A New Solution of the Yang-Baxter Equation
Related to the Adjoint Representation of $U_qB_2$

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Abstract

A new solution of the Yang-Baxter equation, that is related to the adjoint representation of the quantum enveloping algebra $U_qB_2$, is obtained by fusion formulas from a non-standard solution.
1. INTRODUCTION

There are three typical methods [1] for finding the trigonometric solutions of the Yang-Baxter equation [2]. The main one is based on Jimbo’s theorem [3,4]. The necessary condition for using this method is existence of the quantum generator $e_0$, corresponding to the negative lowest root. The second method for finding solutions is so-called Yang-Baxterization, namely to embed appropriately the spectral parameter $x$ into a solution $\tilde{R}_q$ of the simple Yang-Baxter equation such that $\tilde{R}_q(x)$ satisfies the Yang-Baxter equation. This method is useful for the cases where the spectrum-independent solution $\tilde{R}_q$ has only two or three different eigenvalues [5,1]. The third method is the fusion formulas [6,1] where an appropriate project operator is needed.

Unfortunately, firstly, the explicit form of $e_0$, that satisfies the quantum algebraic relations, does not exist for the adjoint representation of any quantum enveloping algebra $U_qG$, except for $U_qA_t$. Secondly, the spectrum-independent solution $\tilde{R}_q$ for the adjoint representation usually has much more different eigenvalues than three. For instance, in the simplest case, the solution $\tilde{R}_q$ for the adjoint representation of $U_qB_2$ has six different eigenvalues. At last, from the solution $\tilde{R}_q(x)$ related to the minimal representation, obtained based on Jimbo’s theorem, the needed project operator for the fusion formulas does not exist for this case. It is the reason why no solution of the Yang-Baxter equation related to the adjoint representation of $U_qG$, except for $U_qA_t$, was found up to now.

On the other hand, by Yang-Baxterization when $\tilde{R}_q$ has three different eigenvalues, there is an additional solution, so-called non-standard one, that happens to provide the needed project operator for the fusion formulas. In this way we are able to compute the solution related to the adjoint representations of $U_qB_t$, $U_qC_t$ and $U_qD_t$. In order to realize this idea, in this paper we compute explicitly the simplest example of those cases: the trigonometric and rational solutions of the Yang-Baxter equation related to the adjoint representation of $U_qB_2$, that is equivalent to $U_qC_2$. The rest of solutions can be computed straightforwardly, but more complicatedly.

This paper is organized as follows. In Sec. 2, we show that the explicit form of
matrix for the adjoint representation of $U_qB_2$, that satisfies the quantum algebraic relations, does not exist. In order to use the fusion formulas, we have to compute firstly the solution $\tilde{R}_q(x)$ of the Yang-Baxter equation related to the minimal representation in Sec. 3. From it we obtain the project operator $\tilde{R}_q(q^{-4})$ that maps the direct product spaces $V_{(1 \ 0)} \otimes V_{(1 \ 0)}$ onto the representation space $V_{\text{adj}} = V_{(0 \ 2)}$ of the adjoint representation, where $V_{(1 \ 0)}$ is the representation spaces of the minimal representation $(1 \ 0)$. In Sec. 4 we sketch the proof for the fusion formulas. The explicit form of $\tilde{R}_{\text{adj}}^q(x)$ is computed in Sec. 5 in terms of the quantum Clebsch-Gordan coefficients for the coproduct in the direct product of two representation spaces of the adjoint representation. The corresponding rational solution of the Yang-Baxter equation is obtained in Sec. 6 by a standard limit process [1].

2. Non-Existence of $e_0$ Matrix

The Cartan matrix for the algebra $B_2$ is

$$a = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad a^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1 & 1 \end{pmatrix}$$  \quad (1)$$

From it we have the relation between the simple roots $r_j$ and the fundamental dominant weight $\lambda_j$:

$$r_1 = 2\lambda_1 - 2\lambda_2, \quad r_2 = -\lambda_1 + 2\lambda_2$$
$$\lambda_1 = r_1 + r_2, \quad \lambda_2 = r_1/2 + r_2$$  \quad (2)$$

An irreducible representation of $U_qB_2$ is denoted by its highest weight $\mathbf{M} = (M_1, M_2)$ and the states by $\mathbf{m} = (m_1, m_2)$:

$$\mathbf{M} = M_1 \lambda_1 + M_2 \lambda_2, \quad \mathbf{m} = m_1 \lambda_1 + m_2 \lambda_2$$  \quad (3)$$

The minimal representation is denoted by $(1 \ 0)$, and the adjoint representation by $(0 \ 2)$. The Casimir $C_2(\mathbf{M})$ is calculated by the following formula:

$$C_2(\mathbf{M}) = M_1^2 + M_1M_2 + M_2^2/2 + 3M_1 + 2M_2$$  \quad (4)$$
Through a standard method [1] we draw the block weight diagrams for the representations (1 0) and (0 2) in Fig. 1.

\begin{align*}
\text{a) Minimal representation} & \quad \text{b) Adjoint representation} \\
\end{align*}

**Fig. 1.** Block weight diagrams for the minimal and adjoint representations of algebra $U_qB_2$

In order to simplify the notation we enumerate the states in those two representations as shown near the blocks in Fig. 1. In terms of the enumerations for the states and the matrix bases $E_{a \ b}$:

\[(E_{a \ b})_{c \ d} = \delta_{ac} \delta_{bd}\]  \hspace{1cm} (5)

we obtain the quantum representation matrices for two representations as follows. For the minimal representation (1 0) we have:

\begin{align*}
D_q(e_1) &= \tilde{D}_q(f_1) = E_{2 \ 1} + E_{1 \ 2} \\
D_q(e_2) &= \tilde{D}_q(f_2) = [2]^{1/2} (E_{1 \ 0} + E_{0 \ 1}) \\
D_q(k_1) &= q E_{2 \ 2} + q^{-1} E_{1 \ 1} + E_{0 \ 0} + q E_{1 \ 1} + q^{-1} E_{2 \ 2} \\
D_q(k_2) &= E_{2 \ 2} + q E_{1 \ 1} + E_{0 \ 0} + q^{-1} E_{1 \ 1} + E_{2 \ 2}
\end{align*}

(6)
and for the adjoint representation \((0 \ 2)\) we have:

\[
D_q(e_1) = \tilde{D}_q(f_1) = E_{3\ 1} + E_{2\ 0} + \left( \frac{[6]}{[3][2]} \right)^{1/2} (E_{2\ 0'} + E_{0'\ 2}) + E_{0\ 2} + E_{1\ 3}
\]

\[
D_q(e_2) = \tilde{D}_q(f_2) = [2]^{1/2} (E_{4\ 3} + E_{3\ 2} + E_{1\ 0} + E_{0\ 1} + E_{2\ 3} + E_{3\ 4})
\]

\[
D_q(k_1) = E_{4\ 4} + q E_{3\ 3} + q^2 E_{2\ 2} + q^{-1} E_{1\ 1} + E_{0\ 0} + E_{0'\ 0'} + q E_{1\ 1} + q^{-2} E_{2\ 2} + q^{-1} E_{3\ 3} + E_{4\ 4}
\]

\[
D_q(k_2) = q E_{4\ 4} + E_{3\ 3} + q^{-1} E_{2\ 2} + q E_{1\ 1} + E_{0\ 0} + E_{0'\ 0'} + q^{-1} E_{1\ 1} + q E_{2\ 2} + E_{3\ 3} + q^{-1} E_{4\ 4}
\]

where, as usual, \([m]\) denotes:

\[
[m] = \frac{q^m - q^{-m}}{q - q^{-1}}
\]

Since the negative lowest root \(r_0\) of \(B_2\) is:

\[
r_0 = -2\lambda_2 = -r_1 - 2r_2
\]

the possible forms of the representation matrices of \(e_0\) and \(f_0\), that correspond to \(r_0\), are as follows:

\[
D_q(e_0) = a_1 E_{0\ 4} + a_2 E_{0'\ 4} + a_3 E_{1\ 3} + a_4 E_{3\ 1} + a_5 E_{4\ 0} + a_6 E_{4\ 0'}
\]

\[
D_q(f_0) = b_1 E_{4\ 0} + b_2 E_{4\ 0'} + b_3 E_{3\ 1} + b_4 E_{1\ 3} + b_5 E_{0\ 4} + b_6 E_{0'\ 4}
\]

From the quantum algebraic relations:

\[
[D_q(e_0), D_q(f_j)] = 0, \quad [D_q(f_0), D_q(e_j)] = 0, \quad j = 1, 2
\]

we obtain

\[
- \left( \frac{[6]}{[3][2]} \right)^{1/2} a_2 = - \left( \frac{[6]}{[3][2]} \right)^{1/2} a_6 = a_1 = a_3 = a_4 = a_5
\]

\[
- \left( \frac{[6]}{[3][2]} \right)^{1/2} b_2 = - \left( \frac{[6]}{[3][2]} \right)^{1/2} b_6 = b_1 = b_3 = b_4 = b_5
\]
It is easy to check that the quantum Serre relations are not satisfied:

\[ D_q(e_0)^2 D_q(e_2) - \left( \frac{[4]}{[2]} \right) D_q(e_0) D_q(e_2) D_q(e_0) + D_q(e_2) D_q(e_0)^2 = - (q^{-1} - q)^2 \left( \frac{[4]}{[2]} \frac{3}{[6]} \right) a_1^2 \left[ \frac{1}{2} \right]^{1/2} (E_{4\ 3} + E_{3\ 4}) \neq 0 \]  

\[ D_q(f_0)^2 D_q(f_2) - \left( \frac{[4]}{[2]} \right) D_q(f_0) D_q(f_2) D_q(f_0) + D_q(f_2) D_q(f_0)^2 = - (q^{-1} - q)^2 \left( \frac{[4]}{[2]} \frac{3}{[6]} \right) b_1^2 \left[ \frac{1}{2} \right]^{1/2} (E_{3\ 4} + E_{4\ 3}) \neq 0 \]  

The commutator of \( D_q(e_0) \) and \( D_q(f_0) \) does not satisfy the quantum algebraic relations, either. Therefore, the representation matrix \( D_q(e_0) \) does not exist for the adjoint representation of \( U_qB_2 \).

3. Solutions for the Minimal Representation

In the fusion formulas, the \( \tilde{R}_q^{\text{adj}}(x) \) matrix for the adjoint representation is expressed in terms of the \( \tilde{R}_q(x) \) matrix for the minimal representation. In this section we compute the \( \tilde{R}_q(x) \) matrix for the minimal representation firstly. As a matter of fact, the \( e_0 \) matrix exists in the minimal representation of \( U_qB_2 \) so that the corresponding solution \( \tilde{R}_q(x) \) was computed \([3,1]\) by the standard method based on Jimbo’s theorem.

The Clebsch-Gordan series for the direct product of two minimal representations is as follows:

\[ (1\ 0) \otimes (1\ 0) = (2\ 0) \oplus (0\ 2) \oplus (0\ 0) \]  

Denote by \( P_N \) the project operator that is the product of two quantum Clebsch-Gordan coefficients \([1]\):

\[ P_N = (C_q)_N \left( \tilde{C}_q \right)_N \]  

By making use of the standard method based on Jimbo’s theorem, we obtain the \( \tilde{R}_q'(x) \) matrix for the minimal representation as follows \([3,1]\):

\[ \tilde{R}_q'(x) = (1 - xq^4)(1 - xq^6) P_{(2\ 0)} + (x - q^4)(1 - xq^6) P_{(0\ 2)} + (x - q^4)(x - q^6) P_{(0\ 0)} \]  

(16)
where a prime is added on \( \tilde{R}'_q(x) \) in order to distinguish it from the additional solution \( \tilde{R}_q(x) \) given in Eq.(17). In this form of \( \tilde{R}'_q(x) \), it cannot be proportional to the projector operator \( \mathcal{P}_{(0\ 2)} \) that maps the direct product space onto the space of the adjoint representation. In the same paper [3] Jimbo pointed out that there is another solution related to the algebra \( U_q A_4^{(2)} \):

\[
\tilde{R}(x) = (1 - xq^4)(1 + xq^{10}) \mathcal{P}_{(2\ 0)} \\
+ (x - q^4)(1 + xq^{10}) \mathcal{P}_{(0\ 2)} \\
+ (1 - xq^4)(x + q^{10}) \mathcal{P}_{(0\ 0)}
\]  

(17)

Now, we know [5,1] that because the Clebsch-Gordan series (14) contains only three representations including an identity representation (0 0), we can obtain two independent solutions of the Yang-Baxter equation given in Eqs.(16) and (17) in terms of Yang-Baxterization. The solution (17), called non-standard one, has a good property:

\[
\tilde{R}(q^{-4}) = (q^{-4} - q^4)(1 + q^6) \mathcal{P}_{(0\ 2)}
\]  

(18)

namely, \( \tilde{R}(q^{-4}) \) is proportional to the project operator \( \mathcal{P}_{(0\ 2)} \) onto the adjoint representation:

\[
\tilde{R}(q^{-4}) \left( V_{(1\ 0)} \otimes V_{(1\ 0)} \right) = V_{(0\ 2)}
\]  

(19)

It is the key point for computing the solution related to the adjoint representation from Eq.(17).

Solution (17) is a 25×25 symmetric matrix on the direct product space \( V_{(1\ 0)} \otimes V_{(1\ 0)} \). The row (column) indices are denoted by \( m_1m_2 \), where both \( m_1 \) and \( m_2 \) take the values 2, 1, 0, \( \bar{1} \), and 2. \( \tilde{R}_q(x) \) has the following symmetries:

\[
\tilde{R}(x)_{m_1m_2 \ m_3m_4} = \tilde{R}(x)_{m_3m_4 \ m_1m_2} \\
= \tilde{R}(x)_{m_2m_1 \ m_4m_3} \\
= -x^2 q^{14} \tilde{R}_{q^{-1}}(x^{-1})_{m_2m_1 \ m_4m_3} \\
\tilde{R}(1)_{m_1m_2 \ m_3m_4} = (1 - q^4)(1 + q^{10}) \delta_{m_1m_3} \delta_{m_2m_4}
\]  

(20)

where \( \bar{0} = 0 \).
\( \check{R}_q(x) \) given in Eq.(17) satisfies the weight conservation condition, namely, \( \check{R}_q(x) \) is a block matrix with four 1×1, eight 2×2 and one 5×5 submatrices. Through straightforward calculation we obtain the explicit form for \( \check{R}_q(x) \). Owing to the symmetries (20) we only need to list the results as follows:

a) Four 1×1 submatrices.
\[
\check{R}_q(x)_{22} \ 22 = \check{R}_q(x)_{11} \ 11 = (1 - x q^4)(1 + x q^{10}) \tag{21a}
\]

b) Eight 2×2 submatrices.
\[
\check{R}_q(x)_{21} \ 21 = \check{R}_q(x)_{20} \ 20 = \check{R}_q(x)_{21} \ 21 = \check{R}_q(x)_{10} \ 10
= (1 - q^4)x(1 + x q^{10}) \tag{21b}
\]
\[
\check{R}_q(x)_{21} \ 12 = \check{R}_q(x)_{20} \ 02 = \check{R}_q(x)_{21} \ i2 = \check{R}_q(x)_{10} \ 01
= q^2(1 - x)(1 + x q^{10})
\]

c) One 5×5 submatrix.
\[
\check{R}_q(x)_{22} \ 22 = (1 - q^4)x \{ (1 + q^4) - x q^4(1 - q^6) \}
\check{R}_q(x)_{11} \ 11 = (1 - q^4)x \{ (1 + q^8) - x q^8(1 - q^2) \}
\check{R}_q(x)_{00} \ 00 = q^2(1 - x)(1 + x q^{10}) + x(1 - q^4)(1 + q^{10})
\check{R}_q(x)_{22} \ 11 = - x(1 - x)q^6(1 - q^4)
\check{R}_q(x)_{22} \ 00 = x(1 - x)q^7(1 - q^4)
\check{R}_q(x)_{22} \ 11 = - x(1 - x)q^8(1 - q^4)
\check{R}_q(x)_{11} \ 00 = - x(1 - x)q^9(1 - q^4)
\check{R}_q(x)_{22} \ 22 = \check{R}_q(x)_{11} \ 11 = q^4(1 - x)(1 + x q^6) \tag{21c}
\]

4. Fusion Formulas

The project operator \( \check{R}_q(q^{-4}) \) maps the direct product space \( V_{(1 \ 0)} \otimes V_{(1 \ 0)} \) of two minimal representations onto the representation space \( V_{(0 \ 2)} \) of the adjoint representation. The solution \( \check{R}_q^{\text{adj}}(x) \) of the Yang-Baxter equation related to the adjoint representation of \( U_q B_2 \) is applied on the direct product space \( V_{(0 \ 2)} \otimes V_{(0 \ 2)} \):
\[
V_{(0 \ 2)} \otimes V_{(0 \ 2)} = \left\{ \check{R}_q(q^{-4}) \otimes \check{R}_q(q^{-4}) \right\} \left\{ V_{(1 \ 0)} \otimes V_{(1 \ 0)} \otimes V_{(1 \ 0)} \otimes V_{(1 \ 0)} \otimes \right\} \tag{22}
\]
According to the fusion formulas, $\tilde{R}^\text{adj}_q(x)$ can be expressed as the following product [6,1]:

$$
\tilde{R}^\text{adj}_q(x) = \left(1 \otimes \tilde{R}_q(x q^4) \otimes 1\right) \left(\tilde{R}_q(x) \otimes \tilde{R}_q(x)\right) \left(1 \otimes \tilde{R}_q(x q^{-4}) \otimes 1\right) \quad (23)
$$

Now, we are going to sketch the proof. First of all, we show that $\tilde{R}^\text{adj}_q(x)$ given in Eq.(23) is a matrix on the space (22). From the Yang-Baxter equation satisfied by $\tilde{R}_q(x)$:

$$
\left(1 \otimes \tilde{R}_q(x)\right) \left(\tilde{R}_q(xy) \otimes 1\right) \left(1 \otimes \tilde{R}_q(y)\right) = \left(\tilde{R}_q(y) \otimes 1\right) \left(1 \otimes \tilde{R}_q(xy)\right) \left(\tilde{R}_q(x) \otimes 1\right) \quad (24)
$$

we have:

$$
\tilde{R}^\text{adj}_q(x) \left\{V_{(0\ 2)} \otimes V_{(0\ 2)}\right\}
= \left(1 \otimes \tilde{R}_q(x q^4) \otimes 1\right) \left(1 \otimes 1 \otimes \tilde{R}_q(x)\right)
\cdot \left(\tilde{R}_q(x) \otimes 1 \otimes 1\right) \left(1 \otimes \tilde{R}_q(x q^{-4}) \otimes 1\right) \left(\tilde{R}_q(q^{-4}) \otimes 1 \otimes 1\right)
\cdot \left(1 \otimes 1 \otimes \tilde{R}_q(q^{-4})\right) \left\{V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)}\right\}
= \left(1 \otimes \tilde{R}_q(x q^4) \otimes 1\right) \left(1 \otimes 1 \otimes \tilde{R}_q(q^{-4}) \otimes 1\right)
\cdot \left(\tilde{R}_q(x q^{-4}) \otimes 1 \otimes 1\right) \left(1 \otimes \tilde{R}_q(x) \otimes 1\right) \left(1 \otimes 1 \otimes \tilde{R}_q(q^{-4}) \otimes 1\right)
\cdot \left(1 \otimes 1 \otimes \tilde{R}_q(q^{-4})\right) \left\{V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)}\right\}
= \left(1 \otimes 1 \otimes \tilde{R}_q(x q^{-4})\right) \left(\tilde{R}_q(q^{-4}) \otimes 1 \otimes 1\right) \left(1 \otimes \tilde{R}_q(x q^{-4}) \otimes 1\right)
\cdot \left(\tilde{R}_q(x) \otimes 1 \otimes 1\right) \left(1 \otimes 1 \otimes \tilde{R}_q(x)\right) \left(1 \otimes \tilde{R}_q(x q^4) \otimes 1\right)
\cdot \left(\tilde{R}_q(q^{-4}) \otimes 1 \otimes 1\right) \left\{V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)}\right\}
\subset \left(R_q(q^{-4}) \otimes \tilde{R}_q(q^{-4})\right) \left\{V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)}\right\}
= V_{(0\ 2)} \otimes V_{(0\ 2)}
By making use of Eq.(24) successively, it is straightforward to prove that $\tilde{R}^{\text{adj}}_q(x)$ satisfies the Yang-Baxter equation, that is an equation on the direct product space $V_{(1\ 0)}^{\otimes 6}$:

$$
\begin{align*}
&\left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(xq^4) \otimes 1\right)\left(1 \otimes 1 \otimes \tilde{R}_q(x) \otimes \tilde{R}_q(x)\right)\left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(xq^{-4}) \otimes 1\right) \\
&\cdot \left(1 \otimes \tilde{R}_q(xyq^4) \otimes 1 \otimes 1 \otimes 1\right)\left(\tilde{R}_q(xy) \otimes \tilde{R}_q(xy) \otimes 1 \otimes 1 \otimes 1\right)\left(1 \otimes \tilde{R}_q(xyq^{-4}) \otimes 1 \otimes 1 \otimes 1\right) \\
&\cdot \left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(yq^4) \otimes 1\right)\left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(y) \otimes \tilde{R}_q(y)\right)\left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(yq^{-4}) \otimes 1\right) \\
&= \left(1 \otimes \tilde{R}_q(yq^4) \otimes 1 \otimes 1 \otimes 1\right)\left(\tilde{R}_q(y) \otimes \tilde{R}_q(y) \otimes 1 \otimes 1 \otimes 1\right)\left(1 \otimes \tilde{R}_q(yq^{-4}) \otimes 1 \otimes 1 \otimes 1\right) \\
&\cdot \left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(xyq^4) \otimes 1\right)\left(1 \otimes 1 \otimes \tilde{R}_q(xy) \otimes \tilde{R}_q(xy)\right)\left(1 \otimes 1 \otimes 1 \otimes \tilde{R}_q(xyq^{-4}) \otimes 1\right) \\
&\cdot \left(1 \otimes \tilde{R}_q(xq^4) \otimes 1 \otimes 1 \otimes 1\right)\left(\tilde{R}_q(x) \otimes \tilde{R}_q(x) \otimes 1 \otimes 1 \otimes 1\right)\left(1 \otimes \tilde{R}_q(xq^{-4}) \otimes 1 \otimes 1 \otimes 1\right)
\end{align*}
$$

(25)

5. Explicit Form of the Solution for the Adjoint Representation

The Clebsch-Gordan series for the direct product of two adjoint representations of $B_2$ is:

$$
(0\ 2) \otimes (0\ 2) = (0\ 4) \oplus (1\ 2) \oplus (2\ 0) \oplus (0\ 2) \oplus (1\ 0) \oplus (0\ 0)
$$

(26)

Both the solution $\tilde{R}^{\text{adj}}_q$ of the simple Yang-Baxter equation and the solution $\tilde{R}^{\text{adj}}_q(x)$ of the Yang-Baxter equation, related to the adjoint representation of $U_qB_2$, can be expanded by the project operators as follows:

$$
\tilde{R}^{\text{adj}}_q = \mathcal{P}_{(0\ 4)} - q^4 \mathcal{P}_{(1\ 2)} + q^6 \mathcal{P}_{(2\ 0)} - q^{10} \mathcal{P}_{(0\ 2)} + q^{12} \mathcal{P}_{(1\ 0)} + q^{16} \mathcal{P}_{(0\ 0)}
$$

(27)

$$
\tilde{R}^{\text{adj}}_q(x) = \Lambda_{(0\ 4)}(x, q) \mathcal{P}_{(0\ 4)} + \Lambda_{(1\ 2)}(x, q) \mathcal{P}_{(1\ 2)} + \Lambda_{(2\ 0)}(x, q) \mathcal{P}_{(2\ 0)} \\
+ \Lambda_{(0\ 2)}(x, q) \mathcal{P}_{(0\ 2)} + \Lambda_{(1\ 0)}(x, q) \mathcal{P}_{(1\ 0)} + \Lambda_{(0\ 0)}(x, q) \mathcal{P}_{(0\ 0)}
$$

(28)

$$
\tilde{R}^{\text{adj}}_q(0) = \tilde{R}^{\text{adj}}_q
$$

(29)

where, as usual, the project operators are the product of two quantum Clebsch-Gordan matrices:

$$
\mathcal{P}_N = \begin{pmatrix} C_q^{(0\ 2)(0\ 2)} \end{pmatrix}_N \begin{pmatrix} \tilde{C}_q^{(0\ 2)(0\ 2)} \end{pmatrix}_N
$$

(30)
Now, we are going to compute the coefficients $\Lambda_N(x, q)$:

$$\tilde{R}^\text{adj}_q(x) \left| N, N \right\rangle = \Lambda_N(x, q) \left| N, N \right\rangle \quad (31)$$

In the computation, we need the quantum Clebsch-Gordan coefficients to combine the states $|m_1, m_2, m_3, m_4\rangle \equiv |m_1\rangle |m_2\rangle |m_3\rangle |m_4\rangle$ in the space $V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)} \otimes V_{(1\ 0)}$ into the state $|N, N\rangle$.

Firstly, through the standard calculation, we obtain the quantum Clebsch-Gordan coefficients for the adjoint representation of $U_q B_2$. Denote by $|(0\ 2), m\rangle$ the states in the adjoint representation, and by $|m_1 m_2\rangle \equiv |m_1\rangle |m_2\rangle$ the states in the space $V_{(1\ 0)} \otimes V_{(1\ 0)}$, where the states is described by the enumerations given in Fig.1. Owing to the symmetry of the quantum Clebsch-Gordan coefficients:

$$(C_q)_{m_1 m_2 (0\ 2) m} = - (C_{q^{-1}})_{m_2 m_1 (0\ 2) m}$$

we only need to list the following Clebsch-Gordan coefficients:

$$
\begin{align*}
| (0\ 2), 4 \rangle &= \left( [2]/[4] \right)^{1/2} \left\{ q^{-1} | 2 \ 1 \rangle - q | 1 \ 2 \rangle \right\} \\
| (0\ 2), 3 \rangle &= [2]^{-1/2} f_2 | (0\ 2), 4 \rangle = \left( [2]/[4] \right)^{1/2} \left\{ q^{-1} | 2 \ 0 \rangle - q | 0 \ 2 \rangle \right\} \\
| (0\ 2), 2 \rangle &= [2]^{-1/2} f_2 | (0\ 2), 3 \rangle = \left( [2]/[4] \right)^{1/2} \left\{ q^{-1} | 2 \ \bar{1} \rangle - q | \bar{1} \ 2 \rangle \right\} \\
| (0\ 2), 1 \rangle &= f_1 | (0\ 2), 3 \rangle = \left( [2]/[4] \right)^{1/2} \left\{ q^{-1} | 1 \ 0 \rangle - q | 0 \ 1 \rangle \right\} \\
| (0\ 2), 0 \rangle &= [2]^{-1/2} f_2 | (0\ 2), 1 \rangle \\
 &= \left( [2]/[4] \right)^{1/2} \left\{ |1 \ \bar{1} \rangle + (q^{-1} - q) | 0 \ 0 \rangle - | \bar{1} \ 1 \rangle \right\} \\
| (0\ 2), 0' \rangle &= \left( [3][2]/[6] \right)^{1/2} \left\{ f_1 | (0\ 2), 2 \rangle - | (0\ 2), 0 \rangle \right\} \\
 &= [2] \left( [3]/[6][4] \right)^{1/2} \left\{ |2 \ \bar{2}\rangle + (q^{-2} - 1) |1 \ \bar{1}\rangle \\
 &\quad - (q^{-1} - q) |0 \ 0\rangle + (1-q^2) | \bar{1} \ 1 \rangle - | \bar{2} \ 2 \rangle \right\} \\
\end{align*}
$$

(33)

From Eq.(33) we are able to compute the expansive expressions for the highest
weight states in the Clebsch-Gordan series (26):

\[
| (0 4) , (0 4) \rangle = | (0 2) , 4 \rangle | (0 2) , 4 \rangle \\
= ([2]/[4]) \left\{ q^{-2} | 2 1 2 1 \rangle - \frac{1}{2} | 2 1 1 2 \rangle \right. \\
\left. - \frac{1}{2} | 1 2 2 1 \rangle + q^2 | 1 2 1 2 \rangle \right\} \\
\tag{34a}
\]

\[
| (1 2) , (1 2) \rangle \\
= ([2]/[4])^{1/2} \left\{ q^{-1} | (0 2) , 4 \rangle | (0 2) , 3 \rangle - q | (0 2) , 3 \rangle | (0 2) , 4 \rangle \right\} \\
= ([2]/[4])^{3/2} \left\{ q^{-3} | 2 1 2 0 \rangle - q^{-1} | 2 1 0 2 \rangle - q^{-1} | 1 2 2 0 \rangle \\
+ q | 1 2 0 2 \rangle - q^{-1} | 2 0 2 1 \rangle + q | 0 2 2 1 \rangle \\
+ q | 2 0 1 2 \rangle - q^3 | 0 2 1 2 \rangle \right\} \\
\tag{34b}
\]

where we see that the second half terms of Eqs.(34a) and (34b) can be obtained from the first half terms by exchanging:

\[
F(q) | m_1 m_2 m_3 m_4 \rangle \rightarrow \pm F(q^{-1}) | m_4 m_3 m_2 m_1 \rangle \\
\tag{35}
\]

where the plus sign stands for Eq.(34a), and the minus sign for Eq.(34b). In the following we will use the abbreviatory notation (S terms) (for eq.(35) with plus sign) or (A terms) (minus sign) to replace the second half terms, respectively. In this way equations (34a) and (34b) are rewritten as follows:

\[
| (0 4) , (0 4) \rangle = ([2]/[4]) \left\{ q^{-2} | 2 1 2 1 \rangle - \frac{1}{2} | 2 1 1 2 \rangle \\
- \frac{1}{2} | 1 2 2 1 \rangle \right. \\
\left. + \{ \text{S terms} \} \right\} \\
| (1 2) , (1 2) \rangle = ([2]/[4])^{3/2} \left\{ q^{-3} | 2 1 2 0 \rangle - q^{-1} | 2 1 0 2 \rangle \\
- q^{-1} | 1 2 2 0 \rangle + q | 1 2 0 2 \rangle \right. \\
\left. + \{ \text{S terms} \} \right\} \\
\tag{34c}
\]

In the same way we have:

\[
| (2 0) , (2 0) \rangle \\
= ([3])^{-1/2} \left\{ q^{-1} | (0 2) , 4 \rangle | (0 2) , 2 \rangle - | (0 2) , 3 \rangle | (0 2) , 4 \rangle \right\} \\
+ q | (0 2) , 2 \rangle | (0 2) , 4 \rangle \right\} \\
= ([2]/[4]) [3]^{-1/2} \left\{ q^{-3} | 2 1 2 \bar{1} \rangle - q^{-1} | 2 1 \bar{1} 2 \rangle - q^{-1} | 1 2 2 \bar{1} \rangle \\
+ q | 1 2 \bar{1} 2 \rangle - q^2 | 0 2 0 2 \rangle + \frac{1}{2} | 2 0 0 2 \rangle \\
+ \frac{1}{2} | 0 2 2 0 \rangle \right. \\
\left. + \{ \text{S terms} \} \right\} \\
\tag{34c}
\]
\[
\begin{align*}
\langle (0\ 2)\ ,\ (0\ 2) \rangle &= [3]^{-1} ([6][5][2]/[10][4])^{1/2} \{ q^{-3} | (0\ 2)\ ,\ 4 \rangle | (0\ 2)\ ,\ 0 \rangle \\
&\quad - q^{-3} ([3][2]/[6])^{1/2} | (0\ 2)\ ,\ 4 \rangle | (0\ 2)\ ,\ 0' \rangle - q^{-1} | (0\ 2)\ ,\ 3 \rangle | (0\ 2)\ ,\ 1 \rangle \\
&\quad + q | (0\ 2)\ ,\ 1 \rangle | (0\ 2)\ ,\ 3 \rangle + q^3 ([3][2]/[6])^{1/2} | (0\ 2)\ ,\ 0' \rangle | (0\ 2)\ ,\ 4 \rangle \\
&\quad - q^3 | (0\ 2)\ ,\ 0 \rangle | (0\ 2)\ ,\ 4 \rangle \} \\
&= ([2]^2/[4]) ([5][2]/[10][6][4])^{1/2} \{ - q^{-4} | 2\ 1\ 2\ 2 \rangle + q^{-2} | 2\ 1\ 1\ 1 \rangle \\
&\quad - q^{-6} | 2\ 1\ 1\ 1 \rangle + q^{-4} | 2\ 1\ 2\ 2 \rangle + q^{-2} | 1\ 2\ 2\ 2 \rangle - | 1\ 2\ 1\ 1 \rangle \\
&\quad + q^{-4} | 1\ 2\ 1\ 1 \rangle - q^{-2} | 1\ 2\ 2\ 2 \rangle + (q^{-1} - q) ([4]/[2]) ( q^{-4} | 2\ 1\ 0\ 0 \rangle \\
&\quad - q^{-2} | 1\ 2\ 0\ 0 \rangle ) + ([6]/[3][2])( - q^{-3} | 2\ 0\ 1\ 0 \rangle + q^{-1} | 2\ 0\ 0\ 1 \rangle \\
&\quad + q^{-1} | 0\ 2\ 1\ 0 \rangle - q | 0\ 2\ 0\ 1 \rangle ) + (\text{A terms}) \} \\
&= ([4]/[8][3])^{1/2} \{ q^{-3} | (0\ 2)\ ,\ 4 \rangle | (0\ 2)\ ,\ 1 \rangle - q^{-2} | (0\ 2)\ ,\ 3 \rangle | (0\ 2)\ ,\ 0 \rangle \\
&\quad + q^{-1} | (0\ 2)\ ,\ 2 \rangle | (0\ 2)\ ,\ 1 \rangle + q | (0\ 2)\ ,\ 1 \rangle | (0\ 2)\ ,\ 2 \rangle \\
&\quad - q^2 | (0\ 2)\ ,\ 0 \rangle | (0\ 2)\ ,\ 3 \rangle + q^3 | (0\ 2)\ ,\ 1 \rangle | (0\ 2)\ ,\ 4 \rangle \} \\
&= [2] ([8][4][3])^{-1/2} \{ q^{-5} | 2\ 1\ 0\ 1 \rangle - q^{-3} | 2\ 1\ 1\ 0 \rangle - q^{-3} | 1\ 2\ 0\ 1 \rangle \\
&\quad + q^{-1} | 1\ 2\ 1\ 0 \rangle - q^{-3} | 2\ 0\ 1\ 1 \rangle - (q^{-4} - q^{-2}) | 2\ 0\ 0\ 0 \rangle + q^{-3} | 2\ 0\ 1\ 1 \rangle \\
&\quad + q^{-1} | 0\ 2\ 1\ 1 \rangle + (q^{-2} - 1) | 0\ 2\ 0\ 0 \rangle - q^{-1} | 0\ 2\ 1\ 1 \rangle + q^{-3} | 2\ 1\ 0\ 0 \rangle \\
&\quad - q^{-1} | 2\ 1\ 0\ 1 \rangle - q^{-1} | 1\ 2\ 1\ 0 \rangle + q | 1\ 2\ 0\ 1 \rangle ) + (\text{S terms}) \} \\
&= (34d) \\
\langle (1\ 0)\ ,\ (1\ 0) \rangle &= ([4]/[8][3])^{1/2} \{ q^{-3} | (0\ 2)\ ,\ 4 \rangle | (0\ 2)\ ,\ 1 \rangle - q^{-2} | (0\ 2)\ ,\ 3 \rangle | (0\ 2)\ ,\ 0 \rangle \\
&\quad + q^{-1} | (0\ 2)\ ,\ 2 \rangle | (0\ 2)\ ,\ 1 \rangle + q | (0\ 2)\ ,\ 1 \rangle | (0\ 2)\ ,\ 2 \rangle \\
&\quad - q^2 | (0\ 2)\ ,\ 0 \rangle | (0\ 2)\ ,\ 3 \rangle + q^3 | (0\ 2)\ ,\ 1 \rangle | (0\ 2)\ ,\ 4 \rangle \} \\
&= [2] ([8][4][3])^{-1/2} \{ q^{-5} | 2\ 1\ 0\ 1 \rangle - q^{-3} | 2\ 1\ 1\ 0 \rangle - q^{-3} | 1\ 2\ 0\ 1 \rangle \\
&\quad + q^{-1} | 1\ 2\ 1\ 0 \rangle - q^{-3} | 2\ 0\ 1\ 1 \rangle - (q^{-4} - q^{-2}) | 2\ 0\ 0\ 0 \rangle + q^{-3} | 2\ 0\ 1\ 1 \rangle \\
&\quad + q^{-1} | 0\ 2\ 1\ 1 \rangle + (q^{-2} - 1) | 0\ 2\ 0\ 0 \rangle - q^{-1} | 0\ 2\ 1\ 1 \rangle + q^{-3} | 2\ 1\ 0\ 0 \rangle \\
&\quad - q^{-1} | 2\ 1\ 0\ 1 \rangle - q^{-1} | 1\ 2\ 1\ 0 \rangle + q | 1\ 2\ 0\ 1 \rangle ) + (\text{S terms}) \} \\
&= (34e)
\end{align*}
\]
\[ |00\rangle (00) \]

\[ \frac{1}{(4/8[5])^{1/2}} \{ q^{-4} |02, 4\rangle |02\rangle & q^{-3} |02, 3\rangle |02\rangle + q^{-2} |02, 2\rangle |02\rangle + q^{-1} |02, 1\rangle |02\rangle, \bar{1} \}
\]

\[ = [2] (8[5][4])^{-1/2} \{ q^{-6} |2112\rangle - q^{-4} |2121\rangle - q^{-4} |1212\rangle
\]

\[ + q^{-2} |1221\rangle - q^{-5} |2022\rangle + q^{-3} |2002\rangle + q^{-3} |0202\rangle
\]

\[ - q^{-1} |0220\rangle + q^{-4} |2112\rangle - q^{-2} |2121\rangle - q^{-2} |1212\rangle
\]

\[ + |1221\rangle + q^{-3} |1001\rangle - q^{-1} |1010\rangle - q^{-1} |0101\rangle
\]

\[ + q |0110\rangle + ([3][2]/[6]) (-2 |2222\rangle - 2 (q^{-4} - q^{-2} + q^2) |1111\rangle
\]

\[ + |1111\rangle + |1111\rangle + |2222\rangle + |2222\rangle
\]

\[ + (q^{-1} - q) (|2200\rangle + |0022\rangle - q^{-1} |2211\rangle - q^{-1} |1122\rangle
\]

\[ - q^{-2} |1100\rangle - q^{-2} |0011\rangle - q |2211\rangle - q |1122\rangle
\]

\[ - (q^{-1} - q)^2 ([4]/[2]) |0000\rangle + (S \text{ terms}) \}
\]

\[(34f)\]

Now, substituting Eqs.(21), (23)) and (34) into Eq.(31), we obtain \( \Lambda_N(x, q) \), and then, the solution \( \tilde{R}_q^{\text{adj}}(x) \) of the Yang-Baxter equation related to the adjoint representation of \( U_qB_2 \) as follows:

\[ \tilde{R}_q^{\text{adj}}(x) = (q^4 - x)(1 - x)(1 + xq^{10})(1 + xq^{14}) \]

\[ \cdot \left\{ (1 - xq^4)(1 + xq^6)(1 - xq^8)(1 + xq^{10}) \mathcal{P}_{(04)}
\]

\[ + (x - q^4)(1 + xq^6)(1 - xq^8)(1 + xq^{10}) \mathcal{P}_{(12)}
\]

\[ + (1 - xq^4)(z + xq^6)(1 - xq^8)(1 + xq^{10}) \mathcal{P}_{(20)}
\]

\[ + (x - q^4)(x + q^6)(1 - xq^8)(1 + xq^{10}) \mathcal{P}_{(02)}
\]

\[ + (x - q^4)(1 + xq^6)(x - q^8)(1 + xq^{10}) \mathcal{P}_{(10)}
\]

\[ + (1 - xq^4)(x + q^6)(1 - xq^8)(x + q^{10}) \mathcal{P}_{(00)} \}
\]

\[(36)\]

where the common factor \((q^4 - x)(1 - x)(1 + xq^{10})(1 + xq^{14})\) can be removed. In principle, this method can be generalized to the solutions of the Yang-Baxter equation related to the adjoint representations of \( U_qB_\ell, U_qC_\ell \) and \( U_qD_\ell \).
6. Rational Solution for the Adjoint Representation

Through a standard limit process [1] we obtain the corresponding rational solution \(R_{adj}(u/\eta)\) for the adjoint representation of \(U_qB_2\):

\[
R_{adj}(u/\eta) = \lim_{q \to 1} \frac{P \tilde{R}^{adj}_q(q^{2u/\eta})}{(1 - q^{2u/\eta})^2} = 4 \left\{ \begin{array}{l}
(1 + 2u/\eta)(1 + 4u/\eta) \left( P_{(0 \ 4)} + P_{(2 \ 0)} + P_{(0 \ 0)} \right) \\
+ (1 - 2u/\eta)(1 + 4u/\eta) \left( P_{(1 \ 2)} + P_{(0 \ 2)} \right) \\
+ (1 - 2u/\eta)(1 - 4u/\eta) P_{(1 \ 0)} \end{array} \right. \tag{37}
\]

where \(P\) is the transposition operator, and

\[
P_N = \lim_{q \to 1} P_N
\]

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