Schwarz-type lemmas for generalized holomorphic maps between pseudo-Hermitian manifolds and Hermitian manifolds

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Abstract

In this paper, we consider some generalized holomorphic maps between pseudo-Hermitian manifolds and Hermitian manifolds. By Bochner formulas and comparison theorems, we establish related Schwarz-type results. As corollaries, the Liouville theorem and little Picard theorem for basic CR functions are deduced. Finally, we study CR Carathéodory pseudo-distance on CR manifolds.

1. Introduction

The classical Schwarz–Pick lemma [15] states that every holomorphic map from the unit disc $D$ of $\mathbb{C}$ into itself is distance-decreasing with respect to the Poincaré metric. This was later generalized by Ahlfors [1] to holomorphic maps from the unit disc $D$ into a Riemannian surface with curvature bounded above by a negative constant. This lemma plays an important role in complex analysis and differential geometry, and has been extended to holomorphic maps between higher dimensional complex manifolds (cf. [4, 5, 14, 17, 21], etc.) and harmonic maps between Riemannian manifolds (cf. [12, 18], etc.). An extremely useful generalization is Yau’s Schwarz lemma [21], which says that every holomorphic map from a complete Kähler manifold with Ricci curvature bounded from below by a constant $-K_1 \leq 0$ into a Hermitian manifold with holomorphic bisectional curvature bounded from above by a constant $-K_2 < 0$ is distance-decreasing up to a constant $K_1/K_2$. Later, Tosatti [19] generalized Yau’s result to the almost Hermitian case. Recently, the authors in [9] extended this lemma to generalized holomorphic maps between pseudo-Hermitian manifolds.

The main purpose of this paper is to generalize Yau’s Schwarz lemma to two classes of generalized holomorphic maps between pseudo-Hermitian manifolds and Hermitian manifolds, which are called $(J, J^N)$-holomorphic maps (see Definition 3) and $(J^N, J)$-holomorphic maps (see Definition 4), respectively. By computing the Bochner formulas for these maps and using the maximum principle, we derive the following Schwarz-type lemmas.

**Theorem 1.1.** Let $(M^{2m+1}, HM, J, \theta)$ be a complete pseudo-Hermitian manifold with pseudo-Hermitian Ricci curvature bounded from below by $-K_1 \leq 0$ and $\|A\|_{C^1}$ bounded from above where $A$ is the pseudo-Hermitian torsion. Let $(N^n, J^N, h)$ be a Hermitian manifold with holomorphic bisectional curvature bounded from above by $-K_2 < 0$. Then for any $(J, J^N)$-holomorphic map $f : M \to N$, we have

$$f^*h \leq \frac{K_1}{K_2} G_{\theta}.$$ 

In particular, if $K_1 = 0$, every $(J, J^N)$-holomorphic map from $M$ into $N$ is constant.
Theorem 1.2. Let \((N^n, J^N, h)\) be a complete Kähler manifold with Ricci curvature bounded from below by \(-K_1 \leq 0\). Let \((M^{2m+1}, HM, J, \theta)\) be a Sasakian manifold with pseudo-Hermitian bisectional curvature bounded from above by \(-K_2 < 0\). Then for any \((J^N, J)\)-holomorphic map \(g : N \to M\), we have

\[ g^* G_{\theta} \leq \frac{K_1}{K_2} h. \]

In particular, if \(K_1 = 0\), any \((J^N, J)\)-holomorphic map is horizontally constant.

We remark that from Theorem 1.1, we can deduce the Liouville theorem and little Picard theorem for basic CR functions (see Corollaries 4.2 and 4.3 for details). In Theorem 1.2, when \(\text{dim}_C N = 1\), the hypothesis of pseudo-Hermitian bisectional curvature on \(M\) can be replaced by pseudo-Hermitian sectional curvature (see Corollary 5.3).

As an application of Schwarz lemmas, we introduce CR Carathéodory pseudo-distance on CR manifolds, which is invariant under CR isomorphisms. Making use of the relationships between Carathéodory pseudo-distance and CR Carathéodory pseudo-distance as well as the Schwarz lemma, we can give another Liouville theorem for \((J, J^N)\)-holomorphic maps.

2. Preliminaries

In this section, we will present some notations and facts of pseudo-Hermitian geometry and Hermitian geometry (cf. [2, 10] for details).

Definition 1. Let \(M\) be a real \(2m + 1\)-dimensional orientable \(C^\infty\) manifold. A CR structure on \(M\) is a complex sub-bundle \(T_{1,0}M\) of complex rank \(m\) of the complexified tangent bundle \(TM \otimes \mathbb{C}\) satisfying

(i) \(T_{1,0}M \cap T_{0,1}M = \{0\}\), \(T_{0,1}M = \overline{T_{1,0}M}\);
(ii) \([\Gamma(T_{1,0}M), \Gamma(T_{1,0}M)] \subseteq \Gamma(T_{1,0}M)\).

Then the pair \((M, T_{1,0}M)\) is called a CR manifold.

The complex sub-bundle \(T_{1,0}\) corresponds to a real sub-bundle of \(TM\):

\[ HM = \text{Re}\{T_{1,0}M \oplus T_{0,1}M\} \tag{2.1} \]

which is called Levi distribution and carries a natural complex structure \(J\) defined by \(J(X + \bar{X}) = i(X - \bar{X})\) for any \(X \in T_{1,0}M\). The CR structure can also be said to be \((HM,J)\). Let \((M,HM,J)\) and \((\bar{M},\bar{H}M,\bar{J})\) be two CR manifolds. A smooth map \(f : M \to \bar{M}\) is a CR map if it preserves the CR structures, that is, \(df(HM) \subseteq \bar{H}M\) and \(df \circ J = \bar{J} \circ df\) on \(HM\). Moreover, if \(f\) is a \(C^\infty\) diffeomorphism, it is referred to as a CR isomorphism.

Since \(M\) is orientable and \(HM\) is oriented by its complex structure \(J\), it follows that there exist a global nowhere vanishing 1-form \(\theta\) such that \(HM = \ker(\theta)\). Such \(\theta\) is called a pseudo-Hermitian structure on \(M\). The Levi form \(L_\theta\) of a given pseudo-Hermitian structure \(\theta\) is defined by

\[ L_\theta(X,Y) = d\theta(X,JY) \tag{2.2} \]

for any \(X,Y \in HM\). The integrability condition (ii) in Definition 1 implies \(L_\theta\) is \(J\)-invariant and symmetric. If \(L_\theta\) is positive definite, then \((M,HM,J)\) is said to be strictly pseudo-convex. The quadruple \((M,HM,J,\theta)\) is called a pseudo-Hermitian manifold.

Since \(L_\theta\) is positive definite, there is a step-2 sub-Riemannian structure \((HM, L_\theta)\) on \(M\), and all sections of \(HM\) together with their Lie brackets span \(T_xM\) at each point \(x \in M\).
We say that a Lipschitz curve $\gamma : [0, l] \to M$ is horizontal if $\gamma'(t) \in HM$ a.e. $t \in [0, l]$. For any two points $p, q \in M$, by the theorem of Chow–Rashevsky (cf. [8, 16]), there always exist such horizontal curves joining $p$ and $q$. Consequently, we may define the so-called Carnot–Carathéodory distance:

$$d_{cc}^M(p, q) = \inf \left\{ \int_0^l \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} \, dt \mid \gamma : [0, l] \to M \text{ is a horizontal curve with } \gamma(0) = p \text{ and } \gamma(l) = q \right\}$$

which induces to a metric space structure on $(M, HM, L_\theta)$.

For a pseudo-Hermitian manifold $(M, HM, J, \theta)$, there is a unique globally defined nowhere zero tangent vector field $\xi$ on $M$ called Reeb vector field such that $\theta(\xi) = 1$, $\xi \lfloor d\theta = 0$. Consequently there is a splitting of the tangent bundle $TM$:

$$TM = HM \oplus L,$$  \hspace{1cm} (2.3)

where $L$ is the trivial line bundle generated by $\xi$.

Let $\pi_H : TM \to HM$ denote the natural projection morphism. Set

$$G_\theta(X, Y) = L_\theta(\pi_H X, \pi_H Y)$$  \hspace{1cm} (2.4)

for any $X, Y \in TM$. We extend $J$ to an endomorphism of $TM$ by requiring that

$$J\xi = 0.$$  \hspace{1cm} (2.5)

The $J$-invariance of $L_\theta$ implies $G_\theta$ is also $J$-invariant. Since $G_\theta$ coincides with $L_\theta$ on $HM \times HM$, Levi form $L_\theta$ can be extended to a Riemannian metric $g_\theta$ on $M$ by

$$g_\theta = G_\theta + \theta \otimes \theta.$$  \hspace{1cm} (2.6)

This metric is usually called the Webster metric. On a pseudo-Hermitian manifold, there is a canonical connection preserving the CR structure and the Webster metric, which is usually called the Tanaka–Webster connection.

**Theorem 2.1** (cf. [10]). Let $(M, HM, J, \theta)$ be a pseudo-Hermitian manifold with the Reeb vector field $\xi$ and Webster metric $g_\theta$. Then there is a unique linear connection $\nabla$ on $M$ satisfying the following axioms:

(i) $HM$ is parallel with respect to $\nabla$;

(ii) $\nabla J = 0$, $\nabla g_\theta = 0$;

(iii) the torsion $T_\nabla$ of $\nabla$ satisfies

$$T_\nabla(X, Y) = 2d\theta(X, Y)\xi \text{ and } T_\nabla(\xi, JX) + J T_\nabla(\xi, X) = 0$$

for any $X, Y \in HM$.

The pseudo-Hermitian torsion, denoted by $\tau$, is a $TM$-valued 1-form defined by $\tau(X) = T_\nabla(\xi, X)$ for any $X \in TM$. Set

$$A(X, Y) = g_\theta(\tau(X), Y)$$  \hspace{1cm} (2.7)

for any $X, Y \in TM$. A pseudo-Hermitian manifold is called a Sasakian manifold if $\tau \equiv 0$. Note that the properties of $\nabla$ in Theorem 2.1 imply that $\tau(T_{1,0}M) \subset T_{0,1}M$ and $A$ is a trace-free symmetric tensor field.

Suppose that $(M^{2m+1}, HM, J, \theta)$ is a real $2m + 1$-dimensional pseudo-Hermitian manifold with the Webster metric $g_\theta$. Let $(e_i)_{i=1}^{m}$ be a unitary frame of $T_{1,0}M$ with respect to $g_\theta$ and
Let $(\theta^i)_{i=1}^n$ be its dual frame. Then the ‘horizontal component of $g_\theta$’ may be expressed as

$$G_\theta = \sum_{i=1}^m \theta^i \dot{\theta}^i. \quad (2.8)$$

From [20], we know the following structure equations for the Tanaka–Webster connection:

$$d\theta = 2\sqrt{-1} \sum_j \theta^j \wedge \bar{\theta}^j \quad (2.9)$$

$$d\theta^i = \sum_j \theta^j \wedge \theta^i_j + \theta \wedge \tau^i \quad (2.10)$$

$$d\theta^i_j = \sum_k \theta^k_j \wedge \theta^i_k + \Pi^i_j \quad (2.11)$$

$$\theta^i + \bar{\theta}^i = 0, \quad (2.12)$$

where $\theta^i_j$ is the Tanaka–Webster connection 1-form with respect to $(e_i)_{i=1}^n$, $\tau^i = \sum_j A^i_j \bar{\theta}^j$, and

$$\Pi^i_j = 2\sqrt{-1} \theta^i \wedge \bar{\tau}^j - 2\sqrt{-1} \tau^i \wedge \bar{\theta}^j + \sum_{k,l} R^i_{jk}\theta^k \wedge \bar{\theta}^l + \sum_{k,l} (W^i_{jk}\theta^k \wedge \theta - W^i_{jkl}\theta^k \wedge \bar{\theta}), \quad (2.13)$$

where $W^i_{jk} = A^i_{jk,\bar{i}}$, $W^i_{jkl} = A^i_{k,jl}$ and $R^i_{jkl}$ are the components of curvature tensor with respect to Tanaka–Webster connection. Set $R^i_{jkl} = R^i_{klj}$, then we know that

$$R^i_{jkl} = -R^i_{jkil} = -R^i_{ijlk},$$

$$R^i_{jkl} = R^j_{kl} = R^i_{klij}. \quad (2.14)$$

Suppose $X = \sum_i X^i e_i$ and $Y = \sum_j Y^j e_j$ are two non-zero vectors in $T_{1,0}M$, then the pseudo-Hermitian bisectional curvature determined by $X$ and $Y$ is defined by

$$\sum_{i,j,k,l} R^i_{jkl} X^i X^j Y^k Y^l \left( \sum_i X^i X^i \right) \left( \sum_j Y^j Y^j \right). \quad (2.15)$$

If $X = Y$, the above quantity is referred to as the pseudo-Hermitian sectional curvature in the direction $X$ (cf. [20]). The pseudo-Hermitian Ricci tensor is defined as

$$R_{ij} = \sum_k R_{kij} \quad (2.16)$$

and thus the pseudo-Hermitian scalar curvature is given by

$$R = \sum_i R_{ii} \quad (2.17)$$

Analogous to Laplace operator in Riemannian geometry, there is a degenerate elliptic operator in CR geometry which is called sub-Laplace operator. For a $C^2$ function $u : M \to \mathbb{R}$, $du$
is a smooth section of $T^*M$. Let $\nabla du$ be the covariant derivative of $du \in \Gamma(T^*M)$ with respect to the Tanaka–Webster connection. Therefore, the sub-Laplace operator can be defined by

$$\Delta_b u = tr_H(\nabla du) = \sum_i (u_{i\bar{i}} + u_{\bar{i}i}), \quad (2.18)$$

where $u_{i\bar{i}} = (\nabla du)(e_i, e_{\bar{i}})$.

In [7], the authors give a sub-Laplacian comparison theorem in pseudo-Hermitian geometry which plays a similar role as the Laplacian comparison theorem in Riemannian geometry.

**Lemma 2.2.** Let $(M^{2m+1}, HM, J, \theta)$ be a complete pseudo-Hermitian manifold with pseudo-Hermitian Ricci curvature bounded from below by $-k \leq 0$ and $\|A\|_{C^1} \leq k_1$ ($k_1 \geq 0$). Let $x_0$ be a fixed point in $M$. Then for any $x \in M$ which is not in the cut locus of $x_0$, there exists $C = C(m)$ such that

$$\Delta_b \gamma(x) \leq C(1/\gamma + \sqrt{1 + k + k_1 + k_1^2}),$$

where $\gamma(x)$ is Riemannian distance of $g_\theta$ between $x_0$ and $x$ in $M$, and $\|A\|_{C^1} = \max_{y \in M} \{\|A(y)\|, \|\nabla A(y)\|\}$. Moreover, $\Delta_b \gamma$ is uniformly bounded from above when $\gamma \geq 1$.

Let $N$ be a Hermitian manifold of complex dimension $n$. Let $(\eta_\alpha)_{\alpha=1}^n$ be a unitary frame field of $N$ and $(\omega_\alpha)_{\alpha=1}^n$ be its coframe field. Then the Hermitian metric $h$ of $N$ is given by

$$h = \sum_\alpha \omega_\alpha \bar{\omega}_\alpha. \quad (2.19)$$

It is well known that there are connection 1-forms $(\omega_\beta^\alpha)$ such that

$$d\omega_\alpha = \sum_\beta \omega_\beta \wedge \omega_\beta^\alpha + \Omega_\alpha \quad (2.20)$$

$$\omega_\bar{\alpha}^\beta + \omega_\alpha^{\bar{\beta}} = 0, \quad (2.21)$$

where

$$\Omega_\alpha = \frac{1}{2} \sum_{\beta, \gamma} T_{\beta \gamma}^\alpha \omega_\beta \wedge \omega_\gamma \quad (2.22)$$

$$T_{\beta \gamma} = -T_{\bar{\gamma} \bar{\beta}}. \quad (2.23)$$

Note that this connection is usually called the Chern connection. The curvature forms $(\Omega_\beta^\alpha)$ are defined by

$$\Omega_\beta = d\omega_\beta - \sum_\gamma \omega_{\bar{\gamma}} \wedge \omega_\gamma^\alpha \quad (2.24)$$

and according to (2.21), we have

$$\Omega_\beta^\alpha = -\Omega_\alpha^\beta = \sum_\gamma R_{\alpha \bar{\gamma} \bar{\beta}} \omega_\gamma \wedge \omega_\beta. \quad (2.25)$$

We set $R_{\alpha \beta \gamma \bar{\delta}} = R_{\alpha \bar{\gamma} \bar{\beta} \bar{\delta}}$, then the skew-Hermitian symmetry of $\Omega_\beta^\alpha$ is equivalent to $R_{\alpha \bar{\gamma} \bar{\beta} \bar{\delta}} = R_{\beta \alpha \gamma \delta}$. If $Z = \sum_\alpha Z^\alpha \eta_\alpha$ and $W = \sum_\beta W^\beta \eta_\beta$ are two tangent vectors in $T_{1,0}N$, then the holomorphic bisectional curvature determined by $Z$ and $W$ is defined by

$$\sum_{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} Z^\alpha \bar{Z}_{\bar{\gamma}} \bar{Z}_{\bar{\delta}} \bar{W}_{\gamma} W_{\delta} \leq \frac{(\sum_\alpha Z^\alpha \bar{Z}_{\bar{\alpha}})(\sum_\beta W^\beta \bar{W}_{\bar{\beta}})}{(\sum_\alpha \bar{Z}_{\bar{\alpha}} Z^\alpha)} \quad (2.26)$$
If $Z = W$, the above quantity is called the holomorphic sectional curvature in the direction $Z$. The Ricci tensor is defined as

$$R_{\alpha \bar{\beta}} = \sum_{\gamma} R_{\gamma \bar{\gamma} \alpha \bar{\beta}}$$

(2.27)

and the scalar curvature is given by

$$S = \sum_{\alpha} R_{\alpha \bar{\alpha}}.$$  

(2.28)

For a $C^2$ function $v : N \to \mathbb{R}$ on the Hermitian manifold $N$, the Laplacian of $v$ is defined as the trace of the Hessian matrix of $v$ with respect to the Chern connection $\nabla^c$. In terms of the local frame fields $(\eta_{\alpha})_{\alpha=1}^{n}$, it is a well-known fact that

$$\Delta v = \sum_{\alpha} (v_{\alpha \bar{\alpha}} + v_{\bar{\alpha} \alpha}) = 2 \sum_{\alpha} v_{\alpha \bar{\alpha}},$$

(2.29)

where $v_{\alpha \bar{\alpha}} = (\nabla^c dv)(\eta_\alpha, \eta_{\bar{\alpha}})$.

3. **Bochner formulas**

This section will derive the Bochner formulas of generalized holomorphic maps between pseudo-Hermitian manifolds and Hermitian manifolds.

In pseudo-Hermitian geometry, there is an analogue of the holomorphic function on a complex manifold, which is called the CR function.

**Definition 2** (cf. [10]). Let $(M, HM, J, \theta)$ be a pseudo-Hermitian manifold. A smooth function $f : M \to \mathbb{C}$ is said to be a CR function, if $Zf = 0$ for all $Z \in T_{0,1}M$. Moreover, $f$ is called a basic CR function if the CR function $f$ satisfies $df(\xi) = 0$, where $\xi$ is Reeb vector field.

Remark 1. (i) When the target manifold $N$ is complex plane $\mathbb{C}$ endowed with the canonical complex structure $J^\mathbb{C}$. A smooth function $f : M \to \mathbb{C}$ is a $(J, J^\mathbb{C})$-holomorphic map if and only if it is a basic CR function, that is, $df(\xi) = 0$ and $Zf = 0$ for any $Z \in \Gamma(T_{0,1}M)$.

(ii) The authors in [6] discussed Siu–Sampson type theorem for $(J, J^\mathbb{C})$-holomorphic maps.

Let $f : (M, HM, J, \theta) \to (N, J^N, h)$ be a smooth map with $\dim_{\mathbb{R}} M = 2m + 1$ and $\dim_{\mathbb{C}} N = n$. Then in terms of local frames in Section 2, we can write $df$ as

$$df = \sum_{A, B} f_{AB}^A \theta^B \otimes \eta_A,$$

(3.2)
where
\[ A = 1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n} \]
\[ B = 0, 1, 2, \ldots, m, \bar{1}, \bar{2}, \ldots, \bar{m}, \]
\[ \theta^0 = \theta. \]

Assume \( f \) is \((J, J_N)\)-holomorphic. Clearly the condition (3.1) leads \( f_0^\alpha = f_0^\alpha = f_0^\bar{\alpha} = 0 \) and
\[ f^* \omega^\alpha = \sum_i f_i^\alpha \theta^i. \tag{3.3} \]

For simplification, denote 
\[ \hat{\omega}_\beta^\alpha = f^* \omega_\beta^\alpha, \hat{T}_\beta^\gamma = f^* T_\beta^\gamma, \hat{\Omega}_\beta^\gamma = f^* \Omega_\beta^\gamma, \text{ etc.} \]
Taking the exterior derivative of (3.3) and using (2.10), (2.20), (2.22), we get
\[ \sum_i Df_i^\alpha \wedge \theta^i = \frac{1}{2} \sum_{\beta, \gamma, i} \hat{T}_\beta^\gamma f_i^\beta \theta^i \wedge \theta^j - \sum_i f_i^\alpha \theta^i \wedge \tau^i, \tag{3.4} \]
where
\[ Df_i^\alpha = df_i^\alpha + \sum_{\beta} f_i^\beta \hat{\omega}_\beta^\alpha - \sum_j f_j^\alpha \theta_j^i = f_{i0}^\alpha + \sum_k (f_{ik}^\alpha \theta^k + f_{i\bar{k}}^\alpha \theta^\bar{k}). \tag{3.5} \]

From (3.4), it follows that
\[ f_{i0}^\alpha = f_{ij}^\alpha = 0 \tag{3.6} \]
\[ f_{ij}^\alpha = f_{ji}^\alpha - \sum_{\beta, \gamma} \hat{T}_\beta^\gamma f_i^\beta f_j^\gamma. \tag{3.7} \]

By (3.5) and (3.6), we have
\[ df_i^\alpha = \sum_k f_{ik}^\alpha \theta^k + \sum_j f_j^\alpha \theta_j^i - \sum_{\beta} f_i^\beta \hat{\omega}_\beta^\alpha. \tag{3.8} \]

Taking the exterior derivative of (3.8), and applying structure equations (2.11) and (2.24), one finds that
\[ \sum_k Df_{ik}^\alpha \wedge \theta^k = \sum_{\beta} f_{ik}^\beta \hat{\Omega}_\beta^\alpha - \sum_j f_j^\alpha \Pi_j^i - \sum_k f_{ik}^\alpha \theta^k \wedge \tau^k, \tag{3.9} \]
where
\[ Df_{ik}^\alpha = df_{ik}^\alpha - \sum_j (f_{ij}^\alpha \theta_k^j - f_{jk}^\alpha \theta_i^j) + \sum_{\beta} f_{ik}^\beta \hat{\omega}_\beta^\alpha = f_{ik0}^\alpha + \sum_j (f_{ikj}^\alpha \theta^j + f_{i\bar{k}j}^\alpha \theta^\bar{j}). \]

Hence,
\[ f_{ikj}^\alpha = \sum_l f_l^\alpha R_{ikj}^\alpha - \sum_{\beta, \gamma, \delta} f_{ik}^\beta f_j^\gamma f_{\delta}^\alpha \hat{R}_{\beta\gamma\delta}^\alpha. \tag{3.10} \]

Set
\[ u = \sum_{\alpha, i} f_{ii}^\alpha f_{i\bar{i}}^\alpha. \tag{3.11} \]

Since (3.6), the horizontal differential of \( u \) is given by
\[ d_H u = \sum_{\alpha, i, k} (f_{ikj}^\alpha \theta^k + f_{i\bar{k}j}^\alpha \theta^\bar{k}), \]
thus,
\[ u_k = \sum_{\alpha,i} f_{ik}^\alpha f_i^\alpha. \]

By computing \( d_H u_k \), we obtain
\[ u_{\bar{k}k} = \sum_{\alpha,i} (f_{ik}^\alpha f_i^\alpha + |f_{ik}|^2). \]

Using (3.10), we get
\[ u_{\bar{k}k} = \sum_{\alpha,i} |f_{ik}^\alpha|^2 + \sum_{\alpha,i,l} f_{i}^\alpha f_{i}^\alpha R_{ikk} - \sum_{\alpha,\beta,\gamma,\delta,i} f_{i}^\alpha f_{i}^\alpha f_{k}^\alpha f_{k}^\alpha R_{i\beta\gamma\delta}. \]

Hence, we have the following lemma.

**Lemma 3.1.** Suppose \( f : (M, HM, J, \theta) \to (N, J^N, h) \) be a \((J, J^N)\)-holomorphic map, then we have
\[
 f^* h \leq uG_\theta, \tag{3.12}
\]
and
\[
 \frac{1}{2} \Delta_b u = \sum_{\alpha,i,k} |f_{ik}^\alpha|^2 + \sum_{\alpha,i,l} f_{i}^\alpha f_{i}^\alpha R_{ikl} - \sum_{\alpha,\beta,\gamma,\delta,i} f_{i}^\alpha f_{i}^\alpha f_{k}^\alpha f_{k}^\alpha R_{i\beta\gamma\delta}. \tag{3.13}
\]

In the rest of this section, we turn to generalized holomorphic maps from Hermitian manifolds to pseudo-Hermitian manifolds.

**Definition 4.** Let \((N, J^N)\) be a complex manifold and \((M, HM, J, \theta)\) be a pseudo-Hermitian manifold. A smooth map \( g : N \to M \) is called a \((J^N, J)\)-holomorphic map if it satisfies
\[
 dg_H \circ J^N = J \circ dg_H, \tag{3.14}
\]
where \( dg_H = \pi_H \circ dg, \pi_H : TM \to HM \) is the natural projection. Moreover, the \((J^N, J)\)-holomorphic map \( g \) is said to be horizontally constant if \( dg_H \equiv 0 \).

**Remark 2.** Every \((J^N, J)\)-holomorphic map \( g : N \to M \) satisfying \( dg(TN) \subseteq HM \), that is, \( dg \circ J^N = J \circ dg \), is constant. Indeed, if not, there exists a local vector field \( e \in T^{1,0} N \) such that \( dg(e) \neq 0 \), so \( HM \ni dg[e, \bar{e}] = [dg(e), dg(\bar{e})] \notin HM \), which leads to a contradiction.

Suppose \((N, J^N)\) is a complex manifold and \((M, HM, J, \theta)\) is a pseudo-Hermitian manifold. Let \( g : N \to M \) be a \((J^N, J)\)-holomorphic map. Under the local frames in Section 2, we can express the differential of \( g \) as
\[
 dg = \sum_{A,B} g_A^B \omega^A \otimes e_B, \tag{3.15}
\]
where \( e_0 = \xi \) and the values of \( A, B \) are the same as those of (3.2). Clearly, the condition (3.14) in Definition 4 is equivalent to
\[
 g_0^i = g_i^0 = 0. \tag{3.16}
\]
From (3.14) and (3.16), we have
\[ g^* \theta^i = \sum_\alpha g^i_\alpha \omega^\alpha \] (3.17)
\[ g^* \theta = \sum_\alpha (g^0_\alpha \omega^\alpha + g^0_\bar{\alpha} \bar{\omega}^\bar{\alpha}). \] (3.18)

To simplify the notations, we set \( \hat{\theta}^i_j = g^* \theta^i_j, \hat{A}^i_j = g^* A^i_j, \hat{R}^i_{jkl} = R^i_{jkl}, \) etc. By taking the exterior derivative of (3.17) and using the structure equations in \( M \) and \( N, \) we get
\[ \sum_\alpha Dg^i_\alpha \wedge \omega^\alpha + \sum_\alpha g^i_\alpha \Omega^\gamma - \sum_{\alpha, \beta, j} g^0_\alpha g^j_\beta \hat{A}^i_j \wedge \omega^\alpha \wedge \omega^\beta - \sum_{\alpha, \beta, j} g^0_\alpha g^j_\beta \hat{A}^i_j \omega^\alpha \wedge \omega^\beta = 0, \] (3.19)
where
\[ Dg^i_\alpha = dg^i_\alpha - \sum_\gamma g^i_\alpha \gamma \omega^\gamma + \sum_j g^j_\alpha \hat{\theta}^i_j = \sum_\beta (g^i_\alpha \beta \omega^\beta + g^i_{\bar{\alpha}} \bar{\beta} \bar{\omega}^\bar{\beta}). \] (3.20)

Then (3.19) gives
\[ g^i_{\alpha \beta} = g^i_{\beta \alpha} - \sum_\gamma g^i_\gamma T^\gamma_{\beta \alpha} \] (3.21)
\[ g^i_{\bar{\alpha} \bar{\beta}} = - \sum_\gamma g^0_\gamma g^j_\beta \hat{A}^i_j \]
\[ \sum_j (g^0_\alpha g^j_{\bar{\beta}} - g^0_\bar{\beta} g^j_\alpha) \hat{A}^i_j = 0. \]

Taking the exterior derivative of (3.20) and using the structure equations again, we obtain
\[ \sum_\beta Dg^i_{\alpha \beta} \wedge \omega^\beta + \sum_\beta Dg^i_{\bar{\alpha} \bar{\beta}} \wedge \omega^\bar{\beta} = - \sum_\gamma g^i_\gamma \Omega^\gamma_\alpha + \sum_j g^j_\alpha g^* \Pi^i_j, \] (3.22)
where
\[ Dg^i_{\alpha \beta} = dg^i_{\alpha \beta} - \sum_\gamma (g^i_\gamma \omega^\beta_\alpha) + \sum_j g^j_\alpha \hat{\theta}^i_j = \sum_\gamma (g^i_{\alpha \beta \gamma} \omega^\gamma + g^i_{\alpha \bar{\beta} \bar{\gamma}} \bar{\omega}^\bar{\gamma}) \] (3.23)
\[ Dg^i_{\bar{\alpha} \bar{\beta}} = dg^i_{\bar{\alpha} \bar{\beta}} - \sum_\gamma (g^i_{\bar{\alpha} \bar{\beta} \bar{\gamma}} \omega^\bar{\gamma} + g^i_{\bar{\alpha} \gamma \gamma} \bar{\omega}^\gamma) \]
\[ \sum_j g^j_{\bar{\alpha} \bar{\beta}} \hat{\theta}^i_j = \sum_\gamma (g^i_{\bar{\alpha} \bar{\beta} \gamma} \omega^\gamma + g^i_{\bar{\alpha} \gamma \bar{\gamma}} \bar{\omega}^\bar{\gamma}). \] (3.24)

Compare the (1,1)-form of (3.22) and get
\[ g^i_{\alpha \bar{\gamma}} = g^i_{\alpha \gamma} + \sum_\delta g^i_{\delta} R^\delta_{\alpha \beta \gamma} - \sum_{j, k, l} g^j_\alpha g^k_l \hat{R}^i_{jkl} - \sum_{j, k} (g^0_\alpha g^k_l g^0_\gamma \hat{W}^i_{jk} + g^0_\alpha g^0_\gamma g^k_l \hat{W}^i_{jk}). \] (3.25)

Set
\[ v = \sum_{i, \alpha} g^i_\alpha g^i_{\bar{\alpha}}. \] (3.26)

By direct computation, we have
\[ \frac{1}{2} \Delta v = \sum_{i, \alpha, \gamma} (|g^i_\alpha|^2 + |g^i_{\bar{\alpha}}|^2) + \sum_{i, \alpha, \gamma} (g^i_\alpha g^i_{\alpha \gamma} + g^i_{\alpha} g^i_{\alpha \bar{\gamma}}). \] (3.27)
Using (3.21) and (3.25), we perform the following computation:

\[g^i_{\alpha\gamma\bar{\gamma}} = g^{i}_{\gamma\alpha\bar{\gamma}} - \left( \sum_{\beta} g^{j}_{\beta T_{\gamma}} \right) \] (3.28)

and

\[g^{i}_{\alpha\gamma\bar{\gamma}} = - \left( \sum_{j} g^{0}_{\beta} g^{j}_{\gamma} \hat{A}^{j}_{\beta} \right) \] (3.29)

From (3.21), (3.27), (3.28), (3.29), we obtain the Bochner formula as follows.

**Lemma 3.2.** Suppose \(f : (N, J^N, h) \rightarrow (M, HM, J, \theta)\) is a \((J^N, J)\)-holomorphic map, then we have

\[g^i G_\theta \leq vh,\] (3.30)

and

\[
\frac{1}{2} \Delta v = \sum_{i, \alpha, \gamma} (|g^i_{\alpha\gamma}|^2 + |g^i_{\alpha\bar{\gamma}}|^2) + \sum_{i, \alpha, \delta} g^\delta_{\alpha \gamma} R_{\alpha \delta} - \sum_{i, j, k, l, \alpha, \gamma} g^j_{\alpha \gamma} g^k_{\alpha \bar{\gamma}} \hat{R}^{j}_{lijk} - \sum_{i, j, k, \alpha, \gamma} (\hat{g}^j_{\alpha \gamma} g^k_{\alpha \bar{\gamma}} W^{j}_{lijk} + g^j_{\alpha \gamma} g^k_{\alpha \bar{\gamma}} W^{j}_{lijk}) + \sum_{i, \alpha, \gamma} \left( \sum_{j} g^0_{\beta} g^{j}_{\gamma} \hat{A}^{j}_{\beta} \right) - \sum_{i, \alpha, \gamma} \left( \sum_{\beta} g^{l}_{\beta} T_{\gamma} \right) \] (3.31)

4. **Schwarz-type lemma for \((J, J^N)\)-holomorphic maps**

In this section, we will establish the Schwarz-type lemma for \((J, J^N)\)-holomorphic maps. As the corollaries of this lemma, the Liouville theorem and little Picard theorem for basic CR functions are given.

Suppose that \(M, HM, J, \theta\) is a complete pseudo-Hermitian manifold and \((N, J^N, h)\) is a Hermitian manifold. Let \(f : M \rightarrow N\) be a \((J, J^N)\)-holomorphic map. To give the Schwarz-type lemma for \(f\), it is sufficient to estimate the upper bound of \(u\) defined by (3.11) due to (3.12). Define

\[\phi(x) = (a^2 - \gamma^2(x))^2 u,\] (4.1)

where \(\gamma(x)\) is the Riemannian distance from a fixed point \(z\) to \(x\) in \(M\). Let \(B_a(z)\) be an open geodesic ball in \(M\) with its center at \(z\) and radius of \(a\). It is obvious that \(\phi(x)\) attains its maximum in \(B_a(z)\). Suppose \(x_0\) is a maximum point. For maximum principle, \(\gamma\) is required to be twice differentiable near \(x_0\). This may be remedied by the following consideration (cf. [4]): Let \(\tau : [0, \gamma(x_0)] \rightarrow M\) be a minimizing geodesic joining \(z\) and \(x_0\) such that \(\tau(0) = z\) and...
\( \tau(\gamma(x_0)) = x_0 \). If \( x_0 \) is a cut point of \( z \), then for a small number \( \varepsilon > 0 \), \( x_0 \) is not a conjugate point of \( \tau(\varepsilon) \) along \( \tau \). It is well known that there is a cone \( C \) with its vertex at \( \tau(\varepsilon) \) and containing a neighborhood of \( x_0 \). Let \( \gamma(x) \) denote the Riemannian distance from \( \tau(\varepsilon) \) to \( x \), then \( \gamma \) is smooth near \( x_0 \). Let \( \tilde{\gamma}(x) = \varepsilon + \gamma \), then we have \( \gamma \leq \tilde{\gamma} \) and the equality holds at \( x_0 \).

So we can consider the function \((a^2 - \tilde{\gamma}^2)u\), it also attains the maximum at \( x_0 \). Let \( \varepsilon \to 0 \), we may assume that \( \gamma \) is smooth near \( x_0 \). Therefore, applying the maximum principle to \( \phi \), at \( x_0 \), we have

\[
\frac{\nabla H u}{u} = -2 \frac{\nabla H (a^2 - \gamma^2)}{a^2 - \gamma^2} \tag{4.2}
\]

\[
\Delta_b u - 4 \frac{\nabla H (a^2 - \gamma^2)}{(a^2 - \gamma^2)^2} \cdot \nabla H u u + 2 \frac{\Delta_b (a^2 - \gamma^2)}{a^2 - \gamma^2} + 2 \frac{\|\nabla H (a^2 - \gamma^2)\|^2}{(a^2 - \gamma^2)^2} \leq 0, \tag{4.3}
\]

where the inner product \( \cdot \) and the norm \( \| \cdot \| \) is induced by the Webster metric \( g_0 \). Substituting (4.2) into (4.3), we obtain

\[
\frac{\Delta_b u}{u} - 6 \frac{\|\nabla H (a^2 - \gamma^2)\|^2}{(a^2 - \gamma^2)^2} - 2 \frac{\Delta_b \gamma^2}{a^2 - \gamma^2} \leq 0. \tag{4.4}
\]

Let \(-K_1\) be the greatest lower bound of the pseudo-Hermitian Ricci curvature of \( M \), and \(-K_2\) be the least upper bound of the holomorphic bisectional curvature of \( N \). Then it follows from (3.13) that

\[
\frac{1}{2} \Delta_b u \geq -K_1 u + K_2 u^2. \tag{4.5}
\]

Using \( \|\nabla H \gamma\| \leq 1 \), we get

\[
\|\nabla H (a^2 - \gamma^2)\|^2 = 2\gamma \nabla H \gamma \leq 4a^2. \tag{4.6}
\]

If \( \|A\|_{C^1} \) is bounded from above on \( M \), then by Lemma 2.2, we have

\[
\Delta_b \gamma^2 = 2\gamma \Delta_b \gamma + 2\|\nabla H \gamma\|^2 \leq C(1 + a), \tag{4.7}
\]

where \( C \) is a positive constant independent of \( a \). Suppose that \( K_1 \geq 0 \) and \( K_2 > 0 \). Then substituting (4.5), (4.6) and (4.7) into (4.4), we obtain

\[
u(x_0) \leq \frac{K_1}{K_2} + \frac{12a^2}{K_2 (a^2 - \gamma^2(x_0))^2} + \frac{C(1 + a)}{K_2 (a^2 - \gamma^2(x_0))}.
\]

Thus,

\[
(a^2 - \gamma^2(x))^2 u(x) \leq (a^2 - \gamma^2(x_0))^2 u(x_0)
\]

\[
\leq \frac{K_1}{K_2} (a^2 - \gamma^2(x_0))^2 + \frac{12a^2}{K_2} + \frac{C(1 + a)}{K_2} (a^2 - \gamma^2(x_0))
\]

\[
\leq \frac{K_1}{K_2} a^4 + \frac{12a^2}{K_2} + \frac{C(1 + a)a^2}{K_2}
\]

for any \( x \in B_a(z) \). It follows that

\[
u(x) \leq \frac{K_1 a^4}{K_2 (a^2 - \gamma^2(x))^2} + \frac{12a^2}{K_2 (a^2 - \gamma^2(x))^2} + \frac{C(1 + a)a^2}{K_2 (a^2 - \gamma^2(x))^2}.
\]

Let \( a \to \infty \), we deduce that

\[
\sup_M u \leq \frac{K_1}{K_2}.
\]
Due to (3.12), we obtain

\[ f^*h \leq \frac{K_1}{K_2} G_{\theta}. \]

Therefore, we have the Schwarz-type lemma as follows.

**Theorem 4.1.** Let \((M^{2m+1}, HM, J, \theta)\) be a complete pseudo-Hermitian manifold with pseudo-Hermitian Ricci curvature bounded from below by \(-K_1 \leq 0\) and \(||A||_{C^1}\) bounded from above. Let \((N^n, J^N, h)\) be a Hermitian manifold with holomorphic bisectional curvature bounded from above by \(-K_2 < 0\). Then for any \((J, J^N)\)-holomorphic map \(f : M \to N\), we have

\[ f^*h \leq \frac{K_1}{K_2} G_{\theta}. \]  

(4.8)

In particular, if \(K_1 = 0\), every \((J, J^N)\)-holomorphic map from \(M\) into \(N\) is constant.

**Remark 3.** Using Royden’s lemma (cf. [17]), we can weaken the hypothesis on \(N\) by assuming the holomorphic sectional curvature of \(N\) is bounded above by \(-K_2 < 0\), but the constant \(\frac{K_1}{K_2}\) in (4.8) will be replaced by \(\frac{2v}{v+1} \frac{K_1}{K_2}\), where \(v\) is the maximal rank of \(df\).

Since a unit disk in complex plane equipped with Poincaré metric is a Kähler manifold with constant negative holomorphic curvature, we have the following Liouville-type theorem for basic CR functions.

**Corollary 4.2.** Let \((M, HM, J, \theta)\) be a complete pseudo-Hermitian manifold with non-negative pseudo-Hermitian Ricci curvature and \(||A||_{C^1}\) bounded from above. Any bounded basic CR function on \(M\) is constant.

Using the fact that the complex plane \(\mathbb{C}\) minus two distinct points admits a complete Hermitian metric with holomorphic curvature less than a negative constant [13], we derive little Picard theorem for basic CR functions.

**Corollary 4.3.** Let \((M, HM, J, \theta)\) be a complete pseudo-Hermitian manifold with non-negative pseudo-Hermitian Ricci curvature and \(||A||_{C^1}\) bounded from above. Any basic CR function \(u : M \to \mathbb{C}\) missing more than one point in its image is constant.

### 5. Schwarz-type lemma for \((J^N, J)\)-holomorphic maps

In this section, we will establish the Schwarz-type lemma for \((J^N, J)\)-holomorphic maps. In [21], Yau has proved that

**Proposition 5.1.** Let \(N\) be a complete Kähler manifold with Ricci curvature bounded below. A non-negative smooth function \(u\) on \(N\) satisfies the following inequality:

\[ \Delta u \geq -k_1 u + k_2 u^2, \]

where \(k_1 \geq 0, k_2 > 0\). Then \(\sup_M u \leq \frac{k_1}{k_2}\).

It is notable that Tosatti has generalized this proposition to almost Hermitian manifold in [19].
Let $g : (N^n, J^N, h) \to (M^{2m+1}, HM, J, \theta)$ be a $(J^N, J)$-holomorphic map. If $N$ is a complete Kähler manifold and $M$ is a Sasakian manifold, by (3.31), we obtain

$$\frac{1}{2} \Delta v = \sum_{i,\alpha,\gamma} (g^i_{\alpha\gamma} |^2 + |g^i_{\alpha\gamma}|^2) + \sum_{i,\alpha,\beta} \overline{g^i_{\alpha\beta}} R_{\alpha\beta} - \sum_{\alpha,\gamma,\iota,\lambda, j, k, l} \overline{g^i_{\alpha\gamma}} g^j_{\iota\lambda} g^k_{\lambda\gamma} \hat{R}_{jikl}, \quad (5.1)$$

where $v$ is defined by (3.26). Let $-K_1$ be the greatest lower bound of the Ricci curvature of $N$, and $-K_2$ be the least upper bound of the pseudo-Hermitian bisectional curvature of $M$, where $K_1 \geq 0, K_2 > 0$. It follows from (5.1) that

$$\frac{1}{2} \Delta v \geq -K_1 v + K_2 v^2. \quad (5.2)$$

By Proposition 5.1, we deduce that $\sup v \leq \frac{K_1}{K_2}$, which, combining with (3.30), yields the Schwarz-type lemma for $(J^N, J)$-holomorphic maps as follows.

**Theorem 5.2.** Let $(N^n, J^N, h)$ be a complete Kähler manifold with Ricci curvature bounded from below by $-K_1 \leq 0$. Let $(M^{2m+1}, HM, J, \theta)$ be a Sasakian manifold with pseudo-Hermitian bisectional curvature bounded from above by $-K_2 < 0$. Then for any $(J^N, J)$-holomorphic map $g : N \to M$, we have

$$g^* G_\theta \leq \frac{K_1}{K_2} h. \quad (5.3)$$

In particular, if $K_1 = 0$, any $(J^N, J)$-holomorphic map is horizontally constant.

**Remark 4.** By Royden’s lemma (cf. [17]), we can weaken the hypothesis on $M$ by assuming the pseudo-Hermitian sectional curvature of $M$ is bounded above by $-K_2 < 0$, but the constant $\frac{K_1}{K_2}$ in (5.3) will be replaced by $\frac{2v K_1}{v+1 K_2}$, where $v$ is the maximal rank of $dg_H$.

If $\dim_{\mathbb{C}} N = 1$, one can also weaken the hypothesis on $N$.

**Corollary 5.3.** Let $(N, J^N, h)$ be a complete one-dimensional Kähler manifold with Ricci curvature bounded from below by $-K_1 \leq 0$. Let $(M^{2m+1}, HM, J, \theta)$ be a Sasakian manifold with pseudo-Hermitian sectional curvature bounded from above by $-K_2 < 0$. Then for any $(J^N, J)$-holomorphic map $g : N \to M$, (5.3) holds.

6. **Invariant pseudo-distance on CR manifolds**

In this section, we will give an invariant pseudo-distance on pseudo-Hermitian manifolds, which are analogous to the Carathéodory pseudo-distance on complex manifolds (cf. [3]). Note that this notion holds true for more general CR manifolds.

Let $(M^{2m+1}, HM, J, \theta)$ be a pseudo-Hermitian manifold. Let $D$ denote the unit disk in the complex plane and $\rho$ denote the Bergman distance of $D$. Analogous to the Carathéodory pseudo-distance on complex manifolds, we may also define CR Carathéodory pseudo-distance on $M$ by

$$c_M(p, q) = \sup_f \rho(f(p), f(q)), \quad (6.1)$$

where the supremum is taken for all possible CR functions $f : M \to D$. Note that $f : M \to D$ is a CR function if and only if it satisfies $df \circ J = J^D \circ df$ on $HM$. It is easy to verify the following axioms for the pseudo-distance:

$$c_M(p, q) \geq 0, \quad c_M(p, q) = c_M(q, p), \quad c_M(p, r) + c_M(r, q) \geq c_M(p, q). \quad (6.2)$$

The most important property of $c_M$ is given as follows, whose proof is trivial.
Proposition 6.1. Let $M, \tilde{M}$ be two pseudo-Hermitian manifolds and let $f : M \to \tilde{M}$ be a CR map. Then
\[
c\tilde{M}(f(p), f(q)) \leq c_M(p, q).
\]

Corollary 6.2. Let $f : M \to \tilde{M}$ be a CR isomorphism, then
\[
c\tilde{M}(f(p), f(q)) = c_M(p, q).
\]

In order to apply the Schwarz lemma in Section 4, we may define the basic pseudo-distance by
\[
c'_M(p, q) = \sup_f \rho(f(p), f(q)), \tag{6.3}
\]
where the supremum is taken for all $(J, J^D)$-holomorphic maps $f : M \to D$. It’s clear that $c'_M \leq c_M$.

Example 1. Let $D$ be the unit disc in $\mathbb{C}$. Set $\theta = dt + i(\bar{\theta} - \partial) \log (1 - |z|^2)^{-1}$, $\xi = \frac{\partial}{\partial t}$ and $H = \ker \theta$. We define an almost complex structure $J$ on $H$ to be the horizontal lift of the complex structure $J^D$ on $D$. Then $D^3(-1) = (D \times \mathbb{R}, H, J, \theta)$ is the three-dimensional Sasakian space form with pseudo-Hermitian sectional curvature $-1$. (cf. [2, 9]). Choose $T = \frac{\partial}{\partial z} + i \frac{\bar{z}}{1 - |z|^2} \frac{\partial}{\partial t}$ as the frame field of $T_{1,0}D^3(-1)$. It is easy to see that $f = f(z,t) : D^3(-1) \to D$ is a basic CR function if and only if $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} = 0$. Thus, for any $(z_1, t_1), (z_2, t_2) \in D^3(-1)$, $c'_{D^3(-1)}((z_1, t_1), (z_2, t_2)) = c'_{D^3(-1)}((z_1, 0), (z_2, 0)) = c_D(z_1, z_2)$, where $c_D$ is the Carathéodory distance of unit disk $D$. For $c_{D^3(-1)}$, since $f_1(z,t) = z$ and $f_2(z,t) = \frac{t - i(\log(1 - |z|^2) + 1)}{t - i(\log(1 - |z|^2) - 1)}$ are CR functions from $D^3(-1)$ to $D$, $c_{D^3(-1)}((z_1, t_1), (z_2, t_2)) \geq \rho(z_1, z_2) > 0$ for any $z_1 \neq z_2$ in $D$, and $c_{D^3(-1)}((z, t_1), (z, t_2)) \geq \rho(f_2(z, t_1), f_2(z, t_2)) > 0$ for $z \in D, t_1 \neq t_2$. Therefore, $c_{D^3(-1)}((z_1, t_1), (z_2, t_2)) = 0$ if and only if $(z_1, t_1) = (z_2, t_2)$. Consequently, for two distinct points $p$ and $q$ in $D^3(-1)$, there is a bounded CR function $f$ on $D^3(-1)$ such that $f(p) \neq f(q)$.

Using Theorem 4.1, we have

Theorem 6.3. Let $(M, H, M, J, \theta)$ be a complete pseudo-Hermitian manifold with pseudo-Hermitian Ricci curvature bounded from below by a constant $-K \leq 0$ and $\|A\|_{C^1}$ bounded from above. Then
\[
c'_M(p, q) \leq \sqrt{K} d_{cc}^M(p, q) < \infty
\]
for $p, q \in M$. In particular, if $K = 0, c'_M \equiv 0$.

Proof. Let $ds_D^2$ denote the Poincaré metric on $D$, and the curvature of $(D, ds_D^2)$ is $-1$. For any $(J, J^D)$-holomorphic map $f : M \to D$, by Theorem 4.1, we have
\[
f^* ds_D^2 \leq Kg_{\theta}.
\]
Assume that $p, q$ are two points in $M$ and $\tau : [0, 1] \to M$ is a horizontal Lipschitz curve between them. Then,
\[
\rho(f(p), f(q)) \leq \int_0^1 \sqrt{ds_D^2(f_*(\tau'), f_*(\tau'))} \, dt \leq \sqrt{K} \int_0^1 \sqrt{L_\theta(\tau', \tau')} \, dt, \tag{6.5}
\]
where the second inequality follows from (6.4). Taking the supremum with respect to $f$ and infimum with respect to $\tau$, the theorem follows. \qed
By the Definitions of the Carathéodory and CR Carathéodory pseudo-distances, it is easy to see the relationships between them.

**Proposition 6.4.** Let $(M^{2m+1}, HM, J, \theta)$ be a pseudo-Hermitian manifold and $(N, J^N)$ a Hermitian manifold.

(i) For any $(J^N)$-holomorphic map $f : M \to N$, we have
\[
e^N(f(p), f(q)) \leq c'_M(p, q) \leq c_M(p, q)
\]
for any $p, q \in M$, where $e^N$ is the Carathéodory pseudo-distance on $N$.

(ii) For any $(J^N, J)$-holomorphic map $g : N \to M$, we have
\[
c'_M(g(x), g(y)) \leq e^N(x, y)
\]
for any $x, y \in N$.

Combining Theorem 6.3 and Proposition 6.4(i), we derive another Liouville theorem for $(J, J^N)$-holomorphic maps.

**Theorem 6.5.** Let $(M, HM, J, \theta)$ be a complete pseudo-Hermitian manifold with non-negative pseudo-Hermitian Ricci curvature and $\|A\|_{C^1}$ bounded from above. Let $(N, J^N)$ be a complex manifold whose Carathéodory pseudo-distance is a distance. Then any $(J, J^N)$-holomorphic map $f : M \to N$ is constant.

**Remark 5.** We can also deduce Corollary 4.2 from Theorem 6.5, since the Carathéodory pseudo-distance of unit disc $D$ is a distance.

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