C}$ be a function in $\dot{H}^{1/2}$, we note that
\[
\| f \|^2_{\dot{H}^{1/2}} = \int i\mathbb{H}\partial_x f(x)\overline{T}(x)\, dx = \frac{1}{2\pi} \iint \frac{|f(x) - f(y)|^2}{(x-y)^2} \, dx \, dy. \quad \text{(A.5)}
\]
We have the following result on $\dot{H}^{1/2}$ functions.

**Proposition A.4.** Let $f, g \in C^1(\mathbb{R})$. Then
\[
\| g \|_{\dot{H}^{1/2}} \lesssim \| f^{-1}\|_{L^\infty}(\| fg \|_{\dot{H}^{1/2}} + \| f'\|_{L^2} \| g \|_{L^2}). \quad \text{(A.6)}
\]

The proof is straightforward from the definition of $\dot{H}^{1/2}$ and the Hardy’s inequality. We omit the details.

Let $A_i \in C^1(\mathbb{R})$, $i = 1, \ldots , m$. Define
\[
C_1(A_1, \ldots , A_m, f)(x) = \text{pv.} \int \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x-y)^{m+1}} f(y) \, dy. \quad \text{(A.7)}
\]

**Proposition A.5.** There exist constants $c_1 > 0$, $c_2 > 0$, such that
\begin{enumerate}
\item For any $f \in L^2$, $A'_i \in L^\infty$, $1 \leq i \leq m$,
\[
\| C_1(A_1, \ldots , A_m, f) \|_{L^2} \leq c_1 \| A'_1 \|_{L^\infty} \ldots \| A'_m \|_{L^\infty} \| f \|_{L^2}. \quad \text{(A.8)}
\]
\item For any $f \in L^\infty$, $A'_i \in L^\infty$, $2 \leq i \leq m$, $A'_i \in L^2$,
\[
\| C_1(A_1, \ldots , A_m, f) \|_{L^2} \leq c_1 \| A'_1 \|_{L^2} \| A'_2 \|_{L^\infty} \ldots \| A'_m \|_{L^\infty} \| f \|_{L^\infty}. \quad \text{(A.9)}
\]
\end{enumerate}

(\textit{A.8}) is a result of Coifman, McIntosh and Meyer [10]. (\textit{A.9}) is a consequence of the Tb Theorem, a proof is given in [32].

Let $A_i$ satisfies the same assumptions as in (A.7). Define
\[
C_2(A, f)(x) = \int \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x-y)^m} \partial_y f(y) \, dy. \quad \text{(A.10)}
\]
We have the following inequalities.

**Proposition A.6.** There exist constants $c_3$, $c_4$ and $c_5$, such that
\begin{enumerate}
\item For any $f \in L^2$, $A'_i \in L^\infty$, $1 \leq i \leq m$,
\[
\| C_2(A, f) \|_{L^2} \leq c_3 \| A'_1 \|_{L^\infty} \ldots \| A'_m \|_{L^\infty} \| f \|_{L^2}. \quad \text{(A.11)}
\]
\item For any $f \in L^\infty$, $A'_i \in L^\infty$, $2 \leq i \leq m$, $A'_i \in L^2$,
\[
\| C_2(A, f) \|_{L^2} \leq c_4 \| A'_1 \|_{L^2} \| A'_2 \|_{L^\infty} \ldots \| A'_m \|_{L^\infty} \| f \|_{L^\infty}. \quad \text{(A.12)}
\]
\item For any $f \in L^2$, $A'_i \in L^\infty$, $1 \leq i \leq m$,
\[
\| C_2(A, f) \|_{L^2} \leq c_5 \| A'_1 \|_{L^\infty} \| A'_2 \|_{L^\infty} \ldots \| A'_m \|_{L^\infty} \| f' \|_{L^2}. \quad \text{(A.13)}
\]
\end{enumerate}

Using integration by parts, the operator $C_2(A, f)$ can be easily converted into a sum of operators of the form $C_1(A, f)$. (\textit{A.11}) and (\textit{A.12}) follow from (\textit{A.8}) and (\textit{A.9}). To get (\textit{A.13}), we rewrite $C_2(A, f)$ as the difference of the two terms $A_1 C_1(A_2, \ldots , A_m, f')$ and $C_1(A_2, \ldots , A_m, A_1 f')$ and apply (\textit{A.8}) to each term.

**Proposition A.7.** There exists a constant $C > 0$ such that for any $f, g \in C^1(\mathbb{R})$ with $f' \in L^2$ and $g' \in L^2$,
\begin{align*}
\| f \ast \mathbb{H} g \|_{L^2} & \leq C \| f \|_{\dot{H}^{1/2}} \| g \|_{L^2} \quad \text{(A.14)}
\| f \ast \mathbb{H} \partial_x g \|_{L^2} & \leq C \| f' \|_{L^2} \| g \|_{\dot{H}^{1/2}} \quad \text{(A.15)}
\end{align*}
(\textit{A.14}) is straightforward by Cauchy-Schwarz and the definition of $\dot{H}^{1/2}$. (\textit{A.15}) follows from integration by parts, then Cauchy-Schwarz, Hardy’s inequality, the definition of $\dot{H}^{1/2}$ and (\textit{A.13}).

Recall $[f, g; h]$ as given in (2.4).
Appendix B. Identities

B.1. Basic identities. Here we derive a few basic identities from the system \([2.9]-(2.8)\) without assuming \(Z = Z(\cdot, t)\) being non-self-intersecting. These identities provide an alternative way of deriving the quasi-linearization of the system \([2.9]-(2.8)\) in this more general context, they also show that the argument in \([21]\) can be modified, so that the a priori estimate of \([21]\) and the characterization of the energy in \(\S 10\) of \([21]\) hold for solutions of the system \([2.9]-(2.8)\) without the non-self-intersecting requirement.

Let \(Z = Z(\cdot, t)\) be sufficiently regular\(^{24}\) and satisfy \([2.9]-(2.8)\):

\[
\begin{aligned}
Z_{tt} + i = iAZ_{\alpha' t},
\{Z_t = \mathbb{H}Z_t, \\
Z_{\alpha' t} - 1 = \mathbb{H}(Z_{\alpha' t} - 1), \quad \frac{1}{Z_{\alpha'}} - 1 = \mathbb{H}(\frac{1}{Z_{\alpha'}} - 1);
\end{aligned}
\]

where \(Z\) and \(Z_t\) are related through \([2.3]-(2.7)\):

\[
z(\alpha, t) = Z(h(\alpha, t), t), \quad z_t(\alpha, t) = Z_t(h(\alpha, t), t) \tag{B.2}
\]

for some (sufficiently regular) homeomorphism \(h(\cdot, t) : \mathbb{R} \to \mathbb{R}\). Let \(ah_\alpha := A \circ h, A_1 := A|Z_{\alpha'}|^2\). Precomposing the first equation of \([B.1]\) with \(h\) gives \([B.3]\):

\[
z_{tt} + i = i\alpha z_\alpha \tag{B.3}
\]

We first show that \([2.18]\) can be derived from \([B.1]\) and \([B.2]\). Let \(\Psi\) be a holomorphic function on \(P_-\), continuously differentiable on \(\overline{P_-}\), such that

\[
\Psi(\alpha', t) = Z(\alpha', t), \quad \Psi_z(\alpha', t) = Z_{\alpha'}(\alpha', t).
\]

Therefore \(z(\alpha, t) = \Psi(h(\alpha, t), t)\) and by the chain rule, \(z_t = \Psi_t \circ h + h_t \Psi_z \circ h\). Precomposing with \(h^{-1}\) then gives

\[
Z_t = \Psi_t + Z_{\alpha'} h_t \circ h^{-1};
\]

dividing by \(Z_{\alpha'}\) yields

\[
h_t \circ h^{-1}(\alpha', t) = \frac{Z_t(\alpha', t)}{Z_{\alpha'}(\alpha', t)} - \frac{\Psi_t}{\Psi_z}(\alpha', t). \tag{B.4}
\]

\(^{24}\)Here we do not specify what precisely "sufficiently regular" means, but assume it is enough so that the calculations make sense.
Notice that $\frac{\partial}{\partial t}$ is a holomorphic function on $P_-$. By Proposition A.1, applying $(I - \mathbb{H})$ to (B.4) then taking the real parts and using the second and third equations of (B.1) to rewrite into the commutator gives (2.18). Conversely, if $h$ satisfies (2.18) for a function $Z$ satisfying the second and third equations of (B.1), then expanding the commutator yields
\begin{equation}
 h_t \circ h^{-1} = \text{Re}(I - \mathbb{H})(\frac{Z_t}{Z_{\alpha'}}) = \frac{Z_t}{2Z_{\alpha'}} + \frac{1}{2}(I + \mathbb{H})(\frac{Z_t}{Z_{\alpha'}}) - \frac{Z_t}{Z_{\alpha'}}. \tag{B.5}
\end{equation}
By Proposition A.1, $\frac{1}{2}(I + \mathbb{H})(\frac{Z_t}{Z_{\alpha'}} - \frac{Z_t}{Z_{\alpha''}})$ is the boundary value of a holomorphic function on $P_-$, tending to zero at the spatial infinity.

In what follows we use the following notations. We write $U_1 \equiv U_2$, if $(I - \mathbb{H})(U_1 - U_2) = 0$; that is if $U_1 - U_2$ is the boundary value of a holomorphic function on $P_-$ that tends to zero at infinity.

Assume $Z$ satisfies the second and third equations of (B.1) and $h$ satisfies (2.18), so (B.5) holds.

**Proposition B.1.** Let $U(\cdot, t) : \mathbb{R} \to \mathbb{C}$ be sufficiently regular, and $u = U \circ h$. Assume $U \equiv 0$. We have 1.
\begin{equation}
 u_t \circ h^{-1} \equiv Z_t D_{\alpha} U; \tag{B.6}
\end{equation}
2.
\begin{equation}
 u_{tt} \circ h^{-1} \equiv Z_{tt} D_{\alpha} U + 2Z_t D_{\alpha'}(u_t \circ h^{-1} - Z_t D_{\alpha'} U) + Z_t^2 D_{\alpha'}^2 U. \tag{B.7}
\end{equation}
3.
\begin{equation}
 U_h^{-1}(u_{tt} + i\alpha \partial_{\alpha} u) \equiv 2Z_{tt} D_{\alpha} U + 2Z_t D_{\alpha'}(u_t \circ h^{-1} - Z_t D_{\alpha'} U) + Z_t^2 D_{\alpha'}^2 U. \tag{B.8}
\end{equation}

**Proof.** Applying the chain rule to $u = U \circ h$ and precompose with $h^{-1}$ gives
\begin{equation}
 u_t \circ h^{-1} = \partial_t U + \partial_{\alpha} U h_t \circ h^{-1}. \tag{B.9}
\end{equation}
Observe that $U \equiv 0$ gives $\partial_t U \equiv 0$ and $\partial_{\alpha} U \equiv 0$. (B.6) follows from (B.5) and the fact that product of holomorphic functions is holomorphic.

Now we apply (B.6) to $u_t \circ h^{-1} - Z_t D_{\alpha} U$. This gives
\begin{equation}
 U_h^{-1}(u_t - Z_t D_{\alpha} u) \equiv Z_t D_{\alpha'}(u_t \circ h^{-1} - Z_t D_{\alpha'} U). \tag{B.10}
\end{equation}
Expanding the left hand side by the product rule, and observe that $\partial_t D_{\alpha} u = D_{\alpha}(u_t - z_t D_{\alpha} u) + z_t D_{\alpha}^2 u$, so
\begin{equation}
 \partial_t (u_t - z_t D_{\alpha} u) = u_{tt} - z_t D_{\alpha} u - z_t \partial_t D_{\alpha} u = u_{tt} - z_t D_{\alpha} u - z_t D_{\alpha}(u_t - z_t D_{\alpha} u) - z_t^2 D_{\alpha}^2 u. \tag{B.11}
\end{equation}
Precomposing with $h^{-1}$ and substituting in (B.10) gives (B.7). (B.8) follows from (B.7) and the fact that $i\alpha \partial_{\alpha} u = (z_t^\alpha + i) D_{\alpha} u$ and $D_{\alpha} U \equiv 0$.

Now assume $Z$ satisfies (B.1).\footnote{Here $Z = Z(\cdot, t)$ need not be non-self-intersecting.} Applying (B.6) to $\overline{Z_t}$ gives $\overline{Z_{\alpha}} \equiv Z_t D_{\alpha'} \overline{Z_t}$. Following the rest of the argument in section 2.2.1 of [35] gives (2.19). Similarly, applying (B.8) to $\overline{Z_t}$ and following the rest of the argument in section 2.2.3 of [35] gives
\begin{equation}
 \frac{a_t}{a} \circ h^{-1} = -\text{Im}(2[Z_t, \mathbb{H}] Z_{ttt, \alpha'} + 2[Z_{ttt}, \mathbb{H}] \partial_{\alpha'} Z_t - [Z_t, Z_{tt}; D_{\alpha'} Z_t]) \tag{B.10}
\end{equation}
where
\begin{equation}
 [Z_t, Z_{tt}; D_{\alpha'} Z_t] := \frac{1}{\pi t} \int \frac{(Z_t(\alpha', t) - Z_t(\beta', t))^2}{(\alpha' - \beta')^2} D_{\beta'} Z_t(\beta', t) d\beta'. \tag{B.11}
\end{equation}
For the periodic case studied in [21], the same computations above and Proposition B.1 hold, and the corresponding equations for (2.18), (2.19), (B.10) can be derived without
the non-self-intersecting assumption. The periodic version of Proposition \[B.1\] shows that the argument in \[21\] can be modified so that the a priori estimate, Theorem 2 of \[21\] and the characterization of the energy in §10 of \[21\] hold more generally without the non-self-intersecting assumption. Proposition \[B.1\] and a small modification of the argument in \[21\] show that a similar a priori estimate and a similar characterization of the energy as in \[21\] hold in the whole line case for solutions of (2.9)–(2.8).

**B.2. Commutator identities.** We include here for reference the various commutator identities that are necessary. The first set: \[B.12\]–\[B.15\] has already appeared in \[21\].

\[
\begin{align*}
[\partial_t, D_\alpha] &= -(D_\alpha z_t)D_\alpha; \\
[\partial_t, D_\alpha^2] &= -2(D_\alpha z_t)D_\alpha^2 - (D_\alpha^2 z_t)D_\alpha; \\
[\partial_t^2, D_\alpha] &= -(D_\alpha z_{tt})D_\alpha + 2(D_\alpha z_t)D_\alpha - 2(D_\alpha z_t)D_\alpha \partial_t; \\
[\partial_t^2 + ia \partial_\alpha, D_\alpha] &= -(2D_\alpha z_t)D_\alpha - 2(D_\alpha z_t)\partial_t D_\alpha.
\end{align*}
\]

We need some additional commutator identities. In general for operators \(A, B\) and \(C\),

\[
[A, BC^k] = [A, B]C^k + B[A, C^k] = [A, B]C^k + \sum_{i=1}^k BC^{i-1} [A, C]C^{k-i}.
\]

We note that for \(f = f(t, \alpha, U_{\alpha-1} f = \frac{\partial_\alpha}{h_{\alpha}} f\). So

\[
[\partial_t, \frac{\partial_\alpha}{h_{\alpha}}] f = -\frac{h_{\alpha}}{h_{\alpha}} \frac{\partial_\alpha}{h_{\alpha}} f = -U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha, U_{\alpha-1} f \};
\]

\[
[U_{\alpha-1} \partial_t U_h, \partial_\alpha] g = U_{\alpha-1} \{ \partial_t \frac{\partial_\alpha}{h_{\alpha}} U_h g = -(h_t \circ h^{-1})_\alpha \partial_\alpha g.
\]

Applying \[B.13\] yields

\[
\begin{align*}
\left[ \partial_t, \left( \frac{\partial_\alpha}{h_{\alpha}} \right)^2 \right] f &= \frac{\partial_\alpha}{h_{\alpha}} \{ \partial_t, \frac{\partial_\alpha}{h_{\alpha}} \} f + \frac{\partial_\alpha^2}{h_{\alpha}^2} f \\
&= -2U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha^2, U_{\alpha-1} f \} - U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha, U_{\alpha-1} f \};
\end{align*}
\]

\[
\begin{align*}
\left[ \partial_t^2, \frac{\partial_\alpha}{h_{\alpha}} \right] f &= \partial_t \{ \partial_t, \frac{\partial_\alpha}{h_{\alpha}} \} f + \frac{\partial_\alpha}{h_{\alpha}} \partial_t f \\
&= -\partial_t U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha, U_{\alpha-1} f \} - U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha, U_{\alpha-1} f \}.
\end{align*}
\]

To calculate \([i \alpha \partial_\alpha, \frac{\partial_\alpha}{h_{\alpha}}] f\), we use the definition \(A \circ h := ah\), and \(i \alpha \partial_\alpha := iA \circ h \frac{\partial_\alpha}{h_{\alpha}}\). We have

\[
[i \alpha \partial_\alpha, \frac{\partial_\alpha}{h_{\alpha}}] f = [iA \circ h \frac{\partial_\alpha}{h_{\alpha}}, \frac{\partial_\alpha}{h_{\alpha}}] f = -iU_h \{ A_\alpha \circ A_\alpha, U_{\alpha-1} f \}.
\]

Adding \[B.13\] and \[B.14\], we conclude that

\[
\left[ \partial_t^2 + i \alpha \partial_\alpha, \frac{\partial_\alpha}{h_{\alpha}} \right] f = -\partial_t U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha U_{\alpha-1} f \} - U_h \{ (h_t \circ h^{-1})_\alpha \partial_\alpha U_{\alpha-1} f \} = -iU_h \{ A_\alpha \circ A_\alpha, U_{\alpha-1} f \}.
\]

We note that \(U_{\alpha-1} \partial_t U_h = \partial_t + b \partial_\alpha\) where \(b := h_t \circ h^{-1}\). Therefore

\[
[U_{\alpha-1} \partial_t U_h, \mathbb{H}] = [h_t \circ h^{-1}, \mathbb{H}] \partial_\alpha.
\]

A straightforward differentiation gives

\[
U_{\alpha-1} \partial_t U_h[f, \mathbb{H}] g = [U_{\alpha-1} \partial_t U_h f, \mathbb{H}] g
\]

\[
+ [f, \mathbb{H}] (U_{\alpha-1} \partial_t U_h g + (h_t \circ h^{-1})_\alpha g) - [f, h_t \circ h^{-1}; g];
\]
with an application of \((B.18)\) yields
\[
U_h^{-1} \partial_t U_h[f, \mathbb{H}] \partial_{\alpha'} g = [U_h^{-1} \partial_t U_h f, \mathbb{H}] \partial_{\alpha'} g \\
+ [f, \mathbb{H}] \partial_{\alpha'} U_h^{-1} \partial_t U_h g - [f, h_t \circ h^{-1}; \partial_{\alpha'} g].
\] (B.25)

The following commutators are straightforward from the product rule. We have
\[
[Z, \alpha'; U_h^{-1} \partial_t U_h] f = [U_h^{-1} \frac{z_\alpha}{h_\alpha}, U_h^{-1} \partial_t U_h] f \\
= -\{U_h^{-1} \partial_t \left( \frac{z_\alpha}{h_\alpha} \right) \} f = -Z, \alpha' (D_\alpha' Z_t - (h_t \circ h^{-1})_{\alpha'} f); \]
(B.26)
\[
[\partial_t, \frac{h_\alpha}{z_\alpha}] f = \partial_t \left( \frac{h_\alpha}{z_\alpha} \right) f = \frac{h_\alpha}{z_\alpha} (U_h(h_t \circ h^{-1})_{\alpha'} - D_\alpha z_t) f;
\] (B.27)
by \(iaz_\alpha = z_t + i\).
\[
[ia\partial_t, \frac{h_\alpha}{z_\alpha}] f = [(z_t + i)D_\alpha \frac{h_\alpha}{z_\alpha}] f = (z_t + i)D_\alpha \frac{h_\alpha}{z_\alpha} f;
\] (B.28)
by \((B.16), (B.27), (B.28)\) and the product rule,
\[
[\partial_t^2 + ia\partial_t, \frac{h_\alpha}{z_\alpha}] f = 2\frac{h_\alpha}{z_\alpha} (U_h(h_t \circ h^{-1})_{\alpha'} - D_\alpha z_t) f_t + \frac{h_\alpha}{z_\alpha} (U_h(h_t \circ h^{-1})_{\alpha'} - D_\alpha z_t)^2 f \\
+ \frac{h_\alpha}{z_\alpha} (\partial_t U_h(h_t \circ h^{-1})_{\alpha'} - \partial_t D_\alpha z_t) f + (z_t + i)D_\alpha \frac{h_\alpha}{z_\alpha} f.
\] (B.29)

**Appendix C. Main quantities controlled by \(\mathcal{E}\)**

We list here the various quantities that we have shown in [21] are controlled by polypro- 
minals of \(\mathcal{E}(t)\). \[26\]
\[
\|D_\alpha^2 \overline{Z}_{tt}\|_{L^2}, \|D_\alpha^2 Z_{tt}\|_{L^2}, \|D_\alpha^2 \overline{Z}_{t}\|_{L^2}, \|D_\alpha^2 Z_{t}\|_{L^2}, \|D_\alpha \partial_t D_\alpha \overline{Z}_{tt}\|_{L^2(h, dt)}, \\
\|\frac{1}{Z, \alpha'} D_\alpha^2 \overline{Z}_{tt}\|_{H^{1/2}}, \|D_\alpha \overline{Z}_{tt}\|_{L^\infty}, \|D_\alpha Z_{tt}\|_{L^\infty}, \|D_\alpha Z_{t}\|_{L^\infty}, \|D_\alpha Z_t\|_{L^\infty}, \\
\|Z_{tt, \alpha'}\|_{L^2}, \|\overline{Z}_{tt, \alpha'}\|_{L^2}, \int |D_\alpha \overline{Z}_{tt}|^2 \frac{d\alpha}{a}, \int |D_\alpha Z_{tt}|^2 \frac{d\alpha}{a}, \frac{1}{\|Z, \alpha'\|_{L^\infty}}, \|Z_{tt} + i\|_{L^\infty}, \|A_1\|_{L^\infty};
\] (C.1)
\[
\cdot \frac{a_\alpha}{a} \|_{L^\infty} = \frac{a_\alpha}{a} \|_{L^\infty};
\]
\[
\frac{\partial_{\alpha'} \overline{Z}_{tt}}{Z, \alpha'} \|_{L^2};
\]
\[
\frac{h_\alpha}{h_{\infty}} \|_{L^\infty};
\]
\[
\frac{1}{(I + \mathcal{H})} D_\alpha Z_t \|_{L^\infty};
\]
\[
\frac{1}{Z, \alpha'} D_\alpha Z_t \|_{L^\infty};
\]
\[
\frac{1}{Z, \alpha'} \overline{Z}_{tt} \|_{L^\infty};
\]
\[
\frac{1}{Z, \alpha'} \overline{Z}_{tt} \|_{L^\infty};
\]
In addition from (179), (186) of [21],
\[
\|D_\alpha (h_t \circ h^{-1})_{\alpha'}\|_{L^2} \lesssim C(\mathcal{E}).
\]

\[26\] The same proof for the symmetric periodic setting in [21] applies to the whole line setting. We leave it to the reader to check the details.
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