FINITE GROUPS OF AUTOMORPHISMS OF ENRIQUES SURFACES AND THE MATHIEU GROUP $M_{12}$

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Abstract. An action of a group $G$ on an Enriques surface $S$ is called Mathieu if it acts on $H^0(2K_S)$ trivially and every element of order 2, 4 has Lefschetz number 4. A finite group $G$ has a Mathieu action on some Enriques surface if and only if it is isomorphic to a subgroup of the symmetric group $S_6$ of degree 6 and the order $|G|$ is not divisible by $2^4$. The 'if' part is proved constructing explicit actions of three groups $S_5, N_{72}$ and $H_{192}$ on polarized Enriques surfaces of degree 30, 18 and 6, respectively and using the result by Keum–Oguiso–Zhang for the alternating group $A_6$.

A (holomorphic) action of a group on a $K3$ surface $X$ is symplectic if it acts on $H^0(K_X) \simeq \mathbb{C}$ trivially. The finite groups which can act symplectically on $K3$ surfaces are classified in [15], relating with the Mathieu group $M_{23}$. There are exactly eleven maximal groups

(1) $L_2(7), \mathfrak{A}_6, \mathfrak{S}_5, M_{20}, F_{384}, \mathfrak{A}_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}$

among them. In this article we give a similar classification for Enriques surfaces, relating with the symmetric group $S_6$ of degree 6 embedded in the Mathieu group $M_{12}$.

A (minimal) Enriques surface $S$ is a smooth complete algebraic surface with $\varrho = h^1(O_S) = 0$, $p_g = h^0(K_S) = 0$ and $2K_S \sim 0$. Equivalently, $S$ is the quotient of a $K3$ surface $X$ by a (fixed point) free involution $\varepsilon$. An action of a group on an Enriques surface $S$ is semi-symplectic if it acts on $H^0(2K_S) \simeq \mathbb{C}$ trivially. If a finite automorphism $\sigma$ is semi-symplectic, then we have $\text{ord}(\sigma) \leq 6$ (Corollary 3.7) and the Lefschetz number $L(\sigma)$ takes the following values.

(2) \[
\begin{array}{c|cccccc}
\text{order} & 1 & 2 & 3 & 4 & 5 & 6 \\
L(\sigma) & 12 & -4, -2, \ldots, 12 & 3 & 4 & 2 & 1, 3 \\
\end{array}
\]

Although the Lefschetz number of a finite symplectic automorphism depends only on its order for $K3$ surfaces, the above table disproves the similar statement for semi-symplectic automorphisms of Enriques surfaces (see also Example 7 below). In this paper we study the most natural subcases from

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the viewpoint of the small Mathieu group $M_{12}$. A classification of general semi-symplectic actions will be discussed elsewhere.

The Mathieu group $M_{12}$ is a finite simple group of sporadic type. $M_{12}$ acts on $\Omega_+ = \{1, \ldots, 12\}$ quintuply transitively and is of order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$. The stabilizer subgroup of $M_{12}$ at a point $\star \in \Omega_+$ is denoted by $M_{11}$. The number of fixed points of the permutation action $M_{11} \ltimes \Omega_+$ depends only on the order and is given as follows.

| order | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 11 |
|-------|---|---|---|---|---|---|---|----|
| fixed points | 12 | 4 | 3 | 4 | 2 | 1 | 2 | 1 |

After Table (3), we make the following

**Definition 1.** A semi-symplectic action of a group $G$ on an Enriques surface is *Mathieu* if the Lefschetz number of $g \in G$ depends only on $\text{ord}(g)$ and coincides with the table (3).

By Lemma 3.10, it suffices for elements $g \in G$ of order 2, 4 to have the Lefschetz number 4. Mathieu actions are automatically effective by definition. Our main result is as follows.

**Theorem 2.** For a finite group $G$, the following two conditions are equivalent to each other.

1. $G$ has a Mathieu action on some Enriques surface.
2. $G$ can be embedded into the symmetric group $S_6$ and the order $|G|$ is not divisible by $2^4$.

The following examples are the key of the proof of (2) $\Rightarrow$ (1).

**Example 3.** Let $X$ be the minimal resolution of the complete intersection

$$
\sum_{i<j} x_ix_j = \sum_{i<j} \frac{1}{x_ix_j} = 0
$$

in $\mathbb{P}^4$ and $S$ its quotient by the involution induced by the Cremona transformation $\varepsilon : (x_i) \mapsto (1/x_i)$. Then the natural action of $S_5$, the third group of (1), on $S$ is (semi-symplectic and) Mathieu (§§1.2).

**Example 4.** Let $S = X/\varepsilon$ be the quotient of the complete intersection

$$
X : x_i^2 - (1 + \sqrt{3})x_{i-1}x_{i+1} = y_i^2 - (1 - \sqrt{3})y_{i-1}y_{i+1}, \quad i = 0, 1, 2 \in \mathbb{Z}/3
$$

in $\mathbb{P}^5$ by the involution $\varepsilon : (x : y) \mapsto (x : -y)$. A subgroup $C_4^2 : C_4$ of the Hessian group $G_{216}$ acts on $X$ linearly and also on $S$. This action extends to that of $N_{72}$, the ninth of (1). The extended action is Mathieu (§§1.3).

**Example 5.** Keum–Oguiso–Zhang [11, 12] constructs a $K3$ surface with a group action by $\mathbb{A}_6 : C_4$. The quotient by the (unique) central involution of $\mathbb{A}_6 : C_4$ is an Enriques surface. The induced action on it by $\mathbb{A}_6$, the second of (1), is Mathieu (§§1.4).
Remark 6. In fact, we can show that the Enriques surfaces in Examples 4, 5 are isomorphic to each other. Thus $S$ has both actions by $\mathfrak{A}_6$ and $N_{72}$. The details will be published elsewhere (cf. Remark A.10).

The eighth group $H_{192}$ of (I), not a subgroup of $M_{11}$, has the following action.

**Example 7. The surface**

$$X : v^2w^2 + w^2u^2 + u^2v^2 + 1 + \sqrt{-1}(u^2 + v^2 + w^2 + u^2v^2w^2) = 0$$

of tri-degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth K3 surface with a symplectic action of the group $H_{192}$, where $u, v, w$ are inhomogeneous coordinates of three projective lines. The involution $\varepsilon : (u, v, w) \mapsto (-u, -v, -w)$ is free, commutes with the action and hence the quotient Enriques surface $S = X/\varepsilon$ has a semi-symplectic action of $H_{192}$.

The involutions

$$(u, v, w) \mapsto (u, -v, -w) \quad \text{and} \quad (u, v, w) \mapsto (-\sqrt{-1}/u, -\sqrt{-1}/v, -\sqrt{-1}/w)$$

are Mathieu, but $(u, v, w) \not\mapsto (u, v, w)$ is not. (In fact the Lefschetz number equals $2$ on $S$.) Thus Example 7 is not Mathieu. But we can find two Mathieu sub-actions by $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$, see [16] and also Section 2. Though neither is a subgroup of $M_{11}$, both are subgroups of $\mathfrak{S}_6$.

**Theorem 8.** The two conditions in Theorem 4 are equivalent to the following:

1. $G$ is a subgroup of one of the five maximal groups $\mathfrak{A}_6, S_5, N_{72}, C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$.
2. $G$ has a small Mathieu representation with $\dim V^G \geq 3$, its $2$-Sylow subgroup is embeddable into $\mathfrak{S}_6$ and $G \not\cong Q_{12}$, the quaternion group of order $12$.

The construction of the paper is as follows. We prove (3) $\Rightarrow$ (1) of Theorems 2 and 8 for the three groups $S_5, N_{72}, \mathfrak{A}_6$ in a refined form (Theorem 1.2) in Section 1. Mathieu actions for the other groups $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$ are constructed in Section 2. They are nothing but the groups studied in detail in [16], but here we give a slightly different treatment. We give a preliminary study of semi-symplectic and Mathieu automorphisms in Section 3. In Section 4 we study groups with small Mathieu representations. Finally in Section 5 we prove the other implications of Theorem 2 and Theorem 8. Especially in Subsection 5.2 we classify all finite groups satisfying the equivalent conditions of main theorems. In the appendix an Enriques analogue of [14, Appendix], from which this article stems, is presented to give an alternative proof of Theorem 1.2.

**Notation and conventions.**

Algebraic varieties $X$ are considered over the complex number field $\mathbb{C}$. For a smooth variety $X$, $K_X$ denotes the canonical divisor class.
Notation of finite groups follows [15]. In particular, $C_n$, $D_{2n}$, $Q_{4n}$, $S_n$, $A_n$ denotes the cyclic group of order $n$, the dihedral group of order $2n$, the quaternion group of order $4n$, the symmetric group of degree $n$, the alternating group of degree $n$ respectively. For groups $A$ and $B$, $A : B$ denotes a split extension with normal subgroup $A$.

The Mathieu group $M_{24}$ acts on the operator domain $\Omega$ consisting of 24 points and the small Mathieu group $M_{12}$ is the stabilizer of a dodecad, denoted by $\Omega +$. The symbol $U$ denotes the rank 2 lattice given by the symmetric matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Root lattices $A_n$, $D_n$ and $E_n$ are considered negative definite. The lattice obtained from a lattice $L$ by replacing the bilinear form $(\cdot, \cdot)$ with $r(\cdot, \cdot)$, $r$ being a rational number, is denoted by $L(r)$.

1. Examples of Mathieu actions

We recall that the symmetric group $S_6$ is a subgroup of $M_{24}$ and decomposes the operator domain $\Omega$ into four orbits $\Omega_2, \Omega_{10}, \Omega_6, \Omega'_6$ of length 2, 10, 6 and 6 ([5]). Two permutation representations on the orbits $\Omega_6$ and $\Omega'_6$ of length 6 differ by the nontrivial outer automorphism of $S_6$. The union $\Omega_6 \cup \Omega'_6$ is a special dodecad, and so is its complement $\Omega_2 \cup \Omega_{10}$. Thus $S_6$ is embedded into $M_{12}$ in two different ways.

Among the eleven groups (1), three groups $S_5, N_{72}, A_6$ are subgroups of $S_6$. (The second one, $N_{72} \simeq C_3^2 : D_8$, is the normalizer of a 3-Sylow subgroup.) By the embedding $S_6 \hookrightarrow M_{24}$ above mentioned, they are also subgroups of $M_{11}$. More precisely we have the following:

**Lemma 1.1.** Each of three groups $S_5, N_{72}, A_6$ is embedded into $M_{11}$ so that it decomposes the operator domain $\Omega_+ \setminus \{\ast\}$ into two orbits. The orbit length $\{a, b\}$ are $\{5, 6\}$, $\{2, 9\}$, $\{1, 10\}$, respectively.

In fact, the three groups are isomorphic to the stabilizer of $S_6 \curvearrowright \Omega_i$ with $i = 6, 10, 2$, respectively.

As is well-known, the free part $H^2(S, \mathbb{Z})_f \simeq \mathbb{Z}^{10}$ of the second cohomology group of an Enriques surface $S$, equipped with the cup product, is isomorphic to the lattice associated with the diagram $T_{2,3,7}$ (Figure 1). The Weyl group of $T_{2,3,7}$ contains $S_5 \times S_6, S_2 \times S_9$ and $S_{10}$ as Weyl subgroups. (The three subgroups become visible by removing the corresponding vertices shown in Figure 1.) Hence, via $M_{11}$ and the Weyl groups, each of the three groups $G =$
\( \mathfrak{S}_5, N_{72}, \mathfrak{A}_6 \) acts isometrically on \( H^2(S, \mathbb{Z})_f \). The invariant part \( H^2(S, \mathbb{Z})_{G_f} \) is generated by an element of square length \( ab \) and its orthogonal complement is isomorphic to the root lattice \( A_{a-1} \oplus A_{b-1} \). In what follows, we construct a semi-symplectic action of \( G \) on an Enriques surface \( S \) which is not only Mathieu but also realizes the above \( G \)-action on \( H^2(S, \mathbb{Z})_f \). Note that, since these groups are generated by involutions, the action is automatically semi-symplectic by Proposition 3.5.

**Theorem 1.2.** The cohomological action \( G \curvearrowright H^2(S, \mathbb{Z})_f \) of three groups in Examples 3, 4 and 5 are \( G \)-equivariantly isomorphic to the one described above. In particular, the actions \( G \curvearrowright S \) in Examples 3, 4 and 5 are Mathieu, proving (3) \( \Rightarrow \) (1) of Theorems 2 and 8. Furthermore, the \( G \)-invariant (primitive) polarization is unique (up to numerical equivalence) and is of degree \( ab \).

We begin with some lattice-theoretic lemmas and then proceed to construct the group actions, proving the theorem for each group.

**1.1. Some lattice theory.** For an ample divisor \( h \) on an Enriques surface \( S \), we define the *gonality function* \( \Phi \) by

\[
\Phi(h) = \min \{(f,h) \mid f \in H^2(S, \mathbb{Z})_f \text{ is primitive with } (f^2) = 0 \text{ and } (f,h) > 0 \}.
\]

A gonality half-pencil \( f \) for \( h \) is an isotropic element \( f \in H^2(S, \mathbb{Z})_f \) satisfying \( (f,h) = \Phi(h) \). It is easy to see that every such \( f \) is nef and defines an elliptic pencil \( |2f| \) on \( S \). It follows that if \( h \) is a \( G \)-invariant polarization, \( G \) permutes gonality half-pencils for \( h \).

**Lemma 1.3.** We have the following.

1. Polarisations \( h \) with \( (h^2) = 10 \) and \( \Phi(h) = 3 \) are unique up to the orthogonal group \( O(T_{2,3,7}) \). Moreover, there are exactly ten gonality half-pencils for such \( h \).
2. The same holds for \( (h^2) = 18 \) and \( \Phi(h) = 4 \). Moreover, there are exactly nine gonality half-pencils for such \( h \).
3. The same holds for \( (h^2) = 30 \) and \( \Phi(h) = 5 \). Moreover, there are exactly six gonality half-pencils for such \( h \).

**Proof.** In each item, the former assertion is an easy consequence of the construction of the dual basis \( b_i \) (\( 0 \leq i \leq 9 \)) of \( r_i \) (in Figure 1) as in [9] (1.3)]. We have \( h = b_0, h = b_2 \) and \( h = b_4 \) respectively. For the latter assertions, we note the following decompositions of \( h \) into isotropic elements in terms of the isotropic sequence \( f_1, \ldots, f_{10} \):

\[
b_0 = \frac{f_1 + \cdots + f_{10}}{3}, \quad b_2 = \frac{(b_0 - f_1 - f_2) + f_3 + \cdots + f_{10}}{2}, \quad b_4 = f_5 + \cdots + f_{10}.
\]

By Cauchy-Schwarz inequality, we see that the sets \{\( f_i \mid 1 \leq i \leq 10 \}\}, \{\( b_0 - f_1 - f_2, f_3, \ldots, f_{10} \)\} and \{\( f_i \mid 5 \leq i \leq 10 \)\} give gonality half-pencils respectively. \( \square \)
1.2. Mathieu action of $S_5$. We consider the surface in $\mathbb{P}^4$ defined by

$$X: \sum_{1 \leq i < j \leq 5} x_i x_j = \sum_{1 \leq i < j \leq 5} \frac{1}{x_i x_j} = 0,$$

which has five nodes at the coordinate points and whose minimal desingularization $X$ is a $K3$ surface. The Cremona transformation $\varepsilon: (x_i) \mapsto (1/x_i)$ induces a free involution and we let $S$ be the Enriques surface $X/\varepsilon$. The symmetric group $S_5$ acts on $X$ and $S$ by permutations of coordinates.

Proof of Theorem 1.2 for $G = S_5$. The key is to construct a good (rational) generator set of the second cohomology. The exceptional curves at nodes are interchanged with the rational curves $X \cap \{x_i = 0\}$ by $\varepsilon$. Thus we get five smooth rational curves $r_1, \ldots, r_5$ on $S$ whose dual graph is the complete graph with doubled edges and five vertices, denoted $K^{[2]}_5$. Also, for each even involution $\sigma = (ij)(kl) \in S_5$, we can find lines

\[ l'_\sigma: x_i : x_j : x_k : x_l = 1 : -1 : \sqrt{-1} : -\sqrt{-1}, \]
\[ l''_\sigma: x_i : x_j : x_k : x_l = 1 : -1 : -\sqrt{-1} : \sqrt{-1}, \]

lying in $X$ and interchanged by $\varepsilon$. Hence we have another 15 smooth rational curves $l_\sigma$ on $S$. The incidence relation is given by

\[ (l_\sigma, l_\tau) = \begin{cases} -2 & \text{if } \sigma = \tau, \\ 1 & \text{if } \sigma \tau \text{ has order 3}, \\ 0 & \text{otherwise}, \end{cases} \]

and their dual graph is isomorphic to the line graph $L(P)$ of the famous Petersen graph $P$ with 10 vertices and 15 edges. The connections between

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{petersen_graph.png}
\caption{Petersen graph}
\end{figure}

\footnote{The line graph, or edge graph, $L(\Gamma)$ of a graph $\Gamma$ is the one whose vertices correspond to edges of $\Gamma$ and two vertices are connected by an edge if they share a vertex in $\Gamma$.}
The 20 curves in $K_5^{[2]} + L(P)$ generate $H^2(S, \mathbb{Z})_f$ up to index two. The overlattice structure is given, for example, by adding the half-pencil of the elliptic fibration defined by a pair of disjoint pentagons in $L(P)$. There are six such pairs in $L(P)$, hence we have six elliptic pencils on $S$ with reducible fibers of type $I_5 + I_5$. We denote them by $|2f_j|$ ($1 \leq j \leq 6$). These classes satisfy $(f_i, f_j) = 1 - \delta_{ij}$.

Now, since the symplectic group action by $S_5$ is maximal (for $K3$ surfaces), we have the relation

\[ \sum_{i=1}^{5} r_i = \sum_{j=1}^{6} f_j \in H^2(S, \mathbb{Z})_f \]

and this gives the $S_5$-invariant polarization $h$ of degree 30. Those $|2f_j|$ are exactly the gonality pencils for $h$. In particular, the orthogonal complement of $h$ is spanned by the nine elements

\[ (r_i - r_{i+1})/2 \ (1 \leq i \leq 4), \quad f_j - f_{j+1} \ (1 \leq j \leq 5). \]

The action of $S_5$ on them is isomorphic to that on the root lattice $A_4 + A_5$ and therefore we obtain Theorem 1.2 for Example 3. (Every $r_i - r_{i+1}$ is divisible by 2 since $r_1 + r_2$ is equivalent to $2(l_{(12)(34)} + l_{(12)(35)} + l_{(12)(45)})$ and so on.)

**Remark 1.4.** Similar to (4), an Enriques surface with a (semi-symplectic) action of $S_5$ is obtained also from the quartic surface

\[ \sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \frac{1}{x_i} = 0. \]

But this action is not Mathieu. (The quartic surface is the Hessian of the cubic surface $\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i^3 = 0$, and this Enriques surface is Kondo’s of type VI in [13]. See [8, Remark 2.4].)

1.3. **Mathieu action of $\mathbb{N}_{72}$.** For simplicity of notation, we put $\lambda = 1 + \sqrt{3}$ and $\mu = 1 - \sqrt{3}$. Let us consider the $K3$ surface

\[ X: \begin{cases} x_0^2 - \lambda x_1 x_2 = y_0^2 - \mu y_1 y_2 \\ x_1^2 - \lambda x_0 x_2 = y_1^2 - \mu y_0 y_2 \\ x_2^2 - \lambda x_0 x_1 = y_2^2 - \mu y_0 y_1 \end{cases} \]

in $\mathbb{P}^5$ and its Enriques quotient $S$ by the free involution $\varepsilon: (x : y) \mapsto (x : -y)$. This surface is closely related to the rational elliptic surface $R$ given by the Hesse pencil of cubics $z_0^3 + z_1^3 + z_2^3 - 3k z_0 z_1 z_2 = 0$ and its two fibers at
where $\lambda, \mu$. The nine sections of $R$ give rise to the action of the group $C_3^2$ on $X$ and $S$, explicitly given by

$$\alpha: (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \mapsto (x_1 : x_2 : x_0 : y_1 : y_2 : y_0),$$

$$\beta: (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \mapsto (x_0 : \omega x_1 : \omega^2 x_2 : y_0 : \omega y_1 : \omega^2 y_2),$$

where $\omega = (-1 + \sqrt{-3})/2$. At fibers $k = \lambda, \mu$ the cubic has the complex multiplication of order 4, which defines the further automorphism

$$\gamma: \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} x_0 + x_1 + x_2 & -\sqrt{-1}(y_0 + y_1 + y_2) \\ x_0 + \omega x_1 + \omega^2 x_2 & -\sqrt{-1}(y_0 + \omega y_1 + \omega^2 y_2) \\ x_0 + \omega^2 x_1 + \omega x_2 & -\sqrt{-1}(y_0 + \omega^2 y_1 + \omega y_2) \end{pmatrix}.$$

These linear automorphisms generate a group action on $S$ by the group $C_3^2 : C_4$ of order 36.

The extension of this action of $C_3^2 : C_4$ to $N_{72}$ is difficult, not linear and done as follows. We consider the matrix $A = (a_{kl})_{0 \leq k, l \leq 2}$ defined by

$$A = \begin{pmatrix} \mu x_0 & x_2 + cy_2 & x_1 - cy_1 \\ x_2 - cy_2 & \mu x_1 & x_0 + cy_0 \\ x_1 + cy_1 & x_0 - cy_0 & \mu x_2 \end{pmatrix},$$

where $c$ is a constant satisfying $c^2 = 1 - \mu^2$. We regard $(a_{kl})$ as the homogeneous coordinates of $\mathbb{P}^8$. Then the above expression defines an embedding $\mathbb{P}^5 \subset \mathbb{P}^8$ whose linear equations are given by

$$2a_{kk} = \mu(a_{lm} + a_{ml}), \quad \{k, l, m\} = \{0, 1, 2\}.$$

Let $\delta$ be the correspondence $A \mapsto A^{\text{adj}} = (\bar{a}_{kl})_{0 \leq k, l \leq 2}$, the adjoint matrix of $A$. By $(A^{\text{adj}})^{\text{adj}} = (\det A) A$, $\delta$ defines a birational involution of $\mathbb{P}^8$. Moreover, we can check that the defining equations of the K3 surface $X$ is nothing but the equations

$$2\bar{a}_{kk} = \mu(\bar{a}_{lm} + \bar{a}_{ml}), \quad \{k, l, m\} = \{0, 1, 2\}$$

for the adjoint matrix. This shows that $\delta$ restricts to an involution on $X$. In terms of new coordinates, $\varepsilon$ on $\mathbb{P}^5$ is identified with the involution $A \mapsto A$. Since the coordinate sets $\{x_i\}$ and $\{y_i\}$ form the basis of the positive and negative eigenspaces of this involution respectively, we see that $\delta$ commutes with $\varepsilon$ and acts on $S$, too. Therefore we have the group $G = \langle \alpha, \beta, \gamma, \delta \rangle$ acting on $S$.

To see the relations, we note that $\alpha, \beta$ and $\tau$ can be identified with the automorphisms $A \mapsto BAB$, where matrices $B$ are given by

$$B_\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, B_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

From this observation we easily obtain the equalities

$$\delta \alpha = \alpha \delta, \quad \delta \beta = \beta^2 \delta, \quad \delta \gamma = \gamma^{-1} \delta.$$

\footnote{For example, the $(0, 1)$-entry $a_{01}$ is given by $-\mu x_2(x_2 + cy_2) + (x_0 - cy_0)(x_1 - cy_1)$.}
Thus, the group $G$ is really the one $C_2^2: D_8 \simeq N_{72}$.

**Proof of Theorem 1.2** for $G = N_{72}$. We denote by $h \in H^2(S, \mathbb{Z})$ the natural polarization of degree 4. The difference of the 1st and 2nd defining equations of $X$ reads as

$$(x_0 - x_1)(x_0 + x_1 + \lambda x_2) = (y_0 - y_1)(y_0 + y_1 + \mu y_2),$$

so the rational functions $F_{00} = (y_0 - y_1)/(x_0 - x_1)$ and $F_{00}' = (y_0 - y_1)/(x_0 + x_1 + \lambda x_2)$ are elliptic parameters on $X$ and their squares $F_{00}^2, (F_{00}')^2$ define elliptic pencils $|2f_{00}|, |2f_{00}'|$ on $S$. By applying the $C_2^2$ action, we get 18 elliptic parameters on $X$ as follows.

$$F_{kl} = \frac{y_k - \omega^j y_{k+1}}{x_k - \omega^j x_{k+1}}, \quad F_{kl}' = \frac{y_k - \omega^j y_{k+1}}{x_k + \omega^j x_{k+1} + \lambda \omega^{2l} x_{k+2}} (k, l \in \{0, 1, 2\} = \mathbb{Z}/3).$$

We denote by $|2f_{kl}|, |2f_{kl}'|$ the induced elliptic pencils on $S$. By definition, we have the relations

$$h = f_{kl} + f_{kl}', \quad (h, f_{kl}) = (f_{kl}, f_{kl}') = 1 - \delta_{kk'; ll'}$$

where $\delta_{kk'; ll'} = 1$ only if $k = k'$ and $l = l'$. We see that $\{h, f_{00}, \ldots, f_{22}\}$ is a set of rational generators of $H^2(S, \mathbb{Z})$.

Next, let us study the action of the Cremona involution $\delta$ on cohomology. Let $F_\infty$ be the indeterminacy locus of $\delta$ as a transformation of $\mathbb{P}^5$. It is obtained by cutting the Segre variety

$$\Sigma_{2,2} = \{A \in \mathbb{P}^8 \mid \text{rank} A \leq 1\}$$

three times by hyperplanes, hence is an elliptic curve of degree 6. (The smoothness of the linear section is easily seen by using two projections $F_\infty \to \mathbb{P}^2$.) It is clear that $F_\infty$ is contained in $X$ as a divisor and is preserved by $\varepsilon$. Therefore the image of $F_\infty$ on $S$ defines an elliptic curve $f_\infty$ which is a half-pencil.

**Lemma 1.5.**

1. Let $C_k$ ($k = 0, 1, 2$) be the conic in $X$ defined by $x_i = y_i$ ($i \neq k$), $\lambda x_k = \mu y_k$. Then $C_0 + C_1 + C_2$ is a Kodaira fiber of type $I_3$ which is disjoint from $F_\infty$. In particular, $F_\infty$ and $C_0 + C_1 + C_2$ are linearly equivalent on $X$.

2. We have $(h, f_\infty) = 3$, $(f_\infty, f_{kl}) = 1$ ($\forall k, l$).

**Proof.** (1) and the first equality in (2) are easy to check. For the last equalities, it suffices to compute $(f_\infty, f_{00})$ since $f_\infty$ is invariant under the subgroup $C_2^2$. Recall that the pullback $F_{00}$ of $f_{00}$ is defined by the elliptic parameter $F_{00} = (y_0 - y_1)/(x_0 - x_1)$. We see that $(F_{00}, C_2) = 0$ and $(F_{00}, C_i) = 1$ ($i = 0, 1$). Therefore $(f_\infty, f_{00}) = (F_\infty, F_{00})/2 = 1$.

Since $\delta$ is defined by the linear system of quadrics containing $F_\infty$, we obtain $\delta^*(h) = 2h - f_\infty$. It follows that $\tilde{h} = h + \delta^*(h)$ is the polarization
of degree 18 invariant under $N_{72}$. Using the relation $2f_{\infty} = -3h + \sum_{k,l} f_{kl}$, we see
$$\tilde{h} = 3h - f_{\infty} = \frac{1}{2} \sum_{k,l} (h - f_{kl}) = \frac{1}{2} (f'_{00} + \cdots + f'_{22}).$$
Therefore, $f'_{kl}$ ($0 \leq k, l \leq 2$) are the gonality half-pencils for $\tilde{h}$. The orthogonal complement of $\tilde{h}$ is spanned by $h - f_{\infty}$ and the differences $f'_{kl} - f'_{k'l'}$ ($k, l, k', l' \in \{0, 1, 2\}$). They generate the lattice $A_1 + A_9$ and we obtain Theorem 1.2 for Example 4.

1.4. Mathieu action of $A_6$. In [11, 12] Keum, Oguiso and Zhang constructed a $K3$ surface with an action by a group $\tilde{A}_6 : C_4$ and determined the abstract structure of the group. Here we show that it contains a fixed point free involution $\varepsilon$ and the action by $A_6$ descends to $S = X/\varepsilon$.

We start with recalling their results.

**Theorem 1.6.** There exists a $K3$ surface $X$ with the following properties.

1. $X$ is a smooth $K3$ surface with Picard number $\rho = 20$ and the transcendental lattice $T_X$ is given by the Gram matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.

2. $X$ is acted on by a group $\tilde{A}_6 : A_6 : C_4$. Here $A_6$ is the subgroup of symplectic automorphisms and satisfies $NS(X)_{A_6} = \mathbb{Z}H$, $(H^2) = 20$.

3. The image of the natural homomorphism $c: \tilde{A}_6 \to Aut(A_6)$ is $M_{10}$.

We put $\varepsilon$ to be the nontrivial element in $\ker c$ (namely the element $(1, -1) \in M_{10} \times C_4$ in the notation of [12 Theorem 2.3]). The corresponding automorphism on $X$ is denoted by the same letter.

**Lemma 1.7.** The automorphism $\varepsilon$ is fixed point free.

**Proof.** By the construction, $\varepsilon$ is a non-symplectic involution on $X$, hence its fixed locus is a disjoint union of smooth curves. Assume it is not empty. We look at the divisor $D$ given by the sum of fixed curves. Since $\varepsilon$ commutes with $A_6$, $D$ belongs to the sublattice $NS(X)_{A_6} = \mathbb{Z}H$ by Theorem 1.6 (2). Since $H$ is ample, $D$ is connected. Then $(H^2) = 20$ shows that the genus of $D$ is (at least) 11, but there are no such fixed curves for non-symplectic involutions. Thus $\varepsilon$ is free. □

Therefore, the Enriques surface $S = X/\varepsilon$ has an action by $\tilde{A}_6$ (or by $M_{10}$, more precisely).

**Proof of Theorem 1.2 for $G = \tilde{A}_6$.**

Let $h$ be the polarization on $S$ induced from $H$; then we have $(h^2) = 10$ and the invariant lattice $H^2(S, \mathbb{Z})_{f}^{A_6}$ is spanned by $h$. It is easy to see that $|h|$ has no fixed components, hence generic members of $|h|$ are irreducible by Bertini’s theorem.

If the gonality $\Phi(h)$ is less than 3, then the gonality pencil for $h$ is unique by [7 III-Section 6], but it contradicts to $\text{rank} H^2(S, \mathbb{Z})_{f}^{A_6} = 1$. Therefore
\( \Phi(h) = 3 \). In this case, there are exactly ten gonality pencils \(|2f_i| (i = 1, \ldots, 10) by Lemma 1.3 (1) and \( \mathfrak{A}_6 \) permutes them. They are given by the isotropic sequence of length 10 and we have \( h = (f_1 + \cdots + f_{10})/3 \). Therefore the orthogonal complement of \( h \) in \( H^2(S, \mathbb{Z}) \) is generated by elements 
\[
f_1 - f_2, \ldots, f_9 - f_{10}
\]
which is \( \mathfrak{A}_6 \)-equivariantly isomorphic to the \( A_9 \) lattice. We obtain Theorem 1.2 for Example 5.

2. Proof of (3) \( \Rightarrow \) (1) of the Main Theorem

We prove that (3) of Theorem 8 implies (1) of Theorem 2. By Theorem 1.2, it suffices to prove for two groups \( C_2 \times A_4 \) and \( C_2 \times C_4 \). We realize their Mathieu actions on the Enriques surface \( S \) in Example 7. The \( K3 \)-cover of \( S \) is a surface \( X : F(u, v, w) = 0 \) of tri-degree \((2, 2, 2)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). A nonzero 2-form on \( X \) is obtained as residue of the rational 3-form \( du \wedge dv \wedge dw/F(u, v, w) \). Hence \( X \) has the following symplectic actions.

(a) The automorphism \((u, v, w) \mapsto (\pm u, \pm v, \pm w)\) with even number of \(-1\)'s is symplectic. Hence \( C_2 \) acts symplectically.

(b) \((u, v, w) \mapsto (-\sqrt{-1}/u, -\sqrt{-1}/v, -\sqrt{-1}/w)\) is a symplectic involution.

(c) For a permutation \( \sigma \) of \( \{u, v, w\} \),
\[
(u, v, w) \mapsto (\sigma(u) \pm 1, \sigma(v) \pm 1, \sigma(w) \pm 1)
\]
is a symplectic automorphism if the parity of the number of \(-1\)'s agree with that of \( \sigma \). Hence \( \mathfrak{S}_4 = C_2^2 : \mathfrak{S}_3 \) acts symplectically.

(a) and (b) generate a group isomorphic to \( C_2^3 \). Hence the semi-direct product \( H = C_2^2 : \mathfrak{S}_4 \) acts on \( X \) symplectically. It is easily checked that this is \( H_{192} \) (cf. the remark on p. 192 in [15]). Since the involution \( \varepsilon : (u, v, w) \mapsto (-u, -v, -w) \) commutes with the above automorphisms, \( H \) acts semi-symplectically on the quotient Enriques surface \( S \). The actions (a) are Mathieu since all involutions have only elliptic curves as fixed curves. So is the action by (b) and the composite of (a) and (b) since they have only isolated fixed points. Hence the action (a)\( \cdot \) (b) of \( C_2^3 \) and the automorphism \((u, v, w) \mapsto (v, w, u)\) generate a Mathieu action of \( C_2 \times \mathfrak{A}_4 \).

The automorphism

\[
(9) \quad (u, v, w) \mapsto (\sqrt{-1}u, \sqrt{-1}v, -\sqrt{-1}/w)
\]
is of order 4 and Mathieu. In fact, it has exactly four fixed points, two of which are symplectic and the rest of which are anti-symplectic. The automorphism \( (9) \) and the involution (a) generate a Mathieu action of \( \mathfrak{S}_4 \times C_2 \). \( \square \)

Remark 2.1. The Enriques surface of Example 7 is the normalization of the sextic surface \( \sum_{i=1}^4 x_i^2 + \sqrt{-1}x_1x_2x_3x_4 \sum_{i=1}^4 x_i^2 = 0 \) in \( \mathbb{P}^3 \). The Mathieu actions of the two groups can be seen from this expression also. See [16, §6].
Remark 2.2. The complete intersection of three diagonal quadrics
\[
\begin{align*}
x_1^2 + x_3^2 + x_5^2 &= x_2^2 + x_4^2 + x_6^2 \\
x_1^2 + x_4^2 &= x_2^2 + x_5^2 = x_3^2 + x_6^2
\end{align*}
\]
in \(\mathbb{P}^5\) is given in [15] as a (smooth) K3 surface with a symplectic action of \(H_{192}\). The automorphism \((x_i) \mapsto (\pm 1)^i x_i\) is a free involution and commutes with the action. But the induced semi-symplectic action of \(H_{192} = C_4 \times D_{12}\) on the Enriques quotient is far from Mathieu. In fact, any (diagonal) involution in \(C_4 \times C_2\) is not Mathieu. Hence any sub-action of \(C_2 \times \mathbb{A}_4\) or \(C_4 \times C_2\) is not Mathieu either.

3. Semi-symplectic and Mathieu automorphisms

Any Enriques surface is canonically doubly covered by a K3 surface. We always denote an Enriques surface by \(S\) and the K3-cover by \(X\). Let \(\omega_X\) be a nowhere vanishing holomorphic 2-form on \(X\). An automorphism \(\phi\) is symplectic if it preserves the symplectic form \(\omega_X\). Equivalently, they are the elements in the kernel of the canonical representation \(\text{Aut}(X) \to \text{GL}(H^0(\mathcal{O}_X(K_X)))\). Along the same line of ideas, we define the following.

Definition 3.1. Let \(S\) be an Enriques surface. An automorphism \(\sigma \in \text{Aut}(S)\) is semi-symplectic if it acts trivially on the space \(H^0(S, \mathcal{O}_S(2K_S))\).

The sections of \(\mathcal{O}_S(2K_S)\) are identified with those of \(\mathcal{O}_X(2K_X)\). Recall that the covering involution \(\varepsilon\) of \(X/S\) negates \(\omega_X\). Since for a given \(\sigma \in \text{Aut}(S)\) we have two lifts \(\phi_1\) and \(\phi_2 = \phi_1 \varepsilon\) to automorphisms of \(X\), we get the following proposition.

Proposition 3.2. \(\sigma \in \text{Aut}(S)\) is semi-symplectic if and only if one, say \(\phi_1\), of the two lifts is symplectic. Moreover, the other lift \(\phi_2 = \phi_1 \varepsilon\) negates \(\omega_X\).

Since we can uniformly choose the symplectic lift, we have also

Corollary 3.3. Let \(G\) be a group of semi-symplectic automorphisms of \(S\). Then the lifts of automorphisms in \(G\) to the K3-cover \(X\) constitute a group isomorphic to \(G \times C_2\), where \(C_2\) is generated by \(\varepsilon\) and whose subgroup \(G \times \{\text{id}\}\) is the subgroup of symplectic automorphisms.

Example 3.4. Let \(S\) be a generic Enriques surface in the sense of [3]. Then the whole automorphism group \(\text{Aut}(S)\) acts on \(S\) semi-symplectically. More generally, let \(S\) be an Enriques surface whose covering K3 surface \(X\) has the following genericity property:

- The transcendental lattice \(T_X\) of \(X\), considered as a lattice equipped with Hodge structure, has only automorphisms \(\{\pm \text{id}_{T_X}\}\).

Then the whole group \(\text{Aut}(S)\) acts on \(S\) semi-symplectically. For example, this is the case when the Picard number \(\rho(X)\) is odd, since the value of the Euler function \(\phi(n)\) is an even number for all \(n \geq 3\) and by [18].
3.1. Semi-symplectic automorphisms of finite order. From now on, we study automorphisms of Enriques surfaces of finite order, which we simply call finite automorphisms as a matter of convenience. First we have the following criterion.

**Proposition 3.5.** Let $\sigma$ be a finite automorphism of an Enriques surface $S$. If $\text{ord}(\sigma)$ is not divisible by 4, then it is automatically semi-symplectic.

**Proof.** We separately argue two cases (1) $\text{ord}(\sigma) = 2$ and (2) $\text{ord}(\sigma)$ is odd. The rest is easily deduced.

(1) Let $\varphi$ be one of two lifts to $X$. Then $\varphi^2$ is a lift of $\text{id}_S$ hence is either $\text{id}_X$ or $\varepsilon$. In the former case, either $\varphi$ or $\varphi \varepsilon$ is symplectic on $X$ and we are done by Proposition 3.2. Suppose the latter occurs. Then $\varphi$ is an automorphism of order 4 on $X$ which has no nontrivial stabilizer subgroup over $X$. Thus $X \to X/\varphi := Y$ is an étale covering of degree 4. But then $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)/4 = 1/2$ must be an integer, a contradiction.

(2) Let $n$ be the order of $\sigma$. Arguing as in (1), we see that the two lifts of $\sigma$ have orders $n$ and $2n$. We denote by $\varphi$ the one with order $n$ and we show that it is symplectic. Suppose that $\varphi$ is non-symplectic. Then $Y := X/\varphi$ is a rational surface with at most quotient singularities. The free involution $\varepsilon$ induces an involution $\varepsilon_Y$ on $Y$, which must have a fixed point $P \in Y$ because $\chi(\mathcal{O}_Y) = 1$. Let $\pi_Y : X \to Y$ be the quotient morphism. Then the fiber $\pi_Y^{-1}(P)$ consists of odd number of points and has an action by $\varepsilon$ by construction. However this is impossible since $\varepsilon$ is fixed-point-free of order 2. Thus $\varphi$ is symplectic.

**Remark 3.6.** The case (2) in the proof above also shows that if $\sigma$ is an automorphism of odd order, then $S/\sigma \simeq X/\langle \varphi, \varepsilon \rangle$ is an Enriques surface (with singularities in general).

By [18, 15], a finite symplectic automorphism $\varphi$ of a $K3$ surface has $\text{ord}(\varphi) \leq 8$ and the following table holds.

| order of $\varphi$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---------------------|---|---|---|---|---|---|---|
| number of fixed points | 8 | 6 | 4 | 4 | 2 | 3 | 2 |

**Corollary 3.7.** A finite semi-symplectic automorphism $\sigma$ has $\text{ord}(\sigma) \leq 6$.

**Proof.** By Corollary 3.3 $\sigma$ has a symplectic lift $\varphi$ of the same order, hence the order is at most 8. If $\text{ord}(\varphi) = 7$, then $\varepsilon$ cannot act freely on the fixed point set $\text{Fix}(\varphi)$ by (10), a contradiction. If $\text{ord}(\varphi) = 8$, $\varphi$ has exactly two fixed points $P$ and $Q$ and they are exchanged by $\varepsilon$. However, by applying the holomorphic Lefschetz formula, the local linearized actions $(d\varphi)_P$, $(d\varphi)_Q$ are given by

$$
\begin{pmatrix}
\zeta_8 & 0 \\
0 & \zeta_8
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\zeta_8^3 & 0 \\
0 & \zeta_8^5
\end{pmatrix}
$$

where $\zeta_8 = e^{2\pi \sqrt{-1}/8}$.

These matrices are not conjugated by $d\varepsilon$ and we get a contradiction. Therefore we see that $\text{ord}(\sigma) \leq 6$. \qed
Next we want to look more closely at the fixed point set $\text{Fix}(\sigma)$. Let $\varphi$ be the symplectic lift and $\varphi_\varepsilon$ be the non-symplectic one. From the relation $S = X/\varepsilon$, $\text{Fix}(\sigma)$ has the decomposition $\text{Fix}(\sigma) = \text{Fix}^+(\sigma) \cup \text{Fix}^-(\sigma)$, where $\text{Fix}^+(\sigma) = \text{Fix}(\varphi)/\varepsilon$ is the set of symplectic fixed points and $\text{Fix}^-(\sigma) = \text{Fix}(\varphi_\varepsilon)/\varepsilon$ is the set of anti-symplectic fixed points. Geometrically, they are distinguished by the determinant ($= \pm 1$) of the local linearized action $(d\sigma)_P$.

The number of symplectic fixed points is deduced from the table [10] by $\# \text{Fix}^+(\sigma) = \# \text{Fix}(\varphi)/2$. On the other hand, the anti-symplectic fixed point set $\text{Fix}^-(\sigma)$ has a variation. Here we use the topological Lefschetz formula

\begin{equation}
\sum_{i=0}^{4} (-1)^i \text{tr}(\sigma^* |_{H^i(S,Q)}) = \chi_{\text{top}}(\text{Fix}(\sigma)).
\end{equation}

\[\text{(11)}\]

to give a rough classification\[3\]. The quantity (11) is called the Lefschetz number and denoted by $L(\sigma)$.

**Proposition 3.8.** Let $\sigma$ be a semi-symplectic automorphism of order $n \leq 6$.

- If $n = 2$, then $\text{Fix}^- (\sigma)$ is a disjoint union of smooth curves. The Lefschetz number $L(\sigma)$ takes (every) even number from $-4$ to $12$.
- If $n = 3$, then $\text{Fix}^- (\sigma) = \emptyset$ and we have $L(\sigma) = 3$.
- If $n = 4$, then $\text{Fix}^- (\sigma)$ is either empty or $2$ points. Accordingly $L(\sigma)$ equals either $2$ or $4$.
- If $n = 5$, then $\text{Fix}^- (\sigma) = \emptyset$ and we have $L(\sigma) = 2$.
- If $n = 6$, then $\text{Fix}^- (\sigma)$ is either empty or $2$ points. Accordingly $L(\sigma)$ equals either $1$ or $3$.

**Proof.** We have only to compute $\text{Fix}^- (\sigma)$ in each case.

$(n = 2)$ The first assertion follows since the local action $(d\sigma)_P$ at $P \in \text{Fix}^- (\sigma)$ is of the form $\text{diag}(1, -1)$. On the other hand, the action $\sigma^* \cap H^2(S, Q)$ is identified with $\varphi^* \cap H^2(X, Q)_{\varphi_\varepsilon = 1}$, the invariant subspace with respect to $\varphi^*$. It is known (also easily deduced from Table [10] via Lefschetz formula) that the action $\varphi^*$ has an $8$-dimensional negative eigenspace on $H^2(X, Q)$. Therefore, by counting the eigenvalues, we get the assertion (2).

The existence of every value is shown in [10].

$(n = 3, 5)$ If the order $n$ is odd, $((d\sigma)_P)^n \neq 1$ for $P \in \text{Fix}^- (\sigma)$. Therefore there are no anti-symplectic points.

$(n = 4, 6)$ Since $(\varphi_\varepsilon)^2 = \varphi^2$, we have $\text{Fix}(\varphi_\varepsilon) \subset \text{Fix}(\varphi^2)$. Since $\varepsilon$ is free, we get $\text{Fix}(\varphi_\varepsilon) \subset \text{Fix}(\varphi^2) - \text{Fix}(\varphi)$. This latter set $T$ consists of 4 points in both cases $n = 4, 6$ and $\varphi$ and $\varepsilon$ both acts freely of order 2 on $T$. Thus we see that the fixed points of $\varphi_\varepsilon$ are either whole $T$ or empty. \[\square\]

3.2. **Mathieu automorphisms.** The left-hand-side of (11) can be regarded as the character of the representation $\sigma^* \cap H^*(S, Q)$ since odd dimensional (rational) cohomology vanishes. In contrast to symplectic automorphisms

\[\text{For a detailed classification of involutions, see [10].}\]
of K3 surfaces, Proposition 3.8 shows that we cannot treat semi-symplectic automorphisms uniformly from the viewpoint of characters. Nevertheless, we can make the following definition.

**Definition 3.9.**

1. A 12-dimensional representation $V$ of a finite group $G$ over a field of characteristic zero is called a small Mathieu representation if its character $\mu(g)$ depends only on $\text{ord}(g)$ and coincides with that of (the permutation representation on $\Omega_+$ of) the small Mathieu group $M_{11}$.

2. Let $G$ be a finite group of automorphisms of an Enriques surface $S$. The action is called Mathieu if it is semi-symplectic and the induced representation $G \rtimes H^*(S, \mathbb{Q})$ is a small Mathieu representation.

For the characters of $M_{11}$, see (3) in Introduction. We note that $G$ acts on $S$ effectively and without numerically trivial automorphisms if it is Mathieu. Comparing (3) and Proposition 3.8, we see that Mathieu condition has effects only on elements of even orders. A little stronger statement holds as follows.

**Lemma 3.10.** A semi-symplectic group action of $G$ on an Enriques surface $S$ is Mathieu if for every element $\sigma$ of order 2 or 4, we have $L(\sigma^2) = 4$.

**Proof.** We prove that under the condition, any element $\sigma \in G$ of order 6 have $L(\sigma) = 1$. Let $a_k$ $(k = 0, \cdots, 5)$ be the number of the eigenvalue $e^{2\pi k\sqrt{-1}/6}$ in the representation $\sigma^* \rtimes H^*(S, \mathbb{Q})$. Since $\chi_{\text{top}}(S) = 12$ and the representation is over $\mathbb{Q}$, we have

$$a_0 + \cdots + a_5 = 12, \quad a_1 = a_5, \quad a_2 = a_4.$$  

By assumption, we also have $L(\sigma^2) = 3$ and $L(\sigma^3) = 4$, which translates into

$$a_0 - a_1 - a_2 + a_3 = 3, \quad a_0 - 2a_1 + 2a_2 - a_3 = 4.$$  

On the other hand, by Proposition 3.8 (6),

$$L(\sigma) = a_0 + a_1 - a_2 - a_3 = 1 \text{ or } 3.$$  

The only integer solution to these equations is given by $a_0 = 4, a_1 = a_5 = 1, a_2 = a_4 = 2, a_3 = 2$. Therefore we get $L(\sigma) = 1$. \hfill $\square$

By Proposition 3.8 and Definition 3.9, we have

**Proposition 3.11.** Let $\sigma$ be a Mathieu automorphism of an Enriques surface of order $n \geq 2$. Then the fixed locus $\text{Fix}(\sigma) = \text{Fix}^+(\sigma) \sqcup \text{Fix}^-(\sigma)$ is as follows.

| $n$ | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|
| $\text{Fix}^+(\sigma)$ | 4 pts. | 3 pts. | 2 pts. | 2 pts. | 1 pts. |
| $\text{Fix}^-(\sigma)$ | $\sqcup_i C_i$ | $\emptyset$ | 2 pts. | $\emptyset$ | $\emptyset$ |

Here $\sqcup_i C_i$ is a disjoint union of smooth curves whose Euler number $\sum_i \chi_{\text{top}}(C_i)$ is zero.
4. Finite groups with small Mathieu representations

In this section we prove

**Proposition 4.1.** Let $G$ be a finite group which has a small Mathieu representation $V$ with character $\mu$. Then the order of $G$ is

$$2^{a_2}3^{a_3}5^{a_5}11^{a_{11}}$$

for non-negative integers $a_2 \leq 4$, $a_3 \leq 2$, $a_5$, $a_{11} \leq 1$. Moreover, this bound is sharp since $M_{11}$ has the order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$.

The definition of a small Mathieu representation is in Definition 3.9. In what follows we use the notation $\mu$ for $A \in G$ and we have $\dim V^G = \mu(G)$. In particular,

$$(\ast) \quad \mu(G) \text{ is a non-negative integer.}$$

Note that a subgroup $H$ inherits a small Mathieu representation and we have $\mu(H) \geq \mu(G)$. Also for a normal subgroup $N$, we can define the function $\mu$ on $G/N$ and we have $\mu(G/N) \geq \mu(G)$. The equality holds if and only if $N = \{1\}$.

**Proof.** We begin the proof of Proposition 4.1. From Table (3), every element $g \in G$ has $\text{ord}(g) \in \{1, 2, 3, 4, 5, 6, 8, 11\}$. For a prime number $p$, let $G_p$ be a Sylow $p$-subgroup of $G$. For odd $p$, $G_p$ does not contain elements of order $p^2$. Hence by $(\ast)$,

$$\mu(G_p) = (12 + e_p(|G_p| - 1))/|G_p| \in \mathbb{Z} \quad (e_3 = 3, e_5 = 2, e_{11} = 1)$$

shows that $|G_3| \leq 3^2$, $|G_5| \leq 5$, $|G_{11}| \leq 11$.

In the rest, let us replace $G$ by $G_2$ and show $|G| \leq 2^4$ for $p = 2$.

**Lemma 4.2.** Abelian groups of order $2^4$ have no small Mathieu representations.

**Proof.** An abelian group of order $2^4$ is isomorphic to either $C_{16}$, $C_8 \times C_2$, $C_4 \times C_4$, $C_4 \times C_2^2$ or $C_2^4$. By definition $C_{16}$ has no small Mathieu representation. For other groups we can easily compute $\mu(G)$ to see that they don’t satisfy the condition $(\ast)$. Hence we get the lemma.

Let $A$ be a maximal normal abelian subgroup of $G = G_2$. We have $|A| \leq 2^3$ by Lemma 4.2. Since $G$ is a $2$-group, the centralizer $C_G(A)$ coincides with $A$ and the natural homomorphism $\varphi: G/A \to \text{Aut}(A)$ is injective. Thus for $A \simeq C_2, C_4, C_2^2$, we have $\text{Aut}(A) \simeq \{1\}, C_2, S_3$ and we see that $|G| \leq 2^3$.

It remains to consider the case $|A| = 2^3$. There are three abelian groups $A \simeq C_8$, $C_4 \times C_2$ and $C_2^3$. We treat them separately.

**Case** $A \simeq C_8$ : Here we have $\text{Aut}(A) \simeq C_2^2$. To show $|G| \leq 2^4$, it suffices to show that $\varphi$ is not surjective. Assume the contrary. Then there exists $x \in G$ which acts on the generator $g$ of $A$ by $xgx^{-1} = g^5$. We set $H = \langle A, x \rangle$. Since $x^2$ commutes with $A$, we get $x^2 \in A$ and $|H| = 2^4$. The equality
$(g\cdot x)^4 = g^4x^4$ shows that $(g\cdot x)^4 = 1$ if $(x^4 = 1$ and $i$ is even) or $(x^4 = g^4$ and $i$ is odd), and ord$(g\cdot x) = 8$ otherwise. This enables us to compute

$$\mu(H) = \frac{1}{2}(\mu(A) + \mu(Ax)) = \frac{1}{2}(4 + 3) \not\in \mathbb{Z}.$$ 

Therefore $\varphi$ is not surjective.

**Case $A \simeq C_4 \times C_2$:** Let $g, h$ be generators of $A$ with ord$(g) = 4$ and ord$(h) = 2$. Then Aut$(A)$ is generated by

$$\alpha: (g\ h) \mapsto (g + h\ 2g + h),$$ $$\beta: (g\ h) \mapsto (g\ 2g + h),$$

and is isomorphic to $D_8$. Assume that $|\text{Im } (\varphi)| \geq 4$. Then $G$ has an element $x$ such that $\varphi(xA) = \alpha^2$ and $x^2 \in A$. By $(ax)^2 = x^2(a \in A)$, it follows that all elements in the coset $Ax$ have the same order ord$(x) \in \{2, 4, 8\}$. Hence $\mu(Ax)$ is even, while $\mu(A) = 5$. Thus the group $H = \langle A, x \rangle$ cannot have a small Mathieu representation by $(\ast)$, a contradiction. Hence $|\text{Im } (\varphi)| \leq 2$.

**Case $A \cong C_3^2$:** We have Aut$(A) = \text{GL}(3, \mathbb{F}_2)$ which is the simple group of order 168. (One of) its 2-Sylow subgroups consist of elements

$$\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}, \quad (\alpha, \beta, \gamma \in \mathbb{F}_2)$$

and is isomorphic to $D_8$. Let us assume that the subgroup Im$(\varphi)$ has at least four elements. Then $G$ contains an element $x$ whose image by $\varphi$ is conjugate to

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2$$

and $x^2 \in A$. In suitable coordinates of $A$, it is easy to see that every element in the coset $Ax$ has order at most 4. Hence we get the contradiction

$$\mu(\langle A, x \rangle) = \frac{1}{2}(\mu(A) + \mu(Ax)) = \frac{1}{2}(5 + 4) \not\in \mathbb{Z}.$$ 

Thus in all cases we get $|G| \leq 2^4$ and we obtain Proposition 4.1. □

5. **Proofs of the Main Theorem**

We give the proofs to the main results, Theorems 2 and 8.

**Theorem 5.1.** The following conditions are equivalent to each other for a finite group $G$.

1. $G$ has a Mathieu action on some Enriques surface.
2. $G$ can be embedded into the symmetric group $\mathfrak{S}_6$ and the order $|G|$ is not divisible by $2^4$.
3. $G$ is a subgroup of one of the following five maximal groups:
   $$\mathfrak{A}_6, \mathfrak{S}_5, N_{72} = C_3^2: D_8, C_2 \times \mathfrak{A}_4, C_2 \times C_4.$$
(4) $G$ is a group with a small Mathieu representation $V$ with $\dim V^G \geq 3$, whose $2$-Sylow subgroup is embeddable into $\mathfrak{S}_6$ and $G \neq Q_{12}$.

**Remark 5.2.** (1) There are 25 isomorphism classes of $G \neq \{1\}$ which satisfy the conditions (1)-(4) of the theorem. They are the groups exhibited in Propositions 5.8, 5.9 and 5.10.

(2) The group $Q_{12}$ has a subtle behavior in the condition (4) of the theorem. Its 2-Sylow subgroup is obviously embedded into $\mathfrak{S}_6$. Moreover, using the notation in [17, Table 2], the character $4\chi_0 + \chi_2 + 2\chi_3 + \chi_4 + \chi_5$ is a small Mathieu representation with $\dim V_{Q_{12}} = 4$. But it has no Mathieu actions by Lemma 5.4. It is thus necessary to put the extra condition on this group in the condition (4).

We have already shown (3)$\Rightarrow$(1) in Section 2. In the following three subsections we prove the rest in the order (1)$\Rightarrow$(4)$\Rightarrow$(2)$\Rightarrow$(3).

5.1. **Proof of (1)$\Rightarrow$(4).** By definition, $H^*(S, \mathbb{Q})$ is a small Mathieu representation. Obviously $H^i(S, \mathbb{Q}) (i = 0, 4)$ are invariant subspaces and for any ample divisor $H$ on $S$ the sum $H' = \sum_{g \in G} gH$ is a $G$-invariant ample divisor, showing $H^2(S, \mathbb{Q})^G \neq 0$. Putting together, we find $\dim H^*(S, \mathbb{Q})^G \geq 3$.

Next we show that the 2-Sylow subgroup $G_2$ is embeddable in $\mathfrak{S}_6$. By Corollary 3.7 every element $g \in G_2$ has $\text{ord}(g) \leq 4$. By the character table [3], we see that $\mu(g) = 4$ unless $g = 1$. Thus the condition $(\ast)$ (Section 4),

$$\mu(G_2) = \frac{1}{|G_2|} (12 + 4(|G_2| - 1)) \in \mathbb{Z}$$

gives $|G_2| \leq 2^3$. It is easy to check that all 2-groups of order at most 8, except for $G_2 = C_8$ and $Q_8$, can be embedded in the group $C_2 \times D_8$, the 2-Sylow subgroup of $\mathfrak{S}_6$. The cyclic group $C_8$ is impossible by Corollary 3.7. The group $Q_8$ is also impossible by the following lemma, which concludes $G_2 \subset \mathfrak{S}_6$.

**Lemma 5.3.** No Mathieu actions on Enriques surfaces by the quaternion group, $Q_8 = \langle g, h \mid g^4 = 1, hgh^{-1} = g^{-1}, g^2 = h^2 \rangle$, exist.

**Proof.** By means of contradiction, suppose that we had one. We denote by $i = g^2$ the unique and central involution in $Q_8$. By Proposition 3.11 we see that the fixed point sets of $g$ and $h$ both coincide with the set of four isolated fixed points of $i$. In particular, these four points are fixed by the whole group.

Let $P$ be one of anti-symplectic fixed points of $g$, which exists by Proposition 3.11. By looking at differentials, we obtain a map $d_P : Q_8 \to \mathbb{GL}(T_P S)$, which is injective by the complete reducibility for finite groups. But since any embedding of $Q_8$ into $\mathbb{GL}(2, \mathbb{C})$ factors through $\mathbb{SL}(2, \mathbb{C})$, this contradicts to that $P$ was an anti-symplectic fixed point. This proves the lemma.

(Proof of the latter fact: Note that the diagonal form of the involution $d_P(i)$ is either $\text{diag}(1, -1)$ or $\text{diag}(-1, -1)$. In the former case, its centralizer in $\mathbb{GL}(2, \mathbb{C})$ is the commutative group of diagonal matrices, hence we get a
contradiction. In the latter case, from $g^2 = h^2 = i$ and $gh \neq hg$, we must have that both $d_P(g)$ and $d_P(h)$ have $\text{tr} = 0$ and $\text{det} = 1$.)

Finally we show $G \not\cong Q_{12}$.

**Lemma 5.4.** No Mathieu actions on Enriques surfaces by the group $Q_{12} = \langle g, h \mid g^6 = 1, h^2 = g^3, hgh^{-1} = g^{-1} \rangle$ exist.

**Proof.** Assume we had one. Since $g^3$ is the unique involution in $Q_{12}$, by Proposition 3.11, we see that the fixed point sets of $h$ and $gh$ both coincide with the set of four isolated fixed points of $g^3$. We denote them by $P_i$ ($i = 1, \ldots, 4$). However from the equations $h(P_i) = P_i, gh(P_i) = P_i$ we get $g(P_i) = P_i$, contradicting to that $g$ of order 6 has a unique isolated fixed point by Proposition 3.11. □

5.2. **Proof of** $(4) \Rightarrow (2)$.

**Lemma 5.5.** Let $G$ have a small Mathieu representation and assume that $\mu(G) \geq 3$. Then 11 does not divide $|G|$.

**Proof.** If $G$ has a nontrivial 11-Sylow subgroup $G_{11}$, the dimensions of invariant subspaces satisfy $2 = \mu(G_{11}) \geq \mu(G)$, a contradiction. □

**Lemma 5.6.** Let $G$ have a small Mathieu representation and assume that $G_2$ is embeddable into $S_6$. Then $G$ has no elements of order 8 and $|G_2| \leq 2^3$.

**Proof.** Since the 2-Sylow subgroup $(S_6)_2$ is isomorphic to $C_2 \times D_8$, $G_2$ has no elements of order 8. By the definition of small Mathieu character $\mu$, every non-identity element $g \in G_2$ has character $\mu(g) = 4$. Thus the condition

$$\mu(G_2) = \frac{1}{|G_2|} (12 + 4(|G_2| - 1)) \in \mathbb{Z}$$

gives $|G_2| \leq 8$. □

**Corollary 5.7.** Let $G$ be a finite group that satisfies the condition (4) of Theorem 5.1. Then for all $g \in G$ we have $\text{ord}(g) \leq 6$. Moreover we have

$$|G| = 2^{a_2}3^{a_3}5^{a_5}$$

for non-negative integers $a_2 \leq 3, a_3 \leq 2, a_5 \leq 1$.

**Proof.** This follows immediately by combining Proposition 4.1 and lemmas above. □

In particular, we have proved the latter part of (2) of Theorem 5.1.

In the following, we classify all groups that satisfy the condition (4). First we consider non-solvable groups. Recall that $G$ is non-solvable if and only if at least one of its composition factors is a non-abelian finite simple group.

**Proposition 5.8.** Let $G$ be a finite group that satisfies the condition (4) of Theorem 5.1. Assume that $G$ is non-solvable. Then $G$ is isomorphic either to $A_5, S_5$ or $A_6$. 

Proof. Let $N$ be a composition factor of $G$ which is a non-abelian simple group. By Corollary 5.7 and the table of finite simple groups (see [2]), $N$ is either $\mathfrak{A}_5$ or $\mathfrak{A}_6$. In the latter case we see $G = N \simeq \mathfrak{A}_6$ by Corollary 5.7.

Let us continue the case $N \simeq \mathfrak{A}_5$. By order reason, $N$ is the only non-abelian simple factor. We have subgroups $H \subset G$ and $T \lhd H$ such that $H/T \simeq \mathfrak{A}_5$. Since $H$ inherits the small Mathieu representation and $\mathfrak{A}_5$ is a quotient of $H$, we have

$$3 = \mu(\mathfrak{A}_5) = \mu(H/T) \geq \mu(H) \geq \mu(G)$$

by the discussion after Proposition 4.1. By the condition $\dim V^G \geq 3$, we have equalities. It follows that $T$ is trivial and $H \simeq \mathfrak{A}_5$.

By Corollary 5.7, the index $[G : H]$ is a divisor of 6. Hence the composition series of $G$ looks either (i) $H \lhd G$ or (ii) $H \lhd G' \lhd G$ (if it has more than one terms). Let us begin with (i). We consider the natural homomorphism $\varphi : G \to \text{Aut}(H)$. Since $G$ does not contain elements of order $5n$ ($n \geq 2$) and $H$ is center-free, we see that $\varphi$ is injective into $\text{Aut}(\mathfrak{A}_5) \simeq S_5$. Therefore $G$ is isomorphic to $S_5$. In the case (ii), we get $G' \simeq S_5$ by (i). We again consider the natural homomorphism $\psi : G \to \text{Aut}(G')$. By the same reasoning as before, $\psi$ is injective. But this is not the case since $G \supset G' \simeq S_5$ is a proper inclusion.

Thus we obtain the classification of non-solvable groups. \(\square\)

We recall that $G$ is nilpotent if and only if $G$ is the direct product of its Sylow subgroups.

Proposition 5.9. Let $G$ be a nilpotent group that satisfies the condition (4) of Theorem 5.7. Then $G \simeq C_n^a$ ($2 \leq n \leq 6$ if $a = 1$ and otherwise $(n,a) = (2,2),(3,2),(2,3)$), $C_2 \times C_4$ or $D_8$.

Proof. Corollary 5.7 and the given condition on 2-Sylow subgroups classify the Sylow subgroups of $G$ as follows: $G_2$ is isomorphic to $C_2^3$, $C_2 \times C_4$, $D_8$ or has order at most 4, $G_3$ is isomorphic to $C_3^2$ or $C_3$ and $G_5$ is isomorphic to $C_5$ (if not trivial).

We claim that neither groups $H_1 = C_2 \times C_3^2$ nor $H_2 = C_3^2 \times C_3$ have small Mathieu representations. In fact, the former has $\mu(H_1) = 8/3 \notin \mathbb{Z}$. For the latter, let us choose generators $g,h \in H_2$ with $g^6 = h^2 = ghg^{-1}h^{-1} = 1$. Let $\psi$ be the character of $H_2$ assigning $h \mapsto 1$ and $g \mapsto \zeta_6$, where $\zeta_6$ is the primitive 6-th root of unity. Then the inner product of characters $(\psi, \mu)$ is $1/2 \notin \mathbb{Z}$. Thus $H_2$ does not have small Mathieu representations.

Recall that all elements $g \in G$ have $\text{ord}(g) \leq 6$ by Corollary 5.7. This fact with the non-existence of subgroups $H_i$ above leads us to the list. \(\square\)

Finally we treat the case $G$ is non-nilpotent and solvable. Recall that any finite group has the maximal normal nilpotent subgroup $F$, called the Fitting subgroup. When $G$ is non-nilpotent and solvable, $F$ is a proper subgroup and it is known that the centralizer $C_G(F)$ coincides with the center $Z(F)$ of $F$. In particular the natural homomorphism $\varphi : G/F \to \text{Out}(F)$ is injective,
where \( \text{Out}(F) = \text{Aut}(F)/\text{Inn}(F) \) is the group of outer automorphism classes. Moreover, by the extended Sylow’s theorem for solvable groups, the exact sequence

\[
(12) \quad 1 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 1
\]
splits if \(|F|\) and \(|G/F|\) are coprime.

**Proposition 5.10.** Let \( G \) be a non-nilpotent and solvable group that satisfies the condition (4) of Theorem 5.1. Then \( G \) belongs to the following list.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
G & D_6 & D_{10} & D_{12} & \mathfrak{A}_4 & \mathfrak{S}_3 \times \mathfrak{S}_3 \\
|G| & 6 & 10 & 12 & 12 & 18 \\
\text{Hol}(C_5) & C_2 \times \mathfrak{A}_4 & \mathfrak{S}_4 & C_2 \times C_4 & \mathfrak{S}_3 \times \mathfrak{S}_3 & N_{72} \\
|G| & 20 & 24 & 24 & 36 & 36 & 72 \\
\hline
\end{array}
\]

**Proof.** Let \( F \) be the Fitting subgroup. The isomorphism class of \( F \) belongs to the list of Proposition 5.9, so we give separate considerations. We note that \( \text{Out}(F) = \text{Aut}(F) \) if \( F \) is abelian.

**Case:** \( F \) is cyclic

In the table below, \(-1\) denotes the inversion \( g \mapsto g^{-1} (g \in F) \).

\[
\begin{array}{c|c|c|c|c|c|c|}
G & \text{Aut}(F) & \{\{1\}\} & \{\{\pm1\}\} & \{\{\pm1\}\} & \{\{\pm1\}\} & \{\{\pm1\}\} \\
\hline
\text{D}_6 & \text{D}_{10} & \text{D}_{12} & \text{Hol}(C_5) & \text{D}_{12} \\
\hline
\end{array}
\]

For \( F = C_2, C_4 \) all extensions \((12)\) are nilpotent by order reasoning. For \( F = C_3, C_5 \), \((12)\) splits and we get the table. Here \( \text{Hol}(C_5) \) denotes the holomorph \( C_5 \rtimes \text{Aut}(C_5) \). For \( F = C_6 \), we get a split extension \( D_{12} \). The other non-split extension \( Q_{12} \) is not allowed by the assumption.

**Case:** \( F = C_2^2 \)

We have \( \text{Aut}(F) = \mathfrak{S}_3 \). Since \( G \) is non-nilpotent, it has an element \( x \) of order 3. Then \( F \) and \( x \) generate a subgroup \( H \) isomorphic to \( \mathfrak{A}_4 \). If further the inclusion \( H \subset G \) is proper, \( H \) has index two and is normal. In this case, such \( G \) is isomorphic to \( \mathfrak{S}_4 \) or \( C_2 \times \mathfrak{A}_4 \), but in the latter group \( C_2^2 \) is not the Fitting subgroup.

**Case:** \( F = C_2^2 \)

We have \( \text{Aut}(F) \simeq \text{GL}(2, \mathbb{F}_3) \) which has order 48 = 2\(^4\)3. This group has the semi-dihedral group \( SD_{16} \) as its 2-Sylow subgroup,

\[
SD_{16} = \langle g, x \mid g^8 = x^2 = 1, xgx^{-1} = g^3 \rangle.
\]

By Corollary 5.7, the extension \((12)\) splits and \( G/F \) is a subgroup of \( SD_{16} \). Since the maximal subgroups of \( SD_{16} \) are \( C_8, D_8, Q_8 \) and all are characteristic, we see that isomorphic subgroups of order 8 in \( \text{GL}(2, \mathbb{F}_3) \) are conjugate. In view of Proposition 5.9 we get the unique extension \( G \simeq C_2^2 \times D_8 = N_{72} \) if \(|G|\) is maximal. If \(|G/F| = 4\), we get unique extensions \( C_2^2 \times C_4, C_2^2 \times C_4 \simeq \mathfrak{S}_3 \). If \(|G/F| = 2\), we have two extensions which are isomorphic to \( \mathfrak{S}_3 \times C_3 \times \mathfrak{S}_3 \).

**Case:** \( F = C_2^3 \)

We have \( \text{Aut}(F) = \text{GL}(3, \mathbb{F}_2) \), which is the simple group of order 168 = 2\(^3\) \cdot 3 \cdot 7. By Corollary 5.7 \((12)\) splits with \( G/F \simeq C_3 \). Since 3-Sylow subgroups in \( \text{GL}(3, \mathbb{F}_2) \) are conjugate, we get the unique extension \( G \simeq C_2 \times \mathfrak{A}_4 \).
Case: $F = C_2 \times C_4, D_8$ In these cases we have $\text{Aut}(F) \cong D_8$, hence we get no non-nilpotent groups.

This completes the classification of possible groups. It is not difficult to check that every groups are in $\mathfrak{S}_6$, and we have proved our theorem. □

5.3. Proof of (2)⇒(3). By condition, $G \neq \mathfrak{S}_6$. By [2], $\mathfrak{S}_6$ has four isomorphism classes of maximal subgroups $\mathfrak{A}_6, \mathfrak{S}_5, N_{72}, C_2 \times C_4$, the first three of which readily satisfy (3). Thus we may assume $G \subset C_2 \times C_4$. Again by the order condition $G$ is a proper subgroup. A standard argument shows that $C_2 \times C_4$ has maximal subgroups $\mathfrak{S}_4, C_2 \times \mathfrak{S}_3, C_2 \times \mathfrak{A}_4, C_2 \times D_8$. The first two are subgroups of $\mathfrak{S}_5$. The third one is in the list of (3). Thus we may assume $G \subset C_2 \times D_8$. Again by the order condition $G$ is a proper subgroup, and the maximal subgroups of $C_2 \times D_8$ are $C_2^3, D_8, C_2 \times C_4$. The first two groups are subgroups of $C_2 \times \mathfrak{A}_4$ and $N_{72}$ respectively. The last group $C_2 \times C_4$ is the final ingredient in (3), so our assertion holds.

Appendix A. Lattice theoretic construction of Mathieu actions

We give a lattice theoretic proof of Theorem 1.2.

In [13, Appendix], for each $G$ of the eleven groups (1), a symplectic action on a K3 surface is constructed using

1. the Niemeier lattice $N$ of type $(A_1)^{24}$,
2. the action of the Mathieu group $M_{24}$ on $N$,
3. an embedding of $G$ into the Mathieu group $M_{23}$, and
4. the Torelli type theorem for K3 surfaces.

Here $N$ is even, unimodular and contains the root lattice

\[(13) \bigoplus_{i \in \Omega} \mathbb{Z}e_i, \quad (e_i^2) = -2\]

as a sublattice of finite ($= 2^{12}$) index, where $\Omega$ is the operator domain of $M_{24}$. The action $M_{24} \curvearrowright \Omega$ extends (isometrically) on $N$. The key of the proof is to show the existence of a primitive embedding of $N_G$ in the K3 lattice $\Lambda \simeq 3U + 2E_8$, where $N_G$ is the orthogonal complement of the invariant lattice $N_G \subset N$.

In this appendix, making this construction $C_2$-equivariant, we give another proof of Theorem 1.2. Namely we decompose (13) in two parts

\[(14) \bigoplus_{i \in \Omega_+} \mathbb{Z}e_i \text{ and } \bigoplus_{i \in \Omega_-} \mathbb{Z}e_i, \quad (e_i^2) = -1, \quad \forall i \in \Omega = \Omega_+ \sqcup \Omega_-\]

with scaling by $1/2$. Let $N_\pm$ be the lattices obtained by adding $(\sum_{i \in \Omega_\pm} e_i)/2$ to these. $N_\pm$ is the dual of the root lattice of type $D_{12}$, and $N_\pm(2)$ is an integral lattice of discriminant $2^{10}$. 
Let $G_{(6)}, G_{(9)}, G_{(10)} \subset M_{11}$ be the image of the embedding of the three groups $S_5, N_{72}, A_6$ in Lemma 1.1 respectively. $G_{(n)}$ decomposes the operator domain $\Omega_+ \setminus \{\star\}$ of $M_{11}$ into two orbits of length $n$ and $11 - n$. Let $\Omega_-$ be the complementary dodecad of $\Omega_+$. The following is immediate from the proof of Lemma 1.1.

**Lemma A.1.** Each $G_{(n)}$ decomposes $\Omega_- \setminus \{\star\}$ into two orbits. Their length are $\{6, 6\}$ when $n = 9, 10$, and $\{2, 10\}$ when $n = 6$.

We consider the orthogonal complement of the invariant lattice for two actions $G_{(n)} \acts N_{\pm}$. The following is obvious.

**Lemma A.2.** Let $G = G_{(n)} \acts N_{\pm} (n = 6, 9, 10)$ be as above.

1. The orthogonal complement $N_{+, G}$ of the invariant lattice $N^G_+ \subset N_+$ is the root lattice of type $A_{n-1} + A_{10-n}$.

2. The orthogonal complement $N_{-, G}$ of the invariant lattice $N^G_- \subset N_-$ is a negative definite odd integral lattice of rank 10. $N_{-, G}$ contains an index-two sublattice which is of type $A_5 + A_5$ when $n = 9, 10$ and $A_1 + A_9$ when $n = 6$.

The Niemeier lattice $N$ contains $N_\pm(2)$ as a primitive sublattice. Since $N$ is unimodular we have an isomorphism

$$\text{Disc } N_+(2) \simeq \text{Disc } N_-(2)$$

of discriminant groups. This isomorphism is compatible with the actions of $M_{12}$.

Now we recall the modulo 2 reduction $l := L/2L$ of an integral lattice $L$. When $L$ is even, $l$ is endowed with the quadratic form

$$q : l \to \mathbb{Z}/2\mathbb{Z}, \quad x \mapsto (x^2)/2$$

with value in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. When $L$ is odd, $l$ is endowed with the bilinear form

$$b : l \times l \to \left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z}, \quad (x, y) \mapsto (x \cdot y)/2.$$  

The alternating part $l^{alt} := \{x | b(x, x) = 0\}$ of $l$ is a subspace of codimension one, and carries

$$q : l^{alt} \to \mathbb{Z}/2\mathbb{Z}, \quad x \mapsto (x^2)/2$$

which is a quadratic refinement of $b$, that is, $q(x + y) - q(x) - q(y) = b(x, y)$ holds for every $x, y \in l^{alt}$.

We need an Enriques counterpart of the isomorphism (15). The $K3$ lattice $\Lambda$ decomposes in two parts by the action of free involution $\varepsilon$. The invariant part is the Enriques lattice $\Lambda_+$ of type $T_{2,3,7}$ scaled by 2. The anti-invariant part $\Lambda_-$, called the anti-Enriques lattice, is isomorphic to $U + U(2) + E_8(2)$. Since $\Lambda$ and $\Lambda_+$ are unimodular and since $\Lambda$ contains the orthogonal direct sum $\Lambda_+(2) + \Lambda_-$, we have the isomorphism

$$\Lambda_+/2\Lambda_+ \simeq \text{Disc } \Lambda_-$$
of 10-dimensional quadratic spaces over $\mathbb{F}_2$.

Returning to the action $G = G_{(q)} \times N$, we put $L_+ := N_{1+G}$ and denote its modulo 2 reduction by $l_+$. Restricting the isomorphism $[15]$ to $l_+$ we have

**Lemma A.3.** Two 9-dimensional quadratic spaces $(l_+, q_+)$ and $(l_-^{alt}, q_-)$ over $\mathbb{F}_2$ are isomorphic to each other including their $G$-actions.

**Remark A.4.** The bilinear form on $(l_+, q_+)$ has 1-dimensional radical, and $q_+$ takes value 1 at the nonzero element in the radical.

The lattice $N_G$, the orthogonal complement of $N_G \subset N$, is obtained by patching two lattices $L_+$ by the isomorphism in the lemma. $N_G$ is an even lattice of Leech type, that is, the induced action of $G$ on the discriminant group $\text{Disc} N_G$ is trivial and $N_G$ does not have a $(-2)$ element.

As is observed in the introduction $L_+$ have a primitive embedding into the lattice of type $T_{2,3,7}$. The following is the counterpart of $L_-$.

**Proposition A.5.** The lattice $L_-(2)$ has a primitive embedding into the anti-Enriques lattice $\Lambda_-$. The essential part is this.

**Lemma A.6.** $L_-$ has a primitive embedding into the odd unimodular lattice $I_{2,10} := \langle 1 \rangle^2 + (-1)^{10}$ of signature $(2,10)$.

**Proof.** We take $\{h_1, h_2, e_1, \ldots, e_{10}\}$ with $(h_1^2) = (h_2^2) = 1$ and $(e_i^2) = -1$ ($1 \leq i \leq 10$) as an orthogonal basis of $I_{2,10}$.

In the case $n = 6$, $L_-$ is the unique (odd) integral lattice containing $A_1 + A_9$ as a sublattice of index 2. $e_i - e_{i+1}$ ($i = 1, \ldots, 9$) and $v = 2(h_1 + h_2) - \sum_1^{10} e_i$ generate a root sublattice of type $A_9 + A_1$ in $I_{2,10}$. Since the half sum of $e_{2j-1} - e_{2j} \in A_9$ ($j = 1, \ldots, 5$) and $v$ belongs to $I_{2,10}$, the primitive hull of $A_9 + A_1 \subset I_{2,10}$ is isomorphic to $L_-$.

In the case $n = 9, 10, L_-$ is the unique integral lattice containing $A_9 + A_5$ as a sublattice of index 2. $L_-$ is isomorphic to the orthogonal complement of $2(h_1 + h_2) - e_1 - e_2 - e_3 - e_4 - e_5$ and $2(h_1 - h_2) - e_6 - e_7 - e_8 = e_9 - e_10$ in $I_{2,10}$. In fact, the orthogonal complement is generated by

$$e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, h_1 + h_2 - e_1 - e_2 - e_3 - e_4;$$

$$e_6 - e_7, e_7 - e_8, e_8 - e_9, e_9 - e_{10}, h_1 - h_2 - e_6 - e_7 - e_8 - e_9$$

which form a root lattice $R$ of type $A_9 + A_5$, and $h_1 - e_2 - e_4 - e_6 - e_8 \notin R$. $\square$

**Remark A.7.** In the above proof we make use of the fact that the blow-up of the projective space $\mathbb{P}^3$ at five points has a Cremona symmetry of type $A_5$, which is described by Dolgachev [9] in terms of root systems.

**Proof of Proposition A.5.** The anti-Enriques lattice $\Lambda_-$ is obtained from $I_{2,10}(2)$ by adding $(-2)$-element $(h_1 + h_2 - \sum_1^{10} e_i)/2$. Since $h_1 + h_2 - \sum_1^{10} e_i$ does not belong to the modulo 2 reduction of the image of $L_- \rightarrow I_{2,10}$.
constructed in the lemma, the induced embedding \( L_\minus(2) \hookrightarrow \Lambda_\minus \) is also primitive. \( \square \)

**Remark A.8.** The above relation between the anti-Enriques lattice \( \Lambda_\minus \) and \( I_{2,10} \) is observed in Allcock[1].

Patching together two primitive embeddings \( L_\plus(2) \hookrightarrow \Lambda_\plus(2) \), determined by \( A_9, A_4 + A_5, A_1 + A_8 \subset T_{2,3,7} \), and \( L_\minus(2) \hookrightarrow \Lambda_\minus \), we have the following.

**Proposition A.9.** There exist \( s \) a primitive embedding of \( N_G \) into the K3 lattice \( \Lambda \) such that \( N_G \cap \Lambda_\plus(2) = L_\plus(2) \) and \( L \cap \Lambda_\minus = L_\minus(2) \).

**Proof of Theorem A.2.** Let \( G \) be one of the three groups \( G_{(6)}, G_{(9)}, G_{(10)} \) (or equivalently \( \mathcal{S}_5, N_{72}, \mathcal{A}_6 \)). Since \( N_G \) is of Leech type the action \( G \) on \( N_G \) extends to that on the K3 lattice. By our construction it preserves \( \Lambda_\plus(2) \) and \( \Lambda_\minus \). Since \( L_\minus(2) \) does not contain a \((-2)\)-element, there exists an Enriques surface \( S_{(n)} \) \((n = 6, 9, 10)\) such that \( H^{1,1}(S_{(n)}, \mathbb{Z})^- \simeq L_\minus(2) \) by the subjectivity theorem (1). Let \( h \in H^2(S_{(n)}, \mathbb{Z})_f \) be a primitive element perpendicular to \( L_\plus \). \( h \) is unique up to sign. Replacing with \(-h\) if necessary, we may assume that \( h \) belongs to the positive cone, that is, the connected component of \( \{ x \in H^2(S, \mathbb{R}) \mid \langle x^2 \rangle > 0 \} \) which contains ample classes. There exists a composition \( w \in O(H^2(S_{(n)}, \mathbb{Z})_f) \) of reflections with respect to smooth rational curves on \( S_{(n)} \) such that \( w(h) \) is nef. By the strong Torelli type theorem (3), the cohomological action of \( G \acts H^2(S_{(n)}, \mathbb{Z})_f \) twisted by \( w \) is realized by an algebraic action. \( \square \)

**Remark A.10.** By construction and by Lemma [A.1] and the Torelli type theorem, two Enriques surfaces \( S_{(9)} \) and \( S_{(10)} \) are isomorphic to each other. The Enriques surface \( S_{(6)} \) is \( \mathcal{S}_5 \)-equivariantly isomorphic to that of type VII in Kondo[13]. In particular, Aut \( S_{(6)} \) is the symmetric group \( \mathcal{S}_5 \).

**Remark A.11.** A Mathieu action of \( G = C_2 \times \mathcal{A}_4 \) on an Enriques surface can be constructed lattice theoretically also. Via the embedding \( \mathcal{G}_6 \hookrightarrow M_{12} \), \( G \) is embedded into \( M_{12} \) and decomposes \( \Omega_\pm \) into three orbits of length 2, 4 and 6. Hence the lattice \( N_{\pm, G} \) contains the root lattice of type \( A_1 + A_3 + A_5 \) as a sublattice of index two. \( N_{+, G} \) has a primitive embedding into the Enriques lattice \( \Lambda_+ \) since \( \Lambda_+ \) contains the lattice \( T_{2,4,6} \) as a sublattice of index 2. \( N_{-, G} \) has a primitive embedding into \( I_{2,10} \) since \( N_{-, G} \simeq N_{+, G} \) and since \( I_{2,10} \simeq \Lambda_+ + (1) + (-1) \). Hence the same argument shows the existence of (a 1-dimensional family of) Enriques surfaces with Mathieu actions of \( G \).

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