ON THE EQUIVARIANT COHOMOLOGY OF SUBVARIETIES OF A $\mathcal{B}$-REGULAR VARIETY

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To Bert Kostant on his 80th birthday.

Abstract. By a $\mathcal{B}$-regular variety, we mean a smooth projective variety over $\mathbb{C}$ admitting an algebraic action of the upper triangular Borel subgroup $\mathcal{B} \subseteq SL_2(\mathbb{C})$ such that the unipotent radical in $\mathcal{B}$ has a unique fixed point. A result of M. Brion and the first author [4] describes the equivariant cohomology algebra (over $\mathbb{C}$) of a $\mathcal{B}$-regular variety $X$ as the coordinate ring of a remarkable affine curve in $X \times \mathbb{P}^1$. The main result of this paper uses this fact to classify the $\mathcal{B}$-invariant subvarieties $Y$ of a $\mathcal{B}$-regular variety $X$ for which the restriction map $i_Y : H^*(X) \to H^*(Y)$ is surjective.

1. Introduction

A nilpotent element $e$ in the Lie algebra $\mathfrak{g}$ of a complex semi-simple Lie group $G$ is regular if it lies in a unique Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. If we let $B$ be the unique Borel subgroup of $G$ with $\text{Lie}(B) = \mathfrak{b}$ and recall that the flag variety $G/B$ of $G$ parameterizes the family of all Borel subalgebras of $\mathfrak{g}$, it follows that the one parameter group $\exp(te)$ ($t \in \mathbb{C}$) of $G$ acts on $G/B$ by left translation with unique fixed point the identity coset $B$ or, equivalently, the unique Borel subalgebra $\mathfrak{b}$ containing $e$. By the Jacobson-Morosov Lemma, $e$ determines a two dimensional solvable subalgebra $\mathfrak{e}$ of $\mathfrak{g}$ isomorphic to the Lie algebra of the upper triangular Borel subgroup $\mathcal{B}$ of $SL_2(\mathbb{C})$. The two dimensional solvable subgroup $E$ of $G$ determined by $e$ is thus isomorphic to a Borel either in $SL_2(\mathbb{C})$ or $\mathbb{P}SL_2(\mathbb{C})$ and hence is a homomorphic image of $\mathcal{B}$. Consequently, $\mathcal{B}$ acts on $G/B$, via $E$, such that its unipotent radical $\mathcal{U}$ has exactly one fixed point, namely $B$.

In view of this, one may generalize the notion of a regular nilpotent by considering an algebraic action $\mathcal{B} \circ X$ of the upper triangular group $\mathcal{B}$ on a smooth complex projective variety $X$ such that its unipotent radical $\mathcal{U}$ has a unique fixed point $o \in X$. For such an action, the maximal torus $\mathcal{T}$ on the diagonal of $\mathcal{B}$ is known to have a finite fixed point set (see [5]). In [4,5], such an action is called regular and $X$ is called a regular variety. Slightly changing this terminology, we will henceforth call the action $\mathcal{B}$-regular and say that $X$ is a $\mathcal{B}$-regular variety. By the above remarks, the flag variety $G/B$ and, more generally, all algebraic homogeneous spaces $G/P$, $P$ a parabolic in $G$, give a rich class of $\mathcal{B}$-regular varieties.

The main goal of this paper is to study the cohomology algebras of $\mathcal{B}$-invariant subvarieties of a $\mathcal{B}$-regular variety. For example, Schubert varieties in a $G/B$ or $G/P$ (that is, closures of $B$-orbits) form an important class of
examples of \( \mathfrak{B} \)-invariant subvarieties to which our results will apply. Moreover, certain nilpotent Hessenberg varieties (including the Peterson variety) \cite{12} and certain Springer fibres corresponding to nilpotents in the centralizer of a regular nilpotent \( e \in \mathfrak{g} \) give further interesting examples of \( \mathfrak{B} \)-invariant subvarieties in \( G/B \) which we hope to investigate in a future work.

If \( X \) is a \( \mathfrak{B} \)-regular variety, then one knows the remarkable fact that its cohomology algebra \( H^\ast(X) \) over \( \mathbb{C} \) is isomorphic with the coordinate ring \( A(Z) \) of the zero scheme \( Z \) associated to the algebraic vector field on \( X \) generated by \( \mathfrak{u} \) (cf. \cite{1, 2, 5}). Moreover, its \( \mathcal{T} \)-equivariant cohomology \( H_\mathcal{T}^\ast(X) \) is isomorphic to the coordinate ring of a canonical \( \mathcal{T} \)-stable affine curve \( \mathcal{Z}_X \) in \( X \times \mathbb{P}^1 \). That is, \( \text{Spec}(H_\mathcal{T}^\ast(X)) = \mathcal{Z}_X \). These isomorphisms extend to what we will call principal subvarieties: namely, \( \mathfrak{B} \)-invariant subvarieties \( Y \) of \( X \) for which the natural restriction map \( i_Y^\ast : H_\mathcal{T}^\ast(X) \to H_\mathcal{T}^\ast(Y) \) is surjective (cf. Theorem 1). In particular, if \( Y \) is principal, then \( \text{Spec}(H_\mathcal{T}^\ast(Y)) \) is the (reduced) affine curve \( \mathcal{Z}_X \cap (Y \times \mathbb{C}) \). Furthermore, if \( Y \cap Z \) denotes the schematic intersection of \( Y \) and \( Z \), then the coordinate ring \( A(Y \cap Z) \) is isomorphic to \( H^\ast(Y) \) as long as \( \dim A(Y \cap Z) = \dim H^\ast(Y) \).

With these facts as motivation, our aim is to classify the principal subvarieties of a \( \mathfrak{B} \)-regular variety. This will follow from a general result which describes the image of \( H_\mathcal{T}^\ast(X) \) in \( H_\mathcal{T}^\ast(Y) \) under \( i_Y^\ast \) for any \( \mathfrak{B} \)-invariant subvariety \( Y \) of a \( \mathfrak{B} \)-regular variety \( X \).

The curve \( \mathcal{Z}_X \) admits a natural description. Consider the diagonal action of \( \mathfrak{B} \) on \( X \times \mathbb{P}^1 \), where \( \mathfrak{B} \) acts on \( \mathbb{P}^1 \) by the standard action of \( SL_2(\mathbb{C}) \) on \( \mathbb{C}^2 \). Then the irreducible components of \( \mathcal{Z}_X \) have the form \( \mathfrak{B} \cdot (\zeta, \infty) \setminus (\zeta, \infty) \), where \( \zeta \) ranges over all of \( X^{\mathcal{T}} \) and \( \infty \) denotes the point \([0, 1] \in \mathbb{P}^1 \). The complete description of the equivariant cohomology of a \( \mathfrak{B} \)-regular variety \( X \) and principal subvariety \( Y \) proved in \cite{4} is given by

\textbf{Theorem 1.} (cf. \cite{4}) If \( X \) is a \( \mathfrak{B} \)-regular variety, then there exists a graded \( \mathbb{C} \)-algebra isomorphism \( \rho_X : H_\mathcal{T}^\ast(X) \to \mathbb{C}[\mathcal{Z}_X] \). Furthermore, if \( Y \) is a principal subvariety and \( \mathcal{Z}_Y \) is the (reduced) affine curve \( \mathcal{Z}_X \cap (Y \times \mathbb{C}) \), then there is also a graded \( \mathbb{C} \)-algebra isomorphism \( \rho_Y : H_\mathcal{T}^\ast(Y) \to \mathbb{C}[\mathcal{Z}_Y] \) making the diagram

\[
\begin{array}{ccc}
H_\mathcal{T}^\ast(X) & \xrightarrow{\rho_X} & \mathbb{C}[\mathcal{Z}_X] \\
\downarrow i_Y^\ast & & \downarrow \overline{i_Y}^\ast \\
H_\mathcal{T}^\ast(Y) & \xrightarrow{\rho_Y} & \mathbb{C}[\mathcal{Z}_Y],
\end{array}
\]

commutative, where the vertical maps are natural restrictions. Moreover, the horizontal maps are \( \mathbb{C}[v] \)-module maps under the standard \( \mathbb{C}[v] \)-module structure on \( H_\mathcal{T}^\ast(X) \) and \( H_\mathcal{T}^\ast(Y) \) and the \( \mathbb{C}[v] \)-module structure on \( \mathbb{C}[\mathcal{Z}_X] \) and \( \mathbb{C}[\mathcal{Z}_Y] \) induced by the second projection.

One easily sees from the definitions that if \( v \) is an affine coordinate on \( \mathbb{C} = \mathbb{P}^1 \setminus [1, 0] \), then \( \mathbb{C}[\mathcal{Z}_X]/(v)\mathbb{C}[\mathcal{Z}_X] \cong A(Z) \). Hence, Theorem 1 allows one to see the isomorphism \( H^\ast(X) \cong A(Z) \) in a natural way from elementary properties of equivariant cohomology. Indeed, it turns out that \( \rho_X \) maps the augmentation ideal \((v)H_\mathcal{T}^\ast(X)\) in \( H_\mathcal{T}^\ast(X) \) to \((v)\mathbb{C}[\mathcal{Z}_X] \), so \( A(Z) \cong H^\ast(X) \).
since $H^*_T(X)/(v)H^*_T(X) \cong H^*(X)$. Similarly, $H^*(Y) \cong \mathbb{C}[Z_Y]/(v)\mathbb{C}[Z_Y]$ if $Y$ is principal. However, it is not in general true that $\mathbb{C}[Z_Y]/(v)\mathbb{C}[Z_Y]$ is isomorphic to $A(Y \cap Z)$.

Since surjectivity of $i_Y^* : H^*(X) \to H^*(Y)$ is equivalent to surjectivity of $i_Y^* : H^*_T(X) \to H^*_T(Y)$, \[1\] suggests an approach to the surjectivity question. Namely, since $i_Y^* : \mathbb{C}[Z_X] \to \mathbb{C}[Z_Y]$ is surjective for any $\mathfrak{B}$-invariant subvariety $Y$, the question of surjectivity of $i_Y^*$ boils down to determining if there exists an injective map $\rho_Y : H^*_T(Y) \to \mathbb{C}[Z_Y]$ such that \[1\] is commutative.

We will resolve this question by showing

**Theorem 2.** A $\mathfrak{B}$-invariant subvariety $Y$ of a $\mathfrak{B}$-regular variety $X$ is principal if and only if $H^*_T(Y)$ (equivalently, $H^*(Y)$) is generated by Chern classes of $\mathfrak{B}$-equivariant algebraic vector bundles on $Y$.

An example of a $\mathfrak{B}$-invariant subvariety which is not principal is easily obtained. In fact, let $X$ be $\mathfrak{B}$-regular, and let $Y$ denote the union of all the $\mathfrak{B}$-stable curves in $X$. Then $X^\mathfrak{T} = Y^\mathfrak{T}$, and it is not hard to see that $i_Y^*$ is not surjective if dim $X > 1$. However, $H^*_T(Y)$ is not generated by $\mathfrak{B}$-equivariant vector bundles. We will verify this claim for $X = \mathbb{P}^2$ in Section 5.

Schubert varieties in $G/B$ or a $G/P$ are well known to be principal. Indeed, the $B$-orbits are affine cells, and every Schubert variety is a union of $B$-orbits. Moreover, certain Springer fibres in $SL_n(\mathbb{C})/B$ are principal. To see this, first recall that a Springer fibre in $G/B$ is by definition the fixed point set of a unipotent element of $G$. Viewing $G/B$ as the variety of Borel subalgebras of $\mathfrak{g}$, this definition is equivalent defining a Springer fibre to be the set of all Borels in $\mathfrak{g}$ containing a fixed nilpotent in $\mathfrak{g}$. By classical result of Spaltenstein [11], the cohomology map $H^*(G/B) \to H^*(Y)$ is surjective for every Springer fibre $Y$ provided $G = SL_n(\mathbb{C})$. Let $e$ denote a regular nilpotent in $\text{Lie}(B)$, and let $\mathfrak{B} \circ G/B$ denote the regular action determined by $e$ as explained in the first paragraph. Any Springer fibre in $G/B$ corresponding to a nilpotent in the centralizer $\mathfrak{g}^e$ of $e$ which is also a $\mathfrak{T}$-weight vector is $\mathfrak{B}$-invariant. Thus such Springer fibres form a class of principal subvarieties of $SL_n(\mathbb{C})/B$. This example shows that principal subvarieties need not be irreducible.

2. Preliminaries

Let $\lambda : \mathbb{C}^* \to \mathfrak{T}$ and $\varphi : \mathbb{C} \to \mathfrak{U}$ denote the one parameter subgroups

$$
\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \varphi(v) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.
$$

Then $\lambda(t)\varphi(v)\lambda(t)^{-1} = \varphi(t^2v)$ for all $t \in \mathbb{C}^*$ and $v \in \mathbb{C}$. Suppose $X$ is a $\mathfrak{B}$-regular variety with $X^\mathfrak{U} = \{o\}$, and note that $o \in X^\mathfrak{T}$. Put

$$
X_o = \{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x = o\}.
$$

Recall from [2] (also see [3]) that $X_o$ is an open neighborhood of $o$ isomorphic with $T_o(X)$. Thus, if $a_1, \ldots, a_n$ ($n = \text{dim } X$) are the weights of $\lambda$ on
$T_0(X)$ repeated with multiplicities, then all $a_i < 0$ and there exist affine coordinates $u_1, \ldots, u_n$ on $X$, which are quasi-homogeneous of positive degrees $-a_1, \ldots, -a_n$ with respect to $\lambda$. That is,

$$\lambda(t) \cdot u_i = t^{-a_i} u_i,$$

for all $i$. The induced positive grading on $\mathbb{C}[X_0] = \mathbb{C}[u_1, \ldots, u_n]$ is frequently called the principal grading. Since the fixed point set $X^T$ of the torus $T$ is finite and contains $o$ (2), we will write $X^T = \{\zeta_1, \ldots, \zeta_r\}$, where $\zeta_1 = o$.

Note that the natural action of $SL_2(\mathbb{C})$ on $\mathbb{P}^1$ induces a regular action. If $\mathfrak{B}$ denotes the upper triangular matrices and $\mathfrak{T}$ the diagonals, then $\zeta_1 = [1,0], \zeta_2 = [0,1]$ and the big cell is $\{(1,v) \mid v \in \mathbb{C}\}$.

From now on, $\mathfrak{B}$ will denote the upper triangular Borel in $SL_2(\mathbb{C})$ and $\mathfrak{T}$ will be the diagonal torus; $X$ will always denote a $\mathfrak{B}$-regular variety. To simplify the notation, we will put $0 = [1,0]$, and let $X^T = \mathbb{P}^1 = \{0, \infty\}$.

Notice that the diagonal action $\mathfrak{B} \subset (X \times \mathbb{P}^1)$ is also regular with $\mathfrak{B}$-fixed point $(o,0)$. Define a projective curve $Z_X \subset X \times \mathbb{P}^1$ as follows: let $Z_i$ be Zariski closure of the orbit $\mathfrak{B}(\zeta_i, \infty)$, where $i \geq 1$, and let

$$Z_X = \bigcup_{1 \leq i \leq r} Z_i.$$

Thus $Z_X$ is a $\mathfrak{B}$-stable curve with $r = |X^T|$ irreducible components. Moreover, the second projection $p_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ induces an isomorphism on each component. Hence $Z_X$ is a bouquet of $r \mathbb{P}^1$s passing through $(o,0)$.

Now define an affine curve

$$Z_X = Z_X \cap (X_0 \times \mathbb{C}),$$

where the intersection is assumed to be in the sense of varieties, i.e. reduced. This curve is $\mathfrak{T}$-stable but not $\mathfrak{B}$-stable. In fact, $Z_X$ is obtained from $X$ by removing the point at infinity on each irreducible component. In particular, the coordinate ring $\mathbb{C}[Z_X]$ is a graded $\mathbb{C}$-algebra via the principal grading, and the projection $p_2$ induces a $\mathbb{C}[v]$-module structure on $\mathbb{C}[Z_X]$, where $v$ denotes the affine coordinate on $\mathbb{C}$. For later use, we observe (cf. [4, p.192])

$$\text{if } v \neq 0, \text{ then } (x,v) \in Z_X \text{ if and only if } \varphi(-v^{-1}) \cdot x \in X^T.$$

We now recall the basic isomorphism $\rho_X : H^*_T(X) \to \mathbb{C}[Z_X]$. Since the odd cohomology of a $\mathfrak{B}$-regular variety is trivial, the localization theorem in equivariant cohomology implies $i^* : H^*_T(X) \to H^*_T(X^T)$ is injective, where $i : X^T \to X$ is the inclusion map. Thus, to each $\alpha \in H^*_T(X)$, one may assign an $r$-tuple $(\alpha_1, \ldots, \alpha_r) \in \bigoplus_i \mathbb{C}[v]$. Now define a function $\rho_X(\alpha)$ on $Z_X$ by putting

$$\rho_X(\alpha)(x, v) = \alpha_i(v),$$

if $(x,v) \in Z_i$. A key fact is that $\rho_X(\alpha)$ is a regular function on $Z_X$.

3. Some applications of surjectivity

We will now use Theorem 1 to draw some conclusions about surjectivity. Suppose that $Y$ is another $\mathfrak{B}$-regular variety and $F : Y \to X$ a $\mathfrak{B}$-equivariant map. We claim $F$ induces a $\mathfrak{T}$-equivariant map $\overline{F} : Z_Y \to Z_X$ by putting
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$\overline{F}(y,v) = (F(y),v)$. For $F(Y) \subset X$, so if $(y,v) \in Z_Y$, say $(y,v) = (b \cdot y_i, b \cdot \infty)$ for some $y_i \in Y$ and $b \in \mathfrak{B}$, we see that

$$(F(y),v) = (F(b \cdot y_i), b \cdot \infty) = (b \cdot F(y_i), b \cdot \infty) = b \cdot (F(y_i), \infty) \in Z_X.$$ But $b \cdot (F(y_i), \infty) \neq (F(y_i), \infty)$ unless $b = 1$, hence the claim. Thus there is a commutative diagram

$$
\begin{array}{ccc}
H^*_T(X) & \overset{\rho_X}{\longrightarrow} & C[Z_X] \\
F^* \downarrow & & \downarrow F^*
\end{array}

(5)

H^*_T(Y) \overset{\rho_Y}{\longrightarrow} C[Z_Y].

We now derive some consequences of this.

**Theorem 3.** Assume $F : Y \to X$ is a $\mathfrak{B}$-equivariant map of $\mathfrak{B}$-regular varieties such that the differential $dF_o$ of $F$ at $o$ is injective. Suppose also that $F|Y$ is injective. Then the restriction map $F^* : H^*_T(X) \to H^*_T(Y)$ is surjective. Consequently, $F^* : H^*(X) \to H^*(Y)$ is also surjective.

**Proof.** First suppose $Y$ is a $\mathfrak{B}$-invariant subvariety of $X$ which is also $\mathfrak{B}$-regular and $F = i_Y$. By Theorem 1, the morphism $\rho_T$ in the diagram (5) is an isomorphism. Thus $i_Y^*$ is surjective. Now consider the general case. Since $dF_o$ is injective, it follows that $F(Y)$ is smooth at $o$. By the Borel Fixed Point Theorem applied to the singular locus of $F(Y)$, it follows that $F(Y)$ is smooth, hence $\mathfrak{B}$-regular. Hence $i^*_Y : H^*_T(X) \to H^*_T(F(Y))$ is surjective. As $F$ is injective on $Y$ and $\mathfrak{B}$-equivariant, it follows that $\overline{F} : Z_Y \to Z_{F(Y)}$ is also injective. Since $F$ has a local holomorphic inverse $G$ in a neighborhood of $o$, $G$ in fact induces (by equivariant extension) a holomorphic map $\overline{G} : Z_{F(Y)} \to Z_Y$ which is an inverse (in the analytic category) to $\overline{F}$. Note that $F(Y)^T = F(Y)$; that is, $F|Y$ is a bijection with $F(Y)^T$. For, if $w \in F(Y)^T$, then, by the Borel Fixed Point Theorem, the subvariety $F^{-1}(w)$ of $Y$ contains a $T$-fixed point due to the fact that it is $T$-invariant. Therefore $\overline{F} : C[Z_{F(Y)}] \to C[Z_Y]$ is an isomorphism, giving the result. □

Moreover, we also get

**Theorem 4.** Two principal subvarieties of a $\mathfrak{B}$-regular variety with the same fixed point set have isomorphic cohomology algebras (both equivariant and classical). In particular, two such subvarieties have the same dimension.

**Proof.** This is an immediate consequence of Theorem 1. □

Consequently, regular actions have a rather special property.

**Corollary 5.** If $X$ is $\mathfrak{B}$-regular and $Y$ is a $\mathfrak{B}$-invariant subvariety of $X$ such that $Y \cap X \cap X = X$, then either $Y = X$ or $Y$ is singular.

**Proof.** By the previous theorem, if $Y$ is smooth, hence $\mathfrak{B}$-regular, then $\dim Y = \dim X$. Since a $\mathfrak{B}$-regular variety is necessarily irreducible, $Y = X$. □
Of course, there are examples of torus actions on smooth projective varieties $Y \subseteq X$ for which $X^T = Y^T$ and the conclusion of Corollary 5 fails. For example, if $X$ is the wonderful compactification of a semi-simple algebraic group $G$ (over $\mathbb{C}$) with maximal torus $T$, then all the fixed points of the (torus) action of $T \times T$ on $X$ lie on the unique closed $G \times G$-orbit $Y$ (cf. [5]). This gives a proper smooth $T \times T$-stable subvariety $Y$ of $X$ for which $X^T \times T = Y^T \times T$. It also shows that a wonderful compactification doesn’t admit a regular action.

Finally, we have

**Theorem 6.** Let $X$ and $Y$ be $\mathcal{B}$-regular varieties or principal subvarieties, and let $F : Y \to X$ be a $\mathcal{B}$-equivariant map such that $F(Y^\mathcal{B}) = X^\mathcal{B}$. Then $F^* : H^*_\mathcal{B}(X) \to H^*_\mathcal{B}(Y)$ is injective. In particular, if $F$ is surjective, then $F(Y^\mathcal{B}) = X^\mathcal{B}$, so $F^*$ is injective on equivariant cohomology.

**Proof.** Since $F(Y^\mathcal{B}) = X^\mathcal{B}$, $F : Z_Y \to Z_X$ is also surjective. Since $Z_X$ and $Z_Y$ are affine, it follows that the comorphism $F^* : \mathbb{C}[Z_X] \to \mathbb{C}[Z_Y]$ is injective. Therefore, $F^* : H^*_\mathcal{B}(X) \to H^*_\mathcal{B}(Y)$ is injective. If $F$ is surjective, then the Borel Fixed Point Theorem implies $F(Y^\mathcal{B}) = X^\mathcal{B}$. $\square$

**Remark.** If $X$ and $Y$ are smooth projective varieties and $F : Y \to X$ is surjective, it is well known that $F^*$ is always injective on ordinary cohomology.

4. A REMARK ON A FORMULA OF KOSTANT AND MACDONALD

Let $G$ be a complex semi-simple algebraic group, $B$ a Borel subgroup of $G$ and put $\mathfrak{b} = \text{Lie}(B)$. We will now describe an interesting class of $\mathcal{B}$-regular subvarieties of $G/B$ which includes all smooth Schubert varieties. Let $e \in \mathfrak{b}$ be a regular nilpotent in $\mathfrak{g}$, and recall from Section 1 that $e$ determines a regular action $\mathcal{B} \circ G/B$ on the flag variety of $G$ by left translation such that the identity coset $B \in G/B$ is the unique $\mathfrak{b}$-fixed point. To simplify notation, let us identify $\mathfrak{g}$ with its homomorphic image in $G$ such that $e \in \text{Lie}(\mathcal{B})$ (see the comment in the first paragraph of the Introduction). Let $\mathfrak{h}$ denote a Lie(\mathcal{B})-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$, and put $Y_\mathfrak{h} = \exp(\mathfrak{h})B$.

**Lemma 7.** $Y_\mathfrak{h}$ is a $\mathcal{B}$-invariant subvariety of $G/B$.

**Proof.** Since the exponential map $\exp : \mathfrak{g} \to G$ is $G$-equivariant for the adjoint action of $G$ on $\mathfrak{g}$, for any $x \in \text{Lie}(\mathcal{B})$, we have

$$\exp \left( \text{Ad} \left( \exp(tx) \right) y \right) B = \exp(tx) \exp(y) \exp(-tx) B = \exp(tx) \exp(y) B.$$ 

But for any $x, y \in \mathfrak{g}$, $\text{Ad} \left( \exp(tx) \right) (y) = e^{ad(tx)}(y)$, so the term on the left side of the above identity is in $\exp(\mathfrak{h})B$ if $y \in \mathfrak{h}$. Thus $Y_\mathfrak{h}$ is stable under $\mathcal{B}$. $\square$

**Remark:** If $\mathfrak{h}$ is $B$-stable, then $Y_\mathfrak{h}$ is also $B$-stable and hence is a Schubert variety.

When the variety $Y_\mathfrak{h}$ is smooth, it is a $\mathcal{B}$-regular variety and one can now apply the formula in [2] for the Poincaré polynomial of a $\mathcal{B}$-regular variety $X$ to find an interesting class of polynomials associated to any complex
semisimple Lie algebra \( g \). Let us first recall the formula. Let \( a_1, \ldots, a_n \) 
\((n = \dim X)\) denote the weights of \( \lambda \) on \( T_\alpha(X) \) introduced in Section 2, and 
recall the \( a_i \) are negative integers. Then 

\[
P(X, t^{1/2}) = \prod_{1 \leq i \leq n} \frac{(1 - t^{-a_i+1})}{(1 - t^{-a_i})}.
\]

In the case \( X = Y_\alpha = G/B \), the weights \( a_i \) are the negatives of the heights of 
the positive roots with respect to any maximal torus \( T \) of \( G \) contained in 
\( B \), so we recover a well known formula 

\[
P(G/B, t^{1/2}) = \prod_{\alpha > 0} \frac{(1 - t^{ht(\alpha)+1})}{(1 - t^{ht(\alpha)})}
\]

of Kostant and Macdonald.

5. A Lemma and an Example

We will now prove some facts about \( \mathfrak{g} \)-invariant subvarieties of a \( \mathfrak{g} \)-
regular variety \( X \). First note

\textbf{Lemma 8.} Let \( Y \) be a \( \mathfrak{g} \)-invariant subvariety of \( X \) with vanishing odd 
cohomology, and let \( I(Z_Y) \subset \mathbb{C}[Z_X] \) denote the ideal of \( Z_Y \). Then:

(i) \( \rho_X(H^*_X(X,Y)) = I(Z_Y) \); and 

(ii) \( \) there exists a \( \mathbb{C}[v] \)-algebra isomorphism 

\[
\psi_Y : \mathbb{C}[Z_Y] \rightarrow i_Y^*(H^*_X(X)) \subset H^*_X(Y).
\]

In fact, \( \psi_Y = i_Y^* \rho_X^{-1}(i_Y)^{-1} \).

\textbf{Proof.} We first show that putting \( \psi_Y = i_Y^* \rho_X^{-1}(i_Y)^{-1} \) gives a well defined 
map. Since \( Z_Y \) is the union of the irreducible components of \( Z_X \) which 
meet \( Y \times \mathbb{C} \), the indeterminacy introduced by \((i_Y)^{-1}\) is supported on the 
complement of \( Z_Y \). Thus \( \psi_Y \) is indeed well defined since \( i_Y \) is surjective, 
and two classes in \( H^*_X(X) \) which have the same image under \( i_Y^* \rho_X \) have the 
same image under \( i_Y^* \), by the localization theorem and the definition of \( \rho_X \).

We now show \( \rho_X(H^*_X(X,Y)) = I(Z_Y) \). Since \( Z_Y \) is an affine curve with 
\( |Y^\mathbb{Z}| \) components, and \( p_2 : Z_Y \rightarrow \mathbb{C} \) is a flat map whose restriction to each 
component is an isomorphism (by \textbf{[4] Prop. 2}), the rank of \( \mathbb{C}[Z_Y] \) over \( \mathbb{C}[v] \) 
is \( |Y^\mathbb{Z}| \). Furthermore, by the long exact sequence of cohomology and the 
localization theorem, the rank of \( H^*_X(X,Y) \) is \( |X^\mathbb{Z}| - |Y^\mathbb{Z}| \). But the rank 
of \( \ker \overline{i_Y^*} \) is also \( |X^\mathbb{Z}| - |Y^\mathbb{Z}| \). As shown above, \( \ker \overline{i_Y^*} \subset \rho_X(H^*_X(X,Y)) \), so 
it follows that \( \overline{i_Y^*}(\rho_X(H^*_X(X,Y))) \) has rank zero in the free module \( \mathbb{C}[Z_Y] \), 

hence is trivial. Therefore, \( I(Z_Y) = \rho_X(H^*_X(X,Y)) \). To finish, we only need 
to show \( \psi_Y \) is injective. But this follows immediately from part (i). \( \square \)

The following example shows that non-principal subvarieties exist. Let 

\[
\lambda(t) = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}
\]
Then $\lambda(t)\varphi(v)(\lambda(t))^{-1} = \varphi(t^2v)$, so $\lambda$ and $\varphi$ determine a two-dimensional solvable subgroup of $SL_3(\mathbb{C})$, and hence a regular action $\mathcal{B} \cap \mathbb{P}^2$. The fixed points are $o = [1,0,0]$, $\zeta_2 = [0,1,0]$, and $\zeta_3 = [0,0,1]$. Let $w_1$ and $w_2$ be the usual affine coordinates around $o$. Then the closures of $Y_1 = \{w_2 = 0\}$ and $Y_2 = \{2w_2 = w_1^2\}$ are the two $\mathcal{B}$-curves in $\mathbb{P}^2$. Let $Y = \overline{Y_1} \cup \overline{Y_2}$. Then $Y$ is a $\mathcal{B}$-invariant subvariety such that $H^*(\mathbb{P}^2) \to H^*(Y)$ is not surjective. Indeed, $\dim H^2(\mathbb{P}^2) < \dim H^2(Y)$. On the other hand, $H^2_T(Y)$ is not generated by Chern classes of equivariant vector bundles. To see this, note that $Y$ has vanishing odd cohomology, so one can use the localization theorem to compute its equivariant cohomology. In fact, by a well-known result of Goresky, Kottwitz and MacPherson [9] (also see [3]), the image of $H^*_T(Y)$ in $H^*_T(Y^\mathcal{B})$ consists of triples $(f_1(t), f_2(t), f_3(t))$ with all $f_i \in \mathbb{C}[t]$ such that $f_1(0) = f_2(0) = f_3(0)$. But not all classes of this form arise as polynomials in Chern classes of equivariant line bundles on $Y$, due to the fact that equivariance forces the further condition $f_2 = f_3 = -f_1$.

**Remark.** We do not know of an example of an irreducible $\mathcal{B}$-invariant subvariety of a $\mathcal{B}$-regular variety which is not principal.

## 6. Equivariant Chern classes

Let $Y$ be a $\mathcal{B}$-invariant subvariety of a $\mathcal{B}$-regular variety $X$. The purpose of this section is to consider when the fundamental isomorphism $\rho_X : H^*_T(X) \to \mathbb{C}[Z_X]$ defined in [14] can be defined for $Y$. For this, we need to consider $\mathcal{B}$-equivariant vector bundles on $Y$.

Recall that if $V$ is an algebraic variety with an action of an algebraic group $G$, then a $G$-linearization of an algebraic vector bundle $E$ on $V$ is an action $G \times E \to E ((g, h) \to g \cdot h)$ such that for all $y \in V$, the restriction of $g \in G$ is a $\mathbb{C}$-linear map $E_y \to E_{g \cdot y}$. In particular if $y \in V^G$, one obtains a representation of $G$ on $E_y$ and hence a representation of $\text{Lie}(G)$ on $E_y$. For $\xi \in \text{Lie}(G)$, we let $\xi_y$ denote the corresponding endomorphism of $E_y$.

Suppose the vector bundle $E$ admits a $G$-linearization. Recall that the $k$-th equivariant Chern class $c^G_k(E) \in H^G_{2k}(V)$ of $E$ is the $k$-th Chern class of the vector bundle $E_G = (E \times \mathcal{E})/G$, where $\mathcal{E}$ is a contractible free $G$-space. The restriction of $c^G_k(E)$ to each $y \in V^G$ is the polynomial on $\text{Lie}(G)$ defined as

$$c^G_k(E)_y(\xi) = \text{Tr}_{\mathcal{E}/y}^E(\xi_y).$$

We now turn to the case $G = \mathcal{B}$, $X$ $\mathcal{B}$-regular and $Y$ a $\mathcal{B}$-invariant subvariety. Put

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
By [4, Lemma 1], if $E$ is $\mathfrak{B}$-equivariant and $(y,v) \in \mathcal{Z}_X$, then

$$\rho_X(c^\mathfrak{B}_k(E))(y,v) = \text{Tr}_{\mathcal{H}^k E_y}(v\mathcal{W} - 2\mathcal{V})_y,$$

where $(v\mathcal{W} - 2\mathcal{V})_y$ is the endomorphism determined by $v\mathcal{W} - 2\mathcal{V} \in \text{Lie}(\mathfrak{B})$. This is a key fact since the smoothness of $X$ implies $H^*_X(X)$ is generated by Chern classes of $\mathfrak{B}$-equivariant vector bundles on $X$ [5, Prop. 3].

Let $H^*_\mathfrak{B}(Y)$ denote the subalgebra of $H^*_Y(Y)$ generated by equivariant Chern classes of $\mathfrak{B}$-equivariant vector bundles on $Y$. As just noted, $H^*_\mathfrak{B}(X) = H^*_X(X)$. The goal of the remainder of this section is to prove that if $E$ is a $\mathfrak{B}$-linearized vector bundle on $Y$ then $\rho_Y(c^\mathfrak{B}_k(E))$ is a regular function on the affine curve $\mathcal{Z}_Y$. Thus, there exists a well defined map

$$\rho_Y : H^*_\mathfrak{B}(Y) \to \mathbb{C}[\mathcal{Z}_Y]$$

such that the diagram

$$(8) \quad \begin{array}{ccc}
H^*_X(X) & \xrightarrow{\rho_X} & \mathbb{C}[\mathcal{Z}_X] \\
\downarrow \iota^*_Y & & \downarrow \iota^*_Y \\
H^*_\mathfrak{B}(Y) & \xrightarrow{\rho_Y} & \mathbb{C}[\mathcal{Z}_Y].
\end{array}$$

is commutative.

Before beginning the proof, assume $v \neq 0$ and define

$$\mathfrak{H}_v = \varphi(1/v)\overline{\varphi}(-1/v).$$

By [4], $(y,v) \in \mathcal{Z}_Y$ if and only if $y \in Y^{\mathfrak{H}_v}$. Also, put

$$\mathcal{H} = \bigcup_{v \neq 0} \mathfrak{H}_v \times \{v\} \subset \mathfrak{B} \times \mathbb{C}.$$

**Lemma 9.** Let $\pi : \mathcal{H} \to \mathbb{C}$ be projection on the second factor. Then:

$$\pi^{-1}(0) = \mathfrak{U} \cup -(\mathfrak{U}).$$

**Proof.** First note the identity

$$(9) \quad \begin{pmatrix}
1 & v^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & 0 \\
0 & a^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & -v^{-1} \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
a & (1 - a^2)/(av)^{-1} \\
0 & a^{-1}
\end{pmatrix}.$$

One must find all the possible finite limits of the right-hand side as $v \to 0$. Take a sequence $(a_i, v_i) \neq (0,0)$ such that $v_i \to 0$, and the right-hand side of (9) has a finite limit. In order for this to happen, $(1 - a_i^2)/a_i \to 0$, so $a_i^2 \to 1$. Hence, if a limit exists, it lies in $\mathfrak{U} \cup -(\mathfrak{U})$. The reverse inclusion is similar.

Consequently, if we put $\mathfrak{h}_v = \text{Lie}(\mathfrak{H}_v)$ for $v \neq 0$ and set $\mathfrak{h}_0 = \text{Lie}(\mathfrak{U})$, then the family of Lie algebras of the fibres of $\mathcal{H}$ is

$$\mathfrak{h} = \bigcup_{v \in \mathbb{C}} \mathfrak{h}_v \times \{v\} \subset \text{Lie}(\mathfrak{B}) \times \mathbb{C},$$

and, moreover, $v\mathcal{W} - 2\mathcal{V}$ is a non-vanishing regular section of $\pi$. For convenience, let $s(v) = v\mathcal{W} - 2\mathcal{V}$.
Now let $E$ be a $\mathcal{B}$-linearized vector bundle on $Y$, and let $\tilde{E}$ be the pullback of $E$ to $Y \times \mathbb{C}$. Assume $v \in \mathbb{C}$ and $y \in Y^{\mathcal{B}e}$. That is, $(y, v) \in Z_Y$. Then $E_y$ admits an $\mathfrak{h}_v$-representation and hence an $\mathfrak{h}_e$-representation. For $\xi \in \mathfrak{h}_v$ let $\xi_y$ denote the corresponding endomorphism on $E_y$. Then the upshot of our discussion is the following:

**Lemma 10.** The map from $Z_Y$ to $\text{End}(\tilde{E}|_{Z_Y})$ given by

$$(y, v) \mapsto s(v)_y,$$

is a regular section of the bundle $\text{End}(\tilde{E}|_{Z_Y})$. Consequently the map

$$(y, v) \mapsto \text{Tr}_{\wedge^k E_y}(s(v)_y)$$

is a regular function on $Z_Y$.

**Lemma 11.** Let $E$ be a $\mathcal{B}$-linearized vector bundle on $Y$. Then $\rho_Y(c^\mathcal{B}_k(E))$ is a regular function on $\mathbb{C}[Z_Y]$.

**Proof.** View $v$ as an element of $\text{Lie} (\mathfrak{T})$. As above let $v_{\xi_j}$ denote the corresponding endomorphism on $E_{\xi_j}$. By Equation (7) we have

$$c^\mathcal{B}_k(E)_{\xi_j} = \text{Tr}_{\wedge^k E_y}(s(v)_{\xi_j}).$$

Put $\zeta_j = \varphi(v^{-1})\zeta_j$. Then the $\zeta_j(v)$ are the fixed points of $\mathfrak{h}_v$. Recall that $s(v) \in \mathfrak{h}_v$, hence it gives an endomorphism $s(v)_{\zeta_j(v)}$ of the fibre $E_{\zeta_j(v)}$. Since $E$ is $\mathcal{B}$-linearized, the element $\varphi(v^{-1})$ gives an isomorphism $E_{\zeta_j} \to E_{\zeta_j(v)}$; moreover the endomorphism $v_{\zeta_j(v)}$ on $E_{\zeta_j(v)}$ is conjugate to the endomorphism $v_{\zeta_j}$ on $E_{\zeta_j}$. Put $y = \zeta_j(v)$. Then $(y, v) \in Z_Y$ and

$$\text{Tr}_{\wedge^k E_{\zeta_j}}(v_{\zeta_j}) = \text{Tr}_{\wedge^k E_y}(s(v)_y).$$

But by Lemma 11, the function $(y, v) \mapsto \text{Tr}_{\wedge^k E_y}(s(v)_y)$ is a regular function on $Z_Y$.

To summarize the discussion of this section, we state

**Theorem 12.** If $Y$ is a $\mathcal{B}$-invariant subvariety of a $\mathcal{B}$-regular variety $X$, then we obtain a $\mathbb{C}[v]$-algebra homomorphism $\rho_Y : \mathcal{H}^*_\mathcal{B}(Y) \to \mathbb{C}[Z_Y]$ such that the diagram (8) is commutative.

**7. Classification of Principal Subvarieties**

We now classify principal subvarieties. First, we describe the image of $H^*_\mathcal{B}(X)$ in $H^*_\mathcal{T}(Y)$ for a $\mathcal{B}$-invariant subvariety $Y$ with vanishing odd cohomology.

**Theorem 13.** Suppose $Y$ is $\mathcal{B}$-invariant subvariety of a $\mathcal{B}$-regular variety $X$ with vanishing odd cohomology. Then

(i) $\rho_Y : \mathcal{H}^*_\mathcal{B}(Y) \to \mathbb{C}[Z_Y]$ is an isomorphism, and

(ii) $i^*_Y(H^*_\mathcal{T}(X)) = \mathcal{H}^*_\mathcal{B}(Y)$. 


Consequently, $\mathcal{H}_B^*(Y)$ is exactly the subalgebra generated by Chern classes of $B$-equivariant vector bundles on $Y$ which are pull backs of $B$-equivariant vector bundles on $X$.

**Proof.** It is clear from the commutativity of (8) that $\rho_Y$ is surjective. The definition of $\rho_Y$ and localization (applied to $Y$) implies it is injective. This proves (i). We now show (ii). By Lemma 8 we have an isomorphism $\psi_Y : \mathbb{C}[Z_Y] \to i_Y^*(H^*_T(X))$. We claim that $\rho_Y \psi_Y = 1$. In fact, by (8) and the definition of $\psi_Y$,

$$\rho_Y \psi_Y = \rho_Y i_Y^* \rho_X^{-1} i_Y^{-1} = i_Y^* \rho_X \rho_X^{-1} i_Y^{-1},$$

which is clearly the identity. Thus

$$\rho_Y (i_Y^* (H^*_T(X)) = \mathbb{C}[Z_Y],$$

which certainly implies (ii). $\square$

**Corollary 14.** A $B$-invariant subvariety $Y$ of a $B$-regular variety $X$ is principal if and only if $\mathcal{H}_B^*(Y) = H^*_T(Y)$.

**Proof.** The necessity is clear. But if $\mathcal{H}_B^*(Y) = H^*_T(Y)$, then $Y$ has vanishing odd cohomology, so the result follows from Theorem 13 (b). $\square$

Theorem 2 follows immediately from this corollary. We conclude with an application.

**Corollary 15.** Suppose $Y$ is a normal $B$-invariant subvariety of a $B$-regular variety $X$ such that $H^*(Y)$ is generated by Chern classes of line bundles. Then $Y$ is principal.

**Proof.** Let $L$ be an algebraic line bundle on $Y$. By [10], some power $L^m$ is $B$-equivariant. Hence the first Chern classes of $B$-equivariant lines bundles on $Y$ generate $H^*(Y)$. This is equivalent to saying that $\mathcal{H}_B^*(Y) = H^*_T(Y)$. $\square$

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