Boundary Layer Study for an Ocean Related System with a Small Viscosity Parameter

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Abstract

We study an ocean related system with a small viscosity parameter, which is the linearized version of the modified Primitive Equations. As the parameter goes to zero, an $L^\infty$ convergence result is obtained together with the estimation on the thickness of the boundary layer.

Keywords. Primitive Equations, $L^\infty$ convergence, boundary layer.

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1 Introduction

We consider the ocean related system with a small parameter $\varepsilon$ in the space-time domain $(0, L) \times (0, T)$:

\begin{equation}
\begin{cases}
\partial_t u^\varepsilon + \overline{U}_0 u_x^\varepsilon + \psi_x^\varepsilon - 2\varepsilon u_{xx}^\varepsilon = f, \\
\psi_t^\varepsilon + \overline{U}_0 \psi_x^\varepsilon + \lambda^{-2}u_x^\varepsilon = g,
\end{cases}
\end{equation}

with the boundary and initial conditions:

\begin{equation}
\begin{cases}
u^\varepsilon(0, t) = u^\varepsilon(L, t) = 0, \quad \psi^\varepsilon(0, t) = 0, \quad 0 < t < T, \\
u^\varepsilon(x, 0) = u_0(x), \quad \psi^\varepsilon(x, 0) = \psi_0(x), \quad 0 < x < L,
\end{cases}
\end{equation}

where $0 < \varepsilon \ll 1, \overline{U}_0 > 0$ and $\lambda > 0$ are constants with $\overline{U}_0 < \lambda^{-1}$. This system was derived from the Primitive Equations (PEs) of the ocean with mild viscosity thanks to a modal decomposition in the vertical direction, see [1] (cf. [2, 3]).

The PEs are one of the fundamental models for geophysical flows and they are used to describe oceanic and atmospheric dynamics, see [4, 5]. In the presence of viscosity, the study of the PEs through analytical means was started by Lions, Temam and Wang in [6, 7], and some recent advances for the PEs have been obtained, see [8] and the survey [9]. In the absence of viscosity, it is known that the PEs are not well-posed for any set of local boundary conditions, see [10]. To overcome this difficulty, a modified PEs (i.e. $\delta$-PEs) was proposed in [1]. System \textsuperscript{(1.1)} is the linearized version of the modified PEs.
Formally setting $\varepsilon = 0$ in (1.1), one obtains the following system in the space-time domain $(0, L) \times (0, T)$:

$$
\begin{cases}
  u_t^0 + \mathcal{U}_0u_x^0 + \psi_0^0 = f, \\
  \psi_t^0 + \mathcal{U}_0\psi_x^0 + \lambda^{-2}u_x^0 = g.
\end{cases}
$$

(1.3)

Like in [2, 3], the boundary and initial conditions are chosen as follows:

$$
\begin{cases}
  u^0(0, t) + \mathcal{U}_0\lambda^2\psi^0(0, t) = 0, & u^0(L, t) = 0, \quad 0 < t < T, \\
  u^0(x, 0) = u_0(x), & \psi^0(x, 0) = \psi_0(x), \quad 0 < x < L.
\end{cases}
$$

(1.4)

In [3], the existence and uniqueness of solutions of both problem (1.1)-(1.2) and problem (1.3)-(1.4) was proved for some initial data. As numerically shown in [2], some boundary layers appear at the boundary $x = 0$ when $\varepsilon$ goes to zero. Actually, it was shown in [3, Theorem 1.3] that $U^\varepsilon$ and $\Psi^\varepsilon$ are $O(\varepsilon^{1/2})$ in $L^\infty(0, T; L^2(0, L))$, where

$$
U^\varepsilon = u^\varepsilon - u^0 - \theta_u^\varepsilon, \quad \Psi^\varepsilon = \psi^\varepsilon - \psi^0 - \theta_\psi^\varepsilon,
$$

where $\theta_u^\varepsilon$ and $\theta_\psi^\varepsilon$ are the boundary layer correctors and are the same as in Section 2. This implies that $(u^\varepsilon, \psi^\varepsilon)$ converges to $(u^0, \psi^0)$ in $L^\infty(0, T; L^2(0, L))$ as $\varepsilon \to 0^+$. Recently, the analysis of the boundary layers for the linearized viscous PEs has been presented in [11] for 2D and in [12, 14] for 3D.

In the present paper, we study the $L^\infty$ convergence of $(U^\varepsilon, \Psi^\varepsilon)$ as $\varepsilon \to 0^+$. Our main result is as follows.

**Theorem 1.1.** Assume that $(u^\varepsilon, \psi^\varepsilon) \in \mathcal{K}$ and $(u^0, \psi^0) \in \mathcal{K}^0$ are solutions of problem (1.1)-(1.2) and problem (1.3)-(1.4), respectively, where $\mathcal{K}$ and $\mathcal{K}^0$ are the same as in Section 3. Then there exists some constant $C$ independent of $\varepsilon$ such that

$$
\| (U^\varepsilon, \Psi^\varepsilon) \|_{L^\infty(0,T;L^2(0,L))} + \varepsilon^{1/2} \| (U^\varepsilon, \Psi^\varepsilon) \|_{L^2((0,L) \times (0,T))} \leq C \varepsilon,
$$

$$
\| (U^\varepsilon, \Psi^\varepsilon) \|_{L^\infty(0,T;L^\infty(0,L))} \leq C \varepsilon^{1/2}.
$$

From Theorem 1.1 and the definitions of $\theta_u^\varepsilon$ and $\theta_\psi^\varepsilon$ (see Section 2), we immediately obtain

**Corollary 1.1.** Under the assumptions of Theorem 1.1, we have $\lim_{\varepsilon \to 0^+} \| (u^\varepsilon - u^0, \psi^\varepsilon - \psi^0) \|_{L^\infty(0,T;L^\infty(0,L))} = 0$ for any nonnegative function $\delta(\varepsilon)$ satisfying $\delta(\varepsilon) \to 0$ and $\delta(\varepsilon)/\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, and $\lim_{\varepsilon \to 0^+} \| (u^\varepsilon - u^0, \psi^\varepsilon - \psi^0) \|_{L^\infty(0,T;L^\infty(0,L))} > 0$ for any nonnegative function $\delta(\varepsilon)$ satisfying $\delta(\varepsilon) \to 0$ and $\delta(\varepsilon)/\varepsilon \to c$ (nonnegative constant) as $\varepsilon \to 0^+$ whenever $(u^0(0,t), \psi^0(0,t)) \neq (0,0)$ in $(0,T)$. This implies that the boundary layer thickness is of the order $O(\varepsilon)$.

Theorem 1.1 will be proved in Section 2. The main difficulty in the proof is that system (1.1) is an incompletely parabolic perturbation of a hyperbolic system. To overcome the difficulty, a key observation is to obtain the equation (2.9). With this, one can deduce the required estimates, for example, (2.25). In Section 3, some remarks on the regularities of $(u^\varepsilon, \psi^\varepsilon)$ and $(u^0, \psi^0)$ will be presented. The regularity conditions will be used to show the equation (2.8).

## 2 Proof of Theorem 1.1

Let $(u^0, \psi^0) \in \mathcal{K}^0$ be a solution of problem (1.3)-(1.4). As in [3], $\theta_u^\varepsilon$ and $\theta_\psi^\varepsilon$ are defined as follows:

$$
\begin{pmatrix}
  \theta_u^\varepsilon(x,t) \\
  \theta_\psi^\varepsilon(x,t)
\end{pmatrix} = e^{-rx/\varepsilon} \begin{pmatrix}
  A^\varepsilon(t) \\
  B^\varepsilon(t)
\end{pmatrix} + \begin{pmatrix}
  C^\varepsilon(t) \\
  D^\varepsilon(t)
\end{pmatrix},
$$

(2.1)
where \( r = \frac{1}{2 \lambda^2} - \frac{\gamma_0}{2} > 0 \), and
\[
\begin{pmatrix}
A^\varepsilon \\
B^\varepsilon \\
C^\varepsilon \\
D^\varepsilon 
\end{pmatrix} = \begin{pmatrix}
-(1 - e^{-rL/\varepsilon})^{-1}u^0(0, t) \\
\lambda^{-2}e^{-rL/\varepsilon} - 1u^0(0, t) \\
e^{-rL/\varepsilon} - 1u^0(0, t) \\
-\lambda^{-2}e^{-rL/\varepsilon} - 1u^0(0, t)
\end{pmatrix}.
\tag{2.2}
\]

Due to \( u^0(0, t) = -U_0 \lambda^2 \psi^0(0, t) \), it is easy to verify that \( \theta_u^\varepsilon \) and \( \theta_\psi^\varepsilon \) satisfy
\[
\begin{array}{l}
\{ \begin{array}{l}
U_0 \theta_u^\varepsilon + \theta_\psi^\varepsilon - 2\varepsilon \theta_{u\psi}^\varepsilon = 0, \\
U_0 \theta_\psi^\varepsilon + \lambda^{-2} \theta_u^\varepsilon = 0,
\end{array}\end{array}
\tag{2.3}
\]
\[
\theta_u^\varepsilon(0, t) = -u^0(0, t), \quad \theta_\psi^\varepsilon(0, t) = -\psi^0(0, t), \quad \theta_u^\varepsilon(L, t) = 0.
\]

From now on, we use \( C \) to denote a positive generic constant independent of \( \varepsilon \).

**Lemma 2.1.** Assume that \( (u^0, \psi^0) \in K^0 \) is a solution of problem (1.3)-(1.4). Then
\[
\begin{array}{l}
\{ \begin{array}{l}
\| (\theta_u^\varepsilon, \theta_\psi^\varepsilon, \theta_{u\psi}^\varepsilon) \|_{L^\infty(0, T; L^2(0, L))} \leq C\varepsilon^{1/2}, \\
\| (\theta_u^\varepsilon, \theta_\psi^\varepsilon, \theta_{u\psi}^\varepsilon) \|_{L^\infty(0, T; L^2(0, L))} \leq C, \\
\| x(\theta_u^\varepsilon, \theta_{u\psi}^\varepsilon, \varepsilon \theta_{u\psi}^\varepsilon) \|_{L^\infty(0, T; L^2(0, L))} \leq C\varepsilon^{3/2}.
\end{array}\end{array}
\tag{2.4}
\]

**Proof.** Due to \( u^0(L, t) = 0, u^0(0, t) = -\int_0^L u^0_x(x, t) dx \). Thus, \( u^0(0, t) \in L^\infty(0, T) \) since \( u^0_x \in L^\infty(0, T; L^2(0, L)) \). This together with \( u^0(0, t) = -U_0 \lambda^2 \psi^0(0, t) \) gives \( \psi^0(0, t) \in L^\infty(0, T) \). Using the inequality \( e^{-rL/\varepsilon} \leq 6/\varepsilon \) (\( \forall \varepsilon > 0 \)), we have \( e^{-2rL/\varepsilon} \leq C\varepsilon^3 \). Noticing \( \| x e^{-rL/\varepsilon} \|_{L^2(0, L)} \leq C\varepsilon^{3/2} \), we deduce that \( \| x \theta_{u\psi}^\varepsilon \|_{L^\infty(0, T; L^2(0, L))} \leq C\varepsilon^{3/2} \). The other estimates of (2.4) can be deduced similarly. The proof is completed. \( \square \)

**Proof of Theorem 1.1** For simplicity, write \( u = U^\varepsilon \) and \( \psi = \Psi^\varepsilon \). It follows from (1.1), (1.3) and (2.3) that \( u \) and \( \psi \) satisfy
\[
\begin{array}{l}
\{ \begin{array}{l}
u_t + U_0 u_x + \psi_x - 2\varepsilon u_{xx} = 2\varepsilon u^0_x - \theta_u^\varepsilon, \\
\psi_t + U_0 \psi_x + \lambda^{-2} u_x = -\theta_\psi^\varepsilon,
\end{array}\end{array}
\tag{2.5}
\]
with the boundary and initial conditions
\[
\begin{array}{l}
\{ \begin{array}{l}
u(0, t) = u(L, t) = 0, \quad \psi(0, t) = 0, \quad 0 < t < T, \\
u(x, 0) = 0, \quad \psi(x, 0) = 0, \quad 0 < x < L.
\end{array}\end{array}
\tag{2.6}
\]

From (2.5), we obtain
\[
(u - \lambda \psi)_t + U_0 (u - \lambda \psi)_x - \lambda^{-1}(u - \lambda \psi)_x - 2\varepsilon u_{xx} = 2\varepsilon u^0_x - \theta_u^\varepsilon + \lambda \theta_\psi^\varepsilon.
\tag{2.7}
\]
Differentiating (2.5) in \( x \) and multiplying the resulting equation by \( 2\varepsilon \lambda^2 \), we have
\[
2\varepsilon \lambda^2 \psi_{xt} + 2\varepsilon U_0 \lambda^2 \psi_{xx} + 2\varepsilon u_{xx} = -2\varepsilon \lambda^2 \theta_{u\psi}^\varepsilon.
\tag{2.8}
\]
Denote \( W = u - \lambda \psi + 2\varepsilon \lambda^2 \psi_\varepsilon \). Adding (2.7) and (2.8) yields
\[
W_t + U_0 W_x - \lambda^{-1}(u - \lambda \psi)_x = 2\varepsilon u^0_x - \theta_u^\varepsilon + \lambda \theta_\psi^\varepsilon - 2\varepsilon \lambda^2 \theta_{u\psi}^\varepsilon =: 2\varepsilon u^0_x + F^\varepsilon.
\tag{2.9}
\]
Multiplying (2.9) by \(W\), integrating over \((0, L) \times (0, t)\) and using (2.6), we have

\[
\frac{1}{2} \int_0^L W^2 dx + \frac{\lambda}{2} \int_0^t W^2 |_{x=L} ds + 2\varepsilon\lambda^2 \int_0^t \int_0^L \psi_x^2 dxds
\]

\[
= \frac{1}{2} \int_0^L W^2 |_{x=0} ds + \frac{\lambda}{2} \int_0^t W^2 |_{x=0} ds + \frac{1}{2\lambda} \int_0^t (u - \lambda \psi)^2 |_{x=0} ds
\]

\[
+ 2\varepsilon\lambda \int_0^t \int_0^L u_x \psi_x dxds + \int_0^t \int_0^L (2\varepsilon u_x^0 + F^\varepsilon) W dxds
\]

\[
= \frac{\lambda}{2} \int_0^t \psi^2 |_{x=L} ds + 2\varepsilon^2 \lambda \int_0^t \psi_x^2 |_{x=0} ds + 2\varepsilon\lambda \int_0^t \int_0^L u_x \psi_x dxds
\]

\[
+ \int_0^t \int_0^L 2\varepsilon u_x^0 W dxds + \int_0^t \int_0^L F^\varepsilon (u - \lambda \psi) dxds + 2\varepsilon \lambda \int_0^t \int_0^L F^\varepsilon \psi_x dxds
\]

\[
= \frac{\lambda}{2} \int_0^t \psi^2 |_{x=L} ds + \sum_{i=1}^5 E_i.
\]

Using the Hardy inequality (cf. [15]) and noticing (2.6), we obtain

\[
\int_0^L \frac{u_x^2}{x^2} dx \leq 4 \int_0^L u_x^2 dx, \quad \int_0^L \frac{\psi_x^2}{x^2} dx \leq 4 \int_0^L \psi_x^2 dx.
\]

By Lemma 2.1, we have

\[
\|F^\varepsilon\|_{L^\infty(0, T; L^2(0, L))}^2 \leq C\varepsilon, \quad \|xF^\varepsilon\|_{L^2(0, T; L^2(0, L))}^2 \leq C\varepsilon^3.
\]

Applying the Young inequality, (2.12) and (2.11), we deduce that

\[
E_4 \leq \frac{C}{\varepsilon} \|xF^\varepsilon\|_{L^2((0, T) \times (0, L))}^2 + \varepsilon \frac{1}{32} \int_0^t \int_0^L \frac{(u - \lambda \psi)^2}{x^2} dxds
\]

\[
\leq C\varepsilon^2 + \frac{\varepsilon}{4} \int_0^t \int_0^L (u_x^2 + \lambda^2 \psi_x^2) dxds,
\]

and

\[
E_5 \leq C\varepsilon^2 + \frac{\varepsilon \lambda^2}{4} \int_0^t \int_0^L \psi_x^2 dxds.
\]

Using (2.5), we have \(\psi_x |_{x=0} = -\frac{\lambda}{\lambda_0} - 2 u_x |_{x=0} - \frac{\lambda}{\lambda_0} \theta_{\psi} |_{x=0}\). Noticing \(\psi_x^0(0, t) \in L^\infty(0, T)\) (see the proof of Lemma 2.1), we have

\[
E_1 \leq C\varepsilon^2 + \frac{4\varepsilon^2}{\lambda_0} \int_0^t u_x^2(0, s) ds.
\]

Since \(u(0, t) = u(L, t) = 0\), there exists some \(\xi = \xi_t \in (0, L)\) such that \(u_x(\xi, t) = 0\), thus,

\[
u_x^2(0, t) = -\int_0^\xi (u_x^2)_x dx.\]

Substituting it into (2.13) and using the Young inequality, we obtain

\[
E_1 \leq C\varepsilon^2 + \frac{8\varepsilon^2}{\lambda_0} \int_0^t \int_0^L |u_x u_{xx}| dxds
\]

\[
\leq C\varepsilon^2 + \frac{64\varepsilon \lambda^2}{\lambda_0^2} \int_0^t \int_0^L u_x^2 dxds + \frac{\varepsilon \lambda^2}{4} \int_0^t \int_0^L u_x^2 dxds.
\]
Using \( u_{xx}^0 \in L^\infty(0, T; L^2(0, L)) \) and Lemma 2.1, we derive from (2.5) that

\[
\int_0^t \int_0^L (4\varepsilon^2 u_{xx}^2 + u_t^2) dx ds + 2\varepsilon \int_0^t u_x^2 dx = \int_0^t \int_0^L (u_t - 2\varepsilon u_{xx})^2 dx ds
\]

(2.17)

\[
\leq C\varepsilon + 4U_0^2 \int_0^t \int_0^L u_x^2 dx ds + 4 \int_0^t \int_0^L \psi_x^2 dx ds
\]

consequently,

\[
\varepsilon^2 \int_0^t \int_0^L u_{xx}^2 dx ds \leq C\varepsilon + \frac{64}{U_0^2\lambda^2} + \frac{U_0^2\lambda^2}{4} \varepsilon \int_0^t \int_0^L u_x^2 dx ds + \frac{\varepsilon \lambda^2}{4} \int_0^t \int_0^L \psi_x^2 dx ds.
\]

(2.18)

Substituting (2.18) into (2.16) yields

\[
E_1 \leq C\varepsilon^2 + \frac{(64 + U_0^2\lambda^2)}{(U_0^2\lambda^2 + 4)} \varepsilon \int_0^t \int_0^L u_x^2 dx ds + \frac{\varepsilon \lambda^2}{4} \int_0^t \int_0^L \psi_x^2 dx ds.
\]

(2.19)

By the Young inequality, we have

\[
E_2 + E_3 \leq C\varepsilon^2 + \frac{1}{2} \int_0^t \int_0^L W^2 dx ds + 4\varepsilon \int_0^t \int_0^L u_x^2 dx ds + \frac{\varepsilon \lambda^2}{4} \int_0^t \int_0^L \psi_x^2 dx ds.
\]

(2.20)

Plugging (2.13), (2.14), (2.19) and (2.20) into (2.10), we obtain

\[
\frac{1}{2} \int_0^t \int_0^L W^2 dx + \varepsilon \lambda^2 \int_0^t \int_0^L \psi_x^2 dx ds
\]

\[
\leq C\varepsilon^2 + \frac{\lambda}{2} \int_0^t \frac{\psi^2}{x=L} ds + C_0\varepsilon \int_0^t \int_0^L u_x^2 dx ds + \frac{1}{2} \int_0^t \int_0^L W^2 dx ds,
\]

(2.21)

where \( C_0 = \frac{17}{4} + \frac{64}{U_0^2\lambda^2} + \frac{U_0^2\lambda^2}{4} \).

On the other hand, multiplying (2.5) by \( 2\varepsilon u_{xx}^0 \) and \( 2\varepsilon^2 \psi \) respectively, integrating over \((0, L) \times (0, t)\), using (2.6) and (2.11), and performing a similar argument to (2.13), we obtain

\[
\int_0^L (u^2 + \lambda^2 \psi^2) dx + U_0\lambda^2 \int_0^t \psi^2 \big|_{x=L} ds + 4\varepsilon \int_0^t \int_0^L u_x^2 dx ds
\]

\[
= 2 \int_0^t \int_0^L (2\varepsilon u_{xx}^0 - \theta \psi u - \lambda^2 \theta \psi \psi) dx ds
\]

\[
\leq C\varepsilon^2 + C \int_0^t \int_0^L u_x^2 dx ds + 2\varepsilon \int_0^t \int_0^L u_x^2 dx ds + \frac{\varepsilon \lambda^2}{2(C_0 + 1/(2U_0))} \int_0^t \int_0^L \psi_x^2 dx ds.
\]

Hence,

\[
\int_0^L (u^2 + \lambda^2 \psi^2) dx + U_0\lambda^2 \int_0^t \psi^2 \big|_{x=L} ds + 2\varepsilon \int_0^t \int_0^L u_x^2 dx ds
\]

\[
\leq C\varepsilon^2 + C \int_0^t \int_0^L u_x^2 dx ds + \frac{\varepsilon \lambda^2}{2(C_0 + 1/(2U_0))} \int_0^t \int_0^L \psi_x^2 dx ds.
\]

(2.23)
Multiplying (2.23) by $C_0 + 1/(2\lambda U_0)$ and adding the resulting equations to (2.21), we have
\[
\frac{1}{2} \int_0^L W^2 dx + C_0 \int_0^L (u^2 + \lambda^2 \psi^2) dx + \varepsilon \int_0^T \int_0^L \left(\frac{\lambda^2}{2} \psi_x^2 + C_0 u_x^2\right) dx ds \\
\leq C\varepsilon^2 + C \int_0^T \int_0^L (u^2 + W^2) dx ds.
\]
Then, the Gronwall inequality gives
\[
\sup_{0 < t < T} \int_0^L (W^2 + u^2 + \psi^2) dx + \varepsilon \int_0^T \int_0^L (\psi_x^2 + u_x^2) dx ds \leq C\varepsilon^2.
\tag{2.24}
\]
Recalling (2.17) and $W = u - \lambda \psi + 2\varepsilon \lambda^2 \psi_x$ and using (2.24), we deduce that
\[
\sup_{0 < t < T} \int_0^L (u_x^2 + \psi_x^2) dx \leq C.
\tag{2.25}
\]
Thanks to $u(0, t) = 0$, we have $u^2(x, t) = \int_0^x (u^2)_x dx$. Then, using the Hölder inequality, (2.24) and (2.25), we obtain
\[
u^2(x, t) \leq 2 \left( \int_0^L u^2 dx \int_0^L u_x^2 dx \right)^{1/2} \leq C\varepsilon,
\]
thus, $\|U^\varepsilon\|_{L^\infty(0,T;L^\infty(0,L))} \leq C\varepsilon^{1/2}$. Similarly, $\|\Psi^\varepsilon\|_{L^\infty(0,T;L^\infty(0,L))} \leq C\varepsilon^{1/2}$. The proof of Theorem 1.1 is completed.

### 3 Remarks on Regularity of Solutions

By the Hille-Yosida theorem, the authors in [39] proved the following results on existence and uniqueness of both problem (1.1)-(1.2) and problem (1.3)-(1.4):

(i) If $(f, g) \in L^1(0, T; H), (u_0, \psi_0) \in D(A)$, and $(f, g)$ is continuous in $H$ at $t = 0$, then for every $\varepsilon > 0$ problem (1.1)-(1.2) admits a unique solution $(u^\varepsilon, \psi^\varepsilon)$ in $F = C([0, T]; H) \cap L^\infty(0, T; D(A))$ with $(u_t^\varepsilon, \psi_t^\varepsilon) \in L^\infty(0, T; H)$, where $H = L^2(0, L) \times L^2(0, L)$ and $D(A) = \{(u, \psi) \in \mathcal{H}| u_x, \psi_x, u_{xx} \in L^2(0, L), u(0) = \psi(0) = u(L) = 0\}$.

(ii) If $(f, g) \in L^1(0, T; H)$ and $(u_0, \psi_0) \in D(A^0)$, then problem (1.1)-(1.2) admits a unique solution $(u_0^0, \psi_0^0) \in \mathcal{F}^0 = C([0, T]; H) \cap L^\infty(0, T; D(A^0))$ with $(u_t^0, \psi_t^0) \in L^\infty(0, T; H)$, where $D(A^0) = \{(u, \psi) \in \mathcal{H}| u_x, \psi_x \in L^2(0, L), u(0) + \Upsilon_0 \lambda^2 \psi(0) = 0, u(L) = 0\}$.

To get $(u^\varepsilon, \psi^\varepsilon) \in \mathcal{K}$, where
\[
\mathcal{K} = \{(u, \psi) \in \mathcal{F}| u_{xx} \in L^\infty(0, T; L^2(0, L)), (u_t, \psi_t) \in C([0, T]; H)\},
\]
some additional conditions on $f, g, u_0, \psi_0$ must be imposed, for example, the following conditions: $(f_t, g_t) \in L^1(0, T; H), (u_0, \psi_0) \in D(A) \cap (H^4(0, L) \times H^3(0, L)), (f_t, g_t)$ is continuous in $H$ at $t = 0$, and
\[
\left\{
\begin{aligned}
u_{0xx}(0) &= u_{0xx}(L) = 0, \\
U_0 u_{0x}(0) + \psi_{0x}(0) &= f(0, 0), \\
U_0 u_{0x}(L) + \psi_{0x}(L) &= f(L, 0), \\
\right.
\end{aligned}
\right.
\tag{3.1}
\]
Indeed, we observe by differentiating the equations in (1.1) with respect to $t$ that $(u_t^\varepsilon, \psi_t^\varepsilon)$ satisfies (1.1) with $(f_t, g_t)$ instead of $(f, g)$ and the initial condition $(u_t^\varepsilon|_{t=0}, \psi_t^\varepsilon|_{t=0}) = (f(x, 0) -$
\[ \mathcal{U}_0 u_{0x} - \psi_{0x} + 2 \varepsilon u_{0xx} - g(x,0) - \mathcal{U}_0 \psi_{0x} - \lambda^{-2} u_{0x} \]. From the above assumptions, we find that \((u^\varepsilon_t|_{t=0}, \psi^\varepsilon_t|_{t=0}) \in D(A)\). Thus, by a similar argument to (i), one has \((u^\varepsilon, \psi^\varepsilon) \in C([0,T]; H) \cap L^\infty(0,T; D(A))\) and then, one deduces from (1.1) that \(\psi^\varepsilon_x \in L^\infty(0,T; H^1(0,L))\) if \(g_x \in L^\infty(0,T; L^2(0,L))\). Consequently, \((u^\varepsilon, \psi^\varepsilon) \in \mathcal{K}\).

To get \((u^0, \psi^0) \in \mathcal{K}^0\), where

\[
\mathcal{K}^0 = \{ (u, \psi) \in \mathcal{F}^0 | \psi_{xt}, u_{xx}, \psi_{xx} \in L^\infty(0,T; L^2(0,L)), (u_t, \psi_t) \in C([0,T]; H) \},
\]

we observe that under \([3]\), \(f, g, u_0, \psi_0\) satisfy the compatibility conditions (1.72) of \([3]\). Consequently, if in addition we assume that \((f_x, g_x) \in L^\infty(0,T; H)\) and \(u_{0xx}, \psi_{0xx}, f_t|_{t=0}, g_t|_{t=0} \in L^2(0,L)\), then \((u^0, \psi^0) \in \mathcal{K}^0\), see \([3\) Remark 1.4] for the detail.

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