Mixed spin-1/2 and 3/2 Ising model with multi-spin interactions on a decorated square lattice

V. Štubňa*, M. Jaščur*a,*

aDepartment of Theoretical Physics and Astrophysics, Institute of Physics, P.J. Šafárik University in Košice, Park Angelinum 9, 040 01 Košice, Slovakia

Abstract

A mixed spin-1/2 and spin-3/2 Ising model on a decorated square lattice with a nearest-neighbor interaction, next-nearest-neighbor bilinear interaction, three-site four-spin interaction and single-ion anisotropy is exactly investigated using a generalized decoration-iteration transformation, Callen-Suzuki identity and differential operator technique. The ground-state and finite-temperature phase boundaries are obtained by identifying all relevant phases corresponding to minimum internal or free energy of the system. The thermal dependencies of magnetization, correlation functions, entropy and specific heat are also calculated exactly and the most interesting cases are discussed in detail.

Keywords: mixed-spin Ising model, many-body interactions, exact results, decorated lattice, phase transitions.

1. Introduction

The investigation of higher-order spin couplings has been initiated many decades ago by Anderson [1] and Kittel [2] who have studied the role of biquadratic exchange interactions of the form $S_i^z S_j^z$, in connection with the superexchange interaction and elastic properties of magnetic materials. Later, the higher-order spin interactions have been experimentally found in magnetic compounds MnO and NiO [3]-[5]. Since then, various types of multi-spin interactions have been intensively studied in different physical systems, in order to explain diverse physical phenomena (see for example [6] and references therein). It has been found that these interactions are usually much weaker than the standard pair Heisenberg exchange coupling, however, due to their non-conventional symmetries they may significantly modify many physical quantities in the systems under investigation.

In this work we will study a special kind of higher-order spin interactions, that are usually called as three-site four-spin interactions due to their geometry. These interactions have been originally introduced by Iwashita and Uryu [7], in order to describe

*This work has been supported under grant VEGA No. 1/0234/14 and APVV-14-0073
*Corresponding author

Email addresses: villamstubna@yahoo.com (V. Štubňa), michal.jascur@upjs.sk (M. Jaščur)
magnetic properties of some clustered complex systems. In general these interaction take the form of \((S_i S_j)(S_j S_k)\) and as a rule, they principally modify physical properties of various localized magnetic system [8-17]. Among recent works in this research field one should also notice our study of a decorated exactly solvable mixed spin-1/2 and spin-1 Ising model with three-site fours-spin interaction [6]. In that work we have found that the such a decorated planar system may exhibit unusually interesting and rich magnetic behavior, among others including also the existence of phases with non-zero ground state entropy. Owing to many interesting phenomena found in our previous study [6] it is of interest to understand the role of varying spin value in the three-site four-spin exchange interaction and to clarify which magnetic properties will be significantly changed due to varying spin of decorating atoms. For this purpose, we will study in this work a bond-decorated square lattice consisting of nodal atoms with spin 1/2 and decorating atoms with spin 3/2. The Hamiltonian of the systems will include, except of standard bilinear interactions terms, also an unconventional three-site four-spin interactions. The role of single-ion anisotropy will be also taken into account. The outline of this paper is as follows. In Section 2 we briefly summarize the application of decoration-iteration transformation to obtain exact relations for all relevant physical quantities. In Section 3 we discuss the most interesting numerical results and finally main conclusions are summarized in the last section.

2. Formulation

The subject of our study is a mixed spin-1/2 and spin-3/2 Ising model with bilinear, three-site four-spin interactions and single ion anisotropy on a decorated square lattice as it is depicted in Fig 1. As one can see from the figure, the system consists of \(N\) spin-1/2 atoms located on the sites of square lattice and \(2N\) decorating spin-3/2 atoms that occupy all bonds of the original square lattice.

Thus, as a whole the system can be treated as a two-sublattice mixed-spin system with unequal number of sub-lattice atoms which can be described by the Hamiltonian

\[
\mathcal{H} = \sum_{k=1}^{2N} \mathcal{H}_k, \tag{1}
\]

where the summation runs over all bonds of the original square lattice and \(\mathcal{H}_k\) represents the Hamiltonian of \(k\)-th bond which takes the following explicit form

\[
\mathcal{H}_k = -J'\mu_{k1}\mu_{k2} - JS_k (\mu_{k1} + \mu_{k2}) - J_4 S_k^2 \mu_{k1}\mu_{k2} - DS_k^2. \tag{2}
\]

Here the parameter \(J\) denotes the bilinear exchange interaction between nearest neighbors (n.n) \(\mu_{k1} - S_k\), parameter \(J' > 0\) represents the bilinear interaction between next-nearest neighbors (n.n.n) \(\mu_{k1} - \mu_{k2}\), the parameter \(J_4\) stands for the three-site four-spin interaction between \(\mu_{k1} - S_k^2 - \mu_{k2}\) spins and \(D\) denotes a single-ion anisotropy. The interaction parameters \(J, J_4\) and \(D\) are allowed to take arbitrary positive or negative values and the spin variables take obviously the following values: \(\mu_{k1} = \pm 1/2\) and \(S_k = \pm 1/2, \pm 3/2\). One should notice here that the summation in (1) must be performed in a way that accounts for different spin terms only once.
Using Eq. (1) the partition function for the present system can be written as

\[ Z = \sum_{\{\mu_{ki}\}} \sum_{\{s_k\}} \exp \left( -\beta \sum_{k=1}^{2N} \mathcal{H}_k \right) \]

\[ = \sum_{\{\mu_{ki}\}} \prod_{k} \sum_{s_k=\pm \frac{1}{2}} \sum_{s_k=\pm \frac{3}{2}} \exp (-\beta \mathcal{H}_k), \quad (3) \]

where \( \beta = 1/k_B T \), \( k_B \) is the Boltzmann constant, \( T \) is absolute temperature and lastly \( \sum_{\{\mu_{ki}\}} \) and \( \sum_{\{s_k\}} \) mean summation over degrees of freedom of \( \mu_{ki} \) and \( s_k \) spins, respectively.

Now introducing the following generalized decoration-iteration transformation [18]-[21],

\[ \sum_{s_k=\pm \frac{1}{2}} \exp(-\beta \mathcal{H}_k) = A e^{\beta R \mu_{k1} \mu_{k2}}, \quad (4) \]

one may recast the partition function of the system in the form

\[ Z = A^{2N} Z_0(\beta R) \quad (5) \]

where \( Z_0 \) represents the partition function of the original (undecorated) square lattice with \( N \) spin-1/2 atoms that interact via nearest-neighbor effective exchange interaction \( R \). Here one should recall that exact analytical expression of \( Z_0 \) is well known from the Onsager seminal work [22]. The expression \( A^{2N} \) represents the contribution of the decorating spins to the total partition function.
Both unknown parameters $A$, $R$ entering Eq. (5) can be straightforwardly evaluated performing the summation on the l.h.s. in Eq. (4) and substituting $\mu_k = \pm 1/2$ and $\mu_{k2} = \pm 1/2$ into resulting expression. In this way one easily gets the following relations
\begin{align*}
A &= (V_1V_2)^{1/2}, \\
\beta R &= 2 \ln \frac{V_1}{V_2},
\end{align*}
where
\begin{align*}
V_1 &= 2e^{(3/2)\beta J} e^{\beta J_4} e^{\beta D} K_1, \\
V_2 &= 2e^{-(3/2)\beta J} e^{\beta J_4} e^{\beta D} K_2,
\end{align*}
with
\begin{align*}
K_1 &= \cosh \left( \frac{3}{2} \beta J \right) + e^{-\frac{1}{2} \beta J_4} e^{-2\beta D} \cosh \left( \frac{1}{2} \beta J \right) \\
K_2 &= 1 + e^{\frac{1}{2} \beta J_4} e^{-2\beta D}.
\end{align*}
Having obtained exact mapping relations (6)-(11), and exact expression for the partition function of the system (5), we are now able to gain exact equations for phase boundaries and exact analytical relations for all physical quantities of interest.

At first, after substituting the value of inverse critical temperature of the square lattice, $\beta_c R = 2 \ln \left(1 + \sqrt{2} \right)$, into l.h.s of (7), we obtain formula for finite-temperature phase diagrams of the decorated system in the form
\begin{equation}
1 + \sqrt{2} = \frac{V_{1c}}{V_{2c}},
\end{equation}
where $V_{1c} = V_1(\beta_c)$, $V_{2c} = V_2(\beta_c)$ and $\beta_c = 1/k_B T_c$.

Next, using the relation $F = -k_B T \ln Z$ one simply obtains from Eq. (5) the following relation for the Helmholtz free energy of the entire system
\begin{equation}
F(\beta, J, J_4, J', D) = -2N \beta^{-1} \ln A(\beta, J, J_4, J', D) + F_0(\beta, R),
\end{equation}
where parameters $A$, $R$ are given by Eq. (6)-(7) and $F_0(\beta, R)$ represents the Helmholtz free energy of the original undecratered Ising square lattice \cite{22}
\begin{equation}
F_0 = -\frac{N}{\beta} \left[ \ln \left( 2 \cosh \frac{\beta R}{2} \right) + \frac{1}{2\pi} \int_0^\pi \zeta(\phi) d\phi \right],
\end{equation}
with
\begin{equation}
\zeta(\phi) = \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1 - \kappa^2 \sin^2 \phi} \right) \right],
\end{equation}
and
\begin{equation}
\kappa = \frac{2 \sinh \left( \frac{\beta R}{2} \right)}{\cosh^2 \left( \frac{\beta R}{2} \right)}.
\end{equation}
Now, the entropy $S$ and specific heat $C$ can be calculated from Eq. (13) using the following thermodynamic relations

\[ S = -\left( \frac{\partial F}{\partial T} \right)_V, \quad C_V = -T \left( \frac{\partial^2 F}{\partial T^2} \right)_V, \]  

and consequently the internal energy can be also easily obtained using equation

\[ U = F + TS. \]  

In addition to the above mentioned thermodynamic quantities, we will also investigate the spin-ordering in all possible phases of the system. For this purpose it is necessary to analyze the total and sub-lattice magnetization along with various spin-correlation functions.

The total magnetization per one site of the decorated lattice is given by

\[ m = \frac{m_A + 2m_B}{3}, \]  

where the sub-lattice magnetization $m_A$ and $m_B$ are respectively given by

\[ m_A = \langle \mu_k \rangle = \frac{1}{Z} \sum_{\{\mu_k\}} \sum_{\{s_k\}} \mu_k \exp \left( -\beta \sum_{k=1}^{2N} \mathcal{H}_k \right) \]  

\[ m_B = \langle S_k \rangle = \frac{1}{Z} \sum_{\{\mu_k\}} \sum_{\{s_k\}} S_k \exp \left( -\beta \sum_{k=1}^{2N} \mathcal{H}_k \right) \]  

The calculation of $m_A$ is a particularly simple task, since using Eqs. (4) and (5) one obtains from (20) the following relation \[ f(\mu_{k1}, \mu_{k2}, \ldots, \mu_{ki}) = f(\mu_{k1}, \mu_{k2}, \ldots, \mu_{ki})_0. \]  

Here $f$ represents an arbitrary function depending exclusively on the spin variables of a sublattice. Thus, setting $f(\mu_{k1}, \mu_{k2}, \ldots, \mu_{ki}) = \mu_{ki}$ one obtains $\langle \mu_{ki} \rangle = \langle \mu_{ki} \rangle_0 = m_0$, where $m_0$ represents the magnetization per one lattice site of the original square lattice which has been exactly calculated by Yang \[ \langle \mu_{ki} \rangle_0 = \frac{1}{2} \left( 1 - \frac{16 e^{-2\beta R}}{1 - e^{-2\beta R}} \right)^{\frac{1}{8}}. \]  

On the other hand, for the calculation of $m_B$ i.e., the mean value of $\langle S_k \rangle$, one can use the exact Callen-Suzuki identity \[ \langle f(\mu_{k1}, \mu_{k2}, \ldots, \mu_{ki}) \rangle = \langle f(\mu_{k1}, \mu_{k2}, \ldots, \mu_{ki})_0 \rangle, \]  

as a starting point.

\[ \langle S_k \rangle = \left( \frac{\sum_{S_k} S_k e^{-\beta \mathcal{H}_k}}{\sum_{S_k} e^{-\beta \mathcal{H}_k}} \right), \]  

where $\mathcal{H}_k$ is defined in Eq. (2). After performing the summation over $S_k$ in previous equation one obtains

\[ \langle S_k \rangle = \frac{1}{2} \left( 3 \sinh \left( \frac{3\beta \hbar}{2} \right) + e^{-2\beta(h_4+D)} \sinh \left( \frac{3\beta \hbar}{2} \right) \right) \]  

\[ \cosh \left( \frac{3\beta \hbar}{2} \right) + e^{-2\beta(h_4+D)} \cosh \left( \frac{3\beta \hbar}{2} \right), \]
where we have denoted effective fields acting on the $k$-th lattice site as
\[ h = J(\mu_k + \mu_{k2}), \quad h_4 = J_4 \mu_k \mu_{k2}. \] (26)

In order to calculate the ensemble average in last equation it is very comfortable to utilize the differential operator method which is based on the following relations
\[ f(x + \lambda x, y + \lambda y) = e^{(\lambda \nabla_x + \lambda \nabla_y)} f(x, y) \] (27)
\[ e^{\alpha \mu} = \cosh \left( \frac{\alpha}{2} \right) + 2 \mu \sinh \left( \frac{\alpha}{2} \right), \quad \mu = \pm \frac{1}{2} \] (28)
where $\nabla_x = \partial/\partial x$, $\nabla_y = \partial/\partial y$ are standard differential operators and $\alpha$ stands for an arbitrary parameter.

Now, with the help of (27) and (28) one obtains for the sublattice magnetization $m_B$ the following simple expression
\[ \langle S_k \rangle = \langle \mu_k \rangle A_1, \] (29)
where
\[ A_1 = \frac{3 \sinh \left( \frac{\beta J}{2} \right) + e^{-\frac{\beta J}{2} e^{-2 \beta D} \sinh \left( \frac{\beta J}{2} \right)}}{\cosh \left( \frac{\beta J}{2} \right) + e^{-\frac{\beta J}{2} e^{-2 \beta D} \cosh \left( \frac{\beta J}{2} \right)}}. \] (30)

Subsequently, the total magnetization per one lattice site (19) for the present system can be explicitly written in the form
\[ m = \frac{1}{3} \langle \mu_k \rangle (1 + 2 A_1). \] (31)

Having obtained the total reduced magnetization, one can now calculate the compensation temperature $T_k$ from the condition $m = 0 \land m_A \neq 0 \land m_B \neq 0$ [30]. By means of this definition, we derive the condition for $T_k$ in the following form
\[ 7w^2 - 5w^{-\frac{1}{2} \alpha} + 3w^{\gamma+} - w^{\gamma-} = 0 \] (32)
where we have defined the following terms
\[ w = e^{\beta J_4} \]
\[ \gamma_+ = -\frac{1}{2} (4d + 1 + \alpha) \]
\[ \gamma_- = -\frac{1}{2} (4d + 1 - \alpha). \]

Of course, the equation (32) has to be used in line with inequality $T_k < T_c$. In a similar way, one also obtains equations for the quadrupolar moment and various spin-correlation...
functions:

$$q_B = \langle S_k^2 \rangle = \frac{1}{8} (A_2 + A_3) + \frac{1}{2} \langle \mu_{k1} \mu_{k2} \rangle (A_2 - A_3),$$  \hspace{1cm} (33)

$$\langle S_k \mu_{k1} \rangle = \left( \frac{1}{8} + \frac{1}{2} \langle \mu_{k1} \mu_{k2} \rangle \right) A_1,$$  \hspace{1cm} (34)

$$\langle S_k^2 \mu_k \rangle = \frac{1}{4} \langle \mu_k \rangle A_2,$$  \hspace{1cm} (35)

$$\langle S_k^2 \mu_{k1} \mu_{k2} \rangle = \frac{1}{32} (A_2 - A_3) + \frac{1}{8} \langle \mu_{k1} \mu_{k2} \rangle (A_2 + A_3),$$  \hspace{1cm} (36)

where the coefficients $A_2$ and $A_3$ are defined as

$$A_2 = \frac{9 \cosh \left( \frac{3}{2} \beta J \right) + e^{-\frac{1}{2} \beta J} e^{-2\beta D} \cosh \left( \frac{3}{2} \beta J \right)}{\cosh \left( \frac{3}{2} \beta J \right) + e^{-\frac{1}{2} \beta J} e^{-2\beta D} \cosh \left( \frac{3}{2} \beta J \right)},$$  \hspace{1cm} (37)

$$A_3 = \frac{9 + e^{\frac{3}{2} \beta J} e^{-2\beta D}}{1 + e^{\frac{3}{2} \beta J} e^{-2\beta D}}.$$  \hspace{1cm} (38)

We recall here that average values entering r.h.s. of (29), (33)-(36) can be simply evaluated, since on the basis of Eq. (22) one obtains $\langle \mu_{k1} \rangle = \langle \mu_{k1} \rangle_0$ and $\langle \mu_{k1} \mu_{k2} \rangle = \langle \mu_{k1} \mu_{k2} \rangle_0$.

3. Numerical results

In this part we will present the most interesting results obtained numerically from equations presented in the previous section. For the sake of simplicity, we introduce the following dimensionless parameters $\alpha = J/J_4$, $d = D/J_4$, $\lambda = J'/J_4$.

3.1. Ground-state phase diagrams

The ground-state phase diagram of the present system has been obtained by investigating the internal energy of all relevant spin configurations at $T = 0$. Our findings are summarized in Fig. 2 where we have depicted the phase diagram in $\alpha - d$ plane which is valid for arbitrary $\lambda$. As one can see, the whole parameter space is divided into four regions in which different ordered magnetic phases can be found. Namely:

I. the ferrimagnetic phase with $m_A = 1/2$, $m_B = -3/2$ and $q_B = 9/4$, for $\alpha < 0$ and $d > \alpha/2 - 1/4$.

II. the ferromagnetic phase with $m_A = 1/2$, $m_B = 3/2$ and $q_B = 9/4$, for $\alpha > 0$ and $d > -\alpha/2 - 1/4$.

III. the ferrimagnetic phase with $m_A = 1/2$, $m_B = -1/2$ and $q_B = 1/4$, for $\alpha < 0$ and $d < \alpha/2 - 1/4$.

IV. the ferromagnetic phase with $m_A = 1/2$, $m_B = 1/2$ and $q_B = 1/4$, for $\alpha > 0$ and $d < -\alpha/2 - 1/4$.

The boundaries separating these regions represent the lines of first-order phase transitions along which relevant couples of phases co-exist. Therefore, for $d = \alpha/2 - 1/4$ and $\alpha < 0$, we have found the phase with $m_A = 1/2$, $m_B = -1$ and $q_B = 5/4$, while for
Figure 2: The ground-state phase diagram in $\alpha - d$ space for arbitrary $\lambda$. In different sectors there are phases with different spin configurations indicated in the figure.

d = -\alpha/2 - 1/4 and $\alpha > 0$, one finds $m_A = 1/2$, $m_B = 1$ and $q_B = 5/4$. Similarly, for the case of pure three-site four spin interaction, (i.e. $\alpha = 0$), we have again the co-existence of relevant phases, but now the resulting phase will be only partially ordered, since each decorating atom occupies equally likely $\pm 3/2$ or $\pm 1/2$ spin states, respectively. Consequently, for $\alpha = 0$ and $d > -1/4$ one gets $m_A = 1/2$, $m_B = 0$ and $q_B = 9/4$, while for $\alpha = 0$ and $d < 1/4$ one finds $m_A = 1/2$, $m_B = 0$ and $q_B = 1/4$. Finally, the point with co-ordinates $(\alpha, d) = (0, -1/4)$ represents a special point in which coexist all four ordered phases, so that the minimum of the internal energy now corresponds to the partially ordered phase with $m_A = 1/2$, $m_B = 0$ and $q_B = 5/4$, since now all the states $S_k = \pm 1/2, \pm 3/2$ on B sublattice are occupied equally likely. Here one should notice that the disorder appearing along the phase boundary $\alpha = 0$ will naturally generate non-zero values of the entropy at the ground state. This interesting phenomenon will be discussed in detail in Subsection 3.2.

3.2. Finite-temperature phase diagrams and thermodynamic properties

In order to investigate thermal properties of the system, we have at first calculated critical and compensation temperatures using Eq. (12) and Eq. (32). In Figs. 3-5 we have depicted representative results by selecting various combinations of model parameters. At first, in Fig. 3 we have shown the results in the $\alpha - T_c$ space for $\lambda = 1.0$ and some characteristic values of the single-ion anisotropy parameter $d$. In the figure, the solid and dotted curves correspond to critical and compensation temperatures, respectively. In agreement with the ground-state analysis, one finds the standard ferri- or ferromagnetic phases to be stable at low temperatures for $\alpha < 0$ or $\alpha > 0$, respectively. On the other hand, the standard paramagnetic phase exists above each phase boundary. It is therefore clear that the normalized spontaneous magnetization of these phases will take its saturation value at $T = 0$ and then will gradually decrease with increasing temperature, until it continuously vanishes at the corresponding critical temperature. Moreover one can see that for fixed values of $\lambda$ and $d$ one can observe the compensation effect by choosing appropriate negative values of the parameter $\alpha$. We have also investigated other cases
and we have found that independently of the values of $d$ and $\lambda$ all phase boundaries exhibit a symmetric U-shape form with minimum values at $\alpha = 0$. Here one should recall that the phase boundary for $\alpha = 0$ represents the critical temperatures of partially ordered phase. Intuitively one can simply understand the minimum value of $T_c$, since in the partially ordered phase (i.e. for $\alpha = 0$) only the sublattice A exhibits a non-zero magnetization and therefore in this case it is easier to destroy the long-range order than that one appearing in the fully ordered ferrimagnetic or ferromagnetic phases. Next, in

To demonstrate the influence of the n.n.n. bilinear interaction on physical behavior of our system, we have studied critical boundaries and compensation temperatures in $\lambda - T$ space. Our findings are summarized in Fig. 4 for $|\alpha| = 0.1$ and characteristic
values of the parameter $d$. As expected, all phase boundaries exhibit almost a perfect linear dependence with the increasing strength of the parameter $\lambda$ and the compensation temperatures are again clearly visible for $\alpha = -0.1$ and appropriate combinations of the parameters $\alpha$ and $d$.

Finally, we have studied the critical and compensation temperatures in $d - T_c$ space. As one can see from Figs. 5 the phase boundaries have very similar shapes as a phase diagram of the standard spin-$3/2$ Blume-Capel model [31]. This typical behavior of the system is mainly driven by the variation of the crystal-field parameter $d$ and it is clear that on the sublattice B positive values of $d$ promote the spin states $\pm 3/2$, while the negative values prefer the occupation of $\pm 1/2$ spin states. One should emphasize here that such suppressing or favoring of the relevant spin states on the B sublattice has a principal influence on the three-site four-spin interaction term which which takes the form of $-J_4 S_2^z \mu_1 \mu_2$. As a matter of fact, one easily identifies that at the ground state this three-site term reduces to $-9J_4 \mu_1 \mu_2/4$ or $-J_4 \mu_1 \mu_2/4$ for $d \to \infty$ or $d \to -\infty$, respectively. Thus it is clear that for strong values of the crystal field the three-site four-spin interaction acts as an effective n.n.n pair interaction which substantially reinforces the effect of parameter $\lambda$ and consequently keeps high values of the sublattice magnetization $m_A$ even at higher temperature region. Due to this interesting effect, the compensation temperatures may exist in very large regions of $d$ and they can exhibit very interesting behavior. In fact, we have found that for a non-zero $\lambda$ and some appropriate values of $\alpha$ the compensation temperature always takes its saturation value for $d \to -\infty$ and it very slowly increases with increasing the crystal-field parameter. On the other hand, the change of relevant compensation curves becomes more dramatic in the region of $d > -2.0$, where for $\alpha < -0.211$ each compensation temperature terminates at the relevant critical boundary, while for $\alpha > -0.211$ the existence of compensation temperatures extends up to infinite values of the crystal-field parameter. Numerical analysis of the critical and compensation temperatures has clearly revealed several interesting physical phenomena that appear in our system due to the presence of unconventional three-site four-spin interaction. To put further insight on thermodynamic properties of

Figure 5: The critical (solid curves) and compensation temperatures (dashed curves) in $d - T$ space for $\lambda = 1.0$ and different values of $\alpha$. 

Finally, we have studied the critical and compensation temperatures in $d - T_c$ space. As one can see from Figs. 5 the phase boundaries have very similar shapes as a phase diagram of the standard spin-$3/2$ Blume-Capel model [31]. This typical behavior of the system is mainly driven by the variation of the crystal-field parameter $d$ and it is clear that on the sublattice B positive values of $d$ promote the spin states $\pm 3/2$, while the negative values prefer the occupation of $\pm 1/2$ spin states. One should emphasize here that such suppressing or favoring of the relevant spin states on the B sublattice has a principal influence on the three-site four-spin interaction term which which takes the form of $-J_4 S_2^z \mu_1 \mu_2$. As a matter of fact, one easily identifies that at the ground state this three-site term reduces to $-9J_4 \mu_1 \mu_2/4$ or $-J_4 \mu_1 \mu_2/4$ for $d \to \infty$ or $d \to -\infty$, respectively. Thus it is clear that for strong values of the crystal field the three-site four-spin interaction acts as an effective n.n.n pair interaction which substantially reinforces the effect of parameter $\lambda$ and consequently keeps high values of the sublattice magnetization $m_A$ even at higher temperature region. Due to this interesting effect, the compensation temperatures may exist in very large regions of $d$ and they can exhibit very interesting behavior. In fact, we have found that for a non-zero $\lambda$ and some appropriate values of $\alpha$ the compensation temperature always takes its saturation value for $d \to -\infty$ and it very slowly increases with increasing the crystal-field parameter. On the other hand, the change of relevant compensation curves becomes more dramatic in the region of $d > -2.0$, where for $\alpha < -0.211$ each compensation temperature terminates at the relevant critical boundary, while for $\alpha > -0.211$ the existence of compensation temperatures extends up to infinite values of the crystal-field parameter. Numerical analysis of the critical and compensation temperatures has clearly revealed several interesting physical phenomena that appear in our system due to the presence of unconventional three-site four-spin interaction. To put further insight on thermodynamic properties of
the system, let us discuss the most interesting thermal variation of the magnetization, entropy, specific heat and Helmholtz free energy. As far as concerns the magnetization, we restrict ourselves to the ferrimagnetic case, in order to illustrate existence of compensation points at finite temperatures. For this purpose, in Fig. 6 we have depicted several temperature dependencies of the absolute value of total magnetization per one atom by selecting suitable combinations of all relevant parameters. The presented curves are in perfect agreement with our results presented in Figs. 2-5. Here one should mention that the detailed theoretical investigation of magnetic systems exhibiting compensation temperatures is also of interest in connection with development of new recording media.

Now let us turn our attention to the analysis of the thermal variations of entropy and specific heat. Here we will preferably discuss the partially ordered phase i.e. $\alpha = 0$
and also the phases that are stable along the phase boundaries given by equations $d = \pm 0.5\alpha - 0.25$. As we have already mentioned above, in the case of $\alpha = 0$ the sublattice B remains strongly disordered down to zero temperature, while the sublattice A exhibits the standard long-range order at $T = 0$, thus the non-zero entropy appears at the ground state. The situation is shown in Fig. 7 where we have presented the thermal variations of the entropy for $\alpha = 0$ and $\lambda = 1.0$ and several typical negative and positive values of $d$. As one can see from the figure, for $d = -0.25$ we have obtained $S/Nk_B = \ln(2^4) = 2.7726$ while for all other values of $d \neq -0.25$ one finds $S/Nk_B = \ln(2^2) = 1.3863$. Here one should notice that for $\alpha = 0$ remain the ground-state values of the entropy unchanged even for arbitrary non-negative $\lambda$.

![Figure 8: Thermal variations of the reduced magnetic entropy for $\alpha = \pm 1$, $\lambda = 1.0$ and several characteristic values of the parameter $d$.](image)

Next, in order to demonstrate the role of n.n. pair interaction, we have in Fig. 8 depicted temperature dependencies of the entropy for $\alpha = \pm 1$, $\lambda = 1.0$ and some generic values of $d$. In this case only one non-zero value of the entropy appears for $d = -0.75$, which corresponds to the ground-state phase point located exactly on the line given by the equation $d = \pm 0.5\alpha - 0.25$ and again, this situation does not change for non-negative values of $\lambda$. Moreover, our results also indicate that the entropy does not depend on the sign of parameter $\alpha$, thus for arbitrary fixed values of $\lambda$ and $d$, the system takes exactly the same values of entropy for the ferromagnetic as well as for the ferrimagnetic equilibrium thermodynamic states. Of course, the described behavior of entropy is in a full agreement with our previous discussion. To complete our investigation of thermal properties of our system we have calculated the magnetic part of the specific heat. Our main finding are illustrated in Fig. 9 and Fig. 10 for several representative combinations of parameters. As one can see, all curves go to zero value for $T \to 0$ in agreement with the Third Law of Thermodynamics and also each curve exhibits Onsager type singularity at the corresponding critical temperature. We can also observe that several thermal dependencies of the specific heat exhibit a very clear local maximum at low-temperature region. This phenomenon is observable whenever the relevant set of parameters is selected from the close neighborhood of ground-state phase boundaries. In such a case there appears a strong mixing of different spin states on the B sublattice, since the relevant
Figure 9: Temperature dependencies of the reduced magnetic specific heat for $\alpha = 0$, $\lambda = 0$ and several characteristic values of the parameter $d$.

Figure 10: Temperature dependencies of the reduced magnetic specific heat for $\alpha = \pm 1$, $\lambda = 0$ and several characteristic values of the parameter $d$.

stets are very easily thermally excited.

4. Conclusion

In this paper we have investigated phase diagrams and thermal properties of the complex mixed spin-1/2 and spin-3/2 Ising model on decorated square lattice. We have mainly concentrated on understanding of the influence of three-site four-spin interaction on magnetic properties of the system. Applying the generalized decoration-iteration transformation we have obtained exact results for phase diagrams and all relevant thermodynamic quantities of the model. We have also clearly demonstrated that due to multi-spin interactions the model exhibits several unexpected features, for example, the existence of a partially ordered phase or non-zero ground state entropy. Comparing the
present results with those obtained in our previous work for the decorating spin value of $S_B = 1$, one can then formulate the following general physical statements:

1. On the contrary to four-site four-spin coupling, the three-site four spin interaction is able to generate a partially ordered phase even in the magnetic systems without bilinear interactions. This partially ordered phase can be stable in a wide temperature region and it exhibits a second-order phase transition at some critical temperature, which can depend on other physical parameters of the model, such as the crystal field.

2. Due to its special symmetry with respect to spatial reversal of spins, the three-site four-spin interaction always suppresses the long-range ordering in arbitrary magnetic phase.

3. In the mixed-spin magnetic systems with three-site four-spin interactions the paramagnetic phase can by stable at $T = 0$ for negative values of the crystal field, whenever atoms of the one sublattice are integer (i.e. $S_B = 1, 2, ...$). This behavior is impossible to observe in the systems with a half-integer values of $S_B$.

In general, the theoretical investigation of the systems with many-body interactions is extraordinarily complicated task, however, the localized-spin models represent an excellent basis for deep understanding of various many-body interactions going beyond the standard pair-wise picture. For that reason, we hope that the present study may initiate a wider interest in investigation of magnetic systems with multi-spin interactions.

References

[1] P. W. Anderson, Phys. Rev. B 115, 2 (1959).
[2] C. Kittel, Phys. Rev. B 120, 335 (1960).
[3] E.A. Harris and J. Owen, Phys. Rev. Lett. 11, 9 (1963).
[4] D. S. Rodbell, I.S. Jacobson, J. Owen and E.A. Harris, Phys. Rev. Lett. 11, 10 (1963).
[5] D. S. Rodbell and J. Owen, J. Appl. Phys. 35, 1002 (1964).
[6] M. Jaščur, V. Štubňa, K. Szalowski, T. Balcerzak, J. Magn. Magn. Mat. 417, 92 (2016).
[7] T. Iwashita and N. Uryu, J. Phys. Soc. Jpn. 36, 48 (1974).
[8] J. Adler and J. Oitmaa, J. Phys. C: Solid State Phys. 12, 575 (1979).
[9] T. Iwashita and N. Uryu, J. Phys. C: Solid State Phys. 17, 855 (1984).
[10] T. Iwashita and N. Uryu, J. Phys. C: Solid State Phys. 21, 4783 (1988).
[11] T. Iwashita and N. Uryu, Phys. Lett. A 125, 5 (1987).
[12] T. Iwashita and N. Uryu, Phys. Lett. A 155, 241 (1991).
[13] T. Iwashita and N. Uryu, J. Phys.: Condens. Matter 3, 8257 (1991).
[14] Y. Muraoka, M. Ochai, T. Idogaki and N. Uryu, J. Phys. A: Math. Gen. 26, 1811 (1993).
[15] T. Iwashita, J. Oitmaa and T. Idogaki, J. Magn. Magn. Mat. 177, 159 (1998).
[16] Y. Muraoka, Phys. Rev. B 64, 134416 (2001).
[17] T. Iwashita, R. Satou, T Imada, Y. Myoshi and T. Idogaki, J. Magn. Magn. Mat. 226, 577 (2001).
[18] I. Syzoi, Prog. Theor. Phys. 6, 341 (1952).
[19] I. Syzoi, in *Phase Transition and Critical Phenomena*, edited by C. Domb and M.S. Green, Academic Press, New York, 1972.
[20] M. Fisher, Phys. Rev. 113 (1959) 960.
[21] S. Lacková and M. Jaščur, Phys. Rev. E 64 036126 (2001).
[22] L. Onsager, Phys.Rev. 65, 117 (1944).
[23] J. H. Barry, T. Tanaka, and M. Khatun, Phys. Rev. B 37, 5193 (1988).
[24] J. H. Barry, T. Tanaka, M. Khatun, and C. H. Munera, Phys. Rev. 44, 2595 (1991).
[25] C. N. Yang, Phys. Rev. 85, 809 (1952).
[26] H. B. Callen, Phys. Lett. 4, 161 (1963).
[27] M. Suzuki, Phys. Lett. 19, 267 (1965).
[28] J. Strečka and M. Jaščur, Acta Physica Slovaca 65, 235 (2015).
[29] R. Honmura and T. Kaneyoshi, J. Phys. C 12, 3979 (1979).
[30] L. Neél, Ann. Phys. (Paris) 3, 137 (1948).
[31] T. Kaneyoshi, J. W. Tucker, M. Jašur, Physica A 186, 495 (1992).