Non-Commutative Differential Geometry
on Discrete Space $M_4 \times Z_N$ and Gauge Theory

Yoshitaka Okumura

Department of Natural Science, Chubu University, Kasugai, 487, Japan

The algebra of non-commutative differential geometry (NCG) on the discrete space $M_4 \times Z_N$ previously proposed by the present author is improved to give the consistent explanation of the generalized gauge field as the generalized connection on $M_4 \times Z_N$. The nilpotency of the generalized exterior derivative $d$ is easily proved. The matrix formulation where the generalized gauge field is denoted in matrix form is shown to have the same content with the ordinary formulation using $d$, which helps us understand the implications of the algebraic rules of NCG on $M_4 \times Z_N$. The Lagrangian of spontaneously broken gauge theory which has the extra restriction on the coupling constant of the Higgs potential is obtained by taking the inner product of the generalized field strength. The covariant derivative operating on the fermion field determines the parallel transformation on $M_4 \times Z_N$, which confirms that the Higgs field is the connection on the discrete space. This implies that the Higgs particle is a gauge particle on the same footing as the weak bosons. Thus, it is expected that the mass relation $m_H = \frac{4}{\sqrt{3}} m_W \sin \theta_W$ proposed by the present author holds without any correction in the same way as the mass relation $m_W = m_Z \cos \theta_W$. The Higgs kinetic and potential terms are regarded as the curvatures on $M_4 \times Z_N$.

§1. Introduction

In order to understand the essence of the Higgs mechanism we have developed the formulation based on the non-commutative differential geometry (NCG) on the discrete space $M_4 \times Z_N$ and in general $M_4 \times Z_N$. Our formulation is a generalization of ordinary differential geometry on a continuous manifold where the one-form connection on the principal bundle is defined, and then two-form curvature is deduced. If the structure group of the principal bundle is non-abelian, the Lagrangian of the non-abelian gauge theory results from the inner product of the curvature. We proceed in the same story as that in the ordinary differential geometry though our space is discrete and a set of one form base consists of $dx^\mu$ and $\chi_k (k = 1, 2, ..., N)$. This is the essential difference between our formulation and others such as Connes’s and Chamseddine, Felder and Fröhlich’s formulations. They use the Clifford algebra of the Dirac matrices $\gamma^\mu$ and $\gamma^5$ instead of $dx^\mu$ and $\chi_k$. We can define the exterior derivative satisfying the nilpotency which is very important to obtain the gauge covariant formulation. This point is rather unclear in Refs. and though their works are pioneering. In any way, the Higgs field is regarded as a kind of gauge field on the discrete space in the same way as the ordinary gauge field, so that the raison d’être of Higgs field is apparent and any extra physical modes are not yielded.

In this article, we revise the algebraic rule of $d\chi_k$ operation on $M_{nl}\chi_l$ which is an important factor responsible for the symmetry breakdown. We can achieve easier proof of the nilpotency of the generalized exterior derivative $d = d + d\chi$ with $d\chi = \sum k d\chi_k$ than that in Ref. It should be noticed that in our formulation two-form basis $\chi_k \wedge \chi_l$ is independent of $\chi_l \wedge \chi_k$ for $k \neq l$ contrary to the ordinary commutative algebra. The name of our NCG results from this non-commutativity. In addition, through the covariant derivative of fermion field, this revision makes it possible to define the parallel transformation of fermion field on the discrete space $Z_N$ in the same way as that on the ordinary continuous manifold. This definition confirms the Higgs field to be the connection on the discrete space. Konisi and Saito indicated that the Higgs field is a connection governing the parallel transformation on the discrete space. Our discussions

* e-mail address: okum@isc.chubu.ac.jp
are inspired by their indication.

Chamseddine, Felder and Frölich extended NCG formulation in the matrix form by use of the Dirac matrices $\gamma_\mu$, $\gamma_5$. We also show in this article that the same type matrix formulation is possible by use of one-form base $dx^\mu$ and $\chi_k$ instead of the Dirac matrices. d operations on matrices are properly defined in conformity with our formulation stated above. As a result, the nilpotency of $d$ holds, which leads to the consistent gauge covariant formulation. The corresponding operation in Ref.9) to $d$ in ours is the commutation relation of matrices and therefore the corresponding equation to the nilpotency of $d$ doesn’t hold, so that it is vague whether their formulation is consistent or not. It seems that their formulation is not applicable to $N \geq 4$ case because they have only one $\gamma_5$. On the contrary, our method is applicable to any $N$ case because we have many $\chi_k$ ($k = 1, 2 \cdots N$). The shifting rule used in the algebraic calculation is very important in our framework stated in §3. However, it is somewhat hard to understand the implication of that rule. The matrix formulation makes it very clear since matrix manipulations bring the resultant equations after the shifting rule has already performed.

This article consists of five sections and one appendix. The next section presents the fundamental framework of NCG on $M_4 \times Z_2$ with the emphasis on the nilpotency of $d$. In third section, the matrix formulation of our NCG is developed, which leads to the same result as that in second section. In fourth section, fermion sector of NCG is treated with the emphasis on the parallel transformation of fermion field. The last section is devoted to concluding remarks. The relationship between the NCG on $M_4 \times Z_2$ stated in §2 and that of our previous framework with only one $\chi$ becomes clear in appendix A.

§2. Non-commutative differential geometry on $M_4 \times Z_N$

The algebra of the previous formulation of NCG on the discrete space $M_4 \times Z_N$ is improved to yield more reasonable understanding of the generalized gauge field $A(x,n)$ in which $x_\mu$ and $n$ are the arguments of fields in $M_4$ and $Z_N$, respectively.

Let us start with the definition of the generalized gauge field $A(x,n)$ written in one-form on the discrete space $M_4 \times Z_N$:

$$A(x,n) = \sum_i a_i(x,n) d\alpha_i(x,n). \quad (2.1)$$

$a_i(x,n)$ is the square-matrix-valued functions. The subscript $i$ is a variable of the extra internal space which we can not now identify. Now, we simply regard $a_i(x,n)$ as the more fundamental field from which to construct gauge field and the Higgs fields. They have only mathematical meaning because $a_i(x,n)$ does not appear in final stage. $d$ in Eq.(2.1) is the generalized exterior derivative defined as follows.

$$d = d + d_\chi = d + \sum_k d_{\chi_k}, \quad (2.2)$$

$$da_i(x,n) = \partial_\mu a_i(x,n) dx^\mu, \quad (2.3)$$

$$d_{\chi_k} a_i(x,n) = [-a_i(x,n)M_{nk}\chi_k + M_{nk}\chi_k a_i(x,n)], \quad (2.4)$$

where $dx^\mu$ is a ordinary one-form basis, taken to be dimensionless, in Minkowski space $M_4$ and $\chi_k$ is the one-form basis, assumed to be also dimensionless, in the discrete space $Z_N$. We have introduced $x$-independent matrix $M_{nm}$ expressed in the rectangular matrix whose hermitian conjugation is given by $M^\dagger_{nk} = M_{kn}$. If $a_i(x,n)$ and $a_i(x,k)$ are $L \times L$ and $K \times K$ square matrices, respectively, $M_{nk}$ is $L \times K$ type matrix. $M_{nn} = 0$ is assumed but it should be noted that this equation must not be used until the calculation of $d_{\chi_k}$ operation is finished because it contradicts with the shifting rule as stated later. The matrix $M_{nk}$ turns out to determine the scale and pattern of the spontaneous breakdown of the gauge symmetry. Thus, the symmetry breaking mechanism is encoded in the $d_\chi$ operation. Before finding the explicit
forms of gauge and the Higgs fields and generalized field strength according to Eqs. (2.1) and (2.2)~(2.4), we determine the several important algebraic rules in non-commutative geometry. We first investigate the condition that the Leibniz rule with respect to \( d_{\chi_k} \) consistently holds. The Leibniz rule is written as (the subscript \( i \) of \( a_i(x,n) \) is abbreviated for a while)

\[
d_{\chi_k} (a(x,n)b(x,n)) = (d_{\chi_k} a(x,n))b(x,n) + a(x,n)(d_{\chi_k} b(x,n)),
\]

(2.5)

where the \( b(x,n) \) is also the same type matrix as \( a(x,n) \). We have to assume the rather unusual rule for \( \chi_k \) that

\[
\chi_k b(x,n) = b(x,k)\chi_k,
\]

(2.6)

which makes the product of matrix valued functions consistently calculable. Eq. (2.6) is only an example to show how to change the discrete variable when \( \chi_k \) is shifted from left to right in an equation. For example,

\[
\chi_k M_{nl} = M_{kl}\chi_k.
\]

(2.7)

Such a change is necessary to assure the consistent product of matrices. We call this rule the shifting rule which reflects the noncommutative nature of the differential calculus just now treated. This rule is reasonably understood and its justification is confirmed in the matrix formulation of NCG described in the next section. According to this shifting rule, the third equation in Eq. (2.4) is rewritten as

\[
d_{\chi_k} a(x,n) = [-a(x,n)M_{nk} + M_{nk}a(x,k)]\chi_k.
\]

(2.8)

Eq. (2.8) and the shifting rule enable us to prove the Leibniz rule for \( d_{\chi} \) in Eq. (2.3).

We next investigate the nilpotency of \( d \) which is very important to obtain the generalized field strength. From the definition of \( d \) in Eq. (2.2),

\[
d^2 a(x,n) = (d^2 + d \cdot d_{\chi} + d_{\chi} \cdot d + d_{\chi}^2) a(x,n).
\]

(2.9)

d\(^2\) = 0 naturally holds in ordinary differential forms. \( dx^\mu \wedge \chi_k = -\chi_k \wedge dx^\mu \) is reasonably assumed and it yields

\[
(d \cdot d_{\chi} + d_{\chi} \cdot d) a(x,n) = 0.
\]

(2.10)

However, \( d^2 a(x,n) = 0 \) is not so easily proved that we need to add a following algebraic rule\(^*\) for \( d_{\chi} \) operation on \( M_{nk} \):

\[
d_{\chi l} M_{nk} \chi_k = M_{nl} \chi_l \wedge M_{nk} \chi_k = M_{nl} M_{lk} \chi_l \wedge \chi_k.
\]

(2.11)

In addition, the following rule is taken into account; whenever the \( d_{\chi_k} \) operation jumps over \( M_{nl} \chi_l \), minus sign attached, for example

\[
d_{\chi_k} (M_{nl} \chi_l a(x,n)) = (d_{\chi_k} M_{nl} \chi_l) a(x,n) - M_{nl} \chi_l (d_{\chi_k} a(x,n)).
\]

(2.12)

In this equation, we should notice that \( \chi_l \wedge \chi_k \) is independent of \( \chi_k \wedge \chi_l \) for \( k \neq l \) due to the noncommutative property of geometry on the discrete space. With these considerations, we can easily calculate:

\[
d_{\chi l} (d_{\chi_k} a(x,n)) = d_{\chi l} (-a(x,n)M_{nk} \chi_k + M_{nk} \chi_k a(x,n))
\]

\[
= - (d_{\chi l} a(x,n)) \wedge M_{nk} \chi_k - a(x,n) (d_{\chi l} M_{nk} \chi_k)
\]

\[
+ (d_{\chi l} M_{nk} \chi_k) a(x,n) - M_{nk} \chi_k \wedge (d_{\chi l} a(x,n))
\]

\[
= (-M_{jl} a(x,l) M_{lk} + M_{lk} M_{jl} a(x,k)) \chi_l \wedge \chi_k
\]

\[
- (-M_{nk} a(x,k) M_{kl} + M_{nk} M_{kl} a(x,l)) \chi_k \wedge \chi_l,
\]

(2.13)

\(^*\) This rule is revised from that in Ref. which was written as

\[
d_{\chi_k} (M_{nl} \chi_l) = M_{nl} M_{lk} \chi_l \wedge \chi_k.
\]

This revision yields simpler proof of the nilpotency of \( d \), which is only one difference between this section and that in Ref.
which yields
\[
(d_{\chi}^\dagger d_{\chi} + d_{\chi} d_{\chi}^\dagger) a(x, n) = 0.
\] (2.14)

From Eq.(2.14) we obtain the nilpotency of \(d_{\chi}^2 a(x, n) = 0\). Then, according to Eqs. (2.9) and (2.10), the nilpotency of \(d_{\chi} a(x, n) = 0\) is followed. In the similar way, we can prove
\[
(d_{\chi}^\dagger d_{\chi} + d_{\chi} d_{\chi}^\dagger) M_{nm}\chi_m = 0,
\] (2.15)

which together with Eq. (2.12) helps us prove, for example, \(d_{\chi}^2 \{M_{mm}\chi_{m}(x, m)\} = 0\). However, we don’t need such a general case but only quote \(d_{\chi}^2 a(x, n) = 0\) in this article.

Inserting Eqs.(2.2)~(2.4) into Eq.(2.1) and using Eq.(2.5), \(\mathcal{A}(x, n)\) is rewritten as
\[
\mathcal{A}(x, n) = A_\mu(x, n) dx^\mu + \sum_k \Phi_{nk}(x)\chi_k,
\] (2.16)

where
\[
A_\mu(x, n) = \sum_i a_i^\dagger(x, n) \partial_\mu a_i(x, n),
\] (2.17)

\[
\Phi_{nk}(x) = \sum_i a_i^\dagger(x, n) \{ -a_i(x, n) M_{nk} + M_{nk} a_i(x, k) \},
\] (2.18)

\(A_\mu(x, n)\) and \(\Phi_{nk}(x)\), are identified with the gauge field in the flavor symmetry and the Higgs field, respectively. In order to identify \(A_\mu(x, n)\) with true gauge fields, the following condition has to be imposed:
\[
\sum_i a_i^\dagger(x, n) a_i(x, n) = 1.
\] (2.19)

Before constructing the gauge covariant field strength, we address the gauge transformation of \(a_i(x, n)\) which is defined as
\[
a_i^q(x, n) = a_i(x, n)g(x, n),
\] (2.20)

where \(g(x, n)\) is the gauge function with respect to the corresponding flavor unitary group specified by \(n\). Let us define the \(d_{\chi}\) operation on \(g(x, n)\) by
\[
dg(x, n) = (d + \sum_k d_{\chi_k}) g(x, n)
\]
\[
= \partial_\mu g(x, n) dx^\mu + \sum_k \{ -g(x, n) M_{nk} + M_{nk} g(x, k) \} \chi_k.
\] (2.21)

Then, we can find from Eq.(2.1) the gauge transformation of \(\mathcal{A}(x, n)\) to be
\[
\mathcal{A}^g(x, n) = g^{-1}(x, n)\mathcal{A}(x, n)g(x, n) + g^{-1}(x, n)dg(x, n),
\] (2.22)

which using (2.17) and (2.18) leads to the gauge transformations of gauge, and Higgs fields as
\[
A^g_\mu(x, n) = g^{-1}(x, n)A_\mu(x, n)g(x, n) + g^{-1}(x, n)\partial_\mu g(x, n),
\] (2.23)

\[
\Phi^g_{nk}(x) = g^{-1}(x, n)\Phi_{nk}(x)g(x, k) + g^{-1}(x, n)\{ -g(x, n) M_{nk} + M_{nk} g(x, k) \}.
\] (2.24)

Equation(2.24) is very similar to Eq.(2.23) if the inhomogeneous term is written as
\[
g^{-1}(x, n)\{ -g(x, n) M_{nk} + M_{nk} g(x, k) \} = g^{-1}(x, n)\partial_{nk} g(x, n).
\] (2.25)

In this context, \(\partial_{nk}\) corresponds to \(\partial_\mu\). Thus, \(\partial_{nk}\) seems to be the difference operator on the discrete space. \(M_{nk}\) in Eq.(2.24) is inserted to insure the consistent calculation of matrix products. The resemblance between Eqs.(2.23) and (2.24) strongly indicates that the Higgs field is a kind of gauge field on the discrete space \(M_4 \times Z_N\). Equation(2.24) is rewritten as
\[
\Phi^g_{nk}(x) + M_{nk} = g^{-1}(x, n) (\Phi_{nk}(x) + M_{nk}) g(x, k),
\] (2.26)
which makes it obvious that

$$H_{nk}(x) = \Phi_{nk}(x) + M_{nk}$$

(2.27)

is un-shifted Higgs field whereas $\Phi_{nk}(x)$ denotes shifted one with the vanishing vacuum expectation value.

With these considerations we can construct the gauge covariant field strength:

$$\mathcal{F}(x, n) = dA(x, n) + A(x, n) \wedge A(x, n)$$

(2.28)

According to the nilpotency of $d$, we can write

$$dA(x, n) = \sum_i da_i^1(x, n) \wedge da_i^1(x, n)$$

which along with Eqs.(2.3), (2.4) and (2.16) serves us to introduce the explicit expression of $\mathcal{F}(x, n)$ as

$$\mathcal{F}(x, n) = \frac{1}{2} F_{\mu\nu}(x, n) dx^\mu \wedge dx^\nu + \sum_{k \neq n} D_\mu \Phi_{nk}(x) dx^\mu \wedge \chi_k$$

$$+ \sum_{k \neq n} V_{nk}(x) \chi_k \wedge \chi_n$$

$$+ \sum_{k \neq n} \sum_{l \neq n} V_{nkl}(x) \chi_k \wedge \chi_l,$$

(2.29)

where

$$F_{\mu\nu}(x, n) = \partial_\mu A_\nu(x, n) - \partial_\nu A_\mu(x, n) + [A_\mu(x, n), A_\mu(x, n)],$$

(2.30)

$$D_\mu \Phi_{nk}(x) = \partial_\mu \Phi_{nk}(x) - (\Phi_{nk}(x) + M_{nk}) A_\mu(x, k)$$

$$+ A_\mu(x, n)(M_{nk} + \Phi_{nk}(x)),$$

(2.31)

$$V_{nk}(x) = (\Phi_{nk}(x) + M_{nk})(\Phi_{kn}(x) + M_{kn}) - Y_{nk}(x),$$

(2.32)

$$V_{nkl}(x) = (\Phi_{nk}(x) + M_{nk})(\Phi_{kl}(x) + M_{kl}) - Y_{nkl}(x),$$

(2.33)

$Y_{nk}(x)$ and $Y_{nkl}(x)$ are the auxiliary fields written as

$$Y_{nk}(x) = \sum_i a_i^1(x, n) M_{nk} M_{kn} a_i(x, n),$$

(2.34)

$$Y_{nkl}(x) = \sum_i a_i^1(x, n) M_{nk} M_{kl} a_i(x, l).$$

(2.35)

In order to obtain the gauge invariant Yang-Mills-Higgs Lagrangian, we address the gauge transformation of $\mathcal{F}(x, n)$. From Eq.(2.22) we can easily find the gauge transformation of $\mathcal{F}(x, n)$ as

$$\mathcal{F}^g(x, n) = g^{-1}(x, n)\mathcal{F}(x, n)g(x, n).$$

(2.36)

The metric structure of one-forms are defined as

$$<dx^\mu, dx^\nu> = g^{\mu\nu}, \quad <\chi_n, dx^\mu> = 0, \quad <\chi_n, \chi_k> = -\alpha^2 \delta_{nk},$$

(2.37)

which determines the inner products of two-forms such as

$$<dx^\mu \wedge dx^\nu, dx^\rho \wedge dx^\sigma> = g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho},$$

(2.38)

$$<dx^\mu \wedge \chi_n, dx^\nu \wedge \chi_k> = -\alpha^2 g^{\mu\nu} \delta_{nk},$$

(2.39)

$$<\chi_n \wedge \chi_k, \chi_m \wedge \chi_l> = \alpha^4 \delta_{nm} \delta_{kl},$$

(2.40)

while other inner products among the basis two-forms vanish. It should be noted that $\chi_m \wedge \chi_l$ is a two-form basis independent of $\chi_1 \wedge \chi_m$ for $m \neq l$, which reflects the non-commutative property
of geometry on the discrete space. Thus, Eq. (2.40) is followed. In this paper, we assume the coefficient of $\delta_{nm}\delta_{kl}$ to be $\alpha^4$ on the right-hand side in Eq. (2.40). In general, it seems that the coefficient cannot be determined only by NCG.

According to these metric structures and Eq. (2.29) we can obtain the formula for gauge-invariant Yang-Mills-Higgs Lagrangian

$$L_{YMH}(x) = -\sum_{n=1}^{N} \frac{1}{g_n^2} <F(x,n), F(x,n)>$$

$$= -\text{Tr} \sum_{n=1}^{N} \frac{1}{g_n^2} F_{\mu\nu}^\dagger(x,n) F^{\mu\nu}(x,n)$$

$$+ \text{Tr} \sum_{n=1}^{N} \sum_{k \neq n} \frac{\alpha^2}{g_n^2} (D_\mu \Phi_{nk}(x))\dagger D^\mu \Phi_{nk}(x)$$

$$- \text{Tr} \sum_{n=1}^{N} \frac{\alpha^4}{g_n^2} \sum_{k \neq n} V_{nk}(x) V_{nk}(x) - \text{Tr} \sum_{n=1}^{N} \frac{\alpha^4}{g_n^2} \sum_{k \neq n} \sum_{l \neq n,k} V_{nk}(x) V_{nl}(x), \quad (2.41)$$

where $g_n$ is the gauge coupling constant and Tr denotes the trace over internal symmetry matrices. The third term is the potential term of Higgs particle and the fourth term is the interaction term between Higgs particles. Thus, the fourth term does not appear when $N = 2$.

§3. Matrix formulation of NCG

In this section we explain the formulation on the discrete space $M_4 \times Z_3 (N = 3)$ since it is very easy to generalize it to that on $M_4 \times Z_N$. In addition, $x_\mu$ as the argument in fields is abbreviated for simplicity. The fundamental field $a_i$ is introduced as

$$a_i = \text{diag}(a_i(1), a_i(2), a_i(3)), \quad (3.1)$$

where $a_i(n) \ (n = 1, 2, 3)$ is the same fundamental field as in Eq. (2.1). The generalized gauge field $A$ is defined as

$$A = \sum_i a_i^\dagger d a_i, \quad (3.2)$$

with the generalized exterior derivative $d$ whose operation on $a_i$ is described as

$$d a_i = (d + d_\chi) a_i = d a_i + d_\chi a_i. \quad (3.3)$$

da_i in Eq. (3.3) is explicitly written in matrix form as

$$d a_i = \begin{pmatrix} \partial_\mu a_i(1) dx^\mu & 0 & 0 \\ 0 & \partial_\mu a_i(2) dx^\mu & 0 \\ 0 & 0 & \partial_\mu a_i(3) dx^\mu \end{pmatrix} \quad (3.4)$$

and $d_\chi a_i$ is defined as

$$d_\chi a_i = -a_i M + Ma_i, \quad (3.5)$$

where $M$ is given as the matrix with one-form base $\chi_k \ (k = 1, 2, 3)$:

$$M = \begin{pmatrix} 0 & M_{12} \chi_2 & M_{13} \chi_3 \\ M_{21} \chi_1 & 0 & M_{23} \chi_3 \\ M_{31} \chi_1 & M_{32} \chi_2 & 0 \end{pmatrix}. \quad (3.6)$$

Then, we can find the matrix element of $d_\chi a_i$ as

$$(d_\chi a_i)_{nk} = -a_i(n) M_{nk} \chi_k + M_{nk} \chi_k a_i(k), \quad \text{for} \ n \neq k,$$

$$(d_\chi a_i)_{nn} = 0. \quad (3.7)$$
With these equations, we obtain the explicit form of $A$ as

$$A = \begin{pmatrix} A_\mu(1)dx^\mu & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & A_\mu(2)dx^\mu & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & A_\mu(3)dx^\mu \end{pmatrix},$$

(3.8)

where the gauge field $A_\mu(k) (k = 1, 2, 3)$ and the Higgs field $\Phi_{nk} (n, k = 1, 2, 3)$ are denoted in the same form as in Eqs.(2.17) and (2.18), respectively. It should be noted that in introducing Eq.(3.8) the shifting rule in Eqs.(2.6) or (2.7) is not necessary because in the matrix formulation of NCG, the expressions such as

$$M_{nk}\chi_k a_i(k) = M_{nk}a_i(k)\chi_k, \quad M_{nk}\chi_k M_{kl} = M_{nk}M_{kl}\chi_k$$

(3.9)

appear. Eq.(3.9) has the forms after the shifting rule has been already applied. This fact makes the meaning of the shifting rule expressed in Eqs.(2.6) and (2.7) very clear and confirms the justification of that rule.

The gauge transformation is given as

$$a^g_i = a_i g,$$

(3.10)

where the gauge transformation function $g$ is expressed in matrix form as

$$g = \text{diag}(g(1), g(2), g(3))$$

(3.11)

and the $d_\chi$ operation on it is

$$d_\chi g = -gM + Mg = [-g, M].$$

(3.12)

From these equations, the gauge transformation of $A$ is given as

$$A^g = g^{-1}Ag + g^{-1}dg$$

(3.13)

which yields the same gauge transformations of $A_\mu(n) (n = 1, 2, 3)$ and $\Phi_{nk} (n, k = 1, 2, 3)$ as those in Eqs.(2.23) and (2.24), respectively.

The generalized field strength is in this matrix formulation written as

$$F = dA + A \wedge A.$$  

(3.14)

In obtaining the explicit form of $F$ we must again address the nilpotency of $d$. $(dd_\chi + d_\chi d)a_i = 0$ easily results under the reasonable condition that $dx^\mu \wedge \chi_k = -\chi_k \wedge dx^\mu$. With the definition

$$d_\chi M = M \wedge M$$

(3.15)

$d_\chi^2 a_i$ is calculated as follows:

$$d_\chi (d_\chi a_i) = d_\chi (-a_i M + Ma_i)$$

$$= -(d_\chi a_i) \wedge M - a_i (d_\chi M) + (d_\chi M)a_i - M \wedge (d_\chi a_i)$$

$$= -(-a_i M + Ma_i) \wedge M - a_i (M \wedge M) + (M \wedge M)a_i - M \wedge (-a_i M + Ma_i)$$

$$= 0,$$

(3.16)

where the rule is again adopted that whenever the $d_\chi$ operation jumps over $M$, minus sign attached. We call this rule jumping rule, hereafter. Then, we can rewrite Eq.(3.14) as

$$F = \sum_i da_i^\dagger \wedge da_i + A \wedge A.$$  

(3.17)
With the help of Eqs.(3.7) and (3.8) we can find the matrix elements of $\mathcal{F}$ as

$$
\mathcal{F}_{nn} = \frac{1}{2} F_{\mu \nu} (n) dx^\mu \wedge dx^\nu + \sum_{k \neq n} (H_{nk} H_{kn} - Y_{nk}) \chi_k \wedge \chi^n \tag{3.18}
$$

and

$$
\mathcal{F}_{nk} = (\partial_\mu H_{nk} + A_\mu (n) H_{nk} - H_{nk} A_\mu (k)) dx^\mu \wedge \chi_k + \sum_{l \neq n, k} (H_{nl} H_{lk} - Y_{nlk}) \chi_l \wedge \chi_k, \quad \text{for} \ n \neq k. \tag{3.19}
$$

$H_{nk}, Y_{nk}$ and $Y_{nlik}$ are written in Eqs.(2.27), (2.34) and (2.35), respectively.

With the same metric structures in Eqs.(2.38)~(2.40), we can obtain the Yang-Mills-Higgs Lagrangian from the following equation;

$$
L_{YMH} = - \text{Tr} < \mathcal{F}, C \mathcal{F} >, \tag{3.20}
$$

where $C$ is a matrix relating to the coupling constant which is given as

$$
C = \text{diag}(\frac{1}{g_1^2}, \frac{1}{g_2^2}, \frac{1}{g_3^2}). \tag{3.21}
$$

It should be noted that $\text{Tr}$ in Eq.(3.20) is the trace not only over the $3 \times 3$ matrix such as in Eq.(3.8) but also over the internal symmetry matrices. It is easily understand that Eq.(3.20) yields the same expression as that in Eq.(2.41).

§4. Fermion sector and parallel transformation

We have already performed in the previous paper[1] to reconstruct the Dirac Lagrangian in NCG by use of the differential forms in the same way as the Yang-Mills-Higgs Lagrangian in Eq.(2.41). However, it was not done to discuss the parallel transformation of fermion field because the algebra of NCG on $M_4 \times Z_N$ was not fully satisfactory. In this section, we review how to construct the Dirac Lagrangian and discuss the parallel transformation of fermion field based on the revised algebra stated in §2. In addition, the matrix formulation of the fermion sector is also presented.

The covariant derivative acting on the spinor $\psi(x,n)$ is defined by

$$
\mathcal{D}_x \psi(x,n) = (d + A^f (x,n)) \psi(x,n), \tag{4.1}
$$

where $A^f (x,n)$ is the differential representation with respect to the fermions such that

$$
A^f (x,n) = A^f_\mu (x,n) dx^\mu + \sum_m \Phi^f_{nm} (x) \chi_m. \tag{4.2}
$$

It should be noticed that $A^f_\mu (x,n)$ is the differential representation for $\psi(x,n)$ and $\Phi^f_{nm} (x)$ also has the expression corresponding to $\psi(x,n)$. However, in almost all model building we can construct $A(x,n)$ in §3 so as to coincide with $A^f_\mu (x,n)$ in Eq.(4.2). Thus, the superscript $f$ in Eq.(4.2) is removed, hereafter. We further define $d_\chi$ operation on $\psi(x,n)$ as

$$
d_\chi \psi(x,n) = \sum_k d_\chi_k \psi(x,n) = \sum_k M_{nk} \chi_k \psi(x,n) = \sum_k M_{nk} \psi(x,k) \chi_k, \tag{4.3}
$$

which helps us rewrite Eq.(4.1) as

$$
\mathcal{D}_x \psi(x,n) = \left( \partial_\mu + A_\mu (x,n) dx^\mu + \sum_k H_{nk} (x) \chi_k \right) \psi(x,n), \tag{4.4}
$$

with \( H_{nk}(x) \) in Eq. (2.27).

Since the covariant derivative of fermion field has been defined in Eq. (4.4), we can address the parallel transformation of fermion field on the discrete space \( M_4 \times Z_N \). If Eq. (4.3) is denoted as

\[
d_\chi \psi(x, n) = \sum_k \partial_{nk} \chi_k \psi(x, n),
\]

the covariant derivative \( \mathcal{D} \psi(x, n) \) is rewritten as

\[
\mathcal{D} \psi(x, n) = \left\{ (\partial_\mu + A_\mu(x, n)) dx_\mu \psi(x, n) + \sum_k (\partial_{nk} + \Phi_{nk}(x)) \chi_k \psi(x, n) \right\}.
\]

Eq. (4.6) implies that \( \Phi_{nk}(x)\chi_k \psi(x, n) \) expresses the variation accompanying the parallel transformation from \( k \)-th to \( n \)-th points on the discrete space just as

\[
A_\mu(x, n) dx_\mu \psi(x, n)
\]

is the variation of parallel transformation on the Minkowski space \( M_4 \). This makes it determinant that the shifted Higgs field \( \Phi_{nk}(x) \) is the gauge field on the discrete space.

The nilpotency of \( d_\chi \) in this case is also important to obtain the consistent explanations of covariant derivative and parallel transformation. From Eq. (4.3), we can easily calculate

\[
d_{\chi_l}(d_{\chi_k} \psi(x, n)) = (d_{\chi_l}(M_{nk} \chi_k \psi(x, n)))
\]

\[
= (d_{\chi_l} M_{nk} \chi_k \psi(x, n)) - M_{nk} \chi_k \wedge d_{\chi_l} \psi(x, n)
\]

\[
= M_{nl} \chi_l \wedge M_{nk} \chi_k \psi(x, n) - M_{nk} \chi_k \wedge M_{nl} \chi_l \psi(x, n)
\]

which leads to

\[
(d_{\chi_l} d_{\chi_k} + d_{\chi_k} d_{\chi_l}) \psi(x, n) = 0.
\]

Equation (4.8) implies that if the Higgs field \( \Phi_{nk}(x) \) as the gauge field on the discrete space becomes vanishing, the curvature on the discrete space also vanishes. In Eq. (4.7), it should be again noted that two-form basis \( \chi_l \wedge \chi_k \) is independent of \( \chi_k \wedge \chi_l \) for \( k \neq l \) because of the noncommutativity of our algebra. From Eq. (4.8), the nilpotency of \( d \) is evident. Thus, we can obtain the generalized field strength as the curvature on the discrete space \( M_4 \times Z_N \) as follows:

\[
\mathcal{F}(x, n) = (d + A(x, n)) \wedge (d + A(x, n))
\]

\[
= dA(x, n) + A(x, n) \wedge A(x, n).
\]

Thus, we can regard the Higgs kinetic term in Eq. (2.31), the Higgs potential term in Eq. (2.32) and the Higgs interacting term in Eq. (2.33) as the curvatures accompanying the parallel transformation on \( M_4 \times Z_N \).

In the follows, we investigate the gauge transformation property of \( \mathcal{D} \psi(x, n) \). Let the gauge transformation of \( \psi(x, n) \) to be

\[
\psi^g(x, n) = g^{-1}(x, n) \psi(x, n),
\]

with the same gauge transformation function in Eq. (2.20). Putting an attention on the equation that \( d g(x, n)^{-1} = -g^{-1}(x, n)(d g(x, n)) g^{-1}(x, n) \), we easily obtain the gauge covariance of \( \mathcal{D} \psi(x, n) \).

\[
\mathcal{D} \psi^g(x, n) = g^{-1}(x, n) \mathcal{D} \psi(x, n).
\]

In order to get the Dirac Lagrangian by use of the inner products of differential forms such as Eq. (2.41) we introduce the associated spinor one-form by

\[
\tilde{\mathcal{D}} \psi(x, n) = \gamma_\mu \psi(x, n) dx^\mu - ic_\gamma \psi(x, n) \sum_m \chi_m,
\]

where \( c_\gamma \) is a real, dimensionless constant and invariant against the gauge transformation. It is evident that \( \tilde{\mathcal{D}} \psi(x, n) \) is gauge covariant

\[
\tilde{\mathcal{D}} \psi^g(x, n) = g^{-1}(x, n) \tilde{\mathcal{D}} \psi(x, n).
\]
The Dirac Lagrangian invariant under the Lorentz and gauge transformations is obtained by taking the inner product

$$L_D^{(x,n)} = i \text{Tr} \langle \bar{D} \psi(x,n), D \psi(x,n) \rangle = i \text{Tr} \left[ \bar{\psi}(x,n) \gamma^\mu (\partial_\mu + A_\mu(x,n)) \psi(x,n) + ig_Y \bar{\psi}(x,n) \sum_m H_{nm}(x) \psi(x,m) \right],$$

(4.14)

where $-c_Y \alpha^2$ is replaced simply by $g_Y$, and we have used the definitions of the inner products for spinor one-forms due to Eq.(2.37),

$$< A(x,n) dx^\mu, B(x,n) dx^\nu > = \text{Tr} \bar{A}(x,n) B(x,n) g^{\mu\nu},$$

(4.15)

$$< A(x,n) \chi_k, B(x,n) \chi_l > = -\alpha^2 \delta_{kl} \text{Tr} \bar{A}(x,n) B(x,n),$$

(4.16)

with vanishing other inner products. The total Dirac Lagrangian is the sum over $n = 1, 2 \cdots N$.

$$L_D(x) = \sum_{n=1}^N L_D^{(x,n)},$$

(4.17)

which evidently satisfies the Hermiticity condition.

We can develop the similar discussions also in matrix formulation of NCG stated in §3. Fermion field is expressed in the same notation in §3 as

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \\ \psi^{(3)} \end{pmatrix}.$$  

(4.18)

The covariant derivative $D \psi$ is described to be

$$D \psi = (d + A) \psi.$$  

(4.19)

Since $d_\chi$ operation on $\psi$ is

$$d_\chi \psi = M \psi,$$  

(4.20)

the nilpotency of $d_\chi$ is easily proved by

$$d_\chi (d_\chi \psi) = d_\chi (M \psi) = (d_\chi M) \psi - M (d_\chi \psi) = (MM) \psi - M (M \psi) = 0,$$  

(4.21)

where the jumping rule is applied. Thus, the nilpotency of $d$ is followed. From these considerations, the generalized field strength $F$ is written as

$$F = (d + A) \wedge (d + A)$$

$$= dA + A \wedge A$$  

(4.22)

which amounts to Eq.(1.11).

Though the associated spinor one-form in Eq.(1.12) has rather unclear expression in matrix notation, we define it so as to have the elements expressed in Eq.(4.12). Then, we can obtain the Dirac Lagrangian as follows:

$$L_D = i \text{Tr} \langle \bar{D} \psi, D \psi \rangle,$$  

(4.23)

which is equal to that in Eq.(1.14). Therefore, we obtain the same results also in the fermion sector as those in the non-matrix formulation.
§5. Concluding remarks

The algebra of NCG on $M_4 \times Z_N$ previously proposed by the present author was improved by revising the $\chi_k$ operation on $M_{nl}\chi_l$. This revision makes it possible that the nilpotency of the exterior derivative $d$ is easily proved, and then the parallel transformation of fermion field is defined through the covariant derivative of fermion field. The discussions of the parallel transformation make it decisive that the Higgs field is a connection on the discrete space in the same way as the ordinary gauge field on $M_4$ and the Higgs kinetic, potential and interacting terms in Lagrangian are the curvatures accompanying the parallel transformation on $M_4 \times Z_N$. Konisi and Saito pointed out that the un-shifted Higgs field $H_{nk}(x)$ is a connection accompanying the parallel transformation of the fermion field on the discrete space without recourse of NCG, and therefore without the introductions of the generalized exterior derivative $d$ and the matrix $M_{nk}$ in our formulation. Their conclusion is different from ours. That is, in the present formulation the shifted Higgs field $\Phi_{nk}(x)$ is a connection on the discrete space. This is apparent from the remembrance between Eqs.(2.23) and (2.24) and also the expression of the covariant derivative in Eq.(4.6). This difference seems to come from differences between two formulations.

The matrix formulation of NCG similar to Chamseddine, Felder and Fröhlich’s work was also developed by use of $d$ and $d\chi$ instead of $\gamma_\mu$ and $\gamma_5$, by which the shifting rule was confirmed to be reasonable. The shifting rule is very important in our calculations. The matrix formulation naturally yields the same results as that of the non-matrix approach not only in Yang-Mills-Higgs sector and also in fermionic sector.

Our NCG on $M_4 \times Z_N$ has already applied to reconstruct $SU(5)$ and $SO(10)$ grand unified theory (GUT) and the left-right symmetric gauge theory. Generally speaking, the restrictions on the Higgs potential terms and interacting terms in Eq.(2.41) are so stringent that we have to devise the model constructions of such theory. For $SU(5)$ GUT, the model construction is fairly successful because the interaction term of the adjoint and 5-dimensional Higgs fields yields colored Higgs with GUT scale mass. However, the Higgs interacting terms are so limited that it might contradict with the quantization. In the case of $SO(10)$ GUT unusual Higgs field must be incorporated to maintain the Higgs potential term which is responsible for the seesaw mechanism. In order to remove such kind of unnaturalness of the reconstruction of the various models, we must continue to improve NCG approach.

It is still unknown whether the quantization of the Lagrangian derived from NCG in a way to be compatible with NCG approach is possible or not. However, It is confirmed that the Higgs field is a gauge particle with the equal footing to the ordinary gauge field such as weak bosons in the Weinberg-Salam theory. Thus, it might be expected that the naturalness problem with respect to the quadratic divergence of the Higgs field is removed if one calculates it with much attention on that the Higgs field is a gauge field. If so, the mass relation $m_H = \frac{1}{\sqrt{3}} m_W \sin \theta_W$ proposed by the present author may hold without any correction in the same way as $m_W = m_Z \cos \theta_W$. These circumstances may be investigated by use of the BRST invariant Lagrangian of spontaneously broken gauge theory in NCG on the discrete space $M_4 \times Z_2$ recently proposed by the present author and Lee, Hwang and Ne’eman.

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Appendix A

Comparison between NCG in this article and our previous NCG

When we reconstructed the standard mode in noncommutative geometry on the discrete space $M_4 \times Z_2[\frac{3}{4}]$ we use only one $\chi$, not $\chi_1$ and $\chi_2$ appearing in this article. Thus, we investigate in this appendix the relation between two formulations. Let us first expand the related expressions in the formulation with $\chi_1$ and $\chi_2$. From Eq.(2.14), the generalized gauge fields are written as

$$A(x, 1) = A_\mu(x, 1)dx^\mu + \Phi_{12}(x)\chi_2,$$

$$A(x, 2) = A_\mu(x, 2)dx^\mu + \Phi_{21}(x)\chi_1.$$

From Eq.(2.29), the generalized field strengths are written as

$$\mathbf{F}(x, 1) = \frac{1}{2}F_{\mu\nu}(x, 1)dx^\mu \land dx^\nu + D_\mu \Phi_{12}(x)dx^\mu \land \chi_2 + V_{12}(x)\chi_2 \land \chi_1,$$

$$\mathbf{F}(x, 2) = \frac{1}{2}F_{\mu\nu}(x, 2)dx^\mu \land dx^\nu + D_\mu \Phi_{21}(x)dx^\mu \land \chi_1 + V_{21}(x)\chi_1 \land \chi_2,$$

where

$$F_{\mu\nu}(x, k) = \partial_\mu A(x, k) - \partial_\nu A(x, k) + [A_\mu(x, k), A_\nu(x, k)], \quad \text{for} \quad k = 1, 2,$$

$$D_\mu \Phi_{12}(x) = \partial_\mu \Phi_{12} + A_\mu(x, 1)\Phi_{12}(x) - \Phi_{12}(x)A_\mu(x, 2),$$

$$D_\mu \Phi_{21}(x) = \partial_\mu \Phi_{21} + A_\mu(x, 1)\Phi_{21}(x) - \Phi_{21}(x)A_\mu(x, 1),$$

and

$$V_{12}(x) = (\Phi_{12}(x) + M_{12})(\Phi_{21}(x) + M_{21}) - \sum_i a_i^\dagger(x, 1)M_{12}M_{21}a_i(x, 1),$$

$$V_{21}(x) = (\Phi_{21}(x) + M_{21})(\Phi_{12}(x) + M_{12}) - \sum_i a_i^\dagger(x, 2)M_{21}M_{12}a_i(x, 2).$$

In these equation, if $\chi_1$ and $\chi_2$ are equally set to be $\chi$ and the replacements such as $A_\mu(x, 1) \rightarrow A_\mu(x, +)$, $A_\mu(x, 2) \rightarrow A_\mu(x, -)$, $\Phi_{12}(x) \rightarrow \Phi(x, +)$ and $\Phi_{21}(x) \rightarrow \Phi(x, -)$ are performed, we can obtain the equations introduced in the formulation with only one $\chi$ in Refs[2]~[3]. We can obtain the same Lagrangian also in Refs[2]~[3] as that in Eq.(2.35). Therefore, we can say that two formulations are equal.

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