On the Motion of Zeros of Zeta Functions

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The motion in the complex plane of the zeros to various zeta functions is investigated numerically. First the Hurwitz zeta function is considered and an accurate formula for the distribution of its zeros is suggested. Then functions which are linear combinations of different Hurwitz zeta functions, and have a symmetric distribution of their zeros with respect to the critical line, are examined. Finally the existence of the hypothetical non-trivial Riemann zeros with $Re\ s \neq 1/2$ is discussed.

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1 Introduction

The Riemann zeta function $\zeta(s)$ and its zeros is a source of inspiration in the field of quantum chaos. It is well known that the imaginary parts of the zeros to $\zeta$ on the critical line $\text{Re } s = 1/2$ have some striking similarities with a spectra of an Hermitian operator with time reversal symmetry broken. The famous Riemann hypothesis states that in fact all non-trivial zeros of $\zeta$ lie on the critical line and the search for the Hermitian operator, and its classical counterpart, has been intensified recently.

With help of powerful computers the Riemann zeros have been explored in great detail. In this paper we present much more modest numerical experiments for the zeros of Hurwitz zeta function $\zeta(s, \alpha)$.

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)s}, \quad 0 < \alpha \leq 1, \quad \sigma > 1, \quad s = \sigma + it,$$

where $\zeta(s, 1) = \zeta(s)$, and for linear combinations of them,

$$\Psi(s) = \sum_{l=1}^{m} c_l \zeta(s, \alpha_l).$$

Important examples of $\Psi(s)$ are the L-functions for which $\alpha_l = l/m$ and $c_l = \chi(l)/m^s$, $\chi(l)$ are Dirichlet characters modulo $m$. The motivation for the study is that it is often very useful to study how a system behaves as function of a parameter, see e.g. chapter 6 in [11]. We will, inspired by the similarities mentioned above, use the word ”spectra” for the imaginary parts of the zeros, not only for $\zeta(s, 1)$ but also for $\zeta(s, \alpha)$ and $\Psi(s)$. In sect. 2 we consider the spectra for $\zeta(s, \alpha)$ as function of $\alpha$ while we in sect. 3 discuss the motion of the zeros for some $\Psi(s)$ with $m = 5$, and symmetric distribution of the zeros. In the latter case the coefficients, $c_l$, depend on one or two parameters and it is the motion with respect to their variations that is studied. Bombieri and Hejhal has shown that in the limit of large $t$ almost all zeros for these symmetric functions are simple and lie on the critical line. Finally in sect. 4 some results on symmetric zeta functions with $m > 5$ are presented together with some speculations on the existence of non-trivial Riemann zeros with $\text{Re } s \neq 1/2$. The numerical calculations presented are done with Mathematica on a Sun Ultra machine.

2 Hurwitz zeta function

In this section we focus on the motion of the zeros of $\zeta(s, \alpha)$ with respect to $\alpha$. Since along a zero $d\zeta = \zeta' ds + \partial \zeta/\partial \alpha \, d\alpha = 0$ we get the following equation for the motion of a zero at $s = z$

$$\frac{dz}{d\alpha} = -\frac{\partial \zeta(z, \alpha)}{\partial \alpha} \cdot \frac{1}{\zeta'(z, \alpha)} = \frac{z \cdot \zeta(z + 1, \alpha)}{\zeta'(z, \alpha)}.$$

(3)
The second equality in (3) is obtained by derivating (1) with respect to $\alpha$. For the extension of $\zeta(s, \alpha)$ to the whole complex plane by analytical continuation see [3, 4]. Note that when $\alpha$ varies zeros can merge to multiple zeros or go away to infinity but never disappear or be created. At multiple zeros $\zeta' = 0$ so the equation of motion must be modified, see sect. 3. A spectra for the Hurwitz zeta function can now be obtained by regarding $t = \text{Im } z(\alpha)$ as an eigenvalue. To get a spectra at $\alpha = 1$ with unit mean spacing one has to use scaled eigenvalues $N(t)$. With $N(T)$ we denote the number of zeros in the range $0 < \text{Im } z \leq T$, $0 \leq \text{Re } z \leq 1$, and the following asymptotic formula [1, 2, 3]:

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi},$$

(4)

is used. To obtain a spectra with approximately unit mean spacing in the whole region $0 < \alpha \leq 1$ we suggest that (4) should be modified in the following way

$$N(T, \alpha) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} - \frac{T}{2\pi} \log(\alpha),$$

(5)

where the zeros now can be located outside the critical strip. Numerical experiments in the range $10^{-10} \leq \alpha \leq 1$, $0 < T \leq 10^4$ indicate that (5) is very accurate but we have not been able to prove it. It is certainly correct for $\alpha = 1/2$ since

$$\zeta(s, 1/2) = (2^s - 1) \cdot \zeta(s, 1).$$

(6)

The zeros of $2^s - 1$ have $\sigma = 0$ with spacings $\Delta t = \frac{2\pi}{\log(2)}$ and their number up to $\text{Im } z = T$ is given by the last term in (5). The formula also seems to be accurate for $L$-functions if $m = p$, $p$ is a prime number, and $\alpha = 1/p$ is used in (5), see also [4].

In fig. 1 a part of the spectra for the Hurwitz zeta function is presented. The 30:th to the 43:th zero in the range $10^{-2} \leq \alpha \leq 1$ are shown. We can observe that the spectra share some properties with a generic quantal spectra since there are many avoided crossings but also degeneracies. Note that these degeneracies are only in $t$, not in $\sigma$. However, we see also that the crossings occur in a somewhat peculiar way. They take place when $N(t(\alpha))$ for one of the zeros is strongly decreasing with increasing $\alpha$ and then usually more crossings occur for nearby $\alpha$-values. This effect is much more pronounced higher up in the spectra. It can also be seen from fig. 1 that the spectra becomes more equidistant when $\alpha \to 0$. This can be understood qualitatively since $\zeta(s, \alpha) \approx \frac{1}{\alpha^s} + \zeta(s, 1)$ for small $\alpha$-values. The zeros seem thus to move towards $s = 0$ when $\alpha \to 0$. From the analytical continuation of (1) to the whole complex plane we know that $\zeta(s, 1)$ has trivial zeros at $s = -2, -4, -6, \ldots$. From (3) we expect that these zeros move to the right on the $\sigma$-axis when $\alpha$ decreases and eq. (6) gives that these zeros have moved two units at $\alpha = 0.5$. When $\alpha \to 0$ the zeros approach $s = 1, 0, -2, -4, -6, \ldots$.

3 Symmetric functions $\Psi(s)$ for $m = 5$

Symmetric functions here means functions of the type given by eq. 2 for which the non-trivial zeros are located symmetrically around $\sigma = 1/2$, i.e., if $\Psi(z) = 0$ also $\Psi(1-\bar{z}) = 0$. 

The symmetric $\Psi(s)$ which also are L-functions can be expressed as Euler products and then it is straightforward to show that $\Psi(s) \neq 0$ when $\sigma > 1$ \[3\]. The three functions of this type with the lowest $m$-values are $\zeta(s, 1)$, $\frac{1}{5} (\zeta(s, 1/3) - \zeta(s, 2/3))$ and $\frac{1}{5} (\zeta(s, 1/4) - \zeta(s, 3/4))$.

For $m \geq 5$ opens up the possibility of constructing symmetric $\Psi(s)$ which are linear combinations of L-functions. For these linear combinations there is generally no Euler product and zeros can appear outside the critical strip. For $m = 5$ we can for example make the following ansatz

$$\Psi_{5o}(s, \beta) = \frac{1}{5} (\zeta(s, 1/5) - \zeta(s, 4/5) + \beta (\zeta(s, 2/5) - \zeta(s, 3/5))),$$

\[7\]

$o$ denotes odd since $c_l = -c_{m-l}$ in (7). Following the lines at p. 212 in \[3\] a condition for $\Psi_{5o}(s, \beta)$ being symmetric can be obtained,

$$\sin \frac{4\pi}{5} - \beta \sin \frac{2\pi}{5} = \bar{\beta} (\sin \frac{2\pi}{5} + \beta \sin \frac{4\pi}{5}).$$

\[8\]

The two odd L-functions with $m = 5$ have $\beta = \pm i$. In fig. 2 a part of the spectra for $\Psi_{5o}(s, \beta)$, with $\beta$ lying on the circle (8), is shown. To obtain this figure the equation of motion (3) is used but with $\alpha$ and $\zeta(s, \alpha)$ replaced by $\beta$ and $\Psi_{5o}(s, \beta)$. In contrast to the spectra for the Hurwitz zeta function double zeros now appear frequently and in the figure four of them can be seen. It happens when two zeros meet on the critical line and move outside $\sigma = 1/2$ and then again when these two zeros join on the critical line. At multiple zeros $\Psi'_{5o}(s, \beta) = 0$ so $\frac{d\Psi}{d\beta}$ becomes infinite there. To overcome this difficulty a small complex constant can be added to the parameter. For each turn on the circle (8) in the counter-clockwise sense a zero from the lower half-plane moves to the upper half-plane and this is why most zeros in fig. 2 move upwards. It is supposed, but not proven, that for the two L-functions at $\beta = \pm i$ all non-trivial zeros lie on the critical line. It seems hard to gain insight to that problem from the study of this one-parameter family of symmetric functions. It becomes more interesting when we now consider to a two-parameter family.

In the same way as above we can try to construct even, i.e. $c_l = c_{m-l}$, and symmetric functions with $m = 5$,

$$\Psi_{5e}(s, \beta) = \frac{1}{5} (\zeta(s, 1/5) + \zeta(s, 4/5) + \beta (\zeta(s, 2/5) + \zeta(s, 3/5))).$$

\[9\]

However, it now turns out that besides the condition, see again p.212 in \[3\]

$$\cos \frac{4\pi}{5} + \beta \cos \frac{2\pi}{5} = \bar{\beta} (\cos \frac{2\pi}{5} + \beta \cos \frac{4\pi}{5})$$

\[10\]

$\beta$ must also fulfill $1 + \beta = 0$ so $\beta = -1$ is the only possibility, which is a L-function. To make a continuous change of the spectra, as for $\Psi_{5o}$, we can take a linear combination of $\Psi_{5e}$ with $\zeta(s, 1)$,

$$\Psi_e(s, \beta, \gamma) = \Psi_{5e}(s, \beta) + \frac{\gamma}{5^s} \zeta(s, 1).$$

\[11\]
The symmetry requirement leads to the following equations for the coefficients $\beta$ and $\gamma$

\[
\cos \frac{4\pi}{5} + \beta \cos \frac{2\pi}{5} + \frac{\gamma}{2} = \bar{\beta} \left( \cos \frac{2\pi}{5} + \beta \cos \frac{4\pi}{5} + \frac{\gamma}{2} \right) \]

\[1 + \beta + \frac{\gamma}{2} = \bar{\gamma} \left( \cos \frac{2\pi}{5} + \beta \cos \frac{4\pi}{5} + \frac{\gamma}{2} \right). \tag{12}
\]

For $\beta = 1$ the first equation is fulfilled for any $\gamma$ while the second equation gives $\gamma = 1 + \sqrt{5} e^{i\theta}$ where $\theta$ is an angle between 0 and $2\pi$. Besides the symmetric functions given by (12) there is one more possibility, namely $\beta = \gamma = 1$. At this point, where $\cos \frac{2\pi}{5} + \beta \cos \frac{4\pi}{5} + \frac{\gamma}{2} = 0$, $\Psi_e(s, 1, 1)$ reduces to $\zeta(s, 1)$ but it is not possible to continuously vary this spectra since $\beta = \gamma = 1$ is not a solution to (12). On the circle $\beta = 1, \gamma = 1 + \sqrt{5} e^{i\theta}$ we have

\[\Psi_e(s, \theta) = (1 + \frac{e^{i\theta}}{5^{s-1/2}})\zeta(s, 1). \tag{13}\]

When $\gamma$ moves counter-clockwise along the circle the zeros of $1 + \frac{e^{i\theta}}{5^{s-1/2}}$ (in the following denoted as trivial), separated by $\Delta t = 2\pi/\text{Log}(5)$, move upwards with $\frac{d\theta}{dt} = 1/\text{Log}(5)$ while the zeros of $\zeta(s, 1)$ remain fixed.

Let us now put $\beta = 1 + \epsilon e^{i\phi}$ and consider when $\beta$ moves on a circle with radius $\epsilon$ around $\beta = 1$ in the counter-clockwise sense. Then eq. 12 gives

\[\gamma = 1 + \sqrt{5} e^{i2\phi} + \epsilon \frac{1 + \sqrt{5}}{2} e^{i\phi}, \tag{14}\]

$0 \leq \phi < 2\pi$. With these coefficients we get

\[\Psi_e(s, \epsilon, \phi) = (1 + \frac{e^{i2\phi}}{5^{s-1/2}})\zeta(s, 1) + \epsilon \frac{e^{i\phi}}{5^{s}} g_5(s), \tag{15}\]

where

\[g_5(s) = \frac{1 + \sqrt{5}}{2} \zeta(s, 1) + \zeta(s, 2/5) + \zeta(s, 3/5). \tag{16}\]

Compared to (13), $\Psi_e(s, \epsilon, \phi)$ in (15) have no trivial zeros in the critical strip. For small values of $\epsilon$ the two cases must, however, be similar, see fig. 3. The difference is that when an upsloping ”almost trivial” zero meet an ”almost Riemann” zero the two zeros bifurcate out in the complex plane or they interact and exchange character on the critical line.

The equation of motion for a simple zero, $z_e(\phi)$, to (15) for a fixed positive $\epsilon$ is

\[
\frac{dz_e}{d\phi} = -\frac{\partial \Psi_e(z_e, \epsilon, \phi)}{\partial \phi} \cdot \frac{1}{\Psi_e(z_e, \epsilon, \phi)} = -i \zeta(z_e, 1) \left( \frac{e^{i2\phi}}{5^{z_e-1/2}} - 1 \right) \cdot \frac{1}{\Psi_e'(z_e, \epsilon, \phi)}. \tag{17}
\]

The fix points of this dynamical system are the Riemann zeros. On the critical line, where $\text{Re} \ z_e = 1/2$, turning points also appear. For large values of $\epsilon$ the second term in (15) dominates and for the symmetric function $g_5(s)$ zeros outside the critical line
appears frequently. If \( z_\infty \) denotes a simple zero of \( g_5(s) \) and if \( \Delta z(\varphi) = z_\epsilon(\varphi) - z_\infty \) the equation of motion (17) becomes to lowest order

\[
\Delta z'(\varphi) = i \Delta z \left( \frac{e^{i2\varphi}}{5^{z_\infty-1/2} - 1} - 1 \right) / \left( \frac{e^{i2\varphi}}{5^{z_\infty-1/2} + 1} \right)
\]

This linearised system gives rise to straight line motion if \( \text{Re} \ z_\infty = 1/2 \) and cycles when \( \text{Re} \ z_\infty \neq 1/2 \). These cycles are counter-clockwise for \( \text{Re} \ z_\infty < 1/2 \) and clockwise for \( \text{Re} \ z_\infty > 1/2 \). Thus, for large values of \( \epsilon \) the zeros of \( \Psi_\epsilon(s, \epsilon, \varphi) \) near the zeros of \( g_5(s) \) perform vibrational or circulating motion. However, far outside the critical strip there is still an upward flow of almost trivial zeros since \( \Psi_\epsilon(s, \epsilon, \varphi) \approx 1 + \epsilon e^{i\phi} \) for large values of \( \sigma \). When \( \epsilon \) decreases this upward flow moves towards the critical strip. For \( \epsilon > 2 \) the flow from the lower plane to the upper half-plane takes mainly place outside the critical strip. Note that \( \Psi_\epsilon(s, \epsilon, \varphi) \) has a pole at \( s = 1 \) except for \( \epsilon = 2, \varphi = \pi \), which is a L-function. This shows that a zero from the upward flow is located at \( s = 1 \) for these parameter values. For \( \epsilon < 2 \) the flow from lower to upper half-plane takes mainly place inside the critical strip. When \( \epsilon \) decreases the ”train” of zeros moves thus to the left. Then zeros of \( g_5(s) \) with \( \sigma > 1/2 \) are approached. If \( \zeta(z_\infty, 1) \neq 0 \) these zeros must be circumvented. The upward moving zeros can pass by these obstacles by a bifurcation, i.e. merge to a double zero, with the zero which form a circuit in the clockwise sense around the zero, \( z_\infty \), to \( g_5(s) \). So for each simple zero \( z_\infty \) with \( \sigma > 1/2 \) one extra zero can be added to the upward flow.

There is however another possible obstacle for the moving zeros, namely the famous hypothetical zeros, \( z_0 \), of \( \zeta(s, 1) \) outside the critical line. The linearised motion around these zeros is also of the form (18), except for a change of sign, so if \( \text{Re} \ z_0 > 1/2 \) the circuits are now in the counter-clockwise sense. When the flow of zeros comes in to the vicinity of such a zero, \( z_0 \), one zero leaves the ”train” and starts to circulate around \( z_0 \). Thus, a bifurcation is needed if \( z_0 \) is not a common zero of \( \zeta(s, 1) \) and \( g_5(s) \). The parameter values for the bifurcation that ”creates” the zero circulating around \( z_0 \) are below denoted by \( \epsilon_B \) and \( \varphi_B \).

4 Symmetric functions with \( m > 5 \)

Symmetric functions of the form (15) can be constructed for higher \( m \) values. Here we concentrate on primes. For \( m = 7 \) there are three complex parameters and with the ansatz \( \beta_1 = 1 + \epsilon x_1 e^{i\varphi}, \beta_2 = 1 + \epsilon x_2 e^{i\varphi} \) and \( \gamma = 1 + \sqrt{7} e^{i2\varphi} + \epsilon x_3 e^{i\varphi} \) the symmetry conditions lead to the following linear equation for the real parameters \( X = (x_1, x_2, x_3) \),

\[
\begin{pmatrix}
\cos \frac{4\pi}{7} & \cos \frac{6\pi}{7} & \frac{1}{2} \\
\cos \frac{6\pi}{7} - \sqrt{7} & \cos \frac{2\pi}{7} & \frac{1}{2} \\
\cos \frac{2\pi}{7} & \cos \frac{4\pi}{7} - \sqrt{7} & \frac{1}{2}
\end{pmatrix} X = 0.
\]
The rank, $r$, of this matrix is two and the null space is one dimensional so there is only one possible construction, namely

$$\Psi_e(s, \epsilon, \varphi) = (1 + \frac{e^{2i\varphi}}{I(s-1/2)})\zeta(s, 1) + \epsilon e^{i\varphi} g_7(s, x), \quad (20)$$

where

$$g_7(s, x) = x_1(\zeta(s, 2/7) + \zeta(s, 5/7)) + x_2(\zeta(s, 3/7) + \zeta(s, 4/7)) + x_3\zeta(s, 1). \quad (21)$$

Numerical investigations of the corresponding matrices for all prime numbers $p < 1000$ shows that $p = 4r \pm 1$ and the order of the matrices is $\frac{p-1}{2}$ so the dimension of the null space increases with increasing $p$.

An interesting case is $p = 13$. Here the null space is three dimensional so for each point $X$ on the sphere $S^2$ there is a symmetric function

$$\Psi_e(s, \epsilon, \varphi, X) = (1 + \frac{e^{2i\varphi}}{I(s-1/2)})\zeta(s, 1) + \epsilon e^{i\varphi} g_{13}(s, X). \quad (22)$$

Let us as in sect 3 assume the existence of simple Riemann zeros, $z_0$, outside the critical line. To each point $X$ there is one point $(\epsilon_B, \varphi_B)$ where the bifurcation near $z_0$ (see sect. 3) takes place and for $-X$ the corresponding point is $(\epsilon_B, \varphi_B + \pi)$. If now $X$ moves one circuit on the equator of $S^2$ we expect that a closed curve $(\epsilon_B(X), \varphi_B(X))$ will be traced out. Let us assume that $\epsilon_B = 0$ does not lie on the curve. If then the equator is continuously deformed to the north pole, say, the closed curve must for some $X$ cross $\epsilon_B = 0$ which is impossible for simple zeros! The situation is different for the bifurcations near $z_\infty$ since the location of $z_\infty$ in the complex plane depends on $X$ and $z_\infty$ is generally located on the critical line for some $X$ values. Further work on these interesting symmetric functions is in progress.
Figure 1: The imaginary part of the 30:th to the 43:th zero of $\zeta(s, \alpha)$ in the range $10^{-2} \leq \alpha \leq 1$ is shown. Scaled values $N(t, \alpha)$ of $t = \text{Im} \ z$ have been used. For this, and all other figures, a 4:th order Runge-Kutta method has been used to solve the equation of motion. The suggested formula (5) for $N(T, \alpha)$ is seen to work well. The somewhat downsloping overall structure is probably due to higher order terms in (5).
Figure 2: A part of the spectra for the symmetric function $\Psi_{50}(s, \beta)$ in (7). The parameter $\beta$ lies on the circle (8) and an angle $\phi$ is used in the figure to parametrise this circle. For $\beta = +i$ and $-i$, which corresponds to left (and right) edge and $\phi \approx 5.73$ respectively, $\Psi_{50}$ are L-fuctions and all zeros for them are supposed to lie on the critical line. Note the two upsloping segments which join the double zeros on the critical line. To remedy the singular equation of motion at the bifurcation points a small complex constant has been added to the parameter $\phi$. 
Figure 3: A low lying part of the spectra for $\Psi_\varepsilon(s, \epsilon, \phi)$, given by (15), is shown for $\epsilon = 0.01$ and $0 \leq \phi < 2\pi$. At $\epsilon = 0$ the spectra consists of upsloping trivial zeros crossing horizontal Riemann zeros. Here these crossings are replaced by bifurcations out in the complex plane or interactions on the critical line. As for fig. 2 a small complex constant has been added to $\phi$ to overcome the bifurcations.
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