THE CAUCHY PROBLEM FOR THE VIBRATING PLATE EQUATION IN MODULATION SPACES

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Abstract. The local solvability of the Cauchy problem for the nonlinear vibrating plate equation is showed in the framework of modulation spaces. In the opposite direction, it is proved that there is no local wellposedness in Wiener amalgam spaces even for the solution to the homogeneous vibrating plate equation.

1. Introduction and results

The study of the wellposedness of the Cauchy problem for the vibrating plate equation (and, more generally, \( p \)-evolution equations) in Sobolev spaces and in Gevrey classes has been performed extensively by many authors (see, e.g., [1,2,14] and references therein). The techniques employed there essentially use classical calculus for pseudodifferential operators and obtain the wellposedness applying a fixed point argument.

In this note we study the Cauchy problem for the nonlinear vibrating plate equation (NLVP) in the framework of modulation spaces and Wiener amalgam spaces. Modulation and Wiener amalgam spaces were introduced by Feichtinger in the 80s [10,11] and soon they revealed to be the natural framework for the Time-Frequency Analysis [13].

Recently, they have been employed in the study of PDE's. In particular, let us recall their applications to local wellposedness for the Schrödinger and wave equation [3,4]. Moreover, let us highlight the deep and pioneering works [22,23] on the global wellposedness for nonlinear Schrödinger, wave and Klein-Gordon equations.

The modulation spaces are a family of Banach spaces, which contains the Sobolev spaces, and can be “arbitrary” close to the Schwartz spaces as well as the spaces of tempered distributions. A Cauchy datum in a modulation space can be rougher than any given one in the standard fractional Bessel potential setting (see definition below). Indeed, the roughness of the initial data is useful in many applications. Contrary to what happens in the case of modulation spaces, we shall show that there is no wellposedness when we consider initial data in Wiener amalgam spaces.

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Precisely, we shall prove the existence and uniqueness of solutions in modulation spaces to the Cauchy problem for NLVP:

\[
\begin{align*}
\partial_t^2 u + \Delta^2_x u &= F(u) \\
u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x),
\end{align*}
\]

with \( t \in \mathbb{R}, \ x \in \mathbb{R}^d, \ d \geq 1, \ \Delta_x = \partial^2_{x_1} + \ldots \partial^2_{x_d}, \ \Delta^2 u = \Delta(\Delta u). \) \( F \) is a scalar function on \( \mathbb{C} \), with \( F(0) = 0. \) The solution \( u(t, x) \) is a complex valued function of \((t, x) \in \mathbb{R} \times \mathbb{R}^d.\) We will consider the case in which \( F \) is an entire analytic function (in the real sense), and we shall highlight the special case \( F(u) = \lambda |u|^{2k}u, \ \lambda \in \mathbb{C}, \ k \in \mathbb{N}, \) where we have better results.

The arguments mainly rely on recent results for Fourier multipliers. Indeed, the integral version of the problem (1) has the form

\[
u(t, \cdot) = K'(t)u_0 + K(t)u_1 + BF(u),
\]

where

\[
K'(t) = \cos(t\Delta), \quad K(t) = \frac{\sin(t\Delta)}{\Delta}, \quad B = \int_0^t K(t - \tau) \cdot d\tau.
\]

Here, for every fixed \( t, \) the operators \( K'(t), K(t) \) in (3) are Fourier multipliers with symbols

\[
\sigma_0(\xi) = \cos(4\pi^2 t|\xi|^2), \quad \sigma_1(\xi) = \frac{\sin(4\pi^2 t|\xi|^2)}{4\pi^2|\xi|^2}, \ \xi \in \mathbb{R}^d.
\]

We recall that given a function \( \sigma \) on \( \mathbb{R}^d \) (the so-called symbol of the multiplier or, simply, multiplier), the corresponding Fourier multiplier operator \( H_\sigma \) is formally defined by

\[
H_\sigma f(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi x} \sigma(\xi) \hat{f}(\xi) d\xi.
\]

Here the Fourier transform is normalized to be \( \hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(y)e^{-2\pi i \xi y} dy. \) So, continuity properties for multipliers in suitable spaces yield estimates for the linear part of the equation. These latter are then combined with classical fixed point arguments to obtain local wellposedness in modulation spaces for (1).

Contrary to what happens for other equations such as the wave equation [7, Thm. 4.4], there is no local wellposedness of (1) in the framework of Wiener amalgam spaces. Precisely, we shall show that the latter already fails for the homogeneous case \( F \equiv 0. \)

Although the techniques used are essentially standard, we think it is worth detailing the study of the Cauchy problem for the NLVP equation, in view of its many applications in architecture and engineering, see for example [15].
In order to state our results, we first introduce the spaces we deal with \((10, 13, 4)\). Let \(T_x\) and \(M_\xi\) be the so-called translation and modulation operators, defined by \(T_x g(y) = g(y - x)\) and \(M_\xi g(y) = e^{2\pi i \xi y} g(y)\). Let \(g \in \mathcal{S}(\mathbb{R}^d)\) be a non-zero window function in the Schwartz class and consider the so-called short-time Fourier transform (STFT) \(V_g f\) of a function/tempered distribution \(f\) with respect to the window \(g\):

\[
V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int e^{-2\pi i \xi y} f(y) g(y-x) dy,
\]

i.e., the Fourier transform \(\mathcal{F}\) applied to \(f T_x g\).

For \(s \in \mathbb{R}\), we consider the weight function \(\langle x \rangle^s = (1 + |x|^2)^{s/2}, x \in \mathbb{R}^d\). If \(1 \leq p, q \leq \infty\), \(s \in \mathbb{R}\), the modulation space \(M^{p,q}_s(\mathbb{R}^d)\) is defined as the closure of the Schwartz class \(\mathcal{S}(\mathbb{R}^d)\) with respect to the norm

\[
\|f\|_{M^{p,q}_s} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p dx \right)^{q/p} \langle \xi \rangle^{sq} d\xi \right)^{1/q}
\]

(with obvious modifications when \(p = \infty\) or \(q = \infty\)). We set \(M^p_s = M^{p,p}_s\), \(M^{p,q}_s = M^{p,q}_0\) and \(M^p = M^p_0\).

Modulation spaces are Banach spaces whose definition is independent of the choice of the window \(g \in \mathcal{S}(\mathbb{R}^d)\). Among them, the following well-known function spaces occur: \(M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)\) and the Sobolev spaces:

\[
M^2_s(\mathbb{R}^d) = H^s(\mathbb{R}^d) = \{ f : \hat{f}(\xi) \langle \xi \rangle^s \in L^2(\mathbb{R}^d) \}.
\]

We also recall the following properties: \(M^{p_1,q_1}_s \hookrightarrow M^{p_2,q_2}_s\), if \(p_1 \leq p_2\) and \(q_1 \leq q_2\), \((M^{p,q}_s)' = M_{-s}^{p,q}\).

Other properties and more general definitions of modulation spaces can now be found in textbooks \([13]\).

The Wiener amalgam spaces \([12]\) can be defined using the STFT as well. Namely, for a fixed non-zero window function \(g \in \mathcal{S}(\mathbb{R}^d)\), the Wiener amalgam space \(W(FL^p_g, L^q_\gamma)\) is the closure of the Schwartz class \(\mathcal{S}(\mathbb{R}^d)\) with respect to the norm

\[
\|f\|_{W(FL^p_g, L^q_\gamma)} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(z, \xi)|^p \langle \xi \rangle^{sp} d\xi \right)^{q/p} \langle z \rangle^{sq} dz \right)^{1/q}
\]

(with obvious modifications when \(p = \infty\) or \(q = \infty\)). This definition is independent of the test function \(g \in \mathcal{S}(\mathbb{R}^d)\).

For more general definitions and properties of Wiener amalgam spaces we refer to \([12]\).
Observe that, for $p = q$, $s = 0$ and $\gamma = 0$, we have
\begin{equation}
\|f\|_{W(F\ell^p, L^p)} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g f(z, \zeta)|^p dz d\zeta\right)^{1/p} \asymp \|f\|_{\mathcal{M}^p},
\end{equation}
that is, $W(F\ell^p, L^p) = \mathcal{M}^p$.

The local wellposedness results for modulation spaces read as follows:

**Theorem 1.1.** Assume $s \geq 0$, $1 \leq p \leq \infty$, $(u_0, u_1) \in \mathcal{M}^{p,1}_s(\mathbb{R}^d) \times \mathcal{M}^{p,1}_{s-2}(\mathbb{R}^d)$ and $F(z) = \sum_{j,k=0}^{\infty} c_{j,k} z^j \bar{z}^k$, an entire real-analytic function on $\mathbb{C}$ with $F(0) = 0$.

For every $R > 0$, there exists $T > 0$ such that for every $(u_0, u_1)$ in the ball $B_R$ of center 0 and radius $R$ in $\mathcal{M}^{p,1}_s(\mathbb{R}^d) \times \mathcal{M}^{p,1}_{s-2}(\mathbb{R}^d)$ there exists a unique solution $u \in C^0([0, T]; \mathcal{M}^{p,1}_s(\mathbb{R}^d))$ to (2). Furthermore, the map $(u_0, u_1) \mapsto u$ from $B_R$ to $C^0([0, T]; \mathcal{M}^{p,1}_s(\mathbb{R}^d))$ is Lipschitz continuous.

For better results concerning the nonlinearity $F(u) = \lambda |u|^{2k}u$ we refer to Theorem 3.2.

Here the tools employed follow the pattern of similar Cauchy problems studied for other equations such as the Schrödinger, wave and Klein-Gordon equations [3, 4, 7]. Let us quote [21, 22, 23] as inspiring works on this topic. We remark that wave-front properties in the context of modulation space theory have been achieved for non-linear PDE’s in [16].

The norm of $K'(t), K(t)$ in [3], as bounded operators on modulation spaces, is controlled by the Wiener amalgam norm of the symbols (see Proposition 2.1 below). A new tool for computing the latter, is given by considering the time variable as a dilation parameter and applying the dilation properties for weighted Wiener amalgam spaces contained in [9].

The negative result concerns the unboundedness of the multiplier $K'(t)$ into the (unweighted) Wiener amalgam spaces $W(F\ell^p, L^q)$, when $p \neq q$. The case $p = q$ gives $W(F\ell^p, L^p) = \mathcal{M}^p$ (see [7]) and we come back to modulation spaces. Indeed, the related Fourier multiplier $T_{\tau}$, having symbol $\tau(\xi) = e^{\pi i \xi^2}$, is unbounded on $W(F\ell^p, L^q)$, when $p \neq q$, see Proposition 2.7. Hence, in contrast to what happens for other equations such as the wave equation [7, Thm. 4.4], there is no wellposedness of (1) in Wiener amalgam spaces.

This negative result and dispersive estimates for the multiplier $K'(t)$, obtained in the study of Schrödinger equation [5, 6], suggest to look for Strichartz estimates in Wiener amalgam spaces. We plan to address this issue in a future work.

2. Preliminary results and multiplier estimates

For the local wellposedness in modulation spaces we need to establish linear and nonlinear estimates on suitable modulation spaces that contain the solution $u$. First of all, we will use estimates for Fourier multipliers on modulation spaces.
The following result [7] shows that the multiplier norm, as bounded operator on weighted modulation spaces, can be controlled by a suitable Wiener amalgam norm of the corresponding symbol.

**Proposition 2.1.** Let \( s, t \in \mathbb{R}, 1 \leq p, q \leq \infty \). Let \( \sigma \) be a function on \( \mathbb{R}^d \) and consider the Fourier multiplier operator defined in (5). If \( \sigma \in W(\mathcal{F}L^1_s, L^\infty_t) \), then the operator \( H_\sigma \) extends to a bounded operator from \( \mathcal{M}^{p,q}_s \) into \( \mathcal{M}^{p,q}_{s+t} \), with

\[
\|H_\sigma f\|_{\mathcal{M}^{p,q}_{s+t}} \lesssim \|\sigma\|_{W(\mathcal{F}L^1_s, L^\infty_t)} \|f\|_{\mathcal{M}^{p,q}_s}.
\]

We shall use the previous criterion to establish the boundedness of the multipliers \( K(t) \) and \( K'(t) \). This simply amounts to looking for the right weighted Wiener norm of the corresponding symbols. To chase this goal we shall apply the lemmata below (see [10] and [7, Lemma 3.1], respectively for their proofs).

**Lemma 2.1.** For \( i = 1, 2, 3 \), let \( B_i \) be one of the Banach spaces \( \mathcal{F}L^q_s \) (\( 1 \leq q \leq \infty \), \( s \in \mathbb{R} \)), \( C_i \) be one of the Banach spaces \( \mathcal{L}^p_\gamma \) (\( 1 \leq p \leq \infty \), \( \gamma \in \mathbb{R} \)). If \( B_1 \cdot B_2 \hookrightarrow B_3 \) and \( C_1 \cdot C_2 \hookrightarrow C_3 \), we have

\[
W(B_1, C_1) \cdot W(B_2, C_2) \hookrightarrow W(B_3, C_3).
\]

**Lemma 2.2.** Let \( R > 0 \) and \( f \in C^\infty_0(\mathbb{R}^d) \) such that \( \text{supp} \ f \subset B(y, R) := \{x \in \mathbb{R}^d, |x - y| \leq R\} \), with \( y \in \mathbb{R}^d \). Then, for every \( 0 < p \leq \infty \), there exist an index \( k = k(p) \in \mathbb{N} \) and a constant \( C_{R,p} > 0 \) (which depends only on \( R \) and \( p \)) such that

\[
\|f\|_{\mathcal{F}L^p} \leq C_{R,p} \sup_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty}.
\]

Let us now introduce the symbols

\[
\tilde{\sigma}_0(\xi) = \cos |\xi|^2 \quad \text{and} \quad \tilde{\sigma}_1(\xi) = \frac{\sin |\xi|^2}{|\xi|^2}.
\]

The boundedness of the Fourier multipliers having symbols (11) is a consequence of the issue below.

**Proposition 2.2.** (i) The multiplier \( \tilde{\sigma}_0 \) in (11) is in the space \( W(\mathcal{F}L^1_s, L^\infty) \). (ii) The multiplier \( \tilde{\sigma}_1 \) in (11) is in \( W(\mathcal{F}L^1_s, L^\infty_2) \).

**Proof.** (i). Since \( \tilde{\sigma}_0(\xi) = \frac{e^{i|\xi|^2} + e^{-i|\xi|^2}}{2} \), the result follows from [3] Theorem 9).

(ii). The proof follows the pattern of [7] Proposition 3.1. Consider a function \( \chi \in C^\infty_0(\mathbb{R}^d), 1 \leq \chi(\xi) \leq 2 \), such that \( \chi(\xi) = 1 \) if \( |\xi| \leq 1 \), whereas \( \chi(\xi) = 0 \) if \( |\xi| \geq 2 \). Then,

\[
\tilde{\sigma}_1 = \chi(\xi)\tilde{\sigma}_1(\xi) + (1 - \chi(\xi))\tilde{\sigma}_1(\xi) := \sigma_{\text{sing}}(\xi) + \sigma_{\text{osc}}(\xi).
\]

**Singularity at the origin.** Since \( \sigma_{\text{sing}} \in C^\infty_0(\mathbb{R}^d) \subset W(\mathcal{F}L^1_s, L^\infty_s) \), for every \( s \in \mathbb{R} \), the claim is proved.
Oscillation at infinity. We can split \( \sigma_{osc} \) into
\[
\sigma_{osc}(\xi) = \sin |\xi|^2 \cdot \frac{1 - \chi(\xi)}{|\xi|^2}.
\]
the first term \( \sin |\xi|^2 \) is in \( W(FL^1, L^\infty) \) (see (i)), hence, if we show that the multiplier \( \frac{1 - \chi(\xi)}{|\xi|^2} \) is in \( W(FL^p, L^\infty) \), the pointwise multiplication properties for Wiener amalgam spaces (cf. Lemma 2.1) give the claim. To prove the latter inclusion we use Lemma 2.2, applied to the function \( \frac{1 - \chi(\xi)}{|\xi|^2} T_x g \in C_0^\infty(\mathbb{R}^d) \), with \( g \in C_0^\infty(\mathbb{R}^d) \).

Precisely,
\[
|\partial^\alpha \left( \frac{1 - \chi(\xi)}{|\xi|^2} \right) | \lesssim \langle \xi \rangle^{-2}, \quad |\partial^\alpha g(\xi-x)| \lesssim \langle x-\xi \rangle^{-N}, \quad \forall x, \xi \in \mathbb{R}^d, \forall N \in \mathbb{N}, \forall \alpha \in \mathbb{Z}_+^d.
\]
Combining the preceding estimates with the weight property \( \langle \xi \rangle^{-\delta} \langle x-\xi \rangle^{-|\delta|} \leq \langle x \rangle^{-\delta} \), we conclude the proof.

Corollary 2.3. Let \( s \in \mathbb{R}, j = 0, 1 \). For every \( 1 \leq p, q \leq \infty \), the Fourier multiplier \( H_{\tilde{\sigma}_j} \), with symbol \( \tilde{\sigma}_j \) defined in (11), extends to a bounded operator from \( \mathcal{M}^{p,q}_s(\mathbb{R}^d) \) into \( \mathcal{M}^{p,q}_{s+2j}(\mathbb{R}^d) \), with
\[
(13) \quad \|H_{\tilde{\sigma}_j}f\|_{\mathcal{M}^{p,q}_{s+2j}(\mathbb{R}^d)} \lesssim \|\tilde{\sigma}_j\|_{W(FL^1, L^\infty)} \|f\|_{\mathcal{M}^{p,q}_s(\mathbb{R}^d)}.
\]

Proof. The desired result follows from Propositions 2.2 and 2.1.

We remark that the boundedness result for \( H_{\tilde{\sigma}_0} \) was first proved in [20].

We first shall show the local wellposedness in modulation spaces of the Cauchy problem for the homogeneous vibrating plate equation:
\[
(14) \quad \begin{cases}
\partial_t^2 u + \Delta_x^2 u = 0 \\
u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x).
\end{cases}
\]
This is obtained by means of the previous estimates, combined with dilation properties for weighted Wiener amalgam spaces.

For \( 1 \leq p \leq \infty \), let \( p' \) be the conjugate exponent of \( p \) (\( 1/p + 1/p' = 1 \)). For \( (1/p, 1/q) \in [0, 1] \times [0, 1] \), using the notation introduced in [17], we define the subsets
\[
I_1 = \max(1/p, 1/p') \leq 1/q, \quad I_1^* = \min(1/p, 1/p') \geq 1/q,
I_2 = \max(1/q, 1/2) \leq 1/p', \quad I_2^* = \min(1/q, 1/2) \geq 1/p',
I_3 = \max(1/q, 1/2) \leq 1/p, \quad I_3^* = \min(1/q, 1/2) \geq 1/p,
\]
as shown in Figure 1:
We introduce the indices:

$$\mu_1(p, q) = \begin{cases} 
-1/p & \text{if } (1/p, 1/q) \in I_1^*, \\
1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*, \\
-2/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*, 
\end{cases}$$

and

$$\mu_2(p, q) = \begin{cases} 
-1/p & \text{if } (1/p, 1/q) \in I_1, \\
1/q - 1 & \text{if } (1/p, 1/q) \in I_2, \\
-2/p + 1/q & \text{if } (1/p, 1/q) \in I_3.
\end{cases}$$

Dilation properties for un-weighted modulation spaces have been completely studied in [17]. The dilation properties for weighted modulation spaces and Wiener amalgam spaces have recently been developed in [9]. In particular, let us recall the following result:

**Proposition 2.4.** Let $1 \leq p, q \leq \infty$, $t, s \in \mathbb{R}$. Then the following are true:

1. There exists a constant $C > 0$ such that $\forall f \in W(\mathcal{F}L_p^s, L_l^q)$, $\lambda \geq 1$,

   $$C^{-1} \lambda^{d_\mu_2(p', q')} \min\{1, \lambda^t\} \min\{1, \lambda^{-s}\} \|f\|_{W(\mathcal{F}L_p^s, L_l^q)} \leq \|f_\lambda\|_{W(\mathcal{F}L_p^s, L_l^q)} \leq C \lambda^{d_\mu_1(p', q')} \max\{1, \lambda^t\} \max\{1, \lambda^{-s}\} \|f\|_{W(\mathcal{F}L_p^s, L_l^q)}.$$

2. There exists a constant $C > 0$ such that $\forall f \in W(\mathcal{F}L_p^s, L_l^q)$, $0 \leq \lambda \leq 1$,

   $$C^{-1} \lambda^{d_\mu_1(p', q')} \min\{1, \lambda^t\} \min\{1, \lambda^{-s}\} \|f\|_{W(\mathcal{F}L_p^s, L_l^q)} \leq \|f_\lambda\|_{W(\mathcal{F}L_p^s, L_l^q)} \leq C \lambda^{d_\mu_2(p', q')} \max\{1, \lambda^t\} \max\{1, \lambda^{-s}\} \|f\|_{W(\mathcal{F}L_p^s, L_l^q)}.$$

![Figure 1. The index sets.](image)
The symbols $\sigma_0, \sigma_1$ in \cite{3} can be rewritten as time dilations of the symbols $\tilde{\sigma}_0 \in W(\mathcal{F}L^1, L^\infty), \tilde{\sigma}_1 \in W(\mathcal{F}L^1, L^2) \in \mathrm{II}$. Precisely, for $t > 0$, we can write $\sigma_0(\xi) = (\tilde{\sigma}_0)_{2\pi \sqrt{t}}, \sigma_1(\xi) = t(\tilde{\sigma}_1)_{2\pi \sqrt{t}}$. Using Proposition 2.4 with $\mu_1(\infty, 1) = 1, \mu_2(\infty, 1) = 0$, we have, for every $R > 0$,

$$
\|\tilde{\sigma}_0\|_{2\pi \sqrt{t}} \|W(\mathcal{F}L^1, L^\infty) \leq \begin{cases} C_{0,R} \|\tilde{\sigma}_0\|_{W(\mathcal{F}L^1, L^\infty)}, & t \leq R \\ C_{0,R} t^2 \|\tilde{\sigma}_0\|_{W(\mathcal{F}L^1, L^\infty)}, & t \geq R. \end{cases}
$$

and

$$
\|\tilde{\sigma}_1\|_{2\pi \sqrt{t}} \|W(\mathcal{F}L^1, L^2) \leq \begin{cases} C_{1,R} \|\tilde{\sigma}_1\|_{W(\mathcal{F}L^1, L^2)}, & t \leq R \\ C_{1,R} t^{\frac{1}{2} + 1} \|\tilde{\sigma}_1\|_{W(\mathcal{F}L^1, L^2)}, & t \geq R. \end{cases}
$$

Hence the Cauchy problem (14) admits a solution $u(t, x)$ satisfying:

$$
\|u(t, \cdot)\|_{\mathcal{M}_{p,q}^d} \leq C_0 (1 + t)^{\theta} \|u_0\|_{\mathcal{M}_{p,q}^d} + C_1 t (1 + t)^{\frac{1}{2} + 1} \|u_1\|_{\mathcal{M}_{p,q}^d}, \quad t > 0,
$$

for every $1 \leq p, q \leq \infty, s \in \mathbb{R}$.

Contrary to what happens for other equations such as the wave equation (see \cite[Thm. 4.4]{7}), there is no wellposedness of (1) or (14) into the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$. The reason being the unboundedness of the multiplier $K'(t)$ into the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$, when $p \neq q$; the case $p = q$ gives $W(\mathcal{F}L^p, L^p) = \mathcal{M}^p$ (see (7)). In this case the boundedness of $K'(t)$ was first proved in \cite{3} Theorem 1.

To prove the unboundedness of $K'(t)$, we first recall the relationship between modulation and Wiener amalgam spaces when $p \neq q$ \cite{12}:

**Proposition 2.5.** The Fourier transform establishes an isomorphism $\mathcal{F} : \mathcal{M}^{p,q} \rightarrow W(\mathcal{F}L^p, L^q)$.

The crucial tool concerns the unboundedness of the pointwise multiplication for modulation spaces \cite{8} Proposition 7.1:

**Proposition 2.6.** The multiplication $U_{t_0} f = e^{\pi i |\xi|^2} f, f \in \mathcal{S}(\mathbb{R}^d)$, is unbounded on $\mathcal{M}^{p,q}$, for every $1 \leq p, q \leq \infty$, with $p \neq q$.

Now, we are ready to prove the unboundedness of the multiplier $K'(t)$. This immediately follows by the unboundedness of the multiplier $T_r$ below.

**Proposition 2.7.** The Fourier multiplier $T_r$, having symbol $\tau(\xi) = e^{\pi t |\xi|^2}$, is unbounded on every $W(\mathcal{F}L^p, L^q), 1 \leq p, q \leq \infty$, with $p \neq q$.

**Proof.** For every $f \in \mathcal{S}(\mathbb{R}^d)$, using Proposition 2.3 we have

$$
\|T_r f\|_{W(\mathcal{F}L^p, L^q)} \lesssim \|\mathcal{F}(T_r f)\|_{\mathcal{M}^{p,q}} = \|\tau f\|_{\mathcal{M}^{p,q}}.
$$
Hence $T_r$ is bounded on $W(FL^p, L^q)$ if and only if the multiplication $U_{L_r}$ is bounded on $\mathcal{M}^{p,q}$, and this happens if and only if $p = q$, thanks to Proposition 2.6 (necessary conditions) and [3, Theorem 1] (sufficient conditions).

3. Local wellposedness of NLVP on modulation spaces

In this section we present the wellposedness result on modulation spaces. To establish nonlinear estimates on appropriate modulation spaces we shall use the lemma below. It was first proved in [10] (see also [23, Corollary 4.2]).

**Lemma 3.1.** Let $s \geq 0$, $1 \leq p \leq p_i \leq \infty$, $1 \leq q, q_i \leq \infty$, $N \in \mathbb{N}$, satisfy

$$
\sum_{i=1}^{N} \frac{1}{p_i} = \frac{1}{p}, \quad \sum_{i=1}^{N} \frac{1}{q_i} = N - 1 + \frac{1}{r},
$$

then we have

$$
\left\| \prod_{i=1}^{N} u_i \right\|_{\mathcal{M}^{p,r}_s} \leq \prod_{i=1}^{N} \left\| u_i \right\|_{\mathcal{M}^{p_i,q_i}}.
$$

In particular, for $p_i = Np$, $q_i = q$, $i = 1, \ldots, N$, we get

$$
\left\| \prod_{i=1}^{N} u_i \right\|_{\mathcal{M}^{p,r}_s} \leq \prod_{i=1}^{N} \left\| u_i \right\|_{\mathcal{M}^{p,q}} , \quad \frac{N}{q} = N - 1 + \frac{1}{r}.
$$

The proof of the local existence theory uses the following variant of the contraction mapping theorem (see, e.g., [19, Proposition 1.38]).

**Proposition 3.1.** Let $\mathcal{N}$ and $\mathcal{T}$ be two Banach spaces. Suppose we are given a linear operator $B : \mathcal{N} \to \mathcal{T}$ with the bound

$$
\|Bf\|_{\mathcal{T}} \leq C_0 \|f\|_{\mathcal{N}}
$$

for all $f \in \mathcal{N}$ and some $C_0 > 0$, and suppose that we are given a nonlinear operator $F : \mathcal{T} \to \mathcal{N}$ with $F(0) = 0$, which obeys the Lipschitz bounds

$$
\|F(u) - F(v)\|_{\mathcal{N}} \leq \frac{1}{2C_0} \|u - v\|_{\mathcal{T}}
$$

for all $u, v$ in the ball $B_\mu := \{u \in \mathcal{T} : \|u\|_{\mathcal{T}} \leq \mu\}$, for some $\mu > 0$. Then, for all $u_{\text{lin}} \in B_{2\mu}$ there exists a unique solution $u \in B_{\mu}$ to the equation

$$
u = u_{\text{lin}} + BF(u),
$$

with the map $u_{\text{lin}} \mapsto u$ Lipschitz with constant at most 2 (in particular, $\|u\|_{\mathcal{T}} \leq 2\|u_{\text{lin}}\|_{\mathcal{T}}$).

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1. We first observe that, by (15), for every $1 \leq p \leq \infty$, the multiplier $K'(t)$, in (3) can be extended to a bounded operator on $\mathcal{M}_s^{p,1}$, with

$$K'(t)u_0 \leq C_0(1 + t)^{\frac{d}{2}}\|u_0\|_{\mathcal{M}_s^{p,1}}, \quad t > 0.$$  

Similarly, the multiplier operator $K(t)$ satisfies the estimate

$$K(t)u_1 \leq C_1(t(1 + t))^{\frac{d+1}{2}}\|u_1\|_{\mathcal{M}_s^{p,1}}, \quad t > 0,$$

for every $1 \leq p \leq \infty$.

Now we are going to apply Proposition 3.1 with $T = \mathcal{N} = C^0([0, T]; \mathcal{M}_s^{p,1})$, where $T > 0$ will be chosen later on, with the nonlinear operator $\mathcal{B}$ given by the Duhamel operator in (3). Here $u_{in} := K'(t)u_0 + K(t)u_1$ is in the ball $B_{\mu/2} \subset \mathcal{T}$ by (20), (21), if $\mu$ is sufficiently large, depending on $R$. Using Minkowski integral inequality and (21), we obtain (18). Namely,

$$\|\mathcal{B}u\|_{\mathcal{M}_s^{p,1}} \leq TC_T\|u\|_{\mathcal{M}_s^{p,1}} \leq TC_T\|u\|_{\mathcal{M}_s^{p,1}},$$

with $C_T = C_1 \max_{t \in [0,T]} t(1 + t)^{\frac{d+1}{2}}$ and using the inclusion $\mathcal{M}_s^{p,1} \hookrightarrow \mathcal{M}_s^{p,1}$.

Condition (19) is already proved in [7, Theorem 4.1]. There, applying the relation (17) for $q = r = 1$, the following estimate is obtained:

$$\|F(u) - F(v)\|_{\mathcal{M}_s^{p,1}} \leq \|u - v\|_{\mathcal{M}_s^{p,1}} \sum_{j,k,l,m \geq 0} (|c_{j,k,l,m}^1| + |c_{j,k,l,m}^2|)\|u\|_{\mathcal{M}_s^{p,1}}\|v\|_{\mathcal{M}_s^{p,1}}^{l+m} < \infty,$$

for $u, v \in \mathcal{M}_s^{p,1}$. This expression is $\leq C_\mu\|u - v\|_{\mathcal{M}_s^{p,1}}$ if $u, v \in B_\mu$. Hence, by choosing $T$ sufficiently small we conclude the proof of existence, and also that of uniqueness among the solution in $\mathcal{T}$ with norm $O(R)$. Finally, this last constraint can be eliminated by a standard continuity argument (cf. the proof of Proposition 3.8 in [19]).

A better result can be obtained when considering the nonlinearity

$$F(u) = F_k(u) = \lambda|u|^{2k}u = \lambda u^{k+1/2}, \quad \lambda \in \mathbb{C}, \quad k \in \mathbb{N}.$$  

Theorem 3.2. Let $F(u)$ be as in (22), $1 \leq p \leq \infty$, $s \geq 2$, and

$$q' > kd.$$

For every $R$ there exists $T > 0$ such that for every $(u_0, u_1)$ in the ball $B_R$ of center 0 and radius $R$ in $\mathcal{M}_s^{p,q}(\mathbb{R}^d) \times \mathcal{M}_s^{p,q}(\mathbb{R}^d)$ there exists a unique solution $u \in C^0([0, T]; \mathcal{M}_s^{p,q}(\mathbb{R}^d))$ to (2). Furthermore the map $(u_0, u_1) \mapsto u$ from $B_R$ to $C^0([0, T]; \mathcal{M}_s^{p,q}(\mathbb{R}^d))$ is Lipschitz continuous.
Proof. Here we set $\mathcal{T} = \mathcal{C}^0([0, T]; \mathcal{M}^{p,q}_0(\mathbb{R}^d))$, $\mathcal{N} = \mathcal{C}^0([0, T]; \mathcal{M}^{p,q}_{2-2}(\mathbb{R}^d))$. Now

$$F_k(z) - F_k(w) = (z - w)p_k(z, w) + (\overline{z} - \overline{w})q_k(z, w),$$

where $p_k, q_k$ are polynomials of degree $2k$ in $z, w, \overline{z}, \overline{w}$ ($q_0(z, w) \equiv 0$). Using (17) for $1 \leq p \leq \infty$, we obtain

$$\|F(u) - F(v)\|_{\mathcal{M}^{p,r}_{2-2}} \leq C|\lambda|\|u - v\|_{\mathcal{M}^{p,q}_{2-2}} (\|u\|_{\mathcal{M}^{p,q}_{2-2}}^{2k} + \|v\|_{\mathcal{M}^{p,q}_{2-2}}^{2k}),$$

with

$$r = \frac{q}{2k(1 - q) + 1}.$$  

The inclusion relations for modulation spaces [10] [22] fulfill

$$\mathcal{M}^{p,r}_{q-2} \hookrightarrow \mathcal{M}^{p,q}_{2-2} \quad \text{if} \quad \frac{d}{q} - \frac{d}{r} < 2,$$

which, combined with (24), yields (23). So (19) is verified and we are done. \(\square\)

We observe that condition (23) let us consider initial data $u_0, u_1$ in rougher spaces than those in the Cauchy problem for the wave equation [7, Theorem 4.3].

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