Approximate Pseudospin and Spin Solutions of the Dirac Equation for a Class of Exponential Potentials

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Abstract

Dirac equation is solved for some exponential potentials, hypergeometric-type potential, generalized Morse potential and Poschl-Teller potential with any spin-orbit quantum number κ in the case of spin and pseudospin symmetry, respectively. We have approximated for non s-waves the centrifugal term by an exponential form. The energy eigenvalue equations, and the corresponding wave functions are obtained by using the generalization of the Nikiforov-Uvarov method.

Keywords: Pseudospin symmetry, Spin symmetry, Dirac Equation, Hypergeometric Potential, generalized Morse Potential, Poschl-Teller Potential, Nikiforov-Uvarov Method

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I. INTRODUCTION

The solutions of the Dirac equation having the pseudospin, and spin symmetry have been strongly studied in the last years. The concept of the pseudospin symmetry [1, 2, 3] is an important theme in nuclear theory because of its features related to construct an effective shell-model coupling scheme [4], to study the structures of the deformed nuclei [5, 6]. The pseudospin symmetry occurs in nuclei when the magnitude of the scalar and vector potentials are nearly equal, but opposite sign, i.e., $V_v(r) \sim -V_s(r)$, where the scalar potential is negative (attractive) and the vector potential is positive (repulsive) in the relativistic region. Further, the spin symmetry appears when the magnitude of the scalar and vector potentials are nearly equal, i.e., $V_v(r) \sim V_s(r)$ [7-12]. The Dirac equation under the exact pseudospin and/or spin symmetry has been studied by with different type of potentials such as the Hulthén potential [13], the Morse potential [14], the Woods-Saxon potential [15, 16], and harmonic oscillator [17-21].

In the present work, we deal with the solutions of the Dirac equation if the exact pseudospin, and spin symmetry occur in the theory under the effect of a class of potentials, which have exponential form, i.e. the hypergeometric-type potential [22], the generalized Morse potential and the Pöschl-Teller potential [23].

In order to obtain the energy eigenvalue equation and the corresponding wave functions, we apply a new approximation scheme [24], which is the parametric generalization of the Nikiforov-Uvarov (NU) method [25], by using an approximation to the centrifugal-like term. So, we obtain the energy spectra of the above potentials for the spin-orbit quantum number $\kappa = 0$, or for any $\kappa$-value for the case of pseudospin and spin symmetry, respectively.

The organization of the present work is as follows. Firstly, we give briefly the equations for the Dirac spinors including the centrifugal term, and the spin-orbit quantum number $\kappa$. Than we present the basics of the parametric generalization of the NU method. We find the bound states and the corresponding wave functions of the Dirac equation with the above potentials in the case of pseudospin and spin symmetry, respectively. Finally, we give our conclusions.
II. DIRAC EQUATION

The Dirac equation for a particle with rest mass \( m \) in the absence of the scalar and vector potential is written (\( \hbar = c = 1 \)) [15]

\[
\{ \bar{\alpha} \cdot \vec{p} + \beta [m + V_s(r)] \} \Psi(r) = (E - V_v(r)) \Psi(r) ,
\]
where \( \vec{p} \) is the momentum operator, \( \bar{\alpha} \), and \( \beta \) are 4 \times 4 Dirac matrices, i.e. \( \bar{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma}^T & 0 \end{pmatrix} \), \( \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \), \( I \) is the 2 \times 2 unit matrix, and \( \vec{\sigma}_i (i = 1, 2, 3) \) are the Pauli matrices. The operator \( \hat{K} \) is the spin-orbit matrix operator and written in terms of the orbital angular momentum operator \( \hat{L} \) as \( \hat{K} = -\beta(\hat{\sigma} \cdot \hat{L} + 1) \), which commute with the Dirac Hamiltonian. The Dirac spinors can be labelled by the quantum number set \((n, \kappa)\), where \( \kappa \) is the eigenvalue of the spin-orbit operator, and written as

\[
\Psi_{n\kappa}(r) = \begin{pmatrix} f_{n\kappa} \\ g_{n\kappa} \end{pmatrix} ,
\]
where \( f_{n\kappa} = [F_{n\kappa}(r)/r]Y_j^\ell (\theta, \phi) \) is the upper, and \( g_{n\kappa} = [iG_{n\kappa}(r)/r]Y_j^\ell (\theta, \phi) \) is the lower component, and \( Y_j^\ell (\theta, \phi) \) , and \( Y_j^\ell (\theta, \phi) \) are the spherical harmonics, respectively. The total angular momentum, the orbital angular momentum, and pseudo-orbital angular momentum can be written in terms of the spin-orbit quantum number \( \kappa = \pm 1, \pm 2, \ldots \), such as \( j = |\kappa| - 1/2 \), \( \ell = |\kappa + 1/2| - 1/2 \), and \( \tilde{\ell} = |\kappa - 1/2| - 1/2 \), respectively.

Substituting Eq. (2) into Eq. (1), and eliminating \( F_{n\kappa}(r) \), we obtain two uncoupled differential equations for the lower, and upper components of the Dirac equation

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - \frac{1}{M_\Sigma(r)} \frac{d\Sigma(r)}{dr} \left( \frac{d}{dr} - \frac{\kappa}{r} \right) \right\} G_{n\kappa}(r) = M_\Delta(r) M_\Sigma(r) G_{n\kappa}(r) ,
\]
and

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} + \frac{1}{M_\Delta(r)} \frac{d\Delta(r)}{dr} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \right\} F_{n\kappa}(r) = M_\Delta(r) M_\Sigma(r) F_{n\kappa}(r) .
\]
where \( M_\Delta(r) = m + E - \Delta(r) \), \( \Delta(r) = V_v(r) - V_s(r) \), \( M_\Sigma(r) = m - E + \Sigma(r) \), and \( \Sigma(r) = V_v(r) + V_s(r) \).
III. PARAMETRIC GENERALIZATION OF THE METHOD

Let us give briefly the parametric generalization of the NU method. By using an appropriate coordinate transformation, the Schrödinger equation can be transformed into the following form

$$\sigma^2(z) \frac{d^2\Psi(z)}{dz^2} + \sigma(z) \tilde{\tau}(z) \frac{d\Psi(z)}{dz} + \tilde{\sigma}(z) \Psi(z) = 0,$$  \hspace{1cm} (5)

where \(\sigma(z)\), and \(\tilde{\sigma}(z)\) are polynomials, at most, second degree, and \(\tilde{\tau}(z)\) is a first-degree polynomial. By writing the general solution as \(\Psi(z) = \psi(z)\varphi(z)\), we obtain a hypergeometric type equation [24]

$$\frac{d^2\varphi(z)}{dz^2} + \frac{\tau(z)}{\sigma(z)} \frac{d\varphi(z)}{dz} + \frac{\lambda}{\sigma(z)} \varphi(z) = 0,$$  \hspace{1cm} (6)

where \(\psi(z)\) and \(\varphi_n(z)\) are defined as [24]

$$\frac{1}{\psi(z)} \frac{d\psi(z)}{dz} = \frac{\pi(z)}{\sigma(z)},$$  \hspace{1cm} (7)

$$\varphi_n(z) = \frac{a_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z) \rho(z)],$$  \hspace{1cm} (8)

where \(a_n\) is a normalization constant, and \(\rho(z)\) is the weight function satisfying the following equation [24]

$$\frac{d\sigma(z)}{dz} + \frac{\sigma(z)}{\rho(z)} \frac{d\rho(z)}{dz} = \tau(z).$$  \hspace{1cm} (9)

The function \(\pi(z)\), and the parameter \(\lambda\) in the above equation are defined as

$$\pi(z) = \frac{1}{2} [\sigma'(z) - \tilde{\tau}(z)] \pm \left[ \frac{1}{4} [\sigma'(z) - \tilde{\tau}(z)]^2 - \tilde{\sigma}(z) + k \sigma(z) \right]^{1/2},$$  \hspace{1cm} (10)

$$\lambda = k + \pi'(z).$$  \hspace{1cm} (11)

In the NU method, the square root in Eq. (10) must be the square of the polynomial, so the parameter \(k\) can be determined. Thus, a new eigenvalue equation becomes
\[ \lambda = \lambda_n = -n\tau'(z) - \frac{1}{2}(n^2 - n)\sigma''(z). \]  \hspace{1cm} (12)

where prime denotes the derivative and the derivative of the function \( \tau(z) = \tilde{\tau}(z) + 2\pi(z) \) should be negative.

Now, in order to clarify the parametric generalization of the NU method [25], let us take the following Schrödinger-like equation written for any potential

\[ z^2(1 - \alpha_3 z)^2 \frac{d^2\Psi(z)}{dz^2} + z(1 - \alpha_3 z)(\alpha_1 - \alpha_2 z) \frac{\Psi(z)}{dz} + [-\xi_1 z^2 + \xi_2 z - \xi_3] \Psi(z) = 0. \]  \hspace{1cm} (13)

Comparing Eq. (13) with Eq. (5), we obtain

\[ \tilde{\tau}(z) = \alpha_1 - \alpha_2 z \; ; \; \sigma(z) = z(1 - \alpha_3 z) \; ; \; \tilde{\sigma}(z) = -\xi_1 z^2 + \xi_2 z - \xi_3. \]  \hspace{1cm} (14)

Substituting these into Eq. (10), we obtain

\[ \pi(z) = \alpha_4 + \alpha_5 z \pm \left[ (\alpha_6 - k\alpha_3) z^2 + (\alpha_7 + k) z + \alpha_8 \right]^{1/2}, \]  \hspace{1cm} (15)

with the following parameters

\[ \begin{align*}
\alpha_4 &= \frac{1}{2} (1 - \alpha_1), \\
\alpha_5 &= \frac{1}{2} (\alpha_2 - 2\alpha_3), \\
\alpha_6 &= \alpha_5^2 + \xi_1, \\
\alpha_7 &= 2\alpha_4\alpha_5 - \xi_2, \\
\alpha_8 &= \alpha_4^2 + \xi_3.
\end{align*} \]  \hspace{1cm} (16)

We obtain the parameter \( k \) from the condition that the function under the square root should be the square of a polynomial

\[ k_{1,2} = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \]  \hspace{1cm} (17)

where \( \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6 \). The function \( \pi(z) \) becomes

\[ \pi(z) = \alpha_4 + \alpha_5 z - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) z - \sqrt{\alpha_8}]. \]  \hspace{1cm} (18)
for the $k$-value $k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9}$. We also have from $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$,

$$
\tau(z) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)z - 2[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})z - \sqrt{\alpha_8}].
$$

(19)

Thus, we impose the following condition to fix the $k$-value

$$
\tau'(z) = -(\alpha_2 - 2\alpha_5) - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) < 0.
$$

(20)

From Eqs. (11), (18) and (19) and by using $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ and equating Eq. (11) with the condition that $\lambda$ must satisfy given by Eq. (12), we obtain the energy eigenvalue equation for the potential under the consideration

$$
n[(n - 1)\alpha_3 + \alpha_2 - 2\alpha_5] - \alpha_5 + (2n + 1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0.
$$

(21)

By using Eq. (9)

$$
\rho(z) = z^{\alpha_{10} - 1}(1 - \alpha_3 z)^{\alpha_{11}} - \alpha_3 - \alpha_{10} - 1,
$$

(22)

and together with Eq. (8), we obtain

$$
\varphi_n(z) = P_n^{(\alpha_{10} - 1, \alpha_{11} - \alpha_{10} - 1)}(1 - 2\alpha_3 z),
$$

(23)

where $\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}$, $\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}$, and $P_n^{(\alpha, \beta)}(1 - 2\alpha_3 z)$ are Jacobi polynomials. By using Eq. (7), we obtain

$$
\psi(z) = z^{\alpha_{12}}(1 - \alpha_3 z)^{-\alpha_{12} - \alpha_{13} - \alpha_3},
$$

(24)

and the total wave function become
\[ \Psi(z) = z^{\alpha_{12}}(1 - \alpha_3 z)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} P_n^{(\alpha_{10} - 1, \frac{\alpha_{13}}{\alpha_3}, -\alpha_{10} - 1)} (1 - 2\alpha_3 z), \tag{25} \]

where \( \alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}). \)

In some problems the situation appears where \( \alpha_3 = 0. \) For this type of the problems, the solution given in Eq. (25) becomes as

\[ \Psi(z) = z^{\alpha_{12}} e^{\alpha_{13} z} L_n^{\alpha_{10} - 1}(\alpha_{11} z), \tag{26} \]

and the energy spectrum is

\[ \alpha_2 n - 2\alpha_5 n + (2n + 1)(\sqrt{\alpha_9} - \alpha_3 \sqrt{\alpha_8}) + n(n - 1)\alpha_3 + \alpha_7 \]
\[ + 2\alpha_3 \alpha_8 - 2\sqrt{\alpha_8 \alpha_9} + \alpha_5 = 0. \tag{27} \]

when the limits become \( \lim_{\alpha_3 \to 0} P_n^{(\alpha_{10} - 1, \frac{\alpha_{13}}{\alpha_3}, -\alpha_{10} - 1)} (1 - \alpha_3 z) = L_n^{\alpha_{10} - 1}(\alpha_{11} z) \) and \( \lim_{\alpha_3 \to 0}(1 - \alpha_3 z)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13} z}. \)

**IV. BOUND STATES**

**A. The Hypergeometric-Type Potential**

The Dirac equation has the exact pseudospin symmetry if \( \Sigma(r) = C = \text{const.}, \) so Eq. (3) becomes under that condition

\[ \left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - (m - E + C)M_\Delta(r) \right\} G_{\kappa}(r) = 0, \tag{28} \]

where \( \kappa = \tilde{\ell} + 1 \) for \( \kappa > 0, \) and \( \kappa = -\tilde{\ell} \) for \( \kappa < 0. \) The hypergeometric-type potential is given by

\[ V(r) = D[1 - \sigma \coth(\alpha r)]^2 = \left( \frac{D_1 + D_2 e^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2, \tag{29} \]

where the real parameters \( D, \sigma, \) and \( \alpha \) represent the potential [22], and \( D_1 = \sqrt{D}(1 - \sigma), \) and \( D_2 = \sqrt{D}(1 + \sigma). \)
Eq. (28) can not be solved analytically for any $\kappa$ values because of $\kappa(\kappa - 1)/r^2$ term, so we use the approximation $1/r^2 \simeq 4\alpha^2 e^{-2\alpha r}/(1 - e^{-2\alpha r})^2$ [26] to solve the equation for any spin-orbit quantum number $\kappa$.

By using this approximation to centrifugal-like term, setting $\Delta(r)$ to the potential given in Eq. (29), and inserting into Eq. (28), we obtain

$$\left\{ \frac{d^2}{dr^2} - 4\alpha^2 \kappa(\kappa - 1) \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} + \mu \frac{(D_1 + D_2 e^{-2\alpha r})^2}{(1 - e^{-2\alpha r})^2} - \epsilon \right\} G_{nk}(r) = 0,$$

(30)

where $\mu = m - E + C$, and $\epsilon = m(m + C) + E(C - E)$. At this point, it is worthwhile to note that Eq. (29) becomes for $\sigma = 1(D_1 = 0)$

$$V(r) = D_2 \left( \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2,$$

(31)

This form of the potential corresponds to the Manning-Rosen potential for $A = 0$ [27] if we set $\sqrt{\frac{1}{2\alpha r} \alpha(\alpha - 1)} \rightarrow D_2$ (here, $\kappa$ and $\alpha$ are the parameters in Ref. [27]), and $\frac{1}{\alpha} \rightarrow 2\alpha$. This means that we could also obtain the energy eigenvalue equation of the Dirac equation for the Manning-Rosen potential in the case of the exact spin symmetry, if we set the parameter $D_1 = 0$ in the equations.

By using the new variable $z = e^{-2\alpha r}(0 < z < 1)$, we obtain from Eq. (30)

$$\frac{d^2 G_{nk}(z)}{dz^2} + \frac{1 - z}{z(1 - z)} \frac{dG_{nk}(z)}{dz} + \frac{1}{[z(1 - z)]^2} \left\{ \beta^2 (\mu D_1^2 - \epsilon) + 2 \beta^2 [D_1 D_2 \mu - \epsilon - 2\alpha^2 \kappa(\kappa - 1)] z + \beta^2 (\mu D_2^2 - \epsilon) z^2 \right\} G_{nk}(z) = 0.$$

(32)

By comparing Eq. (32) with Eq. (13), we get the parameter set

$$\begin{align*}
\alpha_1 &= 1, \quad &\alpha_2 &= 1, \quad &\xi_1 &= -\beta^2 (\mu D_2^2 - \epsilon), \\
\alpha_3 &= 1, \quad &\alpha_4 &= 0, \quad &\xi_2 &= 2\beta^2 [D_1 D_2 \mu - \epsilon - 2\alpha^2 \kappa(\kappa - 1)], \\
\alpha_5 &= -\frac{1}{2}, \quad &\alpha_6 &= \xi_1 + \frac{1}{4}, \quad &\xi_3 &= -\beta^2 (\mu D_1^2 - \epsilon), \\
\alpha_7 &= -\xi_2, \quad &\alpha_8 &= \xi_3, \quad &\alpha_9 &= \xi_1 - \xi_2 + \xi_3 + \frac{1}{4}, \\
\alpha_{10} &= 1 + 2\sqrt{\xi_3}, \quad &\alpha_{11} &= 2 + 2\left( \sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4} + \sqrt{\xi_3}} \right), \quad &\alpha_{12} &= \sqrt{\xi_3}, \\
\alpha_{13} &= -\frac{1}{2} - \left( \sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4} + \sqrt{\xi_3}} \right).
\end{align*}$$

(33)
The energy eigenvalue equation becomes

\[
\left( \sqrt{\beta^2[4\alpha^2\kappa(\kappa - 1) - \mu(D_1 + D_2)^2] + \frac{1}{4}} + \beta\sqrt{\epsilon - \mu D_1^2} \right) \left( 2n + 1 + 2\beta\sqrt{\epsilon - \mu D_1^2} \right) + \beta^2 \left[4\alpha^2\kappa(\kappa - 1) - 2(\epsilon + \mu D_1 D_2)\right] = -n(n + 1) - 1/2. \quad (34)
\]

In this case, we use only the negative energy eigenvalues, because negative energy states exist in the pseudospin symmetry [28].

Now, let us give the corresponding Dirac spinors from Eq. (25)

\[
G_{n\kappa}(z) = z^\beta\sqrt{\epsilon - \mu D_1^2} \times (1 - z)^{\frac{\kappa}{2}} \sqrt{1 - \frac{\mu D_1 D_2}{\epsilon}} P_n^{(2\beta\sqrt{\epsilon \mu D_1^2}, 2\sqrt{\epsilon - \mu D_1^2} + \kappa(\kappa - 1) + \frac{1}{4})} (1 - 2z). \quad (35)
\]

Finally, we briefly give the energy eigenvalue equation for the special case \(\sigma = 1\), which gives the energy spectra of the Manning-Rosen potential with \(A = 0\)

\[
\left( \sqrt{4\beta^2[\alpha^2\kappa(\kappa - 1) - D\mu] + \frac{1}{4}} + \beta\sqrt{\epsilon} \right) \left( 2n + 1 + 2\beta\sqrt{\epsilon} \right) + \beta^2 \left[\alpha^2\kappa(\kappa - 1) - \frac{\epsilon}{2}\right] + \frac{1}{4} ((2n + 1)^2 + 1) = 0. \quad (36)
\]

Under the exact spin symmetry, i.e. \(\Delta(r) = C = const.\), Eq. (4) becomes

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - (m + E - C)\Sigma(r) \right\} F_{n\kappa}(r) = 0, \quad (37)
\]

where \(\kappa = -(\ell + 1)\) for \(\kappa > 0\) and \(\kappa = \ell\) for \(\kappa < 0\). By setting \(\Sigma(r)\) to the potential given in Eq. (30), using the above approximation to the \(\kappa(\kappa + 1)/r^2\) term, and using the new variable \(z = e^{-2\alpha r}(0 < z < 1)\), we obtain

\[
\frac{d^2 F_{n\kappa}(z)}{dz^2} + \frac{1 - z}{z(1 - z)} \frac{d F_{n\kappa}(z)}{dz} + \frac{1}{[z(1 - z)]^2} \left\{ \beta^2(\epsilon' - \mu' D_1^2) - \beta^2[2\epsilon' + 2D_1 D_2\mu' + 4\alpha^2\kappa(\kappa + 1)]z + \beta^2(\epsilon' - \mu' D_2^2)z^2 \right\} F_{n\kappa}(r) = 0. \quad (38)
\]
where $\mu' = m + E - C$, $\epsilon' = m(C - m) + E(E - C)$, and $\beta^2 = 1/4\alpha^2$. By comparing Eq. (39) with Eq. (13), we obtain the parameter set given in Eq. (34) where $\xi_1 = -\beta^2(\epsilon' - \mu'D_2^2)$, $\xi_2 = -2\beta^2[\epsilon' + D_1D_2\mu' + 2\alpha^2\kappa(\kappa + 1)]$, and $\xi_3 = -\beta^2(\epsilon' - \mu'D_1^2)$. The energy eigenvalue equation of the hypergeometric potential for the exact spin symmetry is written from Eq. (21) as

$$
\left(\sqrt{\beta^2 \mu'(D_1 + D_2)^2 + \kappa(\kappa + 1) + \frac{1}{4} + \beta \sqrt{\mu'D_1^2 - \epsilon'}}\right)\left(2n + 1 + 2\beta \sqrt{\mu'D_1^2 - \epsilon'}\right)
+ \beta^2 \left[4\alpha^2\kappa(\kappa + 1) + 2(\epsilon' + \mu'D_1D_2)\right] = -n(n + 1) - 1/2 ,
$$

(39)

The last equation can give negative, and positive eigenvalues, but we choose only positive energy eigenvalues, because in the case of the exact spin symmetry appears only the positive energy eigenstates [28]. The corresponding Dirac spinors are obtained from Eq. (25), and given

$$
F_{nn}(z) = z^{\beta \sqrt{\mu'D_1^2 - \epsilon'}}
\times (1 - z)^{\frac{1}{2} + \sqrt{\beta^2(D_1^2 + D_2^2) + \kappa(\kappa + 1) + \frac{1}{4}}} P_n^{(2\beta \sqrt{\mu'D_1^2 - \epsilon'}}, 2\sqrt{\beta^2(D_1^2 + D_2^2) + \kappa(\kappa + 1) + \frac{1}{4}}) (1 - 2z) .
$$

(40)

Now, we briefly give the energy eigenvalue equation for the Manning-Rosen potential with $A = 0$ ($\sigma = 1$)

$$
\left(\sqrt{4\beta^2D(m + E - C) + \kappa(\kappa + 1) + \frac{1}{4} + \beta \sqrt{m(m - C) + E(C - E)}}\right)
\times \left(2n + 1 + 2\beta \sqrt{m(m - C) + E(C - E)}\right)
+ \beta^2 \left[4\alpha^2\kappa(\kappa + 1) + 2(m(C - m) + E(E - C))\right] + \frac{1}{4} ((2n + 1)^2 + 1) = 0 .
$$

(41)

B. The Generalized Morse Potential

Assuming that the potential $\Delta(r) = V_v(r) + V_s(r)$ is the generalized Morse potential given by

$$
V(r) = V_1e^{-2\alpha r} - V_2e^{-\alpha r} ,
$$

(42)
and substituting Eq. (42) into Eq. (3), and taking \( \Sigma(r) = \Sigma = \text{const.} \), we have the following equation in the exact pseudospin symmetry for \( \kappa = 0 \) \((z = e^{-\alpha r})\)

\[
\left\{ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \frac{1}{z^2} \left[ 4\beta^2 \left( \mu^2 - E^2 + E(\mu + \Sigma) \right) + 4\beta^2 V_1(\mu - E - \Sigma) z \right] \right\} G_{n\kappa}(z) = 0. \tag{43}
\]

Comparing Eq. (43) with Eq. (13), we obtain the following parameter set

\[
\begin{align*}
\alpha_1 &= 1, & \alpha_2 &= 0, & \xi_1 &= 4\beta^2 V_1(\mu - E + \Sigma), \\
\alpha_3 &= 0, & \alpha_4 &= 0, & \xi_2 &= 4\beta^2 V_2(\mu - E + \Sigma), \\
\alpha_5 &= 0, & \alpha_6 &= \xi_1, & \xi_3 &= 4\beta^2 (E^2 - \mu^2 - E(\mu + \Sigma)), \\
\alpha_7 &= -\xi_2, & \alpha_8 &= \xi_3, & \alpha_9 &= \xi_1 \\
\alpha_{10} &= 1 + 2\sqrt{\xi_3}, & \alpha_{11} &= 2\sqrt{\xi_1}, \\
\alpha_{12} &= \sqrt{\xi_3}, & \alpha_{13} &= -\sqrt{\xi_1}.
\end{align*}
\tag{44}
\]

From Eq. (27), we obtain the energy eigenvalue equation for \( \kappa = 0 \)

\[
E^2 - E(\mu + \Sigma) - \mu^2 = \frac{1}{16\beta^2} \left( 2n + 1 - \frac{2\beta V_1}{\sqrt{\mu - E + \Sigma}} \right)^2, \tag{45}
\]

where \( \beta^2 = 1/4\alpha^2 \). We should choose the negative energy solution in Eq. (46) because the negative energy states exist only in the exact pseudospin limit. The corresponding lower spinor component can be obtained from Eq. (26)

\[
G_{n\kappa}(z) = z^{2\beta \sqrt{E^2 - \mu^2 - E(\mu + \Sigma)}} e^{-2\beta \sqrt{V_1(\mu - E + \Sigma)}} z \times L_n^{4\beta \sqrt{E^2 - \mu^2 - E(\mu + \Sigma)}} \left( 4\beta \sqrt{V_1(\mu - E + \Sigma)} \right)\tag{46}
\]

In the case of exact spin symmetry the potential \( \Delta(r) = V_v(r) - V_s(r) \) is a constant, let say \( \Delta(r) = \Delta = \text{const.} \). We set the potential \( \Sigma(r) \) as the Morse potential in Eq. (42). Substituting the potential into Eq. (4), and using the same variable \( z = e^{-\alpha r} \), we obtain

\[
\left\{ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \frac{1}{z^2} \left[ 4\beta^2 \left( E^2 - \mu^2 + \Delta(\mu - E) \right) + 4\beta^2 V_2(\mu + E - \Delta) z + 4\beta^2 V_1(-E - \mu + \Delta) z^2 \right] \right\} F_{n\kappa}(z) = 0, \tag{47}
\]
Comparing Eq. (47) with Eq. (13), we obtain the parameter set given in Eq. (44) where
\[ \xi_1 = 4\beta^2 V_1 (E + \mu - \Delta), \xi_2 = 4\beta^2 V_2 (E + \mu - \Delta) \] and \[ \xi_3 = 4\beta^2 (\mu^2 - E^2 + \Delta(E - \mu)) \]. From Eq. (27), we obtain the energy eigenvalue equation in the case of exact spin symmetry for \( \kappa = 0 \)

\[ \mu^2 - E^2 + \Delta(E - \mu) = \frac{1}{16\beta^2} \left( 2n + 1 - \frac{2\beta V_2}{\sqrt{V_1}} \sqrt{E + \mu - \Delta} \right)^2. \] (48)

where \( \beta^2 = 1/4\alpha^2 \). We should choose the positive energy solution in Eq. (49) because the positive energy states exist only in the exact spin limit. The corresponding Dirac spinor can be written as

\[
F_{nk}(z) = z^{2\beta \sqrt{\mu^2 - E^2 + \Delta(E - \mu)}} e^{-2\beta \sqrt{V_1(E + \mu - \Delta)}} z^{-\frac{1}{2}} L_n^{\sqrt{\mu^2 - E^2 + \Delta(E - \mu)}} (4\beta \sqrt{V_1(E + \mu - \Delta)}) z. \] (49)

C. The Pöschl-Teller Potential

By taking the potential \( \Delta(r) = V_v(r) + V_s(r) \) is the Pöschl-Teller potential [23] given by

\[ V(r) = -4V_0 \frac{e^{-2ar}}{(1 + e^{-2ar})^2}, \] (50)

and substituting Eq. (50) into Eq. (3), taking into account \( \Sigma(r) = \Sigma = \text{const.} \), we have the following equation in the exact pseudospin symmetry for \( \kappa = 0 \) (\( z = -e^{-2ar} \))

\[
\frac{d^2}{dz^2} G_{nk}(z) + \frac{1 - z}{z(1 - z)} \frac{d}{dz} G_{nk}(z) + \frac{1}{[z(1 - z)]^2} \left\{ \beta^2 (\mu - E + \Sigma)[\mu + E - [2\mu + 2E + 4V_0]z + (\mu + E)z^2] \right\} G_{nk}(z) = 0. \] (51)

Following the same procedure, we obtain the parameter set
\[ \alpha_1 = 1, \quad \xi_1 = \beta^2(\mu + E)(-\mu - \Sigma + E), \]
\[ \alpha_2 = 1, \quad \xi_2 = 2\beta^2(-\mu - \Sigma + E)[\mu + E + 2V_0], \]
\[ \alpha_3 = 1, \quad \xi_3 = \beta^2(\mu + E)(-\mu - \Sigma + E), \]
\[ \alpha_4 = 0, \quad \alpha_5 = -\frac{1}{2}, \]
\[ \alpha_6 = \xi_1 + \frac{1}{4}, \quad \alpha_7 = -\xi_2, \]
\[ \alpha_8 = \xi_3, \quad \alpha_9 = \xi_1 - \xi_2 + \xi_3 + \frac{1}{4}, \]
\[ \alpha_{10} = 1 + 2\sqrt{\xi_3}, \quad \alpha_{11} = 2 + 2\left(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}\right), \]
\[ \alpha_{12} = \sqrt{\xi_3}, \quad \alpha_{13} = -\frac{1}{2} - \left(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}\right). \]

and the energy eigenvalue equation of Pöschl-Teller potential under the exact pseudospin symmetry for \( \kappa = 0 \) from Eq. (21)

\[ E^2 - \mu^2 - \Sigma(\mu + E) = \frac{1}{4} \left((2n + 1)\alpha + \sqrt{4V_0(\mu - E + \Sigma)} + \alpha^2\right)^2. \]

The last energy eigenvalue equation has a quadratic form in terms of energy \( E \). We take the negative energy values in the exact pseudospin limit. The corresponding Dirac spinor can be written in terms of Jacobi polynomials, i.e., \( P_n^{(\alpha, \beta)}(x) \),

\[ G_{nk}(z) = z^\beta\sqrt{(\mu + E)(-\mu - \Sigma + E)} (1 - z)^{\frac{1}{2}} \left[ 1 + \sqrt{1 + 16V_0\beta^2(\mu + \Sigma - E)} \right] \times P_n^{(2\beta\sqrt{(\mu + E)(-\mu - \Sigma + E)}, \sqrt{1 + 16V_0\beta^2(\mu + \Sigma - E)})(1 - 2z). \]

In the case of exact spin symmetry, we set the potential \( \Sigma(r) \) as Pöschl-Teller potential given in Eq. (50), \( \Delta(r) = \Delta = \text{const.} \), and by using the coordinate transformation \( z = -e^{-2\alpha r} \), we obtain from Eq. (4)

\[ \frac{d^2F_{nk}(z)}{dz^2} + \frac{1 - z}{z(1 - z)} \frac{dF_{nk}(z)}{dz} + \frac{\Delta - \mu - E}{z(1 - z)^2} \left\{ \beta^2(\mu - E) - 2\beta^2(\mu - E - 2V_0)z \right\} F_{nk}(z) = 0, \]

which gives the parameter set given in Eq. (52) where \( \xi_1 = \beta^2(E - \mu)(\Delta - \mu - E) \), \( \xi_2 = 2\beta^2(\Delta - \mu - E)(E - \mu + 2V_0) \) and \( \xi_3 = \beta^2(E - \mu)(\Delta - \mu - E) \). The energy eigenvalue
equation of the Pöschl-Teller potential under the exact spin symmetry for \( \kappa = 0 \) from Eq. (21) is obtained

\[
E^2 + \mu^2 - \Delta(\mu - E) = \frac{1}{4} \left( (2n + 1)\alpha + \sqrt{4V_0(\mu + E - \Delta) + \alpha^2} \right)^2.
\] (56)

Finally, we obtain the corresponding Dirac spinor from Eq. (25)

\[
F_{n\kappa}(z) = z^\beta \sqrt{(E-\mu)(\Delta-\mu-E)} (1 - z)^\frac{1}{2} \left[ 1 + \sqrt{1 + 16V_0\beta^2(\mu + E - \Delta)} \right] P_n^{(2\beta \sqrt{(E-\mu)(\Delta-\mu-E)} \sqrt{1 + 16V_0\beta^2(\mu + E - \Delta)})} (1 - 2z).
\] (57)

V. CONCLUSION

We have studied the energy eigenvalues and the corresponding eigenfunctions of the Dirac equation with the hypergeometric potential, the Morse potential, and the Pöschl-Teller potential in the case of pseudospin, and spin symmetry. We have used the parametric generalization of the NU method to obtain the results. The energy eigenvalues of all potentials are real and the wave functions are written in terms of the Laguerre (Jacobi) polynomials. We have also investigated the special case \( \sigma = 1 \) in the case of the hypergeometric potential, which corresponds to the case of the Manning-Rosen potential with \( A = 0 \). So we have obtained the energy eigenvalue equation of the Manning-Rosen potential in the pseudospin, and spin symmetry case, respectively.
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