The Lindelöf Hypothesis for almost all Hurwitz’s Zeta-Functions holds true

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March 23, 2010

Abstract
By probability theory we prove here that the Lindelöf hypothesis holds for almost all Hurwitz’s zeta-functions, i.e.
\[ \zeta \left( \frac{1}{2}+it, \omega \right) = o_{\omega, \epsilon} \{(\log t)^{\frac{3}{2}}+\epsilon\} \]
for almost everywhere \( 0 < \omega < 1 \), and for any small \( \epsilon > 0 \), where \( o_{\omega, \epsilon} \) denotes the Landau small \( o \)-symbol which depends on \( \omega \) and \( \epsilon \) and \( \zeta(s, \omega) \) denotes the Hurwitz zeta-function. The details will be given elsewhere.

Key words; The Riemann zeta function, the Hurwitz zeta function, the Lindelöf hypothesis, law of large numbers, law of the iterated logarithm.

Mathematics Subject Classification;
11M06, 11M26, 11M35, 60F15.

Let \( \zeta(s, \omega) \) be the Hurwitz zeta function which is meromorphically extended to the whole complex plane from the Dirichlet series
\[ \sum_{n=0}^{\infty} (n+\omega)^{-s} \quad (s = \sigma + it, \sigma = \Re s > 1, \ 0 < \omega \leq 1). \]

We should note that
\[ \zeta(s,1) = \zeta(s), \]
\[ \zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s), \]

where \( \zeta(s) \) denotes the Riemann zeta function.

In analytic number theory, there are three famous conjectures which are related each other as follows.

**The Riemann Hypothesis** (1859, by B.Riemann):
\[ \rho \notin \mathbb{R}, \quad \zeta(\rho) = 0 \Rightarrow \Re \rho = \frac{1}{2} \]

**The Lindelöf Hypothesis** (1908, by E.Lindelöf):
\[ \zeta\left(\frac{1}{2} + it\right) = O_\varepsilon(t^\varepsilon) \text{ for any small } \varepsilon > 0, \]
where \( O_\varepsilon \) denotes the Bachmann-Landau large O-symbol which depends on \( \varepsilon \).

**The Density Hypothesis**:
\[ N(\sigma, T) = O_\varepsilon(T^{2 - 2\sigma + \varepsilon}) \]
\[ \text{for any small } \varepsilon > 0 \text{ and } \frac{1}{2} \leq \sigma \leq 1, \]

where \( N(\sigma, T) \) denotes the number of zeros of \( \zeta(s) \) in the rectangle whose four vertices are \( \sigma, 1, 1 + iT \) and \( \sigma + iT \).

It is well known that
- the Riemann Hypothesis \( \Rightarrow \) the Lindelöf Hypothesis
- \( \Rightarrow \) the Density Hypothesis.

(\text{It is not known whether the Lindelöf Hypothesis implies the Riemann Hypothesis or not.})

And also as it is well known, the Riemann Hypothesis is the most important and the strongest conjecture that has serious influences on many branches of mathematics including number theory. But it is less known that in fact the Lindelöf Hypothesis has almost the same effects on number theory as the Riemann Hypothesis \[ \text{[1] [9] [10]}. \]

About the Lindelöf Hypothesis there are many studies which improve the power \( L \) of \( t \) in \( \zeta\left(\frac{1}{2} + it\right) = O(t^L) \). These studies in this direction have their long history and story. The recent results in this direction are due to G.Kolesnik, E.Bombieri, H.Iwaniec, M.N.Huxley, N.Watt and others, for example, \( \zeta\left(\frac{1}{2} + it\right) = O(t^{9/56}) \) due to Bombieri and Iwaniec in 1986 and the best up to the present time is \( = O(t^{32/205}) \) due to Huxley in 2005.

In 1952, Koksma and Lekkerkerker \[ \text{[7]} \] proved that
\[ \int_0^1 |\zeta_1\left(\frac{1}{2} + it, \omega\right)|^2d\omega = O(\log t) \]
where \( \zeta_1(s, \omega) := \zeta(s, \omega) - \omega^{-s} \) whose term \(-\omega^{-s}\) makes keeping out the singularity at \( \omega = 0 \).

From this mean value results, by using Čebyšev’s inequality in probability theory, we easily have

\[
\mu\{0 < \omega \leq 1; |\zeta(\frac{1}{2} + it, \omega)| \geq C\sqrt{\log t}\} \leq \frac{O(1)}{C^2}
\]

for any \( t > 1 \) and any large \( C > 0 \),

where \( \mu\{B\} \) denotes the Lebesgue measure of measurable set \( B \), which shows that the Lindelöf Hypothesis holds in the sense of weak law in probability theory.

In this short note we give the following strong law version of the Lindelöf Hypothesis, that is,

**Theorem 1**

\[
\zeta(\frac{1}{2} + it, \omega) = o_{\omega, t}\{(\log t)^{\frac{3}{2} + \epsilon}\}
\]

for almost everywhere \( \omega \in \Omega := (0, 1) \) and for any small \( \epsilon > 0 \).

In order to prove this theorem, we need some definitions and some results in probability theory.

Let \((\Omega, \mathcal{F}, P)\) be some probability space, \( X, Y, Z, \cdots \) be complex valued random variables on this space, \( E[X] \) be the expectation value of the random variable \( X \) and \( V[X] = E[|X - E[X]|^2] \) be the variance of \( X \).

**Lemma 1** Let \( Z \) be a complex valued random variable. If \( E[|Z|^2] < +\infty \), then we have

\[
|Z| < +\infty \text{ almost surely (abbreviated by a.s.),}
\]

i.e. \( P\{|Z| < +\infty\} = 1 \).

**proof.** From \( |Z| \geq 0 \), we have

\[
0 \leq |Z| = |Z(\omega)| < +\infty \text{ or } |Z| = |Z(\omega)| = +\infty.
\]
We define the set $A \subset \Omega$ by $A := \{\omega; |Z(\omega)| = +\infty\}$ and the indicator function of the set $A$;

$$1_A := 1_A(\omega) := \begin{cases} 1 & (\omega \in A) \\ 0 & (\omega \notin A). \end{cases}$$

If we assumed that $P\{\omega \in \Omega; |Z(\omega)| < +\infty\} < 1$, we would have $P\{A\} > 0$ and

$$E[|Z|^2] \geq E[|Z|^21_A] = (+\infty)P\{A\} = +\infty,$$

which is the contradiction to the assumption $E[|Z|^2] < +\infty$. So we have the lemma.

**Lemma 2** Let $Z_n$ be a complex valued random variables $(n = 1, 2, 3, \cdots)$. If $\sum_{n=1}^{\infty} E[|Z_n|^2] < +\infty$, then we have

$$Z_n \to 0 \text{ a.s. (as } n \to +\infty).$$

**proof.** By Lemma 1, we have

$$P\{\sum_{n=1}^{\infty} |Z_n|^2 < \infty\} = 1,$$

which shows that $|Z_n|^2 \to 0 \text{ a.s. (as } n \to +\infty)$, that is, $Z_n \to 0 \text{ a.s. (as } n \to +\infty)$.

**Lemma 3** (Rademacher-Menchoff’s lemma [5] [8])

Let $a(p)$ $(p = 1, 2, \cdots, 2^{n+1} - 1)$ be complex numbers and $a(0) := 0$, then we have

$$\max_{1 \leq p < 2^{n+1}} |a(p)|^2 \leq (n + 1) \sum_{k=0}^{n} \sum_{j=0}^{2^{n-k}-1} |a(2^k + j2^{k+1}) - a(j2^{k+1})|^2.$$

**proof.** For the natural number $p$ which satisfy $1 \leq p < 2^{n+1}$, we have its binomial expansion;

$$p = \sum_{j=0}^{n} \epsilon_j 2^j \ (\epsilon_j = 0 \text{ or } 1).$$
With respect to the above $p$, we define $p_{k+1}$, $p_{n+1}$, $p_0$ respectively by

$$p_{k+1} := \sum_{j=k+1}^{n} \epsilon_j 2^j \ (k = 0, 1, 2, \cdots, n-1),$$

$$p_{n+1} := 0, \ p_0 := p.$$

From these definitions we have

$$p_0 = p \geq p_1 \geq p_2 \geq \cdots \geq p_n \geq p_{n+1} = 0, \quad (1)$$

$$p_k - p_{k+1} = \epsilon_k 2^k,$$

$$p_{k+1} = \sum_{j=k+1}^{n} \epsilon_j 2^j = \sum_{j=0}^{n-k-1} \epsilon_{k+1+j} 2^{j+k+1}$$

$$= \sum_{j=0}^{n-k-1} (\epsilon_{k+1+j} 2^{j}) 2^{k+1} =: \delta_{k+1} 2^{k+1}, \quad (2)$$

$$0 \leq \delta_{k+1} \leq \sum_{i=0}^{n-k-1} 2^i = 2^{n-k} - 1. \quad (3)$$

From

$$a(p) = a(p_0) = a(p_0) - a(p_{n+1}) = \sum_{k=0}^{n} (a(p_k) - a(p_{k+1})),$$

we have

$$|a(p)|^2 = \left| \sum_{k=0}^{n} 1 \cdot (a(p_k) - a(p_{k+1})) \right|^2$$

$$\leq \sum_{k=0}^{n} 1 \sum_{k=0}^{n} |a(p_k) - a(p_{k+1})|^2$$

$$= (n + 1) \sum_{k=0}^{n} |a(p_k) - a(p_{k+1})|^2$$

$$= (n + 1) \sum_{k=0}^{n} |a(\epsilon_k 2^k + p_{k+1}) - a(p_{k+1})|^2$$

(by (1))

$$= (n + 1) \sum_{k=0}^{n} |a(\epsilon_k 2^k + \delta_{k+1} 2^{k+1}) - a(\delta_{k+1} 2^{k+1})|^2$$

(by (2))

$$\leq (n + 1) \sum_{k=0}^{n} \sum_{j=0}^{2^{n-k-1}} |a(2^k + j2^{k+1}) - a(j2^{k+1})|^2, \quad (4)$$
because we take the summation with respect to $k$ into account only when $\epsilon_k = 1$, and we sum up $j$ in place of $\delta_k$ by (3). By the fact that the right hand side of (4) is independent of $p$, we have the lemma.

**Definition 1** Let $X, Y$ be complex valued random variables which satisfy $E[|X|^2], E[|Y|^2] < \infty$. If $E[\bar{X}Y] = E[X]E[Y]$, we call $X, Y$ (pairwise) uncorrelated.

**Definition 2** Let $X, Y$ be complex valued random variables which satisfy $E[|X|^2], E[|Y|^2] < \infty$. If $E[\bar{XY}] = 0$, we call $X, Y$ (pairwise) orthogonal.

**Lemma 4** (Rademacher-Menchoff \[5\] \[8\]) Let $X_1, X_2, \cdots$ be pairwise uncorrelated complex valued random variables which satisfy $E[X_i] = 0$, $\sigma^2_i := V[X_i] (i = 1, 2, \cdots)$, $\sigma_i \geq 0$ and let $S_n := S_n(\omega) := X_1 + X_2 + \cdots + X_n$. Then we have

$$E[\max_{2^m < k \leq 2^{m+1}} |S_k - S_{2^m}|^2] \leq (m^2 + 1) \sum_{i=1}^{2^m} \sigma^2_{2^{m+i}} \text{ for } m = 0, 1, 2, \cdots.$$  

**proof.** In Lemma 3, we put

$$n = m - 1, \quad a(p) = X_{2^m+1} + X_{2^m+2} + \cdots + X_{2^m+p}.$$  

From this, we have

$$E[\max_{2^m < k \leq 2^{m+1}} |S_k - S_{2^m}|^2] \leq E[\max_{1 \leq k < 2^m} |S_{2^m+k} - S_{2^m}|^2] + E[|S_{2^{m+1}} - S_{2^m}|^2]$$

$$= E[\max_{1 \leq p < 2^m} |X_{2^m+1} + X_{2^m+2} + \cdots + X_{2^m+p}|^2]$$

$$+ E[|X_{2^m+1} + X_{2^m+2} + \cdots + X_{2^{m+1}}|^2]$$

$$\leq E[m \sum_{k=0}^{m-1} \sum_{j=0}^{2^{m-1}-k-1} |(X_{2^m+1} + X_{2^m+2} + \cdots + X_{2^m+2^k+j2^{k+1}})|$$

$$\leq \sum_{k=0}^{m-1} \sum_{j=0}^{2^{m-1}-k-1} |(X_{2^m+1} + X_{2^m+2} + \cdots + X_{2^m+2^k+j2^{k+1}})|$$
\[-(X_{2m+1} + X_{2m+2} + \cdots + X_{2m+j2^{k+1}})^2\]
\[+ E[(X_{2m+1} + X_{2m+2} + \cdots + X_{2m+1})^2] \]
(by Lemma 3)

\[= m \sum_{k=0}^{m-1} \sum_{j=0}^{2m^{1-k}-1} E[(X_{2m+j2^{k+1}+1} + X_{2m+j2^{k+1}+2} + \cdots + X_{2m+j2^{k+1}+2^k})^2]
+ E[(X_{2m+1} + X_{2m+2} + \cdots + X_{2m+1})^2] \]

\[= m \sum_{k=0}^{m-1} \sum_{j=0}^{2m^{1-k}-1} V[X_{2m+j2^{k+1}+1} + X_{2m+j2^{k+1}+2} + \cdots + X_{2m+j2^{k+1}+2^k}]
+ V[(X_{2m+1} + X_{2m+2} + \cdots + X_{2m+1})^2] \]

\[= m \sum_{k=0}^{m-1} \sum_{j=0}^{2m^{1-k}-1} \sum_{i=1}^{2^k} \sigma_{2^m+j2^{k+1}+i}^2 + \sum_{i=1}^{2^m} \sigma_{2^m+i}^2 \]
\[\leq m \sum_{k=0}^{m-1} \sum_{i=1}^{2^m} \sigma_{2^m+i}^2 + \sum_{i=1}^{2^m} \sigma_{2^m+i}^2 \]
\[\leq m \cdot m \sum_{i=1}^{2^m} \sigma_{2^m+i}^2 + \sum_{i=1}^{2^m} \sigma_{2^m+i}^2 \]
\[= (m^2 + 1) \sum_{i=1}^{2^m} \sigma_{2^m+i}^2, \]

which completes the proof of the lemma.

By using these lemmas, we have

**Theorem 2** ([12])

Let $X_1^{(n)}, X_2^{(n)}, \cdots, X_k^{(n)}, \cdots$ be pairwise uncorrelated complex valued random variables which may depend on $n$ and satisfy

\[\mathbb{E}[X_k^{(n)}] = 0, \quad \sigma_k^2 := \mathbb{V}[X_k^{(n)}] = O(k^{-2\alpha}), \quad |X_k^{(n)}| < +\infty\]

\[(k, n = 1, 2, \cdots, \alpha \in \mathbb{R}, \forall \omega \in \Omega)\]

, where $\sigma_k \geq 0$ do not depend on $n$. Also let

\[S_n^{(l)} := S_n^{(l)}(\omega) := X_1^{(l)} + X_2^{(l)} + \cdots + X_n^{(l)},\]
\[ \varphi(n) := n^{\beta} (\log n)^{1/2 + \epsilon} \text{ with any small } \epsilon > 0 \]

\[ \beta := \begin{cases} 0 & (\alpha \geq \frac{1}{2}) \\ \frac{1}{2} - \alpha & (\alpha < \frac{1}{2}) \end{cases} \]

Then we have

\[ S_m^{(n)} = S_m^{(n)}(\omega) = o_{\omega,\epsilon}(\varphi(n)) \text{ a.s. } \omega \in \Omega. \]

**proof.** We choose any natural number sequence \( \{n_k\}_{k=1}^{\infty} \) with \( 2^k < n_k \leq 2^{k+1} \) and \( X_1^{(l)}, X_2^{(l)}, \ldots, X_{2^m+1}^{(l)}, \ldots \; (l, m \in \mathbb{N}) \) are pairwise uncorrelated complex valued random variables for any \( l \in \mathbb{N} \). We have

\[ \mathbb{E}\left[ \left| \sum_{k=1}^{\infty} \varphi(2^k) \right|^2 \right] = \sum_{k=1}^{\infty} 2^{-2k\beta} (\log 2^k)^{-3-2\epsilon} (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_{2^k}^2). \quad (5) \]

In case of \( \alpha > \frac{1}{2} \), then \( \beta = 0 \) and we have

\[ (5) = O(\sum_{k=1}^{\infty} \frac{1}{(\log 2^k)^{-3-2\epsilon} \sum_{l=1}^{2^k} 2^{-2\alpha}}) \]

\[ = O(\sum_{k=1}^{\infty} k^{-3-2\epsilon}) < +\infty. \]

In case of \( \alpha = \frac{1}{2} \), then \( \beta = 0 \) and we have

\[ (5) = O(\sum_{k=1}^{\infty} (\log 2^k)^{-3-2\epsilon} \sum_{l=1}^{2^k} l^{-1}) \]

\[ = O(\sum_{k=1}^{\infty} k^{-3-2\epsilon} \log 2^k) \]

\[ = O(\sum_{k=1}^{\infty} k^{-2-2\epsilon}) < +\infty. \]

In case of \( \alpha < \frac{1}{2} \), then \( \beta = \frac{1}{2} - \alpha \) and we have

\[ (5) = O(\sum_{k=1}^{\infty} 2^{-k(1-2\alpha)}(\log 2^k)^{-3-2\epsilon} \sum_{l=1}^{2^k} l^{-2\alpha}) \]

8
\[
\sum_{k=1}^{\infty} 2^{-k(1-2\alpha)} k^{-3-2\epsilon} 2^{k(1-2\alpha)}
\]
\[
= O(\sum_{k=1}^{\infty} k^{-3-2\epsilon}) < +\infty.
\]

Then in any case, we have
\[
\mathbb{E}[\sum_{k=1}^{m} \left| S_{2k}(\omega) \varphi(2^k) \right|^2 + \sum_{k=m+1}^{\infty} \left| S_{2k}(\omega) \varphi(2^k) \right|^2] < +\infty,
\]
which means, by Lemma 2, with some \(A((n_1, \ldots, n_m)) \subset \Omega\),
\[
\sum_{k=1}^{m} \left| S_{2k}(n_k)(\omega) \varphi(2^k) \right|^2 + \sum_{k=m+1}^{\infty} \left| S_{2k}(\omega) \varphi(2^k) \right|^2 < +\infty
\]
for \(\forall \omega \in A((n_1, \ldots, n_m))\) with \(P\{A((n_1, \ldots, n_m))\} = 1\). We put
\[
A(m) := \bigcap_{(n_1, \ldots, n_m)} A((n_1, \ldots, n_m))
\]
where \((n_1, \ldots, n_m)\) under \(\cap\) runs through all \((n_1, \ldots, n_m) \in \mathbb{N}^m\) with \(2^k < n_k \leq 2^{k+1}(k = 1, 2, \ldots, m)\).

Since
\[
\bigcap_{(n_1, \ldots, n_m)}
\]
is finitely many intersections of the sets, we have
\[
P\{A(m)\} = 1.
\]

Therefore we have
\[
\sum_{k=1}^{m} \left| S_{2k}(n_k)(\omega) \varphi(2^k) \right|^2 + \sum_{k=m+1}^{\infty} \left| S_{2k}(\omega) \varphi(2^k) \right|^2 < +\infty
\]
for \(\forall \omega \in A(m)\) with \(P\{A(m)\} = 1\).

We show that
\[
A(m) = A(m + 1) (m = 1, 2, \ldots).
\]

In fact, if \(\omega \in A(m)\) which means
\[
\sum_{k=1}^{m} \left| S_{2k}(n_k)(\omega) \varphi(2^k) \right|^2 + \sum_{k=m+1}^{\infty} \left| S_{2k}(\omega) \varphi(2^k) \right|^2 < +\infty
\]
, then we immediately have
\[
\sum_{k=1}^{m} \left| \frac{S_{2k}^{(n_k)}(\omega)}{\varphi(2^k)} \right|^2 + \left| \frac{S_{2m+1}^{(n_{m+1})}(\omega)}{\varphi(2m+1)} \right|^2 + \sum_{k=m+2}^{\infty} \left| \frac{S_{2k}^{(2k+1)}(\omega)}{\varphi(2k)} \right|^2 < +\infty
\]
for \(2^{m+1} < \forall n_{m+1} \leq 2^{m+2}\), because \(|X_k^{(l)}(\omega)| < +\infty\) for \(\forall \omega \in \Omega\). This means \(\omega \in A(m + 1)\). Inversely \(\omega \in A(m + 1)\) implies \(\omega \in A(m)\) by the same argument.
So, There exists
\[
\lim_{m \to +\infty} A(m) =: A = A(1) \quad \text{and} \quad P\{A\} = 1.
\]
This means
\[
\sum_{k=1}^{\infty} \left| \frac{S_{2k}^{(n_k)}(\omega)}{\varphi(2^k)} \right|^2 < +\infty \quad \text{for} \forall \{n_k\}_{k=1}^{\infty}, \forall \omega \in A \quad \text{with} \quad P\{A\} = 1
\]
and
\[
\lim_{k \to +\infty} \frac{S_{2k}^{(n_k)}(\omega)}{\varphi(2^k)} = 0 \quad \text{a.s.} \quad \omega \quad \text{for} \forall \{n_k\}_{k=1}^{\infty}
\]
Next we put
\[
Y_{2k}^{(n_k)} := \max_{1 \leq l \leq 2k} |X_{2^{k+1}}^{(n_k)} + X_{2^{k+2}}^{(n_k)} + \cdots + X_{2^{k+l}}^{(n_k)}|
\]
By Lemma 4, we have for any \(l \in \mathbb{N}\)
\[
E[|Y_{2k}^{(l)}|^2] \leq (k^2 + 1) \sum_{i=1}^{2k} \sigma_{2k+i}^2
\]
\[
= \begin{cases} 
O(k^2 + 1) & (\alpha \geq \frac{1}{2}) \\
O(2(k+1)(1-2\alpha)(k^2 + 1)) & (\alpha < \frac{1}{2}) 
\end{cases}
\]
and
\[
E[\sum_{k=1}^{m} \left| \frac{Y_{2k}^{(n_k)}}{\varphi(2^k)} \right|^2 + \sum_{k=m+1}^{\infty} \left| \frac{Y_{2k}^{(2k+1)}}{\varphi(2k)} \right|^2] = O(\sum_{k=1}^{\infty} 2^{-2k} \beta k^{-3-2\epsilon} E[|Y_{2k}^{(l)}|^2]) \quad \text{with} \forall l \in \mathbb{N} \quad (7)
\]
In case of \(\alpha \geq \frac{1}{2}\), then \(\beta = 0\) and we have
\[
(7) = O(\sum_{k=1}^{\infty} k^{-3-2\epsilon}(k^2 + 1)) = O\left(\sum_{k=1}^{\infty} k^{-1-2\epsilon}\right) < +\infty.
\]
In case of $\alpha < \frac{1}{2}$, then $\beta = \frac{1}{2} - \alpha$ and we have

$$ (7) = O(\sum_{k=1}^{\infty} 2^{-k(1-2\alpha)} k^{-3-2\epsilon} \cdot (k^2 + 1)2^{(k+1)(1-2\alpha)}) $$

$$ = O(2^{-2\alpha} \sum_{k=1}^{\infty} k^{-1-2\epsilon}) < +\infty. $$

In any case, we have

$$ E\left[\sum_{k=1}^{m} \frac{|Y_{k}(n)|^2}{\varphi(2k)} + \sum_{k=m+1}^{\infty} \frac{|Y_{k}(2^{k+1})}{\varphi(2k)}|^2\right] < +\infty. $$

The same argument as that of $\frac{S_{(nk)}^{(n)}}{\varphi(2^k)}$ leads

$$ \lim_{k \to \infty} \frac{Y_{k}(nk)(\omega)}{\varphi(2^k)} = 0 \text{ a.s. for } \forall\{n_k\}_{k=1}^{\infty}. \quad (8) $$

Then, for any $n$ with $2^m < n \leq 2^{m+1}$, we have, by (6) and (8),

$$ \frac{|S_{n}^{(n)}|}{\varphi(n)} \leq \frac{|S_{2^m}^{(n)}|}{\varphi(2^m)} \leq \frac{|S_{2^m}^{(n)}| + Y_{m}(n)}{\varphi(2^m)} $$

$$ = \frac{|S_{2^m}^{(n)}|}{\varphi(2^m)} + \frac{Y_{m}(n)}{\varphi(2^m)} \to 0 \text{ a.s. (as } n \to \infty), $$

which means

$$ S_{n}^{(n)} = S_{n}^{(n)}(\omega) = o_{\omega}(\varphi(n)) \text{ a.s. } \omega \in \Omega, \text{ with any small } \epsilon > 0. $$

This completes the proof.

Remark 1: This theorem is a generalization of the strong limit theorem the position of which may be placed between laws of large numbers and laws of the iterated logarithm in probability theory. (Therefore we would like to call these types of theorems quasi laws of the iterated logarithm.)

Remark 2: This is also a new proof of the strong law of large numbers without using the Borel-Cantelli theorem. We can prove other limit theorems in probability theory by this method.
We yet need some lemmas for proving Theorem 1.

**Lemma 5** (Functional equation for the Hurwitz zeta function [1][6][16])

\[ \zeta(s, \omega) = \frac{\Gamma(s)}{(2\pi)^s} \{ e^{-\frac{\pi}{2}is} F(\omega, s) + e^{\frac{\pi}{2}is} F(-\omega, s) \} \]

for \( 0 < \omega < 1, \sigma > 0 \) or \( 0 < \omega \leq 1, \sigma > 1 \),

where \( \Gamma(s) \) is the gamma function of Euler and \( F(\omega, s) := \sum_{k=1}^{\infty} k^{-s} e^{2\pi i k \omega} \).

**Lemma 6** [6][9][15][16]

\[ |\Gamma(s)| = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \{ 1 + O_{\sigma_1, \sigma_2, \delta}(\frac{1}{|t|}) \} \]

for \( \sigma_1 \leq \sigma \leq \sigma_2, |t| \geq \delta > 0 \).

**Lemma 7**

\[ F(\omega, s) = \sum_{k \leq t^2} k^{-s} e^{2\pi i k \omega} + O(\frac{1}{|1 - e^{2\pi i \omega}|^{t^{1-2\sigma}}} t^{1-2\sigma}) \]

for \( \sigma \geq \frac{1}{2} \).

**proof.**

\[ F(\omega, s) = \sum_{k \leq t^2} k^{-s} e^{2\pi i k \omega} + \sum_{k > t^2} k^{-s} e^{2\pi i k \omega}. \]

By applying the partial summation to the second term of the above \( F(\omega, s) \),

\[ \sum_{k > t^2} k^{-s} e^{2\pi i k \omega} \]

\[ = -A(t^2)(t^2)^{-s} + s \int_{t^2}^{\infty} A(u) u^{-s-1} du \]

\[ = O(\frac{1}{|1 - e^{2\pi i \omega}|^{t^{-2\sigma}}} t^{-2\sigma}) + O(t \frac{1}{|1 - e^{2\pi i \omega}|^{t^{-2\sigma}}}) \]

\[ = O(\frac{t^{1-2\sigma}}{|1 - e^{2\pi i \omega}|}), \]

where \( A(t^2) := \sum_{k \leq t^2} e^{2\pi i k \omega} \), which completes the proof of the lemma.

From Lemma 5,6 we easily have
Lemma 8

\[ |\zeta(\frac{1}{2} + it, \omega)| = O(F(\omega, \frac{1}{2} + it)) \text{ for } 0 < \omega < 1. \]

Proof of Theorem 1. From Lemma 7, we have

\[ F(\omega, \frac{1}{2} + it) = O_{\delta}(\sum_{k \leq t^2} k^{-\frac{1}{2} - it} e^{2\pi k\omega}) \]

for \(0 < \delta < \omega < 1 - \delta\) with any small \(\delta > 0\).

In Theorem 2, put \(\Omega = (0, 1), P = \mu\) (Lebesgue measure), \(n = [t^2]\) ([x] denotes the integral part of real number \(x\).) and

\[ X_k^{(t)} = k^{-\frac{1}{2} - it} e^{2\pi k\omega} \quad (k = 1, 2, \cdots), \]

which satisfy all the conditions in Theorem 2. Then we have

\[ \sum_{k \leq t^2} k^{-\frac{1}{2} - it} e^{2\pi k\omega} = o_{\omega, \epsilon}(\varphi([t^2])) = o_{\omega, \epsilon}((\log t)^{\frac{3}{2} + \epsilon}) \]

With Lemma 7,8, this completes the proof of the theorem.

Remark 3 The exact expression of the Lindelöf Hypothesis is

\[ \mu_{\omega}(\sigma) = \begin{cases} 
0 & (\sigma \geq 1) \\
\frac{1}{2} - \sigma & (\sigma < \frac{1}{2}).
\end{cases} \]

where \(\mu_{\omega}(\sigma) := \limsup_{t \to +\infty} \frac{\log |\zeta(\sigma + it, \omega)|}{\log t}\),

which is the same form as

\[ \beta = \begin{cases} 
0 & (\alpha \geq 1) \\
\frac{1}{2} - \alpha & (\alpha < \frac{1}{2}),
\end{cases} \]

in Theorem 2.
Remark 4 In 1936, Davenport and Heilbronn \[4\] has already proved that the Riemann Hypothesis fails for \( \zeta(s, \omega) \) with transcendental number \( \omega \) and rational number \( \omega \neq \frac{1}{2}, 1 \) in contrast with our Theorem 1, which shows that the Lindelöf Hypothesis by itself, for example, without the Euler product, does not imply the Riemann Hypothesis.

Remark 5 It seems that the behaviour of \( \zeta(s, \omega) \) as \( \omega \) varies in the interval \((0, 1)\) is very complicated because of the following facts;

(1) Barasubramanian-Ramachandra \[2\] (the case \( \omega = 1 \)) and Ramachandra-Sankaranarayanan \[14\] proved the following \( \Omega \)-theorem;

\[
\zeta\left(\frac{1}{2} + it, \omega\right) = \Omega(\exp(C_\omega \sqrt{\frac{\log t}{\log \log t}}))
\]

with some \( C_\omega > 0 \) and \( \omega \in \mathbb{Q} \), which shows

\[
\{0 < \omega < 1; \text{Theorem 1, holds}\} \cap \mathbb{Q} = \emptyset.
\]

(2) It is well known that divisor problems and circle problems are closely related each other and so are shifted divisor problems and shifted circle problems. The Hurwitz zeta function naturally appears in shifted divisor problems \[10\]. And Bleher-Cheng-Dyson-Lebowitz \[3\] pointed out that the value distributions of the error terms of the number of lattice points inside shifted circles behave very differently when the shift varies by their numerical studies. Therefore it seems that the behaviour of \( \zeta(s, \omega) \) including its value distribution is very complicated as \( \omega \) varies. (For the value distribution of \( \zeta(s, \omega) \) with transcendental number \( \omega \), see \[11\].)

(3) Our numerical studies by ”Mathematica” show also the complexity of the behaviour of \( \zeta(s, \omega) \) as follows, for example,

The graph of \( \zeta(s, x) \) which plots the points \((x, y) \in \mathbb{R}^2\) such that

\[
y = \frac{|\zeta\left(\frac{1}{2} + it, x\right) - x^{-\left(\frac{1}{2} + it\right)}|}{(\log t)^2} (0 \leq x \leq 1, \ t = 10^8).\]

seems to be a kind of white noise.

Acknowledgment The author thanks to Prof. Jyoichi Kaneko of the University of the Ryukyus for his careful reading of the previous manuscript.
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