Symmetry and symmetry breaking in particle physics
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Abstract

Symmetry, in particular gauge symmetry, is a fundamental principle in theoretical physics. It is intimately connected to the geometry of fibre bundles. A refinement to the gauge principle, known as “spontaneous symmetry breaking”, leads to one of the most successful theories in modern particle physics. In this short talk, I shall try to give a taste of this beautiful and exciting concept.

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1 Introduction

The concept of symmetry is one of the very few on which mathematicians and physicists agree, namely that

\[ \text{SYMMETRY} \equiv \text{GROUPS} \]

Hence we shall use these terms interchangeably.

In particle physics, there are two main uses of groups:

1. as transformation groups under which a theory is invariant;
2. as group representations for classifying the many particles we see.

In a sense, the first is all important, just like the main characters of a play. The second is more like the supporting cast, without which the theory, although it can stand on its own, is much less interesting and also much less realistic.

The next question is: which groups does one use or need? Generally speaking, finite-dimensional compact semi-simple Lie groups. In this talk, in order to simplify the presentation but without losing the essentials, I shall consider almost exclusively only the following: for abelian groups \( U(1) \), and for nonabelian groups the unitary groups \( U(N) \) and \( SU(N) \). At the end I shall mention an example where a discrete group figures.

2 The particles: a lightning view

Particles used to be called elementary particles, which made good sense when we knew only the electron, the proton and the neutron, and they were adequate for forming all the elements in the Periodic Table. Then Einstein proved the existence of the photon as a particle. Also Dirac postulated the existence of anti-particles, which was well borne out by later experiments. ... All in all, there are now more than 150 of them listed, and the number keeps on increasing! It would be highly unsatisfactory if we had to put them all in one or more representations or ‘multiplets’ without a good theoretical guidance.

Fortunately, we do now have a theoretical basis, the gauge principle, which we shall study in the next section. In the light of the gauge principle, particles can be classified under three headings:
• Vector bosons: $\gamma$ (the photon), $W^+, W^-, Z^0$.

• Leptons: $e, \nu_e; \mu, \nu_\mu; \tau, \nu_\tau$. (In words, the electron, the electron neutrino, etc.)

• Quarks: these are not observable themselves, but they form most of the other particles by combining two or three together. Each quark $q$ is in the 3-dimensional or fundamental representation, and directly observable particles occur in the 1-dimensional or singlet representation as follows:

$$qqq: \quad 3 \otimes 3 \otimes 3 = 1 \oplus \cdots$$

$$q\bar{q}: \quad 3 \otimes \overline{3} = 1 \oplus \cdots$$

Note that only singlets can be observed as free particles, as will be explained later.

## 3 The gauge principle

We said at the beginning that the invariance of a theory under certain group transformations is the most important aspect of symmetry. Let us study it now in greater detail.

Recall classical electromagnetism. The skew rank 2 field tensor $F_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) has as its components the electric $E$ and magnetic $B$ fields:

$$F_{\mu\nu} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -B_3 & B_2 \\
-E_2 & B_3 & 0 & -B_1 \\
-E_3 & -B_2 & B_1 & 0
\end{pmatrix}.$$ 

These are directly measurable quantities and hence do not transform under any symmetries. However, one can and does introduce a vector potential $A_\mu$, related to $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu,$$

so that there is a freedom in changing $A_\mu$ without affecting $F_{\mu\nu}$:

$$A_\mu \mapsto A_\mu + i e \partial_\mu \Lambda,$$

where $\Lambda(x)$ is a scalar field, and $e$ is a ‘coupling’ constant representing the strength of interaction. In classical theory, there is no need to consider the
potential $A_\mu$. However, in quantum theory, it was demonstrated that $F_{\mu\nu}$ is not enough to describe the physics and one needs $A_\mu$. This is the famous Bohm–Aharonov experiment.

The ‘gauge freedom’ in $A_\mu$ is in fact linked to the arbitrary phase of the electron wave function:

$$\psi \mapsto e^{ie\Lambda} \psi.$$ 

Hence the relevant group for the symmetry of electromagnetism is:

$$\boxed{G = U(1)}$$

In 1954, Yang and Mills extended this gauge principle to a nonabelian group $G$:

$$A_\mu \mapsto SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1},$$

$$\psi \mapsto S\psi,$$

where $S \in G$.

This is the famous Yang–Mills theory. In the last 20 years or so, it has been generally accepted that Yang–Mills theory is the basis of all of particle physics:

$$\textbf{YANG–MILLS THEORY = BASIS OF ALL PARTICLE PHYSICS}$$

A refinement of gauge symmetry is called symmetry breaking, where the whole theory (including equations of motion) is invariant under a group $G$ but a particular solution (or ‘vacuum’) is invariant only under a subgroup $H \subset G$. This will be important for later applications.

4 The geometry of gauge theory

Although it was not realized at the time, gauge theory is intimately linked with geometry. In fact it is as geometric a theory as Einstein’s general relativity. Table[I], borrowed from a paper by Yang, underlines this fact.

Recall the definition of a principal fibre bundle, as illustrated in the accompanying sketch (Figure[II]).

Thus a principal fibre bundle consists of a manifold $P$ (total space), a manifold $X$ (base space or spacetime), a projection $\pi$ and a group $G$ (structure
or gauge group). Above any point \( x \in X \) the inverse image \( \pi^{-1}(x) \subset P \) is called the typical fibre \( F \), and is homeomorphic to \( G \). Above an open set \( U_\alpha \) of \( X \), the inverse image \( \pi^{-1}(U_\alpha) \subset P \) is homeomorphic to the product \( U_\alpha \times F \):

\[
\phi_\alpha: U_\alpha \times F \to \pi^{-1}(U_\alpha).
\]

Thus in a sense, the manifold \( P \) is a ‘twisted’ product of \( G \) and \( X \), the twisting being done by the action of the group:

\[
\phi_{\alpha,x}: F \to \pi^{-1}(x)
\]

\[
y \mapsto \phi_\alpha(x,y),
\]

with

\[
\phi^{-1}_{\beta,x} \phi_{\alpha,x}: F \to F
\]

giving the relevant action of the group \( G \).

A trivial bundle is then just the product \( X \times G \). The most well-known example of a nontrivial bundle is the Möbius band, where twisting is done by the 2-element group \( \mathbb{Z}_2 \). An example which is useful in physics is the magnetic monopole, which can be represented topologically by \( S^3 \), which in turn is a nontrivial \( S^1 \) bundle over \( S^2 \) (the Hopf bundle, of Chern class 1, for the experts). Here spacetime is thought of as \( S^2 \times \mathbb{R}^2 \), where the second factor is just a vector space with no topology, and can thus be ignored for the present purpose. Ordinary electromagnetism without magnetic monopoles is given topologically by the trivial bundle bundle \( \mathbb{R}^4 \times S^1 \). In both cases, the typical fibre is the circle \( S^1 \), which is homeomorphic to the group \( U(1) \).

To proceed further we need to introduce a connection on the principal bundle \( P \). This is a 1-form \( A \) on \( P \) with values in the Lie algebra \( \mathfrak{g} \) of \( G \), satisfying certain properties and giving a prescription for differentiating
vectors and tensors. Locally it combines with the usual partial derivative to give the covariant derivative:

\[ D_\mu = \partial_\mu - ig[A_\mu, \cdot]. \]

In differential geometry and in gauge theory one has to replace the partial derivative by the covariant derivative so as to preserve the invariance or symmetry of the system.

From the connection one can define the curvature:

\[ F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\mu, A_\nu]. \]

One recognizes immediately that these are respectively the gauge potential and the gauge field introduced in the last section, where the extra commutators (in the Lie algebra) take into account that now the group is in general nonabelian.

With this language, the mechanism of symmetry breaking can be stated as the case when the twisting of the bundle are by elements of a subgroup \( H \) of \( G \), and when the connection 1-form takes values in the corresponding Lie subalgebra. One says then that the bundle with connection is reducible to the subgroup \( H \). An important example is the 't Hooft–Polyakov magnetic
monopole, which is a nontrivial $U(1)$ reduction of a trivial $SU(2)$ bundle, given by the exact sequence (for those who are fond of such things):

$$
\cdots \to \pi_2(SU(2)) \to \pi_2(SU(2)/U(1)) \to \pi_1(U(1)) \to \pi_1(SU(2)) \to \cdots.
$$

The first and last terms being zero, one gets the isomorphism

$$
\pi_2(SU(2)/U(1)) \cong \pi_1(U(1)).
$$

5 Briefest summary of the Standard Model

Following the gauge principle, we can now try to fit the three types of particles of section 2 into a more systematic pattern, the better to exhibit their symmetry properties.

The vector bosons, also known as gauge bosons, are the potential $A_\mu(x)$ when considered as fields. Note that in the language of quantum field theory, the concept of “particles” and “fields” are interchangeable: particles interact by influencing the spacetime in their neighbourhood and thus giving rise to fields, that is, functions of spacetime with a definite tensor property (whether scalar, vector, rank 2 skew tensor, etc.). According to the interaction, we have a specific symmetry or gauge group. The other two types of particles are usually thought of as “matter fields” belonging to representations of the corresponding groups.

We now recognize that, other than gravitation, there are two fundamental forces of Nature: the strong and the electroweak. The electroweak theory is an example of a gauge theory with symmetry breaking. The idea, called the Weinberg–Salam model, is that at high energies when the Universe was much younger the symmetry was not broken, but as the Universe cooled down the $U(2)$ gauge group broke down to the $U(1)$ subgroup which is the electromagnetism of today. The rest of the $U(2)$ interaction manifests itself in the present-day weak interaction, of which radioactivity is the most commonly known aspect. The breaking also leaves some remnant fields called the Higgs fields which are yet to be discovered.

As mentioned already, each quark is in a 3-dimensional representation of $SU(3)$. Hence a quark has in fact three states, fancifully called colour. This “colour” is not directly observable, as only states in the singlet representation can exist free. We say that the $SU(3)$ symmetry is exact and confined.
Table 2: Forces and Fields in the Standard Model

Table 2 summarizes these ingredients of the so-called Standard Model of particle physics. The particles in square brackets are not (or have not been) directly observed, but they are part of the theory.

The standard model can in fact be schematically represented as:

$$( \text{QCD} + \text{Weinberg–Salam} ) \times 3$$

the gauge group being $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$. Most physicists neglect the six-fold identification, but it is important for identifying the correct particle representations.

The multiplication by 3 above is necessary to model another aspect of the particle spectrum known as generations. Take the charged leptons as an example. There are 3 of them: the electron $e$, the muon $\mu$ and the tauon $\tau$. Except for their very different masses, they behave in extremely similar fashion. The same pattern is repeated for their neutral ‘partners’ the neutrinos $\nu_e, \nu_\mu, \nu_\tau$. The quarks also come in three generations: the ‘up’ and ‘down’ as the lightest generation, the ‘charm’ and ‘strange’ as the next in mass, and the ‘top’ and ‘bottom’ as the heaviest. Table 3 arranges the 3 generations as 3 rows. The subscripts L and R refer to the left-handed and right-handed field components, a refinement we shall not have time to go into.

The role of the Higgs fields in the standard model is crucial. They break the $U(2)$ symmetry, give masses to the gauge bosons $W, Z$ and also give masses to the quarks and charged leptons. Without them, all particles would be massless. Notice that the neutrinos are supposed to be massless, although some recent experiments in particle physics and astrophysics indicate that they may have extremely small masses.

Even with this briefest of summaries of the Standard Model we can already see how symmetry plays a crucial organizing role in our understanding of particle physics. And in this gauge symmetry is of prime importance.
Table 3: Generations of Quarks and Leptons

6 Electric–magnetic duality: example of a discrete symmetry

It is well-known that electromagnetism has a discrete $\mathbb{Z}_2$ symmetry, that is, the equations are invariant under the change from ‘electric’ to ‘magnetic’ and vice versa. Let us look at this in a little more detail.

As described in section 3, we can start with the potential $A_\mu$ and define the field tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$ 

Further introduce the Hodge star operator, which in this case goes from 2-forms to 2-forms:

$$^*F_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}.$$ 

This operation interchanges electric fields and magnetic fields. We then have the identity:

$$\partial_\mu ^*F^\mu\nu = 0,$$

which always holds for $F_{\mu\nu}$ defined as above in terms of an $A_\mu$. On the other hand, by Gauss’ theorem, this ‘divergen–free’ condition is equivalent to the absence of magnetic monopoles, because $^*F_{\mu\nu}$ gives the magnetic flux out
of such an object if present. This very significant link between a geometric statement and a physical statement can be schematically represented as:

\[
\begin{align*}
\text{A}_\mu & \text{ exists} & \text{Poincaré} & \quad \partial_\mu *F^{\mu\nu} = 0 & \text{Gauss} \quad \text{no magnetic monopoles} \\
\text{geometry} & \quad \iff & & \quad \text{physics} & \\
\end{align*}
\]

In the language of differential forms, the geometric statement is no other than

\[
F \text{ exact} \iff F \text{ closed}
\]

Now in the absence of electric charges (remember: only the main characters and no supporting cast!), we have

\[
\partial_\mu F^{\mu\nu} = 0,
\]

just as for the case of magnetic monopoles above, only this time we have \(F^{\mu\nu}\) instead of \(*F^{\mu\nu}\). So we have the ‘dual’ of the scheme above:

\[
\begin{align*}
\text{A}'_\mu & \text{ exists} & \text{Poincaré} & \quad \partial_\mu F^{\mu\nu} = 0 & \text{Gauss} \quad \text{no electric charges} \\
\text{geometry} & \quad \iff & & \quad \text{physics} & \\
\end{align*}
\]

We see that the electric–magnetic discrete symmetry indeed holds.

It can further be shown that electromagnetism is dual symmetric in the above sense even in the presence of charges.

What is even more interesting—and this is what I am currently working on—is that Yang–Mills theory (or nonabelian gauge theory) is also dual symmetric, but the proof is not all that straightforward. One has to use techniques involving infinite-dimensional loop variables and the dual transform is no longer just the Hodge star but a loop space generalization of it. What is interesting, and intriguing, is that this discrete symmetry is clearly linked to the continuous gauge symmetry. One consequence is that the gauge symmetry is now doubled:

\[
G \times \tilde{G},
\]

where as groups the two factors are identical, only the physical aspects they refer to are not identical but dual to each other. Now ’t Hooft proved a theorem which can be stated as follows: the \(G\) symmetry is exact and confined if and only if the \(\tilde{G}\) symmetry is broken and massive. Compare this to the actual symmetries of the Standard Model:

\[
\begin{align*}
SU(3) & \text{ exact and confined} \\
U(2) & \text{ broken and massive}
\end{align*}
\]

10
Applying ’t Hooft’s theorem to these symmetries lead to very interesting consequences which I do not have time to talk about.

7 Conclusions

Let me summarize the salient points about symmetry in particle physics that I have mentioned:

1. Symmetry is all important in physics. For lack of time (and expertise) I have omitted to treat many symmetries, such as Lorentz symmetry, diffeomorphism symmetry, supersymmetry, . . . .

2. There are two main uses of groups:
   
   (a) in the gauge principle as invariance, and
   
   (b) for particle classification using representations.

3. The Standard Model is a triumph of the gauge principle.

4. Electric–magnetic duality (a discrete $\mathbb{Z}_2$ symmetry), when generalized to Yang–Mills theory, leads to very interesting results.

If, however, you wish to take away with you just one point, then I recommend:

\[
\text{SYMMETRY } \equiv \text{ GROUPS}.
\]

Acknowledgements

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References

There are many excellent textbooks and semi-popular books on modern particle physics which emphasize its symmetry properties. There are also excellent articles in Scientific American which are most suitable to give a taste of the beauty of the subject. Below is a random selection of such, the first
being a more general appreciation of symmetry in physics by the originator of Yang–Mills theory:

1. Chen Ning Yang, Symmetry and Physics, in Oskar Klein Memorial Lectures, Vol. 1, World Scientific Publishing Company, Singapore, 1991.

2. G. ’t Hooft, Scientific American, June 1980, p. 90.

3. S. Weinberg, Scientific American, June 1981, p. 52; ibid. July 1974, p. 50.

4. C. Quigg, Scientific American, April 1985, p. 64.

5. S. Weinberg, Dreams of a Final Theory, Hutchison 1993.

For the reader who might want to know more about the last part of this lecture, here are a few of my recent articles (the last with an amusing application from the Serret–Frenet formulae for space curves):

1. Chan Hong-Mo and Tsou Sheung Tsun, Physical Consequences of Non-abelian Duality in the Standard Model, hep-th/9701120, Phys. Rev. D57 (1998) 2507–2522.

2. Chan Hong-Mo and Tsou Sheung Tsun, Standard Model with Duality: Theoretical Basis, hep-th/9712171; Standard Model with Duality: Physical Consequences, hep-ph/9712436; invited lectures at the Cracow Summer School on Theoretical Physics, May–June 1997, Zakopane, Acta Phys. Pol. B28 (1997) 3027–3040; 3041–3056.

3. José Bordes, Chan Hong-Mo, Jakov Pfaudler and Tsou Sheung Tsun, Features of Quark and Lepton Mixing from Differential Geometry of Curves on Surfaces, hep-ph/9802436, February 1998.