The TF Limit for Rapidly Rotating Bose Gases in Anharmonic Traps

J.-B. Bru\textsuperscript{a}, M. Correggi\textsuperscript{b}, P. Pickl\textsuperscript{a}, J. Yngvason\textsuperscript{a,c}

\textsuperscript{a}Fakultät für Physik, Universität Wien, Boltzmanngasse 5, 1090 Vienna, Austria
\textsuperscript{b}Scuola Normale Superiore SNS, Piazza dei Cavalieri 7, 56126 Pisa, Italy
\textsuperscript{c}Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Vienna, Austria

May 07, 2007

Abstract

Starting from the full many body Hamiltonian we derive the leading order energy and density asymptotics for the ground state of a dilute, rotating Bose gas in an anharmonic trap in the 'Thomas Fermi' (TF) limit when the Gross-Pitaevskii coupling parameter and/or the rotation velocity tend to infinity. Although the many-body wave function is expected to have a complicated phase, the leading order contribution to the energy can be computed by minimizing a simple functional of the density alone.

1 Introduction

Rotating Bose-Einstein condensates exhibit fascinating quantum phenomena like superfluidity and quantization of vorticity and their study is currently an active area of both experimental and theoretical research. Much of the theoretical work (see, e.g., the monograph [A] where an extensive list of references can be found) is based on an effective description of the ground state in terms of the Gross-Pitaevskii (GP) equation that has recently been proved [LSe, S1] to be exact in a suitable limit. The corresponding result for the non-rotating case was obtained in [LSY1], but the rotating case was a long-standing problem whose solution required different techniques from those of [LSY1]. In fact, the rotating case differs markedly from the non-rotating one since the absolute many-body ground state is in general not the same as the bosonic ground state [S1].

The limit considered in [LSe, S1] is the GP limit of the many-body ground state which means that the particle number \( N \) tends to infinity while the GP parameter \( N a \), with \( a \) the scattering length of the interaction potential, as well as the rotational velocity, \( \Omega \), are kept fixed. In several experiments the GP parameter can be quite large, however, and also the rotational velocity can be so large that the effect of the rotation becomes comparable with that of the interactions. Some of the interesting phenomena expected under such conditions are discussed in [B, BP, ECHSC, F, FB, FZ, KTU, KB, Lu, WGBP]. To describe such cases theoretically it is natural to consider the limit when \( N a \) and/or \( \Omega \) tend to infinity. Since the error estimates in [LSe, S1] are not uniform in these parameters the results of [LSe, S1] do not apply to this situation and a complementary investigation is called for. In order to allow arbitrarily large rotational velocities we shall require that the trap potential increases more rapidly at infinity than quadratically (anharmonic traps).

In a recent paper [CRY1] the limit of GP theory (but not the many-body problem) for large \( N a \) and large \( \Omega \) was studied for a rotating 2D gas in a 'flat' trap with walls that confine the gas to a fixed bounded region. (See also [CRY2] for an extension to more general 2D traps.) It was proved that the leading order energy and density asymptotics is correctly described by an energy functional of the density without a gradient term. This functional for rapidly rotating gases was first introduced and studied in
Because of its formal analogies with the Thomas-Fermi (TF) theory for fermions it is also referred to as a TF functional although the physical situation is quite different.

In the present paper we derive the TF description from the many-body problem in 3D. This might at first sight appear to be a simple combination of the results of [LSY5] with those of [CRY1] (extended to 3D and more general traps): first take the GP limit of the many-body theory and then the TF limit of the GP theory. But such an argument is not valid because the error estimates in [LSY5] blow up if $Na$ and/or $\Omega$ tend to infinity. In fact, it is clear that it will at least be necessary to require explicitly that the gas is dilute in the sense that the average particle distance is much larger than the scattering length. (This condition is automatically fulfilled if $Na$ is kept fixed.) It is also instructive to compare with the proof of dimensional reduction of Bose gases in tightly confining traps [LSY4, SY] where one also has the option of taking a limit in two steps. Here the 1D or 2D limits of the 3D GP theory do not, in fact, cover all cases that can occur when one starts with the 3 many-body theory and takes its tight confinement limit directly. In the present situation, however, the TF limit of the GP theory exhausts the leading order asymptotics for the many-body ground state, provided only the gas remains dilute as the limit is taken. Because the many-body wave function will have a complicated phase it is not self-evident that the ground state energy can be computed accurately by a functional of the density alone. That this is indeed possible is a basic message of the present paper.

The main techniques applied in our derivation of the TF limit are on the one hand an extension of [CRY1] to 3D and anharmonic, homogeneous potentials, and on the other hand the techniques described in [LSY5] for treating the many-body problem in the non-rotating case. Additional tools are the diamagnetic inequality as well as the method used in [SI] to bound the many-body energy from above by the GP energy in the rotating case. Our results concern the ground state energy and density to leading order in small parameters (the reciprocal of the coupling constant or the rotational velocity as well as the ratio of $a$ to the mean particle spacing). In contrast to [LSY5] Bose Einstein condensation (BEC) is not proved. In fact, even in the non-rotating case a proof of BEC in the TF limit is still an open problem.

The variational wave functions we employ for deriving upper bounds to the energy reflect the expected short-scale structure due to interactions as well as the quantized vortices generated by the rotation. In 3D the exact vortex structure is presumably rather complicated, e.g., due to bending of vortex lines [AL]. Our trial functions do not model such details but they are sufficient for leading order calculations and leave room for improvements.

The paper is organized as follows. In the next section we describe the general setting and state our main results about the energy and density asymptotics. Section 3 is concerned with the TF limit of GP theory, generalizing the results of [CRY1] to 3D and homogeneous, anharmonic trapping potentials, as well as some basic properties of the TF theory needed for the proofs of the quantum mechanical (QM) limit theorems. In Section 4 we prove the upper and lower bounds on the many-body quantum-mechanical energy and obtain limit theorems for the density as corollaries. Some additional properties of the TF density are discussed in an Appendix.

## 2 General Setting and Main Results

We consider a system of $N$ spinless bosons in $\mathbb{R}^3$ with mass $m$, trapped in an external potential $V$ and interacting with a nonnegative, radially symmetric pair potential $v$ of finite range. The basic quantum mechanical Hamiltonian in a reference frame that rotates with angular velocity $\Omega$ is

$$H_N \equiv \sum_{j=1}^{N} \left( -\Delta_j - \vec{L}_j \cdot \vec{\Omega} + V(\vec{x}_j) \right) + \sum_{1 \leq i < j \leq N} v(|\vec{x}_i - \vec{x}_j|).$$

(2.1)

where units have been chosen so that $2m = \hbar = 1$. Also $\vec{x}_j \in \mathbb{R}^3$ and $\vec{L}_j \equiv -i\vec{x}_j \wedge \nabla_j$ for $j = 1, \ldots, N$ are respectively the positions and the angular momentum operators of the particles. The Hamiltonian operates on symmetric (bosonic) wave functions in $L^2(\mathbb{R}^{3N}, d\vec{x}_1 \ldots d\vec{x}_N)$.

\[\text{We use the following notation: } \vec{x} \equiv (x, y, z) \text{ will always denote a point in } \mathbb{R}^3, \text{ while } \vec{r} \text{ and } z \text{ is its projections on the } x, y \text{-plane and the } z \text{-axis respectively (cylindrical coordinates), namely } \vec{x} \equiv (\vec{r}, z). \text{ Moreover } | \vec{x} | \text{ and } r \equiv | \vec{r} | \text{ will denote the moduli of the corresponding vectors, whereas } \zeta \equiv x + iy \text{ will be the complex number associated with the } x, y \text{-coordinates of } \vec{x}.\]
We keep the external potential fixed and choose the associated length scale (the extension of the ground state of $-\Delta + V$) as a length unit. In order to be able to vary the scattering length of the interaction potential $v$ with $N$ we write $v(|\vec{x}|) = a^{-2}v_1(|\vec{x}|/a)$ where $v_1$ is a potential with scattering length 1. Then $v$ has scattering length $a$ and for fixed $v_1$ the Hamiltonian is parametrized by $N$ and $a$ besides $\Omega$. For convenience, we include a factor $4\pi$ in the definition of the GP parameter

$$g \equiv 4\pi Na.$$  

(2.2)

The ground state energy of $\mathcal{E}_{\Omega}^{GP}$, i.e., the infimum of its spectrum, will be denoted by $E_{g,\Omega}^{QM}(N)$.

Introducing the vector potential $\vec{A}_\Omega \equiv \frac{1}{4}\Omega(\vec{e}_z \wedge \vec{x})$ associated with a rotation $\vec{\Omega} = \Omega\vec{e}_z$, where $\vec{e}_z$ is the unit vector in $z$-direction, the Hamiltonian $H_N$ can be rewritten in the form

$$H_N = \sum_{j=1}^{N} \left( -i\nabla_j - \vec{A}_\Omega(\vec{x}_j) \right)^2 + V(\vec{x}_j) - \frac{1}{4}\Omega^2 r_j^2 + \sum_{1 \leq i < j \leq N} v(|\vec{x}_i - \vec{x}_j|),$$  

(2.3)

where $r \equiv |\vec{e}_z \wedge \vec{x}|$ is the distance from the axis of rotation. The splitting of the term $-\vec{L} \cdot \vec{\Omega}$ into the contribution of the vector potential and the term $-\Omega^2 r^2/4$ corresponds respectively to the Coriolis and the centrifugal force in the rotating frame. The vector potential is primarily responsible for the formation of vortices while the centrifugal potential affects the overall density profile if the rotational velocity is high enough.

In order that (2.3) is bounded from below and trapping for all $\Omega$ we require the external potential $V$ to be bounded from below and moreover that

$$V(\vec{x}) - \frac{1}{4}\Omega^2 r^2 \to \infty \quad \text{for} \quad |\vec{x}| \to \infty.$$  

(2.4)

Due to the particle repulsion and the centrifugal force the gas cloud expands as $g$ or $\Omega$ tend to infinity so essentially only the behavior of $V$ for large $|\vec{x}|$ matters. For simplicity we shall assume that $V$ is a homogeneous function of order $s > 2$, i.e. $V(\lambda \vec{x}) = \lambda^s V(\lambda \vec{x})$ for any $\lambda > 0$ and $\vec{x} \in \mathbb{R}^3$, but we need not assume that $V$ is symmetric w.r.t. rotations about the $z$-axis. In order to obtain explicit error estimates in Section 3 we shall assume that $V$ is twice continuously differentiable but Hölder continuity would in fact be sufficient for the main results.

The case of a ‘flat’ trapping potential, i.e., $V = 0$ within a bounded, open set $B$ with a smooth boundary and $\infty$ outside, can also be treated by our methods. This case corresponds formally to $s = \infty$ and the formulas for the energy and density asymptotics can be obtained as the $s \to \infty$ limit of the formulas for finite $s$. At the end of Section 4 we shall comment on the results for flat traps and the minor modifications required of the proofs.

The GP functional in the rotating frame is defined as

$$\mathcal{E}_{g,\Omega}^{GP}[\phi] = \int_{\mathbb{R}^3} \mathrm{d}\vec{x} \left\{ \left( -i\nabla - i\vec{A}_\Omega \right) \phi \right|^2 + V|\phi|^2 - \frac{1}{4}\Omega^2 r^2 |\phi|^2 + g |\phi|^4 \right\}$$  

(5.5)

on the domain

$$\mathcal{D}_{\Omega}^{GP} \equiv \left\{ \phi \in L^4(\mathbb{R}^3) : V|\phi|^2 \in L^1(\mathbb{R}^3) \quad \text{and} \quad \left( -i\nabla - i\vec{A}_\Omega \right) \phi \in L^2(\mathbb{R}^3) \right\}.$$  

(5.6)

The corresponding energy is

$$E_{g,\Omega}^{GP} \equiv \inf_{\phi \in \mathcal{D}_{\Omega}^{GP}, \|\phi\|_2 = 1} \mathcal{E}_{g,\Omega}^{GP}[\phi].$$  

(2.7)

The infimum is, in fact, a minimum and we denote any normalized minimizer by $\phi_{g,\Omega}^{GP}$. The corresponding density is $\rho_{g,\Omega}^{GP} \equiv |\phi_{g,\Omega}^{GP}|^2$. The minimizer may not be unique because vortices can break rotational symmetry, but any minimizer satisfies the variational (GP) equation

$$\left( -\left( -i\nabla - i\vec{A}_\Omega \right)^2 + V - \frac{1}{4}\Omega^2 r^2 + 2g |\phi_{g,\Omega}^{GP}|^2 \right) \phi_{g,\Omega}^{GP} = \mu_{g,\Omega}^{GP} \phi_{g,\Omega}^{GP}$$  

(5.8)

$^2$Alternatively, one could keep a fixed and vary the external potential and hence the length scale. What matters is only the ratio of $a$ to this scale.

$^3$Asymptotic homogeneity in the sense of [LSY2] would also suffice.
where $\mu_{g,\Omega}^{GP}$ is the GP chemical potential. Multiplying (2.8) with $\phi_{g,\Omega}^{TF}$ and integrating gives

$$\mu_{g,\Omega}^{GP} = E_{g,\Omega}^{GP} + 2g\|\rho_{g,\Omega}^{GP}\|_2. \quad (2.9)$$

In [LSe, S1] it is proved that $E_{g,\Omega}^{QM}(N)/N E_{g,\Omega}^{GP} \to 1$ as $N \to \infty$ if $g$ and $\Omega$ are fixed. In the present paper we are concerned with the situation where $g$ and/or $\Omega$ tend to infinity together with $N$. As we shall show, the first term in (2.5) is negligible in this limit and the ground state energy and density can be described in terms of the TF functional

$$E_{g,\Omega}^{TF}[\rho] = \int_{\mathbb{R}^3} d\vec{x} \left\{ V\rho - \frac{1}{4}\Omega^2 r^2 \rho + g\rho^2 \right\} \quad (2.10)$$

defined on the domain

$$D_{TF} \equiv \{ \rho \in L^2(\mathbb{R}^3) : \rho \geq 0, V\rho \in L^1(\mathbb{R}^3) \} \quad (2.11)$$

with the energy

$$E_{g,\Omega}^{TF} \equiv \inf_{\rho \in D^{TF}_{||\rho||_1=1}} E_{g,\Omega}^{TF}[\rho]. \quad (2.12)$$

The minimization problem (2.12) has a unique solution given by

$$\rho_{g,\Omega}^{TF}(\vec{x}) = \frac{1}{2g} \left[ \mu_{g,\Omega}^{TF} + \frac{1}{4}\Omega^2 r^2 - V(\vec{x}) \right]_+ , \quad (2.13)$$

where $[\cdot]_+$ denotes the positive part and $\mu_{g,\Omega}^{TF}$ is the TF chemical potential determined by the normalization $\|\rho_{g,\Omega}^{TF}\|_1 = 1$. Multiplying (2.13) by $\rho_{g,\Omega}^{TF}$ and integrating gives

$$\mu_{g,\Omega}^{TF} = E_{g,\Omega}^{TF} + 2g\|\rho_{g,\Omega}^{TF}\|_2^2. \quad (2.14)$$

By simple rescaling (explained at the beginning of Section 3) we obtain the relations

$$g^{-s/(s+3)} E_{g,\Omega}^{TF} = E_{1,\omega}^{TF} \quad \text{and} \quad g^{-s/(s+3)} \mu_{g,\Omega}^{TF} = \mu_{1,\omega}^{TF} \quad (2.15)$$

with

$$\omega \equiv g^{-(s-2)/(2s+6)} \Omega \quad (2.16)$$

and likewise

$$g^{3/(s+3)} \rho_{g,\Omega}^{TF} \left( g^{1/(s+3)} \vec{x} \right) = \rho_{1,\omega}^{TF}(\vec{x}). \quad (2.17)$$

The case $\omega = 0$ corresponds to the standard TF functional without rotation whose relation to the many-body problem was already discussed in [LSYS]. Note also that $E_{1,\omega}^{TF}$ is a decreasing function of $\omega \geq 0$ with range $(-\infty, E_{1,0}^{TF}]$ with $E_{1,0}^{TF} > 0$. In particular, there is an $\omega > 0$ such that $E_{1,\omega}^{TF} = 0$.

In the case that $\omega$ tends to infinity the rotational term completely dominates the interaction term. In this case it is appropriate to scale lengths with $\Omega^{\gamma/(s-2)}$ rather than $g^{\gamma/(s+3)}$ (cf. Section 3), obtaining

$$\Omega^{-2s/(s-2)} E_{g,\Omega}^{TF} = E_{\gamma,1}^{TF} \quad \text{and} \quad \Omega^{-2s/(s-2)} \mu_{g,\Omega}^{TF} = \mu_{\gamma,1}^{TF} \quad (2.18)$$

with

$$\gamma \equiv \Omega^{-(s+3)/(s-2)} g = \omega^{-2(s+3)/(s-2)} \quad (2.19)$$

and

$$\Omega^{\gamma/(s-2)} \rho_{g,\Omega}^{TF} \left( \Omega^{2/(s-2)} \vec{x} \right) = \rho_{\gamma,1}^{TF}(\vec{x}). \quad (2.20)$$

Moreover, as $\omega \to \infty$, i.e., $\gamma \to 0$, we have

$$\lim_{\gamma \to 0} E_{\gamma,1}^{TF} = E_{0,1}^{TF} = \inf_{\vec{x} \in \mathbb{R}^3} \{ V(\vec{x}) - \frac{1}{4}r^2 \} < 0 \quad (2.21)$$

while $\rho_{\gamma,1}^{TF}$ converges to a measure supported on the set $\mathcal{M}$ of minima of the function $W(\vec{x}) \equiv V(\vec{x}) - \frac{1}{4}r^2$. This together with the other facts mentioned about the TF theory is discussed further in Section 3.2.

The scaling properties of the TF theory already suggest that one should distinguish between the following three cases when the $N \to \infty$ limit of the many-body ground state with $g$ and/or $\Omega$ also tending to infinity is considered:
• **Slow or moderate rotation**, $\omega \ll 1$: The effect of the rotation is negligible to leading order.

• **Rapid rotation**, $\omega \sim 1$: Rotational effects are comparable to those of the interactions.

• **Ultrarapid rotation**, $\omega \gg 1$: Rotational effects dominate.

Moreover, a description in terms of the simple density functional can only be expected in a dilute limit. A convenient measure for diluteness turns out to be smallness of the parameter $a^3 N \| \rho_{g, \Omega}^{\text{TF}} \|_\infty \sim N^{-2} g^3 \| \rho_{g, \Omega}^{\text{TF}} \|_\infty$. (2.22)

Our main result about the energy asymptotics is the following

**Theorem 2.1 (QM energy asymptotics)**

Let $V$ be a homogeneous potential of order $s > 2$, define $g = 4 \pi a N$ and $\omega = g^{-(s-2)/(2s+6)} \Omega$, and assume that $N^{-2} g^3 \| \rho_{g, \Omega}^{\text{TF}} \|_\infty \to 0$ as $N \to \infty$.

(i) If $g \to \infty$ and $\omega \to 0$ as $N \to \infty$, then \( \lim_{N \to \infty} \left\{ g^{-s/(s+3)} N^{-1} E_{g, \Omega}^{\text{QM}} (N) \right\} = E_{1,0}^{\text{TF}} \).

(ii) If $g \to \infty$ and $\omega > 0$ is fixed as $N \to \infty$, then \( \lim_{N \to \infty} \left\{ g^{-s/(s+3)} N^{-1} E_{g, \Omega}^{\text{QM}} (N) \right\} = E_{1, \omega}^{\text{TF}} \).

(iii) If $\Omega \to \infty$ and $\omega \to \infty$ as $N \to \infty$, then \( \lim_{N \to \infty} \left\{ \Omega^{-2s/(s-2)} N^{-1} E_{g, \Omega}^{\text{QM}} (N) \right\} = E_{0,1}^{\text{TF}} \).

Although stated separately, cases (i) and (ii) can, in fact, be treated together because the convergence of the scaled energy is uniform in $\omega$ as long as $\omega$ stays bounded.

Besides the energy asymptotics we shall also consider the convergence of the quantum mechanical particle density of ground states or approximate ground states. We say that a sequence of bosonic, normalized wave functions $\Psi_N \in L^2(\mathbb{R}^{3N})$ (depending also on the parameters $g$ and $\Omega$) is an approximate ground state if

\[
\langle \Psi_N , H_N \Psi_N \rangle / E_{g, \Omega}^{\text{QM}} (N) \to 1 .
\]  

(2.23)

(Recall that $E_{g, \Omega}^{\text{QM}} (N)$ is, by definition, the infimum of the spectrum of (2.1).) The particle density, normalized so that its integral is 1, is defined by

\[
\rho_{N,g, \Omega}^{\text{QM}} (\vec{x}) = \int_{\mathbb{R}^{3(N-1)}} |\Psi_N (\vec{x}, \vec{x}_2, ..., \vec{x}_N)|^2 d\vec{x}_2 ... d\vec{x}_N .
\]  

(2.24)

To ensure convergence of $\rho_{N,g, \Omega}^{\text{QM}} (\vec{x})$ we have to rescale it in accord with (2.17). For bounded $\omega$ convergence of the density follows from the convergence of the energy using standard arguments [G, LSi].

**Theorem 2.2 (QM density asymptotics for $\omega < \infty$)**

Under the conditions of Theorem (2.1) (i) or (ii) we have

\[
g^{3/(s+3)} \rho_{N,g, \Omega}^{\text{QM}} (g^{1/(s+3)} \vec{x}) \to \rho_{1, \omega}^{\text{TF}} (\vec{x})
\]  

(2.25)

in weak $L^1$ sense.

The case of ultrarapid rotations, $\omega \to \infty$, is a little more delicate. Recall that $\mathcal{M} \subset \mathbb{R}^3$ was defined as the set where the function $V(\vec{x}) - \frac{1}{2} r^2$ attains its minimum. In Lemma 3.2 it is shown that this set is a subset of a cylinder and that the scaled TF density (2.20) is eventually concentrated on $\mathcal{M}$. In physical terms, the ultrastrong centrifugal forces outweigh the repulsion between particles and constrain them to the boundary of the available region (in the scaled variables). We show that the same holds for the scaled QM density.

\[\text{4This regime could be subdivided further into 'slow' rotations, where vortices do not yet form, and 'moderate' rotations where vortices are present. This finer division goes, however, beyond the leading order considerations of the present paper.}\]
Theorem 2.3 (QM density asymptotics for \( \omega \to \infty \))

Under the conditions of Theorem 2.1 (iii) the scaled particle density

\[
\Omega^{6/(s-2)} \rho_n^{QM}(\Omega^{2/(s-2)} x)
\]

becomes concentrated on the set \( M \) as \( N \to \infty \) in the sense that the integral of \( (2.26) \) over any measurable set that has a strictly positive distance from \( M \) tends to zero.

The proofs of Theorems 2.1, 2.3 will be given in several steps. In the next section, we analyze the asymptotic properties of the GP and TF theories when \( g \) and/or \( \Omega \) tend to infinity. Section 4 contains the upper and lower bounds to the quantum mechanical energy that complete the proof of Theorem 2.1. Theorem 2.2 is derived from the convergence of the energy by the arguments of \( \{1, L S\} \) and Theorem 2.3 is proved by showing that any mass outside the set \( M \) would be in conflict with Theorem 2.1 (iii).

3 From GP to TF

In this section we consider the TF limit of GP theory. Since for \( s < \infty \) the GP and TF minimizers spread out over an increasingly large region as \( g \) and/or \( \Omega \) tend to infinity we write \( \bar{x} = \lambda \bar{z} \) and \( \phi'(\bar{z}) = \lambda^{-3/2} \phi'(\bar{z}') \) with a suitable length scale \( \lambda \) depending on \( g \) or \( \Omega \) so that the relevant \( \bar{z}' \) remain essentially bounded. We have \( \| \phi' \|_2 = \| \phi \|_2 = 1 \), and since \( V(\bar{x}) = \lambda \pi V(\bar{x}) \) by assumption, the GP functional can be written

\[
E_{\omega}^{GP} [\phi] = \lambda^{-2} \int_{\mathbb{R}^3} d\bar{z}' \left\{ \left[ \nabla^2 - i A_{\lambda^2 \Omega}(\bar{z}') \right] \phi' \right\}^2 + \lambda^{s+2} V(\bar{x}) |\phi'|^2 - \frac{1}{4}(\lambda^2 \Omega)^2 r^2 |\phi'|^2 + g \lambda^{-1} |\phi'|^4 \right\}.
\]

(3.1)

We now distinguish two cases. When the rotational contribution to the energy is smaller than or at most comparable to the interaction energy we equate \( \lambda^{s+2} \) with \( g \lambda^{-1} \), i.e., choose

\[
\lambda = g^{1/(s+3)}.
\]

(3.2)

For convenience and comparison with \( \{C R Y 1, C R Y 2\} \) and \( \{A\} \) we define a small parameter \( \varepsilon \) by

\[
1/\varepsilon^2 = \lambda^{s+2} = \lambda^{-1} g = g^{(s+2)/(s+3)}.
\]

(3.3)

The parameter \( \omega = g^{-(s-2)/(2s+6) \Omega} \), that measures the relative strength of the rotation with respect to the interactions can then be written

\[
\omega = \varepsilon^{(s-2)/(s+3) \Omega}.
\]

(3.4)

We now define a rescaled GP functional \( \bar{E}_{\varepsilon, \omega}^{GP} \) by writing \( E_{\omega}^{GP} [\phi] = \lambda^{-2} \bar{E}_{\varepsilon, \omega}^{GP} [\phi'] \), or explicitly, dropping the primes,

\[
\bar{E}_{\varepsilon, \omega}^{GP} [\phi] = \int_{\mathbb{R}^3} d\bar{z}' \left\{ \left[ \nabla^2 - i A_{\varepsilon \omega}(\bar{z}') \right] \phi \right\}^2 + \frac{1}{\varepsilon^2} \left( V |\phi|^2 - \frac{\omega^2 r^2}{4} |\phi|^2 + |\phi|^4 \right) \right\}.
\]

(3.5)

In the next subsection we study the \( \varepsilon \to 0 \) limit of this functional with \( \omega < \infty \) fixed or tending to zero.

When the rotation dominates the interaction we take \( \lambda^{s+2} \) to be equal to \( (\lambda^2 \Omega)^2 \) in \( \{B 1\} \), i.e., we take

\[
\lambda = \Omega^{2/(s-2)}.
\]

(3.6)

Proceeding as before we now write \( E_{\omega}^{GP} [\phi] = \lambda^{-2} \bar{E}_{\Omega, \omega}^{GP} [\phi'] \) with

\[
E_{\Omega, \omega}^{GP} [\phi] = \int_{\mathbb{R}^3} d\bar{z}' \left\{ \left[ \nabla^2 - i A_{\Omega}(\bar{z}') \right] \phi \right\}^2 + \Omega' \left[ V - \frac{1}{4} r^2 \right] |\phi|^2 + \gamma |\phi|^4 \right\}
\]

(3.7)

where

\[
\Omega' \equiv \Omega^{(s+2)/(s-2)} \quad \text{and} \quad \gamma = \omega^{-2(s+3)/(s-2)}.
\]

(3.8)

The limit \( \Omega \to \infty \), \( \omega \to \infty \) (i.e., \( \Omega' \to \infty \), \( \gamma \to 0 \)) will be considered in Subsection 3.2.
3.1 The Regime $\omega < \infty$

For $\omega < \infty$ fixed or tending to zero, we study the rescaled GP functional introduced in (3.3) and define $\tilde{E}_{\varepsilon,\omega}^{\text{GP}}$ by

$$\tilde{E}_{\varepsilon,\omega}^{\text{GP}} \equiv \inf_{\phi \in \mathcal{D}_{\text{GP}}, \|\phi\|_1 = 1} \tilde{E}_{\varepsilon,\omega}^{\text{GP}}[\phi] = g^{-2/(s+3)} E_{g,\Omega}^{\text{GP}}. \quad (3.9)$$

The asymptotic behavior of $g^{-2/(s+3)} E_{g,\Omega}^{\text{GP}}$ as $g \to \infty$ is then given by the limit of $\tilde{E}_{\varepsilon,\omega}^{\text{GP}}$ as $\varepsilon \to 0$, see (3.3). We state this result about $E_{g,\Omega}^{\text{GP}}$ in the following theorem.

**Theorem 3.1 (GP energy asymptotics for $\omega < \infty$)**

Under the conditions of Theorem 2.1 (i) or (ii) one has, as $g \to \infty$,

$$g^{-s/(s+3)} E_{g,\Omega}^{\text{GP}} = E_{1,\omega}^{\text{TF}} + O \left( g^{-(s+2)/(2s+6)} \log g \right). \quad (3.10)$$

**Proof:** The proof is obtained by a comparison between suitable lower and upper bounds for $\tilde{E}_{\varepsilon,\omega}^{\text{GP}}$, following closely the proof of Theorem 2.1 in [CRY1].

The two regimes of slow (i) and rapid (ii) rotations can be treated together. Actually, the lower and upper bounds in the case (ii) are sufficient to get the result in the case (i) because the estimates are uniform on bounded intervals of $\omega$ and the TF ground state energy is a continuous function of $\omega$. To simplify our notation, we denote by $C_{\omega}$ any constant independent of $\varepsilon$. In particular, $C_{\omega}$ needs not be the same from one equation to another, but $C_{\omega}$ is always uniformly bounded for $\omega$ bounded.

By simply neglecting the positive contribution of the kinetic energy in (3.3) one obtains the lower bound

$$\varepsilon^2 \tilde{E}_{\varepsilon,\omega}^{\text{GP}}[\phi] \geq E_{1,\omega}^{\text{TF}}[|\phi|^2] \geq E_{1,\omega}^{\text{TF}}. \quad (3.11)$$

For an upper bound we test the functional (3.5) with a trial function of the form

$$\hat{\phi}(\vec{x}) = c_{\varepsilon} \sqrt{\hat{\rho}_\varepsilon(\vec{r})} \chi_{\varepsilon}(\vec{r}) g_{\varepsilon}(\vec{r}), \quad (3.12)$$

where $g_{\varepsilon}$ is a phase factor, $\chi_{\varepsilon}(\vec{r})$ a function that vanishes at the singularities of $g_{\varepsilon}$ and $\hat{\rho}_\varepsilon$ a suitable regularization of $\rho_{1,\omega}^{\text{TF}}$. Note that both $g_{\varepsilon}$ and $\chi_{\varepsilon}$ depend only on the 2d coordinate $\vec{r}$. For simplicity of notation we have suppressed the dependence on $\omega$ that is regarded as fixed.

The regularization of $\rho_{1,\omega}^{\text{TF}}$ is analogous to the one used in [LSY2], Lemma 2.3 and is explicitly given by $\hat{\rho}_\varepsilon \equiv j_{\varepsilon} \ast \rho_{1,\omega}^{\text{TF}}$, with

$$j_{\varepsilon}(\vec{x}) = \frac{1}{4\pi \varepsilon^3} \exp \left\{ -\frac{|\vec{x}|}{\varepsilon} \right\}. \quad (3.13)$$

Since $\|j_{\varepsilon}\|_1 = 1$, $\sqrt{\hat{\rho}_\varepsilon}$ is $L^2$–normalized. It is also clear that $\hat{\rho}_\varepsilon$ converges uniformly to $\rho_{1,\omega}^{\text{TF}}$ as $\varepsilon \to 0$ and it is uniformly bounded in $\varepsilon$, i.e., there exists a constant $C_{\omega}$ such that $\hat{\rho}_\varepsilon \leq C_{\omega}$. Furthermore, although $\hat{\rho}_\varepsilon$ is not compactly supported, it is exponentially small in $\varepsilon$ for $\vec{x}$ sufficiently far from the support of $\rho_{1,\omega}^{\text{TF}}$. More precisely, denoting

$$R_{\omega} \equiv \sup\{|\vec{x}| : \rho_{1,\omega}^{\text{TF}}(\vec{x}) > 0\} < \infty, \quad (3.14)$$

one has for any $\vec{x} \in \mathbb{R}^3$, $|\vec{x}| > R_{\omega}$,

$$\hat{\rho}_\varepsilon(\vec{x}) = \frac{1}{4\pi \varepsilon^3} \int_{\text{supp}(\rho_{1,\omega}^{\text{TF}})} d\vec{x}' \exp \left\{ -\frac{|\vec{x} - \vec{x}'|}{\varepsilon} \right\} \rho_{1,\omega}^{\text{TF}}(\vec{x}') \leq \frac{1}{4\pi \varepsilon^3} \exp \left\{ -\frac{|\vec{x}| - R_{\omega}}{\varepsilon} \right\}. \quad (3.15)$$

We also observe that the gradient of $\hat{\rho}_\varepsilon$ can be bounded in two ways. By using the fact that $|\nabla j_{\varepsilon}| = \varepsilon^{-1} j_{\varepsilon}$, one can easily prove that

$$|\nabla \hat{\rho}_\varepsilon| \leq \varepsilon^{-1} |\hat{\rho}_\varepsilon| \quad (3.16)$$

whereas, by exploiting the regularity of $\rho_{1,\omega}^{\text{TF}}$, i.e., $\|\nabla \rho_{1,\omega}^{\text{TF}}\|_1 \leq C_{\omega}$, one has

$$\int_{\mathbb{R}^3} d\vec{x} \ |\nabla \hat{\rho}_\varepsilon(\vec{x})| \leq \int_{\mathbb{R}^3} d\vec{x} \int_{\mathbb{R}^3} d\vec{x}' \ |\nabla \rho_{1,\omega}^{\text{TF}}(\vec{x} - \vec{x}')| j_{\varepsilon}(\vec{x}') = \|\nabla \rho_{1,\omega}^{\text{TF}}\|_1 \leq C_{\omega}. \quad (3.17)$$

$^5$By $\rho_{1,\omega}^{\text{TF}}$ the support of $\rho_{1,\omega}^{\text{TF}}$ is a compact set, uniformly bounded in $\omega$, if $\omega$ remains bounded.
Writing points \( \vec{r} = (x, y) \in \mathbb{R}^2 \) as complex numbers \( \zeta \equiv x + iy \), the phase \( \varphi \) is defined by

\[
g_\varepsilon(\zeta) = \prod_{\zeta_j \in \mathcal{L}} \frac{\zeta - \zeta_j}{|\zeta - \zeta_j|},
\]

where \( \mathcal{L} \) is a square lattice of spacing \( \ell \) defined in the following way:

\[
\mathcal{L} \equiv \left\{ \vec{r}_j = (m\ell, n\ell), \ m, n \in \mathbb{Z} \left| \ r < 2R_\omega - \ell \right. \right\},
\]

with \( R_\omega \) defined by (3.14). We also assume that the spacing is of order \( \sqrt{\varepsilon} \), i.e., \( \ell = \delta \sqrt{\varepsilon} \), for some \( \delta > 0 \) independent of \( \varepsilon \). Note that the phase \( g_\varepsilon \) carries lines of vortices of degree 1 passing through the lattice points and it coincides with the trial function chosen in [CRY1]. Moreover the number, \( N_\varepsilon \), of lines of vortices included in the support of \( \tilde{\phi} \) is bounded by \( C_\omega/\varepsilon \), because this support is contained in the finite region \( x \leq R_\omega \).

The function \( \chi_\varepsilon \) is given by

\[
\chi_\varepsilon(\vec{r}) \equiv \begin{cases} 1 & \text{if } |\vec{r} - \vec{r}_j| \geq \varepsilon^\eta \text{ for all } \vec{r}_j \in \mathcal{L}, \\ \frac{|\vec{r} - \vec{r}_j|}{\varepsilon^\eta} & \text{if } |\vec{r} - \vec{r}_j| \leq \varepsilon^\eta, \end{cases}
\]

for some \( \eta > 5/2 \).

Finally the constant \( c_\varepsilon \) is fixed by the normalization condition, \( \| \tilde{\phi} \|_2 = 1 \), and, since \( \chi_\varepsilon \leq 1 \) and \( N_\varepsilon \leq C/\varepsilon \),

\[
1 \leq c_\varepsilon^2 \leq 1 + O(\varepsilon^{2\eta-1}) = 1 + o(\varepsilon^4).
\]

The rescaled functional \( \tilde{\mathcal{E}}_{GP} \) evaluated on the trial function (3.12) is given by

\[
\tilde{\mathcal{E}}_{GP}[\tilde{\phi}] = c_\varepsilon^2 \int_{\mathbb{R}^3} d\vec{x} \left[ \nabla \left( \chi_\varepsilon \sqrt{\tilde{\phi}_\varepsilon} \right) \right]^2 + c_\varepsilon^2 \int_{\mathbb{R}^3} d\vec{x} \tilde{\phi}_\varepsilon \chi_\varepsilon^2 \left[ \nabla - i\hat{A}_\omega/\varepsilon \right] \varphi_\varepsilon + \frac{\mathcal{E}_{TF}[\tilde{\phi}]^2}{\varepsilon^2}.
\]

We estimate the three energy contributions separately. Using that \((a + b)^2 \leq 2a^2 + 2b^2\) the first term can be bounded by \( \| \tilde{\phi} \|_2 \leq C_\omega/\varepsilon \),

\[
c_\varepsilon^2 \int_{\mathbb{R}^3} d\vec{x} \left| \nabla \left( \chi_\varepsilon \sqrt{\tilde{\phi}_\varepsilon} \right) \right|^2 \leq 2c_\varepsilon^2 \int_{\mathbb{R}^3} d\vec{x} \left| \nabla \sqrt{\tilde{\phi}_\varepsilon} \right|^2 + c_\varepsilon^2 C_\omega \int_{B_{R_\omega}} d\vec{r} \left| \nabla \chi_\varepsilon \right|^2,
\]

where we have used the trivial bound

\[
\int_{\mathbb{R}} d\zeta \tilde{\phi}_\varepsilon(\vec{x}) \leq C_\omega.
\]

Denoting by \( B_\ell \) a two-dimensional disc of radius \( \varepsilon^\eta \) centered at \( \vec{r}_j \in \mathcal{L} \cap B_{R_\omega} \), one has

\[
\int_{B_{R_\omega}} d\vec{r} \left| \nabla \chi_\varepsilon \right|^2 \leq \frac{\left| \bigcup_{j \in \mathcal{L}} B_\ell \right|}{\varepsilon^{2\eta}} \leq N_\varepsilon \leq \frac{C_\omega}{\varepsilon},
\]

while, by using both (3.16) and (3.17), we get

\[
\int_{\mathbb{R}^3} d\vec{x} \left| \nabla \sqrt{\tilde{\phi}_\varepsilon} \right|^2 = \int_{\mathbb{R}^3} d\vec{x} \frac{\left| \nabla \tilde{\phi}_\varepsilon \right|^2}{4\tilde{\phi}_\varepsilon} \leq \frac{1}{4\varepsilon} \int_{\mathbb{R}^3} d\vec{x} \left| \nabla \tilde{\phi}_\varepsilon \right| \leq \frac{C_\omega}{\varepsilon}.
\]

Hence (3.21) implies the bound

\[
c_\varepsilon^2 \int_{\mathbb{R}^3} d\vec{x} \left| \nabla \left( \chi_\varepsilon \sqrt{\tilde{\phi}_\varepsilon} \right) \right|^2 \leq \frac{C_\omega}{\varepsilon}.
\]

---

6This requirement on \( \eta \) is needed in order to apply Theorem 3.1 in [CRY1] (see below in the proof).

7In this proof \( B_R, R > 0 \), will always denote a two-dimensional disc of radius \( R \), centered at the origin.
In order to estimate the second term in (3.22), we first need to restrict the integration to a suitable two-dimensional compact set, by exploiting the exponential smallness of $\tilde{g}_\varepsilon$ given by (3.16):

$$
\int_{\mathbb{R}^3} d\vec{x} \tilde{g}_\varepsilon \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 = \int_{|\vec{x}| \leq 2R_\omega} d\vec{x} \tilde{g}_\varepsilon \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 + \int_{|\vec{x}| \geq 2R_\omega} d\vec{x} \tilde{g}_\varepsilon \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 \leq 
$$

$$
\leq C_\omega \int_{B_{2R_\omega}} d\vec{r} \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 + C_\omega \int_{x \geq 2R_\omega} d\vec{x} \exp \left\{ - \frac{|\vec{x}| - R_\omega}{\varepsilon} \right\} \leq C_\omega \int_{\Lambda} d\vec{r} \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 + C_\omega \int_{\cup_{j \in C} B_{\varepsilon}^j} d\vec{r} \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 + C_\omega \exp \left\{ - \frac{R_\omega}{\varepsilon} \right\},
$$

where

$$
\Lambda \equiv B_{2R_\omega} \setminus \bigcup_{j \in C} B_{\varepsilon}^j,
$$

and we have used the pointwise estimate (see also (3.10) in [CRY1]),

$$
\left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right| \leq |\nabla g_\varepsilon| + \frac{C_\omega}{\varepsilon} \leq \frac{C_\omega}{\varepsilon^{3/2}}.
$$

which holds for any $\vec{x} \in \mathbb{R}^3$ such that $|\vec{x}| \geq 2R_\omega$.

The second term of the right hand side of the above expression can be bounded in the following way (see also the proof of Theorem 2.1 in [CRY1]):

$$
\int_{\cup_{j \in C} B_{\varepsilon}^j} d\vec{r} \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 \leq 2 \int_{\cup_{j \in C} B_{\varepsilon}^j} d\vec{r} \chi_\varepsilon |\nabla g_\varepsilon|^2 + 2 \int_{\cup_{j \in C} B_{\varepsilon}^j} d\vec{r} \left| \vec{A}_{\omega/\varepsilon} \right|^2 \leq \frac{C_\omega}{\varepsilon} + \frac{C_\omega}{\varepsilon^{1-2\eta}} + \frac{C_\omega}{\varepsilon^{3-8\eta}} \leq \frac{C_\omega}{\varepsilon}.
$$

On the other hand we can apply Theorem 3.1 in [CRY1] to get

$$
\int_{\Lambda} d\vec{r} \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 \leq \pi \frac{R_\omega^2}{\varepsilon^2} \left( \frac{\omega}{2} - \frac{\varepsilon}{8} \right)^2 + \frac{C_\omega |\log \varepsilon|}{\varepsilon}
$$

and, by choosing $\delta = \sqrt{2\pi/\omega},

$$
c_\varepsilon^2 \int_{\mathbb{R}^3} d\vec{x} \tilde{g}_\varepsilon \chi_\varepsilon \left| \left( \nabla - i \vec{A}_{\omega/\varepsilon} \right) g_\varepsilon \right|^2 \leq \frac{C_\omega |\log \varepsilon|}{\varepsilon} + \frac{C_\omega}{\varepsilon} + \frac{C_\omega}{\varepsilon^9} \exp \left\{ - \frac{R_\omega}{\varepsilon} \right\} \leq \frac{C_\omega |\log \varepsilon|}{\varepsilon}.
$$

(3.26)

It remains then to estimate the last term in (3.22), namely the TF energy of $|\tilde{\phi}|^2$. We have

$$
\mathcal{E}_{1,\omega}^{TF} \left[ |\tilde{\phi}|^2 \right] \leq \mathcal{E}_{1,\omega}^{TF} \left[ \tilde{g}_\varepsilon \right] + o(\varepsilon)
$$

and, defining $W_\omega(\vec{x}) \equiv V(\vec{x}) - \omega^2 r^2/4$,

$$
\mathcal{E}_{1,\omega}^{TF} \left[ \tilde{g}_\varepsilon \right] - \mathcal{E}_{1,\omega}^{TF} \left[ \rho_{1,\omega}^{TF} \right] = \int_{\mathbb{R}^3} d\vec{x} W_\omega \left( j_\varepsilon \star \rho_{1,\omega}^{TF} - \rho_{1,\omega}^{TF} \right) = \int_{\text{supp}(\rho_{1,\omega}^{TF})} d\vec{x} \rho_{1,\omega}^{TF} \left( j_\varepsilon \star W_\omega - W_\omega \right).
$$

Differentiability of $V$ implies

$$
|\left( j_\varepsilon \star W_\omega \right)(\vec{x}) - W_\omega(\vec{x})| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} d\vec{x} \left| W_\omega(\vec{x} - \varepsilon \vec{y}) - W_\omega(\vec{x}) \right| e^{-|\vec{y}|} \leq C_\omega \varepsilon \int_{\mathbb{R}^3} d\vec{x} r^2 e^{-|\vec{x}|} \leq C_\omega \varepsilon,
$$

so that

$$
\mathcal{E}_{1,\omega}^{TF} \left[ |\tilde{\phi}|^2 \right] \leq \mathcal{E}_{1,\omega}^{TF} \left[ \rho_{1,\omega}^{TF} \right] + C_\omega \varepsilon = E_{1,\omega}^{TF} + C_\omega \varepsilon.
$$

(3.27)

8By scaling the expression can be reduced to an integral over a set contained in a ball of radius 1. The only change with respect to Theorem 3.1 in [CRY1] is then the multiplying factor $R_\omega^2$. 
Putting all the three estimates (3.25), (3.26) and (3.27) together we obtain the upper bound

$$
\tilde E_{\varepsilon, \omega}^{\text{GP}} \leq \frac{E_{1, \omega}^{\text{TF}}}{\varepsilon^2} + \frac{C_\omega \log \varepsilon}{\varepsilon}.
$$

(3.28)

The upper bound and the lower bound together give the desired result in the rapid rotation regime (ii). Indeed, writing \( \varepsilon^2 \) as \( g^{-(s-2)/(s+3)} \) and using (3.3), we get, as \( \omega \to \infty \),

$$
E_{1, \omega}^{\text{TF}} \leq g^{-s/(s+3)} E_{2, \Omega}^{\text{GP}} \leq E_{1, \omega}^{\text{TF}} + C_\omega g^{-(s+2)/(2s+6)} \log g.
$$

The same estimates prove the convergence in the slow rotation regime (i), by uniformity of the bounds and continuity of \( E_{1, \omega}^{\text{TF}} \).

\( \square \)

As in the two-dimensional case (see CRY1, CRY2), a simple corollary of this result is the \( L^1 \)-convergence of the (scaled) GP density \( |\phi_{\varepsilon, \Omega}^{\text{GP}}|^2 \) to the TF minimizer \( \rho_{1, \omega}^{\text{TF}} \). In particular, in the slow rotation regime (i) this gives the convergence of the GP density to \( \rho_{1, 0}^{\text{TF}} \). In that case the proof of the upper bound could have been simplified a lot by taking as trial function a suitable regularization of the (real) TF minimizer without rotation (\( \omega = 0 \)).

Also in the rapid rotation regime (ii) other trial functions could be chosen with the same leading order contribution to the energy as (3.12). One possibility is to cover the support of \( \tilde A_{\omega/\varepsilon} \) to the kinetic energy is gauged away to leading order. The reason we have chosen a trial function of the form (3.12) is that the error term in (3.10) has the expected dependence on the parameters of a next to leading order term, albeit with an unspecified constant in front. In fact, based on considerations of special cases (see, e.g., A) the true GP minimizer is expected to contain, like (3.12), a large number \( N_\varepsilon \sim \omega/\varepsilon \) of vortex lines, each giving a kinetic contribution of order \( |\log \varepsilon| \) beyond the leading order term. We note that the optimal vortex lattice is expected to be triangular rather than rectangular and bending of lattice lines A1 will also occur but such details only affect the constant factor and higher order contributions in the error term in (3.10).

### 3.2 The Regime \( \omega \to \infty \)

We now consider the case of ultrarapid rotations. In the proofs of the quantum mechanical limit theorems in Section 4 we shall not make direct use of the GP minimizers to construct trial functions for an upper bound to the energy, but the asymptotics of the TF energy and minimizer as \( \omega \to \infty \) will be important. In this subsection we first derive these properties. The limit theorem for the GP energy for ultrarapid rotations can be proved by a simple extension of the corresponding TF result and is included for completeness.

We start with a lemma on the structure of the set \( \mathcal{M} \) of minimizing points for the sum of the external potential \( V \) and the centrifugal potential \( -r^2/4 \). As always, \( V \) is assumed to be homogeneous of order \( s > 2 \).

**Lemma 3.2 (\( \mathcal{M} \) is a subset of a cylinder)** The set \( \mathcal{M} \) of minimizing points for \( W(\vec{x}) = V(\vec{x}) - r^2/4 \) is a compact subset of a cylindrical surface with a fixed radial coordinate

$$
 r_0 = 2 \frac{s m}{s - 2}.
$$

(3.29)

where \( m \equiv -\min W \geq 0 \). By scaling it follows that all points of the set \( \mathcal{M}_\Omega \) of minimizing points for \( W_\Omega(\vec{x}) \equiv V(\vec{x}) - \Omega^2 r^2/4 \) have the same radial coordinate \( r_\Omega = \Omega^{2/(s-2)} r_0 \).

**Proof:** That \( \mathcal{M} \) is compact follows from continuity of \( V \) and the assumption that \( W \) tends to \( \infty \) as \( |\vec{x}| \to \infty \). On \( \mathcal{M} \) we have \( \nabla W = 0 \) and \( W = -m \), and thus \( \partial_r V = r/2 \), \( \partial_\theta V = 0 \) and \( V = r^2/4 - m \). On the other hand, since \( V \) is homogeneous of order \( s \), Euler’s relation gives \( r \partial_r V + z \partial_z V = sV \) and hence (3.29).
If \( V \) is rotationally symmetric, i.e., \( V(\vec{x}) = V(r, z) \), and strictly monotonously increasing in \( |z| \) (examples: \( V(\vec{x}) = a|\vec{x}|^s \), or \( V(\vec{x}) = ar^s + b|z|^s \)), then \( \mathcal{M} \) is clearly a circle in the \( z = 0 \) plane. It is, however, also possible that \( \mathcal{M} \) consists of a two-dimensional subset of the cylinder \( r = r_0 \) (example: \( V(\vec{x}) = r^s f(|z|/r) \) with \( f = 1 \) on some interval but \( > 1 \) and increasing outside the interval), or of discrete points (example: \( V(\vec{x}) = a|x|^s + b|y|^s + c|z|^s \) with \( a \neq b \)).

Next we consider the convergence of the TF energy \( E_{\gamma,\Omega}^{\text{TF}} \) to \( E_{0,1}^{\text{TF}} = \inf W \) as \( \gamma = \omega^{-2(s+3)/(s-2)} \rightarrow 0 \).

**Theorem 3.3 (TF energy and density for \( \omega \rightarrow \infty \))**

For \( \gamma \rightarrow 0 \)

\[
E_{\gamma,1}^{\text{TF}} = E_{0,1}^{\text{TF}} + O(\gamma^{2/5}).
\]  

Moreover, the TF minimizer \( \rho_{\gamma,1}^{\text{TF}} \) satisfies the bound

\[
\|\rho_{\gamma,1}^{\text{TF}}\|_{\infty} \leq \text{const.} \gamma^{-3/5}
\]

and for any \( \epsilon > 0 \) there is a \( \gamma_\epsilon \) such that for \( \gamma < \gamma_\epsilon \) the support of \( \rho_{\gamma,1}^{\text{TF}} \) is contained in \( \mathcal{M}_\epsilon \equiv \{ \vec{x} : |\vec{x} - \vec{x}_0| \leq \epsilon \text{ for all } \vec{x}_0 \in \mathcal{M} \} \).

**Proof:** It is clear that \( E_{\gamma,1}^{\text{TF}} \geq E_{0,1}^{\text{TF}} \). For an upper bound we provide a trial function with energy at most \( E_{0,1}^{\text{TF}} + O(\gamma^{2/5}) \). Let \( h \) be any continuous, nonnegative function with support in the unit ball in \( \mathbb{R}^3 \) with \( \int h = 1 \). For \( \delta > 0 \) and a point \( \vec{x}_0 \in \mathcal{M} \) define \( \rho_h(\vec{x}) = \delta^{-3} h((\vec{x} - \vec{x}_0)/\delta) \). Then, using that \( W \) is \( C^2 \) and that \( \rho_h \) is supported in a ball of radius \( \delta \) around \( \vec{x}_0 \in \mathcal{M} \), we have

\[
E_{\gamma,1}^{\text{TF}} \leq E_{\gamma,1}^{\text{TF}}[\rho_h] = \int_{\mathbb{R}^3} d\vec{x} \left\{ W \rho_h + \gamma \rho_h^2 \right\} \leq E_{0,1}^{\text{TF}} + C \delta^2 + \gamma \delta^{-3} \|h\|_2^2.
\]  

Choosing \( \delta = \gamma^{1/5} \) now proves (3.30).

The TF minimizer is explicitly given by

\[
\rho_{\gamma,1}^{\text{TF}}(\vec{x}) = \frac{1}{2\gamma} \left[ \mu_{\gamma,1}^{\text{TF}} - W(\vec{x}) \right]_+. 
\]  

Since \( \rho_{\gamma,1}^{\text{TF}} \) remains normalized as \( \gamma \rightarrow 0 \), it is clear by continuity of \( W \) that \( \mu_{\gamma,1}^{\text{TF}} \) must converge to the minimum of \( W \) and the support of \( \rho_{\gamma,1}^{\text{TF}} \) shrinks to the set \( \mathcal{M} \) as stated in the lemma. Moreover, \( E_{\gamma,1}^{\text{TF}} \geq E_{0,1}^{\text{TF}} + \gamma \|\rho_{\gamma,1}^{\text{TF}}\|_2^2 \) and, by (2.14), \( \mu_{\gamma,1}^{\text{TF}} = E_{\gamma,1}^{\text{TF}} + g\|\rho_{\gamma,1}^{\text{TF}}\|_2^2 \) and this together with (3.30) implies

\[
0 \leq \left[ \mu_{\gamma,1}^{\text{TF}} - W(\vec{x}) \right]_+ \leq \mu_{\gamma,1}^{\text{TF}} - E_{0,1}^{\text{TF}} \leq O(\gamma^{2/5})
\]

and hence \( \|\rho_{\gamma,1}^{\text{TF}}\|_{\infty} \leq \text{const.} \gamma^{-3/5} \).

**Remark:** In the case that \( \mathcal{M} \) does not consist of discrete points but is one- or two-dimensional the power of \( \gamma \) in the optimal error terms are of higher order than \( \gamma^{2/5} \). For instance if \( V \) is radially symmetric and \( \mathcal{M} \) is a circle, the trial function can be taken to be radially symmetric and the error is \( O(\gamma^{1/2}) \).

As a complementary statement to Theorem 3.3 we now prove the convergence of the scaled GP ground state energy to \( E_{0,1}^{\text{TF}} \) as \( \Omega \) and \( \omega \rightarrow \infty \).

**Theorem 3.4 (GP energy asymptotics for \( \omega \rightarrow \infty \))**

As \( \Omega \rightarrow \infty \) and \( \omega \rightarrow \infty \)

\[
\Omega^{-2s/(s-2)} E_{\Omega,\Omega'}^{\text{GP}} = E_{0,1}^{\text{TF}} + O\left(\Omega'^{-1} + \gamma^{2/5}\right),
\]

with \( \Omega' = \Omega^{(s+2)/(s-2)} \) and \( \gamma = \omega^{-2(s+3)/(s-2)} \).
Proof: Note first that $\Omega^{-2s/(s-2)}E_{g,\Omega}^{\text{GP}} = \Omega^{-2}\tilde{E}_{\Omega,\omega}^{\text{GP}}$ where $\tilde{E}_{\Omega,\omega}^{\text{GP}}$ is the ground state energy of the scaled GP functional $\tilde{E}_{\Omega,\omega}^{\text{GP}}$, c.f. (3.7). Dropping positive terms from the Hamiltonian. In the case of ultra rapid rotations, the lower bound in the case of ultrarapid rotations is simply obtained by using Theorem 3.1 of the previous section to relate it to the TF energy. In the regime $\delta \to 0$ we put $h_\delta(\vec{x}) = \delta^{-3}h(\vec{x}/\delta)$. Let $\vec{x}_0 \in \mathcal{M}$ and define

$$\phi(\vec{x}) = \sqrt{h_\delta(\vec{x} - \vec{x}_0)} \exp \{ i \Omega \vec{x} \cdot (\vec{e}_z \wedge \vec{x}_0)/2 \}. \quad (3.37)$$

Testing $\tilde{E}_{\Omega,\omega}^{\text{GP}}$ with this function we obtain

$$\Omega^{-2}\tilde{E}_{\Omega,\omega}^{\text{GP}} \leq 2\Omega^{-2}\parallel \nabla \sqrt{h_\delta} \parallel^2_2 + \gamma \parallel h_\delta \parallel^2_2 + \int_{\mathbb{R}^3} d\vec{x} \left( \frac{1}{4} |\vec{e}_z \wedge (\vec{x}_0 - \vec{x})|^2 + W(\vec{x}) \right) h_\delta (\vec{x} - \vec{x}_0). \quad (3.38)$$

Since $h_\delta(\vec{x}) \equiv \delta^{-3}h(\vec{x}/\delta)$, the first term is $O(\Omega^{-2}\delta^{-2})$. Since $\parallel h_\delta \parallel^2_2 \leq \parallel h_\delta \parallel_\infty = O(\delta^{-3})$, the second term is $O(\gamma \delta^{-3})$. In the last integral we use that $W \in C^2$, that supp $h_\delta$ is contained in a ball of radius $\delta$ around $\vec{x}_0$ with $W(\vec{x}_0) = E_{0,1}^{\text{TF}}$, and $\parallel h_\delta \parallel_1 = 1$ to get

$$\int_{\mathbb{R}^3} d\vec{x} \left( \frac{1}{4} |\vec{e}_z \wedge (\vec{x}_0 - \vec{x})|^2 + W(\vec{x}) \right) h_\delta (\vec{x} - \vec{x}_0) \leq E_{0,1}^{\text{TF}} + O(\delta^2). \quad (3.39)$$

We thus have

$$\Omega^{-2}\tilde{E}_{\Omega,\omega}^{\text{GP}} \leq E_{0,1}^{\text{TF}} + O(\Omega^{-2}\delta^{-2} + \delta^2 + \gamma \delta^{-3}). \quad (3.40)$$

Equating the second and the last error term leads to the choice $\delta = \gamma^{1/5}$ and an error $O(\Omega^{-2}\gamma^{2/5} + \gamma^{2/5}) = O(\gamma^{2/5})$ provided $\Omega^{-1} \leq \gamma^{2/5}$. For $\Omega^{-1} > \gamma^{2/5}$ we choose $\delta = \Omega^{-1/2}$ (this corresponds to equating the first and the second error term in (3.40)). Then the errors are $O(\Omega^{-1})$. Altogether we obtain (3.35).

Remark: The true GP density is in general not concentrated around a single point in $\mathcal{M}$ and the trial function (3.37) is not designed to give optimal error bounds. Another obvious possibility is to replace $h_\delta$ by a regularization of the TF density and choose as phase factor of the ‘giant vortex’ type like in [CRY1]. In fact, in the proof of (3.38) we use a function of this form as an ingredient of the many-body trial function. Since Theorem 3.4 is not directly used for the proof of the corresponding many-body result we do not elaborate on this point further here.

### 4 Proofs of the QM Limit Theorems

In this section we derive the bounds on the quantum mechanical ground state energy $E_{g,\Omega}^{\text{QM}}$ that lead to the proofs of Theorems 2.1-2.3. The lower bound in the case of ultrarapid rotations is simply obtained by dropping positive terms from the Hamiltonian. In the case $\omega < \infty$ one uses first the diamagnetic inequality [LL] to eliminate the vector potential from the Hamiltonian (2.3) and then proceeds with the techniques described in [LSY5] for the non-rotating case. The upper bound for $\omega < \infty$ is obtained by first bounding the QM energy by the GP energy. The method, that is a generalization of [LSY1], is described briefly in [S1] for fixed $g$ and $\Omega$, but in order to keep track of the error terms as $g$ and/or $\Omega$ tend to $\infty$ and for completeness we carry it out in more detail. Once a bound in terms of the GP energy has been obtained, we can use Theorem 3.1 of the previous section to relate it to the TF energy. In the regime of ultrarapid rotation, $\omega \to \infty$, we use a slightly different method that gives an estimate in terms of the TF energy and error terms involving directly the TF density whose relevant properties were described in Theorem 3.3. The limit Theorems 2.2-2.3 for the density are simple consequences of the energy bounds and are discussed in Subsection 4.2.
4.1 Bounds on the QM energy

Proposition 4.1 (Lower bound for the QM energy)

Let the potential $V$ be homogenous of order $s < 2$. Then

$$\Omega^{-2s/(s-2)} N^{-1} E_{g,\Omega}^{\text{QM}} (N) \geq E_{0,1}^{\text{TF}}.$$  \hfill (4.1)

Furthermore, if $\omega = g^{-(s-2)/(2s+6)} \Omega$ is fixed and $N^{-2} g^3 \| \rho_{g,\Omega}^{\text{TF}} \| \to 0$ as $N \to \infty$ then

$$\liminf_{N \to \infty} \left\{ g^{-s} N^{-1} E_{g,\Omega}^{\text{QM}} (N) \right\} \geq E_{1,\omega}^{\text{TF}}$$  \hfill (4.2)

uniformly in $\omega$ on any bounded interval.

Proof: To prove (4.1) consider a normalized $N$-particle wave function $\Psi_N$ and let $\hat{\rho}_N (\vec{x}) = \lambda^3 \rho_N (\lambda \vec{x})$ with $\lambda = \Omega^{2/(s-2)}$ be the corresponding scaled density. Then we can write

$$\Omega^{-2s/(s-2)} N^{-1} \langle \Psi_N, H_N \Psi_N \rangle = C_{\Psi_N} + \inf_{\mathbb{R}^3} W,$$  \hfill (4.3)

with $W (\vec{x}) = V (\vec{x}) - r^2/4$ and

$$C_{\Psi_N} = \Omega^{-2s/(s-2)} \left[ \left\| \nabla - i \vec{A} \right\| \Psi_N \right]^2_2 + \int_{\mathbb{R}^3} \hat{\rho}_N (\vec{x}) \left( W (\vec{x}) - \inf_{\mathbb{R}^3} W \right) d\vec{x} +
\Omega^{-2s/(s-2)} N^{-1} \sum_{1 \leq i < j \leq N} \langle \Psi_N, v (|\vec{x}_i - \vec{x}_j|) \Psi_N \rangle.$$  \hfill (4.4)

Since the interaction potential $v$ is by assumption nonnegative the same holds for $C_{\Psi_N}$ so the left hand side of (4.1) is $\geq \inf W = E_{0,1}^{\text{TF}}$.

Now, let us consider the case when $\omega < \infty$ is fixed as $N \to \infty$. By the diamagnetic inequality, $|\langle \nabla - i \vec{A} (\vec{x}) | f (\vec{x}) \rangle | \geq |\nabla | f (\vec{x}) | |$, cf. [LL], and the bound

$$V (\vec{x}) \geq \rho_{g,\Omega}^{\text{TF}} + \frac{1}{4} \Omega^2 r^2 - 2 g \rho_{g,\Omega}^{\text{TF}} (\vec{x})$$  \hfill (4.5)

that follows from Eq. (2.13) we obtain

$$E_{g,\Omega}^{\text{QM}} (N) \geq N \rho_{g,\Omega}^{\text{TF}} + \inf_{\Psi, \| \Psi \| = 1} Q (\Psi),$$  \hfill (4.6)

with

$$Q (\Psi) \equiv \sum_{i=1}^N \left\| \nabla_i \Psi \right\|^2 + \sum_{1 \leq i < j \leq N} \int v (|\vec{x}_i - \vec{x}_j|) |\Psi|^2 d^3N \vec{x} - 2g \sum_{i=1}^N \int \rho_{g,\Omega}^{\text{TF}} (\vec{x}_i) |\Psi|^2 d^3N \vec{x}.$$  \hfill (4.7)

Since $\rho_{g,\Omega}^{\text{TF}} (\vec{x})$ tends to zero for every $\vec{x}$ as $g \to \infty$, cf. (2.20) it is convenient at this point to carry out a rescaling by writing $\vec{x} = \lambda \vec{y}$ with $\lambda = g^{1/(s+3)}$, cf. (3.2). The scaled interaction potential $v' (\vec{y}) = \lambda^2 v (\lambda \vec{y})$ has scattering length $a' = g^{-1/(s+3)} a$ and the corresponding coupling parameter is

$$g' = 4 \pi Na' = g^{(s+2)/(s+3)}.$$  \hfill (4.8)

Using (2.14) and (2.15) we obtain

$$g^{-s/(s+3)} N^{-1} E_{g,\Omega}^{\text{QM}} - E_{1,\omega}^{\text{TF}} \geq \int (\rho_{1,\omega}^{\text{TF}})^2 + \inf_{\Psi, \| \Psi \| = 1} Q' (\Psi),$$  \hfill (4.9)

where, dropping the primes on the integration variables,

$$Q' (\Psi) \equiv \sum_{i=1}^N \left\| \nabla_i \Psi \right\|^2 + \sum_{1 \leq i < j \leq N} \int v' (|\vec{x}_i - \vec{x}_j|) |\Psi|^2 d^3N \vec{x} - 2g' \sum_{i=1}^N \int \rho_{1,\omega}^{\text{TF}} (\vec{x}_i) |\Psi|^2 d^3N \vec{x}.$$  \hfill (4.10)
We are now exactly in the situation discussed in [LSY5] for the nonrotating case, cf. Eq. (6.61) in [LSY5]. Like there, the next step is to divide space into boxes, labeled by $\alpha$ and of side length $l$, with Neumann boundary conditions and use the lower bound of [LY] for the homogeneous gas in each box. The result is (cf. Eq. (6.62) in [LSY5])

$$g^{-\frac{1}{4\pi}}N^{-1}E_{g,\Omega}^{QM}(N) - E_{1,\omega}^{TF} \geq \int (\rho_{1,\omega}^{TF})^2 - \sum_{\alpha} d_{\alpha}^2 \int (1 - CY'^{1/17}). \quad (4.11)$$

Here $d_{\alpha}$ is the maximum value of $\rho_{1,\omega}^{TF}$ in the box $\alpha$ and $Y' = a^3 N/l^3 \sim g' N^{-2}/l^3$. Since

$$gN^{-2}||\rho_{1,\omega}^{TF}||_{\infty} = g'N^{-2}||\rho_{1,\omega}^{TF}||_{\infty} \quad (4.12)$$

the diluteness condition implies that $Y' \to 0$ for fixed $l$. If we now first take $N \to \infty$ and then $l \to 0$, the Riemann approximation of $\int (\rho_{1,\omega}^{TF})^2$ implies that the right hand side of (4.11) tends to zero, proving (4.2) for fixed $\omega$. It is also clear that all estimates are uniform in $\omega$ on bounded sets, so in particular one can take $\omega$ to 0.

**Proposition 4.2 (Upper bound on the QM energy for $\omega < \infty$)**

Let the potential $V$ be homogenous of order $s > 2$ and suppose the diluteness condition, $N^{-2}g^3||\rho_{1,\omega}^{TF}||_{\infty} \to 0$ as $N \to \infty$, is fulfilled. If $\omega$ is fixed, then

$$\limsup_{N \to \infty} \left\{ g^{-\frac{1}{4\pi}}N^{-1}E_{g,\Omega}^{QM}(N) \right\} \leq E_{1,\omega}^{TF} \quad (4.13)$$

uniformly in $\omega$ on bounded intervals.

**Proof:** The proof is a combination of a variational bound on the QM energy in terms of the GP energy and the bounds of the GP energy in terms of the TF energy that were discussed in Section 3. For the former we can use the same method as in [LSY1] and [S2] and not all details will be repeated here, but we shall keep track of the error terms and their dependence on the various parameters.

The main step is to show that under the stated assumptions

$$\frac{E_{g,\Omega}^{QM}(N) - NE_{g,\Omega}^{GP}}{Ng||\rho_{g,\Omega}^{GP}||_{\infty}} \leq o(1), \quad (4.14)$$

by exhibiting a sequence of $N$ particle trial functions $\Psi_N$, $N = 1, 2, \ldots$, such that

$$\frac{\langle \Psi_N, H, \Psi_N \rangle \langle \Psi_N, \Psi_N \rangle^{-1} - NE_{g,\Omega}^{GP}}{Ng||\rho_{g,\Omega}^{GP}||_{\infty}} \leq o(1). \quad (4.15)$$

We write the trial functions in the form

$$\Psi_N = F(\bar{x}_1, \ldots, \bar{x}_N)G(\bar{x}_1, \ldots, \bar{x}_N) \quad (4.16)$$

with

$$G(\bar{x}_1, \ldots, \bar{x}_N) \equiv \prod_{i=1}^{N} \phi_{g,\Omega}^{GP}(\bar{x}_i) \quad (4.17)$$

and a real function $F$. Partial integration, using the variational equation (2.8) and the reality of $F$, leads to

$$\langle \Psi_N, H, \Psi_N \rangle = N\mu_{g,\Omega}^{GP}(\Psi_N, \Psi_N) + \sum_{1 \leq i \leq N} \int_{\mathbb{R}^3} |\nabla_i F|^2 |G|^2 + \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^3} v(|\bar{x}_i - \bar{x}_j|) |F|^2 |G|^2 - 2g \sum_{1 \leq i \leq N} \int_{\mathbb{R}^3} \rho_{g,\Omega}^{GP}(\bar{x}_i) |F|^2 |G|^2. \quad (4.18)$$

The second line of (4.18) is a real quadratic form in $F$ and we shall make use of the fact for an upper bound on the bosonic ground state energy is not necessary to require that the trial function $F$ is symmetric.
under permutations of the variables. This can be seen by a simple adaption of an argument of Lieb \cite{L1} which implies that the infimum over all functions $F$ is the same as the infimum over all nonnegative, symmetric functions.

Like in \cite{LSY1} we shall take a trial function of the Dyson form \cite{D}

$$F(x_1, \ldots, x_N) = \prod_{i=1}^{N} F_i(x_1, \ldots, x_i)$$

where

$$F_i(x_1, \ldots, x_i) = f(t_i), \quad t_i = \min (|x_i - x_j|, j = 1, \ldots, i - 1),$$

with a function $f$ satisfying

$$0 \leq f \leq 1, \quad f' \geq 0.$$

The function $f$ will be specified shortly. Our estimates involve the quantities

$$I \equiv \int_{\mathbb{R}^3} (1 - f^2), \quad J \equiv \int_{\mathbb{R}^3} (f'' + \frac{1}{2} v f^2), \quad K \equiv \int_{\mathbb{R}^3} f f'.$$

By exactly the same computation as leads to Eq. (3.29) in \cite{LSY1} we obtain, provided $N\|\rho_{g,\Omega}\|_{\infty} I < 1$,

$$\|FG\|_{L^2}^{-2} \int_{\mathbb{R}^3} |\nabla F|^2 |G|^2 + \sum_{i<j} \int_{\mathbb{R}^3} v(|x_i - x_j|) F^2 |G|^2 \leq \frac{1}{(1 - N\|\rho_{g,\Omega}\|_{\infty} I)^2} \left\{ N^2 J \int_{\mathbb{R}^3} \rho_{g,\Omega} (x)^2 + \frac{2}{3} N^3 K^2 \|\rho_{g,\Omega}\|^2_{L^\infty} \right\}$$

and the same technique gives also a bound on the last term in (4.18)

$$-2g\|FG\|_{L^2}^{-2} \sum_{0<i\leq N} \int_{\mathbb{R}^3} \rho_{g,\Omega} (x_i) |F|^2 |G|^2 \leq -2g N \int_{\mathbb{R}^3} d\tilde{x} \rho_{g,\Omega} (x)^2 + 2g N^2 I \|\rho_{g,\Omega}\|^2_{L^\infty}.$$ 

We now choose the function $f$. For a parameter $b > a$ that will soon be fixed we define

$$f(r) = \begin{cases} (1 + \epsilon_1) u(r)/r & \text{for } r \leq b \\ 1 & \text{for } r > b \end{cases}$$

where $u(r)$ is the solution of the scattering equation

$$-u''(r) + \frac{1}{2} v(r) u(r) = 0 \quad \text{with } u(0) = 0, \lim_{r \to \infty} u'(r) = 1$$

and $\epsilon_1$ is determined by requiring $f$ to be continuous. Convexity of $u$ gives

$$r \geq u(r) \geq \begin{cases} 0 & \text{for } r \leq a \\ r - a & \text{for } r > a \end{cases}, \quad 1 \geq u'(r) \geq \begin{cases} 0 & \text{for } r \leq a \\ 1 - \frac{a}{r} & \text{for } r > a \end{cases}.$$ 

These estimates imply

$$J \leq (1 + \epsilon_1)^2 4\pi a$$

$$I \leq 4\pi \left( \frac{a^3}{3} + ab(b - a) \right)$$

$$K \leq 4\pi (1 + \epsilon_1) a \left( b - \frac{a}{2} \right)$$

$$0 \leq \epsilon_1 \leq \frac{a}{b - a}.$$
Before proceeding further, we need to relate the supremum of the GP density, $||\phi_{g,\Omega}^{GP}||_\infty$ to $||\rho_{\Omega}^{TF}||_\infty$ since the diluteness condition is stated in terms of the latter. For this purpose we write $\phi_{g,\Omega}^{GP} = Re^{iS}$ with real $S$ and the nonnegative amplitude $R$. A straightforward computation, using $\nabla \cdot \vec{A} = 0$, gives

$-(\nabla - i\vec{A})^2 \phi_{g,\Omega}^{GP} = (-\Delta + 2i\vec{A} \cdot \nabla + A^2)\phi_{g,\Omega}^{GP}
= -(\Delta R)e^{iS} - 2i(\nabla R) \cdot (\nabla S)e^{iS} + (\nabla S)^2 Re^{iS} - i(\Delta S)Re^{iS} + 2i\vec{A} \cdot (\nabla R)e^{iS} - 2\vec{A} \cdot (\nabla S)Re^{iS} + A^2Re^{iS}$

and from the GP equation (2.8) one obtains

$(-\Delta + (\nabla S)^2 - 2\vec{A} \cdot (\nabla S) + A^2 + 2g\rho_{g,\Omega}^{GP} + V - \frac{1}{2} \Omega^2 r^2) R = \mu_{g,\Omega}^{GP}$.

In any point $\vec{x} \in \mathbb{R}^3$ where $R$ is maximal $\rho_{g,\Omega}^{GP}(\vec{x}) = ||\rho_{g,\Omega}^{GP}||_\infty$ and $\Delta R(\vec{x}) \leq 0$. Thus,

$2g||\rho_{g,\Omega}^{GP}||_\infty \leq -(\nabla S(\vec{x}))^2 + 2\vec{A}(x) \cdot \nabla S(\vec{x}) - A^2(\vec{x}) - V(\vec{x}) + \frac{1}{4} \Omega^2 r^2 + \rho_{g,\Omega}^{GP}$.

and since $-(\nabla S)^2 + 2\vec{A} \cdot \nabla S \leq A^2$ we obtain

$2g||\rho_{g,\Omega}^{GP}||_\infty \leq ||\rho_{g,\Omega}^{GP}||_\infty \leq \inf_{\vec{x} \in \mathbb{R}^3} \{V(\vec{x}) - \frac{1}{4} \Omega^2 r^2\}$.

On the other hand, by (2.13) we have

$2g||\rho_{g,\Omega}^{TF}||_\infty = \mu_{g,\Omega}^{TF} = \inf_{\vec{x} \in \mathbb{R}^3} \{V(\vec{x}) - \frac{1}{4} \Omega^2 r^2\}$

and therefore, using (2.9) and (2.14),

$2g||\rho_{g,\Omega}^{GP}||_\infty \leq \mu_{g,\Omega}^{GP} - \mu_{g,\Omega}^{TF} + 2g||\rho_{g,\Omega}^{TF}||_\infty \leq \mu_{g,\Omega}^{GP} - \mu_{g,\Omega}^{TF} + \inf_{\vec{x} \in \mathbb{R}^3} \{V(\vec{x}) - \frac{1}{4} \Omega^2 r^2\}$.

By using (2.14) and Theorem 3.1 one sees that

$\frac{E_{g,\Omega}^{GP} - E_{g,\Omega}^{TF}}{g||\rho_{g,\Omega}^{TF}||_\infty} = ||\rho_{g,\Omega}^{TF}||_\infty^{-1} \left(g^{-s/(s+3)} E_{g,\Omega}^{GP} - E_{g,\Omega}^{TF}\right) = o(1)$,

so

$||\rho_{g,\Omega}^{GP}||_\infty \leq 2 ||\rho_{g,\Omega}^{TF}||_\infty \{1 + o(1)\}$.

The diluteness condition $N^{-2}g^3||\rho_{g,\Omega}^{TF}||_\infty \to 0$ thus implies the corresponding condition for the GP density, i.e., $N^{-2}g^3||\rho_{g,\Omega}^{GP}||_\infty \to 0$. Therefore, by choosing

$b = (N||\rho_{g,\Omega}^{GP}||_\infty)^{-\frac{1}{4}}$,

it follows from (4.25-4.28) that $\epsilon_1 \to 0$, $\epsilon_2 \equiv N||\rho_{g,\Omega}^{GP}||_\infty I \to 0$ and $\epsilon_3 \equiv g^{-1}N^2K^2||\rho_{g,\Omega}^{GP}||_\infty \to 0$ for $N \to \infty$. Altogether one gets from (4.18), using (2.3), (4.22) and (4.23), that

$N^{-1}E_{g,\Omega}^{QM} \leq E_{g,\Omega}^{GP} + g||\rho_{g,\Omega}^{GP}||_\infty^2 \{O(\epsilon_1) + O(\epsilon_2)\} + g||\rho_{g,\Omega}^{GP}||_\infty O(\epsilon_3) \leq E_{g,\Omega}^{GP} + o(1)g||\rho_{g,\Omega}^{GP}||_\infty$,

i.e. (4.15). By (2.17) we have

$g^{-s/(s+3)} \{g||\rho_{g,\Omega}^{TF}||_\infty\} = g^{3/(s+3)}||\rho_{g,\Omega}^{TF}||_\infty = ||\rho_{g,\Omega}^{TF}||_\infty$,

and by (4.33) and Theorem 3.1 we can conclude that

$g^{-s/(s+3)}N^{-1}E_{g,\Omega}^{QM} \leq g^{-s/(s+3)}E_{g,\Omega}^{GP} + o(1)||\rho_{g,\Omega}^{TF}||_\infty = E_{g,\Omega}^{TF} + o(1)$.
For ultrapid rotations the proof of (4.33) given above is not valid because the error term may blow up as \( \omega \to \infty \). We shall therefore treat this case separately, using a trial function different from (4.17). If a general proof of \( \| \rho_{g,1}^\Omega \|_\infty \leq (\text{const.}) \| \rho_{g,1}^\Omega \|_\infty \) can be found, then Eq. (4.15) is verified also for \( \omega \to \infty \) and the proof of the next proposition would follow in the same way as the previous one.

**Proposition 4.3 (Upper bound on the QM energy for \( \omega \to \infty \))**

Let the potential \( V \) be homogenous of order \( s > 2 \) and suppose the diluteness condition, \( N^{-2} g^3 \| \rho_{g,1}^\Omega \|_\infty \to 0 \) as \( N \to \infty \), is fulfilled. If \( \Omega \to \infty \) and \( \omega \to \infty \) as \( N \to \infty \), then

\[
\limsup_{N \to \infty} \left\{ \Omega^{-2s/(s-2)} N^{-1} E_{N,g,\Omega}^{QM}(N) \right\} \leq E_{0,1}^{TM}.
\]

**Proof:** The first step is to choose a suitable phase factor for the trial function to compensate the vector potential in the kinetic term as far as possible. As shown in Lemma 3.2 the set of minimizers of

\[
W_\Omega (\vec{x}) = V (\vec{x}) - \frac{1}{2} \Omega^2 r^2
\]

is a subset of a cylinder with radius \( r_\Omega > 0 \). We define the phase factor as follows:

\[
\Theta(x_1, ..., x_N) = \prod_{j=1}^N \theta(x_j),
\]

with

\[
\theta(x) = \exp \left\{ i \left[ \frac{1}{2} r_\Omega^2 \right] \theta \right\} \text{ for } \vec{x} = (r, \theta, z),
\]

and where \([ \cdot ]\) stands for the integer part. Any function \( \Psi \in L^2 \) can be written as \( \Psi = \Theta \Phi \) with \( \Phi \in L^2 \) and a straightforward computation gives

\[
E_{N,g,\Omega}^{QM}(\Psi) = E_{N,g,0}^{QM}(\Phi) + \sum_{j=1}^N \int_{\mathbb{R}^3} \left( -\frac{\Omega^2 r_j^2}{4} |\Phi|^2 + \left| \left( -i \nabla_j - A_\Omega(x_j) \right) \Theta \right|^2 |\Phi|^2 \right)
\]

\[
-2 \sum_{j=1}^N \Re \left( \int_{\mathbb{R}^3} \Phi \Theta^* \left( \left( -i \nabla_j - A_\Omega(x_j) \right) \Theta \right) \nabla_j \Phi^* \right).
\]

Since

\[
\left( -i \nabla - A_\Omega \right) \theta = \left( \frac{1}{r} \left[ \frac{1}{2} r_\Omega^2 \right] - \frac{r_\Omega}{2} \right) \theta e_\theta,
\]

we get, using the Cauchy-Schwartz inequality combined with \( 2ab \leq a^2 + b^2 \),

\[
E_{N,g,\Omega}^{QM}(\Psi) \leq E_{N,g,0}^{QM}(\Phi) + \sum_{j=1}^N \int_{\mathbb{R}^3} \left( -\frac{\Omega^2 r_j^2}{4} |\Phi|^2 + \frac{1}{r_j} \left[ \frac{1}{2} r_\Omega^2 \right] - \frac{r_\Omega}{2} |\Phi|^2 + |\nabla_j \Phi|^2 \right).
\]

In particular, since \( \left[ \frac{1}{2} r_\Omega^2 \right] = \frac{1}{2} r_\Omega^2 \Omega^2 + \kappa \) with \( |\kappa| < 1 \), we then have

\[
E_{N,g,\Omega}^{QM}(\Psi) - N \inf W_\Omega \leq \tilde{E}_{N,g,\Omega}^{QM}(\Phi),
\]

with

\[
\tilde{E}_{N,g,\Omega}^{QM}(\Phi) = \sum_{j=1}^N \int_{\mathbb{R}^3} \left( 2 |\nabla_j \Phi|^2 + \left( \tilde{W}_\Omega - \inf W_\Omega \right) |\Phi|^2 \right) + \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^3} v(|\vec{x}_i - \vec{x}_j|) |\Phi|^2
\]

and

\[
\tilde{W}_\Omega (\vec{x}) = V (\vec{x}) + \Omega^2 \left( -\frac{r^2}{4} + \left| \frac{r_\Omega^2}{r} - r \right|^2 + \frac{4}{\Omega^2 r^2} \right).
\]
The functional (4.44) describes the QM energy of a non-rotating system of particles with mass 1/2 in the positive external potential \((\bar{W}_\Omega - \inf W_\Omega)\) and with a two-body interaction potential \(v \geq 0\).

We need to chose a trial function for (4.44). As in the proof of the previous proposition we are dealing with a \textit{real} quadratic form so the infimum over all \(\Phi \in L^2(\mathbb{R}^3)\) is the same as the infimum over all symmetric \(\Phi \in L^2(\mathbb{R}^3)\). Thus, by (4.43),

\[
E^{\text{QM}}_{N,g,\Omega} - N \inf W_\Omega \leq \inf_{\Phi} \frac{\tilde{E}^{\text{QM}}_{N,g,\Omega}(\Phi)}{||\Phi||_2^2}.
\]

We denote by \(\rho \equiv j_* \ast \rho_{g,\Omega}^\text{TF}\) the regularized TF density with \(j_*\) defined by (3.13) for any \(\epsilon > 0\). Observe that \(||\rho\||_1 = 1\) and \(||\nabla \sqrt{\rho}|| = \epsilon^{-2}\) since \(|\nabla j_*| = \epsilon^{-1} j_*\) and \(||j_*||_1 = 1\) for any \(\epsilon > 0\). Our trial function is defined as

\[
\tilde{\Phi}(\vec{x}_1, ..., \vec{x}_N) \equiv F(\vec{x}_1, ..., \vec{x}_N)G(\vec{x}_1, ..., \vec{x}_N),
\]

with

\[
G(\vec{x}_1, ..., \vec{x}_N) \equiv \prod_{j=1}^N \sqrt{\rho}(\vec{x}_j)
\]

while the function \(F\) is of the Dyson form, cf. (4.19)-(4.20) and (4.21).

The estimation of \(\tilde{E}^{\text{QM}}_{N,g,\Omega}(\Phi)/||\Phi||_2^2\) follows closely the computations in [LSY1], Eqs. (3.11)-(3.29), but with the regularized TF density \(\rho\) instead of the GP density. The diluteness condition \(N^{-2} g^3 ||\rho_{g,\Omega}^\text{TF}||_\infty \to 0\) implies that the same condition is also fulfilled for \(\rho\) because

\[
||\rho||_\infty \leq ||\rho_{g,\Omega}^\text{TF}||_\infty ||j_*||_1 = ||\rho_{g,\Omega}^\text{TF}||_\infty.
\]

The GP equation, that was used in the computations in [LSY1] to obtain Eq. (3.28) in that paper, is not at our disposal for \(\rho\), but we can instead use the Cauchy-Schwarz inequality combined with \(2ab \leq a^2 + b^2\). In this way, using also the positivity of \(v\), we obtain

\[
E^{\text{QM}}_{N,g,\Omega} - N \inf W_\Omega \leq \sum_{j=1}^N ||FG||_2^{-2} \left\{ 4 \int_{\mathbb{R}^{3N}} |\nabla_j G|^2 F^2 + \int_{\mathbb{R}^{3N}} \left( \bar{W}_\Omega (\vec{x}_j) - \inf W_\Omega \right) F^2 G^2 \right\} + 4 \sum_{j=1}^N ||FG||_2^{-2} \left\{ \int_{\mathbb{R}^{3N}} |\nabla_j F|^2 G^2 + \sum_{i<j} \int_{\mathbb{R}^{3N}} v(|\vec{x}_i - \vec{x}_j|) F^2 G^2 \right\}.
\]

Since \(|\nabla j_*| = \epsilon^{-1} j_*\), we have for the first term of (4.50) the bound

\[
4 \int_{\mathbb{R}^{3N}} |\nabla_j G|^2 F^2 = 4 \int_{\mathbb{R}^{3N}} \frac{1}{\rho} \left| \frac{\nabla_j \rho}{2 \sqrt{\rho}} \right|^2 \leq \frac{1}{\epsilon^2}.
\]

Using Eqs. (3.15)-(3.16) and (3.21) in [LSY1] as well as \((\bar{W}_\Omega - \inf W_\Omega) \geq 0\), we obtain the bound

\[
||FG||_2^{-2} \int_{\mathbb{R}^{3N}} \left( \bar{W}_\Omega (\vec{x}_j) - \inf W_\Omega \right) F^2 G^2 \leq \frac{1}{1 - N ||\rho||_\infty I} \int_{\mathbb{R}^3} d^3 \vec{x} \left( \bar{W}_\Omega - \inf W_\Omega \right) \rho
\]

with \(I\) defined by (4.21), provided \(N||\rho||_\infty I < 1\) that is guaranteed by the diluteness condition. The last two terms in (4.50) are bounded in exactly the same way which leads to (4.22) with \(\rho\) in the place of \(\rho_{g,\Omega}\). We omit the details. Altogether we have the upper bound

\[
N^{-1} E^{\text{QM}}_{N,g,\Omega} - \inf W_\Omega \leq \frac{1}{\epsilon^2} + (1 + o(1)) \int_{\mathbb{R}^3} d^3 \vec{x} \left\{ \left( \bar{W}_\Omega - \inf W_\Omega \right) \rho + 4 g^2 \rho^2 \right\} + o(1) g||\rho||_\infty.
\]

Since \(\rho\) is normalized,

\[
||\rho||_2 \leq ||\rho||_\infty \leq ||\rho_{g,\Omega}^\text{TF}||_\infty
\]

\footnote{This \(\epsilon\) is unrelated to the \(\epsilon\) defined in (4.33).}
by (4.49). Also,
\[ \Omega^{-2s/(s-2)} \inf W_\Omega = E_{0,1}^{\text{TF}}, \]  
(4.55)
and by Lemma 3.2 all minimizing points have the same radial coordinate \( r_\Omega = r_0 \Omega \to \infty \) with \( r_0 \) the radius of the set \( \mathcal{M} \). The inequality (4.53) now implies
\[
\Omega^{-2s/(s-2)} N^{-1} E_{N,g,\Omega}^{\text{QM}} - E_{0,1}^{\text{TF}} \leq \Omega^{-2s/(s-2)} \left\{ \int_{\mathbb{R}^3} d\vec{x} \left( \tilde{W}_\Omega - \inf W_\Omega \right) \rho_\epsilon \right\} (1 + o(1)) 
+ 5 \Omega^{-2s/(s-2)} g ||\rho_{g,\Omega}^{\text{TF}}||_\infty (1 + o(1)) 
+ \Omega^{-2s/(s-2)} \epsilon^{-2}. \tag{4.56}
\]

We now have to bound each term of the right hand side. For the second term we use that
\[ \Omega^{-2s/(s-2)} g ||\rho_{g,\Omega}^{\text{TF}}||_\infty = \gamma ||\rho_{\gamma,1}^{\text{TF}}||_\infty \leq o(1) \tag{4.57} \]
by (2.20) and because \( ||\rho_{\gamma,1}^{\text{TF}}||_\infty \leq O(\gamma^{-3/5}) \) by Lemma 3.2. The last term in (4.56) is \( o(1) \) as long as \( \epsilon \gg \Omega^{-s/(s-2)} \).

It remains to consider the first term in (4.56). By scaling we have
\[
\Omega^{-2s/(s-2)} \int_{\mathbb{R}^3} d\vec{x} \left( \tilde{W}_\Omega - \inf W_\Omega \right) \rho_\epsilon = \int_{\mathbb{R}^3} d\vec{x} \left\{ V(\vec{x}) - \frac{1}{2} r^2 + \left| \frac{r_0^2}{r} - r \right|^2 + \frac{4}{\Omega^2 r^2} \right\} \tilde{\rho}_\epsilon(\vec{x}) - E_{0,1}^{\text{TF}}, \tag{4.58}
\]
with
\[ \tilde{\rho}_\epsilon(\vec{x}) = \Omega^{-\frac{2}{s-2}} \rho_\epsilon \left( \Omega^{-\frac{2}{s-2}} \vec{x} \right) = j_\vec{x} \ast \rho_{\gamma,1}^{\text{TF}}(\vec{x}), \quad \tilde{\epsilon} \equiv \Omega^{-\frac{s}{s-2}} \epsilon. \tag{4.59} \]

As \( \gamma \to 0 \) the support of \( \rho_{\gamma,1}^{\text{TF}} \) becomes concentrated on the set \( \mathcal{M} \) where \( V(\vec{x}) - \frac{1}{2} r^2 \) is minimized and all points in \( \mathcal{M} \) have the same radial coordinate, \( r_0 \). Moreover, outside of the support of \( \rho_{\gamma,1}^{\text{TF}} \) the regularized density \( \tilde{\rho}_\epsilon \) decreases exponentially if \( d(\vec{x})/\tilde{\epsilon} \to \infty \) where \( d(\vec{x}) \) is the distance of \( \vec{x} \) from the support of \( \rho_{\gamma,1}^{\text{TF}} \) (see also the proof of Theorem 3.1). This implies that the right hand side of (4.58) tends to zero as \( \gamma \to 0, \Omega \to \infty \) and \( \epsilon \to 0 \) with \( \epsilon^{-1} \Omega^{-s/(s-2)} \to 0 \).

\[ \square \]

4.2 Convergence of the QM particle density

4.2.1 The case \( \omega < \infty \)

Proof of Theorem 2.2 We use Griffiths’ argument [G] in the same way as for an analogous problem in [LSI]. Take any bounded function \( f : \mathbb{R}^3 \to \mathbb{R} \) and for any \( \sigma \in [\delta, \delta'] (\delta > 0) \) perturb the Hamiltonian \( H_N \) with the external potential \( \sigma g^{s/(s+3)} f(g^{-1/(s+3)} \vec{x}) \). Because \( f \) is bounded, the statements (i)-(ii) of Theorem 2.1 can also be proven for the perturbed external potential \( \{ V(\vec{x}) + \sigma g^{s/(s+3)} f(g^{-1/(s+3)} \vec{x}) \} \).

Namely, if \( g \to \infty \) and \( \omega \geq 0 \) is fixed then the corresponding ground state \( E_{g,\Omega,\sigma}^{\text{QM}}(N) \) converges to
\[ \lim_{N \to +\infty} \left\{ g^{-s/(s+3)} N^{-1} E_{g,\Omega,\sigma}^{\text{QM}} (N) \right\} = E_{1,\omega,\sigma}^{\text{TF}} \text{ for any } \sigma \in [\delta, \delta'], \delta > 0 \tag{4.60} \]
where \( E_{1,\omega,\sigma}^{\text{TF}} \) is the TF energy with \( V \) replaced by \( V + \sigma f \). Consequently, for any approximated ground states \( \Psi_N \) of \( H_N \) we have
\[ \langle \Psi_N, [H_{N,\sigma} - H_N] \Psi_N \rangle = N g^{s/(s+3)} \sigma \langle f, \tilde{\rho}_N \rangle, \tag{4.61} \]
with \( \tilde{\rho}_N(\vec{x}) = g^{3/(s+3)} \rho_N(g^{1/(s+3)} \vec{x}) \). Hence, the Rayleigh-Ritz principle for any \( \sigma \in [\delta, \delta'] \) leads to
\[ E_{g,\Omega,\sigma}^{\text{QM}} (N) - E_{g,\Omega,0}^{\text{QM}} (N) \leq N g^{s/(s+3)} \sigma \langle f, \tilde{\rho}_N \rangle + \langle \Psi_N, [H_N - E_{g,\Omega}^{\text{QM}} (N)] \Psi_N \rangle. \tag{4.62} \]
Because \(2.15\) and \(2.17\) are still valid with \(V(\vec{x})\) replaced by \(V(\vec{x}) + \sigma g^{s/(s+3)} f(g^{-1/(s+3)} \vec{x})\), the previous inequality combined with \(4.60\) implies in the limit \(N \to \infty\) that
\[
\sigma^{-1} [E_{1,\omega,\sigma}^{\mathrm{TF}} - E_{1,\omega,\sigma}^{\mathrm{TF}}] \leq \lim_{N \to +\infty} \langle f, \tilde{\rho}_N \rangle \text{ for any } \sigma \in (0, \delta],
\]
whereas a negative parameter \(\sigma \in [-\delta, 0)\) reverses the inequality. The proof of the differentiability of \(E_{1,\omega,\sigma}^{\mathrm{TF}}\) at \(\sigma = 0\) is deduced from similar estimations as from \(4.61\) to \(4.63\) combined with the continuity of the function \(\sigma \to \langle f, \rho_{1,\omega,\sigma}^{\mathrm{TF}} \rangle\), where \(\rho_{1,\omega,\sigma}^{\mathrm{TF}}\) is the minimizer of the variational problem \(E_{1,\omega,\sigma}^{\mathrm{TF}}\). We omit the details. In other words, we have \(\partial_\sigma E_{1,\omega,\sigma}^{\mathrm{TF}} = \langle f, \rho_{1,\omega}^{\mathrm{TF}} \rangle\). Consequently, by \(4.63\) and its reversed inequality, we obtain Theorem 2.2.

\[\Box\]

### 4.2.2 The case \(\omega \to \infty\)

In the case of ultrarapid rotations Griffiths’ argument is not as easily applicable as in the previous situation. There are two complications. First, perturbing \(V\) with a scaled additional term \(\sigma f\) leads to a potential that is not homogeneous and the proof of the upper bound for the QM energy has to be modified in order to get \(4.63\) with \(E_{0,1,\sigma}^{\mathrm{TF}}\) replacing \(E_{1,\omega,\sigma}^{\mathrm{TF}}\). Secondly, if \(\mathcal{M}\) consists of more than one point the variational problem for \(E_{0,1,\sigma}^{\mathrm{TF}}\) does not have a unique minimizer and \(E_{0,1,\sigma}^{\mathrm{TF}}\) is not differentiable at \(\sigma = 0\) in general. To get around this complication, we would need to perturb the Hamiltonian with an additional term to avoid the degeneracy of the variational problem. An example of such an argument is given in \(\text{LS}1\). But since we are content with proving that the density is concentrated on \(\mathcal{M}\) in the limit and not striving to obtain the exact limiting measure on \(\mathcal{M}\) we can ignore all these problems and use the following simpler argument.

**Proof of Theorem 2.5:** If \(\mathcal{K}\) is any set with a positive distance from \(\mathcal{M}\), then \(W(\vec{x}) - E_{0,1}^{\mathrm{TF}} \geq c > 0\) on \(\mathcal{K}\) with some strictly positive number \(c\). Hence, by Eqs. \(4.3\)–\(4.4\),
\[
\Omega^{-2s/(s-2)} N^{-1} \langle \Psi_N, H_N \Psi_N \rangle - E_{0,1}^{\mathrm{TF}} \geq c \int_{\mathcal{K}} \hat{\rho}_N.
\]
On the other hand, by Proposition 4.3 we know that the left hand side of \(4.64\) tends to zero as \(N \to \infty\), so \(\int_{\mathcal{K}} \hat{\rho}_N \to 0\).

\[\Box\]

### 4.3 Remarks on ‘flat’ trapping potentials

As mentioned in the Section 2 the case of a ‘flat’ trap, that corresponds formally to \(s = \infty\), can be treated in essentially the same way as we have done for \(s < \infty\). By a flat potential we mean that \(V\) is 0 inside some open, bounded set \(\mathcal{B}\) with a smooth boundary and \(\infty\) outside. More precisely, the kinetic term in the many-body Hamiltonian \(2.3\) and the GP functional \(2.5\) are defined with Dirichlet conditions on the boundary of \(\mathcal{B}\), but Neumann conditions lead, in fact, to the same results in the large \(\Omega\) limit.

For \(s = \infty\) Eq. \(2.10\) resp. \(2.11\) reduces to \(g^{-1/2}\Omega\) resp. \(\gamma = \Omega^{-2} g\) and the TF energy scales as \(g^{-1} E_{0,1}^{\mathrm{TF}} = E_{1,\omega,\sigma}^{\mathrm{TF}},\) resp. \(\Omega^{-2} E_{0,1}^{\mathrm{TF}} = E_{1,\omega,\sigma}^{\mathrm{TF}}\). Theorem 2.1 becomes in case (i) \(\{ g^{-1} N^{-1} E_{0,1}^{\mathrm{QM}} (N) \} \to E_{1,0}^{\mathrm{TF}}\), in case (ii) \(\{ g^{-1} N^{-1} E_{0,1}^{\mathrm{QM}} (N) \} \to E_{1,0}^{\mathrm{TF}}\), and in case (iii) \(\{ \Omega^{-2} N^{-1} E_{0,1}^{\mathrm{QM}} (N) \} \to E_{1,0}^{\mathrm{TF}}\).

The proofs require only some minor modifications. For instance, in the proofs of Theorem 3.1 and Propositions 4.2–4.3 the trial functions have to be modified in order to take the boundary condition into account. In the case of ultrarapid rotations it has to be noted that the set \(\mathcal{M}\) is now a subset of the boundary \(\partial \mathcal{B}\) of \(\mathcal{B}\), consisting of the points on \(\partial \mathcal{B}\) where the centrifugal potential \(\sigma g^{s/(s+3)} f(g^{-1/(s+3)} \vec{x})\) is minimal, i.e., where \(r\) is maximal. If the boundary is smooth one can still use a Taylor expansion for the proof of an analogue of Theorem 3.3 (with different error terms).
Appendix A  The TF density at large rotational velocities

The TF density is explicitly given by (2.13), i.e.,
\[ \rho_{\gamma,\Omega}^{\text{TF}}(\vec{x}) = \frac{1}{2g} \left[ \mu_{\gamma,\Omega}^{\text{TF}} + \frac{1}{2} \Omega^2 r^2 - V(\vec{x}) \right]_+ . \] (A.1)

To get a picture how it changes with the parameters, in particular as \( \omega = g^{-(s-2)/(2s+6)} \to \infty \), it is convenient to use the scaling
\[ \Omega^{6/(s-2)} \rho_{\gamma,\Omega}^{\text{TF}} \left( \Omega^{2/(s-2)} \vec{x} \right) = \rho_{\gamma,1}^{\text{TF}}(\vec{x}) \] (A.2)
to eliminate the dependence of the potential on \( \Omega \) and consider
\[ \rho_{\gamma,1}^{\text{TF}}(\vec{x}) = \frac{1}{2\gamma} \left[ \mu_{\gamma,1}^{\text{TF}} + \frac{1}{2} r^2 - V(\vec{x}) \right]_+ \] (A.3)
with \( \gamma = \omega^{-2(s+3)/(s-2)} \). As \( \omega \) increases from 0 to \( \infty \), \( \gamma \) decreases from \( +\infty \) to zero and, due to the normalization, the chemical potential \( \mu_{\gamma,1}^{\text{TF}} \) decreases monotonically from \( +\infty \) to \( E_{0,1}^{\text{TF}} = \min \{ V(\vec{x}) - \frac{1}{4} r^2 \} \).

Since \( V \) is homogeneous of order \( s > 2 \) it is clear that \( E_{0,1}^{\text{TF}} < 0 \). By continuity there is a \( \gamma_c \) (and a corresponding \( \omega_c \)) such that \( \mu_{\gamma,1}^{\text{TF}} = 0 \). Explicitly,
\[ \gamma_c = \frac{1}{s} \int \left[ \frac{1}{2} r^2 - V(\vec{x}) \right]_+ \, d\vec{x} . \] (A.4)

Since \( V(0) = 0 \) and \( V \) is continuous, we have \( \rho_{\gamma,1}^{\text{TF}}(\vec{x}) > 0 \) in a neighborhood of 0 for \( \gamma \to \gamma_c \) (i.e., \( \omega \to \omega_c \)). For \( \gamma < \gamma_c \), on the other hand, \( \rho_{\gamma,1}^{\text{TF}}(\vec{x}) = 0 \) in a cylinder around the \( z \) axis, because \( \mu_{\gamma,1} < 0 \) and \( V \geq 0 \). In other words, for \( \gamma < \gamma_c \) (i.e., \( \omega < \omega_c \)), the centrifugal force creates a ‘hole’ of radius \( \geq 2\sqrt{\mu_{\gamma,1}} \) in the density.

To describe this in a little more detail we introduce cylindrical coordinates \( (r, z, \vartheta) \) and consider the density as a function of \( r \) at fixed \( (z, \vartheta) \), first for \( z = 0 \). Using homogeneity of \( V \) we see that the boundary of the support of \( \rho_{\gamma,1} \) is determined by the solutions to the equation
\[ a(\vartheta) r^s - r^2 / 4 = \mu_{\gamma,1}^{\text{TF}} \] (A.5)
with \( a(\vartheta) = V(1, 0, \vartheta) > 0 \). If \( \gamma > \gamma_c \) (i.e., \( \omega < \omega_c \)) so \( \mu_{\gamma,1}^{\text{TF}} > 0 \), the equation has one solution, \( r(\vartheta)^+_0 > 0 \), and \( \rho_{\gamma,1}^{\text{TF}}(r, 0, \vartheta) > 0 \) for \( r < r(\vartheta)^+_0 \) but \( \rho_{\gamma,1}^{\text{TF}} = 0 \) outside. Eq. (A.5) has two solutions, \( r(\vartheta)^+_0 \), if \( \gamma_c > \gamma > \gamma_0 \) (i.e., \( \omega_c < \omega < \omega_0 \)) where \( \gamma_0 \geq 0 \) is determined by solving (A.5) together with
\[ sa(\vartheta) r^{s-1} - r^2 / 2 = 0 . \] (A.6)

The density \( \rho_{\gamma,1}^{\text{TF}}(r, 0, \vartheta) \) is nonzero for \( r \) inside the interval \( (r(\vartheta)^-_0, r(\vartheta)^+_0) \) but vanishes outside. For \( \gamma = \gamma_0 \) the interval shrinks to a point with radial coordinate
\[ r(\vartheta)^{\text{lim}} = (2sa(\vartheta))^{-1/(s-2)} = \sup_{\gamma \geq \gamma_0} r(\vartheta)^-_0 = \inf_{\gamma \geq \gamma_0} r(\vartheta)^+_0 \] (A.7)
The chemical potential \( \mu_{\gamma,1}^{\text{TF}} \), and hence \( \gamma_0 \), is determined by inserting (A.7) into (A.5). If \( \gamma_0 > 0 \), then \( \rho_{\gamma,1}^{\text{TF}} \) is identically zero in the \( \vartheta \) direction for \( \gamma_0 \geq \gamma \geq 0 \). If \( \gamma_0 = 0 \), on the other hand, then (A.5) implies that \( (r^0, \vartheta, 0) \in \mathcal{M} \) and thus \( r(\vartheta)^{\text{lim}} = r_0 \) by Lemma 5.2.

For \( z \neq 0 \) we can use the homogeneity of \( V \) to write \( V(\vec{x}) = r^s V(1, z/r, \vartheta) \) and apply the considerations above to a ray with fixed \( z/r \) and \( \vartheta \). It is, however, more interesting to consider what happens at fixed \( z \neq 0 \) in the case that \( V \) is monotonically increasing in \( |z| \), for instance if \( V(\vec{x}) = V_0(\vec{r}) + c|z|^{3s} \). At fixed \( z \) the term \( c|z|^{3s} \) acts as a shift of the chemical potential, \( \mu_{\gamma,1}^{\text{TF}} - \mu_{\gamma,1}^{\text{TF}} \). Hence for a ‘hole’ to appear at \( z > 0 \) it is not necessary that \( \mu_{\gamma,1}^{\text{TF}} \) becomes negative, it appears already when \( \mu_{\gamma,1}^{\text{TF}} < 0 \). For any \( \gamma < \infty \) the density has therefore a ‘hole’ for sufficiently large \( |z| \), the width increasing with \( |z| \). As \( \gamma \) decreases from an initial value \( > \gamma_c \) the ‘hole’ moves down to lower values of \( z \), reaching \( z = 0 \) at \( \gamma = \gamma_c \).
As an example, consider \( V(r, z, \vartheta) = r^s(1 - \varepsilon \sin^2 \vartheta) + |z|^{s} \) with \( 0 < \varepsilon < 1 \). Here \( \gamma_{\vartheta} > 0 \) for all \( \vartheta \in [0, 2\pi) \) except for \( \vartheta = 0 \) and \( \vartheta = \pi \). As \( \gamma \to 0 \) the density converges to \( \left\{ \frac{1}{2}\delta(x - 1) + \frac{1}{2}\delta(x + 1) \right\} \delta(y)\delta(z) \). If \( \varepsilon = 0 \), i.e., \( V = r^s + |z|^s \), then \( \gamma_{\vartheta} = 0 \) for all \( \vartheta \) and the limiting density for \( \gamma \to 0 \) is \( (2\pi)^{-1}\delta(r - 1)\delta(z) \).

**Acknowledgments.** This work was supported by an Austrian Science Fund (FWF) grant P17176-N02. JY would like to thank the Niels Bohr International Academy and Nordita, Copenhagen, for hospitality and Chris Pethick, Gentaro Watanabe and Gordon Baym for discussions.

**References**

[A] A. Aftalion, *Vortices in Bose-Einstein Condensates*, Birkhäuser, 2006.

[A1] A. Aftalion, T. Rivière Vortex energy and vortex bending for a rotating Bose-Einstein condensate, Phys. Rev. A 64 (2001), 043611.

[B] G. Baym, Rapidly Rotating Bose-Einstein Condensates, *J. Low Temp. Phys.* 138 (2005), 601-610.

[BP] G. Baym, C.J. Pethick, Vortex Core Structure and Global Properties of Rapidly Rotating Bose-Einstein Condensates, Phys. Rev. A 69 (2004), 043619.

[CRY1] M. Correggi, T. Rindler-Daller, J. Yngvason, Rapidly Rotating Bose-Einstein Condensates in Strongly Anharmonic Traps, ArXiv math-ph/0606058, *J. Math. Phys.* (in press).

[CRY2] M. Correggi, T. Rindler-Daller, J. Yngvason, Rapidly Rotating Bose-Einstein Condensates in Homogeneous Traps. Preprint

[D] F. Dyson, Ground-State Energy of a Hard Sphere Gas, Phys. Rev. (1957), 20-26.

[ECHSC] P. Engels, I. Coddington, P.C. Haljan, V. Schweikhardt, E.A. Cornell, Observation of Long-lived Vortex Aggregates in Rapidly Rotating Bose-Einstein Condensates, Phys. Rev. Lett. 90 (2003), 170405.

[F] A.L. Fetter, Rotating Vortex Lattice in a Bose-Einstein Condensate Trapped in Combined Quadratic and Quartic Radial Potentials, Phys. Rev. A 64 (2001), 063608.

[FB] U.R. Fischer, G. Baym, Vortex States of Rapidly Rotating Dilute Bose-Einstein Condensates, Phys. Rev. Lett. 90 (2003), 140402.

[FZ] H. Fu, E. Zaremba, Transition to the Giant Vortex State in a Harmonic-plus-Quartic Trap, Phys. Rev. A 73 (2006), 013614.

[G] R. Griffiths, A Proof that the Free Energy of a Spin System is Extensive, J. Math. Phys. 5 (1964), 1215.

[KTU] K. Kasamatsu, M. Tsubota, M. Ueda, Giant Hole and Circular Superflow in a Fast Rotating Bose-Einstein Condensate, Phys. Rev. A 66 (2002), 053606.

[KB] G.M. Kavoulakis, G. Baym, Rapidly Rotating Bose-Einstein Condensates in Anharmonic Potentials, New J. Phys. 5 (2003), 51.

[L1] E.H. Lieb, The Stability of Matter: From Atoms to Stars, Bull. Am. Math. Soc. 22 (1990), 1-49.

[L2] E.H. Lieb, Thomas-fermi and related theories of atoms and molecules, Rev. Mod. Phys. 53 (1981), 603-641.

[LL] E.H. Lieb, M. Loss, *Analysis (2nd. ed.),* American Mathematical Society (2001).

[LSe] E.H. Lieb, R. Seiringer, Derivation of the Gross-Pitaevskii Equation for Rotating Bose Gases, Commun. Math. Phys. 264 (2006), 505-537.
E.H. Lieb, B. Simon, The Thomas-Fermi theory of atoms, molecules and solids, *Adv. Math.* **23** (1977), 22-116.

E.H. Lieb, R. Seiringer, J. Yngvason, Bosons in a Trap: A Rigorous Derivation of the Gross-Pitaevskii Energy Functional, *Phys. Rev. A* **61** (2000), 0436021-13.

E.H. Lieb, R. Seiringer, J. Yngvason, A Rigorous Derivation of the Gross-Pitaevskii Energy Functional for a Two-dimensional Bose Gas, *Commun. Math. Phys.* **224** (2001), 17-31.

E.H. Lieb, R. Seiringer, J. Yngvason, One-Dimensional Bosons in Three-Dimensional Traps, *Phys. Rev. Lett.* **91** (2003), 1504011-4.

E.H. Lieb, R. Seiringer, J. Yngvason, One-Dimensional Behavior of Dilute, Trapped Bose Gases, *Commun. Math. Phys.* **244** (2004), 347-393.

E.H. Lieb, R. Seiringer, J.P. Solovej, J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, Oberwolfach Seminars **34**, Birkhäuser (2005).

E.H. Lieb, J. Yngvason, Ground State Energy of the Low Density Bose Gas, *Phys. Rev. Lett.* **80** (1998), 2504-2507.

E. Lundh, Multiply Quantized Vortices in Trapped Bose-Einstein Condensates, *Phys. Rev. A* **65** (2002), 043604.

K. Schnee, J. Yngvason, Bosons in Disc-Shaped Traps: From 3D to 2D, *Commun. Math. Phys.* **269** (2007), 659-691.

R. Seiringer, Ground State Asymptotics of a Dilute, Rotating Gas, *J. Phys. A: Math. Gen.* **36** (2003), 9755-9778.

R. Seiringer, Dilute, Trapped Bose Gases and Bose-Einstein Condensation, in *Large Coulomb Systems*, Lect. Notes Phys. **695**, 251-276, J. Dereziński, H. Siedentop, eds., Springer (2006).

G. Watanabe, G. Baym, S.A. Gifford, C.J. Pethick, Global Structure of vortices in rotating Bose-Einstein condensates, *Phys. Rev. A*, **74** (2006), 063621.