Locality-sensitive Hashing without False Negatives

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Abstract
We consider a new construction of locality-sensitive hash functions for Hamming space
that is covering in the sense that it is guaranteed to produce a collision for every pair of
vectors within a given radius $r$. The construction is efficient in the sense that the expected
number of hash collisions between vectors at distance $cr$, for a given $c > 1$, comes close to
that of the best possible data independent LSH without the covering guarantee, namely,
the seminal LSH construction of Indyk and Motwani (FOCS ’98). The efficiency of the
new construction essentially matches their bound if $cr = \log(n)/k$, where $n$ is the number
of points in the data set and $k \in \mathbb{N}$, and differs from it by at most a factor $\ln(4) < 1.4$ in
the exponent for general values of $cr$. As a consequence, LSH-based similarity search in
Hamming space can avoid the problem of false negatives at little or no cost in efficiency.

1 Introduction
Similarity search in high dimensions has been a subject of intense research for the last
decades in several research communities including theory, databases, machine learning,
and information retrieval. In this paper we consider nearest neighbor search in Hamming
space, where the task is to find a vector in a preprocessed set $S \subseteq \{0,1\}^d$ that has minimum
Hamming distance to a query vector $y \in \{0,1\}^d$. It is known that efficient data structures
for this problem, i.e., whose query and preprocessing time does not increase exponentially
with $d$, would disprove the strong exponential time hypothesis [15, 16]. For this reason
the algorithms community has studied the problem of finding a $c$-approximate nearest
neighbor, i.e., a point whose distance to $y$ is bounded by $c$ times the distance to a nearest
neighbor, where $c > 1$ is a user-specified parameter. All existing $c$-approximate nearest
neighbor data structures that have been rigorously analyzed have one or more of the
following drawbacks:

1. Worst case query time linear in the number of points in the data set, or
2. Worst case query time that grows exponentially with $d$, or
3. Multiplicative space overhead that grows exponentially with $d$, or
4. No unconditional guarantee to return a ($c$-approximate) nearest neighbor.

Arguably, the data structures that come closest to overcoming these drawbacks are based
on locality-sensitive hashing (LSH). For many metrics, including the Hamming metric
discussed in this paper, LSH yields sublinear query time (even for $d \gg \log(n)$) and space
usage that is polynomial in $n$ and linear in the number of dimensions [9, 15]. If the

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approximation factor \( c \) is larger than a certain constant (currently known to be at most 3) the space can even be made \( O(nd) \), still with sublinear query time \([13,10]\). However, these methods come with a Monte Carlo-type guarantee: A \( c \)-approximate nearest neighbor is returned only with high probability, and there is no way of detecting if the computed result is incorrect. This means that they do not overcome the 4th drawback above.

In this paper we investigate the possibility of Las Vegas-type guarantees for \( c \)-approximate nearest neighbor search. Traditional LSH schemes pick the sequence of hash functions independently, which inherently implies that we can only hope for high probability bounds. Extending and improving results by Greene et al. \([6]\) and Arasu et al. \([3]\) we show that in Hamming space, by suitably correlating hash functions we can “cover” all possible positions of \( r \) differences and thus eliminate false negatives, while achieving performance bounds comparable to those of traditional LSH methods.

1.1 Notation

For a set \( S \) and function \( f \) we let \( f(S) = \{ f(x) \mid x \in S \} \). We use \( \mathbf{0} \) and \( \mathbf{1} \) to denote vectors of all 0s and 1s, respectively. For \( x, y \in \{0, 1\}^d \) we use \( x \land y \) and \( x \lor y \) to denote bit-wise conjunction and disjunction, respectively, and \( x \oplus y \) to denote the bitwise exclusive-or. Let \( I(x) = \{ i \mid x_i = 1 \} \). We use \( \|x\| = |I(x)| \) to denote the Hamming weight of a vector \( x \), and \( ||x - y|| = |I(x \oplus y)| \) to denote the Hamming distance between \( x \) and \( y \). For \( S \subseteq \{0, 1\}^d \) let \( d_S \) be an upper bound on the time required to produce a representation of the nonzero entries of a vector in \( S \) in the standard (word RAM) model \([7]\). Observe that in general \( d_S \) depends on the representation of vectors (e.g., bit vectors for dense vectors, or sparse representations if \( d \) is much larger than the largest Hamming weight). We use “\( x \) mod \( b \)” to refer to the integer in \( \{0, \ldots, b - 1\} \) whose difference from \( x \) is divisible by \( b \). Finally, let \( \langle x, y \rangle \) denote the purity of \( ||x \land y|| \), i.e., the dot product of \( x \) and \( y \) when interpreted as vectors over the field \( \mathbb{F}_2 \).

2 Related work

Given \( S \subseteq \{0, 1\}^d \) the problem of searching for a vector in \( S \) within Hamming distance \( r \) from a given query string \( y \) was introduced by Minsky and Papert as the approximate dictionary problem \([12] \). The generalization to arbitrary spaces is now known as the near neighbor problem (or sometimes as point location in balls). It is known that a solution to the approximate near neighbor problem implies a solution to the nearest neighbor problem with comparable performance \([9,8]\). In our case this is somewhat simpler to see, so we give the argument for completeness in Appendix A. For this reason, in the following we focus on the near neighbor problem in Hamming space where \( r \) is assumed to be known when the data structure is created.

2.1 Deterministic algorithms

For simplicity we will restrict attention to the case \( r \leq d/2 \). A baseline is the brute force algorithm that looks up all \( \binom{d}{r} \) bit strings of Hamming distance at most \( r \) from \( y \). The time usage is at least \( \binom{d}{r} \cdot \frac{d}{r} \), assuming \( r \leq d/2 \), so this method is not attractive unless \( \frac{d}{r} \) is quite small. The dependence on \( d \) was reduced by Cole et al. \([4]\) who achieve query time \( O_r(n d + \log^* n) \) and space \( O_r(n d + n \log^* n) \). Again, because of the exponential dependence on \( r \) this method is interesting only for small values of \( r \).

2.2 Randomized filtering with false negatives

In a seminal paper \([9]\), Indyk and Motwani presented a randomized solution to the \( c \)-approximate near neighbor problem where the search stops as soon as a vector within
while at the same time achieving nontrivial filtering efficiency for larger distances.

family: if we consider in this paper, uses a set of functions from a Hamming projection family:

\[ \mathcal{H}_A = \{ x \mapsto x \land a \mid a \in A \} \]  

(1)

where \( A \subseteq \{0,1\}^d \). The vectors in \( A \) will be referred to as bit masks. Given a query \( y \), the idea is to iterate through all functions \( h \in \mathcal{H} \) and identify collisions \( h(x) = h(y) \) for \( x \in S \), e.g., using a hash table. This procedure covers a query \( y \) if at least one collision is produced when there exists \( x \in S \) with \( ||x - y|| \leq r \), and it is efficient if the number of hash function evaluations and collisions with \( ||x - y|| > cr \) is not too large. The procedure can be thought of as a randomized filter that attempts to catch data items of interest while filtering away data items that are not even close to being interesting. The filtering efficiency with respect to vectors \( x \) and \( y \) is the expected number of collisions \( h(x) = h(y) \) summed over all functions \( h \in \mathcal{H}_A \), with expectation taken over any randomness in the choice of \( A \). We can argue that without loss of generality it can be assumed that the filtering efficiency depends only on \( ||x - y|| \) and not on the location of the differences. To see this, using an idea from \[ 3 \], consider replacing each \( a \in A \) by a vector \( \pi(a) \) defined by \( \pi(a) = a_{\pi(j)} \), where \( \pi : \{1,\ldots,d\} \to \{1,\ldots,d\} \) is a random permutation used for all vectors in \( A \). This does not affect the covering guarantee, but means that collision probabilities will depend solely on \( ||x - y|| \), \( d \) and the Hamming weights of vectors in \( A \).

Remark. If vectors in \( A \) are sparse it is beneficial to work with a sparse representation of the input and output of functions in \( \mathcal{H} \), and indeed this is what is done by Indyk and Motwani who consider functions that concatenate a suitable number of 1-bit samples from \( x \). However, we find it convenient to work with \( d \)-dimensional vectors, with the understanding that a sparse representation can be used if \( d \) is large. \( \triangle \)

Indyk and Motwani use a collection \( \mathcal{A}(R) = \{ a(v) \mid v \in R \} \), where \( R \subseteq \{1,\ldots,d\}^k \) is a set of uniformly random and independent \( k \)-dimensional vectors, and \( a(v)_i = 1 \) if and only if \( v_j = i \) for some \( j \in \{1,\ldots,k\} \). The intuition is that each vector \( v \) encodes a sequence of \( k \) samples from \( \{1,\ldots,d\} \). By choosing \( k \) appropriately we can achieve a trade-off that balances the size of \( R \) (i.e., the number of hash functions) with the expected number of collisions at distance \( cr \). It turns out that \( |R| = O(n^{1/3}\log(1/\delta)) \) suffices to achieve collision probability \( 1 - \delta \) at distance \( r \) while keeping the expected total number of collisions for “far” pairs of vectors (at distance \( cr \) or more) linear in \( |R| \).

In a recent advance by Andoni and Razenshteyn \[ 2 \], extending preliminary ideas from \[ 1 \], it was shown how data dependent LSH can achieve the same guarantee with a smaller family (having \( n^{o(1)} \) space usage and evaluation time). Specifically, it suffices to check collisions of \( O(n^d\log(1/\delta)) \) hash values, where \( \rho = \frac{1}{2^{1/4}} \). We will not attempt to generalize the new method to the data dependent setting, though that is certainly an interesting possible extension.

2.3 Filtering methods without false negatives

The literature on filtering methods for Hamming distance that do not introduce false negatives, but still yield formal guarantees, is relatively small. As in section \[ 2.2 \] the previous results can be stated in the form of Hamming projection families \[ 1 \]. We consider constructions of sets \( A \) that ensure collision for every pair of vectors at distance at most \( r \), while at the same time achieving nontrivial filtering efficiency for larger distances.

Choosing error probability \( \delta < 4^{-d} \) in the construction of Indyk and Motwani, we see that there must exist a set \( R^* \) of size \( O(dn^{1/3}) \) that ensures collision under some \( h \in \mathcal{A}(R^*) \) for all (less than \( 4^d \)) pairs of vectors within distance \( r \). However, this existence
argument is of little help to design an algorithm, and hence we will be interested in explicit constructions of LSH families without false negatives.

Greene et al. [6] linked the question to the Turan problem in extremal graph theory. While optimal Turan numbers are not known in general, Greene et al. construct a family \( \mathcal{A} \) (based on corrector hypergraphs) that will incur few collisions with random strings, i.e., strings at distance about \( d/2 \) from the query point.

Arasu et al. [3] give a construction that is able to achieve, for example, \( o(1) \) filtering efficiency for approximation factor \( c > 7.5 \) with \( |\mathcal{A}| = \mathcal{O}(1^{2.39}) \). Observe that there is no dependence on \( d \) in these bounds, which is crucial for high-dimensional (sparse) data. The technique of [3] allows a range of trade-offs between \( |\mathcal{A}| \) and the filtering efficiency, determined by parameters \( n_1 \) and \( n_2 \). No theoretical analysis is made of how close to 1 the filtering efficiency can be made for a given \( c \), but it seems difficult to significantly improve the constant 7.5 mentioned above.

Independently of the work of Arasu et al. [3], “lossless” methods for near neighbor search have been studied in the contexts of approximate pattern matching [11] and computer vision [13]. The analytical part of these papers differs from our setting by focusing on filtering efficiency for random strings or vectors, which means that differences between a data vector and the query appear in random locations. In particular there is no need to permute the dimensions as described in section 2.2.

3 Basic construction

Our basic correlated LSH construction is a Hamming projection family of the form (1).

We start by observing the following simple property of Hamming projection families:

Lemma 1. For every \( \mathcal{A} \subseteq \{0, 1\}^d \), every \( h \in \mathcal{H}_\mathcal{A} \), and all \( x, y \in \{0, 1\}^d \) we have \( h(x) = h(y) \) if and only if \( h(x \oplus y) = 0 \).

Proof. Let \( a \in \mathcal{A} \) be the vector such that \( h(x) = x \land a \). We have \( h(x) = h(y) \) if and only if \( a_i \neq 0 \Rightarrow x_i = y_i \). Since \( x_i = y_i \Leftrightarrow (x \oplus y)_i = 0 \) the claim follows.

Thus, to make sure all pairs of vectors within distance \( r \) collide for some function, we need our family to have the property (implicit in the work of Arasu et al. [3]) that every vector with 1s in \( r \) bit positions is mapped to zero by some function, i.e., the set of 1s is “covered” by zeros in a vector from \( \mathcal{A} \).

Definition 1. For \( \mathcal{A} \subseteq \{0, 1\}^d \), the Hamming projection family \( \mathcal{H}_\mathcal{A} \) is \( r \)-covering if for every \( x \in \{0, 1\}^d \) with \( ||x|| \leq r \), there exists \( h \in \mathcal{H}_\mathcal{A} \) such that \( h(x) = 0 \). The family is said to have weight \( w \) if \( ||a|| \geq wd \) for every \( a \in \mathcal{A} \).

A trivial \( r \)-covering family uses \( \mathcal{A} = \{\mathbf{0}\} \). We are interested in \( r \)-covering families that have nonzero weight chosen to make collisions rare among vectors that are not close. Vectors in our basic \( r \)-covering family, which aims at weight around 1/2, will be indexed by nonzero vectors in \( \{0, 1\}^{r+1} \). The family depends on a function \( m : \{1, \ldots, d\} \rightarrow \{0, 1\}^{r+1} \) that maps bit positions to bit vectors of length \( r + 1 \). Define a family of bit vectors \( a(v) \in \{0, 1\}^d \) by

\[
a(v)_i = \langle m(i), v \rangle, \quad \text{for } i \in \{1, \ldots, d\}, \quad v \in \{0, 1\}^{r+1},
\]

where \( \langle m(i), v \rangle \) is the dot product modulo 2 of vectors \( m(i) \) and \( v \). We will consider the family of all such vectors with nonzero \( v \):

\[
\mathcal{A}(m) = \{a(v) \mid v \in \{0, 1\}^{r+1}\backslash\{\mathbf{0}\}\}.
\]

Figure 4 shows the family \( \mathcal{A}(m) \) for \( r = 2 \) and \( m(i) \) being the vector representing \( i \) in binary.
The resulting Hamming projection family $\mathcal{H}_{A_r}$, see \cite{footnote}, is 2-covering since for every pair of columns there exists a row with 0s in these columns. It has weight $4/7$ since there are four 1s in each row. Every row covers 3 of the 21 pairs of columns, so no smaller 2-covering family of weight $4/7$ exists.

**Lemma 2.** For every $m : \{1, \ldots, d\} \to \{0, 1\}^{r+1}$, the Hamming projection family $\mathcal{H}_{A(m)}$ is $r$-covering.

**Proof.** Let $x \in \{0, 1\}^d$ satisfy $\|x\| \leq r$ and consider $a(v) \in A(m)$ as defined in \cite{footnote}. It is clear that whenever $i \in \{1, \ldots, d\} \setminus I(x)$ we have $(a(v) \wedge x)_i = 0$ (recall that $I(x) = \{i \mid x_i = 1\}$). To consider $(a(v) \wedge x)_i$ for $i \in I(x)$ let $M_z = m(I(x))$, where elements are interpreted as $r + 1$-dimensional vectors over the field $\mathbb{F}_2$. The span of $M_z$ has dimension at most $|M_z| \leq \|x\| \leq r$, and since the space is $r + 1$-dimensional there exists a vector $v_z \neq 0$ that is orthogonal to span$(M_z)$. In particular $(v_z, m(i)) = 0$ for all $i \in I(x)$. In turn, this means that $a(v_z) \wedge x = 0$, as desired. \hfill \square

If the values of function $m$ are “balanced” over nonzero vectors the family $\mathcal{H}_{A(m)}$ has weight close to $1/2$ for $d \gg 2^r$. More precisely we have:

**Lemma 3.** Suppose $|m^{-1}(v)| \geq \lceil d/2^{r+1} \rceil$ for each $v \in \{0, 1\}^{r+1}$ and $m^{-1}(0) = \emptyset$. Then $\mathcal{H}_{A(m)}$ has weight at least $2\lceil d/2^{r+1} \rceil/d > (1 - 2^{-r})/2$.

**Proof.** It must be shown that $|a(v)| \geq 2\lceil d/2^{r+1} \rceil$ for each nonzero vector $v$. Note that $v$ has a dot product of 1 with a set $V \subseteq \{0, 1\}^{r+1}$ of exactly $2^r$ vectors (namely the nontrivial coset of $v$’s orthogonal complement). For each $v' \in V$ the we have $a(v)_i = 1$ for all $i \in m^{-1}(v')$. Thus the number of 1s in $a(v)$ is:

$$\sum_{v' \in V} |m^{-1}(v')| \geq 2\lceil d/2^{r+1} \rceil > (1 - 2^{-r})d/2.$$

\hfill \square

We note that the size of $\mathcal{H}(m)$ is near-optimal among $r$-covering families of weight $1/2$, see Appendix \cite{footnote}. Lemmas \cite{footnote} and \cite{footnote} leave open the choice of mapping $m$. We will analyze the setting where $m$ maps to values chosen uniformly and independently from $\{0, 1\}^{r+1}$. In this setting the condition of Lemma \cite{footnote} will in general not be satisfied, but it turns out that it suffices for $m$ to have balance in an expected sense. We can relate collision probabilities to Hamming distances as follows:

**Theorem 1.** For every choice of $x, y \in \{0, 1\}^d$ and for random $m : \{1, \ldots, d\} \to \{0, 1\}^{r+1}$,

1. If $\|x - y\| \leq r$ then $Pr[\exists h \in \mathcal{H}_{A(m)} : h(x) = h(y)] = 1$.
2. $E\left[ (h \in \mathcal{H}_{A(m)} \mid h(x) = h(y)) \right] < 2^{r+1-\|x-y\|}$.

**Proof.** Let $z = x \oplus y$. For the first part we have $\|x - y\| = \|z\| \leq r$. Lemma \cite{footnote} states that there exists $h \in \mathcal{H}_{A(m)}$ such that $h(z) = 0$. By Lemma \cite{footnote} this implies $h(x) = h(y)$.

To show the second part we fix $v \in \{0, 1\}^{r+1} \setminus \{0\}$. Now consider $a(v) \in A(m)$, defined in \cite{footnote}, and the corresponding function $h(x) = x \wedge a(v) \in \mathcal{H}_{A(m)}$. For $i \in I(z)$ we have
$h(z)_i = 0$ if and only if $a(v)_i = 0$. Since $m$ is random and $v \neq 0$ the $a(v)_i$ values are independent and random, so the probability that $a(v)_i = 0$ for all $i \in I(z)$ is $2^{-\|z\|} = 2^{-\|x-y\|}$. By linearity of expectation, summing over the $2^{r+1} - 1$ choices of $v$ the claim follows. 

**Comments.** A few remarks on Theorem 1 (that can be skipped if the reader wishes to proceed to the algorithmic results):

- The vectors in $\mathcal{A}(m)$ can be seen as samples from a Hadamard code consisting of $2^{r+1}$ vectors of dimension $2^{r+1}$, where bit $i$ of vector $j$ is defined by $\langle i, j \rangle \mod 2$. Nonzero Hadamard codewords have Hamming weight and minimum distance $2^{r+1}$. However, it does not seem that error-correcting ability in general yields nontrivial $r$-covering families.

- At first glance it appears that the ability to avoid collision for the correlated LSH ("filtering") is not significant when $\|x-y\| = r+1$. However, we observe that for similarity search in Hamming space it can be assumed without loss of generality that either all distances from the query point are even or all distances are odd. This can be achieved by splitting the data set into two parts, having even and odd Hamming weight, respectively, and handling them separately. For a given query $y$ and radius $r$ we then perform a search in each part, one with radius $r$ and one with radius $r-1$ (in the part of data where distance $r$ to $y$ is not possible). This reduces the expected number of collisions at distance $r+1$ to at most $1/2$.

- A final observation is that if we consider the subfamily of $\mathcal{A}(m)$ indexed by vectors of the form $0^{r+1-r_1}v$, where $v \in \{0,1\}^{\log n}$ for some $r_1 \leq r$, then collision is guaranteed up to distance $r_1$. That is, we can search for a nearest neighbor in an unknown distance in a natural way, by letting $m$ map randomly to $\{0,1\}^{\log n}$ and choosing $v$ as the binary representation of $1,2,3,\ldots$ (or alternatively, the vectors in a Gray code for $\{0,1\}^{\log n}$). In either case Theorem 1 implies the invariant that the nearest neighbor has distance at least $\log v$, where $v$ is interpreted as an integer. This means that when a point $p$ at distance at most $c \log v$ is found, we can stop and return $p$ as a $c$-approximate nearest neighbor. △

### 3.1 Approximate near neighbor search for $cr = \log(n)$

Note that approximation factor $c = \log(n)/r$ corresponds to any vector at distance at most $\log n$ from $y$ being a valid answer. This may be an appropriate assumption for high-entropy data sets of dimension $d > 2\log n$ where most distances tend to be large (see [11, 13] for discussion of such settings). In this case Theorem 1 implies efficient $c$-approximate near neighbor search in expected time $O(d_2 2^r) = O(d_2 n^{1/c})$, where $d_2$ bounds the time to compute the Hamming distance between query vector $y$ and a vector $x \in S$. This matches the asymptotic time complexity of Indyk and Motwani [9].

To show this bound observe that the expected total number of collisions $h(x) = h(y)$, summed over all $h \in \mathcal{H}_A(m)$ and $x \in S$ with $\|x-y\| \geq \log n$, is at most $2^{r+1}$. This means that computing $h(y)$ for each $h \in \mathcal{H}_A(m)$ and computing the distance to the vectors that are not within distance $cr$ but collide with $y$ under some $h \in \mathcal{H}_A(m)$ can be done in expected time $O(d_2 2^r)$. (The expected time bound can be turned into a high probability bound by the standard technique of restarting the computation if the expectation is exceeded by a factor of 2.) What we have bounded is in fact performance on a worst case data set in which most data points are just above the threshold for being a $c$-approximate near neighbor. In general the amount of time needed for a search will depend on the distribution of distances between $y$ and data points, and may be significantly lower.

The space required is $O(2^r n) = O(n^{1+1/c})$ plus the space required to store the vectors in $S$, again matching the bound of Indyk and Motwani. In a straightforward implementation we need additional space $O(d)$ to store the function $m$, but if $d$ is large (for sets of
Suppose we have a set \( S \) of \( n = 2^{30} \) vectors from \( \{0, 1\}^{128} \) and wish to search for a vector at distance at most \( r = 10 \) from a query vector \( y \). A brute-force search within radius \( r \) would take much more time than linear search, so we settle for 3-approximate similarity search. Vectors at distance larger than \( 3r \) have collision probability at most \( 1/(2n) \) under each of the \( 2^{r+1} - 1 \) functions in \( h \in \mathcal{H}_A(n) \), so in expectation there will be less than \( 2^{r} = 1024 \) hash collisions between \( y \) and vectors in \( S \). The time to answer a query is bounded by the time to compute 2047 hash values for \( y \) and inspect the hash collisions.

It is instructive to compare to the family \( \mathcal{H}_A(n) \) of Indyk and Motwani, described in section 2.2 with the same performance parameters (2047 hash evaluations, collision probability \( 1/(2n) \) at distance 31). A simple computation shows that for \( k = 78 \) samples we get the desired collision probability, and collision probability \((1 - r/128)^{78} \approx 0.0018\) at distance \( r = 10 \). This means that the probability of a false negative by not producing a hash collision for a point at distance \( r \) is \((1 - (1 - r/128)^{78})^{2047} > 0.027\). So the risk of a false negative is nontrivial given the same time and space requirements as our “covering” LSH scheme. \( \triangle \)

4 Generalization

We now generalize the result above to arbitrary values of \( r \), \( cr \), and \( n \):

- For an arbitrary choice of \( cr \) (even much larger than \( \log n \)) we can achieve performance that differs from classical LSH by a factor of \( \ln(4) < 1.4 \) in the exponent.
- We can match the exponent of classical LSH for the \( c \)-approximate near neighbor problem whenever \( \lfloor \log n \rfloor/(cr) \) is (close to) integer.

We still use a Hamming projection family \( \{1\} \), changing only the set \( A \) of bit masks used. Our data structure will depend on parameters \( c \) and \( r \), i.e., these can not be specified as part of a query. Without loss of generality we assume that \( cr \) is integer.

Intuition. When \( cr < \log n \) we need to increase the average number of 1s in the bit masks to reduce collision probabilities. The increase should happen in a correlated fashion in order to maintain the guarantee of collision at distance \( r \). The main idea is to increase the fraction of 1s from \( 1/2 \) to \( 1 - 2^{-t} \), for \( t \in \mathbb{N} \), by essentially repeating the sampling from the Hadamard code \( t \) times and selecting those positions where at least one sample hits a 1.

On the other hand, when \( cr > \log n \) we need to decrease the average number of 1s in the bit masks to increase collision probabilities. This is done using a refinement of the partitioning method of Arasu et al. \( \[3\] \) which distributes the dimensions across partitions in a balanced way. The reason this step does not introduce false negatives is that for each data point \( x \) there will always exist a partition in which the distance between query \( y \) and \( x \) is at most the average across partitions. \( \triangle \)

We use \( b, q \in \mathbb{N} \) to denote, respectively, the number of partitions and the number of partitions to which each dimension belongs. Observe that if we distribute \( q \) copies of \( r \) “mismatching” dimensions across \( b \) partitions, there will always exist a partition with at most \( r' = \lfloor rq/b \rfloor \) mismatches. Let Intervals\((b, q)\) denote the set of intervals in \( \{1, \ldots, b\} \) of length \( q \), where intervals are considered modulo \( b \) (i.e., with wraparound). We will use two random functions,

\[
m: \{1, \ldots, d\} \rightarrow \left(\{0, 1\}^{r' + 1}\right)^t
\]
to define a family of bit vectors \( a(v, k) \in \{0,1\}^d \), indexed by vectors \( v \in \{0,1\}^{tr+1} \) and \( k \in \{1,\ldots,b\} \). We define a family of bit vectors \( a(v, k) \in \{0,1\}^d \) by

\[
a(v, k)_i = p^{-1}(k)_i \wedge \left( \bigvee_j (m(i)_j, v) \right), \tag{4}
\]

where \( p^{-1}(k) \) is the preimage of \( k \) under \( p \) represented as a vector in \( \{0,1\}^d \) (that is, \( p^{-1}(k)_i = 1 \) if and only if \( p(i) = k \)), and \( (m(i)_j, v) \) is the dot product modulo 2 of vectors \( m(i)_j \) and \( v \). We will consider the family of all such vectors with nonzero \( v \):

\[
A(m, p) = \{ a(v, k) \mid v \in \{0,1\}^{tr+1}\{0\}, \ k \in \{1,\ldots,b\} \}.
\tag{5}
\]

Note that the size of \( A(m, p) \) is \( b(2^{tr+1} - 1) < b2^{qr/b} \).

**Lemma 4.** For every choice of \( b, d, q, t \in \mathbb{N} \), and every choice of functions \( m \) and \( p \) as defined above, the Hamming projection family \( \mathcal{H}_A(m, p) \) is \( r \)-covering.

**Proof.** Let \( x \in \{0,1\}^d \) satisfy \( \|x\| \leq r \). We must argue that there exist \( v^* \in \{0,1\}^{tr+1}\{0\} \) and \( k^* \in \{1,\ldots,b\} \) such that \( a(v^*, k^*) \wedge x = 0 \), i.e., by [6]

\[
\forall i : x_i \wedge p^{-1}(k)_i \wedge (\bigvee_j (m(i)_j, v)) = 0.
\]

We let \( k^* = \arg \min \| x \wedge p^{-1}(k) \| \) by the pigeonhole principle, \( \| x \wedge p^{-1}(k) \| \leq \lfloor rq/b \rfloor = r' \). Now consider the “problematic” set \( I(x \wedge p^{-1}(k^*)) \) of positions of 1s in \( x \wedge p^{-1}(k^*) \), and the set of vectors that \( m \) associates with it:

\[
M_x = \{ m(I(x \wedge p^{-1}(k^*)))_j \mid j \in \{1,\ldots,t\} \}.
\]

The span of \( M_x \) has dimension at most \( |M_x| \leq tr' \). This means that there must exist \( v^* \in \{0,1\}^{tr+1}\{0\} \) that is orthogonal to all vectors in \( M_x \). In particular this implies that for every \( i \in I_x, \bigvee_j (m(i)_j, v) = 0 \), as desired. \( \square \)

We are now ready to show the following extension of Theorem [1]

**Theorem 2.** For random \( m \) and \( p \), for every \( b, d, q, t \in \mathbb{N} \) and \( x, y \in \{0,1\}^d \):

1. If \( \|x - y\| \leq r \) then \( \Pr \left[ \exists h \in \mathcal{H}_A(m, p) : h(x) = h(y) \right] = 1. \)

2. \( \mathbb{E} \left[ \|h \in \mathcal{H}_A(m, p) \mid h(x) = h(y) \| \right] \leq (1 - (1 - 2^{-t})q/b)^{\|x - y\|} b2^{qr/b}. \)

**Proof.** By Lemma [1] we have \( h(x) = h(y) \) if and only if \( h(z) = 0 \) where \( z = x \oplus y \). So the first part of the theorem is a consequence of Lemma [3]. For the second part consider a particular vector \( a(v, k) \), where \( v \) is nonzero, and the corresponding hash value \( h(z) = z \wedge a(v, k) \). We argue that over the random choice of \( m \) and \( p \) we have, for each \( i \):

\[
\Pr [a(v, k)_i = 0] = \Pr [p^{-1}(k)_i = 0] + \Pr [p^{-1}(k)_i = 1 \wedge \forall j : (m(i)_j, v) = 0]
= (1 - q/b) + 2^{-t}q/b
= 1 - (1 - 2^{-t})q/b. \tag{6}
\]

The second equality uses independence of the vectors \( \{m(i)_j \mid j = 1,\ldots,t\} \) and \( p(i) \), and that for each \( j \) we have \( \Pr [m(i)_j, v = 0] = 1/2 \). Observe also that \( a(v, k) \) depends only on \( p(i) \) and \( m(i) \). Since function values of \( p \) and \( m \) are independent, so are the values.
\[ \{a(v, k)_i | i \in \{1, \ldots, d\}\}. \] This means that the probability of having \(a(v, k)_i = 0\) for all \(i\) where \(z_i = 1\) is a product of probabilities from (5):

\[
\Pr[h(x) = h(y)] = \prod_{i \in I_x} (1 - (1 - 2^{-\alpha})q/b) = (1 - (1 - 2^{-\alpha})q/b)^{|x-y|}.
\]

The second part of the theorem follows by linearity of expectation, summing over the \(2^{r+1} - 1 < 2^{rtr+1}\) choices of \(v\) and the \(b\) choices of \(k\). □

**Discussion.** The expected time complexity of \(c\)-approximate near neighbor search with radius \(r\) is bounded by the size \(|A|\) of the hash family plus the expected number \(\kappa_A\) of hash collisions between the query \(y\) and vectors in the set of vectors that are not valid answers: \(S_{far} = \{x \in S | ||x-y|| > cr\}\). That is,

\[
\kappa_A = \mathbb{E}[((x, h) \in S_{far} \times H_A | h(x) = h(y))]
\]

where the expectation is over the choice of family \(A\). Choosing parameters \(t, b, q\) in Theorem 2 in order to get a family \(A\) that minimizes \(|A| + \kappa_A\) is nontrivial. Ideally we would like to balance the two costs, but integrality of the parameters means that there are “jumps” in the possible sizes and filtering efficiencies of \(H_A(m, p)\). We give some choices of interest below. △

**Corollary 1.** For every \(c > 1\) there exist explicit, randomized \(r\)-covering Hamming projection families \(H_{A_1}, H_{A_2}\) such that for every \(y \in \{0, 1\}^d\):

1. \(|A_1| \leq 2^r n^{1/c}\) and \(\kappa_{A_1} < 2^r n^{1/c}\).
2. If \(\log(n)/(cr) + \varepsilon \in \mathbb{N}\), for \(\varepsilon > 0\), then \(|A_1| \leq 2^r n^{1/c}\) and \(\kappa_{A_1} < 2^r n^{1/c}\).
3. If \(r > [\log(n)/c]\), \(|A_2| \leq 4r n^4/c\) and \(\kappa_{A_2} < 4r n^4/c\).

Proof. We let \(A_1 = A(m, p)\) with \(b = q = 1\) and \(t = [\log(n)/(cr)]\). Then

\[
|A_1| < b2^{2r}r < 2^{2r+1} \leq 2^{rtr+1}.
\]

Summing over \(x \in S_{far}\) the second part of Theorem 2 yields \(\kappa_{A_1} < n2^{-tr}2^{rtr} \leq 2^{rtr} 2^{rtr+1}\).

For the second bound on \(A_1\) we notice that the factor \(2^r\) is caused by the rounding in the definition of \(t\), which can cause \(2^{rtr}\) to jump by a factor \(2^r\). When \(\log(n)/(crr) + \varepsilon\) is integer we instead get a factor \(2^{rtr}\).

Finally, we let \(A_2 = A(m, p)\) with \(b = r\), \(q = 2[\log(n)/c]\), and \(t = 1\). The size of \(A_2\) is bounded by \(b2^{2r}r < 2^{2r[\log(n)/c]}r = 4r n^4/c\). Again, by Theorem 2 and summing over \(x \in S_{far}\):

\[
\kappa_{A_2} < n(1-qf(2r))2^fr < n \exp(-qc/2)2^fr < r2^fr < 4r n^4/c,
\]

where the second inequality follows from the fact that \(1 - \alpha < \exp(-\alpha)\) when \(\alpha > 0\). □

## 5 Conclusion and open problems

We have seen that, at least in Hamming space, LSH-based similarity search can be implemented to avoid the problem of false negatives at little or no cost in efficiency compared to conventional LSH-based methods. The methods presented are simple enough that they may be practical. An obvious open problem is to completely close the gap, or show that a certain loss of efficiency is necessary. Another interesting question is whether it is possible to get nontrivial results in this direction for other spaces and distance measures e.g., \(\ell_1\), \(\ell_2\), or \(\ell_{\infty}\). Finally, the proposed method uses superlinear space and uses a data independent family of functions. Is it possible to achieve covering guarantees in linear or near-linear space and/or improve performance by using data dependent techniques?
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A  Reduction of nearest neighbor to near neighbor

Two reductions are of interest, depending on the size of $d$. If $d$ is small we can obtain a nearest neighbor data structure by having a data structure for every radius $r$, at a cost of factor $d$ in space and $\log d$ in query time. Alternatively, if $d$ is large we can restrict the set of radii to the $O(\log(n) \log(d))$ radii of the form $\left\lfloor (1 + 1/\log n)^i \right\rfloor < d$. This decreases the approximation factor needed for the near neighbor data structures by a factor $1 + 1/\log n$, which can be done with no asymptotic cost in the data structures we consider.

B  Lower bound on the size of $r$-covering families

We note that the size $|H_{A(m)}| = 2^{r+1} - 1$ is close to the smallest possible for an $r$-covering family with projection vectors of Hamming weight $d/2$. To see this, observe that $\binom{d}{r}$ possible sets of errors need to be covered, and each hash function can cover at most $\binom{d/2}{r}$ such sets. This means that the number of hash functions needed is at least

$$\binom{d}{r} \binom{d/2}{r} > 2^r$$

which is within a factor of 2 from the upper bound. \triangle