Coordinated Multicell Multicast Beamforming
Based on Manifold Optimization

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Abstract—Multicast beamforming is a key technology for
next-generation wireless cellular networks to support high-rate
content distribution services. In this paper, the coordinated
downlink multicast beamforming design in multicell networks
is considered. The goal is to maximize the minimum signal-to-
interference-plus-noise ratio of all users under individual base
station power constraints. We exploit the fractional form of
the objective function and geometric properties of the con-
straints to reformulate the problem as a parametric manifold
optimization program. Afterwards we propose a low-complexity
Dinkelbach-type algorithm combined with adaptive exponential
smoothing and Riemannian conjugate gradient iteration, which
is guaranteed to converge. Numerical experiments show that
the proposed algorithm outperforms the existing SDP-based
method and DC-programming-based method and achieves near-optimal
performance.

Index Terms—Multicast beamforming, max-min fair, manifold
optimization, Riemannian conjugate gradient.

I. INTRODUCTION

EXPLOSIVE demands for high-rate wireless content distri-
bution services, such as audio and video streaming, software updates, and Internet TV, have motivated extensive
research on advanced physical layer techniques to boost
the capacity of wireless networks [1]. Downlink multicast
beamforming is a powerful technique to improve the wireless
throughput for next generation cellular networks.

A variety of multicast beamforming problems have been
investigated for different scenarios. For single-cell system,
single-group multicast beamforming was first discussed in [2],
where all users request a common information from the base
station (BS), and then extended to multi-group multicast in
[3]. Recently, the multi-group multicast beamforming under
per-antenna power constraints was further investigated in [4].
Moreover, the coordinated multicast beamforming with indi-
vidual BS power constraints in multi-cell networks has been
considered in [5]. Some other issues, such as energy efficient
design and user selection, were also studied in [6] [7] [8].

In this paper, we revisit the max-min fair coordinated multicast beamforming problem in [5]. The existing approach
of semidefinite relaxation (SDR) and Gaussian randomization
in [3][5] has following drawbacks. First, SDR is not scalable
to large-scale antenna systems as the number of involved
variables is quadratic in the number of antennas. Second,
extracting a rank-one component from the optimum solution
to the SDR problem is NP-hard in general. The polynomial-
time approximation method of Gaussian randomization in
[3][5] needs to solve a large number of multicast feasibility
power control subproblems and such approximation degrades
considerably as the number of antennas increases [2]. In this
paper, we present a new max-min fair multicast beamforming
design that outperforms the existing methods in [5] and [8]
yet with a much lower complexity.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Consider a multicell multicast scenario consisting of $L$ cells
and $K$ single-antenna users per cell, sharing a common time-
frequency resource. Each cell has a BS equipped with $M$
antennas. The BS in the $l$-th cell uses an $M \times 1$ beamforming
vector $\tilde{W}_l$ to send a zero-mean and unit-variance multicast
signal $s_l$ to all users in the $l$-th cell. The signal received by
the $k$-th user in the $l$-th cell is

$$y_{l,k} = \tilde{h}_{l,k}^H \tilde{w}_l s_l + \sum_{j=1, j \neq l}^{L} \tilde{h}_{j,k}^H \tilde{w}_j s_j + n_{l,k},$$

where $\tilde{h}_{l,k} \in \mathbb{C}^{M \times 1}$ is the channel between the $k$-th user in
the $l$-th cell and the BS in the $j$-th cell. $n_{l,k} \sim \mathcal{N}(0, \sigma^2_{l,k})$ is
the additive white Gaussian noise (AWGN) at the $k$-th user in
the $l$-th cell and is independent of $\tilde{h}_{l,k}$ and $s_l$.

Assume that a central processing unit collects the channel
state information between all BSs and all users in the system.
Based on the received signal model in (1), the performance of
each user can be characterized by the signal-to-interference-
plus-noise ratio (SINR). The problem of interest is to max-
imize the minimum weighted SINR among all users under
individual BS power constraints

$$(\mathcal{F}) : \begin{align*}
\max_{\tilde{W} \in \mathbb{C}^{M \times L}} & \min_{i,k} \frac{1}{\Gamma_i} \sum_{j=1}^{L} \| \tilde{h}_{i,l,k}^H \tilde{w}_j \|^2 + \sigma^2_{l,k} \\
\text{s.t.} & \| \tilde{w}_i \|^2 \leq P_i \forall i,
\end{align*}$$

where $\tilde{W} = [\tilde{w}_1, \ldots, \tilde{w}_L] \in \mathbb{C}^{M \times L}$, $\Gamma = [\Gamma_1, \Gamma_2, \ldots, \Gamma_L]^T$ with
each entry $\Gamma_i$ being the target SINR for all users in the $l$-th
cell, and $P = [P_1, \ldots, P_L]^T$ is the power budget vector for all
BSs. Since the multicast information rate for users within one
cell is the same, we set a common target SINR value for all
users in the same cell.
III. ALGORITHM DESIGN

A. Preliminary Analysis

We first map the feasible region (2b) onto spheres by introducing \(L\) complex slack variables \(\tilde{w}_{l,M+1}, l = 1, \ldots, L\), such that \(\|\tilde{w}_{l,M+1}\|^2 + \|\tilde{w}_l\|^2 = P_l, \forall l\). Let \(w_l = \frac{P_l}{\|\tilde{w}_l\|^2}[\tilde{w}_l]^{\dagger}, \tilde{w}_{l,M+1} \in \mathbb{C}^{(M+1)\times 1}\) and \(W = [w_1, \ldots, w_L] \in \mathbb{C}^{(M+1)\times L}\), then the feasible region (2b) becomes \(S = \{W \in \mathbb{C}^{(M+1)\times L}; \|w_l\|^2 = 1, \forall l\}\). Next, channel vector \(b_{j,l,k}\) is normalized by \(P_j\) and \(\sigma_{j,k}\) as follows, \(b_{j,l,k} = \frac{P_j^{\dagger}}{\sigma_{j,k}}[b^{\dagger}_{j,l,k}0]^T \in \mathbb{C}^{(M+1)\times 1}, \forall j, l, k\). Denote

\[
\rho(W) = \min_{l,k} \frac{1}{\Gamma_l} \left[ \sum_{j \neq l} \left| h_{l,l,k}^H w_j \right|^2 + 1 \right].
\]

The multicast beamforming problem (\(F_t\)) is rewritten as

\[
(F_t) : \quad \max_W \rho(W) \quad \text{s.t.} \quad W \in S. \quad (4)
\]

Denote \(F(W, t) = \min_{l,k} f_{l,k}(W, t)\), where

\[
f_{l,k}(W, t) = \frac{1}{\Gamma_l} \left| h_{l,l,k}^H w_l \right|^2 - t \left( \sum_{j \neq l} \left| h_{l,l,k}^H w_j \right|^2 + 1 \right)
\]

represents the received power shortage or redundancy of the \(k\)-th user in the \(l\)-th cell to achieve a weighted SINR value of \(t\). We reformulate the problem (\(F_t\)) as the following parametric programming problem

\[
(F_t) : \quad F(t) = \max_W F(W, t) \quad \text{s.t.} \quad W \in S. \quad (6)
\]

Note \(\rho(W)\) has a fractional form, while \(F(W, t)\) is the pointwise minimum of quadratic functions. In general, \(F(W, t)\) is easier to handle than \(\rho(W)\). Maximizing \(\rho(W)\) over \(S\) is equivalent to maximizing \(F(W, t)\) over \(S\), i.e., calculating the single-variable function \(F(t)\) defined in (6), since \(F(t)\) is continuous and strictly decreasing over \([0, \infty)\), and the unique zero of \(F(t)\) is the optimal objective value of (\(F_t\)). Thus solving (\(F_t\)) reduces to determining the unique value of \(t\) such that \(F(t) = 0\), which can be achieved by solving a sequence of parametric subproblems (\(F_{t,\mu}\)) for different values of \(t\).

It is desirable that the sequence of parameters \(t_k\) increases monotonically, since this guarantees that each solution of (\(F_{t,\mu}\)) is a feasible point and an appropriate initial point as well for the next subproblem. Bisection methods do not have this property, and therefore we employ the Dinkelbach-type algorithm to find the root of \(F(t)\). The basic idea is to first find \(t_k\) such that \(F(W^{(k)}, t_k) = 0\) for a given \(W^{(k)} \in S\) and then find a solution \(W^{(k+1)}\) to (\(F_{t,\mu}\)) [9].

Computing \(F(t)\) or solving (\(F_t\)) is central to solving the original problem (\(F_t\)). However, (\(F_{t,\mu}\)) for a given \(t\) is equivalent to a non-convex quadratically constrained quadratic program (QCQP), which is difficult to solve [3]. Instead, we propose to compute a stationary solution of (\(F_t\)) that provides an achievable low bound of \(F(t)\). Since \(F(W, t)\) is non-smooth, we turn to the following smoothed surrogate problem

\[
(F_{t,\mu}) : \quad \max_W F(W, t, \mu) \quad \text{s.t.} \quad W \in S, \quad (7)
\]

where \(\mu > 0\) is a smoothing parameter and

\[
F(W, t, \mu) = -\mu \log \sum_{l,k} \exp (-f_{l,k}(W, t)/\mu)
\]

is the exponential smoothing of \(F(W, t)\) that satisfies [10]

\[
F(W, t, \mu) \leq F(W, t) \leq F(W, t, \mu) + \mu \log(KL). \quad (9)
\]

Moreover, \(F(W, t, \mu)\) increases while \(F(W, t, \mu) + \mu \log(KL)\) decreases, as \(\mu\) decreases. Therefore, a small \(\mu\) leads to high approximation accuracy. However, when \(\mu\) is small, the problem (\(F_{t,\mu}\)) is nearly ill-conditioned, which is difficult to solve. An effective strategy is to solve a sequence of gradually more accurate approximations [10]. Taking into consideration the structure of the constraint set \(S\), a Riemannian conjugate gradient (RCG) method is well suitable for obtaining a stationary solution to (\(F_{t,\mu}\)) with low complexity, which will be detailed in the next subsection.

As mentioned above, we use a Dinkelbach-type procedure to solve (\(F_t\)) as follows: given \(W^{(k)} \in S\), set \(t_k = \rho(W^{(k)})\) and then find a solution \(W^{(k+1)} \in S\) to (\(F_{t,\mu}\)) for some \(\mu\) such that \(F(W^{(k+1)}) > F(W^{(k)}; t_k)\). The latter is always possible by the RCG method for a sufficiently small \(\mu\) as long as \(W^{(k)}\) is not a stationary point of (\(F_t\)), due to (9). To obtain such a solution, the RCG method would be repeatedly applied on the problems (\(F_{t,\mu}\)) for a decreasing sequence of \(\mu\) values. An important feature of the proposed Dinkelbach-type procedure is the monotonicity of the generated sequence \(\{t_k\}_{k=0}^{\infty}\) which guarantees convergence and makes the algorithm numerically more stable. For clarity, the proposed Dinkelbach-type procedure that employs the RCG method (line 1) is summarized in Algorithm 1.

Algorithm 1: DT-RCG algorithm for problem (\(F_t\))

input: initial point \(W^{(0)} \in S, \mu_0\)
output: \(W^{(k)}\) and \(t_k\)

Initialization Set \(t_0 = \rho(W^{(0)})\).

for \(k = 1, 2, \ldots \) do

1. \(W^{(k)} = \text{RiemannianConGrad}(W^{(k-1)}, t_{k-1}, \mu_{k-1})\).

2. If \(F(W^{(k)}, t_{k-1}) > F(W^{(k-1)}, t_{k-1})\) then \(t_k = \rho(W^{(k)}), \mu_k = \mu_{k-1}\); else \(W^{(k)} = W^{(k-1)}, t_k = t_{k-1}, \mu_k = \mu_{k-1}/2\).

3. If \(t_\mu < \varepsilon\) then STOP.

end for

B. Riemannian Conjugate Gradient Algorithm

Motivated by the superior performance of nonlinear conjugate gradient methods to large-scale unconstrained optimization problems [11], we treat the surrogate problem (\(F_{t,\mu}\)) as an unconstrained optimization problem on a complex oblique manifold and devise a RCG algorithm to find an approximate stationary solution by using the framework of retraction-based manifold optimization [12][13]. To simplify notation, \(F(W, t, \mu)\) will be simply denoted by \(F(W)\) in this subsection.

Conceptually, the RCG algorithm has three stages in each iteration: (i) Compute the Riemannian gradient, i.e., the tangent
vector in the tangent space corresponding to the direction of the 

steepest ascent of \( F(W) \); (ii) Find a tangent vector that is 

conjugate to the Riemannian gradient as the search direction; 

(iii) Invoke the metric projection as a retraction that maps a 

tangent vector to a point on the manifold. We will next detail 

some key ingredients of the RCG algorithm. Background on 

manifold optimization can be found in [13].

A manifold \( \mathcal{M} \) is a topological space that resembles a Eu-

clidean space near each point. For our problem, the feasible set 

\( \mathcal{S} \) defines a complex oblique manifold, namely, the Cartesian 

product of unit spheres

\[
\mathcal{M} = \left\{ W \in \mathbb{C}^{(M+1) \times L} \mid \text{diag}(W^H W) = \mathbf{I}_L \right\},
\]

where \( \text{diag}(Z) \) forms a diagonal matrix, whose diagonal 

elements are those of \( Z \). The tangent vector of any smooth 

curve through the point \( W \) characterizes the direction along 

which it can move. All tangent vectors at a given point on 

manifold form a linear subspace, called tangent space. In our 

case, the tangent space \( T_W \mathcal{M} \) at the point \( W \in \mathcal{M} \) is described 

by

\[
T_W \mathcal{M} = \left\{ U \in \mathbb{C}^{(M+1) \times L} \mid \text{diag}(\text{Re}(W^H U)) = 0 \right\}.
\]

To measure distances and angles on tangent space and use 

calculus on manifold, the canonical inner product \( (U, V)_W = \text{Re} \{ \text{Tr}(U^H V) \} \) is chosen as the 

Riemannian metric on the tangent space \( T_W \mathcal{M} \), which makes \( \mathcal{M} \) a Riemannian manifold.

Hence, the Riemannian gradient of \( F(W) \) on \( \mathcal{M} \), which is 

the unique tangent vector in the tangent space \( T_W \mathcal{M} \) that 

gives the largest increase in \( F(W) \), is given by the orthogonal projection 

of the Euclidean gradient \( \nabla_W F(W) \) onto \( T_W \mathcal{M} \), i.e.,

\[
\nabla_W F(W) = \nabla_W F(W) - W \text{diag}(\text{Re}(W^H \nabla_W F(W))).
\]

The Euclidean gradient \( \nabla_W F(W) \) is expressed as

\[
\nabla_W F(W) = \left[ \frac{\partial F(W)}{\partial w_1}, \ldots, \frac{\partial F(W)}{\partial w_L} \right],
\]

where the complex-valued partial derivative \( \frac{\partial F(W)}{\partial w_l} \) is computed as

\[
\frac{\partial F(W)}{\partial w_l} = 2 \sum_{m,k} a_{l,m} b_{m,k}(W) h_{l,m,k} h_i^H, \quad (14)
\]

with \( a_{l,m} = -t \) for \( l \neq m \) and \( a_{l,m} = 1/\Gamma_l \) for \( l = m \), and 

\( b_{m,k}(W) = \sum_{i,j} e^{-jW_i W_i^T/2} e^{-jW_j W_j^T/2} \).

The conjugate search direction is the weighted sum of the 

Riemannian gradient at present iteration and the search 

direction used at the previous iteration. However, two vectors 

in different tangent spaces cannot be added directly. This is 

accomplished by introducing the following vector transport 

to map a tangent vector \( U \in T_W \mathcal{M} \) to \( T_W \mathcal{M} \),

\[
T_W \mathcal{M}(U) = U - W_+ \text{diag}(\text{Re}(W_+^H U)).
\]

Specifically, given the previous and current Riemannian 

gradients, \( \mathbf{G} \) and \( \mathbf{G} \), and the previous conjugate direction \( \mathbf{D}_n \), the 

current conjugate direction \( \mathbf{D} \) is given by

\[
\mathbf{D} = \mathbf{G} + \nu \mathbf{Y},
\]

where \( \mathbf{Y} = T_{W\mathcal{M}}(\mathbf{D}_n) \), \( \mathbf{Z} = T_{W\mathcal{M}}(\mathbf{G}_n) \) and

\[
\nu = \max \left( 0, \frac{\langle \mathbf{G} - \mathbf{Z}, \mathbf{G} \rangle_W}{\langle \mathbf{G} - \mathbf{Z}, \mathbf{Y} \rangle_W} \right)
\]

is the combination coefficient according to the modified 

Hestenes-Stiefel rule [11].

Given the conjugate direction, a so-called retraction 

mapping is used to map an element from the tangent space back to 

the manifold. Retractions are essentially first-order approximations 

of the exponential map of the manifold. For our problem, 

the following retraction, indeed the metric projection, is chosen 

to map a tangent vector \( U \in T_W \mathcal{M} \) to \( \mathcal{M} \),

\[
R_W(U) = (W + U) \left( \text{diag} \left( (W + U)^H (W + U) \right) \right)^{-1/2}.
\]

For completeness, the proposed RCG algorithm is summa-

rized in Algorithm 2. To avoid confusion with the sequence 

generated by Algorithm 1, the sequence in Algorithm 2 is 

denoted by \( \{X_n\} \). To guarantee the objective function to be 

non-decreasing in each iteration, the RCG algorithm utilizes 

the Armijo line search in line 4 [12]. According to the 

convergence results for line-search method in [13, Theorem 

4.3.1], the RCG algorithm is guaranteed to globally converge 

to a stationary point of the surrogate problem \( (\mathcal{F}_{T,U}) \), namely, 

the point where the smoothed objective function \( F(W, t, \mu) \) 

has vanishing Riemannian gradient.

Algorithm 2 \( X_n = \text{RiemannianConGrad}(X_{0}, t, \mu) \)

input: initial point \( X_0, t, \mu \)

output: \( X_n \)

Initialization Set \( F(X) = F(X, t, \mu) \) according to (8) and 

compute \( G_0 = \text{grad} F(X_0) \), \( D_0 = G_0 \), \( X_1 = X_0 \).

for \( n = 1, 2, \ldots \) do

1. Compute \( G_n = \text{grad} F(X_n) \) according to (12).

2. If \( \|G_n\|_F \leq \varepsilon_0 \) then STOP.

3. Compute \( D_n = G_n + \nu_n Y_n \) according to (16) and (17).

4. If \( \langle G_n, D_n \rangle_{X_n} < 0 \) then \( D_n = G_n \).

4. Compute \( X_{n+1} = R_{X_n}(\alpha_n D_n) \) using retraction in (18) 

where \( \alpha_n = \text{ArmijoLineSearch}(X_n, D_n, X_{n-1}) \).

end for

Algorithm 3 \( \alpha = \text{ArmijoLineSearch}(X, D, X_0) \)

input: \( X, D, X_0 \) (optional)

output: \( \alpha \)

If \( X_0 \) is absent then \( \alpha = 1/\|D\|_F \); 

else \( \alpha = 2 \frac{f(X) - f(X_0)}{(\text{grad} F(X_0))_X} \).

If \( \alpha \|D\|_F \leq 10^{-1} \) then \( \alpha = 1/\|D\|_F \).

while \( F(R_X(\alpha D)) - F(X) < 10^{-4} \|\text{grad} F(X)\|_X \) do

\( \alpha = \alpha/2 \).

end while

Incorporating the RCG subroutine into the Dinkelbach-
type procedure in Algorithm 1, the overall algorithm provides 

monotonically improving approximations to a stationary solution 

to the multicast beamforming problem \( (\mathcal{F}_\tau) \) and guarantees 

feasibility with a low complexity.
the performance of the proposed DT-RCG algorithm. We consider a multicast scenario consisting of 3 cells and 10 single-antenna users per cell. The intracell and intercell channels are i.i.d. generated from \( CN(0,1) \) and \( CN(0,\epsilon) \), respectively. We set \( \epsilon = 1/4 \) which means that the intercell channels undergo 6 dB stronger large-scale fading than the intracell channels. The noise variance of each user is set as \( \sigma^2_{l,k} = 1,\forall l,k \). For the sake of simplicity, we assume that all users in the system have a common SINR target \( \Gamma = 1 \) and the power budget vector for the three BSs is \( \mathbf{P} = [P, P, 2P] \). All results below are averaged over 500 channel realizations.

In Fig. 1, we illustrate the convergence curve of the DT-RCG algorithm. The results validate the monotonicity and convergence of the proposed algorithm. We can also see that at the first few iterations, the DT-RCG algorithm converges very fast and achieves the major part of the limiting value.

In Fig. 2, we compare the average minimum SINR of the DT-RCG algorithm with that of the bisection-based SDR-G scheme [3][5] and the Difference of Convex-functions Algorithm (DCA) [8]. The SDR upper bound (SDR-UB) of the minimum SINR is also presented. It can be seen that the average minimum SINR achieved by DT-RCG is close to DCA, and substantially higher than SDR-G. The gap between the achievable minimum SINR for DT-RCG and the SDR upper bound is less than 1 dB for \( M = 16 \) and less than 0.6 dB for \( M = 8 \). The required number of arithmetic operations per inner iteration for DT-RCG, SDR-G and DCA is \( O(L^2MK) \), \( O(L^2M^3 + L^2M^2K) \) [3] and \( O(L^2M^3 + L^4MK) \) [14], respectively. Hence, DT-RCG is expected to considerably outperform SDR-G and DCA for a fixed number of outer and inner iterations.

### V. Conclusions

In this paper, the max-min fair coordinated multicast beamforming under individual BS power constraints was investigated. The original problem was first recast in a tractable parametric programming form. Afterwards, an efficient Riemannian conjugate gradient algorithm was developed for each parametric subproblem. The overall algorithm features monotonically improving approximations to a stationary solution of the original problem and therefore guarantees convergence. Numerical results validate the effectiveness of the proposed multicast beamforming algorithm and show significant advantages of it over the SDP-based method and the DC-programming-based method in terms of both better performance and lower computational complexity.

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