COMPARING THE ISOMORPHISM TYPES OF EQUIVALENCE STRUCTURES AND PREORDERS

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Abstract. A general theme of computable structure theory is to investigate when structures have copies of a given complexity $\Gamma$. We discuss such problem for the case of equivalence structures and preorders. We show that there is a $\Pi^0_1$ equivalence structure with no $\Sigma^0_1$ copy, and in fact that the isomorphism types realized by the $\Pi^0_1$ equivalence structures coincide with those realized by the $\Delta^0_2$ equivalence structures. We also construct a $\Sigma^0_1$ preorder with no $\Pi^0_1$ copy.

1. Introduction

A primary question of computable structure theory is to ask, given a familiar class of countable structures $\mathfrak{A}$, which of the isomorphism types of $\mathfrak{A}$ can be realized by computable structures. Sometimes the answer is trivial: for instance, all countable vector spaces over $\mathbb{Q}$ have a computable copy \cite{1}; on the other hand, Peano Arithmetic has only one computable model \cite{2}. In other cases, such as abelian $p$-groups, one (partially) answers in terms of a class of invariants \cite{3}. But many natural classes (e.g., linear orders, Boolean algebras, or the class of all graphs) are simply too rich to hope for a nice characterization of which members of the class have a computable copy.

An obvious way of generalizing the above problem is to investigate when structures have copies of some complexity $\Gamma$ — where $\Gamma$ is some level of the arithmetical, hyperarithmetical, analytical, or Ershov hierarchy — and then studying the relation between the isomorphism types realized by structures of different complexity. Interestingly, it can be the case that all the isomorphism types realized by structures of a certain complexity are already realized by structures of considerably less complexity: e.g., Spector famously proved that every hyperarithmetic ordinal has a computable copy \cite{4}. In this paper, we contribute to this research thread by focusing on two case-studies: equivalence structures and preorders.

Equivalence structures are among the simplest structures which are non-trivial. For this reason, they form a class which is often reasonably tame but

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still worth examining. In fact, despite of their structural simplicity, equivalence structures have also remarkably deep effective properties: Calvert, Cenzer, Harizanov, and Morozov [5] proved that the index set of the computable equivalence structures isomorphic to a given one may be \( \Pi^0_4 \)-complete; Downey, Melnikov, and Ng showed \([6]\) that there is an effective list of all computable equivalence structures, thus classifying them in a sense proposed by Goncharov and Knight \([7]\). Cenzer, Harizanov, and Remmel \([8]\) started the study of the isomorphism types of \( \Sigma^0_1 \) and \( \Pi^0_1 \) equivalence structures; our work here extends their study.

Another piece of motivation for focusing on equivalence structures comes from the fact that measuring the complexity of equivalence relations with domain \( \omega \) has been a longstanding endeavour in the literature. In particular, researchers investigated for decades the effective properties of computably enumerable equivalence relations (called ceers, after Gao and Gerdes \([9]\)) and they compared them via a natural effectivization of Borel reducibility called computable reducibility: see, e.g., \([10, 11, 12, 13, 14]\). In comparison with ceers, the study of equivalence relations of higher complexity has been less extensive. Yet, computable reducibility has been used to analyze equivalence relations of various complexity, including some that are not even hyperarithmetical (such as the isomorphism relations for familiar classes of computable structures \([15]\)). In recent times, following the work of Ng and Yu \([16]\), we initiated a systematic study of \( \Delta^0_2 \) equivalence relations: we proved that theory of ceers, co-ceers, and \( \Delta^0_2 \) equivalence relations behave quite differently \([17]\). This motivates the following question (from which the present paper originated):

*Are ceers and co-ceers distinguishable in terms of the isomorphism types that they realize?*

Theorem 2.1 below positively answers to such question. In fact, we give a full description of the comparison between the isomorphism types realized at different levels of the arithmetical hierarchy: in particular, we show that for a non-zero \( n \), the isomorphism types of \( \Delta^0_{n+1} \) equivalence structures are precisely the same as those of \( \Pi^0_n \) equivalence structures (Theorems 2.3 and 2.5). Furthermore, we separate different levels of the analytical hierarchy via the corresponding realizable types of equivalence structures (Proposition 2.6).

Our second case-study is also inspired by the research on computable reducibility. Preorders occur naturally in logic (e.g., think of the structure induced by \( \vdash_{PA} \) on the formulas of arithmetic) and, in fact, their complexity has been studied since the 1980s \([18]\) (see \([19]\) for more recent results). By Theorem 3.1 below we show that the case of preorders differs from that of the equivalence structures.

### 1.1. Preliminaries on equivalence structures and preorders

Equivalence structures (resp. preorders) are structures of the form \( \mathcal{A} = \langle \omega, R \rangle \) where \( R \) is an equivalence relation (a preorder, i.e., a reflexive and transitive relation); \( \mathcal{A} \in \Sigma^i_\alpha \), for \( i \in \{-1, 0, 1\} \), if \( R \in \Sigma^i_\alpha \). Note that, for our
interests, assuming that all structures have domain $\omega$ is not a limitation: every isomorphism type of an equivalence structure with a countably infinite domain is realized by an equivalence structure with domain $\omega$.

We denote by $\mathcal{EQ}$ and by $\mathcal{PO}$ the class of all countable equivalence structures and all countable preorders respectively. Moreover, for any class of structures $\mathcal{K}$, we denote by $\text{IsoType}(\mathcal{K}, \Gamma)$ the collection of all isomorphism types realized by the structures from $\mathcal{K}$ of complexity $\Gamma$.

We denote by $\{R_e\}_{e \in \omega}$ some fixed uniform enumeration of all ceers. We say that an equivalence class $[x]_R$ is older than an equivalence class $[y]_R$ if $\min[x]_R < \min[y]_R$. This relation obviously induces a linear ordering on the $R$-classes.

1.2. **Warning about our terminology.** There is a well-established way of thinking of a $\Sigma^0_1$ (resp. $\Pi^0_1$) presentation of structure $\mathcal{A}$ as the combination of a pre-structure $\mathcal{A}^*$ and a ceer (co-ceer) $R$ such that the quotient structure $\mathcal{A}^*/R$ is isomorphic to $\mathcal{A}$. This approach generalizes the way in which many algebraic structures are presented (e.g., the homomorphism theorem says that every countable group is a quotient of the free group on countably many generators). The study of quotient presentations dates back to the very beginning of computable model theory (see, e.g., the work of Metakides and Nerode on c.e. vector spaces [20]) and it gave rise to a rich research program; the interested reader is referred to [21, 22, 23]. However, it should be clear that in this paper we do not deal with quotients, but rather with the (pre-)structure themselves. We emphasize this point with one example. Let $R$ be a preorder on $\omega$. One can regard at the algebraic properties of $R$ in two ways:

(a) First, considering the quotient structure $\mathcal{A} := \langle R/\text{supp}(R), \leq_R \rangle$, where

- $\text{supp}(R) := \{(x, y) : (x R y) \& (y R x)\};$
- the domain of $\mathcal{A}$ consists of the equivalence classes $[x]_{\text{supp}(R)}$,
  $x \in \omega;$
- and $[x]_{\text{supp}(R)} \leq_R [y]_{\text{supp}(R)}$ if and only if $(x R y)$.

(b) Secondly, considering directly the structure $\langle \omega, R \rangle$.

Feiner’s celebrated result [24] that there is $\Sigma^0_1$-presented linear ordering which is not isomorphic to any computable one shall be understood within the first setting. On the contrary, in the present paper we will always work according to the second setting. For example, if we say that $R$ is a $\Sigma^0_1$ linear order, then this means the following:

- $R \in \Sigma^0_1$, and
- the ordering $R$ is linear (in particular, $R$ is an antisymmetric relation).

Clearly, these conventions imply that in our setting, every $\Sigma^0_1$ linear order is computable.
1.3. Preliminaries on limitwise monotonicity. The system of invariants which characterize the isomorphism types of computable equivalence structures can be nicely represented in terms of limitwise monotonic functions.

Let \( d \) be a Turing degree. A function \( F : \omega \to \omega \cup \{ x \} \) is \( d \)-limitwise monotonic if there is a total \( d \)-computable function \( f(x, s) \) such that

(a) \( f(x, s) \leq f(x, s + 1) \) for all \( x \) and \( s \),
(b) \( F(x) = \lim_s f(x, s) \) for all \( x \).

When one is talking about \( 0 \)-limitwise monotonic functions, the prefix “\( 0 \)-” is typically omitted.

A set \( A \subseteq \omega \) is limitwise monotonic if either \( A = \emptyset \) or there is a limitwise monotonic function \( F : \omega \to \omega \) with \( \text{range}(F) = A \).

**Theorem 1.1** (Calvert, Cenzer, Harizanov, and Morozov [5]). Let \( A \) be a countable equivalence structure with infinitely many classes. Then the following conditions are equivalent:

(a) \( A \) has a computable copy.
(b) There is a limitwise monotonic function \( F : \omega \to \omega \cup \{ x \} \) such that for any \( \kappa \in \omega \cup \{ x \} \), the structure \( A \) has precisely \( \text{card}(\{ x \in \omega : F(x) = \kappa \}) \) classes of size \( \kappa \).

We conclude this brief discussion about limitwise monotonicity by stating two results that we will use later. The reader is referred to, e.g., [25] for further results on limitwise monotonic sets and functions.

**Theorem 1.2** (Harris [26, Lemma 5.2] and, independently, Kach [27, Theorem 1.9]). A function \( F : \omega \to \omega \) is \( 0' \)-limitwise monotonic if and only if there exists a computable function \( g : \omega \to \omega \) such that

\[
F(x) = \liminf_s g(x, s) \quad \text{for all } x.
\]

**Theorem 1.3** (Khoussainov, Nies, and Shore [28, Lemma 2.6]; see also Theorem 2.2 in [25]). There is a d.c.e. set \( A \subseteq \omega \), which is not limitwise monotonic.

2. Isomorphism types of equivalence structures

Cenzer, Harizanov, and Remmel [8] proved that there is a \( \Sigma^0_1 \) equivalence structure with no computable copy. It is then natural to ask whether the \( \Sigma^0_1 \) and the \( \Pi^0_1 \) equivalence structures realize the same isomorphism types. The next theorem shows that this is not the case: \( \Pi^0_1 \) equivalence structures are, in a sense, more expressive than \( \Sigma^0_1 \) equivalence structures.

**Theorem 2.1.** There is a \( \Pi^0_1 \) equivalence structure \( A \) which is not isomorphic to any \( \Sigma^0_1 \) equivalence structure. That is, \( \text{IsoType}(\mathbb{EQ}, \Pi^0_1) \nsubsetneq \text{IsoType}(\mathbb{EQ}, \Sigma^0_1) \).

**Proof.** To prove the theorem, it is enough to build a co-ceer that it is not isomorphic to any ceer. To do so, we construct in stages a co-ceer \( S \) which, for all \( e \), satisfies the following requirements:
There is an $S$-class of size $e + 1$ if and only if there are no $R_e$-classes of such size.

This obviously guarantees that $S$ cannot be isomorphic to any of the $R_e$'s.

**The construction.** In the construction we will define the set $Y_e$ of $e$-witnesses; at any stage, the number of $e$-witnesses will be either $e$ or $e + 2$. The role of such $e$-witnesses will be that of realizing an equivalence class (i.e., $[e,0])_S$) which will eventually ensure that $S \not\equiv R_e$. During the construction we will often say that we transform some $z$ into a $S$-singleton; by this we mean that we let $y \not S z$ and $z \not S y$, for all $z \neq y$. Finally, we say that $P_e$ turns on at some stage $s$ if the oldest $R_e[s]$-class of size $e + 1$ differs, for all $t < s$, from the oldest $R_e[t]$-class of the same size.

**Stage 0.** For all $e$, let $Y_e[0] := \{x : 0 < x \leq e + 1\}$ and let all the requirements $P_e$ be off. Moreover, let $S[0]$ be the following computable partition of $\omega$ in infinitely many classes:

\[
\langle a, b \rangle S[0]\langle c, d \rangle \iff a = c.
\]

**Stage $s + 1 = \langle e, n \rangle$.** We focus on $P_e$. First, let

\[
u_e := \min\{x : x \in Y_e[s] \text{ and } x > e + 1\}
\]

(if such set is not empty), and let

\[
v_e := \min\{x : x > \max Y_e[s] \text{ and } \langle e, 0 \rangle S[s]\langle e, x \rangle\}.
\]

Now, we distinguish four cases:

1. $\text{card}(Y_e[s]) = e + 1$, $R_e[s + 1]$ has some class of size $e + 1$, and $P_e$ is off:
   - If so, we want to declare a new $e$-witness. We do so by setting $Y_e[s + 1] := Y_e[s] \cup \{v_e\}$. We also transform $\langle e, v_e + 1 \rangle$ into a $S$-singleton;

2. $\text{card}(Y_e[s]) = e + 2$, $R_e[t]$ has no classes of size $e + 1$, and $P_e$ is off:
   - If so, we want to decrease the number of $e$-witnesses by 1. We do so by setting $Y_e[s + 1] := Y_e[s] \setminus \{u_e\}$ and transforming $\langle e, u_e \rangle$ into a $S$-singleton;

3. If $P_e$ is on:
   - If so, we want to change the current set of $e$-witnesses. We do so by setting $Y_e[s + 1] := (Y_e[s] \setminus \{u_e\}) \cup \{v_e\}$ and transforming $\langle e, u_e \rangle$ into an $S$-singleton. We also turn $P_e$ off;

4. Any other case:
   - We transform $\langle e, v_e \rangle$ into a $S$-singleton.

Finally, let $Y_e := \lim_{s \to \infty} Y_e[s]$ and let $S := \bigcap_{s \in \omega} S[s]$. 
The verification. By construction, $S$ is obviously a co-ceer. Moreover, by induction it is not difficult to see that, for all $e$, $Y_e[0] \subseteq Y_e$ and the following identity holds,

\[(*) \quad \text{card}([\langle e, 0 \rangle]_S) = \text{card}(Y_e).\]

The rest of the verification is based on the following lemma.

**Lemma 2.2.** For all $e$, $S \not\equiv R_e$.

**Proof.** Suppose first that $R_e$ has an equivalence class of size $e + 1$. So, there exists a stage $s_0$ such that any $R_e[s]$, with $s \geq s_0$, realizes the oldest equivalence class of size $e + 1$ of $R_e$. Let $t$ be the least number such that $\langle e, t \rangle \geq s_0$. It is not difficult to see that, since $s_0$ is chosen minimal, at stage $\langle e, t \rangle$ we enter in case (1) of the construction, by which we set the number of $e$-witnesses to $e + 2$. Next, observe that in all further stages in which we focus on $P_e$ we always enter in case (4) (case (1) is excluded because $\text{card}(Y_e) \neq e + 1$; case (2) is excluded because $R_e$ has some class of size $e + 1$; case (3) is excluded because $R[s_0]$ already realized the oldest $R$-class of size $e + 1$ and therefore $P_e$ stays off). This means that we will not modify the set of $e$-witnesses. Therefore, we have that $Y_e = Y_e[\langle e, t \rangle]$ and, by equation $\text{(*)}$, this means that $\text{card}([\langle e, 0 \rangle]_S) \neq e + 1$.

On the other hand, suppose that $R_e$ has no equivalence classes of size $e + 1$. This can happen in two cases that shall be considered separately.

First, assume that there is a least stage $s_0$ such that any $R_e[s]$, with $s \geq s_0$, has no equivalence classes of size $e + 1$. If so, by reasoning as above, we obtain that there is a least stage $\langle e, t \rangle$ after which we have $e + 1$ many $e$-witnesses and, in all further stages focusing on $P_e$, we enter in case (4). Therefore, $\text{card}([\langle e, 0 \rangle]) = \text{card}(Y_e) = e + 1$, while $R_e$ has no equivalence classes of size $e + 1$.

Secondly, assume that there are infinitely many stages at which $R_e[s]$ has an equivalence class of size $e + 1$, even though there is no $R_e$-class of such size. This can happen only if the approximation to $R_e$ has infinitely many mindchanges regarding the oldest equivalence class of size $e + 1$. That is, only if $P_e$ turns on infinitely often, and therefore the construction enters in case (3) infinitely often when dealing with $P_e$. We claim that this makes $Y_e = Y_e[0]$, and therefore $\text{card}(Y_e) = e + 1$. To see this, suppose that there is a least $z$ such that $z \in Y_e \setminus Y_e[0]$. This means that there is stage $s_0$ such that, for all $s \geq s_0$, $z \in Y_e[s]$. Yet, it is not hard to see that there must be a stage $\langle e, n \rangle \geq s_0$ in which $P_e$ is on and $u_e$ takes value $z$. Hence, in this stage the construction enters in case (3) which implies that $z \not\in Y_e[s + 1]$, a contradiction. So, $\text{card}(Y_e) = e + 1$. By equation $\text{(*)}$, it follows that $\text{card}([\langle e, 0 \rangle]_S) = e + 1$, while $R_e$ has no equivalence class of such size.

This concludes the proof of Theorem 2.1. \[\square\]
Having shown that there is a $\Pi^0_1$ equivalence structure which has no $\Sigma^0_1$ copy, it comes natural to investigate the relation between the $\Pi^0_1$ and the $\Delta^0_2$ equivalence structures. Obviously, every $\Pi^0_1$ is isomorphic to a $\Delta^0_0$ equivalence structure. One might conjecture that the inclusion is strict, i.e., that there is a $\Delta^0_0$ copy, it comes natural to investigate the relation between the $\Pi^0_1$ and $\Delta^0_0$ equivalence structures. This is not the case.

**Theorem 2.3.** The isomorphism types of $\Pi^0_1$ and $\Delta^0_0$ equivalence structures coincide. That is, $\text{IsoType}(\mathbb{E}, \Pi^0_1) = \text{IsoType}(\mathbb{E}, \Delta^0_0)$.

**Proof.** Let $E$ be a $\Delta^0_0$ equivalence relation. If $E$ has only finitely many equivalence classes, then it is obvious that $E$ is isomorphic to a computable structure. Hence, without loss of generality, we assume that $E$ has infinitely many classes.

A relativized version of Theorem 1.1 implies that there is a $\mathbf{0}'$-limitwise monotonic function $H : \omega \to \omega \cup \{\infty\}$ with the following property: For any $\alpha \in \omega \cup \{\infty\}$, the number of $E$-equivalence classes of size $\alpha$ is equal to

$$\gamma(\alpha) := \text{card}(\{k \in \omega : H(k) = \alpha\}).$$

Note that $H(k)$ cannot be equal to zero.

By Theorem 1.2, one can choose a computable function $g : \omega^2 \to \omega$ such that for any $k \in \omega$, we have

$$H(k) = \liminf_s g(k, s).$$

Since $H(k) \neq 0$, we may assume that $g(k, s) \geq 1$ for all $s$ (i.e. if $g(k, s)$ equals zero, then one just replaces this zero with one).

We give a construction of a $\Pi^0_1$ equivalence relation $R$. As per usual, we define $R$ by enumerating its complement, i.e. at any stage of the construction, we are allowed to declare that $(x R y)$, where $x \neq y$. Surely, if $(x R y)$, then we always (implicitly) assume $(y R x)$. In other words, we can only provide negative information about $R$, and we never give positive pieces of $R$-data.

At a non-zero stage $s$, we define a finite set $A[s]$ and a function $\ell[s]$ acting from $A[s]$ onto $\{0, 1, \ldots, s - 1\}$. The intuition behind $\ell[s]$ is as follows — at the stage $s$, we think that the first $s$ classes of $R[s]$ are precisely the following sets:

$$\{x \in A[s] : \ell(x)[s] = k\}, \quad k = 0, 1, \ldots, s - 1.$$

Thus, we will assume that we “automatically” declare the following $R$-information: if $\ell(x)[s] \downarrow \neq \ell(y)[s] \downarrow$, then $(x R y)$.

At a stage $s$, an element $x$ is called fresh if $x \notin \bigcup_{t<s} A[s]$.

**The construction.**

**Stage 0.** Set $A[0] := \emptyset$.

**Stage $s + 1$**. First, choose an element $w$ as follows:

- If the set $\{x \text{ is non-fresh} : x \notin A[s]\}$ is non-empty, then $w$ is the least element of this set.
- Otherwise, $w$ is the least fresh element.
Declare \(\ell(w)[s+1] := s\).

After that, for each \(k < s\), we act according to the following instructions. We define

\[
\delta(k) := \text{card}(\{x \in A[s] : \ell(x)[s] = k\}).
\]

1. If \(g(k, s+1) > \delta(k)\), then choose the least \((g(k, s+1) - \delta(k))\) fresh elements. For each of these elements \(y\), set \(\ell(y)[s+1] := k\).
2. If \(g(k, s+1) < \delta(k)\), then find the greatest \((\delta(k) - g(k, s+1))\) elements \(z\) such that \(\ell(z)[s] = k\). For each of these \(z\), set \(\ell(z)[s+1]\) undefined.

Surely, we also assume \(z \notin A[s+1]\).

The description of the construction is finished.

The verification. It is clear that the constructed relation \(R\) is co-c.e. Moreover, it is not hard to verify the following claim (recall that \(g(k, s) \geq 1\) for all \(k\) and \(s\)).

**Claim 2.4.** Every \(x \in \omega\) satisfies precisely one of the following two cases:

(a) There is a stage \(s_0\) such that \(\ell(x)[s_0]\) is defined, and \[
\ell(x)[s] = \begin{cases} 
\text{undefined}, & \text{if } s < s_0, \\
\ell(x)[s_0], & \text{if } s \geq s_0.
\end{cases}
\]

(b) There are stages \(s_0 < s_1 < s_2\) such that \(\ell(x)[s_0] \downarrow < \ell(x)[s_2] \downarrow\), and \[
\ell(x)[s] = \begin{cases} 
\text{undefined}, & \text{if } s < s_0, \\
\ell(x)[s_0], & \text{if } s_0 \leq s < s_1, \\
\ell(x)[s_1], & \text{if } s_1 \leq s < s_2, \\
\ell(x)[s_2], & \text{if } s \geq s_2.
\end{cases}
\]

For an element \(x \in \omega\), let \(\ell^*(x) := \lim_s \ell(x)[s]\). It is not difficult to show that the following conditions are equivalent:

1. \((x \mathbin{R} y) \iff \ell^*(x) = \ell^*(y)\).

Therefore, we deduce that \(R\) is an equivalence relation.

At the end of a stage \(s + 1\), the relation \(R[s+1]\) satisfies the following: for any \(k < s\),

\[
\text{card}(\{x \in A[s+1] : \ell(x)[s+1] = k\}) = g(k, s+1).
\]

This fact and the description of the construction together imply that for any \(k \in \omega\) and any \(m \geq 1\), the following are equivalent:

1. \(\text{card}(\{x \in \omega : \ell^*(x) = k\}) \geq m\);
2. \(\exists s_0(\forall s \geq s_0)(g(k, s) \geq m)\).

Thus, we obtain that

\[
\text{card}(\{x \in \omega : \ell^*(x) = k\}) = \lim \inf_s g(k, s) = H(k).
\]

By (\(\mathbb{I}\)), for each \(\alpha \in \omega \cup \{\infty\}\), the relations \(R\) and \(E\) have the same number of equivalence classes of size \(\alpha\). Therefore, \(R\) is isomorphic to \(E\). **Theorem 2.3** is proved. \(\square\)
Proof. Ad (i).
Consider a computable partition of \( S \) into infinitely many finite blocks: for a natural number \( i \), the \( i \)th block contains precisely \( 2i + 4 \) elements — \( a^i_0, a^i_1, \ldots, a^i_{2i+3} \).

Let \( X \subseteq \omega \) be a set belonging to \( \Sigma^0_n \setminus \Delta^1_n \). We define an equivalence structure \( S \) as follows:

(a) For every \( i \in \omega \), the elements \( a^i_0, a^i_1, \ldots, a^i_{2i+2} \) are \( S \)-equivalent. If \( i \neq j \), then \( a^i_k \) and \( a^j_k \) are not \( S \)-equivalent.

(b) \( a^i_{2i+3} \in [a^i_0]_S \) if and only if \( i \in X \).

A standard application of the Tarski–Kuratowski algorithm shows that the relation \( S \) is \( \Sigma^1_n \).

Note that the character \( \chi(S) \) satisfies the following:

\[
\chi(S) = \{(1, k) : k \geq 1\} \cup \{(2i + 4, 1) : i \in X\} \cup \{(2j + 3, 1) : j \neq X\}.
\]

Thus, \( X \leq_T \chi(S) \) and the set \( \chi(S) \) cannot be \( \Delta^1_n \).

On the other hand, for an arbitrary countable equivalence structure \( R \), the character \( \chi(R) \) is \( \Sigma^0_3 \) in the atomic diagram of \( R \). Hence, if \( R \) is a \( \Delta^1_n \) structure, then \( \chi(R) \in \Delta^0_3 \). Therefore, we deduce that our structure \( S \) is not isomorphic to any \( \Delta^1_n \) equivalence structure.

The proof of the case \( \Pi^1_n \) vs. \( \Delta^1_n \) is essentially the same: just choose \( X \) belonging to \( \Pi^1_n \setminus \Delta^1_n \).

Ad (ii). Let \( A \) be an \( m \)-complete \( \Sigma^1_n \) set. We define \( X := A^{(3)} \), and we build an equivalence structure \( S \) by applying the previous construction to the set \( X \). Since \( A \in \Delta^1_{n+1} \), we deduce that both \( X \) and \( S \) are also \( \Delta^1_{n+1} \).

**Theorem 2.5.** For all \( n > 0 \), we have that

\[
\text{IsoType}(\mathbb{E}Q_0, \Sigma^0_n) \subseteq \text{IsoType}(\mathbb{E}Q, \Pi^0_n) = \text{IsoType}(\mathbb{E}Q, \Delta^0_{n+1}).
\]

We conclude our study of equivalence structures by lifting our focus to the analytical hierarchy. We show that for all \( n > 0 \),

\[
\text{IsoType}(\mathbb{E}Q, \Delta^1_n) \subseteq \text{IsoType}(\mathbb{E}Q, \Sigma^1_n);
\]

\[
\text{IsoType}(\mathbb{E}Q, \Delta^1_n) \subseteq \text{IsoType}(\mathbb{E}Q, \Pi^1_n);
\]

\[
\text{IsoType}(\mathbb{E}Q, \Sigma^1_n) \cup \text{IsoType}(\mathbb{E}Q, \Pi^1_n) \subseteq \text{IsoType}(\mathbb{E}Q, \Delta^1_{n+1}).
\]

**Proposition 2.6.** Let \( n \) be a non-zero natural number.

(i) There is a \( \Sigma^1_n \) equivalence structure which is not isomorphic to any \( \Delta^1_n \) equivalence structure. A similar result holds for \( \Pi^1_n \) vs. \( \Delta^1_n \).

(ii) There is a \( \Delta^1_{n+1} \) equivalence structure such that its isomorphism type cannot be realized neither by a \( \Sigma^1_n \) structure, nor by a \( \Pi^1_n \) structure.

**Proof.** Ad (i). Consider a computable partition of \( \omega \) into infinitely many finite blocks: for an index \( i \in \omega \), the \( i \)th block contains precisely \( 2i + 4 \) elements — \( a^i_0, a^i_1, \ldots, a^i_{2i+3} \).

Let \( X \subseteq \omega \) be a set belonging to \( \Sigma^1_n \setminus \Delta^1_n \). We define an equivalence structure \( S \) as follows:

(a) For every \( i \in \omega \), the elements \( a^i_0, a^i_1, \ldots, a^i_{2i+2} \) are \( S \)-equivalent. If \( i \neq j \), then \( a^i_k \) and \( a^j_k \) are not \( S \)-equivalent.

(b) \( a^i_{2i+3} \in [a^i_0]_S \) if and only if \( i \in X \).

A standard application of the Tarski–Kuratowski algorithm shows that the relation \( S \) is \( \Sigma^1_n \).

Note that the character \( \chi(S) \) satisfies the following:

\[
\chi(S) = \{(1, k) : k \geq 1\} \cup \{(2i + 4, 1) : i \in X\} \cup \{(2j + 3, 1) : j \neq X\}.
\]

Thus, \( X \leq_T \chi(S) \) and the set \( \chi(S) \) cannot be \( \Delta^1_n \).

On the other hand, for an arbitrary countable equivalence structure \( R \), the character \( \chi(R) \) is \( \Sigma^0_3 \) in the atomic diagram of \( R \). Hence, if \( R \) is a \( \Delta^1_n \) structure, then \( \chi(R) \in \Delta^0_3 \). Therefore, we deduce that our structure \( S \) is not isomorphic to any \( \Delta^1_n \) equivalence structure.

The proof of the case \( \Pi^1_n \) vs. \( \Delta^1_n \) is essentially the same: just choose \( X \) belonging to \( \Pi^1_n \setminus \Delta^1_n \).

Ad (ii). Let \( A \) be an \( m \)-complete \( \Sigma^1_n \) set. We define \( X := A^{(3)} \), and we build an equivalence structure \( S \) by applying the previous construction to the set \( X \). Since \( A \in \Delta^1_{n+1} \), we deduce that both \( X \) and \( S \) are also \( \Delta^1_{n+1} \).
Towards a contradiction, assume that \( R \) is a \( \Sigma_1^n \) equivalence structure, which is isomorphic to \( S \). By \( D(R) \) we denote the atomic diagram of \( R \). It is clear that \( D(R) \) is computable in the \( \Sigma_1^n \)-complete set \( A \). Therefore,

\[
A^{(3)} \leq_T \chi(S) = \chi(R) \leq_T D(R)^{(2)} \leq_T A^{(2)},
\]

which gives a contradiction. Hence, the isomorphism type of \( S \) cannot be realized by a \( \Sigma_1^n \) structure. Moreover, a similar argument applies for \( \Pi_1^n \) structures. \( \square \)

3. Isomorphism Types of Preorders

This section is devoted to the case of preorders. First, observe that there is a \( \Pi_0^1 \) preorder that is not isomorphic to any \( \Sigma_0^1 \) preorder: this is an immediate consequence of Theorem 2.1. The next theorem says that the converse also holds, thus distinguishing the case of preorders from that of equivalence structures.

**Theorem 3.1.** There is a \( \Sigma_0^1 \) preorder \( R \), which is not isomorphic to any \( \Pi_0^1 \) preorder.

**Proof.** First, we describe how one can construct a sufficiently large class of \( \Sigma_0^1 \) preorders. After that, inside this class, we will choose a preorder satisfying the theorem.

Let \( B \subseteq \omega \) be an arbitrary \( \Delta_2^0 \) set with \( 0 \notin B \). Choose a total computable function \( g(x, s) \) such that \( B(x) = \lim_s g(x, s) \).

We define a \( \Sigma_0^1 \) preorder \( S_B \) as follows. Fix a computable partition of \( \omega \) into

\[
\{c, d\} \cup \{a_i : i \in \omega\} \cup \{b_j : j \in \omega\}.
\]

By \( x \prec_{S_B} y \) we denote the formula \( \langle x \ S_B \ y \rangle \land \langle y \ S_B \ x \rangle \).

Beforehand, we set:

- The relation \( S_B \) is reflexive.
- All \( a_i, i \in \omega \), are pairwise \( S_B \)-incomparable. For every \( i \), \( c \prec_{S_B} a_i \), and \( a_i \) is \( S_B \)-incomparable with \( d \).
- The element \( d \) is \( S_B \)-incomparable with \( c \). For every \( j \in \omega \), we have \( b_{j+1} <_{S_B} b_j <_{S_B} d \), and \( b_j \) is incomparable with \( c \).

In order to finish the definition of \( S_B \), our construction will define the following finite values: for every \( i \in \omega \), one needs to recover

\[
v(i) := \min\{j \in \omega : b_j <_{S_B} a_i\}.
\]

The \( \Sigma_0^1 \)-ness of the preorder \( S_B \) will be ensured by the step-by-step definition of \( v(i) \), which proceeds as follows:

- First, set \( v(i)[0] \) undefined.
- Then at some stage \( s_0 \), we put \( v(i)[s_0] := k_0 \), i.e. we enumerate the formulas saying

\[
(b_{k_0} \ S_B \ a_i), \ (b_{k_0+1} \ S_B \ a_i), \ (b_{k_0+2} \ S_B \ a_i), \ldots
\]

into our (approximation of) \( S_B \).
After that, the value \( v(i)[s] \) can change at most one time: at a stage \( s_1 > s_0 \), we put \( v(i)[s_1] := 0 \) — this means that we enumerate the facts \((b_0 S_B a_i), (b_1 S_B a_i), \ldots, (b_{k_0-1} S_B a_i)\) into \( S_B \).

At a stage \( s + 1 \), we say that an element \( a_i \) is fresh if the value \( v(i)[s] \) is undefined.

**The construction.** Stage 0. Set \( v(i)[0] \) undefined for all \( i \in \omega \).

Stage \( s + 1 = 2t + 1 \). Find the least fresh \( a_i \), and define \( v(i)[s + 1] := 0 \).

Stage \( s + 1 = 2t + 2 \). For each non-zero \( x \leq s \), proceed as follows:

- If \( g(x, s) = 0 \) and there is \( a_i \) with \( v(i)[s] = x \), then set \( v(i)[s + 1] := 0 \) (for all such \( a_i \)).
- If \( g(x, s) = 1 \) and there is no \( a_i \) with \( v(i)[s] = x \), then choose the least fresh \( a_k \) and define \( v(k)[s + 1] := x \).

**The verification.** It is clear that for every \( i \in \omega \), there exists the limit \( v(i) := \lim_s v(i)[s] \). Moreover, consider the first stage \( s_0 \) such that \( v(i)[s_0] \) is defined. Then precisely one of the following two cases holds:

(i) either for all \( s \geq s_0 \), we have \( v(i)[s] = v(i)[s_0] = v(i) \);

(ii) or \( v(i)[s_0] \neq 0 \) and there is a stage \( s_1 > s_0 \) such that

\[
v(i)[s] = \begin{cases} v(i)[s_0], & \text{if } s_0 \leq s < s_1, \\ 0, & \text{if } s \geq s_1. \end{cases}
\]

Therefore, one can deduce that the preorder \( S_B \) is c.e.

The fact above and the description of the construction together imply the following:

**Claim 3.2.** Let \( x \) be a non-zero natural number.

- If \( x \in B \), then there is a unique \( i \) with \( v(i) = x \).
- If \( x \notin B \), then \( v(i) \neq x \) for all \( i \in \omega \).

Moreover, there are infinitely many \( k \) with \( v(k) = 0 \).

This claim shows that the isomorphism type of the preorder \( S_B \) depends only on the set \( B \), but not on the choice of its approximation \( g \).

By Theorem 3 there is a d.c.e. set \( A \), which is not limitwise monotonic. Without loss of generality, we may assume that \( 0 \notin A \). We prove that the preorder \( R := S_A \) is not isomorphic to any \( \Pi^0_1 \) preorder.

Towards a contradiction, assume that \( Q \) is a \( \Pi^0_1 \) preorder, which is isomorphic to \( R \). Fix an isomorphism \( f: R \cong Q \). We will slightly abuse our notations, and identify the elements \( c \) and \( f(c) \), \( b_i \) and \( f(b_i) \), etc.

In a non-uniform way, one can find the elements \( c \) and \( d \) inside \( Q \). We note that:

- If \( w \in \{ a_i : i \in \omega \} \) inside \( Q \), then \( w \) is \( Q \)-incomparable with \( d \).
- If \( w \in \{ b_j : j \in \omega \} \), then \( w \) is \( Q \)-incomparable with \( c \).
- Suppose that \( u \) and \( v \) are different elements from \( \{ b_j : j \in \omega \} \). Then \( u, w \) satisfy precisely one of the following: either \( u <_Q w \), or \( w <_Q u \).
Since the preorder $Q$ is co-c.e., we deduce the following: one can effectively check whether a given $w$ is an $f$-image of some $a_i$ or of some $b_j$. Moreover, the ordering of $b_j$-s inside $Q$ is recursive.

Consider a computable set $U := \{f(a_i) : i \in \omega\}$, and for each $w \in U$, define the value

$$p(w) := \text{the number of } b_j\text{-s, which are } Q\text{-incomparable with } w.$$ 

Since $Q$ is $\Pi^0_1$, the function $p$ is limitwise monotonic. Hence, the set

$$C := \{p(w) : w \in U, \ p(w) \geq 1\}$$

is also limitwise monotonic.

On the other hand, recall that $Q$ is isomorphic to $R = S_A$, and hence, by Claim 3.3, $C$ must be equal to $A$, which contradicts the choice of $A$. Therefore, the preorder $R$ is not isomorphic to a $\Pi^0_1$ preorder. \hfill $\Box$

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