Deciders for tangles of set separations

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Abstract

Tangles, as introduced by Robertson and Seymour, were designed as an indirect way of capturing clusters in graphs and matroids. They have since been shown to capture clusters in much broader discrete structures too. But not all tangles are induced by a cluster. We characterise those that are: the tangles that have a decider.

We offer two such characterisations. The first is in terms of how many small sides of a tangle’s separations it takes to cover the graph or matroid. The second uses a new notion of duality for sets of separations that do not necessarily come from graphs or matroids.

1 Introduction

Tangles were introduced by Robertson and Seymour as a tool in their graph minors project [17]. They provided a novel, indirect, way to capture highly cohesive substructures, or ‘clusters’, in graphs. The idea is that since clusters cannot be divided into significantly large parts by graph separations of low order, any given cluster implicitly orients every low-order separation towards its ‘big’ side, the side that contains most of the cluster. It turned out that this induced orientation of all the low-order separations collectively contains all the information needed to prove fundamental theorems about the cluster structure of a graph, which has made tangles a powerful tool in the connectivity theory of graphs.

Over the last decade, the notion of tangles has been significantly generalised to other discrete structures. These include matroids, but also bespoke structures that come with concrete clustering applications [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15]. This has been made possible by re-casting tangle theory in terms of a purely algebraic framework of ‘separation systems’, which encompass the notions of separation from all these various different contexts. Although these separation systems are very general, the central tangle theorems are still valid in this framework.

In this paper we are concerned with only the most basic type of separation systems, those of sets. A separation of a set $V$ is an unordered pair $s = \{A, B\}$ of subsets of $V$ such that $A \cup B = V$. It has two orientations: the ordered pair $(A, B)$, which we think of as pointing towards $B$, and its inverse $(B, A)$. We usually denote the two orientations of a separation $s$ by arrows: one of them is denoted as $\rightarrow s$, the other as $\leftarrow s$, but it does not matter which is which.

Now consider a particular set $S$ of separations of a set $V$, and a subset $X \subseteq V$ that is a ‘cluster’ in the sense that the separations in $S$ cannot divide it evenly: let us assume that for every $\{A, B\} \in S$ more than two thirds, say, of $X$ lies in $A \setminus B$ or in $B \setminus A$. If most of $X$ lies in $B \setminus A$, say, then $X$ orients this separation towards $B$, as $(A, B)$.
Any orientation $\tau$ of (all the separations in) $S$ induced by a cluster $X \subseteq V$ in this way has the following property, which no longer refers to $X$: whenever $\tau$ orients three separations $\{A_i, B_i\} \in S$ ($i = 1, 2, 3$) towards $B_i$, their ‘small sides’ $A_i$ cannot cover $V$, because each contains less than a third of $X$. This is the most basic definition of a tangle of $S$: any orientation of $S$ such that no three small sides cover $V$. Note that the meaning of ‘small’ here is intrinsic to $\tau$: the side of a separation towards which $\tau$ orients it is now called ‘big’, its other side ‘small’. No reference to a ‘cluster’ $X$ is made in this definition of a tangle.

This abstract definition of a tangle has made it possible to investigate clusters in a graph or data set without referring to them directly in the usual concrete way, as sets of vertices or data points. In particular, one can investigate the relative structure of clusters without even having found them in this concrete sense – a sense which, moreover, may be inadequate given the fuzzy nature of many real-world clusters, which often do not enable us to decide easily which data points belong to a cluster and which do not.

However, tangles are not equivalent to clusters but weaker: while every cluster, including fuzzy clusters, gives rise to a tangle, not every tangle is induced by a cluster. Tangles that do not come from clusters can still be interesting; the cluster, including fuzzy clusters, gives rise to a tangle, not every tangle is induced by a cluster.

To be a little more formal, let us say that a set $X \subseteq V$ decides an orientation $\tau$ of $S$ if, for every $(A, B) \in \tau$, there are more elements of $X$ in $B$ than in $A$. It is an open question whether every tangle of a graph is decided by some set of its vertices.

Tangles of more general set separations, however, need not have such decider sets. Let us construct a simple example.

The basic idea of our construction is that we start with $S$ and an ‘orientation’ $\tau = \{ \vec{s} \mid s \in S \}$ of $S$ as just a collection of names: every $\vec{s}$ will eventually become a pair $(A, B)$ of subsets of some set $V$ that form a separation of $V$, but we carefully construct $V$, element by element, after fixing $\tau$ notationally, assigning every $v \in V$ to either $A$ or $B$ for every $\vec{s} = (A, B) \in \tau$. When we are done, we shall verify that $\tau$ is a tangle of $S$ not decided by any subset of $V$.

To implement this formally, we create for every 3-set $\sigma \subseteq \tau$ one element $v_\sigma$ for $V$ so that these $v_\sigma$ are distinct for different $\sigma$, and we let $V$ be the set of all these $v_\sigma$. For each $(A, B) \in \sigma$ we put $v_\sigma$ in $B$, and for each $(A, B) \in \tau \setminus \sigma$ we put $v_\sigma$ in $A$. Then for every $\vec{s} = (A, B) \in \tau$ we have $B = \{ v_\sigma \mid \vec{s} \in \sigma \}$ and $A = V \setminus B$. Now if $|S| = m$, say, then $|V| = \binom{m}{3}$, and for every $(A, B) \in \tau$ we have $|B| = \binom{m-1}{2}$. Let us choose $S$ so that $m \geq 6$.

By construction, $\tau = \{ \vec{s} \mid s \in S \}$ is a tangle of $S$: the big sides of any three $\vec{s} \in \tau$ have an element in common. In fact,

$$\text{every } v \in V \text{ lies on the big side of exactly three elements of } \tau. \quad (*)$$

A simple double count now shows that $\tau$ has no decider set $X \subseteq V$. Indeed, suppose it does and let $$d := \sum_{(A, B) \in \tau} |X \cap B|.$$ Then $|X \cap B| > |X|/2$ for every $(A, B) \in \tau$, because $X$ decides $\tau$, and hence $d > m |X|/2$. On the other hand, by \footnote{It was shown in \cite{13} that graph tangles are decided by sets of vertices with weights assigned to them.}, each $x \in X$ lies in $B$ for exactly...
three \((A, B) \in \tau\), so \(d\) counts it three times: \(d = 3|X|\). Putting these together we obtain \(3|X| > m|X|/2\). This implies \(m < 6\), contrary to our assumption.

Is it possible to distil from this example some property of \(\tau\) that identifies all the tangles without a decider set? We grappled with this question for quite a while, until we found the following solution.

Given an integer \(k\), we say that an orientation \(\tau\) of \(S\) is \(k\)-resilient if it takes more than \(k\) elements of \(\tau\) to obtain \(V\) as the union of their small sides. Every tangle, by definition, is 3-resilient.

Our earlier example of a tangle \(\tau\) without a decider set is not 4-resilient. In fact, given any four distinct separations in \(\tau\), by \(\square\) every \(v \in V\) lies on the small side of at least one of them, so the four small sides have union \(V\). At the other extreme, every principal tangle of a set of bipartitions of a set \(V\), one consisting of all bipartitions \((A, B)\) of \(V\) whose big side \(B\) contains some fixed element \(x \in V\), is infinitely resilient in that it is \(k\)-resilient for every \(k \in \mathbb{N}\). Note that \(\{x\}\) is a decider set for this tangle. In Section \(\square\) we shall see that the unique 5-tangle of the \((n \times n)\)-grid, which is decided by its entire vertex set, is \(\Omega(n^2)\)-resilient.

These examples seem to suggest that tangles of set separations that are \(k\)-resilient for large \(k\) are more likely to have decider sets. We can indeed prove such a fact, with an interesting additional twist: ‘large’ has to be measured not in terms of \(|V|\) or \(|S|\), but relative to the number of maximal elements of the tangle in the usual partial order of oriented separations (see Section \(\square\)). This dependence on the number of maximal elements in a tangle is not even surprising: in a \(k\)-resilient tangle with at most \(k\) maximal elements, the intersection of all their big sides is non-empty, and is clearly a decider set for this tangle.

Let us say that a function \(w: V \to \mathbb{R}_{\geq 0}\), which we may think of as placing weights on the elements of \(V\), decides a separation \(s = \{A, B\}\) of \(V\) if \(s\) has an orientation \(\bar{s} = (A, B)\) such that \(w(A) < w(B)\); we then also say that \(w\) decides \(s\) as \(\bar{s}\). More generally, \(w\) decides a set \(S\) of separations if it decides each of its elements, thus defining an orientation \(\tau\) of \(S\).

Conversely, if \(\tau\) is an orientation of a set \(S\) of separations of \(V\) and \(w: V \to \mathbb{R}_{\geq 0}\) decides \(S\) so as to define \(\tau\), then we say that \(w\) decides \(S\) like \(\tau\) and call \(w\) a decider for \(\tau\). If \(w\) takes values in \(\{0, 1\}\), then clearly \(X = w^{-1}(1)\) is a decider set for \(\tau\) as considered earlier.

For this more general notion of deciders, our earlier observation that \(k\)-resilient tangles with at most \(k\) maximal elements have decider sets has the following strengthening:

**Theorem 1.** A tangle with \(m\) maximal elements has a decider if it is \(k\)-resilient for some \(k > \frac{m}{3}\).

We shall also see that, as a general bound for all tangles, this is best possible.

For our proof of Theorem 1 we introduce the notion of local decidability, which generalises the idea of resilience. We show in Theorem \(\square\) that an orientation \(\tau\) of a set \(S\) of separations has a decider if and only if there exist suitable parameters \(k\) and \(\ell\) such that \(\tau\) is \(k\)-locally \(\ell\)-decidable. In particular, there are such suitable parameters \(k\) and \(\ell\) if \(\tau\) is highly resilient compared with its number of maximal elements, our Theorem 1 above.

For our second characterisation of tangles with deciders, we exploit the recent notion of duality of sets of separations. Dual sets of separations naturally arise
in applications of tangle theory. For example, consider an online shop with a set $V$ of items on sale and a history $P$ of purchases made last year $[3,5]$. In this setting, two different sets of separations occur. Every purchase in $P$ induces a bipartition of $V$ into the items bought versus those not bought. Equally, every item in $V$ defines a bipartition of $P$ into those purchases that included it versus those that did not. The tangles of these two sets of separations can be shown to interact $[6]$, and they will help us to obtain a second characterisation of tangles with deciders, Theorem 8. This will also imply Theorem 1.

Our third contribution in this paper shows the existence of decider sets for some tangles of set separations endowed with an order function, a function that assigns an integer $|s| := |A, B| ≥ 0$ to every separation $s = \{A, B\} \in U$ where $U = U(V)$ is the set of all separations of a set $V$ (see Section 2 for the precise definitions). Given such an order function on $U$ and $k \in \mathbb{N}$, the $k$-tangles in $U$ are the tangles of its subset $S_k := \{s \in U : |s| < k\}$.

Elbracht, Kneip, and Teegen $[13]$ showed that the $k$-tangles in $U$ have deciders, though not necessarily decider sets, when their order is defined as $|A, B| := |A \cap B|$. In Section 5 we strengthen their result for $k$-tangles in $U$ which extend to $2k$-tangles in $U$ in that such $k$-tangles even have decider sets. It would be interesting to know whether similar results hold for other submodular order functions on $U$ than the above. (Submodularity will be defined in Section 2.)

2 Preliminaries

This section collects together the definitions we need in this paper. While we shall work only with separations of sets as considered in the introduction, all definitions given here fit into the more general framework of ‘abstract separation systems’ $[2]$. From this framework we shall borrow some notations which we will also introduce in what follows.

In this paper we consider separations of finite sets $V$, this finiteness assumption on $V$ will not be mentioned explicitly in the remainder of this paper. Throughout we will consider various finite sets $V$; if $V$ is not specified explicitly, then $V$ is any arbitrary finite set.

For definitions around graphs we refer the reader to $[1]$. For every $k \in \mathbb{N}$ we write $[k] := \{1, \ldots, k\}$, we call a set with $k$ elements a $k$-set, and we denote the set of all $k$-element subsets of a set $X$ as $X^{(k)}$.

2.1 Separations of sets

An (unoriented) separation of $V$ is an unordered pair $\{A, B\}$ of subsets $A$ and $B$ of $V$, its sides, such that $A \cup B = V$. The two orientations of $\{A, B\}$ are the ordered pairs $(A, B)$ and $(B, A)$ which is the inverse of $(A, B)$. We write $U(V)$ for the set of all unoriented separations of $V$.

Every ordered pair $(A, B)$ of subsets of $V$ with $A \cup B = V$ is an oriented separation of $V$. Its underlying unoriented separation is $\{A, B\}$, and $(B, A)$ is its inverse. Given an oriented separation $(A, B)$ of $V$, we refer to $A$ as its small side and call $B$ its big side. We shall informally use the term ‘separation’ also

Note that $(A, B)$ and $(B, A)$ coincide if and only if $A = B = V$. In particular, the oriented separation $(V, V)$ equals its own inverse.
as a short term for oriented separations, but we will only do so if the meaning is unambiguous.

As indicated in the introduction, we fix the following notational conventions for separations for better readability: unoriented separations will be denoted as lowercase letters, such as \( s \). Given an unoriented separation \( s \) of a set, we denote its two orientations as \( \tilde{s} \) and \( \breve{s} \). There is no default orientation: once we have called one of the two orientations \( \tilde{s} \), the other one will be \( \breve{s} \), and vice-versa. Oriented separations will be denoted as lowercase letters with a forward or backward arrow on top, such as \( \tilde{s} \) and \( \breve{s} \). Given an oriented separation \( \tilde{s} \) of \( V \), its underlying unoriented separation is denoted as \( s \), and its inverse as \( \breve{s} \).

We define a partial order \( \leq \) on the set of oriented separations of \( V \) as follows: for two oriented separations \( (A, B) \) and \( (C, D) \) of \( V \), we let \( (A, B) \leq (C, D) \) if \( A \subseteq C \) and \( B \supseteq D \); we write \( (A, B) < (C, D) \) if and only if \( (A, B) \leq (C, D) \) and \( (A, B) \neq (C, D) \). With this definition we in particular have

\[
(A, B) \leq (C, D) \iff (B, A) \geq (D, C).
\]

The \textit{maximal elements} of a set \( \sigma \) of oriented separations of \( V \) are always those separations in \( \sigma \) which are maximal with respect to this partial order \( \leq \).

Given two oriented separations \( \tilde{r} = (A, B) \) and \( \tilde{s} = (C, D) \) of \( V \), their supremum \( \tilde{r} \lor \tilde{s} \) with respect to \( \leq \) is the oriented separation \( (A \cup C, B \cap D) \) whereas their infimum \( \tilde{r} \land \tilde{s} \) is \( (A \cap C, B \cup D) \). Note that the supremum and the infimum satisfy DeMorgan’s law in that \( \tilde{r} \lor \tilde{s} \) is the inverse of \( \tilde{r} \land \tilde{s} \) for every two oriented separations \( \tilde{r} \) and \( \tilde{s} \) of \( V \).

A set \( U \) of unoriented separations of \( V \) is a \textit{universe of separations of} \( V \) if the set \( \tilde{U} := \{ \tilde{s}, \breve{s} \mid s \in U \} \) of all orientations of separations in \( U \) is closed under taking suprema and infima, i.e. if \((\tilde{U}, \leq)\) is a lattice. Note that \( U(V) \) is a universe of separations of \( V \) by definition.

An oriented separation \( \tilde{s} = (A, B) \) of \( V \) is \textit{small} if \( B = V \), and \textit{co-small} if \( A = V \); thus, \( \tilde{s} \) is small if and only if its inverse \( \breve{s} \) is co-small. Note that \( \tilde{s} \) is small if and only if \( \tilde{s} \leq \tilde{s} \). If \( A = V = B \) or, equivalently, if \( \tilde{s} = \breve{s} \), then both \( \tilde{s} \) and \( s \) are called \textit{degenerate}; otherwise, \( \tilde{s} \) and \( s \) are non-degenerate. A set \( \sigma \) of oriented non-degenerate separations of \( V \) is a \textit{star} if \( \tilde{r} \leq \tilde{s} \) for every two distinct \( \tilde{r}, \tilde{s} \in \sigma \).

Given a universe \( U \) of separations of \( V \), an \textit{order function} \( | \cdot | \) on \( U \) assigns to each \( s \in U \) its \textit{order} \( |s| \in \mathbb{Z}_{\geq 0} \); the order of an orientation \( \tilde{s} \) of a separation \( s \in U \) is defined to be the order of \( s \). Given \( k \in \mathbb{N} \) we write \( S_k \) for the set of all separations in \( U \) of order less than \( k \).

An order function on \( U \) is \textit{submodular} if, for every two \( r, s \in U \) and arbitrary orientations \( \tilde{r} \) of \( r \) and \( \tilde{s} \) of \( s \), we have \( |\tilde{r} \lor \tilde{s}| + |\tilde{r} \land \tilde{s}| \leq |\tilde{r}| + |\tilde{s}| \). Unless explicitly specified otherwise, we consider the \textit{(standard) order} \( [A, B] \) of a separation \( \{A, B\} \) of \( V \) as the cardinality of its \textit{separator} \( A \cap B \); this order function can easily be seen to be submodular.

There are two special classes of separations which we will consider in this paper: a \textit{bipartition} of \( V \) is a separation of \( V \) whose sides are disjoint. We denote the set of all bipartitions of \( V \) by \( U_{\text{bip}}(V) \); note that \( U_{\text{bip}}(V) \) is again a universe of separations of \( V \). Since all bipartitions have order 0 with respect to our
standard order function, we usually consider other order functions on $U_{\text{up}}(V)$ (see [3] for various examples).

Another example of separations arises in graphs: a separation $\{A, B\}$ of a graph $G = (V, E)$ is a separation of its vertex set $V$ such that $G$ has no edges between $A \setminus B$ and $B \setminus A$. The set of all separations of $G$ then forms a universe of separations of $V$.

### 2.2 Orientations

Let $S$ be a set of unoriented separations of a set $V$. Assigning to every $s \in S$ either $\vec{s}$ or $\bar{s}$ is called orienting $S$ (or the $s \in S$). So an orientation of $S$ is a set $\tau$ of orientations of separations in $S$ satisfying $|\tau \cap \{\vec{s}, \bar{s}\}| = 1$ for every $s \in S$. We say that $\tau$ orients $\{A, B\} \in S$ towards $B$ if $(A, B) \in \tau$. Given an orientation $\tau'$ of a subset $S' \subseteq S$, we say that $\tau'$ extends to an orientation $\tau$ of $S$ if $\tau' \subseteq \tau$.

An orientation $\tau$ of $S$ is consistent if there exist no two $\vec{r}, \bar{s} \in \tau$ with $\vec{r} < \bar{s}$. If $\tau$ is consistent and for every two distinct $\vec{s}, \vec{t} \in \tau$, we have $(\vec{s} \wedge \vec{t}) \notin \tau$, then $\tau$ is a profile of $S$. A profile of $S$ is regular if it does not contain any co-small separation. Given a universe $U$ of separations of $V$, an order function on $U$, and $k \in \mathbb{N}$, we call a profile of the corresponding $S_k$ a $k$-profile in $U$.

Writing $\bar{S} := \{\vec{s}, \bar{s} | s \in S\}$ for the set of all orientations of separations in $S$, let $F \subseteq \bar{S}$. An orientation $\sigma$ of $S$ is an $F$-tangle of $S$ if $\sigma$ is consistent and $\sigma \notin \tau$ for every $\tau \in F$. Let $T$ consist of all sets $\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ of (not necessarily distinct) oriented separations in $\bar{S}$ with $A_1 \cup A_2 \cup A_3 = V$, i.e. the supremum of the $(A_i, B_i)$ is co-small. The $T$-tangles of $S$ are called (abstract) tangles of $S$; they are examples of regular profiles of $S$. Given an order function on a universe $U$ of separations of $V$ and $k \in \mathbb{N}$, we call a tangle of $S_k \subseteq U$ a $k$-tangle in $U$.

In the case that all separations in $S$ are even bipartitions of $V$, we will also consider $\mathcal{F}^\ell$-tangles where $\ell \in \mathbb{R}_{\geq 0}$. Here, $\mathcal{F}^\ell \subseteq \bar{S}$ consists of all sets $\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ of (not necessarily distinct) oriented bipartitions in $\bar{S}$ with $|B_1 \cap B_2 \cap B_3| < \ell$. Note that $T = \mathcal{F}^1$ here since $S$ consists of bipartitions of $V$.

### 2.3 Weight functions and deciders

A weight function on a set $V$ is a map $w$ from $V$ to $\mathbb{R}_{\geq 0}$. For a subset $Z \subseteq V$ we write $w(Z) = \sum_{v \in Z} w(v)$. If there exists $v \in V$ with $w(v) > 0$, then $w$ is non-zero. If $w$ takes values in $\{0, 1\}$ only, then it can equivalently be formulated as an indicator function of the set $X = X_w = w^{-1}(1)$ in that $w(Z) = |X \cap Z|$ for every $Z \subseteq V$; we shall use this equivalence freely throughout.

For any weight function $w$ on $V$ and any separation $\{A, B\}$ of $V$, we have $w(B) - w(A) = w(B \setminus A) - w(A \setminus B)$, a fact we shall also use freely throughout. We say that $w$ decides a separation $\{A, B\}$ of $V$ if $w(A) \neq w(B)$. If $w(A) < w(B)$, then $w$ decides $\{A, B\}$ as $(A, B)$. More generally, $w$ decides a set $S$ of separations of $V$ if it decides each of its elements, thus defining the orientation $\tau = \{(A, B) | \{A, B\} \in S, w(A) < w(B)\}$ of $S$.

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[3] We slightly deviate here from [3] in that our $\mathcal{F}^\ell$ correspond to their $\mathcal{F}^\ell_3$ and in that we consider $\ell \in \mathbb{R}_{\geq 0}$ instead of $\ell \in \mathbb{N}$. 
Conversely, let $S$ be a set of separations of a set $V$, and let $\tau$ be an orientation of $S$. If a weight function $w$ on $V$ decides $S$ defining $\tau$, then we say that $w$ decides $S$ like $\tau$, that it witnesses $\tau$ and all its elements, and call $w$ a decision for $\tau$. If $w$ takes values in $\{0,1\}$, we call $X = w^{-1}(1)$ a decider set for $\tau$. If there exists a decider (set) for $\tau$, then we say that $\tau$ has a decider (set).

Let us note some basic observations about deciders. First, let $w$ be a weight function on $V$ and let $\lambda > 0$ be a scalar. If we scale $w$ by $\lambda$, i.e. if we consider the weight function $v \mapsto \lambda w(v)$ on $V$, then this scaled weight function agrees with $w$ on the sign of $w(B) - w(A)$ for any separation $\{A,B\}$ of $V$. In particular, if an orientation $\tau$ of a set $S$ of separations of $V$ has a decider $w$, then for any given $K > 0$ there exists a decider $w_K$ for $\tau$ with $w_K(B) - w_K(A) \geq K$ for all $(A,B) \in \tau$. This is because $w_K$ can be chosen as an appropriate scaling of $w$, i.e. by a factor $\lambda \geq K/(\min_{(A,B) \in \tau}(w(B) - w(A)))$.

This fact directly implies that if an orientation $\tau$ of a set $S$ of separations of $V$ has a decider $w$, then there also exists a decider for $\tau$ which takes values in $\mathbb{Z}_{\geq 0}$ instead of $\mathbb{R}_{\geq 0}$. Indeed, there exists $\varepsilon > 0$ such that $w(B) - w(A) \geq \varepsilon$ for all $(A,B) \in \tau$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can replace for every $v \in V$ the scalar $w(v) \in \mathbb{R}_{\geq 0}$ with a rational number $w'(v) \in \mathbb{Q}_{\geq 0}$ which satisfies $|w(v) - w'(v)| < \varepsilon/|V|$. By construction, the resulting weight function $w'$ on $V$ still witnesses $\tau$. Now we can scale $w'$ by an appropriate $\lambda \in \mathbb{N}$ to obtain the desired decider for $\tau$ which takes values in $\mathbb{Z}_{\geq 0}$.

For the final observation in this section, let $S$ be a set of separations of a set $V$, and let $\tau$ be an orientation of $S$. Then a weight function on $V$ witnesses $\tau$ as soon as it witnesses the maximal elements of $\tau$. We include a proof of this observation from [15] for the reader’s convenience.

**Lemma 2.** Let $w$ be a weight function on a set $V$, and let $(A,B)$ and $(C,D)$ be separations of $V$ with $(A,B) \leq (C,D)$. Then $w(B) - w(A) \geq w(D) - w(C)$. In particular, $w$ witnesses $(A,B)$ if it witnesses $(C,D)$.

*Proof.* Since $(A,B) \leq (C,D)$, we have $A \subseteq C$ and $B \supseteq D$. So $w(A) \leq w(C)$ and $w(B) \geq w(D)$ as $w$ is a weight function on $V$. This directly implies that $w(B) - w(A) \geq w(D) - w(C)$. The ‘in particular’-part then follows immediately. 

### 3 Deciders and resilience

In this section we use the novel notion of resilience to prove a sufficient criterion for an orientation of a set $S$ of separations to have a decider. After that, we further generalise the concept of resilience towards the notion of $k$-local $\ell$-decidability which allows us to give a characterisation of those orientations of $S$ which have deciders. We begin this section by giving all the definitions around the concept of resilience.

Let $S$ be a set of separations of a set $V$, and let $k \in \mathbb{N}$. An orientation $\tau$ of $S$ is $k$-resilient if no set of at most $k$ separations in $\tau$ has a co-small supremum. So $\tau$ is $k$-resilient if and only if for all sets $\sigma \subseteq \tau$ of size at most $k$, we have that $\bigcup_{(A,B) \in \sigma} A \neq V$. If $S$ consists only of bipartitions of $V$, then this is equivalent to $\bigcap_{(A,B) \in \sigma} B \neq \emptyset$ for all sets $\sigma \subseteq \tau$ of size at most $k$ since $(V,\emptyset)$ is
the only co-small bipartition of $V$. Note that in order to determine whether $\tau$ is $k$-resilient, it is always enough to consider sets $\sigma$ of maximal elements of $\tau$.

If $\tau$ is $k$-resilient, then $\tau$ is also $k'$-resilient for every $k' < k$. We call $\tau$ infinitely resilient if $\tau$ is $k$-resilient for all $k \in \mathbb{N}$. The resilience of $\tau$ is the maximal $k \in \mathbb{N}$ such that $\tau$ is $k$-resilient if such $k$ exists, 0 if $\tau$ is not $k$-resilient for any $k \in \mathbb{N}$, and $\infty$ otherwise.

In addition to the examples on resilience given in the introduction, let us here illustrate the concept once more with a less extreme example. Consider the $(n \times n)$-grid for some $n \geq 5$, and let $S$ be the set of all separations of this graph which have order at most 4. It is easy to see that the orientation $\tau$ of $S$ which is defined by the entire vertex set of the grid is a tangle. Let us show that $\tau$ is $\Omega(n^2)$-resilient. Every element $(A, B)$ of $\tau$ satisfies $|A| \leq 10$; indeed, most satisfy $|A| \leq 5$. Thus, any set of separations in $\tau$ with a co-small supremum has at least $n^2/10$ elements as the $(n \times n)$-grid has precisely $n^2$ vertices.

Why can the notion of resilience help us with constructing a decider for a given orientation? Consider an orientation $\tau$ of a set $S$ of separations of a set $V$. Write $\mu = \mu(\tau)$ for the set of maximal elements of $\tau$. Let us see how it helps us to build a decider for $\tau$ if the resilience of $\tau$ is large compared with $|\mu|$.

Assume that $\tau$ is $k$-resilient for some $k \in \mathbb{N}$, and recall that $\mu^{(k)}$ denotes the set of $k$-element subsets of $\mu$. Then the resilience of $\tau$ implies that for every $\mu' \in \mu^{(k)}$, there exists some $v_{\mu'} \in V$ which is not contained in the small side of any separation in $\mu'$; in particular, $\{v_{\mu'}\}$ is a decider set for $\mu'$. It seems natural to construct a decider for $\mu$ (and thus for $\tau$) by combining all these local decider sets $\{v_{\mu'}\}$, i.e. by assigning to each $v \in V$ as its weight the number of sets $\mu' \in \mu^{(k)}$ with $v_{\mu'} = v$.

It turns out that the weight function $w$ on $V$ defined in this way need not in general be a decider for $\mu$. This is because each $v_{\mu'}$, while adding its weight to the big sides of the separations in $\mu'$, can also add weight to the small sides of separations in $\mu \prec \mu'$. But as soon as $k$ is large enough in that each fixed separation $(A, B) \in \mu$ is contained in the majority of the sets in $\mu^{(k)}$, which will happen as soon as $\mu$ has more $(k-1)$-subsets to form a $k$-subset with $(A, B)$ than it has $k$-subsets not including $(A, B)$, the orientation $(A, B)$ of $(A, B)$ will be witnessed by the majority of the local decider sets $\{v_{\mu'}\}$ for $\mu' \in \mu^{(k)}$. We can then deduce from this that $w$ is a decider for $\mu$, and hence for $\tau$ by Lemma 2.

More precisely, we have the following generalisation of Theorem 1 to arbitrary orientations $\tau$:

**Theorem 3.** Let $S$ be a set of separations of a set $V$, and let $\tau$ be an orientation of $S$. Let $m$ be the number of maximal elements of $\tau$. If $\tau$ is $k$-resilient for some integer $k > \frac{m}{2}$, then $\tau$ has a decider.

We will formally obtain Theorem 3 and hence Theorem 1 below as a corollary of the more general Theorem 4. But before we do so, let us first show that both Theorem 1 and Theorem 3 are in some sense optimal as witnessed by a tangle without decider; we can find such a tangle by using a more general version of the construction from the introduction.

**Proposition 4.** For all $m, k \in \mathbb{N}$ with $3 \leq k \leq \frac{m}{2}$, there exist a set $V$, a submodular order function $\cdot \mid \cdot$ on $\mathcal{U}_{bip}(V)$ and an $m$-tangle $\tau_{m,k}$ in $\mathcal{U}_{bip}(V)$ that has $m$ maximal elements and is $k$-resilient, but does not have a decider.
An example of the tangles in Proposition 4 is given by a certain type of hypergraph edge tangles introduced in [13]. We now describe their construction, and then show that these tangles do indeed have all the desired properties.

Proof of Proposition 4. Let $V = [m]^{(k)}$ consist of all $k$-element subsets of $[m]$. For every $i \in [m]$ let $V_i = \{X \in V \mid i \in X\}$ be the set of all $k$-element subsets of $[m]$ containing $i$. We assign to each bipartition $\{A, B\}$ of $V$ as its order $|A, B|$ the number of sets $V_i$ meeting both $A$ and $B$. This order function $|\cdot|$ on $U_{\text{big}}(V)$ is easily seen to be submodular (see [13] for a formal proof).

Every bipartition $\{A, B\}$ of $V$ of order less than $m$ has precisely one side which contains $V_i$ for some $i \in [m]$. Indeed, one such side exists by the definition of the order function $|\cdot|$, and since $V_i \cap V_j \neq \emptyset$ for distinct $i, j \in [m]$, this side is unique. Hence,

$$\tau_{m,k} := \{(A, B) \mid \{A, B\} \in S_m \text{ and } \exists i \in [m]: V_i \subseteq B\}$$

is a well-defined orientation of $S_m \subseteq U_{\text{big}}(V)$. The maximal elements of $\tau_{m,k}$ are precisely the $s'_i = (V \setminus V_i, V_i)$ for $i \in [m]$. So in order to see that $\tau_{m,k}$ is indeed an $m$-tangle in $U_{\text{big}}(V)$, it is enough to observe that $s'_i \lor s'_j \lor s'_l$ is not co-small for any $1 \leq i, j, l \leq m$. But since $k \geq 3$, we always have $V_i \cap V_j \cap V_l \neq \emptyset$.

As shown in [13], it is immediate from double counting (as in the example from the introduction) that $\tau_{m,k}$ does not have a decider for $m \geq 6$ and $k \leq \frac{4}{3}$. So it remains to check that $\tau_{m,k}$ is $k$-resilient. But this is immediate from the construction: for every collection $\bar{s}_i_1, \ldots, \bar{s}_i_k$ of $k$ distinct maximal elements of $\tau_{m,k}$, we have $\{i_1, \ldots, i_k\} \cap \bigcap_{j \in [k]} V_{i_j}$. Hence for every collection of at most $k$ maximal elements of $\tau_{m,k}$, the intersection of their big sides is non-empty.

The $\tau_{m,k}$ constructed in our proof of Proposition 4 are abstract tangles, but the construction can easily be modified to find $F^\ell$-tangles for arbitrary $\ell > 1$ with the same properties: instead of taking the $k$-element subsets of $[m]$ as the set $V$, we can take $V$ as the disjoint union of $[\ell]$-element sets, one for every $k$-element subset of $[m]$.

Before we proceed towards a proof of Theorem 5, our generalisation of Theorem 3, let us briefly investigate the above examples of $F^\ell$-tangles without deciders in some more detail. Note that our construction of $F^\ell$-tangles does not necessarily work when the value of $\ell$ is not constant, but large in terms of $|V|$, e.g. of size at least $\varepsilon |V|$ for some constant $\varepsilon > 0$. The following proposition shows that there exists a sharp lower bound for those $\varepsilon > 0$ for which $\ell \geq \varepsilon |V|$ guarantees the existence of decider.

Proposition 5. Let $V$ be an $n$-set, and let $0 < \varepsilon < 1$. If $\varepsilon \geq 1/8$, then every $F^\varepsilon$-tangle $\tau$ of a set $S$ of bipartitions of $V$ has a decider; if $\varepsilon > 1/8$, then $\tau$ even has a decider set.

Conversely, for every $\varepsilon < 1/8$ there exist $n \in \mathbb{N}$ and a set $S$ of bipartitions of an $n$-set such that some $F^\varepsilon$-tangle of $S$ has no decider.

Proof. Let $\tau$ be an $F^\ell$-tangle of a set $S$ of bipartitions of a set $V$ with $\ell \geq |V|/8$. If all of $V$ is a decider set for $\tau$, then we are done; so suppose not. Then there exists $(A_1, B_1) \in \tau$ with $|B_1| \leq |V|/2$. Again we are done if $B_1$ is a decider set for $\tau$. If this is not the case, then there is some $(A_2, B_2) \in \tau$ with $|B_1 \cap A_2| \geq |B_1 \cap B_2|$; in particular, we have $|B_1 \cap B_2| \leq |V|/4$. 9
It turns out that if $\ell > |V|/8$, then $B_1 \cap B_2$ needs to be the desired decider set since otherwise, there exists some $(A_3, B_3) \in \tau$ such that
\[|(B_1 \cap B_2) \cap B_3| \leq |(B_1 \cap B_2) \cap A_3| \leq |V|/8\]
This implies $|(B_1 \cap B_2) \cap B_3| \leq |V|/8$ which contradicts the fact that $\tau$ is an $\F^\ell$-tangle.

If $\ell = |V|/8$, then the same arguments as above result in a decider set if at least one of the occurring inequalities is strict. So suppose that all the above inequalities are satisfied with equality. In particular, every $(A, B) \in \tau$ satisfies $|A| \leq |B|$.

With a similar reasoning as above, we can at least obtain a decider: note first that it is enough to find a weight function $w$ on $V$ which witnesses the set $\tau' \subseteq \tau$ consisting of all $(A, B) \in \tau$ with $|A| = |B|$. Indeed, given a decider $w$ for $\tau'$, we obtain a decider for $\tau$ by adding large enough constant weight to all vertices in $V$.

Suppose that there are $(A, B), (C, D) \in \tau'$ with $|B \cap C| > |B \cap D|$. Then this yields $|B \cap D| < |V|/4$ which in turn implies the existence of a decider set with the same arguments as above. Consequently, for every $(A, B), (C, D) \in \tau'$, we have $|B \cap C| \leq |B \cap D|$.

Hence, the weight function $w$ defined by counting for every $v \in V$ the number of $(A, B) \in \tau'$ with $v \in B$ is a decider for $\tau'$: given $(C, D) \in \tau'$, we have that $w(C) = \sum_{(A, B) \in \tau'} |B \cap C|$ and $w(D) = \sum_{(A, B) \in \tau'} |B \cap D|$. As above, we have $|B \cap C| \leq |B \cap D|$ for every $(A, B) \in \tau'$, and as $\tau$ is an $\F^\ell$-tangle, we clearly have $D \neq \emptyset$ and hence $|D \cap C| < |D \cap D|$. This yields $w(C) < w(D)$, and thus, $w$ is a decider for $\tau$.

For the second part of the proposition, consider $V$ and the $m$-tangle $\tau_{m,k}$ in $U_{\text{bip}}(V)$ as constructed in our proof of Proposition 4 for some $m \geq 2k \geq 6$. Then for any three maximal elements of $\tau_{m,k}$, their intersection contains exactly $\binom{m-3}{k-3}$ elements of $V$. In particular, $\tau_{m,k}$ is an $\F^\ell$-tangle for all $\ell < \binom{k-3}{m-3}$.

Recall that $|V| = \binom{m}{k}$. So for $k = m/2$, we have $\lim_{m \to \infty} \frac{\binom{m-3}{k-3}}{\binom{m}{k}} = 1/8$.

Thus, we find for any rational $\varepsilon < 1/8$ some $n = \binom{m}{k} \in \mathbb{N}$ such that the tangle $\tau_{m,k}$ from Proposition 4 is an $\F^n$-tangle of a set of bipartitions of an $n$-set which has no decider.

Back to Theorem 3, recall that this theorem is sharp in terms of the parameter $k$ in $k$-resilience as shown by Proposition 4. However, the converse of Theorem 3 fails, i.e. not even every tangle with a decider set has high resilience compared to the number of its maximal elements.

Example 6. Let $V$ be a set of size $n \geq 4$, and let $S$ be the set of all bipartitions of $V$ that have a side of size less than $n/3$. Let $\tau$ be the orientation of $S$ which orients every $s \in S$ towards the side which contains more elements. In particular, $V$ is a decider set for $\tau$.

This orientation $\tau$ of $S$ is a tangle, since no three big sides of bipartitions in $\tau$ have empty intersection. However, four big sides can, so the supremum of four bipartitions in $\tau$ can be co-small. Thus, $\tau$ has resilience 3.

Now $\tau$ has $m = \binom{n}{\lfloor (n/3)-1 \rfloor}$ maximal elements, namely those bipartitions of $V$ whose small side has maximum size. In particular, the resilience of $\tau$ is low compared with $m$, although $\tau$ has a decider and even a decider set.
It turns out that we can generalise the notion of resilience in a way which includes the tangle from the previous Example 5 without invalidating Theorem 3. In fact, our more general notion leads to a more general result, Theorem 7, which actually characterises the orientations with deciders.

Let $\tau$ be an orientation of a set of separations of a set $V$. Our more general notion of resilience is based on our earlier observation that if $\tau$ is $k$-resilient, then the $k$-resilience provides a local one-element decider set $\{\mu, \mu'\}$ for every $\mu \in \mu^{(k)}$ where $\mu = \mu(\tau)$ is the set of maximal elements of $\tau$. In the following definition we ask, instead of $k$-resilience, that for every $\mu' \in \mu^{(k)}$ there exists a “local” decider $w_{\mu'}$ which decides $\mu'$ correctly and simultaneously is not too badly wrong on the separations in $\mu \setminus \mu'$.

More precisely, an orientation $\tau$ of a set $S$ of separations of a set $V$ is $k$-locally $\ell$-decidable for given $k \in \mathbb{N}$ and $\ell \geq 0$ if for every set $\mu' \subseteq \tau$ of size $|\mu'| \leq k$, there is a weight function $w_{\mu'}$ on $V$ satisfying

(i) $\forall (A, B) \in \mu' : w_{\mu'}(B) - w_{\mu'}(A) \geq 1$;
(ii) $\forall (A, B) \in \tau : w_{\mu'}(A) - w_{\mu'}(B) \leq \ell$.

Observe that by Lemma 2 one needs only consider sets $\mu' \subseteq \mu$ in the above definition where $\mu = \mu(\tau)$ is the set of maximal elements of $\tau$. In addition, we can equivalently strengthen the condition of $|\mu'| \leq k$ to $|\mu'| = k$ (i.e. $\mu' \in \mu^{(k)}$), as long as $\tau$ has at least $k$ maximal elements.

The above definition is indeed a generalisation of our earlier notion of resilience since a $k$-resilient orientation $\tau$ of $S$ is $k$-locally 1-decidable: for $\mu' \subseteq \tau$ with $|\mu'| \leq k$, take $w_{\mu'}$ assigning 1 to a single element in $V \setminus \bigcup_{(A, B) \in \mu'} A$, which is non-empty since $\tau$ is $k$-resilient, and 0 to all other elements in $V$.

If $\tau$ has a decider, then $\tau$ is $k$-locally $\ell$-decidable for all $k \in \mathbb{N}$ and $\ell \geq 0$. Indeed, as described in Section 2,3 there exists a decider for $\tau$ which satisfies (ii) for all separations in $\tau$, and any decider for $\tau$ clearly satisfies (i) in particular, the tangle in Example 3 is $k$-locally 1-decidable for all $k \in \mathbb{N}$ and $\ell \geq 0$.

Here, then, is our generalisation of Theorem 3.

**Theorem 7.** Let $\tau$ be an orientation of a set $S$ of separations of a set $V$, and suppose that $\tau$ has $m$ maximal elements. Then $\tau$ has a decider if and only if it is $k$-locally $\ell$-decidable for some $k \in \mathbb{N}$ and $\ell > 0$ with $k > \frac{m}{1+1/k}$.

Since every $k$-resilient orientation $\tau$ is $k$-locally 1-decidable, Theorem 3, and hence Theorem 1 as well, are direct corollaries of Theorem 7.

**Proof of Theorem 7.** We have seen above that if $\tau$ has a decider $w$, then it is $k$-locally $\ell$-decidable for every $k \in \mathbb{N}$ and $\ell \geq 0$. In particular, $\tau$ is $m$-locally $\ell$-decidable for every $\ell > 0$, and we have $m > \frac{m}{1+1/m}$ in this case.

For the converse, recall that by Lemma 2 it is enough to show that the set $\mu = \mu(\tau)$ of maximal elements of $\tau$ has a decider. If $k \geq m$ the statement is true immediately; so suppose for the following that $k < m$. We construct a decider $w$ for $\mu$ as follows: for every $\mu' \in \mu^{(k)}$ let $w_{\mu'}$ be a weight function on $V$ as in the definition of $k$-local $\ell$-decidability. Then we combine all these $w_{\mu'}$ to define the weight function $w$ on $V$ as

$$w : V \to \mathbb{R}_{\geq 0}, w(v) = \sum_{\mu' \in \mu^{(k)}} w_{\mu'}(v).$$

We show that $w$ is the desired decider for $\mu$. 

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For an arbitrary separation \( \vec{s} = (A, B) \in \mu \), let \( \mu_{\vec{s}}^{(k)} \) consist of all \( \mu' \in \mu^{(k)} \) containing \( \vec{s} \). Then \( \mu_{\vec{s}}^{(k)} \) has size \( \binom{m-1}{k-1} \) as \( \vec{s} \) is contained in \( \binom{m-1}{k-1} \) sets \( \mu' \in \mu^{(k)} \). Similarly, \( \mu_{\vec{s}}^{(k)} \setminus \mu_{\vec{s}}^{(k')} \) has size \( \binom{m-1}{k} \). Therefore, \( \mu \) in the definition of \( k \)-local \( \ell \)-decidability yields

\[
\sum_{\mu' \in \mu^{(k)}} w_{\mu'}(A) \leq \sum_{\mu' \in \mu^{(k)}} (w_{\mu'}(B) - 1) = \sum_{\mu' \in \mu^{(k)}} w_{\mu'}(B) - \binom{m-1}{k-1}.
\]

Similarly, we obtain by \( \mu \) that

\[
\sum_{\mu' \in \mu^{(k)} \setminus \mu_{\vec{s}}^{(k)}} (w_{\mu'}(B) + \ell) = \sum_{\mu' \in \mu^{(k)}} w_{\mu'}(B) + \ell \cdot \binom{m-1}{k}.
\]

These inequalities combine to

\[
w(A) = \sum_{\mu' \in \mu^{(k)}} w_{\mu'}(A) = \sum_{\mu' \in \mu^{(k)}} w_{\mu'}(A) + \sum_{\mu' \in \mu^{(k)} \setminus \mu_{\vec{s}}^{(k)}} w_{\mu'}(A) \leq \sum_{\mu' \in \mu^{(k)}} w_{\mu'}(B) - \binom{m-1}{k-1} + \sum_{\mu' \in \mu^{(k)} \setminus \mu_{\vec{s}}^{(k)}} w_{\mu'}(B) + \ell \cdot \binom{m-1}{k} = w(B) - \binom{m-1}{k-1} + \ell \cdot \binom{m-1}{k}.\]

Now since \( k > \frac{m}{1+1/\ell} \) and \( k < m \), we have that \( \ell < \frac{k}{m-k} \) and thus

\[
\ell \cdot \binom{m-1}{k} < \binom{m-1}{k-1},
\]

as \( \binom{m-1}{k} = \frac{(m-1)!}{k!(m-1-k)!} \) and \( \binom{m-1}{k-1} = \frac{(m-1)!}{(k-1)!(m-k)!} \) differ precisely by the factor \( \frac{k}{m-k} \). This then implies \( w(A) < w(B) \), so \( w \) witnesses \( \vec{s} = (A, B) \). Thus, since \( \vec{s} \in \mu \) was arbitrarily chosen, \( w \) decides \( \mu \) and hence \( \tau \) by Lemma \( \box{2} \) \( \square \)

4 Deciders and duality

In this section we present a second characterisation of the orientations that have a decider: one in terms of a duality between sets of separations. As an unexpected corollary, we obtain an independent second proof of Theorem \( \box{3} \)

The duality of sets of separations, which was introduced in \( \box{3} \) and first studied in \( \box{4} \), is defined as follows. Let \( S \) be a set of separations of a set \( V \), and let \( \sigma \) be an orientation of \( S \). For every \( v \in V \), the sets

\[
C_{\sigma}(v) := \{ (A, B) \in S \mid (A, B) \in \sigma, v \in A \}
\]

and \( D_{\sigma}(v) := \{ (A, B) \in S \mid (A, B) \in \sigma, v \in B \} \)

form the sides of the separation \( \{C_{\sigma}(v), D_{\sigma}(v)\} \) of \( S \). The map \( \varphi_{\sigma} : V \rightarrow U(S) \) then associates with \( v \in V \) the separation \( \varphi_{\sigma}(v) := \{C_{\sigma}(v), D_{\sigma}(v)\} \) of \( S \). We write \( V_{\sigma} := \varphi_{\sigma}(V) \) for the dual set of separations of \( S \) with respect to \( \sigma \).
Suppose now that \( \varphi_\sigma \) is injective.\(^4\) Then the dual set \( V_\sigma \) of separations of \( S \) with respect to \( \sigma \) has a natural default orientation \( \tau_\sigma \) which orients every \( \varphi_\sigma(v) = \{ C_\sigma(v), D_\sigma(v) \} \in V_\sigma \) as \( (C_\sigma(v), D_\sigma(v)) \), i.e., towards the side containing those \( \{ A, B \} \in S \) with \( (A, B) \in \sigma \) and \( v \in B \). It is easy to see that the dual set of separations of \( V_\sigma \) with respect to \( \tau_\sigma \) is again \( S \) with default orientation \( \sigma \). In simple terms, ‘dualising the dual yields the primal’ \(^5\).

So if \( \varphi_\sigma \) is injective, then we can ask in the context of deciders whether the existence of a decider for \( \sigma \) relates to any property of the natural default orientation \( \tau_\sigma \) of the dual set \( V_\sigma \) of separations of \( S \) with respect to \( \sigma \). It turns out that it does, and it does so in an intriguing way: \( \sigma \) has a decider if and only if every non-zero weight function \( w' \) on \( S \) decides some separation in \( V_\sigma \) like \( \tau_\sigma \).

More generally, we have the following theorem:

**Theorem 8.** Let \( S \) be a set of separations of a set \( V \), and let \( \sigma \) be an orientation of \( S \). Then the following two assertions are equivalent:

(i) There exists a decider for \( \sigma \).

(ii) For every non-zero weight function \( w' \) on \( S \), there exists \( v \in V \) with \( w'(C_\sigma(v)) < w'(D_\sigma(v)) \).

In particular, if \( \varphi_\sigma \) is injective and \( \tau_\sigma \) is the default orientation of the dual set \( V_\sigma \) of separations of \( S \) with respect to \( \sigma \), then \( \sigma \) has a decider if and only if every non-zero weight function \( w' \) on \( S \) decides some separation in \( V_\sigma \) like \( \tau_\sigma \).

The proof of Theorem \(^8\) will be done in terms of pure linear algebra and can be followed without any further knowledge about the duality of sets of separations. The key tool in our proof will be the following variant of Farkas’ Lemma (see e.g. \(^16\) 6. Theorem \(^8\)).

**Lemma 9** (Farkas’ Lemma). Let \( Q \in \mathbb{R}^{n \times \ell} \) and \( b \in \mathbb{R}^\ell \). Then exactly one of the following two assertions holds:

(i) There exists \( x \in \mathbb{R}^n_{\geq 0} \) with \( Q^T x \geq b \).

(ii) There exists \( y \in \mathbb{R}^\ell_{\geq 0} \) with \( Qy \leq 0 \) and \( b^T y > 0 \).

**Proof of Theorem \(^8\)** Fix enumerations \( V = \{ v_1, \ldots, v_n \} \) and \( S = \{ s_1, \ldots, s_\ell \} \), and let \( \tau_j^\ell = (A_j, B_j) \in \sigma \) for \( j \in [\ell] \). Using these enumerations, we shall, for the course of this proof, identify a weight function \( w \) on \( V \) with a vector \( x = x(w) \) in \( \mathbb{R}^n_{\geq 0} \) and a weight function \( w' \) on \( S \) with a vector \( y = y(w') \in \mathbb{R}^\ell_{\geq 0} \).

Let us define a matrix \( Q = Q(\sigma) \in \mathbb{R}^{n \times \ell} \) via

\[
Q_{ij} = \begin{cases} 
1, & v_i \in B_j \setminus A_j; \\
0, & v_i \in A_j \cap B_j; \\
-1, & v_i \in A_j \setminus B_j. 
\end{cases}
\]

\(^4\)The map \( \varphi_\sigma \) is injective if and only if for every two distinct elements \( v, v' \in V \), there exist two separations in \( S \) for one of which \( v \) and \( v' \) are contained in the same side and for the other in different sides. If this is not the case for \( v, v' \in V \), then \( v \) and \( v' \) ‘carry the same information’ about how the separations in \( S \) separate \( V \) and can hence be seen as redundant.

\(^5\)Our version of Farkas’ Lemma follows from \(^16\) 6. Theorem by applying their theorem to \( A = Q^T \) and \(-b\) instead of \( b \).
Recall from the definition of $\varphi_\sigma$ that $C_i := C_\sigma(v_i)$ (respectively $D_i := D_\sigma(v_i)$) consists of all those $\{A_j, B_j\} \in S$ with $v_i \in A_j$ (respectively $v_i \in B_j$). So given a weight function $w'$ on $S$, we obtain for $y = y(w') \in \mathbb{R}_\geq 0^n$ and all $i \in [n]$ that

$$(Qy)_i = \sum_{B_j \ni v_i} y_j - \sum_{A_j \ni v_i} y_j = w'(D_i) - w'(C_i).$$

So a non-zero weight function $w'$ on $S$ is not as in Theorem 8 if and only if $Qy \leq 0$ for $y = y(w') \in \mathbb{R}_\geq 0^n$ where $\leq$ is meant coordinate-wise.

Similarly, let $w$ be a weight function on $V$ and $x = x(w) \in \mathbb{R}_\geq 0^n$. Then we compute for all $j \in [\ell]$ that

$$(Q^T x)_j = \sum_{v_i \in B_j} x_i - \sum_{v_i \in A_j} x_i = w(B_j) - w(A_j).$$

So a weight function $w$ on $V$ is a decider for $\sigma$ if and only if $Q^T x > 0$ for $x = x(w) \in \mathbb{R}_\geq 0^n$. Recall from Section 2.3 that $\sigma$ has a decider if and only if it has a decider $w$ with $w(B) - w(A) \geq 1$ for all $(A, B) \in \sigma$. Hence, there exists a decider for $\sigma$ (as in Theorem 8) if and only if $Q^T x \geq 1$ where $x = x(w) \in \mathbb{R}_\geq 0^n$ for some non-zero weight function $w$ on $V$ and $1$ is the constant 1 vector in $\mathbb{R}^\ell$.

The result then follows by applying Lemma 9 to $Q$ and $b = 1 \in \mathbb{R}^\ell$, and denoting $x = x(w)$ in Lemma 10 and $y = y(w')$ in Lemma 11. The ‘in particular’-part is immediate from the definition of $V_\sigma$ and $\tau_\sigma$.

As mapping a separation in $S$ to its orientation in $\sigma$ is a bijection between $S$ and $\sigma$, we could equivalently define the weight function $w'$ in Theorem 8 on the orientation $\sigma$ of $S$; for notational simplicity we will freely switch between these two definitions in what follows.

By Lemma 2 an orientation $\sigma$ of $S$ has a decider if and only if its set of maximal elements has a decider. So applying Theorem 8 to the set $\mu = \mu(\sigma)$ of maximal elements of $\sigma$ and its underlying set of unoriented separations, we have the following corollary.

**Corollary 10.** Let $S$ be a set of separations of a set $V$. Let $\sigma$ be an orientation of $S$, and let $\mu = \mu(\sigma)$ be the set of maximal elements of $\sigma$. Then the following two assertions are equivalent:

(i) There exists a decider for $\sigma$.

(ii) For every non-zero weight function $w'$ on $\mu$, there exists $v \in V$ with

$$\sum_{(A, B) \in \mu \text{ with } v \in A} w'((A, B)) < \sum_{(A, B) \in \mu \text{ with } v \in B} w'((A, B)).$$

As an illustration of the power of Corollary 10 and hence Theorem 8 let us reprove Theorem 4 about the existence of deciders for highly resilient orientations.

**Proposition 11.** Let $S$ be a set of separations of a set $V$, and let $\sigma$ be an orientation of $S$ with $m$ maximal elements. If $\sigma$ is $k$-resilient for some integer $k > \frac{m}{2}$, then $\sigma$ has a decider.
Proof. Let \( \mu = \mu(\sigma) \) be the set of maximal elements of \( \sigma \). We apply Corollary 10 to \( \sigma \) in that we consider an arbitrary non-zero weight function \( w' \) on \( \mu \) and show that case (a) in Corollary 10 holds. Let \( \mu' \subseteq \mu \) consist of those \( k \) separations in \( \mu \) which have the highest weight with respect to \( w' \). Since \( k > \frac{m}{2} \), this immediately yields \( w'(\mu') > w'(\mu \setminus \mu') \).

Now \( \sigma \) is \( k \)-resilient, so there exists some \( v \in V \) which is not contained in the small side of any separation in \( \mu' \). By the choice of \( \mu' \), this \( v \) satisfies

\[
\sum_{(A,B) \in \mu \setminus \{v\}} w'((A,B)) \geq w'(\mu') > w'(\mu \setminus \mu') \geq \sum_{(A,B) \in \mu \setminus \{v\}} w'((A,B)).
\]

Thus, Corollary 10 holds for \( w' \), and since \( w' \) was arbitrarily chosen, this implies that \( \sigma \) has a decider. \( \square \)

5 Deciders for extendable tangles

Let \( S \) be a set of separations of a set \( V \), and let \( \tau \) be an orientation of \( S \). In Section 3 we analysed different properties of \( \tau \) which ensure the existence of a decider for \( \tau \). All these properties required us to consider large subsets of \( \tau \) instead of the usual triples which are required for the definition of a tangle of \( S \). In particular, all the notions considered above may be viewed as strengthenings of the triple condition in the definition of a tangle, i.e., we give a stronger condition that an orientation needs to satisfy in order to be a tangle with a decider.

But how can we guarantee the existence of a decider for a tangle \( \tau \) of \( S \) if we do not want to strengthen the definition of a tangle in the above sense? We know that there exist tangles without deciders (see e.g. Proposition 4). So instead of looking for a decider for \( \tau \) itself, we may try to find a decider for some \( \tau' \subseteq \tau \). Ideally, we can do this in such a way that the decider for \( \tau' \) is still, in some sense, related to the original tangle \( \tau \).

Given an order function on a universe \( U \) of separations of a set \( V \), one natural such subset of a \( k \)-tangle \( \tau \) in \( U \), say, consists of all separations in \( \tau \) of order less than some \( k' < k \). In other words, we would like to obtain, given a \( k \)-tangle \( \tau \) in \( U \), a decider for the \( k' \)-tangle \( \tau' \subseteq \tau \) in \( U \).

One way in which we could try to achieve this consists in proving the following: there exists a function \( f : \mathbb{N} \to \mathbb{N} \) with \( f(k) \geq k \) for all \( k \in \mathbb{N} \) such that if a \( k \)-tangle \( \tau' \) in \( U \) extends to a \( f(k) \)-tangle \( \tau \) in \( U \), then \( \tau' \) has a decider (set). In this case, we may view the decider \( w \) for \( \tau' \) as an approximation of a decider for its extension \( \tau \) – although \( w \) will in general not decide all the separations in \( S_{f(k)} \subseteq U \) like \( \tau \).

Consider for example the \( m \)-tangle \( \tau_{m,k} \) in \( U_{bip}(V) \) constructed in Proposition 4 for some \( 3 \leq k \leq \frac{m}{2} \). This tangle \( \tau_{m,k} \) does not have a decider, but if we consider only those separations of order less than \( \frac{m}{2} \) in this example, then they even have a decider set: \( V \) decides all separations in \( U_{bip}(V) \) of order less than \( \frac{m}{2} \) like \( \tau_{m,k} \).

This leads us to the question whether tangles which extend to tangles of twice their order always have deciders, or even decider sets, i.e., whether \( f(k) := 2k \) is suitable. We show that this is indeed the case in that \( k \)-profiles in \( U \) which extend to regular \( 2k \)-profiles in \( U \) have decider sets – as long as we work in the universe \( U = U(V) \) of separations of a set \( V \) equipped with our standard order
function on $U$ which assigns to a separation the cardinality of its separator as its order. Recall that for this order function, all $k$-profiles in $U$ have deciders \[13\], but those as above even have decider sets:

**Theorem 12.** Let $U = U(V)$ be the universe of all separations of a set $V$ and let $| \cdot |$ be the standard order function on $U$. If $\tau'$ is a $k$-profile in $U$ for some $k \in \mathbb{N}$ that extends to a regular $2k$-profile $\tau$ in $U$, then $\tau'$ has a decider set $X \subseteq U$ of size at least $2k$.

If $V$ has less than $2k$ elements, then there exists no regular $2k$-profile $\tau$ in $U$, since $(V, V) \in \tau$ contradicts its regularity. Thus, Theorem 12 always holds for $|V| < 2k$, and we may assume $|V| \geq 2k$ for the proof of Theorem 12.

In the proof of Theorem 12, we will find a star $\sigma$ contained in $\tau$ whose interior $\bigcap_{(A, B) \in \sigma} B$ is the desired decider set for $\tau'$. Let us first show that the interior of any star contained in $\tau$ has size at least $2k$.

**Lemma 13.** The interior of any star contained in a regular $2k$-profile in $U$ has at least $2k$ elements.

**Proof.** Suppose not, let $\tau$ be a regular $2k$-profile in $U$, and let $\sigma \subseteq \tau$ be a star whose interior $X = \bigcap_{(A, B) \in \sigma} B$ has size $|X| < 2k$. Note that $\sigma$ is non-empty as otherwise the interior of $\sigma$ would be $V$ which by assumption has size at least $2k$.

Let us write $\sigma = \{(A_1, B_1), \ldots, (A_\ell, B_\ell)\}$. We claim that for any $i \leq \ell$ we have $|(A_1, B_1) \lor \cdots \lor (A_i, B_i)| < 2k$. By definition, we have

$$|(A_1, B_1) \lor \cdots \lor (A_i, B_i)| = |(A_1 \uplus \cdots \uplus A_i) \cap (B_1 \cap \cdots \cap B_i)|.$$ 

Since $\sigma$ is a star, we have $(A_1 \uplus \cdots \uplus A_i) \subseteq B_j$ for every $j > i$. So in particular, we have

$$(A_1 \uplus \cdots \uplus A_i) \cap (B_1 \cap \cdots \cap B_i) \subseteq (B_1 \uplus \cdots \uplus (B_i)) = X.$$ 

Therefore, $|(A_1, B_1) \lor \cdots \lor (A_i, B_i)| \leq |X| < 2k$. Since $\tau$ is a profile, it follows inductively that $(A_1, B_1) \lor \cdots \lor (A_i, B_i) \in \tau$ for every $i \leq \ell$. In particular, we have $(Y, X) = (A_1, B_1) \lor \cdots \lor (A_\ell, B_\ell) \in \tau$ where $Y = \bigcup_{(A, B) \in \sigma} A$.

Since $|X| < 2k$, the separation $(X, V)$ has order $< 2k$ and hence an orientation in $\tau$. By the regularity of $\tau$, this orientation must be $(X, V)$ because $(V, X)$ is co-small. But this leads to a contradiction since this would imply $(Y, X) \lor (X, V) = (V, X) \in \tau$ as $\tau$ is a profile.

**Proof of Theorem 12** Let $\sigma \subseteq \tau$ be a star whose interior $X = \bigcap_{(C, D) \in \sigma} D$ is of smallest size among all stars contained in $\tau$. By Lemma 13, we have $|X| \geq 2k$. We claim that $X$ is the desired decider set for $\tau'$.

To prove this, we show that $|X \cap A| < k$ for every $(A, B) \in \tau'$. Since $|X| \geq 2k$, this immediately implies that $X$ is a decider for $\tau'$. So suppose for a contradiction that there exists $(A, B) \in \tau'$ with $|X \cap A| \geq k$, and assume that $(A, B)$ has minimal order among all such separations in $\tau'$. Note that $(A, B)$ may be witnessed by $X$.

Now for every $(C, D) \in \sigma$, the separation $(A \cap D, B \cup C)$ has at least the order of $(A, B)$. Indeed, we have $X \subseteq D$ by construction, and therefore we get $|(A \cap D) \cap X| = |A \cap X| \geq k$. Moreover, if $(A \cap D, B \cup C)$ would have order less than $(A, B)$, then $(A \cap D, B \cup C)$ would be contained in $\tau$ (since $\tau$ is a profile) and hence in $\tau' \subseteq \tau$ because $(A, B)$ has order less than $k$. Then, however, $(A \cap D, B \cup C)$ would contradict the minimal choice of $(A, B)$.
By the submodularity of the standard order function, $(B \cap C, A \cup D)$ has order at most $|C, D|$. Thus, $(B \cap C, A \cup D) \in \tau$ since $(B \cap C, A \cup D) \leq (C, D) \in \tau$ and $\tau$ is consistent. Hence, the star

$$\hat{\sigma} := \{(A, B)\} \cup \{(B \cap C, A \cup D) \mid (C, D) \in \sigma\}$$

is contained in $\tau$.

We claim that the interior $\hat{X}$ of $\hat{\sigma}$ is smaller than $X$ contradicting the choice of $\sigma$. Indeed, by definition, we have

$$\hat{X} = B \cap \bigcap_{(C, D) \in \sigma} (A \cup D) = (A \cap B) \cup (B \cap X) = ((A \cap B) \setminus X) \cup (B \cap X) = ((A \cap B) \setminus X) \cup (B \cap X).$$

Since $(A, B)$ is a separation of $V$, the set $X \subseteq V$ is the disjoint union of $B \cap X$ and $(A \cap X) \setminus B$. So we are done if

$$|(A \cap B) \setminus X| < |(A \cap X) \setminus B|.$$ 

Let $h = |A \cap B \cap X|$. Since $|A \cap X| \geq k$, we have $|(A \cap X) \setminus B| \geq k - h$. However, we have $(A, B) \in \tau$, so $|A \cap B| < k$ and hence $|(A \cap B) \setminus X| < k - h$, completing the proof.

Our proof of Theorem 12 heavily relies on the assumption that the order of a separation in $U(V)$ is given by the size of its separator. We do not know if a similar result holds for other or even all submodular order functions on $U(V)$.

Problem 14. Let $V$ be a set, and consider any submodular order function on $U(V)$. Is it true that if $\tau'$ is a $k$-profile in $U(V)$ for some $k \in \mathbb{N}$ which extends to a regular $2k$-profile $\tau$ in $U(V)$, then $\tau'$ has a decider set $X$? What happens for other universes of separations of $V$ such as $U_{bip}(V)$?

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