S-dual gravity in the axial gauge

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Abstract

We investigate an action that includes simultaneously original and dual gravitational fields (in the first-order formalism), where the dual fields are completely determined in terms of the original fields through axial gauge conditions and partial (non-covariant) duality constraints. We introduce two kinds of matter, one that couples to the original metric, and dual matter that couples to the dual metric. The linear response of both metrics to the corresponding stress–energy tensors coincides with Einstein’s equations. In the presence of nonvanishing standard and dual cosmological constants a stable solution with a time independent dual scale factor exists that could possibly solve the cosmological constant problem, provided our world is identified with the dual sector of the model.

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1. Introduction

The known concept of Hodge duality for form fields (antisymmetric tensor fields), i.e. the permutation of free equations of motion and Bianchi identities, can be generalized to fields of mixed symmetry [1–3]. Here again, linear combinations of free equations of motion and Bianchi identities imply Bianchi identities and free equations of motion for the dual fields or their field strengths. In particular, this generalized Hodge duality can be applied to linearized gravity in $d = 4$ or higher dimensions [1–11].

One motivation for the introduction of a dual graviton are attempts to realize hidden symmetries in $d = 11$ supergravity/$M$-theory [12]. However, only in $d = 4$ the dual of a graviton is again a symmetric two component tensor field [1–9], that could be used to construct a dual gravitational theory.

Any attempt in this direction has to face no-go theorems on local Lorentz invariant dual models of gravity [13–14]. The way out considered here is not to insist on a duality symmetry, and—most of all—to break Lorentz symmetry twice: once through a non-covariant (axial) gauge fixing for the dual gauge fields, and through non-covariant duality constraints on the field strengths.
Non-covariant gauges have the particular property that, in the context of Yang–Mills theories, the gauge field can be reconstructed from the field strength \[ 15, 16 \]. This feature has been used in \[ 15 \] in order to construct dual models of Yang–Mills theories that are not, however, duality symmetric.

Here we study a similar program in the context of 4D gravity. In order to formulate gravity as closely as possible to Yang–Mills theories, we employ the first-order Cartan formalism where vierbeins and (spin) connections are treated as independent fields, and the original local symmetries are diffeomorphisms and an ‘internal’ \( O(3, 1) \) Lorentz symmetry (the gauge fields of the latter are the spin connections).

Let us first outline the rough idea behind the present approach: the starting point is a standard gravitational action involving vierbeins \( e_{\mu}^a \) and connections \( \omega^{ab}_{\mu} \) (here and in the following Latin indices indicate representations under the internal Lorentz group, whereas Greek indices are spacetime indices).

The Riemann–Cartan tensor \( R_{ab\mu\nu} \) is the field strength of the connections, and its dual \( \tilde{R}_{ab\mu\nu} \) can be constructed by the contraction of a pair of (antisymmetric) indices with two of the four indices of the 4D completely antisymmetric epsilon tensor.

Once \( \tilde{R}_{ab\mu\nu} \) is given through a duality relation in terms of \( R_{ab\mu\nu} \), one can ask under which circumstances \( \tilde{R}_{ab\mu\nu} \) can be written as a field strength of a dual connection \( \tilde{\omega}^{ab}_{\mu} \).

First, for each fixed choice of \([ab]\), \( \tilde{R}_{ab\mu\nu} \) contains six antisymmetric combinations of indices \([\mu \nu]\). Deducting the gauge degrees of freedom from \( \tilde{\omega}^{ab}_{\mu} \), we have however only three independent connections \( \tilde{\omega}^{ab}_{\mu} \) (for given \([ab]\)) at our disposal. This leaves us with two possibilities: (1) \( \tilde{R}_{ab\mu\nu} \) has to satisfy certain constraints; (2) we only use three out of the six antisymmetric combinations of indices \([\mu \nu]\) in \( \tilde{R}_{ab\mu\nu} \) in order to construct \( \tilde{\omega}^{ab}_{\mu} \).

The constraints (1) would be the (three independent) Bianchi identities. In fact, in a weak field expansion the validity of the Bianchi identities for \( \tilde{R}_{ab\mu\nu} \) can be deduced from those for \( R_{ab\mu\nu} \) under the condition that the Ricci tensor \( R_{\mu\nu} \) vanishes \[1–9\]. Beyond a weak field expansion (where the covariant derivatives acting on \( \tilde{R}_{ab\mu\nu} \) would involve the not yet known connection \( \tilde{\omega}^{ab}_{\mu} \), that differ in any case from \( \omega^{ab}_{\mu} \)), or for non-vanishing \( R_{\mu\nu} \), the Bianchi identities for \( \tilde{R}_{ab\mu\nu} \) can no longer be employed. We are left with the possibility (2) above: we construct \( \tilde{\omega}^{ab}_{\mu} \) from the three combinations of indices \([\mu \nu]\) that are left over after a contraction of \( \tilde{R}_{ab\mu\nu} \) with a fixed vector \( n_{\nu} \). Note that we still assume that all six combinations of indices \([\mu \nu]\) of \( \tilde{R}_{ab\mu\nu} \) are identified with the field strength of the connection \( \tilde{\omega}^{ab}_{\mu} \); we do not assume, however, that the three additional combinations of indices \([\mu \nu]\) of \( \tilde{R}_{ab\mu\nu} \) are determined through a duality relation in terms of \( R_{ab\mu\nu} \).

First, for a fixed vector \( n^\mu \),\(^1\) we define only three combinations of indices \([\mu \nu]\) of \( \tilde{R}_{ab\mu\nu} \) in terms of \( R_{ab\mu\nu} \):

\[
n^\mu \tilde{R}_{ab\mu\nu} = \frac{1}{2} \varepsilon^{abcd} n^\mu R_{ab\mu\nu}.
\]  

(1.1)

Next we introduce a dual connection \( \tilde{\omega}^{ab}_{\mu} \). On the dual connections we impose the axial gauge condition

\[
\tilde{\omega}^{ab}_{\mu} n^\mu = 0.
\]  

(1.2)

Then we interpret \( \tilde{R}_{ab\mu\nu} \) as a field strength for \( \tilde{\omega}^{ab}_{\mu} \), which implies—in the gauge (1.2)—the relation

\[
n^\mu \partial_\mu \tilde{\omega}^{ab}_{\nu} = n^\mu \tilde{R}_{ab\mu\nu}.
\]  

(1.3)

Once \( n^\mu \tilde{R}_{ab\mu\nu} \) is a known expression in terms of \( R_{ab\mu\nu} \), the enforcement of both conditions (1.2) and (1.3) determines the dual connections completely (up to boundary conditions on a hypersurface perpendicular to \( n^\mu \)) in terms of \( R_{ab\mu\nu} \).

\(^1\) In section 2 we actually employ a Lorentz vector \( n^a \), but the essential idea is more easily formulated with \( n^\mu \).
A similar reasoning can be employed for the construction of a dual vierbein $\tilde{e}_a^\mu$, out of which a dual metric $\tilde{g}_{\mu\nu} = \tilde{e}_a^\mu \tilde{e}_a^\nu$ can be constructed: For given $\tilde{\omega}^{ab}_{\mu}$, the dual vierbein could be constructed from the vanishing of the torsion tensor

$$T^a_{\mu\nu} = \partial_{[\mu} \tilde{e}^a_{\nu]} + \tilde{\omega}^{ab}_{[\mu} \tilde{e}^b_{\nu]}.$$  \hspace{1cm} (1.4)

Again—for each fixed value of $a$—$T^a_{\mu\nu}$ contains six antisymmetric combinations of indices $[\mu \nu]$, but (modulo gauge degrees of freedom) we have only three degrees of freedom in $\tilde{e}_a^\mu$ at our disposal. Again we can restrict ourselves to the vanishing of $T^a_{\mu\nu}$, which allows us to construct the vierbeins $\tilde{e}_a^\mu$ (in the axial gauge), but the three extra components of $T^a_{\mu\nu}$ (for each $a$) will vanish no longer in general. (One can prove that they would vanish in a weak field expansion, but our aim is to go beyond it.)

Hence, we impose a gauge condition on $\tilde{e}_a^\mu$ (i.e. a choice of the coordinate system) such that

$$\tilde{e}_a^\mu n^\mu = \delta^\mu_a,$$  \hspace{1cm} (1.5)

where $\delta^\mu_a$ is the Kronecker symbol. A coordinate system where (1.5) holds can always be constructed locally.

Contracting (1.4) over $n^\mu$ and using both equations (1.2) and (1.5) one obtains

$$n^\mu \partial_\mu \tilde{e}_a^\nu = \tilde{\omega}^{ab}_{\mu} \delta^b_a n^\mu.$$  \hspace{1cm} (1.6)

Similar to the case of the connection, equations (1.5) and (1.6) determine the vierbeins $\tilde{e}_a^\mu$ completely up to boundary conditions, now in terms of the connections $\tilde{\omega}^{ab}_{\mu}$ constructed before. Again, equations (1.5) and (1.6) do not imply the vanishing of all components of $T^a_{\mu\nu}$ (again only half of them), i.e. the dual theory is not entirely torsionless.

On the other hand we have now achieved our goal: the introduction and definition of dual gravitational fields, including a dual metric $\tilde{g}_{\mu\nu}$, beyond a weak field expansion. Moreover, all the previous conditions (1.1)–(1.3), (1.5) and (1.6) can be cast into the form of an action involving Lagrange multipliers.

The possibility of constructing an action for dual, and generally interacting, fields is highly non-trivial: First, attempts to impose (covariant) duality relations for all components of field strengths using Lagrange multipliers lead generically to equations of motions for these Lagrange multipliers that have non-trivial solutions corresponding to new degrees of freedom. Attempts to eliminate all new degrees of freedom through new multipliers result in the need for an infinite tower of Lagrange multipliers [17]. The corresponding problem can be cured quite easily here, with just one extra multiplier required (see section 2).

Second, the complete permutation of nonlinear equations of motion and Bianchi identities (between original and dual fields) resulting from an action is generally impossible, since an action allows for the addition of sources for fields that modify the equations of motion, but leave Bianchi identities unchanged. Here duality relations are satisfied by only some of (half of) the field strengths—that satisfy the second Bianchi identities by construction—and, furthermore, the first-order formalism allows for torsion that allows to violate the first Bianchi (cyclic) identities of the Riemann tensor.

In fact, the possibility of constructing an action involving simultaneously the original and the dual gravitational fields as in the following section 2 allows in particular to couple the dual metric, in a standard fashion, to matter and to study its reactions to a stress energy tensor. We will carry out this analysis in the weak field limit in chapter 3, with the result that this reaction is the same as in standard general relativity. This implies that vacuum solutions of standard general relativity remain valid also for the dual metric in the weak field limit, but we have to expect modifications of the solutions of the combined set of equations of motion (for both the original and the dual fields) beyond lowest order. In particular any attempt to
integrate out completely the original gravitational fields will result in non-local interactions for the dual fields, hence their corresponding effective theory differs definitely from standard general relativity.

The present approach is manifestly asymmetric between the original and the dual gravitational fields; it seems generally impossible, however, to implement a duality symmetry into interacting gravitational theories [13, 14].

In section 4 we study cosmological solutions of the equations of motion derived in section 2. We consider simultaneous cosmological constants in the standard and the dual sector and obtain a stable solution with a static dual scale factor. This scenario could provide a solution of the cosmological constant problem, provided our world is identified with the dual sector of the model.

Finally, a summary and outlook is given in section 5.

2. The action and its variation

The basic idea for the construction of an action involving simultaneously original and dual gravitational fields has been outlined in the introduction: dual gravitational fields (the connection and the vierbein) are introduced together with constraints that determine them completely in terms of the original fields, up to boundary conditions. This procedure guarantees the absence of new degrees of freedom propagating in 4D spacetime, but the price to pay is the explicit breaking of both internal (Lorentz) and spacetime (local coordinate reparametrizations) gauge symmetries. (The fixation of spacetime gauge symmetries could, in principle, be omitted, but then the absence of new degrees of freedom is less obvious.)

In this section we present an action, and discuss the various steps of symmetry breaking. In order to keep track of local coordinate reparametrizations it turns out to be convenient to proceed slightly differently than outlined in the introduction (although the final result is the same): instead of introducing a fixed constant vector \( n^\mu \), that would not be invariant under local coordinate reparametrizations, we introduce a fixed constant Lorentz vector (but spacetime scalar) \( n^a \) that is not invariant under local or global Lorentz transformations.

We start with the standard Lagrangian for general relativity in the first-order Cartan formalism, where the independent fields are the connection \( \omega^{ab\mu} \) and the vierbein \( e^a_\mu \) (or its inverse \( e_\mu^a \)). Lorentz indices \( a, b, \ldots \) are raised and lowered with the flat Lorentz metric

\[
\eta_{ab} = \text{diag}(1, -1, -1, -1). \tag{2.1}
\]

The Riemann–Cartan curvature tensor is the field strength of the connection:

\[
R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab\nu} - \partial_\nu \omega^{ab\mu} + \omega^{ac\mu} \omega^{cb\nu} - \omega^{ac\nu} \omega^{cb\mu} = 2(\partial_{[\mu} \omega^{ab}_{\nu]} + \omega^{a}_{\nu [\mu} \omega^{b}_{\nu]}). \tag{2.2}
\]

The inverse vierbeins allow us to construct a spacetime scalar version of the Riemann–Cartan curvature tensor,

\[
R^{ab}_{cd} = \tilde{e}^e_\mu \tilde{e}^d_\nu R^{ab}_{\mu\nu}. \tag{2.3}
\]

Then the standard Einstein Lagrangian is

\[
\mathcal{L}_E = \frac{1}{2\kappa} \det(e) R^{ab}_{\mu\nu}(e, \omega). \tag{2.4}
\]

(Our convention here and below is that all terms in the Lagrangian transform as densities under local coordinate reparametrizations, such that the final action is simply \( \int d^4x \mathcal{L} \).)
Next we introduce a new ‘dual’ connection $\tilde{\omega}^{ab}$, and a new ‘dual’ vierbein $\tilde{e}^{a}$ (or its inverse $\tilde{e}_{a}$). At the first level concerning $\tilde{\omega}^{ab}$, we have to add three Lagrange multiplier fields:

(i) $K^{(1)}_{ab}$ will serve to impose the axial gauge condition on $\tilde{\omega}^{ab}$.

(ii) $L^{(1)}_{ab}$ will serve to impose that the dual of the Riemann tensor (2.3), contracted with $n^{d}$ over its last index, is equal to the field strength $\tilde{R}^{ab}_{cd}$ of the dual connection $\tilde{\omega}^{ab}$ (defined in analogy to (2.2)), again contracted with $n^{d}$.

A few comments are in order here: in the case of standard duality between (abelian) antisymmetric tensor fields, the dual field strength is obtained by contracting an epsilon tensor (with spacetime indices) over the spacetime indices of the original field strength, which would correspond to the lower pair of indices of the Riemann tensors here. Below we need, however, one of the lower indices—contracted with $n^{d}$—in order to employ a relation similar to equation (1.3). Therefore we employ the upper pair of indices in order to define a dual Riemann tensor. In the absence of torsion, this would make no difference, since then the Riemann tensor would be symmetric with respect to an exchange of these two pairs of indices. Below we will find, however, that torsion is generally non-vanishing here, i.e. not all components of the connections $\omega^{ab}$ and $\tilde{\omega}^{ab}$ are related in the standard fashion to the corresponding vierbeins. Hence the present definition of a dual Riemann tensor in the sense of duality with respect to the internal $O(3, 1)$ Lorentz symmetry differs somewhat from standard S-duality.

Finally we found it convenient to equate the Riemann tensors multiplied with the determinants of the corresponding vierbeins, which simplifies some of the equations of motion below and allows $L^{(1)}_{ab}$ to transform as a spacetime scalar.

(iii) It turns out that the components of $L^{(1)}_{ab}$ with the index $c$ in the direction of $n^{c}$, i.e. $n_{c}L^{(1)}_{ab}$, are not constraint by the action. In order to eliminate these unwanted degrees of freedom we employ another Lagrange multiplier $N^{(1)ab}$. No further Lagrange multipliers are needed in order to cope with unwanted degrees of freedom of $N^{(1)ab}$.

Hence at this first level, which treats the constraints on the dual connection $\tilde{\omega}^{ab}$, the following three terms are added to the Lagrangian:

\[
\mathcal{L}^{(1)} = K^{(1)}_{ab} n^{d} \tilde{e}^{d}_{a} \tilde{\omega}^{ab} + L^{(1)}_{ab} \frac{1}{2} \det(e) e^{ab}_{ef} R^{ef}_{cd}(e, \omega) - \det(\tilde{e}) \tilde{R}^{ab}_{cd}(\tilde{e}, \tilde{\omega}) + N^{(1)ab} n_{c} L^{(1)}_{ab}.
\]

(2.5)

(Here $K^{(1)}$ and $L^{(1)}$ transform as densities like $\det(\tilde{e})$ under coordinate reparametrizations).

For a given dual vierbein $\tilde{e}^{a}$, all components of $\tilde{\omega}^{ab}$ are now fixed by the constraints following from the variation of (2.5) with respect to $K^{(1)}_{ab}$ and $L^{(1)}_{ab}$.

At the next level, we introduce constraints on the dual vierbein $\tilde{e}^{a}$ (or, for convenience, on its inverse $\tilde{e}_{a}^{a}$). Again we will not follow exactly the procedure outlined in the introduction (although the final result will be the same), since we employ the constant Lorentz vector $n^{d}$ rather than a spacetime vector $n^{d}$.

Otherwise, the roles of the three Lagrange multipliers introduced at this second level resemble to those in (2.5):

(i) $K^{(2)}_{ab}$ serves to impose an axial gauge condition on $\tilde{e}$.

(ii) $L^{(2)}_{ab}$ serves to impose the vanishing of the torsion tensor $T^{a}_{bc}$ contracted with $n^{a}$, with

\[
T^{a}_{bc} = \tilde{e}^{b}_{a} \tilde{e}^{c}_{d} T^{a}_{d}.
\]

(2.6)

and $T^{a}_{c}$, as in (1.4).
(iii) Since the components $L^{(2)}_a n_b$ are not determined by varying the action, $N^{(2)a}$ serves to eliminate these degrees of freedom.

Hence at the second level, the following three terms are added to the Lagrangian:

$$L^{(2)} = K^{(2)}_\mu n^\mu \omega^\mu_a + L^{(2)}_a T^{a}_{bc} n^c + N^{(2)a} L^{(2)}_a n_b.$$  \hspace{1cm} (2.7)

(Here $K^{(2)}$ and $L^{(2)}$ transform as densities under coordinate reparametrizations).

Finally we can couple both kinds of gravitational fields to matter, after defining the metrics

$$g_{\mu\nu} = e^{\mu a} e_{\nu a}$$ \hspace{1cm} (2.8a)

$$\tilde{g}_{\mu\nu} = \tilde{e}^{\mu a} \tilde{e}_{\nu a}.$$ \hspace{1cm} (2.8b)

For instance, one can introduce matter that couples to $g_{\mu\nu}$, and another kind of matter that couples only to $\tilde{g}_{\mu\nu}$; hence two kinds of matter Lagrangians

$$-L_M(g) + L_M(\tilde{g}).$$ \hspace{1cm} (2.9)

(The relative minus sign in (2.9) serves just to reproduce the standard signs in the corresponding Einstein equations, see section 3.)

Next we will derive the constraints imposed by the various Lagrange multipliers.

Starting with $L^{(1)}$ in (2.5), its variation with respect to $K^{(1)}_{ab}$ imposes a gauge condition of the axial type on $\tilde{\omega}^{ab}_\mu$:

$$n^c \tilde{\alpha}^a_c \tilde{\omega}^{ab}_\mu \equiv n^\mu \tilde{\omega}^{ab}_\mu = 0.$$ \hspace{1cm} (2.10)

Below we will find that for constant $n^a$, the vector

$$n^\mu = n^\alpha \tilde{e}^\mu_a$$ \hspace{1cm} (2.11)

is constant as well.

The variation of $L^{(1)}$ with respect to $L^{(1)}_c$ includes, at first sight, a term proportional to $N^{(1)ab}$. However, contracting the variation with $n^c$ and using the antisymmetry of the curly bracket in (2.5) in $[cd]$, one obtains

$$N^{(1)ab} = 0.$$ \hspace{1cm} (2.12)

Furthermore we can use that, with the definition analogously to (2.2) and (2.3) of $\tilde{R}^{ab}_{cd}$ and (2.10),

$$\tilde{R}^{ab}_{cd} n^d = -\tilde{\alpha}^a_c n^c \partial_\mu \tilde{\omega}^{ab}_\mu.$$ \hspace{1cm} (2.13)

Hence one finally obtains the constraint from the variation w.r.t. $L^{(1)}_c$,

$$\det(\tilde{\gamma}) \tilde{\alpha}^a_c n^c \partial_\mu \tilde{\omega}^{ab}_\mu = -\frac{1}{2} \det(e) e^{ab}_{ef} R^{ef}_{cd}(e, \omega) n^d.$$ \hspace{1cm} (2.14)

The variation with respect to $N^{(1)ab}$ trivially implies

$$L^{(1)}_c n^c = 0.$$ \hspace{1cm} (2.15)

Now we turn to $L^{(2)}$ in (2.7), with the torsion tensor $T^{a}_{bc} = T^{a}_{[bc]}$ defined in (2.6) and (1.4). The variation with respect to $K^{(2)}_\mu$ gives

$$\partial_\nu (n^\nu \tilde{\omega}^{ab}_\mu) \equiv \partial_\nu n^\mu = 0.$$ \hspace{1cm} (2.16)

Contracting the variation with respect to $L^{(2)}_a n_b$ with $n^b$ gives

$$N^{(2)a} = 0.$$ \hspace{1cm} (2.17)
Using (2.17) and contracting the variation w.r.t. $L^{(2) a b}$ with $\tilde{e}^b \mu$ gives

\[
n^\nu \left( \partial_\nu \tilde{e}^a \mu - \partial_\mu \tilde{e}^a \nu + \tilde{\omega}^{ab} \mu \tilde{e}^b \nu - \tilde{\omega}^{ab} \nu \tilde{e}^b \mu \right) = 0.
\] (2.18)

Using (2.16) and (2.11) one finds that the first term in (2.18) vanishes, whereas (2.10) implies the vanishing of the last term. Hence (2.18) collapses to

\[
n^\nu \partial_\nu \tilde{e}^a \mu = n^b \tilde{\omega}^{ab} \mu.
\] (2.19)

Finally the variation with respect to $N^{(2) a}$ trivially implies

\[
L^{(2) a} n_c = 0.
\] (2.20)

Let us study the consequences of the constraints (2.16) and (2.19) on the dual metric $\tilde{g}_{\mu \nu}$ in (2.8b). Contracting (2.19) with $n_a$ gives

\[
n^\nu \partial_\nu (n_a \tilde{e}^a \mu) = 0.
\] (2.21)

From $n^\mu \tilde{g}_{\mu \nu} = n_a \tilde{e}^a \mu$ one finds that (2.21) is equivalent to

\[
n^\nu \partial_\nu (n^\mu \tilde{g}_{\mu \nu}) = 0.
\] (2.22)

Axial gauges of the form (2.22)—with constant $n^\mu$ as in (2.16)—have been used in [18] in order to construct graviton propagators and to analyse one loop diagrams in quantized gravity. They can constrain the global properties of spacetime; the Schwarzschild solution in such a gauge—with $n^\mu$ timelike, $g_{00} = -1, g_{0i} = 0$, i.e. in a comoving frame (or Novikov coordinates)—is known, however, and given e.g. in [19]. In cosmology, on the other hand, this gauge is standard.

Next we give the equations of motion that follow from the variations of the complete action with respect to the vierbeins and the connections.

First, from the variation with respect to $e_a \mu$, contracted with $\tilde{e}^\mu \nu$ and after division by $\det(e)$ one obtains

\[
\frac{1}{\kappa} \left[ -\frac{1}{2} \delta^a \nu R^{bc} \mu_{bc} + R^{ab} \nu_{gb} \right] + L^{(1)} \left[ e^a \mu \right] \epsilon^{mn} \epsilon_{ef} \left[ -\frac{1}{2} \delta^a \nu R^{ef} \mu_{cd} + \delta^a \nu R^{ef} \mu_{gd} \right] - T_{Mg} \nu = 0,
\] (2.23)

where

\[
T_{Mg} \nu = \frac{\tilde{g}_{\mu}}{\det(e)} \frac{\delta}{\delta \tilde{e}_\mu} L_M.
\] (2.24)

The variations with respect to $\omega^{ab} \mu$ are best expressed in terms of the two tensors

\[
E_{ab} \mu \nu = E_{[ab]} \mu \nu = \det(e) \tilde{g}_{[\mu} \tilde{e}_{\nu]},
\] (2.25)

and

\[
S_{ab} \mu \nu = S_{[ab]} \mu \nu = \det(e) L^{(1)} \left[ e^c \right] \delta_{ef} \tilde{e}_{ab} \tilde{e}_{\nu} \tilde{e}_{\nu}.
\] (2.26)

Then one obtains

\[
\frac{1}{\kappa} \left[ \partial_\nu E_{ab} \mu \nu - 2 E_{[ef} \mu \nu \omega_{gb]} e \right] + \partial_\nu S_{ab} \mu \nu - 2 S_{[e[d} \mu \nu \omega_{f]} c] e = 0.
\] (2.27)

For $S_{ab} \mu \nu = 0$, equation (2.27) would determine $\omega^{ab} \mu$ in terms of $e^a \mu$, as usual, through the vanishing of the covariant derivative of $e^a \mu$, or the vanishing of torsion. Generically, however, $S_{ab} \mu \nu$ will not vanish (since $L^{(1)} \left[ e \right]$ will not vanish), hence torsion is generically non-vanishing for configurations $\omega^{ab} \mu$ that solve equation (2.27).

Concerning the variation with respect to $\tilde{e}_a \mu$, it is convenient to consider the combination

\[
\tilde{e}_b \mu \left( \delta^a \mu - \frac{n^a n_d}{n^c} \right) \frac{\delta}{\delta \tilde{e}_a \mu}.
\] (2.28)
that is independent of the Lagrange multiplier field $K^{(2)\nu\mu}$ in (2.7). (The remaining components just determine $K^{(2)\nu\mu}$, that appears nowhere else.) Then one finds

$$\left(\delta_{ab} - \frac{n_a n_b}{n^2}\right) \left[ 2 L^{(1)}_{ef} \delta_{[b} d \tilde{R}^{ef}_{c]} n^g - \frac{1}{2} \epsilon^{\mu}_{\nu\sigma\tau} \epsilon^g_{\xi} \partial_\xi \left( L^{(2)}_{\mu\nu\sigma\tau} e^\xi_{\nu\sigma\tau} + \tilde{T}_{Mb}^d \right) \right] = 0,$$

with

$$\tilde{T}_{Mb}^d = \frac{\tilde{e}^{\nu\mu}}{\det(\tilde{e})} \frac{\delta}{\delta e^{\nu\mu}_d} \tilde{F}_M.$$

Finally, the variation with respect to $\tilde{\omega}^{ab\mu}$ gives, again after an elimination of $K^{(1)ab}$ through a contraction with $n^d \tilde{e}^{\nu\mu}_d$,

$$n^d \tilde{e}^{\nu\mu}_d \partial_\nu \left( \det(\tilde{e}) L^{(1)}_{ab} e^{\nu\mu}_c \right) - \frac{1}{2} L^{(2)}_{[a} n_b] e^{\nu\mu}_c = 0.$$

Some consequences of these equations of motions will be studied in the next sections.

Before concluding this section we have to make some comments on the Lorentz symmetry breaking induced by the constant Lorentz vector $n^a$. Apart from the terms that impose axial gauge conditions (the first terms in (2.5) and (2.7)), $n^a$ appears in various other terms in (2.5) and (2.7). In the next section we study the equations of motion in a weak field expansion (around a Minkowski vacuum, and considering all Lagrange multipliers as weak fields). At this level we find no explicit Lorentz symmetry breaking in the equations of motion.

Gravitational self-interactions, that start to play a role in higher order in a weak field expansion, will generally not respect Lorentz invariance, however. In the worst case this could induce violations of unitarity and/or strong gravitational self-couplings at unacceptable length scales.

It may then be advisable to promote $n^a$ to a field $n^a(x)$, and to replace explicit Lorentz symmetry breaking by spontaneous Lorentz symmetry breaking via a potential $\lambda(n^a n_a \pm 1)^2$ in the Lagrangian for $n^a(x)$. Apart from a massive radial mode, this scenario will add three Nambu–Goldstone (NG) modes to the model.

The possible fate of such NG modes has recently been reviewed in [20], and depends heavily on the Lagrangian (e.g. additional $R^2$ terms, that generate a propagating spin connection): the NG modes could remain massless, but could also become massive degrees of freedom. Some phenomenological consequences of such NG modes have been studied in [21], but even the quadratic part of an effective Lagrangian would have been to be constructed here.

Also, unless the Lorentz gauge symmetry is restored this way, the axial gauge condition (2.5) imposed on $\tilde{\omega}^{ab\mu}$ are not just gauge conditions that can, in principle, be chosen at will (in contrast to the gauge conditions (2.7) on $\tilde{e}^{\nu\mu}_a$): observables will depend on the present choice of the gauge condition imposed on $\tilde{\omega}^{ab\mu}$—however, only to higher order in a weak field expansion, see the next chapter.

3. The linearized gravitational field equations

The aim of this section is to study the linearized response of the gravitational fields to matter sources, notably to a dual stress energy tensor $\tilde{T}_{Ma}^b(\tilde{g})$.

The point is that, as long as $T_{Ma}^b(g)$ does not depend on $\tilde{g}$ and vice versa (what we will assume in the following), standard matter described by $T_{Ma}^b(g)$ ‘sees’ a spacetime geometry described by $g_{\mu\nu}$, whereas dual matter described by $\tilde{T}_{Ma}^b(\tilde{g})$ ‘sees’ a spacetime geometry described by $\tilde{g}_{\mu\nu}$, that will generally be quite different! An interesting question is then whether $\tilde{g}_{\mu\nu}$ reacts to $\tilde{T}_{Ma}^b$ in a way that resembles the standard Einstein equations; if this is the case, our world could possibly correspond to the dual matter of the model.
An answer to this question is not quite trivial even to lowest order, where (counting Lagrange multipliers as weak fields) terms \( \sim L \cdot R \) in equations (2.23) and (2.29) can be neglected: whereas the effect of \( T_{Ma}{}^b \) on \( g_{\mu\nu} \) is obvious from (2.23) (and agrees with Einstein’s gravity), the effect of \( \tilde{T}_{Ma}{}^b \) is

(i) a non-vanishing value of \( L^{(2)}_{(b)} \) from (2.29), which induces

(ii) a non-vanishing value of \( L^{(1)}_{(c)} \) from (2.31), which induces

(iii) non-vanishing torsion for \( \omega^{\mu}{}_{ab} \) from (2.27), hence non-vanishing components of \( R^{ab}{}_{\mu\nu} \), which induce

(iv) a non-vanishing \( \tilde{\omega}^{\mu}{}_{ab} \) from (2.14), which finally generates

(v) a non-vanishing \( \tilde{e}^{\mu}{}_{a} \) from (2.19).

Subsequently we will carry out these steps explicitly. First it is convenient, however, to adopt a convention concerning the direction of \( n^a \), that we assume to be timelike:

\[
n^a = (1, 0, 0, 0) \tag{3.1}
\]

and the first component will be denoted by 0, i.e. \( n^0 = 1 \). All space and Lorentz coordinates perpendicular to \( n^\mu \sim n^a \) (to lowest order) will be denoted by latin letters \( i, j, k, \ldots \) from the middle of the alphabet.

Next we have to comment the fact that equation (2.29) contains a projector such that the components \( \tilde{T}_{Ma}{}^d \) do not appear. Such a situation is actually familiar from Yang–Mills theories in axial gauges: After imposing the axial gauge condition, the corresponding components of the currents (here: the stress energy tensor) decouple from the gauge field. They contribute nevertheless to the dynamics of the theory through the equations associated with (covariant) current conservation, that can be derived via Noether’s theorem. Likewise, we have to use the covariant conservation of \( \tilde{T}_{Ma}{}^b \) (and its symmetry) here, which gives to lowest order

\[
\partial_0 L^{(2)}_{(j)} = 2 \tilde{T}_{Ma}{}^j. \tag{3.2}
\]

The index combination \((ab) = (j0)\) of equation (2.31) gives, with (3.3) for \( L^{(2)} \),

\[
\partial_0 \partial_0 L^{(1)}_{(j0)} = \frac{1}{2} \tilde{T}_{Ma}{}^j. \tag{3.4}
\]

whereas the combination \((ab) = (jk)\) implies the vanishing of \( L^{(1)}_{(jk)} \) up to terms linear in \( t \). Equation (3.4) implies a non-vanishing value for \( S_{ab}{}^{\mu\nu} \) in (2.26), which gives

\[
\begin{align*}
S_{ij}{}^{k0} &= \epsilon^{i0}{}_{kj} L^{(1)}_{(j0)} = \epsilon^{j0}{}_{ij} \frac{1}{2 \tilde{T}_{Ma}{}^k} \\
&= \frac{1}{2 \tilde{T}_{Ma}{}^k} 
\end{align*}
\tag{3.5}
\]

where we allowed ourselves to represent integrals with respect to \( x^0 = t \) by an inverse derivative \( 1/\partial_0 \). Next we have to determine \( \omega^{\mu}{}_{ab} \) from (2.27), and it is convenient to decompose \( \omega_{ab\mu} \) into

\[
\omega_{ab\mu} = \Omega_{ab\mu}(e) + \omega^{T}_{ab\mu} \tag{3.6}
\]

where

\[
\Omega_{ab\mu}(e) = \frac{1}{2} \left( \partial_0 (\epsilon_{ab\mu} + e_{\mu ab}) - \partial_\mu (e_{ba} + e_{\mu ab}) + \partial_\mu (e_{ba} - e_{ab}) \right) \tag{3.7}
\]

and \( \omega^{T}_{ab\mu} \) represents torsion. If one would replace \( \omega \) by \( \Omega \) in (2.27), its first line would vanish identically. Hence (2.27) determines \( \omega^{T} \) in terms of \( S_{ab}{}^{\mu\nu} \), and \( \Omega \) (or \( e \)) has to be determined.
elsewhere (by equation (2.23)). Exploiting the various index combinations of equation (2.27) one obtains after some calculation
\[ \omega_T^{ijk} = -\frac{\kappa}{2\delta_0} \left( \epsilon^{mn}_{ik} \tilde{T}_{Mjn} + \epsilon^{mn}_{ij} \tilde{T}_{Mkn} + \epsilon^{mn}_{kj} \tilde{T}_{Min} \right), \]
\[ \omega_T^{ij0} = -\frac{\kappa}{2\delta_0} \epsilon^{mn}_{ij} \tilde{T}_{Mkn}, \] (3.8)
Next we wish to construct \( \tilde{R}^{ab}_{\ c\ d}(\tilde{\omega}) \) from \( R^{ab}_{\ cd}(\omega) \) via (2.14) or, better, directly from the constraint from (2.5):
\[ \tilde{R}^{ab}_{\ cd} = \frac{1}{2} \epsilon^{ab}_{\ ef} R^{ef}_{\ cd}. \] (3.9)
At first sight we have a problem, since \( R^{ef}_{\ cd} \) depends on \( \omega = \tilde{\omega} + \omega_T \), and \( \tilde{\omega} \) is not yet known. However, it turns out that the linearized dual Ricci tensor does not depend on \( \tilde{\omega} \), since \( \epsilon^{ab}_{\ ef} R^{ef}_{\ cd}(\tilde{\omega}) = 0 \) identically. This allows us to construct
\[ \tilde{R}^a_{\ c\ b} = \tilde{R}^{ab}_{\ cd} = \frac{1}{2} \epsilon^{ab}_{\ ef} R^{ef}_{\ cd}(\omega_T) \] (3.10)
with (3.8) for \( \omega_T \), and using (3.2) in order to rewrite spacial derivatives. The result is
\[ \tilde{R}^a_{\ b\ c} = \kappa \left( \tilde{T}^a_{\ b\ c} - \frac{1}{2} \delta^a_{\ bc} \tilde{R}^{cd} \right) \] (3.11)
which seems to coincide with the standard Einstein equations.
However, \( \tilde{R}^{ab}_{\ \mu\ \nu} \) is the Riemann–Cartan tensor defined in terms of \( \tilde{\omega}^{ab}_{\ \mu} \), and coincides with the Riemann tensor \( \tilde{R}^{ab}_{\ \mu\ \nu}(\tilde{e}) \) only if the dual connection \( \tilde{\omega}^{ab}_{\ \mu} \) is torsionless, i.e. if
\[ \tilde{\omega}^{ab}_{\ \mu} = \Omega^{ab}_{\ \mu}(\tilde{e}) \] (3.12)
with \( \Omega^{ab}_{\ \mu}(\tilde{e}) \) as in (3.7). In order to study this question, we have to construct all components of \( \tilde{\omega}^{ab}_{\ \mu} \) from equation (2.14), and subsequently \( \tilde{\omega}^{0}_{\ i} \) from (2.19). The result is that (3.12) holds indeed, provided
(i) the original Ricci tensor \( R^{ab}_{\ \mu\ \nu}(\omega = \tilde{\omega} + \omega_T) \) satisfies
\[ R^{0\ i}_{\ j} = 0, \quad \partial_0 R^{i\ j}_0 = 0; \] (3.13)
(ii) the off-diagonal components \( \tilde{\omega}^{0}_{\ i} \) of the dual vierbein, that are required to be \( t \)-independent from equation (2.21), satisfy
\[ \partial_t \tilde{\omega}^{0}_{\ i} = 0. \] (3.14)
(Actually, we could have imposed \( n_a \tilde{\omega}^{0}_{\ i} = n_a \delta^0_{\ i} \), i.e. \( \tilde{\omega}^{0}_{\ i} = 0 \), from the beginning).
Equations (3.13) follow from the not yet considered equation (2.23) if \( T_{M\ a}^\mu = 0 \), but they also allow for a cosmological constant \( T_{M\ a}^\mu = \delta^a_\mu \Lambda \) in the ‘standard’ matter Lagrangian.
Note finally that for a vanishing dual stress energy tensor, equations (3.8) above imply vanishing torsion for the standard spin connection, and the linearized equations (2.23) can be interpreted as standard torsionless Einstein equations.
To summarize, we have learned about two important features of the present model in this chapter:
(i) the possibility of reproducing the linearized Einstein equations including matter for the dual gravitational fields, in spite of the Lorentz-non-covariant action,
(ii) the crucial role played by torsion within the present first-order formalism: ‘standard’ torsion is induced by ‘dual’ matter (and vice versa); this allows us to generalize the known correspondence between vacuum equations of motion and Bianchi identities [1–7] to equations of motion with sources, a possibility already advocated (in a cosmological context) in [22].
4. Cosmological solutions

In this section we return to the full nonlinear equations of motion of section 2 and study cosmological solutions. The aim is to check under which circumstances the standard Freedman–Robertson–Walker (FRW) equations are reproduced, but also to see whether the model could provide a hint for a solution of the cosmological constant problem (CCP). A solution to the CCP would correspond to a (stable) solution of the equations of motion where \( \Lambda \) and/or \( \tilde{\Lambda} \) are non-vanishing, but the standard and/or dual metric remains nearly time independent (does not explode exponentially with cosmic time). Indeed we will find such solutions for the dual metric, with non-vanishing \( \tilde{\Lambda} \) and \( \tilde{\Lambda} \), below.

First we have to make a general ansatz for all fields and Lagrange multipliers of the model, that is consistent with a homogeneous and isotropic universe (i.e. depend on \( x^0 = t \) only).

For the Lorentz vector \( n^\mu \) we will make the same choice as in equation (3.1). Note that we do not have enough gauge symmetries in order to gauge the component \( \tilde{\epsilon}_{00} \) of the standard vierbein to 1, whereas the component \( \tilde{\epsilon}_{00} \) of the dual vierbein is constraint to be constant (that can be chosen as 1) by equation (2.21). Then the most general ansatz is as follows (where \( a, b, r, s, \tilde{a}, \tilde{r}, \tilde{s}, \ell (i) \) are functions of \( t \)):

\[
\begin{align*}
 e^a_{\mu} &= \text{diag}(b, a, a, a) \\
 \omega^{ab}_{\mu} : \omega^{00} &= r \delta^0_j, \omega_0^j = s \epsilon^0_j \\
 \tilde{e}^a_{\mu} &= \text{diag}(1, \tilde{a}, \tilde{a}, \tilde{a}) \\
 \tilde{\omega}^{ab}_{\mu} : \tilde{\omega}^{00} &= \tilde{r} \delta^0_j, \tilde{\omega}_0^j = \tilde{s} \epsilon^0_j \\
 L^{(1)}_{ab} : L^{(1)}_{00} &= \epsilon^{(1)j} \delta^0_j, L^{(1)}_{ij} = \epsilon^{(1)j} \epsilon^{i} \\
 L^{(2)}_{ab} : L^{(2)}_{00} &= \epsilon^{(2)j} \delta^0_j, L^{(2)}_{ij} = \epsilon^{(2)j} \epsilon^{i} 
\end{align*}
\]

(concerning the Lagrange multipliers, we have taken care of the constraints (2.15) and (2.20)).

Plugging these ansätze into the equations (and constraints) of section 2, we obtain from equation (2.19) (where dots denote time derivatives):

\[
\dot{a} = -\tilde{r},
\]

from (2.14) with \( (abc) = (i0j) \) and \( (ijk) \):

\[
\begin{align*}
 \ddot{a} &= -a^2 \tilde{s}, \\
 \ddot{\tilde{a}} &= -a^2 \tilde{r}.
\end{align*}
\]  \( \quad (4.3a) \) \( \quad (4.3b) \)

from (2.31) with \( (ab) = (i0) \) and \( (ij) \):

\[
\begin{align*}
 \dot{\ell}^{(2)} &= 4a \partial_0 (\epsilon^{(1)2}) \\
 2a \partial_0 (\ell^{(1)} a^2) &= 0. 
\end{align*}
\]  \( \quad (4.4a) \) \( \quad (4.4b) \)

These equations are used to simplify some of the equations below, notably to eliminate \( \ell^{(2)} \).

For the stress energy tensor \( T_{M0} \) we will make, to start with, the general ansätze

\[
T_{M0} = \delta_j^i T_{M,S},
\]

and for \( \tilde{T}_{M0} \),

\[
\tilde{T}_{M0} = \frac{a^2}{\tilde{a}} \tilde{T}_{M,S}.
\]

(4.5)

(4.6)

Note that \( \tilde{T}_{M0} \) does not contribute to equation (2.29). As stated before, this does not constitute a paradox, since \( \tilde{T}_{M0} \) has to be determined by the conservation law that assumes, in the present context, the form

\[
\dot{\tilde{T}}_{M0} + 3 \frac{a}{\tilde{a}} (\tilde{T}_{M0} - \tilde{T}_{M,S}) = 0.
\]  \( \quad (4.7) \)
Next equation (2.23) gives, for \((ag) = (00)\) and \((ij)\),
\[
3(\rho^2 - s^2) = \kappa a^2 T_{M00},
\]
\[
-2\dot{r} + \frac{b}{a}(\rho^2 - s^2) + 4\kappa (\ell^{(1)} \dot{s} - \tilde{\ell}^{(1)} \dot{r}) = \kappa ab T_{M,S}. \tag{4.8b}
\]

Equation (2.27) gives, for \((ab\mu) = (0ij)\) and \((ijk)\) (the other index combinations just give \(0 = 0\)):
\[
r + \frac{\dot{a}}{b} = -\frac{\kappa}{ab} \partial_\mu (\ell^{(1)} a^2),
\]
\[
s = \frac{\kappa}{ab} \partial_\nu (\ell^{(1)} a^2). \tag{4.9b}
\]

Finally equation (2.29) gives
\[
4\tilde{g}(\ell^{(1)} \dot{r} - \tilde{\ell}^{(1)} \dot{s}) - 2\partial_\mu \partial_\nu (\ell^{(1)} \dot{a}^2) = -\tilde{a}^2 \tilde{T}_{M,S}. \tag{4.10}
\]

No further equations can be derived, and we are left with indeed ten equations (4.2)–(4.4), (4.8)–(4.10) for ten functions in the ansatz (4.1).

First it can be checked that, in the absence of dual matter \((\tilde{T}_{M,S} = 0\) in equation (4.10)), we can put \(s = \ell^{(1)} = \tilde{\ell}^{(1)} = 0\) (and \(\tilde{a} = \text{const.}\)) and equations (4.8) collapse to (using (4.9a))
\[
\frac{3}{a^2 b^2} = \kappa T_{M00}, \tag{4.11a}
\]
\[
\frac{2}{ab} \frac{\dot{a}}{a^2 b^2} + \frac{a^2}{a^2 b^2} - 2\frac{\dot{a}b}{ab^3} = \kappa T_{M,S}. \tag{4.11b}
\]

These equations are invariant under timelike diffeomorphisms that allow the gauge \(b(t) = 1\), after which they turn into the standard FRW equations that enforce the conservation of the standard stress energy tensor.

Next we have analysed the system for arbitrary standard cosmological constant,
\[
T_{M00} = T_{M,S} = \Lambda, \tag{4.12}
\]
and arbitrary dual cosmological constant,
\[
\tilde{T}_{M,S} = \tilde{\Lambda}. \tag{4.13}
\]

The system of equations can be reduced by eliminating \(\tilde{\ell}\) using (4.2), \(\tilde{s}\) using (4.3b), and introducing
\[
\tilde{\ell}^{(1)} = \ell^{(1)} \tilde{a}^2 = \text{const.} \tag{4.14}
\]
that solves (4.4b). Next, \(r\) can be eliminated using (4.8a), and \(b\) using (4.9b). We are left with four equations for \(a, \tilde{a}, \tilde{s}\) and \(\tilde{\ell}^{(1)}\), where the maximal time derivatives are \(\dot{a}, \ddot{a}, \dot{s}\) and \(\dddot{\ell}^{(1)}\):
\[
\dot{a} = \frac{1}{2\ell^{(1)}} \left( \frac{\dot{b}}{\kappa} - a \dot{\ell}^{(1)} \right),
\]
\[
\ddot{a} = \frac{1}{\ell^{(1)} a^2} \left( \dot{r} a^2 \left( \frac{1}{2\kappa} + \frac{\ell^{(1)}}{a^2} \right) + \frac{a^3 b}{6} \Lambda \right),
\]
\[
\dot{s} = \frac{\tilde{a}^2}{a^2} \ddot{a},
\]
\[
\dddot{\ell}^{(1)} = -2\frac{a^2}{\tilde{a}^3} \left( \frac{1}{\kappa} + \frac{\ell^{(1)}}{a^2} \right) - 4\ell^{(1)} \frac{\ddot{a}}{a} - 2\ell^{(1)} \frac{\dot{a}^2}{a^2} - \frac{2a^3 b}{3\tilde{a}^3} \Lambda + \frac{1}{2} \tilde{\Lambda}.
\]
where one has to replace

\[ b = -\frac{\dot{a}}{r} + \frac{2\kappa \ell^{(1)}}{r a^3} (a\ddot{a} - \dot{a} \dddot{a}), \quad (4.16a) \]

\[ r = \pm \sqrt{s^2 + \frac{2}{3} a^2 \Lambda}. \quad (4.16b) \]

This system can be brought into normal form (i.e. be solved for \( \dot{\tilde{a}}, \dddot{\tilde{a}}, \dot{s} \) and \( \dddot{\ell}^{(1)} \) after the replacements (4.16)) which is suitable for analytic and numerical stability analyses.

Remarkably we found for a wide range of initial conditions, and arbitrary cosmological constants \( \Lambda \) and \( \tilde{\Lambda} \) (and \( \ell^{(1)} \)), an asymptotically stable (constant) solution for the dual scale factor \( \tilde{a} \):

\[ \tilde{a} \rightarrow \tilde{a}_0 \quad a \rightarrow a_0/t \quad s \rightarrow s_0 \quad \ell^{(1)} \rightarrow t^2 \tilde{\Lambda}/4, \quad (4.17) \]

where the constants \( \tilde{a}_0, a_0 \) and \( s_0 \) depend on the initial values. Another stable solution is given by

\[ \tilde{a} \rightarrow t \tilde{a}_0 \quad a \rightarrow a_0/t \quad s \rightarrow s_0 \quad \ell^{(1)} \rightarrow t^2 \tilde{\Lambda}/24 \quad (4.18) \]

that is separated from (4.17) mainly through the initial values for \( \tilde{a} \) and \( \dot{\ell}^{(1)} \). Note that the \( \tilde{\Lambda} \)-dependent values of \( \ell^{(1)} \) are assumed dynamically.

Provided that we identify our known matter with the dual matter of the model, these solutions come very close to a solution of the CCP, since the dual scale factor \( \tilde{a} \) remains asymptotically constant (or increases just linearly in \( t \)) for arbitrary \( \tilde{\Lambda} \), without fine tuned initial conditions.

Let us compare the above solutions to the equations of the previous section 3, in particular to the points (i) to (v) just before equation (3.1), in order to investigate at which point the impact of \( \tilde{\Lambda} \) on the dual scale factor \( \tilde{a} \) gets lost. First, the impact of \( \tilde{\Lambda} \) on \( L^{(1)}_{ab,c} \) in the form of \( \dot{\ell}^{(1)} \) is evident. The crucial equation is equation (4.9b), where the impact of \( \dot{\ell}^{(1)} \sim t^2 \) on \( s \) (which corresponds to torsion \( \omega^T \)) is cancelled through the decay of \( a(t) \sim 1/t \), up to subleading terms that allow for \( s \rightarrow s_0 \). (Remarkably this does not imply that the ‘original’ universe is contracting: from (4.16a) one obtains \( b(t) \sim b_0/t^2 \), hence \( t \) does not correspond to the cosmological time in this universe. Its cosmological time is rather given by \( t' = -b_0/t \), in terms if which the signs of all components of \( e^\mu_\nu \) change and \( a(t') \) increases as \( \left| a(t') \right| \sim \frac{a_0}{b_0} t' \). Note furthermore that increasing \( \left| t' \right| \) corresponds to decreasing \( \left| t \right| \), i.e. the relative arrows of time are reversed.)

However, before this can be considered as a fully acceptable solution of the CCP, the following tasks have to be performed:

(a) dual matter has to be added in order to check, whether the known part of the evolution of our universe can be reproduced in the dual sector,

(b) a weak field expansion around such a solution has to be performed in order to see whether the (linearized) Einstein equations for the dual metric do not deviate too much from its standard form.

We have performed a preliminary analysis in this direction by adding a cosmological perturbation \( \Delta \tilde{a} \) to \( \tilde{a} = \Lambda \). In the case of the validity of the standard Einstein equations this perturbation should induce a perturbation

\[ 2\dddot{a} = \tilde{k} \Delta \tilde{a}. \quad (4.19) \]
of the dual scale factor $\tilde{a}$ (cf (4.11b)). The good news is that, neglecting terms of relative order $\sim t^{-1}$, the induced perturbation of $\tilde{a}$ can indeed be written in the form (4.19). The bad news is that in both cases $\tilde{k}$ is time dependent as $\tilde{k} \sim t^{-1}$ in the case of (4.17), even $\tilde{k} \sim t^{-3}$ in the case of (4.18). (A similar problem persists for the ‘original’ universe, which differs also from a de Sitter universe for arbitrary $\Lambda$, unless $\tilde{\Lambda} = 0$ and fine tuned initial conditions $s = \tilde{a} = \ell^{(i)} = 0$ are used, i.e. the CCP seems to be solved also here: however, the $t^2$ dependent value for $\ell^{(i)}$, plugged into the equations of motion (2.23), shows that the effective gravitational coupling for the ‘original’ universe—its dependence on the Lorentz vector $n^d$—is also time dependent.) Such time-dependent gravitational couplings seem to be in conflict with the cosmological standard model, hence further studies of possible modifications of the model are required.

5. Summary and outlook

We have constructed an action including dual gravitational fields, without adding new degrees of freedom. An amazing feature of the model is that some kind of matter can be coupled to the original gravitational fields (and ‘see’ a spacetime geometry described by the original metric), whereas another kind of matter can be coupled to the dual metric and, hence, propagate in a generally different spacetime geometry. In spite of the manifest breaking of Lorentz symmetry in the action, the reactions of the two different spacetime geometries to the corresponding stress energy tensors coincide with Einstein’s gravity in both cases, to lowest order in a weak field expansion around Minkowski spacetime.

Beyond lowest order, this phenomenon persists only for the standard gravitational fields (in the absence of dual matter): whereas fluctuating standard gravitational fields imply fluctuating dual gravitational fields from equations (2.14) and (2.19), a vanishing $\tilde{T}_{\mu \nu}$ allows for vanishing $L^{(1)}$ and $L^{(2)}$ from (2.29) and (2.31), as a consequence of which equations (2.23) and (2.27) turn into the standard equations for the standard vierbein and connection.

The opposite statement is not true: an attempt to integrate out the standard gravitational fields (even for $\tilde{T}_{\mu \nu} = 0$) will generate non-local effective interactions for the dual fields, that are moreover expected to break Lorentz symmetry manifestly in the form of (positive) powers of $\partial^2 \Box$. Hence it is an open question up to now, whether our world could be identified with the dual sector of the model.

This question is not purely academic, since the cosmological evolution of the dual sector differs dramatically from standard cosmology: A cosmological constant $\tilde{\Lambda}$ does not imply an exponential increase (with cosmic time) of the dual scale factor $\tilde{a}$, but can be ‘absorbed’ completely into a time dependence of the original scale factor. As stated at the end of section 4, such a solution of the cosmological constant problem requires further investigations.

Furthermore, various ways to generalize the model could be studied:

(a) As stated near the end of section 2, the fixed Lorentz vector $n^d$ could be replaced by a field $n^d(x)$ that breaks Lorentz symmetry spontaneously. The present approach to gravitational $S$-duality would then be similar in spirit to the PST approach [23] (reviewed in [11]), that has been applied to $d = 10/11$ supergravities.

(b) In higher dimensional, e.g. five dimensional, spacetimes the dual of $R^{ab}_{\mu \nu}$ would be, after contracting the 5D epsilon tensor with the Lorentz indices, a tensor $\tilde{R}^{abc}_{\mu \nu}$ that can be interpreted as a field strength of a field $\tilde{D}^{abc}_{\mu}$ (antisymmetric in $[abc]$). As before, this field could be fixed completely by an axial gauge condition and the duality constraint contracted with $n^\mu = n^5$. Subsequently the field strength of $\tilde{D}^{ab}_{\mu}$ can be obtained from $n^\nu \tilde{R}^{abc}_{\mu \nu \nu}$, and $\tilde{D}^{\mu}_{\mu}$ from the torsion tensor contracted with $n^\nu$ as before (imposing the
axial gauge condition on all these fields). A systematic procedure for arbitrarily high-dimensional spacetimes could be developed along these lines, and it would be interesting to study the duals of brane universes in such models. (The 4D metric that is S-dual to 2-brane universes could be investigated already in the present model).

(c) A weak point of the present model is that, in spite of the introduction of dual gravitational fields, we did not manage to make additional symmetries (as of the Ehler’s type [24]) manifest, even after dimensional reduction along a coordinate along $n^\mu$. The technical problem here is that the standard $t$ independent dimensional reduction ansatz for the metric does not coincide with our axial gauge conditions on $\tilde{e}_\mu^a$ and $\tilde{\omega}_\mu^{ab}$.

It may then be advisable to carry out the essential steps of the concept presented here—the introduction of dual gravitational fields together with non-covariant gauge conditions and partial (non-covariant) duality constraints—in terms of different variables as nonlinear realizations of gravity [25], or to give up the axial gauge condition on $\tilde{e}_\mu^a$ (which would complicate the analysis considerably, once $n^\mu$ can no longer be assumed to be constant).

In view of the interesting properties of the present model we believe that these various open questions merit corresponding studies.

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