Schulze and Ranked-Pairs Voting Are Fixed-Parameter Tractable to Bribe, Manipulate, and Control*

Lane A. Hemaspaandra, Rahman Lavaee
Department of Computer Science
University of Rochester
Rochester, NY 14627, USA

Curtis Menton†
Google Inc.
Mountain View, CA 94043

October 25, 2012; revised June 21, 2014

Abstract

Schulze and ranked-pairs elections have received much attention recently, and the former has quickly become a quite widely used election system. For many cases these systems have been proven resistant to bribery, control, or manipulation, with ranked pairs being particularly praised for being NP-hard for all three of those. Nonetheless, the present paper shows that with respect to the number of candidates, Schulze and ranked-pairs elections are fixed-parameter tractable to bribe, control, and manipulate: we obtain uniform, polynomial-time algorithms whose degree does not depend on the number of candidates. We also provide such algorithms for some weighted variants of these problems.

1 Introduction

Schulze voting [30], though relatively recently proposed, has quickly been rather widely adopted. Designed in part to well-handle candidate cloning, its users include the Wikimedia foundation, the Pirate Party in a dozen countries, Debian, KDE, the Free Software Foundation Europe, and dozens of other organizations, and Wikipedia even asserts that “currently the Schulze method is the most widespread Condorcet method” [33].

Although the winner-choosing process in Schulze voting is a bit complicated to describe, involving minima, maxima, and comparisons of paths in the so-called weighted majority graph, Schulze [30] proved that finding who won a Schulze election nonetheless is polynomial-time computable, and Parkes and Xia [27] for the so-called destructive case and Gaspers et al. [15] for the so-called constructive case (extending a one-manipulator result for that case by Parkes and Xia [27]) proved that the (unweighted coalitional) manipulation problem for Schulze elections is polynomial-time computable. On the other hand, Parkes and Xia [27] proved that for Schulze elections bribery is NP-hard, and the work of Parkes and Xia [27] and Menton and Singh [24] established that for Schulze elections 15 of the 22 benchmark control attacks are NP-hard.

Parkes and Xia also note that, by the work of [27,33,34], the ranked-pairs election system, which is not widely popular but like Schulze has a polynomial-time winner-determination problem and like

*A two-page extended-abstract version of this paper appeared in the Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2013) [18].

†Work done in part while at the University of Rochester’s Department of Computer Science.
Schulze is based on the weighted majority graph, is resistant to (basically, NP-hard with respect to) bribery, control under each of the control types they study in their paper, and manipulation. Based on their discovery that ranked pairs is more broadly resistant to attacks than Schulze, the fact that Schulze itself “is in wide use,” and the fact that there is “broad axiomatic support for both Schulze and ranked pairs,” Parkes and Xia \[27\] quite reasonably conclude that “there seems to be good support to adopt ranked pairs in practical applications.”

However, in this paper we show that the resistances-to-attack of Schulze and ranked pairs are both quite fragile.

For each of the bribery and control cases studied by Parkes and Xia, Menton and Singh, and Gaspers et al. for which they did not already prove Schulze voting to be in P, we prove that Schulze voting is fixed-parameter tractable with respect to the number of candidates. (The (unweighted) manipulation cases were already all put into P by these papers.) Fixed-parameter tractable (see \[26\]) means there is an algorithm for the problem whose running time is \(f(j)I^{o(1)}\), where \(j\) is the number of candidates and \(I\) is the input’s size. This of course implies that for each fixed number of candidates, the problems are in polynomial time, but it says much more; it implies that there is a global bound on the degree of the polynomial running time, regardless of what the fixed number of candidates is.

That result might lead one to even more strongly suggest the adoption of ranked pairs as an attractive alternative to Schulze. However, although for ranked pairs Parkes and Xia proved all the types of bribery, control, and manipulation they studied to be NP-hard, we show that every one of those cases is fixed-parameter tractable (with respect to the number of candidates) for ranked pairs. So even ranked pairs does not offer a safe haven from fixed-parameter tractability.

Our final results section, Section 7, looks at bribery and manipulation in the case of weighted voting, and proves a number of results for that case. For example, for ranked pairs, we prove that weighted constructive coalitional manipulation is fixed-parameter tractable with respect to the combined parameter “number of candidates” and “cardinality of the manipulators’ weight set.” We give evidence that this fixed-parameter tractability result cannot be extended to a general P result, namely, we prove that weighted constructive coalitional manipulation is NP-complete for five or more candidates. We also show that this “five” is optimal unless \(P = NP\), by proving that this problem is in polynomial time for four or fewer candidates.

2 Presentation of the Key Idea

Our fixed-parameter tractability proofs are of interest in their own right, because they face a very specific challenge, which at first might not even seem possible to handle. We now describe in relatively high-level terms what that challenge is and how we handle it.

Before we start that explanation, we need to present the definition of Schulze voting. Voters will always vote by linear orders over the candidates; in doing that, we adopt the complete, tie-free ordering case of Schulze used in the papers most related to this one \[27\].\[21\]\[15\]. Given the input set of candidates and the set of votes over them (as linear orders), the weighted majority graph (WMG) is the graph that for each ordered pair of candidates \(c\) and \(d\), \(c \neq d\), has an edge from \(c\) to \(d\) having weight equal to the number of voters who prefer \(c\) to \(d\) minus the number of voters who prefer \(d\) to \(c\). Clearly, either all WMG edges have even weight or all WMG edges have odd weight, and the weight of the edge from \(c\) to \(d\) is negative one times the weight of the edge from \(d\) to \(c\). The “strength” of a directed path between two nodes in the WMG is the minimum weight of all the edges along that path. The strength can be negative. The Schulze election system is that candidate \(c\) is a winner exactly if for each other candidate \(d\) it holds that there is some simple path from \(c\) to \(d\) whose strength is at least as great as that of every simple path from \(d\) to \(c\). A lovely result is that
Figure 1: WMG for the election examples, with one edge from each pair left implicit. Those edges are the reverse edges of the displayed edges, with a weight equal to negative one times the weight of their displayed counterpart, e.g., the implicit edge from candidate 2 to candidate 4 has weight -4 and the implicit edge from candidate 3 to candidate 1 has weight 0.

the set of winners, under this definition, is always nonempty \[30\].

We now give a small Schulze-election example from Parkes and Xia \[27\], over the candidate set \{1,2,3,4\}. Although the votes are not specified here, using McGarvey’s method \[22\] we can build a profile of votes realizing the WMG of Figure 1. We will carefully explain McGarvey’s method in Appendix B; for now we ask the reader to briefly take our claim on faith. Candidate 4 is the sole Schulze winner, strictly beating each other candidate in best-path strength. For each other candidate \(i\), candidate 4 has a path to \(i\) of strength 6, but \(i\)’s strongest path to 4 has strength 2.

One of the most powerful tools to use in building algorithms establishing fixed-parameter tractability is a result due to Lenstra, showing that the integer linear programming feasibility problem (henceforth, ILPFP) is in P if the number of variables is fixed \[19\]. Lenstra’s result, based on the geometry of numbers, is very deep and so strong that intuition whispers it should not even be true; yet it is.

Now, if within an appropriate-sized integer linear program with a number of variables that was bounded by some function of the number of candidates we could capture our bribery/manipulation/control challenges and the action of the election system, we would be home free. Indeed, this has been done, for control, for such systems as plurality, veto, Borda, Dodgson, and others (see the discussion on p. 338 of \[10\]). However, Schulze and ranked-pairs elections have such extremely demanding definitions that they seem well beyond such an approach, and we have not been able to make that approach work. So we have a challenge.

Fortunately, the literature provides a way to hope to approach even systems that are too hard to directly wedge—together with the manipulative action—into an ILPFP. That approach is to define some sort of structure associated with subcases of behavior/outcomes of an election system, such that for each fixed number of candidates the number of such structures is bounded as a function of the number of candidates (independent of the number of voters), yet such that for each such structure we can wedge into an ILPFP the question of whether the given action can be made to succeed in the system in a way that is consistent with that structure. If that can be done, then we just loop over all such structures (for the given number of candidates), and for each of them build and run the appropriate ILPFP.

This has not been done often, but it has been done for example by Faliszewski et al. \[12\], with respect to some control problems, for the election system known as Copeland voting. And the structure they used is what they called a Copeland Output Table, which is a collection of bits
associated with the outcomes of the pairwise majority contests between the candidates.

Unfortunately, such output tables don’t seem to have enough information to support the case of Schulze or ranked pairs. (However, see the comments at the start of Appendix A.) The natural structure that would allow us to tackle our systems is the one the systems are based on, namely, the WMG, and looping over all of those would allow us within the loop to easily write/run an appropriate ILPFP to check the given case. However, that falls apart because the number of WMGs is not bounded as a function of the number of candidates; the number also grows as a function of the number of voters. The impossibility of looping over WMGs leaves us still faced with the challenge of how to tackle our problems.

A key contribution of this paper is to show that the needle described above can be threaded—and to thread it—for Schulze and ranked-pairs elections. In particular, we need to, for each of those election systems, find a (winner-set certification) structure that on one hand is rich enough that for each structure instance we can within an ILPFP check whether the given manipulative action can lead to success in a way consistent with the case of which the particular instance of the structure is speaking. Yet on the other hand, the structure must be so restrictive that the number of such structures is bounded purely as a function of the number of candidates (independent of the number of voters). In brief, we need to find, if one exists, a “sweet spot” that meets both these competing needs.

We achieve this with structures we call Schulze winner-set certification frameworks (SWCFs) and ranked pairs winner-set certification frameworks (RPWCFs). A Schulze winner-set certification framework contains a “pattern” for how we can prove that a given set of candidates is the winner set of a Schulze election. To do that, the structure for each winner a specifies, for each other candidate b, a “strong path” \( \gamma_{ab} \) from a to b in the WMG (recall that victory in Schulze elections is based on having strong paths), and then—to establish that the other candidate b has no stronger path back to a—for every simple path from b back to our candidate a the structure identifies a “weak link” (a directed edge on that path) that will keep the path from being too strong; to be more specific, we mean an edge on that path in the WMG such that its weight is less than or equal to that of every edge in our allegedly quite strong path \( \gamma_{ab} \). (Now, keep in mind, at the time we are looping through the structure, we will not even know how strong each link is, as the manipulation/bribery/control will not yet even have happened; rather, the structure is specifying a particular pattern of victory, and the ILPFP will have to check whether the given type/amount of manipulation/bribery/control can bring to life that victory pattern.) Additionally, for each candidate a the structure claims is not a winner, the structure will specify what rival b eliminates that candidate from the winner set and then will outline a pattern for a proof that that is the case, in particular giving a “strong path” from b to a and for each simple path from a to b our structure will specify a “weak link,” i.e., an edge on that path from a to b whose weight in the WMG we hope will be strictly less than the weight of all edges in the selected strong path from b to a; if all our hopes of this sort turn out to be true (and that is among what the integer linear program will be testing, for each of our certification framework’s structures), this proves that b eliminates a. Crucially, the number of structures in that Schulze winner-set certification framework, though large, is bounded as a function of the number of candidates. The certification framework, however, does not itself have its hands on the weights of the WMG, and so the paths and edges it specifies are all given in terms of the self-loop-free graph, on nodes named \( 1, 2, \ldots, ||C|| \), that between each pair of distinct nodes has edges in both directions. Since the candidate names are irrelevant in Schulze voting, we can change to those canonical names, so that our Schulze structures are always in terms of those names.

Crucially, as noted above the number of structures in our Schulze winner-set certification framework, though large, is bounded as a function of the number of candidates. Yet, also crucially, this approach provides enough structure to allow a polynomial-sized (technically, it is actually a
uniform-over-all-numbers-of-candidates polynomial multiplied by a constant that may depend on the number of candidates; this does not trivialize the claim, since the ILPFP must work for all numbers of voters) ILPFP to do the “rest” of the work, namely, to see whether by a given type of attack we can bring to life the proof framework that a given instance of the structure sets out, as to who the winners/nonwinners are in the Schulze election and why. Appendix A gives an alternate winner certification framework that readers may wish to look at to get more of a sense of what certification frameworks do and how they may look. That appendix’s certification framework is closer in flavor to Copeland Output Tables than the approach above is, although as the appendix discusses the alternative approach is both quite clear and quite subtle.

For ranked pairs, the entire approach is what we just described above, except the certification framework we use is completely different than that used for Schulze. Ranked pairs is a method that is defined in highly sequential terms, through successive rounds some of which add a relationship between two candidates, and so our certification framework will be making extensive guesses about what happens in each round (and about a number of other things). But again, we will ensure that the number of such certification structures is bounded as a function of the number of candidates (independent of the number of voters), yet each structure will give enough information that the rest of the work can be done by an integer linear programming feasibility problem. Our notion of a ranked pairs winner-set certification framework will be given in detail in Section 5.1.2.

3 Definitions

Schulze elections were defined in the previous section. We now define the quite different system known as ranked pairs, due to Tideman (see [32]). The ranked-pairs winner is defined by a sequential process that uses the weighted majority graph (WMG). We choose the edge in the WMG of greatest weight, say from \( a \) to \( b \), and fix in the eventual output that \( a \) must beat \( b \); cases of ties, either regarding what edge has the greatest weight, or regarding cases where (\( a, b \)) and (\( b, a \)) both are weight zero, are handled as will be specified in footnote 1. We then remove the edges between \( a \) and \( b \) from the WMG. We then iterate this process, except if the greatest remaining edge is one between two candidates who are already ordered by earlier fixings of output ordering (this can happen, due to transitivity applied to earlier fixings), then we discard the pair of edges between those candidates. We continue until we have completely fixed a linear order. The candidate at the top of this linear

\footnote{There are two different types of ties that must be handled. One is when we get to a case when we are considering an edge, and we don’t discard it, and the candidates tie (the edges between them are both 0); here, we break ties using some simple ordering among the candidates. By simple, we mean feasible; there is a polynomial-time machine that, given the candidates, outputs a linear ordering of them that is the ordering to use when breaking ties of this sort. The second type of tie is when there is a tie as to what is the largest edge remaining in the WMG. In ties of that sort, we use a simple—again, by simple we mean feasible, analogously to the first case—ordering among all unordered pairs of candidates to decide which pair having a highest-weight edge still left is the one to next consider. If that pair is \( \{a, b\} \) and both \( (a, b) \) and \( (b, a) \) have weight zero, either edge can be chosen to consider next, since which we consider at this point among \( (a, b) \) and \( (b, a) \) makes no difference in the result of this step. An at first seemingly tempting alternate approach to breaking ties would be to require as part of the input the two types of tie-breaking orders discussed above. But that is highly unattractive, since that would require changing the definitions of long-defined problems (manipulation, control, bribery), in order to add that extra input part. In truth, the tie-breaking is being made, by us and the earlier papers, to be a part of ranked pairs; and so it should be a feature or setting that is part of one’s version of ranked pairs, and should not be built in by hacking the notions of manipulative actions. So to us, if one wants to speak about ranked pairs, one to be clear and complete must also specify the two feasible tie-breaking functions that are needed to completely define the system. However, our main results for ranked pairs, which are fixed-parameter tractability results, will all hold for all feasible tie-breaking functions. Here is an example of a pair of feasible tie-breaking functions. One could break ties between two candidates in favor of the lexicographically larger. And one could break ties between two candidate pairs in favor of the pair with}
order is the winner under ranked pairs. Even if the first removed edge is from \( a \) to \( b \) and that edge has positive weight, it is possible that \( a \) will not be the ranked pairs winner.

We give a small example of selecting the winner under ranked pairs. We again consider the election with candidate set \( \{1, 2, 3, 4\} \) and votes such that Figure 1 is the WMG. We will break order-of-consideration ties (due to tied edge weights) between \( \{a, b\} \) and \( \{c, d\} \) in favor of which pair has the lexicographically-larger larger-candidate-of-the-pair, and if they tie in that, on which has the lexicographically-larger smaller-candidate-of-the-pair. Thus we handle the edges in the following order: 3\( \rightarrow \)2, 2\( \rightarrow \)1, 4\( \rightarrow \)1, 1\( \rightarrow \)3, 4\( \rightarrow \)2, 3\( \rightarrow \)4. The output ordering will be set by those (3\( \rightarrow \)2, 2\( \rightarrow \)1, etc.), except with 1\( \rightarrow \)3 discarded due to transitivity. So under ranked pairs, 3 is the sole winner.

As mentioned earlier, our elections are specified by a set of candidates and voters (each vote is a tie-free linear ordering of the candidates). The standard (also called “nonsuccinct”) approach to the votes is that each comes in separately. In the succinct approach, which is meaningful only for systems such as Schulze and ranked pairs that don’t care about voters’ names, each tie-free linear ordering that is cast by at least one voter comes with, as a binary integer, the number of voters that voted that way.

In our problems we will speak of making a candidate \( p \) a winner or precluding \( p \) from being a winner. This is known as the nonunique-winner model or, in some papers, the co-winner model. If one changes “a winner” into “the one and only winner,” that is what is known as the unique-winner model.

The problem definitions we are about to give present the definitions for the nonsuccinct, nonunique-winner case. However, the above two paragraphs make clear the very slight changes needed to define the succinct, nonunique-winner case, the nonsuccinct, unique-winner case, and the succinct, unique-winner case. For consistency with the literature, the wording of many of our attack-problem definitions is taken directly from or modeled on the definitions given in [12]. In these definitions, \( \mathcal{E} \) will represent the election system.

**Name:** The constructive bribery problem for \( \mathcal{E} \) elections and the destructive bribery problem for \( \mathcal{E} \) elections [9].

**Given:** A set \( C \) of candidates, a collection \( V \) of voters (specified via their tie-free linear orders over \( C \)), a distinguished candidate \( p \in C \), and a nonnegative integer \( k \).

**Question (constructive):** Is it possible to change the votes of at most \( k \) members of \( V \) in such a way that \( p \) is a winner of the election whose voter collection is \( V \) and whose candidate set is \( C \).

**Question (destructive):** Is it possible to change the votes of at most \( k \) members of \( V \) in such a way that \( p \) is not a winner of the election whose voter collection is \( V \) and whose candidate set is \( C \).

In the literature, the above problem is often called the unweighted, unpriced bribery problem.

**Name:** The constructive manipulation problem for \( \mathcal{E} \) elections and the destructive manipulation problem for \( \mathcal{E} \) elections [10].
Given: A set $C$ of candidates, a collection $V$ of voters (specified via their tie-free linear orders over $C$), a voter collection $W$ (each starting as a blank slate), and a distinguished candidate $p \in C$.

Question (constructive): Is it possible to assign (tie-free linear order) votes to the members of $W$ in such a way that $p$ is a winner of the election whose voters are the members of $V$ and $W$, and whose candidate set is $C$.

Question (destructive): Is it possible to assign (tie-free linear order) votes to the members of $W$ in such a way that $p$ is not a winner of the election whose voters are the members of $V$ and $W$, and whose candidate set is $C$.

In the literature, this is often called the unweighted coalitional manipulation problem. We will sometimes refer to the members of the manipulative coalition $W$ as manipulators and to the members of $V$ as nonmanipulators.

As benchmarks, over time eleven “standard” types of control questions have emerged [2,17,12], each with a constructive version and a destructive version. Four of the eleven are each of adding/deleting (at most $k$, with $k$ part of the input) candidates/voters. A fifth is so-called unlimited adding of candidates. The remaining six are partition of candidates, runoff partition of candidates, and partition of voters, each in both the model where first-round ties promote and in the model where first-round ties eliminate. The benchmark types are modeled on situations from the real world. For example, constructive control by adding voters models highly targeted get-out-the-vote drives and constructive control by adding candidates models trying to draw into elections spoiler candidates who will hurt your opponents. See for example [2,17,12] for a detailed discussion of the motivations of the benchmark control types. We mention in passing that recent work [16] has shown that in the nonunique-winner model two pairs of the eleven destructive types are pairwise identical as to which instances they can be carried out on, while one pair is identical in the unique-winner model.

Due to there being many control types, giving their definitions can take up a large amount of space. In order to keep the paper self-contained for those who want more formal definitions, as Appendix C we provide in their typically stated forms the definitions of all the control types. For those who already know the definitions or at this point simply want a brief treatment (knowing Appendix C is available with formal definitions), the rest of the present paragraph gives a short overview of the flavor of the different benchmark control attacks. All these problems have as their input an election, $(C, V)$, and a distinguished candidate $p \in C$. Constructive (destructive) control by deleting voters—for a given election system, of course—also has a nonnegative integer $k$ in the input and asks whether there is a subset of $V$ of cardinality at most $k$ such that with that subset removed $p$ is (is not) a winner. Control by adding voters is analogous, except the input is the election, $k$, and a set $W$ of voters who can be added (but at most $k$ can be added). Deleting candidates and adding candidates are analogous to the voter cases, with a $k$ as part of the input, and the only twist is that in destructive control by deleting candidates, it is forbidden to delete $p$. Unlimited adding of candidates is the same except there is no limit $k$. Constructive (destructive) partition of voters, in the ties-promote model, asks whether there is a way of partitioning the voters into two groups so that if all winners under the election system of each of those first-round elections compete in a final election under the same election system in which all voters vote (with their votes masked down to the remaining candidates), $p$ is (is not) a winner. In its ties-eliminate variant, only unique-winners of a first round election move forward. The runoff partition of candidates types are analogous, except in the first round it is the candidates that are partitioned and all voters vote in each of those subelections. Partition of candidates has just one side of the partition participating in the first-round election, while the others get a bye to the final round.
A problem is said to belong to the class FPT (is said to be fixed-parameter tractable, see [26]) with respect to a parameter if there is an algorithm for the problem whose running time is $f(j)I^{O(1)}$, where $j$ is the input’s value of that parameter, $f$ is a computable function, and $I$ is the input’s size. Note that this means that although the algorithm for larger values of $j$ can have a bigger multiplicative constant, the degree of the polynomial running time is uniformly bounded from above—there is some single integer $k$ such that regardless of the fixed $j$ the algorithm for that parameter bound runs in time $O(I^k)$. Our parameter will almost always be the most natural one—the number of candidates. In Section 7 we will have cases where our parameter is a tuple of features of the input rather than a single feature, i.e., is a “combined” parameter (see [3, Chapter 9]). We stress that FPT should not be confused with the class, of which it is a subset, XP. XP problems are in P for each fixed bound on the parameter, but their degree can grow as that bound grows. Thus XP is a far less attractive class, even though by brute force it is often clear that parameterized versions of voting problems fall into it. However, our focus is firmly on the more demanding goal of establishing FPT results.

4 Related Work

The computational complexity of manipulation, bribery, and control for Schulze voting and ranked pairs has been studied previously by Parkes and Xia, Xia et al., Menton and Singh, and Gaspers et al. [27,35,24,15]. The work of Xia et al. and Parkes and Xia establishes (see also the table in [27]) that for Schulze elections constructive and destructive bribery, constructive and destructive control by adding and deleting voters, and constructive control by adding candidates are NP-complete, and that ranked pairs has not only all these hardnesses but also has NP-completeness results for constructive and destructive manipulation, for destructive control by adding candidates, and for constructive and destructive control by deleting candidates. Menton and Singh (see Table 1 of [24]) studied all remaining types of constructive and most types of destructive control for Schulze voting, and their work establishes that NP-completeness holds additionally for constructive control by unlimited adding of candidates, constructive control by deleting candidates, each variant of constructive control by partition or runoff partition of candidates, and each variant of constructive and destructive control by partition of voters, and for the four cases of destructive control by partition or runoff partition of candidates they show that polynomial-time algorithms exist.

The (unweighted coalitional) manipulation problem is shown to be in P for the destructive case by Parkes and Xia [27] and for the constructive case by Gaspers et al. [15]. Gaspers et al. [15] prove that the weighted constructive coalitional manipulation problem for Schulze elections is in polynomial time for each fixed number of candidates. We observe that inspection of their paper immediately makes clear that they indeed have even established the stronger claim that the weighted constructive coalitional manipulation problem for Schulze elections is in the class FPT.

For Schulze, three of the 22 benchmark control cases—each clearly belonging to NP for Schulze—were left open by the abovementioned papers and indeed even now these cases remain open as to whether they are in P, are NP-complete, or have some other complexity: destructive control by adding candidates, destructive control by deleting candidates, and destructive control by unlimited adding of candidates. However, for each of those three (and all other of the benchmark control cases), we obtain membership in FPT.

Briefly, the reason their algorithm for the weighted constructive coalitional manipulation problem for Schulze elections is clearly even an FPT algorithm is as follows. Their algorithm is using the fact, observed independently by Menton and Singh [23] and Gaspers et al. [15], that for the nonunique-winner model, weighted constructive coalitional manipulation problem for Schulze elections, if one can make a given candidate a winner then there is a set of manipulative votes in which all manipulators vote the same way and that candidate is selected as a winner. Once one has this, an FPT algorithm is obtained simply by cycling over all possible preference orders, for each seeing whether, if that is what all the manipulators cast as their vote, the given candidate becomes a winner. And that is precisely what their short, elegant algorithm is doing for this case.
All the results in the two papers involving Xia are in the unique-winner model. Ranked pairs is resolute (has exactly one winner), as Parkes and Xia frame it, and we follow their framing. And so the nonunique-winner and the unique-winner models are in effect the same for ranked pairs. Schulze is not resolute, but although Parkes and Xia’s results on that are in the unique-winner model, they comment that their results all also hold in the nonunique-winner model. Gaspar et al. study both the nonunique-winner model and the unique-winner model. Menton and Singh use the nonunique-winner model as their basic model, as do we in the present paper. To us the nonunique-winner model is more attractive in not requiring a tie-breaking that, especially in symmetric cases, is often arbitrary and can change the flavor of the system. However, our main FPT results are proven by a loop approach over ILPFPs (integer linear programming feasibility problems), and it is clear that a straightforward adjustment to these will also handle the unique-winner cases. The key difference between our work and all the abovementioned work is that our work is in general looking at the complexity of these problems when parameterized by the number of candidates, and for this we give FPT algorithms. The earlier papers primarily looked at unbounded numbers of candidates and obtained both P and NP-completeness results; our contribution is that for all their NP-complete cases, we show membership in FPT.

As to technique, the closest precursors of this paper are two papers by Faliszewski et al. Those, like us, use a loop over ILPFPs. The main differences between that work and ours is that (a) they deal with control, and we also are concerned with bribery and manipulation, and (b) as explained in detail in Section 2 their type of loop-over structure isn’t flexible enough for our cases, and the natural structure for us to loop over generates a number of objects not bounded in the number of candidates, and so in this paper we find a middle ground that allows the loop to be over a bounded-in-∥C∥ number of objects yet provides enough information in the objects so as to allow the ILPFPs to complete the checking of whether success is possible. For a different type of attack known as “swap bribery” and a different election system, Dorn and Schlotter have recently employed what in effect (although implicitly) is a loop over ILPFPs, and they mention in passing without details that that swap bribery approach should apply to ranked pairs.

Taking an even broader perspective, this work is part of a line that looks at the complexity of elections in the context of bounds on the number of candidates, a study that for example has been pursued famously by Conitzer, Sandholm, and Lang regarding at what candidate numbers complexity jumps from P to NP-complete. The particular focus on FPT algorithms, and maintaining a uniform degree bound over all values of bounds on the number of candidates, is part of the important field of parameterized complexity (see [26], see [4] for a survey on this approach for

4Dorn and Schlotter separately mention in passing and without definitions or details that their approach should apply to manipulation and some unspecified variants of control. If one takes as implied there the combination of their ranked-pairs aside and their other attacks-aside (and they in their paper don’t explicitly assert that), then that without-details assertion pair, combined, overlaps some of our ranked-pairs results regarding control (though they do not specify which control types they are speaking of). However, in contrast, we here actually provide a certification framework handling the ranked pairs election system. We believe that their results, even if looking at the combination of their asides, don’t overlap our main work on manipulation, since they speak of “weighted and unweighted manipulation,” but by that undefined-there use they seem to mean a weaker notion of manipulation than that used in the present paper, namely, we are looking at coalitional manipulation, but they seem to be referring to noncoalitional manipulation. (The reason we say this is that for the weighted case of coalitional manipulation, the natural ILPFPs one would generate have neither their number of variables nor their number of constraints “bounded in the number of candidates independently of the number of voters.” Beyond this, for ranked pairs we will prove, as Theorem 7.3 that weighted coalitional manipulation is NP-complete for each fixed number of candidates starting at five; this easily implies that unless P = NP no FPT algorithm can exist. There we will discuss how far toward such algorithms one can seem to get within the ILPFP approach and its extensions, namely, by looking at the special case of bounded weights and even of bounded weight-set-cardinality. That section notes that we can handle even the coalitional case for bounded weights, and indeed even for unbounded weights but bounded manipulator-weight-set-cardinality, and that latter case itself is a generalization of weighted noncoalitional manipulation.)
elections, and see [28] for a survey of an alternate approach to bypassing complexity results).

5 Results by Looping over Frameworks

We now present our results that are established by our looping-over-frameworks idea. We will handle in separate sections bribery, control, and manipulation, showing how to achieve FPT results for each.

In the bribery section, Section 5.1, we will first prove the bribery result for Schulze elections, so that the reader quickly gets to seeing how the proof goes without having to have first seen how the approach works for ranked pairs. We then will give our ranked pairs winner-set certification framework, and will note how to convert our proof into a proof for that case also. Then later in the control section, Section 5.2, we will state and prove together the Schulze-elections case and the ranked-pairs case.

Since (unweighted) Schulze manipulation has been shown to be in P in general, both for the constructive [15] and the destructive [27] cases, we do not need to handle Schulze elections in our manipulation section, Section 5.3. (For the nonunique-winner model, weighted constructive coalitional manipulation for Schulze elections, Gaspers et al. [15] provide what as mentioned earlier is an FPT algorithm. And in Section 7 we will show, as part of Theorem 7.2, that for Schulze elections our approach provides an FPT algorithm for special cases of both weighted destructive coalitional manipulation and unique-winner model, weighted constructive coalitional manipulation.)

5.1 Bribery Results and Specification of Ranked Pairs Winner-Set Certification Framework

In this section, we will first state and prove the bribery result for Schulze, then will give our winner-set certification framework for ranked pairs, and then will state and prove our bribery result for ranked pairs.

5.1.1 Bribery Result for Schulze

We state and prove the bribery result for Schulze.

Theorem 5.1 For Schulze elections, bribery is in FPT (is fixed-parameter tractable) with respect to the number of candidates, in both the succinct and nonsuccinct input models, for both constructive and destructive bribery, in both the nonunique-winner model and the unique-winner model.

Proof. We first give the proof for the constructive, nonunique-winner model case. We will handle simultaneously the succinct and nonsuccinct cases.

Our FPT algorithm works as follows. It gets as its input an instance of the bribery problem, and so gets the candidates (with a distinguished candidate noted), the votes (or for the succinct version, a list of which types of votes occur at least once, along with the multiplicities of each), and the limit \( k \) on how many voters can be bribed. Let \( j \) be the number of candidates in the input instance. To mesh with the naming scheme within our SWCFs, we immediately rename all the candidates (including within the votes) to be \( 1, \ldots, j \), with the distinguished candidate becoming candidate 1. Now, the top-level programming loop of the algorithm is as specified in Algorithm 1 (recall that we are showing the constructive, nonunique-winner model case in this algorithm).

All that remains is to specify the ILPFPPs that we build inside the loop, for each given \( j \)-SWCF \( K \). Suppose we are doing that for some particular \( K \). We do it as follows.
There are $j!$ possible votes over $j$ candidates; let us number them from 1 through $j!$ in any natural computationally simple-to-handle way. We call the $i$th of these the $i$th “vote type.” We will have constants $n_i$, $1 \leq i \leq j!$, denoting how many voters start with vote type $n_i$. Our ILPFP will have integer variables (which we will ensure are nonnegative) $m_{i,\ell}$, $1 \leq i \leq j!$, $1 \leq \ell \leq j!$. $m_{i,\ell}$ is the number of voters who start with vote type $i$ but are bribed to instead cast vote type $\ell$. (Having $m_{i,i} \neq 0$ is pointless but allowed, as is having simultaneously $m_{i,\ell} > 0$ and $m_{i,i} > 0$.) So the number of ILPFP variables is $(j!)^2$, which is large but is bounded with respect to $j$, so Lenstra’s algorithm [19] can be used to deliver an FPT performance overall.

Also, not as direct parts of the ILPFP but as tools to help us build it, we define two boolean predicates, $\text{Bigger}(a,b,c,d)$ and $\text{StrictlyBigger}(a,b,c,d)$, where the arguments each vary over $1, \ldots, j$. Let us use $D(a,b)$ to indicate the weight of the WMG edge (after our manipulative actions) that points from $a$ to $b$. Recall, from Section 2 that what a $j$-SWCF does—and so in particular what our under-consideration $j$-SWCF, $K$, does—is specify (for a very large number of such quadruples, though that number actually is bounded as a function of $j$ through the SWCF) that $D(a,b) \geq D(c,d)$ or that $D(a,b) > D(c,d)$, i.e., in the WMG, a certain edge is greater than or equal to another edge in weight, or is strictly greater in weight. For each such specified relation that explicitly appears in $K$, set to true that bit in the appropriate predicate ($\text{Bigger}$ or $\text{StrictlyBigger}$), and leave all the other bits set to false. We of course will have to enforce these specifications through our constraints.

Now we can specify all the constraints of our ILPFP. There will be three types of constraints. The first are the housekeeping constraints to make sure that the number of bribes and the $m_{i,\ell}$s are all reasonable. Our constraints of this sort are: For each $1 \leq i, \ell \leq j!$, we have a constraint $m_{i,\ell} \geq 0$. For each $1 \leq i \leq j!$, we have a constraint $n_i \geq \sum_{1 \leq z \leq j!} m_{i,z}$; that is, we do not try to bribe away from vote type $i$ more votes than initially exist of vote type $i$. And we have the constraint $k \geq \sum_{1 \leq \ell \leq j!} m_{i,\ell}$; that is, our total number of bribes does not exceed the bribe limit $k$.

The second type of constraint consists of those constraints used to enforce the bits set to “true” in $\text{StrictlyBigger}$. For each such bit, we will generate one constraint: for a bit that is saying that $D(a,b) > D(c,d)$, we will enforce that with the constraint shown in Figure 2 in that figure and for the rest of this paper, in order to make the representation of the constraints more concise, we introduce the shorthand notation $\text{pref}(a,b)$ as the set of vote types $i$, $1 \leq i \leq j!$, in which candidate $a$ is preferred to candidate $b$. All that the bulky-looking constraint of the figure says is that after all the gains and losses due to bribing happen, the number of voters who prefer $a$ to $b$ minus the number who prefer $b$ to $a$ is strictly larger than the number of voters who prefer $c$ to $d$ minus the number...
\[
\left( \sum_{i \in \text{pref}(a,b)} \left( n_i - \left( \sum_{1 \leq \ell' \leq j} m_{i,\ell'} + \sum_{1 \leq \ell' \leq j} m_{i',\ell'} \right) \right) \right)
- \left( \sum_{i \in \text{pref}(b,a)} \left( n_i - \left( \sum_{1 \leq \ell' \leq j} m_{i,\ell'} + \sum_{1 \leq \ell' \leq j} m_{i',\ell'} \right) \right) \right)
\geq 1 + \left( \sum_{i \in \text{pref}(c,d)} \left( n_i - \left( \sum_{1 \leq \ell' \leq j} m_{i,\ell'} + \sum_{1 \leq \ell' \leq j} m_{i',\ell'} \right) \right) \right)
- \left( \sum_{i \in \text{pref}(d,c)} \left( n_i - \left( \sum_{1 \leq \ell' \leq j} m_{i,\ell'} + \sum_{1 \leq \ell' \leq j} m_{i',\ell'} \right) \right) \right).
\]

Figure 2: Constraint enforcing that, after the bribes happen, \( D(a,b) > D(c,d) \).

who prefer \textit{d} to \textit{c}. If \textit{StrictlyBigger}(\textit{a}, \textit{b}, \textit{c}, \textit{d}) is set to “false,” that does not mean we generate a constraint ensuring that \( D(a,b) \not> D(c,d) \). Rather, if a given bit is set to “false,” that just means that that particular bit-setting does not itself create a constraint. In contrast, bits set to “true” in \textit{StrictlyBigger} and \textit{Bigger} mean that we generate a constraint to enforce the stated relation.

The third type of constraint consists of those constraints used to enforce the bits set to “true” in \textit{Bigger}. For each such bit, we will generate one constraint: for a bit that is saying that \( D(a,b) \geq D(c,d) \), we will enforce that with precisely the constraint shown in Figure 2 except with the “1+” removed from the right-hand side of the inequality.

That completes our statement of the ILPFP, which indeed captures what it seeks to capture. And using Lenstra’s algorithm [19] for each of our ILPFPs, the overall loop over the ILPFPs has the desired running time. (Although for each fixed \( j \) the multiplicative constant is very large, the degree of the polynomial, which is uniform over all \( j \), isn’t terrible; Lenstra’s algorithm uses just a linear number of arithmetic operations on linear-sized integers [26]. Still, even within the good news that we have placed the problem within FPT, there is the bad news that the multiplicative constant is so large that this FPT algorithm does not provide an algorithm for practical use.) To be clear, since what is a constant and what is a variable is a bit subtle here, let us say a bit more about the use of Lenstra here. What we in effect are using is that Lenstra’s work ensures that there is a \( k \) such that for each fixed number of candidates and each of the (large but bounded as a function of the number of candidates) ILPFPs generated in our loop, if we view that ILPFP as an object whose running time for solution is being evaluated asymptotically as the number of voters increases without bound, the ILPFP’s running time is \( O(n^k) \). (That same value \( k \) holds for all numbers of candidates and for all ILPFPs that our loop generates for that number of candidates. However, for different numbers of candidates the multiplicative constant represented by the “big O” may differ.) Note that each such ILPFP object in effect has as its set of variables (regarding the asymptotics of its running time) the \textit{constants} of the ILPFP; and a big part of what our looping algorithm does is to set those constants based on the votes in the election.

That was the proof for the constructive, nonunique-winner model case. To change the above proof from the constructive to the destructive case and/or from the nonunique-winner case to the unique-winner case, in the main loop we will simply create ILPFPs for only those SWCFs whose set of who wins and loses reflects a sought outcome. For example, for the destructive case in the unique-winner model, that would be having the distinguished candidate not be a unique winner, i.e., the start of Algorithm 1’s “if” statement would become “if candidate 1 is not a unique winner.”

\[\square\]
The same proof approach applies to ranked pairs. However, we first must do some work to define an appropriate winner-set certification framework for ranked pairs. We turn to that now.

5.1.2 Specification of Ranked Pairs Winner-Set Certification Framework

In this section, we describe the winner-set certification framework that makes our approach work for ranked pairs.

Basically, an instance of that framework will be a story that tells us what happens at each stage of the iterative process that defines ranked pairs. We could actually tell this story without fixing up front, for each pair \( \{a, b\} \) of distinct candidates, whether \( a \) is preferred to \( b \) by a majority of the voters, or whether \( b \) is preferred to \( a \) by a majority of the voters, or whether \( a \) and \( b \) exactly tie as to how many voters prefer one to the other. Not fixing that information up front would improve our multiplicative factor that depends on but is fixed for each fixed number of candidates. But we are not focused on that factor. So to make things particularly simple to describe, we are here just going to toss into our framework a fixing of all such pairwise-outcomes-in-the-WMG. As in the Schulze case, we will have changed all the names of the candidates to be \( 1 \) through \( \|C\| \) (and will have remapped our tie-breaking function in the same way). So, one part of our framework is, for each (unordered) pair of distinct candidates \( \{a, b\} \), a claim as to which one of these holds: the WMG edge from \( a \) to \( b \) is strictly positive, the edge from \( b \) to \( a \) is strictly positive, or both edges are \( 0 \). (We do not include any claim about the precise value of those edge weights; that would create a framework whose number of instances, for a fixed number of candidates, grew with the number of voters—something we must firmly avoid.) And an instance of the framework then goes step by step through the process the ranked-pairs algorithm goes through, but in a somewhat ghostly way in terms of what it specifies. For each step of the process, it makes a claim as to what pair of candidates is considered next, and a claim as to whether that pair of candidates will be skipped permanently due to it having been already set (due to transitivity) by earlier actions of our flow through ranked pairs (that isn’t an on-the-fly thing in that the instance itself has all its earlier claims and so we can even make sure to loop only over instances of the framework that are internally consistent regarding this), and if it is not skipped, a claim about which of the two outcomes happens (which is placed above the other in our ranked-pairs outcome; again, we can read this from those choice-of-3-possibilities settings we did up front, plus the feasible tie-breaking if needed). So that is the story the framework provides, and a given instance of the framework will (if properly formed) set an ordering over all the candidates. As before, the algorithms will loop over instances of these frameworks, doing so over only instances that have the desired outcome (e.g., “\( p \) is a unique winner”) and that aren’t obviously internally inconsistent. (For our partition by voter cases, there is a double-loop over such frameworks, to handle both subelections.)

As in the Schulze case, we will use the ILPFPs to see if the given kind of control can create a case where the given framework can be made to hold. All the housekeeping work in the ILPFPs as to tallying how the votes are bribed/controlled/manipulated is still needed here (so the variable sets are the same as the ones for Schulze). But note, crucially, that we now must enforce not things about paths, but rather we must enforce that the framework’s guesses about whether the edge from \( a \) to \( b \) is negative, positive, or \( 0 \) after the bribery/control/manipulation are all correct (this is very natural to enforce with constraints, within the ILPFP framing), and must also enforce that the framework’s claim about which candidate pair is considered next is what would actually happen under the votes that emerged from the bribery/control/manipulation. But that latter claim, for each step in the story, can be checked by appropriate, carefully built constraints, written with close attention paid to the tie-breaking rule among pairs. These constraints will be pretty much our favorite sort of constraint—seeing whether a WMG edge is greater than or equal to another, or
seeing whether it is strictly greater than another. This will be made clearest by an example. Suppose our candidates are named 1, 2, 3, and 4. And suppose the tie-breaking order on unordered pairs is $\{4, 3\} > \{4, 2\} > \{4, 1\} > \{3, 2\} > \{3, 1\} > \{2, 1\}$, and on candidates is $4 > 3 > 2 > 1$. (This is exactly a case of the sample feasible rule pair we gave in Section 3 where for example tied pairs are tie-broken in favor of the pair with the lexicographically-larger larger-candidate-of-the-pair, and when the larger members are the same in both pairs then the tie is broken in favor of whichever pair has the lexicographically-larger smaller-candidate-of-the-pair. We’ve written our unordered pairs, in the tie-breaking order above, with the lexicographically-larger element first simply to make it clear why Section 3’s example tie-breaking rule would put them in the order shown above.) Suppose the RPWCF says the first pair to be compared is $\{1, 4\}$ and that the outcome is $4 \succ 1$. Let $D(a, b)$ be defined as before. To check that $4 \succ 1$ is the right outcome, since $4 > 1$ in the tie-breaking function we need to check that $D(4, 1) \geq D(1, 4)$ (if $1 \succ 4$ in the tie-breaking order, we’d check that $D(4, 1) \geq 1 + D(1, 4)$; we can read this right off the 3-way-claim as to how 1 and 4 compare in their head-to-head contest, which itself we’ll enforce in constraints. And as to the claim that $(1, 4)$ was the first pair to be compared, in light of the tie-breaking order, that can be enforced using 10 constraints: 6 saying that our pair ties or beats those below us in the tie-breaking ordering $(D(4, 1) \geq D(a, b))$ for the $(a, b)$ values $(3, 2)$, $(2, 3)$, $(3, 1)$, $(1, 3)$, $(2, 1)$, $(1, 2)$, and 4 saying that our pair strictly beats those above us in the tie-breaking ordering $(D(4, 1) > D(a, b))$ for the $(a, b)$ values $(4, 3)$, $(3, 4)$, $(4, 2)$, $(2, 4)$. We could cut those 6 + 4 constraints to 3 + 2 if we wish, by using the value of the 3-way-claim for each of those 5 other pairs. Note that all these comparisons are about post-bribe/manipulation/control vote numbers—things we do know how to easily put into an ILPFP constraint, basically, by appropriate summations. Moving on, if the framework says the next unordered pair after $\{1, 4\}$ to be considered is $\{1, 3\}$ and that the outcome is $1 \succ 3$, we of course will not need to enforce any comparisons with $D(4, 1)$ or $D(1, 4)$; we will generate and put into the ILPFP just the needed/appropriate comparisons. If the framework after that says the third pair to consider is $\{3, 4\}$ but it also says that (due to $4 \succ 1 \succ 3$ already being set) the pair $\{3, 4\}$ gets skipped, we under our RPWCF framework still must generate the constraints to check that $\{3, 4\}$ truly under our votes as they now are did deserve to come up next (we mention in passing that we could skip that check as long as we adjust our ILPFP to not check anything regarding pairs that are already related, even transitively, under the $\succ$’s so far—a slightly different approach than ours but also quite fine), but the generated comparisons-to-check-that will not do comparisons against things the existing order so far ($4 \succ 1 \succ 3$) takes out of play. This completes our description of our RPWCF notion.

5.1.3 Bribery Result for Ranked Pairs

Having specified the ranked pairs winner-set certification framework, the bribery case for ranked pairs can now be stated and justified.

**Theorem 5.2** For ranked pairs (with any feasible tie-breaking functions), bribery is in FPT (is fixed-parameter tractable) with respect to the number of candidates, in both the succinct and nonsuccinct input models, for both constructive and destructive bribery, in both the nonunique-winner model and the unique-winner model.

**Proof.** We use the programming loop of Algorithm 1 to now loop over not $j$-SWCFs, but instead over $j$-RPWCFs, with $j$ again being the number of candidates. The variables of the ILPFPs will be the same as those for Schulze. As for the constraints of the ILPFPs, all the housekeeping constraints for bribery remain intact. Additionally an RPWCF, as described in Section 5.1.2, guesses for each edge whether it is positive, negative, or zero in weight. We can easily handle these possibilities
with the constraints \( D(a, b) \geq 1, D(b, a) \geq 1, \) and \( D(a, b) = 0 \) respectively. The other constraints enforced by an RPWCF are those between pairs of edges, and they can be successfully captured by the \textit{StrictlyBigger} and \textit{Bigger} predicates defined in the proof of Theorem 5.1. Thus we clearly can build an ILPFP encoding an RPWCF and the constraints of the bribery problem in the same way we did for Schulze elections, and bribery is in FPT for ranked pairs as well.

\[ \Box \]

5.2 Control Results

In this section, we prove our results for control gained through the looping-over-frameworks approach. Our proofs for these cases will closely follow our general looping-over-frameworks structure, and we will just have to appropriately build the constraints of our ILPFPs to handle the details of the control problems.

\textbf{Theorem 5.3} For Schulze elections and for (with any feasible tie-breaking functions) ranked pairs, control by adding voters is in FPT with respect to the number of candidates, in both the succinct and nonsuccinct input models, for both constructive and destructive control, and in both the nonunique-winner model and the unique-winner model.

\textit{Proof.} We will handle together all the cases of this theorem.

Again, let the candidates in the control problem be \( 1, \ldots, j \) with the distinguished candidate being \( 1. \) The control problem has as its input a set of initial votes \( V \) and a set of additional votes \( W \) (or for the succinct version, one list for each of these two sets describing which types of votes occur at least once in that set, along with the multiplicities of each), the latter of which contains the votes that can be added by the control action. Also there is a limit \( k \) on the number of votes that can be added from \( W. \)

The top-level programming loop is as described in Algorithm 2 (with WCF being either SWCF for Schulze or RPWCF for ranked pairs).

\begin{algorithm}
\textbf{Start}
\textbf{for} each \( j \)-WCF \( K \) \textbf{do}
\textbf{if} candidate 1 (is/is not) (a winner/a unique winner) according to \( K \) and \( K \) is an internally consistent, well-formed \( j \)-WCF \textbf{then}
\hspace{1em} (1) build an ILPFP that checks whether there is a set of votes \( W' \subseteq W, \) with \( \|W'\| \leq k, \) such that \( K \)'s winner-set certification framework is realized by the set of votes \( V \cup W' \)
\hspace{1em} (2) run that ILPFP and if it can be satisfied then halt and accept (note: the satisfying settings will even let us output the precise added set that succeeds)
\textbf{end if}
\textbf{end for}
\textbf{declare that the given goal cannot be reached by adding at most} \( k \) \textbf{voters}
\textbf{End}
\end{algorithm}

The two binary selections in the algorithm's “if” statement are made as in the proof of Theorem 5.6.

All we have to do now is show how we build the ILPFP inside the loop. Again, as with manipulation, we will have two groups of constraints: those corresponding to the WCF-enforcing predicates and those corresponding to the structure of the control problem. For every \( i, 1 \leq i \leq j!, \) we will have
a variable $v_i$ representing the number of votes of type $i$ in $W'$ (i.e., the $W'$ sought by Algorithm 2), a constant $n_i$ representing the number of votes of type $i$ in $V$, and a constant $h_i$ representing the number of votes of type $i$ in $W$.

As we described in the proof of Theorem 5.6 if we can represent $D(a, b)$ as a linear expression in terms of our constants and variables, we can implement all the WCF-enforcing constraints (those of the StrictlyBigger and Bigger predicates and those of the 3-way possibilities for ranked pairs) as linear constraints in our ILPFP. Thus, using the shorthand notation $\text{pref}(a, b)$ as described in Section 5.3, we express $D(a, b)$ as follows:

$$\sum_{i \in \text{pref}(a, b)} (n_i + v_i) - \sum_{i' \in \text{pref}(b, a)} (n_{i'} + v_{i'}).$$

As for the constraints ensuring the validity of the control action, we first need to ensure that for every type of vote, the number of votes of that type in $W'$ is bounded by the number in $W$. For every vote type $i$, $1 \leq i \leq j!$, the constraint $v_i \leq h_i$ will enforce this in our ILPFP. We also make the following constraint to enforce the adding bound:

$$\sum_{1 \leq i \leq j!} v_i \leq k. \quad (5.1)$$

Here again all of our variables have to be nonnegative, and thus we have the constraint $v_i \geq 0$ for every $i$, $1 \leq i \leq j!$

This suffices to describe how we can build a WCF-enforcing ILPFP for the control by deleting voters problem in a way appropriate for our looping-over-frameworks technique. Thus we have an algorithm that will run in uniform polynomial time for every fixed parameter value, putting this problem in FPT.

**Theorem 5.4** For Schulze elections and for (with any feasible tie-breaking functions) ranked pairs, control by deleting voters is in FPT with respect to the number of candidates, in both the succinct and nonsuccinct input models, for both constructive and destructive control, and in both the nonunique-winner model and the unique-winner model.

**Proof.** We will handle together all the cases of this theorem.

The input for this problem is a control instance with a set of votes $V$ over $j$ candidates $1, \ldots, j$, with candidate 1 being the distinguished candidate, and with a bound $k$ on the number of votes to be deleted. Algorithm 3 specifies the top-level loop.

The two binary selections in the algorithm’s “if” statement are made as in the proof of Theorem 5.6.

For each $i$, $1 \leq i \leq j!$, the ILPFP we build inside the loop will include a constant $n_i$, representing how many votes of vote type $i$ are in the initial election. We will include in the ILPFP variables $v_i$, $1 \leq i \leq j!$, representing the number of votes of type $i$ that are deleted. With this new variable, we will formulate $D(a, b)$ in the ILPFP as follows:

$$\sum_{i \in \text{pref}(a, b)} (n_i - v_i) - \sum_{i' \in \text{pref}(b, a)} (n_{i'} - v_{i'}).$$

With this we can easily build the Bigger and StrictlyBigger predicates we use to enforce the WCF, as well as the additional constraints necessary for an RPWCF.
Algorithm 3 Top-level loop for control by deleting voters

Start
for each \( j \)-WCF \( K \) do
    if candidate 1 (is/is not) (a winner/a unique winner) according to \( K \) and \( K \) is an internally consistent, well-formed \( j \)-WCF then
        (1) build an ILPFP that checks whether there is a subset of the voters \( V' \), with \( \|V'\| \leq k \), such that \( K \)'s winner-set certification framework is realized by the set of votes \( V - V' \)
        (2) run that ILPFP and if it can be satisfied then halt and accept (note: the satisfying settings will even let us output the precise deleted set that succeeds)
    end if
end for
declare that the given goal cannot be reached by deleting at most \( k \) voters
End

Additionally, we need to include constraints that ensure that the control action chosen through the assignment to the variables is a legal one. We include a constraint \( v_i \geq 0 \) for each \( i, 1 \leq i \leq j! \), enforcing that we cannot delete a negative number of voters of any type. We include constraints \( n_i \geq v_i \) for each \( i, 1 \leq i \leq j! \), enforcing that we cannot delete more voters of any type than there were in the first place. And finally we add a constraint \( k \geq \sum_{1 \leq i \leq j!} v_i \), bounding the total number of deletions by the bound \( k \).

The above modifications to our basic approach embed the problem of deciding what subset of voters to delete to satisfy a given WCF into an ILPFP, and there will be only a constant number of ILPFPs to test for each number of candidates. Thus the algorithm specified by the outer loop above is uniformly polynomial for every fixed parameter value, putting this problem in FPT.

Theorem 5.5 For Schulze elections and for (with any feasible tie-breaking functions) ranked pairs, control by partition of voters is in FPT with respect to the number of candidates, in both the ties-eliminate and ties-promote models, in both the succinct and non succinct input models, for both constructive and destructive control, and in both the non unique winner model and the unique winner model.

Proof. We will handle together all the cases of this theorem.

The cases of this theorem can again be handled with our looping-over-frameworks approach, and the constraints and modifications necessary to properly implement the WCF-enforcing ILPFP are similar to the previous cases, though we have to loop over not just single WCFs but rather pairs of them, one for each part of the partition. As before we will assume the candidate names are 1, ..., \( j \) with 1 as the distinguished candidate. We specify the outer loop for these cases in Algorithm 4.

The two binary selections in the algorithm’s step (2) are made analogously to the decision in the “if” line within the proof of Theorem 5.4.

Now we must specify the new details of the partition-handling ILPFPs. For each \( i, 1 \leq i \leq j! \), we will include a constant \( n_i \) representing how many votes of vote type \( i \) are in the initial election and a variable \( v_i \) representing how many votes of type \( i \) are put into \( V_1 \) (i.e., the \( V_1 \) sought by Algorithm 4).

Though we must use two WCFs, we can implement each as before, and the constraints for each of a given pair of WCFs will appear together as part of a single ILPFP. Let us use the predicates \( \text{Bigger}_1 \) and \( \text{StrictlyBigger}_1 \) (\( \text{Bigger}_2 \) and \( \text{StrictlyBigger}_2 \)) to handle the constraints of WCF \( K_1 \) (\( K_2 \)) over the weights of its WMG edges, which we will denote by \( D_1 \) (\( D_2 \)).
Algorithm 4 Top-level loop for control by partition of voters

Start
for each j-WCF $K_1$ do
  for each j-WCF $K_2$ do
    if $K_1$ and $K_2$ are internally consistent, well-formed j-WCFs then
      (1) build an ILPFP that checks whether there is a partition of the voters $V$ into $V_1, V_2$, such that $K_1$’s winner-set certification framework is realized by the set of votes $V_1$ and $K_2$’s winner-set certification framework is realized by the set of votes $V_2$
      (2) run that ILPFP and if it can be satisfied, and if candidate 1 (is/is not) (a winner/a unique winner) in the election with voters $V$ and the surviving candidates from $K_1$ and $K_2$ (according to the tie-handling rule in use, namely ties-eliminate or ties-promote), halt and accept (note: the satisfying settings will even let us output the precise partition that succeeds)
    end if
  end for
end for

declare that the given goal cannot be reached by any partition of voters

End

$Bigger_1$ and $StrictlyBigger_1$ can be formulated appropriately in terms of $D_1$, while $Bigger_2$ and $StrictlyBigger_2$ can be formulated in terms of $D_2$. The extra constraints for ranked pairs will be implemented in terms of $D_1$ and $D_2$ as appropriate, as well. We will now show how to handle $D_1$ and $D_2$. $D_1(a, b)$ denotes how many voters in $V_1$ prefer candidate $a$ to candidate $b$ and we can formulate it as follows:

$$\sum_{i \in \text{pref}(a, b)} v_i - \sum_{i \in \text{pref}(b, a)} v_i.$$  

$D_2(a, b)$ denotes how many voters in $V_2$ prefer $a$ to $b$, and we formulate it as follows:

$$\sum_{i \in \text{pref}(a, b)} (n_i - v_i) - \sum_{i \in \text{pref}(b, a)} (n_i - v_i).$$

Finally, we must add constraints to ensure the chosen partition is a legal one. We include a constraint $v_i \geq 0$ for every $i, 1 \leq i \leq j!$, enforcing that a nonnegative number of voters of each type are chosen for the first partition. We include a constraint $n_i \geq v_i$ for every $i, 1 \leq i \leq j!$, enforcing that we do not take more voters than exist of each type for the first partition.

The above modifications to our basic approach embed the problem of deciding what partition of voters to use to satisfy a pair of WCFs into an ILPFP, and there will be only a constant number of ILPFPs to test for each number of candidates. Thus the algorithm specified by the outer loop above is uniformly polynomial for every fixed parameter value, putting this problem in FPT.

5.3 Manipulation Results

Unweighted manipulation in Schulze elections has recently been shown to be in P by Gaspers et al. [15] for the constructive case. So, since the destructive case was itself handled even earlier by Parkes and Xia [27], there is no need to cover (unweighted) manipulation for Schulze elections in this paper.
We now present our FPT result for (unweighted) manipulation in ranked pairs elections. Although for this case we could use the looping-over-framework approach to obtain an FPT algorithm, we instead obtain an FPT algorithm here simply as a consequence of Theorem 5.3.

Very loosely put, the idea behind this proof is that, in a certain sense, for fixed numbers of candidates manipulation nicely reduces—not just regarding ranked pairs but in fact as a general matter—to control by adding voters, simply by putting into the set of potential voters to add enough copies of every possible vote. Unfortunately, this approach doesn’t quite work, due to the standard definition of control by adding candidates bounding rather than exactly setting the number of additions, which is what one needs as one’s reduction’s target to make this connection work. However, this worry is easy enough to smooth over that we still can prove the manipulation case here as a quick consequence of groundwork done for control.

**Theorem 5.6** For ranked pairs (with any feasible tie-breaking functions), manipulation is in FPT with respect to the number of candidates, in both the succinct and nonsuccinct input models, for both constructive and destructive manipulation, and in both the nonunique-winner model and the unique-winner model.

**Proof.** We will handle together all the cases of this theorem. Let us again assume without loss of generality that the candidates in the manipulation problem are 1, . . . , j with the distinguished candidate being 1. The manipulation problem has as its input a set of nonmanipulative votes V (or for the succinct version, a list of which types of votes occur at least once, along with the multiplicities of each) and the set of manipulators W.

We give an fpt-reduction—for readers not familiar with their definition, it can be found near the start of Section 6.4—from this problem to a slightly modified version of control by adding voters. The difference is that instead of having a bound on the number of votes which we can add, we will need an exact number of votes to be added. This modification will change only one of the constraints in our ILPFP (the constraint labeled 5.1 in the proof of Theorem 5.3) from being an inequality to an equality, and so it is clear that the thus-altered control by adding voters problems remain in FPT (by the thus-altered version of Theorem 5.3 and its proof). So an fpt-reduction to that problem will establish our desired FPT result for manipulation.

Our (modified) control instance will have the same set of candidates, and the same distinguished candidate. The set of initial votes in the control instance will be identical to the set of nonmanipulative votes in the manipulation instance. As for the additional vote set in the control instance, it will include ||W|| copies of each possible vote (out of all j! votes) (or for the succinct version, a list of all j! vote types, each with multiplicity of ||W||). Finally, the “exact” number of votes to be added is equal to ||W||. This completes the specification of the reduction.

Clearly, a manipulation instance is a Yes instance of manipulation if and only if this construction map to a Yes instance of the modified control problem. Furthermore, the mapped-to control instance will have the same number of candidates as the mapped-from manipulation instance. Finally, the reduction runs in time |x|(j!), where |x| is the manipulation problem’s input size. Thus the reduction meets the requirement of an fpt-reduction. This proves that manipulation is in FPT with respect to the number of candidates.

6 Other Results for the Unweighted Case

Section 7 will discuss the weighted case, but we first present some additional results regarding the unweighted case.
6.1 Candidate Control Parameterized on the Number of Candidates

Under our primary parameterization of interest, parameterizing on the number of candidates, and when considering manipulation, bribery, and voter control, we achieved FPT results using the looping-over-frameworks technique, and thus involving Lenstra’s algorithm. In contrast, when considering candidate control problems parameterized on the number of candidates, we need not use such a powerful technique. Instead it is sufficient to brute-force search over all possible control solutions to see if any of them are successful. At every possible value for the parameter, there are only a constant number of possible solutions to any of the candidate control problems, and checking the success of the possible solution will require only a simple polynomial-time task. Thus we have algorithms that at each fixed parameter value will have a running time that is a (large, parameter-value-dependent) constant times a small uniform polynomial. This puts these problems in FPT. We mention that for the adding/unlimited adding of candidates cases, the parameter can even be taken to just be the size of the pool of potential additional candidates, regardless of how many original candidates there were.

Theorem 6.1 For Schulze elections and for (with any feasible tie-breaking functions) ranked pairs, control is in FPT with respect to the number of candidates, in both the succinct and nonsuccinct input models, for both constructive and destructive control, for all standard types of candidate control (adding/unlimited adding/deleting candidates and, in both the ties-eliminate and ties-promote first-round promotion models, partition and runoff partition of candidates), in both the nonunique-winner model and the unique-winner model.

Proof. In the case of adding candidates, at most all the $2^{|D|}$ possible subsets of the auxiliary candidate set $D$ need be considered. In the case of deleting candidates, at most all subsets of $C$ that contain the distinguished candidate $p$ need be considered, and so we need look at at most $2^{|C|} - 1$ subsets. (In the destructive case, the definition of this control type forbids trivially satisfying the goal by deleting $p$. In the constructive case, deleting $p$ would make success impossible and so it need not be considered.) In the case of runoff partition, there are $2^{|C|-1}$ interestingly distinct partitions, while in the partition case there are $2^{|C|} - 1$. The difference between these two different types of partition cases is because in runoff partition of candidates case the two parts of the partition are handled symmetrically, and so the partitions $(A, B)$ and $(B, A)$ are not interestingly distinct from each other, and we need consider just one among them. However, in the partition of candidates case, where one side of the partition is getting a bye, no such general symmetry can be claimed. For each of these cases, for each setting of its item that we are cycling through above, we have to call the voting system’s winner problem between one and three times. So all these cases will be in FPT for any voting system with a polynomial-time winner problem.

6.2 Voter Control Parameterized on the Number of Voters

Although we feel that the number of candidates is the most natural parameterization for manipulative action problems, it is natural to ask about parameterizing on the number of voters. We do not exhaustively handle this case for all manipulative action problems, but we note that voter control problems parameterized on the number of voters can be shown to be in FPT through simple brute-force. Again, as in the case of candidate control problems parameterized on the number of candidates, we note that the number of possible solutions to these problems is bounded by a constant for each parameter value, and checking each solution is easily done in polynomial time, giving us an FPT algorithm for each of the voter control problems. We mention that for the adding of voters
cases, the parameter can even be taken to just be the size of the pool of potential additional voters, regardless of how many original voters there were.

**Theorem 6.2** For Schulze elections and for (with any feasible tie-breaking functions) ranked pairs, control is in FPT with respect to the number of voters, in both the succinct and nonsuccinct input models, for both constructive and destructive control, for all standard types of voter control (adding/deleting voters and, in both the ties-eliminate and ties-promote first-round promotion models, partition of voters), in both the nonunique-winner model and the unique-winner model.

*Proof.* In the case of adding voters we will have to try at most all of the $2^{|W|}$ possible subsets of the auxiliary voter set $W$. In the case of deleting voters we will have to consider at most all the $2^{|V|}$ subsets of the voter set $V$. And in the partition of voters cases we will have to consider the $2^{|V|}-1$ interestingly distinct partitions of the voter set (again, we need consider just one among the partitions $(A, B)$ and $(B, A)$). In all of these cases the large exponential term of the complexity will be constant with fixed parameter values. And beyond that term, we will just have to perform a few iterations of the voting system’s winner function along with a few other simple checks, putting all these cases in FPT for any voting system with a polynomial-time winner problem. $\square$

### 6.3 WMG Edge Bound Parameterization

Let us return to considering the case of parameterization by number of candidates. What drove us to our approach of looping over the winner “frameworks” we defined, rather than just looping over all WMGs? It was the fact that even for fixed numbers of candidates, the number of WMGs blows up as the number of voters increases. We mention in passing, though, that if one, in addition to parameterizing on the number of candidates, requires that the absolute value of the edge weights in the WMG be bounded by some fixed constant independent of the number of voters, then for that particular special case, one could loop over all WMGs. Is this natural and important? We would tend to say “no,” because assuming that all edges in the WMG have weights bounded by $k$ is to assume that even as the number of voters grows, every single head-on-head contest between pairs of candidates is very evenly matched. That simply is not the case in most natural elections.

On the other hand, perhaps surprisingly, there is something theoretical to be gained from the strange approach just mentioned of considering elections in which all edges of the WMG turn out to have relatively low weights. In particular, we observe that in the NP-hardness-establishing reductions to Schulze control problems used by Menton and Singh [24], all edges in the WMG have absolute value at most 6 (and for some types of control, at most 4 or 2). That, along with the fact that all weights of edges in the WMG have the same parity, gives us Corollaries 6.3 and 6.4.

**Corollary 6.3** (Corollary to the proofs of Menton and Singh [25]) Even when restricted to instances having all pairwise contests so equal that each WMG edge\(^5\) has absolute value at most 2, Schulze elections are NP-complete (in the nonunique-winner model) for constructive control by deleting candidates. The same holds for constructive control by adding candidates, unlimited adding of candidates, partition of candidates in the ties-eliminate model, and runoff partition of candidates in

---

\(^5\)In Corollaries 6.3 and 6.4 when we speak of restricting a WMG, the WMG we are speaking of as having to obey the restriction is the WMG involving all candidates involved in the problem. So for adding candidates, the WMG this is speaking of is, using the votes in the problem instance, the WMG whose nodes are all the initial candidates and all the candidates in the pool of candidates that can potentially be added. Since membership in NP for all the problems discussed is obvious, what is most interesting in this theorem is NP-hardness. And the fact that our restriction is applying to the broadest WMG involved in these control-by-candidates problems makes the restriction harsher than if it were applying to some sub-WMG, and so makes the results stronger.
the ties-eliminate model, except with a bound of 4. The same holds for constructive control by partition of candidates in the ties-promote model and runoff partition of candidates in the ties-promote model, except with a bound of 6.

Clearly, this immediately implies the following result (still keeping in mind that the values of all edge weights are of the same parity).

**Corollary 6.4** (Corollary to the proofs of Menton and Singh [25]) Even when restricted to instances having all pairwise contests so equal that the total cardinality of the set of absolute values of WMG edges (in the WMG involving all candidates in the instance) is at most 2, Schulze elections are NP-complete (in the nonunique-winner model) for constructive control by deleting candidates. Even when restricted to instances having all pairwise contests so equal that the total cardinality of the set of values of WMG edges is at most 3, Schulze elections are NP-complete (in the nonunique-winner model) for constructive control by deleting candidates. The same claims hold for constructive control by adding candidates, unlimited adding of candidates, partition of candidates in the ties-eliminate model, and runoff partition of candidates in the ties-eliminate model, except with the 2 and 3 above replaced by 3 and 5. The same holds for constructive control by partition of candidates in the ties-promote model and runoff partition of candidates in the ties-promote model, except with the 2 and 3 above replaced by 4 and 7.

### 6.4 Adding/Deleting Bound Parameterization

For those control problems having as part of their inputs a limit on how many candidates or voters can be added/deleted, it is natural to consider parameterizing on that limit. This parameterization has been studied in some voting systems and the relevant problems have often been found to be W[1]-hard or W[2]-hard, and thus very unlikely to be fixed-parameter tractable. For example, under this parameterization, Betzler and Uhlmann [5] showed, for what are known as Copeland^α elections, that constructive control by adding candidates and constructive control by deleting candidates are W[2]-complete, Liu and Zhu [21] proved, for maximin elections, that constructive control by adding candidates is W[2]-hard, and Liu and Zhu also achieved W[1]-hardness results for the relevant voter control problems. For additional control results parameterized on the problem’s internal addition/deletion limit, see Table 8 of Betzler et al. [4].

Hardness for these classes is defined in terms of fpt-reductions (by which we will always mean many-one fpt-reductions). Thus one typically shows a problem is, for instance, W[2]-hard by providing such a reduction from a known W[2]-hard problem. We now give the standard definition of fpt-reductions for the case of reductions from a problem, call it Q, with respect to a parameter j (let j(x) be the function that given an input to Q gives the value of that parameter on input x), to a problem, call it Q', with respect to a parameter j' (let j'(x) be the function that given an input to Q' gives the value of that parameter on input x). A function R is an fpt-reduction from Q to Q' if the following three conditions hold [13]: (i) For each x, x \( \in \) Q if and only if R(x) \( \in \) Q'. (ii) There exist a polynomial p and a computable function f such that R is computable in time f(j(x))p(|x|). (iii) There exists a computable function g such that, for all x, j'(R(x)) \leq g(j(x)).

We observe that, with respect to parameterizing on the internal addition/deletion bound, for Schulze elections it holds that constructive control by adding voters, constructive control by deleting voters, and constructive control by adding candidates are all W[2]-hard.

**Theorem 6.5** Parameterized on the adding bound, each of

1. constructive control by adding voters,
2. constructive control by deleting voters, and

3. constructive control by adding candidates

is $W[2]$-hard for Schulze elections, in the nonunique-winner model.

These problems have already been shown to be NP-complete by Parkes and Xia [27] and Menton and Singh [23] through reductions from exact cover by 3-sets (X3C). In order to prove $W[2]$-hardness, we need to reduce from a parameterized problem that is known to be $W[2]$-hard. One such problem is hitting set. Thus we modify the NP-hardness proofs given by Parkes and Xia to reduce from hitting set instead of from X3C. All that is necessary is to replace the role of candidates modeling X3C elements with the role of candidates modeling hitting set sets, and to replace the role of voters modeling the X3C sets with the role of voters with the same preferences modeling hitting set elements. For the sake of completeness, we give the complete reduction for one of the above three cases, namely, control by adding candidates. First, we give the definition of hitting set.

**Definition 6.6 (Hitting Set)** Given a set of elements $U$, a collection $\mathcal{F}$ of subsets of $U$, and a positive integer $k$, does there exist $H \subseteq U$, with $|H| \leq k$, such that for every $S \in \mathcal{F}$, we have $S \cap H \neq \emptyset$ (i.e., $H$ hits every set in $\mathcal{F}$)?

**Proof of part 3 of Theorem 6.5.** The standard NP-complete problem hitting set is also known to be $W[2]$-complete (see p. 464 of [8]). We give an fpt-reduction from hitting set to our problem.

Given a hitting set instance $(U, \mathcal{F}, k)$ as described in the definition, we will construct a control instance $(C, D, V, p, k)$ where $C$ is a set of original candidates, $D$ is a set of auxiliary candidates, $V$ is a set of voters, $p$ is a distinguished candidate, and $k$ is an adding bound. The original candidate set $C$ will contain the following candidates:

- The distinguished candidate $p$.
- A candidate $S$ for every $S \in \mathcal{F}$.

The auxiliary candidate set $D$ will contain the following:

- A candidate $u$ for every $u \in U$.

The voter set $V$ will be as follows. We will not explicitly construct the entire voter set, but rather we will specify the weight of the WMG edges between the candidates and let the voter set be as constructed by McGarvey’s method [22]. For readers not familiar with McGarvey’s method, and especially since in the proof of Theorem 7.3 we use McGarvey’s method in a weighted-voting setting, and the method’s mechanics are very slightly different in such a setting, we provide as Appendix B a self-contained presentation of McGarvey’s method for both the unweighted and the weighted cases.

- For every $S \in \mathcal{F}$, $D(S, p) = 2$ (and so $D(p, S) = -2$).
- For every $u \in U$, $D(p, u) = 2$ (and so $D(u, p) = -2$).
- For every $u \in U$, and for every $S \in \mathcal{F}$ such that $u \in S$, $D(u, S) = 2$ (and so $D(S, u) = -2$).
- All WMG edges not set above will be of weight 0.
The distinguished candidate will be $p$ and the adding limit will be the same limit $k$ as in the hitting set instance. This completes the specification of the reduction. Note that initially—i.e., before any adding of candidates—the winners are exactly the “$S$” candidates (unless $F = \emptyset$, in which case $p$ is initially a winner in the control problem, and the hitting set instance is a positive instance, so in the case we are already done).

If we map from a positive hitting set instance, we claim that we will have a positive instance of the control problem. Why? Let $H \subseteq U$, $\|H\| \leq k$, be a solution to the hitting set instance. We will show that the set of candidates $D'$ corresponding to the elements from $H$ will be a solution to the control instance. First, $\|D'\| \leq k$, so we are within the adding bound. Also, since the hitting set solution includes members of every set in $F$, there will be a path of strength two from $p$ to each candidate corresponding to those sets, as including a candidate $u \in U$ creates paths from $p$ to every “$S$” candidate hit by $u$. It is easy to see that $p$ will have paths to every other candidate just as strong as they have back to $p$, and so $p$ will be a Schulze winner.

If our reduction maps to a successful control instance, we claim it must have mapped from a positive hitting set instance. Why? Recall that we earlier handled the case $\|F\| = 0$. Initially, each “$S$” candidate will have a path of strength two to $p$, but $p$’s strongest path to each of them is of strength negative two. Adding “$u$” candidates will leave those strength-two paths from each “$S$” candidate to $p$ intact. Since all edges have weight at most two, regardless of how many candidates we add, no path can have a strength of more than two. So for a control instance to succeed it must give $p$ a strength-two path to each “$S$” candidate. In our setting, that means adding a set of $k$ “$u$” (auxiliary) candidates such that each “$S$” candidate is pointed to (with a weight-two edge) by at least one of them. However, given our current setup, that itself says that the hitting set instance being mapped from is a positive instance.

Note also that our reduction clearly runs in polynomial time in the size of the entire input, easily meeting the running-time limit for an fpt-reduction. Additionally, the parameter in the mapped-to instance will always be bounded by—in fact, identical to—the parameter in the mapped-from instance.

So this clearly is an fpt-reduction from hitting set to our problem. Thus we have established that constructive control by adding candidates, parameterized on the adding bound, is $\text{W}[2]$-hard for Schulze elections, in the nonunique-winner model.

7 Weighted Case

Finally, although in this paper we have focused on manipulation problems without weights, and on bribery problems without weights or prices, we mention in passing that (keep in mind we still are also parameterizing on the number of candidates, and that when weights and prices are used in problems, they are typically taken to be—and here we do take them to be—nonnegative integers) if one parameterizes by also bounding the maximum voter weight (if there are weights) and the maximum voter price (if there are prices), our main theorems hold even in the context of weights and prices. That is because when weights and prices are bounded, one can clearly still carry out the approach we use.

In fact, we can go slightly further, although at the outer edge of things doing so will require some surgery on our approach. Not just for the cases of bounded weights and prices, but even for the case where there is a bound on the cardinality of the set of weights (if there are weights) and there is a bound on the cardinality of the set of prices (if there are prices), all theorems (again, still parameterizing also on number of candidates) of Section 5 in bribery and manipulation still hold (as
Theorem 7.1 For Schulze elections and for (with any feasible tie-breaking functions) ranked pairs, weighted bribery, priced bribery, and weighted, priced bribery each are in FPT with respect to the combined parameter “number of candidates” and “cardinality of the set of voter weights” (for cases with weights) and “cardinality of the set of voter prices” (for cases with prices), in both the succinct and nonsuccinct input models, for both constructive and destructive bribery, and in both the nonunique-winner model and the unique-winner model.

Proof. We will describe how to handle the weighted, priced case. This is a generalization of both the weighted, unpriced case and the unweighted, priced case. So our algorithm will also handle both those cases. The top-level loop will be as in the standard bribery proof in Section 5.1. We will specify the extensions to the ILPFP construction that are necessary to handle the weights and prices.

Let 1, . . . , j be the candidates, let w = {w1, . . . , wj} be the set of weights of the voters, and let p = {p1, . . . , pj} be the set of prices. We will have a constant nα,β i denoting how many voters there are of type i with weight wα and with price pβ for every i, 1 ≤ i ≤ j!, α, 1 ≤ α ≤ y, and β, 1 ≤ β ≤ z. We will have a variable mα,β i,ℓ, for every i, 1 ≤ i ≤ j!, ℓ, 1 ≤ ℓ ≤ j!, α, 1 ≤ α ≤ y, and β, 1 ≤ β ≤ z. mα,β i,ℓ describes the number of voters with weight wα and price pβ that are bribed from vote type i to vote type ℓ. Our total number of variables is larger than in the unweighted unpriced case, but it is still bounded as a function of only j, y, and z, which make up our combined parameter.

Now we will describe how to build the constraints in our ILPFP. As before, we can implement the predicates Bigger and StrictlyBigger—and for ranked pairs the additional constraints to handle ranked pairs—in terms of $D(a, b)$, the weight of the edge from a to b in the WMG. So we need only specify how to formulate $D(a, b)$ and how to formulate appropriate housekeeping constraints. As to $D(a, b)$, we need to take into account both voters that are bribed away from preferring a over b and voters that are bribed into preferring a over b. And we need to keep track of voters of every type, weight, and price. Still, what we need is very much in line with our simpler implementation of this function in the standard bribery case. So our new formulation of $D(a, b)$ will to capture it with the following expression:

\[
\sum_{1≤α≤γ, 1≤β≤z} \sum_{i∈pref(a, b)} w_α \left( n_α,β_i - \sum_{1≤ℓ≤j!} \sum_{ℓ≠i} \sum_{1≤j≤j!} \sum_{ℓ≠i} m_α,β_j,ℓ \right) \]

\[
- \sum_{1≤α≤γ, 1≤β≤z} \sum_{i∈pref(b, a)} \sum_{1≤ℓ≤j!} \sum_{ℓ≠i} \sum_{1≤j≤j!} \sum_{ℓ≠i} m_α,β_i,ℓ \]

We have not excluded the unneeded and rather unhelpful case $m_α,β_i,ℓ = 0$. Also, the four “ℓ≠i”s above could be dropped as they cancel each other. However, we include them as they are the natural way to express what is being modeled.
Additionally we need to implement constraints to make sure that the bribery action we select is a legal one. Thus, besides the natural constraints \( m_{\alpha,\beta}^{\alpha,\beta} \geq 0 \), ensuring that we do not bribe a negative number of voters of a certain type, we have to enforce that we do not bribe more voters than exist of every type. Thus for every \( i, 1 \leq i \leq j! \), \( \alpha, 1 \leq \alpha \leq y \), and \( \beta, 1 \leq \beta \leq z \), we have the following constraint:

\[
\sum_{1 \leq \ell \leq j!} m_{\alpha,\beta}^{\alpha,\beta} i,\ell \geq 0.
\]

And also we must restrict the total cost of all our bribes to the bribe-cost limit \( k \). Thus we have the following constraint:

\[
k \geq \sum_{1 \leq \alpha \leq y} \sum_{1 \leq \beta \leq z} \sum_{1 \leq i \leq j} \sum_{1 \leq \ell \leq j!} p_{\beta} m_{\alpha,\beta}^{\alpha,\beta} i,\ell.
\]

These additional constraints will enforce a legal selection of voters to bribe. These, together with our specification of \( D(a, b) \), complete the specification of the WCF-enforcing ILPFP. Overall we have a structure that will have a constant number of constraints in terms of the parameters, and that will have a solution if and only if there is a successful bribery action. Also there will be only a constant number of WCFs for every fixed set of parameter values; in fact it will be constant in terms of just the number of candidates, and there will be no more WCFs than in the unweighted, unpriced case.

Thus we have established that the weighted variant, the priced variant, and the weighted, priced variant of bribery are each in FPT when parameterized on the combined parameter “number of candidates” and “cardinality of the set of voter weights” (if there are weights) and “cardinality of the set of voter prices” (if there are prices).

Pushing beyond this, for the case of manipulation (still parameterized by the number of candidates), we can handle even the case of there not being any bound on the cardinality of the weight set of all the voters, but rather there simply being a bound on the cardinality of the set of weights over all manipulative voters.

**Theorem 7.2** For (with any feasible tie-breaking functions) ranked pairs, it holds that weighted coalitional manipulation is in FPT with respect to the combined parameter “number of candidates” and “cardinality of the manipulators’ weight set” in both the succinct and nonsuccinct input models, for both constructive and destructive manipulation, and in both the nonunique-winner model and the unique-winner model. The same claim holds for Schulze elections, except limited to the destructive case (in both the succinct and nonsuccinct input models, and in both the nonunique-winner model and the unique winner model) and the unique-winner constructive case (in both the succinct and nonsuccinct input models).

---

\[6\] The nonunique-winner model, constructive case for Schulze elections—which the above theorem is carefully avoiding claiming—also holds. But as noted in Section 4, a construction of Gaspers et al. establishes that the weighted constructive coalitional manipulation problem for Schulze elections is in FPT, with respect to the parameter “number of candidates,” in the nonunique-winner model. That is a broader result for nonunique-winner model, weighted constructive coalitional manipulation than would be the nonunique-winner model, constructive, Schulze version of Theorem 7.2 and so it would not make sense to include the nonunique-winner model, constructive Schulze case in the statement of Theorem 7.2.

To avoid a potential worry that expert readers might have, we now briefly address a rather subtle, technical issue related to why we have claimed the destructive Schulze case in Theorem 7.2. Although Gaspers et al. for the case of weighted manipulation of Schulze elections address only the constructive case (and even that, only for the nonunique-winner model), it might be natural to think that their constructive result—and we have commented that
Proof. Without loss of generality, let the candidates be 1, . . . , j, with candidate 1 being the distinguished candidate. The input specifies the nonmanipulators as a collection of votes along with their weights (with, in the succinct case, identical-vote identical-weight votes not listed separately but rather listed by the vote, weight, and a positive binary integer giving the number of that-vote that-weight voters) and it specifies (for the more demanding case, that of succinct inputs) the set of manipulators as a list of pairs, \(((w_1, t_1), \ldots, (w_s, t_s))\), with there being \(t_i\) manipulators having weight \(w_i\) (with each \(t_i > 0\)). Our top-level loop will be as follows.

Algorithm 5 Top-level loop for weighted manipulation

Start
for each \(j\)-WCF \(K\) do
if candidate 1 (is/is not) (a winner/a unique winner) according to \(K\) and \(K\) is an internally consistent, well-formed \(j\)-WCF then
(1) build an ILPFP that checks whether there is an assignment to the votes of \(W\) such that \(K\)'s winner-set certification framework is realized by the set of votes \(V \cup W\)
(2) run that ILPFP and if it can be satisfied then halt and accept (note: the satisfying settings will even let us output the precise manipulation that succeeds)
end if
end for
declare that the given goal cannot be reached through setting the votes of \(W\)
End

The “if” line at the start of the algorithm should be set to “is” (“is not”) for the constructive (destructive) case, and to “a winner” (“a unique winner”) for the nonunique-winner model (the unique-winner model). So all we need to do is to specify the ILPFP that we build inside the loop, for each given \(j\)-WCF \(K\).

Now, to handle this, we must be careful; since there is no limit on the overall number of weights, we can’t have variables capturing how many voters of each occurring voter weight there are before manipulation. Rather, we in our looping algorithm that is building the ILPFPs use the power of our algorithm to itself precompute all the parts of the sums (appearing in the constraints) regarding all the nonmanipulators—so it is our looping algorithm that is putting in place constants (of the ILPFP) that express the sums of the weights of nonmanipulative voters that have various properties, in particular, it will build the constant \(n_{a,b}\) describing the total weight of the nonmanipulative voters casting votes that prefer candidate \(a\) to candidate \(b\), for each pair of candidates \(a, b\). The values they provide even an FPT algorithm—will also hold for the destructive case and yield there an FPT result stronger than that of Theorem 7.2. However, as mentioned earlier in this paper, their constructive result is crucially based on the lovely fact that in the case of the weighted constructive coalitional manipulation problem for Schulze elections, if one can make a given candidate a winner then there is a set of manipulative votes in which all manipulators vote the same way and that candidate is selected as a winner. This does not imply, in any obvious way, that one can make the same claim regarding the destructive case—preventing a candidate from becoming a winner. Thinking that such an implication holds might be very tempting, based on the (untrue) thought that a candidate can be kept from winning exactly if some other candidate can be made to win. This thought, however, is untrue in the nonunique-winner model, as winners can coexist. In the unique-winner model, something close to this does hold though with a rather substantial twist (see footnote 5 of [17]). But the “all can vote the same” insight used by Gaspers et al. [15] seems potentially fragile, and is currently known only for the constructive, nonunique-winner case. Indeed, Menton and Singh [25] state that the analogue fails for the constructive, unique-winner case. It thus remains an interesting open issue whether the weighted destructive coalitional manipulation problem for Schulze elections is in FPT when parameterized only on the number of candidates. This completes our explanation of why Theorem 7.2's claim about the destructive Schulze case (and similarly, its unique-winner model, constructive Schulze case) does not seem to be subsumed by existing results.
((w_1, t_1), \ldots, (w_s, t_s)) describing the weights and how many manipulators there are of each weight will be constants of the ILPFP as well. The variables of the ILPFP describe how many manipulators of each weight get manipulated to each particular vote type: \( m^\alpha_i \) for every \( i, 1 \leq i \leq j! \) and \( \alpha, 1 \leq \alpha \leq s \), describes how many manipulators of weight \( w_\alpha \) are assigned to vote type \( i \).

The WCF can be implemented in terms of the Bigger and StrictlyBigger predicates and \( D(a, b) \) as usual, with Bigger and StrictlyBigger themselves implemented in terms of \( D(a, b) \). In this case we will express \( D(a, b) \) as the following:

\[
 n_{a,b} + \left( \sum_{1 \leq \alpha \leq s} \sum_{i \in \text{pref}(a,b)} w_\alpha m^\alpha_i \right) - n_{b,a} - \left( \sum_{1 \leq \alpha \leq s} \sum_{i \in \text{pref}(b,a)} w_\alpha m^\alpha_i \right).
\]

Besides this we just need a few extra constraints to ensure that the manipulation chosen is a reasonable one. We include a constraint \( m^\alpha_i \geq 0 \) for every \( i, 1 \leq i \leq j! \), and \( \alpha, 1 \leq \alpha \leq s \), ensuring that we do not try to manipulate to some type a negative number of manipulative voters of some weight. And we include a constraint \( \sum_{1 \leq i \leq j!} m^\alpha_i = t_\alpha \) for every \( \alpha, 1 \leq \alpha \leq s \), ensuring that we use the same number of manipulators of every weight as are present.

With these additions, we can build an ILPFP to encode our WCF that will be polynomially bounded in size for fixed values of the two parameters, and there will be only a constant number of such WCFs with those fixed parameter values. Thus our algorithm specified above will run in uniformly polynomial time with fixed parameter values, and this problem is in FPT.

This still is all a valid framework for our many-uses-of-Lenstra-based approach. Note that for the case of the cardinality of the set of weights being 1, that gives the case of weighted nonco-\( \text{template manipulation mentioned as an aside by } \)Dorn and Schlotter [7], though here we’re handling even any fixed-constant number of manipulators (since any fixed-constant number have at most a fixed-constant cardinality of their weight set), and indeed, even a number of manipulators whose cardinality isn’t bounded but who among them in total have a fixed-constant cardinality of occurring weights.

Recall that we already were proving our unweighted manipulation result (Theorem 5.6) nearly “for free,” by drawing on a related control problem’s groundwork (and in doing so, we were pointing out an interesting connection between manipulation and control). The strength of Theorem 7.2 is such that it gives us an alternative and even more “for free” path to seeing that Theorem 5.6 holds: Theorem 5.6 follows from Theorem 7.2 simply by setting all the weights to 1.

It seems intuitively necessary to bound the cardinality of the manipulator weight set to achieve results like the above. For ranked pairs we show that that is in some sense necessary: We prove that weighted constructive coalitional manipulation (with no bound on the number of manipulator weights) is NP-complete in ranked pairs for each fixed number of candidates starting at five. Thus there cannot be any algorithm that is polynomial for any fixed number of (at least five) candidates, unless \( \text{P = NP} \). As a weaker consequence, this will also block the existence of an FPT algorithm for this problem parameterized solely on the number of candidates.

**Theorem 7.3** For any feasible tie-breaking functions, and for each fixed number of candidates, \( j, j \geq 5 \), weighted constructive coalitional manipulation is NP-complete for ranked pairs (under the given feasible tie-breaking functions), in both the nonunique-winner model and the unique-winner model. Even without the assumption that the tie-breaking functions are feasible, NP-hardness will still hold.

**Proof.** Containment in NP is immediately clear if the tie-breaking functions are feasible. To prove NP-hardness we will reduce from a known NP-hard problem, partition. Our construction will not
make any assumptions about the tie-breaking functions. Thus we will establish NP-hardness even for tie-breaking functions that are not feasible. As mentioned earlier, the Parkes-Xia [27] framing of ranked pairs, which we are following in this paper, always selects precisely one winner (i.e., is resolute). The unique-winner model and the nonunique-winner model are in effect the same in this setting, and our result will apply for both the unique-winner model and the nonunique-winner model.

We first note that if there is a successful (constructive) ranked-pairs manipulation, then there is a successful (constructive) ranked-pairs manipulation in which each manipulator puts their initially winning candidate, p, immediately below another candidate a, then if we modify the election by swapping in this voter’s preference list the position of candidates p and a while keeping the rest of his ordering unchanged, p is still a winner under this modified election. (The monotonicity of ranked pairs is due to Tideman, but for clarity as to the result holding even in our flexible-tie-breaking, weighted model/case, and to naturally introduce a notion of “encountering” that we will extensively use in this proof and that of Theorem 7.5 and to be self-contained, we prove monotonicity here.

Our proof of course is deeply indebted to and inspired by Tideman’s. The property we mentioned earlier will then be derived simply by repeatedly applying this monotonicity property on all the manipulators until they all rank p in the top position.

We now prove monotonicity. Assume now that a voter ranks the initially winning candidate, p, immediately below some candidate a. Swapping p and a in this voter’s preference list (while keeping the rest of the ordering unchanged) will affect the WMG of the election only by increasing the weight of the edge (p, a) by two and decreasing the weight of the edge (a, p) by two.

For the rest of this proof, and also for the proof of Theorem 7.5, we use a notion of “encoun-

\footnote{Let us comment on this in a bit more detail. Several formulations of ranked pairs have been shown to be monotonic [31,32]. The formulation of ranked pairs in [32] is quite similar to ours, differing in deriving the tie-breaking ordering over the pairs from the one over the candidates rather than allowing other orderings, and in allowing voters to provide nonstrict rankings rather than requiring strict linear preferences. However, we argue that if one looks very carefully at the monotonicity proof given in that work [32], it can even be seen to yield a proof valid for our formulation. As to the difference in preference models, Tideman’s proof clearly holds when preferences are required to always be strict, since that is a subcase of the more flexible vote model used there. As to why his proof holds for our more flexible tie-breaking model, note that the argument in his proof relies on no details of the tie-breaking function over the pairs except to deliver a strict ordering over the unordered pairs of candidates from the nonstrict ordering over those pairs based on the weights of the WMG edges; crucially, it can be seen by very careful inspection of Tideman’s proof that which particular such strict ordering the tie-breaking function on pairs delivers makes no difference to his proof as long as the ordering is consistent with the WMG-edge-weights-based nonstrict ordering of the pairs. Thus Tideman’s proof would apply just as well even when using our more flexible tie-breaking function over the pairs, since even our functions indeed provide the required action of unambiguously (and efficiently) providing a strict order for encountering pairs, and one that does respect what, as just mentioned, it must respect. (The just-made comments were about the tie-breaking function over pairs. Our tie-breaking function over the individual candidates is in the same model as the one used in that work, and so requires no specific discussion here.) Besides that, although that proof is given for the unweighted case, it is clear from the definition of monotonicity that if monotonicity holds for the unweighted case, it also holds for the weighted case. Thus, taking all of these points together, one can see that that proof can be argued to apply in our case as well (and of course it applies for both the nonunique-winner model and the unique winner model, since they coincide, as ranked pairs always selects exactly one winner). Nonetheless, to be self-contained, and to naturally introduce a notion of “encountering” that we will extensively use in the rest of the proof of Theorem 7.5 and also in the proof of Theorem 7.5 in our main text we give a proof of monotonicity (phased to clearly work in our model, which as mentioned above is more flexible that Tideman’s regarding tie-breaking functions).}
tering," defined as follows: The pair \( \{x, y\} \) is said to be encountered when the ranked pairs winner determination algorithm first considers either of the edges \( (x, y) \) and \( (y, x) \). Each pair is encountered exactly once (although if when it is encountered the relationship between its candidates has already been transitively set, both edges regarding the pair are simply discarded). Clearly, the order in which the pairs are encountered by the algorithm is fixed once the tie-breaking functions and the weights of the WMG edges are specified.

Recall that the weights of all the WMG edges except \( (a, p) \) and \( (p, a) \) remain unchanged after our modification to the initial election. This implies that the relative order in which pairs are encountered can change regarding only \( \{a, p\} \)'s relation to other pairs. In particular, if under the modified election the order of encountering pairs differs from the initial election, it differs only in moving forward or moving backward the encountering of the pair \( \{a, p\} \).

We decompose the rest of the proof of monotonicity into three cases.

**Case 1:** The edge \( (p, a) \) has positive weight in the initial WMG, or the edge \( (p, a) \) has zero weight in the initial WMG but \( p \) is preferred to \( a \) according to the tie-breaking function among candidates. In this case, the shifting of \( p \) from immediately below \( a \) to immediately above \( a \) in the vote of one voter means that the maximum weight ("max"ed of course over the actual values, not over the absolute values) among \( (p, a) \) and \( (a, p) \) in the modified graph will be two greater than in the original graph, and will in fact occur on \( (p, a) \), which will now certainly have strictly positive weight. And so the order in which pairs are encountered will be exactly the same except possibly for \( \{a, p\} \) shifting to an earlier point in that order.

Now, under the initial WMG (and the tie-breaking functions, but we will now mostly stop mentioning those except when important, as those functions don’t change between the initial and the modified cases), \( p \succ a \) in the final outcome, as \( p \) was the winner.

If \( p \succ a \) was set directly at the moment \( \{a, p\} \) was encountered in the initial ordering, then under our modified ordering \( p \succ a \) will certainly also be set directly when \( \{a, p\} \) is encountered, since the encounter is no later (and so no prefix before the encounter, as the process unfolds under the modified graph, can relate \( p \) and \( a \) transitively, since if one did, that same prefix would have existed in the initial graph and would have set their relation transitively), and \( (p, a) \) will have positive weight when it is encountered.

If \( p \succ a \) was set transitively (i.e., before \( \{a, p\} \) was encountered) under the initial WMG's ordering of encounters, then since in the modified case \( \{a, p\} \) is encountered in the same place or earlier, if it now still comes after the prefix that transitively set \( p \succ a \) the modified case will still have \( p \succ a \) (due to that identical prefix), and if it now comes before the prefix that transitively set \( p \succ a \) the modified case will have \( p \succ a \) because \( (p, a) \) is positive. \( p \) will remain the winner.

So under Case 1, \( p \succ a \) will still hold under ranked pairs with respect to the modified WMG. Furthermore, in the modified case, the pairs that are encountered after \( \{p, a\} \), but were encountered in the initial case before \( \{p, a\} \) and were not discarded, will not be inconsistent with the order \( p \succ a \) as this order was established in the initial case too, at latest when \( \{p, a\} \) was encountered. Thus, for each such pair, at the moment when that pair is encountered in the modified case, the same order established by its encounter in the initial case either has already been transitively set by a prefix involving \( p \succ a \) or will be fixed by that encounter. Also, as to the pairs that were encountered in the initial case before \( \{p, a\} \) and were discarded, they will also be discarded in the modified case due to the identical prefix that transitively set the order for those pairs in the initial case. Thus the same orders will be fixed in the modified case as in the initial case, and the final ordering will be the same. As a consequence, \( p \) will remain the winner with respect to the modified WMG.

**Case 2:** The edge \( (p, a) \) has zero weight in the initial WMG and \( a \) is preferred to \( p \) according to the tie-breaking function among candidates. The maximum weight of \( (p, a) \) and \( (a, p) \) in the original
WMG was zero, as both were zero, but the maximum is two in the modified WMG. And so the order in which pairs are encountered will be exactly the same except possibly for \( \{a, p\} \) shifting to an earlier point in that order.

If the relationship between \( a \) and \( p \) was not set transitively in the initial case, then \( a \succ p \) would be the outcome of the encounter (due to the specification of Case 2), contradicting our assumption that \( p \succ a \) in the initial case's overall outcome.

So in the initial case, \( p \) versus \( a \) must have been set transitively (and thus, necessarily, before \( \{a, p\} \) was encountered), and must have been set transitively so as to yield \( p \succ a \). In the modified case, we have that \( \{p, a\} \) has positive weight and the ordering of encounters is unchanged except for \( \{a, p\} \) possibly moving earlier. So since \( \{p, a\} \) is positive, if the relationship between \( p \) and \( a \) is not transitively set before \( \{a, p\} \) is encountered in the modified ordering, then \( p \succ a \) gets set at that encounter. And if the relationship between \( a \) and \( b \) is transitively set before \( \{a, p\} \) is encountered, then that same prefix must have set \( p \succ a \) transitively in the initial case, since that prefix will be identical in both cases (because the encounter didn’t move forward by the modification), and so it will be set as \( p \succ a \) in our modified case.

So under Case 2, \( p \succ a \) will still hold under ranked pairs with respect to the modified WMG. Furthermore, similarly to the argument we made at the end of Case 1, the same orders will be established for pairs that in the initial case were encountered before \( \{a, p\} \) but now are encountered after \( \{a, p\} \) in the modified case. Thus the same orders will be fixed in the modified case as in the initial case, and the final ordering will be the same. As a consequence, \( p \) will remain the winner.

**Case 3:** The edge \( \{p, a\} \) has negative weight in the initial WMG. Note that, unlike our other two cases, in this case the modification decreases by two the maximum among the weights of \( \{p, a\} \) and \( \{a, p\} \), and so the modified ordering of encounters will be identical to the original except the pair \( \{a, p\} \) may be pushed back to a later location in the ordering.

Now, if the relationship between \( a \) and \( p \) was not set transitively in the original, then we would have had \( a \succ p \) in the original case’s outcome, since \( \{a, p\} \) has positive weight in the original case. But we know that \( p \succ a \) in the original case’s outcome, and so in the original, \( p \succ a \) must be fixed transitively before \( \{a, p\} \) is encountered.

But \( \{a, p\} \) is encountered in the same place or later, so whatever prefix transitively set \( p \succ a \) in the original will also occur, and transitively sets \( p \succ a \), in the modified case, all before \( \{a, p\} \) is encountered.

So under Case 3, the pair \( \{a, p\} \) gets discarded in both the initial and the modified election, and since we already noted that the order of encountering pairs does not change by the modification except possibly for \( \{a, p\} \)’s relation to other pairs, the orders will be fixed in the modified case in exactly the same way as in the initial case and as a consequence, \( p \) will remain the winner.

This concludes our proof of monotonicity for ranked pairs, for both the weighted and unweighted cases. And as we discussed earlier, we can repeatedly apply the above-mentioned modification to the preference list of the manipulators until they all rank \( p \) in the top position (while keeping the rest of the ordering unchanged); so for ranked pairs, if there is a manipulation making \( p \) win, then there is a manipulation making \( p \) win in which all manipulators rank \( p \) first.

We now move on to give this proof’s reduction, which is from the partition problem. (Recall that a partition of a set \( A \) is a pair of disjoint sets \( A_1 \) and \( A_2 \) such that \( A_1 \cup A_2 = A \).)

**Definition 7.4 (Partition, see [14])** Given a list of \( n \) integers, \( k_1, \ldots, k_n \), does there exist a partition of \( \{1, \ldots, n\} \) into two sets, \( I_1 \) and \( I_2 \), such that \( \sum_{i \in I_1} k_i = \sum_{j \in I_2} k_j \)?

Fix a \( j \geq 5 \). Given a partition instance, \( k_1, \ldots, k_n \), we will construct a weighted constructive coalitional manipulation instance \((C, V, W, p)\), where \( C \) is a candidate set with \( |C| = j \), \( V \) is a
nonmanipulative voter set (each having a weight and voting by a tie-free linear order), $W$ is a collection of manipulative voters (each starting as a blank-slate vote and having a weight), and $p$ is a distinguished candidate.

The candidate set, $C$, will contain five important candidates: the distinguished candidate $p$ and four other candidates: $a_1$, $a_2$, $b_1$, and $b_2$. There will also be $j - 5$ extra candidates that we will ensure are always easily beaten by $p$. The extra candidates are added just for the purpose of reaching the specified number of candidates. (Recall that we fixed $j$ in the beginning. Thus we present a construction that will directly work for each $j$, $j \geq 5$. Alternatively, we could have presented the construction for five candidates first, and then extend it to larger numbers of candidates.)

Let $S = \sum_{i=1}^{n} k_i$. We will have $n$ manipulators and their weights will be $6k_1, 6k_2, \ldots, 6k_n$. The nonmanipulators will, in polynomial time using McGarvey’s method (this claim is justified by Appendix B, be assigned votes and weights such that the important candidates induce the WMG shown in Figure 3, and for every extra candidate $e$ and each important candidate $c$ (i.e., for each $c \in \{a_1, a_2, b_1, b_2, p\}$), $D(c, e) = 18S + 4$ (so that these edges remain large—at least $12S + 4$—regardless of the manipulators’ votes).

Now we will show that there is a solution to the partition instance if and only if there is a solution to the manipulation instance.

We will show that if there is a solution to the mapped-to manipulation instance, there must be a solution to the mapped-from partition instance. Suppose there is a solution for the manipulation instance. Namely, there is a set $V'$ of $n$ votes with weights as per $W$ such that $p$ wins the election $(C, V \cup V')$. Due to the claim established earlier in this proof, we know that if such a $V'$ exists then such a $V'$ exists in which each manipulator ranks $p$ first; so we will take $V'$ to have $p$ at the top of each manipulator’s vote.

Let us use $D'(a, b)$ to denote the weight of the WMG edge $(a, b)$ after manipulation. Thus we will have $D'(p, a_1) = D'(p, a_2) = 4$ and $D'(b_1, p) = D'(b_2, p) = 2$. Also the weight of the edges from $p$ to the extra candidates will become so large that under ranked pairs their relationship with $p$ is fixed (namely, $p$ beats each) before the relationships between $p$ and the important candidates are set. So, in the final ranking, $p$ is clearly placed higher than all extra candidates (we do not care about the relative order between pairs of extra candidates, since that does not affect $p$’s performance).
Also, we will have \( D'(b_1, a_2) \geq 12S + 4 \) and \( D'(b_2, a_1) \geq 12S + 4 \), regardless of the manipulators’ votes. These edges will thus be considered before every other edge in the subgraph induced by the important candidates. Thus the orders \( b_1 \succ a_2 \) and \( b_2 \succ a_1 \) will necessarily be fixed in the final ranking (and, by the way, we’ll have that directly, not transitively). We argue that at least one of the edges \( (a_1, b_1) \) or \( (a_2, b_2) \) must achieve a nonnegative weight. Otherwise there will not be any path of nonnegative weight edges from \( p \) to \( b_1 \) and \( b_2 \), and since \( D'(b_1, p) = 2 \), that means no sequence of candidate pairs that can transitively establish \( p \succ b_1 \) or \( p \succ b_2 \) in the final ranking can possibly all be encountered come before \((b_1, p)\) and \((b_2, p)\) are considered, and so we will eventually have \( b_1 \succ p \) and \( b_2 \succ p \) in the final ranking. But since the manipulator weights are all multiples of 6 (and the total manipulation weight is \( 6S \) and \( D(a_1, b_1) = D(a_2, b_2) = (-6S) + 4 \)), the only nonnegative weight the edges \((a_1, b_1)\) and \((a_2, b_2)\) can possibly achieve is 4. In particular \( D'(a_1, b_1) \geq 0 \) only when \( D'(a_1, b_1) = 4 \), and this happens only if all the manipulators prefer \( a_1 \) to \( b_1 \). Analogously \( D'(a_2, b_2) \geq 0 \) only when \( D'(a_2, b_2) = 4 \), and this happens only if all the manipulators prefer \( a_2 \) to \( b_2 \).

We have established that at least one of the edges \((a_1, b_1)\) or \((a_2, b_2)\) must achieve a nonnegative weight. Now, without loss of generality (as the other case—namely that \((a_2, b_2)\) achieves nonnegative weight—is symmetric), assume that \((a_1, b_1)\) achieves nonnegative weight. And so, as we have argued, it even holds that \( D'(a_1, b_1) = 4 \). We argue that \( D'(b_1, b_2) = 0 \). Otherwise, we would have one of two cases (note that the manipulator weights are all multiples of 6 and \( D(b_1, b_2) = 0 \), so \( D'(b_1, b_2) \) must be a multiple of 6):

1. \( D'(b_1, b_2) \geq 6 \): In this case, \((b_1, b_2)\) will be considered before \((a_1, b_1)\) and the order \( b_1 \succ b_2 \) will be fixed in the final ranking. Since we already have the order \( b_2 \succ a_1 \), by transitivity, we have \( b_1 \succ a_1 \) and \((a_1, b_1)\) gets discarded. Thus the order between \( p \) and \( b_1 \) will not be specified transitively by any sequence of fixed orders before the edge \((b_1, p)\) is considered. And when \((b_1, p)\) is considered, \( b_1 \succ p \) would be set. So \( p \) would not be a winner, contradicting our assumption that we have a successful manipulation on the given instance.

2. \( D'(b_2, b_1) \geq 6 \): We have two subcases here (keeping in mind still that \( D(b_2, a_2) = 6S - 4 \), the manipulators total \( 6S \) in weight, and each manipulator’s weight is a multiple of 6).

   (a) \( D'(a_2, b_2) = 4 \): In this case, \((b_2, b_1)\) will be considered before \((a_2, b_2)\) and the order \( b_2 \succ a_2 \) will be fixed by transitivity before considering that edge. Then the only nonnegative incoming edge to \( b_2 \), which is \((a_2, b_2)\), gets discarded, and eventually, when \((b_2, p)\) is considered, \( p \) will be ranked lower than \( b_2 \).

   (b) \( D'(b_2, a_2) \geq 2 \): In this case, since there will not be any nonnegative incoming edge to \( b_2 \), \( p \) will be ranked lower than \( b_2 \) when considering \((b_2, p)\).

Thus it must hold that \( D(b_1, b_2) = 0 \). We now can build a solution for the partition problem as follows. For every manipulator \( i \) (which has the weight \( 6k_i \)) having ranked \( b_1 \) higher than \( b_2 \), we make \( I_1 \) include \( i \), and for every manipulator \( j \) (which has the weight \( 6k_j \)) having ranked \( b_2 \) higher than \( b_1 \), we make \( I_2 \) include \( j \). Then \((I_1, I_2)\) is clearly a solution to the partition problem.

We will show that if there is a solution to the mapped-from partition instance, then there must be a solution to the mapped-to manipulation instance. Suppose there is a solution for the partition problem. That is, we have a partition of \( \{1, \ldots, n\} \) into \( I_1 \) and \( I_2 \), such that \( \sum_{i \in I_1} k_i = \sum_{j \in I_2} k_j \). We make every manipulator \( i \) with \( i \in I_1 \) cast the following vote:

\[
p > a_1 > a_2 > b_1 > b_2 > \ldots
\]

33
where \([\ldots]\) denotes the extra candidates in any arbitrary order), and make every manipulator \(j\) with \(j \in I_2\) cast the following vote:

\[p > a_2 > a_1 > b_2 > b_1 > \ldots.\]

After the manipulation, we will have \(D'(a_1, a_2) = D'(b_1, b_2) = 0\) and \(D'(a_1, b_1) = D'(a_2, b_2) = 4\). So, after having fixed in the final ranking the orders \(b_1 \succ a_2, b_2 \succ a_1, p \succ a_1,\) and \(p \succ a_2\), the edges \((a_1, b_1)\) and \((a_2, b_2)\) will be considered, giving the final orders \(p \succ a_1 \succ b_1\) and \(p \succ a_2 \succ b_2\). So, we will have a transitive order from \(p\) to both \(b_1\) and \(b_2\) and \(p\) will be a winner.

One might naturally wonder whether the constant five is tight for the NP-hardness result presented in Theorem 7.3. Theorem 7.5 proves that five indeed is tight here, by showing that for each number of candidates smaller than five, we can solve the manipulation problem under consideration in polynomial time. The key idea is to show that, for each ranked-pairs instance with fewer than five candidates, it holds that if there exists a successful manipulation, then for that same instance there exists a successful manipulation in which all the manipulators vote the same.

**Theorem 7.5** For any feasible tie-breaking functions, and for each number of candidates, \(j, j < 5\), weighted constructive coalitional manipulation is solvable in polynomial time for ranked pairs (under the given feasible tie-breaking functions), in both the nonunique-winner model and the unique-winner model.

**Proof.** We will show that in instances with fewer than five candidates, if a successful manipulation exists, then there exists a manipulation where all the manipulators cast the same vote.

In light of this fact, there are only a constant number of vote types to check regardless of the number of manipulators. Thus we merely need, for each vote type, to run the ranked-pairs winner determination procedure on the election that consists of the nonmanipulators having cast their votes, and the manipulators all having cast a vote of that particular type. Thus we have a polynomial-time algorithm.

All that remains is to prove, for every number of candidates smaller than five, that if a successful manipulation exists, then there exists a manipulation where all the manipulators cast the same vote. For each number of candidates less than five, we first assume that there exists a successful manipulation and then argue that we can still achieve the goal, even if all the manipulators cast identical votes. (In fact, we will even argue that if the number of candidates is less than five and a successful manipulation exists, then there exists a successful manipulation in which all the manipulators cast the same vote and that vote has \(p\) as the most preferred candidate.) As before our result immediately applies to both winner models, since ranked pairs always selects exactly one winner.

Recall that, regardless of the number of candidates, if there exists a manipulation, then there exists a manipulation where all the manipulators rank \(p\) at the top position (this claim is discussed and justified in the proof of Theorem 7.3). So we can safely assume that, in the successful manipulation, all the manipulators rank the distinguished candidate at the top position. This immediately proves our claim for \(j \leq 2\). For the remaining values of \(j\) that we need to handle, namely \(j = 3\) and \(j = 4\), we will show that if we have a successful manipulation, we can make \(p\) win with all the manipulators agreeing on the rest of their rankings in addition to all ranking \(p\) first.

For \(j = 3\), let the set of candidates be \(C = \{p, a, b\}\) with \(p\) being the distinguished candidate. As we discussed earlier, all the manipulators rank \(p\) at the top position. This lets us know, beforehand, the weight of the WMG edges \((p, a)\) and \((p, b)\) after the manipulation, regardless of the manipulators’ preferences over \(a\) and \(b\).
We now consider the moment when the pair \{a, b\} is encountered (the notion of encountering is as defined in the proof of Theorem 7.3). At that time, the order between a and b has already been fixed in the final ranking only if it has been fixed transitively by a sequence of orders including p. However, this can happen only if \( a \succ p \) or \( b \succ p \) has been already set at that time, which cannot happen since we have assumed a successful manipulation. Thus the order between a and b, in the final ranking, will be determined right at the time when the pair \{a, b\} is encountered. Let us assume without loss of generality (as the two cases are symmetric) that the order being fixed is \( a \succ b \). We claim that all the manipulators can vote \( p \succ a \succ b \) with p still winning the manipulated election. This change of the manipulators’ votes will not weaken the weight of the edge \((a, b)\) and will not change the weight of the edges involving \( p \). Thus when \{a, b\} is encountered, the order \( a \succ b \) will be fixed in the final ranking as before, except that this pair might be encountered earlier, before the pairs \{a, p\} or \{b, p\}. However, earlier fixing of the order \( a \succ b \) cannot prevent \( p \) from winning.

Finally, for \( j = 4 \), let the set of candidates be \( C = \{p, a, b, c\} \) with p being the distinguished candidate. Again we are assuming that each manipulator puts \( p \) in his or her top position.

Let us assume without loss of generality that after the manipulation has occurred, among the pairs not including \( p \), the ranked-pairs winner determination algorithm encounters the pairs in the order \{a, b\}, \{a, c\}, and \{b, c\}.

In a similar fashion to the argument we made for the case of \( j = 3 \), at the moments when \{a, b\} and \{a, c\} are encountered, the edges between these pairs would get discarded by a transitivity fixed order only if the sequence of orders establishing that order includes \( p \), which cannot happen since we have assumed a successful manipulation. So for the first two pairs that are encountered among these three pairs, their relative order is fixed right at that time and this can be done in four different ways:

1. The orders \( a \succ b \) and \( a \succ c \) are fixed.
2. The orders \( b \succ a \) and \( c \succ a \) are fixed.
3. The orders \( b \succ a \) and \( a \succ c \) are fixed.
4. The orders \( a \succ b \) and \( c \succ a \) are fixed.

In the first two cases, when the pair \{b, c\} is being encountered, the relative order of \( b \) and \( c \) has not been fixed yet. Thus it will be fixed according to whichever of \((b, c)\) and \((c, b)\) is considered first and according to the tie-breaking function that handles zero-weight edges if these edges are of weight zero. Since \( b \) and \( c \) are symmetric in both cases, let us assume without loss of generality that \( b \succ c \) is what gets fixed. We now claim that all the manipulators can cast the vote \( p \succ a \succ b \succ c \) in the first case (of our above list of four cases) and \( p \succ b \succ c \succ a \) in the second case, with \( p \) still winning the manipulated election. To argue for the first case, we note this change will not weaken any of the edges among \((a, b)\), \((a, c)\), and \((b, c)\). What might be affected instead is the order in which the pairs are encountered, either among the pairs \{a, b\}, \{a, c\}, and \{b, c\} or between these pairs and the pairs involving \( p \) (the pairs not involving \( p \) may be encountered before some of the pairs involving \( p \) that they used to be encountered after). The latter is obviously of no concern if we can show that for each pair of the candidates not involving \( p \), the same order will be fixed as before at latest when that pair is encountered. We now argue that the orders being fixed for the pairs not involving \( p \) will be exactly the same as before. Note that since the weights of the edges \((a, b)\), \((a, c)\), and \((b, c)\) are not weakened by the modification to the manipulators’ votes, for the first two pairs being encountered among \{a, b\}, \{a, c\}, and \{b, c\} (please keep in mind the “without loss of generality” assumptions, two and four paragraphs before the present one, as to the order of encountering among these), the encounter of those pairs establishes the same order as before (i.e. \( a \succ b \) for \{a, b\}, \( a \succ c \) for \{a, c\},
and \( b \succ c \) for \( \{b, c\} \)). Furthermore, when the last pair among these three is encountered, no sequence of orders has transitively set the order between the candidates in that pair. So that encounter will also establish the same order as before. This proves that the final ordering is the same as before and all the manipulators can safely cast the vote \( p > a > b > c \) in the first case. The argument for the second case is analogous.

The second two cases, though, are a bit different since the order between \( b \) and \( c \) is set transitively by the orders \( b \succ a \) and \( a \succ c \) in the third case, or by the orders \( c \succ a \) and \( a \succ b \) in the fourth case. We claim that in the third case all the manipulators can vote \( p > b > a > c \) and \( p \) will still win the manipulated election. Note that this set of manipulative votes might strengthen the weight of the edges \( (b, a) \) and \( (a, c) \) and \( (b, c) \) and allow their corresponding pairs to be encountered in an arrangement different from the initial manipulation (plus the pairs not involving \( p \) may be encountered before some of the pairs involving \( p \) that they used to be encountered after, but again this is not important if we can show that for each pair of the candidates not involving \( p \), the same order will be fixed as before at latest when that pair is encountered). However, under the modified manipulation, this will not change the orders being fixed assuming that the pair \( \{b, c\} \) is the last pair being encountered among the three. But even if \( \{b, c\} \) is no longer the last pair, we argue that the final ordering will be exactly the same. Note that since the weights of the edges \( (b, a) \) and \( (a, c) \) are not weakened by the modification to the manipulators’ votes, the encounter of the first pair among \( \{a, b\} \) and \( \{a, c\} \) establishes the same order as before (i.e. \( b \succ a \) for \( \{a, b\} \) and \( a \succ c \) for \( \{a, c\} \)), regardless of when \( \{b, c\} \) is encountered. And the encounter of the second pair establishes the same order as before, except possibly if that second pair gets discarded due to the order being transitively set (of course by a sequence of orders not involving \( p \)). So the only way the final ordering might differ from the initial manipulation is either when the encounter of the pair \( \{b, c\} \) establishes the order \( c \succ b \) or when the second-encountered pair among \( \{a, b\} \) and \( \{a, c\} \) gets discarded by a sequence of orders including an order between \( b \) and \( c \). The latter itself could happen only if the encounter of the pair \( \{b, c\} \) establishes the order \( c \succ b \) (so that along with \( b \succ a \) it transitively fixes the order \( c \succ a \), or that along with \( a \succ c \) it transitively fixes the order \( a \succ b \)). However, the weight of the edge \((c, b)\) under the initial manipulation must have been at least as large as its weight under the modified manipulation. Furthermore, the weights of the edges \( (b, a) \) and \( (a, c) \) under the initial manipulation have to be at most as large as their weights under the modified manipulation. Thus if under the modified manipulation the order \( c \succ b \) is ever set, \( \{b, c\} \) could not have been the last pair that was encountered among the three under the initial manipulation, which contradicts our assumption. So all the manipulators can cast the vote \( p > b > a > c \) and \( p \) will still win. In the fourth case, the manipulators can analogously vote \( p > c > a > b \) and \( p \) will continue to win.

This proves that for each number of candidates \( j, j \leq 4 \), if a successful manipulation exists, then there exists a manipulation where all the manipulators vote the same. This in turn proves that weighted constructive coalitional manipulation is solvable in polynomial time for each number of candidates \( j, j < 5 \).

\[ \blacksquare \]

8 Open Problems

For all cases of (unweighted) bribery, control, and manipulation, including many where general-case hardness results exist, this paper proves that Schulze elections and ranked-pairs elections are fixed-parameter tractable with respect to the number of candidates.

In order to concisely represent our unweighted results, we list in Figure 8.1 the best currently known tractability and intractability results regarding the unweighted case. All the figure’s FPT results hold for both the constructive and destructive cases, in both the succinct and nonsuccinct
input models, and for both the unique-winner and nonunique-winner models. Due to some of the other NPC cases not having been explored in the literature, though, Figure 4 limits itself to speak just about the constructive case in the nonsuccinct input model and in the nonunique-winner model.

The most striking remaining open direction regards the weighted cases. For example, Gaspers et al. [15] proved that the constructive, nonunique-winner case of weighted coalitional manipulation is in FPT. Can their result be extended to the constructive, unique-winner case, or the destructive, unique-winner case, or the destructive, nonunique-winner case? Can one even provide a fixed-parameter tractability result for these three cases? These all remain open questions, although in Theorem 7.2 we give fixed-parameter tractability results for special cases of all three of these issues. The analogous issues are not open for ranked pairs: Theorem 7.3 shows that unless P = NP, the ranked-pairs analogue of the Gaspers et al. result cannot hold.

**Acknowledgments**

A two-page extended-abstract version of this paper appeared in AAMAS 2013 [18]. We thank anonymous referees for helpful comments and suggestions, including the alternate certification framework of Appendix A and the suggestion to obtain Theorem 5.6 as a consequence of Theorem 5.3. This work was supported in part by NSF grants CCF-0915792, CCF-1101479, and CCF-1116104.

**References**

[1] J. Bartholdi, III, C. Tovey, and M. Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6(3):227–241, 1989.

[2] J. Bartholdi, III, C. Tovey, and M. Trick. How hard is it to control an election? *Mathematical and Computer Modeling*, 16(8/9):27–40, 1992.

[3] N. Betzler. *A Multivariate Complexity Analysis of Voting Problems*. PhD thesis, Friedrich-Schiller-Universität Jena, Jena, Germany, 2010.
[4] N. Betzler, R. Bredereck, J. Chen, and R. Niedermeier. Studies in computational aspects of voting—A parameterized complexity perspective. In The Multivariate Algorithmic Revolution and Beyond, pages 318–363. Springer-Verlag Lecture Notes in Computer Science #7370, 2012.

[5] N. Betzler and J. Uhlmann. Parameterized complexity of candidate control in elections and related digraph problems. Theoretical Computer Science, 410(52):43–53, 2009.

[6] V. Conitzer, T. Sandholm, and J. Lang. When are elections with few candidates hard to manipulate? Journal of the ACM, 54(3):Article 14, 2007.

[7] B. Dorn and I. Schlotter. Multivariate complexity analysis of swap bribery. Algorithmica, 64(1):126–151, 2012.

[8] R. Downey and M. Fellows. Parameterized Complexity. Springer-Verlag, 1999.

[9] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. How hard is bribery in elections? Journal of Artificial Intelligence Research, 35:485–532, 2009.

[10] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. Multimode control attacks on elections. Journal of Artificial Intelligence Research, 40:305–351, 2011.

[11] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. Weighted electoral control. In Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems, pages 367–374. International Foundation for Autonomous Agents and Multiagent Systems, May 2013.

[12] P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Llull and Copeland voting computationally resist bribery and constructive control. Journal of Artificial Intelligence Research, 35:275–341, 2009.

[13] J. Flum and M. Grohe. Parameterized Complexity Theory. Springer-Verlag, 2006.

[14] M. Garey and D. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, 1979.

[15] S. Gaspers, T. Kalinowski, N. Narodytska, and T. Walsh. Coalitional manipulation for Schulze’s rule. In Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems, pages 431–438. International Foundation for Autonomous Agents and Multiagent Systems, May 2013.

[16] E. Hemaspaandra, L. Hemaspaandra, and C. Menton. Search versus decision for election manipulation problems. In Proceedings of the 30th Annual Symposium on Theoretical Aspects of Computer Science, pages 377–388. Leibniz International Proceedings in Informatics (LIPIcs), February/March 2013.

[17] E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Anyone but him: The complexity of precluding an alternative. Artificial Intelligence, 171(5–6):255–285, 2007.

[18] L. Hemaspaandra, R. Lavaee, and C. Menton. Schulze and ranked-pairs voting are fixed-parameter tractable to bribe, manipulate, and control. In Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems, pages 1345–1346. International Foundation for Autonomous Agents and Multiagent Systems, May 2013.
[19] H. Lenstra, Jr. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8(4):538–548, 1983.

[20] A. Lin. *Solving Hard Problems in Election Systems*. PhD thesis, Rochester Institute of Technology, Rochester, NY, 2012.

[21] H. Liu and D. Zhu. Parameterized complexity of control problems in maximin election. *Information Processing Letters*, 110(10):383–388, 2010.

[22] D. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21(4):608–610, 1953.

[23] C. Menton and P. Singh. Manipulation and control complexity of Schulze voting. Technical Report arXiv:1206.2111 (version 1) [cs.GT], Computing Research Repository, arXiv.org/corr/, June 2012.

[24] C. Menton and P. Singh. Control complexity of Schulze voting. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence*, pages 286–292. AAAI Press, August 2013.

[25] C. Menton and P. Singh. Manipulation and control complexity of Schulze voting. Technical Report arXiv:1206.2111 (version 4) [cs.GT], Computing Research Repository, arXiv.org/corr/, May 2013.

[26] R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006.

[27] D. Parkes and L. Xia. A complexity-of-strategic-behavior comparison between Schulze’s rule and ranked pairs. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence*, pages 1429–1435. AAAI Press, August 2012.

[28] J. Rothe and L. Schend. Challenges to complexity shields that are supposed to protect elections against manipulation and control: A survey. *Annals of Mathematics and Artificial Intelligence*, 68(1–3):161–193, 2013.

[29] N. Russell. Complexity of control of Borda count elections. Master’s thesis, Rochester Institute of Technology, 2007.

[30] M. Schulze. A new monotonic, clone-independent, reversal symmetric, and Condorcet-consistent single-winner election method. *Social Choice and Welfare*, 36:267–303, 2011.

[31] T. Tideman. Independence of clones as a criterion for voting rules. *Social Choice and Welfare*, 4(3):185–206, 1987.

[32] T. Tideman. *Collective Decisions and Voting: The Potential for Public Choice*. Ashgate Publishing, 2006.

[33] Wikipedia. Schulze method. en.wikipedia.org/wiki/Schulze_method, 2013.

[34] L. Xia. Computing the margin of victory for various voting rules. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pages 982–999, June 2012.

[35] L. Xia, M. Zuckerman, A. Procaccia, V. Conitzer, and J. Rosenschein. Complexity of unweighted manipulation under some common voting rules. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence*, pages 348–353, July 2009.
A An Alternative Schulze Winner Certification Framework

The following discussion may certainly be skipped, except by curious readers or readers wanting to get more of a sense of the flavors that winner certification frameworks can have. In particular, the following discussion presents an alternative Schulze winner certification framework to the one described above; the above-described framework is the one we used throughout this paper. The alternative framework we will now describe in some ways is very clear (as to information it holds and counting how many such structures there are) but in some ways is a bit subtle (as to the time-cost of seeing whether they really certify that a given person is a winner).

Our SWCFs, as just described and as we use throughout the paper, are very explicitly certifying why a given candidate is a Schulze winner. We mention that one could alternatively loop over a different type of structure that many may find more easier to understand as a clean, clear structure to loop over since it is basically a triangular matrix with 3-valued entries, namely, $\{<, =, >\}$. Namely, consider a table that for each pair of edges-between-candidates says whether the first has a weight greater than, equal to, or less than the other. This structure certainly has enough information to tell whether a given candidate is in the Schulze winner set. So we could loop over all such tables making our candidate of interest a winner (or if in the “destructive” case, not a winner).

The fact that some such tables can never be realized since they set up violations of transitivity—e.g., $n_1 < n_2 < n_3 < n_4$—is not a problem for the way we’ll use our SWCFs; we’ll use them in ILPFPs related to attack types, and unrealizable tables of course can never be realized by any attack actions since they can’t be realized at all, and so they will not harm our algorithms. In some sense, this attractive approach is making the SWCFs more oblique as to why they ensure that a given candidate is a winner, but since these structures are simply a big ternary triangular matrix, they are very simple structures to loop through and are very simple structures for which to see that their number is bounded for each fixed number of candidates. In particular—and we make no attempt to try to take advantage here of the “negative one factor” relationship between $(a, b)$ and $(b, a)$ weights—there are certainly at most $3(p^2 - p)/2$ such structures, where $p = 2^{\|C\|}$.

Since we won’t return to a discussion of this alternate approach, we mention now two issues that are relevant to why it would work as an approach. In the ILPFPs we’ll build later, we’ll loop through the SWCFs and will have to in the ILPFP capture whether the structure can be realized. But the tools we build for the framework we actually use make it very clear that in our ILPFPs we can ensure that a collection of $<, =, >$ relationships hold, and so it would be easy to modify our ILPFPs to enforce the claims of our ternary triangular matrices. The more subtle and interesting issue is that in our ILPFPs we need to tell whether a given SWCF makes a given candidate (or precludes a given candidate from being) a winner. For the SWCFs we are using in our paper, this is a trivial, immediate check. But for the ternary triangular matrix approach, which gives us just $<\text{vs.}=\text{vs.}>$ information, it is not even clear we can check winnership in polynomial time in the size of the matrix, given that there are an exponential number of potential simple paths between two points. However, the key insight to have is that quells that worry is that that issue is irrelevant to ensuring membership in the class FPT, which is the goal of this paper. Even if determining whether a given ternary triangular matrix makes someone a winner takes time exponential in the size of the matrix (which itself is as noted above of size roughly quartic in the number of candidates), this is a constant for each fixed number of candidates, and so does not in any way interfere with our FPT claims. Again, in our paper we are not using the ternary triangular approach, and so all our ILPFPs are in terms of our primary approach. But this appendix is here to make clear that the ternary triangular approach certainly would work, if implemented with the care and cautions just mentioned.
B  McGarvey’s Method

McGarvey’s method [22], is quite simple. However, to be self-contained, and since in one of the three times we draw on it we are using it in a weighted-voting setting (namely, in the proof of Theorem 7.3), where it—see below—works very slightly differently, we now provide a description of McGarvey’s method. Suppose we are given some WMG, and we wish to generate a set of votes realizing it. The one-candidate case isn’t interesting, so we assume that there are at least two candidates. Since we are speaking of a WMG, for each \((a, b)\), \(a \neq b\), it must hold that the weight on \((a, b)\) is negative one times the weight on \((b, a)\). Also, either all of the edges are even in weight or all are odd. In particular, in the case we are now discussing, the unweighted case, the parity of each edge weight is the parity of the number of voters; in the weighted case, which we will discuss later in this appendix, the parity of each edge will be the parity of the total vote weight. Suppose the candidates are \(a\), \(b\), and \(c_1, \ldots, c_z\), \(z \geq 0\). Consider adding the following two votes: \(a > b > c_1 > \cdots > c_z\) and \(c_z > c_{z-1} > \cdots > c_1 > a > b\). This vote pair increases by two the weight of the edge from \(a\) to \(b\) in the WMG, decreases by two the weight of the edge from \(b\) to \(a\) in the WMG, and leaves all other edges unchanged in the WMG. Of course, this trick can be used not just on \(a\) and \(b\), but on any pair of distinct candidates.

So in the case of unweighted voting, if the (target) edge weights are even we can, for each pair of candidates, simply keep adding a pair of votes to shift in the right direction, by two, the weights on the edges between them, until the desired values are reached. If the (target) edge weights are odd, we can initially add one arbitrarily chosen vote, thus making all edges odd in weight, and then can appropriately add pairs of votes as before to bring the weights to the desired values. This clearly can be done in time polynomial in the sum of (a) the sum of the absolute values of the edge weights, and (b) the size of the problem (i.e., the total size of the candidate names and the total size of the edge weight descriptions, though the latter is in fact handled by (a)). Of course, if the edge weights are large, this could take an exponential amount of time unless the edge weights are coded in unary. However, in this paper, everywhere where we are using McGarvey’s method for unweighted voting, the absolute values of the edge weights are bounded by a small constant, so this is not an issue.

However, when we invoke McGarvey’s method in the proof of Theorem 7.3, it is used in a weighted-voting context, and for WMGs where the sum of the absolute values of the edge weights often will be exponential in the problem size. That is no problem, though, since in the weighted-voting case, in the even-weight case, if the desired values of the WMG edges \((e, f)\) and \((f, e)\) are \(2w\) and \(-2w\), with \(w\) a nonnegative integer, we simply add the two votes mentioned above, using \(e\) and \(f\) in the roles of \(a\) and \(b\), with each vote being a weight-\(w\) vote. And we do this for each pair of distinct candidates. In the odd-weight case we can again just initially add an arbitrary weight-one vote to make the weights odd, and then for every pair we can add the appropriately chosen pair of votes (of weights of either both \(w\) or both \(w+1\), depending on the current edge weights set by the one initial vote) to bring the edge weights to the desired values of \(2w+1\) and \(-2w-1\).

So for the weighted-vote case, we can clearly generate a realizing set of votes in time polynomial in the size of the problem instance (the sum of the sizes of all the weights—not the sum of their values but the sum of the sizes of their binary encodings—plus the sum of the sizes of the candidate names).

C  Control Definitions

In this section, we provide more formal definitions of the benchmark control types. Since such papers as [21712] provide a detailed discussion of the motivations of and real-world inspirations for the benchmark control types, we don’t repeat the discussion here.
The statements below are taken, sometimes identically, from Faliszewski et al. [12]. We first define
the voter-control types, since those are more central in this paper. Then we define the candidate-
control types, which in this paper appear only in Section 6. All definitions below are stated for the
nonsuccinct, nonunique-winner model. Section 6’s comments specify how to modify these to define
the succinct case and/or the unique-winner case.

C.1 Voter-Control Definitions

The adding-voter control problems are the following.

Name: The constructive control by adding voters problem for \( E \) elections and the destructive control
by adding voters problem for \( E \) elections.

Given: A set \( C \) of candidates, two disjoint collections of voters, \( V \) and \( W \) (each voter’s vote is a
tie-free linear order), a distinguished candidate \( p \), and a nonnegative integer \( k \).

Question (constructive): Is there a subset \( Q \), \( ||Q|| \leq k \), of voters in \( W \) such that the voters in
\( V \cup Q \) jointly elect \( p \in C \) as a winner according to system \( E \)?

Question (destructive): Is there a subset \( Q \), \( ||Q|| \leq k \), of voters in \( W \) such that the voters in
\( V \cup Q \) do not elect \( p \) as a winner according to system \( E \)?

The deleting-voter control problems are the following.

Name: The constructive control by deleting voters problem for \( E \) elections and the destructive
control by deleting voters problem for \( E \) elections.

Given: A set \( C \) of candidates, a collection \( V \) of voters (each voter’s vote is a tie-free linear order),
a distinguished candidate \( p \in C \), and a nonnegative integer \( k \).

Question (constructive): Is it possible to by deleting at most \( k \) voters ensure that \( p \) is a winner
of the resulting \( E \) election?

Question (destructive): Is it possible to by deleting at most \( k \) voters ensure that \( p \) is not a winner
of the resulting \( E \) election?

As mentioned in Section 3 each partition problem has two variants, based on how ties are handled
in the preliminary (i.e., first) round. In the ties-eliminate tie-handling model, if a first-round election
does not have a unique winner then no one from that subelection moves forward to the second round.
In the ties-promote tie-handling model, if a first-round election does not have a unique winner then
all winners (if any) from that subelection move forward to the second round. This issue of which
tie-handling rule is used is what is referred to in the definition below (and later, in the definitions
of candidate partitioning) by the phrase “that survive the tie-handling rule.”

Name: The constructive control by partition of voters problem for \( E \) elections and the destructive
control by partition of voters problem for \( E \) elections.

Given: A set \( C \) of candidates, a collection \( V \) of voters (each voter’s vote is a tie-free linear order),
and a distinguished candidate \( p \in C \).

Question (constructive): Is there a partition of \( V \) into \( V_1 \) and \( V_2 \) such that \( p \) is a winner
of the two-stage election where the winners of election \( (C, V_1) \) that survive the tie-handling rule
compete against the winners of \( (C, V_2) \) that survive the tie-handling rule? Each subelection
(in both stages) is conducted using election system \( E \).
Question (destructive): Is there a partition of $V$ into $V_1$ and $V_2$ such that $p$ is not a winner of the two-stage election where the winners of election $(C, V_1)$ that survive the tie-handling rule compete against the winners of $(C, V_2)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $E$.

C.2 Candidate-Control Definitions

The unlimited-adding candidate control problems are the following.

Name: The constructive control by adding an unlimited number of candidates problem for $E$ elections and the destructive control by adding an unlimited number of candidates problem for $E$ elections.

Given: Disjoint sets $C$ and $D$ of candidates, a collection $V$ of voters (each voter’s vote is a tie-free linear order over the candidates in the set $C \cup D$), and a distinguished candidate $p \in C$.

Question (constructive): Is there a subset $E$ of $D$ such that $p$ is a winner of the $E$ election with voters $V$ and candidates $C \cup E$?

Question (destructive): Is there a subset $E$ of $D$ such that $p$ is not a winner of the $E$ election with voters $V$ and candidates $C \cup E$?

The adding-candidate control problems are the following.

Name: The constructive control by adding candidates problem for $E$ elections and the destructive control by adding candidates problem for $E$ elections.

Given: Disjoint sets $C$ and $D$ of candidates, a collection $V$ of voters (each voter’s vote is a tie-free linear order over the candidates in the set $C \cup D$), and a distinguished candidate $p \in C$, and a nonnegative integer $k$.

Question (constructive): Is there a subset $E$ of $D$ such that $\|E\| \leq k$ and $p$ is a winner of the $E$ election with voters $V$ and candidates $C \cup E$?

Question (destructive): Is there a subset $E$ of $D$ such that $\|E\| \leq k$ and $p$ is not a winner of the $E$ election with voters $V$ and candidates $C \cup E$?

The deleting-candidate control problems are the following.

Name: The constructive control by deleting candidates problem for $E$ elections and the destructive control by deleting candidates problem for $E$ elections.

Given: A set $C$ of candidates, a collection $V$ of voters (each voter’s vote is a tie-free linear order), a distinguished candidate $p \in C$, and a nonnegative integer $k$.

Question (constructive): Is it possible to by deleting at most $k$ candidates ensure that $p$ is a winner of the resulting $E$ election?

Question (destructive): Is it possible to by deleting at most $k$ candidates other than $p$ ensure that $p$ is not a winner of the resulting $E$ election?

The runoff-partition candidate control problems are the following. In runoff-partition candidate control problems, each candidate must participate in precisely one of the two first-round preliminary elections.
Name: The constructive control by runoff partition of candidates problem for $\mathcal{E}$ elections and the destructive control by runoff partition of candidates problem for $\mathcal{E}$ elections.

Given: A set $C$ of candidates, a collection $V$ of voters (each voter’s vote is a tie-free linear order), and a distinguished candidate $p \in C$.

Question (constructive): Is there a partition of $C$ into $C_1$ and $C_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against the winners of subelection $(C_2, V)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $\mathcal{E}$.

Question (destructive): Is there a partition of $C$ into $C_1$ and $C_2$ such that $p$ is not a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against the winners of subelection $(C_2, V)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $\mathcal{E}$.

The partition candidate control problems are the following. In partition candidate control problems, the partition is between candidates who participate in a first-round preliminary election and candidates who get a “bye” through the first round.

Name: The constructive control by partition of candidates problem for $\mathcal{E}$ elections and the destructive control by partition of candidates problem for $\mathcal{E}$ elections.

Given: A set $C$ of candidates, a collection $V$ of voters (each voter’s vote is a tie-free linear order), and a distinguished candidate $p \in C$.

Question (constructive): Is there a partition of $C$ into $C_1$ and $C_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against all candidates in $C_2$? Each subelection (in both stages) is conducted using election system $\mathcal{E}$.

Question (destructive): Is there a partition of $C$ into $C_1$ and $C_2$ such that $p$ is not a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against all candidates in $C_2$? Each subelection (in both stages) is conducted using election system $\mathcal{E}$.