Abstract. The spine of two-particles Fleming-Viot process driven by Brownian motion is not a Bessel-3 process.

1. Introduction

We start with an informal outline of the main idea of this note. A more detailed review, including history and citations, will be presented later in the introduction.

A Fleming-Viot process is a branching process. Under very mild assumptions, it has a unique spine. When the number of individuals in the population is very large, the distribution of the spine is expected to be very close to the distribution of the driving process conditioned on survival forever. There is an example showing that the distribution of the spine may be different from the distribution of the driving process conditioned on survival forever. The published example is rather artificial so we present in this note a different example illustrating the same claim. Our new example is more natural in the sense that it is based on a model examined in a number of papers on Fleming-Viot processes.

1.1. Literature review. Fleming-Viot-type processes were originally defined in [5]. In this model, there is a population of fixed size. Every individual moves independently from all other individuals according to the same Markovian transition mechanism, in a domain with a boundary. When an individual hits the boundary, an individual chosen randomly (uniformly) from the survivors splits into two individuals and the process continues in this manner. The question of whether the process can be continued for all times was addressed in [5, 4, 3, 11]. All of these papers studied, among other process, Fleming-Viot processes driven by Brownian motion.

A very special case of a Fleming-Viot process is when there are only two individuals driven by Brownian motion on [0, ∞) and 0 plays that role of the boundary; this model was studied in [4, 11]. In particular, it was shown in both papers that this process has infinite lifetime.

Every Fleming-Viot process has a unique spine, i.e., a trajectory inside the branching tree that never hits the boundary of the domain where the process is confined; this was proved under strong assumptions in [11, Thm. 4] and later in the full generality in [2].
It was proved in [2] that if the state space is finite and the number of individuals in the population goes to infinity then the distributions of spine processes converge to the distribution of the driving Markov process conditioned on survival forever. The same paper contains an example of a Fleming-Viot process driven by a Markov process on a three-element state space (one of the elements plays the role of the boundary). In that example, the population consists of two individuals and the distribution of the spine is not equal to the distribution of the driving Markov process conditioned on survival forever. A Markov process with a three-element state space seems to be a rather artificial example in the context of Fleming-Viot models. We will show that the spine of the Fleming-Viot process with two individuals driven by Brownian motions on $[0, \infty)$ has a spine with a distribution different from the distribution of Brownian motion conditioned to stay positive, i.e., the distribution of the 3-dimensional Bessel process. The point of this note is to show that proving that the spine does not have the distribution of the 3-dimensional Bessel process is somewhat tricky. In hindsight, this does not seem to be difficult because our proof is quite elementary, Nevertheless, our previous attempts in [6, 7] generated some new results but failed to show the difference. In a sense that will be made more precise later on in the paper, the spine is quite “close” to a 3-dimensional Bessel process and therefore it is quite hard to distinguish the two. The problem has been open for some time and while the solution is not really difficult we now have a better understanding as to why the spine is nevertheless similar to a 3-dimensional Bessel process in certain respects.

2. Model and main result

We will now define a Fleming-Viot process and other elements of the model. Informally, the process consists of two independent Brownian particles starting at the same point in $(0, \infty)$. At the time when one of them hits 0, it is killed and the other one branches into two particles. The new particles start moving as independent Brownian motions and the scheme is repeated.

2.1. Notation and definitions. On the formal side, let $(W_1(t) : t \geq 0)$ and $(W_2(t) : t \geq 0)$ be two independent Brownian motions starting from $W_1(0) = W_2(0) = 1$. Let

\begin{align*}
T_0 &= 0, \\
Y_0 &= 1, \\
\tau_j &= \inf\{t \geq 0 : W_j(t) = 0\}, \quad j = 1, 2, \\
T_1 &= \min(\tau_1, \tau_2), \\
Y_1 &= \max(W_1(T_1), W_2(T_1)),
\end{align*}

and for $k \geq 2$,

\begin{align*}
T_k &= \inf\{t > T_{k-1} : \min(W_1(t) - W_1(T_{k-1}) + Y_{k-1}, W_2(t) - W_2(T_{k-1}) + Y_{k-1}) = 0\}, \\
Y_k &= \max(W_1(T_k) - W_1(T_{k-1}) + Y_{k-1}, W_2(T_k) - W_2(T_{k-1}) + Y_{k-1}).
\end{align*}

It follows from [3, Thm. 5.4] or [11, Thm. 1] that, a.s.,

\begin{equation}
T_k \to \infty.
\end{equation}
Hence, for any $t \geq 0$ we can find $j$ such that $t \in [T_{j-1}, T_j)$. Then we set
\begin{equation}
V(t) = (V_1(t), V_2(t)) = (W_1(t) - W_1(T_{j-1}) + Y_{j-1}, W_2(t) - W_2(T_{j-1}) + Y_{j-1}).
\end{equation}
This completes the definition of $\{V(t), t \geq 0\}$, an example of a Fleming-Viot process.

Let $J_t = J(t)$ denote the spine, i.e., $J_t = V_1(t)$ if $t \in [T_{k-1}, T_k)$ and
\[ W_1(T_k) - W_1(T_{k-1}) + Y_{k-1} > W_2(T_k) - W_2(T_{k-1}) + Y_{k-1}. \]
If the last condition fails, we let $J_t = V_2(t)$ for $t \in [T_{k-1}, T_k)$.

Note that $J(T_k) = Y_k$ for all $k \geq 1$.

Recall that $d$-dimensional Bessel process $X_t$ is defined by
\begin{equation}
dX_t = dB_t + \frac{d-1}{2X_t}dt,
\end{equation}
where $B$ is Brownian motion; see [12, Sect. 3.3 C]. It is well known that Brownian motion on $[0, \infty)$ conditioned to never hit 0 has the transition probabilities of 3-dimensional Bessel process; this theorem was first proved in [8].

2.2. Main result.

**Theorem 2.1.** The distributions of $\{J_t, 0 \leq t < \infty\}$ and $\{X_t, 0 \leq t < \infty\}$ starting from 1 are singular with respect to each other.

We will explicitly define an event that has a strictly positive probability according to the first distribution but not according to the second one, and vice versa.

We will now review two attempts to prove Theorem 2.1 that failed.

The following version of the Law of the Iterated Logarithm was proved in [7].

**Theorem 2.2.** Almost surely,
\begin{equation}
\limsup_{n \to \infty} \frac{Y_n}{\sqrt{2T_n \log \log T_n}} = 1.
\end{equation}

The Law of Iterated Logarithm stated in (4) is the same as that for the 3-dimensional Bessel process (see [14]), which has the same distribution as the one-dimensional Brownian motion conditioned not to hit 0. Hence, Theorem 2.2 does not eliminate the possibility that the spine $J_t$ has the distribution of Brownian motion conditioned not to hit 0.

In this note, we will prove the following result.

**Theorem 2.3.** For every $u > 0$,
\[ \frac{1}{u} \log P \left( \inf_{s \geq 0} X(s) < -u \right) = -1 \leq \lim_{t \to \infty} \frac{1}{u} \log P \left( \inf_{n \geq 0} J(T_n) < -t \right). \]

The general message from Theorems 2.2 and 2.3 is that it is hard to distinguish between the spine and 3-dimensional Bessel process by studying “extreme” behavior of the two processes. The proof of Theorem 2.1 will be based on the analysis of the processes on the “logarithmic scale.”
3. Bessel processes

Recall the stochastic differential equation (3) defining Bessel processes. Let \( f(x) = \log x \). Then \( f'(x) = 1/x \) and \( f''(x) = -1/x^2 \). Let \( A_t = f(X_t) \). Then by the Ito formula

\[
dA_t = df(X_t) = \frac{d-1}{2X_t} \cdot \frac{1}{X_t} - \frac{1}{2} \cdot \frac{1}{X_t^2} \ dt = \frac{1}{X_t} dB_t + \frac{d-2}{2X_t^2} dt
\]

\[
e^{-A_t} dB_t + \frac{d-2}{2} e^{-2A_t} dt.
\]

For 3-dimensional Bessel process, i.e., when \( d = 3 \), the formula is

\[
dA_t = e^{-A_t} dB_t + \frac{1}{2} e^{-2A_t} dt.
\]

We see that the process \( A_t \) is a time change of the process \( B_t + \frac{1}{2} t \), if we use the clock

\[
\rho(t) := \int_0^t e^{-2A_s} ds = \int_0^t \frac{1}{X_s^2} ds.
\]

In other words, \( \{A_{\rho(t)}, t \geq 0\} = \{\log X_{\rho(t)}, t \geq 0\} \) has the same distribution as \( \{B_t + \frac{1}{2} t, t \geq 0\} \). Hence, we have

**Lemma 3.1.** For the 3-dimensional Bessel process \( X_t \) we have a.s.,

\[
\lim_{t \to \infty} \frac{1}{t} \log X_{\rho(t)} = \lim_{t \to \infty} \frac{1}{t} \left( B_t + \frac{1}{2} t \right) = \frac{1}{2}.
\]

**Lemma 3.2.** Suppose that \( X = \{X(t) : t \geq 0\} \) is the 3-dimensional Bessel process with \( X(0) = 1 \). Let \( M = \inf_{t \geq 0} \log X(t) \). Then \(-M\) has the exponential distribution with mean 1.

**Proof.** Time change does not affect the distribution of the infimum of a process, hence \( M \) has the same distribution as \( \min_{t \geq 0} \left( B_t + \frac{1}{2} t \right) \). According to [12, Sect. 3.3, Exercise 5.9], \(-M\) is exponential with mean 1.

4. Logarithmic transformation of Fleming-Viot process

We will use complex representation \( V_1(t) + iV_2(t) \) of the process \( V(t) = (V_1(t), V_2(t)) \) defined in [2]. We apply the complex mapping \( z \mapsto \log z \) to this process so that it is transformed into a process in the strip \( D := \{(x, y) : 0 < y < \pi/2\} \) (see Fig. 1). Consider the following time change,

\[
\phi(t) = \int_0^t \frac{1}{|V(s)|^2} ds.
\]

It follows from conformal invariance of two-dimensional Brownian motion (see [13, Thm. V (2.5)]) that the process \( Z(t) = (Z_1(t), Z_2(t)) := \log V(\phi(t)) \) is two-dimensional Brownian motion jumping from the boundary of \( D \) to an appropriate point in \( D \) every time it exits \( D \). Let \( R_1, R_2, \ldots \) be the times of jumps of \( Z \), and let \( R_0 = 0 \).
Lemma 4.1. We have the following representation
\begin{equation}
Z_1(t) = \log |V(\phi(t))| = B_t + (N_t - 1) \log \sqrt{2},
\end{equation}
where \( \{B_t : t \geq 0\} \) is Brownian motion and \( \{N_t : t \geq 0\} \) is a renewal process independent of \( B \) such that \( N_0 = 1 \), and, a.s.,
\[
\lim_{t \to \infty} \frac{N_t}{t} = \left( \frac{\pi}{4} \right)^{-2}.
\]

Proof. A jump takes \( Z \) from \((Z_1(R_k), Z_2(R_k)) \in \partial D\), i.e., the point at which \( Z \) exits \( D \), to \((\log (\sqrt{2}) + Z_1(R_k), \pi/4)\).

Brownian motions driving \( Z_1 \) and \( Z_2 \) between jumps are independent of each other. The times \( R_k \) are the times when \( Z_2 \) exits \([0, \pi/2]\). Hence, \( \{R_k, k \geq 0\} \) is independent of the Brownian motion \( B_t \) driving \( Z_1(t) \).

Let \( N_t = \inf\{k : R_k > t\} \). Then
\[
Z_1(t) = \log |V(\phi(t))| = B_t + (\inf\{k : R_k > t\} - 1) \log \sqrt{2}
\]
\[
= \log |V(\phi(t))| = B_t + (N_t - 1) \log \sqrt{2}.
\]

The intervals \( \{R_k - R_{k-1}, k \geq 1\} \) are i.i.d. The distribution of each interval is the distribution of the exit time of Brownian motion from \([0, \pi/2]\) starting from its center. It is well known that \( \mathbb{E}[R_k - R_{k-1}] = (\pi/4)^2 \). Thus, by renewal theory (see, [9]), we have a.s.,
\[
\lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[R_2 - R_1]} = \left( \frac{\pi}{4} \right)^{-2}.
\]
It follows from Lemma 4.1 that
\[
\lim_{t \to \infty} \frac{Z_1(t)}{t} = \lim_{t \to \infty} \frac{B_t}{t} + \lim_{t \to \infty} \frac{N_t - 1}{t} \log \sqrt{2} = 0 + \left(\frac{\pi}{4}\right)^2 \log \sqrt{2} = \left(\frac{4}{\pi}\right)^2 \log \sqrt{2} =: \kappa \approx 0.561844. \tag{7}
\]

**Corollary 4.2.** The process \( \{\log J(T_n), n \geq 0\} \) is a random walk, such that \( \log J(T_0) = 0 \), and satisfying
\[
\log J(T_n) = \log J(T_{n-1}) + \log \sqrt{2} + K_n, \quad n \geq 1,
\]
where \( \{K_n, n \geq 1\} \) is an i.i.d. sequence. The distribution of \( K_n \) is that of \( Z_1(R_1) \).

**Proof.** Recall that the process \( B \) in (6) is Brownian motion and \( R_1 \) is the exit time from the interval \([0, \pi/2]\) for a Brownian motion starting at \( \pi/4 \) and independent of \( B \).

Note that
\[
\log J(T_n) - \log J(T_{n-1}) = \log \sqrt{2} + Z_1(R_n) - Z_1(R_{n-1}).
\]
The lemma follows from independence of \( \{R_n, n \geq 1\} \) and \( B \), and the fact that \( R_n - R_{n-1} \overset{d}{=} R_1 \). \( \square \)

**Theorem 4.3.** We have
\[
\lim_{t \to \infty} e^{\gamma t} \mathbb{P}\left( \inf_{n \geq 0} \log J(T_n) < -t \right) = c \in (0, \infty).
\]

**Proof.** Since \( \{\log J(T_n), n \geq 0\} \) is a random walk, the function
\[
t \to \mathbb{P}\left( \inf_{n \geq 0} \log J(T_n) < -t \right) = \mathbb{P}\left( \sup_{n \geq 0} (-\log J(T_n)) > t \right)
\]
satisfies the Wiener-Hopf equation; see [10, point 2, bottom of page 191] for general overview, and [10, Theorem 3.1]. It follows from these references that
\[
\lim_{t \to \infty} e^{\gamma t} \mathbb{P}\left( \inf_{n \geq 0} \log J(T_n) < -t \right) = \lim_{t \to \infty} e^{\gamma t} \mathbb{P}\left( \sup_{n \geq 0} (-\log J(T_n)) > t \right) = c, \tag{8}
\]
where \( \gamma \) is the positive solution to the equation
\[
\mathbb{E}\left[ e^{\gamma(-\log J(T_1))} \right] = \mathbb{E}[J(T_1)^{-\gamma}] = 1.
\]
It follows from [11, (6.21)] that
\[
\mathbb{P}(J(T_1) \in dy) = \frac{2}{\pi} \left[ \frac{1}{(1 - y)^2 + 1} - \frac{1}{(1 + y)^2 + 1} \right].
\]
It is not difficult to check that \( \mathbb{E}[J(T_1)^{-1}] = 1 \). Hence \( \gamma = 1 \) and, therefore, the theorem follows from (8). \( \square \)

**Proof of Theorem 2.3.** The theorem follows from Lemma 3.2 and Theorem 4.3. \( \square \)
5. Comparing the spine and 3-dimensional Bessel process

Recall definition (5) of $\phi$ and let

$$\sigma(t) = \int_0^t \frac{1}{J(s)^2} ds.$$ 

Let $S_0, S_1, \ldots$ be such that $\phi(S_n) = T_n$ and let $H_0, H_1, \ldots$ be such that $\sigma(H_n) = T_n$.

**Lemma 5.1.** For $t_1, t_2 > 0$, if $\sigma(t_1) = \phi(t_2)$ then $t_1 \leq t_2$.

**Proof.** For all $s > 0$ we have $0 < J^2(s) \leq |V(s)|^2 < \infty$. If $t_1 > t_2$ then

$$\sigma(t_1) = \int_0^{t_1} \frac{1}{J(s)^2} ds \geq \int_0^{t_1} \frac{1}{|V(s)|^2} ds > \int_0^{t_2} \frac{1}{|V(s)|^2} ds = \phi(t_2).$$

□

The following lemma gives us the comparison between time $(H_n)$ and $(S_n)$.

**Corollary 5.2.** For all $n \geq 0$ we have $H_n \leq S_n$.

**Proof.** The case $n = 0$ is trivial, since, $H_0 = S_0 = 0$. For $n \geq 1$ we have $\sigma(H_n) = T_n = \phi(S_n)$, hence by Lemma 5.1 we have $H_n \leq S_n$. □

**Lemma 5.3.** Almost surely, $H_n \to \infty$ and $S_n \to \infty$.

**Proof.** Since $(T_n)_{n \geq 0}$ is a non-decreasing sequence, the definition of $H_n$ implies that $(H_n)_{n \geq 1}$ is a non-decreasing sequence. Hence, $L_H := \lim_{n \to \infty} H_n$ exists in $[0, \infty]$.

If the event $\{L_H < \infty\}$ holds then $T_n = \sigma(H_n) \leq \sigma(L_H) < \infty$ for all $n$, and, therefore, $\limsup_{n \to \infty} T_n < \infty$. The probability of the last event is zero according to (1) so $P(L_H < \infty) = 0$. This shows that $H_n \to \infty$, a.s. Corollary 5.2 implies that $S_n \to \infty$, a.s. □

**Lemma 5.4.** If $\kappa$ is defined as in (7) then, a.s.,

$$\limsup_{t \to \infty} \frac{\log J(\sigma(t))}{t} \geq \kappa > \frac{1}{2}.$$ 

**Proof.** Since $S_n \to \infty$, we can use (6) and (7) to conclude that,

$$\lim_{n \to \infty} \frac{\log |V(\phi(S_n))|}{S_n} = \lim_{t \to \infty} \frac{Z_1(t)}{t} = \kappa, \text{ a.s.}$$

Note that $\sqrt{2}J(\sigma(H_n)) = \sqrt{2}J(T_n) = |V(T_n)| = |V(\phi(S_n))|$ and recall that $S_n \leq H_n$. Thus

$$\limsup_{n \to \infty} \frac{\log J(\sigma(H_n))}{H_n} = \limsup_{n \to \infty} \frac{\log |V(\phi(S_n))|}{S_n} - \log \sqrt{2} = \kappa.$$ 

The lemma follows since $H_n \to \infty$. □

**Proof of Theorem 2.1.** For a process $U_t > 0$, let

$$\alpha(t) = \int_0^t \frac{1}{U_s^2} ds,$$ 

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\[ F(U) = \left\{ \lim_{t \to \infty} \frac{\log U(\alpha(t))}{t} = \frac{1}{2} \right\}. \]

According to Lemmas 3.1 and 5.4, \( P(F(X)) = 1 \) while \( P(F(J)) = 0 \). \( \square \)

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