A simple bijection between permutation matrices and descending plane partitions without special parts.

Markus Fulmek*
Faculty of Mathematics
University of Vienna
Vienna, Austria
Markus.Fulmek@univie.ac.at

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Abstract

We present a simple bijection between permutation matrices and descending plane partitions without special parts. This bijection is already mentioned in [7] (without giving the details); it involves the inversion words of permutations and the (well–known) representation of descending plane partitions as families of non–intersecting lattice paths.

(Taking a short detour, we will also exhibit how the (well–known) enumeration of descending plane partitions follows easily from the evaluation of Andrew’s determinant.)

1 Introduction

It is (nowadays) a well–known fact that the enumeration of descending plane partitions with parts not exceeding \( m \) and of alternating sign matrices of

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dimension $m$ gives the same number:

$$\prod_{k=0}^{m-1} \frac{(3 \cdot k + 1)!}{(m + k)!}. \quad (1)$$

On the other hand, it is apparently very hard to find a simple bijection between alternating sign matrices and descending plane partitions in general. Even the bijection between the much simpler special cases of permutation matrices (i.e., alternating sign matrices without entries $-1$) and descending plane partitions with no special parts (to be explained in a moment) led to complicated constructions. In [3], Ayyer presented an inductively constructed bijection “which does not relate the number of parts of the descending plane partition with the number of inversions of the permutation as one might have expected from the conjecture of Mills, Robbins and Rumsey” ([9, Conjecture 3], to be explained in a moment). In [10], Striker presented another bijection involving monotone triangles as “intermediate” combinatorial objects, which maps descending plane partitions with $k$ parts, all of which are non-special, to permutations with $k$ inversions.

In this note, we shall present a simple bijection which relies only on the (obvious) representation of descending plane partitions as families of non-intersecting lattice paths and on the (obvious) encoding of permutations by inversion words: This bijection was mentioned in [7] without giving the details. It also maps descending plane partitions with $k$ parts to permutations with $k$ inversions.

This note is organized as follows:

In section 2, we present the basic definitions and background information. In section 3 we present the interpretation of descending plane partitions as families of non-intersecting lattice paths. In section 4, we present our simple bijection.

## 2 Background information

For reader’s convenience, we recall some background information needed for our presentation.

### 2.1 Descending plane partitions

Here is the definition of descending plane partitions (see [9, Definition 4]):
Definition 1. A descending plane partition is an array $\pi = (a_{i,j})$, $1 \leq i < j < \infty$, of positive integers

$$
\begin{array}{ccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & \cdots & \cdots & a_{1,\mu_1} \\
  & a_{2,2} & a_{2,3} & \cdots & \cdots & a_{2,\mu_2} \\
  &  &  & \cdots & \cdots &  \\
  &  &  &  & \cdots &  \\
  &  &  &  &  & a_{k,k} & \cdots & a_{k,\mu_k}
\end{array}
$$

such that

1. rows are weakly decreasing, i.e., $a_{i,j} \geq a_{i,j+1}$ for all $i = 1, \ldots, k$ and $i \leq j < \mu_i$,
2. columns are strictly decreasing, i.e., $a_{i,j} > a_{i+1,j}$ for all $i = 1, \ldots, k-1$ and $i < j \leq \mu_{i+1}$,
3. $a_{i,i} > \mu_i - i + 1$ for all $i = 1, \ldots, k$,
4. $a_{i,i} \leq \mu_{i-1} - (i - 1) + 1$ for all $i = 2, \ldots, k$.

It is easy to see that these conditions imply

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq k.$$ 

The parts of a descending plane partition are the numbers (with repetitions) that appear in the array. The empty array, which we denote by $\emptyset$, is explicitly allowed.

A descending plane partition $\pi$ where no part is greater than $m$ (i.e., $\pi$ has at most $m - 1$ rows) is said to have dimension $m$.

We denote the $i$-th row of a descending plane partition by $r_i$. The length of $r_i$ is the number of parts it contains, which is $\mu_i - i + 1$. So we may rephrase the last two conditions as

(A) The first part of $r_i$ is greater than the length of $r_i$ for $i = 1, \ldots, k$,

(B) The first part of $r_i$ is less or equal than the length of the preceding row $r_{i-1}$ for $i = 2, \ldots, k$.

A part $a_{i,j}$ in a descending plane partition is called special if it does not exceed the number of parts to its left (in its row $r_i$), i.e.,

$$a_{i,j} \leq j - i.$$ 

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Example 2. A typical example is the array

\[
\begin{array}{cccccc}
6 & 6 & 6 & 4 & 2 \\
5 & 3 & 2 & 1 \\
2 & & & & & \\
\end{array}
\]

with 3 rows and 10 parts (written in descending order)

6, 6, 6, 5, 4, 3, 2, \underline{2}, \underline{2}, 1.

three of which are special parts (indicated as underlined numbers; note that the 2 in the last row is not a special part):

\[\underline{2}, \underline{2}, 1.\]

(This is example \(D_0\) in [8], see [8, Fig. 1].)

2.2 Alternating sign matrices

Here is the definition of alternating sign matrices (see [9, Definition 1]):

Definition 3. An alternating sign matrix of dimension \(m\) is an \(m \times m\) square matrix which satisfies

- all entries are 1, −1 or 0,
- every row and column has sum 1,
- in every row and column the nonzero entries alternate in sign.

Suppose that \(M = (a_{i,j})^m\) is an alternating sign matrix of dimension \(m\). Then the number of inversions in \(M\) is defined to be

\[
\sum_{1 \leq i < j \leq n} a_{i,j} \cdot a_{k,l}. \tag{2}
\]

(See [9, p. 344].)

Example 4. The following matrix is an example of an alternating sign matrix of dimension 5:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]
2.3 The Mills–Robbins–Rumsey conjecture

Here is the Conjecture of Mills, Robbins and Rumsey [9, Conjecture 3]:

**Conjecture 5.** Suppose that \( m, k, n, p \) are nonnegative integers, \( 1 \leq k \leq m \).
Let \( A(m, k, n, p) \) be the set of alternating sign matrices such that

1. the size of the matrix is \( m \times m \),
2. the 1 in the top row occurs in position \( k \),
3. the number of \(-1\)'s in the matrix is \( n \),
4. the number of inversions in the matrix is \( p \).

On the other hand, let \( D(m, k, n, p) \) be the set of descending plane partitions such that

1. no part exceeds \( m \),
2. there are exactly \( k - 1 \) parts equal to \( m \),
3. there are exactly \( n \) special parts,
4. there are a total of \( p \) parts.

Then \( A(m, k, n, p) \) and \( D(m, k, n, p) \) have the same cardinality.

2.4 Permutation matrices and inversions

Let \( \sigma \in S_m \) be a permutation of the first \( m \) natural numbers \( \{1, 2, \ldots, m\} \).
Recall that an inversion of \( \sigma \) is a pair \((i, j)\) such that \( i < j \) but \( \sigma(i) > \sigma(j) \).
(For the number \( \text{inv}(\sigma) \) of all inversions of \( \sigma \) we have \( 0 \leq \text{inv}(\sigma) \leq \frac{m(m+1)}{2} \).)

We may assign to \( \sigma \) its inversion word \((a_1, a_2, \ldots, a_{m-1})\), where \( a_k \) is the number of inversions \((i, j)\) with \( \sigma(j) = k \), \( k = 1, 2, \ldots, m - 1 \). Clearly we have \( 0 \leq a_k \leq m - k \) and \( a_1 + a_2 + \cdots + a_{m-1} = \text{inv}(\sigma) \).

Considering the permutation word

\[(\sigma(1), \sigma(2), \ldots, \sigma(m))\],

of \( \sigma \), the inversion word’s \( k \)–th entry \( a_k \) is simply the number of elements to the left of \( k \) (in the permutation word) which are greater than \( k \), and
it is easy to see that every word \((b_1, b_2, \ldots, b_{m-1})\) with \(0 \leq b_k \leq m - k\) determines a unique permutation: Inversion words are, in this sense, just another “encoding” for permutations.

A permutation \(\sigma \in S_n\) can be represented by an \(m \times m\)-matrix \(M\) with entries 0 or 1, namely

\[
M_{i,j} = \delta_{\sigma(j),i}
\]

(where \(\delta_{x,y}\) denotes Kronecker’s delta: \(\delta_{x,y} = 1\) if \(x = y\), \(\delta_{x,y} = 0\) if \(x \neq y\)). We call this matrix the permutation matrix of \(\sigma\): Clearly, it contains precisely one entry 1 in every row and column.

**Example 6.** Let \(\sigma \in S_6\) be the permutation with permutation word

\[631425.\]

The corresponding permutation matrix is

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

and the corresponding inversion word is

\[(2, 3, 1, 1, 1).\]

Note that every permutation matrix is also an alternating sign matrix (which does not contain entries \(-1\)), and the definition of inversions (2) for alternating sign matrices is a generalization of the number of inversions of a permutation.

### 3 Representation of descending plane partitions as lattice paths

We shall use the well-known encoding of (shifted) tableaux as non-intersecting lattice paths, with small additions/modifications; i.e., we shall encode descending plane partition \(\pi = (a_{i,j})\) of dimension \(m\) as a set of \(m - 1\)
Consider the following descending plane partition $\pi$ of dimension 6:

\[
\begin{array}{cccccc}
6 & 6 & 6 & 4 & 2 \\
5 & 3 & 2 & 1 \\
2
\end{array}
\]

The picture shows the visualization of $\pi$ as a family of $5 = 6 - 1$ non-intersecting lattice paths:

Here, the starting point $S_i$ corresponds to the point numbered $i$ on the vertical axis, and the ending point $E_i$ corresponds to the point numbered $i$ on the horizontal axis. The 3 real paths corresponding to the 3 rows of $\pi$ are shown as solid lines coloured green, and the 2 remaining virtual paths (connecting $S_2$ to $E_2$ and $S_3$ to $E_3$) are shown as dashed lines coloured red. Note that the special parts of $\pi$ correspond to the horizontal steps in the “special range” below the main diagonal $y = x$. 

Figure 1: Illustration
non-intersecting lattice paths in a particular sub-lattice $\mathcal{L}_m$ of $\mathbb{Z}^2$. This encoding was already considered in [8], but for reader’s convenience, we shall describe its details below. Certainly, it will be helpful to look at an illustrative example; see Figure 1.

The starting points of these lattice paths are the points

$$
\mathcal{S} = \{ S_1 := (0, 2), S_2 := (0, 3), \ldots, S_{m-1} := (0, m) \} \subset \mathbb{Z}^2
$$
on the vertical axis.

The ending points of these lattice paths are the points

$$
\mathcal{E} = \{ E_1 := (1, 0), E_2 := (2, 0), \ldots, E_{m-1} := (m - 1, 0) \} \subset \mathbb{Z}^2
$$
on the horizontal axis.

Moreover, the following rectangle of points also belongs to $\mathcal{L}_m$:

$$
\mathcal{R} = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\} \subset \mathbb{Z}^2.
$$

If two points $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$ in the union of

$$
\mathcal{S} \cup \mathcal{E} \cup \mathcal{R},
$$

which are not both starting points and not both end points, are “horizontally or vertically adjacent”, then there is a (directed) arc “from left to right or from top to bottom” between them, i.e.,

- if $p_2 = (x_2, y_2) = (x_1 + 1, y_1)$, then there is an arc from $p_1$ to $p_2$,
- if $p_2 = (x_2, y_2) = (x_1, y_1 - 1)$, then there is an arc from $p_1$ to $p_2$.

We shall call the directed graph consisting of these points and arcs the real part of our lattice $\mathcal{L}_m$, and we use it in the obvious way to encode the rows of $\pi$ as a family of real non-intersecting lattice paths: Row $i$ of $\pi$ directly corresponds to a path in (the real part of) the lattice $\mathcal{L}_m$ by interpreting the $j$–th part $a_{i,j+i-1}$ of this row as the height of the $j$–th horizontal step of the corresponding path (see Figure 1): Note that this path has starting point $S_{a_{i,i-1}}$, which is not necessarily equal to $S_{m-i}$.

We define the length of some real path to be the number of its horizontal steps: So the length of the path is equal to the length of the row it encodes.

In general, this “interpreting rows as lattice paths” will not give $m - 1$ paths: The missing (virtual) paths shall use the virtual part of our lattice $\mathcal{L}_m$, which contains the starting and ending points $\mathcal{S} \cup \mathcal{E}$ together with the union of two “diagonals”, namely
• \( D_1 = \{ (-x, x) : 2 \leq x \leq m - 1 \} \),
• \( D_2 = \{ (-x, x + 1) : 1 \leq x \leq m - 1 \} \).

The (directed) arcs in this virtual part of our lattice \( \mathcal{L}_m \) are
• from \( S_i = (0, i + 1) \) to \( (-i, i + 1) \) for \( i = 1, 2, \ldots, m - 1 \),
• from \( (-i, i + 1) \) to \( (-i - 1, i + 1) \) for \( i = 1, 2, \ldots, m - 2 \),
• from \( (-i - 1, i + 1) \) to \( E_{i+1} = (i + 1, 0) \) for \( i = 1, 2, \ldots, m - 2 \),
• from \( (-1, 2) \) to \( E_1 = (1, 0) \).

Note that there are unique virtual lattice paths connecting the points
• \( S_i \) and \( E_i \), for \( i = 1, 2, \ldots, m - 1 \),
• \( S_i \) and \( E_{i+1} \) for \( i = 1, 2, \ldots, m - 2 \).

If there are starting points and ending points which are not connected by real paths, then we shall connect them with non-intersecting virtual paths. In fact, this is always possible in a unique way:

Condition 3, rephrased as condition (A) in section 2.1, in the definition of a descending plane partition states that \( a_{k,k} \) is strictly greater than the length of the real path starting at height \( a_{k,k} \), so a real path starting at point \( S_i = (0, i + 1) \) has length \( j \leq i \) and therefore must end in some point \( E_j = (j, 0) \) with \( j \leq i \).

Condition 4, rephrased as condition (B) in section 2.1, in the definition of a descending plane partition states that \( a_{k,k} \) is less or equal than the length of the path starting at height \( a_{k-1,k-1} \), so a real path starting in \( S_i = (0, i + 1) \) and ending in some point \( E_j = (j, 0) \) with \( j \leq i \) implies that the real path below (if any) starts in some starting point \( S_\ell \) with \( \ell < j \).

So, if there are \( d \) real lattice paths connecting the points \( S_{s_i} \) and \( E_{e_i} \) for \( i = 1, 2, \ldots, d \) in their “natural order” (i.e., \( s_d > s_{d-1} > \cdots > s_1 \) and \( e_d > e_{d-1} > \cdots > e_1 \)) then there holds

\[
s_d \geq e_d > s_{d-1} \geq e_{d-1} > \cdots > s_2 \geq e_2 > s_1 \geq e_1.
\]

Now assume that all starting and ending points \( S_i, E_i, a < i \leq m - 1 \), are properly connected by real or virtual paths as described above, and that the next row \( r \) of the descending plane partition (not yet encoded as a real
path) starts with part \( b \leq a \) (at the beginning, this assumption is fulfilled for \( a = m - 1 \)). If \( b < a \), then for \( i = b + 1, b + 2, \ldots a \) we connect starting point \( S_i \) with ending point \( E_i \) by a virtual path: Clearly, there is one and only one way to achieve this. Let \( c \leq b \) be the length of row \( r \) and connect \( S_b \) and \( E_c \) with the real lattice path corresponding to row \( r \). If \( c < b \), then for \( i = c, c + 1, \ldots, b - 1 \) we connect \( S_i \) with \( E_{i+1} \) by a virtual path: Clearly, there is one and only one way to achieve this. Observe that by now we achieved a proper connection of all starting and ending points \( S_i, E_i \) for \( c \leq i \leq m - 1 \).

Repeating this step for every row \( r \) of the given descending plane partition might leave \( q \) pairs of starting points and ending points \((S_i, E_i), i = 1, 2, \ldots q < m\), which are not yet connected properly: If \( q > 0 \), then for \( i = 1, 2, \ldots q \), we connect starting point \( S_i \) with ending point \( E_i \) by a virtual path in the only possible way.

It is easy to see that the \( m - 1 \) lattice paths thus constructed are non-intersecting, so we obtained an encoding of a descending plane partition of dimension \( m \) as a family of \( m \) non-intersecting lattice paths in \( \mathcal{L}_m \).

We call starting (or ending) points real (virtual) if they belong to a real (virtual) path.

Now we show that every family \( F \) of non-intersecting lattice paths connecting the starting points and the ending points in our lattice \( \mathcal{L}_m \) determines a unique descending plane partition: Arranging the heights \( h_{i,j} \) of the \( j \)-th horizontal step of the \( i \)-th real path (counted from above) as follows...

\[
\begin{array}{cccccc}
h_{1,1} & a_{1,2} & h_{1,3} & \cdots & \cdots & h_{1,\mu_1} \\
h_{2,2} & h_{2,3} & \cdots & \cdots & h_{2,\mu_2} \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
h_{l,1} & \cdots & h_{l,\mu_l} \\
\end{array}
\]

...gives a (unique) array of positive integers (maybe empty) which fulfils conditions (1) and (2) in Definition 1, since the real paths are non-intersecting.

In order to show that this array is indeed a descending plane partition, we must check conditions (3) and (4) (rephrased as (A) and (B) in section 2.1).

Assume that \( F \) contains a real lattice path \( p \) connecting \( S_a \) with \( E_b \). Consider the set of (all) lattice paths “above \( p \)”, i.e., starting in points \( S_i \), for \( a < i < m \). Let \( \ell \) be the number of real paths in this set. Then this set contains \((m - 1 - a) - \ell \) virtual paths, which, of course, must have \((m - 1 - a) - \ell \) virtual ending points: By construction, for such virtual ending point \( E_j \) we must have \( j > a \). If \( b > a \), then \( \ell + 1 \) of these possible
ending points would, in fact, be real ending points (since the paths are non-intersecting), leaving only \((m - 1 - a) - \ell - 1\) possible virtual ending points, which simply is not enough. Hence there must hold \(b \leq a\), which is equivalent to condition (3) (or rephrased condition (A)) from Definition 1.

Now consider the set of (all) lattice paths ending in points \(E_i\), for \(b < i < m\). Again, let \(\ell\) be the number of real paths in this set, so there must be \((m - 1 - b) - \ell\) virtual paths in this set, which, of course, must have \((m - 1 - b) - \ell\) virtual starting points: By construction, for every such virtual starting point \(S_j\) we must have \(j \geq b\). Since \(a \geq b\), among these possible starting points there are at least \(\ell + 1\) real points, leaving at most \((m - 1) - (b - 1) - (\ell + 1)\) possible points, which is precisely the required number. Therefore, for every real path with starting point \(S_c\) with \(c < a\) there must hold \(c < b\), which is equivalent to condition 4 (or rephrased condition (B)) from Definition 1.

So we established the bijection between descending plane partitions of dimension \(m\) and families of \((m - 1)\) non–intersecting lattice paths in our lattice \(\mathcal{L}_m\).

### 3.1 Detour: A determinantal formula

From this representation, we immediately obtain the following determinantal expression for the number of descending plane partitions:

**Corollary 7.** The number of descending plane partitions of dimension \(m\) is

\[
\det \left( \binom{i+j-1}{j-1} + \delta_{i,j} - \delta_{i+1,j} \right)_{1 \leq i,j \leq m-1}. \tag{3}
\]

**Proof.** Note that

\[
a_{i,j} := \binom{i+j-1}{j-1} + \delta_{i,j} + \delta_{i+1,j}
\]

is precisely the number of lattice paths in \(\mathcal{L}_m\) connecting starting point \(S_i\) and ending point \(E_j\).

The straightforward application of the *Lindström–Gessel–Viennot Theorem*\(^1\) (see [4, 5]) shows that in the expansion of the determinant \(\det (a_{i,j})\), all terms corresponding to *intersecting* families of \((m - 1)\) lattice paths cancel; and each *non–intersecting* family connecting \(S_i\) to \(E_{\sigma(i)}\), \(i = 1, \ldots, m - 1,\)

\(^1\)Using Krattenthaler’s [6, footnote 10 on page 76] name for this well–known result.
will be counted with the sign of the permutation $\sigma$. Unfortunately, due to the particular construction of our lattice, there are permutations with a negative sign which “survive” the Lindström–Gessel–Viennot–cancellation of intersecting lattice paths, and the corresponding families of lattice paths will be subtracted from instead of added to the number we want to determine.

But it is easy to see that the sign of the permutation $\sigma$ is precisely $(-1)^k$, where $k$ is the number of virtual paths connecting some starting point $S_i$ with an ending point $E_{i+1}$. So giving all such paths weight $(-1)$, i.e., considering the corresponding “weighted” number of lattice paths

$$b_{i,j} := \binom{i+j-1}{j-1} + \delta_{i,j} - \delta_{i+1,j}$$

instead of $a_{i,j}$, will cancel out all the negative signs; and thus the determinant (3) provides the correct number of descending plane partitions of dimension $m$.

3.2 Detour, continued: Andrew’s determinant

Note that this determinant is closely related to the famous determinant considered by Andrews [1]:

$$a_n(x) := \det \left( \binom{x+i+j}{j} + \delta_{i,j} \right)_{0 \leq i,j \leq n-1}.$$  

The analogous generalization of the determinant in (3) would be

$$d_n(x) := \det \left( \binom{x+i+j}{j} + \delta_{i,j} - \delta_{i+1,j} \right)_{0 \leq i,j \leq n-1}.$$  

Obviously, $d_{m-1}(1)$ is precisely the determinant in (3).

But since

$$\left( \binom{x+i+j}{j} + \delta_{i,j} \right) - \left( \binom{x+i+(j-1)}{j-1} + \delta_{i,j-1} \right)$$

equals

$$\binom{(x-1)+i+j}{j} + \delta_{i,j} - \delta_{i+1,j}$$
for all $j \geq 1$, we see that
\[ d_n(x) = a_n(x + 1) \] for all $n > 0$.

So we obtained that the number of descending plane partitions of dimension $m$ is equal to $a_{m-1}(2)$.

Andrews ([1, Theorem 8]; see [2] for a short proof) showed that
\[ a_m(x) = \prod_{k=0}^{m-1} \Delta_k(x), \]
where $\Delta_0(x) \equiv 2$ and for all $j > 0$
\[ \Delta_{2j}(x) = \frac{(x + 2j + 2)_{j/2} (x/2 + 2j + 3/2)_{j-1}}{(j)_{j/2} (x/2 + j + 3/2)_{j-1}}, \]
\[ \Delta_{2j-1}(x) = \frac{(x + 2j)_{j-1/2} (x/2 + 2j + 1/2)_{j}}{(j)_{j/2} (x/2 + j + 1/2)_{j-1}}. \]
(Here, we used Pochhammer’s symbol: $(x)_j = x \cdot (x+1) \cdots (x+j-1)$.)

Note that
\[ \Delta_{k-1}(2) = \frac{(3k + 1)!}{(2k + 1)! (k + 1)_k} \]
for all $k \geq 1$, and
\[ \frac{(3 \cdot 0 + 1)!}{(2 \cdot 0 + 1)! (1)_0} = 1, \]
whence we obtain the following expression for the number of descending plane partitions of dimension $m$:
\[ a_{m-1}(2) = \prod_{k=0}^{m-1} \frac{(3k + 1)!}{(2k + 1)! (k + 1)_k}. \]

It is easy to see that this, in fact, is equal to (1).

4 The bijection between descending plane partitions without special parts and permutations

Now we shall present the promised bijection between descending plane partitions and inversion words (which are in bijection with permutations and with permutation matrices, as outlined in section 2):
If we are given a descending plane partition $\pi$ of dimension $m$ without special parts, we can easily derive from it the inversion word $(a_1, a_2, \ldots, a_{m-1})$ of a permutation $\sigma \in \mathfrak{S}_m$: Simply set
\[ a_i := \text{number of parts } (m - i + 1) \text{ in } \pi. \]

Looking at the representation of $\pi$ as a family of non–intersecting lattice paths in $L_m$, it is easy to see that the number of (non–special) horizontal steps at height $h$ cannot exceed $h - 1$, whence we have $0 \leq a_i \leq m - i$ for all $i$. So the word $(a_1, a_2, \ldots, a_{m-1})$ is indeed an inversion word which encodes some unique permutation $\sigma \in \mathfrak{S}_m$.

The inverse mapping is also quite simple: If we have an inversion word $(a_1, a_2, \ldots, a_{m-1})$, we start with the empty descending plane partition and insert successively (i.e., for , $i = 1, 2, \ldots, m-1$) $a_i$ parts $(m - i + 1)$ into the rows of a “growing” descending plane partition, subject to the simple rule, that we never start a new row that would violate condition (4) (rephrased as (B)) from Definition 1: Table 1 gives the corresponding algorithm in “pseudo–code” notation. The correctness of this “insertion of horizontal steps corresponding to the parts of some descending plane partition without special parts” is easily seen by observing

- that the real path “currently under construction” must reach the line $y = x$ before the “next” (i.e., “lower”) real path may start (according to condition (4), rephrased as (B), from Definition 1 for descending plane partitions);
- and if some path reached the line $y = x$, then it has a unique continuation (by vertical steps only, since there are no special parts) and can thus be “finished”.

Clearly, this bijection is in line with the Mills–Robbins–Rumsey–Conjecture (given here as Conjecture 5): The number of parts of the descending plane partition equals the number of inversions of the permutation, and if the number of parts of the descending plane partition which are equal to $m$ is $k - 1$, then the position of the 1 in the first row of the permutation matrix is $k$.

We conclude this presentation with an illustrating example:

**Example 8.** Consider the inversion word
\[ a = (0, 0, 2, 3, 1, 1, 1, 1). \]
/* Construct $m$-DPP from inversion word $w = (w_1, \ldots, w_{m-1})$. */
/* Input: $w = (w_1, \ldots, w_{m-1})$ ($0 \leq w_i \leq m - i$). */
/* Output: DPP of dimension $m$ (as array of rows). */

$i \leftarrow 0$ /* $i$: current index of inversion word $w$. */
r $\leftarrow 1$ /* $r$: index of the (yet empty) row to be filled. */

repeat
  $i \leftarrow i + 1$ /* Find next index $i$ such that $w_i > 0$: */
  while $i < m$ and $w_i = 0$
    $i \leftarrow i + 1$
  end while
  if $i \geq m$
    break /* Jump out of loop */
  end if
  $e \leftarrow m - i + 1$ /* We have to insert $w_i$ steps at height $e$: */
  $l \leftarrow$ (length of row $r$) /* Do not start a new row if $e > l!$ */
  if $l \leq e$
    $r \leftarrow r + 1$ /* Start a new row: Room for $m - i$ steps! */
    (start new row $r$ and insert $w_i$ entries $e$ into it)
  else
    $a = \min(e - l, w_i)$
    (append $a$ entries $e$ to row $r$)
    $a \leftarrow w_i - a$
    if $a > 0$
      $r \leftarrow r + 1$ /* Start a new row: Room for $m - i$ steps! */
      (start new row and insert $a$ entries $e$ into it)
    end if
  end if
until $i \geq m$.

return (array of rows thus constructed).
We shall illustrate the algorithm by showing the “successively growing” family of (real) non-intersecting lattice paths corresponding to a descending plane partition of dimension 8 without special parts. We start with the empty lattice. Since $a_1 = a_2 = 0$, the first horizontal steps to be inserted are $a_3 = 2$ steps at height $8 - 3 + 1 = 6$:

Now we have to insert $a_4 = 3$ steps at height $8 - 4 + 1 = 5$: Since we did not reach the line $y = x$ yet, we must not start a new path, but append these steps to the current path — by doing this, we reach the line $y = x$ and are thus able to “finish” this path (since we must not insert horizontal steps below $y = x$).

Now we have to insert $a_5 = 1$ step at height $8 - 5 + 1 = 4$: Since the “preceding” path is finished, we start a new one.
Now we have to insert \( a_6 = 1 \) step at height \( 8 - 5 + 1 = 3 \): Since the “preceding” path is not yet finished, we append this step to it: We see that this path will reach the line \( y = x \) in the next iteration, so we may “finish” it already.

Finally, we have to insert \( a_7 = 1 \) step at height \( 8 - 7 + 1 = 2 \): Since the “preceding” path is finished, we start a new one.

Reading off the rows from the (heights of the horizontal steps of the) non-intersecting lattice paths, we obtain the following descending plane partition.
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