FRW Cosmology with Non-positively Defined Higgs Potentials

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Abstract

We discuss the classical aspects of dynamics of scalar models with non-positive Higgs potentials in the FRW cosmology. These models appear as effective local models in non-local models related with string field theories. After a suitable field redefinition these models have the form of local Higgs models with a negative extra cosmological term and the total Higgs potential is non-positively defined and has rather small coupling constant. The non-positivity of the potential leads to the fact that on some stage of evolution the expansion mode gives place to the mode of contraction, due to that the stage of reheating is absent. In these models the hard regime of inflation gives place to inflation near the hill top and the area of the slow roll inflation is very small. Meanwhile one can obtain enough e-foldings before the contraction to make the model under consideration admissible to describe inflation.

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1 Introductory remarks

Scalar fields play a central role in current models of the early Universe – the majority of present models of inflation requires an introduction of an additional scalar – the inflaton [1, 2, 3, 4, 5]. Though the mass of the inflaton is not fixed, typical considerations deal with a heavy scalar field with mass $10^{13}$ GeV [6]. The interaction of the inflaton (self-interacting and interacting with matter fields) is not fixed too, moreover, in typical examples the self-interaction potential energy density of the inflaton is undiluted by the expansion of the universe and it acts as an effective cosmological constant. However the detailed evolution depends on the specific form of the potential $V$. In the last years, in particular, Higgs inflation has attracted a lot of attention [9]-[14]. The kinetic term usually has the standard form, however models with nonstandard kinetic terms also have attracted attention in the literature [15]-[17]. The present paper deals with models mainly related with a nonlocal kinetic term of a special form [17]-[38].

The distinguish property of these models is their string field (SFT) origin [39]. The original motivation to deal with this type of nonlocal cosmological models was related with the dark energy problems [17]. The relevance of this type of models to the early Universe problems has been pointed out in [25, 26]. In this case the scalar field is the tachyon of the NSR fermion string and the model has the form of a nonlocal Higgs model. These nonlocal factors produce essential changes in cosmology. Due to the effect of stretching the potential with the kinetic energy advocated by Lidsey [25], Barnaby and Cline [26] nonlocality destroys the relation between the coupling constant in the potential, the mass term and the value of the vacuum energy (cosmological constant) and produces the total Higgs potential with the negative extra cosmological term. This total Higgs potential is non-positively defined.

We discuss the classical aspect of dynamics of scalar models with non-positive Higgs potentials in the FRW cosmology. As it has been mentioned above these models appear as effective local models in nonlocal models of a special form related with string field theory.

Since nonlocality can produce an effective local theory with rather small coupling constant some stages of evolution can be described by the free tachyon approximation. Due to this reason we start from consideration of dynamics of the free tachyon in the FRW metric. We show that generally speaking there are 4 stages of evolution of the tachyon in the FRW metric: reaching the perturbative vacuum from the past cosmological singularity; rolling from the perturbative vacuum; transition from inflation to contraction and in the ending contraction to the future cosmological singularity. We present approximations suitable to study evolution near the past cosmological singularity, evolution near the top of the hill and evolution near the transition from expansion to contraction.

As to the case of the quartic potential the situation depends on the constant relations. For very small coupling constant the hard regime of inflaton gives place to inflation near the hill top making the area of the slow-roll inflation very small. Meanwhile the model can perform enough e-foldings to make the model under consideration admissible do describe inflation.

The plan of the paper is the following. We start, Sect. 2.1, from the short exposition why the nonlocality of the SFT type produces non-positively defined potentials. In Sect. 2.2 we remind the known facts about massive scalar field and the positively defined Higgs
potential.

In Sect.3 we consider dynamics of the free tachyon in the FRW metric. First, we show the presence of the cosmological singularity \[51\]. Then we show that starting from the top of the hill the tachyon first evolves as an inflaton, i.e. it keeps the inflation regime. After performing a finite number of e-foldings \( N_{\text{max}} \) the tachyon reaches the boundary of the forbidden domain at some point \( (\phi_{\text{max}}, \dot{\phi}_{\text{max}}) \) of the phase diagram. At this point the Hubble \( H \) becomes equal to zero and the tachyon continues to evolve with negative \( H \), i.e. the tachyon becomes in fact a “contracton”. We study the phase portrait of the system and conclude that near the top we can use the de Sitter approximation and, moreover, we can restrict ourself to consideration of one mode of the tachyon evolution, so called \( C_1 \)-mode, and find corrections to the dS approximation. This regime has been previously studied in \[40, 42\]. The \( C_1 \)-mode is a more viable mode and in the case of the zero cosmological constant describes exponentially increasing solution at \( t \to \infty \). To study dynamics of the system near the boundary of the accessible area in more details we use a special parametrization, that is in fact a generalization of a parameterizations that one has used previously to study dynamics of the inflaton with a positively defined potential in the FRW metric \[51\] and refs. therein. Using this parametrization we find asymptotic behavior near the boundary of the accessible region.

In Sect.4 we consider dynamics of the Higgs field in the FRW metric in the presence of an extra negative cosmological constant. This constant makes the total potential non-positively defined and produces the forbidden area in the phase space of the Higgs field.

In conclusion we briefly summarize new cosmological features for Higgs potentials with extra negative cosmological constant. The perturbation analysis will be the subject of forgoing paper.

2 Setup

2.1 Non-positively defined potentials as a consequence of nonlocality

In this section we make few remarks why scalar matter nonlocality leads to essential changes of properties of corresponding cosmological models in comparison with those of pure local cosmological models (we mean early cosmological models). These changes appear due to an effective stretch of the kinetic part of the matter Lagrangian.

We deal with the nonlocal action originated from SFT \[17\]

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{m^2}{2} R + \left( \frac{1}{2} \phi (\xi^2 \Box + \mu^2) e^{-\lambda \Box} \phi - V(\phi) - \Lambda \right) \right],
\]

where \( \Box \) is the D’Alembertian operator, \( \Box \equiv \sqrt{-g} \partial^\mu (\sqrt{-g} \partial_\mu) \), \( \phi \) is a dimensionless scalar field and all constants are dimensionless.

The spatially isotropic dynamics of this field in the FRW metric, with the interval:

\[
ds^2 = -dt^2 + a^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right),
\]
is given by nonlocal equation of motion
\[
e^{\lambda(\partial^2 + 3H\partial)} \left( \xi^2 (\partial^2 + 3H\partial) - \mu^2 \right) \phi(t) = -V_\phi',
\]
\[
3H^2 = 8\pi G \mathcal{E},
\]
(2.1.2)
where the Hubble parameter \( H = \dot{a}/a \), \( \mathcal{E} \) is a sum of nonlocal modified kinetic and potential energies, that are explicit functionals of \( \phi \) and \( \dot{\phi} \). \( \mathcal{E} \) is given by [17]-[33] (see also [47]-[49])
\[
\mathcal{E} = 2E_00 - g_{00} (g^{\rho\sigma}E_{\rho\sigma} + W),
\]
where
\[
E_{00} \equiv \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{l=0}^{n-1} \partial_0 \square_g^l \phi \partial_0 \square_g^{n-1-l} \phi,
\]
\[
W \equiv \frac{1}{2} \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \sum_{l=1}^{n-1} \square_g^{l} \phi \square_g^{n-l} \phi + \frac{f_0}{2} \phi^2 + V(\phi).
\]
In the main consideration we deal with \( V(\phi) = \epsilon \phi^4 \) and \( f_0 = -\mu^2 \) (the tachyon case).

In comparison with the local scalar field dynamics in the FRW metric
\[
\ddot{\phi} + 3H \dot{\phi} = -V_\phi',
\]
\[
3H^2 = 8\pi G \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
\]
(2.1.4)
(2.1.5)
there are two essential differences: there is a nonlocal kinetic term in the equation for scalar field \( 2.1.2 \) as well as the form of the energy is different. Generally speaking the system \( 2.1.2, 2.1.3 \) is a complicated system of nonlocal equations and its study is a rather nontrivial mathematical problem. There are several approaches to study this system. Let us first mention the results of study of equation \( 2.1.2 \) in the flat metric, i.e. equation \( 2.1.2 \) with \( H = 0 \). In the flat metric for the quartic potential it is known that for \( \xi < \xi_c \) there exists an interpolating solution of \( 2.1.2 \) between two vacua. There is a numerical evidence for existence of oscillator solutions for \( \xi > \xi_c \). In the LHS of \( 2.1.2 \) there is a nonlocal term that is related with the diffusion equation. This relation provides rather interesting physical consequences [50].

The case of the FRW metric in the context of the DE problem has been studied in [17,20,28,29]. In the context of the inflation problem this system (especially in the case \( \xi = 0 \)) has been studied in [25,26]. In [25] it has been pointed out that nonlocal operators lead to the effects that can be quantified in terms of a local field theory with a potential whose curvature around the turning point is strongly suppressed.

In the case of a quadratic potential a cosmological model with one nonlocal scalar field can be presented as a model with local scalar fields and quadratic potentials [22,23,24,29,34,36,37]. For this model it has been observed [25,26] that in the regime suitable for the Early Universe the effect of the effective stretch of the kinetic terms takes place (see

\[\text{1}\] There is no interpolating solution for the case of cubic potential [47].
One can generalize this result to a nonlinear case and assume that a nonlocal nonlinear model can be described by an effective local nonlinear theory with the following energy density

\[ E \approx e^{\lambda \Omega^2} \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \mu^2 \phi^2 \right) + \frac{1}{4} \epsilon \phi^4 + \Lambda, \]  

(2.1.6)

and the tachyon dynamics equation becomes

\[ \partial^2 \phi + 3H \partial \phi - \mu^2 \phi = -ee^{-\lambda \Omega^2} \phi^3, \]  

(2.1.7)

where

\[ H = \frac{1}{\sqrt{3}m_p} \sqrt{e^{\lambda \Omega^2} \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \mu^2 \phi^2 \right) + \frac{1}{4} \epsilon \phi^4 + \Lambda} \]  

(2.1.8)

and we take \( \xi = 1 \) for simplicity. \( \Omega \) is a modified “frequency” of the asymptotic expansion for equation (2.1.7) in the flat case. In the first approximation \( \Omega = \mu \).

Or in other words, one can say that the higher powers in \( \Box \) cause the factor \( e^{\lambda \Omega^2} \), where \( \Omega^2 \) in the first approximation is just equal to \( \mu^2 \). Therefore, the corresponding effective action becomes

\[ S = \int d^4x \sqrt{-g} \left[ \frac{m_p^2}{2} R + \left( \frac{1}{2} \phi(\Box + \mu^2)e^{\lambda \Omega^2} \phi - \frac{\epsilon \phi^4}{4} - \Lambda \right) \right]. \]  

(2.1.9)

In term of the canonical field \( \varphi \equiv e^{\lambda \Omega^2/\epsilon} \phi \) the action (2.1.9) becomes

\[ S = \int d^4x \sqrt{-g} \left[ \frac{m_p^2}{2} \varphi^2 + \left( \frac{1}{2} \varphi(\Box + \mu^2)\varphi - \frac{\epsilon}{4} e^{-2\lambda \Omega^2} \varphi^4 - \Lambda \right) \right]. \]  

(2.1.10)

Just this form is responsible for equation (2.1.7) for \( V(\phi) = -\frac{\epsilon \phi^4}{4} \). This equation is nothing more than equation for the Higgs scalar field considered as an inflaton. There is only one difference, that in fact, as we will see, is very important. The potential for the Higgs model being considered usually in the context of the cosmological applications is

\[ V(\phi) = \frac{\epsilon}{4} (\phi^2 - a^2)^2, \]  

(2.1.11)

i.e. the cosmological constant \( \Lambda = \frac{\epsilon}{4} a^4 \) is such that the potential is the positively defined potential. Let us remind that in the SFT inspired nonlocal model the value of the cosmological constant is also fixed by the Sen conjecture [39, 43, 44] that exactly puts the potential energy of the system in the nontrivial vacua equal to zero

\[ V(\phi) = -\frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \epsilon \phi^4 + \Lambda = \frac{1}{4} \epsilon \left( \phi^2 - \frac{\mu^2}{\epsilon} \right)^2, \]  

(2.1.12)

that gives us he value of \( \Lambda = \frac{\mu^4}{4 \epsilon} \). The appearance of the extra factor \( e^{\lambda \Omega^2} \) destroys this positivity

\[ V_{EL}(\phi) = -\frac{1}{2} \mu^2 e^{\lambda \Omega^2} \phi^2 + \frac{1}{4} \epsilon \phi^4 + \frac{\mu^4}{4 \epsilon} = \frac{1}{4} \epsilon \left( \phi^2 - \frac{\mu^2}{\epsilon} e^{\lambda \Omega^2} \right)^2 + \frac{\mu^4}{4 \epsilon} \left( 1 - e^{2\lambda \Omega^2} \right), \]  

(2.1.13)

(here "EL" means "effective local model"). Since \( \lambda > 0 \), we have to deal with non-positively defined potential.
2.2 Phase portraits for positively defined potentials

Cosmological models with one scalar field with positively defined potential are well studied models, see for example [2, 51, 54, 55, 56]. Let us remind the main features of the phase portraits of these models. The phase portrait for a free massive field that is a solution of

\[ \ddot{\phi} + 3 \sqrt{\frac{8\pi G}{3}} \left( \frac{1}{2} \dot{\phi}^2 + \frac{m^2}{2} \phi^2 \right) \dot{\phi} = -m^2 \phi, \]  

(2.2.1)

is presented in Fig.1A. The same phase portrait is presented in Fig.1B in compactified variables. The advantage of this picture is that we see the existence of the cosmological singularity. The phase portrait for the Higgs field with positively defined potential

\[ \ddot{\phi} + 3 \sqrt{\frac{8\pi G}{3}} \left( \frac{1}{2} \dot{\phi}^2 + \frac{\epsilon}{4} (\phi^2 - a^2)^2 \right) \dot{\phi} = -\epsilon \phi^3 + \epsilon a^2 \phi, \]  

(2.2.2)

is presented in Fig.2A.

Figure 1: (color online) A. The phase portrait of equation (2.2.1). B. Inflaton flows in compactified variables \( \psi \) vs \( \rho \) are given by the phase portrait of equations (2.2.6) and (2.2.7).

Comparing the inflaton flows in Fig.1A, Fig.2A, we see that there is no big difference between the phase portrait presenting the chaotic inflation (2.2.1) and the phase portrait of eq. (2.2.2). This is in accordance with the general statement about a weak dependence of the inflaton flow on the potentials. Only explicit formulae describing the slow-roll regime as well as the oscillation regime are different. In all these cases there are 3 stages (eras) of evolution. Just after cosmological singularity the ultra-hard regime starts. Then in the case of any positive/negative initial position \( \phi(0) > 0/\phi(0) < 0 \) the inflaton reaches very fast the “right”/“left” attractor. Then the inflaton moves along the “right”/“left” slow-roll line. In particular, in Fig.1A there is almost horizontal line that corresponds to the “right” slow-roll
Figure 2: (color online) A. The phase portrait of equation (2.2.2). B. The Phase portrait of the system of equations (2.2.8), (2.2.9).

line and it is seen in the plot as a common part of the red, pink and magenta lines. Then oscillations near the point $\phi = 0, \dot{\phi} = 0$ Fig. 1.A, or near one of two nonperturbative vacua Fig.2.A start.

It is well known that it is convenient to present the phase portraits in the compactified variables $[51] \ (G = \frac{1}{m_p^2})$

\[
\begin{align*}
\phi & = \frac{3m_p}{\sqrt{12\pi}} \frac{\rho}{1-\rho} \sin \theta \cos \psi, \\
\dot{\phi} & = \frac{3mm_p}{\sqrt{12\pi}} \frac{\rho}{1-\rho} \sin \theta \sin \psi, \\
H & = m \rho \frac{1}{1-\rho} \cos \theta.
\end{align*}
\]  

(2.2.3)  
(2.2.4)  
(2.2.5)

With the help of these variables we can rewrite equations (2.2.1) and $\dot{H} = -\frac{4\pi \dot{\phi}^2}{m_p^2}$ in the following form (here we use the time variable $\sigma$ such that $\frac{d\sigma}{d\tau} = \frac{1}{1-\rho}$ where $\tau = mt$)

\[
\begin{align*}
\rho_\sigma & = -\frac{3\rho^2(1-\rho)}{\sqrt{2}} \sin^2 \psi, \\
\psi_\sigma & = -(1-\rho) - \frac{3\rho}{2\sqrt{2}} \sin 2\psi,
\end{align*}
\]  

(2.2.6)  
(2.2.7)
and equation for the Higgs inflaton (2.2.2) has the form \((g = \frac{3m^2}{4\pi a^2})\):

\[
\rho_\sigma = (1 - \rho)^2 \rho \sin^2 \theta \sin 2\psi - \rho^2 (1 - \rho) \cos \theta \left(1 + 4 \sin^2 \theta \sin^2 \psi\right)
\]
\[+ \frac{1}{2g} (1 - \rho)^3 \cos \theta + g \rho^3 \sin^4 \theta \cos^3 \psi \left(\frac{\rho}{1 - \rho} \cos \psi \cos \theta - \sin \psi\right),
\]

\[
\psi_\sigma = -\frac{3}{2} \rho \cos \theta \sin 2\psi + (1 - \rho) \cos 2\psi - g \frac{\rho^2}{1 - \rho} \sin^2 \theta \cos^4 \psi.
\]

\[
\theta_\sigma = (1 - \rho) \sin 2\psi \sin 2\theta - \frac{1}{2g} (1 - \rho)^2 \sin \theta + \rho \sin \theta \left[1 - 3 \sin^2 \psi\right]
\]
\[+ 2 \sin^2 \psi \sin^2 \theta \] - \[g \frac{\rho^2}{1 - \rho} \sin^3 \theta \cos^3 \psi \left[\sin \psi \cos \theta + \frac{\rho}{1 - \rho} \sin^2 \theta \cos \psi\right] (2.2.8)
\]

Phase portraits for systems of equations (2.2.6), (2.2.7) and (2.2.8), (2.2.9) are presented in Fig.1.B. and Fig.2.B.

3 Free Tachyon in the FRW space

3.1 The exact phase portrait of the free tachyon

The dynamics of the free tachyon in the FRW metric corresponding to the expansion \((H > 0)\) is described by the following equation

\[
\ddot{\phi} + 3\sqrt{\frac{8\pi G}{3}} \left(\frac{1}{2} \dot{\phi}^2 - \frac{\mu^2}{2} \phi^2 + \Lambda\right) \dot{\phi} = \mu^2 \phi.
\]

(3.1.1)

The phase portraits of eq. (3.1.1) is presented in Fig.3.A. We see that there are two forbidden regions in the phase space. The distance between these forbidden regions depends on the value of the cosmological constant. The boundary of the accessible domain is given by the relation \(\dot{\phi} = \pm \sqrt{\mu^2 \phi^2 - 2\Lambda}\). The accessible region is divided by 4 domains, I, II, III and IV, that are separated by two separatrices: \(S_1\) and \(S_2\), see Fig.3.A. The upper half of the separatrix \(S_1\) is an attractor (the “right” attractor), and the low part of the separatrix \(S_1\) is the other attractor (the “left” attractor). The separatrix \(S_2\) consists of two repulsers.

In Fig.3.A, we see several eras of the tachyon evolution. Let us consider trajectories starting at region I. First, the tachyon reaches the area of the right attractor. In particular, starting from zero initial value \(\phi(0) = 0\) (the corresponding trajectories are presented in Fig.3.A. by colored lines) the tachyon decreasing the value of \(H\) reaches the area near the boundary of the accessible domain, where \(H = 0\). In Fig.4.A, the lines with fixed positive values of \(H\) are denoted by the blue dashed lines. In Fig.4.B we present the 3d plot of values of \(H\) depending on the position in the phase diagram. In the end of this era of evolution the tachyon reaches the line corresponding to \(H = 0\). Then the tachyon keeps moving in the area of the negative \(H\), see Fig.3.B, regions V and VI. Therefore, starting at region I the tachyon continues its evolution in region V. The tachyon EOM in regions V and VI has the
Figure 3: (color online) A. The phase portrait of equation (3.1.1). The accessible region is divided by 4 domains, that are separated by two separatrices: $S_1$ (the red dotted line) and $S_2$ (the magenta dotted line). Colored lines present trajectories starting from different points of the phase diagram. B. The phase portrait of equation (3.1.2). Continuations of the trajectories presented in the left panel to solutions of the Friedman equation with negative values of $H$ are indicated by the same numbers with primes added.

We see from the phase portrait Fig.3 that for finite $\phi$ and $\dot{\phi}$ there are no critical points. To search for critical points in the infinite phase space as well as to justify approximations in different domains of the phase space (in particular, in neighborhood of the forbidden domains) it is convenient following to [51] to rewrite the scalar field equation in FRW space-time as a system of equations for 3-dimensional dynamical system. There are two types of slightly different recipes how to do this. One is given in [51] and uses variables (3.3.2)-(3.3.4) (see below) and the second one is given in [53] for positively defined potentials and is given by formula (3.7.1)-(3.7.3). The first recipe is convenient to study the cosmological singularity (see sect.3.3) and the second one – to study a transition from expansion to contraction (see sect.3.7).

3.2 Free Tachyon in FRW in the slow-roll regime

In the slow-roll regime one usually ignores the second derivative of the field on time in the equation of motion and the kinetic term in the equation for $H$. In this case the tachyon equation becomes

$$3\sqrt{\frac{8\pi G}{3}} \left( \frac{1}{2} \ddot{\phi} - \frac{\mu^2}{2} \phi^2 + \Lambda \right)^{\dot{\phi}} = \mu^2 \phi.$$  

(3.2.1)
A. The same phase portrait as in Fig. 3A. with marked lines (dashed blue lines) of the constant values of $H$. The thick dashed line corresponds to $H^2 = 3\kappa\Lambda$ ($\kappa = \frac{8\pi G}{9}$). The solid blue line corresponds to $H = 0$ and defines the boundary of the accessible domain. B. The plot of the functional $\kappa(\frac{1}{2}\dot{\psi}^2 - \frac{\mu^2}{2}\dot{\phi}^2 + \Lambda)$ that defines the value of $H^2$ (here $\dot{\phi} = \psi$). The intersection with the yellow plane corresponds to the boundary of the accessible region.

This equation obviously has a meaning only for

$$|\phi| < \frac{\sqrt{2\Lambda}}{\mu}.$$  

According to the common approach we can use the slow-roll approximation when slow-roll parameters $\varepsilon$ and $\eta$ are very small

$$\varepsilon = \frac{m_p^2}{16\pi} \left( \frac{V'}{V} \right)^2 \ll 1,$$

$$|\eta| = \left| \frac{m_p^2}{8\pi} \left( \frac{V''}{V} \right) \right| \ll 1.$$  

We can see the typical behaviour of these parameters for the model being considered in Fig. 5A and B in the case when the slow-roll conditions are satisfied. As we see in some interval of $\phi$ values near 0 the slow-roll parameters are small enough. That means that one
Figure 5: (color online) A. The dependence of slow-roll parameter $\varepsilon$ on $\phi$ in the case when the slow-roll conditions are satisfied. B. The dependence of slow-roll parameter $\eta$ on $\phi$ in the case when the slow-roll conditions are satisfied. C. The phase portraits for the model when the slow-roll conditions are satisfied on some interval of $\phi$ values near 0. The green solid lines correspond to different exact trajectories, the blue dashed and dot line corresponds to the slow-roll trajectory. D. The zoomed part of the Fig.C near (0,0) where the slow-roll conditions are highly satisfied.
should expect a good approximation of slow-roll solution for the exact phase trajectories on this interval.

We can see in the phase portraits shown in Fig. 5 C and D that really in this case on some interval of $\phi$ values near 0 the exact phase trajectories are close enough to the slow-roll trajectory. When the slow-roll parameters become large enough we see the difference between the exact trajectories and the slow-roll trajectory.

In the case when the slow-roll conditions are not satisfied the slow-roll parameters evolve as it’s shown in Fig. 6 A and B. The phase portraits for this case are shown in Fig. 6 C and D. As we see the exact trajectories cross the slow-roll trajectory but don’t keep on evolving along it. It means that the slow-roll approximation really cannot be used for the description of the exact trajectories in this case. We can note that in the region where $\dot{\phi}$ is small enough the minima of the exact phase trajectories are situated on the slow-roll trajectory. It is connected with the fact that the minimum of the exact phase trajectory corresponds to the point where $\ddot{\phi} = 0$ and if we also can neglect $\phi$ in the expression for $H$ then from the exact equation of motion we obtain the point of the slow-roll trajectory.

### 3.3 Tachyon dynamics near cosmological singularity

To study tachyon dynamics near the cosmological singularity let us consider for simplicity the case of the zero cosmological constant

$$\ddot{\phi} + 3\sqrt{\frac{8\pi G}{3}} \left( \frac{1}{2} \dot{\phi}^2 - \frac{\mu^2}{2} \phi^2 \right) \dot{\phi} = \mu^2 \phi.$$  \hfill (3.3.1)

This approach is valid when $\phi$ and $\dot{\phi}$ are large enough.

In terms of dimensionless variables

$$X = \frac{\sqrt{12\pi}}{3m_p} \phi,$$ \hfill (3.3.2)

$$Y = \frac{\sqrt{12\pi}}{3\mu m_p} \dot{\phi},$$ \hfill (3.3.3)

$$Z = \frac{H}{\mu},$$ \hfill (3.3.4)

$$\tau = \mu t,$$ \hfill (3.3.5)

system (3.3.1) has the form

$$X_\tau = Y,$$ \hfill (3.3.6)

$$Y_\tau = +X - 3ZY,$$ \hfill (3.3.7)

$$Z_\tau = -X^2 - 2Y^2 - Z^2.$$ \hfill (3.3.8)
Figure 6: (color online) A. The dependence of slow-roll parameter $\varepsilon$ on $\phi$ in the case when the slow-roll conditions are not satisfied. B. The dependence of slow-roll parameter $\eta$ on $\phi$ in the case when the slow-roll conditions are not satisfied. C. The phase portraits for the model when the slow-roll conditions are not satisfied. The green solid lines correspond to different exact trajectories, the blue dashed and dot line corresponds to the slow-roll trajectory. D. The zoomed part of the Fig.C near (0,0).
To get compactification of the phase space we use the coordinates of the unit ball

\[
X = \frac{\rho}{1 - \rho} \sin \theta \cos \psi, \quad (3.3.9)
\]

\[
Y = \frac{\rho}{1 - \rho} \sin \theta \sin \psi, \quad (3.3.10)
\]

\[
Z = \frac{\rho}{1 - \rho} \cos \theta, \quad (3.3.11)
\]

and in these new coordinates the equation of motion takes the form

\[
\rho_{\sigma} = \rho (1 - \rho)^2 \left[ \sin 2\psi \sin^2 \theta - \frac{\rho}{1 - \rho} \cos \theta \left( 1 + 4 \sin^2 \psi \sin^2 \theta \right) \right], \quad (3.3.12)
\]

\[
\psi_{\sigma} = -\frac{3}{2} \rho \sin 2\psi \cos \theta + (1 - \rho) \cos 2\psi, \quad (3.3.13)
\]

\[
\theta_{\sigma} = \frac{1}{2} (1 - \rho) \sin 2\psi \sin 2\theta + \rho \sin \theta \left( 1 + \sin^2 \psi - 4 \cos^2 \theta \sin^2 \psi \right), \quad (3.3.14)
\]

Here

\[
\frac{d\sigma}{d\tau} = \frac{1}{1 - \rho}.
\]

This system has 4 critical points with \( \rho = 1 \) lying on the cone

\[
Y^2 = X^2 + Z^2,
\]

that in the spherical coordinates has the form

\[
\tan^2 \theta = -\frac{1}{\cos 2\psi}. \quad (3.3.15)
\]

These critical points are

\[
\theta_0 = \frac{\pi}{4}, \quad \psi_0 = \frac{\pi}{2}, \quad (3.3.16)
\]

\[
\theta_0 = \frac{\pi}{4}, \quad \psi_0 = -\frac{\pi}{2}, \quad (3.3.17)
\]

\[
\theta_0 = \frac{3\pi}{4}, \quad \psi_0 = \frac{\pi}{2}, \quad (3.3.18)
\]

\[
\theta_0 = \frac{3\pi}{4}, \quad \psi_0 = -\frac{\pi}{2}. \quad (3.3.19)
\]

Let us now consider the behavior of the system near the critical point (3.3.16), i.e. \( \rho = 1, \ \theta_0 = \frac{\pi}{4}, \ \psi_0 = \frac{\pi}{2} \). The have near this point in the linear approximation

\[
\frac{d}{d\sigma} \Delta \rho (\sigma) = \frac{3}{2} \Delta \rho (\sigma) \sqrt{2}, \quad (3.3.20)
\]

\[
\frac{d}{d\sigma} \Delta \psi (\sigma) = -\Delta \rho (\sigma) + \frac{3}{2} \sqrt{2} \Delta \psi (\sigma), \quad (3.3.21)
\]

\[
\frac{d}{d\sigma} \Delta \theta (\sigma) = 2 \sqrt{2} \Delta \theta (\sigma). \quad (3.3.22)
\]
Figure 7: (color online) A. The 3-dimensional picture of the dynamics of the tachyon (system of equations (3.3.12)-(3.3.14)). We see that all trajectories starting on the conus keep moving on the conus. The red parts of trajectories correspond to the expansion and the blue part to the contraction. B. The projection of the previous picture on the \(XY\)-plane. This picture is the tachyon analogue of the picture for the free particle in the FRW presented in the Fig.1

The eigenvalues are \((\sqrt{2}, \sqrt{2}, \sqrt{2})\), and solutions to this system are

\[
\begin{align*}
\Delta \rho (\sigma) &= C_2 e^{\frac{\sqrt{2}}{2} \sigma}, \\
\Delta \psi (\sigma) &= (C_1 - C_2 \sigma) e^{\frac{\sqrt{2}}{2} \sigma}, \\
\Delta \theta (\sigma) &= C_3 e^{\frac{\sqrt{2}}{2} \sigma}.
\end{align*}
\]

(3.3.23) (3.3.24) (3.3.25)

3.4 Approximation of tachyon dynamics in the FRW by dynamics in the dS space

During the evolution of the tachyon between two dashed lines, see Fig.4A, corresponding to \(H = H_0 \pm \Delta H_0\), we can ignore the dependence of \(H\) on time and we take an approximate

\[
\ddot{\phi} + 3H_0 \dot{\phi} = \mu^2 \phi.
\]

(3.4.1)

Near the thick blue dashed line we can use

\[
H_0 = \sqrt{\frac{8\pi G}{3}} \Lambda,
\]

(3.4.2)

and near the starting point with given \(\phi_0\) and \(\dot{\phi}_0\) we can use the approximation

\[
H_0(\phi_0, \dot{\phi}_0) = \sqrt{\frac{8\pi G}{3} \left(\frac{1}{2} \phi_0^2 - \frac{\mu^2}{2} \phi_0^2 + \Lambda\right)}.
\]
In Fig. 8A we present the phase diagram for the tachyon evolution in the dS space, equation (3.4.1). We see that there are two attractors and two repulsers. The phase portraits for dynamics of the tachyon in the dS space with two different $\Lambda$ are presented in Fig. 8B by blue and magenta vectors (the blue vectors for small $\Lambda$ and the magenta vectors for large $\Lambda$).

Figure 8: (color online) A. The phase portrait of the tachyon evolving in the dS space. There are two attractors and two repulsers. B. A comparison of the tachyon phase portraits in the dS space with different $\Lambda$ (the blue vectors for small $\Lambda$, the magenta vectors for large $\Lambda$). The blue/cyan and red/pink solid lines present the dynamics of the tachyon with the same initial data in the dS space-time with different values of the cosmological constant $\Lambda$. The blue/red and cyan/pink lines have the same initial coordinates, but different initial velocities.

Equation (3.4.1) has solutions

$$\phi(t) = C_1 e^{r_+ t} + C_2 e^{r_- t},$$  \hspace{1cm} (3.4.3)

where

$$r_\pm = -\frac{3}{2} H_0 \pm \frac{3}{2} \sqrt{H_0^2 + \frac{4}{9} \mu^2},$$ \hspace{1cm} (3.4.4)

i.e. $r_+ > 0$, $r_- < 0$. We see that for $H_0 > 0$ the $C_2$-mode decreases faster in comparison with the $C_1$-mode that increases for large $t$.

For the initial date $\phi(0) = 0$ we have

$$\phi(t) = C e^{-3/2 H_0 t} \sinh \left( \frac{3H_0}{2} \sqrt{1 + \frac{4}{9} \frac{\mu^2}{H_0^2}} t \right), \hspace{1cm} C_1 = C/2$$ \hspace{1cm} (3.4.5)

and

$$\lim_{t \to \infty} \frac{\dot{\phi}(t)}{\phi(t)} = -\frac{3}{2} H_0 + \frac{3H_0}{2} \sqrt{1 + \frac{4}{9} \frac{\mu^2}{H_0^2}}.$$ \hspace{1cm} (3.4.6)
This is exactly the same answer if we keep only the first term in (3.4.3). One can say that $C_1$-mode is an attractor for all solutions of the tachyon dS equation with zero initial coordinates. $C_1$-mode is important for large times, but we cannot ignore the $C_2$-mode for small times. We also see that the $C_1$-mode is a dominating mode for $t > 0$ for the case when $H_0 = 0$. One can say that the $C_1$-mode is a viable mode and the $C_2$-mode is a transient one.

To select the $C_1$-mode one can also assume that one deals with the solution such as if the tachyon was sitting on the top of the hill at $t = -\infty$.

Let’s note that the $C_1$-mode corresponds to the slow-roll trajectory on the interval where we can consider $H$ as an approximate constant. It means that the slow-roll trajectory is a quasi-attractor for the exact trajectories.

To illustrate how the approximation (3.4.1) works, in Fig.9 we compare phase portraits for equations (3.1.1) and (3.4.1). We see that for small times and small initial velocities the approximation works, but it is failed for large times.

Figure 9: (color online) A. A comparison of the exact evolution (numerical solution of eq. (3.1.1)) with an approximate evolution (3.4.1) with $H_0$ given by (3.4.2). The solid green and khaki lines in the left panel represent the exact trajectories and the blue and cyan dashed lines represent the approximate trajectories. Magenta vectors in the left panel represent the phase portrait of the exact equation. Blue vectors in the right panel represent the phase portrait in the dS space. B. The solution and the phase portraits for the approximate equation.

To estimate a deviation of $H$ from the initial $H_0$ we calculate the energy of solution (3.4.3) of the tachyon equation in the dS space-time. The energy of this solution is a sum of the energies of two modes [24]

$$E = E_{C_1} + E_{C_2}.$$
where

\[ E_{C_1} = \frac{1}{2} \frac{\dot{\phi}^2}{\phi C_1} - \frac{\mu^2}{2} \frac{\phi^2}{\phi C_1} = C_1^2 \frac{9}{4} H_0^2 \left( 1 - \sqrt{1 + \frac{4}{9} \frac{\mu^2}{H_0^2}} \right) e^{2r_+ t}, \] (3.4.7)

\[ E_{C_2} = \frac{1}{2} \frac{\dot{\phi}^2}{\phi C_2} - \frac{\mu^2}{2} \frac{\phi^2}{\phi C_2} = C_2^2 \frac{9}{4} H_0^2 \left( 1 + \sqrt{1 + \frac{4}{9} \frac{\mu^2}{H_0^2}} \right) e^{2r_- t}. \] (3.4.8)

We see that \( C_1 \) gives a negative contribution to the energy and \( C_2 \)-mode gives a positive one. Since the \( C_1 \)-mode is more viable mode its negative energy contribution dominates and the tachyon loses the initial energy in the FRW space up to zero value.

### 3.5 Inflation over the hill

In this subsection we remind the consideration of a scalar field rolling over the top of a local maximum of a potential following the paper by Tzirakis and Kinney [42]. This evolution needs a special attention as it cannot be considered in framework of the slow-roll approximation. It follows from the fact that in the slow-roll limit

\[ \dot{\phi} \propto V' (\phi) = 0, \]

at the maximum of the potential.

The tachyon equation (3.4.1) can be rewritten in terms of the number of e-foldings \( N \), \( N \equiv \ln a/a_{\text{initial}} \) (we omit \( a_{\text{initial}} \) ) as

\[ \ddot{\phi} + 3 \dot{\phi}' = \nu^2 \phi, \] (3.5.9)

where \( \nu^2 = \frac{\mu^2}{H_0^2} \) and \( \phi' = \frac{d\phi}{dN} \). The general solution of (3.5.9) is

\[ \phi(N) = \phi_+ e^{\nu_+ N} + \phi_- e^{\nu_- N}, \] (3.5.10)

with \( \nu_\pm = -\frac{3}{2} \left( 1 \mp \sqrt{1 + \frac{4}{9} \nu^2} \right) \). This form of the tachyon evolution is just the reparametrization of (3.4.3).

If \( \nu^2 \) is small, the constants \( \nu_\pm \) become in the first approximation

\[ \nu_+ = \frac{\nu^2}{3}, \]
\[ \nu_- = -3 - \frac{\nu^2}{3}. \] (3.5.11)

As in the previous consideration the general solution (3.5.10) can be decomposed into two branches: a transient branch \( \nu_- \) which dominates at early times, and a “slow-rolling” branch \( \nu_+ \) which can be identified as the late-time attractor.

If we consider these branches separately we see that the slow-roll parameters are [42]

\[ \varepsilon = 4\pi \nu^2 \left( \frac{\dot{\phi}}{m_p} \right)^2, \]
\[ \eta = -\nu_\pm + \varepsilon. \] (3.5.12)
Near the maximum of the potential being considered, \( \phi \approx 0 \) and \( \varepsilon \ll 1 \), so that
\[
\eta \approx -\nu_\pm = \text{constant.} \tag{3.5.13}
\]
Even though we assume \( \varepsilon \) to be small, it is obvious from eqs. (3.5.11) and (3.5.13) that the \( \nu_- \) branch will never satisfy the slow-roll conditions, since \( \eta > 3 \) always. For the \( \nu_+ \) branch, the slow-roll limit is obtained for \( \nu \ll 1 \) for which
\[
|\eta_{SR}| \approx \frac{\nu^2}{3} \ll 1.
\]
It was also shown in [42] that \( \nu_- \) branch corresponds to a field rolling up the potential.

If we consider the whole solution (3.5.10), we see that in order for the field to be a monotonic function of time, its derivative must not change a sign. It means
\[
\frac{d\phi}{dN} = \nu_+ \phi_+ e^{\nu_+ N} + \nu_- \phi_- e^{\nu_- N} \neq 0, \tag{3.5.14}
\]
we assume that the derivative of the field with respect to \( N \) is a smooth function throughout its evolution. Since we know the signs of the constants \( \nu_\pm \), the above condition can be achieved only if one of the coefficients \( \phi_\pm \) is negative. If we take \( \phi_- < 0 \), it will result in a positive time derivative for the field
\[
\dot{\phi} = H \frac{d\phi}{dN} > 0.
\]
If here we consider the field rolling over the hill then at some moment when \( N = N_0 \) the field must get the value equal to 0 that corresponds to the maximum of the potential. It means that
\[
\phi_+ e^{\nu_+ N_0} = -\phi_- e^{\nu_- N_0}.
\]
We can rewrite the expression for the general solution as follows
\[
\phi = A \left[ e^{\nu_+(N-N_0)} - e^{\nu_-(N-N_0)} \right],
\]
where
\[
A = \phi_+ e^{\nu_+ N_0} = -\phi_- e^{\nu_- N_0}.
\]
As we see in the left part of potential the \( \nu_- \) branch dominates that means that slow-roll approximation cannot be used in this region and after rolling over the top the \( \nu_+ \) branch dominates and slow-roll mode can start if necessary conditions are satisfied.

In the next subsection we consider the next approximation of the tachyon in the FRW metric.

### 3.6 Next-to-leading terms in the approximate dynamics of the tachyon in the FRW metric

Starting from the dS approximation, \( H = H_0 \), where \( H_0 \) is given by (3.4.2), and selecting the \( C_1 \)-mode one can find a solution of equation (3.1.1) performing an analytical expansion
in powers of the $C_1$-mode \[21, 26, 30\],

\[
\phi(t) = \sum_{n=1}^{\infty} \phi_n e^{nr_+ t}, \tag{3.6.15}
\]

\[
H(t) = H_0 - \sum_{n=1}^{\infty} H_n e^{nr_+ t}. \tag{3.6.16}
\]

In the next to the leading approximation

\[
H(t) = H_0 - H_2 e^{2r_+ t} + ... \tag{3.6.17}
\]

with

\[
H_2 = -3\pi G \phi_1^2 H_0 \left(1 - \sqrt{1 + \frac{4}{9} \frac{\mu^2}{H_0^2}}\right). \tag{3.6.18}
\]

In this approximation the tachyon equation (3.1.1) has the form

\[
\ddot{\phi} + 3(H_0 - H_2 e^{2r_+ t}) \dot{\phi} = \mu^2 \phi \tag{3.6.19}
\]

and expanding the solution in the series (3.6.15) we get

\[
\phi_{\text{pert.sol}}(t) \approx \phi_1 e^{r_+ t} + \phi_3 e^{3r_+ t} + \ldots, \tag{3.6.20}
\]

where

\[
\phi_3 = \frac{3H_2}{8r_+ + 6H_0} \phi_1. \tag{3.6.21}
\]

Note that the denominator in (3.6.21) has no zeros.

Note also that in the accepted scheme of calculation we get an one-parametric set of solutions and we deal only with the more viable mode. We can check this explicitly. To this purpose we note that the general solution to (3.6.19) has the form

\[
\phi(t) = (c_1 M(a, a, x) + c_2 W(a, a, x)) e^{\frac{2}{9} e^{-2a r_+ t}}, \tag{3.6.22}
\]

where $M$ is the Whittaker $M$ function and $W$ is the Whittaker $W$ function and

\[
a = \frac{2r_+ + 3H_0}{4r_+} = \sqrt{\frac{9H_0^2 + 4\mu^2}{4r_+}}, \quad x = \frac{3H_2 e^{2r_+ t}}{2r_+}.
\]

Using the relation of the Whittaker function with the hypergeometrical function $\Phi(\alpha, \beta, z) \equiv F_1(\alpha, \beta, z)$ we get

\[
\phi(t) = e^{r_+ t} \left(c_1 \left(\frac{3H_2}{2r_+}\right)^{a+\frac{1}{2}} \Phi\left(\frac{1}{2}, 1 + 2a, x\right) + c_2 \frac{\Gamma(-2a)}{\Gamma\left(\frac{1}{2} - 2a\right)} \left(\frac{3H_2}{2r_+}\right)^{a+\frac{1}{2}} \Phi\left(\frac{1}{2}, 1 + 2a, x\right)\right)
\]

\[
+ c_2 \frac{\Gamma(2a)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{3H_2}{2r_+}\right)^{-a+\frac{1}{2}} e^{-4ar_+ t} \Phi\left(\frac{1}{2} - 2a, 1 - 2a, x\right) \right) \tag{3.6.23}
\]
and the series expansion

\[ \Phi(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma} z + \frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)} z^2 + \ldots \]

we get

\[
\phi(t) \approx \left( c_1 + c_2 \frac{\Gamma(-2a)}{\Gamma(\frac{1}{2} - 2a)} \left( \frac{3H_2}{2r_+} \right)^{a + \frac{1}{2}} e^{r_+ t} + \left( c_1 + c_2 \frac{\Gamma(-2a)}{\Gamma(\frac{1}{2} - 2a)} \left( \frac{3H_2}{2r_+} \right)^{a + \frac{3}{2}} \left( \frac{1}{2(1 + 2a)} \right) e^{3r_+ t} \right) \right)
\]

\[
+ \frac{\Gamma(2a)}{\Gamma(\frac{1}{2})} \left( \frac{3H_2}{2r_+} \right)^{-a + \frac{1}{2}} e^{(1 - 4a)r_+ t} + c_2 \Gamma(2a) \left( \frac{3H_2}{2r_+} \right)^{-a + \frac{3}{2}} \left( \frac{1 - 4a}{2(1 - 2a)} \right) e^{(3 - 4a)r_+ t}. \quad (3.6.24)
\]

We see that to select the solution that starts from zero at \( t = -\infty \) we have to remove the \( c_2 \)-mode

\[
\phi_{s.b.c.}(t) = c_1 M(a, a, x) \approx c_1 \left( \left( \frac{3H_2}{2r_+} \right)^{a + \frac{1}{2}} e^{r_+ t} + \left( \frac{3H_2}{2r_+} \right)^{a + \frac{3}{2}} \left( \frac{1}{2(1 + 2a)} \right) e^{3r_+ t} \right), \quad (3.6.25)
\]

here the subscript \( s.b.c. \) means the special boundary condition that selects the \( c_1 \)-mode. Denoting

\[
\phi_1 \equiv c_1 \left( \frac{3H_2}{2r_+} \right)^{a + \frac{1}{2}}, \quad (3.6.26)
\]

from (3.6.25) we get (3.6.20) where \( \phi_3 \) is given by (3.6.21).

Numerical solutions to equation (3.6.19) are presented in Fig.10 A by dotted red and magenta lines. We can also see in Fig.10 B a comparison of the evolution described by equation (3.6.19) with the exact evolution (numerical solution to equation (3.1.1)) of the tachyon dynamics in the FRW space (green and khaki solid lines). We see that near the top there is no essential difference between the first approximation (corresponding solutions are presented by dashed blue and cyan lines) and the next approximations as well as a difference between the first approximation and the exact solutions. We also see that for the small initial velocity the next to leading approximation (the magenta dotted line) to the exact solution (the khaki solid line) works better in comparison with the same approximation (the red dotted line) to the solution with the large initial velocity (the green solid line) as well as the correction to the first approximation is more essential for the case of the small velocity. However near the forbidden region the approximations do not work at all. In the next subsection we study the dynamics near the boundary of the forbidden region in more details.
Figure 10: (color online) A. First order approximation (dashed blue and cyan lines) and the third order approximation (dotted red and magenta lines) to solutions of (3.1.1). The first order approximation is given by the dynamics of tachyon in the dS space and third order approximation is calculated as a solution of (3.6.19) with $H^2$ given by (3.6.18). Blue vectors represent the phase portrait of the tachyon in the dS space. B. A comparison with the exact solutions of (3.1.1) (khaki and green solid lines). Red vectors represent the phase portrait of the tachyon in the FRW space.

3.7 Expansion to contraction

3.7.1 Tachyon dynamics on e-foldings as 3-dim dynamical system

As it has been mentioned before to study the transition from expansion to contraction it is convenient to use the analogue of the coordinates proposed in [53, 54]

$$x \equiv \sqrt{\frac{4\pi G}{3}} \frac{\dot{\phi}}{H}, \quad (3.7.1)$$

$$y \equiv \sqrt{\frac{4\pi G}{3}} \frac{\mu \phi}{H}, \quad (3.7.2)$$

$$z \equiv \frac{\mu}{H}, \quad (3.7.3)$$

and rewrite eq. (3.1.1) as a system of ordinary differential equations on number of e-foldings $N \equiv \ln a/a_{\text{initial}}$ (we omit $a_{\text{initial}}$)

$$x' = 3x(y^2 - b^2z^2) + yz, \quad (3.7.4)$$

$$y' = 3x^2y + zx, \quad (3.7.5)$$

$$z' = 3x^2z, \quad (3.7.6)$$

here $X' = \frac{dX}{dN}$. Note that due to $\frac{dN}{dt} = \frac{\dot{a}}{a} = H$ we have the following relation $X' = \dot{X} \cdot \frac{1}{H}$. We can check that

$$I(x, y, z) \equiv x^2 - y^2 + b^2z^2, \quad (3.7.7)$$
where \( b^2 = \Lambda \frac{8\pi G}{3\mu^2} \) is an integral of motion. Therefore, if we take the initial conditions \( x(0), y(0), z(0) \) such that
\[
x(0)^2 - y(0)^2 + b^2 z(0)^2 = 1,
\]
this condition is true for ever.

We present the 3-dimensional phase portrait of the system (3.7.4)-(3.7.6) in Fig.11A. This is a very remarkable phase portrait. First of all we see the constraint (3.7.7) in Fig.11A. We also see, Fig.11C, that for each trajectory there is a fixed number of maximal e-foldings, \( N_{\text{max}} \), that can make the tachyon with given initial data. Each trajectory reaches this number when \( z \to \infty \).

### 3.7.2 Tachyon dynamics on e-foldings as 2-dim dynamical system

It is instructive to solve firstly the constraint (3.7.7) and then to study the dynamics of the reduced 2-dimensional system. Let us introduce the following parametrization
\[
x = \sin \psi \cosh \vartheta, \quad (3.7.8) \\
y = \sinh \vartheta, \quad (3.7.9) \\
z = \frac{1}{b} \cos \psi \cosh \vartheta, \quad (3.7.10)
\]
where
\[-\pi < \psi < \pi, \\
-\infty < \vartheta < \infty.
\]
Let us note that expansion solutions, \( H > 0 \), correspond to \(-\pi/2 < \psi < \pi/2\). The contracting solution corresponds to \(-\pi < \psi < -\pi/2\) and \(\pi/2 < \psi < \pi\). Relations between \( \psi \) and \( \vartheta \)
and initial $\phi$, $\dot{\phi}$ and $H$ are given by the following formulae

\begin{align*}
\phi' &= \frac{\sqrt{2}}{l} \sin \psi \cosh \vartheta, \quad (3.7.11) \\
\dot{\phi} &= \sqrt{2}\Lambda \tan \psi, \quad (3.7.12) \\
\phi &= = \frac{\sqrt{2}b}{l} \tanh \vartheta, \quad (3.7.13) \\
H &= \frac{\mu b}{\cos \psi \cosh \vartheta}, \quad (3.7.14)
\end{align*}

here $l \equiv \frac{b\sqrt{\pi}}{\Lambda}$.

Equations (3.7.4) – (3.7.6) in these new variables have the form

\begin{align*}
\vartheta' &= \frac{3}{2} \sin^2 \psi \sinh 2\vartheta + \frac{1}{2b} \sin 2\psi \cosh \vartheta, \quad (3.7.15) \\
\psi' &= \frac{1}{b} \cos^2 \psi \sinh \vartheta - \frac{3}{2} \sin 2\psi. \quad (3.7.16)
\end{align*}

3.7.3 The phase portraits of the system of equations (3.7.15) and (3.7.16)

The phase portrait of the system of eqs. (3.7.15) and (3.7.16) is presented in Fig.12. The 3d plot of $\psi$ and $\vartheta$ as functions of $N$ found by numerical solution of the system of eqs. (3.7.15) and (3.7.16) is presented in Fig.13.

In Fig.12 A and 12 C we see that $\psi$ is almost constant for large $\vartheta$, i.e. for $\vartheta(N) > \vartheta_0 = \vartheta(N_0)$ we have $\psi(N) \approx \psi(N_0)$.

3.7.4 Critical points

We can see in the phase portrait Fig.12 that there are the critical points. They solve the equations

\begin{align*}
\frac{3}{2} \sin^2 \vartheta_0 \sinh 2\vartheta_0 + \frac{1}{2b} \sin 2\vartheta_0 \cosh \vartheta_0 &= 0, \quad (3.7.17) \\
\frac{1}{b} \cos^2 \vartheta_0 \sinh \vartheta_0 - \frac{3}{2} \sin 2\vartheta_0 &= 0. \quad (3.7.18)
\end{align*}

There are only the following real solutions to the system of equations (3.7.17) and (3.7.18)

\begin{align*}
\vartheta_0 &= \pm \frac{\pi}{2}, \quad \vartheta_0 = 0; \quad (3.7.19) \\
\vartheta_0 &= 0, \quad \vartheta_0 = 0; \quad (3.7.20) \\
\vartheta_0 &= \pi, \quad \vartheta_0 = 0. \quad (3.7.21)
\end{align*}
Figure 12:  A. (color online) The phase portrait of the system of equations (3.7.15) and (3.7.16) for $-\pi < \psi < \pi$ and $-2 < \vartheta < 2$. B. The phase portrait of the same system for small and positive $\psi$ and small $\vartheta$. We see an attractor and trajectories go to the attractor from both sides of the part of the blue line. C. The phase portrait of system (3.7.15)-(3.7.16) for $0 < \psi < \pi/2$ and large $\vartheta$. We see that starting at enough large values of $\vartheta$ the value of $\psi$ is almost constant along the trajectories.

Figure 13:  (color online) A solution to (3.7.15) and (3.7.16).
3.7.5 Near critical points

Let us examine the system near critical points \( \psi_0 = \pm \frac{\pi}{2}, \theta_0 = 0 \). Let us take, for example, \( \psi \approx \frac{\pi}{2} - \Delta \psi, \theta \approx \Delta \theta \) and linearize (3.7.15) and (3.7.16). We get

\[
\Delta \theta' = 3\Delta \theta + \frac{1}{b} \Delta \psi, \quad (3.7.22)
\]
\[
\Delta \psi' = 3\Delta \psi. \quad (3.7.23)
\]

Solutions to (3.7.22) and (3.7.23) are given by

\[
\Delta \theta = \frac{1}{2} \left( C_2 N + 2C_1 b \right) e^{3N}, \quad (3.7.24)
\]
\[
\Delta \psi = C_2 b^2 e^{3N}. \quad (3.7.25)
\]

Let us write \( \phi, \dot{\phi} \) and \( H \) near this point. Taking into account (3.7.11)-(3.7.14) we get

\[
\phi' \approx \sqrt{2} l + O(\Delta \psi^2), \quad (3.7.26)
\]
\[
\dot{\phi} \approx \sqrt{2\Lambda} \left( \frac{2}{C_2 b e^{3N}} - \frac{b}{6} C_2 e^{3N} \right), \quad (3.7.27)
\]
\[
\phi \approx \frac{\sqrt{2} b}{l} \left( \frac{1}{2} (C_2 N + 2C_1 b) e^{3N} \right) \left( \frac{2}{C_2 b e^{3N}} + \frac{b}{12} C_2 e^{3N} \right), \quad (3.7.28)
\]
\[
H \approx b \mu \left( \frac{2}{C_2 b e^{3N}} + \frac{b}{12} C_2 e^{3N} \right) \left( 1 - \frac{1}{2} \left( \frac{1}{2} (C_2 N + 2C_1 b) e^{3N} \right)^2 \right). \quad (3.7.29)
\]

3.7.6 Solution for the large \( \theta \) limit

Let us keep in (3.7.15) and (3.7.16) only the terms that dominate in the large \( \theta \) region. We have

\[
\phi' = \frac{3}{4} \sin^2 \psi e^{2\theta}, \quad (3.7.30)
\]
\[
\psi' = \frac{1}{2b} \cos^2 \psi e^\theta. \quad (3.7.31)
\]

In Fig.14 we can compare the phase portraits for the system of equations (3.7.15) and (3.7.16) with that of equations (3.7.30) and (3.7.31). From these plots we see that for the large \( \theta \) limit the approximation (3.7.30), (3.7.31) works rather well and for small it does not work at all.

From (3.7.30) and (3.7.31) we have

\[
\frac{d\phi}{e^\theta} = \frac{3b \sin^2 \psi}{2 \cos^2 \psi} \frac{d\psi}{e^\theta}.
\]

That gives

\[
e^{-\theta} = \frac{3b}{2} (\psi - \tan \psi) + C, \quad (3.7.32)
\]
where
\[ C \equiv \frac{3b}{2} (\tan \psi_0 - \psi_0) + e^{-\vartheta_0}. \]

Let us now substitute (3.7.32) into (3.7.31). We get
\[ \psi' = \frac{1}{2b} \frac{\cos^2 \psi}{\frac{3b}{2} (\psi - \tan \psi) + C}. \]  
(3.7.33)

We see that due to possible zeros of the denominator in the RHS of (3.7.33) singular points can appear. Since \( \psi - \tan \psi \) is negative, a positive \( C \) leads to a singularity.

For us it is important that \( N \) increases during the dynamics (this corresponds to expansion), meanwhile \( \psi \) can increase or decrease, this does not matter. We can say, that if the RHS of (3.7.33) is positive, then \( \psi \) increases with increasing \( N \), or if the RHS of (3.7.33) is negative, then \( \psi \) decreases with increasing \( N \). In the case of a singularity, before the singularity \( \psi \) increases, and after \( \psi \) decreases, see Fig. 15.

We can solve (3.7.33) explicitly to get an implicit form of a dependence of \( \psi \) on \( N \)
\[ \Upsilon(\psi, C, b) - \Upsilon(\psi_0, C, b) = N - N_0, \]  
(3.7.34)

where
\[ \Upsilon(\psi, C, b) \equiv 3b^2 \psi \tan \psi + 3b^2 \ln (\cos \psi) - \frac{3b^2}{2 (\cos \psi)^2} + 2Cb \tan \psi. \]
From Fig.16 we see that for positive values of $C$ there are maximal values of $\Upsilon(\psi, C, b)$. These maxima are reached at some point $0 < \psi_{\text{max}} < \pi/2$, that satisfies the relation

$$3b(\psi_{\text{max}} - \tan \psi_{\text{max}}) + 2C = 0 \quad \Rightarrow \quad \tan \psi_{\text{max}} - \psi_{\text{max}} = \frac{2C}{3b}. \quad (3.7.35)$$

If $\psi_{\text{max}}$ is close to zero, we can expand the LHS of the second formula in (3.7.35) in the Taylor expansion and get $\psi_{\text{max}} \approx \left(\frac{2C}{b}\right)^{1/3}$. This obviously works for $\frac{2C}{b} < 1$.

For negative values of $C$ the maxima are at $\psi_{\text{max}} = 0$ and these solutions correspond to decreasing $N$ for increasing $\psi$.

Let us now examine what this approximation gives for the variables $\phi, \dot{\phi}$. In the accepted
approximation (3.7.11)-(3.7.14) gives

\[ \phi' = \approx \frac{\sqrt{2}}{2l} \sin \psi e^\vartheta, \]  \hfill (3.7.36)

\[ \dot{\phi} = \sqrt{2\Lambda} \tan \psi, \]  \hfill (3.7.37)

\[ \phi \approx \frac{\sqrt{2\Lambda}}{\mu} \frac{1}{\cos \psi}, \]  \hfill (3.7.38)

\[ H \approx \frac{2l\sqrt{\Lambda}}{\cos \psi} e^{-\vartheta}. \]  \hfill (3.7.39)

Note that instead of the Friedmann equation we have

\[ \frac{8\pi G \mu^2}{3} \frac{\phi^2}{2} - \frac{8\pi G}{3} \frac{1}{2} \dot{\phi}^2 \approx \frac{8\pi G}{3} \Lambda. \]  \hfill (3.7.40)

From formulae (3.7.37), (3.7.38) we see that increasing \( \psi \) within \( 0 < \psi < \pi/2 \) corresponds to increasing \( \phi \) and \( \dot{\phi} \). From the phase portrait Fig.3 we see that for increasing time keeping on the part of the solution corresponding to \( H > 0 \) (but very small) we increase \( \dot{\phi} \) and \( \phi \). From this consideration we conclude that to know the behavior near the “separatrix” with positive \( H \) we can take the regime near \( \psi_{\max} \) from the left. We can also say that the field and its time derivative reach the values

\[ \phi_{\text{boundary}} = \frac{\sqrt{2\Lambda}}{\mu} \frac{1}{\cos \psi_{\max}}, \]  \hfill (3.7.41)

\[ \dot{\phi}_{\text{boundary}} = \sqrt{2\Lambda} \tan \psi_{\max}, \]  \hfill (3.7.42)

(that belongs to the boundary of the forbidden region) and then dynamics becomes contracting.
So, we have the following picture: We start from some $N_0$ and some $\psi_0$. Suppose that we take a suitable $\vartheta_0$ (let us remind that we assume that $\vartheta_0$ is large enough) to guarantee that

- $C > 0$. Since we study dynamics with $H > 0$, i.e. the number of foldings should increase, $N_0 < N_{\text{max}}(C)$. In this case we have increasing $N$ during the time evolution till $N$ reaches $N_{\text{max}}(C)$. $N_{\text{max}}$ can be found from (3.7.34)

$$N_{\text{max}} = \Upsilon(\psi_{\text{max}}, C, b) - \Upsilon(\psi_0, C, b) + N_0. \quad (3.7.43)$$

- $C = 0$. In this case the number of foldings decreases with increasing $\psi$, and as we have seen before, increasing $\psi$ corresponds to increasing $\phi$.

- $C < 0$, here we have also a decreasing number of e-foldings.

We can estimate the dynamics near $\psi = \psi_{\text{max}}$. For this purpose let us expand $\Upsilon$ near $\psi = \psi_{\text{max}}$. We have

$$\Upsilon(\psi, C, b) = \Upsilon(\psi_{\text{max}}, C, b)$$

$$+ \frac{3b}{2} \tan \psi_{\text{max}} \left( 2b\psi_{\text{max}} + \frac{4}{3} C - 3b \tan \psi_{\text{max}} \right) (\psi - \psi_{\text{max}})^2 + O((\psi - \psi_{\text{max}})^3) \quad (3.7.44)$$

From the condition that we are near the maximum and $0 < \psi_{\text{max}} < \pi/2$ it follows that

$$2b\psi_{\text{max}} + \frac{4}{3} C - 3b \tan \psi_{\text{max}} = -b \tan \psi_{\text{max}} < 0$$

Now eq. (3.7.34) takes the form

$$v(\psi - \psi_{\text{max}})^2 \approx -\Upsilon(\psi_0, C, b) + \Upsilon(\psi_{\text{max}}, C, b) - N + N_0, \quad (3.7.45)$$

where

$$v \equiv -\frac{1}{2} \tan \psi_{\text{max}} \left( 6b^2 \psi_{\text{max}} + 4bC - 9b^2 \tan \psi_{\text{max}} \right) > 0$$

and we get

$$\psi = \psi_{\text{max}} - \frac{1}{v^{1/2}} \sqrt{\Upsilon(\psi_{\text{max}}, C, b) - \Upsilon(\psi_0, C, b) - N + N_0}. \quad (3.7.46)$$

As $\psi_{\text{max}}$ and $C$ are functions of the initial data denoting

$$N_0 \equiv \Upsilon(\psi_{\text{max}}, C, b) - \Upsilon(\psi_0, C, b) + N_0,$$

we can rewrite (3.7.46) as

$$\psi(N) = \psi_{\text{max}} - \frac{1}{v^{1/2}} \sqrt{N_0 - N}. \quad (3.7.47)$$

Let’s now find the dependence on the time. Due to (3.7.39) and (3.7.32) we have

$$\frac{dt}{dN} \approx \frac{1}{2l\sqrt{\Lambda}} \frac{\cos \psi}{\frac{3b}{2} (\psi - \tan \psi) + C}.$$
Substituting here the explicit dependence of $\psi$ on $N$, eq. (3.7.47), we get
\[
t - t_0 \approx \int_{N_0}^{N} \frac{dN}{2l\Lambda} \frac{\cos \psi(N)}{3b} (\psi(N) - \tan \psi(N)) + C,
\]
where $\psi(N)$ is given by (3.7.47). Making the change of variables
\[
u \equiv \frac{1}{\sqrt{N_0 - N}},
\]
we get
\[
t - t_0 \approx -\int_{N_0}^{\sqrt{N_0 - N}} \frac{uvdu}{l\Lambda} \frac{\cos(\psi_{max} - u)}{\tan \psi_{max}} + C.
\]
Expanding the integrand near $\psi_{max}$ up to $u$ we get
\[
t - t_0 \approx \frac{1}{l\Lambda} \left( \sqrt{N_0 - N_0} - \sqrt{N_0 - N} \right) \cos(\psi_{max} - u).
\]
(3.7.48)
Let’s keep only the leading term (linear in $u$). Then equation (3.7.48) can be rewritten as
\[
t \approx t_0 - \nu \sqrt{N_0 - N}, \quad N_0 - N \approx \frac{(t - t_0)^2}{\nu^2},
\]
(3.7.49)
where $\nu$ is the function of the initial data and has the form
\[
\nu = \frac{1}{l\Lambda \sqrt{3} \tan \psi_{max}}.
\]
Substituting this expression into the asymptotic for $\phi$ near $\phi_{boundary}$ given by (3.7.41) we get
\[
\phi(t) \approx \frac{\sqrt{2\Lambda}}{\mu} \frac{1}{\cos(\psi_{max} - \frac{2L - t}{\nu^2 \rho})}.
\]
(3.7.50)

4 Stretched Higgs in FRW

4.1 Stretched coupling constant

In this section we consider the case of non-positively defined Higgs potential. As it was mentioned in Introduction due to non-locality of the fundamental theory the effective stretch of a coupling constant occurs. We will write the equation of evolution in the following form
\[
e^{\alpha_{eff}^2} \left( \ddot{\phi} + 3H \dot{\phi} - \mu^2 \phi \right) = -\epsilon \phi^3,
\]
(4.1.1)
\[
H^2 = \frac{8\pi G}{3} \left( \frac{e^{\alpha_{eff}^2}}{2} \left( \phi^2 - \mu^2 \phi^2 \right) + \frac{1}{4} \epsilon \phi^4 + \Lambda \right).
\]
(4.1.2)
Therefore, we have the ordinary theory with redefined constants

\[ \ddot{\phi} + 3H \dot{\phi} - \mu^2 \phi = -\epsilon_{eff} \phi^3, \quad (4.1.3) \]

\[ H^2 = \frac{8\pi G_{eff}}{3} \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \mu^2 \phi^2 + \frac{\epsilon_{eff}}{4} \phi^4 + \Lambda_{eff} \right), \quad (4.1.4) \]

where

\[ G_{eff} = G e^{\alpha_{eff}}, \quad (4.1.5) \]

\[ \epsilon_{eff} = \epsilon e^{-\alpha_{eff}}, \quad (4.1.6) \]

\[ \Lambda_{eff} = \Lambda e^{-\alpha_{eff}}. \quad (4.1.7) \]

We have also

\[ \dot{H} = -4\pi G_{eff} \dot{\phi}^2. \quad (4.1.8) \]

### 4.2 Phase portraits for different \( \Lambda_{eff} \)

In this subsection we consider the phase portraits for the Higgs field for different values of the total cosmological constant. The Friedmann equation has the form (here and below we write \( G, \epsilon \) instead of \( G_{eff} \) and \( \epsilon_{eff} \))

\[ H^2 = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\phi}^2 + \frac{\epsilon}{4} (\phi^2 - a^2)^2 + \Lambda \right), \quad (4.2.9) \]

where

\[ \Lambda = \Lambda_{eff} - \frac{\epsilon a^4}{4}. \]

We will consider the cases \( \Lambda > 0, \Lambda = 0 \) and \( \Lambda < 0 \). As we just mentioned the unusual case \( \Lambda < 0 \) appears due to the effect of the stretch of coupling constant in nonlocal theory.

#### 4.2.1 The case of Higgs potential without extra cosmological constant, \( \Lambda_{eff} = \frac{\epsilon a^4}{4} \)

In this case we deal with the potential presented in Fig.17.A. and the phase portrait doesn’t have a forbidden domain. This case has been considered in many works and we discuss it in Sect.2.

#### 4.2.2 \( \Lambda_{eff} = 0 \)

In the case \( \Lambda_{eff} = 0 \), see Fig.17.B., the phase portrait does have the forbidden region and the reheating domain is totally or partially removed (dependently on the trajectory). The characteristic feature of this case is an appearance of the contraction regime. To see this let’s consider this case in more details.
Our goal is to see the future and the past singularities of the Higgs scalar field in the FRW metric. EOM describing inflation is

\[
\ddot{\phi} + 3\sqrt{\frac{8\pi G}{3}} \left( \frac{1}{2} \dot{\phi}^2 - \frac{\mu^2}{2} \phi^2 + \frac{\epsilon}{4} \phi^4 \right) \dot{\phi} = +\mu^2 \phi - \epsilon \phi^3. \tag{4.2.10}
\]

The phase portrait of this system is presented in Fig. 18A. The contraction phase satisfies
\[
\ddot{\phi} - 3 \sqrt{\frac{8\pi G}{3}} \left( \frac{1}{2} \dot{\phi}^2 - \frac{\mu^2}{2} \phi^2 + \frac{\epsilon}{4} \phi^4 \right) \dot{\phi} = +\mu^2 \phi - \epsilon \phi^3
\] (4.2.11)

and its phase portrait is presented in Fig.18. B. To study the infinite phase space following [51] let us use dimensionless variables \((X, Y, Z)\) (3.3.2)-(3.3.4). In these variables the dynamical system has the form

\[
\begin{align*}
X_\tau &= Y, \\
Y_\tau &= X - 3ZY - gX^3, \\
Z_\tau &= -X^2 - 2Y^2 - Z^2 + \frac{g}{2}X^4,
\end{align*}
\] (4.2.12-4.2.14)

where \(g = \epsilon \frac{m_p^2}{4\pi \mu^2}\).

The integral of motion is

\[-X^2 + Y^2 - Z^2 + \frac{g}{2}X^4.\]

To describe \(\Lambda_{\text{eff}} = 0\) we take

\[-X^2 + Y^2 - Z^2 + \frac{g}{2}X^4 = 0.\] (4.2.15)

The Friedmann equation keeps the dynamics of the system on the surface (4.2.15). This surface has two parts: one corresponds to expansion \((H > 0)\) and the second one corresponds to contraction.

The phase portrait of the dynamical system (4.2.12)-(4.2.14) is presented in Fig.18. To study the infinite phase space we use the spherical coordinates \((\rho, \theta, \psi)\) (3.3.9)-(3.3.11), see Section 3. In these coordinates EOM have the form

\[
\begin{align*}
\rho_\sigma &= \rho (1 - \rho)^2 \left[ \sin 2\psi \sin^2 \theta - \frac{\rho}{1 - \rho} \cos \theta (1 + 4 \sin^2 \psi \sin^2 \theta) \\
&\quad - \frac{g}{(1 - \rho)^2} \cos^3 \psi \sin^3 \theta \left( \sin \psi \sin \theta - \frac{1}{2} \sin 2\theta \cos \psi \right) \right] \\
\psi_\sigma &= -\frac{3}{2} \rho \sin 2\psi \cos \theta + (1 - \rho) \cos 2\psi - \frac{g}{1 - \rho} \sin^2 \theta \cos^4 \psi \\
\theta_\sigma &= \frac{1}{2} (1 - \rho) \sin 2\psi \sin 2\theta + \rho \sin \theta (1 + \sin^2 \psi - 4 \sin^2 \psi \cos^2 \theta) \\
&\quad - \frac{g}{2 (1 - \rho)^4} \sin^5 \theta \cos^4 \psi
\end{align*}
\] (4.2.16-4.2.18)

The phase portrait of this system of equations is presented in Fig.19. We see that trajectories are symmetric under the reflection with respect to the boundary of the forbidden area. We also see the two singular points – at the past (cosmological singularity) and the same in the future.
Figure 19: (color online) A. The phase portrait of equations (4.2.16) – (4.2.18) in the $(\rho, \theta)$-plane. The black thin line corresponds to $\theta = \pm \pi/2$. Blue parts of lines represent contraction and red parts represent expansion. We see the symmetry with respect to replacing the red parts by the blue parts of lines. FS and PS are future and past cosmological singularities. B. The 3-dimensional plot in $(\rho, \theta, \psi)$ coordinates of the same phase portrait.

4.2.3 $0 < \Lambda_{\text{eff}} < \frac{\epsilon \alpha^4}{4}$

In this case the potential is presented in Fig.17 C., the forbidden domain contains two non-connected parts.

Near the top of the potential the Higgs potential can be approximated with the tachyon potential

$$V(\phi) \approx -\frac{\epsilon \alpha^2}{2} \phi^2 + \Lambda_{\text{eff}}.$$ (4.2.19)

We can note that near the top of the hill the slow-roll conditions can be satisfied and the scenario of “new inflation” is possible. To illustrate this we present the plot for slow-roll parameters $\varepsilon$ and $|\eta|$ depending on the value of $\phi$ in Fig. 20 A and B. As we see near the top the tachyon approximation can represent the exact model.

To get some estimation in the region near the top one can use the same technique as in the Section 3.

5 Conclusion

To conclude, let us remind that our logic was the following. We started from nonlocality. Using the “stretched” arguments presented by [25, 26] we have got non-positive defined potential. There exist a region where we can use the tachyon approximation. For free tachyon there are 4 eras of evolution: i) from the past cosmological singularity to reaching
the hill, ii) rolling from the hill, iii) transition from expansion to contraction, iv) contraction to the future cosmological singularity.

For the Higgs model with an extra negative cosmological constant and small coupling constant produces 5 eras of evolution: i) the ultra-hard regime starting from the past cosmological singularity, ii) reaching the hill (here we can use the free tachyon approximation), iii) rolling from the hill (here we can use the free tachyon approximation), iv) the transition from expansion to contraction (here we can use the free tachyon approximation), v) contraction to the future cosmological singularity.

Let us note that there are the following differences between dynamics for positively defined and non-positively defined potentials in the FRW cosmology. For non-positively defined potential there is a stage of transition from expansion to contraction and there is no the oscillation regime ending with the zero field configuration. To obtain the era of reheating for the Higgs potential with an extra negative cosmological constant one can introduce an extra matter interacting with the Higgs field.

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