Long-term 3D-MHD Simulations of Black Hole Accretion Disks formed in Neutron Star Mergers

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ABSTRACT
We examine the long-term evolution of accretion tori around black hole (BH) remnants of compact object mergers involving at least one neutron star, to better understand their contribution to kilonovae and the synthesis of r-process elements. To this end, we modify the unsplit magnetohydrodynamic (MHD) solver in FLASH4.5 to work in non-uniform three-dimensional spherical coordinates, enabling more efficient coverage of a large dynamic range in length scales while exploiting symmetries in the system. This modified code is used to perform BH accretion disk simulations that vary the initial magnetic field geometry and disk compactness, utilizing a physical equation of state, a neutrino leakage scheme for emission and absorption, and modeling the BH’s gravity with a pseudo-Newtonian potential. Simulations run for long enough to achieve a radiatively-inefficient state in the disk. We find robust mass ejection with both poloidal and toroidal initial field geometries, and suppressed outflow at high disk compactness. With the included physics, we obtain bimodal velocity distributions that trace back to mass ejection by magnetic stresses at early times, and to thermal processes in the radiatively-inefficient state at late times. The electron fraction distribution of the disk outflow is broad in all models, and the ejecta geometry follows a characteristic hourglass shape. We test the effect of removing neutrino absorption or nuclear recombination with axisymmetric models, finding ∼50% less mass ejection and more neutron-rich composition without neutrino absorption, and a subdominant contribution from nuclear recombination. Tests of the MHD and neutrino leakage implementations are included.

Key words: accretion, accretion disks – MHD – neutrinos – nuclear reactions, nucleosynthesis, abundances — stars: black holes — stars: neutron

1 INTRODUCTION
Neutron star (NS) mergers have long been predicted to produce r-process elements (Lattimer & Schramm 1974) as well as electromagnetic (EM) transients powered by relativistic jets (Paczynski 1986; Eichler et al. 1989) and/or by the radioactive decay of newly formed heavy elements (Li & Paczyński 1998; Metzger et al. 2010). The detection of a short gamma-ray burst (SGRB) and a kilonova associated with the NS merger GW170817 (Abbott et al. 2017a,b) has confirmed these events as progenitors of SGRBs, and placed them as important production sites for the r-process (e.g., Kasen et al. 2013; Tanaka & Hotokezaka 2013; Barnes & Kasen 2013; Fontes et al. 2015) and is the crucial piece linking NS mergers to r-process nucleosynthesis. A blue component had been anticipated as a possible consequence of a finite-lived hypermassive NS (HMNS) irradiating the ejecta with neutrinos (Metzger & Fernández 2014), but the low amount of dynamical ejecta expected (e.g., Shibata et al. 2017; Most et al. 2019) and the low velocities of HMNS disk outflows in viscous hydrodynamics (Fahlman & Fernández 2018; Nedora et al. 2020) rule out a straightforward interpretation. Remaining explanations include magnetically-driven outflows (e.g., Metzger et al. 2018), opacity effects (e.g., Waxman et al. 2018), or connection to a jet cocoon (e.g., Gottlieb et al. 2018; Piro & Kollmeier 2018).

More generally, the third observing run of Advanced LIGO & Virgo (Abbott et al. 2021; The LIGO Scientific Collaboration et al. 2021) found NS-NS and/or NS-black hole (BH) candidates for which no EM counterpart was detected. The lack of counterparts can be explained by either observational factors (e.g., Foley et al. 2020), smaller amounts of mass...
Newtonian potential to model the BH, and a neutrino leakage scheme that includes both emission and absorption. We use this method to explore the role of magnetic field geometry, disk compactness, nuclear recombination, and neutrino absorption on mass ejection in MHD. We perform simulations for initial conditions relevant to GW170817, as well as to systems that could feasibly result from a NS-BH merger. The main limitation of our approach is the absence of relativistic jets, thus our focus is on the sub-relativistic outflows that contain most of the ejected mass, and which are therefore most relevant to the kilonova emission and r-process nucleosynthesis.

The structure of this paper is the following. Section 2 presents a description of the numerical methods employed and models evolved. In §3 the results of our simulations are presented, analyzed, and compared to previous work. We conclude and summarize in §4. The appendices describe the testing and implementation of the MHD (§A, §B) and neutrino leakage (§C) modules employed.

2 METHODS

2.1 Numerical MHD

Our simulations employ a customized version of FLASH version 4.5 (Fryxell et al. 2000; Dubey et al. 2009), in which we have modified the unsplit MHD solver of Lee (2013) to work in 3D curvilinear coordinates with non-uniform spacing (see Appendix A for details). We use this code to numerically solve the Newtonian equations of mass, momentum, energy, and lepton number conservation in MHD supplemented by the induction equation. Additional source terms include the pseudo-Newtonian gravitational potential of a BH and the emission and absorption of neutrinos:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  
\[ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \rho \mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\mathbf{B} \otimes \mathbf{B}) + \nabla P = -\rho \mathbf{v} \mathbf{v} \cdot \nabla \Phi_A \]  
\[ \frac{\partial (\rho E)}{\partial t} + \nabla \cdot [(\rho E + P) - \mathbf{B} \otimes \mathbf{B}] = -\rho \mathbf{v} \mathbf{v} \cdot \nabla \Phi_A + Q_{\text{net}} \]  
\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{v}) = 0, \]  
where \( \rho \) is the density, \( \mathbf{v} \) is the velocity, \( \mathbf{B} \) is the magnetic field (including a normalization factor \( \sqrt{4\pi} \)), \( Y_e \) is the electron fraction, \( E \) is the total specific energy of the fluid

\[ E = \frac{1}{2} (\mathbf{v} \cdot \mathbf{v} + \mathbf{B} \cdot \mathbf{B}) + e_{\text{int}}, \]  
with \( e_{\text{int}} \) the specific internal energy, and \( P \) is the sum of gas and magnetic pressure

\[ P = P_{\text{gas}} + P_{\text{mag}}, \]  
\[ P_{\text{mag}} = \frac{1}{2} B \cdot B. \]

The induction equation (5) is discretized using the Constrained Transport (CT) method (Evans & Hawley 1988) and conserved quantities are evolved using the HLLD Riemann solver (Miyoshi & Kusano 2005) with a piecewise linear
MUSCL–Hancock reconstruction method (Colella 1985). The gravity of the BH is modeled with the pseudo-Newtonian potential ΦA of Artemova et al. (1996), ignoring the self-gravity of the disk (see also Fernández et al. 2015). The equation of state (EOS) is that of Timmes & Swesty (2000), with abundances of protons, neutrons and α-particles in nuclear statistical equilibrium (NSE) so that \( P_{\text{gas}} = P_{\text{gas}}(\rho, e_{\text{int}}, Y_e) \), and accounting for the nuclear binding energy of α-particles as in Fernández & Metzger (2013).

We implement the framework for neutrino leakage emission and annular light bulb absorption described in Fernández & Metzger (2013) and Metzger & Fernández (2014). The scheme includes emission and absorption of electron neutrinos and antineutrinos due to charged-current weak interactions on nucleons, and with improvements in the calculation of the neutrino diffusion timescale in high-density regions following the prescription of Ardevol-Pulpillo et al. (2019). A detailed description of the implementation and verification tests (comparing to the Monte Carlo scheme of Richers et al. 2015) are presented in Appendix B.

The leakage and absorption scheme outputs scalar source terms for the net rate of change of energy per unit volume \( Q_{\text{net}} \), and net rate of change per baryon number \( \Gamma_{\text{net}} \), which are respectively applied to \( E \) and \( Y_e \) (equations 3 and 4) in operator-split way. We neglect the contribution of neutrinos to the momentum equation.

Finite-volume codes fail when densities in the simulation become too low. We impose a radial- and time-dependent density floor, designed to prevent unreasonably low simulation timesteps in highly magnetized regions (e.g., near the rotation axis) while also not affecting the dynamics of outflow. It has a functional form approximately following that in Fernández et al. (2019a)

\[
\rho_{\text{floor}} = \rho_{\text{init}} \left( \frac{r}{20\text{km}} \right)^{-3} \left( \frac{\max[r,0.1\text{s}]}{0.1\text{s}} \right)^{-1.5},
\]

where \( \rho_{\text{init}} = 2 \times 10^4 \text{ g cm}^{-3} \) and \( r \) is the spherical radius. The time dependence is modelled after empirically determining the rate of change of the maximum torus density in 2D runs of the baseline model. When the density undershoots the floor value, it is topped up to the floor level with material tagged as ambient, such that we can keep track of it and discard it when assessing outflows and accretion. Keeping the density above the floor is generally enough to prevent the internal energy (and gas pressure) at levels that do not crash the code. Nevertheless, we also impose explicit floors for these quantities, following the same form as in equation (9), but with normalizations \( P_{\text{init}} = 2 \times 10^{42} \text{ erg cm}^{-3} \) and \( e_{\text{init}} = 2 \times 10^{11} \text{ erg g}^{-1} \) for gas pressure and internal energy, respectively.

2.2 Computational Domain and Initial Conditions

Equations (1)–(5) are solved in spherical polar coordinates \((r, \theta, \phi)\) centered at the BH and with the \( z \)-axis aligned with the disk and BH angular momentum. The computational domain extends from an inner radius, \( r_{\text{in}} \), located halfway between the innermost stable circular orbit (ISCO) and the BH horizon, both dependent on the BH mass and spin, to an outermost radius \( r_{\text{out}} \) located at \( 10^4 r_{\text{in}} \). The polar and azimuthal angular ranges are \( [5^\circ, 175^\circ] \) and \([0, 180^\circ]\), respectively, corresponding to a half-sphere with a \( 5^\circ \) cutout around the \( z \)-axis. The radial grid is discretized with 512 logarithmically-spaced cells satisfying \( \Delta r / r \sim 0.018 \), the meridional grid has 128 cells equally spaced in \( \cos \theta \), corresponding to \( \Delta \theta \sim 0.92^\circ \) at the equator, and the azimuthal grid is uniformly discretized with 64 cells.

The boundary conditions are set to outflow at the polar cutout and at both radial limits, and to periodic at the \( \phi \) boundaries. The cutout around the polar axis is used to mitigate the stringent time step constraints arising from the small size of \( \phi \) cells next to the \( z \) axis. We do not expect our polar boundary conditions to affect our analysis, as the sub-relativistic outflow is well separated from the jet by a centrifugal barrier (Hawley & Krolik 2000). Any outflow along the polar axes without the use of full GR is unreliable anyway, as many of the proposed mechanisms for jet formation involve general relativistic energy extraction from the BH spin energy (e.g., the Lense-Thirring and Blandford-Znajek effects: Bardeen & Petterson 1975; Blandford & Znajek 1977). These processes also involve the formation of a baryon-free funnel along the rotation axis, which means outflow along the polar axes contains minimal mass. Evolution tests using reflecting, transmitting (with no azimuthal symmetry), and outflow polar boundary conditions showed little to no difference over short times after initialization (~0.5 orbits).

The initial condition for all of our models is an equilibrium torus with constant specific angular momentum, entropy, and composition, consistent with the pseudo-Newtonian potential for the BH (Fernández & Metzger 2013; Fernández et al. 2015). The input parameters are the BH mass, torus mass, radius of density peak, entropy (i.e. thermal content or vertical extent), and \( Y_e \). In all cases, the latter two parameters are set to \( s_B = 8 \text{ keV} \) baryon and \( Y_e = 0.1 \), respectively, with other parameters changing between models (§2.3). Initial maximum tori densities \( \rho_{\text{max}} \) are typically \( \sim 10^{-10} \text{ g cm}^{-3} \).

Recent studies have shown that tori formed in NS mergers have a doubly peaked distribution of \( s_B \) and \( Y_e \) (Nedora et al. 2020; Most et al. 2021), however, the use of more realistic initial conditions for these quantities has little impact on the resulting outflows (e.g., Fujibayashi et al. 2020b), and is expected to be smaller than differences due to our approximate handling of neutrino interactions and gravity.

Models that start with a poloidal field are initialized from an azimuthal magnetic vector potential which traces the density contours, such that \( A_\phi \propto \max(\rho - \rho_0, 0) \), where \( \rho_0 \) is defined as 0.009\( \rho_{\text{max}} \), ensuring the field is embedded well within the torus (e.g., Hawley 2000). This yields an initially poloidal field topology, commonly known as “standard and normal evolution” (SANE) in the literature. The normalization is chosen such that the maximum magnetic field strength \( \sim 4 \times 10^{14} \text{ G} \) is dynamically unimportant, with an average gas to magnetic pressure ratio of

\[
\langle \beta \rangle = \frac{\int P_{\text{gas}} dV}{\int P_{\text{mag}} dV} = 100
\]

with \( \min(\beta) \sim 5 \) at the inner edges and \( \max(\beta) \sim 10^5 \) at the initial density maximum. We also evolve a model that starts with a toroidal field, which is initialized by imposing a constant \( B_\theta = 4 \times 10^{14} \text{ G} \) wherever \( \rho > \rho_0 \). The magnetic field strength and mass density set the Alfvén velocities in the meridional and azimuthal directions,

\[
v_{\theta, \phi} = \frac{B_{\theta, \phi}}{\sqrt{\rho}}.
\]
which in turn determine the respective wavelengths of the most unstable MRI modes (e.g., Balbus & Hawley 1992; Duez et al. 2006),
\[ \lambda_{MRl,\theta,\phi} \sim \frac{2\pi|v_{\theta,\phi}'|}{\Omega_s}, \]
where \( \Omega_s \) is the cylindrical angular velocity. All of our simulations resolve the relevant MRI modes with least 10 cells within the torus. Resolution tests with 2D models indicate that our mass ejection results have an uncertainty of \( \lesssim 10\% \) due to spatial resolution (§3.3.2).

### 2.3 Models

Table 1 shows all the models we evolve and the parameters used. Our base model employs the most likely BH mass (\( M_{bh} = 2.65 M_\odot \), dimensionless spin 0.8), initial torus mass (\( M_t = 0.13 M_\odot \)), and initial radius of density peak (\( R_t = 50 \) km) for GW170817 (e.g., Abbott et al. 2017c; Shibata et al. 2017; Fahlman & Fernández 2018), using a poloidal field geometry. Model bhns uses a typical parameter combination expected from a BH-NS merger (\( M_{bh} = 8 M_\odot \) with spin 0.8; \( M_t = 0.03 M_\odot \); \( R_t = 60 \) km), also with a poloidal initial field, to probe the effect of a higher disk compactness (e.g., Fernández et al. 2020).

Three additional simulations test the influence of key physical effects on the base model. Model base-tor employs an initial toroidal magnetic field geometry instead of poloidal. The other two are explored in axisymmetry (2.5 dimensions): Model base-norec sets the nuclear binding energy of \( \alpha \)-particles to zero, and model base-noirr turns off neutrino absorption. These two simulations are compared to model base-2D, an axisymmetric version of the base model. All 3D models are evolved for at least 3 s, or until a time at which there is a clear power-law decay with time in the ejected mass at large radius, allowing for an analytic extrapolation until completion of mass ejection (§2.4). To achieve this phase, the base model needs to be evolved to 4 s. The axisymmetric models are evolved until 1.4 s, when accretion onto the BH stops due to a build up of magnetic pressure: continuing evolution causes feedback which disrupts the torus. The MRI is expected to dissipate in axisymmetry after \( \sim 100 \) orbits at the initial torus density peak, corresponding to a few 100 ms (e.g., Cowling 1933; Shibata et al. 2007).

### 2.4 Outflow Characterization

The mass flux at a given radius is computed as
\[ \dot{M}(r) = \int_{A_r} (\rho v_{r} dA_r), \]
where the spherical area \( A_r \) is given by
\[ A_r = \int r^2 \sin \theta d\theta d\phi. \]

For outflows, the extraction radius is \( r = 10^4 \) km, whereas for accretion onto the BH we take the radius of the ISCO. We only consider unbound outflows, which we quantify with a positive Bernoulli parameter at the extraction radius
\[ \Phi_g + \epsilon_{\text{int}} + \epsilon_k + \epsilon_{\text{mag}} + \frac{P_{\text{gas}}}{\rho} > 0. \]

We also require that both outflowing and accreting matter have an atmospheric mass fraction \( \chi_{\text{atmo}} < 0.2 \), and subtract off any remaining atmospheric mass so that
\[ \rho = \rho_{\text{atmo}} (1 - \chi_{\text{atmo}}). \]

The total ejected mass is computed by temporally integrating the outflow mass flux, such that
\[ M_{\text{out}} = \int t \int_{A_r} (\rho v_{r} dA_r) dt. \]

The mass weighted averages of electron fraction and radial velocity,
\[ \langle Y_e \rangle = \frac{\int t \int_{A_r} (\rho v_{r} Y_e dA_r) dt}{M_{\text{out}}}, \]
\[ \langle v_{r} \rangle = \frac{\int t \int_{A_r} (\rho v_{r} v_{r} dA_r) dt}{M_{\text{out}}}, \]
are provided as a summary of our model results in Table 2. We further subdivide outflows based on their electron fraction into “red” (\( Y_e < 0.25 \)) and “blue” (\( Y_e \geq 0.25 \)), based on kilonova models which predict a sharp cutoff between lanthanide-rich and lanthanide-poor matter (e.g., Kasen et al. 2015; Lippuner & Roberts 2015).

Once the torus reaches a quasi-steady phase following freezeout of weak interactions, \( (t_{ss} \sim 1.1 s) \), the mass outflow rate enters a phase of power-law decay, \( \dot{M}_{\text{out}} (t > t_{ss}) \propto t^{-\delta}. \) We can therefore estimate the completed mass ejection over timescales of \( \sim 10 s \) by extrapolating from a power-law fit to the mass outflow rate (e.g., Margalit & Metzger 2016; Fernández et al. 2019b),
\[ M_{\text{out}}^{\text{extr}} = M_{\text{out}}(t_{ss}) + \frac{1}{\delta - 1} \dot{M}_{\text{out}}(t_{ss}) t_{ss}, \]
where the integral in equation 17 is computed until \( t = t_{ss} \). The choice of \( t_{ss} \) is made based on visual inspection of when the cumulative mass outflows begin to plateau. Varying this choice in response to episodic mass ejection events results in an uncertainty in the exponent of \( |\Delta \delta| \lesssim 0.5 \) corresponding to about 5-15% difference in total \( M_{\text{out}}^{\text{extr}} \).

### 3 RESULTS

#### 3.1 Overview of Torus Evolution in MHD

Our base and bhns runs, with a poloidal field embedded in the torus, show very similar evolution to previous runs in

| Model       | \( M_{bh} \) (\( M_\odot \)) | \( M_t \) (\( M_\odot \)) | \( R_t \) (km) | \( B \) | \( \alpha \)-rec | \( \nu \)-abs | dim |
|-------------|-----------------|-----------------|-------------|-----|--------|--------|-----|
| base        | 2.65            | 0.10            | 50          | pol | yes    | yes    | 3   |
| bhns        | 8.00            | 0.03            | 60          |     |        |        |     |
| base-tor    | 2.65            | 0.10            | 50          | tor |        |        |     |
| base-2D     | 2.65            | 0.10            | 50          | pol | yes    | yes    | 2.5 |
| base-norec  |                |                 |             |     | no     | yes    |     |
| base-noirr  |                |                 |             |     | no     | no     |     |

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In contrast to hydrodynamic models, mass ejection begins on a timescale of $\sim$ns, forming “wings” of ejected material away from the midplane and rotation axis. More isotropic, thermally-driven ejecta takes over at $\sim$1s, as neutrino emission has subsided and the disk enters an advective state. As thermally-driven ejecta takes over, the flow becomes more isotropic and radial velocity broken down by electron fraction (superscript blue – lanthanide-poor: $Y_e \gtrsim 0.25$, red – lanthanide-rich: $Y_e < 0.25$). The last column shows the range of total unbound ejected mass extrapolated to infinity with a power-law fit to the mass outflow rate (equation 20). Note that 2D and 3D simulations run for different times, so direct comparisons of simulations with different dimensionality should not be made (See §3.3.1).

### 3.2 Mass Ejection in 3D Models

Table 2 shows the total unbound mass ejected by the end of each simulation (equation 17), and the extrapolation of the mass outflow rate to infinity in time (equation 20), for all of our 3D models. The base and base-tor models eject $\sim 28\%$ and $\sim 42\%$ of the initial torus mass at the same time required for the toroidal MRI to generate poloidal field, which then drives angular momentum transport. This is illustrated in Figure 3, which shows the evolution of the volume-integrated Maxwell stress for all 3D models. The bhns run has a deeper gravitational potential at the initial density maximum than the other runs (see §3.2.2), leading to more total energy input required to begin mass ejection (e.g., Fernández et al. 2020).

We also find that mass ejection peaks earlier in the base model than in the other two runs. The base-tor run reaches peak mass ejection around 0.1s later than the base model at a somewhat larger outflow rate, but then decays with time following a power law slope of $\delta_{\text{base}} = 1.45$, only $\sim$3% different than the base model slope $\delta_{\text{base}} = 1.50$. This qualitatively similar behaviour between poloidal and toroidal models is also found by Christie et al. (2019), although the initial conditions of their simulations lead to different quantitative values of $\delta$. Each 3D model has a different quantitative value of $\delta$, despite having the same late-time mass ejection mechanism. This variation can be attributed to physical differences in the disks and the timing of mass ejection. Each disk reaches maximum outflow at a phase in its evolution when the remaining disk mass, wind loss rate, and accretion rate are different relative to the initial disk mass and timescale of angular momentum transport in the disk. The initial time of mass ejection is also related to the range of electron fractions in the outflow. All runs produce a broad range of $Y_e$ in the ejecta, with a lower limit $Y_e \gtrsim 0.05$ (See §3.3.4).

#### 3.2.1 Morphology

Kilonova emission is dependent on the ejecta morphology (e.g., Kasen et al. 2017; Kawaguchi et al. 2020, 2021; Kobobkin et al. 2021; Heinzel et al. 2021), which can vary depending on the type of binary and mass ejection mechanism. The morphology of the disk outflow ejecta for the base and base-tor models is shown in the rightmost panel of Figure 1, showing a characteristic “hourglass” shape found in previous GRMHD simulations (e.g., Fernández et al. 2019a; Christie et al. 2019). This feature is robust across all our 3D models, as can be seen from the angular mass outflow histograms in Figure 4. Model base-tor ejects 50% less mass with $v \gtrsim 0.25 \, c$ and within $\sim \pi/4$ of the rotation axis than model base, and no ejecta in this velocity range is produced within $\sim \pi/4$ of the rotation axis.
equatorial plane. In contrast, models base and bhns have a much wider distribution of fast/early ejecta, extending down to within $\pi/6$ of the rotation axis. This implies that the morphology of the highest velocity ejecta is dependent on the initial condition of the torus, with the compactness of the disk having little effect on outflow geometry. This result is supported by the results of (Christie et al. 2019), as well as those of (Siegel & Metzger 2018) in their discussion of the initial transient phase.

3.2.2 Compactness

We find that the fraction of the initial disk mass ejected decreases with increasing disk compactness (model bhns), following the same trend as the hydrodynamic results of
Fernández et al. (2020). This shows that the additional mass ejected by MHD effects relative to pure viscous hydrodynamics also decreases with increasing disk compactness. Figure 4 shows that the fastest ejecta ($t < 0.1 s$) becomes a smaller fraction of the total mass ejected, with the majority of the disk outflow ensuing after 1 s. This change can be attributed to multiple differences with the base model: the gravitational potential is deeper by a factor $\sim 2$ at the density maximum, the ISCO of the BH is closer to the torus density maximum (see, e.g., Figure 2), and the initially lower density torus emits an order of magnitude less neutrino luminosity and is more transparent to neutrinos. Nevertheless, the different disk structure results in a shorter neutrino cooling time at the initial density peak in the bhns model (1 ms) relative to the base model (2 ms). Thus, at early times ($t < 0.1 s$) when neutrino cooling is strong, disk material is more bound in the bhns than in the base case.

We note a sharp drop in mass ejected with $Y_e \sim 0.3$ for the base model, also found in Fernández et al. (2020), which can be attributed to the larger relative importance of neutrino absorption in more massive tori. For a neutron-rich disk where neutrino absorption dominates the evolution of $Y_e$, the process $\nu_e + n \rightarrow e^- + p$ occurs more frequently than its inverse, increasing the net $Y_e$ (Siegel & Metzger 2018; Fernández et al. 2020; Most et al. 2021). This trend is also found by Just et al. (2022) with a significantly more advanced neutrino scheme - a broader distribution in $Y_e$ corresponds to more absorption (see their Figure 13). Similar 2D axisymmetric simulations by Shibata & Sekiguchi (2012) yield neutrino luminosities that decrease from $10^{52}$ erg s$^{-1}$ to $10^{51}$ erg s$^{-1}$ as the BH mass increases by from $3 M_\odot$ to $6 M_\odot$. We find a very similar trend as we change the compactness.

### 3.3 Mass Ejection in 2D: Sensitivity to Physics Inputs

#### 3.3.1 Dimensionality

Measuring unbound ejecta by the end of each simulation, the base-2D model ejects a factor of $\sim 2$ more mass than the equivalent base run in 3D, despite running for 1.4 s instead of 3 s. The evolution is qualitatively similar for the first $\sim 1$ s, until the axisymmetric torus becomes dominated by magnetic pressure and is disrupted, at which point we end the simulation. Quantitatively, model base-2D produces a higher mass outflow rate at all times, in particular more ejecta with velocities $\gtrsim 0.25 c$. We attribute this enhanced mass ejection to the lower accretion rate onto the BH in axisymmetry given the suppression of the MRI, with divergence in the evolution from the 3D case starting at $\sim 20$ ms. With less accretion, the larger amount of matter in the torus results in a higher outflow rate, given that the same outflow driving processes operate in 2D and in 3D.
3.3.2 Spatial Resolution

To quantify uncertainties due to spatial resolution, we run versions of model \textit{base-2D} at half and twice the resolution in both the radial and polar directions. The high-resolution model is evolved until 0.5 s, probing the early, magnetically-driven phase, and the half-resolution model is evolved until 1 s, which includes the radiatively-inefficient phase of mass ejection. Mass ejection up to 1 s is \(\sim 10\%\) higher in the standard resolution model relative to the low-resolution model. We thus associate an uncertainty of 10\% to our mass ejection numbers due to spatial resolution. The growth of the toroidal magnetic field is identical during both the magnetic winding and MRI growth phase in all models until 3 ms, when the maximum value of toroidal magnetic field saturates at \(2 \times 10^{15}\) G. We find a 0.5\% difference in the saturation value of \(|B_\phi|\) between the standard and high resolution runs, and a 5\% lower saturation value comparing the low to standard resolution run. Thereafter, the maximum of \(|B_\phi|\) undergoes stochastic fluctuations with amplitude of order unity until a dissipation phase begins at \(\sim 30\) ms. The standard and high-resolution models remain consistent within fluctuations.

3.3.3 Nuclear Recombination

Comparing mass ejection from models \textit{base-2d} (with recombination) and \textit{base-norec} (without recombination), we find that nuclear recombination remains a subdominant effect until the end of our 2D simulation at 1.4 s. Before 0.5 s, energy input from nuclear recombination increases the mass-averaged velocity of ejecta, resulting in a noticeable decrease in mass ejected at \(\sim 0.3\) c and an increase at \(\sim 0.25\) c in model \textit{base-norec} compared to model \textit{base}. After \(\sim 0.5\) s, comparatively less mass is ejected in model \textit{base-norec} due to the lack of recombination heating. In other words, mass which would have been ejected in the initial MHD-driven phase is instead ejected slower and at a later time, indicating that the net effect of nuclear recombination is to make matter less gravitationally bound and thus easier to eject by magnetic forces at early times. We find an almost identical distribution in electron fraction, skewed to a slightly lower average value since less mass is ejected later when the charged current weak interactions have already raised the \(Y_e\).
3.3.4 Neutrino Absorption

Inclusion of neutrino absorption results in additional mass ejection by a factor of $\sim 2$ relative to a model without it (base-noirr), and a negligible effect on the mass ejected at $t < 0.1$ s, when magnetic stresses dominate. The energy input from neutrino absorption causes additional mass to become marginally unbound, extending the distribution in velocity space to slower outflow. However, since more mass is ejected, neutrino absorption produces a decrease by 0.05 c in the mass averaged velocity. Turning off neutrino absorption skews the electron fraction of the ejecta to lower values, with more mass (factor of 2) being ejected at all times with $Y_e < 0.1$, and 2 orders of magnitude less ejecta with $Y_e > 0.4$.

3.4 Comparison to previous work

The ejected masses and velocities from our 3D models are in broad agreement with comparable simulations (Siegel & Metzger 2018; Miller et al. 2019; Fernández et al. 2019a; Christie et al. 2019; Just et al. 2022). The base run is qualitatively closest to the model of Siegel & Metzger (2018), which lacks neutrino absorption, and to the MHD model of Just et al. (2022), which has a less massive torus (0.01$M_\odot$). Siegel & Metzger (2018) find that 16% of the torus is ejected during 381 ms of evolution, and Just et al. (2022) that 20% of the initial torus mass is ejected during 2.1 s. Our base model ejects a higher fraction of the initial disk mass due to the difference in compactness as well as a longer simulation time. Relative to the long-term GRMHD simulation of Fernández et al. (2019a), which employed an initial field geometry conducive to a magnetically-arrested disk (MAD, e.g., Tchekhovskoy et al. 2011), ran for a longer time ($\sim 10$ s), and did not include neutrino absorption, our base run ejects $\sim 20$% less mass by the end of the simulation at 3 s. Our extrapolated ejected masses are comparable to that from this longer run, with other differences explainable by the difference in compactness and initial field geometry. The weak poloidal (SANE) model from Christie et al. (2019) is run for the same amount of time as our base run but with an initially weaker field, and also ejects $\sim 30$% of the torus mass during the simulation, despite not including neutrino absorption.

The toroidal run of Christie et al. (2019) ejects 3% less mass than their weak poloidal (MAD) run, whereas we find 15% more mass ejection in our base-tor (toroidal) model relative to our base (poloidal) model. We do find a lower average velocity (by $\sim 0.02$ c) in the base-tor model relative to our base model, same as they do. Christie et al. (2019) find that their toroidal model begins mass ejection at almost the same time as their weak poloidal run, and find a more sustained period of mass ejection from $\sim 0.01$ – 0.05 s in the toroidal model. Comparatively, we find that mass ejection in our base-tor model begins later and quickly rises to peak at a value higher than that of the poloidal simulation. This difference in dynamics could be attributed to a comparatively stronger toroidal magnetic field ($\beta \sim 0.01 - 2$ in the initial torus) and the effect of neutrino absorption. We do not vary the initial field strength in our simulations, as a lower field strength would require more cells to properly resolve the MRI. We can speculate on how lowering the field strength would change our outflows by comparing to the results of Christie et al. (2019). They find that lowering the field strength reduces the initial ($t \lesssim 0.5$ s) outflows driven by magnetic stresses, but the late-time thermal outflows are nearly identical. The effects on our (SANET) field configuration would likely be similar, but less prominent, given that their MAD configuration is optimized for producing magnetic outflows.

The work of Miller et al. (2019) utilizes the same initial conditions as our base model but with a more advanced neutrino scheme to treat neutrino emission and absorption. We find a similar amount of mass ejected by 100 ms of evolution ($\sim 2 \times 10^{-3} M_\odot$), indicating broad agreement despite the difference in neutrino schemes.

Utilizing a mean field dynamo to address the suppression of MRI in 2D, Shibata et al. (2021b) run resistive MHD simulations of high compactness toroidal disks. They find $\sim 10 - 20$% of the initial disk mass is ejected over $\gtrsim 4$ s with an average electron fraction ($Y_e \sim 0.25 - 0.35$), in broad agreement with our findings. Notably, they find that mass ejection begins $\gtrsim 500$ ms later than in our toroidal run, although this delay can be attributed to the high compactness of their models.

The recent GRMHD simulations of Hayashi et al. (2021) start from the inspiral of a 1.35$M_\odot$ NS and a 5.4 or 8.1 $M_\odot$ BH and evolve the remnant for up to 2 s, including neutrino leakage and absorption. They find qualitatively similar results when compared to our runs, albeit with much larger tori masses post-merger ($\sim 0.2 - 0.3 M_\odot$). The fraction of the torus mass ejected is also comparable to our bhns runs, as $\sim 10$% of their tori is ejected in the first 1 s, with a broad distribution in both electron fraction and velocity. Discounting dynamical ejecta, they find a peak electron fraction $Y_e \sim 0.25 - 0.35$ and post merger outflow velocities of $v \lesssim 0.08$ c, in good agreement with our results. They find outflows starting at $\gtrsim 200$ ms later than our bhns model, however the initial tori in their simulations are in a deeper potential well, and form with an initially toroidal field, both of which we find delay outflows in comparison to our base model.

By analyzing the net specific energy of tracer particles, Siegel & Metzger (2018) find that nuclear recombination of $\alpha$-particles plays a key role in unbinding matter in the disk outflow (in a simulation that does not include neutrino absorption). Our 2D model base-norec which has nuclear recombination turned off but includes neutrino absorption, ejects only slightly less mass than our base-2D run indicating that under these circumstances nuclear recombination is a sub-dominant effect. It remains to be tested whether recombination will remain sub-dominant in a fully 3D simulation that includes neutrino absorption and runs for a long time ($\gg 1$ s).

Our 2D model without neutrino absorption (base-noirr) ejects a factor $\sim 2$ less mass than the base-2D model. This difference is significantly larger than that found in models that employ viscous hydrodynamics, which typically find that neutrino absorption is dynamically sub-dominant for mass ejection (e.g., Fernández & Metzger 2013; Just et al. 2015). This also inconsistent with the 3D MHD run of Just et al. (2022), who find that turning off neutrino absorption results in a 2.5% increase in mass ejection relative to the initial torus mass, although they use a different neutrino leakage scheme and a less massive torus that is more transparent to neutrinos. Our results are limited by the use of axisymmetry for these simulations, but suggest that neutrino absorption could indeed be more significant for the dynamics of mass ejection and motivates further studies in 3D.
4 SUMMARY AND DISCUSSION

We have run long-term 3D MHD simulations to explore mass ejection from BH-tori systems formed in neutron star mergers. The publicly available code \textsc{FLASH}5 has been extended to allow its unsplit MHD solver to work on non-uniform spherical coordinates in 3D (Appendix A, B). All of our models include a physical EOS, neutrino emission and absorption via a leakage scheme with disk-lightbulb irradiation (Appendix C), and treat the gravity of the BH with a pseudo-Newtonian potential. Our 3D models employ different initial magnetic field geometries and disk compactnesses. We have also carried out axisymmetric models that suppress the nuclear recombination and neutrino absorption source terms.

The disk outflows from our 3D models exhibit a broad distribution in electron fraction and ejection polar angle (Figure 4), with a typical hourglass morphology (Figure 1). The tori eject matter with a bimodal distribution in velocity (Figure 4), with a typical hourglass morphology (Figure 1). The disk outflows from our 3D models exhibit a broad distribution in electron fraction and ejection polar angle (Figure 4), with a typical hourglass morphology (Figure 1). The tori eject matter with a bimodal distribution in velocity (Figure 4) associated with two different mass ejection phases: MHD stresses power early time ($t < 0.1 \text{ s}$) high velocity ($v \gtrsim 0.25 \text{ c}$) ejecta, and late-time ($t \sim s$) “thermal” ejection provides the majority of mass outflows centered around $v \sim \text{c}$.1.

We find that imposing an initially toroidal field configuration ejects $\sim 15\%$ more of the initial torus mass than the standard SANE poloidal field of similar maximal field strength (Figure 2, Table 2). However, the toroidal model ejects an order of magnitude less ejecta in the first 100 ms of evolution (Figure 4), comprising all of the ejecta travelling at velocities $v \gtrsim 0.25 \text{ c}$. The high-velocity ejecta is suppressed by the additional time it takes for dynamo action to convert toroidal into large-scale poloidal fields, which then drives radial angular momentum transport.

Increasing the disk compactness to values expected for typical BH-NS mergers results in significantly less mass ejection relative to our \textit{base} (NS-NS) model, beginning at a later time and decaying at a faster rate. Comparing to the viscous hydrodynamic models of Fernández et al. (2020), we find the same overall trend of decreased mass ejection with increasing disk compactness. Model \textit{bhs} has the same initial torus and BH configuration as their model \textit{b08d03}: by 4 s, our 3D MHD simulation has ejected 5.5% of the initial torus mass, while the viscous hydrodynamic equivalent has ejected only 1.9% of the initial disk mass by the same time. The hydrodynamic model goes on to eject 5.0% of the initial disk mass after 12 s of evolution, with mass ejection peaking at 2.3 s but remaining non-negligible until later times. The extrapolated outflow for our 3D MHD model \textit{bhs} predicts 6–8% of the initial torus mass, which is consistent with the previously-found enhancement in mass ejection by MHD relative to viscous hydrodynamics at smaller compactnesses when evolving both to $\sim 10\text{ s}$ (Fernández et al. 2019a). Our results inform analytic fits to fractions of the initial disk mass ejected like that of Raaijmakers et al. (2021), which has the enhancement in mass ejection due to magnetic effects relative to viscous hydrodynamics as a free parameter. More 3D MHD simulations at different compactness and with various initial magnetic field geometries are needed to improve the predictive power of these fits.

Our axisymmetric models that vary the physics show that nuclear recombination is a sub-dominant effect, while neutrino absorption can make a significant difference in mass ejection. Inclusion of neutrino absorption produces a shift of the velocity distribution at late times, down by $\sim 0.05 \text{ c}$, with negligible effects at early times ($t < 0.1 \text{ s}$). In the absence of neutrino absorption, the distribution of electron fraction shifts to include additional material with $Y_e < 0.1$ and exclude $Y_e > 0.4$. Nuclear recombination deposits additional energy into the already unbound outflows at $t \sim 0.1 \text{ s}$, resulting in ejecta with moderately higher velocities, but its absence only decreases the total ejecta mass by 1%. Inclusion of $r$-process heating by the formation of heavier nuclei can further speed up the ejecta at late times (Klion et al. 2022). Proper characterization of the effect of neutrino absorption and nuclear recombination on the mass ejection dynamics and composition must be done with full 3D simulations, which unfortunately still remain expensive computationally.

Our \textit{base} and \textit{base-tor} models have initial conditions consistent with the post-merger system of the observed NS-NS merger GW170817. We find that although these 3D models can eject lanthanide-free material ($Y_e > 0.25$) with velocities inferred from kilonova modelling, $v \gtrsim 0.25 \text{ c}$, there is insufficient mass in the outflows to match observations (e.g., Kasen et al. 2017; Villar et al. 2017) with the disk outflow alone. The inclusion of a finite-lived remnant in our \textit{base} model is a promising way to produce more lanthanide-free ejecta at the required velocities (e.g., Fahlman & Fernández 2018). The main limitations of our work are the approximations made for modelling neutrino radiation transport on the necessary timescales. Prior research into the effectiveness of neutrino schemes have shown that the differences between two-moment (M1) and Monte Carlo (MC) schemes can result in a $\sim 20\%$ uncertainty in neutrino luminosity, translating to a difference of 10% in outflow electron fraction (Foucart et al. 2020). Comparison between M1 and the leakage scheme of Ardevol-Pulpillo et al. (2019) (which our scheme is based on, see Appendix C) shows a further 10% uncertainty in neutrino luminosities, and a comparison of leakage+M0 to M1 schemes shows that leakage schemes tend to decrease the average $Y_e$, but with a minimal effect on nucleosynthetic yields (Radice et al. 2021). The exclusion of relativistic effects implies that we cannot accurately model jet formation, but the effects of these approximations on mass ejection and composition are likely minimal, since the relevant processes operate far from the BH. The leading order special relativistic corrections to the MHD equations are $\sim v/c$ for $v < c$, hence we estimate uncertainties associated with our fastest ejecta to be at least of the same order (e.g., 50% for matter with $v/c = 0.5$, etc.). The bulk of mass ejection has $v/c < 0.1$, and thus uncertainties due to Newtonian physics should be on the order of 10%, comparable to those due to spatial resolution.

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A1 Governing Equations of Magnetohydrodynamics

The ideal MHD equations are those of mass, momentum, and energy conservation (equations 1-3 without source terms) together with the induction equation (5). Note that in FLASH, the induction equation is implemented as

\[
\frac{d\mathbf{B}}{dt} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0 \quad (A1)
\]

which has the opposite sign as equation (5). For the rest of this Appendix, we will use this form of the induction equation for consistency with the available literature on FLASH.

The conservation equations are written in terms of a vector of conserved variables \( \mathbf{U} \), a tensor of associated fluxes \( \mathcal{F} \), and a vector of source terms \( \mathbf{S} \).

\[
\frac{d\mathbf{U}}{dt} + \nabla \cdot \mathcal{F} = \mathbf{S} \quad (A2)
\]

Writing out the associated fluxes in the radial, polar, and azimuthal direction as \( \mathbf{F}, \mathbf{G}, \) and \( \mathbf{H} \) respectively, we obtain

\[
\frac{d\mathbf{U}}{dt} + \frac{1}{r} \frac{\partial (r^2 \mathbf{F})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta \mathbf{G})}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \mathbf{H}}{\partial \phi} = \mathbf{S} \quad (A3)
\]
where (Lee 2013)

\[
\mathbf{U} = \begin{pmatrix}
\rho \\
\rho v_r \\
\rho v_\theta \\
\rho v_\phi \\
B_r \\
B_\theta \\
B_\phi \\
pE
\end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix}
\rho v_r \\
\rho v_r v_r + P - B_r^2 \\
\rho v_r v_\theta - B_r B_\theta \\
\rho v_r v_\phi - B_r B_\phi \\
v_r B_\theta - v_\theta B_r = -E_\phi \\
v_r B_\phi - v_\phi B_r = E_\theta \\
v_r (\rho E + P) - B_r (\mathbf{v} \cdot \mathbf{B})
\end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix}
\rho v_\theta \\
\rho v_\theta v_r - B_\theta B_r \\
\rho v_\theta v_\phi - B_\phi B_\theta \\
\rho v_\theta v_\phi - B_\phi B_\phi \\
v_\theta B_r - v_r B_\theta = -E_\phi \\
v_\theta B_\phi - v_\phi B_\theta = E_\theta \\
v_\theta (\rho E + P) - B_\theta (\mathbf{v} \cdot \mathbf{B})
\end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix}
\rho v_\phi \\
\rho v_\phi v_r - B_\phi B_r \\
\rho v_\phi v_\theta - B_\theta B_\phi \\
\rho v_\phi v_\phi - B_\phi B_\phi \\
v_\phi B_r - v_r B_\phi = -E_\theta \\
v_\phi B_\theta - v_\theta B_\phi = E_\phi \\
v_\phi (\rho E + P) - B_\phi (\mathbf{v} \cdot \mathbf{B})
\end{pmatrix}.
\]  

A2 Geometric Source Terms in Spherical Coordinates

The geometric source terms arise when taking covariant derivatives of second rank tensors, often referred to as a tensor divergence. These terms take the physical form of fictitious forces, and only arise in the equations for vector quantities. The scalar energy and density evolution equations therefore do not have source terms. For a tensor, \(T\), in spherical coordinates the divergence is written as (see Fahlman 2019, or Mignone et al. 2005 for a separate derivation)

\[
\vec{\nabla} \cdot T = \begin{pmatrix}
\nabla_r T \\
\nabla_\theta T \\
\nabla_\phi T
\end{pmatrix},
\]

where we can define the divergences in each individual direction as

\[
\nabla_r T = \left( \frac{1}{r^2} \frac{\partial (r^2 T^r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta T^\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta T^\phi)}{\partial \phi} - \frac{T^\theta}{r} + \frac{T^\phi}{r \cot \theta} \right),
\]

\[
\nabla_\theta T = \left( \frac{1}{r^2} \frac{\partial (r^2 T^\theta)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta T^r)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta T^\phi)}{\partial \phi} + \frac{T^r}{r} + \frac{T^\phi}{r \cot \theta} \right),
\]

\[
\nabla_\phi T = \left( \frac{1}{r^2} \frac{\partial (r^2 T^\phi)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta T^r)}{\partial \phi} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta T^\theta)}{\partial \phi} + \frac{T^r}{r} - \frac{T^\phi}{r \cot \theta} \right).
\]

Explicitly applying the divergences (A8)-(A10) to the dyads in the momentum and induction equations (2 and 5) yields the source term vector in equation (A2)

\[
\mathbf{S} = \begin{pmatrix}
0 \\
\rho (v_r^2 + v_\theta^2) - B_\theta^2 - B_\phi^2 \\
\frac{\rho v_r v_\theta - B_r B_\theta}{r} + \frac{\cot \theta (\rho v_\theta^2 - B_\phi^2)}{r} \\
\frac{\rho v_r v_\phi - B_r B_\phi}{r} - \frac{\cot \theta (\rho v_\phi^2 - B_\theta^2)}{r} \\
0 \\
(B_\theta v_r - B_r v_\theta) \\
(B_\phi v_r - B_r v_\phi) + \frac{(B_\phi v_\phi - B_\theta v_\phi) \cot \theta}{r} \\
0
\end{pmatrix}.
\]

The source terms have to be volume averaged according to

\[
\langle \mathbf{S} \rangle = \frac{1}{V} \iiint_V \mathbf{S}(r, \theta) \, dv.
\]

for use in the discretized conservative update equation (A5).

Since conservative mesh schemes store hydrodynamic variables as volume averages, we make the approximation of taking them out of the integral in equation (A12). The remaining terms contain either a \(r^{-1}\) or \(\cot \theta\) dependence, which are integrated analytically to get the volume-averaged source term.
vector

\[
\mathbf{S} = \frac{3\Delta r^2}{2\Delta r^3} \begin{pmatrix}
\rho(v_r^2 + v_\theta^2) - B_r^2 - B_\phi^2 \\
-\rho v_r v_\theta - B_r B_\theta + \frac{\Delta \sin \theta}{\Delta \cos \theta} (\rho v_\theta^2 - B_\phi^2) \\
-\rho v_r v_\phi - B_r B_\phi - \frac{\Delta \sin \theta}{\Delta \cos \theta} (\rho v_\phi v_\theta - B_\phi B_\theta) \\
(B_\theta v_r - B_r v_\theta) \\
(B_\phi v_r - B_r v_\phi) + \frac{\Delta \sin \theta}{\Delta \cos \theta} (B_\phi v_\theta - B_\theta v_\phi)
\end{pmatrix},
\]

where the coordinate differences are defined as

\[
\Delta r^2 = r_{i+1/2}^2 - r_{i-1/2}^2, \quad (A14)
\]
\[
\Delta r^3 = r_{i+3/2}^3 - r_{i-1/2}^3, \quad (A15)
\]
\[
\Delta \cos \theta = \cos \theta_{j+1/2} - \cos \theta_{j-1/2}, \quad (A16)
\]
\[
\Delta \sin \theta = \sin \theta_{j+1/2} - \sin \theta_{j-1/2}. \quad (A17)
\]

These are the source terms used in the update of conserved variables (equation A5).

### A3 Fluxes and Primitive Variables in Spherical Coordinates

Before the conservative update (equation A5) can be performed, the fluxes at the face must be known. Here we provide only the information needed to adjust the reconstruction of fluxes for spherical coordinates. The discretization and methods used in FLASH to extend variables to the face and construct appropriate Riemann states can be found in Lee (2013). By default, FLASH employs a piecewise linear MUSCL-Hancock method (Colella 1985) to reconstruct the so-called cell-centered primitive variables,

\[
\mathbf{V} = \begin{pmatrix}
\rho \\
v_r \\
v_\theta \\
v_\phi \\
B_r \\
B_\theta \\
B_\phi \\
P_{gas}
\end{pmatrix}^T,
\]

(A18)

instead of the conserved variables, to the faces. The corresponding primitive system of equations is derived using the chain rule to expand the divergence in the conservative system. The primitive system is utilized in part because it has a quasi-linear form, which is written out in spherical coordinates as

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{M}_{r} \frac{\partial \mathbf{V}}{\partial r} + \mathbf{M}_{\theta} \frac{1}{r} \frac{\partial \mathbf{V}}{\partial \theta} + \mathbf{M}_{\phi} \frac{1}{r \sin \theta} \frac{\partial \mathbf{V}}{\partial \phi} = \mathbf{S}_p,
\]

(A19)

where the matrices \( \mathbf{M}_{r}, \mathbf{M}_{\theta}, \mathbf{M}_{\phi} \) can be found in Lee & Deane (2009) and Lee (2013). Explicitly expanding all the spatial and temporal derivatives of the conservative equations (1, 2, 3, A1) in spherical coordinates results in a source term

vector for the primitive system of variables given by

\[
\mathbf{S}_p = \begin{pmatrix}
-\rho \frac{2v_r + v_\theta \cot \theta}{r} \\
(\frac{v_\theta^2 + v_\phi^2}{r}) - (B_r^2 + B_\phi^2)/\rho \\
-\frac{v_r v_\theta - B_r B_\theta}{r} + \frac{\cot \theta (v_\theta^2 - B_\theta^2/\rho)}{r} \\
-\frac{v_r v_\phi - B_r B_\phi}{r} + \frac{\cot \theta (v_\phi v_\theta - B_\phi B_\theta/\rho)}{r} \\
-\frac{B_\theta v_r - B_r v_\theta}{r} \\
-\frac{B_\phi v_r - B_r v_\phi}{r} \\
-\gamma_1 P_{gas} \frac{2v_r + v_\theta \cot \theta}{r}
\end{pmatrix}.
\]

These are the source terms implemented in the reconstruction of cell-centered variables to the faces, along with the geometrically correct lengths, in equation (A19).

### A4 Constrained Transport

Magnetic fields require an update method different from conserved quantities, as the induction equation does not explicitly require that the solenoidal constraint,

\[
\nabla \cdot \mathbf{B} = 0,
\]

(A21)

is satisfied. To satisfy this constraint, fields are taken to be averaged over cell faces, and the induction equation is written as

\[
\oint \oint \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dA} = \oint \oint -\nabla \times \mathbf{E} \cdot \mathbf{dA}.
\]

(A22)

Applying Stokes’ theorem, using the definition of magnetic fields as area-averaged, and discretizing the resulting line integral as a sum, we arrive at the Newtonian form of CT (Evans & Hawley 1988)

\[
\mathbf{B} \cdot \mathbf{A} = -\frac{\Delta t}{2} \sum_{\text{edges}} \mathbf{E} \cdot \Delta \ell.
\]

(A23)

To perform the summation, we need the electric fields around each edge of the cell. These are found by Taylor expanding the face centered electric fields, which are obtained from the fluxes of conserved magnetic variables (equation A4) in the reconstruction step, and using Ohm’s law in ideal MHD

\[
\mathbf{E} = -\nabla \times \mathbf{B}.
\]

(A24)

Each flux contains 2 electric fields, one in each transverse direction. At each edge there will be 4 independent fluxes that can be used to find the electric field at that point (Figure A1 shows two of the fluxes) Conventionally, all 4 contributions are included, which can be numerically unstable (e.g., Tóth 2000; Mignone & Del Zanna 2021). Fortunately, FLASH includes an option to only use the upwind fluxes (flux vectors which have a positive mass flux) to construct the electric fields (e.g., White et al. 2016). We find that this upwinded method is necessary to preserve numerical stability when using FLASH to evolve a magnetized torus in 3D spherical coordinates, despite not being a default option. This is
also found by Kuroda et al. (2020), who note that in the regions of their core-collapse supernova setup where matter is supersonically advecting (analogous to the torus), the upwind method is necessary for stability. While no additional work is needed to implement the upwinding in spherical coordinates, the Taylor expansion for expanding the face fluxes to cell edges must be updated to account for non-uniform grid spacing. The Taylor expansion uses second-order finite differences to determine the approximate value at face edges. For example, the expansion in an arbitrary $i^{th}$ direction is

$$E_{i \pm \frac{1}{2}, j, k} = E_{i, j, k} \pm \Delta \ell \pm \frac{1}{2} \frac{\partial E}{\partial \ell} + \left( \Delta \ell \pm \frac{1}{2} \right)^2 \frac{1}{2} \frac{\partial^2 E}{\partial \ell^2},$$

(A25)

where $\Delta \ell \pm \frac{1}{2}$ corresponds to the distance from the center of the cell to the $i^{th}$ face. Writing the spatial derivatives as cell-centered finite differences, where $\Delta \ell \pm$ is the distance from the center of the cell $i$ to the cell centers above and below ($i \pm 1$), yields

$$\frac{\partial E}{\partial \ell} = \frac{E(\ell + \Delta \ell) - E(\ell - \Delta \ell)}{2 \Delta \ell},$$

(A26)

$$\frac{\partial^2 E}{\partial \ell^2} = \frac{E(\ell + 2\Delta \ell) - 2E(\ell) + E(\ell - 2\Delta \ell)}{2(\Delta \ell)^2}. \tag{A27}$$

For uniform spacing, $\Delta \ell_+ = \Delta \ell_- = \Delta \ell$, and the first and second order derivative finite differences simplify to

$$\frac{\partial E}{\partial \ell} = \frac{E(\ell + \Delta \ell) - E(\ell - \Delta \ell)}{2 \Delta \ell}, \tag{A28}$$

$$\frac{\partial^2 E}{\partial \ell^2} = \frac{E(\ell + 2\Delta \ell) - 2E(\ell) + E(\ell - 2\Delta \ell)}{2(\Delta \ell)^2}. \tag{A29}$$

Since the public version of FLAS only supports a uniform grid, the equations in the code (A28-A29) must be modified so that the more general case (A26-A27) is used for non-uniform grid spacing. Once the electric field construction is complete, the magnetic fields are updated to half timestep with equation (A23) by adding up electric fields around the cell faces (Figure A1).
We perform two torus tests. The first is carried out in 2.5D in both cylindrical and spherical coordinates, and employs a standard and normal evolution (SANE) initial magnetic field configuration. Results can be compared to multiple previous implementations (Hawley 2000; Mignone et al. 2007; Tzeferacos et al. 2012). The second test is the extension to 3D spherical coordinates of the same SANE torus.

We set up the tori as in Fahlman (2019), using the gravitational potential of (Paczynski & Wiita 1980) and setting the gravitating mass, $M = 1$. We normalize units such that $G = c = 1$. We choose the same parameters as model GT1 from Hawley (2000), which creates a thin, constant angular momentum torus with a maximum density of 10 and an orbital timescale of $\sim 50$ at the initial circularization radius $r_{\text{circ}} = 4.7$, and a minimum radius of $r = 3$. The initial vector potential follows the density distribution,

$$A_\phi = \max(\rho - \frac{1}{2}\rho_{\text{max}}, 0),$$  \hspace{1cm} (B1)

which creates poloidal field loops threaded well within the torus. Initially, the field is normalized to $\langle \beta \rangle = 100$, so the field is dynamically unimportant and does not disturb the equilibrium condition. The tori are then evolved for a total of $\sim 3$ orbits at $r_{\text{circ}}$.

We compare the evolution of hydrodynamic variables in the 2.5D runs quantitatively and qualitatively in Figures (B3-B5). One notable difference between the two axisymmetric runs is the resolution and the inner boundary. In cylindrical coordinates, the entire inner $z$-boundary is set to be absorbing, while in spherical coordinates the inner radial boundary is set to be absorbing, and the polar boundaries are reflecting. This affects the dynamics near the inner edges of the torus, which is noticeable in the magnetic fields. The resolution in the two runs is comparable, but not exactly the same as necessitated by the difference in grid structures. The spherical run resolves the torus better, with $\Delta r = 0.028$ and the highest angular resolution in the midplane, $r_{\text{circ}} \min(\Delta \theta) = 0.024$, resulting in nearly square cells at that location, compared to the constant cylindrical resolution of $\Delta r_{\text{cyl}} = 0.035$ and $\Delta z = 0.035$. The two grids are shown in Figure B4 for reference.

The axisymmetric runs follow the same qualitative evolution in cylindrical and spherical coordinates: the azimuthal
Figure B5. Results from the 2.5D axisymmetric torus tests, evolved for 3 orbits at the initial radius of maximum density $r_{\text{circ}}$. The left column shows the height-averaged density, angular momentum, and magnetic pressure (all in code units) in the cylindrical simulation, while the right column shows the same quantities for the spherical case but angle-averaged instead. Lines of increasing thickness show the torus evolution from its initial to final state at each orbit ($n_{\text{orb}}$). Numerical differences between the angle- and height-averaging appear even at the 0th orbit, but the general trends are not affected.

Figure B6. Comparison between the 2.5D axisymmetric and 3D torus test in spherical coordinates, showing the same angle-averaged quantities as in Figure B5 at various times. Both runs follow the expected evolution: the 3D torus sustains a more powerful MRI for a longer time due to the additional azimuthal turbulence, with the evolution being otherwise identical during the first $\sim 2$ orbits. Differences in spatial resolution manifest as smoother profiles in the 3D runs, which has coarser cell sizes.
magnetic field grows due to winding, and the magnetic pressure causes the torus to expand and accrete. The radial angular momentum profile flattens as matter approaches the ISCO, and the magnetic field grows largest in the central regions and then accretes, frozen in with the mass flow. The first large divergences between the two simulations begin to appear in the magnetic field after $\sim 1$ orbit at $r_{\text{circ}}$, when the accretion stream reaches the inner boundary. Feedback from the reflecting boundary changes the expansion of the torus into the ambient between the two cases. Furthermore, additional turbulent structures manifest in the spherical torus, noticeable as less smooth profiles in Figure B5. We attribute this in part to the differences in spatial resolution, and also note that Mignone et al. (2007) see similar effects in their PLUTO code tests with spherical and cylindrical coordinates and the exact same setup (see their Figure 8).

The 3D SANE torus follows the same evolution as the axisymmetric spherical case for the first few orbits, as shown in Figure B6. Notable differences begin to appear after $\sim 3$ orbits at $r_{\text{circ}}$, as the MRI begins to die down in the axisymmetric run. In 3D, the MRI creates stronger magnetic fields and is sustained for longer through the additional turbulence in the azimuthal direction (Hawley 2000). The profiles in the 3D run appear smoother, as we run it with a lower resolution than in the 2D case due to computational limitations. To make up for this difference in resolution, we use a logarithmic grid in radius in the 3D model, so that the region containing the torus is still resolved well. This corresponds to resolutions in the radial, polar, and azimuthal directions being $\Delta r_{\text{circ}} \sim 0.040$, $r_{\text{circ}} \sin(\Delta \theta) = 0.10$, and $\Delta \phi = 0.03$.

**APPENDIX C: NEUTRINO LEAKAGE SCHEME**

Neutrinos change the composition of disk outflows through charged current weak interactions that alter the ratio of protons to neutrons. These transformations proceed via emission or absorption of electron neutrinos and antineutrinos. A common approach to modeling emission of neutrinos is the so-called “leakage scheme” (Ruffert et al. 1996), which interpolates between the diffusive and transparent regimes of radiative transport. Leakage schemes have been shown to capture the dominant effects of neutrinos in post-merger tori around compact objects, especially for BH disks, for which they are subdominant energy sources (Foucart et al. 2019; Fernández & Metzger 2013; Siegel & Metzger 2018; Fernández et al. 2019a). However, significant differences appear when compared quantitatively to more advanced Monte-Carlo or two-moment (M1) schemes (Richers et al. 2015; Foucart et al. 2015; Perego et al. 2016; Ardevol-Pulpillo et al. 2019; Radice et al. 2021). For this reason it is necessary to make improvements to the previous leakage-scheme implemented in FLASH (Fernández & Metzger 2013; Metzger & Fernández 2014), while retaining computational efficiency.

### C1 Leakage Overview

The key components of a leakage scheme are the two source terms that describe the effective neutrino energy and number loss rate per unit volume for each neutrino species,

$$Q_{\nu_i}^{\text{eff}} = Q_{\nu_i} \chi_{\nu_i,E}, \quad R_{\nu_i}^{\text{eff}} = R_{\nu_i} \chi_{\nu_i,N},$$

where $i$ represents a species ($\nu_e$ or $\bar{\nu}_e$ in our scheme), the subscripts $E$ and $N$ refer to energy and number, respectively. $Q$ and $R$ are the energy and number production rates per unit volume, respectively, which are obtained from analytic expressions (Ruffert et al. 1996). The scaling factors $\chi_{\nu_i,E}$ and $\chi_{\nu_i,N}$ interpolate between the free-streaming and optically thick (diffusive) regimes for neutrinos in both energy and number,

$$\chi_{\nu_i,E,N} = \left(1 + \frac{t_{\nu_i}^{\text{loss}}}{t_{\nu_i}^{\text{diff}}} \right)^{-1},$$

where $t_{\nu_i}^{\text{diff}}$ and $t_{\nu_i}^{\text{loss}}$ are the diffusion and loss timescales for each species, respectively. The source terms for equations (3)-(4) are then obtained as follows

$$Q_{\text{net}} = Q_{\nu_i}^{\text{eff}} + Q_{\nu_i}^{\text{abs}}, \quad \Gamma_{\text{net}} = m_n \rho Q_{\nu_i}^{\text{eff}} + R_{\nu_i}^{\text{eff}} + \Gamma_{\nu_i}^{\text{abs}},$$

where $Q_{\nu_i}^{\text{abs}}$ and $\Gamma_{\nu_i}^{\text{abs}}$ are the contributions from neutrino absorption (treated separately) and $m_n$ is the neutron mass. The diffusion and loss timescales in equation (C3) are central to the accuracy of the scheme, and thus we will discuss them in more detail.

#### C1.1 Loss timescale

Once the direct energy and number production rates in (C1-C2) are found, the loss times are obtained as

$$t_{\nu_i}^{\text{loss}} = \frac{Q_{\nu_i}}{E_{\nu_i}}, \quad t_{\nu_i}^{\text{loss}} = \frac{R_{\nu_i}}{N_{\nu_i}},$$

where $E_{\nu_i}$ and $N_{\nu_i}$ are the neutrino energy density and number density, respectively. These quantities are obtained using analytic fits to Fermi integrals from Takahashi et al. (1978).

#### C1.2 Diffusion timescale

The diffusion timescale is approximately given by

$$t_{\nu_i}^{\text{diff}}(E,N) \sim \frac{3\kappa_{\nu_i}(E,N) d^2}{c},$$

where $\kappa_{\nu_i}(E,N)$ is the energy or number opacity for species $i$, and $d$ is a characteristic diffusion distance. A more accurate expression involves calculation of the optical depth in various directions, which many leakage schemes incorporate (e.g., Rosswog & Liebendörfer 2003), but which is a global calculation that is computationally expensive. Our previous leakage implementation (Fernández & Metzger 2013; Metzger & Fernández 2014) approximates $d$ as the pressure scale height assuming hydrostatic equilibrium in the cylindrical $z$-direction, which is the preferential direction for neutrinos to escape the torus,

$$d \approx \frac{P}{(\frac{\partial P}{\partial r})} = \frac{P}{\rho \cos \theta |g|}.$$
This is a local calculation which yields a neutrino optical depth correct to within a factor of $\sim 2$.

Recently, Ardevol-Pulpillo et al. (2019) have developed a novel method for determining the diffusion timescale which is local (computationally efficient) and accurate. In this method, the diffusion timescale is determined analogously to the loss timescales:

$$\tau_{\nu_i} = \frac{1}{\nu_i} \left[ 1 + \frac{1}{3 \nu_i} \frac{|\nabla \{E, N\}|}{\{E, N\}} \right]^{-1}, \quad \text{(C14)}$$

In contrast to Ardevol-Pulpillo et al. (2019) who integrate quantities over the neutrino distribution, we use energy-averaged (over a Fermi-Dirac distribution) opacities, energy densities, and number densities in equation (C13), computing only the spatial gradient.

C2 Implementation in FLASH4.5

We extend the leakage and light bulb absorption scheme of Fernández & Metzger (2013) and Metzger & Fernández (2014) by computing the diffusion time with equation (C13) and implement it in FLASH4.5. Neutrino energy and number gradients are obtained using second order finite differences in each spatial direction. The flux limiters are calculated for each species as in (C14). The flux limiters are then combined with the gradients, energy/number densities, and opacities to form the fluxes in (C11). The fluxes are then linearly interpolated to the cell faces, such that we can take a numerical divergence using Gauss’s theorem (A6) and the face areas of a given cell, analogous to the unsplit update in §A1.

C2.1 Comparison with 2D hydrodynamic simulations

We test the effect of the new diffusion time on the long-term disk evolution by running a 2D axisymmetric hydrodynamic simulation of an accretion torus using the previous leakage scheme and the new one. A full description of the methodology is available in Fernández & Metzger (2013); Metzger & Fernández (2014), here we briefly describe the initial setup. The compact object is a BH with mass $3M_\odot$, and the orbiting equilibrium torus is chosen to be optically thin with a mass of 0.03$M_\odot$. The entropy, electron fraction, and torus distortion parameter are chosen to be $8k_B$/baryon, 0.1, and 1.911, respectively. This yields a well-studied initial condition compatible with dynamical merger simulations (Fernández & Metzger 2013; Metzger & Fernández 2014; Richers et al. 2015; Lippuner et al. 2017). Angular momentum transport is handled with the viscous stress parameterization of Shakura & Sunyaev (1973), with $\alpha$ set to 0.05. The tori are evolved for 1 second.

The average mass-flux-weighted electron fraction of the outflow increases by 5% when using the new diffusion time formulation, as shown in the histogram of torus outflows in Figure C1. This result corresponds to a decrease in diffusion time by a factor of $\sim 10$ in the high-density regions of the torus. Neutrinos are still preferentially trapped near the midplane, but the new methodology no longer overestimates this effect for the optically thin torus. The new method yields about 50% less mass outflows, predominately due to the same effect: the energy deposited into early time outflows is cooled more efficiently, unbinding less material. The major difference occurs in the neutrino-driven outflow ejected from the back of the torus, with electron fraction $Y_e \sim 0.25$. We caution that this difference will likely not extend to the late-time, radiatively inefficient evolution, as the energy and density of late time outflows appear identical, and affects mainly the initial winds.
Figure C2. Comparison of neutrino source terms (Top: energy rate per unit mass in cgs, Bottom: number rate per baryon) obtained using a leakage scheme with the diffusion time prescription from Ardevol-Pulpillo et al. (2019) and light bulb absorption (Left: FLASH New), Center: SedonuGR Monte-Carlo transport, and the leakage scheme plus light bulb absorption previously implemented in Metzger & Fernández (2014) (Right: FLASH Old). The background fluid quantities correspond to a snapshot of the FLASH New simulation at 0.6 ms. The new leakage scheme no longer artificially suppresses the source terms in the midplane of the torus, coming closer to the results of SedonuGR. Contours of $10^5 \text{g cm}^{-3}$ in density are shown as thick dashed lines, corresponding to the density cutoff at which SedonuGR no longer performs neutrino calculations.

C2.2 Comparison with SedonuGR in Hydro

We compare the emission and absorption of neutrinos to the Monte-Carlo neutrino transport code SedonuGR (Richers et al. 2015, 2017). We are interested in the rate of change of lepton number, which governs the change in electron fraction, and the rate of change of internal energy, which governs changes in internal energy of the ejecta due to neutrinos. Following Richers et al. (2015), we load a fluid snapshot from the FLASH4.5 simulation using the new method in SedonuGR. We use the Helmholtz EOS in SedonuGR for consistency.

At late times, the neutrino scheme for the optically thin torus shows minimal differences with the full Monte-Carlo results (Richers et al. 2015). The changes to the neutrino scheme are most evident at very early times, so we choose a time of $\sim 0.6$ ms for the comparison, the result of which is shown in Figure C2. Most of the differences in source terms occur in the regions where absorption becomes important, which in our approach is handled by an approximate light bulb implementation (Fernández & Metzger 2013; Metzger & Fernández 2014), which we did not modify here. Importantly, the new neutrino leakage scheme no longer suppresses neutrino emission in the midplane of the torus, where the pressure scale height is comparatively large. This is shown quantitatively with slices in the equatorial plane and along the z-direction at the torus density maximum in Figure C3.

C2.3 MHD Comparison with SedonuGR

Since our base torus is more massive than those used in the previous hydrodynamic test runs, and MHD evolution differs in comparison to viscous hydrodynamics, we also show a comparison of our leakage scheme in 2D-MHD with results from SedonuGR (Figure C4). Equatorial and vertical slices through the density maximum of a torus identical to our base model are shown in Figure C5 at 30 ms, when the neutrinos...
Figure C4. Same as Figure C2, but for a 0.1$M_\odot$ torus evolved with 2D-MHD at $\sim 30$ ms using the updated leakage scheme and comparing with SedonuGR. There are no previous MHD results to compare with. Overall, SedonuGR and the new leakage scheme show similar trends in both heating/cooling and change in lepton number. SedonuGR does not discriminate between ambient and torus material, while our scheme does not perform neutrino calculations on ambient matter. This appears as purple areas in the poloidal regions of the FLASH simulations.

Figure C5. Same as Figure C3, but for a 0.1$M_\odot$ torus evolved with 2D-MHD and the updated leakage scheme at $\sim 30$ ms. There are no previous MHD results to compare with, but the overall effects of neutrinos are captured well, with significant deviations only in small localized regions.

are important for setting the electron fraction of the outflows. Importantly, the source term modifying the electron fraction remains of the same order of magnitude across most of the torus.

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