Abstract

The bright soliton in optical fiber is generally investigated via its spatial evolution in the time domain, where its waveform is considered in many studies. To be consistent with the well-established picture of the dynamics of solitons in other systems, in this letter, we propose it is helpful to study the temporal evolution of the bright soliton by examining its waveshape propagating along the space coordinate axis. We develop a singular theory. Equations governing the evolution of the parameters of the bright soliton in the slow time and the radiated field are explicitly formulated for the first time. In addition, localized modes are found to appear.

PACS (numbers): 03.40.Kf, 52.35.Mw, 42.65.Tg

Owing to the promising application to long distance soliton-based communication and the great fundamental interest of physics of the process involved, solitary waves and solitons in the nonlinear monomode optical fiber have received intensive studies in recent years [1,2]. The generalized propagation equation of optical field in the fiber takes the form

\[ iu_x' + ik_1 u_t' - \frac{1}{2}k_2 u_{tt'} + \sigma |u'|^2 u' = i\varepsilon P'[u] \]  

(1)

in which \( x' \) represents the propagation distance, \( t' \) the time and \( u' \) the complex field envelope. Usually, \( \varepsilon P'[u] \), including linear loss, high-order dispersion and other nonlinear effects, is
assumed to be small and treated as perturbations to place emphasis on important phenomena of the bright and dark solitons in the fiber [3]. In the region of anomalous group-velocity dispersion (GVD), by introducing the retarded time $T' = t' - k_1 x' = t' - x'/v_g$, Eq. (1) is normalized as

$$iu_{x''}'' + \frac{1}{2} u_{TT}'' + |u''|^2 u'' = i\varepsilon P''[u] \quad (2)$$

in terms of $T = T'/T_0$, $x'' = x'/L_D = x'|k_2|/T_0^2$ and $u'' = \sqrt{|k_2|/\sigma T_0^2} u'$. Customarily, Eq. (2) is referred to as optical nonlinear Schrödinger equation (NLSE), and its unperturbed version supports distortionless propagation of a type of solitary wave called the bright or temporal soliton [3].

Generally, waves travelling along the $x$-axis at speed $v$ are expressible as functions of $(x - vt)$. A wave $F(x, t)$ may be thought of as formed from the shape $f(\zeta)$ by the substitution $\zeta = (x - vt)$, or else as built from the time signal $h(\tau)$ by the substitution $\tau = (t - x/v)$. Here, the function $f(\zeta)$ with $f(x) = F(x, 0)$ characterizes the “waveshape”, and $h(\tau)$ with $h(t) = F(0, t)$ depicts the “waveform” [4]. Resulting pictures from the two standpoints for the wave $F(x, t)$ are that the “waveshape” changes and propagates along the $x$-axis as time elapses and the “waveform” distorts versus the retarded time $\tau$ as the distance $x$ keep increasing. These actually presents two different point of views for the visualization of scenario of soliton under perturbations.

The bright soliton propagating in the fiber governed by Eq. (2) was typically investigated by interchanging the roles of the retarded time $T$ and the space $x''$ and defining an “initial-value” problem, or equivalently by directly treating the space $x''$ as the evolution coordinate and defining a boundary-value problem. Accordingly, the aspect of waveform of the bright soliton was taken into consideration and studies could benefit from the direct application [5,6] of the celebrated frameworks developed by Zakharov and Shabat (ZS) and Ablowitz-Kaup-Newell-Segur (AKNS). Nevertheless, to avoid complication of the ZS and AKNS schemes, other elaborate approaches were developed in the framework of direct expansion as well [7,8].

In contrast to the studies of bright soliton, the aspect of waveshape is extensively ex-
examined in other soliton problems with perturbations \[6\], including envelope soliton of the integrable cubic NLSE in water and other applications \[9,10\]. Although results for the understanding of the waveform of bright soliton have been achieved, a natural question, how the waveshape evolves in the real time or what the dynamics of bright soliton is, is inevitable to arise. To answer this question, the corresponding mathematical model is essentially different from the one investigated in previous theories, and is also intractable in the ZS and AKNS schemes. Consequently, a new theoretical challenge turns up. In this letter, we introduce our theory for the subject.

Let’s start from the dimensionless form of Eq. (1) in the anomalous dispersion regime of the fiber

\[ iu_x + iu_t + \frac{1}{2}u_{tt} + |u|^2 u = i\varepsilon P[u] \]  

where \( t = t'/t_0 \), \( x = x'/l = x'|k_2|/t_0^2 \), \( u = \sqrt{|k_2|/\sigma t_0^2}u' \) and \( t_0 = |k_2|/k_1 = v_g^{-1}|dv_g/d\omega| \). Obviously, instead of the usual \( T_0 \) that is determined by the width of the input waveform in existing theories, a characteristic time, namely \( t_0 \) that is determined by the working wavelength and nature of the fiber, is used in the normalization. Formally, Eq. (3) differs from Eq. (2) or the normal form of optical NLSE only by an additional term \( iu_t \) due to invalidation of the retarded time, but the essential difference lies in that the time \( t \) here must be treated as the evolution coordinate and an initial-value problem is consequently defined, since the waveshape of bright soliton is to be taken into account. In the absence of perturbations, the bright soliton admitted by Eq. (3) is given by

\[ u_{sol}(x, t) = 2\eta \text{sech}2\eta(2\zeta + 1)[x - \frac{1}{(2\zeta + 1)}t - \chi'] \]
\[ \times \exp\{-i[2(\zeta^2 - \eta^2 + \zeta)x - 2\zeta t - \theta_1]\} \]  

provided the initial waveshape is in the form

\[ u_{sol}(x, 0) = 2\eta \text{sech}2\eta(2\zeta + 1)[x - \chi'] \]
\[ \times \exp\{-i[2(\zeta^2 - \eta^2 + \zeta)x - \theta_1]\} \]  

provided the initial waveshape is in the form

\[ u_{sol}(x, 0) = 2\eta \text{sech}2\eta(2\zeta + 1)[x - \chi'] \]
\[ \times \exp\{-i[2(\zeta^2 - \eta^2 + \zeta)x - \theta_1]\} \]  


From the general point of view of solitons under perturbations, if the perturbations turn on, the bright soliton with a starting state of Eq. (5) to propagate in the fiber governed by Eq. (3) can not be described by Eq. (4), it undergoes a slow change in the time via the variation of its parameters. Moreover, other wave modes come on to appear \[6,11\]. To characterize the picture, we introduce a slow time scale \(t_1 = \varepsilon t\) and assume that the solution of Eq. (3) is of the form

\[
u(t, z, t_1) = [2\eta f(z) + \varepsilon v(t, z, t_1)]e^{-i\theta(z, t_1)}
\]  

where \(f(z) = \text{sech} z, z = 2\eta(2\zeta + 1)(x - \varepsilon^{-1} \chi - \chi')\) and \(\theta = (Kz - \varepsilon^{-1}\theta_0 - \theta_1).\) Also, we need further to assume that \(\eta, \zeta, \chi, \chi', K, \theta_0, \theta_1\) are dependent directly on \(t_1.\) Obviously, \(z\) is the coordinate variable in the reference frame tied up to the bright soliton. Here, if we take \(t, z\) and \(t_1\) in place of \(t\) and \(x\) as new independent variables, the derivatives with respect to time and space in Eq. (3) are thus replaced by

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - 2\eta(2\zeta + 1)\Lambda \frac{\partial}{\partial z} + \varepsilon \frac{\eta t_1}{\eta} \frac{\partial}{\partial z} + \varepsilon \frac{2\zeta t_1}{(2\zeta + 1)} z \frac{\partial}{\partial z} - \varepsilon 2\eta(2\zeta + 1)\chi t_1 \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_1} \]

and

\[
\frac{\partial}{\partial x} = 2\eta(2\zeta + 1) \frac{\partial}{\partial z}
\]

where \(\chi t_1 = \Lambda\) is defined. Introducing Eqs. (6)-(8) into Eq. (3), we transform from the laboratory frame into the soliton’s one and get two equations from \(O(1)\) and \(O(\varepsilon),\) respectively. Examining the zeroth-order equation for \(O(1),\) we derive \(\Lambda = (2\zeta + 1)^{-1},\)

\[
\theta_{0t_1} = \Omega = 2(\zeta^2 + \eta^2)\Lambda\]  

and \(K = (\zeta^2 - \eta^2 + \zeta)\Lambda\eta^{-1}.\) By virtue of these relations, we simplify the first-order equation for \(O(\varepsilon)\) as

\[
\frac{1}{2} v_{tt} + i\Lambda^{-1}v_t - 2\eta v_{zt} + 2\eta^2 v_{zz} + 8\eta^2 h^2(z)v + 4\eta^2 h^2(z)v^* - 2\eta^2 v = R(z)
\]

where asterisk * denotes the complex conjugate. And the “source term” \([11] R(z) = R_r(z) + iR_i(z)\) is given by
$$R_r = -\text{Im}(P e^{i\theta}) - 4\eta(2\eta_t + \eta\Lambda\zeta_t)\varphi_2(z)$$

$$+ 2\eta\Lambda^{-1}[2\eta\Lambda^{-1}K'\chi_t' - 4\eta^2\chi_t' + \theta_{1t1}]\phi_1(z)$$

$$- 2\eta[\Lambda^{-1}(\eta^{-1}\zeta_t - 2\Lambda\eta_t) - 2(2\eta\Lambda\zeta_t + \eta_t)]\phi_2(z)$$

$$+ 16\eta^3\Lambda^{-1}\chi_t'\phi_1^3(z) - 8\eta[2\eta\Lambda\zeta_t + \eta_t]z\phi_1^3(z)$$

(10a)

and

$$R_i = \text{Re}(P e^{i\theta}) + 4\eta^2(\eta^{-1}\zeta_t - \Lambda\eta_t)\phi_1(z)$$

$$- 2[\Lambda^{-1}\eta_t + 4\eta^2(\eta^{-1}\zeta_t - \Lambda\eta_t)]\varphi_1(z)$$

$$- 4\eta^2[\theta_{1t1} + 2\eta\Lambda^{-1}\chi_t' K + \Lambda^{-2}\chi_t'']\varphi_2(z)$$

(10b)

where \(\phi_1(z) = \text{sech}z\), \(\phi_2(z) = z\text{sech}z\), \(\varphi_1(z) = \text{sech}z(1 - z\text{tanh}z)\), \(\varphi_2(z) = \text{sech}z\text{tanh}z\)

are defined for simplicity and later use. Expectably, a fresh equation comes out after the linearization. Here, we should note that although the basic idea of the present linearization is a natural extension of the normal scheme of multiple scale expansion [14], the implementation of the idea in handling such soliton problems as of the second order derivative with respect to time is original. As usual, extra freedoms for the purpose of preventing the occurrence of secular terms are introduced and included in the source term. Taking advantage of Laplace transform to solve Eq. (9) yields

$$\frac{1}{2}s^2\tilde{v} + i\Lambda^{-1}s\tilde{v} - 2\eta s\tilde{v}_z + 2\eta^2\tilde{v}_{zz} + 8\eta^2h^2(z)\tilde{v} + 4\eta^2h^2(z)\tilde{v}^* - 2\eta^2\tilde{v} = s^{-1}R(z)$$

(11)

where \(\tilde{v}\) stands for the Laplace transform of \(v\). Putting \(v = v_1 + iv_2\) and \(\tilde{v} = \tilde{v}_1 + i\tilde{v}_2 = (w_1 + iw_2)e^{\frac{sz}{2}}\), we derive from the real and imaginary parts of Eq. (11)

$$sw_1 + 2\eta^2\Lambda\hat{L}_1w_1 = s^{-1}\Lambda R_i e^{-\frac{sz}{2\eta}}$$

(12a)

$$sw_2 - 2\eta^2\Lambda\hat{L}_2w_1 = -s^{-1}\Lambda R_i e^{-\frac{sz}{2\eta}}$$

(12b)

where two Hermitian operators \(\hat{L}_1 = d^2/dz^2 + (2\text{sech}^2z - 1)\) and \(\hat{L}_2 = d^2/dz^2 + (6\text{sech}^2z - 1)\)

are defined.

To solve Eq. (12) by virtue of eigen-expansion, a complete set of basis is needed. Considering the homogeneous counterpart of Eq. (12), we derive the following eigen-value problem
Now, if we define a non-Hermitian operator \( \hat{H} = \hat{L}_2 \hat{L}_1 \), then the corresponding adjoint operator is \( \hat{H}^\dagger = \hat{L}_1 \hat{L}_2 \). Using the operator \( \hat{L}_2 \) to act on both sides of Eq. (13a) and then the \( \hat{L}_1 \) on Eq. (13b) gives

\[
\hat{H} \phi = \lambda^2 \phi \quad (14a)
\]

\[
\hat{H}^\dagger \varphi = \lambda^2 \varphi. \quad (14b)
\]

Eigenstates of operators \( \hat{H} \) and \( \hat{H}^\dagger \) are composed of a continuous spectrum with eigenvalue \( \lambda = -(k^2 + 1) \) and doubly degenerated discrete states with eigenvalue \( \lambda = 0 \), respectively. Under the definition of inner product in the Hilbert space, their eigenstates \( \phi = \{\phi(z,k), \phi_1(z), \phi_2(z)\} \) and \( \varphi = \{\varphi(z,k), \varphi_1(z), \varphi_2(z)\} \) turn out to be a biorthogonal basis (BB) with the completeness relation

\[
\int_{-\infty}^{+\infty} \phi(z,k) \varphi^* (z',k) dk + \phi_1(z) \varphi_1(z') + \phi_2(z) \varphi_2(z') = \delta(z - z') \quad (15)
\]

where

\[
\phi(z,k) = \frac{1}{\sqrt{2\pi(k^2 + 1)}} (1 - 2ik \tanh z - k^2) e^{ikz} \quad (16)
\]

and

\[
\varphi(z,k) = \frac{1}{\sqrt{2\pi(k^2 + 1)}} (1 - 2\text{sech}^2 z - 2ik \tanh z - k^2) e^{ikz} \quad (17)
\]

represent the continuous spectrum and \( \phi_1(z), \phi_2(z), \varphi_1(z), \varphi_2(z) \) that are defined above stand for the discrete states. BB is popular in the studies of non-Hermitian Hamiltonian problems [13]. With the set of BB, we can expand the solutions of Eq. (12) as

\[
w_1(t, z, t_1) = \int_{-\infty}^{+\infty} \tilde{w}_1(t, k, t_1) \phi(z,k) dk + \tilde{w}_11(t, t_1) \phi_1(z) + \tilde{w}_{12}(t, t_1) \phi_2(z) \quad (18a)
\]

\[
w_2(t, z, t_1) = \int_{-\infty}^{+\infty} \tilde{w}_2(t, k, t_1) \phi(z,k) dk + \tilde{w}_{21}(t, t_1) \phi_1(z) + \tilde{w}_{22}(t, t_1) \phi_2(z). \quad (18b)
\]
Introducing Eq. (18) into Eq. (12) and solving by means of orthogonality of the basis, we derive $w_1$ and $w_2$. Thus, $v_1$ and $v_2$ are determined from the inverse Laplace transformation. Some terms directly proportional to $t$ and $t^2$ are found to appear in $v_1$ and $v_2$, they are non-physical and called secular terms. But if we require

$$
\int_{-\infty}^{+\infty} R_i(z) \phi_1(z) dz = 0
$$

(19)

$$
\int_{-\infty}^{+\infty} R_i(z) \phi_2(z) dz + 2\eta\Lambda \int_{-\infty}^{+\infty} R_r(z) z \varphi_2(z) dz = 0
$$

(20)

$$
\int_{-\infty}^{+\infty} R_r(z) \varphi_2(z) dz = 0
$$

(21)

$$
\int_{-\infty}^{+\infty} R_r(z) \varphi_1(z) dz + 2\eta\Lambda \int_{-\infty}^{+\infty} R_i(z) z \phi_1(z) dz = 0,
$$

(22)

those terms vanish and we then get the final solution

$$
v_1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\eta^2 \lambda} (\sin \beta) R_i(z') \phi^* (z', k) \varphi(z, k) dz' dk
\quad + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\eta^2 \lambda} (1 - \cos \beta) R_r(z') \varphi^* (z', k) \varphi(z, k) dz' dk
\quad - \int_{-\infty}^{+\infty} \frac{\Lambda}{2\eta} R_i(z') z' \phi_1(z') dz' \varphi_1(z)
\quad - \int_{-\infty}^{+\infty} \frac{\Lambda}{2\eta} R_i(z') z' \phi_2(z') dz' + \int_{-\infty}^{+\infty} \frac{\Lambda^2}{2} R_r(z') z^2 \varphi_2(z') z \varphi_2(z)
\quad + \int_{-\infty}^{+\infty} \frac{\Lambda}{2\eta} R_i(z') \phi_2(z') dz' + \int_{-\infty}^{+\infty} \Lambda^2 R_r(z') z' \varphi_2(z') dz' z \varphi_2(z)
$$

(23)

and

$$
v_2 = -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\eta^2 \lambda} (\sin \beta) R_r(z') \phi^* (z', k) \phi(z, k) dz' dk
\quad + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\eta^2 \lambda} (1 - \cos \beta) R_r(z') \phi^* (z', k) \phi(z, k) dz' dk
\quad + \int_{-\infty}^{+\infty} \frac{\Lambda}{2\eta} R_i(z') z' \phi_1(z') dz' + \int_{-\infty}^{+\infty} \frac{\Lambda^2}{2} R_i(z') z^2 \phi_1(z') dz' \phi_1(z)
\quad - \int_{-\infty}^{+\infty} \frac{\Lambda}{2\eta} R_r(z') \phi_1(z') dz' + \int_{-\infty}^{+\infty} \Lambda^2 R_i(z') z' \phi_1(z') dz' \phi_1(z)
\quad + \int_{-\infty}^{+\infty} \frac{\Lambda}{2\eta} R_r(z') z' \phi_2(z') dz' \phi_2(z)
$$

(24)
where we define $\beta = 2\eta^2\Lambda(t - \frac{(z' - z)}{2\eta})$. It is noteworthy that localized modes turn out to appear in the solution, which is essentially different from the envelope soliton of the integrable cubic NLSE with the first-order temporal derivative $[14]$. From our viewpoint, the localized modes here is a kind of internal modes whose occurrence is acknowledged to be intrinsic for nonintegrable models, for instance, the $\phi^4$ model $[15]$. Although it is not sufficient to conclude that the model we consider here is nonintegrable, the corresponding Lax representation is really difficult to find. Returning to the restriction condition imposed on the solution, we indicate that they can be satisfied by the extra freedoms we introduce in advance. In fact, they result in a sequence of novel equations

$$\eta_t = \frac{\Lambda}{2} \int_{-\infty}^{+\infty} \text{Re}(Pe^{i\theta}) \text{sech} z \text{d}z$$

$$\zeta_t = -\frac{\Lambda}{2} \int_{-\infty}^{+\infty} \text{Im}(Pe^{i\theta}) \text{tanh} z \text{sech} z \text{d}z$$

$$4\eta^2[\Lambda^{-2} + \frac{4}{3}\eta^2] \chi'_1$$

$$= \int_{-\infty}^{+\infty} \text{Re}(Pe^{i\theta}) \text{sech} z \text{d}z - 2\eta \Lambda \int_{-\infty}^{+\infty} \text{Im}(Pe^{i\theta}) z \text{tanh} z \text{sech} z \text{d}z$$

$$2\eta[\Lambda^{-1} - 4\eta^2\Lambda] \times [\theta_1 + 2\eta^{-1}K \chi'_1]$$

$$= \int_{-\infty}^{+\infty} \text{Im}(Pe^{i\theta}) \text{sech} z (1 - z \text{tanh} z) \text{d}z - 2\eta \Lambda \int_{-\infty}^{+\infty} \text{Re}(Pe^{i\theta}) \text{sech} z \text{d}z,$$

which govern the dynamic evolution of bright soliton in the time. In accordance with the usual definition of the width $w = 1/2\eta(2\zeta + b)$, we can derive a useful equation

$$w_t = 4\eta^2w^3 \int_{-\infty}^{+\infty} \text{Im}(Pe^{i\theta}) \text{tanh} z \text{sech} z \text{d}z$$

$$-w^2 \int_{-\infty}^{+\infty} \text{Re}(Pe^{i\theta}) \text{sech} z \text{d}z.$$
From Eq. (25), we compute $\eta_1 = -2\alpha_1 \eta \Lambda$, and then we obtain $\eta = \eta_0 e^{-2\alpha_1 \Lambda t_1} = \eta_0 e^{-2\alpha_1 \Lambda t}$ by integration. In this case, $\Lambda$ remains constant, thus, the propagation distance of a fixed point of the soliton is calculated by $x = \Lambda t$. As a result, we can write

$$\eta = \eta_0 e^{-2\alpha_1 x},$$

which recovers a well-known result in previous theories \[\[.\]

Secondly, we give a brief study of the perturbation $P[u] = -i\alpha_2 u \partial |u|^2 / \partial t$ accounting for the Raman effect. Using Eq. (7), we derive $\text{Im}(P e^{i\theta}) = -32\eta^4 \alpha_2 \tanh \text{sech}^3 z$, which has influence on the soliton’s width and velocity. By Eq. (29), we get $w_{t_1} = -8\alpha_2 (2\eta)^6 w^3 / 15$, integrating this equation yields

$$w = w_0[1 + \frac{16}{15} \alpha_2 (2\eta)^6 w_0^2 t_1]^{-\frac{1}{2}},$$

which exhibits that the soliton is narrowed under this effect. As well, we can derive that the velocity decreases, obeying

$$\Lambda = \Lambda_0[1 + \frac{16}{15} \alpha_2 (2\eta)^4 \Lambda_0^2 t_1]^{-\frac{1}{2}}.$$

Under the picture of waveform, the width is depicted differently, and the velocity of dynamic sense can not be defined.

In conclusion, we think that the waveshape presents a more transparent picture of directly physical significance than the waveform, especially in the study of soliton under perturbations. Hence, we believe that our theory is a nontrivial and necessary alternative for the subject. Moreover, the mathematical development in this paper is distinct and normal, its idea is helpful for the study of other soliton problems as well.

Acknowledgments. This work was support by the NNSF (No.19625409) and Nonlinear Project of the NSTC.
REFERENCES

[1] G. P. Agrawal, Nonlinear Fiber Optics (San Diego, Academic Press, INC. 1989)

[2] H. A. Haus and W. S. Wong, Rev. Mod. Phys. 68, 423 (1996)

[3] Yu. S. Kivshar and B. Luther-Davies, Phys. Rep. 298, 81 (1998)

[4] P. Diament, Wave Transmission and Fiber Optics (New York, Macmillan Publishing Co., 1990)

[5] J. N. Elgin, Phys. Rev. A47, 4331 (1993)

[6] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. 61, 763 (1989)

[7] D. J. Kaup, Phys. Rev. A42, 5689 (1990); A44, 4582 (1991)

[8] S. Burtsev and D. J. Kaup, J. Opt. Soc. Am. B14, 627 (1997)

[9] G. L. Lamb, Element of Soliton Theory (Wiley, New York, 1980)

[10] R. Scharf and A. R. Bishop, Phys. Rev. E47, 1375 (1993)

[11] D. W. McLaughlin and A. C. Scott, Phys. Rev. A18, 1652 (1978)

[12] M. H. Holmes, Introduction to Perturbation Methods (Springer-Verlag, New York, Inc. 1995)

[13] P. T. Leung, W. M. Suen, C. P. Sun and K. Young, Phys. Rev. E57, 6101 (1998)

[14] J. P. Keener and D. W. McLaughlin, Phys. Rev. A16, 777 (1977)

[15] Yu. S. Kivshar, D. E. Pelinovsky, T. Cretegny and M. Peyrard, Phys. Rev. Lett. 80, 5032 (1998); N. R. Quintero, A. Sánchez and F. G. Mertens, Phys. Rev. Lett. 84, 871 (2000)