Establishing Conservation Laws in Pair Correlated Many Body theories: T matrix Approaches

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We address conservation laws associated with current, momentum and energy and show how they can be satisfied within many body theories which focus on pair correlations. Of interest are two well known t-matrix theories which represent many body theories which incorporate pairing in the normal state. The first of these is associated with Nozieres Schmitt-Rink theory, while the second involves the t-matrix of a BCS-Leggett like state as identified by Kadanoff and Martin. T-matrix theories begin with an ansatz for the single particle self energy and are to be distinguished from Φ-derivable theories which introduce an ansatz for a particular contribution to the thermodynamical potential. Conservation laws are equivalent to Ward identities which we address in some detail here. Although Φ-derivable theories are often referred to as “conserving theories”, a consequence of this work is the demonstration that these two t-matrix approaches similarly can be made to obey all conservation laws. Moreover, simplifying approximations in Φ-derivable theories, frequently lead to results which are incompatible with conservation.

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I. INTRODUCTION

In this paper we consider approximate many body fermionic theories which emphasize pairing fluctuation effects in the normal state. Although there are a host of different scenarios for the centrally important “pseudogap”, such theories are of potential interest in high temperature superconductors. It has been conjectured that this gap in the fermionic excitation spectrum of the cuprates reflects pairing in advance of condensation and is to be associated with stronger than BCS attraction\textsuperscript{1,2}. Even more definitively, pairing (amplitude) fluctuation theories are of interest in ultracold Fermi gases where the interaction strength is tuneable and pairing necessarily occurs in the normal phase\textsuperscript{3}. Among the topics of particular current interest in these cold gases are transport phenomena. In this context, recent attention has focused on the shear viscosity both experimentally\textsuperscript{4–8} and theoretically\textsuperscript{9,10}. However, no calculation of a transport property can be considered meaningful without establishing conservation laws.

The goal of the present paper is to establish what are the requirements for arriving at an approximate “conserving theory” in the context of transport. Approximate many body theories of pairing correlations are of two types: either one begins with an ansatz for the self energy [t-matrix approach] or an ansatz for a component of the thermodynamical potential [“Φ-derivability” approach]. The latter category is more frequently associated with “conserving theories”\textsuperscript{11,12} and characterized as such in the literature. Here we emphasize that Φ-derivability is a sufficient but not necessary condition for arriving at a proper conserving theory. Moreover, the Φ-derivability conditions are often only approximately satisfied so that conservation laws cannot be proved to hold. Alternative theories known as t-matrix theories are the simplest category of pairing many body theories. Here one incorporates pairing effects between the fermions by considering the summation of a series of ladder diagrams in the particle-particle channel which then feeds back into a fermionic self energy. In this paper we consider the simplest t-matrix approach of Nozieres Schmitt-Rink theory\textsuperscript{11,12} as well as the t-matrix introduced by Kadanoff and Martin\textsuperscript{13} which includes more interaction effects and is chosen to be appropriate to BCS theory and its BCS-Leggett generalizations.\textsuperscript{14} We show here how to arrive at a proper conserving t-matrix-based transport theory.

In contrast to t-matrix schemes, the emphases of Φ-derivable theories is more directly on including multiple classes of many body diagrams, which are subject to internal consistency. This approach is based on the observation\textsuperscript{12} that if one starts with a contribution to the thermodynamical potential, Φ, of a certain form, then conservation laws follow. More specifically, consistency is represented by a key equation relating the self energy Σ to Φ. An additional consequence of precise Φ-derivability beyond the implications for transport, is that such theories, when specialized to the equilibrium case, obey the integrated form of conservation laws. Nevertheless, often there are approximations or truncations involved so that one may violate conservation laws. In principle, then, conservation law tests should also be applied to approximate Φ-derivable theories given there is no guarantee that these approaches are fully consistent.

A. Overview of Ward Identities (WI)

In this sub-section we provide a brief overview of conservation laws or the equivalent Ward identities which form the basis for validating many body theories and the
basis for the present paper. Our goal here is to introduce some of the concepts and notation which are later addressed in more detail. Here we envision systems subject to an external perturbation. The conservation laws of interest are local conservation laws:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \]
\[ \frac{\partial \mathbf{J}}{\partial t} + \nabla_i T_{ij} = 0, \]
\[ \frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J}_c = 0 \]

Here \( \rho \) and \( \rho_c \) are particle number and energy density, \( \mathbf{J} \), \( \mathbf{J}_c \), and \( T_{ij} \) are particle number, energy and momentum flow current respectively.

In the language of 4-vectors the conservation laws for particle current and the stress tensor take a more concise form

\[ \partial_\mu j^\mu = 0 \]
\[ \partial_\mu T^{\mu\nu} = 0 \]

Here the last equation can be written as \( \partial_i T^{0i} + \partial_i T^{ij} = 0 \). This 4-vector notation is convenient and we will use it throughout. In this way current and charge are combined, as are space and time and momentum and energy.

One can alternatively address many body systems in the absence of perturbations. Here we integrate the above expressions in the whole space to find global conservation laws which must be satisfied by many body systems. Comparing this to the analogue (Eq. (8)) for the number equation (the "contact terms") we see that for the stress tensor there are extra momentum factors multiplying each inverse Green’s function. For the interacting case, these extra momentum factors enter as the “contact terms”. Applying the above general expression to the 3 point correlation function case, we have

\[ \partial_\mu \langle T^{\mu\nu}(x)\phi(x_1)\cdots\phi(x_n)\rangle = -\sum_i \delta(x-x_i) \frac{\partial}{\partial y^{\nu}} \langle \phi(x_1)\cdots\phi(x_n)\rangle \]

Transferring to momentum and frequency space, we find

\[ q_\mu \Gamma^{\mu\nu}(K+Q,K)G(Q,K)G(K) = k^\nu G(K) - (k+q)^\nu G(K+Q) \]

or

\[ q_\mu \Gamma^{\mu\nu}(K+Q,K) = k^\nu G^{-1}(K+Q) - (k+q)^\nu G^{-1}(K) \]

Here we define 4-momentum as \( K = (\omega, \mathbf{k}) \) and \( Q = (q_0, \mathbf{q}) \). This last equation, to which we shall return later in the paper, is the general Ward Identity (WI) for the stress tensor vertex for both non-interacting or interacting systems. Comparing this to the analogue (Eq. (8)) for the number equation (the "U(1) current vertex") we see that for the stress tensor there are extra momentum factors multiplying each inverse Green’s function. For the interacting case, these extra momentum factors enter in subtle ways into the Feynman diagrams and make the establishment of WI for the interacting case more difficult than the U(1) current WI.

II. COMPARING T-MATRIX THEORIES WITH \( \Phi \)-DERIVATIVE THEORIES

For strongly correlated systems, such as ultra-cold Fermi gases in the unitary limit, perturbation calculations are not reliable because of the lack of small parameters. To capture the strong fluctuations, various approximation methods have been invented, one example is...
t-matrix theory. Our interest here will be on two models for the t-matrix in which the ladder contains one or more bare Green’s functions. The third alternative involving two dressed Green’s function in the ladder (which we refer to as “GG theory”) has many different versions. They appear generally distinct from the other two schemes and are more or less based on $\Phi$-derivable schemes.

We label the t-matrix-based approaches by “$G_0G$” (associated with Kadanoff and Martin) and “$G_0G_0$” (associated with Nozieres and Schmitt-Rink). More specifically, in a $G_0G$ t-matrix theory, the self-energy is dressed by the pair propagator as $\Sigma(K) = \sum_P t_{pg}(P) G_0(P - K)$. The pair propagator is given by the summation of infinite ladders made by bare and full Green’s functions as

$$t_{pg}(P) = \frac{g}{1 + g\chi(P)}$$  \hspace{1cm} (13)

$$\chi(P) = \sum_K G_0(P - K)G(K)$$  \hspace{1cm} (14)

The behavior in Nozieres Schmitt-Rink theory is rather similar except that all Green’s functions are bare

$$t^0_{pg}(P) = \frac{g}{1 + g\chi^0(P)}$$  \hspace{1cm} (15)

$$\chi^0(P) = \sum_K G_0(P - K)G_0(K)$$  \hspace{1cm} (16)

One might ask for the justification in considering one bare and one dressed Green’s function in the ladder series, as in the first case. This justification derives from its equivalence to BCS-like theories. To see this, we note that BCS theory can be viewed as incorporating virtual non-condensed pairs which are in equilibrium with the condensate and so have a vanishing “pair chemical potential”; that is, their excitation spectrum is gapless. We may interpret the t-matrix $t_{pg}(Q)$ as simply related to the propagator for non-condensed pairs. This t-matrix satisfies the Hugenholtz-Pines condition in the form

$$t_{pg}(Q = 0) = \infty \to \mu_{pair} = 0, T \leq T_c$$  \hspace{1cm} (17)

Moreover since,$ \Sigma_{ac}(K) = -\Delta_{ac}^2 G_0(-K)$ \hspace{1cm} (18)

one can use Eq. (17) to re-derive the BCS gap equation

$$\Delta_{ac}(T) = -U \int_0^\infty \frac{1 - 2f(E_k)}{2E_k}$$

with $E_k = (\epsilon_k - \mu)^2 + \Delta_{ac}^2, T \leq T_c$.  \hspace{1cm} (19)

In $G_0G$ theory, the self-energy reflects a dressing of one of the propagators. The connected part of the 2 particle Green’s function is also the t-matrix under this approximation. Higher order Green’s functions can be decomposed as 1 and 2 particle Green’s functions.

In evaluating the stress tensor we introduce an effective classical field and focus on the associated stress tensor vertex. By including certain vertex corrections associated with the self-energy, we demonstrate the WI is satisfied for this vertex. This should also imply the WI for the 1 and 2 particle Green’s functions. While the scheme is tractable we stress that a t-matrix approximation is clearly oversimplified as one can see that no full dressed internal vertex is introduced.

By contrast, in schematic form, the central equation of a $\Phi$-derivable theory is given by a constraint on the self energy in terms of $\Phi$ defined in terms of the thermodynamical potential $\Omega$ by

$$\Omega = \text{tr} \ln(-G) - \text{tr}(G_0^{-1}G - 1) + \Phi[G]$$  \hspace{1cm} (20)

such that

$$\Sigma(1') = \frac{\delta\Phi[G]}{\delta G(1')}$$  \hspace{1cm} (21)

Here we have introduced shorthand notation $1 \equiv (x_1, \tau_1)$, etc. Frequently, approximations need to be made. Throughout this paper when we refer to approximate $\Phi$-derivable theories we are not referring to given class of diagrams chosen to represent the thermodynamical potential. Rather we refer to the adoption of further approximations made within this scheme. Frequently these approximate theories omit some of the terms which should be present in the vertex function.

Without approximations, in this approach the one-particle Green’s functions satisfy the conservation laws and, because $\Phi$ is related to the thermodynamical potential, the two-particle Green’s functions satisfy thermodynamical consistency. In the most general conserving approximation, $\Phi$ can be represented as 2-particle-irreducible skeleton vacuum diagrams. In practice one has to choose a particular sub-class of diagrams and because of this truncation, not all the WI will necessarily be satisfied. Often approximations violate an important symmetry such as the crossing symmetry determined by the Pauli principle.

In another approach proposed by de Dominicis and Martin, one introduces the full dressed 2 particle scattering vertex which is determined via the parquet equations. In this approach, the crossing symmetry is respected but it does not guarantee the conservation laws. Motivated by these ideas, Bickers and Scalapino proposed the fluctuation exchange approximation (FLEX), which is based on a certain choice of $\Phi[G]$. This approximation has the advantage that it satisfies the conservation laws by construction. However, the disadvantage of FLEX is that the vertex which satisfies Ward identities is obtained at a different level of approximation than that at which the self-energy is computed. Therefore the calculation of the self-energy is performed with the vertices which do not satisfy Ward identities (see e.g. the discussion in Vilk and Tremblay). Similar ideas were formulated by Haussmann.

In summary, approximate conserving theories do not guarantee the satisfaction of conservation laws. That is, not all WI are automatically satisfied. In order to respect crossing symmetry, one has to treat the full vertex on the
III. MOMENTUM CURRENT WI FOR FREE GAS

The focus of this paper is the stress tensor Ward identity. To build our understanding we begin with the non-interacting gas. The Lagrangian for the non-interacting system treated as a Schrödinger field is

$$L(x) = \frac{i}{2} \bar{\psi} \frac{\partial}{\partial x} \psi - \frac{i}{2} \partial_x \bar{\psi} \psi - \frac{1}{2m} \partial_x \bar{\psi} \partial_x \psi + \mu \bar{\psi} \psi$$ (22)

Here we have taken a symmetric form for the time derivative. It follows that the equation of motion is

$$\frac{\partial L}{\partial \bar{\psi}} - \partial_t \left( \frac{\partial L}{\partial (\partial_t \psi)} \right) - \partial_x \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) = -i \partial_t \bar{\psi} + \frac{1}{2m} \partial^2 \bar{\psi} + \mu \bar{\psi} = 0$$ (23)

which is equivalent to the Schrödinger equation for a free Fermi gas.

The components of the canonical stress tensor involving momentum density and current momentum are given by

$$T^{0j} = -\left(\frac{i}{2} \bar{\psi} \frac{\partial}{\partial x} - \frac{i}{2} \partial_x \bar{\psi} \psi \right)$$ (24)

$$T^{ij} = \frac{1}{2m} \left(\partial_x \bar{\psi} \partial_j \psi + \partial_j \bar{\psi} \partial_x \psi \right) + \delta^{ij} L$$ (25)

which satisfy momentum current conservation

$$\partial_t T^{0j} + \partial_j T^{ij} = 0.$$ (26)

One can see that in a non-relativistic theory, the momentum density is essentially the same as the U(1) current $J^{0} = T^{0j} / m$.

In momentum space, if we assume $T^{0j}$ carries external momentum $Q$, then the bare vertices are

$$\gamma^{0j}(K + Q, K) = k^j + \frac{Q_j}{2}$$ (27)

$$\gamma^{ij}(K + Q, K) = \frac{(k + q)^i k^j + (k + q)^j k^i}{2m}$$

$$+ \delta^{ij} \left[ -\frac{(k + q) \cdot k}{2m} + (\omega + \frac{q^0}{2}) + \mu \right].$$ (28)

Taking a dot product with external momentum, we find

$$q^i \gamma^{ij} = \frac{(k + q) \cdot q k^j + k \cdot q (k + q)^j}{2m}$$

$$+ \mu q^j + (\omega + \frac{q^0}{2}) q^j \right].$$ (29)

Then it is straightforward to verify the WI for the bare vertex as

$$q_\mu \gamma^{0j}(K + Q, K) = k^j G_0^{-1}(K + Q) - (k + q)^j G_0^{-1}(K)$$ (30)

which is the result we cited earlier in the paper.

A. Response Functions of the stress tensor: the free gas

The physical properties of interest are the response functions. Once one establishes the proper form for a conserving theory of the stress tensor, it is possible to evaluate the general stress tensor response function given by

$$Q^{\mu j,ab}(x - y) = -i \theta(x^0 - y^0) \langle [T^{\mu j}(x), T^{ab}(y)] \rangle$$ (31)

We next explore the consequences of momentum conservation for the stress-stress correlations. The divergence of the stress-stress correlation function in coordinate space is

$$\partial_\mu Q^{\mu j,ab}(x - y) = -i \delta(x^0 - y^0) \langle [T^{0j}(x), T^{ab}(y)] \rangle$$ (32)

That the right hand side is not zero arises from the so-called “contact terms” which in turn arise from the time-ordering in the response function definition. The commutator with $T^{0j}(x)$ will generate spatial translation $[T^{0j}(x, t), \psi(y, t)] = i \nabla_y \psi(y, t) \delta^3(x - y)$. Then in momentum space, the above equation is

$$q_\mu Q^{\mu j,ab}(Q) = \sum_K [k^j G_0(K) - (k + q)^j G_0(K + Q)]$$

$$\times \gamma^{ab}(K + Q, K)$$ (33)

For the free Fermi gas, we can directly evaluate the correlation by diagrammatic methods

$$Q_0^{\mu \nu, \rho \lambda}(Q) = \sum_K \gamma^{\mu \nu}(K, K + Q) G_0(K + Q) \gamma^{\rho \lambda}(K + Q, K) G_0(K)$$ (34)

That these equations are consistent can be confirmed by making use of the WI for the bare vertex which yields,

$$q_\mu Q_0^{\mu j,ab}(Q) = \sum_K q_\mu \gamma^{0j}(K, K + Q) G_0(K + Q)$$

$$\times \gamma^{ab}(K + Q, K) G_0(K)$$

$$= \sum_K \left[ k^j G_0(K) - (k + q)^j G_0(K + Q) \right] \gamma^{ab}(K + Q, K)$$ (35)

which agrees with the general result Eq. (32).

B. A simplified momentum current vertex

In general, the stress tensor is not uniquely defined. Different forms for the stress tensor will lead to different forms of WI. The canonical stress tensor contains a time derivative which will make the frequency summation quite complex. A more convenient form for the stress tensor can be obtained by making use of the equation of
motion to get rid of the time derivative in $T^{ij}$. Then one finds
\begin{equation}
T^{0j} = -\frac{i}{2} \psi^\dagger \partial_j \psi - \frac{i}{2} \partial_j \psi^\dagger \psi
\end{equation}
\begin{equation}
T^{ij} = \frac{1}{2m} (\partial_i \psi^\dagger \partial_j \psi + \partial_j \psi^\dagger \partial_i \psi) - \delta^{ij} \frac{\partial^2 (\psi^\dagger \psi)}{4m}
\end{equation}
The corresponding vertices are given by
\begin{equation}
\lambda^{0j}(K + Q, K) = k^j + \frac{q^j}{2}
\end{equation}
\begin{equation}
\lambda^{ij}(K + Q, K) = \frac{(k + q)^i k^j + (k + q)^j k^i}{2m} + \delta^{ij} \frac{q^2}{4m}
\end{equation}
We refer to this representation as introducing the $\Lambda$ vertex.
For the bare $\Lambda$ vertex, one can verify that
\begin{equation}
q^i \lambda^{ij} = \frac{\langle k + q \rangle \cdot q k^j + k \cdot q (k + q)^j}{2m} - \frac{q^2 q^j}{4m}
\end{equation}
\begin{equation}
= (k^j + \frac{q^j}{2})(\xi_{k+q} - \xi_k)
\end{equation}
Thus we write the WI for this $\Lambda$ vertex as
\begin{equation}
q_\mu \lambda^{ij}(K + Q, K)
= (k^j + \frac{q^j}{2})[G^{-1}_0(K + Q) - G^{-1}_0(K)]
\end{equation}
which is importantly different from the WI discussed earlier.
Indeed, these two representations of the stress tensors are related by
\begin{equation}
T^{ij}_{\text{new}} = T^{ij}_{\text{old}} - \delta^{ij} \frac{1}{2} \left[ \psi^\dagger (i \partial_t + \frac{\nabla^2}{2m} + \mu) \psi \\
+ (-i \partial_t + \frac{\nabla^2}{2m} + \mu) \psi^\dagger \right]
\end{equation}
Thus we have
\begin{equation}
\partial_\mu \left\langle T^{ij}_{\text{new}}(x) \psi^\dagger(y) \psi(z) \right\rangle
= \partial_\mu \left\langle T^{ij}_{\text{old}}(x) \psi^\dagger(y) \psi(z) \right\rangle
+ \frac{1}{2} \partial_j \left\langle \psi^\dagger(x) (i \partial_t + \frac{\nabla^2}{2m} + \mu) \psi(x) \psi^\dagger(y) \psi(z) \right\rangle
+ \frac{1}{2} \partial_i \left\langle (-i \partial_t + \frac{\nabla^2}{2m} + \mu) \psi^\dagger(x) \psi(x) \psi^\dagger(y) \psi(z) \right\rangle
\end{equation}
In momentum space, we find
\begin{equation}
q_\mu \lambda^{ij} G_0(K + Q) G_0(K) = q_\mu \gamma^{ij} G_0(K + Q) G_0(K)
+ \frac{q^j}{2} \left[ G_0(K) + G_0(K + Q) \right]
\end{equation}
This equation connects the first and second versions of the stress tensor Ward identities.

In the $\Lambda$ vertex representation, the divergence of the stress-stress correlation function is again non-zero, but introduces somewhat different “contact terms”.
\begin{equation}
q_\mu Q^{ij,ab}(Q)
= \langle \left[ T^{0j}(q, t), T^{ab}(-q, t) \right] \rangle
= \sum_{p, k} \left( p + \frac{q}{2} \right) \lambda^{ab}(k + q, k) \langle [c_p^\dagger c_{p+q}, c_{k+q}^\dagger c_k] \rangle
\end{equation}
In the specific case of a free Fermi gas, the divergence of stress-stress correlation can be obtained by diagrammatic methods as
\begin{equation}
q_\mu Q^{ij,ab}_0(Q)
= \sum_{K} q_\mu \lambda^{ij}(K, K + Q) G_0(K + Q) \lambda^{ab}(K + Q, K) G_0(K)
= \sum_{K} \left( k + \frac{q}{2} \right) \lambda^{ab}(k + q, k) \langle c_{k}^\dagger c_{-k} - c_{k+q}^\dagger c_{-k-q} \rangle
\end{equation}
which is consistent with the WI for the $\Lambda$ vertex.

IV. STRESS TENSOR WARD IDENTITY FOR INTERACTING FERMIONS: NOZIERES SCHMITT-RINK AND $G_0G$ T-MATRIX THEORY

We now turn to addressing the stress tensor and WI in the interacting case. The contribution to the Lagrangian from interaction terms and the and the equation of motion in the presence of contact interactions are
\begin{equation}
L_{\text{int}}(x) = g \psi^\dagger \psi \psi \psi
- i \partial_t \psi^\dagger + \frac{1}{2m} \nabla^2 \psi^\dagger + \mu \psi^\dagger + 2g \psi^\dagger \psi^\dagger \psi = 0
\end{equation}
In interacting systems, we have to introduce a new term in the stress tensor which is first order in $g$
\begin{equation}
T_{1}^{ij} = g \delta^{ij} \psi^\dagger \psi^\dagger \psi \psi
\end{equation}
The remaining contribution to $T^{ij}$ is the same as in the free gas, which we refer to as $T_0$.
To verify the WI, we follow the standard textbook approach. We insert the stress tensor vertex in the self-energy diagram in all possible ways. Then for a specific class of diagrams we can establish whether or not the WI is satisfied. In what follows we will present results for the more complex $GGG_0$ case and note it is straightforward to extend these to the Nozieres Schmitt-Rink (NSR) case.
In order to handle the extra contribution, $T_1$ which is one order higher in $g$ than other terms, we have to insert $T_1$ into the appropriate lower order diagrams. Since $T_1$ has four field operators, it is sufficient to consider only insertions into the pair propagators.
Figure 1: The diagrams contributing to the stress tensor vertex $\Gamma$. The wiggly lines represent the $T$-matrix, thin (thick) solid lines are bare (dressed) Green’s functions, dashed lines are external stress tensor field. The small black dots represent bare vertex $T_0$ at zero order of $g$ and the small black square is bare vertex $T_1$ at first order of $g$. The larger open circle represents full vertex. Labels $MT$, $AL1$ and $AL2$ and $\delta \Gamma_1$ are defined in text. The lower panel is the corresponding diagrams contributing to the stress tensor correlation function. While the results are shown for the Kadanoff Martin $t$-matrix, one can readily deduce the counterpart diagrams for the Nozieres Schmitt-Rink $t$-matrix, by assuming that the $AL1$ and $AL2$ diagrams are equivalent.

A. The WI for the stress tensor ($\Gamma$) vertex

When we insert the bare vertex $\gamma$ into the self-energy in all possible ways this leads to three types of vertex corrections. Two of these are associated with known literature contributions: the Aslamazov Larkin (AL) diagrams and the Maki-Thompson (MT) diagrams which are defined as

\begin{align}
\delta \Gamma_{MT}^{\mu j}(K+Q,K) &= \sum_P t_{pg}(P) G_0(P-K-Q) \gamma_{\mu j}(P-K-Q,P-K) G_0(P-K) \\
\delta \Gamma_{AL1}^{\mu j}(K+Q,K) &= -\sum_P t_{pg}(P+Q) \left[ \sum_{P_3,P_4} G_0(P_3+Q) \gamma_{\mu j}(P_3+Q,P_3) G_0(P_3) G(P_4) \right] t_{pg}(P) G_0(P-K) \\
\delta \Gamma_{AL2}^{\mu j}(K+Q,K) &= -\sum_P t_{pg}(P+Q) \left[ \sum_{P_3,P_4} G_0(P_3) G(P_4+Q) \Gamma^{\mu j}(P_4+Q,P_4) G(P_4) \right] t_{pg}(P) G_0(P-K)
\end{align}

In establishing the diagram set for simple ($U(1)$) number current conservation, three types of vertex corrections are sufficient to prove the WI. We will see that for the stress tensor WI, we need more diagrams; these come from inserting the $T_1$ operator into the self-energy diagram.

Our derivation of the WI involves a $q$ dot product. We write this down first for the sum of AL diagrams which
In summary, by construction we have established a conserving diagram set for the Kadanoff-Martin series (as well as for the simpler NSR 

Finally, collecting all terms we find the stress tensor vertex satisfies a simple relationship (the associated Ward

As noted above, $T_1$ has four legs. Thus we can directly insert it into $t_{pg}$. This leads to the following new correction terms as

There is an extra $-1/g$ factor because $T_1$ must be inserted into a lower level diagram. Combining this with the AL diagrams, we find

For the MT diagram the dot product yields

Finally, collecting all terms we find the stress tensor vertex satisfies a simple relationship (the associated Ward identity):

The upper panel in Figure 1 shows the right hand side of a self consistent equation for the vertex appearing in the stress tensor correlation functions. The lower panel shows the diagrammatic series (which includes the self consistently determined vertex) which must be evaluated to obtain the stress tensor- stress tensor correlation. This correlation function would enter into the shear viscosity as obtained from Eq. (64). While the figure is explicitly for the Kadanoff-Martin t-matrix, the results associated with the t-matrix of Nozieres and Schmitt-Rink can be readily obtained by taking the AL2 diagram to be equivalent to AL1. Thus, for a $G_0G_0$ t-matrix there is no self consistency required to obtain the corresponding vertex function. It should be noted that, if one is interested in the shear viscosity only, a simpler approach is to start with the current-current correlation functions as in Eq. (59). However, for the case of NSR theory, because there is no self consistency required, the stress tensor correlation functions are somewhat more tractable.
V. RELATION TO VISCOSITY

Once we have a WI for the momentum flux current we have a means of evaluating viscosity in terms of stress tensor correlations. One can equally well address the viscosity in terms of current-current correlations first following Luttinger and then find the correspondence with the stress tensor correlations.

Assuming that an external vector potential is applied to the fluid, one can readily deduce the conductivity from the Kubo formula as

$$\sigma_{ij}(\omega) = \frac{im}{m\omega^2} \delta_{ij} + \frac{1}{\omega^2} \int_0^{\infty} e^{i\omega t} \lim_{q \to 0} \langle [J_i(q), J_j(-q)] \rangle$$

(52)

A Kubo formulation of the viscosity is, however, more subtle. Here one makes use of the linearized hydrodynamic equations in momentum space

$$-\omega n + n_0 q_i v_i = 0$$

(53)

$$i\omega (\varepsilon - w_0 n) \to 0 \text{ at small } q$$

(54)

$$-imn_0 v_i = q_i E_i - iq_i p$$

(55)

where $n_0, w_0$ are the equilibrium value of density and enthalpy per particle. We define $n, p, \varepsilon$ as fluctuations around equilibrium values of density, pressure and energy density, respectively. These fluctuations and the velocity $v_i$ are considered as first order quantities when we linearize hydrodynamic equations.

In the uniform or $q \to 0$ limit, we have

$$n = n_0 q_i v_i, \quad v_i = -\frac{E_i}{im\omega}, \quad \varepsilon = w_0 n,$$

$$p = \left(\frac{\partial p_0}{\partial n_0}\right)_T \varepsilon + \left(\frac{\partial p_0}{\partial E_0}\right)_n \varepsilon = (n_0 \kappa_S)^{-1} n$$

(56)

where $\kappa_S = n_0^{-1} (\partial n_0 / \partial p_0)_S$ is the adiabatic compressibility. If we substitute all the above into Eq. (52) and also use $J_i = n_0 v_i$, we find

$$J_i = \frac{im_0}{m\omega} E_i + \frac{i\kappa_S^{-1} q_i (q_i E_k)}{m^2 \omega^3}$$

$$+ \frac{1}{m^2 \omega^2} \left[ q_i^2 E_i + (\zeta + \frac{1}{3})q_i (q_i E_k) \right]$$

(57)

We can decompose any correlation function $\chi$ into transverse and longitudinal components as

$$\chi^{ij}_{JJ} = \chi^{T}_{JJ} \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right) + \chi^{L}_{JJ} \frac{q^i q^j}{q^2}$$

(58)

If we take $q_i \perp E_i$ and $q_i \parallel E_i$ respectively, we find shear and bulk viscosity as

$$\eta = \lim_{q \to 0} m^2 \omega \chi^{T}_{JJ},$$

$$\zeta + \frac{4}{3} \eta = \lim_{q \to 0} m^2 \omega \chi^{L}_{JJ} - \frac{i\kappa_S^{-1}}{\omega}$$

(59)

(60)

From the momentum flux current WI, we have $\partial_i T^{0j} + \partial_j T^{0i} = 0$ and $T^{0i} = mJ^i$. Thus we find an important relation between the current current and stress tensor correlation functions which is given by

$$m^2 \omega^2 \chi^{ij}_{JJ} = q^i q^j \chi_T^{T} + m q^i \langle [J^i(q), T^{j}(q)] \rangle$$

(61)

Eq. (61) is more subtle than one might have inferred owing to the extra commutator, which is sometimes ignored in the literature.

If we introduce the viscosity tensor

$$\eta_{ia,jb} = \eta (\delta_{ij} \delta_{ab} + \delta_{ib} \delta_{aj} - \frac{2}{3} \delta_{ia} \delta_{jb}) + \zeta \delta_{ia} \delta_{jb}$$

(62)

then we have

$$\eta_{ia,jb} \frac{q^a q^b}{q^2} = \lim_{q \to 0} \left( \frac{q^a q^b \chi_{ia,jb}^{T} + m q^b}{\omega q^2} \langle [J^i(q), T^{j}(q)] \rangle \right)$$

$$- \frac{i n_{s}^{-1}}{\omega} \frac{q^a q^b}{q^2}$$

(63)

For arbitrary $q$, we thus have arrived at an expression for the shear viscosity in terms of stress tensor correlation functions

$$\eta_{ia,jb} = \lim_{q \to 0} \left( \frac{\chi_{ia,jb}^{T}}{\omega} + \frac{m}{\omega} \frac{\partial}{\partial q_a} \langle [J^i(q), T^{j}(q)] \rangle \right)$$

$$- \frac{i n_{s}^{-1}}{\omega} \delta_{ia} \delta_{jb}$$

(64)

We see that the shear viscosity is dependent not only on the stress tensor correlation function but also on two additional terms involving the adiabatic compressibility and the additional commutator (or contact terms). This expression was derived earlier by N. Read and colleagues.

VI. ENERGY CURRENT WI

Establishing the Ward identity associated with energy conservation is essential for addressing transport coefficients such as thermopower and thermal conductivity. Within the 4-vector notation energy and energy current involve components of $T^\mu\nu$. These are respectively given by

$$T^{00} = \frac{1}{2m} \partial_i \psi \partial^i \psi - \mu \psi \partial^i \psi$$

(65)

$$T^{ij} = -\frac{1}{2m} (\partial_j \psi \partial^i \psi + \partial_i \psi \partial^j \psi)$$

(66)

From Eq. (23), they are interconnected through the conservation law $\partial_i T^{0j} + \partial_j T^{0i} = 0$. In computing a thermal response, one needs the bare and dressed vertices. The bare vertex is given by

$$\gamma^{00} (K + Q, K) = \frac{(k + q) \cdot k}{2m} - \mu$$

(67)

$$\gamma^{ij} (K + Q, K) = \frac{1}{2m} (\Omega + \omega) k^j + \omega (k + q)^j$$

(68)
which can be shown to satisfy
\[ q_\mu \gamma^{\mu 0}(K + Q, K) = \xi_k \Omega - \omega(\xi_{k+q} - \xi_k) \]
\[ = \omega G_0^{-1}(K + Q) - (\omega + \Omega)G_0^{-1}(K) \]

This, then presents a form of “template” for the form of the Ward identity associated with energy conservation in the dressed vertex.

A. Energy current WI for the interacting case: t-matrix theory

We can prove the energy current WI in a very similar way as was done for the momentum current WI. Here the vertex \(\gamma^{\mu j}\) will be replaced by \(\gamma^{\mu 0}\). For the sum of AL diagrams, we have

\[ q_\mu \left[ \delta \Gamma^{\mu 0}_A L_1(K + Q, K) + \delta \Gamma^{\mu 0}_A L_2(K + Q, K) \right] = \sum_p \left[ (p + q)^0 t_{pg}(P)G_0(P - K) - p^0 t_{pq}(P + Q)G_0(P - K) \right] \tag{69} \]

Combining this with the AL diagrams, we find

\[ q_\mu \left[ \delta \Gamma^{\mu 0}_A L_1(K + Q, K) + \delta \Gamma^{\mu 0}_A L_2(K + Q, K) \right] = \sum_p \left[ (p + q)^0 t_{pg}(P)G_0(P - K) - p^0 t_{pq}(P + Q)G_0(P - K) \right] \tag{70} \]

For the MT diagrams we have

\[ q_\mu \delta \Gamma^{\mu 0}_M T(K + Q, K) = \sum_p t_{pg}(P) \left[ (p - k - q)^0 G_0(P - K - Q) - (p + k - q)^0 G_0(P - K) \right] \tag{71} \]

Collecting all results we find

\[ q_\mu \delta \Gamma^{\mu 0}(K + Q, K) = \sum_p \left[ (k + q)^0 t_{pg}(P)G_0(P - K) - k^0 t_{pq}(P)G_0(P - K - Q) \right] \tag{72} \]

Here \(\delta \Gamma^{\mu 0} = \delta \Gamma^{\mu 0}_A L_1 + \delta \Gamma^{\mu 0}_A L_2 + \delta \Gamma^{\mu 0}_A L_3\). This equation which is closely analogous to Eq. \(51\) for the momentum current Ward identity is the desired WI for energy conservation.

VII. CONCLUSION

In this paper we have examined the requirements for arriving at a consistent theory of transport. As noted in seminal earlier work, “In describing transport it is vital to build the conservation laws of number, energy, momentum and angular momentum into the structure of the approximation used to determine the thermodynamic many-particle Green’s functions.” As a sequel to this earlier study, Baym was led to formulate a \(\Phi\)-derivable theory. What we have emphasized here is that \(\Phi\)-derivability is sufficient, but not necessary. When approximations are made to the self consistency conditions within this scheme, there is no guarantee that a theory is conserving. A somewhat more tractable approach, which we apply here, is to begin with an ansatz for the self energy. Here we take as an example a t-matrix theory which incorporates pairing fluctuations relevant to the ultracold Fermi gases, and more general strongly correlated superconductors and superfluids. We have considered two simple t-matrix theories, that of Nozieres and Schmitt-Rink and the more self consistent t-matrix of Kadanoff and Martin.

We show how to construct the diagrammatic series for
the response functions in order to be consistent with local conservation laws, via Ward identities. These Ward identities become particularly complicated and not as well known to the condensed matter community for the case of momentum conservation (which relates to the viscosity calculations). This is the reason we have devoted more attention to the stress tensor here. Nevertheless, we have addressed local number, and energy conservation Ward identities as well. The central finding of this work was the demonstration that these t-matrix theories, which are not $\Phi$-derivable, are indeed "conserving" as required for a consistent theory of transport.

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Appendix A: The stress tensor WI for $\Lambda$ vertex

It is often more convenient to work with the $\Lambda$ vertex introduced in the text. While the Ward identities of the bare vertices are different, we will see that the same set of diagrams are also sufficient to show the WI is satisfied for the $\Lambda$ vertex. Notably the first order vertex term $T^i_1$ is different from the previous case, but the Lagrangian and equation of motion in the presence of interactions are the same. The equation of motion for $T^{ij}$ is

$$T^{ij} = \frac{1}{2m} (\partial_i \psi^\dagger \partial_j \psi + \partial_j \psi^\dagger \partial_i \psi) - \delta^{ij} \left( \frac{\partial^2 (\psi^\dagger \psi)}{4m} + g \psi^\dagger \psi \psi^\dagger \psi \right)$$  

(A1)

The interaction vertex

$$T_{1}^{ij} = -g \delta^{ij} \psi^\dagger \psi^\dagger \psi \psi$$  

(A2)

has a different sign, as compared to its counterpart.

By inserting the bare vertex $\lambda$ into the t-matrix self-energy, we still find three types of vertex corrections $\delta \Lambda^{\nu_1}_{AL1}$, $\delta \Lambda^{\nu_2}_{AL2}$ and $\delta \Lambda^{\mu_1}_{MT}$ which are the same as before but with $\Gamma$ vertex replaced by $\Lambda$ vertex. Now the $q$ dot product with the sum of AL diagrams gives

\[ q_{\mu} \left[ \delta \Lambda^{\nu_1}_{AL1}(K + Q, K) + \delta \Lambda^{\nu_2}_{AL2}(K + Q, K) \right] \]

\[ = - \sum_{P} t_{pq}(P + Q) \sum_{P_3 P_4} \left[ (p_3 + \frac{q}{2}) G_0(P_3)G_0(P_4) + (p_4 + \frac{q}{2}) G_0(P_3)G_0(P_4) \right] t_{pq}(P)G_0(P - K) \]

\[ + \sum_{P} t_{pq}(P + Q) \sum_{P_3 P_4} \left[ (p_3 + \frac{q}{2}) G_0(P_3 + Q)G_0(P_4) + (p_4 + \frac{q}{2}) G_0(P_3)G_0(P_4 + Q) \right] t_{pq}(P)G_0(P - K) \]

\[ = \sum_{P} t_{pq}(P + Q) \left[ (p + q)^2 \chi(P) - p^2 \chi(P + Q) \right] t_{pq}(P)G_0(P - K) \]

(A3)

By inserting $T_1$ into $t_{pq}$, we find

$$\delta \Lambda^{ij}_{1}(K + Q, K) = -\frac{\delta^{ij}}{g} \sum_{P} t_{pq}(P + Q)t_{pq}(P)G_0(P - K)$$  

(A4)

which is the same as before except for an overall sign change. This term can be rewritten in two different ways as

$$\delta \Lambda^{ij}_{1}(K + Q, K) = -\delta^{ij} \sum_{P} \left[ 1 - t_{pq}(P + Q)\chi(P + Q) \right] t_{pq}(P)G_0(P - K)$$

\[ = -\delta^{ij} \sum_{P} \left[ 1 - t_{pq}(P)\chi(P) \right] t_{pq}(P + Q)G_0(P - K) \]

Taking the average of the above two equations, we have

$$\delta \Lambda^{ij}_{1}(K + Q, K) = \delta^{ij} \left[ t_{pq}(P + Q) \chi(P + Q) + \chi(P) \right] t_{pq}(P)G_0(P - K) - \frac{\Sigma(P + Q) + \Sigma(P) + \Sigma(P) - \Sigma(P)}{2}$$

(A5)
Combining $\delta \Lambda_{\mu i}^{\nu j}$ with the AL diagrams, we find
\[
q_\mu \left[ \delta \Lambda_{AL1}^{\mu j}(K + Q, K) + \delta \Lambda_{AL2}^{\mu j}(K + Q, K) + \delta \Lambda_{AL1}^{\nu j}(K + Q, K) \right]
= \sum_p t_{pg}(P_1 + Q)(p + \frac{q}{2})^j \left[ \chi(P + Q) - \chi(P) \right] t_{pg}(P) G_0(P - K)
= \sum_p \left[ (p + \frac{q}{2})^j t_{pg}(P) G_0(P - K) - t_{pg}(P + Q) G_0(P - K) \right]
= \sum_p \left[ (p + \frac{q}{2})^j t_{pg}(P) G_0(P - K) - (p - \frac{q}{2})^j t_{pg}(P) G_0(P - Q - K) \right]
\] (A6)

For the MT diagrams we have
\[
q_\mu \delta \Lambda_{MT}^{\mu j}(K + Q, K) = \sum_p (p - k - \frac{q}{2})^j \left[ t_{pg}(P) G_0(P - K - Q) - t_{pg}(P) G_0(P - K) \right]
\] (A7)

Collecting all results we find
\[
q_\mu \left( \delta \Lambda_{AL1}^{\mu j} + \delta \Lambda_{AL2}^{\mu j} + \delta \Lambda_{AL1}^{\nu j} + \delta \Lambda_{AL1}^{\nu j} + \delta \Lambda_{MT}^{\mu j} \right)(K + Q, K)
= \sum_p \left[ (k + q)^j t_{pg}(P) G_0(P - K) - k^j t_{pg}(P) G_0(P - Q - K) \right] - \frac{q^j}{2} \left[ \Sigma(P + Q) + \Sigma(P) \right]
= (k + \frac{q}{2})^j \left[ \Sigma(K) - \Sigma(K + Q) \right]
\] (A8)

Combining this equation with the bare WI, we find our desired WI for the t-matrix self-energy.
\[
q_\mu \Lambda_{\mu j}(K + Q, K) = (k + \frac{q}{2})^j \left[ G^{-1}(K + Q) - G^{-1}(K) \right]
\] (A9)

**Appendix B: A simplified vertex $\lambda$ for energy current**

As in the momentum current case, we can use the equation of motion to get rid of the time derivative in $T^{00}$ and $T^{ij}$. In this way, we find a simplified vertex which will make the frequency summation easier when computing the energy current response functions. The result is
\[
T^{00} = \frac{1}{2m} \partial_t \psi^\dagger \partial_k \psi - \mu \psi^\dagger \psi
\] (B1)
\[
T^{ij} = \frac{i}{2m} \left[ \partial_j \psi^\dagger \left( -\frac{\partial^2 \psi}{2m} - \mu \psi \right) + \left( \frac{\partial^2 \psi^\dagger}{2m} + \mu \psi^\dagger \right) \partial_j \psi \right]
\] (B2)

The bare vertex is
\[
\lambda^{00}(K + Q, K) = \frac{(k + q) \cdot k}{2m} - \mu
\] (B3)
\[
\lambda^{ij}(K + Q, K) = \frac{1}{2m} \left( (k + q)^j \xi_k + k^j \xi_{k+q} \right)
\] (B4)

Then it can be seen that
\[
q^j \lambda^{00} = \frac{1}{2m} \left[ q \cdot (k + q) \xi_k + q \cdot k \xi_{k+q} \right] = (\xi_{k+q} - \xi_k)(\xi_k + \frac{q \cdot k}{2m})
\] (B5)

from which we obtain the WI for this bare vertex
\[
q_\mu \lambda^{00}(K + Q, K) = \left( \frac{(k + q) \cdot k}{2m} - \mu \right) \left[ G_0^{-1}(K + Q) - G_0^{-1}(K) \right]
\] (B6)

We next use Eq. (B1) to write the divergence of the energy current correlation function as
\[
q_\mu Q^{\mu 0,00}(Q) = \langle [T^{0j}(q, t), T^{00}(-q, t)] \rangle
\] (B7)
and evaluate the commutator as

\[ \langle [T^{00}(q, t), T^{a0}(-q, t)] \rangle = \sum_k \left( \frac{(k + q) \cdot k}{2m} - \mu \right) \lambda^{ab}(k + q, k) (c_{k+q}^a k - c_{k+q}^b) \]

To validate this result, we may use diagrammatic methods to directly obtain

\[ q_\mu Q^{\mu,0}_0(Q) = \sum_K q_\mu \lambda^{\mu0}(K, K + Q) G_0(K + Q) \lambda^{0a}(K + Q, K) G_0(K) \]

\[ = \sum_K \left( \frac{(k + q) \cdot k}{2m} - \mu \right) \left[ G_0(K) - G_0(K + Q) \right] \lambda^{a0}(K + Q, K) \]

which, with the WI, establishes consistency.

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