Exponentially convergent distributed Nash equilibrium seeking for constrained aggregative games

Shu Liang\textsuperscript{1,2}, Peng Yi\textsuperscript{1,2}, Yiguang Hong\textsuperscript{1,2} and Kaixiang Peng\textsuperscript{3}

\textsuperscript{1}Department of Control Science & Engineering, Tongji University, Shanghai, 200092, China.
\textsuperscript{2}Shanghai Research Institute for Intelligent Autonomous Systems, Shanghai, 201210, China.
\textsuperscript{3}School of Automation and Electrical Engineering, University of Science and Technology Beijing Beijing, 100083, China.

Contributing authors: sliang@tongji.edu.cn; yipeng@tongji.edu.cn; yghong@iss.ac.cn; kaixiang@ustb.edu.cn;

Abstract

Distributed Nash equilibrium seeking of aggregative games is investigated and a continuous-time algorithm is proposed. The algorithm is designed by virtue of projected gradient play dynamics and distributed average tracking dynamics, and is applicable to games with constrained strategy sets and weight-balanced communication graphs. We obtain an exponential convergence of the proposed algorithm to the Nash equilibrium. Numerical examples illustrate the effectiveness of our methods.

Keywords: Distributed algorithms, aggregative games, projected gradient play, weight-balanced graph, exponential convergence

1 Introduction

Distributed Nash equilibrium seeking with game-theoretic formulation and multi-agent system consideration has received research attention from the control and optimization communities, partially due to its applications in smart grids, communication networks and artificial intelligence. Various distributed
algorithms for Nash equilibrium or generalized Nash equilibrium seeking have been developed, which guide a group of discrete-time or continuous-time agents to achieve the equilibrium based on local data and information exchange over a network graph [1–6].

Aggregative games have become an important type of games since the well-known Cournot duopoly model was proposed [7], where the strategic interaction is clearly characterized via an aggregation term. Recently, aggregative games have been considered in congestion control of communication networks [8], public environmental models [9], demand response management of power systems [10], and multiproduct-firm oligopoly [11]. Because of the large-scale systems involved in these problems, seeking or computing the Nash equilibrium in a distributed manner is of practical significance.

We consider distributed Nash equilibrium seeking of aggregative games, where the aggregation information is unavailable to each local player and the communication graph can be directed with balanced weights. Similar problems have also been investigated in [10, 12–16]. In this work, an exponentially convergent algorithm design is proposed for the considered problem. First, a distributed projected gradient play dynamics is designed, where we replace the global aggregation by its local estimation to calculate the gradient. Then an average tracking dynamics is augmented, where the distributed tracking signals are local parts of the aggregation. We analyze these interconnected dynamics and prove that our distributed algorithm achieves an exponential convergence to the Nash equilibrium. The contributions are as follows:

• A distributed Nash equilibrium seeking algorithm for aggregative game is developed. The algorithm is designed with two interconnected dynamics: a projected gradient play dynamics for equilibrium seeking and a distributed average tracking dynamics for estimation of the aggregation. The projected part can deal with local constrained strategy sets, which generalizes those in [10, 15]. Also, the distributed average tracking dynamics applies to weight-balanced directed graphs, which improves the algorithm in [13].

• Exponential convergence of the proposed distributed algorithm is obtained, which is consistent with the convergence results in [10, 14, 17] for unconstrained problems and is stronger than those in [14, 17] for constrained ones. In other words, this is a first work, to our knowledge, to propose an exponentially convergent distributed algorithm for aggregative games with local feasible constraints.

The rest of paper is organized as follows. Section 2 shows some basic concepts and preliminary results, while Section 3 formulates the distributed Nash equilibrium seeking problem of aggregative games. Then Section 4 presents our main results including algorithm design and analysis. Section 5 gives a numerical example to illustrate the effectiveness of the proposed algorithm. Finally, Section 6 gives concluding remarks.
2 Preliminaries

In this section, we give basic notations and related preliminary knowledge.

Denote \( \mathbb{R}^n \) as the \( n \)-dimensional real vector space; denote \( \mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n \), and \( \mathbf{0}_n = (0, \ldots, 0)^T \in \mathbb{R}^n \). Denote \( \text{col}(x_1, \ldots, x_n) = (x_1^T, \ldots, x_n^T)^T \) as the column vector stacked with column vectors \( x_1, \ldots, x_n \), \( \| \cdot \| \) as the Euclidean norm, and \( I_n \in \mathbb{R}^{n \times n} \) as the identity matrix. Denote \( \nabla f \) as the gradient of \( f \).

A set \( C \subseteq \mathbb{R}^n \) is convex if \( \lambda z_1 + (1 - \lambda) z_2 \in C \) for any \( z_1, z_2 \in C \) and \( 0 \leq \lambda \leq 1 \). For a closed convex set \( C \), the projection map \( P_C : \mathbb{R}^n \rightarrow C \) is defined as

\[
P_C(x) \triangleq \arg\min_{y \in C} \|x - y\|.
\]

The projection map is 1-Lipschitz continuous, i.e.,

\[
\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.
\]

A map \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be \( \mu \)-strongly monotone on a set \( \Omega \) if

\[
(x - y)^T(F(x) - F(y)) \geq \mu \|x - y\|^2, \quad \forall x, y \in \Omega.
\]

Given a subset \( \Omega \subseteq \mathbb{R}^n \) and a map \( F : \Omega \rightarrow \mathbb{R}^n \), the variational inequality problem, denoted by \( \text{VI}(\Omega, F) \), is to find a vector \( x^* \in \Omega \) such that

\[
(y - x^*)^T F(x^*) \geq 0, \quad \forall y \in \Omega,
\]

and the set of solutions to this problem is denoted by \( \text{SOL}(\Omega, F) \) [18]. When \( \Omega \) is closed and convex, the solution of \( \text{VI}(\Omega, F) \) can be equivalently reformulated via projection as follows:

\[
x \in \text{SOL}(\Omega, F) \iff x = P_\Omega(x - \alpha F(x)), \quad \forall \alpha > 0.
\]

It is known that the information exchange among agents can be described by a graph. A graph with node set \( \mathcal{V} = \{1, 2, \ldots, N\} \) and edge set \( \mathcal{E} \) is written as \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) [19]. The adjacency matrix of \( \mathcal{G} \) can be written as \( \mathbf{A} = [a_{ij}]_{N \times N} \), where \( a_{ij} > 0 \) if \( (j, i) \in \mathcal{E} \) (meaning that agent \( j \) can send its information to agent \( i \), or equivalently, agent \( i \) can receive some information from agent \( j \)), and \( a_{ij} = 0 \), otherwise. A graph is said to be strongly connected if, for any pair of vertices, there exists a sequence of intermediate vertices connected by edges. For \( i \in \mathcal{V} \), the weighted in-degree and out-degree are \( d_{in}^i = \sum_{j=1}^N a_{ij} \) and \( d_{out}^i = \sum_{j=1}^N a_{ji} \), respectively. A graph is weight-balanced if \( d_{in}^i = d_{out}^i \), \( \forall i \in \mathcal{V} \).

The Laplacian matrix is \( \mathbf{L} = \mathbf{D}_{in} - \mathbf{A} \), where \( \mathbf{D}_{in} = \text{diag}\{d_{in}^1, \ldots, d_{in}^N\} \in \mathbb{R}^{N \times N} \).

The following result is well known.

**Lemma 1** Graph \( \mathcal{G} \) is weight-balanced if and only if \( \mathbf{L} + \mathbf{L}^T \) is positive semidefinite; it is strongly connected only if zero is a simple eigenvalue of \( \mathbf{L} \).
3 Problem Formulation

Consider an $N$-player aggregative game as follows. For $i \in \mathcal{V} \triangleq \{1, \ldots, N\}$, the $i$th player aims to minimize its cost function $J_i(x_i, x_{-i}) : \Omega \to \mathbb{R}$ by choosing the local decision variable $x_i$ from a local strategy set $\Omega_i \subset \mathbb{R}^{n_i}$, where $x_{-i} \triangleq \text{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, $\Omega \triangleq \Omega_1 \times \cdots \times \Omega_N \subset \mathbb{R}^n$ and $n = \sum_{i \in \mathcal{V}} n_i$.

The strategy profile of this game is $x \triangleq \text{col}(x_1, \ldots, x_N) \in \Omega$. The aggregation map $\sigma : \mathbb{R}^n \to \mathbb{R}^m$, to specify the cost function as $J_i(x_i, x_{-i}) = \vartheta_i(x_i, \sigma(x))$ with a function $\vartheta_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}$, is defined as

$$\sigma(x) \triangleq \frac{1}{N} \sum_{i=1}^{N} \varphi_i(x_i),$$

where $\varphi_i : \mathbb{R}^m \to \mathbb{R}^m$ is a map for the local contribution to the aggregation.

The concept of Nash equilibrium is introduced as follows.

**Definition 1** A strategy profile $x^*$ is said to be an Nash equilibrium of the game if

$$J_i(x^*_i, x_{-i}^*) \leq J_i(y_i, x_{-i}^*), \forall y_i \in \Omega_i, \forall i \in \mathcal{V}.$$  

Condition (2) means that all players simultaneously take their own best (feasible) responses at $x^*$, where no player can further decrease its cost function by changing its decision variable unilaterally.

We assume that the strategy sets and the cost functions are well-conditioned in the following sense.

**A1:** For any $i \in \mathcal{V}$, $\Omega_i$ is nonempty, convex and closed.

**A2:** For any $i \in \mathcal{V}$, the cost function $J_i(x_i, x_{-i})$ and the map $\varphi(x_i)$ are differentiable with respect to $x_i$.

In order to explicitly show the aggregation of the game, let us define map $G_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^{n_i}, i \in \mathcal{V}$ as

$$G_i(x_i, \eta_i) \triangleq \nabla_{x_i} J_i(\cdot, x_{-i}) |_{\sigma(x)=\eta_i}$$

$$= (\nabla_x \vartheta_i(\cdot, \sigma) + \frac{1}{N} \nabla_{\sigma} \vartheta_i(x_i, \cdot)^T \nabla \varphi_i) |_{\sigma=\eta_i}. \quad (3)$$

Also, let $G(x, \eta) \triangleq \text{col}(G_1(x_1, \eta_1), \ldots, G_N(x_N, \eta_N))$. Clearly, $G(x, 1_N \otimes \sigma(x)) = F(x)$, where the pseudo-gradient map $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$F(x) \triangleq \text{col}\{\nabla_{x_1} J_1(\cdot, x_{-1}), \ldots, \nabla_{x_N} J_N(\cdot, x_{-N})\}.$$

Under **A1** and **A2**, the Nash equilibrium of the game is a solution of the variational inequality problem VI$(\Omega, F)$, referring to [18]. Moreover, we need the following assumptions to ensure the existence and uniqueness of the Nash equilibrium and also to facilitate algorithm design.
Distributed Nash equilibrium seeking for constrained aggregative games

Theorem 2 Under A1-A4, the considered game admits a unique Nash equilibrium \( x^* \).

In the distributed design for our aggregative game, the communication topology for each player to exchange information is assumed as follows.

A5: The network graph \( G \) is strongly connected and weight-balanced.

The goal of this paper is to design a distributed algorithm to seek the Nash equilibrium for the considered aggregative game over weight-balanced directed graph.

4 Main Results

In this section, we first propose our distributed algorithm and then analyze its convergence.

4.1 Algorithm

Our distributed continuous-time algorithm for Nash equilibrium seeking of the considered aggregative game is designed as the following differential equations:

\[
\begin{align*}
\dot{x}_i &= P_{\Omega_i}(x_i - \alpha G_i(x_i, \eta_i)) - x_i, \quad x_i(0) \in \Omega_i \\
\dot{\theta}_i &= \beta \sum_{j=1}^{N} a_{ij}(\eta_j - \eta_i), \quad \theta_i(0) = 0_m \\
\eta_i &= \theta_i + \varphi_i(x_i)
\end{align*}
\]

Algorithm parameters \( \alpha \) and \( \beta \) satisfy

\[
0 < \alpha < \frac{2\mu \beta \lambda_2 - 4\kappa_2 \kappa_3}{\kappa^2 \beta \lambda_2 + 2\mu \kappa_2 \kappa_3},
\]

\[
\beta > \frac{2\kappa_2 \kappa_3}{\mu \lambda_2},
\]

where

\[
\kappa \triangleq \kappa_1 + \kappa_2 \cdot \kappa_3,
\]

and \( \lambda_2 \) is the smallest positive eigenvalue of \( \frac{1}{2}(L + L^T) \) (\( L \) is the Laplacian matrix).
Distributed Nash equilibrium seeking for constrained aggregative games

The compact form of (4) can be written as
\[
\begin{align*}
\dot{x} &= P_\Omega(x - \alpha G(x, \eta)) - x, \quad x(0) \in \Omega \\
\dot{\theta} &= -\beta L \otimes I_m \eta, \quad \theta(0) = 0_{mN} \\
\dot{\eta} &= \theta + \varphi(x)
\end{align*}
\]
where \(\varphi(x) = \text{col}(\varphi_1(x), \ldots, \varphi_N(x_N))\). Furthermore, we can rewrite (7) as
\[
\begin{align*}
\dot{x} &= P_\Omega(x - \alpha G(x, \eta)) - x, \quad x(0) \in \Omega \\
\dot{\eta} &= -\beta L \otimes I_m \eta + \frac{d}{dt} \varphi(x), \quad \eta(0) = \varphi(x(0))
\end{align*}
\]
The dynamics with respect to \(x\) can be regarded as distributed projected gradient play dynamics with the global aggregation \(\sigma(x)\) replaced by local variables \(\eta_1, \ldots, \eta_N\). The dynamics with respect to \(\eta\) is distributed average tracking dynamics that estimates the value of \(\sigma(x)\). The design idea is similar to [10, 13]. Here, we use projection operation to deal with local feasible constraints, and replace the nonsmooth tracking dynamics in [13] by this simple one to cope with weight-balanced graphs.

4.2 Analysis

First, we verify that the equilibrium of dynamics (8) coincides with the Nash equilibrium \(x^*\).

**Theorem 1** Under A1 - A5, the equilibrium of dynamics (8) is
\[
\begin{bmatrix} x \\ \eta \end{bmatrix} = \begin{bmatrix} x^* \\ \eta^* \end{bmatrix} = \begin{bmatrix} x^* \\ 1_N \otimes \sigma(x^*) \end{bmatrix}.
\]

**Proof** The equilibrium of (8) should satisfy
\[
\begin{align*}
0_n &= P_\Omega(x - \alpha G(x, \eta)) - x \\
0_{mN} &= -L \otimes I_m \eta
\end{align*}
\]
which are obtained by setting \(\dot{x}, \dot{\eta}\) and \(\frac{d}{dt} \varphi(x)\) as zeros. Since \(G\) is strongly connected, \(L \otimes I_m \eta = 0\) implies \(\eta_1 = \eta_2 = \cdots = \eta_N = \eta^*\) for some \(\eta^*\) to be further determined.

Since \(G\) is weight-balanced, \(1_N^T L = 0_N^T\). Combining this property with dynamics (8) yields
\[
\frac{1}{N} \sum_{i=1}^N \dot{\eta}_i = \frac{d}{dt} \sigma(x), \quad \frac{1}{N} \sum_{i=1}^N \eta_i(0) = \sigma(x(0)).
\]
As a result,
\[
\frac{1}{N} \sum_{i=1}^N \eta_i = \sigma(x),
\]
which implies that any equilibrium pair \((x^*, 1_N \otimes \eta^*)\) should also satisfy \(\eta^* = \sigma(x^*)\).
Substituting $x^\diamond, 1_N \otimes \eta^\diamond$ into the projected equation for the equilibrium yields

$$0_n = P_\Omega(x^\diamond - \alpha G(x^\diamond, 1_N \otimes \eta^\diamond)) - x^\diamond$$

$$= P_\Omega(x^\diamond - \alpha F(x^\diamond)) - x^\diamond,$$

which indicates $x^\diamond = x^\ast$. Therefore, the point given in (9) is the equilibrium of (8). This completes the proof. □

In view of the identity (10) derived from (8), let

$$y \triangleq \eta - 1_N \otimes \sigma(x).$$

Then it follows from $L1_N = 0_N$ and (8) that

$$\dot{x} = P_\Omega \left(x - \alpha G(x, 1_N \otimes \sigma(x) + y)\right) - x \quad (11)$$

$$\dot{y} = -\beta L \otimes I_m y + \frac{d}{dt} \left(\varphi(x) - 1_N \otimes \nabla \sigma(x)\right)^T.$$ 

$$= -\beta L \otimes I_m y + \left(\nabla \varphi(x) - 1_N \otimes \nabla \sigma(x)\right)^T.$$

$$= -\beta L \otimes I_m y + \left(P_\Omega \left(x - \alpha G(x, 1_N \otimes \sigma(x) + y)\right) - x\right)$$

The whole dynamics with respect to $x$ and $y$ consists of two interconnected subsystems as shown in Fig. 1. Each dynamical subsystem has its own state

![Diagram](image)

**Fig. 1** The interconnection of two subsystems (11) and (12).

variable, equilibrium point and external input.

Our convergence results are given in the following theorem.

**Theorem 2** Under $A1$-$A5$, the distributed continuous-time algorithm (4) with parameters satisfying (5) converges to the Nash equilibrium with an exponential convergence rate.

**Proof** Let

$$\omega_1 \triangleq \frac{2\alpha \cdot \mu - \alpha^2 \cdot \kappa^2}{2 + \alpha \cdot \kappa},$$
Distributed Nash equilibrium seeking for constrained aggregative games

\[ \omega_2 \triangleq \beta \cdot \lambda_2 - \alpha \cdot \kappa_2 \cdot \kappa_3, \]

\[ \xi_1 \triangleq \alpha \cdot \kappa_2, \]

\[ \xi_2 \triangleq \kappa_3(2 + \alpha \cdot \kappa), \]

and

\[ \gamma^* \triangleq \omega_1 + \omega_2 - \sqrt{(\omega_1 - \omega_2)^2 + 4\xi_1\xi_2}. \]

We will show that the rate of exponential convergence of our algorithm is \( \gamma^* \). It follows from (5) that \( \gamma^* > 0 \), and

\[ (\omega_1 - \frac{\gamma^*}{2}) \cdot (\omega_2 - \frac{\gamma^*}{2}) = \xi_1\xi_2. \]

Let

\[ H(x) \triangleq x - P_{\Omega}(x - \alpha F(x)), \]

\[ \tilde{H}(x, y) \triangleq x - P_{\Omega}(x - \alpha G(x, 1_N \otimes \sigma(x) + y)), \]

\[ \xi(x, y) \triangleq \tilde{H}(x, y) - H(x). \]

We verify the following three properties.

1) \( \| \xi(x, y) \| \leq \alpha \cdot \kappa_2 \| y \|. \)

2) The map \( F \) is \( \kappa \)-Lipschitz continuous.

3) The map \( H \) is \( \omega_1 \)-strongly monotone.

Property 1) holds because

\[ \| \xi(x, y) \| = \| \tilde{H}(x, y) - H(x) \| \]

\[ = \| P_{\Omega}(x - \alpha F(x)) \]

\[ - P_{\Omega}(x - \alpha G(x, 1_N \otimes \sigma(x) + y)) \|

\[ \leq \alpha \| F(x) - G(x, 1_N \otimes \sigma(x) + y) \| \]

\[ \leq \alpha \cdot \kappa_2 \| y \|. \]

Property 2) follows from the fact that

\[ \| F(y) - F(x) \| \]

\[ = \| G(y, 1_N \otimes \sigma(y)) - G(x, 1_N \otimes \sigma(x)) \| \]

\[ \leq \| G(y, 1_N \otimes \sigma(x)) - G(x, 1_N \otimes \sigma(x)) \| \]

\[ + \| G(y, 1_N \otimes \sigma(y)) - G(y, 1_N \otimes \sigma(x)) \| \]

\[ \leq \kappa_1 \| y - x \| + \kappa_2 \cdot \kappa_3 \| y - x \|. \]

Property 3) holds because

\[ (x - y)^T(H(x) - H(y)) \]

\[ = \| x - y \|^2 - (x - y)^T \cdot (P_{\Omega}(x - \alpha F(x)) - P_{\Omega}(y - \alpha F(y))) \]

\[ \geq \| x - y \| \cdot (\| x - y \| - \| P_{\Omega}(x - \alpha F(x)) - P_{\Omega}(y - \alpha F(y)) \|) \]

\[ \geq \| x - y \| \cdot (\| x - y \| - \| x - \alpha F(x) - (y - \alpha F(y)) \|). \]
In addition, there holds the identity $H(x^*) = 0$, since $x^*$ is the Nash equilibrium.

Consider the following Lyapunov candidate function

$$V_1(x) = \frac{1}{2}\|x - x^*\|^2.$$ 

Its time derivative along the trajectory of (11) is

$$\dot{V}_1 = -(x - x^*)^T \tilde{H}(x, y)$$

$$= -(x - x^*)^T (H(x) + \xi(x, y))$$

$$= -(x - x^*)^T (H(x) - H(x^*)) - (x - x^*)^T \xi(x, y)$$

$$\leq -\omega_1 \|x - x^*\|^2 + \|x - x^*\| \|\xi(x, y)\|$$

$$\leq -\omega_1 \|x - x^*\|^2 + \xi_1 \|x - x^*\| \|y\|.$$ 

Next, we focus on dynamics (12). Let

$$\zeta(x, y) \triangleq \frac{d}{dt} (\varphi(x) - 1_N \otimes \sigma(x))$$

$$= (\nabla \varphi(x) - 1_N \otimes \nabla \sigma(x))^T (P_\Omega (x - \alpha G(x, 1_N \otimes \sigma(x) + y)) - x),$$

where the time derivative $\dot{x}$ is along the dynamics (11).

Clearly, $1_N^T \otimes I_m \zeta(x, y) = 0$. Also, since

$$\|P_\Omega (x - \alpha G(x, 1_N \otimes \sigma(x) + y)) - x^*\|$$

$$\leq \|P_\Omega (x - \alpha G(x, 1_N \otimes \sigma(x) + y)) - x\|$$

$$- P_\Omega (x - \alpha G(x, 1_N \otimes \sigma(x) + y))$$

$$+ \|P_\Omega (x - \alpha F(x)) - P_\Omega (x^* - \alpha F(x^*))\|$$

$$\leq \alpha \cdot \kappa_2 \|y\| + \|x - x^*\| + \alpha \cdot \kappa \|x - x^*\|,$$

there holds

$$\|\zeta(x, y)\| \leq \|\nabla \varphi(x) - 1_N \otimes \nabla \sigma(x)\|.$$ 

$$\|P_\Omega (x - \alpha G(x, 1_N \otimes \sigma(x) + y)) - x\|$$

$$\leq \kappa_3 \|P_\Omega (x - \alpha G(x, 1_N \otimes \sigma(x) + y)) - x^*\|$$

$$+ \kappa_3 \|x - x^*\|$$

$$\leq \kappa_3 \cdot (2 + \alpha \cdot \kappa) \|x - x^*\| + \alpha \cdot \kappa_2 \cdot \kappa_3 \|y\|.$$ 

Let

$$\tilde{y} \triangleq \frac{1}{N} 1_N 1_N^T \otimes I_m y,$$

$$\tilde{y} \perp \triangleq (I_N - \frac{1}{N} 1_N 1_N^T) \otimes I_m y.$$
Then $y = \hat{y} + \hat{y}^\perp$. Since $1_N^T L = 0_N^T$, it follows from (12) that
$$\dot{\hat{y}} = 0, \quad \hat{y}(0) = 0.$$ As a result,
$$\hat{y}(t) = 0, \quad y(t) = \hat{y}^\perp(t), \quad \forall \ t \geq 0.$$ Consider the following Lyapunov candidate function
$$V_2(y) = \frac{1}{2} \|y\|^2.$$ The time derivative of $V_2$ along the trajectory of (12) is
$$\dot{V}_2 = -\beta y^T (L \otimes I_m) y + y^T \zeta(x, y)$$
$$= -\beta y^T \left( \frac{1}{2} (L + L^T) \otimes I_m \right) y + y^T \zeta(x, y)$$
$$= -\beta (\hat{y}^\perp)^T \left( \frac{1}{2} (L + L^T) \otimes I_m \right) \hat{y}^\perp + y^T \zeta(x, y)$$
$$\leq -\beta \cdot \lambda_2 \|\hat{y}^\perp\|^2 + y^T \zeta(x, y),$$
where the last inequality follows from Rayleigh quotient theorem [20, Page 234]. Also, since $y(t) = \hat{y}^\perp(t), \forall \ t \geq 0$,
$$\dot{V}_2 \leq -\beta \cdot \lambda_2 \|\hat{y}^\perp\|^2 + y^T \zeta(x, y)$$
$$= -\beta \cdot \lambda_2 \|y\|^2 + y^T \zeta(x, y)$$
$$\leq -\left( \beta \cdot \lambda_2 - \alpha \cdot \kappa_2 \cdot \kappa_3 \right) \|y\|^2$$
$$+ \kappa_3 \cdot (2 + \alpha \cdot \kappa) \|y\| \|x - x^*\|$$
$$= -\omega_2 \|y\|^2 + \xi_2 \|y\| \|x - x^*\|.$$ Combining $V_1$ and $V_2$, let
$$V \triangleq \xi_2 V_1 + \xi_1 V_2$$
The time derivative of $V$ along the trajectory of (11) and (12) is
$$\dot{V} \leq -\xi_2 \omega_1 \|x - x^*\|^2 + 2 \xi_2 \xi_1 \|x - x^*\| \|y\| - \xi_1 \omega_2 \|y\|^2$$
$$= -\gamma^* V - (\omega_1 - \frac{\gamma^*}{2}) \xi_2 \|x - x^*\|^2 - (\omega_2 - \frac{\gamma^*}{2}) \xi_1 \|y\|^2 + 2 \xi_1 \xi_2 \|x - x^*\| \|y\|$$
$$\leq -\gamma^* V.$$ Therefore, the algorithm converges to the Nash equilibrium with the exponential convergence rate $\gamma^*$. \hfill \square

Remark 1 Exponential convergence of distributed algorithms has become a research topic in recent years. [21] has designed a distributed discrete-time optimization algorithm and proves its exponential convergence via a small-gain approach, while [22] has introduced a criterion for the exponential convergence of distributed primal-dual gradient algorithms in either continuous or discrete time. Theorem 2 provides an exponential convergence result by analyzing the interconnected subsystems.
5 Numerical Example

Consider a Cournot game played by $N = 20$ competitive players. For $i \in \mathcal{V} = \{1, ..., N\}$, the cost function $\vartheta_i(x_i, \sigma)$ and strategy set $\Omega_i$ are

$$\vartheta_i(x_i, \sigma) = a_i x_i^2 + b_i x_i + c_i x_i \sigma(x),$$

$$\Omega_i = \left[ -1 - \frac{1}{2i}, \frac{i}{10} + \frac{1}{\sqrt{i}} \right],$$

where

$$a_i = 0.1 + 0.01 \times \sin(i), \quad b_i = \frac{i - \ln(i)}{1 + i + i^3},$$

$$c_i = 0.003 \times \cos(i),$$

and

$$\sigma(x) = \frac{1}{N} \sum_{j=1}^{N} x_j.$$ 

It can be verified that the game mode satisfies $A1$-$A4$ with constants $\mu = 0.1770, \kappa_1 = 0.2199, \kappa_2 = 0.0030, \kappa_3 = 1$. We adopt a network graph as shown in Fig. 2, which satisfies $A5$.

![Fig. 2](image)

Fig. 2 The communication graph of the agents.

To render condition (5), we assign $\alpha = 3$ and $\beta = 1$. The trajectory of strategy profile generated by our algorithm is shown in Fig. 3.

In order to make some comparisons, we also use directed cycle graph and undirected Erdos-Renyi (ER) graph for the algorithm. The performance of the algorithm with these graphs is shown in Fig. 4.

These results indicate that our distributed algorithm exponentially converges to the Nash equilibrium.
Finally, we use the undirected ER graph to compare our algorithm with the one given in [13]. The numerical results are shown in Fig. 5. It indicates that our algorithm converges faster than that algorithm. In addition, only our algorithm applies to directed graphs such as the original graph and the directed cycle graph.

### 6 Conclusions

A distributed algorithm has been proposed for Nash equilibrium seeking of aggregative games, where the strategy set can be constrained and the network is described by a weight-balanced graph. The exponential convergence has
been established. The effectiveness of our method has also been illustrated by a numerical example. Further work may consider generalized Nash equilibrium seeking problem for aggregative games with coupled constraints.

**Declarations**

The authors confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

This paper was supported in part by National Natural Science Foundation of China under Grant 61903027, 72171171, 62003239, and in part by Shanghai Municipal Science and Technology Major Project under grant 2021SHZDZX0100, and in part by Shanghai Sailing Program under Grant Nos. 20YF1453000.

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