Külshammer ideals of graded categories and Hochschild cohomology.

Yury Volkov and Alexandra Zvonareva

Abstract

We generalize the notion of Külshammer ideals to the setting of a graded category. This allows us to define and study some properties of Külshammer type ideals in the graded center of a triangulated category and in the Hochschild cohomology of an algebra, providing new derived invariants. Further properties of Külshammer ideals are studied in the case where the category is $d$-Calabi-Yau.

1 Introduction

Let $\Lambda$ be a symmetric algebra over a field of positive characteristic $p$ with a symmetrizing form $(-,-)$. The sequence of Külshammer ideals

$$K^d_r = \{a \in \Lambda | (a, b) = 0 \text{ for } b \in \Lambda \text{ such that } b^{p^r} \in [\Lambda, \Lambda]\}$$

in the center of $\Lambda$ is a fine invariant of the derived category of an algebra [20, 11]. These ideals were applied to distinguish various algebras up to derived equivalence [8, 9]. With the use of trivial extensions the definition of Külshammer ideals was extended to arbitrary algebras [3]. Also various attempts to generalize Külshammer ideals to higher Hochschild (co)homology were taken (see [22, 23]).

In this paper we propose to consider the same type of ideals in the center of a graded category $A$. These ideals are defined using the module structure of the graded abelianization of the category $A$ over its graded center. For a graded category $A$ the center of $A$ is the graded $k$-algebra $A^A$, whose $i$-th component $A^A_i$ is formed by elements $(m_{i,x})_{x \in A} \in \Pi_{x \in A} \text{A}(x,x)$, such that $f m_{i,x} = (-1)^{ij} m_{i,y} f$ for any $x,y \in A$ and any $f \in \text{A}(x,y)$. The abelianization of $A$ is the graded $k$-module $A_A = \oplus_{x \in A} \text{A}_x^x/[\text{A}, \text{A}]$, where $[\text{A}, \text{A}]$ denotes the subspace of $\oplus_{x \in A} \text{A}(x,x)$ formed by the elements $fu - (-1)^{ij} uf$, for all $x,y \in A$, $u \in \text{A}(y,x)$ and $f \in \text{A}(y,x)$.

The ideals $K_{r,s}A$ are defined as the annihilators of the appropriate homogeneous component of the kernel of the map $(-)^{p^r}$. For precise definitions see Sections 2 and 3. It turns out that the ideals defined in this way are invariant under graded equivalences.

In this construction is then applied in the particular situation of a category $A$ with an automorphism $\Sigma$. In this case the orbit category $A/\Sigma$ is graded and we can consider ideals
in the graded center of $A/\Sigma$. If $A$ is d-Calabi-Yau, we establish a duality between the Hochshild-Mitchel homology and cohomology of $A$. This generalizes the well known duality between Hochshild homology and cohomology for symmetric algebras. Using the pairing provided by this duality we recover the usual definition of the ideals $K_{r,d}(A/\Sigma)$ in this general context. Thus,

$$K_{r,d}(A/\Sigma) = \{ a \in (A/\Sigma)^{A/\Sigma} | (a,b) = 0 \text{ for } b \in \oplus_{x \in A}(A/\Sigma)_x \text{ such that } b^{p^r} \in [A/\Sigma, A/\Sigma] \}.$$ 

If $A$ is the homotopy category of complexes of finitely generated projective modules $K_p^b\Lambda$ over some algebra $\Lambda$ and $[1]$ is the shift functor, then $(K_p^b\Lambda/[1])^{K_p^b\Lambda/[1]}$ is the graded center $Z^*(K_p^b\Lambda)$ of $K_p^b\Lambda$, i.e. the graded ring of natural transformations from the identity functor to $[n]$ which commute modulo the sign $(-1)^n$ with $[1]$. Graded centers of triangulated categories are studied and used by various authors and attract some interest (see [4, 15, 2, 12]). Thus, the ideals defined for $K_p^b\Lambda/[1]$ belong to the graded center of $K_p^b\Lambda$.

The characteristic map from the Hochshild cohomology of $\Lambda$ to the graded center of $K_p^b\Lambda$ allows us to define Küchhammer ideals $HK_{r,s}(A)$ in the higher Hochshild cohomology as the inverse image of the ideals $K_{r,s}(K_p^b\Lambda/[1])$. If $\Lambda$ is a symmetric algebra, then $HK_{r,0}(\Lambda)$ coincide with the classical Küchhammer ideals $K_{r}^d$. So the ideals $HK_{r,s}(\Lambda)$ can be considered as a generalization of Küchhammer ideals to higher Hochshild cohomology. Both ideals $K_{r,s}(K_p^b\Lambda/[1])$ and $HK_{r,s}(\Lambda)$ are invariant under derived equivalence.

For arbitrary algebras we provide an alternative description of $HK_{r,0}(\Lambda)$.

$$HK_{r,0}^0(\Lambda) = \{ a \in Z(\Lambda) | ab \in [\Lambda, \Lambda] \text{ for } b \text{ such that } b^{p^r} \in [\Lambda, \Lambda] \}.$$ 

This provides an alternative generalization of Küchhammer ideals for non-symmetric algebras.

We finish the paper computing all the defined ideals for the algebra $k[x]/x^2$.

## 2 Graded center and abelianization

All categories and functors are assumed to be $k$-linear for some fixed field $k$. Moreover, all categories are assumed to be small. We write simply $\otimes$ and $\text{Hom}$ instead of $\otimes_k$ and $\text{Hom}_k$.

In this section we recall some definitions and introduce some notation. From here on we will write $A^y_x$ instead of $\text{Hom}_A(x, y)$ for Hom-sets in a category $A$.

**Definition 2.1.** A category $A$ is called a graded category if, for any $x$, $y$ in $A$, there is a fixed decomposition of graded spaces $A^y_x = \oplus_{i \in \mathbb{Z}}(A^y_x)_i$ such that $fg \in (A^y_x)_{i+j}$ for any $f \in (A^y_x)_j$ and $g \in (A^y_x)_j$. We write $|f|$ for the degree of $f \in A^y_x$, i.e. $|f| = i$ if and only if $f \in (A^y_x)_i$.

**Definition 2.2.** The tensor product $A \otimes B$ of categories $A$ and $B$ is a $k$-linear category defined in the following way. Its objects are pairs $(x, y)$ where $x \in A$, $y \in B$. Its morphism spaces are

$$(A \otimes B)_{(x_1, y_1)}^{(x_2, y_2)} = A_{x_1}^{x_2} \otimes B_{y_1}^{y_2}.$$ 

The composition in $A \otimes B$ is given by the formula

$$(f_2 \otimes g_2)(f_1 \otimes g_1) = f_2f_1 \otimes g_2g_1,$$

where $f_1 \in A_{x_1}^{x_2} , f_2 \in A_{x_2}^{x_3} , g_1 \in B_{y_1}^{y_2} , g_2 \in B_{y_2}^{y_3}$ and $x_1, x_2, x_3 \in A, y_1, y_2, y_3 \in B$. 

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Definition 2.3. A \( k \)-linear contravariant functors from \( A \) to \( \text{Mod-}k \) are called \( A \)-modules. We denote by \( \text{Mod-}A \) the category of \( A \)-modules. An \( A \)-bimodule is by definition an \( A^{op} \Box A \)-module. We denote by \( \text{Bimod-}A \) the category of \( A \)-bimodules.

If it does not cause any confusion, we will write \( fug \) instead of \( M(f \otimes g)u \) for an \( A \)-bimodule \( M \) and \( g \in A^w_x, f \in A^y_z, u \in M(x, w) \), where \( w, x, y, z \in A \).

If \( A \) is graded, then the \( A \)-bimodule \( M \) is called \textit{graded} if there are some fixed decompositions \( M(x, y) = \bigoplus_{i \in \mathbb{Z}} M(x, y)_i \) such that \( fug \in M(y, z)_{i+j+l} \) for all \( g \in (A^w_x)_j, f \in (A^y_z)_l \), and \( u \in M(x, w)_i \). A homogeneous morphism of degree \( m \) from a graded bimodule \( M \) to a graded bimodule \( N \) is a collection of maps \( \phi_{x,y,k} : M(x, y)_k \rightarrow N(x, y)_{k+m} \) such that \( N(f \otimes g)\phi_{x,w,k} = (-1)^m \phi_{y,z,k+l}M(f \otimes g) \) for \( g \in (A^w_x)_j, f \in (A^y_z)_l \). A morphism between graded bimodules is by definition a finite sum of homogeneous morphisms. Any nongraded category \( A \) can be considered as a graded category with all morphisms of degree 0. In this case a graded bimodule is simply a module \( M \) with \( A \)-bimodule decomposition \( M = \bigoplus_{i \in \mathbb{Z}} M_i \). We define the graded bimodule \( M[n] \) as a graded \( A \)-bimodule such that \( M[n](x, y)_i = M(x, y)_{i+n} \) and \( M[n](f \otimes g)u = (-1)^m M(f \otimes g)u \) for all \( g \in (A^w_x)_j, f \in (A^y_z)_l \), and \( u \in M(x, w)_i \). If \( \phi_{x,y,j} \) is a homogeneous morphism form \( M \) to \( N \), then \( \phi_{x,y,j}[n] := \phi_{x,y,j} + n \).

One can consider \( A \) as an \( A \)-bimodule defined by the equality \( A(x, y) = A^y_x \) on objects and in the obvious way on morphisms.

Definition 2.4. An \( A \)-linear category is an \( A \)-bimodule \( M \) together with a structure of a \( k \)-linear category (which will be also denoted by \( M \)) compatible with the bimodule structure. Namely, the class of objects of \( M \) is the same as the class of objects of \( A \), and the morphism spaces of \( M \) are \( M^x_y = M(x, y) \). Thus, there are bilinear maps

\[
M(y, x) \times M(x, z) = M^v_x \times M^x_z \xrightarrow{-\circ-} M^y_z = M(y, z)
\]

that satisfy all the conditions of a categorical composition. The compatibility conditions are

\[
f(v \circ u)g = (fv) \circ (ug) \text{ and } u \circ (fv) = (uf) \circ v,
\]

\[
\begin{array}{ccc}
M(y, x) \times M(x, z) \xrightarrow{-\circ-} M(y, z) & & \xrightarrow{-\circ-} M_z \times M (y, x) \\
\downarrow M(f \otimes id) \times M(id \otimes g) & & \downarrow M(f \otimes id) \\
M(w, x) \times M(x, \tau) \xrightarrow{-\circ-} M(w, \tau) & & \xrightarrow{-\circ-} M(z, w) \times M(y, x)
\end{array}
\]

where \( f \in A^w_x \) and \( g \in A^z_x \) are morphisms in \( A \) and \( u \) and \( v \) are morphisms in \( M \).

We say that an \( A \)-linear category \( M \) is graded if \( M \) is a graded category and a graded bimodule with respect to the same family of decompositions \( M^x_y = \bigoplus_{i \in \mathbb{Z}} (M^x_y)_i \).

Definition 2.5. Let \( A \) be a graded category and let \( M \) be a graded \( A \)-bimodule. The \textbf{\( A \)-center} of \( M \) is the graded \( k \)-module \( \text{Ass} \), whose \( i \)-th component \( \text{Ass}^i \) is formed by such elements \( (m_{i,x})_{x \in A} \in \Pi_{x \in A} M(x, x) \), that \( fm_{i,x} = (-1)^{ij}m_{i,y}f \) for any \( x, y \in A \) and any \( f \in (A^y_z)_j \). If \( A \) and \( M \) are not graded, then we can consider them as a graded category and a bimodule concentrated in degree 0. So we can talk about the center of a nongraded bimodule over a nongraded category.
Note that Definition [2.4] guarantees that if M is a A-linear category, then $M^A$ has a structure of a unital associative $k$-algebra, with multiplication induced by the composition in M. In particular, the compatibility of the the bimodule and the categorical structure ensures that $M^A$ is closed under multiplication.

**Definition 2.6.** Let A be a (nongraded) category and $\Sigma : A \to A$ be an automorphism of A. The orbit category $A/\Sigma$ is a graded category defined as follows.

- The class of objects of $A/\Sigma$ is equal to that of A;
- The sets of morphisms are $((A/\Sigma)^x) = A^{xy}_x$ for $x, y \in A/\Sigma$ and $n \in \mathbb{Z}$;
- The composition $\circ$ in $A/\Sigma$ is given by the formula $g \circ f = \Sigma^n(g)f$ for $f \in ((A/\Sigma)^y)_n$ and $g \in ((A/\Sigma)^x)_m$.

Note that $A/\Sigma$ becomes a graded A-linear category if we define $(A/\Sigma)(f \otimes g)u = \Sigma^n(f)ug$ for $f \in A^{y_1}_{x_1}$, $g \in A^{y_2}_{x_2}$ and $u \in ((A/\Sigma)^{y_2})_n$. Given an automorphism $\alpha$ of the category A and $M \in \operatorname{Bimod}-A$, we define the bimodule $\alpha M$ as the composition of functors $M \circ (\alpha \otimes \operatorname{Id}_A) \in \operatorname{Bimod}-A$. It is easy to see that this defines an action of $(\operatorname{Aut} A)^{op}$ on $\operatorname{Bimod}-A$. Note that $A/\Sigma \cong \oplus_{n \in \mathbb{Z}} (\Sigma^n A)[-n]$ as a graded A-bimodule.

If $\alpha$ is an automorphism of A, then we say that $\alpha$ acts on the graded bimodule M if there is a homogeneous isomorphism $M \xrightarrow{\alpha} M \circ (\alpha \otimes \operatorname{Id}_A)$ of degree 0. Such action induces an automorphism $\alpha_{M^A}$ of the graded center $M^A$. If M is an A-linear category and $\alpha$ acts on it by a category automorphism, then $\alpha_{M^A}$ is an automorphism of the graded algebra $M^A$. Note that if $\alpha$ acts on M by $\alpha_M$, then we can define $\alpha_{M[n]} = (-1)^n \alpha_M[n]$. The automorphism $\Sigma$ of A acts on $\Sigma^n A$ by the rule $\Sigma_{(\Sigma^n A)}(u) = \Sigma(u)$ for $u \in A^{xy}_x$. We fix this action of $\Sigma$ on $\Sigma^n A$ and write simply $\Sigma$ instead of $\Sigma_{(\Sigma^n A)}$. Moreover, for any integer $m$, we denote by $\Sigma$ the natural transformation $\Sigma_{(\Sigma^n A)[m]}$ and the automorphism $\Sigma_{(\Sigma^n A)[m]}^A$. Thus, $\Sigma$ acts on the bimodule $A/\Sigma$ and this action determines an automorphism of the graded A-linear category $A/\Sigma$. For any space V with an action of some automorphism $\alpha$ we can consider the subspace of invariants $V^\alpha$, i.e. $\{m \in V|\alpha(m) = m\}$.

**Definition 2.7.** Let A be a (nongraded) category and let $\Sigma : A \to A$ be some fixed automorphism of A. We define the graded rings $\operatorname{Nat}^*(A) = \operatorname{Nat}^*(A, A)$ and $Z^*(A) = Z^*(A, \Sigma)$ as follows. Let $\operatorname{Nat}^*(A)$ ($n \in \mathbb{Z}$) be the abelian group formed by all natural transformations $\eta : \operatorname{Id}_A \to \Sigma^n$ and $Z^*(A)$ be its subgroup formed by $\eta$ that satisfy the equality $\eta \Sigma = (-1)^n \eta \Sigma$. Given natural transformations $\eta : \operatorname{Id}_A \to \Sigma^n$ and $\theta : \operatorname{Id}_A \to \Sigma^m$, we define the product of $\eta$ and $\theta$ by the formula $\eta \theta = \Sigma^m(\eta) \circ \theta : \operatorname{Id}_A \to \Sigma^{n+m}$. We call $Z^*(A)$ the graded center of A. Note that $\eta \theta = (-1)^mn \theta \eta$ if $\eta \in Z^*(A)$.

**Remark 2.8.** The definition of a graded center has sense if $\Sigma$ is an autoequivalence, but further we need it to be an automorphism. On the other hand, we can replace any autoequivalence of a category by an automorphism (the category is changed during this process) by the results of [3].

**Lemma 2.9.** There is an isomorphism of graded algebras $\operatorname{Nat}^*(A) \cong (A/\Sigma)^A$ that induces isomorphisms $Z^*(A) \cong ((A/\Sigma)^A)_{\Sigma} \cong (A/\Sigma)^{A/\Sigma}$. 

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Definition 2.10. Given a graded A-bimodule M, \([A, M]\) denotes the subspace of \(\bigoplus_{x \in A} M^x\) formed by the elements \(fu - (-1)^{iy}uf\), for all \(x, y \in A, u \in M(y, x), \) and \(f \in (A^x_y)\). The A-abelianization of M is the graded \(k\)-module \(M_A = \bigoplus_{x \in A} M^x/[A, M]\). As in the case of the center, we can talk about the abelianization of a nongraded bimodule over a nongraded category.

For any space \(V\) with an action of some automorphism \(\alpha\) we can consider the space of co-invariants \(V_\alpha\), i.e. the quotient space of \(V\) modulo the subspace generated by the classes of the elements of the form \(v - \alpha(v)\) for \(v \in V\). If \(M\) is an \(A\)-linear category, then the composition in \(M\) induces a structure of a graded \(M^A\)-bimodule on \(M_A\). Note that an action of an automorphism on \(M\) induces an action on \(M_A\). So in this case one can define the graded space \((M_A)_\Sigma\) as the quotient space of \(M_A\) modulo the subspace generated by the classes of the elements of the form \(m - \Sigma m\) for \(m \in \bigoplus_{x \in A} M^x\). The \(M^A\)-bimodule structure on \(M_A\) induces the \((M^A)_\Sigma\)-bimodule structure on \((M_A)_\Sigma\). Note also that if the action of the automorphism \(\alpha\) of \(A\) on the bimodule \(M\) is given by \(\alpha_M\), then \(\alpha\) acts on \(M\) by \(a\alpha_M\) for any \(a \in k^*\) as well.

Definition 2.11. Let \(A\) be a (nongraded) category and let \(\Sigma : A \to A\) be some fixed automorphism of \(A\). We define the graded spaces \(\text{Nat}_\Sigma(A) = \text{Nat}_\Sigma(A, \Sigma)\) and \(\text{Ab}_\Sigma(A, \Sigma)\) as follows. Let \(\text{Nat}_\Sigma(A)\) be the quotient space of the space \(\bigoplus_{x \in A} A^x\Sigma^x\) modulo the subspace generated by the elements of the form \(fg - \Sigma^n(g)f\) for all \(f \in A^x\Sigma^y, g \in A^y\). We define \(\text{Ab}_\Sigma(A)\) as the quotient space of \(\text{Nat}_\Sigma(A)\) modulo the subspace generated by the classes of the elements of the form \(f + (-1)^n\Sigma(f)\) for all \(f \in A^x\Sigma^n\). We call \(\text{Ab}_\Sigma(A)\) the graded abelianization of \(A\).

It is easy to see that \(\text{Nat}_\Sigma(A)\) is a \(\text{Nat}_\Sigma(A)\)-bimodule. Moreover, the corresponding \(Z^*(A)\)-bimodule structure on \(\text{Nat}_\Sigma(A)\) induces a \(Z^*(A)\)-bimodule structure on \(\text{Ab}_\Sigma(A)\). Note also
that the isomorphisms from Lemma 2.9 induce a \( \text{Nat}^*(A) \)-bimodule structure on \((A/\Sigma)_{\Lambda}\) and a \( \mathbb{Z}^*(A) \)-bimodule structure on \((A/\Sigma)_{\Lambda/\Sigma}\) and \(((A/\Sigma)_{\Lambda})_{-\Sigma}\) (in fact, as it was noted above, on \((A/\Sigma)_{\Lambda}\) for any \( a \in \mathbb{k}^* \)).

**Lemma 2.12.** There is an isomorphism of graded \( \text{Nat}^*(A) \)-bimodules \( \text{Nat}_*(A) \cong (A/\Sigma)_{\Lambda} \) that induces isomorphisms \( \text{Ab}_*(A) \cong \left( (A/\Sigma)_{\Lambda} \right)_{-\Sigma} \cong (A/\Sigma)_{\Lambda/\Sigma} \).

**Proof.** By definition, \( \text{Nat}_*(A) \) and \( (A/\Sigma)_{\Lambda} \) are both quotients of the \( \text{Nat}^*(A) \)-bimodule \( \bigoplus_{n \in \mathbb{Z}} \bigoplus_{x \in \Lambda} A_x^{\Sigma^n y} \) and it suffices to check that we take the quotient space modulo the same submodule.

It is easy to see that

\[
f g - \Sigma^n(g)f = (A/\Sigma)(\text{Id}_y \otimes g)(f) - (A/\Sigma)(g \otimes \text{Id}_x)(f)
\]

for all \( f \in A_x^{\Sigma^n y} \) and \( g \in A_y^x \), i.e. \( \text{Nat}_n(A) = \left( (A/\Sigma)_{\Lambda} \right)_n = \bigoplus_{x \in \Lambda} A_x^{\Sigma^n y}/U_n \), where \( U_n \) is generated by \( f g - \Sigma^n(g)f \) for \( f \in A_x^{\Sigma^n y} \) and \( g \in A_y^x \).

Now we have \( \text{Ab}_n(A) = \bigoplus_{x \in \Lambda} A_x^{\Sigma^n y}/(U_n + V_n) \), where \( V_n \) is generated by \( f + (-1)^n \Sigma(f) \) for all \( f \in A_x^{\Sigma^n y} \). It is easy to see that \( \left( (A/\Sigma)_{\Lambda} \right)_{-\Sigma}^n = \bigoplus_{x \in \Lambda} A_x^{\Sigma^n y}/(U_n + V_n) \) too. Let now consider \( \left( (A/\Sigma)_{\Lambda/\Sigma} \right)_n = \bigoplus_{x \in \Lambda} A_x^{\Sigma^n y}/W_n \), where \( W_n \) is generated by \( \Sigma^j(f)g - (-1)^j \Sigma^j(g)f \) for \( f \in A_x^{\Sigma^n y} \) and \( g \in A_y^{\Sigma^j x} \), \( i + j = n \). Taking \( j = 0 \), we get that \( U_n \subseteq W_n \). Taking \( i = n + 1, j = -1 \), \( y = \Sigma^{-1} x \), and \( g = \text{Id}_x \), we get that \( V_n \subseteq W_n \). Thus, \( U_n + V_n \subseteq W_n \).

Since \( \Sigma^j(f)g - \Sigma^n(g)\Sigma^j(f) = \Sigma^j(g)f - (-1)^j \Sigma^j(\Sigma^j(g)f) \in V_n \), we have \( W_n = U_n + V_n \).

\( \square \)

## 3 Külshammer ideals in the center of a graded category

In this section we define Külshammer ideals in the center of a graded category. From now on we assume that \( \mathbb{k} \) is a field of characteristic \( p > 0 \). For a graded category \( A \) we define the map \( \xi_p : \bigoplus_{x \in \Lambda} A_x^p \rightarrow \bigoplus_{x \in \Lambda} A_x^p \)

by the equality \( \xi_p \left( \sum_{n \in \mathbb{Z}} f_n \right) = \sum_{n \in \mathbb{Z}} f_n^p \), where \( f_n \in \left( \bigoplus_{x \in \Lambda} A_x^p \right)_n \) for \( n \in \mathbb{Z} \). Note that \( \xi_p \) maps \( (A_x^n)_n \) to \( (A_x^n)_{np} \).

**Lemma 3.1.** Let \( A \) be a graded category. The map \( \xi_p : \bigoplus_{x \in \Lambda} A_x^p \rightarrow \bigoplus_{x \in \Lambda} A_x^p \) induces a well-definite map \( \xi_p : A_A \rightarrow A_A \).

**Proof.** It is enough to show that \( \xi_p ([A, A]_n) \subseteq [A, A]_{np} \). Let us first prove that for \( f, g \in \left( \bigoplus_{x \in \Lambda} A_x^p \right)_n \) one has \((f + g)^p - f^p - g^p \in [A, A]_{np} \). Fix \( s_i \in \left( \bigoplus_{x \in \Lambda} A_x^p \right)_n \) for \( 1 \leq i \leq p \). By
Here, for an algebra $\Lambda$, a $\Lambda$-bimodule $M$ and $V \subseteq \Lambda$. Since $(-1)^{n^2(p-1)} = 1$ if $2 \nmid p$ or $\text{char } K = p = 2$, we have $s_1 \ldots s_p - s_2 \ldots s_p s_1 \in [\Lambda, \Lambda]$. Now the condition $(f + g)^p - f^p - g^p \in [\Lambda, \Lambda]$ can be proved in the same way as in [21] Lemma 1.1.

Any element of $[\Lambda, \Lambda]_n$ has the form $u = \sum_{t=1}^{k} (f_t g_t - (-1)^{(n-t)} g_t f_t)$ for some integer $k \geq 0$ and some $f_t \in (\Lambda^{y_t})_{i_t}$ and $g_t \in (\Lambda^{y_t})_{n-i_t}$ for $1 \leq t \leq k$. Then we have

$$\xi_p(u) + [\Lambda, \Lambda]_{np} = \sum_{t=1}^{k} ((f_t g_t)^p - (-1)^{(n-t)}(g_t f_t)^p) + [\Lambda, \Lambda]_{np}$$

$$= \sum_{t=1}^{k} (f_t((g_t f_t)^{p-1} g_t) - (-1)^{(n-t)}((g_t f_t)^{p-1} g_t) f_t) + [\Lambda, \Lambda]_{np} \in [\Lambda, \Lambda]_{np},$$

i.e. $\xi_p([\Lambda, \Lambda]_n) \subseteq [\Lambda, \Lambda]_{np}$.

Definition 3.2. Let $A$ be a graded category. Then we define $T_s A \subseteq A_A$ as the kernel of the map $\xi^s_p$. Let us define the graded subspace $K_{r,s} A \subseteq A_A$ by the equality

$$(K_{r,s} A)_n = \text{Ann}_{A_A}((T_s A)_{s-n}).$$

Here, for an algebra $\Lambda$, a $\Lambda$-bimodule $M$ and $V \subseteq M$, we denote by $\text{Ann}_A(V)$ the set $\{a \in A \mid aV = V a = 0\}$. We will call $K_{r,s} A$ the $(r, s)$-th Kulshammer ideal of $A$ and $K_r A = \cap_{s \in \mathbb{Z}} K_{r,s} A = \text{Ann}_{A_A}(T_r A)$ the $r$-th Kulshammer ideal of $A$. Also we will call $R_s A = \cap_{r \geq 0} K_{r,s} (A)$ the $s$-th Reynolds ideal of $A$ and will call $\mathcal{R} A = \cap_{r \geq 0} K_{r,s} (A)$ the Reynolds ideal of $A$.

We say that $F : A \rightarrow A'$ is a degree preserving equivalence if $F$ induces an isomorphism of graded spaces $F : A^{x}_y \rightarrow (A')^{x'}_{y'}$ for any $x, y \in A$, and for any $x' \in A'$ there exists an isomorphism $\xi_{x'} \in A^{x'}_{x'}$ of degree zero for some $x, x' \in A$.

Theorem 3.3. If $A$ is a graded category and $s \in \mathbb{Z}$, then

$$A^A = K_{0,s} A \supseteq K_{1,s} A \supseteq \cdots \supseteq K_{r,s} A \supseteq \cdots \supseteq R_s A$$

is a decreasing sequence of graded ideals. In particular,

$$A^A = K_{0} A \supseteq K_{1} A \supseteq \cdots \supseteq K_{r} A \supseteq \cdots \supseteq \mathcal{R} A$$

is a decreasing sequence of graded ideals. Moreover, if $F : A \rightarrow A'$ is a degree preserving equivalence, then $F$ induces an isomorphism of graded algebras $\varphi_F : A^A \rightarrow (A')^{A'}$ such that $\varphi_F(K_{r,s} A) = K_{r,s} A'$ for any $r \geq 0, s \in \mathbb{Z}$.

Proof. It follows directly from the definition that $R_s A \subseteq K_{r+1,s} A \subseteq K_{r,s} A$ for all $r \geq 0$. Since $A^A$ is graded commutative, it is easy to see that $\bigoplus_{n \in \mathbb{Z}} (A^A_n \cap \text{Ann}_{A_A}(V_{s-n}))$ is a graded ideal of $A^A$ for any graded $A^A$-bimodule $M$ and any graded subbimodule $V \subseteq M$. For $f \in (A_A)_m$ such that $f^p = 0$, $\theta \in (A_A)_{d-m-n}$ we have $(\theta f)^p = (-1)^{m} \theta f^p = 0$, where $a = \frac{p(p^2-1)}{2} m(d - m - n)$, and hence $T_r A$ is a graded subbimodule of $A_A$. Thus, the first part of the theorem is proved.
Let us now prove the second part. We can choose a quasi inverse equivalence $F'$ for $F$ and natural isomorphisms $\varphi : \text{Id}_A \to FF'$ and $\beta : \text{Id}_{A'} \to FF'$ in such a way that $\alpha_x$ and $\beta_{x'}$ are of degree zero for all $x \in A$ and $x' \in A'$.

Suppose that $f = (f_x)_{x \in A} \in \Pi_{x \in A}(A^n_x)^n$ belongs to $A^n_n$. We define $\varphi_F(f)_{x'} = \beta_{x'}^{-1}F(f_{x'})_{x'}$. It is easy to see that $\varphi_F(f) \in (A^n_n)^{x'}$. It is clear also that $\varphi_F : A^n \to (A')^{n'}$ is a homomorphism of graded algebras. Let us define $\varphi_F^{-1} : (A')^{n'} \to A^n$ in the same way (using $\alpha$ instead of $\beta$). Take $f = (f_x)_{x \in A} \in A^n_n$. We have $f_{x'}F\alpha_x = \alpha_x f_x$ and $F'F(f_x)\alpha_x = \alpha_x f_x$ since $f$ belongs to the center and $\alpha$ is a natural transformation. Hence, $f_{x'}F\alpha_x = F'F(f_x)\alpha_x$. Thus,

$$\varphi_F \varphi_F(f)_x = \alpha_x^{-1}F'(\beta_{x'}^{-1}F(f_{x'})\beta_{x'})\alpha_x$$

$$= \alpha_x^{-1}F'(\beta_{x'}^{-1}FF'(f_x)\beta_{F_x})\alpha_x = \alpha_x^{-1}F'F(f_x)\alpha_x = f_x.$$

Thus, $\varphi_F \varphi_F = \text{Id}_{A^n}$. Analogously, $\varphi_F \varphi_F^- = \text{Id}_{(A')^{n'}}$. Consequently, $\varphi_F$ is an isomorphism.

Suppose now that $f = (f_x)_{x \in A} \in \Pi_{x \in A}(A^n_x)^n$ belongs to $(K_{r,s}A)_n$. Let us take some $u = (u_{x'})_{x' \in A'} \in (\oplus_{x' \in A'}(A^n_{x'})_{x'})_{x'}$ such that $u_{x'} \in [A', A']$. Let $(F'(x))^{-1}(x)$ be the inverse image of $x$, i.e. the set $\{x' \in A' \mid F'(x') = x\}$. We define $F'(u)_x = \sum_{x' \in (F'(x))^{-1}(x)} F'(u_{x'})$. By Lemma 3.1 we have $(F'(u)_x)' = \sum_{x' \in (F'(x))^{-1}(x)} F'(u_{x'})'$. Since $F'([A', A']) \subseteq [A, A]$, we see that $F'(u) \in T_{r,A}$. Then $f_{F'(u)} \in [A, A]$. As before, we have $f_{F'(u)}F'(u) \in [A', A']$, where $(F(f)F'(u)_{x'}) = \sum_{F'(u') = x'} F'(f_{F'(u')}F'(u')).$ Since

$$\sum_{y' \in A'} (F(f_{F'(y')}F'(u_{y'}) - \beta_{y'}^{-1}F(f_{F'(y')}F'(u_{y'})\beta_{y'})) \in [A', A']$$

and $F'(u_{y'})\beta_{y'} = \beta_{y'}u_{y'}$, we have

$$\varphi_F(f)u = F(f)F'(u) - \sum_{y' \in A'} (F(f_{F'(y')}F'(u_{y'}) - \beta_{y'}^{-1}F(f_{F'(y')}\beta_{y'}u_{y'})) \in [A', A'].$$

Consequently, $\varphi_F(f) \in (K_{r,s}A')_n$, i.e. $\varphi_F(K_{r,s}A) \subseteq K_{r,s}A'$. In the same way one can prove that $\varphi_F^{-1}(K_{r,s}A') = \varphi_F^{-1}(K_{r,s}A) \subseteq K_{r,s}A$, i.e. $\varphi_F(K_{r,s}A) = K_{r,s}A'$.

$\square$

**Remark 3.4.** Note that the map $\varphi_F$ constructed in the proof does not depend on $\beta$ and $F'$, i.e. it is really induced by $F$. Indeed, if $\beta, \tilde{\beta} : \text{Id}_{A'} \to FF'$ are two natural isomorphisms, then $\tilde{\beta}^{-1}$ is a natural isomorphism from $\text{Id}_A$ to itself and hence

$$(\tilde{\beta}^{-1})_{x'}F(f_{x'})_{x'} = F(f_{x'})_{x'}\tilde{\beta}_{x'}^{-1}\beta_{x'} = F(f_{x'})_{x'}\tilde{\beta}_{x'},$$

i.e. $\beta_{x'}^{-1}F(f_{x'})_{x'} = \tilde{\beta}_{x'}^{-1}F(f_{x'})_{x'}$. Now, if $F''$ is another quasi inverse of $F$, then there is a natural isomorphism $\gamma : F' \to F''$. Then $F(\gamma)\beta : \text{Id}_{A'} \to FF''$ is a natural isomorphism too and we have

$$(F(\gamma_{x'})_{x'})^{-1}F(f_{x'})_{x'}F(\gamma_{x'})_{x'} = \beta_{x'}^{-1}F(f_{x'}f_{x'}\gamma_{x'})_{x'}\beta_{x'} = \beta_{x'}^{-1}F(f_{x'})_{x'}\beta_{x'}$$

since $f \in A^n$. 

8
Corollary 3.5. If $A$ is a category with the automorphism $\Sigma$ and $s \in \mathbb{Z}$, then
$$Z^*(A) = K_{0,s}(A/\Sigma) \supseteq K_{1,s}(A/\Sigma) \supseteq \cdots \supseteq K_{r,s}(A/\Sigma) \supseteq \cdots \supseteq R_s(A/\Sigma)$$
is a decreasing sequence of graded ideals. In particular,
$$Z^*(A) = K_{0}(A/\Sigma) \supseteq K_{1}(A/\Sigma) \supseteq \cdots \supseteq K_{r}(A/\Sigma) \supseteq \cdots \supseteq R(A/\Sigma)$$
is a decreasing sequence of graded ideals. Moreover, if $A'$ with the automorphism $\Sigma'$ is another category and there is an equivalence $F : A \rightarrow A'$ such that $F \Sigma \sim \Sigma' F$, then $F$ induces an isomorphism of graded algebras $\varphi_F : Z^*(A) \rightarrow Z^*(A')$ such that $\varphi_F(K_r(A/\Sigma)) = K_r(A'/\Sigma')$.

Proof. Suppose that $\psi : \Sigma' \rightarrow F \Sigma$ is a natural isomorphism. It induces a natural isomorphism $\psi_n : (\Sigma')^n F \rightarrow F (\Sigma)^n$ for any $n \in \mathbb{Z}$. Then we define $F^*_\Sigma : (A/\Sigma) \rightarrow (A'/\Sigma')$ by the equality $F^*_\Sigma x = F x$ on objects and by the equality $F^*_\Sigma(f) = (\psi_n)^{-1} f(x) \in ((A'/\Sigma')^{\Sigma y}_n)^{\Sigma y}_n$ on morphisms $f \in ((A/\Sigma)^y_n) = A^{\Sigma y}_n$. It is easy to see that $F^*_\Sigma$ is a degree preserving equivalence. Now the stated result follows from Theorem 3.3 and Lemma 2.9.

\[\square\]

4 Calabi-Yau categories

In this section we recall the definition of a (weakly) Calabi-Yau category and establish some dualities arising for such categories. We will give an alternative definition of Külsammer ideals for a Calabi-Yau category and show that Külsammer ideals in such categories satisfy additional properties.

Definition 4.1. Let $A$ be a category with a fixed automorphism $\Sigma$ and let $d$ be some integer. The category $A$ is called $d$-Calabi-Yau if $A^x_y$ is finite dimensional for any $x, y \in A$ and there is a family of linear maps $tr_x : A^{\Sigma^d x}_x \rightarrow k$ ($x \in A$) such that

- the pairing $(\cdot, \cdot) : A^{\Sigma^d x}_x \times A^y_y \rightarrow k$ given by the formula $(f, g) = tr_x(fg)$ is nondegenerate, and

- for all $m \in \mathbb{Z}$, $g \in A^{\Sigma^m x}_x$, and $f \in A^{\Sigma^d-m x}_y$ one has
$$tr_x(\Sigma^{m}(f)g) = (-1)^{m(d-m)}tr_y(\Sigma^{d-m}(g)\Sigma^{m-m}(f)). \quad (4.1)$$

If the first condition is fulfilled and the second condition is true for $m = 0$, then $A$ is called a weakly $d$-Calabi-Yau category. Usually Calabi-Yau categories are assumed to be triangulated and $\Sigma$ is assumed to be the shift functor. For more details on Calabi-Yau categories see \cite{10}.

If $A$ is a (weakly) $d$-Calabi-Yau category, then we define the map $tr_A : \oplus_{x \in A, n \in \mathbb{Z}} A^{\Sigma^n x}_x \rightarrow k$ by the equality
$$tr_A|_{A^{\Sigma^n x}_x} = \begin{cases} tr_x, & \text{if } n = d, \\ 0 & \text{otherwise.} \end{cases}$$
Lemma 4.2. Let $A$ with the automorphism $\Sigma$ be a weakly $d$-Calabi-Yau category. Then the map $tr_A$ induces a map $tr_A : \text{Nat}_s(A) \to k$. If $A$ is $d$-Calabi-Yau, then $tr_A$ induces also a map $tr_A : \text{Ab}_s(A) \to k$.

Proof. If $A$ is weakly Calabi-Yau, then $tr_A(fg) = tr_A(\Sigma^n(g)f)$ for $f \in A^{d+n}_x$ and $g \in A^d_x$ by definition. Thus, the first assertion is true. For the second assertion it suffices to show that $tr_A(f) = (-1)^{n+1}tr_A(\Sigma(f))$ for all $f \in A^{d+n}_x$. If $n \neq d$, then both sides are zero. For $n = d$, let us take $m = 1, y = \Sigma^{-1} x, g = \text{Id}_x$. By (4.1), we have

$$tr_A(\Sigma(f)) = tr_x(\Sigma(f)) = (-1)^{d-1}tr_y(f) = (-1)^{d+1}tr_A(f)$$

for all $f \in A^{\Sigma^{-d-1}x}$. Replacing $x$ by $\Sigma x$, we deduce the required equality.

Let us now recall the definitions of Hochschild-Mitchel homology and cohomology. Since in this paper we use this notion only for a graded bimodule over a nongraded category, we restrict our definition only to this case.

Definition 4.3. Let $A$ be a nongraded category (i.e. a graded category with all morphisms of degree zero) and let $M$ be a graded $A$-bimodule. For $n \geq 0$, we define the set of $n$-cochains

$$C^n(A, M) = \prod_{x_0, \ldots, x_n \in A} \text{Hom}(A^{x_0}_{x_1} \otimes \cdots \otimes A^{x_{n-1}}_{x_n}, M(x_0, x_n))$$

and the set of $n$-chains

$$C_n(A, M) = \bigoplus_{x_0, \ldots, x_n \in A} M(x_n, x_0) \otimes A^{x_0}_{x_1} \otimes \cdots \otimes A^{x_{n-1}}_{x_n}.$$

For $n \geq 0$, let us define the linear maps $d_{C^n(A, M)} : C^n(A, M) \to C^{n+1}(A, M)$ and $d_{C_n(A, M)} : C_{n+1}(A, M) \to C_n(A, M)$ by the equalities

$$d_{C^n(A, M)}(\alpha)(f_0 \otimes \cdots \otimes f_n) = f_0 \alpha(f_1 \otimes \cdots \otimes f_n) + \sum_{i=1}^{n} (-1)^i \alpha(f_0 \otimes \cdots \otimes f_{i-1}f_i \otimes \cdots \otimes f_n) + (-1)^{n+1} \alpha(f_0 \otimes \cdots \otimes f_{n-1})f_n,$$

$$d_{C_n(A, M)}(u \otimes f_0 \cdots \otimes f_n) = uf_0 \otimes f_1 \otimes \cdots \otimes f_n + \sum_{i=1}^{n} (-1)^i uf_0 \otimes f_0 \otimes \cdots \otimes f_{i-1}f_i \otimes \cdots \otimes f_n + (-1)^{n+1} uf_n \otimes f_0 \otimes \cdots \otimes f_{n-1},$$

where $\alpha \in C^n(A, M), u \in M(x, y)$ for some $x, y \in A$ and $f_0, \ldots, f_n$ are morphisms in $A$ such that the composition $f_0 \ldots f_n$ is a well-defined element of $A^y_x$. We define the $n$-th Hochschild-Mitchel cohomology and homology of $A$ with coefficients in $M$ by the equalities

$$\text{HH}^n(A, M) = \text{Ker} d_{C^n(A, M)}/\text{Im} d_{C^{n-1}(A, M)} \quad \text{and} \quad \text{HH}_n(A, M) = \text{Ker} d_{C_{n-1}(A, M)}/\text{Im} d_{C_n(A, M)}.$$
Remark 4.4. For more details on Hochschild-Mitchell cohomology and homology see [10], derived invariance of Hochschild-Mitchell homology and cohomology is proved in [7]. In general, the formulas for the differentials above have to be more complicated (see, for example, [14]), but in the case where A is nongraded the formulas above are valid. In fact, in this case we can even define $\operatorname{HH}_n(A, M)$ and $\operatorname{HH}^n(A, M)$ for a nongraded M and then set $\operatorname{HH}_n(A, M) = \bigoplus_{i \in \mathbb{Z}} \operatorname{HH}_n(A, M_i)$ and $\operatorname{HH}^n(A, M) = \bigoplus_{i \in \mathbb{Z}} \operatorname{HH}^n(A, M_i)$ for a graded bimodule M.

As usually, we have $\operatorname{HH}^0(A, M) = M^A$ and $\operatorname{HH}_0(A, M) = M_A$. Thus, by Lemmas 2.4 and 2.12 we have $\operatorname{Nat}^*(A, A/\Sigma) \cong \operatorname{HH}^0(A/\Sigma)$ and $\operatorname{Nat}_*(A) \cong \operatorname{HH}_0(A, A/\Sigma)$ for a category A with an automorphism $\Sigma$.

If M is a graded A-linear category, then the sets defined above carry a lot of additional structure. Here we need only the so-called contraction map

$$i : C^n(A, M) \to \operatorname{Hom}(C_n(A, M), C_{n-m}(A, M)).$$

In general $i$ is defined for all integers $n \geq m$ and induces a well-defined map from $\operatorname{HH}^m(A, M)$ to $\operatorname{Hom}(\operatorname{HH}_n(A, M), \operatorname{HH}_{n-m}(A, M))$ that is denoted by $i$ as well. In the present paper we need only the case $n = m$. Let us write $i_\alpha$ for the image of $\alpha \in C^n(A, M)$ in $\operatorname{Hom}(C_n(A, M), C_0(A, M))$ under the map $i$. Then $i : C^n(A, M) \to \operatorname{Hom}(C_n(A, M), C_0(A, M))$ can be defined by the equality

$$i_\alpha(u \otimes f_1 \otimes \cdots \otimes f_n) = u \circ \alpha(f_1 \otimes \cdots \otimes f_n)$$

for $\alpha \in C^n(A, M)$, $u \in M(x, y)$ and morphisms $f_1, \ldots, f_n$ in A such that the composition $f_1 \ldots f_n$ is a well-defined element of $A^n$, where $x, y \in A$. As was mentioned above, $i$ induces a well-defined map $i : \operatorname{HH}^n(A, M) \to \operatorname{Hom}(\operatorname{HH}_n(A, M), \operatorname{HH}_0(A, M))$, i.e. we have a map $i_\alpha : \operatorname{HH}_n(A, M) \to M_A$ for each $\alpha \in \operatorname{HH}^n(A, M)$.

Suppose now that $\Sigma$ is an automorphism of A that acts on the graded bimodule M. Then there is an action of $\Sigma$ on $C^n(A, M)$ and $C_n(A, M)$ defined by the equalities

$$(\Sigma \alpha)(f_1 \otimes \cdots \otimes f_n) = \Sigma(\alpha^{\Sigma^{-1}} f_1 \otimes \cdots \otimes \alpha^{\Sigma^{-1}} f_n) \quad \text{and} \quad \Sigma(u \otimes f_1 \otimes \cdots \otimes f_n) = \Sigma u \otimes f_1 \otimes \cdots \otimes f_n.$$

It is easy to see that this action induces an action of $\Sigma$ on $\operatorname{HH}^n(A, M)$ and $\operatorname{HH}_n(A, M)$, respectively. As before, $a\Sigma$ acts on $C_n(A, M)$ and $\operatorname{HH}_n(A, M)$ for any $a \in k^*$. Moreover, $i$ induces maps

$$i : C^n(A, M)^\Sigma \to \operatorname{Hom}(C_n(A, M)_{a\Sigma}, C_0(A, M)_{a\Sigma})$$

and

$$i : \operatorname{HH}^n(A, M)^\Sigma \to \operatorname{Hom}(\operatorname{HH}_n(A, M)_{a\Sigma}, (M_A)_{a\Sigma}).$$

There are gradings on $C^n(A, M)$ and $C_n(A, M)$ defined by the equalities $C^n(A, M)_i = C^n(A, M)_{i_1}$ and $C_n(A, M)_i = C_n(A, M)_{i_1}$. These gradings induce gradings on $\operatorname{HH}^n(A, M)$, $\operatorname{HH}_n(A, M)_{\Sigma}$, $\operatorname{HH}^n(A, M)_{\Sigma}$, $\operatorname{HH}_n(A, M)_{a\Sigma}$, and $\operatorname{HH}_n(A, M)_{a\Sigma}$. If $V$ is a graded space, then we define $V^*$ as a graded space with the degree $i$ component $(V^*)_i = (V_{-i})^*$ for $i \in \mathbb{Z}$. If we fix some isomorphism $\Theta : X \to Y^*$ between graded spaces X and Y, then, for $U \subseteq Y$, we define $U^\perp = \{ x \in X \mid \Theta(x)[U] = 0 \}$. If $U \subseteq Y$ is a graded subspace, then it is easy to see that $\Theta$ induces an isomorphism from $U^\perp$ to $(Y/U)^*$.

Since $(A/\Sigma)_A \cong \operatorname{Nat}_*(A)$ and $\left( (A/\Sigma)_A \right)_{-\Sigma} \cong \operatorname{Ab}_*(A)$, we have a map $tr_A : (A/\Sigma)_A \to k$ in the case where A is weakly Calabi-Yau and a map $tr_A : \left( (A/\Sigma)_A \right)_{-\Sigma} \to k$ in the case where A is Calabi-Yau.
**Theorem 4.5.** Let $A$ with the automorphism $\Sigma$ be a weakly $d$-Calabi-Yau category. Then, for any $n \geq 0$, the map $\Theta_n : HH^n(A, A/\Sigma) \to HH_n(A, A/\Sigma)^*[-d]$ defined by the equality $\Theta_n(\alpha) = tr_A \mathbf{i}_n$ for $\alpha \in HH^n(A, A/\Sigma)$ is an isomorphism of graded spaces. Moreover, if $A$ is $d$-Calabi-Yau, then $\Theta_n$ induces an isomorphism of graded spaces $\Theta_n : HH^n(A, A/\Sigma)^\Sigma \to (HH_n(A, A/\Sigma)^\Sigma)^*[-d]$. 

**Proof.** Let us consider the map $\Theta_n : C^n(A, A/\Sigma) \to C_n(A, A/\Sigma)^*[-d]$ defined by the equality $\Theta_n(\alpha) = tr_A \mathbf{i}_n$. Its $i$-th component equals to the composition of isomorphisms

$$
\prod_{x_0, \ldots, x_n \in A} \text{Hom}(A_{x_1}^{x_0} \otimes \cdots \otimes A_{x_n}^{x_{n-1}}, A_{x_n}^{\Sigma x_0}) \cong \prod_{x_0, \ldots, x_n \in A} \left( A_{x_n}^{\Sigma x_0} \otimes \text{Hom}(A_{x_1}^{x_0} \otimes \cdots \otimes A_{x_n}^{x_{n-1}}, k) \right) 
$$

$$
\cong \prod_{x_0, \ldots, x_n \in A} \left( (A_{x_0}^{\Sigma d-i x_n})^* \otimes (A_{x_1}^{x_0} \otimes \cdots \otimes A_{x_n}^{x_{n-1}})^* \right) \cong \left( \bigoplus_{x_0, \ldots, x_n \in A} A_{x_0}^{\Sigma d-i x_n} \otimes A_{x_1}^{x_0} \otimes \cdots \otimes A_{x_n}^{x_{n-1}} \right)^* ,
$$

where the isomorphism $A_{x_0}^{\Sigma x_0} \cong (A_{x_0}^{\Sigma d-i x_n})^*$ is induced by the pairing $(,) : A_{x_0}^{\Sigma d-i x_n} \times A_{x_0}^{\Sigma x_0} \to k$ defined by the equality $(f, g) = tr_x(\Sigma^i fg)$. Thus, $\Theta_n$ is an isomorphism.

Let us now prove that $\Theta_{n+1} d_{C^n(A, A/\Sigma)} = (d_{C_n(A, A/\Sigma)})^* \Theta_n$ for any $n \geq 0$. Fix some $\alpha \in C^n(A, A/\Sigma)$, $x, y \in A$, $u \in A/\Sigma(x, y)_j$, and some morphisms $f_0, \ldots, f_n$ in $A$ such that the composition $f_0 \ldots f_n$ is a well-defined element of $A_y^x$. Then

$$
(\Theta_{n+1} d_{C^n(A, A/\Sigma)}(\alpha))(u \otimes f_0 \otimes \cdots \otimes f_n) = tr_A (ud_{C^n(A, A/\Sigma)}(\alpha)(f_0 \otimes \cdots \otimes f_n))
$$

and

$$
((d_{C_n(A, A/\Sigma)})^* \Theta_n(\alpha))(u \otimes f_0 \otimes \cdots \otimes f_n) = tr_A \mathbf{i}_n d_{C_n(A, A/\Sigma)}(u \otimes f_0 \otimes \cdots \otimes f_n) .
$$

It is easy to see that

$$
ud_{C^n(A, A/\Sigma)}(\alpha)(f_0 \otimes \cdots \otimes f_n) - \mathbf{i}_n d_{C_n(A, A/\Sigma)}(u \otimes f_0 \otimes \cdots \otimes f_n)
$$

$$
= (-1)^{n+1} (u \alpha(f_0 \otimes \cdots \otimes f_{n-1}) f_n - f_n u \alpha(f_0 \otimes \cdots \otimes f_{n-1})).
$$

If $i + j \neq d$, then it follows from the definition of $tr_A$ that

$$
tr_A(u \alpha(f_0 \otimes \cdots \otimes f_{n-1}) f_n) = tr_A(f_n u \alpha(f_0 \otimes \cdots \otimes f_{n-1})) = 0 .
$$

For $i + j = d$, we have

$$
tr_A(u \alpha(f_0 \otimes \cdots \otimes f_{n-1}) f_n) = tr_A(f_n u \alpha(f_0 \otimes \cdots \otimes f_{n-1}))
$$

by the weak Calabi-Yau property. Thus, the maps $\Theta_i$ induce an isomorphism of graded spaces

$$
\Theta_n : HH^n(A, A/\Sigma) \cong \text{Ker}(d_{C_n(A, A/\Sigma)})^*[d]/\text{Im}(d_{C_{n-1}(A, A/\Sigma)})^*[d] \cong HH_n(A, A/\Sigma)^*[d]
$$

for each $n \geq 0$. So the first part of the theorem is proved.

Suppose now that $A$ is $d$-Calabi-Yau. Note that it follows from the definition of $tr_A$ that $\Theta_n(\alpha)(\gamma) = 0$ for $\alpha \in C^n(A, A/\Sigma)$ and $\gamma \in C_n(A, A/\Sigma)$ if $i + j \neq d$. Then, for
\( \alpha \in C^n(A, A/\Sigma), \ x, y \in A, \ u \in A/\Sigma(x, y), \) and morphisms \( f_1, \ldots, f_n \) in \( A \) such that the composition \( f_1 \ldots f_n \) is a well-defined element of \( A^*_x \), we have

\[
\Theta_n(\alpha)(u \otimes f_1 \otimes \ldots f_n + \Sigma(u \otimes f_1 \otimes \ldots f_n)) = tr_A(u \alpha(f_1 \otimes \ldots f_n) + \Sigma u(\Sigma f_1 \otimes \ldots \Sigma f_n)) = tr_A(u(\alpha - \Sigma^{-1}\alpha)(f_1 \otimes \ldots f_n)) = \Theta_n(\alpha - \Sigma^{-1}\alpha)(u \otimes f_1 \otimes \ldots f_n).
\]

This holds by the Calabi-Yau property and since \( \Sigma \in \gamma \) have (\( U_n \)). Let us consider \( U_n \) and \( \gamma \). It is clear that \( K \) is perfect. Then, for each \( r \geq 0 \), there exists a unique linear map \( \xi_p : Z^*(A) \to Z^*(A) \) such that \( (\xi_p f, g)^{\Sigma} = (f, \xi_p g) \) for all \( f \in Z^*(A) \) and \( g \in Ab_s(A) \). Moreover, \( K_r(A/\Sigma) = \text{Im} \xi_r \).

Lemma 4.6. Let \( A \) with the automorphism \( \Sigma \) be a \( d \)-Calabi-Yau category. Then \( K_r(A/\Sigma) = K_{r,d}(A/\Sigma) = T_r(A/\Sigma)^\perp \).

Proof. It is clear that \( K_r(A/\Sigma) \subseteq K_{r,d}(A/\Sigma) \subseteq T_r(A/\Sigma)^\perp \). Let us prove that \( T_r(A/\Sigma)^\perp \subseteq K_r(A/\Sigma) \). Suppose that \( \eta \in Z^m(A) \) does not belong to \( K_r(A/\Sigma) \). Then there is \( f \in Ab_m(A) \) such that \( f^{\Theta^r} = 0 \) and \( \eta f \neq 0 \). Then there exists \( \theta \in Z^{d-m-n}(A) \) such that

\[
0 \neq (\theta, \eta f) = tr_A(\theta \eta f) = (-1)^{n(d-m-n)}tr_A(\eta \theta f) = (-1)^{n(d-m-n)}(\eta, \theta f).
\]

Since \( \theta f \in T_r(A/\Sigma) \), we have \( \eta \notin T_r(A/\Sigma)^\perp \). Consequently, \( K_r(A/\Sigma) = T_r(A/\Sigma)^\perp \).

Lemma 4.6 gives an alternative definition of Külshammer ideals for a Calabi-Yau category. In particular, it implies that \( T_r(A/\Sigma)^\perp \) does not depend on the choice of \( \Theta_{tr} \). As usually, such a definition can be reformulated in terms of the adjoint map.

Proposition 4.7. Let \( A \) with the automorphism \( \Sigma \) be a \( d \)-Calabi-Yau category. Assume that the field \( k \) is perfect. Then, for each \( r \geq 0 \), there is a unique linear map \( \zeta_r : Z^*(A) \to Z^*(A) \) such that \( (\zeta_r f, g)^{\Sigma} = (f, \zeta_r g) \) for all \( f \in Z^*(A) \) and \( g \in Ab_s(A) \). Moreover, \( K_r(A/\Sigma) = \text{Im} \zeta_r \).
such that $k$ is an isomorphism for $u \in (\text{Ab}_n(A))^\bullet$. The equality that $\zeta$ has to satisfy can be rewritten in the form $\Theta_{tr} \zeta_r = (\phi_r^{-1})_s(\xi_p^r)^* \Theta_{tr}$. Thus, the unique map satisfying the required condition is $\zeta_r = \Theta_{tr}^{-1}((\phi_r^{-1})_s(\xi_p^r)^*) \Theta_{tr}$. Note that $(\phi_r^{-1})_s(\xi_p^r)^*(u) = 0$ for $u \in (\text{Ab}_n(A))^\bullet$ if $p^r \nmid n$ and hence $\zeta_r(f) = 0$ for $f \in Z^\bullet(A)$ if $p^r \nmid d - m$.

Let us now prove that $K^r(A/\Sigma) = \text{Im} \zeta_r$. It is clear that $\text{Im} \zeta_r \subseteq (\text{Ker} \xi_p^r)_m$. Let us prove the inverse inclusion. Note that $\Theta_{tr}(\text{Im} \zeta_r) = \text{Im} ((\phi_r^{-1})_s(\xi_p^r)^* \Theta_{tr} = \text{Im} ((\phi_r^{-1})_s(\xi_p^r)^*)$. Let us prove that $\Theta_{tr}((\text{Ker} \xi_p^r)_m) \subseteq \text{Im} ((\phi_r^{-1})_s(\xi_p^r)^*)$. Suppose that $h \in \Theta_{tr}((\text{Ker} \xi_p^r)_m) \subseteq (\text{Ab}_{d-m}(A))^\bullet$.

Let $u_i, i \in I \cup J$ be a basis of $\text{Ab}_{d-m}(A)$ such that $u_i$, $i \in J$ is a basis of $\text{Ker} \xi_p^r|_{\text{Ab}_{d-m}(A)}$. Then $(\xi_p^r(u_i), i \in I$ is a set of linearly independent elements and there exists $u \in (\text{Ab}_{d-m})^\bullet$ such that $u \xi_p^r(u_i) = \phi_r h(u_i)$. Then $h = (\phi_r^{-1})_s(\xi_p^r)^* u \in \text{Im} ((\phi_r^{-1})_s(\xi_p^r)^*)$. Hence, $K^*(A/\Sigma) = \text{Im} \zeta_r$ and the proposition is proved.

\[ \Box \]

5. Küllshammer ideals in the Hochschild cohomology

In this section we will define Küllshammer ideals in the Hochschild cohomology of an algebra. From here on $\Lambda$ denotes a $k$-algebra.

Let $DA$ be the derived category of the module category over $\Lambda$ and $K^b_p \Lambda$ be the full subcategory of $DA$ formed by objects isomorphic to complexes of finitely generated projective modules. Let $[1] : DA \rightarrow DA$ be the shift functor. Then we can define the graded center of $K^b_p \Lambda$ with the automorphism $[1]$. Let us recall the definition of the homomorphism $\chi_A : HH^*(\Lambda) \rightarrow Z^*(K^b_p \Lambda)$, which is called the characteristic homomorphism. It is well known that there is an isomorphism of algebras $HH^*(\Lambda) \cong \oplus_{n \geq 0} \text{Hom}_{D(A^op \otimes \Lambda)}(\Lambda, \Lambda[n])$ and so any element of $HH^*(\Lambda)$ corresponds to a unique morphism $f \in \text{Hom}_{D(A^op \otimes \Lambda)}(\Lambda, \Lambda[n])$. Then, for each $X \in K^b_p \Lambda$, we define $\chi_A(f)_X = \text{Id}_X \otimes f : X \cong X \otimes \Lambda \rightarrow X \otimes \Lambda[n] \cong X[n]$.

It is not hard to see that $\chi_A(f)$ is a natural transformation satisfying the equality $\chi_A(f)_{X[1]} = (-1)^n \chi_A(f)_X[1]$. Thus, $\chi_A : HH^*(\Lambda) \rightarrow Z^*(K^b_p \Lambda)$ is a homomorphism of graded algebras.

It is well known that $HH^*(\Lambda)$ is invariant under derived equivalences. Since $K^b_p \Lambda$ is invariant under derived equivalences too, it is clear that $Z^*(K^b_p \Lambda)$ is a derived invariant. In fact, we can say a little more. Let us recall that by [17] if $\Lambda$ and $\Gamma$ are derived equivalent algebras over a field, then there exist $U \in D(\Lambda^op \otimes \Gamma)$ and $V \in D(\Gamma^op \otimes \Lambda)$ such that $U \otimes \Lambda V \cong \Lambda$ in $D(\Lambda^op \otimes \Lambda)$ and $V \otimes \Lambda U \cong \Gamma$ in $D(\Gamma^op \otimes \Gamma)$. In this case $F_U = - \otimes \Lambda U$ and $F_V = - \otimes \Lambda V$ induce a pair of quasi inverse equivalences between $K^b_p \Lambda$ and $K^b_p \Gamma$. Moreover, $U$ and $V$ induce an isomorphism $\varphi_{U,V} : HH^*(\Lambda) \rightarrow HH^*(\Gamma)$ in the following way. For $f \in \text{Hom}_{D(A^op \otimes \Lambda)}(\Lambda, \Lambda[n])$ we define $\varphi_{U,V}(f) = \alpha_{U,V}^{-1}[n](\text{Id}_V \otimes \Lambda f \otimes \Lambda \text{Id}_U)\alpha_{U,V}$, where $\alpha_{U,V} : \Gamma \rightarrow V \otimes \Lambda U$ is an isomorphism and we use the identifications $V \otimes \Lambda U \cong V \otimes \Lambda A \otimes \Lambda U$ and $V \otimes \Lambda U[n] \cong V \otimes \Lambda \Lambda[n] \otimes \Lambda U$. Note also that due to Corollary [15], the equivalence $F_U$ induces an isomorphism $\varphi_{F_U} : Z^*(K^b_p \Lambda) \rightarrow Z^*(K^b_p \Gamma)$. 

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Lemma 5.1. Let $\Lambda$, $\Gamma$, $U$ and $V$ be as above. Then $\varphi_{F_U} \chi_\Lambda = \chi_\Gamma \varphi_{U,V}$.

Proof. For $X \in K^b_p \Gamma$, we define
$$\alpha_X = \text{Id}_X \otimes_{\Gamma} \alpha_{U,V} : X \cong X \otimes_{\Gamma} \Gamma \to X \otimes_{\Gamma} V \otimes_{\Lambda} U = F_U F_V(X).$$

Then $\alpha : \text{Id}_{K^b_p \Gamma} \to F_U F_V$ is a natural isomorphism. Using the construction of $\varphi_{F_U}$ from Theorem 3.3, we get, for $f \in \text{Hom}_{D(A^{op} \otimes \Lambda)}(\Lambda, \Lambda[n]) \cong \text{HH}^* (\Lambda)$ and $X \in K^b_p \Gamma$, that
$$(\varphi_{F_U} \chi_\Lambda (f))_X = (\text{Id}_X \otimes_{\Gamma} \alpha_{U,V}^{-1} [n]) (\chi_\Lambda (f) \chi_{\otimes_{\Gamma} V} \otimes_{\Lambda} \text{Id}_U)(\text{Id}_X \otimes_{\Gamma} \alpha_{U,V})$$
$$= (\text{Id}_X \otimes_{\Gamma} \alpha_{U,V}^{-1} [n]) (\text{Id}_X \otimes_{\Gamma} f \otimes_{\Lambda} \text{Id}_U)(\text{Id}_X \otimes_{\Gamma} \alpha_{U,V})$$
$$= \text{Id}_X \otimes_{\Gamma} (\alpha_{U,V}^{-1} [n] (\text{Id}_V \otimes f \otimes \text{Id}_U) \alpha_{U,V}) = (\chi_\Gamma \varphi_{U,V} (f))_X.$$ 

\[ \square \]

Lemma 5.1 immediately implies derived invariance of the following ideal in the Hochschild cohomology.

Corollary 5.2. The ideal $\operatorname{Ker}(\chi_\Lambda) \subseteq \text{HH}^* (\Lambda)$ is invariant under derived equivalences.

For any $s \in \mathbb{Z}$, Corollary 3.5 gives a decreasing sequence of ideals
$$Z^*(K^b_p \Lambda) = K_{0,s} (K^b_p \Lambda/[1]) \supseteq K_{1,s} (K^b_p \Lambda/[1]) \supseteq \cdots \supseteq K_{r,s} (K^b_p \Lambda/[1]) \supseteq \cdots \supseteq R_s (K^b_p \Lambda/[1]).$$

Definition 5.3. The $(r, s)$-th higher Kulshammer ideal $\text{HK}^*_{r,s} (\Lambda)$, the $r$-th higher Kulshammer ideal $\text{HK}^*_r (\Lambda)$, the $s$-th higher Reynolds ideal $\text{HR}^*_s (\Lambda)$ and the higher Reynolds ideal $\text{HR}^* (\Lambda)$ of $\Lambda$ are the ideals in $\text{HH}^* (\Lambda)$ defined by the equalities
$$\text{HK}^*_{r,s} (\Lambda) = \chi_\Lambda^{-1} (K_{r,s} (K^b_p \Lambda/[1])), \quad \text{HK}^*_r (\Lambda) = \chi_\Lambda^{-1} (K_r (K^b_p \Lambda/[1])),$$
$$\text{HR}^*_s (\Lambda) = \chi_\Lambda^{-1} (R_s (K^b_p \Lambda/[1])), \quad \text{and} \quad \text{HR}^* (\Lambda) = \chi_\Lambda^{-1} (R(K^b_p \Lambda/[1])).$$

Theorem 5.4. If $\Lambda$ is an algebra and $s$ is an integer, then
$$\text{HH}^* (\Lambda) = \text{HK}^*_{0,s} (\Lambda) \supseteq \text{HK}^*_1 (\Lambda) \supseteq \cdots \supseteq \text{HK}^*_s (\Lambda) \supseteq \cdots \supseteq \text{HR}^*_s (\Lambda)$$
is a decreasing sequence of graded ideals. In particular,
$$\text{HH}^* (\Lambda) = \text{HK}^*_0 (\Lambda) \supseteq \text{HK}^*_1 (\Lambda) \supseteq \cdots \supseteq \text{HK}^*_s (\Lambda) \supseteq \cdots \supseteq \text{HR}^* (\Lambda)$$
is a decreasing sequence of graded ideals. Moreover, if $\Gamma$ is derived equivalent to $\Lambda$, then there is an isomorphism of graded algebras $\varphi : \text{HH}^* (\Lambda) \to \text{HH}^* (\Gamma)$ such that $\varphi (\text{HK}^*_{r,s} (\Lambda)) = \text{HK}^*_{r,s} (\Gamma)$.

Proof. $\text{HK}^*_{r,s} (\Lambda)$ is an ideal, since it is an inverse image of the ideal $K_{r,s} (K^b_p \Lambda/[1])$ under the algebra homomorphism $\chi_\Lambda$. All inclusions follow from the corresponding inclusions for the ideals in $Z^*(K^b_p \Lambda)$.

Let now $U \in D(\Lambda^{op} \otimes \Gamma)$ and $V \in D(\Gamma^{op} \otimes \Lambda)$ be as above. By Lemma 5.1 and Corollary 3.5, we have
$$\chi_\Gamma \varphi_{U,V} (\text{HK}^*_{r,s} (\Lambda)) = \varphi_{F_U} \chi_\Lambda (\text{HK}^*_{r,s} (\Lambda)) \subseteq \varphi_{F_U} (K_{r,s} (K^b_p \Lambda/[1])) = K_{r,s} (K^b_p \Gamma/[1]),$$
i.e. $\varphi_{U,V} (\text{HK}^*_{r,s} (\Lambda)) \subseteq \text{HK}^*_{r,s} (\Gamma)$, and
$$\chi_\Lambda \varphi_{U,V} (\text{HK}^*_{r,s} (\Gamma)) = \varphi_{F_U} \chi_\Gamma (\text{HK}^*_{r,s} (\Gamma)) \subseteq \varphi_{F_U} (K_{r,s} (K^b_p \Gamma/[1])) = K_{r,s} (K^b_p \Lambda/[1]),$$
i.e. $\varphi_{U,V}^{-1} (\text{HK}^*_{r,s} (\Gamma)) \subseteq \text{HK}^*_{r,s} (\Lambda)$. Thus, $\varphi_{U,V}$ satisfies the required conditions as desired.

\[ \square \]
6 Zero degree

In this section we fix some algebra $\Lambda$ over a field $k$ of characteristic $p$. We will be considering the finite dimensional case, but the first construction is valid for any algebra. We define $[\Lambda, \Lambda] = \{ab - ba \mid a, b \in \Lambda\}$. It is well known and can be proved analogously to Lemma 3.1 that the map $\xi_p : \Lambda \to \Lambda$ defined by the equality $\xi_p(a) = a^p$ for $a \in \Lambda$ induces a well defined map $\xi_p : \Lambda/[[\Lambda, \Lambda] \to \Lambda/[[\Lambda, \Lambda]$, i.e. an endomorphism of $\text{HH}_0(\Lambda)$.

Let us recall the notion of the so-called Hattori-Stallings trace (see [6, 18, 14]). For the map $f \in \text{End}_\Lambda(\Lambda^n)$, we define $\text{tr}(f) = \sum_{i=1}^n \pi_i(f(e_i))$, where $e_i \in \Lambda^n$ is an element that has 1 in the $i$-th component and zeros in all others, and $\pi_i : \Lambda^n \to \Lambda/[[\Lambda, \Lambda]$ is the composition of the canonical projection to the $i$-th component and the canonical projection $\Lambda \to \Lambda/[[\Lambda, \Lambda]$. For a finitely generated projective module $P$ and $f \in \text{End}_\Lambda(P)$, we choose some pair of maps $\iota : P \to \Lambda^n$ and $\pi : \Lambda^n \to P$ such that $\pi \iota = 1_P$ and define $\text{tr}(f) = \text{tr}(\iota \pi f)$. One can check that this definition does not depend on $\iota$ and $\pi$. Now, for a bounded complex $C$ with finitely generated projective terms and a map $f \in \text{End}_{K^b_p}(C)$, we define $\text{tr}(f) = \sum_{i\in \mathbb{Z}} (-1)^i \text{tr}(f_i)$, where $f_i$ is the $i$-th component of $f$. Among other $\text{tr}(f)$ has the following properties: $\text{tr}(f + h) = \text{tr}(f) + \text{tr}(h)$, $\text{tr}(fh) = \text{tr}(hf)$. One can check that $\text{tr} : \bigoplus_{i \in \mathbb{Z}} K^b_p(\Lambda) \to \Lambda/[[\Lambda, \Lambda]$ is a well-defined map that, moreover, induces a map $\text{tr} : \text{Ab}_0(K^b_p(\Lambda)) \to \Lambda/[[\Lambda, \Lambda]$. One can easily check also that there is a well defined map $\phi : \Lambda/[[\Lambda, \Lambda] \to \text{Ab}_0(K^b_p(\Lambda))$ that sends the class of $a \in \Lambda$ to the element $\phi(a) \in \bigoplus_{i \in \mathbb{Z}} K^b_p(\Lambda)$ that has only one nonzero component $\phi(a)_i : \Lambda \to \Lambda$ defined by the equality $\phi(a)_i(1_A) = a$. Note that $\Lambda/[[\Lambda, \Lambda]$ and $\text{Ab}_0(K^b_p(\Lambda))$ are $\text{Z}(\Lambda)$-bimodules, where the second $\text{Z}(\Lambda)$-bimodule structure is induced by the inclusion $\chi_\Lambda : \text{Z}(\Lambda) \to \text{Z}^0(\Lambda)$.

**Lemma 6.1.** The maps $\phi$ and $\text{tr}$ are homomorphisms of $\text{Z}(\Lambda)$-bimodules such that $\xi_p \phi = \phi \xi_p$, $\text{tr} \xi_p = \xi_p \text{tr}$ and $\text{tr} \phi = \text{Id}_{\text{HH}_0(\Lambda)}$.

**Proof.** All the assertions can be easily verified.

Let us now describe $\text{Ab}_0(K^b_p(\Lambda))$ for a finite dimensional algebra $\Lambda$. This will allow us to obtain alternative descriptions of $\text{HK}^0_{r,0}(\Lambda)$ and $\text{HR}^0_{r,0}(\Lambda)$. Let us consider the set of indecomposable objects $U \in K^b_p(\Lambda)$ that are not isomorphic to direct summands of $\Lambda$ and satisfy the condition $\max_{x_i(U) \neq 0} i = 0$. Let us choose one object in each isomorphism class contained in this set and denote by $C$ the obtained collection of objects of $K^b_p(\Lambda)$.

**Theorem 6.2.** Let $\Lambda$ be a finite dimensional algebra. Then $\text{Ab}_0(K^b_p(\Lambda)) = \text{Im} \phi \oplus \bigoplus_{x \in C} kT_x$, where $T_x$ denotes the class of $\text{Id}_x$ in $\text{Ab}_0(K^b_p(\Lambda))$. In particular, $\text{Ker} \text{tr} = \bigoplus_{x \in C} k(T_x - \phi \text{tr}(T_x))$.

**Proof.** Let us prove that $\text{Im} \phi \oplus \bigoplus_{x \in C} kT_x$ is really a subspace of $\text{Ab}_0(K^b_p(\Lambda))$, i.e. that if $f + \sum_{i=1}^k a_i T_{x_i} = 0$ for some $f \in \text{Im} \phi$, $a_i \in k$ and distinct $x_i \in C$ ($1 \leq i \leq k$), then $f = 0$ and $a_i = 0$ for all $1 \leq i \leq k$. Since $\phi$ is injective, it is enough to show that $\sum_{i=1}^k a_i T_{x_i} \notin \text{Im} \phi$. The proof is similar to the proof of Lemma 3.1.
if at least one of the elements \( a_i \) is nonzero. We may assume that \( a_1 \neq 0 \). Let us consider \( I \subseteq \oplus_{x \in K_p^b} (K_p^b)^x \) generated by all the nilpotent maps and all the maps that can be factored throw an indecomposable element not isomorphic to an element of the form \( x_1[i] \) (\( i \in \mathbb{Z} \)). Then it is easy to see that any element \( g \), whose class in \( \text{Ab}_0(K_p^b) \) coincides with the class of \( \sum_{i=1}^{k} a_i \mathcal{T}_{x_i} \), can be represented in the form

\[
g = \sum_{x \equiv x_1, i \in \mathbb{Z}} c_{x,i} \text{Id}_{x[i]} + T
\]

for some \( T \in I \) and \( c_{x,i} \in k \) almost all zero such that \( \sum_{x \equiv x_1, i \in \mathbb{Z}} (-1)^i c_{x,i} = a_0 \). It is easy to see that \( g \neq 0 \).

Now, note that any element of \( \text{Ab}_0 \) can be represented by \( (f_x)_{x \in K_p^b} \in \oplus_{x \in K_p^b} (K_p^b)^x \) such that \( f_x = 0 \) for \( x \notin C \cup \{ \Lambda \} \). Indeed, if \( x = y \oplus z \), then, for any \( f : x \rightarrow x \), we have

\[
\bar{f} = \bar{f}((\pi_y y + \pi_z z)) = \bar{f}_y \pi_y + \bar{f}_z \pi_z,
\]

where \( \pi_y : x \rightarrow y, \pi_z : x \rightarrow z \), \( \bar{f}_y : y \rightarrow x, \bar{f}_z : z \rightarrow x \) are the canonical projections and inclusions, and \( \pi \) denotes the class of \( a \) in \( \text{Ab}_0(K_p^b) \). Thus, we may assume that \( f_x = 0 \) for any decomposable \( x \). Then, due to the equality \( \bar{f} + \bar{f}[i] = 0 \), we may assume that \( f_x = 0 \) if \( \max \ i \neq 0 \). Finally, for any \( x \) such that \( f_x \) is still nonzero, we can choose an isomorphism \( \alpha : x \cong y \) \( (y \in C) \) or a direct inclusion \( \alpha : x \hookrightarrow \Lambda \) and change \( f_x \) by \( \alpha f_x \beta \), where \( \beta \) is such a map that \( \beta \alpha = \text{Id}_x \).

The class of any element \( f \in (K_p^b)^\Lambda \) is obviously contained in \( \text{Im} \phi \). Let us now take \( U \in C \). We may assume that the differential \( d_U \) of \( U \) has image contained in \( U J_A \), where \( J_A \) is the Jacobson radical of \( \Lambda \). Since \( \text{End}_{K_p^b}(U) \) is a local algebra, any \( f \in (K_p^b)^U \) can be represented in the form \( f = a_U \text{Id}_U + f_N \), where \( f_N \) is nilpotent, \( a_U \in k \). Thus, it remains to show that \( \bar{f} \in \text{Im} \phi \) for any nilpotent \( f \in (K_p^b)^U \). Since we assume that \( \text{Im} d_U \subseteq U J_A \), it is easy to show that all the components of \( f \) are nilpotent. Let us prove that \( \bar{f} \in \text{Im} \phi \) using induction on the length of \( U \). The assertion is obvious if \( U \) has only one nonzero term. Suppose that the assertion holds for complexes of length \( n \) and \( U \) has length \( n + 1 \), i.e.

\[
U = (\cdots \rightarrow 0 \rightarrow U_{-n} \xrightarrow{d_{-n}} U_{-n+1} \xrightarrow{d_{-n+1}} \cdots \rightarrow U_{-1} \xrightarrow{d_0} U_0 \rightarrow 0 \rightarrow \cdots),
\]

and \( f \) has components \( f_{-n}, \ldots, f_0 \). Let us prove by induction that the class of \( f \) in \( \text{Ab}_0(K_p^b) \) equals to the class of the endomorphism of \( U(i) \) with components \( f_{-n}, \ldots, f_0 \), where

\[
U(i) = (\cdots \rightarrow 0 \rightarrow U_{-n} \xrightarrow{d_{-n} f^i_{-n}} U_{-n} \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_1} U_{-1} \xrightarrow{d_0} U_0 \rightarrow 0 \rightarrow \cdots).
\]

The assertion is vacuous for \( i = 0 \). For the induction step, it is enough to represent the map with components \( f_{-n}, \ldots, f_0 \) from \( U(i - 1) \) to itself as the composition of the map with components \( \text{Id}_{U_{-n}}, f_{-n}, \ldots, f_0 \) from \( U(i - 1) \) to \( U(i) \) and the map with components \( f_{-n}, \text{Id}_{U_{-1}}, \ldots, \text{Id}_{U_0} \) from \( U(i) \) to \( U(i - 1) \). Since the map \( f_{-n} \) is nilpotent,

\[
U(i) = U_{-n}[n] \oplus (\cdots \rightarrow 0 \rightarrow U_{-n} \xrightarrow{d_{-n}} U_{-n} \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_1} U_{-1} \xrightarrow{d_0} U_0 \rightarrow 0 \rightarrow \cdots)
\]

for big enough \( i \), and the induction hypothesis implies \( \bar{f} \in \text{Im} \phi \).
Definition 6.3. The $r$-th Külsheimer ideal $K_r\Lambda$ ($n \geq 0$) is the set of such $a \in Z(\Lambda)$ that $ab \in [\Lambda, \Lambda]$ for all $b \in \Lambda$ such that $b^r \in [\Lambda, \Lambda]$. The Reynolds ideal of $\Lambda$ is the set $R\Lambda = \bigcap_{r \geq 0} K_r\Lambda$.

It is easy to see that

$$Z(\Lambda) = K_0\Lambda \supseteq K_1\Lambda \supseteq \cdots \supseteq K_r\Lambda \supseteq \cdots \supseteq R\Lambda$$

is a decreasing sequence of ideals. It is also not difficult to prove that

$$R\Lambda = \{a \in Z(\Lambda) \mid aJ_\Lambda \subseteq [\Lambda, \Lambda]\}$$

if $\Lambda$ is finite dimensional, where $J_\Lambda$ is the Jacobson radical of $\Lambda$. Now we are ready to describe the ideals $HK^0_{r,0}$. $\Lambda$

Corollary 6.4. If $\Lambda$ is a finite dimensional algebra, then $HK^0_{r,0}(\Lambda) = K_r\Lambda$ for any $r \geq 0$. In particular, $HR^0_{r,0}(\Lambda) = R\Lambda$ in this case.

Proof. Since $\phi$ is injective and respects $\xi_p$, we have by definition

$$K_r\Lambda = \text{Ann}_{Z(\Lambda)} \text{Ker}(\xi_p^r)^r = \text{Ann}_{Z(\Lambda)}(\text{Ker}(\xi_p^r)^r \cap \text{Im } \phi)$$

and

$$HK^0_{r,0}(\Lambda) = \text{Ann}_{Z(\Lambda)} \text{Ker}(\xi_p^r)^r$$

where $\xi_p^\Lambda$ denotes $\xi_p : \Lambda/[[\Lambda, \Lambda] \to \Lambda/[[\Lambda, \Lambda]])$ and $\xi_p^{Ab}$ denotes $\xi_p : \text{Ab}_0(K^b_p\Lambda) \to \text{Ab}_0(K^b_p\Lambda)$. By Theorem 6.2 and since $\xi_p^{Ab}(\text{Im } \phi) \subseteq \text{Im } \phi$ we have $\text{Ker}(\xi_p^{Ab})^r \subseteq \text{Im } \phi$, and hence the assertion follows.

Corollary 6.5. If $\Lambda$ and $\Gamma$ are derived equivalent finite dimensional algebras, then there exists an isomorphism $\varphi : Z(\Lambda) \cong Z(\Gamma)$ such that $\varphi(K_r\Lambda) = K_r\Gamma$ for any $r \geq 0$ and $\varphi(R\Lambda) = R\Gamma$.

Proof. Follows from Theorem 5.4 and Corollary 6.4.

Remark 6.6. It follows from Corollary 6.5 that the set $R\Lambda = \{a \in Z(\Lambda) \mid aJ_\Lambda \subseteq [\Lambda, \Lambda]\}$ is an ideal in $Z(\Lambda)$ invariant under derived equivalences if $\Lambda$ is an algebra over a field of characteristic $p$. In fact, our argument can be adopted to prove the derived invariance of this ideal for a finite dimensional algebra over a field of characteristic 0 as well. For this one uses the fact, that $R\Lambda$ is the annihilator of the set of nilpotent elements in $\text{Ab}_0(K^b_p\Lambda)$.

Let us recall the classical definition of Külsheimer ideals.

Definition 6.7. The finite dimensional algebra $\Lambda$ is called symmetric if there is a nondegenerate bilinear form $(, ) : \Lambda \times \Lambda \to k$ such that $(ab, c) = (a, bc)$ and $(a, b) = (b, a)$ for all $a, b, c \in \Lambda$. 

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For a symmetric algebra \( \Lambda \), the \( r \)-th classical Küllshammer ideal \( K_r^\text{cl} \Lambda \) \((r \geq 0)\) is the set of \( a \in \Lambda \) such that \((a, b) = 0\) for all \( b \in \Lambda \) such that \( b^r \in [\Lambda, \Lambda] \). It is known that
\[
\text{Z}(\Lambda) = K_0^\text{cl} \Lambda \supseteq K_1^\text{cl} \Lambda \supseteq \cdots \supseteq K_r^\text{cl} \Lambda \supseteq \cdots \supseteq R\Lambda
\]
is a decreasing sequence of ideals. Moreover, if the algebra \( \Gamma \) is derived equivalent to \( \Lambda \), then \( \Gamma \) is symmetric \([17]\) and there is an isomorphism from \( \text{Z}(\Lambda) \) to \( \text{Z}(\Gamma) \) that maps \( K_r^\text{cl} \Lambda \) to \( K_r^\text{cl} \Gamma \) for any \( r \geq 0 \) \([20]\). The later fact can be recovered from the following lemma, the proof is analogous to the proof of Lemma \([4,6]\).

**Lemma 6.8.** If \( \Lambda \) is symmetric, then \( K_r^\text{cl} \Lambda = K_r \Lambda \) for any \( r \geq 0 \).

The following lemma is well known \([3]\).

**Lemma 6.9.** If \( \Lambda \) is a symmetric algebra, then \( K_p^b \Lambda \) is a 0-Calabi-Yau category.

**Corollary 6.10.** If \( \Lambda \) is symmetric, then \( HK_0^b(\Lambda) = K_r \Lambda \) for any \( r \geq 0 \). In particular, \( HR^0(\Lambda) = R\Lambda \) is the socle of the algebra \( \Lambda \).

**Proof.** Follows from Lemmas \([4,6]\) and \([6,8]\) and Corollary \([6,4]\) \( \square \)

**Remark 6.11.** Let \( \Lambda \) be symmetric. The bilinear form \((,): \Lambda \times \Lambda \to k\) induces a nondegenerate bilinear form \((,): HH^*(\Lambda) \times HH_*(\Lambda)\) by the equality \((f, u) = \varepsilon(f \preceq u)\). Here the map \( \varepsilon: \Lambda/|\Lambda, \Lambda| \cong HH_0(\Lambda) \to k\) is induced by the map \( \varepsilon: \Lambda \to k\) defined by the equality \( \varepsilon(a) = (1, a)\). Thus, we can define the map \( \lambda^\Lambda: \text{Ab}_s(K_p^b \Lambda) \to HH_*(\Lambda)\) as the unique map satisfying the equality \( (f, \lambda^\Lambda(u)) = (\chi^\Lambda(f), u)\) for all \( f \in HH^*(\Lambda)\) and \( u \in \text{Ab}_s(K_p^b \Lambda)\). It is not difficult to show that \( \lambda^\Lambda\) is a homomorphism of graded \( HH^*(\Lambda)\)-modules. Note that \( \text{Ker}(\lambda^\Lambda) \subseteq HH^*(\Lambda)\). Actually we have \( \text{Ker}(\lambda^\Lambda) = \text{Im}(\lambda^\Lambda)^\perp\) and \( HK_0^*\Lambda = \lambda^\Lambda(T\text{K}^b_p \Lambda)^\perp\).

**Remark 6.12.** One can consider the Tate-Hochschild cohomology \( \widetilde{HH}^*(\Lambda)\) of a selfinjective algebra \( \Lambda \) and the stable category \( \text{mod}\Lambda\) of the category of finitely generated \( \Lambda\)-modules. Note that \( \text{mod}\Lambda\) is a triangulated category with the shift functor \( \Omega^\Lambda\). In this case there exists the characteristic map \( \chi^\Lambda: \widetilde{HH}^*(\Lambda) \to Z^*(\text{mod}\Lambda)\) and one can define Küllshammer and Reynolds ideals in \( \widetilde{HH}^*(\Lambda)\) as preimages of the corresponding ideals in \( Z^*(\text{mod}\Lambda)\). These ideals are invariant under stable equivalences of Morita type and it would be interesting to study their properties. Of course, the case where \( \Lambda \) is stably d-Calabi-Yau for some integer \( d\) is of special interest. Note, in particular, that symmetric algebras are stably \((-1)\)-Calabi-Yau.

7 Example

In this section we are going to compute all notions defined above for the category \( K_p^b \Lambda\), where \( \Lambda = k[x]/x^2\). The classification of indecomposable objects in \( K_p^b \Lambda\) is given in \([13]\), the graded center of \( K_p^b \Lambda\) is computed in \([12]\).

For any \( m \leq n \in \mathbb{Z}\) consider
\[
\Lambda^{[m,n]} = \cdots \to 0 \to \Lambda \xrightarrow{x} \Lambda \to \cdots \to \Lambda \xrightarrow{x} \Lambda \to 0 \cdots,
\]
where nonzero entries are concentrated in the interval \([m, n]\). Each indecomposable object of \(K^b_p\Lambda\) is isomorphic to an object of the form \(\Lambda^{[m, n]}\) for some \(m \leq n \in \mathbb{Z}\). Let us denote by \(x_t\) the element of \(\text{Hom}(\Lambda^{[m,n]}, \Lambda^{[m',n']})\) given by the multiplication by \(x\) in degree \(t\) and zero in all other degrees. It is easy to see that \(\text{Hom}(\Lambda^{[m,n]}, \Lambda^{[m,n]})\) is two dimensional and any map is homotopic to a map of the form \(c_1 \text{Id}_{\Lambda^{[m,n]}} + c_2 x_n\) for some \(c_1, c_2 \in k\). If \([m, n] \neq [m', n']\) and the intervals have a nontrivial intersection, then \(\text{Hom}(\Lambda^{[m,n]}, \Lambda^{[m',n']})\) is one dimensional in the following cases:

1) \(m \leq m', n \leq n'\), any map is homotopic to a map of the form \(c x_n\) \((c \in k)\):

\[
\cdots \to 0 \overset{0}{\longrightarrow} \Lambda \overset{x}{\longrightarrow} \Lambda \overset{0}{\longrightarrow} \Lambda \overset{0}{\longrightarrow} \cdots
\]

2) \(m \geq m', n \geq n'\), any map is homotopic to a map of the form \(c \text{Id}_{\Lambda^{[m,n]}}\) \((c \in k)\), where the map \(\text{Id}_{\Lambda^{[m,n]}}\) is induced by \(\text{Id}_{\Lambda}\) in degrees \(m, \ldots, n'\) and zero in other degrees:

\[
\cdots \to 0 \overset{0}{\longrightarrow} \Lambda \overset{x}{\longrightarrow} \Lambda \overset{0}{\longrightarrow} \cdots
\]

In all other cases there are no nonzero morphisms.

The following description of \(\text{Nat}^t(K^b_p\Lambda)\) and \(Z^t(K^b_p\Lambda)\) for \(t \geq 0\) was obtained in \([12]\).

\(\text{Nat}^0(K^b_p\Lambda)\) consists of natural transformations \(\eta\) given by the data of the form \(\{\mu, \lambda_{[m,n]} \in k, -\infty < m \leq n < \infty\}\), the corresponding natural transformation is given by \(\eta_{\Lambda^{[m,n]}} = \mu \text{Id}_{\Lambda^{[m,n]}} + \lambda_{[m,n]} x_n\).

The transformation \(\eta\) belongs to \(Z^0(K^b_p\Lambda)\) if and only if \(\lambda_{[m,n]} = \lambda_{[m+r,n+r]}\) for any \(r\).

\(\text{Nat}^t(K^b_p\Lambda)\), \(t > 0\) consists of natural transformations \(\eta\) given by the data of the form \(\{c \in k\}\), the corresponding natural transformation is given by \(\eta_{\Lambda^{[m,n]}} = c \text{Id}_{\Lambda^{[m,n]-t}}\) for \(n-t-m \geq 0\) and \(\eta_{\Lambda^{[m,n]}} = 0\), otherwise.

The transformation \(\eta\) belongs to \(Z^t(K^b_p\Lambda)\) if and only if \(\text{char} k = 2\) or \(t\) is even.

\(\text{Nat}^t(K^b_p\Lambda) = 0 = Z^t(K^b_p\Lambda)\) for \(t < 0\).

\(\text{Nat}_0(K^b_p\Lambda) = \oplus_{\Lambda^{[m,n]}} \langle \text{Id}_{\Lambda^{[m,n]}} \rangle \oplus V_0\), where \(V_0\) is one dimensional. \(V_0 = \oplus_{\Lambda^{[m,n]} \in K^0_p\Lambda} \langle x_n \rangle / U_0\), where \(U_0\) is the subspace generated by the elements \(x_n - (-1)^{n'-n} x_{n'}\) for \(x_n : \Lambda^{[m,n]} \to \Lambda^{[m,n]}, x_{n'} : \Lambda^{[m',n']} \to \Lambda^{[m',n']}\).

\(\text{Ab}_0(K^b_p\Lambda) = \oplus_{\Lambda^{[m,n]}} \langle \text{Id}_{\Lambda^{[m,n]}} \rangle / W_0 \oplus V_0\), where \(W_0\) is the subspace generated by the elements \(\text{Id}_{\Lambda^{[m,n]}} - (-1)^t \text{Id}_{\Lambda^{[m+n+r]}}\).

\(\text{Nat}_t(K^b_p\Lambda) = 0 = \text{Ab}_t(K^b_p\Lambda)\), for \(t > 0\).

For \(t < 0\) the space \(\text{Nat}_t(K^b_p\Lambda) = \Lambda_t\) is one dimensional. \(\Lambda_t = \oplus_{\Lambda^{[m,n]}, n \geq m-t} \langle x_n \rangle / U_t\), where \(U_t\) is the subspace generated by the elements \(x_n - (-1)^{n'-n} x_{n'}\) for \(x_n : \Lambda^{[m,n]} \to \Lambda^{[m,n]}, x_{n'} : \Lambda^{[m',n']}, t \to \Lambda^{[m',n']}\). Here, for \(t\) odd, the map \(x_t\) still denotes the multiplication by \(x\) in degree \(t\) and zero in all other degrees.

For \(t < 0\) the space \(\text{Ab}_t(K^b_p\Lambda) = V_t\) if \(\text{char} k = 2\) or \(t\) is even, \(\text{Ab}_t(K^b_p\Lambda) = 0\), otherwise.

Let us compute the ideals \(K_{r,s}(K^b_p\Lambda / \Sigma)\) and \(K_r(K^b_p\Lambda / \Sigma)\).

For any \(r > 1\) we have
Let us denote by $\tilde{Z}^0$ the subset of $Z^0(K_p^b\Lambda)$ given by \{0, $\lambda_{[m,n]} \in k, -\infty < m \leq n < \infty$\}.

$$T_r := T_r(K_p^b\Lambda/\Sigma) = V_0 \oplus \bigoplus_{t < 0} \text{Ab}_t(K_p^b\Lambda).$$

Hence, $R(K_p^b\Lambda/\Sigma) = K_r(K_p^b\Lambda/\Sigma) = \tilde{Z}^0$.

If $\text{char } k \neq 2$, then

$$K_{r,s}(K_p^b\Lambda/\Sigma)_{t} = \begin{cases} 
0, & \text{for } t < 0, s \in \mathbb{Z}, \\
Z^t(K_p^b\Lambda/\Sigma), & \text{for } s > 0, t \geq 0, \\
Z^t(K_p^b\Lambda/\Sigma), & \text{for } (s - t) \text{ odd}, s \leq 0, t \geq 0, \\
\tilde{Z}^0, & \text{for } (s - t) \text{ even}, s \leq 0, t = 0, \\
0, & \text{for } (s - t) \text{ even}, s \leq 0, t > 0.
\end{cases}$$

If $\text{char } k = 2$, then

$$K_{r,s}(K_p^b\Lambda/\Sigma)_{t} = \begin{cases} 
0, & \text{for } t < 0, s \in \mathbb{Z}, \\
Z^t(K_p^b\Lambda/\Sigma), & \text{for } s > 0, t \geq 0, \\
\tilde{Z}^0, & \text{for } s \leq 0, t = 0, \\
0, & \text{for } s \leq 0, t > 0.
\end{cases}$$

$$R_s(K_p^b\Lambda/\Sigma) = K_{r,s}(K_p^b\Lambda/\Sigma).$$

Let us now compute the corresponding ideals in the Hochschild cohomology. The bimodule resolution of $\Lambda$ is

$$\cdots \rightarrow \Lambda \otimes \Lambda \otimes r \oplus \Lambda \otimes \Lambda \otimes r \rightarrow \Lambda \otimes \Lambda \otimes r \oplus \Lambda \otimes \Lambda \otimes r \rightarrow \Lambda$$

$$\text{HH}^l(\Lambda) = \begin{cases} 
\Lambda, & \text{for } l = 0, \\
\Lambda/2r\Lambda, & \text{for even } l > 0, \\
\text{Ann}(2r), & \text{for odd } l.
\end{cases}$$

$\text{HH}^0(\Lambda) = \Lambda$, $\chi_{\Lambda}(c + dx)$ is the natural transformation $\eta_{\Lambda_{[m,n]}}$ given by the data \{c, $\lambda_{[m,n]}$\}, where $\lambda_{m,n} = 0$ for $m - n$ odd and $\lambda_{m,n} = d$ for $m - n$ even.

For greater $l$ we are going to compute $\chi_{\Lambda}$ in the following way: any element of $\text{HH}^l(\Lambda)$ gives a map $f$ from the bimodule resolution of $\Lambda$ to its shift, so first we compute $\Lambda^{[m,m+n]} \otimes_\Lambda^L \Lambda$ as the totalization of a bicomplex, then we compute mutually inverse isomorphisms $\iota_{[m,n]} : \Lambda^{[m,m+n]} \rightarrow \Lambda^{[m,m+n]} \otimes_\Lambda^L \Lambda$ and $\pi_{[m,n]} : \Lambda^{[m,m+n]} \otimes_\Lambda^L \Lambda \rightarrow \Lambda^{[m,m+n]}$, then $\chi_{\Lambda}(f) = \pi_{[m,n]}[l]((\text{Id}_{\Lambda^{[m,m+n]} \otimes f}\iota_{[m,n]}$). Since the shift does not matter for these computations we can assume $m + n = 0$. Let us use the notation $d_- := x \otimes 1 - 1 \otimes x$, $d_+ := x \otimes 1 + 1 \otimes x$. As a right module $\Lambda^{op} \otimes \Lambda$ is isomorphic to $\Lambda \oplus \Lambda$ (the first $\Lambda$ is generated by $1 \otimes 1$, the
second $\Lambda$ is generated by $x \otimes 1$), let us denote by $\iota_1$ the map $\Lambda \xrightarrow{(1,0)^t} \Lambda^{\text{op}} \otimes \Lambda$, by $\pi_1$ the map $\Lambda^{\text{op}} \otimes \Lambda \xrightarrow{(1,0)} \Lambda$ and by $x_2$ the map $\Lambda^{\text{op}} \otimes \Lambda \xrightarrow{(0,x)} \Lambda$.

The complex $C := \Lambda[-n,0] \otimes^L_{\Lambda} \Lambda$ is the totalization of the bicomplex $\tilde{C}$ with $n$ nonzero rows:

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \Lambda \otimes \Lambda & \xrightarrow{d_-} & \Lambda \otimes \Lambda & \xrightarrow{d_+} & \Lambda \otimes \Lambda & \xrightarrow{d_-} & \Lambda \otimes \Lambda \\
\xrightarrow{-x \otimes 1} & \xrightarrow{x \otimes 1} & \xrightarrow{-x \otimes 1} & \xrightarrow{x \otimes 1} \\
\cdots & \Lambda \otimes \Lambda & \xrightarrow{d_-} & \Lambda \otimes \Lambda & \xrightarrow{d_+} & \Lambda \otimes \Lambda & \xrightarrow{d_-} & \Lambda \otimes \Lambda \\
\xrightarrow{-x \otimes 1} & \xrightarrow{x \otimes 1} & \xrightarrow{-x \otimes 1} & \xrightarrow{x \otimes 1} \\
\cdots & \Lambda \otimes \Lambda & \xrightarrow{d_-} & \Lambda \otimes \Lambda & \xrightarrow{d_+} & \Lambda \otimes \Lambda & \xrightarrow{d_-} & \Lambda \otimes \Lambda \\
\end{array}
\]

\[C^{-i} = (\Lambda^{\text{op}} \otimes \Lambda)^{i+1} \text{ for } i \leq n; \quad C^{-i} = (\Lambda^{\text{op}} \otimes \Lambda)^{n+1} \text{ for } i > n.\]

To obtain $C^{-i}$ one takes the sum of the entries of the bicomplex along the diagonal, the numbering of the summands $\Lambda^{\text{op}} \otimes \Lambda$ goes from the lower left entry to the upper right entry. The differential $d^{-i}$ is a matrix with entries $d_{k,k}^{-i} = d_{(-1)^{l+k+1}}$, $d_{k,k+1}^{-i} = (-1)^{l+k}x \otimes 1$, all other entries are zero. The maps $\iota_{[-n,0]} : \Lambda[-n,0] \rightarrow C$ and $\pi_{[-n,0]} : C \rightarrow \Lambda[-n,0]$ can be defined by the equalities

\[
\begin{align*}
\iota_{[-n,0]} &= ((-1)^{\frac{l}{2}} \iota_1, (-1)^{\frac{l+1}{2}} \iota_1, \ldots, \iota_1)^t : \Lambda \rightarrow (\Lambda^{\text{op}} \otimes \Lambda)^{i+1}, \quad -n \leq -i \leq 0; \\
\pi_{[-n,0]} &= ((-1)^{\frac{l}{2}} \pi_1, 0, \ldots, 0) : (\Lambda^{\text{op}} \otimes \Lambda)^{i+1} \rightarrow \Lambda, \quad -n < -i \leq 0; \\
\pi_{[-n,0]} &= ((-1)^{\frac{l}{2}} \pi_1 \pm x_2, \pm x_2, \ldots, \pm x_2) : (\Lambda^{\text{op}} \otimes \Lambda)^{n+1} \rightarrow \Lambda,
\end{align*}
\]

for appropriate signs before the maps $x_2$.

Case 1: $\text{char } k \neq 2$

For $l$ odd $\text{HH}^l(\Lambda)$ is generated by $x$. We have $\chi_{\Lambda}\text{HH}^l(\Lambda) = 0$ since $Z^l(K_p\Lambda/\Sigma) = 0$.

For $l$ even $\text{HH}^l(\Lambda)$ is generated by the class of 1. The corresponding $f$ has entries $f_{-i} = \text{Id}_\Lambda$, $n \geq i \geq l$. The corresponding composition $(\pi_{[-n,0]}[l])(\text{Id}_\Lambda[-n,0] \otimes f)\iota_{[-n,0]} - i = (-1)^{l/2}\text{Id}_\Lambda$ for $n \geq i \geq l$. Hence,

\[
\chi_{\Lambda}\text{HH}^l(\Lambda) = Z^l(K_p\Lambda/\Sigma), \quad \text{for } l > 0.
\]

If $r > 0, s \leq 0$ is even, then $\text{HK}_{r,s}(\Lambda) = \langle x \rangle \oplus \bigoplus_{l \geq 0} \text{HH}^{2l+1}(\Lambda)$ since

\[
\text{HK}_{r,s}^l(\Lambda) = \begin{cases} 
\langle x \rangle, & \text{for } l = 0, \\
\text{HH}^l(\Lambda), & \text{for } l > 0, l \text{ odd}, \\
0, & \text{for } l > 0, l \text{ even}.
\end{cases}
\]

If $r > 0$ and either $s \leq 0$ is odd or $s > 0$, then $\text{HK}_{r,s}(\Lambda) = \text{HH}^r(\Lambda)$.

\[
\text{HR}(\Lambda) = \text{HK}_r(\Lambda) = \langle x \rangle \oplus \bigoplus_{l \geq 0} \text{HH}^{2l+1}(\Lambda); \quad \text{HR}_s(\Lambda) = \text{HK}_{r,s}(\Lambda), \text{ for } r > 0.
\]
Case 2: $\text{char} \, k = 2$

For $l > 0$, $\text{HH}^l(\Lambda)$ is generated by 1 and $x$, for $x$ the corresponding $f$ has entries $f_{-i} = x \otimes 1$, $n \geq i \geq l$. The corresponding composition $\pi_{[-n,0]}[l](\text{Id}_{\Lambda[-n,0]} \otimes f)_{[-n,0]}$ is clearly homotopic to zero, i.e. $\chi_\Lambda(f) = 0$. For $l$ the corresponding $f$ has entries $f_{-n} = \text{Id}_\Lambda$, $n \geq i \geq l$. The corresponding composition $\pi_{[-n,0]}[l](\text{Id}_{\Lambda[-n,0]} \otimes f)_{[-n,0]} = \text{Id}_\Lambda$ for $n \geq i \geq l$. Hence,

$$\chi_\Lambda \text{HH}^l(\Lambda) = Z^l(K^b_p\Lambda/\Sigma), \text{ for } l > 0.$$ 

If $r > 0$, $s \leq 0$, then $\text{HK}_{r,s}(\Lambda) = \bigoplus_{t \geq 0} \langle x \rangle_t$, is the ideal generated by $x \in \text{HH}^0(\Lambda)$. Here $\langle x \rangle_t$ denotes the subspace $\langle x \rangle$ of $\text{HH}^t(\Lambda)$.

If $r > 0$, $s > 0$, then $\text{HK}^{r,s}(\Lambda) = \text{HH}^r(\Lambda)$

$$\text{HK}_r(\Lambda) = \bigoplus_{t \geq 0} \langle x \rangle_t; \text{HR}_s(\Lambda) = \text{HK}_{r,s}(\Lambda), \text{ for } r > 0.$$

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