On the distribution of the Cantor-integers

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Abstract

For any positive integer \(p \geq 3\), let \(A\) be a proper subset of \(\{0, 1, \ldots, p-1\}\) with \(\|A\| = s \geq 2\). Suppose \(h : \{0, 1, \ldots, s-1\} \rightarrow A\) is a one-to-one map which is strictly increasing with \(A = \{h(0), h(1), \ldots, h(s-1)\}\). We focus on so-called Cantor-integers \(\{a_n\}_{n \geq 1}\), which consist of these positive integers \(n\) such that all the digits in the \(p\)-ary expansion of \(n\) belong to \(A\). Let \(\mathcal{C} = \left\{ \sum_{n \geq 1} \frac{\varepsilon_n}{p^n} : \varepsilon_n \in A \text{ for any positive integer } n \right\}\) be the appropriate Cantor set, and denote the classic self-similar measure supported on \(\mathcal{C}\) by \(\mu_\mathcal{C}\). Now that \(\lambda_{\log p, r}\) is the growth order of \(a_n\) and \(\left\{ \frac{a_n}{\lambda_{\log p, r}} : n \geq 1 \right\}\) is precisely the set \(\left\{ \frac{x}{\lambda_{(0,1)}(\log x)} : x \in \mathcal{C} \cap \left[ \frac{h(1)}{p}, 1 \right] \right\}\), where \(E'\) is the set of limit points of \(E\), we show that \(\left\{ \frac{a_n}{\lambda_{\log p, r}} : n \geq 1 \right\}\) is just an interval \([m, M]\) with \(m := \inf \left\{ \frac{a_n}{\lambda_{\log p, r}} : n \geq 1 \right\}\) and \(M := \sup \left\{ \frac{a_n}{\lambda_{\log p, r}} : n \geq 1 \right\}\). In particular, \(\left\{ \frac{x}{\lambda_{(0,1)}(\log x)} : x \in \mathcal{C} \setminus \{0\} \right\} = [m, M]\) if \(0 \in A\), and \(m = \frac{q(s-1)+s}{p-1}, M = \frac{q(p-1)+p}{p-1}\) if the set \(A\) consists of all the integers in \(\{0, 1, \ldots, p-1\}\) which have the same remainder \(r \in \{0, 1, \ldots, q-1\}\) modulus \(q\) for some positive integer \(q \geq 2\) (i.e. \(h(x) = qx + r\)). We further show that the sequence \(\left\{ \frac{a_n}{\lambda_{\log p, r}} \right\}_{n \geq 1}\) is not uniformly distributed modulo 1, and it does not have the cumulative distribution function, but has the logarithmic distribution function (give by a specific Lebesgue integral).

Keywords: Cantor-integers, Uniform distribution modulo 1, Cumulative distribution function, Logarithmic distribution function, Self-similar measure

2010 MSC: 11N64, 28A80

1. Introduction

The behavior of an arithmetical function \(f(n)\) for large values of \(n\) has always been one of the important problems in number theory. For example, the average order of a fluctuating arithmetical function, the growth order of a monotone increasing arithmetical function. The distribution properties of the sequences derived from them have also been studied popularly.

An arithmetical function can be seen as a sequence. Let \(\{r_n\}_{n \geq 0}\) be the Rudin-Shapiro sequence, the properties of the Rudin-Shapiro sums \(s(n) = \sum_{k=0}^{n} r_k\) and \(t(n) = \sum_{k=0}^{n} (-1)^k r_k\) have been developed by Brillhart and Morton in [2], where it is showed that

\[
\frac{\sqrt{3}}{\sqrt{5}} \leq \frac{s(n)}{\sqrt{n}} < \sqrt{3} \quad \text{and} \quad 0 \leq \frac{t(n)}{\sqrt{n}} < \sqrt{3},
\]

for any \(n \geq 1\), and that the sequences \(\{s(n)/\sqrt{n}\}_{n \geq 1}\) and \(\{t(n)/\sqrt{n}\}_{n \geq 1}\) are dense respectively in the intervals \([\sqrt{3}/\sqrt{5}, \sqrt{3}]\) and \([0, \sqrt{3}]\). Five years later, in collaboration with Erdös, the authors further studied the distribution properties of the sequences \(\{s(n)/\sqrt{n}\}_{n \geq 1}\) and \(\{t(n)/\sqrt{n}\}_{n \geq 1}\) in [3], they showed that the
sequences do not have the cumulative distribution functions, but do have the logarithmic distribution functions at each point of the respective intervals $[\sqrt{3/5}, \sqrt{6}]$ and $[0, \sqrt{3}]$.

In 2020, Lü, Chen, Wen etc. [8] introduced the quasi-linear integer sequence $f(n)$, which satisfies

$$\alpha := \inf \left\{ \gamma \geq 0 : \limsup_{n \to \infty} \frac{|f(n)|}{n^\gamma} = 0 \right\} > \beta := \inf \left\{ \gamma > 0 : \limsup_{n \to \infty} \frac{|f(n + 1) - f(n)|}{n^\gamma} = 0 \right\},$$

and

$$\left\{ \frac{f(bn + i) - b^\alpha f(n)}{n^\beta} : n \geq 1, 0 \leq i \leq b - 1 \right\}$$

is bounded for some integer $b \geq 2$.

They showed that the growth order of $f(n)$ is $n^\alpha$ (i.e. there exist $0 < c_1 \leq c_2$ such that $c_1 \leq \frac{f(n)}{n^\alpha} \leq c_2$ for any $n$). They focused on the limit function

$$\lambda(x) := \lim_{k \to \infty} \frac{f(b^kx)}{(b^kx)^\alpha}.$$

Their research shows that $\lambda$ is continuous, self-similar and bounded, further, $\{\lambda(x) : x \geq 0\}$ is dense between any two limit points of the sequence $(f(n)/n^\alpha)_{n \geq 1}$. It is not hard to check that the Rudin-Shapiro sums $s(n)$ and $t(n)$ are both quasi-linear discrete functions.

For integers $p > s \geq 2$ and subset $A \subset \{0, 1, \ldots, p - 1\}$ with $|A| = s$. We call an integer $n$ a Cantor integer if the digits in the $p$-ary expansion of $n$ can only take values in $A$, which is named by analogy to the usual middle thirds Cantor set.

For convenience, let us introduce some notations which will be used throughout the text. For any integer $s \geq 2$ and any non-negative integer $n$, the $s$-ary expansion of $n$ is denoted by

$$n = [e_k e_{k-1} \ldots e_0]_s := \sum_{i=0}^{k} e_i s^i,$$

where the digit $e_i \in \{0, 1, \ldots, s - 1\}$ for any $0 \leq i \leq k$. It would be noticed that the top digit $e_k \neq 0$ if $n \neq 0$. And for any $x \in (0, 1)$, the $s$-ary expansion of $x$ is denoted by

$$x = [0, d_1 d_2 \ldots]_s := \sum_{i=1}^{\infty} d_i/s^i,$$

where the $s$-ary digit $d_i \in \{0, 1, \ldots, s - 1\}$ for any $i \geq 1$. Thus for any real number $x \geq 0$, it can be written as $x = [e_k e_{k-1} \ldots e_0 \ldots d_1 d_2 \ldots]_s$, where $[e_k e_{k-1} \ldots e_0]_s$ equals to $[x]$, known as the integer part of $x$, and $[0, d_1 d_2 \ldots]_s$ equals to $\{x\}$, known as the fractional part of $x$.

Based on the notations above, suppose $h : \{0, 1, \ldots, s - 1\} \to A$ is a one-to-one map which is strictly increasing with $A = \{h(0), h(1), \ldots, h(s - 1)\}$. Then for any positive integer $n = [e_k e_{k-1} \ldots e_0]_s$, the $n$-th Cantor integer is

$$a_n = [h(e_k)h(e_{k-1}) \ldots h(e_0)]_p.$$

Meanwhile, we adopt the convention that $a_0 = 0$. Note that

$$a_n = \sum_{i=0}^{k} h(e_i)p^i = \sum_{i=0}^{k} h(e_i)/p^{k-i} \cdot \left( \sum_{i=0}^{k} e_i s^{k-i} \right)^{-\log_b p} \cdot n^{\log_b p},$$

and $\sum_{i=0}^{k} h(e_i)/p^{k-i} \in [1, p]$, $\sum_{i=0}^{k} e_i s^{k-i} \in [1, s]$, we have that the growth order of $a_n$ is $n^{\log_b p}$. By further calculation, it could be obtained that $\alpha = \beta = \log_b p$ for the Cantor integers sequence. That is to say, the Cantor integers sequence what we concerned with are not quasi-linear discrete arithmetic functions.
Theorem 1.2. The sequence \( \{b_n\}_{n \geq 1} \) defined in (1.1) has been discussed by Gawron and Ulas [6] for the case \( p = 4, A = [0, 2] \), and then by Cao and Li [4] for the case \( p \geq 3, A = [d : \ d \in \{0, 1, \ldots, p - 1\}, \ d \text{ is even} \). Moreover, Cao and Li [4] established a connection between the Cantor integers and the self-similar measure. We will generalize their results and further investigate the distribution of the sequence \( \{b_n\}_{n \geq 1} \), parallel to [3], and get some similar results about the Cantor integers. Moreover, we will consider whether or not the sequence \( \{b_n\}_{n \geq 1} \) would be uniformly distributed modulo 1.

At first, we establish a close connection (in Section 2) between the limit points of the sequence \( \{b_n\}_{n \geq 1} \) and the self-similar probability measure \( \mu_\ell \) which supported on the corresponding missing \( p \)-ary digit set
\[
\mathcal{C} := \left\{ \sum_{n=1}^\infty \frac{e_n}{p^n} : e_n \in A \text{ for any positive integer } n \right\},
\]
with
\[
\mu_\ell = \sum_{i=0}^{t-1} 1_{\mu_\ell \circ S_i^{-1}}, \text{ where } S_i = \frac{x + h(i)}{p} (i = 0, 1, \ldots, s - 1).
\]

**Theorem 1.1.** Let \( E' \) be the set of limit points of \( E \), known as the derived set of \( E \). One has
\[
\left\{ \frac{x}{(\mu_\ell([0, x]))^{\log_p, p}} : x \in \mathcal{C} \cap \left[ \frac{h(1)}{p}, 1 \right] \right\} = \{b_n : n \geq 1\}'.
\]
In particular, \( \left\{ \frac{x}{\mu_\ell([0, x]))^{\log_p, p}} : x \in \mathcal{C} \setminus \{0\} \right\} = \{b_n : n \geq 1\}' \) if \( 0 \in A \) (i.e. \( h(0) = 0 \)).

Then we discuss the limit points of the sequence \( \{b_n\}_{n \geq 1} \), and show that each point between the supremum and infimum of \( \{b_n\}_{n \geq 1} \) is an limit point of the sequence \( \{b_n\}_{n \geq 1} \).

**Theorem 1.2.** The sequence \( \{b_n\}_{n \geq 1} \) is dense in \([m, M]\), where \( m = \inf\{b_n : n \geq 1\}, M = \sup\{b_n : n \geq 1\} \). That is to say, \( \{b_n : n \geq 1\}' = [m, M] \).

Theorem 1.2 urges us to discuss the “homogeneity” of the distribution of the sequence \( \{b_n\}_{n \geq 1} \).

In the process, we introduce the function \( \lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with
\[
\lambda(x) = \lim_{k \to \infty} \frac{a(s^k x)}{(s^k x)^{\log_p, p}},
\]
where \( a(x) := a_{[s]} \). We show that \( \lambda(sx) = \lambda(x), \lambda(x) \in [m, M] \) for any \( x > 0 \), and
\[
\{\lambda(x) : x \geq 0 \text{ and } x \text{ is } s \text{-ary irrational number}\}
\]
is dense in \([m, M]\). We also discuss the continuity and the level sets of the function \( \lambda \) and obtain the general results, show that \( \lambda \) is always continuous from the right at any \( x > 0 \) and continuous from the left at \( s \)-ary irrational number \( x > 0 \), as well as \( \{x > 0 : \ \lambda(x) = a\} \) has measure zero for any \( a \in [m, M] \).

Based on this, we show that the sequence \( \{b_n\}_{n \geq 1} \) is not uniformly distributed modulo 1 and it does not have the natural distribution function.

**Theorem 1.3.** The sequence \( \{b_n\}_{n \geq 1} \) is not uniformly distributed modulo 1.
Here a real sequence \( \{x_n\}_{n \geq 1} \) is said to be uniformly distributed modulo 1 (abbreviated u. d. mod 1) if for any interval \( I \subset [0, 1] \) one has
\[
\lim_{N \to \infty} \frac{\# \{1 \leq n \leq N : \{x_n\} \in I \}}{N} = |I|,
\]
where \(|I|\) is the length of the interval \( I \). The formal definition of u. d. mod 1 was given by Weyl ([9, 10]).

**Theorem 1.4.** *The cumulative distribution function of the sequence \( \{b_n\}_{n \geq 1} \) does not exist at any point \( \alpha \in (m, M) \).*

By this we mean the limit \( \lim_{x \to \infty} x^{-1}D(x, \alpha) \) does not exist and \( D(x, \alpha) \) denotes the number of times \( b_n \leq \alpha \) for \( 1 \leq n \leq x \).

In the positive direction, we prove that a modified distribution function for \( \{b_n\}_{n \geq 1} \) does exist.

**Theorem 1.5.** *For any \( \alpha \in [m, M] \), the logarithmic distribution function of the sequence \( \{b_n\}_{n \geq 1} \) exists at \( \alpha \), and has the value
\[
L(\alpha) = \frac{1}{\ln s} \int_{E_\alpha} \frac{1}{x} dx,
\]
where \( E_\alpha = \{x \in [s^{-1}, 1), \lambda(x) \leq \alpha \} \) and the integral is a Lebesgue integral.*

The definition of the logarithmic distribution function used here comes from [3], which are defined as follows,
\[
L(\alpha) := \lim_{x \to \infty} \frac{1}{\ln x} \sum_{1 \leq n \leq x} \frac{1}{n}.
\]

At last, we consider the Cantor integers sequence consists of the non-negative integers \( n \) whose digits in the \( p \)-ary expansions of them have the same remainder \( r \) modulus \( q \), where positive integers \( q \geq 2 \) and \( p > q + r \) with \( r \in \{0, 1, \ldots, q - 1\} \). It can be seen as the special case of \( h(x) = qx + r \). This is why they are called the “linear” Cantor integers sequences. We give the exact value of the supremum and infimum of \( \{b_n\}_{n \geq 1} \).

**Theorem 1.6.** *For any positive integers \( q \geq 2 \) and \( p > q + r \) with \( r \in \{0, 1, \ldots, q - 1\} \), let \( s = \lceil \frac{p-r}{q} \rceil \) and \( \lceil \cdot \rceil \) be the ceil function. When \( A = \{qi + r : i \in \{0, 1, \ldots, s-1\}\} \), one has
\[
m = \frac{q(s-1)+r}{p-1}, \quad M = \frac{q(p-1)+pr}{p-1}.
\]

2. **The connection of \( \{b_n\}_{n \geq 1} \) and the self-similar measure \( \mu_\xi \)**

Recall that the Cantor set
\[
\mathcal{C} := \left\{ \sum_{n=1}^{\infty} \frac{e_n}{p^n} : e_n \in A \text{ for any positive integer } n \right\}
\]
is the attractor of the family of contracting self-maps \( \{S_i\}_{i=0}^{s-1} \) of \([0, 1]\) with
\[
S_i(x) = \frac{x}{p} + \frac{h(i)}{p}, \quad i \in \{0, 1, \ldots, s-1\}.
\]
Let \( \mathbf{P}(C) \) be the set of the Borel probability measures on \( C \), and
\[
L(v, v') = \sup_{\text{Lip}(g) \leq 1} \left| \int gj dv - \int gj dv' \right|, \quad \text{with } \text{Lip}(g) = \sup_{x \neq y} \left| \frac{g(x) - g(y)}{x - y} \right|
\]
be the dual Lipschitz metric on the space \( \mathbf{P}(C) \). Since the mapping \( F \) defined on \( \mathbf{P}(C) \) by
\[
F(v) = \sum_{i=0}^{s-1} \frac{1}{s} v_S^{-1}
\]
is a contracting self-map of the compact metric space \((\mathbf{P}(C), L(v, v'))\). By the fundamental theorem on iterated function systems and the compact fixed-point theorem, one has
\[
F^k(\delta_0) \to \mu_\xi \text{ as } k \to \infty,
\]
where \( \delta_0 \) is the Dirac measure concentrated at 0. Note that for any \( x \in [0, 1] \), \( \mu_\xi((x)) = 0 \), one has
\[
\lim_{k \to \infty} F^k(\delta_0)([0, x]) = \mu_\xi([0, x]),
\]
by Portmanteau Theorem. Now, let us prove Theorem 1.1.
For any \( x \in C \cap \left[ \frac{h(k)}{p}, 1 \right] \), suppose
\[
x = \sum_{i=1}^{\infty} \frac{h(\varepsilon_i)}{p^i}, \quad \text{where } \varepsilon_i \in \{0, 1, \ldots, s-1\}, \quad \varepsilon_1 \neq 0.
\]
Since
\[
F^k(\delta_0) = \sum_{i_1 \ldots i_k \in [0, 1]^{s-1}} \frac{1}{s^k} \delta_{S_{i_1} \circ \cdots \circ S_{i_k}(0)}.
\]
Then for any \( k \geq 1 \), if we put \( n_k = \sum_{i=1}^{k} \varepsilon_i s^{k-i} \), one has
\[
a_{n_k} = [p^k x], \quad x = \lim_{k \to \infty} \frac{a_{n_k}}{p^k},
\]
and
\[
F^k(\delta_0)([0, x]) = \# \{i_1 \ldots i_k \in [0, 1]^{s-1} : S_{i_1} \circ \cdots \circ S_{i_k}(0) \in [0, x] \} = \frac{n_k + 1}{s^k}.
\]
Thus
\[
\frac{x}{(\mu_\xi([0, x]))^{\log, p}} = \lim_{k \to \infty} \frac{a_{n_k}}{p^k (n_k + 1)^{\log, p}} = \lim_{k \to \infty} \frac{a_{n_k}}{(n_k + 1)^{\log, p}} = \lim_{k \to \infty} \frac{a_{n_k}}{n_k^{\log, p}}.
\]
That is to say, \( \frac{x}{(\mu_\xi([0, x]))^{\log, p}} \) is the limit point of \( \{b_n\}_{n \geq 1} \).

For any limit point \( \gamma \) of \( \{b_n\}_{n \geq 1} \). Assume that \( \{n_k\}_{k \geq 1} \) be a subsequence with
\[
\lim_{k \to \infty} b_{n_k} = \gamma.
\]
Note that \( \left\{ \frac{a_{n_k}}{p^{f_k}} \right\}_{k \geq 1} \) and \( \left\{ \frac{n_k}{s^{f_k}} \right\}_{k \geq 1} \) are both bounded, without loss of generation, we can further ask that
\[
\lim_{k \to \infty} \frac{a_{n_k}}{p^{f_k}} \quad \text{and} \quad \lim_{k \to \infty} \frac{n_k}{s^{f_k}}
\]
both exist and denote the limit values are \( x \) and \( t \) respectively. It is clearly that \( x \in [h(1)/p, 1] \) and \( t \in [1/s, 1] \).

For any \( k \geq 1 \), let \( \ell_k \) be the unique integer such that \( s^{\ell_k-1} \leq n_k < s^{\ell_k} \). At this time,

\[
\gamma = \lim_{k \to \infty} b_m = \lim_{k \to \infty} \frac{a_{n_k}/p^{\ell_k}}{(n_{k}/s^{\ell_k})^\log_{p}, p} = \lim_{k \to \infty} \frac{a_{n_k}/p^{\ell_k}}{((n_{k} + 1)/s^{\ell_k})^\log_{p}, p} = \frac{x}{(\mu_\xi([0, x]))^\log_{p}, p}
\]

for the above \( x \), since

\[
\frac{n_k + 1}{s^{\ell_k}} = \left(F_{s^{\ell_k}}(\delta_0) \left[0, \frac{a_{n_k}}{p^{\ell_k}}\right] - F_{s^{\ell_k}}(\delta_0)([0, x])\right) + \left(F_{s^{\ell_k}}(\delta_0)([0, x]) - \mu_\xi([0, x])\right) + \mu_\xi([0, x]),
\]

and

\[
\lim_{k \to \infty} \left(F_{s^{\ell_k}}(\delta_0) \left[0, \frac{a_{n_k}}{p^{\ell_k}}\right] - F_{s^{\ell_k}}(\delta_0)([0, x])\right) = 0, \quad \lim_{k \to \infty} F_{s^{\ell_k}}(\delta_0)([0, x]) = \mu_\xi([0, x]).
\]

When \( 0 \in A \), i.e., \( h(0) = 0 \), for any \( x \in \mathbb{C} \cap \left(0, \frac{h(1)}{p}\right) \), suppose

\[
x = \sum_{i=1}^{\infty} \frac{h(\varepsilon_i)}{p^i}, \text{ where } \varepsilon_i \in \{0, 1, \ldots, s-1\}, \text{ and } k_0 = \min\{i : \varepsilon_i \neq 0\}.
\]

It is clear that \( k_0 > 1 \) and \( p^{k_0-1}x \in \mathbb{C} \cap \left[\frac{h(1)}{p}, 1\right] \). By the definition of \( \mu_\xi \), one has \( \mu_\xi([0, x]) = s^{-k_0+1}\mu_\xi([0, p^{k_0-1}x]) \), and thus

\[
\frac{x}{(\mu_\xi([0, x]))^\log_{p}, p} = \frac{p^{k_0-1}x}{\mu_\xi([0, p^{k_0-1}x])}.
\]

All of this means the results of Theorem 1.1 are correct.

**Remark 1.** Let \( p = 4, s = 2, A = \{1, 3\} \) (i.e., \( h(x) = 2x + 1 \), \( x = \frac{x}{8} \in \mathbb{C} \cap \left(\frac{1}{M}, \frac{3}{M}\right) = \mathbb{C} \cap \left(\frac{h(0)}{p}, \frac{h(1)}{p}\right) \). Then \( \mu_\xi([0, x]) = \frac{1}{4} \) and \( \frac{1}{\mu_\xi([0, x])^\log_{p}, p} = 6 \notin \{1, \frac{10}{3}\} \). Where \( m = 1, M = \frac{10}{3} \) base on Theorem 1.2.

3. The denseness of \( \{b_n\}_{n \geq 1} \)

We begin the proof of Theorem 1.2 with the facts that (1) \( a_n \geq n \) for any integer \( n \geq 1 \), (2) \( a_i = h(i) \) for any \( i \in \{0, 1, \ldots, s-1\} \), (3) \( a_{s^{\ell} + i} = pa_{n} + h(i) \) for any integer \( n \geq 1 \) and \( i \in \{0, 1, \ldots, s-1\} \), as well as the following properties of \( b_n \).

**Proposition 3.1.** For any positive integer \( n \), one has \( b_{s^{\ell}n+s^{\ell}-1} < b_n \) for any large \( \ell \) with \( 1 - s^{-\ell} > \log_{p} s \).

**Proof.** Note that

\[
b_{s^{\ell}n+s^{\ell}-1} = \frac{p^{\ell}a_{n} + h(s-1)(p^{\ell} - 1)}{(s^{\ell}n + s^{\ell-1} - 1)^\log_{p}, p} \leq \frac{a_{n} + 1}{(n + \log_{p} s)^\log_{p}, p} = b_n \cdot \frac{1 + a_n^{-1}}{(1 + n^{-1} \log_{p} s)^\log_{p}, p},
\]

since \( h(s-1) \leq p - 1 \) and \( 1 - s^{-\ell} > \log_{p} s \). Combine this with the facts that

\[
a_n \geq n \quad \text{and} \quad (1 + n^{-1} \log_{p} s)^\log_{p}, p > 1 + n^{-1},
\]

one has \( b_{s^{\ell}n+s^{\ell}-1} < b_n \). \( \square \)

**Proposition 3.2.** \( b_{s^{\ell}n+1} < b_{sn+s-2} < \cdots < b_{sn+1} < b_n \) when \( n \) large enough, and \( b_n \leq b_{sn} \) for any \( n \geq 1 \). Moreover \( b_n = b_{sn} \) if and only if \( h(0) = 0 \).
Proof. Note that for any \( i \in \{1, 2, \ldots, s-2\} \), \( b_{sn+i+1} < b_{sn+i} \) if and only if

\[
1 + \frac{h(i+1) - h(i)}{pa_n + h(i)} < \left(1 + \frac{1}{sn+i}\right)^{\log p},
\]

and the facts

\[
\left(1 + \frac{1}{sn+i}\right)^{\log p} > 1 + \log p \cdot \frac{1}{sn+i}, \quad 0 \leq h(i) \leq p - 1.
\]

We have \( b_{sn+s-1} < b_{sn+s-2} < \cdots < b_{sn+1} \) for \( n \) large enough since the growth order of \( a_n \) is \( n^{\log p} \) and \( \log p > 1 \).

Similarly,

\[
1 + \frac{h(1)}{pa_n} < \left(1 + \frac{1}{sn}\right)^{\log p}
\]

implies \( b_{sn+1} < b_n \) when \( n \) large enough.

At last, for any positive integer \( n \), \( b_n \leq b_{sn} \) if and only if \( h(0) \geq 0 \), and \( b_n = b_{sn} \) holds if and only if \( h(0) = 0 \).

For convenience, suppose the inequalities \( b_{sn+s-1} < b_{sn+s-2} < \cdots < b_{sn+1} < b_n \leq b_{sn} \) in Proposition 3.2 hold for \( n > N_0 \) and \( s^{k_0-1} \leq N_0 < s^{k_0} \) for some positive integer \( k_0 \).

At first, we show that infimum \( m = \inf\{b_n : n \geq 1\} \) and the supremum \( M = \sup\{b_n : n \geq 1\} \) are both the limit points of \( \{b_n\}_{n \geq 1} \).

Note that \( b_n \neq m \) for any integer \( n \geq 1 \), which follows from Proposition 3.1. Thus the infimum \( m \) must be the limit point of \( \{b_n\}_{n \geq 1} \).

Similarly, the supremum \( M \) must be the limit point of \( \{b_n\}_{n \geq 1} \) in the case of \( h(0) \neq 0 \) since \( b_n < b_{sn} \) for any \( n \geq 1 \).

When \( h(0) = 0 \). Let \( b_{n_0} = \max\{b_n : 1 \leq n < s^{k_0+1}\} \) for some \( 1 \leq n_0 < s^{k_0+1} \). Then for any \( n \geq s^{k_0+1} \), assume \( n = [\varepsilon_k \ldots \varepsilon_1 \varepsilon_0] \) with \( k \geq k_0 + 1 \), by Proposition 3.2 one has

\[
b_n \leq b_{[\varepsilon_k \ldots \varepsilon_1 \varepsilon_0]} \leq \cdots \leq b_{[\varepsilon_k \ldots \varepsilon_{k-k_0}]} \leq b_{n_0},
\]

since \( [\varepsilon_k \ldots \varepsilon_{k-k_0}] \geq s^{k_0} > N_0 \). Thus

\[
M = b_{n_0} = \lim_{k \to \infty} b_{s^{k_0}}.
\]

This is due to the fact that \( b_{sn} = b_n \) for any \( n \geq 1 \) if \( h(0) = 0 \). Therefore, the supremum \( M \) is also the limit point of \( \{b_n\}_{n \geq 1} \) when \( h(0) = 0 \).

On the basis of the above findings we offer the following Lemma which will be used in the construction of the subsequence of \( \{b_n\}_{n \geq 1} \) whose limit exactly belongs to \( (m, M) \).

**Lemma 3.3.** For any \( \gamma \in (m, M) \), there exists an integer \( n_1 \geq s^{k_0} \) such that \( b_{n_1+1} < \gamma \leq b_{n_1} \).

**Proof.** Suppose the result does not hold, then for any \( n \geq s^{k_0} \), \( b_{n+1} \geq \gamma \) or \( b_n < \gamma \). At this moment, for any \( n > s^{k_0} \), \( b_n = b_{(n-1)+1} < \gamma \) implies \( b_{n-1} < \gamma \), and \( b_n \geq \gamma \) implies \( b_{n+1} \geq \gamma \). Thus for any \( n \geq s^{k_0} \),

- If \( b_n < \gamma \), then for any integer \( m \in [s^{k_0}, n] \) one has \( b_m < \gamma \).
- If \( b_n \geq \gamma \), then for any integer \( m \geq n \) one has \( b_m \geq \gamma \).
Put
\[ m_0 := \min\{n \geq s^k : b_n \geq \gamma\}. \]
It is well defined since \( \gamma < M \) and \( M \) is the limit point of \( \{b_n\}_{n \geq 1} \) imply \( \{n \geq s^k : b_n \geq \gamma\} \neq \emptyset. \)

Therefore, for any \( n \geq m_0 \), one has \( b_n \geq \gamma \), which is contradict with \( \gamma > m \) and \( m \) is also the limit point of \( \{b_n\}_{n \geq 1}. \)

Now, we will show that for any point \( \gamma \in (m, M) \), it is the limit point of \( \{b_n\}_{n \geq 1} \). That is to say, there is a subsequence \( \{n_k\}_{k \geq 1} \) such that
\[ \lim_{k \to \infty} b_{n_k} = \gamma. \]

Take \( n_1 \geq s^k \) with \( b_{n_1+1} < \gamma \leq b_{n_1} \). And define \( n_{k+1} \) recursively as follows.
\[ n_{k+1} = \begin{cases} s_{n_k} + i, & \text{if } b_{s_{n_k}+(i+1)} < \gamma \leq b_{s_{n_k}+i} \text{ for some } i \in \{0, 1, \ldots, s-2\}; \\ s_{n_k} + (s-1), & \text{if } b_{s_{n_k}+(s-1)} \geq \gamma. \end{cases} \]

At first, we will prove that the limit \( \lim_{k \to \infty} b_{n_k} \) exists. To do that, we just need to show that \( \sum_{k=1}^{\infty} |b_{n_k} - b_{n_{k+1}}| \)
is convergent. It is sufficient to show that \( \sum_{k=1}^{\infty} |b_{n_k} - b_{n_{k+1}}| < \infty. \)

Suppose \( n_{k+1} = s_{n_k} + i \) for some \( i \in \{0, 1, \ldots, s-1\}. \)
If \( i \in \{1, \ldots, s-1\}, \) by Proposition 3.2 we have
\[ |b_{n_k} - b_{n_{k+1}}| = b_{n_k} - b_{n_{k+1}} = \frac{a_{n_k}}{n_k \log_2 p} - \frac{pa_{n_k} + h(i)}{(s_{n_k} + i) \log_2 p} = \frac{a_{n_k}}{n_k \log_2 p} \left( 1 - \frac{1 + \frac{h(i)}{pa_{n_k}}}{(1 + \frac{b}{s}) \log_2 p} \right). \]

Note that
\[ 1 - \frac{1 + \frac{h(i)}{pa_{n_k}}}{(1 + \frac{b}{s}) \log_2 p} < \left( 1 + \frac{i}{s_{n_k}} \right) \log_2 p - 1 - \frac{h(i)}{pa_{n_k}} < \left( 1 + \frac{i}{s_{n_k}} \right) \log_2 p - 1. \]
And
\[ \left( 1 + \frac{i}{s_{n_k}} \right) \log_2 p - 1 = \log_2 p \cdot \left( 1 + \frac{\theta i}{s_{n_k}} \right) \log_2 p - 1 \cdot \frac{i}{s_{n_k}} \leq \log_2 p \cdot \frac{p \log_2 p}{s}. \]

since \( \theta \in (0, 1) \) implies \( 1 + \frac{\theta i}{s_{n_k}} < s \), and \( n_k \geq s^{k-1}. \) Thus at this case,
\[ |b_{n_k} - b_{n_{k+1}}| < M \cdot \frac{p \log_2 p}{s^k} =: C s^{-k}, \]
where \( C := M p \log_2 p. \)

If \( i = 0 \), by Proposition 3.2 we also have
\[ |b_{n_k} - b_{n_{k+1}}| = b_{n_{k+1}} - b_{n_k} = \frac{h(0)}{p a_{n_k} \log_2 p} \leq \frac{h(0)}{p s^{(k-1)} \log_2 p} = \frac{h(0)}{p^k} < C s^{-k}. \]

since \( p > s, 0 \leq h(0) < p \) and \( M \geq b_1 = \frac{a_1}{1 \log_2 p} = h(1) \geq 1. \) Therefore,
\[ \sum_{k=1}^{\infty} |b_{n_k} - b_{n_{k+1}}| < \sum_{k=1}^{\infty} C s^{-k} = \frac{C}{s-1} < \infty. \]

It is obviously that \( \lim_{k \to \infty} b_{n_k} \geq \gamma. \) It can be seen, the set \( \{k \geq 1 : n_k \neq s-1(\mod s)\} \) is an infinite set.
If not, put \( K = \max\{k \geq 1 : n_k \not\equiv s - 1(\text{mod } s)\} \) if \( k \geq 1 : n_k \not\equiv s - 1(\text{mod } s)\) \( \neq 0 \) and 1 otherwise. Then \( b_{n_k+1} < \gamma \leq b_{n_k} \), and \( n_k \equiv s - 1(\text{mod } s) \) for any \( k > K \). Thus

\[
\lim_{k \to \infty} b_n = \lim_{t \to \infty} \frac{p^t a_{n_k} + h(s-1) \log_p p}{(s^t n_K + s^t - 1) \log_p p} = \frac{a_{n_k} + \frac{h(s-1)}{(n_K + 1) \log_p p}}{\frac{a_{n_k} + 1}{(n_K + 1) \log_p p}} \leq \frac{b_{n_k+1} < \gamma},
\]

which contradicts with \( \lim_{k \to \infty} b_n \geq \gamma \).

Now, we show that \( \lim_{k \to \infty} b_{n_k} \leq \gamma \). Note that \( n_k \not\equiv s - 1(\text{mod } s) \) for infinitely many \( k \geq 1 \). One has

\[
b_{s n_k + s - 1} = \frac{p a_{n_k} + h(s-1)}{(s n_k + s - 1) \log_p p} < \gamma
\]

for infinitely many \( k \). We denote the set of such \( k \) by \( K \), thus

\[
\lim_{k \to \infty} \lambda = \lim_{k \to \infty} \frac{a_m}{s_k \log_p p} = \lim_{k \to \infty} \frac{a_m + \frac{h(s-1)}{(s n_k + s - 1) \log_p p}} {\leq \gamma},
\]

since \( a_m \to \infty, n_k \to \infty \) as \( k \to \infty \). In conclusion, we have \( \lim_{k \to \infty} b_{n_k} = \gamma \).

### 4. The distribution of \( \{b_n\}_{n \geq 1} \)

In this section, we shall come to the distribution of the sequence \( \{b_n\}_{n \geq 1} \). For better research, let us start by expanding the domain of the arithmetical function \( b_n \) to positive real number. Just like what we have done in the proof of the dense of \( \{n_k\}_{k \geq 1} \), if the subsequence \( \{n_k\}_{k \geq 1} \) satisfies \( n_k + 1 = s n_k + i \) for any \( k \geq 1 \) and some \( i \in \{0, 1, \ldots, s - 1\} \), then \( b_{n_k} \) exists. So we can define the function

\[
\lambda(x) := \lim_{k \to \infty} \frac{a_{s^k x}}{(s^k x) \log_p p}
\]

for any \( x \in (0, \infty) \), where \( a(x) := a_{[x]} \). It is clear that \( a(x) = 0 \) for any \( x \in (0, 1) \). At first, we will prove strictly that \( \lambda(x) \) is well-defined, rewrite the express of \( \lambda(x) \) and estimate the convergence speed of the limit.

**Proposition 4.1.** For any \( x > 0 \), the limit \( \lim_{k \to \infty} \frac{a_{s^k x}}{(s^k x) \log_p p} \) exists, \( \lambda(sx) = \lambda(x) \), and \( \lambda(x) \in [m, M] \). Furthermore, the set \( \{\lambda(x) : x \in (s^{-1}, 1]\} \) is dense in \( [m, M] \).

**Proof.** For any \( x > 0 \), suppose \( x = [x] + [0.1]d_1[d_2] \ldots \]. Write \( n_k = [s^k x] \). It is clear that \( n_{k+1} = s n_k + d_k \) for any \( k \geq 1 \). Thus

\[
\lim_{k \to \infty} \frac{a_{s^k x}}{(s^k x) \log_p p} = \lim_{k \to \infty} \frac{a_m}{n_k \log_p p} \cdot \frac{[s^k x]}{(s^k x) \log_p p} = \lim_{k \to \infty} b_{n_k}.
\]

That is to say, the function \( \lambda \) is well defined. The self-similarity of \( \lambda \) can be obtained by the definition of itself. The boundedness of \( \lambda \) and the denseness of \( \{\lambda(x) : x \in (s^{-1}, 1]\} \) follow equality (4.1), the self-similarity of \( \lambda \) and Theorem [1,2] \( \square \)

**Proposition 4.2.** For \( x \geq s^{-1} \), suppose \( x = [x] + [0.1]d_1[d_2] \ldots \), and let \( \phi(x) = \sum_{j=1}^{\infty} h(d_j) p^{-j} \). Then

\[
\lambda(x) = \frac{a(x) + \phi(x)}{x \log_p p}.
\]

And

\[
|\lambda(x) - \frac{a_{s^k x}}{(s^k x) \log_p p}| \leq p^{-k} x^{-\log_p p},
\]

for any \( k \geq 1 \).
Proof. For \( x \geq s^{-1} \) and \( k \geq 1 \), let \( n_k := \lfloor s^k x \rfloor \). Then \( n_k = \lfloor x \rfloor s^k + \sum_{j=1}^{k} d_j s^{k-j} \), and

\[
a(s^k x) = a_{n_k} = p^k a(x) + \sum_{j=1}^{k} h(d_j) p^{-j}.
\]

By the equality (4.1), we have

\[
\lambda(x) = \lim_{k \to \infty} \frac{p^k a(x) + \sum_{j=1}^{k} h(d_j) p^{-j}}{(s^k x) \log_s p} = \lim_{k \to \infty} \frac{a(x) + \sum_{j=1}^{k} h(d_j) p^{-j}}{x \log_s p} = \frac{a(x) + \phi(x)}{x \log_s p}.
\]

It is clearly that \( \lambda(x) - \frac{a(s^k x)}{(s^k x) \log_s p} \leq \frac{1}{x \log_s p} \). Note that \( \lambda(sx) = \lambda(x) \), then

\[
\left| \lambda(x) - \frac{a(s^k x)}{(s^k x) \log_s p} \right| = \left| \lambda(s^k x) - \frac{a(s^k x)}{(s^k x) \log_s p} \right| \leq \frac{1}{(s^k x) \log_s p} = p^{-k} x^{-\log_s p}.
\]

Now, let us further explore the continuity of \( \lambda(x) \).

**Proposition 4.3.** The function \( \lambda \) is always continuous from the right at any \( x > 0 \) and continuous from the left at \( s \)-ary irrational number \( x > 0 \). \( \lambda \) is continuous from the left at the \( s \)-ary rational number \( x = [0.d_1 \ldots d_N] \) with \( d_N \in \{1, \ldots, s-1\} \) for some positive integer \( N \) if and only if \( h(s-1) = p-1 \) and \( h(d_N) = h(d_N-1) = 1 \).

Especially, if \( h(x) = x+r \) for some positive integer \( r \), and \( s+r = p \), that is to say, \( A = \{r, r+1, \ldots, p-1\} \), then the corresponding function \( \lambda \) is continuous at any \( x > 0 \).

**Proof.** By the self-similarity of \( \lambda \), we only need to consider \( x \in [s^{-1}, 1) \). Note that

\[
\lambda(x) = \frac{\phi(x)}{x \log_s p} \text{ for any } x \in [s^{-1}, 1).
\]

It is sufficient to consider the continuity of \( \phi(x) = \sum_{j=1}^{\infty} h(d_j) p^{-j} \) for \( x = [0.d_1 d_2 \ldots]_s \), with \( d_1 \geq 1 \).

For any \( x \in [s^{-1}, 1) \), assume that \( x = [0.d_1 d_2 \ldots]_s \), with \( d_1 \geq 1 \). For any \( n \geq 1 \), take

\[
x_n = [0.d_1 \ldots d_n]_s + \frac{1}{s^n}.
\]

It is clear that \( x < x_n \) and \( \lim_{n \to \infty} x_n = x \). And for any \( x^* \in (x, x_n) \), suppose \( x^* = [0.d'_1 d'_2 \ldots]_s \), then \( d'_k = d_k \) for any \( 1 \leq k \leq n \). Thus,

\[
|\phi(x^*) - \phi(x)| = \left| \sum_{k=n+1}^{\infty} h(d'_k) p^{-k} - \sum_{k=n+1}^{\infty} h(d_k) p^{-k} \right| \leq \frac{1}{p^n}.
\]

Therefore, \( \lambda \) is continuous from the right at any \( x \in [s^{-1}, 1) \).

For any \( s \)-ary irrational number \( x \in [s^{-1}, 1) \), suppose \( x = [0.d_1 d_2 \ldots]_s \) with \( d_1 \geq 1 \). Set \( x_n := [0.d_1 \ldots d_n]_s \). For any \( x^* \in (x_n, x) \) with \( x^* = [0.d'_1 d'_2 \ldots]_s \), one has \( d'_k = d_k \) for any \( 1 \leq k \leq n \). Similar to the discussion above, \( |\phi(x^*) - \phi(x)| \leq \frac{1}{p^n} \). Therefore, \( \lambda \) is continuous from the left at any \( s \)-ary irrational \( x \in [s^{-1}, 1) \).
For any $s$-ary rational number $x \in [s^{-1}, 1)$. Suppose $x = [0.d_1 \ldots d_N]_s$ for some positive integer $N$ with $d_1, d_N \neq 0$. For any integer $n > N$, set
\[
x_n := x - s^{-n} = [0.d_1 \ldots d_{N-1}(d_N - 1)(s - 1)^{n-N}]_s.
\]
Then $x_n \to x$ as $n \to \infty$, and
\[
|\phi(x_n) - \phi(x)| = \phi(x) - \phi(x_n) = \frac{h(d_N) - h(d_N - 1)}{p^N} - h(s - 1) \left(\frac{1}{p^n} - \frac{1}{p^N}\right).
\]
Note that
\[
\begin{align*}
&\text{If } h(s - 1) \neq p - 1, \text{ then } |\phi(x_n) - \phi(x)| > p^{-(N+1)}.
&\text{If } h(s - 1) = p - 1, h(d_N) - h(d_N - 1) \neq 1, \text{ then } |\phi(x_n) - \phi(x)| > p^{-N}.
&\text{If } h(s - 1) = p - 1, h(d_N) - h(d_N - 1) = 1, \text{ then } |\phi(x_n) - \phi(x)| = p^{-n}, \text{ and thus for any } x^* \in (x_n, x),
&|\phi(x^*) - \phi(x)| < p^{-n}.
\end{align*}
\]
Therefore, the results hold.

\[\square\]

**Proposition 4.4.** The set $\{\lambda(x) : x \in [s^{-1}, 1) \text{ and } x \text{ is } s\text{-ary irrational number}\}$ is dense in $[m, M]$. Hence, the set $\{\lambda(x) : x \in [s^{-1}, 1) \text{ and } \lambda \text{ is continuous at } x\}$ is dense in $[m, M]$.

**Proof.** For any $\gamma \in [m, M]$ and any $\delta > 0$. By Theorem 1.2, there exists $n_k$ such that
\[
\frac{M p^2}{n_k} < \delta \quad \text{and} \quad \left|\frac{a_m}{n_k^{\log, p}} - \gamma\right| < \frac{\delta}{2}.
\]
Suppose the $s$-ary expansion of $n_k$ is $n_k = [\varepsilon_1 \varepsilon_2 \ldots \varepsilon_1 \varepsilon_0]_s$. Take $x_k = [0.\varepsilon_1 \varepsilon_2 \ldots \varepsilon_1 \varepsilon_0 (01)^{\infty}]_s$. Then $x_k$ is $s$-ary irrational number which belongs to $[s^{-1}, 1)$. Note that
\[
\frac{a_{n_k}}{n_k^{\log, p}} = \frac{a(s^k x_k)}{(s^k x_k)^{\log, p}} \left(1 + \log, p \cdot \left(1 + \frac{\theta}{s^k - 1} \cdot \frac{1}{n_k}\right)\right),
\]
for some $\theta \in (0, 1)$. Then
\[
\left|\lambda(x_k) - \frac{a_m}{n_k^{\log, p}}\right| \leq \left|\lambda(x_k) - \frac{a(s^k x_k)}{(s^k x_k)^{\log, p}}\right| + \left|\log, p \cdot \frac{a(s^k x_k)}{(s^k x_k)^{\log, p}} \left(1 + \frac{\theta}{s^k - 1} \cdot \frac{1}{n_k}\right)\right|.
\]
The first part of the right in the above inequality is no more than $p^{-t_k}$ by Proposition 4.2. And the second part is no more than
\[
p \cdot M \cdot \frac{p}{s} \cdot \frac{1}{s^2 - 1} \cdot \frac{1}{n_k} < \frac{\delta}{6}.
\]
Thus,
\[
\left|\lambda(x_k) - \gamma\right| < \frac{1}{p^{t_k}} + \frac{\delta}{6} + \frac{\delta}{2} \leq \delta,
\]
which implies the proposition is correct.
\[\square\]

**Lemma 4.5.** For any interval $(e, f) \subset [m, M]$, there exist $x_0 \in [1/s, 1]$, $\eta_0 \in (0, 1)$ and $k_0 \in \mathbb{Z}^+$ such that $b_n \in (e, f)$ for any integer $n \in [s^k (x_0 - \eta_0), s^k (x_0 + \eta_0)]$ with $k > k_0$.
Proof. By Proposition 4.4, there is an s-ary irrational number $x_0 \in (s^{-1}, 1)$ and real numbers $\tilde{e}, \tilde{f}$ such that $e < \tilde{e} < \lambda(x_0) < f < \tilde{f}$. Since $\lambda$ is continuous at $x_0$, there exists $\eta_0 \in (0, x_0 - s^{-1})$ such that for any $x \in [x_0 - \eta_0, x_0 + \eta_0] \subseteq [s^{-1}, +\infty)$, one has $e < \lambda(x) < f$.

Let $\delta = \min(f - \tilde{f}, \tilde{f} - e)$, and choose $k_0$ so large that $p^{-k_0} (x_0 - \eta_0)^{-\log s} \frac{\lambda(x)}{\lambda(x)} < \delta$. Thus for any $k \geq k_0$, by Proposition 4.4, for any $x \in [x_0 - \eta_0, x_0 + \eta_0]$, one has

$$| \lambda(x) - \frac{a_n}{n^{\log s}} | \leq p^{-k} x^{-\log s} \left( \frac{n}{x} \right) + \delta = p^{-k_0} (x_0 - \eta_0)^{-\log s} \frac{\lambda(x)}{\lambda(x)} < \delta.$$

Then for any integer $n \in [s^k(x_0 - \eta_0), s^k(x_0 + \eta_0)]$,

$$b_n = \frac{a_n}{n^{\log s}} \leq \frac{a_n}{n^{\log s}} - \left| \frac{n}{x} \right| + \frac{\lambda(x)}{\lambda(x)} < \delta + \tilde{f} \leq f,$$

and

$$b_n = \frac{a_n}{n^{\log s}} \geq \left| \frac{n}{x} \right| - \frac{a_n}{n^{\log s}} - \frac{\lambda(x)}{\lambda(x)} > \tilde{e} - \delta \geq e.$$

Now let us give the proof of Theorem 2.2. show that $\{b_n\}_{n \geq 1}$ is not u. d. mod 1. Two situations are analyzed, calculated and discussed according to the position relation between $M$ and $m$.

Proof. Case I: $[M] = [m]$ or $[M] = [m]$.

Note that for any $n \geq 1$, $\{b_n\} = b_n - [m] \in [[m], [M]]$ and $[[m], [M]] \neq [0, 1]$, thus $\{b_n\}_{n \geq 1}$ is not u. d. mod 1.

Case II: $[M] \neq [m]$ and $[M] \neq [m]$.

Assume that the sequence $\{b_n\}_{n \geq 1}$ is u. d. mod 1. Then for any $\alpha \in (0, 1)$, one has

$$\lim_{N \to \infty} \frac{\#[1 \leq n \leq N : \{b_n\} \in [0, \alpha)]}{N} = \alpha.$$

Let $1_E$ be the characteristic function of the set $E$, and $||x||$ be the distance from the real number $x$ to the integers, that is the infimum of $|x - n|$ over all $n \in \mathbb{Z}$. Put

$$\gamma := \min \left\{ \frac{1}{2} 1_E(\mathbb{Z}) + ||m||, \frac{1}{2} 1_E(M) + ||M|| \right\}.$$

It is clear that $0 < \gamma \leq \frac{1}{2}$.

For any $\alpha \in (0, \gamma)$, one has $[[m], [m] + \alpha] \subset [M, M]$.

By Lemma 3.1, there exist $x_0 \in (s^{-1}, 1), \eta_0 \in (0, 1), k_0 \in \mathbb{Z}^+$ such that for any integer $n \in [s^k(x_0 - \eta_0), s^k(x_0 + \eta_0)]$ with $k \geq k_0$, one has $b_n \in ([m], [M] + \alpha)$, which implies that $\{b_n\} \in (0, \alpha)$.

Note that for any large enough $k$, the number of integers in interval $[s^k(x_0 - \eta_0), s^k(x_0 + \eta_0)]$ is

$$s^k(x_0 + \eta_0) - s^k(x_0 - \eta_0) = 2s^k \eta_0 + O(1).$$

Thus

$$\#[1 \leq n \leq s^k(x_0 + \eta_0) : \{b_n\} \in [0, \alpha)] = \#[1 \leq n \leq s^k(x_0 - \eta_0) : \{b_n\} \in [0, \alpha)] + 2s^k \eta_0 + O(1).$$

Dividing both sides by $s^k(x_0 + \eta_0)$, and letting $k \to \infty$, then gives that

$$\alpha = \frac{x_0 - \eta_0}{x_0 + \eta_0} \alpha + \frac{2\eta_0}{x_0 + \eta_0},$$

which implies $\alpha = 1$, contradicts with $0 < \alpha < \frac{1}{2}$. Therefore, $\{b_n\}_{n \geq 1}$ is not u. d. mod 1. 

□
Next, recall the definition of the cumulative distribution function in [3].

**Definition 4.6.** Let \( \{u_n\}_{n \geq 1} \) be a sequence of real numbers contained in an interval \( I \). Let \( \alpha \in I \) and let \( D(x, \alpha) \) denote the number of \( 1 \leq n \leq x \) for which \( u_n \leq \alpha \), i.e.

\[
D(x, \alpha) = \sum_{1 \leq n \leq x, u_n \leq \alpha} 1.
\]

If the limit

\[
\lim_{x \to \infty} \frac{D(x, \alpha)}{x} = D(\alpha)
\]

exists, then the sequence \( \{u_n\}_{n \geq 1} \) is said to have the distribution \( D(\alpha) \) at \( \alpha \). \( D(\alpha) \) is called the cumulative distribution function of \( \{u_n\}_{n \geq 1} \).

Let us give the proof of Theorem 1.4. Show that the cumulative distribution function of the sequence \( \{b_n\}_{n \geq 1} \) does not exist.

**Proof.** Assume that the cumulative distribution function of the sequence \( \{b_n\}_{n \geq 1} \) exists at some \( \alpha \in (m, M) \), denoted by

\[
\lim_{x \to \infty} \frac{D(x, \alpha)}{x} = D(\alpha).
\]

On the one hand, by Lemma 4.5, there exist \( x_1 \in [s^{-1}, 1) \), \( \eta_1 \in (0, 1) \), \( k_1 \in \mathbb{Z}^+ \) such that \( n \in [s^k(x_1 - \eta_1), s^k(x_1 + \eta_1)] \) implies that \( b_n \in (m, \alpha) \) for any \( k \geq k_1 \). Then for any large enough \( k \),

\[
D(s^k(x_1 + \eta_1), \alpha) = D(s^k(x_1 - \eta_1), \alpha) + 2s^k \eta_1 + O(1),
\]

and thus \( D(\alpha) = D(\alpha) + \frac{2s^k \eta_1}{s^{-1} + \eta_2} \), which implies \( D(\alpha) = 1 \).

On the other hand, there exist \( x_2 \in [s^{-1}, 1) \), \( \eta_2 \in (0, 1) \), \( k_2 \in \mathbb{Z}^+ \) such that \( n \in [s^k(x_2 - \eta_2), s^k(x_2 + \eta_2)] \) implies that \( b_n \in (\alpha, M) \) for any \( k \geq k_2 \). Then for any large enough \( k \),

\[
D(s^k(x_2 + \eta_2), \alpha) = D(s^k(x_2 - \eta_2), \alpha),
\]

and thus \( D(\alpha) = D(\alpha) + \frac{2s^k \eta_2}{s^{-1} + \eta_2} \), which implies \( D(\alpha) = 0 \).

This is the contradiction. And therefore \( \lim_{x \to \infty} \frac{D(x, \alpha)}{x} \) does not exist.

In order to show that the logarithmic distribution function of the sequence \( \{b_n\}_{n \geq 1} \) does exist, we need the properties of the level set of \( \lambda(x) \).

**Proposition 4.7.** For any \( \alpha \in [m, M] \), the set

\[
S_\alpha = \{ x > 0 : \lambda(x) = \alpha \}
\]

has measure zero.

**Proof.** By the self-similarity of \( \lambda \), we only need to consider the set of \( x \in [s^{-1}, 1) \) with \( \lambda(x) = \alpha \), which is still denoted by \( S_\alpha \). Let \( \mathcal{N} := \{ x : x \) is normal to base \( s \} \).

If \( S_\alpha \cap \mathcal{N} = \emptyset \), then \( S_\alpha \) has measure zero, since almost all real numbers (in the sense of Lebesgue measure) are absolutely normal [7]. The result is correct.

If not, for any \( x = [0.d_1d_2\ldots]_s \in S_\alpha \cap \mathcal{N} \), there are infinitely many positive integers \( n \)'s such that \( d_n \ldots d_{n+s-1} = 0^s \). That is to say, \( I := \{ n : d_n \ldots d_{n+s-1} = 0^s \} \) is an infinite set. For any \( n \in I \), take

\[
x_n = x + s^{-n} = [0.d_1 \ldots d_{n-1}10^{s-1}d_{n+s} \ldots]_s,
\]

13
\[ y_n = x + s^{-n} + s^{-(n+\tau-1)} = [0.d_1\ldots d_{n-1}10^{q-2}1d_{n+\tau} \ldots]. \]

Then for any \( x^* = [0.d_1^* d_2^* \ldots]_s \in (x_n, y_n) \), one has \( d_1^* \cdots d_{n+\tau-2}^* = d_1 \cdots d_{n-1}10^{q-2} \). Thus

\[
|\phi(x^*) - \phi(x)| = \left| \frac{h(1) - h(0)}{p^n} + \sum_{k=n+\tau-1}^{\infty} \frac{h(d_1^* - h(d_k^*))}{p^k} \right| \\
\leq \frac{h(1) - h(0)}{p^n} + \sum_{k=n+\tau-1}^{\infty} \frac{h(s - 1)}{p^k} \leq \frac{b - 1}{p^n} + \frac{1}{p^{n+\tau-2}} \leq 1.
\]

It follows from the above inequality that there exists \( N > 0 \) such that for any \( n > N \) with \( n \in I \),

\[
|\lambda(x^*) - \lambda(x)| = \left| \frac{\phi(x^*) - \phi(x)}{x^{\log s/p}} - \frac{\phi(x^*)}{x^{\log s/p}} \right| \\
\geq m \cdot \log s \cdot \frac{x^* - x}{x} - \frac{1}{p^{n+\tau-1}} \cdot \frac{1}{x^{\log s/p}} \geq ms^n - p^{n+\tau} > 0,
\]

since \( x^* - x > s^{-n} \), \( \log s > 1 \), and \( x \in [s^{-1}, 1) \). That is to say, for sufficiently large \( n \in I \) and any \( x^* \in (x_n, y_n) \), one has \( \lambda(x^*) \neq \alpha \). Therefore, when \( n \in I \) large enough,

\[
\frac{1}{s^{-n} + s^{-(n+\tau-1)}} \mathcal{L}(S_\alpha \cap (x, y_n)) = \frac{1}{s^{-n} + s^{-(n+\tau-1)}} \mathcal{L}(S_\alpha \cap (x, x_n)) \leq \frac{s^{-n}}{s^{-n} + s^{-(n+\tau-1)}} = \frac{1}{1 + s^{1-\tau}} < 1,
\]

where \( \mathcal{L} \) denote the Lebesgue measure. But at the same time,

\[
\frac{1}{h} \mathcal{L}(S_\alpha \cap (x, x + h)) = \frac{1}{h} \int_x^{x+h} \mathbb{1}_{S_\alpha}(t)dt \to \mathbb{1}_{S_\alpha}(x),
\]

as \( h \to 0 \) for almost all real number \( x \), where \( \mathbb{1}_E(x) = 1 \) if and only if \( x \in E \). Combining the above two conclusions, one has that the set \( S_\alpha \cap N \) has measure zero. Therefore

\[
L(S_\alpha) = L(S_\alpha \cap N) + L(S_\alpha \cap N^c) = 0.
\]

\[\square\]

**Lemma 4.8.** For any \( \alpha \in [m, M] \), let \( E_\alpha = \{ x \in [s^{-1}, 1) : \lambda(x) \leq \alpha \} \), then \( \frac{1_{E_\alpha}(x)}{x} \) is Riemann integral.

**Proof.** Since the conclusion is equivalent to that the points in \([s^{-1}, 1)\) at which \( \frac{1_{E_\alpha}(x)}{x} \) fails to be continuous has measure zero. It is sufficient to show that the set

\[
\{ x \in [s^{-1}, 1) : 1_{E_\alpha}(x) \text{ is not continuous at } x \}
\]

has measure zero.

Let \( x \in (s^{-1}, 1) \) be an \( s \)-ary irrational number with \( \lambda(x) \neq \alpha \). By Proposition 4.3, \( \lambda \) is continuous at \( x \). If \( \lambda(x) > \alpha \), there exists \( \delta_1 > 0 \), such that for any \( y \in (x - \delta_1, x + \delta_1) \), \( \lambda(y) > \alpha \), which implies \( 1_{E_\alpha}(y) = 1_{E_\alpha}(x) = 0 \). Similarly, if \( \lambda(x) < \alpha \), there exists \( \delta_2 > 0 \), such that for any \( y \in (x - \delta_2, x + \delta_2) \), \( 1_{E_\alpha}(y) = 1_{E_\alpha}(x) = 1 \). Thus, \( 1_{E_\alpha} \) is continuous at \( x \). Therefore,

\[
\{ x \in [s^{-1}, 1) : x \text{ is } s \text{-ary irrational number with } \lambda(x) \neq \alpha \} \subset \{ x \in [s^{-1}, 1) : 1_{E_\alpha}(x) \text{ is continuous at } x \}. \]

The result is true by Proposition 4.7. \[\square\]

Now, recall the definition of the logarithmic distribution.
Definition 4.9. Let \( \{u_n\}_{n \geq 1} \) be a real sequence contained in an interval \( I \). Let \( \alpha \in I \) and let

\[
L(x, \alpha) = \sum_{1 \leq n \leq x, u_n \leq \alpha} \frac{1}{n}.
\]

If the limit

\[
\lim_{x \to \infty} \frac{1}{\ln x} L(x, \alpha) = L(\alpha)
\]

exists, then the sequence \( \{u_n\}_{n \geq 1} \) is said to have the logarithmic distribution \( L(\alpha) \) at \( \alpha \). \( L(\alpha) \) is called the logarithmic distribution function of the sequence \( \{u_n\}_{n \geq 1} \).

And complete the proof of the Theorem 1.3 show that the logarithmic distribution function does exist for the sequence \( \{b_n\}_{n \geq 1} \).

Proof. For any positive integer \( k \), let \( I_k := \{n \in \mathbb{Z}^+ : s^{k-1} \leq n < s^k\} \). By Proposition 4.2 for any \( n \in I_k \),

\[
|\lambda(n) - b_n| = |\lambda(n \cdot s^{-k}) - b_n| \leq p^{-k}(n \cdot s^{-k})^{-\log p} \leq p^{-(k-1)}.
\]

Thus

\[
\{n \in I_k : \lambda(n) \leq \alpha - p^{-(k-1)}\} \subseteq \{n \in I_k : b_n \leq \alpha\} \subseteq \{n \in I_k : \lambda(n) \leq \alpha + p^{-(k-1)}\},
\]

which implies that

\[
\sigma_k^*(\alpha - p^{-(k-1)}) \leq \sigma_k^*(\alpha) \leq \sigma_k^*(\alpha + p^{-(k-1)}),
\]

where

\[
\sigma_k^*(\alpha) := \sum_{n \in I_k, b_n \leq \alpha} \frac{1}{n}, \quad \text{and} \quad \sigma_k^*(\alpha) := \sum_{n \in I_k, \lambda(n) \leq \alpha} \frac{1}{n}.
\]

We can rewrite \( \sigma_k^*(\alpha) \) as

\[
\sigma_k^*(\alpha) = \sum_{n \in I_k, (n \cdot s^{-k}) \leq \alpha} \frac{1}{n} = \sum_{n = s^{k-1}}^{s^k-1} \sum_{m = 1}^{\frac{E_m(\frac{\alpha}{s})}{s}} \frac{1}{n}.
\]

By Lemma 4.8 \( \frac{E_m(x)}{s} \) is Riemann integral, we have

\[
\lim_{k \to \infty} \sigma_k^*(\alpha) = \int_{s^{k-1}}^{s^k} \frac{1}{E_m(x)} \frac{1}{x} dx = \int_{E_m} \frac{1}{x} dx =: h(\alpha).
\]

Note that

\[
\bigcap_{m \geq 1} E_{\alpha + \frac{1}{m}} = E_\alpha, \quad \bigcup_{m \geq 1} E_{\alpha - \frac{1}{m}} = E_\alpha - S_\alpha.
\]

From the continuity of the integration, and Proposition 4.7, we can know that

\[
\lim_{m \to \infty} \int_{E_{\alpha + \frac{1}{m}}} \frac{1}{x} dx = \int_{E_\alpha} \frac{1}{x} dx = \int_{\bigcup_{m \geq 1} E_{\alpha - \frac{1}{m}}} \frac{1}{x} dx = \lim_{m \to \infty} \int_{E_{\alpha - \frac{1}{m}}} \frac{1}{x} dx.
\]

Therefore \( h(\alpha) \) is continuous, since \( h(\alpha) \) is increasing with respect to \( \alpha \).

For any fixed positive integer \( k_0 \), and any integer \( k > k_0 \),

\[
\sigma_k^*(\alpha - p^{-(k_0-1)}) \leq \sigma_k^*(\alpha - p^{-(k-1)}) \leq \sigma_k^*(\alpha + p^{-(k-1)}) \leq \sigma_k^*(\alpha + p^{-(k_0-1)}).
\]
First, let \( k \to \infty \), we can get
\[
\begin{align*}
    & h(\alpha - p^{-(k_0-1)}) \leq \lim_{k \to \infty} \inf \sigma_k^*(\alpha - p^{-(k-1)}) \leq \lim_{k \to \infty} \sup \sigma_k^*(\alpha - p^{-(k-1)}) \leq h(\alpha + p^{-(k_0-1)}) , \\
    & h(\alpha - p^{-(k_0-1)}) \leq \lim_{k \to \infty} \inf \sigma_k^*(\alpha + p^{-(k-1)}) \leq \lim_{k \to \infty} \sup \sigma_k^*(\alpha + p^{-(k-1)}) \leq h(\alpha + p^{-(k_0-1)}) .
\end{align*}
\]

And then let \( k_0 \to \infty \), by the continuity of \( h(\alpha) \), we have
\[
\lim_{k \to \infty} \sigma_k^*(\alpha - p^{-(k-1)}) = \lim_{k \to \infty} \sigma_k^*(\alpha + p^{-(k-1)}) = h(\alpha) .
\]

Therefore,
\[
\lim_{k \to \infty} \sigma_k(\alpha) = h(\alpha) ,
\]

which can in turn
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} \sigma_k(\alpha) = h(\alpha) .
\]

Then for any \( x > 0 \), choose integer \( m \) such that \( s^{-m-1} \leq x < s^{-m} \),
\[
\lim_{x \to \infty} \frac{1}{\ln x} \sum_{1 \leq \ell \leq x, \sigma(\alpha) \leq \alpha} \frac{1}{n} = \lim_{m \to \infty} \frac{1}{m \ln s} \sum_{k=1}^{m} \sigma_k(\alpha) = \frac{1}{\ln s} h(\alpha) .
\]

Thus, the logarithmic distribution function of the sequence \( \{b_n\}_{n \geq 1} \) exists. \( \square \)

5. The supremum and infimum of \( \{b_n\}_{n \geq 1} \) corresponding to linear Cantor integers

We begin the proof of Theorem 1.6 with the fact \( a_n \geq \left( q + \frac{r}{r-1} \right) n \) and some properties of \( b_n \).

**Proposition 5.1.** For any non-negative integer \( k \) and any integer \( n = [\varepsilon_k \varepsilon_{k-1} \ldots \varepsilon_0]_s \in [s^k, s^{k+1}) \),
\[
b_n = \tilde{b}_n + r \cdot \frac{p^{k+1} - 1}{p-1} \cdot n^{-\log_p p} , \tag{5.1}
\]

where \( \tilde{b}_n = \frac{\tilde{a}_n}{n^{\log_p p}} \), and \( \tilde{a}_n := [(qe_k)(qe_{k-1}) \ldots (qe_0)]_p \).

**Proof.**
\[
b_n = \frac{a_n}{n^{\log_p p}} = \frac{\sum_{i=0}^{k} (qe_i + r)p^i}{n^{\log_p p}} = \frac{\tilde{a}_n + r \frac{p^{k+1} - 1}{p-1}}{n^{\log_p p}} = \tilde{b}_n + r \cdot \frac{p^{k+1} - 1}{p-1} \cdot n^{-\log_p p} .
\]

\( \square \)

**Proposition 5.2.** Given a positive integer \( k \). For any non-negative integer \( \ell < k \) and any \( \varepsilon_k \ldots \varepsilon_{\ell+1} \in \{0, 1, \ldots, s-1\}^{k-\ell} \) with \( \varepsilon_k \neq 0 \), we have that \( b_{[\varepsilon_k \ldots \varepsilon_{\ell+1}]_{s^{(s-1)}\ell}} \) decreases with the increasing of \( \varepsilon_\ell \). That is to say,
\[
b_{[\varepsilon_k \ldots \varepsilon_{\ell+1}]_{s^{(s-1)}\ell}} < b_{[\varepsilon_k \ldots \varepsilon_{\ell+1}]_{s^{(s-1)}\ell-1}} < \cdots < b_{[\varepsilon_k \ldots \varepsilon_{\ell+1}]_{s^{(s-1)}1}} < b_{[\varepsilon_k \ldots \varepsilon_{\ell+1}]_{s^{(s-1)}0}} , \tag{5.2}
\]

Especially, for any integer \( n \geq 1 \),
\[
b_{sn+s-1} < b_{sn+s-2} < \cdots < b_{sn+1} < b_n \leq b_{sn} .
\]

And \( b_{sn} = b_n \) if and only if \( r = 0 \).
Proof. By Proposition 5.1, we only need to show that the inequality (5.2) holds for \( \tilde{b}_n \). That is to say

\[
\tilde{b}_{[x_k...x_{k+1}(s-1)]'} < \tilde{b}_{[x_k...x_{k+1}(s-2)]'} < \cdots < \tilde{b}_{[x_k...x_{k+1}(s-1)]'} < \tilde{b}_{[x_k...x_{k+1}(s)]'}.
\]

(5.3)

Fix 0 \( \leq \ell < k \), write \( n^* = [\varepsilon_k \varepsilon_{k-1} \cdots \varepsilon_{\ell+1}] \). Then \( n^* \geq 1 \) and

\[
\tilde{b}_{[x_k...x_{k+1}(s-1)]'} = \frac{p^{\ell+1}a_{n^*} + qe_{\ell}p^{\ell} + q(s-1)(p^{\ell-1})}{(s^{\ell} + \varepsilon_{\ell}s^{\ell-1} + \ldots - 1)^{\log s} p^{\ell}}.
\]

For \( x \geq 0 \), let

\[
f(x) = \frac{p^{\ell+1}a_{n^*} + qe_{\ell}p^{\ell} + q(s-1)(p^{\ell-1})}{(s^{\ell} + \varepsilon_{\ell}s^{\ell-1} + \ldots - 1)^{\log s} p^{\ell}}.
\]

Now it suffices to show \( f'(x) < 0 \) for any \( x > 0 \).

Through calculation and analysis, one has

\[
\text{sgn}(f'(x)) = \text{sgn} \left( q \cdot p^{\ell} - \log_s p \cdot \frac{p^{\ell+1}a_{n^*} + qe_{\ell}p^{\ell} + q(s-1)(p^{\ell-1})}{sn^* + x + 1 - s^{-\ell}} \right).
\]

Since

\[
\log_s p > 1, \quad \frac{p^{\ell+1}a_{n^*} + qe_{\ell}p^{\ell} + q(s-1)(p^{\ell-1})}{sn^* + x + 1 - s^{-\ell}} > \frac{p^{\ell+1}qn^* + qe_{\ell}p^{\ell}}{sn^* + x + 1},
\]

and

\[
sn^* + x + 1 - s^{-\ell} < sn^* + x + 1,
\]

implies that

\[
q \cdot p^{\ell} - \log_s p \cdot \frac{p^{\ell+1}a_{n^*} + qe_{\ell}p^{\ell} + q(s-1)(p^{\ell-1})}{sn^* + x + 1 - s^{-\ell}} < q \cdot p^{\ell} - \frac{p^{\ell+1}qn^* + qe_{\ell}p^{\ell}}{sn^* + x + 1} = -qe_{\ell}((p-s)n^* - 1) < 0.
\]

Thus, \( f'(x) < 0 \) for any \( x > 0 \). Therefore, the inequality (5.3) holds.

Take \( \ell = 0 \) in the inequality (5.2), we have for any integer \( n \geq 1 \),

\[
b_{sn+s-1} < b_{sn+s-2} < \cdots < b_{sn+1} < b_{sn}.
\]

At this time, the inequality

\[
b_{sn+1} = \frac{a_{sn+1}}{(sn+1)^{\log_s p}} = \frac{pa_n + q + r}{pm^{\log_s p}(1+(sn-1)^{\log_s p})} < \frac{a_n}{n^{\log_s p}} = b_n
\]

holds if and only if \( pa_n((1+(sn-1)^{\log_s p}) - 1) > q + r \). Note that

\[
(1+(sn-1)^{\log_s p}) - 1 > \log_s p \cdot (sn-1) > (sn)^{-1}, \quad p > q(s-1), \quad a_n \geq \left( q + \frac{r}{s-1} \right)n,
\]

since \( \log_s p > 1 \). It is sufficient to show that \( q^2(s-1) + qr - (q+r)s \geq 0 \), which follows from

\[
q^2(s-1) + qr - (q+r)s = (q^2 - q - r)s - q^2 + qr \geq 2q^2 - q^2 - r - q^2 + qr = (q-2)(q+r) \geq 0.
\]

By combining this with Proposition 5.2 one has the results. \( \square \)

It is important to notice that \( b_{[x_k(s-1)]''} \) is not necessarily monotonous with respect to \( \varepsilon_k \). But it also get the minimal value at \( \varepsilon_k = s-1 \).
Proposition 5.3. For any $k \geq 0$, one has
\[
\min \left\{ b_{(\varepsilon_k(s-1)^2)} : \varepsilon_k \in \{1, \ldots, s-1\} \right\} = b_{(s-1)(s-1)^2}.
\]  
(5.4)

Especially, $b_{s-1} \leq b_{s-2} \leq \cdots \leq b_1$.

Proof. By Proposition 5.1, we only need to show that the inequality (5.4) holds for $\tilde{b}_n$. That is to say
\[
\tilde{b}_{(\varepsilon_k(s-1)^2)} \geq \tilde{b}_{(s-1)(s-1)^2}, \text{ for any } \varepsilon_k \in \{1, \ldots, s-1\}.
\]
Since
\[
\tilde{b}_{(\varepsilon_k(s-1)^2)} = \frac{q\varepsilon_k p^k + \frac{q(s-1)(p^k-1)}{p-1}}{(s^k + s^k - 1)\log_p, p}.
\]
We consider the function
\[
f(x) := \frac{q p^k x + \frac{q(s-1)(p^k-1)}{p-1}}{(s^k x + s^k - 1)\log_p, p},
\]
which is defined for any $x \geq 1$. Through simple calculation, we have
\[
\sgn(f'(x)) = \sgn\left(p^k(s^k x + s^k - 1) - \log_p, p \cdot s^k \left(\frac{p^k x + (s-1)(p^k-1)}{p-1}\right)\right).
\]
Note that
\[
p^k(s^k x + s^k - 1) - \log_p, p \cdot s^k \left(\frac{p^k x + (s-1)(p^k-1)}{p-1}\right)
\]
is a linear function with respect to $x$, and the coefficient of $x$ is $-p^k s^k (\log_p, p - 1)$, which is less than zero. Thus,
\[
\min\{f(x) : 1 \leq x \leq s-1\} = \min\{f(1), f(s-1)\}.
\]
Now, we only need to show that for any $s \geq 3$, $f(1) \geq f(s-1)$. Namely
\[
\frac{q p^k + \frac{q(s-1)(p^k-1)}{p-1}}{(2s^k - 1)\log_p, p} \geq \frac{q p^k (s-1) + \frac{q(s-1)(p^{k-1})}{p-1}}{(s^{k+1} - 1)\log_p, p}.
\]  
(5.5)

Let $u := 2s^k - 1$, $v := q \cdot p^k + \frac{q(s-1)(p^{k-1})}{p-1}$. The right part of the inequality (5.5) is converted to
\[
\frac{v}{(\frac{u}{2} + \frac{2s^k}{2})\log_p, p} \leq \frac{v}{(\frac{u}{2} + \frac{2s^k}{2})\log_p, p} = \frac{v}{(\frac{(s-1)p^{k+1} - (s-1)}{(p+s-2)p^k - (s-1)}\log_p, p} = \frac{v}{2 - \log_p, p \cdot p \cdot u\log_p, p}.
\]
We just need to show
\[
\frac{p}{2\log_p, p} \geq \frac{(s-1)p^{k+1} - (s-1)}{(p + s - 2)p^k - (s-1)},
\]
which is equivalent of
\[
(p + s - 2 - 2\log_p, p(s-1))p^{k+1} - (p - 2\log_p, p)(s-1) \geq 0.
\]  
(5.6)

Note that
\[
2\log_p, p = p^2 \log^2, s = s^2 \log^2, (1 + \frac{p - s}{s}) \log^2, s < 2(1 + \log, s) 2 \cdot \frac{p - s}{s} < 2(1 + \frac{3}{7} \cdot \frac{p - s}{s - 1}),
\]  
(5.7)
since \( \log_s 2 < \frac{1}{4} \cdot \frac{3}{s-1} < 1 \) when \( s \geq 3 \). We can obtain that
\[
p + s - 2 - 2 \log_s p(s - 1) > p + s - 2(1 + \frac{3}{7} \cdot \frac{p - s}{s - 1})(s - 1) = \frac{1}{7}(p - s) > 0.
\]
Thus
\[
(p + s - 2 - 2 \log_s p(s - 1))p^{k+1} \geq (p + s - 2 - 2 \log_s p(s - 1))p^2,
\]
and this causes the left part of the inequality (5.6) is not less than
\[
(p + s - 2)p^2 - 2 \log_s p(s - 1)(p^2 - 1) - p(s - 1).
\]
By inequality (5.7), it can be further decreased to
\[
(p + s - 2)p^2 - 2(1 + \frac{3}{7} \cdot \frac{p - s}{s - 1})(s - 1)(p^2 - 1) - p(s - 1) = \frac{1}{7}(p - s)p^2 + \frac{6}{7}(p - s) - (p - 2)(s - 1),
\]
and then is not less than
\[
\frac{1}{7}(q - 1)p^2 + 6(q - 1) - 7p + 14)(s - 1) \geq \frac{1}{7}(p^2 - 7p + 20)(s - 1) > 0,
\]
for any \( q \geq 2 \), since \( p - s \geq q(s - 1) + 1 - s = (q - 1)(s - 1) \). That is to say, the inequality (5.6) holds. \( \square \)

Corollary 5.4. For any non-negative integer \( k \), and any integer \( n \in [s^k, s^{k+1}) \), \( b_{j+1} \leq b_n \leq b_j \).

Proof. For any integer \( n \in [s^k, s^{k+1}) \), suppose \( n = [e_1 e_2 \ldots e_i] \).

At first, by Proposition 5.2 and Proposition 5.3, the value of \( b_n \) decreases if we replace the last digit which does not equal to \( s - 1 \) in the \( s \)-ary expansion of \( n \) with \( s - 1 \) from the heading of the expansion. Repeating this process, we can obtain,
\[
b_n = b_{[e_i e_{i-1} \ldots e_{i-1}]} \geq b_{[e_i e_{i-1} \ldots e_(s-1)]} \geq b_{[e_i e_{i-1} \ldots (s-1)]} \geq \cdots \geq b_{([s-1]^{i+1}]_s} = b_{j+1}.
\]

Secondly, by inequality (5.3) in the proof of Proposition 5.2 and the definition of \( \tilde{b}_n \), we have
\[
\tilde{b}_{s^{m+i}} \leq \tilde{b}_m = \tilde{b}_{s^m} \text{ for any integer } m \geq 1 \text{ and } i \in \{0, 1, \ldots, s - 1\}.
\]
Thus
\[
\tilde{b}_n = \tilde{b}_{[e_i e_{i-1} \ldots e_{i-1}]} \leq \tilde{b}_{[e_i e_{i-1} \ldots e_{1}]} \leq \cdots \leq \tilde{b}_{[1]} = \tilde{b}_{[1]} = \cdots = \tilde{b}_{[10]} = \tilde{b}_j.
\]

Combine this with Proposition 5.1 we have \( b_n \leq b_j \). \( \square \)

Now, let complete the proof of Theorem 1.6.

Using Corollary 5.4, it is sufficient to show that for any non-negative integer \( k \),
\[
\frac{q(s - 1) + r}{p - 1} \leq b_{j+1} \quad \text{and} \quad b_j \leq \frac{q(p - 1) + pr}{p - 1}.
\]
Since
\[
b_j = \frac{a_j}{(s^k) \log_s p} = \frac{(q + r)p^k + \frac{r(p-1)}{p-1}}{p^k} = \frac{q(p - 1) + pr}{p - 1} - \frac{r}{(p - 1)p^k},
\]
which monotonously increases with the increase of \( k \), one has
\[
b_j \leq \lim_{k \to \infty} b_j = \frac{q(p - 1) + pr}{p - 1}.
\]
Similarly,

\[
b_{s^{k+1}-1} = \frac{a_{s^{k+1}-1}}{(s^{k+1} - 1) \log_s p} = \frac{(q(s-1)+r)p^{k+1}-1}{(s^{k+1} - 1) \log_s p} = \frac{q(s-1) + r}{p - 1} \cdot \frac{p^{k+1} - 1}{(s^{k+1} - 1) \log_s p},
\]

and \(b_{s^{k+1}-1} = b_{(s-1)^{k+1}}\) is decreasing with respect to \(k\), which follows from Proposition 5.2. Then

\[
b_{s^{k+1}-1} \geq \lim_{k \to \infty} b_{s^{k+1}-1} = \frac{q(s-1) + r}{p - 1}.
\]

Acknowledgments. The work is supported by the Fundamental Research Funds for the Central University (Grant Nos. 2662020LXPY010).

References

[1] C. J. Bishop and Y. Peres, Fractals in Probability and Analysis, Cambridge studies in advanced mathematics, vol. 162, Cambridge: Cambridge University Press, (2017).

[2] J. Brillhart and P. Morton, Über Summen von Rudin-Shapiro-koefizienten, Illinois J. Math., 22 (1978), 126-148.

[3] J. Brillhart, P. Erdős and P. Morton, On sums of Rudin-Shapiro coefficients II, Pacific J. Math. 107 (1) (1983) 39-69.

[4] C.-Y. Cao, H.-H. Li, Asymptotic behaviour of a class of Cantor-integers, Mathematica Applicata, 36(1), (2023), 258-264.

[5] K. Dajani and C. Kalle, A first Course in Ergodic Theory, CRC press, Boca Raton, FL, 2021.

[6] M. Gawron and M. Ulas, On the formal inverse of the Prouhet-Thue-Morse sequence, Discrete Math. 339 (2016), 1459-1470.

[7] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequence, Wiley-interscience Publication, John Wiley & Sons, Inc. 1974.

[8] X.-T. Lü, J. Chen, Z.-X. Wen and W. Wu, Limit behavior of the quasi-linear discrete functions, Fractals, 28(3), (2020) 2050041.

[9] H. Weyl, Über ein Problem aus dem Debiete der diophantischen Approximationen, Nache. Ges. Will. Göttingen, Math.-phys. Kl., (1914), 234-244.

[10] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77, , Nache. Ges. Will. Göttingen, Math.-phys. Kl., (1916), 315-352.