Signature Transition and Compactification

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Abstract

It is shown that a change in the signature of the space-time metric together with compactification of internal dimensions could occur in a six-dimensional cosmological model. We also show that this is due to interaction with Maxwell fields having support in the internal part of the space-time.

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It is generally supposed in quantum cosmology, either in no-boundary proposal or tunneling model that, the universe has experienced a transition in the signature of its metric from Euclidean to Lorentzian. It is argued that this change of signature prevents the appearance of singularities as a result of fluctuations in the space-time topology.

There have been recent attempts to study this effect in the framework of classical general relativity [1]. There are also studies of the effect in the framework of higher

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dimensional models \cite{4}. In particular in \cite{3} it was suggested that signature transition could serve as a mechanism for compactification of the internal dimensions. This idea was pursued further in \cite{4} in which a change in the signature of the space-time metric induces compactification of the internal dimension in a cosmological model with a negative cosmological constant and no matter fields. In both of these works, it was assumed that the internal space has a compact topology (for example $S^1$ in \cite{4}) ab initio, but its size becomes unobservabley small as a result of the dynamics being induced by signature change. Now it is also interesting to ask whether signature transition could have any role in a true dynamical compactification \cite{5}, that is in a scheme in which one starts with an internal space which may or may not be compact and compactifies it through some dynamics.

In this paper we use the results of \cite{6} and present a model in which both a change in the signature of the four dimensional part of space-time metric and compactification of the internal dimensions occur as a result of interaction of gravity with an electromagnetic field living in the internal space. We start with a noncompact internal manifold. A change in the signature of the metric of the internal manifold occurs and this induces a signature transition in the four dimensional metric via interaction with the Maxwell fields leading to the compactification of the internal dimensions. It should be added that the field equations alone cannot induce compactification since they constrain the manifold only locally. It is then obvious that some extra assumptions are implied here \cite{7}.

Here we consider a model in which Maxwell fields interact with gravity. The action is

$$S = -\frac{1}{16\pi G} \int R\sqrt{-g}d^6x - \frac{1}{4} \int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}d^6x,$$  \hspace{1cm} (1)

leading to the field equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi GT^{\mu\nu},$$ \hspace{1cm} (2)
\[
\frac{1}{\sqrt{-g}} \partial \mu (\sqrt{-g} F^{\mu \nu}) = 0,
\]

where \( T^{\mu \nu} \) are the components of the energy momentum tensor associated with the Maxwell fields

\[
T^{\mu \nu} = F^\mu_{\rho} F^{\nu \rho} - \frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} g^{\mu \nu}.
\]

We are interested in solutions which are of the form of the product of a 2-dimensional (semi-)Riemannian manifold \( M_2 \) and a 4-dimensional one, \( M_4 \), with the following metric

\[
ds^2 = ds^2_2 + ds^2_4,
\]

where choosing the chart \((x^1, ..., x^6)\) we may write \( ds^2_2 \) and \( ds^4_2 \) as

\[
ds^2_2 = g_{mn} dx^m dx^n,
\]

\[
ds^2_4 = g_{pq} dx^p dx^q,
\]

with \( m, n = 5, 6 \) and \( p, q = 1, 2, 3, 4 \). In the above equations \( g_{mn} \) are functions of \( x^5, x^6 \) while \( g_{pq} \) are functions of \( x^1, x^2, x^3, x^4 \) with \((x^1, ..., x^4)\) and \((x^5, x^6)\) being charts on \( M_4 \) and \( M_2 \) respectively.

Maxwell equations ((3) together with \( \partial [\lambda F_{\mu \nu}] = 0 \) ) admit the following solution

\[
F^{\mu \nu} = \frac{f \epsilon^{\mu \nu}}{\sqrt{|g_2|}},
\]

in which \( \epsilon^{\mu \nu} \) is the same as \( \epsilon^{mn} \) whenever \( \mu \) and \( \nu \) take the the values \( m, n \) and vanishes whenever they take the values \( p, q \). Here \( \epsilon^{mn} \) is the alternating tensor with \( \epsilon^{56} = 1 \), \( g_2 \) is the determinant of \( g_{mn} \) and \( f \) is a constant. The above solution represents electromagnetic fields living only in the internal space. For this \( F^{\mu \nu} \) we have

\[
T^{mn} = F^m_{\rho} F^{n \rho} - \frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} g^{mn},
\]

\[
T^{pq} = -\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} g^{pq},
\]
where \( \alpha, \beta = 1, \ldots, 6 \). Thus the Einstein’s field equations (2) become

\[
R^{mn} - \frac{1}{2} g^{mn} R = -8\pi G(F^m_{\rho} F^n_{\rho} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{mn}), \tag{11}
\]

\[
R^{pq} - \frac{1}{2} g^{pq} R = -8\pi G(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{pq}). \tag{12}
\]

The r.h.s of equation (12) and the second term in the r.h.s of (11) are obviously proportional to some overall cosmological terms. The first term in the r.h.s of (11) which may be written as \( F_{\sigma\rho} F^{\sigma\rho} g^{mm} \) will also be proportional to \( g^{mn} \) after taking (8) into account, and hence is a cosmological term. Thus the Maxwell fields induce cosmological constants with different values on \( M_2 \) and \( M_4 \).

Now inserting (8) into (11) and (12) and noting that \( R_4 = g_{pq} R^{pq} \) and \( R_2 = g_{mn} R^{mn} \) are the scalar curvatures of \( M_4 \) and \( M_2 \) respectively, so that \( R = R_2 + R_4 \), we obtain the following relations

\[
R_4 = \lambda \text{sign}(g_2), \tag{13}
\]

\[
R_2 = -\frac{3}{2} \lambda \text{sign}(g_2), \tag{14}
\]

in which \( \lambda = 8\pi G f^2 \) and \( \text{sign}(g_2) = \frac{|g_2|}{g_2} \). The signs of \( R_4 \) and \( g_2 \) are the same while the sign of \( R_2 \) is the opposite to that of \( g_2 \). Therefore when the time dimension is in \( R_4 \), that is \( g_2 > 0 \), the internal space \( M_2 \) is compact and vice versa. This is because a negative (in our notation) scalar curvature implies that the manifold is compact which is itself a consequence of certain theorems in global differential geometry [8]. Therefore equations (13) and (14) imply spontaneous compactification of the internal dimensions.

Now consider the following as the metric of the space-time

\[
g = -tdt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right) + \frac{1}{b^2(\tau)}(\tau d\tau^2 + dx^2), \tag{15}
\]

This metric represents an internal space with \( \tau, x \) as coordinates and \( b^{-1}(\tau) \) as the scale factor and a FRW type space-time as \( M_4 \). The signature of \( ds_4^2 \) is determined by the sign
of \( t \). In other words \( M_4 \) has a Euclidean metric whenever \( t \) is negative and a Lorentzian metric when \( t \) is positive. So the signature of the metric on \( M_4 \) changes whenever \( t \) changes sign. Similarly the metric on \( M_2 \) has a Euclidean signature for positive \( \tau \) and a Lorentzian one for negative \( \tau \) and the signature of the metric on \( M_2 \) changes whenever \( \tau \) changes sign. We have required \( g \) to have only one time-dimension, that is, we only consider regions where \( t < 0, \tau < 0 \) and \( t > 0, \tau > 0 \).

Inserting (8) and (15) into Einstein equations (11) and (12) we obtain the following equations

\[
-\frac{3}{t}(\frac{\dot{a}^2 + kt}{a^2}) - \frac{1}{\tau}(b\ddot{b} - \dot{b}^2 - \frac{b\dot{b}}{2\tau}) = \frac{1}{2}\lambda \text{sign}(g_2),
\]

\[
\frac{1}{t}(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kt}{2at}) + \frac{1}{\tau}(b\ddot{b} - \dot{b}^2 - \frac{b\dot{b}}{2\tau}) = -\frac{1}{2}\lambda \text{sign}(g_2),
\]

\[
\frac{3}{t}(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kt}{a^2} - \frac{\dot{a}}{2at}) = \frac{1}{2}\lambda \text{sign}(g_2),
\]

where a dot denotes differentiation with respect to the argument.

Let us restrict ourselves to the case of \( k = +1 \). Equation (18) contains \( a(t) \) only and hence can be solved easily. A solution is as follows; for \( g_2 > 0 \), (corresponding to \( t > 0 \))

\[
a(t) = \frac{1}{H} \cosh\left(\frac{2}{3}Ht^{3/2}\right),
\]

with \( 12H^2 = \lambda \) and therefore (16) and (17) consistently yield

\[
\frac{1}{\tau}(b\ddot{b} - \dot{b}^2 - \frac{b\dot{b}}{2\tau}) = -\frac{3}{4}\lambda.
\]

A solution to (20) is

\[
b(\tau) = \sqrt{3\lambda} \frac{2h}{\sqrt{3h}} \sin\left(\frac{2}{3}h\tau^{3/2}\right),
\]

with \( h \) being a constant. Thus for positive \( t \) (which corresponds to positive \( \tau \)) we have a compact (one point compactified) internal space decreasing in size with time \( \tau \), up to \( b^{-1}(\tau) = \left(\frac{3\pi}{4h}\right)^{2/3} = \frac{2h}{\sqrt{3\lambda}} \) while the four dimensional space has the de Sitter geometry. If
we do not deploy the one point compactification scheme, we will have a bounded non-
compact internal manifold (a two-sphere with its north pole removed). Of course models
based on bounded non-compact internal manifolds have also been proposed, see e.g., [9].
For \( g_2 < 0 \), (corresponding to \( t < 0 \))
\[
a(t) = \frac{1}{H} \cos \left( \frac{2}{3} H (-t)^{3/2} \right).
\tag{22}
\]
This solution together with (19) correspond to a scale factor which is continuous across
the signature change hypersurface \( t = 0 \). Now from (16) and (17) we obtain
\[
\frac{1}{\tau} (\dot{b}^2 - \ddot{b}^2 - \frac{\dot{b}}{2\tau}) = \frac{3}{4} \lambda,
\tag{23}
\]
and a solution to this equation is
\[
b(\tau) = \sqrt{3\lambda} \frac{2}{2l} \sinh \left( \frac{2}{3} l (-\tau)^{3/2} \right),
\tag{24}
\]
in which \( l \) is a constant. Thus in this region \( M_4 \) has an oscillatory scale factor while \( M_2 \)
is noncompact and its size grows with its time \( \tau \). The (Euclidean time) \( t \)-interval between
the big bang and the time at which signature transition occurred is given by the the first
zero of \( a(t) \) in (22), that is \( t = \frac{(3\pi}{17})^{2/3} \).

We can say in conclusion that there are solutions, which accommodate both com-
packification of internal dimensions and a signature transition in the metric of the four
dimensional part of the space-time due to the interaction of electromagnetic fields with
gravity. In contrast to previous studies in which a signature transition is caused by conver-
sion of a space-like dimension to a time-like one , in our model the transition occurs only
by shifting the time-like dimension. Also in our model there is no dynamical matter field
in the 4-dimensional sector. It should be interesting to study this model in the framework
of quantum cosmology.
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