STABILITY OF THE COSINE-SINE FUNCTIONAL EQUATION ON AMENABLE GROUPS

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Abstract. In this paper we establish the stability of the functional equation

\[ f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in G, \]

where \( G \) is an amenable group.

1. Introduction

The stability problem of functional equations go back to 1940 when Ulam [14] proposed a question concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’s Theorem was generalized by Aoki [3] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. The stability problem of several functional equations have been extensively investigated by a number of authors. An account on the further progress and developments in this field can be found in [5, 7, 8].

In this paper we investigate the stability of the trigonometric functional equation

\[ (1.1) \quad f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in G \]
on amenable groups.

The continuous solutions of the trigonometric functional equations

\[ (1.2) \quad f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in G \]

and

\[ (1.3) \quad f(xy) = f(x)f(y) - g(x)g(y), \quad x, y \in G \]

are obtained by Poulsen and Stetkær [9], where \( G \) is a topological group that need not be abelian. Regular solutions of \( (1.2) \) and \( (1.3) \) were described by Aczél [1] on abelian groups. Chung et al. [4] solved the functional equation \( (1.1) \) on groups. Recently, Ajebbar and Elqorachi [2] obtained the solutions of the functional equation \( (1.1) \) on a semigroup generated by its squares. The stability properties of the functional equations \( (1.2) \) and \( (1.3) \) have been obtained by Székelyhidi [13] on amenable groups.

The aim of the present paper is to extend the Székelyhidi’s results [13] to the functional equation \( (1.1) \).

Key words and phrases. Hyers-Ulam stability; Semigroup; Amenable group; Cosine equation; Sine equation; Additive function; Multiplicative function.

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2. Definitions and preliminaries

Throughout this paper $G$ denotes a semigroup (a set with an associative composition) or a group. We denote by $B(G)$ the linear space of all bounded complex-valued functions on $G$. We call $a : G \to \mathbb{C}$ additive provided that $a(xy) = a(x) + a(y)$ for all $x, y \in G$ and call $m : G \to \mathbb{C}$ multiplicative provided that $m(xy) = m(x)m(y)$ for all $x, y \in G$.

Let $V$ be a linear space of complex-valued functions on $G$. We say that the functions $f_1, \ldots, f_n : G \to \mathbb{C}$ are linearly independent modulo $V$ if $\lambda_1 f_1 + \cdots + \lambda_n f_n \in V$ implies that $\lambda_1 = \cdots = \lambda_n = 0$ for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. We say that the linear space $V$ is two-sided invariant if $f \in V$ implies that that the functions $x \mapsto f(xy)$ and $x \mapsto f(yx)$ belong to $V$ for any $x \in G$.

Notice that the linear space $B(G)$ is two-sided invariant.

3. Basic results

Throughout this section $G$ denotes a semigroup and $V$ a two-sided invariant linear space of complex-valued functions on $G$.

Lemma 3.1. Let $f, g, h : G \to \mathbb{C}$ be functions. Suppose that $f$, $g$ and $h$ are linearly independent modulo $V$. If the function

$$ x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y) $$

belongs to $V$ for all $y \in G$, then there exist two functions $\varphi_1, \varphi_2 \in V$ such that

$$ \psi(x, y) = \varphi_1(x)f(y) + \varphi_2(x)h(y) $$

for all $x, y \in G$, where

$$ \psi(x, y) := f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y) $$

for $x, y \in G$.

Proof. We use a similar computation as the one of the proof of [13, Lemma 2.1]. Since the functions $f$, $g$ and $h$ are linearly independent modulo $V$ so are $f$ and $h$, then $f$ and $h$ are linearly independent. Then there exist $y_0, z_0 \in G$ such that $f(y_0)h(z_0) - f(z_0)h(y_0) \neq 0$, which implies that that $f(y_0)h(z_0) \neq 0$ or $f(z_0)h(y_0) \neq 0$. We can finally assume that $f(y_0) \neq 0$ and $h(z_0) \neq 0$. By applying (3.2) to the pair $(x, y_0)$ we derive

$$ g(x) = \alpha_0 f(x) + \alpha_1 h(x) + \alpha_2 f(xy_0) - \alpha_2 \psi(x, y_0) $$

for all $x \in G$, where $\alpha_0 := -f(y_0)^{-1}g(y_0) \in \mathbb{C}$, $\alpha_1 := -f(y_0)^{-1}h(y_0) \in \mathbb{C}$ and $\alpha_2 := f(y_0)^{-1} \in \mathbb{C}$ are constants. Similarly, by applying (3.2) to pair $(z, x_0)$, we get that

$$ h(x) = \beta_0 f(x) + \beta_1 g(x) + \beta_2 f(xz_0) - \beta_2 \psi(x, z_0) $$

for all $x \in G$, where $\beta_0 := -h(z_0)^{-1}g(z_0) \in \mathbb{C}$, $\beta_1 := -h(z_0)^{-1}f(z_0) \in \mathbb{C}$ and $\beta_2 := h(z_0)^{-1} \in \mathbb{C}$ are constants. Let $x \in G$ be arbitrary. Substituting (3.4) in (3.3) we obtain

$$ g(x) = \alpha_0 f(x) + \alpha_1 \left[ \beta_0 f(x) + \beta_1 g(x) + \beta_2 f(xz_0) - \beta_2 \psi(x, z_0) \right] + \alpha_2 f(xy_0) - \alpha_2 \psi(x, y_0) $$

$$ = (\alpha_0 + \alpha_1 \beta_0) f(x) + \alpha_1 \beta_1 g(x) + \alpha_1 \beta_2 f(xz_0) - \alpha_1 \beta_2 \psi(x, z_0) $$

$$ + \alpha_2 f(xy_0) - \alpha_2 \psi(x, y_0). $$
So that
\[
(1 - \alpha_1\beta_1) g(x) = (\alpha_0 + \alpha_1\beta_0) f(x) + \alpha_1\beta_2 f(xz_0) - \alpha_1\beta_2 \psi(x, z_0)
\]
(3.5)
\[+ \alpha_2 f(xy_0) - \alpha_2 \psi(x, y_0).\]
Since \(f(y_0)h(z_0) - f(z_0)h(y_0) \neq 0\) and \(f(y_0)h(z_0) \neq 0\) we get that \(\alpha_1\beta_1 \neq 1\). So, \(x\) being arbitrary, we derive from (3.5) that there exist \(\gamma_0, \gamma_1, \gamma_2 \in \mathbb{C}\) such that
\[
g(x) = \gamma_0 f(x) + \gamma_1 f(xy_0) + \gamma_2 f(xz_0) - \gamma_1 \psi(x, y_0) - \gamma_2 \psi(x, z_0)
\]
(3.7)
for all \(x \in G\). Similarly we prove that there exist \(\delta_0, \delta_1, \delta_2 \in \mathbb{C}\) such that
\[
h(x) = \delta_0 f(x) + \delta_1 f(xy_0) + \delta_2 f(xz_0) - \delta_1 \psi(x, y_0) - \delta_2 \psi(x, z_0)
\]
(3.8)
for all \(x \in G\). Let \(x, y, z \in G\) be arbitrary. In the following we compute \(f(xyz)\) first as \(f((xy)z)\) and then as \(f(x(yz))\). By applying (3.2) to the pair \((xy, z)\), and taking (3.6) and 3.7 into account, we obtain
\[
f((xy)z) = f(xy)g(z) + g(xy)f(z) + h(xy)h(z) + \psi(xy, z)
\]
\[= [f(x)g(y) + g(x)f(y) + h(x)h(y) + \psi(x, y)]g(z)
\]
\[+ \gamma_0 f(xy_0) + \gamma_1 f(xz_0) - \gamma_1 \psi(xy, y_0) - \gamma_2 \psi(xy, z_0)f(z)
\]
\[+ \delta_0 f(xy_0) + \delta_1 f(xz_0) - \delta_1 \psi(xy, y_0) - \delta_2 \psi(xy, z_0)h(z)
\]
+ \(\psi(xy, z)\).
So that
\[
f((xy)z) = f(xy)g(z) + \gamma_0 f(y)g(z) + \gamma_0 g(y)f(z) + \gamma_1 g(y)g(y_0)f(z) + \gamma_2 g(yz_0)f(z)
\]
\[+ \delta_0 f(y)h(z) + \delta_1 g(y)g(y_0)h(z) + \delta_2 g(yz_0)h(z)\]
\[+ g(x)f(y)g(z) + \gamma_1 f(yz_0)f(z) + \gamma_2 f(xz_0)f(z)
\]
\[+ \delta_0 f(y)h(z) + \delta_1 f(y)h(z) + \delta_2 f(y)h(z)\]
\[+ h(x)h(y)g(z) + \gamma_0 h(y)h(z) + \gamma_1 h(y)g(y_0)f(z) + \gamma_2 h(yz_0)f(z)
\]
\[+ \delta_0 h(y)h(z) + \delta_1 h(y)h(z) + \delta_2 h(y)h(z)\]
\[+ [\gamma_0 \psi(x, y) + \gamma_1 \psi(x, y_0) + \gamma_2 \psi(xy, z_0) - \gamma_1 \psi(xy, y_0)
\]
\[\gamma_2 \psi(xy, z_0)]f(z) + \psi(x, y)g(z) + [\delta_0 \psi(x, y) + \delta_1 \psi(x, y_0)
\]
\[+ \delta_2 \psi(x, z_0) - \delta_1 \psi(xy, y_0) - \delta_2 \psi(xy, z_0)]h(z) + \psi(xy, z).
\]
\[
(3.8)
\]
On the other hand, by applying (3.2) to the pair \((x, yz)\) we get that
\[
f(x(yz)) = f(x)g(yz) + g(x)f(yz) + h(x)h(yz) + \psi(x, yz)
\]
(3.9)
Now, let \( y, z \in G \) be arbitrary. By assumption the functions
\[
x \mapsto \psi(x, y), x \mapsto \psi(x, yz), x \mapsto \psi(x, yz_0), x \mapsto \psi(x, y)
\]
belong to \( \mathcal{V} \). Moreover, since the linear space \( \mathcal{V} \) is two sided invariant the functions
\[
x \mapsto \psi(xy, y), x \mapsto \psi(xy, z), x \mapsto \psi(xy, z_0)
\]
belong to \( \mathcal{V} \). Hence, by using \((3.8)\), \((3.9)\) and the fact that \( f, g \) and \( h \) are linearly independent modulo \( \mathcal{V} \), we get that
\[
(3.10) \quad f(yz) = f(y)g(z) + [\gamma_0 f(y) + \gamma_1 f(yy_0) + \gamma_2 f(yz_0)]f(z)
\]
\[+ [\delta_0 f(y) + \delta_1 f(yy_0) + \delta_2 f(yz_0)]h(z).\]
From \((3.10)\), \((3.7)\) and \((3.11)\) we get
\[
f(yz) = f(y)g(z) + [\gamma_0 f(y) + \gamma_1 \psi(y, y_0) + \gamma_2 \psi(y, z_0)]f(z)
\]
\[+ [\delta_0 f(y) + \delta_1 \psi(y, y_0) + \delta_2 \psi(y, z_0)]h(z) = f(y)g(z) + g(y)f(z) + h(y)h(z) + [\gamma_1 \psi(y, y_0) + \gamma_2 \psi(y, z_0)]f(z)
\]
\[+ [\delta_1 \psi(y, y_0) + \delta_2 \psi(y, z_0)]h(z).
\]
Hence, by using \((3.9)\), we obtain
\[
\psi(y, z) = [\gamma_1 \psi(y, y_0) + \gamma_2 \psi(y, z_0)]f(z) + [\delta_1 \psi(y, y_0) + \delta_2 \psi(y, z_0)]h(z).
\]
So, \( y \) and \( z \) being arbitrary, we deduce \((3.1)\) by putting
\[
\varphi_1(x) := \gamma_1 \psi(x, y_0) + \gamma_2 \psi(x, z_0)
\]
and
\[
\varphi_2(x) := \delta_1 \psi(x, y_0) + \delta_2 \psi(x, z_0)
\]
for all \( x \in G \). This completes the proof of Lemma 3.1. \( \square \)

**Lemma 3.2.** Let \( f, g, h : G \rightarrow \mathbb{C} \) be functions. Suppose that \( f \) and \( h \) are linearly independent modulo \( \mathcal{V} \) and \( g \in \mathcal{V} \). If the function
\[
x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)
\]
belongs to \( \mathcal{V} \) for all \( y \in G \), then \( g \) is multiplicative.

**Proof.** Let \( y, z \in G \) be arbitrary. By using the same computation as the one of the proof of Lemma 3.1 we obtain from \((3.8)\) and \((3.9)\), with the same notations, the following identity
\[
f(x)g(yz) + g(x)f(yz) + h(x)h(yz) + \psi(x, yz)
\]
\[
= f(x)[g(y)g(z) + \gamma_0 g(y)f(z) + \gamma_1 g(yy_0)f(z) + \gamma_2 g(yz_0)f(z) + \delta_0 g(y)h(z)
\]
\[+ \delta_1 g(yy_0)h(z) + \delta_2 g(yz_0)h(z)] + g(x)[f(y)g(z) + \gamma_0 f(y)f(z) + \gamma_1 f(yy_0)f(z)
\]
\[+ \gamma_2 f(yz_0)f(z) + \delta_0 f(y)h(z) + \delta_1 f(yy_0)h(z) + \delta_2 f(yz_0)h(z)] + h(x)[h(y)g(z)
\]
\[+ \gamma_0 h(y)f(z) + \gamma_1 h(yy_0)f(z) + \gamma_2 h(yz_0)f(z) + \delta_0 h(y)h(z) + \delta_1 h(yy_0)h(z)
\]
\[+ \delta_2 h(yz_0)h(z)] + [\gamma_0 \psi(x, y) + \gamma_1 \psi(x, yy_0) + \gamma_2 \psi(x, yz_0) - \gamma_1 \psi(xy, y_0)
\]
\[ - \gamma_2 \psi(xy, z_0)]f(z) - \psi(x, y)g(z) + [\delta_0 \psi(x, y) + \delta_1 \psi(x, yy_0) + \delta_2 \psi(x, yz_0)
\]
\[ - \delta_1 \psi(xy, y_0) - \delta_2 \psi(xy, z_0)]h(z) + \psi(xy, z)
for all $x \in G$. So that
\begin{equation}
(3.11)
f(x)[g(y)g(z) + \gamma_0 g(y)f(z) + \gamma_1 g(yz_0)f(z) + \gamma_2 g(yz_0)f(z) + \delta_0 g(y)h(z)
+ \delta_1 g(yz_0)h(z) + \delta_2 g(yz_0)h(z) - g(yz)] + h(x)[h(y)g(z) + \gamma_0 h(y)f(z)
+ \gamma_1 h(yz_0)f(z) + \gamma_2 h(yz_0)f(z) + \delta_0 h(y)h(z) + \delta_1 h(yz_0)h(z)
+ \delta_2 h(yz_0)h(z) - h(yz)]
= -g(x)[f(y)g(z) + \gamma_0 f(y)f(z) + \gamma_1 f(yz_0)f(z) + \gamma_2 f(yz_0)f(z) + \delta_0 f(y)h(z)
+ \delta_1 f(yz_0)h(z) + \delta_2 f(yz_0)h(z) - f(yz)] - \gamma_0 \psi(x, y) + \gamma_1 \psi(x, yz_0)
+ \gamma_2 \psi(x, yz_0) - \gamma_1 \psi(xy, y_0) - \gamma_2 \psi(xy, z_0)]f(z)
- [\delta_0 \psi(x, y) + \delta_1 \psi(x, yz_0) + \delta_2 \psi(x, yz_0) - \delta_1 \psi(xy, y_0) - \delta_2 \psi(xy, z_0)]h(z)
- \psi(xy, z) + \psi(x, yz)
\end{equation}

for all $x \in G$. Since $g \in V$, the function $x \mapsto \psi(x, t)$ belongs to $V$ for all $t \in G$ and $V$ is a two-sided-invariant linear space of complex-valued functions on $G$, we get that the right hand side of the identity (3.11) belongs to $V$ as a function in $x$, so does the left hand side of (3.11). Since $f$ and $h$ are linearly independent modulo $V$, then we get that
\begin{equation}
(3.12)
g(y)g(z) + \gamma_0 g(y)f(z) + \gamma_1 g(yz_0)f(z) + \gamma_2 g(yz_0)f(z) + \delta_0 g(y)h(z)
+ \delta_1 g(yz_0)h(z) + \delta_2 g(yz_0)h(z) - g(yz) = 0.
\end{equation}

So, $y$ and $z$ being arbitrary, then we get that
\begin{equation}
(3.13)
g(yz) - g(y)g(z) = [\gamma_0 g(y) + \gamma_1 g(yz_0) + \gamma_2 g(yz_0)]f(z)
+ [\delta_0 g(y) + \delta_1 g(yz_0) + \delta_2 g(yz_0)]h(z)
\end{equation}

for all $y, z \in G$. Now, let $y \in G$ be arbitrary. Since $g \in V$ and $V$ is a two-sided-
invariant linear space of complex-valued functions on $G$, we derive from (3.13) that the function
\[z \mapsto [\gamma_0 g(y) + \gamma_1 g(yz_0) + \gamma_2 g(yz_0)]f(z) + [\delta_0 g(y) + \delta_1 g(yz_0) + \delta_2 g(yz_0)]h(z)\]
belongs to $V$. Hence, seeing that $f$ and $h$ are linearly independent modulo $V$, we get that $\gamma_0 g(y) + \gamma_1 g(yz_0) + \gamma_2 g(yz_0) = 0$ and $\delta_0 g(y) + \delta_1 g(yz_0) + \delta_2 g(yz_0) = 0$. Substituting this back into (3.13) we obtain $g(yz) = g(y)g(z)$ for all $z \in G$. So, $y$ being arbitrary, we deduce that $g$ is multiplicative. This completes the proof of Lemma 3.2. \qed

Lemma 3.3. Let $f, g, h : G \to \mathbb{C}$ be functions. Suppose that $f$ and $h$ are linearly
dependent modulo $V$. If the function
\[x \mapsto f(xy) - f(x)g(y) - g(xy)f(y) - h(x)h(y)\]
belongs to $V$ for all $y \in G$, then we have one of the following possibilities:
(1) $f = 0$, $g$ is arbitrary and $h \in V$;
(2) $f, g, h \in V$;
(3) $g + \frac{\lambda^2}{4} f = m - \lambda \varphi$, $h - \lambda f = \varphi$, where $\lambda \in \mathbb{C}$ is a constant, $\varphi \in V$ and $m : G \to \mathbb{C}$ is a multiplicative function such that $m \in V$;
(4) $f = \alpha m - \alpha b$, $g = \frac{1 - \alpha^2}{2} m + \frac{1 + \alpha^2}{2} b - \lambda \varphi$, $h = \alpha \lambda m - \alpha \lambda b + \varphi$, where $\alpha, \lambda \in \mathbb{C}$ are constants, $m : G \to \mathbb{C}$ is a multiplicative function and $b, \varphi \in V$;
that the function $g$ is the result (5) of Lemma 3.3 for a constant $\alpha$ multiplicative function and (iii) $f$ is arbitrary and the function $x \mapsto h(x)h(y)$ belongs to $V$ for all $y$ in $G$. Hence $h \in V$. The result occurs in (1) of Lemma 3.3. In what follows we assume that $f \neq 0$. We have the following cases

**Case 1:** $h \in V$. Then the function $x \mapsto h(x)h(y)$ belongs to $V$ for all $y$ in $G$. So that the function $x \mapsto f(xy) - f(x)g(y) - g(x)f(y)$ belongs to $V$ for all $y$ in $G$. So, according to [13, Lemma 2.2] and taking into account that $f \neq 0$, we get that one of the following possibilities holds

(i) $f, g, h \in V$ which occurs in (2) of Lemma 3.3.
(ii) $g = m$ and $h = \varphi$, where $\varphi \in V$ and $m : G \to C$ is a multiplicative function such that $m \in V$. This is the result (3) of Lemma 3.3 for $\lambda = 0$.
(iii) $f = \alpha m - \alpha b, g = \frac{1}{2}m + \frac{1}{2}b, h = \varphi$, where $\alpha \in C$ is a constant, $m : G \to C$ is a multiplicative function and $b, \varphi \in V$. This is the result (4) of Lemma 3.3 for $\lambda = 0$.
(iv) $f(xy) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$ and $h = \varphi$, where $\varphi \in V$, which is the result (5) of Lemma 3.3 for $\lambda = 0$.

**Case 2:** $h \notin V$. If $f \in V$ then the function $x \mapsto f(xy)$ belongs to $V$ for all $y$ in $G$, because the linear space $V$ is two-sided invariant. As the function $x \mapsto \psi(x, y)$ belongs to $V$ for all $y \in G$ we get that the function $x \mapsto g(x)f(y) + h(x)h(y)$ belongs to $V$ for all $y \in G$. Since $h \notin V$ we have $h \neq 0$. We derive that there exist a constant $\alpha \in C \setminus \{0\}$ and a function $k \in V$ such that

\begin{equation}
(3.14) \quad h = \alpha g + k,
\end{equation}

so that

\begin{align*}
\psi(x, y) &= f(xy) - f(x)g(y) - g(x)f(y) - (\alpha g(x) + k(x))(\alpha g(y) + k(y)) \\
&= f(xy) - f(x)g(y) - g(x)f(y) - \alpha^2 g(x)g(y) - \alpha g(x)k(y) - \alpha k(x)g(y) - k(x)k(y) \\
&= f(xy) - k(x)k(y) - g(x)[f(y) + \alpha^2 g(y) + \alpha k(y)] - g(y)[f(x) + \alpha k(x)] \\
&= f(xy) - k(x)k(y) - g(x)[f(y) + \alpha h(y)] - g(y)[f(x) + \alpha k(x)]
\end{align*}

for all $x, y \in G$. Since the functions $x \mapsto f(xy), x \mapsto k(x)k(y), x \mapsto g(y)[f(x) + \alpha k(x)]$ and $x \mapsto \psi(x, y)$ belong to $V$ for all $y \in G$, we derive from the identity above that the function $x \mapsto g(x)[f(y) + \alpha h(y)]$ belongs to $V$ for all $y \in G$, which implies that $g \in V$ or $f(y) + \alpha h(y) = 0$ for all $y \in G$. Hence, since $\alpha \in C \setminus \{0\}$, we get that $g \in V$ or $h = -\frac{1}{\alpha} f$. So, taking (3.14) into account, we get that $h \in V$; which contradicts the assumption on $h$, hence $f \notin V$. As $f$ and $h$ are linearly dependent modulo $V$ we infer that there exist a constant $\lambda \in C \setminus \{0\}$ and a function $\varphi \in V$ such that

\begin{equation}
(3.15) \quad h = \lambda f + \varphi.
\end{equation}
So we get from (3.2) that
\[
\psi(x, y) = f(xy) - f(x)g(y) - g(x)f(y) - (\lambda f(x) + \varphi(x))(\lambda f(y) + \varphi(y))
\]
\[
= f(xy) - f(x)g(y) - g(x)f(y) - \lambda^2 f(x)f(y) - \lambda f(x)\varphi(y) - \lambda \varphi(x)f(y)
\]
\[
- \varphi(x)\varphi(y)
\]
\[
= f(xy) - \varphi(x)\varphi(y) - f(x)[g(y) + \frac{\lambda^2}{2} f(y) + \lambda \varphi(y)]
\]
\[
- [g(x) + \frac{\lambda^2}{2} f(x) + \lambda \varphi(x)]f(y).
\]
for all \(x, y \in G\), which implies that that
\[
(3.16) \quad \psi(x, y) + \varphi(x)\varphi(y) = f(xy) - f(x)\phi(y) - \phi(x)f(y)
\]
for all \(x, y \in G\), where
\[
(3.17) \quad \phi := g + \frac{\lambda^2}{2} f + \lambda \varphi.
\]
Since \(\varphi \in V\) and the function \(x \mapsto \psi(x, y)\) belongs to \(V\) for all \(y \in G\) we get from (3.16) that the function
\[
x \mapsto f(xy) - f(x)\phi(y) - \phi(x)f(y)
\]
belongs to \(V\) for all \(y \in G\). Moreover \(V\) is a two-sided invariant linear space of complex-valued function. Hence, according to [13, Lemma 2.2] and taking into account that \(f, h \not\in V\), we have one of the following possibilities:

(i) \(\phi = m\) where \(m \in V\) is multiplicative. Then we get, from (3.17) and (3.15), that
\[
g + \frac{\lambda^2}{2} f = m - \lambda \varphi \quad \text{and} \quad h - \lambda f = \varphi, \quad \text{where} \quad \varphi \in V.
\]
The result occurs in (3) of Lemma 3.3.

(ii) \(f = \alpha m - \alpha b, \phi = \frac{1}{2} m + \frac{1}{2} b\), where \(m : G \to \mathbb{C}\) is multiplicative, \(b : G \to \mathbb{C}\) is in \(V\) and \(\alpha \in \mathbb{C}\) is a constant. Taking (3.17) and (3.15) into account, we obtain respectively
\[
g = \frac{1}{2} m + \frac{1}{2} b - \frac{\lambda^2}{2} (\alpha m - \alpha b) - \lambda \varphi
\]
\[
= \frac{1 - \alpha \lambda^2}{2} m + \frac{1 + \alpha \lambda^2}{2} b - \lambda \varphi
\]
and
\[
h = \alpha \lambda m - \alpha \lambda b + \varphi.
\]
So the result (4) of Lemma 3.3 holds.

(iii) \(f(xy) = f(x)\phi(y) + \phi(x)f(y)\) for all \(x, y \in G\). The result (5) of Lemma 3.3 holds easily by using the identities (3.15) and (3.17). This completes the proof of Lemma 3.3. \(\square\)

**Lemma 3.4.** Let \(f, g, h : G \to \mathbb{C}\) be functions. Suppose that \(f\) and \(h\) are linearly independent modulo \(V\). If the functions
\[
x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)
\]
and
\[
x \mapsto f(xy) - f(yx)
\]
belong to \(V\) for all \(y \in G\), then we have one of the following possibilities:

(1) \(f = -\lambda^2 f_0 + \lambda^2 \varphi, \quad g = \frac{1+\rho^2}{2} f_0 + \rho g_0 + \frac{1-\rho^2}{2} \varphi, \quad h = \lambda \rho f_0 + \lambda g_0 - \lambda \rho \varphi, \quad \text{where} \)

\[
\lambda, \rho \in \mathbb{C}, \quad \lambda \neq 0, \quad \lambda \rho = 1.
\]
Subcase A.1

for all \( x, y \in G \);

(2)

for all \( x, y \in G \),

\[
g = \frac{1}{2} \beta^2 f + \beta h + m
\]

and

\[
\beta f + h = \lambda M - \lambda m,
\]

where \( \beta \in \mathbb{C}, \lambda \in \mathbb{C} \setminus \{0\} \) are constants, \( m, M : G \to \mathbb{C} \) are multiplicative functions such that \( m \in \mathcal{V}, M \notin \mathcal{V} \) and \( \psi \) is the function defined in \((3.2)\);

(3)

\[f(xy) = f(x)m(y) + m(x)f(y) + H(x)H(y) + \psi(x,y),\]

\[g = \frac{1}{2} \beta^2 f + \beta h + m\]

and

\[H(xy) - m(x)H(y) - H(x)m(y) = \eta_1 \psi(x,y) + \eta_2 m(x)L_1(y) + \eta_3 m(x)L_2(y)
+ \eta_4 \psi(x,l_1(y)) + \eta_5 \psi(x,l_2(y)) + \eta_6 L_1(xy) + \eta_7 L_2(xy)\]

for all \( x, y \in G \), where \( \beta, \eta_1, \cdots, \eta_7 \in \mathbb{C} \) are constants, \( m : G \to \mathbb{C} \) is a multiplicative function in \( \mathcal{V}, L_1, L_2 \in \mathcal{V}, l_1, l_2 : G \to G \) are mappings, \( H = \beta f + h \) and \( \psi \) is the function defined in \((3.2)\);

(4) \( f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y) \) for all \( x, y \in G \).

Proof. We split the discussion into the cases \( f, g, h \) are linearly dependent modulo \( \mathcal{V} \) and \( f, g, h \) are linearly independent modulo \( \mathcal{V} \).

Case A: \( f, g, h \) are linearly dependent modulo \( \mathcal{V} \). Since \( f \) and \( h \) are linearly independent modulo \( \mathcal{V} \) we get that there exist a function \( \varphi \in \mathcal{V} \) and two constants \( \alpha, \beta \in \mathbb{C} \) such that

(3.18)

\[g = \alpha f + \beta h + \varphi.\]

By substituting \((3.18)\) in \((3.2)\) we obtain

\[
\psi(x,y) = f(xy) - f(x)[\alpha f(y) + \beta h(y) + \varphi(y)] - [\alpha f(x) + \beta h(x) + \varphi(x)]f(y)
- h(x)h(y)
= f(xy) - 2 \alpha f(x)f(y) - f(x)\varphi(y) - \varphi(x)f(y) - \beta f(x)h(y) - \beta h(x)f(y)
- h(x)h(y),
\]

for all \( x, y \in G \), which implies that

(3.19)

\[
\psi(x,y) = f(xy) - (2 \alpha - \beta^2) f(x)f(y) - f(x)\varphi(y) - \varphi(x)f(y)
- h(x)h(y)]
\]

for all \( x, y \in G \). We have the following subcases

Subcase A.1: \( 2 \alpha \neq \beta^2 \). Let \( x, y \in G \) be arbitrary and let \( \delta \in \mathbb{C} \setminus \{0\} \) such that

(3.20)

\[\delta^2 = -(2 \alpha - \beta^2).\]
Multiplying both sides of (3.19) by $-\delta^2$ and then adding $\varphi(xy) - \varphi(x)\varphi(y)$ to both sides of the identity obtained we derive

$$-\delta^2 \psi(x, y) + \varphi(xy) - \varphi(x)\varphi(y) = -\delta^2 f(xy) + \varphi(xy) - [\delta^4 f(x)f(y) - \delta^2 f(x)\varphi(y) - \delta^2 \varphi(xy) - \varphi(x)\varphi(y)] + \delta^2 [\beta f(x) + h(x)][\beta f(y) + h(y)].$$

So, $x$ and $y$ being arbitrary, we get from the identity above that

$$-\delta^2 \psi(x, y) + \varphi(xy) - \varphi(x)\varphi(y) = f_0(xy) - f_0(x)f_0(y) + g_0(x)g_0(y),$$

for all $x, y \in G$, where

$$f_0 := -\delta^2 f + \varphi$$

and

$$g_0 := \delta (\beta f + h).$$

Notice that $f_0$ and $g_0$ are linearly independent modulo $\mathcal{V}$ because $f$ and $h$ are. Now, let $y$ be arbitrary. As $\varphi \in \mathcal{V}$ the function $x \mapsto \varphi(x)\varphi(y)$ belongs to $\mathcal{V}$, and since the linear space $\mathcal{V}$ is two-sided invariant, we get that the function $x \mapsto \varphi(xy)$ belongs to $\mathcal{V}$. Moreover, by assumption the function $x \mapsto \psi(x, y)$ belongs to $\mathcal{V}$. Hence the left hand side of the identity (3.21) belongs to $\mathcal{V}$ as a function in $x$. So that the function

$$x \mapsto f_0(xy) - f_0(x)f_0(y) + g_0(x)g_0(y)$$

belongs to $\mathcal{V}$. On the other hand, by using (3.22), we have

$$f_0(xy) - f_0(yx) = -\delta^2 (f(xy) - f(yx)) + \varphi(xy) - \varphi(yx)$$

for all $x \in G$. So, $y$ being arbitrary, the function $x \mapsto f_0(xy) - f_0(yx)$ belongs to $\mathcal{V}$ for all $y \in G$ because the functions $x \mapsto f(xy) - f(yx)$ and $x \mapsto \varphi(xy) - \varphi(yx)$ do. Moreover $f_0$ and $g_0$ are linearly independent modulo $\mathcal{V}$. Hence we get, according to [13] Lemma 3.1, that

$$f_0(xy) = f_0(x)f_0(y) - g_0(x)g_0(y)$$

for all $x, y \in G$. By putting $\lambda = \frac{1}{\delta}$ we get, from (3.22), that

$$f = -\lambda^2 f_0 + \lambda^2 \varphi.$$

By putting $\rho = \beta \lambda$ we get, from (3.23), that $h = \lambda g_0 - \beta (-\lambda^2 f_0 + \lambda^2 \varphi)$, which implies that

$$h = \lambda \rho f_0 + \lambda g_0 - \lambda \rho \varphi.$$

So, we derive from (3.18), (3.21) and (3.25) that

$$g = \alpha (-\lambda^2 f_0 + \lambda^2 \varphi) + \beta (\lambda \rho f_0 + \lambda g_0 - \lambda \rho \varphi) + \varphi$$

$$= (-\alpha \lambda^2 + \beta \lambda \rho) f_0 + \beta \lambda g_0 + (\alpha \lambda^2 - \beta \lambda \rho + 1) \varphi$$

$$= (-\alpha \lambda^2 + \rho^2) f_0 + \rho g_0 + (\alpha \lambda^2 - \rho^2 + 1) \varphi$$

Using (3.20) we find, by elementary computations, that $\alpha \lambda^2 = \frac{1}{2} \rho^2 - \frac{1}{2}$. Hence, from the identity above, we get that

$$g = \frac{1 + \rho^2}{2} f_0 + \rho g_0 + \frac{1 - \rho^2}{2} \varphi.$$
The result obtained in this case occurs in (1) of Lemma 3.4.

**Subcase A.2:** $2\alpha = \beta^2$. In this case the identity (3.19) becomes

\[(3.26) \quad \psi(x, y) = f(xy) - f(x)\varphi(y) - \varphi(x)f(y) - H(x)H(y)\]

for all $x, y \in G$, where

\[(3.27) \quad H := \beta f + h.\]

Since $f$ and $h$ are linearly independent modulo $\mathcal{V}$ so are $f$ and $H$. Moreover $\varphi \in \mathcal{V}$.

Hence, according to Lemma 3.2, there exists a multiplicative function $m : G \rightarrow \mathbb{C}$ in $\mathcal{V}$ such that $\varphi = m$. So the identities (3.18) and (3.26) become respectively

\[(3.30) \quad g = \frac{1}{2}\beta^2 f + \beta h + m.\]

and

\[(3.31) \quad \psi(x, y) = f(xy) - f(x)m(y) - m(x)f(y) - H(x)H(y)\]

for all $x, y \in G$. We use similar computations to the ones in the proof of [4, Theorem]. Let $x, y, z \in G$ be arbitrary. First we compute $f(xyz)$ as $f(xyz)$ and then as $f((xy)z)$. From (3.24) we get that

\[f(xyz) = f(xy)\psi(y, z) + m(xy)\psi(f, z) + m(x)f(y) + m(x)H(y)H(z)\]

On the other hand

\[f((xy)z) = f(xy)m(z) + m(xy)f(z) + H(x)H(y) + \psi(x, z)\]

so that

\[(3.32) \quad f((xy)z) = f(x)m(z) + m(x)f(y) + m(xy)f(z) + m(x)H(y)H(z)\]

From (3.30) and (3.31) we get that

\[(3.33) \quad f(z_1)H(z_2) - f(z_2)H(z_1) \neq 0.\]

Let $x, y \in G$ be arbitrary. By putting $z = z_1$ and then $z = z_2$ in (3.32) we get respectively

\[(3.34) \quad H(x)k_i(y) - H(z_i)(H(xy) - H(x)m(y) - m(x)H(y)) = \psi_i(x, y)\]

where

\[k_i(y) := H(yz_i) - H(y)m(z_i) - m(y)H(z_i)\]
\[ (3.35) \quad \psi_i(x, y) := m(z_i)\psi(x, y) - m(x)\psi(y, z_i) - \psi(x, yz_i) + \psi(xy, z_i) \]

for \( i = 1, 2 \). Multiplying both sides of \((3.33)\) by \( f(z_2) \) for \( i = 1 \) and by \( f(z_1) \) for \( i = 2 \), and subtracting the identities obtained we get that

\[ (3.36) \quad H(x)k_3(y) + [f(z_1)H(z_2) - f(z_2)H(z_1)]|H(xy) - H(x)m(y) - m(x)H(y)| = \psi_3(x, y), \]

where

\[ k_3(y) := f(z_2)k_1(y) - f(z_1)k_2(y) \]

and

\[ (3.37) \quad \psi_3(x, y) := f(z_2)\psi_1(x, y) - f(z_1)\psi_2(x, y). \]

So, \( x \) and \( y \) being arbitrary, we get, taking \((3.33)\) and \((3.36)\) into account, that

\[ (3.38) \quad H(xy) - H(x)m(y) - m(x)H(y) = H(x)k(y) + \Phi(x, y) \]

for all \( x, y \in G \), where

\[ k(x) := -[f(z_1)H(z_2) - f(z_2)H(z_1)]^{-1}k_3(x) \]

and

\[ (3.39) \quad \Phi(x, y) := [f(z_1)H(z_2) - f(z_2)H(z_1)]^{-1}\psi_3(x, y) \]

for all \( x, y \in G \). Substituting \((3.38)\) into \((3.32)\) we get that

\[
H(x)[H(y)k(z) + \Phi(y, z)] - H(z)[H(x)k(y) + \Phi(x, y)]
\]

\[ = m(z)\psi(x, y) - m(x)\psi(y, z) + \psi(xy, z) - \psi(x, yz), \]

which implies that

\[ (3.40) \quad H(x)[H(y)k(z) - H(z)k(y) + \Phi(y, z)] = H(z)\Phi(x, y) + m(z)\psi(x, y) \]

\[ - m(x)\psi(y, z) + \psi(xy, z) - \psi(x, yz) \]

for all \( x, y, z \in G \). Now let \( y, z \in G \) be arbitrary. Since \( \mathcal{V} \) is a two-sided invariant linear space of complex-valued functions on \( G \), and the functions \( x \mapsto m(x) \) and \( x \mapsto \psi(x, y) \) belong to \( \mathcal{V} \), we deduce from \((3.35)\), \((3.37)\), and \((3.39)\) that the functions \( x \mapsto \Phi(x, y) \) and \( x \mapsto \psi_i(x, y) \) belong to \( \mathcal{V} \) for \( i = 1, 2, 3 \). Hence the right hand side of \((3.40)\) belongs to \( \mathcal{V} \) as a function in \( x \). It follows that the left hand side of \((3.40)\) belongs to \( \mathcal{V} \) as a function in \( x \). As \( f \) and \( H \) are linearly independent modulo \( \mathcal{V} \), we derive, from \((3.40)\), that \( H(y)k(z) - H(z)k(y) + \Phi(y, z) = 0 \). So, \( y \) and \( z \) being arbitrary, we get that

\[ (3.41) \quad H(z)k(x) = H(x)k(z) + \Phi(x, z) \]

for all \( x, z \in G \).

On the other hand we deduce from \((3.33)\) that \( f(z_1)H(z_2) \neq 0 \) or \( f(z_2)H(z_1) \neq 0 \), so we can assume, without loss of generality, that \( H(z_1) \neq 0 \). Replacing \( z \) by \( z_1 \) in the identity \((3.41)\) we derive that

\[ (3.42) \quad k(x) = \gamma H(x) + \Phi_1(x) \]

for all \( x \in G \), where \( \gamma := H(z_1)^{-1}k(z_1) \) and

\[ (3.43) \quad \Phi_1(x) := H(z_1)^{-1}\Phi(x, z_1) \]

for all \( x \in G \). From \((3.38)\) and \((3.42)\) we get that

\[ (3.44) \quad H(xy) = H(x)m(y) + m(x)H(y) + \gamma H(x)H(y) + H(x)\Phi_1(y) + \Phi(x, y) \]
for all \( x, y \in G \). Since the functions \( m \) and \( x \mapsto \Phi(x, y) \) belongs to \( V \) for all \( y \in G \) we get, from (3.44), that the function
\[
(3.45) \quad x \mapsto H(xy) - H(x)[m(y) + \Phi_1(y) + \gamma H(y)]
\]
belongs to \( V \) for all \( y \in G \). As \( H \notin V \) we get from (3.45), according to [12, Theorem], that there exists a multiplicative function \( M : G \to \mathbb{C} \) such that
\[
(3.46) \quad m + \Phi_1 + \gamma H = M.
\]
We have the following subcases

Case A.2.1: \( \gamma \neq 0 \). Putting \( \lambda = \frac{1}{\gamma} \in \mathbb{C} \setminus \{0\} \) we obtain from (3.46) the identity
\[
(3.47) \quad H = \lambda M - \lambda m - \lambda \Phi_1.
\]
Let \( x, y \in G \) be arbitrary. Since \( m \) and \( M \) are multiplicative we get from the identity above that \( H(xy) - H(yx) = \lambda \Phi_1(yx) - \lambda \Phi_1(xy) \). Taking (3.44) into account we get that
\[
H(x)\Phi_1(y) - H(y)\Phi_1(x) + \Phi(x, y) - \Phi(y, x) = \lambda \Phi_1(yx) - \lambda \Phi_1(xy).
\]
So, \( x \) and \( y \) being arbitrary, we get for all \( x, y \in G \), \( H(x)\Phi_1(y) = H(y)\Phi_1(x) + \Phi(x, y) - \Phi(y, x) - \lambda \Phi_1(yx) + \lambda \Phi_1(xy)\) (3.48)
\[
\text{for all } x, y \in G. \text{ Now let } y \text{ be arbitrary. As seen early the functions } \Phi_1 \text{ and } x \mapsto \Phi(x, y) - \Phi(y, x) \text{ belong to } V. \text{ So, } V \text{ being a two-sided invariant linear space of complex-valued functions on } G, \text{ we get from (3.48) that the function } x \mapsto H(x)\Phi_1(y) \text{ belongs to } V. \text{ Taking into account that } f \text{ and } H \text{ are linearly independent, we get } \Phi_1(y) = 0. \text{ So, } y \text{ being arbitrary, we obtain } \Phi_1 = 0. \text{ Hence, using (3.47), we get that }
\[
(3.49) \quad H = \lambda M - \lambda m.
\]
Substituting this back into (3.29) we get, by an elementary computation, that
\[
(3.50) \quad f(xy) - \lambda^2 M(xy) = (f(x) - \lambda^2 M(x))m(y) + m(x)(f(y) - \lambda^2 M(y)) + \lambda^2 m(xy) + \psi(x, y),
\]
for all \( x, y \in G \). We conclude from (3.27), (3.28), (3.49) and (3.50) that the result (2) of Lemma 3.4 holds.

Case A.2.2: \( \gamma = 0 \). Let \( y \in G \) be arbitrary. The identity (3.32) implies that \( k = \Phi_1 \). Hence we derive from (3.41) that
\[
H(x)\Phi_1(y) = H(y)\Phi_1(x) - \Phi(x, y),
\]
for all \( x \in G \). Since the function \( x \mapsto \Phi(x, y) \) belongs to \( V \) we get, taking the identity above and (3.43) into account, that the function \( x \mapsto H(x)\Phi_1(y) \) belongs to \( V \). As \( f \) and \( H \) are linearly independent modulo \( V \) we infer that \( \Phi_1(y) = 0 \). So, \( y \) being arbitrary, we get that \( \Phi_1 = 0 \). Hence the identity (3.44) becomes
\[
(3.51) \quad H(xy) = m(x)H(y) + H(x)m(y) + \Phi(x, y).
\]
On the other hand, by using (3.35), (3.37) and (3.39) we derive, using the same notations, that there exist \( \eta_i \in \mathbb{C} \) with \( i = 1, \cdots, 7 \) such that
\[
\Phi(x, y) = \eta_1 \text{ } \Psi(x, y) + \eta_2 \text{ } m(x)\Psi(y, z_1) + \eta_3 \text{ } m(x)\Psi(y, z_2) + \eta_4 \text{ } \psi(x, yz_1) + \eta_5 \text{ } \psi(x, yz_2) + \eta_6 \text{ } \psi(xy, z_1) + \eta_7 \text{ } \psi(xy, z_2)
\]
x, y \in G. We get that
\[
(3.52) \quad \Phi(x, y) = \eta_1 \text{ } \Psi(x, y) + \eta_2 \text{ } m(x)\Psi_1(y) + \eta_3 \text{ } m(x)\Psi_2(y) + \eta_4 \text{ } \psi(x, l_1(y))
\]
+ \eta_5 \text{ } \psi(x, l_2(y)) + \eta_6 \text{ } L_1(xy) + \eta_7 \text{ } L_2(xy).
for all \( x, y \in G \), where

\[ L_i(x) := \psi(x, z_i) \]

for \( i = 1, 2 \) and for all \( x \in G \), and \( l_i : G \to G \) is defined for \( i = 1, 2 \) by \( l_i(x) = xz_i \) for all \( x \in G \). Hence we get from (3.51) and (3.48) the identity

\[ (3.53) \]

\[ H(xy) - m(x)H(y) - H(x)m(y) = \eta_1 \psi(x, y) + \eta_2 m(x)L_1(y) + \eta_3 m(x)L_2(y) + \eta_4 \psi(x, l_1(y)) + \eta_5 \psi(x, l_2(y)) + \eta_6 L_1(xy) + \eta_7 L_2(xy) \]

for all \( x, y \in G \).

We conclude from (3.27), (3.28), (3.29) and (3.53) that the result (3) of Lemma 3.4 holds.

**Case B.** \( f, g \) and \( h \) are linearly independent modulo \( \mathcal{V} \). Then, according to Lemma 3.1, there exist two functions \( \varphi_1, \varphi_2 \in \mathcal{V} \) satisfying (3.1), where \( \psi \) is the function defined in (3.2). Let \( y \in G \) be arbitrary. Since the functions \( x \mapsto \psi(x, y) \) and \( x \mapsto f(xy) - f(yx) \) belong to \( \mathcal{V} \) by assumption, so does the function \( x \mapsto \psi(y, x) \). Seeing that \( \psi(y, x) = \varphi_1(y)f(x) + \varphi_2(y)h(x) \), and that \( f \) and \( h \) are linearly independent modulo \( \mathcal{V} \), we get that \( \varphi_1(y) = \varphi_2(y) = 0 \). So, \( y \) being arbitrary, we deduce that \( \psi(x, y) = 0 \) for all \( x, y \in G \). Then the result (4) of Lemma 3.4 holds. This completes the proof of Lemma 3.4.

\[ \square \]

### 4. Stability of the Cosine-Sine Functional Equation on Amenable Groups

Throughout this section \( G \) is an amenable group with an identity element that we denote by \( e \). We will extend the Székelyhidi’s results [13, Theorem 2.3], about the stability of the functional equation (1.2), to the functional equation (1.1).

**Theorem 4.1.** Let \( f, g, h : G \to \mathbb{C} \) be functions. The function

\[ (x, y) \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y) \]

is bounded if and only if one of the following assertions holds:

1. \( f = 0, g \) is arbitrary and \( h \in \mathcal{B}(G) \);
2. \( f, g, h \in \mathcal{B}(G) \);
3. \( \left\{ \begin{array}{l} f = am + \varphi, \\ g = (1 - \frac{\varphi}{a})m - \lambda b - \frac{\varphi}{a} \varphi, \\ h = \lambda a m + b + \lambda \varphi, \end{array} \right. \)
   where \( \lambda \in \mathbb{C} \) is a constant, \( a : G \to \mathbb{C} \) is an additive function, \( m : G \to \mathbb{C} \) is a bounded multiplicative function and \( b, \varphi : G \to \mathbb{C} \) are two bounded functions;
4. \( \left\{ \begin{array}{l} f = \alpha m - \alpha b, \\ g = \frac{1-\alpha \lambda^2}{2} m + \frac{1+\alpha \lambda^2}{2} b - \lambda \varphi, \\ h = \alpha \lambda m - \alpha \lambda b + \varphi, \end{array} \right. \)
   where \( \alpha, \lambda \in \mathbb{C} \) are two constants, \( m : G \to \mathbb{C} \) is a multiplicative function and \( b, \varphi : G \to \mathbb{C} \) are two bounded functions;
5. \( \left\{ \begin{array}{l} f = f_0, \\ g = g = g_0 - \frac{\lambda^2}{2} f_0 - \lambda b, \\ h = \lambda f_0 + b, \end{array} \right. \)
where \( \lambda \in \mathbb{C} \) is a constant, \( b : G \to \mathbb{C} \) is a bounded function and \( f_0, g_0 : G \to \mathbb{C} \) are functions satisfying the sine functional equation

\[
f_0(xy) = f_0(x)g_0(y) + g_0(x)f_0(y), \quad x, y \in G;
\]

(6)

\[
\begin{aligned}
  f &= -\lambda^2 f_0 + \lambda^2 b, \\
g &= \frac{1 + \lambda^2}{2} f_0 + \rho g_0 + \frac{1 - \rho^2}{2} b, \\
h &= \lambda \rho f_0 + \lambda g_0 - \lambda \rho b,
\end{aligned}
\]

where \( \rho \in \mathbb{C}, \lambda \in \mathbb{C} \setminus \{0\} \) are two constants, \( b : G \to \mathbb{C} \) is a bounded function and \( f_0, g_0 : G \to \mathbb{C} \) are functions satisfying the cosine functional equation

\[
f_0(xy) = f_0(x)f_0(y) - g_0(x)g_0(y), \quad x, y \in G;
\]

(7)

\[
\begin{aligned}
f &= \lambda^2 M + a m + b, \\
g &= \beta \lambda(1 - \frac{1}{2}\beta \lambda)M + (1 - \beta \lambda) m - \frac{1}{2} \beta^2 a m - \frac{1}{2} \beta^2 b, \\
h &= \lambda(1 - \beta \lambda) M - \lambda m - \beta a m - \beta b,
\end{aligned}
\]

where \( \beta \in \mathbb{C}, \lambda \in \mathbb{C} \setminus \{0\} \) are two constants, \( m, M : G \to \mathbb{G} \) are two multiplicative functions such that \( m \) is bounded, \( a : G \to \mathbb{C} \) is an additive function and \( b : G \to \mathbb{C} \) is a bounded function;

(8)

\[
\begin{aligned}
f &= \frac{1}{2} a^2 m + \frac{1}{2} a_1 m + b, \\
g &= -\frac{1}{2} \beta^2 a^2 m + \beta a m - \frac{1}{2} \beta^2 a_1 m + m - \frac{1}{2} \beta^2 b, \\
h &= -\beta a^2 m + a m - \frac{1}{2} \beta a_1 m - \beta b,
\end{aligned}
\]

where \( \beta \in \mathbb{C} \) is a constant, \( m : G \to \mathbb{C} \) is a nonzero bounded multiplicative function, \( a, a_1 : G \to \mathbb{C} \) are two additive functions such that \( a \neq 0 \) and \( b : G \to \mathbb{C} \) is a bounded function;

(9) \( g = -\frac{1}{2} \beta^2 f + (1 + \beta a)m + \beta b \) and \( h = -\beta f + a m + b \), where \( \beta \in \mathbb{C} \) is a constant and \( a : G \to \mathbb{C} \) is an additive function, \( m : G \to \mathbb{C} \) is a nonzero bounded multiplicative function and \( b : G \to \mathbb{C} \) is a bounded function such that the function

\[
(x, y) \mapsto f(xy)m((xy)^{-1}) - \frac{1}{2} a^2(xy) - (f(x)m(x^{-1}) - \frac{1}{2} a^2(x)) - (f(y)m(y^{-1}) - \frac{1}{2} a^2(y)) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1})
\]

is bounded;

(10) \( f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y) \) for all \( x, y \in G \).

**Proof.** First we prove the necessity. Applying the Lemma 3.3(1), Lemma 3.3(2), Lemma 3.3(4), Lemma 3.3(5), Lemma 3.4(1) and Lemma 3.4(4) with \( V = B(G) \) we get that either one of the conditions (1), (2), (4), (5), (6), (10) in Theorem 4.1 is satisfied, or we have one of the following cases:

**Case A:**

\[
g + \frac{\lambda^2}{2} f = m - \lambda b
\]

and

\[
h - \lambda f = b,
\]
where $\lambda \in \mathbb{C}$ is a constant, $b : G \to \mathbb{C}$ is a bounded function and $m : G \to \mathbb{C}$ is a bounded multiplicative function. From (3.2) and the identities above we obtain, by an elementary computation,

\[(4.1)\quad g = -\frac{\lambda^2}{2} f + m - \lambda b,\]

\[(4.2)\quad h = \lambda f + b\]

and

\[(4.3)\quad f(xy) - f(x)m(y) - m(x)f(y) = \psi(x, y) + b(x)b(y)\]

for all $x, y \in G$. If $m \neq 0$ then, by multiplying both sides of (4.3) by $m((xy)^{-1})$, and using the fact that $m$ is a bounded multiplicative function, and that the functions $b$ and $\psi$ are bounded, we get that the function $(x, y) \mapsto f(xy)m((xy)^{-1}) - f(x)m(x^{-1}) - f(y)m(y^{-1})$ is bounded. Notice that we have the same result if $m = 0$. So, according to Hyers’s theorem [11, Theorem 3.1], there exist an additive function $a : G \to \mathbb{C}$ and a function $\varphi_0 \in \mathcal{B}(G)$ such that $f(x)m(x^{-1}) - a(x) = b_0(x)$ for all $x \in G$. Then, by putting $\varphi = m\varphi_0$, we get that $f = a + \varphi$ with $\varphi \in \mathcal{B}(G)$. Substituting this back into (4.1) and (4.2) we obtain, by an elementary computation, that $g = (1 - \frac{\lambda^2}{2} a)m - \lambda b - \frac{\lambda^2}{2} \varphi$ and $h = \lambda a + b + \lambda \varphi$. So the result (3) of Theorem 4.1 holds.

Case B:

\[
f(xy) - \lambda^2 M(xy) = (f(x) - \lambda^2 M(x))m(y) + m(x)(f(y) - \lambda^2 M(y)) + \lambda^2 m(xy) + \psi(x, y)\]

for all $x, y \in G$,

\[
g = \frac{1}{2} \beta^2 f + \beta h + m\]

and

\[
\beta f + h = \lambda M - \lambda m,\]

where $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants, $M : G \to \mathbb{C}$ are multiplicative functions such that $m \in \mathcal{B}(G)$, $M \notin \mathcal{B}(G)$ and $\psi$ is the function defined in (3.2). If $m \neq 0$ then, by multiplying both sides of the first identity above by $m((xy)^{-1})$ and using that $m$ is multiplicative, we get that

\[
(f(xy) - \lambda^2 M(xy))m((xy)^{-1}) = (f(x) - \lambda^2 M(x))m(x^{-1}) + (f(y) - \lambda^2 M(y))m(y^{-1}) + \lambda^2 + m((xy)^{-1})\psi(x, y)\]

for all $x, y \in G$. Since the functions $m$ and $\psi$ are bounded, then we get from the identity above that

\[
(x, y) \mapsto (f(xy) - \lambda^2 M(xy))m((xy)^{-1}) - (f(x) - \lambda^2 M(x))m(x^{-1}) - (f(y) - \lambda^2 M(y))m(y^{-1})\]

is bounded. Notice that we have the same result if $m = 0$. So, according to Hyers’s theorem [11, Theorem 3.1], there exist an additive function $a : G \to \mathbb{C}$ and a function $b_0 \in \mathcal{B}(G)$ such that

\[
(f(x) - \lambda^2 M(x))m(x^{-1}) - a(x) = b_0(x)\]

for all $x \in G$. Then, by putting $b = m b_0$, we derive that

\[
f = \lambda^2 M + a m + b\]
with $b \in \mathcal{B}(G)$. As $g = \frac{1}{2} \beta^2 f + \beta h + m$ and $\beta f + h = \lambda M - \lambda m$, we obtain

$$h = -\beta(\lambda^2 M + a m + b) + \lambda M - \lambda m$$

$$= \lambda(1 - \beta \lambda) M - \lambda m - \beta a m - \beta b$$

and

$$g = \frac{1}{2} \beta^2 (\lambda^2 M + a m + b) + \beta(\lambda(1 - \beta \lambda)M - \lambda m - \beta a m - \beta b) + m$$

$$= \beta \lambda(1 - \frac{1}{2} \beta \lambda) M + (1 - \beta \lambda)m - \frac{1}{2} \beta^2 a m - \frac{1}{2} \beta^2 b.$$  

The result occurs in (7) of Theorem 4.1.

Case C:

$$f(xy) = f(x)m(y) + m(x)f(y) + H(x)H(y) + \psi(x,y),$$

$$H(xy) - H(x)m(y) - m(x)H(y) = \eta_1 \psi(x,y) + \eta_2 m(x)L_1(y) + \eta_3 m(x)L_2(y)$$

$$+ \eta_4 \psi(x,l_1(y)) + \eta_5 \psi(x,l_2(y)) + \eta_6 L_1(xy) + \eta_7 L_2(xy)$$

for all $x, y \in G$,

$$g = \frac{1}{2} \beta^2 f + \beta h + m$$

and

$$H = \beta f + h$$

and where $\beta, \eta_1, \cdots, \eta_7 \in \mathbb{C}$ are constants, $m : G \to \mathbb{C}$ is a bounded multiplicative function, $L_1, L_2 \in \mathcal{B}(G)$, $l_1, l_2 : G \to G$ are mappings, and $\psi$ is the function defined in [3,2].

If $H \in \mathcal{B}(G)$ then $f$ and $h$ are linearly dependent modulo $\mathcal{B}(G)$. So, according to Lemma 3.3, on of the assertions (1)-(5) of Theorem 4.1 holds.

In what follows we assume that $H \notin \mathcal{B}(G)$. Since the functions $m, L_1, L_2$ and $\psi$ are bounded, we get from the above second identity that the function

$$(x, y) \mapsto H(xy) - H(x)m(y) - m(x)H(y)$$

is bounded. Hence $m \neq 0$ because $H \notin \mathcal{B}(G)$. Then, according to [13, Theorem 2.3] and taking the assumption on $H$ into account, we have one of the following subcases:

Subcase C.1: $H = am + b$, where $a : G \to \mathbb{C}$ is additive and $b \in \mathcal{B}(G)$. Then $\beta f + h = am + b$, which implies that

$$h = -\beta f + am + b.$$  

Moreover, since $g = \frac{1}{2} \beta^2 f + \beta h + m$ we get that

$$g = -\frac{1}{2} \beta^2 f + m + \beta am + \beta b.$$  

Let $x, y \in G$ be arbitrary. By using the first identity in the present case, we get that

$$\psi(x,y) = f(xy) - f(x)m(y) - m(x)f(y) - (a(x)m(x) + b(x))(a(y)m(y) + b(y))$$

$$= f(xy) - f(x)m(y) - m(x)f(y) - a(x)a(y)m(xy) - m(x)a(x)b(y)$$

$$- m(y)a(y)b(x) - b(x)b(y).$$
Since $m$ is a nonzero multiplicative function on the group $G$ we have $m(xy) = m(x)m(y) \neq 0$ and $m((xy)^{-1}) = m(x^{-1})m(y^{-1}) = (m(x))^{-1}(m(y))^{-1}$. Hence, by multiplying both sides of the identity above we get that

$$m((xy)^{-1})[\psi(x,y)b(x)b(y)] = f(xy)m((xy)^{-1}) - f(x)m(x^{-1}) - f(y)m(y^{-1})$$

$$- a(x)a(y) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1})$$

$$= (f(xy)m((xy)^{-1}) - \frac{1}{2}a^2(xy)) - (f(x)m(x^{-1}) - \frac{1}{2}a^2(x))$$

$$- (f(y)m(y^{-1}) - \frac{1}{2}a^2(y)) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1}).$$

So, $x$ and $y$ being arbitrary, and the functions $m$, $b$ and $\psi$ are bounded, we deduce that the function

$$(x, y) \mapsto f(xy)m((xy)^{-1}) - \frac{1}{2}a^2(xy) - (f(x)m(x^{-1}) - \frac{1}{2}a^2(x))$$

$$- (f(y)m(y^{-1}) - \frac{1}{2}a^2(y)) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1})$$

is bounded. The result occurs in (9) of the list of Theorem 4.1.

Subcase C.2: $H(xy) = H(x) + H(y)m(x)$ for all $x, y \in G$. Since $m$ is a nonzero multiplicative function on the group $G$ we have $m(x) \neq 0$ for all $x \in G$. Then, in view of $H \notin B(G)$, we get from the last functional equation that there exists a nonzero additive function $a : G \to \mathbb{C}$ such that $H = a m$. Substituting this back in the first identity in the present case and proceeding exactly as in Subcase C.1, we get that the function

$$(x, y) \mapsto 2f(xy)m((xy)^{-1}) - a^2(xy) - (2f(x)m(x^{-1}) - a^2(x))$$

$$- (2f(y)m(y^{-1}) - a^2(y))$$

is bounded. Hence, according to Hyers’s theorem [11, Theorem 3.1], there exist an additive function $a_1 : G \to \mathbb{C}$ and a function $b_0 \in B(G)$ such that $2f(x)m(x^{-1}) - a^2(x) = a_1(x) + b_0(x)$ for all $x, y \in G$. So, by putting $b = \frac{1}{2}m b_0$ we deduce that $b \in B(G)$ because $m, b_0 \in B(G)$ and

$$(4.4) \quad f = \frac{1}{2}a^2 m + \frac{1}{2}a_1 m + b.$$

Since $H = \beta f + h$ and $g = \frac{1}{2} \beta^2 f + \beta h + m$ we get, by using (4.4) and an elementary computation, that $g = -\frac{1}{4}\beta^2 a^2 m + \beta a m - \frac{1}{2}\beta^2 a_1 m + m - \frac{1}{2}\beta^2 b$ and $h = -\frac{1}{2}\beta a^2 m + a m - \frac{1}{2}\beta a_1 m - \beta b$. The result occurs in (8) of the list of Theorem 4.1.

Conversely, we check by elementary computations that if one of the assertions (1)-(10) in Theorem 4.1 is satisfied then the function $(x, y) \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$ is bounded. This completes the proof of Theorem 4.1.

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