Projective Connections and the Algebra of Densities

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Abstract. Projective connections first appeared in Cartan’s papers in the 1920’s. Since then they have resurfaced periodically in, for example, integrable systems and perhaps most recently in the context of so called projectively equivariant quantisation. We recall the notion of projective connection and describe its relation with the algebra of densities on a manifold. In particular, we construct a Laplace-type operator on functions using a Thomas projective connection and a symmetric contravariant tensor of rank 2 (‘upper metric’).

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INTRODUCTION

This paper is concerned with the geometry of differential operators on a manifold and their relation with projective connections. The notion of projective connection is an old one, first appearing in Cartan’s papers of the 1920s and then in various modified forms throughout the 20th century. They have surfaced periodically in mathematical physics, in particular in integrable systems and more recently in projectively equivariant quantisation. This paper establishes that the algebra of densities introduced in [1] and projective connections on a given manifold are fundamentally linked. We recall the notion of a projective connection; in fact there are two distinct but related notions which will be detailed here. We define the manifold \( \hat{M} \) (cf. \( \hat{M} \) in [1]) for which the algebra of densities \( \mathcal{V}(M) \) may be interpreted as a subalgebra of the algebra of smooth functions \( \mathcal{C}^\infty(\hat{M}) \). Having defined Thomas’ manifold \( \tilde{M} \) (see [2]), we explicitly construct a diffeomorphism \( F : \hat{M} \to \tilde{M} \). By means of this, we show that a projective connection on \( M \) gives rise canonically to a linear connection on \( \hat{M} \). Then we consider a manifold equipped with a Thomas projective connection and a symmetric contravariant tensor field of rank 2 (which may be viewed as an ‘upper metric’, though it need not be non-degenerate). From these data, we construct an upper connection (contravariant derivative) on the bundle of volume forms and an invariant Laplace-type operator acting on functions, a ‘projective Laplacian’. This leads to some results regarding differential operators and brackets on the algebra \( \mathcal{V}(M) \) when \( M \) is equipped with a projective connection.
Here we present a natural definition of projective connection and give the correspondence to the Thomas-Weyl-Veblen definition of projective equivalence classes. Most of this account may be extracted, with a little work, from [3]. Recall that a map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be \textit{projective} or \textit{fractional linear} if it is of the form

$$\phi(v) = \frac{\alpha(v) + \beta}{\gamma(v) + \delta},$$

where $\alpha \in GL_n(\mathbb{R}), \beta \in \mathbb{R}^n, \gamma \in (\mathbb{R}^n)^*, \delta \in (\mathbb{R}), \det \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$.

With this in mind, we make the following natural definition.

\textit{Definition.} A \textit{projective connection} on a vector bundle $E \to M$ is an Ehresmann connection such that the associated parallel transport induces a projective map between fibres.

This intuitive definition is a reformulation, in modern language, of Cartan’s idea as proposed in his seminal paper [4]. In the special case of the tangent bundle $TM \to M$, projective connections are closely related to the following notion.

\textit{Definition.} Two torsion free linear connections on $TM \to M$ are said to be \textit{projectively equivalent} if any of the following hold.

1. $\nabla \sim \bar{\nabla}$ if they define the same geodesics up to reparametrisation.
2. $\nabla \approx \bar{\nabla}$ if there is a 1-form $\vartheta$ s.t. $\bar{\nabla}X Y - \nabla X Y = \vartheta(X) Y + \vartheta(Y) X$ for all vector fields $X, Y$.
3. $\nabla \simeq \bar{\nabla}$ if their coefficients $\Gamma^k_{ij}$ and $\bar{\Gamma}^k_{ij}$ satisfy

$$\Pi^k_{ij} := \Gamma^k_{ij} - \frac{1}{n+1} (\delta^k_i \Gamma^l_{j,l} + \delta^k_j \Gamma^l_{i,l}) = \bar{\Gamma}^k_{ij} - \frac{1}{n+1} (\delta^k_i \bar{\Gamma}^l_{j,l} + \delta^k_j \bar{\Gamma}^l_{i,l}) := \bar{\Pi}^k_{ij}$$

This would not be a well defined notion had these conditions not been equivalent. Details of the proof of (1)$\iff$(2) can be found in [3]; a brute force calculation yields (2)$\iff$(3). Before relating these two notions, we must rephrase the former analytically.

Recall that an Ehresmann connection in a fibre bundle $\pi : E \to M$ is a distribution $\mathcal{H}$ of horizontal linear subspaces of $TE$ such that $TE = \mathcal{H} \oplus \ker \pi$. By \textit{horizontal curves} we will mean integral curves of $\mathcal{H}$. We express $\mathcal{H}$ as the kernel of a collection of 1-forms on $E$. In terms of local coordinates $x^i$ on the base and $\xi^a$ on the fibres, these forms may be written (after a careful choice of coordinates) in the form $\Psi^a = d\xi^a + f^a_i dx^i$ for some functions $f^a_i$ on $M$. These functions together constitute a local connection 1-form on $M$. The condition that a curve $\sigma$ be horizontal is then simply that $\Psi^a(\sigma') = 0$ for each $a$, or explicitly in terms of $f^a_i$,

$$\frac{d}{dt} \xi^a(\sigma(t)) = f^a_i \frac{d}{dt} x^i(\sigma(t)). \quad (1)$$

In this context, we require that the parallel translation defines a projective map between fibres. This is equivalent to requiring that locally, the horizontal curves are projective flows on the typical fibre, that is, of the form

$$\sigma(t) = \frac{\alpha(\sigma) + \beta}{\gamma(\sigma) + \delta}$$
with the aforementioned non-degeneracy condition on the coefficients. Differentiating this, we have the equation in fibre coordinates

\[
d\frac{d}{dt}\xi^a(\sigma(t)) = A^a(t) + B^a_b(t)\xi^b(\sigma(t)) + C^a_b(\sigma(t))\xi^b(\sigma(t))
\]

for some coefficient functions \(A^a, B^a_b, C^a_b\). Comparing this with (1) we come to a description of projective connection in a vector bundle \(E \rightarrow M\) as an Ehresmann connection defined by the annihilating 1-forms

\[
\Psi^a = d\xi^a - (\phi^a_i(x) + \psi^a_{bi}(x)\xi^b + \eta_{bi}(x)\xi^a\xi^b)dx^i
\]

for some functions \(\phi^a_i, \psi^a_{bi}\) and \(\eta_{bi}\).

**Remark.** With respect to (linear) transition functions in the vector bundle \(E\), the forms \(\psi^a_{bi}dx^i\) define a connection 1-form for a linear connection in \(E\), while \(\psi^a_{bi}dx^i\) and \(\eta_{bi}dx^i\) define 1-forms taking values in sections of \(E\) and the dual bundle \(E^*\) respectively. This will be important in the following section.

In the case of the tangent bundle, there is a distinguished choice of \(\phi^i_j\), namely \(\delta^i_j\). This gives \(\Psi^i = d\xi^i - dx^i - \omega^i_j\xi^j - \omega^i_0\xi^i\xi^j\) where now \(\xi^i\) are the fibre coordinates naturally related to the local coordinates \(x^i\). Projective connections also have a related notion of geodesics. These are given in this case by the equations

\[
\frac{d^2\sigma^i}{dt^2} = \frac{d\sigma^a}{dt} + \Gamma^i_{jk} \frac{d\sigma^j}{dt} \frac{d\sigma^k}{dt} + \gamma^i_{jk} \frac{d\sigma^j}{dt} \frac{d\sigma^k}{dt}
\]

where \(\omega^i_k = \Gamma^i_{jk}dx^j\) and \(\omega^i_0 = \gamma^i_{jk}dx^j\). Having established this notion, we may mimic our previous definition and analogously define projective equivalence classes of projective connections.

For arbitrary Ehresmann connections, there is a natural notion of curvature which in this case amounts to a collection of 2-forms:

\[
\Omega^i_j = d\omega^i_j - \omega^i_k \wedge \omega^j_k - \omega^i_j \wedge dx^i - \delta^i_j \omega^0_k \wedge dx^k \text{ and } \Omega^0_i = \omega^0_j \wedge \omega^i_j.
\]

**Definition.** Define the numbers \(A^i_{jkl}\) and \(A_{jk}\) by \(\Omega^i_j = A^i_{jkl}dx^k \wedge dx^l\) and \(A_{jk} := A^i_{jki}\). A projective connection is said to be normal if \(A_{jk} = 0\).

We also consider the Ricci tensor \(R_{jk}\), where \(d\omega^i_j - \omega^i_p \wedge \omega^0_j\) equals \(R^i_{jkld}dx^k \wedge dx^l\) and \(R_{jk} := R^i_{jki}\).

The notion of normality was introduced by Cartan and derives its utility from the following observations. Let \((\omega^0_j, \omega^0_j)\) define a projective connection in \(TM\).

- Given that the Ricci tensor \(R_{ij}\) is symmetric, the projective connection is normal if and only if \(\omega^0_j = -\frac{2}{n-1}R_{jk}dx^k\).
- Given any linear connection \(\omega^0_j\) in the tangent bundle, there is another with symmetric Ricci tensor with the same geodesics up to reparametrisation.
• If \((\omega^i_j, \omega^0_j), (\bar{\omega}^i_j, \bar{\omega}^0_j)\) are two normal projective connections in the tangent bundle whose associated linear connections \(\omega^i_j\) and \(\bar{\omega}^i_j\) are projectively equivalent, then \((\omega^i_j, \omega^0_j)\) and \((\bar{\omega}^i_j, \bar{\omega}^0_j)\) are projectively equivalent.

Proofs of these facts may be be found in [3]. Given these, we have a one-to-one correspondence between projective equivalence classes of linear connections and projective equivalence classes of projective connections on any given manifold \(M\). All of the notions discussed throughout this paper also make sense on supermanifolds. Bearing these relations in mind, we will adopt from now on the (classical) terminology and call a projective equivalence class of linear connections on a manifold a Thomas projective connection.

### RELATIONS WITH DENSITIES

**Definition.** A density of weight \(\lambda\) is a formal expression of the form \(\phi = \phi(x)(Dx)^{\lambda}\) defined in local coordinates, \(Dx\) being the associated local volume form and \(\lambda \in \mathbb{R}\). There is a natural notion of multiplication for densities given by \(\phi \cdot \chi = (\phi(x)(Dx)^{\lambda_1}) \cdot (\chi(x)(Dx)^{\lambda_2}) = \phi(x)\chi(x)(Dx)^{\lambda_1 + \lambda_2}\). The algebra of densities denoted \(\mathcal{U}(M)\), is the algebra of finite formal sums \(\sum_{\lambda} \phi_{\lambda}(x)(Dx)^{\lambda}\) of densities with the multiplication defined previously (see [1]).

A \(\lambda\)-density may equally be thought of as a ‘function’ \(\phi(x)\) which under a change of coordinates, picks up the modulus of the Jacobian of the transformation to the \(\lambda\)-th power as a factor. Having defined densities, it is natural to define a linear operator on \(\mathcal{U}(M)\), namely the weight operator \(w\). This is defined as having each \(\lambda\)-density as a \(\lambda\)-eigenvector, that is \(w(\phi(x)(Dx)^{\lambda}) = \lambda \phi(x)(Dx)^{\lambda}\).

Let \(\dim M = n\). We now define two \((n+1)\)-dimensional manifolds \(\tilde{M}\) and \(\tilde{\tilde{M}}\) which are fibre bundles over \(M\) and significant with respect to the algebra of densities and projective connections respectively.

\(\tilde{M} : \text{Densities as functions.}\) A density \(\sum \phi_{\lambda}(Dx)^{\lambda}\) may be interpreted as a function on the manifold \(\tilde{M}\) defined in some sense as the ‘strictly positive half’ of the determinant bundle (see below). Denoting the fibre coordinate by \(t > 0\), the algebra of densities is the subalgebra of \(C^\infty(\tilde{M})\) consisting of functions of the form \(\sum \phi_{\lambda} t^{\lambda}\).

\(\tilde{\tilde{M}} : \text{Thomas projective connections as linear connections.}\) In [5], T. Y. Thomas details the construction of a manifold \(\tilde{\tilde{M}}\) from a given manifold \(M\) such that any projective equivalence class \(\Pi^{k}_{ij}\) on \(M\) gives rise canonically to linear connection coefficients \(\tilde{\Gamma}^{k}_{ij}\) on \(\tilde{\tilde{M}}\). From the viewpoint of the previous section, each projective equivalence class of projective connections gives rise to a linear connection on \(\tilde{\tilde{M}}\).
If $x^0, \ldots, x^n$ are local coordinates in some neighbourhood of $\tilde{M}$, these are given by

\[
\begin{align*}
\tilde{\Gamma}^k_{ij} &= \tilde{\Gamma}^k_{ji} = \Pi^k_{ij} & \text{for } i, j, k = 1, \ldots, n \quad (2) \\
\tilde{\Gamma}^k_{0i} &= -\frac{\delta_i^k}{n + 1} & \text{for } i, k = 0, \ldots, n \quad (3) \\
\tilde{\Gamma}^0_{ij} &= \tilde{\Gamma}^0_{ji} = \frac{n + 1}{n - 1} \left( \frac{\partial \Pi^r_{ij}}{\partial x^r} - \Pi^r_{ir} \Pi^r_{sj} \right) & \text{for } i, j = 1, \ldots, n. \quad (4)
\end{align*}
\]

The manifolds $\tilde{M}$ and $\hat{M}$ are defined locally in terms of the charts on $M$. Let us for a moment denote local coordinates on $\tilde{M}$ by $t, \hat{x}^1, \ldots, \hat{x}^n$ and on $\hat{M}$ by $\tilde{x}^0, \ldots, \tilde{x}^n$. Then if $x^i = f^i(x^1, \ldots, x^n)$ is a coordinate transformation on $M$, define

\[
\begin{array}{c|c}
\text{On } \tilde{M} & \text{On } \hat{M} \\
\hline
\tilde{x}^0 = \tilde{x}^0 + \log J_f; & t' = t J_f; \\
\tilde{x}^i = f^i(\hat{x}^1, \ldots, \hat{x}^n), & \tilde{x}^i = f^i(\hat{x}^1, \ldots, \hat{x}^n) \quad \text{for } i = 1, \ldots, n,
\end{array}
\]

where $J_f$ denotes the modulus of the Jacobian of $f$ considered as a function of coordinates. We are immediately led to the following observation.

**Theorem 1.** 1. The bundles $\tilde{\pi} : \tilde{M} \to M$ and $\hat{\pi} : \hat{M} \to M$ are isomorphic, that is, there is a diffeomorphism $F : \tilde{M} \to M$ such that $\tilde{\pi} \circ F = \hat{\pi}$.

2. The weight operator $w$ is mapped to $\frac{\partial}{\partial (\tilde{x}^0)}$ under $F$.

3. A projective equivalence class $\Pi^k_{ij}$ on $M$ canonically induces a linear connection on $\hat{M}$, in particular allowing us to consider covariant derivatives of densities along vector fields on $M$.

**Proof.** 1. In coordinates, simply define

\[
\tilde{x}^i(F(x)) = F^i(t, \hat{x}^1, \ldots, \hat{x}^n) = \begin{cases} 
\log t & \text{for } i = 0, \text{ and } t > 0; \\
\hat{x}^i & \text{for } i = 1, \ldots, n.
\end{cases}
\]

2. Written in terms of generating functions $w$ takes the form of a logarithmic derivative $\frac{\partial}{\partial t'}$, which is mapped to $\frac{\partial}{\partial (\tilde{x}^0)}$.

3. A projective equivalence class $\Pi^k_{ij}$ canonically defines, via Thomas’ construction, a linear connection $\tilde{\nabla}$ on $\tilde{M}$ whose coefficients are given by (2)-(4). Now define a linear connection on $\hat{M}$ by pulling $\tilde{\nabla}$ back along $F$.

**Operators, Brackets and Future Developments**

Let $M$ be a manifold endowed with a tensor field $S^{ij}$. This defines a map $S^i : T^*M \to TM$ given in local coordinates by $\omega_i dx^j \mapsto \omega_j S_{ij} \partial_i$.

**Definition.** Let $E \to M$ be a vector bundle over a manifold $M$ endowed with a tensor field $S^{ij}$. An upper connection on $E$ over $S^{ij}$ is an $\mathbb{R}$-bilinear map $\nabla : \Omega^1(M) \times \Gamma(E) \to \Gamma(E)$
satisfying
\[ \nabla^{f^0} \sigma = f \nabla^0 \sigma \text{ and } \nabla^0 f \sigma = f \nabla^0 \sigma + (S^i \omega) \sigma \text{ for } f \in C^\infty(M). \]

If \( S^{ij} \) is an invertible matrix, upper and ordinary connections correspond by raising and lowering indices.

In [1], symmetric biderivations on the algebra of densities (or more briefly brackets) were considered. In a system of local coordinates \( x^1, \ldots, x^n \), a homogeneous bracket \( \{ \cdot, \cdot \} \) is uniquely defined by a triple of quantities \( S^{ij}, \gamma^i, \theta \) via
\[
\{ x^i, x^j \} = S^{ij} (Dx)^\lambda, \quad \{ x^i, Dx \} = \gamma^i (Dx)^{\lambda+1}, \quad \{ Dx, Dx \} = \theta (Dx)^{\lambda+2};
\]
where \( \lambda \in \mathbb{R} \) is the weight of the bracket. Here \( S^{ij} = S^{ji} \) is symmetric. For a bracket of weight zero, \( S^{ij} \) is a tensor and \( \gamma^i \) defines an upper connection over \( S^{ij} \) in the bundle of volume forms on \( M \).

**Theorem 2.** Let \( M \) be a manifold endowed with a projective equivalence class (Thomas projective connection \( \Pi^i_{jk} \)) and a tensor field \( S^{ij} \). Then:

1. These data define an upper connection over \( S^{ij} \) in the bundle of volume forms given by the coefficients
\[
\gamma^i = \frac{n+1}{n+3} (\partial_j S^{ij} + S^{jk} \Pi^i_{jk}). \tag{5}
\]
2. The following expression defines an invariant second order differential operator acting on functions on \( M \)
\[
\Delta = S^{ij} \partial_i \partial_j + \left( \frac{2}{n+3} \partial_j S^{ij} - \left( \frac{n+1}{n+3} \right) S^{jk} \Pi^i_{jk} \right) \partial_i. \tag{6}
\]

**Remark.** In general, a linear connection \( \Gamma^k_{ij} \) gives rise to a connection on the bundle of volume forms by taking the trace \( \gamma := \Gamma^k_{ki} \). Suppose that \( S^{ij} \) is non-degenerate, i.e., \( S_{ij} = g_{ij} \) defines a metric. An upper connection may be obtained in two ways:

- Take the coefficients \( \Gamma^k_{ij} \) of the Levi-Civita connection associated with \( g_{ij} \). Define \( \gamma^i := g^{ij} \Gamma^k_{kj} \).
- Take the class \( \Pi^i_{jk} \) associated with the Levi-Civita connection coefficients \( \Gamma^k_{ij} \) and use (5).

These two constructions give exactly the same coefficients \( \gamma^i \).

Given a Thomas projective connection on \( M \), the principal symbol of a second order differential operator (of arbitrary weight) on the algebra of densities \( \mathfrak{V}(M) \) may similarly be extended to an invariant operator on \( \mathfrak{V}(M) \). From a projective equivalence class \( \Pi^i_{jk} \) on \( M \), Theorem 1 gives a linear connection \( \hat{\Gamma}^k_{ij} \) on \( \hat{M} \). Applying formula (6) to the associated Thomas projective connection on \( \hat{M} \) gives the result.

In the presence of a Thomas projective connection on \( M \), this is a method of constructing, from a bracket on \( \mathfrak{V}(M) \), a canonical differential operator on \( \mathfrak{V}(M) \) generating the
bracket. A similar construction using a natural inner product was one of the main results of [1]. Comparing operators from (6) and [1] gives expressions for $\gamma$ and $\theta$ in terms of $S^{ij}$ and $\Pi_{ij}$. Given a manifold, we have therefore a map from the space of symbols $S^{ij}$ to the space of second order differential operators on the algebra of densities, or using Ovsienko’s terminology (see [6]), a projectively equivariant quantisation (in the non-flat case). Indeed it would no doubt prove fruitful to investigate this relation further.

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