Convex Formulation for Planted Quasi-Clique Recovery

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Received: date / Accepted: date

Abstract In this paper, we consider the planted quasi-clique or $\gamma$-clique problem. This problem is an extension of the well known planted clique problem which is NP-hard. The maximum quasi-clique problem is applicable in community detection, information retrieval and biology. We propose a convex formulation using nuclear norm minimization for planted quasi-clique recovery. We carry out numerical experiments using our convex formulation and the existing mixed integer programming formulations. Results show that the convex formulation performs better than the mixed integer formulations when $\gamma$ is greater than a particular threshold.

Keywords Quasi-clique · Relaxations · Nuclear norm · Edge density

1 Introduction

A clique is the densest subgraph of any undirected graph, $G = (V,E)$. A subgraph, $G[V']$ induced by $V' \subseteq V$, forms a clique if every pair of nodes are adjacent [31]. The problem of finding the largest clique in a graph is known as the Maximum Clique Problem (MCP) [10, 35, 36]. The size of the largest clique in $G$ is known as the clique number and it is denoted by $\omega(G)$. Although this problem is NP-hard [23], it has been well studied due to its wide applications. Verily, cliques possess the ideal properties for cohesiveness [38]. However, the requirement that every pair of nodes are adjacent is too confining for some applications [39]. This motivates the emergence of different clique relaxations. Some of the clique relaxation models emanating from social network analysis include the $k$-clique, $k$-club, and $k$-plex,
A density based relaxation known as quasi-clique or γ-clique was introduced in [1]. Although γ-clique is the most recent of clique relaxations, it is one of the most popular due to its suitability for a range of applications [17]. The quasi-clique model is applicable in community detection [24, 38], data clustering and data mining [6, 48], information retrieval [44], protein-protein network [26], and criminal network analysis [5, 27].

A subgraph, \( G[V'] \), is a γ-clique if \(|V' \times V' \cap E|/\binom{|V'|}{2} \geq \gamma \), where \( \gamma \in (0, 1] \). The problem of finding a γ-clique with maximum cardinality in \( G \) is known as the maximum quasi-clique problem (MQCP). Obviously, the case \( \gamma = 1 \) is equivalent to the maximum clique problem. This problem has been shown to be NP-hard [39].

The size of the maximum quasi-clique in \( G \) is known as γ-clique number and we denote it by \( \omega_\gamma(G) \). A number of existing works on finding the maximum γ-clique focused on developing heuristic methods for finding large quasi-cliques for different instances [2, 9, 30, 40]. Other various heuristic and enumerative algorithms have recently been developed (see e.g. [32, 33, 37, 42, 50]).

The first mathematical model for maximum quasi-clique recovery is the mixed integer programming (MIP) of [39]. This model has been reformulated in [47] to handle larger problems. These are the only known existing maximum quasi-clique recovery models. In this paper, we focus on a special case of this problem, namely, the planted quasi-clique. We propose a novel convex formulation for the planted quasi-clique recovery. We adopt techniques from the matrix decomposition to split the adjacency matrix of the given graph into its low rank and sparse component. We were inspired by the work in [4] where a matrix completion strategy was used for planted maximum clique recovery. Planted clique is a well known problem that has been studied in \([3, 4, 21, 28]\). To the best of our knowledge, this paper presents the first attempt to solve the planted maximum γ-clique problem. Our numerical experiments show that this approach is more robust than the nuclear norm formulation of [4] and more effective for planted quasi-clique recovery than the mixed integer programming models of [39, 47].

The rest of this paper is organised as follows. We present the nuclear norm formulation for the planted clique and planted quasi-clique problem in Section 2. We briefly present the mixed integer programming formulations for maximum quasi-clique problem in Section 4. The report of our numerical experiments is presented in Section 5 while the concluding remarks are made in Section 6.

2 The planted quasi-clique model

An instance of the MCP is the planted (hidden) clique problem. The problem can be formulated in two different ways namely: the randomized case and the adversarial case. For the randomized case, \( n_c \) vertices are chosen at random from \( n \) vertices \((n > n_c)\) and a clique of size \( n_c \) is constructed. The remaining pairs of nodes are then connected depending on a given probability. For the adversarial case, on the contrary, instead of joining the diversionary edges in a probabilistic manner, an adversary is allowed to join the edges. A restriction is placed on the maximum number of edges he can insert so that a clique bigger than the planted clique is not formed. For our planted quasi-clique problem, we have only considered the randomized case. We have formulated the problem using two probabilities, namely: \( p, p \geq \gamma \), is the probability of an edge existing between two nodes belonging
to the planted quasi-clique while \( \rho \) is the probability of an edge between the nodes not belonging to the planted quasi-clique. So, summarily, we generate a graph of size \( n \) and select \( n_c \) nodes randomly (\( n_c < n \)) and connect them with probability \( p \). The remaining \( n - n_c \) nodes form the diversionary nodes as they are connected with probability \( \rho < p \). The smaller the value of \( p \) the more difficulty it is to recover the planted quasi-clique. Conversely, as \( \rho \) grows bigger, tending towards 0.5, the harder it is to recover the planted quasi-clique.

The planted clique problem has previously been studied in \([3, 19, 21, 28]\). The following nuclear norm minimization formulation has been recently proposed for solving the planted clique problem \([4]\):

\[
\begin{align*}
\text{min} & \quad ||X||_*, \\
\text{subject to} & \quad \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n_c^2, \\
X_{ij} & = 0 \forall (i, j) \notin E \text{ and } i \neq j, \\
X & = X^T, \\
X_{ij} & \in [0, 1],
\end{align*}
\]

where \( X \in \mathbb{R}^{n \times n} \) and \( n_c \) is the size of the planted clique. The nuclear norm is defined as \( ||X||_* := \sigma_1(X) + \sigma_2(X) + \ldots + \sigma_r(X) \), where \( \sigma_i(X), i \in \{1, \ldots, r\} \), are the singular values and \( r \) is the rank of the matrix. We denote model (1) as NNM(1) (nuclear norm based model 1).

In our case, we adopt the technique from matrix decomposition \([14]\) to recover the planted quasi-clique in a graph. The planted quasi-clique problem is a more difficult problem than the planted clique problem, as the latter is a special case of the former. The matrix decomposition problem is described as follows. A matrix, \( M \), is formed by adding a low rank matrix, \( L \), to a sparse matrix, \( S \). The objective is to devise a mean to separate \( M \) into its low rank and sparse component. Mathematically, we want to solve:

\[
\begin{align*}
\text{min} & \quad \text{rank}(L) + ||S||_0, \\
\text{subject to} & \quad L + S = M,
\end{align*}
\]

where \( ||S||_0 = \text{card}(S) \) is the number of non-zero entries of \( S \). Both the rank function and \( l_0 \) minimization are non-convex. However, nuclear norm minimization gives a good approximation of the rank minimization problem \([11]\). Furthermore, the matrix \( l_1 \) norm, defined as \( ||X||_1 = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |X_{ij}| \) for \( X \in \mathbb{R}^{n_1 \times n_2} \), is a good replacement for the cardinality minimization problem \([20]\). Hence, problem (2) can be written as

\[
\begin{align*}
\text{min} & \quad ||L||_* + ||S||_1, \\
\text{subject to} & \quad L + S = M.
\end{align*}
\]

This problem has applications in facial recognition and image segmentation \([14]\). The problem has been studied in \([14, 15, 16]\), with application to facial recognition in \([14]\). We apply this technique to planted quasi-clique recovery. Our proposed
formulation is the following:

\[
\begin{align*}
\min & \quad ||Q||_* + \lambda||D||_1 \\
\text{subject to} & \quad \sum_i \sum_j Q_{ij} \geq \gamma \eta^2 \\
& \quad Q + D = A \\
& \quad Q_{ij}, D_{ij} \in [0, 1], \quad \eta \in \mathbb{N},
\end{align*}
\tag{4a}
\]

where \(Q, D \in \mathbb{R}^{n \times n}\) are matrix variables corresponding to the quasi-clique and the diversionary edges, \(A\) is the adjacency matrix of the input graph while the parameter \(\gamma \in (0, 1]\) is the desired edge density of the quasi-clique to be recovered.

The constraint (4b) ensures that the solution satisfies the edge density requirement, while (4c) makes sure that the decomposition agrees with the input matrix. \(\eta\) is a positive integer value variable that determines the size of the recovered quasi-clique.

Since we are only interested in \(Q\), we can eliminate constraint (4c) and write \(D = A - Q\). Therefore, (4) can be reformulated as

\[
\begin{align*}
\min & \quad ||Q||_* + \lambda||A - Q||_1 \\
\text{subject to} & \quad \sum_i \sum_j Q_{ij} \geq \gamma \eta^2 \\
& \quad Q_{ij} \in [0, 1], \quad \eta \in \mathbb{N}
\end{align*}
\tag{5a}
\]

We denote model (5) as NNM(5). Following the approach in Appendix A of [15], the semidefinite (SDP) formulation for (5) is the following:

\[
\begin{align*}
\minimize & \quad \frac{1}{2}(\text{trace}(Z_1) + \text{trace}(Z_2)) + \lambda 1_n^T W 1_n, \\
\text{subject to} & \quad \begin{bmatrix} Z_1 & Q \\ Q^T & Z_2 \end{bmatrix} \succeq 0, \\
& \quad -W_{ij} \leq A_{ij} - Q_{ij} \leq W_{ij}, \quad \forall (ij), \\
& \quad \sum_i \sum_j Q_{ij} \geq \gamma \eta^2, \\
& \quad Q_{ij} \in [0, 1], \quad \eta \in \mathbb{N},
\end{align*}
\tag{6}
\]

where \(1_n \in \mathbb{R}^n\) is an \(n\)-dimensional vector of all entries equal to one and \(Z_1, Z_2, W \in \mathbb{R}^{n \times n}\).

Problems (4) and (5) are convex optimization problems that can be solved using one of the available convex optimization solvers.

### 3 Illustrative Example

Suppose the input graph, \(G\), containing the planted quasi clique is the graph presented in Figure 1 and that we want to recover the planted 0.9-clique from it. \(A^*\) is the adjacency matrix of \(G\). We add a loop to every node of \(G\) to obtain \(A^*\).
as the adjacency matrix. This is necessary for the algorithm to be able to recover a low rank submatrix.

\[
A^* = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\quad A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The matrix decomposition algorithm will perform two tasks, namely; completion of the low-rank matrix and separation of the low-rank matrix from the sparse matrix. However, since we are only interested in the low-rank submatrix, we have reformulated the model to suite this purpose. The reformulation has improved the performance of the algorithm in terms of speed. Therefore, for this particular example, we recover \( Q \) as the largest rank-one matrix and \( \eta = 5 \) in this case. This corresponds to the adjacency matrix of the recovered maximum clique. The adjacency matrix of the planted maximum quasi-clique, \( Q^* \), can then finally be obtained by setting \( Q_{ij} = 0 \) if \( Q_{ij} \neq A^*_{ij} \).

\[
Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\quad Q^* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
Fig. 1: A graph $G$, with the planted quasi-clique using nodes $\{3, 4, 5, 6, 7\}$

### 4 Existing Formulations for Maximum Quasi-Clique Problem

As stated earlier, majority of the existing works on $\gamma$-clique focused on developing heuristics for detection of large quasi-clique. The first deterministic solution for $\gamma$-clique recovery problem is based on the linear mixed integer programming model suggested in [39], where an upper bound was derived. For $i \in V$, define $x_i \in \{0, 1\}$ such that $x_i = 1$ if and only if $i \in V'$ and 0 otherwise, where $V'$ is the vertex set of the maximum quasi-clique. The following linearized MIP formulation was proposed:

$$\omega_{\gamma} = \max \sum_{i \in V} x_i,$$

subject to:

$$\sum_{i \in V} h_i \geq 0$$

$$h_i \leq \nu x_i, \quad h_i \geq -\nu x_i \quad \forall \ i \in V,$$

$$h_i \geq \gamma x_i + \sum_{j \in V} (A_{ij} - \gamma) x_j - \nu (1 - x_i) \quad \forall \ i \in V,$$

$$h_i \leq \gamma x_i + \sum_{j \in V} (A_{ij} + \gamma) x_j - \nu (1 - x_i) \quad \forall \ i \in V,$$

$$x_i \in \{0, 1\} \quad \forall \ i \in V,$$

where, $A_{ij}$ are the entries of the adjacency matrix of the graph, $\nu$ is a constant that is large enough and $h_i$ is defined as

$$h_i = x_i (\gamma x_i + \sum_{(i,j) \in E} (A_{ij} - \gamma) x_j).$$
can only handle problems with small graph size. Because of this drawback, Veremyev et al. [47] reformulate this model by defining \( z_{ij} \) as a binary variable, such that \( z_{ij} = 1 \) if and only if \((i, j) \in E \cap (V' \times V')\). In addition, a binary variable \( s_t, t = 1, \ldots, |V'| \), which determines the size of the quasi-clique is defined. This implies that \( s_t = 1 \) if and only if \(|V'| = t\). With these additional variables and notations, the improved MIP model presented in [47] is:

\[
\begin{align*}
\text{max} & \quad \sum_{i \in V} x_i, \\
\text{subject to} & \quad \sum_{(i,j) \in E} z_{ij} \geq \gamma \sum_{t=\omega_l}^{\omega_u} \frac{t(t-1)}{2} s_t, \\
& \quad z_{ij} \leq x_i, \quad z_{ij} \leq x_j, \quad \forall (i, j) \in E, \\
& \quad \sum_{i \in V} x_i = \sum_{t=\omega_l}^{\omega_u} t s_t, \quad \sum_{t=\omega_l}^{\omega_u} s_t = 1, \\
& \quad x_i \in \{0, 1\}, \quad z_{ij} \geq 0, \quad \forall i, j \in V, i < j, \\
& \quad s_t \geq 0, \quad \forall t \in \{\omega_l, \ldots, \omega_u\},
\end{align*}
\]

where \( \omega_l \) and \( \omega_u \) are the upper and lower bound on the size of quasi-clique that could be found in the input graph. These can be set to 0 and \( |V'| \) respectively if no estimates are available. The constraint (9b) is the edge density requirement while (9c) ensures that \( z_{ij} = 1 \) if and only if \( i \) and \( j \) belong to the quasi-clique. Observe that the left hand side of (9b) can be written as

\[
\sum_{(i,j) \in E} z_{ij} = 1/2 \sum_{i \in V} \sum_{j: (i,j) \in E} x_i x_j = 1/2 \sum_{i \in V} \left( x_i \sum_{j: (i,j) \in E} x_j \right).
\]

Setting \( w_i \) to the quantity in the bracket in equation (10) above, (9) can be reformulated as [47]:

\[
\begin{align*}
\text{max} & \quad \sum_{i \in V} x_i, \\
\text{subject to} & \quad \sum_{i \in V} w_i \geq \gamma \sum_{t=\omega_l}^{\omega_u} t(t-1) s_t, \\
& \quad w_i \leq \psi_i x_i, \quad w_i \leq \sum_{j: (i,j) \in E} x_j, \quad \forall i \in V, \\
& \quad \sum_{i \in V} x_i = \sum_{t=\omega_l}^{\omega_u} t s_t, \quad \sum_{t=\omega_l}^{\omega_u} s_t = 1, \\
& \quad x_i \in \{0, 1\}, \quad z_{ij} \geq 0, \quad \forall i, j \in V, i < j, \\
& \quad s_t \geq 0, \quad \forall t \in \{\omega_l, \ldots, \omega_u\},
\end{align*}
\]

where \( \psi_i \) is a parameter that is sufficiently large. In particular, \( \psi_i = \deg_G(i) \), where \( \deg_G(i) \) is the degree of a node \( i \) in a given graph, \( G \).
5 Numerical Experiments

Recall that the planted quasi-clique problem becomes planted clique problem when \( \gamma \) is equal to one. We performed numerical experiments with our nuclear norm minimization (NNM) formulation (5) for planted maximum quasi-clique to compare its performance with the existing nuclear norm minimization formulation (1) for planted maximum clique recovery. Further, we compare the efficacy of our formulation with the mixed integer programming models (7), (9) and (11) for quasi cliques.

The experiments have been performed on a HP computer with 16GB Ram and Intel core i7 processor. The machine runs on Debian Linux. The simulations are performed using CVXPY \[17\] with NCVX \[18\]. CVXPY is a python package used to solve convex optimization problems with different solvers, e.g SCS, CVXOPT, and XPRESS. Every instance of the experiment has been carried out ten times and the average result is taken. We have used different values of the regularization parameter, \( \lambda \), to ascertain that our choice of \( \lambda \) works well for the problem. We have planted a quasi-clique with \( \omega_\gamma(G) = 35 \) for various values of \( \gamma \) in a graph with 50 nodes. We implement our algorithm for \( \lambda = n, \frac{1}{\sqrt{n}}, \frac{1}{2\sqrt{n}}, \frac{1}{n} \). The results is presented in Table 1 and Figure 2. From Table 1, we discover that when \( \lambda = n \) and \( \frac{1}{n} \), the algorithm fails in all instances considered. This finding has been supported by the relative errors in Figure 2, where the relative errors have been calculated using (12).

\[
\text{Relative Error} = \frac{||\text{recovered } \gamma\text{-clique} - \text{planted } \gamma\text{-clique}||_F}{||\text{planted } \gamma\text{-clique}||_F},
\]

(12)

where \( F \) denotes the Frobenius norm. This is contrary to what can be observed when \( \lambda = \frac{1}{\sqrt{n}} \) and \( \frac{1}{2\sqrt{n}} \). For these values, the exact size of planted quasi-clique has been recovered with zero relative error when \( \gamma \) approaches 1 (\( \gamma \to 1 \)). From this results, we conclude that values of \( \frac{1}{\sqrt{n}} \leq \lambda \leq \frac{1}{2\sqrt{n}} \) will work for our model. This is similar to the recommendation in \[14\]. This is not surprising since the entries of our matrix are also independent and identically distributed (iid) and hence satisfy the incoherence condition (see \[12, 13, 14\]). Chandrasekaran et al. \[15\] also contains a heuristic for choosing \( \lambda \) and our finding agrees with their result, although our approach is different. The detailed report of the experiments is as follows.

| \( \gamma \) | Size of the planted quasi-clique | \( \lambda = n \) | \( \frac{1}{\sqrt{n}} \) | \( \frac{1}{2\sqrt{n}} \) | \( \frac{1}{n} \) |
|---|---|---|---|---|---|
| 0.5 | 35 | 50 | 22.7 | 0 | 0 |
| 0.55 | 35 | 50 | 31 | 0 | 0 |
| 0.6 | 35 | 50 | 33.9 | 0 | 0 |
| 0.65 | 35 | 50 | 34.5 | 12.1 | 0 |
| 0.7 | 35 | 50 | 34.5 | 29.9 | 0 |
| 0.75 | 35 | 50 | 34.7 | 34.6 | 0 |
| 0.8 | 35 | 50 | 35 | 35 | 0 |
| 0.85 | 35 | 50 | 35 | 35 | 0 |
| 0.9 | 35 | 50 | 35 | 35 | 0 |
| 0.95 | 35 | 50 | 35 | 35 | 0 |
| 1 | 35 | 50 | 35 | 35 | 0 |

Table 1: Quasi-clique recovery for different values of \( \lambda \).
5.1 Comparison between NNM(1) and NNM(5)

Our model, NNM(5), represents the planted maximum clique model when $\gamma = 1$. Hence, we compare its performance with NNM(1). We have considered a graph with 100 nodes in this experiment. We have planted clique of size 80 and 50 and then varied the probability, $\rho$, of an edge existing between the remaining nodes. The results of this experiment is contained in Figure 3. In both cases considered, the results show that both (1) and (5) recover planted clique perfectly when the probability of adding a diversionary edge is below certain threshold (roughly 0.45). However, (5) fails to perfectly recover the planted clique when this threshold is exceeded while (1) still solves the problem perfectly. We have observed that the presence of constraint (1c) enables finding the largest rank-one submatrix in the input matrix easier in the formulation (1). However, this constraint can not be imposed in the case of (5), otherwise, solving planted quasi-clique problem with the formulation will be impossible. Figure 4 represents the CPU times for NNM(1) and NNM(5) with planted clique of size $\omega_\gamma(G) = 50$ and 80 and $\gamma = 1$. It can be observed from the figures that NNM(1) is more efficient than NNM(5) in this case.

Figure 5 shows the performance of NNM(1) compared with NNM(5) in finding quasi-clique (with $\gamma < 1$). We have observed from Figure 5a that despite the fact that the planted quasi-clique that we have considered for this case has very few missing edges ($\gamma = 0.99$), NNM(1) failed to recover the quasi-clique for every trial. However, NNM(5) produced similar result as the case $\gamma = 1$ (see Figure 3b and 5a). In addition, NNM(5) is, by far, more efficient than NNM(1) for the case $\gamma = 0.99$ (see Figure 5b).
(a) Probability of planted clique recovery with clique size = 50  
(b) Probability of planted clique recovery with clique size = 80

Fig. 3: Planted clique recovery from a graph with 100 nodes with varied probability of adding a diversionary edge ($\rho$).

(a) CPU time comparison with clique size = 50  
(b) CPU time comparison with clique size = 80

Fig. 4: CPU time comparison for planted clique recovery from a graph with 100 nodes with varied number of diversionary edges.

(a) Recovery probability of planted quasi-clique with $\gamma = 0.99$  
(b) CPU time comparison for planted quasi-clique

Fig. 5: Comparison of the recovery probability and CPU time of NNM for planted quasi-clique.
5.2 Maximum Quasi-Clique Recovery

5.2.1 The planted case

Two types of experiment have been performed in this case. In the first case, we have checked whether the recovered quasi-clique satisfies the edge density requirement or not. The second experiment focuses on the size of the recovered quasi-clique, i.e., to examine whether the size of the planted quasi-clique \((n_c)\) is the same as the size of the recovered quasi-clique \((\eta)\). The detailed report of both experiments is as follows.

The goal of the first experiment is to examine the error in the edge density of the recovered \(\gamma\)-clique with respect to the edge density of the planted maximum \(\gamma\)-clique. We have computed the relative error between the edge density of the recovered \(\gamma\)-clique and the edge density of the planted \(\gamma\)-clique (i.e., the expected edge density) for various \(\gamma\). All the errors computed in this section are relative errors.

We have considered again graphs with 50 and 100 nodes for this case with planted \(\gamma\)-cliques of sizes 40 and 80, respectively. The planted \(\gamma\)-clique corresponds to a dense submatrix of the 50 \(\times\) 50 (respectively, 100 \(\times\) 100) input matrix with 40 (respectively, 80) nonzero rows/columns. We have varied the edge density of the planted \(\gamma\)-clique by setting \(p = 0.6, 0.65, 0.7, \ldots, 1\). The probability, \(p\), determines whether an edge will exist between two nodes in the planted quasi-clique. The smaller the \(p\), the fewer the edges and consequently, the more difficult it is to recover what is planted. The setup follows the Stochastic Block Model (SBM) [29]. Detail is as follows. For the case \(n = 50\), we generate a 50 \(\times\) 50 symmetric matrix, \(M\), with zero entries. We choose a 40 \(\times\) 40 submatrix of this matrix and assign 1 to its indices with probability 0.6 (suppose \(p = 0.6\)), using Bernoulli trial. This forms the dense component of the input matrix (the planted \(\gamma\)-clique). The entries of the remaining 10 rows and columns are also assigned values 1 but with a much smaller probability, (say \(\rho = 0.2\)). This forms the sparse component of the matrix (or the random noise). The goal is to recover the dense submatrix from the input matrix. The results of these experiments are reported in Tables 2 and 3. In both Tables, columns 2 – 4 contain errors in edge density of the planted quasi-cliques recovered using the MIP models (7), (9) and (11) while column 5 contains the errors in edge density of the \(\gamma\)-clique recovered using our nuclear norm minimization approach, NNM(5). The relative error here shows the disparity in the densities of what is planted and what is recovered. If these edge densities coincide, i.e., if the edge densities of what is planted and the recovered quasi-clique are equal, the relative error with respect to the Frobenius norm will be zero. When \(n = 50\), MIP(7) performed better than the two other MIP models for all values of \(\gamma\). However, our model NNM(5) has exhibited the best performance when \(\gamma \geq 0.75\). For graphs with 100 nodes (see Table 3), MIP(11) performed better than other MIP models except for when \(\gamma\) is equal to 0.75, 0.95 and 1 where MIP(7) has shown better performances. Nevertheless, when \(\gamma \geq 0.7\), NNM(5) outperformed all the mixed integer programs. One can also infer from Tables 2 and 3 that as the graph size increases, the lower bound on \(\gamma\) for perfect recovery by NNM(5) decreases. Figure 6 shows the CPU time for each of the methods for the experiments reported in Tables 2 and 3. Our off-the-shelf solver, splitting conic solver (SCS) [54], is faster than the popular SDP solvers like SeDumi [43] and SDP3 [46]. However, it is not
as efficient as the well-developed FICO XPRESS optimizer used to solve the MIP models. Nonetheless, as $\gamma$ increases, there is a drastic drop in the CPU time for NNM(5) in both instances.

Table 2: Errors in the edge density of the planted maximum $\gamma$-clique recovery for a graph with 50 nodes

| $\gamma$ | MIP(7)  | MIP(9)  | MIP(11) | NNM(5)  |
|----------|---------|---------|---------|---------|
| 0.6      | 0.0922  | 0.1279  | 0.2093  | 0.3085  |
| 0.65     | 0.0688  | 0.1607  | 0.1897  | 0.2035  |
| 0.7      | 0.0809  | 0.1646  | 0.2086  | 0.1170  |
| 0.75     | 0.0809  | 0.1933  | 0.1358  | 0.0230  |
| 0.8      | 0.0876  | 0.2044  | 0.1526  | 0       |
| 0.85     | 0.02    | 0.2263  | 0.1299  | 0       |
| 0.9      | 0.0215  | 0.2234  | 0.1806  | 0       |
| 0.95     | 0       | 0.2245  | 0.2026  | 0       |
| 1        | 0       | 0.2236  | 0.2434  | 0       |

Table 3: Errors in the edge density of the planted maximum $\gamma$-clique recovery for a graph with 100 nodes

| $\gamma$ | MIP(7)  | MIP(9)  | MIP(11) | NNM(5)  |
|----------|---------|---------|---------|---------|
| 0.6      | 0.0916  | 0.0736  | 0.054   | 0.2424  |
| 0.65     | 0.0892  | 0.0843  | 0.0492  | 0.1012  |
| 0.7      | 0.0879  | 0.0719  | 0.0634  | 0.0131  |
| 0.75     | 0.0879  | 0.1378  | 0.0905  | 0       |
| 0.8      | 0.0829  | 0.1001  | 0.0766  | 0       |
| 0.85     | 0.0783  | 0.1564  | 0.0693  | 0       |
| 0.9      | 0.0817  | 0.1144  | 0.0975  | 0       |
| 0.95     | 0.0745  | 0.1342  | 0.1376  | 0       |
| 1        | 0.0694  | 0.1581  | 0.1717  | 0       |

The second experiment was to find out if the number of nodes in the planted quasi-cliques, $n_c$, is the same as the number of nodes in the recovered quasi-cliques, $\eta$. For this experiment, we have considered graphs of sizes $n = 50, 100, \ldots, 250$ and $\gamma = 0.6, 0.7, \ldots, 1$. We chose the size of the planted quasi clique, $n_c$, to be $0.8 \times n$. We have again run the experiment 10 times for each case and averaged the recovered quasi-clique size. The results obtained are presented in Table 4. The first column under each method contains the average size of recovered quasi-clique using the method while the second contains the relative error for the method. We compute relative error in this case using

$$\frac{|\text{size of the recovered quasi-clique} - \text{size of the planted quasi-clique}|}{|\text{size of the planted quasi-clique}|}.$$  

Clearly, if the size of the recovered quasi-clique is equal to the size of the planted quasi-clique, this error will be equal to zero. As shown in the last column of Table
the relative errors in the size of quasi-clique recovered via NNM(5) are all zero since $n_c = \eta$ throughout. This shows that the convex formulation always returns correct planted quasi-clique size. MIP(7) has the overall worst performance in this experiment. Based on the results in Table 2 and 4, when $\gamma > 0.75$, $n_c = \eta$ and the error in edge density is equal to zero. This implies that our convex formulation perfectly recovers maximum planted quasi-clique when $\gamma > 0.75$ for $n \geq 50$ and $n_c$ large enough.

5.2.2 Recovery from random graphs

Our last experiment focuses on checking the performance of our model in a scenario that mirrors real-life situation. It has been observed that real networks obey some scaling laws rather than being completely random. Hence, the well-known Erdos Renyi random graph, where edges are generated with a constant probability with degree distribution following a Poisson law, may not be suitable. Hence, we have generated our random graph using the preferential attachment model of Barabasi-Albert [8]. The degree distribution of these graphs follow power-law. In this setting, the rate, $\Pi(k)$, with which a node with $k$ edges acquires new edges is a monotonically increasing function of $k$. The time evolution of the degree $k_i$ of node $i$ can be obtained from the first-order ordinary differential equation [25]:

$$\frac{dk_i}{dt} = m\Pi(k_i),$$

(13)

where $m$ is a constant; it is the number of edges to attach from a new node to the existing nodes. We have considered graphs with 50 and 100 nodes with $m$ set to 15 and 30, respectively. The results of this experiment are presented in Table 4. From Table 4(a) and 4(b), it can be observed that MIP(7) returns the largest quasi-clique while our NNM(5) returns quasi-cliques with the smallest size. However, for $\gamma \geq 0.8$, our formulation and MIP(11) return similar results. Recall, from the first experiment of Section 4.2.1, that the recovery error of our formulation is zero for $\gamma \geq 0.8$. Unfortunately, since the quasi-cliques in this case have not been planted, computing the error in the recovered quasi-clique is not straight-forward.
Table 4: Errors in the size of planted maximum γ-clique recovered using different methods for γ ranging from 0.6 to 1. n is the graph size while \( n_c \) is the size of the planted γ-clique.

| \( \gamma \) | Average Recovered Quasi-clique size/Relative Error |
|-------------|---------------------------------|
| \( \gamma = 0.6 \) |                     |
| n  | \( n_c \) | MIP(7) | MIP(11) | NN M(5) |
| 50 | 40 | 41 | 0.025 | 40.4 | 0.01 | 40.8 | 0.02 | 40 | 0 |
| 100 | 80 | 81.7 | 0.023 | 80.8 | 0.01 | 80.8 | 0.01 | 80 | 0 |
| 150 | 120 | 122.7 | 0.023 | 121 | 0.008 | 120.6 | 0.005 | 120 | 0 |
| 200 | 160 | 165.4 | 0.021 | 161.6 | 0.01 | 160.9 | 0.006 | 160 | 0 |
| 250 | 200 | 204.4 | 0.022 | 201.9 | 0.01 | 200.7 | 0.003 | 200 | 0 |
| \( \gamma = 0.7 \) |                     |
| 50 | 40 | 40.5 | 0.013 | 40.1 | 0.003 | 40.4 | 0.01 | 40 | 0 |
| 100 | 80 | 81.3 | 0.016 | 80.4 | 0.005 | 80.5 | 0.006 | 80 | 0 |
| 150 | 120 | 122.4 | 0.02 | 121 | 0.008 | 120.7 | 0.006 | 120 | 0 |
| 200 | 160 | 163 | 0.019 | 161 | 0.006 | 160.6 | 0.004 | 160 | 0 |
| 250 | 200 | 204 | 0.02 | 201.5 | 0.008 | 200.4 | 0.002 | 200 | 0 |
| \( \gamma = 0.8 \) |                     |
| 50 | 40 | 40.2 | 0.005 | 40 | 0 | 40.5 | 0.013 | 40 | 0 |
| 100 | 80 | 81 | 0.025 | 80.2 | 0.003 | 80.2 | 0.003 | 80 | 0 |
| 150 | 120 | 123 | 0.025 | 120.5 | 0.004 | 120.3 | 0.002 | 120 | 0 |
| 200 | 160 | 163.2 | 0.02 | 161.2 | 0.007 | 160.4 | 0.003 | 160 | 0 |
| 250 | 200 | 204 | 0.02 | 201.3 | 0.007 | 200.4 | 0.002 | 200 | 0 |
| \( \gamma = 0.9 \) |                     |
| 50 | 40 | 40.6 | 0.015 | 40 | 0 | 40.5 | 0.013 | 40 | 0 |
| 100 | 80 | 81.1 | 0.014 | 80.2 | 0.003 | 80.2 | 0.003 | 80 | 0 |
| 150 | 120 | 122 | 0.017 | 120.3 | 0.002 | 120.1 | 0.001 | 120 | 0 |
| 200 | 160 | 163 | 0.019 | 160.9 | 0.006 | 160.3 | 0.002 | 160 | 0 |
| 250 | 200 | 203.9 | 0.02 | 200.9 | 0.005 | 200.1 | 0 | 200 | 0 |
| \( \gamma = 1 \) |                     |
| 50 | 40 | 40 | 0 | 40 | 0 | 40 | 0 | 40 | 0 |
| 100 | 80 | 81 | 0.013 | 80 | 0 | 80 | 0 | 80 | 0 |
| 150 | 120 | 122 | 0.017 | 120.3 | 0.002 | 120 | 0 | 120 | 0 |
| 200 | 160 | 163 | 0.019 | 160.9 | 0.006 | 160 | 0 | 160 | 0 |
| 250 | 200 | 204 | 0.02 | 201 | 0.005 | 200 | 0 | 200 | 0 |

MIP(7) has the worst performance in terms of CPU time for this experiment while MIP(11) has the best performance of the three formulations compared (see Figure 7). Also, both MIP(11) and NNM(5) show no significance difference in CPU time for various value of \( \gamma \).

Table 5: Quasi-clique recovery from random graph

(a) Quasi-clique recovery from a power-law graph with \( n = 50 \) and \( m = 15 \).

| \( \gamma \) | MIP(7) | MIP(11) | NNM(5) |
|-------------|--------|--------|--------|
| 0.6 | 38 | 39 | 35 |
| 0.7 | 34 | 35 | 32 |
| 0.8 | 32 | 30 | 29 |
| 0.9 | 30 | 29 | 28 |
| 1 | 29 | 27 | 27 |

(b) Quasi-clique recovery from a power-law graph with \( n = 100 \) and \( m = 30 \).

| \( \gamma \) | MIP(7) | MIP(11) | NNM(5) |
|-------------|--------|--------|--------|
| 0.6 | 76 | 78 | 68 |
| 0.7 | 68 | 69 | 65 |
| 0.8 | 64 | 63 | 64 |
| 0.9 | 61 | 58 | 59 |
| 1 | 58 | 53 | 54 |
6 Conclusion

We have studied the planted quasi-clique problem in this paper. We have considered a matrix decomposition type of mathematical formulation for the problem. We have used this formulation to solve the planted maximum quasi-clique problem. We have shown, experimentally, the range of values of the regularization parameter, $\lambda$, that works for the model. We have numerically established the superiority of our formulation over the nuclear norm minimization model in [4] by solving a wider range the problem and the three existing mixed integer programming formulations in terms of effectiveness. Our future research will be to establish the theoretical guarantee for perfect recovery and providing a bound on $\gamma$ for which recovery is guaranteed. There are some special algorithms developed for nuclear norm minimization and low-rank plus sparse matrix recovery like the iterative singular value thresholding [11], accelerated proximal gradient [45] and the alternating direction method [49, 22]. It will be interesting to implement these algorithms for planted quasi-clique recovery to compare their performances with the SCS used for this work. Lastly, there is no theory to explain why the values of $\lambda$ that work do. It will be interesting have a better understanding of why they do.

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