On the Equitable Choosability of the Disjoint Union of Stars

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Abstract
Equitable k-choosability is a list analogue of equitable k-coloring that was introduced by Kostochka et al. (J Graph Theory 44:166–177, 2003). It is known that if vertex disjoint graphs $G_1$ and $G_2$ are equitably k-choosable, the disjoint union of $G_1$ and $G_2$ may not be equitably k-choosable. Given any $m \in \mathbb{N}$ the values of $k$ for which $K_{1,m}$ is equitably k-choosable are known. Also, a complete characterization of equitably 2-choosable graphs is not known. With these facts in mind, we study the equitable choosability of $\sum_{i=1}^{n} K_{1,m_i}$, the disjoint union of $n$ stars. We show that determining whether $\sum_{i=1}^{n} K_{1,m_i}$ is equitably choosable is NP-complete when the same list of two colors is assigned to every vertex. We completely determine when the disjoint union of two stars (or $n \geq 2$ identical stars) is equitably 2-choosable, and we present results on the equitable k-choosability of the disjoint union two stars for arbitrary $k$.

Keywords Graph coloring · Equitable coloring · List coloring · Equitable choosability

Mathematics Subject Classification 68R10 · 05C15

1 Introduction

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow [5] and [19] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots \}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. If $G$ is a graph and $S \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$. We write $\Delta(G)$ for the maximum degree of a vertex in $G$. We write $K_{n,m}$ for complete bipartite graphs with partite sets of size $n$ and $m$. When $G_1$ and $G_2$ are vertex disjoint graphs, we write $G_1 + G_2$ or $\sum_{i=1}^{2} G_i$ for the disjoint union of $G_1$ and $G_2$. When $f$ is a function, we use $\text{Ran}(f)$ to denote the range of $f$.

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In this paper we study a list analogue of equitable coloring known as equitable choosability which was introduced in 2003 by Kostochka et al. [12]. More specifically, we study the equitable choosability of the disjoint union of stars. A star is a complete bipartite graph with partite sets of size 1 and $m$ where $m \in \mathbb{N}$ (i.e., a copy of $K_{1,m}$). We will occasionally need to consider complete bipartite graphs that are copies of $K_{1,0}$. In such cases, we assume $K_{1,0} = K_1$ (i.e., a complete bipartite graph with partite sets of size 1 and 0 is a complete graph on one vertex). We will now briefly review equitable coloring and list coloring.

1.1 Equitable Coloring and List Coloring

1.1.1 Equitable Coloring

Equitable coloring is a variation on the classical vertex coloring problem that began with a conjecture of Erdős [2] in 1964 which was proved in 1970 by Hajnál and Szemerédi [6]. In 1973 the notion of equitable coloring was formally introduced by Meyer [14]. A proper $k$-coloring $f$ of a graph $G$ is said to be an equitable $k$-coloring if the color classes associated with $f$ differ in size by at most 1. It is easy to see that for an equitable $k$-coloring, the color classes associated with the coloring are each of size $\frac{|V(G)|}{k}$ or $\lfloor \frac{|V(G)|}{k} \rfloor$. We say that a graph $G$ is equitably $k$-colorable if there exists an equitable $k$-coloring of $G$. Equitable colorings are useful when it is preferable to form a proper coloring without under-using or over-using any color (see [7, 8, 16, 17] for applications).

Unlike the typical vertex coloring problem, if a graph is equitably $k$-colorable, it need not be equitably $(k+1)$-colorable. Indeed, $K_{2m+1,2m+1}$ is equitably $k$-colorable for each even $k$ less than $2m+1$, it is not equitably $(2m+1)$-colorable, and it is equitably $k$-colorable for each $k \geq 2m+2 = \Delta(K_{2m+1,2m+1}) + 1$ (see [13] for further details). In 1970, Hajnál and Szemerédi [6] proved: Every graph $G$ has an equitable $k$-coloring when $k \geq \Delta(G) + 1$. In 1994, Chen et al. [1] conjectured that this result can be improved by 1 for most connected graphs by characterizing the extremal graphs as: $K_m$, $C_{2m+1}$, and $K_{2m+1,2m+1}$. Their conjecture is still open and is known as the $\Delta$-Equitable Coloring Conjecture ($\Delta$-ECC for short).

Importantly, when it comes to the disjoint union of graphs, equitable colorings on components can be merged after appropriately permuting color classes within each component to obtain an equitable $k$-coloring of the whole graph.

**Theorem 1** [20] Suppose $G_1, G_2, \ldots, G_t$ are pairwise vertex disjoint graphs and $G = \sum_{i=1}^{t} G_i$. If $G_i$ is equitably $k$-colorable for all $i \in [t]$, then $G$ is equitably $k$-colorable.

On the other hand, an equitably $k$-colorable graph may have components that are not equitably $k$-colorable; for example, the disjoint union $G = K_{3,3} + K_{3,3}$ with $k = 3$. With this in mind, Kierstead and Kostochka [11] proposed an extension of the $\Delta$-ECC to the disjoint union of graphs.
1.1.2 List Coloring

List coloring is another variation on the classical vertex coloring problem introduced independently by Vizing [18] and Erdős et al. [3] in the 1970s. For list coloring, we associate a list assignment $L$ with a graph $G$ such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$ (we say $L$ is a list assignment for $G$). The graph $G$ is $L$-colorable if there exists a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a proper $L$-coloring of $G$). A list assignment $L$ is called a $k$-assignment for $G$ if $|L(v)| = k$ for each $v \in V(G)$. We say $G$ is $k$-choosable if $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$.

Suppose that $L$ is a list assignment for a graph $G$. A partial $L$-coloring of $G$ is a function $f : D \to \bigcup_{v \in V(G)} L(v)$ such that $D \subseteq V(G), f(v) \in L(v)$ for each $v \in D$, and $f(u) \neq f(v)$ whenever $u$ and $v$ are adjacent in $G[D]$. Also, the palette of colors associated with $L$ is $\bigcup_{v \in V(G)} L(v)$. From this point forward, we use $L$ to denote the palette of colors associated with $L$ whenever $L$ is a list assignment. We say that $L$ is a constant $k$-assignment for $G$ when $L$ is a $k$-assignment for $G$ and $|L| = k$ (i.e., $L$ assigns the same list of $k$ colors to every vertex in $V(G)$).

1.2 Equitable Choosability

In 2003 Kostochka et al. [12] introduced a list analogue of equitable coloring called equitable choosability. They used the word equitable to capture the idea that no color may be used excessively often. If $L$ is a $k$-assignment for a graph $G$, a proper $L$-coloring of $G$ is an equitable $L$-coloring of $G$ if each color in $L$ appears on at most $\lceil |V(G)|/k \rceil$ vertices. We call $G$ equitably $L$-colorable when an equitable $L$-coloring of $G$ exists, and we say $G$ is equitably $k$-choosable if an equitable $L$-coloring of $G$ exists for every $L$ that is a $k$-assignment for $G$. It is conjectured in [12] that the Hajnal-Szemerédi Theorem and the $\Delta$-ECC hold in the context of equitable choosability.

Much of the research on equitable choosability has been focused on these conjectures. There is not much research that considers the equitable $k$-choosability of a graph $G$ when $k < \Delta(G)$. In [12] it is shown that if $G$ is a forest and $k \geq 1 + \Delta(G)/2$, then $G$ is equitably $k$-choosable. It is also shown that this bound is tight for forests. Also, in [9], it is conjectured that if $T$ is a total graph, then $T$ is equitably $k$-choosable for each $k \geq \max \{ \lceil \chi_l(T) \rceil, \Delta(T)/2 + 2 \}$ where $\chi_l(T)$, the list chromatic number of $T$, is the smallest $m$ such that $T$ is $m$-choosable. Finally, in [10], it is remarked that determining precisely which graphs are equitably 2-choosable is open.

Furthermore, most results about equitable choosability state that some family of graphs is equitably $k$-choosable for all $k$ above some constant; even though, as with equitable coloring, if $G$ is equitably $k$-choosable, it need not be equitably $(k + 1)$-choosable. It is rare to have a result that determines whether a family of graphs is equitably $k$-choosable for each $k \in \mathbb{N}$. Two examples of results of this form are: $K_{1,m}$ is equitably $k$-choosable if and only if $m \leq \lceil (m + 1)/k \rceil (k - 1)$, and $K_{2,m}$ is equitably $k$-choosable if and only if $m \leq \lceil (m + 2)/k \rceil (k - 1)$ (see [15]).
It is important to note that the analogue of Theorem 1 does not hold in the setting of equitable choosability. For example, we know $K_{1,6}$ and $K_{1,1}$ are equitably 3-choosable, but $K_{1,6} + K_{1,1}$ is not equitably 3-choosable. This fact along with the fact that the equitable choosability of $K_{1,n}$ has been completely characterized motivated us to study the following question which is the focus of this paper.

**Question 2** Suppose $n \geq 2$. For which $k, m_1, \ldots, m_n \in \mathbb{N}$ is $\sum_{i=1}^{n} K_{1,m_i}$ equitably $k$-choosable?

Since $\sum_{i=1}^{n} K_{1,m_i}$ is a forest, we know that it is equitably $k$-choosable whenever $k \geq 1 + \Delta(\sum_{i=1}^{n} K_{1,m_i})/2 = 1 + \max_{i \in [n]} m_i/2$. Even for this simple class of graphs, we do not know what happens when $k$ is smaller than $1 + \max_{i \in [n]} m_i/2$. In this paper, we make some further progress on Question 2 in the case when $k = 2$ and in the case when $n = 2$. We completely answer Question 2 in the case when $n = k = 2$. This can be seen as progress towards understanding which graphs are equitably 2-choosable.

### 1.3 Outline of the Paper and Open Questions

We begin by studying Question 2 in the case of equitable 2-choosability. In Sect. 2 we study the complexity of the decision problem STARS EQUITABLE 2-COLORING which is defined as follows.

**Instance:** An $n$-tuple $(m_1, \ldots, m_n)$ such that $m_i \in \mathbb{N}$ for each $i \in [n]$.

**Question:** Is $\sum_{i=1}^{n} K_{1,m_i}$ equitably 2-colorable?

Perhaps surprisingly, since most coloring problems with 2 colors tend to be easy, we show that STARS EQUITABLE 2-COLORING is NP-complete. In studying when $\sum_{i=1}^{n} K_{1,m_i}$ is equitably 2-choosable, a possible natural starting point is to try to determine: for which $n$-tuples $(m_1, \ldots, m_n)$ is $\sum_{i=1}^{n} K_{1,m_i}$ not equitably 2-colorable and hence not equitably 2-choosable? The fact that STARS EQUITABLE 2-COLORING is NP-complete tells us that this “natural starting point“ should not be pursued unless P = NP.

STARS EQUITABLE 2-CHOOSABLITY is the decision problem whose instances are the same as STARS EQUITABLE 2-COLORING, but it asks the question: Is $\sum_{i=1}^{n} K_{1,m_i}$ equitably 2-choosable? Clearly, this decision problem is closely related to Question 2 in the case when $k = 2$. The following question is open.

**Question 3** Is STARS EQUITABLE 2-CHOOSABLITY NP-hard?

In Sect. 3 we completely characterize when the disjoint union of 2 stars is equitably 2-choosable by proving the following.

**Theorem 4** Let $G = K_{1,m_1} + K_{1,m_2}$ where $1 \leq m_1 \leq m_2$. $G$ is equitably 2-choosable if and only if $m_2 - m_1 \leq 1$ and $m_1 + m_2 \leq 15$. 

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1 One need only consider a constant 3-assignment to see this.

2 See [4] for an overview of complexity results related to equitable coloring.
Theorem 4 makes progress on the task of identifying which graphs are equitably 2-choosable which in general is open (see [10]). It is also worth noting that $K_{1,m_1} + K_{1,m_2}$ is equitably 2-colorable if and only if $|m_2 - m_1| \leq 1$. So, there are infinitely many equitably 2-colorable graphs that are the disjoint union of two stars, but there are only 14 equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of two stars. We end Sect. 3 by completely determining when the disjoint union of $n$ identical stars is equitably 2-choosable.

Theorem 5 Suppose that $n, m \in \mathbb{N}, n \geq 2$, and $G = \sum_{i=1}^{n} K_{1,m_i}$. When $n$ is odd, $G$ is equitably 2-choosable if and only if $m \leq 2$. When $n$ is even, $G$ is equitably 2-choosable if and only if $m \leq 7$.

With these results in mind, the following open question is natural to ask.

Question 6 Suppose that $n$ is a fixed integer such that $n \geq 2$. Are there only finitely many equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of $n$ stars?

For a fixed integer $N$, $N$-STARS EQUITABLE 2-CHOOSABILITY is the decision problem whose instances are $N$-tuples of natural numbers of the form $(m_1, \ldots, m_N)$, and asks the question: Is $\sum_{i=1}^{N} K_{1,m_i}$ equitably 2-choosable? If the answer to Question 6 is yes for a $n \geq 2$, then $N$-STARS EQUITABLE 2-CHOOSABILITY is not NP-hard for $N = n$ unless P=NP. Note that Theorem 4 shows that this is true for $N = 2$.

Finally, in Sect. 4 we study the equitable $k$-choosability of the disjoint union of two stars for arbitrary $k$. We use an extremal choice of a partial list coloring that minimizes the difference of the cardinalities of the sets of uncolored vertices in the two stars along with a greedy partial list coloring process to show the following.

Theorem 7 Let $k \in \mathbb{N}, 1 \leq m_1 \leq m_2$, and $\rho = \lceil (m_1 + m_2 + 2)/k \rceil$. If $m_2 \leq \rho(k - 1) - 1$ and $m_1 + m_2 \leq 15 + \rho(k - 2)$, then $K_{1,m_1} + K_{1,m_2}$ is equitably $k$-choosable.

We also show that the converse of Theorem 7 does not hold. However, Theorem 7 is sharp in a sense. Lemma 12 in Sect. 3 demonstrates that the first inequality in Theorem 7 is necessary for $K_{1,m_1} + K_{1,m_2}$ to be equitably $k$-choosable. However, this necessary condition alone is not sufficient for $K_{1,m_1} + K_{1,m_2}$ to be equitably $k$-choosable. Indeed Proposition 22 in Sect. 4 implies that $K_{1,(k-1)(k^3-k^2)+1} + K_{1,k^3}$, which satisfies the first inequality but not the second inequality, is not equitably $k$-choosable whenever $k \geq 2$. So, if one wishes to determine precisely when $K_{1,m_1} + K_{1,m_2}$ is equitably $k$-choosable for $k \geq 3$, the characterization needs to be stronger than $m_2 \leq \rho(k - 1) - 1$. We suspect however that the second inequality in Theorem 7 can be relaxed quite a bit for $k \geq 3$. This leads us to ask the following question which is a special case of Question 2.

Question 8 For which $k, m_1, m_2 \in \mathbb{N}$, is $K_{1,m_1} + K_{1,m_2}$ equitably $k$-choosable?
2 A Complexity Result

To prove STARS EQUITABLE 2-COLORING is NP-complete, we will use the following well-known NP-complete problem [5]: PARTITION which is defined as follows.

Instance: An \( n \)-tuple \((m_1, \ldots, m_n)\) such that \( m_i \in \mathbb{N} \) for each \( i \in [n] \).

Question: Is there a partition \( \{A, B\} \) of the set \([n]\) such that \( \sum_{i \in A} m_i = \sum_{j \in B} m_j \)?

The following lemma captures the essence of why STARS EQUITABLE 2-COLORING is NP-hard.

**Lemma 9** Suppose that \( n \in \mathbb{N} \) and \((m_1, \ldots, m_n)\) is an \( n \)-tuple such that \( m_i \in \mathbb{N} \) for each \( i \in [n] \) and \( \sum_{i=1}^{n} m_i \) is even. There is a partition \( \{A, B\} \) of the set \([n]\) such that \( \sum_{i \in A} m_i = \sum_{j \in B} m_j \) if and only if \( G = \sum_{i=1}^{n} K_{1, m_i+1} \) is equitably 2-colorable.

**Proof** Throughout this proof, we assume that for each \( i \in [n] \) the bipartition of the copy of \( K_{1, m_i+1} \) used to form \( G \) is \( A_i, B_i \) where \( A_i \) is the partite set of size 1, and we suppose \( s \in \mathbb{N} \) satisfies \( 2s = \sum_{i=1}^{n} m_i \). Suppose that there is a partition \( \{A, B\} \) of the set \([n]\) such that \( \sum_{i \in A} m_i = \sum_{j \in B} m_j = s \). Now, consider the proper 2-coloring \( f \) of \( G \) defined as follows. For each \( i \in A \) color each vertex in \( A_i \) with 1, and color each vertex in \( B_i \) with 2. Similarly, for each \( j \in B \) color each vertex in \( A_j \) with 2, and color each vertex in \( B_j \) with 1. Now, it is easy to see

\[
|f^{-1}(1)| = |A| + \sum_{j \in B} (m_j + 1) = |A| + |B| + s = n + s.
\]

Similarly, \( |f^{-1}(2)| = n + s \). So, \( f \) is an equitable 2-coloring of \( G \).

Conversely, suppose that \( g : V(G) \to [2] \) is an equitable 2-coloring of \( G \). Since \( |V(G)| = 2n + 2s \), we know that \( |g^{-1}(1)| = |g^{-1}(2)| = n + s \). We also have that for each \( i \in [n] \), \( f(B_i) \) is either \( \{1\} \) or \( \{2\} \). Now, let

\[
A = \{i \in [n] : f(B_i) = \{1\}\}
\]

and \( B = [n] - A \). Clearly, \( \{A, B\} \) is a partition of the set \([n]\). Notice

\[
n + s = |g^{-1}(1)| = \sum_{i \in A} |B_i| + \sum_{j \in B} |A_j| = |B| + \sum_{i \in A} (m_i + 1) = n + \sum_{i \in A} m_i.
\]

This implies that \( \sum_{i \in A} m_i = s \). A similar argument shows \( \sum_{j \in B} m_j = s \) as desired. \( \Box \)

**Theorem 10** STARS EQUITABLE 2-COLORING is NP-complete.

**Proof** We first show that STARS EQUITABLE 2-COLORING is in NP. Suppose \( x = (m_1, \ldots, m_n) \) such that \( m_i \in \mathbb{N} \) for each \( i \in [n] \) is an input that STARS EQUITABLE 2-COLORING accepts. Notice a proper 2-coloring of \( \sum_{i=1}^{n} K_{1, m_i} \) can be represented by a binary string of length \( n \) where the \( i \)th bit indicates which of two possible proper 2-colorings is used to color \( K_{1, m_i} \). Let \( y \) be a certificate that represents an equitable 2-coloring of \( \sum_{i=1}^{n} K_{1, m_i} \). Clearly, the certificate is of size \( n \) and \( x \) is of size \( O(\sum_{i=1}^{n} (\lceil \log_2 (m_i) \rceil + 1)) \). Finally, it is easy to verify \( y \) represents an equitable 2-
coloring in polynomial time.

Now, we will show that STARS EQUITABLE 2-COLORING is NP-Hard by showing there is a polynomial reduction from PARTITION to STARS EQUITABLE 2-COLORING. Suppose \( x = (m_1, \ldots, m_n) \) is an arbitrary \( n \)-tuple such that \( m_i \in \mathbb{N} \) for each \( i \in [n] \). We view \( x \) as an input into PARTITION. If \( \sum_{i=1}^n m_i \) is odd input \( y = (3) \) into STARS EQUITABLE 2-COLORING; otherwise, input \( y = (m_1 + 1, \ldots, m_n + 1) \) into STARS EQUITABLE 2-COLORING. Then, accept if and only if STARS EQUITABLE 2-COLORING accepts. It is obvious that this reduction runs in polynomial time.

We must show that there is a partition \( \{A, B\} \) of the set \( [n] \) such that \( \sum_{i \in A} m_i = \sum_{j \in B} m_j \) if and only if there is an equitable 2-coloring of: \( G = K_{1, 3} \) in the case \( \sum_{i=1}^n m_i \) is odd and \( G = \sum_{i=1}^n K_{1, m_i + 1} \) in the case \( \sum_{i=1}^n m_i \) is even. This statement clearly holds when \( \sum_{i=1}^n m_i \) is odd, and the statement follows from Lemma 9 when \( \sum_{i=1}^n m_i \) is even. \( \square \)

### 3 Equitable 2-Choosability of the Disjoint Union of Stars

From this point forward, for any graph \( G \) and \( k \in \mathbb{N} \), we let \( \rho(G, k) = \lceil |V(G)|/k \rceil \). Additionally, when \( G \) and \( k \) are clear from context, we use \( \rho \) to denote \( \rho(G, k) \). Until we reach Theorem 5 in this section, \( G \) is always \( G_1 + G_2 \) where \( G_1 \) is a copy of \( K_{1, m_1} \) with bipartition \( \{w_0\}, A = \{w_1, \ldots, w_{m_1}\} \) and \( G_2 \) is a copy of \( K_{1, m_2} \) with bipartition \( \{u_0\}, B = \{u_1, \ldots, u_{m_2}\} \). We begin this section with a lemma that gives us a simple necessary condition for the disjoint union of two stars to be equitably \( k \)-choosable. In this section, our primary use of this result will be in the case of equitable 2-choosability.

**Lemma 11** Let \( k \in \mathbb{N} \) and \( G = K_{1, m_1} + K_{1, m_2} \) where \( 1 \leq m_1 \leq m_2 \). If \( m_2 > \rho(G, k)(k - 1) - 1 - \max\{0, m_1 - \rho(G, k) + 1\} \) then \( G \) is not equitably \( k \)-choosable.

**Proof** Note that the result clearly holds when \( k = 1 \). So, we may assume that \( k \geq 2 \). Also note that \( \rho \geq 1 \), and the result is vacuously true when \( \rho = 1 \). So, we may assume that \( \rho \geq 2 \) (i.e., \( k < m_1 + m_2 + 2 \)). Consider the \( k \)-assignment \( L \) for \( G \) given by \( L(v) = [k] \) for all \( v \in V(G) \). To prove the desired result, we will show that \( G \) is not equitably \( L \)-colorable.

Suppose for the sake of contradiction that \( f \) is an equitable \( L \)-coloring of \( G \). Suppose that \( f(w_0) = c_1 \) and \( f(u_0) = c_2 \). We will derive a contradiction in the following two cases: (1) \( c_1 = c_2 \) and (2) \( c_1 \neq c_2 \). For the first case, since \( f \) is proper, the vertices of \( A \cup B \) are colored with colors from \( [k] \setminus \{c_1\} \). Note that

\[
m_1 + m_2 = \sum_{i \in [k] \setminus \{c_1\}} |f^{-1}(i)| \leq \rho(k - 1)
\]

which implies that \( m_2 \leq \rho(k - 1) - m_1 \). Since \( \rho \geq 2 \), \( m_1 \geq m_1 + (2 - \rho) = 1 + m_1 - \rho + 1 \). So, \( -m_1 \leq -1 - \max\{0, m_1 - \rho + 1\} \). Therefore we know that \( m_2 \leq \rho(k - 1) - 1 - \max\{0, m_1 - \rho + 1\} \) which is a
In the second case, we know that the vertices of $A$ are colored with colors from $[k] - \{c_1\}$, and the vertices of $B$ are colored with colors from $[k] - \{c_2\}$. Since $f$ is an equitable $L$-coloring it is clear that $|f^{-1}(c_2) \cap A| \leq \rho - 1$ and $|f^{-1}(c_1) \cap B| \leq \rho - 1$. Suppose that $\max \{0, m_1 - \rho + 1\} = m_1 - \rho + 1$. We have that

$$m_1 + m_2 = |f^{-1}(c_1) \cap B| + |f^{-1}(c_2) \cap A| + \sum_{i \in [k] - \{c_1, c_2\}} |f^{-1}(i)| \leq 2(\rho - 1) + \rho(k - 2).$$

So, it follows that $m_2 \leq \rho(k - 1) - 1 - (m_1 - \rho + 1) = \rho(k - 1) - 1 - \max \{0, m_1 - \rho + 1\}$ which is a contradiction. Now, suppose $\max \{0, m_1 - \rho + 1\} = 0$. Notice that $|f^{-1}(c_1) \cap A| \leq |A| = m_1$, which implies that

$$m_1 + m_2 = |f^{-1}(c_1) \cap B| + |f^{-1}(c_2) \cap A| + \sum_{i \in [k] - \{c_1, c_2\}} |f^{-1}(i)| \leq (\rho - 1) + m_1 + \rho(k - 2).$$

Thus, we have that $m_2 \leq \rho(k - 1) - 1 = \rho(k - 1) - 1 - \max \{0, m_1 - \rho + 1\}$ which is a contradiction.

Lemma 11 gives us a necessary condition for the disjoint union of two stars to be equitably $k$-choosable: if $G = K_{1, m_1} + K_{1, m_2}$ is equitably $k$-choosable, then

$$m_2 \leq \rho(G, k)(k - 1) - 1 - \max \{0, m_1 - \rho(G, k) + 1\}.$$  

Interestingly, $m_2 \leq \rho(G, k)(k - 1) - 1$ implies that $m_2 \leq \rho(G, k)(k - 1) - 1 - \max \{0, m_1 - \rho(G, k) + 1\}$. So, we immediately have an equivalent necessary condition that is a bit easier to state.

Lemma 12 Suppose $G = K_{1, m_1} + K_{1, m_2}$ where $1 \leq m_1 \leq m_2$. If $m_2 \leq \rho(G, k)(k - 1) - 1$, then $m_2 \leq \rho(G, k)(k - 1) - 1 - \max \{0, m_1 - \rho(G, k) + 1\}$. Consequently, the following two statements hold and are equivalent.

(i) If $G$ is equitably $k$-choosable, then $m_2 \leq \rho(G, k)(k - 1) - 1 - \max \{0, m_1 - \rho(G, k) + 1\}$.

(ii) If $G$ is equitably $k$-choosable, then $m_2 \leq \rho(G, k)(k - 1) - 1$.

Proof Suppose for the sake of contradiction that $m_2 > \rho(k - 1) - 1 - \max \{0, m_1 - \rho + 1\}$. We clearly get a contradiction when $0 \geq m_1 - \rho + 1$. So, we suppose that $0 < m_1 - \rho + 1$. Then, we have that $m_2 > \rho(k - 1) - 1 - m_1 + \rho - 1$ which implies that $m_1 + m_2 + 2 > \rho k$ which is clearly a contradiction since $\rho = \lceil (m_1 + m_2 + 2)/k \rceil$. □

When we apply Lemma 12 in this paper, we will always be using the Statement (ii). In the case of equitable 2-choosability, we may immediately deduce the following.

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3 It should be noted that while these statements are equivalent, the inequality in Statement(i) holds with equality more often than the inequality in Statement(ii).
**Corollary 13** Let $G = K_{1,m_1} + K_{1,m_2}$ where $1 \leq m_1 \leq m_2$. If $m_2 = m_1 \geq 2$, then $G$ is not equitably 2-choosable.

**Proof** Note that $\rho(G, 2) \leq \lceil (m_2 + m_2 - 2 + 2)/2 \rceil = m_2$. So, $m_2 \geq \rho > \rho - 1$. Lemma 12 implies $G$ is not equitably 2-choosable. □

We now present three lemmas that we will use to prove Theorem 4.

**Lemma 14** Let $G = K_{1,m_1} + K_{1,m_2}$ where $1 \leq m_1 \leq m_2$. If $m_1 + m_2 \geq 16$, then $G$ is not equitably 2-choosable.

**Proof** Note that by Corollary 13 we may assume that $m_2 - m_1 \leq 1$. It must be the case that $8 \leq m_1 \leq m_2 \leq m_1 + 1$.

We will now construct a 2-assignment $L$ for $G$ for which there is no equitable $L$-coloring. Let $L(u_i) = \{1, 2\}$ for $i \in [m_2] \cup \{0\}$, $L(w_0) = \{3, 4\}$, $L(w_1) = L(w_2) = \{1, 3\}$, $L(w_3) = L(w_4) = \{1, 4\}$, $L(w_5) = L(w_6) = \{2, 3\}$, $L(w_7) = L(w_8) = \{2, 4\}$, and $L(w_i) = \{3, 4\}$ for all $i \in [m_1] - \{8\}$. For the sake of contradiction, suppose that $G$ is equitably $L$-colorable. Let $f$ be an equitable $L$-coloring of $G$. Note $\rho(G, 2) = \lceil (m_1 + m_2 + 2)/2 \rceil = m_2 + 1$. Clearly, $f(u_0)$ is either 1 or 2 and $f(w_0)$ is either 3 or 4. By symmetry, without loss of generality, we can assume that $f(u_0) = 2$ and $f(w_0) = 3$. Then it is clear that $f(u_i) = 1$ for all $i \in [m_2]$. Also, $f(w_1) = f(w_2) = 1$. However, this would imply that $|f^{-1}(1)| \geq m_2 + 2$ which is a contradiction. □

**Lemma 15** Let $G = G_1 + G_2$ where both $G_1$ and $G_2$ are copies of $K_{1,m}$ such that $m \in \{7\}$. If $L$ is a 2-assignment for $G$ such that $L(w_0) \cap L(u_0) = \emptyset$, then $G$ is equitably $L$-colorable.

**Proof** For the sake of contradiction, suppose there is a 2-assignment $K$ for $G$ such that $K(w_0) \cap K(u_0) = \emptyset$ and $G$ is not equitably $K$-colorable. Among all such 2-assignments, choose a 2-assignment, $L$, with the smallest possible palette size. Let $L(w_0) = \{k, c\}$ and $L(u_0) = \{t, d\}$. Clearly, $|L| \geq 4$. We will first show that $|L| > 4$.

Assume $|L| = 4$; that is, $L = \{t, k, c, d\}$. For each $\{c_1, c_2\}$ such that $|\{c_1, c_2\}| = 2$ and $\{c_1, c_2\} \subseteq \{t, k, c, d\}$ let $a_{\{c_1, c_2\}} = |L^{-1}(\{c_1, c_2\}) \cap A|$ and $b_{\{c_1, c_2\}} = |L^{-1}(\{c_1, c_2\}) \cap B|$. We now consider all possible colorings of $w_0$ and $u_0$ while keeping in mind that $\rho = m + 1$. For all $v \in A$ let $L^{(1)}(v) = L(v) - \{k\}$, and for all $v \in B$ let $L^{(1)}(v) = L(v) - \{t\}$. Notice that if $w_0$ were assigned color $k$ and $u_0$ were assigned color $t$, $k$ can only be used to color vertices in $B$ and $t$ can only be used to color vertices in $A$. This means that if $f$ is a proper $L$-coloring of $G$ such that $f(w_0) = k$ and $f(u_0) = t$, then $|f^{-1}(k)| \leq m + 1 = \rho$ and $|f^{-1}(t)| \leq m + 1 = \rho$. Since $G$ is not equitably $L$-colorable, it must be that among the lists $L^{(1)}(u_1), \ldots, L^{(1)}(u_m), L^{(1)}(w_1), \ldots, L^{(1)}(w_m)$ there are $m + 2$ lists that are $\{d\}$ or there are $m + 2$ that are $\{c\}$. Suppose that $m + 2$ of them are $\{d\}$ (the case where $m + 2$ of them are $\{c\}$ is similar). Then, $a_{\{k,d\}} + b_{\{t,d\}} \geq m + 2$ which implies $a_{\{k,d\}} \geq 2$.

Let $L^{(2)}(v) = L(v) - \{c\}$ for all $v \in A$, and let $L^{(2)}(v) = L(v) - \{t\}$ for all $v \in B$. By an argument similar to the one used for $L^{(1)}$, it must be that among the lists
$L^{(2)}(u_1), \ldots, L^{(2)}(u_m), L^{(2)}(w_1), \ldots, L^{(2)}(w_m)$ there are $m+2$ that are $\{d\}$ or there are
$m+2$ that are $\{k\}$. Notice that if $m+2$ of these lists are $\{k\}$, $a_{\{c,k\}} + b_{\{t,k\}} \geq m+2$
which implies $a_{\{c,k\}} + b_{\{t,k\}} + a_{\{k,d\}} + b_{\{t,d\}} \geq 2m + 4 > |V(G)|$ which is a
contradiction. So it must be that $m+2$ of these lists are $\{d\}$, and we have that $a_{\{c,d\}} + b_{\{t,d\}} \geq m+2$
which means $a_{\{c,d\}} \geq 2$.

Let $L^{(3)}(v) = L(v) - \{k\}$ for all $v \in A$, and let $L^{(3)}(v) = L(v) - \{d\}$ for all $v \in B$.
By an argument similar to the one used for $L^{(1)}$, it must be that among the lists
$L^{(3)}(u_1), \ldots, L^{(3)}(u_m), L^{(3)}(w_1), \ldots, L^{(3)}(w_m)$ there are $m+2$ that are $\{c\}$ or there are
$m+2$ that are $\{t\}$. Notice that if $m+2$ of these lists are $\{c\}$, $a_{\{c,k\}} + b_{\{c,d\}} \geq m+2$
which implies $a_{\{c,k\}} + b_{\{c,d\}} + a_{\{k,d\}} + b_{\{t,d\}} \geq 2m + 4 > |V(G)|$ which is a
contradiction. So it must be that $m+2$ of these lists are $\{t\}$, and we have that $a_{\{t,k\}} + b_{\{t,d\}} \geq m+2$
which means $a_{\{t,k\}} \geq 2$.

Let $L^{(4)}(v) = L(v) - \{c\}$ for all $v \in A$, and let $L^{(4)}(v) = L(v) - \{d\}$ for all $v \in B$.
By an argument similar to the one used for $L^{(1)}$, it must be that among the lists
$L^{(4)}(u_1), \ldots, L^{(4)}(u_m), L^{(4)}(w_1), \ldots, L^{(4)}(w_m)$ there are $m+2$ that are $\{t\}$ or there are
$m+2$ that are $\{k\}$. Notice that if $m+2$ of these lists are $\{k\}$, $a_{\{c,k\}} + b_{\{d,k\}} \geq m+2$
which implies $a_{\{c,k\}} + b_{\{d,k\}} + a_{\{k,d\}} + b_{\{t,d\}} \geq 2m + 4 > |V(G)|$ which is a
contradiction. So it must be that $m+2$ of these lists are $\{t\}$, and we have that $a_{\{t,k\}} + b_{\{t,d\}} \geq m+2$
which means $a_{\{t,k\}} \geq 2$.

So, $a_{\{c,t\}} + a_{\{t,k\}} + a_{\{c,d\}} + a_{\{k,d\}} \geq 8$. However this implies that $8 \leq |A| = m$
which is a contradiction.

Now, we have that $|\mathcal{L}| \geq 5$. For every $q \in \mathcal{L} - \{t, k, c, d\}$, let
$\eta(q) = \{|v : q \in L(v)\}|$. In the case there is an $r \in \mathcal{L} - \{k, t, c, d\}$ satisfying
$\eta(r) \leq m+1$, let $L'$ be a new 2-assignment for $G$ given by

$$L'(v) = \begin{cases} 
\{x, k\} & \text{if } L(v) = \{x, r\} \text{ for some } x \neq k \\
\{k, c\} & \text{if } L(v) = \{k, r\} \\
L(v) & \text{if } r \not\in L(v)
\end{cases}$$

By the extremal choice of $L$, we know that $G$ is equitably $L'$-colorable, and we call
such a coloring $f$. We then recolor all $v$ such that $f(v) \not\in L'(v)$ with $r$. Note that since
$r \not\in \{t, k, c, d\}$, this coloring is proper, and it is easy to see that it is also an
equitable $L$-coloring of $G$.

Now, suppose that for every $q \in \mathcal{L} - \{t, k, c, d\}$ that $\eta(q) > m+1$. Since $\mathcal{L} - \{t, k, c, d\}$ is nonempty, we may suppose that $s \in \mathcal{L} - \{t, k, c, d\}$. Note that $|L^{-1}(\{s, t\}) \cap B| + |L^{-1}(\{s, d\}) \cap B| \leq m$
and $|L^{-1}(\{s, k\}) \cap A| + |L^{-1}(\{s, c\}) \cap A| \leq m$. Without loss of generality assume $|L^{-1}(\{s, t\}) \cap B| \leq m/2$ and
$|L^{-1}(\{s, k\}) \cap A| \leq m/2$. Color all $v \in L^{-1}(\{s, t\}) \cap B$ and $v \in L^{-1}(\{s, k\}) \cap A$ with
$s$. In doing this $s$ is used at most $m$ times. Then, arbitrarily color uncolored vertices
that have $s$ in their lists with $s$ until exactly $m+1$ vertices are colored with $s$. Then,
color $w_0$ with $k$ and $u_0$ with $t$. Now, let $U$ be the set of all uncolored vertices in $A \cup B$.
Let $L'(v) = L(v) - \{s, k\}$ for all $v \in U \cap A$, and let $L'(v) = L(v) - \{s, t\}$ for all
$v \in U \cap B$. Clearly, $|U| = m - 1$ and $|L'(v)| \geq 1$ for all $v \in U$. So, we can color each
$v \in U$ with a color in $L'(v)$ to complete an equitable $L$-coloring of $G$. This contradiction completes the proof. \hfill \Box

**Lemma 16** Let $G = K_{1,m} + K_{1,m}$ where $m \in \mathbb{N}$. Then $G$ is equitably 2-choosable.

**Proof** Suppose that the two components that make up $G$ are $G_1$ and $G_2$. We will show that $G$ is equitably 2-choosable by induction on $m$. The result holds when $m = 1$ and when $m = 2$ since $\Delta(G) \leq 2$ in these cases (this was observed immediately after the statement of Question 2). So, suppose that $2 < m \leq 7$ and the desired result holds for all natural numbers less than $m$.

For the sake of contradiction, suppose there is a 2-assignment $L$ for $G$ such that $G$ is not equitably $L$-colorable. Let $G' = G - \{u_m, w_m\}$ and $K(v) = L(v)$ for all $v \in V(G')$. By the inductive hypotheses there is an equitable $K$-coloring $f$ of $G'$ which uses no color more than $m$ times. The strategy of the proof is to now determine characteristics of $L$ and to then show that an equitable $L$-coloring of $G$ must exist. Let $L'(u_m) = L(u_m) - \{f(u_0)\}$ and $L'(w_m) = L(w_m) - \{f(w_0)\}$. \hfill \Box

**Observation 1** $L'(u_m) = L'(w_m)$ and $|L'(u_m)| = 1$. Suppose that $L'(u_m) \neq L'(w_m)$ or $|L'(u_m)| > 1$. Notice it is possible to color $u_m$ and $w_m$ with two distinct colors from $L'(u_m)$ and $L'(w_m)$ respectively. Combining this with $f$ completes an equitable $L$-coloring of $G$ which is a contradiction.

So, we can assume $L'(u_m) = L'(w_m) = \{c\}$.

**Observation 2** $|f^{-1}(c)| = m$. Suppose that $|f^{-1}(c)| < m$. Coloring $u_m$ and $w_m$ with $c$ and the other vertices in $G$ according to $f$ completes an equitable $L$-coloring of $G$ which is a contradiction.

Let $A' = A \cap f^{-1}(c)$ and $B' = B \cap f^{-1}(c)$. Without loss of generality assume $A' = \{w_1, w_2, \ldots, w_a\}$ and $B' = \{u_1, u_2, \ldots, u_b\}$. Since $f(w_0) \neq c$ and $f(u_0) \neq c$, $a + b = m$. Without loss of generality assume $b \leq a$. This implies $1 \leq b \leq a \leq m - 1$ and $m/2 \leq a$.

**Observation 3** For all $v \in A' \cup \{w_m\}$, $L(v) = \{c, f(w_0)\}$, and for all $v \in B' \cup \{u_m\}$, $L(v) = \{c, f(u_0)\}$. Suppose that there is a $u_k \in B'$ such that $L(u_k) = \{x, c\}$ where $x \neq f(u_0)$. Since $|f^{-1}(f(u_0))| \geq 1$, $|f^{-1}(c)| = m$, and $|V(G')| = 2m$, we have that $|f^{-1}(x)| < m$. Now, color all the vertices in $V(G') - \{u_k\}$ according to $f$, and color $u_k$ with $x$. Coloring $u_m$ and $w_m$ with $c$ completes an equitable $L$-coloring which is a contradiction. A similar argument can be used to show that for all $v \in A' \cup \{w_m\}$, $L(v) = \{c, f(w_0)\}$.

Now suppose that $L(u_0) = \{f(u_0), d\}$ and $L(w_0) = \{f(w_0), l\}$.

**Observation 4** $d \neq c$. Suppose that $d = c$. Color all the vertices in $V(G')$ according to $f$. Then, recolor $u_0$ with $c$ and each vertex in $B'$ with $f(u_0)$. Finally, color $u_m$ with $f(u_0)$ and $w_m$ with $c$. Note that the number of times $c$ is used is exactly $a + 1 + 1 \leq m + 1$. Also note that in $G_1$ $f(u_0)$ can either be used to color $w_0$ or vertices in $A - (A' \cup \{w_m\})$. Consequently, the number of times $f(u_0)$ is used is at most $b + 1 + \max\{1, m - (a + 1)\} \leq m + 1$. So, we have constructed an equitable $L$-coloring of $G$ which is a contradiction.
Observation 5 \( l = c \). Suppose that \( l \neq c \). Color all the vertices in \( V(G') \) according to \( f \). Then, recolor \( w_0 \) with \( l \). Also, color \( w_m \) with \( f(w_0) \) and \( u_m \) with \( c \). For each \( v \in (A' - (A' \cup \{w_m\})) \) such that \( f(v) = l \), we recolor \( v \) with the element in \( L(v) - \{l\} \). Let \( r = |\{v \in (A' - (A' \cup \{w_m\}) : f(v) = l\}| \), and note that \( 0 \leq r \leq m - (a + 1) \leq m/2 \leq a \). At this stage, we know that \( c \) is used \( m + 1 + z \) times where \( z \) is an integer satisfying \( 0 \leq z \leq r \leq a \). If \( z \geq 1 \), recolor \( w_1, \ldots, w_z \) with \( f(w_0) \). Note that the resulting coloring is proper. Moreover, the resulting coloring uses \( c \) exactly \( m + 1 \) times. So, it must be an equitable \( L \)-coloring of \( G \) since \( |V(G)| = 2m + 2 \). This is a contradiction.

We now have that \( L(u_0) = \{f(u_0), d\} \), \( L(w_0) = \{f(w_0), c\} \), and \( c \neq d \).

Observation 6 \( f(u_0) \neq f(w_0) \). Suppose that \( f(u_0) = f(w_0) \). Color all the vertices in \( V(G') \) according to \( f \). Then, recolor \( w_0 \) with \( c \), and for each \( v \in A' \), recolor \( v \) with \( f(w_0) \). Finally, color \( w_m \) with \( f(w_0) \) and \( u_m \) with \( c \). Note that since no vertices in \( B \) are colored with \( f(w_0) \), it must be that \( f(w_0) \) is used at most \( m + 1 \) times. Also note that \( c \) is used at most \( b + 1 + 1 \leq m + 1 \) times. So, we have constructed an equitable \( L \)-coloring of \( G \) which is a contradiction.

Observation 7 \( f(w_0) \neq d \). Suppose that \( f(w_0) = d \). Color all the vertices in \( V(G') \) according to \( f \). Then, recolor \( w_0 \) with \( c \) and \( u_0 \) with \( d \). For all \( v \in A' \), recolor \( v \) with \( d \). Also, recolor all \( v \in B - (A' \cup \{u_m\}) \) satisfying \( f(v) = d \) with the element in \( L(v) - \{d\} \). Also, color \( w_m \) with \( d \) and \( u_m \) with \( c \). Our resulting coloring is clearly proper. Note that \( d \) is used exactly \( a + 2 \) times which means it is used at most \( m + 1 \) times. Also note that \( c \) is used at least \( m + 1 \) times and at least \( b + 2 \) times. Thus \( c \) and \( d \) are used at least \( a + b + 4 = m + 4 > m + 1 = |V(G)|/2 \) times. Thus, we have constructed an equitable \( L \)-coloring of \( G \) which is a contradiction.

Observations 6 and 7 allow us to conclude \( L(w_0) \cap L(u_0) = \emptyset \) by which Lemma 15 implies there exists an equitable \( L \)-coloring of \( G \) which is a contradiction.

We are now ready to prove Theorem 4 which we restate.

**Theorem 4** Let \( G = K_{1,m_1} + K_{1,m_2} \) where \( 1 \leq m_1 \leq m_2 \). \( G \) is equitably 2-choosable if and only if \( m_2 - m_1 \leq 1 \) and \( m_1 + m_2 \leq 15 \).

**Proof** We begin by assuming that \( m_2 - m_1 \geq 2 \) or \( m_1 + m_2 \geq 16 \). By Corollary 13 and Lemma 14 we know that in both cases \( G \) is not equitably 2-choosable.

Conversely, suppose that \( m_1 + m_2 \leq 15 \) and \( m_2 - m_1 \leq 1 \). In the case that \( m_1 = m_2 \) we know that the desired result holds by Lemma 16. So we may assume that \( m_2 = m_1 + 1 \). Suppose that \( L \) is an arbitrary 2-assignment for \( G \), and let \( m = m_1 \). Let \( G' = G - \{u_{m+1}\} \), and let \( L'(v) = L(v) \) for all \( v \in V(G') \). We know by Lemma 16 that there exists an equitable \( L' \)-coloring \( f \) of \( G' \). Suppose the vertices in \( V(G') \) are colored according to \( f \). Note that \( \rho(G',2) < \rho(G,2) \). Thus, we can complete an equitable \( L \)-coloring of \( G \) by coloring \( u_{m+1} \) with a color in \( L(u_{m+1}) - \{f(u_0)\} \).

We end this section by proving Theorem 5 which we restate. It should be noted that when \( G = \sum_{i=1}^{n} K_{1,m_i} \), \( G \) is a forest of maximum degree \( m \), and we know that \( G \) is equitably 2-choosable when \( m \in [2] \) (see Sect. 1).
Theorem 5 Suppose that $n,m \in \mathbb{N}$, $n \geq 2$, and $G = \sum_{i=1}^{n} K_{1,m}$. When $n$ is odd, $G$ is equitably 2-choosable if and only if $m \leq 2$. When $n$ is even, $G$ is equitably 2-choosable if and only if $m \leq 7$.

Proof Throughout this argument, let $G_1, G_2, \ldots, G_n$ be the components of $G$. Let $A_i' = \{w_{i,0}\}$ and $A_i = \{w_{i,1}, \ldots, w_{i,m}\}$ be the bipartition of $G_i$ for each $i \in [n]$.

First, suppose that $n$ is odd. We know that $G$ is equitably 2-choosable when $m \leq 2$. For the converse, we suppose that $m \geq 3$, and we will construct a 2-assignment $L$ for $G$ for which there is no equitable $L$-coloring. Let $L(v) = [2]$ for all $v \in V(G)$. For the sake of contradiction, suppose that $G$ is equitably $L$-colorable. Let $f$ be an equitable $L$-coloring of $G$. Note that

$$\max\{|f^{-1}(1)|, |f^{-1}(2)|\} \geq m \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \frac{nm + m + n - 1}{2}.\$$

Also note that $\rho(G, 2) = \left\lfloor (m+1)/2 \right\rfloor \leq (m+1)/2 = (n+1)/2$. Since $m \geq 3$ we know that $m - 1 > 1$. Therefore, we see that $\max\{|f^{-1}(1)|, |f^{-1}(2)|\} > \rho$ which is a contradiction.

Now, suppose that $n$ is even. We begin by assuming that $m \geq 8$. We will now construct a 2-assignment $L$ for $G$ for which there is no equitable $L$-coloring. Let $L(v) = [2]$ for all $v \in \bigcup_{i=2}^{n} (A_i' \cup A_i)$, $L(w_{1,0}) = \{3, 4\}$, $L(w_{1,1}) = L(w_{1,2}) = \{1, 3\}$, $L(w_{1,3}) = L(w_{1,4}) = \{1, 4\}$, $L(w_{1,5}) = L(w_{1,6}) = \{2, 3\}$, $L(w_{1,7}) = L(w_{1,8}) = \{2, 4\}$, and $L(v) = \{3, 4\}$ for all $v \in A_1 - \{w_{1,1}, w_{1,2}, \ldots, w_{1,8}\}$. For the sake of contradiction, suppose $G$ is equitably $L$-colorable, and suppose $f$ is an equitable $L$-coloring of $G$. Note that $\rho(G, 2) = \left\lfloor (m+1)/2 \right\rfloor = (n+1)/2$. We calculate

$$\max\{|f^{-1}(1)|, |f^{-1}(2)|\} \geq m \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lceil \frac{n-1}{2} \right\rceil + 2 = \frac{mn + n - 2}{2} + 2 = \frac{mn + n + 2}{2}.$$

It is easy to see that $\max\{|f^{-1}(1)|, |f^{-1}(2)|\} > \rho$ which is a contradiction. Thus, $G$ is not equitably 2-choosable. Conversely, suppose that $m \leq 7$. Let $G^{(i)} = G_{2i} + G_{2i-1}$ for all $i \in [n/2]$. Suppose that $L$ is an arbitrary 2-assignment for $G$. Let $L^{(i)}$ be $L$ restricted to the vertices of $G^{(i)}$. By Theorem 4 we know that there is a proper $L^{(i)}$-coloring $f_i$ of $G^{(i)}$ that uses no color more than $m + 1$ times for each $i \in [n/2]$. Coloring the vertices of $G$ according to $f_1, \ldots, f_{n/2}$ achieves an equitable $L$-coloring of $G$ since no color could possibly be used more than $(m + 1)n/2 = \rho$ times.

4 Equitable Choosability of the Disjoint Union of Two Stars

We begin by stating an inductive process that will be used throughout the remainder of the paper. Intuitively, the process takes in a graph $G = G_1 + G_2$ where $G_i$ is a copy of $K_{1,m_i}$ for $i \in [2]$, and a $k$-assignment $L$ for $G$ where $k \geq 3$. It then constructs a partial $L$-coloring of $G$ by repeatedly selecting $\rho$ vertices from the partite sets of size $m_1$ and $m_2$ that include a common color $c$, assigns them all color $c$, removes those
vertices from consideration, and removes $c$ from each list of vertices that remain. The process repeats in this way until there is no color in $\rho$ lists of the remaining vertices or until the step is repeated $k - 2$ times; whichever comes first.

We now formally present the process. Note below $\epsilon$ is used to indicate ‘equitable’.

**Process 17 $\epsilon$-greedy process:** The $\epsilon$-greedy process takes as input: a graph $G = G_1 + G_2$ where $G_i$ is a copy of $K_{1,m_i}$ for $i \in [2]$, and a $k$-assignment $L$ where $k \geq 3$. It outputs $G_r$, where $G_r$ is an induced subgraph of $G$, a list assignment $L_r$ for $G_r$, and a partial $L$-coloring $g_\epsilon$ of $G$ that colors the vertices in $V(G) - V(G_r)$.

Suppose the bipartition of $G_1$ is $\{w_0\}, A = \{w_1, \ldots, w_{m_1}\}$ and the bipartition of $G_2$ is $\{u_0\}, B = \{u_1, \ldots, u_{m_2}\}$. To begin we determine whether there is a color that appears in at least $\rho(G,k)$ of the lists associated with the vertices in $A \cup B$. If no such color exists let $G_1 = G, L_\epsilon = L$, and $g_\epsilon$ be a function with an empty domain, then the process terminates. Otherwise there exists a color $c_1$ that is in at least $\rho$ of the lists associated with the vertices in $A \cup B$, and we arbitrarily put $\rho$ of these vertices in a set $C_1$. We consider this the first step of the $\epsilon$-greedy process.

If $k = 3$ let $G_\epsilon = G - C_1$, $L_\epsilon$ be the list assignment for $G_\epsilon$ given by: $L_\epsilon(v) = \{c_1\}$ for all $v \in V(G_\epsilon)$, and $g_\epsilon : C_1 \to \{c_1\}$ be the partial $L$-coloring of $G$ given by $g_\epsilon(v) = c_1$ whenever $v \in C_1$, then the process terminates.

If $k > 3$, we proceed inductively. For each $t = 2, \ldots, k - 2$ if the process has not terminated in the $(t - 1)$th step we determine whether there is a color in $L - \{c_1, \ldots, c_{t-1}\}$ that appears in at least $\rho$ of the lists associated with the vertices in $(A \cup B) - \bigcup_{i=1}^{t-1} C_i$. If no such color exists let $G_\epsilon = G - \bigcup_{i=1}^{t-1} C_i$, $L_\epsilon$ be the list assignment for $G_\epsilon$ given by: $L_\epsilon(v) = \{c_i : i \in [t - 1]\}$ for all $v \in V(G_\epsilon)$, and $g_\epsilon : \bigcup_{i=1}^{t-1} C_i \to \{c_i : i \in [t - 1]\}$ be the partial $L$-coloring of $G$ given by $g_\epsilon(v) = c_i$ whenever $v \in C_i$, then the process terminates. Otherwise there exists a color $c_t \in L - \{c_1, \ldots, c_{t-1}\}$ that is in at least $\rho$ of the lists associated with the vertices in $(A \cup B) - \bigcup_{i=1}^{t-1} C_i$, and we arbitrarily put $\rho$ of these vertices in a set $C_t$.

If the process does not terminate when $t = k - 2$ let $G_\epsilon = G - \bigcup_{i=1}^{k-2} C_i$, $L_\epsilon$ be the list assignment for $G_\epsilon$ given by: $L_\epsilon(v) = \{c_i : i \in [k - 2]\}$ for all $v \in V(G_\epsilon)$, and $g_\epsilon : \bigcup_{i=1}^{k-2} C_i \to \{c_i : i \in [k - 2]\}$ be the partial $L$-coloring of $G$ given by $g_\epsilon(v) = c_i$ whenever $v \in C_i$, then the process terminates.

By the definition of the $\epsilon$-greedy process we easily obtain the following observation and two lemmas.

**Observation 18** Suppose that the $\epsilon$-greedy process is run on $G = K_{1,m_1} + K_{1,m_2}$ with a $k$-assignment $L$ where $k \geq 3$. Then $|L_\epsilon(v)| \geq 2$ for all $v \in V(G_\epsilon)$.

**Lemma 19** Suppose that the $\epsilon$-greedy process is run on $G = K_{1,m_1} + K_{1,m_2}$ with a $k$-assignment $L$. Let $\{w_0\}$ (resp., $\{u_0\}$) denote the partite set of size one for the copy of $K_{1,m_1}$ (resp., $K_{1,m_2}$) used to form $G$. If no color appears in at least $\rho(G,k)$ of the lists associated with the vertices in $V(G_\epsilon) - \{w_0, u_0\}$ by $L_\epsilon$, then $G$ is equitably $L$-colorable.

**Proof** Note that since no color appears in at least $\rho(G,k)$ of the lists associated with the vertices in $V(G_\epsilon) - \{w_0, u_0\}$ by $L_\epsilon$, any proper $L_\epsilon$-coloring cannot possibly use a
color more than $\rho(G,k)$ times. By Observation 18 and the fact that $G_c$ is 2-choosable there must exist a proper $L_c$-coloring of $G_c$. Such a coloring combined with $g_c$ yields an equitable $L$-coloring of $G$.

Lemma 20 Suppose that the $\epsilon$-greedy process is run on $G = K_{1,m_1} + K_{1,m_2}$ with a $k$-assignment $L$. Let $\{w_0\}$ (resp., $\{u_0\}$) denote the partite set of size one for the copy of $K_{1,m_1}$ (resp., $K_{1,m_2}$) used to form $G$. If there is a color that appears in at least $\rho(G,k)$ of the lists associated with the vertices in $V(G_c) - \{w_0,u_0\}$ by $L_c$, then $|\text{Ran}(g_c)| = k - 2$.

Proof We prove the contrapositive. Suppose that $|\text{Ran}(g_c)| \neq k - 2$. It is easy to verify that $|\text{Ran}(g_c)| < k - 2$. For the sake of contradiction, suppose that there exists a color that appears in at least $\rho(G,k)$ of the lists associated with the vertices $V(G_c) - \{w_0,u_0\}$ by $L_c$. This implies that the $\epsilon$-greedy process would have been able to continue to the $(|\text{Ran}(g_c)| + 1)$th step, a contradiction.

Before proving Theorem 7 we prove the following Lemma.

Lemma 21 Suppose that $0 \leq m_1$ and $m_2 \geq \max\{2,m_1\}$. Let $G = K_{1,m_1} + K_{1,m_2}$ with $A$ (resp., $B$) denoting the partite set of size $m_1$ (resp., $m_2$) in the copy of $K_{1,m_1}$ (resp., $K_{1,m_2}$) used to form $G$. Suppose $L$ is a list assignment for $G$ such that: $|L(b)| \geq 3$ for all $b \in B$, $|L(a)| \geq 2$ for all $a \in V(G) - B$, and there is a color $c$ that appears in at least $[(m_1 + m_2 + 2)/2]$ of the lists associated with the vertices in $A \cup B$. Then, there exists a proper $L$-coloring of $G$ that uses no color more than $[(m_1 + m_2 + 2)/2]$ times.

Proof Let $\sigma = [(m_1 + m_2 + 2)/2]$ and $C = \{v \in A \cup B : c \in L(v)\}$. We begin by coloring all the vertices in $C \cap A$ with $c$, and note that less than $\sigma$ vertices are colored in doing this since $|A| = m_1 < \sigma$. We arbitrarily color $\sigma - |C \cap A|$ vertices in $B \cap C$ with $c$. Let $C_1$ be the set of vertices colored with $c$, and let $G' = G - C_1$. Let $L'(v) = L(v) - \{c\}$ for all $v \in V(G')$. Let $\{w_0\}$ (resp., $\{u_0\}$) be the partite set of size $1$ in the copy of $K_{1,m_1}$ (resp., $K_{1,m_2}$) used to form $G$. Note that $|L'(w_0)| \geq 1$, $|L'(u_0)| \geq 1$, and $|L'(v)| \geq 2$ for all $v \in V(G') - \{w_0,u_0\}$. Now, order the vertices of $G'$ in such a way that $u_0$ and $w_0$ are the first two. Then, greedily color the vertices so that a proper $L'$-coloring of $G'$ is achieved. Note that the resulting proper $L'$-coloring either uses at least 2 colors or $|V(G')| = 2$. If the proper $L'$-coloring uses at least 2 colors, it uses any color at most $|V(G)| - 1$ times which implies it uses any color at most $\sigma$ times since $2\sigma \geq |V(G)| - 1$. If $|V(G')| = 2$ the proper $L'$-coloring uses any color at most 2 times, and $m_2 \geq 2$ implies that $\sigma \geq 2$. Consequently, the resulting proper $L'$-coloring uses no color more than $\sigma$ times. It follows that this proper $L'$-coloring of $G'$ completes a proper $L$-coloring of $G$ with the desired property.

We are now ready to prove Theorem 7 which we will restate.

Theorem 7 Let $k \in \mathbb{N}$, $1 \leq m_1 \leq m_2$, and $\rho = \lceil (m_1 + m_2 + 2)/k \rceil$. If $m_2 \leq \rho(k - 1) - 1$ and $m_1 + m_2 \leq 15 + \rho(k - 2)$ then $K_{1,m_1} + K_{1,m_2}$ is equitably $k$-choosable.

Proof Let $G = K_{1,m_1} + K_{1,m_2}$. Suppose the bipartition of the copy of $K_{1,m_1}$ used to form $G$ is $\{w_0\}$, $A = \{w_1, \ldots, w_{m_1}\}$, and suppose the bipartition of the copy of $K_{1,m_2}$
1. $|μ_4 - μ_5| ≤ 12$
2. $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$ or $μ_4 - μ_5 ≥ 2$.
3. $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$.

For the second case first suppose that $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$. Clearly $μ_4 ≥ max(2, μ_4)$. We claim that for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$. To see why this is so, suppose there is some $h ∈ g \setminus U^g_4$. Then, $|L(h)| = 2$. Since for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$, we can construct an element $h$ of $C$ from $g$ by removing the color $g(h)$ from vertex $w$ and coloring $h$ with $g(h)$. Then, $|L(h)| ≥ 3$.

For the second case first suppose that $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$. Clearly $μ_4 ≥ max(2, μ_4)$. We claim that for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$. To see why this is so, suppose there is some $h ∈ g \setminus U^g_4$. Then, $|L(h)| = 2$. Since for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$, we can construct an element $h$ of $C$ from $g$ by removing the color $g(h)$ from vertex $w$ and coloring $h$ with $g(h)$. Then, $|L(h)| ≥ 3$.

For case one notice that $μ_4$ and $μ_5$ are positive since $p ≥ 2$. Now, for each $v ∈ V(G)$ such that $|L(v)| ≥ 2$ we arbitrarily delete colors from $L(v)$ until it is of size 2. After the removal of colors from $L(v)$, the list of colors of $v$ is an $L$-coloring of $G$. Clearly, $|L(v)| ≤ 2$ for all $v ∈ V(G)$. Note that $L$ is a $2$-assignment for $G$. Hence, $L$ is an equitable $L$-coloring of $G$.

For the second case first suppose that $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$. Clearly $μ_4 ≥ max(2, μ_4)$. We claim that for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$. To see why this is so, suppose there is some $h ∈ g \setminus U^g_4$. Then, $|L(h)| = 2$. Since for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$, we can construct an element $h$ of $C$ from $g$ by removing the color $g(h)$ from vertex $w$ and coloring $h$ with $g(h)$. Then, $|L(h)| ≥ 3$.

For the second case first suppose that $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$. Clearly $μ_4 ≥ max(2, μ_4)$. We claim that for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$. To see why this is so, suppose there is some $h ∈ g \setminus U^g_4$. Then, $|L(h)| = 2$. Since for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$, we can construct an element $h$ of $C$ from $g$ by removing the color $g(h)$ from vertex $w$ and coloring $h$ with $g(h)$. Then, $|L(h)| ≥ 3$.

We now show that an equitable $L$-coloring of $G$ exists in each of the following three cases:

1. $|μ_4 - μ_5| ≤ 12$
2. $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$ or $μ_4 - μ_5 ≥ 2$.
3. $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$.

For case one notice that $μ_4$ and $μ_5$ are positive since $p ≥ 2$. Now, for each $v ∈ V(G)$ such that $|L(v)| ≥ 2$ we arbitrarily delete colors from $L(v)$ until it is of size 2. After the removal of colors from $L(v)$, the list of colors of $v$ is an $L$-coloring of $G$. Clearly, $|L(v)| ≤ 2$ for all $v ∈ V(G)$. Note that $L$ is a $2$-assignment for $G$. Hence, $L$ is an equitable $L$-coloring of $G$.

For the second case first suppose that $μ_4 - μ_5 ≥ 2$ and $U^g_4 ≠ A$. Clearly $μ_4 ≥ max(2, μ_4)$. We claim that for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$. To see why this is so, suppose there is some $h ∈ g \setminus U^g_4$. Then, $|L(h)| = 2$. Since for each $h ∈ g \setminus U^g_4$, $|L(h)| ≥ 3$, we can construct an element $h$ of $C$ from $g$ by removing the color $g(h)$ from vertex $w$ and coloring $h$ with $g(h)$. Then, $|L(h)| ≥ 3$.
that there is a proper $L'$-coloring of $G'$ that uses no color more than $\rho$ times. Combining such a coloring with $g$ completes an equitable $L$-coloring of $G$.

If instead we have that $\mu_A - \mu_B \geq 2$, we claim that $U^g_B \neq B$. To see why this is so, note that if $U^g_B = B$, then we have that $m_2 = |B| = \mu_B \leq \mu_A - 2 < |A| = m_1$ which is a contradiction. Since $U^g_B \neq B$ an argument similar to the argument employed at the start of the second case can be used to show that there is an equitable $L$-coloring of $G$.

Finally, we turn our attention to case three, and we suppose that $\mu_B - \mu_A \geq 2$ and $U^g_A = A$. Note that clearly $\mu_B \geq \mu_A > 0$. In this case we let $d = \mu_B - \mu_A$. Also, as in case one, for each $v \in V(G')$ such that $|L'(v)| > 2$ we arbitrarily delete colors from $L'(v)$ until it is of size 2 so that $L'$ becomes a 2-assignment for $G'$. By the given bound on $m_2$ and the fact that $U^g_A = A$, we know that:

$$\mu_B = m_2 - \rho(k - 2) \leq \rho(k - 1) - 1 - \rho(k - 2) = \rho - 1.$$ 

Now, let $G'' = G' - \{b_1, \ldots, b_d\}$, and let $L''$ be the 2-assignment for $G''$ obtained by restricting the domain of $L'$ to $V(G'')$. Clearly, $G''$ is a copy of $K_{1, \mu_A} + K_{1, \mu_A}$. Also, $2\mu_A \leq \mu_A + \mu_B = m_1 + m_2 - \rho(k - 2) \leq 15$. Theorem 4 implies that there is an equitable $L''$-coloring $h$ of $G''$; that is, $h$ is a proper $L''$-coloring of $G''$ that uses no color more than $\mu_A + 1$ times. Now, we extend $h$ to a proper $L'$-coloring of $G'$ by coloring each $b_i \in \{b_1, \ldots, b_d\}$ with an element in $L'(b_i) - \{h(u_0)\}$. The proper $L'$-coloring that we obtain clearly uses no color more than $\mu_A + 1 + d = \mu_B + 1$ times which immediately implies that it uses no color more than $\rho$ times. Combining this proper $L'$-coloring with $g$ completes an equitable $L$-coloring of $G$. \qed

Next, we demonstrate that for $k \geq 2$, we can not drop the second inequality in the statement of Theorem 7.

**Proposition 22** Suppose $k \geq 2$. Then, $K_{1,(k-1)(k^3-k+2)} + K_{1,k^3}$ is not equitably $k$-choosable.

Before we begin the proof, notice that if $G = K_{1,(k-1)(k^3-k+2)} + K_{1,k^3}$, then

$$\rho(G, k) = \left\lceil \frac{2 + (k - 1)(k^3 - k + 2) + k^3}{k} \right\rceil = k^3 - k + 3.$$ 

Also, 

$$(k - 1)(k^3 - k + 2) = (k - 1)(\rho - 1) = \rho(k - 1) + 1 - k \leq \rho(k - 1) - 1$$

which means that $G$ satisfies the first inequality in Theorem 7. But for the second inequality, $m_1 + m_2 - \rho(k - 2) = 2(k^3 - k + 2) \geq 16$.

**Proof** Let $G = K_{1,(k-1)(k^3-k+2)} + K_{1,k^3}$. Suppose $G_1$ and $G_2$ are the components of $G$. Moreover, suppose $G_1$ has bipartition $\{w_0\}, A = \{w_i : i \in [(k - 1)(k^3 - k + 2)]\}$, and suppose $G_2$ has bipartition $\{u_0\}, B = \{u_i : i \in [k^3]\}$. We will now construct a $k$-assignment $L$ for $G$ with the property that there is no equitable $L$-coloring of $G$.

For each $v \in V(G_1)$, let $L(v) = [k]$. Also, let $L(u_0) = \{k + 1, k + 2, \ldots, 2k\}$. Now, let $O = \{O_1, \ldots, O_k\}$ be the set of all $(k - 1)$-element subsets of $[k]$. Then, let $P = \{(k + i) \cup O_j : i \in [k], j \in [k]\}$. Clearly, $|P| = k^2$. So, we can name the elements of $P$ so that $P = \{P_1, \ldots, P_{k^2}\}$. Finally, for each $i \in [k^2]$ and $j \in [k]$, let $L(u_{(i-1)k+j}) = P_i$. \hfill \(\Box\) Springer
Now, for the sake of contradiction, suppose that $f$ is an equitable $L$-coloring of $G$. We know that $f$ uses no color more than $\rho = k^3 - k + 3$ times. Without loss of generality, suppose that $f(w_0) = 1$. Then, for each $i \in \{2, \ldots, k\}$, let $a_i = |f^{-1}(i) \cap A|$. Clearly, $\sum_{i=2}^{k} a_i = (k - 1)(k^3 - k + 2) = (k - 1)(\rho - 1)$. Now, suppose that $f(u_0) = d$, and without loss of generality assume that $u_1, \ldots, u_k$ are the $k$ vertices in $B$ that were assigned the list $\{d\} \cup \{2, \ldots, k\}$ by $L$. Then, for each $i \in \{2, \ldots, k\}$, let $b_i = |f^{-1}(i) \cap \{u_j : j \in [k]\}|$. Since $f(u_j) \neq d$ for each $j \in [k]$, we have that $\sum_{i=2}^{k} b_i = k$. We also have that $a_i + b_i \leq |f^{-1}(i)| \leq \rho$ for each $i \in \{2, \ldots, k\}$. So, we see that

$$(k - 1)\rho \geq \sum_{i=2}^{k} (a_i + b_i) = \sum_{i=2}^{k} a_i + \sum_{i=2}^{k} b_i = (k - 1)(\rho - 1) + k = (k - 1)\rho + 1$$

which is a contradiction. \hfill \Box

Finally, we will show that the converse of Theorem 7 does not hold.

**Proposition 23** $K_{1,8} + K_{1,9(k-1)-1}$ is equitably $k$-choosable for all $k \geq 3$.

Notice that for this Proposition: $m_1 = 8$, $m_2 = 9(k - 1) - 1$, $\rho = 9$, and $8 + 9(k - 1) - 1 > 15 + 9(k - 2)$. So, the graph does not satisfy the second inequality in Theorem 7. The proof illustrates how ideas from the proof of Theorem 7 can be applied even in this situation.

**Proof** Let $G = K_{1,8} + K_{1,9(k-1)-1}$, and let the components of $G$ be $G_1$ and $G_2$. Suppose the bipartition of $G_1$ is $\{w_0\}, A = \{w_1, \ldots, w_8\}$ and the bipartition of $G_2$ is $\{u_0\}, B = \{u_1, \ldots, u_{9(k-1)-1}\}$. Let $L$ be an arbitrary $k$-assignment for $G$. For the sake of contradiction, suppose that $G$ is not equitably $L$-colorable. Let $S$ be the set containing all colors that appear in at least 9 of the lists associated with the vertices in $A \cup B$. Suppose we run the $\epsilon$-greedy process on $G$ and $L$.

Suppose we run the $\epsilon$-greedy process on $G$ and $L$. Since $G$ is not equitably $L$-colorable, Lemmas 19 and 20 imply that $|\text{Ran}(g_i)| = k - 2$. Hence, $|S| \geq k - 2$.

For each $(k - 2)$-element subset $P$ of $S$, let $C_P$ denote the set of partial $L$-colorings $f : D \rightarrow L$ of $G$ such that $D \subset A \cup B, |D| = 9(k - 2)$, $\text{Ran}(f) = P$, and each color class associated with $f$ is of size 9. Let $S$ be the set of all such $P$ such that $C_P \neq \emptyset$. Notice $S$ is non-empty since it contains $\text{Ran}(g_i)$. Also, let $S' = \bigcup_{P \in S} P$.

For each $f \in C_P$, let $U_A^f (U_B^f)$ be the subset of vertices in $A$ (resp., $B$) not colored by $f$. (Note that $U_A^f$ and $U_B^f$ are dependent on the choice of $P$. For each $P \in S$, let $g_P : D_P \rightarrow L$ be a function chosen from the elements of $C_P$ so that $|U_A^{g_P}| - |U_B^{g_P}|$ is as small as possible. We will write $g$ instead of $g_P$ when $P$ is clear from context. Let $G_P = G - D_P$. Note that it is possible for $|U_B^{g_P}| = 0$. Also note that $G_P$ is a copy of $K_{1,|U_A^{g_P}|} + K_{1,|U_B^{g_P}|}$. Let $L_P(v) = L(v) - P$ for all $v \in V(G_P)$.

**Observation 1** If $c \notin S'$ then $c$ is not in 9 of the lists associated with the vertices in $U_A^{g_P} \cup U_B^{g_P}$ by $L_P$ for each $P \in S$. Suppose that $c \notin S'$ and $c$ is in 9 of the lists associated with the vertices in $U_A^{g_T} \cup U_B^{g_T}$ by $L_T$ for some $T \in S$. Also suppose that
$t_1 \in T$. We modify $g_T$ as follows: we uncolor the vertices that are colored with $t_1$ and color 9 of the vertices in $U_A^g \cup U_B^g$ that have $c$ in their original lists with $c$. Let $T' = (T \cup \{c\}) - \{t_1\}$. Clearly we see that this new partial coloring of $G$ is in $C_T$. Therefore it must be that $c \in S'$ which is a contradiction.

**Observation 2** $|S'| \geq k - 1$. Suppose that $|S'| < k - 1$. This implies $|S'| = k - 2$ which implies that $|S| = 1$, and let $P$ be the element in $S$. By Observation 1, there is no color that appears in at least 9 of lists associated with the vertices in $U_A^g \cup U_B^g$ by $L_P$. Also note that $|L_P(v)| \geq 2$ for all $v \in V(G_P)$, and $G_P$ is 2-choosable. So, we know that $G_P$ is equitably $L_P$-colorable. Such a coloring of $G_P$ combined with $g_P$ completes an equitable $L$-coloring of $G$ which is a contradiction.

**Observation 3** For all $P \in S$, $G_P = K_{1,8} + K_{1,8}$. Suppose that there exists a $P \in S$ such that $G_P \neq K_{1,8} + K_{1,8}$. Then, $g_P$ colors at least one vertex in $A$ which means $U_A^g \neq A$ and $|U_B^g| - |U_A^g| \geq 2$. This along with the fact that $||U_A^g| - |U_B^g||$ is as small as possible allows us to use the proof idea from case two in Theorem 7 to deduce that $|L_P(v)| \geq 3$ for all $v \in U_B^g$. In the case that there is no color that appears in at least 9 of lists associated by $L_P$ with the vertices in $U_A^g \cup U_B^g$, we can complete an equitable $L$-coloring of $G$ as we did in Observation 2. Otherwise by Lemma 21 we know that there exists a proper $L_P$-coloring of $G_P$ that uses a color no more than 9 times. Such a coloring of $G_P$ combined with $g_P$ completes an equitable $L$-coloring of $G$ which is a contradiction.

**Observation 4** For all $P \in S$, $|L_P(v)| = 2$ for all $v \in U_A^g \cup U_B^g$. Consequently, $P \subseteq L(v)$ for all $v \in U_A^g \cup U_B^g$. Suppose that for some $P \in S$ there exists a $v' \in U_A^g \cup U_B^g$ such that $|L_P(v')| \geq 3$. Without loss of generality we suppose that $v' \in U_A^g$ (this is permissible by Observation 3). Let $G_P' = G_P - \{v'\}$, and note that $G_P'$ is a copy of $K_{1,7} + K_{1,8}$. Also let $L_P'(v) = L_P(v)$ for all $v \in V(G_P')$. We arbitrarily remove colors from $L_P'(v)$ until $|L_P'(v)| = 2$ for all $v \in V(G_P')$. We know by Theorem 4 that $G_P'$ is equitably 2-choosable which implies that there exists an equitable $L_P'$-coloring $h$ of $G_P'$. Note that there can exist at most one color $c \in h(V(G_P'))$ such that $|h^{-1}(c)| = 9$. If there is such a color remove it from $L_P(v)$, and also remove $h(w_0)$ from $L_P(v)$ if $h(w_0) \in L_P(v)$. Coloring $v'$ with a color still in $L_P(v')$ completes an equitable $L$-coloring of $G$.

**Observation 5** For all $P \in S$, $P \subseteq L(v)$ for each $v \in A \cup B$. Consequently, $S' \subseteq L(v)$ for each $v \in A \cup B$. Suppose $P \in S$. If $v \in A$, it is clear that $P \subseteq L(v)$ by Observation 4 since Observation 3 implies $A = U_A^g$. So, suppose for the sake of contradiction that $v' \in A \cup B$ has the property that $P$ is not a subset of $L(v')$. We know that $v' \in B$, and Observation 4 implies that $v' \in B - U_B^g$. So, $v' \in D_P$, and $g(v') \in P$. Now, modify $g$ as follows. Color an element $w \in U_B^g$ with $g(v')$, and remove the color $g(v')$ from $v'$. We know the resulting coloring is still a partial $L$-coloring of $G$ by Observation 4. Let $G'$ be the subgraph of $G$ induced by the vertices of $G$ not colored by this partial $L$-coloring. Notice $G' = K_{1,8} + K_{1,8}$. Let $L'(v) = L(v) - P$ for each $v \in V(G')$. Clearly, $|L'(v)| \geq 3$. So, we can complete an equitable $L$-coloring of
\( G \) by following the argument in Observation 4. This is a contradiction.

We note that Observation 5 implies that \( |S'| \leq k \).

**Observation 6** \( |S'| \neq k \). Consequently, \( |S'| = k - 1 \). Suppose that \( |S'| = k \), and \( S' = \{c_1, c_2, c_3, \ldots, c_k\} \). By Observation 5 we know that \( L(v) = \{c_1, c_2, c_3, \ldots, c_k\} \) for all \( v \in A \cup B \). Color \( G \) as follows:

\[
\begin{aligned}
&h(v) = \\
&\begin{cases}
  c_i & \text{if } v \in \{u_j : 1 + 9(i - 1) \leq j \leq 9i\} \text{ where } i \in [k - 2] \\
  c_{k-1} & \text{if } v \in A \\
  c_k & \text{if } v \in \{u_j : 1 + 9(k - 2) \leq j \leq 9(k - 1) - 1\} \\
  c' & \text{if } v = w_0 \\
  c'' & \text{if } v = u_0
\end{cases}
\end{aligned}
\]

where \( c' \in L(w_0) - \{c_1, \ldots, c_{k-1}\} \) and \( c'' \in L(u_0) - \{c_1, \ldots, c_{k-2}, c_k\} \). Notice that \( h \) is an equitable \( L \)-coloring of \( G \) which is a contradiction.

Now we will complete the proof. By Observation 6 we may suppose that \( S' = \{c_1, c_2, \ldots, c_{k-1}\} \). We know that either: (1) \( (L(u_0) \cup L(w_0)) \cap S' \neq \emptyset \) or (2) \( (L(u_0) \cup L(w_0)) \cap S' = \emptyset \). We handle the first case by considering sub-cases where \( L(u_0) \) contains an element of \( S' \) and where \( L(w_0) \) contains an element of \( S' \). First, without loss of generality suppose \( c_{k-1} \in L(u_0) \), and color \( G \) according to the function \( h \) defined as follows:

\[
\begin{aligned}
&h(v) = \\
&\begin{cases}
  c_i & \text{if } v \in \{u_j : 1 + 9(i - 1) \leq j \leq 9i\} \text{ where } i \in [k - 2] \\
  c_{k-1} & \text{if } v \in A \cup \{u_0\} \\
  d_j & \text{if } v \in \{u_j : 9(k - 2) + 1 \leq j \leq 9(k - 1) - 1\} \\
  c' & \text{if } v = w_0
\end{cases}
\end{aligned}
\]

where \( c' \in L(w_0) - \{c_1, \ldots, c_{k-1}\} \) and \( d_j \in L(u_j) - \{c_1, \ldots, c_{k-1}\} \) for each \( 9(k - 2) + 1 \leq j \leq 9(k - 1) - 1 \). Clearly, \( h \) is an equitable \( L \)-coloring of \( G \) which is a contradiction. Second, without loss of generality suppose \( c_{k-1} \in L(w_0) \), and color \( G \) according to \( h \) defined as follows:

\[
\begin{aligned}
&h(v) = \\
&\begin{cases}
  c_i & \text{if } v \in \{u_j : 1 + 9(i - 1) \leq j \leq 9i\} \text{ where } i \in [k - 2] \\
  c_{k-1} & \text{if } v \in \{u_j : 9(k - 2) + 1 \leq j \leq 9(k - 1) - 1\} \cup \{w_0\} \\
  d_j & \text{if } v \in A \\
  c' & \text{if } v = u_0
\end{cases}
\end{aligned}
\]

where \( c' \in L(u_0) - \{c_1, \ldots, c_{k-1}\} \) and \( d_j \in L(w_j) - \{c_1, \ldots, c_{k-1}\} \) for each \( j \in [8] \). Clearly \( h \) is an equitable \( L \)-coloring of \( G \) which is a contradiction.

In the second case, suppose that \( L(u_0) = \{c'_1, c'_2, c'_3, \ldots, c'_k\} \) and \( L(w_0) = \{c''_1, c''_2, c''_3, \ldots, c''_k\} \). Without loss of generality assume \( P = \{c_2, c_3, \ldots, c_{k-1}\} \in S \). We begin by coloring vertices in \( A \cup B \) according to \( g_P \). By Observation 3, we know that \( U_A^g = A \). Note that
\[
\sum_{i \in [k]} |L^{-1}_p(\{c_1, c_i'\}) \cap U^g_B| \leq 8 \quad \text{and} \quad \sum_{i \in [k]} |L^{-1}_p(\{c_1, c_i''\}) \cap A| \leq 8.
\]

So without loss of generality assume \( |L^{-1}_p(\{c_1, c_1'\}) \cap U^g_B| \leq 8/k \) and \( |L^{-1}_p(\{c_1, c_1''\}) \cap A| \leq 8/k \). Color all vertices in \( (L^{-1}_p(\{c_1, c_1'\}) \cap U^g_B) \cup (L^{-1}_p(\{c_1, c_1''\}) \cap A) \) with \( c_1 \), color \( u_0 \) with \( c_1' \), and color \( w_0 \) with \( c_1'' \). Note that we used \( c_1 \) at most \( 2 \cdot 8/k \) times which is clearly less than 9. So we arbitrarily color uncolored vertices with \( c_1 \) until exactly 9 vertices are colored with \( c_1 \) (this is possible by Observation 5). Let \( U \) be the set containing all uncolored vertices in \( A \cup U^g_B \). Let

\[
L'_p(v) = \begin{cases} 
L_p(v) - \{c_1, c'_1\} & \text{if } v \in (U^g_B \cap U), \\
L_p(v) - \{c_1, c''_1\} & \text{if } v \in (A \cap U).
\end{cases}
\]

Note that \( |U| = 7 \), and \( |L'_p(v)| \geq 1 \) for each \( v \in U \). So, we can color each \( v \in U \) with a color in \( L'_p(v) \). This completes an equitable \( L \)-coloring of \( G \) which is a contradiction. \( \square \)

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