Ferromagnetism in one-dimensional metals: Breakdown of the Hartree-Fock approximation and possible first-order phase transition

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We calculate the Gibbs potential $\Gamma(M)$ of a one-dimensional metal at constant magnetization $M$ to second order in the screened electron-electron interaction $U$. At zero temperature we find that $\Gamma(M)$ contains non-analytic corrections proportional to $M^2 \ln |M|$ and $|M|^3$, implying that a possible paramagnetic-ferromagnetic quantum phase transition in one-dimensional metals must be first order.

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I. INTRODUCTION

Can the ground state of a one-dimensional (1D) clean metal exhibit spontaneous ferromagnetism? A rigorous theorem due to Lieb and Mattis\cite{Lieb, Mattis} implies that the answer to this question is “no” for 1D continuum models as well as for 1D lattice models with nearest neighbor hopping and interactions involving only densities. However, the Lieb-Mattis theorem does not apply to lattice models with longer range hoppings. Indeed, some time ago Daul and Noack\cite{Daul, Noack} presented numerical evidence that the 1D Hubbard model with nearest- and next-nearest neighbor hopping has a ferromagnetic ground state in a substantial range of densities and on-site interactions $U$. Given the stability of the ferromagnetic ground state in 1D in a certain parameter regime, one might want to know the critical behavior of the system close to the quantum phase transition separating the paramagnetic from the ferromagnetic regime. Some physical properties of ferromagnetic metals in 1D have been studied in several recent works\cite{Belitz, Kirtz}. However, the fluctuation corrections to the Hartree-Fock approximation for the ground state energy have not been thoroughly investigated. In view of the fact that the metallic state in 1D is a Luttinger liquid, it is not clear whether the Hartree-Fock scenario of a second order phase transition to a ferromagnetic state for sufficiently strong interaction is at least qualitatively correct. In this work we shall therefore calculate the leading (second order in $U$) correction to the Hartree-Fock approximation for the Gibbs-potential $\Gamma(M)$ at constant magnetization $M$. We find non-analytic terms which completely invalidate the Hartree-Fock prediction and imply that, if the paramagnetic-ferromagnetic quantum phase transition exists in a 1D metal, then it must be first order. Recently Belitz and Kirkpatrick\cite{Belitz} came to a similar conclusion about the paramagnetic-ferromagnetic transition in clean itinerant ferromagnets in dimensions $1 < D < 3$.

Although most of our considerations are fairly general and independent of any specific model, for explicit calculations we shall use the Hubbard model with nearest neighbor hopping $t$ and next-nearest neighbor hopping $t'$ on a one-dimensional lattice with $N_L = L/a$ sites and lattice spacing $a$. The Hamiltonian is

$$\hat{H} = \sum_{k\sigma} \epsilon_k \hat{c}_k^{\dagger} \hat{c}_{k\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}.$$ (1.1)

Here $\hat{c}_{k\sigma} = N_L^{-1/2} \sum_i e^{-ikx_i} \hat{c}_{i\sigma}$ annihilates an electron with momentum $k$ and spin $\sigma$, the operator $\hat{n}_{i\sigma} = \hat{c}_{i\sigma}^{\dagger} \hat{c}_{i\sigma}$ counts the number of electrons with spin $\sigma$ at lattice site $x_i$, and the energy dispersion is

$$\epsilon_k = -2t \cos(ka) - 2t' \cos(2ka).$$ (1.2)

II. RENORMALIZED PERTURBATION THEORY AT CONSTANT MAGNETIZATION

A. General considerations

In the presence of a uniform magnetic field $h$ the grand canonical potential is

$$\Omega(\mu, h) = -T \ln \left\{ \text{Tr} \exp[-\beta(\hat{H} - \mu \hat{N} - h \hat{M})] \right\},$$ (2.1)

where $\mu$ is the chemical potential, $T = 1/\beta$ is the temperature, $\hat{N} = \sum_{k\sigma} \hat{c}_{k\sigma}^{\dagger} \hat{c}_{k\sigma}$ is the particle number operator, and the operator $\hat{M} = \sum_{k\sigma} \sigma \hat{c}_{k\sigma}^{\dagger} \hat{c}_{k\sigma}$ represents the uniform magnetization. The expectation values of these operators are

$$N = \langle \hat{N} \rangle = -\partial \Omega / \partial \mu, \quad M = \langle \hat{M} \rangle = -\partial \Omega / \partial h.$$ (2.2)

To study spontaneous symmetry breaking, it is more convenient to work with the corresponding Gibbs potential\cite{Lieb, Mattis, Kirtz}

$$\Gamma(N, M) = \Omega(\mu(N, M), h(N, M)) + \mu N + h M,$$ (2.3)

which is a function of $N$ and $M$. Here $\mu(N, M)$ and $h(N, M)$ should be calculated by inverting Eqs. (2.2). Perturbation theory generates an expansion of $\Gamma(N, M)$ in powers of the relevant dimensionless interaction, the so-called Stoner factor\cite{Stoner}

$$I = U \nu_0,$$ (2.4)
where \( \nu_0 = \nu(\xi = 0) \) is the density of states (DOS) per spin projection at the Fermi energy of the non-interacting system in the absence of a magnetic field. We define the energy-dependent DOS via

\[
\nu(\xi) = \frac{1}{N_L} \sum_k \delta(\xi - \epsilon_k + \mu),
\]

which has units of inverse energy. It is convenient to measure particle number and magnetization in reduced units

\[
n = \frac{N}{2N_L}, \quad m = \frac{M}{2N_L},
\]

so that \( 0 \leq n \leq 1 \) and \( -\frac{1}{2} \leq m \leq \frac{1}{2} \). Note that the physical range of \( m \)-values is \( |m| \leq n \) for \( n \leq 1/2 \) and \( |m| \leq 1 - n \) for \( n \geq 1/2 \), because the magnetization \( M \) cannot exceed the number \( N \) of electrons for a less than half-filled band or the number \( 2N_L - N \) of holes for a more than half filled band. Depending on the values of the two dimensionless parameters \( n \) and

\[
\gamma = \frac{4t'}{t},
\]

the Fermi surface of our model without interaction and magnetic field consists of two or four discrete points. The different regimes are shown in Fig. 1 (A similar figure can be found in Ref. [12]). For \( |\gamma| < 1 \) the Fermi surface has two Fermi points for all fillings \( 0 < n < 1 \). For \( \gamma > 1 \), i.e. for \( t' > t/4 \), the Fermi surface has two points for \( 0 < n < n_c(\gamma) \), and four points for \( n_c(\gamma) < n < 1 \). The critical filling separating these regimes is

\[
n_c(\gamma) = \frac{1}{\pi} \arccos \left( 2 - \frac{1}{\gamma^2} \right), \quad \gamma > 1.
\]

On the other hand, for \( \gamma < -1 \), corresponding to \( t' \leq -t/4 \), the Fermi surface has four points at low fillings \( 0 < n < n_c(\gamma) \), and two points at larger fillings \( n_c(\gamma) < n < 1 \), where now

\[
n_c(\gamma) = \frac{1}{\pi} \arccos \left( \frac{2}{|\gamma|} - 1 \right), \quad \gamma < -1.
\]

Besides the usual singularities for \( n \to 0 \) and \( n \to 1 \), for \( |\gamma| > 1 \) the DOS at the Fermi energy exhibits additional one-sided singularities at the critical fillings \( n_c(\gamma) \), which are related to vanishing Fermi velocities when the Fermi surface topology changes discontinuously.

In the regime where the Fermi surface consists of two points the DOS at the Fermi energy is

\[
\nu_0 = \frac{a}{\pi v_F} = \frac{1}{2\pi t \sin(\pi n)[1 + \gamma \cos(\pi n)]},
\]

where \( v_F \) is the Fermi velocity. The DOS is more complicated in the regime where the Fermi surface has four points. For \( \gamma > 1 \) and \( n > n_c \) we find

\[
\nu_0 = \frac{2}{\pi t' r} \left[ \frac{1}{\tan(\pi (1 - n)/2)} \right] \times \frac{1 + \cos[\pi (1 - n)]}{[1 + \cos(\pi (1 - n))][1 + \cos(\pi n)][1 - \cos(\pi n) - 4/\gamma^2]}.
\]

Graphs of the DOS at the Fermi energy for different values of \( \gamma \) are shown in Fig. 2. For simplicity we assume in this work that in the absence of a magnetic field and ferromagnetic symmetry breaking the true Fermi surface consists only two points \( k_F \) and \( -k_F \), each of which is split by the magnetic field \( h \) (or for finite \( m \)) into two spin-dependent points \( k_\sigma \) and \( -k_\sigma \), where \( \sigma = \uparrow, \downarrow \). The calculations in this work are therefore restricted to the shaded regime shown in Fig. 1. We emphasize that \( k_\sigma \) are the true Fermi momenta of the interacting system, which have to be determined self-consistently. In terms of the dimensionless filling \( n \) and the magnetization \( m \) defined in Eq. (2.6), we may write

\[
k_\uparrow + k_\downarrow = 2\pi n/a = \pi N/L, \quad (2.13a)
\]

\[
k_\uparrow - k_\downarrow = 2\pi m/a = \pi M/L. \quad (2.13b)
\]

To study ferromagnetic symmetry breaking, we consider the change of the Gibbs potential due to a finite value of \( M \). For convenience we introduce the dimensionless magnetization-dependent part of the Gibbs potential

\[
g(m) = \frac{\nu_0}{N_L} \left[ \Gamma(n, m) - \Gamma(n, m = 0) \right]
\]

\[
= g_0(m) + I_1(m) + \frac{I^2}{2} g_2(m) + \ldots. \quad (2.14)
\]

We suppress the dependence on the filling factor \( n \), which is implicit in all quantities such as \( v_F \), \( \nu_0 \) and \( I \). In
The natural scale \( \gamma = 4 \) is significant for different values of \( \gamma \). Above, we have normalized the Gibbs potential by \( \gamma \) above. In contrast, for \( \gamma \) from below, but diverges if \( \gamma \) approaches \( n_c \) from above. In contrast, for \( \gamma < -1 \) the DOS is finite if \( n \to n_c \) from above and is singular if \( n \to n_c \) from below.

Eq. (2.14) we have normalized the Gibbs potential by the natural scale \( \frac{N}{\gamma} = \frac{L}{\gamma} \left( \frac{\pi}{g} \right)^2 \) of the kinetic energy. In the regime where the Fermi surface without symmetry breaking has only two points \( N \), it is then sufficient to set up the renormalized perturbation theory by introducing two counterterms quadratic in the fermions with coefficients \( \Delta_T \) and \( \Delta_\gamma \) and rewrite the operator in the exponential of Eq. (2.1) as follows,

\[
\hat{H} - \mu \hat{N} - \hbar \hat{M} = \hat{K}_0 + \hat{V} ,
\]

where

\[
\hat{K}_0 = \sum_{k\sigma} \xi_{k\sigma} \hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} ,
\]

and

\[
\hat{V} = U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - \sum_{k\sigma} \Delta_{\sigma} \hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} ,
\]

with

\[
\xi_{k\sigma} = \epsilon_k + \Delta_{\sigma} - \mu - \sigma \hbar .
\]

In the language of many-body theory, \( \Delta_{\sigma} = \Sigma_{\sigma}(k, \omega) \) is the (a priori unknown) self-energy due to the two-body interaction of our original hamiltonian \( \hat{H}_0 \) for momenta \( k = k_\sigma \) at the Fermi surface and for vanishing frequency. Following the usual procedure, the counterterms \( \Delta_{\sigma} \) can be determined order by order in perturbation theory by demanding that the self-energy generated by the subtracted interaction \( \hat{V} \) in Eq. (2.17) vanishes for \( k = k_\sigma \) and \( \omega = 0 \). Alternatively, the self-consistent determination of the counterterms can also be implemented non-perturbatively within the framework of the renormalization group.

B. Non-interacting limit

Let us first consider the non-interacting limit \( U = 0 \), where \( \Delta_{\sigma} = 0 \) and

\[
\Omega_0(\mu, \hbar) = -T \sum_{k\sigma} \ln[1 - e^{-\beta(\epsilon_k - \mu - \sigma \hbar)}] .
\]

The expressions relating particle number and magnetization to chemical potential and magnetic field are then

\[
N = \sum_{k\sigma} f(\epsilon_k - \mu - \sigma \hbar) ,
\]

\[
M = \sum_{k\sigma} \sigma f(\epsilon_k - \mu - \sigma \hbar) ,
\]

where \( f(\epsilon) = [e^{\beta \epsilon} + 1]^{-1} \) is the Fermi function. We denote the corresponding Gibbs potential by \( \Gamma_0(N, M) \). In general it is not possible to calculate \( \Gamma_0(N, M) \) analytically. However, for weak ferromagnets, where \( |m| < 1 \), we may expand \( \Gamma(N, M) \) in powers of \( M \). For the \( m \)-dependent part of the dimensionless Gibbs potential defined in Eq. (2.14) we obtain in the non-interacting limit at zero temperature,

\[
g_0(m) = m^2 + \frac{C}{12} m^4 + O(m^6) ,
\]
where the dimensionless coefficient $C$ of the quartic term can be written in terms of the derivatives of the DOS at the Fermi energy,

$$C = \frac{1}{\nu_0} \left[ 3 \left( \frac{\nu'_0}{\nu_0} \right)^2 - \frac{\nu''_0}{\nu_0} \right]. \quad (2.22)$$

Here $\nu_0$ and $\nu'_0$ are the first and the second derivative of the energy-dependent DOS $\nu(\xi)$ defined in Eq. (2.20) at $\xi = 0$. Equation (2.22) is valid in any dimension provided we use the $D$-dimensional DOS. Interestingly, if in $1D$ the Fermi surface without magnetic field has only two points $\pm k_F$, then the constant $C$ can be related to the cubic term in the expansion of the energy dispersion around the Fermi surface as follows

$$C = \frac{c_3}{v_F} \left( \frac{\pi^2}{a} \right)^2, \quad (2.23)$$

where $c_3$ is defined by

$$\epsilon_{k_F+q} = \epsilon_{k_F} + v_F q + \frac{q^2}{2m^*} + \frac{c_3}{6} q^3 + O(q^4). \quad (2.24)$$

The quadratic term in the expansion (2.24) cancels on the right-hand side of Eq. (2.22). For the dispersion (1.2) we obtain in the regime where the Fermi surface consists of two points,

$$C = -n \pi^2 \frac{1 + 4\gamma \cos(\pi n)}{1 + \gamma \cos(\pi n)}. \quad (2.25)$$

A graph of this expression is shown in Fig. 3. For clarity, in Fig. 4 we also show some cuts for fixed $\gamma$ through the surface in Fig. 3. While for $|\gamma| < 1/4$ the coefficient $C$ is negative for all densities, for $|\gamma| > 1/4$ there exists always a regime of densities where $C$ is positive. In fact, by choosing $n$ sufficiently close to $n_c$ we can fine tune $C$ to assume any desired positive value.

**C. Hartree-Fock approximation**

To first order in $U$ the self-energy generated by the subtracted interaction $\hat{V}$ is $\Sigma_\sigma = U n_{-\sigma} - \Delta_\sigma$, which is independent of momentum $k$ and frequency $\omega$. Here

$$n_\sigma = \frac{1}{N_L} \sum_k f(\xi_{k\sigma}). \quad (2.26)$$

From the requirement that the subtracted interaction does not generate a momentum- and frequency-independent self-energy, $\Sigma_\sigma = 0$, we find the countertems to first order in $U$, $\Delta_\sigma = U n_{-\sigma} \equiv \Delta + \sigma \tilde{\Delta}$, (2.27)

where we have defined

$$\Delta = \frac{\Delta_\uparrow + \Delta_\downarrow}{2} = U \frac{n_\uparrow + n_\downarrow}{2} = Un, \quad (2.28a)$$

$$\tilde{\Delta} = \frac{\Delta_\uparrow - \Delta_\downarrow}{2} = U \frac{n_\uparrow - n_\downarrow}{2} = Um. \quad (2.28b)$$

The grand canonical potential is in this approximation given by

$$\Omega_0 = \Omega_0(\mu - \Delta, h + \tilde{\Delta}) - U(n^2 - m^2)N_L, \quad (2.29)$$

and the corresponding Gibbs potential is

$$\Gamma(N, M) = \Gamma_0(N, M) + U(n^2 - m^2)N_L. \quad (2.30)$$

The function $g_1(m)$ defined in Eq. (2.14) is therefore $g_1(m) = -m^2$ so that to first order in $I$ we obtain

$$g(m) = (1 - I)m^2 + \frac{C}{12} m^4 + O(m^6, I^2). \quad (2.31)$$

In the regime where $C > 0$ this leads to the usual Hartree-Fock scenario of a second order quantum phase transition to a ferromagnetic state at $I_c = 1$, as shown in Fig. 5.
Of course, extrapolating the Hartree-Fock approximation to \( I \approx 1 \) is an uncontrolled procedure, because in this interaction range there is no reason why the corrections of order \( P^2 \) and higher in Eq. (2.31) should be small. In order to assess the validity of the Hartree-Fock scenario, we shall calculate in the following section the correction to the Gibbs potential to second order in \( I \).

### III. Gibbs Potential to Second Order in the Interaction

The second order correction to the Gibbs potential is given by the Feynman diagram in Fig. 6. In real space and imaginary time this diagram represents the following expression,

\[
\Gamma_2(N, M) = -N_L U^2 a^4 \int_{-\beta/2}^{\beta/2} d\tau \times \sum_i \Pi_\uparrow(x_i, \tau)\Pi_\downarrow(x_i, \tau),
\]

(3.1)

where

\[
\Pi_\sigma(x, \tau) = -G_\sigma(x, \tau)G_\sigma(-x, -\tau)
\]

(3.2)

is the polarization bubble for spin \( \sigma \) electrons without interactions. Here the real-space imaginary-time Green function is

\[
G_\sigma(x, \tau) = \frac{1}{\beta L} \sum_{k, \omega} e^{i(kx - \omega \tau)}
\]

(3.3)

where \( \omega_l = 2\pi(l + \frac{1}{2})T \), \( l = 0, \pm 1, \pm 2, \ldots \) are fermionic Matsubara frequencies. Carrying out the Matsubara sum and taking the zero temperature limit (\( \beta \to \infty \)) and the infinite system limit (\( L \to \infty \)) we obtain

\[
G_\sigma(x, \tau) = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} e^{ikx} G_{k\sigma}(\tau),
\]

(3.4)

with

\[
G_{k\sigma}(\tau) = -e^{-\xi_{k\sigma}\tau} [\Theta(\tau)\Theta(\xi_{k\sigma}) - \Theta(\tau)\Theta(-\xi_{k\sigma})].
\]

(3.5)

To make progress analytically, we linearize the energy dispersion within an interval \( -\Lambda < q < \Lambda \) around the Fermi points. Here \( \Lambda \) is an ultraviolet cutoff of the order of \( 1/a \) and we assume that

\[
m \lesssim \frac{\Lambda a}{\pi} \equiv \lambda,
\]

(3.6)

which is necessary in order to justify the linearisation of the energy dispersion at the Fermi points. The dominant correction term to the Hartree-Fock approximation turns out to depend only logarithmically on \( \Lambda \), so that it is not very sensitive to the numerical value of \( \Lambda \). With these assumptions the integration in Eq. (3.3) can be carried out exactly,

\[
G_\sigma(x, \tau) \approx \frac{1}{2\pi i} \left[ e^{ik_x x} \frac{1 - e^{i\Lambda s_{\tau}(x + iv_{\sigma}\tau)}}{x + iv_{\sigma}\tau} - e^{-ik_x x} \frac{1 - e^{-i\Lambda s_{\tau}(x - iv_{\sigma}\tau)}}{x - iv_{\sigma}\tau} \right],
\]

(3.7)

where \( s_{\tau} = \text{sign}(\tau) \) and \( v_{\sigma} = \partial \xi_{k\sigma}/\partial k|_{k=k_{\sigma}} \). In this approximation the polarization can be written as

\[
\Pi_\sigma(x, \tau) = \Pi_\sigma^0(x, \tau) + \Pi_{2k\sigma}(x, \tau) + \Pi_{2k\sigma}(x, \tau),
\]

(3.8)
with the forward scattering contribution
\[ \Pi^0_\sigma(x, \tau) = -\frac{2}{(2\pi)^2} \text{Re} \left[ \frac{1 - e^{i\Lambda x + iv_\sigma \tau}}{x + iv_\sigma \tau} \right]^2, \quad (3.9) \]
and the backscattering part
\[ \Pi^{2k_\sigma}_\sigma(x, \tau) = \frac{e^{2ik_\sigma x}}{(2\pi)^2} \frac{1 - e^{i\Lambda x + iv_\sigma \tau}}{x^2 + (v_\sigma \tau)^2}. \quad (3.10) \]

Substituting Eq. (3.7) into Eq. (3.1) and taking the limits \( \beta \to \infty \) and \( L \to \infty \), we find

\[ \Gamma_2(N, M) = -N_L U^2 a^3 \frac{4}{(2\pi)^4} \int_0^\infty dx \int_{-\infty}^\infty d\tau \left\{ \text{Re} \left[ \frac{1 - 2g_\uparrow e^{i\Lambda x} + g^2 e^{2i\Lambda x}}{(x + iv_\uparrow \tau)^2} \right] \text{Re} \left[ \frac{1 - 2g_\downarrow e^{i\Lambda x} + g^2 e^{2i\Lambda x}}{(x + iv_\downarrow \tau)^2} \right] - \cos(2k_\uparrow x) \left[ \frac{1 - 2g_\uparrow \cos(\Lambda x) + g^2}{x^2 + (v_\uparrow \tau)^2} \right] \text{Re} \left[ \frac{1 - 2g_\downarrow e^{i\Lambda x} + g^2 e^{2i\Lambda x}}{(x + iv_\downarrow \tau)^2} \right] - \cos(2k_\downarrow x) \left[ \frac{1 - 2g_\downarrow \cos(\Lambda x) + g^2}{x^2 + (v_\downarrow \tau)^2} \right] \text{Re} \left[ \frac{1 - 2g_\uparrow e^{i\Lambda x} + g^2 e^{2i\Lambda x}}{(x + iv_\uparrow \tau)^2} \right] + \cos(2k_\uparrow x) \cos(2k_\downarrow x) \left[ \frac{1 - 2g_\uparrow \cos(\Lambda x) + g^2}{x^2 + (v_\uparrow \tau)^2} \right] \text{Re} \left[ \frac{1 - 2g_\downarrow e^{i\Lambda x} + g^2 e^{2i\Lambda x}}{(x + iv_\downarrow \tau)^2} \right] \right\}. \quad (3.11) \]

Here \( g_\sigma = e^{-v_\sigma \Lambda x}. \) The \( x \)-integration can now be performed using the residue theorem. The resulting \( \tau \)-integration can then be carried out exactly. Keeping in mind that the difference \( v_\uparrow - v_\downarrow \) between the Fermi velocities is proportional to \( mU \), we may approximate \( v_\uparrow \approx v_\downarrow \approx v_F \) in Eq. (3.11), because the prefactor is already of order \( U^2 \). In this approximation we obtain
\[ g_2(m) = \frac{v_\uparrow}{N_L} [\Gamma_2(N, M) - \Gamma_2(N, 0)] = \lambda^2 f(m/\lambda), \quad (3.12) \]
with
\[ f(x) = -x^2 \ln |x| - 3|x| - |x|(1 - 2|x|) \ln(1 - 2|x|) - (1 - x^2) \ln(1 - |x|) + (1 + |x| - 2x^2) \ln(1 + 2|x|). \quad (3.13) \]

Because Eqs. 3.12 and 3.13 have been derived assuming that \( x = m/\lambda \) is small (see Eq. 3.16) it is consistent to expand \( f(x) \) in powers of \( x \),
\[ f(x) = \left[ \frac{5}{2} - \ln |x| \right] x^2 - 6|x|^3 + \frac{13}{12} x^4 + O(x^5). \quad (3.14) \]
Then we obtain the magnetization-dependent part of the normalized Gibbs potential to second order in \( I \) and to fourth order in \( m \),
\[ g(m) = \left\{ 1 - I + \frac{I^2}{2} \left[ \ln \left( \frac{\lambda}{|m|} \right) + \frac{5}{2} \right] \right\} m^2 - \frac{3I^2}{\lambda} |m|^3 + \left( C + \frac{13I^2}{2\lambda^2} \right) \frac{m^4}{12} + O(|m|^5, I^3). \quad (3.15) \]
A reasonable choice for the ultraviolet cutoff \( \Lambda \) defining the interval where the linearization of the energy dispersion is justified is \( \Lambda = n/a \) for \( n \leq 1/2 \) and \( \Lambda = (1-n)/a \) for \( n > 1/2 \). Our dimensionless cutoff \( \lambda \) defined in Eq. 3.6 is then \( \lambda = \pi^{-1} \min\{n, 1-n\} \). A graph of Eq. 3.11 for this choice of \( \lambda \) and \( n = 0.9 \) is shown in Fig. 7. The non-analytic terms proportional to \( m^2 \ln |m| \) and \( |m|^3 \) generated by the second order correction completely change the Hartree-Fock scenario of a second order quantum phase transition depicted in Fig. 8. From Fig. 7 it is obvious that the extrapolation of the second order correction to an interaction strength of the order of unity leads to a first order quantum phase transition, which for our choice of \( \gamma = 0.9 \) and filling \( n = 0.9 \) occurs at a critical value \( I_c \approx 0.6615 \). At this value of \( I \) our function \( g(m) \) develops three separate degenerate minima, which is the characteristic feature of a first order phase transition. The precise numerical value for the critical \( I_c \) depends on our particular choice of the cutoff \( \lambda \), so that the above value of \( I_c \) should not be taken too seriously. However, with any reasonable choice of \( \lambda \) the critical \( I_c \) is smaller than the Hartree-Fock result \( I_c = 1 \), suggesting that correlations can stabilize the ferromagnetic state in certain parameter regimes. For consistency, we should require that the magnetization \( m \) at the phase transition satisfies \( m \lesssim \lambda \approx (1-n)/\pi \), see Eq. 3.6. From Fig. 7 we see that this condition is only marginally satisfied, so that our approximations loose their quantitative accuracy once the curves in Fig. 7 develop minima.
at finite $m$.

It is instructive to examine the physical origin of the dominant logarithmic correction proportional to $m^2 \ln |m|$ in Eq. (3.15). This term arises from a product of two backscattering contributions to the polarization in Eq. (3.1), which for small $m$ contain also a component involving only small momentum transfers. In fact, if we are only interested in the non-analytic contributions to $\Gamma_2(N, M)$, we may replace Eq. (3.1) by the simpler expression

$$\Gamma_2^{sing}(N, M) = -N_L U^2 a^3 \int_{\tau_0}^{\tau_m} d\tau \int_{-\infty}^{\infty} dx \times \left[ \Pi_1^{2k_1} (x, \tau) \Pi_1^{-2k_1} (x, \tau) + \Pi_1^{2k_1} (x, \tau) \Pi_1^{-2k_1} (x, \tau) \right]$$

$$= -N_L U^2 a^3 \frac{2}{(2\pi)^4} \int_{\tau_0}^{\tau_m} d\tau \int_{-\infty}^{\infty} dx \frac{\cos[2(k_\uparrow - k_\downarrow)x]}{[x^2 + (v_F \tau)^2]^2},$$

(3.16)

where $\tau_0 = (v_F \Lambda)^{-1}$ and $\tau_m = (4\pi|m|v_F/a)^{-1}$. Using $k_\uparrow - k_\downarrow = 2\pi m/a$ and expanding under the integral sign for small $m$

$$\cos[4\pi mx/a] = 1 - \frac{(4\pi x)^2}{2a^2} m^2 + O(m^4),$$

(3.17)

it is easy to see from Eq. (3.16) by power counting that the coefficient of $m^2$ is decorated by a non-analytic correction proportional to $\ln(\tau_m/\tau_0) = \ln(a\Lambda/4\pi|m|)$.

IV. DISCUSSION AND CONCLUSIONS

In this work we have evaluated the magnetization-dependent part $g(m)$ of the Gibbs potential $\Gamma(N, M)$ to second order in the relevant dimensionless interaction strength $I = U \nu_0$. Our main result is an explicit expression for $g(m)$ up to order $I^2$ given in Eq. (3.15), which contains non-analytic corrections proportional to $m^2 \ln |m|$ and $|m|^3$. When extrapolated to interactions $I$ of the order of unity, these corrections imply that the paramagnetic-ferromagnetic quantum phase transition in 1D, if it exists, must be first order. Of course, the extrapolation of the weak coupling expansion to values of $I$ of the order of unity is an uncontrolled procedure, so that with our method we cannot proof the existence or the absence of a ferromagnetic ground state in 1D. However, the numerical density-matrix renormalization group calculations by Daul and Noack suggest that in a certain regime of $\gamma = 4t'/t$ and fillings $n$ a ferromagnetic ground state indeed exists. On the other hand, these authors found numerical evidence that the paramagnetic-ferromagnetic quantum phase transition in 1D is second order, which is not supported by our calculation.

Given the fact that the metallic state in 1D is a Luttinger liquid, one could have expected that the perturbative expansion of the Gibbs potential contains non-analytic corrections. Surprisingly, according to a recent calculation by Betouras, Efremov, and Chubukov similar non-analytic corrections exist even in higher dimensions. These authors evaluated the magnetic-field dependent part of the interaction-correction to the grand canonical potential to second order in $U$ in three and two dimensions. They found that the susceptibility is proportional to $h^2 \ln |h|$ in three dimensions, and proportional to $|h|$ in two dimensions. These corrections imply non-analyticities in the corresponding magnetization-dependent part of the Gibbs potential, leading to the breakdown of the Hartree-Fock approximation in dimensions $D > 1$, in agreement with the work of Belitz and Kirkpatrick.

Our result (3.14) for the Gibbs potential implies that the magnetic susceptibility at weak coupling vanishes for $m \to 0$ as

$$\chi(m) = 2\nu_0 \left[ \frac{\partial^2 g(m)}{\partial m^2} \right]^{-1} \sim \frac{2\nu_0}{I^2 \ln(\lambda/|m|)}.$$  (4.1)

Hence, in the paramagnetic state the zero-field susceptibility $\chi(m = 0)$ vanishes for any finite value of the interaction, while it approaches the finite value $\nu_0$ in the non-interacting limit. In one-dimensional metals other quantities are known to exhibit a similar discontinuity. For example, the density of states of a Luttinger liquid vanishes at the Fermi energy, while in the absence of interactions it is finite\textsuperscript{11}. We conjecture that higher order corrections neglected in Eq. (3.15) will transform the logarithmic singularity in Eq. (4.1) into a power law with interaction-dependent exponent.

It is important to emphasize that a ferromagnetic ground state in a 1D lattice model with hopping beyond the nearest neighbors does not contradict the Lieb-Mattis theorem\textsuperscript{17}. Ferromagnetic ground states of the $t - t'$ Hubbard model can therefore not be ruled out a
priori. Whether such a ferromagnetic ground state is relevant to explain recent measurements of the conductance anomaly in quantum wires\textsuperscript{18,19} is not clear at this point. Very recently Klironomos \textit{et al.}\textsuperscript{20} pointed out that electron-electron interactions in quantum wires induce deviations from the strictly one-dimensional geometry, in which case the Lieb-Mattis theorem does not apply and a ferromagnetic ground state is in principle possible.

A detailed evaluation of the second order correction to the Gibbs potential in the regime where the Fermi surface without interaction consists of four points still remains to be done. This calculation is more complicated than in the case of two Fermi points, because a self-consistent treatment within renormalized perturbation theory requires the introduction of four counterterms, one for each Fermi point\textsuperscript{14,15}. Vollhardt \textit{et al.}\textsuperscript{21} pointed out that a large asymmetry in the DOS with a peak at the lower band edge tends to favor ferromagnetism by minimizing the increase of kinetic energy due to the spin polarization. Given the fact that in our model for $|\gamma| > 1$ the DOS exhibits a one-sided singularity at the critical filling $n_c(\gamma)$ where the number of Fermi points changes discontinuously (see Eqs. (2.8, 2.9) and Fig. 2), we suspect that for fillings $n$ close to $n_c$ a ferromagnetic ground state can be stabilized even for values of the interaction strength $I$ that are substantially smaller than unity. Whether this hypothesis is correct requires a detailed calculation, which is beyond the scope of this work.

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