Anharmonicity can enhance the performance of quantum refrigerators

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We explore a thermodynamical effect of anharmonicity present in quantum mechanical oscillators. We show in the context of an exactly solvable model that quartic perturbations to the quantum harmonic oscillator potential lead to the enhancement of performance of quantum refrigerators. A similar nonlinearity driven enhancement of performance is also observed for an analogous spin-qubit model. Our results are illustrated for both the Otto and Stirling quantum refrigeration cycles. Finally, we investigate the energy cost for creating anharmonicity. The robustness of improvement of the coefficient of performance versus the energy cost can be demonstrated for the experimentally realizable Otto refrigerator.

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Introduction:- Microscopic configuration with restricted degrees of freedom enables a system to exhibit quantum mechanical supremacy [1] beyond the limits set by classical thermodynamics [2, 3]. Quantum behaviour of working media, such as quantum harmonic oscillators [4, 5], two-level [6, 7] and multi-level spin systems [8, 9] is inherently connected to the figures of merit belonging to various thermodynamic cycles. Quantum thermodynamics has attracted an upsurge of interest in recent years revealing certain novel features and generalizations over its classical counterpart [10–12], such as various forms of the second law of thermodynamics [2, 13, 14], and linkages with resource theories of quantum coherence [15, 16].

Quantum heat engines or refrigerators are appropriate test grounds for quantum thermodynamics, having potential applications in diverse areas such as nanotechnology [17, 18] and information processing [19, 20]. Single-mode bosonic (or spin-½) systems are widely used as working substances for quantum heat engines [4–6, 11, 12, 21]. Single mode harmonic oscillators have experimental realizations in trapped ions [22] and optomechanical systems [23]. However, implementation of the ideal harmonic oscillator is quite difficult in practice. On the other hand, since no realistic oscillator is perfectly harmonic, a small quartic perturbation term can be introduced in the potential [24, 25], leading to exactly solvable energy eigenvalues. Such anharmonic oscillators are realizable through experiments [26, 27].

Non-linear perturbations in the arena of quantum optical set-ups [28, 29] have been studied to investigate various kinds of non-classical effects [30, 31]. Interesting proposals for generating and stabilizing quantum entanglement aided with non-linearity have been formulated [32, 33]. In the present work we are motivated to investigate the impact of non-linearity on thermodynamic processes. To this end here we specifically consider the quantum Otto [11] and Stirling refrigerators [34]. The Otto and Stirling engines are prototypical thermodynamic cycles extensively studied in the literature [35] with recent progress in experimental implementation at the quantum level [36, 37].

Our approach here is to employ first the exactly solvable and experimentally implementable anharmonic oscillator with quartic correction to the potential [24, 25]. We study the Otto and Stirling refrigeration cycles in the above framework. We find that the co-efficient of performance of the two refrigerators are enhanced through increased non-linearity in the form of larger strength of anharmonicity. We further construct a spin analogue of the anharmonic oscillator as a separate working medium, and show that this unexpected feature of improved performance of refrigeration persists even here for both the Otto and Stirling cycles. The improved co-efficient of performance achieved through a higher magnitude of anharmonicity obviously comes at the cost of the energy supplied, as we next show by evaluating the quantitative change in the average energy fluctuation. However, the generic enhancement of performance for the Otto refrigerator grows surprisingly with increasing energy, thus exhibiting the robustness of anharmonicity as a resource vis-a-vis the energy cost.

Anharmonic Oscillator(AO):- The Hamiltonian for AO with quartic perturbation term upto first-order of λ can be written as [24, 25],

\[ H^{an} = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \lambda x^4 \] (1)

where, \( x, p, \omega \) are position, momentum and frequency of the oscillator. By setting mass of the oscillator, \( m = 1 \) and \( h = k_B = 1 \) (\( k_B \) being Boltzmann constant), we have, \( x = \frac{a + \beta}{\sqrt{2}\omega}, \)

\( p = \frac{a - \beta}{\sqrt{2}\omega} \) in terms of creation(\( a^\dagger \)) and annihilation(\( a \)) operators and the dimension of \( \lambda \sim \) (frequency)\(^3\). Thus, the dimensionless variable, \( \frac{a}{\omega^2} \) \((0 < \frac{a}{\omega^2} << 1)\) serves as the entity of anharmonicity, where \( \omega^2 \) is a constant characteristic frequency pertaining to \( \lambda \). The energy eigenvalues of the Hamiltonian in Eq.(1) have the form,

\[ E_n = (n + \frac{1}{2})\omega + \frac{3\lambda}{2\omega^2}(n^2 + n + \frac{1}{2}) \] (2)

where \( n \) is any non-negative integer. The canonical partition function of AO by taking corrections upto second-order in \( \lambda \)
turns out to be,
\[
Z^{\omega} = \frac{1}{2} \cosh(\frac{\beta \omega}{2}) \left[ 1 - \frac{3 \beta \lambda}{4 \omega^2} \cosh^2(\frac{\beta \omega}{2}) \right] + \frac{9 \beta^2 \lambda^2}{32 \omega^4} \left( 1 + 3 \cosh(\frac{\beta \omega}{2})(1 + \cosh(\beta \omega)) \right)
\]
(3)

**AO-like spin system:** The operator form of AO can be simulated in terms of ladder operators of spin angular momentum for spin-\(\frac{1}{2}\) particles [39]. In order to make the normal ordered form of Hamiltonian consistent with that of AO, we consider that a spin-\(\frac{1}{2}\) particle is placed in a magnetic field \(B_z\) along the \(z\)-direction and a constant driving Hamiltonian \((\Omega \mathbb{I}_2)\) acts on the system. Hence, the total Hamiltonian of the system is,
\[
H^{\omega} = \gamma B_z S_z + \Omega \mathbb{I}_2
\]
(4)
where the constants \(\gamma\) and \(\Omega\) can be represented in terms of \(\lambda\) and \(\omega\) of AO as, \(\gamma = \frac{\lambda}{2} (\omega + \frac{\beta}{2})\) and \(\Omega = (\omega + \frac{\beta}{2})\). By decomposing the \(z\)-component of spin angular momentum, \(S_z\) (= \(\frac{\gamma}{\Omega}\)) by means of ladder operators \(S_+\) and \(S_-\), and taking normal order\(^1\), we get
\[
H^{\omega} = (S_+ S_- + \frac{1}{2})\omega + (4S_+ S_- + 1) \frac{3 \lambda}{4 \omega^2}
\]
(5)
which has two energy eigenvalues corresponding to the two levels, \(E_0 = \frac{\gamma}{4} + \frac{3 \lambda}{4 \omega^2}\) and \(E_1 = \frac{\gamma}{4} + \frac{15 \lambda}{4 \omega^2}\). It is straightforward to obtain the corresponding partition function given by, \(Z^{\omega} = \sum_{n=0}^{\infty} \exp(-\beta E_n)\). Here too \(\lambda\) is of dimension (frequency)\(^3\), and we treat \(\frac{\lambda}{\omega}\) as \((0, 1)\) parameter of anharmonicity, with \(\omega\) a non-negative constant.

**Quantum Otto cycle:** The four-step Otto refrigerator [11, 35, 39, 40] (see Fig.1) can be described as follows: (i) Isochoric-1 (A→B): The system is coupled to a cold reservoir maintained at temperature \(T_c\) while the system Hamiltonian is kept fixed at \(H'\). The amount of heat absorbed from the cold bath during isochoric cooling is given by,
\[
Q_c = \sum_n E_n^c (P_n^h - P_n^c) > 0,
\]
(6)
where \(E_n^c = E_{n|\omega=\omega'}\) is either of the form given by Eq.(2) for AO, or of the form of the eigen-energy of the spin system, depending on the case considered. \(P_n = \exp(-\beta E_n)\) and \(P_n^c = \exp(-\beta E_n)|_{\omega=\omega'}\) are the occupation probabilities of the system in the \(n\)-th eigenstate corresponding to the points A and B, respectively. (ii) Adiabatic-1 (B→C): As the process conserves entropy (S) at points B and C, i.e., \(S_B = S_C\), the occupation distribution remains invariant under the adiabatic evolution which alters the Hamiltonian from \(H^{\omega\omega'}(B) = H'\) to \(H^{\omega\omega'}(C) = H\) (or frequency from \(\omega'\) to \(\omega\) adiabatically. (iii) Isochoric-2 (C→D): The system rejects heat to the hot reservoir at temperature \(T_h\) during isochoric heating, keeping the Hamiltonian fixed at \(H\), by an amount,
\[
Q_h = \sum_n E_n^h (P_n^c - P_n^h) < 0,
\]
(7)
where \(E_n^h = E_{n|\omega=\omega''}\) having the corresponding forms for the AO and spin qubit cases, respectively. \(P_n^h\) and \(P_n^c\) are the occupation probabilities for the system to remain in the \(n\)-th eigenstate corresponding to points C and D, respectively. (iv) Adiabatic-2 (D→A): During this process the Hamiltonian changes from \(H^{\omega\omega'}(D) = H\) to \(H^{\omega\omega'}(A) = H'\) (or frequency from \(\omega\) to \(\omega'\)) quasi-statically keeping the entropy constant for points D and A, \(S_D = S_A\) which, in turn, keeps the occupancies unaltered. The net work done on the system per cycle can be calculated as, \(W_O = Q_h + Q_c < 0\) (|\(Q_h\| < |Q_c|\)). The ratio of heat removed from the cold reservoir (\(Q_c\)) to the total amount of work (\(W_O\)) done on the system is called as the co-efficient of performance (COP) for Otto refrigerator, given by,
\[
\epsilon_O = \frac{Q_c}{|W_O|}
\]
(8)

**Quantum Stirling cycle:** The four-step Stirling refrigerator [34, 41] (see Fig.1) is as follows: (i) Isothermal-1 (A→B): The system is coupled to a cold reservoir maintained at temperature \(T_c\), and the Hamiltonian of the system changes slowly from \(H^{\omega\omega'}(A) = H\) to \(H^{\omega\omega'}(B) = H'\) (or frequency from \(\omega\) to \(\omega'\)) to keep the system in thermal equilibrium with the cold bath. Meanwhile, the entropy of the system changes from \(S_A = S^{\omega\omega'}|_{\omega=\omega''}\) to \(S_B = S^{\omega\omega'}|_{\omega=\omega'}\). The amount of heat thereby absorbed is
\[
Q_{AB} = T_c (S_B - S_A) > 0,
\]
(9)
(ii) Isochoric-1 (B→C): The Hamiltonian is kept fixed at \(H'\) while the temperature of the system increases from \(T_c\) to

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\(^1\) In terms of \((a, a')\), the normal ordered form of \(x^4\) is given by \((a^4)^4 + 4(a^4)^2 + 6(a^2)^2 a^2 + 12a^4 a^2 + 6a^4 a^4 + 6a^4 + a^6 + 3\). We replace \(a^4\) and \(a\) by the spin raising and lowering operators \(S_+\) and \(S_-\), respectively, and use \((S_+ S_- + 1) = 0\).

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**Fig. 1.** (a) Temperature-Entropy (T-S) diagram of 4-step Quantum Otto cycle. (b) Schematic diagram of internal energy vs Hamiltonian(U-H) for 4-step Quantum Stirling cycle.
The mean internal energy of the system increases from $U_R = U^{ao(sp)}|_{\beta_R,\omega_R=\omega_R'}$ to $U_C = U^{ao(sp)}|_{\beta_R,\omega_R=\omega}$. As a result, the system gains heat by an amount,

$$Q_{BC} = U_C - U_R > 0,$$  \hspace{1cm} (10)

(iii) Isothermal-2 ($C\rightarrow D$): The system is now attached to a hot reservoir at temperature $T_h$ and the quasi-static change in the Hamiltonian from $H^{ao(sp)}(C) = H'$ to $H^{ao(sp)}(D) = H$ (or frequency from $\omega'$ to $\omega$) governs the change in entropy from $S_C = S^{ao(sp)}|_{\beta_R,\omega_R=\omega}$ to $S_D = S^{ao(sp)}|_{\beta_R,\omega=\omega}$. Thus, the heat rejected to the bath is given by,

$$Q_{CD} = T_h(S_D - S_C) < 0,$$  \hspace{1cm} (11)

(iv) Isochoric-2 ($D\rightarrow A$): The system Hamiltonian remains constant at $H$, and the temperature changes from $T_h$ to $T_c$ leading to the decrease in mean internal energy from $U_D = U^{ao(sp)}|_{\beta_R,\omega_R=\omega}$ to $U_A = U^{ao(sp)}|_{\beta_R,\omega_R=\omega}$. Thus, the released heat amounts to

$$Q_{DA} = U_A - U_D < 0.$$  \hspace{1cm} (12)

In each case the internal energy and entropy can be derived from the partition function as, $f^{ao(sp)} = -\frac{1}{T} \ln Z^{ao(sp)}$ and $S^{ao(sp)} = \ln Z^{ao(sp)} + \beta U^{ao(sp)}$. The Stirling refrigeration cycle is a regenerative cycle as the input in the isochoric-1 process comes via plugging in the output of the isochoric-2 process. The net work done on the system is $W_S = Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA} < 0$ ($|Q_{AB}| + |Q_{BC}| < |Q_{CD}| + |Q_{DA}|$). The co-efficient of performance (COP) of the Stirling refrigerator is given by,

$$\epsilon_S = \frac{Q_{AB} + Q_{BC}}{|W_S|}$$  \hspace{1cm} (13)

**Improved COP of Otto refrigerator:** Ingraining anharmonicity in the oscillator and spin-$\frac{1}{2}$ system, the amount of heat absorbed from the cold bath during the Isochoric-1 process, is respectively.\(^2\)

$$Q_{AO}^{ao} = \frac{\omega'}{2} \left( \coth \left( \frac{\beta_R\omega}{2} \right) - \coth \left( \frac{\beta_R\omega'}{2} \right) \right)$$

$$Q_{AO}^{sp} = \frac{\omega'}{2} \left( \coth \left( \frac{\beta_R\omega}{2} \right) - \coth \left( \frac{\beta_R\omega'}{2} \right) \right)$$

$$+ \lambda Q_{AO}^{ao}(\omega, \omega', \beta_R, \beta_c) + \lambda^2 Q_{AO}^{sp}(\omega, \omega', \beta_R, \beta_c) + ... ,$$  \hspace{1cm} (14)

$$Q_{AO}^{sp} = \frac{\omega'}{2} \left( \coth \left( \frac{\beta_R\omega}{2} \right) - \coth \left( \frac{\beta_R\omega'}{2} \right) \right)$$

$$+ \lambda Q_{AO}^{ao}(\omega, \omega', \beta_R, \beta_c) + \lambda^2 Q_{AO}^{sp}(\omega, \omega', \beta_R, \beta_c) + ... ,$$  \hspace{1cm} (15)

The heat rejected to the hot reservoir during the Isochoric-2 process by the AO and qubit are respectively,

$$Q_{AO}^{CO} = -\frac{\omega}{2} \left( \coth \left( \frac{\beta_R\omega}{2} \right) - \coth \left( \frac{\beta_R\omega'}{2} \right) \right)$$

$$+ \lambda Q_{AO}^{ao}(\omega, \omega', \beta_R, \beta_c) + \lambda^2 Q_{AO}^{sp}(\omega, \omega', \beta_R, \beta_c) + ... ,$$  \hspace{1cm} (16)

$$Q_{AO}^{sp} = \frac{\omega}{2} \left( \tanh \left( \frac{\beta_R\omega}{2} \right) - \tanh \left( \frac{\beta_R\omega'}{2} \right) \right)$$

$$+ \lambda Q_{AO}^{ao}(\omega, \omega', \beta_R, \beta_c) + \lambda^2 Q_{AO}^{sp}(\omega, \omega', \beta_R, \beta_c) + ... ,$$  \hspace{1cm} (17)

Therefore, the COPs of the Otto refrigerator corresponding to the AO and spin system respectively become,

$$\epsilon_S^{ao} = \frac{\omega}{\omega - \omega'} + \frac{3\lambda}{2\omega^2\omega'^2} \left( \frac{\omega^2 - \omega'^2}{\omega - \omega'} \right) \left( \coth \left( \frac{\beta_R\omega}{2} \right) - \coth \left( \frac{\beta_R\omega'}{2} \right) \right)$$

$$+ \lambda^2 \epsilon_S^{ao}(\omega, \omega', \beta_R, \beta_c) + ... ,$$  \hspace{1cm} (18)

$$\epsilon_S^{sp} = \frac{\omega}{\omega - \omega'} + \frac{3\lambda}{2\omega^2\omega'^2} \left( \frac{\omega^2 - \omega'^2}{\omega - \omega'} \right) + \lambda^2 \epsilon_S^{sp}(\omega, \omega', \beta_R, \beta_c) + ... ,$$  \hspace{1cm} (19)

![FIG. 2](Color Online) COP of Otto refrigerator($\epsilon_S$) versus anharmonicity with $\omega=5$ and $\omega'=4$, $\beta_R = \frac{1}{\omega}$, $\beta_c = 1$ and $\omega_0 = 2$. The upper(blue), middle(red) and lower(black) lines imply COP corresponding to AO, AO-like qubit and harmonic oscillator respectively.

The negative work condition of the Otto refrigerator for both AO and spin systems dictates, $\omega > \omega'$ and $\beta_R, \omega' > \beta_R \omega$, making the co-efficients of $\lambda$ and $\lambda^2$ increase with the anharmonicity parameter (for the harmonic oscillator it is essentially $\frac{\omega}{\omega - \omega'}$). It can be observed from Fig.2 that the COP for AO is higher than the COP for the spin system for all $\lambda$.

**Improved COP of Stirling refrigerator:** Corresponding to change in frequency ($\omega \rightarrow \omega'$) during the Isothermal-1 process at inverse temperature, $\beta_c$, the absorbed heat from the

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\(^2\) In the Eqs.(14-29) below the co-efficients of $\lambda$ and $\lambda^2$ are given in the Supplementary Material [42], and "..." indicate higher order terms.
cold bath by AO and qubit are respectively,

\[ Q_{AB}^{ao} = \frac{\omega'}{2} \coth\left(\frac{\beta_{h} \omega'}{2}\right) - \frac{\omega}{2} \coth^2\left(\frac{\beta_{h} \omega}{2}\right) + \frac{1}{\beta_{c}} \ln \left[ \sinh\left(\frac{\beta_{c} \omega}{2}\right) \right] \]

\[ + \lambda Q_{AB1}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{AB2}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (20) \]

\[ Q_{AB}^{sp} = \frac{\omega'}{2} \tanh\left(\frac{\beta_{h} \omega'}{2}\right) - \frac{\omega'}{2} \tanh^2\left(\frac{\beta_{h} \omega}{2}\right) + \frac{1}{\beta_{c}} \ln \left[ \cosh\left(\frac{\beta_{c} \omega}{2}\right) \right] \]

\[ + \lambda Q_{AB1}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{AB2}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (21) \]

The heat absorbed by AO and qubit during Isochoric-1 process due to rise in temperature turns out to be,

\[ Q_{BC}^{ao} = \frac{\omega'}{2} \left( \coth\left(\frac{\beta_{h} \omega'}{2}\right) - \coth\left(\frac{\beta_{h} \omega}{2}\right) \right) + \lambda Q_{BC1}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{BC2}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (22) \]

\[ Q_{BC}^{sp} = \frac{\omega'}{2} \left( \tanh\left(\frac{\beta_{h} \omega'}{2}\right) - \tanh\left(\frac{\beta_{h} \omega}{2}\right) \right) + \lambda Q_{BC1}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{BC2}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (23) \]

The heat rejected to the bath during Isothermal-2 process at inverse temperature, \( \beta_{h} \) by AO and spin system are respectively,

\[ Q_{CD}^{ao} = -\frac{\omega'}{2} \coth\left(\frac{\beta_{h} \omega'}{2}\right) + \frac{\omega}{2} \coth\left(\frac{\beta_{h} \omega}{2}\right) - \frac{1}{\beta_{c}} \ln \left[ \sinh\left(\frac{\beta_{c} \omega}{2}\right) \right] \]

\[ + \lambda Q_{CD1}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{CD2}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (24) \]

\[ Q_{CD}^{sp} = -\frac{\omega'}{2} \tanh\left(\frac{\beta_{h} \omega'}{2}\right) + \frac{\omega'}{2} \tanh\left(\frac{\beta_{h} \omega}{2}\right) - \frac{1}{\beta_{c}} \ln \left[ \cosh\left(\frac{\beta_{c} \omega}{2}\right) \right] \]

\[ + \lambda Q_{CD1}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{CD2}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (25) \]

Finally, in the Isochoric-2 process, the heat rejected by AO and spin are respectively,

\[ Q_{DA}^{ao} = -\omega' \left( \coth\left(\frac{\beta_{h} \omega}{2}\right) - \coth\left(\frac{\beta_{c} \omega}{2}\right) \right) + \lambda Q_{DA1}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{DA2}^{ao}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (26) \]

\[ Q_{DA}^{sp} = -\omega' \left( \tanh\left(\frac{\beta_{h} \omega}{2}\right) - \tanh\left(\frac{\beta_{c} \omega}{2}\right) \right) + \lambda Q_{DA1}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + \lambda^2 Q_{DA2}^{sp}(\omega, \omega', \beta_{h}, \beta_{c}) + ... \quad (27) \]

From the Eqs.\((20-27)\), we obtain the COP of Stirling refriger-
and for the AO-like spin system,

\[
\delta H := \frac{1}{n} \sum_{n=0}^{\infty} \exp(-\beta E_n) \frac{1}{Z_n} \left[ E_n - \left( n + \frac{1}{2} \right) \omega \right] \\
= \frac{1}{Z_n} \frac{3\lambda}{4\omega^2} \exp\left(-\frac{3\beta\omega}{2}\right) \left[ \left( 5 + \exp(\beta\omega) \right) \right] \\
= \frac{3\beta\lambda}{4\omega^2} \left[ 25 + \exp(\beta\omega) \right]. \tag{31}
\]

We study two cases, namely (i) \( \beta = \beta_h, \omega = \omega \), and (ii) \( \beta = \beta_c, \omega = \omega' \), to observe the variation of COP compared to \( \delta H \). For the choice of parameters: \( \beta_h = \frac{1}{2}, \beta_c = 1, \omega = 5 \) and \( \omega' = 4 \), the variation of \( \delta H \) lies in the range \([0, 0.27]\) for AO, and \([0, 0.28]\) for qubit, corresponding to case (i), and in \([0, 0.36]\) for both AO and qubit corresponding to case (ii). The COP for Otto and Stirling refrigerators are plotted against \( \delta H \) in Fig.4 and Fig.5 respectively, corresponding to the case (i) and case (ii) above. In all the cases, COP increases with \( \delta H \) compared to the harmonic counterparts (with \( \delta H=0 \) or \( \lambda=0 \)). The improvement of COP is higher for AO compared to the AO-like qubit.

**FIG. 4.** (Color Online) In the array of figures, (a) and (b) indicate COP of Otto refrigerator\((c_{EO})\) as a function of \( \delta H \) using case(i) and case(ii) respectively. The upper (blue), middle (red), lower (black) lines imply the working media as AO, AO-like qubit and harmonic oscillator respectively. The parameters are \( \beta_h = \frac{1}{2}, \beta_c = 1, \omega=5, \omega'=4 \) and \( \omega_0 = 2 \).

**FIG. 5.** (Color Online) COP\((c_{ES})\) vs energy cost for Stirling refrigerator, (a) corresponding to case(i) and (b) corresponding to case(ii). The upper (blue) and lower (red) curves, and the upper (black) and lower (brown) straight lines indicate the working media for AO, AO-like qubit system, harmonic oscillator and qubit analogous to harmonic oscillator, respectively. The parameters are \( \beta_h = \frac{1}{2}, \beta_c = 1, \omega=5, \omega'=4 \) and \( \omega_0 = 2 \).

**Conclusions:** To conclude, anharmonic substances outperform harmonic ones in terms of the COP for both Otto and Stirling refrigerators provided the contribution of anharmonicity is small. To certify the robustness between anharmonic oscillator and spin-\( \frac{1}{2} \) particle, the slopes of \( \epsilon \) vs \( \lambda \) plots are compared. In case of Otto refrigerator, \( \frac{d\delta H}{d\lambda} : \frac{d\delta H}{d\lambda} = \lambda(\beta_h, \beta_c, \omega, \omega') > 1 \) for all \( \lambda, \omega > \omega' \) and \( \beta_h < \beta_c \). This implies that, anharmonic oscillator is more useful than qubit by means of growth in COP. Whereas the robustness corresponding to Stirling refrigerator can not be determined in general from the ratio \( \frac{d\delta H}{d\lambda} : \frac{d\delta H}{d\lambda} \) because it changes depending on the parameter values \( \lambda, \omega, \omega': \beta_h, \beta_c \). Moreover energy fluctuations due to anharmonicity through external parameters shows monotonic behaviour as that of \( \lambda \). This establishes anharmonicity as a resource for certain thermodynamic processes. Our results are thus significant from the perspectives discussed in [10, 43].

Proper implementation of Otto refrigerator requires very slow adiabatic processes to maintain no further coherence generation in the eigen states of the Hamiltonian, otherwise mean population will change. Also to achieve thermal equilibrium with the reservoir, the system has to spend much time during thermalisation processes. It may be noted here that although generating non-linearity in Hamiltonian requires some energy cost, one needs to design such non-linearity in appropriate manner – as for example, in the present work – to get a better effect compared to the corresponding linear case. Also note that, we have considered here a specific way of calculating the energy cost for creating anharmonicity – based on which, we have seen (in Fig.4 and 5) enhancement in COP of Otto and Stirling refrigerators as we increase the energy cost. It may be interesting to look for some other (in particular, some experimentally driven) types of energy cost for creating anharmonicity and thereby its effect on COP.

Further investigation by modelling different higher ordered forms of potential [30, 44] is needed under the aforementioned framework to develop nonlinearity as a better resource in the applicable areas [45] of quantum theory. For example, possible application of the same in coupled working media [39, 40, 46, 47] and utilizing non-Markovian reservoirs [48] may be fascinating to explore.

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dynamics of non-markovian reservoirs and heat engines,” Phys. Rev. E 97, 062108 (2018).
SUPPLEMENTARY MATERIAL.

Co-efficient of $\lambda$ in Eq.(14),
\[ Q_{c1}^{\psi}(\omega, \omega', \beta_h, \beta_c) = \frac{3}{4\omega^2\omega'^2} \left( \omega^2(\beta_c\omega' \coth(\frac{\beta_c\omega'}{2}) - 1)\text{csch}(\frac{\omega\beta_h}{2})^2 + \text{csch}(\frac{\omega\beta_h}{2})^2(\omega^2 - \beta_h(\omega')^3 \coth(\frac{\omega\beta_h}{2})) \right) \]  
\[ (32) \]

Co-efficient of $\lambda^2$ in Eq.(14),
\[ Q_{c2}^{\psi}(\omega, \omega', \beta_h, \beta_c) = \frac{9}{32\omega^4\omega'^4} \left( -\omega^4\beta_c^2\omega'(9 \cosh(\frac{\beta_c\omega'}{2}) + \cosh(\frac{3\beta_c\omega'}{2}))\text{csch}(\frac{5\beta_c\omega'}{2}) + 2\omega^2\beta_c(2 \cosh(\beta_c\omega') + 3)\text{csch}(\frac{\beta_c\omega'}{2}) \\
+ 2\beta_h\omega'^2\text{csch}(\frac{\omega\beta_h}{2})(\beta_h(\omega')^3(\sinh(\omega\beta_h) + 5 \coth(\frac{\omega\beta_h}{2})) - \omega^2(2 \cosh(\omega\beta_h) + 3)) \right) \]  
\[ (33) \]

Co-efficient of $\lambda$ in Eq.(15),
\[ Q_{c1}^{\psi}(\omega, \omega', \beta_h, \beta_c) = \frac{3}{4\omega^2\omega'^2} \left( \omega^2(\tanh(\frac{\beta_c\omega'}{2}) + \beta_c\omega' \text{sech}(\frac{\beta_c\omega'}{2}) - \tanh(\frac{\omega\beta_h}{2}) - \beta_h\omega'^3 \text{sech}(\frac{\omega\beta_h}{2}) \right) \]  
\[ (34) \]

Co-efficient of $\lambda^2$ in Eq.(15),
\[ Q_{c2}^{\psi}(\omega, \omega', \beta_h, \beta_c) = \frac{9}{8\omega^4\omega'^4} \left( \beta_h\omega'^2\text{sech}(\frac{\omega\beta_h}{2})(\beta_h(\omega')^3 \tanh\left(\frac{3\omega\beta_h}{2}\right)) - \omega^2 - \omega^2 \beta_c (\beta_c \omega' \tanh(\frac{\beta_c\omega'}{2}) - 1)\text{sech}(\frac{\omega\beta_h}{2})^2 \right) \]  
\[ (35) \]

Co-efficient of $\lambda$ in Eq.(16),
\[ Q_{h1}^{\psi}(\omega, \omega', \beta_h, \beta_c) = -\frac{3}{4\omega^2\omega'^2} \left( \omega^2\beta_c \coth(\frac{\beta_c\omega'}{2}) - \omega^2 \text{csch}(\frac{\omega\beta_h}{2})^2 - \omega^2(\omega\beta_h \coth(\frac{\omega\beta_h}{2}) - 1)\text{csch}(\frac{\omega\beta_h}{2})^2 \right) \]  
\[ (36) \]

Co-efficient of $\lambda^2$ in Eq.(16),
\[ Q_{h2}^{\psi}(\omega, \omega', \beta_h, \beta_c) = -\frac{9}{32\omega^4\omega'^4} \left( -\omega^5\beta_c^2(9 \cosh(\frac{\beta_c\omega'}{2}) + \cosh(\frac{3\beta_c\omega'}{2}))\text{csch}(\frac{5\beta_c\omega'}{2}) + 2\omega^2\beta_c(2 \cosh(\beta_c\omega') + 3)\text{csch}(\frac{\beta_c\omega'}{2}) \\
+ \beta_h\omega'^4\text{csch}(\frac{\omega\beta_h}{2})(\omega\beta_h(9 \cosh(\frac{\omega\beta_h}{2}) + \cosh(\frac{3\omega\beta_h}{2})) - 2(2 \sinh(\frac{3\omega\beta_h}{2}) + \sinh(-\frac{3\omega\beta_h}{2}))) \right) \]  
\[ (37) \]

Co-efficient of $\lambda$ in Eq.(17),
\[ Q_{h1}^{\psi}(\omega, \omega', \beta_h, \beta_c) = -\frac{3}{4\omega^2\omega'^2} \left( \omega^2\beta_c \text{sech}(\frac{\beta_c\omega'}{2}) - \omega^2(-\tanh(\frac{\beta_c\omega'}{2}) + \tanh(\frac{\omega\beta_h}{2}) + \omega\beta_h \text{sech}(\frac{\omega\beta_h}{2})) \right) \]  
\[ (38) \]

Co-efficient of $\lambda^2$ in Eq.(17),
\[ Q_{h2}^{\psi}(\omega, \omega', \beta_h, \beta_c) = -\frac{9}{8\omega^4\omega'^4} \left( \beta_h\omega'^4(\omega\beta_h \tanh(\frac{\omega\beta_h}{2}) - 1)\text{sech}(\frac{\omega\beta_h}{2})^2 - \omega^2\beta_c (\omega^2 \beta_c \beta_h \tanh(\frac{\beta_c\omega'}{2}) + \omega^2 \beta_c \text{sech}(\frac{\omega\beta_h}{2})^2) \right) \]  
\[ (39) \]

Co-efficient of $\lambda^2$ in Eq.(18),
\[ e^{\psi}_{h1}(\omega, \omega', \beta_h, \beta_c) = \frac{1}{128\omega^4\omega'^4(\omega - \omega')} \times \left( 9(\omega^2 + \omega' + \omega)\text{csch}(\frac{\omega}{4})^2\text{csch}(\frac{1}{4}(\omega - 2\omega'))^2 \right) \]  
\[ \times (-8(\omega')^2(\sinh(\frac{\omega}{2}) - 2 \sinh(\frac{\omega'}{2}))\text{sinh}(\frac{\omega'}{2}) + 32\omega' \sinh(\frac{\omega}{4})^2(\omega^2 - 2\omega') \sinh(\frac{1}{4}(2\omega' + \omega)) \sinh(\frac{\omega'}{2}) \\
+ 2\omega(2 \sinh(\frac{\omega}{2}) - 2 \sinh(\omega' + \omega) + \sinh(\omega' + \omega) - \sinh(2\omega' + \omega) - 2\omega^2(3 \text{sinh}(\frac{\omega}{4})^2(\sinh(\frac{\omega}{4}) - \omega' + \omega)) \\
- 6\omega - \sinh(\frac{\omega}{2}) - \sinh(\omega) + 8\omega \cosh(\frac{\omega}{2})) \right) \]  
\[ (40) \]
Co-efficient of $\lambda^2$ in Eq.(19),

$$
\epsilon_{02}^{lp}(\omega, \omega', \beta, \beta_c) = \frac{9(\omega + \omega')(\omega^3 - \omega'^3)}{4\omega^4\omega'^4(\omega - \omega')^2}
$$

(41)

Co-efficient of $\lambda$ in Eq.(20),

$$Q_{AB1}^{p}(\omega, \omega', \beta, \beta_c) = \frac{3}{8} \beta_c \left( \frac{\sinh(\omega\beta_c)\cosh^4\left(\frac{\beta_c}{2}\right)}{\omega} - \frac{\sinh(\beta_c\omega')\cosh^4\left(\frac{\beta_c\omega'}{2}\right)}{\omega'} \right)
$$

(42)

Co-efficient of $\lambda^2$ in Eq.(20),

$$Q_{AB2}^{p}(\omega, \omega', \beta, \beta_c) = \frac{9}{32} \beta_c \left( \frac{\cosh^2\left(\frac{\omega\beta_c}{2}\right)(2\sinh^2\left(\frac{\omega\beta_c}{2}\right) + \sinh(\omega\beta_c)(9\omega\beta_c + \cosh\left(\frac{3\omega\beta_c}{2}\right)))}{\omega^4} \\
+ \frac{\cosh^2\left(\frac{\beta_c\omega'}{2}\right)(-2\sinh(\frac{\beta_c\omega'}{2}) - \sinh(\frac{3\beta_c\omega'}{2}) + \beta_c\omega')(9\cosh\left(\frac{\beta_c\omega'}{2}\right) + \cosh(\frac{3\beta_c\omega'}{2}))}{\omega'^4} \right)
$$

(43)

Co-efficient of $\lambda$ in Eq.(21),

$$Q_{AB1}^{p}(\omega, \omega', \beta, \beta_c) = \frac{3\beta_c}{2\omega\omega'} \left( \frac{\omega'}{\cosh(\omega\beta_c) + 1} - \frac{\omega}{\cosh(\beta_c\omega') + 1} \right)
$$

(44)

Co-efficient of $\lambda^2$ in Eq.(21),

$$Q_{AB2}^{p}(\omega, \omega', \beta, \beta_c) = \frac{9}{8} \beta_c \left( \frac{(1 - \omega\beta_c \tanh\left(\frac{\omega\beta_c}{2}\right))\cosh^4\left(\frac{\omega\beta_c}{2}\right)}{\omega^4} + \frac{(\beta_c\omega' \tanh\left(\frac{\beta_c\omega'}{2}\right) - 1)\cosh^4\left(\frac{\beta_c\omega'}{2}\right)}{\omega'^4} \right)
$$

(45)

Co-efficient of $\lambda$ in Eq.(22),

$$Q_{BC1}^{p}(\omega, \omega', \beta, \beta_c) = \frac{3}{16\omega^2} \left( \sinh(\beta_c\omega')(\sinh(\beta_c\omega') - 2\beta_c\omega')\cosh^4\left(\frac{\beta_c\omega'}{2}\right) - \sinh(\beta_c\omega')(\sinh(\beta_c\omega') - 2\beta_c\omega')\cosh^4\left(\frac{\beta_c\omega'}{2}\right) \right)
$$

(46)

Co-efficient of $\lambda^2$ in Eq.(22),

$$Q_{BC2}^{p}(\omega, \omega', \beta, \beta_c) = \frac{9}{32\omega^4} \left( 2\beta_c(2\cosh(\beta_c\omega') + 3)\cosh^4\left(\frac{\beta_c\omega'}{2}\right) - 2\beta_c^2\omega' (\cosh(\beta_c\omega') + 4) \coth\left(\frac{\beta_c\omega'}{2}\right) \cosh^4\left(\frac{\beta_c\omega'}{2}\right) \\
+ \beta_c^2 \cosh^4\left(\frac{\beta_c\omega'}{2}\right)(9\cosh(\beta_c\omega') + \cosh(3\beta_c\omega')) - 2(2\sinh(\frac{\beta_c\omega'}{2}) + \sinh(\frac{3\beta_c\omega'}{2})) \right)
$$

(47)

Co-efficient of $\lambda$ in Eq.(23),

$$Q_{BC1}^{p}(\omega, \omega', \beta, \beta_c) = \frac{3}{8\omega^2} \left( \frac{\beta_c\omega' \cosh(\beta_c\omega') + \beta_c\omega' + \sinh(\beta_c\omega') - \beta_c\omega' - \sinh(\beta_c\omega')}{8\omega^2} \right)
$$

(48)

Co-efficient of $\lambda^2$ in Eq.(23),

$$Q_{BC2}^{p}(\omega, \omega', \beta, \beta_c) = \frac{9}{8\omega^4} \left( \beta_c(2 - \beta_c\omega' \tanh\left(\frac{\beta_c\omega'}{2}\right))\cosh^2\left(\frac{\beta_c\omega'}{2}\right) - \beta_c(2 - \beta_c\omega' \tanh\left(\frac{\beta_c\omega'}{2}\right))\cosh^2\left(\frac{\beta_c\omega'}{2}\right) \right)
$$

(49)

Co-efficient of $\lambda$ in Eq.(24),

$$Q_{CD1}^{p}(\omega, \omega', \beta, \beta_c) = \frac{3}{8} \beta_c \left( \frac{\sinh(\omega\beta_c)\cosh^4\left(\frac{\omega\beta_c}{2}\right)}{\omega} - \frac{\sinh(\beta_c\omega')\cosh^4\left(\frac{\beta_c\omega'}{2}\right)}{\omega'} \right)
$$

(50)
Co-efficient of $\lambda^2$ in Eq.(24),

$$Q_{CD2}^{\lambda^2}(\omega, \omega', \beta_h, \beta_c) = -\frac{9}{32}\beta_h \left( \frac{\csc^2\left(\frac{\omega_{\beta_h}}{2}\right)(2\sinh\left(\frac{\omega_{\beta_h}}{2}\right) + \sinh\left(\frac{3\omega_{\beta_h}}{2}\right) - \omega_{\beta_h}(9\cosh\left(\frac{\omega_{\beta_h}}{2}\right) + \cosh\left(\frac{3\omega_{\beta_h}}{2}\right)))}{\omega^4} ight.$$

$$+ \frac{\csc^2\left(\frac{\omega_{\beta_c}}{2}\right)(-2\sinh\left(\frac{2\omega_{\beta_c}}{2}\right) - \sinh\left(\frac{3\omega_{\beta_c}}{2}\right) + \beta_h\omega'(9\cosh\left(\frac{\beta_h\omega'}{2}\right) + \cosh\left(\frac{3\beta_h\omega'}{2}\right)))}{\omega^4} \right)$$

(51)

Co-efficient of $\lambda$ in Eq.(25),

$$Q_{CD1}^{\lambda}(\omega, \omega', \beta_h, \beta_c) = -\frac{3\beta_h}{2\omega_{\lambda}} \left( \frac{\omega'}{\cosh(\omega_{\beta_h}) + 1} - \frac{\omega}{\cosh(\beta_h\omega') + 1} \right)$$

(52)

Co-efficient of $\lambda^2$ in Eq.(25),

$$Q_{CD2}^{\lambda^2}(\omega, \omega', \beta_h, \beta_c) = -\frac{9}{8}\beta_h \left( \frac{(1 - \omega_{\beta_h} \tanh(\frac{\omega_{\beta_h}}{2}))\text{sech}^2(\frac{\omega_{\beta_h}}{2}) - (1 - \beta_h\omega' \tanh(\frac{\beta_h\omega'}{2}))\text{sech}^2(\frac{\beta_h\omega'}{2})}{\omega^4} \right)$$

(53)

Co-efficient of $\lambda$ in Eq.(26),

$$Q_{DA1}^{\lambda}(\omega, \omega', \beta_h, \beta_c) = -\frac{3}{16\omega_{\lambda}} \left( \sinh(\omega_{\beta_h})(\sinh(\omega_{\beta_c}) - 2\omega_{\beta_h}\csc^4\left(\frac{\omega_{\beta_h}}{2}\right)) - \sinh(\omega_{\beta_c})(\sinh(\omega_{\beta_c}) - 2\omega_{\beta_c}\csc^4\left(\frac{\omega_{\beta_c}}{2}\right)) \right)$$

(54)

Co-efficient of $\lambda^2$ in Eq.(26),

$$Q_{DA2}^{\lambda^2}(\omega, \omega', \beta_h, \beta_c) = -\frac{9}{32\omega_{\lambda}^2} \left( \beta_h\csc^5\left(\frac{\omega_{\beta_h}}{2}\right)(2\sinh(\omega_{\beta_h}) + \cosh(h(\frac{3\omega_{\beta_h}}{2})) - 2(2\sinh(\omega_{\beta_h}) + \sinh(h(\frac{3\omega_{\beta_h}}{2})) 

- \beta_h\csc^5\left(\frac{\omega_{\beta_c}}{2}\right)(2\sinh(\omega_{\beta_c}) + \cosh(h(\frac{3\omega_{\beta_c}}{2})) - 2(2\sinh(\omega_{\beta_c}) + \sinh(h(\frac{3\omega_{\beta_c}}{2}))) \right)$$

(55)

Co-efficient of $\lambda$ in Eq.(27),

$$Q_{DA1}^{\lambda^2}(\omega, \omega', \beta_h, \beta_c) = -\frac{3\text{sech}^2\left(\frac{\omega_{\beta_h}}{2}\right)\text{sech}^2\left(\frac{\omega_{\beta_c}}{2}\right)}{8\omega_{\lambda}^2} \left( \sinh(\omega_{\beta_c}) + \sinh(\omega_{\beta_c} - \beta_h) + \omega_{\beta_c}(\cosh(\omega_{\beta_c}) + 1) \right. \frac{\omega_{\beta_c}(\cosh(\omega_{\beta_c}) + 1) - \sinh(\omega_{\beta_c})}{\omega_{\lambda}^2}$$

(56)

Co-efficient of $\lambda^2$ in Eq.(27),

$$Q_{DA2}^{\lambda^2}(\omega, \omega', \beta_h, \beta_c) = -\frac{9}{8\omega_{\lambda}^3} \left( \beta_c(2 - \omega_{\beta_c}\tanh(\frac{\omega_{\beta_c}}{2}))\text{sech}^2(\frac{\omega_{\beta_c}}{2}) - \beta_h(2 - \omega_{\beta_h}\tanh(\frac{\omega_{\beta_h}}{2}))\text{sech}^2(\frac{\omega_{\beta_h}}{2}) \right)$$

(57)

Co-efficient of $\lambda$ in Eq.(28),

$$e_{31}^{\alpha}(\omega, \omega', \beta_h, \beta_c) = \frac{3\beta_h\beta_c}{8\omega_{\lambda}^2\omega_{\lambda}^2} \left( \beta_c \ln \left[ \frac{\csc^2\left(\frac{\omega_{\beta_c}}{2}\right)}{\csc^2\left(\frac{\omega_{\beta_h}}{2}\right)} \right] + \beta_h \ln \left[ \frac{\csc^2\left(\frac{\omega_{\beta_h}}{2}\right)}{\csc^2\left(\frac{\omega_{\beta_c}}{2}\right)} \right] \right)$$

$$\times \left( -\omega_{\beta_c}\beta_h\omega^2 \cosh^2(\frac{\omega_{\beta_h}}{2}) + \beta_h\omega^2 \cosh^2(\frac{\omega_{\beta_c}}{2})(2\ln(\csc\left(\frac{\beta_c\omega'}{2}\right))$$

$$- 2\ln(\csc\left(\frac{\omega_{\beta_c}}{2}\right)) + \beta_c\omega' \coth\left(\frac{\beta_c\omega'}{2}\right) - \beta_c\omega' \coth\left(\frac{\beta_c\omega'}{2}\right)(2\ln(\csc\left(\frac{\beta_c\omega'}{2}\right)) - 2\ln(\csc\left(\frac{\omega_{\beta_c}}{2}\right)) + \beta_c\omega' \coth\left(\frac{\beta_c\omega'}{2}\right)$$

$$- \omega_{\beta_c}\beta_h\omega^2 \coth^2(\frac{\omega_{\beta_h}}{2} + 2\omega_{\beta_h} \coth^2(\frac{\beta_c\omega'}{2})(\ln(\csc\left(\frac{\omega_{\beta_c}}{2}\right)) - \ln(\csc\left(\frac{\beta_c\omega'}{2}\right)) + \omega_{\beta_c} \coth^2(\frac{\beta_c\omega'}{2})\beta_h\omega' \coth\left(\frac{\beta_c\omega'}{2}\right)$$

$$+ 2\ln(\csc\left(\frac{\beta_c\omega'}{2}\right)) - 2\ln(\csc\left(\frac{\omega_{\beta_c}}{2}\right)) + \omega_{\beta_c} \ln(\csc\left(\frac{\beta_c\omega'}{2}\right)) - \beta_h \ln(\csc\left(\frac{\omega_{\beta_h}}{2}\right)) - \beta_c \ln(\csc\left(\frac{\omega_{\beta_c}}{2}\right)) + \beta_h \ln(\csc\left(\frac{\beta_c\omega'}{2}\right))$$

$$\omega_{\beta_c} \sinh(\beta_h\omega') \csc^4\left(\frac{\beta_h\omega'}{2}\right) - \beta_c\omega' \sinh(\omega_{\beta_c})\csc^4\left(\frac{\omega_{\beta_c}}{2}\right) + \omega_{\beta_c}\beta_h \coth\left(\frac{\omega_{\beta_h}}{2}\right)\omega^2 \coth^2(\frac{\beta_c\omega'}{2}) - \omega^2 \coth^2(\frac{\beta_c\omega'}{2})) \right)$$

(58)
Co-efficient of $\lambda^2$ in Eq.(28),

$$
\epsilon^{\alpha\beta}_{S2}(\omega, \omega', \beta_h, \beta_c) = -\frac{9\beta^2\beta_h^3}{32\omega^4\omega'^4} \beta_c \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] + \beta_h \ln B_i g\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] \\
\times \left(\coth^2\left(\frac{\omega\beta_c}{2}\right) - \coth^2\left(\frac{\omega\beta_c}{2}\right)\right)\omega^2 + \left(\coth^2\left(\frac{\omega\beta_h}{2}\right) - \coth^2\left(\frac{\omega\beta_h}{2}\right)\right)\omega'^2 \\
+ \frac{9\beta^2\beta_h^3}{64\omega^4\omega'^4} \beta_c \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] + \beta_h \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] \\
\times \left(\sinh\left(\frac{3\omega\beta_c}{2}\right)\beta_c \omega^4 \csc h\left(\frac{\omega\beta_c}{2}\right) + 2\beta_c \omega^4 \csc h\left(\frac{\omega\beta_c}{2}\right)\right) - (2 \cosh(\omega\beta_h) + 3)\csc h^2\left(\frac{\omega\beta_h}{2}\right)\beta_h \omega^4 \\
+ \omega^4 (2 \cosh(\beta_c \omega') + 3)\beta_c \csc h^2\left(\frac{\omega\beta_c}{2}\right) - \omega^2 (2 \cosh(\beta_h \omega') + 3)\beta_h \csc h^2\left(\frac{\omega\beta_h}{2}\right) - 8\csc h^2\left(\frac{\omega\beta_h}{2}\right)\beta_h \\
+ 2\csc h^2\left(\frac{\beta_h \omega'}{2}\right)\beta_h \left(2 \omega \cosh(\beta_h \omega') + 4 \cosh(\beta_h \omega') + 4\cosh(\beta_h \omega') + 4 \cosh(\beta_h \omega')\right) \right) \\
(59)
$$

Co-efficient of $\lambda$ in Eq.(29),

$$
\epsilon^{\alpha\beta}_{S1}(\omega, \omega', \beta_h, \beta_c) = \frac{3\beta^2\beta_h^2}{4\omega^2\omega'^2} \beta_c \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] + \beta_h \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] \\
\times \left(2 \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] - \omega \beta_c \tanh\left(\frac{\omega\beta_c}{2}\right) + \beta_c \omega' \tanh\left(\frac{\beta_h \omega'}{2}\right)\right) \\
3\beta \beta_h \left(-2 \omega \tanh\left(\frac{\beta_h \omega'}{2}\right) - \beta_c \omega^2 \sech^2\left(\frac{\omega\beta_c}{2}\right) + \omega \beta_h \omega' + \sinh(\beta_h \omega') \sech^2\left(\frac{\beta_h \omega'}{2}\right)\right) \\
- \frac{3\beta \beta_h}{4\omega^2\omega'^2} \beta_c \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] + \beta_h \ln\left[\frac{\csc h\left(\frac{\omega'\beta_h}{2}\right)}{\csc h\left(\frac{\omega\beta_h}{2}\right)}\right] \right) \\
(60)
$$
Co-efficient of $\lambda^2$ in Eq.(29),

$$C_{S2}^{\beta}(\omega, \omega^\prime, \beta_h, \beta_c) = -\frac{9\beta_c\beta_h}{8\omega^4 \omega^4(\omega \beta_c \tanh(\frac{\omega \beta_c}{2}) - 1) \sech^2(\frac{\omega \beta_c}{2}) - \omega^4 \left(\beta_c \sech^2(\frac{\omega \beta_c}{2}) + \beta_h (\beta_c \omega^\prime \tanh(\frac{\beta_c \omega^\prime}{2}) - 2 \sech^2(\frac{\beta_c \omega^\prime}{2}))\right)}$$

$$+ \beta_c \beta_h \left(\frac{1}{\beta_c} \ln \left[\frac{\cosh(\frac{\beta_c \omega^\prime}{2})}{\cosh(\frac{\omega \beta_c}{2})}\right] + \frac{1}{2} \omega \tanh(\frac{\omega \beta_c}{2}) - \frac{1}{2} \omega^\prime \tanh(\frac{\beta_c \omega^\prime}{2})\right) + \beta_h \beta_c \left(\frac{1}{\beta_h} \ln \left[\frac{\cosh(\frac{\beta_h \omega^\prime}{2})}{\cosh(\frac{\omega \beta_c}{2})}\right] + \frac{1}{2} \omega^\prime \tanh(\frac{\beta_h \omega^\prime}{2})\right)$$

$$\times \left(\frac{9\beta_c^2 \beta_h^2}{4\omega^4 \omega^4(\omega \beta_c \tanh(\frac{\omega \beta_c}{2}) - 1) \sech^2(\frac{\omega \beta_c}{2}) - \omega^4 \left(\beta_c \sech^2(\frac{\omega \beta_c}{2}) + \beta_h (\beta_c \omega^\prime \tanh(\frac{\beta_c \omega^\prime}{2}) - 2 \sech^2(\frac{\beta_c \omega^\prime}{2}))\right)}{\omega^2 \left(\tanh(\frac{\omega \beta_c}{2}) - \tanh(\frac{\omega \beta_c}{2})\right)} + \omega^2 \left(\tanh(\frac{\beta_c \omega^\prime}{2}) - \tanh(\frac{\beta_c \omega^\prime}{2})\right)\right)^2$$

$$- \frac{9\beta_c^2 \beta_h^2}{8\omega^3 \omega^3(\omega \beta_c \tanh(\frac{\omega \beta_c}{2}) - 1) \sech^2(\frac{\omega \beta_c}{2}) - \omega^4 \left(\beta_c \sech^2(\frac{\omega \beta_c}{2}) + \beta_h (\beta_c \omega^\prime \tanh(\frac{\beta_c \omega^\prime}{2}) - 2 \sech^2(\frac{\beta_c \omega^\prime}{2}))\right)}{\omega^2 \left(\tanh(\frac{\omega \beta_c}{2}) - \tanh(\frac{\omega \beta_c}{2})\right)} + \omega^2 \left(\tanh(\frac{\beta_c \omega^\prime}{2}) - \tanh(\frac{\beta_c \omega^\prime}{2})\right)\right)$$

$$\times \left(2\omega \tanh(\frac{\beta_c \omega^\prime}{2}) + \beta_c \omega^\prime \sech^2(\frac{\omega \beta_c}{2}) - \omega (\beta_h \omega^\prime + \sinh(\beta_h \omega^\prime)) \sech^2(\frac{\beta_h \omega^\prime}{2})\right)$$

(61)