On Linear Convergence of Adaptively Iterative Thresholding Algorithms for Compressed Sensing

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Abstract—This paper studies the convergence of the adaptively iterative thresholding (AIT) algorithm for compressed sensing. We first introduce a generalized restricted isometry property (gRIP). Then we prove that the AIT algorithm converges to the original sparse solution at a linear rate under certain gRIP condition in the noise free case. While in the noisy case, its convergence rate is also linear until attaining certain error bound. Moreover, as by-products, we also provide some sufficient conditions for the convergence of the AIT algorithm based on the two well-known properties, i.e., the coherence property and the restricted isometry property (RIP), respectively. It should be pointed out that such two properties are special cases of gRIP. The solid improvements on the theoretical results are demonstrated as compared with the known results. Finally, we provide a series of simulations to verify the correctness of the theoretical assertions as well as the effectiveness of the AIT algorithm.

Index Terms—Compressed sensing, iterative hard/soft thresholding algorithm, coherence, restricted isometry property

I. INTRODUCTION

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$, compressed sensing [1], [2] solves the following constrained $\ell_0$-minimization problem

$$
\min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s.t. } b = Ax + \epsilon,
$$

where $\epsilon \in \mathbb{R}^m$ is the noise and $\|x\|_0$ denotes the number of the nonzero components of $x$. Due to the NP-hardness of problem [1] [3], approximate methods including the greedy method and relaxed method are introduced. The greedy method approaches the sparse solution by successively incorporating one or more components that yield the greatest improvement in quality [3]. These algorithms include orthogonal matching pursuit (OMP) [4], [5], stagewise OMP (StOMP) [6], regularized OMP (ROMP) [7], compressive sampling matching pursuit (CoSaMP) [8] and subspace pursuit [9]. The greedy algorithms can be quite fast, especially when the optimal solution is ultra-sparse. However, when the signal has many non-zeros or the level of the observational noise is relatively high, the performance of the greedy algorithms is not guaranteed.

The relaxed method converts the combinatorial $\ell_0$-minimization into a more tractable model through replacing the $\ell_0$ norm with a nonnegative and continuous function $P(\cdot)$, that is,

$$
\min_{x \in \mathbb{R}^n} P(x) \quad \text{s.t. } b = Ax + \epsilon.
$$

One of the most important cases is the $\ell_1$-minimization problem (also known as basis pursuit (BP) [10] in the noise free case and basis pursuit denoising (in the noisy case) with $P(x) = \|x\|_1$, where $\|x\|_1 = \sum_{i=1}^n |x_i|$ is called the $\ell_1$ norm. The $\ell_1$-minimization problem is a convex optimization problem that can be efficiently solved. Nevertheless, the $\ell_1$ norm may not induce further sparsity when applied to certain applications [11], [12], [13], [14]. Therefore, many nonconvex functions were proposed as substitutions of the $\ell_0$ norm. Some typical nonconvex examples include the $\ell_q$ $(0 < q < 1)$ norm [11], [12], [13], smoothly clipped absolute deviation (SCAD) [15] and minimax concave penalty (MCP) [16]. Compared with the $\ell_1$-minimization model, the nonconvex relaxed models can often induce better sparsity and reduce the bias, while they are generally more difficult to solve.

The iteratively reweighted method and regularization method are two main classes of algorithms to solve [2] when $P(x)$ is nonconvex. The iteratively reweighted method includes the iteratively reweighted least square minimization (IRLS) [17], [18], and the iteratively reweighted $\ell_1$-minimization (IRL1) algorithms [13]. Specifically, the IRLS algorithm solves a sequence of weighted least squares problems, which can be viewed as some approximations to the original optimization problem. Similarly, the IRL1 algorithm solves a sequence of non-smooth weighted $\ell_1$-minimization problems, and hence it is the non-smooth counterpart to the IRLS algorithm. However, the iteratively reweighted algorithms are slow if the nonconvex penalty cannot be well approximated by the quadratic function or the weighted $\ell_1$ norm function. The regularization method transforms problem [2] into the following unconstrained optimization problem

$$
\min_{x \in \mathbb{R}^n} \{ \|Ax - b\|^2_2 + \lambda P(x) \},
$$

where $\lambda > 0$ is a regularization parameter. For some special penalties $P(x)$ such as the $\ell_q$ norms $(q = 0, 1/2, 2, 3, 1)$, SCAD and MCP, an optimal solution of the model [3] is a fixed point of the following equation

$$
x = H(x - sA^T(Ax - b)),
$$

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where $H(\cdot)$ is a componentwise thresholding operator which will be defined in detail in the next section and $s > 0$ is a step size parameter. This yields the corresponding iterative thresholding algorithm \((13, 19, 20, 21, 22, 23)\)

$$\text{Algorithm}$$

Compared to greedy methods and iteratively reweighted algorithms, iterative thresholding algorithms have relatively lower computational complexities \((24), (25), (26)\). So far, most of the theoretical guarantees of the iterative thresholding algorithms were developed for the regularization model \((3)\) with fixed parameter. One strategy is to set the regularization parameter adaptively so that $\|x(t+1)\|_0$ remains the same at each iteration. This strategy was first applied to the iterative hard thresholding algorithm (called Hard algorithm for short henceforth) \((27)\), and later the iterative soft thresholding algorithm (called Soft algorithm for short henceforth) \((28)\) in \((27)\), and \((28)\) respectively. The convergence of Hard algorithm was justified when $A$ satisfies the restricted isometry property (RIP) with $\delta_{3k^*} < \frac{1}{\sqrt{s}}$ \((27)\), where $k^*$ is the number of the nonzero components of the truly sparse signal. Later, Maleki \((28)\) investigated the convergence of both Hard and Soft algorithms in terms of the coherence. Recently, Zeng et al. \((29)\) generalized Maleki’s results to a wide class of iterative thresholding algorithms. However, most of the guarantees in \((29)\) are coherence-based and focus on the noise free case with the step size equal to 1. While it has been observed that in practice, the AIT algorithm can have remarkable performance for noisy cases with a variety of step sizes. In this paper, we develop the theoretical guarantees of the AIT algorithm with different step sizes in both noise free case and noisy case.

A. Main Contributions

The main contributions of this paper are the following:

i) Based on the introduced gRIP, we give a new uniqueness theorem for the sparsest signal \((see \text{Theorem} \((1)\))\), and then show that the AIT algorithm can converge to the original sparse signal at a linear rate \((see \text{Theorem} \((2)\))\). Specifically, in the noise free case, the AIT algorithm converges to the original sparse signal exponentially fast. While in the noisy case, it also converges to the original sparse signal at a linear rate until reaching an error bound.

ii) The tightness of our analysis is further discussed in two specific cases. The coherence based condition for the Soft algorithm is the same as those required for both OMP and BP. Moreover, the RIP based condition for the Hard algorithm is $\delta_{3k^*+1} < \sqrt{\frac{2}{s}} \approx 0.618$, which is much better than the results in \((29)\).

The reminder of this paper is organized as follows. In section II, we describe the adaptively iterative thresholding (AIT) algorithm. In section III, we introduce the generalized restricted isometry property, and then provide a new uniqueness theorem. In section IV, we prove the convergence of the AIT algorithm. In section V, we compare the obtained theoretical results with some other known results. In section VI, we implement a series of simulations to verify the correctness of the theoretical results as well as the efficiency of the AIT algorithm, and then conclude this paper in section VII.

**Notations.** We denote $N$ and $R$ as the natural number set and one-dimensional real space, respectively. For any vector $x \in \mathbb{R}^n$, $x_i$ is the $i$-th component of $x$ for $i = 1, \ldots, n$. For any matrix $A \in \mathbb{R}^{m \times n}$, $A_i$ denotes the $i$-th column of $A$. $x^T$ and $A^T$ represent the transpose of vector $x$ and matrix $A$ respectively. For any index set $S \subset \{1, \ldots, n\}, |S|$ represents its cardinality. $S^c$ is the complementary set, i.e., $S^c = \{1, \ldots, n\} \setminus S$. For any vector $x \in \mathbb{R}^n$, $x_S$ represents the subvector of $x$ with the components restricted to $S$. Similarly, $A_S$ represents the submatrix of $A$ with the columns restricted to $S$. We denote $x^* \equiv (x^*_1, \ldots, x^*_n)$ as the original sparse signal with $\|x^*\|_0 = k^*$, and $I^* = \{i : |x^*_i| \neq 0\}$ the support set of $x^*$. $I^*_r \subset \mathbb{R}^{r \times r}$ is the $r$-dimensional identity matrix. $sgn(\cdot)$ represents the signum function.

II. ADAPTIVELY ITERATIVE THRESHOLDING ALGORITHM

The AIT algorithm for \((3)\) is the following

$$z^{(t+1)} = z^{(t)} - sA^T(Az^{(t)} - b), \quad \text{ (4)}$$

$$x^{(t+1)} = H_{x^{(t+1)}}(z^{(t+1)}), \quad \text{ (5)}$$

where $s > 0$ is the step size and

$$H_{x^{(t+1)}}(x) = (h_{x^{(t+1)}}(x_1), \ldots, h_{x^{(t+1)}}(x_n))^T \quad \text{ (6)}$$

is a componentwise thresholding operator. The thresholding function $h_\tau(u)$ is defined as

$$h_\tau(u) = \begin{cases} f_\tau(u), & |u| > \tau \\ 0, & \text{otherwise} \end{cases} \quad \text{ (7)}$$

where $f_\tau(u)$ is the defining function. In the following, we give some basic assumptions of the defining function, which were firstly introduced in \((29)\).

**Assumption 1.** Assume that $f_\tau$ satisfies

1) **Odeity.** $f_\tau(u)$ is an odd function of $u$.
2) **Monotonicity.** $f_\tau(u) < f_\tau(v)$ for any $\tau \leq u < v$.
3) **Boundedness.** There exist two constants $0 \leq c_2 \leq c_1 \leq 1$ such that $u - c_1 \tau \leq f_\tau(u) \leq u - c_2 \tau$ for $u \geq \tau$.

Note that most of the commonly used thresholding functions satisfy Assumption 1. In Fig. \((1)\) we show some typical thresholding functions including hard \((21)\), soft \((19)\) and half \((13)\) thresholding functions for $\ell_0, \ell_1, \ell_1/2$ norms respectively, as well as the thresholding functions for $\ell_2/3$ norm \((20)\) and SCAD penalty \((13)\). Their corresponding boundedness parameters are shown in Table \((1)\).

This paper considers a heuristic way for setting the threshold $\tau^{(t)}$, specifically, we let

$$\tau^{(t)} = |z_{[k+1]}^{(t)}|,$$

where $z_{[k+1]}^{(t)}$ is the $(k + 1)$-th largest component of $z^{(t)}$ in magnitude and $k$ is the specified sparsity level, $[k+1]$ denotes ...
A Remark 1. We formalize the AIT algorithm. When $k$ is chosen according to property (gRIP) and then gives the uniqueness theorem.

At the $t$-th iteration, the AIT algorithm yields a sparse vector $x^{(t+1)}$ with $k$ nonzero components. The sparsity level $k$ is a crucial parameter for the performance of the AIT algorithm. While $k > k^*$, the results will get better as $k$ decreases. Once $k < k^*$, the AIT algorithm fails to find the original sparse solution. Thus, $k$ should be specified as an upper bound estimation of $k^*$.

### III. Generalized Restricted Isometry Property

This section introduces the generalized restricted isometry property (gRIP) and then gives the uniqueness theorem.

**Definition 1.** For any matrix $A \in \mathbb{R}^{m \times n}$, and a constant pair $(p, q)$ where $p \in [1, \infty)$, $q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the $(k, p, q)$-generalized restricted isometry constant (gRIC) $\beta_{k, p, q}$ of $A$ is defined as

$$
\beta_{k, p, q} = \sup_{S \subseteq \{1, \ldots, n\}, |S| \leq k} \sup_{z \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|t^T (A^T A - I_n) z\|_2}{\|z\|_p} \geq 1.
$$

We will show that the introduced gRIP satisfies the following proposition.

**Proposition 1.** For any positive constant pair $(p, q)$ with $\frac{1}{p} + \frac{1}{q} = 1$, the generalized restricted isometric constant $\beta_{k, p, q}$ associated with $A$ and $k$ must satisfy

$$
\beta_{k, p, q} \leq \sup_{z \in \mathbb{R}^n \setminus \{0\}, \|z\|_p \leq k} \frac{\|z^T (A^T A - I_n) z\|_2}{\|z\|_p^2} \leq \beta_{k, p, q}. \tag{9}
$$

**Proof:**

For any index set $S \subseteq \{1, \ldots, n\}$ with $|S| \leq k$ and a vector $x \in \mathbb{R}^{|S|}$, since $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell_p$ and $\ell_q$ norms are dual to each other, which implies that

$$
\|t^T (I_{|S|} - A^T S) x\|_q = \sup_{y \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|y^T (I_{|S|} - A^T S) x\|_p}{\|y\|_p}. \tag{10}
$$

By Definition 1 then

$$
\beta_{k, p, q} = \sup_{|S| \leq k} \sup_{x, y \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|y^T (I_{|S|} - A^T S) x\|_p}{\|x\|_p \|y\|_p}. \tag{11}
$$

It is obvious that

$$
\beta_{k, p, q} \geq \sup_{|S| \leq k} \sup_{x, y \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|x^T (I_{|S|} - A^T A S) x\|_p}{\|x\|_p \|y\|_p} \frac{\|y^T (I_{|S|} - A^T A S) x\|_p}{\|y\|_p} \quad \text{subject to } x, y \in \mathbb{R}^{|S|}. \tag{12}
$$

and for any $x, y \in \mathbb{R}^{|S|}$,

$$
\|y^T x - y^T A^T S Ax\|_p \leq \frac{1}{2} \|x\|_2^2 + \|y\|_2^2 - \|x - y\|_2^2.
$$

Furthermore, it can be noted that

$$
\beta_{k, p, q} \leq \sup_{u, v \in \mathbb{R}^{|S|}, \|u\|_p \leq 1} \frac{\|u - v\|_2^2 - \|A_S (u - v)\|_2^2}{\|u\|_p^2 - \|A_S w\|_2^2} \leq 4 \sup_{u, v \in \mathbb{R}^{|S|}, \|u\|_p \leq 1} \frac{\|u\|_2^2 - \|A_S w\|_2^2}{\|u\|_p^2 - \|A_S w\|_2^2}, \tag{14}
$$

since $\|u - v\|_p \leq 2$ for $\|u\|_p \leq 1$ and $\|v\|_p \leq 1$. Plugging (13) and (14) into (12), it yields

$$
\beta_{k, p, q}^A \leq 3 \sup_{|S| \leq k} \sup_{\|x\|_p \leq 1} \frac{\|x\|_2^{p^2} - \|A_S x\|_2^{p^2}}{\|x\|_p^2 - \|A_S w\|_2^2}, \tag{15}
$$

where (8) and (9) can be obtained by setting $\beta_{k, p, q}^A$.
which implies the left-hand side of (3). Therefore, the proof of this proposition is completed.

It can be noted that the gRIP closely relates to the coherence property and restricted isometry property (RIP), whose definitions are listed in the following.

**Definition 2.** For any matrix $A \in \mathbb{R}^{m \times n}$, the coherence of $A$ is defined as
\[
\mu = \max_{i \neq j} \frac{|\langle A_i, A_j \rangle|}{\|A_i\|_2 \cdot \|A_j\|_2},
\]
where $A_i$ denotes the $i$-th column of $A$ for $i = 1, \ldots, n$.

**Definition 3.** For any matrix $A \in \mathbb{R}^{m \times n}$, given $1 \leq k \leq n$, the restricted isometry constant (RIC) of $A$ with respect to $k$, $\delta_k$, is defined to be the smallest constant $\delta$ such that
\[
(1 - \delta)\|z\|_2^2 \leq \|Az\|_2^2 \leq (1 + \delta)\|z\|_2^2,
\]
for all $k$-sparse vector, i.e., $\|z\|_0 \leq k$.

By Definition 2 RIC can also be written as:
\[
\delta_k = \sup_{z \in \mathbb{R}^n \setminus \{0\}, \|z\|_0 \leq k} \frac{\|z^T (A^T A - I_n) z\|_2}{\|z\|_2^2},
\]
which is very similar to the right-hand side of (3). In fact, Proposition 2 shows that coherence and RIP are two special cases of gRIP.

**Proposition 2.** For any column-normalized matrix $A \in \mathbb{R}^{m \times n}$, that is, $\|A_j\|_2 = 1$ for $j = 1, \ldots, n$, it holds
(i) $\beta_{k,1,\infty} \geq \mu$, for $2 \leq k \leq n$.
(ii) $\beta_{2,2,2} = \delta_k$, for $1 \leq k \leq n$.

**Proof:** (i) The definition of gRIP induces $\beta_{k,1,\infty} \geq \beta_{2,1,\infty}$ for all $k \geq 2$. Therefore, if we can claim the following two facts: (a) $\beta_{2,1,\infty} \geq \mu$, and (b) $\beta_{k,1,\infty} \leq \mu$ for all $k \geq 2$, then Proposition 2 (ii) follows.

We first justify the fact (a). Suppose the maximal element of $I_n - A^T A$ in magnitude appears at the $i_0$-th row and the $j_0$-th column. Since the diagonal elements of $I_n - A^T A$ are zero, we know $i_0 \neq j_0$. Without loss of generality, we assume that $i_0 < j_0$. Let $A_{i_0}$ and $A_{j_0}$ be the $i_0$-th and $j_0$-th column vector of $A$, respectively, then Definition 2 gives
\[
\mu = \|A_i A_{j_0}\|_1.
\]
Let $S = \{i_0, j_0\}$ and $e = (0, 1)^T$. Then
\[
\beta_{2,1,\infty} \geq \|(I_2 - A_i^T A_i) e\|_\infty = \|e - A_i^T A_{j_0}\|_\infty = \mu.
\]

Then we prove the fact (b). For any vector $x \in \mathbb{R}^k$ and a subset $S \subset \{1, 2, \ldots, n\}$ with $|S| = k$, let $B = I_k - A_S^T A_S$ and $z = Bx$. Then
\[
|z_i| = \sum_{j=1}^k B_{ij} x_j \leq \sum_{j=1}^k |B_{ij} x_j| \leq \mu \|x\|_1,
\]
for any $i = 1, \ldots, k$. It implies that
\[
\|Bx\|_\infty \leq \mu \|x\|_1.
\]
By the definition of $\beta_{k,1,\infty}$, it implies
\[
\beta_{k,1,\infty} \leq \mu.
\]
According to (19) and (20), for all $2 \leq k \leq n$, it holds
\[
\beta_{k,1,\infty} = \mu.
\]
(ii) The proof of this part can be referred to the Definition 1 in [10].

**A. Uniqueness Theorem Characterized via gRIP**

We first give a lemma to show the relation between two different norms for a $k$-sparse vector space.

**Lemma 1.** For any vector $x \in \mathbb{R}^n$ with $\|x\|_0 = k \leq n$, and for any $1 \leq q \leq p \leq \infty$, then
\[
\|x\|_p \leq \|x\|_q \leq k^{\frac{1}{q} - \frac{1}{p}} \|x\|_p.
\]
This lemma is trivial based on the well-known norm equivalence theorem so the proof is omitted. Note that Lemma 1 is equivalent to
\[
\|x\|_p \leq k^{\max\left\{\frac{1}{p} - \frac{1}{q}, 0\right\}} \|x\|_q, \forall p, q \in [1, \infty].
\]
With Lemma 1 the following theorem shows that a $k$-sparse solution of the equation $Ax = b$ is the unique sparsest solution if $A$ satisfies certain gRIP condition.

**Theorem 1.** Let $x^*$ be a $k$-sparse solution of $Ax = b$. If $A$ satisfies $(2k, p, q)$-gRIP with
\[
0 < \beta_{2k, p, q} < (2k)\min\{\frac{1}{p} - \frac{1}{q}, 0\},
\]
then $x^*$ is the unique sparsest solution.

**Proof:** We prove this theorem by contradiction. Assume $x^{**}$ satisfies $Ax^{**} = b$ and $\|x^{**}\|_0 \leq k$. Then
\[
A(x^* - x^{**}) = 0,
\]
which implies
\[
(I_n - A^T A)(x^* - x^{**}) = x^* - x^{**}.
\]
Let $x = x^* - x^{**}$, $S$ be the support of $x$ and $x_S$ be a subvector of $x$ with the components restricted to $S$. It follows
\[
(I_{|S|} - A_S^T A_S)x_S = x_S,
\]
and further
\[
\|(I_{|S|} - A_S^T A_S)x_S\|_q = \|x_S\|_q,
\]
for any $q \in [1, \infty]$. Since $\|x^*\|_0 \leq k$ and $\|x^{**}\|_0 \leq k$, then $|S| \leq 2k$. For any $p \in [1, \infty)$, and by the definition of gRIP, we have
\[
\|(I_{|S|} - A_S^T A_S)x_S\|_p \leq \beta_{2k, p, q} \|x_S\|_p.
\]
By Lemma 1 there holds
\[
\|x_S\|_p \leq (2k)^{\max\left\{\frac{1}{p} - \frac{1}{q}, 0\right\}} \|x_S\|_q.
\]
By the assumption of this theorem, then
$$\| (I_{|S|} - A_S^T A_S) x_S \|_q \leq \beta_{2k,p,q}(2k)^{\max \{ \frac{1}{p} - \frac{1}{q}, 0 \}} \| x_S \|_q$$
$$< \| x_S \|_q,$$
which contradicts with (23). Therefore, $x^*$ is the unique sparsest solution.

According to Proposition 2 and Theorem 1 we can obtain the following uniqueness results characterized via coherence and RIP, respectively.

**Corollary 1.** Let $x^*$ be a $k$-sparse solution of the equation $Ax = b$. If $\mu$ satisfies
$$0 < \mu < \frac{1}{2k},$$
then $x^*$ is the unique sparsest solution.

It was shown in [31] that when $\mu < \frac{1}{2k}$, the $k$-sparse solution should be unique. In another perspective, it can be noted that the condition $\mu < \frac{1}{2k}$ is equivalent to $k > \frac{1}{2 \mu}$ while $\mu < \frac{1}{2k} \frac{1}{2}$ is equivalent to $k < \frac{1}{2 \mu} + \frac{1}{2}$. Since $k$ should be an integer, these two conditions are almost the same.

**Corollary 2.** Let $x^*$ be a $k$-sparse solution of the equation $Ax = b$. If $\delta_{2k}$ satisfies
$$0 < \delta_{2k} < 1,$$
then $x^*$ is the unique sparsest solution.

According to [32], the RIP condition obtained in Corollary 2 is the same as the state-of-the-art result and more importantly, is tight in the sense that once the condition is violated, then we can construct a sparse signal that cannot be reconstructed by any method.

IV. CONVERGENCE ANALYSIS

In this section, we will study the convergence of the AIT algorithm based on the introduced gRIP.

A. Characterization via gRIP

Before justifying the convergence of the AIT algorithm based on gRIP, we first introduce three lemmas.

**Lemma 2.** For any $x, y \in \mathbb{R}^n$, and $p \in [1, \infty)$, then
$$\|x + y\|_p \leq 2^{p-1}(\|x\|_p + \|y\|_p).$$

Moreover, if $x_i \cdot y_i \geq 0$ for $i = 1, \ldots, n$, then
$$\|x + y\|_p \geq \|x\|_p + \|y\|_p.$$ (25)

**Proof:** To prove these two inequalities, we first consider the special case when $n = 1$. Let $z_1, z_2$ be any two real numbers, then we claim
$$|z_1 + z_2|^p \leq 2^{p-1}(|z_1|^p + |z_2|^p),$$
and furthermore, if $z_1 \cdot z_2 \geq 0$, then
$$|z_1|^p + |z_2|^p \leq |z_1 + z_2|^p.$$ (26)

To prove (26), we first observe that
$$\left|\frac{z_1 + z_2}{2}\right|^p \leq \left|\frac{z_1}{2}\right|^p + \left|\frac{z_2}{2}\right|^p.$$ Since $f(z) = z^p$ is convex for $p \geq 1$ and any $z \geq 0$, Jensen’s inequality implies
$$\left|\frac{z_1}{2}\right|^p + \left|\frac{z_2}{2}\right|^p \leq \frac{|z_1|^p + |z_2|^p}{2}.$$ The above two inequalities yields (26).

Next, we will prove (27). If $z_1 \cdot z_2 \geq 0$, then
$$|z_1 + z_2|^p = (|z_1| + |z_2|)^p.$$ Moreover, $p \geq 1$ induces
$$\sqrt{|z_1|^p + |z_2|^p} \leq |z_1| + |z_2|.$$ Combining the two inequalities yields (27). Applying (26) and (27) to every coordinate gives (24) and (25).

**Lemma 3.** For any $t \geq 1$ and $q \in [1, \infty)$, if $k \geq k^*$, the following inequality holds for the AIT algorithm:
$$\tau(t) \leq \left( \sum_{i \in I_+^k} |x_i(t) - x_i^*|^q \right)^{1/q} = \| x_{I_+^k}^* - x_{I_+^k}^* \|_q,$$ (28)
where $I_+^k$ is the index set of the largest $k + 1$ components of $z(t)$ in magnitude.

**Proof:** When $q = \infty$, we need to show
$$\tau(t) \leq \max_{i \in I_+^k} |x_i(t) - x_i^*|,$$ (29)
then Lemma 1 shows that (28) holds for all $q \in [1, \infty]$.

Let $I^t$ be the index set of the largest $k$ components of $z(t)$ in magnitude, then $I_+^k = I^t \cup \{k + 1\}$, where $k + 1$ represents the index of the $(k + 1)$-th largest component of $z(t)$ in magnitude. We will prove (29) in the following two cases.

Case (i). If $I^* = I^t$, then
$$\tau(t) = |z_{k+1}(t)| = |x_{k+1}^* - x_{k+1}^*| \leq \max_{i \in I_+^k} |x_i(t) - x_i^*|.$$ (30)

Case (ii). If $I^* \neq I^t$, then there exists $i_0 \in I^t$ such that
$$i_0 \notin I^*.$$ Otherwise $I^t \subset I^*$ and $I^t \neq I^*$ which contradicts with $|I^t| \geq k^*$ and $|I^*| = k^*$. Thus, $x_{i_0}^* = 0$ and
$$\tau(t) = |x_{i_0}(t)| \leq |x_{i_0}(0)| - x_{i_0}^* \leq \max_{i \in I_+^k} |x_i(t) - x_i^*|.$$ (31)

Combining (30) and (31) gives (29). To describe the convergence of the AIT algorithm, we first define
$$L_1 = 2^{p-1}(k^*)^{\max \{ 1 - \frac{1}{q}, 0 \} + (2^{p-1} - (c_2)^p + 1)k^*},$$
$$L_2 = 2^p(2k^*)^{\max \{ 1 - \frac{1}{q}, 0 \} + 2^{p-1}(c_1^p)k^*},$$
and 
\[ L = \min\{\sqrt[2]{L_1}, \sqrt[2]{L_2}\}, \]

where \( p \in [1, \infty) \), \( q \in [1, \infty) \) and \( c_1, c_2 \) are the corresponding boundedness parameters.

**Theorem 2.** Let \( \{x(t)\} \) be a sequence generated by the AIT algorithm. Assume that \( A \) satisfies \((3k^* + 1, p, q)\)-gRIP with the constant \( \beta_{3k^*+1, p, q} < \frac{1}{L} \), and let

(i) \( k = k^* \);
(ii) \( \gamma < s < \frac{\kappa}{L} \), where

\[
\kappa = \frac{(2k^*)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} - \frac{1}{L}}{(2k^*)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} - \beta_{3k^*+1, p, q}},
\]

and

\[
\gamma = \frac{(2k^*)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} + \frac{1}{L}}{(2k^*)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} + \beta_{3k^*+1, p, q}}.
\]

Then
\[
\|x(t) - x^*\|_p \leq (\rho_s)^t \|x^* - x(0)\|_p + \frac{sL}{1 - \rho_s} \|A^T \epsilon\|_q,
\]

where \( \rho_s = \gamma < s \leq \frac{\kappa}{L} \) with

\[
\gamma = \frac{1}{L} - (2k^*)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} + \beta_{3k^*+1, p, q}.
\]

Particularly, when \( \epsilon = 0 \), it holds
\[
\|x(t) - x^*\|_p \leq (\rho_s)^t \|x^* - x(0)\|_p.
\]

**Proof:** By the Step 1 of Algorithm 1, for any \( t \in N \),
\[
z(t+1) = x(t) - sA^T (Ax(t) - b),
\]

and we note that \( b = Ax^* + \epsilon \), then
\[
z(t+1) - x^* = (I_n - sA^T A)(x(t) - x^*) + sA^T \epsilon
\]
\[= (1 - s)(x(t) - x^*) + s(I_n - A^T A)(x(t) - x^*) + sA^T \epsilon.
\]

For any \( t \in N \) and \( q \in [1, \infty) \), let \( S^t = I^{t+1} \cup I^t \cup I^* \). Noting that \( I^t, I^* \subset S^t \), it follows
\[
A(x(t) - x^*) = A_{S^t}(x_{S^t}(t) - x_{S^t}^*).
\]

Then we have
\[
z_{S^t}(t+1) - x_{S^t}^* = (1 - s)(x_{S^t}(t) - x_{S^t}^*)
\]
\[+ s(I_{S^t}| - A_{S^t}^T A_{S^t})(x_{S^t}(t) - x_{S^t}^*) + sA_{S^t}^T \epsilon.
\]

Therefore,
\[
\|z_{S^t}(t+1) - x_{S^t}^*\|_q \leq (1 - s)\|x_{S^t}(t) - x_{S^t}^*\|_q
\]
\[+ s\|I_{S^t}| - A_{S^t}^T A_{S^t})(x_{S^t}(t) - x_{S^t}^*)\|_q + s\|A_{S^t}^T \epsilon\|_q. \tag{32}
\]

Since \( \|x(t)\|_0 \leq k = k^* \) and \( \|x^*\|_0 = k^* \) then
\[|I^t| \leq k^*, |I^{t+1}| \leq k^* + 1, |I^*| = k^* \),

and hence \( |S^t| \leq 3k^* + 1 \). For any \( p \in [1, \infty) \), by (22) and the definition of gRIP [8], it holds
\[
\|x(t) - x^*\|_p \leq (2k^*)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} \|x(t) - x^*\|_p, \tag{33}
\]

and
\[
\|I_{S^t}| - A_{S^t}^T A_{S^t})(x_{S^t}(t) - x_{S^t}^*)\|_q
\]
\[\leq \beta_{3k^*+1, p, q} \|x_{S^t}(t) - x_{S^t}^*\|_p = \beta_{3k^*+1, p, q} \|x(t) - x^*\|_p. \tag{34}
\]

Plugging (33) and (34) into (32), then
\[
\|z_{S^t}^{(t+1)} - x_{S^t}^*\|_q
\]
\[\leq \left( 1 - s \right) \left( 2k^* \right)^{\max\left(\frac{1}{2} - \frac{1}{p}, 0\right)} + s\beta_{3k^*+1, p, q} \|x(t) - x^*\|_p
\]
\[+ s\|A_{S^t}^T \epsilon\|_q.
\]

If we can further prove for any \( t \geq 1 \), it holds
\[
\|x(t) - x^*\|_p \leq L \|z_{S^t}^{(t-1)} - x_{S^t}^*\|_q. \tag{36}
\]

Then combining (35) and (36), it holds
\[
\|x(t+1) - x^*\|_p \leq L \|z_{S^t}^{(t)} - x_{S^t}^*\|_q \leq \rho_s \|x(t) - x^*\|_p + sL \|A^T \epsilon\|_q.
\]

Since \( 0 < \rho_s < 1 \) under the assumption of this theorem, then by induction for any \( t \geq 1 \), we have
\[
\|x(t) - x^*\|_p \leq (\rho_s)^t \|x^* - x(0)\|_p + \frac{sL}{1 - \rho_s} \|A^T \epsilon\|_q.
\]

Now we turn to the proof of (36). We will prove it in two steps.

**Step a:** For any \( p \in [1, \infty) \),
\[
\|x(t) - x^*\|_p = \|x_{I^t} - x_{I^t}^*\|_p + \|x_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p. \tag{37}
\]

By Lemma 2
\[
\|x_{I^t} - x_{I^t}^*\|_p \leq \|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p
\]
\[\leq 2\|x_{I^t}^* - x_{I^t}^*\|_p + 2\|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p. \tag{38}
\]

Moreover, by the Step 3 of Algorithm 1 and Assumption 1, for any \( i \in I^t \):
\[
\text{sgn}(x_i(t)) = \text{sgn}(z_i(t)) \text{ and } |x_i(t)| \leq |z_i(t)|.
\]

Thus, for any \( i \in I^t \setminus I^* \), it holds
\[
x_i(t) \cdot (z_i(t) - x_i(t)) \geq 0. \tag{39}
\]

With (39) and by Lemma 2 we have
\[
\|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p \geq \|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p + \|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p. \tag{40}
\]

Plugging (38) and (40) into (37), it becomes
\[
\|x(t) - x^*\|_p \leq 2\|x_{I^t}^* - x_{I^t}^*\|_p + 2\|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p
\]
\[+ \|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p. \tag{41}
\]

Furthermote, by the Step 2 of Algorithm 1, Assumption 1 and Lemma 3 for any \( t \geq 1 \), we have:

(a) if \( i \in I^t \), \( c_2 \tau_i(t) \leq \tau_i(t) \leq c_1 \tau_i(t) \leq \tau_i(t) \);

(b) if \( i \notin I^t \), \( |z_i(t) - x_i(t)| \leq |z_i(t)| \leq \tau_i(t) \);

(c) \( \tau_i(t) \leq |z_i(t) - x_i(t)| \).

By the above facts (a)-(c), it holds
\[
\|z_{I^t \setminus I^*} - x_{I^t \setminus I^*}^*\|_p \leq k^* \max_{i \in I^*} |z_i(t) - x_i(t)| \leq k^* \|\tau_i(t)\|_p. \tag{42}
\]
where $|I^*|$ represents the cardinality of the index set $I^*$. Plugging (43), (44) into (41), it follows
\[
\|x(t) - x^*\|_p^p \leq 2^{p-1}\|z(t)_{I^*'} - x_{I^*'}\|_p^p + \|z(t)_{I^*\setminus I^*'}\|_p^p + (2^{p-1}k^* - (c_2p|I^* |)^\tau(t)^p, (44)
\]

Therefore, we note that
\[
\|z(t)_{I^*'}\|_p^p = \max_{i \in I^*'} \|z(t)_i - x_i\|_p^p \\
\leq |I^*| \cdot \max_{i \in I^*'} \|z(t)_i - x_i\|_p^p \\
= |I^*| \cdot \|z(t)_{I^*'} - x_{I^*'}\|_\infty^p,
\]

where the first equality holds because $x_{I^*'} = 0$, and the second inequality holds because of Lemma 1. Therefore, (44) becomes
\[
\|x(t) - x^*\|_p^p \leq 2^{p-1}\|z(t)_{I^*'} - x_{I^*'}\|_p^p + |I^*| \cdot \|z(t)_{I^*'} - x_{I^*'}\|_\infty^p + (2^{p-1}k^* - (c_2p|I^* |)^\tau(t)^p \|z(t)_{I^*\setminus I^*'}\|_p^p \\
\leq 2^{p-1}\|z(t)_{I^*'} - x_{I^*'}\|_p^p + (2^{p-1}k^* - (c_2p|I^* |)^\tau(t)^p \|z(t)_{I^*\setminus I^*'}\|_p^p \\
\leq 2^{p-1}(k^*\max(1 - \frac{p}{q}, 0))\|z(t)_{I^*'} - x_{I^*'}\|_p^p + (2^{p-1} - (c_2p + 1)k^*\|z(t)_{I^*\setminus I^*'}\|_p^p \\
\leq L_1\|z(t)_{S_{I^*'} - I^*'} - x_{S_{I^*'} - I^*'}\|_p^p.
\]

where the second inequality holds by the fact (c), i.e., $\tau(t) \leq \|z(t)_{I^*'} - x_{I^*'}\|_\infty^p$, the third inequality holds by Lemma 1 and $|I^* | \cdot \|z(t)_{I^*'} - x_{I^*'}\|_\infty^p$. The last inequality holds because $S^t_{I^*} = I^* \cup I^* \setminus I^* \cup I^*$. Thus, it implies
\[
\|x(t) - x^*\|_p^p \leq \sqrt[L]{L_1}\|z(t)_{S_{I^*'} - I^*'} - x_{S_{I^*'} - I^*'}\|_q^q.
\]

**Step b):** By Lemma 2
\[
\|x(t) - x^*\|_p^p = \|z(t)_{I^*'} - x_{I^*'}\|_p^p + \|z(t)_{I^*\setminus I^*'}\|_p^p \\
\leq 2^{p-1}\|z(t)_{I^*'} - x_{I^*'}\|_p^p + 2^{p-1}\|z(t)_{I^*\setminus I^*'}\|_p^p \\
+ 2^{p-1}\|z(t)_{I^*'} - x_{I^*'}\|_p^p + 2^{p-1}\|z(t)_{I^*\setminus I^*'}\|_p^p \\
= 2^{p-1}\|z(t)_{I^*'} - x_{I^*'}\|_p^p + 2^{p-1}\|z(t)_{I^*\setminus I^*'}\|_p^p.
\]

Moreover, by Lemma 1 it holds
\[
\|z(t)_{I^*'} - x_{I^*'}\|_p^p \leq |I^* |\|z(t)_{I^*'} - x_{I^*'}\|_p^p + |I^*\setminus I^*'|\|z(t)_{I^*\setminus I^*'}\|_p^p \\
\leq (2k^*\max(1 - \frac{p}{q}, 0))\|z(t)_{I^*'} - x_{I^*'}\|_p^p + (2k^*\max(1 - \frac{p}{q}, 0))\|z(t)_{I^*\setminus I^*'}\|_p^p.
\]

where the last inequality holds for $|I^* \setminus I^*| \leq 2k^*$. We also have
\[
\|z(t)_{I^*'} - x_{I^*'}\|_p^p \leq k^*\max_{i \in I^*} |z(t)_i - x_i|_p^p \\
\leq k^* (c_1\tau(t))^p \leq k^* (c_1)^p \|z(t)_{I^*'} - x_{I^*'}\|_p^p.
\]

Since $|I^* | = |I^* | = k^*$, then
\[
|I^* \setminus I^*| = |I^* \setminus I^*|.
\]

Thus, it holds
\[
\|z(t)_{I^*'} - x_{I^*'}\|_p^p \leq |I^* | \cdot \|z(t)_{I^*'} - x_{I^*'}\|_p^p \\
\leq |I^* | \cdot \|z(t)_{I^*\setminus I^*'}\|_p^p \\
\leq (k^*)^\max(1 - \frac{p}{q}, 0)\|z(t)_{I^*\setminus I^*'}\|_p^p.
\]

Plugging (48) and (49) into (47), further since $S_t = I^* \setminus I^* \setminus I^* = I^* \setminus I^* \\
\leq k^* \max(1 - \frac{p}{q}, 0)$, and thus $S_t \subseteq S_t^t = I^* \setminus I^* \\
\leq k^* \max(1 - \frac{p}{q}, 0)$, then $S_t \subseteq S_t^t = I^* \setminus I^*$. It becomes
\[
\|x(t) - x^*\|_p^p \leq (2k^*)^\max(1 - \frac{p}{q}, 0) + 2^{p-1}(c_1p^k)\|z(t)_{S^t_{I^*'} - I^*'} - x_{S^t_{I^*'} - I^*'}\|_q^q.
\]

Thus, we have
\[
\|x(t) - x^*\|_p \leq \sqrt[L]{1}\|z(t) - x^*\|_q.
\]

From (46) and (52), for any $t \geq 1$, it holds
\[
\|x(t) - x^*\|_p \leq \min(\sqrt[L]{1}, \sqrt[L]{2})\|z(t) - x^*\|_q \\
= L\|z(t) - x^*\|_q.
\]

Therefore, we end the proof of this theorem.

Under the conditions of this theorem, we can verify that $0 < \rho_s < 1$. We first note that $\beta_{3k^*+1,p,q} < \frac{1}{2} < 1 \leq (2k^*)^\max(1 - \frac{p}{q}, 0)$, then it holds $0 < \rho_s < 1$. The definition of $\gamma_s$ gives $\gamma_s = \left\{ \begin{array}{ll} (1 - s)(2k^*)^\max(1 - \frac{\rho_s}{p}, 0) + s\beta_{3k^*+1,p,q}, & \text{if } s < s \leq 1 \\ (s - 1)(2k^*)^\max(1 - \frac{\rho_s}{p}, 0) + s\beta_{3k^*+1,p,q}, & \text{if } 1 < s < \frac{\rho_s}{\gamma_s} \\ \end{array} \right.$

If $s < 1$, it holds
\[
\gamma_s < (1 - s)(2k^*)^\max(1 - \frac{\rho_s}{p}, 0) + s\beta_{3k^*+1,p,q} = \frac{1}{L}.
\]

Similarly, if $1 < s \leq \frac{\rho_s}{\gamma_s}$
\[
\gamma_s < (s - 1)(2k^*)^\max(1 - \frac{\rho_s}{p}, 0) + s\beta_{3k^*+1,p,q} = \frac{1}{L}.
\]

Therefore, we have $\gamma_s < \frac{1}{L}$ and thus, $\rho_s = \gamma_s L < 1$.

As shown in Theorem 3, it demonstrates that in the noise free case, the AIT algorithm converges to the original sparse signal exponentially fast, while in the noisy case, it also converges exponentially fast until reaching an error bound. Moreover, it can be noted that the constant $\rho_s$ depends on the step size $s$. Since $\beta_{3k^*+1,p,q} < \frac{1}{2} < (2k^*)^\max(1 - \frac{\rho_s}{p}, 0)$, $\rho_s$ reaches its minimum at $s = 1$. The trend of $\rho_s$ with respect to $s$ is shown in Fig. 2.
By Proposition 2, it shows that the coherence and RIP are two special cases of gRIP, thus we can easily obtain some recovery guarantees based on coherence and RIP respectively in the next two subsections.

B. Characterization via Coherence

Let \( p = 1, q = \infty \). In this case, \( L_1 = (3 - c_2)k^* \), \( L_2 = (4 + c_1)k^* \), and \( L = (3 - c_2)k^* \). According to Theorem 2 and Proposition 2, assume that \( \mu < \frac{1}{(3 - c_2)k^*} \), then the AIT algorithm converges linearly with the convergence rate constant

\[
\rho_s = \gamma_s L = (|1 - s| + s\mu)L < 1
\]

if we take \( k = k^* \) and \( \frac{1 - \frac{k}{L}}{1 - \frac{k}{\mu}} < s < \frac{1 + \frac{k}{L}}{1 + \frac{k}{\mu}} \). In the following, we show that the constant \( \gamma_s \) and thus \( \rho_s \) can be further improved when \( p = 1 \) and \( q = \infty \).

**Theorem 3.** Let \( \{x^{(t)}\} \) be a sequence generated by the AIT algorithm for \( b = Ax + \epsilon \). Assume that \( A \) satisfies 0 < \( \mu < \frac{1}{(3 - c_2)k^*} \), and we take

(i) \( k = k^* \);  
(ii) \( 1 - \frac{k}{L} < s < \min\{\frac{1}{\mu}, 1 + \frac{k}{L}\} \),

then it holds

\[
\|x^{(t)} - x^*\|_1 \leq (\rho_s)^t \|x^* - x^{(0)}\|_1 + \frac{sL}{1 - \rho_s} \|AT \epsilon\|_\infty,
\]

where \( \rho_s = \gamma_s L < 1 \) with

\[
\gamma_s = \max\{|1 - s|, s\mu\}.
\]

**Particularly, when** \( \epsilon = 0 \), **it holds**

\[
\|x^{(t)} - x^*\|_1 \leq (\rho_s)^t \|x^* - x^{(0)}\|_1.
\]

_Proof:_ The proof is similar to that of Theorem 2. According to the proof of Theorem 2, we have known that \( 32 \)–\( 34 \) hold for all pairs of \( (p, q) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), and thus obviously hold for \( p = 1 \) and \( q = \infty \). In the following, instead of the inequality \( 35 \), we will derive a tighter upper bound of \( \|z^{(t+1)}_{S_i} - x^*_S\|_\infty \), that is,

\[
\|z^{(t+1)}_{S_i} - x^*_S\|_\infty \leq \max\{\mu S, |1 - s|\} \|x^{(t)} - x^*\|_1 + s \|AT \epsilon\|_\infty.
\]  \( 54 \)

Now we turn to prove the inequality \( 54 \). According to \( 3 \), it can be observed that

\[
\|z^{(t+1)}_{S_i} - x^*\|_\infty \leq \|((1 - s)I_n + s(I_n - AT A))(x^{(t)} - x^*)\|_\infty + s \|AT \epsilon\|_\infty.
\]

Let \( B = (1 - s)I_n + s(I_n - AT A) \) and \( B_{ij} \) be the \((i, j)\)-th element of \( B \). Since \( \|A_j\|_2 = 1 \) for all \( j = 1, \ldots, n \), then

\[
B_{ii} = 1 - s,
\]

for all \( i = 1, \ldots, n \). Moreover, by the definition of the coherence \( \mu \), the absolutes of all the off-diagonal elements of \( I_n - AT A \) are no bigger than \( \mu \). Thus,

\[
|B_{ij}| \leq s\mu,
\]

for any \( i \neq j \). As a consequence, it holds

\[
\max_{i,j \in \{1, \ldots, n\}} |B_{ij}| \leq \max\{|1 - s|, s\mu\} = \gamma_s.
\]

Furthermore, for any \( i = 1, \ldots, n \),

\[
\|z^{(t+1)}_{S_i} - x^*_S\|_\infty \leq \|z^{(t)}_{S_{i-1}} - x^*_S\|_\infty + s \|AT \epsilon\|_\infty.
\]

This implies

\[
\|z^{(t+1)}_{S_i} - x^*_S\|_\infty \leq \gamma_s \|z^{(t)}_{S_{i-1}} - x^*_S\|_\infty + s \|AT \epsilon\|_\infty.
\]

Therefore, we obtain the \( 54 \). According to the proof of Theorem 2, we have that the inequality \( 55 \) still holds when \( p = 1 \) and \( q = \infty \), that is,

\[
\|x^{(t)} - x^*\|_1 \leq L \|z^{(t)}_{S_{i-1}} - x^*_S\|_\infty.
\]

Similar to the rest of the proof of Theorem 2 combining \( 54 \) and \( 55 \), we can conclude the proof of this theorem.

As shown in Theorem 3, the constant \( \gamma_s \) can be improved from \(|1 - s| + s\mu \) to \( \max\{|1 - s|, s\mu\} \), and also the feasible range of the step length parameter \( s \) gets larger from \( \frac{1 - \frac{k}{L}}{1 - \frac{k}{\mu}} \) to \( \frac{1}{\mu}, \min\{\frac{1}{\mu}, 1 + \frac{k}{L}\} \). We list the coherence-based convergence conditions of several typical AIT algorithms in Table II. As shown in Table II, it can be observed that the recovery condition for the Soft algorithm is the same as those of OMP, MP and BP.

| Table II: Coherence Based Conditions for Different AIT Algorithms |
| --- |
| **Algorithm** | **Hard** | **Half** | **Soft** | **SCAD** |
| \( c_2 \) | 0 | 0 | 1 | 0 |
| \( \mu \) | \( \frac{1}{(1 + \frac{k}{L})} \) | \( \frac{1}{(1 + \frac{k}{\mu})} \) | \( \frac{1}{(1 + \frac{k}{\mu})} \) |

C. Characterization via RIP

Let \( p = 2, q = 2 \). In this case, \( L_1 = 2 + (3 - c_2^2)k^* \), \( L_2 = 4 + 2c_2^2k^* \), and thus

\[
L = \min\{\sqrt{4 + 2c_2^2k^*}, \sqrt{2 + (3 - c_2^2)k^*}\}.
\]

According to Theorem 2 and by Proposition 2, we can directly claim the following corollary.
Corollary 3. Let \{x^{(t)}\} be a sequence generated by the AIT algorithm for \(b = Ax + \epsilon\). Assume that \(A\) satisfies \(\delta_{3k^*+1} < \frac{1}{2}\), and if we take

(i) \(k = k^*\);

(ii) \(s \leq s \leq \bar{s}\), where \(s = \frac{1 - \frac{k}{q}}{1 - \delta_{3k^*+1}}\) and \(\bar{s} = \frac{1 + \frac{k}{q}}{1 - \delta_{3k^*+1}}\).

Then
\[
\|x^{(t)} - x^*\|_2 \leq (\rho_s)^t \|x^* - x^{(0)}\|_2 + \frac{sL}{\delta_{3k^*+1}}\|A^T\epsilon\|_2, 
\]
where \(\rho_s = \gamma_sL < 1\) with \(\gamma_s = |1-s| + s\delta_{3k^*+1}\). Particularly, when \(\epsilon = 0\), it holds
\[
\|x^{(t)} - x^*\|_2 \leq (\rho_s)^t \|x^* - x^{(0)}\|_2, 
\]

According to Corollary 3 the RIP based sufficient conditions for some typical AIT algorithms are listed in Table III.

| AIT       | Hard | Half | Soft | SCAD |
|-----------|------|------|------|------|
| \(c_1\)   | 0    | \(\frac{1}{3}\) | \(1\) | \(1\) |
| \(\delta_{3k^*+1}\) | \(\frac{2}{3}\) | \(\sqrt{2}\) | \(\sqrt{2}\) | \(\sqrt{2}\) |

Moreover, we note that the condition in Corollary 3 for Hard algorithm can be further improved via using the specific expression of the hard thresholding operator. This can be shown as the following theorem.

Theorem 4. Let \(\{x^{(t)}\}\) be a sequence generated by Hard algorithm for \(b = Ax + \epsilon\). Assume that \(A\) satisfies \(\delta_{3k^*+1} < \frac{\sqrt{2}}{2}\), and if we take \(k = k^*\) and \(s = 1\), then
\[
\|x^{(t)} - x^*\|_2 \leq \rho^t \|x^* - x^{(0)}\|_2 + \frac{\sqrt{2} + 1}{2} \|A^T\epsilon\|_2, 
\]
where \(\rho = \frac{\sqrt{2} + 1}{2} \delta_{3k^*+1} < 1\). Particularly, when \(\epsilon = 0\), it holds
\[
\|x^{(t)} - x^*\|_2 \leq \rho^t \|x^* - x^{(0)}\|_2. 
\]

Proof: The proof of this theorem is also very similar to that of Theorem 2. According to the proof of Theorem 2, we have known that (55) holds for all pairs of \((p, q)\) with \(\frac{1}{p} + \frac{1}{q} = 1\), and thus obviously holds for \(p = 2\) and \(q = 2\), that is,
\[
\|z_{i}^{(t)} - x_{i}^{*}\|_2 \leq \delta_{3k^*+1}\|x^{(t)} - x^*\|_2 + \|A^T\epsilon\|_2, 
\]
where \(S^t = I^+_1 \cup I^t \cup I^*\) and \(I^*_+\) is the index set of the largest \(k + 1\) components of \(z^{(t+1)}\). \(I^t\) and \(I^*\) represent the support sets of \(x^{(t)}\) and \(x^*\), respectively. In the following, instead of the inequality (56), we will derive a tighter upper bound of \(\|x^{(t)} - x^*\|_2\), that is,
\[
\|x^{(t)} - x^*\|_2 \leq \frac{\sqrt{2} + 1}{2} \|z_{i}^{(t)} - x_{i}^{*}\|_2. 
\]

Now we turn to prove the inequality (57). It can be noted that
\[
\|x^{(t)} - x^*\|_2 = \|x_{i}^{*}\|_2- \|x_{i}^{*}\|_2 + \|x_{i}^{(t)} - x_{i}^{*}\|_2. 
\]

On one hand, since \(z_{i}^{(t)} = x_{i}^{*}\) for any \(i \in I^t\), then
\[
\|x_{i}^{(t)} - x_{i}^{*}\|_2 = \|x_{i}^{*}\|_2 - \|x_{i}^{*}\|_2. 
\]

On the other hand, we can also observe that \(x_{i}^{*} = 0\) for any \(i \in I^* \setminus I^t\), and thus
\[
\|x_{i}^{(t)} - x_{i}^{*}\|_2 = \|x_{i}^{*}\|_2 + \|x_{i}^{*}\|_2. 
\]

Then
\[
\|x^{(t)} - x^*\|_2 \leq \rho^t \|x^* - x^{(0)}\|_2 + \frac{\sqrt{2} + 1}{2} \|A^T\epsilon\|_2. 
\]

The first inequality holds by the following relation
\[
(a + b)^2 = a^2 + 2ab \leq (1 + \frac{\sqrt{2} + 1}{2})a^2 + (1 + \frac{\sqrt{2} + 1}{2})b^2 
\]
for any \(a, b \in \mathbb{R}\). The second inequality holds due to the following facts:
(a) for any \(i \in I^* \setminus I^t\), \(|z_{i}^{(t)}| \leq \tau_{i}^{(t)}\),
(b) for any \(i \in I^t \setminus I^*\), \(|z_{i}^{(t)}| \geq \tau_{i}^{(t)}\),
(c) \(I^* \setminus I^t = |I^*|\),

and hence
\[
\max_{i \in I^* \setminus I^t} |z_{i}^{(t)}| \leq \min_{i \in I^t \setminus I^*} |z_{i}^{(t)}|. 
\]

The last equality holds for \(x_{i}^{*} = 0, \forall i \in I^t \setminus I^*\). Plugging (59) and (60) into (58), we have
\[
\|x^{(t)} - x^*\|_2 \leq \frac{\sqrt{2} + 1}{2} \|z_{i}^{(t)} - x_{i}^{*}\|_2 + \frac{\sqrt{2} + 1}{2} \|x_{i}^{*}\|_2.
\]

V. COMPARISON WITH THE PREVIOUS WORKS

This section discusses some related work of the AIT algorithm, and then compares its computational complexities and sufficient conditions for convergence with other algorithms.
1) On related work of the AIT algorithm: In [28], Maleki provided some similar results for two special AIT algorithms, i.e., Hard and Soft algorithms with $k = k^*$ and $s = 1$ for the noiseless case. The sufficient conditions for convergence are $\mu < \frac{1}{3 + c_1}$ and $\mu < \frac{1}{4 - c_2}$ for Hard and Soft algorithms, respectively. In [29], Zeng et al. improved and extended Maleki’s results to a wide class of the AIT algorithm with step size $s = 1$. The sufficient condition based on coherence was improved to $\mu < \frac{1}{(3 + c_2)k^*}$, where the boundedness parameter $c_1$ can be found in Table I. Compared with these two tightly related works, several significant improvements are made in this paper.

(i) **Weaker convergence condition.** The conditions obtained in this paper is weaker than those in both [28] and [29]. More specifically, we give a unified convergence condition based on the introduced generalized restricted isometry property (gRIP). Particularly, as shown in Theorem 3, the coherence based conditions for convergence are

$$\mu < \frac{1}{(3 + c_2)k^*},$$

which is much better than the condition $\mu < \frac{1}{1 + c_2}$ obtained in [29]. Moreover, except Hard algorithm, we firstly show the convergence of the other AIT algorithms based on RIP.

(ii) **Better convergence rate.** The asymptotic linear convergence rate was justified in both [28] and [29]. In this paper, we show the global linear convergence rate of the AIT algorithm, which means it converges exponentially fast from the first iteration.

(iii) **More general model.** In this paper, besides the noiseless model $b = Ax$, we also consider the performance of the AIT algorithm for the noisy model $b = Ax + \epsilon$, which is very crucial since the noise is almost inevitable in practice.

(iv) **More general algorithmic framework.** In both [28] and [29], the AIT algorithm was only considered unit step size ($s = 1$). While in this paper, we show that the AIT algorithm converges when $s$ is in an appropriate range.

Among these AIT algorithms, Hard algorithm has been widely studied. In [31], it was demonstrated that if $A$ has unit-norm columns and coherence $\mu$, then $A$ has the $(r, \delta_r)$-RIP with

$$\delta_r \leq (r - 1)\mu.$$  

(61)

In terms of RIP, Blumensath and Davies [27] justified the performance of Hard algorithm when applied to signal recovery problem. It was shown that if $A$ satisfies a certain RIP with $\delta_{3k^*} < \frac{1}{\sqrt{4}}$, then Hard algorithm has global convergence guarantee. Later, Foucart [30] improved this condition to $\delta_{3k^*} < \frac{1}{2}$ or $\delta_{2k^*} < \frac{1}{4}$. Now we can improve this condition to $\delta_{3k^*+1} < \frac{1}{\sqrt{4}} \mu \approx 0.618$ as shown by Theorem 4.

2) On comparison with other algorithms: For better comparison, we list the state-of-the-art results on sufficient conditions of some typical algorithms including BP, OMP, CoSaMP, Hard, Soft, Half and general AIT algorithms in Table IV.

From Table IV in the perspective of coherence, the sufficient conditions of AIT algorithms are slightly stricter than those of BP and OMP algorithms except Soft algorithm. However, AIT algorithms are generally faster than both BP and OMP algorithms with lower computational complexities, especially for large scale applications due to their linear convergence rates. As shown in the next section, the number of iterations required for the convergence of the AIT algorithm is empirically of the same order of the original sparsity level $k^*$, that is, $O(k^*)$. At each iteration of the AIT algorithm, only some simple matrix-vector multiplications and a projection on the vector need to be done, and thus the computational complexity per iteration is $O(MN)$. Therefore, the total computational complexity of the AIT algorithm is $O(k^*MN)$. While the total computational complexities of BP and OMP algorithms are generally $O(M^2N)$ and $O\{O(k^*MN), O(\sqrt{k^*}(k^*+1)^2/4)\}$, respectively. It should be pointed out that the computational complexity of OMP algorithm is related to the commonly used halting rule of OMP algorithm, that is, the number of maximal iterations is set to be the true sparsity level $k^*$.

Another important greedy algorithm, CoSaMP algorithm, identifies multicomponents (commonly $2k^*$) at each iteration. From Table IV the RIP based sufficient condition of CoSaMP is $\delta_{4k^*} < 0.384$ and a deduced coherence based sufficient condition is $\mu < 0.384$. In the perspective of coherence, our conditions for AIT algorithms are better than CoSaMP, though this comparison is not very reasonable. On the other hand, our conditions for AIT algorithms except Hard algorithm are generally worse than that of CoSaMP in the perspective of RIP. However, when the true signal is very sparse, the conditions of AIT algorithms may be better than that of CoSaMP. At each iteration of CoSaMP algorithm, some simple matrix-vector multiplications and a least squares problem should be considered. Thus, the computational complexity per iteration of CoSaMP algorithm is generally $O(k^*MN), O((3k^*)^3)$, which is higher than those of AIT algorithms, especially when $k^*$ is relatively large.

Besides BP and greedy algorithms, another class of tightly related algorithms are the reweighted techniques that have been also widely used for solution to $l_q$ regularization with $q \in (0, 1)$. Two well known examples of such reweighted techniques are the iteratively reweighted least squares (IRLS) method [17] and the reweighted $l_1$ minimization (IRL1) method [14]. The convergence analysis conducted in [18] shows that the IRLS method converges with an asymptotically superlinear convergence rate under the assumption $A$ possesses

| Algorithm | $\mu$ | $(r, \delta_r)$ |
|-----------|-------|----------------|
| BP        | $\frac{1}{3} \cdot (2k^*)^2$ | $(2k^*, 0.707)$ |
| OMP       | $\frac{1}{3} \cdot (k^* + 1)$ | $(k^* + 1, 1)$ |
| CoSaMP    | $\frac{1}{3} \cdot (3k^*)^2$ | $(k^* + 1, 0.618)$ |
| Hard      | $\frac{1}{3} \cdot (3k^* + 1)$ | $(3k^* + 1, \sqrt{2})$ |
| Soft      | $\frac{1}{3} \cdot (3k^* + 1)$ | $(3k^* + 1, \sqrt{2})$ |
| Half      | $\frac{1}{3} \cdot (3k^* + 1)$ | $(3k^* + 1, \sqrt{2})$ |
| General AIT | $\frac{1}{3} \cdot (3k^* + 1)$ | $(3k^* + 1, \sqrt{2})$ |

*: a coherence based sufficient condition for CoSaMP derived by the fact that $\delta_{4k^*} < 0.384$ and $\delta_r \leq (r - 1)\mu$. |
a certain null-space property (NSP). However, from Theorem 2 the rates of convergence of AIT algorithms are globally linear. Furthermore, Lai et al. [37] applied the IRLS method to the unconstrained \( l_q \) minimization problem and also extended the corresponding convergence results to the matrix case. It was shown also in [38] that the IRL1 algorithm can converge to a stationary point and the asymptotic convergence speed is approximately linear when applied to the unconstrained \( l_q \) minimization problem. Both in [37] and [38], the authors focused on the unconstrained \( l_q \) minimization problem with a fixed regularization parameter \( \lambda \), while in this paper, we focus on a different model with an adaptive regularization parameter.

VI. NUMERICAL EXPERIMENTS

We conducted a set of numerical experiments in this section to substantiate the validity of the theoretical analysis of the AIT algorithm.

A. Experiment setup

We considered four typical AIT algorithms including Hard [27], Half [13], Soft [19] and SCAD [15] algorithms. In these experiments, we set \( m = 250, n = 400 \) and \( k^* = 15 \). The nonzero components of \( x^* \) were generated randomly according to the standard Gaussian distribution. The matrix \( A \) was generated from i.i.d normal distribution \( N(0, 1/250) \) and was preprocessed via column-normalization, i.e., \( \|A_i\|_2 = 1 \) for any \( i \). Such measurement matrix is known to satisfy (with high probability) the RIP with optimal bounds [39, 40] and thus the gRIP condition of Theorems 2 can also be satisfied.

B. Justification of the linear convergence rate

In this subsection, we justified that the AIT algorithm converges linearly. We considered both the noise free case and noisy case. In these experiments, we set the step size \( s \) to 0.95. As shown in Fig. 3, all of the four AIT algorithms converge to the original sparse signal exponentially fast. In the noiseless case, all AIT algorithms can recover the original sparse signal with high precision, while in the noisy case, \( \|x^{(t)} - x^*\|_2 \) decays exponentially fast until reaching some error bounds. This experiment justified the convergence results in Theorem 2.

![Fig. 3. Linear convergence of AIT algorithms in both noiseless and noisy cases when \( s = 0.95 \). (a) Noiseless case; (b) Noisy case with the signal-to-noise-ratio (SNR) 60 dB.](image)

![Fig. 4. The effect of the step length parameter. (a) The trends of the required iteration numbers of different AIT algorithms in noiseless case. (b) The trends of the required iteration numbers of different AIT algorithms in noisy case. (c) The trends of the rates of convergence of different AIT algorithms in noiseless case. (d) The trends of the rates of convergence of different AIT algorithms in noisy case.](image)

C. On effect of the step size

In this subsection, we verified the effect of the step size parameter \( s \) on the performance of the AIT algorithm in both noise free case and noisy case. To be specific, we considered its effect on the required number of iterations (IterNum) and the rate of convergence. In this experiment, the step size \( s \) was taken in the interval \([0.4, 1.3]\). We implemented four AIT algorithms with different step sizes, and then recorded the required number of iterations and also the estimated rate of convergence of different AIT algorithms to achieve the same recovery precision. For the noise free case, the stopping criterias of all algorithms were set as \( \|x^{(t)} - x^*\|_2 < 10^{-6} \). For the noisy case, the signal-to-noise ratio (SNR) was set to 60 dB and the stopping criterias were set as \( \|x^{(t)} - x^*\|_2 < 10^{-2} \). The experiment results are reported in Fig. 4.

Fig. 4 shows that the four AIT algorithms have almost the smallest rate of convergence, \( \rho_s \) when \( s \) is around 1. Generally, when \( s \) is smaller than 1, the rate of convergence \( \rho_s \) gets smaller as \( s \) increases, and thus the AIT algorithm converges faster. As a consequence, fewer iterations are required to attain a specified recovery precision. On the contrary, when \( s \) is bigger than 1, the rate of convergence \( \rho_s \) gets bigger as \( s \) increases. Therefore, more iterations are required to attain a specified recovery precision. Such trend of the convergence rate constant \( \rho_s \) clearly coincides with the theoretical analysis of Theorem 2 in Subsection IV.A.

D. Robustness of the estimated sparsity level

In the preceding proposed algorithms, the specified sparsity level parameter \( k \) is taken exactly as the true sparsity level \( k^* \), which is generally unknown in practice. Instead, we can often obtain a rough estimate of the true sparsity level. Therefore, in
this experiment, we will explore the performance of the AIT algorithm with a variety of specified sparsity levels. We varied \( k \) from 1 to 150 while kept \( k^* = 15 \).

From Fig. 5, we can observe that these AIT algorithms are efficient for a wide range of \( k \). Interestingly, the point \( k = k^* \) is a break point of the performance of all these AIT algorithms. When \( k < k^* \), all AIT algorithms fail to recover the original sparse signal, while when \( k \geq k^* \), a wide interval of \( k \) is allowed for small recovery errors, as shown in Fig. 5 (b) and (d). In the noise free case, if \( \| x^{(t)} - x^* \|_2 < 10^{-10} \), the feasible interval of the specified sparsity level \( k \) is \([15, 109]\) for SCAD and Soft, \([15, 81]\) for Half and \([15, 65]\) for Hard. This observation is very important for real applications of AIT algorithms because \( k^* \) is usually unknown. In the noisy case, if \( \| x^{(t)} - x^* \|_2 < 10^{-2} \), the feasible interval of sparsity level \( k \) is \([15, 105]\) for SCAD, \([15, 40]\) for Soft, \([15, 37]\) for Half and \([15, 26]\) for Hard. The detailed exploration of their relations could be discussed in the future work.

VII. CONCLUSION

We have conducted a study of a wide class of AIT algorithms for compressed sensing. It should be pointed out that almost all of the existing iterative thresholding algorithm like Hard, Soft, Half and SCAD algorithms are included in such class of algorithms considered in this paper. The main contribution of this paper is the establishment of the convergence analysis of the AIT algorithm. In summary, we have shown when the measurement matrix satisfies certain gRIP condition, the AIT algorithm can converge to the original sparse signal at a linear rate in the noiseless case, and approach to the original sparse signal at a linear rate until achieving an error bound in the noisy case. As two special cases of gRIP, the coherence and RIP based conditions can be directly derived for the AIT algorithm. Moreover, the tightness of our analysis can be demonstrated by two specific cases, that is, the coherence-based condition for Soft algorithm is the same as those of OMP and BP, and the RIP based condition for Hard algorithm is much better than the recent result \( \delta_{2k} < 0.5 \) obtained in [30]. Furthermore, the efficiency of the algorithm and the correctness of the theoretical results are also verified via a series of numerical experiments.

As shown in Fig. 5, the AIT algorithm is robust to the specified sparsity level parameter \( k \), that is, the parameter \( k \) can be specified in a large range to guarantee the well performance of the AIT algorithm. However, the corresponding theoretical guarantee has not been developed in the the present research, which should be studied in the future.

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