Stability tests for a class of switched descriptor systems with non-homogenous indices

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Abstract—In this paper we derive stability conditions for a switched system where switching occurs between linear descriptor systems of different indices. In particular, our results can be used to analyse the stability of the important case when switching between a standard system and an index one descriptor system, and systems where switching occurs between an index one and an index two descriptor system. Examples are given to illustrate the use of our results.

Index Terms—switched systems, descriptor systems, non-linear systems, Lyapunov functions, global uniform exponential stability.

I. INTRODUCTION

Descriptor systems provide a natural framework to model and analyse many dynamic systems with algebraic constraints. They appear frequently in modelling engineering systems: for example in the description of interconnected large scale systems; in economic systems (e.g. the fundamental dynamic Leontief model); network analysis [3] and they are also particularly important in the simulation and design of very large scale integrated (VLSI) circuits.

Recently, motivated by certain applications, some authors have begun the study of descriptor systems that are characterized by switching between a number of descriptor modes [9], [10], [12], [13], [14], [15], [17]. For example, in [10] the authors focus on dwell time arguments, and on conditions on the "consistency projectors", to obtain stability under arbitrary switching. In [11] and [12], the authors, under an assumption of a state-dependent switching condition (to avoid impulses), obtain a condition for stability based on commuting vector fields. These results mimic similar results derived for standard switched systems by [3], [6], [7].

Our approach in this paper differs from that given in the above papers. Our results in this note are based on a fundamental result derived in [1] to recursively reduce the dimension of a switched descriptor system using full rank decomposition. This approach allows us to obtain conditions which can also be checked without resorting to complicated linear algebraic manipulations. This has been achieved for a special system class of index one descriptor systems; namely, a class of switched systems characterised by rank-1 perturbations, for which a simple continuity assumption on the state at the switching instances is satisfied (see [1]). In this note, we now extend the results in [1] to switching between an index one descriptor system and (i) a standard system (which can be described completely by a set of ordinary differential equations); and (ii) an index two descriptor system, some results presented earlier in [2]. The present results provide detailed proofs and explanations for some results presented earlier in [2], and provide new examples of systems to which these results can be applied. In particular, our results apply to systems for which standard assumptions in the descriptor literature, do not apply.

II. PRELIMINARY RESULTS

Consider a linear time invariant (LTI) descriptor system described by

\[ E \dot{x} = Ax, \]  

where \( E, A \in \mathbb{R}^{n \times n} \). When \( E \) is nonsingular, this system is also described by the standard system \( \dot{x} = E^{-1}Ax \).

When \( E \) is singular, then both algebraic equations and differential equations describe the behavior of the system, and the system is known as a descriptor system. Since we shall be interested in switched systems which are constructed by switching between systems that are exponentially stable about zero, we require \( A \) to be nonsingular \cite{19}; note that were \( A \) singular, there would be equilibrium states other than zero.

The following notions are important when studying descriptor systems. First, the system (1) is said to be stable if every eigenvalue of \((E,A)\) has a negative real
part and the system is called \underline{regular} if \( \det [sE - A] \neq 0 \).

With \( A \) invertible, the \underline{index} of system (1) or the pair \((E, A)\) is the smallest integer \( k^* \leq n \) for which

\[
\text{Im}( (A^{-1}E)^{k^*+1} ) = \text{Im}( (A^{-1}E)^{k^*} )
\]

(2)

where \( \text{Im} \) denotes the image of a matrix. Thus, a standard system is an index zero descriptor system.

The \underline{consistency space} for system (1) or \((E, A)\) is defined by

\[
\mathcal{C} = \mathcal{C}(E, A) := \text{Im}( (A^{-1}E)^{k^*} )
\]

(3)

where \( k^* \) is the index of \((E, A)\). Note that \( \mathcal{C} \) is the set of all initial states for which the system has a continuous solution. Since \( \text{Im}( (A^{-1}E)^{k^*+1} ) = \text{Im}( (A^{-1}E)^{k^*} ) \) we see that \( A^{-1}E \mathcal{C} = \mathcal{C} \); this means that \( A^{-1}E \) is a one-to-one mapping of \( \mathcal{C} \) onto itself; hence the kernel of \( E \) and \( \mathcal{C} \) intersect only at zero \([19]\). If \( \mathcal{C} = \{0\} \) the system is trivial and the only continuous solution is the zero solution \( x(t) \equiv 0 \). If \( \mathcal{C} \neq \{0\} \), we let \( \tilde{A} \) be the inverse of the map \( A^{-1}E \) restricted to \( \mathcal{C} \); then (1), or equivalently \( x = A^{-1}E \tilde{x} \), is equivalent to

\[
\dot{\tilde{x}} = \tilde{A}\tilde{x}
\]

(4)

Thus the restriction of the descriptor system to its consistency space is equivalent to the standard system (4) where \( x(t) \) is in \( \mathcal{C} \).

Another way to introduce the consistency space is as follows. When \((E, A)\) is regular with a non-trivial consistency space, it can be shown that there exist nonsingular matrices \( S \) and \( T \) such that \([16]\)

\[
\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}
\]

where the matrix \( N \) is nilpotent, that is, \( N^k = 0 \) for some \( k \geq 1 \). If \( A \) is nonsingular, then \( J \) is nonsingular and for any \( k \geq 1 \),

\[
(A^{-1}E)^k = T \begin{bmatrix} J^{-k} & 0 \\ 0 & N^k \end{bmatrix} T^{-1}
\]

Then the index of \((E, A)\) is the smallest \( k^* \) for which \( N^{k^*} = 0 \) and the consistency space is the range of the consistency projector defined by

\[
\Pi = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}
\]

Also

\[
\tilde{A} = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}
\]

Note that the consistency space for \((E, A)\) is trivial if and only if \( A^{-1}E \) is nilpotent.

**Lyapunov functions:** To obtain stability conditions for switching descriptor systems we shall use Lyapunov functions. Let us first consider Lyapunov functions for system (1). Consider any differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \). Its derivative along solutions of (1) is given by \( \dot{V} = DV(\dot{x})\dot{x} \) which can be expressed as a function of \( x \).

**Definition 1:** A differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \) is a \underline{Lyapunov function} for \((E, A)\) if \( V \) is positive definite on \( \mathcal{C} = \mathcal{C}(E, A) \) and \( \dot{V} \) is negative definite for any non-zero state in \( \mathcal{C}(E, A) \). A symmetric matrix \( P \) is a \underline{Lyapunov matrix} for \((E, A)\) if \( V(x) = x^T P x \) is a Lyapunov function for \((E, A)\).

The following lemma provides a characterization of all Lyapunov matrices for a linear descriptor system.

**Lemma 1:** A symmetric matrix \( P \) is a Lyapunov matrix for \((E, A)\) if and only if \( P \) is positive-definite on \( \mathcal{C} = \mathcal{C}(E, A) \) and \( PA^{-1}E + E^TA^{-T}P \) is negative definite on \( \mathcal{C} \).

**Proof:** It follows from (1) and the invertibility of \( A \) that \( x = A^{-1}E \tilde{x} \); hence

\[
\dot{V} = 2x^T P \dot{x} = 2(A^{-1}E)\dot{x}^T P \dot{x} = \dot{x}^T (PA^{-1}E + E^TA^{-T}P) \dot{x} = -\dot{x}^T Q \dot{x}
\]

where \( Q := -PA^{-1}E - E^TA^{-T}P \). Recall that system description (1) is equivalent to \( \dot{x} = \tilde{A}x \), where \( x \) is in \( \mathcal{C} \) and \( \tilde{A} \) is an invertible map on \( \mathcal{C} \). Hence

\[
\dot{V} = -x^T \tilde{Q} x \quad \text{where} \quad \tilde{Q} = -\tilde{A}^T Q \tilde{A}.
\]

Since \( \tilde{A} \) maps \( \mathcal{C} \) onto \( \mathcal{C} \) and is invertible on \( \mathcal{C} \), \( \tilde{Q} \) is positive-definite on \( \mathcal{C} \) if and only if \( Q \) is positive-definite on \( \mathcal{C} \). Hence, \( \dot{V} \) is negative for any non-zero state in \( \mathcal{C} \) if and only if \( PA^{-1}E + E^TA^{-T}P \) is negative-definite on \( \mathcal{C} \). \( \Box \)

Previous papers such as \([19], [16], [10], [20]\) consider a specific class of Lyapunov matrices of the form \( P = E^T \tilde{P} E \) where \( \tilde{P} \) is a positive definite matrix for which \( E^T \tilde{P} A + A^T \tilde{P} E \) is negative definite on \( \mathcal{C} \). In particular, \([19]\) shows that the existence of a Lyapunov matrix of this type is necessary and sufficient for asymptotic stability of system (1).

Let \( C \) be any matrix with the following property:

\[
x \in \mathcal{C}(E, A) \quad \text{if and only if} \quad Cx = 0
\]

(5)

Then we have the following LMI characterization of Lyapunov matrices for \((E, A)\).
Lemma 2: A symmetric matrix $P$ is a Lyapunov matrix for $(E,A)$ if and only if there exists scalars $\kappa_1, \kappa_2 \geq 0$ such that
\begin{align*}
P + \kappa_1 C^T C & > 0 \quad (6) \\
P A^{-1} E + E^T A^{-T} P - \kappa_2 C^T C & < 0 \quad (7)
\end{align*}
where $C$ is any matrix satisfying (5).

Proof: A vector $x$ is in $\mathcal{C} = \mathcal{C}(E,A)$ if and only if $Cx = 0$ which is equivalent to $x^T C^T C x = 0$. From Lemma 1 we know that $P$ is a Lyapunov matrix for $(E,A)$ if and only if $x^T P x > 0$ and $x^T (PA^{-1} E + E^T A^{-T} P)x < 0$ whenever $x \neq 0$ and $x^T C^T C x = 0$. It follows from Finslers Lemma that there exist scalars $\kappa_1$ and $\kappa_2$ such that (6) and (7) hold; since $C^T C \succeq 0$, (6) and (7) also hold with $\kappa_1, \kappa_2 \geq 0$.

Note that if there exists a Lyapunov matrix $P$ satisfying (6) and (7) then there also exists a Lyapunov matrix satisfying
\begin{align*}
P + C^T C & > 0 \quad (8) \\
P A^{-1} E + E^T A^{-T} P - C^T C & < 0 \quad (9)
\end{align*}

A. Linear switched descriptor systems

The ultimate objective of our work is to analyze the stability of switched descriptor systems described by
\begin{equation}
E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x \quad \text{where} \quad \sigma(t) \in \{1, \ldots, N\}. \quad (10)
\end{equation}

We assume throughout this paper that $\sigma$ is piecewise continuous with a finite number of discontinuities in any bounded time interval. Also, we do not require (10) to hold at points of discontinuity of $\sigma$. Thus, if $\sigma$ is continuous at $t$ and $\sigma(t) = i$, the system must satisfy
\begin{equation}
E_i \dot{x} = A_i x;
\end{equation}
hence $x(t)$ must be in the consistency space of $(E_i, A_i)$.

To complete the description of the switching descriptor systems under consideration one must also specify how the system behaves during switching. If $\sigma$ switches from $i$ to $j$ at $t_\ast$ then $x(t_\ast) := \lim_{t \to t_\ast, t < t_\ast} x(t)$ must be in $\mathcal{C}(E_i, A_i)$ and $x(t_\ast^+) := \lim_{t \to t_\ast, t > t_\ast} x(t)$ must be in $\mathcal{C}(E_j, A_j)$. If $x(t_\ast^-)$ is not in $\mathcal{C}(E_j, A_j)$ then, one has to have a solution which is discontinuous at $t_\ast$ and to complete the description one must specify how $x(t_\ast^+)$ is obtained from $x(t_\ast^-)$. In some switching systems, there is a restriction on $x(t_\ast^-)$, that is, $\sigma$ can only switch from a specific $j$ to a specific $i$ if $x(t_\ast^-)$ is in a restricted subset of the consistency space of $(E_i, A_i)$. This is illustrated in Example 1.

In some treatments of switched descriptor systems, (10) is satisfied for all $t$. In that case, one has to consider $x(\cdot)$ to be a distribution because the solution of (10) may contain impulses. When (10) is satisfied for all $t$ and $\sigma$ is continuous from the right, it can be shown that, if $\sigma$ switches from $i$ to $j$ at $t_\ast$, then
\begin{equation}
x(t_\ast^+) = \Pi_i x(t_\ast^-)
\end{equation}
where $\Pi_j$ is the consistency projector associated with $(E_j, A_j)$. Here we do not require that (10) be satisfied at discontinuity points of $\sigma$ nor do we require (11). This is illustrated in Example 1. When a system satisfies (10) for all $t$ and if switching can occur from any state in the consistency space of $(E_i, A_i)$ then, as in (11) one must assume that
\begin{equation}
E_j (I - \Pi_j) \Pi_i = 0
\end{equation}
in order to guarantee solutions without impulses when switching from from $i$ to $j$. We do not need this assumption here.

B. Lyapunov stability conditions

Our next result contains conditions that are sufficient to guarantee global uniform exponential stability (GUES) of (10).

Theorem 1: Consider a switched descriptor system which satisfies (10) at points of discontinuities of $\sigma$ and suppose that for each $i = 1, \ldots, N$, there is a Lyapunov matrix $P_i$ for $(E_i, A_i)$ such that
\begin{equation}
x(t_\ast^-)^T P_j x(t_\ast^-) \leq x(t_\ast^-)^T P_i x(t_\ast^-). \quad (13)
\end{equation}

whenever $\sigma$ switches from $i$ to $j$ at $t_\ast$. Then the system is GUES.

Proof: Consider any solution $x(\cdot)$ of the system, and let $v(t) = x(t)^T P_{\sigma(t)} x(t)$. If $t$ is a point of discontinuity of $\sigma$, then, according to (13),
\begin{equation}
v(t^+) \leq v(t^-). \quad (14)
\end{equation}

Suppose $t$ is not a point of discontinuity of $\sigma$. If $\mathcal{C}_i = \mathcal{C}(E_i, A_i) = \{0\}$ then $v(t) = 0$ and $v(t) = 0$. Otherwise, $E_i \dot{x} = A_i x$ where $i = \sigma(t)$. Following the proof of Lemma 1
\begin{equation}
\dot{\nu} = -x^T \tilde{Q}_i x \quad \text{where} \quad \tilde{Q}_i = A_i^T Q_i A_i
\end{equation}
\begin{equation}
Q_i = -P_i A_i^{-1} E_i - E_i^T A_i^{-T} P_i \quad (15)
\end{equation}
and $\tilde{Q}_i$ is positive-definite on $\mathcal{C}_i$. Recalling that $P_i$ is positive-definite on $\mathcal{C}_i$, let
\begin{equation}
\alpha_i = \frac{1}{2} \min \{ x^T \tilde{Q}_i x : x \in \mathcal{C}_i \text{ and } x^T P_i x = 1 \}. \end{equation}
Then $\alpha_i > 0$ and $\dot{v} \leq -2\alpha_i v$. Now let $\alpha = \min\{\alpha_1, \cdots, \alpha_n\}$. Then $\alpha > 0$ and
\begin{equation}
\dot{v}(t) \leq -2\alpha v(t)
\end{equation}
when $\sigma$ is continuous at $t$. From this and the discontinuity condition \eqref{14}, we can conclude that $v(t) \leq e^{-2\alpha(t-t_0)}v(t_0)$ for $t \geq t_0$. Since each $P_i$ is positive-definite on $\mathcal{C}_i$, there are constants $\lambda_1, \lambda_2 > 0$ such that, for $i = 1, \cdots, N$, we have $\lambda_1 \|x\|^2 \leq x^T P_i x \leq \lambda_2 \|x\|^2$ whenever $x$ is in $\mathcal{C}_i$. Hence $\lambda_1 \|x(t)\|^2 \leq v(t) \leq \lambda_2 \|x(t)\|^2$ and every solution $x(\cdot)$ satisfies
\begin{equation}
\|x(t)\| \leq \beta e^{-\alpha(t-t_0)}\|x(t_0)\|
\end{equation}
for all $t \geq t_0$, where $\beta = \sqrt{\lambda_2/\lambda_1}$. This means that the system is GUES. Q.E.D.

We have the following corollary to Theorem 1 and Lemma 2.

**Corollary 1**: Consider a switching descriptor system described by \eqref{10} and suppose that there is a symmetric matrix $P$ satisfying
\begin{equation}
P + C_i^T C_i > 0
\end{equation}
for $i = 1, \cdots, N$ where
\begin{equation}
x \in \mathcal{C}(E_i, A_i) \text{ if and only if } C_i x = 0
\end{equation}
Also,
\begin{equation}
x(t_+^+) P x(t_+^+) \leq x(t_-^+) P x(t_-^+)
\end{equation}
if $\sigma$ switches at $t_+$. Then, the system is GUES.

To conclude this section, we present an example to motivate our results. This example illustrates the use of Theorem 1 to analyse stability of switching between a standard system and a descriptor system.

**Example 1 (A simple switched mechanical system)**: Consider the switched system in which the two masses can lock onto each other when their displacements are the same; see Figure 2. When they are locked together their displacements remain equal. When they unlock, their displacements are independent.

![Figure 1. A mechanical system](image1)

![Figure 2. Masses locked together](image2)

When the masses are not locked together, the system is described by
\begin{equation}
m_1 \ddot{q}_1 + c_1 \dot{q}_1 + k_1 q_1 = 0
\end{equation}
\begin{equation}
m_2 \ddot{q}_2 + c_2 \dot{q}_2 + k_2 q_2 = 0
\end{equation}
When the masses are locked together, we have the following description
\begin{equation}
m \ddot{q}_1 + c_1 q_1 + k_1 q_1 = 0
\end{equation}
\begin{equation}
q_1 - q_2 = 0
\end{equation}
where
\begin{equation}
m := m_1 + m_2, \quad c := c_1 + c_2, \quad k := k_1 + k_2.
\end{equation}
Since lock-up is due to internal forces in the system, linear momentum is conserved during lock-up, that is, if lockup occurs at time $t$ then,
\begin{equation}
m \ddot{q}_2(t^+) = m \ddot{q}_1(t^-) = m_1 \ddot{q}_1(t^-) + m_2 \ddot{q}_2(t^-)
\end{equation}
which results in
\begin{equation}
\ddot{q}_2(t^+) = \ddot{q}_1(t^-) = m_1 \ddot{q}_1(t^-) + m_2 \ddot{q}_2(t^-)
\end{equation}
We also have
\begin{equation}
q_1(t^+) = q_2(t^+) = q_1(t^-) = q_2(t^-)
\end{equation}
Introducing state variables $x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2, x_4 = \dot{q}_2$, this system can be described by the switched system \eqref{10} where $N = 2$,
\begin{equation}
E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & m_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & m_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\
-k_1 & -c_1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -k_2 & -c_2 \end{bmatrix}
\end{equation}
\begin{equation}
E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & m & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\
-k & -c & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 \end{bmatrix}
\end{equation}
Observe that $(E_1, A_1)$ is a standard system and it can be readily shown that $(E_2, A_2)$ is an index two descriptor system.
Switching to model one can occur at any state in the consistency space of mode two and if this occurs at time $t_*$, then $x(t_*) = x(t_*)$. Switching to mode two only occurs when $x_1 = x_3$ and if this occurs at time $t_*$,

$$
\begin{align*}
    x_1(t_*) &= x_3(t_*) \\
    x_2(t_*) &= x_4(t_*) \quad \text{if} \quad m_1x_2(t_*) + m_2x_4(t_*) = m
\end{align*}
$$

(26)

Note that this system does not satisfy condition (11) at switching. One can show that this condition requires that $x_2(t_*) = x_4(t_*) = x_2(t_*)$.

As candidate Lyapunov matrices for this system consider

$$
P_1 = P_2 = P := \begin{bmatrix}
    k_1 & \epsilon m_1 & 0 & 0 \\
    \epsilon m_1 & m_1 & 0 & 0 \\
    0 & 0 & k_2 & \epsilon m_2 \\
    0 & 0 & \epsilon m_2 & m_2
\end{bmatrix}
$$

where $\epsilon > 0$; clearly $P > 0$ for $\epsilon$ sufficiently small. Recalling definition (13) of the symmetric matrix $Q$, we obtain

$$
Q_1 = \begin{bmatrix}
    2c_1 - 2\epsilon m_1 & \epsilon m_1 c_1 & 0 & 0 \\
    * & 2\epsilon m_1 & 0 & 0 \\
    0 & 0 & 2c_2 - 2\epsilon m_2 & \epsilon m_2 c_2 \\
    0 & 0 & * & 2\epsilon m_2
\end{bmatrix}
$$

$$
Q_2 = \frac{1}{k} \begin{bmatrix}
    2k_1 c_2 - 2\epsilon m_1 k & m_1 k - m_1 k + \epsilon m_1 c & k_2 c & \epsilon m_2 c \\
    * & 2\epsilon m_1 & m_2 k & \epsilon m_2 m \\
    0 & 0 & * & -2\epsilon k \epsilon - m_2 k \\
    0 & 0 & * & *
\end{bmatrix}
$$

Clearly, $Q_1 > 0$ for $\epsilon$ sufficiently small. When $x$ is in the consistency space of $(E_2, A_2)$, we have $x_3 = x_1$, $x_4 = x_2$; hence

$$
x^T \hat{Q}_2 x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \hat{Q}_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T
$$

where

$$
\hat{Q}_2 = \frac{1}{k} \begin{bmatrix}
    2k_1 c_2 - 2\epsilon m_1 k & \epsilon m_1 c & \epsilon m_2 c \\
    * & 2\epsilon m_1 & \epsilon m_2 m \\
    0 & 0 & \epsilon m_2 \\
    0 & 0 & \epsilon m_2
\end{bmatrix}
$$

and $\hat{Q}_2 > 0$ for $\epsilon$ sufficiently small; hence $Q_2$ is positive definite on the consistency space of $(E_2, A_2)$.

When switching to model one, $x(t_*)^T P_1 x(t_*) = x(t_*)^T P_2 x(t_*)$. When switching to mode two, it follows from the switching conditions in (26) that

$$
x(t_*)^T P_2 x(t_*) - x(t_*)^T P_1 x(t_*) = -\frac{m_1 m_2}{m} [x_2(t_*) - x_4(t_*)]^2 \leq 0
$$

Thus, for $\epsilon$ sufficiently small, the matrices $P_1$ and $P_2$ satisfy the requirements of Lemma (1) hence this system is globally uniformly exponentially stable about zero for any allowable switching sequence.

### III. MAIN RESULTS

We now present some simple tests to check the stability of special classes of switched descriptor systems constructed by (i) switching between a standard system and an index-one system; and (ii) switching between index-one and index-two descriptor systems. These results build on (13) and the following lemma.

**Lemma 3:** Suppose $A, B \in \mathbb{R}^{n \times n}$ with $\text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$ and

$$
A + A^T > 0 \\
B + B^T \geq 0.
$$

Then, the kernels of $B$ and $B + B^T$ are equal.

**Proof:** Since $Q_A > 0$, where $Q_A := A + A^T$, we see that $\text{rank}(A) = n$. Let $r := \text{rank}(B)$. Then, by assumption, we have $\text{rank}(A - B) = n - r$. Recall that the nullity of a matrix is the dimension of its kernel. First, we show that the nullity of $Q_B := B + B^T$ is at most $n - r$. So, suppose that $x \neq 0$ is in the kernel of $Q_B$. Then

$$
0 = x^T Q_B x = x^T (A + A^T) x + 2x^T (B - A) x = x^T Q_A x - 2x^T (A - B) x
$$

Since $Q_A > 0$ and $x \neq 0$, we have $x^T Q_A x > 0$; hence $(A - B)x \neq 0$, that is, $x$ is not in the kernel of $A - B$. Thus, the kernel of $Q_B$ and $A - B$ intersect only at zero. Since the rank of $A - B$ is $n - r$, its nullity is $r$; hence the nullity of $Q_B$ is at most $n - r$. We now show that the kernel of $Q_B$ contains the kernel of $B$. So, suppose that $x$ is in the kernel of $B$, that is, $Bx = 0$. Then $x^T Q_B x = 2x^T B x = 0$. Since $Q_B \geq 0$, it follows that $Q_B x = 0$, that is, $x$ is in the kernel of $Q_B$. Thus, the kernel of $Q_B$ contains the kernel of $B$. Finally, since $B$ has rank $r$, its nullity is $n - r$. Since we also know that the nullity of $Q_B$ is less than or equal to $n - r$, it now follows that the kernel of $Q_B$ is the same as the kernel of $B$. Q.E.D.

The following general result is a consequence of Theorem (11) and Lemma (5).

**Lemma 4:** Consider a switching descriptor system described by (10) and suppose that, for some $N_1 \leq N$, there is a symmetric positive-definite matrix $P$ satisfying

$$
PA_i^{-1} E_i + (A_i^{-1} E_i)^T P < 0, \quad i = 1, \ldots, N_1 \quad (27)
$$

$$
PA_j^{-1} E_j + (A_j^{-1} E_j)^T P \leq 0, \quad j = N_1 + 1, \ldots, N \quad (28)
$$

and for each $j \in \{N_1 + 1, \ldots, N\}$ there is a subscript $i_j \in \{1, \ldots, N_1\}$ such that

$$
\text{rank}(A_{ij}^{-1} E_{ij} - A_{ij}^{-1} E_j) = \text{rank}(A_{ij}^{-1} E_{ij}) - \text{rank}(A_{ij}^{-1} E_j). \quad (29)
$$

The matrices $P_1$ and $P_2$ satisfy the requirements of Lemma (1) hence this system is globally uniformly exponentially stable about zero for any allowable switching sequence.
Also, \[ x(t_+^+)Px(t_+^+) \leq x(t_+^+)^TPx(t_+^+) \] (30)
if \( \sigma \) switches at \( t_s \). Then, the system is GUES.

**Proof:** We prove this result by showing that the hypotheses of Theorem 1 hold. Since \( P \) is positive definite, hypothesis (a) holds. Also, (30) implies that hypothesis (c) holds. To see that hypothesis (b) holds, consider any \( j \in \{N_1 + 1, \ldots, N\} \) and apply Lemma 3 with \( A = -PA_j^{-1}E_i \) and \( B = -PA_j^{-1}E_j \) to obtain that the kernel of \( Q_j := -PA_j^{-1}E_j - (A_j^{-1}E_j)^T P \) is the same as that of \( -PA_j^{-1}E_j \) which also equals the kernel of \( A_j^{-1}E_j \); thus \( Q_j \) and \( A_j^{-1}E_j \) have the same kernel. Since \( Q_j \geq 0 \) and the kernel of \( A_j^{-1}E_j \) and \( \mathcal{G}(E_j, A_j) \) intersect only at zero, we conclude that \( Q_j \) is positive definite on the consistency space of \( (E_j, A_j) \). Hypothesis (b) now follows by taking into account (27). It now follows from Theorem 1 that the switched system (10) is GUES. Q.E.D.

Now we consider a special class of switched descriptor systems described by
\[ E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x, \quad \sigma(t) \in \{1, 2\} \] (31)
where each constituent system is stable, with the first being index zero (standard system) and the second index one; also the rank of \( A_1^{-1}E_1 - A_2^{-1}E_2 \) is one. We show that if the matrix \( A_1^{-1}E_1A_2^{-1}E_2 \) has no negative real eigenvalues, exactly one eigenvalue at zero and some other regularity conditions hold then, the system is GUES. To achieve this result, we recall the following result from [18].

**Theorem 2:** [18] Suppose that \( A \) is Hurwitz and all eigenvalues of \( A - gh^T \) have negative real part, except one, which is zero. Suppose also that \( (A, g) \) is controllable and \((A, h)\) is observable. Then, there exists a matrix \( P = P^T > 0 \) such that
\[ A^T P + PA < 0 \] (32)
\[ (A - gh^T)^T P + P(A - gh^T) \leq 0 \] (33)
if and only if the matrix product \( A(A - gh^T) \) has no real negative eigenvalues and exactly one zero eigenvalue.

The following result follows from Lemma 4 and Theorem 2.

**Theorem 3:** Consider a switching descriptor system described by (31) where \( x(\cdot) \) is continuous during switching and suppose that it satisfies the following conditions
(a) \( (E_1, A_1) \) and \( (E_2, A_2) \) are stable.
(b) \( (E_1, A_1) \) is index zero and \( (E_2, A_2) \) is index one.
(c) There exists column matrices \( g \) and \( h \) such that
\[ A_2^{-1}E_2 = A_1^{-1}E_1 - gh^T, \] (34)
where \( (A_1^{-1}E_1, g) \), \( (A_2^{-1}E_1, h) \) are controllable and observable, respectively.
(d) The matrix \( A_1^{-1}E_1A_2^{-1}E_2 \) has no negative real eigenvalues and exactly one zero eigenvalue.

Then the switching descriptor system (31) is globally exponentially stable about zero.

**Proof:** We first show that the hypotheses of Theorem 2 hold with \( A = A_1^{-1}E_1 \). For \( i = 1, 2 \), \((E_i, A_i)\) is stable; hence the non-zero eigenvalues of \( A_i^{-1}E_i \) have negative real parts. Since \( (A_1, E_1) \) is index zero, \( A_1^{-1}E_1 \) is nonsingular and has no zero eigenvalues. This implies that \( A_1^{-1}E_1 \) is Hurwitz.

Since \( A_1^{-1}E_1A_2^{-1}E_2 \) has exactly one eigenvalue at zero, its nullity is one; the non-singularity of \( A_1^{-1}E_1A_2^{-1} \) now implies that the nullity of \( E_2 \) is one; hence the rank of \( E_2 \) and \( A_2^{-1}E_2 \) is \( n-1 \). Since \( (E_2, A_2) \) has index one and the nullity of \( E_2 \) is one, the matrix \( A_2^{-1}E_2 \) has a single eigenvalue at zero. Thus, all eigenvalues of \( A_2^{-1}E_2 \) have negative real part except one which is zero.

Recalling hypotheses (c) and (d) of this theorem, we see that the hypotheses of Theorem 2 hold with \( A = A_1^{-1}E_1 \). Hence there exists a matrix \( P = P^T > 0 \) such that
\[ PA_1^{-1}E_1 + (A_1^{-1}E_1)^T P < 0, \] (35)
\[ PA_2^{-1}E_2 + (A_2^{-1}E_2)^T P \leq 0. \] (36)

Since \( \text{rank}(A_1^{-1}E_1 - A_2^{-1}E_2) = \text{rank}(gh^T) = 1 = \text{rank}(A_1^{-1}E_1) - \text{rank}(A_2^{-1}E_2) \), Lemma 4 now implies that the switched descriptor system (31) is globally uniformly exponentially stable about zero. Q.E.D.

**Comment 1:** The above result requires \( x(\cdot) \) to be continuous during switching. Since \((E_1, A_1)\) is an index zero system its consistency space is the whole state space. Hence switching to this system can occur at any state. Since \((E_2, A_2)\) is index one, the consistency space of this system is not the whole state space. Thus, the switched system does not switch to the second system from an arbitrary point in the state space. To switch to the second system, the state must be in the consistency space of that system, that is it must be in \( \text{Im}(A_2^{-1}E_2) \).

**Switching between index-one and index-two systems**
Now we consider switching between index-one and index-two descriptor systems. Our results in this subsection are based on an order reduction result from [11]. They result from an application of full rank decomposition to a switched descriptor system in the
form of (10).

Full rank decomposition: A pair of matrices \((X,Y)\) is a decomposition of \(E \in \mathbb{R}^{n \times n}\) if

\[ E = XY^T. \]  

(37)

If, in addition, \(X\) and \(Y\) both have full column rank we say that \((X,Y)\) is a full rank decomposition of \(E\). Note that, if \((X,Y)\) is a full rank decomposition of \(E \in \mathbb{R}^{n \times n}\) and \(\text{rank}(E) = r\) then, \(X,Y \in \mathbb{R}^{n \times r}\) and \(\text{rank}(X) = \text{rank}(Y) = r\). Suppose \(E\) has rank \(r > 0\) and \(\tilde{E} = Y^T A^{-1} X\) where \((X,Y)\) is a full rank decomposition of \(E\); then \((\tilde{E},I)\) is a reduced order descriptor system with \(r < n\) state variables. The original descriptor system \((E,I)\) is stable if and only if the non-zero eigenvalues of \(\tilde{E}\) have negative real parts. Also, if \(k^*\) is the index of \((E,I)\) then the index of the equivalent reduced order system \((\tilde{E},I)\) is \(k^* - 1\).

One can iteratively apply full rank decomposition to achieve further order reduction of \((\tilde{E},I)\), provided that there is a decomposition \((\tilde{X},\tilde{Y})\) of \(\tilde{E}\) with \(\tilde{Y} \in \mathbb{R}^{r \times p}\) and \(r < p\). Since a non-zero square matrix always has a full rank decomposition, one can always iteratively reduce a single linear system \((E,A)\) to a standard system or to a system of algebraic equations, that is a system whose "E-matrix" is zero.

Commonly, the switching condition on the state can be described by:

\[ x(t^+_n) = M_{ji} x(t^-_n) \]  

(38)

when \(\sigma\) switches from \(i\) to \(j\) at \(t_n\). Also, switching may be restricted in the sense that one does not switch from \(i\) to \(j\) at any state \(x(t^-_n)\) in \(\mathcal{C}(E_i,A_i)\). In this case, the restriction may be described by

\[ x(t^-_n) \in \mathcal{S}_{ji} \]  

(39)

Theorem 4 (Order reduction of linear switching descriptor systems \([1]\)): Consider a switching descriptor system described by (10) and switching conditions (38) and (39) when \(\sigma\) switches from \(i\) to \(j\) and suppose that \((X_i,Y_i)\) is a decomposition of \(E_i\) with \(Y_i \in \mathbb{R}^{n \times r}\) for \(i = 1, \ldots, N\). Then, there exist matrices \(T_1, \ldots, T_N\) such that the following holds. A function \(x(\cdot)\) is a solution to system (10) with (38) and only if

\[ x(t) = T_{\sigma(t)} z(t) \]  

(40)

for all \(t\) where \(z(\cdot)\) is a solution to the descriptor system

\[ \tilde{E}_{\sigma(t)} \dot{z} = z \]  

(41)

with switching conditions

\[ z(t^+_n) = Y^n_j M^n_j T z(t^-_n) \]  

(42)

\[ T z(t^-_n) \in \mathcal{S}_{ji} \]  

(43)

when \(\sigma\) switches from \(i\) to \(j\) where

\[ \tilde{E}_i = Y^n_i A^{-1}_i X_i. \]  

(44)

Moreover

\[ z(t) = Y^{T}_{\sigma(t)} X(t) \]  

(45)

for all \(t\), \(\mathcal{C}^\dagger(E_i,I) = Y^{T}_i \mathcal{C}(E_i,A_i)\) and \(z(\cdot)\) is continuous during switching if and only if the same is true of \(Y^{T}_i x\). Hence, global uniform exponential stability (GUES) of the new system (41)-(43) and the original system (10)-(39) are equivalent.

Now we present a general result which is a corollary of Theorem 4 and Lemma 4.

Corollary 2: Consider a switching descriptor system described by \([10]\) where \(Y^n_i x\) is continuous during switching and \((X_i,Y_i)\) is a decomposition of \(E_i\) with \(Y_i \in \mathbb{R}^{n \times r}\) for \(i = 1, \ldots, N\). Suppose that, for some \(N_i \leq N\), there is a symmetric positive-definite matrix \(P\) such that the following conditions are satisfied, where \(\tilde{E}_i = Y^n_i A^{-1}_i X_i\).

\[ P \tilde{E}_i + \tilde{E}_i^T P < 0, \quad i = 1, \ldots, N_1 \]  

(46)

\[ P \tilde{E}_j + \tilde{E}_j^T P \leq 0, \quad j = N_1 + 1, \ldots, N \]  

(47)

and for each \(j \in \{N_1 + 1, \ldots, N\}\) there is a subscript \(i_j \in \{1, \ldots, N_1\}\) such that

\[ \text{rank}(\tilde{E}_i - \tilde{E}_j) = \text{rank}(\tilde{E}_i) - \text{rank}(\tilde{E}_j). \]  

(48)

Then, the system is globally uniformly exponentially stable about zero.

The above result requires that \(Y^n_i x\) be continuous during switching. Continuity of \(Y^n_i x\) during switching is equivalent to the following switching condition. If \(\sigma\) switches from \(i\) to \(j\) at a point of discontinuity \(t^n\) then

\[ Y^n_i(x(t^+_n)) = Y^n_j x(t^-_n). \]  

(49)

Since \(x(t^-_n)\) must be in \(\mathcal{C}_i = \mathcal{C}(E_i,A_i)\), the above switch can only occur at states \(x(t^-_n)\) in \(\mathcal{C}_i = \mathcal{C}(E_i,A_i)\) for which

\[ Y^n_i x(t^-_n) \in \mathcal{C}_j \]  

(49)

If \((E_j,A_j)\) is index-one and \(Y_j \in \mathbb{R}^{r \times n}\) is full column rank where \(r = \text{rank}(E)\) then, switching to this system can occur from any state. To see this, recall that the kernel of \(Y^n_j\) and \(\mathcal{C}_j\) intersect only at the origin, and since the system \((E_j,A_j)\) is index one, the dimension of \(\mathcal{C}_j\) is \(r = \text{rank}(E_j)\). Hence the dimension of \(Y^n_j \mathcal{C}_j\) is \(r\). Since \(Y^n_j \in \mathbb{R}^{r \times n}\) we now see that \(Y^n_j \mathcal{C}_j = \mathbb{R}^r\); hence (49).
is satisfied for any \(x(t^-)\). This means that switching to an index one system can occur from any state. For an index-two system

\[
Y_i^T \mathcal{G}_i = \text{Im}(Y_i^T (A_i^{-1} E_i)^2) = \text{Im}(Y_i^T A_i^{-1} X_i Y_i^T) = \text{Im}(Y_i^T A_i^{-1} X_i^2).
\]

For an index two system, \(Y_i^T A_i^{-1} X_i\) is singular; hence the dimension of \(Y_i^T \mathcal{G}_i\) is strictly less than \(r\). Hence we can always find \(x(t^-)\) such that \(Y_i^T x(t^-) \notin Y_i^T \mathcal{G}_i\). Thus we cannot arbitrarily switch to an index-two system.

Now to conclude we present our next main result: switching between an index-one and an index-two descriptor system. The following result is obtained from Corollary 2 and Theorem 2

**Theorem 5:** Consider a switching descriptor system described by

\[
E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x, \quad \sigma(t) \in \{1, 2\},
\]

where \(Y_i^T x\) is continuous during switching and \((X_i, Y_i)\) is a full rank decomposition of \(E_i\) with \(Y_i \in \mathbb{R}^{r \times r}\) for \(i = 1, 2\). Suppose that the following conditions are satisfied where \(\bar{E}_i = Y_i^T A_i^{-1} X_i\) for \(i = 1, 2\).

(a) \((E_1, A_1)\) and \((E_2, A_2)\) are stable.

(b) \((E_1, A_1)\) is index one and \((E_2, A_2)\) is index two.

(c) There exist vectors \(g\) and \(h\) such that

\[
\bar{E}_2 = \bar{E}_1 - gh^T
\]

with \((\bar{E}_1, g)\) controllable and \((\bar{E}_1, h)\) observable.

(d) \(E_1 \bar{E}_2\) has no negative real eigenvalues and exactly one zero eigenvalue.

Then the switched descriptor system (51) is globally uniformly exponentially stable about zero.

**Proof:** Since \((X_1, Y_1)\) is a full rank decomposition of \(E_1\) and \((A_1, E_1)\) is stable and index-one, its corresponding reduced order system \((\bar{E}_1, I)\) is stable and index-zero. Since \((X_2, Y_2)\) is a full rank decomposition of \(E_2\) and \((A_2, E_2)\) is stable and index-two, its corresponding reduced order system \((\bar{E}_2, I)\) is stable and index one. Theorem 3 now guarantees GUES of the reduced-order switched system. Theorem 4 now implies the same stability properties for the original switched system (51). Q.E.D.

**Comment 2:** Clearly the continuity assumption on \(Y_{\sigma}^T x\) restricts the applicability of our results to certain decompositions of \(E_i\), since full rank decompositions are in general non-unique. Note however that, for any system, if \(E_i = E\) for all \(i\) and \(Ex\) is continuous then \(Y^T x\) is continuous for any full rank decomposition \((X, Y)\) of \(E\).

**IV. NUMERICAL EXAMPLES**

**Example 2 (Switching between an index-zero and an index-one descriptor system):** Consider a switched system of the form (31) where \(x(\cdot)\) is continuous and

\[
E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 4 \pi \\ -4 \pi & -4 \end{bmatrix}
\]

\[
E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 4 \pi \\ -\pi & -1 \end{bmatrix}
\]

Note that \((E_1, A_1)\) is a stable index-zero system whereas \((E_2, A_2)\) is a stable index one descriptor system whose consistency space \(\mathcal{G}_2\) is \(\text{Im}\left[1 - k_1 k_2^T\right]\) where \(k_1 = \pi - 1\) and \(k_2 = 1/(4 \pi + 1)\); note that \(\mathcal{G}_2\) can be represented by the line \((k_1 k_2) x_1 + x_2 = 0\). Note also that \(A_1^{-1} E_1 - A_2^{-1} E_2 = gh^T\), where \(g^T = [1\ 1/4 \pi]\) and \(h = [-4 \pi/k_3 \ x/k_3]^T\) with \(k_3 = 4 \pi^2 + 1\). The pairs \((A_1^{-1} E_1, g)\) and \((A_1^{-1} E_1, h)\) are controllable and observable, respectively. The eigenvalues of \(A_1^{-1} E_1 A_1^{-1} E_2\) are \((0, 0.0042)\). Hence from Theorem 3 the switched system described above is globally uniformly exponentially stable about zero.

To illustrate GUES of this system we consider a special switching signal. The switching signal cannot be arbitrary, because of the assumption that \(x(\cdot)\) is continuous during switching. When switching from the index-zero system \((E_1, A_1)\) to the index-one system \((E_2, A_2)\) at a time \(t_s\), we must have \(x(t^-) \in \mathcal{G}_2\). However, switching from the index-one system to the index-zero can happen at any arbitrary time.

The restriction to switch only when consistency spaces intersect can be enforced through state dependent switching. However, for the purpose of illustration we consider a switching signal which is combination of state dependent switching and periodic switching. To explain, let \(t_s\) be a time when the trajectory of the index-zero system reaches \(\mathcal{G}_2\), i.e., \(x(t_s) \in \mathcal{G}_2\) or equivalently \((k_1 k_2) x_1(t_s) + x_2(t_s) = 0\). Now, we let (31) switch from \((E_1, A_1)\) to \((E_2, A_2)\), i.e., \(\sigma(t^-) = 1\) and \(\sigma(t^+) = 2\). Every time this switch happens we fix \(\sigma(t) = 2\) for a time period \(T\) before the system switches back to \((E_1, A_1)\).

Now we plot the trajectory of (31) with \(T = 0.2\) seconds and the initial state \(x_0 = [1\ -0.1579]^T \in \mathcal{G}(E_2, A_2)\) using MATLAB (code is available online at [22]). The resulting trajectory is illustrated in Figure 3 for \(0 \leq t \leq 2\) seconds. The trajectory for the index-zero system is represented by the red spiral and the trajectory for the index-one system is along the blue line passing through origin.
Example 3 (Switching between an index-one and an index-two descriptor system):

Consider a switched system of the form (51) where

\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 & 4\pi \\ 0 & -1 & 0 \\ -4\pi & 0 & -4 \end{bmatrix}
\]

\[
E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -k_1k_3 & 0 & 0 \\ 0 & -1 & 0 \\ -4k_1k_3 & -1 & -4k_3 \end{bmatrix}
\]

with \(k_1, k_2\) and \(k_3\) as defined in the previous example. Note that \((E_1, A_1)\) and \((E_2, A_2)\) are a stable index one and index two descriptor systems respectively. A full rank decomposition \((X_1, Y_1)\) of \(E_1\) is given by

\[
X_1 = Y_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T
\]

and a full rank decomposition of \((X_2, Y_2)\) of \(E_2\) is given by

\[
X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T, \quad Y_2 = Y_1.
\]

Now we use Theorem 4 to obtain the equivalent reduced order switched system

\[
\tilde{E}_{\sigma(t)}\tilde{z}(t) = z(t), \tag{53}
\]

where \(\tilde{E}_1 = Y_1^TA_1^{-1}X_1, \tilde{E}_2 = Y_2^TA_2^{-1}X_2\) and \(z(t) = Y^Tx(t)\) with \(Y = Y_1 = Y_2\). Upon evaluating \(\tilde{E}_1\) and \(\tilde{E}_2\) we can observe that (53) is the same as the switched descriptor system described in Example 2. Now, if we assume that \(z(\cdot)\) is continuous during switching then it follows from the conclusions in Example 2 and Theorem 4 that (53) is GUES. One can also use Theorem 5 to deduce GUES.

V. CONCLUSIONS

In this paper we derive stability conditions for a switched system where switching occurs between linear descriptor systems of non-homogeneous indices. To the best of our knowledge, these conditions are some of the first to consider the case of switching between modes of different indices. For specific cases, such as switching between two systems whose indices differ by one, spectral conditions are derived that can be used to check stability of such systems in an elementary manner. Examples are also given to illustrate the use of our results. Future work will consider extending our analysis using non-quadratic Lyapunov functions and also consider switching between systems whose indices differ by more than one.

REFERENCES

[1] Sajja, Surya, M. Corless, Ezra Zeheb, and Robert Shorten. “On dimensionality reduction and the stability of a class of switched descriptor systems.” Automatica 49.6 (2013): 1855-1860.

[2] Sajja, Surya, M. Corless, Ezra Zeheb, and Robert Shorten. “Stability of a class of switched descriptor systems.” In American Control Conference (ACC), 2013, pp. 54-58. IEEE, 2013.

[3] L. Dai, Singular Control Systems. Berlin, Germany: Springer-Verlag,1989.

[4] Campbell, Stephen La Vern. Singular systems of differential equations II. Pitman Publishing (UK), 1982.

[5] Mori, Yoshihiro, Takehiro Mori, and Yasuaki Kuroe. “A solution to the common Lyapunov function problem for continuous-time systems.” Decision and Control, 1997, Proceedings of the 36th IEEE Conference on. Vol. 4. IEEE, 1997.

[6] Liberonz, Daniel, Joao P. Hespanha, and A. Stephen Morse. “Stability of switched systems: a Lie-algebraic condition.” Systems & Control Letters 37.3 (1999): 117-122.

[7] Shorten, Robert, and Fiacre O. Cairbre. “A proof of global attractivity for a class of switching systems using a semi quadratic Lyapunov approach.” IMA Journal of Mathematical Control and Information 18.3 (2001): 341-353.

[8] Zeheb, Ezra, Robert Shorten, and S. Shravan K. Sajja. “Strict positive realness of descriptor systems in state space.” International Journal of Control 83.9 (2010): 1799-1809. (see also errata for this paper available online).

[9] S. Trenn, Distributional differential algebraic equations, PhD-dissertation, Ilmenau University of Technology, Germany, 2009.

[10] Liberonz, Daniel, and Stephan Trenn. “On stability of linear switched differential algebraic equations.” Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on. IEEE, 2009.

[11] Liberonz, Daniel, Stephan Trenn, and Fabian Wirth. “Commutativity and asymptotic stability for linear switched DAEs.” 2011 50th IEEE Conference on Decision and Control and European Control Conference. IEEE, 2011.

[12] Zhai, Guisheng, and Xuping Xu. “A commutation condition for stability analysis of switched linear descriptor systems.” Nonlinear Analysis: Hybrid Systems 5.3 (2011): 383-393.

[13] Zhou, Lei, Daniel WC Ho, and Guisheng Zhai. “Stability analysis of switched linear singular systems.” Automatica 49.5 (2013): 1481-1487.
[14] Mironchenko, Andrii, Fabian Wirth, and Kai Wulff. “Stabilization of switched linear differential algebraic equations and periodic switching.” IEEE Transactions on Automatic Control 60.8 (2015): 2102-2113.

[15] Trenn, Stephan, and Fabian Wirth. “Linear switched DAEs: Lyapunov exponents, a converse Lyapunov theorem, and Barabanov norms.” 2012 IEEE 51st IEEE Conference on Decision and Control (CDC). IEEE, 2012.

[16] Trenn, Stephan. “Distributional differential algebraic equations.” Phd, Technische Universitat Ilmenau, Ilmenau, Germany, 2009.

[17] Trenn, Stephan. “Switched differential algebraic equations.” Dynamics and Control of Switched Electronic Systems. Springer London, 2012. 189-216.

[18] Shorten, Robert, Martin Corless, Kai Wulff, Steffi Klinge, and Richard Middleton. “Quadratic stability and singular SISO switching systems.” IEEE Transactions on Automatic Control 54.11 (2009): 2714-2718.

[19] Owens, David H., and Dragutin Lj Debeljkovic. “Consistency and Liapunov stability of linear descriptor systems: A geometric analysis.” IMA Journal of Mathematical Control and Information 2.2 (1985): 139-151.

[20] Stykel, Tatjana, “Analysis and numerical solution of generalized Lyapunov equations”, PhD, Technische Universitat Berlin, Berlin, Germany, 2002.

[21] MATLAB code for Example 2 (http://smarttransport.ucd.ie/wordpress/wp-content/uploads/DescriptorSystemsExample.zip)