A Two Dimensional Fermi Liquid

Part 1: Overview

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Abstract. In a series of ten papers, of which this is the first, we prove that the temperature zero renormalized perturbation expansions of a class of interacting many–fermion models in two space dimensions have nonzero radius of convergence. The models have “asymmetric” Fermi surfaces and short range interactions. One consequence of the convergence of the perturbation expansions is the existence of a discontinuity in the particle number density at the Fermi surface. Here, we present a self contained formulation of our main results and give an overview of the methods used to prove them.

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I. Introduction

The standard model for a gas of weakly interacting fermions in a \(d\)-dimensional crystal at low temperature is given in terms of

- a single particle dispersion relation (shifted by the chemical potential) \(e(k)\) on \(\mathbb{R}^d\),
- an ultraviolet cutoff \(U(k)\) on \(\mathbb{R}^d\),
- an interaction \(V\).

Here \(k\) is the momentum variable dual to the position variable \(x \in \mathbb{R}^d\). The Fermi surface associated to the dispersion relation \(e(k)\) is by definition

\[ F = \{ k \in \mathbb{R}^d \mid e(k) = 0 \} \]

The ultraviolet cutoff is a smooth function with compact support that fulfills \(0 \leq U(k) \leq 1\) for all \(k \in \mathbb{R}^d\). We assume that it is identically one on a neighbourhood of the Fermi surface\(^{(1)}\).

We use renormalization group techniques to show that, for \(d = 2\) and under the assumptions on the dispersion relation \(e(k)\) specified in Hypotheses I.12 below, such a system is a Fermi liquid whenever \(V\) is small enough (the precise statement is given in Theorem I.5 below). Renormalization is necessary since the Fermi surfaces for the noninteracting (that is \(V = 0\)) and interacting systems \((V \neq 0)\) do not, in general, agree. We therefore select \((V\)-dependent\) counterterms \(\delta e(k)\), from the space in Definition I.1, below, in such a way that the Fermi surface of the model with dispersion relation \(e(k) - \delta e(k)\) and interaction \(V\) is equal to \(F\).

**Definition I.1** The space of counterterms, \(E\), consists of all functions \(\delta e(k)\) on \(\mathbb{R}^d\) that are supported in \(\{ k \in \mathbb{R}^d \mid U(k) = 1 \}\) and for which the \(L^1\)-norm of the Fourier transform is finite. That is

\[ \int d^d x \mid \delta e^\wedge(x) \mid < \infty \]

where \(\delta e^\wedge(x) = \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot x} \delta e(k)\).

The temperature Green’s functions at temperature zero (also known as the Euclidean Green’s functions) for this model can be described in field theoretic terms using the anticommuting fields \(\psi_\sigma(x_0, x), \bar{\psi}_\sigma(x_0, x)\), where \(x_0 \in \mathbb{R}\) is the temperature (or Euclidean time) argument and \(\sigma \in \{\uparrow, \downarrow\}\) is the spin argument. For \(x = (x_0, x, \sigma)\) we write \(\psi(x) = \psi_\sigma(x_0, x)\) and \(\bar{\psi}(x) = \bar{\psi}_\sigma(x_0, x)\).

\(^{(1)}\) In particular, we assume that \(F\) is compact.
For a model with dispersion relation $e(k) - \delta e(k)$ and interaction $V = 0$, the Green’s functions are the moments of the Grassmann Gaussian measure, $d\mu_{C(\delta e)}$, whose covariance is the Fourier transform of

$$C(k_0, k; \delta e) = \frac{U(k)}{ik_0 - e(k) + \delta e(k)}$$

Precisely, for $x = (x_0, x, \sigma)$, $x' = (x'_0, x', \sigma') \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$

$$C(x, x'; \delta e) = \int \psi(x) \bar{\psi}(x') \ d\mu_{C(\delta e)}(\psi, \bar{\psi}) = \delta_{\sigma, \sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\langle k, x-x' \rangle} C(k; \delta e)$$

where $\langle k, x-x' \rangle = -k_0(x_0-x'_0) + k \cdot (x-x')$ for $k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d$. To simplify notation we set

$$C(k) = C(k; 0) \quad , \quad C(x, x') = C(x, x'; 0)$$

The interaction between the fermions is determined by the effective potential

$$\mathcal{V}(\psi, \bar{\psi}) = \int_{(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^4} \mathcal{V}(x_1, x_2, x_3, x_4) \bar{\psi}(x_1) \psi(x_2) \bar{\psi}(x_3) \psi(x_4) \ dx_1 dx_2 dx_3 dx_4$$

We assume that $V$ is translation invariant and spin independent. For some results, we also assume that $V$ obeys

$$V(R_0 x_1, R_0 x_2, R_0 x_3, R_0 x_4) = \overline{V(-x_1, -x_2, -x_3, -x_4)} \quad (I.1)$$

and

$$V(-x_2, -x_1, -x_4, -x_3) = V(x_1, x_2, x_3, x_4) \quad (I.2)$$

where $R_0(x_0, x, \sigma) = (-x_0, x, \sigma)$ and $-R_0(x, \sigma) = (-x_0, -x, \sigma)$. We call (I.1) “$k_0$–reversal reality” and (I.2) “bar/unbar exchange invariance”. Precise definitions and a discussion of the properties of these symmetries are given in Appendix B of [FKTo2].

In the case of a two–body interaction $v(x_0, x)$, the interaction kernel is

$$V((x_{1,0}, x_{1,\sigma}), \ldots, (x_{4,0}, x_{4,\sigma})) = -\frac{1}{2} \delta(x_{1,2}) \delta(x_{3,4}) \delta(x_{1,0}-x_{3,0}) v(x_{1,0}-x_{3,0}, x_{1}-x_{3}) \quad (I.3)$$

where $\delta((x_0, x, \sigma), (x'_0, x', \sigma')) = \delta(x_0-x'_0) \delta(x-x') \delta_{\sigma, \sigma'}$. If the Fourier transform, $\tilde{v}(k_0, k)$, of the two–body interaction $v(x_0, x)$ obeys $\tilde{v}(-k_0, k) = \tilde{v}(k_0, k)$, then the interaction kernel $V$ has all four symmetries mentioned above. In addition, $\mathcal{V}$ always conserves particle number.

We briefly discuss the norms imposed on interaction kernels. For a function $f(x, \cdots, x_n)$ on $(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^n$ we define its $L_1$–$L_\infty$–norm as

$$\|f\|_{1, \infty} = \max_{1 \leq j_0 \leq n} \sup_{x_{j_0} \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}} \int \prod_{j=1, j \neq j_0} dx_j \ |f(x_1, \cdots, x_n)|$$

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A multiindex is an element \( \delta = (\delta_0, \delta_1, \cdots, \delta_d) \in \mathbb{N}_0 \times \mathbb{N}_0^d \). The length of a multiindex \( \delta = (\delta_0, \delta_1, \cdots, \delta_d) \) is \( |\delta| = \delta_0 + \delta_1 + \cdots + \delta_d \) and its factorial is \( \delta! = \delta_0! \delta_1! \cdots \delta_d! \). For two multiindices \( \delta, \delta' \) we say that \( \delta \leq \delta' \) if \( \delta_i \leq \delta_i' \) for \( i = 0, 1, \cdots, d \). The spatial part of the multiindex \( \delta = (\delta_0, \delta_1, \cdots, \delta_d) \) is \( \delta = (\delta_1, \cdots, \delta_d) \in \mathbb{N}_0^d \). It has length \( |\delta| = \delta_1 + \cdots + \delta_d \). For a multiindex \( \delta \) and \( x = (x_0, x, \sigma) \), \( x' = (x'_0, x', \sigma') \) is \( \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \) set

\[
(x - x')^{\delta} = (x_0 - x'_0)^{\delta_0} (x_1 - x'_1)^{\delta_1} \cdots (x_d - x'_d)^{\delta_d}
\]

We fix \( r_0, r \geq 6 \) for the numbers of temporal and spatial momentum derivatives that we will control. The norm imposed on an interaction kernel will be

\[
\max_{\delta_i, j \in \mathbb{N}_0 \times \mathbb{N}_0^d \text{ for } 1 \leq i < j \leq 4} \left\| \prod_{1 \leq i < j \leq 4} \frac{1}{\delta_i, j} (x_i - x_j)^{\delta_{i, j}} V(x_1, x_2, x_3, x_4) \right\|_{1, \infty} \leq \max_{\delta_i, j \in \mathbb{N}_0 \times \mathbb{N}_0^d \text{ for } 1 \leq i < j \leq 4} \prod_{1 \leq i < j \leq 4} \frac{1}{\delta_i, j} \int |x^{\delta} v(x)| \, dx
\]

Formally, the generating function for the connected Green’s functions is

\[
\mathcal{G}(\phi, \bar{\phi}; \delta e) = \log \frac{1}{2} \int e^{\phi J \psi} e^{V(\psi, \bar{\psi})} e^{-\mathcal{K}(\psi, \bar{\psi}; \delta e)} d\mu_C(\psi, \bar{\psi})
\]

where the source term is

\[
\phi J \psi = \int dx \, \bar{\phi}(x) \psi(x) + \bar{\psi}(x) \phi(x)
\]

The counterterm is implemented in

\[
\mathcal{K}(\psi, \bar{\psi}; \delta e) = \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \int d\tau d^d x d^d y \, \delta e^{\wedge}(x - y) \, \bar{\psi}_\sigma(\tau, x) \psi_\sigma(\tau, y)
\]

and \( Z = \int e^{V(\psi, \bar{\psi})} e^{-\mathcal{K}(\psi, \bar{\psi}; \delta e)} d\mu_C(\psi, \bar{\psi}) \) is the partition function. The fields \( \phi, \bar{\phi} \) are called source fields and the fields \( \psi, \bar{\psi} \) are called internal fields. The connected Green’s functions themselves are determined by

\[
\mathcal{G}(\phi, \bar{\phi}; \delta e) = \sum_{n=1}^{\infty} \frac{1}{(n!)^{d}} \int \prod_{i=1}^{n} dx_i dy_i \, G_{2n}(x_1, y_1, \cdots, x_n, y_n; \delta e) \prod_{i=1}^{n} \bar{\phi}(x_i) \phi(y_i)
\]
Observe that for $\delta e \in \mathcal{E}$, formally,

$$
G(\phi, \bar{\phi}; \delta e) = \log \frac{1}{Z} \int e^{J \phi} e^{\mathcal{V}(\psi, \bar{\psi})} d\mu_{C(\delta e)}(\psi, \bar{\psi})
$$

where $Z' = \int e^{\mathcal{V}(\psi, \bar{\psi})} d\mu_{C(\delta e)}(\psi, \bar{\psi})$. See [FKTo2, Lemma C.2].

We show in this paper that, for $d = 2$ and under the Hypotheses I.12 on $e(k)$, there exists, for every sufficiently small interaction, a counterterm $\delta e \in \mathcal{E}$ such that connected Green’s functions $G_{2n}(\cdot; \delta e)$ exist and have Fermi surface $F$. This statement needs to be made precise, because the functional integrals in the definition of $G$ are not, a priori, well defined due to the singularities of the propagator. This problem is dealt with by a multiscale analysis.

We introduce scales by slicing momentum space into shells around the Fermi surface. We choose a “scale parameter” $M > 1$ and a function $\nu \in C^\infty_0([1/M, 2M])$ that takes values in $[0, 1]$, is identically 1 on $[2/M, M]$ and obeys

$$
\sum_{j=0}^\infty \nu(M^{2j} x) = 1
$$

for $0 < x < 1$ (see also [FKTo2, §VIII]). The function $\nu$ may be constructed by choosing a function $\varphi \in C^\infty_0((2, 2))$ that is identically one on $[-1, 1]$ and setting $\nu(x) = \varphi(x/M) - \varphi(Mx)$ for $x > 0$ and zero otherwise.

**Definition I.2**

i) For $j \geq 1$, the $j$th scale function on $\mathbb{R} \times \mathbb{R}^d$ is defined as

$$
\nu^{(j)}(k) = \nu(M^{2j}(k_0^2 + e(k)^2))
$$

By construction, $\nu^{(j)}$ is identically one on

$$
\{ k = (k_0, k) \in \mathbb{R} \times \mathbb{R}^d \mid \sqrt{\frac{1}{M} \frac{1}{M^7}} \leq \frac{i k_0 - e(k)}{M} \leq \sqrt{M} \frac{1}{M^7} \}
$$

The support of $\nu^{(j)}$ is called the $j$th shell. By construction, it is contained in

$$
\{ k \in \mathbb{R} \times \mathbb{R}^d \mid \frac{1}{\sqrt{M}} \frac{1}{M^7} \leq \frac{i k_0 - e(k)}{M} \leq \sqrt{2M} \frac{1}{M^7} \}
$$

The momentum $k$ is said to be of scale $j$ if $k$ lies in the $j$th shell.

ii) For real $j \geq 1$, set

$$
\nu^{(2j)}(k) = \varphi(M^{2j-1}(k_0^2 + e(k)^2))
$$
By construction, $\nu^{(\geq j)}$ is identically 1 on
\[
\{ k \in \mathbb{R} \times \mathbb{R}^d \mid |ik_0 - e(k)| \leq \sqrt{M \frac{1}{M^j}} \}
\]
Observe that if $j$ is an integer, then for $|ik_0 - e(k)| > 0$
\[
\nu^{(\geq j)}(k) = \sum_{i \geq j} \nu^{(i)}(k)
\]
The support of $\nu^{(\geq j)}$ is called the $j^{th}$ neighbourhood of the Fermi surface. By construction, it is contained in
\[
\{ k \in \mathbb{R} \times \mathbb{R}^d \mid |ik_0 - e(k)| \leq \sqrt{M \frac{1}{M^j}} \}
\]
The support of $\varphi(M^{2j-2}(k_0^2 + e(k)^2))$ is called the $j^{th}$ extended neighbourhood. It is contained in
\[
\{ k \in \mathbb{R} \times \mathbb{R}^d \mid |ik_0 - e(k)| \leq \sqrt{2M \frac{1}{M^j}} \}
\]
iii) For real $\bar{j} \geq 1$, the covariance with infrared cutoff at scale $\bar{j}$ and counterterm $\delta e$ is
\[
C^{\text{IR}(\bar{j})}(k_0, k; \delta e) = \frac{U(k) - \nu^{(\geq \bar{j})}(k)}{ik_0 - e(k) + \delta e(k)[1 - \nu^{(\geq \bar{j})}(k)]}
\]
\[
C^{\text{IR}(\bar{j})}(x, x'; \delta e) = \delta_{\sigma, \sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i<k,x-x'>} C^{\text{IR}(\bar{j})}(k; \delta e)
\]
Also write
\[
C^{\text{IR}(\bar{j})}(k) = C^{\text{IR}(\bar{j})}(k; 0), \quad C^{\text{IR}(\bar{j})}(x, x') = C^{\text{IR}(\bar{j})}(x, x'; 0)
\]
The Grassmann Gaussian measure $d\mu_{C^{\text{IR}(\bar{j})}(\delta e)}$ is characterized by
\[
\int \psi(x) \bar{\psi}(x') d\mu_{C^{\text{IR}(\bar{j})}(\delta e)}(\psi, \bar{\psi}) = C^{\text{IR}(\bar{j})}(x, x'; \delta e)
\]
\[
\int \bar{\psi}(x) \psi(x') d\mu_{C^{\text{IR}(\bar{j})}(\delta e)}(\psi, \bar{\psi}) = \int \bar{\psi}(x) \psi(x') d\mu_{C^{\text{IR}(\bar{j})}(\delta e)}(\bar{\psi}, \psi) = 0
\]
Remark I.3 As the scale parameter $M > 1$, the shells near the Fermi curve have $j$ near $+\infty$, and the neighbourhoods shrink as $j \to \infty$. Also,
\[
\lim_{j \to \infty} C^{\text{IR}(\bar{j})}(k) = C(k)
\]
pointwise.
Even for the cutoff, and hence bounded, covariance $C^{\text{IR}(\bar{j})}$, it is not a priori clear that the generating functional

$$G_{\bar{j}}(\phi, \bar{\phi}; V, \delta e) = \log \frac{1}{Z} \int e^{\phi J \psi} e^{V(\psi, \bar{\psi})} d\mu_{C^{\text{IR}(\bar{j})}(\delta e)}(\psi, \bar{\psi})$$

where $Z = \int e^{\lambda V(\psi, \bar{\psi})} d\mu_{C^{\text{IR}(\bar{j})}(\delta e)}(\psi, \bar{\psi})$

or the corresponding connected Green’s functions, $G_{2n;\bar{j}}(x_1, y_1, \cdots, x_n, y_n)$, defined by

$$G_{\bar{j}}(\phi, \bar{\phi}; V, \delta e) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} dx_i dy_i \ G_{2n;\bar{j}}(x_1, y_1, \cdots, x_n, y_n) \prod_{i=1}^{n} \bar{\phi}(x_i) \phi(y_i)$$

make sense for a reasonable set of $V$’s and $\delta e$’s. On the other hand, it is easy to see, using graphs, that each term in the formal Taylor expansion of the Grassmann function\(^{(2)}\)

$$G_{\bar{j}}(\phi, \bar{\phi}; V, \delta e(V))$$

in powers of $V$ is well-defined for a large class of $V$’s and $\delta e(V)$’s. The Taylor expansion of $\int e^{\phi J \psi} e^{V(\psi, \bar{\psi})} d\mu_{C^{\text{IR}(\bar{j})}}(\psi, \bar{\psi})$ is $\sum_{n=1}^{\infty} G_{\bar{j},n}(V, \cdots, V)$ where the $n^{\text{th}}$ term is the multilinear form

$$G_{\bar{j},n}(V_1, \cdots, V_n) = \frac{1}{n!} \int e^{\phi J \psi} V_1(\psi, \bar{\psi}) \cdots V_n(\psi, \bar{\psi}) \ d\mu_{C^{\text{IR}(\bar{j})}}(\psi, \bar{\psi})$$

restricted to the diagonal. Explicit evaluation of the Grassmann integral expresses $G_{\bar{j},n}$ as the sum of all graphs with vertices $V_1, \cdots, V_n$ and lines $C^{\text{IR}(\bar{j})}$. The (formal) Taylor coefficient $\left[ \frac{d}{dt_1} \cdots \frac{d}{dt_n} G_{\bar{j}}(\phi, \bar{\phi}; t_1 V_1 + \cdots + t_n V_n, 0) \right]_{t_1 = \cdots = t_n = 0}$ of $G_{\bar{j}}(\phi, \bar{\phi}; V, 0)$ is similar, but with only connected graphs. Choosing $\delta e$ to be an appropriate function of $V$ produces renormalized connected graphs\(^{(3)}\). We prove here that, under suitable hypotheses, for each $j$, the renormalized formal Taylor series for $G_{\bar{j}}(\phi, \bar{\phi}; V, \delta e(V))$ converges to an analytic\(^{(4)}\) function of $V$ with a radius of convergence that is independent of $j$. We further show that the limit as $j \to \infty$ exists.

**Theorem I.4** Assume that $d = 2$ and that $e(k)$ fulfills the Hypotheses I.12 below. There is an open ball, centered on the origin, in the Banach space of translation invariant and spin independent interaction kernels $V$ with norm

$$\max_{\delta_{i,j} \in \mathbb{N}^d \times \mathbb{N}_0^d} \prod_{1 \leq i < j \leq 4} \| \frac{1}{\delta_{i,j}} (x_i - x_j)^{\delta_{i,j}} V(x_1, x_2, x_3, x_4) \|_{1, \infty}$$

\(^{(2)}\) We shall, in Definition VI.7, introduce a norm on the Grassmann algebra generated by $\phi$ and $\bar{\phi}$. All of our generating functionals will in fact have finite norm.

\(^{(3)}\) Under the hypotheses of Theorem I.4, below, when $j$ is finite, both the propagator $C^{\text{IR}(\bar{j})}$ and the vertex $V$ are continuous in momentum space. The values of all connected graphs, whether renormalized or not, are well-defined. However renormalization is essential for the limit $j \to \infty$.

\(^{(4)}\) For an elementary discussion of analytic maps between Banach spaces see, for example, Appendix A of [PT].
and an analytic counterterm function $V \mapsto \delta e(V) \in \mathcal{E}$ on the ball, that vanishes for $V = 0$, such that the following holds:

For any real $j \geq 1$, the formal Taylor series

$$G_j(\phi, \bar{\phi}) = \log \frac{1}{2} \int e^{\phi J \psi} e^{\lambda V(\psi, \bar{\psi})} d\mu_{C^{1,\infty}(x)(\delta e(V))}(\psi, \bar{\psi})$$

converges to an analytic function on the ball. As $j \to \infty$, the Green's functions $G_{2n,j}$ converge uniformly, in $x_1, \ldots, y_n$ and $V$, to a translation invariant, spin independent, particle number conserving function $G_{2n}$ on $(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^{2n}$ that is analytic in $V$.

If, in addition, $V$ is $k_0$-reversal real, as in (1.1), then $\delta e(k;V)$ is real for all $k$.

The proof of Theorem I.4 follows the statement of Theorem VIII.5 in [FKTf2].

**Theorem I.5** Under the hypotheses of Theorem I.4 and the assumption that $V$ obeys the symmetries (I.1) and (I.2), the Fourier transform

$$\tilde{G}_2(k_0, k) = \int dx_0 d^d x \ e^{i(-k_0 x_0 + k \cdot x)} G_2((0, 0, \uparrow), (x_0, x, \uparrow))$$

$$= \int dx_0 d^d x \ e^{i(-k_0 x_0 + k \cdot x)} G_2((0, 0, \downarrow), (x_0, x, \downarrow))$$

of the two-point function exists and is continuous, except on the Fermi surface (precisely, except when $k_0 = 0$ and $e(k) = 0$). Define

$$n(k) = \lim_{\tau \to 0^+} \frac{dk_0}{2\pi} e^{i k_0 \tau} \tilde{G}_2(k_0, k)$$

Then $n(k)$ is continuous except on the Fermi surface $F$. If $\bar{k} \in F$, then $\lim_{k \to \bar{k}} n(k)$ and $\lim_{k \to \bar{k}} n(k)$ exist and obey

$$\lim_{k \to \bar{k}} n(k) - \lim_{k \to \bar{k}} n(k) > \frac{1}{2}$$

The proof of Theorem I.5 follows Lemma XII.4 in [FKTf3].

**Remark I.6** The quantity $n(k)$ is known as the momentum distribution function. The jump discontinuity in $n(k)$ at the Fermi surface exhibited in Theorem I.5 is generally viewed as the most basic characteristic of a Fermi liquid [MCD, §4.1]. The number $\frac{1}{2}$ in the bound of I.5 has no special significance. It may be replaced by any number strictly smaller than one, provided the interaction is made sufficiently weak.
Theorem I.7 Let
\[
\hat{G}_{4;\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1, k_2, k_3) = \int \mathcal{G}_4(x_1, x_2, x_3, (0, 0, \sigma_4)) \prod_{\ell=1}^{3} e^{-(1)^{\ell}k_{\ell}x_{\ell}} \, dx_0, e^{d}x_{\ell}
\]
be the Fourier transform of the four-point function and
\[
\hat{G}_{4,\sigma_1,\sigma_2,\sigma_3,\sigma_4}^A(k_1, k_2, k_3) = \hat{G}_{4;\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1, k_2, k_3) \prod_{\ell=1}^{4} \frac{1}{G_2(k_{\ell})} \bigg|_{k_4 = k_1 - k_2 + k_3}
\]
its amputation by the physical propagator. Under the hypotheses of Theorem I.5, \(\hat{G}_{4}^A\) is continuous on
\[
\{ (k_1, k_2, k_3) \mid k_1 \neq k_2, \, k_2 \neq k_3, \, U(k_1) = U(k_2) = U(k_3) = U(k_1 - k_2 + k_3) = 1 \}
\]
Furthermore, \(\hat{G}_{4}^A\) has a decomposition
\[
\hat{G}_{4,\sigma_1,\sigma_2,\sigma_3,\sigma_4}^A(k_1, k_2, k_3) = N_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1, k_2, k_3)
\]
\[
+ \frac{1}{2} L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(\frac{k_1 + k_2}{2}, \frac{k_3 + k_4}{2}, k_2 - k_1) \bigg|_{k_4 = k_1 - k_2 + k_3}
\]
\[
- \frac{1}{2} L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(\frac{k_3 + k_4}{2}, \frac{k_1 + k_4}{2}, k_2 - k_3) \bigg|_{k_4 = k_1 - k_2 + k_3}
\]
with
- \(N_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\) continuous
- \(L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, t)\) continuous except at \(t = 0\)
- \(L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, (0, t))\) having an extension to \(t = 0\) which is continuous in \((q_1, q_2, t)\)
- \(L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, (t_0, 0))\) having an extension to \(t_0 = 0\) which is continuous in \((q_1, q_2, t_0)\)

The proof of Theorem I.7 is at the end of § XV in [FKTf3].

Remark I.8 \(L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, t)\) consists of particle–hole ladder contributions of the form

\[
\begin{array}{c}
q_1 + \frac{i}{2} \\
q_1 - \frac{i}{2}
\end{array}
\]

\[
\begin{array}{c}
q_2 + \frac{i}{2} \\
q_2 - \frac{i}{2}
\end{array}
\]

Remark I.9 Theorem I.4 defines \(G_{2m;\overline{2}}(V)\) through its renormalized perturbation expansion. Alternatively, one could introduce an additional finite volume cutoff by replacing the space–time \(\mathbb{R} \times \mathbb{R}^d\) with \((\mathbb{R}/L\mathbb{Z}) \times (\mathbb{R}^d/L\mathbb{Z}^d)\). The associated Green’s functions, \(G_{2m;\overline{2},L}(V)\), are trivially meromorphic functions of \(V\) that are analytic at \(V = 0\). The methods of these
papers could be used to show that the domain of analyticity of $G_{2n;\bar{j},L}(V)$ includes the ball of Theorem I.4 and that, as $L \to \infty$, $G_{2n;\bar{j},L}$ converges uniformly to $G_{2n;\bar{j}}$. We do not do so.

We also do not deal with the question of independence of the renormalization prescription. Suppose that $e(k), \delta e(k; V)$ and $e'(k), \delta e'(k; V)$ satisfy the appropriate regularity conditions and that $e(k) - \delta e(k; V) = e'(k) - \delta e'(k; V)$, for some specific $V$. Observe that $G_{2n;\bar{j},L}(k)$ depends on $e(k)$ and $\delta e(k; V)$ only through the combinations $e(k) - \delta e(k; V)[1 - \nu(\geq \bar{j})(k)]$ and $\nu(\geq \bar{j})(k) = \varphi(M^{2\bar{j}-1}(k_0^2 + e(k)^2))$. Hence, it is likely that the limiting Green’s functions constructed using $e(k), \delta e(k; V)$ coincide with those constructed using $e'(k), \delta e'(k; V)$, for the specific $V$, but we do not attempt to prove so here.

We now state the hypotheses on the dispersion relation $e(k)$ used in Theorems I.4, I.5 and I.7. First, we assume that the dispersion relation $e(k)$ is $r + 6$ times continuously differentiable. Furthermore, we assume that the Fermi curve

$$F = \{ k \in \mathbb{R}^2 \mid e(k) = 0 \}$$

is a strictly convex, smooth connected curve with curvature bounded away from zero. Since $F$ is strictly convex, for each point $k \in F$ there is a unique point $a(k) \in F$ different from $k$ such that the tangent lines to $F$ at $k$ and $a(k)$ are parallel. $a(k)$ is called the antipode of $k$. Choose an orientation for $F$.

**Definition I.10**

i) Let $k \in F$, $\mathbf{t}$ the oriented unit tangent vector to $F$ at $k$ and $\mathbf{n}$ the inward pointing unit normal vector to $F$ at $k$. Then there is a function $\varphi_k(s)$, defined on a neighbourhood of 0 in $\mathbb{R}$, such that $s \mapsto k + s\mathbf{t} + \varphi_k(s)\mathbf{n}$ is an oriented parametrization of $F$ near $k$.

ii) We say that $F$ is strongly asymmetric if there is $n_0 \in \mathbb{N}$, with $n_0 \leq r$, such that for each $k \in F$ there exists an $n \leq n_0$ with

$$\varphi_k^{(n)}(0) \neq \varphi_{a(k)}^{(n)}(0)$$

**Remark I.11**

i) By construction, $\varphi_k(0) = \varphi_{a(k)}(0) = 0$ and $\varphi_k(0)$ is the curvature of $F$ at $k$.

ii) If $F$ is symmetric about a point $p \in \mathbb{R}^2$, that is $F = \{ 2p - k \mid k \in F \}$, then $\varphi_k = \varphi_{a(k)}$ for all $k \in F$. Symmetry of the Fermi curve about a point promotes the formation of Cooper pairs and the phase transition to a superconducting state. Theorem I.4 shows that — at
temperature zero — this is the only instability in a broad class of short range many fermion models, at least when $d = 2$. Sufficiently high temperature also blocks the Cooper instability and leads to Fermi liquid behaviour, even for a round Fermi surface. This was shown in [DR1, DR2] using the criterion proposed in [S].

When $d = 1$, fermionic many-body models exhibit Luttinger liquid rather than Fermi liquid behaviour. See Chapter 11 of [BG] and the references therein. We would expect that results like Theorems I.4 and I.5 also hold for $d = 3$. There has been some progress in this direction [MR,DMR].

iii) In [FKTa] we show that independent fermions in a suitably chosen periodic electromagnetic background field have a dispersion relation whose associated Fermi curve, for suitably chosen chemical potential, is smooth, strictly convex, strongly asymmetric and has nonzero curvature everywhere.

**Hypothesis I.12 (on the dispersion relation):** We assume that $e(\mathbf{k})$ is $r + 6$ times continuously differentiable with $r \geq 6$, that the Fermi curve $F$ is a strictly convex, smooth, strongly asymmetric, connected curve whose curvature is bounded away from zero and that $\nabla e(\mathbf{k})$ does not vanish on $F$.

This paper is divided into three parts. This first part, which consists of Sections I through III and Appendix A, contains an overview of the main ideas involved in the construction (§II) and the algebraic structure of the construction (§III). Part 2, [FKTf2], contains Sections IV through X and Appendix B and is concerned with the proof of convergence of the expansion. Part 3, [FKTf3], contains Sections XI through XV and Appendices C and D and is concerned with the proof of the existence of the Fermi surface. Cumulative notation tables are provided at the end of each part. The construction described in these papers was outlined in [FKLT1, FKLT2, FKLT3].
II. An Overview

In this Section, we describe the main difficulties in the proof of Theorems I.4 and I.5 and outline our strategy to overcome them. For simplicity, we omit spins in this discussion. The notation in this section is close, but not always identical to that used in the rest of this paper. As the main theorems are for \( d = 2 \) space dimensions, we describe all constructions only for \( d = 2 \), even those that can be extended to other dimensions.

1. Renormalization of the Fermi Surface and the Dispersion Relation

The Fermi surface of a fermionic many particle system is the locus in momentum space where the two point function \( \hat{G}_2(0, \mathbf{k}) \) has a discontinuity – if such a discontinuity occurs at all.\(^{(1)}\) For a system of non interacting fermions, the Fermi surface coincides with the zero–set of the dispersion relation. In a metal or a crystal, the dispersion relation is a datum derived from first principles; it is determined by the associated periodic Schrödinger operator. On the other hand, the Fermi surface of the system of interacting fermions is accessible to measurement. See, for example, [AM].

As already mentioned in §I, the Fermi surface of a system of interacting fermions is, in general, different from that of the system of non interacting fermions with the same dispersion relation. This shift in the Fermi surface is responsible for the divergence of many coefficients in naive perturbation expansions. It is controlled by renormalizing the dispersion relation. Theorems I.4 and I.5 state that, given a function \( e(\mathbf{k}) \) and an interaction \( V \) fulfilling all of the Theorems’ hypotheses, there is a function \( \delta e(\mathbf{k}) \), called the “counterterm”, such that system with dispersion relation \( e(\mathbf{k}) - \delta e(\mathbf{k}) \) and interaction \( V \) has a Fermi surface and that Fermi surface is precisely \( F = \{ \mathbf{k} | e(\mathbf{k}) = 0 \} \).

We pointed out in the previous paragraph that the data derived from first principles are the dispersion relation and the interaction \( V \). Therefore it is desirable to prove that every reasonable function \( e'(\mathbf{k}) \) is of the form \( e(\mathbf{k}) - \delta e(\mathbf{k}) \) as above. This could be done by proving the invertibility of the map \( e(\mathbf{k}) \mapsto e(\mathbf{k}) - \delta e(\mathbf{k}) \) in an appropriate function space. To all orders in perturbation theory, this has been achieved in [FST4]. The bounds of this paper are not yet strong enough to prove the corresponding result non perturbatively.

Even for \( C^\infty \) functions \( e(\mathbf{k}) \) and \( V \), it is not known how smooth the counterterm \( \delta e(\mathbf{k}) \) is. In this paper, we show that \( \delta e \) is \( C^\epsilon \). In [FST3] it is shown that \( \delta e \) is \( C^{2+\epsilon} \) to all orders in perturbation theory. Later in this overview (subsection 10) we shall point out where this lack of smoothness in the counterterm creates difficulties for the construction.

\(^{(1)}\) For example, in a superconductor there is no such discontinuity.
2. Multi Scale Analysis

We cannot treat the functional integral (I.4) defining the formal Green’s functions in one piece, because the propagator is singular. Similarly it is probably impossible to determine the counterterm \( \delta e(k) \) in one step. Therefore we introduce scales adjusted to the size of the propagator in momentum space and to the infrared cut off propagators \( C^{IR(j)}(k_0, k; \delta e) \) of Definition I.4, construct an appropriate counterterm for each scale \( j \) and take the limit \( j \to \infty \). The limit is controlled by comparing, for each \( j \), the model with covariance \( C^{IR(j+1)}(k_0, k; \delta e) \) to that with covariance \( C^{IR(j)}(k_0, k; \delta e) \). This comparison amounts to “integrating out scale \( j \)”. We give an introduction to “integrating out a scale” in the next subsection. In §III, we describe, formally but in more detail, how the limit \( j \to \infty \) is taken.

3. Integrating out a Scale

The discussion of the previous subsection shows that the essential estimates in our construction concern the effect of integrating out one scale as above. To simplify the discussion, we consider the case in which the covariance \( C^{IR(j)}(k; \delta e) \) is replaced by \( C^{IR(j)}(k; 0) \). That is, for the moment, we ignore the effect of the counterterm. Since \( C^{IR(j+1)}(k; 0) = C^{IR(j)}(k; 0) + C^{(j)}(k) \) with \( C^{(j)}(k) = \frac{\delta^{(j)}(k)}{i k_0 - e(k)} \),

\[
G_{j+1}(\phi, \bar{\phi}) = \log \frac{1}{Z} \int e^{\phi J(\psi + \zeta) + V(\psi + \zeta, \bar{\psi} + \bar{\zeta})} d\mu_{C^{IR(j)}}(\zeta, \bar{\zeta}) d\mu_{C^{(j)}}(\psi, \bar{\psi})
= \log \frac{1}{Z} \int e^{\phi J(\psi + V(\psi, \bar{\psi}))} d\mu_{C^{(j)}}(\psi, \bar{\psi})
\]

where

\[
W(\phi, \bar{\phi}, \psi, \bar{\psi}) = \int e^{\phi J(\psi + \zeta) + V(\psi + \zeta, \bar{\psi} + \bar{\zeta})} d\mu_{C^{IR(j)}}(\zeta, \bar{\zeta})
\]

is the effective interaction at scale \( j \). The partition functions \( Z, Z' \) and \( Z_j \) are chosen so that \( G_{j+1}(0, 0) = W(0, 0, 0, 0) = 0 \). To iterate this construction, we need an effective interaction

\[
W'(\phi, \bar{\phi}, \psi, \bar{\psi}) = \log \frac{1}{Z_{j+1}} \int e^{\phi J(\psi + \zeta) + W(\phi, \bar{\phi}, \psi + \zeta, \bar{\psi} + \bar{\zeta})} d\mu_{C^{IR(j+1)}}(\zeta, \bar{\zeta})
\]

at scale \( j + 1 \). So the problem is to estimate

\[
W'(\phi, \bar{\phi}, \psi, \bar{\psi}) = \log \frac{1}{Z} \int e^{\phi J + W(\phi, \bar{\phi}, \psi + \zeta, \bar{\psi} + \bar{\zeta})} d\mu_{C^{(j)}}(\zeta, \bar{\zeta})
\]

in terms of estimates on \( W \). The main difficulties already occur when \( \phi = \bar{\phi} = 0 \), so we concentrate on this special case. Write

\[
W(0, 0, \psi, \bar{\psi}) = \sum_{n \geq 0} \int dp_1 \cdots dp_n dq_1 \cdots dq_n w_{2n}(p_1, \ldots, p_n, q_1, \ldots, q_n) \delta(p_1 + \cdots + p_n - q_1 - \cdots - q_n) \bar{\psi}(p_1) \cdots \bar{\psi}(p_n) \psi(q_1) \cdots \psi(q_n)
\]
and
\[ \mathcal{W}'(0, 0, \psi, \bar{\psi}) = \sum_{n \geq 0} \int dp_1 \cdots dp_n dq_1 \cdots dq_n \ w_{2n}'(p_1, \ldots, p_n, q_1, \ldots, q_n) \delta(p_1 + \cdots + p_n - q_1 - \cdots - q_n) \bar{\psi}(p_1) \cdots \bar{\psi}(p_n) \ \psi(q_1) \cdots \psi(q_n) \]
where \( \psi(k) \) and \( \bar{\psi}(k) \) are the Fourier transforms of \( \psi(x) \) and \( \bar{\psi}(x) \) respectively\(^\text{(2)}\).

Then \( w_{2n}' \) can be written as a sum of values of connected directed graphs with vertices \( w_2, w_4, \cdots \) and propagator \( C^{(j)}(k) \). See [FW, Chapter 3]. Naive power counting just uses that
\[ \|C^{(j)}(k)\|_\infty = \sup_k |C^{(j)}(k)| \text{ is of order } M^j \] (II.1)

and
\[ \text{volume of the } j^{\text{th}} \text{ shell is of order } \frac{1}{M^{2j}} \] (II.2)

This is because the \( j^{\text{th}} \) shell has width of order \( \frac{1}{M^{2j}} \) in the \( k_0 \) direction and in the \( k \)-direction transversal to \( F \) and has circumference, in the direction along \( F \), of order one. If one assumes that
\[ \|w_{2n}\|_\infty \text{ is of order } M^j(n-2) \text{ for all } n \] (II.3)
then every graph contributing to \( w_{2n}' \) is again of order
\[ M^j \Sigma_i(n_i-2) M^j \Sigma_i((2n_i-2n)/2) M^{-2j} \Sigma_i(2n_i-2n)/2 - \Sigma_i 1+1 = M^j(n-2) \]

The three factors come from the suprema of the vertex functions \( w_{2n_i} \), the suprema of the \( \Sigma_i(2n_i-2n) \) propagators and the volume of the domain of integration respectively. For \( n \geq 3 \) the Condition (II.3) grows nicely in \( j \). Four legged vertices \( (n = 2) \) are marginal — the estimate (II.3) alone would lead to divergences in powers of \( j \) when the sum over \( j \) is performed. We exploit “overlapping loops” and special estimates on ladders to derive better bounds on the four point functions. The counterterm is built up from contributions at each scale chosen so that (II.3) holds for \( n=1 \), too. Overlapping loops will be discussed in subsection 4, ladders in subsections 5 and 6, and the construction of the counterterm in subsection 10.

It is well known that the sum of the norms of all graphs diverges. In this paper we use cancellations between different graphs to derive convergent bounds. There are several schemes to implement such cancellations, all variants of the basic scheme of [C]. In all of these schemes the cancellation is only seen when one writes the model in position space variables. This is a major technical difficulty, since the geometry of the Fermi surface and special effects like “overlapping loops” and ladder estimates are naturally formulated in momentum space. We

\(^{\text{(2)}}\) Precise Fourier transform conventions are formulated in §VI.
use the cancellation scheme developed in [FMRT, FKTcf, FKTr1, FKTr2]. It is sufficiently close to the graphical picture to allow us to implement overlapping loops in collections of diagrams within which cancellations take place.

4. Overlapping Loops

We first describe the effect of overlapping loops in an example. Consider the diagram

contributing to $w'_4(p_1, p_2, q_1, q_2)$. If $w_4 = 1$, its value is

$$\Gamma(p_1, p_2, q_1, q_2) = \int dk_1 dk_2 C^{(j)}(k_1 + q_1 - p_1) C^{(j)}(k_1) C^{(j)}(k_1 - k_2) C^{(j)}(q_2 - k_2)$$

Naive power counting gives

$$\|\Gamma\|_\infty \leq \|C^{(j)}\|_\infty^4 \cdot \text{(volume of the } j^{\text{th}} \text{ shell)}^2 = O(1)$$

However, taking into account that $C^{(j)}$ is supported on the $j^{\text{th}}$ shell,

$$\|\Gamma\|_\infty \leq \|C^{(j)}\|_\infty^4 \cdot \text{(volume of } T)$$

where

$$T = \{(k_1, k_2) \mid k_1, k_1 - k_2, q_2 - k_2 \in j^{\text{th}} \text{ neighbourhood}\}$$

Let $S$ be the set of all $k_2$ for which $q_2 - k_2$ lies in the $j^{\text{th}}$ neighbourhood and $S'$ be the set of all $k_2$ for which $|k_2| \geq \frac{1}{\sqrt{M_j}}$. Then $T \subset T_1 \cup T_2$ where

$$T_1 = \{(k_1, k_2) \mid k_1, k_1 - k_2 \in j^{\text{th}} \text{ neighbourhood}, k_2 \in S \cap S'\}$$

$$T_2 = \{(k_1, k_2) \mid k_1 \in j^{\text{th}} \text{ neighbourhood}, k_2 \in S \setminus S'\}$$

For $k_2 \in S'$ the volume of the set

$$\{k_1 \mid k_1, k_1 - k_2 \in j^{\text{th}} \text{ neighbourhood}\} = (j^{\text{th}} \text{ neighbourhood}) \cap (k_2 + j^{\text{th}} \text{ neighbourhood})$$
is of order \( \frac{1}{M^{j/2}} \cdot \frac{1}{M^{j/2}} \), since this set has width of order \( \frac{1}{M^{j}} \) in \( k_0 \) direction and in the \( k \) direction transversal to \( F \) and width at most of order \( \frac{1}{\sqrt{M^{j}}} \) in the direction along \( F \). Here we use that the Fermi surface \( F \) has curvature bounded away from zero. Therefore the volume of \( T_1 \) is of order

\[
\frac{1}{M^{j/2}} \cdot (\text{volume of } S) = \frac{1}{M^{j/2}} \cdot O\left(\frac{1}{M^{j/2}}\right) = O\left(\frac{1}{M^{j/2}}\right)
\]

Similarly the volume of \( T_2 \) is bounded by

\[
(\text{volume of } j^{th} \text{ neighbourhood}) \cdot (\text{volume of } S \setminus S') = O\left(\frac{1}{M^{j/2}}\right) \cdot \frac{1}{M^{j/2}} = O\left(\frac{1}{M^{j/2}}\right)
\]

Therefore, using (II.1), \( ||\Gamma||_\infty = O\left(\frac{1}{\sqrt{M^{j}}}\right) \). The volume estimate on \( T \) derived above is not optimal. By [FST2, Theorem 1.1], the volume of \( T \) is \( O\left(\frac{1}{M^{j/2}}\right) \).

Similar improvements are possible for all diagrams with “overlapping loops”, i.e. with two different simple loops which have at least one line in common. If \( w_2 = 0 \), the only four legged diagrams without overlapping loops and tadpoles are particle–particle ladders

\[\text{particle–particle ladder}\]

and particle–hole ladders

\[\text{particle–hole ladder}\]

See [FST1, §2.4]. We shall Wick order the effective interactions with respect to the future covariance in order to exclude tadpoles. The condition that \( w_2 = 0 \) is achieved by moving the two legged part of the effective interaction into the covariance.

We have already mentioned that the effect of overlapping loops and special ladder estimates are used to get convergent bounds on the four point functions. The effect of overlapping
loops has to be combined with the cancellation scheme between diagrams mentioned at the end of subsection 3 and discussed in subsection 9 below. Thus the cancellation scheme has to be sensitive enough to detect simultaneous overlapping loops in all mutually cancelling diagrams and to isolate ladder diagrams. Furthermore the geometric estimates on $T$ above have to be exploitable in a position space setting. This is done using sectors. See subsection 8.

5. **Particle–Particle Bubbles**

The strong asymmetry condition of Definition I.10 is used to get improved power counting on particle–particle ladders. We describe the effect in the example of the particle–particle bubble

\[
\begin{array}{c}
p_1 \\
p_2 \\
k \\
q_1 \\
q_2
\end{array}
\]

again assuming that $w_4 = 1$. The value of this graph is

\[
\int dk C^{(j)}(t - k) C^{(j)}(k)
\]

where $t = p_1 + p_2 = q_1 + q_2$ is the transfer momentum. Naive power counting again gives that this value is $O(1)$. On the other hand, taking the support condition of $C^{(j)}$ into account, the value of this graph is bounded by

\[
\|C^{(j)}\|_\infty^2 \cdot \text{(volume of } \{k \mid k \text{ and } t - k \text{ lie in the } j^{\text{th}} \text{ neighbourhood}\}
\]

By the strong asymmetry condition of Definition I.10, $F$ and the shifted reflected Fermi surface $t - F = \{t - k \mid k \in F\}$ have tangency of order at most $n_0$. From this one deduces that

\[
\{k \mid k, t - k \in j^{\text{th}} \text{ neighbourhood}\} = (j^{\text{th}} \text{ neighbourhood}) \cap (t - (j^{\text{th}} \text{ neighbourhood}))
\]

has width of order $\frac{1}{M^j}$ in $k_0$ direction and in the $k$ direction transversal to $F$ and width at most of order $\frac{1}{M^{j/n_0}}$ in the direction along $F$. Therefore its volume is of order $\frac{1}{M^j} \frac{1}{M^{j/n_0}}$ and

\[
\begin{array}{c}
t - (j^{\text{th}} \text{ neighbourhood}) \\
\end{array}
\]

\[
\begin{array}{c}
j^{\text{th}} \text{ neighbourhood}
\end{array}
\]
the value of the particle–particle bubble is of order \( \frac{1}{M^{j/n_0}} \).

As in the case of overlapping loops, this estimate is based on a volume estimate in momentum space. Again sectors will be used to implement this in position space variables.

6. Particle–Hole Ladders

Our estimate on particle–hole ladders is not based on a geometric argument as in the case of particle–particle ladders or overlapping loops, but on cancellations between scales.

The limit as \( j \to \infty \) of the particle–hole bubble \( B_j(p_1, p_2, q_1, q_2) \)

\[
\begin{array}{c}
p_1 \quad q_1 \\
\downarrow \\
p_2 \quad q_2
\end{array}
\]

with \( w_4 = 1 \) and propagator \( \sum_{i \leq j} C^{(i)} \) has a discontinuity in the transfer momentum \( t = p_1 - q_1 \) at \( t = 0 \), but is continuous for \( t \neq 0 \) and smooth in a neighbourhood of the origin in \( \{(t_0, t) \mid t_0 = 0\} \). See the introduction to [FKTl], and in particular Lemma I.1 there. The proof of this Lemma is based on integration by parts and thus cancellation between scales.

In the multi scale analysis, we combine all the contributions of particle–hole ladders created at scales \( \leq j \), and give a uniform estimate on the result. A vertex of a particle–hole ladder at scale \( j \) may be a particle–hole ladder created at a previous scale \( i < j \), as in the diagram

\[
\begin{array}{c}
p_1 \quad q_1 \\
\downarrow \\
p_2 \quad q_2
\end{array}
\]

The iteration of these effects (and of the Wick ordering with respect to future covariances) leads to the concept of iterated particle–hole ladders of Definition VII.7. Uniform estimates on these iterated particle–hole ladders are stated in Theorem VII.8 and proven in [FKTl]. They have to be in position space, as they have to be combined with the other estimates derived from the “cancellation scheme” mentioned in subsection 3. This is technically difficult, because it amounts to taking Fourier transforms of quantities whose limit, as \( j \to \infty \), is discontinuous.
Proving power counting bounds on graphs is usually split into two steps. To illustrate the two steps, we consider the situation that we have two vertices \( \varphi_1 \) and \( \varphi_2 \) with \( n_1 \) and \( n_2 \) legs, respectively. For simplicity we ignore orientation of the lines. We assume we form a diagram \( \Gamma \) by connecting \( \varphi_1 \) and \( \varphi_2 \) by \( 1 \leq r \leq \min\{n_1, n_2\} \) lines

\[
\Gamma = \equiv \varphi_1 \equiv \varphi_2
\]

This can be done in two steps: First connect \( \varphi_1 \) and \( \varphi_2 \) by one line and call the result \( \Gamma' \).

\[
\Gamma' = \equiv \varphi_1 \equiv \varphi_2
\]

Secondly, pairwise contract \( 2r - 2 \) legs of \( \Gamma' \) to form \( r - 1 \) lines.

The first estimate is to bound the norm of \( \Gamma' \) in terms of a constant times the product of the norms of \( \varphi_1 \) and \( \varphi_2 \). We call such a constant a “contraction bound” \( c \). If one uses \( \| \cdot \|_\infty \) norms in momentum space, then \( \| C^{(j)}(k) \|_\infty \) is a contraction bound.

A tadpole bound is a number \( b \) with the following property. Let \( \varphi \) be any graph with at least two legs, and \( \varphi' \) the graph obtained from \( \varphi \) by connecting two legs to form a line.

Then the norm of \( \varphi' \) is bounded by \( b \) times the norm of \( \varphi \). If one uses \( \| \cdot \|_\infty \) norms in momentum space, then \( \| C^{(j)}(k) \|_1 \) is a tadpole bound.

Applying one contraction bound and \( r - 1 \) tadpole bounds, one sees that the norm of \( \Gamma \) is bounded by \( cb^{r-1} \) times the product of the norms of \( \varphi_1 \) and \( \varphi_2 \). By (II.1) and (II.2), for the \( \| \cdot \|_\infty \) norms in momentum space, the contraction bound is of order \( M^j \), while the tadpole bound is of order \( \frac{1}{M^r} \). The power counting for the \( \| \cdot \|_\infty \) norm in momentum space described in subsection 3 can be generalized to the abstract setting of contraction and tadpole bounds: If one assumes that

\[
\text{the norm of } w_{2n} \text{ is of order } \frac{1}{e^{n/\rho}} \text{ for all } n
\]
then every graph contributing to \( w'_{2n} \) is again of order \( \frac{1}{c b^{n}} \). For example, if such a graph \( \Gamma \) has two vertices, \( w_{2n_{1}} \) and \( w_{2n_{2}} \) then there are \( r = n_{1} + n_{2} - n \) connecting lines and the norm of \( \Gamma \) is bounded by

\[
\text{cb}^{n_{1}+n_{2}-n-1} \left( \text{norm of } w_{2n_{1}} \right) \left( \text{norm of } w_{2n_{2}} \right) = O \left( \text{cb}^{n_{1}+n_{2}-n-1} \frac{1}{c b^{n_{1}-1}} \frac{1}{c b^{n_{2}-1}} \right)
\]

\[
= O \left( \frac{1}{c b^{n-1}} \right)
\]

A general graph may be bounded by building it up one vertex at a time. As pointed out in subsection 3, we need to use position space variables. A natural position space norm for translation invariant vertices \( \varphi(x_{1}, \cdots, x_{n}) \), which mimics the \( \| \cdot \|_{\infty} \) norm in momentum space, is

\[
\| \varphi \|_{1,\infty} = \max_{1 \leq p \leq n} \sup_{x_{p}} \int \prod_{j \neq p} dx_{j} \, |\varphi(x_{1}, \cdots, x_{n})|
\]

It is easily seen that the \( L^{1} \) norm, \( \|C^{(j)}(x)\|_{1} \), of the Fourier transform of \( C^{(j)}(k) \) is a contraction bound for this norm and that the \( L^{\infty} \) norm of \( C^{(j)}(x) \) in position space, \( \|C^{(j)}(x)\|_{\infty} \), is a tadpole bound. Clearly, \( \|C^{(j)}(x)\|_{\infty} \leq \|C^{(j)}(k)\|_{1} \), so that we again have a tadpole bound of order \( \frac{1}{M^{j}} \). A naive computation, given in the next paragraph, gives a bound on \( \|C^{(j)}(x)\|_{1} \) that is of order \( M^{2j} \). A more refined argument, sketched in the next but one paragraph, gives a (realistic — see Example A.1) bound of order \( M^{3j/2} \). In any event, \( M^{3j/2} \gg \|C^{(j)}(k)\|_{\infty} \) and naive power counting in position space does not coincide with power counting in momentum space. Substituting \( c = O(M^{3j/2}) \) and \( b = O \left( \frac{1}{M^{j}} \right) \) into (II.5) yields the requirement that \( \|\hat{w}_{2n}\|_{1,\infty} \) be order \( M^{j(n-\frac{2}{j})} \). In particular the norm of the four point function would have to decrease like \( \frac{1}{\sqrt{M^{j}}} \) as \( j \) increased. This is absurd, since the original interaction \( V \) is, at each scale, the dominant part of the four point function. Again, we use sectors to cope with this problem.

We first sketch the standard calculation that gives the naive bound on \( \|C^{(j)}(x)\|_{1} \). For a multi index \( \delta = (\delta_{0}, \delta_{1}, \delta_{2}) \) of non negative integers write \( |\delta| = \delta_{0} + \delta_{1} + \delta_{2} \) and \( x^{\delta} = x_{0}^{\delta_{0}} x_{1}^{\delta_{1}} x_{2}^{\delta_{2}} \). Then, integrating by parts \( |\delta| \) times,

\[
\left( \frac{x}{M^{j}} \right)^{|\delta|} |C^{(j)}(x)| \leq \frac{1}{M^{j|\delta|}} \| \frac{\partial^{|\delta|}}{\partial k^{\delta}} C^{(j)}(k) \|_{1} = O \left( \frac{1}{M^{j}} \right)
\]

since the support of \( \frac{\partial^{|\delta|}}{\partial k^{\delta}} C^{(j)}(k) \) has volume of order \( \frac{1}{M^{2j}} \) and \( \| \frac{\partial^{|\delta|}}{\partial k^{\delta}} C^{(j)}(k) \|_{\infty} \) is of order \( M^{2(|\delta|+1)} \). Therefore

\[
\left( 1 + \left( \frac{x}{M^{j}} \right)^{2} \right) \left( 1 + \left( \frac{x}{M^{j}} \right)^{2} \right) \left( 1 + \left( \frac{x}{M^{j}} \right)^{2} \right) |C^{(j)}(x)| = O \left( \frac{1}{M^{j}} \right)
\]

Dividing by \( \prod_{\nu=0,1,2} \left( 1 + \left( \frac{x}{M^{j}} \right)^{2} \right) \) and integrating over \( \mathbb{R}^{3} \) gives the bound \( \|C^{(j)}(x)\|_{1} = O(M^{2j}) \).
As it motivates the construction of sectors, we indicate how the more refined bound on $\|C^{(j)}(x)\|_1$ is derived. Assume first that $F$ has a straight segment of length $l$ on the $k_2$ axis\(^{(3)}\) and that $e(k) = k_1$ in a neighbourhood of this segment. Choose a cutoff function $\chi(k_2)$ that is identically one on most of the segment, zero outside the the segment and for which $|\frac{\partial^n}{\partial k^{\alpha}} \chi(k_2)|$ is of order $\frac{1}{l^n}$. Set $C^{(j)}_s(k) = \chi(k_2)C^{(j)}(k)$. An argument similar to that of the previous paragraph shows that

$$
\left(1 + \left(\frac{x_1}{M^j}\right)^2\right)\left(1 + \left(\frac{x_2}{M^j}\right)^2\right)\left(1 + (1l_2)^2\right)|C^{(j)}_s(x)| = O\left(\frac{1}{M^j}\right)
$$

and therefore that $\|C^{(j)}_s(x)\|_1 = O(M^j)$. The same argument also works for a realistic Fermi surface, if one cuts out a “sector” of length $l \leq \frac{1}{\sqrt{M^j}}$ as indicated in the figure below.

The precise computation is in [FKTo3, Proposition XIII.1 and Lemma XIII.2]. If the sector is too long, the curvature of the Fermi surface causes deterioration of the bounds on the derivatives parallel to the Fermi surface. One can divide up the $j$th neighbourhood into $O\left(\frac{1}{l}\right)$ sectors and use a partition of unity by functions like $\chi$, to see that

$$
\|C^{(j)}(x)\|_1 \leq (\text{number of sectors}) \cdot O(M^j) = O\left(\frac{1}{l}M^j\right)
$$

If one chooses $l = \frac{1}{\sqrt{M^j}}$, one gets the bound $\|C^{(j)}(x)\|_1 = O(M^{3j/2})$.

8. **Sectors**

We cover the $j$th neighbourhood by slightly overlapping sectors of length $l \leq \frac{1}{\sqrt{M^j}}$ as indicated in the figure below.

The set $\Sigma$ of sectors is called a sectorization of scale $j$ and length $l$. Furthermore we select a partition of unity $\{\chi_s(k) | s \in \Sigma\}$ subordinate to the sectorization, such that each $\chi_s$ has

\(^{(3)}\) This of course contradicts our hypotheses that $F$ is strictly convex. We will remove this assumption immediately.
properties analogous to those of the function χ in the last subsection\(^{(4)}\). Recall that we want to integrate out scale \(j\) and that

\[
\mathcal{W'}(0, 0, \psi, \bar{\psi}) = \frac{1}{2} \int e^{\mathcal{W}(0, 0, \psi + \zeta, \bar{\psi} + \bar{\zeta})} d\mu_{C(j)}(\zeta, \bar{\zeta})
\]

Decompose the kernel \(w_{2n}\) of the part of \(\mathcal{W}\) that is homogeneous of degree 2\(n\) in the fields

\[
w_{2n}(p_1, \ldots, p_n, q_1, \ldots, q_n) = \sum_{s_1, \ldots, s_{2n} \in \Sigma} \omega_{2n}(\{p_1, s_1\}, \ldots, (p_n, s_n), (q_1, s_{n+1}), \ldots, (q_n, s_{2n}))
\]

for \(p_i, q_i\) in the \(j\)th neighbourhood. Here, \(\omega_{2n}\) is a function that vanishes unless \(p_i\) lies in the sector \(s_i\) and \(q_i\) lies in the sector \(s_{n+i}\). \(\omega_{2n}\) is called a \(\Sigma\)–sectorized representative of \(w_{2n}\). A \(\Sigma\)–sectorized representative \(\omega'_{2n}\) of \(w'_{2n}\) can then be written as a sum of values of connected directed graphs with vertices \(\omega_2, \omega_4, \ldots\) and propagator \(C^{(j)}(k)\), where the momentum integrals in the graphs also include sector sums. The main norm we use for the Fourier transforms \(\hat{\omega}_{2n}((x_1, s_1), \ldots, (x_{2n}, s_{2n}))\) of \(\omega_{2n}\) is

\[
\|\hat{\omega}_{2n}\|_{1, \Sigma} = \max_{1 \leq i_0 \leq 2n} \max_{s_{i_0} \in \Sigma} \sum_{s_i \in \Sigma, \text{ for } i \neq i_0} \|\hat{\omega}_{2n}((x_1, s_1), \ldots, (x_{2n}, s_{2n}))\|_{1, \infty}
\] (II.6)

That is, we first fix one sector and then take the sum over all other sectors of the \(\|\cdot\|_{1, \infty}\) norms of the functions \(\hat{\omega}_{2n}((\cdot, s_1), \ldots, (\cdot, s_{2n}))\). With respect to this norm, we have a contraction bound of order \(M^j\) and a tadpole bound of order \(\frac{1}{M^j}\).

We first indicate how the contraction bound is derived. Let \(\varphi_1((x_1, s_1), \ldots, (x_n, s_n))\) and \(\varphi_2((x_1, s_1), \ldots, (x_m, s_m))\) be vertices, and

\[
\Gamma'(x_1, s_1, \ldots, (x_{n-1}, s_{n-1}), (x'_2, s'_2), \ldots, (x'_m, s'_m))
\]

be the graph constructed by connecting \(\varphi_1\) and \(\varphi_2\) with one line. Write \(C^{(j)}(k) = \sum_{s \in \Sigma} C_s^{(j)}(k)\) where \(C_s^{(j)}(k) = \chi_s(k) C^{(j)}(k)\). Let \(C_s^{(j)}(x)\) be the Fourier transform of \(C_s^{(j)}(k)\). Then

\[
\Gamma' = \sum_{s_n, s'_s \in \Sigma} \int dx_n dx'_1 \varphi_1((x_1, s_1), \ldots, (x_n, s_n)) C_s^{(j)}(x_n-x'_1) \varphi_2((x'_1, s'_1), \ldots, (x'_m, s'_m))
\] (II.7)

By conservation of momentum, the integral in (II.7) vanishes if \(s_n \cap s \cap s'_1 = \emptyset\). For fixed sectors \(s_1, \ldots, s_n, s'_1, \ldots, s'_m, s, \) by double convolution

\[
\|\int dx_n dx'_1 \varphi_1((x_1, s_1), \ldots, (x_n, s_n)) C_s^{(j)}(x_n-x'_1) \varphi_2((x'_1, s'_1), \ldots, (x'_m, s'_m))\|_{1, \infty} \leq \|\varphi_1((\cdot, s_1), \ldots, (\cdot, s_n))\|_{1, \infty} \|C_s^{(j)}(x)\|_1 \|\varphi_2((\cdot, s'_1), \ldots, (\cdot, s'_m))\|_{1, \infty}
\] (II.8)

\(^{(4)}\) For precise Definitions of sectors, sectorizations, and the partition of unity see Definition VI.2 through Definition VI.5.
We consider the contribution to \( \|\Gamma'\|_{1,\Sigma} \) having the sector \( s_1 \) fixed. By (II.7), (II.8) and conservation of momentum it is bounded by

\[
\sum_{s_2,\ldots,s_{n-1} \in \Sigma} \sum_{s_n, s_n' \in \Sigma} \|\varphi_1((\cdot,s_1),\ldots,(\cdot,s_n))\|_{1,\infty} \|C_s^{(j)}(x)\|_1 \|\varphi_2((\cdot,s_1'),\ldots,(\cdot,s_{n'}))\|_{1,\infty}
\]

(II.9)

Observe that, for given \( s_n \), there are at most three sectors \( s \) and at most three sectors \( s_n' \) with \( s_n \cap s \neq \emptyset \), \( s_n \cap s_n' \neq \emptyset \). Therefore (II.9) is bounded by

\[
9 \max_{s,s_n' \in \Sigma} \left| \sum_{s_2,\ldots,s_{n-1}, s_n} \|\varphi_1((\cdot,s_1),\ldots,(\cdot,s_n))\|_{1,\infty} \|C_s^{(j)}(x)\|_1 \|\varphi_2((\cdot,s_1'),\ldots,(\cdot,s_{n'}))\|_{1,\infty} \right|
\]

\[\leq 9 \|\varphi_1\|_{1,\Sigma} \|\varphi_2\|_{1,\Sigma} \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_1 \]

Fixing \( s_i \) with \( 2 \leq i \leq n - 1 \) or \( s_{i}' \) with \( 2 \leq i \leq m \) leads to the same bound, so that

\[
\|\Gamma'\|_{1,\Sigma} \leq 9 \|\varphi_1\|_{1,\Sigma} \|\varphi_2\|_{1,\Sigma} \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_1
\]

(II.10)

As at the end of the previous subsection, one sees that \( \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_1 = O(M^j) \). This gives the contraction bound of order \( M^j \).

To derive the tadpole bound, let \( \varphi((x_1,s_1),\ldots,(x_n,s_n)) \) be a vertex and

\[
\Gamma((x_1,s_1),\ldots,(x_{n-2},s_{n-2}))
\]

\[= \sum_{s_{n-1},s_n \in \Sigma} \int dx_{n-1} dx_n \varphi((x_1,s_1),\ldots,(x_{n-2},s_{n-2}),(x_{n-1},s_{n-1}),(x_n,s_n)) C_s^{(j)}(x_{n-1}-x_n)
\]

be obtained by joining the last two legs of \( \varphi \) to form a tadpole. As above, by conservation of momentum, for each choice of sectors \( s_1,\ldots,s_{n-2} \)

\[
\|\Gamma((\cdot,s_1),\ldots,(\cdot,s_{n-2}))\|_{1,\infty}
\]

\[\leq \sum_{s_{n-1},s_n \in \Sigma} \int dx_{n-1} dx_n \varphi((\cdot,s_1),\ldots,(\cdot,s_{n-2}),(x_{n-1},s_{n-1}),(x_n,s_n)) C_s^{(j)}(x_{n-1}-x_n)\|_{1,\infty}
\]

\[\leq 3 \sum_{s_{n-1},s_n \in \Sigma} \|\varphi((\cdot,s_1),\ldots,(\cdot,s_{n-2}),(\cdot,s_{n-1}),(\cdot,s_n))\|_{1,\infty} \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_\infty
\]

as, for a given sector \( s_{n-1} \), there are at most three sectors \( s \) for which \( s \cap s_{n-1} \neq \emptyset \). Consequently

\[
\|\Gamma\|_{1,\Sigma} \leq 3 \|\varphi\|_{1,\Sigma} \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_\infty
\]

As in the previous subsection, \( \|C_s^{(j)}(x)\|_\infty = O(\frac{1}{M^j}) \) and the tadpole bound of order \( \frac{1}{M^j} \) follows.
Substituting \( c = O(M^j) \) and \( b = O\left(\frac{1}{M^r}\right) \) into (II.5) yields the requirement that

\[
\|\hat{\omega}_{2n}\|_{1,\Sigma} \text{ is of order } \frac{M^{j(n-2)}}{t-1} \text{ for all } n \quad (\text{II.11})
\]

In contrast to the norm \( \| \cdot \|_{1,\infty} \) of subsection 7, this norm is compatible with change of scale. In \( \S \text{VI} \), we choose, for each scale \( i \), a sectorization \( \Sigma_i \) of length \( l_i \) where \( l_i \) goes to zero as some power of \( \frac{1}{M} \)\(^{(5)}\). In going from scale \( j \) to scale \( j+1 \), we construct from a \( \Sigma_j \)-sectorized representative \( \omega'_{2n} \) of \( w_{2n}^j \) a \( \Sigma_{j+1} \)-sectorized representative \( \omega''_{2n} \) of \( w_{2n}^j \) that fulfills (II.11) with \( j \) replaced by \( j+1 \) and \( l \) replaced by \( l_{j+1} \). It is constructed using a partition of unity subordinate to \( \Sigma_{j+1} \). See Example A.3.

To give an idea of the underlying mechanism, we show that the problem with the \( \| \cdot \|_{1,\Sigma} \) norm of the four point function described in subsection 7 does not occur for the \( \| \cdot \|_{1,\Sigma} \) norm. To do so, we need a \( \Sigma = \Sigma_j \)-sectorized representative for the original interaction kernel \( V(p_1, p_2, q_1, q_2) \) whose \( \| \cdot \|_{1,\Sigma} \) norm is of order \( \frac{1}{t} \). A natural such representative is

\[
v((p_1, s_1), (p_2, s_2), (q_1, s_3), (q_2, s_4)) = \chi_{s_1}(p_1) \chi_{s_2}(p_2) \chi_{s_3}(q_1) \chi_{s_4}(q_2) V(p_1, p_2, q_1, q_2)
\]

By conservation of momentum, the Fourier transform \( \hat{v}((\cdot, s_1), (\cdot, s_2), (\cdot, s_3), (\cdot, s_4)) \) vanishes if \( (s_1 + s_2) \cap (s_3 + s_4) = \emptyset \). Here, \( s_1 + s_2 = \{ p_1 + p_2 \mid p_1 \in s_1, \ p_2 \in s_2 \} \).

One sees, by the same method that yielded \( \| C_{s}(x) \|_1 = O(M^j) \), that the Fourier transform of each \( \chi_{s}(k) \) fulfills \( \| \hat{\chi}_{s}(x) \|_1 = O(1) \). Therefore, for each choice of sectors \( s_1, s_2, s_3, s_4 \),

\[
\| \hat{\chi}((\cdot, s_1), (\cdot, s_2), (\cdot, s_3), (\cdot, s_4)) \|_{1,\infty} = O(\| \hat{V} \|_{1,\infty})
\]

Therefore, \( \| \hat{v} \|_{1,\Sigma} \) is of the order

\[
\max_{s_1 \in \Sigma} \# \{ (s_2, s_3, s_4) \in \Sigma^3 \mid (s_1 + s_2) \cap (s_3 + s_4) \neq \emptyset \} \quad (\text{II.12})
\]

To estimate this number, fix a sector \( s_1 \). Observe that the map

\[
F \times F \to \mathbb{R}^2, \ (k_1, k_2) \mapsto k_1 + k_2 \quad (\text{II.13})
\]

\(^{(5)}\) This is forced on us by the condition \( l_i \leq \frac{1}{\sqrt{M^i}} \), which is imposed to ensure that the curvature of the sector does not affect Fourier transform estimates.
is locally invertible at every \((k_1, k_2) \in F \times F\) for which the tangent vector to \(F\) at \(k_1\) is not parallel to the tangent vector to \(F\) at \(k_2\). From this one concludes that for a general choice of \(s_2\) (such that \(s_1 + s_2\) is not close to \(2k\) or \(k + \text{antipode}(k)\) for all \(k \in F\)), there are only \(O(1)\) choices of sectors \(s_3, s_4\) such that \((s_1 + s_2) \cap (s_3 + s_4) \neq \emptyset\). Since there are \(O(\frac{4}{l})\) sectors, the contribution to (II.12) from all general sectors \(s_2\) is \(O\left(\frac{4}{l}\right)\). One can see that the few other sectors \(s_2\) also only contribute \(O\left(\frac{4}{l}\right)\). Thus, indeed \(||\hat{v}|||_{1, \Sigma} = O\left(\frac{1}{l}\right)\). The precise argument is given in [FKTo4, Proposition XIX.1].

The estimates on the contraction and tadpole bounds and about change of sectorization in this subsection all are specific to two space dimensions. In three space dimensions, similar arguments would give that

- a contraction bound for \(C^{(j)}\) is of order \(M^j\)
- a tadpole bound for \(C^{(j)}\) is of order \(\frac{l^2}{M^j}\)

\[|||\hat{v}|||_{1, \Sigma} = O\left(\frac{1}{l^3}\right)\], since the map (II.13) has one dimensional fibers in this case.

Thus, for the case of three space dimensions, the power counting suggested by sectors is not compatible with change of sectorization. This is the reason why, in this paper, we restrict ourselves to two space dimensions\(^{(6)}\).

The advantage of sectors is that they allow exploitation of conservation of momentum, while working in position space. One example is the estimate on \(|||\hat{v}|||_{1, \Sigma}\) derived two paragraphs ago. Another important example is that the improvements due to overlapping loops and particle–particle bubbles described in subsections 4 and 5 can be implemented using sectors. To this end we use a variant of the norm (II.6), with three sectors held fixed,

\[|||\hat{\omega}_{2n}|||_{3, \Sigma} = \max_{1 \leq i_1 < i_2 < i_3 \leq 2n} \max_{s_{i_1}, s_{i_2}, s_{i_3} \in \Sigma} \sum_{s_j \in \Sigma \text{ for } j \neq i_1, i_2, i_3} |||\hat{\omega}_{2n}((x_{1,s_1}, \ldots, (x_{2n,s_{2n}}))|||_{1, \infty} \quad (II.14)\]

Its power counting is better by one factor of \(l\) than the power counting of the \(||| \cdot |||_{1, \Sigma}\) norm. That is, we expect

\[|||\hat{\omega}_{2n}|||_{3, \Sigma} \text{ is of order } \left(\frac{M^j}{l}\right)^{n-2} \text{ for all } n \quad (II.15)\]

and in particular that \(|||\hat{\omega}_4|||_{3, \Sigma}\) is of order one\(^{(7)}\). This norm is particularly useful for the four point function. Since a given momentum can lie in at most two sectors,

\[|||\omega_4(p_1, p_2, q_1, q_2)|||_{\infty} \leq 2^3|||\hat{\omega}_4|||_{3, \Sigma}\]

In Example A.2, we show how (II.11) and (II.15) can be used to get improved power counting for the \(||| \cdot |||_{1, \Sigma}\) norm of the diagram discussed in subsection 4.

\(^{(6)}\) Progress in the use of sectorization in three space dimensions has been made in [MR, DMR].

\(^{(7)}\) The argument above that the sectorized representative \(v\) of the original interaction fulfills \(|||\hat{v}|||_{1, \Sigma} = O\left(\frac{1}{l}\right)\) also shows that \(|||\hat{v}|||_{3, \Sigma} = O(1)\).
Cancellation Between Diagrams

In subsections 7 and 8, we described how the power counting bound on a diagram $\Gamma$ formed by connecting two vertices $\varphi_1$ and $\varphi_2$ by $r \geq 1$ lines (8) is obtained. First we connect $\varphi_1$ and $\varphi_2$ by one line and call the resulting graph $\Gamma'$ and apply a contraction bound. Then we form the remaining $r-1$ lines, each time applying a tadpole bound. Iterating this procedure, adding vertex after vertex, leads to the power counting for arbitrary diagrams.

Since we are dealing with fermions, we may assume in our discussion that all vertex functions are antisymmetric. Let $n_1 \geq r$ and $n_2 \geq r$ be the number of legs of $\varphi_1$ and $\varphi_2$, respectively. Assume that $\max\{n_1, n_2\} > r$. Denote by $\mathcal{G}$ the set of all diagrams obtained from joining $\varphi_1$ and $\varphi_2$ by $r$ lines. $\mathcal{G}$ has cardinality $\binom{n_1}{r} \binom{n_2}{r} r!$. If one bounds each individual diagram by power counting, and sums over all diagrams in $\mathcal{G}$, this large number of diagrams leads to divergences (due to the factor $r!$).

We first describe how one finds cancellations between diagrams of $\mathcal{G}$, then describe a blocking of diagrams that allows one to find similar cancellations for arbitrary numbers of vertices, and then show how, using this blocking, one may simultaneously exploit both these cancellations and overlapping loops.

The first step in constructing the diagrams of $\mathcal{G}$ is again to choose one leg of $\varphi_1$ and one leg of $\varphi_2$, form a line between these two legs, call the resulting graph $\Gamma'$ and estimate the norm of $\Gamma'$ in terms of the norms of $\varphi_1$ and $\varphi_2$ using a contraction bound. The second step is to choose $(r-1)$ additional legs of $\varphi_1$ and $(r-1)$ additional legs of $\varphi_2$.

$$\Gamma' = \varphi_1 \quad \varphi_2$$

The third step is to form all possible connections between the $(r-1)$ legs of $\varphi_1$ and the $(r-1)$ legs of $\varphi_2$ chosen in the second step. There are $n_1 n_2$ choices in the first step, $\binom{n_1-1}{r-1} \binom{n_2-1}{r-1}$ choices in the second step and $(r-1)!$ choices in the third step. Each diagram of $\mathcal{G}$ is obtained $r$ times, since with the first step we distinguish one of the $r$ lines. Observe that

$$\frac{1}{r} n_1 n_2 \binom{n_1-1}{r-1} \binom{n_2-1}{r-1} (r-1)! = \binom{n_1}{r} \binom{n_2}{r} r!$$

There are cancellations amongst the diagrams formed in the third step described above. For simplicity, assume that the $(r-1)$ legs of $\varphi_1$ chosen in the second step are labeled by $2, \cdots, r$ and are all incoming, and that the $(r-1)$ legs of $\varphi_1$ are labeled by $n_2-r+2, \cdots, n_2$ and are all outgoing. Then the sum of all diagrams formed in the third step is

$$\int dx_2 \cdots dx_r dx'_{n_2-r+2} \cdots dx'_{n_2} \Gamma'(x_2, \cdots, x_r, \cdots, x'_{n_2-r+2}, \cdots, x'_{n_2}) \int \bar{\psi}(x_2) \cdots \bar{\psi}(x_r) \psi(x_{n_2-r+2}) \cdots \psi(x'_{n_2}) d\mu_{\mathcal{C}(\cup)}$$

(8) Again, for simplicity, we ignore orientation of the lines.
The magnitude of the functional integral is

\[ \| \Gamma' \|_{1,\infty} \sup_{x_2', \ldots, x_r'} \left| \int \tilde{\psi}(x_2) \cdots \tilde{\psi}(x_r) \tilde{\psi}(x_{n_2'-r+2}) \cdots \tilde{\psi}(x'_{n_2}) \, d\mu_{C(j)} \right| \]

The magnitude of the functional integral is

\[ \left| \int \tilde{\psi}(x_2) \cdots \tilde{\psi}(x_r) \tilde{\psi}(x_{n_2'-r+2}) \cdots \tilde{\psi}(x'_{n_2}) \, d\mu_{C(j)} \right| = \left| \det [C^{(j)}(x_{\mu}-x_{n_2'-r+\nu})]_{\mu,\nu=2,\ldots,r} \right| \]

Observe that the \( \mu-\nu \) matrix entry

\[ C^{(j)}(x_{\mu}-x_{n_2'-r+\nu}) = \int dk \, e^{i \langle k, x_{\mu}-x_{n_2'-r+\nu} \rangle - C^{(j)}(k)} \]

\[ = \left\langle e^{i \langle k, x_{\mu} \rangle - \sqrt{|C^{(j)}(k)|}}, e^{i \langle k, x_{n_2'-r+\nu} \rangle - \frac{C^{(j)}(k)}{\sqrt{|C^{(j)}(k)|}}} \right\rangle_{L^2} \]

(we are deliberately ignoring some unimportant factors of \( 2\pi \)) is the \( L^2 \) inner product of the vector \( v_{\mu}(k) = e^{i \langle k, x_{\mu} \rangle - \sqrt{|C^{(j)}(k)|}} \) and the vector \( v'_{\nu}(k) = e^{i \langle k, x_{n_2'-r+\nu} \rangle - \frac{C^{(j)}(k)}{\sqrt{|C^{(j)}(k)|}}} \). The vectors \( v_{\mu}, v'_{\nu} \) all have \( L^2 \) norm \( \sqrt{||C^{(j)}(k)||_{L^2}} \). Therefore, by Gram’s bound on determinants,

\[ \left| \int \tilde{\psi}(x_2) \cdots \tilde{\psi}(x_{n_2}) \, d\mu_{C(j)} \right| \leq \| C^{(j)}(k) \|_{1}^{r-1} \]

Thus the bound on the sum of all diagrams formed at the third step is

\[ \| \Gamma' \|_{1,\infty} \| C^{(j)}(k) \|_{1}^{r-1} \leq c \| C^{(j)}(k) \|_{1}^{r-1} \| \varphi_1 \|_{1,\infty} \| \varphi_2 \|_{1,\infty} \]

where \( c \) is a contraction bound. Consequently, the \( \| \cdot \|_{1,\infty} \) norms of the sum of all graphs in \( \mathcal{G} \) is bounded by

\[ \frac{1}{r} n_1 n_2 \left( \frac{n_1-1}{r-1} \right) \left( \frac{n_2-1}{r-1} \right) \| C^{(j)}(x) \|_{1} \| C^{(j)}(k) \|_{1}^{r-1} \| \varphi_1 \|_{1,\infty} \| \varphi_2 \|_{1,\infty} \]

since, as seen in subsection 7, \( \| C^{(j)}(x) \|_{1} \) is a contraction bound. This is smaller than the bound

\[ (z \mathcal{G}) \| C^{(j)}(x) \|_{1} \| C^{(j)}(k) \|_{1}^{r-1} \| \varphi_1 \|_{1,\infty} \| \varphi_2 \|_{1,\infty} \]

derived by summing the graphwise bounds, by a factor of \( \frac{1}{(r-1)!} \).

In general, we say that \( b \) is an integral bound, if the following holds: Let \( \varphi \) be any antisymmetric vertex, \( 2r \leq n, \) and \( S(x_{2r+1}, \ldots, x_n) \) be the sum of the values of all diagrams obtained from \( \varphi \) by joining each of the legs labeled \( 1, \ldots, r \) to one and only one of the legs labeled \( r+1, \ldots, 2r \).

\[ 2r+1, \ldots, n \}

\begin{center}
\[ \varphi \]
\end{center}

\[ \text{26} \]
Then the norm of $S$ is bounded by $b^{2r}$ times the norm of $\varphi$. The argument in the previous paragraph shows that $C^{(j)}$ has an integral bound with respect to the $\| \cdot \|_{1,\infty}$ norm that is of order $\sqrt{\|C^{(j)}(k)\|_1} = O\left(\frac{1}{\sqrt{M^j}}\right)$. Similarly one sees that $C^{(j)}$ has an integral bound, with respect to the $\| \cdot \|_{1,\Sigma}$ norm, that is of order $O\left(\sqrt{M^j \Sigma}\right)$. Thus, in both cases, the integral bound is of the order of the square root of the tadpole bound found before.

The discussion above shows how to get cancellations between diagrams that have only two vertices. The combinatorics for treating diagrams of arbitrary size was developed in [FMRT, FKTcf, FKTr1, FKTr2]. We sketch it here. As mentioned in subsection 4, we Wick order with respect to future covariances in order to avoid tadpoles. Thus, we write

$$W(0,0,\psi,\bar{\psi}) = :U(\psi,\bar{\psi})C^{(j)}:$$

$$W'(0,0,\psi,\bar{\psi}) = :U'(\psi,\bar{\psi})C^{(j+1)}:$$

where $C^{(\geq j)} = \sum_{i \geq j} C^{(i)}$. Then the kernels of $U'$ are sums of diagrams whose vertices are kernels of $U$ and which have two kinds of lines. The first arises from integrating with respect to $d\mu_{C^{(j)}}$ and has propagator $C^{(j)}$. The second arises in Wick ordering $W'$ and has propagator $C^{(j+1)}$. The subgraph of each diagram obtained by deleting the $C^{(\geq j+1)}$ lines must be connected and tadpole–free. For clarity, we ignore the Wick lines. That is, we discuss a simplified situation in which $W$ is Wick ordered with respect to $C^{(j)}$ and $W'$ is not Wick ordered at all\(^{(9)}\), so that we write

$$W(0,0,\psi,\bar{\psi}) = :\tilde{W}(\psi,\bar{\psi})C^{(j)}:$$

$$\tilde{W}(\psi,\bar{\psi}) = \sum_{n \geq 0} \int dp_1\ldots dp_n \bar{w}_{2n}(p_1,\ldots,p_n) \delta(p_1+\ldots+p_n-q_1-\ldots-q_n)\bar{\psi}(p_1)\cdots\bar{\psi}(p_n)\psi(q_1)\cdots\psi(q_n)$$

Then

$$w_2' = \sum_{\ell} \frac{1}{\ell!} \left( \text{connected diagrams without tadpoles with } 2n \text{ external legs and } \ell \text{ vertices from } \tilde{w}_2, \tilde{w}_4, \ldots \right)$$

All diagrams are labeled. A rooted diagram is a diagram with one distinguished vertex, called the root. Clearly,

$$w_2' = \sum_{\ell} \frac{1}{\ell!} \frac{1}{\ell} \left( \text{connected, rooted, tadpole–free diagrams with } 2n \text{ external legs and } \ell \text{ vertices from } \tilde{w}_2, \tilde{w}_4, \ldots \right)$$

The distance between two vertices in a diagram is the minimal number of lines needed to form a path connecting these two vertices. In a rooted diagram, the $r$th ring is defined as the set of vertices of distance $r$ from the root. Thus, the zeroth ring is the root itself, and the

\(^{(9)}\) General Wick ordering is treated in [FKTr2].
first ring consists of all vertices that are directly connected to the root by a line. Observe that the full subgraph formed by the union of the first \( r \) rings of a diagram \( G \) is again a connected diagram \( G_r \). Each leg emanating from a vertex of the \( r \)th ring that is not part of a line in \( G_r \) (that is, each external leg of \( G_r \)) is either an external leg for the whole diagram or is connected to a vertex of the \((r + 1)\)st ring. Observe that, for each graph, there is an \( r_0 \) such that the \( r \)th ring is empty for all \( r \geq r_0 \).

For simplicity, we only discuss the case that \( n = 0 \), i.e. that there are no external legs. Given a rooted diagram without external legs, we have the following combinatorial data:

- \( \ell_r = \# \) (vertices of the \( r \)th ring)
- For \( i = 1, \ldots, \ell_r \), let \( \begin{cases} \gamma_{i,r}^- \\ \gamma_{i,r}^0 \\ \gamma_{i,r}^+ \end{cases} \) be the number of legs of the \( i \)th vertex in the \( r \)th ring that are connected to a vertex in ring number \( \begin{cases} r - 1 \\ r \\ r + 1 \end{cases} \).

By the definition of “ring”, the \( i \)th vertex in the \( r \)th ring has \( \gamma_{i,r}^- + \gamma_{i,r}^0 + \gamma_{i,r}^+ \) legs and \( \gamma_{i,r}^- \geq 1 \) for all \( 1 \leq r \leq r_0 \). The sequences \( \vec{\ell} = (\ell_1, \ell_2, \cdots) \) and \( \vec{\gamma} = (\gamma_{i,r}^-, \gamma_{i,r}^0, \gamma_{i,r}^+) \) \( i = 1, 2, \cdots \) are called the combinatorial data of the rooted graph.

We only exploit cancellations between connected rooted graphs with the same combinatorial data. So fix some combinatorial data \( \vec{\ell}, \vec{\gamma} \) and let \( \mathcal{G} \) be the set of all diagrams with these combinatorial data. Denote by \( \tilde{G}_r \) the sum of the values of all subgraphs \( G_r \) of graphs \( G \in \mathcal{G} \). View it as a single vertex. It has

\[
\sum_{i=1}^{\ell_r} \gamma_{i,r}^+ = \sum_{i=1}^{\ell_{r+1}} \gamma_{i,r+1}^-
\]

external legs. We describe how \( \tilde{G}_r \) is formed from \( \tilde{G}_{r-1} \) and how the norm of \( \tilde{G}_r \) is bounded in terms of the norm of \( \tilde{G}_{r-1} \): Connect each of the \( \ell_r \) vertices of the \( r \)th ring to \( \tilde{G}_{r-1} \) by one line and apply contraction bounds. Then apply an integral bound for the \( \sum_{i=1}^{\ell_r} (\gamma_{i,r}^- - 1) \)
remaining connections between $\tilde{G}_{r-1}$ and the $r^{th}$ ring.

\[ \gamma_1^+,\gamma_0 1,\gamma_1^+,\gamma_0 2 \quad \gamma_2^+,\gamma_0 r \]

Then, using a variant of the integral bound, form all connections between the $\gamma_1^0,\gamma_0 r$ lines coming from the first vertex, the $\gamma_2^0,\gamma_0 r$ lines coming from the second vertex, $\cdots$ and the $\gamma_\ell^0,\gamma_0 r$ lines coming from the last vertex, always avoiding tadpoles. Repeat this procedure for all $r$ for which $\ell_r \neq 0$. The bounds obtained in this way are summable over all combinatorial data.

Observe that ladders

\[ \cdots \]

have very special combinatorial data: Each $\ell_r$ is either zero, one or two, $\gamma_i^-,\gamma_i^0 2, \gamma_i^-,\gamma_i^0 0$ and $\gamma_i^+,\gamma_i^0 r$ is either zero or two. In the example

\[ \cdots \]

we have $\ell_1 = \ell_2 = 2, \ell_3 = 1, \gamma_1^+,\gamma_2^+,\gamma_1^+,\gamma_2^+,\gamma_1^+,\gamma_2^+,\gamma_1^+,\gamma_2^+,\gamma_1^+,\gamma_2^+ = 0$. Not all diagrams with such combinatorial data are ladders, but there are so few of them that the non–ladder diagrams with these combinatorial data can be bounded individually without generating divergences.

In many cases, the combinatorial data of a diagram alone allow the detection of an overlapping loop. The two basic cases are:

(i) If, for some $r \geq 1$ and $1 \leq i \leq \ell_r$, we have $\gamma_i^-,\gamma_i^0 r + \gamma_i^+,\gamma_i^0 r \geq 3$ then there is an overlapping loop, as indicated in the following figures. Here $v_i$ denotes the $i^{th}$ vertex of the $r^{th}$ ring.

\[ \tilde{G}_{r-1} \quad \gamma_i^-,\gamma_i^0 r \geq 3 \]

\[ \tilde{G}_{r-1} \quad \gamma_i^-,\gamma_i^0 r = 2, \gamma_i^-,\gamma_i^0 r \geq 1 \]

\[ \text{or} \]

\[ \gamma_i^-,\gamma_i^0 r = 1, \gamma_i^-,\gamma_i^0 r \geq 2 \]
(ii) If \( \ell_r = 2, \ell_{r+1} = 1 \) and \( \gamma_{1, r}^{-} + \gamma_{1, r}^{0} \geq 2, \gamma_{1, r}^{+} \geq 1, \gamma_{2, r}^{+} \geq 1 \) then there is an overlapping loop as seen in the following figures.

These overlapping loops can be used to generate improved estimates by the techniques mentioned at the end of subsection 8 without seriously affecting the cancellations described above. If one also takes external legs into account and if \( w_2 = 0 \), then cases (i) and (ii) are enough to identify at least one overlapping loop in each four–legged diagram that does not have the combinatorial data of a ladder diagram. See §VII of [FKTr2].

10. The Counterterm

The counterterm \( \delta e \) is constructed so that the proper self energy is bounded by \( \frac{1}{2} |i k_0 - e(k)| \). That is, let \( \tilde{G}_{2,j}(p) \delta(p - q) \) be the Fourier transform of the two point Green’s function \( G_{2,j}(x, y) \) constructed in Theorem I.4, and define \( \Sigma_j(k) \) by

\[
\tilde{G}_{2,j}(k) = \frac{U(k) - \nu^{(\geq j)}(k)}{i k_0 - e(k) - \Sigma_j(k)}
\]

Then \( |\Sigma_j(k)| \leq \frac{1}{2} |i k_0 - e(k)| \).

To achieve this, we specify, for each scale \( j \), a space \( \mathcal{K}_j \) of “allowed future counterterm contributions for scales after \( j \)”. It consists of functions \( K(k) \) of the vector part of \( k = (k_0, k) \) only. These functions are required to be bounded by a small constant times \( \frac{1+1}{M^{j+1}} \). The numerator reflects overlapping loop volume improvement. We construct a map \( \delta e_j \) from \( \mathcal{K}_j \) to the space \( \mathcal{E} \) of counterterms such that, if one writes

\[
\tilde{G}_{2,j}(k; \delta e_j(K)) = \frac{U(k) - \nu^{(\geq j)}(k)}{i k_0 - e(k) - \Sigma_j(k; K)}
\]

then \( \Sigma_j((0, k); K) = K(k) \) in the \( j \)th neighbourhood. Think of \( \delta e_j(k; K) \) as being of the form \( \delta \tilde{e}_j(k; K) + K(k) \) with \( \delta \tilde{e}_j(k; K) \) implementing cancellations that have already been identified and \( K(k) \) reserved to implement as yet unidentified cancellations. Since \( \Sigma_j(k; K) \) equals \( \delta e_j(k; K) \) plus higher order contributions, it is easy to solve \( \Sigma_j((0, k); K) = K(k) \) for \( \delta \tilde{e}_j(k; K) \). The algebra of this (recursive) construction is presented in detail in §III. We prove that \( \delta e = \lim_{j \to \infty} \delta e_j(0) \) exists and has the required properties.
At each scale, the properties of the counterterms are used to obtain the bound for the $w_2(p, q)$ of order $\frac{1}{M_j}$ necessary for power counting. The choice indicated above guarantees that such a bound holds for momenta $k = q - p$ for which the “temperature part” $k_0$ vanishes. To get this bound for arbitrary $k = (k_0, k)$ in the $j$th neighbourhood, in particular when $|k_0| = O\left(\frac{1}{M_j}\right)$, we show that the $k_0$ derivative of this function is of order one. For this, we have to control derivatives (in momentum space) of all the data in the renormalization construction. Control of derivatives in momentum space is also needed to provide decay in position space. This is used, for example, in proving contraction bounds through $L^1$ norms in position space. We pointed out in the subsection 1 that derivatives of $\delta e_j(0)$ might blow up as $j \to \infty$. Therefore we have to pay special attention to the behaviour of derivatives. This is the reason why we introduce, in Definition V.2, the “norm domain” that gives a convenient notation for bounds on derivatives of functions.
III. Formal Renormalization Group Maps

To simplify notation involving the fields, we define, for $\xi = (x_0, x, \sigma, a) = (x, a) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$, the internal fields

$$\psi(\xi) = \left\{ \begin{array}{ll} \psi(x) = \psi_\sigma(x_0, x) & \text{if } a = 0 \\ \tilde{\psi}(x) = \tilde{\psi}_\sigma(x_0, x) & \text{if } a = 1 \end{array} \right.$$ 

Similarly, we define for an external variable $\eta = (y_0, y, \tau, b) = (y, b) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$, the source fields

$$\phi(\eta) = \left\{ \begin{array}{ll} \phi(y) = \phi_\sigma(y_0, y) & \text{if } b = 0 \\ \tilde{\phi}(y) = \tilde{\phi}_\sigma(y_0, y) & \text{if } b = 1 \end{array} \right.$$ 

$\mathcal{B} = \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ is called the “base space” parameterizing the fields. An antisymmetric function $S(\xi, \xi')$ on $\mathcal{B} \times \mathcal{B}$ is called a covariance and determines a Grassmann Gaussian measure by

$$\int \psi(\xi) \psi(\xi') d\mu_S(\psi) = S(\xi, \xi')$$

A function $S(k)$ on momentum space, $\mathbb{R} \times \mathbb{R}^d$, defines a function $S(x, x')$ on position space $(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^2$ by

$$S(x, x') = \delta_{\sigma, \sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik \cdot (x - x')} - S(k)$$ (III.1)

Any function $S(x, x')$ on position space, $(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^2$ defines a unique antisymmetric function $S(\xi, \xi')$ on $\mathcal{B} \times \mathcal{B}$ by

$$S((x, a), (x', a')) = \left\{ \begin{array}{ll} S(x, x') & \text{if } a = 0, a' = 1 \\ -S(x', x) & \text{if } a = 1, a' = 0 \\ 0 & \text{if } a = a' \end{array} \right.$$ (III.2)

We denote the associated Grassmann Gaussian measure again by $d\mu_S$.

With the notation introduced above the source term of (I.5) is

$$\phi J \psi = \int d\xi \, d\xi' \, \phi(\xi) J(\xi, \xi') \psi(\xi') = \psi J \phi$$

where the operator $J$ has kernel

$$J((x_0, x, \sigma, a), (x'_0, x', \sigma', a')) = \delta(x_0 - x'_0) \delta(x - x') \delta_{\sigma, \sigma'} \left\{ \begin{array}{ll} 1 & \text{if } a = 1, a' = 0 \\ -1 & \text{if } a = 0, a' = 1 \\ 0 & \text{otherwise} \end{array} \right.$$ (III.3)
Definition III.1 (Renormalization Group Maps) Let $S$ be a covariance and $W(\phi, \psi)$ a Grassmann function for which $Z = \int e^{W(0, \zeta)} d\mu_S(\zeta) \neq 0$. We set

$$\Omega_S(W)(\phi, \psi) = \log \frac{1}{Z} \int e^{W(\phi, \psi + \zeta)} d\mu_S(\zeta)$$

$$\tilde{\Omega}_S(W)(\phi, \psi) = \log \frac{1}{Z} \int e^{\phi J \zeta} e^{W(\phi, \psi + \zeta)} d\mu_S(\zeta)$$

$\Omega_S$ and $\tilde{\Omega}_S$ map Grassmann functions in the variables $\phi, \psi$, to Grassmann functions in the same variables. They obey the semigroup property

$$\Omega_{S_1+S_2} = \Omega_{S_1} \circ \Omega_{S_2}, \quad \tilde{\Omega}_{S_1+S_2} = \tilde{\Omega}_{S_1} \circ \tilde{\Omega}_{S_2} \quad (\text{III.4})$$

By Lemma VII.3 of [FKTo2], they are related by

$$\tilde{\Omega}_S(W)(\phi, \psi) = \frac{1}{2} \phi J S J \phi + \Omega_S(W)(\phi, \psi + SJ \phi) \quad (\text{III.5})$$

where, for any covariance $S$, $\phi S \phi = \int d\xi_1 d\xi_2 \phi(\xi_1) S(\xi_1, \xi_2) \phi(\xi_2)$. Clearly

$$G_j(\phi; \delta e) = \tilde{\Omega}_{CIR(\delta e)}(\tilde{\mathcal{V}})(\phi, 0) \quad \text{with} \quad \tilde{\mathcal{V}}(\phi, \psi) = \mathcal{V}(\psi)$$

Observe that

$$C^{IR}(j; 0) = C^{IR}(1; 0) + C^{(1)}(k) + \cdots + C^{(j-2)}(k) + C^{(j-1)}(k)$$

where

$$C^{(i)}(k_0, k) = \frac{\nu^{(i)}(k)}{i k_0 - \nu(k)}$$

Therefore, by induction and the semigroup property,

$$G_j(\phi; 0) = \tilde{\Omega}_{C^{(j-1)}} \circ \tilde{\Omega}_{C^{(j-2)}} \circ \cdots \circ \tilde{\Omega}_{C^{(1)}} \circ \tilde{\Omega}_{C^{IR}(0)}(\tilde{\mathcal{V}})(\phi, 0) \quad (\text{III.6})$$

Since $C^{(i)}(k)$ is supported on the $i^{th}$ shell only, (III.6) would provide a convenient framework for a multiscale analysis of the unrenormalized generating functional $G_j(\phi; 0)$, by recursively controlling the effective interactions

$$G_i(\phi, \psi; 0) = \tilde{\Omega}_{C^{(i-1)}} \circ \cdots \circ \tilde{\Omega}_{C^{(1)}} \circ \tilde{\Omega}_{C^{IR}(0)}(\tilde{\mathcal{V}})(\phi, \psi) = \tilde{\Omega}_{C^{(i-1)}}(G_{i-1}(0))(\phi, \psi) \quad (\text{III.7})$$

We now describe an analog of (III.6) for the case $\delta e \neq 0$. It incorporates a number of technical modifications needed to maintain control over the bounds. These modifications
include the introduction of a scale–dependent Wick ordering and a scale–dependent contribution to the counterterm, periodic shifting of a portion of the interaction into the covariance and the periodic isolation of purely $\phi$ dependent terms. To this end, the effective interaction $G_i(\phi, \psi; 0)$ of (III.7) is replaced by a triple $(W, G, u)$ with

- $G(\phi)$ being the purely $\phi$ dependent part of the effective interaction,
- $W(\phi, \psi)$ being the rest of the interaction and
- $u$ being the kernel of a quadratic Grassmann function that has been moved from the effective interaction into the covariance

To help clarify the algebraic structure of this more complicated setting, we outline the construction in the category of formal power series, without specifying the bounds that will ultimately be proven. To avoid formal power series in infinitely many variables, we introduce a coupling constant $\lambda$ into the interaction

$$V(\psi) = \lambda \int_{(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^4} V(x_1, x_2, x_3, x_4) \bar{\psi}(x_1)\psi(x_2)\bar{\psi}(x_3)\psi(x_4) \, dx_1dx_2dx_3dx_4$$

and deal with Grassmann algebra valued formal power series in $\lambda$. Correspondingly, the final counterterm, $\delta e(k)$ is made $\lambda$–dependent.

**Definition III.2** The space of formal (final) counterterms, $E^{form}$, consists of the space of all formal power series $\delta e(k, \lambda) = \sum_{n=1}^{\infty} \delta e_n(k)\lambda^n$ in $\lambda$ each of whose coefficients is supported in

$$\{ k \in \mathbb{R}^d \mid U(k) = 1 \}.$$

As indicated above the final counterterm $\delta e$ is built up from contributions at each scale.

**Definition III.3** The space $R_j^{form}$ of formal (future) counterterms for scale $j$ is the space of all formal power series in $\lambda$ whose coefficients are antisymmetric, translation invariant functions of $x, x'$. The coefficient of $\lambda^0$ vanishes and the Fourier transform of each coefficient is supported on supp $\nu(\geq j+1)((0, k))$.

**Definition III.4** A formal interaction triple at scale $j$ is a triple $(W, G, u)$ that obeys the following conditions.

- $W(\phi, \psi; K)$ is a formal power series in $\lambda$, whose coefficients are functions of $K \in R_j^{form}$ that take values in the Grassmann algebra generated by the fields $\phi(\xi)$ and $\psi(\xi)$. Furthermore $W(\phi, 0; K) = 0$ and $W(\phi, \psi; K)|_{\lambda=0} = 0$. The coefficients of $W$ are translation invariant, spin independent and particle–number conserving.
\( G(\phi; K) \) is a formal power series in \( \lambda \) whose coefficients are functions of \( K \in \mathbb{R}^{2m}_j \) that take values in the Grassmann algebra generated by the fields \( \phi(\xi) \). The constant term \( G(0; K) = 0 \). The coefficients of \( G \) are translation invariant, spin independent and particle–number conserving.

\( u(\xi_1, \xi_2; K) \) is a formal power series in \( \lambda \) whose coefficients are antisymmetric, spin independent, particle number conserving, translation invariant functions of \( \xi_1, \xi_2 \in \mathcal{B} \) and \( K \in \mathbb{R}^{2m}_j \). The Fourier transform \( \tilde{u}(k; K) \) of \( u \) obeys

\[
\tilde{u}((0, k); K) = -\tilde{K}(k) \quad \tilde{u}(k; K)|_{\lambda=0} = -\tilde{K}(k)
\]

The condition that \( \mathcal{W}(\phi, 0; K) = 0 \) ensures that \( G(\phi; K) \) contains the full pure \( \phi \) part of the effective interaction.

As mentioned above, \( u \) is the kernel of a quadratic Grassmann function that has been moved from the effective interaction into the covariance. Precisely,

**Definition III.5**

(i) Let \( u(\xi_1, \xi_2) \) be a formal power series in \( \lambda \) whose coefficients are antisymmetric, spin independent, particle number conserving, translation invariant functions of \( \xi_1, \xi_2 \in \mathcal{B} \). Then

\[
C_u^{(j)}(k) = \frac{\nu^{(j)}(k)}{ik_0 - e(k) - \tilde{u}(k)}
\]

(ii) Let \( u(\xi_1, \xi_2; K) \) be a formal power series in \( \lambda \) whose coefficients are antisymmetric, spin independent, particle number conserving, translation invariant functions of \( \xi_1, \xi_2 \in \mathcal{B} \). Then, for \( K \in \mathbb{R}^{2m}_j \),

\[
C_J(u; K)(k) = \frac{\nu^{(\geq j)}(k)}{ik_0 - e(k) - \tilde{u}(k; K) - \tilde{K}(k)\nu^{(\geq j + 2)}(k)}
\]

\[
D_J(u; K)(k) = \frac{\nu^{(\geq j + 1)}(k)}{ik_0 - e(k) - \tilde{u}(k; K) - \tilde{K}(k)\nu^{(\geq j + 2)}(k)}
\]

For formal interaction triple \((\mathcal{W}, G, u)\) at scale \( j \), integrating out scale \( j \) involves the evaluation of an integral with respect to the Gaussian measure with covariance \( C_u^{(j)}(K) \). The effective interaction \( \mathcal{W} \) will be Wick ordered with respect to the covariance \( C_J(u(K); K) \). One important property of the Wick ordering covariances is that \( D_J(u; K) = C_J(u; K) - C_u^{(j)}(K) \), so that

\[
\int :f(\psi + \zeta):C_J(u, K) \, dp_{C^{(j)}_u(K)}(\zeta) = :f(\psi):D_J(u, K)
\]

\(A systematic set of Fourier transform conventions will be given in Definition VI.1. In the present context \( \tilde{u}(k; K) \) is the Fourier transform of \( u((0, 0, \uparrow, 1), (x_0, x, \uparrow, 0); K) \) as in Theorem I.5.
for all Grassmann functions $f(\psi)$, by Proposition A.2.i and Lemma A.4.ii of [FKTr1]. This property prevents the formation of Wick self-contractions

and ensures that the effective interaction resulting from integrating out scale $j$ is naturally Wick ordered with respect to the “output Wick ordering covariance” $D_j(u, K)$. Also the Wick ordering covariances have been chosen so that, for $k$ of scale at least $j + 3$, $\tilde{u}(k; K) + \tilde{K}(k)\nu^{(\geq j+2)}(k) = \tilde{u}(k; K) + \tilde{K}(k)$ vanishes for $k_0 = 0$. This property ensures that the denominator still vanishes only on the Fermi surface.

**Definition III.6** Integrating out the fields of scale $j$ is implemented by the map $\Omega_j$, which maps a formal interaction triple $(W, G, u)$ of scale $j$ to the triple $(W', G', u)$ determined by

$$
:W'(\phi, \psi; K) :_{\psi, D_j(u; K)} = \log \frac{1}{Z(\phi)} \int e^{\phi J \zeta} e^{W(\phi, \psi + \zeta; K) :_{\psi, C_j(u; K)}} d\mu_{C_j(u; K)}(\zeta)
$$

$$
G'(\phi) = G(\phi) + \log \frac{Z(\phi)}{Z(0)}
$$

where

$$
\log Z(\phi) = \int \left[ \log \int e^{\phi J \zeta} e^{W(\phi, \psi + \zeta; K) :_{\psi, C_j(u; K)}} d\mu_{C_j(u; K)}(\zeta) \right] d\mu_{D_j(u; K)}(\psi)
$$

In fact $(W', G', u)$ is again a formal interaction triple of scale $j$. That $W'(\phi, 0; K) = 0$ follows by inserting the definitions into

$$
W'(\phi, 0; K) = \int :W'(\phi, \psi; K) :_{\psi, D_j(u; K)} d\mu_{D_j(u; K)}(\psi)
$$

To verify $W'(\phi, \psi; K)\big|_{\lambda=0} = 0$, observe that for $\lambda = 0$

$$
\int e^{\phi J \zeta} e^{W(\phi, \psi + \zeta; K) :_{\psi, C_j(u; K)}} d\mu_{C_j(u; K)}(\zeta) = \int e^{\phi J \zeta} d\mu_{C_j(u; K)}(\zeta) = e^{\frac{1}{2} \phi J C_{\psi}(u; K) J \phi}
$$

is independent of $\psi$. To verify the various symmetries, apply remark B.5 of [FKTo2].

**Remark III.7** Define, for all $1 \leq i \leq j \leq \infty$,

$$
C_{u}^{[i, j]}(k) = \begin{cases} 
\frac{\nu^{(\geq i)}(k) - \nu^{(\geq j)}(k)}{i k_0 - \epsilon(k) - \tilde{u}(k) [1 - \nu^{(\geq j)}(k)]} & \text{if } j < \infty \\
\frac{\nu^{(\geq i)}(k)}{i k_0 - \epsilon(k) - \tilde{u}(k)} & \text{if } j = \infty
\end{cases}
$$

We also write $C_{u}^{(\geq i)} = C_{u}^{[i, \infty]}$. 

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Let \((\mathcal{W}', \mathcal{G}', u) = \Omega_j(\mathcal{W}, \mathcal{G}, u)\). Then, for any infrared cutoff \(j + 2 \leq j \leq \infty\), formally, ignoring the problems engendered by the infrared singularity,

\[
\mathcal{G}(\phi) + \log \frac{\int e^{\phi J \psi} e^{\mathcal{W}(\phi, \psi; K)} \psi_c u_j K d\mu_{C_u^{j, \delta}}(\psi)}{\int e^{\mathcal{W}(0, \psi; K)} \psi_c u_j K d\mu_{C_u^{j, \delta}}(\psi)} = G'(\phi) + \log \frac{\int e^{\phi J \psi} e^{\mathcal{W}'(\phi, \psi; K)} \psi_c u_j K d\mu_{C_u^{j+1, \delta}}(\psi)}{\int e^{\mathcal{W}'(0, \psi; K)} \psi_c u_j K d\mu_{C_u^{j+1, \delta}}(\psi)}
\]

**Proof:** Since \(C_u^{j, \delta} = C_u^{j} + C_u^{j+1, \delta}\),

\[
\int f(\phi, \psi) d\mu_{C_u^{j, \delta}}(\psi) = \int \int f(\phi, \psi + \zeta) d\mu_{C_u^{j}}(\zeta) d\mu_{C_u^{j+1, \delta}}(\psi)
\]

by Proposition I.21 of [FKTff]. Hence, by Proposition A.2.ii of [FKTr1],

\[
\log \int e^{\phi J \psi} e^{\mathcal{W}(\phi, \psi; K)} \psi_c u_j K d\mu_{C_u^{j, \delta}}(\psi) = \log \int \int e^{\phi J(\psi + \zeta)} e^{\mathcal{W}(\phi, \psi + \zeta; K)} \psi_c u_j K d\mu_{C_u^{j}}(\zeta) d\mu_{C_u^{j+1, \delta}}(\psi)
\]

\[
= \log \int e^{\phi J \psi} e^{\mathcal{W}'(\phi, \psi; K)} \psi_c u_j K + \log Z(\phi) d\mu_{C_u^{j+1, \delta}}(\psi)
\]

\[
= \log \int e^{\phi J \psi} e^{\mathcal{W}'(\phi, \psi; K)} \psi_c u_j K d\mu_{C_u^{j+1, \delta}}(\psi) + \log Z(\phi)
\]

Subtracting the same equation with \(\phi = 0\) gives the desired result.

When we derive bounds on the map \(\Omega_j\), we get improvements on the two- and four-legged contributions to \(\mathcal{W}'(0, \psi)\) by exploiting overlapping loops. See subsection 4 of §II and the introduction to [FKTr2]. However to ensure the presence of and to detect sufficiently many overlapping loops, we need that the two-leg part of \(\mathcal{W}(0, \psi)\) vanishes. See the end of subsection 9 of §II and Remark VI.7 of [FKTr2]. Therefore, we wish that the formal interaction triple \((\mathcal{W}, \mathcal{G}, u)\) input to \(\Omega_j\) be an element of

**Definition III.8 (Formal Input Data)** The space \(\mathcal{D}_{in}^{(j, \text{form})}\) of formal input data consists of the set of all formal interaction triples \((\mathcal{W}, \mathcal{G}, u)\) at scale \(j\), in the sense of Definition III.4, obeying

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(i) If the effective interaction $\mathcal{W}(K) = \sum_{m,n \geq 0} \mathcal{W}_{m,n}$ with

$$\mathcal{W}_{m,n} = \int \prod_{i=1}^{m} d\eta_i \prod_{\ell=1}^{n} d\xi_\ell \mathcal{W}_{m,n}(\eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_n) \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1) \cdots \psi(\xi_n)$$

then $\mathcal{W}_{0,2} = 0$.

(ii) The coefficient of $\lambda^0$ in $\mathcal{G}(\phi; K)$ is $\frac{1}{2} \phi J C_{-K}^{(<j)} J \phi$. Here $C_{u}^{(<j)} = \frac{U(k) - \nu^{(\geq j)}(k)}{i k_0^{-\nu(k)} - u(k)}$.

When $\Omega_j$ is applied to an element of $\mathcal{D}^{(j, \text{form})}_{\text{in}}$, the output no longer satisfies condition (i) of Definition III.8. Rather, the output lies in

**Definition III.9 (Formal Output Data)** The space $\mathcal{D}^{(j, \text{form})}_{\text{out}}$ of formal output data consists of the set of all formal interaction triples $(\mathcal{W}, \mathcal{G}, u)$ at scale $j$, in the sense of Definition III.4, for which the coefficient of $\lambda^0$ in $\mathcal{G}(\phi; K)$ is $\frac{1}{2} \phi J C_{-K}^{(\leq j)} J \phi$.

**Lemma III.10** Let $(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}^{(j, \text{form})}_{\text{in}}$. Then $\Omega_j(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}^{(j, \text{form})}_{\text{out}}$.

**Proof:** Set $(\mathcal{W}', \mathcal{G}', u) = \Omega_j(\mathcal{W}, \mathcal{G}, u)$. Then

$$\mathcal{G}'(\phi)|_{\lambda=0} = \mathcal{G}(\phi)|_{\lambda=0} + \log \frac{Z(\phi)}{Z(0)}|_{\lambda=0} = \frac{1}{2} \phi J C_{-K}^{(<j)} J \phi + \log \frac{Z(\phi)}{Z(0)}|_{\lambda=0}$$

Since

$$\log Z(\phi)|_{\lambda=0} = \int \left[ \log \int e^{J \phi} d\mu_{C_{-K}^{(j)}}(\xi) \right] d\mu_{D_j(u; K)}(\psi) = \frac{1}{2} J \phi C_{-K}^{(j)} J \phi$$

the result follows.

Elements of the space $\mathcal{D}^{(j, \text{form})}_{\text{out}}$ are not of the form desired for the application of $\Omega_{j+1}$, the map implementing the integration out of scale $j + 1$. In particular, the two-point part of the effective interaction is nonzero and $\mathcal{R}^{\text{form}}_{j+1}$ is not the appropriate space of counterterms. Below, just before Proposition III.12, we construct maps

$$\mathcal{O}_j : \mathcal{D}^{(j, \text{form})}_{\text{out}} \rightarrow \mathcal{D}^{(j+1, \text{form})}_{\text{in}}$$

and, for each $(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}^{(j, \text{form})}_{\text{out}}$,

$$\text{ren}_{j,j+1}(\cdot, \mathcal{W}, u) : \mathcal{R}^{\text{form}}_{j+1} \rightarrow \mathcal{R}^{\text{form}}_{j}$$
The renormalization group flow is the concatenation of the maps $\text{ren}$. We recursively define maps $\text{ren}$ that describe the renormalization group flow. We start by choosing an arbitrary but fixed infrared cutoff $j_0 \geq 2$ and integrate out all scales from 1 to $j_0$ to arrive at the initial effective interaction triple $(\mathcal{W}_{j_0}^{\text{out}}, \mathcal{G}_{j_0}^{\text{out}}, u_{j_0}) \in \mathcal{D}_{\text{out}}^{(j_0,\infty)}$ with

$$\begin{align*}
\mathcal{W}_{j_0}^{\text{out}} &= \tilde{\Omega}_{C_{j_0}}(\phi, \psi) - \tilde{\Omega}_{C_{j_0}}(\phi, 0) \\
\mathcal{G}_{j_0}^{\text{out}} &= \tilde{\Omega}_{C_{j_0}}(\phi, 0) \\
u_{j_0} &= -K
\end{align*}$$

The renormalization group flow is the concatenation of the maps $\mathcal{O}_{j_0}$, $\Omega_{j_0+1}$, $\Omega_{j_0+2}$, $\ldots$, $\Omega_j$, $\mathcal{O}_j$, $\ldots$ applied to the initial datum. Set

$$\begin{align*}
(\mathcal{W}_j^{\text{in}}, \mathcal{G}_j^{\text{in}}, u_j) &= \mathcal{O}_{j-1} \circ \Omega_{j-1} \circ \mathcal{O}_{j-2} \circ \ldots \circ \Omega_{j_0+1} \circ \mathcal{O}_{j_0} (\mathcal{W}_{j_0}^{\text{out}}, \mathcal{G}_{j_0}^{\text{out}}, u_{j_0}) \in \mathcal{D}_{\text{in}}^{(j,\infty)} \\
(\mathcal{W}_j^{\text{out}}, \mathcal{G}_j^{\text{out}}, u_j) &= \mathcal{O}_j \circ \Omega_{j-1} \circ \mathcal{O}_{j-1} \circ \mathcal{O}_{j-2} \circ \ldots \circ \Omega_{j_0+1} \circ \mathcal{O}_{j_0} (\mathcal{W}_{j_0}^{\text{out}}, \mathcal{G}_{j_0}^{\text{out}}, u_{j_0}) \in \mathcal{D}_{\text{out}}^{(j,\infty)}
\end{align*}$$

We recursively define maps $\text{ren}_{i,j} : \mathcal{R}_j^{\infty} \to \mathcal{R}_i^{\infty}$, $j_0 \leq i \leq j$ by

$$\begin{align*}
\text{ren}_{j,j}(K) &= K \\
\text{ren}_{i,j}(K) &= \text{ren}_{i,j-1}(\text{ren}_{j-1,j}(K)) \quad \text{for } j > i
\end{align*}$$

We define for $K \in \mathcal{R}_j^{\infty}$

$$\delta e_j(K) = \text{ren}_{j_0,j}(K)$$
and show that, for the generating function of the connected Green’s functions at scale $j$ of Theorem I.4,

$$G_j^\ast(\phi, \bar{\phi}; \delta e_j(K)) = G_j^{\text{in}}(\phi; K) + \log \frac{\int e^{\phi J_p} e^{W_j^{\text{in}}(\phi, \bar{\phi}; K)} d\mu_{C_j^{(j, \phi)}}(\psi)}{\int e^{\phi J_p} e^{W_j^{\text{in}}(0, \bar{\phi}; K)} d\mu_{C_j^{(j, \phi)}}(\psi)}$$

$$= G_j^{\text{out}}(\phi; K) + \log \frac{\int e^{\phi J_p} e^{W_j^{\text{out}}(\phi, \bar{\phi}; K)} d\mu_{C_{j_0}^{(j_0, \phi)}}(\psi)}{\int e^{\phi J_p} e^{W_j^{\text{out}}(0, \bar{\phi}; K)} d\mu_{C_{j_0}^{(j_0, \phi)}}(\psi)}$$

(III.10)

for $K \in \mathcal{R}_{j_0}^{\text{form}}$ and $j_0 < j < j - 2$. When $j = \infty$, (III.10) holds with $G_\infty$ being the formal generating function of the connected Green’s functions (I.4).

Equation (III.10) is proven by induction on $j$, combining Remark III.7 and (III.8). To start the induction, observe that $G_{j_0}^{\text{out}}(\phi; K) + W_{j_0}^{\text{out}}(\phi, \bar{\phi}; K) = \tilde{\Omega}_{C_{-K}}^{(\leq j_0)}(\tilde{V})$, so that, by the semigroup property (III.4),

$$G_j^{\ast}(\phi, \bar{\phi}; \delta e_j(K)) = G_{j_0}^{\text{out}}(\phi; K) + \log \frac{\int e^{\phi J_p} e^{W_{j_0}^{\text{out}}(\phi, \bar{\phi}; K)} d\mu_{C_{j_0}^{(j_0, \phi)}}(\psi)}{\int e^{\phi J_p} e^{W_{j_0}^{\text{out}}(0, \bar{\phi}; K)} d\mu_{C_{j_0}^{(j_0, \phi)}}(\psi)}$$

$$= G_{j_0}^{\text{out}}(\phi; K) + \log \frac{\int e^{\phi J_p} e^{W_{j_0}^{\text{out}}(\phi, \bar{\phi}; K)} d\mu_{C_{j_0}^{(j_0, \phi)}}(\psi)}{\int e^{\phi J_p} e^{W_{j_0}^{\text{out}}(0, \bar{\phi}; K)} d\mu_{C_{j_0}^{(j_0, \phi)}}(\psi)}$$

with $D_{j_0} = 0$.

In Theorem VIII.5, we shall prove bounds that show that the limits $\delta e = \lim_{j \to \infty} \delta e_j(0)$ and $G(\phi, \bar{\phi}; \delta e) = \lim_{j \to \infty} G_j^{\text{out}}(\phi; 0) = \lim_{j \to \infty} G_j^{\text{in}}(\phi; 0)$ exist. To prove Theorem I.4 we show that

$$\lim_{j \to \infty} (G_j^{\ast}(\phi, \bar{\phi}) - G_j^{\text{out}}(\phi; 0)) = 0.$$ 

We now describe the passage from output data to input data, that is the maps

$$\mathcal{O}_j : D_{\text{out}}^{(j, \text{form})} \to D_{\text{in}}^{(j+1, \text{form})} \quad \text{ren}_{j, j+1}(\cdot, \mathcal{W}, u) : \mathcal{R}_{j+1}^{\text{form}} \to \mathcal{R}_j^{\text{form}}$$

Let $(\mathcal{W}, \mathcal{G}, u) \in D_{\text{out}}^{(j, \text{form})}$ and write

$$\mathcal{W}(0, \psi; K) = \sum_{m \geq 2} \int d\xi_1 \cdots d\xi_m \int_{W_{0,m}(\xi_1, \cdots, \xi_m; K)} \psi(\xi_1) \cdots \psi(\xi_m)$$

with each $W_{0,m}(\xi_1, \cdots, \xi_m; K)$ antisymmetric under permutation of the $\xi_i$’s.

To perform the reWick ordering, observe that, if $E$ is any covariance and

$$:\mathcal{W}(0, \psi; K):; \mathcal{D}_{j_0}^{(j, \text{out})} = :\tilde{\mathcal{W}}(\phi, \psi; K);; \mathcal{D}_{j_0}^{(j, \text{out})} + E$$
then, by Lemma A.2.i of [FKTr1],

\[ \tilde{W}(\phi, \psi; K) = \int \mathcal{W}(\phi, \psi + \psi'; K) \, d\mu_E(\psi') \]

We wish to choose the covariance \( E \) such that, after we reWick order \( \mathcal{W}(\phi, \psi; K); \psi, D_j(u; K) \) to \( \tilde{W}(\phi, \psi; K); \psi, D_j(u; K) + E \) and move the quadratic part of \( \tilde{W}(\phi, \psi; K) \) into the covariance, replacing \( u(K) \) by \( u'(K') \), then the Wick ordering covariance \( D_j(u; K) + E \) is exactly \( C_{j+1}(u'; K') \). When we replace \( u(K) \) by \( u'(K') \), we choose \( K \in \mathcal{R}_j^{\text{even}} \) as a function of \( K' \in \mathcal{R}_{j+1}^{\text{even}} \) in such a way that \( u'(K') \) fulfills the third condition of Definition III.4. This function will be denoted \( K(K') = \text{ren}_{j,j+1}(K', \mathcal{W}, u) = K' + \delta K(K') \).

The unknowns in the scheme outlined in the last paragraph are \( E \) and \( \delta K \). They are determined implicitly by the requirements of the last paragraph. We choose to express \( E \) and \( \delta K \) in terms of one function \( q(\xi_1, \xi_2; K') \), with \( \frac{1}{2} q \) being the kernel of \( \tilde{W}_{0,2} \). Once, \( q \) is determined, we set

\[ \delta K(k; K'; q) = \tilde{q}((0, k); K') \nu^{(\geq j+1)}((0, k)) \]

\[ K(K'; q) = K' + \delta K(K'; q) \]

\[ u'(k; K'; q) = \tilde{u}(k; K(K'; q)) + \tilde{q}(k; K') \nu^{(\geq j+1)}(k) \]

\[ E(K'; q) = C_{j+1}(u'(\cdot; q); K') - D_j(u; K(K'; q)) \]

Set

\[ \tilde{W}(\phi, \psi; K'; q) = \int \mathcal{W}(\phi, \psi + \psi'; K(K'; q)) \, d\mu_{E(K'; q)}(\psi') \]

and expand

\[ \tilde{W}(0, \psi; K'; q) = \sum_{m \geq 0} \int d\xi_1 \cdots d\xi_m \, \tilde{W}_{0,m}(\xi_1, \cdots, \xi_m; K'; q) \, \psi(\xi_1) \cdots \psi(\xi_m) \]

The requirement that \( \frac{1}{2} q \) be the kernel of \( \tilde{W}_{0,2} \) is now an implicit equation.

**Lemma III.11** There is a unique formal power series \( q_0(\xi_1, \xi_2; K') \) in \( \lambda \) that solves the equation

\[ \frac{1}{2} q(K') = \tilde{W}_{0,2}(K'; q(K')) \]  

(III.12)

The coefficient of \( \lambda^0 \) in \( q_0 \) vanishes.

**Proof:** Equation (III.12) is the form \( q = F(\lambda, q) \), with \( F \) being \( C^\infty \) in \( \lambda \) and \( q \) and with \( F(\lambda, 0) \) being of order at least \( \lambda \). An easy formal power series argument yields the result. \( \blacksquare \)
We define, for each $K' \in \mathcal{R}_{j+1}^{\text{form}}$

\[ \tilde{W}(\phi, \psi; K') = \tilde{W}(\phi, \psi; K'; q_0(K')) \]

\[ W'(\phi, \psi; K') = \tilde{W}(\phi, \psi; K') - \tilde{W}(\phi, 0; K') - \frac{1}{2} \int d\xi_1 d\xi_2 \, q_0(\xi_1, \xi_2; K') \psi(\xi_1) \psi(\xi_2) \]

\[ G'(\phi; K') = G(\phi; K(K')) + \tilde{W}(\phi, 0; K') - \tilde{W}(0, 0; K') \]

\[ u'(K') = u'(K'; q_0(K')) \]

and

\[ \mathcal{O}_j(\mathcal{W}, \mathcal{G}, u) = (\mathcal{W}', \mathcal{G}', u') \]

We also define

\[ \text{ren}_{j,j+1}(K', \mathcal{W}, u) = K(K') = K(K'; q_0(K')) \in \mathcal{R}_j^{\text{form}} \]

**Proposition III.12** Let $j \geq j_0$, $(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}^{(j, \text{form})}_{\text{out}}$ and $(\mathcal{W}', \mathcal{G}', u') = \mathcal{O}_j(\mathcal{W}, \mathcal{G}, u)$. Then

a) 

\[ (\mathcal{W}', \mathcal{G}', u') \in \mathcal{D}^{(j+1, \text{form})}_{\text{in}} \]

b) If $K' \in \mathcal{R}_{j+1}^{\text{form}}$ and $K = \text{ren}_{j,j+1}(K', \mathcal{W}, u)$ then, formally, ignoring the problems engendered by the infrared singularity,

\[ G(\phi; K) + \log \frac{\int e^{\phi \psi} e^{W(\phi, \psi; K)} d\mu_{C^{(j+1, \nu)}}(\psi)}{\int e^{W(0, \psi; K)} d\mu_{C^{(j+1, \nu)}}(\psi)} = \mathcal{G}'(\phi; K') + \log \frac{\int e^{\phi \psi} e^{W'(\phi, \psi; K')} d\mu_{C^{(j+1, \nu)}}(\psi)}{\int e^{W'(0, \psi; K')} d\mu_{C^{(j+1, \nu)}}(\psi)} \]

if $j + 1 < j \leq \infty$.

**Proof:** Let $K' \in \mathcal{R}_{j+1}^{\text{form}}$ and set $K = \text{ren}_{j,j+1}(K', \mathcal{W}, u)$.

a) We first verify that $(\mathcal{W}', \mathcal{G}', u')$ is a formal interaction triple. The only condition of Definition III.4 that is not trivially satisfied is

\[ \dot{u}'((0, k); K') = \ddot{u}'((0, k); K(K'; q_0(K'))) + q_0((0, k); K') \nu^{(\geq j+1)}(0, k) \]

\[ = -\dot{K}(k; K'; q_0(K')) + \delta K(k; K'; q_0(K')) \]

\[ = -\dot{K}(k) \]

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We now verify the conditions of Definition III.8. That $W'_{0,2}$ vanishes amounts to $\tilde{W}_{0,2} = \frac{1}{2}q_0$ which is Lemma III.11. Condition (ii) of Definition III.8 is trivially fulfilled.

b) Observe that

$$C_{u(K')}^{[j+1,j]} = C_{u(K(K'))}^{[j+1,j]} + q_0(K')\nu(\geq j+1) \quad (\text{III.13})$$

Set $U(K') = \frac{1}{2} \int d\xi_1 d\xi_2 \; q_0(\xi_1, \xi_2; K'; \psi(\xi_1)\psi(\xi_2); C_{j+1}(u'; K'))$. By Lemma C.2 of [FKTo2] and (III.13)

$$\mathcal{G}(\phi; K) + \log \int e^{\phi J\psi} e^{\mathcal{W}(\phi, \psi; K)} d\mu_{C_{u(K)}^{[j+1,j]}}(\psi)
= \mathcal{G}'(\phi; K') + \log \int e^{\phi J\psi} e^{\tilde{\mathcal{W}}(\phi, \psi; K') - U} d\mu_{C_{u(K)}^{[j+1,j]}}(\psi)
= \mathcal{G}'(\phi; K') + \log \int e^{\phi J\psi} e^{\mathcal{W}'(\phi, \psi; K') - U} d\mu_{C_{u(K)}^{[j+1,j]}}(\psi) + \text{const}
= \mathcal{G}'(\phi; K') + \log \int e^{\phi J\psi} e^{\mathcal{W}'(\phi, \psi; K') - U} d\mu_{C_{u(K')}^{[j+1,j]}}(\psi) + \text{const}
= \mathcal{G}'(\phi; K') + \log \int e^{\phi J\psi} e^{\mathcal{W}'(\phi, \psi; K') - U} d\mu_{C_{u(K')}^{[j+1,j]}}(\psi) + \text{const}$$

Subtracting the same equation with $\phi = 0$ gives part b).
Appendix A: Model Computations

This appendix provides a number of model computations that illustrate important features of the present construction. Various other model computations are given in the introductory sections of this paper and other papers in this series. Here is a table of model computations and their locations.

| Topic                                    | Location                  |
|------------------------------------------|---------------------------|
| Overlapping loop volume improvement      | §II, subsection 4         |
| Particle–particle bubble volume improvement | §II, subsection 5         |
| Particle–hole bubble sign cancellation   | [FKTl, Lemma I.1]         |
| Sectorization and conservation of momentum | Example A.1               |
| Power counting with sectorization        | §II, subsection 8         |
| Overlapping Loops and Sectors            | Example A.2               |
| Sectorization and change of scale        | Example A.3               |
| Cancellations at high orders of perturbation theory | §II, subsection 9         |
|                                          | [FKTr2, §X]               |
|                                          | [FKTo1, §V]               |

Example A.1 (Sectorization and Conservation of Momentum) Sectors enable us to apply $L^1$ norms to functions for which the $L^1$ norm well–approximates the $L^\infty$ norm of the Fourier transform. We provide a simple illustration of the use of sectorization as a tool for bounding $\| \|_1, \infty$ norms of Green’s functions built from $C^{(j)}$.

Recall that $C^{(j)}(k) = \frac{\nu^{(j)}(k)}{ik_0 - e(k)}$ and that, on the support of $\nu^{(j)}$, $|ik_0 - e(k)| \approx \frac{1}{M_j}$. To make this example as explicit as possible, we suppress $k_0$, choose $d = 3$, choose $e(k) = |k|^2 - 1$, replace $ik_0 - e(k)$ by $\frac{|k|}{M_j}$ and replace $\nu^{(j)}(k)$ by $\varphi(M_j(|k| - 1))$ where $\varphi \in C^\infty([-1, 1])$ is real and even. So we define

$$c^{(j)}(k) = \frac{M_j}{|k|} \varphi(M_j(|k| - 1)) \quad c^{(j)}(x, x') = \int \frac{d^dk}{(2\pi)^d} e^{ik \cdot (x - x')} c^{(j)}(k)$$

We have rigged things so that we can compute $c^{(j)}(x, x')$ relatively explicitly using the Fourier transform $\int_{S^{d-1}} d\sigma(k') e^{irk' \cdot (x - x')} = 2^{\frac{d}{2} - 1} \Gamma\left(\frac{d}{2}\right) \omega_d \left(r |x - x'|\right)^{1 - \frac{d}{2}} J_{\frac{d}{2} - 1}\left(r |x - x'|\right)$ of the unit sphere in $\mathbb{R}^d$. Here $\Gamma$, $\omega_d$ and $J_{\nu}$ are the Gamma function, the surface area of $S^{d-1}$ and the Bessel function of order $\nu$, respectively. The answer is

$$c^{(j)}(x, x') = \frac{1}{2\pi^2 |x - x'|} \sin \left(|x - x'|\right) \hat{\varphi}\left(\frac{|x - x'|}{M_j}\right)$$
In particular
\[ \| c^{(j)}(x, x') \|_{1, \infty} = \int d^3x \frac{1}{2\pi^2|x|} |\sin (|x|)\hat{\varphi}(\frac{|x|}{M})| = \frac{2}{\pi} \int_0^\infty dr \frac{\sin (r)\hat{\varphi}(\frac{r}{M})}{r} \]

For large \( j \), \( |\sin(r)| \) is much more rapidly varying than \( \hat{\varphi}(\frac{r}{M}) \) and replacing \( |\sin(r)| \) by its average value, \( \frac{2}{\pi} \), introduces only a small error.

\[ \| c^{(j)}(x, x') \|_{1, \infty} = \left( 4/\pi + O(1/M) \right) \int_0^\infty dr \frac{\sin (r)\hat{\varphi}(\frac{r}{M})}{r} = M^{2j} \left( 4/\pi + O(1/M) \right) \int_0^\infty dr |\hat{\varphi}(r)| \tag{A.1} \]

Observe that \( \| c^{(j)}(x, x') \|_{1, \infty} \) is a factor of about \( M^j \) larger than \( \sup_k |c^{(j)}(k)| \sim M^j \).

Now introduce a sectorization \( \Sigma \) of the Fermi surface \( \{ k \in \mathbb{R}^3 \mid |k| = 1 \} \) as in subsection 8 of §II (for details, see Definition VI.2) using approximately square sectors of side \( 1/M^{j/2} \). We may construct, also as in subsection 8 of §II (for details, see (XIII.2) of [FKTo3] and Lemma XII.3 of [FKTo3]), a partition of unity, \( \chi_s \), \( s \in \Sigma \), of the support of \( \varphi(M^j(|k| - 1)) \) such that

\[ c^{(j)}_s(k) = \frac{M^j}{|k|} \varphi(M^j(|k| - 1)) \chi_s(k) \quad c^{(j)}_s(x, x') = \int \frac{d^4k}{(2\pi)^d} e^{ik \cdot (x-x')} c^{(j)}_s(k) \]

obeys \( \| c^{(j)}_s(x, x') \|_{1, \infty} \leq \text{const} \, M^j \). The proof of this bound was sketched in subsection 7 of §II. For details, see Proposition XIII.1 and Lemma XIII.2 of [FKTo3]. Observe that, with sectorization,

\[ \| c^{(j)}_s(x, x') \|_{1, \infty} \leq \text{const} \sup_k |c^{(j)}(k)| \]

Had we chosen sectors of size \( 1/M^{j/2} \) with \( \delta < 1/2 \), this would no longer be the case. Also observe that, since \( |\Sigma| \) is of order \( (M^{j/2})^2 \),

\[ \| c^{(j)}(x, x') \|_{1, \infty} \leq \sum_{s \in \Sigma} \| c^{(j)}_s(x, x') \|_{1, \infty} \leq \text{const} \, (M^{j/2})^2 M^j = \text{const} \, M^{2j} \]

recovers (A.1), up to an unimportant constant, by a technique that extends to nonround Fermi surfaces, for which explicit computations of \( c^{(j)}(x, x') \) are not available.

Finally, consider

\[ G_2(x, x') = \int dy \, c^{(j)}(x, y)c^{(j)}(y, x') \]

This would be a (first order) contribution to a model with covariance \( c^{(j)} \) and an ultralocal quadratic interaction. If we attempt to bound \( \| G_2(x, x') \|_{1, \infty} \) just using \( \| c^{(j)}(x, x') \|_{1, \infty} \), we get

\[ \| G_2(x, x') \|_{1, \infty} = \int dx \, |G_2(x, 0)| = \int dx \left| \int dy \, c^{(j)}(x, y)c^{(j)}(y, 0) \right| \leq \int dx \int dy \, |c^{(j)}(x, y)c^{(j)}(y, 0)| = \| c^{(j)}(x, x') \|_{1, \infty}^2 \sim \text{const} \, M^{4j} \]

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which is a terrible answer: \( G_2(x, x') \) is the Fourier transform of \( c^{(j)}(k)^2 = \frac{M^{2j}}{|k|} \varphi(M'(k-1))^2 \) and \( \frac{1}{|k|} \varphi(M'(k-1))^2 \) is very much like \( \frac{1}{|k|} \varphi(M'(k-1)) \), so the real behaviour of \( \|G_2(x, x')\|_{1, \infty} \) is \( M^{3j} \). We may recover this real behaviour using sectors.

\[
\|G_2(x, x')\|_{1, \infty} = \int dx \int dy \ c^{(j)}(x, y)c^{(j)}(y, 0) = \int dx \sum_{s, s' \in \Sigma} \int dy \ c_s^{(j)}(x, y)c_{s'}^{(j)}(y, 0) \leq \sum_{s, s' \in \Sigma} \int dx \int dy \ c_s^{(j)}(x, y)c_{s'}^{(j)}(y, 0) \]

But \( \int dy \ c_s^{(j)}(x, y)c_{s'}^{(j)}(y, 0) \) is the Fourier transform of \( c_s^{(j)}(k)c_{s'}^{(j)}(k) = c^{(j)}(k)\chi_s(k)\chi_{s'}(k) \), which vanishes identically unless the supports of \( \chi_s \) and \( \chi_{s'} \) overlap. For each fixed \( s \in \Sigma \), there are at most 9 sectors \( s' \in \Sigma \) that overlap with \( s \). Hence

\[
\|G_2(x, x')\|_{1, \infty} \leq 9|\Sigma| \max_{s, s' \in \Sigma} \int dx \sum_{s, s' \in \Sigma} \int dy \ c_s^{(j)}(x, y)c_{s'}^{(j)}(y, 0) \leq 9|\Sigma| \max_{s, s' \in \Sigma} \|c_s^{(j)}(x, x')\|_{1, \infty} \|c_{s'}^{(j)}(x, x')\|_{1, \infty} \leq \text{const} |\Sigma|M^{2j} \leq \text{const} M^{3j}
\]

as desired.

**Example A.2 (Overlapping Loops and Sectors)**

Let \( \Gamma \) be the diagram of subsection 4 in §III with vertices \( \omega_4 \) fulfilling the bounds

\[
\|\hat{\omega}_4\|_{1, \Sigma} = O\left(\frac{1}{1}\right), \quad \|\hat{\omega}_4\|_{3, \Sigma} = O(1)
\]

of (II.11) and (II.15). The naive power counting bound (II.11) gives that \( \|\hat{\Gamma}\|_{1, \Sigma} = O\left(\frac{1}{1}\right) \). We show that exploiting overlapping loop volume improvement leads to the bounds \( \|\hat{\Gamma}\|_{1, \Sigma} = O(1) \) and \( \|\hat{\Gamma}\|_{3, \Sigma} = O(1) \).

Let \( \Gamma'(p_1, p_2, p_3', q_1, q_2', q_3') \) be the six legged subgraph consisting of the left two vertices of \( \Gamma \).

![Diagram of \( \Gamma' \)](https://example.com/diagram)

As in (II.10),

\[
\|\hat{\Gamma}'\|_{1, \Sigma} \leq 9 \|\hat{\omega}_4\|_{1, \Sigma} \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_1 \leq O\left(\frac{M^j}{1}\right) \quad \|\hat{\Gamma}'\|_{3, \Sigma} \leq 9 \|\hat{\omega}_4\|_{3, \Sigma} \|\hat{\omega}_4\|_{1, \Sigma} \max_{s \in \Sigma} \|C_s^{(j)}(x)\|_1 \leq O\left(\frac{M^j}{1}\right)
\]

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Clearly, $\Gamma$ is obtained by joining $\Gamma'$ to $w_4$ with three lines.

As in subsection 8 of §II, by conservation of momentum,

$$
\hat{\Gamma}'((x_1, s_1), (x_2, s_2), (x_3, s_3), (x_4, s_4)) = \sum_{\sigma_i, \sigma'_i, \sigma''_i \in \Sigma \atop \sigma_i \cap \sigma'_i \cap \sigma''_i \neq \emptyset \quad \text{for } i = 1, 2, 3} \int dy_1 dy_2 dy_3 dz_1 dz_2 dz_3 \hat{\Gamma}'((x_1, s_1), (x_2, s_2), (y_1, s_1), (x_3, s_3), (y_2, s_2), (y_3, s_3))
$$

so that

$$
\| \hat{\Gamma}'((\cdot, s_1), (\cdot, s_2), (\cdot, s_3), (\cdot, s_4)) \|_{1, \infty} \leq \sum_{\sigma_i, \sigma'_i, \sigma''_i \in \Sigma \atop \sigma_i \cap \sigma'_i \cap \sigma''_i \neq \emptyset \quad \text{for } i = 1, 2, 3} \| C^{(j)}_{\sigma'_1}(y_1 - z_1) C^{(j)}_{\sigma'_2}(z_2 - y_2) C^{(j)}_{\sigma'_3}(z_3 - y_3) \hat{\omega}_4(z_2, \sigma''_2, (z_3, \sigma''_3), (z_1, \sigma''_4), (x_4, s_4)) \|_{1, \infty}
$$

For fixed sectors $s_1, s_2, s_3$

$$
\sum_{s_4} \| \hat{\Gamma}'((\cdot, s_1), (\cdot, s_2), (\cdot, s_3), (\cdot, s_4)) \|_{1, \infty} \leq 3^6 \sum_{\sigma_1, \sigma_2, \sigma_3 \in \Sigma} \| \hat{\Gamma}'((\cdot, s_1), (\cdot, s_2), (\cdot, s_1), (\cdot, s_3), (\cdot, s_2), (\cdot, s_3)) \|_{1, \infty} \left( \max_{\sigma'_1 \in \Sigma} \| C^{(j)}_{\sigma'_1}(x) \|_{1, \infty} \right) 2 \max_{\sigma''_1, \sigma''_2, \sigma''_3 \in \Sigma} \| \hat{\omega}_4((\cdot, \sigma''_1), (\cdot, \sigma''_2), (\cdot, \sigma''_3), (\cdot, s_4)) \|_{1, \infty}
$$

$$
\leq O(M^j \left( \frac{1}{M^\tau} \right)^2) \| \hat{\omega}_4 \|_{3, \Sigma} \sum_{\sigma_1, \sigma_2, \sigma_3 \in \Sigma} \| \hat{\Gamma}'((\cdot, s_1), (\cdot, s_2), (\cdot, s_1), (\cdot, s_3), (\cdot, s_2), (\cdot, s_3)) \|_{1, \infty}
$$

since, for a given sector $\sigma_i$, there are three sectors $\sigma'_i$ and three sectors $\sigma''_i$ with $\sigma_i \cap \sigma'_i \neq \emptyset$, $\sigma_i \cap \sigma''_i \neq \emptyset$. We see that the contribution to $\| \hat{\Gamma}' \|_{3, \Sigma}$ with fixed $s_1, s_2, s_3$ is bounded by

$$
O(M^j \left( \frac{1}{M^\tau} \right)^2) \| \hat{\omega}_4 \|_{3, \Sigma} \| \hat{\Gamma}' \|_{3, \Sigma} = O(M^j \left( \frac{1}{M^\tau} \right)^2 1 \left( \frac{M^j}{M^\tau} \right)) = O(1)
$$

and, taking the sector sum over $s_2, s_3$, the contribution to $\| \hat{\Gamma}' \|_{1, \Sigma}$ with fixed $s_1$ is bounded by

$$
O(M^j \left( \frac{1}{M^\tau} \right)^2) \| \hat{\omega}_4 \|_{3, \Sigma} \| \hat{\Gamma}' \|_{1, \Sigma} = O(M^j \left( \frac{1}{M^\tau} \right)^2 1 \left( \frac{M^j}{M^\tau} \right)) = O(1)
$$

The contributions with other sectors fixed are estimated in the same way.
Example A.3 (Sectorization and Change of Scale)

Each time the renormalization group flows to a new scale, the associated sector decomposition changes. Therefore, in the notation of §II, we have to choose new sectorized representatives for \( w_2, w_4, \ldots \) and compare the norms of these new sectorized representatives with respect to the new sectorization to the norms of the old sectorized representatives with respect to the old sectorization. To isolate this problem, suppose that \( j' > j \), that \( \Sigma \) and \( \Sigma' \), respectively, are sectorizations of scale \( j \) and \( j' \), respectively, of length \( l \) and \( l' \), respectively and that \( l' < l \). Furthermore, let \( \omega_{2n}((p_1, s_1), \ldots, (p_n, s_n), (q_1, s_{n+1}), \ldots, (q_n, s_{2n})) \) be a \( \Sigma \)-sectorized representative of \( w_{2n} \). Using the partition of unity \( \{\chi_{s'}\}_{s' \in \Sigma} \) subordinate so \( \Sigma' \), one constructs the \( \Sigma' \)-sectorized representative

\[
\omega'_{2n}((p_1, s'_1), \ldots, (q_n, s'_{2n})) = \sum_{s'_1 \in \Sigma', s'_1 \cap s_i \neq \emptyset, 1 \leq i \leq 2n} \chi_{s'_1}(p_1) \cdots \chi_{s'_{2n}}(q_n) \omega_{2n}((p_1, s_1), \ldots, (q_n, s_{2n}))
\]

of \( w_{2n} \). The norm \( \|\omega'_{2n}\|_{1, \Sigma'} \) of (II.6) is defined in terms of a supremum over \( s' \in \Sigma' \) of a sum over

\[
\text{Mom}_i(s') = \{(s'_1, \ldots, s'_{2n}) \in \Sigma'^{2n} \mid s'_i = s' \text{ and there exist } p_{\ell} \in s'_{\ell}, q_{\ell} \in s'_{n+\ell}, 1 \leq \ell \leq n \text{ such that } p_1 + \cdots + p_n = q_1 + \cdots + q_n \}
\]

It is natural to partition the sum over \( \text{Mom}_i(s') \) into a sum over \( (s_1, \ldots, s_{2n}) \in \Sigma^{2n} \) followed by a sum over elements of \( \text{Mom}_i(s') \) that obey \( s_\ell \cap s'_\ell \neq \emptyset \) for all \( 1 \leq \ell \leq 2n \). Now, we will try to motivate that, “morally”, for any fixed \( s_1, \ldots, s_{2n} \in \Sigma \), there are at most \( \left[\text{const} \frac{1}{l}\right]^{2n-3} \) elements of \( \text{Mom}_i(s') \) obeying \( s_\ell \cap s'_\ell \neq \emptyset \) for all \( 1 \leq \ell \leq 2n \). We may assume that \( i = 1 \). Then \( s'_1 \) must be \( s' \). Denote by \( I_\ell \) the interval on the Fermi curve \( F \) that has length \( l + 2l' \) and is centered on \( s_\ell \cap F \). If \( s' \in \Sigma' \) intersects \( s_\ell \), then \( s' \cap F \) is contained in \( I_\ell \). Every sector in \( \Sigma' \) contains an interval of \( F \) of length \( \frac{3}{4} l' \) that does not intersect any other sector in \( \Sigma' \). (The specific number \( \frac{3}{4} \) comes from Definition VI.2. It is not important.) At most \( \left[\frac{4}{3} \frac{1 + 2l'}{l}\right]^{2n-3} \) choices for \( s'_{i}, i \neq 1, n, 2n \).

Fix \( s'_{i}, i \neq n, 2n \). Once \( s'_{n} \) is chosen, \( s'_{2n} \) is essentially uniquely determined by conservation of momentum. But the desired bound demands more. It says, roughly speaking, that both \( s'_{n} \) and \( s'_{2n} \) are essentially uniquely determined. As \( p_\ell \) and \( q_\ell \) run over \( s'_{\ell} \) and \( s'_{n+\ell} \), respectively, for \( 1 \leq \ell \leq n - 1 \), the sum \( p_1 + \cdots + p_{n-1} - q_1 - \cdots - q_{n-1} \) runs over a small set centered on some point \( k \). In order for \( (s'_{1}, \ldots, s'_{2n}) \) to be in \( \text{Mom}_1(s') \), there must exist \( p' \in s'_n \cap F \) and \( q' \in s'_{2n} \cap F \) with \( q' - p' \) very close to \( k \). But \( q' - p' \) is a secant joining
two points of the Fermi curve \( F \). We have assumed that \( F \) is strictly convex. Consequently, for any given \( k \neq 0 \) in \( \mathbb{R}^2 \) there exist at most two pairs \((p', q') \in F^2\) with \( q' - p' = k \). So, if \( k \) is not near the origin, \( s'_n \) and \( s'_{2n} \) are almost uniquely determined. If \( k \) is close to zero, then \( p_1 + \cdots + p_{n-1} - q_1 - \cdots - q_{n-1} \) must be close to zero and the number of allowed \( s'_i \), \( i \neq n, 2n \) is reduced. Careful application of these types of arguments yields (Proposition XIX.4 of [FKTo4])

\[
\begin{align*}
\|\hat{\omega}'_{2n}\|_{1, \Sigma'} & \leq \text{const}_n \left[ \frac{1}{T} \right]^{2n-3} \left( \|\hat{\omega}_{2n}\|_{1, \Sigma} + \frac{1}{T} \|\hat{\omega}_{2n}\|_{3, \Sigma} \right) \\
\|\hat{\omega}'_{2n}\|_{3, \Sigma'} & \leq \text{const}_n \left[ \frac{1}{T} \right]^{2n-4} \|\hat{\omega}_{2n}\|_{3, \Sigma}
\end{align*}
\] (A.2)

For this reason, we usually estimate the combination \( \|\hat{\omega}_{2n}\|_{1, \Sigma} + \frac{1}{T} \|\hat{\omega}_{2n}\|_{3, \Sigma} \). The analog of (II.11) and (II.15) for this combination is

\[
\|\hat{\omega}_{2n}\|_{1, \Sigma} + \frac{1}{T} \|\hat{\omega}_{2n}\|_{3, \Sigma} \quad \text{is of order} \quad \frac{M^{(n-2)}}{1^{n-1}} \quad \text{for all} \ n
\]

Thanks to (A.2), this bound is preserved, for \( n > 2 \), when the sector decomposition is refined.

\[
\begin{align*}
M^{-(j+1)(n-2)} t^{n-1} \left( \|\hat{\omega}'_{2n}\|_{1, \Sigma'} + \frac{1}{T} \|\hat{\omega}'_{2n}\|_{3, \Sigma'} \right) \\
\leq \text{const}_n M^{-(j+1)(n-2)} t^{n-1} \left[ \frac{1}{T} \right]^{2n-3} \left( \|\hat{\omega}_{2n}\|_{1, \Sigma} + \frac{1}{T} \|\hat{\omega}_{2n}\|_{3, \Sigma} \right) \\
= \text{const}_n M^{-(n-2)} [\frac{1}{T}]^{n-2} M^{-j(n-2)} t^{n-1} \left( \|\hat{\omega}_{2n}\|_{1, \Sigma} + \frac{1}{T} \|\hat{\omega}_{2n}\|_{3, \Sigma} \right) \\
= \text{const}_n M^{-(1-N)(n-2)} M^{-j(n-2)} t^{n-1} \left( \|\hat{\omega}_{2n}\|_{1, \Sigma} + \frac{1}{T} \|\hat{\omega}_{2n}\|_{3, \Sigma} \right)
\end{align*}
\]

We even have a small factor, \( M^{-(1-N)(n-2)} \), available for eating up constants like \( \text{const}_n \).
References

[AM] N. W. Ashcroft, N. D. Mermin, *Solid State Physics*, Saunders College (1976).

[BG] G. Benfatto and G. Gallavotti, *Renormalization Group*, Physics Notes, Vol. 1, Princeton University Press (1995).

[C] E. R. Caianello, *Number of Feynman Diagrams and Convergence*, Nuovo Cimento 3, 223-225 (1956).

[DMR] M. Disertori, J. Magnen, V. Rivasseau, *Interacting Fermi liquid in three dimensions at finite temperature: Part I: Convergent Contributions*, preprint cond-mat/0012270.

[DR1] M. Disertori, V. Rivasseau, *Interacting Fermi liquid in two dimensions at finite temperature. Part I: Convergent Attributions*, Communications in Mathematical Physics 215, 251–290 (2000).

[DR2] M. Disertori, V. Rivasseau, *Interacting Fermi liquid in two dimensions at finite temperature. Part II: Renormalization*, Communications in Mathematical Physics 215, 291–341 (2000).

[FKLT1] J. Feldman, H. Knörrer, D. Lehmann, E. Trubowitz, *Fermi Liquids in Two-Space Dimensions*, in *Constructive Physics*, V. Rivasseau ed., Springer Lecture Notes in Physics 446, 267-300 (1995).

[FKLT2] J. Feldman, H. Knörrer, D. Lehmann, E. Trubowitz, *Are There Two Dimensional Fermi Liquids?*, in *Proceedings of the XIth International Congress of Mathematical Physics*, D. Iagolnitzer ed., 440-444 (1995).

[FKLT3] J. Feldman, H. Knörrer, D. Lehmann, E. Trubowitz, *A Class of Fermi Liquids*, in *Particles and Fields*, G. Semenoff and L. Vinet eds., CRM Series in Mathematical Physics, 35-62 (1999), Springer-Verlag, New York.

[FKTa] J. Feldman, H. Knörrer, E. Trubowitz, *Asymmetric Fermi Surfaces for Magnetic Schrödinger Operators*, Communications in Partial Differential Equations 25 (2000), 319-336.

[FKTcf] J. Feldman, H. Knörrer, E. Trubowitz, *A Nonperturbative Representation for Fermionic Correlation Functions*, Communications in Mathematical Physics, 195, 465-493 (1998).

[FKTf2] J. Feldman, H. Knörrer, E. Trubowitz, *A Two Dimensional Fermi Liquid, Part 2: Convergence*, preprint.
[FKTf3] J. Feldman, H. Knörrer, E. Trubowitz, A Two Dimensional Fermi Liquid, Part 3: The Fermi Surface, preprint.

[FKTffi] J. Feldman, H. Knörrer, E. Trubowitz, Fermionic Functional Integrals and the Renormalization Group, André Aisenstadt Monograph Series, to appear.

[FKTl] J. Feldman, H. Knörrer, E. Trubowitz, Particle–Hole Ladders, preprint.

[FKTo1] J. Feldman, H. Knörrer, E. Trubowitz, Single Scale Analysis of Many Fermion Systems, Part 1: Insulators, preprint.

[FKTo2] J. Feldman, H. Knörrer, E. Trubowitz, Single Scale Analysis of Many Fermion Systems, Part 2: The First Scale, preprint.

[FKTo3] J. Feldman, H. Knörrer, E. Trubowitz, Single Scale Analysis of Many Fermion Systems, Part 3: Sectorized Norms, preprint.

[FKTo4] J. Feldman, H. Knörrer, E. Trubowitz, Single Scale Analysis of Many Fermion Systems, Part 4: Sector Counting, preprint.

[FKTr1] J. Feldman, H. Knörrer, E. Trubowitz, Convergence of Perturbation Expansions in Fermionic Models, Part 1: Nonperturbative Bounds, preprint.

[FKTr2] J. Feldman, H. Knörrer, E. Trubowitz, Convergence of Perturbation Expansions in Fermionic Models, Part 2: Overlapping Loops, preprint.

[FMRT] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An Infinite Volume Expansion for Many Fermion Green’s Functions, Helvetica Physica Acta, 65 (1992) 679-721.

[FST1] J. Feldman, M. Salmhofer, and E. Trubowitz, Perturbation Theory around Non–Nested Fermi Surfaces, I. Keeping the Fermi Surface fixed, Journal of Statistical Physics, 84 (1996) 1209-1336.

[FST2] J. Feldman, M. Salmhofer, and E. Trubowitz, Regularity of the Moving Fermi Surface: RPA Contributions, Communications on Pure and Applied Mathematics, LI, 1133-1246 (1998).

[FST3] J. Feldman, M. Salmhofer, and E. Trubowitz, Regularity of Interacting Non-spherical Fermi Surfaces: The Full Self–Energy, Communications on Pure and Applied Mathematics, LII, 273-324 (1999).

[FST4] J. Feldman, M. Salmhofer, and E. Trubowitz, An inversion theorem in Fermi surface theory, Communications on Pure and Applied Mathematics, LIII, 1350–1384 (2000).

[FW] A.L. Fetter and J.D. Walecka, Quantum Theory of Many-Particle Systems, McGraw-Hill, 1971.
[MR] J. Magnen, V. Rivasseau, A Single Scale Infinite Volume Expansion for Three-Dimensional Many Fermion Green’s Functions, Mathematical Physics Electronic Journal 1, No. 3 (1995).

[MCD] W. Metzner, C. Castellani and C. Di Castro, Fermi Systems with Strong Forward Scattering, Adv. Phys. 47, 317-445 (1998).

[PT] J. Pöschel, E. Trubowitz, Inverse Spectral Theory, Academic Press (1987).

[S] M. Salmhofer, Continuous renormalization for Fermions and Fermi liquid theory, Communications in Mathematical Physics 194, 249–295 (1998).
## Notation

| Not’n | Description | Reference |
|-------|-------------|-----------|
| $\mathcal{E}$ | counterterm space | Definition I.1 |
| $r_0$ | number of $k_0$ derivatives tracked | following (I.3) |
| $r$ | number of $k$ derivatives tracked | following (I.3) |
| $M$ | scale parameter, $M > 1$ | before Definition I.2 |
| $\nu^{(j)}(k)$ | $j^{th}$ scale function | Definition I.2 |
| $\nu^{(\geq j)}(k)$ | $\sum_{i \geq j} \nu^{(j)}(k)$ | Definition I.2 |
| $n_0$ | degree of asymmetry | Definition I.10 |
| $\|\cdot\|_{1,\Sigma}$ | no derivatives, all but 1 sector summed | (II.6) |
| $\|\cdot\|_{3,\Sigma}$ | no derivatives, all but 3 sectors summed | (II.14) |
| $J$ | particle/hole swap operator | (III.3) |
| $\Omega_S(W)(\phi,\psi)$ | $\log \frac{1}{Z} \int e^{W(\phi,\psi + \zeta)} d\mu_S(\zeta)$ | Definition III.1 |
| $\tilde{\Omega}_C(W)(\phi,\psi)$ | $\log \frac{1}{Z} \int e^{J^c(\phi,\psi + \zeta)} e^{W(\phi,\psi + \zeta)} d\mu_C(\zeta)$ | Definition III.1 |