Stabilizer algebra of adjoint-invariant forms

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Abstract

In this paper we study the stabilizer algebra of adjoint-invariant $l$-forms on a simple Lie algebra over the complex number field. We prove that the stabilizers of most adjoint-invariant $l$-forms on a complex simple Lie algebra $g$ coincide with $\mathfrak{ad}(g)$.

Keywords: Adjoint-invariant forms, stabilizer algebra

1 Introduction

Special geometries associated with a class of differential forms on manifolds are motivated by many known geometries including Riemannian geometry, symplectic geometry and geometry with special holonomy [3]. To study the geometry associated with a class of differential forms on manifold one could study the stabilizer group of those forms. This method is widely used by geometer, for example, see [3], [6], [7]. To study the stabilizer group of a form, one may first study its Lie algebra. Once we know the Lie algebra, we can trace back to compute the group.

Let $g$ be a complex simple Lie algebra. Suppose that $\omega$ is an Adjoint-invariant $l$-form on $g$, $\text{Stab}(\omega)$ is the stabilizer group of $\omega$ and $\text{stab}(\omega)$ is the Lie algebra of $\text{Stab}(g)$. In case $\omega$ is the 3-Cartan form, $\text{Stab}(\omega)$ is studied by Anthony C. Kable in [4] and by Hồng Vân Lê in [6]. Our paper can be thought as a continuation of Kable’s paper, [4]. The main result is the following

**Theorem 1.** The stabilizer algebra of any Adjoint-invariant $l$-form on a simple Lie algebra $g$ coincides with $\mathfrak{ad}(g)$ if $l < \text{dim}(g)$.

This result can be useful in finding the stabilizer group $\text{Stab}(\omega)$ of $\omega$ and further, we may hope to extend the result of Hồng Vân Lê in [6] for Adjoint-invariant forms.

Our plan is as follows
In the first part, we introduce a notion, $\varepsilon$-decomposable form (see Definition 1), and recall a result of Kempf (see Lemma 1).

In the second part, we give a proof of the main theorem and remarks for further researches.

2 Preliminary

In this paper, we assume that $\mathfrak{g}$ is a complex simple Lie algebra.

Let $V$ be a vector space of dimension $n$, and $\varepsilon = \{e^1, e^2, \ldots, e^n\}$ a basis of $V^*$. An $l$-form $\omega$ of $V$ can be written as $\omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq n} a_{i_1 \ldots i_l} e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_l}$, we call it the canonical form of $\omega$ with respect to $\varepsilon$. For each element $A$ of $\mathfrak{gl}(V)$ we define

$$\omega_A := A(\omega) = \sum_{1 \leq i_1 < i_2 < \ldots < i_l} \sum_j a_{i_1 \ldots i_l j} e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_l}$$

We now introduce a notion for later use.

**Definition 1.** ($\varepsilon$-decomposable form)

Let $V$ be a vector space of dimension $n$, and $\varepsilon = \{e^1, e^2, \ldots, e^n\}$ a basis of $V^*$. An $l$-form $\omega$ is called $\varepsilon$-decomposable if it can be written as $\delta = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_l}$ where $1 \leq i_1, i_2, \ldots, i_l \leq n$ are integers.

Define the $\varepsilon$-presentation of a form $\gamma$ to be the expression of $\gamma$ as the sum of $\varepsilon$-decomposable forms.

**Definition 2.** (Equivalent $\varepsilon$-decomposable forms)

For each $X$ in $\mathfrak{gl}(V)$ and a differential form $\beta$ we write $\beta_X := X(\beta)$. Then we say two $\varepsilon$-decomposable forms $\alpha, \beta$ equivalent if $\alpha$ appears as a summand in the $\varepsilon$-presentation of $\beta_X$ in $\varepsilon$.

**Proposition 1.** Given two $\varepsilon$-decomposable $l$-forms $\alpha$ and $\beta$, they are equivalent if and only if we can write $\alpha = a \wedge \gamma$, $\beta = b \wedge \gamma$, for some $a, b \in \varepsilon$, and $\gamma = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_{l-1}}$.

*Proof.* If we can write $\alpha = a \wedge \gamma$, $\beta = b \wedge \gamma$, where $a, b \in \varepsilon$, $\gamma = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_{l-1}}$. Then we can choose an element $X \in \mathfrak{gl}(\mathfrak{g})$ such that it transforms $b$ to $a$. Then, $\alpha$ is a summand in the $\varepsilon$-presentation of $\beta_X$.

Conversely, if $\alpha$ and $\beta$ are equivalent, let $X \in \mathfrak{gl}(V)$ be such that $\alpha$ is a summand in the $\varepsilon$-presentation of $\beta_X$. We write $\beta = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_l}$, then $X(\beta) = \sum_{j=1}^l e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge X(e^{i_j}) \wedge \ldots \wedge e^{i_l}$. Hence, $\alpha$ appears as a summand of $X(\beta)$ only if it has the form $\alpha = a \wedge \gamma$, where $\gamma = e^{i_{j_1}} \wedge e^{i_{j_2}} \wedge \ldots \wedge e^{i_{j_{l-1}}}$, $\{j_1, j_2, \ldots, j_{l-1}\} \subset \{i_1, i_2, \ldots, i_l\}$, $a \in \varepsilon$.

Later, we will use Dynkin’s classification of triples $(\alpha_1, \alpha_2, \rho)$ where $\alpha_2$ is a simple Lie algebra, $\alpha_1$ is a semisimple Lie subalgebra of $\alpha_2$ and $\rho$ is an
irreducible representation of $\alpha_2$ which remains irreducible when restricted to $\alpha_1$. Suppose $\rho_\alpha$ and $\rho_\beta$ are representations of $\alpha_2$ and $\beta_2$ on $V_\alpha$ and $V_\beta$ respectively where $V_\alpha$ and $V_\beta$ are some vector spaces, then two triples $(\alpha_1, \alpha_2, \rho_\alpha)$ and $(\beta_1, \beta_2, \rho_\beta)$ are called equivalent if there is a linear isomorphism $L : V_\alpha \to V_\beta$ such that $\rho_\beta (\beta_2) = L \rho_\alpha L^{-1}$. The classification will be found in Table 5 of [2].

We also introduce a direct consequence of Theorem 3.4, Corollary 3.5 and Theorem 4.4 in [5]. We refer the readers to [5] for the proof of the following

**Lemma 1.** Given $\rho : G \times X \to X$ an action of an affine algebraic group $G$ on an affine variety $X$ and $x$ a point in $X$. If the orbit $O_x$ of $x$ is not closed in $X$ then $G$ possesses a non-trivial one parameter subgroup $\lambda : G_m \to G$. Further, the subgroup

$$P(\lambda) := \{ g \in G : \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } G \}$$

is a parabolic subgroup of $G$ containing the stabilizer subgroup of $x$ in $G$.

### 3 Main result

We state the main result of this article.

**Theorem 1.** The stabilizer algebra of any Adjoint-invariant $l$-form on a simple Lie algebra $g$ coincides with $\mathfrak{ad}(g)$ if $l < \text{dim}(g)$.

In Proposition 2 we will first prove that $\mathfrak{stab}(g)$ is is a simple Lie algebra. To prove that Proposition, we need the following lemmas.

**Lemma 2.** Denote $\mathfrak{SL}(g)(\cdot)$ the standard action of $\mathfrak{SL}(g)$ on $\Lambda^l g^*$. Then the orbit $O(\omega) = \mathfrak{SL}(g)(\omega)$ is closed in $\Lambda^l g^*$ under the Zariski topology, consequently $O(\omega)$ is an affine variety.

**Proof.** Suppose conversely that $O(\omega)$ is not closed. Applying Lemma 1 we can find a non-trivial one-parameter subgroup $\lambda : G_m \to \mathfrak{SL}(g)$ of $\mathfrak{SL}(g)$ and a parabolic subgroup

$$P(\lambda) = \{ g \in \mathfrak{SL}(g) : \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } \mathfrak{SL}(g) \}$$

of $\mathfrak{SL}(g)$ containing the stabilizer subgroup of $\omega$ in $\mathfrak{SL}(g)$.

Because $g$ is a simple Lie algebra, we have

$$\mathfrak{ad}(g) = [\mathfrak{ad}(g), \mathfrak{ad}(g)] \subset [\mathfrak{gl}(g), \mathfrak{gl}(g)] \subset \mathfrak{sl}(g).$$

Hence,

$$\mathfrak{Aut}_0(g) = \exp(\mathfrak{ad}(g)) \subset \exp(\mathfrak{sl}(g)) \subset \mathfrak{SL}(g).$$
In addition, the stabilizer group of $\omega$ contains $Aut_0(\mathfrak{g})$, consequently $P(\lambda)$ contains $Aut_0(\mathfrak{g})$.

As $\lambda(t)$ is a one parameter subgroup of $SL(\mathfrak{g})$, there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathfrak{g}$ such that the action of $\lambda(t)$ on $\mathfrak{g}$ can be written in the matrix form

$$\lambda(t) = \begin{pmatrix}
t^{m_1} & 0 & \cdots & 0 \\
0 & t^{m_2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & t^{m_n}
\end{pmatrix},$$

where $n$ is the dimension of $\mathfrak{g}$ and $m_1 \leq m_2 \leq \ldots \leq m_n$ are integers.

Then for any matrix $A = (a_{ij}) \in SL(\mathfrak{g})$ we have

$$\lambda(t)A\lambda(t)^{-1} = (t^{m_i - m_j}a_{ij}).$$

Thus, if $A \in P(\lambda)$ then $a_{ij} = 0$ for $m_i < m_j$. Let $i_0$ be the greatest number such that $m_1 = m_2 = \ldots = m_{i_0}$. Then

$$a_{ij} = 0$$

for any $i, j$ such that $1 \leq i \leq i_0 < j \leq n$. Therefore, the vector subspace $V$ spanned by $\{e_1, e_2, \ldots, e_{i_0}\}$ is invariant under the action of $P(\lambda)$. As a result, $V$ is stable under the action of $Aut_0(\mathfrak{g})$.

Because $\mathfrak{g}$ is simple, the action of $Aut_0(\mathfrak{g})$ on $\mathfrak{g}$ is irreducible, hence $V$ should be either 0 or $\mathfrak{g}$. Notice that $V \neq 0$ for $e_1 \in V$, we have $V = \mathfrak{g}$. It follows $i_0 = n$, in other words

$$m_1 = m_2 = \ldots = m_n.$$

Further, $P(\lambda)$ is a subset of $SL(\mathfrak{g})$ and this implies $\lambda(t) \subset SL(\mathfrak{g})$. It follows that $det(\lambda(t)) = 1$. As a result,

$$\sum_{i=1}^{n} m_i = 0.$$

In addition, we have

$$m_1 = m_2 = \ldots = m_n,$$

hence

$$m_1 = m_2 = \ldots = m_n = 0.$$

And therefore $\lambda(t) \equiv I_n$ contradicting from the assumption that $\lambda(t)$ is non-trivial.

\[ \square \]

Remark 1. The idea of using Lemma 4 in the above proof comes from Theorem 1 in [4]. In fact, Kable’s proof of Theorem 1 in [4] is applicable in our case. However, our proof is simpler, more specifically, we do not need to compute $\lim_{t \to 0} \lambda(t)\omega$. 

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The following lemma is in [4].

**Lemma 3.** The commutant of $ad(g)$ in $\text{stab}(\omega)$ is zero.

**Proof.** Since $g$ is simple, then $ad(g)$ is simple. It follows that

$$[ad(g), ad(g)] = ad(g),$$

hence

$$ad(g) = [ad(g), ad(g)] \subset [gl(g), gl(g)] \subset sl(g).$$

Let $\phi$ be a non-zero element in $\text{stab}(\omega)$ commuting with $ad(g)$. we have

$$\phi([X,Y]) = \phi(ad_X(Y)) = (\phi \circ ad_X)(Y) = ad_X(\phi(Y)) = [X, \phi(Y)].$$

Consequently, if $\phi(Y) = 0$ then $\phi([X,Y]) = 0$. That is $[X,Y] \in ker(\phi)$ for every $X \in g, Y \in ker(\phi)$. Thus $ker(\phi)$ is an ideal of $g$. Because $g$ is simple, $ker(\phi)$ is either 0 or $g$. Notice that $ker(\phi) \neq g$ as $\phi$ is non-zero, it follows that $ker(\phi) = 0$. As a result, $\phi$ possesses a non-zero eigenvalue $c$.

Denoted by $X$ an eigenvector of $\phi$ corresponding to $c$. Consider $\psi = \phi - cI$, then $\psi(X) = 0$. Further, as $\phi$ commutes with $ad(g)$, then $\psi$ also commutes with $ad(g)$. It yields that if $\psi$ is different from zero then by the same argument as we did with $\phi$, we obtain $ker(\psi) = 0$, a contradiction since $X \in ker(\psi)$. Thus, $\psi = 0$. Therefore

$$\phi(X) = cX$$

for any $X$ in $g$ and for some constant $c \in \mathbb{C}$. It follows that

$$\phi \circ \omega = -c.l.\omega.$$

It implies $c = 0$ as $\phi \circ \omega = 0$. In other words, $\phi = 0$, contradicting the assumption that $\phi \neq 0$, by which the lemma follows. 

As a direct consequence of Lemma 3 we obtain the following

**Corollary 1.** The Lie subalgebra $\text{stab}(\omega)$ has zero center.

**Lemma 4.** The Lie subalgebra $\text{stab}(\omega)$ is semisimple and contained in $sl(g)$.

**Proof.** Let

$$I(\omega) = \text{Stab}(\omega) \cap SL(g)$$

and $i(\omega)$ its Lie algebra. Since the commutant of $ad(g)$ in $\text{stab}(\omega)$ is zero, we have the commutant of $ad(g)$ in $i(\omega)$ is zero. Hence, $i(\omega)$ has zero center.

Lemma 2 shows that $SL(g)/I(\omega)$ is an affine variety. Matsushima’s criterion, [1], then implies $I(\omega)$ is reductive, hence $i(\omega)$ is reductive. Furthermore, $i(\omega)$ has zero center, it is semisimple.
Let $\mathfrak{h} = \text{stab}(\omega) \setminus i(\omega)$ be the set complement of $i(\omega)$ in $\text{stab}(\omega)$. Since
\[
[\mathfrak{gl}(\mathfrak{g}), \mathfrak{gl}(\mathfrak{g})] \subset \mathfrak{sl}(\mathfrak{g}),
\]
then
\[
[i(\omega), \text{stab}(\omega)] \subset \mathfrak{sl}(\mathfrak{g}).
\]
But we already have
\[
[i(\omega), \text{stab}(\omega)] \subset \text{stab}(\omega),
\]
thus
\[
[i(\omega), \text{stab}(\omega)] \subset \mathfrak{sl}(\mathfrak{g}) \cap \text{stab}(\mathfrak{g}) = i(\omega).
\]
Therefore, $i(\omega)$ is an ideal of $\text{stab}(\omega)$.

Suppose that $\text{stab}(\omega)$ has some abelian ideal $\alpha$. We have two cases

1. **Case 1** $\alpha \subset \mathfrak{h}$

   We have
   \[
   [\alpha, i(\omega)] \subset [\mathfrak{h}, i(\omega)] \subset i(\omega)
   \]
   and
   \[
   [\alpha, i(\omega)] \subset \alpha \subset \mathfrak{h}.
   \]
   Since $\mathfrak{h}$ is the set complement of $i(\omega)$ in $\text{stab}(\omega)$, we have $[\alpha, i(\omega)] = 0$.
   In other words, $\alpha$ commutes with $i(\omega)$. In particular, $\alpha$ commutes with $\mathfrak{ad}(\mathfrak{g})$. Lemma 3 then implies $\alpha = 0$.

2. **Case 2** $\alpha \not\subset \mathfrak{h}$

   We have $\alpha \cap i(\omega) \neq 0$ is a non-zero abelian ideal of $i(\omega)$ contradicting with the fact that $i(\omega)$ is semisimple.

Therefore, $\text{stab}(\omega)$ is semisimple. Hence,
\[
[\text{stab}(\omega), \text{stab}(\omega)] = \text{stab}(\omega),
\]
consequently,
\[
\text{stab}(\omega) = [\text{stab}(\omega), \text{stab}(\omega)] \subset [\mathfrak{gl}(\mathfrak{g}), \mathfrak{gl}(\mathfrak{g})] \subset \mathfrak{sl}(\mathfrak{g}).
\]

Remark 2. The idea of using Matsushima’s criterion ([7]) in Proposition 4 comes from the proof of Theorem 2 in Kable’s paper [4]. Indeed, Kable’s proof works in our case but here we have given a different proof.

Proposition 2. The stabilizer algebra $\text{stab}(\omega)$ is simple.
Proof. Since $\text{stab}(\omega)$ is semisimple, we can write it as a sum of non-zero simple ideals, 
$$ \text{stab}(\omega) = S_1 \oplus S_2 \oplus \ldots \oplus S_k. $$
Since $\text{ad}(\mathfrak{g}) \subset \text{stab}(\mathfrak{g})$ and the action of $\text{ad}(\mathfrak{g})$ on $\mathfrak{g}$ is irreducible, $\text{stab}(\omega)$ acts irreducibly on $\mathfrak{g}$. For each $i$, we consider the action of $S_i$ on $\mathfrak{g}$ and denote $V_i := \text{ker}(S_i)$. As $S_i$ is an ideal of $\text{stab}(\omega)$, we have $[\text{ad}(\mathfrak{g}), S_i] \subset S_i$.

We then obtain 
$$ [\text{ad}(\mathfrak{g}), S_i](V_i) = 0, $$
more specifically 
$$ (\text{ad}_X A - A \text{ad}_X)(V_i) = 0, $$
for any $X \in \mathfrak{g}, A \in S_i$.

It follows 
$$ A \text{ad}_X(V_i) = 0, \forall A \in S_i. $$
Consequently 
$$ \text{ad}_X(V_i) \subset \text{ker}(S_i) = V_i, $$

hence $\text{ad}(\mathfrak{g})V_i \subset V_i$. In other words, $V_i$ is a subrepresentation of $\text{ad}(\mathfrak{g})$ on $\mathfrak{g}$, $V_i$ must be either 0 or $\mathfrak{g}$. If $V_i = \mathfrak{g}$ then $\text{ker}(S_i) = \mathfrak{g}$, it implies $S_i = 0$, a contradiction. Thus $\text{ker}(S_i) = 0$, that is $S_i \mathfrak{g} = \mathfrak{g}$.

We now can consider the representations from each $S_i$ on $\mathfrak{g}$. We have the following

**Claim 1.** The representation of $S_1$ on $\mathfrak{g}$ is irreducible.

*Proof (of the claim).* Let $U(\text{stab}(\omega)), U(S_1), U(S_2), \ldots, U(S_k)$ be the smallest associative subalgebras of $\text{gl}(\mathfrak{g})$ containing $\text{stab}(\omega), S_1, S_2, \ldots, S_k$, respectively.

Since, 
$$ \text{stab}(\omega) = S_1 \oplus S_2 \oplus \ldots \oplus S_k, $$
then
$$ U(\text{stab}(\omega)) = U(S_1) + U(S_2) + \ldots + U(S_k). $$
Furthermore, as $S_i$’s are simple ideals of $\text{stab}(\mathfrak{g})$, $S_i$ and $S_j$ are commute under the Lie bracket. It means 
$$ [X_i, X_j] = 0 \ \forall X_i \in S_i, X_j \in S_j, i \neq j. $$
In other words, 
$$ X_iX_j = X_jX_i \ \forall X_i \in S_i, X_j \in S_j, i \neq j. $$
It follows that $U(S_i)$ and $U(S_j)$ are commute.
Now, suppose that $S_1$ does not act irreducibly on $g$. As $S_1$ is a simple Lie algebra, the action of $S_1$ on $g$ reduced completely. We can write $g$ as a sum of non-zero irreducible subrepresentations of $S_1$

\[ g = V_1 \oplus V_2 \ldots \oplus V_m \]

Consider the action of $S_1$ on $V_i$ because $\ker(S_1) = 0$, then $S_1(V_i) = V_i$; hence $U(S_1)(V_i) = V_i$. Let $B$ be any element in $U(S_2)$ such that $B(V_1) \neq 0$. Since $U(S_1)$ and $U(S_2)$ are commute, we have

\[ A(B(V_1)) = B(A(V_1)) = B(V_1), \forall A \in U(S_1). \]

Thus $B(V_1)$ is a subrepresentation of $S_1$, then there exists some $i$ such that $B(V_1) = V_i$, we have two cases.

1. **Case 1:** $i \neq 1$.

   Without loss of generality, we may assume $B(V_1) = V_2$. Let $C$ be an element in $U(S_2)$ such that $C(V_2) \neq 0$ and $C$ is different from $B$, $-B$. Applying the argument above for $C$ and $B + C$, there exist $j$ and $k$ such that $C(V_1) = V_j$ and $(B + C)(V_1) = V_k$. On the other hand

\[ (B + C)(V_1) \subset V_2 \oplus V_j. \]

In addition,

\[ V_k \cap (V_2 \oplus V_j) = \emptyset \text{ if } k \neq 2, j. \]

Consequently, $k$ is either 2 or $j$. If $j \neq 2$ then $(B + C)(V_1)$ is different from $V_2$ and $V_j$, then there will be no such $k$. Hence, $j = 2$ and therefore

\[ B(V_1) = V_2, \forall B \in U(S_2), \]

in other words $U(S_2)(V_1) = V_2$. We then have

\[
\begin{align*}
U(S_2)(V_2) &= U(S_2)(U(S_2)(V_1)) \\
&= (U(S_2)U(S_2))(V_1) \\
&= U(S_2)(V_1) \\
&= V_2.
\end{align*}
\]

Thus, $V_2$ is a subrepresentation of $U(S_2)$, hence it is a subrepresentation of $S_2$. For any $B \in S_2$ we have $BV_2 \subset V_2$. As $V_2$ is a complex vector space, by Schur’s lemma the action of $B$ on $V_2$ is a multiplication by a scalar $c_B$.

As $s_2$ is simple, we have $[S_2, S_2] = 0$. Then, for any $B \in s_2$ there exist $C, D \in S_2$ such that

\[ B = [C, D] = CD - DC, \]
then
\[ c_B = c_Cc_D - c_DC = 0. \]
Thus, the action of \( \mathcal{S}_2 \) on \( V_2 \) is the multiplication by zero, contradicting the fact that \( \ker(\mathcal{S}_2) = 0 \).

2. Case 2: \( i = 1 \).
Using the same argument as above, we have \( \mathcal{S}_2(V_1) = V_1 \) and then the action of \( s_2 \) on \( V_1 \) is the multiplication by zero, that is \( V_1 \in \ker(\mathcal{S}_2) \), a contradiction.

\[ \square \]

We now come back to the proof of Proposition \([2]\). The action of \( \mathcal{S}_1 \) on \( \mathfrak{g} \) is irreducible. For each \( j \) the actions of \( \mathcal{S}_j \) on \( \mathfrak{g} \) are \( \mathbb{C} \)-linear and it commutes with the action of \( \mathcal{S}_1 \) on \( \mathfrak{g} \). Let \( X_j \) be any non-zero element in \( \mathcal{S}_j \), \( X_1 \) a non-zero element in \( \mathcal{S}_1 \). If \( Y \in \ker(X_j) \), i.e, \( X_j(Y) = 0 \), then
\[ X_j(X_1(Y)) = X_1(X_j(Y)) = 0. \]
By Schur’s lemma, \( X_j \) acts on \( \mathfrak{g} \) as a scalar multiplication. Therefore, for any \( X_j \in \mathcal{S}_j \), \( X_j \mathfrak{g} = c_{X_j} \mathfrak{g} \) for some constant \( c_{X_j} \in \mathbb{C} \). But then
\[ X_j(\omega) = -lc_{X_j}\omega = 0. \]
It implies \( c_{X_j} = 0 \). Thus \( \delta_j \mathfrak{g} = 0 \) and therefore \( \mathcal{S}_j = 0 \) if \( j \neq 1 \), hence \( \text{stab}(\omega) = \mathcal{S}_1 \) is simple.

\[ \square \]

In order to use Dynkin’s classification we need the followings

**Lemma 5.** Let \( l \) be an integer smaller than \( \dim(\mathfrak{g}) \), then \( \mathfrak{sl}(\mathfrak{g}) \) can not preserve any \( l \)-form in \( \mathfrak{g} \).

**Proof.** For a \( l \)-form \( \omega \) in \( \mathfrak{g} \) and an element \( A \) in \( \mathfrak{sl}(\mathfrak{g}) \), if \( A(\omega) \neq 0 \) then the lemma is proved. If not, we consider one \( \epsilon \)-decomposable summand \( \gamma \) of \( \omega \), as \( l < \dim(\mathfrak{g}) \) then there exist another \( \epsilon \)-decomposable \( l \)-form \( \delta \) that equivalent to \( \gamma \) . From Proposition \([1]\) we can write \( \gamma = e^1 \wedge \eta, \delta = e^2 \wedge \eta \). Now we can make the entry in \( B \) (when regard it as a matrix in a basis that has \( e^1 \) and \( e^2 \) as component) that transforms \( e^1 \) to \( e^2 \) arbitrary large such that the new transformation is still in \( \mathfrak{sl}(\mathfrak{g}) \) and the coefficient of \( \delta \) in \( B(\omega) \) is different from zero, which contradicts with the fact that \( A(\omega) = 0, \forall A \in \mathfrak{sl}(\mathfrak{g}) \). \( \square \)

**Remark 3.** For any matrix \( A \in \mathfrak{so}(\mathfrak{g}) \), we can also make \( a_{ij} \) and \( a_{ji} \) arbitrarily large such that \( A \) still in \( \mathfrak{so}(\mathfrak{g}) \) . Then we can apply the same argument as in Lemma \([4]\) to prove that \( \mathfrak{so}(\mathfrak{g}) \) can not preserve any \( l \)-form.

**Proposition 3.** If \( l < \dim(\mathfrak{g}) \) then \( \text{stab}(\omega) \) can not be \( \mathfrak{sl}(\mathfrak{g}), \mathfrak{so}(\mathfrak{g}) \) or \( \mathfrak{sp}(\mathfrak{g}) \).
Proof. From Lemma 5 and Remark 3 we have that \( \text{stab}(\omega) \) can not be either \( \mathfrak{sl}(g) \) or \( \mathfrak{so}(g) \).

If \( \text{stab}(\omega) = \mathfrak{sp}(g) \) then it must preserve a non-zero skew-symmetric bilinear form \( \alpha \). Since \( \mathfrak{ad}(g) \subset \text{stab}(\omega) \), the form \( \alpha \) should be preserved by \( \mathfrak{ad}(g) \). But any bilinear form preserved by \( \mathfrak{ad}(g) \) must be a multiple of the Killing form on \( g \), which is a symmetric form, a contradiction.

We now come to the proof of the main theorem.

Proof. (Of the theorem)

As \( \mathfrak{ad}(g) \) is a subalgebra of \( \text{stab}(\omega) \) and \( \text{stab}(\omega) \) is simple, we now consider the triple \( (\mathfrak{ad}(g), \text{stab}(g), id) \). Proposition 3 states that \( \text{stab}(g) \) can not be \( \mathfrak{sl}(g) \), \( \mathfrak{so}(g) \), or \( \mathfrak{sp}(g) \). Hence, if \( (\alpha_1, \alpha_2, \rho) \) be any triple that equivalent to \( (\mathfrak{ad}(g), \text{stab}(g), id) \) then \( \alpha_2 \) can not be \( \mathfrak{sl}(V) \), \( \mathfrak{sp}(V) \) or \( \mathfrak{so}(V) \) with \( V \) is the representation vector space of \( \alpha_2 \) by \( \rho \). So, we can look for possibilities of \( (\mathfrak{ad}(g), \text{stab}(\omega), id) \) in Table 5 of [2].

Furthermore, the restriction of \( \rho \) on \( \alpha_1 \) must isomorphic to the adjoint representation, as the restriction of \( id \) on \( \mathfrak{ad}(g) \) is the adjoint action. We can restrict the triple \( (\mathfrak{ad}(g), \text{stab}(g), id) \) to the cases \( I_1 (n \geq 2, k = 2), I_2 (n \geq 3, k = 2), I_4 (n \geq 4, k = 2) \) by comparing the dimension of cases in Table 5 in [2].

In Table 5 in [2], the models for types \( I_1, I_2, I_4 \) are \( (\mathfrak{sp}(n), \mathfrak{sl}(2n), \vee \rho_{2n}), (\mathfrak{so}(n), \mathfrak{sl}(2n + 1), \wedge \rho_{2n+1}) \) (n is odd), \( (\mathfrak{so}(n), \mathfrak{sl}(2n), \wedge \rho_{2n}) \) (n is even), respectively.

In addition, the triple \( (\mathfrak{ad}(g), \text{stab}(g), id) \) has the property that \( \text{stab}(g) \) preserves a \( \mathfrak{l} \)-form on \( g \). We will show that neither of the above three types satisfy this property if \( l < \text{dim}(g) \). For a prove of this, one only need to show that there exist an element \( X \in \mathfrak{sl}(m, \mathbb{C}) \) \( (m = 2n \) in cases \( I_2 \) and \( I_4 \) and \( m = 2n + 1 \) in case \( I_1 \) \) such that \( X(\omega) \neq 0 \). But the proof of these facts are same as the proof of Lemma 5.

3.1 Final remarks

We can use the same technique as in Kable’s paper, [4], to prove that the stabilizer group of \( \omega \) is isomorphic to

\[
\text{Aut}(g) \ltimes M(g)
\]

where \( M(g) = \{ \phi \in GL(g) \mid \phi^t = \text{id}_g \text{ and } \mathfrak{ad}(X) \circ \phi = \phi \circ \mathfrak{ad}(X) \forall X \in g \} \)

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