Gravitational renormalization of quantum field theory

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Abstract

We propose to include gravity in quantum field theory non-perturbatively by modifying the propagators so that each virtual particle in a Feynman graph move in the space-time determined by the momenta of the other particles in the same graph. By making additional working assumptions, we are able to put this idea at work and obtain a modified Feynman propagator for the massless neutral scalar field which shows a suppression at high momentum strong enough to entail finite results to all loop orders for processes involving at least two virtual particles.

1 Introduction

Pauli, long ago [1], suggested that gravity could act as a regulator for the ultraviolet (UV) divergences that plague quantum field theory (QFT) by providing a natural cut-off at the Planck scale. Later on, classical divergences in the self-mass of point-like particles were indeed shown to be cured by gravity [2], and the general idea has since then resurfaced in the literature many times (see, e.g., Refs. [3, 4, 5, 6, 7]). In spite of that, Pauli’s ambition has never been fulfilled.

As it happens, QFT has been successfully used to describe particle physics in flat [8] (or curved but still fixed [9]) space-time where standard renormalization techniques work very well to produce testable results, notwithstanding the presence of ubiquitous singularities stemming from the very foundations of the theory, that is the causal structure of (free) propagators. We have thus grown accustomed to the idea that the parameters in the Lagrangian of the Standard Model (or generalisations thereof) have no direct physical meaning and infinite contributions may be subtracted to make sense of mathematically diverging integrals. The modern approach to renormalization [10] views the occurrence of such infinities as a measure of our theoretical ignorance of nature and every Lagrangian should, in turn, be considered an effective (low energy) description doomed to fail at some UV energy scale Λ [11]. Moreover, gravitational corrections to the Standard Model amplitudes to a given order in the (inverse of the) Planck mass $m_p$ are negligibly small at experimentally accessible energies [12]. These facts briefly elucidate the main theoretical reason that makes it so difficult to use gravity as a regulator: if it is to provide a natural solution to the problem of UV divergences, gravity must be treated non-perturbatively [7].

Taking a step back to the basics, we should notice that, in the QFT community, gravity is mainly viewed as a spin-2 field which also happens to describe distances and angles (to some extent). As such, the most advanced strategy to deal with it is the background field method for functional integrals [13, 8], according to which one expands the Einstein-Hilbert Lagrangian (or a generalisation thereof) around all the fields’ classical values, including the classical background metric. The latter is reserved the role of defining the causal structure of space-time, whereas the quantum mechanical part yields the graviton

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propagator and matter couplings (of order $m_p^{-2}$). The effect of gravity on matter fields can then be analysed perturbatively by computing the relevant Feynman graphs [14]. A notorious consequence of this approach is that, by simple power counting, pure gravity (all matter fields being switched off) is seen to be non-renormalizable, a “text-book” statement by now [15], which is however debated occasionally. For example, Ref. [7] suggested that perturbative expansions are performed in the wrong variables and that Einstein gravity would appear manifestly renormalizable if one were able to resum logarithmic-like series [5]. In the physically more interesting case with matter, non-perturbative results can be obtained in just a very few cases, one of particular interest being the correction to the self-mass of a scalar particle, which becomes finite once all ladder-like graphs containing gravitons are added [4]. A remarkable approach was developed in Refs. [16], in which a tree-level effective action for gravity at the energy scale $\mu$ is derived within the background field method but without specifying the background metric a priori. The latter is instead a posteriori and self-consistently equated to the quantum expectation value determined by the effective action at that scale. This method does not involve cumbersome loop contributions and hints that gravity might be non-perturbatively renormalizable [17], with a non-Gaussian UV fixed point, thus realising the asymptotic safety conjectured several decades ago by Weinberg [18].

Based on the idea that QFT is an effective approach [11], different attempts have taken a shortcut and addressed the effects of gravity on the propagation of matter field modes directly, e.g., by employing modified dispersion relations or uncertainty principles at very high (usually referred to as trans-Planckian) energy [19]. Some works have postulated such modifications, whereas others have tried to derive them from (effective) descriptions of quantum gravity (see, e.g., Refs. [20]). It is in fact common wisdom that, for energies of the order of $m_p$ or larger, the machinery of QFT fails and one will need a more fundamental quantum theory of gravity, such as String Theory [21] or Loop Quantum Gravity [22]. Quite interestingly, both hint at space-time non-commutativity [23] as an effective implementation of gravity as a regulator, with the scale of non-commutativity of the order of the Planck length $\ell_p$. A new feature which, in turn, follows from space-time non-commutativity is the IR/UV mixing, whereby physics in the infrared (IR) is affected by UV quantities [24]. This feature gives us hope of probing (indirectly) such an extreme energy realm in future experiments or even using available data of very large scale (cosmological) structures.

In the present work, we shall propose a yet different strategy to incorporate gravity along the above line of reasoning. Instead of proposing a new, or relying on an available, fundamental theory of quantum gravity, we shall try to define modified propagators in a very “conservative” (in a sense, minimal) way inspired by the simple semiclassical perspective in which gravity is described by Einstein’s geometrical theory and matter by perturbative QFT. In this approach, gravity is therefore not viewed as a spin-2 field (although with very complicated interactions with itself and with matter fields), but rather as the causal structure of space-time (or the manifestation thereof), a property the background field method reserves to the classical part of the metric only. The modified propagators for matter fields should therefore take into consideration the presence of each and every source, classical or virtual, in a given process mathematically described by Feynman’s diagrams. Of course, philosophical perspectives aside, the relevant question is whether this idea leads to different (or the same) phenomenological predictions with respect to the other approaches to UV physics currently available, but we are in a fairly premature stage to assess that.

The rest of this paper is conceptually divided into two parts: the general proposal is described in the next Section, where we briefly review the idea of semiclassical gravity, the interplay between propagators, the causal structure of space-time and UV divergences, and then list four prescriptions which should serve as guidelines in order to modify the propagators accordingly; in Section 3, we shall try to apply the proposal and obtain a modified scalar field propagator to estimate the UV behaviour of the four-point function to one-loop. Let us remark that the second part of the paper is based on several more working assumptions, in addition to the general guidelines, and the results about transition amplitudes are therefore a consequence of both the general idea and some simplifications which might indeed appear more questionable.

We shall use units with $c = \hbar = 1$ and the Newton constant $G = \ell_p/m_p$. 


2 Geometrical gravity in QFT

In order to make contact with the physics from the very start, let us note that one needs to consider two basic energy scales, one related to phenomenology and one of theoretical origin, namely:

a) the highest energy presently available in experiments, say $E_{\text{exp}} \sim 1$ TeV, and

b) the Planck energy $m_p \sim 10^{16}$ TeV.

It is well assessed that, for energies up to $E_{\text{exp}}$, the Standard Model (without gravity) and renormalization techniques yield results in very good agreement with the data. Further, finite, albeit experimentally negligible, quantum gravitational corrections can be obtained by employing the effective QFT approach [12] (which also yields some – but not all – of the general relativistic corrections to the Newtonian potential).

At the opposite end, for energies of the order of $m_p$ or larger, one presumably needs a new quantum theory which includes gravity in a fundamental manner, like String Theory [21] or Loop Quantum Gravity [22].

We hence expect that gravitational corrections to QFT amplitudes play an increasingly important role for larger and larger energy scale $\mu > E_{\text{exp}}$, and that it should be possible to describe such effects in perturbative QFT directly (at least in the regime $E_{\text{exp}} \lesssim \mu \lesssim m_p$). We shall call this window the realm of “semiclassical gravity”, and that is the range where our proposal is more likely to shed some new light, although we shall also attempt at pushing our predictions even further.

2.1 Semiclassical gravity

As we just mentioned, at intermediate energies $E_{\text{exp}} \lesssim \mu \ll m_p$, we expect that a semiclassical picture holds in which the space-time can be reliably described as a classical manifold with a metric tensor $g_{\alpha\beta}$ that responds to the presence of (quantum) matter sources according to [9]

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{\ell_p}{m_p} \langle \hat{T}_{\alpha\beta} \rangle,$$

where $R_{\alpha\beta}$ ($R$) is the Ricci tensor (scalar) and $\langle \hat{T}_{\alpha\beta} \rangle$ the expectation value of the matter stress tensor obtained from QFT on that background. All the same, if one takes Eq. (2.1) at face value, the way perturbative terms are computed in QFT appears questionable, since loops of virtual particles are included in Feynman’s diagrams whose four-momentum $|k^2| = |k_\alpha k^\alpha|$ formally goes all the way to infinity (i.e., to $m_p$ and beyond) but are still described by the (free) propagators computed on a fixed (possibly flat) background.

For example, let us consider the graph in Fig. 1 for scalar particles with self-interaction $\lambda \phi^4$, which is a pictorial representation of the integral

$$\Gamma^{(4)}(p) \simeq \int \frac{k^3 \, dk}{(2\pi)^4} \hat{G}_F(k) \hat{G}_F(p - k),$$

where $\hat{G}_F$ is the momentum-space Feynman propagator in four dimensions,

$$\hat{G}_F(p) = \frac{1}{p^2 + i \epsilon}.$$ 

Although the external momenta $p_i$ ($i = 1, \ldots, 4$) are taken within the range of experiments (that is, $|p_i^2| \lesssim E_{\text{exp}}^2$ in the laboratory frame), the two virtual particles in the loop have unconstrained momenta $k$ and $p_1 + p_2 - k$ respectively. One might therefore wonder if it is at all consistent to describe those two particles using the above flat-space propagator. The common QFT approach to this problem would result in adding gravity in the form of graviton exchanges (see Fig. 2) and estimate deviations from purely flat-space results. This procedure is however likely to miss non-perturbative contributions that the UV physics might induce into the IR. For sure, it will not render finite diverging integrals, such as the one in Eq. (2.2), unless one is able to resum an infinite number of perturbative terms.
The interplay among propagators, UV divergences and the causal structure of space-time can be better understood by noting that, in any approach in which the space-time structure is given by a fixed background, the short distance behaviour of QFT (in four dimensions) is described by the Hadamard form of the propagators \[ G(x, x') = U(x, x') \frac{1}{\sigma} + V(x, x') \ln(\sigma) + W(x, x') , \] where \( U, V \) and \( W \) are regular functions and \( 2\sigma \) is the square of the geodesic distance between \( x \) and \( x' \). In Minkowski space-time, \[ 2\sigma = (x - x')^2 , \] and the expression of the propagator contains divergences for \( \sigma \to 0 \) (i.e., along the light cone and for \( x \to x' \)). Calculations based on the use of propagators in QFT therefore (implicitly) rely on the formalism of distribution theory and UV divergences appear as a consequence when one tries to compute (mathematically) ill-defined quantities such as the four-point function in Eq. (2.2). One can devise mathematical workarounds for this problem, but what matters here is that, if only the relation (2.5) is modified (like in QFT on a curved space-time), the divergences for \( \sigma \to 0 \) will remain. Nonetheless, a few partial results suggest that deeper modifications of the causal structure might occur at the quantum level. For example, it was shown that the divergence on the light-cone disappears (with a smearing at large momenta of the form considered in Ref. [25]) if graviton fluctuations are in a coherent state [6] and, with the further inclusion of negative norm states, all UV divergences should be cured [26].

It seems sensible to us to tackle this problem by pushing further the validity of the semiclassical Einstein equations. We shall hence assume that virtual particles propagate in a background compatible with Eq. (2.1) at the scale \( \mu \sim k = \sqrt{|k^2|} \) and their propagators be correspondingly adjusted [3]. As we mentioned in the Introduction, our underlying viewpoint here is that gravity is not just another field (although with a very complicated dynamics), but the geometrical view according to which gravity is the
space-time and, in particular, the causal structure obeyed by all (other) fields. Let us remark again that this perspective is partly incorporated into the background field method, whereby the metric is split into two parts,

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (2.6) \]

The classical part \( \eta_{\mu\nu} \) possesses the expected symmetries of General Relativity and determines the causal structure for all (other) classical and quantum fields, whereas \( h_{\mu\nu} \) is just another quantum field which acts on the matter fields via usual (although complicated) interaction terms, hence in a non-geometrical way. In comparison, one could actually view our approach as a step backward, since the gravitational field is not explicitly quantised (there is no analogue of the above \( h_{\mu\nu} \)), and it is in fact not even defined separately (i.e., in the absence of matter).

2.2 Gravity in propagators

Inspired by the previous considerations, we formulate the following basic prescriptions for defining a “gravitationally renormalised” QFT:

- **A1)** perturbative QFT defined by Feynman diagrams is a viable approach to particle physics for energies \( \mu \) below a cut-off \( \Lambda \gg E_{\exp} \);
- **A2)** in a (one-particle irreducible) Feynman diagram with \( N \) internal lines, each virtual particle is described by a Feynman propagator \( G^{(A)}(x, y) \) corresponding to the space-time generated by the other \( N - 1 \) virtual particles in the same graph with momenta \( k_i \) \( (i = 1, \ldots, N - 1) \) and constrained according to **A1**;
- **A3)** Standard Model results are recovered at low energy, \( \mu \lesssim E_{\exp} \ll m_p \).

Several comments on the above guidelines are in order. First of all, we explicitly introduced a cut-off in **A1**, having in mind that our approach is not meant to be the final theory of everything, but should rather be regarded as a computational recipe. A second, essential, simplification was introduced in **A2**, in that each virtual particle is treated like a test particle in the space-time generated by the other particles, its own gravitational backreaction thus being neglected. Another consequence of **A2** is that integration over momenta inside loops can now be viewed as also purporting a (quantum mechanical) superposition of (virtual) metrics, and there is hope that this can smear the usual divergences of (2.3) out (as was shown in Ref. [6] for particular gravitational states).

It should not go unnoticed that we did not mention a Lagrangian (or action) from which the modified propagators satisfying **A2** could be obtained. In this respect, our proposal follows the philosophy of Ref. [14], which gives the Lagrangian a secondary role with respect to Feynman’s rules for computing perturbative amplitudes. It is however true that the symmetries of a system are far more transparent if a Lagrangian is available and it would be interesting to find out whether an action principle can be devised to streamline the derivation and show which symmetries are preserved or broken. The latter kind of analysis can also be performed perturbatively, although, as is well known for the Slavnov-Taylor identities of (non-Abelian) Yang-Mills theory, that task requires a lot more effort.

A final observation is that the Standard Model of particle physics (without gravity) is a rigid theory and it is very likely that a generic modification of the sort we are proposing here has hazardous effects in the range of presently available data, thus compromising **A3**. One should therefore check very careful that none of the assessed predictions of the Standard Model is lost in our approach.

At this point, we cannot proceed ignoring the technical fact that the \( N \)-body problem in General Relativity is extremely complicated, to say the least, already for \( N = 2 \). We therefore make the following “mean field” assumption to deal with graphs containing more than two virtual particles:

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1This is reminiscent of the “relational mechanics” approach to gravity (see, e.g., Ref. [27] and References therein).

2Let us note in passing that this somewhat parallels a perturbative result of non-commutative QFT, according to which there is no tree-level correction to the commutative case.

3Although no such graph will be considered here.
one can approximate the propagator for each virtual particle $G_{(k_i)}^{(\Lambda)}(x, y) \simeq G_q^{(\Lambda)}(x, y)$, where $q \simeq \sqrt{|\sum k_i|^2}$ is the total momentum of the remaining $N-1$ particles.

Since this is intended to be a (necessary) working assumption, the above approximate equalities can be replaced with other expressions of choice, the key point being that the problem is now reduced to study the propagator for a test particle in a background generated by an “average” source. Again, it will be crucial that $A3$ remains valid in order to have a physically sensible construction.

3 Scalar QFT

We shall now apply our prescriptions $A1$–$A4$ to the simple case of a neutral massless scalar field $\phi$ in four dimensions with $\lambda \phi^4$ self-interaction. Although $A3$ (that is, the Standard Model physics) is lost from the outset, this model is quite adequate for studying the UV (or short distance) behaviour of QFT, which is our main concern here.

To begin with, we need a metric that describes the space-time around virtual particles from which a propagator satisfying $A2$ can be obtained. Although $A4$ implies that we can reduce the problem to the relatively simple case of one source, this is not yet enough to single out the metric to use, and we shall need to make more working assumptions hereafter.

3.1 Point-particle metrics

Since we are considering neutral scalar particles, the first option for a point-like source of mass $m$ that comes to mind is, of course, the well-known Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2\bar{m}}{r}\right) dt^2 + \left(1 - \frac{2\bar{m}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2$ is the line element of the unit two-sphere. We also denoted with $\bar{m} = \ell_p m/m_p$ the particle mass in geometric units, which equals the ADM mass of the system, that is the asymptotic limit for $r \to \infty$ of the (in this case, constant) mass function $\bar{m}(r)$. For $m > 0$, the space-time with metric (3.1) represents a black hole. However, the Compton length associated with a massive particle is $m^{-1} \gg \bar{m}$ for $m \ll m_p$, and one might argue whether the horizon really survives quantum mechanical corrections for $m$ comparable to the mass of known elementary particles$.^4$ Moreover, and what is more important for our purpose of computing Feynman diagrams, the causal structure of the Schwarzschild space-time would make the analytic expression for the scalar field propagator overly complicated at short distances (that is, right around $r \sim \bar{m}$).

Luckily, another solution of the Einstein equations generated by a point-like (spherically symmetric and neutral) particle was found in Ref. [2], which is significantly simpler to deal with, namely

$$g_{\mu\nu} = \Omega_m^2 \eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric,

$$\Omega_m(\vec{x}) = \left(1 + \frac{\bar{m}}{r}\right)^2,$$

and $r = |\vec{x}|$ is again the radial coordinate centred at the Dirac $\delta(r)$ source of bare mass $m$. This metric, being regular everywhere for $r > 0$ and having no horizon, is a naked singularity even in the absence of electric charge. It was obtained to show that General Relativity renders the classical energy of point-like sources finite. In fact, its ADM mass is exactly equal to zero$.^5$ which might raise some questions regarding the IR behaviour of the propagator obtained on this background. Let us however note that the standard

$^4$For example, space-time non-commutativity could change the short distance metric into a de Sitter-like background [28].

$^5$Charged sources would have finite ADM mass instead.
propagator (2.3) in Minkowski space-time also corresponds to assigning a vanishing ADM mass to the virtual particles acting as sources, and this suggests that Eq. (3.2) is indeed better suited for analysing UV corrections to standard QFT amplitudes than the metric (3.1). Moreover, compared to the more famous Schwarzschild solution, the simpler causal structure (at short distance, $r \sim \bar{m}$) and, in particular, the conformal flatness of the metric (3.2) will substantially ease the derivation of the propagator.

### 3.2 Modified propagator

Having chosen the metric in Eq. (3.2), the next step is then to relate the bare mass $\bar{m}$ with the momentum of the virtual particles regarded as background sources. According to A4, we shall assume

$$m \simeq \sqrt{|q^2|} \equiv q,$$

where $q$ is the either the total or the average momentum of $N - 1$ virtual particles in a graph with $N$ such particles.

We can now obtain the general form of the propagator starting from the Klein-Gordon equation for the metric (3.2),

$$\Box \phi = \Omega_q^{-3} \Box_M (\Omega_q \phi) = 0,$$

where $\Box_M$ is the D’Alembertian in Minkowski space. The modified Feynman propagator in coordinate space (with $x = (t, \vec{x})$, etc) is thus

$$G^{(A)}_q(x, y) = \frac{1}{\Omega_q(\vec{x}) \Omega_q(\vec{y})} G_F(x - y),$$

where $G_F(x - y)$ is the standard Feynman propagator in Minkowski space-time and the factors of $\Omega_q^{-1}$ are expected to suppress the propagation of scalar modes at short distance, i.e., for $|\vec{x}|, |\vec{y}| \lesssim \ell_p q/m_p$.

In order to see whether this improved behaviour is sufficient to cure UV divergences, we compute the propagator in momentum space by taking the Fourier transform of (3.6), with the cut-off $k < \Lambda$ according to A1,

$$\tilde{G}^{(A)}_q(p; p') = \frac{1}{\Omega_q(\vec{p} - \vec{k}) \Omega_q(\vec{k} - \vec{p}')} G_F(k) \tilde{\Omega}_q(\vec{k}),$$

where $\tilde{G}_F(k)$ is the standard Feynman propagator in momentum space and

$$\tilde{\Omega}_q(\vec{k}) = \frac{1}{(2\pi)^3} \int \frac{e^{-i\vec{k} \cdot \vec{x}}}{\Omega_q(\vec{x})} \, d^3x.$$

One can study this distribution as usual by integrating inside the box $-\bar{L} < \vec{x} < \bar{L}$, and then taking $\bar{L} \to \infty$. By rotating the reference frame so that $k = (k_x, 0, 0)$, we find

$$\tilde{\Omega}_q(\vec{k}) = \delta(k_y) \delta(k_z) \lim_{L_x \to \infty} \rho^{(L_s)}_q(k_x),$$

where $\delta(w)$ is the Dirac $\delta$-function and

$$\rho^{(L)}_q(w) = \frac{1}{2\pi} \int_{-\bar{L}}^{\bar{L}} \frac{x^2 e^{-iwx} \, dx}{(x^2 + \ell_p q/m_p)^2}.$$

Its explicit expression is rather cumbersome,

$$\rho^{(L)}_q(w) = 2\tilde{q} + e^{-i\tilde{q} \bar{q}} \tilde{q} (2 - i k \bar{q}) [Ei(i k \bar{q}) - Ei(i k (L + \bar{q}))] + i e^{-iL \bar{q} / k(L + \bar{q})} + c.c.,$$

where $Ei(z)$ is the exponential integral.
Figure 3: Distribution $\rho_q^{(L)}(w)$ for $q = m_p$ (thick line) and $q = 0$ (dashed line) for $L = 100\ell_p$.

where c.c. stands for complex conjugate,

$$Ei(z) = \int_{-\infty}^{z} t^{-1} e^t \, dt , \quad (3.12)$$

and $\bar{q} = (\ell_p/m_p) q$. Upon closer inspection, one realises that $\rho_q^{(L)}(w)$ is real and even in $w$, and actually resembles the usual approximation of $\delta(w)$ (see Fig. 3), with $\rho_q^{(L)}(0) \approx L$ and the normalisation

$$\lim_{L \to \infty} \int_{-\Lambda}^{+\Lambda} \rho_q^{(L)}(w) \, dw = \rho_\Lambda(q) , \quad (3.13)$$

where $\Lambda$ is again the cut-off introduced in A1. Finally, we obtain

$$\tilde{\Omega}_q(\tilde{k}) = \rho_\Lambda(q) \delta(\tilde{k}) , \quad (3.14)$$

and the relevant propagator is therefore given by

$$\tilde{G}_q^{(\Lambda)}(p) = \rho_\Lambda^2(q) \tilde{G}_F(p) , \quad (3.15)$$

in which the weight $\rho_\Lambda^2$ describes, in momentum space, the previously mentioned suppression at short distance. Remarkably, an explicit dependence on the UV cut-off $\Lambda$ and $m_p$ emerged as manifestation of non-trivial IR/UV mixing at all scales $p \sim \mu > 0$.

It is interesting to note that the limit

$$\lim_{\Lambda \to \infty} \rho_\Lambda^2(q) = \begin{cases} 1 & \text{for } q = 0 \\ 0 & \text{for } q > 0 \end{cases} , \quad (3.16)$$

which is therefore not uniform and must be taken carefully at the end of the computation only. Further, since one has $\rho_\Lambda(0) = 1$ for all values of $\Lambda$, the standard propagator $\tilde{G}_F(p)$ (with no dependence on $\Lambda$ and $m_p$) is recovered for $q/m_p \to 0$ followed by $\Lambda \to \infty$. One can therefore approximate

$$\tilde{G}_q^{(\Lambda \gg p)}(p) \simeq \tilde{G}_F(p) , \quad (3.17)$$

if need be.
Figure 4: Weight $|\rho_{\Lambda}^{(L)}(q)|^2$ with $L = 10^6 \ell_p$ for $\Lambda = 10 m_p$ (thin solid line), $\Lambda = m_p$ (thick solid line) and its approximation (3.19) (dotted line) with $\alpha = 0.55$.

In order to study the UV behaviour of transition amplitudes, we shall need analytically manageable approximations of the propagator (3.15) for $q \lesssim \Lambda$. For $(m_p^2/\Lambda) < q < \Lambda$, we numerically find the bound

$$
\rho_{\Lambda > m_p}^2(q) < \left( \frac{m_p^2}{\Lambda q} \right)^{\beta},
$$

with $\beta \simeq 5.8$, which can be used to estimate quantities in the limit $\Lambda \to \infty$. It is also tempting to relate $\Lambda$ to $m_p$ explicitly [5]. In this case, we numerically checked that the weight can be approximated by

$$
\rho_{\Lambda \approx m_p}^2(q \lesssim m_p) \simeq 1 - \tanh \left[ 2 \left( \frac{\Lambda q}{m_p^2} \right)^{\alpha} \right],
$$

with $\alpha \simeq 0.55$ (see Fig. 4). Note that neither (3.18) nor (3.19) is accurate for $q \to 0$ and we are thus not providing useful approximations to study the IR behaviour.

### 3.3 Scattering amplitudes

From Eq. (3.18), one already suspects that the propagator (3.15) yields finite amplitudes for all the irreducible diagrams involving at least two virtual particles. Let us see this in detail for the one-loop correction to the vertex $\lambda \phi^4$ of Fig. 1.

The standard asymptotic behaviour for the total momentum of the incoming scalars $p \ll \Lambda$ is

$$
\Gamma^{(4)}(p) \simeq \int \frac{\Lambda^3 dk}{(2\pi)^4} \tilde{G}_F(k) \tilde{G}_F(p-k) \simeq C \ln \left( \frac{\Lambda}{\mu} \right),
$$

with $C$ a constant of order one. This amplitude contains the formal (that is, bare) coupling constant $\lambda$, which can be replaced with the physical coupling constant given by the transition amplitude measured at the scale $\mu$,

$$
\lambda_{\mu} \simeq \lambda + \lambda^2 \Gamma^{(4)}(\mu).
$$

Upon solving for $\lambda$ in terms of $\lambda_{\mu}$, the scattering amplitude for $p \sim \mu$ becomes a function of the two physically meaningful quantities $\lambda_{\mu}$ and $\mu$,

$$
\mathcal{M} \simeq \lambda_{\mu} - \lambda_{\mu}^2 C \ln \left( \frac{p}{\mu} \right).
$$
This expression is independent of $\Lambda$, so that the low energy physics ($p \sim \mu \ll \Lambda$) depends on the (otherwise unknown) high energy theory ($k \gtrsim \Lambda$) only through the “renormalized” value of $\lambda_\mu$. The UV cut-off may then be removed safely (formally, taking $\Lambda \to \infty$ at the end of the computation does not affect the result).

The “gravitationally renormalized” amplitude is obtained by replacing each particle’s propagator in Eq. (3.20) with the expression (3.15) and $q$ equal to the momentum of the other virtual particle,

$$\Gamma^{(4)}_{\text{GR}}(p) \simeq \frac{k^3 \, dk}{(2\pi)^4} \tilde{G}^{(\Lambda)}(p-k) \tilde{G}^{(\Lambda)}(k) \tilde{G}^{(\Lambda)}(p-k).$$

(3.23)

The result now depends on $\Lambda$ and we shall consider two cases by making use of the approximate expressions for $\rho_\Lambda^2$ found previously.

### 3.3.1 Infinite cut-off

It is easy to see that the asymptotic behaviour (for $p \ll \Lambda \to \infty$) is now given by

$$\Gamma^{(4)}_{\text{GR}}(p) \lesssim \frac{m_p^3}{\Lambda^{2\beta}} \int_0^\Lambda \frac{dk}{k^{1+2\beta}} \sim \left( \frac{m_p}{\Lambda} \right)^{4\beta},$$

(3.24)

and the integral [3.23] therefore remains finite for $\Lambda \to \infty$. The same occurs in all higher order cases and the only irreducible graph left (potentially diverging is the tadpole, since it only contains one virtual particle propagated by $\tilde{G}^{(\Lambda)}_{(q=0)}(k) = \tilde{G}_F(k)$.

### 3.3.2 Planck scale cut-off

If we instead identify $\Lambda \simeq m_p$ [5], we obtain

$$\Gamma^{(4)}_{\text{GR}}(p) \simeq C \ln \left( \frac{m_p}{p} \right),$$

(3.25)

which shows an explicit dependence on $m_p$ as anticipated. Incidentally, this is exactly the same asymptotic behaviour one finds from the standard expression (3.20) by simply setting $\Lambda = m_p$, which shows that deviations from purely Standard Model UV results are only obtained by pushing the cut-off above the Planck scale.

Let us repeat that the displayed results only pertain to the high energy regime and, to complete the analysis, one should also consider the IR behaviour more carefully.

## 4 Final remarks

Inspired by the observation that a semiclassical description of gravity should be possible in processes that involve energies below the Planck scale, we formulated general properties that a modified QFT should enjoy in order to include gravitational contributions. Such properties were listed in the form of prescriptions that formalise our approach to include gravity within the Standard Model of particle physics in a non-perturbative way. As such, they are of course debatable and subject to possible refinements.

In order to have a first look at what predictions such guidelines imply, we then derived the Feynman propagator (3.15) for a neutral scalar field. However, to carry on the computation analytically required several more working assumptions, starting from the choice of the background metric (3.2). The resulting propagator explicitly depends on the energy (length) scale $m_p$ ($\ell_p$) and cut-off $\Lambda$, which entails a IR/UV mixing, with the high energy scale $\Lambda$ (possibly proportional to $m_p$) that appears explicitly in the low energy scattering amplitudes. From the phenomenological point of view, our approach can therefore be regarded as an attempt to predict the effects of the existence of a fundamental length in QFT.\(^6\)

\(^6\)In the Standard Model, this would only occur for the gluon self-mass [29].

\(^7\)The physical value of the cut-off $\Lambda$ could then be estimated by (high precision) measurements such as the electron or muon $g - 2$.\(^7\)
Results such as (3.19) and (3.25) are consequently illustrative of the magnitude of the UV gravitational corrections one expects in four space-time dimensions, where $m_p \gg E_{\text{exp}}$ and we know a priori that it all must boil down to very small figures.\footnote{The situation might be remarkably different in models with extra-spatial dimensions and $m_p \approx 1 \text{ TeV}$ [30].}

As for the long-standing problem of the UV behaviour of QFT, we need to push our semiclassical scheme by letting $\Lambda \gg m_p$ (like in Ref. [31]) in order to tackle it. Our conclusion using (3.15) is that the dependence on the UV cut-off is much improved over that of the standard QFT propagators and finite results without the need of removing divergences are expected in all cases but the few involving just one virtual particle (like the tadpole diagram for a scalar field). We cannot, however, exclude that the asymptotic behaviour might change again by considering more refined descriptions. For instance, one should likely relax sphericity and conformal flatness of the metric (3.2), since these hardly suit systems of particles with large relative momenta. And, of course, more realistic QFT should be analysed before the final word can be spoken on that old idea of Pauli.

References

[1] See, e.g., in W. Pauli, Letter of Heisenberg to Peierls (1930), in Scientific Correspondence, editor K. von Meyenn (Springer-Verlag, 1985), p. 15, Vol II; O. Klein, Helv. Phys. Acta. Suppl. 4, 58 (1956).

[2] R. Arnowitt, S. Deser and C.W. Misner, Phys. Rev. Lett. 4 (1960) 375.

[3] S. Deser, Rev. Mod Phys. 29, 417 (1957).

[4] B.S. DeWitt, Phys. Rev. Lett. 13, 114 (1964).

[5] C.J. Isham, A. Salam and J. Strathdee, Phys. Rev. D 3 (1971) 1805.

[6] L.H. Ford, “Quantum field theory in curved spacetime,” arXiv:gr-qc/9707062.

[7] R.P. Woodard, “Particles as bound states in their own potentials,” arXiv:gr-qc/9803096.

[8] M.E. Peskin and D.W. Schroeder, An introduction to quantum field theory, Perseus Books, Reading (1995).

[9] N.D. Birrell and P.C.W. Davies, Quantum fields in curved space, Cambridge University Press, Cambridge (1982).

[10] K.G. Wilson and J.B. Kogut, Phys. Rept. 12, 75 (1974).

[11] S. Weinberg, “What is quantum field theory, and what did we think it was?,” arXiv:hep-th/9702027.

[12] J.F. Donoghue, Phys. Rev. Lett. 72 (1994) 2996.

[13] B.S. DeWitt, Phys. Rev. 162, 1195 (1967).

[14] M.J.G. Veltman, Diagrammatica: The Path to Feynman rules, Cambridge University Press, Cambridge (1994).

[15] A. Shomer, “A pedagogical explanation for the non-renormalizability of gravity,” arXiv:0709.3555 [hep-th].

[16] M. Reuter, Phys. Rev. D 57, 971 (1998); W. Souma, Prog. Theor. Phys. 102, 181 (1999).

[17] M. Reuter and H. Weyer, “Background Independence and Asymptotic Safety in Conformally Reduced Gravity,” arXiv:0801.3287 [hep-th].
[18] S. Weinberg, “Ultraviolet divergences in quantum theories of gravitation,” in General relativity: an Einstein centenary survey, edited by S. Hawking and W. Israel, Cambridge University Press (Cambridge, 1979).

[19] W.G. Unruh, Phys. Rev. D 51, 2827 (1995); S. Corley and T. Jacobson, Phys. Rev. D 54, 1568 (1996).

[20] M. Maggiore, Phys. Lett. B 304, 65 (1993); F. Scardigli, Phys. Lett. B 452, 39 (1999); S. Capozziello, G. Lambiase and G. Scarpetta, Int. J. Theor. Phys. 39, 15 (2000); G.L. Alberghi, R. Casadio and A. Tronconi, Phys. Rev. D 74, 103501 (2006); G. Amelino-Camelia, “Quantum Gravity Phenomenology,” arXiv:0806.0339 [gr-qc].

[21] K. Becker, M. Becker and J.H. Schwarz, String theory and M-theory: a modern introduction, Cambridge University Press, Cambridge (2007).

[22] C. Rovelli, Living Rev. Rel. 1, 1 (1998).

[23] R.J. Szabo, Phys. Rept. 378, 207 (2003).

[24] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 0002, 020 (2000); A. Matusis, L. Susskind and N. Toumbas, JHEP 0012, 002 (2000).

[25] A. Smailagic and E. Spallucci, J. Phys. A 36, L517 (2003); J. Phys. A 37, 7169 (2004).

[26] S. Rouhani and M.V. Takook, “A naturally renormalized quantum field theory,” arXiv:gr-qc/0607027.

[27] E. Anderson, “Triangleland. I. Classical dynamics with exchange of relative angular momentum,” arXiv:0809.1168 [gr-qc].

[28] P. Nicolini, “Noncommutative Black Holes, The Final Appeal To Quantum Gravity: A Review,” arXiv:0807.1939 [hep-th].

[29] V. Gogokhia, “The tadpole term and the role of ghosts in QCD,” arXiv:0806.0247 [hep-th].

[30] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 429, 263 (1998); Phys. Rev. D 59, 086004 (1999); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 436, 257 (1998); L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999); Phys. Rev. Lett. 83, 3370 (1999).

[31] S.P. Robinson and F. Wilczek, Phys. Rev. Lett. 96, 231601 (2006).