A Study of Cross-Over Effects For The 2D Random Bond Potts Model

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ABSTRACT

We present results of a numerical simulation of the $q$-state random bond Potts model in two dimensions and for large $q$. In particular, care is taken to study the crossover from the pure model to the random model, as well as the crossover from the percolation to the random model. We show how to determine precisely the random fixed point and measure critical exponents at this point.

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These last years, many studies have been devoted to the problem of the effect of randomness on two dimensional statistical models. At a second order phase transition, the effect of randomness is supposed to be directly related to the critical exponent of the specific heat, $\alpha$, according to the well-known Harris criterion [1]. If $\alpha$ is positive then the disorder will be relevant, *i.e.* under the effect of the disorder, the model will reach a new critical behavior at a new critical point. Otherwise, if $\alpha$ is negative, disorder is irrelevant, the critical behavior will not change.

More recently, attention has turned to models which, before the introduction of disorder, are of first order transition type [2, 3, 4, 5, 6]. As it is well understood now, introducing disorder in a $q$-state Potts model, with $q > 4$, will change the order of the phase transition into a second one [2, 3]. Such a type of disorder has been studied recently, mostly in numerical simulations. In particular, Chen, Ferrenberg and Landau have measured critical exponents for the 8-state Potts model using a Monte Carlo simulation and found them to be close to the ones of the Ising model (with or without disorder, since the disorder produces only log corrections in that case) [7]. More recently, the central charge has been measured for this same model, and it was shown that $c \approx 3/2$ [8, 9]. However, Cardy and Jacobsen also measured the dimension of the magnetization operator and found a value which is very different from the one of the Ising model, *i.e.* $x_1 = \beta/\nu = 0.142 \pm 0.001$ [9]. Since in this study, Cardy and Jacobsen used a transfer matrix method, it is rather difficult to compare their results with the ones of Chen, Ferrenberg and Landau. In a recent Monte Carlo simulation of the 8-state Potts model, Chatelain and Berche obtained a third value for the magnetic exponent $x_1 = 0.153 \pm 0.003$ [10]. We believe that the discrepancy between these values depends only on the choice of the strength disorder with which the model was considered. As already noted by Cardy and Jacobsen the strength of the disorder can influence the value of the exponents. In their simulations, these authors had to choose a rather weak disorder ($R = 2$). On the other hand, Chatelain and Berche, who obtained a different value for $x_1$, considered a model with a stronger disorder ($R = 10$). Here and in the following, $R$ corresponds to the strength of the disorder. $R = 1$ is a model without disorder and $R = \infty$ is the percolation point. On a lattice where each bond is chosen randomly with
a value $J_0$ or $J_1$, $R$ is defined by $R = J_1/J_0$.

Another problem which has not been well studied so far is the effects of cross-over. It is rather well understood that cross-over between the model without disorder and the model with disorder can produce rather subtle effects in the case of the Ising model. In the numerical study of the effect of randomness for the Ising model by Andreichenko, Dotsenko, Selke and Wang, the specific heat has been carefully measured for different disorder strength, and it is only with a strong disorder ($R \simeq 10$) that the cross-over can be ignored [11]. For the Potts models that we study here, the situation is still more complicated, since we also have to take care of the possible cross-over with the percolation. In models with random bond disorder, it is expected that the random fixed point separates two unstable fixed points, the point corresponding to the model without disorder and the percolation point, which is the infinite strength disorder limit [12].

Another reason to care about the possible influence of percolation over the random critical point is that, as first observed by Cardy and Jacobsen, the value of the central charge $c(q)$ for the $q$-state Potts model in presence of disorder depends in $q$ as for the percolation points ($c(q) \propto \ln(q)$). Moreover, the constant of proportionality is very close: 0.721 for the random models and 0.689 at the percolation. As the measurements in [8,9] give values of central charge with errors of few percent, it is not really possible to distinguish the two cases. Thus, it is necessary to understand better the possible influence of the percolation on the random fixed point. In particular, we want to be able to determine if the values of the central charges measured in [8,9] correspond to new random fixed points or just correspond to the percolation points.

A first step in this direction has already been done by Cardy and Jacobsen who used phenomenological RG methods to determine the randomness corresponding to the fixed point. Then a second step is to be able to measure precisely the exponent of the magnetization. There are (at least) three parameters which can characterize a random $q$-state Potts model. The central charge $c(q)$, the specific heat exponent $\alpha(q)$ and the magnetic exponent $x_1(q)$. The first parameter is already well determined and for a large choice of $q$’s, leading to the result $c(q) \propto \ln(q)$. The specific heat exponent takes, in
all the systems simulated so far, a value close to zero or negative (as expected from the Harris criterion). Moreover, it looks like this exponent does not depend much on \( q \). On the contrary, the magnetic exponent depends very strongly in \( q \). Another reason to look the magnetic exponent is that it is very different from the one at the percolation point \((x_1 = 5/48 \simeq 0.1042)\). Thus, since the magnetic exponent increases with \( q \), by studying carefully \( x_1(q) \), we should be able to locate precisely the random fixed point.

The model on which we will focus is the 8-state Potts model. After understanding carefully the cross-over for this model, we will also extend our study to more general cases. By performing simulations for the 8-state Potts model, we expect three possible regimes, depending of the strength of the disorder \( R \). For a very weak disorder \( (i.e. R \simeq 1) \) the simulation will start very close to the unstable fixed point corresponding to the case without disorder. As we increase the lattice size, we should move along the renormalization group flow towards the stable random fixed point and cross-over effects should be very obvious. In practice, it turns out to be very difficult to start with a value of the disorder too small, and this because of too large autocorrelation times. A second regime is when we start with a disorder strength \( R_c \) such that we are close to the random fixed point. There, we should observe a perfect scaling law as this is a stable (attractive) fixed point (apart, of course, of finite size effects). The third regime corresponds to large disorder. In that case, since \( R \to \infty \) corresponds to the percolation point which is unstable, we expect to flow toward the random fixed point as we increase the lattice size. Thus the strategy is just to scan the values of disorder and determine the one for which we have a scaling behavior.

The Hamiltonian of the simulated model is given by

\[
H = - \sum_{\{i,j\}} J_{ij} \delta_{\sigma_i,\sigma_j},
\]

where the coupling constant between nearest neighbor spins takes the value

\[
J_{ij} = \begin{cases} 
J_0 & \text{with probability } p \\
J_1 & \text{with probability } 1 - p
\end{cases}
\]

and \( \sigma_i \) takes \( q \) possible values \( \sigma_i = 0, 1, \ldots, q - 1 \).
Measurements were performed on a square lattice with helical boundary conditions. Without any loss of generality, we can consider the case where \( p = \frac{1}{2} \). Then the model is self-dual and thus the critical temperature is exactly known. It is given by the solution of the equation \([14]\)

\[
1 - e^{-\beta J_0} \over 1 + (g - 1)e^{-\beta J_0} = e^{-\beta J_1},
\]

As explained above, we are mainly interested in understanding how the critical exponents can depend on the strength of the disorder defined by \( R = J_1/J_0 \).

Monte Carlo data were obtained by using the well known Wolff cluster algorithm \([15]\). The parameters that we choose depend very much on the value of the disorder. For very weak disorder \((R \simeq 2)\), the auto correlation times \( \tau \) are very large (385 for \( L=100 \)), while for strong disorder \((R \simeq 100)\), it is 8, again for \( L=100 \). Thus the length of the measurements and thermalisation was adapted to each case of disorder. A typical measurement for the magnetization was performed after discarding for thermalisation \( 100 \times \tau(L) \) and then the same number of update for measurements. It turns out that the length of the measurement is not very important since we also have to perform an average over the disorder. The statistical error \( \delta A \) of a quantity \( A \) has two contributions, one from the thermal fluctuation, with a variance \( \sigma_T \), and one from the disorder fluctuation, with a variance \( \sigma_N \). Thus the statistical error is given by

\[
(\delta A)^2 = \frac{\sigma_N^2}{N} + \frac{\sigma_T^2}{Nt_1/\tau},
\]

where \( N \) is the number of configurations of disorder, \( \tau \) is the autocorrelation time and \( t_1 \) is the number of updates. For the quantity that we measured, it turns out that the two variances are near equal and then

\[
(\delta A)^2 \simeq \frac{\sigma_N^2}{N}(1 + \frac{\tau}{t_1}).
\]

By choosing \( t_1 \) such that \( t_1 \simeq 100\tau \), we can ignore the thermal fluctuations.

Due to the rather strong disorder that we consider, we needed to have huge statistics over the number of configurations of disorder. Simulations were performed for lattices with size ranging from \( L = 5 \) to \( L = 1000 \). A typical number of configurations of disorder
is 100000 for $L = 5 - 100$, and then a smaller number for larger lattice size. All these parameters are summarized in the table 1. Simulations have been performed for a large range of disorder, $R = 2, 5, 8, 10, 20, 100$ and $1000$.

The first result that we present is the Log-Log plot of the magnetization versus the lattice size $L$ at the critical point for different values of disorder (see Fig. 1). We only show the data for some values of the disorder ($R = 2, 10, 100$ and $1000$). We see clearly on this figure that many things happen as we change the disorder $R$. For a very weak disorder $R = 2$ (the upper plot), we see that we are not yet in the asymptotic regime and thus there is no scaling law, since there are cross-over effects. Nevertheless, we see that by increasing the lattice size, the curvature in the plot does decrease. We should also mention that for this value of disorder, we were able to simulate only up to $L = 200$ and with a rather poor statistics in number of configuration. This is due to the very large autocorrelation time ($\tau \simeq 533$ for $L = 200$). The second plot in this figure corresponds to $R = 10$. There, the situation changes drastically. We do see a nice scaling law. Since there are no cross-over effects (we do have a scaling behavior for all lattice sizes) then we must be very close to the critical point and $R_c \simeq 10$. The next two plots correspond to $R = 100$ and $R = 1000$. There, we see that again the behavior changes as we increase the lattice size, indicating that we flow towards the random fixed point. Thus we do again feel the effects from another unstable fixed point, the percolation point, contrary to the case $R = 2$.

To have a more quantitative way of understanding these cross-over effects, we compute the effective values of $x_1(q)$ by changing the lattice size, see table 2. ($x_1$ is defined by $m(L) \simeq L^{-x_1}$ with $m(L)$ the magnetization on a lattice of linear size $L$.) In this table, we can easily see that there are three different regimes. The first one, for $R = 2, 5$ for which at small distances we feel the influences of another fixed point. As we increase the lattice size, we see that the value of $x_1(q)$ decreases (in fact for $R = 2$ it first increases, then decreases due to finite size effects) and tends to some fixed value. Since, due to the large autocorrelation times for small disorder, we were not able to perform simulations for lattices larger than $L = 200$, we can not reach completely the asymptotic regime.
Figure 1: log-log plot of the magnetization versus the lattice size for the 8-state Potts model. From top to bottom, R=2,10,100 and 1000. We performed a vertical shift of the two last plots for the sake of clarity.
The second regime occurs for disorder between $R = 8$ and $R = 20$. There, we do see that the exponent $x_1$ is near constant as we change the lattice size with a value between 0.150 and 0.155 indicating that $R_c$ must be in this range of disorder. Then we observe a third regime for large disorder, i.e. $R = 100,1000$. There, as we increase the lattice size, $x_1$ strongly increases. This third regime corresponds to the one which, at short distance, still feels the influence of the fixed point corresponding to the percolation for which we have $x_1 = 5/48 \simeq 0.1042$. Thus, we are able to locate precisely the random fixed point with $R \simeq 8 - 20$ and the value of the magnetic exponent is $x_1 = 0.150 - 0.155$. These results confirm the one of Cardy and Jacobsen who obtained $R_c \simeq 9$ using phenomenological RG methods [9] as well as the one of Chatelain and Berche who obtained a perfectly compatible result for $x_1$ at $R = 10$ [10].

Finally, as a last result, we want to present some results for larger value of $q$'s. From previous results (in particular the ones of Cardy and Jacobsen [3]) it is expected that $x_1(q)$ will increase with $q$. In the following, we present some results for simulations for the case $q = 64$. Since we just want to check if $x_1$ continues to increase as we increase $q$, these simulations have been done with a rather limited statistics, using only 1000 samples of disorder. The length of the measurements has been chosen as for the 8-state Potts model, i.e. $100 \times \tau(L)$ after a thermalisation of $100 \times \tau(L)$ steps. Here again $\tau(L)$ depends very much of the disorder: $\tau(L = 128) \simeq 951$ for $R = 20$ and $\tau(L = 128) \simeq 94$ for $R = 1000$. Based on our results for the 8-state simulations, we performed the following simulations: First we performed a simulation for very strong disorder, $R = 1000$. As expected, we see a strong cross-over from the percolation point towards the random fixed point with a magnetic exponent which increase from $x_1 \simeq 0.105$ for small lattice size up to $x_1 \simeq 0.150$ for the larger lattice size that we simulated. In the Fig. 2, where we plotted our results for $q = 64$, we see that we are still very far from an asymptotic behavior. Thus we performed simulations for a weaker value of the disorder. Since we know from our previous results that we just need to locate roughly the value of $R_c$ (the result for $x_1$ was consistent between $R = 8$ and $R = 20$ for $q = 8$), we just arbitrary chose a value of $R = 20$ and performed measurements. For this value of disorder, apart finite size effects, we observe a reasonably good scaling behavior as we increase the lattice size,
Figure 2: log-log plot of the magnetization versus the lattice size for the 64-state Potts model. From top to bottom, $R=20$ and 1000. We performed a vertical shift of the second plot for the sake of clarity.
with a value of $x_1 \simeq 0.185 \pm 0.005$, thus indicating that $x_1(q)$ will continue to increase with the lattice size.

To summarize, in this letter, we have studied the magnetic exponent $x_1$ of the $q$-state Potts model in the presence of random bond disorder. We have confirmed that there exists a stable random fixed point, despite strong cross-over effects. By showing that we will flow toward this fixed point, either by starting from a very weak disorder or from a very strong one, we can then believe that the measurements of the central charges obtained in [8, 9] do really correspond to new conformal field theories and are not just the effect of the percolation. A next step in the understanding of the random fixed points will be to give a more explicit construction of these conformal field theories.

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Table 1: Parameters of the simulations. We indicate, for each strength of the disorder $R = 2, 5, 8, 10, 20, 100$ and $1000$, the number of configurations that we simulated (divided by 1000) and the auto correlation time in parentheses. For large disorder, we just computed this auto correlation time for few sizes, extrapolating for others sizes.

| L  | R=2  | 5    | 8    | 10   | 20   | 100  | 1000 |
|----|------|------|------|------|------|------|------|
| 5  | 100 (14) | 100 (6) | 100 (4) | 100 (3) | 100  | 100  | 100  |
| 10 | 100 (36) | 100 (10) | 100 (6) | 100 (5) | 100 (3) | 100 (1) | 100 (1) |
| 20 | 100 (84) | 100 (17) | 100 (10) | 100 (8) | 100  | 100  | 100  |
| 50 | 35 (183) | 100 (30) | 100 (17) | 100 (13) | 100  | 100  | 100  |
| 100| 5 (385) | 35 (42) | 35 (24) | 100 (20) | 100 (11) | 100 (8) | 100 (3) |
| 200| 2 (533) | 4 (59) | 4 (31) | 10 (29) | 10  | 10  | 10  |
| 500|            | 2 (38) | 2 (25) | 2  | 0.5 (55) |
| 1000|        |      |      |      |      |      |      |

Table 2: Magnetic exponent $x_1$ for different values of disorder $R$ and lattice size $L_1 - L_2$. $x_1$ is computed by performing a best fit of the magnetization on three successive lattice size between $L_1$ and $L_2$. Errors on the last quoted digit are indicated in parentheses.

| $L_1 - L_2$ | R=2  | 5    | 8    | 10   | 20   | 100  | 1000 |
|-------------|------|------|------|------|------|------|------|
| L=5-20      | 0.162 (1) | 0.160 (1) | 0.156 (1) | 0.155 (1) | 0.151 (1) | 0.138 (1) | 0.115 (1) |
| 10-50       | 0.175 (1) | 0.159 (1) | 0.155 (1) | 0.153 (1) | 0.150 (1) | 0.142 (1) | 0.119 (1) |
| 20-100      | 0.167 (2) | 0.156 (1) | 0.154 (1) | 0.153 (1) | 0.150 (1) | 0.144 (1) | 0.123 (1) |
| 50-200      | 0.158 (3) | 0.149 (2) | 0.155 (3) | 0.152 (2) | 0.152 (2) | 0.147 (2) | 0.130 (1) |
| 100-500     |           |      |      | 0.151 (4) |      |      |      |
| 200-1000    |           |      |      |      | 0.151 (7) |      |      |
| 1000        |           |      |      |      |      |      |      |