Saddle solutions for the fractional Choquard equation

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Abstract. We study the saddle solutions for the fractional Choquard equation
\[
(\Delta)^s u + u = (K_\alpha * |u|^p)|u|^{p-2} u, \quad x \in \mathbb{R}^N
\]
where \(s \in (0, 1), N \geq 3\) and \(K_\alpha\) is the Riesz potential with order \(\alpha \in (0, N)\). For every finite Coxeter group \(G\) with rank \(1 \leq k \leq N\) and \(p \in \left[2, \frac{N+\alpha}{N-2}\right]\), we construct a \(G\)-saddle solution with prescribed symmetric nodal configurations. This is a counterpart for the fractional Choquard equation of saddle solutions to the Choquard equation and further completes the existence of non-radial sign-changing solutions for this doubly nonlocal problem.

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1. Introduction and main results

We consider the Choquard equation involving a fractional Laplacian:
\[
(\Delta)^s u + u = (K_\alpha * |u|^p)|u|^{p-2} u, \quad x \in \mathbb{R}^N, \quad (1.1)
\]
where \(N \geq 3\), \(K_\alpha : \mathbb{R}^N \to \mathbb{R}\) with \(\alpha \in (0, N)\) is the Riesz potential, defined by
\[
K_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{with} \quad A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^{\alpha}}.
\]
Here, the fractional Laplacian \((\Delta)^s\) with \(s \in (0, 1)\) is defined by
\[
\mathcal{F}((\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi)
\]
with \(\mathcal{F}\) denoting the Fourier transform. When \(u\) is sufficiently smooth, the fractional Laplacian can also be expressed by
\[
(\Delta)^s u(x) = C_{N,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,
\]
with \(C_{N,s} > 0\) being a normalization constant. In the last few years, the fractional Laplacian arises in the description of anomalous diffusion [28] and is treated as the infinitesimal generators of Lévy stable diffusion processes [19], and since then various fractional equations were derived in distinct fields: game theory [3], minimal surfaces [4] and finance [8], to name a few. In the remarkable seminal work [5] of Caffarelli and Silvestre, the \(s\)-harmonic extension technique was introduced which makes it possible to transform the nonlocal problem into a local one via the Dirichlet–Neumann map. When \(s = 1\), equation (1.1) is reduced to the Choquard equation or the Choquard–Pekar equation,
\[
-\Delta u + u = (K_\alpha * |u|^p)|u|^{p-2} u, \quad x \in \mathbb{R}^N. \quad (1.2)
\]
This type of nonlocal equation was adopted in many physical contexts such as the Hartree–Fock theory of one-component plasma [20], the quantum mechanics of a polaron at rest [32], the self-gravitating matter model [33] and so on.

The study of the Choquard equation (1.2) has been going on for many years. The existence and uniqueness of the groundstate solution were first considered by Lieb in 1976 [20] for the Choquard–Pekar equation (1.2) with $N = 3$, $p = 2$ and $\alpha = 2$. Lions [22] later showed the existence of infinitely many radially solutions to the same model. Under some assumptions on $N$, $\alpha$ and $p$, Ma and Zhao [26] proved that every positive solution of (1.2) is radially symmetric and monotone decreasing about some point. The restrictions in [26] were ultimately eliminated by Moroz and Van Schaftingen in [29] and some qualitative properties of the groundstate were also investigated including positivity, radial symmetry and decay behaviors. Since the Choquard equation (1.2) has been widely investigated by employing variational methods, we limit ourselves to citing a few references [30,31,36] and refer to their bibliographies for a broader list.

As for the fractional Choquard equation (1.1), it turns out to be a doubly nonlocal problem due to the appearance of the convolution and the fractional Laplacian. When $s = \frac{1}{2}$, Frank and Lenzmann [13] proved the radial symmetry, uniqueness and positivity of the $L^2$-critical groundstate solution in the model of dynamics of pseudo-relativistic boson stars. For the general setting, d’Avenia, Siciliano and Squassina [10] showed the existence of positive groundstate solutions if the exponent $p$ satisfies

$$\frac{N + \alpha}{N} < p < \frac{N + \alpha}{N - 2s}.$$  \hspace{1cm} (1.3)

In addition, the radial symmetry, Morse index and asymptotic decays for the groundstates were investigated as well, see also [1]. It is also shown in [10] that the above range of $p$ is optimal for the existence of finite energy solutions and the endpoints of the above interval are critical exponent with respect to the Hardy–Littlewood–Sobolev inequality [21]. Shen, Gao and Yang [34] obtained the groundstates for the fractional Choquard equation (1.1) with general nonlinearities in spirit of Berestycki–Lions. Some other results were also established by using variational methods on the existence and multiple solutions to the fractional Choquard equation (1.1), we refer to [6,7,17,24,25,27] and the references therein.

In recent years, the study of the sign-changing solutions to the Choquard equation (1.2) has received a lot of attention. As we all know, the compact embedding plays an important role in searching the entire nodal solutions for a class of variational nonlinear equations. As a result, the existence of sign-changing solutions of equation (1.2) has been established in [16,18,36] under compact settings, and we refer to [43] for the existence of infinitely many radially sign-changing solutions to the fractional Choquard equation (1.1). Recently, Ghimenti and Van Schaftingen constructed in [15] a surprising odd solution with exactly two half-space nodal domains. Such an odd solution demonstrates different features of the Choquard equation because its nodal set consists of hyperplane, whereas it is impossible to admit this type of solution for the nonlinear Schrödinger equation $-\Delta u + u = |u|^{p-2}u$. As a consequence, by employing a minimax procedure on the Nehari nodal set, Ghimenti and Van Schaftingen [15] showed the existence of the minimal nodal solution of the Choquard equation (1.2) for $p \in (2, \frac{N + \alpha}{N - 2s})$, see [14] for the quadratic case. This result is quite surprising since the energy of the minimal nodal solution is strict less than twice the groundstate energy, while nodal solutions for the nonlinear Schrödinger equation obey the rule of doubling energy, see e.g., [38]. Moreover, this equation presents some degeneracy for the case $p < 2$ which turns to be that the minimizer on the Nehari nodal set is still the positive groundstate. Wang and Xia (the corresponding author of the current paper) [37,40] successively constructed some saddle type nodal solutions whose nodal domains are of conical shapes demonstrating symmetric configurations by making use of the dihedral groups and the polyhedral symmetric groups, respectively. As a consequence, more general saddle solutions with prescribed Coxeter symmetries were constructed by a unified approach in [41,42] for the critical and subcritical Choquard equations, respectively.
Motivated by the above-mentioned progresses on the nodal solutions of the Choquard equation (1.2), we are led to a natural question: whether the saddle solutions for \( s = 1 \) can still exist for the full range \( s \in (0, 1) \). In this aspect, some attempts have been made in [9] by the lead author of the current paper and the associated odd solution and the minimal nodal solution were obtained. It also turns out that the least energy on the corresponding Nehari nodal set is still the groundstate level for the case \( p < 2 \). In the present paper, we will give an affirmative answer and construct saddle solutions for any prescribed Coxeter symmetric nodal structure.

To state our results, we first introduce some notations. Let \( G = \langle S \rangle \) be a finite Coxeter group with \( S \) being its generating set and the cardinality \( |S| = k \in \{1, 2, \ldots, N\} \). Since \( s^2 = 1 \) for any \( s \in S \), there exists a unique epimorphism \( \phi : G \to \{\pm 1\} \) induced by \( \phi(s) = -1 \) for each \( s \in S \), see Lemma 2.4. A function \( u : \mathbb{R}^N \to \mathbb{R} \) is said to be \( G \)-symmetry if it satisfies
\[
g \circ u(x) = \phi(g)u(x), \quad \text{for } g \in G, \quad x \in \mathbb{R}^N.
\]
Here, \( \circ \) denotes the group action that will be explained in Sect. 2. In what follows, we say \( u \in H^s(\mathbb{R}^N) \setminus \{0\} \) is a \( G \)-saddle solution if \( u \) solves the Choquard equation (1.1) with \( G \)-symmetry and \( u \) is called \( G \)-groundstate if in addition \( u \) minimizes the energy functional amongst all the \( G \)-saddle solutions.

Our main result can be stated as follows.

**Theorem 1.1.** Assume that \( s \in (0, 1) \), \( N \geq 3 \), \( \alpha \in (0, N) \) and \( 2 \leq p < \frac{N + \alpha}{N - 2s} \). For any finite Coxeter group \( G \) with its rank \( k \) satisfying \( 1 \leq k \leq N \), the fractional Choquard equation (1.1) permits a \( G \)-saddle solution \( u_G \in H^s(\mathbb{R}^N) \) with exactly \( |G| \) nodal domains which turns out to be a \( G \)-groundstate.

Although this seems to be a predictable result compared with the saddle solutions for the Choquard equation (see e.g., [37, 40, 42]), it is worthwhile to remark that there are still some difficulties for the fractional Choquard equation and some new ideas are needed. On one hand, due to the appearance of the fractional Laplacian, solutions for the fractional Choquard equation (1.1) cannot have exponential decay anymore even for the case of \( p \geq 2 \). In fact, its solutions turn out to be of polynomial decay at infinity. This leads to a refine analysis in establishing the strict energy inequalities which shall play a significant role in restoring the compactness, see Proposition 3.1; on the other hand, the signed property of the \( G \)-groundstate on the fundamental domain of \( G \) cannot be deduced directly by similar arguments as in [9, 29, 40]. Actually, it will be proved by applying the strong maximum principle for the fractional Laplacian, see e.g., [2, Corollary 4.12]. To this end, we shall make use of the extension method for fractional Laplacian established by Caffarelli and Silvestre [5], through which instead of the nonlocal operator we can study a local elliptic equation with a Neumann boundary condition in one dimension higher. As a result, our method takes advantage of finite Coxeter groups used in [41, 42] and adapts it to the fractional Laplacian operator. Some ideas of [9] are borrowed and our proofs turn more transparent.

This paper is organized as follows. Section 2 is devoted to introducing some preliminaries on the Coxeter groups and the variational framework space. The proofs of Theorem 1.1 will be completed in Sects. 3 and 4. More precisely, the \( G \)-groundstate will be constructed inductively in Sect. 3 by employing the Lions’ concentration-compactness principle. In the final section, we show the signed property of the \( G \)-groundstate by using the \( s \)-harmonic extension method.

### 2. Coxeter groups and variational framework

#### 2.1. Finite Coxeter groups

In this section, we shall collect some fundamental results about Coxeter groups for the readers’ convenience although it has been done in [41, Section 2]. We refer to [11, 35] for further information. We first recall some notations and definitions. Let \( H \) and \( N \) be the subgroups of the group \( G \). We write \( G \) as \( G = N \rtimes H \),
Lemma 2.2. Assume that the group $G$ acts on $\mathbb{R}^k$ by homeomorphisms. The orbit of $x \in \mathbb{R}^k$ defined by $O_x = \{gx | g \in G\}$. The isotropy subgroup of $x$ is $S_x = \{g \in G | gx = x\}$.

For any set $M$, we denote $|M|$ the cardinality of $M$. The following Lagrange’s theorem is well known.

Lemma 2.2. Assume that the group $G$ acts on $\mathbb{R}^k$. Then, for any $x \in \mathbb{R}^k$, it holds

$$|G| = |O_x||S_x|.$$ 

We now introduce the Coxeter symmetry. We begin with the definition of reflection. Let $V$ be a vector space. A linear reflection on $V$ is a linear automorphism $r : V \to V$ with $r^2 = 1$. And the wall $H_r$ is the set of midpoints of edges flipped by $r$. We then recall a formal definition of the Coxeter group.

Definition 2.3. Let $I$ be a finite indexing set with cardinal number $|I| = k$ and let $S = \{s_i\}_{i \in I}$. Let $M = (M_{ij})_{i,j \in I}$ be a $k \times k$ matrix such that $M_{ii} = M_{jj} \in \{1,2,\cdots,\infty\}$ for all $i, j \in I$ and $M_{ij} = 1$ if and only if $i = j \in I$. Then, $M$ is called a Coxeter matrix. The associated Coxeter group $G_S$ is defined by the presentation

$$G = G_S = \langle S | (s_is_j)^{M_{ij}} = 1, \forall i, j \in I \rangle.$$ 

The pair $(G, S)$ is a Coxeter system and $S$ is a Coxeter generating set of $G$. The cardinality $|I|$ is usually called the rank of $(G, S)$.

The following two results are fundamental for Coxeter groups.

Lemma 2.4. Let $(G, S)$ be a Coxeter system. Then, there exists an epimorphism $\phi : G \to \{-1,1\}$ induced by $\phi(s) = -1$ for all $s \in S$. Particularly, for any $g \in G$, $\phi(g^{-1}) = \phi^{-1}(g) = \phi(g)$.

Lemma 2.5. Let $(G, S)$ be a Coxeter system with rank $k$. Then, there exists a faithful representation $\rho : G \to GL(\mathbb{R}^k)$.

For any $T \subset S$, we can check $G_T$ is a Coxeter group. This leads to the irreducible Coxeter group.

Definition 2.6. A Coxeter system $(G, S)$ is reducible if $S = S' \cup S''$ with $S', S''$ being nonempty and $S' \cap S'' = \emptyset$, such that $(s_is_j)^2 = 1$ for all $s_i \in S'$ and $s_j \in S''$. It then follows that $G = G_{S'} \times G_{S''}$. A Coxeter system $(G, S)$ is irreducible if it is not reducible.

For any finite irreducible Coxeter group, we have

Lemma 2.7. Let $(G, S)$ be a finite irreducible Coxeter system with rank $k$. Then, for any $s \in S$, there exists a normal subgroup $N_k$, such that $G = N_k \rtimes \langle s \rangle$.

Finally, we recall the fundamental domain for the Coxeter groups.

Definition 2.8. Assume the Coxeter group $G$ acts on $\mathbb{R}^k$ by homeomorphisms. A fundamental domain is a closed, connected subset $D$ of $\mathbb{R}^k$ such that $O_x \cap D \neq \emptyset$ for any $x \in \mathbb{R}^k$ and $O_x \cap D = \{x\}$ for any $x$ in the interior of $D$.

Note that $\cup_{g \in G} H_g$ separates $\mathbb{R}^k$ into $|G|$ components. For any component, its closure is a fundamental domain of $G$. Write the $k-i$ dimensional facets of the fundamental domain $D$ as $\partial^iD$ for $i = 0, 1, \ldots, k-1$. We have

Lemma 2.9. Let $G_k$ be the collection of finite Coxeter groups of rank $k$. Assume $D$ to be the fundamental domain of $G \in G_k$. Then, for any $x \in \partial^iD$, $S_x \in S_i$ with $i = 0, 1, \ldots, k-1$. 

if $G = HN$ and $H \cap N = \{1\}$ with $N$ being a normal subgroup of $G$. There are two basic concepts about group actions.
2.2. Variational framework

We start with the fractional Sobolev space. For a measurable function \( u : \mathbb{R}^N \to \mathbb{R} \), the Gagliardo semi-norm is defined by

\[
[u]_s = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

And by [12, Propositions 3.4 and 3.6], we have

\[
2 \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 \, d\xi = 2\|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2 = C_{N,s} [u]^2_s. \tag{2.1}
\]

Then, the fractional Sobolev space can be defined by

\[ H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) | [u]_s < +\infty \}, \]

endowed with the norm

\[ \|u\|_{2,s}^2 = \|u\|_{L^2(\mathbb{R}^N)}^2 + [u]_s^2. \]

Up to a constant, we may assume \( C_{N,s} = 2 \), i.e., we define for \( u \in H^s(\mathbb{R}^N) \) that

\[ \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2 = [u]_s^2. \]

The following fractional embedding theorem has been proved, see [12, Theorem 6.5] for instance.

**Lemma 2.10.** Let \( s \in (0,1) \), \( N > 2s \) and \( 2^*_s = \frac{2N}{N-2s} \). Then, there exists a positive constant \( C = C(N,s) \) such that, for any \( u \in H^s(\mathbb{R}^N) \) it holds

\[ \|u\|_{2^*_s(\mathbb{R}^N)} \leq C \|u\|_{2,s}. \]

Moreover, the embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \) is continuous for any \( r \in [2,2^*_s] \).

We shall study the fractional Choquard equation (1.1) with the help of Coxeter’s symmetries. Let \( G \in \mathcal{G}_k \) be a Coxeter group with its rank \( 1 \leq k \leq N \). For any \( g \in G \), the group action on \( u \in H^s(\mathbb{R}^N) \) is defined by

\[ g \circ u = u(g^{-1} x), \quad \text{with} \quad gx = \text{diag}(g,1_{N-k})x. \]

We shall seek saddle solutions for the fractional Choquard equation (1.1) in the following subspace

\[ H^s_G(\mathbb{R}^N) = \{ u \in H^s(\mathbb{R}^N) | g \circ u(x) = \phi(g)u(x), \forall g \in G \}, \]

where \( \phi \) is the unique epimorphism induced by \( \phi(s) = -1 \) for \( s \in S \), see Lemma 2.4.

The fractional Choquard equation is variational in nature, and it is easy to see weak solutions of (1.1) corresponding to critical points of the following action functional

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{A_\alpha |u(y)|^p |u(x)|^p}{|x - y|^{N-\alpha}} \, dx \, dy. \]

From the Hardy–Littlewood–Sobolev inequality and the fractional embedding theorem, we see that the energy functional \( I : H^s_G(\mathbb{R}^N) \to \mathbb{R} \) is well defined and belongs to the class of \( C^1 \) if \( p \) satisfies (1.3). It is known that the positive groundstate can be described as

\[ c_0 = \inf_{v \in \mathcal{N}} I(v), \]

where \( \mathcal{N} \) is the Nehari constraint \( \mathcal{N} = \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} | \langle I'(u), u \rangle = 0 \} \).
We shall consider the minimization problem
\[ c_G = \inf_{u \in \mathcal{N}_G} I(u), \]
with the $G$-Nehari manifold being defined by $\mathcal{N}_G = \mathcal{N} \cap H^s_G(\mathbb{R}^N)$. The above minimal energy also has a mountain pass type description. In fact, we can conclude as [39, Theorem 4.2] that
\[ c_G = \inf_{\gamma \in \Gamma, t \in [0,1]} \max I(\gamma(t)) \]
where the paths set is
\[ \Gamma = \{ \gamma \in C([0,1]; H^s_G(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0 \}. \]

The qualitative properties including regularity and decay behaviors of solutions for the fractional Choquard equation (1.1) have been obtained in [10, Theorems 3.2-3.3] and it turns out to be very important in seeking saddle solutions. We recall and adopt it as follows.

**Lemma 2.11.** Let $N \geq 3$, $\alpha \in (0, N)$ and $2 \leq p < \frac{N+\alpha}{N-2s}$. If $v \in H^s(\mathbb{R}^N)$ with $s \in (0, 1)$ is a solution of (1.1), then $v \in L^1(\mathbb{R}^N) \cap C^\beta(\mathbb{R}^N)$ for some $\beta \in (0, 1)$. Moreover, there exists $C > 0$ such that for all $x \in \mathbb{R}^N$,
\[ |v(x)| \leq C(1 + |x|^2)^{-(N+2s)/2}. \]

**3. Existence of saddle solutions.**

We begin with an energy estimate.

**Proposition 3.1.** Let $s \in (0, 1)$, $N \geq 3$, $\alpha \in (0, N)$ and $2 \leq p < \frac{N+\alpha}{N-2s}$. For any $G \in \mathcal{G}_k$ with $1 \leq k \leq N$, there exists $c^*_G$ such that
\[ 0 < c_G < c^*_G = \min\{|\mathcal{O}_x|c_{S_x} \mid x \in \partial^{k-1}D\} \]
\[ < \min\{|\mathcal{O}_x|c_{S_x} \mid x \in \partial^{k-i}D, i = 2, \ldots, k\}. \]

**Proof.** Since $\mathcal{N}_G \subset \mathcal{N}$, we easily have $0 < c_0 \leq c_G$. The remaining inequalities will be completed by the method of induction. Assume $k = 1$. By the classification results of the finite Coxeter group (see Theorem 6.9.1 in [11]), $G = A_1$, the cyclic group with order 2, and its fundamental domain is, up to a rotation, $\mathbb{R}^+$. In this case, the inequality to be proved reads as $c^*_G = c_{\text{odd}} < 2c_0$, which is exactly the result in [9].

We now suppose that Proposition 3.1 holds true for $k - 1$. Then, $H$-saddle solutions for the fractional Choquard equation (1.1) do exist for any $H \leq G$ with $H \in \mathcal{S}_k$, this can be proved by repeating our subsequent procedures. In particular, each $H$-saddle solution we obtained is an $H$-ground-state. Fix $q \in \partial^{k-1}D$ such that $|q| = \sqrt{q \cdot q} = 1$. Without loss of generality, we assume that $q$ is perpendicular to $H_{s_1}$. By Lemma 2.9, we see that $S_q \in \mathcal{G}_{k-1}$. The inductive assumption implies that the fractional Choquard equation (1.1) has a saddle solution $u_{S_q}$ with $S_q$-symmetry such that
\[ c_{S_q} = I(u_{S_q}) = \inf_{N_{S_q}} I, \quad \text{and} \quad g \circ u_{S_q} = \phi(g)u_{S_q}, \forall g \in S_q, \]
where $\phi : S_q \rightarrow \{-1, 1\}$ is the restriction of the unique epimorphism induced by $\phi(s) = -1$ for $s \in S$. Moreover, from Lemma 2.11, it follows that
\[ u_{S_q} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \text{and} \quad \limsup_{|x| \rightarrow +\infty} |x|^{N+2s}|u_{S_q}| < +\infty. \]

As a result of Definition 2.6, a finite Coxeter group always can be reduced to the direct product of some finite irreducible Coxeter groups, see [11, Theorem 6.9.1]. Without loss of generality, we write $G = G_{k_1} \times G_r$ where $G_{k_1} \in \mathcal{G}_{k_1}$ is a finite irreducible Coxeter group such that $s_1 \in G_{k_1}$ and $G_r \in \mathcal{G}_{k-k_1}$. 

If $G$ is irreducible, then $G_r = \{1\}$. For the case $k_1 = 1$, we assume that $H_{s_1} = \partial \mathbb{R}^N$. We define a function $u_R : \mathbb{R}^N \to \mathbb{R}$ for each $x = (x', x^N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ by

$$u_R(x) = (\xi_R u_{S_q})(x', x^N - 3R) - (\xi_R u_{S_q})(x', -x^N - 3R),$$

where the function $\xi_R(x) = \xi(x/R)$ and $\xi(x) \in C^\infty(\mathbb{R}^N)$ is radial and satisfies $\xi(x) = 1$ if $|x| \leq 1$, $\xi(x) = 0$ if $|x| \geq 2$ and $0 \leq \xi \leq 1$ on $\mathbb{R}^N$. In this situation, we can easily check $G_r = \langle S_q \rangle$. Since $s_1 g = gs_1$ for any $g \in G_r$. We can treat $G_r$ as a group acting on $\mathbb{R}^{N-1}$ so that $g \circ u_R = \phi(g) u_R$ for any $g \in G_r$. Hence by $s_1 \circ u_R = -u_R$, we conclude that $u_R \in H^s_{G_r}(\mathbb{R}^N)$.

For the case $k_1 \geq 2$, by Lemma 2.7, there exists a normal subgroup $N_{k_1} < G$ such that $G_{k_1} = N_{k_1} \times \langle s_1 \rangle$. Let $N_k = N_{k_1} \times G_r$. We define a function $u_R$ by

$$u_R(x) = \frac{1}{|S_q|} \sum_{g \in N_k} (\xi_R u_{S_q})(g^{-1} s_1 x - l_G R q) - (\xi_R u_{S_q})(g^{-1} x - l_G R q),$$

where $l_G$ is a constant such that $l_G \min_{x \neq y \in S_q} |x - y| \geq 6$. In this case, we have $S_q = G_{S_{k_1}} \times G_r$, where $S_{k_1} = S \setminus \{s_1\} = \{s_2, \ldots, s_k\}$. We now show that $u_R \in H^s_{G_r}(\mathbb{R}^N)$. From the definition of $u_R$, we first observe that $s_1 \circ u_R = -u_R = \phi(s_1) u_R$. For $s \in S_{k_1}$, since $s \in G_{k_1} = N_{k_1} \times \langle s_1 \rangle$, there exist unique $n_s, h_s$ such that $s = n_s s_1 = s_1 h_s$. Hence, we deduce that

$$s \circ u_R = \frac{1}{|S_q|} \sum_{g \in N_k} (\xi_R u_{S_q})(g^{-1} s_1 s^{-1} x - l_G R q) - (\xi_R u_{S_q})(g^{-1} s^{-1} x - l_G R q)$$

$$= \frac{1}{|S_q|} \sum_{g \in N_k} (\xi_R u_{S_q})(g^{-1} n_s^{-1} x - l_G R q) - (\xi_R u_{S_q})(g^{-1} h_s^{-1} s_1 x - l_G R q)$$

$$= -\frac{1}{|S_q|} \sum_{g \in N_k} (\xi_R u_{S_q})(g^{-1} s_1 x - l_G R q) - (\xi_R u_{S_q})(g^{-1} x - l_G R q) = -u_R.$$

Note that $g h = h g$ for $g \in G_r$ and $h \in G_{k_1}$. We deduce that

$$g_r \circ u_R = \frac{1}{|S_q|} \sum_{g \in N_k} (\xi_R u_{S_q})(g^{-1} s_1 g_r^{-1} x - l_G R q) - (\xi_R u_{S_q})(g^{-1} g_r^{-1} x - l_G R q)$$

$$= \phi(g_r) u_R.$$

The conclusion $u_R \in H^s_{G_r}(\mathbb{R}^N)$ follows immediately by combining $s_1 \circ u_R = -u_R$.

Due to the Nehari structure of the functional, we can take $t_R > 0$ such that $\langle I'(t_R u_R), t_R u_R \rangle = 0$ so that $t_R u_R \in \mathcal{N}_G$. We then deduce that,

$$I(t_R u_R) = \left( \frac{1}{2} - \frac{1}{2p} \right) \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_R|^2 + |u_R|^2 \, dx \right)^{\frac{p}{p-1}} \left( \int_{\mathbb{R}^N} (K_\alpha * |u_R|^p)|u_R|^p \, dx \right)^{\frac{1}{p-1}}. \quad (3.1)$$
Therefore, \( c_G < c^*_G \) follows once we establish that for some \( R > 0 \),

\[
\left( \int_{\mathbb{R}^N} |u_R|^2 + |(-\Delta)^{s/2}u_R|^2 \, dx \right)^{\frac{p}{p-1}} < \left( \int_{\mathbb{R}^N} |u_{S_q}|^2 + |(-\Delta)^{s/2}u_{S_q}|^2 \, dx \right)^{\frac{p}{p-1}}.
\]

Let \( u^g_R = (\xi_R u_{S_q})(g^{-1}x - l_GRq) \) and \( D_g = \text{supp} u^g_R \). Here, the notation \( \text{supp} v \) denotes the support of the function \( v \). Direct calculations give us that

\[
\|(-\Delta)^{s/2}u_R\|_{L^2(\mathbb{R}^N)}^2 = \frac{1}{|S_q|} \sum_{g \in G} \|(-\Delta)^{s/2}u^g_R\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{|S_q|} \sum_{g \neq h \in G} \int \langle \xi, (-\Delta)^{s/2}u^g_R(-\Delta)^{s/2}u^h_R \rangle \, dx.
\]

(3.2)

Thanks to (2.1), we obtain

\[
\|(-\Delta)^{s/2}(\xi_R u_{S_q})\|_{L^2(\mathbb{R}^N)}^2 = \|(-\Delta)^{s/2}u^g_R\|_{L^2(\mathbb{R}^N)}^2, \quad \forall g \in G.
\]

(3.3)

By the definition of \( \xi \) and the choice of \( l_G \), we have

\[
2R \leq \min_{x \in D_g, y \in D_h} |x - y| \leq (2l_G + 4)R, \quad \forall g \neq h.
\]

(3.4)

By combining (2.1), (3.2)–(3.4) and the fact \( u_{S_q} \in L^1(\mathbb{R}^N) \), we then conclude that

\[
\|(-\Delta)^{s/2}u_R\|_{L^2(\mathbb{R}^N)}^2 \leq \mathcal{O}_q \|(-\Delta)^{s/2}(\xi_R u_{S_q})\|_{L^2(\mathbb{R}^N)}^2 + \frac{C}{R^{N+2s}} \|u_{S_q}\|_{L^1(\mathbb{R}^N)}^2.
\]

(3.5)

To estimate the semi-norm of \( \xi_R u_{S_q} \), for convenience we denote

\[
u^0_R(x, y) = (\xi_R u_{S_q})(x) - (\xi_R u_{S_q})(y) \quad \text{for} \quad x, y \in \mathbb{R}^N.
\]

We see from (2.1) that

\[
2C_{N,s}^{-1}\|(-\Delta)^{s/2}(\xi_R u_{S_q})\|_{L^2(\mathbb{R}^N)}^2 = \sum_{i=1}^3 \sum_{j=1}^3 \int_{E_i} \int_{E_j} \frac{|u^0_R(x, y)|^2}{|x - y|^{N+2s}} \, dx \, dy = \sum_{i=1}^3 \sum_{j=1}^3 F_{ij},
\]

where \( E_1 = B_R, E_2 = B_2R \setminus B_R, E_3 = \mathbb{R}^N \setminus B_{2R} \). By the definition of \( \xi_R \),

\[
F_{11} = \int_{B_R} \int_{B_R} \frac{|u_{S_q}(x) - u_{S_q}(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
\]

Noticing that \( u_{S_q} \) has a polynomial decay at infinity, we obtain

\[
\int_{\mathbb{R}^N \setminus B_R} |u_{S_q}(y)| \, dy \leq C \int_{\mathbb{R}^N \setminus B_R} \frac{1}{|y|^{N+2s}} \, dy = CR^{-2s}.
\]

(3.6)
Take $r, t > 1$ such that $1/r + 1/t + (N + 2s - 2)/N = 2$, so that $N/r + N/t = N + 2 - 2s$. Note that $|\xi'(t)| \leq 2$. We deduce by the Hardy–Littlewood–Sobolev inequality and (3.6) that
\[
\int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
\leq C \int_{B_R} \int_{B_R} \frac{|x - y|^2 |u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
\leq C \frac{R^2}{R^{2(N + 2s) - N/t}} \leq C \frac{R^{N/r}}{R^{2(N + 2s) - N/t}} \leq C \frac{R^{N/r}}{R^{2(N + 2s) - N/t}} = o(R^{-8s}).
\] 
(3.7)

Similarly, we have
\[
\int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy = o(R^{-8s}),
\] 
(3.8)

and
\[
\int_{\mathbb{R}^N \setminus B_R} \int_{\mathbb{R}^N \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_n}(y)|}{|x - y|^{N + 2s}} \, dx \, dy = O(R^{-4s}).
\] 
(3.9)

We observe that
\[
F_{21} = \int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|u_{S_n}(x) - (\xi_R u_{S_n})(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
= \int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|u_{S_n}(x) - u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
+ \int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
+ 2 \int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{(u_{S_n}(x) - u_{S_n}(y))(\xi_R(x) - \xi_R(y)) |u_{S_n}(y)|}{|x - y|^{N + 2s}} \, dx \, dy.
\]

By combining (3.7) and the Cauchy–Schwarz inequality, we deduce that
\[
\int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{(u_{S_n}(x) - u_{S_n}(y))(\xi_R(x) - \xi_R(y)) u_{S_n}(y)}{|x - y|^{N + 2s}} \, dx \, dy
\leq \left( \int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|u_{S_n}(x) - u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{1/2}
\times \left( \int_{B_R \setminus B_R} \int_{B_R \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_n}(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{1/2}
\]
In a similar manner, we can conclude by (3.8) that

$$\leq C \left( \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2} = o(R^{-4s}).$$

Here, we use the fact that $u_{S_q}$ is bounded in $H^s(\mathbb{R}^N)$. It then follows that

$$F_{21} \leq \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{|u_{S_q}(x) - u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + o(R^{-4s}).$$

In a similar manner, we can conclude by (3.8) that

$$F_{22} = \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{|(\xi_R u_{S_q})(x) - (\xi_R u_{S_q})(y)|^2}{|x-y|^{N+2s}} \, dx \, dy$$

$$= \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{\xi_R(x)\xi_R(y) |u_{S_q}(x) - u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy$$

$$+ \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{|\xi_R(x) - \xi_R(y)|^2 |u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy$$

$$+ \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{\xi_R(x)(\xi_R(x) - \xi_R(y))(u_{S_q}^2(x) - u_{S_q}^2(y))}{|x-y|^{N+2s}} \, dx \, dy$$

$$\leq \int_{B_{2R} \setminus B_R} \int_{B_{2R} \setminus B_R} \frac{|u_{S_q}(x) - u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + o(R^{-4s}).$$

Recall that $\xi_R \equiv 0$ in $\mathbb{R}^N \setminus B_{2R}$. By (3.9) and $u_{S_q} \in L^\infty(\mathbb{R}^N)$, we have

$$F_{32} = \int_{\mathbb{R}^N \setminus B_{2R}} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{|(\xi_R u_{S_q})(x)|^2}{|x-y|^{N+2s}} \, dx \, dy$$

$$= \int_{\mathbb{R}^N \setminus B_{2R}} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{\xi_R^2(x)|u_{S_q}(x) - u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy$$

$$+ \int_{\mathbb{R}^N \setminus B_{2R}} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{|\xi_R(x) - \xi_R(y)|^2 (2u_{S_q}^2(x) - u_{S_q}(y))u_{S_q}(y)}{|x-y|^{N+2s}} \, dx \, dy$$

$$\leq \int_{\mathbb{R}^N \setminus B_{2R}} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{|u_{S_q}(x) - u_{S_q}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + O(R^{-4s}).$$
Note that $|x - y| \geq R$ for $x \in B_R$ and $y \in \mathbb{R}^N \setminus B_{2R}$. We deduce

\[
F_{31} = \int_{\mathbb{R}^N \setminus B_{2R}} \int_{B_R} \frac{|u_s(x)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^N \setminus B_{2R}} \int_{B_R} \frac{|u_s(x) - u_s(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^N \setminus B_{2R}} \int_{B_R} \frac{(2u_s(x) - u_s(y))u_s(y)}{|x - y|^{N+2s}} \, dx \, dy
\]

\[
\leq \int_{\mathbb{R}^N \setminus B_{2R}} \int_{B_R} \frac{|u_s(x) - u_s(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{CR^N}{R^{N+2s}} \sum_{y \in B_{2R}} |u_s(y)| \, dy
\]

\[
= \int_{\mathbb{R}^N \setminus B_{2R}} \int_{B_R} \frac{|u_s(x) - u_s(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + O(R^{-4s}).
\]

Observing that

\[
F_{12} = F_{21} \leq \int_{B_R} \int_{B_R \setminus B_{2R}} \frac{|u_s(x) - u_s(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + o(R^{-4s}),
\]

\[
F_{13} = F_{31} \leq \int_{B_R \setminus B_{2R}} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{|u_s(x) - u_s(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + O(R^{-4s}),
\]

\[
F_{23} = F_{32} \leq \int_{B_{2R} \setminus B_R} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{|u_s(x) - u_s(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + O(R^{-4s}),
\]

we finally obtain by noting $F_{33} = 0$ that

\[
\|(-\Delta)^{s/2}(\xi_R u_s)\|_{L^2(\mathbb{R}^N)}^2 = \frac{C_{N,s}}{2} \sum_{1 \leq i,j \leq 3} F_{ij} \leq \|u_s\|_{H^s}^2 + O(R^{-4s}).
\]

This, together with (2.1) and (3.5), implies that

\[
\|(-\Delta)^{s/2} u_R\|_{L^2(\mathbb{R}^N)}^2 \leq \|O_q\|_{L^2(\mathbb{R}^N)}^2 \|(-\Delta)^{s/2} u_s\|_{L^2(\mathbb{R}^N)}^2 + O(R^{-4s}).
\]

By the construction of $u_R$, we easily obtain

\[
\int_{\mathbb{R}^N} |u_R|^2 \, dx = |O_q| \int_{B_{2R}} |\xi_R u_s|^2 \, dx \leq |O_q| \int_{\mathbb{R}^N} |u_s|^2 \, dx.
\]

Note that $(N + \alpha)/(N - 2s) > 2$. We get an estimate about the numerator of quotient (3.1) that

\[
\|u_R\|_{2,s}^2 \leq |O_q| \int_{\mathbb{R}^N} |u_s|^2 + |(-\Delta)^{s/2} u_s|^2 \, dx + o(R^{\alpha - N}).
\]

(3.10)
We now deal with the denominator of (3.1). By (3.4), we have
\[
\int_{\mathbb{R}^N} (K_\alpha * |u_R|^p)|u_R|^p \, dx
= |O_q| \int_{\mathbb{R}^N} (K_\alpha * |\xi_R u_{S_q}|^p)|\xi_R u_{S_q}|^p \, dx + \frac{1}{|S_q|} \sum_{y \neq h} \int_{\mathbb{R}^N} (K_\alpha * |u_R^q|^p)|u_R^h|^p \, dx
\geq |O_q| \int_{\mathbb{R}^N} (K_\alpha * |\xi_R u_{S_q}|^p)|\xi_R u_{S_q}|^p \, dx + \frac{C}{R^{N-\alpha}} \left( \int_{B_R} |u_{S_q}|^p \, dx \right)^2.
\]
For the first term, we have
\[
\int_{\mathbb{R}^N} (K_\alpha * |\xi_R u_{S_q}|^p)|\xi_R u_{S_q}|^p \, dx
= \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)|u_{S_q}|^p \, dx - 2 \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)(1 - \xi_R^p)|u_{S_q}|^p \, dx
+ \int_{\mathbb{R}^N} (K_\alpha * (1 - \xi_R^p)|u_{S_q}|^p)(1 - \xi_R^p)|u_{S_q}|^p \, dx
\geq \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)|u_{S_q}|^p \, dx - 2 \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)(1 - \xi_R^p)|u_{S_q}|^p \, dx.
\]
By the decay properties of \(u_{S_q}\), we see that \(u_{S_q} \in L^r(\mathbb{R}^N)\) for \(r > \frac{\alpha}{N + 2s}\). Moreover, for every \(r > \frac{\alpha}{N + 2s}\), we conclude similarly as in [41, Proposition 4.1] that \(K_\alpha * |u_{S_q}|^r \in L^\infty(\mathbb{R}^N)\). It then follows that there exists a positive constant \(C > 0\) such that
\[
\limsup_{|x| \to +\infty} \frac{K_\alpha * |u_{S_q}|^p}{K_\alpha(x)} \leq C,
\]
which leads that
\[
2 \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)(1 - \xi_R^p)|u_{S_q}|^p \, dx \leq C \int_{\mathbb{R}^N \setminus B_R} \frac{|u_{S_q}|^p}{|x|^{N-\alpha}} \, dx.
\]
Therefore, by the asymptotic properties of \(u_{S_q}\), we deduce that
\[
\int_{\mathbb{R}^N} (K_\alpha * |u_R|^p)|u_R|^p \, dx
\geq |O_q| \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)|u_{S_q}|^p \, dx + \frac{C}{R^{N-\alpha}} \left( \int_{B_R} |u_{S_q}|^p \, dx \right)^2 - C \int_{\mathbb{R}^N \setminus B_R} \frac{|u_{S_q}|^p}{|x|^{N-\alpha}} \, dx
\geq |O_q| \int_{\mathbb{R}^N} (K_\alpha * |u_{S_q}|^p)|u_{S_q}|^p \, dx + \frac{C}{R^{N-\alpha}} \left( \int_{B_R} |u_{S_q}|^p \, dx \right)^2 + o(R^{\alpha-N}).
\]
We then conclude by combining (3.10), (3.11) and the definition of \(c_G\) that
\[
c_G \leq I(t_Ru_R) = |O_q|c_{S_q} (1 - CR^{\alpha-N} + o(R^{\alpha-N})) < |O_q|c_{S_q}.
\]
We are now in position to complete the proof of Theorem 1.1. To streamline the organization structure, we give the existence of $G$-saddle solutions here and move the $G$-symmetry property for the $G$-groundstate to the next section. □

Proof of Theorem 1.1. Thanks to (2.2), by the general minimax principle (see e.g., [39, Theorem 2.8]), we can obtain a sequence $\{u_n\}_{n \geq 1} \subset H^s_G(\mathbb{R}^N)$ such that

$$I(u_n) \rightarrow c_G, \quad I'(u_n) \rightarrow 0 \quad \text{in} \quad (H^s_G(\mathbb{R}^N))^* \quad \text{as} \quad n \rightarrow \infty.$$ 

It is easy to verify that $\{u_n\}_{n \geq 1} \subset H^s_G(\mathbb{R}^N)$ is bounded, since

$$c_G + o(1)\|u_n\|_{2,s} = I(u_n) - \frac{1}{2p}\langle I'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_{2,s}^2. \quad (3.12)$$

We shall claim that there exist $T > 0$ and $\{a_n\}_{n \geq 1} \subset \mathbb{R}^{N-k}$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_T(a_n)} |u_n|^\frac{2Np}{N+\alpha} \, dx > 0. \quad (3.13)$$

Up to translations and a subsequence, we may assume $u_n$ converges weakly to some function $u_G \in H^s_G(\mathbb{R}^N) \setminus \{0\}$. A standard procedure (e.g., [15], see also [9,40]) implies that $I'(u_G) = 0$ in $(H^s_G(\mathbb{R}^N))^*$ and $I(u_G) = c_G$. We then apply the symmetric criticality principle [39, Theorem 1.28] to conclude that $u_G$ is a critical point of $I$ in $H^s(\mathbb{R}^N)$. □

To prove (3.13), we first assert that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n(x)|^\frac{2Np}{N+\alpha} \, dx = \Lambda \in (0, +\infty). \quad (3.14)$$

Indeed, $\Lambda < +\infty$ follows from the boundedness of $\{u_n\}_{n \geq 1}$ in $H^s(\mathbb{R}^N)$ and the fractional embedding theorem (Lemma 2.10). If $\Lambda = 0$, by combining (3.12), (3.14) and the Hardy–Littlewood–Sobolev inequality, we would deduce a contradiction that

$$0 < c_G = \left(\frac{1}{2} - \frac{1}{2p}\right)\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A_\alpha |u_n(x)|^p |u_n(y)|^p \frac{1}{|x-y|^{N-\alpha}} \, dx \, dy = 0.$$ 

We first observe that the sequence $\{u_n\}_{n \geq 1}$ is non-vanishing, i.e., for any $r > 0$, we have $\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^\frac{2Np}{N+\alpha} \, dx > 0$. Otherwise, we conclude by [10, Lemma 2.3] that $u_n \rightarrow 0$ strongly in $L^r(\mathbb{R}^N)$ for any $r \in (2, \frac{2N}{N-2\alpha})$, which contradicts (3.14) since $\Lambda > 0$. According to Lions’ concentration compactness lemma [23], we shall consider the remaining two cases stated as Compactness and Dichotomy.

Compactness there exists $\{x_n\}_{n \geq 1} \subset \mathbb{R}^N$, such that for any $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_{R_0}(x_n)} |u_n(x)|^\frac{2Np}{N+\alpha} \, dx \geq \Lambda - \varepsilon.$$ 

In this case, we can easily check that for $\{x_n\}_{n \geq 1}$, there exists some $M_1 > 0$ such that

$$|\text{Pr}^k_{x_n}(x)| \leq M_1.$$ 

Here and in the sequel $\text{Pr}^k : \mathbb{R}^N \rightarrow \mathbb{R}^k, (x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_k)$ denotes the projection. Otherwise, for every $g \in G \setminus \{1\}$ and for largely $n$, we have that $B_{R_0}(x_n) \cap gB_{R_0}(x_n) = \emptyset$. By the symmetric setting, we would deduce a contradiction if $\varepsilon$ becomes small enough that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n(x)|^\frac{2Np}{N+\alpha} \, dx \geq \liminf_{n \rightarrow \infty} \int_{B_{R_0}(x_n)} |u_n(x)|^\frac{2Np}{N+\alpha} \, dx$$
\[ + \liminf_{n \to -\infty} \int_{gB_{R_0}(x_n)} |u_n(x)| \frac{2np}{N+2s} \, dx \geq 2\Lambda - 2\varepsilon > \Lambda. \]

**Dichotomy** there exist \( \beta \in (0, 1) \) and \( \{y_n\}_{n \geq 1} \subset \mathbb{R}^N \), such that for any \( \varepsilon > 0 \), there exists \( R_2 = R_2(\varepsilon) > 0 \) such that for all \( r_1 \geq R_2 \) and \( r_2 \geq R_2 \),

\[
\limsup_{n \to -\infty} \left| \int_{B_{r_1}(y_n)} |u_n(x)| \frac{2np}{N+2s} \, dx - \beta \Lambda \right| + \left| \int_{\mathbb{R}^N \setminus B_{r_2}(y_n)} |u_n(x)| \frac{2np}{N+2s} \, dx - (1 - \beta)\Lambda \right| \leq \varepsilon.
\]

In this situation, we will also show that

\[ |\text{Pr}_k(y_n)| \leq M_2, \quad \forall n \geq 1, \quad (3.15) \]

for some appropriate constant \( M_2 > 0 \). Without loss of generality, we may assume \( \{y_n\}_{n \geq 1} \subset \mathcal{D} \), where \( \mathcal{D} \) denotes the fundamental domain of the Coxeter group \( G \). If there exists \( R_1 > 0 \) such that \( \{y_n\}_{n \geq 1} \subset B_{R_1} \cap \mathcal{D} \), conclusion (3.15) can be deduced immediately since \( \beta \Lambda > 0 \).

For the subsequent proofs, we assume by contradiction that \( |\text{Pr}_k(y_n)| \to +\infty \), then by a diagonal process and up to a subsequence, we can choose \( \varepsilon_n \to 0 \), and \( r'_n = 4r_n \to +\infty \), such that

\[ \left| \int_{B_{r_n}(y_n)} |u_n| \frac{2np}{N+2s} - \beta \Lambda \right| + \left| \int_{B'_{r_n}(y_n)} |u_n| \frac{2np}{N+2s} - (1 - \beta)\Lambda \right| \leq \varepsilon_n. \quad (3.16) \]

Particularly, we can choose a subsequence of \( \{y_n\}_{n \geq 1} \) such that \( |x - y| \geq 4r_n \) for all \( x \in B_{3r_n}(z_1) \) and \( y \in B_{3r_n}(z_2) \) with \( z_1 \neq z_2 \in G y_n \). As a result, two cases occur due to the variant of the asymptotic distance between the sequence \( \{y_n\}_{n \geq 1} \) and the boundary \( \partial \mathcal{D} \).

**Case 1:** up to a subsequence, there exists some \( j \in \{1, \ldots, k - 1\} \) such that

\[ \lim_{n \to -\infty} \text{dist} (y_n, \partial^j \mathcal{D}) = 0 \quad \text{and} \quad \lim_{n \to -\infty} \text{dist} (y_n, \partial^j \mathcal{D}) = +\infty \quad \text{for all} \quad \ell > j. \]

This situation contains several subcases and for every subcase, we fix a unit vector \( q \in \partial^j \mathcal{D} \cap \partial B_1 \) and up to a subsequence, we may assume \( y_n = \nu_n q \) with \( \nu_n \to +\infty \).

For any \( g \in G \), we denote \( B^g_{r_n} = B_{r_n}(gy_n) \) and \( B_{r_n} = \bigcup_{g \in G} B^g_{r_n} \). We first observe from (3.16) that

\[ \int_{B_{4r_n} \setminus B_{r_n}} |u_n| \frac{2np}{N+2s} \, dx \leq 2|\mathcal{O}_{y_n}|\varepsilon_n. \quad (3.17) \]

We now prove that

\[ \int_{B_{3r_n} \setminus B_{2r_n}} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dy \, dx + \int_{B_{3r_n} \setminus B_{2r_n}} |u_n|^2 \, dx = o(1). \quad (3.18) \]

Take \( \psi_r(x) = \psi(x/r) \) where \( \psi \in C^\infty(\mathbb{R}^N) \) satisfies that \( \psi(x) = 0 \) for \( |x| \leq 1 \) or \( |x| \geq 4 \), \( \psi(x) = 1 \) for \( 2 \leq |x| \leq 3 \) and \( 0 \leq \psi(x) = \psi(|x|) \leq 1 \). For any \( g \in G \), let \( \psi^g_n(x) = \psi_{r_n}(x - gyn) \) and

\[ \Psi_n(x) = \sum_{z \in \mathcal{O}_{y_n}} \psi_{r_n}(x - z) = \frac{1}{|\mathcal{S}_{y_n}|} \sum_{g \in G} \psi^g_n(x). \]

Observe that for any \( h \in G \),

\[ h \circ (\Psi_n u_n)(x) = \frac{1}{|\mathcal{S}_{y_n}|} \sum_{g \in G} \psi_{r_n}(h^{-1}x - h^{-1}hgy_n)u_n(h^{-1}x) \]

\[ = \frac{1}{|\mathcal{S}_{y_n}|} \sum_{g \in G} \psi_{r_n}(x - hgy_n)\phi(h)u_n(x) = \phi(h)(\Psi_n u_n)(x), \]
we infer that $\Psi_n u_n \in H^s_G(\mathbb{R}^N)$. Since $I'(u_n) \to 0$ in $(H^s_G(\mathbb{R}^N))^*$, we have $\langle I'(u_n), \Psi_n u_n \rangle = o(1)$. And a direct computation shows that

$$
\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\Psi_n(x)u_n(x) - u_n(y)}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} \Psi_n |u_n|^2 \, dx
$$

$$
= \langle I'(u_n), \Psi_n u_n \rangle + \int_{B_{4r_n}\setminus B_{r_n}} (K_\alpha * |u_n|^p)u_n(x)^p \Psi_n(x) \, dx
$$

$$
- \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\Psi_n(x) - \Psi_n(y))(u_n(x) - u_n(y))u_n(y)}{|x-y|^{N+2s}} \, dx \, dy.
$$

Then, by the Cauchy–Schwarz inequality, the Hardy–Littlewood–Sobolev inequality and (3.17), we obtain that

$$
\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\Psi_n(x)u_n(x) - u_n(y)}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} \Psi_n |u_n|^2 \, dx
$$

$$
\leq \left[ |u_n|_s \left( \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\Psi_n(x) - \Psi_n(y)^2}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy \right) \right]^{1/2} + o(1). \tag{3.19}
$$

By the choice of $\Psi_n$, we readily check that

$$
\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\Psi_n(x) - \Psi_n(y)^2}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
= \frac{1}{|S_n|} \sum_{g \in G_{\mathbb{R}^N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi_n^g(x) - \psi_n^g(y)^2|}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
- \frac{1}{|S_n|} \sum_{g \neq h \in G_{\mathbb{R}^N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\psi_n^g(x)\psi_n^h(y)}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
- \frac{1}{|S_n|} \sum_{g \neq h \in G_{\mathbb{R}^N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\psi_n^h(y)\psi_n^g(x)}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
= \frac{1}{|S_n|} \sum_{g \in G_{\mathbb{R}^N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi_n^g(x) - \psi_n^g(y)^2|}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy + C \|u_n\|_{L^2(\mathbb{R}^N)}^{s-2s}. \tag{3.20}
$$

On one hand, we have

$$
\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi_n^g(x) - \psi_n^g(y)^2|}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
\leq \int \int_{\mathbb{R}^N \setminus B_{4r_n}(gyn)} \frac{|\psi_n^g(x) - \psi_n^g(y)^2|}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
+ \int \int_{B_{4r_n}(gyn) \setminus B_{5r_n}(gyn)} \frac{|\psi_n^g(x) - \psi_n^g(y)^2|}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy
$$

$$
+ \int \int_{B_{4r_n}(gyn) \setminus B_{3r_n}(gyn)} \frac{|\psi_n^g(x) - \psi_n^g(y)^2|}{|x-y|^{N+2s}} |u_n(y)|^2 \, dx \, dy. \tag{3.21}
$$
By using the Hardy–Littlewood–Sobolev inequality, we deduce for any $l \in (0, +\infty)$ that,

$$
\int_{\mathbb{R}^N} \int_{B_{r_n}(gy_n)} \frac{|\psi_n^g(x) - \psi_n^g(y)|^2|u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \leq C \int_{\mathbb{R}^N} \int_{B_{r_n}(gy_n)} \frac{|x - y|^2|u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$

$$
\leq \left( \int_{B_{r_n}(gy_n)} 1 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u_n(y)|^{2^*} \, dy \right)^{\frac{1}{2}} \leq \frac{C}{r_n^{2-N/r}} = o(1).
$$

Here, we take $r, t > 1$ such that $1/r + 1/t + (N + 2s - 2)/N = 2$ and $t < N/(N - 2s)$ so that $2 - N/r = 2s - N + N/t > 0$. On the other hand,

$$
\int_{B_{r_n}(gy_n) \setminus B_{2r_n}(gy_n)} \int_{B_{2r_n}(gy_n)} \frac{|\psi_n^g(x) - \psi_n^g(y)|^2|u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$

$$
\leq C \int_{B_{2r_n}(gy_n)} |u_n(y)|^2 \, dy \int_{\mathbb{R}^N \setminus B_{r_n}(y)} \frac{1}{|x - y|^{N+2s}} \, dx
$$

$$
\leq C \int_{r_n}^{+\infty} \frac{\rho^{N-1}}{\rho^{N+2s}} \, d\rho = C \frac{r_n^{2s}}{r_n^{N+2}}.
$$

Inserting these estimates (3.21)–(3.23) into (3.20), we deduce

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\Psi_n(x) - \Psi_n(y)|^2|u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0.
$$

This, together with (3.19) yields conclusion (3.18).

We now take another radial cut-off function $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta(x) = 1$ if $x \in B_2$, $\eta(x) = 0$ if $|x| \geq 3$ and $0 \leq \eta(x) = \eta(|x|) \leq 1$. Let $\eta_n(x) = \eta(|x|/r_n)$. For any $1 \leq i \leq |O_{y_n}|$ and $z^i \in O_{y_n}$, we denote $v^i_n = \eta_n(x - z^i)u_n$. Let $v_n = \sum_i v^i_n$ and $w_n = u_n - v_n$. It is clear that $v_n, w_n \in H^s_G(\mathbb{R}^N)$. Particularly, we have $v^i_n \in H^s_{G, \eta_i} (\mathbb{R}^N)$ with $G_{\eta_i} \subseteq G_j$ for every $i$. Indeed, if $z^i = v_n g_i g^{-1}$, then for every $s \in S_q$, we deduce

$$
g_i g^{-1} v^i_n = \eta(|g_i g^{-1} x - z_i|/r_n)u_n(g_i g^{-1} x) = \psi(g_i g^{-1})v^i_n = \psi(s) v^i_n,
$$

so that $v^i_n \in H^s_{g_i G_{\eta_i} g_i^{-1}} (\mathbb{R}^N)$. The conclusion follows since the group $g_i G_{\eta_i} g_i^{-1}$ is isomorphic to $G_{\eta_i}$ for every $i$.

We now claim that

$$
I(u_n) = \sum_{i=1}^{|O_{y_n}|} I(v^i_n) + I(w_n) + o(1),
$$

and

$$
\langle I'(v^i_n), v^i_n \rangle = o(1), \quad \langle I'(w_n), w_n \rangle = o(1).
$$

We assume (3.25) and (3.26) temporarily and postpone the proofs. By Lemma 2.10 and the Hardy–Littlewood–Sobolev inequality, we have

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|(-\Delta)^{s/2} v^i_n|^2 + |v^i_n|^2}{dx} = \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{A(x)|v^i_n(x)|^p|v^i_n(y)|^p}{|x - y|^{N+\alpha}} \, dx \, dy > 0.
$$
Then, for each $1 \leq i \leq O_{y_n}$, there exists a unique $t^i_n \in (0, +\infty)$ such that $t^i_n v^i_n \in N_{S_n}$ satisfying that $\lim_{n \to \infty} t^i_n = 1$. Hence, by (3.25) and (3.26), we deduce that

$$c_G = \lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} \sum_{i=1}^{\lfloor O_{y_n} \rfloor} I(t^i_n v^i_n) + \lim_{n \to \infty} I(w_n) \geq \lim_{n \to \infty} \sum_{i=1}^{\lfloor O_{y_n} \rfloor} I(t^i_n v^i_n) + \left( \frac{1}{2} - \frac{1}{2p} \right) \lim_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} w_n|^2 + |w_n|^2 \, dx \geq |O_{y_n}| c_{S_n}.$$  

This runs counter to Proposition 3.1 which states that $c_G < |O_{y_n}| c_{S_n}$. We therefore obtain conclusion (3.15).

We now turn our heads to prove claims (3.25) and (3.26). Indeed, we readily verify that

$$I(u_n) = \sum_{i=1}^{\lfloor O_{y_n} \rfloor} I(v^i_n) + I(w_n) + \int_{\mathbb{R}^N} (-\Delta)^{s/2} v_n (-\Delta)^{s/2} w_n \, dx + \int_{\mathbb{R}^N} v_n w_n \, dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2} v^i_n (-\Delta)^{s/2} v^m_n \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(x, y) \, dx \, dy \quad (3.27)$$

where $F_n : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is

$$F_n(x, y) = A_\alpha \left( \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} - \frac{|v_n(x)|^p |v_n(y)|^p}{|x-y|^{N-\alpha}} - \frac{|w_n(x)|^p |w_n(y)|^p}{|x-y|^{N-\alpha}} \right).$$

Note that $|x-y| \geq 4r_n$ for $x \in \text{supp } v^i_n$ and $y \in \text{supp } v^m_n$ with $i \neq m$. By the Cauchy–Schwarz inequality, we obtain for $i \neq m$ that

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} v^i_n (-\Delta)^{s/2} v^m_n \, dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v^i_n(x) - v^i_n(y)||v^m_n(x) - v^m_n(y)|}{|x-y|^{N+2s}} \, dx \, dy \leq 2 \int_{B_{3r_n}^n} \int_{B_{3r_n}^n} \frac{|v^i_n(y)v^m_n(x)|}{|x-y|^{N+2s}} \, dx \, dy \leq \frac{Cr_n^N}{r_n^{N+2s}} \left( \int_{B_{3r_n}^n} |u_n(y)|^2 \, dy \right)^{1/2} \left( \int_{B_{3r_n}^n} |u_n(x)|^2 \, dx \right)^{1/2} \leq \frac{C}{r_n^{2s}} \int_{\mathbb{R}^N} |u_n|^2 \, dx = o(1).$$
By the fractional embedding theorem, we deduce for \( i \neq m \) that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n^i(x)|^p |v_n^m(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \leq \frac{C\|u_n\|_{L^p(\mathbb{R}^N)}^{2p}}{r_n^{N-\alpha}} \leq \frac{C\|u_n\|_{2s}^{2p}}{r_n^{N-\alpha}} = o(1). \tag{3.29}
\]

Let \( \eta_n = \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} \eta_{r_n}(x-z^i) \). By the choice of \( \eta \), we have
\[
\int_{\mathbb{R}^N} (-\Delta)^{s/2} v_n (-\Delta)^{s/2} w_n \, dx + \int_{\mathbb{R}^N} v_n w_n \, dx
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\eta_n(x)(1-\eta_n(x))|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} + \frac{|\eta_n(x) - \eta_n(y)|^2 u_n^2(y)}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(1-\eta_n(x))(u_n(x) - u_n(y))(\eta_n(x) - \eta_n(y))u_n(y)}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\eta_n(x)(u_n(x) - u_n(y))(\eta_n(x) - \eta_n(y))u_n(y)}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} v_n w_n \, dx
\]
\[
\leq \int_{B_{3r_n} \setminus B_{2r_n}} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dy \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta_n(x) - \eta_n(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
+ \left[ u_n \right]_s \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta_n(x) - \eta_n(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2} + \int_{B_{3r_n} \setminus B_{2r_n}} |u_n|^2 \, dx.
\]

Arguing as in the proof of (3.24), we get
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta_n(x) - \eta_n(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy = o(1).
\]

We thus deduce by (3.18) that
\[
\int_{\mathbb{R}^N} (-\Delta)^{s/2} v_n (-\Delta)^{s/2} w_n \, dx + \int_{\mathbb{R}^N} v_n w_n \, dx = o(1). \tag{3.30}
\]

We now turn to the integration of the convolution term \( F_n \). By combining (3.17) and the Hardy–Littlewood–Sobolev inequality, we derive that
\[
\int_{B_{3r_n} \setminus B_{2r_n}} \int_{\mathbb{R}^N} F_n(x,y) \, dx \, dy
\]
\[
\leq 3 \int_{B_{3r_n} \setminus B_{2r_n}} \int_{\mathbb{R}^N} \frac{A_\alpha |u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \tag{3.31}
\]
\[
\leq C \left( \int_{B_{3r_n} \setminus B_{2r_n}} |u_n(y)|^{\frac{2Np}{N+\alpha}} \, dy \right)^{N+\alpha} \left( \int_{\mathbb{R}^N} |u_n(x)|^{\frac{2Np}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} = o(1).
\]
Note that
\[
\int \int_{B_{2r_n} \setminus B_{3r_n}} F_n(x, y) \, dx \, dy = \int \int_{B_{2r_n} \setminus B_{3r_n}} \frac{A_\alpha |u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \\
\leq C \frac{\|u_n\|_{L^p(\mathbb{R}^N)}^2}{r_n^{N-\alpha}} \leq C \frac{\|u_n\|_{L^2}^{2p}}{r_n^{N-\alpha}} = o(1).
\]
(3.32)

We then conclude that
\[
\int \int_{\mathbb{R}^N \setminus \mathbb{R}^N} F_n(x, y) \, dx \, dy \\
= \int \int_{B_{2r_n} \setminus B_{3r_n}} F(x, y) \, dx \, dy + \int \int_{B_{2r_n} \setminus B_{3r_n}} F_n(x, y) \, dx \, dy \\
+ \int \int_{B_{3r_n} \setminus B_{2r_n} \setminus \mathbb{R}^N} F_n(x, y) \, dx \, dy + \int \int_{B_{3r_n} \setminus B_{2r_n} \setminus \mathbb{R}^N} F_n(x, y) \, dx \, dy \\
\leq 2 \int \int_{B_{2r_n} \setminus B_{3r_n} \setminus \mathbb{R}^N} F_n(x, y) \, dx \, dy + 3 \int \int_{B_{3r_n} \setminus B_{2r_n} \setminus \mathbb{R}^N} F_n(x, y) \, dx \, dy = o(1).
\]
(3.33)

Therefore, claim (3.25) follows by a combination of (3.27)–(3.30) and (3.33).

We now show claim (3.26). We define \( K_n : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) such that
\[
K_n(x, y) = \frac{|u_n(x)|^p |u_n(y)|^p - u_n(x)v_n(y) - |v_n(x)|^p |v_n(y)|^p}{|x-y|^{N-\alpha}}.
\]

Arguing as for the integration of \( F_n(x, y) \), by (3.31) and (3.32), we have
\[
\int \int_{\mathbb{R}^N \setminus \mathbb{R}^N} K_n(x, y) \, dx \, dy \\
= \int \int_{\mathbb{R}^N \setminus B_{2r_n} \setminus \mathbb{R}^N} K_n(x, y) \, dx \, dy + \int \int_{B_{2r_n} \setminus B_{3r_n} \setminus \mathbb{R}^N} K_n(x, y) \, dx \, dy \\
\leq 3 \int \int_{\mathbb{R}^N \setminus B_{3r_n} \setminus \mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \\
+ \int \int_{B_{2r_n} \setminus B_{3r_n} \setminus \mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy = o(1).
\]
(3.34)

Since \( I'(u_n) \to 0 \) in \( (H^s_G(\mathbb{R}^N))^* \), and \( v_n \in H^s_G(\mathbb{R}^N) \), we have \( \langle I'(u_n), v_n \rangle = o(1) \). Then, by (3.30) and (3.34), we deduce that
\[
\int |(-\Delta)^{s/2} v_n|^2 + |v_n|^2 \, dx = \int \int_{\mathbb{R}^N \setminus \mathbb{R}^N} \frac{A_\alpha |v_n(x)|^p |v_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy + o(1).
\]
(3.35)

Note that for any \( i \neq m \),
\[
\int |(-\Delta)^{s/2} v^i|^2 + |v^i|^2 \, dx = \int |(-\Delta)^{s/2} v^m|^2 + |v^m|^2 \, dx,
\]
and
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A_\alpha |v_n^i(x)|^p |v_n^i(y)|^p \frac{dx}{|x-y|^{N-\alpha}} dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A_\alpha |v_n^m(x)|^p |v_n^m(y)|^p \frac{dx}{|x-y|^{N-\alpha}} dy. \]

By combining (3.28), (3.29) and (3.35), we then infer that
\[ \langle I'(v_n^i), v_n^i \rangle = o(1), \quad \forall 1 \leq |\mathcal{O}_{v_n}|. \]

By a similar argument as in (3.34), we obtain
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x)|^p |u_n(y)|^{p-2} u_n(y) w_n(y) - |w_n(x)|^p w_n(y) \frac{dx}{|x-y|^{N-\alpha}} dy = o(1). \]

By taking advantage of \( \lim_{n \to \infty} (I'(u_n), w_n) = 0 \) and (3.30), we then conclude \( \langle I'(w_n), w_n \rangle = o(1) \).

**Case 2:** up to a subsequence, for any \( \ell \geq 1 \), \( \lim_{n \to \infty} \text{dist}(y_n, \partial^\ell D) = +\infty \).

We take the test function that
\[ \phi_n^G = \sum_{g \in G} \xi(|x-g y_n|/r_n) u_n. \]

We conclude that \( \phi_n^G \in H^s_0(\mathbb{R}^N) \) since for every \( \bar{g} \in G \),
\[ \bar{g} \phi_n^G = \sum_{g \in G} \xi(|x-g \bar{g} y_n|/r_n) u_n(\bar{g}^{-1} x) = \psi(\bar{g}) \phi_n^G. \]

By repeating the arguments as above, we can also conclude (3.18).

Similarly, we define for each \( z^j \in G y_n \) that
\[ v_n^j(x) = \eta(|x-z^j|/r_n) u_n(x) \quad \text{and} \quad w_n = u_n - \sum v_n^j. \]

Since \( \text{supp} \ v_n^j \subset B_{3r_n}(z^j) \) for each \( j \), we deduce that \( v_n^j \in H^s(\mathbb{R}^N) \). It can also be checked that \( w_n \in H^s_0(\mathbb{R}^N) \). Similarly, we have the decompositions that
\[ \mathcal{E}(u_n) = \sum I(v_n^j) + I(w_n) + o(1), \quad \langle I'(v_n^j), v_n^j \rangle = o(1), \quad \text{and} \quad \langle I'(w_n), w_n \rangle = o(1). \]

It then follows that there exists \( t_n^j \in (0, +\infty) \) such that \( t_n^j v_n^j \in \mathcal{N} \) with \( t_n^j \) satisfying \( \lim_{n \to \infty} t_n^j = 1 \) for each \( 1 \leq j \leq |G| \). We therefrom deduce that
\[ c_G = \lim_{n \to \infty} I(u_n) = \sum \lim_{n \to \infty} I(v_n^j) + \lim_{n \to \infty} I(w_n) \geq \sum \lim_{n \to \infty} I(t_n^j v_n^j) \geq |G| c_0. \]

This is also in contradiction to Proposition 3.1 which gives that \( c_G < |G| c_0 \), so that (3.15) holds.

Therefore, when the compactness case happens, we take \( a_n = (1 - \text{Pr}_{J_k}) x_n \in \{0\} \times \mathbb{R}^{N-k} \) and \( T = M_1 + R_0 \), we get (3.13); alternatively, if the dichotomy holds, we obtain (3.13) by choosing \( a_n = (1 - \text{Pr}_{J_k}) y_n \in \{0\} \times \mathbb{R}^{N-k} \) and \( T = M_2 + R_1 \).

### 4. Nodal structures for the G-groundstate

We shall prove the signed property of the saddle solution \( u_G \) on a fundamental domain \( D \). Consequently, the \( G \)-groundstate has exactly \( |G| \) nodal domains demonstrating Coxeter nodal configurations. To this end, the \( s \)-harmonic extension method developed by Caffarelli and Silvestre [5] will be introduced.

Let \( X^{1,s}(\mathbb{R}^{N+1}_+) \) with \( s \in (0,1) \) be the closure of \( C^\infty_0(\mathbb{R}^{N+1}_+) \) under the norm
\[ \|U\|^2 = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla U|^2 dx + \int_{\mathbb{R}^N} |U(x,0)|^2 dx. \]
Let $\text{Tr}(V)$ be the trace of $V \in X^{1,s}(\mathbb{R}^{N+1}_+)$ on $\mathbb{R}^N \times \{y = 0\}$. We then have
\[
\|(-\Delta)^{s/2} \text{Tr}(V)\|^2_{L^2(\mathbb{R}^N)} \leq k_s^{-1} \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla V|^2 \, dx \, dy,
\] (4.1)

where $k_s = 2^{1-2s} \Gamma(1-s) \Gamma^{-1}(s)$. For a given $u \in H^s(\mathbb{R}^N)$, there exists a unique $U \in X^{1,s}(\mathbb{R}^{N+1}_+)$ such that
\[
\begin{align*}
- \text{div}(y^{1-2s} \nabla U) &= 0, \quad (x, y) \in \mathbb{R}^{N+1}_+, \\
U(x, 0) &= u(x), \quad x \in \mathbb{R}^N,
\end{align*}
\]
and
\[
\lim_{y \to 0} y^{1-2s} \frac{\partial U}{\partial y}(x, y) = -k_s (-\Delta)^s u(x).
\]

We usually call $U = h_s(u)$ the $s$-harmonic extension of $u$. It can also be expressed as
\[
\mathcal{F}(U)(\xi, y) = \mathcal{F}(u)(\xi) \psi(|\xi| y)
\] (4.2)
with $\psi$ minimizing the functional
\[
H(\psi) = \int_{y > 0} (|\psi(y)|^2 + |\psi'(y)|^2) y^{1-2s} \, dy.
\]

Moreover, it is shown that
\[
\int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla h_s(\xi)|^2 \, dx \, dy = \sqrt{k_s} \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}.
\] (4.3)

In what follows, the constant $k_s$ will be omitted for convenience. For any $g \in G$, we define the action $g$ on $U \in X^{1,s}(\mathbb{R}^{N+1}_+)$ by
\[
g \circ U(x, y) = U(g^{-1} x, y), \quad g^{-1} x = \text{diag}\{g, 1_{N-k}\} x.
\]

Recalling the unique epimorphism $\phi : G \to \{\pm 1\}$, we define the subspace
\[
X_G^{1,s}(\mathbb{R}^{N+1}_+) = \{ U \in X^{1,s}(\mathbb{R}^{N+1}_+) | g \circ U(x, y) = \phi(g) U(x, y), \forall g \in G \}.
\]

Let $J : X^{1,s}(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ be a functional defined by
\[
J(U) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla U|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} |U(x, 0)|^2 \, dx
\]
\[
- \frac{1}{2p} \int_{\mathbb{R}^N} (K_\alpha * |U(x, 0)|^p) |U(x, 0)|^p \, dx.
\]

We then define analogously that
\[
C_G = \inf_{U \in \mathcal{N}_G'} J(U),
\]
where $\mathcal{N}_G' = \mathcal{N}' \cap X_G^{1,s}(\mathbb{R}^{N+1}_+)$ with the Nehari manifold being defined by
\[
\mathcal{N}' = \{ U \in X^{1,s}(\mathbb{R}^{N+1}_+) \backslash \{0\} | \|U\|^2 = \int_{\mathbb{R}^N} (K_\alpha * |U(\cdot, 0)|^p) |U(x, 0)|^p \, dx \}.
\]

Now we shall show that $c_G = C_G$ and $U_G = h_s(u_G)$ (the $s$-harmonic extension of $u_G$) is also a minimizer of $J$ on the corresponding Nehari manifold, which may be of independent interest.
Proposition 4.1. Assume that $s \in (0, 1)$, $N \geq 3$, $\alpha \in (0, N)$ and (1.3) hold, then $c_G = C_G$. Moreover, if $u_G \in H^2_G(\mathbb{R}^N)$ is a solution of (1.1) such that $I(u_G) = c_G$, then $U_G$ satisfies

$$J(U_G) = C_G = c_G, \quad (J'(U_G), \varphi) = 0, \quad \forall \varphi \in X^{1,s}(\mathbb{R}^N_+)$$

Proof. We only need to show that $c_G = C_G$. On one hand, for any $u \in \mathcal{N}_G$, from (4.3), we see that $U = h_s(u) \in X^{1,s}(\mathbb{R}^N_+)$ and therefore deduce by the arbitrariness of $u \in \mathcal{N}_G$ that $U \in \mathcal{N}_G$. We then have by (4.1) that

$$\inf_{\mathcal{N}_G} I(t_u U(x, 0)) \leq \frac{1}{2} - \frac{1}{2p} \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} U(x, 0) \right|^2 + \left| U(x, 0) \right|^2 \, dx \right)^{\frac{p}{p-1}}$$

$$\leq \frac{1}{2} \left( \int_{\mathbb{R}^N} y^{1-2s} |\nabla U|^2 \, dy + \int_{\mathbb{R}^N} \left| U(x, 0) \right|^2 \, dx \right)^{\frac{p}{p-1}}$$

Hence, $c_G = \inf_{\mathcal{N}_G} I \leq \inf_{\mathcal{N}_G} J = C_G$. We then see that $U$ has exactly $|G|$ nodal domains.

Proposition 4.2. Let $u_G \in H^2_G(\mathbb{R}^N)$ be a solution of (1.1) such that $I(u_G) = c_G$. Then, $u_G$ has a constant sign on $\mathcal{D}$ and hence $u_G$ has exactly $|G|$ nodal domains.
Proof. Let $U_G$ be the $s$-harmonic extension of $u_G \in H^s(\mathbb{R}^N)$. Then, by Proposition 4.1

$$J(U_G) = C_G = c_G, \quad \langle J'(U_G), \varphi \rangle = 0, \quad \forall \varphi \in X^{1,s}(\mathbb{R}^N_+)$$

Since $u_G \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)(\text{see Lemma 2.11})$, we have $U_G \in L^\infty(\mathbb{R}^N_+)$ (see e.g., [2, Lemma 4.4]).

We now define $\overline{U}_G : \mathbb{R}^N_+ \to \mathbb{R}$ such that

$$\overline{U}_G = \begin{cases} |U_G(x,y)|, & (x,y) \in D \times \{y \geq 0\}, \\ \phi(y)|U_G(g^{-1}x,y)|, & (x,y) \in g(D) \times \{y \geq 0\}. \end{cases}$$

We then verify that $\overline{U}_G \in X^{1,s}(\mathbb{R}^N_+)$. Moreover, direct computations show us that

$$J(\overline{U}_G) = J(U_G) = C_G, \quad \langle J'(U_G), \overline{U}_G \rangle = \langle J'(U_G), U_G \rangle = 0.$$

Then, similar arguments as that in [39, Theorem 4.3], we conclude that $\overline{U}_G$ is a weak solution of

$$\left\{ \begin{array}{ll}
- \text{div}(y^{1-2s}\nabla V) = 0, & \text{in } \mathbb{R}^N_+,

\partial^s_v V = - V(x,0) + (K_\alpha * |V(\cdot,0)|^p) |V(x,0)|^{p-2} V(x,0), & \text{in } \partial \mathbb{R}^N_+, 
\end{array} \right. (4.4)$$

where

$$\partial^s_v V = - \frac{1}{k_s} \lim_{y \to 0} y^{1-2s} \frac{\partial V}{\partial y}(x,y) = (-\Delta)^s V(x,0).$$

Furthermore, we see that $\overline{u}_G = \overline{U}_G(x,0)$ is a weak solution of (1.1). By Lemma 2.11, we have $\overline{u}_G \in C^0(\mathbb{R}^N)$ and then $(K_\alpha * |\overline{u}_G|^p)|\overline{u}_G|^{p-2} \overline{u}_G \in C^0(\mathbb{R}^N)$. Note that $U_G \in L^\infty(\mathbb{R}^N_+)$ infers $\overline{U}_G \in L^\infty(\mathbb{R}^N_+)$. By applying a strong maximum principle (see e.g., [2, Corollary 4.12]) to system (4.4) in $\mathbb{R}^N_+ \cap D \times \{y \geq 0\}$, we conclude that $\overline{U}_G > 0$ in $D \times \{y \geq 0\}$. We thus have $|u_G| = \overline{u}_G > 0$ in $D$. \hfill \square

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