DEGENERATIONS AND FUNDAMENTAL GROUPS RELATED TO SOME SPECIAL TORIC VARIETIES

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1. Introduction

Let $X$ be a projective algebraic surface embedded in a projective space $\mathbb{CP}^N$. Take a general linear subspace $V$ in $\mathbb{CP}^N$ of dimension $N - 3$. Then the projection centered at $V$ to $\mathbb{CP}^2$ defines a finite map $f : X \to \mathbb{CP}^2$. Let $B \subseteq \mathbb{CP}^2$ be the branch curve of $f$. Denote $\pi_1(\mathbb{CP}^2 \setminus B)$ to be the fundamental group of the complement of the branch curve. This group is an invariant of the surface. Closely related to this group is the affine part $\pi_1(\mathbb{C}^2 \setminus B)$.

In this work we compute the above defined groups, related to four toric varieties. The first surface is $X_1 := F_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$, the Hirzebruch surface of degree one in $\mathbb{CP}^2$.

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\( \mathbb{C}P^6 \) embedded by the line bundle with the class \( s + 3g \), where \( s \) is the negative section and \( g \) is a general fiber. The second is \( X_2 := F_0 = \mathbb{C}P^1 \times \mathbb{C}P^1 \), the Hirzebruch surface of degree zero in \( \mathbb{C}P^7 \) embedded by \( \mathcal{O}(1, 3) \). We generalize the results to the case where \( X_2 \) is embedded in \( \mathbb{C}P^{2n+1} \) by \( \mathcal{O}(1, n) \). The third is \( X_3 := F_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) \) in \( \mathbb{C}P^5 \) embedded by the class \( s + 3g \). The fourth is a singular toric surface \( X_4 \) with one \( A_1 \) singular point embedded in \( \mathbb{C}P^6 \). \( A_1 \)-singularity is an isolated normal singularity of dimension two whose resolution consists of one \((-2)\)-curve (i.e., a nonsingular rational curve on a surface with \(-2\) as its self-intersection number). For the first three cases, we use different triangulations of tetragons from those treated in [24] and [25].

This work fits into the program initiated by Moishezon and Teicher to study complex surfaces via braid monodromy techniques. They defined the generators of a braid group from a line arrangement in \( \mathbb{C}P^2 \), which is the branch curve of a generic projection from a union of projective planes [24], namely degeneration. In order to explain the process of such a degeneration, they used schematic figures consisting of triangulations of triangles and tetragons ([20], [23], [24]). Moishezon and Teicher studied the cases when \( X \) is the projective plane embedded by \( \mathcal{O}(3) \) [24], or when \( X \) are Hirzebruch surfaces \( F_k(a, b) \) for \( a, b \) relatively prime [19]. Later works compute the group \( \pi_1(\mathbb{C}P^2 \setminus B) \) related to \( K3 \) surfaces [4], \( \mathbb{C}P^1 \times T \) where \( T \) is a complex torus ([5], [6]), \( T \times T \) ([2], [3]), and Hirzebruch surface \( F_1(2, 2) \) [9]. A very interesting and helpful work concerning degenerations, braid monodromy and fundamental groups, was written by Auroux-Donaldson-Katzarkov-Yotov [11].

We consult the above works and give a geometric meaning to these schematic figures from the point of view of toric geometry ([14], [27]). The work is done along
the following lines. First we degenerate $X$ into a union $X_0$ of planes. Then $X_0$ is composed of $n = \deg(X_0)$ planes. $B_0$ is the union of the intersection lines $1, 2, \ldots, m$ (as depicted in Figures 1, 5, 7, 8). The lines are numerated for future use. It is very complicated to get a presentation of $\pi_1(\mathbb{C}^2 \setminus B)$ directly, therefore we use the regeneration rules from [25] to get a braid monodromy factorization of $B$ from the one of $B_0$. Then we can use the van Kampen Theorem [31] to get a finite presentation of $\pi_1(\mathbb{C}^2 \setminus B)$ with generators $\Gamma_1, \Gamma_1', \ldots, \Gamma_m, \Gamma_m'$ ($2m$ is the degree of $B$). A presentation of $\pi_1(\mathbb{C}\mathbb{P}^2 \setminus B)$ is obtained by adding the projective relation $\Gamma_m\Gamma_m\cdots\Gamma_1\Gamma_1 = e$. The reader might want to check the papers [3], [5], [7] and [9] in order to get the feeling of the type of presentations we are dealing with.

Artin [10] defined the braid group $B_n$ with $n - 1$ generators $\{\sigma_i\}$ and with the following relations

(1) \[ \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i - j| > 1 \]

(2) \[ \sigma_i\sigma_i+1\sigma_i = \sigma_i+1\sigma_i\sigma_i+1. \]

The main results in this work, related to $X_1, X_2, X_3$, appear in Theorems [15] [17] and [20]:

- $\pi_1(\mathbb{C}\mathbb{P}^2 \setminus B_1) \cong B_5/\langle \Gamma_4^2\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2\Gamma_3 \rangle$,
- $\pi_1(\mathbb{C}\mathbb{P}^2 \setminus B_2) \cong B_6/\langle \Gamma_3\Gamma_4\Gamma_5^2\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2 \rangle$,
- $\pi_1(\mathbb{C}\mathbb{P}^2 \setminus B_3) \cong B_4/\langle \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 \rangle$.

Remark 1. The groups $\pi_1(\mathbb{C}\mathbb{P}^2 \setminus B_i)$ are in fact the braid group of points on the sphere. A general geometric interpretation is the following. The surfaces $X_i$ $(i = 1, 2, 3)$ are
ruled surfaces, and if \( p \) is any point of \( \mathbb{CP}^2 \) outside the branch curve, then its \( N \) preimages in \( X_i \) \((N = 5, 6, 4)\) project to distinct points of \( \mathbb{CP}^1 \); this gives a homomorphism from \( \pi_1(\mathbb{CP}^2 \setminus B_i) \) to \( B_N(\mathbb{CP}^1) \).

The result related to \( X_4 \) appears in Theorem 24:

- \( \pi_1(\mathbb{CP}^2 \setminus B_4) \) is isomorphic to a quotient of the group \( \tilde{B}_6 = B_6 / \langle [X, Y] \rangle \) \((X, Y \) are transversal) by \( \langle (92) \rangle \).

In this work we are also interested in two important quotient groups. The first one \( \Pi(B) = \pi_1(\mathbb{CP}^2 \setminus B)/\langle \Gamma_1^2, \Gamma_1^2 \rangle \) is defined to be a quotient of \( \pi_1(\mathbb{CP}^2 \setminus B) \) by the normal subgroup generated by the squares of the generators. This group is a key ingredient in studying invariants of \( X \), and in particular \( \pi_1(\mathbb{CP}^2 \setminus B) \). The braid monodromy technique of Moishezon-Teicher enables one to compute \( \pi_1(X_{Gal}) \), the fundamental group of a Galois cover \( X_{Gal} \) of \( X \), from \( \Pi(B) \). In particular, they showed that there is a natural map from \( \Pi(B) \) to the symmetric group \( S_n \), where \( n \) is the degree of \( X \), and \( \pi_1(X_{Gal}) \) is the kernel of this homomorphism. Moishezon-Teicher proved in [23] that for \( X = \mathbb{CP}^1 \times \mathbb{CP}^1 \) the group \( \pi_1(X_{Gal}) \) is a finite abelian group on \( n - 2 \) generators, each of order g.c.d.\((a, b)\) \((a \) and \( b \) are the parameters of the embedding). In [5] the treated surface is \( X = \mathbb{CP}^1 \times T \) \((T \) is a complex torus) and \( \pi_1(X_{Gal}) = \mathbb{Z}^{10} \). In [6] the same surface was embedded in \( \mathbb{CP}^{2n-1} \) and \( \pi_1(X_{Gal}) = \mathbb{Z}^{4n-2} \). In [7] and [8] the surface \( X = T \times T \) is studied, and \( \pi_1(X_{Gal}) \) is nilpotent of class 3. In [9] this group was computed for the Hirzebruch surface \( F_1(2, 2) \) and it is \( \mathbb{Z}_2^{10} \).

It turns out in this paper (Theorems 15, 17, 20, 24) that

- The group \( \Pi(B_i) \) is isomorphic to \( S_5, S_6, S_4, S_6 \) for \( i = 1, 2, 3, 4 \), respectively.

Hence we have
Corollary 2. The fundamental group $\pi_1((X_i)_{Gal})$ is trivial for $i = 1, 2, 3, 4$.

The second group is a Coxeter group $C = \Pi_{(B)}/\langle \Gamma_i = \Gamma_i' \rangle$, defined as a quotient of $\Pi_{(B)}$ under identification of pairs of generators, see [29]. It is still unclear whether $C$, introduced here, is an invariant of the surface or of the branch curve. It might be conjectured that there exists a dependence on the choice of a pairing between geometric generators $\Gamma_j$ and $\Gamma_j'$ (and hence on the choice of a degeneration to a union of planes). It turns out that $C$ is isomorphic to a symmetric group $S_n$ for Hirzebruch surfaces ([9], [19]) and $\mathbb{C}P^1 \times \mathbb{C}P^1$ ([20], [23]). The cases of $\mathbb{C}P^1 \times T$ ([5]) and $T \times T$ ([7]) are the first examples in which $C$ is a larger group, namely $C \cong \mathbb{Z}_5 \rtimes S_6$ and $C \cong K_C \rtimes S_{18}$ ($K_C$ is a central extension of $\mathbb{Z}^{34}$ by $\mathbb{Z}$), respectively.

Then we have

Corollary 3. The group $C_i$ is isomorphic to $S_5, S_6, S_4, S_6$ for $i = 1, 2, 3, 4$, respectively.

The paper is divided as follows. In Section 2 we study degeneration of toric varieties. In Section 3 we compute the requested groups related to the toric varieties $X_1, X_2$ and $X_3$, and in Section 4 we compute the ones related to $X_4$.

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2. Degeneration of toric surfaces

In their process to calculate the braid monodromy, Moishezon and Teicher studied the projective degeneration of $V_3 = (\mathbb{CP}^2, \mathcal{O}(3))$ [24] and Hirzebruch surfaces [19]. Since $\mathbb{CP}^2$ and the Hirzebruch surfaces are toric surfaces, we shall describe the projective degeneration of toric surfaces in this section.

2.1. Basic notions. We outline definitions needed in toric geometry and refer to [14] and [27] for further statements and proofs.

**Definition 4. Toric variety.** A toric variety is a normal algebraic variety $X$ that contains an algebraic torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an algebraic action $T \times X \to X$ of $T$ on $X$, that is an extension of the natural action of $T$ on itself.

Let $M$ be a free $\mathbb{Z}$-module of rank $n$ ($n \geq 1$) and $M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}$ the extension of the coefficients to the real numbers. Let $T := \text{Spec} \mathbb{C}[M]$ be an algebraic torus of dimension $n$. Then $M$ is considered as the character group of $T$, i.e., $M = \text{Hom}_{\mathbb{R}}(T, \mathbb{C}^*)$. We denote an element $m \in M$ by $e(m)$ as a function on $T$, which is also a rational
function on $X$. Let $L$ be an ample line bundle on $X$. Then we have

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}e(m),$$

where $P$ is an integral convex polytope in $M_\mathbb{R}$ defined as the convex hull $\text{Conv}\{m_0, m_1, \ldots, m_r\}$ of a finite subset $\{m_0, m_1, \ldots, m_r\} \subset M$. Conversely we can construct a pair $(X, L)$ of a polarized toric variety from an integral convex polytope $P$ so that the above isomorphism holds (see [14, Section 3.5] or [27, Section 2.4]). If an affine automorphism $\varphi$ of $M$ transforms $P$ to $P_1$, then $\varphi$ induces an isomorphism of polarized toric varieties $(X, L)$ to $(X_1, L_1)$, where $(X_1, L_1)$ corresponds to $P_1$.

Example 5. Let $M = \mathbb{Z}^2$. Then $V_3 = (\mathbb{C}P^2, \mathcal{O}(3))$ corresponds to the integral convex polytope $P_3 := \text{Conv}\{(0,0), (3,0), (0,3)\}$.

Example 6. The Hirzebruch surface $F_d = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(d))$ of degree $d$ has generators $s, g$ in the Picard group consisting of the negative section $s^2 = -d$ and general fiber $g^2 = 0$. A line bundle $L$ with $[L] = as + bg$ in $\text{Pic}(F_d)$ is ample if $a > 0, b > ad$. Then this pair $(F_d, L)$ corresponds to $P_{d(a,b)} := \text{Conv}\{(0,0), (b - ad, 0), (b,a), (0,a)\}$.

Next we consider degenerations of toric surfaces defined by Moishezon-Teicher. We recall the definition from [24].

Definition 7. Projective degeneration. A degeneration of $X$ is a proper surjective morphism with connected fibers $\pi : V \to \mathbb{C}$ from an algebraic variety $V$ such that the restriction $\pi : V \setminus \pi^{-1}(0) \to \mathbb{C} \setminus \{0\}$ is smooth and that $\pi^{-1}(t) \cong X$ for $t \neq 0$.

When $X$ is projective with an embedding $k : X \hookrightarrow \mathbb{C}P^n$, a degeneration of $X$ $\pi : V \to \mathbb{C}$ is called a projective degeneration of $k$ if there exists a morphism
$F : V \to \mathbb{CP}^n \times \mathbb{C}$ such that the restriction $F_t = F|_{\pi^{-1}(t)} : \pi^{-1}(t) \to \mathbb{CP}^n \times t$ is an embedding of $\pi^{-1}(t)$ for all $t \in \mathbb{C}$ and that $F_1 = k$ under the identification of $\pi^{-1}(1)$ with $X$.

Moishezon and Teicher used the triangulation of $P_3$ consisting of nine standard triangles as a schematic figure of a union of nine projective planes [24]. In the theory of toric varieties, however, the lattice points $P_3 \cap M$ correspond to rational functions of degree 3 on $V_3 \cong \mathbb{CP}^2$. Let $m_0 = (0, 0), m_1 = (1, 0), m_2 = (0, 1), \ldots, m_9 = (0, 3) \in \mathbb{Z}^2$. Then we may write $e(m_0) = x_0^3, e(m_1) = x_0^2x_1, e(m_2) = x_0^2x_2, \ldots, e(m_9) = x_0^3$ with a suitable choice of the homogeneous coordinates of $\mathbb{CP}^2$. The Veronese embedding $V_3 \hookrightarrow \mathbb{CP}^9$ is given by $z_i = e(m_i)$ for $i = 0, 1, \ldots, 9$ with the homogeneous coordinates $[z_0 : z_1 : \cdots : z_9]$ of $\mathbb{CP}^9$. Let $P_1 := \text{Conv}\{((0, 0), (1, 0), (0, 1))\}$, which corresponds to $(\mathbb{CP}^2, O(1))$. The subset $P_1 \subset P_3$ corresponds to the linear subspace $\{z_3 = \cdots = z_9 = 0\} \subset \mathbb{CP}^9$. Thus a triangulation of $P_3$ into a union of nine standard triangles means the subvariety of dimension two consisting of the union of nine projective planes in $\mathbb{CP}^9$ and each standard triangle defines a linear subspace of dimension two with corresponding coordinates.

2.2. **Constructing the degeneration of toric surfaces.** In the following we construct a semistable degeneration of toric surfaces according to Hu [15]. Let $M = \mathbb{Z}^2$. Let $P$ be a convex polyhedron in $M_\mathbb{R}$ corresponding to a polarized toric surface $(X, L)$. The lattice points $P \cap M$ define the embedding $\varphi_L : X \to \mathbb{P}(\Gamma(X, L))$. Let $\Gamma$ be a triangulation of $P$ consisting of standard triangles with vertices in $P \cap M$. Let $h : P \cap M \to \mathbb{Z}_{>0}$ be a function on the lattice points in $P$ with values in positive integers. Let $\tilde{M} = M \oplus \mathbb{Z}$ and let $\tilde{P} = \text{Conv}\{(x, 0), (x, h(x)); x \in P \cap M\}$ the
integral convex polytope in $\tilde{M}_\mathbb{R}$. We want to choose $h$ to satisfy the conditions that $(x, h(x))$ for $x \in P \cap M$ are vertices of $\tilde{P}$ and that for each edge in $\Gamma$ joining $x$ and $y \in P \cap M$ there is an edge joining $(x, h(x))$ and $(y, h(y))$ as a face of $\partial \tilde{P}$. We say that $\tilde{P}$ realizes the triangulation $\Gamma$ if these conditions are satisfied. Now we assume that $\tilde{P}$ realizes the triangulation $\Gamma$. Then $\tilde{P}$ defines a polarized toric 3-fold $(\tilde{X}, \tilde{L})$. From the construction, $\tilde{X}$ has a fibration $p : \tilde{X} \to \mathbb{CP}^1$ satisfying that $p^{-1}(t) \cong X$ with $t \neq 0$ and that $p^{-1}(0)$ is a union of projective planes. Furthermore we see that $p^{-1}(\mathbb{CP}^1 \setminus \{0\}) \cong \mathbb{C} \times X$. Thus the flat family $p : \tilde{X} \to \mathbb{CP}^1$ gives a degeneration of $X$ into a union of projective planes with the configuration diagram $\Gamma$. Hu treats only nonsingular toric varieties of any dimension. The difficulty of this construction is to find a triangulation $\Gamma$. Here we restrict ourselves to toric surfaces. Then we can find a triangulation for any integral convex polygon $P$.

**Example 8.** Let $m_0 = (0, 0), m_1 = (1, 0), m_2 = (0, 1), m_3 = (1, 1) \in M = \mathbb{Z}^2$. Let $P = \text{Conv}\{m_0, m_1, m_2, m_3\}$. Then $P$ defines the polarized surface $(X = \mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(1, 1))$. Let $\Gamma$ be the triangulation of $P$ defined by adding the edge connecting $m_1$ and $m_2$. Define $h(m_0) = h(m_3) = 1, h(m_1) = h(m_2) = 2$. Let $\tilde{M} : = M \oplus \mathbb{Z}$. Set $m_i = (m_i, 0)$ and $m_i^+ = (m_i, h(m_i))$ for $i = 0, \ldots, 3$ and $m_4 = (1, 0, 1), m_5 = (0, 1, 1)$ in $\tilde{M}$. Then the integral convex polytope $\tilde{P} := \text{Conv}\{m_0, \ldots, m_3, m_0^+, \ldots, m_3^+\}$ in $\tilde{M}$ defines the polarized toric 3-fold $(\tilde{X}, \tilde{L})$. By definition, $\tilde{X}$ has a fibration $p : \tilde{X} \to \mathbb{CP}^1$. The global sections of $\tilde{L}$ defines an embedding of $\tilde{X}$ as follows: Let $[z_0 : \cdots : z_9]$ be the homogeneous coordinates of $\mathbb{CP}^9$. The equations $z_i = e(m_i)$ for $i = 0, \ldots, 5$ and $z_{6+j} = e(m_j^+)$ for $j = 0, \ldots, 3$ define the embedding $\tilde{X} \to \mathbb{CP}^9$. The fiber $p^{-1}(\infty)$ is given by $\{z_0z_3 = z_1z_2, z_4 = \cdots = z_9 = 0\}$ which is isomorphic
to $X \subset \mathbb{P}(\Gamma(X, \mathcal{O}(1))) \cong \mathbb{CP}^3 = \{z_4 = \cdots = z_9 = 0\}$, and the fiber $p^{-1}(0)$ is given by $\{z_6z_9 = 0, z_0 = \cdots = z_5 = 0\}$ which is a union of two projective planes in $\mathbb{CP}^3 \cong \{z_0 = \cdots = z_5 = 0\}$.

Lemma 9. The line bundle $\tilde{L}$ on $\tilde{X}$ is very ample.

Proof. Let $m_1, m_2, m_3 \in P \cap M$ be three vertices of a standard triangle in the triangulation $\Gamma$ of $P$. Set $m_i^- = (m_i, 0), m_i^+ = (m_i, h(m_i))$ in $\tilde{M}_R$ for $i = 1, 2, 3$. Denote $Q = \text{Conv}\{m_i^\pm; i = 1, 2, 3\}$ the integral convex polytope with vertices $\{m_i^\pm; i = 1, 2, 3\}$. Then we divide $\tilde{P}$ into a union of triangular prisms like $Q$. We can divide $Q$ into a union of standard 3-simplices. We may assume $h(m_1) \geq h(m_2) \geq h(m_3)$ by renumbering $m_i$ if necessary. Then we can divide $Q$ into a union of $Q_0 = \text{Conv}\{m_1^+, m_2^+, m_3^+, (m_1, h(m_1) - 1)\}$ and $Q_1 = \text{Conv}\{m_1^-, (m_1, h(m_1) - 1), m_2^+, m_3^+\}$. Here $Q_0$ is a standard 3-simplex and $Q_1$ has a similar shape to $Q$ but less volume than that of $Q$. Thus we obtain a division of $\tilde{P}$ into a union of standard 3-simplices. This is not always triangulation of $\tilde{P}$, but this gives a covering of $\tilde{P}$ consisting of standard 3-simplices. From the theory of polytopal semigroup ring (see, for instance, [13] and [30]), we see that $\tilde{L}$ is simply generated, hence very ample. \qed

We claim that $\tilde{X}$ also defines a projective degeneration of $(X, L)$. Denote $\Phi := \varphi_{\tilde{L}} : \tilde{X} \longrightarrow \mathbb{P}(\Gamma(\tilde{X}, \tilde{L})) =: \mathbb{P}$ the morphism defined by global sections of $\tilde{X}$. We see that $p^{-1}(t) \cong X$ for $t \neq 0$ with $[1 : t] \in \mathbb{CP}^1$ and that $p^{-1}(\infty) \cong X$ and $p^{-1}(0)$ are $T$-invariant reduced divisors. Thus the restriction maps $\Gamma(\tilde{X}, \tilde{L}) \longrightarrow \Gamma(p^{-1}(\infty), \tilde{L}|_{p^{-1}(\infty)}) \cong \Gamma(X, L)$ and $\Gamma(\tilde{X}, \tilde{L}) \longrightarrow \Gamma(p^{-1}(0), \tilde{L}|_{p^{-1}(0)})$ are surjective. From the construction of $\tilde{P}$, we see that $\dim \Gamma(X, L) = \dim \Gamma(p^{-1}(0), \tilde{L}|_{p^{-1}(0)})$. Since
$p^{-1}(\mathbb{CP}^1 \setminus \{0\}) \cong X \times \mathbb{C}$, we have $\tilde{L}|_{p^{-1}(t)} \cong L$ for $t \neq 0$. Then $F := \Phi \times p : \tilde{X} \rightarrow \mathbb{P} \times \mathbb{CP}^1$ is a projective degeneration of $k : X \rightarrow \mathbb{P}(\Gamma(X, L)) \hookrightarrow \mathbb{P}$.

**Theorem 10.** Let $P$ be an integral convex polyhedron of dimension 2 corresponding to a polarized toric surface $(X, L)$ and let $\Gamma$ a triangulation of $P$ consisting of standard triangles with vertices in $M$. Assume that $\check{P}$ is an integral convex polytope in $\check{M}_{\mathbb{R}}$ realizing the triangulation $\Gamma$. Then $\check{P}$ defines a polarized toric 3-fold $(\check{X}, \check{L})$, which gives a projective degeneration of $(X, L)$ to a union of projective planes.

2.3. **Degeneration of the four toric surfaces.** In this paper we study four degenerations of polarized toric surfaces, each one of which is defined by integral convex polygon $P$. We choose a triangulation $\Gamma$ for each $P$ and define a function $h : P \cap M \rightarrow \mathbb{Z}_{\geq 0}$ so that the integral convex polytope $\check{P}$ of dimension 3 should realize the triangulation $\Gamma$ of $P$.

The first surface is the Hirzebruch surface $X_1 := F_1$ of degree one embedded in $\mathbb{CP}^6$ by the very ample line bundle $L_1$ whose class is $s + 3g$, where $s$ is the negative section and $g$ is a general fiber. We mentioned this surface as a polarized toric surface in Example 6, which corresponds to the integral convex polygon $P_{1(1,3)}$ in $M = \mathbb{Z}^2$. Let $m_i = (i, 0)$ for $i = 0, 1, 2, 3$ and $m_j = (j - 3, 1)$ for $j = 4, 5, 6$. Then $P_{1(1,3)} = \text{Conv}\{m_0, m_3, m_4, m_6\}$. Let $\Gamma_1$ be the triangulation of $P_{1(1,3)}$ obtained by adding the edges $m_1 m_4, m_2 m_4, m_2 m_5, m_3 m_5$, see Figure 1. This triangulation is slightly different from the one treated in [23]. We define a function $h_1 : P_{1(1,3)} \cap M \rightarrow \mathbb{Z}_{>0}$ as $h_1(m_0) = h_1(m_6) = 1, h_1(m_1) = h_1(m_3) = h_1(m_4) = h_1(m_5) = 3, h_1(m_2) = 4$. Then we can define an integral convex polytope $\check{P}$ in $\check{M} = M \oplus \mathbb{Z}$ realizing the triangulation $\Gamma_1$ of $P_{1(1,3)}$. Hence we have a projective degeneration of $\varphi_1 := \varphi_{L_1} : F_1 \hookrightarrow \mathbb{CP}^6$. 
The second surface is \( X_2 := \mathbb{CP}^1 \times \mathbb{CP}^1 \) embedded in \( \mathbb{CP}^7 \) by \( \mathcal{O}(3, 1) \). This embedded toric surface corresponds to the convex polygon \( P_{3,1} := \text{Conv}\{(0,0), (3,0), (0,1), (3,1)\} \) in \( M = \mathbb{Z}^2 \). Let \( m_i = (i,0) \) for \( i = 0,1,2,3 \) and \( m_j = (j-4,1) \) for \( j = 4,5,6,7 \). Then \( P_{3,1} = \text{Conv}\{m_0, m_3, m_4, m_7\} \). Let \( \Gamma_2 \) be the triangulation of \( P_{3,1} \) obtained by adding the edges \( m_0 \bar{m}_5, m_1 \bar{m}_5, m_1 \bar{m}_6, m_2 \bar{m}_6, m_2 \bar{m}_7 \), see Figure 4. We define a function \( h_2 : P_{3,1} \cap M \rightarrow \mathbb{Z}_{>0} \) as \( h_2(m_4) = 1, h_2(m_0) = h_2(m_3) = 3, h_2(m_5) = h_2(m_7) = 4, h_2(m_1) = h_2(m_2) = h_2(m_6) = 5 \). Then we have a projective degeneration of \( \varphi_2 := \varphi_{\mathcal{O}(3,1)} : \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^7 \) corresponding to the triangulation \( \Gamma_2 \).

The third surface is the Hirzebruch surface \( X_3 := F_2 \) of degree 2 embedded in \( \mathbb{CP}^5 \) by the ample line bundle \( L_2 \) whose class is \( s + 3g \). The corresponding polygon is \( P_{2(1,3)} \). Let \( m_i = (i,0) \) for \( i = 0,1,2,3 \) and \( m_j = (j-3,1) \) for \( j = 4,5 \) in \( M = \mathbb{Z}^2 \). Then \( P_{2(1,3)} = \text{Conv}\{m_0, m_3, m_4, m_5\} \) up to affine automorphism of \( M \). Let \( \Gamma_3 \) be the triangulation of \( P_{2(1,3)} \) obtained by adding the edges \( m_1 \bar{m}_4, m_1 \bar{m}_5, m_2 \bar{m}_5 \), see Figure 7. We define a function \( h_3 : P_{2(1,3)} \cap M \rightarrow \mathbb{Z}_{>0} \) as \( h_3(m_0) = h_3(m_3) = 1, h_3(m_4) = 3, h_3(m_1) = h_3(m_2) = h_3(m_5) = 4 \). Then we have a projective degeneration of \( \varphi_3 := \varphi_{L_2} : F_2 \hookrightarrow \mathbb{CP}^5 \) corresponding to the triangulation \( \Gamma_3 \).

The last surface is a singular toric surface \( X_4 \) embedded in \( \mathbb{CP}^6 \) corresponding to the polygon \( P_4 := \text{Conv}\{(0,0), (2,0), (0,1), (1,2), (2,1)\} \). Let \( m_i = (i,0) \) for \( i = 0,1,2 \), \( m_j = (j-3,1) \) for \( j = 3,4,5 \) and \( m_6 = (1,2) \). Let \( \Gamma_4 \) be the triangulation of \( P_4 \) obtained by adding the edges \( \{m_i \bar{m}_4, m_1 \bar{m}_j; i = 1,3,5,6 \text{ and } j = 3,5\} \), see Figure 8. We define a function \( h_4 : P_4 \cap M \rightarrow \mathbb{Z}_{>0} \) as \( h_4(m_0) = h_4(m_2) = 1, h_4(m_1) = h_4(m_3) = h_4(m_5) = h_4(m_6) = 3, h_4(m_4) = 4 \). Then we have a projective degeneration of \( \varphi_4 : X_4 \hookrightarrow \mathbb{CP}^6 \) corresponding to the triangulation \( \Gamma_4 \).
3. The surfaces $X_1$, $X_2$ and $X_3$

In this section we compute the groups $\pi_1(\mathbb{C}^2 \setminus B_i)$, $\pi_1(\mathbb{CP}^2 \setminus B_i)$, and $\Pi(B_i)$ for $i = 1, 2, 3$. Zariski \cite{zariski33} investigated indirectly complements of the types of curves as $B_1, B_2$ and $B_3$. We compare our methods and results to those of Zariski.

Using degenerations of toric varieties, such as those that we have here, makes these special cases of a more general theory, rather than isolated examples. Having the degenerations of $X_1, X_2$ and $X_3$, we project them onto $\mathbb{CP}^2$ and get line arrangements. By the Regeneration Lemmas of Moishezon-Teicher \cite{moishezon-teicher}, the diagonal lines regenerate to conics which are tangent to the lines with which they intersect. When the rest of the lines regenerate, each tangency (the point of tangency of line and conic) regenerates to three cusps. We end up with cuspidal curves $B_i$, $i = 1, 2, 3$. The existence of nodes in these curves depends on the existence of the ‘parasitic intersections’ (projecting the degenerations onto $\mathbb{CP}^2$ causes extra intersections). By the braid monodromy techniques and regeneration rules of Moishezon-Teicher (\cite{moishezon-teicher}, \cite{moishezon-teicher2}), we get the related braid monodromy factorizations (by \cite{moishezon-teicher}, each braid of a parasitic intersection, say $Z_{ij}^2$, regenerates to $Z_{ij}^2$ in the factorizations), see Notation \ref{notation}. We do not use properties of braid groups, but rather the definition of the factorization \cite{moishezon-teicher}, from which the van Kampen Theorem \cite{van-kampen} for cuspidal curves gives a complete set of relations for the fundamental groups $\pi_1(\mathbb{C}^2 \setminus B_i)$.

Zariski \cite{zariski33} gets a collection of local relations without using degeneration and regeneration, as follows. He uses properties of curves to conclude relations for certain groups, called the Poincaré groups (contemporary fundamental groups). He defines the class of Poincaré groups $G_n$, which practically coincides with the Artin braid
A group of type $G_n$ is also a group of automorphism classes of a sphere with $n$ holes (the points $P_1, \ldots, P_n$ are removed), see [16]. For generators $g_1, \ldots, g_{n-1}$ ($g_1$ connects $P_1$ and $P_2$, $g_2$ connects $P_2$ and $P_3$, etc.), Zariski proves that

\[(4) \quad g_i g_j = g_j g_i \mid i - j \neq 1\]

\[(5) \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}\]

\[(6) \quad g_1 g_2 \cdots g_{n-2} g_{n-1}^2 g_{n-2} \cdots g_2 g_1 = e\]

constitute a complete set of generating relations of $G_n$. He denotes a rational curve with degree $n$ and $k$ cusps as $(n, k)$. He shows how the individual generating relations of $G_n$ correspond to the singularities of a maximal cuspidal curve $(2n - 2, 3(n - 2))$ with $2(n - 2)(n - 3)$ nodes. The $(n - 2)(n - 3)/2$ commutativity relations (4) are the typical relations at nodes, while the $n - 2$ relations (5) are the typical cusp relations [32].

Concerning the results, the cuspidal curves $B_1, B_2$ and $B_3$ ((8, 9), (10, 12) and (6, 6), respectively) fulfill the above statements and they are maximal. Therefore, Zariski gets the groups $G_5, G_6$ and $G_4$, respectively. Here the results related to $X_1, X_2$ and $X_3$, turn out to be the ones of Zariski, i.e., $\pi_1(\mathbb{CP}^2 \setminus B_i)$ is a braid group of points on a sphere.

Since we use the degeneration on toric varieties, which is different from that which Zariski did, it would be worth to give a proof for the groups related to $X_1$. The ones related to $X_2$ and $X_3$ are computed in a similar way, and therefore the proofs are omitted.
Remark 11. A braid monodromy factorization $\Delta^2$ should be written as a product of factors in an actual order (see [25]). Since our goal is to compute fundamental groups, the order of the factors does not matter. Here we list the monodromies with an unmeaningful order, and concentrate on finding the relations in the groups by applying the van Kampen Theorem on the monodromies.

3.1. The surface $X_1$. Let $X_1 = F_1(3,1)$ be the Hirzebruch surface, as defined in Section 2. The construction of the degeneration of Hirzebruch surfaces of type $F_1(p,q)$ (for $p > q \geq 2$) appears in [17] and [18]. In [11], Auroux-Donaldson-Katzarkov-Yotov dedicate Section 6.2 to the construction of degeneration of $F_1$ surfaces and to the presentations of the fundamental groups of complements of branch curves.

The degeneration of $X_1$ into a union of five planes $(X_1)_0$ is embedded in $\mathbb{C}P^6$. The numeration of lines is fixed according to the numeration of the vertices in Section 2, see Figure 1. Note that each of the points $m_2, m_4, m_5$ is contained in three distinct planes, while each of $m_1, m_3$ is contained in two planes.

Take a generic projection $f_1 : X_1 \to \mathbb{C}P^2$. The union of the intersection lines is the ramification locus $R_0$ in $(X_1)_0$ of $f_1^0 : (X_1)_0 \to \mathbb{C}P^2$. Let $(B_1)_0 = f_1^0(R_0)$ be the degenerated branch curve. It is a line arrangement, $(B_1)_0 = \bigcup_{j=1}^4 L_j.$

Denote the singularities of $(B_1)_0$ as $f_1^0(m_i) = m_i, i = 1, \ldots, 5$ (the points $m_0, m_6$ do not lie on numerated lines, hence they are not singularities of $(B_1)_0$). The points

![Figure 1. Degeneration of $X_1$](image-url)
Figure 2. Regeneration around the point $m_2$

$m_1$ and $m_3$ (resp. $m_2, m_4, m_5$) are called 1-points (resp. 2-points). They were studied in [5], [9], [20] and [25]. Other singularities may be the parasitic intersections.

The regeneration of $(X_1)_0$ induces a regeneration of $(B_1)_0$ in such a way that each point on the typical fiber, say $c$, is replaced by two close points $c, c'$. The regeneration occurs as follows. We regenerate in a neighborhood of $m_1, m_3$ to get conics. Now, in a neighborhood of $m_2, m_4, m_5$, the diagonal line regenerates to a conic [22, Regenerations Lemmas]), which is tangent to the line it intersected with [25, Lemma 1]. See Figure[2] for the regeneration around $m_2$. When the line regenerates, the tangency regenerates into three cusps, [22, Regeneration Lemmas].

The resulting curve $B_1$ has degree 8 and nine cusps. The intersection points of the curve with a typical fiber are $\{1, 1', \ldots, 4, 4'\}$. We are interested in the braid monodromy factorization of $B_1$, the groups $\pi_1(\mathbb{C}^2 \setminus B_1), \pi_1(\mathbb{C}P^2 \setminus B_1)$ and $\Pi(B_1)$.

Notation 12. We denote by $Z_{i,j}$ the counterclockwise half-twist of $i$ and $j$ along a path below the real axis. Denote by $Z_{i,j}^2$, the product $Z_{i,j}^2$, and by $Z_{i,j}^2$, the product $Z_{i,j}^2$. Likewise, $Z_{i,j}^3$ denotes the product $Z_{i,j}^3 \cdot Z_{i,j}^3 \cdot Z_{i,j}^3$. Conjugation of braids is defined as $a^b = b^{-1}ab$. 
Theorem 13. The braid monodromy factorization of the curve $B_1$ is the product of

\begin{align*}
\varphi_{m_1} &= Z_1 1' \\
\varphi_{m_2} &= Z_2^3 2',3' \cdot Z_2 2' Z_2^2 2',3' \\
\varphi_{m_3} &= Z_4 4' \\
\varphi_{m_4} &= Z_1^3 1',2' \cdot Z_2 2' Z_1^2 1',2' \\
\varphi_{m_5} &= Z_3^3 3',4' \cdot Z_4 4' Z_4^2 3',4'
\end{align*}

and the parasitic intersections braids

\begin{equation}
Z_1^2 1',3' \cdot Z_1^2 1',4' \cdot Z_2 2' 4'.
\end{equation}

Proof. The monodromies (7) and (9) are derived from the regenerations around 1-points, and the ones related to 2-points are (8), (10), (11), see for example the braids of $\varphi_{m_2}$ in Figure 3. The parasitic intersections were formulated in [21], these

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The braids of $\varphi_{m_2}$}
\end{figure}

are the intersections of the lines $L_1$ and $L_3$, $L_1$ and $L_4$, $L_2$ and $L_4$. See Figure 4

\begin{figure}[h]
\centering
\includegraphics[width=1\textwidth]{figure4.png}
\caption{Parasitic intersections braids in the factorization of $B_1$}
\end{figure}

Summing the degrees of the braids gives 56. Since the degree of the factorization is 56 [21 Cor. V.2.3], no other braids are involved.
**Notation 14.** $\Gamma_{i'}$ stands for $\Gamma_i$ or $\Gamma_{i'}$. The relation $\langle \Gamma_a, \Gamma_b \rangle = e$ means $\Gamma_a \Gamma_b \Gamma_a = \Gamma_b \Gamma_a \Gamma_b$.

**Theorem 15.** The group $\pi_1(\mathbb{C}^2 \setminus B_1)$ is generated by $\{\Gamma_j\}_{j=1}^4$ subject to the relations

\begin{align*}
(13) & \quad \langle \Gamma_i, \Gamma_{i+1} \rangle = e \quad \text{for } i=1,2,3 \\
(14) & \quad [\Gamma_1, \Gamma_i] = e \quad \text{for } i=3,4 \\
(15) & \quad [\Gamma_2, \Gamma_4] = e \\
(16) & \quad [\Gamma_4, \Gamma_3 \Gamma_2 \Gamma^2_1 \Gamma_2 \Gamma_3] = e.
\end{align*}

The group $\pi_1(\mathbb{C}P^2 \setminus B_1)$ is isomorphic to $\mathcal{B}_5/\langle \Gamma_3^2 \Gamma_3 \Gamma_2 \Gamma^2_1 \Gamma_2 \Gamma_3 \rangle$, and the group $\Pi(B_1)$ is isomorphic to $S_5$.

**Proof.** The group $\pi_1(\mathbb{C}^2 \setminus B_1)$ is generated by the elements $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^4$, where $\Gamma_j$ and $\Gamma_{j'}$ are loops in $\mathbb{C}^2$ around $j$ and $j'$, respectively.

By the van Kampen Theorem, the two branch points braids give the following relations

\begin{align*}
(17) & \quad \Gamma_i = \Gamma_{i'} \quad \text{for } i=1,4.
\end{align*}
From the monodromies $\varphi_{m_2}, \varphi_{m_4}$ and $\varphi_{m_5}$, we produce relations (18)-(19), (20)-(21) and (22)-(23) respectively (e.g., from Figure 3 we have (18)-(19):

\[ \langle \Gamma_{2'}, \Gamma_{33'} \rangle = \langle \Gamma_{2'}, \Gamma_{3}^{-1} \Gamma_{3'} \Gamma_{3} \rangle = e \]

\[ \Gamma_{3} \Gamma_{3} \Gamma_{2} \Gamma_{3}^{-1} \Gamma_{3'}^{-1} = \Gamma_{2} \]

\[ \langle \Gamma_{11'}, \Gamma_{2} \rangle = \langle \Gamma_{1}^{-1} \Gamma_{1}, \Gamma_{1}, \Gamma_{2} \rangle = e \]

\[ \Gamma_{2} \Gamma_{1} \Gamma_{1} \Gamma_{2} \Gamma_{1}^{-1} \Gamma_{1'}^{-1} \Gamma_{2}^{-1} = \Gamma_{2'} \]

\[ \langle \Gamma_{33'}, \Gamma_{4} \rangle = \langle \Gamma_{3}^{-1} \Gamma_{3} \Gamma_{3}, \Gamma_{4} \rangle = e \]

\[ \Gamma_{4} \Gamma_{3} \Gamma_{3} \Gamma_{4} \Gamma_{3}^{-1} \Gamma_{3'}^{-1} \Gamma_{4}^{-1} = \Gamma_{4'} \]

The parasitic intersections braids contribute commutative relations

\[ [\Gamma_{11'}, \Gamma_{ii'}] = e \text{ for } i=3,4 \]

\[ [\Gamma_{22'}, \Gamma_{44'}] = e. \]

Using (17), (20) and (22), relations (21) and (23) can be rewritten as $\Gamma_{1}^{-2} \Gamma_{2} \Gamma_{1}^{2} = \Gamma_{2'}$ and $\Gamma_{4}^{-2} \Gamma_{3} \Gamma_{4}^{2} = \Gamma_{3'}$, respectively. Using the fact that $\langle \Gamma_{2}, \Gamma_{3'} \rangle = \langle \Gamma_{2}^{2} \Gamma_{2} \Gamma_{1}^{-2}, \Gamma_{3'} \rangle = 1$, we can rewrite (19) as $\Gamma_{2}^{-1} \Gamma_{3} \Gamma_{2} \Gamma_{1}^{-1} \Gamma_{2} = \Gamma_{3'}$. Substituting these three relations in one another gives (16), and substituting them in (18), (20) and (22) (in (24) and (25), respectively), gives (13) ((14) and (15), respectively).

In order to get $\pi_{1}(\mathbb{C}P^2 \setminus B_1)$, we add the projective relation $\Gamma_4 \Gamma_4 \Gamma_3 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_{1'} = e$, which is transformed to $\Gamma_4 \Gamma_3 \Gamma_2 \Gamma_2 \Gamma_3 = e$. Therefore, relation (16) is omitted and we have $\pi_{1}(\mathbb{C}P^2 \setminus B_1) \cong \mathcal{B}_5 / \langle \Gamma_{4} \Gamma_{3} \Gamma_{3} \Gamma_{2} \Gamma_{2} \Gamma_{3} \rangle$ and $\Pi_{(B_1)} \cong S_5$. \qed
3.2. The surface $X_2$. In [20], Moishezon-Teicher embed the surface $X_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ into a big projective space by the linear system $(\mathcal{O}(i), \mathcal{O}(j))$, where $i \geq 2, j \geq 3$. They use its degeneration to compute the fundamental group of the Galois cover corresponding to the generic projection of the surface onto $\mathbb{CP}^2$.

In this paper the embedding is by the linear system $(\mathcal{O}(3), \mathcal{O}(1))$. The degeneration of $X_2$ is a union of six planes embedded in $\mathbb{CP}^7$, as depicted in Figure 5.

![Figure 5. Degeneration of $X_2$](image)

Now we explain what happens in the regeneration of the branch curve $(B_2)_0$. Each diagonal line regenerates to a conic. That means that in neighborhoods of $m_0$ and $m_7$ we have only conics, while in neighborhoods of $m_1, m_2, m_5, m_6$ the conics are tangent to the lines they intersected with (the vertical lines in the figure). Then each of these lines regenerates, causing a regeneration of each tangency to three cusps. We end up with the curve $B_2$ with degree 10 and 12 cusps.
Theorem 16. The braid monodromy factorization of the curve $B_2$ is the product of

\begin{align}
\varphi_{m_0} &= Z_{1 1'} \\
\varphi_{m_1} &= Z_{2 2',3}^3 \cdot Z_{3 3'}^3 Z_{2 2',3}^2 \cdot Z_{3 3'}^2 \\
\varphi_{m_2} &= Z_{4 4',5}^3 \cdot Z_{5 5'}^2 Z_{4 4',5}^2 \\
\varphi_{m_5} &= Z_{1 1',2 2'}^3 \cdot Z_{1 1'}^2 \cdot Z_{1 1',2 2'} Z_{1 1'}^2 \\
\varphi_{m_6} &= Z_{3 3',4 4'}^3 \cdot Z_{3 3',4 4'} Z_{3 3',4 4'}^2 \\
\varphi_{m_7} &= Z_{5 5'}^3
\end{align}

and the parasitic intersections braids

\begin{align}
Z_{1 1',3 3'}^2, Z_{1 1',4 4'}^2, Z_{2 2',4 4'}^2, Z_{2 2',5 5'}^2, Z_{3 3',5 5'}^2.
\end{align}

Proof. The monodromies $\varphi_{m_0}$ and $\varphi_{m_7}$ are braids of branch points of the conics there. The monodromies $\varphi_{m_1}, \varphi_{m_2}$ (resp. $\varphi_{m_5}, \varphi_{m_6}$) are similar to the monodromies (10) and (11) (resp. (8)). According to this similarity of braids (only modification of indices in Figure 3), we depict only the parasitic intersections braids in Figure 6. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Parasitic intersections braids in the factorization of $B_2$}
\end{figure}
We apply the van Kampen Theorem on the above braids to get a presentation for \( \pi_1(\mathbb{C}^2 \setminus B_2) \), and by omitting the generators \( \Gamma_i, i = 1, \ldots, 5 \), and simplifying the relations, as done in the proof of Theorem 15, we get the following:

**Theorem 17.** The fundamental group \( \pi_1(\mathbb{C}^2 \setminus B_2) \) is generated by \( \{ \Gamma_j \}_{j=1}^5 \) subject to the relations

\begin{align*}
\langle \Gamma_i, \Gamma_{i+1} \rangle &= e \quad \text{for } i=1,2,3,4 \quad (33) \\
[\Gamma_1, \Gamma_i] &= e \quad \text{for } i=3,4,5 \quad (34) \\
[\Gamma_2, \Gamma_i] &= e \quad \text{for } i=4,5 \quad (35) \\
[\Gamma_3, \Gamma_5] &= e \quad (36) \\
\Gamma_2^{-1}\Gamma_1^{-2}\Gamma_2^{-1}\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2 &= \Gamma_4^{-1}\Gamma_5^{-2}\Gamma_4^{-1}\Gamma_3\Gamma_4\Gamma_5^2\Gamma_4 \quad (37)
\end{align*}

The group \( \pi_1(\mathbb{C}P^2 \setminus B_2) \) is isomorphic to \( B_6/\langle \Gamma_3\Gamma_4\Gamma_5^2\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2 \rangle \), and the group \( \Pi(B_2) \) is isomorphic to \( S_6 \).

One can easily generalize this result. Take \( X_2 := \mathbb{C}P^1 \times \mathbb{C}P^1 \) embedded in \( \mathbb{C}P^{n+1} \) by \( O(n, 1) \). This embedded toric surface corresponds to the convex polygon \( P_{n,1} := \text{Conv}\{(0,0), (n,0), (0,1), (n,1)\} \). And we have

**Corollary 18.** The groups \( \Pi(B) \) and \( C \) are isomorphic to \( S_{2n} \), and \( \pi_1((X_2)_{\text{Gal}}) \) is trivial.

### 3.3. The surface \( X_3 \)

The degeneration of \( X_3 \) is a union of four planes embedded in \( \mathbb{C}P^5 \), as depicted in Figure 7.
The branch curve $(B_3)_0$ in $\mathbb{CP}^2$ is a line arrangement. Regenerating it, the diagonal line regenerates to a conic, which is tangent to the lines 1 and 3. When the lines regenerate, each tangency regenerates into three cusps. We obtain the branch curve $B_3$, whose degree is 6 and which has six cusps.

**Theorem 19.** The braid monodromy factorization related to $B_3$ is the product of

\begin{align*}
\varphi_{m_1} &= Z_{1,2}^3 \cdot Z_{2,2'}^{x_{1,2}} \\
\varphi_{m_5} &= Z_{2,3}^3 \cdot Z_{2,2'}^{x_{2,3}} \\
\varphi_{m_2} &= Z_{3,3'} \\
\varphi_{m_4} &= Z_{1,1'}
\end{align*}

and the parasitic intersections braids

\begin{equation}
Z_{1,3}^2 \cdot Z_{1,3'}^2.
\end{equation}

**Proof.** Similar proof as in Theorem 13. \qed

We apply the van Kampen Theorem on the above braids to get a presentation for $\pi_1(\mathbb{C}^2 \setminus B_3)$, and again, by simplifying the relations and omitting generators, we get:
**Theorem 20.** The fundamental group $\pi_1(\mathbb{C}^2 \setminus B_3)$ is generated by $\Gamma_1, \Gamma_2, \Gamma_3$ subject to the relations

\begin{align*}
\langle \Gamma_i, \Gamma_{i+1} \rangle &= e \quad \text{for } i=1,2 \\
[\Gamma_1, \Gamma_3] &= e \\
\Gamma_1^{-2}\Gamma_2\Gamma_1^2 &= \Gamma_3^{-2}\Gamma_2\Gamma_3^2.
\end{align*}

The group $\pi_1(\mathbb{C}P^2 \setminus B_3)$ is isomorphic to $B_4/\langle \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 \rangle$, and the group $\Pi_{(B_3)}$ is isomorphic to $S_4$.

4. **The surface $X_4$**

The degeneration $(X_4)_0$ of $X_4$ is a union of six planes embedded in $\mathbb{C}P^6$ (Figure 8).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8}
\caption{Degeneration of $X_4$}
\end{figure}
The regeneration of \((X_4)_0\) induces a regeneration on the branch curve \((B_4)_0\) (line arrangement, composed of six lines). \(X_4\) has \(A_1\) singularity as explained in the introduction. That means that the regeneration of the top vertex \(m_6\) should yield a node in the branch curve, involving the components labelled 6 and 6’ (so that the double cover possesses an ordinary double point). The vertices \(m_3\) and \(m_5\) are 2-points, and therefore the regeneration around them is already known: the line 1 (4 resp.) regenerates to a conic which is tangent to the line 3 (5 resp.). When these lines regenerate, each tangency regenerates to three cusps. The vertex \(m_4\) is a 4-point, see e.g., [2].

The regeneration is as follows. The lines 3 and 5 regenerate to a hyperbola, and each line among 2 and 6 regenerates to a pair of parallel lines. The hyperbola is then tangent to the lines 2, 2’, 6, 6’, see Figure 9. The hyperbola doubles, therefore we have four branch points, and moreover, each tangency regenerates to three cusps.

However, the vertex \(m_1\) is of new type. The regeneration can be done as follows. Line 4 regenerates to a conic, while 1 is still unregenerated. Figure 10 describes this step. The points \(P_1\) and \(P_2\) are the intersections of 1 with the conic (they are complex). The intersection of lines 1 and 2 can be then locally considered as a 2-point; this means that 1 regenerates to a conic, which is tangent to line 2. At this point \(P_1\) and \(P_2\) are doubled. Line 2 then regenerates to a pair of parallel lines 2 and 2’, and each tangency regenerates to three cusps. Note that keeping a parabola, which we get in the regeneration around \(m_1\), as our picture in the affine part of the conics, we have possibly another branch point further away, possibly at infinity. We prove below the existence of these two extra branch points, which contribute two half-twists to the braid monodromy factorization.
The parasitic intersections are fixed by Figure 8 and this time they are the intersections in $\mathbb{CP}^2$ of line 1 with lines 5 and 6, and 4 with 3 and 6.

Therefore, we have
Figure 10. Regeneration around $m_1$

Theorem 21. The braid monodromies which we get from the regeneration around $m_1, m_3, m_4, m_5, m_6$ are

\begin{align*}
\varphi_{m_1} &= Z_2^{4,4'} \cdot (Z_4^{2,2'})^2 \cdot (Z_2^{1,1'} \cdot (Z_4^{2,2'})^{2,2'}) \cdot (Z_2^{1,1'} \cdot (Z_4^{2,2'})^{2,2'}) \cdot (Z_2^{1,1'} \cdot (Z_4^{2,2'})^{2,2'}) \\
\varphi_{m_3} &= Z_3^{1,1'} \cdot (Z_1^{1,1'})^{2,2'} \\
\varphi_{m_5} &= Z_4^{2,4'} \cdot (Z_4^{2,4'})^{2,2'} \\
\varphi_{m_6} &= Z_6^{2,2'} \\
\varphi_{m_4} &= (Z_2^{3,3'} \cdot Z_5^{2,5'} \cdot h_1 \cdot h_3 \cdot h_4 \cdot (Z_2^{2,6})^{2,2'} \cdot (Z_2^{2,6})^{2,2'}) \\
&\quad \cdot (Z_3^{2,2,3} \cdot (Z_5^{2,5'})^{2,2} \cdot h_3 \cdot h_4 \cdot (Z_2^{2,6})^{2,2} \cdot (Z_2^{2,6})^{2,2}).
\end{align*}

where $h_1, h_2$ are the upper braids and $h_3, h_4$ are the lower ones in Figure 11.

The parasitic intersections braids (Figure 12) are

\begin{align*}
(51) &\quad (Z_1^{2,2,5'} \cdot Z_2^{2,5'} \cdot Z_3^{2,2,3'} \cdot Z_4^{2,2,4'} \cdot Z_6^{2,2,4'}). \end{align*}
Since $B_4$ has degree 12, the total degree of the braid monodromy factorization $\Delta_{12}^2$ should be $12 \cdot 11 = 132$, see [21]. By the above regeneration, $B_4$ has 8 branch points, 24 cusps, and 25 nodes. Their related braids give a total degree of 130. The missing braids correspond to two extra branch points. We explain how to find them.

We look at the preimage in $X_4$ of a vertical line in $\mathbb{CP}^2$ (a fiber of the projection); this is an elliptic curve (a 6-fold cover of $\mathbb{CP}^1$ branched in 12 points). Considering the entire family of vertical lines in $\mathbb{CP}^2$, we get that $X_4$ admits a projection to $\mathbb{CP}^1$, with generic fiber an elliptic curve. The preimage of a vertical line in $\mathbb{CP}^2$ is singular if and only if that vertical line is tangent to the branch curve or if it passes through the intersection of the lines 6 and 6'.
There is a “lifting homomorphism” from the braid group $B_{12}$ to the mapping class group $SL(2, \mathbb{Z})$, obtained by considering the above-mentioned 6-fold cover of $\mathbb{C}P^1$: if the 12 branch points are moved by a braid, this induces a homeomorphism of the covering, [12] Section 5.2. Now, since the abelianization of $SL(2, \mathbb{Z})$ is $\mathbb{Z}/12$ and the quotient homomorphism $SL(2, \mathbb{Z}) \to \mathbb{Z}/12$ takes Dehn twists to the integer 1, the number of Dehn twists we get is a multiple of 12. However, we get 2 from $\mathbb{Z}^2_{6,6'}$, and 1 from each one of the 8 branch points.

In order to check which braids are missing, we consider a homomorphism from the pure braid group on 12 strings to the pure braid group on 2 strings, defined by deleting all the strands except $i$ and $i'$; it should map $\Delta^2_{12}$ to $\Delta^2_2 = Z^2_{i,i'}$. By Lemma 2.I in [25], $Z^3_{i,i',j} = Z^2_{i,j}Z^2_{i',j}Z^2_{i,j}Z^2_{i,i'}$. Therefore, by Theorem 21, we get $\Delta^2_2 = Z^2_{i,i'}$ for $i = 1, 2, 4, 6$. Now, forgetting all indices and remembering 3 and 3' (resp. 5 and 5') gives the half-twist $Z_{3,3'}$ (resp. $Z_{5,5'}$), counted three times. But by Lemma 8.IV in [25], $\varphi_m = \Delta^2_{8}Z^2_{2,2}Z^2_{6,6}Z^2_{3,3}Z^2_{5,5}$. In his thesis [28], Robb discusses existence of extra branch points. According to our results, there is an extra branch point, which contributes the half-twist $Z_{3,3'}$ (resp. $Z_{5,5'}$). By [28] Prop. 3.3.1], the relation in $\pi_1(\mathbb{C}P^2 \setminus B_4)$ should be $\Gamma_3 = \Gamma_{3'}$ (resp. $\Gamma_5 = \Gamma_{5'}$).

**Remark 22.** Another justification for this can be also group-theoretic. Because Moishezon-Teicher’s formulas for arrangements of lines [25] deal only with what happens before each line regenerates to a pair $i, i'$, their global formula ($\Delta^2 = \prod C_i \varphi_i$, $C_i$ are the parasitic braids) is only correct up to half-twists of the form $Z_{i,i'}$, which are not seen at all by configurations at the level of the double lines (before regeneration).
In our case the above product is not $\Delta_{12}^2$ but $\Delta_{12}^2 Z_{3^3}^{-1} Z_{5^5}^{-1}$, and thus implies that there are two extra half-twists which must be $Z_{3^3}$ and $Z_{5^5}$.

**Corollary 23.** The braid monodromy factorization $\Delta_{12}^2$ is a product of the braids from Theorem 21 and the extra branch points braids $Z_{3^3}$ and $Z_{5^5}$.

Now we are ready to compute the group $\pi_1(\mathbb{C}P^2 \setminus B_4)$.

**Theorem 24.** Let $\tilde{B}_6$ be the quotient of the braid group $B_6$ by $\langle [X, Y] \rangle$, where $X, Y$ are transversal. The fundamental group $\pi_1(\mathbb{C}P^2 \setminus B_4)$ is isomorphic to a quotient of $\tilde{B}_6$ by $\langle \varphi_m \rangle$. The group $\Pi(B_4)$ is isomorphic to $S_6$.

**Proof.** Applying the van Kampen Theorem \[31\] on the factorization $\Delta_{12}^2$, we get a presentation of $\pi_1(\mathbb{C}P^2 \setminus B_4)$ with the generators $\{\Gamma_i, \Gamma_{i'}\}_{i=1}^6$.

The monodromy $\varphi_{m_1}$ contributes the relations

\[(52)\quad \langle \Gamma_{22'}, \Gamma_4 \rangle = \langle \Gamma_2 \Gamma_2 \Gamma_2^{-1}, \Gamma_4 \rangle = e\]

\[(53)\quad \Gamma_4^{-1} \Gamma_2 \Gamma_2^{-1} = \Gamma_4\]

\[(54)\quad [\Gamma_{11'}, \Gamma_4^{-1} \Gamma_2 \Gamma_2^{-1} \Gamma_4^{-1}] = [\Gamma_{11'}, \Gamma_4] = e\]

\[(55)\quad \langle \Gamma_{1'}, \Gamma_{22} \rangle = \langle \Gamma_{1'}, \Gamma_2 \Gamma_2 \Gamma_2^{-1} \rangle = e\]

\[(56)\quad \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_2^{-1} = \Gamma_1.\]
From the monodromies $\varphi_{m_3}$ and $\varphi_{m_5}$ we have

\begin{align*}
(57) \quad \langle \gamma_1', \gamma_3' \rangle &= \langle \gamma_1', \gamma_3 \gamma_3^{-1} \gamma_3' \rangle = e \\
(58) \quad \gamma_3 \gamma_3 \gamma_1 \gamma_3^{-1} \gamma_3' &= \Gamma_1 \\
(59) \quad \langle \gamma_4', \gamma_5' \rangle &= \langle \gamma_4', \gamma_5 \gamma_5^{-1} \rangle = e \\
(60) \quad \gamma_5 \gamma_5 \gamma_4 \gamma_5^{-1} \gamma_5' &= \Gamma_4.
\end{align*}

By $\varphi_{m_4}$ we have

\begin{align*}
(61) \quad \langle \gamma_2', \gamma_3 \rangle &= \langle \gamma_2', \gamma_3' \rangle = \langle \gamma_2', \gamma_3 \gamma_3^{-1} \gamma_3' \rangle = e \\
(62) \quad \langle \gamma_5', \gamma_6 \rangle &= \langle \gamma_5 \gamma_5^{-1} \gamma_5', \gamma_6 \rangle = e \\
(63) \quad \langle \gamma_5', \gamma_6^{-1} \gamma_6' \rangle &= \langle \gamma_5 \gamma_5^{-1} \gamma_5', \gamma_6^{-1} \gamma_6' \rangle = e \\
(64) \quad [\gamma_2, \gamma_6] &= [\gamma_2', \gamma_6'] = e \\
(65) \quad [\gamma_2', \gamma_6^{\gamma_3' \gamma_3}] &= [\gamma_2, \gamma_6^{\gamma_3' \gamma_3}] = e \\
(66) \quad \gamma_3 \gamma_3^{-1} \gamma_3^{-1} \gamma_3' &= \gamma_5^{-1} \\
(67) \quad \gamma_3 \gamma_3^{-1} \gamma_3^{-1} \gamma_3' &= \gamma_5^{-1} \gamma_6^{-1} \\
(68) \quad \gamma_3 \gamma_3^{-1} \gamma_3^{-1} \gamma_3' &= \gamma_5^{-1} \gamma_6^{-1} \gamma_6 \\
(69) \quad \gamma_3 \gamma_3^{-1} \gamma_3^{-1} \gamma_3' &= \gamma_5^{-1} \gamma_6^{-1} \gamma_6^{-1} \gamma_6,
\end{align*}

and $\varphi_{m_6}$ contributes

\begin{align*}
(70) \quad [\gamma_6, \gamma_6'] &= e.
\end{align*}
From the parasitic intersections braids, we have

\[
\begin{align*}
[\Gamma_{11'}, \Gamma_{55'}] &= e \\
[\Gamma_{11'}, \Gamma_{66'}] &= e \\
[\Gamma_{44'}, \Gamma_{i''}] &= e \text{ for } i=3,6,
\end{align*}
\]

and the extra branch points contribute

\[
\begin{align*}
\Gamma_3 &= \Gamma_{3'} \\
\Gamma_5 &= \Gamma_{5'}.
\end{align*}
\]

The projective relation is

\[
\begin{align*}
\Gamma_6\Gamma_5\Gamma_5\Gamma_4\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_2\Gamma_2\Gamma_1\Gamma_1 &= e.
\end{align*}
\]

**Lemma 25.** The above presentation is a complete one.

**Proof of the Lemma.** Considering the complex conjugations (details in [20], [25]) of the braids, we get a complete set of relations. Simplifying them gives the same list as above.

We outline now the simplification of the above presentation. We will express the relations in terms of \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\) and \(\Gamma_{6'}\). First we use relations (74) and (75) to omit the generators \(\Gamma_{3'}\) and \(\Gamma_{5'}\) from all the given relations.
The branch points relations (53), (56), (58), (60) and (66) - (69) are rewritten as

\[(77) \quad \Gamma_4' = \Gamma_2^{-1}\Gamma_2'^{-1}\Gamma_4\Gamma_2\Gamma_2'\] (using \((52)\))

\[(78) \quad \Gamma_1' = \Gamma_2^{-1}\Gamma_2'^{-1}\Gamma_1\Gamma_2\Gamma_2'\]

\[(79) \quad \Gamma_1' = \Gamma_3^{-2}\Gamma_1\Gamma_3^2\]

\[(80) \quad \Gamma_4' = \Gamma_5^{-2}\Gamma_4\Gamma_5^2\]

\[(81) \quad \Gamma_6 = \Gamma_5\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_5^{-1}\]

\[(82) \quad \Gamma_6' = \Gamma_5\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_5^{-1}.\]

Now we rewrite the commutations. Using \((82)\), relation \((62)\) gets the form \(\langle \Gamma_3, \Gamma_6 \rangle = e\), and this enables us to prove that \((65)\) is

\[(83) \quad e = [\Gamma_2', \Gamma_3^{-2}\Gamma_6\Gamma_3^2] = [\Gamma_3^{-1}\Gamma_6\Gamma_5\Gamma_6^{-1}\Gamma_3, \Gamma_3^{-2}\Gamma_6\Gamma_3^2] = \]

\[\Gamma_6\Gamma_5\Gamma_6^{-1}, \Gamma_3^{-1}\Gamma_6\Gamma_3]\]

Relation \((71)\) is rewritten as \([\Gamma_1, \Gamma_5] = e\), using \((59)\), \((80)\), \((79)\) and \([\Gamma_3, \Gamma_5] = e\).

This enables us to prove from \((54)\) that \([\Gamma_1, \Gamma_4] = e\).

Using these two resulting relations, together with \((81)\), \((77)\) and \((73)\), relation \([\Gamma_1, \Gamma_6] = e\) is rewritten as follows:

\[e = [\Gamma_1, \Gamma_6] = [\Gamma_1, \Gamma_5\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_5^{-1}] = [\Gamma_1, \Gamma_3\Gamma_2\Gamma_3^{-1}] = [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_2'] =\]

\[\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_4^{-1}\Gamma_2\Gamma_4\Gamma_2^{-1}\Gamma_4]\]

\[= [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_4^{-1}\Gamma_2\Gamma_4\Gamma_2^{-1}\Gamma_4] = [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_2'] = [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_2] = [\Gamma_1, \Gamma_3\Gamma_2\Gamma_3^{-1}].\]
In a similar way, \([\Gamma_1, \Gamma_6] = e\) can be rewritten as \([\Gamma_1, \Gamma_3^{-1}\Gamma_2 \Gamma_3] = e\). Using (80) and \([\Gamma_3, \Gamma_5] = e\), relation \([\Gamma_3, \Gamma_{4'}] = e\) gets the form \([\Gamma_3, \Gamma_4] = e\). Relation (62) is rewritten as

\[
(84) \quad e = \langle \Gamma_5, \Gamma_6 \rangle = \langle \Gamma_5, \Gamma_3 \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_5^{-1} \rangle = \langle \Gamma_5, \Gamma_2 \rangle =
\]

\[
\langle \Gamma_5, \Gamma_1^{-1} \Gamma_2 \Gamma_1^{-1} \Gamma_1 \rangle = \langle \Gamma_5, \Gamma_1^{-1} \Gamma_2 \Gamma_1 \Gamma_1 \rangle = \langle \Gamma_5, \Gamma_2 \rangle.
\]

Thus \([\Gamma_{4'}, \Gamma_6] = e\) are rewritten as

\[
(85) \quad [\Gamma_4, \Gamma_5 \Gamma_2 \Gamma_5^{-1}] = [\Gamma_4, \Gamma_5^{-1} \Gamma_2 \Gamma_5] = e.
\]

Now, relation \([\Gamma_{2'}, \Gamma_{6'}] = e\) gets the form \([\Gamma_2, \Gamma_{6'}] = e\), using (72), (55) and (78).

This relation, together with (77) and (78) enable us to prove that \([\Gamma_{1'}, \Gamma_{6'}] = e\) and \([\Gamma_{4'}, \Gamma_{6'}] = e\) get the forms \([\Gamma_1, \Gamma_6] = e\) and \([\Gamma_4, \Gamma_6] = e\) respectively. Since (63) can be rewritten as \(\langle \Gamma_3, \Gamma_{6'} \rangle = e\), relation \([\Gamma_2, \Gamma_3^{-2} \Gamma_6 \Gamma_3^2] = e\) from (65) gets the form \([\Gamma_3, \Gamma_5] = e\).

Relation \([\Gamma_2, \Gamma_6] = e\) from (64) gets the form

\[
e = [\Gamma_2, \Gamma_6] \overset{(81)}{=} [\Gamma_2, \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_5^{-1}] = [\Gamma_5^{-1} \Gamma_2 \Gamma_5, \Gamma_3 \Gamma_2 \Gamma_3^{-1}] \overset{(77)}{=} \]

\[
[\Gamma_5^{-1} \Gamma_2 \Gamma_5, \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_4 \Gamma_3^{-1}] \overset{(73), (85), (84)}{=} [\Gamma_2^{-1} \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_2, \Gamma_4'] \overset{(61)}{=} \]

\[
[\Gamma_3 \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_5 \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_4] \overset{(83)}{=} [\Gamma_2^{-1} \Gamma_5 \Gamma_2, \Gamma_4'] \overset{(77)}{=} \]

\[
[\Gamma_2^{-1} \Gamma_5 \Gamma_2, \Gamma_2^{-1} \Gamma_2 \Gamma_4 \Gamma_2 \Gamma_2] \overset{(52)}{=} [\Gamma_5, \Gamma_4 \Gamma_2 \Gamma_4^{-1}] \overset{(78)}{=} \]

\[
[\Gamma_5, \Gamma_4 \Gamma_1^{-1} \Gamma_2 \Gamma_1^{-1} \Gamma_1 \Gamma_4^{-1}] \overset{(54), (71), (55)}{=} [\Gamma_5, \Gamma_4 \Gamma_1^{-1} \Gamma_2 \Gamma_1 \Gamma_4^{-1}] =
\]

\[
[\Gamma_5, \Gamma_4 \Gamma_2 \Gamma_4^{-1}] \overset{(52), (84)}{=} [\Gamma_4, \Gamma_5^{-1} \Gamma_2 \Gamma_5].
\]
The only relation which is left for now in its original form is (70). We prove below that \( \Gamma_6 = \Gamma_{6'} \), and this equality will eliminate it.

The triple relations are rewritten as follows. (57) and (59) get the forms \( \langle \Gamma_1, \Gamma_3 \rangle = e \) and \( \langle \Gamma_4, \Gamma_5 \rangle = e \), using (79) and (80). It is also easy to prove that from (52), (55) and (61) we get \( \langle \Gamma_2, \Gamma_4 \rangle = e \), \( \langle \Gamma_1, \Gamma_2 \rangle = e \) and \( \langle \Gamma_2, \Gamma_3 \rangle = e \), respectively.

Relation (76) is now

(86) \[ \Gamma_6' \Gamma_5 \Gamma_3 \Gamma_4^{-1} \Gamma_2 \Gamma_5^{-2} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4 \Gamma_5^{-1} \Gamma_4  \Gamma_5^2 \Gamma_3 \Gamma_2 \Gamma_5^{-2} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4 = e. \]

Now, equating the two expressions of \( \Gamma_{2'} \) given by (77) and (78), we get the relation

(87) \[ \Gamma_1^{-1} \Gamma_2 \Gamma_3^{-2} \Gamma_1 \Gamma_3^2 \Gamma_2^{-1} \Gamma_1 = \Gamma_4^{-1} \Gamma_2 \Gamma_5^{-2} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4, \]

which will be redundant later on.

The relations which we have now are (70), (82), (86), (87) and

(88) \[ \langle \Gamma_i, \Gamma_j \rangle = e, \quad \Gamma_i \text{ and } \Gamma_j \text{ share a common triangle} \]

(89) \[ [\Gamma_i, \Gamma_j] = e, \quad \Gamma_i \text{ and } \Gamma_j \text{ share no common triangle} \]

(90) \[ [\Gamma_1, \Gamma_3^{-1} \Gamma_2 \Gamma_3] = [\Gamma_1, \Gamma_3 \Gamma_2] = e \]

(91) \[ [\Gamma_4, \Gamma_5^{-1} \Gamma_2 \Gamma_5] = [\Gamma_4, \Gamma_5 \Gamma_2] = e. \]

Using (82), we omit \( \Gamma_{6'} \), and therefore the group \( \pi_1(\mathbb{CP}^2 \setminus B_4) \) has the generators \( \{\Gamma_i\}_{i=1}^5 \) and admits the relations (70), (87), (88) - (91) for \( i, j \neq 6' \), and the new form which (86) gets:

(92) \[ \Gamma_5 \Gamma_3 \Gamma_2 \Gamma_4^{-1} \Gamma_2 \Gamma_5^{-2} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4 \Gamma_5^{-1} \Gamma_4 \Gamma_5^2 \Gamma_3 \Gamma_2 \Gamma_5^{-2} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4 \Gamma_5^2 \Gamma_2^{-1} \Gamma_4 = e. \]
Now we show that $\pi_1(\mathbb{C}P^2 \setminus B_4)$ is isomorphic to a quotient of $\mathcal{B}_6/\langle [X, Y] \rangle$, where $X, Y$ are transversal half-twists. We choose a point in each triangle in Figure 8. Then we choose a path $h_i$, connecting two points in neighboring triangles, skipping the one which crosses the edge 6. We get a tree, see Figure 13. The paths represent generators

\[
\{H_i\}_{i=1}^5 \quad \text{of the braid group } \mathcal{B}_6 \quad \text{with the following complete list of relations}
\]

\begin{align*}
(93) \quad \langle H_i, H_j \rangle &= e, \quad H_i \text{ and } H_j \text{ are consecutive} \\
(94) \quad [H_i, H_j] &= e, \quad H_i \text{ and } H_j \text{ are disjoint} \\
(95) \quad [H_4, H_5 H_2 H_5^{-1}] &= e \\
(96) \quad [H_1, H_3^{-1} H_2 H_3] &= e.
\end{align*}

Denote $H_6' = H_5 H_3 H_2 H_3^{-1} H_5^{-1}$, where $H_6'$ corresponds to the missing path $h_6$, being transversal to $H_1$ and $H_2$ and disjoint from $H_4$. Recall the definition.
Section IV]) of the group $\tilde{B}_6$ as $B_6/\langle [X,Y] \rangle$, and $X,Y$ are transversal. Denote the images of $H_i$ as $\tilde{H}_i$ in $\tilde{B}_6$. Then the group $\tilde{B}_6$ is generated by $\tilde{H}_i$, $i = 1, \ldots, 5, 6'$, and the only relations are

(97) $\langle \tilde{H}_i, \tilde{H}_j \rangle = e$, $\tilde{H}_i$ and $\tilde{H}_j$ are consecutive, $i,j \neq 6'$

(98) $[\tilde{H}_i, \tilde{H}_j] = e$, $\tilde{H}_i$ and $\tilde{H}_j$ are disjoint, $i,j \neq 6'$

(99) $[\tilde{H}_4, \tilde{H}_5\tilde{H}_2\tilde{H}_5^{-1}] = [\tilde{H}_4, \tilde{H}_5^{-1}\tilde{H}_2\tilde{H}_5] = e$

(100) $[\tilde{H}_1, \tilde{H}_3^{-1}\tilde{H}_2\tilde{H}_3] = [\tilde{H}_1, \tilde{H}_3\tilde{H}_2\tilde{H}_3^{-1}] = e$

(101) $\tilde{H}_5\tilde{H}_3\tilde{H}_2\tilde{H}_3^{-1}\tilde{H}_5^{-1} = \tilde{H}_{6'}$

where $\tilde{H}_4$ and $\tilde{H}_5^{-1}\tilde{H}_2\tilde{H}_5$ ($\tilde{H}_1$ and $\tilde{H}_3\tilde{H}_2\tilde{H}_3^{-1}$, respectively) are transversal. We note that (101) can be used to remove $\tilde{H}_{6'}$ from the list of generators (in the same way as $\Gamma_{6'}$ has been eliminated from the presentation of $\pi_1(\mathbb{C}P^2 \setminus B_4)$).

According to our result, $\pi_1(\mathbb{C}P^2 \setminus B_4)$ is a quotient of $\tilde{B}_6$. Now we eliminate (87). Since $\Gamma_2(\mathbb{C}P^2 \setminus B_4)$ and $\Gamma_3^{-1}\Gamma_2\Gamma_3$ are transversal, the relations in $\tilde{B}_6$ imply that they commute and so the left hand side of (87) is equal to

$$\Gamma_1^{-1}\Gamma_2(\Gamma_3^{-1}\Gamma_2^{-1}\Gamma_3)\Gamma_3^{-2}\Gamma_1\Gamma_3^{2}(\Gamma_3^{-1}\Gamma_2\Gamma_3)\Gamma_2^{-1}\Gamma_1 = \Gamma_1^{-1}\Gamma_3^{-1}\Gamma_2^{-1}\Gamma_1\Gamma_2\Gamma_3\Gamma_1 = \Gamma_2.$$ 

Similarly, the right hand side of (87) is also equal to $\Gamma_2$. This allows us to eliminate (87). Since both sides of (87) are equal to $\Gamma_2$, we have shown that $\Gamma_2 = \Gamma_{2'}$, therefore $\Gamma_6 = \Gamma_{6'}$ (see (81) and (82)). That means that (70) is redundant too. Thus $\pi_1(\mathbb{C}P^2 \setminus B_4)$ is isomorphic to $\tilde{B}_6/\langle (92) \rangle$.

In order to get the group $\Pi(B_4)$, we take $\Gamma_j^2 = e$ for each $j$. Relation (92) is then redundant. By (29), the rest of the relations in $\pi_1(\mathbb{C}P^2 \setminus B_4)$, together with the ones
\[ \Gamma_j^2 = e, \text{ are the only ones which are required in order to make } \Pi_{(B_4)} \text{ isomorphic to } S_6. \]  

\[ \square \]

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