On The Chromatic Numbers of Integer and Rational Lattices

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Abstract

In the present paper, we give new upper bounds for the chromatic numbers for integer lattices and some rational spaces and other lattices. In particular, we have proved that for any concrete integer number \(d\), the chromatic number of \(Z^n\) with critical distance \(\sqrt{2d}\) has a polynomial growth in \(n\) with exponent less than or equal to \(d\) (sometimes this estimate is sharp). The same statement is true not only in the Euclidean norm, but also in any \(l_p\) norm. Besides, we have given concrete estimates for some small dimensions as well as upper bounds for the chromatic number of \(Q^n_p\), where by \(Q_p\) we mean the ring of all rational numbers having denominators not divisible by some prime numbers.

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1 Introduction

By the chromatic number of a metric space \(M\) with forbidden distance (or critical distance) \(d\) we mean the minimal cardinality of a set \(S\) for which there exists a map \(f : M \to S\) such that for every two points \(x, y \in M\) at distance \(d\) we have \(f(x) \neq f(y)\). Notation: \(\chi(M, d)\). For shorthand we write \(\chi(M)\) for \(\chi(M, 1)\).

The chromatic numbers of Euclidean spaces and linear spaces over the rational numbers (if the norm is Euclidean, we denote them by \(Q^n\)) were studied by many authors, see, e.g., [11, 3, 8, 4, 7, 2] and references therein.

The chromatic numbers for integer lattices in \(l_2\) and \(l_1\) norms were studied, in particular, by Z.Füredi and J.Kang [2], where a lower bound exponential in \(n\) was found for \(\chi(Z^n, \sqrt{r})\) for even \(r\) in \(l_2\)-norm, and similarly, for \(\chi(Z^n, \sqrt{r})\) for even \(r\) in \(l_1\)-norm, however, the result really proved dealt with some special case when \(r\) depends on \(n\) (e.g., \(r = 2q, n = 4q - 1\) for some integer \(q\)). Some better estimates were obtained in [12].

It turns out that the best known upper asymptotic estimates for the chromatic number of rational spaces are exactly those known ones for the Euclidean spaces: the chromatic number of \(Q^n\) is known to be bounded from above by \((3 + o(1))^n\) as \(n\) tends to infinity [8, 11].

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In the present paper, we undertake a systematic treatment of integer lattices and some variations of them: lattices over some completions of the ring of integers, lattices over rational numbers with denominators not divisible by some concrete numbers.

The paper is organised as follows.

In the next section, we deal with low-dimensional cases of integer lattices, in particular we find cases when the chromatic number is equal to 3. These results were not previously found in the literature.

The main result of our paper goes in Section 3: we prove that for a fixed \( m \), the chromatic number \( \chi(Z^n, \sqrt{2m}) \) is estimated from above by \( c \cdot n^m \) in any norm \( l_p \), where \( c \) does not depend on \( n \).

This result goes in contrast with the similar result concerning rational lattices because for the latter, it is known that the lower bound is exponential.

Then we revisit some known lower bounds coming from the Frankl–Wilson theorem, which can be used for obtaining lower bounds for integer lattices.

Another interesting case deals with estimates for the chromatic numbers of rational spaces. We give new upper estimates for lattices over rings of rational numbers whose denominators are coprime with 5 and 3, respectively (Theorems 16 and 15).

As a step towards new estimates for rational lattices, we consider lattices over rational numbers with some forbidden denominators.

The paper is concluded by further discussion and open problems.

1.1 Acknowledgements

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2 Low-dimensional integer lattices

The following theorem is evident, see, e.g., [3].

**Theorem 1.** For every \( k \) which can be represented as a sum of two integer squares, one has \( \chi(Z^2, \sqrt{k}) = 2 \). Otherwise, \( \chi(Z^2, \sqrt{k}) = 1 \). Moreover, the same statement is true for \( Z^n \).

For \( Z^3 \), it makes sense to consider only critical distances of type \( \sqrt{4l+2}, l \) is odd: in the case of odd distances we know that the chromatic number does not exceed two by parity reasons. As for the case when we have \( 4l, l \in Z \), under the square root, it is reduced in \( R^3 \) to the case of \( \sqrt{l} \) since every representation of \( 4l \) as a sum of three squares of integers consists of three even squares.

**Theorem 2** (Upper estimate: the universal colouring). For every \( k = 4l + 2, l \in Z \), we have \( \chi(Z^3, \sqrt{k}) \leq 4 \).
Proof. We consider sets of points of the three-dimensional lattice where the sum of the three coordinates is even. The colouring of the other points will be obtained by shifting this colouring by a vector (1, 0, 0).

Let us consider the following “universal 4-colouring” with four colours (0, 0), (0, 1), (1, 0), (1, 1), where with a triple of integer coordinates in \( \mathbb{Z}^3 \) we associate two numbers, the first being the parity of the first coordinate, and the second being the parity of the third coordinate.

It is clear that if two points in the integer 3-dimensional lattice are at distance \( \sqrt{4l + 2} \), then either they have different parity of the \( z \) coordinate, or they have different parity of the \( x \) coordinate. \( \square \)

To get lower estimates we shall often use the Raiskii–Moser spindle \([9],[14]\), and its generalisations. By the two-dimensional spindle (for critical distance 1) we mean a graph on 7 vertices which looks as follows: one vertex \( A \) forms two unit-distance triangles \( ABC \) and \( AB'C' \), there are also two unit-distance triangles \( BCD \) and \( B'C'D' \) for some points \( D \) and \( D' \) which are at distance 1. When trying to colour this spindle with three colours, we get to a contradiction: \( A \) and \( D \) have to have the same colour, the same is true for \( A \) and \( D' \). On the other hand, \( D \) and \( D' \) are at distance 1, so, their colours should be different. One can take the homothety of this spindle for any critical distance, the resulting graph embedding will be called spindle as well.

In rational spaces or integer lattices for concrete critical distances, one usually can not find spindles directly. Nevertheless, the argument can be corrected if one considers the generalised spindle (or Kupavskii spindle). Assume we have two pairs of triangles \((A, B, C, D), (A, B', C', D')\) for some critical distance as above (in spaces of higher dimensions, two pairs of simplices) and that the points \( D \) and \( D' \) are not at a critical distance. Denote the distance \( l(D, D') \) by \( d \). Assume that for every proper colouring there are two points \( \tilde{D}, \tilde{D}' \) of different colours and an isometry of the space which takes \( D \mapsto \tilde{D}, D' \mapsto \tilde{D}' \). Then taking the image of all the points \((A, B, C, D, B', C', D')\), we see that there is no way to colour them with 3 colours.

This statement (in some more generality) for the case of Euclidean spaces was proved by A.B.Kupavskii \([7]\). Certainly, isometries in the integer case have to be treated in a more delicate way than those in the real case.

Now, to prove the lower estimates for \( \mathbb{Z}^3 \) we shall often use the generalized spindle (Kupavskii spindle) construction. Note that in \( \mathbb{Z}^3 \), the equality of distances \( l(D, D') \) and \( l(\tilde{D}, \tilde{D}') \) does not guarantee the existence of such an isometry taking \( D \) to \( \tilde{D} \) and \( D' \) to \( \tilde{D}' \).

To this end, we shall need the following

Lemma 1 (Integral Analogue of the Kupavskii Lemma). Let \( A = (a_1, a_2, a_3) \) be a point in \( \mathbb{Z}^3 \), so that \( a_1 + a_2 + a_3 \) is even and \( \text{GCD}(a_1, a_2, a_3) = 1 \). Assume further, \( m = b_1^2 + b_2^2 + b_3^2 \) is even for some integers \( b_1, b_2, b_3 \) with \( \text{GCD}(b_1, b_2, b_3) = 1 \). Then for every proper colouring (with at least two distinct colours) of \( \mathbb{Z}^3 \) with forbidden distance \( \sqrt{m} \), there exist \( P, Q \in \mathbb{Z}^3 \) and an isometry of \( \mathbb{Z}^3 \) which takes the origin to \( P \) and \( A \) to \( Q \) such that \( P \) and \( Q \) have different colours.
Proof. Indeed, there is a chain \(X_0, X_1, \ldots, X_k\) of points in \(\mathbb{Z}^3\) from \((0,0,0) = X_0\) to \((b_1, b_2, b_3) = X_k\) with all \(X_i, i = 1, \ldots, k-1\) such that for every two adjacent \(X_i, X_{i+1}\) the vector \(X_{i+1} - X_i\) is obtained from the vector \((a_1, a_2, a_3)\) by an isometry of \(\mathbb{Z}^3\). Now, since \(X_0\) and \(X_k\) are colored differently, there exist two adjacent \(X_i\) and \(X_{i+1}\) having different colours.

The existence of such a chain is left for the reader as an exercise. \(\square\)

**Theorem 3.** \(\chi(\mathbb{Z}^3, \sqrt{2}) = 4\).

One can easily construct the spindle: we take the two triangles \((0,0,0), (0,1,1), (1,0,1), (1,1,2)\) and a similar pair of triangles which is obtained from the first pair by permuting the second and the third coordinates. The points \((1,1,2)\) and \((1,2,1)\) are at the distance \(\sqrt{2}\).

Examples of forbidden distances for which the chromatic number in the three-dimensional space is equal to four were well known, see, e.g., [3]. M.Benda and M.Perles in [3] asked the question whether there exists a forbidden distance in \(\mathbb{Q}^3\) for which the chromatic number is equal to 3.

**Theorem 4.** For \(k = 10 + 12l, l \in \mathbb{Z}\), one has \(\chi(\mathbb{Z}^3, \sqrt{k}) = 3\).

Proof. One can easily check that any decomposition of \(10 + 12l\) into a sum of three integer squares looks like \(a^2 + b^2 + c^2\), where modulo 6 reduction of the triple \((a,b,c)\) coincides with one of the following triples (up to order): \((1,3,0), (5,3,0), (3,3,2), (3,3,4)\).

Then we colour the points with even sum of coordinates as follows: for the point with coordinates \(x, y, z\) we take the residue classes of \(x+y+z\) modulo 6, which provides a three-colouring. Analogously one gets a three-colouring for the set of those points whose sum of coordinates is odd. \(\square\)

Now, we turn to those forbidden distances for \(\mathbb{Z}^3\) for which the chromatic number is equal to four.

**Theorem 5.** If \(m = a^2 + ab + b^2\) for some integers \(a, b\) then \(\chi(\mathbb{Z}^3, \sqrt{2m}) = 4\). In particular, let \(p = 6k + 1\) be a prime number for an integer \(k\). Then \(\chi(\mathbb{Z}^3, \sqrt{2p}) = 4\).

Proof. Indeed, assume first \(m = a^2 + ab + b^2\) for coprime \(a, b\).

We suppose our space is 3-colourable and want to get a contradiction. Let us first assume exactly one of \(a\) and \(b\) is odd, without loss of generality, assume \(a\) is odd, \(b\) is even.

We have \(2m = (a^2 + b^2 + (a + b)^2)\). So, the distance \(\sqrt{2m}\) is realised by vectors with three coordinates, whose difference has coordinates equal to \(\pm a, \pm b, \pm(a+b)\) up to order. So, we shall try different vectors to construct the generalised integer spindle.

First, take the two triangles \(ABC, BCD\) with the following vertices \(A = (0,0,0), B = (a,b,a+b), C = (-b,a+b,a), (a-b,a+2b,2a+b) = D\).

Now, we may get other pairs of triangles with \(A = (0,0,0)\) by permuting the coordinates and taking minus sign for \(a\) and/or \(b\). For example, there is a pair of triangles with the free end \(D' = (-a-b,-2a+b,-a+2b)\). Now, it is easy to see that \(GCD(-a-b,-2a+b,-a+2b)\) is either 1 or 3. If it is 3, then by changing \(b\) to \(-b\), we get \(GCD(a+b,-2a-b,-a-2b) = 1\), and we get the desired spindle.

In the case when both \(a, b\) are odd, we notice that the same pair of triangles can be obtained starting with \((a,a+b)\), where \((a+b)\) is even.
In the case when \( a, b \) are not coprime, take \( c = \text{GCD}(a, b) \), \( a = a'c \), \( b = b'c \) and construct analogous spindles for the sublattice with all coordinates divisible by \( c \).

Now we apply Lemma 1 and see that after applying some isometry to \( \mathbb{Z}^3 \), the images \( \tilde{D} \) and \( \tilde{D}' \) will have different colours. Taking all images of \( A, B, C, D, B', C', D' \) we get a contradiction to 3-colouring of the space. \( \square \)

Collecting the above results about colourings of \( \mathbb{Z}^3 \), we get the following

**Theorem 6.** We have:

1. \( \chi(\mathbb{Z}^3, \sqrt{m}) = 2 \) if and only if \( m \) is odd;
2. for even \( m \) we have \( \chi(\mathbb{Z}^3, \sqrt{m}) \) is either 3 or 4;
3. If \( m \equiv 10(\text{mod } 12) \), then \( \chi(\mathbb{Z}^3, \sqrt{m}) = 3 \);
4. If \( m = 2(a^2 + b^2 + ab) \), \( a, b \in \mathbb{Z} \), then \( \chi(\mathbb{Z}^3, \sqrt{m}) = 4 \);
5. \( \chi(\mathbb{Z}^3, \sqrt{m}) = \chi(\mathbb{Z}^3, 2\sqrt{m}) \).

The only statement of the above theorem, we haven’t yet proved, is 2. We prove it in several steps.

a) It suffices to prove it for \( m = 2p \) for prime \( p \).

b) Let \( m = 2p = a^2 + b^2 + c^2 \) for \( \text{GCD}(a, b, c) = 1 \).

c) From exercise on page 4, it follows that there is a chain in \( \mathbb{Z}^3 \) from the origin to \((0, 1, 1)\) with every two adjacent nodes at distance \( \sqrt{m} \).

d) If there is a chain of odd length \( l \), then we easily construct analogous chains from the origin to \((1, 0, 1)\) and from \((1, 0, 1)\) to \((0, 1, 1)\) which leads to a closed chain of length \( 3l \) which contradicts 2-colourability.

e) Assume the chain from c) has even length. Then, there is a chain in \( \mathbb{Z}^3 \) of even length (with distance \( \sqrt{m} \) between two adjacent points) from the origin to every point with even sum of coordinates.

In particular, there is a chain of even length from the origin to \((a+1, b, c+1)\). Thus, there is a chain of odd length from the origin to \((1, 0, 1)\). From d) we get a contradiction with 2-colourability.

The theorem is proved.

The first critical distance which does not fit into the list above is \( \sqrt{30} \).

**Conjecture 1.** There are no other examples of the chromatic number 3, in other words, \( \chi(\mathbb{Z}^3, \sqrt{m}) = 3 \) only for those \( m \) which can be represented in the form \( 2^{2k} \cdot l \), where \( l \equiv 10 \text{ mod } 12 \).

Now, let us pass to the dimensions 4 and 5.

**Theorem 7.** We have:

1. (A.B.Kupavskii) For \( k = 4l + 2, l \in \mathbb{Z} \), one has \( \chi(\mathbb{Z}^4, \sqrt{k}) \leq 4, \chi(\mathbb{Z}^5, \sqrt{k}) \leq 8 \).
2. \( \chi(\mathbb{Z}^4, \sqrt{8k}) = \chi(\mathbb{Z}^4, \sqrt{2k}) \).
3. \( \chi(\mathbb{Z}^4, \sqrt{l}) \leq 4 \) for odd \( l \).
Proof. To prove the first statement, it suffices to colour the unit cube \( \{0,1\}^4 \) (resp., \( \{0,1\}^5 \)) corresponding to the parities of the coordinates. Indeed, if two points in \( \mathbb{Z}^4 \) (resp., \( \mathbb{Z}^5 \)) have the same parity for all coordinates, then the square of the distance between these points is divisible by four. Besides, it suffices to colour only one half of the cube with the sum of coordinates being even (the “odd” part of the cube is coloured similarly). So, in \( \mathbb{Z}^4 \) (in fact, in \( \{0,1\}^4 \)) we colour 8 points \((a, b, c, d)\), \(a, b, c, d \in \mathbb{Z}^2\), \(a + b + c + d \equiv 0 \mod 2\), with four colours in such a way that every two opposite points \((x, y, z, t)\) and \((1-x, 1-y, 1-z, 1-t)\) have the same colour. For the 5-dimensional case, it suffices to use this four-colouring for the first four coordinates and add an independent colour representing the parity of the fifth coordinate: in total, we get an 8-colouring.

The second result follows from the fact that the sum of squares of four integer numbers, at least one of which is odd, is never divisible by 8, so, the problem is reduced to the case when all coordinates are even.

The upper bound in the third case can be obtained as follows. For the colouring we take two modulo two residue classes. The first class is equal to the parity of the first coordinate. The second one is equal to the sum of parities of \( \left\lfloor \frac{x_i}{2} \right\rfloor \) over all \( i = 1, 2, 3, 4 \).

Remark 1. First note that the estimate given above give a universal covering for all forbidden distances of one of three types listed in the formulation of the theorem. The above proof of the first statement can be generalized for higher dimensions. We shall consider the question of upper bounds for \( \chi(\mathbb{Z}^n, 4k + 2) \) in a separate paper.

3 For every \( m \) the growth of \( \chi(\mathbb{Z}^n, \sqrt{2m}) \) is polynomial in \( n \) of degree at most \( m \)

It is well known, see, e.g., [11], that for rational spaces the lower estimates for the chromatic number grow exponentially as the dimension tends to infinity. Below we prove that in the case of integer lattices it is never so for any concrete forbidden distance.

Let us start with a well known theorem about integer lattices, see, e.g., [3]. The colouring by a scalar product will be used for the proof of the main theorem.

The following theorem is well known.

Theorem 8. The growth of \( \chi(\mathbb{Z}^n, \sqrt{2}) \) is linear as \( n \) tends to infinity.

Proof. To get a lower bound (see [5]), let us take the set of points in \( \mathbb{Z}^n \) with exactly one non-zero coordinate equal to \( \pm 1 \) and the other coordinates all equal to zero, then it can not be coloured with less than \( n \) colours for any \( n \geq 2 \).

The upper estimate is established by the following colouring. In \( \mathbb{Z}^n \), let us consider the following vector: \( v = (1, 3, 5, 7, \ldots, 2n-1) \). For every \( u \in \mathbb{Z}^n \), let us consider the scalar product \( \langle u, v \rangle \). It is clear that if two integer points \( u_1, u_2 \) are at distance \( \sqrt{2} \), we have \( \langle u_1, v \rangle \neq \langle u_2, v \rangle \). More precisely, the difference of values of \( \langle \cdot, v \rangle \) is an even number whose absolute value is between 2 and \( 4n - 4 \). Thus, if we take the residue class of this scalar product modulo \( 4n - 2 \) for the colouring, we get a proper \( (2n - 1) \)-colouring. \( \square \)
The idea of colouring by using scalar products taken modulo some large integer will later be used in some more complicated situations. In particular, it will be used for our main result, the polynomial upper bounds for the chromatic number of integer lattices with a fixed critical distance.

**Theorem 9.** For every fixed $m$ the upper estimate for $\chi(\mathbb{Z}^n, \sqrt{2m})$ in any norm $l_\alpha$ is polynomial in $n$ of degree at most $m$.

Before proving this general estimate which relies on some deep additive combinatorics, we shall give an explicit colouring for the following partial case.

**Statement 1.** $\chi(\mathbb{Z}^n, 2)$ grows quadratically as $n \to \infty$.

Let us prove the upper estimate. The lower estimate is in fact well known and will be proved later. Let $n$ be an integer number. Let $p$ be a prime such that $p \leq n \leq 2p$.

We shall prove the quadratic upper bound for prime $p$ which obviously yields the quadratic upper bound as $n \to \infty$. Consider the set $(k, a^k \mod p)$ of $p$ elements from the abelian group $S = \{0, \ldots, p-1\} \times \{0, \ldots, p-1\}$: where $k$ runs over $\{1, \ldots, p-1\}$ and $a$ is a primitive $(p-1)$-th root of unity in $\mathbb{Z}_p$.

It can be easily seen that for any four distinct elements $a, b, c, d$ from the subset described above we have $a - b \neq c - d$. Indeed, if for some $e, f, g, h \in \mathbb{Z}_p$ we have $f - e = h - g$ and $h \neq f, e \neq f$ we see that $a^f - a^e$ differs from $a^h - a^g$ by multiplication by $a^{f-h}$.

We have constructed a set (abelian group) with no solution to $a - b = c - d$ for distinct $a, b, c, d$. Now, we shall modify this set a little bit to get rid of solutions of some simpler equations.

Now, take the set $S' \subset \mathbb{Z} \times \mathbb{Z}$ of integer numbers $(4k, a^k \mod p)$ where $a^k \mod p$ is treated as an integer between $0$ and $p-1$ (we use the inclusion $\mathbb{Z}_p \subset \mathbb{Z}$).

**Lemma 2.** For every four distinct elements $a, b, c, d \in S'$ we have:

1. none of the sums $\pm a \pm b \pm c \pm d$ is equal to zero.
2. the absolute value of the first coordinate of the sum $\pm a \pm b \pm c \pm d$ does not exceed $16p$, and the absolute value of the second coordinate does not exceed $4p$.

**Proof.** The second statement is evident.

We have proved that $a - b = c - d$ for $a, b, c, d \in S'$ implies $a = c$ or $a = b$. The equation $a + b + c + d = 0$ has no solutions because of positivity of $a, b, c, d$, and $a + b + c - d \neq 0$ follows from a modulo 4 argument.

Now, considering $S'$ as a subset of the abelian group $S'' = \mathbb{Z}_{16p+1} \times \mathbb{Z}_{4p+1}$, we see that for every four distinct elements $a, b, c, d \in S' \subset S''$ we have $\pm a \pm b \pm c \pm d \neq 0 \in S''$. We shall call these $p$ elements constituting the subset $S' \subset S''$ the distinguished elements of $S''$. Denote these distinguished elements from $S''$ by $q_1, \ldots, q_p$. They form a vector which will be used to construct the desired colouring.

Now, for each vector $x = (x_1, \ldots, x_p) \in \mathbb{Z}^p$ let $x'_j$ be the mod $p$ residue class of $x_j$ considered as an integer. With $x$ we associate the element (colour) $f(x) = \sum x'_j \cdot q_j \in S''$ of the group $S''$.

**Lemma 3.** If the distance between two points $x, \tilde{x}$ is equal to 2 then $f(x) \neq f(\tilde{x})$ in $S''$. 

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Proof. Indeed, since all elements $q_i$ are non-zero, two points having all coordinates but one equal and one coordinate which differs by two, get different colours. If two points have all coordinates but four equal and in each of four coordinates the difference is $\pm 1$, these points have different colours by Lemma 2.

Now, taking into account that $|S'|$ grows as $n^2$ as $n$ tends to infinity, we get the claim of Statement 1.

Let us now return to the proof of Theorem 9. We shall prove this theorem for the $l_2$-norm. The construction used in the proof is actually the same for all norms $l_\alpha$.

We shall start with the lower estimate. Let $\mathcal{M}$ be a subset of integers of cardinality $N = |\mathcal{M}|$. Fix an integer number $m$. The following question has been studied by many authors, see, e.g. [10, 15, 16] and references therein.

Which is the largest cardinality of the subset $\mathcal{M}' \subset \mathcal{M}$ for which there are no non-trivial solutions of the equation

$$a_1 + \cdots + a_m - a_{m+1} - \cdots - a_{2m} = 0, \quad (1)$$

where $a_i, i = 1, \ldots, 2m \in \mathcal{M}$? What can one say when $N$ tends to infinity?

The answer of course depends on the definition of non-trivial solution. We shall adopt the definition from [10] (sets with no non-trivial solutions to similar linear equations are called Sidon sets).

A solution to (1) is said to be trivial if there are exactly $l$ different elements among $a_j$, and if we fix one concrete $a_j$ and take all $a_k$ not equal to any of $a_j$ to be 0, we shall still get a solution.

For example, for $m = 4$, the solution $a_1 = a_3 = 1, a_2 = a_4 = 2$ is trivial, whence $a_1 = 0, a_2 = 2, a_3 = a_4 = 1$ is not.

In [10] the following statement is proved

**Statement 2.** There is an infinite sequence of abelian groups $\mathcal{M}_N$ and their subsets $\mathcal{M}'_N$ such that there are no non-trivial solution of (1) for elements from these subsets and $n$ grows as $(1 + o(1))N^{1/m}$ where $N$ and $n$ are cardinalities of $\mathcal{M}_N$ and $\mathcal{M}'_N$, respectively.

Note that if $N$ is large enough, we may assume $n > N \frac{m}{2}$. Consequently, if we take a specific $n$ large enough, then $N$ can be chosen to be not greater than $2n^m = O(n^m)$.

Now notice that non-trivial solutions of (1) can actually serve as solutions of many other equations (2), see below.

For example, if in some set $\mathcal{M}' \subset \mathcal{M}$ we have three elements $a, b, c$ forming an arithmetic progression $c + a = 2b$, then this gives rise to a non-trivial solution of (1): we set $a_1 = a, a_2 = c, a_3 = b, a_4 = b$.

Moreover, we have the following obvious

**Lemma 4.** Let $k < m$ and let $\alpha_1, \ldots, \alpha_k$ be a collection of integer numbers, $\sum |\alpha_i| < m$ and $\sum \alpha_i = 0$. Then every solution to

$$\sum_{i=1}^{k} \alpha_i b_i = 0 \quad (2)$$

gives rise to a solution of (1).
Proof. Indeed, collect all positive \(\alpha_i\) and all negative \(\alpha_j\) separately. Let \((b_1,\ldots,b_k)\) be a solution to \((2)\). For every positive \(\alpha_i\), we take \(\alpha_i\) variables from \(a_1,\ldots,a_m\) to be equal to \(b_i\), and for every negative \(\alpha_j\), we take \(-\alpha_j\) elements from \(a_{m+1},\ldots,a_{2m}\) to be equal to \(b_j\). It is possible to choose coordinates of these elements all distinct because \(\sum |\alpha_i| < m\). We set the remaining coordinates \(a_k\) to be 0. The claim follows.

Now, we can modify the set \(\mathcal{M}'\) as follows. Let \(\tilde{\mathcal{M}}' = \mathcal{M}' + s\), where the addition of \(s\) denotes the shift by a large positive integer number. This number will be chosen in such a way that the ratio between the minimal element of \(\mathcal{M}' + s\) and the maximal element of \(\mathcal{M}' + s\) is strictly greater than \(\frac{m-2}{m}\). We first treat \(\mathcal{M}'\) as a subset of \(\mathbb{N}\). Of course, \(s\) grows linearly with respect to \(n\). This will be needed to avoid solutions of equations \((2)\) where the sum of coefficients is non-zero. This leads to an extension \(\tilde{\mathcal{M}}\) of the group \(\mathcal{M}\) which will be taken to be a cyclic group \(\mathbb{Z}_{2f(s)}\) where \(f(s)\) is larger than the absolute value of the maximal element of \(\mathcal{M}\) multiplied by \(m+1\).

Theorem 10. Let \(\beta_i, i = 1,\ldots,k\) be coefficients such that \(\sum |b_i|\) is even and \(\sum \beta_i \neq 0\). There are no nontrivial solutions to \((7), (2)\) in \(\tilde{\mathcal{M}}'\); neither there are any solutions to any of the equations

\[
\sum_{i=1}^{k} \beta_i c_i = 0.
\]

In other words, having constructed a group and its subset with no solutions of \((1)\) and \((2)\) with the sum of coefficients equal to zero, we can easily forbid solutions to all equations where the sum of coefficients is not equal to zero just by shifting this set by some function \(\mu(m)\) which does not depend on \(n\).

Now we are ready to prove the main theorem. First note that any decomposition of an even \(n\) into sum of squares of integer numbers is a set of numbers which can serve as coefficients of the equations of the type \((2)\) or \((3)\). Moreover, when substituting elements from \(\mathcal{M}'\) treated as integer numbers to \((1), (2)\) or \((3)\) we get an integer number whose absolute value is less than \(\lambda(m) \cdot n^m\), where \(\lambda(m)\) is some function of \(m\) which does not depend on \(n\).

Fix an positive even integer \(m\).

Let us take all possible representations of \(n\) as the sum of squares \(\sum n_i^2\) of integer numbers. Such a representation contains at most \(n\) summands, moreover, the sum of these numbers is even.

Let us choose the set \(\mathcal{M}'\) of cardinality \(n\) and the abelian group \(\mathcal{M}' \supseteq \mathcal{M}\) of cardinality \(|\tilde{\mathcal{M}}'| = O(n^m)\) to avoid solutions of \((1), (2)\). By shifting them by a large integer number we get the group \(\tilde{\mathcal{M}}\) and the set \(\tilde{\mathcal{M}}'\) in it avoiding solutions of \((3)\) as well.

Enumerate elements of \(\tilde{\mathcal{M}}'\) by \(x_1,\ldots,x_n\) and fix the vector \((x_1,\ldots,x_n)\) in \(\mathbb{Z}^n\).

Let us associate with points of \(y = \mathbb{Z}^n\) integer numbers \(\langle x, y \rangle\). If two points \(y, y'\) are at distance \(\sqrt{2m}\) then \(\langle y - y', x \rangle \neq 0\). Indeed, the coordinates of \(y - y'\) form a decomposition of \(n\) into a sum of squares, and \(x_1,\ldots,x_n\) are chosen in such a way that none of the equations of types \((2), (3)\) holds. So, the scalar products are different.

Besides, \(\langle y_1, x \rangle\) does not exceed \((\max_{x \in \tilde{\mathcal{M}}'} |x|) \cdot m\) which grows as \(O(n^m)\).

Thus, taking the residue class of this scalar product modulo \(\lambda(m) \cdot n^m + 1\), we get a colouring of \(\mathbb{Z}^n\) with forbidden distance \(\sqrt{2m}\) in the \(l_1\) norm.

The proof in any other norm \(l_n\) with the same estimate is similar.
4 Lower Estimates for The Chromatic Numbers of Integer Lattices

We have proved upper polynomial estimates for $Z^n$. Now, we are going to prove polynomial lower estimates. We shall show that for many fixed $m$, the exponents $c \cdot n^m$ for $\chi(Z^n, \sqrt{2m})$ are optimal.

Let $S$ be a metric space, let $d$ be a critical distance. By a $(M, D)$-critical configuration we mean a subset $M \subset S$ of cardinality $M$ such that for every subset $M' \subset M$ with no two points $a, b \in M'$ with critical distance $l(a, b) = d$, the cardinality $|M'|$ is at most $D$.

By the pigeon-hole principle, if there is a critical $(M, D)$-configuration in $S$ then $\chi(S, d) \geq \chi(M, d) \geq \frac{M}{D}$.

The lower estimate from $\chi(Z^n, 2)$ is in fact well-known. We present it here for consistency.

We shall present a concrete critical configuration. Fix a natural number $n$, and let $S = Z^n$, $M$ be a set of all points from $Z^n$ having three coordinates equal to 1 and the others equal to zero, and let $M'$ be a subset of $M$ where no two points are at a distance two. Clearly, $|M| = \binom{n}{3}$. Every point from $M'$ can be considered as a triple of those coordinates equal to one. Now, the fact that two points $x$ and $y$ from $M'$ are at distance not equal to two means that the corresponding triples are either disjoint or have exactly two common coordinates. Now, it is easy to see, that the number of such elements from $M'$ can not exceed $n$. So, $M$ is an $(M, D)$-critical configuration, where $M = |M| = \frac{n(n-1)(n-2)}{6}$ and $D = n$.

Thus, the chromatic number for $Z^n$ with critical distance 2 is greater than or equal to $\frac{n(n-1)(n-2)}{6}$.

The methods of finding $(M, D)$-critical configuration are widely used for establishing lower bounds for the chromatic number of lattices in arbitrary dimension, the main tool being the well known Frankl–Wilson theorem [6] with its further modifications (see [11]).

**Theorem 11** (The Frankl–Wilson Theorem). Let us fix an $n$-element set $N = \{1, \ldots, n\}$. Let $p$ be a prime power, and let $a$ be a positive integer number, $a < 2p$. Furthermore, let $M$ be a collection of $a$-element subsets of $N$ such that the cardinality of the intersection of any two of them is not equal to $a - p$. Then $|M| \leq \binom{n}{p-1}$.

For modifications of the Frankl-Wilson Theorem, see [13].

Now, we shall see that for many numbers $2m$ the exponent $m$ in the upper estimate $n^m$ for the chromatic number $\chi(Z^n, \sqrt{2m})$ is optimal for $m$ being a power of a prime number. Indeed, we consider subsets of $N$ as elements of $Z^n$ with coordinates being equal to 1 and 0 ($i$-th coordinate is equal to 1 if and only if $i \in N$).

The fact given below, written in [11]; however, the same argument was treated as a lower estimate for the chromatic number of $R^n$, not of $Z^n$.

**Theorem 12.** Let $p$ be a power of a prime number. Then $\chi(Z^n, \sqrt{2p}) \geq \left( \frac{n}{\binom{n}{p-1}} \right)$.

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Proof. Indeed, it suffices to take all vectors of length $2p - 1$ and forbid the intersection $p - 1$, or, equivalently, forbid the distance $\sqrt{2p}$. The claim follows.

Thus, for $p$ being a power of a prime number, we proved that the growth of $\chi(\mathbb{Z}^n, \sqrt{2p})$ is polynomial in $n$ of degree $p$.

\section{Estimates for rational lattices $\mathbb{Q}^n$.}

\textbf{Theorem 13.} For rational $k$ which can be represented as a sum of two squares of rational numbers one has $\chi(\mathbb{Q}^2, \sqrt{k}) = 2$. Otherwise $\chi(\mathbb{Q}^2, \sqrt{k}) = 1$.

Let $m = \frac{q}{q} \sqrt{l}$, where $l$ is an odd number representable a sum of two squares of integers. Then
\[
\chi(\mathbb{Q}^3, m) = 2;
\chi(\mathbb{Q}^4, m) \leq 4.
\]

Proof. The statements about $\mathbb{Q}^2$ are obvious. Let us pass to $\mathbb{Q}^3$. Without loss of generality we may assume that $m = \sqrt{l}$.

Let us colour $\mathbb{Q}^3$ with two colours, as follows. For a point $(a, b, c) \in \mathbb{Q}^3$, take their minimal common denominator $d$ and write $a = \frac{a'}{d}, b = \frac{b'}{d}, c = \frac{c'}{d}$. For colouring, we take the modulo 2 residue class of $a' + b' + c'$. Obviously, if two such points are at distance $\sqrt{l}$, then they have different colours.

Now, notice the following. If at least one of the numbers $a, b, c \in \mathbb{Q}$ has even denominator in its reduced fraction, then the sum of squares of $a, b, c$ can not be an integer. Likewise, if at least one denominator contains $2^k$, then the sum of three squares can not be a square of a rational number with denominator whose power of 2 is less than $k$.

Now we make the fact that the points with different exponents of 2 in denominators “do not interfere”: the sum of three squares of integer numbers, at least one of which is odd, can not be even.

Thus, the above colouring can be extended to points of the rational lattice having coordinates with even denominators. Indeed, we first shift the initial lattice points with their colours by vectors $(\frac{1}{4}, 0, 0), (0, \frac{1}{4}, 0), (0, 0, \frac{1}{4})$, then we shift our colouring by coordinate vectors of length $\frac{1}{4}$, etc.

To get the estimate for $\mathbb{Q}^4$, we shall first colour points with no coordinate having denominator divisible by four. Every vector $v$ of such sort can be represented as $(\frac{s}{2^7}, \frac{b}{2^7}, \frac{c}{2^7}, \frac{d}{2^7})$, where $s$ is an odd number (possibly, some of $a, b, c, d$ are even). With such a point we associate the colour $\alpha(v)$ which is equal to residue class of $a$ modulo 2.

Let $v_1 = (\frac{a}{2^7}, \frac{b}{2^7}, \frac{c}{2^7}, \frac{d}{2^7})$ and $v_2 = (\frac{a'}{2^7}, \frac{b'}{2^7}, \frac{c'}{2^7}, \frac{d'}{2^7})$ be two such vectors; $t, s$ are odd. If $|v_1 - v_2| = l$ then we have one of two options: either all $a - a', b - b', c - c', d - d'$ are odd (in this case $\alpha(v) \neq \alpha(v')$), or all these numbers are even.

Let us now define $\beta(v)$ as follows. First we define $\beta(v)$ for points from $\mathbb{Q}^4$ with all coordinates having odd denominators: it is just the parity of the sum of numerators. Then we expand it to points with all denominators of coordinates not divisible by 4 by parallel transports by $(\frac{1}{2^7}, 0, 0, 0), (0, \frac{1}{2^7}, 0, 0), (0, 0, \frac{1}{2^7}, 0), (0, 0, 0, \frac{1}{2^7})$.

Now, we see that if two vectors $(v_1, v_2)$ with denominators not divisible by four are at distance $\sqrt{l}$, then either $\alpha(v_1) \neq \alpha(v_2)$ or $\beta(v_1) \neq \beta(v_2)$. So, we have constructed the four-colouring $\alpha, \beta$ for all points with denominators of coordinates not divisible by 4.
Now, we extend the colouring by shifts by vectors $\frac{1}{2} l$, where $l \geq 2$. Here we use the fact that the sum of squares of four integer numbers can not be divisible by 16 if at least one of them is odd.

**Theorem 14.** Let $m = \sqrt{2l/q}$, where $l$ is an odd number such that $2l$ can be represented as a sum of two integer squares. Then we have
\[ \chi(\mathbb{Q}^4, m) \leq 4, \]
hence, \[ \chi(\mathbb{Q}^3, m) \leq 4. \]

**Proof.** The proof for those points in $\mathbb{Q}^4$ for points whose coordinates have odd denominators, repeats the argument for $\mathbb{Z}^4$ from Theorem 7: instead of parities of integer numbers, we take parities of numerators of fractions with odd denominators.

Then this colouring extends to $\mathbb{Q}^4$ just by shifting it along coordinate vectors of lengths $\frac{1}{2} l^k$, $k > 0$ as in the proof of Theorem 13.

Here one should take into account that the sum of four squares of integer numbers can not be divisible by 8 if at least one of these numbers is odd.

6 Colourings of Some Finite Graphs

Let us consider the fields $\mathbb{Z}_3$ and $\mathbb{Z}_5$; we shall construct graphs $\mathbb{Z}_3^n$ and $\mathbb{Z}_5^m$, where for the (pseudo)metric we take the $l_2$-metric taken modulo 3 (resp., modulo 5).

**Theorem 15.** $\chi(\mathbb{Z}_3^n, 1) \leq c(\sqrt{3})^n$.

**Proof.** The proof is by induction on the dimension $n$. It suffices for us to prove that for $n = 2 + 3k$ we have $\chi(\mathbb{Z}_3^n) \leq 3^{2k+1}$ for positive integers $k$.

For $\mathbb{Z}_3^2$, let us use three colours to colour 9 points: we just take the colour to be the modulo three residue class of the sum of coordinates.

Now, assume we have a proper colouring of $\mathbb{Z}_3^{2+3k}$; let us colour $\mathbb{Z}_3^{5+3k}$ as follows. We colour the first $2 + 3k$ coordinates by using $3^{2k+1}$ colours, and take 9 colours for $\mathbb{Z}_3^3$. The colour for $\mathbb{Z}_3^{5+3k}$ will consist of two components, the one for the first $2 + 3k$ coordinates, and the one for the last three coordinates. The last component will have 9 colours, namely, for $(a, b, c) \in \mathbb{Z}_3^3$ we take the colour to be $(b-a, c-a) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$. If for two points $(a, b, c)$ and $(a', b', c')$ from $\mathbb{Z}_3^3$ we have $b-a \equiv b'-a' \mod 3$ and $c-a \equiv c'-a' \mod 3$, then these points either coincide if $a = a'$, or these points $(a, b, c)$ and $(a', b', c')$ are at a distance three if $a \neq a'$. In any case, this colouring forbids distances congruent to 1 and 2 modulo 3.

We claim that for such a colouring of $\mathbb{Z}_3^{5+3k}$ no two points at distance congruent to 1 modulo 3 have the same colour. Indeed, for two points $x, y \in \mathbb{Z}_3^{5+3k}$ at distance congruent to 1 modulo 3, either the distance between the projections to the first $2 + 3k$ coordinates is congruent to 1 modulo 3, or the distance between the projections to the last 3 coordinates is not congruent to 0 modulo 3. In the first case, the colours of these points have different first component; in the second case, the colours have different second component.

This completes the induction step.

**Theorem 16.** $\chi(\mathbb{Z}_5^n, 1) \leq c'(\sqrt{5})^n$. 

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Proof. We proceed in a way similar to the above. We establish the induction base by colouring \( Z_5 \) with five different colours; then we colour \( Z_2 \) in five colours so that two points share one colour whenever they are at distance congruent to 0 modulo five. Namely, for \((a, b) \in Z_2^2\) we take the colour to be \(a - 2b \mod 5\). Then we proceed by induction: for every two new coordinates we need to multiply the number of colours by 5, and the result follows.

Remark 1. The above estimates remain true if instead of 1 we take any forbidden distance congruent to 1 modulo 3 (congruent to 1 modulo 5, respectively).

7 Estimates for Lattices over Some Algebraic Extensions of \( Z \)

Now, let \( p_1, \ldots, p_k \) be a set of integers. By \( \mathbb{Q}_{p_1, p_2, \ldots, p_k} \) we shall denote the ring of rational numbers with denominators coprime with \( p_1 \ldots p_k \). By \( \mathbb{Q}_{\text{odd}} \) we mean the set of rational numbers with odd denominators only.

Theorem 17. \( \chi(\mathbb{Q}_{\text{odd}}^n, 1) = 2 \). Moreover, for every extension \( K \) of the ring of integers for which there exists a homomorphism \( K \to Z_2 \) we have \( \chi(K^n, 1) = 2 \).

Proof. Indeed, assume all denominators of coordinates are odd. Then for the colouring we take the set of modulo 2 residue classes of numerators.

Theorem 18. \( \chi(\mathbb{Q}_3^n, 1) \leq c(\sqrt[3]{9})^n \), where \( c \) is some universal constant. The same remains true if one replaces \( \mathbb{Q}_3 \) with any subring of \( \mathbb{R} \) admitting a homomorphism to \( Z_3 \).

Theorem 19. \( \chi(\mathbb{Q}_5^n, 1) \leq c'(\sqrt[5]{5})^n \), where \( c' \) is some universal constant. Moreover, the same is true if one replaces \( \mathbb{Q}_5 \) with a subring of the ring of integers admitting a homomorphism to \( Z_5 \).

The two last theorems easily follow from Theorems\[15\] and \[16\]. The idea is to take the coordinates \((x_1, \ldots, x_n) \mod 3\) (respectively, modulo 5) and to use the estimate for \( \chi(Z_3^n) \) (respectively, for, \( \chi(Z_5^n) \)). Here “taking the residue class” means considering the corresponding ring homomorphism.

8 Some Open Problems

We conjecture that all possible critical distances where the chromatic numbers for \( Z^d \) is equal to 3, are those of form \( 2^k \sqrt{12l + 10} \); besides, we conjecture that the chromatic number 3 never occurs for integer lattices in higher dimensions.

The best known upper asymptotic estimate for the chromatic number of rational spaces still remains the same as for Euclidean spaces of the same dimension: it is \((3 + o(1))^n\), see \[8\]; the methods of obtaining these estimates are based on some Voronoï tilings of Euclidean spaces; in other words, these known upper estimates come from tilings of the Euclidean spaces into smaller parts.

Moreover, every lower bound for \( \chi(\mathbb{Q}^n, \sqrt{d}) \) for some concrete \( n, \sqrt{d} \) comes from a concrete finite graph \( \Gamma \) in \( \mathbb{Q}^n \) with critical distance \( \sqrt{d} \).

The possibility to get an exact estimate from a finite graph is exactly the de Bruijn–Erdős theorem \[1\]. If we consider such a graph for \( \mathbb{Q}^n \) and take the common denominator \( D \) of all coordinates of all
points of this graph, we get a homothetic graph $D\Gamma$ in $\mathbb{Z}^n$ with critical distance $D\alpha$. So, all lower estimates for rational lattices actually come from integer lattices.

It would be interesting to apply the argument of the present paper to obtain sharper estimates for $\mathbb{Q}^n$. The direct approach fails because when taking some concrete forbidden distance, one will have to take the maximum over all estimates for $D\alpha$ which tends to infinity as $n$ tends to infinity.

Our estimates for lattices with rational coordinates with some restrictions on denominators are somewhat better but use principally different ideas: some number theoretic properties of modulo $p$ reductions. It would be very interesting to get other estimates for $\chi(\mathbb{Q}^n)$ by combining the two approaches: the one from the present paper and the one using Voronoi tilings.

We have found upper estimates for $\chi(\mathbb{Z}^n, \sqrt{d})$ for every fixed $d$ as $n$ tends to infinity. If we fix a concrete $n$, then we have estimates for odd $d$: 2 colours, for $d$ not divisible by 3: $c_1 \cdot (\sqrt{5})^n$ colours, and for $d$ not divisible by 5 we get $c_2 \cdot (\sqrt{5})^n$ colours. All these upper bounds are better than the best known estimates for rational lattices, which is $(3 + o(1))^n$. So, it would be interesting to find an upper bound for $\max_{d \mid 30}(\chi(\mathbb{Z}^n, \sqrt{d}))$, where the maximum is taken over all $d$ divisible by 30. Possibly, there is a way to elaborate similar methods for other prime numbers; however, an argument for numbers whose denominators are not divisible by 7 similar to those given for numbers whose denominators are coprime with 2, 3, 5 will give an estimate which is worse than the well-known one for rational lattices.

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