Multiplication kernels

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Received: 17 May 2021 / Revised: 14 November 2021 / Accepted: 15 November 2021 / Published online: 15 December 2021
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Abstract
We introduce the notion of multiplication kernels of birational and $D$-module type and give various examples. We also introduce the notion of a semi-classical multiplication kernel associated with an integrable system and discuss its quantization. Finally, we discuss geometric and algebraic aspects of method of separation of variables and describe hypothetically a cyclic $D$-module for the generalized multiplication kernels for Hitchin systems for groups $GL_r$.

Keywords Multiplication kernel · Langlands correspondence · Integrable system · Weinstein symplectic monoidal category

Mathematics Subject Classification 14E08 · 16H99 · 33E99 · 34L99

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1 Introduction

1.1 Informal explanation of the problem

Suppose we have a collection of commuting linear operators $T_\alpha$ acting on a finite-dimensional vector space $V$. Assume that the joint spectrum of these operators is simple. Then, we have a basis of $V$ (of joint eigenvectors of $T_\alpha$) defined up to permutation and rescaling. Assume furthermore that we choose a vector $v \in V$, which is cyclic, i.e., generates $V$ as the module over the algebra generated by $T_\alpha$. Then, the basis of eigenvectors $e_i$ is defined only up to permutations if we put the constraint $v = \sum e_i$, i.e., $v$ has coordinates $(1, 1, ..., 1)$. Alternatively, one may assume that a cyclic covector $u \in V^*$ is given and normalize basic elements by the condition $(u, e_i) = 1$.

The basis $\{e_i\}$ up to permutations can be encoded by the structure of a commutative associative initial algebra on $V$ with the multiplication $e_i \cdot e_j = \delta_{i,j}e_i$ where $\delta_{i,j}$ is the Kronecker delta.

In this paper, we deal with a functional analogue of this situation where the vector space $V$ is a space of functions (in a broad sense) in one or several variables, or more generally, $V$ is a space of functions on a smooth or algebraic manifold. A typical example is $V = C^\infty(\mathbb{R}^n)$ where the commuting operators are derivations $\frac{\partial}{\partial x_k}$, $k = 1, ..., n$. The continuous analogue of joint eigenvectors consists of Fourier modes $e_\lambda(x) = e^{ix\cdot\lambda}$, $\lambda \in \mathbb{R}^n$ where the normalization is given by $e_\lambda(0) = 1$. The multiplication is given by $e_\lambda(x) \ast e_\mu(x) = \delta(\lambda - \mu)e_\lambda(x)$ where $\delta(\lambda - \mu)$ is the Dirac

1 A lot of interesting examples of such collections appears in the theory of integrable systems and in representation theory.
Notice that this multiplication is the additive convolution and can be written in terms of the standard multiplication of functions as
\[ f * g(y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta(x_1 + x_2 - y) f(x_1) g(x_2) dx_1 dx_2. \]

Let \( \mathcal{F} \) be a vector space of functions on a manifold \( X \). We denote a typical element of \( \mathcal{F} \) by \( f(x) \in \mathcal{F} \), where \( x \in X \).

We want to study commutative associative multiplications \( \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \) on the vector space \( \mathcal{F} \). We write such multiplication in the form:
\[ f * g(y) = \int_{X \times X} K(x_1, x_2, y) f(x_1) g(x_2) dx_1 dx_2 \quad (1.1) \]
where \( K(x_1, x_2, y) \) is a kernel of our multiplication and \( dx \) is a measure on \( X \) defined by a volume form. Commutativity of our multiplication means:
\[ K(x_1, x_2, y) = K(x_2, x_1, y) \quad (1.2) \]
and associativity (if our multiplication is commutative) means
\[ \int_X K(x_1, x_2, y) K(y, x_3, z) dy = \int_X K(x_1, x_3, y) K(y, x_2, z) dy \quad (1.3) \]
How can one verify the associativity condition (1.3) if the kernel \( K \) is given in a closed form (for example, in terms of elementary functions)? In principle, one can suggest the following possibilities:

1. Compute explicitly the l.h.s. and the r.h.s. of Eq. (1.3).
2. Find a change of variables \( y \mapsto \tilde{y} \), possibly depending on \( x_1, x_2, x_3 \) and \( z \), which transforms the l.h.s. of the equation
\[ K(x_1, x_2, y) K(y, x_3, z) dy = K(x_1, x_3, y) K(y, x_2, z) d\tilde{y} \]
to its r.h.s. In this case, Eq. (1.3) also holds. More generally, it is sufficient to assume that the difference between the l.h.s. and the r.h.s. of the equality above is not zero, but a “total derivative” in variable \( y \), and the integration domain \( X \) is a “cycle”.
3. Prove that the l.h.s and the r.h.s. of Eq. (1.3) satisfy the same holonomic system of differential equations as a function in \( x_1, x_2, x_3, z \). Strictly speaking, this does not mean that Eq. (1.3) holds on the nose, but still can be considered as an associativity condition for a kernel \( K \).

In this paper, we do not deal with analysis as possibility 1 suggests and concentrate on the algebraic side of the problem. For example, assuming that \( X \) is a small circle in

\[ \Delta \]
around zero, when exploring possibility 1 we can reduce the integration to algebraic operations if \( K(x_1, x_2, y) \in \mathbb{C}[\frac{1}{y}][[x_1, x_2]] \), i.e., \( K \) is a power series in \( x_1, x_2 \) with coefficients polynomial in \( \frac{1}{y} \).

Exploring possibility 2, we assume that \( X \) is an algebraic variety, and a change of variables \( y \mapsto \tilde{y} \) defines a birational mapping \( X \to X \).

Finally, in possibility 3 we assume that \( X \) is a cycle, and therefore, \( \int_X \partial h \partial q dq = 0 \) for any \( h \). This assumption reduces computations to algebraic manipulations in differential algebra.

More generally, the kernel \( K \) can be given by integration of an “elementary function” over some auxiliary variables. Namely, let \( Q \) be another manifold with a measure \( dq \) defined by a volume form. Assume that \( K(x_1, x_2, y) = \int_Q K(x_1, x_2, y, q) dq \). (1.4)

In this case, associativity condition takes a form

\[
\int_{X \times Q \times Q} K(x_1, x_2, y, q_1)K(y, x_3, z, q_2)dydq_1dq_2
= \int_{X \times Q \times Q} K(x_1, x_3, y, q_3)K(y, x_2, z, q_4)dydq_3dq_4. \tag{1.5}
\]

Finally, let \( X, Q \) both be algebraic varieties over \( \mathbb{C} \). In this case, we assume that

\[ K = K_1^{s_1}...K_l^{s_l} \tag{1.6} \]

where \( s_1, ..., s_l \in \mathbb{C} \) are arbitrary parameters and \( K_1, ..., K_l \) are either algebraic functions or exponential of algebraic functions\(^3\). Here by algebraic function we mean rational function on a finite cover. These lead to the following:

**Definition 1.1** We say that \( K(x_1, x_2, y, q) \) is a multiplication kernel of birational type if there exists a birational automorphism \( (y, q_1, q_2) \to (\tilde{y}, \tilde{q}_3, \tilde{q}_4) \) of \( X \times Q \times Q \) which transforms the l.h.s. of the equation

\[
K(x_1, x_2, y, q_1)K(y, x_3, z, q_2)dydq_1dq_2
= K(x_1, x_3, \tilde{y}, \tilde{q}_3)K(\tilde{y}, x_2, z, \tilde{q}_4)d\tilde{y}d\tilde{q}_3d\tilde{q}_4
\]

to its r.h.s. If \( K = K_1^{s_1}...K_l^{s_l} \), then this birational automorphism should not depend on \( s_1, ..., s_l \).

The kernel \( K(x_1, x_2, y) \) given by (1.4), (1.6) satisfies a holonomic system of differential equations in \( x_1, x_2, y \), i.e., it gives a holonomic \( D \)-module on \( X^3 \) endowed with

\(^3\) In the case of general ground field, we replace \( K_i^{s_i} \) by \( \chi_i(K_i) \) where \( \chi_i \) are either multiplicative or additive characters of the ground field.
a cyclic vector. The associativity constraint (1.3) can be understood as an isomorphism between two holonomic $D$-modules in $x_1, x_2, x_3$ with cyclic vectors. The operation of integration corresponds to the direct image of $D$-modules.

**Definition 1.2** In the situation as above, we say that $K(x_1, x_2, y)$ is a multiplication kernel of $D$-module type.

**Remark 1.1** The associativity conditions from Definition 1.2 can be formulated less abstractly as follows, in the case when all varieties under considerations are affine. Assume that $K(x_1, x_2, y)$ is a solution of a holonomic system of differential equations in $x_1, x_2, y$. Let $I^{12}$ be the left ideal in the ring of differential operators in $x_1, x_2, y, x_3, z$ generated by differential equations of $K(x_1, x_2, y)$. Let $I^{23}$ be the left ideal in the same ring generated by differential equations of $K(y, x_3, z)$. It is clear that $I^{23}$ is obtained from $I^{12}$ by the change of variables $(x_1, x_2, y) \to (y, x_3, z)$. Define a left ideal $I^{123}$ in the ring of differential operators in $x_1, x_2, x_3, z$ by

$$I^{123} = (I^{12} + I^{23} + J) \cap R_{123}$$

where $J$ is the right ideal generated by $\frac{\partial}{\partial y}$ and $R_{123}$ is the ring of differential operators in $x_1, x_2, x_3, z$. By construction, $I^{123}$ consists of differential equations for

$$\int_X K(x_1, x_2, y)K(y, x_3, z)dy$$

where $X$ is a cycle (or, in other words, we assume that integrals over exact forms vanish). Indeed, let $L \in I^{123}$. Then, we can interchange $L$ and integration because $L \in R_{123}$. Moreover, $L(K(x_1, x_2, y)K(y, x_3, z))dy = 0$ modulo an exact form because $L \in I^{12} + I^{23} + J$. Therefore, $L$ gives a differential equation for our integral.

**Definition 1.1.2’.** We say that $K(x_1, x_2, y)$ is a multiplication kernel of $D$-module type if $I^{123}$ is invariant with respect to interchanging of $x_2$ and $x_3$.

### 1.2 Formal setups for functions and integrations

In the previous informal definitions of multiplication kernel, we use the notions of function, integration, etc., in a non-rigorous way. There are various rigorous formalisms for the notions of “explicit formulas” and “function” for an algebraic variety $X$ defined over a field $k$, none of which is totally satisfactory.

1. If $k$ is a local field (i.e., $k = \mathbb{R}, \mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$) and $\pi : Y \to X$ is a family of $n$-dimensional varieties over $X$, endowed with the volume element $vol \in \Gamma(Y, K_Y/X)$ along fibers of $\pi$. Here $K_Y/X = \Lambda^nT^*_Y/X$ is the relative canonical line bundle. Then, we obtain an $\mathbb{R}$-valued function on the set $\pi_*(\lfloor vol \rfloor)$, the integration of the density $|vol|$ along fibers.

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Note: We understand integral in (1.4) as direct image which means that our system of differential equations for $K(x_1, x_2, y)$ is a consequence of a holonomic system of differential equations for $K(x_1, x_2, y, q)$.
Here we assume that the integral is convergent, at least for the generic point \( x \in X(k) \).

One can twist the integral by additive and multiplicative characters applied to rational functions on \( Y \). (This is a formal replacement of exponentials and fractional powers.) Moreover, it is enough to assume more generally that for some integer \( N \geq 1 \) we have an element of \( \Gamma(Y, K_Y^{\otimes N}) \), e.g., \( \text{vol}^{\otimes N} \) if \( \text{vol} \) is defined up to multiplication by \( N \)-th root of 1.

This definition has several drawbacks. First, the same function on \( X(k) \) can be presented as an integral in different ways, and it is not clear whether the equality always follows from a birational equivalence. Second, one expects that some interesting functions (for example in representation theory) do not have such an integral representation (see [6] for discussion). Third, there is a problem with \( S_n \)-covariance for \( n \)-fold compositions of multiplication kernels, see Sect. 2.6 for details.

(2) If \( \text{char}(k) = 0 \), we can encode a “function” by a holonomic \( D \)-module endowed with a cyclic vector, see Sect. 2.7 for details. The drawback of this definition is that we cannot distinguish functions \( f \) and \( cf \) where \( c \) is a nonzero constant.

(3) We can forget the cyclic vector in the previous definition, and encode a function by an \textit{equivalence class} of holonomic \( D \)-modules.

(3') The previous definition can be transported to arbitrary characteristic, with holonomic \( D \)-modules replaced by motivic constructible sheaves.

(4) Finally, if we are in the case of positive characteristic \( p > 0 \) and \( k = \mathbb{F}_{p^n} \), then one can associate with motivic constructible sheaf a \( \overline{\mathbb{Q}}^{CM} \)-valued function\(^5\) on the finite set \( X(\mathbb{F}_{p^n}) \) given by the trace of Frobenius. Surprisingly, here we have again a well-defined function (as in 1)) although the information on the cyclic vector seems to be lost.

In all formalisms above, one can speak about integrals (as direct images) and hence the associativity constraint (1.3) makes sense. Therefore, one can speak about multiplication kernels in different contexts.

### 1.3 Multiplication formulas for special functions

Let us discuss a dual viewpoint on multiplication kernels. The product \( * \) defined by (1.3) on the space \( \mathcal{F} \) of functions gives by duality a coproduct \( \Delta \) on the dual space \( \mathcal{F}^* \) of densities. The continuous basis \( e_\lambda(x) \) of elementary projectors for \( * \) gives the dual basis \( e_\lambda^*(y) dx \) of \( \mathcal{F}^* \). The property \( \Delta(e_\lambda^*(y) dy) = e_\lambda^*(x_1) dx_1 \otimes e_\lambda^*(x_2) dx_2 \) is equivalent to\(^6\)

\[
e_\lambda^*(x_1)e_\lambda^*(x_2) = \int K(x_1, x_2, y)e_\lambda^*(y) dy
\]

\(^5\) Here \( \overline{\mathbb{Q}}^{CM} \subset \overline{\mathbb{Q}} \) stands for the maximal totally real extension of \( \mathbb{Q} \) with added \( \sqrt{-1} \). This notation comes from the theory of complex multiplication for abelian varieties.

\(^6\) In the examples below, we make an identification between \( \mathcal{F} \) and \( \mathcal{F}^* \) and write this formula in terms of \( e_\lambda \).
for all \( \lambda \), where \( K(x_1, x_2, y) \) is the same kernel as in (1.3) and, in particular, does not depend on \( \lambda \).

Let us give several examples where \( M \) is one-dimensional and eigenfunctions \( e_\lambda(x) \) are classical special functions. In these examples, \( T_\alpha \) consists of one differential operator \( T \).

**Example 1.1** \( M = \mathbb{R} \), \( T = \frac{d}{dx}, \ e_\lambda(x) = e^{\lambda x} \). We have \( T e_\lambda(x) = \lambda e_\lambda(x) \), the normalization is defined by \( e_\lambda(0) = 1 \), and

\[
e_\lambda(x_1)e_\lambda(x_2) = e_\lambda(x_1 + x_2) = \int_\mathbb{R} \delta(x_1 + x_2 - y)e_\lambda(y)dy.
\]

**Example 1.2** \( M = \mathbb{R}_>0 \), \( T = x \frac{d}{dx}, \ e_\lambda(x) = \lambda^x \). We have \( T e_\lambda(x) = \lambda e_\lambda(x) \), the normalization is defined by \( e_\lambda(1) = 1 \), and

\[
e_\lambda(x_1)e_\lambda(x_2) = e_\lambda(x_1x_2) = \int_\mathbb{R} \delta(x_1x_2 - y)e_\lambda(y)dy.
\]

**Example 1.3** \( M = \mathbb{R}_>0 \), \( T = \frac{d^2}{dx^2} + x^{-1} \frac{d}{dx}, \ e_\lambda(x) = J_0(\lambda x) \). We have \( T e_\lambda(x) = -\lambda^2 e_\lambda(x) \), the normalization is defined by \( \lim_{x \to 0} e_\lambda(x) = 1 \), and

\[
e_\lambda(x_1)e_\lambda(x_2) = \int_{|x_2 - x_1|}^{x_2 + x_1} \frac{e_\lambda(y)}{A(x_1, x_2, y)} \frac{ydy}{2\pi}.
\]

Here

\[
J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2} \left( \frac{x}{2} \right)^{2m} = \frac{1}{2\pi i} \oint e^{\frac{x+u-1}{2}} \frac{du}{u}
\]

is the Bessel function and

\[
A(x_1, x_2, y) = \frac{1}{4} \left( 2x_1^2x_2^2 + 2x_1^2y^2 + 2x_2^2y^2 - x_1^4 - x_2^4 - y^4 \right)^{\frac{1}{2}}
\]

is the area of a triangle with sides of the length \( x_1, x_2, y \). This multiplication formula is called Sonine–Gegenbauer formula [4,5].

### 1.4 Relations to Langlands correspondence and integrable systems

The theory of automorphic forms provides examples of commuting operators (Hecke operators). In the case of a curve \( C \) over finite field \( \mathbb{F}_q \), these operators act on the space of functions on the countable set of isomorphism classes of \( G \)-bundles on \( C \) where \( G \) is a reductive group. In the case \( G = \text{GL}_2 \), there are multiplicity one theorems, which guarantee that the joint spectrum is simple. In the old paper [7] of one of us, the multiplication kernel was written explicitly in a special case of rank 2 bundles on
with the parabolic structure in 4 points. Also, in the same paper, a kernel for the case of local field \( k \) was given by the formula:

\[
f \ast g(y) = \int_{y \in k, F_t(x_1, x_2, y) \in (k^*)^2} \frac{f(x_1)g(x_2)}{|F_t(x_1, x_2, y)|^2} |dy|
\]

where

\[
F_t(x_1, x_2, y) = (x_1 x_2 + x_1 y + x_2 y - t)^2 + 4x_1 x_2 y (1 + t - x_1 - x_2 - y).
\]

Recently, in the paper [1], Hecke operators in the case of curves over \( \mathbb{C} \) were defined (but not yet the multiplication kernels).

The joint spectrum of commuting integral Hecke operators for the case of curves over local fields is rather mysterious, and its relation to the usual Langlands program is quite unclear. In the case of a non-archimedean field \( k \) with the residue field \( \mathbb{F}_p \), a finite “low-frequency” part of the spectrum is presumably the same as the spectrum for the case of curves over finite fields and hence is related to Galois representations. For \( k = \mathbb{C} \), the joint spectrum is expected to coincide [1] with the set of opers (roughly speaking, differential equations of rank \( r \)) with real monodromy. Similarly, for \( k = \mathbb{R} \) the joint spectrum is expected to be the spectrum of the algebra of commuting differential operators on the set of \( \mathbb{R} \)-points of algebraic variety \( \text{Bun}_G \), coming from the quantization of Hitchin integrable system. Presumably, this description can be translated to a real algebraic constraint on the monodromy of an oper on the complex curve defined over \( \mathbb{R} \).

In general, if \( H_1, \ldots, H_n \) are independent commuting differential operators on an \( n \)-dimensional manifold, then for any scalar parameters \( \lambda_1, \ldots, \lambda_n \) we have a holonomic system

\[
(H_i - \lambda_i) \psi(x) = 0, \quad i = 1, \ldots, n.
\]

Let us denote \( \psi_{\lambda}(x) \) a solution of this system where \( \lambda = (\lambda_1, \ldots, \lambda_n) \). In order to have unique solution, one has to impose some normalization conditions. See Sect. 4 for details.

In the case \( G = \text{GL}_r \) and arbitrary constraints at singularities, there exists a remarkable birational symplectomorphism between the phase space of the Hitchin integrable system and the cotangent bundle to \( \text{Sym}^g \mathbb{C} \) where \( g \) is the genus of the generic spectral curve or, equivalently, the dimension of the base of the integrable system. This construction is called the method of separation of variables [13]. It is expected that in the case \( G = \text{GL}_r \) there exists an integral operator given by an explicit kernel, identifying functions on \( \text{Bun}_G \) and on \( \text{Sym}^g \mathbb{C} \). Moreover, eigenfunctions of commuting differential operators on \( \text{Bun}_G \) (or of Hecke operators) map to symmetric functions on \( \text{Sym}^g \mathbb{C} \) of the form \( \phi_{\lambda_1}(x_1) \ldots \phi_{\lambda_g}(x_g) \) which are external powers of functions in one variable.

In this presentation, the multiplication kernel can be expressed by subsequent integration in terms of a more elementary kernel, which we denote by \( K_{g+1,g} \). The latter is a function of \( 2g + 1 \) variables, whereas the original multiplication kernel is a function of \( 3g \) variables. We will discuss in detail this approach in Sect. 4.
1.5 Multiplication kernels in other contexts

The first group of questions we want to discuss here is related to commuting families of Hecke operators in the theory of modular forms. The multiplicity one theorems in the theory of automorphic forms for the group $\text{GL}_r$ are valid not only for the case of curves over finite fields, but also in the number field case. This leads, for example, to the following question concerning classical modular forms for the group $\text{SL}(2, \mathbb{Z})$.

For any $n \geq 1$, there are $d_n$ Hecke eigenforms:

$$f_i^{(n)}(q) = q + \sum_{j > 1} a_{i,j}^{(n)} q^j$$

of weight $n$ and level 1, where $a_{i,j}^{(n)}$, $i = 1, \ldots, d_n$ are eigenvalues of Hecke operator $H_j^{(n)}$ and $d_n$ is the dimension of the space of cusp forms of weight $n$. Coefficients of these forms are algebraic integers, not necessarily rational. Let us consider the generating series in four variables

$$K = \sum_{n \geq 1} t^n \sum_{i=1}^{d_n} f_i^{(n)}(q_1) f_i^{(n)}(q_2) f_i^{(n)}(q_3) \in \overline{\mathbb{Q}}[[q_1, q_2, q_3, t]].$$

One can show that $K \in \mathbb{Z}[[q_1, q_2, q_3, t]]$. We can also write $K$ in terms of traces of products of Hecke operators as

$$K = \sum_{n, j_1, j_2, j_3 \geq 1} \text{tr} \left( H_{j_1}^{(n)} H_{j_2}^{(n)} H_{j_3}^{(n)} \right) q_1^{j_1} q_2^{j_2} q_3^{j_3} t^n.$$

It will be interesting to find a closed formula for $K$. A similar question can be asked about higher-level modular forms and about Maass forms.

In order to explain the second group of questions, we start with an example. Let

$$X = [0, 1]$$

be the unit interval in $\mathbb{R}$, and $K : X^3 \to \{0, 1\}$ be the characteristic function of the closed tetrahedron with vertices $(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)$.

$$K(x, y, z) = \begin{cases} 1, & \text{if } x \leq y + z, \ y \leq x + z, \ z \leq x + y, \ x + y + z \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then, the following is true. For any integer $n \geq 1$ introduce the vector space

$$A_n = \mathbb{Q}^{X \cap \frac{1}{n} \mathbb{Z}} = \mathbb{Q}^{[0, \frac{1}{n}, \ldots, 1]} = \mathbb{Q}^{n+1}$$

with the basis $e_x, x \in X \cap \frac{1}{n} \mathbb{Z}$. Define a product in $A_n$ by

$$e_{x_1} \cdot e_{x_2} = \sum_{x_3 \in X \cap \frac{1}{n} \mathbb{Z}} K(x_1, x_2, x_3) e_{x_3}. $$
Then, this product is commutative and associative. In fact, it is the Verlinde algebra for $sl_2$ at level $n$.

The proof of associativity can be made independent of $n$. Namely, it follows from the existence of a piecewise-linear identification with integer coefficients ($\mathbb{Z}$PL in short) of two 5-dimensional polytopes fibered over $X^4$. Let

$$P_1 = \{(x_1, x_2, x_3, x_4, y); \ (x_1, x_2, y) \in K, \ (y, x_3, x_4) \in K\},$$
$$P_2 = \{(x_1, x_2, x_3, x_4, y); \ (x_1, x_3, y) \in K, \ (y, x_2, x_4) \in K\}.$$

Define two maps $\pi_i : P_i \to X^4$, $i = 1, 2$ by $\pi_i(x_1, x_2, x_3, x_4, y) = (x_1, x_2, x_3, x_4)$. One can check that for all $x_1, x_2, x_3, x_4$ the fibers $\pi_i^{-1}(x_1, x_2, x_3, x_4)$ and $\pi_2^{-1}(x_1, x_2, x_3, x_4)$ are closed intervals of the same length and can be identified by a shift. The resulting map $P_1 \to P_2$ is a $\mathbb{Z}$PL homeomorphism. This argument is similar to the cut-and-paste proof of associativity in the case of multiplication kernels for varieties over finite field studied in [7].

This example leads to several questions:

1. Generalize it to the case of other reductive groups,
2. Find other $\mathbb{Z}$PL examples of multiplication kernels,
3. Find the relation with multiplication kernels given by integral operators over non-archimedean fields,
4. Find similar formulas for multiplication kernels where numbers of integer points in polytopes are replaced by volumes of polytopes.

### 1.6 Other questions for functional analogue of tensor algebra

In this paper, we study explicit associative commutative kernels using purely algebraic framework (see discussion after associativity condition (1.3)). This algebraic approach can be applied to other problems in functional analogue of tensor algebra.

1. Let $\mathcal{F}_1, \mathcal{F}_2$ be two spaces of functions, possibly on different manifolds. One can study explicit kernels for mappings

$$R : \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{F}_2$$

subject to constraint $R(f_1, R(f_2, g)) = R(f_2, R(f_1, g))$ where $f_1, f_2, g \in \mathcal{F}_1$. This condition means that all linear operators on $\mathcal{F}_2$ of the form $g \mapsto R(f, g)$ commute. Note that associative commutative kernels provide examples of this structure in the case $\mathcal{F}_1 = \mathcal{F}_2$ because operators of multiplication by a given element commute in commutative associative algebras.

2. Let $e_\lambda(x)$ be the set of joint eigenfunctions of a family of commuting operators on a space of functions $\mathcal{F}$. Here we do not need to choose a normalization. Define a mapping

$$R : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$$
by \( R(e_\lambda(x) \otimes e_\mu(x)) = \delta(\lambda - \mu)e_\lambda(x) \otimes e_\lambda(x). \) One can study explicit kernels for this mapping for some interesting classes of commuting operators. Notice that \( R \) also satisfies to an analogue of associativity and commutativity constraints: \( R \) is invariant with respect to the action of \( S_2 \times S_2 \) and the composition \( R^{13} \circ R^{23} \) is invariant with respect to the action of \( S_3 \times S_3 \). A hypothetical example of such structure is given in Remark 3.6.

(3) Given \( g \geq 1 \), one can study kernels for a mapping

\[
R : \mathcal{F}^\otimes (g+1) \to \mathcal{F}^\otimes g
\]

invariant with respect to the action of \( S_{g+1} \times S_g \) such that \( R \circ (R \otimes Id \mathcal{F}) \) is invariant with respect to the action of \( S_{g+2} \times S_g \). We call this structure a \textit{generalized product}. Such a product induces a structure of an associative commutative algebra on the space \( \text{Sym}^g \mathcal{F} \) (see Proposition 4.1).

(4) One can study kernels for associative but not necessarily commutative multiplications.

(5) Let \( \mathcal{F}_1, \mathcal{F}_2 \) be two spaces of functions. One can study kernels for two inverse linear mappings \( \mathcal{F}_1 \to \mathcal{F}_2 \) and \( \mathcal{F}_2 \to \mathcal{F}_1 \). The problem of finding such explicit kernels can be thought of as a fundamental question of integral geometry in the sense of Gelfand–Gindikin–Graev [2].

### 1.7 Content of the paper

In Sect. 2, we define the notion of multiplication kernel of birational type in more rigorous way and give examples. Most of examples are related to the Hitchin systems for the group \( GL_2 \) on the curve \( \mathbb{P}^1 \) with 4 or more regular singular points. Another way of constructing examples is to solve certain functional equations, which is explained in Remark 2.5. Notice that Sects. 2.2–2.5 can be read independently of other parts of the paper, provided that the reader is fine with informal explanation of Definition 1.1 from Sect. 1.1.

In Sect. 3, we explain that classical integrable systems give commutative monoids in the Weinstein category of symplectic varieties and Lagrangian correspondences. We also discuss quantization, which is a construction of a multiplication kernel of \( D \)-module type starting from a quantum integrable system. Our examples are again related to Hitchin systems. It seems to be an interesting and important problem to find explicitly multiplication kernels for various quantum integrable systems.

In Sect. 4, we describe semi-classical geometry of Sklyanin’s method of separation of variables. The algebraic counterpart of this method is the notion of a generalized product, a map \( \text{Sym}^{g+1} V \to \text{Sym}^g V \) satisfying an analog of associativity constraint. We also introduce a hypothetical construction of quantum generalized products via formal solutions of differential equations expanded in a chosen base point. All considerations in Sect. 4 can be generalized to trigonometric and elliptic difference equations, which is beyond the standard geometric Langlands perspective.

In Sect. 5, we explain and illustrate, by examples, how to construct multiplication kernels satisfying property (1.3) where \( X \) is a small circle \( |z| = \varepsilon, 0 < \varepsilon \ll 1 \) starting
with a differential operator. These kernels can also be considered as lifts of more abstract kernels from Sect. 4. Notice that Sect. 5 can be read independently of other parts of the paper.

2 Multiplication kernels of birational type

2.1 Formulation of the problem

Let us fix a ground field \( k \) of characteristic zero and a commutative algebraic group \( A \) over \( k \).

Consider the following category \( C = C_{k, A} \). Its objects are smooth equidimensional varieties over \( k \). The set of morphisms \( \text{Hom}_C(X_1, X_2) \) is defined as the set of equivalence classes of tuples \((Z, \pi_1 : Z \to X_1, \pi_2 : Z \to X_2, \text{vol}, \rho : Z \to A)\) where \( Z \) is an equidimensional smooth variety over \( k \), \( \pi_1, \pi_2 \) are smooth morphisms (submersions), \( \text{vol} \in \Gamma(Z, K_{Z/X_2})/(\pm 1) \) is a volume element along fibers of \( \pi_2 \) up to a sign\(^8\), and \( \rho : Z \to A \) is an arbitrary map of varieties. The equivalence relation is generated by identifications of tuples:

\[
(Z, \pi_1, \pi_2, \text{vol}, \rho) \sim (U, \pi_1|_U, \pi_2|_U, \text{vol}|_U, \rho|_U)
\]  

(2.1)

for fixed \( X_1, X_2 \) where \( U \subset Z \) is a Zariski open dense subvariety of \( Z \).

Remark 2.1 If the ground field \( k \) is a local field, then the morphisms in \( C \) can be thought of as formal integral operators depending on a generic character of locally compact abelian group \( A(k) \). Namely, for a variety \( X/k \) denote by \( F_X \) the space of \( \mathbb{C} \)-valued continuous functions on \( X(k) \). A tuple \((Z, \pi_1 : Z \to X_1, \pi_2 : Z \to X_2, \text{vol}, \rho : Z \to A)\) as above gives a formal integral operator \( K : F_{X_1} \to F_{X_2} \) given by

\[
K(f)(x_2) = \int_{z \in \pi_2^{-1}(x_2)(k)} f(\pi_2(z)) \cdot \chi(\rho(z)) \cdot |\text{vol}|_{\pi_2^{-1}(x_2)}
\]

where \( \chi : A(k) \to \mathbb{C}^* \) is a character. We ignore the convergence issues here. This heuristics explains the following definition of the composition in \( C \).

For two tuples

\[
(Z, \pi_1 : Z \to X_1, \pi_2 : Z \to X_2, \text{vol}, \rho : Z \to A),
\]
\[
(Z', \pi_2' : Z' \to X_2, \pi_3' : Z' \to X_3, \text{vol}', \rho' : Z' \to A)
\]

their composition is given by the fibered product

\[
Z'' = Z \times_{X_2} Z'
\]

\(^7\) In our examples, \( A \) is the product of the additive group scheme \( \mathbb{G}_a \) and of several copies of the multiplicative group scheme \( \mathbb{G}_m \).

\(^8\) More generally, one can modify the definition by replacing \( \text{vol} \in \Gamma(Z, K_{Z/X_2}) \) by its power \( \text{vol}^\otimes N \in \Gamma(Z, K_{Z/X_2}^\otimes N) \) for \( N \geq 1 \).
endowed with maps \( \pi''_1 = \pi_1 \circ \pi''_2(\pi'_1) : Z'' \to X_1, \pi''_3 = \pi'_3 \circ (\pi'_1)^*(\pi_2) : Z'' \to X_3. \)

This can be represented by the following commuting diagram:

\[
\begin{array}{ccc}
Z'' & \stackrel{\pi''_2}{\longrightarrow} & Z' \\
\downarrow^{\pi''_1} & & \downarrow^{\pi'_2} \\
X_1 & \stackrel{\pi_1}{\longrightarrow} & X_2 \\
\downarrow^{\pi_2} & & \downarrow^{\pi'_1} \\
X_2 & \stackrel{\pi_2}{\longrightarrow} & X_3 \\
\downarrow^{\pi_3} & & \downarrow^{\pi'_3} \\
Z' & \stackrel{\pi''_3}{\longrightarrow} & Z'' \\
\end{array}
\]

The volume element \( vol'' \) along fibers of \( \pi''_3 \) is obtained by the multiplication of volume elements along fibers of maps \( Z'' \to Z' \) and \( Z' \to X_3 \). The map \( \chi'' : Z'' \to A \) is defined as the product in the group scheme \( A \) of the maps \( \rho \circ (Z'' \to Z) \) and \( \rho' \circ (Z'' \to Z') \).

One can check that the composition is well-defined on equivalence classes of representatives of morphisms.

The identity morphism \( \text{id}_C(X) \) is given by \( Z = X, \text{pr}_1 = \text{pr}_2 = \text{id} : X \to X, \text{vol} = 1, \text{and} \rho(x) = 0 \in A \).

One can see that two varieties \( X_1, X_2 \) are isomorphic as objects of \( C \) iff they are birationally equivalent.

**Remark 2.2** In the definition of \( C \), one can omit the condition that \( \pi_1 \) is a submersion and the equivalence relation generated by (2.1). In the modified category, one loses the birational invariance. On the other hand, if we want to keep the equivalence relation (and birational invariance), then the composition is defined if we assume that \( \pi_1 \) is dominant on each component of \( Z \). Passing to a Zariski open dense set \( U \subset Z \), we can replace this condition by the smoothness of \( \pi_1 \).

The category \( C \) carries the natural structure of a symmetric monoidal category. The tensor product on objects is given by the usual product of varieties. In the definition of the tensor product of morphisms, we use the product in group scheme \( A \).

**Definition 2.1** A multiplication kernel of birational type is a commutative semigroup object\(^9\) in \((C, \otimes)\).

**Remark 2.3** This definition seems to be too general, as we get some pathological examples. The issue is related to the fact that in our heuristics with local field, we ignore the question of convergence. As a first approximation to a better definition (which takes the convergence into account), one can suggest the following.

**Definition 2.2** A morphism \((Z, \pi_1 : Z \to X_1, \pi_2 : Z \to X_2, \text{vol}, \rho : Z \to A)\) in \( C \) is called *geometrically convergent* if the following property holds. Consider generic point \( x_2 \in X_2 \). Then \( \pi_2^{-1}(x_2) \) is a smooth variety which is mapped to \( X_1 \times A \) by

---

\(^9\) In the framework of the category \((C, \otimes)\) it is not reasonable to require an object to be a commutative monoid, which is a commutative semigroup object with a unit. The problem with units in our axiomatics is that in natural examples it should be given by a tuple where projection \( \pi_2 \) is not a submersion. In other formalisms (see Sect. 3), we include the existence of a unit.
Let $Y_{x_2}$ denote the image of a connected component of $\pi_2^{-1}(x_2)$. This is not necessarily a smooth variety, but it is nevertheless smooth at its generic point $y \in Y_{x_2}$. On the smooth variety $V_{x_2,y} = (\pi_1, \rho)^{-1}(y) \subset Y_{x_2}$, we have a volume element $\text{vol}_{x_2,y}$ defined up to multiplication by a nonzero constant. Namely, $\text{vol}_{x_2,y}$ is defined as the ratio of $\text{vol}_{\pi_2^{-1}(x_2)}$ and a nonzero element in $\Lambda^d T^*_y Y_{x_2}$ where $d = \dim Y_{x_2}$. We demand that $\text{vol}_{x_2,y}$ extends to a volume form without poles on some (or equivalently, on all) smooth compactification of $V_{x_2,y}$.

This definition of convergence is not completely satisfactory. For example, the composition of geometrically convergent morphisms is not necessarily geometrically convergent. This reflects the fact that in the case of local field and $A = 0$ the integral operator associated with a geometrically convergent morphism maps the space of bounded measurable functions to a larger space of unbounded measurable functions. One cannot compose such operators in general.

On the other hand, there exist situations when the integral operator corresponding to a not geometrically convergent morphism gives a well-defined compact operator. This is related to the fact that integrals
$$\int_{V(k)} |\text{vol}|$$
for a meromorphic volume element $\text{vol}$ on an algebraic variety $V/k$ can be convergent even if $\text{vol}$ has poles on $V(\bar{k})$. For example, the sphere $S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; \sum_{i=1}^n x_i^2 = 1\}$ has finite volume.

For a multiplication kernel of birational type $\mu \in \text{Hom}_C(X \times X, X)$, one can ask that $\mu_2 = \mu$, and $\mu_3 = \mu \circ (\mu \otimes \text{id}_X) \in \text{Hom}_C(X^3, X)$, ..., $\mu_n \in \text{Hom}_C(X^n, X)$, ... are geometrically convergent where $\mu_n$ corresponds to a product of length $n$.

### 2.2 One-dimensional examples without auxiliary integration

Let $X = \mathbb{A}^1$, $Z = X^3$ or a ramified covering of $X^3$. In this case, we do not have auxiliary integration in the formulas for kernels.

**Example 2.1** Let
$$K(x_1, x_2, y) = e^{c(x_1 x_2 y + y)} \frac{1}{y}$$
where $c$ is a constant. Then we have
$$K(x_1, x_2, y) \frac{1}{y} \int K(y, x_3, z) \, dy = K(x_1, x_3, \tilde{y}) \frac{1}{\tilde{y}} \int K(\tilde{y}, x_2, z) \, d\tilde{y}$$
if $\tilde{y} = \frac{x_1 x_2 + x_3 z + 1}{x_1 x_3 + x_2 z + 1} y$.

Example 2.1 looks degenerate because $\mu_n$ are not geometrically convergent for $n \geq 4$. It looks plausible, however, that further examples in this Section, as well as in Sects. 2.3, 2.4, are not degenerate in the sense that all $\mu_n$, $n \geq 2$ are geometrically convergent.

**Example 2.2** Here we assume $A = \mathbb{G}_a$ in notations of Sect. 2.1. Let
$$K(x_1, x_2, y) = e^{c\left(x_1 x_2 y + \frac{x_1}{x_2 y} + \frac{x_2}{y x_1} + \frac{y}{x_1 x_2}\right)} \frac{1}{y}$$
where \( c \) is a constant. Then, we have

\[
K(x_1, x_2, y) \ K(y, x_3, z) \ dy = K(x_1, x_3, \tilde{y}) \ K(\tilde{y}, x_2, z) \ d\tilde{y}
\]

if \( \tilde{y} = \frac{x_1x_2+x_3z}{x_1x_3+x_2c} y \).

A closely related version of this kernel is:

\[
K(x_1, x_2, y) = e^{\frac{x_1x_2x_3+y_1+y_3+y}{\sqrt{1+2}}} \frac{1}{y}.
\]

**Example 2.3** Here \( A = \mathbb{G}_m^2 \). Introduce the notation (see also the formula for \( F_i \) in Sect. 1.4)

\[
f_t(x, y, z) = (xy + yz + zx - t)^2 + 4xyz(1 + t - (x + y + z))
\]

where \( t \neq 0,1 \) is a parameter. Let

\[
K(x_1, x_2, y) = \left( \frac{x_1x_2(2y - 1) - (x_1 + x_2)y + t + w_{12}}{(x_1 - 1)(x_2 - 1)y} \right)^{c_1} \\
\quad \times \left( \frac{x_1x_2(2y - t) - t(x_1 + x_2)y + t^2 + tw_{12}}{(x_1 - t)(x_2 - t)y} \right)^{c_2} \frac{1}{w_{12}}
\]

where \( w_{12} := f_t(x_1, x_2, y)^{1/2} \) and \( c_1, c_2 \) are arbitrary constants. Then,

\[
K(x_1, x_2, y)K(y, x_3, x_4) \ dy = K(x_1, x_3, \tilde{y})K(\tilde{y}, x_2, x_4) \ d\tilde{y}
\]

(2.2)

where \( \tilde{y} \) is a function in \( x_1, x_2, x_3, x_4, y \) independent of \( c_1, c_2 \).

More precisely, let \( E_1 \subset \mathbb{A}^3 \) be an affine elliptic curve given by\(^{10}\)

\[
w_{12}^2 = f_t(x_1, x_2, y), \quad w_{34}^2 = f_t(y, x_3, x_4)
\]

and \( E_2 \subset \mathbb{A}^3 \) be an affine elliptic curve given by\(^ {11}\)

\[
w_{13}^2 = f_t(x_1, x_3, \tilde{y}), \quad w_{24}^2 = f_t(\tilde{y}, x_2, x_4).
\]

Then, there exists a unique birational mapping \( \rho: E_1 \rightarrow E_2 \) which transforms the l.h.s. of (2.2) to its r.h.s. In particular, the \( j \)-invariants of the elliptic curves \( E_1, E_2 \) are equal. This birational mapping has the form

\[
\rho: (y, w_{12}, w_{34}) \mapsto (\tilde{y}, w_{13}, w_{24})
\]

\(^{10}\) Here \( w_{12}, w_{34}, y \) are affine coordinates on \( \mathbb{A}^3 \) and \( t, x_1, x_2, x_3, x_4 \) are parameters.

\(^{11}\) Here \( w_{13}, w_{24}, \tilde{y} \) are affine coordinates on \( \mathbb{A}^3 \) and \( t, x_1, x_2, x_3, x_4 \) are parameters.
where \((y, w_{12}, w_{34}) \in E_1, (\tilde{y}, w_{13}, w_{24}) \in E_2\) and \(\tilde{y}, w_{13}, w_{24}\) are given by
\[
\tilde{y} = \frac{(x_1 - x_2)(x_3 - x_4)y^2 - w_{12}w_{34} + (x_1x_2 - t)(x_3x_4 - t) + \frac{y}{(x_1 - x_4)(x_2 - x_3)}Q_1}{2(x_1 - x_3)(x_2 - x_4) + \frac{2(x_1x_2 - x_3x_4)}{(x_1 - x_4)(x_2 - x_3)}Q},
\]
\[
w_{13} = \frac{(x_3 - x_4)w_{12}y - (x_1 - x_2)w_{34}y + \frac{w_{12}}{(x_1 - x_3)(x_2 - x_3)}Q_2 + \frac{w_{34}}{(x_1 - x_3)(x_1 - x_4)}Q_3}{2(x_2 - x_4)y + \frac{2(x_1x_2 - x_3x_4)}{(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)}Q},
\]
\[
w_{24} = \frac{(x_1 - x_2)w_{34}y - (x_3 - x_4)w_{12}y + \frac{w_{34}}{(x_2 - x_3)(x_2 - x_4)}Q_4 + \frac{w_{12}}{(x_2 - x_4)(x_1 - x_4)}Q_5}{2(x_1 - x_3)y + \frac{2(x_1x_2 - x_3x_4)}{(x_2 - x_3)(x_2 - x_4)(x_1 - x_4)}Q},
\]
here \(Q, Q_1, Q_2, Q_3, Q_4, Q_5\) are irreducible polynomials in \(x_1, x_2, x_3, x_4, t\) defined by the following properties:
\[
(x_2 - x_4)w_{13} - (x_1 - x_3)w_{24} = (x_3 - x_4)w_{12} - (x_1 - x_2)w_{34},
\]
\[
\rho : (0, x_1x_2 - t, x_3x_4 - t) \mapsto (0, x_1x_3 - t, x_2x_4 - t),
\]
\[
\rho : (1, x_1x_2 - x_1 - x_2 + t, x_3x_4 - x_3 - x_4 + t) \mapsto (1, x_1x_3 - x_1 - x_3 + t, x_2x_4 - x_2 - x_4 + t),
\]
\[
\rho : (t, x_1x_2 - tx_1 - tx_2 + t, x_3x_4 - tx_3 - tx_4 + t) \mapsto (t, x_1x_3 - tx_1 - tx_3 + t, x_2x_4 - tx_2 - tx_4 + t)
\]
where
\[
(0, x_1x_2 - t, x_3x_4 - t), (1, x_1x_2 - x_1 - x_2 + t, x_3x_4 - x_3 - x_4 + t),
\]
\[
(t, x_1x_2 - tx_1 - tx_2 + t, x_3x_4 - tx_3 - tx_4 + t) \in E_1,
\]
\[
(0, x_1x_3 - t, x_2x_4 - t), (1, x_1x_3 - x_1 - x_3 + t, x_2x_4 - x_2 - x_4 + t),
\]
\[
(t, x_1x_3 - tx_1 - tx_3 + t, x_2x_4 - tx_2 - tx_4 + t) \in E_2
\]
are rational points of the elliptic curves.

**Remark 2.4** The kernel in Example 2.3 depends on four points of \(\mathbb{P}^1\) which are set to 0, 1, \(t, \infty\). After an arbitrary fractional linear transformation of the variables \(x_1, x_2, x_3, x_4, y, \tilde{y}\), we obtain four pairwise distinct arbitrary points in \(\mathbb{P}^1\). Colliding some of these points, we obtain various degenerations of the kernels in this family, including the kernel in Example 2.2.

**Remark 2.5** Let
\[
K(x_1, x_2, y) = \phi(x_1, x_2, y)^c \psi(x_1, x_2, y)
\]
where \(\phi, \psi\) are symmetric with respect to \(x_1, x_2\). Assume that there exists a function \(\tilde{y}(x_1, x_2, x_3, y, z)\) independent of \(c\) such that
\[
K(x_1, x_2, y)K(y, x_3, z) \, dy = K(x_1, x_3, \tilde{y})K(\tilde{y}, x_2, z) \, d\tilde{y}.
\]
This condition gives a system of functional equations for the functions $\phi$, $\psi$, $\tilde{y}$. For computational purposes, it is convenient to set

$$\tilde{y} = y + q_1(x_1, x_2, y, z) \cdot (x_2 - x_3) + q_2(x_1, x_2, y, z) \cdot (x_2 - x_3)^2 + ...$$

and assume that the equations

$$\phi(x_1, x_2, y)\phi(y, x_3, z) = \phi(x_1, x_3, \tilde{y})\phi(\tilde{y}, x_2, z),$$
$$\psi(x_1, x_2, y)\psi(y, x_3, z) \, dy = \psi(x_1, x_3, \tilde{y})\psi(\tilde{y}, x_2, z) \, d\tilde{y}$$

hold simultaneously.

Examples 2.1, 2.3 are obtained by solving this system of functional equations, and Examples 2.6, 2.7 are obtained by solving the similar functional equations for kernels of the form

$$K(x_1, x_2, x_3, y) = \phi(x_1, x_2, x_3, y)^c \psi(x_1, x_2, x_3, y).$$

It looks that any solution of these functional equations is either listed in Examples 2.1, 2.3, 2.6 and 2.7 or can be obtained as a limit of the family described in Example 2.3. It would be interesting to study similar functional equations for more general kernels.

### 2.3 One-dimensional example with auxiliary integration

Let $X = \mathbb{A}^1$, $Z = \mathbb{A}^5$, $A = \mathbb{G}_m^4$. In this case, we have auxiliary integration over two-dimensional domain in the formulas for kernels. Fix $t \neq 0, 1$ as in Example 4. Let $s_1, s_2, s_3, r$ be arbitrary constants symbolizing a generic character of $A$.

**Example 2.4** Let

$$K(x_1, x_2, y, q_1, q_2) = (x_1 x_2)^{1-s_1}((x_1 - 1)(x_2 - 1))^{1-s_2}((x_1 - t)(x_2 - t))^{1-s_3} F(u, v)$$

where

$$u = \frac{(x_1 - 1)(x_2 - 1)(y - 1)}{(t - 1)^2}, \quad v = \frac{(x_1 - t)(x_2 - t)(y - t)}{t(t - 1)^2}$$

and

$$F(u, v) = (1 - q_1 - q_2)^{s_1 - r - 1} q_1^{s_2 - r} q_2^{s_3 - r} (q_1 q_2 + v q_1 + u q_2)^{r - 2}.$$
depending on \( x_1, x_2, x_3, z \), which transforms the l.h.s. of the equation

\[
K(x_1, x_2, y, q_1, q_2) K(y, x_3, z, q_3, q_4) \, dy \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4
= \pm K(x_1, x_3, \tilde{y}, \tilde{q}_1, \tilde{q}_2) K(\tilde{y}, x_2, z, \tilde{q}_3, \tilde{q}_4) \, d\tilde{y} \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4
\] (2.3)
to its r.h.s.

**Proof** The l.h.s. of Eq. (2.3) can be written as:

\[
f_1 f_2 f_3 f_4^r \cdot h \, \, dy \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4
\]

where

\[
f_1 = \frac{1 - q_1 - q_2}{x_1 x_2} \cdot \frac{1 - q_3 - q_4}{y x_3}
f_2 = \frac{q_1}{(x_1 - 1)(x_2 - 1)} \cdot \frac{q_3}{(y - 1)(x_3 - 1)}
f_3 = \frac{q_2}{(x_1 - t)(x_2 - t)} \cdot \frac{q_4}{(y - t)(x_3 - t)}
\]

\[
f_4 = \left( q_1 q_2 + \frac{(x_1-t)(x_2-t)(y-t)}{t(t-1)} q_1 + \frac{(x_1-1)(x_2-1)(y-1)}{(t-1)^2} q_2 \right) \left( q_3 q_4 + \frac{(y-t)(x_3-t)(z-t)}{t(t-1)} q_3 + \frac{(y-1)(x_3-1)(z-1)}{(t-1)^2} q_4 \right)
\]

\[
h = \frac{x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 - t)(x_2 - t)}{(1 - q_1 - q_2) \left( q_1 q_2 + \frac{(x_1-t)(x_2-t)(y-t)}{t(t-1)^2} q_1 + \frac{(x_1-1)(x_2-1)(y-1)}{(t-1)^2} q_2 \right)^2}
\]

The r.h.s. of Eq. (2.3) can be written as:

\[
\tilde{f}_1 \tilde{f}_2 \tilde{f}_3 \tilde{f}_4^r \cdot \tilde{h} \, d\tilde{y} \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4
\]

where \( \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{h} \) are obtained from \( f_1, f_2, f_3, f_4, h \) by swapping \( x_2 \) and \( x_3 \) and replacing \( y, q_1, q_2, q_3, q_4 \) by \( \tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4 \).

Let \( E \) be a curve in affine space \( \mathbb{A}^5 \) with coordinates \( y, q_1, q_2, q_3, q_4 \) given by

\[
f_1 = C_1, \quad f_2 = C_2, \quad f_3 = C_3, \quad f_4 = C_4.
\]

Let \( \tilde{E} \) be a curve in affine space \( \mathbb{A}^5 \) with coordinates \( \tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4 \) given by

\[
\tilde{f}_1 = C_1, \quad \tilde{f}_2 = C_2, \quad \tilde{f}_3 = C_3, \quad \tilde{f}_4 = C_4.
\]
here \((C_1, C_2, C_3, C_4)\) is a generic point of \(A = \mathbb{G}_m^4\).

Note that equations for the curve \(E\) can be written as:

\[
(1 - q_1 - q_2)(1 - q_3 - q_4) = y \cdot (C_1 x_1 x_2 x_3),
\]

\[
q_1 q_3 = (y - 1) \cdot \left( C_2(x_1 - 1)(x_2 - 1)(x_3 - 1) \right),
\]

\[
q_2 q_4 = (y - t) \cdot \left( C_3(x_1 - t)(x_2 - t)(x_3 - t) \right),
\]

\[
\left( q_1 q_2 + \frac{(x_1 - t)(x_2 - t)(y - t)}{t(t - 1)^2} q_1 + \frac{(x_1 - 1)(x_2 - 1)(y - 1)}{(t - 1)^2} q_2 \right)
\]

\[
\cdot \left( q_3 q_4 + \frac{(y - t)(x_3 - t)(z - t)}{t(t - 1)^2} q_3 + \frac{(y - 1)(x_3 - 1)(z - 1)}{(t - 1)^2} q_4 \right)
\]

\[
= y(y - 1)(y - t) \cdot \left( C_1 C_2 C_3 C_4 x_1 x_2 x_3 (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_1 - t) \right)
\]

\[
\times (x_2 - t)(x_3 - t) \right)
\]

and equations for the curve \(\tilde{E}\) are obtained from Eqs. \((2.4)\) by interchanging \(x_2\) and \(x_3\) and replacing \(y, q_1, q_2, q_3, q_4\) by \(\tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4\). One can show by direct computation that \(E\) and \(\tilde{E}\) are elliptic curves with the same \(j\)-invariant. To show this, we solve the first three equations in system \((2.4)\) with respect to \(q_1, q_2, y\) and substitute the result into the fourth equation. We obtain a plane cubic curve with coordinates \(q_3, q_4\); its genus and \(j\)-invariant can be computed in a usual way. After that, we observe that \(j\)-invariant is symmetric with respect to \(x_1, x_2, x_3\).

Observe that there exist rational points

\[
(y, q_1, q_2, q_3, q_4) = \left( 0, -\frac{C_2(t - 1)}{z - 1}, \frac{C_3 t(t - 1)}{z - t}, \frac{z - 1}{t - 1}, \frac{z - t}{t - 1} \right) \in E,
\]

\[
(y, q_1, q_2, q_3, q_4) = \left( 0, -\frac{C_2(t - 1)}{z - 1}, \frac{C_3 t(t - 1)}{z - t}, \frac{z - 1}{t - 1}, \frac{z - t}{t - 1} \right) \in \tilde{E}.
\]

This gives a birational mapping \(\mu : E \to \tilde{E}\) such that

\[
\mu(0, -\frac{C_2(t - 1)}{z - 1}, \frac{C_3 t(t - 1)}{z - t}, \frac{z - 1}{t - 1}, \frac{z - t}{t - 1}) = \left( 0, -\frac{C_2(t - 1)}{z - 1}, \frac{C_3 t(t - 1)}{z - t}, \frac{z - 1}{t - 1}, \frac{z - t}{t - 1} \right).
\]

One can also check that \(\mu\) transforms \(h dy \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4\) to \(\tilde{h} d\tilde{y} \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4\).
2.4 A hypothetical example of a generalized product

Here we suggest a hypothetical and more complicated example related to the generalized products described in Sect. 4.1. We still have $X = \mathbb{A}^1$ with auxiliary integration over $(n + 1)$-dimensional cycle, but our commutative associative operation has $n + 1$ inputs and $n$ outputs.

**Example 2.5** Let

$$K(x_1, \ldots, x_{2n+1}, q_1, \ldots, q_{n+1}) = (x_1 \ldots x_{2n+1})^s u_1^{-k_1} \ldots u_{n+1}^{-k_{n+1}} q_1^{2k_1-1} \ldots q_{n+1}^{2k_{n+1}-1}$$

$$\times \left( 1 + q_1 + \ldots + q_{n+1} \right)^{s-k_1-\ldots-k_{n+2}} \times \left( 1 + \frac{u_1}{q_1} + \ldots + \frac{u_{n+1}}{q_{n+1}} \right)^{s+k_1+\ldots+k_{n+2}}$$

(2.5)

where

$$u_1 = \frac{(x_1 - 1) \ldots (x_{2n+1} - 1) t_1^2 \ldots t_n^2}{x_1 \ldots x_{2n+1} (t_1 - 1)^2 \ldots (t_n - 1)^2},$$

$$u_{i+1} = \frac{(x_1 - t_i) \ldots (x_{2n+1} - t_i) \prod_{j \neq i} t_j^2}{x_1 \ldots x_{2n+1} (t_i - 1)^2 \prod_{j \neq i} (t_j - t_i)^2}, \ i = 1, \ldots, n.$$ 

here $t_1, \ldots, t_n \neq 0, 1$ are pairwise distinct parameters and $s, k_1, \ldots, k_{n+2}$ are arbitrary constants.

**Conjecture 2.1** There exists a birational mapping

$$(y_1, \ldots, y_n, q_1, \ldots, q_{2n+2}) \rightarrow (\tilde{y}_1, \ldots, \tilde{y}_n, \tilde{q}_1, \ldots, \tilde{q}_{2n+2})$$

which transforms the l.h.s. of the equation

$$K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n, q_1, \ldots, q_{n+1}) K(x_{n+2}, y_1, \ldots, y_n, z_1, \ldots, z_n, q_{n+2}, \ldots, \times q_{2n+2}) d y_1 \wedge \ldots \wedge d y_n \wedge d q_1 \wedge \ldots \wedge d q_{2n+2}$$

$$= \pm K(x_1, \ldots, x_{n+2}, \tilde{y}_1, \ldots, \tilde{y}_n, \tilde{q}_1, \ldots, \tilde{q}_{n+1}) K(x_{n+1}, \tilde{y}_1, \ldots, \tilde{y}_n, z_1, \ldots, z_n, \times \tilde{q}_{n+2}, \ldots, \tilde{q}_{2n+2}) d \tilde{y}_1 \wedge \ldots \wedge d \tilde{y}_n \wedge d \tilde{q}_1 \wedge \ldots \wedge d \tilde{q}_{2n+2}$$

to its r.h.s. Here the r.h.s. is obtained from the l.h.s. by interchanging $x_{n+1}$ and $x_{n+2}$, and replacing the variables $y_1, \ldots, y_n, q_1, \ldots, q_{2n+2}$ by $\tilde{y}_1, \ldots, \tilde{y}_n, \tilde{q}_1, \ldots, \tilde{q}_{2n+2}$.

Notice that in the Conjecture above $n > 1$ because if $n = 1$, then this family of kernels is essentially the same as in Example 5, so for $n = 1$ this is proved.
The kernel in this example can be written in more symmetric form

\[
\int K(x_1, \ldots, x_{2n+1}, q_1, \ldots, q_{n+1}) dq_1 \ldots dq_{n+1} = \int \prod_{i=1}^{n+2} (d_{i,+} q_{i,-}) \cdot \left( \sum_{i=1}^{n+2} q_i, + \right)^{s-k_1-\ldots-k_2+n+2} \left( \sum_{i=1}^{n+2} q_i, - \right)^{s+k_1+\ldots+k_{n+2}} \cdot \frac{\prod_{i=1}^{n+2} \frac{dq_{i,+}}{q_{i,+}}}{\prod_{i=1}^{n+2} \frac{dq_{i,-}}{q_{i,-}}}
\]

where we integrate over the \((n+1)\)-dimensional torus \(\mathbb{G}_m^{n+2}/(\mathbb{G}_m)_{\text{diag}}\) with coordinates \(q_{1,+}, \ldots, q_{n+2,+}\) factorized by the diagonal action \(q_{i,+} \mapsto \lambda q_{i,+}\) of \(\mathbb{G}_m\). The variables \(q_{1,-}, \ldots, q_{n+2,-}\) are defined by:

\[
q_{i,+} q_{i,-} = \frac{\prod_{\alpha=1}^{2n+1} (x_\alpha - t_i)}{P'(t_i)^2}, \quad i = 1, \ldots, n + 2
\]

where \(P(u) = \prod_{i=1}^{n+2} (u - t_i)\). Here, we use arbitrary pairwise distinct parameters \(t_1, \ldots, t_{n+2}\), and the previous formula for the kernel \(K(x_1, \ldots, x_{2n+1}, q_1, \ldots, q_{n+1})\) corresponds to the choice \(t_{n+1} = 0, \ t_{n+2} = 1\).

The expression which we integrate is invariant under the diagonal action of \(\mathbb{G}_m\). It is also invariant with respect to the group \(S_{n+2}\) acting on variables \(q_{1,+}, \ldots, q_{n+2,+}\) by permutations. One can achieve invariance with respect to a larger group \(S_{n+2} \ltimes (\mathbb{Z}/2)^{n+2}\) (where the \(i\)th generator of \((\mathbb{Z}/2)^{n+2}\) acts by interchanging of \(q_{i,+}\) and \(q_{i,-}\)) by adding one extra variable of integration (and losing geometric convergence). We have

\[
\int K(x_1, \ldots, x_{2n+1}, q_1, \ldots, q_{n+2}) dq_1 \ldots dq_{n+2} = C \int \prod_{i=1}^{n+2} \left( \frac{q_{i,+}}{q_{i,-}} \right)^{2x} \prod_{i=1}^{n+2} \frac{dq_{i,+}}{q_{i,+}}
\]

where we integrate over the \((n+2)\)-dimensional torus with coordinates \(q_{1,+}, \ldots, q_{n+2,+}\). Here \(C\) is independent of \(x_1, \ldots, x_{2n+1}\).

Indeed, our two expressions for the kernel can be written as:

\[
F_1 = \int \left( \sum_{i=1}^{n+2} a_i q_i + \sum_{i=1}^{n+2} b_i \frac{q_i}{q_i} \right)^{\alpha + \beta} \prod_{i=1}^{n+2} q_i^{\lambda_i} \prod_{i=1}^{n+2} \frac{dq_i}{q_i},
\]

\[
F_2 = \int \left( \sum_{i=1}^{n+2} a_i q_i \right)^{\alpha} \left( \sum_{i=1}^{n+2} b_i \frac{q_i}{q_i} \right)^{\beta} \prod_{i=1}^{n+2} q_i^{\lambda_i} \prod_{i=1}^{n+2} \frac{dq_i}{q_i}
\]
for some \( a_i, b_i, \lambda_i, \alpha, \beta \) such that

\[
\sum_{i=1}^{n+2} \lambda_i = \beta - \alpha.
\]

Notice that the expressions \( F_1, F_2 \) both satisfy the same holonomic system of differential equations\(^{12}\):

\[
\left( a_i \frac{\partial}{\partial a_i} - b_i \frac{\partial}{\partial b_i} + \lambda_i \right) F = 0, \quad i = 1, \ldots, n + 2,
\]

\[
\left( \sum_{i=1}^{n+2} \left( a_i \frac{\partial}{\partial a_i} + b_i \frac{\partial}{\partial b_i} \right) - \alpha - \beta \right) F = 0,
\]

\[
\frac{\partial^2 F}{\partial a_1 \partial b_1} = \frac{\partial^2 F}{\partial a_2 \partial b_2} = \ldots = \frac{\partial^2 F}{\partial a_{n+2} \partial b_{n+2}}
\]

(2.6)

which means that they should be in a sense equal.

The equality \( F_2 = CF_1 \) can be also shown as follows. Introduce an auxiliary function

\[
\phi(A, B) = \int \delta \left( \sum_{i=1}^{n+2} a_i q_i - A \right) \cdot \delta \left( \sum_{i=1}^{n+2} b_i q_i - B \right) \cdot \prod_{i=1}^{n+2} q_i^{\lambda_i} \cdot \prod_{i=1}^{n+2} dq_i.
\]

The function \( \phi(A, B) \) is homogeneous

\[
\phi(qA, q^{-1}B) = q^{\beta-\alpha} \phi(A, B) \quad \text{for all} \quad q \neq 0,
\]

so we can write

\[
\phi(A, B) = \phi_0(AB) \cdot A^{\beta-\alpha}
\]

where \( \phi_0(u) = \phi(1, u) \). We have

\[
F_1 = \int (A + B)^{\alpha+\beta} \phi(A, B) dA dB = \int (A + B)^{\alpha+\beta} \phi_0(AB) \cdot A^{\beta-\alpha} dA dB
\]

\[
= \int \left( A + \frac{u}{A} \right)^{\alpha+\beta} \phi_0(u) A^{\beta-\alpha} du \cdot \frac{dA}{A} = \int u^\beta \phi_0(u) du \cdot \int \frac{(1 + v^2)^{\alpha+\beta}}{2v^{2\alpha}} \frac{dv}{v}
\]

where we made substitutions \( B = \frac{u}{A} \) and \( A = v \sqrt{u} \).

\(^{12}\) Such expressions and the corresponding \( D \)-modules belong to a class of \( A \)-hypergeometric functions and systems [3].
A similar computation for $F_2$ gives:

$$
F_2 = \int A^\alpha B^\beta \phi(A, B) dA dB = \int A^\alpha \left( \frac{u}{A} \right)^\beta \phi_0(u) A^{\beta-\alpha} d\frac{u}{A} \\
= \int u^\beta \phi_0(u) du \cdot \int \frac{dA}{A}.
$$

Therefore, $F_1$ and $F_2$ are both equal to the integral $\int u^\beta \phi_0(u) du$ multiplied by a constant independent of $a_i, b_i, \ i = 1, ..., n + 2$. Moreover, removing the integral $\int \frac{dA}{A}$ from our final expression for $F_2$, we cure its geometrical divergence.

### 2.5 Some examples with three inputs and one output

Here we give examples of products $\mu_3$ with three inputs and one output which look a bit pathological but still satisfy an analog of associativity and commutativity conditions: $\mu_3$ is $S_3$-invariant and $\mu_3 \circ (\mu_3 \otimes id)$ is $S_5$-invariant.

**Example 2.6** Let

$$
K(x_1, x_2, x_3, y) = e^{c(x_1 x_2 x_3 y + y)} \frac{1}{y}
$$

where $c$ is a constant. Then, we have

$$
K(x_1, x_2, x_3, y) \ K(y, x_4, x_5, z) \ dy = K(x_1, x_2, x_4, \tilde{y}) \ K(\tilde{y}, x_3, x_5, z) \ d\tilde{y}
$$

if $\tilde{y} = \frac{x_1 x_2 x_3 + x_4 x_5 z + 1}{x_1 x_2 x_4 + x_3 x_5 z + 1} y$.

This example is similar to Example 1, and it is degenerate in the same sense.

**Example 2.7** Let

$$
K(x_1, x_2, x_3, y) = \left( \frac{1}{y} + \frac{1}{y} \sqrt{1 + x_1 x_2 x_3 y} \right)^c \frac{1}{\sqrt{1 + x_1 x_2 x_3 y}}
$$

where $c$ is an arbitrary constant. Then,

$$
K(x_1, x_2, x_3, y) \ K(y, x_4, x_5, z) \ dy = K(x_1, x_2, x_4, \tilde{y}) \ K(\tilde{y}, x_3, x_5, z) \ d\tilde{y} \quad (2.7)
$$

where $\tilde{y}$ is a function in $x_1, x_2, x_3, x_4, x_5, y, z$ independent of $c$.

More precisely, let $C_1 \subset \mathbb{A}^3$ be a rational affine curve given by

$$
w_{123}^2 = 1 + x_1 x_2 x_3 y, \quad w_{45}^2 = 1 + y x_4 x_5 z \quad (2.7)
$$

---

13 Here $w_{123}, w_{45}, y$ are affine coordinates on $\mathbb{A}^3$ and $x_1, x_2, x_3, x_4, x_5, z$ are parameters.
and $C_2 \subset \mathbb{A}^3$ be a rational affine curve given by\textsuperscript{14}

$$w^2_{124} = 1 + x_1x_2x_4\tilde{y}, \quad w^2_{35} = 1 + \tilde{y}x_3x_5z.$$  

Then, the formulas

$$w_{124} = \frac{(x_1x_2 - x_5z)x_4w_{123} + (x_3 - x_4)x_1x_2w_{45}}{x_1x_2x_3 - x_4x_5z},$$

$$w_{35} = \frac{(x_3 - x_4)x_5zw_{123} + (x_1x_2 - x_5z)x_3w_{45}}{x_1x_2x_3 - x_4x_5z},$$

$$\tilde{y} = \frac{2(x_3 - x_4)(x_1x_2 - x_5z)(w_{123}w_{45} - 1) + x_3x_4(x_1x_2 - x_5z)^2y + x_1x_2x_5z(x_3 - x_4)^2y}{(x_1x_2x_3 - x_4x_5z)^2}$$

define a birational mapping $C_1 \to C_2$, which transforms the l.h.s. of (2.7) to its r.h.s.

### 2.6 On $S_n$-covariance of higher compositions of kernels

If $K$ is a multiplication kernel of birational type on a variety $X/k$, then for any $n \geq 2$ and any planar binary rooted tree $T$ with $n$ leaves (or, equivalently, choice of bracketing on a product of $n$ symbols) we get a variety $Z_T$ which maps to $X^{n+1} \times A$ and is endowed with a volume element along fibers for the projection to the last factor $X$. This variety $Z_T$ corresponds to the kernel of the $n$-fold product with the chosen bracketing. Associativity implies that there exists a birational identification of varieties $Z_T$ for different trees $T$. Commutativity of our kernel implies that one can lift each permutation $\sigma \in S_n$ of the first $n$ factors in $X^{n+1}$, to the birational identification of the varieties $Z_T$. These identifications, however, are not compatible in general. It is desirable to lift these identifications to an $S_n$-action and construct just one variety $Z_n$ (with an action of $S_n$) which maps to $X^{n+1} \times A$ in an $S_n$-covariant way.

One of the reasons why it is desirable is the following. Let $k$ be a local field and assume that the kernel is geometrically convergent. Then, we expect that an appropriate space $\mathcal{F}(X(k))$ of complex-valued functions on $X(k)$ to be a commutative associative algebra with product $*$ given by our kernel $K$. For any separable finite extension $k' \supset k$, we get another algebra $(\mathcal{F}(X(k')), *)$. The existence of $S_n$-action on $Z_n$ for $n = \text{deg}(k'/k)$ gives rise to a homomorphism

$$(\mathcal{F}(X(k')), *) \to (\mathcal{F}(X(k)), *). \tag{2.8}$$

Namely, an extension $k' \supset k$ gives a transitive action of $\text{Gal}(\bar{k}/k)$ on an $n$-element set (the set of embeddings $k' \hookrightarrow \bar{k}$ over $k$), hence a homomorphism $\rho : \text{Gal}(\bar{k}/k) \to S_n$, up to conjugation. Using $\rho$ we can define a twisted form $Z_{n, \rho}$ of $Z_n$. Recall that the set $Z_{n, \rho}(\bar{k})$ of $k$-points of $Z_{n, \rho}$ coincides with $Z_n(\bar{k})$, but the action of $\text{Gal}(\bar{k}/k)$ is twisted by $\rho$. The variety $Z_{n, \rho}$ maps to the product $(X^n)_{\rho} \times X \times A$ where $(X^n)_{\rho}$ is

\textsuperscript{14} Here $w_{124}$, $w_{35}$, $\tilde{y}$ are affine coordinates on $\mathbb{A}^3$ and $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $z$ are parameters.
the twisted form of \( X^n \), i.e., the Weil restriction \( \text{Res}_{k'/k}(X) \). Recall that the latter is a variety over \( k \) such that its set of \( k \)-points is \( X(k') \). The twisted multiplication kernel gives a map (2.8). One can check that it is a homomorphism of algebras.

In the case of varieties over finite fields (see setup 4 in Sect. 1.2) there are homomorphisms (2.8) for extensions \( k' = F_{q^m} \supset k = F_q \) of finite fields, even without the action of \( S_n \) on \( Z_n \). It looks plausible that in this case there is still an action of \( S_n \) on the corresponding motivic constructible sheaves. The tower of homomorphisms (2.8), or dually, maps (which are inclusions for finite fields)

\[
\text{Spec } (\mathcal{F}(X(k)), *) \hookrightarrow \text{Spec } (\mathcal{F}(X(k')), *)
\]

played an essential role in [7]. The inductive limit

\[
\lim_{\rightarrow m} \text{Spec } (\mathcal{F}(X(F_{q^m})), *)(\mathbb{Q})
\]

is an infinite countable set endowed with commuting actions of two Galois groups: \( \text{Gal}(\overline{F_q}/F_q) \) and \( \text{Gal}((\mathbb{Q})^{CM}/\mathbb{Q}) \).

Unfortunately, in general it seems to be impossible to lift the \( S_n \) action from \( X^n \times X \times A \) to \( Z_n \). Let us describe this problem in the case of Example 2.2. Every binary rooted tree \( T \) with \( n \geq 3 \) leaves gives a family of Calabi–Yau varieties \( V_{x_1,\ldots,x_{n+1},t}^T \) of dimension \( n - 3 \) depending on \( n + 2 \) parameters \( (x_1,\ldots,x_{n+1},t) \in (\mathbb{G}_m)^{n+1} \times \mathbb{G}_a \) in the following way. The variety \( V_{x_1,\ldots,x_{n+1},t}^T \) is a hypersurface in a toric variety, and it is given by the following equation in variables \( (y_1,\ldots,y_{n-2}) \in \mathbb{G}_m^{n-2} \)

\[
\sum_{v \in \{\text{vertices of } T\}} f_v = t,
\]

where the parameters \( x_1,\ldots,x_n \) are attached to the leaves of \( T \), the parameter \( x_{n+1} \) is attached to the root of \( T \), and the parameters \( y_1,\ldots,y_{n-2} \) are attached to the inner edges of \( T \). For each vertex \( v \), we define

\[
f_v = f(z_1,z_2,z_3) = z_1z_2z_3 + \frac{z_1}{z_2z_3} + \frac{z_2}{z_1z_3} + \frac{z_3}{z_1z_2}
\]

where \( z_1, z_2, z_3 \) are variables attached to three edges adjacent to \( v \). For example, if

\[
T = \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
& x_1 & x_2 & x_3 & x_4 \\
& & y_1 & y_2 & \\
& & & & x_5
\end{array}
\]
then $V_{x_1,\ldots,x_5,t}$ is an elliptic curve given by

$$f(x_1, x_2, y_1) + f(x_3, x_4, y_2) + f(y_1, y_2, x_5) = t.$$ 

The variable $t$ is the coordinate on $A = \mathbb{G}_m = \mathbb{A}^1$.

The volume element (up to a sign) on $V_{x_1,\ldots,x_{n+1},t}$ is given by:

$$\frac{\wedge_{i=1}^{n-2} d \log y_i}{d (\sum f_v)}.$$ 

The integral operator corresponding to $Z_T$ is given by a density on $X^{n+1} = (\mathbb{G}_m)^{n+1}$ with coordinates $x_1, \ldots, x_{n+1}$. This density is the Fourier transform of the function $t \mapsto vol(V_{x_1,\ldots,x_{n+1},t})$ in the variable $t$. One can construct a birational identification $V_{x_1,\ldots,x_{n+1},t} \sim V_{x_1,\ldots,x_{n+1},t}'$ where $T'$ is obtained by a flip from $T$.

The composition of five flips corresponding to the pentagon relation is a non-trivial automorphism of the K3 surface $V_{x_1,\ldots,x_5,t}$.

It might be possible to cure this problem by increasing the dimension of $Z_n$ (or equivalently, adding auxiliary integration variables). This might require an extension of our formalism. One such possibility is discussed in the next Section.

### 2.7 Direct images of cyclic $D$-modules at a generic point

In this section, we propose a mixed birational/$D$-module type formalism using which one can speak about multiplication kernels. In this formalism, like in Sect. 2.1, one considers algebraic varieties only up to birational equivalence. On the other hand, we deal here with cyclic $D$-modules. Roughly speaking, we encode a “function” $f$ on an algebraic variety $X$ by a system of algebraic linear differential equations satisfied by $f$ at the generic point of $X$. The most interesting operation with these objects is the direct image (which we denote by $\pi_*$) defined below, which informally corresponds to the integration over an unspecified cycle. Almost all examples in our paper can be understood in this mixed formalism.

Let $X$ be a smooth algebraic variety over field $k$ of characteristic zero, endowed with a line bundle $\mathcal{L}$. Denote by $\text{Diff}_{\mathcal{L},\text{rat}}$ the algebra of differential operators in $\mathcal{L}$ with coefficients in rational functions on $X$. Denote by $\mathcal{I}_{\mathcal{L}}$ the set of left ideals in $\text{Diff}_{\mathcal{L},\text{rat}}$. We can think about elements of $\mathcal{I}_{\mathcal{L}}$ as systems of differential equations on a “section” of $\mathcal{L}$ (e.g., analytic germ of a section for $k = \mathbb{C}$). For any $I \in \mathcal{I}_{\mathcal{L}}$, the quotient $\text{Diff}_{\mathcal{L},\text{rat}}/I$ is a cyclic $\text{Diff}_{\mathcal{L},\text{rat}}$-module. In this paper, we deal only with
examples where this cyclic module is holonomic, i.e., finite-dimensional over \( k(X) \). It is clear that both \( \text{Diff}_{L, \text{rat}} \) and \( L \) depend only on the birational type of \( X \) (and also of \( L \)), i.e., only on the field \( k(X) \) of rational functions, together with a 1-dimensional module over it. Moreover, algebra \( \text{Diff}_{L, \text{rat}} \) is identified with \( \text{Diff}_{X, \text{rat}} := \text{Diff}_{O_X, \text{rat}} \) if one chooses a rational trivialization on \( L \).

There are three basic operations:

1. For a dominant map \( \pi : Y \to X \), and \( L \) on \( X \), we have a pullback \( \pi^* : L \to \pi^* L \).

2. In the same situation, we have pushforward \( \pi_* : \pi_* L \otimes_{k_Y} \pi^{-1}(X) \to L \).

3. For a collection of line bundles \( L_1, \ldots, L_n \) on \( X \), we have product \( \prod_i \pi_{iL} \to \prod_i L_i \).

In order to explain (1) and (2) it is convenient to have a nonlinear flat connection on the fibration \( \pi : Y \to X \). We claim that there exists a Zariski open dense set \( Y' \subset Y \) such that \( \pi|_{Y'} \) is a submersion, and a nonlinear flat connection \( \nabla \) on \( \pi|_{Y'} : Y' \to X \), i.e., an integrable distribution on \( Y' \) transversal to fibers of \( \pi \).

Indeed, passing to an open dense subset of \( Y \) we may assume that fibers of \( \pi \) are smooth locally closed subvarieties of \( \mathbb{A}^N \) for some \( N \). Let us choose a generic affine projection \( \mathbb{A}^N \to \mathbb{A}^n \) where \( n = \dim Y - \dim X \). Then, the fibers \( \pi^{-1}(x) \) for \( x \in X \) are ramified coverings over the same affine space \( \mathbb{A}^n \) and hence are locally identified in etale topology. This gives a nonlinear flat connection outside of the ramification loci \( \pi^{-1}(x) \to \mathbb{A}^n \), \( x \in X \). Any nonlinear flat connection \( \nabla \) gives rise to map of algebras \( \nabla^* : \text{Diff}_{L, \text{rat}} \to \text{Diff}_{\pi^*L, \text{rat}} \). After making the choice of connection \( \nabla \), the maps \( \pi^*, \pi_* \) in (1), (2) are defined as follows. In the case (1), it is convenient to assume \( L = O_X \) (this can be achieved by passing to an open dense set \( X' \subset X \)). For an ideal \( I \in \mathcal{I}_{O_X} \), we define \( \pi^*I \) to be the left ideal in \( \mathcal{I}_{O_X} \) generated by elements \( \nabla^*(P) \) for \( P \in I \), and by vector fields along fiber of \( \pi \).

In the case (2), it is convenient to assume \( L = K_X \) and the left ideals in \( \text{Diff}_{L, \text{rat}} \) are the same as right ideals in \( \text{Diff}_{O_X, \text{rat}} = \text{Diff}_{L, \text{rat}}^\text{op} \). Let \( I \) be a right ideal in \( \text{Diff}_{Y, \text{rat}} \), which can be thought of as an annihilator of a “volume form” on \( Y \) (e.g., analytic volume form in some domain for \( k = \mathbb{C} \)). We want to define \( \pi_*I \) to be the right ideal in \( \text{Diff}_{X, \text{rat}} \) consisting of differential operators annihilating formal integrals \( \pi_* (\text{vol}_Y) \). By definition, \( P \in \text{Diff}_{X, \text{rat}} \) belongs to \( \pi_*I \) if \( \nabla^*(P) \text{mod} \mathcal{I}_Y \in \mathcal{I}_{Y \setminus \text{Diff}_{Y, \text{rat}}} \) belongs to the image of the right submodule in \( \text{Diff}_{Y, \text{rat}} \) generated by vector fields along fibers of \( \pi \).

In both cases (1), (2), the result of these operations does not depend on the choice of the flat connection \( \nabla \).

In the case (3), we assume that all \( L_1, \ldots, L_n \) are trivialized. It is enough to consider the case \( n = 2 \). Define a linear map \( \Delta : \text{Diff}_{O_X, \text{rat}} \to \text{Diff}_{O_X, \text{rat}} \otimes_{k(X)} \text{Diff}_{O_X, \text{rat}} \) (here both factors on the right are understood as left \( k(X) \)-modules), by the condition \( \Delta P = \sum \alpha P_{\alpha}^{(1)} \otimes P_{\alpha}^{(2)} \) if for any \( f, g \in k(X) \) we have \( P(fg) = \sum \alpha P_{\alpha}^{(1)} f \cdot P_{\alpha}^{(2)} g \). For two left ideals \( I_1, I_2 \in \mathcal{I}_{O_X} \), we define their “product” \( J \) by \( J = \{ P \in \text{Diff}_{O_X, \text{rat}} : \Delta P \in I_1 \otimes \text{Diff}_{O_X, \text{rat}} + \text{Diff}_{O_X, \text{rat}} \otimes I_2 \} \). This is the ideal annihilating the product \( f_1 f_2 \) where \( f_1, f_2 \) are general analytic functions annihilated by \( I_1, I_2 \), respectively.
3 Semi-classical kernels and their quantization

3.1 Semi-classical kernels associated with classical integrable systems

Definition 3.1 Let \((M, \omega)\) be a symplectic manifold. A semi-classical multiplication kernel is a tuple \(((M, \omega), N, P)\) where \(N \subset (M, \omega)\), \(P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)\) are Lagrangian submanifolds with the following properties:

1. The projections \(\pi_i : P \to M, \ i = 1, 2, 3\) are submersions.
2. \(P\) is symmetric with respect to the interchanging of the first two components in \(M^3\).
3. The correspondence \(M \times M \to M\) given by \(P\) is associative.
4. The map \((\pi_2 \times \pi_3)\mid_{\pi_1^{-1}(N) \cap P}\) is an inclusion whose image is open dense in the diagonal Lagrangian submanifold \(M_{diag} = \{(m, m) ; \ m \in M\} \subset (M, -\omega) \times (M, \omega)\).

Remark 3.1 In other words, \((M, \omega)\) is a commutative monoid in the symplectic monoidal category introduced by A. Weinstein. The product in this commutative monoid is defined by \(P\), and the neutral element is defined by \(N\). In general, the objects of this category are symplectic manifolds (in algebraic or \(C^\infty\) sense) and morphisms \(Hom((M_1, \omega_1), (M_2, \omega_2))\) are defined as Lagrangian submanifolds in \((M_1, -\omega_1) \times (M_2, \omega_2)\). The tensor product is given by the usual product of symplectic manifolds. Notice that there are transversality problems in the definition of the composition as the composition of correspondences. Strictly speaking, the above definition is a rough sketch, a first approximation to a not yet found more satisfactory and rigorous notion.

Remark 3.2 The constraint on Lagrangian subvariety \(P\) to be smooth (i.e., to be a manifold) seems to be not completely natural. The closure of \(P\) in \(M^3\) is usually singular.

It seems plausible that semi-classical multiplication kernels are essentially the same as Liouville integrable systems endowed with a Lagrangian section.

Let \((M, \omega)\) be a symplectic manifold with a structure of Liouville integrable system. In other words, we have a fibration

\[ f : M \to B \]

where \(\dim B = \frac{1}{2} \dim M\) and such that pullback of functions on \(B\) Poisson commute. Then, \(L_b = f^{-1}(b) \subset M\) is a Lagrangian submanifold of \(M\) for generic point \(b \in B\), endowed with a natural affine structure (a torsion-free flat connection on the tangent bundle \(TL_b\)). Let us assume for simplicity that the generic fiber \(L_b\) is connected and compact, and moreover, a Lagrangian section \(\sigma : B \to M, f \circ \sigma = id\) is chosen. Then, \(L_b\) will have a structure of a commutative group with the identity element given by \(\sigma(b) \in L_b\). In fact, in the algebraic setting \(L_b\) is an abelian variety\(^{15}\). Define

\(^{15}\) In practice, fibers of \(f\) are often noncompact and generically admit compactification to abelian varieties. On the other hand, in the case \(\dim M = 2\), there are many integrable systems for which the generic fiber of \(f\) is a punctured curve of genus \(g > 1\), e.g., one can take \(M = T^*\mathbb{A}^1 = \mathbb{A}^2\) and \(B = \mathbb{A}^1\), with the map \(f\) given by a polynomial in two variables of sufficiently high degree.
This implies that if \( I \) naturally only a structure of an abelian torsor on \( L_b \) is a structure similar to one in Example 2, Sect. 1.5, where we consider commuting operators with simple joint spectrum and without the choice of normalization for the correspondence \( \text{Remark 3.3} \)

If we do not choose the section \( \sigma \) (or submanifold \( N \)), then we will have naturally only a structure of an abelian torsor on \( L_b, b \in B \). Instead of multiplication \( P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega) \), we will have a manifold

\[
\{(u_1, u_2, u_3, u_4) \in M^4; f(u_1) = f(u_2) = f(u_3) = f(u_4), u_1 + u_2 = u_3 + u_4 \}
\]

which is a Lagrangian submanifold in \( (M, -\omega) \times (M, -\omega) \times (M, \omega) \). This is a structure similar to one in Example 2, Sect. 1.5, where we consider commuting operators with simple joint spectrum and without the choice of normalization for the eigenfunctions.

Let \( M = T^*X \) be the cotangent bundle with the canonical symplectic structure, where \( X \) is a smooth complex algebraic variety. Furthermore, to simplify notations we assume that \( X = \mathbb{A}^1 \) with an affine coordinate \( x \). In this case, \( M \cong \mathbb{A}^2 \) with canonical coordinates \( x, p \) and \( \omega = dx \wedge dp \). Let \( f : M \to B \) be given by the formula \((x, p) \mapsto f(x, p)\) where \( f(x, p) \) is a meromorphic function. In this case, the graph of our commutative associative multiplication is defined by the system of equations of the form

\[
f(x_1, p_1) = f(x_2, p_2) = f(x_3, p_3), \quad g(x_1, p_1, x_2, p_2, x_3, p_3) = 0 \quad (3.1)
\]

where \((x_1, p_1, x_2, p_2, x_3, p_3)\) are coordinates on \( M \times M \times M \), and the first two equations mean that \( f(m_1) = f(m_2) = f(m_3) = b \) for some \( b \in B \). Here \((m_1, m_2, m_3) \in M \times M \times M \) and \((x_i, p_i)\) are coordinates of point \( m_i, i = 1, 2, 3 \).

Notice that commutativity of our semi-classical multiplication kernel means that\(^{16}\)

\[g(x_1, p_1, x_2, p_2, x_3, p_3) = g(x_2, p_2, x_1, p_1, x_3, p_3)\]

and associativity means that if we take the system of equations

\[
f(x_1, p_1) = f(x_2, p_2) = f(x_3, p_3) = f(x_4, p_4) = f(x_5, p_5), \quad g(x_1, p_1, x_2, p_2, x_3, p_3) = g(x_3, p_3, x_4, p_4, x_5, p_5) = 0
\]

and eliminate \( x_3, p_3 \) from it, the resulting affine variety will be symmetric with respect to interchanging of the pairs \((x_2, p_2)\) and \((x_4, p_4)\).

Recall also that \( P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega) \) is a Lagrangian submanifold. This implies that if \( I \) is the ideal generated by Eq. (3.1), then

\[
\{I, I\} \subset I
\]

\(^{16}\)Strictly speaking, we need only the equivalence \( g(x_1, p_1, x_2, p_2, x_3, p_3) = 0 \iff g(x_2, p_2, x_1, p_1, x_3, p_3) = 0 \).
where \{, \} is the Poisson bracket defined by \([p_1, x_1] = [p_2, x_2] = -1, [p_3, x_3] = 1\) and other brackets between coordinates are equal to zero. Moreover, for a middle-dimensional reduced subvariety \(P\) the property \(\{I, I\} \subset I\) implies that \(P\) is Lagrangian.

More generally, let \(M = T^*\mathbb{A}^n\) with canonical coordinates \(x_1, ..., x_n, p_1, ..., p_n\) be a phase space of an integrable system. Let \(f_1(x_1, ..., p_n), ..., f_n(x_1, ..., p_n)\) be commuting integrals of this integrable system. In this case, the Lagrangian submanifold\(^\text{17}\) \(P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)\) is defined by a system of equations in the variables \(x_{1,i}, ..., x_{n,i}, p_{1,i}, ..., p_{n,i}, i = 1, 2, 3\) of the form

\[
\begin{align*}
\hat{f}_1(x_{1,1}, ..., x_{n,1}, p_{1,1}, ..., p_{n,1}) &= \hat{f}_1(x_{1,2}, ..., x_{n,2}, p_{1,2}, ..., p_{n,2}), \\
\hat{f}_1(x_{1,2}, ..., x_{n,2}, p_{1,2}, ..., p_{n,2}) &= \hat{f}_1(x_{1,3}, ..., x_{n,3}, p_{1,3}, ..., p_{n,3}), \\
g_i(x_{1,1}, ..., p_{n,3}) &= 0, \ i = 1, ..., n
\end{align*}
\]

for some functions \(g_1, ..., g_n\). This system should satisfy the following properties:

1. Let \(I^{123}\) be the ideal generated by Eqs. (3.2). Then \(I^{123}, I^{123} \subset I^{123}\) where \(\{, \}\) is the canonical Poisson structure on the symplectic manifold \((M, -\omega) \times (M, -\omega) \times (M, \omega)\). This property implies that \(P\) is a Lagrangian submanifold in \((M, -\omega) \times (M, -\omega) \times (M, \omega)\).

2. Let \(I^{213}\) be the ideal obtained from \(I^{123}\) by interchanging of variables \(x_{i,1}, p_{i,1}\) and \(x_{i,2}, p_{i,2}, i = 1, ..., n\). Then \(I^{213} = I^{123}\). This property means commutativity of our semi-classical kernel.

3. Let \(I^{345}\) be the ideal obtained from \(I^{123}\) by replacing of variables \(x_{i,1}, p_{i,1}, x_{i,2}, p_{i,2}, x_{i,3}, p_{i,3}\) by the variables \(x_{i,3}, p_{i,3}, x_{i,4}, p_{i,4}, x_{i,5}, p_{i,5}\) for \(i = 1, ..., n\). Let \(I^{12345}\) be the ideal generated by \(I^{123}\) and \(I^{345}\). Then, the ideal obtained from \(I^{12345}\) by eliminating the variables \(x_{i,3}, p_{i,3}, i = 1, ..., n\) should be symmetric with respect to interchanging of the variables \(x_{i,2}, p_{i,2}\) and \(x_{i,4}, p_{i,4}\), \(i = 1, ..., n\). This property means associativity of our semi-classical kernel.

### 3.2 Quantization of semi-classical kernels and quantum integrable systems

Let us discuss quantization of the picture above. Let us start with the case \(M = T^*\mathbb{A}^1\). As usual, we replace the Poisson algebra in the variables \(x_1, x_2, x_3, p_1, p_2, p_3\) by the ring of differential operators\(^\text{18}\) in \(x_1, x_3, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\). System (3.1) should be replaced by a system of differential equations for a quantum multiplication kernel \(\hat{K}(x_1, x_2, x_3)\)

\[
\hat{f}\left(x_1, \frac{\partial}{\partial x_1}\right)^*K(x_1, x_2, x_3) = \hat{f}\left(x_2, \frac{\partial}{\partial x_2}\right)^*K(x_1, x_2, x_3) = \hat{f}\left(x_3, \frac{\partial}{\partial x_3}\right)
\]

\[
\times K(x_1, x_2, x_3), \hat{g}\left(x_1, \frac{\partial}{\partial x_1}, x_2, \frac{\partial}{\partial x_2}, x_3, \frac{\partial}{\partial x_3}\right)K(x_1, x_2, x_3) = 0
\]

\(^\text{17}\) Here as usual \(\omega = dx_1 \wedge dp_1 + ... + dx_n \wedge dp_n\).

\(^\text{18}\) Informally, we replace \(p_1 \mapsto \frac{d}{dx_1}, p_2 \mapsto \frac{d}{dx_2}, p_3 \mapsto -\frac{d}{dx_3}\) and choose a “correct” ordering of operators. Here we set Planck constant to one. The sign difference between derivatives by \(x_1, x_2\) and \(x_3\) reflects the sign difference in \(\omega\) in the semi-classical case.
here \( \hat{f}, \hat{g} \) are differential operators which are quantizations of \( f, g \) and \( \ast \) is the anti-involution of the algebra of differential operators defined by 
\[
x_i^\ast = x_i, \quad \left( \frac{\partial}{\partial x_i} \right)^\ast = -\frac{\partial}{\partial x_i},
\]
\((AB)^\ast = B^\ast A^\ast\).

We assume that the left ideal \( \hat{I} \) in the algebra of differential operators in 
\( x_1, x_2, x_3, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \) generated by Eq. (3.3) satisfies the property 
\[
[\hat{I}, \hat{I}] \subset \hat{I}.
\]
This is a quantization of the property \( \{I, I\} \subset I \) in the semi-classical case.

We also assume that \( \hat{g}(x_1, \frac{\partial}{\partial x_1}, x_2, \frac{\partial}{\partial x_2}, x_3, \frac{\partial}{\partial x_3}) \) 
is symmetric with respect to interchanging of \( x_1, x_2 \). This means that our quantum multiplication is commutative.

It is less clear, however, how to lift associativity to the quantum case. We see the following two possibilities:

P1. There exists a nonzero solution \( K(x_1, x_2, x_3) \) of system (3.3) such that \( K \) is a multiplication kernel of birational type.

P2. Denote by \( \hat{I}^{123} \) the left ideal in the algebra of differential operators generated by Eq. (3.3). Let \( \hat{I}^{345} \) be obtained from \( \hat{I}^{123} \) by replacing \( x_1, x_2, x_3 \) by \( x_3, x_4, x_5 \). Let 
\[
\hat{I}^{1245} = (\hat{I}^{123} + \hat{I}^{345} + J) \cap R_{1245}
\]
where \( J \) is the right ideal in the ring of differential operators in variables \( x_1, \ldots, x_5 \) generated by \( \frac{\partial}{\partial x_3} \) and \( R_{1245} \) is the ring of differential operators in \( x_1, x_2, x_4, x_5 \).

We assume that \( \hat{I}^{1245} \) is symmetric with respect to interchanging of \( x_2 \) and \( x_4 \). Notice that \( \hat{I}^{1245} \) is the left ideal in the ring \( R_{1245} \).

It is not clear how these two properties are related in general. Informally, if P1 holds, then
\[
\int_{\Gamma} K(x_1, x_2, x_3)K(x_3, x_4, x_5)dx_3 = \int_{\Gamma} K(x_1, x_4, x_3)K(x_3, x_2, x_5)dx_3 \quad (3.4)
\]
for any cycle \( \Gamma \). This means that the l.h.s. and the r.h.s. of (3.4) should satisfy the same system of differential equations as a function in \( x_1, x_2, x_4, x_5 \). Therefore, P2 looks feasible, but it is not clear how to make these considerations rigorous.

On the other hand, if P2 holds, then the l.h.s. and the r.h.s. of (3.4) should satisfy the same system of differential equations. In general, this does not mean that the r.h.s. is obtained from the l.h.s. by a birational mapping, but it would be natural to expect this in examples.

The advantage of P2 is that it can in principle be verified algorithmically. In this paper, however, we concentrate on P1.
Let us discuss quantization of semi-classical kernels in a more general case. Consider a quantum integrable system defined by commuting differential operators $D_1, \ldots, D_n$ in the variables $x_1, \ldots, x_n$. A quantization of system (3.2) has a form

$$D_{i,1}^* K = D_{i,2}^* K = D_{i,3} K, \quad i = 1, \ldots, n,$$

$$G_{i} K = 0, \quad i = 1, \ldots, n \tag{3.5}$$

for some differential operators $G_i$ in the variables $x_{i,j}$, $i = 1, \ldots, n$, $j = 1, 2, 3$. Here $D_{i,j}$, $i = 1, \ldots, n$, $j = 1, 2, 3$ are obtained from $D_i$ by replacing the variables $x_1, \ldots, x_n$ by $x_{1,j}$, $\ldots$, $x_{n,j}$, and $K$ is a function in variables $x_{i,j}$. This system should satisfy the following properties:

1. Let $\hat{I}^{123}$ be the left ideal generated by Eq. (3.5). Then $[\hat{I}^{123}, \hat{I}^{123}] \subset \hat{I}^{123}$ where $[A, B] = AB - BA$ is the usual commutator of differential operators. This property means that system (3.5) is holonomic.

2. Let $\hat{I}^{213}$ be the left ideal obtained from $\hat{I}^{123}$ by interchanging of variables $x_{i,1}$ and $x_{i,2}$, $i = 1, \ldots, n$. Then, $\hat{I}^{213} = \hat{I}^{123}$. This property means commutativity of our kernel.

3. Let $\hat{I}^{345}$ be the left ideal obtained from $\hat{I}^{123}$ by replacing of variables $x_{i,1}, x_{i,2}, x_{i,3}$ by the variables $x_{i,3}, x_{i,4}, x_{i,5}$ for $i = 1, \ldots, n$. Let

$$\hat{I}^{1245} = (\hat{I}^{123} + \hat{I}^{345} + J) \cap R_{1245}$$

where $J$ is the right ideal generated by $\frac{\partial}{\partial x_{i,3}}$, $i = 1, \ldots, n$ and $R_{1245}$ is the ring of differential operators in $x_{i,1}, x_{i,2}, x_{i,4}, x_{i,5}$, $i = 1, \ldots, n$. Then $\hat{I}^{1245}$ should be symmetric with respect to interchanging of the variables $x_{i,2}$ and $x_{i,4}$, $i = 1, \ldots, n$. This property means associativity of our kernel.

3’ There exists a nonzero solution $K$ of system (3.5), which is a multiplication kernel of birational type.

Notice that the informal discussion about the properties (P1) and (P2) of system (3.3) is also applicable here to the properties (3) and (3’), so relation (if any) between these properties is unclear as well.

Remark 3.4 The Weinstein category (in the $C^\infty$ setting) discussed in Sect. 3.1 can be heuristically thought of as a semi-classical limit of the symmetric monoidal category of complex Hilbert spaces, infinite-dimensional in general. Informally, for a real symplectic manifold $(M, \omega)$ and a small positive Planck constant $\hbar \ll 1$ one constructs the corresponding Hilbert space $\mathcal{H}_h(M, \omega)$, the quantization of $M$. The dimension of this space $\dim \mathcal{H}_h(M, \omega) \approx \frac{1}{(2\pi \hbar)^n} \int_M \frac{\omega^n}{n!}$ where $n = \frac{1}{2} \dim M$. Notice that $\dim \mathcal{H}_h(M, \omega)$ is infinite if $M$ has infinite volume, for example if $M = T^* X$ is a cotangent bundle. The Hilbert space $\mathcal{H}_h(M, -\omega)$ is the dual to $\mathcal{H}_h(M, \omega)$. The product of symplectic manifolds corresponds to the tensor product of Hilbert spaces. A Lagrangian submanifold $L \subset (M, \omega)$ corresponds approximately to a vector $\psi_L \in \mathcal{H}_h(M, \omega)$ defined up to multiplication by a phase. This picture relates

\footnote{Notice that $\hat{I}^{1245}$ is a left ideal in the ring $R_{1245}$.}
collections of commuting operators provided by quantum integrable systems with multiplication kernels (see the beginning of the Introduction).

There is another type of quantization of real symplectic manifolds. Namely, with $(M, \omega)$ one can associate an $A_\infty$-category depending on small parameter $e^{-\frac{1}{\hbar}}$, the Fukaya category $\mathcal{F}(M, \omega)$. Passing to the opposite manifold $(M, -\omega)$ corresponds to passing to the opposite category, and the product of manifolds corresponds to the tensor product of $A_\infty$-categories. Hence, a semi-classical multiplication kernel (i.e., the structure of an integrable system with a Lagrangian section) corresponds to a commutative monoid in the symmetric monoidal $(\infty, 1)$-category of small $A_\infty$-categories. In other words, we get a symmetric monoidal $A_\infty$-category. The basic example of such a category is $\text{Perf}(Y)$, the category of perfect complexes of coherent sheaves on $Y$ (where $Y$ is an algebraic variety), endowed with the tensor product over $\mathcal{O}_Y$. In this way, one gets a homological mirror symmetry equivalence $\mathcal{F}(M, \omega) \sim \text{Perf}(Y)$ where $Y$ is the mirror of $(M, \omega)$.

3.3 The example corresponding to Hitchin systems for rank 2 bundles on the projective line with 4 regular singular points

In this section, we describe the semi-classical and quantum multiplication kernels corresponding to Example 2.4 in Sect. 2.3.

Fix an affine coordinate $x$ on $\mathbb{P}^1$ and choose a point on $\mathbb{P}^1$ with coordinate $t \neq 0, 1, \infty$. Define $M_0 = T^*(\mathbb{P}^1 \setminus \{0, 1, t, \infty\})$, it is a subset of $\mathbb{A}^2 = T^* \mathbb{A}^1$ with coordinates $x, p$.

Let $B = \mathbb{A}^1$ and define a fibration $f : M_0 \to B$ by

$$f(x, p) = x(x - 1)(x - t)p^2 - s^2x - \frac{k_1^2 t}{x} + \frac{k_2^2 (t - 1)}{x - 1} - \frac{k_3^2 (t - 1)}{x - t}.$$ 

Any fiber of $f$ is an elliptic curve (maybe degenerate) with four punctures. We define $M \supset M_0$ as a partial compactification obtained by adding four missing points on $f^{-1}(b)$ for each $b \in B = \mathbb{A}^1$, i.e., adding 4 copies of $\mathbb{A}^1$.

Define a Lagrangian submanifold $P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)$ by

$$f(x_1, p_1) = f(x_2, p_2) = f(x_3, p_3),$$

$$\frac{x_1(x_1 - 1)(x_1 - t)}{(x_1 - x_2)(x_1 - x_3)} p_1 + \frac{x_2(x_2 - 1)(x_2 - t)}{(x_2 - x_1)(x_2 - x_3)} p_2 - \frac{x_3(x_3 - 1)(x_3 - t)}{(x_3 - x_1)(x_3 - x_2)} p_3 - s = 0$$

(3.6)

where $t, k_1, k_2, k_3, s$ are arbitrary parameters.

Define a Lagrangian submanifold $N \subset M$ as the glued copy of $B = \mathbb{A}^1$ corresponding to the puncture $x = \infty$ on the generic fiber of the original map $f : M_0 = T^* \mathbb{A}^1 \to \mathbb{P}^1$.

Theorem 3.1 The tuple $((M, \omega), N, P)$ constructed above in this subsection is a semi-classical multiplication kernel.
Proof All properties of a semi-classical multiplication kernel listed in the beginning of this Section can be verified by direct computation. In particular, the last equation in system (3.6) corresponds to addition on the elliptic curve embedded into $\mathbb{A}^2$ with coordinates $x, p$ and defined by the equation

$$x(x - 1)(x - t)p^2 - s^2x - k_1^2t + k_2^2(t - 1)x - k_3^2(t - 1) = b.$$  

To quantize the above system, introduce the differential operators

$$D_x = \frac{d}{dx} x(x - 1)(x - t), \quad D_x = \frac{d}{dx} s(s + 2)x - \frac{k_1^2t}{x} + \frac{k_2^2(t - 1)}{x - 1} - \frac{k_3^2(t - 1)}{x - t},$$

$$L = \frac{x_1(x_1 - 1)(x_1 - t)}{(x_1 - x_2)(x_1 - x_3)} \frac{d}{dx_1} + \frac{x_2(x_2 - 1)(x_2 - t)}{(x_2 - x_1)(x_2 - x_3)} \frac{d}{dx_2} + \frac{x_3(x_3 - 1)(x_3 - t)}{(x_3 - x_1)(x_3 - x_2)} \frac{d}{dx_3} - s.$$  

(3.7)

Consider the following system of equations

$$D_{x_1}K(x_1, x_2, x_3) = D_{x_2}K(x_1, x_2, x_3) = D_{x_3}K(x_1, x_2, x_3),$$

$$L K(x_1, x_2, x_3) = 0.$$  

(3.8)

Theorem 3.2 System (3.8) satisfies the properties (1), (2), (3') listed after system (3.5). Moreover, let

$$K(x_1, x_2, x_3) = (x_1x_2x_3)^s F(u, v)$$

where

$$u = \frac{t^2(x_1 - 1)(x_2 - 1)(x_3 - 1)}{(t - 1)^2x_1x_2x_3}, \quad v = \frac{(x_1 - t)(x_2 - t)(x_3 - t)}{(t - 1)^2x_1x_2x_3},$$  

(3.9)

and

$$F(u, v) = u^{-k_2}v^{-k_3} \int_{\Gamma} \left[ \frac{q_1^{2k_2}q_2^{2k_3}}{(1 + q_1 + q_2)^{s-k_1-k_2-k_3}} \left( \frac{1}{q_1} + \frac{u}{q_2} \right)^{s+k_1+k_2+k_3} \right] dq_1 \wedge dq_2$$

where $\Gamma$ is a cycle. Then, $K(x_1, x_2, x_3)$ satisfies system (3.8). Moreover, if we write

$$K(x_1, x_2, x_3) = \int_{\Gamma} \tilde{K}(x_1, x_2, x_3, q_1, q_2) dq_1 \wedge dq_2$$

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where $\tilde{K}$ is defined by (3.9) where we substitute the integral expression for $F(u, v)$ and put everything under the integral, then

$$
\tilde{K}(x_1, x_2, x_3, q_1, q_2) \tilde{K}(x_3, x_4, x_5, q_3, q_4) \, dx_1 \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4 \\
= \tilde{K}(x_1, x_4, \tilde{x}_3, q_1, \tilde{q}_2) \tilde{K}(x_2, x_5, \tilde{q}_3, \tilde{q}_4) \, d\tilde{x}_3 \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4
$$

for some birational mapping $(x_3, q_1, q_2, q_3, q_4) \mapsto (\tilde{x}_3, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4)$.

**Proof** Property (1) can be verified by direct computation and Property (2) is clear because $L$ is symmetric with respect to interchanging of $x_1$ and $x_2$.

Substituting $K(x_1, x_2, x_3)$ in form (3.9) into system (3.8), one can check that the last equation $LK = 0$ holds for an arbitrary function $F(u, v)$ and the first two equations are equivalent to the following system for the function $F(u, v)$

$$
\frac{\partial^2 G}{\partial u_1 \partial v_1} = \frac{\partial^2 G}{\partial u_2 \partial v_2} = \frac{\partial^2 G}{\partial u_3 \partial v_3}
$$

(3.10)

where

$$G = F\left(\frac{u_1 v_1}{u_3 v_3}, \frac{u_2 v_2}{u_3 v_3}\right) \left(\frac{v_1}{u_1}\right)^{k_2} \left(\frac{v_2}{u_2}\right)^{k_3} \left(\frac{v_3}{u_3}\right)^{k_1}\left( u_3 v_3 \right)^s.
$$

(3.11)

An Euler-type integral representation for a solution of this system can be obtained using the theory of generalized hypergeometric functions. Let us write

$$G = \int P_t^{2k_2} t_1^{2k_3} t_2^{2k_1} (u_1 t_1 + u_2 t_2 + u_3 t_3)^s \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3}.
$$

It is clear that this expression satisfies (3.10). On the other hand, after change of variables $t_1 = \frac{q_1 v_3}{u_1}$, $t_2 = \frac{q_2 v_3}{u_2}$, $t_3 = \frac{q_3}{u_3}$ and integrating out $q_3$, we obtain representation (3.11) where $F(u, v)$ is given by (3.9).

Finally, notice that the family of kernels $\tilde{K}(x_1, x_2, x_3, q_1, q_2)$ is essentially the same as the family of kernels $K(x_1, x_2, y, q_1, q_2)$ in Example 2.4, Sect. 2.3 (see also Theorem 2.1). These kernels are related by a gauge transformation of the form $\tilde{K}(x_1, x_2, x_3, q_1, q_2) \mapsto K(x_1, x_2, x_3, q_1, q_2) \frac{q(x_1)q(x_2)}{q(x_3)}$, a change of variables $(x_1, x_2, x_3, t) \mapsto (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{y}, \frac{1}{t})$, and redefinition of other parameters. \hfill \square

---

20 In fact, system (3.8) is obtained as a quantization of system (3.6) with Properties (1), (2).

21 In this formula, $k_1, k_2, k_3$ are defined up to sign because the differential operator $D_3$ depends on $k_1^2, k_2^2, k_3^2$ only. Different choices may give different solutions of system (3.8) and any solution is a linear combination of these.
Remark 3.5 The differential operators $D_x$ and $L$ in system (3.8) can be written in the form:

$$D_x = \frac{d}{dx} \cdot (x^3 + g_2x + g_3) \cdot \frac{d}{dx} - s(s + 2)x + \frac{k_0 + k_1x + k_2x^2}{x^3 + g_2x + g_3},$$

$$L = \frac{x_1^3 + g_2x_1 + g_3}{(x_1 - x_2)(x_1 - x_3)} \cdot \frac{d}{dx_1} + \frac{x^3 + g_2x_2 + g_3}{(x_2 - x_1)(x_2 - x_3)} \cdot \frac{d}{dx_2} + \frac{x_3^3 + g_2x_3 + g_3}{(x_3 - x_1)(x_3 - x_2)} \cdot \left( \frac{d}{dx} \right) - s,$$

where $g_2, g_3, k_0, k_1, k_2, s$ are arbitrary constants. This form is convenient to study degenerations of this family. Moreover, if $s = -1, k_0 + k_1x + k_2x^2 = (q_1 + q_2x)^2$ for some constants $q_1, q_2$, then a solution of system (3.6) can be written in the form

$$K(x_1, x_2, x_3) = \exp \left( q_1 f_1(x_1, x_2, x_3) + q_2 f_2(x_1, x_2, x_3) \right) \frac{1}{P(x_1, x_2, x_3)^{1/2}},$$

where

$$P(x_1, x_2, x_3) = 2x_1x_2x_3(x_1 + x_2 + x_3) - x_1^2x_2^2 - x_1^2x_3^2 - x_2^2x_3^2 + 2g_2(x_1x_2 + x_1x_3 + x_2x_3) + 4g_3(x_1 + x_2 + x_3) - g_2^2$$

and $f(x_1, x_2, x_3) = q_1 f_1(x_1, x_2, x_3) + q_2 f_2(x_1, x_2, x_3)$ satisfies the system of differential equations

$$\frac{\partial f}{\partial x_3} = \frac{1}{(x_3^3 + g_2x_3 + g_3)P(x_1, x_2, x_3)^{1/2}} \left( q_1 \left( 2x_3^2 + x_1x_3 + x_2x_3 - x_1x_2 + g_2 \right) \right. \left. - q_2 \left( x_1x_2x_3 - x_1x_3^2 - x_2x_3^2 + g_2x_3 + 2g_3 \right) \right)$$

and the other two equations are obtained from this by a cyclic permutation of $x_1, x_2, x_3$. This family of multiplication kernels coincides, up to a gauge transformation and redefining constants, with the family from Example 2.3, Sect. 2.2.

Remark 3.6 Let $D_x$ be the differential operator given by (3.7). Define a function $K_4(x_1, x_2, x_3, x_4)$ by

$$K_4 = F \left( \frac{x_1x_2x_3x_4}{t^2}, \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(t - 1)^2}, \frac{(x_1 - t)(x_2 - t)(x_3 - t)(x_4 - t)}{t^2(t - 1)^2} \right),$$

where $F(u, v, w)$ satisfies a system of differential equations

$$\frac{\partial^2 G}{\partial u_1 \partial v_1} = \frac{\partial^2 G}{\partial u_2 \partial v_2} = \frac{\partial^2 G}{\partial u_3 \partial v_3} = \frac{\partial^2 G}{\partial u_4 \partial v_4}.$$
Here $G$ is given by:

$$G = F\left(\frac{u_2 v_2}{u_1 v_1}, \frac{u_3 v_3}{u_1 v_1}, \frac{u_4 v_4}{u_1 v_1}\right) \cdot \left(\frac{v_2}{u_2}\right)^k_1 \left(\frac{v_3}{u_3}\right)^k_2 \left(\frac{v_4}{u_4}\right)^k_3 \left(\frac{v_1}{u_1}\right)^{s+1} \cdot \frac{1}{u_1 v_1}$$

Then $K_4$ satisfies the equations

$$D_{x_1} K_4 = D_{x_2} K_4 = D_{x_3} K_4 = D_{x_4} K_4.$$

One can obtain an Euler-type integral representation (similar to the one in Theorem 3.2) using the theory of generalized hypergeometric functions [3]. It looks feasible that $K_4$ gives an example of a structure discussed in Sect. 1.6, problem (2). See also Remark 3.8 for the generalization of the kernel $K_4$.

### 3.4 The example corresponding to Hitchin systems for rank 2 bundles on the projective line with more than four regular singular points

Fix an affine coordinate $x$ on $\mathbb{P}^1$ and choose $n > 1$ pairwise distinct points on $\mathbb{P}^1$ with coordinates $t_1, ..., t_n \neq 0, 1, \infty$. Introduce the differential operator

$$D_x = \frac{\partial}{\partial x} \cdot x(x-1)(x-t_1)...(x-t_n) \frac{\partial}{\partial x} - s(s+n+1)x^n - \sum_{i=-1}^{n} k_{i+2}^{2} \prod_{j \neq i} (t_i - t_j) \frac{1}{x - t_i}$$

where $t_{-1} = 0, t_0 = 1$ and $s, k_1, ..., k_{n+2}$ are generic parameters. Define furthermore

$$L_{x_1, ..., x_{n+1}} = \sum_{i=1}^{n+1} \frac{1}{\prod_{j \neq i} (x_i - x_j)} D_{x_i},$$

$$M_{x_1, ..., x_{n+1}} = \sum_{i=1}^{n+1} \frac{x_i(x_i - 1)(x_i - t_1)...(x_i - t_n)}{\prod_{j \neq i} (x_i - x_j)} \cdot \frac{\partial}{\partial x_i} - s.$$

**Theorem 3.3** Define the kernel

$$K_n(x_1, ..., x_{2n+1}) = \int K(x_1, ..., x_{2n+1}, q_1, ..., q_{n+1}) dq_1 ... dq_{n+1}$$

where the kernel in the r.h.s. is given by (2.5). Then, the kernel $K_n$ satisfies the following system of holonomic differential equations
\[ L_{x_1, \ldots, x_{n+1}} K_n = 0, \quad (3.12) \]
\[ M_{x_1, \ldots, x_{n+1}} K_n = 0 \quad (3.13) \]

where \( 1 \leq i_1 < \ldots < i_{n+1} \leq 2n + 1 \).

**Proof** It is similar to the proof of Theorem 3.2 from the previous section. System (3.12) has the following general solution in terms of an arbitrary function \( F(w_1, \ldots, w_{n+1}) \):

\[ K_n = (x_1 \ldots x_{2n+1})^s F(w_0, \ldots, w_n), \]

where

\[
w_0 = \frac{(x_1 - 1)(x_2n+1) - 1) t_1^2 \ldots t_n^2}{x_1 \ldots x_{2n+1} (t_1 - 1)^2 \ldots (t_n - 1)^2}, \quad w_i = \frac{(x_1 - t_i)(x_2n+1 - t_i) \prod_{j \neq i} t_j^2}{x_1 \ldots x_{2n+1} (t_i - 1)^2 \prod_{j \neq i} (t_i - t_j)^2}, \]

\( i = 1, \ldots, n \).

System (3.13) can be written in terms of \( F \) as

\[
\frac{\partial^2 G}{\partial u_1 \partial v_1} = \frac{\partial^2 G}{\partial u_2 \partial v_2} = \ldots = \frac{\partial^2 G}{\partial u_{n+2} \partial v_{n+2}},
\]

where

\[
G = F \left( \frac{u_2 v_2}{u_1 v_1}, \ldots, \frac{u_{n+2} v_{n+2}}{u_1 v_1} \right) \cdot \left( \frac{u_1}{v_1} \right)^{k_1} \ldots \left( \frac{u_{n+2}}{v_{n+2}} \right)^{k_{n+2}} \cdot (u_1 v_1)^s.
\]

Our integral representation for the kernel \( K \) is the Euler-type representation in the theory of generalized hypergeometric functions [3]. See also the proof of Theorem 3.2 and system (2.6).

**Remark 3.7** The operator \( D_x \) can be also written in the form

\[
D_x = \frac{\partial}{\partial x} \cdot P_{n+2}(x) \frac{\partial}{\partial x} - s(s + n + 1)x^n + \frac{Q_{n+1}(x)}{P_{n+2}(x)}
\]

where \( P_{n+2}(x), Q_{n+1}(x) \) are arbitrary polynomials in \( x \) of degrees \( n + 2 \) and \( n + 1 \), respectively. In this case, the operator \( L_{x_1, \ldots, x_{n+1}} \) is given by the same formula and

\[
M_{x_1, \ldots, x_{n+1}} = \sum_{i=1}^{n+1} \frac{P_{n+2}(x_i)}{\prod_{j \neq i} (x_i - x_j)} \cdot \frac{\partial}{\partial x_i} - s.
\]

**Conjecture 3.1** The holonomic cyclic \( D \)-module given by (3.12), (3.13) defines a generalized product in the sense of Sect. 4.1.

This \( D \)-module seems to have the semi-classical limit described in abstract terms in Sect. 4.2 and corresponds to the Hitchin integrable system for rank 2 bundles with
One can write solutions of this system in terms of Euler-type integrals \([3]\).

**Remark 3.8** Define a family of differential operators by:

\[
D_x = \prod_{i=1}^{n} (x - t_i) \cdot \left( \frac{\partial^2}{\partial x^2} - \sum_{j=1}^{n} b_{j,1} + b_{j,2} - 1 \cdot \frac{\partial}{\partial x} \right) + \sum_{i=1}^{n} \frac{b_{i,1} b_{i,2} \prod_{j \neq i} (t_i - t_j)}{x - t_i},
\]

where we assume \(\sum_{i=1}^{n} (b_{i,1} + b_{i,2}) = n - 2\).

The operator \(D_x\) has \(n\) regular singular points at \(x = t_1, \ldots, t_n\) and solutions of the equation \(D_x f(x) = 0\) near \(x = t_i\) have a form \(f(x) = (x - t_i)^{b_{i,j}}(1 + O(x - t_i))\) for \(i = 1, \ldots, n, j = 1, 2\). Moreover, any differential operator with these properties and the same symbol as \(D_x\) has form \(D_x + \lambda_1 + \lambda_2 x + \ldots + \lambda_{n-3} x^{n-4}\) where \(\lambda_1, \ldots, \lambda_{n-3}\) are arbitrary parameters.

Fix an integer \(l\) such that \(2 \leq l \leq n\) and construct a function \(K_{n,l}(x_1, \ldots, x_{l+n-4})\) in \(n + l - 4\) variables as follows:

\[
K_{n,l} = F\left( \prod_{1 \leq j \leq l-1} (x_1 - t_j) \prod_{l+1 \leq k \leq n} \frac{(x_1 - t_k)(x_{l+n-4} - t_k)}{(x_1 - t_k)(x_{l+n-4} - t_k)} \right) \times \prod_{i=1}^{l-1} \frac{(x_1 - t_i)(x_{l+n-4} - t_i)}{(x_1 - t_i)(x_{l+n-4} - t_i)} \times \prod_{1 \leq j \leq n} \frac{t_j - t_k}{t_j - t_k}.
\]

and \(F\) satisfies the system of differential equations\(^{22}\)

\[
\frac{\partial^2 G}{\partial u_{1,1} \partial u_{1,2}} = \frac{\partial^2 G}{\partial u_{2,1} \partial u_{2,2}} = \ldots = \frac{\partial^2 G}{\partial u_{l,1} \partial u_{l,2}}.
\]

here

\[
G = F\left( \frac{u_{1,1} u_{1,2}}{u_{1,1} u_{1,2}}, \ldots, \frac{u_{l-1,1} u_{l-1,2}}{u_{l-1,1} u_{l-1,2}}, \frac{u_{l,1} u_{l,2}}{u_{l,1} u_{l,2}} \right) \cdot \prod_{1 \leq \alpha \leq l, 1 \leq \beta \leq 2} u_{\alpha, \beta}^{-b_{\alpha, \beta}} \cdot (u_{l,1} u_{l,2})^{-b_{l+1,1} - \ldots - b_{n,1}}.
\]

Notice that if \(l = n\), then in the formulas above we have \(b_{l+1,1} + \ldots + b_{n,1} = 0\) and \(\prod_{l+1 \leq j \leq n} = 1\). Moreover, in this case the kernel \(K_{n,l}\) is a function of \(2g + 2\) variables

\(^{22}\) One can write solutions of this system in terms of Euler-type integrals [3].
(here \( g = n - 3 \) is the genus of the spectral curve) and is covariant with respect to the full symmetry group \( S_n \ltimes (\mathbb{Z}/2)^n \) of the problem.

Define furthermore
\[
L_{x_1, \ldots, x_{n-2}} = \sum_{i=1}^{n-2} \frac{1}{\prod_{j \neq i} (x_i - x_j)} D_{x_i}.
\]

Then, the function \( K_{n,l} \) satisfies the following system of holonomic differential equations
\[
L_{x_{i_1} \ldots x_{i_{n-2}}} K_{n,l} = 0 \quad (3.14)
\]
where \( 1 \leq i_1 < \ldots < i_{n-2} \leq l + n - 4 \).

It will be interesting to understand for which \( n, l \) the kernel \( K_{n,l} \) gives an example of structures discussed in Sect. 1.6.

### 4 Separation of variables

#### 4.1 Families of functions in one variable, and generalized products

This section can be considered as a continuation of Sect. 1.4. We will describe the general framework for the Sklyanin method of separation of variables in a broad and informal way.

Recall that in Sect. 1.4 we consider a situation when we have a collection of functions \( \psi_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) \) in \( n \) variables \( x_1, \ldots, x_n \) depending on \( n \) parameters \( \lambda_1, \ldots, \lambda_n \). These functions are the normalized eigenfunctions of a family of commuting operators. Moreover, these functions form in a sense a continuous basis of the algebra of functions in \( x_1, \ldots, x_n \), thus giving a new product \( * \) on functions.

Now let us consider a different situation: suppose we have a collection \( \phi_{\lambda_1, \ldots, \lambda_n}(x) \) of functions in one variable \( x \) depending on \( n \) parameters \( \lambda_1, \ldots, \lambda_n \). Then, we can construct a collection of functions in \( n \) variables \( x_1, \ldots, x_n \) (symmetric under permutations) and depending again on \( n \) parameters by
\[
\psi_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) = \phi_{\lambda_1, \ldots, \lambda_n}(x_1) \cdots \phi_{\lambda_1, \ldots, \lambda_n}(x_n).
\]

Hence, one can expect that the functions \( \psi_{\lambda_1, \ldots, \lambda_n} \) form a continuous basis of the space of \( S_n \)-invariant functions in \( x_1, \ldots, x_n \).

Conversely, the functions \( \psi_{x_1, \ldots, x_n}(\lambda_1, \ldots, \lambda_n) = \psi_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) \) form a continuous basis of the space of functions in \( \lambda_1, \ldots, \lambda_n \); this is an analogue of the inverse Fourier transform.

**Example 4.1** Let \( \phi_{\lambda_1, \ldots, \lambda_n}(x) = e^{\lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_n x^n} \). Then
\[
\psi_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) = e^{\lambda_1 x_1 + \ldots + x_n} e^{\lambda_2 (x_1^2 + \ldots + x_n^2)} \cdots e^{\lambda_n (x_1^n + \ldots + x_n^n)}.
\]

Notice that \( h_1 = x_1 + \ldots + x_n, \ldots, h_n = x_1^n + \ldots + x_n^n \) are coordinates on \( Sym^n \mathbb{A}^1 = \mathbb{A}^n \). Hence, we see that the functions \( \psi_{\lambda_1, \ldots, \lambda_n} \) form the continuous basis of Fourier modes.
For any \(x_1, \ldots, x_{n+1}\), consider the function in \(\lambda_1, \ldots, \lambda_n\) given by

\[
(\lambda_1, \ldots, \lambda_n) \mapsto \phi_{\lambda_1, \ldots, \lambda_n}(x_1) \cdots \phi_{\lambda_1, \ldots, \lambda_n}(x_{n+1}).
\]

Let us expand this function in the continuous basis \(\psi_1^{x_1}, \ldots, \psi_n^{x_n}(\lambda_1, \ldots, \lambda_n)\). Then, we get the generalization of multiplication formulas in Sect. 1.3.

\[
\phi_{\lambda_1, \ldots, \lambda_n}(x_1) \cdots \phi_{\lambda_1, \ldots, \lambda_n}(x_{n+1}) = \int K_{n+1,n}(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n)\phi_{\lambda_1, \ldots, \lambda_n}(y_1) \cdots \phi_{\lambda_1, \ldots, \lambda_n}(y_n) dy_1 \cdots dy_n
\]

for some kernel \(K_{n+1,n}\) depending on \(2n + 1\) variables. This kernel satisfies a generalized associativity condition (see also Sect. 1.5):

\[
\int K_{n+1,n}(x_1, \ldots, x_{n+1}, z_1, \ldots, z_n)K_{n+1,n}(z_1, \ldots, z_n, x_{n+2}, y_1, \ldots, y_n)dz_1 \cdots dz_n
\]

is symmetric with respect to permutations of \(x_1, \ldots, x_{n+2}\). Formally, this property implies that the kernel in \(3n\) variables given by

\[
K_{2n,n}(x_1, \ldots, x_n, x'_1, \ldots, x'_n, y_1, \ldots, y_n) = \int K_{n+1,n}(x_1, \ldots, x_n, x'_1, z_1^{(1)}, \ldots, z_n^{(1)}) \cdot
K_{n+1,n}(z_1^{(1)}, \ldots, z_n^{(1)}, x'_2, z_1^{(2)}, \ldots, z_n^{(2)}) \cdots K_{n+1,n}(z_1^{(n-1)}, \ldots, z_n^{(n-1)}, x'_n, y_1, \ldots, y_n)\]

\[
\prod_{1 \leq i \leq n, 1 \leq j \leq n} d z_{(i)}^j
\]

gives a commutative associative product on the space of \(S_n\)-invariant functions in \(n\) variables. In particular, \(K_{2n,n}\) is symmetric with respect to permutations of \(x'_1, \ldots, x'_n\).

The above informal considerations can be done formally in the linear algebra framework.

**Definition 4.1** For \(n \geq 1\), the generalized commutative associative product on a vector space \(V\) is a linear map:

\[
\mu_{n+1,n} : V^\otimes(n+1) \to V^\otimes n
\]

which is invariant under the \(S_{n+1} \times S_n\)-action, i.e., it factorizes as

\[
V^\otimes(n+1) \to \left(V^\otimes(n+1)\right)_{S_{n+1}} \to \left(V^\otimes n\right)_{S_n} \hookrightarrow V^\otimes n,
\]

and the map \(\mu_{n+2,n} : V^\otimes(n+2) \to V^\otimes n\) given by \(\mu_{n+1,n} \circ (\mu_{n+1,n} \otimes id_V)\) is \(S_{n+2} \times S_n\)-invariant. For \(n = 2\), this map can be represented by the picture

\[\text{Diagram}\]
This definition of a generalized commutative associative product makes sense in an arbitrary symmetric monoidal category.

**Proposition 4.1** Let us define $\mu_{2n,n} : V^{\otimes 2n} \to V^{\otimes n}$ as the composition of $\mu_{n+1,n} \otimes \text{id}_V \otimes^k$ for $k = n - 1, \ldots, 1, 0$. For example, for $n = 3$ we have

$$\mu_{6,3} = \mu_{4,3} \circ (\mu_{4,3} \otimes \text{id}_V) \circ (\mu_{4,3} \otimes \text{id}_V^2).$$

This map can be also represented by the picture

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

Then $\mu_{2n,n}$ defines a structure of a commutative associative (possibly non-unital) algebra on $\text{Sym}^n V$.

### 4.2 Semi-classical generalized products

In this section, we describe a class of Poisson compactifications (introduced in Section 8.3 in [9]) of cotangent bundles to curves and a geometric construction of semi-classical generalized products.

Let $C$ be a smooth, not necessarily compact, algebraic curve over $\mathbb{C}$. Write $C = \overline{C} \setminus S$ where $S = \{s_1, \ldots, s_m\}$ is a finite subset of the smooth compact curve $\overline{C}$. Denote by $\mathcal{P}_0$ the compact surface which is the total space of the $\mathbb{P}^1$-bundle over $\overline{C}$, given by $\mathbb{P}(\mathcal{O}_C \oplus T^*_{\overline{C}, \log S})$. The surface $\mathcal{P}_0$ carries a natural Poisson structure $\gamma_0$ with the symplectic leaf isomorphic to $T^*C$. The Poisson structure $\gamma_0$ vanishes on the complement $\mathcal{P}_0 \setminus T^*C$, which is the union of smooth divisors $\overline{C}_\infty \cong \overline{C}$ and $T^*_{s_iC} \cong \mathbb{A}^1$, $i = 1, \ldots, m$ intersecting transversally.

The symplectic form $\omega = \gamma_0^{-1}$ has poles of order 2 along the horizontal divisor $\overline{C}_\infty$, and of order one along vertical divisors $T^*_{s_iC}$. Starting with the Poisson surface $\mathcal{P}_0$, let us construct a sequence of Poisson surfaces $(\mathcal{P}_i, \gamma_i)$, $i = 0, 1, \ldots, N$ for some $N \geq 0$ recursively by applying a sequence of blowups of the following type. Let $p = p_i$ be a point in $\mathcal{P}_i$ such that the Poisson tensor $\gamma_i$ vanishes at $p$, and there are local coordinates $x, y$ near $p$ such that $\gamma_i = x^a y^b (1 + O(x, y)) \partial_x^a \partial_y^b$, point $p$ has coordinates $x = y = 0$, and $a + b \geq 2$ for $a, b \in \mathbb{Z}_{\geq 0}$. Then, we make a blowup at $p$ and obtain a new Poisson surface $\mathcal{P}_{i+1} = Bl_p \mathcal{P}_i$ with Poisson tensor $\gamma_{i+1}$. The Poisson tensor $\gamma_{i+1}$ vanishes to order $a + b - 1$ at the exceptional divisor.

Let $\mathcal{P} = \mathcal{P}_N$ be the final term of our sequence. The divisor $D$ of zeros of the Poisson structure $\gamma = \gamma_N$ on $\mathcal{P}$ has simple normal crossings [24]. The open dense symplectic leaf $\mathcal{P} \setminus D$ is equal to $T^*C$.

---

[23] Our sequence $\mathcal{P}_i$ depends on choices of these points.

[24] This means that every irreducible component is smooth.
Denote by \( \{ D_\alpha \} \) the set of irreducible components of \( D \) at which \( \gamma \) vanishes with multiplicity one (or, equivalently, the meromorphic symplectic form \( \omega = \gamma^{-1} \) has a pole of order one)\(^{25}\). It follows by induction from the construction that divisors \( D_\alpha \) do not intersect each other. Moreover, each \( D_\alpha \) contains exactly one double point of \( D \), and the complement \( D_\alpha^0 \) to this point is isomorphic to \( \mathbb{A}^1 \). In fact, there is a canonical coordinate on \( D_\alpha^0 \) given by the residue of the restriction of the Liouville 1-form on \( T^* C \) to a small disc in \( P \) transversally intersecting \( D_\alpha^0 \).

**Remark 4.1**

One can associate with each component \( D_\alpha \) as above, a point \( \nu \in \overline{C} \) and a Puiseux series \( f(x) \in \bigcup_{n \geq 1} \mathbb{C}[[x^{1/n}]] = \overline{\mathbb{C}(x)} \) where \( x \) is a local coordinate at \( \nu \), up to certain identification. First, we identify two series differing by an element of \( \bigcup_{n \geq 1} \mathbb{C}[[x^{1/n}]] \). In other words, we can keep only terms with strictly negative exponents. Second, we identify series which differ by the action of the Galois group \( \mathbb{Z} \) of \( \overline{\mathbb{C}(x)} \) over \( \mathbb{C}(x) \), with the topological generator \( x^{1/n} \mapsto e^{2\pi i/n} x^{1/n} \).

For a given component \( D_\alpha \), the corresponding Puiseux series is defined as follows. Let us choose a germ \( \Sigma_\alpha \) of smooth curve in \( P \) intersecting \( D_\alpha^0 \) transversely at one point, and such that \( \Sigma_\alpha \) projects to \( \overline{C} \) non-trivially, i.e., not to a point. Then, the punctured disc \( \Sigma_\alpha \setminus \Sigma_\alpha \cap D_\alpha^0 \) contained in \( T^* C \) can be considered as the graph of a meromorphic multivalued 1-form at a point \( \nu \in \overline{C} \). This form can be written in a local coordinate \( x \) at \( \nu \) as \( df(x) + \lambda d \log x \) where \( \lambda \in \mathbb{C} \), \( f(x) \in \overline{\mathbb{C}(x)} \). One can show that different choices of the germ \( \Sigma_\alpha \) give equivalent series \( f(x) \) in the above sense. Incidentally, the equivalence classes of such series correspond to all the possible irregular terms for formal meromorphic connections on \( \mathbb{C}(x) \), see, e.g., \([10]\).

Let us choose a collection of points \( \sum_i n_i u_i, n_i \geq 1 \) in \( \bigcup_{i} D_\alpha^0 = \bigcup_{\alpha} \mathbb{A}^1 \) with multiplicities (i.e., an effective divisor) and an integer \( r \geq 1 \). With the tuple \((P, \sum_i n_i u_i, r)\), we associate a classical integrable system. The base \( B \) of this system will be the set of smooth connected curves \( \Sigma \subseteq P \) (spectral curves) such that \( \Sigma \not\subseteq D \) and \( \Sigma \cap D = \sum_i n_i u_i \), and the projection \( \Sigma \to \overline{C} \) has degree \( r \).

Let us assume that \( B \neq \emptyset \) (this assumption considered as a property of \((P, \sum_i n_i u_i, r)\) is related to the solvability of the additive Deligne–Simpson problem \([12]\)). One can show that \( B \) is an open dense subset of \( \mathbb{A}^g \) where \( g \) is the genus of any spectral curve \( \Sigma \). The tangent space \( T_\Sigma B \) is canonically identified with \( \Gamma(\Sigma, \Omega^1_{\Sigma}) \).

Let us fix an integer \( d \in \mathbb{Z} \). Define \( M_d \) to be the space of pairs \((\Sigma, [L])\) where \([L] \in Pic_d(\Sigma) \) is a class of a line bundle of degree \( d \).

There is a natural symplectic form \( \omega_{M_d} \) on \( M_d \), and the natural projection \( M_d \to B \) is a Liouville integrable system.

The above construction gives an alternative description of a Zariski open dense part of Hitchin integrable systems for the group \( GL \) on \( \overline{C} \) with possibly irregular singularities.

Notice that among integrable systems \( M_d \to B \) only one, corresponding to \( d = 0 \), has an obvious Lagrangian section, which is the zero section.

On the other hand, we can define a birational symplectomorphism\(^{26}\) \( M_g \sim Sym^g T^* C \sim T^* Sym^g C \). Indeed, we have \( Pic_g(\Sigma) \sim Sym^g \Sigma \). Hence, a generic

\(^{25}\) We will not use other components of \( D \) in our considerations.

\(^{26}\) We use the notation \( \sim \) for birational equivalence.
point in $M_g$ is a spectral curve $\Sigma$ and a collection of $g$ points $(t_1, \ldots, t_g)$ of $\Sigma$ up to a permutation. Generically, all points $t_i$ are distinct and do not belong to $\Sigma \cap D$, hence giving $g$ points in $T^*C$. Conversely, given $g$ generic points in $T^*C$, there exists a unique spectral curve $\Sigma$ passing through them, and therefore, we obtain a point in $M_g$.

This is a geometric version of the method of separation of variables. This is different from the usual Hitchin systems, where the total space is identified birationally with the cotangent bundle to the moduli space $Bun_G(\overline{C})$ of $G$-bundles on $\overline{C}$, or its version associated with marked points and irregular singularities.

Now we can define a semi-classical generalized product. Let us pick a point $u_{i_0}$ among the collection $\{u_i\}$. Notice that $u_{i_0} \in \Sigma$ for any spectral curve $\Sigma \in B$. Using $u_{i_0}$, we can identify $M_d$ for all $d \in \mathbb{Z}$ by adding multiples of $u_{i_0}$. Also, we define a Lagrangian subvariety $L_{g+1}^1, g \subset Sym^{g+1}(T^*C, -\omega) \times Sym^g(T^*C, \omega)$ by the formula

$$L_{g+1}^1, g = \{(a_1, \ldots, a_{g+1}, b_1, \ldots b_g); \text{there exists } \Sigma \in B \text{ such that}$$

$$a_1, \ldots, a_{g+1}, b_1, \ldots b_g \in \Sigma, \quad a_1 + \ldots + a_{g+1} = b_1 + \ldots + b_g + u_{i_0} \in \text{Pic}(\Sigma)\}.$$ 

This variety might be singular, so we treat it only as a first rough approximation. It is clear that $L_{g+1}^1, g$ considered as a morphism in the Weinstein category, which is a generalized product.

Similarly, one can define Lagrangian correspondences $L_{g+k}^1, g$ for any $k \geq 1$. In the case $k = g$, we obtain a semi-classical multiplication kernel for the symplectic variety $M_0 = M_g$.

**Remark 4.2** The choice of a point $u_{i_0}$ corresponds in a sense to the choice of a normalization of functions $\phi_{\lambda_1, \ldots, \lambda_n}(x)$ in Sect. 4.1, and therefore, to the choice of a normalization of functions $\psi_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n)$.

### 4.3 Toward quantized generalized product

In this section, we will propose a hypothetical construction of the quantization of the semi-classical generalized product introduced in Sect. 4.2, understood as a holonomic $D$-module on $C^{2g+1}$ together with a cyclic vector.

Let $(\mathcal{P}, \sum_i n_i u_i, r)$ be a tuple as in Sect. 4.2. We now assume that for any two different points $u_i \neq u_j$ belonging to the same component $D^0_\alpha = \mathbb{A}^1$ the difference $\mu_i - \mu_j$ between their canonical coordinates is not an integer. Starting with such a tuple we construct a family of cyclic $D$-modules on $\overline{C}$ depending on $g$ parameters $\lambda_1, \ldots, \lambda_g$. Namely, consider meromorphic differential operators $L$ of order $r$ acting from the trivial line bundle $\mathcal{O}_{\overline{C}}$ to $K^{\otimes r}_{\overline{C}}$, with symbol $\left(\frac{\partial}{\partial x}\right)^r$ in local coordinates, and such that singularities of solutions correspond to $\sum_i n_i u_i$. To explain the latter condition more precisely, recall that a point $u_i \in D$ belongs to a component $D_{\alpha_i}$ of the divisor $D$. In Sect. 4.2, we explained that the divisor $D_{\alpha_i}$ gives a Puiseux series $f_i(x)$ at point $v_i = pr_{D \to \overline{C}}(u_i) \in \overline{C}$. Denote by $x = x_i$ the local coordinate at $v_i$.
and by $\mu_i \in \mathbb{C}$ the position of the point $u_i \in D$ on $D_{\alpha_i}^0 = \mathbb{A}^1$. Then, we say that differential equation $L \phi = 0$ has a singularity at $v_i$ corresponding to $u_i$ and $n_i \geq 1$ if this equation has solutions with asymptotic behavior

$$\phi(x) = x^{\mu_i} e^{f_i(x)} \log(x)^k(1 + \ldots), \quad k = 0, \ldots, n_i - 1.$$ 

One can show that differential operators $L$ satisfying these properties form an affine space of dimension $g$ (a version of the variety of opers for the Hitchin system for the group $GL_r$) and can be written as $L = L_0 + \sum_{i=1}^g \lambda_i L_i$ where $\deg L_0 = r$, $\deg L_i < r, i = 1, \ldots, g$. Here $(\lambda_1, \ldots, \lambda_g)$ are parameters (coordinates on the space of opers).

Recall that in Sect. 4.2 we made a choice $u_{i_0}$ of one point of $\Sigma \cap D$. Let us assume that the corresponding Puiseux series $f_{i_0}$ is unramified 27, i.e., belongs to $\mathbb{C}((x)) \subset \bigcup_{n \geq 1} \mathbb{C}((x^{1/n}))$. We define the normalized solution corresponding to $u_{i_0}$ as the unique formal solution at $v_{i_0}$ of the form

$$\phi_{\lambda_1, \ldots, \lambda_g}(x) = x^{\mu_{i_0}} e^{f_{i_0}(x)} \left(1 + \sum_{j \geq 1} P_j x^j\right)$$

where $P_1, P_2, \ldots$ depend on $\lambda_1, \ldots, \lambda_g$. One can see that $P_j$ are polynomials in $\lambda_1, \ldots, \lambda_g$, and we set $P_0 = 1$.

Without loss of generality, in order to simplify the exposition, we assume that $\mu_{i_0} = 0, f_{i_0}(x) = 0$. This can be achieved by a conjugation of $L$.

Examples suggest that the following is true:

The set $\{P_{i_1} \ldots P_{i_g}; i_1 \leq \ldots \leq i_g\}$ forms a linear basis of $\mathbb{C}[\lambda_1, \ldots, \lambda_g]$.

Assuming this property, we can define the coefficients

$$C_{i_1, \ldots, i_{g+1}}^{j_1, \ldots, j_g} = \text{Coeff}_{P_{j_1} \ldots P_{j_g}}(P_{i_1} \ldots P_{i_{g+1}}).$$

These coefficients are structure constants of a generalized commutative associative product on an infinite-dimensional space with a basis $e_0, e_1, \ldots$ where $e_j$ corresponds to $P_j$.

**Remark 4.3** In general, suppose that $A$ is a commutative associative algebra over a field $k$ of characteristic zero, and $V \subset A$ is a vector subspace such that the composition $\text{Sym}^g V \rightarrow V^\otimes g \rightarrow A^\otimes g \rightarrow A$ is an isomorphism of vector spaces, where the last map is induced by multiplication in $A$. Then $V$ carries a structure of commutative associative generalized product $\text{Sym}^{g+1} V \rightarrow \text{Sym}^g V = A$.

Let us consider generating series

$$K_{g+1,g}(x_1, \ldots, x_{g+1}, y_1, \ldots, y_g) = \sum_{i_1, \ldots, j_g \geq 0} C_{i_1, \ldots, i_{g+1}, j_1, \ldots, j_g}^{1, \ldots, g} \frac{dy_k}{y_k}.$$

\(^{(4.1)}\)

27 We do not know how to extend our construction to the case of ramified $f_{i_0}$.
We consider $K_{g+1,g}$ as an element of $W/W_+$ where
\[
W = \mathbb{C}[[y_1, ..., y_g]] \left[ y_1^{-1}, ..., y_g^{-1} \right] \left[ x_1, ..., x_{g+1} \right] \prod_{k=1}^{g} dy_k
\]
and $W_+ \subset W$ is the subspace consisting of series which do not have poles in the variable $y_i$ for some $i = 1, ..., g$.

Consider the algebraic variety $X = \mathbb{C}^{2g+1}$ with the line bundle $\mathcal{L} = \otimes_{i=1}^{2g+1} Pr_i^* T_{\mathbb{P}^1}^*$ and a point $p = (v_{i_0}, ..., v_{i_0})$. Then we have an algebra $\text{Diff}_{rat} := \text{Diff}_L, \text{rat}$ of differential operators on $L$ with coefficients in rational functions on $X$ (i.e., differential operators at the generic point of $X$). It has a subalgebra $\text{Diff}_{rat,p}$ of differential operators without poles at $p$. Vector spaces $W$, $W_+$ and hence $W/W_+$ are $\text{Diff}_{rat,p}$-modules.

**Conjecture 4.1** The cyclic $\text{Diff}_{rat,p}$-module $K_{g+1,g}$ generated by $K_{g+1,g}$ is holonomic, i.e., isomorphic to the restriction of a holonomic $D$-module on $X$ to a Zariski neighborhood of $p \in X$. Moreover, $\text{Diff}_{rat} \otimes \text{Diff}_{rat,p}$ $K_{g+1,g}$ is the kernel of the generalized product localized at the generic point of $X = \mathbb{C}^{2g+1}$.

Examples suggest that this kernel can be written in an explicit form, see Eqs. (1.4), (1.6) in the Introduction where the integration is understood as the direct image of holonomic $D$-modules. (Here, in order to speak about direct images, we should work with honest $D$-modules, defined not only at the generic point).

**Remark 4.4** The example of a family of functions $\phi_{\lambda_1, ..., \lambda_g}(x) = e^{\lambda_1 x + ... + \lambda_g x^g}$ does not fit into the above scheme. Here $\phi_{\lambda_1, ..., \lambda_g}(x)$ is the unique solution of the equation
\[
(L_0 + \lambda_1 L_1 + ... + \lambda_g L_g)\phi(x) = 0
\]
in $\mathbb{C}[[x]]$ with constant term 1. Here $L_0 = \frac{\partial}{\partial x}$, $L_i = -ix^{i-1}$, $i = 1, ..., g$. It will be interesting to find a general framework which includes this example.

Conjecture 4.1 is formulated in terms of the quotient $W/W_+$, which is a complicated object not suitable for explicit computations. It is more convenient to view $K_{g+1,g}$ in (4.1) literally as a function of $x_i$, $y_i$. For example, we can assume that $\mathbb{C} = \mathbb{P}^1$ and $x$ is a global coordinate.

The examples studied in the next section indicate that the cyclic $D$-module for the lifted kernel is still holonomic, and it maps epimorphically to the expected generalized multiplication kernel.

**Remark 4.5** The quantum generalized multiplication kernel should produce isomorphisms of the following type. For all $\bar{\lambda} = (\lambda_1, ..., \lambda_g)$, we expect that
\[
(C^{g+1} \to C^{g+1})_\# (K_{g+1,g} \otimes (E_{\bar{\lambda}} \boxtimes ... \boxtimes E_{\bar{\lambda}})) \simeq E_{\lambda_1} \boxtimes ... \boxtimes E_{\lambda_g}
\]
(4.2)
where $\mathcal{E}_\lambda$ is a cyclic\textsuperscript{28} holonomic $D$-module (oper) parameterized by $\lambda$. It is known in the theory of integrable systems that the locus of opers is a Lagrangian subvariety in the symplectic manifold parameterizing all (non-cyclic) holonomic $D$-modules on $C$ with given singularities (de Rham moduli space). We claim that Eq. (4.2) holds also for more general modules. The rough reason is that (4.2) makes sense in the Betti realization, in which the locus of opers is Zariski dense.

### 4.4 A generalization to other Poisson surfaces

In Sect. 4.2, we made a sequence of blowups of the Poisson surface $\mathcal{P}_0 = \mathbb{P}(\mathcal{O}_C \oplus T^*_{C, \text{log} S})$. Assume that $S \neq \emptyset$, hence $C$ is affine. The quantization of the open dense symplectic leaf of $\mathcal{P}_0$ is the algebra of differential operators on $C$. Let us refer to this class of Poisson surfaces as to “rational”. There are two other classes of compact Poisson surfaces, which we will call “trigonometric” and “elliptic”, with an open dense symplectic leaf $M$. In all three cases, the open dense symplectic leaf $M$ is an affine variety.

In the trigonometric case, $\mathcal{P}$ is any toric compactification of its symplectic leaf $M = \mathbb{C}^* \times \mathbb{C}^*$ with $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$. The quantization of $M$ is the quantum torus $A_q = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / XY = qYX$.

In the elliptic case, $\mathcal{P} = \mathbb{C}P^2$ with the symplectic leaf $M = \mathcal{P} \setminus \{\text{cubic curve}\}$. The quantization is an inhomogeneous version of the Sklyanin algebra with three generators [11].

In the elliptic case, there is also a version with non-affine $M$ similar to $T^*C$ for compact curve $C$. Namely, let $\mathcal{P} = \mathbb{P}^1 \times E$ where $E$ is an elliptic curve and $M = \mathbb{C}^* \times E$. In this case instead of algebras, one should consider abelian or triangulated categories (analogs of categories of modules).

In our considerations in Sect. 4.2, the principal role was played by divisors on the blowup at which $\omega$ has pole of order one.

In the trigonometric case, such divisors do not have continuous parameters and correspond to pairs of coprime integers $(a, b)$, i.e., these divisors are toric divisors. Each $D_0^a$ is isomorphic to $\mathbb{C}^*$.

In the elliptic case, there is only one such divisor, the initial cubic curve $E$ (and there are no non-trivial blowups).

Similarly to Sect. 4.2, we choose a sequence of blowups (trivial in the elliptic case) and a collection of points with multiplicities on $\bigsqcup \alpha D_0^a$.

In this way, we obtain a classical integrable system. Choosing one point $u_{j0}$, we get a semi-classical kernel $K_{g+1,g}$. The analog of opers will be cyclic holonomic $A$-modules where $A$ is the quantization of $M$. The quantum kernel should be a cyclic holonomic $A^{\otimes (g+1)} \otimes (A^{op})^{\otimes g}$-module.

Notice that the rational case $M = T^*C$ is related to the geometric Langlands correspondence for groups $GL_r$. In the trigonometric (resp. elliptic) cases, the multiplication kernels $K_{g+1,g}$ are also expected to be a kind of motivic in trigonometric (resp. elliptic) sense. For example, the $q$-analog of motivic holonomic modules is

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\textsuperscript{28} By a cyclic $D$-module, we mean a $D$-module together with a choice of a cyclic vector.
discussed in Section 6.1 of [8]. Roughly, these modules are built from the basic $A_q$-modules with cyclic vector $A_q/A_q \cdot (X + Y - 1)$ by external tensor products, actions of $Sp(2n, \mathbb{Z})$ and pushforwards.

5 Multiplication kernels associated with differential operators

In this section, we always work in the global coordinate on $\mathbb{A}^1$.

5.1 General setup

Let $P_0(\lambda), P_1(\lambda), \ldots \in \mathbb{C}[\lambda]$ be a basis of the vector space $\mathbb{C}[\lambda]$ such that $\deg P_i = i$ and $P_0 = 1$. We have

$$P_i(\lambda)P_j(\lambda) = \sum_{k=0}^{\infty} C_{i,j}^k P_k(\lambda) \quad (5.1)$$

where $C_{i,j}^k \in \mathbb{C}$ are structure constants of polynomial multiplication in the basis $P_i(\lambda)$. Construct generating functions:

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda)x^i, \quad (5.2)$$

$$K(x_1, x_2, y) = \sum_{i,j,k \geq 0} C_{i,j}^k x_1^i x_2^j y^{k+1}. \quad (5.3)$$

Notice that $f_\lambda(x) = 1 + O(x)$ and $K(x, 0, y) = \frac{1}{y-x}$.

We have by construction

$$f_\lambda(x_1)f_\lambda(x_2) = \frac{1}{2\pi i} \oint K(x_1, x_2, y)f_\lambda(y)dy \quad (5.4)$$

where integral is taken over a small circle around zero. This means that the following associativity condition holds:

$$\oint K(x_1, x_2, y)K(y, x_3, z)dy = \oint K(x_1, x_3, y)K(y, x_2, z)dy. \quad (5.5)$$

Let $D_x$ be a differential operator in $x$. Assume that our generating function is a solution of the differential equation

$$D_x f_\lambda(x) = \lambda f_\lambda(x).$$

In this case, the kernel $K(x_1, x_2, y)$ satisfies the equations

$$D_{x_1}K(x_1, x_2, y) = D_{x_2}K(x_1, x_2, y)$$
and

\[ D_x, K(x_1, x_2, y) - D_y K(x_1, x_2, y) \in \mathbb{C}[[y]]. \]

**Example 5.1** Let \( D_x = \frac{d}{dx} \) and \( f_\lambda(x) = e^{\lambda x} \). In this case, we have

\[ K(x_1, x_2, y) = \frac{1}{y - x_1 - x_2}. \]

The construction above can be generalized to the case of polynomials \( P_i(\lambda_1, \ldots, \lambda_g) \), \( i = 0, 1, \ldots \) such that \( P_0(\lambda_1, \ldots, \lambda_g) = 1 \) and \( \{P_{i_1} \cdots P_{i_g} : 0 \leq i_1 \leq i_2 \leq \cdots \leq i_g\} \) is a basis in the vector space \( \mathbb{C}[\lambda_1, \ldots, \lambda_g] \). We have

\[ P_1 \cdots P_{ig+1} = \sum_{j_1, \ldots, j_g \geq 0} C_{j_1, \ldots, j_g}^{i_1, \ldots, i_g} P_{j_1} \cdots P_{j_g} \quad (5.6) \]

where we assume that \( C_{j_1, \ldots, j_g}^{i_1, \ldots, i_g} \) are independent of \( \lambda_1, \ldots, \lambda_g \) and symmetric with respect to indexes \( j_1, \ldots, j_g \).

Construct generating functions:

\[ f_{\tilde{\lambda}}(x) = \sum_{i=0}^{\infty} P_i(\lambda_1, \ldots, \lambda_g)x^i, \quad \text{where} \quad \tilde{\lambda} = (\lambda_1, \ldots, \lambda_g), \quad (5.7) \]

\[ K(x_1, \ldots, x_{g+1}, y_1, \ldots, y_g) = \sum_{i_1, \ldots, i_g \geq 0} C_{i_1, \ldots, i_g}^{j_1, \ldots, j_g} x_1^{i_1} \cdots x_{g+1}^{i_{g+1}} y_1^{-j_1-1} \cdots y_g^{-j_g-1}. \quad (5.8) \]

Notice that \( f_{\tilde{\lambda}}(x) = 1 + O(x) \) and

\[ K(x_1, \ldots, x_n, 0, y_1, \ldots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(y_{\sigma_1} - x_1) \cdots (y_{\sigma_n} - x_n)}. \]

We have by construction

\[ f_{\tilde{\lambda}}(x_1) \cdots f_{\tilde{\lambda}}(x_{n+1}) = \frac{1}{(2\pi i)^n} \oint \cdots \oint K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) f_{\tilde{\lambda}}(y_1) \cdots f_{\tilde{\lambda}}(y_n)dy_1 \cdots dy_n \quad (5.9) \]

where we integrate over small circles around zero with respect to each \( y_i \). We also have an associativity condition: the expression

\[ \oint \cdots \oint K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) K(x_{n+2}, y_1, \ldots, y_n, z_1, \ldots, z_n)dy_1 \cdots dy_n \quad (5.10) \]

is symmetric with respect to \( x_1, \ldots, x_{n+2} \).
Assume that
\[ D_x f_\lambda(x) = (\lambda_1 + \lambda_2 x + \ldots + \lambda_g x^{g-1}) f_\lambda(x) \]
for a differential operator \( D_x \). In this case, the kernel \( K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) \) satisfies the equation
\[
\sum_{i=1}^{n+1} \frac{1}{(x_1 - x_i) \cdots (x_{n+1} - x_i)} D_{x_i} K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) = 0.
\]

### 5.2 A kernel associated with first-order differential operators

Here we return to the basic example introduced in Sect. 4.1. Define polynomials \( P_i(\lambda_1, \ldots, \lambda_g) \) by
\[
e^{\lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_g x^g} = \sum_{i \geq 0} P_i(\lambda_1, \ldots, \lambda_g) x^i.
\]
Notice that if
\[
D_x = \frac{d}{dx} - (\lambda_1 + 2\lambda_2 x + \ldots + g\lambda_g x^{g-1}),
\]
then \( D_x e^{\lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_g x^g} = 0. \)

Define structure constants \( C_{i_1, \ldots, i_g} \in \mathbb{Q} \) by (5.6).

Define the kernel \( K(x_1, \ldots, x_{g+1}, y_1, \ldots, y_g) \) as the generating function for these structure constants by (5.8).

**Theorem 5.1**

\[
K(x_1, \ldots, x_{g+1}, y_1, \ldots, y_g) = \frac{1}{g!} \sum_{\sigma \in S_g} \frac{1}{(y_{\sigma_1} - q_1) \cdots (y_{\sigma_g} - q_g)}
\]

where \( q_1, \ldots, q_g \) are roots of the polynomial
\[
Q(t) = t^g - (x_1 + \ldots + x_{g+1}) t^{g-1} + (x_1 x_2 + \ldots + x_g x_{g+1}) t^{g-2} + \ldots \pm (x_1 \ldots x_g + \ldots + x_2 \ldots x_{g+1}).
\]

The coefficients of this polynomial in \( t \) are elementary symmetric polynomials in \( x_1, \ldots, x_{g+1} \).

---

29 In the notations of Sect. 4.3, we have \( L_0 = D_x \), \( L_i = -x^{i-1}, i = 1, \ldots, g \).
Proof To simplify notations let \( g = 2 \), the general case is similar. We have

\[
e^{\lambda_1(x_1+x_2+x_3)+\lambda_2(x_1^2+x_2^2+x_3^2)} = \sum_{i_1,i_2,i_3 \geq 0} P_{i_1} P_{i_2} P_{i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}
\]

\[
= \sum_{i_1, \ldots, i_2 \geq 0} C_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} P_{i_1} P_{i_2}
\]

\[
= \frac{1}{(2\pi i)^2} \sum_{i_1, \ldots, i_2 \geq 0} C_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \oint \oint K(x_1, x_2, x_3, y_1, y_2) e^{\lambda_1(y_1+y_2)+\lambda_2(y_1^2+y_2^2)} dy_1 dy_2
\]

\[
\times dy_1 dy_2
\]

where integrals are taken over small circles around 0. Expanding in power series in \( \lambda_1, \lambda_2 \) and equating coefficients at \( \lambda_1^{n_1} \lambda_2^{n_2} \), we get

\[
(x_1 + x_2 + x_3)^{n_1} (x_1^2 + x_2^2 + x_3^2)^{n_2} = \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2) (y_1 + y_2)^{n_1}
\]

\[
\times (y_1^2 + y_2^2)^{n_2} dy_1 dy_2
\]

It follows that

\[
\phi(x_1 + x_2 + x_3, x_1^2 + x_2^2 + x_3^2) = \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2)
\]

\[
x \phi(y_1 + y_2, y_1^2 + y_2^2) dy_1 dy_2
\]

where \( \phi(u, v) \) is an arbitrary function analytic near \( u = v = 0 \). We can write this condition as

\[
\phi(x_1 + x_2 + x_3, x_1 x_2 + x_2 x_3 + x_1 x_3) = \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2)
\]

\[
x \phi(y_1 + y_2, y_1 y_2) dy_1 dy_2
\]

by changing variables in \( \phi \) (because elementary symmetric functions can be written in terms of sums of powers).

Let \( \phi = \phi_{m_1,m_2} \) where \( \phi_{m_1,m_2}(y_1 + y_2, y_1 y_2) = y_1^{m_1} y_2^{m_2} + y_1^{m_2} y_2^{m_1} \), it is clear that \( \phi_{m_1,m_2}(u, v) = s_1^{m_1} s_2^{m_2} + s_2^{m_1} s_1^{m_2} \) where \( s_1, s_2 \) are roots of the polynomial \( t^2 - ut + v \).

With this choice of \( \phi \), we obtain

\[
\frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2) (y_1^{m_1} y_2^{m_2} + y_1^{m_2} y_2^{m_1}) dy_1 dy_2 = q_1^{m_1} q_2^{m_2} + q_2^{m_1} q_1^{m_2}
\]
where $q_1, q_2$ are roots of the polynomial $t^2 - (x_1 + x_2 + x_3)t + x_1x_2 + x_2x_3 + x_1x_3$. It follows

$$K(x_1, x_2, x_3, y_1, y_2) = \frac{1}{2} \sum_{m_1, m_2 \geq 0} \left( q_1^{m_1} q_2^{m_2} y_1^{-m_1-1} y_2^{-m_2-1} + q_2^{m_1} q_1^{m_2} y_1^{-m_1-1} y_2^{-m_2-1} \right)$$

and summing up geometric series we obtain the statement of the Theorem. □

5.3 The case of second-order differential operators with four regular singular points

Let

$$D_x = x(x-1)(x-t) \frac{d^2}{dx^2} + x(x-1)(x-t) \left( \frac{s_1}{x} + \frac{s_2}{x-1} + \frac{s_3}{x-t} \right) \frac{d}{dx} + r_1 r_2 x + \lambda$$

where $t, s_1, s_2, s_3, r_1, r_2, \lambda$ are parameters such that $t \neq 0, 1$ and

$$r_1 + r_2 = s_1 + s_2 + s_3 - 1.$$

This is the most general second-order differential operator\(^{30}\) with regular singularities at $x = 0, 1, t, \infty$ and with analytic solutions near $x = 0, 1, t$.

There exists a unique solution $f_\lambda(x)$ of the differential equation

$$D_x f_\lambda(x) = 0$$

such that $f_\lambda(x)$ is analytic near $x = 0$ and $f_\lambda(0) = 1$. We have

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda) x^i$$

where $P_i(\lambda)$ are polynomials in $\lambda$ of degree $i$ and $P_0(\lambda) = 1$.

Since $P_i(\lambda), i = 0, 1, \ldots$ is a basis of the vector space $\mathbb{C}[\lambda]$, we can define structure constants of polynomial multiplication in this basis by (5.1).

Define the kernel $K(x_1, x_2, y)$ as a generating function of these structure constants by (5.3).

Recall that the Gauss hypergeometric function is given by

$$F(a, b, c, u) = \sum_{n=0}^{\infty} \frac{a(a+1)...(a+n-1) \cdot b(b+1)...(b+n-1)}{c(c+1)...(c+n-1)} \cdot \frac{u^n}{n!} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 p^{b-1}(1-p)^{c-b-1}(1-pu)^{-a} \, dp.$$

\(^{30}\) This example already appeared in Sects. 3.3 and 4.3, with slightly different notations for parameters.
Theorem 5.2 The kernel \( K(x_1, x_2, y) \) is given by:
\[
K(x_1, x_2, y) = \frac{1}{y - 1} \sum_{i=0}^{\infty} \frac{(r_1 + r_2 - s_1 - s_3 + 1)...(r_1 + r_2 - s_1 - s_3 + i)}{(r_1 + r_2 - s_1 + i)...(r_1 + r_2 - s_1 + 2i - 1)}
\times F\left(i + 1, r_1 + r_2 - s_1 - s_3 + i + 1, r_1 + r_2 - s_1 + 2i + 1, \frac{t - 1}{y - 1}\right)
\times F\left(-i, r_1 + r_2 - s_1 + i, r_1 + r_2 - s_1 - s_3 + 1, \frac{t(x_1 - 1)(x_2 - 1)}{(t - 1)(x_1 x_2 - t)}\right)
\times F\left(r_1 + i, r_2 + i, s_1, \frac{x_1 x_2}{t}\right)\left(\frac{x_1 x_2}{t} - 1\right)^i \frac{(t - 1)^i}{(y - 1)^r}.
\]

The proof is based on the following

Lemma 5.1 The function \( K(x_1, x_2, y) \) is the unique function characterized by the following properties:

1. \( K(x_1, x_2, y) = K(x_2, x_1, y) \).
2. \( K(x_1, x_2, y) \) has Laurent series expansion in nonnegative powers of \( x_1, x_2 \) and negative powers of \( y \).
3. \( D_{x_1} K(x_1, x_2, y) = D_{x_2} K(x_1, x_2, y) \).
4. \( K(x_1, 0, y) = \frac{1}{y - x_1} \).

Proof These properties of \( K(x_1, x_2, y) \) follow from the definition, see the discussion in Sect. 5.1. The proof of uniqueness of the solution of the differential equation \((D_{x_1} - D_{x_2}) K = 0\) with the properties (1), (2), (4) is omitted. \( \square \)

Based on this explicit series representation for the kernel \( K(x_1, x_2, y) \), one can derive the following holonomic system of differential equations.

Theorem 5.3 The function \( K(x_1, x_2, y) \) satisfies the following differential equations:
\[
D_{x_1} K - D_{x_2} K = 0,
\]
\[
D^*_y K - D_{x_1} K = (r_1 - 1)(r_2 - 1) F\left(r_1, r_2, s_1, \frac{x_1 x_2}{t}\right)
\]
\[
L K(x_1, x_2, y) = \frac{(s_1 - 1)t}{x_1 x_2(x_1 - y)(x_2 - y)} F\left(r_1 - 1, r_2 - 1, s_1 - 1, \frac{x_1 x_2}{t}\right)
\]

where
\[
D^*_y = \frac{d^2}{dy^2} \cdot y(y - 1)(y - t) - \frac{d}{dy} \cdot y(y - 1)(y - t) \left(\frac{s_1}{y} + \frac{s_2}{y - 1}\right)
+ \frac{s_3}{y - t} + r_1 r_2 y + \lambda
\]
is the differential operator conjugate to \( D_y \) and

\[
L = \frac{(x_1 - 1)(x_1 - t)}{(x_1 - x_2)(x_1 - y)} \cdot \frac{d}{dx_1} + \frac{(x_2 - 1)(x_2 - t)}{(x_2 - x_1)(x_2 - y)} \cdot \frac{d}{dx_2} + \frac{(y - 1)(y - t)}{(y - x_1)(y - x_2)} \cdot \frac{d}{dy}
- \frac{(r_1 + r_2 - 2)x_1 x_2 y + (s_2 + 1 - r_1 - r_2)x_1 x_2 - (s_1 + s_2 - 2)t x_1 x_2 + (s_1 - 1)t(x_1 + x_2 - y)}{x_1 x_2(x_1 - y)(x_2 - y)}
\]
Theorem 5.4 The kernel $K(x_1, x_2, y)$ admits the following integral representation

$$K(x_1, x_2, y) = \frac{s_1 - 1}{1 - t} \left( \frac{2(x_1 - 1)(x_2 - 1)}{1 - t} \right)^{1-s_2} \left( \frac{2(x_1 - t)(x_2 - t)}{t(t - 1)} \right)^{1-s_3}$$

$$\times \int_0^1 F(r_1 - 1, r_2 - 1, s_1 - 1, uq) q^{s_1 - 2} Q^{-1/2}$$

$$\times \left( - uq - v + w + 1 + Q^{1/2} \right)^{s_2 - 1}$$

$$\times \left( - uq + v - w + 1 - Q^{1/2} \right)^{s_3 - 1} \ dq$$

where

$$u = \frac{x_1 x_2}{t}, \quad v = \frac{(x_1 - 1)(x_2 - 1)(y - 1)}{(t - 1)^2},$$

$$w = \frac{(x_1 - t)(x_2 - t)(y - t)}{t(t - 1)^2},$$

$$Q = (v - w)^2 - 2(v + w)(uq - 1) + (uq - 1)^2.$$ 

and we assume that $Q^{1/2} \sim v - w$ for $y \to \infty$.

After substitution of the integral representation of Gauss hypergeometric function and the change of variables

$$q_1 = \frac{-uq - v + w + 1 + Q^{1/2}}{2(1 - pqu)}, \quad q_2 = \frac{-uq + v - w + 1 - Q^{1/2}}{2(1 - pqu)} \quad (5.11)$$

we obtain another integral formula for the kernel

$$K(x_1, x_2, y) = \left( \frac{x_1 x_2}{t} \right)^{1-s_1} \left( \frac{(x_1 - 1)(x_2 - 1)}{1 - t} \right)^{1-s_2} \left( \frac{(x_1 - t)(x_2 - t)}{t(t - 1)} \right)^{1-s_3}$$

$$\times \frac{\Gamma(s_1)}{\Gamma(r_1 - 1)\Gamma(s_1 - r_1)(1 - t)} \int_D q_1^{s_1 - 1} q_2^{s_3 - 1} (1 - q_1 - q_2)^{s_1 - r_1 - 1}$$

$$\times \left( 1 + \frac{v}{q_1} + \frac{w}{q_2} \right)^{r_1 - 2} \frac{dq_1}{q_1} \cdot \frac{dq_2}{q_2}$$

where $D = \{(q_1, q_2), \ 0 \leq p, q \leq 1\}$ and $q_1, q_2$ are parameterized by (5.11).

Theorem 5.5 Let $s_1 = r_1 = 1$. In this case, the kernel $K(x_1, x_2, y)$ is given by:

$$K(x_1, x_2, y) = \left( \frac{2t - t(x_1 + x_2 + y) + x_1 x_2 y + ty P^{1/2}}{2t(1 - x_1)(1 - x_2)} \right)^{s_2 - 1}$$

$$\times \left( \frac{2t^2 - t(x_1 + x_2 + y) + x_1 x_2 y + ty P^{1/2}}{2(t - x_1)(t - x_2)} \right)^{s_3 - 1} \frac{1}{y P^{1/2}}$$
where

\[
P = 1 - \frac{2x_1}{y} - \frac{2x_2}{y} + \frac{x_1^2}{y^2} + \frac{x_2^2}{y^2} - \frac{2x_1^2x_2}{ty} - \frac{2x_2^2}{ty} + \frac{x_1^2x_2^2}{t^2} + \frac{2(2ty - y^2 - t + 2y)x_1x_2}{ty^2}.
\]

**Proof** Set \( r_2 = 1 \) in the integral formula for the kernel above. After that write

\[
(s_1 - 1)q^{s_1-2}dq = d(q^{s_1-1}),
\]

integrate by parts and set \( s_1 = 1 \).

\[\square\]

5.4 The case of second-order differential operators with more than four regular
singular points

Fix a natural number \( n \geq 1 \), pairwise distinct points \( t_1, \ldots, t_n \in \mathbb{C} \) such that \( t_i \neq 0, 1 \) for \( i = 1, \ldots, n \) and parameters \( s_1, \ldots, s_{n+2}, r_1, r_2 \in \mathbb{C} \) such that

\[
s_1 + \ldots + s_{n+2} = r_1 + r_2 + 1.
\]

Let

\[
D_x = x(x-1)(x-t_1)\ldots(x-t_n)\left(\frac{d^2}{dx^2} + \left(\frac{s_1}{x} + \frac{s_2}{x-1} + \frac{s_3}{x-t_1} + \ldots + \frac{s_{n+2}}{x-t_n}\right)\frac{d}{dx}\right)
\]

\[
+ \lambda_1 + \lambda_2 x + \ldots + \lambda_n x^{n-1} + r_1 r_2 x^n.
\]

This is the most general second-order differential operator with regular singularities
at \( x = 0, 1, t_1, \ldots, t_n, \infty \) and having analytic solutions near \( x = 0, 1, t_1, \ldots, t_n \).

Let \( f_\lambda^x(x) \) be the unique solution of the equation \( D_x f_\lambda^x(x) = 0 \) analytic at \( x = 0 \)
and such that \( f_\lambda^x(x) = 1 + O(x) \). Write

\[
f_\lambda^x(x) = \sum_{i=0}^{\infty} P_i x^i
\]

where \( P_i, i = 0, 1, \ldots \) are polynomials in \( \lambda_1, \ldots, \lambda_n \) and \( P_0 = 1 \). One can show that the
products \( \{P_{i_1} \ldots P_{i_n}, 0 \leq i_1 \leq \ldots \leq i_n\} \) form a basis in the vector space \( \mathbb{C}[\lambda_1, \ldots, \lambda_n] \).

Define structure constants \( C_{j_1 \ldots j_{n+1}}^{i_1 \ldots i_n} \) by (5.6).

Define the kernel \( K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) \) as the generating function by (5.8).

**Lemma 5.2** The kernel \( K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) \) is the unique function characterized
by the following properties:

(1) It is symmetric with respect to \( x_1, \ldots, x_{n+1} \) and with respect to \( y_1, \ldots, y_n \).
(2) It has Laurent series expansion in nonnegative powers of \( x_1, \ldots, x_{n+1} \) and in negative powers of \( y_1, \ldots, y_n \).

(3) The following differential equation holds:

\[
\sum_{i=1}^{n+1} \frac{1}{(x_1 - x_i) \ldots (x_{n+1} - x_i)} D_{x_i} K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) = 0
\]

(4) \( K(x_1, \ldots, x_n, 0, y_1, \ldots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(y_{\sigma_1} - x_1) \ldots (y_{\sigma_n} - x_n)}. \)

**Theorem 5.6** The following equation holds:

\[
K(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) = \sum_{i_1, \ldots, i_n \geq 0} F(r_1 + i_1 + \ldots + i_n, r_2 + i_1 + \ldots + i_n, s_1, \frac{x_1 \ldots x_{n+1}}{t_1 \ldots t_n})
\times u_0^{i_1 + \ldots + i_n} P_{i_1, \ldots, i_n}(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0}) Q_{i_1, \ldots, i_n}(y_1, \ldots, y_n),
\]

where \( F \) is the Gauss hypergeometric function,

\[
P_{i_1, \ldots, i_n}(v_1, \ldots, v_n) = \sum_{j_1, \ldots, j_n \geq 0} \frac{\prod_{j=1}^{i_1} (s_2 + j_1 + \ldots + j_n - l) \prod_{j=1}^{i_1} (s_1 - j_1 + l + 1) \prod_{j=1}^{i_n} (s_n - j_n + l + 1)}{\prod_{j=1}^{i_1} (s_3 + l - 1) \prod_{j=1}^{i_n} (s_{n+2} - j_n + l + 1) j_1! \ldots j_n!}
\times \frac{(x_1 - t_1) \ldots (x_{n+1} - t_1)}{t_1(t_1 - 1) \ldots (t_n - t_1)},
\]

Note that \( u_0^{i_1 + \ldots + i_n} P_{i_1, \ldots, i_n}(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0}) \) are polynomials in \( u_0, u_1, \ldots, u_n \) for nonnegative \( i_1, \ldots, i_n \in \mathbb{Z} \) and the sum in the definition of \( P_{i_1, \ldots, i_n} \) is finite in this case.

The functions \( Q_{i_1, \ldots, i_n}(y_1, \ldots, y_n) \) are determined by the system of equations

\[
\sum_{i_1, \ldots, i_n \geq 0} \tilde{u}_{i_1}^{i_1 + \ldots + i_n} P_{i_1, \ldots, i_n}(\frac{\tilde{u}_1}{u_0}, \ldots, \frac{\tilde{u}_n}{u_0}) Q_{i_1, \ldots, i_n}(y_1, \ldots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(y_{\sigma_1} - x_1) \ldots (y_{\sigma_n} - x_n)}
\]

where \( \tilde{u}_i = u_i |_{x_{n+1} = 0} \).

We expect that the holonomic \( D \)-module generated by \( K \) maps epimorphically at the generic point to the one described in Sect. 3.4 with a possible change of the cyclic vector.
5.5 The case of third-order differential operators with three regular singular points

Let

\[ D_x = x^2(x - 1)^2 \frac{d^3}{dx^3} + x(x - 1)(a_1 + a_2x) \frac{d^2}{dx^2} + (a_3 + a_4x + a_5x^2) \frac{d}{dx} + a_6x + \lambda \]

where \( a_1, \ldots, a_6, \lambda \) are parameters. This is the most general third-order differential operator with regular singularities at \( x = 0, 1, \infty \) and with analytic solutions near \( x = 0, 1 \).

There exists a unique solution \( f_\lambda(x) \) of the differential equation

\[ D_x f_\lambda(x) = 0 \]

such that \( f_\lambda(x) \) is analytic near \( x = 0 \) and \( f_\lambda(0) = 1 \). We have

\[ f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda)x^i \]

where \( P_i(\lambda) \) are polynomials in \( \lambda \) of degree \( i \) and \( P_0(\lambda) = 1 \).

Since \( P_i(\lambda), i = 0, 1, \ldots \) is a basis of the vector space \( \mathbb{C}[\lambda] \), we can define structure constants of polynomial multiplication in this basis by (5.1).

Define kernel \( K(x_1, x_2, y) \) as the generating function of these structure constants by (5.3)

Introduce new parameters \( b_1, b_2, c_1, c_2, c_3 \) by

\[
\begin{align*}
  b_1 + b_2 &= -a_1 - 1, \quad b_1b_2 = a_3, \quad c_1 + c_2 + c_3 = a_2 - 3, \\
  c_1c_2 + c_1c_3 + c_2c_3 &= a_5 - a_2 + 2, \quad c_1c_2c_3 = a_6.
\end{align*}
\]

Lemma 5.3 The kernel \( K(x_1, x_2, y) \) satisfies the following differential equations

\[ D_{x_1}K = D_{x_2}K, \]

\[ D_y^*K - D_{x_1}K = (c_1 - 1)(c_2 - 1)(c_3 - 1) \sum_{i=0}^{\infty} \prod_{l=0}^{i-1} \frac{(c_1 + l)(c_2 + l)(c_3 + l)}{(b_1 + l)(b_2 + l)} \cdot \frac{(x_1x_2)^i}{i!} \]

where \( D_y^* \) is the conjugate differential operator given by

\[
D_y^* = -\frac{d^3}{dy^3} \cdot y^2(y - 1)^2 + \frac{d^2}{dy^2} \cdot y(y - 1)(a_1 + a_2y) - \frac{d}{dy} \cdot (a_3 + a_4y + a_5y^2)
+ a_6y + \lambda
\]
Theorem 5.7 The kernel \( K(x_1, x_2, y) \) is given by:

\[
K(x_1, x_2, y) = \frac{1}{y - 1} \sum_{0 \leq j < k \leq i + j, \ 0 \leq i} \frac{(-1)^j k!}{j!(k-j)!(i+j-k)!} \cdot \frac{(x_1x_2)^j ((x_1 - 1)(x_2 - 1))^j}{(y - 1)^k}
\]

\[
\times \prod_{l=j+1}^{k} (b_1b_2 + c_1c_2 + c_1c_3 + c_2c_3 + a_4 + l(c_1 + c_2 + c_3))
\]

\[
\times + (1 - l)(b_1 + b_2) + l^2 - l + 1
\]

\[
\times \prod_{l=k}^{i+j-1} (c_1 + l)(c_2 + l)(c_3 + l)
\]

\[
\prod_{l=0}^{l-1} (b_1 + l)(b_2 + l).
\]

Remark 5.1 Let \( D_x \) be third-order differential operator with symbol \( (x - t_1)^2(x - t_2)^2 \frac{\partial^3}{\partial y^3} \), with regular singular points at \( x = t_1, t_2, t_3 \), and solutions near these points of the form \( f(x) = (x - t_i)^{b_{i,j}}(1 + O(x - t_i)) \) for \( i, j = 1, 2, 3 \). Here \( t_i, b_{i,j} \) are generic parameters such that \( \sum_{1 \leq i, j \leq 3} b_{i,j} = 3 \). Notice that any two differential operators with these properties differ by a constant.

Define a function \( K_3(x, y, z) \) as

\[
K_3 = F\left(q_1 \frac{(x - t_1)(y - t_1)(z - t_1)}{(x - t_2)(y - t_2)(z - t_2)}, q_2 \frac{(x - t_2)(y - t_2)(z - t_2)}{(x - t_3)(y - t_3)(z - t_3)}\right)
\]

where \( q_1 = \frac{(t_2-t_3)^3}{(t_1-t_2)^3} \), \( q_2 = \frac{(t_1-t_3)^3}{(t_2-t_1)^3} \) and \( F \) satisfies the system of differential equations

\[
\frac{\partial^3 G}{\partial u_{11} \partial u_{12} \partial u_{13}} = \frac{\partial^3 G}{\partial u_{21} \partial u_{22} \partial u_{23}} = \frac{\partial^3 G}{\partial u_{31} \partial u_{32} \partial u_{33}}.
\]

Here

\[
G = F\left(u_{11}u_{12}u_{13}, u_{21}u_{22}u_{23}, u_{31}u_{32}u_{33}\right) \prod_{1 \leq i, j \leq 3} u_{i,j}^{-b_{i,j}}.
\]

Then, \( K_3(x, y, z) \) satisfies the differential equations

\[
\frac{\partial K_3}{\partial x} = \frac{\partial K_3}{\partial y} = \frac{\partial K_3}{\partial z}.
\]

It will be interesting to understand if \( K_3 \) is somehow connected with the kernel \( K \) constructed in Theorem 5.7. It looks feasible that \( K_3 \) also satisfies to some kind of associativity condition. The kernel \( K_3 \) is similar to kernels constructed in Remarks 3.8 for \( l = n, \) and 3.6.
Acknowledgements  We thank V. Golyshev, V. Rubtsov, D. van Straten for the discussion at early stages of this project. We are also grateful to V. Drinfeld, P. Etingof, D. Kazhdan and other participants of Geometric Langlands Seminar for their remarks and comments. A.O. is grateful to IHES for invitations and excellent working atmosphere.

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