DISORDERED ARCS AND HARER STABILITY

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Abstract. We give a new proof of homological stability with the best known isomorphism range for mapping class groups of surfaces with respect to genus. The proof uses the framework of Randal-Williams–Wahl and Krannich applied to disk stabilization in the category of bidecorated surfaces, using the Euler characteristic instead of the genus as a grading. The monoidal category of bidecorated surfaces does not admit a braiding, distinguishing it from previously known settings for homological stability. Nevertheless, we find that it admits a suitable Yang–Baxter element, which we show is sufficient structure for homological stability arguments.

1. Introduction

Let $S_{g,r}$ be a surface of genus $g$ with $r$ boundary components and $s$ punctures. The mapping class group $\Gamma(S_{g,r}) := \pi_0 \text{Homeo}(S_{g,r} \text{ rel } \partial S)$ of $S$ satisfies homological stability: the homology group $H_i(\Gamma(S_{g,r}); \mathbb{Z})$ is independent of $g$ and $r$ when $g$ is large relative to $i$. This stability result was originally proved by Harer in [Har85], and later improved by Ivanov, Boldsen and Randal-Williams [Iva89, Bol12, RW16], see also [Har93, Wah13, HV17, GKRW19]. We recast the result here as a stability theorem in the category of bidecorated surfaces, and give a new proof of the best know stability range using the most straightforward inductive argument originally designed by Quillen, and formalized in [RWW17, Kra19]. Our proof at the same time illustrates how little is needed to run the stability machines of these two papers.

Our main stability result is the following, recovering precisely the ranges of [Bol12, Thm 1] and [RW16, Thm 7.1 (i),(ii)]:

**Theorem A.** Let $S_{g,b}$ be a surface of genus $g \geq 0$, with $r \geq 1$ marked boundary components and $s \geq 0$ punctures, and let $\Gamma(S_{g,r}) = \pi_0 \text{Homeo}(S_{g,r} \text{ rel } \partial S)$ denote its mapping class group. The map

$$H_i(\Gamma(S_{g,r}); \mathbb{Z}) \to H_i(\Gamma(S_{g,r+1}); \mathbb{Z})$$

induced by gluing a pair or pants along one boundary component is always injective, and an isomorphism when $i \leq \frac{2g}{3}$, and the map

$$H_i(\Gamma(S_{g,r+1}); \mathbb{Z}) \to H_i(\Gamma(S_{g+1,r}); \mathbb{Z})$$

induced by gluing a pair of pants along two boundary components is an epimorphism when $i \leq \frac{2g+1}{3}$ and an isomorphism when $i \leq \frac{2g-2}{3}$.

Combining the two maps in the theorem gives a genus stabilization that is known to be close to optimal by a computation of Morita [Mor03] and low dimensional computations, see Remarks 2.5 and 4.11. While we do not know whether the two ranges in the above statement can be individually improved, it is remarkable that three rather different proofs (those of Boldsen [Bol12], Randal-Williams [RW16], and ours) end up with the exact same ranges.

A particular feature of our proof is that the two maps occurring in the theorem will be for us “the same map”, namely a disk stabilization in the category $\mathbf{M}_2$ of bidecorated surfaces. A bidecorated surface is a surface $S$ with two marked intervals $I_0, I_1$ in its boundary. The two intervals may lie on the same or on different boundary components. Morphisms in $\mathbf{M}_2$ are mapping classes, i.e. isotopy classes of homeomorphisms, and $\mathbf{M}_2$ admits a monoidal structure # defined by identifying the marked intervals in pairs.

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Our main example of a bidecorated surface will be the bidecorated disk \( D \). As shown in Lemma 3.1, taking sums of the disk with itself in \( M_2 \) produces surfaces of any genus: \( D^{2g+1} \) is a surface \( S_{g,1} \) of genus \( g \) with a single boundary component, while \( D^{2g+2} \) is a surface \( S_{g,2} \) of genus \( g \) with two boundary components, each containing a marked interval. To obtain any surface \( S_{g,r} \) with \( r \geq 1 \), we will consider the object \( S \# D^{2g} \) in \( M_2 \), for \( S = S_{g,r} \) a genus 0 surface with \( r \) boundary components and \( s \) punctures. Now the maps in Theorem A are precisely the disk stabilization maps in \( M_2 \):

\[
\text{Aut}_{M_2}(S \# D^{2g}) \xrightarrow{\#D} \text{Aut}_{M_2}(S \# D^{2g+1}) \xrightarrow{\#D} \text{Aut}_{M_2}(S \# D^{2g+2})
\]

for these particular choices of surfaces.

Theorem A is thus the statement that disk stabilization \( \#D \) in \( M_2 \) induces isomorphisms on the homology of these automorphism groups in a range. We show in the present paper that this result can be obtained as a direct application of the main result of [Kra19], from which an additional stability statement with twisted coefficients automatically follows. We start by stating this additional result.

**Twisted coefficients.** Fix \( r \geq 1 \) and \( s \geq 0 \). In our setting, a coefficient system \( F \) for the mapping class group \( \Gamma(S^s_{g,r}) \) is a collection of \( \mathbb{Z}[\Gamma(S^s_{g,r})] \)-modules \( F_{2g} \) and \( \mathbb{Z}[\Gamma(S^s_{g, r+1})] \)-modules \( F_{2g+1} \), for each \( g \geq 0 \), together with maps

\[
F_n \longrightarrow F_{n+1}
\]
equivariant with respect to the disk stabilization and satisfying that a certain Dehn twist acts trivially on the image of \( F_n \) in \( F_{n+2} \) under double stabilization (see Definition 4.6). Given a coefficient system, one can define a notion of degree; a constant coefficient systems has degree 0 and for example the coefficient system \( F_{2g+i} = H_1(S^s_{g, r+i}; \mathbb{Z})^\otimes k \), \( i \in \{0, 1\} \), has degree \( k \) (see Example 4.7).

We obtain the following twisted stability result:

**Theorem B.** Let \( \Gamma(S^s_{g, r}) \) be as in Theorem A, and \( F \) be a coefficient system of degree \( k \). The stabilization map

\[
H_i(\Gamma(S^s_{g,r}); F_{2g}) \longrightarrow H_i(\Gamma(S^s_{g,r+1}); F_{2g+1})
\]
is an epimorphism for \( i \leq \frac{2g-3k-2}{4} \) and an isomorphism for \( i \leq \frac{2g-3k-5}{3} \), and the map

\[
H_i(\Gamma(S^s_{g,r+1}); F_{2g+1}) \longrightarrow H_i(\Gamma(S^s_{g+1,r}); F_{2g+2})
\]
is an epimorphism for \( i \leq \frac{2g-3k-1}{4} \) and an isomorphism for \( i \leq \frac{2g-3k-4}{3} \). In these bounds, \( 3k \) can be replaced by \( k \) if \( F \) is in addition split in the sense of Definition 4.6.

Stability theorems for mapping class groups with twisted coefficients can be found in the work of Ivanov, Boldsen, Randal-Williams–Wahl, and Galatius–Kupers–Randal-Williams [Blo12, Iva93, RWW17, GKRW19]. The results are not easy to compare as the types of coefficient system that are permitted depend on the paper, but some classical examples such as the one described above fit all frameworks (see Remarks 4.8 and 4.11 for more details).

**Braided action and Yang–Baxter operators.** We want to obtain Theorems A and B as consequences of Theorems A and C of [Kra19]. For this, we first have to show that disk stabilization in the monoidal category \( (M_2, \#) \) comes from an action of a braided monoidal groupoid.

Let \( B \) denote the groupoid of braid groups, with object the natural numbers and the braid group \( B_n \) as automorphisms of \( n \). We will construct an action of \( B \) on \( M_2 \) using an appropriate *Yang–Baxter operator* in \( M_2 \): The sum of bidecorated disks \( D \# D \) in \( M_2 \) is a cylinder, whose mapping class group is an infinite cyclic group generated by the Dehn twist \( T \) along the core circle of the cylinder. It turns out that this morphism \( T \in \text{Aut}_{M_2}(D \# D) \) is a Yang–Baxter operator in \( M_2 \), in the sense that it satisfies the equation

\[
(T \# 1)(1 \# T)(T \# 1) = (1 \# T)(T \# 1)(1 \# T)
\]
in \( \text{Aut}_{M_2}(D^3) \). The same holds for the inverse twist \( T^{-1} \), that will turn out more convenient for us. As explained in Section 5.1, we get an associated strong monoidal functor \( B \longrightarrow M_2 \).
taking the object $n$ to $D^\# n$. The corresponding homomorphism $B_n \to \text{Aut}_{\mathcal{M}_2}(D^\# n)$ can be identified with the geometric embedding in the sense of [Wa99], associated to the chain of curves $a_1, \ldots, a_{n-1}$ in

$$D^\# n = D \# D \# \cdots \# D,$$

where the $i$th curve $a_i$ is the core circle in the $i$th cylinder $D\# D$ in the above sum, see Lemma 3.5 and Example 3.3.

The strong monoidal functor $\mathcal{B} \to \mathcal{M}_2$ from above endows $\mathcal{M}_2$ with the structure of an $E_1$-module over the braid groupoid $\mathcal{B}$, and since the latter is braided monoidal, we can apply the results of [Kra19] to study disk stabilization in $\mathcal{M}_2$.

**Remark 1.1.** Homological stability frameworks such as [RWW17, Kra19, GKR19] require an $E_2$-algebra, or the weaker structure of $E_1$-module over an $E_2$-algebra, as input. This is a priori a lot of data, and it may be that the most natural choice in a given context simply does not admit an $E_2$-structure. This turns out to be the case for the monoidal category of bidecorated surfaces $\mathcal{M}_2$: In the context of categories, $E_2$-structures are given by braided monoidal structures and we show in Section 5.3 that even the full monoidal subcategory of $\mathcal{M}_2$ generated by our stabilizing object, the disk $D$, does not admit a braiding. This distinguishes our situation from most previous examples of homological stability.

On the other hand, it does not take much to equip a given monoidal category $\mathcal{X}$ with the structure of an $E_1$-module over a braided monoidal category. In fact, as shown in Section 5.1 any Yang–Baxter operator in $\mathcal{X}$ determines a strong monoidal functor $\mathcal{B} \to \mathcal{X}$ from the braid groupoid $\mathcal{B}$, and thus endows $\mathcal{X}$ with the structure of an $E_1$-module over $\mathcal{B}$. This perspective also makes sense if $\mathcal{X}$ itself acts on a category $\mathcal{M}$, and one is interested in the stabilization

$$\mathcal{M} \oplus \mathcal{X} \xrightarrow{\oplus \tau} \mathcal{M} \oplus \mathcal{X} \to \cdots$$

induced by acting with an object $X$ of $\mathcal{X}$ admitting a Yang-Baxter operator $\tau \in \text{Aut}_{\mathcal{X}}(X \oplus X)$. The category $\mathcal{M}$ becomes this way likewise a module over $\mathcal{B}$, where the object $n$ of $\mathcal{B}$ acts on $A \in \mathcal{M}$ via $A \oplus n = A \oplus X^\oplus n$.

**Disordered arcs.** Given a category $\mathcal{M}$ as above, with the structure of an $E_1$-module over a monoidal category $\mathcal{X}$ with a distinguished Yang–Baxter operator $(X, \tau)$, such that acting by $X$ satisfies a certain injectivity property (see Proposition 3.4), the main result of [Kra19] implies that homological stability for stabilization with $X$ is controlled by the connectivity of certain complexes of destabilizations. In the category of bidecorated surfaces $\mathcal{M}_2$, stabilizing with the bidecorated disk $D$ corresponds homotopically to attaching an arc, and we show in Proposition 4.4 that the relevant complex of destabilizations for stabilizing a surface $S$ with a disk $n$ times identifies with the “disordered arc complex” of the surface $S\# D^\# n$. This is a simplicial complex whose vertices are isotopy classes of non-separating arcs in the surface with endpoints $b_0 = I_0(\frac{1}{2})$ and $b_1 = I_1(\frac{1}{2})$, and where a collection of isotopy classes forms a simplex if the classes can be represented by arcs that are disjoint away from the endpoints, are jointly non-separating, and such that the arcs have the same ordering at $I_0$ and $I_1$.

Writing $\mathcal{D}^\nu(S_{g,r}, b_0, b_1)$ for the disordered arc complex of a surface $S_{g,r}$ with marked points $b_0$ and $b_1$ in $\nu = 1$ or $\nu = 2$ boundary components, the main ingredient of our proof of homological stability is the following connectivity result:

**Theorem C.** (Theorem 2.4) The disordered arc complex $\mathcal{D}^\nu(S_{g,r}, b_0, b_1)$ is $(\frac{2g+r-3}{3})$-connected.

**Remark 1.2.** It is conjectured in [RWW17] that the complex of destabilizations is highly connected if and only if stability holds with all appropriate twisted coefficients. The slope $2/3$ bounds in Theorems 3.3 and 4.3 is precisely dictated by the same slope $2/3$ in Theorem C in the connectivity of the arc complex, which is the complex of destabilizations in that case. This connectivity bound is best possible among linear bounds as a better bound would prove an incorrect stability statement, see Remark 2.5.

\footnote{We called those disordered arcs because it is the opposite ordering convention than the one used in the “ordered arc complex” of [RW16].}
**Organization of the paper.** In Section 2, we prove the high connectivity of the disordered arc complex. In Section 3, we define the monoidal category of bidecorated surfaces \((\mathcal{M}_2, \#)\), as well as the action of the braid groupoid \(B\) on this category. In Section 4, we show Theorems A and B by showing that the disordered arc complex agrees with the complex of destabilizations, and applying the main result of Kra19. Finally, in Section 5 we explain the relationship between homological stability and Yang–Baxter operators, and show the non-braidedness of the category of bidecorated surfaces.

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2. High connectivity of the disordered arc complex

In this section, we prove that the disordered arc complex is highly connected. It will be defined as a subcomplex of the following simplicial complex of non-separating arcs:

**Definition 2.1.** Let \(S\) be an orientable surface with nonempty boundary, and let \(b_0, b_1\) be distinct points in \(\partial S\). The complex of non-separating arcs \(\mathcal{B}(S, b_0, b_1)\) is the simplicial complex whose \(p\)-simplices are collections of \(p+1\) distinct isotopy classes of arcs between \(b_0, b_1\) that admit representatives \(a_0, \ldots, a_p\) such that

(a) \(a_i \cap a_j = \{b_0, b_1\}\) for each \(i \neq j\) and

(b) \(S - (a_0 \cup \cdots \cup a_p)\) is connected.

For convenience, we will add a superscript \(\nu\) to the notation of the complex, with \(\nu = 1\) indicating that \(b_0, b_1\) lie on the same boundary component and \(\nu = 2\) indicating that they do not.

Note that the orientation of the surface defines orderings of the arcs \(a_0, \ldots, a_p\) representing a simplex at both \(b_0\) and \(b_1\).

**Definition 2.2.** Let \((S, b_0, b_1)\) be as before. The disordered arc complex is the subcomplex \(\mathcal{D}^\nu(S, b_0, b_1) \subseteq \mathcal{B}^\nu(S, b_0, b_1)\) consisting of those simplices \(\sigma\) that admit arc representatives \(a_0, \ldots, a_p\), again subject to (a), (b) satisfying in addition

(c) the ordering of the arcs at \(b_0\) agrees with the ordering of the arcs at \(b_1\).

The name “disordered” was chosen to contrast with the pre-existing ordered arc complex used by Ivanov Iva89 in the case \(\nu = 1\) and Randall-Williams RW16 in their proofs of homological stability for the mapping class group of surfaces; the “ordered” version is also a subcomplex of the \(\mathcal{B}^\nu(S, b_0, b_1)\), but with the requirement that the order of the arcs at \(b_1\) is reversed compared to the order at \(b_0\). Fixing an ordering condition has the effect that the action of the mapping class group is transitive on the set of \(p\)-simplices for each \(p\), see Har85 Lem 3.2. The ordered and disordered arc complexes represent the two extremes of how fast the genus of the surface decreases when cutting along larger and larger simplices: for the ordered arc complex, the genus goes down as fast as possible, essentially every time one removes an arc, while for the disordered arc complex, the genus goes down as slow as possible, only every other time:

**Proposition 2.3.** For a \(p\)-simplex \(\sigma = \langle a_0, \ldots, a_p \rangle \in \mathcal{D}^\nu(S_{g,r}, b_0, b_1)\), the surface \(S_{g,r} \setminus \sigma\) obtained by removing a tubular neighborhood of \(a_i\) for each \(i\) has genus \(g'\) with \(r'\) boundary components for

\[
g' = g - \left[ \frac{p+3-\nu}{2} \right]
\]

and

\[
r' = \begin{cases} 
    r + (-1)^\nu, & \text{if } p \text{ is even,} \\
    r, & \text{else.}
\end{cases}
\]

By surface we mean a topological 2-manifold \(S\) which is compact except for a finite number of punctures, i.e. there is a compact topological 2-manifold \(\overline{S}\) and an embedding \(i: S \hookrightarrow \overline{S}\) so that \(\overline{S} \setminus i(S)\) is a (possibly empty) finite union of points.

For convenience, we will add a superscript \(\nu\) to the notation of the complex, with \(\nu = 1\) indicating that \(b_0, b_1\) lie on the same boundary component and \(\nu = 2\) indicating that they do not.

Note that the orientation of the surface defines orderings of the arcs \(a_0, \ldots, a_p\) representing a simplex at both \(b_0\) and \(b_1\).

**Definition 2.2.** Let \((S, b_0, b_1)\) be as before. The disordered arc complex is the subcomplex \(\mathcal{D}^\nu(S, b_0, b_1) \subseteq \mathcal{B}^\nu(S, b_0, b_1)\) consisting of those simplices \(\sigma\) that admit arc representatives \(a_0, \ldots, a_p\), again subject to (a), (b) satisfying in addition

(c) the ordering of the arcs at \(b_0\) agrees with the ordering of the arcs at \(b_1\).

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**Proposition 2.3.** For a \(p\)-simplex \(\sigma = \langle a_0, \ldots, a_p \rangle \in \mathcal{D}^\nu(S_{g,r}, b_0, b_1)\), the surface \(S_{g,r} \setminus \sigma\) obtained by removing a tubular neighborhood of \(a_i\) for each \(i\) has genus \(g'\) with \(r'\) boundary components for

\[
g' = g - \left[ \frac{p+3-\nu}{2} \right]
\]

and

\[
r' = \begin{cases} 
    r + (-1)^\nu, & \text{if } p \text{ is even,} \\
    r, & \text{else.}
\end{cases}
\]
Proof. The number of boundary components $r'$ can be obtained by a direct inductive computation, with the genus $g'$ then deduced using the Euler characteristic. The computation is a special case of [Bol12] Prop 2.11, applied to the case where the permutation $\alpha$ is the inversion \([p(p-1)\ldots 0]\), once one computes that the genus $S(\alpha)$ of a neighborhood of the arcs is \([\frac{p+2g-2}{2}]\), e.g. using Corollary 2.15 of the same paper.

The complex $\mathcal{D}^\nu(S,b_0,b_1)$ is known to be $(2g+\nu-3)$-connected. (This was first stated in [Har85]; see [Wah08 Thm 3.2] or [Wah13 Thm 4.8] for a complete proof.) We will here use this fact to deduce that $\mathcal{D}^\nu(S_{g,r},b_0,b_1)$ is also highly-connected. While the ordered arc complex is $(g-2)$-connected [RW16 Thm A.1], the following result shows that the disordered arc complex is only slope $\frac{2}{3}$ connected with respect to the genus, despite being $\sim 2g$-dimensional.

**Theorem 2.4.** The disordered arc complex $\mathcal{D}^\nu(S_{g,r},b_0,b_1)$ is $(\frac{2g+\nu-5}{3})$-connected.

To prove the result, we use essentially the same argument as the one given in [RW16] in the ordered case.

Proof. Let $S = S_{g,r}$. In the case $g = 0$, the statement for $\mathcal{D}^1(S)$ is vacuous, and for $\mathcal{D}^2(S)$ it states that the complex is $(-1)$-connected, i.e. nonempty, which holds as any arc in the surface connecting $b_0$ and $b_1$ defines a vertex in $\mathcal{D}^2(S)$. We prove the remaining cases by induction on $g$.

Let $g > 0$. Suppose we are given $f : \partial D^{k+1} \rightarrow \mathcal{D}^\nu(S,b_0,b_1)$ for some $k \leq (2g+\nu-5)/3$. We wish to exhibit a nullhomotopy of this map. Since $(2g+\nu-5)/3 \leq 2g+\nu-3$, Theorem 3.2 in [Wah08] enables us to choose a map $\hat{f}$ such that the outer diagram

\[
\begin{array}{ccc}
\partial D^{k+1} & \xrightarrow{f} & \mathcal{D}^\nu(S,b_0,b_1) \\
\downarrow & \searrow & \downarrow \\
D^{k+1} & \xrightarrow{\hat{f}} & \mathcal{D}^\nu(S,b_0,b_1),
\end{array}
\]

commutes. Using PL-approximation, we may assume that $\hat{f}$ and $f$ are simplicial with respect to some PL-triangulation of $D^{k+1}$. We will repeatedly replace $\hat{f}$ until the dotted arrow exists, thereby giving the desired nullhomotopy.

Write $<_0$ and $<_1$ for the anti-clockwise orderings at $b_0$ and $b_1$. We call a $p$-simplex $\sigma$ in $D^{k+1}$ regular bad if $\hat{f}(\sigma) = (a_0, \ldots, a_{p'})$, indexed in such a way that $a_0 <_0 \cdots <_0 a_{p'}$ are anticlockwise at $b_0$, and there is $j > 0$ with $a_j <_1 a_0$ at $b_1$. Here $p' \leq p$ is the dimension of the image simplex $\hat{f}(\sigma)$, and $p' \geq 1$ if $\sigma$ is regular bad. This condition is “dense” in the sense that any simplex $\sigma$ in $D^{k+1}$ with image not included in $\mathcal{D}^\nu(S,b_0,b_1)$ must contain a regular bad simplex as a face. Thus it suffices to give a procedure for exchanging $\hat{f}$ with a map having strictly fewer regular bad simplices, while maintaining commutativity of the outer diagram (2.1).

Let $\sigma$ be a regular bad simplex of $D^{k+1}$ of maximal dimension $p$ and consider its link $Lk\sigma \subset D^{k+1}$. Maximality implies that $\hat{f}|_{Lk\sigma}$ factors as

$$\hat{f}|_{Lk\sigma} : Lk\sigma \rightarrow \mathcal{D}^\nu(S \setminus \hat{f}(\sigma), b'_0, b'_1) \rightarrow \mathcal{D}^\nu(S,b_0,b_1) \subseteq \mathcal{D}^\nu(S,b_0,b_1),$$

where $S \setminus \hat{f}(\sigma)$ is the closure of the surface obtained from $S$ by cutting out the collection of arcs $\hat{f}(\sigma)$, and $b'_0$ and $b'_1$ in $S \setminus \hat{f}(\sigma)$ are the first copies of $b_0$ and $b_1$ in the cut surfaces as depicted in Figure 1. Indeed, suppose that $\tau \in Lk\sigma$ and write $\hat{f}(\tau) = (a'_0, \ldots, a'_{p'})$. If $a_0 \leq a'_i$ at $b_0$ for any $a'_i$, then the simplex $\sigma * \langle a'_i \rangle$ is regular bad of a larger dimension, contradicting maximality. So we must have $a'_i < a_0$ at $b_0$ for each $i$, i.e. the arcs of $\tau$ are at $b'_0$ in the cut surface. Now we must also have that each $a'_j < a_0$ as otherwise $\sigma * \langle a'_j \rangle$ would again be regular bad. Finally, maximality of $\sigma$ would also be contradicted if the orderings of the arcs $a'_0, \ldots, a'_{p'}$ does not agree at $b_0$ and $b_1$ as, if $a'_i < a'_j$ with $a'_j < a'_i$ for some $i, j$, then $\sigma * \langle a'_i, a'_j \rangle$ would again be regular bad, of larger dimension. Thus $\hat{f}(\tau)$ must be disordered and, after cutting the surface at the arcs of $\sigma$, can be viewed as a simplex of $\mathcal{D}^\nu(S \setminus \hat{f}(\sigma), b'_0, b'_1)$. 

The link $\mathrm{Lk}(\sigma)$ is a simplicial sphere $S^{k-p} \subset D^{k+1}$. We want to show that the map $\hat{f}|_{\mathrm{Lk}(\sigma)}$ extends to a simplicial map

$$F : D^{k-p+1} \rightarrow \mathcal{D}^\mu(S - \hat{f}(\sigma), b'_0, b'_1) \rightarrow \mathcal{D}^\nu(S, b_0, b_1) \subset \mathcal{B}(S, b_0, b_1)$$

for $D^{k-p+1}$ a disk with some PL-structure extending that of $\mathrm{Lk}(\sigma)$. This will follow if we can show that the complex $\mathcal{D}^\mu(S \setminus \hat{f}(\sigma), b'_0, b'_1)$ is at least $(k-p)$-connected. Note that necessarily $g(S \setminus \hat{f}(\sigma)) < g$ as $f(\sigma)$ is a non-separating $p'$-simplex with $p' \geq 1$. Hence we can use our induction hypothesis. We consider the cases $\nu = 1$ and $\nu = 2$ separately.

**Case 1: $\nu = 1$.** We have that $g(S \setminus \hat{f}(\sigma)) \geq g - p' - 1 \geq g - p - 1$, as removing $p' + 1$ arcs reduces the genus by at most $p' + 1 \leq p + 1$. Hence by induction we have that $\mathcal{D}^\mu(S \setminus \hat{f}(\sigma), b'_0, b'_1)$ is at least \( \left( \frac{2(g-p)-4}{3} \right) \)-connected, using also that $\mu \geq 1$. If $p \geq 2$, we have

$$k - p \leq \frac{2g - 4}{3} - p = \frac{2g - 3p - 4}{3} \leq \frac{2(g - p - 1) - 4}{3}.$$

For $p = p' = 1$, note that $b'_0, b'_1$ necessarily lie in different boundary components, so that $\mu = 2$ in that case. (See Figure 2.) Hence in that case $\mathcal{D}^\mu(S \setminus \hat{f}(\sigma), b'_0, b'_1)$ is \( \left( \frac{2(g-2)-4}{3} \right) \)-connected, and

$$k - 1 \leq \frac{2g - 4}{3} - 1 = \frac{2g - 7}{3} = \frac{2(g - 2) - 3}{3},$$

so we get the desired extension in both subcases.

**Case 2: $\nu = 2$.** The fact that $b_0, b_1$ lie in different components implies that $g(S \setminus \hat{f}(\sigma)) \geq g - p' \geq g - p$ as cutting along the first arc has no effect on the genus. Hence induction here gives that $\mathcal{D}^\mu(S \setminus \hat{f}(\sigma), b'_0, b'_1)$ is at least \( \left( \frac{2(g-p)-4}{3} \right) \)-connected. Now for all $p \geq 1$,

$$k - p \leq \frac{2g - 4}{3} - p = \frac{2g - 3p - 4}{3} \leq \frac{2(g - p) - 4}{3}$$

yielding the desired connectivity.
We will use the map $F$ of (2.2) to modify $\hat{f}$ in the star $\text{St}(\sigma)$. For this purpose, note that as simplicial subcomplexes of $D^{k+1}$,

$$\text{St}(\sigma) = \sigma \ast \text{Lk}(\sigma),$$

$$\partial \text{St}(\sigma) = \partial \sigma \ast \text{Lk}(\sigma).$$

In particular, we get an identification $\partial(\partial \sigma \ast D^{k-p+1}) \cong \partial \text{St}(\sigma)$ for $D^{k-p+1}$ the simplicial disk that is the source of the map $F$ above.

We replace $\hat{f} \mid_{\text{St}(\sigma)}$ by the unique simplicial map

$$\hat{f} \ast F : \partial \sigma \ast D^{k-p+1} \longrightarrow \mathcal{B}^\nu(S, b_0, b_1).$$

It remains to show that this has improved the situation. Indeed, suppose that $\tau = \tau_0 \ast \tau_1$ is a regular bad simplex in $\partial \sigma \ast D^{k-p+1}$. By construction, $\tau_1$ has image in $\mathcal{D}^\nu(S \setminus \hat{f}(\sigma), b_0', b_1') \subset \mathcal{D}^\nu(S, b_0, b_1)$, so the ordering of the arcs of $\tau$ at $b_0$ and $b_1$ starts with the arcs of $\tau_1$, all in anti-clockwise order. Hence, if $\tau$ is regular bad, we must have $\tau = \tau_0$ is a strict face of $\sigma$. In particular, no new regular bad simplices have been added. As the simplex $\sigma$ has been removed, we have thus reduced the total number of regular bad simplices in the disk. Repeating this procedure, we will after finitely many stages remove every regular bad simplex, thus making the dashed arrow exist, which proves the result.

**Remark 2.5.** The connectivity estimate above can be shown to be optimal in certain low-genus examples, corresponding to known computations of the unstable homology of mapping class groups. Indeed, $\mathcal{D}^2(S_{1,r})$ is disconnected. To see this, consider the spectral sequence associated to the action of the mapping class group $\Gamma(S_{1,r})$ on the simplicial complex $\mathcal{D}^2(S_{1,r})$. This is the spectral sequence arising from the vertical filtration of the double complex $Z \mathcal{D}^2(S_{1,r})_\ast \otimes_{\Gamma(S_{1,r})} F$, where $F_\ast \longrightarrow Z$ is a free resolution of the trivial $\Gamma(S_{1,r})$-module. By a standard argument using Shapiro’s lemma (see e.g. [HW10] Thm 5.1 or [HV17] Sec 1), one finds that the first page of this spectral sequence is given by

$$E^{1}_{p,q} \cong \begin{cases} \tilde{H}_q(\Gamma(S_{1,r})) & \text{if } p = -1, \\ \tilde{H}_q(\Gamma(S_{1,r-1})) & \text{if } p = 0, \\ \tilde{H}_q(\Gamma(S_{0,r})) & \text{if } p = 1, \\ \tilde{H}_q(\Gamma(S_{0,r-1})) & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Assume for contradiction that $\mathcal{D}^2(S_{1,r})$ is connected. Then an analysis of the horizontal filtration of the double complex $Z \mathcal{D}^2(S_{1,r})_\ast \otimes_{\Gamma(S_{1,r})} F$ shows that $E^{\infty}_{p,q} = 0$ for $p + q \leq 0$, so the differential $d^1 : H_1(\Gamma(S_{1,r-1})) \longrightarrow H_1(\Gamma(S_{1,r}))$ must be surjective. This contradicts the fact that $H_1(\Gamma(S_{1,s})) \cong \mathbb{Z}^s$ for $s \geq 1$ (see [Kor02] Thm 5.1]). Hence it is not true that $\mathcal{D}^\nu$ is $(2g + \nu - 2)$-connected when $\nu = 2$.

Similarly, one finds that $H_1(\mathcal{D}^1(S_{3,r})) \neq 0$ by considering the spectral sequence associated to the action of $\Gamma(S_{3,r})$ on $\mathcal{D}^1(S_{2,r+1})$ and noting that the differential $d^1 : H_1(\Gamma(S_{2,r+1})) \longrightarrow H_1(\Gamma(S_{3,r}))$ cannot be injective since the source identifies with $\mathbb{Z}/10\mathbb{Z}$ and the target is zero (see [Kor02] Thm 5.1]). Thus $\mathcal{D}^\nu$ fails to be $(2g + \nu - 2)$-connected when $\nu = 1$ also.

Note that these low dimensional computations also show that the first and last ranges in Theorem $\mathbf{A}$ cannot be improved by a constant.

**3. The monoidal category of bidecorated surfaces**

In this section, we describe a monoidal groupoid $(\mathcal{M}_2, \#)$ of surfaces decorated by two intervals in their boundary, where the monoidal structure glues the intervals in pairs. We show that this groupoid is a module over the braided monoidal groupoid $\mathbf{B}$ of braid groups, giving, on classifying spaces, the structure of an $E_1$-module over an $E_2$-algebra in the sense of [Kra19].
3.1. Bidecorated surfaces and the monoidal structure. The groupoid $\mathbb{M}_2$ has objects bidecorated surfaces, that are, informally, surfaces with two intervals marked in their boundary. To give a precise definition of the objects that is convenient for the monoidal structure, we start by constructing a special sequence of bidecorated surfaces $X_n$, built out of disks, and defined inductively.

Let $X_1 = D^2 \subset \mathbb{C}$ denote the unit disk in the complex plane, and define the embeddings $\iota_0^1, \iota_1^1 : I \rightarrow X_1$ by

$$\iota_0^1(t) = e^{i(\pi/4 + t\pi/2)}$$

and

$$\iota_1^1(t) = e^{i(5\pi/4 + t\pi/2)}.$$

We denote by $\iota_i^1 : I \rightarrow X_1$ the reversed map $t \mapsto \iota_i^1(1 - t)$ for $i = 0, 1$.

Recursively, suppose we have defined $(X_m, \iota_0^m, \iota_1^m)$ for some $m \geq 1$. We construct $X_{m+1}$ from $X_m$ by gluing an additional disk along two half intervals, with new markings $\iota_0^{m+1}, \iota_1^{m+1}$ coming from the first half of the markings of $X_m$ and the second half of the markings of the attached disk:

$$X_{m+1} := X_m \sqcup_{\iota_0^m(t) \sim \iota_1^m(t), \, t \in [1/2, 1]} X_1$$

with

$$\iota_i^{m+1}(t) = \begin{cases} 
\iota_i^m(t), & \text{if } t \leq 1/2, \\
\iota_1^1(t), & \text{else.}
\end{cases}$$

for $i = 0, 1$. Note that the marked intervals in the boundary of $X_m$ might live in different boundary components (in fact this will happen every other time). Figure 3 shows what happens when a disk is glued to a surface in the above described manner, in each of these two possible cases.

Lemma 3.1. Let $m \geq 1$. Then $X_m \cong S_{g,r}$ is a surface of genus $g$ with $r$ boundary components, where

$$(g, r) = \begin{cases} 
(m/2 - 1, 2), & \text{if } m \text{ is even}, \\
(m/2 - 1, 1), & \text{if } m \text{ is odd}.
\end{cases}$$

Proof. Note first that $X_m$ is a connected surface for each $m$, since $X_1$ is a disk and $X_m$ is obtained from $X_1$ by successively adding disks (or strips), attached along two disjoint intervals in the boundary. For the same reason, we get that the Euler characteristic of $X_m$ is

$$\chi(X_{m+1}) = \chi(X_m) - 1 = \cdots = 1 - m.$$

By the classification of surfaces, we are left to compute the number of boundary components of $X_m$. For this, observing Figure 3 we notice that if we glue a disk along two intervals of $S$ that lie in the same boundary component, the new marked intervals given by the above procedure will give new intervals in different boundary components and vice versa, and no boundary component without marked intervals are ever created. It follows that the number of boundary components of $X_m$ alternates between 1 and 2. The result follows.

We are now ready to define the objects of the groupoid $\mathbb{M}_2$. We will use the boundary of the above defined surfaces $X_m$ to parametrize the boundary components of the surfaces that contain the marked intervals, to allow us to work with parametrized boundary components instead of parametrized arcs, in order to simplify some definitions.
Definition 3.2. A bidecorated surface is a tuple $(S, m, \varphi)$ where $S$ is a surface, $m \geq 1$ is an integer, and
$$\varphi: \partial X_m \sqcup (\sqcup_k S^1) \rightarrow \partial S$$
is a homeomorphism, giving a parametrization of the boundary of $S$. We think of $(S, m, \varphi)$ as a surface with two parametrized arcs
$$I_0 := \varphi \circ i_m^0 \quad \text{and} \quad I_1 := \varphi \circ i_m^1$$in its boundary, and $k$ additional parametrized boundaries. The surface $S$ may also have punctures.

The monoidal groupoid $(\mathbf{M}_2, \# , U)$ has objects the bidecorated surfaces together with a formal unit $U$. There are no morphisms between two bidecorated surfaces $(S, m, \varphi)$ and $(S', m', \varphi')$ unless $S$ and $S'$ are homeomorphic and $m = m'$, in which case we define the set of morphisms to be all the mapping classes of homeomorphisms that preserve the boundary parametrizations
$$\text{Hom}_{\mathbf{M}_2}((S, m, \varphi), (S', m, \varphi')) := \pi_0 \text{Homeo}_0(S, S') = \pi_0 \{ f \in \text{Homeo}(S, S') | f \circ \varphi = \varphi' \},$$where Homeo$(S, S')$ is endowed with the compact-open topology, and Homeo$_0(S, S')$ with the subspace topology. The only morphism involving the unit $U$ is the identity $\text{id}_U$.

Remark 3.3. Our definition of the morphisms in the category $\mathbf{M}_2$ is such that punctures in a surfaces $S$ can be permuted by automorphisms of $S$ in $\mathbf{M}_2$. Our argument works just as well with labeled punctures, that are not permutable by homeomorphisms, or both labeled and unlabeled punctures, just like we could also have additional boundary components that are only marked up to a permutation. The only changes this would cause to the argument would be that it would make the notations and conventions more cumbersome.

The monoidal structure $\#$ is defined as follows. The object $U$ is by definition a unit for $\#$. For the remaining objects, the monoidal product $\#$ is defined by
$$(S, m, \varphi) \# (S', m', \varphi') := \left( \frac{S \sqcup S'}{I(t) \sim T(t), t \in [\frac{1}{2}, 1]}, m + m', \varphi \# \varphi' \right),$$where $i = 0, 1$, and where
$$\varphi \# \varphi': \partial X_{m+m'} \sqcup (\sqcup_k S^1) \rightarrow \partial (S \# S'),$$is obtained using the canonical identification $\partial X_{m+m'} \cong (\partial X_n \sqcup I_m(\frac{1}{2}, 1)) \sqcup (\partial X_{m'} \sqcup I_{m'}(0, \frac{1}{2}))$. On morphisms, the monoidal product is given by juxtaposition.

The monoidal category $\mathbf{M}_2$ has the following injectivity property with respect to gluing a disk, that will be useful in the proof of our stability result.

Proposition 3.4. For any object $S = (S, m, \varphi)$ of $\mathbf{M}_2$, and any $p \geq 0$, the map
$$\text{Aut}_{\mathbf{M}_2}(S) \xrightarrow{\# D^{p+1}} \text{Aut}_{\mathbf{M}_2}(S \# D^{p+1})$$is injective, where $D = (X_1, 1, \text{id})$ is our chosen disk.

Proof. Recall that the underlying surface of $D^{p+1}$ is the surface $X_{p+1}$ defined above. Picking a smooth representative of the underlying surface of $S \# X_{p+1}$, with $S$ a smooth subsurface in its interior, we can model the map in the statement using the description of the mapping class group of surfaces in terms of isotopy classes of diffeomorphisms rather than homeomorphisms. (See e.g., [Bol09] Thm 1.2 for a detailed account of the classical isomorphism $\pi_0 \text{Homeo}_0(S) \cong \pi_0 \text{Diff}_0(S)$ when $S$ is compact.) Now the result follows by essentially the same argument as the case of attaching of surface along a single arc instead of two, as treated in [RWW17] Prop 5.18, using the fibration
$$\text{Diff}(S \# X_{p+1} \text{ rel } \partial S \cup X_{p+1}) \rightarrow \text{Diff}(S \# X_{p+1} \text{ rel } \partial (S \# X_{p+1}))$$
$$\rightarrow \text{Emb}((X_{p+1}, I_0|_{\frac{1}{2}, 1} \cup I_1|_{\frac{1}{2}, 1}), (S \# X_{p+1}, I_0|_{\frac{1}{2}, 1} \cup I_1|_{\frac{1}{2}, 1}))$$
where the fiber identifies with Diff(S rel ∂₀S) and where we note that I₀|₁/₂,₁ ∪ I₁|₁/₂,₁ = ∂X₀ ∩ ∂(S # X₁). Injectivity of the first map on π₀ follows if we can show that the base is simply-connected. In fact the base can be shown inductively to have contractible components, using that X₀ is built inductively by attaching disks along two intervals, or homotopically attaching arcs, and using the contractibility of the components of embeddings of arcs in a surface, as proved in [Gra73, Thm 5].

3.2. Braided action. We want to apply the homological stability machine of [Kra19] to stabilization in M₂ with the bidecorated disk

\[ D := (X₁, 1, \text{id}) \]

For this, we need that the classifying space of M₂ is an E₁–module over an E₂–algebra. This will follow if we can show on the categorical level that M₂ admits an appropriate action of a braided monoidal groupoid. We will build such an action in this section, using as braided monoidal groupoid the groupoid of braid groups. In contrast with most classical examples of homological stability, we will show in Section 5.3 that this action of the braid groupoid does not come from a braided structure on M₂, or the full monoidal subcategory generated by D. It is instead constructed using a Yang–Baxter element in M₂, associated to a braid subgroup of the mapping class group of Xₘ, that we will describe now.

Write

\[ D^{#m} = D₁ # ... # (D_i # D_{i+1}) # ... # D_m \]

where we use subscript to enumerate the disks, and where the underlying surface is Xₘ. We let \( a_i \) denote the isotopy class of a curve in the interior \( D_i # D_{i+1} \approx S^1 \times I \) that is parallel to its boundary components, as shown in Figure 4.

**Lemma 3.5.** The curves \( a_1, \ldots, a_{m-1} \) form a chain in \( D^{#m} \), i.e. \( a_i \) and \( a_{i+1} \) have intersection number 1 for each i, and \( a_i \cap a_j = \emptyset \) if \( |i - j| > 1 \).

**Proof.** The curve \( a_i \) lives in the disks \( D_i \) and \( D_{i+1} \), so it can only intersect \( a_{i-1} \) and \( a_{i+1} \) non-trivially, and hence it suffices to consider the subsurface of \( D^{#m} \) corresponding to \( D_i # D_{i+1} # D_{i+2} \). Here the claim can be checked by hand, see Figure 5.

Let \( T_i \in Aut_{M₂}(D^{#m}) \) denote the Dehn twist along the curve \( a_i \) in \( D^{#m} \). A classical fact states that the Dehn twists along a chain of embedded curves satisfy the braid relations (see e.g. [FMM11, 3.9 and 3.11]):

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for all } i,
\]

\[
T_i T_j = T_j T_i \quad \text{if } |i - j| > 1,
\]

\[ (3.1) \]
Note that the same relations are satisfied by the inverse twists $T_i^{-1}$, that will turn out to be more convenient for us. Also, adding a disk to the right or left of $D^n_m$ gives the relations

$$T_i \# \text{id}_D = T_i \quad \text{and} \quad \text{id}_D \# T_i = T_{i+1}$$

in $\text{Aut}_{\mathcal{M}_2}(D^n_m)$. In particular (3.1) includes the relation

$$(T_1^{-1} \# \text{id}_D)(\text{id}_D \# T_1^{-1})(T_1^{-1} \# \text{id}_D)(\text{id}_D \# T_1^{-1})$$

in $\text{Aut}_{\mathcal{M}_2}(D^n_3)$, so in other words, the inverse Dehn twist $T_1^{-1} \in \text{Aut}_{\mathcal{M}_2}(D^n_3)$ is a Yang–Baxter operator in the sense of Section 5.1.

Recall from the introduction that $\mathcal{B}$ denotes the groupoid of braid groups, with objects the natural numbers $\{0, 1, 2, \ldots\}$, automorphisms of the braid group $B_n$, and no other non-trivial morphisms. In Section 5.1 we show that, being a Yang-Baxter operator, $T_1^{-1}$ yields a strong monoidal functor

$$\Phi = \Phi_{D, T_1^{-1}} : (\mathcal{B}, \oplus) \longrightarrow (\mathcal{M}_2, \#),$$

uniquely determined up to monoidal natural isomorphism by the fact that $\Phi(1) = D$ and, for the standard generator $\sigma_1 \in B_2 = \text{Aut}_{\mathcal{B}}(1)$, $\Phi(\sigma_1) = T_1^{-1}$.

Such a functor $\Phi$ endows $\mathcal{M}_2$ with the structure of an $E_1$-module over $\mathcal{B}$ via the associated functor

$$\alpha = (- \# \Phi(-)) : \mathcal{M}_2 \times \mathcal{B} \longrightarrow \mathcal{M}_2,$$

given on objects by $\alpha(S, n) = S \# \Phi(n) = S \# D^n_n$, and likewise for morphisms. On classifying spaces, this yields exactly the kind of input needed in Kramnich’s homological stability framework, see [Kra19] Lem 7.2.

**Remark 3.6.** For each $m$, the restriction of the functor $\Phi : \mathcal{B} \longrightarrow \mathcal{M}_2$ to $B_m = \text{Aut}_{\mathcal{B}}(m)$ maps the standard generator $\sigma_i$ to the inverse Dehn twist $T_i^{-1} \in \text{Aut}_{\mathcal{M}_2}(D^n_m)$.

By Birman–Hilden theory [BH72, BH73] the homomorphisms $\Phi|_{B_m} : B_m \longrightarrow \text{Aut}_{\mathcal{M}_2}(D^n_m)$ are actually injective.

4. **Homological stability**

Generalizing the main result of [RWW17], Kramnich associates to an $E_1$-module $\mathcal{M}$ over an $E_2$-algebra $\mathcal{X}$ with a chosen stabilizing object $X \in \mathcal{X}$, a space of destabilizations at every $A \in \mathcal{M}$, whose high connectivity implies homological stability at $A$ when stabilizing by $X$. We are interested in the case where $\mathcal{M} = BM_2$ is the classifying space of $\mathcal{M}_2$ and $\mathcal{X} = BB$, acting...
on $BM_2$ via the map $\alpha \colon M_2 \times B \to M_2$ defined in Section 3.2. We will pick $A = S \in M_2$ to be some surface, with $X = 1 \in B$ modelling stabilization with the disk as $\alpha(-, X) = - \# D$ is the sum with the bidecorated disk $D = (X_1, 1, \text{id})$ of Section 3.1.

Generally, the space of destabilizations is a semi-simplicial space, but in settings such as ours, it is actually levelwise homotopy discrete. Indeed, by [Kra19, Lem 7.6]), when the structure of $E_2$-module over an $E_2$-algebra is induced by an action of a braided monoidal category on a groupoid, and under the injectivity condition given in Proposition 3.4, the space of destabilizations is equivalent to the following semi-simplicial set, defined just as in [RWW17] in the case of a braided monoidal groupoid acting on itself.

**Definition 4.1.** ([Kra19 Def 7.5]) Let $\mathcal{M}$ be a right module over a braided monoidal groupoid $(\mathcal{X}, \oplus, b)$, where we denote also by $\oplus$ the module action. Let $A$ and $X$ be objects of $\mathcal{M}$ and $\mathcal{X}$ respectively. The space of destabilizations $W_n(A, X)_\bullet$ is the semi-simplicial set with set of $p$-simplices

$$W_n(A, X)_p = \{ (B, f) \mid B \in \text{Ob}(\mathcal{M}) \text{ and } f : B \oplus X^{\oplus p+1} \to A \oplus X^{\oplus n} \in \mathcal{M} \}/\sim$$

where $(B, f) \sim (B', f')$ if there exists an isomorphism $g : B \to B'$ in $\mathcal{M}$ satisfying that $f = f' \circ (g \oplus \text{id}_{X^{\oplus p+1}})$. The face map $d_i : W_n(A, X)_p \to W_n(A, X)_{p-1}$ is defined by $d_i[B, f] = [B \oplus X, d_i f]$ for

$$d_i f : B \oplus X \oplus X^p \xrightarrow{id_B \oplus b_{X^{\oplus i} X} \oplus \text{id}_{X^{\oplus p-i}}} B \oplus X^{\oplus i} \oplus X \oplus X^{\oplus p-i} \xrightarrow{f} A \oplus X^{\oplus n},$$

for $b_{X^{\oplus i} X} : X \oplus X^{\oplus i} \to X^{\oplus i} \oplus X$ coming from the braiding in $\mathcal{X}$.

### 4.1. Disk destabilizations and disordered arcs

Given a bidecorated orientable surface $S = (S, m, \varphi)$, with $I_0, I_1$ compatibly oriented, let $\mathcal{D}(S) = \mathcal{D}''(S, b_0, b_1)$ denote the disordered arc complex of $S$ as in Section 2, where

$$b_0 = I_0(1/2) \quad \text{and} \quad b_1 = I_1(1/2)$$

are the midpoints of the marked intervals, and $\nu = 1$ if $I_0$ and $I_1$ lie on the same boundary component and $\nu = 2$ otherwise. The vertices of a simplex in $\mathcal{D}(S)$ are canonically ordered by the anti-clockwise ordering at $b_0$ (or equivalently at $b_1$). Hence we can associate to this simplicial complex a semi-simplicial set that we denote $\mathcal{D}(S)_\bullet$, with same set of $p$-simplices and whose $i$th face map is given by forgetting the $(i + 1)$st arc with respect to that ordering. As $\mathcal{D}(S)$ and $\mathcal{D}(S)_\bullet$ have homeomorphic realizations, they have the same connectivity.

Write $W_n(S, D)_\bullet$ for the space of destabilization of Definition 4.1 associated to the module $\mathcal{M} = M_2$ over the $E_2$-algebra $\mathcal{X} = B$ acting on $M_2$ as above, with $X = 1 \in B$, and $A = S = (S_{g, r}, m, \varphi)$ some bidecorated orientable surface of small genus $g \geq 0$, with $r$ boundary components and $s$ punctures. The space $W_n(S, D)_\bullet$ is then the space of destabilizations of the stabilization map

$$\text{Aut}_{M_2}(S \# D^{n-1}) \xrightarrow{\ast D} \text{Aut}_{M_2}(S \# D^n)$$

that attaches an additional disk to the surface along the two marked intervals.

We want to identify $W_n(S, D)_\bullet$ with $\mathcal{D}(S \# D^n)_\bullet$. For this, we start by constructing a particular disordered collection of arcs in $D^n$. Write again

$$D^n = D_1 \# \cdots \# D_i \# \cdots \# D_n,$$

and let $\rho_i$ denote the unique isotopy class of arc in the $i$th disk $D_i$ going from $b_0 = I_0(1/2)$ to $b_1 = I_1(1/2)$.

**Lemma 4.2.** The arcs $\rho_1, \ldots, \rho_m$ are ordered anti-clockwise at both $b_0$ and $b_1$.

---

3The definition of the disordered arc complex naturally extend to non-orientable bidecorated surfaces, ordering the arcs according to the orientations of $I_0$ and $I_1$; but we will only consider orientable surfaces here.
Proof. It suffices to show that $\rho_i$ and $\rho_{i+1}$ are ordered anti-clockwise at $b_0$ and $b_1$ for each $i$. Thus we need only consider what happens in the subsurface $D_i \# D_{i+1}$. The gluing being defined in exactly the same way at $I_0$ and $I_1$, the arcs are ordered in the same way at both endpoints, and the particular choice of gluing gives the anti-clockwise ordering, see Figure 6. □

Recall from Section 3.2 the Dehn twist $T_i$ along the curve $a_i$ in $D_i \# D_{i+1}$. The union of the arcs $\rho_i$ in $D_i \# D_{i+1}$ define a deformation retract of the surface, as each disk $D_i$ retracts onto the corresponding arc $\rho_i$, and we can understand the action of the twists $T_i$ on the surface by considering their action on the arcs $\rho_i$. The action is given by the following result, that will be needed to compare the face maps in the semi-simplicial sets $W_n(S,D)_\bullet$ with $\mathcal{Q}(S \# D^{\# n})_\bullet$.

Lemma 4.3. The action of the Dehn twist $T_i$ along the curve $a_i$ on the homotopy classes of the arcs $\rho_i$, relative to their endpoints, is

$$T_i(\rho_j) = \begin{cases} \rho_i \overline{\rho_{i+1}} \rho_i & \text{if } j = i, \\ \rho_i & \text{if } j = i + 1, \\ \rho_j & \text{else.} \end{cases}$$

Equivalently,

$$T_i^{-1}(\rho_i) = \rho_{i+1} \quad \text{and} \quad T_i^{-1}(\rho_{i+1}) = \rho_{i+1} \overline{\rho_i} \rho_i$$

and $T_i^{-1}$ leaves the other $\rho_j$ invariant.

Proof. The Dehn twist $T_i$ can only affect $\rho_i$ and $\rho_{i+1}$ as the curve $a_i$ only intersects these two arcs, from which the last case in the statement follows. The computation for the arcs $\rho_i$ and $\rho_{i+1}$ is local to $D_i \# D_{i+1}$, where, as shown in Figure 7, we have $T_i(\rho_i) \cong \rho_i \overline{\rho_{i+1}} \rho_i$, giving the first case in the statement, and $T_i(\rho_{i+1}) \cong \rho_i$, giving the second case. □

Proposition 4.4. Let $S = (S,m,\varphi)$ be an object of $\mathbf{M}_2$. There is an isomorphism of semi-simplicial sets

$$W_n(S,D)_\bullet \cong \mathcal{Q}^{\nu}(S \# D^{\# n})_\bullet$$

where the marked points $b_0$ and $b_1$ are the midpoints of the intervals $I_0$ and $I_1$ in $S \# D^{\# n}$ and with $\nu = \text{parity}(m+n)$, that is $\nu = 1$ if $I_0$ and $I_1$ lie in the same boundary component of $S \# D^{\# n}$ and $\nu = 2$ otherwise.

Proof. We first show that both $W_n(S,D)_p$ and $\mathcal{Q}^{\nu}(S \# D^{\# n})_p$ are isomorphic to $\text{Aut}_{M_2}(S \# D^{\# n}) \cap \text{Aut}_{M_2}(S \# D^{\# n-p-1})$ for every $p \geq 0$. This holds by definition for the first semi-simplicial set. For $\mathcal{Q}^{\nu}(S \# D^{\# n})_p$, it will follow from two facts: (1) the natural action of $\text{Aut}_{M_2}(S \# D^{\# n}) = \pi_0 \text{Homeo}_0(S \# D^{\# n})$...
on this set of \( p \)-simplices is transitive, and (2) the stabilizer of a \( p \)-simplex is isomorphic to 
\( \text{Aut}_{\mathcal{M}}(S \# D^{\#n-p-1}) \). The first fact follows because the homeomorphism type of the complement \( S \setminus \sigma \) of a collection of non-separating arcs \( \sigma = \langle a_0, \ldots, a_p \rangle \) is determined by the orderings of the arcs at the endpoints as this determines the number of boundary components of the complement (see [Har85, Lem 3.2]), and the second from the fact that this complement is precisely diffeomorphic to \( S \# D^{\#n-p-1} \) for any \( p \)-simplex in the disordered arc complex. Indeed, this diffeomorphism type does not depend on the simplex by transitivity of the action, so it is enough to check the claim for any chosen simplex. Let

\[
\sigma_p = \langle \rho_{n-p}, \ldots, \rho_n \rangle
\]

be the collection of arcs in \( S \# D^{\#n} \) consisting of the cores \( \rho_i \) of the last \( p + 1 \) disks. Recall from Lemma 4.2 that this is a disordered simplex, once we note additionally that the arcs are also non-separating. Now Figure 8 shows that the operation of cutting along the core \( \rho \) of a disk exactly undoes the gluing operation, which proves the claim in that case.

Note that the actions on both sets of simplices are given by post-composition with mapping classes, where we think here of an arc as an isotopy class of embedding. There is then a unique equivariant isomorphism \( \varphi_p : W_n(S, D)_p \xrightarrow{\cong} \mathcal{G}^p(S \# D^{\#n})_p \) taking the \( p \)-simplex

\[
f_p = (S \# D^{\#n-p-1}, \text{id}_{S \# D^{\#n}})
\]

of \( W_n(S, D) \) to the \( p \)-simplex \( \sigma_p = \langle \rho_{n-p}, \ldots, \rho_n \rangle \) of the target already considered above.

We are left to check that the face maps \( d_i \) correspond to each other under the isomorphisms \( \varphi_p \). Because the face maps are equivariant with respect to the \( \text{Aut}_{\mathcal{M}}(S \# D^{\#n}) \)-action in both cases, and the actions are transitive, it is enough to check that the face maps agree for the simplices \( f_p \) and \( \sigma_p = \varphi_p(f_p) \). By definition,

\[
d_i f_p = ((S \# D^{\#n-p-1}) \# D, \text{id}_{S \# D^{\#n-p-1}} \# b_{D^{\#n}, D}^{-1} \# \text{id}_{D^{\#n-p-1}})
\]
while
\[ d_i(\sigma) = \langle \rho_{n-p}, \ldots, \rho_{n-p+i}, \ldots, \rho_n \rangle \]
is the simplex obtained by forgetting the \((i+1)\)st arc. In particular, we immediately have that
\[ d_0(f_p) = f_{p-1} \text{ and } d_0(\sigma) = \varphi_{p-1}(f_{p-1}) \]
giving that the face maps agree in that case.

For the remaining face maps, note that
\[ \text{id}_{\#D^n \rightarrow \#D^n} \oplus \text{id}_{\#D^n \rightarrow \#D^n} = T_{n-p+i-1} \circ \cdots \circ T_{n-p} : S \# D^n \rightarrow S \# D^n \]
as composition of Dehn twists \( T_i \) of Section 3.2. We need to compute the image of \( n_{n-p+1}, \ldots, n_p \) under this map. By Lemma 4.3, we have that for \( 1 \leq j \leq i \),
\[ T_{n-p+i-1} \circ \cdots \circ T_{n-p}(\rho_{n-p+j}) = T_{n-p+i-1} \circ \cdots \circ T_{n-p+j-1}(\rho_{n-p+j}) = T_{n-p+i-1} \circ \cdots \circ T_{n-p+j-1}(\rho_{n-p+j-1}) = \rho_{n-p+j-1} \]
while for \( i + 1 \leq j \leq p \),
\[ T_{n-p+i-1} \circ \cdots \circ T_{n-p}(\rho_{n-p+j}) = \rho_{n-p+j} \]
Hence \( d_i(f_p) \) takes the arcs \( n_{n-p+1}, \ldots, n_p \) to the arcs
\[ n_{n-p+1} \ldots n_{n-p+i-1} n_{n-p+i+1} \ldots n_p, \]
and precisely to the arcs of \( d_i(\sigma) \). So we indeed have that \( \varphi_{p-1}(d_i(f_p)) = d_i(\varphi_{p-1}(f_p)) \), which finishes the proof.

4.2. **Coefficient systems.** Having identified the space of destabilizations with the semisimplicial set of disordered arcs in Proposition 4.4, we can now input the connectivity computation of the disordered arc complex of Section 3.2 into the general stability theorem of [Kra19]. To state the resulting stability theorem in full generality, we need to introduce the notions of (split) finite degree coefficient systems. We follow [Kra19 Sec 4], which generalizes [RWW17] 4.1-4 that unify the earlier definitions of Dwyer for the general linear groups [Dwy80] and Ivanov for the mapping class groups [Iva93]. (The papers [Kra19] [RWW17] consider in addition abelian coefficient systems, but these are not relevant here, because the abelianization of the mapping class group of surfaces of large enough genus is trivial by a theorem of Mumford– Birman–Powell, see Lemma 1.1 in [Har83].)

Fix a bidecorated surface \( S = (S,m,\varphi) \), and let \( D \) be the bidecorated disk as above. Definition 4.1 of [Kra19] becomes in our case:  

**Definition 4.5.** A **coefficient system** for the groups \( \text{Aut}_{M_n}(S \# D^n) \) with respect to the stabilization by \( D \) is a collection of \( \mathbb{Z}[\text{Aut}(S \# D^n)] \)-modules \( M_n \) for \( n \geq 0 \), together with maps \( s_n : M_n \rightarrow M_{n+1} \) that are equivariant with respect to the stabilization map \( \text{Aut}(S \# D^n) \rightarrow \text{Aut}(S \# D^{n+1}) \), satisfying the following condition:

\[ T_{n+1} \in \text{Aut}(S \# D^{n+2}) \text{ acts trivially on the image of } M_n \xrightarrow{s_{n+1} \circ s_n} M_{n+2} \]
for \( T_{n+1} \) the Dehn twist of Section 3.2 with support the last two disks in \( S \# D^{n+2} \).

We will encode the data of a coefficient system as a pair \((F,\sigma^F)\) with
\[ F : M_2|_{S,D} \rightarrow \text{Mod}_2 \]
a functor from the full subcategory of \( M_2 \) on the objects \( S \# D^n \) for \( n \geq 0 \) to abelian groups, where \( M_n = F(S \# D^n) \) with its \( \text{Aut}(S \# D^n) \)-action induced by \( F \), and
\[ \sigma^F : F(-) \rightarrow F(- \# D) \]
is a natural transformation encoding the suspension maps \( s_n \), where we assume that \( F(\text{id} \# T) \) acts trivially on the image of \( (\sigma^F)^2 : F(-) \rightarrow F(- \# D^2) \) for \( T \) the Dehn twist supported on the added disks \( D^2 \).
Given a coefficient system $F$, we define its suspension $\Sigma F$: $M_2|S,D \to \text{Mod}_\mathbb{Z}$ by $\Sigma F(-) = F(- \# D)$ with

$$\sigma^{\Sigma F}: \Sigma F(-) = F(- \# D) \xrightarrow{\sigma^F} F(- \# D^{\#2}) \xrightarrow{\text{id} \# \tau} F(- \# D^{\#2}) = \Sigma F(- \# D),$$

where one checks that the triviality condition 4.1 is satisfied with this choice of structure map $\sigma^{\Sigma F}$. (See [Kra19, Def 4.4].)

The structure map $\sigma^F$ induces a natural transformation $F \to \Sigma F$, called the suspension map. We define the kernel $\ker F$ and cokernel $\text{coker} F$ to be the kernel and cokernel functors of that natural transformation. We call $F$ split if the suspension map is split injective in the category of coefficient systems.

**Definition 4.6.** [RWW17 Def 4.10] A coefficient system $F$ is

(a) of (split) degree $-1$ at $N$ if $F(S \# (D^{\#n})) = 0$ for all $n \geq N$;
(b) of degree $k \geq 0$ at $N$ if $\ker(F)$ has degree $-1$ at $N$ and $\text{coker}(F)$ has degree $(k - 1)$ at $(N - 1)$;
(c) of split degree $k \geq 0$ at $N$ if $F$ is split and $\text{coker}(F)$ is of split degree $(k - 1)$ at $(N - 1)$.

**Example 4.7.**

(a) A coefficient system $F$ is of degree 0 at 0 if and only if $\sigma^F$ is a natural isomorphism. This is in particular the case for constant coefficient systems.
(b) The functor $F_k: M_2 \to \text{Mod}_\mathbb{Z}$ defined by

$$F_k(S) = H_1(S, \mathbb{Z})^\otimes k$$

is a split coefficient system of degree $k$ at 0. (This is essentially a result of Ivanov [Iva93 Sec 2.8], who considers a version of the composite stabilization $\# D^{\#2}$. See also [Bol12 Ex 4.3] for the case $k = 1$, and [Son20] Lem 2.9 that proves this in a very general set-up, though in the case of a braided groupoid acting over itself only.)
(c) Given a $k$-connected space $X$, the coefficient system $F_n^k: M_2 \to \text{Mod}_\mathbb{Z}$ defined by

$$F_n^k(S) = H_n(\text{Map}(S/\partial S), X),$$

which appears in the work of Cohen–Madsen [CM09], is a coefficient system of degree $[n/k]$ (see [Bol12 Ex 4.3]).

**Remark 4.8.** Although the above examples all makes sense in the different set-ups considered in the literature, one should keep in mind that there are variations in what precisely a finite degree coefficient system for the mapping class groups of surfaces means in e.g. the papers [Iva93 CM09 Bol12 RWW17] and [Kra19]. This is due to two facts: first, the definition of the coefficient system depends on the category of surfaces considered and on the stabilization map(s) one works with, and second, the triviality condition 4.1 arising from Krannich’s framework is actually weaker than the one used in earlier frameworks, see e.g. [Kra19 Rem 7.9].

In addition, the paper [GKRW19] uses a homological condition instead of a finite degree condition (see 5.5.1 in that paper). The relationship between that condition and finite degree conditions is discussed in [GKRW19] Rem 19.11.

**4.3. The stability theorem.** We are now ready to state our main theorem:

**Theorem 4.9.** Let $S = (S,m,\varphi)$ be an object of $M_2$ with $m$ odd, i.e. such that $I_0, I_1$ are in the same boundary component. Let $F: M_2|S,D \to \text{Mod}_\mathbb{Z}$ be a coefficient system and write $F_n = F(S \# D^{\#n})$. The map

$$H_i(\text{Aut}_{M_2}(S \# D^{\#n}); F_n) \to H_i(\text{Aut}_{M_2}(S \# D^{\#n+1}); F_{n+1})$$

is

(a) an epimorphism for $i \leq \frac{n}{3}$ and an isomorphism for $i \leq \frac{n-3}{3}$ if $F$ is constant.
(b) an epimorphism for $i \leq \frac{n-3k-2}{3}$ and an isomorphism for $i \leq \frac{n-3k-5}{3}$ if $F$ has degree $k$ at $N \geq 0$ and $n > N$. 


(c) an epimorphism for $i \leq \frac{n-k-2}{3}$ and an isomorphism for $i \leq \frac{n-k-5}{3}$ if $F$ has split degree $k$ at $N \geq 0$ and $n > N$.

**Remark 4.10.** We have stated the theorem in the case of an initial surface $S$ with $I_0$ and $I_1$ in the same boundary component for simplicity. The case of a surface $S'$ where the two intervals lie in different components is actually also included in the statement, by writing $S' = S \# D$ for $S$ of the previous type, or considering $S' \# D$ if $S'$ does not admit such a decomposition. Indeed, as we have already seen in Section [3] (see Figure [3]), gluing in a disk exactly changes whether $I_0$ and $I_1$ are in the same boundary or not.

We will first show that the above results implies the two main theorems stated in the introduction.

**Proof of Theorems A and B from Theorem 4.9.** Let $S^\nu_{r,s}$ be a surface of genus 0 with $r \geq 1$ boundary components and $s$ punctures, and consider the associated object $S = (S^\nu_{r,s},1,\varphi)$ of $M_2$, with two marked intervals in the first boundary component. Then $S \# D^{2g}$ has the form $(S^\nu_{r,s},1 + 2g, \varphi)$ while $S \# D^{2g+1}$ has the form $(S^\nu_{r,s+1},2 + 2g, \varphi)$. Moreover, the maps

$$S \# D^{2g} \xrightarrow{\partial D} S \# D^{2g+1} \xrightarrow{\partial D} S \# D^{2g+2}$$

precisely induce on automorphism groups in $M_2$, the two maps appearing in Theorems A and B.

The fact that the first map is always injective in homology follows from the fact that postcomposing the map $S^\nu_{r,s} \longrightarrow S^\nu_{r+1,s}$, defined by the sum $\partial D$, with the map $S^\nu_{r+1,s} \longrightarrow S^\nu_{r+1,s} \cup I_s D^2 \simeq S^\nu_{r,s}$ filling in one of the newly created boundary component, is homotopic to the identity. Now Theorem 4.9(a) gives that the map

$$H_i(\text{Aut}_{M_2}(S \# D^{2g})) \xrightarrow{\partial D} H_i(\text{Aut}_{M_2}(S \# D^{2g+1}))$$

is surjective for $i \leq \frac{3}{2}$ in homology with constant coefficients. Given that the map is always injective, we can get an isomorphism in that same range, proving the first part of Theorem A.

Applying (b) and (c) instead gives Theorem B for the first map.

For the second map, we now apply Theorem 4.9 in the case $n = 2g + 1$, but in that case, there is no additional argument for injectivity, so the bounds translate directly to surjectivity and isomorphism bounds.

**Proof of Theorem 4.9.** Proposition 4.4 together with Theorem 2.4 give that $W_n(S,D)_\bullet$ is $(\frac{2g + \nu - 1}{3})$-connected, for $g$ the genus of $S \# D^n$ and $\nu = 1$ if $I_0$ and $I_1$ are in the same boundary component of $S \# D^n$, which is the case precisely when $n$ is even, and $\nu = 2$ otherwise. The surface $S \# D^n$ has genus greater than or equal to the genus of $D \# D^n$, that is $\frac{2}{3}$ if $n$ is even and $\frac{2 + 1}{2}$ if $n$ is odd (see Lemma 3.1). Hence $2g + \nu \geq n + 1$ is both cases, and $W_n(S,D)_\bullet$ is at least $(\frac{2 + 1}{3})$-connected.

Now $W_n(S,D)_\bullet$ is the semi-simplicial set denoted $W^\text{RW}(S \# D^n)_\bullet$ in [Kra19] (see Definition 7.5 in that paper). By Lemma 7.6 in the same paper, using Proposition 3.4 this semi-simplicial set has the same connectivity as the semi-simplicial space $W(S \# D^n)_\bullet$ of [Kra19], which by Remark 2.7 of that paper determines the connectivity assumption of Theorem A in that paper: the canonical resolution of the assumption of the theorem is $m$-connected, if and only if the space $W(S \# D^n)_\bullet$ is $(m - 1)$-connected. Given that $W(S \# D^n)_\bullet$ is $(\frac{2g + \nu}{3})$-connected, we have that the canonical resolution of is $(\frac{2g + \nu}{3} + 1)$-connected. Hence we can apply [Kra19 Thm A] with $k = 3$ and grading $g_{M_2} : M_2[S,D] \longrightarrow N$ given by $g_{M_2}(S \# D^n) = n - 2$; see also [Kra19 Rem 2.24], where we can take $m = 4$. The theorem, with the improvement given by (i) in the remark, then gives that

$$H_i(\text{Aut}_{M_2}(S \# D^n); Z) \longrightarrow H_i(\text{Aut}_{M_2}(S \# D^{n+1}); Z)$$

is an isomorphism for $i \leq \frac{n-3}{3}$ and an epimorphism for $i \leq \frac{n}{3}$, giving the stated result in the case of constant coefficients. For a coefficient system $F$ of degree $k$ at $N$, [Kra19 Thm C] gives that

$$H_i(\text{Aut}_{M_2}(S \# D^n); F_n) \longrightarrow H_i(\text{Aut}_{M_2}(S \# D^{n+1}); F_{n+1})$$
is an isomorphism for \(i \leq \frac{n-3k-5}{3}\) and an epimorphism for \(i \leq \frac{n-3k-2}{3}\) for \(n > N\), improved to an isomorphism for \(i \leq \frac{n-k-5}{3}\) and an epimorphism for \(i \leq \frac{n-k-2}{3}\) if \(F\) is split.

\[\text{Remark 4.11 (Optimality of the stability bounds).} \text{ Combining the two maps in Theorem A we obtain that the genus stabilization} \]

\[H_i(\Gamma(S^*_g,r)); \mathbb{Z}) \longrightarrow H_i(\Gamma(S^*_{g+1,r}); \mathbb{Z})\]

is an epimorphism when \(i \leq \frac{2g}{3}\) and an isomorphism when \(i \leq \frac{2g-2}{3}\). The slope \(\frac{2}{3}\) is known to be optimal by a computation of Morita [Mor03], with optimal isomorphism range since for instance \(H_1(\Gamma(S_2,r)); \mathbb{Z}) \longrightarrow H_1(\Gamma(S_3,r)); \mathbb{Z})\) is not injective as the source is isomorphic to \(\mathbb{Z}/12\) and the target is trivial, see e.g. [Kor02] Theorem 5.1. Our combined genus epimorphism range, on the other hand, falls short of the range \(i \leq \frac{2g+1}{3}\), as given in [GKRW19], a range that is optimal by Morita’s computation (see Theorem B (i) of [GKRW19]).

Our results for twisted coefficients are most easily compared with those of Boldsen [Bol12] Thm 3], whose coefficient systems are coefficient systems of finite split degree in our sense, though with a stricter triviality condition upon double stabilization. For these coefficient systems, he obtains slightly better ranges, with improvement \(+\frac{\gamma_i}{3}\) for the first map and \(+\frac{\gamma_i}{3}\) for the second. The papers [RWW17, GKRW19] only consider genus stability. In [RWW17], the stability slope obtained is only \(\frac{1}{3}\), while in [GKRW19] Sec 5.5.1, the finite degree condition is replaced by a more general homological condition that applies to some finite coefficient systems [GKRW18] Sec 19.2]. In the particular case of the \(k\)th tensor power of the first homology of the surface, they do however only get the epimorphism range \(i \leq \frac{2g-2k+1}{3}\) and isomorphism range \(i \leq \frac{2g-2k-2}{3}\), see Example 5.22 in that paper.

5. Braiding and Homological Stability for Groups

In order to use the framework of Krannich [Kra19] to prove homological stability for a sequence of groups, one needs the structure of an “\(E_1\)-module over an \(E_2\)-algebra”. We give in Proposition 5.1 below a simple way to construct such a module structure, in terms of Yang–Baxter operators. Compared to earlier approaches to homological stability such as [RWW17], which Krannich’s work generalizes, this has the advantage of being very lightweight. Instead of having to provide the structure of a braiding on the monoidal category whose automorphism groups one is interested in, it suffices to provide a single morphism satisfying a simple equation.

Our main example of a Yang–Baxter operator is the inverse Dehn twist \(T^{-1}_{1}\) \(\in \text{Aut}_{\mathcal{M}_2}(D \# D)\), defined in Section 3.2 and used to prove our main result. In Section 5.3, we show that this Yang–Baxter operator is not part of a braided monoidal structure on the category \(\mathcal{M}_2\), but gives instead a twisted version of such a structure.

5.1. Yang–Baxter operators and braid groupoid actions. Let \(\mathcal{X} = (\mathcal{X}, \oplus, 1)\) be a monoidal category. A Yang–Baxter operator in \(\mathcal{X}\) is a pair \((X, \tau)\) consisting of an object \(X \in \mathcal{X}\) and a morphism \(\tau \in \text{Aut}_\mathcal{X}(X \oplus X)\), satisfying the Yang–Baxter equation

\[(\tau \oplus 1)(1 \oplus \tau)(\tau \oplus 1) = (1 \oplus \tau)(\tau \oplus 1)(1 \oplus \tau) \in \text{Aut}_\mathcal{X}(X \oplus X \oplus X),\]

where we suppress associators from the notation.

Yang–Baxter operators are closely related to the braid groupoid: Recall from Section 3.2 the braid groupoid \(\mathcal{B}\), with objects the natural numbers and only non-trivial morphisms \(\text{Aut}_{\mathcal{B}}(n) = B_n\). A variant of the coherence theorem for braided monoidal categories says that the category of strong monoidal functors from the braid groupoid into \(\mathcal{X}\) is equivalent to a naturally defined category of Yang–Baxter operators in \(\mathcal{X}\) [JS93] Prop 2.2.\footnote{In other words, the pair consisting of the braid groupoid \(\mathcal{B}\) and the Yang–Baxter operator \(\sigma_1 \in \text{Aut}_\mathcal{B}(2)\), is the initial monoidal category with a distinguished Yang–Baxter element.} To a Yang–Baxter operator \((X, \tau)\) in \(\mathcal{X}\), this equivalence associates the strong monoidal functor \(\Phi_{X,\tau} : \mathcal{B} \longrightarrow \mathcal{X}\) given by \(\Phi_{X,\tau}(n) = X^\oplus n\) on objects, and on morphisms by letting

\[\Phi_{X,\tau} : B_n \longrightarrow \text{Aut}_\mathcal{X}(X^\oplus n)\]
send the $i$th standard generator $\sigma_i$ to $\text{id}_X \oplus \tau \oplus \text{id}_X \oplus \cdots \oplus \text{id}_X \oplus \cdots$, where the required maps $\Phi_{X,\tau}(m) \oplus \Phi_{X,\tau}(n) \rightarrow \Phi_{X,\tau}(m+n)$ are given by the monoidal structure of $\mathcal{X}$.

Suppose now that the monoidal category $\mathcal{X}$ acts on a category $\mathcal{M}$ via a functor $\mathcal{M} \times \mathcal{X} \rightarrow \mathcal{M}$, which we also denote by $\oplus$, compatible with the monoidal sum in $\mathcal{X}$. The following result shows that the choice of a Yang–Baxter operator defines an action of the braid groupoid $\mathcal{B}$ on $\mathcal{M}$, and hence is appropriate data to apply the stability framework of [Kra19]:

**Proposition 5.1.** Let $(\mathcal{X}, \oplus, 1)$ be a monoidal category with $\tau \in \text{Aut}_\mathcal{X}(X \oplus X)$ a Yang–Baxter operator in $\mathcal{X}$. Suppose $\mathcal{X}$ acts on a category $\mathcal{M}$. Then there is an action of the braid groupoid $\mathcal{B}$ on $\mathcal{M}$.

**Proof.** If $\mathcal{M}$ is defined as the composite functor $\mathcal{M}(\mathcal{M}, \mathcal{X}) \times \mathcal{B} \rightarrow \mathcal{M}$ given on objects by $\alpha_{\tau}(A, n) = A \oplus X^{\oplus n}$ and determined on morphisms by

$$\alpha_{\tau}(f, \sigma_i) = f \oplus \text{id}_X \oplus \tau \oplus \text{id}_X \oplus \cdots,$$

for $\sigma_i$ the $i$th elementary braid in $\mathcal{B}$. Furthermore, taking classifying spaces this endows $\mathcal{B}$ with the structure of an $E_1$-module over the $E_2$-algebra $BB$.

Note that if we are interested in homological stability for stabilization by $X$ for the automorphism groups $G_n := \text{Aut}_\mathcal{M}(A \oplus X^{\oplus n})$ for some object $A$ of $\mathcal{M}$, only the full subcategory $\mathcal{M}_{A,X} \subseteq \mathcal{M}$ spanned by objects of the form $A \oplus X^{\oplus n}$, is relevant. So for stability purposes, it is enough to consider the subfunctor $\mathcal{M}_{A,X} \times \mathcal{B} \rightarrow \mathcal{M}_{A,X}$.

In fact, to make sure that the structure of $E_1$-module over the $E_2$-algebra $BB$ is graded, one can even replace the category $\mathcal{M}_{A,X}$ by a category with objects the natural numbers and setting $\text{Aut}(n) = \text{Aut}_{\mathcal{M}_{A,X}}(A \oplus X^{\oplus n})$, avoiding any potential issue coming from unwanted equalities $A \oplus X^{\oplus n} = A \oplus X^{\oplus m}$ for $m \neq n$.

**Example 5.2.** If $\mathcal{X} = (\mathcal{X}, \oplus, 1)$ admits a braiding $b$, then $\tau = b_{X,X} \in \text{Aut}_\mathcal{X}(X \oplus X)$ is a Yang–Baxter operator for any object $X$. For $\mathcal{X}$ a groupoid acting on itself or $\mathcal{X}$ acting on a category $\mathcal{M}$, this recovers the basic set-up for homological stability of the paper [RWW17], or Section 7 of [Kra19].

**Example 5.3** (Mapping class groups of surfaces). As explained above, a Yang–Baxter operator $\tau \in \text{Aut}_\mathcal{X}(X \oplus X)$ gives in particular a collection of homomorphisms $\Phi_{X,\tau} : B_n \rightarrow \text{Aut}_\mathcal{X}(X^{\oplus n})$ from the braid groups to the automorphism group of $n$ copies of $X$. There are two standard ways to embed braid groups in mapping class groups of surfaces, and we explain here how they both come from Yang–Baxter elements in appropriate categories of surfaces.

(a) Let $\mathcal{Y}$ denote the category of bidecorated surfaces of Section 3. As explained in Section 5.2, the Dehn twist $\tau \in \text{Aut}_{\mathcal{M}_{1}}(D \# D) \cong \pi_0 \text{Homeo}_0(S^1 \times I) \cong \mathbb{Z}$, or its inverse $^{-1}_D$, is a Yang–Baxter operator. The associated map $\Phi_{D,T} : B_n \rightarrow \text{Aut}_{\mathcal{M}_{1}}(D^{\oplus n})$ is the embedding of braid group in the mapping class groups of $S_{g,1}$ (when $n = 2g + 1$) and of $S_{g,2}$ (when $n = 2g + 2$) associated to Dehn twists along the chain of embedded curves in the surfaces described in Lemma 3.5. This embedding goes back at least to the work of Birman and Hilden [BH72, BH73].

(b) Let $\mathcal{Y}$ denote instead the category of surfaces decorated by a single interval, with monoidal structure $\oplus$ defined just as in the case of $\mathcal{M}_{1}$ but gluing only along one interval. Then $\mathcal{Y}$ in braided monoidal, see [RWW17], Sec 5.6.1. Hence by Example 5.2 for any object $X$ of $\mathcal{Y}$, we have a Yang–Baxter element $\tau_X \in \text{Aut}_{\mathcal{Y}}(X \oplus X)$. For $X = S_{1,2}$, this can be used to prove genus stabilization (albeit with the suboptimal slope $\frac{1}{2}$), and in the case $X = S^1 \times I$ marked by an interval in one of its boundary components, we
have that $X^\oplus n$ has underlying surface an $n$-legged pair of pants $D^2\backslash(\sqcup_i D^2)$ and the associated morphism
\[ \Phi_{X,\tau_X}: B_n \longrightarrow \text{Aut}_\mathcal{M}_1(X^\oplus n) = \pi_0\text{Homeo}_\partial(D^2\backslash(\sqcup_i D^2)) \]
is the standard embedding of the braid group as the subgroup of the mapping class group of the multi-legged pants that does not twist the legs, see e.g. [RWW17] Sec 5.6.1.

We will show in Proposition 5.7 below that the Yang–Baxter operator $T$ of the first example, in the category $\mathcal{M}_2$, does not come from a braiding in $\mathcal{M}_2$.

5.2. Homological stability from Yang–Baxter elements. Suppose we are given the data of a monoidal category $(\mathcal{X}, \oplus, 1)$ acting on a category $\mathcal{M}$, along with a choice of stabilizing object $X \in \mathcal{X}$ and Yang–Baxter operator $\tau \in \text{Aut}_\mathcal{X}(X \oplus X)$. Proposition 5.1 above allows to apply [Kra19, Thm A], which in this case says that for any $A \in \mathcal{M}$, there is a sequence of simplicial spaces $W_n(X,A)_\bullet$, for $n \geq 0$, so that if $W_n(X,A)$ is highly-connected for large $n$, then the sequence
\[ \text{Aut}_\mathcal{M}(A) \overset{-\oplus X}{\longrightarrow} \text{Aut}_\mathcal{M}(A \oplus X) \overset{-\oplus X}{\longrightarrow} \text{Aut}_\mathcal{M}(A \oplus X \oplus X) \overset{-\oplus X}{\longrightarrow} \ldots \]
satisfies homological stability. Theorem B of the same paper gives in addition a stability statement with twisted coefficients. Under an injectivity assumption of the form of Proposition 5.4 this simplicial space is homotopy discrete, and modeled by the space of destabilizations as described in Definition 4.1.

Remark 5.4. The fact that $(X,\tau)$ is a Yang–Baxter operator is precisely what is needed for the collection of sets $W_n(A,X)_p$ and maps $d_i: W_n(A,X)_p \longrightarrow W_n(A,X)_{p-1}$, defined as in Definition 4.1, to assemble into a semi-simplicial set; indeed, the Yang–Baxter equation implies the necessary simplicial identities.

For a fixed monoidal category $\mathcal{X} = (\mathcal{X}, \oplus, 1)$ acting on a category $\mathcal{M}$, and a stabilizing object $X \in \mathcal{X}$, the choice of Yang–Baxter element will not affect the stabilizing map, but it will affect the spaces $W_n(X,A)_\bullet$. The identity map $1 \in \text{Aut}_\mathcal{X}(X \oplus X)$ is a trivial choice of Yang–Baxter operator. But, as is to be expected, this trivial twist is not useful for proving stability:

Proposition 5.5. Let $\mathcal{X},\mathcal{M}, A$ and $X$ be as above. If we choose the Yang–Baxter operator $\tau \in \text{Aut}_\mathcal{X}(X \oplus X)$ to be the identity element, then the semi-simplicial set $W_n(A,X)_\bullet$ is connected if and only if the map
\[ G_{n-1} = \text{Aut}_\mathcal{M}(A \oplus X^\oplus n-1) \overset{-\oplus X}{\longrightarrow} \text{Aut}_\mathcal{M}(A \oplus X^\oplus n) = G_n \]
is an isomorphism.

Proof. If $\tau$ is the identity element, all face maps $d_i$ are equal to the canonical map $G_n/G_{n-p-1} \longrightarrow G_n/G_{n-p}$. In particular, the vertices of any $p$-simplex are all equal, so the semi-simplicial set $W_n(A,X)_\bullet$ is isomorphic to a disjoint union of semi-simplicial sets, one for each 0-simplex. The result follows from the fact that the set of 0-simplices is precisely the quotient $G_n/G_{n-1}$. \qed

In fact, Barucco proved in his master thesis a result that translates to the following stronger statement (stated in the thesis in the context of a groupoid acting on itself, i.e. $\mathcal{M} = \mathcal{X}$):

Lemma 5.6. [Bar17, Lem 3.1] The space $W_n(A,X)$ is connected if and only if $1^\oplus n-2 \oplus 1$ and $G_{n-1} \oplus 1$ together generate $G_n = \text{Aut}(A \oplus X^\oplus n)$.

The connectivity of the semi-simplicial set $W_n(A,X)$ (or of the associated simplicial complex defined in [RWW17] Def 2.8]) can be thought of as a measure a form of higher generation of the group $G_n$ by the cosets of the subgroups $G_{n-p}$ for $p \geq 1$ and braid subgroups generated by the chosen Yang–Baxter element $t$, in a way similar to the notion of higher generation for a family of subgroups of a group defined in [AH93] 2.1.
5.3. Braidings and bidecorated surfaces. We show in this section that the Yang–Baxter operator $T$ on the bidecorated disk $D$ in the groupoid $M_2$ does not come from a braiding on the subcategory of $M_2$ generated by the disk. In fact, we will show that this subcategory does not admit a braiding.

Let $D = (D^2, 1, \text{id})$ be the standard bidecorated disk of Section 3, where we recall that $X_1 = D^2$. We define a “rotated” bidecorated disk $\overline{D} = (\overline{D^2}, 1, r_\pi)$, where $r_\pi$ is the rotation of $\partial X_1 = \partial D^2$ by $\pi$ radians, which has the effect of interchanging the intervals $I_0$ and $I_1$. Rotating all of $D^2$ by $\pi$ then induces a morphism $\iota: D \to \overline{D}$ in $M_2$, and likewise morphisms $\iota^m: D^m \to \overline{D}^m$ for every $m \geq 1$, each which we will by abuse of notation also denote by $\iota$. The morphism $\iota$ can be identified with the hyperelliptic involution of the underlying surface depicted in Figure 9 for the two cases $m = 2g$ and $m = 2g + 1$, where in the latter case the boundary components are exchanged by $\iota$. The morphism $\iota$ induces an identification

\[ \text{Aut}_{M_2}(D^m) \xrightarrow{\cong} \text{Aut}_{M_2}(\overline{D}^m) \]

\[ f \mapsto \iota \circ f \circ \iota^{-1} \]

In order to precisely state the failure of $T$ to extend to a braiding, we will also need the identification

\[ I: \text{Aut}_{M_2}(D^m) \xrightarrow{\cong} \text{Aut}_{M_2}(\overline{D}^m) \]

\[ f \mapsto f \]

that comes from the fact that an element $f \in \text{Aut}_{M_2}(D^m)$ is just a mapping class for the underlying surface of $D^m$, which is the same as the underlying surface of $\overline{D}^m$, so $f$ can just as well be viewed as an element of $\text{Aut}_{M_2}(\overline{D}^m)$. In contrast with the identification induced by $\iota$, the second identification is “external”, in the sense that it does not come from a morphism in $M_2$.

Viewing $\iota$ as a diffeomorphism of the underlying surface $X_m$ of $D^m$ that does not fix the boundary, and specifically exchanges the marked points $b_0 = I_0(\frac{1}{2})$ and $b_1 = I_1(\frac{1}{2})$, we see that it takes the isotopy class of arc $\rho_i$ of Section 4.1 to the reversed arc $\overline{\rho_i}$. We will use in the proof of the following result that the homotopy classes $\rho_i$ generate the fundamental groupoid of $X_m$ based at the points $b_0, b_1$. The mapping class $\iota$ is in fact completely determined by the fact that $\iota(\rho_i) = \overline{\rho_i}$.

---

5 As a full subgroupoid of the ordinary fundamental groupoid of $X_m$, this groupoid is the one spanned by the objects corresponding to the points $b_0, b_1 \in X_m$. 
Proposition 5.7. Let $D \subset M_2$ denote the full monoidal subcategory generated by $D$.

(i) The monoidal category $D$ does not admit a braiding. In particular, the monoidal functor

$$\Phi: (B, \oplus) \rightarrow (D, \#) \subset (M_2, \#)$$

does not come from a braiding on $D$.

(ii) Let $f \in \text{Aut}_{M_2}(\mathcal{D}^{\star m})$ and $g \in \text{Aut}_{M_2}(\mathcal{D}^{\star n})$, and put $\beta_{m,n} = \Phi(b_{m,n})$, where the block braid $b_{m,n}$ is the braid which passes the last $n$ strands over the first $m$ strands. Then

$$\beta_{m,n} \circ (f \# g) \circ \beta_{n,m}^{-1} = \begin{cases} g \# (\iota^{-1} \circ f \circ \iota) & \text{if } n \text{ is odd,} \\ g \# f & \text{else,} \end{cases}$$

for $i: \mathcal{D}^{\star m} \rightarrow \mathcal{D}^{\star m}$ the involution defined above, and where $f$ in the rightmost expression is the map $f$ considered as an element of $\text{Aut}_{M_2}(\mathcal{D}^{\star m})$ via the isomorphism $I$ defined above.

Proof. We start by proving (ii). It is enough to check the statement when $f$ and $g$ are Dehn twists, as those generate the mapping class group. Note that if $c$ is a curve in the underlying surface $X_{m+n}$ of $\mathcal{D}^{\star m+n}$, and $T_c$ denotes the Dehn twist along $c$, then conjugating $T_c$ by a diffeomorphism $\varphi$ of the surface gives

$$\varphi \circ T_c \circ \varphi^{-1} = T_{\varphi(c)}.$$

Recall further that the isotopy class of a Dehn twist $T_c$ depends only on the free homotopy class of the curve $c$. We are therefore to compute the images of curves in $\mathcal{D}^{\star m}$ and $\mathcal{D}^{\star n}$ under the map $\beta_{m,n}$, as free homotopy classes. A curve $c$ can be written, up to free homotopy, as a concatenation of the arcs $\rho_i$ and their inverses $\overline{\rho}_i$, as the homotopy classes of these arcs generate the fundamental groupoid of the surface $X_{m+n}$ based at $b_0, b_1$. In particular, write

$$c \simeq \rho_{i_1} \ast \overline{\rho}_{i_2} \ast \rho_{i_3} \ast \cdots \ast \overline{\rho}_{i_k}.$$  

The mapping class $\beta_{m,n}$ can be written as the composition

$$\beta_{m,n} = (T_n \circ \cdots \circ T_{m+n-1}) \circ \cdots \circ (T_2 \circ \cdots \circ T_{m+1}) \circ (T_1 \circ \cdots \circ T_m)$$

and hence we can compute the image of each $\rho_i$ using Lemma 4.3. For $r > 0$, denote by $T_{i,r}$ the composition of Dehn twists $T_i \circ T_{i+1} \circ \cdots \circ T_{i+r}$. Note first that

$$T_{i,j} = T_{i,j-1} \ast T_j \simeq \cdots \simeq \rho_i.$$  

From this, it follows that for $i \geq 1$,  

$$\beta_{m,n}(\rho_{i+1}) \simeq (T_n \circ \cdots \circ (T_{i,m})(\rho_{i+1})$$

$$\simeq (T_n \circ \cdots \circ (T_{i,m})(\rho_{m+i})$$

$$\simeq (T_{n,m+n-1} \circ \cdots \circ (T_{i+1,m+i})(\rho_i)$$

$$\simeq \rho_i.$$  

On the other hand, for $i \leq k \leq j$, we have

$$T_{i,j} = T_{i,k}(\rho_k) \simeq T_{i,k-1}(\rho_k \ast \overline{\rho}_{k+1} \ast \rho_k) \simeq \rho_i \ast \overline{\rho}_{k+1} \ast \rho_i,$$

from which we can deduce that for $i \leq m$,

$$\beta_{m,n}(\rho_i) \simeq (T_n \circ \cdots \circ (T_{i,m})(\rho_i)$$

$$\simeq (T_{n,m+n-1} \circ \cdots \circ (T_{i,m+1})(\rho_i \ast \overline{\rho}_{i+1} \ast \rho_i)$$

$$\simeq (T_{n,m+n-1} \circ \cdots \circ (T_{i,m+2})(\rho_i \ast \overline{\rho}_{i+2} \ast \rho_{i+2} \ast \rho_1)$$

$$\simeq \cdots$$

$$\simeq \rho_1 \ast \iota(\rho_2) \ast \cdots \ast \iota^{n-1}(\rho_n) \ast \iota^n(\rho_{i+n}) \ast \iota^{n-1}(\rho_{i+n}) \ast \cdots \ast \iota(\rho_2) \ast \rho_1$$

since $\iota(\rho_i)$ is $\rho_i$ when $j$ is even and $\overline{\rho}_i$ when $j$ is odd.
If the curve $c$ lies in the last $n$ disks $D^m$ inside $D^m + n$, it can be written as a product (5.1) with each $i_j > m$. Then the above computation gives that
\[
\beta_{m,n}(c) \simeq \rho_1 \ast \rho_2 \ast \cdots \ast \rho_{i_1-n} \ast \rho_{i_1-m} \ast \cdots \ast \rho_{i_k-m},
\]
that is, $c$ is mapped to the corresponding curve in the first $n$ disks $D^m$ inside $D^m + n$.

If the curve $c$ instead lies in the first $m$ disks $D^m$ inside $D^m + n$, it can be written as a product (5.1) with each $i_j \leq m$. Then the above computation gives that
\[
\beta_{m,n}(c) \simeq \rho_1 \ast \rho_2 \ast \cdots \ast \rho_{i_1-n} \ast \rho_{i_1-m} \ast \cdots \ast \rho_{i_k-m} \ast \rho_{i_k+1} \ast \cdots \ast \rho_{i_k+n},
\]
\[
\simeq \epsilon_1 \ast \epsilon_2 \ast \cdots \ast \epsilon_{i_1-n} \ast \epsilon_{i_1-m} \ast \cdots \ast \epsilon_{i_k-m} \ast \epsilon_{i_k+1} \ast \cdots \ast \epsilon_{i_k+n},
\]
\[
\simeq \epsilon_1 \ast \epsilon_2 \ast \cdots \ast \epsilon_{i_1-n} \ast \epsilon_{i_1-m} \ast \cdots \ast \epsilon_{i_k-m} \ast \epsilon_{i_k+1} \ast \cdots \ast \epsilon_{i_k+n},
\]
\[
\simeq t^n(\beta_{i_1-n} \ast \rho_{i_1-m} \ast \cdots \ast \rho_{i_k-m} \ast \rho_{i_k+1} \ast \cdots \ast \rho_{i_k+n}).
\]

Hence $c$ is mapped to the curve $t^n(c)$ in the last $m$ disks $D^m$ inside $D^m + n$, from which the statement follows.

We are left to prove (i). To see that the images $\beta_{m,n}$ of block braids under $\beta$ do not define a braiding in $D$, using (ii) it is enough to find a curve $c$ in $D^m$ for some $m$ so that $\iota(c) \neq c$, and such curves are plentiful. The same argument shows that the inverses $\beta_{m,n}^{-1}$ likewise do not define a braiding.

Now suppose that $\overline{\beta}$ is a braiding on $D$. The braiding is determined by $\overline{\beta}_{1,1} \in \text{Aut}_{\mathcal{M}_2}(D^2) \cong \mathbb{Z}$, a group generated by the Dehn twist $T_1$. We have excluded the possibilities $\overline{\beta}_{1,1} = T_1$, and $\overline{\beta}_{1,1} = \text{id}$ is similarly ruled out using now the fact that curves are not moved at all by the identity. So assume that $\overline{\beta}_{1,1} = T_1^k$, with $|k| > 1$. Then $\overline{\beta}_{1,1} = T_1^k T_2^k$ would have to satisfy $T_1^k T_2^k(a_1) = a_2$ in order for naturality to hold, where $T_1$ is the Dehn twist along the curve $a_4$ as in Section 3.2. Applying Proposition 3.2 in [FM11] twice, we get that the intersection number $i(a_2, T_2^k(a_1)) = i(T_1^k(T_2^k(a_1)), T_2^k(a_1)) = |k|i(a_1, T_1^k a_2) = |k| |i(a_1, a_2)|^2 = |k|^2$. On the other hand, using Proposition 3.4 in [FM11] we obtain
\[
|k|^2 = i(a_2, T_2^k(a_1)) = |i(T_2^k(a_1), a_2) - |k|i(a_1, a_2)|i(a_1, a_2)| \leq |i(a_1, a_2)| = 1,
\]
where we have also used that $i(a_1, a_1) = 0$. This contradicts our assumption of $\overline{\beta}_{1,1}$. \qed

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