THE CONLEY INDEX FOR DISCRETE DYNAMICAL SYSTEMS AND THE MAPPING TORUS

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Abstract. The Conley index for flows is a topological invariant describing the behavior around an isolated invariant set $S$. It is defined as the homotopy type of a quotient space $N/L$, where $(N,L)$ is an index pair for $S$. In the case of a discrete dynamical system, i.e., a continuous self-map $f: X \to X$, the definition is similar. But one needs to consider the index map $f_{(N,L)}: N/L \to N/L$ induced by $f$. The Conley index in this situation is defined as the homotopy class $[f_{(N,L)}]$ modulo shift equivalence. The shift equivalence relation is rarely used outside this context and not well understood in general. For practical purposes like numerical computations, one needs to use weaker algebraic invariants for distinguishing Conley indices, usually homology. Here we consider a topological invariant: the homotopy type of the mapping torus of the index map $f_{(N,L)}$. Using a homotopy type offers new ways for comparing Conley indices – theoretically and numerically. We present some basic properties and examples, compare it to the definition via shift equivalence and sketch an idea for its construction using rigorous numerics.

1. Introduction

The classical Conley index for flows as introduced by Conley (1978) describes the local behavior of a flow around an isolated invariant set $S$. It is defined using a so-called index pair $L \subset N$ of compact sets in the phase space: The set $S$ is the invariant part of $\text{cl}(N \setminus L)$ and $S \subset \text{int}(N \setminus L)$. A trajectory that leaves $N$ has to pass through $L$. The Conley index of $S$ is then the pointed homotopy type of the quotient $N/L$.

Starting with Robbin and Salamon (1988), an analogue of this index for discrete dynamical systems (continuous self-maps) has been developed. The definition by Franks and Richeson (2000), equivalent to the one presented by Szymczak (1995), uses the homotopy class of a so-called index map $[f_{(N,L)}: N/L \to N/L]$ up to shift equivalence. We call this quite general version of the Conley index the shift equivalence index in this article. Even though this index is homotopy theoretic in spirit, it is not defined as the homotopy type of a topological space – in contrast to the version for flows.

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Algebraic versions were already introduced before the shift equivalence index: One can use functors like homology and then equivalence relations on linear maps (Mrozek 1990).

In this article, we consider the following invariant of the shift equivalence index: the homotopy type of the mapping torus of the index map $f_{(N,L)}$. The main features of this mapping torus index are: Since it is the homotopy type of a space, it offers additional ways of extracting information than the homological Conley index usually used for numerical computations. A shift equivalence between maps induces a homotopy equivalence between their mapping tori. The reduced mapping torus index is the classical flow Conley index for the suspension semiflow. The fundamental group of the mapping torus index contains information not apparent in the homological Conley index. And its construction as a cell complex seems to be feasible using rigorous numerics as described in Kaczynski et al. (2004) – at least as a cell complex with the correct homology.

The definition of the mapping torus index and why it makes sense is presented in Sect. 2 and Sect. 3. Its main properties are shown in Sect. 4. In Sect. 5, we compare our definition with the shift equivalence index. Under strong assumptions, a homotopy equivalence of mapping tori can yield a shift equivalence of self-maps. After recalling algebraic invariants of the mapping torus, we consider some examples in Sect. 6. In Sect. 7, we show that the reduced mapping torus index coincides with the flow Conley index of the suspension semiflow. Sect. 8 sketches how mapping tori could be constructed without having full information about a map, but only certain enclosures of its graph.

Despite being apparently a coarse invariant, the mapping torus offers new ways for understanding shift equivalence. Representing the Conley index as a space, one can potentially use methods and algorithms developed for comparing homotopy types.

Basic definitions. We use the following basic notions of pointed and unpointed topology. For a topological space $X$ and a subspace $Y \subset X$ with $Y \neq \emptyset$, we consider the quotient space $X/Y$ as a pointed space with base point $[Y]$. We let $X/\emptyset := X \coprod \{\ast\}$, the disjoint union of $X$ and the one-point space $\{\ast\}$. An asterisk $\ast$ usually denotes the respective base point.

Given a pointed space $(X,x_0)$ and an unpointed space $Y$, they form a reduced product:

$$X \times Y := (X,x_0) \times Y := \frac{X \times Y}{\{x_0\} \times Y}.$$  

A homotopy from $f$ to $g$ is a continuous map $H : X \times [0,1] \to Y$ such that $H(\cdot,0) = f$ and $H(\cdot,1) = g$. Given pointed (base point preserving) maps on pointed spaces, a pointed homotopy is a pointed continuous map $\tilde{H} : (X,x_0) \times [0,1] \to (Y,y_0)$ with analogous properties. We often omit the base point when this does not lead to confusion.
2. Definition of the mapping torus index

From here throughout this article, we let $X$ be a locally compact metric space, and we let $f: X \to X$ be a discrete dynamical system, i.e., a continuous map. Let $M \subset X$ and $x \in M$. A solution of $f$ in $M$ through $x$ is a sequence $\gamma: \mathbb{Z} \to M$ such that for all $n \in \mathbb{Z}: \gamma(n+1) = f(\gamma(n))$ and $\gamma(0) = x$. The invariant subset of $M$ is

$$\text{Inv}(M,f) = \{x \in M \mid \text{there is a solution of } f \text{ in } M \text{ through } x\}.$$

We call a set $S \subset X$ isolated invariant if it is compact and has a neighborhood $M$ such that $\text{Inv}(M,f) = S$.

We use the definition of index pairs from Robbin and Salamon (1988): If $A \subset B \subset X$, where $A$ and $B$ are compact, we call $(B,A)$ a compact pair. Given an isolated invariant set $S \subset X$, a compact pair $(N,L)$ is an index pair for $(S,f)$ if

$$\text{Inv}(\text{cl}(N \setminus L),f) = S \subset \text{int}(N \setminus L)$$

and the map $f_{(N,L)}: N/L \to N/L$,

$$f_{(N,L)}([x]) := \begin{cases} [f(x)] & \text{if } x,f(x) \in N \setminus L, \\ [L] & \text{otherwise,} \end{cases}$$

is continuous. In this case, we call $f_{(N,L)}$ the index map.

For a continuous map $\kappa: P \to P$ on some space $P$, we let its (unreduced) mapping torus be

$$T(\kappa) := \frac{P \times [0,1]}{(x,1) \sim (\kappa(x),0)}.$$  

For a pointed continuous map $\kappa: (P,p_0) \to (P,p_0)$ on some pointed space $(P,p_0)$, let its reduced mapping torus be

$$T_\bullet(\kappa) := \frac{(P,p_0) \times [0,1]}{(x,1) \sim (\kappa(x),0)}.$$  

Its homotopy type depends only on the homotopy class of $\kappa$ (see Ranicki 1987, Prop. 6.1(i)). This gives us two ways of defining a mapping torus (Conley) index.

**Definition 2.1.** Let $(N,L)$ be an index pair for $(S,f)$. The (unreduced) mapping torus index of $(S,f)$ is the homotopy type of $T(f_{(N,L)})$ (an unpointed space). We write

$$\text{CT}(S,f) := [T(f_{(N,L)})].$$

The reduced mapping torus index of $(S,f)$ is the pointed homotopy type of $T_\bullet(f_{(N,L)})$, we abbreviate this as

$$\text{CT}_\bullet(S,f) := [T_\bullet(f_{(N,L)})].$$

We start with basic examples (more are given in Sect. 6). The empty invariant set $S = \emptyset$ has an index pair $(\emptyset,\emptyset)$ with index map the pointed map on the one-point space $\{^*\}$. We call the mapping torus index of $(\emptyset,f)$ trivial. For the definitions above, this means: The unreduced mapping torus index
$CT(S,f)$ is trivial if it is the homotopy type of the circle $S^1$. The reduced mapping torus index $CT_\bullet(S,f)$ is trivial if it is the pointed homotopy type of the one-point space $\{\ast\}$.

Even though the index map is pointed, we mainly consider the unreduced mapping torus in this article. It can contain finer information: For example, if $f_{(N,L)}$ is the degree-2 map on the circle $S^1$, then $\pi_1(T_{\#}(f_{(N,L)}))$ is the trivial group, whereas $\pi_1(T(f_{(N,L)}))$ is not isomorphic to the fundamental group of the circle $S^1$, as we show in Example 6.1.

3. THE MAPPING TORUS INDEX IS WELL-DEFINED

Using that the shift equivalence index is well-defined (Franks and Richeson 2000), one could apply Theorem 5.2 to show that the mapping torus index is also well-defined. In this section, we present a more direct proof. We first recall Theorem 3.5, which was already shown by Robbin and Salamon (1988). But there it is assumed that $X$ is a manifold and $f$ a diffeomorphism. We recall details of the theory therein to show that the theorem also holds in our context of a self-map on the metric space $X$. For a continuous map $\kappa: P \to P$, consider the map

$$(P \times [0, 1]) \times [0, \infty) \to T(\kappa),$$

where $\lfloor \cdot \rfloor: [0, \infty) \to \mathbb{N}$ denotes the floor function. This map is continuous and sends $((x, 1), t)$ and $((\kappa(x), 0), t)$ to the same point

$$(\kappa^{[1+t]}(x), 1 + t - [1 + t]) = (\kappa([t](\kappa(x)), t - [t])) \in P \times [0, 1).$$

Hence, it induces the continuous suspension semiflow

$$\varphi_\kappa: T(\kappa) \times [0, \infty) \to T(\kappa),$$

$$( [x, \theta], t) \mapsto [\kappa^{[\theta+t]}(x), \theta + t - [\theta + t]].$$

Let $j_\kappa: P \to T(f)$, $j_\kappa(x) = [x, 0]$. Given maps $\kappa: P \to P$, $\lambda: Q \to Q$ and a map $r: P \to Q$ such that $\lambda r = r \kappa$, let the induced map $r_{\#}: T(\kappa) \to T(\lambda)$ be given by $r_{\#}[x, \theta] = [r(x), \theta]$. This definition makes the following diagram commute.

$$\begin{array}{ccc}
P & \xrightarrow{\kappa} & P \\
\downarrow{r} & & \downarrow{r_{\#}} \\
Q & \xrightarrow{\lambda} & Q \\
\end{array}$$

$$\varphi_\kappa([x, \theta], t) = [\kappa^{[\theta+t]}(x), \theta + t - [\theta + t]].$$

**Lemma 3.1.** If $P = Q$, $\kappa = \lambda$ and $r = \kappa^n$ for some $n \geq 0$, then the induced map $r_{\#} = \kappa_{\#}^n$ is homotopic to the identity on $T(\kappa)$. In particular, $j_\kappa \kappa^n \simeq j_\kappa$.

**Proof.** The suspension semiflow defines a homotopy because

- $\text{id}_{T(\kappa)}[x, \theta] = [x, \theta] = \varphi_\kappa([x, \theta], 0)$ and
- $\kappa_{\#}^n[x, \theta] = [\kappa^n(x), \theta] = \varphi_\kappa([x, \theta], n)$. 

$$\xymatrix{ P \ar[r]^\kappa \ar[d]_r & P \ar[d]^{r_{\#}} \\
Q \ar[r]^\lambda & Q \ar[r]^{j_\lambda} & T(\lambda) }$$
We want to relate two indices given different index pairs for \((S,f)\). We re-
call the proof of Theorem 3.4, which is basically Theorem 6.3 from Robbin and Salamon (1988). There it was originally stated for flows and invertible discrete sys-
tems. We only need it for discrete systems here and the invertibility assump-
tion was not used in the original proof.

**Lemma 3.2** (Robbin and Salamon 1988, Theorem 4.3). For a compact pair
\((N,L)\), the map \(f_{(N,L)}\) as defined above is continuous if and only if both of
the following conditions are fulfilled for every \(x_0 \in f^{-1}(N \setminus L)\):

(i) If \(x_0 \in L\), then there is an open set \(U \subset X\) such that \(x_0 \in U\) and
\(f(U \cap N \setminus L) \subset X \setminus N\).

(ii) If \(x_0 \in N \setminus L\), then there is an open set \(U \subset X\) such that \(x_0 \in U\) and
\(f(U \cap N \setminus L) \subset N \setminus L\).

Define for an arbitrary subset \(M \subset X\) and \(n \geq 0\):

\[\text{Inv}^n(M,f) = \{f^n(x) | x, f(x), \ldots, f^{2n}(x) \in M\}.\]

Let \((N_\alpha, L_\alpha)\) and \((N_\beta, L_\beta)\) be index pairs for \((S,f)\). Then there is a number
\(n \geq 0\) such that

\[\text{Inv}^n(N_\beta \setminus L_\beta, f) \subset N_\alpha \setminus L_\alpha\]

and \(\text{Inv}^n(N_\alpha \setminus L_\alpha, f) \subset N_\beta \setminus L_\beta\).

Let \(u = u(\alpha, \beta)\) be the smallest \(n \geq 0\) with this property. Obviously, \(u(\alpha, \beta) = u(\beta, \alpha)\), and we get the following property right from the definition of \(u = u(\alpha, \beta)\).

**Lemma 3.3.** For any \(x \in X\), if \(f^{[0,2u]}(x) \subset N_\alpha \setminus L_\alpha\), then \(f^n(x) \in N_\beta \setminus L_\beta\).

Now we define

\[C_{\alpha\beta} := \{x \in N_\alpha \setminus L_\alpha | f^{[0,2u]}(x) \subset N_\alpha \setminus L_\alpha\} \text{ and } f^{[u+1,3u+1]}(x) \subset N_\beta \setminus L_\beta\}\]

and the (not necessarily continuous) map

\[f_{\beta\alpha} : N_\alpha / L_\alpha \to N_\beta / L_\beta, \quad x \mapsto \begin{cases} f^{3u+1}(x) & \text{if } x \in C_{\alpha\beta}, \\ [L_\beta] & \text{otherwise}. \end{cases}\]

A special case is \(\alpha = \beta\). Then \(u(\alpha, \alpha) = 0\) and \(f_\alpha := f_{\alpha\alpha} = f_{(N_\alpha, L_\alpha)}\). The following theorem allows us to compare index maps.

**Theorem 3.4** (Robbin and Salamon 1988, Theorem 6.3).

(i) \(f_{\beta\alpha}\) is continuous,

(ii) \(f_{\alpha\beta} \circ f_{\beta\alpha} = f_{\alpha}^{6u(\alpha, \beta)+2}\),

(iii) \(f_{\beta\alpha} \circ f_{\alpha} = f_{\beta} \circ f_{\beta\alpha}\).

**Proof.** The idea for the proof of (i) is to consider five cases depending on
where \(x_0\) lies within \(N_\alpha / L_\alpha\), and to show for each case that \(f_{\beta\alpha}\) is continuous
in \(x_0\). We do not recall all cases here, but only present one difficult case from
the proof in Robbin and Salamon 1988 slightly adapted to our needs here.
The proofs of the other four cases are similar or shorter. We mainly use Lemma 3.3 and the continuity of the index maps $f_\alpha$ and $f_\beta$.

Case $x_0 \in C_{\alpha \beta}$. We mainly need to show that there is an open set $U \subset X$ such that $x_0 \in U$ and $U \cap N_\alpha \setminus L_\alpha \subset C_{\alpha \beta}$.

Note that $f^i(x_0) \in \{x_0 \setminus L_\alpha\}$ for $i \in \{0, \ldots, 2u-1\}$ by definition of $C_{\alpha \beta}$. Using Lemma 3.2(ii), there are open sets $U_i \subset X$ such that $f^i(x_0) \in U_i$ and

$$f(U_i \cap N_\alpha \setminus L_\alpha) \subset N_\alpha \setminus L_\alpha.$$ 

Since $x_0 \in C_{\alpha \beta}$, Lemma 3.3 yields $f^u(x_0) \in N_\beta \setminus L_\beta$. Now, applying Lemma 3.2(ii) to the index pair $(N_\beta, L_\beta)$ yields: For each $i \in \{u, \ldots, 3u\}$, there is an open set $V_i \subset X$ such that $f^i(x_0) \in V_i$ and

$$f(V_i \cap N_\beta \setminus L_\beta) \subset N_\beta \setminus L_\beta.$$ 

For $0 \leq i \leq 3u$, we define open sets

$$W_i := \begin{cases} 
U_i & \text{if } 0 \leq i \leq u-1, \\
U_i \cap V_i & \text{if } u \leq i \leq 2u-1, \\
V_i & \text{if } 2u \leq i \leq 3u.
\end{cases}$$ 

Now we let

$$U := \bigcap_{i=0}^{3u} f^{-i}(W_i) \subset X,$$ 

an open set. Let $x \in U \cap N_\alpha \setminus L_\alpha$. For $0 \leq i \leq 2u-1$, we have the implication

$$f^i(x) \in U_i \cap N_\alpha \setminus L_\alpha \implies f^{i+1}(x) \in U_{i+1} \cap N_\alpha \setminus L_\alpha.$$ 

Overall, $f^{[0,2u]}(x) \subset N_\alpha \setminus L_\alpha$, and therefore, by Lemma 3.3, $f^u(x) \in U_u \cap N_\beta \setminus L_\beta$. This implies $f^{[u,3u+1]}(x) \in N_\beta \setminus L_\beta$. Hence, $x \in C_{\alpha \beta}$.

Therefore, $f_{\beta \alpha}(x) = f^{3u+1}(x)$ for every $x \in U \cap N_\alpha \setminus L_\alpha$. Note that $U \cap N_\alpha \setminus L_\alpha$ is open in $N_\alpha \setminus L_\alpha$ and therefore open in $N_\alpha \setminus N_\alpha$. Since $f$ is continuous, $f_{\beta \alpha}$ is continuous in $x_0$. This finishes the proof of the case $x_0 \in C_{\alpha \beta}$.

The statements (ii) and (iii) are special cases of Theorem 6.3(iii) in Robbin and Salamon (1988). □

Theorem 3.5. The mapping torus index of $(S, f)$ is independent of the choice of an index pair $(N, L)$.

Proof. Let $(N_\alpha, L_\alpha)$ and $(N_\beta, L_\beta)$ be index pairs for $(S, f)$. Now let $r := f_{\beta \alpha}$, $s := f_{\alpha \beta}$ and $n := 6u(\alpha, \beta) + 2$. Theorem 3.4 shows that

(i) $rf_\alpha = f_\beta r$ and $sf_\beta = f_\alpha s$,
(ii) $sr = f^n_\alpha$ and $rs = f^n_\beta$.

Then, Lemma 3.1 yields a homotopy equivalence

$$s_# r_# = (s r)_# = (f^n_\alpha)_# \simeq \text{id}_{T(f_\alpha)},$$

and similarly for $r_# s_#$. Therefore $T(f_\alpha) \simeq T(f_\beta)$. □

Similarly, $\text{CT}_*(S, f)$ is a well-defined pointed homotopy type.
4. Main properties

It is possible to replace the index pair by a homotopy equivalent one in the following sense.

**Proposition 4.1.** If the map $r$ in Diagram (3.1) is a homotopy equivalence, then so is $r_{\#}$. 

**Proof.** Diagram (3.1) is a special case of Diagram (5.1) by putting $H(x,\theta) = [r(x),\theta]$ and then $r_{\#} = (r,H)_{\#}$. Theorem 5.2 yields the result. $\square$

An important property of the usual Conley index definitions is the invariance under continuation. Consider a collection $\{(S_t,f_t) | t \in [0,1]\}$ of sets $S_t \subset X$ and maps $f_t: X \to X$, such that the dynamical system

$$F: X \times [0,1] \to X \times [0,1],$$

$$(x,t) \mapsto (f_t(x),t),$$

is continuous and the set $\Sigma \subset X \times [0,1]$ given by

$$\Sigma = \{(x,t) | x \in S_t\}$$

is an isolated invariant set for $F$. The collection $\{S_t,f_t\}$ is called a continuation from $(S_0,f_0)$ to $(S_1,f_1)$. Note that $f_{s} \simeq f_{t}$ for all $s,t \in [0,1]$.

**Theorem 4.2.** If there is a a continuation from $(S_0,f_0)$ to $(S_1,f_1)$, then $\text{CT}(S_0,f_0) = \text{CT}(S_1,f_1)$.

**Proof.** Let $\{(S_t,f_t) | t \in [0,1]\}$ be a continuation of isolated invariant sets. Then, applying Corollary 5.5 in Robbin and Salamon (1988), there are open sets $I_1,\ldots,I_n$ covering the unit interval $[0,1]$ and pairs $(N_1,L_1),\ldots,(N_n,L_n)$ such that each $(N_i,L_i)$ is an index pair when $(S_t,f_t)$ for $t \in I_i$. We assume $0 \in I_1$ and $1 \in I_n$. Now one can observe:

(i) If $s,t \in I_i$, then $T(f_{s,(N_i,L_i)}) \simeq T(f_{t,(N_i,L_i)})$ since $f_{s} \simeq f_{t}$.

(ii) If $t \in I_i \cap I_j$, then $(N_i,L_i)$ and $(N_j,L_j)$ are index pairs for $(S_t,f_t)$, hence Theorem 3.5 yields $T(f_{t,(N_i,L_i)}) \simeq T(f_{t,(N_j,L_j)})$.

Since $[0,1]$ is connected and every $I_i$ is open in $[0,1]$, this shows that

$$\text{CT}(S_0,f_0) = [T(f_{0,(N_1,L_1)})] = [T(f_{1,(N_n,L_n)})] = \text{CT}(S_1,f_1).$$

$\square$

The following result about compositions is an analogue of Theorem 1.12 from Mrozek (1994).

**Theorem 4.3** (Commutativity). Let $\varphi: X \to Y$ and $\psi: Y \to X$ be continuous maps. Consider the dynamical systems $f = \psi \varphi$ and $g = \varphi \psi$. Let $S \subset X$ be an isolated invariant set for $f$. Then $\varphi(S)$ is an isolated invariant set for $g$ and $\text{CT}(S,f) = \text{CT}(\varphi(S),g)$.

**Proof.** Using the proof of Theorem 1.12 from Mrozek (1994), index pairs $(N,L)$ for $(S,f)$ and $(M,K)$ for $(\varphi(S),g)$ exist such that there are continuous maps $\tilde{\varphi}: N/L \to M/K$ and $\tilde{\psi}: M/K \to N/L$ with $\tilde{\psi}\tilde{\varphi} = f_{(N,L)}$ and $\tilde{\varphi}\tilde{\psi} = g_{(M,K)}$. 

By the homotopy invariance, the diagram

$$\begin{array}{ccc}
S & \xrightarrow{\varphi} & T(S) \\
\downarrow \varphi & & \downarrow \varphi
\end{array}$$

induces a homotopy equivalence $\tilde{\varphi} \tilde{\psi} = \varphi \psi : T(S) \to T(S)$. Hence $T(S)$ is a continuation of isolated invariant sets.
5. Definition via shift equivalence

Homotopy classes of self-maps \([\kappa: P \to P]\) and \([\lambda: Q \to Q]\) are called shift equivalent if there are continuous maps \(r: P \to Q\) and \(s: Q \to P\) such that \(\lambda r \simeq r\kappa\), \(s\lambda \simeq \kappa s\), \(sr \simeq \kappa^n\) and \(rs \simeq \lambda^n\) for some \(n \in \mathbb{N}\). Here we call the shift equivalence class of \([f_{(N,L)}]\) the shift equivalence (Conley) index. It was introduced by Franks and Richeson (2000). In this section, we show that the mapping torus index is strictly coarser, but sometimes allows statements about shift equivalence if \(N/L\) is compact and connected.

Assume we are given maps \(\kappa: P \to P\) and \(\lambda: Q \to Q\) and a map \(r: P \to Q\) such that \(\lambda r \simeq r\kappa\). This means that \(j_\lambda r\kappa \simeq j_\lambda\kappa r\). Hence, there is a homotopy \(H: P \times [0,1] \to T(\lambda)\) with \(H(x,0) = j_\lambda r(x)\) and \(H(x,1) = j_\lambda r\kappa(x)\) for all \(x \in P\). Let the induced map \((r,H)_\#: T(\kappa) \to T(\lambda)\) be given by \((r,H)_\#: [x,\theta] \mapsto H(x,\theta)\). This is well-defined because \(H(x,1) = [r\kappa(x),0] = H(\kappa(x),0)\). In the diagram

\[
P \xrightarrow{\kappa} P \xrightarrow{\lambda} T(\kappa)
\]

(5.1)

the left square is homotopy commutative and the right square is strictly commutative. First we observe the following generalization of Lemma 3.1.

Lemma 5.1. In Diagram (5.1), assume that \(P = Q\), \(\kappa = \lambda\) and \(r = \kappa^n\) for some \(n \geq 0\). If \(H: P \times [0,1] \to T(\kappa)\) is a homotopy from \(j_\kappa r\) to \(j_\kappa r\kappa\), i.e., \(H(\cdot,0) = j_\kappa\kappa^n\) and \(H(\cdot,1) = j_\kappa\kappa^{n+1}\), then \((\kappa^n,H)_\# \simeq \text{id}_{T(\kappa)}\).

Proof. A homotopy \(H': T(\kappa) \times [0,1] \to T(\kappa)\) is given by:

\[
H'(][x,\theta],t) = \begin{cases} 
\varphi_\kappa([x,\theta],2t(n-\theta)) & \text{for } 0 \leq t \leq 1/2, \\
H(x,(2t-1)\theta) & \text{for } 1/2 \leq t \leq 1. 
\end{cases}
\]

Then \(H'(][x,\theta],0) = [x,\theta]\) and \(H'(][x,\theta],1) = H(x,\theta) = (\kappa^n,H)_#[x,\theta]\).

Lemma 3.1 is the special case \(H(x,\theta) = \varphi_\kappa([\kappa^n(x),0],\theta)\) of Lemma 5.1.

Theorem 5.2. If the homotopy classes \([\kappa]\) and \([\lambda]\) are shift equivalent, then \(T(\kappa)\) and \(T(\lambda)\) are homotopy equivalent.

Proof. This works similarly to the proof of Theorem 3.5. But here the induced maps depend on the chosen homotopies. By assumption, there are a homotopy \(H: P \times [0,1] \to T(\lambda)\) such that \(H(x,0) = j_\lambda r(x)\) and \(H(x,1) = j_\lambda r\kappa(x)\) for all \(x \in P\), and a homotopy \(H': Q \times [0,1] \to T(\kappa)\) such that \(H'(x,0) = j_\kappa s(x)\) and \(H'(x,1) = j_\kappa s\lambda(x)\) for all \(x \in Q\). We show that the composition \(K = (s,H')_\# \circ (r,H)_\#\) is homotopic to the identity on \(T(\kappa)\).
Since $[\kappa]$ and $[\lambda]$ are shift equivalent, there is a homotopy $L: P \times [0, 1] \to P$ together with an $n \in \mathbb{N}$ such that $L(x, 0) = sr(x)$ and $L(x, 1) = \kappa^n(x)$ for all $x \in P$.

Using a retraction from the square $[0, 1] \times [0, 1]$ to three of its boundary edges, there is a map

$$F: P \times [0, 1] \times [0, 1] \to T(\kappa)$$

such that for all $\theta \in [0, 1]$ and $t \in [0, 1]$:

(i) $F(x, \theta, 0) = K([x, \theta])$,

(ii) $F(x, 0, t) = j_{\kappa}L(x, t)$, and

(iii) $F(x, 1, t) = j_{\kappa}L(\kappa(x), t)$.

Note that $F$ is well-defined, e.g.,

$$K([x, 1]) = K([\kappa(x), 0]) = [sr\kappa(x), 0] = j_{\kappa}L(\kappa(x), 0).$$

Additionally, $F(x, 1, t) = F(\kappa(x), 0, t)$, This means that $F$ induces a continuous map $F': T(\kappa) \times [0, 1] \to T(\kappa)$. Let

$$K': P \times [0, 1] \to T(\kappa),$$

$$F'(x, \theta) \to F([x, \theta], 1).$$

Observe that $K'(x, 0) = j_{\kappa}\kappa^n$ and $K'(x, 1) = j_{\kappa}\kappa^{n+1}$ by construction. We get the following homotopies, where the first one is given by $F'$ and the second one by Lemma 5.1:

$$(s, H')_# \circ (r, H)_# = K \simeq (\kappa^n, K')_# \simeq \text{id}_{T(\kappa)}.$$

An analogous argument shows that $(r, H)_# \circ (s, H')_# \simeq \text{id}_{T(\lambda)}$. $$\square$$

The converse can easily be shown to be false. For example, if $P = \{1\}$, $Q = \{1, 2\}$ with the discrete topology and $\lambda: Q \to Q, \lambda(1) = 2, \lambda(2) = 1$. Then $T(\kappa) = T(\lambda) = S^1$. Suppose there is a map $r: P \to Q$ such that $\lambda r \simeq r\kappa$. Then $\lambda r(1) = \kappa r(1) = r(1)$, but $\lambda(x) \neq x$ for all $x$. A contradiction.

The rest of this section deals with a specific situation in which the converse is true, as described in Theorem 5.5. As a tool in the following proof, we use the mapping telescope (see also Hatcher 2002, Sect. 3.F): Let $P$ be a topological space and let $\kappa: P \to P$ be a continuous map. Then let

$$\text{Tel}(\kappa) = \coprod_{i \in \mathbb{Z}} (P \times [0, 1] \times \{i\}) / (x, 1, i) \sim (\kappa(x), 0, i + 1),$$

i.e., countably many mapping cylinders of $\kappa$ are glued together. It is a covering space of $T(\kappa)$ via

$$\pi: \text{Tel}(\kappa) \to T(\kappa), (x, t, i) \mapsto (x, t) \text{ for } t \in [0, 1).$$

For $n \in \mathbb{N}$, let

$$\text{Tel}_n(\kappa) = \left( \coprod_{i=-n}^{n-1} (P \times I \times \{i\}) \coprod P \times \{0\} \times \{n\} \right) / \sim,$$
using the same identifications as in $\text{Tel}(\kappa)$. Then $\text{Tel}_n(\kappa)$ deformation retracts to $P$ along a lift of the suspension semiflow $\varphi(\cdot)$ via a map $\rho: \text{Tel}_n(\kappa) \to P$. Note that any compact subset of $\text{Tel}(\kappa)$ is contained in $\text{Tel}_n(\kappa)$ for some $n \in \mathbb{N}$. We first show two lemmas.

**Lemma 5.3.** Let $\kappa P \to P$ be a continuous map on a compact and connected topological space $P$, and let $\alpha: P \to P$ be a continuous map such that $\alpha \kappa \simeq \kappa \alpha$. Hence, there is a homotopy $H: P \times [0, 1] \to \text{Tel}(\kappa)$ such that $H(\cdot, 0) = j_\kappa \alpha$ and $H(\cdot, 1) = j_\kappa \alpha \kappa$. If $(\alpha, H) \# \simeq \text{id}_{\text{Tel}(\kappa)}$, then there are $n,k \in \mathbb{N}$ such that $\kappa^n \simeq \kappa^k \alpha$.

**Proof.** The assumptions yield that $(\alpha, H) \# \simeq \text{id}_{\text{Tel}(\kappa)}$, hence

$$j_\kappa = \text{id}_{\text{Tel}(\kappa)} j_\kappa \simeq (\alpha, H) \# j_\kappa = j_\kappa \alpha,$$

i.e., there is a homotopy $H': P \times [0, 1] \to \text{Tel}(\kappa)$ such that $H'(\cdot, 0) = j_\kappa$ and $H'(\cdot, 1) = j_\kappa \alpha$. The map $H'$ fits into the following commutative diagram with solid arrows:

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{i_0} & \text{Tel}(\kappa) \\
\downarrow & & \downarrow \pi \\
P \times [0, 1] & \xrightarrow{H'} & \text{Tel}(\kappa),
\end{array}
$$

where $i_0(x,0) = (x,0,0)$. The dashed lift $\overline{H}$ exists and is the unique map making the diagram commute because $\pi$ is a covering projection (Hatcher 2002, Prop. 1.30). Since the domain of $\overline{H}$ is compact, its image is a compact subset of $\text{Tel}(\kappa)$, hence there is an $n \in \mathbb{N}$ such that $\overline{H} = i \circ h$, where $h: P \times [0, 1] \to \text{Tel}_n(\kappa)$ and $i: \text{Tel}_n(\kappa) \to \text{Tel}(\kappa)$ is the inclusion. Let $\rho: \text{Tel}_n(\kappa) \to P$ be the deformation retraction from following the flow lines of the suspension semiflow. This yields a map $\rho h: P \times [0, 1] \to P$ with $\rho h(x,0) = \rho(x,0,0) = \kappa^n(x)$. Since $\pi h(x,1) = H(x,1) = [\alpha(x),0]$, the image $h(P \times \{1\})$ is in the fiber $\prod_{i=-n}^n P \times \{0\} \times \{i\} \subset \text{Tel}_n(\kappa)$ over $P \times \{0\} \subset \text{Tel}(\kappa)$. Since $P$ is connected, $h(P \times \{1\}) \subset P \times \{0\} \times \{n-k\}$ for some $k \in \{0, \ldots, 2n\}$. Therefore $\rho h(x,1) = \kappa^k(\alpha(x))$, and $\rho h$ is a homotopy $\kappa^n \simeq \kappa^k \alpha$. \hfill $\Box$

Now we show that the exponents in the definition of shift equivalence are allowed to differ in the following sense.

**Lemma 5.4.** Let $\kappa: P \to P$ and $\lambda: Q \to Q$ be continuous maps, and assume there are $r: P \to Q$ and $s: Q \to P$ such that $r \kappa \simeq \lambda r$, $\kappa s \simeq s \lambda$. Assume also that there are $n,m \in \mathbb{N}$ such that $sr \simeq \kappa^n$ and $rs \simeq \lambda^m$. Then $[\kappa]$ and $[\lambda]$ are shift equivalent.
Proof. The assumptions yield a homotopy commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{s} & P \\
\downarrow{\kappa^{n+m}} & & \downarrow{\lambda^{n+m}} \\
Q & \xrightarrow{r} & Q
\end{array}
\]

We do not distinguish the upper and lower row (alternatively, one can think of three copies of this diagram “glued” together). There are several paths from the left \(P\) to itself. First going along the horizontal composition \(rsrsr\) and then to the lower left, we get the homotopy

\[
(rs\kappa^n srs)(rsrsr) \simeq \kappa^{3(n+m)}.
\]

In a similar manner, starting from the upper right \(Q\), one sees

\[
(rsr)(rs\kappa^n srs) \simeq \lambda^{3(n+m)}.
\]

Hence, \([\kappa]\) and \([\lambda]\) are shift equivalent. \(\square\)

Theorem 5.5. Let \(\kappa: P \to P\), \(\lambda: Q \to Q\), \(r: P \to Q\) and \(s: Q \to P\) such that \(r\kappa \simeq \lambda r\) and \(s\lambda \simeq k\kappa s\). Now suppose that \((sr)\# \simeq \text{id}_{T(\kappa)}\) and \((rs)\# \simeq \text{id}_{T(\lambda)}\) and that \(P\) and \(Q\) are compact and connected. Then \([\kappa]\) and \([\lambda]\) are shift equivalent.

Proof. Applying Lemma 5.3 to \(\alpha = rs\) and \(\alpha = sr\), respectively, we see that there are \(k,n,m \in \mathbb{N}\) such that \(\kappa^n \simeq \kappa^k sr\) and \(\lambda^m \simeq \lambda^k rs\). Hence the following diagram commutes up to homotopy.

\[
\begin{array}{ccc}
P & \xrightarrow{\kappa^n} & P \\
\downarrow{r} & & \downarrow{r} \\
Q & \xrightarrow{\lambda^m} & Q
\end{array}
\]

Now the result follows from Lemma 5.4. \(\square\)

6. Examples

In order to discuss examples, we first recall the following results about the homology and the fundamental group of mapping tori.

The following statement about homology is shown in Hatcher (2002), Example 2.48: A continuous map \(\kappa: P \to P\) induces a map \(\kappa_*\) in homology and this fits into a long exact sequence in homology.

\[
\cdots \to H_n(P) \xrightarrow{id - \kappa_*} H_n(P) \xrightarrow{j_*} H_n(T(\kappa)) \xrightarrow{\partial} H_{n-1}(P) \to \cdots
\]

In the following examples, the spaces \(P\) are finite wedges of circles and \(\kappa\) sends the base point to the base point. Computing the fundamental group in this case works as follows. Let \(P = \bigvee_{i=1}^n S^1\). Each of the circles contributes
a generator $a_i \in \pi_1(P)$ and $\pi_1(P) = \langle a_1, \ldots, a_n \rangle$. In order to build $T(\kappa)$, one adds a circle, say $z$, and for each $i$ a 2-cell attached to the 1-skeleton along $a_iz\kappa_s(a_i)^{-1}z^{-1}$. Hence, $\pi_1(T(\kappa))$ is a quotient of the free group on $n + 1$ generators as follows (cf. Hatcher 2002, Prop. 1.26):

$$\pi_1(T(\kappa)) = \langle a_1, \ldots, a_n, z \mid a_1z = z\kappa_s(a_1), \ldots, a_nz = z\kappa_s(a_n) \rangle.$$  

The space $T_\bullet(\kappa)$ for a pointed map $\kappa$ is constructed similarly, but without adding the 1-cell $z$. This yields

$$\pi_1(T_\bullet(\kappa)) = \langle a_1, \ldots, a_n \mid a_1 = \kappa_s(a_1), \ldots, a_n = \kappa_s(a_n) \rangle.$$  

**Example 6.1.** Let $P = S^1$ and let $\kappa$ be a degree-2 map. Then $\pi_1(S^1) \cong \langle a \rangle$ and the group $\pi_1(T_\bullet(f(N,L)))$ is trivial because $\kappa_s(a) = a^2$. Therefore, $\pi_1(T_\bullet(f(N,L))) \cong \langle a \mid a = a^2 \rangle = \{e\}$, the trivial group. But $\pi_1(T(f(N,L))) \cong \langle a, z \mid az = za^2 \rangle$, which is not the free group on one generator.

Since we would like to be able to compare Conley indices numerically, we show how this can be done from the presentation of a group with finitely many generators as in Example 6.1. We use the software package GAP. Given $G$, the fundamental group from the example, the software lists all the subgroups $S$ of $G$ with index $[G : S] \leq 3$ and then computes the abelianization for each of these $S$. Executing the GAP code

```
F:=FreeGroup("a","z");
G:=F/ParseRelators(F,"az = za^2");
subgroups:=LowIndexSubgroupsFpGroup(G,3);
Print(List(subgroups,AbelianInvariants)); # abelianizations
Print(List(subgroups,IndexInWholeGroup),"\n"); # indices
```

yields the output

```
[ 1, 2, 3, 3 ]
[ [ 0 ], [ 0, 3 ], [ 0, 7 ], [ 0 ] ]
```

The groups are represented by giving torsion coefficients, e.g., $[0,3]$ represents the abelianization $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ of a subgroup $S$ with index 2. In particular, the Conley index in our example is not trivial.

A commonly used algebraic invariant of the Conley index is the **homological Conley index**, by which we mean the shift equivalence class of reduced homology $\tilde{H}(f(N,L))$, where linear maps $\kappa, \lambda$ are shift equivalent if there are linear maps $r,s$ such that $r\kappa = \lambda s$, $s\lambda = r\kappa$, $sr = \kappa^n$ and $rs = \lambda^n$ for some $n \in \mathbb{N}$. This is not the only useful algebraic invariant as we show in the following example.

**Example 6.2.** The mapping torus index contains information which the homological Conley index cannot represent. Let $P = S^1 \vee S^1$ with circles $a$ and $b$, and let $\kappa: P \to P$ such that $a \mapsto aba^{-1}b^{-1}$ and $b \mapsto a^{-1}bab^{-1}$. This induces the trivial (zero) homomorphism in reduced homology. The fundamental group of its mapping torus is

$$\pi_1(T(\kappa)) = \langle a, b, z \mid az = zaba^{-1}b^{-1}, bz = za^{-1}bab^{-1} \rangle.$$
Similarly to the example above, GAP computes that $\pi_1(T(\kappa))$ has a subgroup with index 5 and abelianization $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. The shift equivalence class of $[\kappa]$ is therefore not trivial. But this information is not visible when using homology $\overline{H}_*(\kappa)$, which is shift equivalent to the graded module homomorphism $0 \to 0$. It also seems hard to see that $[\kappa]$ is not shift equivalent to the trivial map $\{\ast\} \to \{\ast\}$ directly from the definition of shift equivalence.

**Example 6.3.** Let $f_1 : \mathbb{R} \to \mathbb{R}, f_1(x) = 2x$. Then $\{0\}$ is an isolated fixed point with index pair $([-2,2],[-2,-2] \cup [1,2])$, and the index map is homotopic to the identity on $S^1$. Its mapping torus index is $CT(\{0\}, f_1) = [S^1 \times S^1]$, and $CT_\bullet(S, f_1) = [S^1 \times S^1]$.

Let $f_2 : \mathbb{R} \to \mathbb{R}, f_2(x) = -x^3$. Then $\{-1,1\}$ is an isolated invariant set with $CT_\bullet((-1,1), f_2) = [S^1 \times S^1] = CT_\bullet(\{0\}, f_1)$.

This equality can be seen using Theorem 7.3. But the shift equivalence indices of $\{(0), f_1\}$ and $\{(-1,1), f_2\}$ differ. Indeed, choosing some field $\mathbb{F}$, the homological Conley index of $\{(0), f_1\}$ in first homology is the identity on $\mathbb{F}$, whereas the corresponding homological Conley index of $\{(-1,1), f_2\}$ is an automorphism of $\mathbb{F}^2$. They cannot be shift equivalent (Misiakow and Mrozek 2002).

**Example 6.4.** Let $g : \mathbb{R} \to \mathbb{R}, g(x) = -2x$. Then $\{0\}$ is an isolated fixed point with index pair $([-2,2],[-2,-2] \cup [1,2])$ and $g_{(N,L)}$ is a map on $S^1$ with degree $-1$. Hence the mapping torus index of $\{(0), g\}$ is (the homotopy type of) the Klein bottle.

**Example 6.5.** We recall Example 6.1 from Szymczak (1995). This example has non-trivial shift equivalence index, whereas the indices defined by Mrozek (1990) and Robbin and Salamon (1988) are trivial. The mapping torus index offers some more insight: Let $X = (-\infty,0] \cup \{2^{-k} | k \in \mathbb{N}\} \subset \mathbb{R}$ and $h : X \to X,$

$$h(x) = \begin{cases} 2x & \text{if } x \leq 0, \\ x/2 & \text{if } x \geq 0. \end{cases}$$

Then $(N,L) = (X \cap [-2,1], [-2, -1])$ is an index pair for $\{(0), h\}$. Its mapping torus index $CT(\{0\}, h)$ is the union of a helix approaching $\{0\} \times S^1$ and the cylinder $[-1,0] \times S^1$. This space is compact and connected, but neither pathwise connected nor locally connected. In particular, $CT(\{0\}, h) \neq [S^1]$.

**Example 6.6** (Smale’s horseshoe). Consider Smale’s $U$-horseshoe $f : \mathbb{R}^2 \to \mathbb{R}^2$, a homeomorphism which bends the unit square $N = [0,1] \times [0,1]$ to form a horseshoe, which is sketched in Fig. 1. We consider the following index pair used in Mrozek (1990), Example 8.1, and Misiakow and Mrozek (2002).

Let $L_1 = [0,1] \times [0,1/5], L_2 = [0,1] \times [2/5, 3/5], L_3 = [0,1] \times [4/5, 1]$ and $L = L_1 \cup L_2 \cup L_3$. Then $(N,L)$ is an index pair for $\text{Inv}(N,f)$ and $N/L \cong S^1 \cup S^1$ with fundamental group the free group on two generators $a$ and $b$, where $f_{(N,L)_+}(a) = ab$ and $f_{(N,L)_+}(b) = b^{-1}a^{-1}$. Now

$$\pi_1(T(f_{(N,L)_+})) \cong \langle a, b, z \mid az = zab, bz = zb^{-1}a^{-1} \rangle \cong \langle z \rangle \cong \mathbb{Z}.$$
This can be seen by observing that the left relation is equivalent to \( b^{-1}a^{-1} = z^{-1}a^{-1}z \) and inserting this into the right relation.

Similarly \((N,L)\) is an index pair for \((\text{Inv}(N,g),g)\), where \(g : \mathbb{R}^2 \to \mathbb{R}^2\) is the \(G\)-horseshoe. The images of \(N\) and \(L\) are sketched in the right subfigure of Fig. 1. Here, \(g(N,L)\) is the quotient map; i.e.,

\[
\pi_1(T(g(N,L))) \cong \langle a, b, z \mid az = zab, bz = zab \rangle \cong \langle a, z \mid az = za^2 \rangle,
\]

the group from Example 6.1. Hence, its (unreduced) mapping torus index is non-trivial. On the contrary, \(\pi_1(T(g(N,L)))\) is trivial.

7. DEFINITION VIA SUSPENSION SEMIFLOW ON \(X\)

The reduced mapping torus index is equivalent to the flow Conley index of the suspension semiflow in the sense presented here. The results of this section are also included in Sect. 4 of the unpublished paper *Morse inequalities and zeta functions*\(^1\) written by J. W. Robbin, D. A. Salamon and E. C. Zeeman in 1989. The idea of considering the Conley index for the suspension semiflow also appeared in *Floer* (1990).

Let \(f : X \to X\) be a discrete dynamical system. For a set \(A \subset X\), let \(I_f A := q(A \times [0,1]) \subset T(f)\), where \(q : X \times [0,1] \to T(f)\) is the quotient map; i.e.,

\[
I_f A = \{[x,\theta] \in T(f) \mid x \in A, \theta \in [0,1]\} = \frac{A \times [0,1] \cup f(A) \times \{0\}}{(x,1) \sim (f(x),0)}.
\]

Note that, given an invariant set \(S\) of \(f\), \(I_f S = T(f|_S)\), the mapping torus of the restriction of \(f\) to \(S\).

For the following proof of Theorem 7.3, we recall a special kind of index pair for maps. A compact pair \((N,L)\) is a strong index pair for an isolated invariant set \(S\) of \(f\) if

(i) \(S = \text{Inv}(\text{cl}(N \setminus L), f) \subset \text{int}(N \setminus L)\),

\(^1\) currently accessible at [https://people.math.ethz.ch/~salamon/PREPRINTS/zeta.pdf](https://people.math.ethz.ch/~salamon/PREPRINTS/zeta.pdf)
(ii) \( f(L) \cap N \subset L \), and
(iii) \( f(N \setminus L) \subset N \).

A strong index pair exists for every isolated invariant set \( S \) (Szymczak 1995, Theorem 3.1; Mischaikow and Mrozek 2002, Theorem 3.25) and a strong index pair is an index pair (Robbin and Salamon 1988, Corollary 4.4).

We recall the definition of an index pair for a semiflow \( \varphi \colon X \times [0, \infty) \to X \) given by Conley (1978). Given a subset \( M \subset X \), its invariant part is

\[ \text{Inv}(M, \varphi) = \{ x \in M \mid \text{there is a solution } \gamma \colon \mathbb{R} \to M \text{ of } \varphi \text{ with } \gamma(0) = x \}. \]

A compact pair \( (\tilde{N}, \tilde{L}) \) is an index pair for \( (S, f) \) if the following conditions are fulfilled:

(i) \( \text{Inv}(\text{cl}(\tilde{N} \setminus \tilde{L}), \varphi) \subset \text{int}(\tilde{N} \setminus \tilde{L}) \);
(ii) if \( x \in \tilde{L}, t > 0, \varphi(x, [0, t]) \subset \tilde{N} \), then \( \varphi(x, [0, t]) \subset \tilde{L} \);
(iii) if \( x \in \tilde{N}, t > 0, \varphi(x, t) \not= \tilde{N} \), then there is a \( t' \in [0, t] \) such that \( \varphi(x, t') \in \tilde{L} \) and \( \varphi(x, [0, t']) \subset \tilde{N} \).

**Lemma 7.1.** Let \( (N, L) \) be a strong index pair for \( (S, f) \). Then \( (I_f N, I_f L) \) is an index pair for \( (I_f S, \varphi_f) \).

**Proof.** For (i), we first show that \( I_f S \subset \text{int}(I_f N \setminus I_f L) \): Let \( x \in S, 0 \leq \theta \leq 1 \). It suffices to consider \( \theta < 1 \) (since \( [x, 1] = [f(x), 0] \)). Pick an open neighborhood \( U \) of \( x \) with \( U \subset N \setminus L \). If \( \theta \in (0, 1) \), one gets \( [x, \theta] \in q(U \times (0, 1)) \subset I_f N \setminus I_f L \). If \( \theta = 0 \), let \( V = f^{-1}(U) \subset N \setminus L \). Then \( q(U \times (0, 1) \cup V \times (0, 1)) \subset I_f N \setminus I_f L \) is an open neighborhood of \( [x, 0] \).

Now we show \( I_f S = \text{Inv}(\text{cl}(I_f N \setminus I_f L), \varphi_f) \): The inclusion “\( \subset \)” holds because \( I_f S \) is obviously an invariant set for \( \varphi_f \). For the other inclusion, first observe that

\[ I_f N \setminus I_f L \subset I_f(N \setminus L) \subset I_f \text{cl}(N \setminus L) \]

The left inclusion follows because \( [x, \theta] \in I_f N \setminus I_f L \) implies \( x \in N \setminus L \). These sets differ in general because if \( x \in N \setminus L \) and \( f(x) \in L \), then \( [x, 1] = [f(x), 0] \in I_f L \). Since the set on the right is compact, we get \( \text{cl}(I_f N \setminus I_f L) \subset I_f \text{cl}(N \setminus L) \).

Now let \( [x, \theta] \in \text{Inv}(\text{cl}(I_f N \setminus I_f L), \varphi_f) \) with \( 0 \leq \theta \leq 1 \), i.e., there is a solution \( \gamma \colon \mathbb{R} \to \text{cl}(I_f N \setminus I_f L) \) of \( \varphi_f \) with \( \gamma(0) = [x, 0] \in \text{cl}(I_f N \setminus I_f L) \). The curve \( \gamma \) yields a solution \( \tilde{\gamma} \colon \mathbb{Z} \to \text{cl}(I_f N \setminus I_f L) \) of \( f \) with \( \tilde{\gamma}(0) = x \) and \( \tilde{\gamma}(n), 0]) \in \text{cl}(I_f N \setminus I_f L) \subset I_f \text{cl}(N \setminus L) \) for all \( n \in \mathbb{Z} \). Therefore, \( \tilde{\gamma}(n) \in \text{cl}(N \setminus L) \). Since \( \text{cl}(N \setminus L) \) is an isolating neighborhood for \( S \), this yields \( x \in S \) and hence \( [x, \theta] \in I_f S \).

(ii) and (iii) follow similarly: for (ii), consider \( x \in L, 0 \leq \theta \leq 1 \), and a trajectory \( \varphi_f([x, \theta], [0, t]) \subset I_f N \) for some \( t > 0 \). Then \( \varphi_f([x, 0], [0, \theta + t]) \subset I_f N \). Let \( n := |\theta + t| \). Then \( f^n(x) \in L \) for each \( 0 \leq k \leq n \) since \( f(L) \cap N \subset L \). Therefore, \( \varphi_f([x, \theta], [0, t]) \subset I_f N \).

For (iii), consider some \( x \in N \) with \( 0 \leq \theta \leq 1 \), and suppose that \( \varphi_f([x, \theta], t) = \varphi_f([x, 0], \theta + t) \not= \tilde{N} \) for some \( t > 0 \). Let \( n := |\theta + t| \). Then \( f^n(x) \not= N \). Let \( m := \max\{ k \in \mathbb{N} \mid f^i(x) \in N \text{ for all } 0 \leq i \leq k \} < n \).
Then $\varphi_f([x, \theta], [0,m]) \subset I_f N$. Now assume $f^m(x) \notin L$. Then $f^{m+1}(x) \in N$ because $f(N \setminus L) \subset N$. This contradicts the definition of $m$. Overall, $f^m(x) \in L$, and therefore $\varphi_f([x, \theta], m) \in I_f L$. \hfill $\Box$

**Lemma 7.2.** If $(N, L)$ is a strong index pair for $(S, f)$, then $T_\bullet(f_{(N, L)}) = I_f N/I_f L$.

**Proof.** Using $f(N \setminus L) \subset N$, we have

$$I_f N = \frac{N \times [0,1] \cup f(L) \times \{0\}}{(x, 1) \sim (f(x), 0) \text{ for } x \in N} \subset T(f).$$

Taking the quotient yields

$$I_f N/I_f L = \frac{N \times [0,1]/L \times [0,1]}{(x, 1) \sim (f(x), 0) \text{ for } x, f(x) \in N} = T_\bullet(f_{(N, L)}).$$ \hfill $\Box$

Since we are free to choose an arbitrary index pair for $(S, f)$ and $(I_f S, \varphi_f)$, respectively, Lemmas 7.1 and 7.2 yield

**Theorem 7.3.** For every isolated invariant set $S$ of $f$, $CT_\bullet(S, f)$ is the flow Conley index of $(I_f S, \varphi_f)$.

**8. Numerical Representation**

In this section, we sketch an idea for the numerical representation of the mapping torus of some self-map $f : X \to X$. Using interval arithmetic in rigorous numerics (see Kaczynski et al. 2004; or Bush et al. 2012), one can construct a numerical representation of a covering $Z \subset X \times X$ of the graph $G(f)$ of $f$. The sets $Z$ and hence the maps $p$ and $q$ can be represented on a computer, even though $f$ is not directly known. Letting $p(x, y) = x$ and $q(x, y) = y$, the map $f$ factors through $G(f)$ and hence through $Z$ as follows:

Let

$$\tilde{f} : X \to Z,$$

$$x \mapsto (x, f(x)).$$

Then $f = q \circ \tilde{f}$. For the diagram $p, q : Z \rightrightarrows X$, we consider its homotopy colimit, which we also call the mapping torus of $p$ and $q$ here (cf. Hatcher 2002, Example 2.48),

$$T(p, q) := \frac{(Z \times [0,1]) \coprod X}{(z, 0) \sim p(z), (z, 1) \sim q(z)}.$$  

Analogously, we define $T(id_X, f)$, the mapping torus of $id_X$ and $f$, as the quotient of $(X \times [0,1]) \coprod X$. It is is homotopy equivalent to $T(f)$ as defined in Sect. 2. Now let

$$(\tilde{f}, id)_\#: T(id_X, f) \to T(p, q)$$

be the map induced on the summands by

$$\tilde{f} \times id_{[0,1]} : X \times [0,1] \to Z \times [0,1]$$
and the identity on $X$, respectively. Now, $T(p,q)$ is potentially useful for representing $T(f) \simeq T(id,f)$ because of the following property.

**Proposition 8.1.** If $\tilde{f}$ is a homotopy equivalence, then $\tilde{f},id_X\#_\ast$ is a homotopy equivalence.

**Proof.** This follows from the main property of homotopy colimits (Kozlov 2008, Lemma 15.12; Hatcher 2002, Prop. 4G.1).

From the representations in rigorous numerics, it seems hard to show that $\tilde{f}$ is a homotopy equivalence. But the algorithms therein can construct an enclosure $Z$ of the graph of $f$ such that $p^{-1}(x)$ has the homology of the one-point space for all $x \in X$. In this case, the Vietoris mapping theorem (Vietoris 1927) shows that $\tilde{f}$ induces an isomorphism in homology (similar theorems for homotopy groups exist, cf. Smale (1957)). Then the following proposition offers a way to compute the homology of $T(f)$.

**Proposition 8.2.** If $\tilde{f}$ induces an isomorphism in homology, then $\tilde{f},id_X\#_\ast$ induces an isomorphism in homology $H_\ast(T(f)) \cong H_\ast(T(p,q))$.

**Proof.** Let $j(x) = [x,0]$ be the inclusion of $X$ into the respective mapping torus. Example 2.48 from Hatcher (2002) shows the construction of homomorphisms $\partial$ such that the upper and lower sequence in the following diagram are exact:

\[
\begin{array}{cccccccc}
\cdots & \to & H_n(X) & \xrightarrow{id-f_\ast} & H_n(X) & \xrightarrow{j_\ast} & H_n(T(id_X,f)) & \xrightarrow{\partial} & H_{n-1}(X) & \to & \cdots \\
\cong & \downarrow{\tilde{f}_\ast} & \id & \downarrow{id} & \xrightarrow{(\tilde{f},id_X\#)_\ast} & \cong & \tilde{f}_\ast \\
\cdots & \to & H_n(Z) & \xrightarrow{p_\ast-q_\ast} & H_n(X) & \xrightarrow{j_\ast} & H_n(T(p,q)) & \xrightarrow{\partial} & H_{n-1}(Z) & \to & \cdots
\end{array}
\]

The left and middle square commute because the underlying squares of continuous maps commute. In order to apply the 5-lemma, it remains to check that the right square commutes. By construction,

\[
\partial: H_n(T(p,q)) \to \{(\alpha,-\alpha) \mid \alpha \in H_{n-1}(Z)\} \cong H_{n-1}(Z)
\]

is the composition of homomorphisms which are either part of the long exact sequence of some pair of spaces or induced by a continuous map. The desired commutativity follows from the naturality of these long exact sequences in homology.

Note that the numerical computation of the homology groups of some space (here, $T(p,q)$) requires less machinery than computing the induced map in homology $H_\ast(f)$. This might be useful in situations where it is not obvious how to extract $H_\ast(f)$ from the representation of $f$, for example when only a noisy sample of pairs $(x,f(x))$ is given as in Edelsbrunner et al. (2015). But also if one has an enclosure $Z$ of the graph as above, one could avoid computing the induced homology $H_\ast(f)$ if the homology $H_\ast(T(f))$ of the mapping torus already contains the relevant information.

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