A Technical Review of Penning Trap based Investigations in Neutron Decay

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July 14, 2014

Abstract

This review is concerned with a detailed analysis of some of the technical problems which arise in the application of the Penning trap method to the experimental study of neutron β-decay, a technique which was first successfully tested on the low-flux swimming-pool reactor LIDO (capture flux = 3·10^6 cm^−2 s^−1) at AERE, Harwell in the 1970’s. It does not discuss the scientific merits or demerits of these studies. Of particular importance are the trapping and release of neutron decay protons, and the influence of magnetic mirror effects and radial drifting on the trapped particles. Since these have energies <1 keV they must be accelerated to energies of order 20-30 keV following release, at which point they are recorded in a silicon surface barrier detector. However serious difficulties were encountered in the post-release acceleration process associated with vacuum breakdown in the presence of crossed electric and magnetic fields.

#1 The Role of the Neutron Lifetime in Astrophysics and Particle Physics.

The availability of a precise value for the lifetime of the free neutron is of major importance in astrophysics because it is this quantity which ultimately determines the rate at which hydrogen is transmuted into helium by thermonuclear processes in the sun [1]. According to big-bang scenarios the neutron lifetime also influences the rate of primordial helium production, but in this case persistent disagreements between cosmologists and nuclear experimentalists have been satisfactorily resolved [2]. Thus the continuing interest in neutron lifetime measurements centres on the crucial role this number plays in fixing precise values for the weak coupling constants in beta-decay [3], and for arriving at a nuclear-structure-independent value for the V_{ud} element in the CKM quark mixing matrix [4−5].

The important relationship connecting the lifetime \( \tau_n = t/\ln(2) \), where \( t \) is the half-life, with the vector and axial vector weak coupling constants \( G_V \) and \( G_A \) is given by

\[
\mathcal{F}t = \frac{2\pi^3 (\ln 2) \hbar^7/m_\gamma e^4}{(G_V^2 + 3G_A^2)} = \frac{K}{(G_V^2 [1 + 3 | \lambda |^2])},
\]

where \( \lambda = G_A/G_V \), \( K/(hc)^6 = (8.1202787 \pm 0.000011) \cdot 10^{-7}\text{GeV}^{-4}s \) and \( \mathcal{F} = 1.71489 \pm 0.000002 \) [6] is the integrated Fermi phase-space factor including model-dependent and model-independent radiative corrections.
Neutron lifetime experiments are notoriously difficult, and this is so for essentially three reasons: (i) neutron decay is a rare process which is difficult to isolate against an intense background of $\gamma$-rays from nuclear interactions, (ii) absolute neutron counting relies for its precision on an array of physical and chemical data, e.g. cross sections, surface densities, isotopic ratios etc., (iii) absolute counting of the charged decay products requires a detailed understanding of the electromagnetic forces to which these particles are subjected during transport from the source volume to the detector. Many of these difficulties are avoided in stored neutron experiments, in which ultra-cold neutrons are confined in suitable magnetic field configurations \[7 − 8\], or in material bottles \[9 − 10\]. However these techniques have their own sources of systematic error which have not as yet been entirely clarified. In this communication we confine attention to the technical aspects of the Penning trap method for studying neutron decay, and examine in some detail some of the technical problems which have arisen in successive versions of the method \[11−14\].

# 2 The Penning Trap Method.

The operating principle of the Penning trap is based on the Penning cold-cathode vacuum gauge \[15\] and the description ”Penning trap” was introduced by Dehmelt \[16\] in his study of the electron g-factor anomaly. The Penning trap has proved to be an extremely versatile instrument in fundamental physics \[17\], and its application to the neutron lifetime problem goes back some four decades \[11\]. In this method protons from neutron decay, which have energies less than about 0.75 keV, are stored in a Penning trap before being ejected and counted in a silicon surface barrier counter, maintained at a negative potential of 20-30 kV. This technique has the double advantage that the source volume can be precisely determined, and that the background is reduced in the ratio of detection time to storage time. In the ideal Penning trap an axially symmetric electrostatic quadrupole potential is superimposed on a coaxial uniform magnetic field, a combination in which the trapped charged particles undergo harmonic oscillations along the axis and epicycloidal motion in the transverse plane \[18\]. However these ideal conditions are unnecessarily restrictive in a trap for protons from neutron decay, where there is no requirement for the axial motion to be harmonic. Thus suitable quasi-Penning traps may be formed in a wide range of axially symmetric electric field configurations, based on the two-cylinder electrostatic lens \[19\]. This system of proton trap and detector functions when a minimal set of conditions is met. In particular it is essential to have (a) a magnetic field of sufficient strength that the radius of cyclotron motion is small in comparison with the dimensions of the apparatus, (b) an electrostatic potential well approximately 1kV in depth, (c) a fast negative pulse to open the trap, and (d) a negative accelerating potential $> 20kV$.

The adiabatic invariants associated with motion in the ideal Penning trap may be calculated by application of the Hamilton Jacobi equation \[20\]. Evaluation of the adiabatic invariants associated with the $\phi$- and $z$-coordinates goes ahead in a simple manner and we find

\[
(a) \ J_\phi = \oint p_\phi d\phi = \pi m \omega_r (a^2 - R^2) = J_a - J_R \ ; \quad (b) \ J_z = \oint p_z dz = \pi m \omega_z Z^2
\]

where $p_\phi$, $p_r$, and $p_z$ are canonical momenta, $a$ is the radius of the cyclotron orbit, $R$ is the radial coordinate of the guiding centre and $Z$ is the amplitude of the axial oscillation. For the $r$-coordinate the situation is slightly more complicated in that the two cases: (a) $a>R$ (cyclic accelerator), and (b) $a<R$ (Penning trap), must be treated separately. In the latter case we find then that

\[
J_r = \oint p_r dr = \pi m \omega_r R^2 = J_R
\]
Since, assuming strict cylindrical symmetry, 
\[ p_\phi = m r^2 (\dot{\phi} - \omega_c / 2) \]  
(2.3)
is a constant of the motion, it follows that \( J_a \) is also an adiabatic invariant.

Both the quantities \( J_a \) and \( J_R \) have simple physical interpretations. Thus \( e J_a / 2\pi \) is the magnetic moment of the cyclotron orbit traced out by the particle circulating in the magnetic field, while \( J_R \) is proportional to the magnetic flux linking the circle of radius \( R \) centred at the origin. If we write \( r = x + iy \) for the position vector in the plane transverse to the magnetic field then the exact solution expressed in Cartesian coordinates is given by \([18]\)

\[ r = \sqrt{\frac{J_a}{\pi m \omega_r}} \exp\left[ i \left( \frac{\omega_c + \omega_r}{2} \right) t + \delta_a \right] + \sqrt{\frac{J_R}{\pi m \omega_r}} \exp\left[ i \left( \frac{\omega_c - \omega_r}{2} \right) t + \delta_A \right] \]  
(2.4)

where \( \omega_r = (\omega_c^2 - 2\omega_z^2)^{1/2} \), \((\omega_c + \omega_r) / 2 \approx \omega_c + \omega_z^2 / 2\omega_c \) and \((\omega_c - \omega_r) / 2 \approx \omega_z^2 / 2\omega_c \). These results can evidently be understood in terms of the azimuthal drift velocity 
\[ v_\phi = \frac{cE_r}{B_z} = \frac{\omega_z^2 r}{2\omega_c} \]

which is a feature of charged particle motion in crossed electric and magnetic fields. Expressed in the language of special relativity, the electric field vanishes in a frame of reference rotating about the \( z \)-axis with angular velocity \( \omega_z^2 / 2\omega_c \), and the particle behaves as it would in a homogeneous magnetic field.

It needs to be borne in mind that the sudden lowering of the confining potential on the "gate electrode" facing the detector during the release phase is a non-adiabatic process whose influence on the trapped particles may require further exploration. Since this failure of adiabaticity corresponds to an injection of heat into the system there remains the possibility that particles trapped in the vicinity of the gate may be lifted into orbits such that they are lost on the electrode itself.

**#3 First Experiments at AERE Harwell 1970-75**

The apparatus, which is shown in Figure 1, was initially designed to operate in magnetic fields up to 5T and accelerating potentials up to 50 kV. Some unexpected problems were encountered in attempting to achieve these conditions of operation. In the first experiments with the trapping system it was found that:

(i) The superconducting magnet would operate in persistent mode without danger of quenching at currents \( \leq 32 \) amps in all four coils. In order to reach higher currents, progressively longer run-up times were required (1.5-2 hours), otherwise the magnet would go normal with immediate loss of the liquid helium charge. In practice all four coils would run without difficulty at 30 amps corresponding to magnetic fields of 1.6 T in the centre of the trap and 4.0 T at the detector.

(ii) At zero magnetic field the accelerating electric field could be safely raised to \( \simeq -40 \) kV; at higher potentials transitory breakdown pulses occurred with increasing frequency.

(iii) With maximum magnetic field electrical breakdown was immediate and total on application of 2-3 kV post-acceleration.

(iv) With an accelerating potential \( \simeq -30 \) kV electrical breakdown was immediate and total when the magnet current reached 2-3 amps.
Figure 1: The Penning trap used to detect low energy protons from neutron $\beta$-decay at AERE, Harwell [Ref 21], and subsequently at the Institut Laue-langevin, Grenoble [Ref 11]. The ceramic insulator was added to the original design in order to protect against vacuum breakdown due to the magnetron effect in the presence of crossed electric and magnetic fields. The protons were detected in a silicon surface barrier detector, a technique introduced here for the first time in the study of neutron decay, which today is standard practice [Refs 30-32]
The most obvious feature of the observed pattern of breakdown in the combined system of electric and magnetic fields was its very rapidity and smoothness, quite unlike normal electrical breakdown whose onset is usually preceded by periods of instability. Since elementary considerations would indicate that breakdown transverse to a magnetic field should be hindered, it was originally concluded that electrons, generated by some means or other within the apparatus, were being channelled along the magnetic field from the detector to the upper ("pulsed" or "gate") trapping electrode. However a study of the breakdown characteristics revealed that the dependence on the electrode-detector separation was minimal and the discharge was taking place in the space between the 2 cm diameter cylindrical tube containing the detector signal and power leads and the 9.0 cm diameter cryostat wall which is at ground potential. Thus the breakdown was associated with the presence of crossed rather than parallel electric and magnetic fields.

The ultimate explanation for these observations appears to derive from the magnetron effect, whereby electrons, generated in the annular gap between the cylindrical tube carrying the detector leads and the cylindrical cryostat wall, move in cyclotron orbits which then precess at right angles to both electric and magnetic fields. At sufficiently high magnetic field this precessional motion is unimpeded and the electron orbits carry out a free motion about the axis of the system. Ionization occurs in the residual background gas producing more electrons which contribute to an amplification of the process leading ultimately to breakdown following electron diffusion to the cryostat wall.

The solution which has been successfully applied to this problem is to enclose the detector tube in a coaxial beryllium oxide insulating cylinder as shown in Figure 1. A procedure is then adopted whereby the high magnetic field is established first and the high voltage is raised in steps of about 0.5-1.0 kV every few minutes. The reasoning behind this technique is that, when the voltage is raised, the production of a single ion pair will be followed by a mini-avalanche and the subsequent diffusion current will deposit charge on the insulator rather than on the cryostat wall. Eventually the point is reached where the electric field in the annular gap between insulator and detector tube is reduced to a low value and avalanche generation stops. When the discharge has ceased the potential is raised again and the procedure is repeated until the final voltage is reached.

In the initial search for trapped protons from neutron decay the magnetic field reached its minimum value of 1.6 T at the centre of the trap, rising to $\approx 4.0$ T, about 12.5 cms above the upper (pulsed) electrode, and again below the lower (mirror) electrode. This is a typical 'magnetic mirror' configuration (see # 5) although the significance of this point was not fully appreciated at the time. This feature revealed itself in the observation of a large number of magnetically trapped decay electrons with an intensity essentially independent of the accelerating voltage. It was therefore necessary to re-configure the magnetic field profile such that the field decreased uniformly from the detector, through the trap and beyond, thereby eliminating the magnetic mirror effect[^21]. The magnetic field in the trap in the re-configured system attained a value of 1.2 T.

It should also be pointed out that decay electrons of energy $< 1$ keV can be stored in the space inside the mirror electrode which, while providing a potential barrier for protons, is a potential well for electrons, which can generate background protons by ionization of residual hydrogen. This effect may be identified from the non-statistical rate of arrival of the spurious protons and is eliminated by reducing the potential on the mirror to zero and resetting the trap before beginning each trapping cycle.

[^21]: This reference is not provided in the text but is assumed to be a citation or further reading.
#4. Vacuum Breakdown in Crossed Electric and Magnetic Fields

This is a phenomenon which has been explored experimentally in the greatest detail by Penning[22]. In the specific case of current interest we consider the motion of an electron of mass $m_e$ and charge $-e$ moving under the action of a uniform magnetic field $B_z$ in the cylindrical annulus between a cathode of radius $r_i$ fixed at a potential $-V_0$, and an anode of radius $r_o > r_i$. The electrostatic potential at radius $r$ is then given by

$$V(r) = -V_0 \left( \frac{\ln(r_o/r)}{\ln(r_o/r_i)} \right)$$

(4.1)

We shall assume that the electron was initially emitted from the cathode with zero kinetic energy so that $\dot{r} = r^2 \dot{\phi} = 0$ when $r = r_i$. Since the conditions of cylindrical symmetry still apply we retain the conservation of angular momentum about the axis

$$p_\phi = m_e r_i^2 (\dot{\phi} + \frac{1}{2} \omega_{ce}) = m_e r_i^2 \left( \frac{1}{2} \omega_{ce} \right)$$

(4.2)

where $\omega_{ce} = (eB_z/m_e)$ has a positive value for an electron. The total energy equation is then

$$E_e = \frac{1}{2} m_e \dot{r}^2 + \frac{1}{2} m_e (\omega_{ce} r_i)^2 (r^2 - r_i^2)^2 - e(V(r) - V_0)$$

(4.3)

The magnetron effect is initiated at that potential at which the electron turns back, i.e. $\dot{r} = 0$ at, say, $r = r_m < r_o$ and at lower potentials the cyclotron orbits can precess freely about the axis thereby generating avalanches in the background gas. At this point

$$e(V(r_m) - V_0) = \frac{1}{2} m_e \omega_{ce}^2 \left( \frac{(r_o^2 - r_m^2)^2}{4r_o^2} \right) = \frac{1}{2} m_e \omega_{ce}^2 a^2$$

(4.4)

where $a$ is the cyclotron radius of electron motion in the magnetic field $B_z$. In the original neutron lifetime experiments $r_i = 0.01$ m and $r_o = 0.045$ m, thus $a = 0.0214$ m and $B_z a = 0.107$ Tm. From tabulated values of the Bρ-parameters in electron spectroscopy we may conclude that the electron energy at the magnetron transition point has a value close to 3 MeV which is well into the region of relativistic energies. We may also invert the question and ask at what value of $B_z$ does the magnetron transition take place when $V_0 = 30$ kV? For $a = 0.0214$ m the answer is $B_z = 0.028$ T = 280 gauss. The precise value is not important since it is clear that the potential $V_0 \simeq 2$ kV above which breakdown was observed was not related to the magnetron effect which is already in full operation, but rather to the electron energy required to generate an avalanche which in the case of the proportional counter is typically of the order of few keV.

The charge to mass ratio for the electron has the value

$$\left( \frac{|e|}{m_e} \right) = 1.758796 \cdot 10^{11} C(kG)^{-1}$$

Thus the angular frequency of non-relativistic cyclotron motion in a 5 T magnetic field is

$$\omega_c = 8.794 \cdot 10^{11} \text{ sec}^{-1}$$

and the radial electric field in the annulus is
\[ E_r(r) = -\left( \frac{\partial V}{\partial r} \right) = -\frac{V_0}{r \ln(r_o/r_i)} \]

which, for \( V_0 = 30 \text{ kV} \), gives the value

\[ E_r(r)_{\text{max}} = -1.995 \cdot 10^6 \text{ V m}^{-1} \]

In order that the electron motion be oscillatory in the annular region it is required that

\[ \left( \frac{\omega_r}{2} \right)^2 = \left( \frac{\omega_c}{2} \right)^2 + \left( \frac{eE_r(r)}{m_e} \right) > 0 \tag{4.5} \]

This requirement is easily satisfied since

\[ \left( \frac{\omega_r}{2} \right)^2 = 1.993 \cdot 10^{23} - 3.509 \cdot 10^{19} \text{ sec}^{-2} \gg 0 \]

According to the Diethorn theory \cite{23} of avalanche generation, these occur under conditions of pressure \( p \) where

\[ \frac{V_0}{p r_i \ln(r_o/r_i)} > (2 - 10) \cdot 10^6 (V/m)(\text{bar})^{-1} \tag{4.6} \]

which at the observed breakdown point of \( V_0 \approx 2 \text{ kV} \) indicates a pressure range of 1.6-6.7 millibar. Application of Langevin theory \cite{24} shows that, when \( E \) and \( B \) are orthogonal, the radial and azimuthal drift velocities are given by

\[ v_r = \left( \frac{e}{m_e} \right) \left[ \frac{\lambda_{\text{coll}}}{\lambda_{\text{coll}}^2 + \omega_c^2} \right] E_r \quad v_\phi = \left( \frac{e}{m_e} \right) \left[ \frac{\omega_c}{\lambda_{\text{coll}}^2 + \omega_c^2} \right] E_r \tag{4.7} \]

where \( \lambda_{\text{coll}} \) is the collision frequency between electrons and gas molecules. For \( \lambda_{\text{coll}} = 0 \), \( v_r = 0 \) and \( v_\phi = E_r / B_z \), which is the usual condition for motion in a vacuum. However, for \( \lambda_{\text{coll}} \neq 0 \), electrons will diffuse from cathode to anode and the system breaks down. In practice the radial drift current can be cut off by inserting a ceramic insulator in the annular region between cathode and anode, as described in \# 3, and further breakdown is inhibited.

It is also important to understand that the fields may be crossed at the point where the detector is positioned to record the accelerated protons, putting the detector itself in danger of breakdown. This danger can be avoided by recessing the detector approximately one diameter into its containing tube, which is also at high negative voltage, where the radial component of electric field approaches zero \cite{14}.

\#5 The Magnetic Mirror Effect

A proton of energy \( E \approx 0.75 \text{ keV} \) moving in a magnetic field \( B = 5T \) carries out a cyclotron motion with angular frequency

\[ \omega_c = 4.80 \cdot 10^8 \text{ sec}^{-1} , \]

on a circular orbit of radius \( a < 0.8 \text{ mm} \). In a non-harmonic axial potential the oscillation angular frequency is of course amplitude dependent, but, approximating the potential on the axis by a quadratic dependence \( V(z) = V''(0) z^2/2 \), such that \( V(z) = 1.0 \text{ kV} \) at \( z = 10 \text{ cm} \), we may estimate

\[ \omega_z = 4.38 \cdot 10^6 \text{ sec}^{-1} . \]
Thus $\omega_z$ is about 1% of $\omega_c$, but is 100 times greater that the angular frequency of the magnetron drift motion. Since the conditions required for the conservation of the adiabatic invariants in the motion are then easily fulfilled, the longitudinal magnetic force exerted on the trapped particle is then given by

$$F_{mz} = -\frac{eB_r}{r} < r^2 \dot{\phi} > = e \frac{\partial B_z}{\partial z} < r^2 \dot{\phi} > = -m \omega_c \left( \frac{1}{B_z} \frac{\partial B_z}{\partial z} \right) < r^2 \dot{\phi} >$$

(5.1)

where the factor

$$< r^2 \dot{\phi} > = \frac{J_a}{2\pi} = \frac{m \omega_c a^2}{2} + O \left( \frac{\omega_z}{\omega_c} \right)^2$$

(5.2)

is an adiabatic invariant of the motion.

Equation 5.1 defines the so-called ‘magnetic mirror’ force which repels charged particles from regions of high magnetic field irrespective of the sign of the charge, and which has led to difficulties of interpretation in a number of experiments on neutron beta-decay. Essentially what happens is that the conservation of the longitudinal adiabatic invariant brings about a transfer of energy between the transverse and longitudinal degrees of freedom, consistent with the requirement that the magnetic force does no work. The magnetic force may also be viewed as arising from a pseudo-potential

$$F_m = -\mu \cdot \nabla B, \quad \mu = \frac{e}{(-2\pi/\omega_c)} \pi a^2 = \left( \frac{m}{2} \right) a^2 \omega_c^2 / B$$

(5.3)

derived from the coupling of the magnetic moment $\mu$ of the cyclotron orbit and the magnetic field.

In the most recent version of the Penning trap method the magnetic field was designed to decrease by about 5% between the trap exit and the detector in order to ensure that the exiting protons were impelled in the direction of the detector when the confining potential barrier was lowered. In this way there was no possibility that protons remained permanently trapped, thereby increasing the measured neutron lifetime.

Subsequently it was noted that the magnetic mirror effect could be exploited in reverse to measure the proton spectrum by setting the confining potential barrier in a region where the magnetic field was only about 10% of its value at the centre of the trap, an arrangement which transfers most of the proton’s kinetic energy into its longitudinal degree of freedom. This phenomenon, known as ‘adiabatic focusing’, also provides the basis for the Fermi process for the acceleration of cosmic rays by a moving magnetic mirror. The process has many features in common with the betatron accelerator.

# 6 Radial Drifting

For the study of neutron decay it is important to know that a proton which is produced at a certain point in space, moving in a cyclotron orbit with its guiding centre on a given magnetic field line, will move its guiding centre onto an equivalent field line obtained by rotation through an arbitrary angle about the z-axis. The concept of a guiding centre is valid only when the motion is averaged over a period of time of order $|2\pi/\omega_c|$. If, however, there are substantial departures from cylindrical symmetry, the guiding centre may end up on quite a different quite field line having drifted away from the axis, and perhaps out of the trap. The same phenomenon is of considerable significance in plasma physics.
In the case of the Penning trap such azimuthal asymmetries may come about mainly by
(a) a misalignment of electric and magnetic fields:
(b) an intrinsic asymmetry in the electric field due to slight deformation of the electrodes
into an elliptical shape:
(c) an intrinsic asymmetry in the magnetic field arising from asymmetric coil winding.

In the case (a) of a misalignment the radial velocity is given by
\[
\left( \frac{dR(E)}{dt} \right)_1 = -\left( \frac{n \times E}{B_z} \right)_r
\]
where \( n \) is a unit vector in the direction of the magnetic field \( B \) which we assume to be
cylindrically symmetric about the \( z \)-axis, whereas \( E \) is cylindrically symmetric about a \( z' \)-axis
which is set at a small angle \( \theta \) with respect to the \( z \)-axis. Thus we have the relations
\[
x' = x \cos(\theta) + z \sin(\theta), \quad y' = y, \quad z' = -x \sin(\theta) + z \cos(\theta)
\]
Assuming that the electric field in the \( z' \)-system is derived from a first order potential
\[
V^{(1)}(r') = -\left( \frac{m \omega_z^2}{e} \right) \left[ \left( \frac{1}{2} \right) r'^2 - z'^2 \right] \approx -\left( \frac{m \omega_z^2}{e} \right) \left[ \left( \frac{1}{2} \right) r^2 - z^2 + 3rz \theta \cos(\phi) \right]
\]
it follows that
\[
E_\phi = -\left( \frac{1}{r} \right) \left( \frac{\partial V}{\partial \phi} \right) \approx \left( \frac{m \omega_z^2}{e} \right) 3z \theta \sin(\phi)
\]
and
\[
\left( \frac{dR(E)}{dt} \right)^{(1)}_1 = -\left( \frac{\omega_z^2}{2 \omega_c} \right) 6z \theta \sin(\phi) = -\omega_p 6z \theta \sin(\phi)
\]
Assuming that \( \omega_z \approx 10^7 \text{sec}^{-1} \) and \( \omega_p \approx 10^5 \text{sec}^{-1} \) and therefore \( \omega_p/\omega_z \approx 10^{-2} \), the angle \( \phi \) changes by an amount of order 1% in half a precession of \( z \), after which time the drift velocity changes sign. Therefore the maximum total drift is given when \( \phi = \pi/2 \) and \( z \) goes from \(-Z\) to \(+Z\)
\[
\Delta R = -\omega_p 6\theta \int_{-Z}^{Z} z dz = 12 \theta Z \omega_p/\omega_z
\]
Assuming \( \theta = 1\% \) and \( Z = 3\) cm, this yields the value \( \Delta R \approx 4 \cdot 10^{-2} \text{mm} \) which is negligible. Also, since this drift velocity changes sign every \( 10^{-7} \) seconds it is impossible for a substantial drift to build up.

We may repeat the calculation taking into account the second order correction to the potential
\[
V^{(2)}(r') \approx -\theta^2 \left( \frac{m \omega_z^2}{e} \right) \left[ \left( \frac{-3}{2} \right) (x^2 - z^2) \right]
\]
from which, by a similar procedure, we may derive the result
\[
\left( \frac{dR(E)}{dt} \right)^{(2)}_1 = -3\theta^2 \omega_p r \sin(2\phi)
\]
This equation may now be integrated to give
\[
R(\phi) = R_0 \exp\left( (3\theta^2/2) \cos(2\phi) \right)
\]
Since the maximum value of $\cos(2\phi)$ is unity and the minimum value is zero the maximum radial displacement is

$$\Delta R = |R - R_0| = \left(\frac{3}{2}\right) \theta^2 r_{\text{min}} \approx 10^{-3} \text{mm}, \quad \theta = 1\% \quad R_0 = 5 \text{mm}$$

We conclude that a small misalignment of the electric field produces minimal radial drift.

A second possibility for finding a non-zero value for $E_\phi$ is a slight deformation of the electrode into an elliptical shape. A potential which is constant on the elliptical boundary

$$\frac{x^2}{\rho^2(1-\varepsilon)^2} + \frac{y^2}{\rho^2} = 1$$

is obtained by adding a term of the form

$$V_\varepsilon(r) = -\varepsilon \left(\frac{m\omega_z^2}{e}\right) [r^2\cos^2(\phi) - z^2]$$

so that the total potential is

$$V_{el}(r) = -\left(\frac{m\omega_z^2}{e}\right) \left[\frac{1}{2}r^2(1 + 2\varepsilon \cos^2(\phi)) - z^2(1 + \varepsilon)\right]$$

This potential satisfies Laplace’s equation $\nabla^2 V_{el}(r) = 0$ and is constant on the elliptical electrode. This corresponds to a relative deformation in the radius $\rho$ of order $\varepsilon$. It may be noted that $V_{el}(r)$ is identical in form to $V^{(2)}(r')$, except that $\varepsilon$ replaces $3\theta^2/2$, i.e.

$$\left(\frac{dR/E}{dt}\right)_1 = -2\varepsilon\omega_p R \sin(2\phi)$$

an equation which can be integrated as before. Assuming that $\varepsilon \approx 10^{-3}$ it follows that

$$\Delta R \approx 10^{-2} \text{mm}$$

and the deformation term is an order of magnitude larger that the second order contribution due to a misalignment of electric and magnetic fields.

There are, in addition, two terms which describe radial drifting in a cylindrically asymmetric $B$-field. These are

$$\left(\frac{dR(B)}{dt}\right)_1 = \left(\frac{n \times (\mu/e)\nabla B}{B}\right)_r$$

and

$$\left(\frac{dR(B)}{dt}\right)_2 = \left(\frac{n \times (p_t^2/em)\partial n/\partial s}{B}\right)_r$$

Here $p_t$ is the component of momentum parallel to $B$, $p_\nu$ is the transverse component and

$$\mu = \frac{p_t^2}{(2mB)} = \left(\frac{m}{2}\right) a^2 \omega_c^2 / B$$

is the magnetic moment of the cyclotron orbit (see eqn.5.3)
The magnetic field in the trapping volume is designed to be uniform only to within about 1% and we may assume that on the axis

\[ B_z(z,0) = B_0[1 + \alpha \left(\frac{z}{Z}\right)^2] \]

where \(|\alpha| \approx 10^{-2}\). Therefore off-axis we have the results

\[ B_z(z,r) = B_z(z,0) - \left(\frac{r}{2}\right)^2 B''_z(z,0) + \ldots = B_0[1 + \alpha \left(\frac{z}{Z}\right)^2 - (1/2)(\frac{r}{Z})^2] + \ldots \]

\[ B_r(z,r) = -rB'_z(z,0) + \ldots = -2\alpha B_0 \left(\frac{r}{Z}\right) \left(\frac{z}{Z}\right) + \ldots \]

The first magnetic radial drift velocity is then given by

\[ \left(\frac{dR(B)\{B\}}{dt}\right)_1 = \left(\frac{\mu}{Be}\right) \left(\frac{1}{R}\right) \left(\frac{\partial B}{\partial \phi}\right) = \left(\frac{\mu}{Be}\right) \left(\frac{-\eta^2}{B_z}\right)^2 2R \sin(2\phi) \approx \left(\frac{\mu}{Be}\right) \left(\frac{\eta}{Z}\right)^2 B_0 R \frac{dcos(2\phi)}{d\phi} \right) \]

Writing

\[ \frac{dcos(2\phi)}{d\phi} = \frac{dcos(2\phi)}{dt} / \frac{d\phi}{dt} = \frac{dcos(2\phi)}{dt} / \omega_p \]

the radial drift equation can now be integrated to give

\[ R = R_0 e^{\exp\left(\frac{ma^2 \omega_c^2 \eta^2 Z^2 \cos(2\phi)}{2mc^2 \omega_c \omega_p}\right)} \]

The maximum drift then occurs when \(\cos(2\phi) = 1\). Also, since for neutron-decay protons \(ma^2 \omega_c^2 / 2mc^2 < 10^{-6}\), it follows that for \(\omega_c = 5 \times 10^8\), \(\omega_p = 10^5\) and \(Z=3\) cm, \(R/R_0 \approx 2\eta^2\). Since the field is designed to be uniform to within 1% we may assume that \(\eta \ll 0.1\%\) and \(\Delta R \ll 10^{-5} mm\).

The second magnetic drift velocity is given by

\[ \left(\frac{dR(B)\{B\}}{dt}\right)_2 = \left(\frac{mz^2}{eB}\right) \left(\frac{B_\phi}{B}\right) \left(\frac{\partial B_z}{\partial s}\right) \left(\frac{B_z}{B}\right) - \left(\frac{B_\phi}{B}\right) \left(\frac{\partial B_z}{\partial s}\right) \left(\frac{B_z}{B}\right) \right) \]

\[ \approx \left(\frac{mz^2}{eB_0}\right) \left(\frac{\eta R \sin(2\phi)}{Z}\right) \left(\frac{2\alpha z - 2\eta}{Z^2}\right) \]

\[ \approx \left(\frac{mz^2}{eB_0}\right) \left(\frac{\eta R \sin(2\phi)}{Z}\right) \left(\frac{2\alpha z - 2\eta}{Z^2}\right) \]

\[ \approx \left(\frac{mz^2}{eB_0}\right) \left(\frac{\eta R \sin(2\phi)}{Z}\right) \left(\frac{2\alpha z - 2\eta}{Z^2}\right) \]

\[ \approx \left(\frac{mz^2}{eB_0}\right) \left(\frac{\eta R \sin(2\phi)}{Z}\right) \left(\frac{2\alpha z - 2\eta}{Z^2}\right) \]
The term proportional to $2\alpha \frac{z}{Z^2}$ changes sign every period of $z$-oscillation and may therefore be ignored. We then find that

$$\left( \frac{d \ln(R(B))}{dt} \right)_2 = \left( \frac{m\dot{z}^2}{mc^2} \right) \left( \frac{\eta}{Z} \right)^2 \frac{d}{dz} \cos(2\phi)$$

As before this equation can be integrated to give

$$R = R_0 \exp \left( \frac{1}{2} \frac{m\dot{z}^2}{mc^2} \right) \left( \frac{2c^2}{\omega_c \omega_p} \right) \left( \frac{\eta}{Z} \right)^2 \cos(2\phi)$$

Apart from the additional factor of 2 this result implies that the two magnetic drift terms are about the same and equally negligible. Of course this is not true in the case that the magnetic field lines are deliberately designed to bend [12–14].

**#7 Proton Loss by Transverse Diffusion**

Charged particles contained in a Penning trap in conditions of perfect vacuum will stay trapped forever. Unfortunately perfect vacuum cannot be achieved in practice and the trapped particles will undergo collisions with atoms of residual gas. In the case of a cryo-pumped system helium atoms are likely to be the most important scattering centres. As a result the guiding centres of the cyclotron orbits will suffer random displacements. Since there is an applied electrostatic field in the form of a longitudinal potential well, the protons are prohibited by energy conservation from escape along the magnetic field whether or not they undergo collisions. However, since there is an outward directed radial electric field in the trap these protons can be transported by successive collisions transverse to the magnetic field lines and must eventually be lost on the electrode walls.

To make further progress it is necessary to inquire into the details of the individual collision processes. There are two extreme situations corresponding to the conventional classification of collisions into close and distant encounters (i) In a close encounter the guiding centre may be displaced through the maximum amount, equal to twice the radius of gyration, corresponding to Poissonian modulation of the free motion. (ii) In a distant collision the displacement is infinitesimal for a single collision but, since the number of such collisions is large the net displacement is finite. This situation may be described as Gaussian modulation of the free motion, and seems likely to dominate assuming that individual collisions between protons and residual atoms are governed by a shielded Coulomb potential, whose differential cross section

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{1}{16} \left( \frac{Ze^2}{4\pi\varepsilon_0 E_p} \right)^2 \left[ \sin^2 \theta + \eta^2 \right]^{-2}$$

(7.1)

is strongly peaked in the forward direction. Here $\eta = \hbar/2r_s p_p$ where $r_s$ is the shielding radius. This latter is a somewhat uncertain quantity but for rare gas atoms is typically of the order of half of one Bohr radius and therefore $< 0.5$ Å. Assuming that the mean free path between collisions is small in comparison with the radius of gyration, then the mean density of trapped protons $F(r, t)$ at radius $r$ at time $t$ satisfies the diffusion equation

$$\frac{\partial F(r, t)}{\partial t} = D \nabla^2 F(r, t)$$

(7.2)

where

$$D = \left( \frac{\nu}{2} \right) \langle x^2 \rangle$$

(7.3)
is the diffusion coefficient, $\nu$ is the collision rate per unit time and $<x^2>$ is the mean square displacement per collision transverse to the magnetic field. $F(r,t)$ is subject to the spatial boundary condition $F(r, t) = 0$ when $r = r_e$ where $r_e$ is the electrode radius.

The diffusion equation has to be solved subject to a second boundary condition which specifies the density $F(r, 0)$ of trapped particles at zero time. The solution of this equation is quite lengthy and leads to the result that, if the trap is filled at a uniform rate $n_0/\tau$, where $\tau$ is the trapping time, the number of trapped protons at time $\tau$ is given by

$$n(\tau) = \int_0^{r_e} F(r, \tau) 2\pi r dr = 4n_0 \sum_{m=0}^{\infty} \frac{r J_1(\alpha_m r_e)}{r_e J_1(\alpha_m)} \left( \alpha_m^4 D\tau/r_e^2 \right)^{-1} \left[ 1 - \exp(\alpha_m^2 D\tau/r_e^2) \right]$$

(7.4)

For scattering in the centre of mass frame of protons of energy $E_p$ and momentum $p_p$ on residual atoms of atomic number $Z$, via a shielded Coulomb potential, the diffusion coefficient is found to be

$$D = \frac{\pi}{6} \left( \frac{Ze^2}{4\pi \varepsilon_0 E_p} \right)^2 N v a^2 \left( \ln[1 + 1/\eta^2] - [1 + 1/\eta^2] \right)$$

(7.5)

where $N$ is the number density of residual atoms and $v$ is the proton velocity. Assuming a background pressure of $10^{-8} \text{torr}$, corresponding to a number density of helium atoms of $3.5 \times 10^{14} \text{m}^{-3}$ it has been estimated that the root mean square position of the guiding centre drifts by about 1 mm in 180 seconds which means that for trapping times $\tau \leq 10 \text{ ms}$ proton loss by diffusion across the magnetic field lines is negligible.

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Acknowledgement

I should like to express my gratitude to Ferenc Glück at the Karlsruhe Institute of Technology for his advice, support and encouragement in the preparation of this review.