Force sensing in hybrid Bose-Einstein condensate optomechanics based on parametric amplification

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In this paper, the scheme of a force sensor is proposed which has been composed of a hybrid optomechanical cavity containing an interacting cigar-shaped Bose-Einstein condensate (BEC) where the s-wave scattering frequency of the BEC atoms as well as the spring coefficient of the cavity moving end-mirror (the mechanical oscillator) are parametrically modulated. It is shown that in the red-detuned regime and under the so-called impedance-matching condition, the mechanical response of the system to the input signal is enhanced substantially, which leads to the amplification of the weak input signal while the added noises of measurement (backaction noises) can be suppressed and lowered much below the standard quantum limit (SQL). In this way, such a hybrid system operates as an ultra-sensitive force sensor which can amplify the input signal and simultaneously suppress the added noises by controlling the amplitudes of modulation and the system cooperativity. The advantage of the presented nonlinear hybrid system accompanied with the mechanical and atomic modulations in comparison to the bare optomechanical cavities is the enhancement of signal amplification as well as the extension of amplification bandwidth.

I. INTRODUCTION

As is well-known, every measurement, at either the classical or quantum level, is affected by noise which reduces the accuracy of the measurement. Therefore, finding methods, especially in quantum systems, for noise suppression, noise cancellation, or signal amplification is of particular interest and importance in quantum measurements and quantum metrology. For example, the so-called coherent quantum noise cancellation (CQNC) scheme has been recently introduced [1, 2] in which the “anti-noise” path in the quantum dynamics of the system can be employed to cancel the original noise path via destructive quantum interference.

During the past decade, optomechanical systems (OMSs), in which the electromagnetic radiation pressure is coupled to a mechanical oscillator (MO) as a macroscopic object, have been developed [3–6] for the purpose of testing the fundamentals of physics like the Bell test [7] and emergence of quantum effects in macro scale. Also, OMSs have been applied to a wide variety of research fields including ultra precision force sensing [8], MO ground-state cooling [9–11], generation of bipartite entanglement [12–14], synchronization of MOs [15–19], generation of mechanical/optical nonclassical states [20–24], quantum simulation of the parametric dynamical Casimir effect (DCE) [25–30] as well as the curved space-time [31], and generation of squeezing [32–38].

In optomechanical force sensors, the competition between the shot noise and the radiation pressure backaction noise which have opposite dependence on the input power, determines the standard quantum limit (SQL). In fact, increasing the input power makes the shot noise decrease, but nevertheless it causes an increase of the backaction noise. Therefore, in order to improve the force sensing precision one has to find methods to suppress or evade the backaction noise.

There are different theoretical and experimental proposals for backaction noise reduction to overcome the SQL in ultra precision force measurements [35, 39–42]. In addition to these proposals which are based on noise reduction, the CQNC proposals are based on the noise cancellation via quantum interference [43–48]. It should be noted that although in these methods the backaction noise of measurement is reduced or even canceled but the signal is not amplified at all. In a more recent proposal [49] it has been shown that it is possible to suppress the added noise of measurement while amplifying the input signal simultaneously in a bare optomechanical system through the parametric modulation of the spring coefficient of the MO. On the other hand, recently proposed hybrid optomechanical cavities containing Bose-Einstein condensates (BECs) [50–52] in which the fluctuation of the collective excitation of the BEC, i.e., Bogoliubov mode, behaves like an effective mechanical mode [50] and the nonlinear atom-atom interaction simulates an atomic amplifier [53, 54], have more controllability and can increase the quantum effects at macroscopic level [55–59]. Besides, such hybrid systems are suitable for reduction of quantum noise [60] or can act as a quantum amplifier/squeezer [61]. Moreover, by considering the quadratic optomechanical coupling in such hybrid systems, one can generate robust entanglement and strong mechanical squeezing beyond the SQL [62].

Here, inspired by the above-mentioned investigations on optomechanical force sensors and the properties of the hybrid OMSs, we propose an experimentally feasible scheme for the weak force measurement beyond the SQL based on simultaneous signal amplification and backaction noise suppression via parametric amplification of the mechanical and Bogoliubov modes. We consider a hybrid optomechanical
frequency $\omega_m$, and damping rate $\gamma_m$ whose spring coefficient is parametrically modulated at twice its natural frequency. The cavity which is driven through the fixed mirror by a laser with frequency $\omega_L$ and wavenumber $k_0 = \omega_L/c$ contains a BEC of $N$ ultracold two-level atoms with mass $m_a$ and transition frequency $\omega_a$. Furthermore, we assume that the collision frequency of the condensate atoms is parametrically modulated through the modulation of the electromagnetic trap or the density of the BEC by changing the trap stiffness [63]. The total Hamiltonian of the system in the frame rotating at the driving laser frequency $\omega_L$ can be written as

$$\hat{H} = \hbar \Delta_c \hat{a}^\dagger \hat{a} + i \hbar E_L (\hat{a}^\dagger - \hat{a}) + \hbar \omega_m \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a} \hat{b} + \hat{b}^\dagger)$$

$$+ \frac{i \hbar}{2} \left( \lambda_m \hat{b}^\dagger e^{-2i\omega_m t} - \lambda_m^* \hat{b} e^{2i\omega_m t} \right) + \hat{H}_F + \hat{H}_{BEC}. \quad (1)$$

The first three terms in the Hamiltonian describe, respectively, the free energy of the cavity mode, the coupling between the cavity mode and the driving laser, and the free energy of the MO. Here, $\Delta_c = \omega_c - \omega_a$ is the detuning of the optical mode from the driving laser frequency, $E_L$ is the pump rate of the external laser, and $\hat{a} (\hat{b})$ is the annihilation operator of the cavity (MO) mode. The canonical position and momentum of the MO are $\hat{x}_p = x_{cp} (\hat{b} + \hat{b}^\dagger)$ and $\hat{p} = \hbar (\hat{b} - \hat{b}^\dagger)/2i \hat{x}_p$, respectively, with $x_{cp} = \sqrt{\hbar/2m_0 \omega_m}$ being the zero-point position fluctuation.

The fourth term in Hamiltonian (1) is the optomechanical interaction between the mechanical and optical modes with the single-photon optomechanical coupling $g_0 = x_{cp} \omega_c / L$. The fifth term describes the parametric driving of the MO spring coefficient at twice its natural frequency $[K(t) = K + \delta K \cos(2\omega_m t + \varphi_m)$ with $\varphi_m$ being the phase of external modulation] which is written in the rotating wave approximation (RWA) over time scales longer than $\omega_m^{-1}$ where $\lambda_m = |\lambda_m|^2 e^{i\varphi_m}$ with $|\lambda_m| = \delta K x_{cp}^2/2 \hbar$ [30, 49]. Note that by fixing the phase of modulation $\varphi_m$, it is always possible to take $\lambda_m$ as a real number. It is worth to point out that this term can be considered as the mechanical phonon analog of the degenerate parametric amplification (DPA) which may lead to the DCE of mechanical phonons [30]. The sixth term, $\hat{H}_F$, accounts for the coupling of the MO to the input classical-force $F$ to be measured which is given by

$$\hat{H}_F = F(t) \hat{x} = x_{cp} F(t) (\hat{b} + \hat{b}^\dagger). \quad (2)$$

The last term of Eq. (1) is the Hamiltonian of the atomic BEC. If the atom-laser detuning $\Delta_a = \omega_a - \omega_L$ is much greater than the atomic linewidth, then the excited electronic state of the atoms can be adiabatically eliminated and the Hamiltonian of the BEC can be written as [64]

$$\hat{H}_{BEC} = \int_{-L/2}^{L/2} dx \hat{\psi}^\dagger (x) \left[ -\frac{\hbar^2}{2m_a} \frac{d^2}{dx^2} + \hbar U_0 \cos^2 (k_0 x) \hat{a}^\dagger \hat{a} + \frac{1}{2} U_s \hat{\psi}^\dagger (x) \hat{\psi}^\dagger (x) \hat{\psi} (x) \right] \hat{\psi} (x), \quad (3)$$

where $\hat{\psi} (x)$ is the quantum field operator of the atomic BEC, $U_0 = -g_0^2/\Delta_a$ is the optical lattice barrier height per photon, $g_a$...
where $\Delta$ is the Bogoliubov mode of the BEC which corresponds to the quantum fluctuations of the atomic field around the quantum field operator of the BEC under the Bogoliubov approximation. The Hamiltonian $\hat{H}_{\text{BEC}}$ is obtained as follows:

$$\hat{H}_{\text{BEC}} = \hbar \delta \hat{a}^\dagger \hat{a} + \hbar \omega_d \hat{a}^\dagger \hat{a} + \hbar G_0 \hat{a}^\dagger (\hat{a} + \hat{d}^\dagger) + \hat{H}_{\text{sw}},$$

where $\delta = NU_0/2$, $\omega_d = 4\omega_R + \omega_{\text{sw}}$ is the effective frequency of the Bogoliubov mode of the BEC, and $G_0 = \sqrt{2N}U_0/4$ is the radiation pressure coupling between the Bogoliubov mode of the BEC and the optical mode.

The Hamiltonian $\hat{H}_{\text{sw}}$ in Eq. (5) refers to the atom-atom interaction energy. In the presence of time modulation of the s-wave scattering frequency of atomic collisions at twice the frequency of the Bogoliubov mode, i.e., $\omega_{\text{sw}}(t) = \omega_{\text{sw}}[1 + e \cos(2\omega_d t + \varphi_d)]$ where $e$ and $\varphi_d$ are, respectively, the amplitude and the phase of modulation, and $\omega_{\text{sw}} = \delta Nh\lambda_{\text{eff}}/(m_L \nu^2)$ with $\nu$ being the beam waist of the optical mode, $\hat{H}_{\text{sw}}$ in the RWA is given by

$$\hat{H}_{\text{sw}}(t) = \frac{i\hbar}{2}(\lambda_d \hat{a}^\dagger e^{-2i\omega_d t} - \lambda_d^* \hat{a} e^{2i\omega_d t}),$$

where $\lambda_d = -i e \omega_{\text{sw}} e^{-i\varphi_d}/4$ can be taken real by fixing the phase $\varphi_d$.

It should be noted that the s-wave scattering frequency, $\omega_{\text{sw}}$, can be controlled experimentally by manipulating the transverse trapping frequency of the BEC through changing the waist radius of the optical mode $w$ [67]. Besides, as has been shown in Ref. [63] the time modulation of the atomic collisions can be experimentally realized by the time modulation of the scattering length via the modulation of the electromagnetic trap, or the modulation of the density of the BEC by changing the trap stiffness via the intensity modulation of the pump laser.

The Hamiltonian of Eq. (6) is a Bogoliubov-phonon analog of the DPA which can give rise to the generation of phonons [30]. Here, it should be mentioned that in the Hamiltonian (5), we have ignored the Bogoliubov-type Casimir phonons [30].

Substituting Eqs. (2), (5), and (6) into the Hamiltonian of Eq. (1) the total Hamiltonian of the system takes the form

$$\hat{H}_{\text{tot}} = \hbar \Delta_0 \hat{a}^\dagger \hat{a} + \hbar \omega_d \hat{b}^\dagger \hat{b} + \hbar \omega_d \hat{d}^\dagger \hat{d} + i\hbar E_L(\hat{a}^\dagger - \hat{a}) - i\hbar G_0 \hat{a}^\dagger (\hat{a} + \hat{d}^\dagger) + \frac{\hbar}{2} \lambda_d \hat{a}^\dagger e^{-2i\omega_d t} - \lambda_d^* \hat{a} e^{2i\omega_d t} + \frac{\hbar}{2} \lambda_d \hat{b}^\dagger e^{-2i\omega_d t} - \lambda_d^* \hat{b} e^{2i\omega_d t},$$

where $\Delta_0 = \Delta_c + NU_0/2$ is the cavity Stark-shifted detuning.

### III. Dynamics of the System

The linearized quantum Langevin equations (QLEs) of the system can be derived from the Hamiltonian of Eq. (7). Here, it is desirable to work in the red-detuned regime of cavity optomechanics where $\Delta_0 \approx \omega_m \approx \omega_d$. For this purpose, the frequency of the Bogoliubov mode of the BEC, i.e., $\omega_d$, should be matched to the mechanical frequency $\omega_m$ which is possible through the manipulation of the s-wave scattering frequency of the Bogoliubov mode via controlling the transverse frequency of the BEC trap [67]. Besides, the effective detuning $\Delta_0$ can be set in the red-detuning regime through the pump laser frequency. As has been shown in Ref. [30] in the red detuned regime and within the RWA where the two optomechanical and opto-atomic couplings are analogous to the beam-splitter interaction, the linearized QLEs describing the dynamics of the quantum fluctuations are given by

$$\delta \hat{a} = \frac{k}{2} \delta \hat{a} + i g \delta \hat{b} - i G \delta \hat{a} + \sqrt{\gamma_d} \hat{a}_{in},$$

$$\delta \hat{b} = -\frac{\gamma_m}{2} \delta \hat{b} + i g \delta \hat{a} + \lambda_d \delta \hat{d}^\dagger - i \frac{\omega_d}{\hbar} F e^{i\omega_d t} + \sqrt{\gamma_m} \hat{b}_{in},$$

$$\delta \hat{d} = -\frac{\gamma_d}{2} \delta \hat{d} - i G \delta \hat{a} + \lambda_d \delta \hat{d}^\dagger + \sqrt{\gamma_d} \hat{d}_{in},$$

where $g = g_0 \delta$ and $G = G_0 \delta$ are, respectively, the enhanced-optomechanical and opto-atomic coupling strengths in which $\delta = E_L/\sqrt{\omega^2/4 + \Delta_0^2}$ is the steady-state mean value of the optical mode. Here, $\Delta_0 = \Delta_0 - 2g_0 \delta + 2G_0 \delta$ is the effective cavity detuning where $\delta \approx \delta_0^\text{in}/\omega_m$ and $\delta \approx -G_0 \delta^\text{in}/\omega_d$ are, respectively, the steady-state values of the mechanical and atomic mean fields in the RWA and in the high quality factors limit. Besides, $\gamma_m$ and $\gamma_d$ are the dissipation rates of the mechanical and Bogoliubov modes, respectively.

Furthermore, the optical input vacuum noise $\hat{a}_{in}$ as well as the Brownian noises $\hat{b}_{in}$ and $\hat{d}_{in}$ affecting, respectively, the MO and the Bogoliubov mode of the BEC, satisfy the Markovian correlation functions $\langle \hat{a}_{in}(t) \hat{a}_{in}^\dagger(t') \rangle = \langle \hat{b}_{in}(t) \hat{b}_{in}^\dagger(t') \rangle = \langle \hat{d}_{in}(t) \hat{d}_{in}^\dagger(t') \rangle = (1 + \bar{n}_j)(\delta(t-t'))$, where $\bar{n}_j = [\exp(h\omega_j/k_B T) - 1]^{-1}$ with $j = c, m, d$ and $\delta$ are the mean number of thermal excitations of the cavity, mechanical, and Bogoliubov modes at temperature $T$. The quantum noise $\hat{d}_{in}$ originates from the other extra modes of the BEC as well as the fluctuations in the electromagnetic trap as has been shown in Ref. [68].

Now by defining the quadratures $\delta \hat{X}_c = (\delta + \delta^\dagger)/\sqrt{2}$ and $\delta \hat{P}_c = (\delta - \delta^\dagger)/\sqrt{2}i$ (with $c, a, b, d$ the set of Eqs. (8a)-(8c) can be written as the following compact matrix form

$$\delta \hat{u}(t) = \Lambda \delta \hat{u}(t) + \hat{u}_{in}(t),$$

where the vector of continuous-variable fluctuation operators and the corresponding vector of noises are, respectively, given by $\delta \hat{u} = (\delta \hat{X}_a, \delta \hat{P}_a, \delta \hat{X}_b, \delta \hat{P}_b, \delta \hat{X}_d, \delta \hat{P}_d)^T$ and $\hat{u}_{in}(t) = (\sqrt{\nu} \hat{X}_a, \sqrt{\nu} \hat{P}_a, \sqrt{\nu} \hat{X}_b, \sqrt{\nu} \hat{P}_b, \sqrt{\nu} \hat{X}_d, \sqrt{\nu} \hat{P}_d)^T$ in which $\hat{X}_c = (\hat{a}_{in} + \hat{a}_{in}^\dagger)/\sqrt{2}$ and $\hat{P}_c = (\hat{a}_{in} - \hat{a}_{in}^\dagger)/\sqrt{2}i$.


(o = a, b, d). Besides,
\[
\hat{X}_b^{\text{in}}(t) = \hat{X}_b^{\text{in}}(t) + \sqrt{\frac{2}{\gamma_m \hbar}} F(t) \sin \omega_m t, \quad (10a)
\]
\[
\hat{P}_b^{\text{in}}(t) = \hat{P}_b^{\text{in}}(t) - \sqrt{\frac{2}{\gamma_m \hbar}} F(t) \cos \omega_m t, \quad (10b)
\]
are the modified mechanical noises. Furthermore, the time-independent drift matrix \(A\) is given by
\[
A = \begin{pmatrix}
-\frac{1}{2} & 0 & 0 & -g & 0 & 0 & G \\
0 & -\frac{1}{2} & g & 0 & -G & 0 & 0 \\
g & 0 & -\lambda_m - \frac{z_p}{2} & 0 & 0 & 0 & 0 \\
0 & G & 0 & 0 & 0 & \lambda_d - \frac{z_p}{2} & 0 \\
-G & 0 & 0 & 0 & 0 & -\lambda_d + \frac{z_p}{2} & 0
\end{pmatrix}. \quad (11)
\]
As has been shown in Ref. [30], based on the Routh-Hurwitz criterion for the optomechanical stability condition, the parameters \(\lambda_m\) and \(\lambda_d\) should satisfy the condition
\[
\lambda_{\text{md}} \leq \frac{\gamma_{\text{md}}}{2} \left[ 1 + C_{\text{md}} \right] := \lambda_{\text{max}}^{\text{md}}, \quad (12)
\]
in which \(C_{\text{md}}(C_d)\) is the collective optomechanical cooperativity associated with the mechanical mode (Bogoliubov mode) given by
\[
C_{\text{md}} = C_{\text{d}(1)} (1 + C_{\text{d}(0)} - \xi_{\text{d}m}^2) - \xi_{\text{d}m}^2 \xi_{\text{d}m}, \quad (13)
\]
where \(C_0 = 4g^2/\kappa \gamma_m\) and \(C_1 = 4G^2/\kappa \gamma_d\) are the optomechanical and opto-atomic cooperativities, respectively, and \(\xi_{\text{d}m} = 2\lambda_{\text{md}}/\gamma_{\text{md}}\) plays the role of an effective dimensionless-amplitude of modulation.

The solution to the QLEs, i.e., Eq. (9), in the Fourier space can be written as \(\hat{\delta}(\omega) = \chi(\omega)\hat{\delta}(\omega)\) where \(\chi(\omega)\) is the susceptibility matrix and the Fourier transforms of the modified mechanical noises, i.e., those of Eqs. (10a) and (10b) are as follows
\[
\hat{X}_b^{\text{in}}(\omega) = \hat{X}_b^{\text{in}}(\omega) - \sqrt{\frac{2}{\gamma_m \hbar}} \frac{i}{2} [F(\omega + \omega_m) - F(\omega - \omega_m)],
\]
\[
\hat{P}_b^{\text{in}}(\omega) = \hat{P}_b^{\text{in}}(\omega) - \sqrt{\frac{2}{\gamma_m \hbar}} \frac{i}{2} [F(\omega + \omega_m) + F(\omega - \omega_m)].
\]
Now, using the input-output theory for the field operators, the output P-quadrature of the cavity field, i.e., \(\delta \hat{P}_a^{\text{out}}(\omega) = -\sqrt{\kappa} \delta \hat{\chi}_a(\omega) + \hat{P}_m^{\text{in}}(\omega),\) is obtained as follows
\[
\delta \hat{P}_a^{\text{out}}(\omega) = \mathcal{A}(\omega) \hat{P}_a^{\text{in}}(\omega) + \mathcal{B}(\omega) \hat{X}_b^{\text{in}}(\omega) + \mathcal{D}(\omega) \hat{X}_d^{\text{in}}(\omega), \quad (15)
\]
where \(\mathcal{A}(\omega) = 1 - \kappa \chi_{22}(\omega), \mathcal{B}(\omega) = \sqrt{\kappa \gamma_m} \chi_{23}(\omega),\) and \(\mathcal{D}(\omega) = \sqrt{\kappa \gamma_d} \chi_{25}(\omega)\) and the relevant elements of the susceptibility matrix are given by
\[
\chi_{22}(\omega) = \left[ \chi_{0}^{-1}(\omega) + G^2 \chi_{-m} + G^2 \chi_{-d}(\omega) \right]^{-1},
\]
\[
\chi_{23}(\omega) = \kappa \chi_{0}^{-1}(\omega) \chi_{-m}(\omega) + g^2 + G^2 \chi_{-m}(\omega) \chi_{-m}(\omega),
\]
\[
\chi_{25}(\omega) = -G \left[ \chi_{0}^{-1}(\omega) + G^2 \chi_{-m}(\omega) \chi_{-m}(\omega) \right]^{-1},
\]
with \(\chi_{0}^{-1}(\omega) = \kappa/2 - i \omega\) and \(\chi_{-m}^{-1}(\omega) = \gamma_{md}/2 - \lambda_{md} - i \omega.\) It is clear that \(\chi_{0}(\omega) = \chi_{0}^{*}(\omega)\) and \(\chi_{-m}^{-1}(\omega) = \chi_{-m}^{*}(\omega).\)

IV. SINGLE QUADRATURE FORCE SENSING

In this section by calculating the spectrum of the optical output phase quadrature, we will show how coherent modulations of both the atomic collisions frequency and the mechanical spring coefficient lead to the simultaneous signal amplification and backaction noise suppression which provides the best conditions for an ultra precision force sensing.

In the optomechanical force sensor demonstrated in Fig. (1), the imprint of the input mechanical signal is manifested in the cavity output field through the optomechanical interaction. In other words, the MO position shift exerted by the external force leads to a change of the effective cavity length and therefore causes the variation of the optical cavity output phase. As a consequence, the signal corresponding to the exerted external force can be detected by measuring the spectrum of the optical output phase quadrature, \(\hat{P}_a^{\text{out}},\) through methods like heterodyne, homodyne or synodyne detections [46]. In the following, we will show how the proposed hybrid optomechanical system allows us for single-quadrature force sensing with noise suppression and signal amplification which helps to surpass the SQL on force detection.

In order to measure and detect the input mechanical force, one should calculate the optical output phase quadrature spectrum,
\[
S_{P_a}^{\text{out}}(\omega) = \frac{1}{4\pi} \int d\omega' e^{i\omega' \omega} \kappa \delta \hat{P}_a^{\text{out}}(\omega) \delta \hat{P}_a^{\text{out}}(\omega') + \delta \hat{P}_a^{\text{out}}(\omega) \delta \hat{P}_a^{\text{out}}(\omega), \quad (16)
\]
Since the signal has been coded in the input mechanical noise quadrature, for an efficient force-sensing, we should manipulate the system parameters such that the mechanical response to the input quadrature \(\hat{X}_b^{\text{in}}\) is amplified while the optical and atomic responses to the input noise quadratures \(\hat{P}_b^{\text{in}}\) and \(\hat{X}_d^{\text{in}}\) are attenuated. Since the mechanical response of the system is independent of the classical input signal force and depends only on the quantum properties of the system, in the following, we set aside the classical function \(F\) and calculate the optical output phase quadrature spectrum by considering just the input quantum noises. So the output optical power spectrum is obtained as
\[
S_{P_a}^{\text{out}}(\omega) = (\hat{n}_c^T + \frac{1}{2})|\mathcal{A}(\omega)|^2 + (\hat{n}_d^T + \frac{1}{2})|\mathcal{B}(\omega)|^2 + (\hat{n}_d^T + \frac{1}{2})|\mathcal{D}(\omega)|^2. \quad (17)
\]
After some algebraic manipulations, one can rewrite the spectrum of the output optical phase quadrature as follows
\[
S_{P_a}^{\text{out}}(\omega) = R_m(\omega) \left[ (\hat{n}_m^T + 1/2)^2 + n_{\text{mod}}(\omega) \right], \quad (18)
\]
where
\[ R_m(\omega) = |B(\omega)|^2 = \kappa \gamma_m |\gamma_{23}(\omega)|^2, \tag{19} \]
\[ n_{add}(\omega) = (\bar{\eta}^T + 1)(\tilde{\eta}^T - \frac{1}{2}) |B^2(\omega)|^2 + 2 |D(\omega)|^2. \tag{20} \]

Here, \( R_m(\omega) \) is the mechanical response to the input signal and \( n_{add}(\omega) \) is the added noise of measurement which originates from the contributions of the input optical and atomic vacuum noises to the phase quadrature of the output cavity field. As is seen from Eq. (18), the added noise can be considered as an effective increase in the number of the thermal excitations of the mechanical reservoir due to the backaction of the optical and atomic modes. For a high precision force sensing and surpassing the SQL, one should amplify the mechanical response and suppress the added noise spectrum simultaneously. The SQL on force-sensing is defined as \( n_{add}^{SQL}(\omega) = 1/2 \) [69, 70] which has already been achieved experimentally [71]. In the following it is shown that through the mechanical and atomic modulations the SQL can be surpassed by suppressing the added backaction noise especially near the on-resonance frequency of the output \( P \)-quadrature while the input force signal is amplified through the enhancement of the system mechanical response.

The on-resonance added noise and mechanical response are, respectively, given by
\[ n_{add}(0) = \left(1 - \xi_m\right)^2 \left[ \frac{G_a}{C_0} \left( \left(\sqrt{G_a} - 1\right)^2 \bar{\eta}^T + \frac{1}{2} \right) \right] + \frac{C_1}{(1-\xi_d)^2} \left( \bar{\eta}^T + \frac{1}{2} \right), \tag{21} \]
\[ R_m(0) = C_0 \left( \frac{\sqrt{G_a} - 1}{1 - \xi_m} \right)^2, \tag{22} \]
where the optical gain \( G_a \), which is defined in the context of the linear amplifiers as the ratio of photons number in the output of the amplifier to that in the input [70], is given by [61]
\[ \sqrt{G_a} = \frac{C_0 - (1 - \xi_m) + C_1 \frac{1 - \xi_m}{1 - \xi_d}}{C_0 + (1 - \xi_m) + C_1 \frac{1 - \xi_m}{1 - \xi_d}} \tag{23} \]

As is seen from Eq. (21), the added noise is suppressed in the limit of \( \xi_m \rightarrow 1 \). However, as is seen from Eq. (22), in order to have signal amplification, the mechanical response should be increased simultaneously which is only possible when the optical gain is negligibly small or equal to zero, i.e., when \( G_a = 0 \). To achieve zero gain, the impedance-matching condition given by
\[ C_0 + (\xi_m - 1)(1 - C_1)/(1 - \xi_d) = 0, \quad C_0 + C_1 \leq 1, \tag{24} \]
should be satisfied. In other words, in order to have simultaneous noise suppression together with signal amplification, the numerical values of the cooperativities (\( C_0 \) and \( C_1 \)) and also the atomic modulation \( \xi_a \) should be chosen so that the impedance-matching condition of Eq. (24) is satisfied for any specified value of the mechanical modulation in the limit of \( \xi_m \rightarrow 1 \).

In the case where there is neither mechanical nor atomic modulation (off-modulations), i.e., \( \xi_d = \xi_m = 0 \), we have
\[ n_{add}^{off}(0) = \frac{1}{C_0} \left[ \frac{(C_0 + C_1 - 1)^2}{4} \left( \bar{\eta}^T + \frac{1}{2} \right) + C_1 \left( \bar{\eta}^T + \frac{1}{2} \right) \right], \tag{25} \]
\[ R_m^{off}(0) = \frac{4C_0}{(1 + C_0 + C_1)^2}. \tag{26} \]

As is evident, in this case the mechanical response is always smaller than unity under the impedance-matching condition (\( C_0 + C_1 = 1 \)) while the added noise is fairly large. This means that in the off-modulations case, the system is able to transduce the mechanical force but cannot amplify the signal and suppress the added noise. In other words, it cannot operate as a high precision measurement device.

In the other special case where there is no atomic modulation (\( \xi_d = 0 \)) while the mechanical modulation is turned on, the impedance-matching condition reads \( \xi_m + C_0/(1 - C_1) = 1 \), and consequently
\[ n_{add}(0) = \frac{C_1}{C_0} \left(1 - \xi_m\right)^2 \left( \bar{\eta}^T + \frac{1}{2} \right), \tag{27} \]
\[ R_m(0) = C_0 \left( \frac{1 - \xi_m}{1 - \xi_d} \right)^2. \tag{28} \]

In this case, it is clear that in the limit of \( \xi_m \rightarrow 1 \) there is a large mechanical response to the input signal with no added optical noise while there is a small residual backaction noise due to the Bogoliubov mode of the BEC.

In order to see how the mechanical and atomic modulations affect the signal amplification and noise suppression, in Fig. (2) we have plotted the added noise \( n_{add}(\omega) \) [Fig. 2(a)] and the mechanical response to the signal \( R_m(\omega) \) [Fig. 2(b)] versus the normalized frequency \( \omega/\gamma_m \) in the largely different cooperativities regime with \( C_0 = 0.04 \) and \( C_1 = 0.5 \) under the impedance-matching condition [curves indicated by 1 to 5]. Here, the effective modulation amplitudes, i.e., \( \xi_m \) and \( \xi_d \), corresponding to the above specified values of cooperativities have been calculated based on the impedance-matching condition [Eq. (24)] together with the stability condition [Eq. (12)]. Besides, we have demonstrated the case of off-modulations (\( \xi_m = 0, \xi_d = 0 \)) under the impedance-matching condition of \( C_0 + C_1 = 1 \) with \( C_1 = 0.5 \) [curve indicated by 6] and also the case of the absence of the impedance-matching condition [curve indicated by 7] for the sake of comparison with the other ones.

As is seen from Fig. 2(b), in the case of “off-modulations” (\( \xi_m = \xi_d = 0 \)) [the densely dotted curve indicated by 6] the mechanical response is lower than unity which means that there is no signal amplification while in the presence of modulations [curves indicated by 1 to 5, and 7] the mechanical response gets larger than unity which leads to the signal amplification. Furthermore, the most efficient situation of signal amplification occurs under the impedance-matching condition. As is seen, the signal amplification in the absence of the impedance-matching condition [curve indicated by 7] is not as efficient as those under this condition.

On the other hand, in the absence of the BEC when the mechanical modulation is on [the red curve indicated by 1]
FIG. 2. (Color online) (a) The added noise, $n_{\text{add}}(\omega)$, and (b) the mechanical response to the input signal, $R_{m}(\omega)$, vs dimensionless frequency $\omega/\gamma_{m}$. The curves indicated by 1 to 5 and 7 have been plotted under the impedance-matching condition of Eq. (24) with $C_{0} = 0.04$ and $C_{1} = 0.5$. The red solid very thick curve 1 corresponds to the absence of BEC with $\xi_{m} = 0.96$, and $\gamma_{m}/\gamma_{d} = 1$; the black dotted curve 2 corresponds to $\xi_{m} = 0.98, \xi_{d} = 1.42,$ and $\gamma_{m}/\gamma_{d} = 10^{2}$; the blue solid thick curve 3 corresponds to $\xi_{m} = 0.98, \xi_{d} = 1.42,$ and $\gamma_{m}/\gamma_{d} = 1$; the green dashed curve 4 corresponds to $\xi_{m} = 0.98, \xi_{d} = 1.42$ and $\gamma_{m}/\gamma_{d} = 10^{-2};$ the orange solid thick curve 5 corresponds to $\xi_{m} = 0.92, \xi_{d} = 0$, and $\gamma_{m}/\gamma_{d} = 1$; the purple densely dotted curve 6 corresponds to $\xi_{m} = 0, \xi_{d} = 0$, and $\gamma_{m}/\gamma_{d} = 1$, i.e., the off-modulations case with the impedance-matching condition $C_{0} + C_{1} = 1 \ (C_{0} = C_{1} = 0.5);$ and the brown dashed-double-dotted curve 7 corresponds to the no impedance-matching condition with $\xi_{m} = 0.9, \xi_{d} = 0.2,$ and $\gamma_{m}/\gamma_{d} = 1$. The gray region (under the SQL line) in panel (a) and the cyan region in panel (b) correspond, respectively, to the situations where there are noise suppression and signal amplification. Here, we have assumed $\kappa/\gamma_{m} = 10^{3}, \tilde{n}_{m}^{2} = 10^{3},$ and $\tilde{n}_{d}^{2} = \tilde{n}_{d} = 0.$

which is similar to the situation studied in Ref. [49], there is a strong noise suppression together with a fairly good signal amplification notably near the on-resonance frequency ($\omega \approx 0$). This shows how the presence of the mechanical modulation can lead to an ultra sensitive force measurement. However, the presence of the BEC together with atomic modulation improve the signal amplification substantially through the increase of the mechanical response of the system near the on-resonance frequency [compare the black, blue and green curves indicated, respectively, by 2, 3 and 4 with the red curve indicated by 1]. Naturally, the price paid for this strong improvement of the signal amplification is an increase in the added noise of the measurement because the presence of the BEC, as an extra phononic mode, induces an additional back-action noise. Nevertheless, the increment of the added noise due to the presence of the BEC is not so large to affect a precise measurement. As is seen from Fig. 2(a), the added noise remains much below the SQL near the on-resonance frequency for the curves indicated by 2, 3 and 4.

Another advantage of the presence of the BEC with atomic modulation is the possibility of the “off-resonance” force sensing. As is seen from Fig. 2, the proposed optomechanical force sensor can amplify the signal [the cyan region in Fig. 2(b)] and attenuate the added noise [the gray region in Fig. 2(a)] in a wide range as large as $\Delta \omega_{\text{measurement}} \sim \gamma_{m}/5$ about the on-resonance frequency. In fact, by controlling the ratio of phononic damping rates such that $\gamma_{m}/\gamma_{d} < 1$ [see the green curve indicated by 4 for which $\gamma_{m}/\gamma_{d} = 0.01$] the signal can be amplified strongly in a much wider range around the off-resonance region (the bandwidth of amplification gets much larger). Therefore, the presence of the BEC together with atomic modulation improve the signal amplification effectively while in the absence of atomic modulation the BEC by itself does not enhance the signal amplification considerably [see the orange curves indicated by 5 in Fig. 2(b)].

On the other hand, in order to see how the ratio of the atomic and the mechanical cooperativities affects the signal amplification and noise suppression, in Fig. (3) we have plotted the added noise [Fig. 3(a)] and the mechanical response [Fig. 3(b)] versus the normalized frequency $\omega/\gamma_{m}$ for different ratios of cooperativities $C_{1}/C_{0}$ under the impedance-matching condition. For each curve represented in Fig. (3), the effective amplitudes of modulations ($\xi_{d}$ and $\xi_{m}$) can be obtained from the impedance-matching condition (24) for the specified values of cooperativities. Here, the red solid thick and red loosely dashed curved indicated, respectively, by $C_{0} = 0.04$ [with $\xi_{m} = 0.96$] and $C_{0} = 0.4$ [with $\xi_{m} = 0.6$] correspond to the absence of the BEC, i.e., $C_{1} = 0$. Besides, the black solid thin curve indicated by $C_{0} = 0.04, C_{1} = 0.5$ [with $\xi_{m} = 0.98, \xi_{d} = 1.42$], the black densely dashed curve indicated by $C_{0} = 0.4, C_{1} = 0.5$ [with $\xi_{m} = 0.84, \xi_{d} = 1.32$], and the black dotted curve indicated by $C_{0} = 0.04, C_{1} = 0.05$ [with $\xi_{m} = 0.30, \xi_{d} = 0.94$] correspond to the presence of the BEC when both atomic and mechanical modulations are turned on.

As is seen clearly in Fig. (3), in the absence of the BEC an acceptable amount of signal amplification is achievable near the on-resonance frequency through the mechanical modulation for small values of mechanical cooperativities while the added noise is nearly equal to zero which is due to the absence of an extra mode (the red solid thick curve indicated by $C_{0} = 0.04$). Nevertheless, the presence of the BEC with a large ratio of $C_{1}/C_{0}$ together with both atomic and mechanical modulations lead to much stronger signal amplification while the added noise does not increase very much and stays much below the SQL (see the black solid thin curve indicated by $C_{0} = 0.04, C_{1} = 0.5$).

However, the signal amplification is reduced substantially by decreasing the ratio of $C_{1}/C_{0}$ (the black densely dashed curve indicated by $C_{0} = 0.4, C_{1} = 0.5$). Especially, for
FIG. 3. (Color online) (a) The added noise, $n_{\text{add}}(\omega)$, and (b) the mechanical response to the input signal, $R_m(\omega)$, vs dimensionless frequency $\omega/\gamma_m$ for different ratios of cooperativities $C_1/C_0$ under the impedance-matching condition of Eq. (24). The red solid thick (indicated by $C_0 = 0.04$) and the red loosely dashed (indicated by $C_0 = 0.4$) curves correspond to the absence of BEC. The black solid thin (indicated by $C_0 = 0.04, C_1 = 0.5$), black densely dashed (indicated by $C_0 = 0.4, C_1 = 0.5$), and black dotted (indicated by $C_0 = 0.04, C_1 = 0.05$) curves correspond to the presence of the BEC when both atomic and mechanical modulations are turned on. The gray region (under the SQL line) in panel (a) and the cyan region in panel (b) correspond, respectively, to the situations where there are noise suppression and signal amplification. Here, we have set $\gamma_m/\gamma_d = 1$. The other parameters are the same as those of Fig. (2).

lower values of $C_1$ (the black dotted curve indicated by $C_0 = 0.04, C_1 = 0.05$) not only there is no signal amplification (the signal is attenuated) but also the added noise increases significantly. Therefore, equipping the system with an extra atomic mode of a BEC together with atomic modulation can enhance the ability of signal amplification substantially while the extra added noise can be kept much below the SQL in a specific parametric regime which is based on the so-called impedance-matching condition with a large ratio of $C_1/C_0$.

Finally, it is worth to compare the presented method of force sensing which is based on parametric modulations with those based on the backaction-evasion [35, 69] and CQNC techniques [1, 2, 43, 45, 47]. The former is able to surpass the SQL by producing a large signal without suppressing the added noise while the latter can cancel the backaction noise completely without amplifying the signal. However, the force sensing scenario proposed in the present work, which is based on simultaneous signal amplification (mechanical response amplification) and noise suppression possesses the advantages of both the above-mentioned methods in that it provides a large signal to noise ratio. It should be pointed out that the improvement of force sensing in our scheme relies on small cooperativities with a large difference, i.e., $C_0 + C_1 < 1$ and $C_0 \ll C_1$, which is achievable by taking a weak red-detuned driving together with the validity of the RWA.

V. SUMMARY AND CONCLUSION

In this work, it has been proposed a scheme for an optomechanical force sensor composed of a hybrid optomechanical cavity containing an interacting cigar-shaped BEC where both the atomic collisions frequency of the BEC and the spring coefficient of the MO are coherently modulated. It has been shown that under these conditions the mechanical response of the system to the input signal is enhanced substantially which leads to the amplification of the weak input signal while the added noises of measurement can be maintained much below the SQL. In this way, such a hybrid system can operate as an ultra sensitive force sensor which can amplify the input signal without increasing the noise of measurement.

The advantage of the presented hybrid system in comparison to the bare optomechanical cavities is that the presence of the BEC together with atomic modulation improves the signal amplification substantially through the increase of the mechanical response of the system. Naturally, the price paid for this strong improvement of the signal amplification is an increase in the added noise of the measurement because the presence of the BEC, as an extra mode, induces an additional backaction noise. Nevertheless, the increment of the added noise due to the presence of the BEC is not so large to affect a precise measurement. It has been shown that by controlling the system cooperativities and modulation amplitudes, the backaction noises of measurement can be suppressed and lowered much below the SQL.

The presented optomechanical force sensor has the optimum functionality near the on-resonance frequencies in the largely different cooperativities and red-detuned regimes where the impedance-matching condition is satisfied. Nevertheless, there exists the possibility of ultra precise measurement in the off-resonance region by controlling the BEC parameters and amplitudes of modulations which can enlarge the detection bandwidth.

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