Random approximation and the vertex index of convex bodies

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Abstract

We prove that there exists an absolute constant $\alpha > 1$ with the following property: if $K$ is a convex body in $\mathbb{R}^n$ whose center of mass is at the origin, then a random subset $X \subseteq K$ of cardinality $|X| = \lceil \alpha n \rceil$ satisfies with probability greater than $1 - e^{-n}$

$$K \subseteq c_1 n \text{conv}(X),$$

where $c_1 > 0$ is an absolute constant. As an application we show that the vertex index of any convex body $K$ in $\mathbb{R}^n$ is bounded by $c_2 n^2$, where $c_2 > 0$ is an absolute constant, thus extending an estimate of Bezdek and Litvak for the symmetric case.

1 Introduction

The starting point of this article is the following result of Barvinok from [3]: If $C \subseteq \mathbb{R}^n$ is a compact set then, for every $d > 1$ there exists a subset $X \subseteq C$ of cardinality $\text{card}(X) \leq dn$ such that for any $z \in \mathbb{R}^n$ we have

$$\max_{x \in X} |\langle z, x \rangle| \leq \max_{x \in C} |\langle z, x \rangle| \leq \gamma_d \sqrt{n} \max_{x \in X} |\langle z, x \rangle|,$$

where $\gamma_d := \frac{\sqrt{d+1}}{\sqrt{d-1}}$. For the proof of this fact, Barvinok assumes that the Euclidean unit ball $B_2^n$ is the ellipsoid of minimal volume containing the convex hull conv$(C)$ of $C$, and makes essential use of a theorem of Batson, Spielman and Srivastava [4] on extracting an approximate John’s decomposition with few vectors from a John’s decomposition of the identity. From (1.1) one can easily conclude that if $K$ is an origin symmetric convex body in $\mathbb{R}^n$ then for any $d > 1$ there exist $N \leq dn$ points $x_1, \ldots, x_N \in K$ such that

$$\text{absconv}\{x_1, \ldots, x_N\} \subseteq K \subseteq \gamma_d \sqrt{n} \text{absconv}\{x_1, \ldots, x_N\}.$$

A generalization of Barvinok’s lemma was recently obtained by the first named author in [7]: There exists an absolute constant $\alpha > 1$ with the following property: if $K$ is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there exist $N \leq an$ points $x_1, \ldots, x_N \in K \cap S^{n-1}$ such that

$$\text{absconv}\{x_1, \ldots, x_N\} \subseteq K \subseteq c n^{3/2} \text{conv}(X),$$

where $c > 0$ is an absolute constant. The proof involves a more delicate theorem of Srivastava from [21]. Using (1.3) one can establish the following “quantitative diameter version” of Helly’s theorem (see [7]): If $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\text{diam}\left(\bigcap_{i \in I} P_i\right) = 1$, then there exist $s \leq an$ and $i_1, \ldots, i_s \in I$ such that

$$\text{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leq cn^{3/2},$$

where $c > 0$ is an absolute constant. Our first main result provides a random version of (1.3) with an improved dependence on the dimension.
Theorem 1.1. There exists an absolute constant \(\alpha > 1\) with the following property: if \(K\) is a convex body in \(\mathbb{R}^n\) whose center of mass is at the origin, if \(N = \lceil \alpha n \rceil\) and if \(x_1, \ldots, x_N\) are independent random points uniformly distributed in \(K\) then, with probability greater than \(1 - e^{-n}\) we have
\[
K \subseteq c_1 n \text{conv}(\{x_1, \ldots, x_N\}),
\]
where \(c_1 > 0\) is an absolute constant.

For the proof we may assume that \(K\) is an isotropic convex body (see Section 2 for background information) and we use the so-called one-sided \(L_q\)-centroid bodies of \(K\); these are the convex bodies \(Z_q^+(K), q \geq 1\), with support functions
\[
h_{Z_q^+(K)}(y) = \left(2 \int_K (x, y)^q dx\right)^{1/q},
\]
where \(a_+ = \max\{a, 0\}\). We show that if \(N \geq \alpha n\), where \(\alpha > 1\) is an absolute constant, then \(N\) independent random points \(x_1, \ldots, x_N\) uniformly distributed in \(K\) satisfy
\[
\text{conv}(\{x_1, \ldots, x_N\}) \supseteq c_1 Z_q^+(K) \supseteq c_2 L_K B_2^n
\]
with probability greater than \(1 - \exp(-n)\), where \(c_1, c_2 > 0\) are absolute constants. Since \(K\) is contained in \((n + 1)L_K B_2^n\), Theorem 1.1 follows.

We were led to Theorem 1.1 by the question to estimate the vertex index of a not necessarily symmetric \(n\)-dimensional convex body. The vertex index of a symmetric convex body \(K\) in \(\mathbb{R}^n\) was introduced in [5] as follows:
\[
\text{vi}(K) = \inf \left\{ \sum_{j=1}^N \|y_j\|_K : K \subseteq \text{conv}(\{y_1, \ldots, y_N\}) \right\},
\]
where \(\|\cdot\|_K\) is the norm with unit ball \(K\) in \(\mathbb{R}^n\). This index is closely related to the illumination parameter of a convex body and to a well-known conjecture of Boltyanski and Hadwiger about covering of an \(n\)-dimensional convex body by \(2^n\) smaller positively homothetic copies (see [5] and [11]). Bezdek and Litvak proved that
\[
\frac{c_1 n^{3/2}}{\text{ovr}(K)} \leq \text{vi}(K) \leq c_2 n^{3/2},
\]
where \(c_1, c_2 > 0\) are absolute constants and ovr\((K)\) is the outer volume ratio of \(K\) (see Section 2 for the definition). To the best of our knowledge the notion of vertex index has not been studied in the not necessarily symmetric case. A way to define it for an arbitrary convex body \(K\) in \(\mathbb{R}^n\) is to consider first any \(z \in \text{int}(K)\) and to set
\[
\text{vi}_z(K) = \inf \left\{ \sum_{j=1}^N p_{K,z}(y_j) : K \subseteq \text{conv}(\{y_1, \ldots, y_N\}) \right\},
\]
where
\[
p_{K,z}(x) = p_{K-z}(x) = \inf\{t > 0 : x \in t(K - z)\}
\]
is the Minkowski functional of \(K\) with respect to \(z\). Then, one may define the (generalized) vertex index of \(K\) by
\[
\text{vi}(K) = \text{vi}_{\text{bar}(K)}(K),
\]
where \(\text{bar}(K)\) is the center of mass of \(K\). With this definition, we clearly have \(\text{vi}(K) = \text{vi}(K - \text{bar}(K))\), and hence we may restrict our attention to centered convex bodies (i.e. convex bodies whose center of mass is at the origin). In Section 4 we establish some elementary properties of this index and using Theorem 1.1 we obtain the following general estimate.
Theorem 1.2. There exist two absolute constants $c_1, c_2 > 0$ such that for every $n \geq 2$ and for every centered convex body $K$ in $\mathbb{R}^n$,

$$
(1.13) \quad \frac{c_1 n^{3/2}}{\operatorname{ovr}(\operatorname{conv}(K, -K))} \leq \operatorname{vi}(K) \leq c_2 n^2.
$$

A natural question, which is closely related to Theorem 1.1, is to fix $N \geq \alpha n$ and to ask for the largest value $t(N, n)$ for which $N$ independent random points $x_1, \ldots, x_N$ uniformly distributed in $K$ satisfy

$$
(1.14) \quad \operatorname{conv}(\{x_1, \ldots, x_N\}) \supseteq t(N, n) K
$$

with probability “exponentially close” to 1. A sharp answer to this question would unify Theorem 1.1 and the following result from [10] which deals with the case where $N = 0$ for which

$$
(1.15) \quad K \supseteq \operatorname{conv}(\{x_1, \ldots, x_N\}) \supseteq c(\delta) \gamma K,
$$

where $c(\delta)$ is a constant depending on $\delta$. We prove the following.

Theorem 1.3. Let $\beta \in (0, 1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on $\beta$ and an absolute constant $c_1 > 0$ with the following property: if $K$ is a centered convex body in $\mathbb{R}^n$, if $\alpha n \leq N \leq e^n$ and if $x_1, \ldots, x_N$ are independent random points uniformly distributed in $K$, then

$$
(1.16) \quad \operatorname{conv}(\{x_1, \ldots, x_N\}) \supseteq \frac{c_1 \beta \log(N/n)}{n} K.
$$

with probability greater than $1 - e^{-N^{1-\beta} n^\beta}$.

In fact, Theorem 1.1 is a special case of Theorem 1.3. The proof of both theorems is given in Section 3.

2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $(\cdot, \cdot)$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write $B_2^n$ for the Euclidean unit ball and $S^{n-1}$ for the unit sphere. Volume is denoted by $| \cdot |$. We use the same notation $|X|$ for the cardinality of a finite set $X$. We write $\omega_n$ for the volume of $B_2^n$ and $\sigma$ for the rotationally invariant probability measure on $S^{n-1}$.

The letters $c, c', c_1, c_2, \ldots$ denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

We refer to the book of Schneider [20] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

A convex body in $\mathbb{R}^n$ is a compact convex subset $K$ of $\mathbb{R}^n$ with non-empty interior. We say that $K$ is symmetric if $x \in K$ implies that $-x \in K$, and that $K$ is centered if its center of mass

$$
(2.1) \quad \operatorname{bar}(K) = \frac{1}{|K|} \int_K x \, dx
$$

is at the origin. The circumradius of $K$ is the radius of the smallest ball which is centered at the origin and contains $K$:

$$
(2.2) \quad R(K) = \max\{\|x\|_2 : x \in K\}.
$$
If $0 \in \text{int}(K)$ then the polar body $K^\circ$ of $K$ is defined by
\begin{equation}
K^\circ := \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K \},
\end{equation}
and the Minkowski functional of $K$ is defined by
\begin{equation}
p_K(x) = \inf \{ t > 0 : x \in tK \}.
\end{equation}
Recall that $p_K$ is subadditive and positively homogeneous.

We say that a convex body $K$ is in John’s position if the ellipsoid of maximal volume inscribed in $K$ is the Euclidean unit ball $B_2^n$. John’s theorem (see [1, Chapter 2]) states that $K$ is in John’s position if and only if $B_2^n \subseteq K$ and there exist $v_1, \ldots, v_m \in \text{bd}(K) \cap S^{n-1}$ (contact points of $K$ and $B_2^n$) and positive real numbers $a_1, \ldots, a_m$ such that
\begin{equation}
\sum_{j=1}^m a_j v_j = 0
\end{equation}
and the identity operator $I_n$ is decomposed in the form
\begin{equation}
I_n = \sum_{j=1}^m a_j v_j \otimes v_j,
\end{equation}
where $(v_j \otimes v_j)(y) = \langle v_j, y \rangle v_j$. We say that a convex body $K$ is in Löwner’s position if the ellipsoid of minimal volume containing $K$ is the Euclidean unit ball $B_2^n$. One can check that this holds true if and only if $K^\circ$ is in John’s position; in particular, we have a decomposition of the identity similar to (2.6). The outer volume ratio of a convex body $K$ in $\mathbb{R}^n$ is the quantity
\begin{equation}
\text{ovr}(K) = \inf \left\{ \left( \frac{|\mathcal{E}|}{|K|} \right)^{1/n} : \mathcal{E} \text{ is an ellipsoid and } K \subseteq \mathcal{E} \right\}.
\end{equation}
If $K$ is in Löwner’s position then $(|B_2^n|/|K|)^{1/n} = \text{ovr}(K)$.

A convex body $K$ in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that
\begin{equation}
\int_K \langle x, \theta \rangle^2 dx = L_K^2
\end{equation}
for every $\theta$ in the Euclidean unit sphere $S^{n-1}$. It is known that if $K$ is isotropic then
\begin{equation}
cL_K B_2^n \subseteq K \subseteq (n+1)L_K B_2^n,
\end{equation}
where $c > 0$ is an absolute constant. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that
\begin{equation}
L_n := \max \{ L_K : K \text{ is isotropic in } \mathbb{R}^n \} \leq C
\end{equation}
for all $n \geq 1$. Bourgain proved in [6] that $L_n \leq c\sqrt{n} \log n$, while Klartag [14] obtained the bound $L_n \leq c\sqrt{n}$. A second proof of Klartag’s bound appears in [15]. We refer the reader to the article of V. Milman and Pajor [17] and to the book [8] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

The $L_q$-centroid body $Z_q(K)$ of $K$ is the centrally symmetric convex body with support function
\begin{equation}
h_{Z_q(K)}(y) = \left( \int_K |\langle x, y \rangle|^q dx \right)^{1/q}.
\end{equation}
Note that $K$ is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in SL(n)$ then $Z_q(T(K)) = T(Z_q(K))$. From Hölder’s inequality it follows that $Z_1(K) \subseteq Z_q(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \text{conv}(K, -K)$. Using Borell’s lemma (see [8, Chapter 1]) one can check that

$$Z_q(K) \subseteq \frac{q}{p} Z_p(K) \tag{2.12}$$

for all $1 \leq p < q$, where $\tau_1 > 0$ is an absolute constant. In particular, if $K$ is isotropic then

$$R(Z_q(K)) \leq \tau_1 q L_K. \tag{2.13}$$

One can also check that if $K$ is centered, then $Z_q(K) \supseteq c_2 Z_\infty(K)$ for all $q \geq n$. For a proof of all these assertions see [8, Chapter 5]. The class of $L_q$-centroid bodies of $K$ was introduced (with a different normalization) by Lutwak, Yang and Zhang in [16]. An asymptotic approach to this family was developed by Paouris in [18] and [19].

For the proof of Theorem [13] we generalize the arguments from [9] who used $L_q$-centroid bodies in order to describe the asymptotic shape of the absolute convex hull of $N$ random points chosen from a convex body. The use of one-sided $L_q$-centroid bodies allows one to consider the convex hull itself.

### 3 Random approximation of convex bodies

Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ we consider the one-sided $L_q$-centroid body $Z^+_q(K)$ of $K$ with support function

$$h_{Z^+_q(K)}(y) = \left( \frac{2}{K} \int_K (x, y)^q dx \right)^{1/q}, \tag{3.1}$$

where $a_+ = \max\{a, 0\}$. When $K$ is symmetric, it is clear that $Z^+_q(K) = Z_q(K)$. In any case, we easily verify that

$$Z^+_q(K) \subseteq 2^{1/q} Z_q(K). \tag{3.2}$$

Note that $Z^+_q(K) \subseteq 2^{1/q} K$ for all $q \geq 1$. Using Grünbaum’s lemma (see [4, Proposition 1.5.16]) one can check that if $1 \leq q \leq r < \infty$ then

$$\left( \frac{2}{e} \right)^{\frac{1}{1-q}} Z^+_q(K) \subseteq Z^+_r(K) \subseteq \frac{C r}{q} \left( \frac{2e - 2}{e} \right)^{\frac{1}{1-q}} Z^+_q(K), \tag{3.3}$$

where $C > 0$ is an absolute constant. The next lemma is due to Guédon and E. Milman (see [13]).

**Lemma 3.1.** There exists an absolute constant $\tau_0 > 0$ such that, for every isotropic convex body $K$ in $\mathbb{R}^n$,

$$Z^+_2(K) \supseteq \tau_0 L_K B_2^n. \tag{3.4}$$

Equivalently, for any $\theta \in S^{n-1}$,

$$h_{Z^+_2(K)}(\theta) = \left( \frac{2}{K} \right)^{\frac{1}{2}} \int_K (x, y)^2 dx \right)^{1/2} \geq \tau_0 L_K. \tag{3.5}$$

We also need the next lemma, which appears in [13] (see also [8, Theorem 13.2.7]).

**Lemma 3.2.** Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. We fix $\theta \in S^{n-1}$ and define $f_0(t) = |K \cap \{x : (x, \theta) = t\}|$, $t \in \mathbb{R}$. Then,

$$\left( \frac{2}{e^2} \right)^{1/q} \left( \frac{\Gamma(n) \Gamma(q+1)}{\Gamma(n+q+1)} \right)^{1/q} h_K(\theta) \leq h_{Z^+_q(K)}(\theta) \leq 2^{1/q} h_K(\theta). \tag{3.6}$$
Proof. We sketch the proof of the left hand side inequality. Let
\begin{equation}
H_\theta^+ = \{ x \in \mathbb{R}^n : \langle x, \theta \rangle \geq 0 \}.
\end{equation}

First observe that, by the Brunn-Minkowski inequality, \( f_\theta^+ \) is concave on its support, and hence we have
\begin{equation}
f_\theta(t) \geq \left( 1 - \frac{t}{h_K(\theta)} \right)^{n-1} f_\theta(0)
\end{equation}
for all \( t \in [0, h_K(\theta)] \). Therefore,
\begin{equation}
h_q^q Z_q^+(K)(\theta) = 2 \int_0^{h_K(\theta)} t^{q} f_\theta(t) dt \geq 2 \int_0^{h_K(\theta)} t^{q} \left( 1 - \frac{t}{h_K(\theta)} \right)^{n-1} f_\theta(0) dt
\end{equation}
\begin{align*}
&= 2 f_\theta(0) h_q^q h_K^{q+1}(\theta) \int_0^1 s^q (1 - s)^{n-1} ds \\
&= \frac{\Gamma(n) \Gamma(q + 1)}{\Gamma(q + n + 1)} 2 f_\theta(0) h_q^q h_K^{q+1}(\theta).
\end{align*}

Observe that
\begin{equation}
2 f_\theta(0) h_K(\theta) = \frac{f_\theta(0)}{\| f_\theta \|_\infty} 2 \| f_\theta \|_\infty h_K(\theta) \geq \frac{f_\theta(0)}{\| f_\theta \|_\infty} (2 |K \cap H_\theta^+|).
\end{equation}

We know that \( \| f_\theta \|_\infty \leq e f_\theta(0) \) by a result of Fradelizi (see e.g. \cite{F} Theorem 2.2.2]) and that \( |K \cap H_\theta^+| \geq e^{-1} \) by Grünbaum’s lemma (see \cite{G} Proposition 1.5.16). Combining the above we get the result. \( \square \)

Theorem \((\ref{thm:3.3})\) and Theorem \((\ref{thm:1.2})\) will follow from the next fact, which generalizes work of Dafnis, Giannopoulos and Tsolomitis \cite{G} to the not necessarily symmetric setting.

**Theorem 3.3.** Let \( \beta \in (0, 1) \). There exists a constant \( \alpha = \alpha(\beta) > 1 \) depending only on \( \beta \) and absolute constants \( c_1, c_2 > 0 \) with the following property: if \( K \) is a centered convex body in \( \mathbb{R}^n \), if \( N \geq \alpha n \) and if \( x_1, \ldots, x_N \) are independent random points uniformly distributed in \( K \) then there exists \( q \geq c_1 \beta \log(N/n) \) such that
\begin{equation}
\text{conv}(\{x_1, \ldots, x_N\}) \supseteq c_2 Z_q^+(K)
\end{equation}
with probability greater than \( 1 - e^{-N^{1-\beta} n^{\beta}} \).

Our proof of \((\ref{thm:3.3})\) is using the family of one-sided \( L_q \)-centroid bodies of \( K \). In particular, we need the following estimate.

**Lemma 3.4.** There exists an absolute constant \( C > 1 \) with the following property: for every \( n \geq 1 \), for every centered convex body \( K \) in \( \mathbb{R}^n \) and for every \( q \geq 2 \),
\begin{equation}
\inf_{\theta \in S^{n-1}} \mu_K \left( \{ x : \langle x, \theta \rangle > \frac{1}{2} h_q^q Z_q^+(K)(\theta) \} \right) \geq C^{-q}.
\end{equation}

**Proof.** Let \( K \) be a centered convex body in \( \mathbb{R}^n \), let \( q \geq 2 \) and let \( \theta \in S^{n-1} \). We apply the Paley-Zygmund inequality
\begin{equation}
P \left( g \geq tE(g) \right) \geq (1 - t)^2 \frac{[E(g)]^2}{E(g^2)}
\end{equation}
for the non-negative random variable
\begin{equation}
g_\theta(x) = 2 \langle x, \theta \rangle_q^n
\end{equation}
on \((K, \mu_K)\), where \(\mu_K\) is Lebesgue measure on \(K\). Applying (3.3) with \(r = 2q\) we see that

\[
\mathbb{E}(g_\theta^2) = h_{Z_q^+(K)}^2(\theta) \leq C_1^q h_{Z_q^+(K)}^2(\theta) = C_1^q |\mathbb{E}(g_\theta)|^2,
\]

where \(C_1 > 0\) is an absolute constant. From (3.13) we get

\[
\mu_K(\{x : (x, \theta) > t h_{Z_q^+(K)}(\theta)\}) = \mu_K(\{x : (x, \theta) > t |\mathbb{E}(g_\theta)|^{1/q}\}) = \mu_K(\{x : (x, \theta) > t q \mathbb{E}(g_\theta)\}) = \mu_K(\{x : g_\theta(x) > 2t q \mathbb{E}(g_\theta)\}) \geq (1 - 2t q)^2 \frac{|\mathbb{E}(g_\theta)|^2}{\mathbb{E}(g_\theta)} \geq (1 - 2t q)^2 \frac{C_1}{q} \tag{3.16}
\]

for every \(t \in (0, 2^{-1/2})\). Choosing \(t = \frac{1}{2}\) we get the lemma with \(C = 4C_1\). □

**Proof of Theorem 3.3** Let \(q \geq 2\) and consider the random polytope \(C_N := \text{conv}\{x_1, \ldots, x_N\}\). With probability equal to one, \(C_N\) has non-empty interior and, for every \(J = \{j_1, \ldots, j_n\} \subset \{1, \ldots, N\}\), the points \(x_{j_1}, \ldots, x_{j_n}\) are affinely independent. Write \(H_J\) for the affine subspace determined by \(x_{j_1}, \ldots, x_{j_n}\) and \(H_J^+\), \(H_J^-\) for the two closed halfspaces whose bounding hyperplane is \(H_J\).

If \(\frac{1}{2} Z_q^+(K) \not\subseteq C_N\), then there exists \(x \in \frac{1}{2} Z_q^+(K) \setminus C_N\), and hence, there is a facet of \(C_N\) defining some affine subspace \(H_J\) as above that satisfies the following: either \(x \in H_J^-\) and \(C_N \not\subset H_J^+\), or \(x \in H_J^+\) and \(C_N \not\subset H_J^-\). Observe that, for every \(J\), the probability of each of these two events is bounded by

\[
\left(\sup_{\theta \in S^{n-1}} \mu_K\left(\left\{x : (x, \theta) \leq \frac{1}{2} h_{Z_q^+(K)}(\theta)\right\}\right)\right)^{N-n} \leq (1 - C^{-q})^{N-n}, \tag{3.17}
\]

where \(C > 0\) is the constant in Lemma 3.4. It follows that

\[
\mathbb{P}\left(\frac{1}{2} Z_q^+(K) \not\subseteq C_N\right) \leq 2 \left(\frac{N}{n}\right)^{1-\beta} \exp(-N^{1-\beta} n^\beta) \tag{3.18}
\]

Since \(\left(\frac{N}{n}\right) \leq \left(\frac{\alpha_1(\beta)}{n}\right)^n\), this probability is smaller than \exp\((-N^{1-\beta} n^\beta)\) if

\[
\left(\frac{2\epsilon N}{n}\right)^n (1 - C^{-q})^{N-n} < \left(\frac{2\epsilon N}{n}\right)^n e^{-C^{-q}(N-n)} < \exp(-N^{1-\beta} n^\beta), \tag{3.19}
\]

and the second inequality is satisfied if

\[
\frac{N}{n} - 1 > C^q \left[\left(\frac{N}{n}\right)^{1-\beta} + \log\left(\frac{2\epsilon N}{n}\right)\right]. \tag{3.20}
\]

We choose \(q = \frac{\beta}{2 \log C \log \left(\frac{N}{n}\right)}\) and \(\alpha_1(\beta) := C^4 / \beta\). Note that if \(N \geq \alpha_1(\beta)n\) then \(q \geq 2\) if and that

\[
\frac{N}{n} - 1 > \left(\frac{N}{n}\right)^{1-\beta} + \left(\frac{N}{n}\right)^{\frac{\beta}{2}} \log\left(\frac{2\epsilon N}{n}\right). \tag{3.21}
\]

Since

\[
\lim_{t \to +\infty} \left[ t - 1 - t^{1-\frac{\beta}{4}} - t^{\frac{\beta}{4}} \log(2\epsilon t) \right] = +\infty,
\]

we may find \(\alpha_2(\beta)\) such that (3.21) is satisfied for all \(N \geq \alpha_2(\beta)n\). Setting \(\alpha = \max\{\alpha_1(\beta), \alpha_2(\beta)\}\) we see that the assertion of the theorem is satisfied with probability greater that \(1 - e^{-N^{1-\beta} n^\beta}\) for all \(N \geq \alpha n\), with \(q \geq c_2 \beta \log \left(\frac{N}{n}\right)\), where \(c_2 > 0\) is an absolute constant. □

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Proof of Theorem 1.3. Let \( \beta \in (0, 1) \) and let \( \alpha = \alpha(\beta) \) be the constant from Theorem 3.3. Let \( \alpha n \leq N \leq e^n \) and let \( x_1, \ldots, x_N \) be independent random points uniformly distributed in \( K \). Applying Lemma 3.2 with \( q = n \) we see that \( h_{Z_n^+}(K) \geq c_1 h_K(\theta) \) for all \( \theta \in S^{n-1} \), and hence

\[
(3.23) \quad Z_n^+ (K) \supseteq c_1 K,
\]

where \( c_1 > 0 \) is an absolute constant. From Theorem 3.3 we know that if \( q = c_2 \beta \log(N/n) \) (note also that \( q \leq n \)) then

\[
(3.24) \quad Z_n^+ (K) \supseteq c_3 Z_q^+ (K)
\]

with probability greater than \( 1 - \exp(-N^{1-\beta}n^\beta) \), where \( c_2, c_3 > 0 \) are absolute constants. From (3.3) we see that

\[
(3.25) \quad Z_n^+ (K) \subseteq \frac{c_4 n}{q} \left( \frac{2e - 2}{e} \right)^{\frac{1}{q-1}} Z_q^+ (K) \subseteq \frac{2c_4 n}{q} Z_q^+ (K),
\]

where \( c_4 > 0 \) is an absolute constant. Combining the above we get that

\[
(3.26) \quad Z_n^+ (K) \supseteq \frac{c_5 q}{n} K \supseteq \frac{c_6 \beta \log(N/n)}{n} K
\]

with probability greater than \( 1 - \exp(-N^{1-\beta}n^\beta) \), where \( c_5, c_6 > 0 \) are absolute constants.

Choosing \( N = \lceil \alpha n \rceil \) in Theorem 1.3 we immediately get Theorem 1.1:

**Theorem 3.5.** There exists an absolute constant \( \alpha > 1 \) with the following property: if \( K \) is a centered convex body in \( \mathbb{R}^n \) then a random subset \( X \subset K \) of cardinality \( \text{card}(X) = \lceil \alpha n \rceil \) satisfies

\[
(3.27) \quad K \subseteq C n \text{ conv}(X)
\]

with probability greater than \( 1 - e^{-n} \), where \( C > 0 \) is an absolute constant.

### 4 Generalized vertex index

Let \( K \) be a convex body in \( \mathbb{R}^n \). From the definition of the vertex index that we gave in the introduction, we may clearly assume that \( K \) is centered, and then

\[
(4.1) \quad \text{vi}(K) = \inf \left\{ \sum_{j=1}^{N} p_K(y_j) : K \subseteq \text{conv}\{y_1, \ldots, y_N\} \right\},
\]

where \( p_K \) is the Minkowski functional of \( K \). Since every origin symmetric convex body is centered, our definition coincides with the one given by Bezdek and Litvak in [5] for the symmetric case.

It is also easy to check that the vertex index is invariant under invertible linear transformations. For every convex body \( K \) in \( \mathbb{R}^n \) and any \( T \in GL(n) \) one has

\[
(4.2) \quad \text{vi}(T(K)) = \text{vi}(K).
\]

To see this, note that \( T(K) \subseteq \text{conv}\{y_1, \ldots, y_N\} \) if and only if \( K \subseteq \text{conv}\{x_1, \ldots, x_N\} \) where \( T(x_j) = y_j \), therefore

\[
(4.3) \quad \text{vi}(T(K)) = \inf \left\{ \sum_{j=1}^{N} p_{T(K)}(T(x_j)) : K \subseteq \text{conv}\{x_1, \ldots, x_N\} \right\}
\]

\[
= \inf \left\{ \sum_{j=1}^{N} p_K(x_j) : K \subseteq \text{conv}\{x_1, \ldots, x_N\} \right\}
\]

\[
= \text{vi}(K).
\]
Another useful observation is that the vertex index is stable under a variant of the Banach-Mazur distance. Recall that the Banach-Mazur distance between two convex bodies $K$ and $L$ in $\mathbb{R}^n$ is the quantity
\begin{equation}
(4.4) \quad d(K, L) = \inf\{ t > 0 : T(L + y) \subseteq K + x \subseteq t(T(L + y)) \},
\end{equation}
where the infimum is over all $T \in GL(n)$ and $x, y \in \mathbb{R}^n$. Given two centered convex bodies $K$ and $L$, we set
\begin{equation}
(4.5) \quad \tilde{d}(K, L) = \inf\{ t > 0 : T(L) \subseteq K \subseteq tT(L) \},
\end{equation}
where the infimum is over all $T \in GL(n)$. Note that if $K$ and $L$ are symmetric convex bodies then $\tilde{d}(K, L) = d(K, L)$. With this definition we easily check that if $K$ and $L$ are centered convex bodies in $\mathbb{R}^n$ then
\begin{equation}
(4.6) \quad \operatorname{vi}(K) \leq \tilde{d}(K, L) \operatorname{vi}(L).
\end{equation}

The main result of this section is the upper bound in Theorem 1.2.

**Proposition 4.1.** There exists an absolute constant $C_1 > 0$ such that, for every $n \geq 2$ and for every centered convex body $K$ in $\mathbb{R}^n$,
\begin{equation}
(4.7) \quad \operatorname{vi}(K) \leq C_1 n^2.
\end{equation}

**Proof.** We may assume that $K$ is isotropic. By Theorem 3.5 we can find $N \leq \alpha n$ and $x_1, \ldots, x_N \in K$ such that
\begin{equation}
(4.8) \quad K \subseteq C n \operatorname{conv}(\{x_1, \ldots, x_N\}),
\end{equation}
where $\alpha, C > 0$ are absolute constants. We set $y_j = Cnx_j$, $1 \leq j \leq N$. Then, $K \subseteq \operatorname{conv}(\{y_1, \ldots, y_N\})$ and $p_K(y_j) = Cnp_K(x_j) \leq Cn$, therefore
\begin{equation}
(4.9) \quad \operatorname{vi}(K) \leq \sum_{j=1}^{N} p_K(y_j) \leq CnN \leq Cn^2.
\end{equation}

The result follows with $C_1 = C\alpha$. \qed

For the lower bound we just check that the argument of [5] remains valid in the not necessarily symmetric case.

**Proposition 4.2.** There exists an absolute constant $c > 0$ such that, for every $n \geq 2$ and for every centered convex body $K$ in $\mathbb{R}^n$,
\begin{equation}
(4.10) \quad \operatorname{vi}(K) \geq \frac{cn^{3/2}}{\operatorname{ovr}(\operatorname{conv}(K, -K))}.
\end{equation}

**Proof.** By the linear invariance of the vertex index we may assume that $B_2^n$ is the ellipsoid of minimal volume which contains $\operatorname{conv}(K, -K)$. In other words, $K \subseteq \operatorname{conv}(K, -K) \subseteq B_2^n$ and
\begin{equation}
(4.11) \quad \left( \frac{|B_2^n|}{|\operatorname{conv}(K, -K)|} \right)^{1/n} = \operatorname{ovr}(\operatorname{conv}(K, -K)).
\end{equation}

For any $N \in \mathbb{N}$ and $y_1, \ldots, y_N$ such that $K \subseteq \operatorname{conv}(\{y_1, \ldots, y_N\})$ we consider the absolute convex hull $Q = \operatorname{conv}(\{\pm y_1, \ldots, \pm y_N\}) \supseteq \operatorname{conv}(K, -K)$ of $y_1, \ldots, y_N$. Then,
\begin{equation}
(4.12) \quad Q^o = \{ x \in \mathbb{R}^n : |\langle x, y_j \rangle| \leq 1 \text{ for all } j = 1, \ldots, N \},
\end{equation}
and a result of Ball and Pajor [2] provides the lower bound
\begin{equation}
(4.13) \quad |Q^o| \geq \left( \frac{n}{\sum_{j=1}^{N} \|y_j\|_2} \right)^{1/n}.
\end{equation}
for its volume. Using the Blaschke-Santaló inequality we get

\[(4.14) \quad |\text{conv}(K, -K)| \leq |Q| \leq \frac{|B_2^n|^2}{|Q|^2} \leq \left(\frac{\sum_{j=1}^{N} \|y_j\|_2}{n}\right)^n.\]

It follows that

\[(4.15) \quad 1 \leq \left(\frac{|B_2^n|}{|\text{conv}(K, -K)|}\right)^{1/n} \left(\frac{|B_2^n|}{|\text{conv}(K, -K)|}\right)^{1/n} \frac{\sum_{j=1}^{N} \|y_j\|_2}{n} \leq \frac{\text{ovr}(\text{conv}(K, -K))}{cn^{3/2}} \sum_{j=1}^{N} \|y_j\|_2\]

for some absolute constant \(c > 0\). Since \(K \subseteq B_2^n\), we have \(\|y_j\|_2 \leq p_K(y_j)\) for all \(j = 1, \ldots, N\). Therefore,

\[(4.16) \quad \sum_{j=1}^{N} p_K(y_j) \geq \frac{cn^{3/2}}{\text{ovr}(\text{conv}(K, -K))},\]

and taking the infimum over all \(N\) and all such \(N\)-tuples \((y_1, \ldots, y_N)\) we get the lower bound for \(\text{vi}(K)\). \(\square\)

**Remark 4.3.** The lower bound of Proposition 4.2 is not sharp, even in the symmetric case. Gluskin and Litvak [12] have proved that for every \(n \geq 1\) there exists a symmetric convex body \(K\) in \(\mathbb{R}^n\) such that

\[(4.17) \quad \text{ovr}(K) \geq c \sqrt{n \log(2n)} \quad \text{and} \quad \text{vi}(K) \geq cn^{3/2}.\]

It would be interesting to provide alternative lower bounds for \(\text{vi}(K)\); in particular, it would be interesting to decide whether, in the non-symmetric case, the upper bound \(\text{vi}(K) \leq Cn^2\) of Proposition 4.1 is sharp or not.

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