Inflaton in anisotropic higher derivative gravity

W. F. Kao

Institute of Physics, Chiao Tung University, Hsinchu, Taiwan

Existence and stability analysis of the Kantowski-Sachs type inflationary universe in a higher derivative scalar-tensor gravity theory is studied in details. Isotropic de Sitter background solution is shown to be stable against any anisotropic perturbation during the inflationary era. Stability of the de Sitter space in the post inflationary era can also be realized with proper choice of coupling constants.

PACS numbers: 98.80.-k, 04.50.+h

I. INTRODUCTION

Our universe is known to be homogeneous and isotropic to a very high degree of precision [1, 2]. Such an universe can be described by the well known Friedmann-Robertson-Walker (FRW) metric [3]. There had been, however, some cosmological problems associated with the standard big bang model responsible for the evolution of our present universe. Inflationary models resolves many problems including the flatness, monopole, and horizon problem [4].

Moreover, gravitational physics is expected to be different from the Einstein-Hilbert models near the Planck scale [5, 6]. For example, quantum gravity or string corrections indicate that higher derivative terms could have some interesting cosmological applications [5]. In addition, higher derivative terms also arise as the quantum corrections of the matter fields [7]. Therefore, the possibility of deriving inflation from higher derivative corrections have been an focus of research interest [7, 8, 9, 10, 11].

Recently, there are also growing interests in the study of Kantowski-Sachs (KS) type anisotropic spaces [17, 18, 19]. We will hence try to study the problem of existence and stability associated with the inflationary solution an de Sitter final state. In particular, we will study the effects of the higher derivative terms in Kantowski-Sachs spaces. In fact, a large class of pure gravity models and induced gravity models with inflationary KS/FRW solutions was presented in Ref. [20]. Any KS type solution leading itself to an asymptotic FRW final state will be referred to as the KS/FRW solution in this paper for convenience.

It is shown that the stability of the de Sitter background space is closely related to the choice of the coupling constants [20]. Indeed, the pure gravity model given below:

$$L_g = -R - \alpha R^2 - \beta R_{\mu\nu} R^{\mu\nu} + \gamma R^{\mu\nu} R^{\rho\sigma} R_{\rho\sigma}^{\mu\nu}$$ (1)

admits an inflationary solution with a constant Hubble parameter given by $H_0^4 = 1/4\gamma$. This will requires that $\gamma > 0$. Here $\alpha$, $\beta$, and $\gamma$ are coupling constants. This shows that (a) the $\gamma$ factor determines the scale of the inflation characterized by the Hubble parameter $H_0$ and (b) $\alpha$ and $\beta$ factors are irrelevant to the scale $H_0$ in the de Sitter phase. The quadratic terms are, however, important to the stability of the de Sitter phase.

Indeed, perturbing the KS type metric with $H_i \rightarrow H_0 + \delta H_i$, we can show that

$$\delta H_i = c_i \exp\left[\frac{-3H_0 t}{2}(1 + \delta_1)\right] + d_i \exp\left[\frac{-3H_0 t}{2}(1 - \delta_1)\right]$$ (2)

for

$$\delta_1 = \sqrt{1 + 8/[27 - 9(6\alpha + 2\beta)H_0^2]}$$ (3)
and some arbitrary constants $c_i, d_i$ to be determined by the initial perturbations. Here $H_i \equiv \dot{a}_i/a_i$ with $a_i(t)$ the scale factor in $i$-direction. We will describe the notation shortly in section II. It is easy to see that any small perturbation $\delta H_i$ will be stable against the de Sitter background if both modes characterized by the exponents

$$\Delta_\pm \equiv -[3H_0^2/2][1 \pm \delta_1]$$

(4)

are all negative. This will happen if $\delta_1 < 1$. In such case, the inflationary de Sitter space will remain a stable background as the universe evolves.

It can be shown that the stability equation for the anisotropic KS space and the stability equation for the isotropic FRW space in the presence of the same inflationary de Sitter background turns out to be identical $\text{[11, 12, 13]}$. Therefore, the stability of isotropic perturbations also ensures the stability of the anisotropic perturbations. The stability of the isotropic perturbations for the FRW space is important for any physical models. Unfortunately, inflationary models that are stable against any isotropic perturbations will have problem with the graceful exit process. Therefore, the pure gravity model may have troubles dealing with the stability and exit mechanism all together.

Instead of the pure gravity theory, a slow rollover scalar field may help resolving this problem. An inflationary de Sitter solution in a scalar-tensor model is expected to have one stable mode (against the perturbation in $\delta H_i$ direction) and one unstable mode (against the perturbation in $\delta \delta$ direction). As a result, the inflationary era will come to an end once the unstable mode takes over after a brief period of inflationary expansion. Therefore, we propose to study the effect of such theory.

In particular, we will show in this paper that the roles played by the higher derivative terms are dramatically different in the inflationary phase of our physical universe in both pure gravity theory and scalar-tensor theory. First of all, third order term will be shown to determine the expansion rate $H_0$ for the inflationary de Sitter space. The quadratic terms will be shown to have nothing to do with the expansion rate of the background de Sitter space. They will however affect the stability condition of the de Sitter phase. Their roles played in the existence and stability condition of the evolution of the de Sitter space are dramatically different.

II. NON-REDUNDANT FIELD EQUATION AND BIANCHI IDENTITY IN KS SPACE

Given the metric of the following form:

$$ds^2 = -dt^2 + c^2(t)dr^2 + a^2(t)(d^2\theta + f^2(\theta)d\varphi^2)$$

(5)

with $f(\theta) = (\theta, \sinh \theta, \sin \theta)$ denoting the flat, open and close anisotropic space known as Kantowsky-Sachs type anisotropic spaces. More specifically, Bianchi I (BI), III(BIII), and Kantowski-Sachs (KS) space corresponds to the flat, open and closed model respectively. This metric can be rewritten as

$$ds^2 = -dt^2 + a^2(t)((d\varphi^2 (1 - kr^2) + r^2 d\theta^2) + a_z^2(t)dz^2$$

(6)

with $r, \theta$, and $z$ read as the polar coordinates and $z$ coordinate for convenience and for easier comparison with the FRW metric. Note that $k = 0, 1, -1$ stands for the flat, open and closed universes similar to the FRW space.

Writing $H_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu}$, Einstein equation can be written as $D_\mu H^{\mu\nu} = 0$ incorporating the Bianchi identity $D_\mu G^{\mu\nu} = 0$ and the energy momentum conservation $D_\mu T^{\mu\nu} = 0$. Here $G^{\mu\nu}$ and $T^{\mu\nu}$ represent the Einstein tensor and the energy momentum tensor coupled to the system respectively. With the metric (6), it can be shown that the $r$ component of the equation $D_\mu H^{\mu\nu} = 0$ implies that

$$H''_r = H^0_0.$$ 

(7)

This result also says that any matter coupled to the system has the symmetric property $T''_r = T^0_0$. In addition, the equations $D_\mu H^{\mu\theta} = 0$ and $D_\mu H^{\mu z} = 0$ both vanish identically for all kinds of energy momentum tensors. More interesting information comes from the $t$ component of this equation. It says:

$$(\partial_t + 3H)H^t_t = 2H_1H'_r + H_zH'_z.$$ 

(8)

This equation implies that (i) $H'_t = 0$ implies that $H''_r = H'_z = 0$ and (ii) $H'_t = H'_z = 0$ only implies $(\partial_t + 3H)H^t_t = 0$ instead of $H'_t = 0$. Case (ii) can be solved to give $H'_t = \text{constant} \exp[-a_2a_z]$ which approaches zero when $a_2a_z \to \infty$. For the anisotropic KS spaces, the metric contains two independent variables $a$ and $a_z$. The Einstein field equations have, however, three non-vanishing components: $H'_t = 0$, $H'_r = H^0_0 = 0$ and $H'_z = 0$. The Bianchi identity implies
that the $tt$ component is not redundant and will hence be retained for complete analysis. Ignoring either one of the $rr$ or $zz$ components will not affect the final result of the system. In short, the $H^i_t = 0$ equation, known as the generalized Friedman equation, is a non-redundant field equation as compared to the $H^r_r = 0$ and $H^z_z = 0$ equations.

In addition, restoring the $g_{tt}$ component $b^2(t) = 1/B_1$ will be helpful in deriving the non-redundant field equation associated with $G_{tt}$ that will be shown shortly. More specifically, the generalized KS metric will be written as:

$$ds^2 = -b^2(t)dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2d\theta^2) + a_z^2(t)dz^2. \quad (9)$$

In principle, the Lagrangian of the system can be reduced from a functional of the metric $g_{\mu\nu}$, $\mathcal{L}(g_{\mu\nu})$, to a simpler function of $a(t)$ and $a_z(t)$, namely $L(t) \equiv a^2a_z\mathcal{L}(g_{\mu\nu}(a(t), a_z(t)))$. The equation of motion should be reconstructed from the variation of the reduced Lagrangian $L(t)$ with respect to the variable $a$ and $a_z$. The result is, however, incomplete because, the variation of $a$ and $a_z$ are related to the variation of $g_{rr}$ and $g_{zz}$ respectively. The field equation from varying $g_{tt}$ can not be derived without restoring the variable $b(t)$ in advance. This is the motivation to introduce the metric (14) such that the reduced Lagrangian $L(t) \equiv ba^2a_z\mathcal{L}(g_{\mu\nu}(b(t), a(t), a_z(t)))$ retains the non-redundant information of the $H^i_t = 0$ equation. Non-redundant Friedmann equation can be reproduced resetting $b = 1$ after the variation of $b(t)$ has been done.

After some algebra, all non-vanishing components of the curvature tensor can be computed [20]

$$R^{ti}_{\, tij} = \left[\frac{1}{2}B_1H_t + B_1(\dot{H}_t + H^2_t)\right]d_i^j, \quad (10)$$

$$R^{ij}_{\, kli} = B_1H_tH_j \epsilon^{ijm}\epsilon_{klm} + \frac{k}{a^2}\epsilon^{ijz}\epsilon_{klz} \quad (11)$$

with $H_i \equiv (\dot{a}/a, \dot{a}/a, \dot{a}_z/a_z) = (H_1, H_2, H_3)$ for $r, \theta$, and $z$ component respectively.

Given a Lagrangian $L = \sqrt{g}\mathcal{L} = L(b(t), a(t), a_z(t))$, it can be shown that

$$L = \frac{a^2a_z}{\sqrt{B_1}}\mathcal{L}(H_i, \dot{H}_i, a^2). \quad (12)$$

The variational equations for this action can be shown to be: [20]

$$L + H_t(\frac{d}{dt} + 3H)L^i = H_tL_i + \dot{H}_tL^i \quad (13)$$

$$L + (\frac{d}{dt} + 3H)^2L^z = (\frac{d}{dt} + 3H)L^z \quad (14)$$

Here $L_i \equiv \delta\mathcal{L}/\delta H_i$, $L^i \equiv \delta\mathcal{L}/\delta \dot{H}_i$, and $3H \equiv \sum_i H_i$. For simplicity, we will write $\mathcal{L}$ as $L$ from now on in this paper. As a result, the field equations can be written in a more comprehensive form:

$$DL \equiv L + H_t(\frac{d}{dt} + 3H)L^i - H_tL_i - \dot{H}_tL^i = 0 \quad (15)$$

$$D_zL \equiv L + (\frac{d}{dt} + 3H)^2L^z - (\frac{d}{dt} + 3H)L^z = 0 \quad (16)$$

III. HIGHER DERIVATIVE GRAVITY MODEL WITH A SCALAR FIELD

In this section, we will study the higher derivative induced gravity model:

$$L = -R - \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\beta\gamma}^\sigma R^{\beta\gamma}_{\sigma\rho}R^{\sigma\rho}_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^n\phi - V(\phi) \equiv L_g + L_{\phi} \quad (17)$$

with $L_g = -R - \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\beta\gamma}^\sigma R^{\beta\gamma}_{\sigma\rho}R^{\sigma\rho}_{\mu\nu}$, and $L_{\phi} = -\frac{1}{2}\partial_\mu\phi\partial^n\phi - V(\phi)$ denoting the pure gravity terms and the scalar field Lagrangian respectively.

The corresponding Lagrangian can be shown to be:

$$L = 2(2A + B + 2C + D) - 4\alpha \left[4A^2 + B^2 + 4C^2 + D^2 + 4AB + 8AC + 4AD + 4BC + 2BD + 4CD\right] - 2\beta \left[3A^2 + B^2 + 3C^2 + D^2 + 2AB + 2AC + 2AD + 2BC + 2CD\right] + 8\gamma \left[2A^3 + B^3 + 2C^3 + D^3\right] + \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (18)$$
with
\[ A = \dot{H}_1 + H^2, \]  
\[ B = H^2 + \frac{k}{a^2}, \]  
\[ C = H_1 H_z, \]  
\[ D = H_z + H^2. \]  

This Lagrangian can be shown to reproduce the de Sitter models when we set \( H_i \rightarrow H_0 \) in the isotropic limit. The Friedmann equation reads:
\[ DL_g = \frac{1}{2} \dot{\phi}^2 + V(\phi) \]  
for the induced gravity model. In addition, the scalar field equation can be shown to be:
\[ \ddot{\phi} + 3H_0 \dot{\phi} + V' = 0. \]  

The leading order de Sitter solution with \( \phi = \phi_0 \) and \( H_i = H_0 \) for all directions can be shown to be:
\[ V_0 \equiv V(\phi_0) = 6[1 - 4\gamma H_0^4]H_0^2. \]  

Note that the coupling constants \( \alpha \) and \( \beta \) do not affect the strength of inflation determined by the Hubble constant \( H_0 \). This polynomial equation can be written as an cubic equation in terms of the variable \( x = H_0^2 \):
\[ f(x) \equiv x^3 - \frac{1}{4\gamma} x + \frac{V_0}{24\gamma} = 0. \]  

It can be shown that the cubic polynomial \( f(x) \) attains its maximum and minimum at \( x = x_M \) and \( x = x_m \) respectively with \( x_m = -x_M = 1/[2\sqrt{3} \gamma] \). Plotting \( C \equiv (x, y = f(x)) \) as a curve \( C \) on the \( x-y \) plane, it is easy to show that the curve \( C \) intersects with the \( y \)-axis at the point \( Y = (0, V_0/[24\gamma]) \).

If \( f(y) \) attains its maximum and minimum at the points \( M = (x_M, f(x_M)) \) and \( N = (x_m, f(x_m)) \) respectively, it can be shown that \( f(x) = \pm x \) as \( x \rightarrow \pm \infty \). Therefore the function \( f(x) \rightarrow \pm \infty \) as \( x \rightarrow \pm \infty \). Therefore curve \( C \) will have two intersection points \( x_+ = (x_+, 0) \) and \( x_- = (x-, 0) \) with the positive \( x \)-axis if the the minimum point \( N \) locates at the quadrant IV, or equivalently, \( f(x_m) < 0 \). This implies that the cubic equation \( f(x) = 0 \) has two positive roots if \( 3\gamma V_0^2 < 4 \). There is, however, only one positive root when \( x_+ = x_- \), or equivalently, when \( 3\gamma V_0^2 = 4 \).

Indeed, the cubic equation \( f(x) = 0 \) can be solved by using the triple angle formula of cosine function:
\[ \cos[3\theta] = 4 \cos^3 \theta - 3 \cos \theta. \]  

As a result, the solutions to the cubic function can be shown to be
\[ x_1 = -\left(\frac{1}{3\gamma}\right)^{1/2} \cos \left(\frac{\theta_0}{3}\right), \]  
\[ x_\pm = \left(\frac{1}{3\gamma}\right)^{1/2} \cos \left(\frac{\theta_0 \pm \pi}{3}\right). \]  

with \( \theta_0 \) defined by the identity \( \cos \theta_0 \equiv \sqrt{3\gamma} V_0/2 \leq 1 \). The notation \( x_\pm \) is defined according to its orientation with respect to the \( x \)-coordinate, i.e. \( x_- \leq x_m \leq x_+ \). These solutions can be easily converted to the well known solutions when \( 3\gamma V_0^2 \geq 4 \) remains valid.

Note that \( H_0 = x_\pm^{1/2} > 0 \) can have two different choices as long as \( 3\gamma V_0^2 < 4 \) is satisfied. These two solutions become identical to each other when \( 3\gamma V_0^2 = 4 \), or equivalently, when \( \cos \theta_0 = 1 \). It is also straightforward to verify that \( x_i \) given above does solve the cubic polynomial equation \( f(x) = 0 \) in consistent with the constraint: \( 3\gamma V_0^2 \leq 4 \). In addition, \( x_1 \) can not be a physical solution because it is negative.

IV. STABILITY OF HIGHER DERIVATIVE INFLATIONARY SOLUTION

Our universe could start out anisotropic and evolves to the present highly isotropic state in the post inflationary era. Therefore, a stable KS/FRW solution is necessary for any physical model of our universe.
Given an effective action of the sort described by Eq. (15), the scale factor $H_0$ is determined by the solution $H_0^\pm = x_\pm$ in the presence of the de Sitter solution $H_\pm = H_0$ and the static condition $\phi = \phi_0$. In addition to these constraints, small perturbations, $H_\pm = H_0 + \delta H_j$ and $\phi = \phi_0 + \delta \phi$, against the background de Sitter solution $(H_0, \phi_0)$ may also put a few more constraints to the system. This perturbation will enable us to understand whether the background solution is stable or not.

It can be shown that the perturbation equation of the Bianchi models are identical to the perturbation equation of the FRW models. Therefore, any inflationary solutions with a stable mode and an unstable mode could provide us with a natural resolution to problem of graceful exit. Such models will, however, also be unstable against anisotropic perturbations. Therefore, any inflationary solutions with a stable mode and an unstable mode is also negative to our search for a stable and isotropic inflationary model. It will be shown shortly that the higher derivative gravity theory with an inflaton scalar field could hopefully resolve this problem all together.

In practice, perturbing the background de Sitter solution along the $\delta H_i$ direction should be stable for at least a brief moment such that around 60 e-fold inflation can be induced before the de Sitter phase collapses. And the resulting universe should also be stable against isotropic and anisotropic perturbations. Therefore, the perturbation along the $\delta \phi$ direction is expected to be unstable favoring the system for a natural mechanism of graceful exit. Hence, we will try to study the stability equations of the system for small perturbations against the de Sitter background solutions in this section.

The first order perturbation equation for $DL$, with $H_\pm \rightarrow H_0 + \delta H$, can be shown to be [20]:

$$
\delta(DL) = <H_iL^{ij}\delta H_j> + 3H <H_iL^{ij}\delta H_j> + \delta <H_iL^i> + 3H <(H_iL^i + L^i)\delta H_j> \\
+ <H_iL^i> \delta(3H) - <H_iL^i> \delta H_j
$$

(30)

for any $DL$ defined by Eq. (15) with all functions of $H_i$ evaluated at some FRW background $H_i = H_0$. The notation $<A_iB_i> \equiv \sum_{i=1,z} A_iB_i$ denotes the summation over $i = 1$ and $z$ for repeated indices. Note that we have absorbed the information of $i = 2$ into $i = 1$ since they contribute equally to the field equations in the KS type spaces. In addition, $L^i_j \equiv \delta^i_j \delta H_i\delta H_j$ and similarly for $L^i_i$ and $L^i_j$ with upper index $i$ and lower index $j$ denoting variation with respect to $H_i$ and $H_j$ respectively for convenience. In addition, perturbing Eq. (16) can also be shown to reproduce the Eq. (30) in the de Sitter phase [20].

In addition, it can be shown that

$$
<H_iL^{i1}> = 2 <H_iL^{iz}>
$$

(31)

$$
<H_iL^i_1> = 2 <H_iL^i_2>
$$

(32)

$$
L^1 = 2L^2
$$

(33)

$$
<H_iL^i_{i1}> = 2 <H_iL^i_{iz}>
$$

(34)

in the inflationary de Sitter background with $H_0$ = constant. Therefore, the stability equations (30) can be greatly simplified. For convenience, we will define the operator $DL$ as

$$
DL\delta H \equiv <H_iL^{i1}> \delta H + 3H <H_iL^{i1}> \delta H + 3H <H_iL^i_1 + L^i_1> \delta H + 2 <H_iL^i_1> \delta H - <H_iL^i_{i1}> \delta H.
$$

(35)

As a result, the stability equation (30) becomes

$$
\delta(DL) = DL(\delta H_1 + \delta H_2/2) = \frac{3}{2} DL(\delta H)
$$

(36)

with $H = (2H_1 + H_2)/3$ as the average of $H_i$.

Hence the leading order perturbation equation in $\delta H$ and $\delta \phi$ of the Friedmann equation can be shown to be:

$$
\delta(DL_0) = \frac{3}{2} DL_0\delta H = V'\delta\phi
$$

(37)

with $DL_0\delta H \equiv DL_0\delta H$ as short-handed notations. This equation can further be shown to be:

$$
12H_0 \left\{ 2 [6\gamma H_0^2 - 3\alpha - \beta] (\delta \dot{H} + 3H_0\delta H) + [1 - 12\gamma H_0^2] \delta H \right\} = V_0''\delta \phi.
$$

(38)

Similarly, the leading perturbation of the scalar field equation can be shown to be:

$$
\ddot{\delta \phi} + 3H_0\delta \dot{\phi} + V_0''\delta \phi = 0
$$

(39)
The variational equation of $a_\pm$ can also be shown explicitly to be redundant in the limit $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta\phi$ following the Bianchi identity.

Assuming that $\delta H = \exp[hH_0t]\delta H_0$ and $\delta\phi = \exp[pH_0t]\delta\phi_0$ for some constants $h$ and $p$, one can write above equations as:

$$12H_0 \left\{ 2 \left[ 6\gamma H_0^2 - 3\alpha - \beta \right] H_0^2 (h^2 + 3h)\delta H + \left[ 1 - 12\gamma H_0^4 \right] H_0^2 \delta H \right\} = V'_0 \delta\phi. \quad (40)$$

$$H_0^2 (p^2 + 3p + \frac{V''_0}{H_0^2}) \delta\phi = 0. \quad (41)$$

These equations are consistent when all coefficients vanish simultaneously. This implies that $V'_0 = 0$ and

$$h^2 + 3h + \frac{1 - 12\gamma H_0^2}{2H_0^2 \left[ 6\gamma H_0^2 - 3\alpha - \beta \right]} = 0, \quad (42)$$

$$(p^2 + 3p + \frac{V''_0}{H_0^2}) = 0. \quad (43)$$

As a result, the solutions to $p$ are $p = p_\pm = -3/2 \pm \sqrt{9 - 4V''_0/H_0^2}/2$. Since $V'_0 = 0$, the scalar field equation can be solved in the de Sitter background $H_i = H_0$. Indeed, the solution to the equation $\frac{\delta\phi}{H_0} = 0$ is:

$$(\delta\phi)_0 \equiv \phi_0 + \frac{\delta\phi}{3H_0} [1 - \exp(-3H_0t)]. \quad (44)$$

This result indicates that the scalar field does change very slowly similar to the slow roll-over assumption in various scalar field models.

An appropriate effective spontaneously symmetry breaking (SSB) potential $V$ of the following form

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 + V_m \quad (45)$$

with arbitrary coupling constant $\lambda$ can be shown to be a good candidate for our physical universe. Here $V_m$ is a small cosmological constant dressing the SSB potential. The local extremum of this effective potential can be shown to be $\phi = 0$ (local maximum) and $\phi = v$ (local minimum). We expect that the scalar field starts off from the initial state $\phi = 0$ and rolls slowly down to its local minimum locating at $\phi = v$.

When the scalar field eventually rolls down to the minimum of $V$ at $\phi = v$, the system will oscillate around this local minimum with a friction term related to the effective Hubble constant $H_m$ at this stage. Reheating process is expected to take away the kinetic energy of the scalar field. The scalar field will eventually becomes a constant background field losing all its dynamics.

The value of $H_0$ can be chosen to induce enough inflation for a brief moment as long as the slowly changing scalar field remains close to the initial state $\phi = \phi_0 = 0$. The de Sitter phase will hence remain valid and drive the inflationary process for a brief moment determined by the decaying speed of the scalar field.

In addition, at the initial stage,

$$p_\pm = -3/2 \pm \sqrt{9 - 4\lambda v^2/H_0^2}/2 \quad (46)$$

for the SSB $\phi^4$ potential model. Therefore, it is easy to prove that $p_+ > 0$ and $p_- < 0$ indicating an unstable mode and a stable mode do exist when we perturb the scalar field in this model. Hence the system will be unstable against the perturbation of the scalar field consistent with the slowly changing scalar field solution [14].

On the other hand, the solutions to the $h$-equation are $h = h_\pm = -3(1 \pm \delta_2)/2$ with

$$\delta_2^2 = 1 + 2(12\gamma H_0^4 - 1)/\{9H_0^2(6\gamma H_0^2 - 3\alpha - \beta)\} \quad (47)$$

indicating that $h_+ < 0$. Therefore, the $h$-perturbation can have two stable modes only when $h_- < 0$ or $\delta_2^2 < 0$. This implies that either

(a) $1 < 12\gamma H_0^4 < 2(3\alpha + \beta)H_0^2$ \quad (48)

or

(b) $2(3\alpha + \beta)H_0^2 < 12\gamma H_0^4 < 1 \quad (49)$

has to hold. Let us define the function $\Gamma_\pm = 12\gamma H_0^4 - 1$ with $H_0^2 \equiv x_\pm = [\cos(\theta \mp \pi)/3]/\sqrt{37}$ for arbitrary $\theta$. $\Gamma_\pm$ function can be written as

$$\Gamma_\pm = \pm 4\sin \frac{\theta}{3} \cos \frac{\theta}{3} \mp \frac{\pi}{6}. \quad (50)$$
Therefore, it is easy to show that $\Gamma_{+} \geq 0$ and $\Gamma_{-} \leq 0$ for all $0 \leq \theta \leq \pi/2$. As a result, solution $x_{+}$ is consistent only with the constraint (a) in Eq. (49). Similarly, the solution $x_{-}$ is consistent only with the constraint (b) in Eq. (49).

In addition, the equation for the perturbation on anisotropic KS type space and isotropic FRW space are the same. Therefore, the stable $h$-perturbation also ensures that isotropy of the de Sitter space can be made stable against any small anisotropic perturbation $H_{i} = H_{m} + 6H_{i}$.

Once the scalar field falls close to its local minimum at $\phi = v$, the scalar field will start to oscillate around this local minimum. The Hubble parameter at this stage will read $H_{i} = H_{m}$ with $H_{m}$ given by the solution to the leading order Friedmann equation at this stage with $\phi = v$ and $H_{i} = H_{m}$:

$$V_{m} \equiv V(v) = 6(1 - 4\gamma H_{m}H_{m}^{2}).$$

(51)

The cubic equation in $x = H_{m}^{2}$ also reads:

$$g(x) = x^{3} - \frac{1}{4\gamma} x + \frac{V_{m}}{24\gamma} = (x - x_{2})(x - x_{2+})(y - x_{2-}) = 0,$$

(52)

with

$$x_{2} = -\left(\frac{1}{3\gamma}\right)^{1/2} \cos \left(\frac{\theta_{m}}{3}\right),$$

(53)

$$x_{2\pm} = \left(\frac{1}{3\gamma}\right)^{1/2} \cos \left(\frac{\theta_{m} \pm \pi}{3}\right).$$

(54)

and $\theta_{m}$ defined by the identity $\cos \theta_{m} = \sqrt{3\gamma} V_{m}/2 \leq 1$. Note again that only positive roots $x_{2\pm}$ can be physical solutions to $H_{m}^{2}$. In addition, $H_{0}$ should be much larger than $H_{m}$ in order to generate inflation. Moreover, any physical solution should also respect the fact that $V_{0} > V_{m}$ such that these solutions can be made consistent with the SSB potential discussed here.

Indeed, the function $y_{\pm} \equiv (3\gamma)^{1/2} x_{\pm} \equiv \cos (\theta \pm \pi)/3$ (with $\cos \theta \equiv \sqrt{3\gamma} V/2$) can be shown to be a monotonically increasing/decreasing function in $\theta$. Therefore, when $\theta$ decreases, corresponding to the increasing of $V$, $y_{\pm}$ will decrease/increase accordingly. Hence the relations $H_{m}^{2} \geq H_{0}^{2}$ and $V_{0} > V_{m}$ are consistent if we choose the set of solutions as $(x_{-}, x_{2-})$. Therefore, the Hubble constant $H_{m}$ at the final stage can be chosen to be much smaller than the Hubble constant $H_{0}$ at the inflationary era with proper choice of coupling constants.

We can also choose the set of solutions as $(x_{+}, x_{2+})$. In any case, the initial condition $\theta_{0} \sim 0$ and the final condition $\theta_{m} \sim \pi/2$ can be adjusted with the proper choice of $V_{0}$ and $V_{m}$. Note that $\theta_{0}$ and $\theta_{m}$ can be chosen to be close to $0$ and $\pi/2$ respectively.

In addition, the first order perturbation equations also have two modes: $p = p_{\pm} = -3/2 \pm [9 - 4V_{\prime\prime}/H_{m}^{2}]^{1/2}/2 = -3/2 \pm [9 - 8\lambda^{2}/H_{m}^{2}]^{1/2}/2$ for the SSB $\phi^{4}$ potential model (45). Stability is also ensured even if the discriminant becomes imaginary. Therefore, there is no further constraints to be imposed on the parameters from the $p$-mode perturbation.

There are also two modes for the $h$-equation: $h = h_{\pm} = -3(1 \pm \delta_{h})/2$ with

$$\delta_{h}^{2} = 1 + 2(12\gamma H_{m}H_{m}^{4} - 1)/\{9H_{m}^{2}[6\gamma H_{m}^{2} - 3\alpha - \beta]\}.$$

(55)

Both $h$-modes must be stable for the existence of a stable isotropic de Sitter space in the post-inflationary era. This implies that either

$$1 < 12\gamma H_{m}H_{m}^{4} < 2(3\alpha + \beta)H_{m}^{2}$$

(56)

or

$$2(3\alpha + \beta)H_{m}^{2} < 12\gamma H_{m}H_{m}^{4} < 1$$

(57)

has to hold. Note that constraint (a) and constraint (b) are organized in a similar way for both $H_{0}$ and $H_{m}$. This will make our analysis and comparison more easily.

If we take the set of solutions as $(x_{-}, x_{2-})$ consistent with $H_{0} \gg H_{m}$, both $H_{0}$ and $H_{m}$ should obey the constraint (b). Indeed, Eqs. (49) imply that $12\gamma H_{0}^{4} \leq 1$ and $12\gamma H_{m}^{4} \leq 1$. Similarly, if we take the set of solutions as $(x_{+}, x_{2+})$, it can be shown that $12\gamma H_{0}^{4} \geq 1$ and $12\gamma H_{m}^{4} \leq 1$. Therefore, we should choose constraint (a) for $H_{0}$ and constraint (b) for $H_{m}$. Explicit calculation shows, however, that this will lead to inconsistent constraint on the factor $k_{1} \equiv (3\alpha + \beta)/(6\gamma)$. Indeed, the inequality $12\gamma H_{0}^{4} < 2(3\alpha + \beta)H_{0}^{2}$ implies that $H_{0}^{2} < k_{1}$, but the inequality $12\gamma H_{m}^{4} > 2(3\alpha + \beta)H_{m}^{2}$ implies instead that $H_{m}^{2} > k_{1}$. This is inconsistent with the fact that $H_{0} > H_{m}$. Therefore this case can be excluded. As a result, the only consistent set of solutions for $H_{0}$ and $H_{m}$ are derived from the set of solutions $(x_{-}, x_{2-})$.  


Consider the special case where $V_m = 0$. The cubic equation becomes $x(4\gamma x^2 - 1) = 0$. This implies that $4\gamma H^4_m = 1$. Or equivalently, this solution can also be obtained by taking the limit $\theta_m = \pi/2$ in Eqs. (53, 54). Therefore, the solution to $h$ equation is $h = h_\pm = -3(1 \pm \delta_m)/2$ with

$$\delta_m = \sqrt{1 + 8/[27 - 9(6\alpha + 2\beta)H^2_m]}. \quad (58)$$

The situation at this stage is similar to the case of pure gravity model with a vanishing cosmological constant [20].

For the special case $V_m = 0$, we also expect that these two anisotropic perturbation modes against the isotropic background solution $H_i = H_m + \delta H_i$ are both stable modes. As a result, $\Delta_\pm \equiv [-3H_0t/2][1 \pm \delta_m] < 0$ can be shown to be the necessary condition, implying that $\delta_m < 1$ or, equivalently, $(3\alpha + \beta)H^2_m > 35/18$. In particular, both modes become oscillatory triangular functions when $35/18 > (3\alpha + \beta)H^2_m > 3/2$. In both cases, the de Sitter space will remain a stable background as the universe evolves.

V. CONCLUSION

The existence of a stable de Sitter background is closely related to the choices of the coupling constants. The pure higher derivative gravity model with quadratic and cubic interactions [20] admits an inflationary solution with a constant Hubble parameter. Proper choices of the coupling constants allow the de Sitter phase to admit one stable mode and one unstable mode for the anisotropic perturbation.

The stable mode favors a strong inflationary period and the unstable mode provides a natural mechanism for the graceful exit process. It is also found that the perturbation against the isotropic FRW background space and the perturbation against the anisotropic KS type background space obey the same perturbation equations. This is true for both pure and induced gravity models. As a result, the unstable mode in pure gravity model also means that the isotropic de Sitter background is unstable against anisotropic perturbations. Therefore, small anisotropic could be generated during the de Sitter phase for pure gravity model.

We have shown that, for gravity models with an additional inflaton scalar field, stable mode for perturbations along the anisotropic $\delta H_i$ directions do exist with proper constraints to be imposed on the coupling constants. In addition, another unstable mode also exist for the perturbation against the scalar field background $\phi_0$ with proper constraints.

Therefore, the de Sitter background can remain stable and isotropic during the inflationary process for this inflaton gravity models.

Explicit model with a spontaneously symmetry breaking $\phi^4$ potential is presented as an example. Proper constraints are derived for reference. It is shown that the these conditions must hold for the initial de Sitter phase solutions in order to admit a constant background solution during the inflationary phase for the system.

Once the scalar field rolls down to the minimum at $\phi = v$, the system picks up a small Hubble constant $H_m$ characterized by the small cosmological constant $V_m$. We also show that the resulting de Sitter space can be a stable final state with proper choices of coupling constants. Therefore, the scalar-tensor theory proposed in this paper is capable of inducing inflation and maintaining a stable FRW de Sitter background space all together.

In summary, we have shown that a stable mode for (an)isotropic perturbation against the de Sitter background does exist for the gravity model with a scalar field. The problem of graceful exit can rely on the unstable mode of the scalar field perturbation against the constant phase $\phi_0$.

It is also found that the quadratic terms will not affect the the inflationary solution characterized by the Hubble parameter $H_0$ and $H_m$. These quadratic terms play, however, critical role in the stability of the de Sitter background. In addition, it is also interesting to find that their coupling constants $\alpha$ and $\beta$ always show up as a linear combination of $3\alpha + \beta$ in these stability equations. Implications of these constraints deserve more attention for the applications to the inflationary models.

Acknowledgments

This work is supported in part by the National Science Council of Taiwan.

[1] S. Gulkis, P. M. Lubin, S. S. Meyer and R. F. Silverberg, The cosmic background explorer, Sci. Am. 262(1) 122-129 (1990);
[2] D. J. Fixsen, et al, The cosmic microwave background spectrum from the full COBE firs data set, Astrophys. J. 473 (1996) 576-587.
