A new approach to the relativistic treatment of the fermion-boson system, based on the extension of the $SL(2, C)$ group

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Abstract

A new technique for constructing the relativistic wave equation for the two-body system composed of the spin-1/2 and spin-0 particles is proposed. The method is based on the extension of the $SL(2, C)$ group to the $Sp(4, C)$ one. The obtained equation includes the interaction potentials, having both the Lorentz-vector and Lorentz-tensor structure, exactly describes the relativistic kinematics and possesses the correct one-particle limits. The comparison with results of other approaches to this problem is discussed.

1 Introduction

Necessity of taking into account the relativistic effects in two-body systems arises in many problems in particle and nuclear physics. In recent years, there has been much interest in the relativistic treatment of the two-body problem concerning the spin-1/2 fermion and the spin-0 boson. For such systems, finite size effects and relativistic corrections in the pionic and kaonic hydrogen have been evaluated [1], effects of the retarded interaction between an electron and a spin-0 nucleus have been studied [2]. These investigations apply the technique of the Breit equation [3], so that expansions in powers of $1/c^2$ are used. Alternatively, there exist other approaches to the fermion-boson problem that describe the relativistic kinematics exactly. Namely, the phenomenological two-body equation proposed by Królikowski [4], the Barut method [5, 6], the reductions of the Bethe-Salpeter equation [7, 8] and the relativistic quantum mechanics with constraints [9–12] are known.

Note that the above-mentioned investigations deal with potentials of the Lorentz-vector structure. However, the tensor coupling receives considerable attention of late. For instance, within the framework of the one-particle Dirac
equation, the Lorentz-tensor potentials have been used for describing the nuclear properties [13–15] and for constructing the relativistic harmonic-oscillator models [16, 17].

Recently, the possibility of deriving the relativistic wave equation for the two-body problem by means of the extension of the \( SL(2, C) \) group to the \( Sp(4, C) \) one has been proved [18]. The goal of the present work is to elaborate the proposed technique and to construct the relativistic equation for the two-body system consisted of the spin-1/2 and spin-0 particles. This equation must be suitable for including not only the commonly used Lorentz-vector potentials but the Lorentz-tensor ones, too, that will permit us to go beyond the one-particle approximation in the consideration of the tensor coupling effects in the fermion-boson systems.

The paper is organized as follows. Section 2 is devoted to a symplectic spacetime extension generated by the extension of the \( SL(2, C) \) group to the \( Sp(4, C) \) one. In Section 3 this approach is applied to construct the relativistic wave equation for a fermion-boson system with interaction introduced by means of the Lorentz-vector and -tensor potentials. The comparison with results of other approaches is given in Section 4. At last, our conclusions are summarized in Section 5.

### 2 Symplectic spacetime extension

It is known that the homogeneous Lorentz group \( SO(1, 3) \) is covered by the \( Sp(2, C) \equiv SL(2, C) \) group. As a consequence, there exists the one-to-one correspondence between the \( Sp(2, C) \) Hermitian spin-tensors of the second rank and the Minkowski four-vectors. Therefore, the relativistic field theory in the Minkowski space can be equivalently formulated entirely through the \( Sp(2, C) \) Weyl spinors [19].

Developing this approach, we may consider the symplectic \( Sp(2l, C) \), \( l > 1 \) group which is linked to a \((2l)^2\)-dimensional metric spacetime with a signature corresponding to \(l(2l - 1)\) timelike and \(l(2l + 1)\) spacelike dimensions [20]. Note that this extension of the pseudo-Euclidean spacetime is alternative to those having one time and \(d > 3\) spatial dimensions and used in the Kaluza-Klein theory. For description of the two-particle systems, the suitable symplectic spacetime extension has been proved to be the extension with the \( Sp(4, C) \) group [18]. For completeness and consistency of the following consideration, let us now outline the main ideas of this approach.

The symplectic \( Sp(4, C) \) group is the group of \(4 \times 4\) matrices, with complex elements and determinant equal to one, acting on the four-component Weyl spinors \( \varphi_{\alpha} \) and preserving an antisymmetric bilinear form \( \eta_{\alpha\beta} = -\eta_{\beta\alpha} \) [21]. This form plays the role of the “metrics” in the spinor space in the sense that the Weyl spinors with lower and upper indices are related by transformation \( \varphi_{\alpha} = \eta_{\alpha\beta} \varphi^{\beta} \). This means that the representations \( \varphi_{\alpha} \) and \( \varphi^{\alpha} \) are equivalent, but the complex conjugative spinors \( \bar{\varphi}_{\dot{\alpha}} = (\varphi_{\alpha})^{\ast} \) belong to the other representation. Thus, we have two spinor representations, say \( \varphi_{\alpha} \) and \( \chi^{\alpha} \), which are suitable for describing the wave functions.
In order to introduce the spacetime, in which the wave functions will be defined, we consider the irreducible representation of the $Sp(4, C)$ group described by Hermitian spin-tensor of the second rank, $P_{\alpha \dot{\alpha}}$. As in the case of the $Sp(2, C)$ group, such spin-tensors also correspond to real vectors but having already 16 components. Let us define this correspondence by the relations

$$P_{\alpha \dot{\alpha}} = \mu^M_{\alpha \dot{\alpha}} P_M, \quad P^M = \frac{1}{4} \tilde{\mu}^{M \dot{\alpha} \alpha} P_{\alpha \dot{\alpha}}$$  \hspace{1cm} (1)$$

where $\mu^M_{\alpha \dot{\alpha}} (M = 1, 2, ..., 16)$ are matrices of the basis in the space of $4 \times 4$ Hermitian matrices and tilde labels the transposed matrix with upper spinor indices. Hereafter we will suppress the spinor indices when possible.

For clarifying the connection of the discussed vector space $\mathbb{R}^{16}$ to the Minkowski space $\mathbb{R}^4$, let us represent 16 values of the vector index of $P_M$ through $4 \times 4$ combinations of two indices, $M = (a, m)$, with both $a$ and $m$ running from 0 to 3. Then the matrices $\mu^M$ and $\tilde{\mu}^N$ can be chosen as follows

$$\mu^M \equiv \mu^{(a,m)} = \Sigma^a \otimes \sigma^m, \quad \tilde{\mu}^N \equiv \tilde{\mu}^{(b,n)} = \tilde{\Sigma}^b \otimes \tilde{\sigma}^n,$$  \hspace{1cm} (2)$$

where explicit expressions for matrices $\Sigma^a$ and $\sigma^m$, written in terms of $2 \times 2$ unit matrix $I$ and the Pauli matrices $\tau^i$, are

$$\Sigma^0 = \tilde{\Sigma}^0 = I, \quad \Sigma^1 = \tilde{\Sigma}^1 = \tau^1, \quad \Sigma^2 = -\tilde{\Sigma}^2 = \tau^2, \quad \Sigma^3 = \tilde{\Sigma}^3 = \tau^3,$$

$$\sigma^0 = \tilde{\sigma}^0 = I, \quad \sigma^1 = -\tilde{\sigma}^1 = \tau^1, \quad \sigma^2 = -\tilde{\sigma}^2 = \tau^2, \quad \sigma^3 = -\tilde{\sigma}^3 = \tau^3.$$  \hspace{1cm} (3)$$

With such the representation for matrices $\mu^M$ and $\tilde{\mu}^N$, the metrics $g^{MN} = \frac{1}{4} \mu^M_{\alpha \dot{\alpha}} \tilde{\mu}^{N \dot{\alpha} \alpha}$ of the discussed vector space $\mathbb{R}^{16}$ explicitly takes the factorized form

$$g^{MN} \equiv g^{(a,m)(b,n)} = h^{ab} h^{mn}$$  \hspace{1cm} (4)$$

where $h^{mn} = diag(1, -1, -1, -1)$ is the usual Minkowski metrics and the factor $h^{ab} = diag(1, 1, -1, 1)$ is caused by the group extension.

The factorization of the metrics implies that a vector from $\mathbb{R}^{16}$ may be decomposed into four Minkowski four-vectors. As a consequence, the $Sp(4, C)$ momentum spin-tensor [1] can be employed to construct the wave equation for a two-body system.

### 3 Wave equation for a fermion-boson system

We are going to construct the wave equation for the system consisted of one spin-1/2 and one spin-0 particle. With the total spin of the system being equal to 1/2, the wave function of this system have to be represented by a Dirac spinor or, alternatively, by two Weyl spinors [19]. Then, in terms of the $Sp(4, C)$ Weyl spinors, the corresponding wave equation must have the form of the Dirac equation

$$P_{\alpha \dot{\alpha}} \chi^\dot{\alpha} = m \varphi_\alpha, \quad \tilde{P}^{\dot{\alpha} \alpha} \varphi_\alpha = m \chi^\alpha$$  \hspace{1cm} (5)$$
where \( m \) is a mass parameter and \( P_{α\bar{α}} \) is the \( Sp(4, C) \) momentum spin-tensor which, in view of Eqs. (1) and (2), can be written as
\[
P = \mu (a, m) P(a, m) = \Sigma^0 \otimes \sigma^m w_m + \Sigma^1 \otimes \sigma^m p_m + \Sigma^2 \otimes \sigma^m u_m + \Sigma^3 \otimes \sigma^m q_m \tag{6}
\]
where \( w_m, p_m, u_m, q_m \) are the Minkowski four-momenta. As it has been shown [18], this wave equation, supplemented with subsidiary conditions, describes the fermion-boson system with the equal mass constituents.

Now we generalize the wave equation (5) to the case of particles with unequal masses. For this end, let us replace the mass parameter on the right-hand side of this equation by a suitable matrix term which breaks the \( Sp(4, C) \) symmetry of the wave equation but retains the Lorentz \( SO(1, 3) \subset Sp(4, C) \) symmetry. A natural ansatz for this term is a combination of direct products of matrices as in Eq. (6). Then the Lorentz symmetry is retained if and only if the second matrix in this direct product is the unit matrix. In this case, we have two equivalent possibilities to obtain the plus sign for the quantity \( h_{ab} \), which are realized by choosing the first matrix as \( \tau^1 \) or \( \tau^3 \). In view of this fact, we replace the mass parameter as follows:
\[
m \to (m_1 + m_2)/2 + \tau^1 \otimes I(m_1 - m_2)/2, \tag{7}
\]
so that the additional term vanishes if \( m_1 = m_2 \).

Thus, the wave equation without interaction for the fermion-boson system with unequal masses takes the form
\[
P_{\bar{α}} \chi = (m_+ + \tau^1 \otimes I m_-) \varphi, \quad \bar{P} \varphi = (m_+ + \tau^1 \otimes I m_-) \bar{\chi} \tag{8}
\]
where \( m_\pm = (m_1 \pm m_2)/2 \).

To proceed, let us consider the structure of the \( Sp(4, C) \) momentum spin-tensor given by Eq. (6). It should be stressed that the description of the two-particle system requires only two four-momenta whereas the \( Sp(4, C) \) momentum spin-tensor involves four four-momenta, \( w_m, p_m, u_m, \) and \( q_m \). Hence, the number of their independent components must be decreased that can be implemented by means of subsidiary conditions.

For deriving the subsidiary conditions, we transform Eqs. (5) into the form of the Klein-Gordon equation that guarantees the correct dynamical relation between energy and momentum for free particles. Upon eliminating \( \bar{\chi} \) and inserting Eq. (6), we arrive at
\[
\left( w^2 + p^2 - u^2 + q^2 - \frac{2m_-}{m_+} (wp - iuq) - m_+^2 + m_-^2 + \sum_{A=1}^{5} \Gamma_A K^A \right) \varphi = 0 \tag{9}
\]
where \( w^2 = (w^0)^2 - w^2, \) \( wp = w^0 p^0 - wp, \) etc, \( \Gamma_A \) are direct products of the Pauli matrices, and \( K^A \) are quadratic forms with respect to the four-momenta.

Because the non-diagonal terms \( \Gamma_A K^A \) in this equation do not occur in the case of the ordinary Klein-Gordon equation, we put \( K^A = 0 \) on the wave
functions $\varphi$ and $\bar{\chi}$ that yields
\[
[(m_+^2 - m_-^2)(wp - m_+m_-) - m_+m_-(u^2 - q^2)]\psi = 0,
\]
\[
(m_+wq - m_-pq)\psi = 0,
\]
\[
(m_+up - m_-uw)\psi = 0,
\]
\[
uq\psi = 0,
\]
\[
[w_+(u^m w^n - v^m w^n - \epsilon^{\mnkl} p_k q_l) - m_-(u^m p^n - u^n p^m - \epsilon^{\mnkl} w_k q_l)]\psi = 0
\]}

where, for brevity, we denote $\psi = \begin{pmatrix} \varphi \\ \bar{\chi} \end{pmatrix}$ and $\epsilon^{\mnkl}$ is the totally antisymmetric tensor ($\epsilon^{0123} = +1$).

Thus, the imposed conditions and the Klein-Gordon-like equation (9) set ten components of $w_m$, $p_m$, $u_m$, $q_m$ to be the independent ones. The connection of these four-momenta with the four-momenta, $p_{1m}$ and $p_{2m}$, of the constituent particles is established by assuming
\[
w_m = \frac{1}{2}(p_{1m} + p_{2m}), \quad p_m = \frac{1}{2}(p_{1m} - p_{2m}), \quad u_m = 0, \quad q_m = 0.
\]

With these expressions for the four-momenta, the only one condition in Eqs. (10) remains nontrivial that reads
\[
(wp - m_+m_-)\psi = 0.
\]

This constraint, on the one hand, together with the Klein-Gordon-like equation guarantees that for the free particle case, since $wp - m_+m_- = (p_1^2 - p_2^2 - m_1^2 + m_2^2)/4$, the particles are on the mass shell. But on the other hand it removes the dependence of the wave function on the relative time.

Note that for the isolated system the total four-momentum $w_m$ is conserved and so can be treated as the eigenvalue rather than the operator. As the relative four-momentum can be represented by the operator $p_m = i\partial/\partial x^m$, with $x^m = x_{1m} - x_{2m}$ being the relative coordinate divided in the longitudinal and transverse parts
\[
x^m_\perp = (h^{mn} - w^m w^n/w^2)x_n, \quad x^m = x^m_\parallel - x^m_\perp,
\]

the subsidiary condition (12) determines the dependence of the wave function on the relative coordinates in the form
\[
\psi(x) = e^{-im_+m_-(wx_\parallel)/w^2} \psi(x_\perp).
\]

Here the longitudinal part $x^m_\parallel$ plays the role of the “relative time” of the particles, because in the center-of-mass frame it reduces to the difference of the usual time components of $x^m_1$ and $x^m_2$, whereas the transverse coordinates $x^m_\perp$ describe the interparticle separation in space. However, the relative-time variable is contained only in the phase factor which may be suppressed. Thus, the dynamics of the relative motion of this system is described only with the transverse coordinates $x^m_\perp$ and does not depend on the relative time $x^m_\parallel$ of the
particles. This property also holds in the case with the potential interaction included.

In order to clarify the two-particle interpretation of the wave equation (8), let us reduce it to the one-particle Dirac and Klein-Gordon equations which must describe the constituents of this system. With decomposing the spinor wave functions into the projections

$$\varphi_{\pm} = \frac{1}{2}(1 \pm \tau^1 \otimes I)\varphi, \quad \bar{\chi}_{\pm} = \frac{1}{2}(1 \pm \tau^1 \otimes I)\bar{\chi}$$

which are two-component $Sp(2, C)$ Weyl spinors, Eqs. (8) and (12) are reduced to two uncoupled sets of equations:

$$p_{1m}\sigma^m\bar{\chi}_+ = m_1\varphi_+, \quad p_{1m}\tilde{\sigma}^m\varphi_+ = m_1\bar{\chi}_+$$

(16)

$$\begin{align*}
(p_2^2 - m_2^2)\varphi_+ &= 0, \\
(p_2^2 - m_2^2)\bar{\chi}_+ &= 0
\end{align*}$$

(17)

and

$$p_{2m}\sigma^m\bar{\chi}_- = m_2\varphi_-, \quad p_{2m}\tilde{\sigma}^m\varphi_- = m_2\bar{\chi}_-$$

(18)

$$\begin{align*}
(p_1^2 - m_1^2)\varphi_- &= 0, \\
(p_1^2 - m_1^2)\bar{\chi}_- &= 0
\end{align*}$$

(19)

These equations contain the free one-particle Dirac equations written in the Weyl spinor formalism and the free Klein-Gordon ones. Consequently, the wave equation (8) supplemented with the subsidiary condition (12) describes two systems composed of the spin-1/2 and spin-0 particles, which differ from each other only in the permutation of masses of the particles.

As a next step, we must incorporate a potential interaction in the consideration. A widely accepted receipt for introducing the interaction consists in the replacement of the constituent four-momenta in the equation without interaction by the generalized momenta in the minimal manner, so that each particle is in the presence of a field of an external potential raised by the other particle. However, there also exists the non-minimal scheme [16, 17] which uses the momentum substitution having the matrix structure.

Because the free particle case in our approach is described by the the equation (8) and the subsidiary condition (12), we introduce the interaction in those through the combination of the minimal and non-minimal substitutions on the four-momenta. In terms of the total and relative four-momenta, $w_m$ and $p_m$, this reads as

$$\begin{align*}
w_m\varphi &\rightarrow (w_m + A_m)\varphi + iC_{mn}I \otimes \sigma^n\bar{\chi}, \\
w_m\bar{\chi} &\rightarrow (w_m + A_m)\bar{\chi} + iC_{mn}I \otimes \tilde{\sigma}^n\varphi, \\
p_m\varphi &\rightarrow (p_m + B_m)\varphi + iD_{mn}I \otimes \sigma^n\bar{\chi}, \\
p_m\bar{\chi} &\rightarrow (p_m + B_m)\bar{\chi} + iD_{mn}I \otimes \tilde{\sigma}^n\varphi
\end{align*}$$

(20)

where the involved potentials $A_m$, $B_m$, $C_{mn}$ and $D_{mn}$ in the general case are the functions of $w_m$, $p_m$ and the relative coordinate $x_m$. The non-minimal part of these substitutions is implemented through the matrices $\sigma^n$ and $\tilde{\sigma}^n$ which are the blocks of the Dirac gamma-matrices in the Weyl spinor formalism.
With Eqs. (20) inserted, the wave equation (8) takes the form
\[
[I \otimes \sigma^m (w_m + A_m) + \tau^1 \otimes \sigma^m (p_m + B_m)] \bar{\chi} \\
= (m_+ + \tau^1 \otimes I m_- - i I \otimes \sigma^m \tilde{\sigma}^n C_{mn} - i \tau^1 \otimes \sigma^m \tilde{\sigma}^n D_{mn}) \varphi,
\]
\[
[I \otimes \tilde{\sigma}^m (w_m + A_m) + \tau^1 \otimes \tilde{\sigma}^m (p_m + B_m)] \varphi \\
= (m_+ + \tau^1 \otimes I m_- - i I \otimes \tilde{\sigma}^m \sigma^n C_{mn} - i \tau^1 \otimes \tilde{\sigma}^m \sigma^n D_{mn}) \bar{\chi}.
\]

(21)

For elucidating the meaning of the Lorentz-tensor interaction, let us consider the matrix structure of the terms \(iC_{mn} I \otimes \tilde{\sigma}^m \sigma^n\) and \(iD_{mn} I \otimes \sigma^m \tilde{\sigma}^n\). The algebra of the matrices \(\sigma^m, \tilde{\sigma}^m\)
\[
\sigma^m \tilde{\sigma}^n = h^{mn} I + i 2 \epsilon^{mkl} \tilde{\sigma}^k \tilde{\sigma}^l,
\]
\[
\tilde{\sigma}^m \sigma^n = h^{mn} I - i 2 \epsilon^{mkl} \sigma^k \sigma^l,
\]
implies that the symmetric parts of \(C_{mn}\) and \(D_{mn}\), being contracted with the Minkowski metrics \(h^{mn}\), result in the known Lorentz-scalar interaction. In one’s turn, the contribution of the antisymmetric parts of \(C_{mn}\) and \(D_{mn}\) is similar to that of the non-minimal coupling term introduced by Pauli [22] within the framework of the one-particle Dirac equation. As the Pauli term describes the coupling of the anomalous magnetic moment of a fermion with an external field, the antisymmetric Lorentz-tensor potentials \(C_{mn}\) and \(D_{mn}\) can be regarded as being responsible for the interactions with the anomalous magnetic moment in the fermion-boson system.

As regards the subsidiary condition, upon inserting (20) into (12) and symmetrizing the resulted expression for each its piece to become self-adjoint, we obtain
\[
\left[ \frac{1}{2} (\omega_m \pi^m + \pi^m \omega_m) + \frac{1}{2} (\omega^m D_{mn} - D_{mn} \omega^m + C_{mn} \pi^m - \pi^m C_{mn}) \gamma^n \\
- \frac{1}{2} (C_{mk} D^{mn} + D_{mk} C^{mn}) \gamma^k \gamma^l - m_+ m_- \right] \psi = 0
\]
(23)

where the following abbreviations are used
\[
\omega_m = w_m + A_m, \quad \pi_m = p_m + B_m, \quad \gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \tilde{\sigma}^m & 0 \end{pmatrix}
\]
(24)

and the gamma-matrices are self-adjoint in the usual Dirac sense, that is \(\tilde{\gamma}^m \equiv \gamma^0 (\gamma^m)^\dagger \gamma^0 = \gamma^m\).

It should be stressed that the subsidiary condition (23) must be compatible with the wave equation (21). Let us first consider the case when only the Lorentz-vector potentials are present. Then Eq. (23) becomes
\[
\left[ \frac{1}{2} (\omega \pi + \pi \omega) - m_+ m_- \right] \psi = 0
\]
(25)

and the compatibility takes place when the operator \((\omega \pi + \pi \omega)\) in the subsidiary condition commutes with all operators in the wave equation:
\[
[\omega \pi + \pi \omega, w_m + A_m] = 0, \quad [\omega \pi + \pi \omega, p_m + B_m] = 0.
\]
(26)

The last conditions, because \([wp, x_m] \neq 0\) but \([wp, x_{\perp m}] = 0\), require that the potentials \(A_m\) and \(B_m\) depend on the relative coordinate only through its
transverse part $x_{\perp m}$ defined by Eq. (13). Then the simplest solution to the compatibility conditions (26) comes from the ansatz

$$\omega \pi + \pi \omega = 2wp$$

(27)

that at once results in vanishing commutators (20), and the subsidiary condition (25) takes the same form as Eq. (12) describing the case without interaction. Therefore, in the presence of the interaction, the dependence of the wave function on the relative-time variable can be excluded in the same manner as in the case without interaction.

The ansatz (27) obviously restricts the shape of the potentials $A_m$ and $B_m$. As it will be shown, with the potentials so involved, the interaction is described both by the time-component of the Lorentz vector and its spacelike part, as proposed in the two-particle relativistic quantum mechanics with constraints [9–11]. However, the wave equation (21) enables us to treat not only the commonly used Lorentz-vector potentials but also the novel interaction, involved through the Lorentz-tensor potentials $C_{mn}$ and $D_{mn}$.

In the case of the Lorentz-tensor interaction, from Eqs. (23) and (25) it follows that the shape of the tensor potentials is restricted by the relations

$$\omega^m D_{mn} - D_{mn} \omega^m + C_{mn} \pi^m - \pi^m C_{mn} = 0, \quad C_{mk} D_{mn} + D_{mk} C_{mn} = 0.$$  (28)

Then the requirement of commuting the operator $(\omega \pi + \pi \omega)$ from the subsidiary condition with the Lorentz-tensor potentials $C_{mn}$ and $D_{mn}$ in the wave equation is easily satisfied by the demand that the potentials $C_{mn}$ and $D_{mn}$ depend on the relative coordinate only through its transverse part $x_{\perp m}$.

Thus, the subsidiary condition, supplementing the wave equation, has the same form (12) in both cases, with and without interaction. It is the condition that enables us to exclude the dependence of the wave function on the relative-time variable.

However, it is to be pointed that in spite of having the same form, this condition has the different physical content. So, in the presence of the interaction the particles are not on the mass shell because the form of the wave equation is changed. Furthermore, the relative four-momentum $p_m$ is not conserved since it does not commute with the potentials of the wave equation (for instance, $[p_m, A_n] = i\partial A_n/\partial x_{\perp m} \neq 0$ though $[wp, A_n] = 0$).

4 Comparison with results of other approaches

Now let us compare the proposed equation (21) with those obtained with some other approaches to this problem.

At first we discuss the connection with the equations derived within the framework of the relativistic quantum mechanics with constraints [9–11]. Because the wave equation (21) describes two fermion-boson systems which differ from each other only in the permutation of masses of the particles, we define the first particle as a Dirac fermion with the mass $m_1$ and the second one as a Klein-Gordon boson with the mass $m_2$. According to the decomposition (15),
the wave functions of this system are the spinor projections \( \varphi_+ \) and \( \bar{\chi}_+ \). If we now use the representation

\[
\Psi = \left( \begin{array}{c} \varphi_+ \\ \bar{\chi}_+ \end{array} \right), \quad \gamma^m = \left( \begin{array}{cc} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{array} \right)
\]

and take into account definitions (11) our equation (21), in the lack of the Lorentz-tensor interaction, can be readily transformed into the Dirac-like form

\[
(p_{1m} \gamma^m - m_1 - V) \Psi = 0
\]

where \( V = -(A_m + B_m) \gamma^m \). Together with the subsidiary condition (12), this yields

\[
[p_2^2 - m_2^2 - (p_{1m} \gamma^m + m_1)V] \Psi = 0.
\]

These equations are just those obtained for the fermion-boson system within the framework of the two-particle relativistic quantum mechanics with constraints [9, 10].

However, in the relativistic quantum mechanics, there is another approach [11] for involving the interaction in the pair of the Dirac and Klein-Gordon equations that uses the generalized four-momenta of the particles \( \pi_{im} = p_{im} + A_{im} (i = 1, 2) \) instead of the single potential \( V \). In such procedure, the shape of the potentials \( A_{1m} \) and \( A_{2m} \) is restricted by the compatibility requirement and the demand to give correct one-particle limits for these equations.

In our treatment, such method is easily reproduced with the replacement of the Lorentz-vector potentials \( A_m \) and \( B_m \), introduced by the substitution (20) on the total and relative four-momenta, by their expressions in terms of \( A_{1m} \) and \( A_{2m} \) as

\[
A_m = (A_{1m} + A_{2m})/2, \quad B_m = (A_{1m} - A_{2m})/2.
\]

Note that the above-mentioned restrictions on the shape of the potentials \( A_{1m} \) and \( A_{2m} \) are reduced to the constraint (27) in our method.

Then, adopting the ansatz [11]

\[
A_m = \left( \left( 1 - \frac{2A}{E} \right)^{1/2} - 1 \right) w_m, \quad B_m = \left( \left( 1 - \frac{2A}{E} \right)^{-1/2} - 1 \right) p_m + \frac{i}{2E} \left( 1 - \frac{2A}{E} \right)^{-3/2} \frac{\partial A}{\partial x_{\perp}^m}
\]

where \( E = 2\sqrt{w^2} \) is the total energy and \( A = A(x_{\perp}^2) \) is a scalar potential function, and inserting these potentials in our Eqs. (31) and (32) results in the coupled Dirac and Klein-Gordon equations with the “electromagneticlike” Lorentz-vector interaction derived in the framework of the relativistic quantum mechanics with constraints [11].

Another approach to describing the fermion-boson system consists in the use of the effective single-body equation in the center-of-mass frame [4, 7].

For obtaining such an equation within the framework of our formalism, we consider the center-of-mass frame with \( \mathbf{w} \equiv (\mathbf{p}_1 + \mathbf{p}_2)/2 = 0 \) and \( E = 2w_0 \). From
Eq. (12) it follows that the zeroth component of the relative four-momentum is
\[ p_0 = \frac{(m_1^2 - m_2^2)}{2E}. \]
Then, using the standard notation for the Dirac matrices, \( \gamma^0 = \beta, \gamma = \beta\alpha \), the equation (30) with the potentials (33) is rewritten as
\[
\left( 1 - \frac{2A}{E} \right)^{-1/2} \alpha \mathbf{p} + \beta m_1 - \frac{E^2 - 2EA + m_1^2 - m_2^2}{2E(1 - 2A/E)^{1/2}} - \frac{i\alpha \nabla A}{2E(1 - 2A/E)^{3/2}} \right] \Psi = 0.
\]
(34)

On the other hand, if we take the other form of the potentials
\[
A_0 = -\frac{V}{E}w_0, \quad A = 0, \quad B_0 = \frac{V}{E - V}p_0, \quad B = \frac{i\nabla V}{2(E - V)}
\]
(35)
we arrive at the equation [4]
\[
\left( \alpha \mathbf{p} + \beta m_1 - \frac{E - V}{2} - \frac{m_1^2 - m_2^2 - i\alpha \nabla V}{2(E - V)} \right) \Psi = 0
\]
(36)
which can also be derived by reducing the Bethe-Salpeter equation [7].

It is interesting to study the difference between Eqs. (34) and (36) caused by the different structure of the involved potentials. Note that Eq. (36) includes only the timelike Lorentz-vector interaction, whereas Eq. (34) involves the potentials \( A_m \) and \( B_m \) having both the timelike and spacelike parts, as seen from their manifestly covariant form (33). If we set the potential \( A \) in Eq. (34) as \( A = V - V^2/2E \), the above two equations will have the same form, except for the derivative terms with \( \nabla A \) and \( \nabla V \), and the factor before \( \alpha \mathbf{p} \).

The difference in the derivative terms is unessential because these terms can be removed by passing to new wave functions, namely, \( \Psi_1 = (1 - 2A/E)^{-1/4}\Psi \) in Eq. (34). On the contrary, the difference between Eqs. (34) and (36) caused by the factor before \( \alpha \mathbf{p} \) is meaningful and has its origin in the Lorentz-structure of the potential interaction in these equations.

However, in spite of this, Eqs. (34) and (36) coincide in either one-particle limits, i.e., when the mass of one of particles is much larger than the other. In the case of the heavy spin-0 particle, putting \( E = m_2 + E_1 \) and taking the limit \( m_2/m_1 \rightarrow \infty \), Eq. (34) as well as Eq. (36) reduces to the one-particle Dirac equation
\[
(\alpha \mathbf{p} + \beta m_1 + V - E_1)\Psi = 0.
\]
(37)

In the other limit, \( m_1/m_2 \rightarrow \infty \), if we write \( E = m_1 + E_2 \), both Eqs. (33) and (36) give rise to the one-particle Klein-Gordon equation
\[
[p^2 + m_2^2 - (E_2 - V)^2] \Psi_L = 0
\]
(38)
for the large component, \( \Psi_L = (1 + \beta)\Psi/2 \), of the Dirac spinor.

Thus, we conclude that the proposed wave equation based on the extension of the \( SL(2,C) \) group has correct one-particle limits and coincide with those obtained in other approaches to describing the relativistic fermion-boson system.
5 Conclusions

In the present work, the extension of the $SL(2,C)$ group to the $Sp(4,C)$ one has been used to construct the relativistic wave equation for the system consisted of the spin-$1/2$ and spin-$0$ particles. The obtained equation involves the potentials of both the Lorentz-vector and Lorentz-tensor structure. With the Lorentz-vector potentials introduced by a minimal substitution on the four-momenta, this equation reproduces the known equation by Królikowski [4], which is also derived by reducing the Bethe-Salpeter equation [7], and the equations of the relativistic quantum mechanics with constraints [9–11].

However, including the Lorentz-tensor potentials in our equation, in addition to the Lorentz-vector ones, permits us to treat more wide range of problems concerning fermion-boson interactions. It should be pointed that the Lorentz-tensor interaction is involved in other means than in the Breit approach [1]. Instead of expanding the scattering amplitude in the $1/c^2$-powers, that leads to nonlocal terms in the potential [1], we introduce the local Lorentz-tensor potentials through the non-minimal substitution on the four-momenta. In the fermion-boson equation, the contribution of the so involved interaction resembles the tensor coupling term appeared in the effective Dirac Lagrangian of the relativistic mean-field theories of nuclei [13] and in the relativistic models of the Hartree approach [14, 15]. Thus, the constructed fermion-boson equation is suitable for divorcing from each other the effects of the tensor coupling and the relative motion. At last, the obtained equation possesses the exact solutions of the harmonic-oscillator type that will be published elsewhere.

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References

[1] N. G. Kelkar, M. Nowakowski, Phys. Lett. B 651, 363 (2007).
[2] S. N. Datta, A. Misra, J. Chem. Phys. 125, 084111 (2006).
[3] G. Breit, Phys. Rev. 34, 553 (1929).
[4] W. Królikowski, Acta Phys. Pol. B 10, 739 (1979).
[5] A. O. Barut, S. Komy, Fortschr. Phys. 33, 309 (1985).
[6] W. Garczyński, M. Klimek, Acta Phys. Pol. B 20, 755 (1989).
[7] T. Tanaka, A. Suzuki, and M. Kimura, Z. Phys. A 353, 79 (1995).
[8] J. Bijtebier, J. Broekaert, J. Phys. G 22, 1727 (1996).
[9] H. Sazdjian, *Phys. Rev. D* **33**, 3401 (1986).

[10] H. Sazdjian, *Phys. Rev. D* **33**, 3435 (1986).

[11] H. W. Crater, P. Van Alstine, *Phys. Rev. D* **36**, 3007 (1987).

[12] H. W. Crater, P. Van Alstine, *Phys. Rev. D* **70**, 034026 (2004).

[13] R. J. Furnstahl, J. J. Rusnak, and B. D. Serot, *Nucl. Phys. A* **632**, 607 (1998).

[14] G. Mao, *Phys. Rev. C* **67**, 044318 (2003).

[15] P. Alberto, R. Lisboa, M. Malheiro, and A. S. de Castro, *Phys. Rev. C* **71**, 034313 (2005).

[16] M. Moshinsky, A. Szczepaniak, *J. Phys. A* **22**, L817 (1989).

[17] V. I. Kukulin, G. Loyola, M. Moshinsky, *Phys. Lett. A* **158**, 19 (1991).

[18] D. A. Kulikov, R. S. Tutik, and A. P. Yaroshenko, *Phys. Lett. B* **644**, 311 (2007).

[19] R. Penrose, W. Rindler, *Spinors and Space-Time* (Cambridge University Press, 1984).

[20] Yu. F. Pirogov, *Yad. Fiz.* **66**, 138 (2003) [*Sov. J. Nucl. Phys. 66*, 136 (2003)].

[21] H. Weyl, *The Classical Groups: Their Invariants and Representations* (Princeton University Press, 1946).

[22] W. Pauli, *Rev. Mod. Phys.* **13**, 203 (1941).