The Topology of $\text{Out}(F_n)$

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Abstract

We will survey the work on the topology of $\text{Out}(F_n)$ in the last 20 years or so. Much of the development is driven by the tantalizing analogy with mapping class groups. Unfortunately, $\text{Out}(F_n)$ is more complicated and less well-behaved.

Culler and Vogtmann constructed Outer Space $X_n$, the analog of Teichmüller space, a contractible complex on which $\text{Out}(F_n)$ acts with finite stabilizers. Paths in $X_n$ can be generated using “foldings” of graphs, an operation introduced by Stallings to give alternative solutions for many algorithmic questions about free groups. The most conceptual proof of the contractibility of $X_n$ involves folding.

There is a normal form of an automorphism, analogous to Thurston’s normal form for surface homeomorphisms. This normal form, called a “(relative) train track map”, consists of a cellular map on a graph and has good properties with respect to iteration. One may think of building an automorphism in stages, adding to the previous stages a building block that either grows exponentially or polynomially. A complicating feature is that these blocks are not “disjoint” as in Thurston’s theory, but interact as upper stages can map over the lower stages.

Applications include the study of growth rates (a surprising feature of free group automorphisms is that the growth rate of $f$ is generally different from the growth rate of $f^{-1}$), of the fixed subgroup of a given automorphism, and the proof of the Tits alternative for $\text{Out}(F_n)$. For the latter, in addition to train track methods, one needs to consider an appropriate version of “attracting laminations” to understand the dynamics of exponentially growing automorphisms and run the “ping-pong” argument. The Tits alternative is thus reduced to groups consisting of polynomially growing automorphisms, and this is handled by the analog of Kolchin’s theorem (this is one instance where $\text{Out}(F_n)$ resembles $\text{GL}_n(\mathbb{Z})$ more than a mapping class group).

Morse theory has made its appearance in the subject in several guises. The original proof of the contractibility of $X_n$ used a kind of “combinatorial” Morse function (adding contractible subcomplexes one at a time and studying the intersections). Hatcher-Vogtmann developed a “Cerf theory” for graphs. This is a parametrized version of Morse theory and it allows them to prove homological stability results. One can “bordify” Outer Space (by analogy with the Borel-Serre construction for arithmetic groups) to make the action

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of $Out(F_n)$ cocompact and then use Morse theory (with values in a certain ordered set) to study the connectivity at infinity of this new space. The result is that $Out(F_n)$ is a virtual duality group.

Culler-Morgan have compactified Outer Space, in analogy with Thurston’s compactification of Teichmüller space. Ideal points are represented by actions of $F_n$ on $\mathbb{R}$-trees. The work of Rips on group actions on $\mathbb{R}$-trees can be used to analyze individual points and the dynamics of the action of $Out(F_n)$ on the boundary. The topological dimension of the compactified Outer Space and of the boundary have been computed. The orbits in the boundary are not dense; however, there is a unique minimal closed invariant set. Automorphisms with irreducible powers act on compactified Outer Space with the standard North Pole – South Pole dynamics. By first finding fixed points in the boundary of Outer Space, one constructs a “hierarchical decomposition” of the underlying free group, analogous to the Thurston decomposition of a surface homeomorphism.

The geometry of Outer Space is not well understood. The most promising metric is not even symmetric, but this seems to be forced by the nature of $Out(F_n)$. Understanding the geometry would most likely allow one to prove rigidity results for $Out(F_n)$.

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1. Introduction

The aim of this note is to survey some of the topological methods developed in the last 20 years to study the group $Out(F_n)$ of outer automorphisms of a free group $F_n$ of rank $n$. For an excellent and more detailed survey see also [69]. Stallings’ paper [64] marks the turning point and for the earlier history of the subject the reader is referred to [55]. $Out(F_n)$ is defined as the quotient of the group $Aut(F_n)$ of all automorphisms of $F_n$ by the subgroup of inner automorphisms. On one hand, abelianizing $F_n$ produces an epimorphism $Out(F_n) \to Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$, and on the other hand $Out(F_n)$ contains as a subgroup the mapping class group of any compact surface with fundamental group $F_n$. A leitmotiv in the subject, promoted by Karen Vogtmann, is that $Out(F_n)$ satisfies a mix of properties, some inherited from mapping class groups, and others from arithmetic groups. The table below summarizes the parallels between topological objects associated with these groups.

Outer space is not a manifold and only a polyhedron, imposing a combinatorial character on $Out(F_n)$.

2. Stallings’ Folds

A graph is a 1-dimensional cell complex. A map $f : G \to G'$ between graphs is simplicial if it maps vertices to vertices and open 1-cells homeomorphically to open
1-cells. The simplicial map $f$ is a fold if it is surjective and identifies two edges that share at least one vertex. A fold is a homotopy equivalence unless the two edges share both pairs of endpoints and in that case the induced homomorphism in $\pi_1$ corresponds to killing a basis element.

**Theorem 1 (Stallings [63]).** A simplicial map $f : G \to G'$ between finite connected graphs can be factored as the composition

$$G = G_0 \to G_1 \to \cdots \to G_k \to G'$$

where each $G_i \to G_{i+1}$ is a fold and $G_k \to G'$ is locally injective (an immersion). Moreover, such a factorization can be found by a (fast) algorithm.

In the absence of valence 1 vertices the last map $G_k \to G'$ can be thought of as the core of the covering space of $G'$ corresponding to the image in $\pi_1$ of $f$. The following problems can be solved algorithmically using Theorem 1 (these were known earlier, but Theorem 1 provides a simple unified argument). Let $F$ be a free group with a fixed finite basis.

- Find a basis of the subgroup $H$ generated by a given finite collection $h_1, \cdots, h_k$ of elements of $F$.
- Given $w \in F$, decide if $w \in <h_1, \cdots, h_k>$.
- Given $w \in F$, decide if $w$ is conjugate into $<h_1, \cdots, h_k>$.
- Given a homomorphism $\phi : F \to F'$ between two free groups of finite rank, decide if $\phi$ is injective, surjective.
- Given finitely generated $H < F$ decide if it has finite index.
- Given two free subgroups $H_1, H_2 < F$ compute $H_1 \cap H_2$ and also the collection of subgroups $H_1 \cap H_2^g$ where $g \in F$. In particular, is $H_1$ malnormal?
- Represent a given automorphism of $F$ as the composition of generators of $\text{Aut}(F)$ of the following form:
  - Signed permutations: each $a_i$ maps to $a_i$ or to $a_i^{-1}$.
  - Change of maximal tree: $a_1 \mapsto a_1, a_i \mapsto a_1^{\pm 1} a_i$ or $a_i \mapsto a_i a_1^{\pm 1}$ ($i > 1$).
- Todd-Coxeter process [65].
3. **Culler-Vogtmann’s Outer space**

Fix the wedge of $n$ circles $R_n$ and a natural identification $\pi_1(R_n) \cong F_n$ in which oriented edges correspond to the basis elements. Thus any $\phi \in \text{Out}(F_n)$ can be thought of as a homotopy equivalence $R_n \to R_n$. A marked metric graph is a pair $(G, g)$ where

- $G$ is a finite graph without vertices of valence 1 or 2.
- $g : R_n \to G$ is a homotopy equivalence (the marking).
- $G$ is equipped with a path metric so that the sum of the lengths of all edges is 1.

**Outer space** $X_n$ is the set of equivalence classes of marked metric graphs under the equivalence relation $(G, g) \sim (G', g')$ if there is an isometry $h : G \to G'$ such that $gh$ and $g'$ are homotopic $^{28}$.

If $\alpha$ is a loop in $R_n$ we have the length function $l_\alpha : X_n \to \mathbb{R}$ where $l_\alpha(G, g)$ is the length of the immersed loop homotopic to $g(\alpha)$. The collection $\{l_\alpha\}$ as $\alpha$ ranges over all immersed loops in $R_n$ defines an injection $X_n \to \mathbb{R}^\infty$ and the topology on $X_n$ is defined so that this injection is an embedding. $X_n$ naturally decomposes into open simplices obtained by varying edge-lengths on a fixed marked graph. The group $\text{Out}(F_n)$ acts on $X_n$ on the right via

$$(G, g)\phi = (G, g\phi).$$

**Theorem 2 (Culler-Vogtmann $^{28}$).** $X_n$ is contractible and the action of $\text{Out}(F_n)$ is properly discontinuous (with finite point stabilizers). $X_n$ equivariantly deformation retracts to a $(2n-3)$-dimensional complex $(n > 1)$.

If $(G, g)$ and $(G', g')$ represent two points of $X_n$, there is a “difference of markings” map $h : G \to G'$ such that $h g$ and $g'$ are homotopic. Representing $h$ as a composition of folds (appropriately interpreted) leads to a path in $X_n$ from $(G, g)$ to $(G', g')$. Arranging that these paths vary continuously with endpoints leads to a proof of contractibility of $X_n$ $^{66, 60, 71}$.

**Corollary 3.** The virtual cohomological dimension $\text{vcd}(\text{Out}(F_n)) = 2n-3$ $(n > 1)$.

**Theorem 4 (Culler $^{26}$).** Every finite subgroup of $\text{Out}(F_n)$ fixes a point of $X_n$.

Outer space can be equivariantly compactified $^{27}$. Points at infinity are represented by actions of $F_n$ on $\mathbb{R}$-trees.

4. **Train tracks**

Any $\phi \in \text{Out}(F_n)$ can be represented as a cellular map $f : G \to G$ on a marked graph $G$. We say that $\phi$ is reducible if there is such a representative where

- $G$ has no vertices of valence 1 or 2, and
- there is a proper $f$-invariant subgraph of $G$ with at least one non-contractible component.
Otherwise, we say that $\phi$ is irreducible.

A cellular map $f : G \to G$ is a train track map if for every $k > 0$ the map $f^k : G \to G$ is locally injective on every open 1-cell. For example, homeomorphisms are train track maps and Culler’s theorem guarantees that every $\phi \in \text{Out}(F_n)$ of finite order has a representative $f : G \to G$ which is a homeomorphism. More generally, we have

**Theorem 5 (Bestvina-Handel [12]).** Every irreducible outer automorphism $\phi$ can be represented as a train track map $f : G \to G$.

Any vertex $v \in G$ has a cone neighborhood, and the frontier points can be thought of as “germs of directions” at $v$. A train track map (or any cellular map that does not collapse edges) $f$ induces the “derivative” map $Df$ on these germs (on possibly different vertices). We declare two germs at the same vertex to be equivalent (and the corresponding “turn” illegal) if they get identified by some power of $Df$ (and otherwise the turn is legal). An immersed loop in $G$ is legal if every turn determined by entering and then exiting a vertex is legal. It follows that $f$ sends legal loops to legal loops. This gives a method for computing the growth rate of $\phi$, as follows. The transition matrix $(a_{ij})$ of $f$ (or more generally of a cellular map $G \to G$ that is locally injective on edges) has $a_{ij}$ equal to the number of times that the $f$-image of $j$th edge crosses $i$th edge. Applying the Perron-Frobenius theorem to the transition matrix, one can find a unique metric structure on $G$ such that $f$ expands lengths of edges (and also legal loops) by a factor $\lambda \geq 1$. For a conjugacy class $\gamma$ in $F_n$ the growth rate is defined as

$$GR(\phi, \gamma) = \limsup_{k \to \infty} \frac{\log(||\phi^k(\gamma)||)}{k}$$

where $||\gamma||$ is the word length of the cyclically reduced word representing $\gamma$. Growth rates can be computed using lengths of loops in $G$ rather than in $R_n$.

**Corollary 6.** If $\phi$ is irreducible as above, then either $\gamma$ is a $\phi$-periodic conjugacy class, or $GR(\phi, \gamma) = \log \lambda$. Moreover, $\limsup$ can be replaced by $\lim$.

The proof of Theorem 5 uses a folding process that successively reduces the Perron-Frobenius number of the transition matrix until either a train track representative is found, or else a reduction of $\phi$ is discovered. This process is algorithmic (see [13],[21]).

Another application of train tracks is to fixed subgroups.

**Theorem 7 (Bestvina-Handel [12]).** Let $\Phi : F_n \to F_n$ be an automorphism whose associated outer automorphism is irreducible. Then the fixed subgroup $\text{Fix}(\Phi)$ is trivial or cyclic. Without the irreducibility assumption, the rank of $\text{Fix}(\Phi)$ is at most $n$.

It was known earlier by the work of Gersten [39] that $\text{Fix}(\Phi)$ has finite rank (for simpler proofs see [42],[25]). The last sentence in the above theorem was conjectured by Peter Scott. Subsequent work by Collins-Turner [24], Dicks-Ventura [31], Ventura [68], Martino-Ventura [58], imposed further restrictions on a subgroup of $F_n$ that occurs as the fixed subgroup of an automorphism. To analyze reducible automorphisms, a more general version of a train track map is required.
**Definition 8.** A cellular map $f: G \to G$ on a finite graph with no vertices of valence 1 that does not collapse any edges is a relative train track map if there is a filtration

$$\emptyset = G_0 \subset \cdots \subset G_m = G$$

into $f$-invariant subgraphs with the following properties. Denote by $H_r$ the closure of $G_r \setminus G_{r-1}$, and by $M_r$ the part of the transition matrix corresponding to $H_r$. Then $M_r$ is the zero matrix or an irreducible matrix. If $M_r$ is irreducible and the Perron-Frobenius eigenvalue $\lambda_r > 1$ then:

- the derivative $Df$ maps the germs in $H_r$ to germs in $H_r$,
- if $\alpha$ is a nontrivial path in $G_{r-1}$ with endpoints in $G_{r-1} \cap H_r$ then $f(\alpha)$, after pulling tight, is also a nontrivial path with endpoints in $G_{r-1} \cap H_r$, and
- every legal path in $H_r$ is mapped to a path that does not cross illegal turns in $H_r$.

As an example, consider the automorphism $a \mapsto a, b \mapsto ab, c \mapsto caba^{-1}b^{-1}d, d \mapsto dbcd$ represented on the rose $R_4$. The strata are $\emptyset \subset G_1 = \{a\} \subset \{a, b\} \subset G$. $H_1$ and $H_2$ have $\lambda = 1$ while $H_3$ has $\lambda_3 > 1$. The following is an analog of Thurston’s normal form for surface homeomorphisms.

**Theorem 9.** Every automorphism of $F_n$ admits a relative train track representative.

Consequently, automorphisms of $F_n$ can be thought of as being built from building blocks (exponential and non-exponential kinds) but the later stages are allowed to map over the previous stages. This makes the study of automorphisms of $F_n$ more difficult (and interesting) than the study of surface homeomorphisms. Other non-surface phenomena (present in linear groups) are:

- stacking up non-exponential strata produces (nonlinear) polynomial growth,
- the growth rate of an automorphism is generally different from the growth rate of its inverse.

### 5. Related spaces and structures

Unfortunately, relative train track representatives are far from unique. As a replacement, one looks for canonical objects associated to automorphisms that can be computed using relative train tracks. There are 3 kinds of such objects, all stemming from the surface theory: laminations, $\mathbb{R}$-trees, and hierarchical decompositions of $F_n$.

**Laminations.** Laminations were used in the proof of the Tits alternative for $\text{Out}(F_n)$. To each automorphism one associates finitely many attracting laminations. Each consists of a collection of “leaves”, i.e. biinfinite paths in the graph $G$, or alternatively, of an $F_n$-orbit of pairs of distinct points in the Cantor set of ends of $F_n$. A leaf $\ell$ can be computed by iterating an edge in an exponentially growing stratum $H_r$. The other leaves are biinfinite paths whose finite subpaths appear as subpaths of $\ell$. Some of the attracting laminations may be sublaminations of other
attracting laminations, and one focuses on the maximal (or topmost) laminations. It is possible to identify the basin of attraction for each such lamination. Let $\mathcal{H}$ be any subgroup of $\text{Out}(F_n)$. Some of the time it is possible to find elements $f, g \in \mathcal{H}$ that attract each other’s laminations and then the standard ping-pong argument shows that $\langle f, g \rangle \cong F_2$. Otherwise, there is a finite set of attracting laminations permuted by $\mathcal{H}$, a finite index subgroup $\mathcal{H}_0 \subset \mathcal{H}$ that fixes each of these laminations and a homomorphism (“stretch factor”) $\mathcal{H}_0 \to A$ to a finitely generated abelian group $A$ whose kernel consists entirely of polynomially growing automorphisms. There is an analog of Kolchin’s theorem that says that finitely generated groups of polynomially growing automorphisms can simultaneously be realized as relative train track maps on the same graph (the classical Kolchin theorem says that a group of unipotent matrices can be conjugated to be upper triangular, or equivalently that it fixes a point in the flag manifold). The main step in the proof of the analog of Kolchin’s theorem is to find an appropriate fixed $\mathbb{R}$-tree in the boundary of Outer space. This leads to the Tits alternative for $\text{Out}(F_n)$:

**Theorem 10 (Bestvina-Feighn-Handel [9], [10], [7])**. Any subgroup $\mathcal{H}$ of $\text{Out}(F_n)$ either contains $F_2$ or is virtually solvable.

A companion theorem [8] (for a simpler proof see [1]) is that solvable subgroups of $\text{Out}(F_n)$ are virtually abelian.

-$\mathbb{R}$-trees. Points in the compactified Outer space are represented as $F_n$-actions on $\mathbb{R}$-trees. It is then not surprising that the Rips machine [5], which is used to understand individual actions, provides a new tool to be deployed to study $\text{Out}(F_n)$. Gaboriau, Levitt, and Lustig [37] and Sela [59] find another proof of Theorem 10. Gaboriau and Levitt compute the topological dimension of the boundary of Outer Space [36]. Levitt and Lustig show [51] that automorphisms with irreducible powers have the standard north-south dynamics on the compactified Outer space. Guirardel [43] shows that the action of $\text{Out}(F_n)$ on the boundary does not have dense orbits; however, there is a unique minimal closed invariant set. For other applications of $\mathbb{R}$-trees in geometric group theory, the reader is referred to the survey [2].

**Cerf theory.** An advantage of $\text{Aut}(F_n)$ over $\text{Out}(F_n)$ is that there is a natural inclusion $\text{Aut}(F_n) \to \text{Aut}(F_{n+1})$. One can define $\text{Auter Space}$ $AX_n$, similarly to Outer space, except that all graphs are equipped with a base vertex, which is allowed to have valence 2. The degree of the base vertex $v$ is $2n - \text{valence}(v)$. Denote by $D_k^n$ the subcomplex of $AX_n$ consisting of graphs of degree $\leq k$. Hatcher-Vogtmann [47] develop a version of Cerf theory and show that $D_k^n$ is $(k - 1)$-connected. Since the quotient $D_k^n/\text{Aut}(F_n)$ stabilizes when $n$ is large, one sees that (rational) homology $H_i(\text{Aut}(F_n))$ also stabilizes when $n$ is large ($n \geq 3i/2$). Hatcher-Vogtmann show that the same is true for integral homology and in the range $n \geq 2i + 3$. They also make explicit computations in low dimensions [49] and all stable rational homology groups $H_i$ vanish for $i \leq 7$.

**Bordification.** The action of $\text{Out}(F_n)$ on Outer space $X_n$ is not cocompact. By analogy with Borel-Serre bordification of symmetric spaces [14] and Harer’s bordification of Teichmüller space [34], Bestvina and Feighn [6] bordify $X_n$, i.e. equivariantly add ideal points so that the action on the new space $BX_n$ is cocompact. This is done by separately compactifying every simplex with missing faces in $X_n$.
and then gluing these together. To see the idea, consider the case of the theta-graph in rank 2. Varying metrics yields a 2-simplex $\sigma$ without the vertices. As a sequence of metrics approaches a missing vertex, the lengths of two edges converge to 0. Restricting a metric to these two edges and normalizing so that the total length is 1 gives a point in $[0, 1]$ (the length of one of the edges), and a way to compactify $\sigma$ by adding an interval for each missing vertex. The compactified $\sigma$ is a hexagon. This procedure equips the limiting theta graph with a metric that may vanish on two edges, in which case a “secondary metric” is defined on their union. In general, a graph representing a point in the bordification is equipped with a sequence of metrics, each defined on the core of the subgraph where the previous metric vanishes.

Lengths of curves (at various scales) provide a “Morse function” on $BX_n$ with values in a product of $[0, \infty)$'s with the target lexicographically ordered. The sublevel and superlevel sets intersect each cell in a semi-algebraic set and it is possible to study how the homotopy types change as the level changes. A distinct advantage of $BX_n$ over the spine of $X_n$ (an equivariant deformation retract) is that the change in homotopy type of superlevel sets as the level decreases is very simple – via attaching of cells of a fixed dimension.

**Theorem 11 (Bestvina-Feighn [6]).** $BX_n$ and $\text{Out}(F_n)$ are $(2n - 5)$-connected at infinity, and $\text{Out}(F_n)$ is a virtual duality group of dimension $2n - 3$.

**Mapping tori.** If $\phi : F_n \to F_n$ is an automorphism, form the mapping torus $M(\phi)$. This is the fundamental group of the mapping torus $G \times [0, 1]/(x, 1) \sim (f(x), 0)$ of any representative $f : G \to G$, and it plays the role analogous to 3-manifolds that fiber over the circle. Such a group is always coherent [33]. A quasi-isometry classification of these groups seems out of reach, but the following is known. When $\phi$ has no periodic conjugacy classes, $M(\phi)$ is a hyperbolic group [20]. When $\phi$ has polynomial growth, $M(\phi)$ satisfies quadratic isoperimetric inequality [57] and moreover, $M(\phi)$ quasi-isometric to $M(\psi)$ for $\psi$ growing polynomially forces $\psi$ to grow as a polynomial of the same degree [54]. Bridson and Groves announced [16] that $M(\phi)$ satisfies quadratic isoperimetric inequality for all $\phi$.

**Geometry.** Perhaps the biggest challenge in the field is to find a good geometry that goes with $\text{Out}(F_n)$. The payoff would most likely include rigidity theorems for $\text{Out}(F_n)$. Both mapping class groups and arithmetic groups act isometrically on spaces of nonpositive curvature. Unfortunately, the results to date for $\text{Out}(F_n)$ are negative. Bridson [15] showed that Outer space does not admit an equivariant piecewise Euclidean $\text{CAT}(0)$ metric. $\text{Out}(F_n)$ ($n > 2$) is far from being $\text{CAT}(0)$ [17], [40].

An example of a likely rigidity theorem is that higher rank lattices in simple Lie groups do not embed into $\text{Out}(F_n)$. A possible strategy is to follow the proof in [11] of the analogous fact for mapping class groups. The major missing piece of the puzzle is the replacement for Harvey’s curve complex; a possible candidate is described in [48].
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