Convective Transport in Nanofluids: Regularity of Solutions and Error Estimates for Finite Element Approximations

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Abstract. We study the stationary version of a thermodynamically consistent variant of the Buongiorno model describing convective transport in nanofluids. Under some smallness assumptions it is proved that there exist regular solutions. Based on this regularity result, error estimates, both in the natural norm as well as in weaker norms for finite element approximations can be shown. The proofs are based on the theory developed by Caloz and Rappaz for general nonlinear, smooth problems. Computational results confirm the theoretical findings.

AMS Subject Classifications. 35B65, 35K55, 65N30.

Keywords. Nanofluid, Thermophoresis, Heat transfer, Weak solution, Regularity, \(L_p\) estimates, Finite elements, Error estimates.

1. Introduction

Nanofluids, i.e. a dilute mixture of a conventional base fluid and particles of submicron size, have received much attention for instance as a cooling liquid. This is due to their superior heat transfer properties. The enhanced heat transfer cannot solely be understood by the altered heat conducting coefficients of the mixture, but rather by effects of a heterogeneous distribution of the particles. Among the mathematical models to explain such behavior, the Buongiorno model [8] has become rather popular. By now, many simulations are based on this model, see [1, 4, 11, 14, 16, 17] for a by far not complete list of applications. In [5] the mechanism of the enhanced heat transfer for laminar flow conditions was revealed: strong temperature gradients at a hot wall lead to reduction of concentration of particles by thermophoresis there and this in turn reduces the concentration dependent viscosity of the dispersion. This then alters the flow profile leading to a stronger convective heat transfer.

To the best of our knowledge, despite its relevance in applications, there is hardly any rigorous mathematical analysis of the Buongiorno model. In [5] existence of weak solutions was shown using energy techniques. It was shown that solutions of a decoupled semi-implicit time-discretization converge to a solution of the continuous system, thereby also suggesting an effective numerical method.

In [6] existence of solutions to the stationary system was shown. Interestingly, the proof is somewhat technically more demanding than for the time-dependent problem.

The objective of the present work is to first show (under some smallness assumptions) regularity for the stationary problem and then use these regularity results to prove quasi-optimal error estimates for finite element approximations of the system. To this end it is shown that the system can be cast into the general framework of Caloz and Rappaz [9] for nonlinear (smooth) problems.

It turns out that the right space for the scalar quantities concentration and temperature is \(W_p^1\) with \(p > d\), \(d\) the space dimension, whereas for the fluid part we can stay in a Hilbert space setting.

The rest of the paper is organized as follows. In Sect. 2 we present the mathematical model and set some notation. In Sect. 3 we present the regularity results for the solutions of the system of PDEs. In
Sect. 4 we present a linearization of the problem which will allow us to prove the error estimates for a finite element discretization in Sect. 5. We close this article with some numerical experiments in Sect. 6 where we illustrate the orders of convergence, as well as the interesting effect of thermophoresis as a means to enhance the heat transfer properties.

2. The Mathematical Model

We consider the stationary system of a variant of the four equations, two-phase Buongiorno model \[8\] describing the motion of a nanofluid including concentration transport by thermophoresis. The model has been slightly modified to make it thermodynamically consistent, see \[5,6\]. In non-dimensional form it reads as follows: Let \(\Omega \subseteq \mathbb{R}^d, d \in \{2,3\}\) be an open, bounded domain with \(C^2\) boundary. We look for a concentration field \(\phi\), a temperature \(T\) as well as a velocity \(u\) and a pressure \(p\) fulfilling the following system of equations in \(\Omega\) (in the distributional sense)

\[
\begin{align*}
  u \cdot \nabla \phi + \frac{1}{\text{Re} \times \text{Sc}} \nabla \cdot j &= 0, \\
  u \cdot \nabla (\eta T) + \frac{1}{\text{Re} \times \text{Pr} \times \text{Le}} \nabla \cdot (T j) - \frac{1}{\text{Re} \times \text{Pr}} \nabla \cdot (k(\phi) \nabla T) &= f, \\
  u \cdot \nabla (\rho u) + \frac{1}{\text{Re} \times \text{Sc}_f} \nabla \cdot (u \otimes j) - \frac{1}{\text{Re}} \nabla \cdot (\mu(\phi) D(u)) + \nabla p + \beta T e_g &= g, \\
  \nabla \cdot u &= 0, \\
\end{align*}
\]

with \(D(u) = \nabla u + \nabla u^T, \eta = 1 + \phi\) and the particles’ flux given by

\[
  j := -\left(\nabla \phi + (1 - \phi) \frac{1}{N_{BT}} \frac{\nabla T}{T_0}\right),
\]

with \(N_{BT}\) the ratio of Brownian diffusivity/thermophoretic diffusivity, \(T_0\) a non-dimensional ambient temperature, \(\text{Re}\) the Reynolds number, \(\text{Pr}\) the Prandtl number, \(\text{Sc}_f\) the Schmidt and fluid Schmidt number, respectively and \(\text{Le}\), the Lewis number. Buoyancy effects through a Boussinesq approximation are considered with \(\beta > 0\) and \(e_g\) denoting a unit vector in the direction of gravity. The above system must of course be supplemented by appropriate boundary conditions.

The flux \(j\) is the non convective slip flux consisting on a Brownian part \(-\nabla \phi\) and the so called thermophoretic part \(\phi(1 - \phi) \frac{1}{N_{BT}} \frac{\nabla T}{T_0}\) that drives particles from hot to cold. Phenomenologically this can be explained by the fact that collisions of the particles with molecules from the base fluid are stronger on the hot side of the particle than on the cold part, resulting in a net flux from hot to cold. For more on thermophoresis we refer for instance to \[13\].

The mathematical challenge with the above system lies in the rather strong nonlinearity. A good space for \(\phi, T\) is therefore \(W^1_p(\Omega)\) with \(p > d\). However, an energy estimate is only available in \(H^1(\Omega)\), see \[5,6\]. To overcome this problem, in Sect. 3 we prove regularity estimates in \(W^2_p(\Omega)\) based on some bootstrap arguments and a smallness assumption.

This paves the way to cast the problem in the general framework developed in \[9\] for nonlinear problems.

**Notation** As usual, Lebesgue spaces are denoted by \(L_p(\Omega)\), \(1 \leq p \leq \infty\) and Sobolev spaces by \(W^m_p(\Omega)\), \(m \in \mathbb{N}_0\). If \(p = 2\), the notation \(H^m(\Omega)\) is used. In what follows, scalar quantities will be denoted by normal characters, whereas vector and tensor valued functions will be denoted by bold characters. Consequently, for instance \(L^p(\Omega) := L^p(\Omega)^d\). Define \(V := \{v \in H^1_0(\Omega) \mid \nabla \cdot v = 0\}\), where \(H^1_0(\Omega)\) is the closure of \(C_0^\infty(\Omega)\) (the space of test functions) in \(H^1(\Omega)\), as well as \(\tilde{V} := V \cap H^{4,2}(\Omega)\) with corresponding norm. As usual, the pressure space is chosen to be \(L_{2,0}(\Omega) := \{q \in L_2(\Omega) \mid \int_{\Omega} q(x) \, dx = 0\}\). The expression \(A \lesssim B\) will denote \(A \leq CB\) with a constant \(C\) that might depend on the regularity of the domain \(\Omega\), the dimension \(d\) of the underlying space and also on the norms involved in the expressions \(A\) and \(B\).
3. Regularity

Since in the next sections we are concerned with analytical problems, we set all non-dimensional constants to one for ease of presentation. Let \( f \in L^p(\Omega) \), \( p > d \), \( g \in L^2(\Omega) \) and \( j := -\nabla \phi - h(\phi) \nabla T \), where \( h(s) = s(1 - s) \). Now the problem reads (in the distributional sense): Find \( \phi, T, u, p \) such that

\[
-\Delta \phi = \nabla \cdot (h(\phi) \nabla T) - u \cdot \nabla \phi \quad \text{in } \Omega,
\]

\[
-\nabla \cdot (k(\phi) \nabla T) = f - \nabla \cdot (T j) - u \cdot \nabla (\eta T) \quad \text{in } \Omega,
\]

\[
-\nabla \cdot (\mu(\phi) D(u)) + \nabla p = g - \nabla \cdot (u \otimes j) - u \cdot \nabla (\rho u) - T e_y \quad \text{in } \Omega,
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega.
\]

(3.1)

For the boundary conditions we choose

\[
\phi = \phi_D, \quad T = 0, \quad u = 0 \quad \text{on } \partial \Omega.
\]

In the above equations, \( \eta = 1 + \phi, \rho = 1 + \phi \) and we assume \( \phi_D \in W^2_p(\Omega) \) with \( 0 \leq \phi_D \leq 1 \) and \( \Delta \phi_D = 0 \).

The coefficients \( k(\phi), \mu(\phi) \) fulfill \( k(\cdot), \mu(\cdot) \in C^2([0, 1]) \) and

\[
0 < k_0 \leq k(s), \quad 0 < \mu_0 \leq \mu(s) \quad \text{for all } s \in [0, 1].
\]

We also denote by \( h(\cdot), k(\cdot), \mu(\cdot) \) their extensions to \( C^2(\mathbb{R}) \) satisfying

\[
0 < k_0 \leq k(s), \quad 0 < \mu_0 \leq \mu(s) \quad \text{for all } s \in \mathbb{R}
\]

and

\[
|D^\ell h(s)| \leq C, \quad |D^\ell k(s)| \leq C \quad \text{for } \ell = 0, 1, 2 \quad \text{and all } s \in \mathbb{R}.
\]

We define a vector-valued cut-off function, for \( R > 0 \) as follows

\[
\sigma_R : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \sigma_R(y) = \begin{cases} y, & \text{if } |y| \leq R, \\ \frac{y}{|y|} R, & \text{if } |y| > R. \end{cases}
\]

Note that \( \sigma_R \) is Lipschitz and

\[
\partial_j (\sigma_R(y)_i) = \begin{cases} \delta_{ij}, & \text{if } |y| \leq R, \\ \frac{R}{|y|} \left( \delta_{ij} - \frac{y_i y_j}{|y|^2} \right), & \text{if } |y| > R. \end{cases}
\]

For \( R > 0 \) we now consider the regularized problem

\[
-\Delta \phi_R = \nabla \cdot \sigma_R(h(\phi_R) \nabla T_R) - u_R \cdot \nabla \phi_R \quad \text{in } \Omega,
\]

\[
\phi_R = \phi_D \quad \text{on } \partial \Omega,
\]

\[
-\nabla \cdot (k(\phi_R) \nabla T_R) = f - \nabla \cdot \left( T_R \left( -\sigma_R(h(\phi_R) \nabla T_R) - \nabla \phi_R \right) \right) - u_R \cdot \nabla (\eta_R T_R) \quad \text{in } \Omega,
\]

\[
\underbrace{\nabla \cdot u_R}_R = 0 \quad \text{in } \Omega,
\]

\[
u_R = 0 \quad \text{on } \partial \Omega.
\]

(3.2a)

(3.2b)

(3.2c)

Following the regularization procedure detailed in [6] one can prove the existence of one solution triplet \( \phi_R, T_R \in H^{1,2}(\Omega), u_R \in V := \{ v \in H_{0}^{1,2}(\Omega) \mid \nabla \cdot v = 0 \} \) which satisfy \( 0 \leq \phi_R \leq 1, \ a.e., \) and also the estimate

\[
\|\phi_R\|_{1,2} + \|T_R\|_{1,2} + \|u_R\|_V \leq C(\|\phi_D\|_{2,2} + \|f\|_{0,2} + \|g\|_{0,2})
\]

(3.3)
with $C$ independent of $R$. It is worth mentioning that these bounds are not obtained by usual energy-like estimates, and also that uniqueness is not guaranteed for this problem; the proof in [6] entails the construction of a solution triplet satisfying those bounds.

The following lemma will be instrumental in proving our regularity results.

**Lemma 3.1.** Let $j \in \mathbb{N}_0$ and let $k(\cdot), \mu(\cdot) \in C^{j+1}(\mathbb{R})$ with all derivatives up to order $j+1$ bounded and $\partial \Omega$ of class $C^{j+2}$. Furthermore, let $r > d$ and $1 < M, M' < r$, with $M' = M/(M-1)$, the Lobesgue dual exponent to $M$. Let $\tilde{f} \in W^j_M(\Omega)$, $\tilde{g} \in W^j_M(\Omega)$ and $\phi \in W^{d+j}_r(\Omega)$. If $T \in H^1(\Omega)$, $u \in \mathcal{V}$ fulfill
\[
\int_\Omega k(\phi) \nabla T \cdot \nabla \psi = \int_\Omega \tilde{f} \psi, \quad \text{for all } \psi \in C_0^\infty(\Omega),
\]
\[
\int_\Omega \frac{\mu(\phi)}{2} D(u) : D(v) = \int_\Omega \tilde{g} \cdot v, \quad \text{for all } v \in \mathcal{D}(\Omega) = \{ v \in (C_0^\infty(\Omega) \mid \nabla \cdot v = 0 \},
\]
then $T, u \in W^{2+j}_M(\Omega)$ and
\[
\|T\|_{2+j,M} \leq CM \|\tilde{f}\|_{W^j_M(\Omega)}, \quad \|u\|_{2+j,M} \leq CM \|\tilde{g}\|_{W^j_M(\Omega)}.
\]
where $C_M = C_M(\|\phi\|_{1+j,r})$ is a non-decreasing function of $\|\phi\|_{1+j,r}$.

**Proof.** The regularity for $u$ follows from the proof of [2, Lemma 4]. There, regularity in $H^2(\Omega)$ for $j = 0, 1$ was shown. However, a closer inspection of the proof shows that the assertion is also valid for $M$ as above and arbitrary $j \in \mathbb{N}_0$. The regularity for $T$ can be shown in the same way with even some simplifications.

The first regularity result for the solution of (3.1) holds under a smallness assumption on the data of the problem. The precise result is the following.

**Theorem 3.2.** Given $p > d$ there exists a constant $F_0 > 0$ such that, if $\|\phi_D\|_{2,p} + \|f\|_{0,p} + \|g\|_{0,p} < F_0$, then there exists $R > 0$ and a solution $(\phi_R, T_R, u_R)$ of (3.2) which satisfies $\|\nabla T_R\|_{0,\infty} \leq R$, and consequently $(\phi, T, u) = (\phi_R, T_R, u_R)$ is also a solution to (3.1).

Moreover, given $\delta > 0$, there exists $F_\delta > 0$ such that whenever $\|\phi_D\|_{2,p} + \|f\|_{0,p} + \|g\|_{0,2} \leq F_\delta \leq F_0$ there exists a solution of (3.1) with $\|\nabla T\|_{0,\infty} \leq \delta$.

**Proof.** The proof is based on a couple of bootstrap arguments.

We fix $p > d$ and denote $D := \|\phi_D\|_{2,p} + \|f\|_{0,p} + \|g\|_{0,p} < \infty$. For the sake of simplicity of the proof we assume $D \leq 1$. For each $R > 0$, let $(\phi_R, T_R, u_R)$ be a solution to (3.2) satisfying (3.3), and emphasize that the constants involved in the symbols $\lesssim$ below do not depend on $R, D$, or the problem data $\phi_D, f, g$, but only on Sobolev embeddings and the constant $C$ from (3.3).

By embedding, $\|u_R\|_{0,6} \lesssim \|u_R\|_{1,2} \lesssim D \leq 1$. Hence, by Hölder’s inequality $u_R \cdot \nabla \phi_R \in L^3_2(\Omega)$ and $\|u_R \cdot \nabla \phi_R\|_{0,3/2} \lesssim \|u_R\|_{0,6}\|\nabla \phi_R\|_{0,2} \leq D$ whence the right-hand side for the $\phi_R$-equation (3.2a) is the sum of a function in $L^3_2(\Omega)$ and the divergence of a function in $L^\infty(\Omega)$ (bounded by $R$). Then by regularity [3, Theorem 3.29], $\nabla \phi_R \in L^3(\Omega)$ with
\[
\|\nabla \phi_R\|_{0,3} \lesssim \|\sigma_R(h(\phi_R)\nabla T_R)\|_{0,\infty} + \|u_R \cdot \nabla \phi_R\|_{0,3/2} \lesssim R + D.
\]

Applying again Hölder’s inequality, $u_R \cdot \nabla \phi_R \in L^2(\Omega)$ and $\|u_R \cdot \nabla \phi_R\|_{0,2} \lesssim \|u_R\|_{0,6}\|\nabla \phi_R\|_{0,3} \lesssim D(R + D) \lesssim R + D$. Now, the right-hand side of (3.2a) is the sum of a function in $L^2(\Omega)$ and the divergence of a function in $L^\infty(\Omega)$. By the same regularity result [3, Theorem 3.29], $\nabla \phi_R \in L^6(\Omega)$ with
\[
\|\nabla \phi_R\|_{0,6} \lesssim \|\sigma_R(h(\phi_R)\nabla T_R)\|_{0,\infty} + \|u_R \cdot \nabla \phi_R\|_{0,2} \lesssim R + D.
\]

Repeating this argument once again, $\nabla \phi_R \in L^M(\Omega)$ for all $1 \leq M < \infty$ with the estimate
\[
\|\nabla \phi_R\|_{0,M} \lesssim R + D,
\]
which thereby implies
\[
\|\phi_R\|_{0,M} \lesssim R + D.
\]
Let us now turn to the equations for $T_R$ and $\mathbf{u}_R$. Thanks to (3.2a) and the definition of $\eta, \rho$ the right-hand sides can be written as

$$\tilde{f}_R = f - (\mathbf{j}_R + \eta_R \mathbf{u}_R) \cdot \nabla T_R,$$

$$\tilde{g}_R = g - (\mathbf{j}_R + \rho_R \mathbf{u}_R) \cdot \nabla \mathbf{u}_R - T_R \mathbf{e}_g.$$ 

First, we need an intermediate regularity result for $\mathbf{u}_R$, namely $\mathbf{u}_R \in L_M(\Omega)$ for all $1 \leq M < \infty$. To this end, we observe

$$\|\tilde{g}_R\|_{0,3/2} \lesssim \|\mathbf{g}\|_{0,3/2} + (\|\mathbf{j}_R\|_{0,6} + \|\mathbf{u}_R\|_{0,6})\|\nabla \mathbf{u}_R\|_{0,2} + \|T_R\|_{0,3/2}.$$ 

Lemma 3.1 (for $j = 0, M = 3/2, r > 3$) and (3.3) with (3.6) yield

$$\|\mathbf{u}_R\|_{2,3/2} \lesssim \|\tilde{g}_R\|_{0,3/2} \lesssim R + D$$

and therefore $\mathbf{u}_R \in L_M$ for all $1 \leq M < \infty$ (since $W^{2,3/2}_{3/2}(\Omega) \hookrightarrow L_M$) and

$$\|\mathbf{u}_R\|_{0, M} \lesssim R + D. \quad (3.8)$$

Now fix any $M > 2d/(4 - d)$. Using Hölder’s inequality for $\tilde{M} = \frac{2M}{M+2} < 2 < p$, we get

$$\|\tilde{f}_R\|_{0, \tilde{M}} \lesssim \|f\|_{0, \tilde{M}} + \|\nabla T_R\|_{0,2} (\|\mathbf{j}_R\|_{0, M} + \|\mathbf{u}_R\|_{0, M}),$$

whence (3.3) and the bounds (3.8) (3.7) imply

$$\|\tilde{f}_R\|_{0, \tilde{M}} \lesssim \|f\|_{0, \tilde{M}} + D(R + D).$$

Therefore, by Lemma 3.1 applied to $T_R$ we have $\|T_R\|_{2, \tilde{M}} \lesssim \|f\|_{0, \tilde{M}} + D(R + D)$, which by Sobolev embedding implies

$$\|\nabla T_R\|_{0, \tilde{M}^*} \lesssim \|f\|_{0, \tilde{M}} + D(R + D)$$

for $\tilde{M}^* = \frac{\tilde{M} d}{d - \tilde{M}} > d$ (see Lemma 3.3 below).

Let now $d < t < \tilde{t} := \min\{p, \tilde{M}^*\} \leq p$ and $1/q := 1/t - 1/\tilde{t}$ then

$$\|\tilde{f}_R\|_{0, t} \lesssim \|f\|_{0, t} + \|\nabla T_R\|_{0, \tilde{t}} (\|\mathbf{j}_R\|_{0, q} + \|\mathbf{u}_R\|_{0, q}) \lesssim \|f\|_{0, p} + \|f\|_{0, p} R + DR^2,$$

where we have used (3.8).

Finally, using again Lemma 3.1 and the embedding $W^2_1(\Omega) \hookrightarrow W^1_\infty(\Omega)$, we have $\|\nabla T_R\|_{0, \infty} \lesssim \|T_R\|_{W^2_1(\Omega)}$ so that

$$\|\nabla T_R\|_{0, \infty} \lesssim D(C_1 + C_2 R + C_3 R^2). \quad (3.9)$$

We notice that $D(C_1 + C_2 R + C_3 R^2)$ can be made smaller than $R$ by choosing $D < R/(C_1 + C_2 R + C_3 R^2)$, so that under this assumption, $\|\nabla T_R\|_{0, \infty} \lesssim R$ and the first assertion follows with $F_0 := \min\{1, 1/C_2\}$ because $1/C_2 = \sup_{R > 0} R/(C_1 + C_2 R + C_3 R^2)$.

The second assertion is an immediate consequence of (3.9). \hfill \Box

Lemma 3.3. If $M > \frac{2d}{4-d}$ then $\tilde{M}^* = \frac{2M}{M+2}$, the Sobolev conjugate of $\tilde{M} := \frac{2M}{M+2}$ is larger than $d$.

Proof. Let $M > \frac{2d}{4-d}$, then

$$\tilde{M}^* = \left(\frac{2M}{M+2}\right)^* = \frac{2M d}{d - \frac{2M}{M+2}} = \frac{2Md}{d(M+2) - 2M} > d$$

because $2M > d(M+2) - 2M$ if and only if $M(4-d) > 2d$. \hfill \Box

Combining the previous theorem with regularity results we obtain the following corollary. From now on we fix $0 < \tilde{F}_0 < F_0$, with $F_0 \leq 1$ as in Theorem 3.2 and throughout the rest of this article we use $A \lesssim B$ to denote $A \leq CB$ with a constant $C$ that may depend on $\tilde{F}_0$, besides regularity of the domain $\Omega$, the dimension $d$ of the underlying space and also on the norms involved in the expressions $A$ and $B$.  

Corollary 3.4. If \( p > d \), there exists a positive constant \( C_p \) such that, if \( \| \phi_D \|_{2,p} + \| f \|_{0,p} + \| g \|_{2,0} \leq \tilde{F}_0 \), then problem \( (3.1) \) has a (possibly non-unique) solution \( (\phi, T, u) \) satisfying
\[
\| \phi \|_{2,p} + \| T \|_{2,p} + \| u \|_{2,2} \leq C_p(\| \phi_D \|_{2,p} + \| f \|_{0,p} + \| g \|_{2,0}).
\]

Proof. Let the problem data satisfy \( D := \| \phi_D \|_{2,p} + \| f \|_{0,p} + \| g \|_{2,0} \leq \tilde{F}_0 < F_0 \). Then, there exists \( R > 0 \) and a solution \( (\phi_R, T_R, u_R) \) of \( (3.2) \) which satisfies \( \| \nabla T_R \|_{0,\infty} \leq R \) and \( \| u \|_{2,0}\| \nabla \phi \|_{0,2} \leq \| u \|_{0,2} \| \nabla \phi \|_{0,2} \leq D^2 \leq D \). Proceeding with the same bootstrapping argument as in the beginning of the proof of Theorem 3.2 we find that
\[
\| \nabla \phi \|_{0,M} \leq D \quad \text{and} \quad \| j \|_{0,M} \leq D,
\]
for any \( 1 \leq M < \infty \). Proceeding further as in the proof of Theorem 3.2 we arrive at the analogous of \( (3.8) \), which now reads
\[
\| u \|_{0,M} \leq D,
\]
for any \( 1 \leq M < \infty \). Finally, by regularity
\[
\| \phi \|_{2,p} \leq \| RSH(\phi) \|_{0,p} \leq \| \nabla \phi \|_{0,p} \| \nabla T \|_{0,\infty} + \| u \|_{0,2p} \| \nabla \phi \|_{0,2p} \leq D.
\]

Also, \( T \) is a solution of \( (3.4) \) with \( \tilde{f} = f - \nabla \cdot (T j) - u \cdot \nabla (\eta T) \in L_p(\Omega) \) whence
\[
\| T \|_{2,p} \leq D.
\]

It remains to show \( H^2 \)-regularity for \( u \), which is an immediate consequence of the fact that \( u \) is a solution to \( (3.5) \) with \( \tilde{g} = g - (j + (1 + \phi)u) \cdot \nabla u - T e_g \), which satisfies
\[
\| \tilde{g} \|_{0,2} \leq \| g \|_{0,2} + (\| j \|_{0,6} + \| u \|_{0,6}) \| \nabla u \|_{0,3} + \| T \|_{0,2}.
\]

Note that in the proof of Theorem 3.2 above we have already shown that \( u_R \in W^{2}_{3/2}(\Omega) \hookrightarrow W^{1}_{3}(\Omega) \). \( \square \)

Once regularity in \( W^2_p(\Omega), H^2(\Omega) \) is established, it is not difficult to get higher regularity, provided data is more regular. This is stated in the next corollary.

Corollary 3.5. (Higher regularity). For \( j \in \mathbb{N}_0 \) let \( k(\cdot), \mu(\cdot) \in C^{j+1}(\mathbb{R}) \) with all derivatives up to order \( j + 1 \) bounded, \( \partial \Omega \in C^{j+2} \) and \( p > d \). If the solution of \( (3.1) \) is regular, i.e. \( \phi, T \in W^2_p(\Omega), u \in H^2(\Omega) \) and data \( \phi_D \in W^2_{2+j}(\Omega), f \in W^j_p(\Omega), g \in H^j(\Omega) \) then \( \phi, T \in W^2_{2+j}(\Omega), u \in H^{j+2}(\Omega) \) and
\[
\| \phi \|_{2+j,p} + \| T \|_{2+j,p} + \| u \|_{2+j,2} \leq \| \phi_D \|_{2+j,p} + \| f \|_{j,p} + \| g \|_{j,2}.
\]

Proof. The proof is based on induction over \( j \). The case \( j = 0 \) is the assumption of this corollary, which holds under the smallness assumption of Corollary 3.4. So let us assume that the statement is correct for \( j \). We shall then show that it also holds for \( j + 1 \). First, note that by the induction assumption and because \( p > d \)
\[
\partial_\beta \phi, \partial_\beta T \in L_\infty(\Omega)
\]
for any multiindex \( \beta \in \mathbb{N}_0^d \) with \( |\beta| \leq 1 + j \) and
\[
\partial_\beta \phi, \partial_\beta T \in L_p(\Omega)
\]
for \( \beta \in \mathbb{N}_0^d \) with \( |\beta| \leq 2 + j \). Likewise,
\[
\partial_\beta u \in H^2(\Omega) \hookrightarrow L_\infty(\Omega)
\]
for \( |\beta| \leq j \) and
\[
\partial_\beta u \in H^1(\Omega) \hookrightarrow L_6(\Omega)
\]
for $|\beta| \leq 1 + j$. Recall that $T$ is a solution to (3.4) with $\tilde{f} = f - (j + (1 + \phi)u) \cdot \nabla T = f + \nabla T \cdot \nabla \phi + h(\phi)|\nabla T|^2 - (1 + \phi)u \cdot \nabla T$. Let us now check the regularity of $\partial_\alpha \tilde{f}$ for $|\alpha| = 1 + j$:

$$
\partial_\alpha \tilde{f} = \partial_\alpha f + \partial_\alpha (\nabla \phi \cdot \nabla T) + \partial_\alpha (h(\phi) \nabla T \cdot \nabla T) - \partial_\alpha ((1 + \phi)u \cdot \nabla T).$$

(3.10)

Clearly, the first term on the right-hand side is in $L_p(\Omega)$. By Leibniz’ formula, the second term can be written as

$$
\partial_\alpha (\nabla \phi \cdot \nabla T) = \sum_{|\beta| \leq \alpha} \binom{\alpha}{\beta} \nabla \partial_\beta \phi \cdot \nabla \partial_{\alpha - \beta} T.
$$

Our first observation yields $\partial_\alpha (\nabla \phi \cdot \nabla T) \in L_p(\Omega)$. The last term can be written by Leibniz’ formula as

$$
\partial_\alpha ((1 + \phi)u \cdot \nabla T) = \sum_{|\beta| \leq \alpha} \binom{\alpha}{\beta_1, \beta_2, \beta_3} \partial_{\beta_1}(1 + \phi) \partial_{\beta_2} u \cdot \nabla \partial_{\beta_3} T.
$$

In view of our first observation, the worst summand in the sum above (if $6 < p$) is attained for $|\beta_2| = 1 + j$ with $\partial_{\beta_2} u \in H^1(\Omega) \hookrightarrow L_6(\Omega)$, showing that altogether $\partial_\alpha ((1 + \phi)u \cdot \nabla T) \in L_{\min(6,p)}(\Omega)$. Since $|\alpha| = 1 + j$ was arbitrary, this shows $\partial_\alpha \tilde{f} \in L_{\min(6,p)}(\Omega)$. As an intermediate result one gets

$$
T \in W^{3+j}_{\min(6,p)}(\Omega).
$$

Note that, if $d < p \leq 6$ this is already the desired regularity for $T$.

Now we turn our attention to the $\phi$-equation:

$$
-\Delta \phi = \nabla \cdot (h(\phi) \nabla T) - u \cdot \nabla \phi =: \text{RHS}(\phi).
$$

With the same arguments as above, one concludes $\partial_\alpha \text{RHS}(\phi) \in L_{\min(6,p)}(\Omega)$ for all $|\alpha| \leq j + 1$ and thus

$$
\phi \in W^{3+j}_{\min(6,p)}(\Omega).
$$

The velocity $u$ is a solution to (3.5) with $\tilde{g} = g + (\nabla \phi + h(\phi) \nabla T - (1 + \phi)u) \cdot \nabla u - Te_g$. The derivative of the right hand side $\tilde{g}$ for this momentum equation reads

$$
\partial_\alpha \tilde{g} = \partial_\alpha g + \partial_\alpha [(\nabla \phi + h(\phi) \nabla T - (1 + \phi)u) \cdot \nabla u] - \partial_\alpha Te_g.
$$

Expanding again the derivative by Leibniz’ formula and observing $W^{1}_{\min(6,p)}(\Omega) \hookrightarrow L_\infty(\Omega)$ the worst term is identified to be

$$
(1 + \phi)u \cdot \nabla \partial_\alpha u \in L_2(\Omega)
$$

for all $|\alpha| \leq 1 + j$ and then

$$
u \in H^{3+j}(\Omega),
$$

which is already the desired regularity for $u$.

One more sweep of the above arguments, but now using the intermediate regularity results, concludes the proof. \hfill \Box

**Lemma 3.6.** (Existence and regularity of the pressure). \textit{Let the assumptions of Lemma 3.1 hold for some $j, r, M$ and let $u \in V$ be a solution of Eq. (3.5). Then there exists a unique pressure $p \in L_{2,0}(\Omega) \cap W_h^{1+j} M(\Omega)$ fulfilling}

$$
\nabla \cdot (\mu(\phi) D(u)) + \nabla p = \tilde{g}
$$

and $\|p\|_{1+j,M} \leq C\|\tilde{g}\|_{j,M}.$

\textbf{Proof.} From standard theory it is clear that there exists a unique pressure $p \in L_{2,0}(\Omega)$ such that the above equation is fulfilled in the distributional sense. Now, shifting the term $\nabla \cdot (\mu(\phi) D(u))$ to the right hand side, differentiating the right hand side successively up to the desired order and using the regularity of $\phi$ and $u$ it follows that $\nabla p \in W_M^j(\Omega)$ and the estimate is then also immediate. \hfill \Box
4. Linearization

Let $p > 3 \geq d$. Define $X := \tilde{W}_{p}^{1}(\Omega) \times \tilde{W}_{p}^{1}(\Omega) \times H_{0}^{1,2}(\Omega) \times L_{2,0}(\Omega)$, $X_{D} := (\phi_{D} + \tilde{W}_{p}^{1}(\Omega)) \times \tilde{W}_{p}^{1}(\Omega) \times H_{0}^{1,2}(\Omega) \times L_{2,0}(\Omega)$ and $Y := \tilde{W}_{p}^{1}(\Omega) \times \tilde{W}_{p}^{1}(\Omega) \times H_{0}^{1,2}(\Omega) \times L_{2,0}(\Omega)$ with $p' = p/(p-1)$ the dual Lebesgue exponent to $p$.

We introduce the nonlinear operator $\mathcal{F} : X \to Y'$ by

$$
\langle \mathcal{F}(\phi, T, u, p), (\psi, \varphi, v, q) \rangle := \int_{\Omega} \nabla \phi \cdot \nabla \psi + \int_{\Omega} h(\phi) \nabla T \cdot \nabla \psi + \int_{\Omega} u \cdot \nabla \psi \\
+ \int_{\Omega} k(\phi) \nabla \varphi + \int_{\Omega} (j + \eta u) \cdot \nabla T \varphi - \int_{\Omega} f \varphi \\
+ \int_{\Omega} \frac{\mu(\phi)}{2} D(u) : D(v) + \int_{\Omega} (j + \rho u) \cdot \nabla u \cdot v - \int_{\Omega} p \nabla \cdot v \\
- \int_{\Omega} g \cdot v - \int_{\Omega} T e_{g} \cdot v \\
+ \int_{\Omega} q \nabla \cdot u
$$

for all $(\psi, \varphi, v, q) \in Y$. Here, $\phi := \phi_{D} + \tilde{\phi}$.

Since $W_{p}^{1}(\Omega) \hookrightarrow L_{q}(\Omega)$ with $1/q = 1 - 1/(d-1/p)$ and $p > d$ we have $2/p + 1/q < 1$. Also $1/6 + 1/p + 1/q < 1$ as well as $1/p + 1/2 + 1/6 < 1$. Thus the above integrals are well defined and also taking into account the definition of the coefficients, one concludes that $\mathcal{F} : X \to Y'$ is continuous. Clearly, $(\phi, T, u, p) \in X_{D}$ is a solution of system (3.1), iff $\mathcal{F}(\phi, T, u, p) = 0$ and $0 \leq \phi \leq 1$.

Due to the properties of the coefficients and since $p > 3 \geq d$ (which in particular implies $W_{p}^{1}(\Omega) \hookrightarrow L_{\infty}(\Omega)$) $\mathcal{F}$ is Frechet differentiable with derivative given by

$$
\langle D\mathcal{F}(\phi, T, u, p)(\chi, \Theta, w, r), (\psi, \varphi, v, q) \rangle := \int_{\Omega} \nabla \chi \cdot \nabla \psi + \int_{\Omega} h'(\phi) \chi \nabla T \cdot \nabla \psi + \int_{\Omega} h(\phi) \nabla \Theta \cdot \nabla \psi \\
+ \int_{\Omega} u \cdot \nabla \psi + \int_{\Omega} w \cdot \nabla \phi \psi \\
+ \int_{\Omega} \nabla (k(\phi) \Theta) \cdot \nabla \varphi + \int_{\Omega} k'(\phi)(\chi \nabla T - \Theta \nabla \phi) \cdot \nabla \varphi \\
+ \int_{\Omega} (\partial j + \chi u + \eta w) \cdot \nabla T \varphi + \int_{\Omega} (j + \eta u) \cdot \nabla \Theta \varphi \\
+ \int_{\Omega} \frac{\mu(\phi)}{2} D(w) : D(v) + \int_{\Omega} \frac{\mu'(\phi)}{2} \chi D(u) : D(v) \\
+ \int_{\Omega} (\partial j + \chi u + \rho w) \cdot \nabla u \cdot v + \int_{\Omega} (j + \rho u) \cdot \nabla w \cdot v - \int_{\Omega} r \nabla \cdot v - \int_{\Omega} \Theta e_{g} \cdot v \\
+ \int_{\Omega} q \nabla \cdot w
$$

(4.1)

for all $(\chi, \Theta, w, r) \in X, (\psi, \varphi, v, q) \in Y$. Here, $\partial j$ is an abbreviation for $\partial j = -\nabla \chi - h'(\phi) \chi \nabla T - h(\phi) \nabla \Theta$.

We state the differentiability and also the Lipschitz continuity of $D\mathcal{F}$ in the next lemma.

**Lemma 4.1.** Let $X, Y$ and $\mathcal{F}$ be as above. Then $\mathcal{F} : X \to Y'$ is Frechet differentiable with $D\mathcal{F}$ given in (4.1) above. Moreover, $D\mathcal{F}$ is locally Lipschitz continuous, i.e. for $U = (\phi, T, u, p) \in X$ and $r_{0} > 0$ there is an $L = L(U, r_{0}) \geq 0$ such that

$$
\|D\mathcal{F}(U) - D\mathcal{F}(V)\|_{\mathcal{L}(X, Y')} \leq L\|U - V\|_{X}
$$

for all $V \in B_{r_{0}}(U) \subseteq X$. 


Proof. The differentiability was already discussed above. To show the Lipschitz continuity we have to estimate
\[
\langle (DF(U) - DF(V))(\chi, \Theta, w, r), (\psi, \varphi, v, q) \rangle \leq L\|U - V\|_X \| (\chi, \Theta, w, r) \|_X \| (\psi, \varphi, v, q) \|_Y
\]
which again follows by inspecting the individual integrals and noting that \( p > d \).

The goal of these linearization results is to obtain estimates for a finite element discretization using the framework from [9]. The crucial step now is to show that under some smallness assumptions \( DF(U) \) is an isomorphism from \( X \) to \( Y' \).

**Proposition 4.2.** There exists \( \delta > 0 \) such that if \( \|(\phi, T, u, p)\|_X < \delta \), then \( DF(\tilde{\phi}, T, u, p) : X \rightarrow Y' \) is an isomorphism.

Proof. Given \( U = (\tilde{\phi}, T, u, p) \), we first consider the reduced operator \( T : X \rightarrow Y' \) defined by
\[
\langle T(\chi, \Theta, w, r), (\psi, \varphi, v, q) \rangle := \int_{\Omega} \nabla \chi \cdot \nabla \psi + \int_{\Omega} h(\phi) \nabla \Theta \cdot \nabla \psi + \int_{\Omega} \nabla (R\Theta) \cdot \nabla \varphi + \int_{\Omega} \frac{\mu(\phi)}{2} D(w) : D(v) - \int_{\Omega} r \nabla \cdot v + \int_{\Omega} q \nabla \cdot w
\]
with \( R : \tilde{W}^1_p(\Omega) \rightarrow \tilde{W}^1_p(\Omega), R\Theta := k(\phi)\Theta, \) see [9]. Due to the properties of \( k(\cdot) \) and since \( \phi \in W^1_p(\Omega) \), \( R \) is well defined and an isomorphism.

It is easy to show that \( T \) is an isomorphism. For this, let \( (l_\phi, l_T, l_u, l_p) \in Y' \) be given. Clearly, by standard theory for the Stokes equations, there is a unique \( (w, r) \in H^{1,2}_0(\Omega) \times L_{2,0}(\Omega) \) such that
\[
\langle T(0, 0, w, r), (0, 0, v, q) \rangle = \langle (0, 0, l_u, l_p), (0, 0, v, q) \rangle
\]
for all \( (v, q) \in H^{1,2}_0(\Omega) \times L_{2,0}(\Omega) \). Next, since the Laplace operator with Dirichlet boundary condition is an isomorphism from \( \tilde{W}^1_p(\Omega) \) to \( (\tilde{W}^1_p(\Omega))' \) [12, Theorem 1.1] and since \( R \) is an isomorphism, there is a unique \( \Theta \in \tilde{W}^1_p(\Omega) \) fulfilling
\[
\langle T(0, \Theta, 0, 0), (0, \varphi, 0, 0) \rangle = \langle (0, l_T, 0, 0), (0, \varphi, 0, 0) \rangle
\]
for all \( \varphi \in \tilde{W}^1_p(\Omega) \). Given these \( \Theta, w, q \) we finally can solve the first equation to get a unique \( \chi \in \tilde{W}^1_p(\Omega) \) so that eventually
\[
\langle T(\chi, \Theta, w, r), (\psi, \varphi, v, q) \rangle = \langle (l_\phi, l_T, l_u, l_p), (\psi, \varphi, v, q) \rangle
\]
for all \( (\psi, \varphi, v, q) \in Y \). Thus \( T \) is bijective. Since \( T \) is continuous, its inverse is also continuous, hence \( T \) is an isomorphism. Because \( X, Y \) are reflexive Banach spaces, it follows that there exists \( \alpha > 0 \) such that (see [10])
\[
\inf_{U \in X} \sup_{V \in Y} \langle TU, V \rangle = \inf_{V \in Y} \sup_{U \in X} \langle TU, V \rangle = \alpha > 0.
\]

The remaining part of the operator \( \mathcal{N} := DF(\phi, T, u, p) - T \) reads
\[
\langle \mathcal{N}(\chi, \Theta, w, r), (\psi, \varphi, v, q) \rangle = \int_{\Omega} h'(\phi) \chi \nabla T \cdot \nabla \psi + \int_{\Omega} u \cdot \nabla \chi \psi + \int_{\Omega} w \cdot \nabla \phi \psi + \int_{\Omega} k'(\phi) (\chi \nabla T - \Theta \nabla \phi) \cdot \nabla \varphi + \int_{\Omega} (\partial j + \chi u + \eta w) \cdot \nabla T \varphi + \int_{\Omega} (j + \eta u) \cdot \nabla \Theta \varphi
\]
\[ + \int_{\Omega} \frac{\mu'(\phi)}{2} \chi D(u) : D(v) + \int_{\Omega} (\partial j + \chi u + \rho w) \cdot \nabla u \cdot v + \int_{\Omega} (j + \rho u) \cdot \nabla w \cdot v - \int_{\Omega} \Theta e_s \cdot v. \]

The norm of \( \mathcal{N} \) depends continuously on the \( X \)-norm of \((\phi, T, u, p)\). Thus, for \((\phi, T, u, p)\) sufficiently small we get \( \| \mathcal{N} \|_{\mathcal{L}(X',X')} \leq \alpha/2 \) and

\[
\inf_{U \in X} \sup_{V \in Y} \langle D\mathcal{F}(\phi, T, u, p)U, V \rangle = \inf_{V \in Y} \sup_{U \in X} \langle D\mathcal{F}(\phi, T, u, p)U, V \rangle \geq \alpha - \| \mathcal{N} \|_{\mathcal{L}(X,X')} = \frac{\alpha}{2}.
\]

This shows that \( D\mathcal{F}(\phi, T, u, p) \) is an isomorphism. \( \square \)

### 5. Finite Element Discretization and Error Estimates

Let \( \{T_h\}_{h>0} \) be a quasiform, shape regular family of conforming triangulations of \( \Omega \) with \( \max_{T \in T_h} \text{diam}(T) \leq h \).

To avoid technical details estimating the mismatch of the triangulation with the exact geometry we (unrealistically) assume that elements on the boundary are curved and match the boundary exactly. Hence

\[ \bigcup_{T \in T_h} T = \bar{\Omega}. \]

**Remark 5.1.** The quasiformity of \( \{T_h\}_{h>0} \) is required to guarantee the \( W^1_p \) stability of the Ritz operator (see below).

To discretize \( W^1_p(\Omega), W^1_p(\Omega) \) we choose Lagrange elements of polynomial order \( k \geq 1 \). Denote this space by \( S_h = S_h(T_h) \). Other choices, however, are possible and are restricted only by the assumptions in [7, Chapter 8]. Furthermore, for the Navier–Stokes part of the system we choose an inf-sup stable pair of elements \( V_h \times Q_h \) with \( V_h \subseteq H^{1,2}(\Omega), Q_h \subseteq L_{2,0}(\Omega) \) with the approximation property

\[
\inf_{v_h \in V_h} \| u - v_h \|_{1,2} + \inf_{q_h \in Q_h} \| p - q_h \|_{0,2} \leq h^k(\| u \|_{k+1,2} + \| p \|_{k,2}). \quad (5.1)
\]

The discrete problem reads: Find \( U_h = (\tilde{\phi}, T_h, u_h, p_h) \in X_h = S_h \times S_h \times V_h \times Q_h \) such that

\[
\langle \mathcal{F}(U_h), V_h \rangle = 0 \quad (5.2)
\]

for all \( V_h = (\psi_h, \varphi_h, v_h, q_h) \in Y_h := S_h \times S_h \times V_h \times Q_h \).

### 5.1. Error Estimates in the Norm of \( X \)

Let \( U = (\tilde{\phi}, T, u, p) \in X \) be a solution of \( \mathcal{F}(U) = 0 \). To be able to apply the general results from [9] regarding error estimates, we have to show the following.

1. \( \mathcal{F} : X \to Y' \) is differentiable;
2. \( D\mathcal{F} \) is locally Lipschitz continuous at \( U \);
3. \( D\mathcal{F}(U) : X \to Y' \) is an isomorphism;
4. \( \dim X_h = \dim Y_h \);
5. the following discrete inf-sup condition holds: \( \inf_{U_h \in X_h} \sup_{V_h \in Y_h} \langle D\mathcal{F}(U)U_h, V_h \rangle = \beta > 0 \).

Properties (1)–(3) have been shown in the previous section under some smallness assumption and (4) holds by construction. The remaining point thus is the inf-sup condition (5).
As in the proof of Proposition 4.2 we first consider the reduced operator \( T \). Since we chose a pair of finite element spaces \( V_h \times Q_h \) which is inf-sup stable for Navier-Stokes and due to Korn’s inequality, for the \((u, p)\) part of \( T \) one has:

\[
\sup_{\|v_h\|_{1, 2} = 1, \|q_h\|_{0, 2} = 1} \langle T(0, 0, w_h, r_h), (0, 0, v_h, q_h) \rangle = \sup_{\|v_h\|_{1, 2} = 1, \|q_h\|_{0, 2} = 1} \int_{\Omega} \frac{\mu(\phi)}{2} D(w_h) : D(v_h) - \int_{\Omega} r_h \nabla \cdot v_h + \int_{\Omega} q_h \nabla \cdot w_h
\]

\[
\geq \beta_1 \left( \|w_h\|_{1, 2} + \|r_h\|_{0, 2} \right)
\]

for all \( w_h \in V_h, r_h \in Q_h \) and some \( \beta_1 > 0 \).

Next we consider the \( \Theta \)-part of \( T \). From [9, Thm. 10.1] we infer that

\[
\sup_{\|\varphi_h\|_{1, \prime} = 1} \langle T(0, \Theta_h, 0, 0), (0, \varphi_h, 0, 0) \rangle = \sup_{\|\varphi_h\|_{1, \prime} = 1} \int_{\Omega} \nabla(\Theta_h) \cdot \nabla \varphi_h \geq \beta_2 \|\Theta\|_{1, p}
\]

for all \( \Theta_h \in S_h \) and some \( \beta_2 > 0 \). The crucial point in the proof in [9] was the stability of the Ritz-operator \( R_h : W^1_p(\Omega) \to S_h \) in the \( W^1_p \)-norm:

\[
\|R_h \Theta\|_{1, p} \lesssim \|\Theta\|_{1, p}.
\]

In [9] this result was cited from [15], where it was proved for dimension \( d = 2 \). A much more general result valid for \( d = 2 \) as well as for \( d = 3 \) and a variety of finite element spaces can be found in [7].

In order to get an inf-sup estimate for the \( \chi \)-equation, we rescale the \( T \)-equation for \( F \): For \( \lambda > 0 \) define

\[
\langle F_\lambda(\tilde{\phi}, T, u, p), (\psi, \varphi, v, q) \rangle := \langle F(\tilde{\phi}, T, u, p), (\psi, \lambda \varphi, v, q) \rangle.
\]

All what have been shown for \( F \) and \( DF \) remains valid also for \( F_\lambda \) and \( DF_\lambda \) except that \( \beta_2 \) becomes \( \lambda \beta_2 \).

Now for the \( \chi \)-equation we use the same result from [9] as for the \( \Theta \)-equation and infer for \( \chi_h, \Theta \in S_h \)

\[
\sup_{\|\psi_h\|_{1, \prime} = 1} \langle T_\lambda(\chi, \Theta_h, 0, 0), (\psi_h, 0, 0, 0) \rangle \geq \sup_{\|\psi_h\|_{1, \prime} = 1} \left( \int_{\Omega} \nabla \chi_h \cdot \nabla \psi_h - \int_{\Omega} h(\phi) \nabla \theta \cdot \nabla \psi_h \right)
\]

\[
\geq \beta_1 \|\chi_h\|_{1, p} - c \|\Theta_h\|_{1, p},
\]

where the last step follows from the boundedness of \( h(\cdot) \) and Hölder’s inequality.

Putting everything together we arrive at

\[
3 \times \sup_{\|V_h\|_{Y} = 1} \langle T(U_h, V_h) \rangle \geq \beta_1 \|\chi_h\|_{1, p} + (\lambda \beta_2 - c) \|\Theta_h\|_{1, p} + \beta_3 \left( \|w_h\|_{1, 2} + \|r_h\|_{0, 2} \right) \geq \beta \|U_h\|_X
\]

for all \( U_h = (\chi_h, \Theta_h, w_h, r_h) \in X_h \) and some \( \beta > 0 \), provided \( \lambda \) is sufficiently big.

The rest of the inf-sup estimate follows exactly as in the proof of Proposition 4.2: define \( N_{\lambda} := DF_\lambda(\phi, T, u, p) - T_\lambda \). Then we readily have

\[
\inf_{U_h \in X_h} \sup_{V_h \in Y_h} \langle DF_\lambda(\phi, T, u, p)U_h, V_h \rangle \geq \beta - \|N\|_{L(X,Y')} = \tilde{\beta} > 0
\]

if \( \|T(\phi, T, u, p)\|_X \) is small enough.

We are now in a state to apply [9, Thm. 7.1].

**Theorem 5.2.** Let \( p > d \) and \( S_h, V_h, Q_h \) as above. Let \( U = (\tilde{\phi}, T, u, p) \in X \) be a solution of \( F(U) = 0 \). Then there exist constants \( \delta, h_0, r_0, C > 0 \) such that if \( \|T(\phi, T, u, p)\|_X < \delta \) for \( 0 < h \leq h_0 \) the discrete problem (5.2) has a locally unique solution \( X_h \supseteq U_h \in B_{r_0}(U) \subseteq X \) and

\[
\|U - U_h\|_X \leq C \inf_{V_h \in X_h} \|U - V_h\|_X.
\]
Corollary 5.3. Let \( p > 3 \geq d \) and \( j \in \mathbb{N}_0 \). Furthermore, let \( k = 1 + j \). The spaces \( X, Y, X_h = Y_h \) are as above. Let \( \partial \Omega, k(\cdot), \mu(\cdot) \) fulfill the regularity assumptions of Corollary 3.5 and also

\[
\phi_D \in W^{2+j}_p(\Omega), \quad f, g \in W^j_p(\Omega).
\]

Then there are constants \( \delta, h_0, r_0, C > 0 \) such that the following holds: if

\[
||\phi_D||_{2,p} + ||f||_{0,p} + ||g||_{0,2} < \delta
\]

then there is a solution \( U \in X \) of problem (3.1), i.e. \( F(U) = 0 \), fulfilling

\[
||\phi||_{2+j,p} + ||T||_{2+j,p} + ||u||_{2+j,2} \leq C \left( ||\phi_D||_{2+j,p} + ||f||_{j,p} + ||g||_{j,2} \right).
\]

Moreover, for \( 0 < h \leq h_0 \) the discrete problem (5.2) has a locally unique solution \( X_h \ni U_h \in B_{r_0}(U) \subseteq X \) and the following error estimate holds:

\[
||\phi - \phi_h||_{1,p} + ||T - T_h||_{1,p} + ||u - u_h||_{1,2} + ||p - p_h||_{0,2} \lesssim h^k.
\]

Proof. By the above theorem we know that for sufficiently small \( \delta > 0 \) there is a solution \( (\phi, T, u) \in X \cap W^{2+j}_p(\Omega) \times W^j_p(\Omega) \times H^2(\Omega) \). Possibly reducing \( \delta \) further, the above theorem guarantees the existence of a locally unique discrete solution \( U_h \) and its quasi-optimality \( ||U - U_h||_X \leq \inf_{V_h \in X_h} ||U - V_h||_X \). The rest follows by the approximation property of the finite element spaces and the (possibly) higher regularity shown in Corollary 3.5 and Lemma 3.6.

5.2. Error Estimates in Weaker Norms

Error estimates in \( L_p, L_2 \) can be proved by using rather standard duality techniques. To this end, let us introduce the bilinear form \( b(\cdot, \cdot) : X \times Y \rightarrow \mathbb{R} \) for the solution \( U = (\phi, T, u,p) \):

\[
b(U, V) := \langle DF(\phi, T, u,p) \bar{U}, V \rangle
\]

and the associated operators \( B : X \rightarrow Y' \), \( B^* : Y \rightarrow X' \) by

\[
\langle B U, V \rangle_{Y', Y} = b(U, V) = \langle \bar{U}, B^* V \rangle_{X, X'}
\]

for all \( \bar{U} \in X, V \in Y \).

Recall that \( B \) is an isomorphism, iff \( B^* \) is an isomorphism. As shown in Proposition 4.2 this is for instance the case, if \( ||(\phi, T, u, p)||_X \) is small enough, which we assume hereafter.

Now, choose \( G = (g_\phi, g_T, g_u, 0) \in L^p(\Omega) \times L^p(\Omega) \times L^2(\Omega) \times L_2(0) \subseteq X' \) such that \( ||g_\phi||_{0,p'} = ||g_T||_{0,p'} = ||g_u||_{0,2} = 1 \) and

\[
\langle g_\phi, \phi - \phi_h \rangle = ||\phi - \phi_h||_{0,p}, \quad \langle g_T, T - T_h \rangle = ||T - T_h||_{0,p}, \quad \langle g_u, u - u_h \rangle = ||u - u_h||_{0,2}.
\]

Let \( W \in Y \) be the unique solution of the dual problem (which exists, since \( B^* \) is an isomorphism)

\[
b(\bar{U}, W) = \langle \bar{U}, B^* W \rangle_{X, X'} = \langle \bar{U}, G \rangle_{X, X'}
\]

for all \( \bar{U} \in X \).

Let \( W_h \in Y_h \). As in [9] we calculate

\[
||\langle U - U_h, G \rangle|| = ||b(U - U_h, W)| \leq ||b(U - U_h, W - W_h)| + ||b(U - U_h, W_h)|
\]

for \( U, U_h \) the continuous and discrete solution, respectively. In order to bound the last term on the right-hand side we define \( \zeta(t) = (1-t)U + tU_h, t \in [0,1] \) and apply the mean value theorem to the scalar-valued function \( t \rightarrow \langle F(\zeta(t)), W_h \rangle \) to obtain

\[
b(U - U_h, W_h) = \langle DF(\phi, T, u,p)(U - U_h), W_h \rangle
\]

\[
= \langle -F(U) + F(U_h) - DF(U)(U_h - U), W_h \rangle
\]

\[
= \langle (DF(\zeta(t)) - DF(U))(U_h - U), W_h \rangle
\]
for some $t \in [0, 1]$. Thus we get
\[ |b(U - U_h, W_h)| \leq L\|U - U_h\|^2_X\|W_h\|_Y, \]
if $\|U - U_h\|_X$ is sufficiently small and with $L$ the Lipschitz constant of $D\mathcal{F}$. The above estimate may be viewed as a substitute for the orthogonality of the error in the linear case. Using this estimate in Eq. (5.4) and the boundedness of $b(\cdot, \cdot)$ (which follows from Lemma 4.1) we arrive at
\[ |\langle U - U_h, G \rangle| \leq \|U - U_h\|_X\|W - W_h\|_Y + \|U - U_h\|^2_X(\|W - W_h\|_Y + \|W\|_Y). \] (5.5)
It remains to estimate $\|W - W_h\|_Y$, which will be accomplished by a regularity result for $W$.

Lemma 5.4. Let the solution $U = (\dot{\phi}, T, u, p)$ fulfill
\[ \dot{\phi}, T \in W^2_p(\Omega) \cap W^1_p(\Omega), \quad u \in H^2(\Omega) \cap V. \]
Then the dual solution $W = (\psi, \varphi, v, q)$ from Eq. (5.3) is regular, i.e.
\[ \psi, \varphi \in W^2_{p'}(\Omega), \quad v \in H^2(\Omega), q \in H^1(\Omega) \]
and
\[ \|\psi\|_{2,p'} + \|\varphi\|_{2,p'} + \|v\|_{2,2} + \|q\|_{1,2} \leq C \]
independent of $h > 0$.

Proof. The dual solution $W = (\psi, \varphi, v, q)$ fulfills $B^*W = G$. This implies in the distributional sense:
\[ -\Delta \psi + h'(\phi)\nabla T \cdot \nabla \psi - u \cdot \nabla \psi \]
\[ + k'(\phi)\nabla T \cdot \nabla \varphi + u \cdot \nabla T \varphi + \nabla \cdot ((\varphi \nabla T) - h'(\phi)\nabla T)^2 \varphi \]
\[ - h'(\phi)\nabla T^T (\nabla u \nabla T) + (u \cdot \nabla \varphi) \cdot v + \nabla \cdot (\nabla T \nabla u) + \frac{\mu'(\phi)}{2} D(u) : D(v) = g_{\phi}, \] (5.6)
\[ - k(\varphi)\Delta \varphi - k'(\phi)\nabla \psi \cdot \nabla \varphi + \nabla \cdot (h(\phi)\varphi \nabla T) - e_p \cdot v \]
\[ - (\mu(T) D(v)) - \nabla q + \psi T + q + \psi \nabla \psi + \eta \varphi \nabla T = g_{u}, \quad -\nabla \cdot v = 0, \] (5.7)
or equivalently
\[ -\Delta \phi = l_\phi := g_{\phi} - h'(\phi)\nabla T \cdot \nabla \psi + u \cdot \nabla \psi \]
\[ - k'(\phi)\nabla T \cdot \nabla \varphi - u \cdot \nabla T \varphi - \nabla \cdot ((\varphi \nabla T) - h'(\phi)\nabla T)^2 \varphi \]
\[ + h'(\phi)\nabla T^T (\nabla u \nabla T) - (u \cdot \nabla \varphi) \cdot v - \nabla \cdot (\nabla T \nabla u) - \frac{\mu'(\phi)}{2} D(u) : D(v), \] (5.9)
\[ -\Delta \varphi = l_T := \frac{1}{k(\phi)} \left( g_T + k'(\phi)\nabla \psi \cdot \nabla \varphi - \nabla \cdot (h(\phi)\varphi \nabla T) + (\mu(T) D(v)) + (\mu(T) D(v)) \cdot e_p \cdot v \right. \]
\[ \left. + \nabla \cdot (h(\phi)\nabla \psi) - \nabla \cdot (h(\phi)\nabla T \nabla u) \right), \] (5.10)
\[ -\nabla \cdot (\mu(T) D(v)) - \nabla q = l_u := g_{u} + \nabla q + \rho \nabla u + \rho \nabla u^T \nabla v - \psi \nabla \psi - \eta \varphi \nabla T, \quad -\nabla \cdot v = 0. \] (5.11)

We already know that there is a unique solution $W \in Y$ fulfilling $\|W\|_Y \leq C$, which means $\|\psi\|_{1,p'} + \|\varphi\|_{1,p'} + \|v\|_{1,2} + \|q\|_{0,2} \leq C$. Let us start inspecting Eq. (5.11). Following from the assumption on $(\phi, T, u)$ and because $W^1_{p'}(\Omega) \hookrightarrow L_q(\Omega)$ with $q \geq 3/2$ we conclude $l_u \in L^1_{3/2}(\Omega)$. From Lemma 3.1 one infers $v \in W^2_{3/2}(\Omega)$.

Next, it is readily seen that $l_\phi \in L^p_{p'}(\Omega)$ (note that $p' < 2$). We show this for the worst term occurring in $l_\phi$, namely $\nabla \cdot (\varphi \nabla T)$:
\[ \nabla \cdot (\varphi \nabla T) = \nabla \varphi \cdot \nabla T + \nabla \varphi \cdot \nabla T \]
\[ \in L^p_{p'}(\Omega) \quad \in L^\infty(\Omega) \quad \in L^q(\Omega) \in L^p_p(\Omega) \]
with \( \frac{1}{q} = \frac{1}{p'} - \frac{1}{d} \) by embedding and then
\[
\varphi \Delta T \in L_s(\Omega)
\]
with \( \frac{1}{s} = \frac{1}{q} + \frac{1}{p} = \frac{1}{p'} - \frac{1}{d} + \frac{1}{p} = 1 - \frac{1}{d} \leq 2/3 \) so that \( s \geq 3/2 \geq p' \). From \( l_\varphi \in L_{p'}(\Omega) \) we conclude \( \varphi \in W^2_{p'}(\Omega) \).

With this information one checks that also \( l_T \in L_{p'}(\Omega) \) and therefore \( \varphi \in W^2_{p'}(\Omega) \).

As a last step, we go back to Eq. (5.11). Knowing that \( \psi, \varphi \in W^2_{p'}(\Omega) \hookrightarrow L_q(\Omega) \) with \( q > 3 \) we finally conclude \( l_u \in L_2(\Omega) \) and so \( \nu \in H^2(\Omega) \).

In the above arguments it is of course understood that the corresponding norms are bounded by the right-hand side.

Putting everything together, we arrive at the following error estimate.

**Theorem 5.5.** Let \( p > 3 \) and \( U = (\tilde{\phi}, T, u, p) \in X \) be a solution of the continuous system Eq. (3.1) with \( \tilde{\phi}, T \in W^2_{p}(\Omega) \cap W^1_p(\Omega), \ u \in H^2(\Omega) \cap V \)
and \( U_h = (\tilde{\phi}_h, T_h, u_h, p_h) \in X_h \) the corresponding discrete solution from Eq. (5.2), sufficiently close to \( U \). Then
\[
\|\phi - \phi_h\|_{0,p} + \|T - T_h\|_{0,p} + \|u - u_h\|_{0,2} \lesssim \|U - U_h\|_X + h\|U - U_h\|_X.
\]

**Proof.** In view of the definition of the dual solution \( W \) from Eq. (5.3) and the estimate Eq. (5.5), take the infimum over all \( W_h \in Y_h \) in the latter estimate. The assertion then follows by the regularity of \( W \) from Lemma 5.4 and the approximation properties of the finite element spaces. \( \square \)

### 6. Computational Results

For the computations presented in this section piecewise quadratic finite elements are used for \( \phi \) and \( T \) as well as the \( P_2 \times P_1 \)-Taylor–Hood element for \( u \) and \( p \). In order to get an interesting computational example, where one can see the effect of thermophoresis, we slightly deviate from the set of boundary conditions imposed in the theoretical part. Set \( \Omega = [0,2] \times [0,1] \). For the concentration \( \phi \) we impose homogeneous Neumann boundary conditions on the whole of \( \partial\Omega \) and a mean concentration \( \phi_m = 0.1 \).

The temperature \( T \) is set to \( T = 1 \) at the left side wall and \( T = 0 \) at the right side wall. On the remaining parts of \( \partial\Omega \) homogeneous Neumann boundary conditions are imposed. For the velocity \( u \) homogeneous no-slip conditions are chosen except for the upper boundary, where a slip condition is enforced, i.e. \( u_2 = 0 \) together with vanishing tangential stress \( \mu(\phi)(\partial_{x_2} u_1 + \partial_{x_1} u_2) = 0 \). Note that this condition can be realized as a natural boundary condition for the space \( \{v \in H^1(\Omega) \mid v_2 = 0 \text{ on the upper part of } \partial\Omega, v = 0 \text{ else on } \partial\Omega \} \).

For the coefficients \( \mu(\cdot), k(\cdot) \) we set
\[
\mu(\phi) = 1 + 39.11\phi + 533.9\phi^2, \quad k(\phi) = 1 + 4.5503\phi,
\]
similar to those in [8], representing fittings from experimental data for alumina \( \text{Al}_2\text{O}_3 \) particles.

Let us first consider the case without thermophoretic effects (i.e. \( \phi \equiv \phi_m \)). The flow is driven by buoyancy forces: the liquid heats up at the left lateral wall inducing an upward flow field that turns to the right at the top of the container, transporting warm liquid to the right, cold wall, where it cools down and flows downwards. The cold liquid is flowing back at the bottom of the container to the left hot wall, see Fig. 1, left picture.

Switching on thermophoretic effects, the flow field is strongly enhanced on the upper boundary. This can be understood by inspecting the concentration field, see Fig. 2. The thermophoretic flux \( j_{\text{therm}} = -\phi(1 - \phi) \frac{1}{N_{BT}} \sum \nabla \phi \) pushes concentration away from the left, hot and upper walls (the flux is in direction from hot to cold), thus decreasing the viscosity there. The opposite effect takes place at the right, cold wall: concentration is pushed to the cold wall.
Fig. 1. Magnitude of velocity without (left) and with (right) thermophoretic effect (Re = 700, Pr = 6, $N_{BT} \times T_0 = 0.586$, $Sc = 1$, $Sc_f = 1 \times 10$, $Le = -1 \times 10$, $\phi_m = 0.1$). Switching on thermophoretic effects, the flow field is strongly enhanced on the upper boundary.

Fig. 2. Concentration $\phi$ (Re = 700, Pr = 6, $N_{BT} \times T_0 = 0.586$, $Sc = 1$, $Sc_f = 1 \times 10$, $Le = -1 \times 10$). The thermophoretic flux $j_{therm} = -\phi (1 - \phi) \frac{1}{N_{BT} T_0} \nabla T$ pushes concentration away from the left, hot and upper walls (the flux is in direction from hot to cold), thus decreasing the viscosity there. The opposite effect takes place at the right, cold wall: concentration is pushed to the cold wall.

Table 1. Errors and EOCs; $nt =$ number of elements

| $nt$ | $\| \phi - \phi_h \|_{0.6}$ | EOC | $\| T - T_h \|_{0.6}$ | EOC | $\| u - u_h \|_{0.2}$ | EOC |
|------|-----------------|-----|-----------------|-----|-----------------|-----|
| 256  | $3.0296e-04$    | $-$ | $5.3766e-05$    | $-$ | $2.1233e-04$    | $-$ |
| 1024 | $2.6758e-05$    | $3.50$ | $5.2109e-06$    | $3.37$ | $2.7126e-05$    | $2.97$ |
| 4096 | $2.6659e-06$    | $3.33$ | $5.8438e-07$    | $3.16$ | $3.3971e-06$    | $3.00$ |
| 16384| $3.7459e-07$    | $2.83$ | $7.4084e-08$    | $2.98$ | $4.2396e-07$    | $3.00$ |

Re = 100, Pr = 1, $N_{BT} \times T_0 = 0.586$, $Sc = 1$, $Sc_f = 1 \times 4$, $Le = 1 \times 4$, $\phi_m = 0.1$

In order to quantitatively assess the convergence, the same setting as above, however with different parameters, is used. Since the exact solution is unknown, the computational solution on a very fine grid with $nt = 262$, 144 triangles is used as reference instead. Starting from a coarse triangulation, the grid
is successively refined by two bisection steps each. The corresponding errors and the experimental order of convergence (EOC) are listed in Table 1. As expected one gets a convergence order of 3 (although the boundary of the domain is not of class $C^3$).

Acknowledgements. Pedro Morin was partially supported by Agencia Nacional de Promoción Científica y Tecnológica, through grants PICT-2014-2522, PICT-2016-1983, by CONICET through PIP 2015 11220150100661, and by Universidad Nacional del Litoral through grants CAI+D 2016-50420150100022LI. A research stay at Universität Erlangen was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach as well as by the DFG–RTG 2339 IntComSin.

Compliance with Ethical Standards

Conflict of interest The authors declare that there is no conflict of interest.

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(accepted: December 31, 2020; published online: March 22, 2021)