Gravitational Lenses and Unconventional Gravity Theories

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We study gravitational lensing by clusters of galaxies in the context of the generic class of unconventional gravity theories which describe gravity in terms of a metric and one or more scalar fields (called here scalar–tensor theories). We conclude that if the scalar fields have positive energy, then whatever their dynamics, the bending of light by a weakly gravitating system, like a galaxy or a cluster of galaxies, cannot exceed the bending predicted by general relativity for the mass of visible and hitherto undetected matter (but excluding the scalar field’s energy). Thus use of general relativity to interpret gravitational lensing observations can only underestimate the mass present in stars, gas and dark matter. The same conclusion obtains within general relativity if a nonnegligible part of the mass in clusters is in the form of coherent scalar fields, *i.e.* Higgs fields. The popular observational claim that clusters of galaxies deflect light much more strongly than would be expected from the observable matter contained by them, if it survives, cannot be interpreted in terms of some scalar-tensor unconventional gravity theory with no dark matter. And if the observations eventually show that the matter distribution inferred via general relativity from the lensing is very much like that determined from the dynamics of test objects, then scalar–tensor unconventional gravity will be irrelevant for understanding the mass discrepancy in clusters. However, even a single system in which the dynamical mass determined from virial methods significantly exceeds the lensing mass as determined by general relativity, would be very problematic for the dark matter picture, but would be entirely consistent with unconventional scalar–tensor gravity theory.

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1. INTRODUCTION

The matter content of extragalactic systems, e.g., galaxies and clusters of galaxies, can be assessed in two independent ways: from the dynamics of test objects, e.g., extended rotation curves of disk galaxies delineated by neutral hydrogen, or the velocity dispersion profile of cluster delineated by galaxies, and from the deflection and focusing of electromagnetic radiation, e.g., gravitationally lensing cluster of galaxies. Because the “bending of light” by gravitational fields is intrinsically a relativistic effect, the second approach provides a way to test the relativistic aspects of gravitation at the extragalactic level. This is particularly important in light of a variety of proposals (Tohline 1982, Milgrom 1983a,b,c, Bekenstein & Milgrom 1984, Sanders 1984, Bekenstein 1987, Kuhn & Kruglyak 1987, Sanders 1986a,b, 1988, 1989, Manneheim & Kazanas 1989, Kazanas & Manneheim 1991, Sanders 1991, Bekenstein 1992, Romatka 1992, Milgrom 1993, Kennouku et al. 1993) that seek to replace the dark matter paradigm with one or another deviation of gravity from standard theory at extragalactic scales. Whatever the success of such schemes, and there have been many in the case of Milgrom’s, they must still face the challenge of providing a theory of light bending that correctly predicts the features of gravitational lenses, arcs, etc.

Now in the standard gravitational theory (general relativity [GR] and its Newtonian limit), the two approaches to measure the matter content of a system use the same tool to describe the predictions: the Newtonian gravitational field $\vec{g}$. Thus the acceleration of a test object in a galaxy or in a cluster of galaxies is just this $\vec{g}$ evaluated at the position of the object. And the bending angle of a light ray as it passes near the system along an approximately straight path may be stated to lowest order in $v^2/c^2$, where $v$ denotes the typical velocity in the system, as

$$\vartheta = \frac{2}{c^2} \int |g_\perp|dz$$

where $\perp$ denotes the component normal to the ray’s direction, and $dz$ is the element of length along the ray. In this context do observations of cosmic gravitational lenses affirm the dark matter paradigm in the context of standard gravity theory and rule out novel gravitational theories as alternatives to dark matter?

Unfortunately, up to now both observational and theoretical ambiguities have stood in the way of a sharp answer to this question. As detailed below, the observations of lensing systems, principally clusters of galaxies, do not yet permit a very accurate comparison of the lensing mass with the standard dynamical mass derived from kinematical measurements. And unconventional gravity theories have not, up to now, led to model independent predictions for the lensing that can be confronted with the evidence and the predictions from the dark matter picture. For instance, it has not been known in any generality whether in scalar–tensor (ST) type gravitational theories without dark matter, the scalar field is expected to bend light beyond what is accomplished by the luminous matter. We now review the status of the observations and unconventional gravity theoretical frameworks.

Regarding observations of gravitational lenses, in those cases of strong imaging events i.e., the formation of multiple images or Einstein rings, where the lens is an
individual galaxy (see Blandford & Naryan 1992), there is no indication of a discrepancy between the lensing and luminous masses (Breimer & Sanders 1993). This is not surprising because, typically, the impact parameter of photons forming multiple images is only a few kpc, and there is convincing evidence that the bright inner regions of galaxies are typically dominated by the visible component (Kent 1987, van Albada & Sancisi 1987). True, Maoz and Rix (1993) have claimed that the observed frequency and image separation of multiply-lensed quasars can only be achieved if galaxies – in particular elliptical galaxies – have dark halos i.e., the dark component makes a contribution to the image magnification and separation. However, this result depends upon rather uncertain model parameters and scaling relations for elliptical galaxies.

The issue of lensing by dark haloes has also been examined by observations of the lensing effects of foreground galaxies on background galaxies within projected separations between 10 kpc and 200 kpc i.e., weak gravitational imaging (Tyson et al. 1984). In this situation of large projected separations, the presence of the foreground mass does not create multiple images of a background galaxies, but does systematically distort their shapes. This is very relevant to the problem of the mass discrepancy because it is a light–bending probe of the mass distribution in the outer parts of galaxies where the dark halo is thought to dominate. In the present context it is interesting that the results of Tyson et al. are consistent with less extensive dark haloes than implied by measures of the extended rotation curves. For example, these results imply that a typical lens galaxy has a total mass less than $3 \times 10^{11} M_\odot$ within a radius of 80 kpc; this is comparable to the luminous mass of a typical elliptical galaxy i.e. a galaxy with the fiducial luminosity, $L^*$, in the Schechter (1976) luminosity function. Is this result a falsification of dark matter hypothesis? Unfortunately the Tyson et al. result remains controversial. For example, Kovner & Milgrom (1987) argue that the assumption of infinite distance for the background sources underestimates the total mass of the foreground galaxies by a factor of three. On the other hand, Breimer (1993), using a realistic distribution of redshifts for the background sources, has confirmed the result of Tyson et al. provided that the intrinsic source size is larger than the seeing disk. Clearly, further observations of this sort under conditions of good seeing would be most useful. But it can be said that, up to now, there is no conclusive evidence for lensing by dark matter in individual galaxies.

The most persuasive evidence in favor of lensing by dark matter is provided by the luminous arcs seen in the central regions of a dozen rich clusters (Blandford & Naryan 1992). It is now well-established that these are background galaxies imaged into partial Einstein rings by the foreground cluster (Lynds & Petrosian 1986, Soucail et al. 1987). The presence of one of these arcs implies, via eq. (1), the existence of more than $10^{13} M_\odot$ projected within the inner 200 kpc (the typical Einstein ring radius), masses which exceed, by a factor of ten or twenty, the projected luminous mass (Bergmann, Petrosian & Lynds 1990). Moreover, the observations of systematically distorted images of background galaxies within projected radii of 1/2 to one Mpc in two distant clusters (Tyson, Valdez, & Wenk, 1990) apparently requires a substantial extended dark component. In the cluster A 370, the most well–studied of the arc
systems, the cluster velocity dispersion predicted by the observed arc is consistent within the observational errors with the observed velocity dispersion of the cluster galaxies, assuming that the visible galaxies trace the dark mass distribution and that the velocity distribution is isotropic (Mellier et al. 1988, Breimer & Sanders 1992). There thus appears to be consistency between the virial mass and the lensing mass. This would seem to be evidence in favor of the dark matter hypothesis.

However, these observations do not yet rule out generic unconventional gravity theories without dark matter for two reasons. First of all, determinations of the mass and mass distribution by standard dynamical methods are notoriously crude. Virial methods, including those based upon the Jeans equation, which use measurements of the line-of-sight velocities of cluster galaxies are beset by uncertainties resulting from limited statistics, the unknown degree of anisotropy in the velocity distribution, the effects of subclustering and contamination, and the unknown distribution of the total (observed and unobserved) mass (The and White 1986, Merritt 1987). The potentially powerful method of using the observed density and temperature distributions of the hot X-ray emitting gas has been limited by the lack of high resolution spectral information; i.e., the temperature distribution is very uncertain (Sarazin 1988). For these reasons, dynamical masses are typically determined to an accuracy of a factor of two at best. Secondly, recent reanalyses of the gaseous component of clusters on the basis of results from ROSAT and earlier X-ray observatories indicate that the contribution of the hot gas to the total cluster mass may be greater than previously supposed – up to 40% in some cases (Hughes 1989, Eayles et al. 1991, Briel, Henry & Bohringer 1992, Bohringer, Schwarz & Briel 1993). This goes a long way towards relieving the discrepancy in clusters. Moreover, many rich clusters show evidence for cooling flows which can deposit up to \(10^{13} M_\odot\) in the central regions in a Hubble time, presumably in the form of cold gas or low mass stars (Sarazin 1988). This raises the possibility that, in clusters containing giant arcs, the lensing mass is nothing more exotic than hot gas or objects formed out of the hot gas. Future X-ray satellites, which will probe the temperature distribution of the gas in clusters, should clarify the issue of the mass distribution in the central regions and the contribution of the hot gas to the total mass in the lensing clusters.

In light of the above discussion, claims sometimes heard, e.g. Dar (1993), that the extant observations of gravitational lenses, when compared with dynamical determinations of the virial mass, certify the validity of general relativity in the extragalactic regime are evidently premature. They do not reckon with the intrinsically crude determination of \(\vec{g}\) provided by the dynamical measurements. Neither do they take cognizance of the dearth, in the extant literature, of clearcut predictions for gravitational light bending in theories of gravity, which aim at explaining the data without an appeal to dark matter. There is a good reason for this dearth. Bending of light in a gravitational theory depends on details of the theory: what fields constitute gravity, how are they coupled to gravitating matter, what equations are satisfied by these fields, \ldots? On the face of it, predictions of light bending would have to be worked out separately for each theory, and it certainly could not be ruled out \textit{a priori} that some model theory could produce light bending of the same strength as GR.
The actual situation is not so bad. In many known gravity theories where a scalar field plays a gravitational role, it does so by entering as a conformal factor in the relation between the primitive (Einstein) metric and the physical one. This fact implies, in a rough way, that the scalar field will not induce any light bending of itself (Bekenstein 1992, Romatka 1992). Two advances were required in order to turn this observation into the solid conclusion (this paper) that ST unconventional gravity theories cannot reconcile the lensing and dynamical (velocity dispersions) data for clusters without requiring much ordinary or exotic unobserved matter.

One advance was to establish the generic way to couple a scalar field to the metric. Bekenstein (1993) has shown that the relation between Einstein and physical metric may be generalized to the form of a disformal transformation in which the stretch of length in the special spacetime direction delineated by the scalar field gradient is either larger or smaller than the average one. Because of the anisotropic stretch, it is no longer automatic that the scalar field cannot cause bending of light. The disformal transformation, discussed in Sec. 2., seems to be the generic way to introduce a scalar field into a gravity theory which has a fighting chance to pass the tests of GR. Considerations of causality dictate (Bekenstein 1992) that the special direction undergoes a smaller stretch than the average over all directions. This particular sign turns out to be critical in the results to be described presently.

The second advance required was a relativistic calculation of light bending for a generic ST gravitational theory. The rough insight that, as a conformal factor in the physical metric, a scalar field cannot bend light, does not reckon with the fact that the stress energy of the scalar field can, by way of its contribution to the sources of the Einstein metric, contribute to bending of light. This contribution cannot be ignored: the evidence is that there is a significant non-luminous mass in clusters, which could very well turn out to be energy of coherent fields. A truly relativistic calculation of light bending will take into account this effect. Such is carried out in this paper, in Secs. 3. and 4. for the bending due to a spherically symmetric system in a generic gravitational theory involving scalar fields with positive energy densities.

The result (Sec. 5.) is surprising: despite all the subtleties mentioned, the bending angle cannot exceed that predicted by the standard theory for the actual matter (excluding the scalar field) present in the system. Thus it is a theorem that in a generic ST theory of gravity, the scalar field cannot enhance lensing. However, if a theory is presented as a candidate for gravity with no dark matter, it will, almost by construction, predict that the visible matter in the system generates a stronger gravitational field than would be predicted by standard theory. Therefore, modified gravitational theories based on metric and scalar fields and assuming no dark matter, and standard theory with dark matter must give distinctly different predictions for the comparison of dynamical and light bending data. That is, a modified gravity theory of the type we have in mind cannot mimic all predictions of standard theory with dark matter.

Thus, as we conclude in Sec.6., if future measurements of lensing and of velocity dispersions in clusters (or of the temperature distribution of the hot gas) certify the presently popular view that gravitational lensing requires the same mass discrepancies
in galaxy clusters as are implied by test body dynamical measurements, then unobserved matter makes up a large fraction of the mass in such clusters, whether gravity is best described by conventional theory or by some unconventional ST theory. In the later case the indication would be that in the cluster regime the predictions of the unconventional theory differ little from those of conventional gravity theory, with the difference between the theories being significant, if at all, only on scale of galaxies. By contrast, if future observations show that the lensing by clusters suggest masses below those implied by the velocity dispersions of galaxies or the distribution of hot gas, then dark matter would lose much of its appeal as an explanation for the entire mass discrepancy, while unconventional gravity theories of ST type would become quite relevant. In this light gravitational lensing is seen to be a crucial tool for the issue of dark matter vs. unconventional gravity.

In what follows we choose units such that the speed of light \( c = 1 \), but we shall retain Newton’s constant explicitly. Our conventions for gravitational theory are those of Misner, Thorne and Wheeler (1973). In particular we assume metric signature \((-, +, +, +)\).

### 2. CHARACTERIZING GRAVITATIONAL THEORIES

#### 2.1. From General Relativity to Scalar–Tensor Theory

Since we shall discuss gravitational theories which involve a scalar field, it is in order to start with a discussion of why a scalar field is the most natural entity that can be used to modify standard gravitational theory. The full characterization of a gravitational theory must be relativistic. The simplest such theory, GR, is formulated in terms of the metric of spacetime, \( g_{\alpha\beta} \), which enters into all dynamical equations for matter in gravitational fields in the same manner that Minkowski’s metric enters in gravity’s absence. The simplicity of GR is manifested in the absence of other gravitational fields apart from the metric, and in the fact that the dynamical action for the metric is the simplest invariant action that can be built in 4-D curved spacetime:

\[
S_g = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x
\]  

(2)

Here \( R \) is the scalar curvature built from the metric \( g_{\alpha\beta} \) and its first and second derivatives, and \( g \) is the metric’s determinant.

The success of GR in confronting the results of solar system precision experiments (Will 1992) suggests that the correct gravitational theory, if not identical to GR, must nevertheless share with it some key features. Therefore, any theory which attempts to displace dark matter as an explanation of the mass discrepancy in large astrophysical systems must show clear kinship with GR. The action (2) and the underlying metric \( g_{\alpha\beta} \) are such key features of GR that it is tempting to see them as required of any competing theory. We shall assume that a theory of gravity, to be credible, must have action (2) in some choice of local units (in some conformal frame).

Similarly, the role of a curved spacetime metric in rewriting special relativistic physics to include the effects of gravity seems a required feature of theories that are to compete successfully with GR. We shall assume, therefore, that the generic theory to
be considered uses some such metric. However, it does not follow that this last metric, call it $\tilde{g}_{\alpha\beta}$, is identical to $g_{\alpha\beta}$. In GR that is the case, and the equivalence is a facet of the strong equivalence principle which is part and parcel of GR. However, it is only the Einstein equivalence principle, a weaker principle, which is supported by many experiments. As stressed by Dicke long ago (Dicke 1962), those experiments are evidence only for the existence of the metric $\tilde{g}_{\alpha\beta}$, and it is logically consistent with all extant experiments for the metrics $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ to be distinct. The metric $g_{\alpha\beta}$, often referred to as the Einstein metric, shall here be called the gravitational metric; $\tilde{g}_{\alpha\beta}$ shall here be referred to as the physical metric. This last is the metric determined by rods and clocks built of matter.

In the ancient Nordström theory of gravity (Nordström 1913), which actually preceded GR, the metric $g_{\alpha\beta}$ was taken as flat, whereas $\tilde{g}_{\alpha\beta}$ was taken as conformal to it, with the conformal factor being the square of a scalar field which obeyed the standard massless scalar equation written with $g_{\alpha\beta}$. Thus Nordström’s theory is a pure scalar gravity theory. Despite its inherent simplicity, the theory was rejected early. It is amusing, given our concerns here, that its failing was predicting that there is no bending of light.

The Bergmann–Wagoner scalar tensor gravitational theories (Will 1992), of which Brans–Dicke theory is the simplest, are a natural marriage of Nordström’s theory with GR. The gravitational metric $g_{\alpha\beta}$ is taken to satisfy Einstein–like equations, whereas the physical metric $\tilde{g}_{\alpha\beta}$ is taken to be conformally related to it, with the conformal factor being intimately related to the dynamical scalar field of the theory, which together with the metric constitutes gravity.

2.2. Disformal Transformation

Of course, once a scalar field, $\psi$, is admitted into gravitational theory, the relation between physical and gravitational metrics may be more general. For example, consider a disformal relation (Bekenstein 1992) between the metrics,

$$\tilde{g}_{\alpha\beta} = \exp(2\psi) [A(I)g_{\alpha\beta} + L^2B(I)\psi,\alpha \psi,\beta]$$

(3)

where $L$ is a scale of length, and $A$ and $B$ are two functions of the invariant

$$I \equiv L^2g^{\alpha\beta}\psi,\alpha \psi,\beta$$

(4)

The form of the factor $\exp(2\psi)$ is dictated by the reasonable requirement that a shift in the zero of $\psi$ shall have no physical consequences. Indeed, adding a constant to $\psi$ merely changes the global units of length paced out by the physical metric.

The relation (3) is the most general between $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ based on a scalar field $\psi$ which involves only the first derivatives of $\psi$. Generalizations of the relation (3) involving second derivatives of $\psi$ obviously entail a higher derivative theory, i.e. one in which the matter field equations involve field derivatives of third or higher order. It is well known that such theories tend to display causality problems such as preacceleration and runaway solutions, and it seems wise to exclude them at the outset. It has also been shown (Bekenstein 1993) that the disformal relation (3) is the most general relation
between a given gravitational metric \( g_{\alpha\beta} \) and a Finsler geometry for the matter equations which is allowed by the requirements of causality and universality of free fall.

Let us recapitulate the structure of the generic ST gravitational theory. The gravitational action (2) is responsible for dynamics of the gravitational metric \( g_{\alpha\beta} \). The full Einstein–like equations are obtained from the variation of the full action

\[
S_{\text{tot}} = S_g + S_m + S_\psi
\]

with respect to \( g_{\alpha\beta} \). Here \( S_m \) is the action of the matter fields obtained by replacing the Minkowski metric in the special relativistic action for matter by \( \tilde{\eta}_{\alpha\beta} \), and partial derivatives by covariant derivatives constructed with \( \tilde{g}_{\alpha\beta} \). And \( S_\psi \) is an action for \( \psi \) built out of \( \psi \), its first derivatives, and \( g_{\alpha\beta} \). The dynamics of \( \psi \) comes from its field equation which is obtained by varying \( S_{\text{tot}} \) with respect to \( \psi \). They combine with the dynamics of \( g_{\alpha\beta} \) coming from the Einstein–like equations, to determine how \( \tilde{g}_{\alpha\beta} \) varies in spacetime.

### 2.3. Examples and Generic Theory

Let us look at some examples of theories encompassed by the stated framework. First, GR has \( S_\psi = 0 \) with \( \psi = 0 \) by fiat, with \( A = B = 1 \). Here the two metrics are equivalent. Brans–Dicke theory has \( A = 1, B = 0 \) and

\[
S_\psi = -\frac{1}{16\pi G} \int (\omega + 3/2)g^{\alpha\beta}\psi_{,\alpha} \psi_{,\beta}\sqrt{-g}d^4x
\]

with \( \Lambda = e^\psi \) being the scalar field in Dicke’s (nonmetric) form of the theory (Dicke 1962), and \( \omega \) being the usual Brans–Dicke parameter (really an inverse coupling constant in modern jargon). As is well known, Brans–Dicke theory is in good accord with all solar system tests of gravitational theory for \( \omega > 10^3 \). The Bergmann-Wagoner theories differ from the special Brans–Dicke case in that \( \omega = \omega(\psi) \) and a potential term is added to the kinetic part of the action (6). In the extragalactic context all the above theories are essentially indistinguishable.

Several ST theories have been proposed as alternatives to the dark matter hypothesis coupled to standard gravity theory. First in simplicity is Sanders’ two scalar field theory (Sanders 1986), which implicitly assumes \( A = 1 \) and \( B = 0 \), and in which a linear massless scalar field is accompanied by a massive scalar field whose kinetic action has a sign opposite to the usual one. This negative energy field is repulsive. The theory may pass the various tests of GR, and is not obviously in contradiction with the universality of free fall. Yet, the negative energy is a detractive feature.

Following Sander’s theory in complexity is the quadratic lagrangian (AQUAL) theory (Bekenstein and Milgrom 1984). It has \( A = 1, B = 0 \) and

\[
S_\psi = -\frac{1}{8\pi G L^2} \int F(I)\sqrt{-g}d^4x
\]

with \( F(I) \propto I^{3/2} \) for small \( I \) and \( F \propto I \) for large \( I \). This theory agrees with all solar system tests of relativity provided \( F(I) \) approaches its linear form rapidly enough as \( I \)
rises. It also reproduces much of the mass discrepancy phenomenology of spiral galaxies as encapsulated in Milgrom’s MOND formula (Milgrom 1983a,b). Sanders (1986) has proposed a variant of AQUAL in which $F(I)$ also turns linear in $I$ for sufficiently low $I$.

One problem with AQUAL and its variants is that the scalar field can propagate superluminally (Bekenstein 1987, 1988, 1992). This acausality is eliminated in the phase coupled gravity (PCG) theory (Bekenstein 1987, 1988, 1992). It has $A = 1$ and $B = 0$ and instead of one scalar field, it has two, the usual $\psi$ and a second one $\mathcal{A}$. The PCG dynamics comes from the action

$$
S_{\psi,\mathcal{A}} = -\frac{1}{2} \int g^{\alpha\beta}[\mathcal{A}_{,\alpha} \mathcal{A}_{,\beta} + \eta^{-2} \mathcal{A}^2 \psi_{,\alpha} \psi_{,\beta} + V(\mathcal{A}^2)] \sqrt{-g} d^4x
$$

where $\eta$ is a small coupling constant and $V$ is a potential. For small $\eta$ the term in $S_{\psi,\mathcal{A}}$ involving $\psi_{,\alpha}$ dominates the one involving $\mathcal{A}_{,\alpha}$. Neglecting the latter term eliminates $\mathcal{A}$ as a dynamical variable. Indeed, extremization of the action with respect to $\mathcal{A}$ establishes that $\mathcal{A}$ is a function of $g^{\alpha\beta}\psi_{,\alpha} \psi_{,\beta}$, so that to lowest order in $\eta$, the PCG action reduces to AQUAL’s, eq. (7). This suggests that PCG should be as successful as AQUAL in explaining the phenomenology of the mass discrepancy. Several in–depth studies (Bekenstein 1987, Sanders 1988, 1989) have shown that this is the case, but have also uncovered potential problems with PCG in the solar system (Bekenstein 1987), and in galaxies (Sanders 1988a).

From all the above examples we may write down the generic scalar action of interest:

$$
S_{\psi,\mathcal{A}} = -\int \mathcal{E}(\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{A}) \sqrt{-g} d^4x
$$

Here $\mathcal{E}$ is a function, and $\mathcal{I} \equiv g^{\alpha\beta}\psi_{,\alpha} \psi_{,\beta}$, $\mathcal{J} \equiv g^{\alpha\beta}\mathcal{A}_{,\alpha} \mathcal{A}_{,\beta}$ and $\mathcal{K} \equiv g^{\alpha\beta}\mathcal{A}_{,\alpha} \phi_{,\beta}$ are the three invariants that can be formed from first derivatives of $\mathcal{A}$ and $\phi$. Although it would not change any of the discussion to come, $\psi$ is not included as an argument of $\mathcal{E}$ because it would spoil the invariance of the theory under a shift of the zero of $\psi$. At little cost we could generalize our action to include any number of $\mathcal{A}$–type fields. Note that we do not assume in what follows that the kinetic part of the scalar’s action can be separated out, and is a quadratic form. We shall, however, assume that the function $\mathcal{E}(\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{A})$ is such that the scalar field always bears positive energy density. In Sec. 5 we translate this requirement into conditions on $\mathcal{E}$ and its derivatives.

### 3. Light Bending in General

#### 3.1. Equation for Light Rays

In all modified gravity theories proposed so far as alternatives to dark matter, $B = 0$ and the two metrics are taken as conformally related. Because Maxwell’s equations are conformally invariant, this seems to say that the scalar field, because it enters only as a conformal factor, has no influence on the propagation of light (Bekenstein 1992, Romatka 1992). Since the gravitational metric comes from Einstein–like equations, this would seem to say that all the ST theories with $B = 0$ will give the same bending of light
as GR. However, as mentioned in Sec. 1., observations of arcs suggest that the lensing in clusters of galaxies is much stronger than can be ascribed, via GR and formula (1), to the visible matter. This could mean that the modified gravity theories fail, and there is much dark matter in clusters. Alternatively, one can hope (Bekenstein 1992) that theories with $B \neq 0$ could play a role in reconciling the large lensing with the idea of unconventional gravity by breaking the conformal relation between the metrics. Below we shall see that this hope is dashed because causality requires the “wrong sign” for $B$.

The argument from conformality of the metrics to the absence of a light bending ascribable to the scalar field (Bekenstein 1992) leaves out an important factor. The scalar field, by virtue of its energy–momentum tensor, must make a contribution to gravitational metric $g_{\alpha\beta}$, and thus must lead to some extra light bending. The calculations to follow are designed to take this extra bending into account. To accomplish this they are carried out at the fully relativistic level, rather than starting from results like formula (1) which are already stated in nonrelativistic terms. The results are surprising in that if the scalar field bears strictly positive energy, the extra light bending is negative, i.e., if anything it decrements the general relativistic bending angle. As a result, no scalar–based gravity theory can make exactly the same predictions as GR for both dynamics of galaxies and clusters, and light bending. The opportunity to discriminate sharply between the two approaches thus arises.

We shall discuss “light bending” of either electromagnetic or neutrino radiation. The Maxwell, massless Dirac and Weyl equations are all conformally invariant. Hence the physics of light bending may be discussed equally well by using the physical metric $\tilde{g}_{\alpha\beta}$ or the reduced metric

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + \omega \psi_{\alpha} \psi_{\beta},$$

where $\omega \equiv L^2 B(I)/A(I)$. It is important in all that follows that elementary considerations of signature and causality require that $A(I) > 0$ while $B(I) \leq 0$ (Bekenstein 1992, 1993). Hence we may take $\omega \leq 0$.

For astrophysical systems the dimensions of the lensing system are large compared to the wavelengths of interest. Hence, we are only interested in geometric optics results. This means that our central question concerns the form of a null geodesic in the spacetime defined by the reduced metric $\bar{g}_{\alpha\beta}$ (we assume that “masses” like the neutrino mass or the plasma frequency in intergalactic space are small). Therefore, the tracks of light rays through a gravitating cluster, $x^{\mu} = x^{\mu}(\xi)$, where $\xi$ is a parameter along the ray, must satisfy

$$\bar{g}_{\alpha\beta} \frac{dx^{\alpha}}{d\xi} \frac{dx^{\beta}}{d\xi} = 0$$

together with any restrictions arising from symmetries and conservation laws.

### 3.2. Bending in Spherical Symmetry

It is sufficient for the points to be made to restrict attention to light bending by a static spherically symmetric configuration. As is well known, on symmetry grounds the metric of the corresponding spacetime, say $g_{\alpha\beta}$, may always be put in the form

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(12)
with $\nu$ and $\lambda$ functions of $r$ only. Similarly, we expect $\psi = \psi(r)$ only. Hence eq. (11) takes the form

$$-e^\nu \left( \frac{dt}{d\xi} \right)^2 + (e^\lambda + \omega \psi^2) \left( \frac{dr}{d\xi} \right)^2 + r^2 \left( \frac{d\theta}{d\xi} \right) = 0$$

(13)

where a prime denotes derivative with respect to $r$, and we have assumed that by angular momentum conservation, the path is confined to the equatorial plane.

Conservation of energy (static configuration) tells us that (Misner, Thorne and Wheeler 1973)

$$e^\nu \frac{dt}{d\xi} = E$$

(14)

where $E$ is the constant “energy” of the “photon”. We write “energy” because $E$’s value depends on the scaling of the parameter $\xi$, which is arbitrary. Likewise, conservation of angular momentum (spherical symmetry) leads to

$$r^2 \frac{d\theta}{d\xi} = \ell$$

(15)

with $\ell$ the “photon”’s “angular momentum”. $\ell$ is likewise changed by redefinition of $\xi$. However, the ratio $b \equiv \ell/E$ is independent of such redefinition, and may be interpreted as the impact parameter of the ray from the center of the configuration.

Substituting eqs. (14,15) into eq. (13) and replacing $\xi$ by $\theta$ as independent variable we cast the equation as

$$\left( \frac{dr}{d\theta} \right)^2 \frac{\ell^2}{r^2} (e^\lambda + \omega \psi^2) = e^{-\nu} E^2 - \ell^2 \frac{1}{r^2}$$

(16)

Its first quadrature is

$$\theta = \pm \ell \int \frac{(e^\lambda + \omega \psi^2)^{1/2} \, dr}{(e^{-\nu} E^2 - \ell^2 / r^2)^{1/2} \, r^2}$$

(17)

This is the full description of the path of the ray in the form $\theta = \theta(r)$. The integral depends only on the ratio $b = \ell/E$, but not separately on $E$ and $\ell$, thus realizing the independence on the parameter $\xi$.

How is the bending angle $\vartheta$ determined? Let $\theta = 0$ stand for the direction of incidence of the ray being bent so that the lower limit of the integral in eq. (17) is $r = \infty$. The integral from $r = \infty$ down to the turning point value, $r = r_{\text{turn}}$ where $d\theta/dr = 0$, with minus sign choice in the integral, is just half the rotation undergone by the radius vector as the photon comes in from infinity to closest approach. Were there no bending, the radius vector at closest approach would be at $\theta = \pi/2$. Hence, when there is bending, the net bending angle is given by

$$\vartheta = 2 \int_{r_{\text{turn}}}^{\infty} \frac{(e^\lambda + \omega \psi^2)^{1/2} \, dr}{(e^{-\nu} (r/b)^2 - 1)^{1/2} \, r} - \pi$$

(18)

with the $\pi$ representing the overall change of $\theta$ when the path is perfectly straight.

3.3. Bending in Linear Approximation
Equation (18) is exact, but the integral cannot be evaluated analytically for the generic metric of interest. However, some approximations that can make the integration tractable are in order. Recall that \( \lambda \) and \( \nu \) arise from a solution \( g_{\alpha\beta} \) of Einstein–like equations whose source is the \( T_{\alpha\beta} \) of the matter in the system as well as of the scalar field generated by that matter. Because the system in question is weakly gravitating, we expect the metric \( g_{\alpha\beta} \) to be nearly flat. Hence we may assume \(|\lambda|, |\nu| \ll 1\). Likewise, the physical metric \( \tilde{g}_{\alpha\beta} \) must be nearly flat by everyday experience in extragalactic astronomy. Of course, the flatness of the two metrics has to be evident in the same set of coordinates. Looking at eq. (3), we see that the conditions for common flatness of both metrics are that \(|\varpi|/\psi^2 \ll 1\) as well as \(A(I) \exp(2\psi) \approx 1\). From \(|\lambda|, |\nu|, |\varpi|/\psi^2 \ll 1\) we can obtain to linear order the following approximation to eq. (18):

\[
\vartheta = 2\int_{r_{\text{turn}}}^{\infty} \left(1 + \lambda/2 + \varpi\psi^2/2\right) \frac{dr}{(r^2/b^2)(1-\nu) - 1} r - \pi
\]

(19)

It would, however, be premature to expand the integrand in \( \nu \) because an integral with \((r^2 - b^2)^{3/2}\) in the denominator diverges at the turning point. Appendix A shows how to handle this subtlety, and how to compute the integral to linear order in \( \lambda, \nu \) and \( \varpi \). The result is

\[
\vartheta = \frac{b}{2} \int_{-\infty}^{\infty} \left(\frac{\lambda + \varpi\psi^2}{r^2} + \frac{\nu'}{r}\right) dz
\]

(20)

where an integration of the various quantities along a straight path with impact parameter \( b \) is meant. Here \( z \) stands for linear distance.

4. USING THE GRAVITATIONAL EQUATIONS

4.1. Correction to Standard Theory Bending

We now determine \( \lambda, \nu \) from the \( T_{\alpha\beta} \) distribution through the Einstein–like equations. These are

\[
e^{-\lambda}(r^{-2} - r^{-1}\lambda') - r^{-2} = 8\pi GT_t^t
\]

(21)

\[
e^{-\lambda}(r^{-1}\nu' + r^{-2}) - r^{-2} = 8\pi GT_r^r
\]

(22)

Here \( T_{\alpha\beta} = g_{\alpha\gamma} T^{\gamma\beta} \), and \( T^{\alpha\beta} \), the energy–momentum tensor, is given by the variational derivative

\[
T^{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(S_m + S_{\psi})}{\delta g_{\alpha\beta}}.
\]

(23)

The solution of eq. (21) is

\[
e^{-\lambda} = 1 - 2Gm(r)/r
\]

(24)

with

\[
m(r) = -4\pi \int_0^r T_t^t r^2 dr
\]

(25)

Usually \( m(r) \) is interpreted as the gravitating mass up to radius \( r \). Here, since \( T_t^t \) is defined with respect to the gravitational metric and not to the physical one, \( m(r) \) is mass in unconventional local units which do not correspond to those measured by
physical metersticks. Appendix B. develops the relation between the components of $T_{\alpha\beta}$ in the two frames of units. We find there that the factor that must be included in the integrand of eq. (25) to give the physical mass is very nearly unity for the situations of interest, so that $m(r)$ is very nearly the physical mass.

If we now expand $e^{-\lambda}$ in eq. (22) to $O(\lambda)$, cancel a $r^{-2}$ term, and neglect $\lambda \nu'/r$ in comparison with $\nu'/r$, we get

$$\nu' = \frac{\lambda}{r} + 8\pi GrT_r^r$$

(26)

Substituting this and the linear approximation $\lambda = 2Gm(r)/r$ in (26) gives

$$\vartheta = 2b \int_{-\infty}^{\infty} \frac{Gm(r)}{r^3} dz + b \int_{-\infty}^{\infty} \left( \frac{\varpi \psi'^2}{2r^2} + 4\pi GT_r^r \right) dz$$

(27)

Note that $Gm(r)/r^2$ is just the magnitude of the formal Newtonian gravitational field $\vec{g}$ produced by all the mass energy interior to the radius $r$. The factor $b/r$ is just the cosine of the angle between the radial direction and that normal to the ray at its nearest approach to the center. Hence the first integral in eq. (27) is identical to the standard prediction for $\vartheta$, eq. (1) (except for the fact that the scalar field’s energy has to be included here in calculating $\vec{g}$). The second integral gives the correction due to the scalar field which is not associated with its energy.

We next recall that $\varpi \leq 0$ by causality. Thus, even without knowing anything about the equation that determines $\psi$, we can state categorically that

$$\vartheta \leq 2 \int_{-\infty}^{\infty} |g^1|^2 dz + 4\pi Gb \int_{-\infty}^{\infty} T_r^r dz$$

(28)

We now proceed to break the integral over the stress $T_r^r$ into a contribution $\delta \vartheta$ from the matter (stars, gas, dark) and another, $\Delta \vartheta$, from the scalar field, and to show that the first is always negligible for the systems we have in mind.

4.2. Evaluating the Stress Term

The usual laws of energy–momentum conservation in special relativity must here take the covariant form

$$T_{\mu\nu} ;_{\nu} = 0$$

(29)

where the semicolon signifies covariant differentiation with respect to the gravitational metric $g_{\alpha\beta}$. The law (29) follows from coordinate invariance of the action $S_m + S_\psi$ (Landau and Lifshitz 1975), or from the Bianch identities combined with the gravitational field equations derived by varying $S_g + S_m + S_\psi$ with respect to $g_{\alpha\beta}$. Now the $r$ component of eq. (29) may be written in the form (Landau and Lifshitz 1975)

$$(\sqrt{-g}T_r^r)' - \frac{1}{2} \sqrt{-g}(\partial g_{\alpha\beta}/\partial r)T^\alpha\beta = 0$$

(30)

The assumed stationary and spherical symmetry forces $g_{\alpha\beta}$ and $T^\alpha\beta$ to be symmetric tensors. Thus eq. (30) takes the form

$$(e^{\frac{\lambda+\nu}{2} r^2 T_r^r})' - \frac{1}{2} e^{\frac{\lambda+\nu}{2} r^2} \left[ \nu'T_t^t + \lambda'T_r^r + 2(T_\theta^\theta + T_\phi^\phi)/r \right] = 0$$

(31)
Spherical symmetry also demands $T_\theta^\theta = T_\phi^\phi$. Substituting this and noting that the terms involving $\lambda'$ cancel each other, we obtain the important result

$$(e^{\nu/2}r^2 T_r^r)' = \frac{1}{2} e^{\nu/2}r^2 \left[ \nu' T_t^t + 4T_\theta^\theta / r \right]$$  \hspace{1cm} (32)$$

As the radial component of the energy–momentum conservation equations, eq. (32) is the exact statement of hydrostatic equilibrium.

Let us integrate this last equation over $r$ in the interval $[0,r]$. Assuming that $T_r^r$ is bounded as befits a pressure, we see that the boundary term at $r = 0$ vanishes. Hence

$$T_r^r(r) = \frac{e^{-\nu/2}}{2r^2} \int_0^r r^2 e^{\nu/2}(\nu' T_t^t + 4T_\theta^\theta / r) dr$$  \hspace{1cm} (33)$$

In our weakly gravitating cluster we may replace the factors $e^{\nu/2}$ by unity to get

$$T_r^r(r) \approx \frac{1}{2r^2} \int_0^r r^2 (\nu' T_t^t + 4T_\theta^\theta / r) dr$$  \hspace{1cm} (34)$$

which is our main working equation.

4.3. Matter and Scalar Contributions

There are two distinct contributions to $T_r^r$ which enter ultimately into the expression for light bending, eqs. (27,28). One belongs to the ordinary matter, and the other to the scalar field. First the ordinary matter (gas, stars, dark) contributes to $T_t^t$ and $T_\theta^\theta$. Because this matter has a velocity dispersion $\sim v \ll 1$, we expect its contribution to the stress $|T_t^t|$ to be of order $v^2|T_t^t|$. To assess the magnitude of the first term in the integral in eq. (34), let us write down the physical line element that follows from eq. (12) in accordance with the transformation, eq. (3):

$$d\tilde{s}^2 = -e^{\nu+2\psi} dt^2 + e^{\lambda+2\psi} dr^2 + r^2 e^{2\psi}(d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (35)$$

It is evident from this that in the nonrelativistic limit $\phi \equiv \nu/2 + \psi$ plays the role of physical gravitational potential. The anomalous term $\psi$ here encapsulates the effects of unconventional gravity. Any theory of gravity whose task is to explain the mass discrepancy in terms of anomalous gravity must be such that $\psi' > 0$ (anomalous contributions strengthens the gravitational force). We may conclude from this that since $v^2/r \sim \phi'$, $0 < \nu' < v^2/r$. Hence, according to eq. (34), the matter’s contribution to $T_r^r(r)$ is $kv^2 \langle \rho \rangle$, where $\rho$ denotes the matter’s energy density, $(-T_t^t)_m$, $k$ is a number of order unity and either sign, and the average used is defined by

$$\langle O \rangle_r \equiv \frac{2}{r^2} \int_0^r r O dr$$  \hspace{1cm} (36)$$

The scalar field makes equal contributions to $T_t^t$ and $T_\theta^\theta$ in eqs. (33,34). For according to eqs. (9,23) the scalar’s energy–momentum tensor is

$$\tau_\alpha^\beta = -E g_\alpha^\beta + 2(\partial E/\partial I)\psi_\alpha \psi_\beta + 2(\partial E/\partial J)A_\alpha A_\beta + (\partial E/\partial K)(A_\alpha \psi_\beta + \psi_\alpha A_\beta)$$  \hspace{1cm} (37)$$
Because of the symmetries, neither $\psi$ nor $A$ depend on $t$, $\theta$ or $\phi$. Therefore,

$$\tau_t^t = \tau_\theta^\theta = \tau_\phi^\phi = -\mathcal{E}$$  (38)

where it is understood that $\mathcal{E}$ is to be evaluated using the solution to the field equations for $\psi$ and $A$. Our assumption (Sec. 2) of positive scalar field energy density obviously requires that $-\tau_{tt} > 0$, which in view of eq. (12) implies that $\tau_t^t < 0$ and $\mathcal{E} > 0$, at least for a static situation.

Because $\tau_t^t$ appears in eq. (34) multiplied by $\nu' < \nu^2/r$, the first term in the integral in eq. (34) is negligible compared to the second. Replacing in this last $\tau_\theta^\theta$ by $\tau_t^t$, and recalling the definition of averaging, we see that the scalar’s contribution to $T_r^r(r)$ is $\langle \tau_t^t \rangle_r$. We may summarize all the above in the equation

$$T_r^r(r) = k\nu^2 \langle \rho \rangle_r + \langle \tau_t^t \rangle_r$$  (39)

where $|k|$ may vary from system to system, and even with $r$ in accordance with the distribution of velocity and density, but should usually be of order unity.

5. CONSEQUENCES OF POSITIVE ENERGY

5.1. Scalar Field Decreases Bending

To assess the size of the line integral over $\langle \rho \rangle_r$ which enters into eq. (28) by virtue of eq. (39) we shall appeal to the Poisson equation for the Newtonian gravitational field $\vec{g}_m$ generated by the matter. In cylindrical coordinates this is

$$\frac{dg_m^\perp}{db} + \frac{g_m^\perp}{b} + \frac{dg_m^z}{dz} = -4\pi G\rho$$  (40)

where we assume spherical symmetry to drop derivatives with respect to $\varphi$, and denote by $b$ the coordinate in the direction of $\perp$. Multiplying this equation by $b$ and integrating over $z$ gives

$$\left(b \frac{d}{db} + 1\right) \int_{-\infty}^{\infty} g_m^\perp dz = -4\pi Gb \int_{-\infty}^{\infty} \rho dz$$  (41)

where we have assumed that $g^z \to 0$ as $|z| \to \infty$. Hence, taking into account that $\langle \rho \rangle_b$ is of the same order as $\rho(b)$, we see that the contribution of the matter to the last term in eq. (28) is

$$\delta \vartheta \sim -k\nu^2 \left(b \frac{d}{db} + 1\right) \int_{-\infty}^{\infty} g_m^\perp dz$$  (42)

Now, whatever the $b$ dependence of the last integral, its logarithmic derivative in eq. (42) is unlikely to differ from itself by more than an order of magnitude. Since $\nu$ in a cluster is typically $10^3$ km s$^{-1} \approx 3 \times 10^{-3}c$, it is clear that $\delta \vartheta$ may be neglected as compared with the first contribution to $\vartheta$ in eq. (28). (In a relativistic cluster, e.g. cluster of neutron stars at galactic center, $\nu \sim 1$ and $\delta \vartheta$ would no longer be negligible).

According to eqs. (28,36,39), the scalar field’s contribution to the bending angle through the last term in eq. (28) is

$$\Delta \vartheta = 8\pi Gb \int_{-\infty}^{\infty} \frac{dz}{r^2} \int_0^r \tau_t^t r dr$$  (43)
It is already clear from the positivity of scalar field energy \(-\tau_t^t\) that \(\Delta \vartheta < 0\). As we shall see, this negative contribution dominates the positive one of the scalar field to the first term in \(\vartheta\) as given by eq. (28). We may transform the inner integral in eq. (43) by appealing to the Poisson equation for the formal Newtonian potential \(\Phi_s\) produced by the scalar’s energy density. In spherical coordinates this is

\[
\frac{1}{r} \frac{d^2(r\Phi_s)}{dr^2} = -4\pi G \tau_t^t
\]

Multiplying through by \(r\), integrating from \(r = 0\) to \(r\), and noting that \(d\Phi_s/dr\) must be finite at \(r = 0\), we find

\[
4\pi G \int_0^r \tau_t^t r dr = -\frac{d\Phi_s(r)/dr}{r} + \frac{\Phi_s(0) - \Phi_s(r)}{r^2}
\]

(45)

Of course, since \(\Phi_s\) comes from eq. (44), \(d\Phi_s(r)/dr = Gm_s(r)/r^2\), where \(m_s(r)\) is the scalar’s contribution to the total mass, c.f. eq. (25). Thus if we substitute eq. (45) in eq. (43) and that into eq. (28), and compare the last one with eq. (27), we see that the scalar’s contribution to the first integral in eq. (28) is exactly cancelled out. We are thus left with

\[
\vartheta \leq 2 \int_{-\infty}^{\infty} |g_{m} | dz + 2b \int_{-\infty}^{\infty} \frac{\Phi_s(0) - \Phi_s(r)}{r^2} dz
\]

(46)

which makes it clear that the bending angle is negative. Such a scalar–energy dominated equilibrium configuration would thus behave as a divergent lens when bending light!

5.2. Divergent Gravitational Lenses?

What happens in the hypothetical case when the scalar field energy becomes the dominant part of the gravitational lens’s mass? We continue to assume that the configuration is nonrelativistic in the sense that \(\lambda, |\nu|, \text{etc.}\) are small compared to unity. Then the same arguments as above lead to the cancellation of the scalar’s contribution to the first integral in eq. (28). However, now we may neglect the matter’s contributions \(\delta \vartheta\) and \(\int_{-\infty}^{\infty} |g_{m} | dz\) as compared to the last term in eq. (45). The result is

\[
\vartheta < 2 \int_{-\infty}^{\infty} |g_{m} | dz
\]

(47)

This important result tells us that the scalar fields of the theory fail to augment the light bending ability of the matter, and may even weaken it.
without making the nonrelativistic approximation by returning to eq. (33), and replacing $T_\alpha^\beta \rightarrow \tau_\alpha^\beta$. By using eq. (38) we may recast eq. (33) into the form

$$\tau_r^r(r) = \frac{e^{-\nu/2}}{r^2} \int_0^r (r^2 e^{\nu/2})' \tau_t^t dr$$

(49)

Now the factor $r^2 e^{\nu/2}$ should grow with $r$ because the potential $\nu/2$ is expected to increase with $r$ (attractive gravitational force), and because of the increasing factor $r^2$. Even if over some range $\nu$ were to decrease with $r$, it seems highly unlikely that $e^{\nu/2}$ would decrease faster than $r^{-2}$. Thus positive energy density of the scalar field means here that at all radii $\tau_r^r < 0$.

We now multiply eq. (49) by $r^2 e^{\nu/2}$ and differentiate with respect to $r$. The result may be put in the form

$$\left(\tau_r^r\right)' = \left(r^2 e^{\nu/2}\right)' e^{-\nu/2} r^{-2} \left(\tau_t^t - \tau_r^r\right)$$

(50)

However, according to eq. (37)

$$\tau_t^t - \tau_r^r = -2 e^{-\lambda} \left[\frac{\partial E}{\partial I} \psi, r^2 + \frac{\partial E}{\partial J} A, r^2 + \frac{\partial E}{\partial K} A, r^2 \psi, r^2\right]$$

(51)

We now argue that positive energy density of the scalar field with respect to any observer implies that the quantity in square brackets in eq. (51) is positive. If $U^\alpha$ denotes the timelike 4-velocity of an observer, the scalar field will exhibit positive energy density to him if $T_\alpha^\beta U^\alpha U^\beta > 0$. Since $U^\alpha U_\beta = -1$, substituting eq. (37) into this condition we obtain the condition

$$\mathcal{E} + 2 \left[\left(\frac{\partial E}{\partial I}\right) (\psi, \alpha U^\alpha)^2 + \left(\frac{\partial E}{\partial J}\right) (A, \alpha U^\alpha)^2 + \left(\frac{\partial E}{\partial K}\right) A, \alpha U^\alpha \psi, \beta U^\beta\right] > 0$$

(52)

Now, we already know that $\mathcal{E} > 0$. Because $U^\alpha$ occurs in the quadratic form but not in $\mathcal{E}$, positivity of this expression for any timelike $U^\alpha$ is guaranteed only if the quadratic form is positive definite. This requires

$$\frac{\partial E}{\partial I} > 0; \quad \frac{\partial E}{\partial J} > 0$$

(53)

and

$$\left(\frac{\partial E}{\partial K}\right)^2 < 4 \left(\frac{\partial E}{\partial I}\right) \left(\frac{\partial E}{\partial J}\right)$$

(54)

From these three conditions it follows that the quadratic form in the square brackets in eq. (51) is positive for any $\psi, r$ and $A, r$. But then $\tau_t^t - \tau_r^r < 0$, so that it follows from eq. (50) that not only is $\tau_r^r$ negative for all $r$, but it also decreases outward. In particular, $\tau_r^r(r) < -\left|\tau_r^r(0)\right|$. This, of course, means that $\tau_r^r$ can never reach zero, as would befit a finite configuration. This behavior also means, by eq. (51) that

$$-\tau_t^t(r) > \left|\tau_r^r(0)\right| + 2 e^{-\lambda} \left[\frac{\partial E}{\partial I} \psi, r^2 + \frac{\partial E}{\partial J} A, r^2 + \frac{\partial E}{\partial K} A, r^2 \psi, r^2\right] > 0$$

(55)

which makes it clear that the energy density cannot vanish asymptotically as would be required for a configuration of finite mass. We conclude that a static spherically symmetric self–gravitating configuration of scalar field is untenable. It seems likely that
the result would still hold for a configuration containing matter in which the scalar field dominates the mass of the matter. This makes the existence of divergent gravitational lenses very doubtful.

6. CONCLUSIONS AND CAVEATS

Comparing eqs. (1,47) we conclude that in a gravitational theory where gravity is mediated only by a metric and some scalar fields, the bending of light by a weakly gravitating system, like a galaxy or a cluster of galaxies, cannot exceed the bending predicted by GR for the mass of visible and hitherto undetected matter (but excluding the scalar’s energy). This means that if gravity is properly described by a ST theory, and one uses the standard theory’s formula (1) to interpret gravitational lensing observations, one can only underestimate the mass present in stars, gas and dark matter.

The same conclusion obtains within GR if matter includes one or more scalar fields, i.e. Higgs fields. We obtain this case in our formalism by setting $A(I) = 1$, $B(I) = 0$ and $\psi = 0$. Then $\tilde{g}_{\alpha \beta} = g_{\alpha \beta}$. The $\mathcal{A}$ fields may stand for the scalar fields in question, and we may take over the results of our previous discussion: if gravity is described by GR, but matter includes various scalar fields, use of formula (1) to interpret gravitational lensing observations can only underestimate the mass of matter present not in scalar fields.

Thus the observational claim that clusters of galaxies deflect light more strongly than would be expected from the observable matter contained by them, if it survives, cannot be interpreted in terms of scalar-tensor gravity without dark matter. Specifically if follow-up observations eventually certify that the matter distribution inferred via standard theory from the lensing is very much like that determined from the dynamics of test objects or the temperature distribution of the X–ray emitting gas, then departures from standard gravity of ST form cannot play a role in the inner parts of clusters of galaxies which act as the lenses.

A similar conclusion cannot be drawn for galactic gravitational lenses since lensing by single galaxies, if observed at all, is produced by the inner regions where, in all likelihood, the visible matter is all there is. Neither is our last conclusion necessarily at variance with Milgrom’s MOND scheme since the accelerations in the cores of rich clusters seem to be large compared to $10^{-8}$ c.g.s., the scale at which anomalous gravity effects set in according to MOND. In fact, what we have demonstrated is that ST unconventional gravity theories are irrelevant for understanding gravitational lensing by galaxy clusters. This does not exclude the possibilities that this type of theory may be of relevance for understanding the mass discrepancy in disk galaxies, or that the proper relativistic formulation of ideas like MOND involves more esoteric gravitational physics, e.g. modification of Newton’s second law (Milgrom 1983a, 1993).

In fact, it must be stressed that the very fact that a lensing mass, when determined by standard theory, exceeds the directly detectable mass in stars or gas does not necessarily falsify unconventional gravity theories in general. There may be an as yet undetected component in clusters. A significant fraction, perhaps as large as 40%, of the mass in clusters, which was once considered to be dark, is now known to be in the
form of hot gas. The matter in cooling flows must go somewhere, and such flows can deposit a substantial mass in the central regions of clusters. It would be interesting if clusters containing arcs all showed evidence for cooling flows.

A system in which the classical dynamical mass, determined from virial methods or the distribution of hot gas, significantly exceeds the lensing mass as determined by GR, would be very problematic for the dark matter picture, but would be entirely consistent with unconventional ST gravity. This is not an entirely hypothetical situation. As mentioned in Sec. 1., the Tyson et. al measurements may be an example of this at the galactic scale. At cluster scales we may point out that what has been noticed in the sky are the striking examples of distant rich clusters containing luminous arcs or arclets; of equal importance may be the many examples of distant rich clusters without arcs or arclets (too low lensing masses).

Our main results, e.g. eqs. (46,47) were obtained under the assumption that the scalar field configuration is truly static. There is thus a loophole in our conclusions. An equilibrium configuration containing a coherent complex scalar field oscillating harmonically in time, as in the boson star models (Liddle and Madsen 1992), is not covered by eq. (46). Neither is an equilibrium configuration in the PCG theory where scalar energy is nonnegligible, and where the cosmological expansion causes the field $\psi$ to vary in time in approximately linear fashion (Sanders 1989). In future work we shall examine these situations.

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APPENDIX A.

Let us rewrite eq. (19) as

$$\vartheta = -4 \left[ \frac{\partial}{\partial \alpha} \int_{r_{\text{turn}}}^{\infty} \left( 1 + \frac{\lambda}{2} + \omega \psi'^2/2 \right) \left( r^2/b^2 \right) \left( 1 - \nu - \alpha \right)^{1/2} r^{-1} dr \right]_{\alpha = 1} - \pi$$  \hfill (56)

and then expand to first order in $\nu$. The results are

$$\vartheta = \vartheta_1 + \vartheta_2$$  \hfill (57)

with

$$\vartheta_1 = \int_{b}^{\infty} \frac{2 + \lambda + \omega \psi'^2}{(r^2/b^2 - 1)^{1/2} r} dr - \pi$$  \hfill (58)

$$\vartheta_2 = 2 \left[ \frac{\partial}{\partial \alpha} \int_{b/\alpha}^{\infty} \frac{r \nu / b^2}{(r^2/b^2 - \alpha)^{1/2}} \right]_{\alpha = 1} \hfill (59)$$

We proceed to simplify the expression for $\vartheta_1$. Two terms cancel out because of the integral

$$2 \int_{b}^{\infty} \frac{dr/r}{(r^2/b^2 - 1)^{1/2}} = \pi$$  \hfill (60)

Next, we transform the independent variable from $r$ to $z = \pm (r^2 - b^2)^{1/2}$ which is just the Euclidean length along a ray whose impact parameter is $b$, with zero length taken at the point of closest approach to the center, $r = b$. Thus

$$\vartheta_1 = \frac{b}{2} \int_{-\infty}^{\infty} \frac{\lambda + \omega \psi'^2}{b^2 + z^2} dz$$  \hfill (61)

where we have extended the integral to $z = -\infty$ and divided by 2.

We now turn to $\vartheta_2$. Integration by parts converts it to

$$\vartheta_2 = \frac{b}{2} \left\{ \frac{\partial}{\partial \alpha} \left[ \lim_{r \to \infty} \nu (r^2 - \alpha b^2)^{1/2} \right] - \int_{\sqrt{\alpha b}}^{\infty} \nu' (r^2 - \alpha b^2)^{1/2} dr \right\}$$  \hfill (62)

By expanding the square root under the limit in $\alpha b^2/r^2$, we verify that after differentiation with respect to $\alpha$, the corresponding term vanishes. Carrying out the $\alpha$ differentiation of the integral, setting $\alpha = 1$, and passing from variable $r$ to $z$ we have

$$\vartheta_2 = \frac{b}{2} \int_{-\infty}^{\infty} \frac{\nu'}{(b^2 + z^2)^{1/2}} dz$$  \hfill (63)

Therefore, the full bending angle is

$$\vartheta = \frac{b}{2} \int_{-\infty}^{\infty} \left( \frac{\lambda + \omega \psi'^2}{r^2} + \frac{\nu'}{r} \right) dz$$  \hfill (64)

where again $r \equiv (b^2 + z^2)^{1/2}$. 
APPENDIX B.

We begin by writing the tilde analog of eq. (23).

\[ \tilde{T}^{\alpha\beta} \equiv -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta(S_m + S_\psi)}{\delta \tilde{g}_{\alpha\beta}}. \] (65)

In order to simplify the algebra we restrict ourselves to the case where \( A \) and \( B \) vary slowly, and neglect their derivatives. Since according to eq. (3)

\[ \frac{\delta g_{\alpha\beta}}{\delta \tilde{g}_{\alpha\beta}} = A^{-1} e^{-2\psi}, \] (66)

we find by the chain rule that

\[ \tilde{T}^{\alpha\beta} = (g/\tilde{g})^{1/2} A^{-1} e^{-2\psi} T^{\alpha\beta}. \] (67)

Of greater interest are the mixed components \( \tilde{T}^{\alpha\beta} \). Contracting both sides of eq. (67) with \( \tilde{g}_{\gamma\alpha} \) and using the relation (3) and the definition \( \psi^\prime \equiv B/A \) we find that

\[ \tilde{T}^{\gamma\beta} = (g/\tilde{g})^{1/2} \left( T^{\gamma\beta} + \varpi T^{\alpha\beta} \psi_\alpha \psi_\gamma \right) \] (68)

where indices of \( T^{\alpha\beta} \) are lowered with \( g_{\alpha\beta} \) and those of \( \tilde{T}^{\alpha\beta} \) with \( \tilde{g}_{\alpha\beta} \).

The above result is general. For a static spherically symmetric situation the two metrics are given by eqs. (12,35) from which we infer that

\[ (g/\tilde{g})^{1/2} = A^{-2} e^{-4\psi} (1 + e^{-\lambda \varpi \psi^\prime 2})^{-1/2} \] (69)

Further, because of the symmetries, \( \psi_\alpha = \psi_r \delta_\alpha^r \), and the mixed tensor \( T^{\gamma\beta} \) is diagonal. Hence we find

\[ \tilde{T}_l^l / T_l^l = \tilde{T}_\theta^\theta / T_\theta^\theta = \tilde{T}_\varphi^\varphi / T_\varphi^\varphi = A^{-2} e^{-4\psi} (1 + e^{-\lambda \varpi \psi^2})^{-1/2} \] (70)

and

\[ \tilde{T}_r^r / T_r^r = A^{-2} e^{-4\psi} (1 + e^{-\lambda \varpi \psi^2})^{1/2} \] (71)

Of course all the above results apply to the scalar field energy momentum tensor \( \tau^{\alpha\beta} \) alone.

We may also use eq. (70) to reexpress eq. (25) for the mass function \( m(r) \) in terms of the physical energy density \( -\tilde{T}_l^l \):

\[ m(r) = -4\pi \int_{0}^{r} A^2 e^{4\psi} (1 + e^{-\lambda \varpi \psi^2})^{1/2} \tilde{T}_l^l r^2 dr \] (72)

Since in the weak field situations under consideration \( Ae^{2\psi} \approx 1 \), \( \lambda \ll 1 \) and \( |\varpi|\psi^2 \ll 1 \), it is clear that \( m(r) \) is very nearly the physical mass interior to radius \( r \).
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