Spherical collapse in Generalized Dark Matter models

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The influence of considering a generalized dark matter (GDM) model, which allows for a non-pressure-less dark matter and a non-vanishing sound speed in the non-linear spherical collapse model is discussed for the Einstein-de Sitter-like (EdSGDM) and AGDM models. By assuming that the vacuum component responsible for the accelerated expansion of the Universe is not clustering and therefore behaving similarly to the cosmological constant $\Lambda$, we show how the change in the GDM characteristic parameters affects the linear density threshold for collapse of the non-relativistic component ($\delta_c$) and its virial overdensity ($\Delta_V$). We found that the generalized dark matter equation of state parameter $w_{\text{gdm}}$ is responsible for lower values of the linear overdensity parameter as compared to the standard spherical collapse model and that this effect is much stronger than the one induced by a change in the generalized dark matter sound speed $c_{s,\text{gdm}}^2$. We also found that the virial overdensity is only slightly affected and mostly sensitive to the generalized dark matter equation of state parameter $w_{\text{gdm}}$. These effects could be relatively enhanced for lower values of the matter density. Finally, we found that the effects of the additional physics on $\delta_c$ and $\Delta_V$, when translated to non-linear observables such as the halo mass function, induce an overall deviation of about 40% with respect to the standard $\Lambda$CDM model at late times for high mass objects. However, within the current linear constraints for $c_{s,\text{gdm}}^2$ and $w_{\text{gdm}}$, we found that these changes are the consequence of properly taking into account the correct linear matter power spectrum for the GDM model while the effects coming from modifications in the spherical collapse model remain negligible.

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I. INTRODUCTION

Nowadays, most of the cosmological data suggests a cosmic expansion history with a flat geometry and some sort of dark energy, usually in the form of the cosmological constant $\Lambda$, in order to explain the recent accelerating expansion of the universe. Assuming that large scale structure formed thanks to the gravitational interaction of the cold dark matter (CDM) component, the resulting standard model of cosmology is then dubbed $\Lambda$CDM (see [1, 2] for a review). In this model, the cosmological constant is a fluid with constant equation of state $w = -1$ and energy density $\rho_{\Lambda}$, both constant in time and space, that is usually associated to the vacuum energy density. The CDM component is instead described as a non-relativistic fluid whose influence is only gravitational. Together, the cosmological constant and the CDM amount to approximately 95% of the total energy budget, with the remaining 5% in form of baryons and a negligible amount, today, of relativistic particles (photons and neutrinos) [3].

However, with the advent of stage IV surveys like DESI [4], Euclid [5], LSST [6], WFIRST [7], and the SKA [8], providing high accuracy data especially on small scales, one of the most challenging problems is to understand the role played by the different cosmic components in the non-linear regime of gravitational clustering. This aspect could be tackled through different approaches, among which we recall the halo model [9], where one of the issues is to understand the interplay of different possible physical effects that contribute to determine the properties of virialized halos. One of the powerful tools to study the non-linear evolution of perturbations and formation of haloes is given by the popular spherical collapse model (SCM), introduced in a seminal paper by [10] to deal with a system made only of CDM. This model has been later extended and applied to study the evolution of density perturbations and structure formation in the presence of dark energy (DE), both homogeneous [11–16] and clustering [17,19]. In this work, we investigate further the non-linear evolution of matter perturbations by focusing on the generalized dark matter (GDM) model, which considers the dark matter fluid augmented by pressure, parametrized by a background equation of state $w_{\text{gdm}}$ and a non-vanishing sound speed $c_{s,\text{gdm}}^2$. The effects of this modelling on the expansion and linear perturbations have been recently studied in [21,25].

However, the standard SCM needs to be appropriately modified to have a recipe for the GDM to be able to explore the small scales that next stage surveys will probe and extract the maximum information from them. A recent approach, based on heuristic scaling of the spherical collapse model, was used to constrain cosmology [24]. Here we want to determine the effect of GDM on non-linear perturbations starting from first principles within...
the framework of the spherical collapse model.

We restrict our analysis to an Einstein-de-Sitter-like (EdSGDM) model where \( \Omega_m = 1 \) and \( \Omega_\Lambda = 0.0 \), and a flat ΛCDM cosmology. For the ΛCDM model, we assume the following cosmological parameters: \( \Omega_m = 0.3 \), \( \Omega_\Lambda = 0.7 \) and \( h = 0.7 \). In particular, we discuss how the linear overdensity threshold for collapse (\( \delta_c \)) and the virial overdensity (\( \Delta_v \)) change while changing the properties of the dark matter component.

The paper is organised as follows: in Section II we give a brief description of the spherical collapse model for generalized dark matter and derive the appropriate equations describing the evolution of non-linear perturbations, by specialising on the virial overdensity \( \Delta_v \) and the linearly-extrapolated overdensity \( \delta_c \). In Section III we present our findings on the evolution of these two quantities as a function of the equation of state and effective sound speed of the matter component and translate our results into observable quantities such as the mass function. We conclude in Section IV.

II. THE GDM MODEL

In this work we assume that dark matter only interacts gravitationally with the other components and all fluid components satisfy the standard continuity equation \( \nabla_i T_{\mu\nu}^{i} = 0 \), where \( T_{\mu\nu}^{i} \) is the stress-energy tensor and for a perfect fluid reads as

\[
T_{\mu\nu}^{i} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},
\]

where \( \rho \), \( p \) and \( u_i \) are the density, the pressure, and the 4-velocity of each fluid, respectively, and \( g^{\mu\nu} \) the metric.

Contracting the continuity equation once with \( u_i \) and once with the projection operator \( h_{\mu\alpha} = g_{\mu\alpha} + u_\mu u_\alpha \), one obtains the relativistic expressions for the continuity and the Euler equations, respectively:

\[
\frac{\partial \rho_i}{\partial t} + \nabla_i (\rho_i \vec{v}_i) + \frac{P_i}{c^2} \nabla \cdot \vec{v}_i = 0, \tag{2}
\]

\[
\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \nabla) \vec{v}_i + \nabla \Phi + \frac{\nabla P_i}{\rho_i + P_i/c^2} = 0. \tag{3}
\]

Here \( \vec{v}_i \) is the three-dimensional velocity of each species, \( \Phi \) the Newtonian gravitational potential and \( \vec{r} \) denotes physical coordinates.

The 00 component of Einstein’s field equations gives the relativistic Poisson equation

\[
\nabla^2 \Phi = 4\pi G \sum_k \left( \rho_k + \frac{3P_k}{c^2} \right), \tag{4}
\]

where the potential is sourced by all the fluid components and \( \rho_k \) and \( P_k \) are the total density and pressure of each fluid. We define, in fact \( \rho_k = \bar{\rho}_k + \delta \rho_k \) and \( P_k = \bar{P}_k + \delta P_k \), where overbarred quantities represent the background.

The background continuation equation for the fluid \( i \) is

\[
\dot{\bar{\rho}}_i + 3H (\bar{\rho}_i + \bar{P}_i/c^2) = 0, \tag{5}
\]

where \( \bar{\rho}_i = \frac{3H^2 \Omega_i}{8\pi G} \) and \( \Omega_i \) is the fluid density parameter. To solve the previous expression, it is necessary to specify a relation between pressure and density. This is usually done by introducing the background equation-of-state parameter \( w_i = P_i/\rho_i c^2 \), so that one solves the equation \( \bar{\rho}_i + 3H (1+w_i) \bar{\rho}_i = 0 \), once the time-dependency of \( w_i \) is provided.

To study the perturbations, we introduce comoving coordinates \( \bar{\vec{x}} = \vec{r}/a \), with \( a \) the scale factor, and define

\[
\rho_i(\bar{\vec{x}}, t) = \bar{\rho}_i(1 + \delta_i(\bar{\vec{x}}, t)), \tag{6}
\]

\[
P_i(\bar{\vec{x}}, t) = \bar{P}_i + \delta P_i, \tag{7}
\]

\[
\Phi(\bar{\vec{x}}, t) = \Phi_0(\bar{\vec{x}}, t) + \phi(\bar{\vec{x}}, t), \tag{8}
\]

\[
\vec{u}_i(\bar{\vec{x}}, t) = a[H(a)\bar{\vec{x}} + \vec{u}_i(\bar{\vec{x}}, t)], \tag{9}
\]

where \( H(a) \) is the Hubble function and \( \vec{u}_i(\bar{\vec{x}}, t) \) the comoving peculiar velocity. We relate pressure perturbations to density perturbations by introducing the effective sound speed \( c_{s,i}^2 = \delta P_i/\delta \rho_i c^2 \). In a standard cold dark matter model, \( c_{s,i}^2 = 0 \), as there are no pressure perturbations.

Inserting Eqs. (6)–(9) into Eqs. (2)–(4), and taking into account the background, we derive the following equations for the perturbed quantities:

\[
\dot{\delta}_i + 3H (c_{s,i}^2 - w_i) \delta_i = - [1 + w_i] - [c_{s,i}^2 \delta_k \vec{\nabla} \cdot \vec{u}_k], \tag{10}
\]

\[
\dot{\vec{u}}_i + 2H \vec{u}_i + (\vec{\nabla} \cdot \vec{\nabla}) \vec{u}_i + \frac{\vec{\nabla} \Phi}{a^2} = 0, \tag{11}
\]

\[
\nabla^2 \phi = 4\pi G a^2 \sum_k \bar{\rho}_k (1 + 3c_{s,k}^2) \delta_k. \tag{12}
\]

Note that, as commonly done, we assumed a top-hat profile for the density perturbations. This leads to \( \nabla^2 \delta_i = 0 \), which considerably simplifies the equations. In addition, both \( w_i \) and \( c_{s,i}^2 \) are functions of time only. While this is justified for the equation of state, it is a simple approximation for the sound speed, but nevertheless in agreement with current literature [21].

The previous sets of equations allows us to study the evolution of the linearly-extrapolated overdensity \( \delta_c \), which represents an important ingredient for the mass function, a tool used to infer the effects of dark energy and modified gravity on some observables like cluster abundance. In this work, we will follow a similar line of thinking, but use the mass function to test properties of the dark matter component, rather than the gravitational sector.

In full generality, to derive the equation of motion of the (generalised) dark matter component, one takes the time derivative of Eq. (10) and substitutes in it the divergence of Eq. (11) and Eq. (12). Nevertheless, when doing so, the final expression becomes very complicated as both the equation of state and the effective sound speed can be time dependent. This expression will give very little insight to understand the physics of the problem. We will, therefore, first derive the full equation by defining additional coefficients which will help to write the final result.
in a rather compact form and, subsequently, we will specialise it to the simpler case where \( c_s^2 = 0 \), but \( w \neq 0 \). This will correspond to the case where dark matter fully clusters.

Following [26], we define the following quantities:
\[
A_i \equiv 3H \left( c_s^2 - w_i \right) \delta_i, \quad B_i \equiv 1 + w_i + (1 + c_s^2) \delta_i,
\]
so that Eq. (10) can be written as
\[
\ddot{\delta}_i + A_i + B_i \theta_i = 0,
\]
where \( \theta_i \equiv \nabla \cdot \vec{v}_i \).

At the same time, the divergence of Eq. (11) can be written as
\[
\dot{\theta}_i + 2H \theta_i + \frac{1}{3} \theta_i^2 + \nabla^2 \phi = 0,
\]
where spherical symmetry is assumed.

Taking the time derivative of (13) and using Eq. (14) to replace \( \dot{\theta}_i \) and Eq. (13) for \( \theta_i \), we finally get
\[
\ddot{\delta}_i + A_i + \left( 2H - \frac{B_i}{\bar{B}_i} \right) \left( A_i + \dot{\delta}_i \right) - \frac{1}{3} \left( \frac{\delta_i + A_i}{B_i} \right)^2 - \frac{B_i}{\bar{B}_i} \nabla^2 \phi = 0.
\]

For \( c_s^2 = w = 0 \), \( A_i = 0 \) and \( B_i = 1 + \delta_i \), leading to the standard equation describing matter perturbations in the presence of the cosmological constant or smooth dark energy:
\[
\ddot{\delta}_i + 2H \dot{\delta}_i - \frac{4}{3} \frac{\delta_i^2}{1 + \delta_i} - 4\pi G \sum_k \delta_k \delta_k = 0.
\]

Note that here the sum over the perturbed species is done for baryons (considered to be a pressureless fluid) and (generalised) dark matter. Therefore, we need to solve two differential equations of motion for the perturbations, one for baryons and one for generalised dark matter. Nevertheless, since baryons are subdominant at all times, considering only the GDM component would not alter our conclusions.

Let us now consider a specific case where \( c_s^2, i = w_i \) to grasp more understanding of the evolution of matter perturbations. The previously defined coefficients simplify to \( A_i = 0 \) and \( B_i = 1 + w_i(1 + \delta_i) \), and perturbations are adiabatic, with \( P = w_{\text{gdm}} c_s^2 \rho \) also at the perturbative level. Similarly to what was shown in [13] for homogeneous dark energy models, the equation of motion now reads
\[
\ddot{\delta}_i + \left( 2H - \frac{\dot{w}_{\text{gdm}}}{1 + w_{\text{gdm}}} \right) \dot{\delta}_i - \frac{1}{3} \frac{1 + 3 w_{\text{gdm}}}{1 + w_{\text{gdm}}} \frac{\delta_i^2}{1 + \delta_i} - (1 + w_{\text{gdm}}) (1 + \delta_i) \frac{\nabla^2 \phi}{\bar{a}^2} = 0,
\]
where, for simplicity, from now on, we drop the index \( i \) and consider only the expressions for generalised dark matter.

These expressions show that the non-linear dynamics of matter perturbations can be heavily affected by the presence of a background equation-of-state parameter \( w_{\text{gdm}} \) and therefore we expect its value to be severely constrained. Similar conclusions can be reached for the effective sound speed \( c_s^2 \), as a value different from zero defines a sound horizon scale associated to perturbations which generally implies that the fluid is not fully clustering.

To determine the virial overdensity \( \Delta_V \), we assume energy conservation during the collapse. This condition leads to a relation between the potential and kinetic energy of the collapsing sphere at turn-around and virialization time [27]:
\[
U_{\text{gdm,ta}} + U_{\Lambda,\text{ta}} = U_{\text{gdm,vir}} + T_{\text{gdm,vir}} + U_{\Lambda,\text{vir}} + T_{\Lambda,\text{vir}},
\]
where \( U \) and \( T \) are the potential and kinetic energy, respectively, of the GDM and dark energy \( \Lambda \) component. The subscripts \( \text{ta} \) and \( \text{vir} \) refer to turn-around and virialization, respectively. For simplicity, we will assume the dark energy component to be in the form of a cosmological constant, but our results can be easily extended to more general models.

The potential energy for a fluid endowed with pressure as the GDM is \( U_{\text{gdm}} = -\frac{3}{5} \left( 1 + 3 w_{\text{gdm}} \right) \frac{G M^2}{R^2} \) and for the cosmological constant is \( U_{\Lambda} = \frac{4\pi}{3} G M \rho_a R^2 \), where \( M \) and \( R \) are the mass and the radius of the spherical perturbation, respectively. For a system with the potential energy of the form \( U \propto R^n \), the kinetic energy will be \( T = n U/2 \) [29]. Then, according to the virial theorem, we find
\[
U_{\text{gdm,ta}} + U_{\Lambda,\text{ta}} = \frac{1}{2} U_{\text{gdm,vir}} + 2 U_{\Lambda,\text{vir}}.
\]

Defining \( \theta = \frac{\rho_{\text{vir}}}{\rho_{\text{gdm}}} \) and \( \eta = \frac{\rho_{\text{vir}}}{\rho_{\text{ta}}} \) as in [12, 29], we find a cubic equation describing the evolution of \( \eta \)
\[
\theta \eta^3 + \left( 1 + \frac{\theta}{2} \right) \eta - 1/2 = 0,
\]
where we used
\[
\left( \frac{\rho_{\text{vir}}}{\rho_{\text{ta}}} \right)_{\text{vir}} = \theta \eta^3 \left( \frac{\rho_{\text{vir}}}{\rho_{\text{ta}}} \right)^{-3(1+w_{\Lambda})}.
\]
In the previous expression, \( w_{\Lambda} = -1 \) and \( \rho_{\text{vir}} = \rho_{\text{X}} + 3 \rho_{\text{X}}/c_s^2 \).

Solving for \( \eta \), the virial overdensity at collapse redshift \( z_c \) is
\[
\Delta_V(z_c) = \frac{\rho_{\text{vir}}}{\rho_{\text{vir}}} - \eta - 3 \rho_{\text{cluster}} \left| \frac{\rho_{\text{cluster}}}{\rho_{\text{ta}}} \right| \left( 1 + \frac{z_{\text{ta}}}{1 + z_{\text{coll}}} \right)^3,
\]
where \( \rho_{\text{cluster}} = \rho(1 + \delta) \) is the total density of the perturbation. Since the constrained values for \( w_{\text{gdm}} \ll 1 \) and \( c_s^2 \ll 1 \) [23, 24], we assumed, for simplicity, that matter scales as in the standard model CDM model.
III. RESULTS

In this section we present some results for the spherical collapse model for the generalised dark matter models previously discussed, taking into account the effects of both the background equation-of-state parameter \( w_{\text{gdm}} \) and the effective sound speed \( c_{s,\text{gdm}}^2 \). We concentrate on the linear overdensity parameter \( \delta_c \) and the virial overdensity \( \Delta_V \).

To evaluate their evolution, we follow [13, 19] and we look for an initial overdensity \( \delta_{\text{ini}} \) such that the non-linear equation (17), in the general case where \( w_{\text{gdm}} \neq c_{s,\text{gdm}}^2 \) and both not null, diverges at the chosen collapse time. This same value is then used as initial condition of the linearised version of (17), which describes the evolution of \( \delta_c \). The value of \( \Delta_V \), instead, simply follows by evaluating \( \eta \) as explained in [12], which, as said above, is an approximation to the true behaviour of the GDM, but due to the strong constraints, this does not introduce a significant bias.

In Fig. 1 we show the evolution of the critical overdensity \( \delta_c \) as a function of redshift \( z \) assuming as GDM models an EdS- and ΛCDM-like flat cosmological model. Different curves refer to different values of the effective sound speed, while keeping \( w_{\text{gdm}} = 0 \) as for the standard cold dark matter model. This setup allows us to study the effect of the modified clustering properties of dark matter. Note that for stability reasons, \( c_{s,\text{gdm}}^2 > 0 \). The values chosen for the effective sound speed are motivated by the constraints obtained studying the evolution of linear perturbations [24, 25]. For comparison, in cyan, we also show the evolution of the reference ΛCDM cosmology.

As expected, all the models asymptotically approach the EdS limit at high redshifts, regardless of the sound speed value. Differences for \( \delta_c \) between the AGDM and the standard ΛCDM model are absolutely negligible, and likely due to numeric, except for very high values of the sound speed, i.e. \( c_{s,\text{gdm}}^2 \gtrsim 10^{-3} \). This shows that to modify the evolution of \( \delta_c \), relatively high values of the sound speed are required.

As the sound speed \( c_{s,\text{gdm}}^2 \) influences how much the fluid collapses, we can understand the dependence of \( \delta_c \) if we vary this parameter. As the sound speed increases, a higher \( \delta_c \) is needed, because there is an additional pressure effect that resist the collapse and opposes to structure formation. We remind the reader that a higher value of the sound speed implies a smoother component.

We do not show the effect of the sound speed on the virial overdensity \( \Delta_V \) as this parameter is not directly included in the definition of \( \Delta_V \), but it enters in it through the non-linear evolution of matter perturbations.

In Fig. 2 we present the evolution of \( \delta_c \) (top panel) and \( \Delta_V \) (bottom panel) as a function of redshift \( z \) for different values of the generalized dark matter equation-of-state parameter \( w_{\text{gdm}} \), while setting \( c_{s,\text{gdm}}^2 = 0 \). It is immediately clear that \( w_{\text{gdm}} \) has a much stronger effect than that induced by the sound speed and it is more pronounced for \( \delta_c \) than \( \Delta_V \), in particular for \( w_{\text{gdm}} = 10^{-3} \), where the relative difference with the ΛCDM result is 0.3%. For smaller values of \( w_{\text{gdm}} \), the relative difference is a few times smaller and likely due to numerical artifacts.

To see why the equation-of-state parameter \( w_{\text{gdm}} \) has a stronger effect than \( c_{s,\text{gdm}}^2 \), we remind the reader that when \( w_{\text{gdm}} = 0 \), the background expansion history is modified and pressure effects are not negligible even at early times, while \( c_{s,\text{gdm}}^2 \) only affects the perturbations. This also explains why increasing \( w_{\text{gdm}} \) leads to a decrease of \( \delta_c \): a positive \( w_{\text{gdm}} \) makes the contribution of the dark matter component less important (as it decreases faster) than that of the cosmological constant at late times and to overcome the additional contribution to the expansion one needs lower overdensities to achieve the collapse.

So far, we have considered the influence of \( w_{\text{gdm}} \) and \( c_{s,\text{gdm}}^2 \) separately, but to span the full parameter space of the model, we need to consider their combined effect and we do so by solving the full equation of motion (15). We verified that for realistic values of the two parameters, the resulting \( \delta_c \) is in agreement with ΛCDM. For the highest values considered for both \( w_{\text{gdm}} \) and \( c_{s,\text{gdm}}^2 \), the relative difference is still below the percent level.

Our analysis led us to the conclusion, largely expected, that the values allowed for \( w_{\text{gdm}} \) and \( c_{s,\text{gdm}}^2 \) from previous works on the evolution of linear perturbations have a negligible impact on \( \delta_c \) and \( \Delta_V \). Nevertheless, these two quantities are not directly observable and therefore it is important to study how the mass function is influenced.

Before we study the impact on the mass function, though, we want to take a step further and investigate the combined action of varying the background matter den-
place for low matter density parameters for both $\Omega_m$ and $\delta_c$. We present our results in Fig. 3. Stronger effects take place for low matter density parameters for both $\delta_c$ and $\Delta_V$, with deviations up to a few percent for accepted values of the matter density parameter $\Omega_m$. Although these numbers are small, we remind the reader that their combined effect enters exponentially into the evaluation of the halo mass function, therefore even small differences can be amplified and lead to appreciable differences, therefore making it a very sensible probe for cosmology when its high-mass end is investigated (i.e., massive galaxy clusters).

The halo mass function is defined as \[ \frac{dn(M)}{dM} = \frac{\bar{\rho}}{M} \frac{d\nu}{dM} \mathcal{F}(\nu), \] where $\bar{\rho}$ is the mean density today, $\mathcal{F}(\nu)$ the multiplicity function and $\nu = \delta_c/\sigma(M)$ with $\sigma(M)$ the variance within a sphere of radius $R$ and mass $M = 4\pi/3 \bar{\rho} R^3$ for a cosmology described by a linear matter power spectrum $P(k)$ which we computed by adapting a modified version of the Einstein-Boltzmann code CLASS \[22, 23, 25\]. The mass variance is defined as $\sigma^2(M, z) = \frac{1}{\nu} \int k^3 P(k, z) W^2(k R) dk$, where $W(k R)$ is an appropriate window function representing the Fourier transform of the top-hat function in real space.

For the multiplicity function, we adopt the functional form proposed by \[31, 32\]
\[ \nu F_{ST}(\nu) = A \sqrt{\frac{2\pi}{\nu}} \left[ 1 + \left( \frac{1}{\nu \sigma^2} \right)^p \right] \nu \exp \left\{ -\frac{\nu \sigma^2}{2} \right\}, \] with the parameters $A$, $a$ and $p$ more recently fitted by \[36\] using the abundance matching technique in $N$-body simulations. While the mass function depends explicitly on $\delta_c$, the fitted parameters are a function of the virial overdensity $\Delta_V$ and read
\[ a = 0.4332 x^2 + 0.2263 x + 0.7665, \]
\[ p = -0.1151 x^2 + 0.2554 x + 0.2488, \]
\[ A = -0.1362 x + 0.3292, \]
where $x = \log (\Delta(z)/\Delta_V(z))$ and $\Delta(z)$ is a given overdensity, such as a multiple of the critical density$^1$.

$^1$ Note that our definition of the virial overdensity refers to the background density rather than the critical one. Therefore, in the evaluation of the mass function we scale it by $\Omega_m$, where necessary.
FIG. 3. In the left panels we show the evolution of the linear critical density parameter $\delta_c$ (upper panel) and of the virial overdensity $\Delta_V$ (bottom panel) as a function of the collapse redshift $z$ for different values of the matter density parameter $\Omega_m$ assuming $w_{gdm} = 5 \times 10^{-4}$ and $c_{gdm}^2 = 5 \times 10^{-7}$ as reference values for the parameters of the GDM model. In the right panels, we show the relative difference between $\Lambda$CDM and $\Lambda$GDM.

Thus, the overall effect on the mass function is given by the combination of a few factors: a different background expansion induced by $w_{gdm}$, the evolution of structures given by the linear growth factor $D_s(a)$ and $\delta_c$, the evolution of $\Delta_V$ and the linear matter power spectrum $P(k)$.

We present the results of our investigation in Fig. 4, where we show the mass function at different redshifts, considering both $\Lambda$CDM and $\Lambda$GDM models assuming $w_{gdm} = 5 \times 10^{-4}$ and $c_{gdm}^2 = 5 \times 10^{-7}$ for the latter, two values within the constraints obtained from probes of large scale structures formation [24] [25].

We note immediately that despite $\delta_c$ is hardly affected and $\Delta_V$ only at the percent level by the combined action of the equation-of-state parameter $w_{gdm}$ and the sound speed $c_{gdm}^2$ as we previously discussed, the differential mass function (top left panel) shows strong signatures due to the additional physics investigated. One of the main reason is the strong suppression of power in the linear matter power spectrum due to $c_{gdm}$ especially on small scales, as shown by [21] and as we checked but do not show here by changing the value of $\Delta_V$. As we will discuss more in detail later, this directly explains why there is, in general, a lower number of structures, especially for small mass objects. Finally, a part of the contribution could also comes from $\Delta_V$, as we showed that at the non-linear level this is the quantity more affected.

More quantitatively, as evinced from the top right panel, there is a decrement of about 75% and 80% for objects of $\approx 5 \times 10^{13} M_\odot h^{-1}$ for $z = 0$ and $z = 2$, respectively. At higher masses, differences between the $\Lambda$CDM and $\Lambda$GDM models are comparable and of the order of 40%. At low masses though, according to expectations, differences steadily increase with redshift.

To see why we obtain the counter-intuitive result of stronger effects at low masses, in Fig. 5 we show the evolution of the square root of the variance $\sigma(M)$ as a function of the perturbation mass $M$ for different redshifts $z$. We consider both the $\Lambda$CDM and the $\Lambda$GDM
models with the same set of parameters we used to study the halo mass function. We immediately see that $\sigma_{\Lambda GDM} < \sigma_{\Lambda CDM}$ at all masses and redshifts, thus explaining the smaller number of halos in the GDM model. In addition, and this is the key to explain the results for the halo mass function, stronger differences occur at low masses and low redshifts, as the $\Lambda GDM$ model approaches $\Lambda CDM$ at higher redshifts.

To disentangle the effect of the matter power spectrum from that of the virial overdensity $\Delta_V$ entering in the definition of the parameters of the mass function, we evaluate the mass function for $\Lambda CDM$ and $\Lambda GDM$ by assuming the same $\Lambda CDM$ linear matter power spectrum for both models, but keeping the other quantities relative to each model. We show this in the bottom left panel of Fig. 4 with the relative difference in the bottom right panel of the same figure.

With respect to before, we now see a completely different situation, which is more in line with usual expectations as the major differences occur, as one would expect, for high-mass objects. Nevertheless, differences are very small and probably more likely due to numerical effects rather than genuine physical effects.

These results thus show that the additional physics of the dark matter sector has a strong impact on the observables, not only on the linear evolution of perturbations but on the non-linear evolution of the formation of structure through the halo mass function (top panel). However, within the current linear constraints on $c_{s,gdm}^2$ and $w_{gdm}$, there is no significant modification coming from changes to $\delta_c$ and $\Delta_V$, and this is confirmed in the bottom panel where we replaced the correct linear matter power spectrum for the GDM with that expected from the $\Lambda CDM$ cosmology.

Before concluding, we would like to comment upon the non-linear evolution of the matter power spectrum. We did not investigated it as we do not have any recipe to reliably evaluate it, rather than using some fitting func-
tion for the ΛCDM model [37] or the halo model [9]. One possible heuristic approach would be to consider the non-linear evolution of the ΛCDM model, derive a non-linear transfer function based on the linear spectrum and use this to derive the GDM non-linear matter power spectrum. From the analysis of the variance in Fig. 5, we expect strong differences, which would propagate to other observables such as the shear power spectrum. Due to the limited validity of the approach, based on the strong differences already seen at the linear level, we prefer not to pursue the study of the non-linear matter power spectrum for GDM models.

IV. CONCLUSIONS

In this work we discussed how the equation-of-state parameter $w_{\text{gdm}}$ and the sound speed $c^2_{s,\text{gdm}}$ for generalised dark matter (GDM) affect the properties of the two main quantities of the spherical collapse model, the linear overdensity parameter $\delta_c$ and the virial overdensity $\Delta_V$.

We compared them with the corresponding quantities derived for the standard ΛCDM model for different values of $w_{\text{gdm}}$ and $c^2_{s,\text{gdm}}$. We demonstrated that the parameter mostly affecting their evolution is the background equation of state $w_{\text{gdm}}$, while the sound speed $c^2_{s,\text{gdm}}$, within the constraints from linear probes, has a negligible contribution to the overall non-linear evolution.

The effect of the two additional quantities describing the GDM properties is strongly dependent on the matter density parameter $\Omega_m$. We saw that the lower the matter density parameter, the stronger are the deviations from a ΛCDM model, as higher initial overdensities are required to overcome the accelerated expansion of the universe to allow structures to collapse.

Since the spherical collapse parameters are not directly observable, we used their evolution, together with the linear matter power spectrum $P(k)$ obtained for the GDM model, as building blocks for the halo mass function. We found that major deviations take place on smaller mass objects, rather than at higher masses and this is a direct consequence of the modifications on the linear matter power spectrum, as we verified by using the linear matter power spectrum of the ΛCDM model for both cosmologies (bottom panels of Fig. 1). In this case in fact, differences are much smaller and in line with expectations: the decrement is stronger at higher masses, albeit in general at the subpercent level. The overall effect of the GDM dynamics is that of decreasing the number of halos, as additional pressure terms kick in in the equations of motion. This is a strong effect, up to 70% already at $z = 0$ and 80% at $z = 2$ and of the order of 40% for objects of $M \approx 10^{15} M_\odot h^{-1}$. Due to the abundance of galactic objects, this effect should be easily seen and therefore put strong constraints on the GDM parameters.

We finally note that, while both important to completely characterize the dynamics of GDM models, the equation-of-state parameter $w_{\text{GDM}}$ and the sound speed $c_{s,\text{GDM}}$ act on two different sets of observables. The first is important mainly at the non-linear level, while the latter is very important for the evolution of the linear matter power spectrum, which reflects, of course, on the halo mass function. Therefore, while heavily constrained at the linear level already, the two GDM parameters can be further strongly constrained if probes at the non-linear level are combined.

With the effect on the mass function being detectable with present and most importantly, near-future deep surveys, we remark that a proper analysis of the additional degrees of freedom in a particular cosmological model need to be taken into full consideration, especially if we do not fix the matter density parameter to the fiducial value. As a practical example for GDM models, a robust measurement of the matter power spectrum and of the mass function, or more generally in the non-linear regime, can lead to stronger constraints on the GDM parameters, as it is now the case for modified gravity models, such as $f(R)$ cosmologies [38]. Our analysis can also serve to explain the difference in the value of $\sigma_8$ between local and deep probes, that common modifications to ΛCDM failed to achieve [39].

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