BOUNDS ON ZIMIN WORD AVOIDANCE

JOSHUA COOPER* AND DANNY RORABAUGH*

Abstract. How long can a word be that avoids the unavoidable? Word $W$ encounters word $V$ if there is a homomorphism $\phi$ defined by mapping letters to nonempty words such that $\phi(V)$ is a subword of $W$. Otherwise, $W$ is said to avoid $V$. If, on any arbitrary finite alphabet, there are finitely many words that avoid $V$, then we say $V$ is unavoidable. Zimin (1982) proved that every unavoidable word is encountered by some word $Z_n$, defined by:

\[ Z_1 = x \]
\[ Z_{n+1} = Z_n x_{n+1} Z_n \]

Here we explore bounds on how long words can be and still avoid the unavoidable Zimin words.

In 1929, Frank Ramsey proved that, for any fixed $r, n, \mu \in \mathbb{Z}^+$, every sufficiently large set $\Gamma$ with its $r$-subsets partitioned into $\mu$ classes is guaranteed to have a subset $\Delta_n \subseteq \Gamma$ such that all the $r$-subsets of $\Delta_n$ are in the same class. This was the advent of a major branch of combinatorics that became known as Ramsey theory. Often applied to graph theoretic structures, Ramsey theory looks at how large a random structure must be to guarantee that a given substructure exists or a given property is satisfied. Here we apply this paradigm to an existence result from the combinatorics of words.

Definition 0.1. A \textit{q-ary word} is a string of characters, at most $q$ of them distinct.

Over a fixed $q$-letter alphabet, the set of all finite words forms a semigroup with concatenation as the binary operation (written multiplicatively) and the empty word $\varepsilon$ as the identity element. We also have a binary subword relation $\leq$ where $V \leq W$ when $W = UVU'$ for some words $U, V,$ and $U'$. That is, $V$ appears contiguously in $W$.

Definition 0.2. We call word $W$ an \textit{instance} of $V$ provided

- $V = x_0 x_1 \cdots x_{m-1}$ where each $x_i$ is a letter;
- $W = A_0 A_1 \cdots A_{m-1}$ with each $A_i \neq \varepsilon$ and $A_i = A_j$ whenever $x_i = x_j$.

Equivalently, $W$ is a $V$-\textit{instance} provided there exists some semigroup homomorphism $\phi$ such that $\phi(x_i) = A_i \neq \varepsilon$ for each $i$.

Example 0.3. $W = abceabbcxdec$ is an instance of $V = xyxzzy$, with $\phi$ defined by $\phi(x) = abb$, $\phi(y) = c$, and $\phi(z) = xd$.

*University of South Carolina
Definition 0.4. A word $U$ encounters word $V$ provided some subword $W \leq U$ is an instance of $V$. If $U$ fails to encounter $V$, then $U$ avoids $V$.

![Figure 1. Binary words that avoid $xx$.](image)

We see in Figure 1 that $xx$ is avoided by only finitely many words over a two-letter alphabet. However, it has been known for over a century [5] that $xx$ can be avoided by arbitrarily long (even infinite) ternary words.

Definition 0.5. A word $V$ is unavoidable provided for any finite alphabet, there are only finitely many words that avoid $V$.

A. I. Zimin proved an elegant classification of all unavoidable words [6].

Definition 0.6. Define the $n$th Zimin word recursively by $Z_0 := \varepsilon$ and, for $n \in \mathbb{N}$, $Z_{n+1} := Z_n x_n Z_n$. Using the alphabet rather than indexed variables:

$Z_1 = a, \quad Z_2 = aba, \quad Z_3 = abacaba, \quad Z_4 = abacabadacaba, \quad \ldots$

Equivalently, $Z_n$ can be defined over the natural numbers as the word of length $2^n - 1$ such that the $i$th letter is the 2-adic order of $i$ for $1 \leq i < 2^n$.

Theorem 0.7 (Zimin, 1982). A word $V$ with $n$ distinct letters is unavoidable if and only if $Z_n$ encounters $V$.

1. Avoiding the Unavoidable

From Zimin’s explicit classification of unavoidable words, a natural question arises in the Ramsey theory paradigm: for a fixed unavoidable word $V$, how long can a word be that avoids $V$? Our approach to this question is to start with avoiding the Zimin words, which gives upper bounds for all unavoidable words. Define $f(n, q)$ to be the smallest integer $M$ such that every $q$-ary word of length $M$ encounters $Z_n$. 

Theorem 1.1. For \( n, q \in \mathbb{Z}^+ \) and \( Q := 2q + 1 \),
\[
f(n, q) \leq n^{-1} Q := Q^Q,
\]
with \( Q \) occurring \( n - 1 \) times in the exponential tower.

Proof. We proceed via induction on \( n \). For the base case, set \( n = 1 \). Every nonempty word is an instance of \( Z_1 \), so \( f(1, q) = 1 \).

For the inductive hypothesis, assume the claim is true for some positive \( n \) and set \( T := f(n, q) \). That is, every \( q \)-ary word of length \( T \) encounters \( Z_n \). Concatenate any \( q^T + 1 \) strings \( W_0, W_1, \ldots, W_{q^T} \) of length \( T \) with an arbitrary letter \( a_i \) between \( W_{i-1} \) and \( W_i \) for each positive \( i \leq q^T 

\[
U := W_0 a_1 W_1 a_2 W_2 \cdots W_{q^T - 1} a_{q^T} W_{q^T}.
\]

By the pigeonhole principle, \( W_i = W_j \) for some \( i < j \). That string, being length \( T \), encounters \( Z_n \). Therefore, we have some word \( W \leq W_i \) that is an instance of \( Z_n \) and shows up twice, disjointly, in \( U \). The extra letter \( a_{i+1} \) guarantee that the two occurrences of \( W \) are not consecutive. This proves that an arbitrary word of length \( (T + 1)(q^T + 1) - 1 \) witnesses \( Z_{n+1} \), so
\[
f(n + 1, q) \leq (T + 1)(q^T + 1) - 1 \leq (2q + 1)^T = Q^T.
\]

There is clearly a function \( Q(n, q) \) such that \( f(n + 1, q) \leq Q(n, q)^{f(n, q)} \) and \( Q(n, q) \to q \) as \( n \to \infty \). No effort has been made to optimize the choice of function, as such does not decrease the tetration in the bound.

The technique used to prove Theorem 1.1 is first found in Lothaire’s proof of unavoidability of \( Z_n \) ([1], 3.1.3). The technique in Zimin’s original proof [6] implicitly gives that for \( n \geq 2 

\[
f(n + 1, q + 1) \leq f(n + 1, q + 2 |Z_{n+1}|) f(n, |Z_{n+1}|^2 q^{f(n+1, q)}).
\]

This is an Ackermann-type function for an upper bound, which is much larger than the primitive recursive bound from Theorem 1.1.

Table 1 shows known values of \( f(n, 2) \). Supporting word-lists and Sage code are found in the Appendix.

| \( n \) | \( Z_n \) | \( f(n, 2) \) |
|---|---|---|
| 0 | \( \varepsilon \) | 0 |
| 1 | a | 1 |
| 2 | aba | 5 |
| 3 | abacaba | 29 |
| 4 | abacabadabacaba | \( \geq 10483 \) |
2. Finding a Lower Bound with the First Moment Method

Throughout this section, $q$ is a fixed integer greater than 1. Given a fixed alphabet of $q$ letters, $C(n, q, M)$ denotes the set of length-$M$ instances of $Z_n$. That is

$$C(n, q, M) := \{W \mid W \in \{x_0, \ldots, x_{q-1}\}^M \text{ is a } Z_n\text{-instance}\}.$$

**Lemma 2.1.** For all $n, M \in \mathbb{Z}^+$,

$$|C(n, q, M+1)| \geq q \cdot |C(n, q, M)|.$$

**Proof.** Take arbitrary $W \in C(n, q, M)$. We can write $W = W_1W_0W_1$ with $W_1 \in C(n, q, N)$, where $2N < M$. Choose the decomposition of $W$ to minimize $|W_1|$. Then $W_1W_0xW_1 \in C(n, q, M+1)$ for each $i < q$.

The lemma follows, unless a $Z_n$-instance of length $M+1$ can be generated in two ways – that is, if $W_1W_0aW_1 = V_1V_0bV_1$ for some $V_1V_0V_1 = V$, where $|V_1|$ is also minimized. If $|V_1| < |W_1|$, then $V_1$ is a prefix and suffix of $W_1$, so $|W_1|$ was not minimized. But if $|V_1| > |W_1|$, then $W_1$ is a prefix and suffix of $V_1$, so $|V_1|$ was not minimized. Therefore, $|V_1| = |W_1|$, so $V_1 = W_1$, which implies $a = b$ and $V = W$. □

**Corollary 2.2** (Monotonicity). For all $n, M \in \mathbb{Z}^+$,

$$\Pr(W \in C(n, q, M + 1) \mid W \in \{x_0, \ldots, x_{q-1}\}^{M+1}) \geq \Pr(W \in C(n, q, M) \mid W \in \{x_0, \ldots, x_{q-1}\}^{M}),$$

assuming uniform probability on words of a fixed length.

**Lemma 2.3.** For all $n, M \in \mathbb{Z}^+$,

$$|C(n, q, M)| \leq \left(\frac{q}{q-1}\right)^{n-1} q^{(M-2^n+n+1)}.$$

**Proof.** The proof proceeds by induction on $n$. For the base case, set $n = 1$. Every non-empty word is an instance of $Z_1$, so $|C(1, q, M)| = q^M$.

For the inductive hypothesis, assume the claim is true for some positive $n$. The first inequality below derives from the following way to overcount the number of $Z_{n+1}$-instances of length $M$. Every such word can be written as $UVU$ where $U$ is a $Z_n$-instance of length $j < M/2$. Since an instance of $Z_n$ can be no shorter than $Z_n$, we have $2^n - 1 \leq j < M/2$. For each possible $j$, there are $|C(n, q, j)|$ ways to choose $U$ and $q^{M-2j}$ ways to choose $V$. This is an overcount, since a Zimin-instance may have multiple decompositions.
\[|C(n + 1, q, M)| \leq \sum_{j=2^n-1}^{(M-1)/2} |C(n, q, j)| q^{M-2j} \]
\[\leq \sum_{j=2^n-1}^{(M-1)/2} \left( \frac{q}{q-1} \right)^{n-1} q^{(j-2^n+n+1)} q^{M-2j} \]
\[= \left( \frac{q}{q-1} \right)^{n-1} q^{(M-2^n+n+1)} \sum_{j=2^n-1}^{(M-1)/2} q^{-j} \]
\[< \left( \frac{q}{q-1} \right)^{n-1} q^{(M-2^n+n+1)} \sum_{j=2^n-1}^{\infty} q^{-j} \]
\[= \left( \frac{q}{q-1} \right)^{(n-1)+1} q^{(M-2^n+n+1)+1}.\]

\[\square\]

**Corollary 2.4.** For all \(n, M \in \mathbb{Z}^+\),
\[\Pr \left( W \in C(n, q, M) \mid W \in \{x_0, \ldots, x_{q-1}\}^M \right) \leq \left( \frac{q}{q-1} \right)^{n-1} q^{(-2^n+n+1)},\]
assuming uniform probability on words of length \(M\).

**Theorem 2.5.**
\[f(n, q) \geq q^{2(n-1)(1+o(1))} \quad (q \to \infty, n \to \infty).\]

**Proof.** Let word \(W\) consist of \(M\) uniform, independent random selections from the alphabet \(\{x_0, \ldots, x_{q-1}\}\). Define the random variable \(X\) to count the number of subwords of \(W\) that are instances of \(Z_n\) (including repetition if a single subword occurs multiple times in \(W\)):
\[X = |\{V \mid W \geq V \in C(n, q, |V|)\}|.\]
By monotonicity with respect to word length:
\[E(X) \leq |\{V \mid V \leq W\}| \cdot \Pr(W \in C(n, q, M)) \leq \left( \frac{M+1}{2} \right) \left( \frac{q}{q-1} \right)^{n-1} q^{(-2^n+n+1)} \leq M^2 e^{(n-1)/(q-1)} q^{(-2^n+n+1)}.\]
There exists a word of length $M$ that avoids $Z_n$ when $E(X) < 1$. It suffices to show that:

$$M^2 \left( e^{(n-1)/(q-1)} q^{-2^n+n+1} \right) \leq 1.$$

Solving for $M$:

$$M \leq \left( e^{(n-1)/(q-1)} q^{-2^n+n+1} \right)^{-1/2} = q^{2(n-1)} \left( e^{(n-1)/(q-1)} q^{(n+1)} \right)^{-1/2} = q^{2(n-1)}(1+o(1)).$$

□

**CONTINUING WORK**

Current efforts to improve bounds on the probability that a word is an instance of $Z_n$ will help close the gap between the lower and upper bounds on $f(n,q)$.

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Appendix: Binary Words that Avoid $Z_n$

All binary words that avoid $Z_2$. The following 13 words are the only words over the alphabet \{0, 1\} that avoid $Z_2 = aba$.

$$
\varepsilon, 0, 00, 001, 0011, 01, 011, 1, 10, 100, 11, 110, 1100.
$$

Maximum-length binary words that avoid $Z_3$. The following 48 words are the only words of length $f(3, 2) - 1 = 28$ over the alphabet \{0, 1\} that avoid $Z_3 = abacaba$. All binary words of length $f(3, 2) = 29$ or longer encounter $Z_3$. This result is easily, computationally verified by constructing the binary tree of words on \{0, 1\}, eliminating branches as you find words that encounter $Z_3$.

0010010011011011111100000011, 11000000100110110111111100,
001001001111110000001110111, 110000001001111111110011100011,
0010101100110011111100000011, 1100000010110011001111111000011,
001010111111000001100110011, 11000000110011001100111111110011,
0010101111110011001100000011, 11000000110011001100111111110011,
00110011001111110000000101011, 11000000110110111111001110011011011011,
001100110011111100110010101111, 11000000110110111111001110011011011011,
001100110011111101001100110011, 11000000110110111111001110011011011011,
00110011001111110110100100110011, 11000000110110111111001110011011011011,
00110011001111110110100100110011, 11000000110110111111001110011011011011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
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0011111100000101001101101100000011, 1100101000000011111111111111110011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
0011111100000101001101101100000011, 1100101000000011111111111111110011,
A long binary word that avoid $Z_4$:
The following binary word of length 10482 avoids $Z_4 = abacabadabacaba$. This implies that $f(4, 2) \geq 10483$. The word is presented here as an image with each row, consisting of 90 squares, read left to right. Each square, black or white, represents a bit. For example, the longest string of black in the first row is 14 bits long. We cannot have the same bit repeated $15 = |Z_4|$ times consecutively, as that would be a $Z_4$-instance. A string of 14 white bits is found in the 46th row.
Verifying that a word avoids $Z_n$:
The code to generate a $Z_4$-avoiding word of length 10482 is messy. The
following, easy-to-validate, inefficient, brute-force, Sage code was used
for verification of the word above. It took about half a day, running on an
Intel® Core™ i5-2450M CPU @ 2.50GHz \times 4.

```python
#Recursive function to test if V is an instance of Z_n
def inst(V,n):
    if n==1:
        if len(V)>0:
            return 1
        return 0
    else:
        top = ceil(len(V)/2)
        for i in range(2**(n-1)-1,top):
            if V[:i]==V[-i:]:
                if inst(V[:i],n-1):
                    return 1
        return 0

#Paste word here as a string
W =
L = len(W)
n = 4

#Check every subword V of length at least 2^n-1
for b in range(L+1):
    for a in range(b-(2**n-1)):
        if inst(W[a:b],n):
            print a,b,W[a:b]
```