Distinguishability of the symmetric states

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Abstract

In this paper, the distinguishability of multipartite geometrically uniform quantum states obtained from a single reference state is studied in the symmetric subspace. We specially focus our attention on the unitary transformation in a way that the produced states remain in the symmetric subspace, so rotation group with $J_y$ as the generator of rotation is applied. The optimal probability and measurements are obtained for the pure and some special mixed separable states and the results are compared with those obtained at the previous articles for the special cases. The results are valid for linearly dependent states. The discrimination of these states is also investigated using the

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separable measurement. We introduce appropriate transformation to gain the optimal separable measurements equivalent to the optimal global measurements with the same optimal probability.
1 INTRODUCTION

Discrimination of nonorthogonal quantum states is a fundamental and important problem in quantum information theory. In distinguishing a quantum state that belongs to the set of known quantum states with given prior probabilities, one possibility is to find a set of positive operator valued measure (POVM) that maximizes the probability of correct detection, which is called minimum error discrimination [1, 2, 3]. In the 1970s, necessary and sufficient conditions for an optimum measurement have been derived by Holevo, Helstrom, and Yuen et al. [4, 5, 6]. However, solving problems by means of them, except for some particular cases, is a difficult task. Jafarizadeh et al. [1] presented optimality conditions, by using Helstrom family of ensembles, which is not only powerful in solving problems but also easy to apply. Accordingly, using this technique, we obtain the optimal measurements.

In many discrimination problems, considerable attention is paid to use local quantum operations and classical communication between the components (LOCC) [2, 3, 7]. However, using LOCC does not have a simple mathematical structure to give analytical optimization, and for some cases can’t achieve the optimal probability which obtained by global measurements; therefore, obtained information reduce [7, 3]. In order to partially overcome the defects, some researchers began to solve alternative problem by separable operation, to investigate the local distinguishability [8, 9, 10, 11]. These operators are free of entanglement and made strict superset of LOCC [7].

Dimension of Hilbert space for \( n \)-qubit systems grows exponentially by \( n \). In order to decrease the complexity of discrimination problem for these particles, we restrict the problem to a set of states that possesses sufficient symmetry. Symmetric subspace contains the states which are invariant under the permutations of particles. This symmetric subspace is spanned by the \( n + 1 \) Dicke states. Dicke states are produced and detected experimentally [12, 13, 14]. In addition, Dicke states are proposed for certain tasks in quantum information theory [15, 15].
In this paper, we investigate the minimum-error discrimination of the bosonic state in many-particle spin 1/2 systems. Selected States have geometrically uniform (GU) symmetry in the bosonic subspace. The set of GU states are in the form of \( \{\rho_k = U_k \rho_0 (U_k)^\dagger, k = 0, 1, ..., m\} \), where \( U \) is unitary matrices \([17]\). GU states have well-known examples such as Quadrature amplitude modulation (QAM), pulse-position modulated (PPM) and phase-shift-keyed (PSK) that discrimination of them investigated extensively \([18, 19]\). We select \( \rho_0 \) as pure or special mixed separable state in symmetric subspace and as \( U_k = \exp(-i 2k \frac{\pi}{m}) \) that rotates a spin-\( j \) state by \( 2k \pi / m \) with respect to the \( J_y \)-axis. We obtain optimal probability of correct detection and optimal global measurement, while, results are valid for arbitrary \( k \), even for linearly dependent states. The set of pure GU states in the symmetric subspace which are perfectly discriminated by the obtained measurements, are identified. Also, separable form of optimal global measurements is obtained. By Mapping the optimal measurement from the symmetric space to entire space of \( n \)-qubit, we succeed in obtain optimal separable measurements equivalent to optimal global measurements with the same error probability.

In Sec. II a brief review of the minimum error discrimination is presented. In Sec. III and IV optimal detection of GU pure and mixed states in the Symmetric subspace are investigated and the optimal probability and the optimal global measurements are obtained, respectively. In Sec. V appropriate transformation is introduced to gain the optimal separable measurements equivalent to the optimal global measurements with the same optimal probability, and Sec. VII is devoted to the conclusions.

2 Minimum error discrimination

We assume a quantum system is prepared from a collection of given states which represented by \( m \) density operators \( \{\rho_i; \rho_i \geq 0, Tr(\rho_i) = 1, i = 0, 1, ..., m\} \), and transmission probability to the receiver for each of them is \( p_i \) is \( \sum_i^m p_i = 1 \). The aim is to obtain the set of positive semidefinite
operators, \( \{ \Pi_i, \sum \Pi_i = I \} \), in the way that the output state of operator \( \Pi_i \), represent state \( \rho_i \). Therefore, the probability of correct discrimination for each \( \rho_i \) is \( Tr(\rho_i \Pi_i) \). In the minimum error approach, the set of measurement operators are looked for which provide maximum probability of correct discrimination as follows

\[
p_{\text{opt}} = 1 - p_{\text{error}} = \sum_{i=1}^{m} p_i Tr(\rho_i \Pi_i). \tag{2-1}
\]

The necessary and sufficient conditions of discrimination with the maximum-success probability is

\[
\sum_{i=1}^{m} p_i \Pi_i \rho_i - p_j \rho_j \geq 0, \forall j = 1, ..., m. \tag{2-2}
\]

In the Ref. [1] has been demonstrated that the necessary and sufficient conditions are equivalent to a Helstrom family of ensembles; then a more suitable form of the conditions of the minimum error discrimination is presented as

\[
M = p_j \rho_j + (p - p_j) \tau_j, \forall j. \tag{2-3}
\]

where \( M = \sum_{i=1}^{m} p_i \rho_i \Pi_i \) and \( \{ \tau_i, \tau_i \geq 0 \} \) is the conjugate state of \( \rho_i \). Also, eigenvector of \( \tau_i \) with zero eigenvalue is proportional to \( \Pi_i \) [1],

\[
\Pi_i \tau_i = 0. \tag{2-4}
\]

In the following two sections, the new technique is applied for optimal detection of GU pure and mixed states in the Symmetric subspace.

### 3 Optimal detection of GU pure states in the Symmetric subspace

In this section we derive the maximum attainable value of the success probability in the method of the minimum error discrimination probability for GU Symmetric states of \( n \)-qubit with equal the priori probabilities.
States that are invariant under permutation of particles, \( \{ P|\psi \rangle = |\psi \rangle, P \in S_n \} \), are called symmetric subspace states. For the \( n \)-qubit in the Hilbert space, \( \otimes^n H_2 \), common eigenvectors \( J_z, J^2 \) are standard orthogonal bases and the symmetric subspace \( H_s \) is indicated with \( j = n/2 \).

Therefore, any state in this subspace is expressed as

\[
|\psi_0 \rangle = \sum_{q=-n/2}^{n/2} c_q |j, q \rangle_z. \tag{3-5}
\]

where \( c_q \) is the probability amplitude. We distinguish the set of states \( \{ \rho_k = U^k \rho_0 (U^k)^\dagger \} \), which \( \rho_0 \) is in the symmetric subspace and \( U^k \) is a unitary operator. This set is well-known as GU states. We specially are interest in the unitary transformation which the produced states remain in the symmetric subspace, in a way that, \( \sigma_y \) and \( J_y \) are selected as generator of rotation for each qubit and generator for \( n \)-qubit, respectively. Therefore, unitary transformation is written as

\[
U = \exp(-i\frac{\pi}{m} \sigma_y) \Rightarrow U = \overline{U} \otimes \overline{U} \otimes \cdots \otimes \overline{U} = e^{-i\frac{2\pi}{m} J_y}, \tag{3-6}
\]

the number of states, \( m \), may be equal or greater than the dimension of \( H_s \), in other words, it is not necessary to states be linear independent. We consider states with equal initial probability, \( 1/m \), hence, from Eq. (2-3) for all \( k \)

\[
M = U^k [\frac{1}{m} \rho_0 + (p - \frac{1}{m}) \tau_0](U^k)^\dagger = U^k M (U^k)^\dagger \quad k = 0, 1, \ldots, m - 1. \tag{3-7}
\]

Thus, \( M \) and \( U^k \) commute, in addition, by Cayley-Hamilton theorem in the subspace \( j = n/2 \), \( M \) is written as \( M = \sum_{i=0}^{n-1} a_i J^i \) and from Eq. (2-3) one obtains

\[
p = \sum_{i=0}^{n} a_i Tr(J^i_y), \tag{3-8}
\]

Which \( p \) is Helestrom ratio and \( p_{opt} < p \) [1]. Then optimization problem are given by

\[
\text{min} \quad p = \sum_{i=0}^{n} a_i Tr(J^i_y), \tag{3-9}
\]

subject to \( -\tau_0 = -(\sum_{i=0}^{n} a_i J^i_y - \frac{1}{m}|\psi_0 \rangle \langle \psi_0 |) \leq 0 \), \( \tag{3-10} \)
and the dual problem is

\[ \max g(Z_0) = \frac{1}{m} \langle \psi_0 | Z_0 | \psi_0 \rangle, \]  

(3-11)

subject to \( Z_0 \geq 0 \)

\[ \text{Tr}(J_y^i) - \text{Tr}(Z_0^i J_y^i) = 0 \quad i = 0, 1, ..., n - 1, \]  

(3-12)

From slackness conditions \( \tau_0 Z_0 = 0 \) and Eq. (2-4) \( \Pi_i \) is concluded,

\[ \Pi_0 = Z_0 = |z_0\rangle \langle z_0|, \]  

(3-13)

where \( |z_0\rangle \) is expressed by eigenvector of \( J_y \), \( |z_0\rangle = \sum_{q=-\frac{n}{2}}^{\frac{n}{2}} \alpha_q |j, q\rangle_y \). Using Eq. (3-12), \( \sum q^i (1 - |\alpha_q|^2) = 0 \). So, for all \( i \),

\[
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
(\frac{n}{2}) & (\frac{n}{2} - 1) & (\frac{n}{2} - 2) & \ldots & (-\frac{n}{2}) \\
(\frac{n}{2})^2 & (\frac{n}{2} - 1)^2 & (\frac{n}{2} - 2)^2 & \ldots & (-\frac{n}{2})^2 \\
\vdots & & & \ldots & \\
(\frac{n}{2})^n & & & \ldots & (-\frac{n}{2})^n
\end{bmatrix}
\begin{bmatrix}
1 - |\alpha_{\frac{n}{2}}|^2 \\
\vdots \\
1 - |\alpha_{-\frac{n}{2}}|^2
\end{bmatrix} = 0,
\]  

(3-14)

matrix of coefficients is the same of the vandermonde matrix, since there is no two equal rows, determinant of the matrix of coefficients is non-zero, thus, we conclude that \( |\alpha_q|^2 = 1 \) and \( |z_0\rangle = \sum_q e^{i\theta_q} |j, q\rangle_y \). Inserting the above result into the equation \( \tau_0 |z_0\rangle = 0 \), one obtains

\[
\sum_{i=0}^{n-1} a_i J_y^i - \frac{1}{m} |\psi_0\rangle \langle \psi_0| \sum_{q=-\frac{n}{2}}^{\frac{n}{2}} e^{i\theta_q} |j, q\rangle_y = 0
\]

\[
\sum_{q=-\frac{n}{2}}^{\frac{n}{2}} [e^{i\theta_q} \sum_{i=0}^{n-1} a_i q^i - \frac{\lambda}{m} y \langle j, q | \psi_0 \rangle | j, q \rangle_y = 0
\]

\[
e^{i\theta_q} \sum_{i=0}^{n-1} a_i q^i = \frac{\lambda}{m} y \langle j, q | \psi_0 \rangle,
\]

(3-15)

and

\[
| \sum_{i=0}^{n-1} a_i q^i | = |\lambda| y |j, q | \psi_0 \rangle |,
\]

(3-16)
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where \( \lambda = \langle \psi_0 | z_0 \rangle \).

After this, all coefficients in the initial state are real. Equation \( J_y | J, q \rangle_y^* = -q | J, q \rangle_y^* \), yields

\[
| y \langle j, q | \psi_0 \rangle | = | y \langle j, -q | \psi_0 \rangle |,
\]

and

\[
| \sum_{i=0}^{n} a_i q^i | = | \sum_{i=0}^{n} a_i (-q)^i |. \tag{3-18}
\]

Hence, Helestrom ratio,

\[
\lambda = \sum_{q=-\frac{m}{2}}^{\frac{m}{2}} \langle \psi_0 | j, q \rangle_y e^{i \theta_q},
\]

is zero for the odd numbers of \( i \), in the Eq. (3-9), and only even numbers of \( i \) have non-zero terms. For \( \lambda = |\lambda| e^{i \theta_\lambda} \), and \( y \langle j, q | \psi_0 \rangle = | y \langle j, q | \psi_0 \rangle | e^{i \theta_q} | y \langle j, q | \psi_0 \rangle | \), last term of Eq. (3-15) yields \( \theta_q = \theta_\lambda + \theta_q \langle j, q | \psi_0 \rangle \), and implies:

\[
\lambda = \sum_{q=-\frac{m}{2}}^{\frac{m}{2}} (\psi_0 | j, q \rangle_y e^{i \theta_q}, \tag{3-19}
\]

so

\[
|\lambda| = \sum_{q=-\frac{m}{2}}^{\frac{m}{2}} | y \langle j, q | \psi_0 \rangle |. \tag{3-20}
\]

Strong duality for the optimal value of dual problem, \( p_{opt} \), yield \( p_{opt} = p \), thus

\[
p_{opt} = \sum_{i=0}^{n} a_i Tr(J_i^y) = \sum_{q=-\frac{m}{2}}^{\frac{m}{2}} e^{-i \theta_q} \frac{|\lambda| e^{i \theta_q}}{m} y \langle j, q | \psi_0 \rangle | e^{i \theta_q} \langle j, q | \psi_0 \rangle | \langle j, q | \psi_0 \rangle | = \frac{|\lambda|^2}{m}. \tag{3-21}
\]

Eq. (3-21) is valid for all of the number of states, \( m \). The unnormalized vector \( |z_0 \rangle \) in the optimal measurement operator, \( \Pi_0 = |z_0 \rangle \langle z_0 | \), is

\[
|z_0 \rangle = \sum_{q=-\frac{m}{2}}^{\frac{m}{2}} e^{i \theta_q} | j, q \rangle_y = \frac{1}{\sqrt{m}} \sum_{q=-\frac{m}{2}}^{\frac{m}{2}} e^{i \theta_q \langle j, q | \psi_0 \rangle} | j, q \rangle_y. \tag{3-22}
\]

Therefore, one obtains the set of the optimal measurements as \( \{ Z_k = U^k Z_0 (U^k)^\dagger, \ k = 0, 1, ..., m - 1 \} \). In the case that the set of GU states are linearly independent, the optimal measurements, \( |z_k \rangle = U^k |z_0 \rangle \), are projective and discrete Fourier transformation of \( |z_0 \rangle \)
and a projective set,

\[ \langle z_k | z_h \rangle = \frac{1}{m} \sum_{q, q'} e^{-i\theta y (j, q | \psi_0)} \left( y \langle j, q | \left( U^\dagger \right)^k \right) \left( U^k | j, q' \rangle \rangle \right) y \] 

\[ = \frac{1}{m} \sum_{q = \pm \frac{n}{2}} e^{i\frac{2\pi}{m} q (k-h)} \]

\[ = \frac{1}{m} e^{-\frac{2\pi (n/2 + 1)}{m} (k-h)} \sum_{p=1}^{n+1} e^{i\frac{2\pi}{m} p (k-h)} = \delta_{k,h} \quad \forall \quad n + 1 = m. \quad (3-23) \]

This result is in agreement with pretty good measurement.

If we suppose reference state as \(|\psi_0\rangle = |j, q\rangle_z\), it is always possible to make \(\theta y (j, q | \psi_0)\), constant, therefore, \(|z_0\rangle\) has a simple form as

\[ |z_0\rangle = \frac{1}{\sqrt{m}} \sum_{q = -\frac{n}{2}}^{\frac{n}{2}} |j, q\rangle_y \]

\[ (3-24) \]

This result is consistent with Ref [2].

For the perfect discrimination, Eq. (3-21) is written as

\[ \sum_{q = -\frac{n}{2}}^{\frac{n}{2}} e^{i\theta y} |y \langle j, q | \psi_0\rangle| = \sqrt{m}. \quad (3-25) \]

For example, for generating state \(|\psi_0\rangle = \frac{1}{\sqrt{n+1}} \sum_{q = -\frac{n}{2}}^{\frac{n}{2}} e^{i\theta y} |j, q\rangle_y\), the set of GU states, which are linearly independent, are discrete Fourier transformation of \(|\psi_0\rangle\) and orthogonal, so perfectly is discriminated.

4 Optimal detection of GU mixed states in the Symmetric subspace

For discrimination of GU mixed states in the symmetric subspace, we select separable mixed state,

\[ \rho_0 = r |\frac{n}{2}, \frac{n}{2}\rangle \langle \frac{n}{2}, \frac{n}{2}| + (1-r) |\frac{n}{2}, -\frac{n}{2}\rangle \langle \frac{n}{2}, -\frac{n}{2}|, \quad (4-26) \]
as the reference state. Such as the previous section, optimal probability is obtained from the minimization of Eq. (3-9),

$$(p - \frac{1}{m})\tau_0 = \sum_{i=0}^{n-1} a_i J^i - \frac{1}{m}(r|\frac{n}{2}, \frac{n}{2}, \frac{n}{2}| + (1 - r)|\frac{n}{2}, -\frac{n}{2}, -\frac{n}{2}|),$$  

(4-27)

Thus, the dual problem is written as follows:

$$\max \quad g(Z_0) = \frac{1}{m}(r|\frac{n}{2}, \frac{n}{2}, \frac{n}{2}| + (1 - r)|\frac{n}{2}, -\frac{n}{2}, -\frac{n}{2}|)$$

subject to

$$Z_0 \geq 0$$

$$Tr(J^i_y) - Tr(Z_0 J^i_y) = 0 \quad i = 0, 1, ..., n - 1,$$

(4-28)

From $\tau_0|z_0\rangle = 0$, and $|z_0\rangle = \sum_{q=-\frac{n}{2}}^{\frac{n}{2}} \alpha_q |j, q\rangle_y$, Eq. (3-16) for each $q$ is written as

$$e^{i\theta} \sum_{i=0}^{n-1} a_i q^i = \frac{r}{m}\langle j, q|\frac{n}{2}\rangle \sum_{\tilde{q}}\langle \frac{n}{2}|j, \tilde{q}\rangle e^{i\theta\tilde{q}} + \frac{1 - r}{m} \langle j, q| -\frac{n}{2}\rangle \sum_{\tilde{q}}\langle -\frac{n}{2}|j, \tilde{q}\rangle e^{i\theta\tilde{q}},$$

(4-29)

$$e^{-i\pi J^y_{j, \frac{n}{2}}}|j, j, \frac{n}{2}\rangle_z = |j, -\frac{n}{2}\rangle_z$$

and

$$g(j, q|j, -\frac{n}{2}|) = e^{i\pi q} e^{-i\pi J^y_{j, \frac{n}{2}}}|j, q|j, \frac{n}{2}\rangle$$

are concluded From $z\langle j, m|e^{-i\pi J^y_{j, \frac{n}{2}}}|j, m\rangle_z = \delta_{m,m,0.}$ Thus, Eq. (4-29) is given by

$$e^{i\theta} \sum_{i=0}^{n-1} a_i q^i = \frac{\langle j, q|\frac{n}{2}\rangle}{m} \Omega_q,$$

(4-30)

where $\Omega_q = \sum_{q}\langle \frac{n}{2}|j, \tilde{q}\rangle e^{i\theta\tilde{q}}[r + (1 - r) e^{i\pi(q - \theta)}]$. It is always possible to make $\theta_{y,j,q|\frac{n}{2}, \frac{n}{2}} = 0$, for all $q_s$, and from $\Omega_q = \Omega_{q+2}$, is concluded which $\exp(i\theta_q)$ takes only two different values. Without loss the generality, phase of $|j, -j + 2s\rangle_y$ and $|j, -j + (2s + 1)\rangle_y$ for $s = 0, 1, ...$ is given 1 and $\exp(i\theta)$, respectively. Therefore,

$$\Omega_q = \sum_{s}\langle \frac{n}{2}|j, -j + 2s\rangle[r + (1 - r)e^{i\pi(q + \frac{\theta}{2})}e^{-i\pi 2s}]$$

$$+ e^{i\theta} \sum_{s}\langle \frac{n}{2}|j, -j + (2s + 1)\rangle[r + (1 - r)e^{i\pi(q + \frac{\theta}{2})}e^{-i\pi(2s + 1)}]$$

$$= (r + (1 - r)e^{i\pi(q + \frac{\theta}{2})})A + e^{i\theta}(r + (1 - r)e^{i\pi(q + \frac{\theta}{2})})B,$$

(4-31)

where $\sum_{s}\langle \frac{n}{2}|j, -j + 2s\rangle = A, \sum_{s}\langle \frac{n}{2}|j, -j + (2s + 1)\rangle = B$, and $\Omega_{q=-j+2s} = A + e^{i\theta}(2r - 1)B$, $e^{-i\theta}\Omega_{q=-j+(2s+1)} = (2r - 1)e^{-i\theta}A + B$. 

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According to obtained equations, optimal success probability is given by

\[
p_{\text{opt}} = \sum_{q=-\frac{n}{2}}^{\frac{n}{2}} e^{-i\eta_q \langle j, q| \frac{n}{2} \rangle} \Omega_q = \frac{\Omega_{q=-j+2s}}{m} A + \frac{e^{-i\eta} \Omega_{q=-j+(2s+1)}}{m} B
\]

\[
= \frac{1}{m} \left[ (\sum_{q} \langle j, q| \frac{n}{2} \rangle)^2 + 2AB(\cos(\theta)(2r - 1) - 1) \right],
\]

(4-32)

From Eq. (4-32), \(\cos(\theta) = 1\) and \(\cos(\theta) = -1\) for \(r > \frac{1}{2}\) and \(r < \frac{1}{2}\), respectively. For the systems with odd number of qubit, \(A = B\). Therefore, \(p_{\text{opt}}\) is simplified to the following form

\[
p_{\text{opt}} = \frac{\cos(\theta)(2r - 1) - 1}{2m} (\sum_{q} \langle j, q| \frac{n}{2} \rangle)^2.
\]

(4-33)

As a special case, for mixed three-qubit states, \(p_{\text{opt}}\) becomes

\[
p_{\text{opt}} = \frac{1}{3} (1 + |2r - 1|).
\]

(4-34)

This is in agreement with the results in the Ref. [20].

Like the pure states for the linearly independent states, \(m = n + 1\), the optimal measurements are discrete Fourier transformations of \(|z_0\rangle\) and projective.

## 5 Finding optimal separable measurements

In the Majorana representation any symmetric state of \(n\)-qubit, \(|\phi_S\rangle\), which is invariant under the permutation, is uniquely made from the sum of all permutations of \(n\) single qubit state as

\[
|\phi_s\rangle = \frac{1}{\sqrt{k}} \sum_{g \in S_n} g|\varphi_1\rangle|\varphi_2\rangle\ldots|\varphi_n\rangle,
\]

(5-35)

which \(k\) is the normalization factor. The vector of \(|\varphi_i\rangle\) is made from the roots of the following function

\[
\Phi(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{4} a_k t^k,
\]

(5-36)

where \(t = e^{i\theta} \tan(\theta), \ |\varphi_i\rangle = \cos \theta|0\rangle + e^{i\varphi} \sin \theta|1\rangle\) and \(a_k\) is expansion coefficient of \(|\phi_S\rangle\) in the Dick basis.
Since $|z_0\rangle$ is in the symmetric subspace thus it is always possible to write $|z_0\rangle$ in the Majorana representation as

$$|z_0\rangle = \frac{1}{\sqrt{k}} \sum_{g \in S_n} g |\varphi_1\rangle |\varphi_2\rangle ... |\varphi_n\rangle = \frac{1}{\sqrt{k}} \sum_{g \in S_n} g |\varphi_0\rangle,$$

(5-37)

the term of $|\varphi_1\rangle |\varphi_2\rangle ... |\varphi_n\rangle$ is inserted by $|\varphi_0\rangle$.

Similar to the general Fourier transformation, we introduce the following map on quantum state $|\varphi\rangle$

$$|\tau_0\rangle = \sqrt{\frac{d_\rho}{n!}} \sum_{g \in S_n} \rho_{ij}(g) g |\varphi\rangle,$$

(5-38)

where $S_n$ is Symmetric group and $\rho(g)$ is an irreducible representation of $S_n$ with the dimension of $d_\rho$. From Eq. (3-21), Majorana representation is equivalent, up to a constant, to transformed form of $|\varphi_0\rangle$, by the trivial representation, $|\tau_0\rangle = \frac{1}{\sqrt{n!}} \sum_{g \in S_n} (g |\varphi_0\rangle)$.

Here, we prove one of the important properties of this map which $|\tau_0\rangle$ and each symmetric state $|\psi_0\rangle$, is orthogonal to the transformed form of non-trivial representation of $S_n$. By multiplying of $|\psi_0\rangle$, into the Eq. (5-38) one obtains

$$\langle \tau_0 | \tau_0\rangle = \langle \tau_0 | \sqrt{\frac{d_\rho}{n!}} \sum_{g \in S_n} \rho_{ij}(g) g |\varphi_0\rangle$$

$$= \sqrt{\frac{d_\rho}{n!}} \sum_{g \in S_n} \rho_{ij}(g) \langle \tau_0 | g |\varphi_0\rangle$$

$$= \langle \tau_0 | |\varphi_0\rangle \sqrt{\frac{d_\rho}{n!}} \sum_{g \in S_n} \rho_{ij}(g).$$

(5-39)

In this step, we show that $\sum_{g \in S_n} \rho(g)$ equal to zero. If $\sum_{g \in S_n} \rho(g) = A$ then for all $\hat{g} \in G$ is obtained

$$\rho(\hat{g}) A = A \rho(\hat{g}),$$

(5-40)

and from the Shor lemma for an irreducible representation

$$\sum_{g \in S_n} \rho(g) = \lambda I_{d_\rho}.$$

(5-41)
Taking the trace of both sides leads to

$$\sum_{g \in S_n} \chi_{\rho}(g) = \lambda_{d_{\rho}}, \quad (5-42)$$

where $$\chi_{\rho}(g)$$ is the character of group elements. For the finite group $$\sum_{g \in S_n} \chi_{\rho}(g)\chi_{\rho}^*(g) = 0$$, and for the trivial representation $$\chi_{\rho}^*(g) = 1$$. Therefore, one obtains $$\lambda = 0$$, and consequently $$\sum_{g \in S_n} \rho(g) = 0$$. This indicates that $$\sum_{g \in S_n} \rho_{ij}(g) = 0$$, thus,

$$\langle \tau^0 | \tau_{ij}^{\rho} \rangle = 0 \quad \forall \rho \neq 1. \quad (5-43)$$

So, $$|\tau_{ij}^{\rho}\rangle$$ by non-trivial representations, is not in the symmetric space. In fact from the Schur-Weyl duality theorem [21] the Hilbert space of $$n$$-qubit, $$H_2^\otimes n$$, expressed in term of subspaces which are invariant under irreducible representation (irrep) of $$S_n, U_2$$, i.e,

$$H_2^\otimes n = \bigoplus H_{\text{irrep} S_n, U_2}. \quad (5-44)$$

Then, $$|\tau^0\rangle$$, Belongs to the Symmetric Hilbert subspace and from (5-43) all $$|\tau_{ij}^{\rho}\rangle$$, which $$\rho \neq 1$$, are the states in the other subspaces.

In following, in order to find optimum separable operation, the orthogonal terms are added to the trivial representation.

$$\sum_{ij} \sum_{\rho} |\tau_{ij}^{\rho}\rangle \langle \tau_{ij}^{\rho}| = \sum_{\hat{g},g \in S_n} \sum_{ij,\rho} \frac{d_{\rho}}{n!} \rho_{ij}(g) \rho_{ij}^*(\hat{g}) g |\varphi_0\rangle \langle \varphi_0| \hat{g}$$

$$= \sum_{\hat{g},g \in S_n} \sum_{\rho} \frac{d_{\rho}}{n!} Tr(\rho(g) \rho_0^*(\hat{g})) g |\varphi_0\rangle \langle \varphi_0| \hat{g}$$

$$= \sum_{\hat{g},g \in S_n} \sum_{\rho} \frac{d_{\rho}}{n!} Tr(\rho(\hat{g}g^{-1}) g |\varphi_0\rangle \langle \varphi_0| \hat{g}$$

$$= \sum_{\hat{g},g \in S_n} [\sum_{\rho} \frac{d_{\rho}}{n!} \chi_{\rho}(\hat{g}g^{-1})] g |\varphi_0\rangle \langle \varphi_0| \hat{g}$$

$$= \sum_{\hat{g},g \in S_n} \delta_{g,\hat{g}} g |\varphi_0\rangle \langle \varphi_0| g$$

$$= \sum_{g \in S_n} g |\varphi_0\rangle \langle \varphi_0| g$$

$$= \sum_{g \in S_n} g |\varphi_0\rangle \langle \varphi_0| g$$

$$= \sum_{g \in S_n} g |\varphi_1\rangle \langle \varphi_1| \varphi_2\rangle \langle \varphi_2| \ldots \langle \varphi_n| \varphi_1\rangle \langle \varphi_2| \ldots \langle \varphi_n| g$$
Discrimination

\[ \begin{align*}
&= (\langle \varphi_1 | \otimes | \varphi_2 \rangle \langle \varphi_2 | \otimes ... | \varphi_n \rangle \langle \varphi_n |) + (\langle \varphi_2 | \otimes | \varphi_1 \rangle \langle \varphi_1 | \otimes ... | \varphi_n \rangle \langle \varphi_n |) + ... \\
&+ (\langle \varphi_n | \otimes | \varphi_1 \rangle \langle \varphi_1 | \otimes ... ) + ..... \\
&+ ... \quad (5-45)
\end{align*} \]

According to the last term in the Eq. (5-45) is yielded each term of last summation is separable, and the probability of correct discrimination pure state, in the symmetric subspace, by this local operator is

\[ p_{\text{local opt}} = \frac{1}{m} \text{Tr} \left( c \sum_{ij} \sum_{\sigma} | \tau_{ij}^\sigma \rangle \langle \tau_{ij}^\sigma | \rho_0 \right) = \frac{1}{m} \text{Tr} \left( \sqrt{c} | \tau^0 \rangle \langle \tau^0 | \sqrt{c} \rho_0 \right) + \frac{c}{m} \sum_{ij} \sum_{\sigma \neq 1} \text{Tr} \left( | \tau_{ij}^\sigma \rangle \langle \tau_{ij}^\sigma | \rho_0 \right) \]

\[ = p_{\text{global opt}} + c \sum_{ij} \sum_{\sigma \neq 1} | \langle \tau_{ij}^\sigma | \psi_0 \rangle | = p_{\text{global opt}}. \quad (5-46) \]

where, \( | z_0 \rangle = \sqrt{c} | \tau^0 \rangle \). Therefore, the optimum separable operation equivalent to the optimum global operation is achieved.

6 CONCLUSION

The discrimination of GU states in the symmetric subspace of \( n \)-qubit particles is investigated. In this subspace for the general pure states, the optimal probability and the optimal global measurements are obtained by the presented method in the Ref. [1]. Also, as a special case, the mixed reference states which Included the convex combination of up and down states are studied and the results are consistent with the results in the Ref. [2] and [20]. The following, we introduce a mapping to gain the optimal separable measurements equivalent to the optimal global measurements with the same optimal probability. The achieved results are always valid for make of the separable operations which are in the symmetric subspace. We expect to the same results are expressed in the other subspaces and this paper offers a good starting point. To make the states which are orthogonal to symmetric subspace, we use Majorana representation, however, this representation is not valid for general case of qudit state, so, finding of the appropriate map to make the separable measurement from each symmetric measurement, is difficult and under investigation.
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