Non-perturbative improvement of the vector current in Wilson lattice QCD

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Abstract

Many observables of interest in lattice QCD are extracted from correlation functions involving the vector current. If Wilson fermions are used, it is therefore of practical importance that, besides the action, the current be $O(a)$ improved in order to remove the leading discretization errors from the observables. Here we introduce and apply a new method to determine the improvement coefficient for the two most widely used discretizations of the current.

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I. INTRODUCTION

Lattice QCD is a powerful tool to calculate the predictions of Quantum Chromodynamics in its non-perturbative regime. While the quantum field theory is regularized by discretizing it on a lattice, ultimately the quantities of interest – for instance, ratios of hadron masses – must be determined in the limit where the cutoff is removed. For numerical purposes, it is computationally advantageous to accelerate the approach to the continuum by removing the leading cutoff effects. In particular, Symanzik’s continuum effective theory [1, 2] can be used to remove the $O(a)$-cutoff effects which appear generically when using the Wilson fermion action in lattice QCD simulations [3]. To eliminate $O(a)$-cutoff effects in the hadronic spectrum it suffices to improve the action by introducing the dimension-five Sheikholeslami-Wohlert term [4] with a non-perturbatively determined coefficient $c_{sw}$ [5]. However, the addition of higher-dimensional counterterms to local operators is also necessary for the improvement of their matrix elements, along with the determination of the corresponding improvement coefficients.

In the following, we focus on the vector current, which requires a single $O(a)$-improvement term. Estimates of the relative contribution of the improvement term evaluated with the perturbative improvement coefficient may suggest that the effect of the improvement in correlation functions would be small for the local vector current [6]. However, an improvement condition based on chiral Ward identities previously used to determine the improvement coefficient $c_V$ non-perturbatively in pure gauge theory with [6] and without [7] Schrödinger functional boundary conditions indicated significant deviations from the tree-level result.

In this work, we describe a simple prescription for the non-perturbative determination of the improvement coefficients, $c_V$ and $c_{CV}$, for the local and conserved (point-split) isovector vector currents, defined below. In the following section we report large differences between the lowest-order perturbative estimates and our non-perturbative evaluation of the improvement coefficients with $N_f = 2$ Wilson clover fermions. In section [IV] we demonstrate the effects of the improvement on the scaling of an observable in the continuum limit.
II. THEORY BACKGROUND AND A NEW IMPROVEMENT CONDITION

We use the $O(a)$-improved Wilson fermion action with the non-perturbatively tuned value of $c_{sw}$ [8]. The two discretizations of the continuum vector current that we employ are

\begin{equation}
(V)^l_\mu(x) = \overline{\psi}(x)\gamma_\mu \frac{\tau_3}{2} \psi(x),
\end{equation}

(1)

\begin{equation}
(V)^c_\mu(x) = \frac{1}{2} \left( \overline{\psi}(x + a\hat{\mu})(1 + \gamma_\mu)U_\mu(x)\frac{\tau_3}{2} \psi(x) ight. \\
- \left. \overline{\psi}(x)(1 - \gamma_\mu)U_\mu(x)\frac{\tau_3}{2} \psi(x) \right).
\end{equation}

(2)

The renormalized improved current for $i = l, c$ is defined by [5]

\begin{equation}
(V_R)^i_\mu(x) = Z^i_V (1 + b^i_V am_q)(V)^i_\mu(x),
\end{equation}

(3)

\begin{equation}
(V_I)^i_\mu(x) = (V)^i_\mu(x) + ac^i_V \partial_\mu T_{\mu\nu}(x),
\end{equation}

(4)

where the lattice discretization of $\partial_\mu T_{\mu\nu}(x)$ will be discussed later and $Z^l_V \equiv Z_V$ in the notation of [5], while $Z^c_V = 1$. The on-shell improvement of the vector current is required in many lattice studies: hadronic form factors, the hadronic contributions to $(g-2)_\mu$ and thermal correlation functions related to the dilepton production rate, to mention a few.

While the local vector current requires improvement only at one-loop order [9] the conserved vector current requires improvement at tree level also in the massless limit,

\begin{equation}
c^l_V = -0.01225(1) \times C_F \times g_0^2 + O(g_0^4),
\end{equation}

(5)

\begin{equation}
c^e_V = \frac{1}{2} + O(g_0^2),
\end{equation}

(6)

where $C_F = (N^2 - 1)/(2N)$ is the quadratic Casimir in the fundamental representation for gauge group SU(N).

A. Improvement condition for $c^l,c_V$

The proposed improvement condition is based on two discretizations of the vector current defined in eq. (1) ($l$) and eq. (2) ($c$). The main observable we consider is the vector current correlator

\begin{equation}
(G_I)^{ij}_{\mu\nu}(x_0) \equiv (G_U)^{ij}_{\mu\nu}(x_0)
+ Z^l_V Z^c_V \int d^3x \left\langle ac^l_V \overline{\psi}(V)^i_\mu(x_0, x)\partial_\rho T_{\nu\rho}(0) \\
+ ac^c_V \partial_\rho T_{\mu\rho}(x_0, x)(V)^j_\nu(0) \right\rangle.
\end{equation}

(7)
with

\[(G_U)^{ij}_{\mu\nu}(x_0) \equiv Z^i_V Z^j_V \int d^3x \langle (V)^i_{\mu}(x_0, x)(V)^j_{\nu}(0) \rangle,\]

where we have indicated its dependence on the discretization of the current at finite lattice spacing, \(ij = ll, cl, cc\). We will use periodic boundary conditions in space and thermal boundary conditions in time, though our method is more generally applicable, for instance to open boundary conditions in time. In ref. [10], two discretizations of the unimproved vector current correlator demonstrated significant differences in the region of \(t \lesssim 0.5\text{fm}\) at intermediate lattice spacings corresponding to bare lattice couplings \(\beta = 5.3\) with \(N_f = 2\) Wilson clover fermions. This suggests that the three independent discretizations of the temporal vector current correlator could be used to formulate an improvement condition.

In figure 1, we illustrate the discrepancies between the three discretizations for two lattice spacings corresponding to bare lattice couplings of \(\beta = 5.2\) and \(\beta = 5.5\) in the left and right panels respectively. The details of the ensembles and the number of measurements are listed in table I. The non-perturbative renormalization constant, \(Z_V\), is taken from ref. [11]. Note that a more precise non-perturbative result for \(Z_V\) has been reported in ref. [12]. As expected, the differences between the two discretizations are reduced as the lattice spacing decreases.

Demanding that three discretizations of the temporal vector current correlator agree at
a particular Euclidean time \( x_0 \),

\[
(G_1)^{IJ}_{xx}(x_0) \doteq (G_1)^{el}_{xx}(x_0) \doteq (G_1)^{cc}_{xx}(x_0),
\]

allows one to solve the following \( 2 \times 2 \) linear system for the improvement coefficients \( c_{V}^{(l,e)}(x_0) \):

\[
\begin{pmatrix}
2Z_V G^{TT}_{xx}(x_0) - G^{ct}_{xx}(x_0) & -G^{IT}_{xx}(x_0) \\
-Z_V G^{ct}_{xx}(x_0) & 2G^{IT}_{xx}(x_0) - Z_V G^{TT}_{xx}(x_0)
\end{pmatrix}
\times
\begin{pmatrix}
\hat{c}_V \\
\hat{c}_V
\end{pmatrix}
= \frac{1}{a}
\begin{pmatrix}
(G_U)^{el}_{xx}(x_0) - (G_U)^{tt}_{xx}(x_0) \\
(G_U)^{el}_{xx}(x_0) - (G_U)^{cc}_{xx}(x_0)
\end{pmatrix},
\]

where

\[
G^{IT}_{\mu\nu}(x_0) = \int d^3x \langle (V)^i_{\mu}(x_0, x) \partial_\nu T_{\nu\rho}(0) \rangle.
\]

Translation invariance and time-reversal antisymmetry of the vector-tensor correlation function was used to simplify the system (10). The correlators appearing in (10) may be averaged over the spatial components to improve the signal. We define the improvement coefficient to be \( \hat{c}_V = c_{V}^{(l,e)}(x_0) \) for some choice of \( x_0 \). The method is viable in practice provided that a signal exists both for the r.h.s. and the determinant of the linear operator on the l.h.s. of eq. (9). The results for \( c_{V} \) obtained using different legitimate prescriptions will in general differ by \( O(a) \) corrections. Ideally, one would choose \( x_0 \) in a region where there is both a signal and higher-order lattice artifacts are highly suppressed. This improvement condition can be implemented directly in a finite volume and is straightforward to compute, not requiring the three-point functions of ref. [6]. Although the quark mass-dependence of the renormalization factor is neglected in this improvement condition, namely the \( b_V \) term in eq. (3), due to the smallness of the quark mass these effects ought to be small. Furthermore, in the following section numerical evidence demonstrates that no dependence on the quark mass is likely to be observed in these improvement coefficients.

1. **Discretization of \( \partial_\mu \) and \( T_{\mu\nu} \)**

In the improvement term, we use the local discretization of the tensor current,

\[
T_{\mu\nu}(x) = \frac{-1}{2} \bar{\psi}(x) [\gamma_\mu, \gamma_\nu] \frac{\tau_3}{2} \psi(x),
\]

5
TABLE I. Details of CLS $N_f = 2$ ensembles and number of measurements used in this work.

with the same spacetime argument as the vector current. The choice of the discretization of $\partial_\mu$ affects only higher-order lattice artifacts, which nevertheless can be large. In ref. [13], the improvement of the conserved current (2) was considered. The effect of using the symmetric derivative $\tilde{\partial}_\nu$ and the tensor current averaged over sites $x$ and $(x + a\hat{\mu})$ was examined at one-loop level in lattice perturbation theory. While with this choice the identity $\partial_\mu^c(V_R)_\mu^c = 0$ still holds in on-shell correlation functions, it was noted to introduce large higher-order lattice artifacts to the connected part of the hadronic vacuum polarization tensor. Therefore, for the tensor current with time argument $x_0$ (assumed positive) in correlation function (7), we choose the forward finite-difference derivative, while for the tensor current at the origin we used the backward derivative. We remark that it is admissible to use different discretizations of $\partial_\mu$ in the determination of the improvement coefficient and subsequently in matrix elements of the improved operator [14].

III. EVALUATION OF $\tilde{c}_V^{c,l}$

Using the improvement condition eq. (9) we determined $c_V^{l,c}(x_0)$ at three lattice spacings on ensembles with $N_f = 2$ Wilson clover fermions with non-perturbatively tuned value of $c_{sw}$ [8] and the plaquette action. The most relevant parameters are given in table I for more details on the ensembles, see [15], where the action is also given explicitly. In figure 2 we show the dependence of $c_V^{l,c}$ on the choice of $x_0$ for ensembles A5, F6 and N6 with bare lattice couplings $\beta = 5.2, 5.3$ and 5.5, respectively, and almost identical pion masses. Additionally, we compare with another ensemble, E5, with $\beta = 5.3$ and a larger quark mass. In figure 2 (left), some evidence can be seen for a plateau in the value of $c_V^l$ at the smallest lattice
FIG. 2. Non-perturbatively determined improvement coefficient $c_V^l$ (left) and $c_V^c$ (right). The blue bursts have been displaced horizontally for clarity.

| Ensemble | $c_V^l$  | $c_V^c$  |
|----------|---------|---------|
| A5       | -0.434(5) | 0.203(7) |
| E5       | -0.400(6) | 0.229(8) |
| F6       | -0.401(6) | 0.232(9) |
| N6       | -0.377(7) | 0.243(10) |

TABLE II. Results for improvement coefficients for the local and conserved vector currents for the ensembles listed in table I. The statistical error is estimated by single-elimination jackknife resampling.

spacing corresponding to $\beta = 5.5$ (black squares). For $c_V^l$, no significant changes can be observed as a function of the lattice spacing. The right-hand panel of figure 2 shows there is a greater dependence on the cutoff for $c_V^c$. We are unable to distinguish any dependence on the quark mass.

Our choice for the improvement coefficients is $c_V^{lc} = c_V^c(x_0/a = 3)$ which is used in the rest of this work. Our results are given in table II.

This choice leads to values of $c_V^l$ which deviate significantly from the perturbative estimate of eq. (5). We can make a rough comparison with the value of $c_V^l$ determined in the quenched
theory [6] based on the observation that, at fixed bare coupling, the improvement coefficients are independent of $N_f$ to one-loop in perturbation theory. We note that the value we obtain at $\beta = 5.3$ is quite similar to the value obtained at $\beta = 6.0$ in the quenched theory.

In figure 3 we show the effect of the improvement on the vector correlators with the given improvement condition for the A5 (left) and N6 (right) ensembles. By definition, the central values now coincide at $x_0/a = 3$. The effect of the improvement appears to be smallest for the local-conserved vector current correlator, due to the contributions of the improvement of each current entering with opposite signs.

2. Interpolation in $g_0^2$

The following polynomial interpolation formulas in $g_0^2 = 6/\beta$ can be used to determine the improvement coefficients for $N_f = 2$ flavours of non-perturbatively improved Wilson fermions

\begin{align}
c_V^l(g_0^2) &= -0.01225 C_F g_0^2 \left( 1 + 7.19 g_0^2 + 10.15 g_0^4 \right), \\
c_V^c(g_0^2) &= \frac{1}{2} \left( 1 + 0.33 g_0^2 - 0.728 g_0^4 \right). \tag{14}
\end{align}

Conservatively, these parametrizations should be used in the interval $5.2 \leq \beta \leq 5.5$, even though the known behavior at small $g_0^2$ is built in. It would be interesting to extend the present calculations to smaller values of $g_0^2$ to make explicit contact with the one-loop result.

IV. CONTINUUM LIMIT OF IMPROVED OBSERVABLES

In order to quantify the effect of the improvement we examine the scaling of the observable

\begin{equation}
\hat{I}_1^{ij} = \int_{t_a}^{t_b} dx_0 \ x_0^4 (G_1^{ij}(x_0)) \tag{15}
\end{equation}

and its unimproved counterpart $\hat{I}_U^{ij}$, which is defined analogously with the unimproved two-point function $(G_U)^{ij}_{xx}(x_0)$ (eq. (8)), toward the continuum limit. The limits $t_a/a = 8$ and $t_b/a = 26$ are fixed at the smallest lattice spacing, corresponding to $t_a \approx 0.4$fm and $t_b \approx 1.3$fm. Although the contact term does not contribute to such an observable when
FIG. 3. Improved correlators A5 (left) and N6 (right).

\( t_a = 0 \), the lower limit explicitly removes very short-distance contributions. A lattice estimate for this observable is obtained from the discretized correlation function by quadrature with an improved integration scheme. An interpolation is needed at the limits for the coarser two lattice spacings. This observable is related to the slope of the Adler function through the mixed representation of the hadronic vacuum polarization function [16]. Therefore, it may serve as a useful proxy to quantify the effect of the improvement on phenomenologically relevant observables which are dominated by the long-distance physics of the vector correlation function.

The values of the Sommer scale, \( r_0/a \), used to set the relative scale and perform the continuum limit were taken from ref. [15].

In figure 4 (left) the three discretizations \( I_{U}^{ll} \), \( I_{U}^{cl} \), \( I_{U}^{cc} \) are shown in red, green and blue, respectively. The scaling of the unimproved observables is modelled linearly in \( a \) to obtain the continuum limit. While the continuum limits of the different discretizations agree within the statistical precision, and could be constrained to agree in a simultaneous fit, the use of a single discretization demonstrates a significant fraction of the uncertainty in the continuum result is due to the long extrapolation in \( a \).

In the right-hand panel of figure 4 the analogous plot for the improved observable is shown. A quadratic model in \( a \) describes the continuum scaling well, and the error in continuum limit is correspondingly reduced. Furthermore, the residual scaling violations appear to be
FIG. 4. Linear scaling of observable (15) for unimproved currents (left) and quadratic scaling of improved currents (right). The points in the right-hand panel have been displaced horizontally for clarity.

FIG. 5. Scaling of the ratio of local-local and conserved-conserved observable (15).

small. Note that the observable defined from the local-conserved vector correlation function appears to have the mildest scaling of all three unimproved discretizations.

Another illustration of the improved scaling behaviour of the O($a$)-improved observable is shown in figure 5. The ratio of the observable defined using the local-local vector correlation function and the conserved-conserved vector correlation function is shown for the unimproved
(red) and improved currents (green). This discrepancy should vanish in the continuum limit and the fits (solid lines) are constrained to vanish there likewise. Here, owing to the additional constraint imposed in the continuum limit we can use a quadratic model for both unimproved and improved discretizations.

V. CONCLUSIONS

The basic idea of the improved strategy used here is that if \( n_d \) discretizations of a current are considered, there are \( n_d(n_d + 1)/2 \) lattice versions of its two-point function. Requiring the equality of these two-point functions thus allows one to determine the improvement coefficients of each discretization, for \( n_d \) sufficiently large. The method however requires that the relative normalization of the operators be known at the outset. For the program to go through consistently one thus needs to determine the renormalization factors in a situation where the improvement terms do not contribute. This is the case for the vector current, because the improvement term does not affect the conserved charge.

Since the improvement of spectral quantities only depends on the improvement of the action, the effect of the vector current improvement is expected to be less significant for the effective mass at moderate distances, where the vector correlator is dominated by the rho meson. This has implications for e.g. the scale-setting procedure defined through the effective mass of the vector correlator \[10\]. We explicitly observed the effect of the improvement in this quantity to be small.

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