COARSE HOMOLOGY OF LEAVES OF FOLIATIONS

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Abstract. We investigate the coarse homology of leaves in foliations of compact manifolds. This is motivated by the observation that the non-leaves constructed by Schweitzer and by Zeghib all have non-finitely generated coarse homology. This led us to ask whether the coarse homology of leaves in a compact manifold always has to be finitely generated. We show that this is not the case by proving that there exist many leaves with non-finitely generated coarse homology. Moreover, we improve Schweitzer’s non-leaf construction and produce non-leaves with trivial coarse homology.

Contents

1. Introduction 1
2. Schweitzer’s bounded homology property 3
3. Computations of coarse homology 5
4. Non-leaves with trivial coarse homology 12
References 16

1. Introduction

Given a Riemannian manifold $L$, it is in general very hard to determine whether there exists a foliation of a compact manifold $M$ such that $L$ is quasi-isometric to one of the leaves equipped with the induced metric from $M$. However, we can rule out certain Riemannian manifolds. 6-dimensional examples of non-leaves were given by Attie and Hurder [AH], and Zeghib [Z] modified their construction to produce 2-dimensional non-leaves. Schweitzer showed in 1994 [S1] that every open surface carries a metric of bounded geometry that cannot be bi-Lipschitz equivalent to a leaf in a foliation of a compact 3-manifold, and in 2009 he was able to extend his results from dimension 2 to any dimension [S2]. He showed that every non-compact manifold carries a metric of bounded geometry such that the resulting Riemannian manifold cannot be diffeomorphically quasi-isometric to a leaf in a codimension one foliation of a compact manifold.

The above-mentioned non-leaves are constructed through manifolds that violate certain conditions met by leaves in foliations of compact manifolds, and which are preserved under quasi-isometries. Schweitzer’s bounded homology
property of a Riemannian manifold \((M, g)\) is a condition on certain types of volumes of nullhomologous hypersurfaces in \(M\). We give a brief description of the bounded homology property in Section 2, for more details we refer the reader to [S2]. Attie-Hurder and Zeghib use the so-called geometric entropy of a metric space, a measure of the complexity of coverings of the space. The bounded homology property is specifically tailored to foliations and might at first be difficult to grasp. It is thus an interesting question whether the bounded homology property can be expressed through established quasi-isometry invariants.

Schweitzer’s and Zeghib’s non-leaf constructions deform a metric on a Riemannian manifold (in Schweitzer’s case an arbitrary metric on an arbitrary manifold, in Zeghib’s case the hyperbolic metric on \(\mathbb{H}^2\)) by inserting balloons of radius tending towards infinity. Computing the coarse homology of the non-leaves, one notices that it is never finitely generated (Proposition 3.7). This suggests that the bounded homology property and possibly other conditions on leaves might be expressible through the number of generators of the coarse homology. The only known non-leaf with finitely generated coarse homology is the 6-dimensional example given by Attie and Hurder but no non-leaves with trivial coarse homology were known before. What we want to find out is hence whether the coarse homology of a leaf in a compact manifold must always be finitely generated, and conversely, whether there exist non-leaves with trivial coarse homology. As we will now describe, the first question is to be answered in the negative while we can give a positive answer to the latter.

Due to a connection between coarse homology and ends of manifolds, we can show that there exist leaves in foliations of compact manifolds that have non-finitely generated coarse homology.

**Theorem 1.1.** In every dimension greater than or equal to 2 there exist Riemannian manifolds \(L\) with the degree 1 coarse homology \(HX_1(L)\) containing an Abelian subgroup of infinite rank, such that \(L\) can be realized as a leaf in a foliation of a compact manifold of any codimension.

All known non-leaf constructions on an arbitrary manifold produced metrics with non-finitely generated coarse homology. Building on the work of Schweitzer we are able to give a non-leaf construction starting with any non-compact Riemannian manifold that does not affect the coarse homology.

**Theorem 1.2.** On every non-compact Riemannian manifold of bounded geometry \((M, g)\), \(\dim M \geq 3\), there exists a deformation of \(g\) to a bounded geometry metric \(g'\) such that \((M, g')\) cannot be diffeomorphically quasi-isometric to a leaf of a codimension one \(C^{2,0}\)-foliation of a compact manifold. This deformation can be performed in such a way that the coarse homology and the growth type of \((M, g)\) remain unchanged.
Applying the theorem to any Riemannian manifold with trivial coarse homology, e.g. a one-ended cylinder $\partial D^n \times \mathbb{R}_{\geq 0} \cup D^n \times \{0\}$ we find the following corollary.

**Corollary 1.3.** In every dimension $n \geq 3$, there exist Riemannian manifolds with trivial coarse homology which are not diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$-foliation of a compact manifold.

Theorem 1.2 is optimal in the sense that we cannot expect to find a non-leaf metric of bounded geometry with trivial coarse homology on every open manifold since the degree 1 coarse homology might be non-trivial for topological reasons as is shown in Proposition 3.8.

**Convention:** Throughout this article, all foliated manifolds are compact and the foliations are of codimension one. In particular, the statement "$(L, g)$ is not quasi-isometric to a leaf" means "There doesn’t exist a compact manifold $M$ and a codimension 1 foliation $\mathcal{F}$ of $M$ such that $(L, g)$ is quasi-isometric to a leaf of $\mathcal{F}$". Moreover, we take all Riemannian metrics to be of bounded geometry and all manifolds to be connected and without boundary.

**Acknowledgements:** This article presents results from my dissertation [S], which was supervised by D. Kotschick. I would like to thank him for his advice and continuous support.

## 2. Schweitzer’s bounded homology property

This section gives a brief overview of the results of Schweitzer’s article [S2] which we will use in Section 4 to give a construction of non-leaves with trivial coarse homology. The following definitions and results are all from that article. We will also briefly recall Schweitzer’s construction of the non-leaf metric $g_S$ at the end of this section.

**Theorem 2.1** (Theorem 2.8, [S2]). Let $(L, g)$ be an open Riemannian manifold of bounded geometry. Then there exists a bounded geometry metric $g_S$ on $L$ such that $(L, g_S)$ cannot be diffeomorphically quasi-isometric to a leaf in a codimension $C^{2,0}$-foliation of codimension one of a compact manifold. Moreover, $g_S$ can be chosen such that $g$ and $g_S$ have the same growth type.

Theorem 2.1 follows from the fact that every manifold which is diffeomorphically quasi-isometric to a leaf has to satisfy the bounded homology property. The definition of the bounded homology property requires a little preparation:

**Definition 2.2** ($\beta$-volume). Let $S$ be a subset of a metric space and $\beta > 0$. The $\beta$-volume $\text{vol}_\beta(S)$ of $S$ is defined as the minimal number of balls of radius $\beta$ needed to cover $S$. We have $\text{vol}_\beta(S) \in \mathbb{N} \cup \{\infty\}$. 
Definition 2.3 (Morse-\(\beta\)-volume). Let \((C, g)\) be a compact Riemannian manifold with boundary and \(f : C \to [0, \infty)\) a Morse function satisfying \(f|_{\partial C} \equiv 0\). For \(\beta > 0\), the Morse-\(\beta\)-volume of \(C\) with respect to \(f\) is defined to be the smallest natural number \(\text{MVol}_\beta(C, f)\) such that the \(\beta\)-volume of every level set of \(f\) is bounded by \(\text{MVol}_\beta(C, f)\), that is \(\text{vol}_\beta(f^{-1}(t)) \leq \text{MVol}_\beta(C, f)\) for all \(t \geq 0\).

The Morse-\(\beta\)-volume of \(C\) is then defined to be the minimum of \(\text{MVol}_\beta(C, f)\) taken over all Morse functions vanishing on \(\partial C\). In formulae, the Morse-\(\beta\)-volume is defined as

\[
\text{MVol}_\beta(C) = \min_{\beta|_{\partial C} \equiv 0} \max_{t \geq 0} \text{vol}_\beta \left( f^{-1}(t) \right).
\]

Definition 2.4 (bounded homology property). A Riemannian manifold \(M\) has the bounded homology property if for all \(k > 0\) and all sufficiently large \(\beta > 0\), there exists a constant \(K(\beta, k)\) such that the Morse-\(\beta\)-volume \(\text{MVol}_\beta(C)\) of all compact codimension 0 submanifolds \(C\) with smooth boundary is bounded by \(K(\beta, k)\), provided they satisfy the following conditions:

i) \(\text{vol}_\beta(\partial C) \leq k\),

ii) \(C\) and \(\partial C\) are connected and simply connected,

iii) \(\partial C\) has a tubular neighbourhood \(V\) that contains \(B_\beta(\partial C) = \{ x \in M \mid \text{dist}(x, \partial C) < \beta \}\).

Theorem 2.5 (Theorem 2.6, [S2]). Every \(n\)-manifold, \(n \geq 3\), that is diffeomorphically quasi-isometric to a leaf of a codimension one \(C^{2,0}\)-foliation of a compact manifold satisfies the bounded homology property.

The non-leaves by Schweitzer are constructed by deforming any given metric to a metric not satisfying the bounded homology property. The deformation is done by inserting balloons: Since \((L, g)\) has bounded geometry, the injectivity radius is bounded from below by some \(d > 0\). Hence every metric \(d\)-ball is topologically a ball. Now choose a sequence of real numbers \(d_i\) such that \(d_i + 2d < d_{i+1}\) and let \(x_i\) be points in \(L\) such that \(d(x_0, x_i) = d_i\). Choose moreover a sequence of real numbers \(r_i\) converging to infinity. And let \(S^n(r_i) \setminus B_{\sqrt{2}}(S)\) be the spheres of radius \(r_i\), where a ball of radius \(d/2\) has been removed around the south pole. Now replace every ball \(B(x_i, d/2)\) in \(L\) by the “balloon” \(S^n(r_i) \setminus B_{\sqrt{2}}(S)\) and on the annulus \(B_\beta(x_i) \setminus B_{\sqrt{2}}(x_i)\) interpolated smoothly between the round metric of \(S^n(r_i)\) and the original metric \(g\) on \(L\). This defines the balloon metric \(g_S\) on \(L\) (see Figure 1). Note that topologically we have just replaced a ball by a ball and hence the new manifold is diffeomorphic to \(L\).

It is now not hard to see that \((L, g_S)\) does not satisfy the bounded homology property, for the balloons \(S^n(r_i) \setminus B_{\sqrt{2}}(S)\) form a sequence of submanifolds \(C_i\) with \(\text{vol}_\beta(\partial C_i) = \text{vol}_\beta(\partial B_{\sqrt{2}}(S))\) but \(\text{MVol}_\beta(C_i)\) tends to infinity as \(i\) does.

\[\text{Figure 1}\] The picture is taken from Schweitzer’s article [S2]. I kindly thank him for giving his permission to reproduce it here.
3. Computations of coarse homology

In this section we briefly recall the most important definitions and facts about coarse homology and show that the coarse homology of the non-leaves constructed by Schweitzer is non-finitely generated (Proposition 3.7). For more details about coarse homology, we refer the reader to [HR] and [R2]. Whenever possible, we will omit to mention the coefficient ring of the homology theories. All the results we prove hold for arbitrary coefficients, though.

Recall that the locally finite homology $H_k^f(Z)$ of a locally compact topological space $Z$ is the homology of the chain complex based locally finite chains, i.e. possibly infinite formal sums of singular simplices $\sum \alpha_\sigma \sigma: \Delta^k \to Z$ such that every compact subset of $Z$ intersects at most finitely many $\sigma$ with $\alpha_\sigma \neq 0$. The ordinary boundary map on singular simplices induces a well-defined boundary map on locally finite chains.

A coarsening sequence of a proper metric space $X$ is a sequence of locally finite open coverings $\mathcal{U}_1, \mathcal{U}_2, \ldots$ such that the diameter of the sets in $\mathcal{U}_i$ is bounded from above by a constant $R_i$ and that the Lebesgue number of $\mathcal{U}_{i+1}$ is at least $R_i$. Moreover, $R_i$ tends to infinity as $i$ does. We denote by $|\mathcal{U}_i|$ the nerve of the covering $\mathcal{U}_i$, that is the simplicial complex with vertices $(U)$ given by the sets $U \in \mathcal{U}_i$ and $k$-simplices $(U_0, \ldots, U_k)$ spanned by $U_0, \ldots, U_k \in \mathcal{U}_i$ with $U_0 \cap \ldots \cap U_k \neq \emptyset$. Note that each $U_j \in \mathcal{U}_i$ lies in some $V_l \in \mathcal{U}_{i+1}$ and
the choice of such an assignment $U_j \leftrightarrow V_i$ induces a proper map $|U_i| \to |U_{i+1}|$. In what follows, we will fix such an assignment and the induced maps will be called the coarsening maps. We will use term coarsening sequence both for the sequence of locally finite coverings $U_1, U_2, \ldots$ and for the sequence of their geometric realizations together with the coarsening maps $|U_1| \to |U_2| \to \ldots$. Since the coarsening maps are proper, the induced homomorphisms on locally finite homology give rise to a direct system

$$H^I_*(|U_1|) \to H^I_*(|U_2|) \to H^I_*(|U_3|) \to \ldots$$

A very practical coarsening sequence is given as follows: Let $Y$ be a 1-dense subset of $X$, that is for any $x \in X$, there exists a $y \in Y$ such that $d_X(x, y) < 1$. Assume further that $Y$ has no accumulation points. Then for any radius $i \geq 1$, the collection of open balls of radius $i$, $B_i(Y) := \{B_i(y)\}_{y \in Y}$ forms a locally finite open covering of $X$ with Lebesgue number at least $i - 1$. By letting the radii range over all natural numbers, we obtain a coarsening sequence $\{B_i(Y)\}_{i \in \mathbb{N}}$. We set $|B_i(Y)| = R_i(X; Y)$, but we will henceforth simply write $R_i(X)$ whenever the choice of a 1-dense subset $Y$ as above is implicit. We have natural inclusions $B_i(y) \hookrightarrow B_{i+1}(y)$ and the induced coarsening maps hence are just the inclusion of a subcomplex $R_i(X) \hookrightarrow R_{i+1}(X)$. (The notation $R_i(X)$ is slightly abusive, since it usually denotes the $i$th Rips complex of $X$ in which the simplices are spanned by all elements of $X$. In our notation $R(X; Y) = R_i(X)$ is the $i$th Rips complex of $Y$.)

**Definition 3.1** (coarse homology). The coarse homology groups of a proper metric space $(X, d)$ are defined as

$$HX_*(X) = \lim_{\to} H^I_*(|U_i|),$$

One can show that up to natural isomorphism $HX_*(X)$ does not depend on the choice of the coarsening sequence. Moreover, quasi-isometries induce isomorphisms on coarse homology.

For coarse homology there exists an analogue of the Mayer-Vietoris sequence, but we must require the decomposition of a metric space $X = C \cup D$ to be coarsely excisive, that is for every $r > 0$ there exists an $R > 0$ such that

$$B_r(C \cap D) \subset B_R(C \cap D).$$

**Proposition 3.2** (Lemma 3.9, [M]). Let $X$ be a proper metric space and $X = C \cup D$ be a coarsely excisive decomposition. Then there exists a coarse Mayer-Vietoris sequence

$$\ldots \to HX_n(C \cap D) \to HX_n(C) \oplus HX_n(D) \to HX_n(X) \to HX_{n-1}(C \cap D) \to \ldots$$

It is often quite hard and very inconvenient to compute the coarse homology via an explicit coarsening sequence. For certain spaces, though, Higson and Roe [HR] showed that the coarse homology already equals the locally finite homology.
Definition 3.3 (uniform contractibility). A metric space $X$ is called *uniformly contractible* if for every $r > 0$ there exists an $R \geq r$ such that $B_r(x)$ is contractible in $B_R(x)$ for every $x \in X$.

Definition 3.4 (bounded coarse geometry). A proper metric space has *bounded coarse geometry* if there exists some $\varepsilon > 0$ such that the $\varepsilon$-capacity of any ball of radius $r$ (i.e. the maximal number of disjoint $\varepsilon$-balls in $B_r$) is bounded by some $c_r$.

Following the terminology of [HR], we understand a *bounded geometry complex* to be a metric simplicial complex, i.e. a simplicial complex equipped with the path metric induced from the canonical metric on the simplices, which has bounded coarse geometry.

Proposition 3.5 (Proposition 3.8, [HR]). If $(X, d)$ is a uniformly contractible bounded geometry complex, then $H_*^X(X, d)$ and $H^I_*(X)$ are isomorphic.

Remark 3.6. In the following sections we will use the fact that $H_*^I([0, \infty)) = \{0\}$. This is most easily seen by introducing a locally finite $\Delta$-homology analogous to the singular $\Delta$-homology presented in [H] and showing that it is isomorphic to the ordinary locally finite homology. The details can be found in [S], Chapter 3.

3.1. Coarse homology of Schweitzer’s non-leaves. In this section, we show that the coarse homology of the non-leaves constructed by Schweitzer in [S2] is non-finitely generated. Very similar arguments show that also the coarse homology of the non-leaves constructed by Zeghib in [Z] is non-finitely generated.

Schweitzer’s construction presented in Section 2 is closely related to the following balloon space consisting of spheres with radii tending towards infinity attached to the real line, which is defined in [HKRS]

$$B = [0, \infty) \bigcup_{i \in N} (\cup_i S^n(i)),$$

where $S^n(i)$ denotes the $n$-dimensional sphere of radius $i$ (see Figure 2). They compute the coarse homology of $B$ to be

$$HX_n(B) \simeq \left( \prod_i \mathbb{Z} \right) / \left( \bigoplus_i \mathbb{Z} \right).$$

The following proposition shows that an adapted balloon space $B'$ coarsely embeds into the non-leaves $(L, g_S)$ constructed by Schweitzer and shows that this embedding is injective in coarse homology.

Proposition 3.7. Let $(L, g)$ be a complete connected open Riemannian manifold, of dimension $n \geq 2$ and let $g_S$ be the deformation of $g$ to the above
described balloon metric performed along a ray in $L$. Then $H^\ast_n(L, g_S)$ is not finitely generated. In fact

$$H^\ast_n(B) = \left( \prod_i \mathbb{Z} \right) / \left( \bigoplus_i \mathbb{Z} \right) \subset H^\ast_n(L, g_S).$$

Proof. We shall use the notation of Section 2. Let $\gamma$ be a ray in $(L, g)$ and let $x_i = \gamma(d_i), i = 0, 1, \ldots$ be the sequence of points in $L$ where we have inserted balloons of radius $r_i$ tending towards infinity. Note that $(L, g_S)$ and the space $L'$ which results from gluing the south poles of the $S^n(r_i)$ to the $x_i$ are quasi-isometric if we equip $L'$ with the path metric induced from $(L, g)$ and the $S^n(r_i)$. Hence their coarse homology groups are isomorphic and we can work with $L'$ instead of $(L, g_S)$. Let $B'$ be the balloon space which is given by attaching an $n$-sphere $S^n(r_i)$ of radius $r_i$ to $d_i \in [0, \infty)$ for every $i \in \mathbb{N}$. In particular, we can consider $B'$ as a subspace of $L'$. In what follows, $L$ is understood to be the ‘original’ Riemannian manifold $(L, g)$.

Since $L'$ carries the path metric, $L' = L \cup B'$ is a coarsely excisive decomposition with $L \cap B' = [0, \infty)$ and by Proposition 3.2 we have a Mayer-Vietoris sequence

$$\ldots \rightarrow H^\ast_k([0, \infty)) \rightarrow H^\ast_k(L) \oplus H^\ast_k(B') \rightarrow H^\ast_k(L') \rightarrow H^\ast_{k-1}([0, \infty)) \rightarrow \ldots$$

But $[0, \infty)$ is a uniformly contractible bounded geometry complex and thus Proposition 3.3 implies that $H^\ast_k([0, \infty)) \simeq H^H_k([0, \infty))$. Since $H^H_k([0, \infty)) = \{0\}$ for all $k \in \mathbb{N}$ (see Remark 3.6), the above sequence breaks down to

$$0 \rightarrow H^\ast_k(L) \oplus H^\ast_k(B') \rightarrow H^\ast_k(L') \rightarrow 0$$

and thus

$$H^\ast_k(L, g_S) \simeq H^\ast_k(L') \simeq H^\ast_k(L, g) \oplus H^\ast_k(B').$$

As in [HKRS], we can choose a coarsening sequence $U_k$ for $B'$ such that $|U_k|$ is properly homotopic to $B'$ with the first $k$ spheres collapsed to their respective south soles. Hence we find again that

$$H^\ast_n(B') \simeq \lim_{k\to\infty} \left( \prod_i \mathbb{Z} \right) / \left( \bigoplus_{i=1}^k \mathbb{Z} \right) \simeq \left( \prod_i \mathbb{Z} \right) / \left( \bigoplus_i \mathbb{Z} \right).$$
3.2. Leaves with non-finitely generated coarse homology. In this section we will prove Theorem 1.1. This will follow from the fact that there exist foliations of compact manifolds with leaves that have infinitely many ends. We will show that the degree 1 coarse homology of such leaves with any metric induced from the foliated manifold is non-finitely generated. This follows from the following proposition according to which $k$ distinct ends in a proper geodesic space span a free Abelian subgroup of rank $k - 1$ in the degree 1 coarse homology. (Cf. Prop. 2.25, [R1] for the analogous statement for coarse cohomology.)

Proposition 3.8. Let $(X, d)$ be a proper, connected, geodesic space with $k \in \mathbb{N} \cup \{\infty\}$ ends. Then $HX_1((X, d); \mathbb{Z})$ contains a subgroup isomorphic to $\bigoplus_{i=1}^{k-1} \mathbb{Z}$.

Note that not all elements in the degree one coarse homology originate from the ends of the space. Take for example $X$ to be the 1-dimensional balloon space from [HKRS], then $X$ has just one end, but the coarse homology $HX_1(X) = \prod_j \mathbb{Z}/\bigoplus_j \mathbb{Z}$ is not even finitely generated.

The proof of Proposition 3.8 will occupy the remainder of this section. Before, we show how the proposition implies Theorem 1.1:

Proof of Theorem 1.1. By a result of Cantwell and Conlon (Theorem A, [CC2]) every compact, totally disconnected, metrizable space $E$ can be realized as the endspace of a 2-dimensional leaf $\Sigma$ in a codimension one foliation of a compact 3-manifold $M$. In particular there exist surfaces with infinitely many ends which are diffeomorphic to a leaf in a foliation of a compact 3-manifold. (Such leaves can also be constructed more elementarily by turbulizing a linear foliation of $T^3$ by dense cylinders (Example 4.3.10, [CC1]).) Simply by taking products $M \times N^d \times N^c$ with compact manifolds and considering the leaf $\Sigma \times N^d$, we find that in every leaf-dimension at least 2 and every codimension at least 1, there exist leaves in foliations of compact manifolds with leaves that have infinitely many ends.

Let $(M, \mathcal{F})$ be such a foliation and $L$ be a leaf of $\mathcal{F}$ with infinitely many ends. Then every metric $g$ on $M$ induces a complete metric $\iota^*g$ on $L$. In particular $(L, \iota^*g)$ is a proper geodesic space with infinitely many ends. Hence we can apply Proposition 3.8 to see that $HX_1(L, \iota^*g)$ contains a free Abelian subgroup of infinite rank and hence cannot be finitely generated. □

3.2.1. Ends of a topological space. Let $X$ be a topological space. By an end of $X$ we mean the equivalence class of a proper ray $r: [0, \infty) \rightarrow X$, where two rays $r, r'$ are equivalent if for each compact subset $K \subset X$ there exist a $t_K > 0$ such that $r((t_K, \infty))$ and $r'((t_K, \infty))$ lie in the same path component of $X \setminus K$. We denote by $\mathcal{E}(X)$ the set of ends and let $e(X)$ be the cardinality of $\mathcal{E}(X)$.

The number of ends is in general not a quasi-isometry invariant of metric spaces and thus is in general not detected by the coarse homology. Let for
example $X = (\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ with the metric induced from $\mathbb{R}^2$. Then $X$ is a connected metric space and there exists a bijection from $\mathcal{E}(X)$ to $\mathbb{Q} \cup \mathbb{Q}$. But $X$ is quasi-isometric to $\mathbb{R}^2$ which has only one end. For proper geodesic spaces this does not happen.

**Lemma 3.9** (8.29 Proposition, [31]). For proper geodesic spaces $X$ and $Y$, every quasi-isometry $f : X \to Y$ induces a bijection $\mathcal{E}(X) \to \mathcal{E}(Y)$.

It turns out that if $f$ is just the isometric embedding of a subspace, the induced map on the end spaces can be chosen as the embedding of rays. It is moreover not hard to see that a metric space is quasi-isometric to any of its coarsenings. Thus we get the following remark:

**Remark 3.10.** Let $U_1, U_2, \ldots$ be a coarsening sequence for $X$, then the above lemma implies that $e(X) = e([U_i])$. That is, by coarsening a geodesic space we do not create or lose ends. Moreover, for $|U_i| = R_i(X)$ the bijections $\mathcal{E}(R_i(X)) \to \mathcal{E}(R_{i+1}(X))$ are induced by the inclusion of rays from $R_i(X)$ into $R_{i+1}(X)$.

3.2.2. **Proof of Proposition 3.8.** We will prove Proposition 3.8 as follows. First, we will show how two distinct ends of a space give rise to a locally finite 1-chain and when we are dealing with a locally finite simplicial complex $S$ components of $S$ can be viewed as a 1-chain, which we again denote by $r$.

**Proof.** Let $H_1(\mathbb{Z})$ contain a free rank $e(X) - 1$ Abelian subgroup. In particular, for infinitely many ends, $H_X(X)$ cannot be finitely generated.

**Lemma 3.11.** Let $S$ be a locally finite simplicial complex with $k \in \mathbb{N} \cup \{\infty\}$ ends. Then $H_1^I(S; \mathbb{Z})$ contains a subgroup isomorphic to $\bigoplus_{j=1}^{k-1} \mathbb{Z}$.

**Proof.** Let $r_1, r_2, \ldots$ be the ends of $S$. By subdividing $r_j$ as $\sum_n r_j|_{[n,n+1]}$ each end can be viewed as a 1-chain, which we again denote by $r_j$. These chains are locally finite because the maps $r_j : [0, \infty) \to$ are proper. Without loss of generality, we may assume that all rays $r_j$ emanate from the same point $x \in S$. Then $z_{j,j+1} := -r_j + r_{j+1}$ are locally finite 1-cycles for all $j = 1, \ldots, k-1$. We claim that the $z_{j,j+1}$ generate a free Abelian subgroup of rank $k-1$ in $H_1^I(S)$.

Let $K \subset S$ be chosen such that $r_1, \ldots, r_k \in \mathcal{E}(S)$ eventually map to distinct components of $S \setminus K$ and let $K' \supset K$ be a compact neighbourhood of $K$ that properly deformation retracts onto $K$ (this is possible since $S$ is a locally finite simplicial complex) and consider the locally finite Mayer-Vietoris sequence for $S = K' \cup (S \setminus K)$:

$$\ldots \to H_1^I(S) \xrightarrow{\partial} H_1^I(S \setminus K \cap K') \xrightarrow{\nu} H_1^I(S \setminus K) \bigoplus H_1^I(K') \to \ldots$$
Then \( S \setminus K \cap K' \) has \( k \) compact components corresponding to the \( r_1, \ldots, r_k \) and hence \( H^I_0(S \setminus K \cap K') \) contains a free Abelian subgroup of rank \( k \), which is, without loss of generality, generated by points of the from \( r_j(n_j), j = 1, \ldots, k \) and some \( n_j \in \mathbb{N} \). The first component of \( \varphi \) is the zero map since \( r_j(n_j) = \partial(\sum_{n \geq n_j} r_j|_{n, n+1}) \) the second component of \( \varphi \) is the map \((m_1, \ldots, m_k) \mapsto m_1 + \ldots + m_k \). Hence \( \text{im}(\partial) = \ker(\varphi) \supset \bigoplus_{j=1}^{k-1} \mathbb{Z}\{r_{j+1}(n_{j+1}) - r_j(n_j)\} \).

Recall that the boundary map of the Mayer-Vietoris sequence is defined by subdividing 1-chains on \( S \) into a sum of 1-chains on \( K' \) and on \( S \setminus K \) and mapping to the boundary of either summand. Since \( z_{j,j+1} \) can be decomposed as

\[
\left( - \sum_{n < n_j} r_j|_{n,n+1} + \sum_{n \geq n_j} r_j|_{n,n+1} \right) + \left( - \sum_{n \geq n_j} r_j|_{n,n+1} + \sum_{n > n_j} r_j|_{n,n+1} \right),
\]

where the first summand lies in \( K' \) and the second in \( S \setminus K \), we find that

\[
r_{j+1}(n_{j+1}) - r_j(n_j) = \partial z_{j,j+1}.
\]

In particular, the \( z_{j,j+1} \) map to the generators of \( \bigoplus_{j=1}^{k-1} \mathbb{Z}\{r_{j+1}(n_{j+1}) - r_j(n_j)\} \) and thus generate a free Abelian subgroup of rank \( k - 1 \) in \( H^I_1(S) \). Thus we have an increasing, and if \( e(S) < \infty \) eventually stationary, sequence

\[
\mathbb{Z}\{z_{1,2}\} \subset \mathbb{Z}\{z_{1,2}\} \oplus \mathbb{Z}\{z_{2,3}\} \subset \ldots \subset \bigoplus_{j=1}^{l-1} \mathbb{Z}\{z_{j,j+1}\} \subset \ldots \subset H^I_1(S; \mathbb{Z}).
\]

Hence \( \bigoplus_{j=1}^{e(S)-1} \mathbb{Z}\{z_{j,j+1}\} \subset H^I_1(S; \mathbb{Z}) \).

The proof of Proposition 3.8 is now an easy consequence of the above facts.

**Proof of Proposition 3.8** Recall that for a direct system of groups \( G_1 \rightarrow G_2 \rightarrow \ldots \), elements \( a_1, \ldots, a_n \in \varinjlim G_i \) generate a rank \( n \) Abelian subgroup in \( \varinjlim G_i \) if for all sufficiently large \( i \), the representatives \( a_j \in G_i \) of the \( \alpha_j \) generate a rank \( n \) Abelian subgroup in \( G_i \).

For simplicity and geometric clearness, we take \( \{R_i(X)\} \) as coarsenings of \( X \) and compute \( HX_1(X) \) via \( \varinjlim H^I_1(R_i(X)) \). By Remark 3.10 the \( R_i(X) \) all have \( k = e(X) \) ends and Lemma 3.11 then shows that the degree one locally finite homology groups \( H^I_1(R_i(X)), i \geq 1 \), each contain a rank \( k - 1 \) free Abelian subgroup. Moreover, we have seen that the coarsening maps \( R_i(X) \rightarrow R_{i+1}(X) \) map the ends of \( R_i(X) \) to the ends of \( R_{i+1}(X) \) and hence the generators of the rank \( k - 1 \) free Abelian subgroup in \( H^I_1(R_i(X)) \) constructed in Lemma 3.11 to the generators of the respective subgroup in \( H^I_1(R_{i+1}(X)) \). Thus the equivalence classes \( [z_{j,j+1}] \in \varinjlim H^I_1(R_i(X)) = HX_1(X) \) generate a free Abelian subgroup of rank \( k - 1 \). \( \square \)
4. Non-leaves with trivial coarse homology

In this section we prove Theorem 1.2. By the results of Schweitzer presented in Section 2 of this article, every manifold that is diffeomorphically quasi-isometric to a leaf of a codimension one $C^{2,0}$ foliation must satisfy the bounded homology property. Thus Theorem 1.2 follows directly from the following lemma.

**Lemma 4.1.** On every non-compact manifold $(M, g)$ of bounded geometry, $\dim M \geq 3$, there exists a metric $g'$ not satisfying the bounded homology property such that $HX_\ast(M, g') \simeq HX_\ast(M, g)$. Moreover, we can choose $g'$ such that it has the same growth type as $g$.

The proof of Lemma 4.1 is carried out in three steps. We first show how to construct the metric $g'$, then show that it doesn’t satisfy the bounded homology property and finally prove that its coarse homology is isomorphic to that of the original metric.

**Construction of the metric $g'$:** The construction of $g'$ is analogous to that of the non-leaf metric constructed by Schweitzer in [S2], Section 4. Instead of balloons we use a kind of tree-manifolds $T_k$ constructed below.

Recall that a **rooted tree** is a tree with a distinguished vertex, which we call the **root**. The **leaves** of a connected tree are the vertices of degree 1, while we do not want to consider the root as a leaf. The **height** of a leaf is its distance from the root, where we let each edge have length equal to 1. By the **perfect binary tree of height $k$** we mean the rooted tree, where the root and the leaves have degree 1, while every other vertex has exactly 2 children (i.e. has degree 3) and every leaf has height $k$ (see Figure 3). (Note that commonly the root in binary trees is also required to have 2 children.) Denote the perfect binary tree of height $k$ by $T_k$.

By gluing cylinders $S^{n-1} \times [-1, 1]$ to the edges and connecting them through spheres with three punctures at the vertices and finally adding disks to the leaves, we thicken the trees $T_k$ to get Riemannian manifolds of bounded geometry, which we denote by $\tilde{T}_k$ (see Figure 4). For technical reasons, we rescale the edge emanating from the root to have length $2k$ and glue a cylinder...
$S^{n-1} \times [-k, k]$ to it. Then $\Xi_k$ is quasi-isometric to $T_k$ with a rescaled edge, but topologically $\Xi_k$ is just a ball.

The tree-manifolds $\Xi_k$ are glued to $M$ as in Section 2. We use the same notation. Let $\gamma$ be a ray in $(M, g)$ and set $x_k = \gamma(d_k), i = 0, 1, \ldots$ such that $d(x_{k-1}, x_k) > 3d$, where $1/2 > d > 0$ is a lower bound for the injectivity radius of $M$. Then the distance balls $B_d(x_k)$ are also topological balls. We replace the original metric $g$ on $B_d(x_k)$ by the metric induced from $T_k$ (after possibly interpolating the metric on $S^{n-1} \times [-k, 0)$ between the diameter of $\partial B_d(x_k)$ and the diameter of the sphere factor of $S^{n-1} \times [-k, k]$). By the same arguments as in [S2] this can be performed in such a way, that the new resulting metric $g'$ on $M$ is smooth and again of bounded geometry. By placing the $T_k$ sufficiently far apart, we can make sure that $g$ and $g'$ have the same growth type.

**Proof that** $(M, g')$ **does not satisfy the bounded homology property:** In what follows, denote by $\Xi'_k$ the manifold $\Xi_k$ without the lower part $S^{n-1} \times [-k, 0)$ of the cylinder starting from the root of $T_k$. Thus $\partial \Xi'_k = S^{n-1} \times \{0\}$. It suffices to show that the Morse-$\beta$-volume of the $\Xi'_k$ is unbounded for every $\beta > 0$ as $k$ goes to infinity. For then, the $\Xi'_k$ form a sequence of closed codimension 0 submanifolds with $\text{vol}_\beta(\partial \Xi'_k) = \text{vol}_\beta(S^{n-1})$, while there exists no constant $L > 0$ such that $\text{MVol}_\beta(\Xi'_k) \leq L$. It is easy to verify that the $\Xi'_k$ satisfy the conditions i)-iii) in Definition 2.4. We have already seen that $\text{vol}_\beta(\partial \Xi'_k)$ is constant, and as $n \geq 3$ both $\Xi'_k \approx D^n$ and
\[ \partial \Sigma_k \approx S^{n-1} \] are connected and simply connected, thus conditions i) and ii) are fulfilled. To see that the boundary of \( \Sigma_k \) has a large tubular neighbourhood, i.e. satisfies condition iii), we use that the length of the cylinder at the root of \( T_k \) has length \( 2k \). Thus for every \( k > \beta \), this cylinder \( S^{n-1} \times [-k, k] \) is a tubular neighbourhood of \( \partial \Sigma_k \) that contains the \( \beta \)-neighbourhood \( B_\beta(\partial \Sigma_k) \).

To prove that \( \text{MVol}_\beta(\Sigma_k) \) goes to infinity, we show that for every continuous (and in particular, for every Morse function) \( f : \Sigma_k' \to \mathbb{R} \) there exists an \( x \in \mathbb{R} \) such that \( f \) takes the value \( x \) on at least \( \lfloor \frac{k}{2} \rfloor \) cylinders \( S^{n-1} \times [-1, 1], S^{n-1} \times [0, k] \), disks or punctured spheres in \( \Sigma_k \). We will henceforth refer to these four types of pieces as building blocks of \( \Sigma_k \). This statement is sufficient to prove that \( \text{MVol}_\beta(\Sigma_k') \) goes to infinity because every ball of radius \( \beta \) in \( \Sigma_k \) contains at most a bounded number of building blocks, say \( c_\beta \), and for \( x \) as above

\[ \text{vol}_\beta(f^{-1}(x)) \geq \frac{\lfloor \frac{k}{2} \rfloor}{c_\beta} \]

holds and hence

\[ \text{MVol}_\beta(\Sigma_k') \geq \frac{\lfloor \frac{k}{2} \rfloor}{c_\beta}. \]

Note that we do neither require our functions to vanish on \( \partial \Sigma_k \). This will enable us to do induction to the trees of lower height lying inside of \( T_k \). We define \( u(k) \) to be the smallest natural number such that for any continuous function \( f : \Sigma_k \to \mathbb{R} \) there exists some \( x_f \in \mathbb{R} \) such that \( f \) takes the value \( x_f \) on at least \( u(k) \) building blocks and make the following claim:

**Lemma 4.2.** \( u(k + 2) > u(k) \).

**Proof.** Let \( f : \Sigma_{k+2}' \to \mathbb{R} \). Note that \( \Sigma_k' \) contains four copies \( \Sigma_k^{(1)}, \ldots, \Sigma_k^{(4)} \) of \( \Sigma_k \) (up to scaling the length of the edge leading to the root) as shown in Figure 5. Let \( t_{\text{max}} \in \Sigma_{k+2} \) be such that \( f(t_{\text{max}}) = \max f \) and \( t_{\text{min}} \) analogously. Then there exists a \( \Sigma_k^{(s)} \), say \( \Sigma_k^{(1)} \), such that the geodesic between \( x_{\text{min}} \) and \( x_{\text{max}} \) does not intersect \( \Sigma_k^{(1)} \). But for \( f \mid_{\Sigma_k^{(1)}} : \Sigma_k^{(1)} \to \mathbb{R} \) there exists some \( x \in \mathbb{R} \) and \( u(k) \) building blocks in \( \Sigma_k^{(1)} \) such that \( f \mid_{\Sigma_k^{(1)}} \) takes the value \( x \) on these blocks. But by construction \( f(t_{\text{min}}) \leq x \leq f(t_{\text{max}}) \) and since the geodesic between \( t_{\text{min}} \) and \( t_{\text{max}} \) does not intersect \( \Sigma_k^{(1)} \), there exists an additional edge in \( \Sigma_{k+2}' \setminus \Sigma_k^{(1)} \) such that \( f \) takes the value \( x \) on this edge. Hence \( u(k + 2) \geq u(k + 1) \). \( \square \)

**Proof that** \( HX_*(M, g') \cong HX_*(M, g) \): Note that \( (M, g') \) is coarsely quasi-isometric to \( (M, g) \) with the trees \( T_k \) attached to the \( x_k \). Denote this metric space by \( \mathfrak{M} \). Then \( HX_*(M, g') \cong HX_*(\mathfrak{M}) \). But \( \mathfrak{M} \) has a coarsely exciseive decomposition into \( (M, g) \) and the ray \( \gamma \) with the trees \( T_k \) attached to
Figure 5. $\Sigma_{k+2}$ with four copies of $\Sigma_k$. 

$x_k := \gamma(d_k)$, which in turn is isometric to 

$$T' := [0, \infty) \bigcup \bigcup_{d_k} \cup_k \tau_k,$$

where $T'$ carries the path metric induced from the $\tau_k$ and $[0, \infty)$. Since $(M, g) \cap T'$ is isometric to $[0, \infty)$, Proposition 3.5 yields a coarse Mayer-Vietoris sequence

$$\ldots \rightarrow HX_k([0, \infty)) \rightarrow HX_k(M, g) \oplus HX_k(T') \rightarrow HX_k(M) \rightarrow \ldots$$

Since $HX_k([0, \infty)) = \{0\}$ (see the proof of Proposition 3.7), we have isomorphisms $HX_k(M) \simeq HX_k(M, g) \oplus HX_k(T')$. It thus remains to show that $HX_k(T')$ vanishes.

$T'$ naturally has the structure of a metric simplicial complex and this space has bounded coarse geometry in the sense of Definition 3.4. Any ball of radius $r$ in $T'$ intersects at most $2^{|r|+2}$-many edges. Since every subset of diameter 1 intersects at least one edge and since every edge is intersected by at most two disjoint sets of diameter 1, the 1-capacity of every ball of radius $r$ in $T'$ is bounded for any given $r$. Hence $T'$ is a bounded geometry complex. Moreover every ball in $T'$ is contractible within itself, consequently $T'$ is in particular uniformly contractible. Thus we can apply Proposition 3.5 to find
that $HX_*(\mathcal{T}') \simeq H^I_*(\mathcal{T}')$. Though collapsing each $\mathcal{T}_k$ to its respective root is not a quasi-isometry, it is a proper homotopy equivalence and thus $H^I_*(\mathcal{T}') \simeq H^I_*([0, \infty)) = \{0\}$.

Summarizing, we have the following sequence of isomorphisms:

$$HX_*(M, g') \simeq HX_*(\Omega \mathcal{R}) \simeq HX_*(M, g) \oplus HX_*(\mathcal{T}') \simeq HX_*(M, g).$$

This finishes the proof of Theorem 1.2. \qed

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