Power Generalized DUS Transformation of Exponential Distribution

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Abstract

DUS transformation of lifetime distributions received attention by engineers and researchers in recent years. The present study introduces a new class of distribution using exponentiation of DUS transformation. A new distribution using the Exponential distribution as the baseline distribution in this transformation is proposed. The statistical properties of the proposed distribution have been examined and the parameter estimation is done using the method of maximum likelihood. The fitness of the proposed model is established using real data analysis.

Keywords: DUS transformation, Exponential Distribution, Failure Rate, Moments, Maximum likelihood estimator

1. Introduction

Modeling and analysis of lifetime distribution have been extensively used in many fields of science like engineering and statistics. Fitting of appropriate distribution is essential for a proper data analysis. Different methods are available that propose new classes of distributions using existing distributions, see Gupta et al.[5], Nadarajah and Kotz[11], Cordeiro and Castro[1], Cordeiro et al.[2], Kumar et al.[8], etc. Kumar et al.[7] proposed a method called DUS transformation to obtain a new parsimonious class of distribution. Recently,
Maurya et al.\cite{10} proposed a generalization to DUS transformation to make it more flexible. Deepthi and Chacko\cite{3} introduced DUS Lomax distribution. Gauthami and Chacko\cite{4} introduced DUS Inverse Weibul distribution. But the existing approach is not appropriate for some data. A search for distributions with better fit is quite essential for data analysis in statistics and reliability engineering.

The current research work aims to introduce a new class of distribution using the exponentiation of DUS transformation, called Power generalized DUS (PGDUS) tranformation. The new PGDUS transformed distribution can be obtained as follows: Let $X$ be a random variable with baseline cumulative distribution function (cdf) $F(x)$ and corresponding probability density function (pdf) $f(x)$. Then the cdf of the proposed PGDUS distribution is defined as,

$$G(x) = \left( \frac{e^{F(x)} - 1}{e - 1} \right)^{\theta}, \theta > 0, x > 0.$$  \hspace{1cm} (1)

and the corresponding pdf is,

$$g(x) = \frac{\theta}{(e - 1)^{\theta}} (e^{F(x)} - 1)^{\theta - 1} e^{F(x)} f(x), \theta > 0, x > 0.$$  \hspace{1cm} (2)

The associated survival function is,

$$S(x) = 1 - \left( \frac{e^{F(x)} - 1}{e - 1} \right)^{\theta}, \theta > 0, x > 0.$$

The failure rate function is,

$$h(x) = \frac{\theta f(x) e^{F(x)} (e^{F(x)} - 1)^{\theta - 1}}{(e - 1)^{\theta} - (e^{F(x)} - 1)^{\theta}}, \theta > 0, x > 0.$$

Application of the new transformation to the existing distributions has to be investigated. Using Exponential distribution as baseline distribution, Power Generalized DUS Exponential (PGDUSE) distribution is proposed, in this paper. It has to be studied in detail.

The rest of the paper is organized as follows. In Section 2, the PGDUSE is proposed. Sections 3 discussed the statistical properties of the proposed distribution. In Section 4, the maximum likelihood estimation procedure is applied for estimation of parameters. Real data set is analyzed in Section 5. Concluding remarks are given in Section 6.
2. Power Generalized DUS Exponential Distribution

Here, Power Generalized DUS transformation to the baseline distribution, Exponential distribution, is considered. Consider the Exponential distribution with parameter $\lambda$ as the baseline distribution. Invoking the PGDUS transformation given in equation (1), the cdf of the PGDUSE distribution is obtained as

$$G(x) = \left( \frac{e^{1-e^{-\lambda x}} - 1}{e - 1} \right)^\theta, \lambda > 0, \theta > 0, x > 0. \quad (3)$$

and the corresponding pdf is given by,

$$g(x) = \theta \lambda e^{1-\lambda x - e^{-\lambda x}} \frac{(e^{1-e^{-\lambda x}} - 1)^{\theta-1}}{(e - 1)^{\theta}}, \lambda > 0, \theta > 0, x > 0. \quad (4)$$

Then, the associated failure rate function is,

$$h(x) = \frac{\theta \lambda e^{1-\lambda x - e^{-\lambda x}} (e^{1-e^{-\lambda x}} - 1)^{\theta-1}}{(e - 1)^{\theta} - (e^{1-e^{-\lambda x}} - 1)^{\theta}}, \lambda > 0, \theta > 0, x > 0. \quad (5)$$

We denote $PGDUSE(\lambda, \theta)$ for PGDUSE distribution with parameters $\lambda$ and $\theta$. Figure 1 shows that the density function of PGDUSE distribution is likely to be unimodal.

3. Statistical Properties

For a distribution, the statistical properties are inevitable. In this section, a few statistical properties like moments, moment generating function, characteristic function, cumulant generating function, quantile function, order statistics, and entropy of the proposed PGDUSE distribution are derived.
Figure 1: Density plot
3.1. Moments

The rth raw moment of the $PGDUS - E(\lambda, \theta)$ distribution is given by

$$\mu'_r = E(X^r)$$

$$= \int_0^\infty x^r \frac{\theta \lambda}{(e - 1)^\theta} e^{1-\lambda x} e^{-\lambda x} (e^{1-e^{-\lambda x}} - 1) dx$$

$$= \frac{\theta \lambda}{(e - 1)^\theta} \int_0^\infty x^r e^{2-\lambda x} - 2e^{-\lambda x} dx - \frac{\theta \lambda}{(e - 1)^\theta} \int_0^\infty x^r e^{1-\lambda x} - e^{-\lambda x} dx$$

$$= \frac{\theta \lambda e^2}{(e - 1)^\theta} \sum_{m=0}^\infty \frac{(-1)^m}{m!(1+m)^{r+1}} \lambda^{r+1} \frac{m!^2}{m!} - \frac{\theta \lambda e}{(e - 1)^\theta} \sum_{m=0}^\infty \frac{(-1)^m}{m!(1+m)^{r+1}} \lambda^{r+1}$$

$$= \frac{\theta \lambda e}{(e - 1)^\theta} \left[ e \sum_{m=0}^\infty \frac{(-1)^m}{m!(1+m)^{r+1}} \frac{m!}{5} - \sum_{m=0}^\infty \frac{(-1)^m}{m!(1+m)^{r+1}} \right]$$
By putting \( r=1, 2, 3 \ldots \) the raw moments can be viewed.

### 3.2. Moment Generating Function

The moment generating function (MGF) of \( \text{PGE}(\lambda, \theta) \) distribution is given by

\[
M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{\theta \lambda}{(e-1)^\theta} e^{\lambda x - e^{-\lambda x}} (e^{1-e^{-\lambda x}} - 1) dx
\]

\[
= \frac{\theta \lambda}{(e-1)^\theta} \int_0^\infty e^{tx-\lambda x-2e^{-\lambda x}} dx - \frac{\theta \lambda}{(e-1)^\theta} \int_0^\infty e^{1+tx-\lambda x-e^{-\lambda x}} dx
\]

\[
= \frac{\theta \lambda e^2}{(e-1)^\theta} \sum_{m=0}^\infty \frac{(-1)^m 2^m}{m!(\lambda + \lambda m - t)} - \frac{\theta \lambda e}{(e-1)^\theta} \sum_{m=0}^\infty \frac{(-1)^m}{m!(\lambda + \lambda m - t)}
\]

3.3. Characteristic Function

The characteristic function (CF) is given by

\[
\phi_X(t) = \frac{\theta \lambda e}{(e-1)^\theta} \left[ e \sum_{m=0}^\infty \frac{(-1)^m 2^m}{m!(\lambda + \lambda m - it)} - \sum_{m=0}^\infty \frac{(-1)^m}{m!(\lambda + \lambda m - it)} \right]
\]

where \( i = \sqrt{-1} \) is the unit imaginary number.

### 3.4. Cumulant Generating Function

The cumulant generating function (CGF) is given by

\[
K_X(t) = \log \phi_X(t) = \log \left[ \frac{\theta \lambda e}{(e-1)^\theta} \left[ e \sum_{m=0}^\infty \frac{(-1)^m 2^m}{m!(\lambda + \lambda m - it)} - \sum_{m=0}^\infty \frac{(-1)^m}{m!(\lambda + \lambda m - it)} \right] \right]
\]

\[
= \log \left[ \frac{\theta \lambda e}{(e-1)^\theta} \right] + \log \left[ e \sum_{m=0}^\infty \frac{(-1)^m 2^m}{m!(\lambda + \lambda m - it)} - \sum_{m=0}^\infty \frac{(-1)^m}{m!(\lambda + \lambda m - it)} \right]
\]

where \( i = \sqrt{-1} \) is the unit imaginary number.
3.5. Quantile Function

The qth quantile \( Q(q) \) is the solution of the equation \( G(Q(q)) = q \). Hence,

\[
Q(q) = -\frac{1}{\lambda} \log(1 - \log(q^{\frac{1}{\theta}}(e - 1) + 1)).
\]

The median is obtained by setting \( q = 0.5 \) in the above equation. Thus,

\[
\text{Median} = -\frac{1}{\lambda} \log(1 - \log(0.5^{\frac{1}{\theta}}(e - 1) + 1))
\]

3.6. Order Statistic

Let \( X(1), X(2), \ldots, X(n) \) be the order statistics corresponding to the random sample \( X_1, X_2, \ldots, X_n \) of size \( n \) from the proposed PGDUSE distribution.

The pdf and cdf of \( r \)th order statistics of the proposed PGDUSE distribution are given by

\[
g_r(x) = \frac{n! \theta \lambda}{(r-1)!(n-r)!} \frac{e^{1-\lambda x} - e^{\lambda x}}{(e-1)^{2\theta}} \left[ 1 - \left( \frac{e^{1-\lambda x} - 1}{e - 1} \right)^\theta \right]
\]

and

\[
G_r(x) = \sum_{i=1}^{n} \binom{n}{i} \left( \frac{e^{1-\lambda x} - 1}{e - 1} \right)^{\theta i} \left[ 1 - \left( \frac{e^{1-\lambda x} - 1}{e - 1} \right)^\theta \right]^{n-i}
\]

Then, the pdf and cdf of \( X(1) \) and \( X(2) \) are obtained by substituting \( r = 1 \) and \( r = n \) respectively in \( g_r(x) \) and \( G_r(x) \). It is nothing but, the distribution of Minimum and Maximum in series and parallel reliability systems, respectively.

3.7. Entropy

Entropy quantifies the measure of information or uncertainty. An important measure of entropy is Rényi entropy. Rényi entropy is defined as

\[
H_R(\delta) = \frac{1}{1-\delta} \log \left( \int g^\delta(x)dx \right)
\]
where $\delta > 0$ and $\delta \neq 1$.

$$
\int_0^\infty g^\delta(x)\,dx = \frac{(\theta\lambda)^\delta}{(e - 1)^\delta} \int_0^\infty (e^{\delta x} - \delta e^{-\lambda x})(e^{1-e^{-\lambda x}} - 1)\,dx
$$

$$
= \frac{(e\theta\lambda)^\delta}{(e - 1)^\delta} \int_0^\infty e^{-\delta \lambda x} e^{-\delta e^{-\lambda x}} \sum_{k=0}^{\infty} \binom{\delta}{k} (e^{1-e^{-\lambda x}})^{\delta-k}(1)^k\,dx
$$

$$
= \frac{(\theta\lambda e)^\delta}{(e - 1)^\delta} \sum_{k=0}^{\infty} \binom{\delta}{k} (-1)^k e^{\delta-k} e^{-(\delta-k)\lambda x} e^{-\delta \lambda x} e^{-\delta e^{-\lambda x}}\,dx
$$

$$
= \frac{(\theta\lambda e)^\delta}{(e - 1)^\delta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{\delta}{k} (-1)^k e^{\delta-k} e^{-\lambda \delta x} e^{-\lambda x} \int_0^\infty e^{-\lambda(2\delta-k)x}\,dx
$$

$$
= \frac{(\theta\lambda e)^\delta}{(e - 1)^\delta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{\delta}{k} (-1)^k e^{\delta-k} e^{-\lambda(2\delta-k)x}\,dx
$$

The Rényi entropy takes the form

$$
\mathcal{H}_R(\delta) = \frac{1}{1 - \delta} \log \left[ \frac{(\theta\lambda e)^\delta}{(e - 1)^\delta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{\delta}{k} (-1)^k e^{\delta-k} e^{-\lambda(2\delta-k)x}\,dx \right]
$$

$$
= \frac{\delta}{1 - \delta} \log \left[ \frac{\theta\lambda e}{(e - 1)^\delta} \right] + \frac{1}{1 - \delta} \log \left[ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{\delta}{k} (-1)^k e^{\delta-k} e^{-\lambda(2\delta-k)x}\,dx \right]
$$

4. Estimation

Here, the estimation of parameters by the method of maximum likelihood is discussed. For this, consider a random sample of size $n$ from $PGDUSE(\lambda, \theta)$
distribution. Then the likelihood function is given by,
\[ L(x) = \prod_{i=1}^{n} g(x) = \prod_{i=1}^{n} \frac{\theta \lambda}{(e - 1) \theta} e^{1 - \lambda x_i - e^{-\lambda x_i}} (e^{1 - e^{-\lambda x_i}} - 1)^{\theta - 1} \]

Then the log-likelihood function becomes,
\[ \log L = n \log \theta + n \log \lambda - \theta n \log (e - 1) - \lambda \sum_{i=1}^{n} x_i + n \sum_{i=1}^{n} e^{-\lambda x_i} + (\theta - 1) \sum_{i=1}^{n} \log (e^{1 - e^{-\lambda x_i}} - 1) \]

The maximum likelihood estimator (MLE) is obtained by maximizing the log-likelihood with respect to the unknown parameters \( \lambda \) and \( \theta \).
\[ \frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\lambda x_i} + (\theta - 1) \sum_{i=1}^{n} \frac{x_i e^{1 - \lambda x_i - e^{-\lambda x_i}}}{e^{1 - e^{-\lambda x_i}} - 1} \]
\[ \frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n \log (e - 1) + \sum_{i=1}^{n} \log (e^{1 - e^{-\lambda x_i}} - 1) \]

These non-linear equations can be numerically solved through statistical softwares like R with arbitrary initial values.

5. Data Analysis

In this section, a real data analysis is given to assess how well the proposed distribution works has been performed. The data given in Lawless [9] that contains the number of million revolutions before failure of 23 ball bearings put on life test is considered, see Table 1.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
|17.88|28.92|33.00|41.52|42.12|45.60|
|48.80|51.84|51.96|54.12|55.56|67.80|
|68.64|68.64|68.88|84.12|93.12|98.64|
|105.12|105.84|127.92|128.04|173.40|

Table 1: Lawless Data

Further, the proposed distribution has been compared with generalized DUS exponential (GDUSE), DUS exponential (DUSE), exponential (ED), and KM exponential distributions. AIC (Akaike Information Criterion), BIC (Bayesian
Table 2: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics and p-values of the fitted models

| Model | MLEs                  | log L  | AIC    | BIC    | KS         | p-value |
|-------|-----------------------|--------|--------|--------|------------|---------|
| PGDUSE| $\hat{\lambda} = 0.03362141$ | -113.003 | 230.006 | 232.277 | 0.11025   | 0.9425  |
|       | $\hat{\theta} = 3.80657627$  |         |        |        |            |         |
| GDUSE | $\hat{\alpha} = 4.73914452$ | -113.0466 | 230.0931 | 232.3641 | 0.11793   | 0.9064  |
|       | $\hat{\beta} = 0.03553247$  |         |        |        |            |         |
| DUSE  | $\hat{a} = 0.01824005$    | -127.4622 | 256.9244 | 261.1954 | 0.2774    | 0.05804 |
| KME   | $\hat{\theta} = 0.009544456$ | -123.1065 | 248.2129 | 252.4839 | 0.31102   | 0.02337 |
| ED    | $\hat{\theta} = 0.01384327$ | -121.4393 | 244.8786 | 246.0141 | 0.30673   | 0.02639 |

Information Criterion), value of Kolmogorov–Smirnov (KS) statistic, p-value, and log-likelihood value have been used for model selection. Table 2 elucidates that the proposed distribution gives the lowest AIC, BIC, KS values, greatest log-likelihood and p-value. Thus, it can be concluded that the proposed PGDUSE distribution provides a better fit for the given data set when compared with other competing distributions. The empirical cumulative density plot is depicted in Figure 3.

6. Conclusion

In this article, a new class of distribution by generalizing the DUS transformation, called the PGDUS transformation is introduced. A new lifetime distribution called the PGDUSE distribution with exponential as the baseline distribution is also proposed. The generalized form provides greater flexibility in modelling real datasets. Different statistical properties such as moments, moment generating function, characteristic function, quantile function, cumulant generating function, order statistic and entropy are derived. The parameter
estimation has been done through the method of maximum likelihood. Lastly, a real data analysis is performed to show that the proposed generalization can be used effectively to provide better fits.

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