Escobar type theorems for elliptic fully nonlinear degenerate equations

D. P. Abanto, J. M. Espinar

American Journal of Mathematics, Volume 141, Number 5, October 2019, pp. 1179-1216 (Article)

Published by Johns Hopkins University Press

DOI: https://doi.org/10.1353/ajm.2019.0030

For additional information about this article
https://muse.jhu.edu/article/732652/summary
Abstract. In this paper we prove non-existence and classification results for elliptic fully nonlinear elliptic degenerate conformal equations on certain subdomains of the sphere with prescribed constant mean curvature along its boundary. We also consider non-degenerate equations. Such subdomains are geodesic balls in $S^m$, punctured balls and annular domains.

Our results extend those of Escobar when $m \geq 3$, and Hang-Wang and Jiménez when $m = 2$.

1. Introduction. Let $(\mathcal{M}^m, g_0)$ be a closed orientable Riemannian manifold of dimension $m \geq 3$. The following problem is called the Yamabe Problem:

Is there a conformal metric $g = e^{2\rho}g_0$ such that the scalar curvature with respect to this metric is constant?

It was solved by a sequence of works, beginning with H. Yamabe, Trudinger, Aubin and Schoen (cf. [33] and references therein). When the manifold $(\mathcal{M}, g_0)$ is complete but not compact, the existence of a conformal metric solving the Yamabe Problem does not hold in general, as we can see in the work of J. Zhiren [24].

When $(\mathcal{M}^m, g_0)$ is a compact orientable Riemannian manifold with boundary, $m \geq 3$, an analogous problem was proposed:

Is there a conformal metric $g = e^{2\rho}g_0$ such that the scalar curvature with respect to this metric is constant and the boundary has constant mean curvature?

Almost all the cases were solved in the works of J. Escobar [12, 13, 14, 15], continued by F. Marques [32] among others. We will refer to this problem as the Escobar Problem. By far, the most important case is when $\mathcal{M}$ is the closed unit Euclidean ball or, equivalently, the closed hemisphere $S^m_+$ endowed with the standard round metric $g_0$.

Regarding to the existence of solutions, J. Escobar proved that given a bounded domain in $\mathbb{R}^m$, $m > 6$, with smooth boundary, there exists a conformal metric $g$ to the standard Euclidean metric with zero scalar curvature and constant positive
mean curvature along its boundary with respect to $g$ [13]. Also, he showed existence when $g$ has non-zero scalar curvature and minimal boundary for $m \geq 3$ [14]. The existence of solutions to the Yamabe problem on noncompact manifolds $(\mathcal{M}, g_0)$ with compact boundary was proved for a large class of manifolds in the work of F. Schwartz. He proved that the Riemannian manifolds that are positive and their ends are large have a conformal metric of zero scalar curvature and constant mean curvature on its boundary (see [35] for details).

In terms of classification of solutions, J. Escobar showed that the solution to the Escobar Problem must have constant sectional curvatures when $(\mathcal{M}, g_0)$ is the closed Euclidean ball $\mathbb{B}^m$. Also, he proved that the space of solutions in the Euclidean ball is empty when the scalar curvature is zero and the mean curvature is a non-positive constant [12]. This result is generalized by Theorem 3.1.

The Yamabe Problem opened the door to a rich subject in the last few years: the study of conformally invariant equations. More precisely,

Given a smooth functional $f(x_1, \ldots, x_m)$, does there exist a conformal metric $g = e^{2\rho} g_0$ in $\mathcal{M}$ such that the eigenvalues $\lambda_i$ of its Schouten tensor satisfy $f(\lambda_1, \ldots, \lambda_m) = c$ in $\mathcal{M}$?

Let $(\mathcal{M}^m, g)$, $m \geq 3$, be Riemannian manifold, the Schouten tensor of $g$ is given by

$$\text{Sch}(g) := \frac{1}{m-2} \left( \text{Ric}(g) - \frac{R(g)}{2(m-1)} g \right)$$

where $\text{Ric}(g)$ and $R(g)$ are the Ricci tensor and the scalar curvature function of $g$ respectively. Note that $f(x_1, \ldots, x_m) = x_1 + \cdots + x_m$ reduces to the Yamabe Problem. For example, a conformal metric with scalar curvature equal to 1 is equivalent to

$$\lambda_1 + \cdots + \lambda_m = \frac{1}{2(m-1)}, \text{ in } \mathcal{M},$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of the Schouten tensor. It is of special interest to consider $f(\lambda) \equiv \sigma_k(\lambda)^{1/k}$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, where $\sigma_k(\lambda)$ is the $k$-th elementary symmetric polynomial of its arguments $\lambda_1, \ldots, \lambda_m$ and set it to be a constant, i.e., $\sigma_k(\lambda) = \text{constant}$, such problem is known as the $\sigma_k$-Yamabe Problem [36, 37]. This is an active research topic and has interactions with other fields as Mathematical General Relativity [4, 21]. Interesting problems arose in this context of conformally invariant equations. One of them is the classification of conformal metrics satisfying a Yamabe type equation on a subdomain of the sphere, in the line of Y. Y. Li and collaborators [25, 26, 27, 28, 29, 30]. Also, it is interesting to find non-trivial solutions to conformal metrics on subdomains of the sphere prescribing the scalar curvature in the interior, or other elliptic combination of the Schouten tensor, and the mean curvature along the boundary. Such problem is related to the Min-Oo conjecture when we consider the scalar curvature inside. S. Brendle, F. C.
Marques and A. Neves [5] showed the existence of such non-trivial metric in the hemisphere, however such metric is not conformal to the standard one. In other words, could one find conditions on the interior and the boundary that imply that such conformal metric is unique (cf. [30, 38])? In this work we will focus in the case $\mathcal{M}$ is a subdomain of the $m$-dimensional sphere $\mathbb{S}^m$. The definition of elliptic data in order to define conformally invariant elliptic equations will be explicitly given in subsection 2.6.

M. P. Cavalcante and J. M. Espinar [6] have shown the following using geometric methods:

**Theorem [6].** If $g = e^{2\rho}g_0$ is a conformal metric in $\overline{\mathbb{S}^m}_+$ that satisfies

\[
\begin{aligned}
  f(\lambda(g)) &= 1, \quad \text{in } \mathbb{S}^m_+,
  \\
  h(g) &= c, \quad \text{on } \partial \mathbb{S}^m_+,
\end{aligned}
\]

then, there is a conformal diffeomorphism $\Phi : \mathbb{S}^m \to \mathbb{S}^m$, preserving $\mathbb{S}^m_+$, such that $g = \Phi^*(g_0|_{\mathbb{S}^m_+})$.

Using analytic methods, A. Li and Y. Y. Li [27] proved the result above. Nevertheless, M. P. Cavalcante and J. M. Espinar went further and they dealt with annular domains, as J. Escobar did [12] for the scalar curvature, in the fully non-linear elliptic case. They classified conformal metrics on the closure of the annulus $A(r) = \{ q \in \mathbb{S}^m : r < d_{\mathbb{S}^m}(q, e_{m+1}) < \frac{\pi}{2} \}, 0 < r < \pi/2$, such that

\[
\begin{aligned}
  f(\lambda(g)) &= 1, \quad \text{in } A(r), \\
  h(g) &= 0, \quad \text{on } \partial A(r).
\end{aligned}
\]

In the above situation, they obtained:

**Theorem [6].** Let $\rho \in C^2,\alpha(\overline{A(r)})$ be a solution to (1). Then, $g = e^{2\rho}g_0$ is rotationally symmetric metric in $\overline{A(r)}$.

In this work we combine the analytic methods of A. Li and Y. Y. Li and the geometric methods of M. P. Cavalcante and J. M. Espinar. Specifically, we use the Maximum Principle for fully non-linear conformally invariant equations of Y. Y. Li and collaborators but we achieve the conditions in order to use those Maximum Principles by geometric ideas. This is important in two ways; on the one hand, we do not need to define elliptic problems for hypersurfaces in the Hyperbolic space as in [3, 6] and, on the other hand, the geometric point of view, in our idea, simplifies to verify the requirements of the Maximum Principle for fully non-linear elliptic functionals.

In the following, let us denote by $\mathbf{n} \in \mathbb{S}^m_+ \subset \mathbb{S}^m$ the north pole and $B_r(\mathbf{n})$ the geodesic ball in $\mathbb{S}^n$ with center $\mathbf{n}$ and radius $r$. 
The paper is organized as follows. To facilitate access, the sections are rendered as self-contained as possible. In Section 2, we first review the local relationship between horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$ and conformal metrics in $\mathbb{S}^m$. We will recall the Local Representation Theorem [17] and Global Representation Theorem [3] in Section 2, which plays a critical role in this work. Such results are of great importance because we can obtain horospherically concave hypersurfaces with injective Gauss map from conformal metrics defined in domains $\Omega$ of the sphere $\mathbb{S}^m$ if one imposes certain natural conditions. The correspondence results were originated in [9, 10, 11].

Section 2 contains all the geometric cornerstone results for horospherically concave hypersurfaces associated to conformal metric will be presented. Our first main result gives necessary and sufficient conditions for such horospherically concave hypersurface to be proper in terms of the associated conformal metric (see Theorem 2.2). Using Theorem 2.2, we can give a condition on a complete conformal metric that guarantees that the associated map is proper (see Theorem 2.5). Next, we define the parallel flow, $\phi_t : \Omega \to \mathbb{H}^{m+1}$ for $t > 0$, of a horospherically concave hypersurface, such flow is defined using the opposite to the canonical orientation of the hypersurface. The horospherical metric of $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is the conformal metric $g_t = e^{2t}g$, where $g$ is the horospherical metric of $\phi$. It is remarkable that the property of properness is invariant under the parallel flow (cf. Proposition 2.6).

Another important issue about horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$ is embeddedness. We will get embeddedness along the parallel flow under certain conditions on the conformal factor (see Theorem 2.9). In this theorem is not assumed that $\rho$ is proper and we use different techniques of those used in [3, Theorem 3.6]. When $\rho$ is proper in Theorem 2.9, then the Schouten tensor of the conformal metric is bounded, so we get a proper hypersurface as in [3].

Also, it is natural to relate analytic conditions along the boundary of a complete conformal metric with boundary (cf. Definition 2.7) and the boundary of the associated horospherically concave hypersurface, this is our next step. We recover in this part of the paper some results in [6]. Finally, we see how the parallel flow affects to elliptic problems for conformal metrics. Hence, we will be ready to prove our main results on conformal metrics by means of horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$. In what follows, we will deal with degenerate and non-degenerate elliptic problems for conformal metrics on either closed balls, punctured balls or compact annuli in the sphere $\mathbb{S}^m$.

In Section 3, we generalize Escobar’s result [12] to fully nonlinear degenerate conformally invariant equations in Theorem 3.1. In fact, Theorem 3.1 can be extended to $m$-dimensional compact, simply-connected, locally conformally flat manifolds $(\mathcal{M}, g_0)$ with umbilic boundary $\partial \mathcal{M}$ and $R(g_0) \geq 0$ on $\mathcal{M}$, using a result of F. M. Spiegel [38] (cf. Theorem 3.2). Also, using [38], we can extend the
result of Cavalcante-Espinhar [6] to locally conformally flat manifolds (cf. Theorem 3.3).

In Section 4, we study the degenerate case in the punctured geodesic ball with minimal boundary. Observe that up to a conformal diffeomorphism acting on \( S^m \) we can consider the punctured geodesic ball as the punctured northern hemisphere, see Theorem 4.1. Moreover, if we assume that there is a solution \( g = e^{2\rho}g_0 \) in \( S^m \setminus \{n\} \) of

\[
\begin{cases}
  f(\lambda(g)) = 0 & \text{in } S^m \setminus \{n\}, \\
  h(g) = 0 & \text{on } \partial S^m,
\end{cases}
\]

such that \( \sigma = e^{-\rho} \) can be extended to a \( C^2 \) function \( \tilde{\sigma} \) on \( S^m \) with \( \tilde{\sigma}(n) = 0 \). Such solution is called punctured solution. In particular, Theorem 4.1 says that a punctured solution is rotationally symmetric. Finally, we study the non-degenerate case in the punctured closed geodesic ball in Theorem 4.5. Theorems 4.1 and 4.5 can be recovered by analytic methods using [25, 26, 27, 29]. Here, we give a different proof using the horospherically concave hypersurface associated to a conformal metric.

Next, in Section 5, we deal with degenerate problems in the compact annulus \( \mathbb{A}(r), 0 < r < \pi/2 \). We first observe that every solution to the degenerate problem in \( \mathbb{A}(r) \) with minimal boundary is rotationally invariant (see Theorem 5.1). Also, if there is a solution to such problem then it is unique up to dilations (see Theorem 5.2).

In Section 6, we focus on different boundary conditions on the annulus. Our next result will say that any conformal metric \( g = e^{2\rho}g_0 \) in

\[ \mathbb{A}(r, \pi/2) := \{ x \in \mathbb{S}^m : r < d_{\mathbb{S}^m}(x, n) \leq \pi/2 \} \]

solution to a degenerate problem satisfying certain property at its end and with non-negative constant mean curvature on its boundary, has unbounded Schouten tensor. In other words, we establish non-existence result for degenerate (and non-degenerate) elliptic equations in \( \mathbb{A}(r, \pi/2) \) (Theorem 6.1). In Theorem 6.1 we do not assume that the conformal metric is complete. If the conformal metric is complete, due to the Hausdorff dimension estimates of Schoen and Yau [34] and the fact that boundary at infinity of the associated hypersurface is large, one can easily obtain the result. We also study the non-degenerate case in Theorem 6.2.

In the last Section 7, we see that we can extend the definition of the Schouten tensor to conformal metrics to the standard one in domains of the sphere \( \mathbb{S}^2 \). So, we can extend the notion of eigenvalues of the Schouten tensor and we can also speak of elliptic problems for conformal metrics in domains of the sphere \( \mathbb{S}^2 \). There, we observe that the Yamabe Problem reduces to the classical Liouville Problem. Hence, fully nonlinear equations for conformal metrics in domains on \( \mathbb{S}^2 \) can be regarded as a generalization of the Liouville Problem and we obtain analogous
results in the two dimensional case. It is remarkable that there is a solution to the Yamabe Problem on the compact annulus with zero scalar curvature and minimal boundary, however, in dimension higher there does not exist such a solution.

2. Hypersurfaces via conformal metrics. In this section we first briefly sketch how a conformal metric defined in a subdomain of the sphere gives rise to an immersion into the Hyperbolic space and its geometric consequences (cf. [2, 3, 6, 16, 17]). Let \((S^m, g_0)\) be the standard \(m\)-sphere, \(\Omega \subset S^m\) be a relatively compact domain and \(g = e^{2\rho} g_0\) be a \(C^{2,\alpha}\) conformal metric in \(\Omega\). Assume that

\[
\text{Sch}_g(p) < \frac{1}{2} \quad \text{for all } p \in \Omega,
\]

i.e., each eigenvalue of the Schouten tensor is less than 1/2. Observe that we only need to assume that \(\text{Sch}_g < +\infty\) since we can always achieve this condition by a dilation \(g_t = e^{2t} g\).

Denote by \(\mathbb{L}^{m+2}\) the standard Lorentz-Minkowski space, i.e., \(\mathbb{L}^{m+2} = (\mathbb{R}^{m+2}, \langle \cdot, \cdot \rangle)\), where \(\langle \cdot, \cdot \rangle\) is the standard Lorentzian metric given by

\[
\langle \cdot, \cdot \rangle = -dx_0^2 + \sum_{i=1}^{m+1} dx_i^2.
\]

In this model one can consider

\[
\mathbb{H}^{m+1} = \{ x \in \mathbb{L}^{m+2} : \langle x, x \rangle = -1, x_0 > 0 \},
\]

\[
\mathbb{S}^{m+1}_1 = \{ x \in \mathbb{L}^{m+2} : \langle x, x \rangle = 1 \},
\]

\[
\mathbb{N}^{m+1}_+ = \{ x \in \mathbb{L}^{m+2} : \langle x, x \rangle = 0, x_0 > 0 \},
\]

that is, the Hyperbolic space, the de Sitter space and the positive Light cone respectively. We recall that an immersion is horospherically concave if and only if the principal curvatures at any point are bigger than \(-1\) with respect to the prescribed orientation \(\eta\) (the inward orientation for a totally umbilical sphere). Also \(g := \langle d\psi, d\psi \rangle = e^{2\rho} g_0\) is a Riemmanian metric, being \(\psi := \phi - \eta : \Omega \to \mathbb{N}^{m+1}_+\) the light cone map, and it satisfies

\[
\psi = e^{\rho}(1, x), \quad x \in \Omega,
\]

that is, \(g\) is nothing but the First Fundamental Form of \(\psi\). Moreover, the principal curvatures, \(\kappa_i\), of \(\Sigma\) and the eigenvalues, \(\lambda_i\), of the Schouten tensor of \(g\) are related by

\[
\lambda_i = \frac{1}{2} - \frac{1}{1 + \kappa_i}.
\]
Following [3, 17], one can construct a representation of \((\Omega, g)\) as an immersion 
\(\phi : \Omega \to \mathbb{H}^{m+1} \subset (\mathbb{L}^{m+2}, \langle \cdot, \cdot \rangle)\), endowed with a canonical orientation \(\eta : \Omega \to \mathbb{S}_{1}^{m+1} \subset \mathbb{L}^{m+2}\), that is:

**Local Representation Theorem [17].** Let \(\phi : \Omega \subseteq \mathbb{S}^{m} \to \mathbb{H}^{m+1} \subseteq \mathbb{L}^{m+2}\) be a piece of horospherically concave hypersurface with Gauss map \(G(x) = x\). Then, it holds

\[
\phi = \frac{e^{\rho}}{2} \left(1 + e^{-2\rho} \left(1 + |\nabla \rho|^2\right)\right)(1, x) + e^{-\rho}(0, -x + \nabla \rho).
\]

Moreover, the eigenvalues \(\lambda_i\) of the Schouten tensor of the horospherical metric 
\(g = e^{2\rho}g_0\) and the principal curvatures \(\kappa_i\) of \(\phi\) are related by

\[
\lambda_i = \frac{1}{2} - \frac{1}{1 + \kappa_i}.
\]

Conversely, given a conformal metric \(g = e^{2\rho}g_0\) defined on a domain of the sphere \(\Omega \subseteq \mathbb{S}^{m}\) such that the eigenvalues of its Schouten tensor are all less than 1/2, the map \(\phi\) given above defines an immersed, horospherically concave hypersurface in \(\mathbb{H}^{m+1}\) whose Gauss map is \(G(x) = x\) for \(x \in \Omega\) and whose horospherical metric is the given metric \(g\).

Here, \(|\cdot|\) and \(\nabla \rho\) represent the norm and the gradient with respect to \(g_0\). Recall that the hyperbolic Gauss map is defined as follows. Let \(x \in \Omega\) be a point, consider \(p := \phi(x) \in \mathbb{H}^{m+1}\) and \(v := -\eta(x) \in T_p \mathbb{H}^{m+1}\). Then, \(G : \Omega \to \mathbb{S}^{m}\) is defined by

\[
G(x) := \lim_{t \to i \cdot \infty} \gamma_{p,v}(t) \in \mathbb{S}^{m},
\]

where \(\gamma_{p,v} : \mathbb{R} \to \mathbb{H}^{m+1}\) is the complete geodesic parametrized by arc-length in \(\mathbb{H}^{m+1}\) passing through \(p\) in the direction \(v\).

**Remark 2.1.** Note that, from (3), if \(\rho \in C^{k+1}(\Omega)\) then the immersion \(\phi\) is \(C^k\), and also, the First and Second Fundamental Forms are \(C^{k-1}\). It is worth noting that we only need \(\rho \in C^{2}(\Omega)\).

Throughout this work we will use different models of the Hyperbolic space. We recall now the representation formula (3) in the different models:

- **Poincaré ball model:** Given \(x \in \Omega\), the representation formula (3) is

\[
\varphi_p(x) = \frac{1 - e^{-2\rho(x)} + |\nabla e^{-\rho(x)}|^2}{(1 + e^{-\rho(x)})^2 + |\nabla e^{-\rho(x)}|^2} x - \frac{1}{(1 + e^{-\rho(x)})^2 + |\nabla e^{-\rho(x)}|^2} \nabla(e^{-2\rho})(x).
\]

- **Klein model:** Given \(x \in \Omega\), the representation formula (3) is

\[
\varphi(x) = \frac{1 - e^{-2\rho(x)} + |\nabla e^{-\rho(x)}|^2}{1 + e^{-2\rho(x)} + |\nabla e^{-\rho(x)}|^2} x - \frac{1}{1 + e^{-2\rho(x)} + |\nabla e^{-\rho(x)}|^2} \nabla(e^{-2\rho})(x).
\]
Also, set \( \sigma = e^{-\rho} \), then the representation formula (3) in the Klein model can be written as

\[
\varphi_K(x) = \frac{1 - \sigma^2(x) + |\nabla \sigma(x)|^2}{1 \sigma^2(x) + |\nabla \sigma(x)|^2} x - \frac{1}{1 + \sigma^2(x) + |\nabla \sigma(x)|^2} \nabla (\sigma^2)(x).
\]

\( (5) \)

2.1. Properness. In this subsection we give our first main result. We characterize the properness of the associated horospherically concave hypersurface in terms of the behaviour of \( \rho \) along the boundary.

**Theorem 2.2.** Given \( \rho \in C^1(\Omega) \), the map \( \phi : \Omega \to \mathbb{H}^{m+1} \) is proper if, and only if, \( |\rho|_{1,\infty}^2(x) \to \infty \) when \( x \to p \), for every \( p \in \partial \Omega \). Here,

\[
|\rho|_{1,\infty}^2(x) = \rho^2(x) + |\nabla \rho(x)|^2.
\]

**Proof.** From (5), taking the Euclidean norm of \( \varphi_K \) we obtain

\[
|\varphi_K(x)|^2 = 1 - \left(\frac{2\sigma(x)}{1 \sigma^2(x) + |\nabla \sigma(x)|^2}\right)^2 \quad \text{for all } x \in \Omega,
\]

hence, \( \varphi_K \) is proper if, and only if,

\[
\lim_{x \to p} \left(\frac{1}{\sigma(x)} + \sigma(x) + \frac{|\nabla \sigma(x)|^2}{\sigma(x)}\right) = +\infty \quad \text{for all } p \in \partial \Omega,
\]

which is equivalent to

\[
\lim_{x \to p} \left(2 \cosh(\rho(x)) + \frac{|\nabla \rho(x)|^2}{e^{\rho(x)}}\right) = +\infty \quad \text{for all } p \in \partial \Omega.
\]

Finally, the above is equivalent to

\[
\lim_{x \to p} \left[\rho(x)^2 + |\nabla \rho(x)|^2\right] = +\infty \quad \text{for all } p \in \partial \Omega. \quad \square
\]

In particular, we have:

**Corollary 2.3.** If \( \rho : \Omega \to \mathbb{R} \) is a proper smooth function then \( \phi : \Omega \to \mathbb{H}^{m+1} \) is proper.

**Remark 2.4.** In [3, Theorem 3.6], it is shown that if the conformal metric is complete and the scalar curvature is bounded, then it is proper. The hypothesis in [3, Theorem 3.6] imply that \( \rho \) is proper (cf. [8]), so it can be seen as a particular case of Theorem 2.2. Also, in [3, Lemma 3.2], the condition that gives the properness is stronger that the one given here in Theorem 2.2.

Also, as a consequence of the proof of the above theorem, we obtain another condition on \( \rho \) that makes \( \phi \) proper when \( g = e^{2\rho} g_0 \) is complete.
Theorem 2.5. Let $g = e^{2\rho} g_0$ be a complete metric in $\Omega$ such that $\sigma = e^{-\rho}$ is the restriction of a continuous function defined in $\Omega$. Then $\phi : \Omega \to \mathbb{H}^{m+1}$ is a proper map.

Proof. Since $g = e^{2\rho} g_0$ is a complete metric in $\Omega \subset S^m$, we have $\limsup_{x \to p} \rho(x) = +\infty$ for all $p \in \partial \Omega$, that is equivalent to $\liminf_{x \to p} [-\rho(x)] = -\infty$ for all $p \in \partial \Omega$. Let $H : \Omega \to \mathbb{R}$ the continuous extension of $\sigma : \Omega \to \mathbb{R}$, then

$$H(p) = \lim_{x \to p} \sigma(x) = \liminf_{x \to p} \sigma(x) = 0 \quad \text{for all } p \in \partial \Omega.$$ 

Thus, $\lim_{x \to p} \rho(x) = +\infty$ for all $p \in \Omega$, which implies that

$$\lim_{x \to p} \left[ \rho(x)^2 + |\nabla \rho(x)|^2 \right] = +\infty \quad \text{for all } p \in \partial \Omega,$$

that is, $\phi : \Omega \to \mathbb{H}^{m+1}$ is proper. \qed

We note that the above theorem also follows from [3, Lemma 2].

2.2. Geodesic flow and dilations. An interesting relation between conformal metrics and horospherically concave hypersurfaces is how they are related by dilations and geodesic flow (cf. [3, 17]). Let us explain this in more detail. We assume that $\phi : \Omega \to \mathbb{H}^{m+1}$ is an horospherically concave hypersurface in $\mathbb{H}^{m+1}$. When we move the hypersurface $\phi : \Omega \to \mathbb{H}^{m+1}$ using the unit normal vector field $-\eta$, we have a family of horospherically concave hypersurfaces $\{\phi_t : \Omega \to \mathbb{H}^{m+1} : t > 0\}$. For every $t > 0$,

$$\phi_t(x) = \cosh(t) \phi(x) - \sinh(t) \eta(x) \quad \text{for all } x \in \Omega,$$

i.e.,

$$\phi_t(x) = \frac{e^{t+\rho(x)}}{2} \left[ 1 + e^{-2(t+\rho(x))} \left( 1 + |\nabla \rho(x)|^2 \right) \right] (1, x) + e^{-(t+\rho(x))}(0, -x + \nabla \rho(x)),$$

for all $x \in \Omega$. Then the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is well defined with horospherical metric $g_t = e^{2t} g = e^{2(t+\rho)} g_0$. That is, the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is just obtained from the conformal metric $g_t = e^{2t} g$ by the representation formula. Since the eigenvalues of the Schouten tensor $\lambda_{i,t}$ of $g_t$ are just the dilation by a factor of $e^{-2t}$ of the eigenvalues of the Schouten tensor $\lambda_i$ of $g$, i.e.,

$$\lambda_{i,t} = e^{-2t} \lambda_i \leq \lambda_i < \frac{1}{2} \quad \text{for all } i = 1, \ldots, m,$$

then the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is a horospherically concave hypersurface, and clearly its horospherical metric is $g_t = e^{2t} g = e^{2(t+\rho)} g_0$. In conclusion, if we take $t > 0$,
the conformal metric \( g_t = e^{2t}g \) give rise to a horospherically concave hypersurface \( \phi_t : \Omega \to \mathbb{H}^{m+1} \) with the natural orientation \( \eta_t \) given by

\[
\eta_t(x) = \phi_t(x) - e^{t+\rho(x)}(1, x) \quad \text{for all } x \in \Omega.
\]

Then, as an immediate consequence we have:

**Proposition 2.6.** Assume that \( \phi : \Omega \to \mathbb{H}^{m+1} \) is proper, then \( \phi_t : \Omega \to \mathbb{H}^{m+1} \) is also proper for every \( t \in \mathbb{R} \).

### 2.3. From immersed to embedded.

Here we show our second main result for horospherically concave hypersurfaces. So far we have seen that the geodesic flow preserves the regularity of a horospherically concave hypersurface. Now, we study how an immersed horospherically concave hypersurface becomes embedded under the geodesic flow. We start by defining the meaning of a complete conformal metric with boundary in our situation.

**Definition 2.7.** Let \( \Omega \subset S^m \) be an open domain such that \( \partial \Omega = V_1 \cup V_2 \) where \( V_1 \) and \( V_2 \) are disjoint compact submanifolds. We say that a conformal metric \( g = e^{2\rho}g_0, \rho \in C^{2,\alpha}(\Omega \cup V_1) \), is complete with boundary if given a divergent path \( \gamma : [0, 1) \to \Omega \) then either

- \( \int_0^1 |\gamma'(t)|_g dt < +\infty \) and then \( \lim_{t \to 1} \gamma(t) \in V_1 \), or
- \( \int_0^1 |\gamma'(t)|_g dt = +\infty \) and then \( \lim_{t \to 1} \gamma(t) \in V_2 \).

In other words, \( g \) is a complete metric on the manifold with boundary \( \Omega \cup V_1 \).

**Remark 2.8.** In the above definition, \( V_2 \) can contain points, i.e., it is permitted submanifolds that have dimension zero, or \( V_2 \) might be empty.

The next main result shows that if we impose conditions on certain functions related to \( \sigma = e^{-\rho} \) we can move along the geodesic flow and then we get an embedded hypersurface.

**Theorem 2.9.** Let \( \rho \in C^{2,\alpha}(\Omega \cup V_1) \) be such that \( \sigma = e^{-\rho} \in C^{2,\alpha}(\Omega \cup V_1) \) satisfies:

1. \( \sigma^2 \) can be extended to a \( C^{1,1} \) function on \( \overline{\Omega} \).
2. \( |\nabla \sigma|^2 \) can be extended to a Lipschitz function on \( \overline{\Omega} \).

Then, there is \( t_0 > 0 \) such that for all \( t > t_0 \) the map \( \phi_t : \Omega \cup V_1 \to \mathbb{H}^{m+1} \) associated to \( \rho_t = \rho + t \) is an embedded horospherically concave hypersurface.

**Proof.** Let \( \zeta : S^m \to \mathbb{R} \) be a \( C^{1,1} \)-extension of \( \sigma^2 \) such that \( \zeta > -\frac{1}{3} \), and \( l \in \text{Lip}(S^m) \) be a Lipschitz extension of \( |\nabla \sigma|^2 \) such that also \( l > -\frac{1}{3} \). Then, for \( t > 0 \), we have the following Lipschitz extension of \( \varphi_t \) (we work on the Klein model):

\[
\Phi_t(x) = x - 2 \frac{e^{-2t}\zeta(x)}{1 + e^{-2t}[\zeta(x) + l(x)]} x - \frac{e^{-2t}}{1 + e^{-2t}[\zeta(x) + l(x)]} \nabla \zeta(x), \quad x \in S^m.
\]
Since \( \{\Phi_t\}_{t>0} \) converges to the inclusion \( \mathbb{S}^m \hookrightarrow \mathbb{R}^{m+1} \) uniformly on \( \mathbb{S}^m \), there is \( t_0 > 0 \) such that for every \( t > t_0 \), the map \( \Phi_t \) is embedded (cf. [20]). Then there is \( t_0 > 0 \) such that for every \( t > t_0 \) the map \( \varphi_t \) is embedded.

Also, from the equation:

\[
-g^{-1} \text{Sch}(g) + \frac{1}{2} |\nabla \sigma|^2 \text{Id} + \langle \nabla \sigma, \cdot \rangle \nabla \sigma = \frac{1}{2} \sigma^2 \text{Id} + \nabla^2 \sigma^2 \quad \text{in } \Omega
\]

and the hypothesis that the eigenvalues of the Schouten tensor of \( g = e^{2\rho} g_0 \) are bounded in \( \Omega \), we can choose \( t_0 > 0 \) large such that for every \( t > t_0 \), the map \( \varphi_t : \Omega \rightarrow \mathbb{H}^{m+1} \) (in the Hyperboloid model) is a horospherically concave hypersurface (see (6)). This concludes the proof. \( \square \)

**Remark 2.10.** In Theorem 2.9 is not assumed that \( \rho \) is proper and we use different techniques of those used in [3, Theorem 3.6]. When \( \rho \) is proper in Theorem 2.9, then the Schouten tensor of the conformal metric is bounded, so we get a proper hypersurface as in [3].

Let us see the difference between the Klein and Poincaré models with a simple example. Let \( \Sigma \) be the horospherically concave hypersurface associated to the function \( e^{-\rho} = \sigma : \Omega = \{(x,y,z) \in \mathbb{S}^2 : |z| < \cos(\pi/4)\} \rightarrow \mathbb{R} \) given by

\[
\sigma(x,y,z) = \frac{2}{2 + \sqrt{2}} \left( \sqrt{1 - z^2} - \sin(\pi/4) \right).
\]

One can easily observe (cf. Figure 1(a)) that in the Poincaré model \( \Sigma \) is transversal to the ideal boundary. Nevertheless, in the Klein model, \( \Sigma \) is tangential to the ideal boundary (cf. Figure 1(b)).
As we can see, the function $\sigma$ can be smoothly extended to $\mathbb{S}^2 \setminus \{\pm e_3\}$, in fact, Figure 1(b) is a smooth extension of the surface in $\mathbb{R}^3$ in order to see the tangency of the surface with $\mathbb{S}^2$, the ideal boundary of $\mathbb{H}^3$.

2.4. Conditions along the boundary. Let $\Omega \subset \mathbb{S}^m$ be a domain such that $\partial \Omega = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\rho \in C^{2,\alpha}(\Omega \cup \mathcal{V}_1)$. Assume that $g := e^{2\rho}g_0$ is complete in $\Omega \cup \mathcal{V}_1$ (see Definition 2.7). We study how conditions on either $\rho$ along $\mathcal{V}_1$, or geometric conditions on $\mathcal{V}_1$, influence the boundary of the horospherically concave hypersurface $\Sigma$ associated to $\rho$.

In general, if $(\mathcal{M},g_0)$ is a Riemannian manifold with boundary $\partial \mathcal{M}$, $\nu$ is a unit normal vector field along $\partial \mathcal{M}$, $g = e^{2\rho}g_0$ is a conformal metric to $g_0$, $h_0$ the mean curvature of $\partial \mathcal{M}$ with respect to the metric $g_0$ and the unit normal vector field $\nu$, $h(g)$ the mean curvature of $\partial \mathcal{M}$ with respect to the metric $g$ and the normal vector field $\nu_g = e^{-\rho}\nu$, then

$$e^\rho \cdot h(g) + \frac{\partial \rho}{\partial \nu} = h_0 \quad \text{on} \quad \partial \mathcal{M}.$$  

Take $\sigma = e^{-\rho}$, then we have the following relation

$$\frac{\partial \sigma}{\partial \nu} + h_0 \cdot \sigma = h(g) \quad \text{on} \quad \partial \mathcal{M}.$$  

If we consider the dilated metric $g_t = e^{2t}g$ in $\mathcal{M}$, $t \in \mathbb{R}$, then the mean curvature of $\partial \mathcal{M}$ with respect to the unit normal vector field $\nu_t = e^{-t}\nu_g$ is

$$h(g_t) = e^{-t}h(g) \quad \text{on} \quad \partial \mathcal{M}. (8)$$

Therefore, if $\partial \mathcal{M}$ is compact then $h(g_t)$ goes to 0 when $t$ goes to infinity. In our case $\mathcal{M} = \Omega \cup \mathcal{V}_1$ and $\partial \mathcal{M} = \mathcal{V}_1$.

Given $p \in \mathbb{S}^m$ and $r \in (0, \frac{\pi}{2}]$, the geodesic ball of $\mathbb{S}^m$ with center $p$ and radius $r$ is given by

$$B_r(p) = \{ q \in \mathbb{S}^m : d_{\mathbb{S}^m}(q,p) < r \}$$

and its inward unit normal along $\partial B_r(p)$ is given by

$$\nu(x) = \csc(r)p - \cot(r)x, \quad x \in \partial B_r(p).$$

Any geodesic ball $B_r(p)$ has associated a unique totally geodesic hypersurface $E(a,0) \subset \mathbb{H}^{m+1}$, here we use the Hyperboloid model to define $E(a,0)$, such that

$$\partial B_r(p) = \partial E(a,0),$$

where $a \in \mathbb{L}^{m+2}$ is a spacelike unit vector and $E(a,0)$ is defined by

$$E(a,0) = \{ y \in \mathbb{H}^{m+1} : \langle\langle y, a \rangle\rangle = 0 \}.$$
We can explicitly recover the vector $a$ from the center $p$ and the radius $r$, and vice-versa. Specifically, if $a = (a_0, \bar{a})$, $\langle\langle a, a \rangle\rangle = 1$, then

$$p = \frac{1}{|a|}a \quad \text{and} \quad \cot(r) = a_0, \quad r \in \left(0, \frac{\pi}{2}\right).$$

Now, we will study the boundary $\phi(\partial B_r(p))$ of the associated horospherically hypersurface $\phi$ to $\rho$ when the boundary of $B_r(p)$ has constant mean curvature with respect to the metric $g = e^{2\rho}g_0$. Consider a complete conformal metric $g = e^{2\rho}g_0$ in a domain $\Omega \cup \mathcal{V}_1 \subset B_r(p)$ such that $\partial B_r(p) \subset \mathcal{V}_1$. Let $h(g)$ be the mean curvature of $\partial B_r(p)$ with respect to $g$ and the inward unit normal vector field $\nu_g = e^{-\rho}\nu$ along $\partial B_r(p)$, and $h_0 = \cot(r)$.

Let $\phi: \Omega \cup \mathcal{V}_1 \rightarrow \mathbb{H}^{m+1}$ be the horospherically concave hypersurface associated to the complete conformal metric $g$ in $\Omega \cup \mathcal{V}_1$. Then, a straightforward computation shows

$$\langle\langle \phi(x), h_0(1, x) + (0, \nu(x)) \rangle\rangle = -h(g) \quad \text{for all } x \in \partial B_r(p).$$

Assume that $h(g) = c$, then

$$\langle\langle \phi(x), h_0(1, x) + (0, \nu(x)) \rangle\rangle = -c \quad \text{for all } x \in \partial B_r(p),$$

where

$$\nu(x) = \csc(r)p - \cot(r)x \quad \text{for all } x \in \partial B_r(p).$$

Set $a = h_0(1, x) + (0, \nu(x))$, since $h_0 = \cot(r)$, we have

$$a = \left(\cot(r), \frac{1}{\sin(r)}p\right) \quad \text{for all } x \in \partial B_r(p),$$

i.e., $a$ only depends of $p$ and $r$.

**Remark 2.11.** In the particular case that $r = \pi/2$ and $p = n$, the north pole, we have

$$a = (0, \ldots, 0, 1) = (0, e_{m+1}).$$

Then, from (9) and (10) we have that

$$\phi(\partial B_r(p)) \subset E(a, -c) = \{y \in \mathbb{H}^{m+1}: \langle\langle y, a \rangle\rangle = -c\}$$

which is an equidistant hypersurface to $E(a, 0)$. In the case that $a = (0, \ldots, 0, 1)$, we just denote $E(-c) = E(a, -c)$. Summarizing, we have (see [6, Claim E]):
Proposition 2.12. Under the conditions above, assuming that $V_1$ contains a component which is the boundary of a geodesic ball $\partial B_r(p)$, $p \in S^m$, $r \in (0, \pi/2]$, and $h(g) = c$ along $\partial B_r(p)$, then

$$\phi(\partial B_r(p)) \subset E(a, -c),$$

where $E(a, -c)$ is the totally geodesic hypersurface equidistant to $E(a, 0)$ given by

$$E(a, -c) = \{ y \in \mathbb{H}^{m+1}_+ : \langle \langle y, a \rangle \rangle = -c \}$$

and $a = (\cot(r), \csc(r)p)$.

We can say even more, in fact, $\Sigma = \phi(\Omega)$ makes a constant angle with $E(a, -c)$ along $\phi(\partial B_r(p))$. Without loss of generality and for simplicity, we will assume $V_1 = \partial B_r(p)$. A unit normal vector field along $E(a, -c)$ is given by

$$N(y) = \frac{1}{\sqrt{1 + c^2}}(a - cy),$$

for all $y \in E(a, -c)$. Since $\partial \Sigma \subset E(a, -c)$, we have the following result (see also [6, Claim D]):

Proposition 2.13. Under the above conditions, it holds

$$\langle \langle N, \eta \rangle \rangle = \frac{-c}{\sqrt{1 + c^2}} \text{ along } \phi(\partial B_r(p)).$$

In other words, the angle $\alpha$ between $\Sigma$ and $E(a, -c)$ along $\phi(\partial B_r(p))$ is constant and it satisfies

$$\cos(\alpha) = \frac{-c}{\sqrt{1 + c^2}}.$$

2.5. Moving the hypersurface along the geodesic flow. We will show the following boundary half-space property: Let $\Omega \subset S^m_+$ be an open connected domain such that $\partial \Omega = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, where the subset $V_1$ is a compact hypersurface (not necessary connected) of $S^m$ that contains $\partial S^m_+$, and $V_2$ is a finite union of disjoint compact submanifolds of $S^m$, it might contain points. Let $\rho \in C^{2,\alpha}(\Omega \cup V_1)$ be such that

$$\lim_{x \to q} \left( e^{2\rho(x)} + |\nabla \rho(x)|^2 \right) = +\infty \text{ for all } q \in V_2.$$

Set

$$V_1' = V_1 \setminus \partial S^m_+,$$

we will show that, if $h(g) = c$ on $\partial S^m_+$, then there exists $t_1 \geq 0$ such that

$$\Sigma_t = \varphi_t(\Omega \cup V_1') \subset C_t \text{ for all } t \geq t_1,$$
where $C_t \subset \mathbb{H}^{m+1}$ is the half-space determined by the equidistant $E(-e^{-t}c)$ that contains $n$ at its boundary at infinity. Here, $\varphi_t$ stands for the immersion along the parallel flow in the Klein model (see (5)). Specifically (see [6, Claim C] in the compact case, i.e., for $V_2 = \emptyset$):

**Theorem 2.14.** Let $g = e^{2\rho}g_0$, $\rho \in C^{2,\alpha}(\Omega \cup V_1)$, be a conformal metric in $\Omega$ such that $\partial S^m_+ \subset V_1$ and

$$h(g) = c \quad \text{on } \partial S^m_+,$$

where $c \in \mathbb{R}$ is a constant. Assume that

$$\lim_{x \to q} \left(e^{2\rho(x)} + |\nabla \rho(x)|^2\right) = +\infty \quad \text{for all } q \in V_2.$$

Then, there exists $t_0 \geq 0$ such that for every $t > t_0$, the set $\varphi_t(\Omega \cup V'_1)$ is contained in the half-space determined by $E(-e^{-t}c)$ that contains $n$ at its ideal boundary.

**Proof.** First, we work in the Klein model. Set $K = S^m_+ \setminus \Omega$ and $\text{int}(K) = K \setminus \partial K$. For every $t > 0$ we define the continuous extension $\Phi_t : S^m_+ \setminus \text{int}(K) \to \mathbb{R}^{m+1}$ of $\varphi_t : S^m_+ \setminus K \to \mathbb{H}^{m+1}$ given by

$$\Phi_t(x) = \begin{cases} \varphi_t(x), & x \in \Omega \cup V_1, \\ x, & x \in V_2. \end{cases}$$

Observe that $\{\Phi_t\}_{t>0}$ converges to the inclusion $\overline{S^m_+ \setminus \text{int}(K)} \hookrightarrow \mathbb{R}^{m+1}$ when $t \to \infty$. We take an open set $V$ such that

$$K \subset V \subset \overline{V} \subset S^m_+.$$

Since $\overline{V} \setminus \text{int}(K)$ is compact, there exists $t_1 > 0$ such that the set $\varphi_t(\overline{V} \setminus K)$ is in the half-space determined by the equidistant $E(|c|)$ and contains $n$ at its ideal boundary for all $t > t_1$. Then $\varphi_t(\partial V)$ is contained in the half-space. It should be noted that the equidistant $E(-ct)$ is nearer to $E(0)$ than $E(-c)$, i.e., we have a portion of the hypersurface satisfying the theorem.

Now we consider the map $\varphi : \overline{S^m_+ \setminus V} \to \mathbb{H}^{m+1}$. We will prove that there is $t_0 \geq t_1$ such that the set $\varphi_t(S^m_+ \setminus V)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $n$ at its ideal boundary for every $t > t_0$. That will finish the proof.

From (8), the mean curvature of $\partial S^m_+$ with respect to the scaled metric $g_t = e^{2t}g$, $t \in \mathbb{R}$, is $h(g_t) = e^{-t}c$ along $\partial S^m_+$.

Now, we work in the Hyperboloid model. From Proposition 2.12, $\phi_t(\partial S^m_+) \subset E(-e^{-t}c)$. Recall that $\phi_t$ stands for the immersion along the positive geodesic flow in the Hyperboloid model. We consider the following unit normal vector field along
$E(-e^{-t}c)$:

$$N(y) = \frac{1}{\sqrt{1 + s_1^2}} \left[(0, n) - (e^{-t}c) \cdot y\right], \quad y \in E(-e^{-t}c),$$

then the principal curvatures of the umbilic hypersurface $E(-e^{-t}c)$ with respect to $N$ are equal to $\frac{ce^{-t}}{1 + c^2 e^{-2t}}$.

Let $\kappa_{1,t}, \ldots, \kappa_{m,t}$ be the principal curvatures of $\phi_t$. Then, for all $t > 0$, we have

$$\frac{1}{2} = e^{-2t}\lambda_i + \frac{1}{1 + \kappa_{i,t}} \text{ on } \mathbb{S}^m_{+} \setminus V, \quad \text{for all } i = 1, \ldots, m,$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of the Schouten tensor of $g$. Since $\kappa_{i,t}$ goes to 1 uniformly on $\mathbb{S}^m_{+} \setminus V$ as $t$ goes to infinity, for $i = 1, \ldots, m$, then there exists $t_0 \geq t_1$ such that

$$\kappa_{i,t} > \frac{1}{2} > -\frac{ce^{-t}}{\sqrt{1 + c^2 e^{-2t}}} \quad \text{for all } t > t_0 \text{ and for all } i = 1, \ldots, m,$$

on $\mathbb{S}^m_{+} \setminus V$.

We claim that for every $t > t_0$, the set $\phi_t(\mathbb{S}^m_{+} \setminus V)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $n$ at its ideal boundary. If this were not the case, we consider the foliation by equidistant hypersurfaces $\{E(s)\}_{s \in \mathbb{R}}$ of the Hyperbolic space $\mathbb{H}^{m+1}$, given by

$$E(s) = \{ y \in \mathbb{H}^{m+1} : \langle (y, (0, e_{m+1})) \rangle = s \}.$$

We consider the first equidistant hypersurface $E(s_1)$ that intersects $\phi_t(\mathbb{S}^m_{+} \setminus V)$ (cf. Figures 2(a) and 2(b)), i.e.,

$$E(s_1) \cap \phi_t(\mathbb{S}^m_{+} \setminus V) \neq \emptyset \quad \text{and} \quad E(s) \cap \phi_t(\mathbb{S}^m_{+} \setminus V) = \emptyset \quad \text{for all } s < s_1.$$

Clearly $s_1 \leq -c_t = -e^{-t}c$. We note that $E(s_1) \cap \phi_t(\partial V) = \emptyset$ since $s_1 \leq |c|$. Then

$$E(s_1) \cap \phi_t(\mathbb{S}^m_{+} \setminus V) \neq \emptyset.$$

We claim that $E(s_1) \cap \phi_t(\mathbb{S}^m_{+} \setminus V) = \emptyset$, otherwise there would exist an interior contact point of $\phi_t(\mathbb{S}^m_{+} \setminus V)$, say $x \in \mathbb{S}^m_{+} \setminus V$ such that $w = \phi_t(x) \in E(s_1)$.

Consider the normal vector field along $E(s_1)$ that is defined in the Hyperboloid Model by $N(y) = \frac{1}{\sqrt{1 + s_1^2}} [(0, n) + s_1 \cdot y]$, for all $y \in E(s_1)$. The principal curvatures of $E(s_1)$ with respect to this orientation are equal to $-\frac{s_1}{\sqrt{1 + s_1^2}}$.

Along the horospherically concave hypersurface $\phi_t$, we consider the opposite orientation to the canonical one, i.e., the unit normal vector field $\xi_t = -\eta_t$ (see (7)). Then the principal curvatures are $\tilde{\kappa}_{i,t} = -\kappa_{i,t}$ with respect to that normal (cf. Figure 2(c)).
(a) The equidistant $E(s)$ does not intersect the hypersurface.

(b) The equidistant $E(s_1)$ intersects the hypersurface.

(c) Opposite orientation of $\Sigma = \text{Im}(\phi_t)$.

Figure 2. Getting the first contact equidistant in the Poincaré ball model.

Since the Hyperbolic Gauss map is the inclusion $\overline{S^m_+ \setminus V} \hookrightarrow \mathbb{R}^{m+1}$, the normal vector field $\xi$ coincides with the normal $N$ along the equidistant $E(s)$ at the point $w$. The principal curvatures of $E(s_1)$ with respect to $N$ at the point $w$ satisfy

$$-\frac{s_1}{\sqrt{1+s_1^2}} \geq \frac{c_t}{\sqrt{1+c_t^2}} \text{ since } -s_1 \geq c_t.$$

Moreover, since $\phi_t(S^m_+ \setminus V)$ is more convex than $E(s_1)$ at the point $w$ with the orientation $N(w)$, we have

$$\tilde{K}_{i,t} \geq -\frac{s_1}{\sqrt{1+s_1^2}} \text{ at the point } w.$$
That is, 
\[ \kappa_{i,t} \leq \frac{s_1}{\sqrt{1 + s_1^2}} \] 
at the point \( w \).

So, at the point \( w \), we have
\[ \kappa_{i,t} \leq -\frac{c_t}{\sqrt{1 + c_t^2}}, \]

but this contradicts (11). Then, for every \( t > t_0 \), the set \( \phi_t(S^m_+ \setminus V) \) is contained in the half-space determined by \( E(-e^{-t}c) \) that contains \( n \) at its ideal boundary. \( \square \)

2.6. Elliptic problems for conformal metrics in domains of \( S^m \). Consider \( g = e^{2\rho}g_0 \) a conformal metric, \( \rho \in C^{2,\alpha}(\Omega), \Omega \subseteq S^m \), and denote by \( \lambda(g) = (\lambda_1, \ldots, \lambda_m) \) the eigenvalues of the Schouten tensor of \( g \).

We want to study partial differential equations associated to \( \rho \) relating the eigenvalues of the Schouten tensor. For instance, the simplest example one may consider is the \( \sigma_k \)-Yamabe problem in \( \Omega \subseteq S^m \), i.e., the \( k \)-symmetric function of the eigenvalues of the Schouten tensor. We are interested in the fully nonlinear case of this problem (see [25, 26, 27, 28, 29] and references therein).

Namely, given \( (f, \Gamma) \) an elliptic data and a constant \( b \geq 0 \), find \( \rho \in C^{2,\alpha}(\Omega) \) so that \( g = e^{2\rho}g_0 \) is a solution to the problem
\[ f(\lambda(g)) = b \quad \text{in} \ \Omega. \]

We must properly define the meaning of elliptic data \((f, \Gamma)\) for conformal metrics. Define
\[ \Gamma_m = \{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i > 0, \ i = 1, \ldots, m \} \]
and
\[ \Gamma_1 = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i > 0 \right\}. \]

Let \( \Gamma \subset \mathbb{R}^m \) be a convex open set satisfying:

(C1) It is symmetric. If \( (x_1, \ldots, x_m) \in \Gamma \), then \( (x_{i_1}, \ldots, x_{i_m}) \in \Gamma \), for every permutation \( (i_1, \ldots, i_m) \) of \( (1, \ldots, m) \).

(C2) It is a cone. For every \( t > 0 \), we have that \( t(x_1, \ldots, x_m) \in \Gamma \) for every \( (x_1, \ldots, x_m) \in \Gamma \).

(C3) \( \Gamma_m \subset \Gamma \subset \Gamma_1 \).

Then, we are ready to define:
**Definition 2.15.** We say that \((f, \Gamma)\) is an *elliptic data* for conformal metrics if \(\Gamma \subset \mathbb{R}^m\) is a convex cone satisfying \((C1), (C2)\) and \((C3)\) and \(f \in C^0(\overline{\Gamma}) \cap C^1(\Gamma)\) is a function satisfying

1. \(f\) is symmetric in \(\Gamma\).
2. \(f|_{\partial \Gamma} = 0\).
3. \(f|_{\Gamma} > 0\).
4. \(f\) is homogeneous of degree 1.
5. \(f(1, \ldots, 1) = 2\),
6. \(\nabla f(x) \in \Gamma_m\) for every \(x \in \Gamma\).

**Remark 2.16.** Condition \((5)\) is just technical, we can always achieve this by multiplying by a suitable positive constant.

This elliptic data is necessary for the definition of non-degenerate and degenerate elliptic problems.

**Definition 2.17.** (Elliptic problems for conformal metrics) Given an elliptic data \((f, \Gamma)\) for conformal metrics and \(\Omega \subseteq S^m\) a domain:

1. The non-degenerate elliptic problem is to find a conformal metric \(g = e^{2\rho} g_0\) such that

\[
 f(\lambda(g)) = 1 \text{ in } \Omega
\]

where \(\lambda(g) = (\lambda_1, \ldots, \lambda_m)\) is composed by the eigenvalues of the Schouten tensor of \(g\).

2. The degenerate elliptic problem is to find a conformal metric \(g = e^{2\rho} g_0\) such that

\[
 f(\lambda(g)) = 0 \text{ in } \Omega
\]

where \(\lambda(g) = (\lambda_1, \ldots, \lambda_m)\) is composed by the eigenvalues of the Schouten tensor of \(g\).

There are two remarks concerned to these equations. First, we have a particular solution in each case of special interest:

- **Non-degenerate elliptic problems:** The standard metric \(g_0\) is a solution to \(f(\lambda(g_0)) = 1\) in \(\Omega\). This follows since \(f\) is homogeneous of degree one and the normalization \(f(1, \ldots, 1) = 2\). Recall that the Schouten tensor of \(g_0\) is given by \(\text{Sch}_{g_0} = \frac{1}{2} g_0\). Such a solution is used mainly in [6].

- **Degenerate elliptic problems:** A horosphere endowed with the inward orientation is horospherically concave as defined here. Then, the image of the Hyperbolic Gauss map is the whole sphere minus one point, say \(x \in S^m\), the point at infinity where is based the horosphere. Hence, such horosphere defines a conformal metric \(g := e^{2\rho_H} g_0\) in \(S^m \setminus \{x\}\), \(x \in \mathbb{S}^m\) (cf. [16]). Thus, for a domain \(\Omega \subset S^m \setminus \{x\}\),
the metric \( g := e^{2\rho H} g_0 \) is a solution to \( f(\lambda(g)) = 0 \) in \( \Omega \) since all the eigenvalues of the Schouten tensor are identically zero. This follows from (4).

Second, it is about the Maximum Principle.

- **Non-degenerate elliptic problems**: These equations satisfy the usual version of the Maximum Principle, Interior and Boundary (cf. [25, 26]).
- **Degenerate elliptic problems**: These equations do not always satisfy the usual Maximum Principle [31]. Nevertheless, we will use a version developed by Y. Y. Li in [29].

In the degenerate case there are two results of Y. Y. Li [29] that are crucial in our proofs. One of them is:

**Theorem 2.18.** (YanYan Li, [29]) Set \( m \geq 2 \) and \( \Omega \) is a domain in \( S^m \) such that \( \overline{\Omega} \neq S^m \). Let \( g_1 = e^{2\rho_1} g_0 \) and \( g = e^{2\rho} g_0 \) be solutions to the degenerate problem associate to \((f, \Gamma)\) on \( \overline{\Omega} \). If

\[
\rho_1 > \rho \quad \text{on} \quad \partial \Omega
\]

then

\[
\rho_1 > \rho \quad \text{on} \quad \overline{\Omega}.
\]

Geometrically, we can rephrase the above result for hypersurfaces in the following way: given two horospherical hypersurfaces with boundary that are solutions to the same degenerate Weingarten equation [6], with injective hyperbolic Gauss maps and the same image under the hyperbolic Gauss maps, if the boundary of one of them is at one side to the other boundary (comparison between support functions), then one hypersurface is at one side to the other.

The second result is:

**Theorem 2.19.** (YanYan Li, [29]) Set \( m \geq 3 \). Let \( g = e^{2\rho_1} g_0 \) be a conformal metric on \( S^m \setminus \{n, s\} \) such that

\[
f(\lambda(g)) = 0 \quad \text{on} \quad S^m \setminus \{n, s\},
\]

then \( g \) is rotationally invariant.

Geometrically, the above says: given a Weingarten hypersurface \( \phi : \Sigma^m \to \mathbb{H}^{m+1} \) solution to a degenerate Weingarten equation (cf. [6]), where \( m \geq 3 \), if its Gauss map is injective with image \( S^m \setminus \{n, s\} \), then the hypersurface is rotationally invariant.

**2.7. Dilation and elliptic problems for conformal metric.** Given a conformal metric \( g = e^{2\rho} g_0, \rho \in C^{2,\alpha}(\Omega) \), that satisfies an elliptic problem for conformal metrics in \( \Omega \subseteq S^m \), i.e.,

\[
f(\lambda(g)) = cte \quad \text{in} \ \Omega,
\]
where \( f : \Gamma \to \mathbb{R} \) is an elliptic function for conformal metrics (see Subsection 2.6), one can naturally ask the following question: Given \( t_0 \in \mathbb{R} \), is the metric \( g_{t_0} = e^{2t_0} g \) a solution to an elliptic problem for conformal metrics in \( \Omega \)? The answer is affirmative in both, the non-degenerate and degenerate case. Let see this in the non-degenerate case.

**Proposition 2.20.** Given a solution \( g = e^{2\rho} g_0, \rho \in C^{2,\alpha}(\Omega) \), to a non-degenerate elliptic problem with elliptic data \((f, \Gamma)\) and \( t_0 \in \mathbb{R} \), then \( g_{t_0} = e^{2t_0} g \) is a solution to a non-degenerate elliptic problem with elliptic data \((f_{t_0}, \Gamma)\) where

\[
f_{t_0}(x) = f(e^{2t_0} x) \quad \text{for all } x \in \Gamma.
\]

**Proof.** Since \( \Gamma \subset \mathbb{R}^m \) is a cone then \( e^{-2t_0} \Gamma = \Gamma \). Also \( \partial \Gamma = e^{-2t_0} \partial \Gamma \), then

\[
f_{t_0}(x) = f(e^{2t_0} x) = 1 \quad \text{for all } x \in \partial \Gamma,
\]

and

\[
\nabla f_{t_0}(x) = e^{2t_0} \nabla f(e^{2t_0} x) \in \Gamma_m \quad \text{for all } x \in \Gamma.
\]

It is clear that \( f_{t_0} : \overline{\Gamma} \to \mathbb{R} \) is symmetric and

\[
f_{t_0}(\lambda(g_0)) = f_{t_0}(e^{-2t_0} \lambda(g)) = f(e^{2t_0} e^{-2t_0} \lambda(g)) = 1 \quad \text{in } \Omega.
\]

Then, the conformal metric \( g_{t_0} = e^{2t_0} g \) is a solution to the elliptic problem given by the elliptic data \((f_{t_0}, \Gamma)\). \(\square\)

In the degenerate case we have:

**Proposition 2.21.** Let \( g = e^{2\rho} g_0 \) be a solution to the degenerate problem \( f(\lambda(g)) = 0 \) and \( t_0 \in \mathbb{R} \), then \( e^{2t_0} g \) is solution to the same degenerate problem.

**Proof.** We have \( \lambda(g) \in \partial \Gamma \), then \( e^{-2t_0} \lambda(g) \in \partial \Gamma \). Since \( \lambda(e^{2t_0} g) = e^{-2t_0} \lambda(g) \), we have \( f(\lambda(e^{2t_0} g)) = 0 \). \(\square\)

3. **A non-existence Theorem on \( \mathbb{S}^m_+ \).** As we have said at the Introduction, Escobar [12] proved that there is no conformal metric \( g = e^{2\rho} g_{\text{Eucl}} \) in the Euclidean unit ball \( \mathbb{B}^m \) with zero scalar curvature and non-positive constant mean curvature along boundary, i.e., \( h(g) \leq 0 \) constant. That result follows from the Maximum Principle. We extend here such result for degenerate elliptic equations.

**Theorem 3.1.** Let \((f, \Gamma)\) be an elliptic data for conformal metrics and let \( c \leq 0 \) be a constant. Then, there is no conformal metric \( g = e^{2\rho} g_0 \) in \( \mathbb{S}^m_+ \), where \( \rho \in C^{2,\alpha}(\mathbb{S}^m_+) \), such that

\[
\begin{aligned}
f(\lambda(g)) &= 0 \quad \text{in } \mathbb{S}^m_+, \\
h(g) &= c \quad \text{on } \partial \mathbb{S}^m_+,
\end{aligned}
\]
where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of $g = e^{2\rho}g_0$.

Proof. The proof will be done by contradiction. Assume that there exists a conformal metric in $\mathbb{S}^m_+$, $g = e^{2\rho}g_0$, $\rho \in C^{2,\alpha}(\mathbb{S}^m_+)$, that solves the above problem. Since $\mathbb{S}^m_+$ is compact, up to a dilation of $g$, we can assume (see Subsection 2.7), without loss of generality, that all the eigenvalues of $\text{Sch}(g)$ are less than $1/2$. By Theorems 2.9 and 2.14, we can assume that the associated horospherically concave hypersurface $\phi : \mathbb{S}^m_+ \to \mathbb{H}^{m+1}$ is embedded and it is contained in the half-space determined by the equidistant hypersurface $E(-c)$ to the totally geodesic hypersurface $E(0)$, and such component contains $n$ at its ideal boundary. We notice that, in the Poincaré ball model, since $-c \geq 0$, the equidistant $E(-c)$ is contained in the half-space determined by the totally geodesic hyperplane $E(0)$ that contains $n$ at its ideal boundary.

We consider the horospherically concave hypersurface $\Sigma = \phi(\mathbb{S}^m_+) \subset \mathbb{H}^{m+1}$ that is associated to $\rho$. Moreover, the angle between the hypersurface $\Sigma$ and the equidistant hypersurface $E(-c)$ along $\partial \Sigma$ is constant equals to

$$\cos(\alpha) = -\frac{c}{\sqrt{1+c^2}},$$

this follows from Proposition 2.13. Recall that the support function associated to horospheres are solutions to the above degenerate elliptic problem.

Now, we consider the foliation of the Hyperbolic space $\mathbb{H}^{m+1}$ by horospheres, $\{H(s)\}_{s \in \mathbb{R}}$, that have the same point at the ideal boundary of the Hyperbolic Space, $p_\infty = s$. We parametrize this foliation by the signed distance, $s \in \mathbb{R}$, to the origin at the Poincaré ball model.

Since $\Sigma$ is compact, for $s$ large enough negatively, $\Sigma$ is completely contained in the mean-convex side of the horosphere $H(s)$ (cf. Figure 3(a) when $c = 0$). We continue increasing $s$ until we reach the first contact point with $\Sigma$, let us say $s_1$ (cf. Figures 3(b), 3(c), and 3(d) when $c = 0$). This means that, for every $s < s_1$, the hypersurface $\Sigma$ is in the interior of $H(s)$, $H(s) \cap \Sigma = \emptyset$, and $H(s_1) \cap \Sigma$ is not empty.

Recall that

$$\cos(\alpha) = -\frac{c}{\sqrt{1+c^2}} \geq 0,$$

since the angle $\alpha$ between the normal $\eta$ and the upward normal of $E(-c)$ is acute or $\pi/2$, we get that the contact point is at the interior of $\Sigma$.

The horosphere $H(s_1)$ with its natural orientation has its own representation formula over $\mathbb{S}^m \setminus \{s\}$. We consider such formula restricted to $\mathbb{S}^m_+$ and we denote its support function by $\rho_1$. Since this horosphere is the first contact horosphere, we have that $\rho_1 \geq \rho$ on $\mathbb{S}^m_+$. Also, there is no boundary contact point of $\Sigma$, so there is
(a) The hypersurface $\Sigma$ is in the convex side of the horosphere $H(s)$.

(b) The horosphere $H(s)$ does not intersect $\Sigma$.

(c) The horosphere $H(s)$ does not intersect the hypersurface $\Sigma$.

(d) The horosphere $H(s_1)$ touches the hypersurface $\Sigma$.

Figure 3. Getting the first contact horosphere in the Poincaré ball model. Case $c = 0$.

Let $x \in S_m^+$ such that $\rho_1(x) = \rho(x)$ and $\rho_1 > \rho$ on $\partial S_m^+$, then, from Theorem 2.18, $\rho_1 > \rho$ in $\overline{S_m^+}$, which is a contradiction, since there is $x \in S_m^+$ such that $\rho_1(x) = \rho(x)$. \hfill $\square$

In the case that we consider a $m$-dimensional compact, simply-connected, locally conformally flat manifold $(\mathcal{M}, g_0)$ with umbilic boundary and $R(g_0) \geq 0$ on $\mathcal{M}$, we have,

**Theorem 3.2.** Set $(f, \Gamma)$ an elliptic data for conformal metrics and $c \leq 0$ a constant. Let $(\mathcal{M}, g_0)$ be a $m$-dimensional compact, simply-connected, locally conformally flat manifold with umbilic boundary and $R(g_0) \geq 0$ on $\mathcal{M}$. Then, there
is no conformal metric $g = e^{2\rho}g_0$, $\rho \in C^{2,\alpha}(\mathcal{M})$, such that

$$\begin{align*}
f(\lambda(g)) &= 0 \quad \text{in } \mathcal{M}, \\
h(g) &= c \quad \text{on } \partial \mathcal{M},
\end{align*}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho}g_0$.

**Proof.** By a result of F. M. Spiegel [38], there exists a conformal diffeomorphism between the Riemannian manifold $(\mathcal{M},g_0)$ and the standard closed hemisphere $S^m_+$. This is done by using the developing map $\Phi$ of Schoen-Yau [34] and observing that umbilicity is preserved under conformal transformations. Hence, the boundary $\partial \mathcal{M}$ maps into a hypersphere in $S^m$ by $\Phi$ and therefore, $\Phi$ maps $\mathcal{M}$ into the interior of a ball whose boundary has constant mean curvature. Moreover, the elliptic problem on $\mathcal{M}$ pass to an elliptic problem for the pushforward metric on the geodesic ball. Thus, the statement follows from Theorem 3.1. \(\square\)

Observe that the above argument can be used into the non-degenerate case studied by Cavalcante-Espinar [6]. Specifically,

**Theorem 3.3.** Set $(f, \Gamma)$ an elliptic data for conformal metrics and $c \leq 0$ a constant. Let $(\mathcal{M},g_0)$ be a $m$-dimensional compact, simply-connected, locally conformally flat manifold with umbilic boundary and $R(g_0) \geq 0$ on $\mathcal{M}$. If there exists a conformal metric $g = e^{2\rho}g_0$, $\rho \in C^{2,\alpha}(\mathcal{M})$, such that

$$\begin{align*}
f(\lambda(g)) &= 1 \quad \text{in } \mathcal{M}, \\
h(g) &= c \quad \text{on } \partial \mathcal{M},
\end{align*}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho}g_0$, then $\mathcal{M}$ is isometric to a geodesic ball in the standard sphere $S^m$.

**Proof.** So, as above, there exists a conformal diffeomorphism from $\mathcal{M}$ to a ball into $S^m$ whose boundary has constant mean curvature. Moreover, the elliptic problem on $\mathcal{M}$ pass to an elliptic problem for the pushforward metric on the geodesic ball. Thus, by [6, Theorem 1.1], such pushforward metric has constant Schouten tensor, i.e., all the eigenvalues of the Schouten tensor are equal to the same constant. Hence, $\mathcal{M}$ is isometric to the geodesic ball in the sphere by the work of F. M. Spiegel [38]. \(\square\)

4. **Punctured geodesic ball.** Now, we see that any solution to a degenerate elliptic problem in the punctured geodesic ball $S^m_+ \setminus \{n\}$ with minimal boundary is rotationally invariant.
THEOREM 4.1. Let \( g = e^{2\rho} g_0 \) be a conformal metric in \( \mathbb{S}^m_+ \setminus \{ n \} \) that is solution to the following degenerate elliptic problem:

\[
\begin{cases}
  f(\varphi) = 0 & \text{in } \mathbb{S}^m_+ \setminus \{ n \}, \\
  h(\varphi) = 0 & \text{on } \partial \mathbb{S}^m_+, 
\end{cases}
\]

then \( g \) is rotationally invariant.

**Proof.** Let us define \( \tilde{\varphi} : \mathbb{S}^m_+ \setminus \{ n, s \} \to \mathbb{R} \) as

\[
\tilde{\varphi}(x) = \begin{cases}
  \varphi(x_1, \ldots, x_m) & x \in \mathbb{S}^m_+ \setminus \{ n \}, \\
  \varphi(x_1, \ldots, -x_m) & x \in \mathbb{S}^m_+ \setminus \{ s \}.
\end{cases}
\]

First, we show that \( \tilde{\varphi} \) is \( C^1 \). Since

\[
\frac{\partial \varphi}{\partial x_{m+1}} = 0 \quad \text{on } \partial \mathbb{S}^m_+
\]

then \( \tilde{\varphi} \in C^1(\mathbb{S}^m_+ \setminus \{ n, s \}) \). Therefore, the vector field \( \nabla \tilde{\varphi} : \mathbb{S}^m_+ \setminus \{ n, s \} \to T \mathbb{S}^m_+ \) given by

\[
\nabla \tilde{\varphi}(x) = \begin{cases}
  \nabla \varphi(x) & x \in \mathbb{S}^m_+ \setminus \{ n \}, \\
  R \nabla \varphi(R(x)) & x \in \mathbb{S}^m_+ \setminus \{ s \},
\end{cases}
\]

where \( R : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} \) is the Euclidean reflection \( R(x_1, \ldots, x_m) = (x_1, \ldots, -x_m) \), \( (x_1, \ldots, x_m) \in \mathbb{R}^{m+1} \), is continuous.

Now, let us see that \( \tilde{\varphi} \in C^2 \), that is, \( \nabla \tilde{\varphi} \) is \( C^1 \). Let \( X_1 = \nabla \varphi|_{\mathbb{S}^m_+ \setminus \{ n \}} \) and \( X_2 = \nabla \tilde{\varphi}|_{\mathbb{S}^m_+ \setminus \{ s \}} \). Since \( X_1 = X_2 \) on \( \partial \mathbb{S}^m_+ \), given \( x \in \partial \mathbb{S}^m_+ \) and \( v \in T_x(\partial \mathbb{S}^m_+) \) we have \( \nabla_v X_1 = \nabla_v X_2 \).

Also, assume for a moment that \( R(\nabla^2 \varphi(x)(e_{m+1})) = -\nabla^2 \varphi(x)(e_{m+1}) \) for all \( x \in \partial \mathbb{S}^m_+ \). We have that for \( x \in \partial \mathbb{S}^m_+ \) and \( v_m = e_{m+1} \),

\[
\nabla_{-v_m} X_2 = \frac{\partial X_2}{\partial (-v_m)}(x) + \langle X_2(x), -v_m \rangle x = \frac{\partial X_2}{\partial (-v_m)}(x)
\]

\[
= R \nabla_{v_m} X_1 = R(\nabla^2 \varphi(x)(e_{m+1})) = -\nabla_{v_m} X_1
\]

then \( \nabla \tilde{\varphi} \) is \( C^1 \), hence \( \tilde{\varphi} \) is \( C^2 \).

Since \( \tilde{\varphi} : \mathbb{S}^m_+ \setminus \{ n, s \} \to \mathbb{R} \) is \( C^2 \) and it is a solution to a degenerate problem, from Theorem 2.19, \( e^{2\tilde{\varphi}} g_0 \) is rotationally invariant, thus \( g \) is rotationally invariant.

In order to conclude the proof, let us see that

\[
R(\nabla^2 \varphi(x)(e_{m+1})) = -\nabla^2 \varphi(x)(e_{m+1}) \quad \text{for all } x \in \partial \mathbb{S}^m_+.
\]
For every \( v \in T_x \partial S^m_+ \) we have that \( \langle \nabla^2 \rho(x) (v), e_{m+1} \rangle = 0 \) then \( \nabla^2 \rho(x) \subset V \) where \( V = T_x \partial S^m_+ \). Since \( \nabla^2 \rho(x) : V \to V \) is symmetric, \( \nabla^2 \rho(x)(e_{m+1}) \| e_{m+1} \). Thus \( R(\nabla^2 \rho(x)(e_{m+1})) = -\nabla^2 \rho(x)(e_{m+1}) \).

**Definition 4.2.** We say that a conformal metric \( g = e^{2\rho} g_0 \) in \( \overline{S^m_+} \setminus \{n\} \) is a punctured solution to the problem

\[
\begin{aligned}
& f(\lambda(g)) = 0 \quad \text{in} \quad S^m_+ \setminus \{n\}, \\
& h(g) = 0 \quad \text{on} \quad \partial S^m_+,
\end{aligned}
\]

if it is a solution to (12) and \( \sigma = e^{-\rho} \) admits a \( C^2 \)-extension \( \tilde{\sigma} : \overline{S^m_+} \to \mathbb{R} \) with \( \tilde{\sigma}(n) = 0 \).

As a first observation, we have:

**Proposition 4.3.** Let \( g = e^{2\rho} g_0 \) be a punctured solution to (12), then the associated map \( \varphi_P : \overline{S^m_+} \setminus \{n\} \to \mathbb{R}^{m+1} \) in the Poincaré ball model, i.e.,

\[
\varphi_P(x) = \frac{1 - e^{-2\rho(x)} + \left| \nabla e^{-\rho(x)} \right|^2}{(1 + e^{-\rho(x)})^2 + \left| \nabla e^{-\rho(x)} \right|^2} x - \frac{1}{(1 + e^{-\rho(x)})^2 + \left| \nabla e^{-\rho(x)} \right|^2} \nabla (e^{-2\rho}(x)),
\]

for all \( x \in \overline{S^m_+} \setminus \{n\} \), admits a \( C^1 \)-extension to \( \overline{S^m_+} \) and it is regular at \( n \).

The following corollary follows from Theorem 4.1.

**Corollary 4.4.** Punctured solutions are rotationally invariant.

In the case of a non-degenerate elliptic problem, we have the following:

**Theorem 4.5.** Let \( g = e^{2\rho} g_0 \) be a conformal metric in \( \overline{S^m_+} \setminus \{n\} \) that is solution to the following non-degenerate elliptic problem:

\[
\begin{aligned}
& f(\lambda(g)) = 1 \quad \text{in} \quad \overline{S^m_+} \setminus \{n\}, \\
& h(g) = 0 \quad \text{on} \quad \partial S^m_+,
\end{aligned}
\]

then \( g \) is rotationally invariant.

**Proof.** If \( \rho \) admits a smooth extension \( \tilde{\rho} : \overline{S^m_+} \to \mathbb{R} \) (just \( C^2 \)) then the conformal metric \( g_1 = e^{2\tilde{\rho}} g_0 \), defined in \( \overline{S^m_+} \), is a solution to a non-degenerate problem with constant mean curvature on its boundary. Then \( g_1 \) is rotationally invariant, so, \( g \) is rotationally invariant.

If \( \rho \) does not admit smooth extension on \( \overline{S^m_+} \), then by [28], \( \rho \) is rotationally invariant, then \( g \) is rotationally invariant. This concludes the proof. \( \square \)
(a) The hypersurface $\Sigma$ is between two totally geodesic hypersurfaces: $E(a,0)$ and $E(0)$.

(b) Extension $\tilde{\Sigma}$ of $\Sigma$ by reflection w.r.t. totally geodesic hypersurfaces.

Figure 4. Extension of the compact hypersurface $\Sigma$.

5. Compact annulus. We consider an annulus in $\mathbb{S}^m$ whose boundary components are geodesic spheres, that is, the domains we will consider in this section are the sphere $\mathbb{S}^m$ minus two geodesic balls. Observe that, up to a conformal diffeomorphism, we can assume that the compact annulus is $\mathbb{A}(r)$, $r \in (0, \pi/2)$, with minimal boundary. Here,

$$\mathbb{A}(r) = \{ x \in \mathbb{S}^m : r < d_{\mathbb{S}^m}(x, n) < \pi/2 \}.$$

We prove:

**Theorem 5.1.** Set $r \in (0, \pi/2)$. If there is a solution $g = e^{2\rho}g_0$ of the following problem

$$\begin{cases}
f(\lambda(g)) = 0 & \text{in } \mathbb{A}(r), \\
h(g) = 0 & \text{on } \partial \mathbb{A}(r),
\end{cases}$$

then $g$ is rotationally invariant and unique up to dilations.

**Proof.** Let $g_1$ be a solution to the above degenerate problem. First, we show that $g_1$ is rotationally invariant. Using the same techniques from Cavalcante-Espinar in [6], we get a conformal metric in $\mathbb{S}^m \setminus \{n, s\}$ that is a solution to the degenerate problem. Let us sketch this for the reader convenience. Using the representation formula, the horospherical hypersurface associated to $g_1$ is contained in the slab determined by two parallel hyperplanes and, by the boundary condition, the boundaries meet orthogonally such hyperplanes (cf. Figure 4(a)). Thus, we can reflect this annulus with respect to the hyperplanes and we obtain a properly embedded horospherically concave hypersurface $\tilde{\Sigma}_1$ whose boundary at infinity are
the north and south pole and, by construction, the extension $\tilde{\Sigma}_1$ of $\Sigma_1$ is contained in the interior of an equidistant hypersurface to the geodesic joining the north and south pole, the radius of such equidistant is determined by the original annulus.

Then its horospherical metric $\tilde{g}_1$ is a solution to the degenerate problem in $S^m \setminus \{n, s\}$ (cf. Figure 4(b)). In fact, the metric $\tilde{g}_1$ is $C^2$ in $S^m \setminus \{n, s\}$ (one can proceed as in Theorem 4.1). From Theorem 2.19, we have that $\tilde{g}_1$ is rotationally invariant, thus $g_1$ is rotationally invariant.

Now, we show that it is unique up to dilations. Again, using the parallel flow, we can assume that the hypersurface $\Sigma_1$ associated to $g_1$ is an embedded horospherically concave hypersurface.

By Theorems 2.9 and 2.14, we can suppose that the hypersurface is between the totally geodesic hypersurfaces $E(0)$ and $E(a, 0)$, where $a = (\cot(r), \frac{1}{\sin(r)} n)$. Also, the hypersurfaces along the parallel flow will remain in such slab as embedded horospherically concave hypersurfaces.

Let $\Sigma$ be the hypersurface associated to $g$ and $\Sigma_1'$ the parallel flow of $\Sigma_1$. Since $\Sigma$ and $\Sigma_1$ are compact, there exists $t_0 \geq 0$ so that $\Sigma_1'$ has no intersection with $\Sigma$. That is, in the Poincaré ball model, $\Sigma_1'$ is in the exterior of $\Sigma$, thus we can decrease $t$ until $\Sigma$ touches $\Sigma_1'$ for the first time. We denote this first contact hypersurface as $\Sigma_1$. We have a contact point.

We claim that there is a boundary contact point. If not, let $\rho_1$ be the support function of $\Sigma_1$ and $\rho$ be the support function of $\Sigma$. We have that there is $x \in \mathbb{A}(r)$ such that $\rho_1(x) = \rho(x)$, and, also,

$$\rho_1 > \rho \quad \text{on } \partial \mathbb{A}(r),$$

by Theorem 2.18, $\rho_1(x) > \rho(x)$, which is a contradiction.

Even more, we claim that both components of $\partial \Sigma_1 = \partial_1 \cup \partial_2$ have a contact point with $\partial \Sigma = \partial_1 \cup \partial_2$, that is, $\partial_i \cap \partial_i \neq \emptyset$, $i = 1, 2$. Suppose that one component of $\partial \Sigma_1$ does not have a contact point with $\partial \Sigma$, we reflect both hypersurfaces with respect to the hyperplane that contains the component of $\partial \Sigma_1$ that touches $\partial \Sigma$. We have an extension $\Sigma_1'$ of $\Sigma_1$ and an extension $\Sigma'$ of $\Sigma$ such that they have an interior contact point and there is no boundary contact point. Then, repeating the arguments of the above paragraph, we get a contradiction.

Similar arguments show that at each level of $\mathbb{A}(r)$ there is a contact point between $\Sigma_1'$ and $\Sigma$, i.e., for every $s \in (r, \pi/2)$ there is $x \in \mathbb{A}(r)$, with $d_{S^m}(x, n) = s$, such that the support function of $\Sigma_1$ and the function support of $\Sigma$ take the same value at $x$. The proof is also by contradiction. If there is no such $x$, then we use the reflection to obtain a contradiction as above.

We know that $\rho_1$ and $\rho$ are rotationally invariant. Since at every level of $\mathbb{A}(r)$, there is a point at that level such that $\rho_1$ and $\rho$ are equal, we have that $\rho_1 = \rho$ on that level, then $\rho_1 = \rho$ on $\mathbb{A}(r)$. So, they are equal up to parallel flow. That is, $g_1$ is a dilation of $g$. This concludes the proof.
In case that the above elliptic data admits a punctured solution then there is no solution to the above problem. Specifically:

**Theorem 5.2.** Set \( r \in (0, \pi/2) \). If the degenerate elliptic data \((f, \Gamma)\) admits a punctured solution, then there is no solution to the following degenerate elliptic problem:

\[
\begin{aligned}
& f(\lambda(g)) = 0 \quad \text{in } \overline{A}(r), \\
& h(g) = 0 \quad \text{on } \partial A(r).
\end{aligned}
\]

**Proof.** We will prove this by contradiction. Suppose that there is a solution to the above problem. Arguing as in Theorem 5.1, such solution can be extended to a solution \( g = e^{2s}g_0 \) on \( \mathbb{S}_m^+ \setminus \{n\} \). Let \( \Sigma \) be the associated hypersurface to \( g \). Using the parallel flow, we can assume that it is a horospherically concave hypersurface contained in the component \( C \) of \( \mathbb{H}^{m+1} \setminus E(0) \) that contains \( n \).

Let \( g_P \) be a punctured solution to the problem (12). Since \( \partial \Gamma \) is a cone, the conformal metric \( e^{2s}g_P \) is also a solution to the problem (12), for every \( s \in \mathbb{R} \).

Since \( \sigma'' \) admits \( C^2 \)-extension with \( \tilde{\sigma}(n) = 0 \), by Theorems 2.9 and 2.14 there is \( s_0 > 0 \) such that for all \( s \geq s_0 \) the associated horospherical hypersurface \( \phi_s \) associated to the punctured solution \( e^{2s}g_P \) is embedded and its interior is contained in the same component \( C \). The family of hypersurfaces \( \{Q(s)\}_{s \geq s_0} = \{\text{Im}(\phi_s)\}_{s \geq s_0} \) converges, in the Poincaré ball model, to the inclusion \( \mathbb{S}_m^+ \setminus \{n\} \hookrightarrow \mathbb{S}^m \). Also, every \( Q(s) \) admits a \( C^1 \)-extension to \( n \) in the Poincaré ball model contained in \( \mathbb{R}^{m+1} \) and their tangent hyperplanes at \( n \) are parallel to the hyperplane \( x_{m+1} = 0 \).

There is a \( t_1 > 0 \) such that the associated hypersurface \( \Sigma_1 \) to \( g_1 = e^{2t_1}g \) intersects the family \( \{Q(s)\}_{s \geq s_0} \). Without loss of generality we assume that the associated hypersurface \( \Sigma \) to \( g \) intersects the family \( \{Q(s)\}_{s \geq s_0} \). Also, there is \( s_1 \geq s_0 \) such that:

1. \( Q(s_1) \cap \Sigma \neq \emptyset \),
2. for \( s > s_1 \) we have \( Q(s) \cap \Sigma = \emptyset \).

Hence, we have found a first contact point. Observe that such contact point cannot be at infinity, this follows since \( \Sigma \) is contained in the interior of the equidistant to a geodesic and the family \( Q(s) \) extends to \( n \) smoothly. Now, if the first contact point is an interior point then \( Q(s_1) \) and \( \Sigma \) are tangent at such point. If the first contact point occurs at the boundary, \( Q(s_1) \) and \( \Sigma \) are tangent at that point too, because all the hypersurfaces \( Q(s) \) are orthogonal to the totally geodesic hypersurface \( E(0) \) and \( \Sigma \) is orthogonal to \( E(0) \) too.

We reflect \( Q(s_1) \) and \( \Sigma \) with respect to the totally geodesic hypersurface \( E(0) \) and we obtain the horospherically concave hypersurfaces \( \tilde{Q} \) and \( \tilde{\Sigma} \) (cf. Figure 5). Their support functions are defined in \( \mathbb{S}^m \setminus \{n, s\} \). Let \( \rho_1 \) be the support function of \( \tilde{Q} \) and \( \tilde{\rho} \) be the support function of \( \tilde{\Sigma} \). Since there is a contact point in the interior
of \( \tilde{Q} \), there is \( 0 < \delta < \pi/2 \) such that

\[
\rho_1 > \tilde{\rho} \quad \text{on} \, \partial \Omega,
\]

where \( \Omega = \{ x \in S^m : \delta < d_{S^m}(x, n) < \pi - \delta \} \) and there exists \( x_0 \in \Omega \) such that \( \rho_1(x_0) = \tilde{\rho}(x_0) \), but this contradicts Theorem 2.18. This concludes the proof. \( \square \)

It is important to say that the \( \sigma_k \)-Yamabe problem in \( S^m_+ \{ n \} \) admits a punctured solution when \( 1 \leq k < m/2 \). That punctured solution is associated to

\[
e^{-\rho(x)} = \sigma(x) = \left( (1 + x_{m+1})^\beta + (1 - x_{m+1})^\beta \right)^{\frac{1}{\beta}} \quad x \in S^m_+ \{ n \},
\]

where \( \beta = 1 - m/(2k) < 0 \), these solutions were constructed by S.-Y. A. Chang, Z. Han, and P. Yang [7]. When \( m \) is even and \( k = m/2 \), the \( \sigma_k \)-Yamabe problem on the compact annulus has a solution \( g \) with \( \sigma_k(\lambda(g)) = 0 \) and minimal boundary (cf. Figure 6 when \( m = 3 \) and \( k = 1 \)).

Also, it is good to say that the assumption on the existence of the punctured solution is not a necessary condition for the non-existence of solutions to degenerate problems in the compact annulus with minimal boundary. We have seen that punctured solutions are rotationally invariant (cf. Corollary 4.4). For the \( \sigma_k \)-Yamabe problem when \( k > m/2 \), there is no solution to the degenerate problem with minimal boundary and, also, there is no punctured solution to these problems (cf. [7]).
6. Noncompact annulus with boundary. Now we focus on different boundary conditions in the annulus. At one boundary component we will impose mild conditions on the metric and at the other we will impose constancy of the mean curvature of the conformal metric. Our next result will say that any conformal metric \( g = e^{2\rho}g_0 \) in

\[ A(r, \pi/2) := \{ x \in S^m : r < d_{S^m}(x,n) \leq \pi/2 \} \]

satisfying a certain property at its end and solution to a degenerate problem with non-negative constant mean curvature on its boundary, has unbounded Schouten tensor. In other words, we establish a non-existence result for degenerate (and non-degenerate) elliptic equations in \( A(r, \pi/2) \). Specifically:

**Theorem 6.1.** Let \( r \in (0, \pi/2) \), \( c \geq 0 \) be a non-negative constant and \( g = e^{2\rho}g_0 \) be a conformal metric in \( A(r, \pi/2) \) that is solution to the following degenerate elliptic problem:

\[
\begin{cases}
    f(\lambda(g)) = 0 & \text{in } A(r, \pi/2), \\
    h(g) = c & \text{on } \partial S^m_+.
\end{cases}
\]

If \( e^{2\rho} + |\nabla \rho|^2 : A(r, \pi/2) \to \mathbb{R} \) is proper then \( \lambda(g) \) is unbounded.

**Proof.** The proof is by contradiction. Suppose that \( \lambda(g) \) is bounded. Using the parallel flow we can assume that \( \varphi_\rho : A(r, \pi/2) \to \mathbb{R}^{m+1} \subset \mathbb{R}^{m+1} \) is a proper
horospherically concave hypersurface. Recall that this property is invariant under the parallel flow. Consider the continuous extension $\Phi : \mathbb{A}(r) \to \mathbb{R}^{m+1}$ of $\varphi_P : \mathbb{A}(r, \pi/2) \to \mathbb{R}^{m+1} \subset \mathbb{R}^{m+1}$, defined by

$$
\Phi(x) = \begin{cases} 
\varphi_P(x) & x \in \mathbb{A}(r, \pi/2), \\
x & x \in S_r(n).
\end{cases}
$$

Consider the foliation of $\mathbb{H}^{m+1}$ by horospheres $\{H(s)\}_{s \in \mathbb{R}}$ having the north pole $\{n\}$ as the boundary at infinity, $s$ is the signed distance between $H(s)$ and the origin of the Poincaré ball model. Since $r > 0$, we have that there is $s_1 \in \mathbb{R}$ such that $H(s_1) \cap \text{Im}(\Phi) \neq \emptyset$ and $H(s) \cap \text{Im}(\Phi) = \emptyset$ for $s > s_1$.

Also, since $c \geq 0$, the angle between $\Sigma = \text{Im}(\Phi)$ and the equidistant $E(-c)$ is non-acute and the angle between the horosphere $H = H(s_1)$ and the equidistant is acute, hence the first contact point is at the interior of $\Sigma$ (cf. Figure 7 when $c = 0$). That is, there is $x \in \mathbb{A}(r)$ where $\phi_P(x) \in H$. Also $\phi_P^{-1}(H) \subset \mathbb{A}(r)$ is compact and there is $r < r_1 < \pi/2$ such that

$$x \in \phi_P^{-1}(H) \subset \mathbb{A}(r_1).$$

Let $\rho_0$ be the support function associated to the horosphere $H$ restricted to $\mathbb{A}(r_1)$. Then, $\rho_0(x) = \rho(x)$ and

$$\rho > \rho_0 \quad \text{on} \quad \partial \mathbb{A}(r_1),$$
hence, by Theorem 2.18, \( \rho(x) > \rho_0(x) \), which is a contradiction. This concludes the proof. \( \square \)

A similar conclusion can be obtained for conformal metrics that are solutions to a non-degenerate elliptic problem that satisfy certain mild conditions.

**Theorem 6.2.** Let \( 0 < r < \pi/2 \), \( c \in \mathbb{R} \) be a constant and \( g = e^{2\rho}g_0 \) be a conformal metric in \( A(r, \pi/2) \) that is solution to the following non-degenerate elliptic problem:

\[
\begin{aligned}
&f(\lambda(g)) = 1 \quad \text{in} \ A(r, \pi/2), \\
h(g) = c \quad \text{on} \ \partial S^{m+1}_{+}, \\
&\lim_{x \to q} \rho(x) = +\infty \quad \forall q \in \partial B_r(n).
\end{aligned}
\]

Set \( \sigma = e^{-\rho} \), if \( |\nabla \sigma|^2 \) is Lipschitz then \( \nabla^2(\sigma^2) \) is unbounded.

**Proof.** The proof is by contradiction. We suppose that \( \nabla^2 \sigma^2 \) is bounded. Using the parallel flow, we can assume that \( \phi : A(r, \pi/2) \to \mathbb{H}^{m+1} \) is a properly embedded horospherically concave hypersurface. Since \( h(g) = c \), we have that the boundary \( \partial \Sigma \) is contained in \( E(-c) \).

Take a closed ball \( Q \) with center the origin of the Poincaré model of radius big enough so that \( \partial \Sigma \) is in the interior of \( Q \). Since \( f \) is homogeneous of degree one and \( f(1, \ldots, 1) = 2 \), for each \( t_0 > 0 \) there is a constant \( 0 < \lambda_0 < 1/2 \) such that

\[
f(\lambda_0, \ldots, \lambda_0) = e^{-t_0}.
\]

We work in the Poincaré ball model. Consider the family of totally umbilic spheres in the Hyperbolic space centred at the \( x_{m+1} \)-axis, \( \{Z(s)\}_{s \in (-1, 1)} \), such that the principal curvatures are equal to

\[
k_0 = \frac{1 + 2\lambda_0}{1 - 2\lambda_0} > 1.
\]

Observe that the support function of all these totally umbilic spheres are solutions to the non-degenerate elliptic problem.

Consider the continuous extension \( \Phi : A(r) \to \mathbb{R}^{m+1} \) of \( \varphi_P : A(r, \pi/2) \to \mathbb{R}^{m+1} \), defined by

\[
\Phi(x) = \begin{cases} 
\varphi_P(x) & x \in A(r, \pi/2), \\
x & x \in S_r(n).
\end{cases}
\]

Since \( r > 0 \), there is \( \delta > 0 \) such that for all \( s \in (1 - \delta, 1) \):

1. \( Z(s) \cap \Sigma = \emptyset \),
2. \( Z(s) \cap Q = \emptyset \).
We take one of them, say $Z_0 = Z(s)$. In the Poincaré ball model, consider a circle centered at the origin 0 with radius $s$ and passing through the center of $Z_0$. Move $Z_0$ along this circle until we have the first totally umbilic hypersurface $\tilde{Z}_0$ touching the hypersurface. By item (2), the contact point is at the interior (cf. Figure 8 when $c = 0$). That is, there is $x \in \mathbb{B}(r)$ such that $\varphi_P(x) \in \tilde{Z}_0$. At such contact point, the canonical orientations of $\Sigma$ and $\tilde{Z}_0$ agree.

Let $\rho_0$ be the support function of $Z_0$ restricted to $\mathbb{B}(r)$, then we have that
\[
\rho \geq \rho_0 \text{ on } \mathbb{B}(r) \quad \text{and} \quad \rho(x) = \rho_0(x),
\]
then by the strong maximum principle, $\rho = \rho_0$. So, $\Sigma$ is part of a sphere, but $\Sigma$ has non-empty ideal boundary. This contradiction concludes the proof. \(\Box\)

7. The 2-dimensional case. As we have seen, the Schouten tensor is defined for Riemannian manifolds $(\mathcal{M}^m, g_0)$ when $m \geq 3$. Let us consider the conformal metric $g = e^{2\rho}g_0$, where $\rho \in C^{2, \alpha}(\mathcal{M})$, then we have the following relation:
\[
\text{Sch}(g) + \nabla^2 \rho + \frac{1}{2} |\nabla \rho|^2 g_0 = \text{Sch}(g_0) + \nabla \rho \otimes \nabla \rho,
\]
where $\nabla$, $\nabla^2$ are the gradient and the Hessian with respect the metric $g_0$ respectively, and $|\cdot|$ the norm with respect of $g_0$. 

\textbf{Figure 8.} $Z_0$ touching at the interior of $\Sigma$, case $c = 0$. 

In the case of the standard sphere \((S^m, g_0)\), we know that \(\text{Sch}(g_0) = \frac{1}{2} g_0\), then for every conformal metric \(g = e^{2\rho}g_0\), we have that

\[
\text{Sch}(g) + \nabla^2 \rho + \frac{1}{2} |\nabla \rho|^2 g_0 = \frac{1}{2} g_0 + \nabla \rho \otimes \nabla \rho. \tag{13}
\]

So, we can take the above expression as a definition of the Schouten tensor for a conformal metric to the standard one in domains of the sphere \(S^2\). From (13), we have

\[
\text{Tr}(g^{-1} \text{Sch}(g)) = e^{-2\rho} (1 - \Delta \rho),
\]

\[
\lambda_1 + \lambda_2 = R(g)/2,
\]

\[
\lambda_1 + \lambda_2 = K,
\]

where \(K\) is the Gaussian curvature of \(g = e^{2\rho}g_0\), then, if we ask for a conformal metric with constant scalar curvature (Liouville Problem), we get an equation that is similar to a conformally invariant equation in dimension \(m \geq 3\). This example says that the definition of the Schouten tensor for conformal metrics w.r.t. the standard metric in domains of the sphere \(S^2\), given by (13), makes sense. Then, we can consider more general elliptic problems for conformal metrics in \(S^2\).

We establish the analogous result we can obtain in the case of domains of \(S^2\) without proof. First, for geodesic disks we have:

**Theorem 7.1.** Let \((f, \Gamma)\) be a degenerate elliptic data for conformal metrics and let \(c \leq 0\) be a constant. Then, there is no conformal metric \(g = e^{2\rho}g_0\) in \(S^2_+\), \(\rho \in C^{2,\alpha}(\overline{S^2_+})\), satisfying

\[
\begin{cases}
 f(\lambda(g)) = 0 & \text{in } S^2_+, \\
 h(g) = c & \text{on } \partial S^2_+, 
\end{cases}
\]

where \(\lambda(g) = (\lambda_1, \lambda_2)\) is composed by the eigenvalues of the Schouten tensor of the metric \(g = e^{2\rho}g_0\).

Second, for compact annulus, we have the following non-existence result:

**Theorem 7.2.** If the problem (12) with \(m = 2\) admits a punctured solution, then there is no solution to the following degenerate elliptic problem:

\[
\begin{cases}
 f(\lambda(g)) = 0 & \text{in } A(r), \\
 h(g) = 0 & \text{on } \partial A(r),
\end{cases}
\]

where \(\lambda(g) = (\lambda_1, \lambda_2)\) is composed by the eigenvalues of the Schouten tensor of \(g\).

In this part, it is good to say that it is possible that the punctured solution in Theorem 7.2 might not exist. For example, the Yamabe problem, or Liouville Problem in the annulus \(A(r) \subset S^2\), \(0 < r < \pi/2\), has a solution with zero scalar curvature
and minimal boundary, then there is no punctured solution for the Yamabe problem on \( S^2_+ \setminus \{ n \} \).

The solution to that problem is given by the conformal metric \( g = e^{2\rho} g_0 \) in \( \overline{A}(r) \), \( 0 < r < \pi/2 \), where (cf. Figure 9)

\[
e^{2\rho(x,y,z)} = \frac{1}{\sigma^2(x,y,z)} = \frac{1}{1 - z^2} \quad \text{for all} \quad (x,y,z) \in \overline{A}(r).
\]

Analogously, in dimension \( m > 2 \), we can define the conformal metric \( g = e^{2\rho} g_0 \) in \( \overline{A}(r) \), \( 0 < r < \pi/2 \), given by

\[
e^{2\rho(x_1,\ldots,x_{m+1})} = \frac{1}{\sigma^2(x_1,\ldots,x_{m+1})} = \frac{1}{1 - x^2_{m+1}} \quad \text{for all} \quad (x_1,\ldots,x_{m+1}) \in \overline{A}(r),
\]

but, in this case, it has constant scalar curvature equals to \((m - 1)(m - 2) > 0\).

When \( m \) is even and \( k = m/2 \), this conformal metric is a solution to the degenerate \( \sigma_k \)-Yamabe problem in the compact annulus \( \overline{A}(r) \) with minimal boundary. Hence, we have

**Theorem 7.3.** Let \( r \in (0, \pi/2) \), \( c \geq 0 \) be a non-positive constant and \( g = e^{2\rho} g_0 \) be a conformal metric in \( \overline{A}(r, \pi/2) \) that is solution to the following degenerate elliptic problem:

\[
\begin{align*}
\left\{ \begin{array}{ll}
    f(\lambda(g)) = 0 & \text{in} \ A(r, \pi/2), \\
    h(g) = c & \text{on} \ \mathbb{S}^m_+ \cap \ A_r(\pi/2),
\end{array} \right.
\end{align*}
\]

If \( e^{2\rho} + |\nabla \rho|^2 : \mathbb{A}(r, \pi/2) \to \mathbb{R} \) is proper then \( \lambda(g) \) is not bounded.
REFERENCES

[1] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl. (9)* **55** (1976), no. 3, 269–296.

[2] V. Bonini, J. M. Espinar, and J. Qing, Correspondences of hypersurfaces in hyperbolic Poincaré manifolds and conformally invariant PDEs, *Proc. Amer. Math. Soc.* **138** (2010), no. 11, 4109–4117.

[3] H. L. Bray, The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature, Ph.D. thesis, Stanford University, 1997.

[4] S. Brendle and F. C. Marques, Scalar curvature rigidity of geodesic balls in $\mathbb{S}^n$, *J. Differential Geom.* **88** (2011), no. 3, 4109–4117.

[5] M. P. Cavalcante and J. M. Espinar, Uniqueness theorems for fully nonlinear conformal equations on subdomains of the sphere, *J. Math. Pures Appl.* **122** (2019), 49–66.

[6] S.-Y. A. Chang, Z.-C. Han, and P. Yang, Classification of singular radial solutions to the $\sigma_k$ Yamabe equation on annular domains, *J. Differential Equations* **216** (2005), no. 2, 482–501.

[7] S.-Y. A. Chang, F. Hang, and P. C. Yang, On a class of locally conformally flat manifolds, *Int. Math. Res. Not. IMRN* **2004** (2004), no. 4, 185–209.

[8] C. L. Epstein, The hyperbolic Gauss map and quasiconformal reflections, *J. Reine Angew. Math.* **372** (1986), 96–135.

[9] W. J. Firey, The determination of convex bodies from their mean radius of curvature functions, *Mathematika* **14** (1967), 7–21.

[10] K. Fukui and T. Nakamura, A topological property of Lipschitz mappings, *Topology Appl.* **148** (2005), no. 1-3, 143–152.

[11] M. J. Gursky and J. A. Viaclovsky, Volume comparison and the $\sigma_k$-Yamabe problem, *Adv. Math.* **187** (2004), no. 2, 447–487.

[12] F. Hang and X. Wang, A new approach to some nonlinear geometric equations in dimension two, *Calc. Var. Partial Differential Equations* **26** (2006), no. 1, 119–135.

[13] A. Jiménez, The Liouville equation in an annulus, *Nonlinear Anal.* **75** (2012), no. 4, 2090–2097.

[14] Z. R. Jin, A counterexample to the Yamabe problem for complete noncompact manifolds, *Partial Differential Equations (Tianjin, 1986)*, Lecture Notes in Math., vol. 1306, Springer-Verlag, Berlin, 1988, pp. 93–101.
[25] A. Li and Y. Y. Li, On some conformally invariant fully nonlinear equations, *Comm. Pure Appl. Math.* 56 (2003), no. 10, 1416–1464.

[26] ________, On some conformally invariant fully nonlinear equations. II. Liouville, Harnack and Yamabe, *Acta Math.* 195 (2005), 117–154.

[27] ________, A fully nonlinear version of the Yamabe problem on manifolds with boundary, *J. Eur. Math. Soc. (JEMS)* 8 (2006), no. 2, 295–316.

[28] Y. Y. Li, Conformally invariant fully nonlinear elliptic equations and isolated singularities, *J. Funct. Anal.* 233 (2006), no. 2, 380–425.

[29] ________, Degenerate conformally invariant fully nonlinear elliptic equations, *Arch. Ration. Mech. Anal.* 186 (2007), no. 1, 25–51.

[30] Y. Y. Li and L. Nguyen, A fully nonlinear version of the Yamabe problem on locally conformally flat manifolds with umbilic boundary, *Adv. Math.* 251 (2014), 87–110.

[31] ________, Harnack inequalities and Bôcher-type theorems for conformally invariant, fully nonlinear degenerate elliptic equations, *Comm. Pure Appl. Math.* 67 (2014), no. 11, 1843–1876.

[32] F. C. Marques, Existence results for the Yamabe problem on manifolds with boundary, *Indiana Univ. Math. J.* 54 (2005), no. 6, 1599–1620.

[33] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Differential Geom.* 20 (1984), no. 2, 479–495.

[34] R. Schoen and S.-T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, *Invent. Math.* 92 (1988), no. 1, 47–71.

[35] F. Schwartz, The zero scalar curvature Yamabe problem on noncompact manifolds with boundary, *Indiana Univ. Math. J.* 55 (2006), no. 4, 1449–1459.

[36] W. Sheng, N. S. Trudinger, and X.-J. Wang, The $k$-Yamabe problem, *Surveys in Differential Geometry. Vol. XVII*, Surv. Differ. Geom., vol. 17, Int. Press, Boston, MA, 2012, pp. 427–457.

[37] W.-M. Sheng, N. S. Trudinger, and X.-J. Wang, The Yamabe problem for higher order curvatures, *J. Differential Geom.* 77 (2007), no. 3, 515–553.

[38] F. M. Spiegel, Scalar curvature rigidity for locally conformally flat manifolds with boundary, preprint, [https://arxiv.org/abs/1511.06270](https://arxiv.org/abs/1511.06270).

[39] N. S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Scuola Norm. Sup. Pisa (3)* 22 (1968), 265–274.

[40] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* 12 (1960), 21–37.