Reaching Nonlinear Consensus via Non-Autonomous Polynomial Stochastic Operators

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Abstract. This paper is a continuation of our previous studies on nonlinear consensus which unifies and generalizes all previous results. We consider a nonlinear protocol for a structured time-varying synchronous multi-agent system. We present an opinion sharing dynamics of the multi-agent system as a trajectory of non-autonomous polynomial stochastic operators associated with multidimensional stochastic hyper-matrices. We show that the multi-agent system eventually reaches to a nonlinear consensus if either one of the following two conditions is satisfied: (i) every member of the group people has a positive subjective distribution on the given task after some revision steps or (ii) all entries of some multidimensional stochastic hyper-matrix are positive.

1. Introduction

This paper is a continuation of our previous studies on nonlinear consensus which unifies and generalizes all previous results. We consider a nonlinear protocol for a structured time-varying synchronous multi-agent system. We present an opinion sharing dynamics of the multi-agent system as a trajectory of non-autonomous polynomial stochastic operators associated with multidimensional stochastic hyper-matrices. In the paper [12] the nonlinear consensus problem was studied for a single cubic triple stochastic hyper-matrix meanwhile in the paper [14] the nonlinear consensus problem was studied for a sequence of cubic triple stochastic hyper-matrices. The nonlinear consensus problem for a single \( k \)-dimensional \( k \)-tuple stochastic hyper-matrix was studied in [13]. In this paper, we are aiming to study the nonlinear consensus problem for a sequence of \( k \)-dimensional \( k \)-tuple stochastic hypermatrices which unifies and generalizes all previous results.

The novelty of the paper is that a new nonlinear protocol for a structured time-varying and synchronous environment was presented as a trajectory of higher dimensional stochastic hyper-matrices. One of the classical results in the theory of Markov chains states that if every element of the square stochastic matrix is positive then its trajectory converges to its unique fixed distribution. To the best of our knowledge, the similar problem for the higher dimensional hyper-matrix was open. In this sense, our paper is the pioneering study for the higher dimensional hyper-matrices. \textit{One of our main results states that if every element of the \( k \)-dimensional \( k \)-tuple stochastic hyper-matrix is positive then its trajectory converges to its unique fixed distribution.} Since the Markov chains and the consensus problems are dual to each other, we present the application of our results into the nonlinear consensus problem.
For the sake of the completeness, we first review the linear consensus problem for an estimate-modification process of a structured time-invariant and synchronous environment which was presented in [1, 3].

An idea of reaching consensus through repeated averaging was first introduced by DeGroot [3] for a structured time-invariant and synchronous environment. Since that time, the consensus which is the most ubiquitous phenomenon of multi-agent systems became popular in various scientific communities, such as biology, physics, control engineering and social science [10, 17]. To our best knowledge, the most research papers are concerned with the consensus problem under linear protocols. However, the many systems, such as the well-known Kuramoto oscillator, exhibit nonlinear locally passive dynamics. The consensus problems for some nonlinear protocols were studied in [7, 18, 19].

Let us consider a group of \( m \) individuals \( I = \{1, 2, \cdots, m\} \), each of whom can specify his or her own subjective probability distribution for some given task. For \( i = 1, \cdots, m \), let \( x_i^{(0)} \) denote the subjective distribution that the individual \( i \) is assigned to a given task. The subjective distributions, \( x_i^{(0)} = (x_{i1}^{(0)}, \cdots, x_{im}^{(0)})^T \), will be based on the different backgrounds and different levels of expertise of the members of the group. It is assumed that if the individual \( i \) is informed of the distributions of each of the other members of the group, he/she might wish to revise his/her subjective distribution to accommodate this information. In the DeGroot’s model [3], it was assumed that when the individual \( i \) makes this revision, his/her revised distribution is a linear combination of the distributions \( x_1^{(0)}, \cdots, x_m^{(0)} \). Let \( p_{ij} \) denote the weight that the individual \( i \) assigns to \( x_j^{(0)} \) when he/she makes this revision. It was assumed that the \( p_{ij} \geq 0 \) and \( \sum_{j=1}^{m} p_{ij} = 1 \). So, after being informed of the subjective distributions of the other members of the group, the individual \( i \) revises his/her own subjective distribution from \( x_i^{(0)} \) to \( x_i^{(1)} = \sum_{j=1}^{m} p_{ij} x_j^{(0)} \).

Let \( P \) denote an \( m \times m \) matrix whose \((i, j)\)th element is \( p_{ij} \). It is clear that \( P \) is a row-stochastic matrix since the elements are all non-negative and the row-sums are equal to one. Let \( x_i^{(0)} = (x_{i1}^{(0)}, \cdots, x_{im}^{(0)})^T \) and \( x_i^{(1)} = (x_{i1}^{(1)}, \cdots, x_{im}^{(1)})^T \) be vectors. Then the vector of revised subjective distributions can be written as \( x_i^{(1)} = P x_i^{(0)} \). The critical step in this process is that the above revision is iterated. It is assumed that after the individual \( i \) is informed of the first-revised subjective distributions \( x_i^{(1)} = (x_{i1}^{(1)}, \cdots, x_{im}^{(1)})^T \) of the members of the group, he/she revises his/her subjective distribution from \( x_i^{(1)} \) to \( x_i^{(2)} = \sum_{j=1}^{m} p_{ij} x_j^{(1)} \). The process continues in this way. Let \( x_i^{(n)} = (x_{i1}^{(n)}, \cdots, x_{im}^{(n)})^T \) denote the subjective distribution of the members of the group after \( n \) revisions. Then \( x_i^{(n)} = P x_i^{(n-1)} = P^n x_i^{(0)} \). DeGroot states [3] that a consensus is reached if and only if all \( m \) components of \( x_i^{(n)} \) converge to the same limit as \( n \to \infty \).

In this paper, our main assumption is that the subjective distribution \( x_i^{(n)} = (x_{i1}^{(n)}, \cdots, x_{im}^{(n)})^T \) of the members is probabilistic in every step, i.e., \( \sum_{k=1}^{m} x_k^{(n)} = 1 \) and \( x_k^{(n)} \geq 0 \) for any \( k = 1, m \) and \( n \in \mathbb{N} \). Let \( S^{m-1} \) be an \((m - 1)\)-dimensional simplex, where

\[
S^{m-1} = \left\{ x \in \mathbb{R}^m : \sum_{k=1}^{m} x_k = 1, \; x_k \geq 0, \; \forall \; k = 1, m \right\} .
\]

In this case, one has that \( x^{(n)} \in S^{m-1} \) for any \( n \in \mathbb{N} \) in DeGroot’s model if and only if
\[ \sum_{i=1}^{m} p_{ij} = \sum_{j=1}^{m} p_{ij} = 1 \text{ and } p_{ij} \geq 0, \text{ i.e., } \mathbb{P} \text{ is a doubly stochastic matrix. Consequently, we may conclude that a trajectory } \{x^{(n)}\}_{n=0}^{\infty} \text{ of the double stochastic matrix } \mathbb{P} \text{ presents the DeGroot model of a structured time-invariant synchronous environment with the probabilistic subjective distribution for some given task.} \]

In [2], Chatterjee and Seneta consider a generalization of DeGroot’s model in which the individuals can change their weights \( p_{ij} \) at each iteration. More precisely, let \( \{\mathbb{P}_n\}_{n \in \mathbb{N}} \) be a sequence of double stochastic matrices (a non-homogeneous Markov chain) and \( x^{(0)} \in S^{m-1} \). A sequence \( x^{(n+1)} = \mathbb{P}_{n+1} x^{(n)} \) presents the Chatterjee-Seneta model of a structured time-varying and synchronous environment with the probabilistic subjective distribution for some given task.

In this paper, we shall consider a nonlinear model for the estimate modification process of the structured time-varying synchronous environment which generalizes both the DeGroot and Chatterjee-Seneta models.

In general, we suppose that doubly stochastic matrices (in the Chatterjee-Seneta model) depend on subjective distributions \( x^{(n)} \) and \( n \) in every step, i.e., entries of doubly stochastic matrices are not constants but functions of \( x^{(n)} \) and \( n \),

\[ \mathbb{P}_{x(n)} := \left( p_{ij} \left( n, x^{(n)} \right) \right)_{i,j=1}^{m}. \tag{1.1} \]

A general model of the structured time-varying synchronous environment with probabilistic subjective distributions is defined as follows

\[ x^{(n+1)} = \mathbb{P}_{x(n)} x^{(n)} \tag{1.2} \]

where, \( \mathbb{P}_{x(n)} \) is a doubly stochastic matrix defined by (1.1).

By choosing of \( \mathbb{P}_{x(n)} \), we may get different models of multi-agent systems. For instance, if \( \mathbb{P}_{x(n)} = \mathbb{P}_0 \) (the matrices are free of \( n \) and \( x^{(n)} \) then we get the DeGroot model. If \( \mathbb{P}_{x(n)} = \mathbb{P}_n \) (the matrices are free of \( x^{(n)} \) but depended on \( n \)) then we get the Chatterjee-Seneta model.

We shall study a consensus problem in the multi-agent system.

**Definition 1.1.** We say that a consensus is reached in a structured time-varying synchronous multi-agent system given by (1.2) if \( x^{(n)} \) converges to the center \( C = \left( \frac{1}{m}, \ldots, \frac{1}{m} \right)^T \) of the simplex \( S^{m-1} \) as \( n \to \infty \).

The following notations are used in this paper. Let \( I = \{1, \ldots, n\} \) be an index set, \( \mathbb{R} \) be a set of real numbers, \( \mathbb{R}^m \) be an \( m \)-dimensional Euclidean space with the standard inner product \( (x, y) = \sum_{i=1}^{m} x_i y_i \). Every element of \( \mathbb{R}^m \) is a column vector. Let \( x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m \) and \( e_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})^T, \ i = \overline{1, m} \) be the standard basis of \( \mathbb{R}^n \) where \( \delta_{ij} \) is Kronecker’s delta symbol. Let \( S^{m-1} = \left\{ x \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1, \ x_i \geq 0, \ \forall i = \overline{1, m} \right\} \) be an \((m-1)\)-dimensional simplex and \( \text{int} S^{m-1} = \{x \in S^{m-1} : x_i > 0 \ \forall i = \overline{1, m} \} \) be its interior. Let \( C = \left( \frac{1}{m}, \ldots, \frac{1}{m} \right)^T \) be the center of the simplex \( S^{m-1} \). Let \( M(x) = \max_{i \in I} x_i, \ m(x) = \min_{i \in I} x_i, \) and \( d(x) = M(x) - m(x) \) be functions. Let \( \mathbb{P} = (p_{ij})_{i,j=1}^{m} \) be a square matrix and \( \mathcal{P} = (P_{i_1 \cdots i_k})_{i_1 \cdots i_k=1}^{m \cdots m} \) be a \( k \)-dimensional hyper-matrix where \( k \geq 3 \) is any natural number.

2. The Main Model

In this section, we shall provide some nonlinear protocols of multi-agent systems. We need some preliminary notions and notations.
**Definition 2.1.** A $k$–dimensional hyper-matrix $\mathcal{P} = (P_{i_1 \cdots i_k})_{i_1, \cdots, i_k=1}^{m, \cdots, m}$ is called a $k$–tuple stochastic if all its entries are non-negative and it is stochastic in all directions, i.e.,

$$\sum_{i_1=1}^{m} P_{i_1 \cdots i_k} = \cdots = \sum_{i_k=1}^{m} P_{i_1 \cdots i_k} = 1, \quad P_{i_1 \cdots i_k} \geq 0, \quad \forall \, i_1, \cdots, i_k = 1, m.$$

Let $\mathcal{P} = (P_{i_1 \cdots i_k})_{i_1, \cdots, i_k=1}^{m, \cdots, m}$ be a $k$–dimensional $k$–tuple stochastic hyper-matrix and $\mathcal{P}_l = (P_{i_1 \cdots i_k \_l})_{i_1, \cdots, i_k=1}^{m, \cdots, m}$ be its $(k - 1)$– dimensional $l$–th subhyper-matrix for fixed $l = 1, m$. It is clear that $\mathcal{P}_l = (P_{i_1 \cdots i_k \_l})_{i_1, \cdots, i_k=1}^{m, \cdots, m}$ is also the $(k-1)$–tuple stochastic hyper-matrix for every $l = 1, m$. In the sequel, we write the $k$–dimensional $k$–tuple stochastic hyper-matrix $\mathcal{P}$ as follows $\mathcal{P} = (\mathcal{P}_1 | \mathcal{P}_2 | \cdots | \mathcal{P}_m)$. We define a polynomial operator $V : \mathbb{R}^m \rightarrow \mathbb{R}^m$ associated with the $k$–dimensional $k$–tuple stochastic hyper-matrix $\mathcal{P} = (\mathcal{P}_1 | \mathcal{P}_2 | \cdots | \mathcal{P}_m)$ as follows:

$$(V(x))_l = \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} P_{i_1 \cdots i_k \_l} x_{i_1} \cdots x_{i_k}, \quad (2.1)$$

for all $l = 1, m$. Since $\sum_{l=1}^{m} P_{i_1 \cdots i_k \_l} = 1$ and $P_{i_1 \cdots i_k \_l} \geq 0$, we have that $V(S^{m-1}) \subset S^{m-1}$. In this sense, the polynomial operator $V : S^{m-1} \rightarrow S^{m-1}$ defined by (2.1) is called stochastic.

Let $\mathcal{P}_l = (P_{i_1 \cdots i_k \_l})_{i_1, \cdots, i_k=1}^{m, \cdots, m}$ be the $l$–th subhyper-matrix of the $k$– dimensional $k$–tuple stochastic hyper-matrix $\mathcal{P} = (P_{i_1 \cdots i_k})_{i_1, \cdots, i_k=1}^{m, \cdots, m}$ for fixed $l = 1, m$. We also define a polynomial stochastic operator $V_l : S^{m-1} \rightarrow S^{m-1}$ associated with the hyper-matrix $\mathcal{P}_l$ as follows:

$$(V_l(x))_j = \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} P_{i_1 \cdots i_k \_j} x_{i_1} \cdots x_{i_k}, \quad (2.2)$$

for all $j = 1, m$.

Due to (2.1) and (2.2), it is easy to see that

$$(V(x))_l = \sum_{j=1}^{m} \left( \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} P_{i_1 \cdots i_k \_jl} x_{i_1} \cdots x_{i_k} \right) x_j = \sum_{j=1}^{m} (V_l(x))_j x_j = (V_l(x), x)$$

where $(\cdot, \cdot)$ stands for the standard inner product of two vectors. Therefore, the polynomial stochastic operator $V : S^{m-1} \rightarrow S^{m-1}$ given by (2.1) can be written as follows:

$$V(x) = \left( (V_1(x), x), \cdots, (V_m(x), x) \right)^T, \quad (2.3)$$

where $V_l : S^{m-1} \rightarrow S^{m-1}$ is defined by (2.2) for $l = 1, m$. In the sequel, we write the polynomial stochastic operator $V$ as follows $V = (V_1 | V_2 | \cdots | V_m)$.

We now define an $m \times m$ square matrix as follows:

$$\mathbb{P}_x = \begin{pmatrix} V_1(x)^T \\ V_2(x)^T \\ \vdots \\ V_m(x)^T \end{pmatrix} = \begin{pmatrix} (V_1(x))_1 & (V_1(x))_2 & \cdots & (V_1(x))_m \\ (V_2(x))_1 & (V_2(x))_2 & \cdots & (V_2(x))_m \\ \vdots & \vdots & \ddots & \vdots \\ (V_m(x))_1 & (V_m(x))_2 & \cdots & (V_m(x))_m \end{pmatrix}$$

(2.4)
where an image $V_l(x)$ of $x$ under $V_l$ defined by (2.2) is a column vector for $l = 1, m$. It follows from (2.3) and (2.4) that

$$V(x) = P_xx.$$  

(2.5)

We call (2.5) a matrix form of the polynomial stochastic operator.

We now show that the square matrix $P_x$ is doubly stochastic for any $x \in S^{m-1}$. Since $V_l(x) \in S^{m-1}$ for any $x \in S^{m-1}$, the matrix $P_x$ is row stochastic. On the other hand, it follows from (2.2) that $P_x = (p_{ij}(x))_{i,j=1}^m$ where

$$p_{ij}(x) = (V_l(x))_j = \sum_{i_1=1}^m \cdots \sum_{i_{k-2}=1}^m P_{i_1 \cdots i_{k-2} ji} x_{i_1} \cdots x_{i_{k-2}}$$

(2.6)

Consequently, we have that

$$\sum_{i=1}^m p_{ij}(x) = \sum_{i=1}^m \left( \sum_{i_1=1}^m \cdots \sum_{i_{k-2}=1}^m P_{i_1 \cdots i_{k-2} ji} x_{i_1} \cdots x_{i_{k-2}} \right)$$

$$= \sum_{i_1=1}^m \cdots \sum_{i_{k-2}=1}^m \left( \sum_{i=1}^m P_{i_1 \cdots i_{k-2} ji} \right) x_{i_1} \cdots x_{i_{k-2}}$$

$$= \sum_{i_1=1}^m \cdots \sum_{i_{k-2}=1}^m x_{i_1} \cdots x_{i_{k-2}} = (x_1 + \cdots + x_m)^{k-2} = 1$$

Thus, the polynomial stochastic operator $V : S^{m-1} \rightarrow S^{m-1}$ defined by (2.1) can be written in the matrix form (2.5) where $P_x$ is the doubly stochastic matrix. The polynomial stochastic operator (2.1) has an incredible application in physics, biology, economics (see [16]).

The simplest nonlinear polynomial stochastic operator is a quadratic stochastic operator associated with a cubic stochastic matrix. The quadratic stochastic operator has a fascinating application in population genetics [6, 8]. The quadratic stochastic operator describes a distribution of the next generation in the population system if the distribution of the current generation was given. For example, in the paper [4], the mathematical model of the transmission of human ABO blood groups was described as the quadratic stochastic operator on 7-dimensional simplex and based on some numerical investigations the future ABO blood group distribution of Malaysian people was predicted. The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators. This problem was not fully finished even in the class of the quadratic stochastic operators (for more details see [5, 9]). A fixed point set, an omega limiting set, ergodicity and chaotic dynamics of the quadratic stochastic operators defined on the finite dimensional simplex were deeply studied (for references see [5, 9]). In [5, 9], it was given a long self-contained exposition of the recent achievements and open problems in the theory of the quadratic stochastic operators.

We figured out that the matrix form (2.5) of the polynomial stochastic operator $V$ defined by (2.1) gives an advantage during the study of its dynamics. More precisely, on the one hand, the trajectory $\{x^{(n)}\}_n^{\infty}$ of $V$ starting from $x^{(0)} \in S^{m-1}$, where $x^{(n+1)} = V(x^{(n)}) = P_{x^{(n)}} x^{(n)}$, presents a nonlinear protocol of an opinion sharing dynamics of the multi-agent system. On the other hand, due to the matrix form (2.5), the trajectory of the polynomial stochastic operator is nothing but non-homogeneous Markov chains $\{P_{x^{(n)}}\}_n^{\infty}$ with some initial distribution $x^{(0)}$. In this paper, by means of the theory of non-homogeneous Markov chains, we shall establish a consensus in the multi-agent system (see also [11, 15]).
A trajectory \( x^{(n+1)} = V(x^{(n)}) \) of a single polynomial stochastic operator \( V \) (it is called an autonomous system) presents a time-invariant nonlinear protocol of the multi-agent system. Meanwhile, a trajectory \( x^{(n+1)} = V_n+1(x^{(n)}) \) of a sequence of polynomial stochastic operators \( \{V_n\}_{n=1}^\infty \) (it is called a non-autonomous system) presents a time-varying nonlinear protocol of the multi-agent system. Now, we are ready to give our nonlinear protocol of multi-agent systems.

**PROTOCOL A.** Let \( \{P_n\}_{n=1}^\infty \), where \( P_n = (P_i^{(n)})_{i_1,\ldots,i_k=1}^{m_1,\ldots,m_k} \), be a sequence (a non-homogeneous system) of \( k \)-dimensional \( k \)-tuple stochastic hypermatrices and \( \{V_n\}_{n=1}^\infty \), where \( V_n : S_1^{m-1} \rightarrow S_1^{m-1} \), be a sequence (a non-autonomous system) of associated polynomial stochastic operators defined by (2.1). Suppose that an opinion sharing dynamics of the multi-agent system is given as a trajectory of non-autonomous polynomial stochastic operators

\[
x^{(n+1)} = V_{n+1}(x^{(n)}), \quad x^{(0)} \in S_1^{m-1}
\]  

where \( x^{(n)} = (x_1^{(n)}, \ldots, x_m^{(n)})^T \) is the probabilistic subjective distribution after \( n \) revisions.

**Remark 2.2.** In Protocol A, if we choose a single or a sequence of square stochastic matrices we then obtain the classical linear DeGroot and Chatterjee-Seneta models, respectively (see [2, 3]). If we choose a singe or a sequence of cubic stochastic matrices we then get the nonlinear model proposed in [13]. In this sense, Protocol A generalizes all previous models presented in the papers [12, 13, 14].

3. Results and Discussion

Let \( M(x) = \max_{i \in I} x_i, \quad m(x) = \min_{i \in I} x_i \) and \( d(x) = M(x) - m(x) \) for any \( x \in S_1^{m-1} \), where \( I = \{1, 2, \ldots, m\} \). It is clear that all functions \( M, m, d : S_1^{m-1} \rightarrow \mathbb{R} \) are continuous and \( d(x) = 0 \) if and only if \( x = C = (\frac{1}{m}, \ldots, \frac{1}{m})^T \). We need the following simple but crucial lemma.

**Lemma 3.1 ([12, 13, 14]).** Let \( \{x^{(n)}\}_{n=0}^\infty \subset S_1^{m-1} \) be any sequence. A sequence \( \{x^{(n)}\}_{n=0}^\infty \) converges to the center \( C = (\frac{1}{m}, \ldots, \frac{1}{m})^T \) of the simplex \( S_1^{m-1} \) if and only if \( \lim_{n \rightarrow \infty} d(x^{(n)}) = 0 \).

**Theorem 3.2.** Suppose that an opinion sharing dynamics in the multi-agent system is given by nonlinear Protocol A. If every member of the people has a positive subjective distribution on the given task after some revision steps then the multi-agent system eventually reaches to a consensus.

**Proof.** The theorem states that if \( x^{(n_0)} \in \text{int} S_1^{m-1} \) for some \( n_0 \) then the sequence \( \{x^{(n)}\}_{n=0}^\infty \) defined by (2.7) converges to the center \( C = (\frac{1}{m}, \ldots, \frac{1}{m})^T \) of the simplex \( S_1^{m-1} \). In order to prove it, due to Lemma 3.1, it is enough to show that \( \lim_{n \rightarrow \infty} d(x^{(n)}) = 0 \). Without loss of any generality, we may suppose that \( x^{(0)} \in \text{int} S_1^{m-1} \).

Let \( \{P_n\}_{n=0}^\infty \), where \( P_n = (p_{i_1 \ldots i_k}^{(n)})_{i_1, \ldots, i_k=1}^{m_1, \ldots, m_k} \), be a non-homogeneous system of \( k \)-dimensional \( k \)-tuple stochastic hypermatrices and \( \{V_n\}_{n=0}^\infty \) be a non-autonomous system of associated polynomial stochastic operators defined by (2.1). Let \( V_n(x) = P_n x \) be a matrix form of the polynomial operator \( V_n \), where \( P_n = (p_{i_1 \ldots i_k}^{(n)})_{i_1=1}^{m_1} \ldots \sum_{i_k=1}^{m_k} \) be the doubly stochastic matrix such that

\[
p_{i_1 \ldots i_k}^{(n)}(x) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=2}^{m_k} p_{i_1 \ldots i_{k-2} i_k}^{(n)} x_{i_1} \cdots x_{i_{k-2}}.
\]  

(3.1)
In this case, the opinion sharing dynamics (2.7) can be written as follows

\[ x^{(n+1)} = P^{(n+1)} x^{(n)}. \]  

(3.2)

We want to show that

\[ M\left(x^{(0)}\right) \geq \cdots \geq M\left(x^{(n)}\right) \geq \cdots \]  

(3.3)

\[ m\left(x^{(0)}\right) \leq \cdots \leq m\left(x^{(n)}\right) \leq \cdots \]  

(3.4)

where \( M(x) = \max x_i \) and \( m(x) = \min x_i \).

In fact, since \( P^{(n+1)} \) is the doubly stochastic matrix, it follows from (3.2) that

\[ M\left(x^{(n+1)}\right) = x^{(n+1)}_{i_0} = \sum_{i=1}^{m} \sum_{i_{k-3}=1}^{m} x^{(n)}_{i_1} \cdots x^{(n)}_{i_{k-3}} \left( \sum_{i_{k-2}=1}^{m} P_{i_1 \cdots i_{k-2} j} x^{(n)}_{i_{k-2}} \right) \]

\[ \leq M\left(x^{(0)}\right) \sum_{i=1}^{m} \sum_{i_{k-3}=1}^{m} x^{(n)}_{i_1} \cdots x^{(n)}_{i_{k-3}} \left( \sum_{i_{k-2}=1}^{m} P_{i_1 \cdots i_{k-2} j} \right) \]

\[ = M\left(x^{(0)}\right) \sum_{i=1}^{m} x^{(n)}_{i_1} \cdots x^{(n)}_{i_{k-3}} \]

\[ = M\left(x^{(0)}\right), \]

\[ p^{(n+1)}_{ij}\left(x^{(n)}\right) = \sum_{i_1=1}^{m} \sum_{i_{k-3}=1}^{m} x^{(n)}_{i_1} \cdots x^{(n)}_{i_{k-3}} \left( \sum_{i_{k-2}=1}^{m} P_{i_1 \cdots i_{k-2} j} x^{(n)}_{i_{k-2}} \right) \]

\[ \geq m\left(x^{(0)}\right) \sum_{i_1=1}^{m} \sum_{i_{k-3}=1}^{m} x^{(n)}_{i_1} \cdots x^{(n)}_{i_{k-3}} \left( \sum_{i_{k-2}=1}^{m} P_{i_1 \cdots i_{k-2} j} \right) \]

\[ = m\left(x^{(0)}\right) \sum_{i_1=1}^{m} x^{(n)}_{i_1} \cdots x^{(n)}_{i_{k-3}} \]

\[ = m\left(x^{(0)}\right), \]

for any \( i, j = 1, m \) and \( n \in \mathbb{N} \).

This means that for any \( n \in \mathbb{N} \), all entries of the doubly stochastic matrices \( P^{(n+1)} \) lie in the segment \([ m\left(x^{(0)}\right), M\left(x^{(0)}\right) ]\), i.e., one has that \( m\left(x^{(0)}\right) \leq p^{(n+1)}_{ij}\left(x^{(n)}\right) \leq M\left(x^{(0)}\right) \) for any \( i, j = 1, m \) and \( n \in \mathbb{N} \). Since \( x^{(0)} \in int S^{m-1} \), we have that \( m\left(x^{(0)}\right) > 0 \).
We then obtain from the last argument that
\[
x_i^{(n+1)} = \sum_{j=1}^{m} P_{ij}^{(n+1)}(x^{(n)}) (x_j^{(n)} - M(x^{(n)})) + M(x^{(n)})
\]
\[
\leq m \left( x^{(0)} \right) \left( M(x^{(n)}) - M(x^{(n)}) \right) + M(x^{(n)})
\]
\[
= \left( 1 - m \left( x^{(0)} \right) \right) M(x^{(n)}) + m \left( x^{(0)} \right) M(x^{(n)}) + m \left( x^{(0)} \right) M(x^{(n)}) ,
\]
(3.5)
\[
x_i^{(n+1)} \geq m \left( x^{(0)} \right) M(x^{(n)}) - m \left( x^{(0)} \right) M(x^{(n)}) + m \left( x^{(0)} \right) M(x^{(n)})
\]
\[
= m \left( x^{(0)} \right) M(x^{(n)}) + \left( 1 - m \left( x^{(0)} \right) \right) m \left( x^{(0)} \right) M(x^{(n)}) .
\]
(3.6)
for any \( i = 1, m \). Consequently, we obtain from (3.5) and (3.6) that
\[
d(x^{(n+1)}) = M(x^{(n+1)}) - m(x^{(n+1)})
\]
\[
\leq \left( 1 - 2m(x^{(0)}) \right) \left( M(x^{(n)}) - m(x^{(n)}) \right) + \left( 1 - 2m(x^{(0)}) \right) d(x^{(n)}) .
\]
(3.7)
for any \( n \in \mathbb{N} \). Then, it follows from (3.7) that
\[
d(x^{(n+1)}) \leq \left( 1 - 2m(x^{(0)}) \right)^{n+1} d(x^{(0)}) .
\]
Since \( m(x^{(0)}) > 0 \) and \( 1 - 2m(x^{(0)}) < 1 \), we get that \( \lim_{n \to \infty} d(x^{(n)}) = 0 \). This completes the proof. \( \square \)

**Remark 3.3.** It is worth mentioning that the similar result (Theorem 3.2) does not hold for the classical DeGroot or Chatterjee-Seneta models. Indeed, we may choose a permutation or a sequence of permutation square (doubly) stochastic matrices in the DeGroot or Chatterjee-Seneta models, respectively, for which regardless of any initial positive subjective distribution the multi-agent system never reaches to a consensus (we may observe a permutation of opinions of people). Seemingly, the nonlinear model has an advantage in the modeling of consensus problems.

**Corollary 3.4.** Suppose that an opinion sharing dynamics in the multi-agent system is given by nonlinear Protocol A. If all entries of some \( k \)-dimensional \( k \)-tuple stochastic hyper-matrix \( P_{n0} = (P_{i_1 \ldots i_k}^{(n)})_{i_1 \ldots i_k = 1}^{m \ldots m} \) are positive, i.e., \( P_{i_1 \ldots i_k}^{(n)} > 0 \) for any \( i_1, \ldots, i_k = 1, m \), then the multi-agent system eventually reaches to a consensus.

**Proof.** It is easy to check that if all entries of some \( k \)-dimensional stochastic hyper-matrix \( P_{n0} = (P_{i_1 \ldots i_k}^{(n)})_{i_1 \ldots i_k = 1}^{m \ldots m} \) are positive, then \( x^{(n_0)} = V_{n0}(x^{(n_0-1)}) = V_{n0}(x^{(0)}) \in int S^{m-1} \). Then, thanks to Theorem 3.2, we reach to a consensus in the multi-agent system. \( \square \)

**Remark 3.5.** We know from the theory of Markov chains that if all entries of a doubly stochastic matrix \( P \) are positive then its trajectory \( \{x^{(n)}\}_{n=0}^{\infty} \), where \( x^{(n)} = P^n x^{(0)} \), starting from any initial point \( x^{(0)} \in S^{m-1} \) converges to the center \( C = (\frac{1}{m}, \ldots, \frac{1}{m}) \) of the simplex \( S^{m-1} \) (i.e., it is regular). The similar result was open for stochastic multidimensional hyper-matrices. Thanks to Corollary 3.4, this result is generalized for stochastic multidimensional hyper-matrices. To the best of our knowledge, Corollary 3.4 is a pioneering result for stochastic multidimensional hyper-matrices.
Remark 3.6. In the Protocol A, we suppose that all $k$-dimensional hyper-matrices are $k$-tuple stochastic (i.e., elements are nonnegative and they are stochastic in all directions). However, in the proof of Theorem 3.2, we only utilize the following conditions

$$
\sum_{i_k=1}^{m} p^{(n)}_{i_1\cdots i_k} = \sum_{i_{k-1}=1}^{m} p^{(n)}_{i_1\cdots i_k} = \sum_{i_k=1}^{m} p^{(n)}_{i_1\cdots i_k} = 1, \quad p^{(n)}_{i_1\cdots i_k} \geq 0, \quad \forall i_1, \cdots, i_k = 1, m.
$$

Therefore, all our results remain true if we only require that all $k$-dimensional hyper-matrices are triple stochastic, i.e., having nonnegative entries and stochastic only in three fixed directions (without loss of any generality, we may assume that one of these three fixed directions is the last $k^{th}$-direction).

4. Conclusions

In this paper, we have studied a nonlinear protocol for a structured time-varying and synchronous multi-agent system which generalizes both the DeGroot and Chatterjee-Seneta classical models. The opinion sharing dynamics in the multi-agent system is given by the trajectory of non-autonomous polynomial stochastic operators associated with the $k$-dimensional $k$-tuple stochastic hyper-matrices (Protocol A). We showed that the multi-agent system eventually reaches to a consensus if either of the following two conditions is satisfied: (i) every member of the people has a positive subjective distribution on the given task after some revision steps or (ii) all entries of some multidimensional stochastic hyper-matrix are positive.

Acknowledgments

This work has been done under the MOHE grant FRGS14-141-0382. The first Author (M.S.) is grateful to the Junior Associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

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