ILL-POSEDNESS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN $\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3)$

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Abstract

In this paper, authors show the ill-posedness of 3D incompressible Navier-Stokes equations in the critical Triebel-Lizorkin spaces $\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3)$ for any $q > 2$ in the sense that arbitrarily small initial data of $\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3)$ can lead the corresponding solution to become arbitrarily large after an arbitrarily short time. Thus extends Bourgain and Pavlović’s work [1]. In view of the well-posedness of 3D-incompressible Navier-Stokes equations in $BMO^{-1}$ (i.e. the Triebel-Lizorkin space $\dot{F}_{\infty}^{-1,2}(\mathbb{R}^3)$ ) by Koch and Tataru, our work completes a dichotomy of well-posedness and ill-posedness in the Triebel-Lizorkin space framework depending on $q = 2$ or $q > 2$.

Keywords: Navier-Stokes equations; Triebel-Lizorkin space; Well-posedness; Ill-posedness.
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1 Introduction

In this article, we are concerned with the following incompressible 3D Navier-Stokes equations (NS):

\[
\begin{aligned}
\partial_t u + \Delta u + u \cdot \nabla u + \nabla p &= 0, \\
\text{div } u &= 0, \\
u(x,0) &= u_0(x),
\end{aligned}
\]

where $(x,t) \in \mathbb{R}^3 \times (0, \infty)$, $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ are unknown vector functions, $p(x,t)$ is unknown scaler function, and $u_0(x)$ is a given vector function satisfying divergence free condition $\nabla \cdot u_0 = 0$.

Mathematical study on the existence and uniqueness of the incompressible Navier-Stokes equations has a long history. In 1934, Leray [9] first proved existence of global weak solution associated with any $L^2(\mathbb{R}^3)$ initial data by some weak compactness arguments. Until now, whether such a weak solution is unique and smooth or not
is still a great open problem, See Fefferman [4], also Caffarelli, Kohn and Nirenberg [3] and F.-H. Lin [10] for partial regularity of suitable weak solution. Beginning with a different method by semigroup and Picard’s iteration, Fujita-Kato [5] in 1964 established the local well-posedness of the N-S in $H^s(\mathbb{R}^3)$ for any $s \geq \frac{3}{2} - 1$, and also global well-posedness for any small initial data of $H^s(\mathbb{R}^3)$. This remarkable approach can be adapted to other various function spaces of initial data, see [2, 6, 8, 11] for expositions and references therein. In particular, an interesting result that should be mentioned is due to Koch-Tataru [7], where they proved that the N-S equation is well-posed in $BMO^{-1}$ (i.e. the Triebel-Lizorkin space $\dot{F}_{\infty}^{-1,2}(\mathbb{R}^3)$).

On the other hand, recently, Bourgain and Pavlović in [1] proved the ill-posedness of the incompressible 3D-Navier-Stokes equations in the largest scaling invariant Besov function space $B^{-1,\infty}_{\infty}(\mathbb{R}^3)$ in the sense that arbitrarily small initial data in the space $\dot{B}^{-1,\infty}_{\infty}(\mathbb{R}^3)$ can lead the solution to become arbitrarily large after an arbitrarily short time. Notice that $BMO^{-1} \subset \dot{B}_{\infty}^{-1,\infty}(\mathbb{R}^3)$, clearly, there exists a slight difference between the well-posed space and the ill-posed space. Therefore, it would be an interesting problem whether there exist strictly smaller ill-posed spaces than $\dot{B}_{\infty}^{-1,\infty}(\mathbb{R}^3)$. Such an improvement of illposedness was obtained by Yoneda [14] in a logarithmic type Besov space near $BMO^{-1}$. Motivated by that $BMO^{-1}$ is identical to the end-pointed Triebel-Lizorkin space $\dot{F}_{\infty}^{-1,2}(\mathbb{R}^3)$, in the current paper, the authors further study the interesting problem in a critical Triebel-Lizorkin space framework and show that the the NS equation \((1.1)\) is ill-posed in $\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3)$ for any $q > 2$, which are all strictly smaller than $\dot{B}_{\infty}^{-1,\infty}(\mathbb{R}^3)$ (See Theorem 1.3 below). Hence our work along with [7] establishes a dichotomy of well-posedness and ill-posedness in the Triebel-Lizorkin space framework depending on $q = 2$ or $q > 2$.

For the end, let us first recall the definitions of homogeneous Besov spaces/Triebel-Lizorkin spaces. Let $\varphi(\xi) = \varphi(|\xi|) \geq 0$ be a real-valued smooth function such that

\[
\begin{align*}
&\text{supp } \varphi \subset \{ \xi \in \mathbb{R}^3; \ 5/8 \leq |\xi| \leq 7/4 \}, \\
&\varphi \equiv 1 \text{ in } \{ \xi \in \mathbb{R}^3; \ 7/8 \leq |\xi| \leq 5/4 \}, \\
&\sum_j \varphi(2^{-j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\}.
\end{align*}
\]

(1.2)

For any tempered distribution $f$ and $i, j \in \mathbb{Z}$, define the dyadic block as follows:

\[
\Delta_j f(x) = \varphi(2^{-j} \nabla) f(x) \quad \text{and} \quad \Delta_i \Delta_j f \equiv 0 \text{ if } |i - j| \geq 2.
\]

(1.3)

In order to exclude nonzero polynomials in homogeneous Besov spaces and Triebel-Lizorkin spaces, it is natural to use $Z'(\mathbb{R}^3)$ to denote the subset of tempered distribution $f \in S'(\mathbb{R}^3)$ modulo all polynomials set $P(\mathbb{R}^3)$, i.e. $Z'(\mathbb{R}^3) = S'(\mathbb{R}^3)/P(\mathbb{R}^3)$.

Now we are ready to give the definitions of Triebel-Lizorkin spaces $\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3)$ and Besov space $\dot{B}_{\infty}^{-1,\infty}(\mathbb{R}^3)$, also see [13] for a detailed exposition about other general spaces $\dot{F}^{s,q}_p(\mathbb{R}^3)$ and $\dot{B}^{s,q}_p(\mathbb{R}^3)$.

**Definition 1.1.** For $1 < q < \infty$, we define $\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3)$ as the following set so that

\[
\dot{F}_{\infty}^{-1,q}(\mathbb{R}^3) = \left\{ f \mid f \in Z'(\mathbb{R}^3), \exists \{f_k(x)\}_{k \in \mathbb{Z}} \text{ s.t. } f = \sum_{k \in \mathbb{Z}} \Delta_k f_k(\cdot) \right\}
\]
in \( Z'(\mathbb{R}^3) \) and \( \left\| \{2^{-k}|f_k(\cdot)|\} \right\|_{L^\infty(\mathbb{R}^3)} < \infty \) \hspace{1cm} (1.4) 

and the corresponding norm is defined by

\[
\| f \|_{\dot{F}^{-1,q}_\infty(\mathbb{R}^3)} = \inf \| \{2^{-k}|f_k(\cdot)|\} \|_{L^\infty(\mathbb{R}^3)},
\]

where the infimum is taken over all admissible representations in the sense of (1.4). Meanwhile, we denote by \( \dot{B}^{-1,\infty}_\infty(\mathbb{R}^3) \) the set of distribution \( f \in Z'(\mathbb{R}^3) \) such that

\[
\| f \|_{\dot{B}^{-1,\infty}_\infty(\mathbb{R}^3)} = \sup_{t>0} \sqrt{t} \| e^{t\Delta} f \|_{L^\infty(\mathbb{R}^3)} < \infty.
\] \hspace{1cm} (1.5)

**Remark 1.2.** It is known that \( \dot{F}^{-1,2}_\infty(\mathbb{R}^3) = BMO^{-1}(\mathbb{R}^3) \) and \( BMO^{-1}(\mathbb{R}^3) \) has the following equivalent Carleson measure characterization (cf. [2]):

\[
\| f \|_{BMO^{-1}(\mathbb{R}^3)} = \sup_{x \in \mathbb{R}^3; R > 0} \left( \frac{1}{|B_R(x)|} \int_{B_R(x)} | e^{t\Delta} f(y) |^2 \, dy \right)^{1/2} < \infty. \] \hspace{1cm} (1.6)

Moreover, we remark that for any \( 1 < q < \infty, \)

\( \dot{F}^{-1,q}_\infty(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1,\infty}_\infty(\mathbb{R}^3) \).

As usual, we first write (1.1) into the following equivalent integral equations:

\[
u = e^{t\Delta} u_0 - B(u, u), \] \hspace{1cm} (1.7)

where \( \mathcal{P} \) denotes the Leray projection operator \( Id - \nabla \frac{1}{2} \mathcal{P} \)div, and the bilinear term \( B(u, v) \) is defined by

\[
B(u, v) := \int_0^t e^{(t-\tau)\Delta} \mathcal{P}(u \cdot \nabla v) \, d\tau. \] \hspace{1cm} (1.8)

For any \( u \in L^2_{loc}(\mathbb{R}^3 \times \mathbb{R}) \), we define that

\[
\| u \|_{X_T} := \sup_{x_0 \in \mathbb{R}^3; 0 < R < T} \left( \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} | u |^2 \, dx \right)^{1/2}, \] \hspace{1cm} (1.9)

and

\[
\| u \|_{\mathcal{E}_T} := \sup_{0 < t < T} t^{1/2} \| u \|_{L^\infty(\mathbb{R}^3)} + \| u \|_{X_T}. \] \hspace{1cm} (1.10)

Recall that \( B(u, v) \) satisfies the following a-priori bilinear estimates:

\[
\| B(u, v) \|_{\mathcal{E}_T} \lesssim \| u \|_{\mathcal{E}_T} \| v \|_{\mathcal{E}_T}. \] \hspace{1cm} (1.11)

Applying boundedness property of \( \mathcal{P} \) in \( \dot{F}^{-1,2}_\infty(\mathbb{R}^3) \) and decay estimate for heat kernel, following similar argument as in [3] Lemma 16.3], we have

\[
\| B(u, v) \|_{L^\infty_T \dot{F}^{-1,2}_\infty} \lesssim \| u \|_{\mathcal{E}_T} \| v \|_{\mathcal{E}_T}. \] \hspace{1cm} (1.12)

Based on (1.11), Koch and Tataru [7] established the existence of solutions to the Navier-Stokes equation in \( BMO^{-1}(\mathbb{R}^3) \). By using (1.11) and (1.12), we will further prove the following ill-posedness in \( \dot{F}^{-1,q}_\infty(\mathbb{R}^3) \) for any \( q > 2 \).
Theorem 1.3. For any $q > 2$ and $\delta > 0$, there exists a solution $u$ to system (1.1) with some initial data $u_0 \in \dot{F}_\infty^{-1,q}(\mathbb{R}^3)$ satisfying

$$\|u_0\|_{\dot{F}_\infty^{-1,q}(\mathbb{R}^3)} \lesssim \delta$$

and $\text{div} \, u_0 = 0$ such that for some $0 < T < \delta$,

$$\|u(T)\|_{\dot{F}_\infty^{-1,q}(\mathbb{R}^3)} \gtrsim \frac{1}{\delta}.$$ 

This paper is organized as follows: In Section 2, we first construct a very special initial data and list some necessary remarks and lemmas. In section 3, we establish all the desired estimates about the first and second approximation terms which will be used in controlling the remainder term. Finally, combining all the a-priori estimates we prove ill-posedness of the Navier-Stokes equations.

Notations: Throughout this paper, we shall use $C$ and $c$ to denote generic constants and may change from line to line. Both $\mathcal{F}f$ and $\hat{f}$ stand for Fourier transform of $f$ with respect to space variable, while $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. We denote $A \leq CB$ by $A \lesssim B$ and $A \gtrsim B$ by $A \sim B$. For any $1 \leq p \leq \infty$, we denote $L^q(0,T), L^q(0,\infty)$ and $L^p(\mathbb{R}^3)$ by $L^q_t, L^q_t$ and $L^p_x$, respectively. Later on, we also use $\dot{F}^{s,q}_p$ to denote $\dot{F}^{s,q}_p(\mathbb{R}^3)$ if there is no confusion about the domain, and similar conventions are applied. For simplicity, we denote by $B_R(x)$ the ball centered at $x$ of radius $R$.

2 Construction of initial data

For any $\delta > 0$, we define the initial data as follows:

$$u_0(x) = \frac{Q}{\sqrt{I}} \sum_{s=1}^r \left( \cos(k_s x)\Psi_1 - |k_s|\sin(k_s x)\Psi_2 + \cos(k'_s x)\Psi_3 + |k_s|\sin(k'_s x)\Psi_4 \right), \quad (2.1)$$

where

$$k_s = (0, 2\frac{(s+1)(s+2m_0)}{2}, 0), \quad k'_s = (2^3, -2\frac{(s+1)(s+2m_0)}{2}, 0) \quad \text{with} \quad s = 0, 1, 2, \ldots, r,$$

$$\begin{aligned}
\Psi_1 &= (0, -\partial_3 \psi, \partial_2 \psi), \quad \Psi_2 = (0, 0, \psi), \quad \Psi_3 = (\partial_2 \psi, -\partial_1 \psi, 0), \quad \Psi_4 = (\psi, \frac{2^3 \psi}{|k_s|}, 0)
\end{aligned} \quad \text{(H)}$$

$\psi(x)$ satisfies: $\hat{\psi}(\xi) = \hat{\psi}(|\xi|) \geq 0$, $\text{supp} \, \hat{\psi} \subset B_{\frac{3}{4}}(0)$ and $\|\hat{\psi}\|_{L^1_{\xi}} = 1$

and $Q, r, m_0$ will be chosen sufficiently large according to the size of $\delta$.

Remark 2.1. From the above assumptions we have the following observations:

(i) From (2.1) and (H), it is easy to check that $u_0$ is smooth, real-valued, divergence free and $L^2_x$ finite.
(ii) For $\forall g \in S(\mathbb{R}^3)$ and $\forall k \in \mathbb{Z}^3$ we have $\hat{g}(\xi - k) = \mathcal{F}(e^{ikx}g(x))$, which shows that
\[
\text{supp } \mathcal{F}(\cos(kx) \pm \sin(kx)\psi(x)) \subset B_{\frac{1}{4}}(k) \cup B_{\frac{1}{4}}(-k).
\] (2.2)

(iii) Recall that $|k_s| = 2^{\frac{(s+1)(s+2m_0)}{2}}$, for any $s \in [0, r] \cap \mathbb{N}$, we denote $j_s = \log_2 |k_s|$. Then for large enough $m_0$ ($m_0 \geq 7$),
\[
(\cos(k_s x)\psi, \cos(k'_s x)\psi) = (\Delta_{j_s}(\cos(k_s x)\psi), \Delta_{j_s}(\cos(k'_s x)\psi)).
\] (2.3)
Similar arguments work well for $\sin(k_s x)\psi$ and $\sin(k'_s x)\psi$.

(iv) From (2.1) and (iii), we denote
\[
f_{\ell} = \begin{cases} \cos(k_s x)\psi_1 - |k_s|\sin(k_s x)\psi_2 + \cos(k'_s x)\psi_3 + |k_s|\sin(k'_s x)\psi_4 & \text{if } \ell = j_s, \\ 0 & \text{otherwise}, \end{cases}
\] (2.4)
then $u_0 = \frac{Q}{\sqrt{r}} \sum_{\ell \in \mathbb{Z}} \Delta_{\ell} f_{\ell}$ and $\left\{ \frac{Q}{\sqrt{r}} \right\}_{\ell}$ is a decomposition of $u_0$ in the sense of (1.4).

Lemma 2.2. Let $a(D)$ be a 3-dimensional Fourier multiplier operator corresponding to the homogeneous symbol $a(\xi)$ of degree $m \geq 0$. Then there exists some $c > 0$ such that for any $j \in \mathbb{Z}$ and $t \geq 0$, the following point-wise estimate holds
\[
|\langle a(D)e^{t\Delta} \Delta^j f(x) \rangle| \lesssim 2^{mj}e^{-ct2^j}(Mf)(x),
\] (2.5)
where $\Delta^j$ is defined in (1.3) and $Mf$ is the Hardy-Littlewood maximal function of $f$.

Proof. Let $K_j(t, x)$ be the kernel of $a(D)e^{t\Delta} \Delta^j$. Then by scaling we have
\[
K_j(t, x) = 2^{(m+3)j} \int_{\mathbb{R}^3} e^{i2^j x \xi} e^{-t2^j |\xi|^2} a(\xi) \varphi(\xi) d\xi,
\]
which immediately yields $|K_j(t, x)| \lesssim 2^{(m+3)j}e^{-ct2^j}$. By integrating by parts we get
\[
|2^j x|^4 |K_j(t, x)| \lesssim 2^{(3+m)j} \sum_{|\mu|=4} \left| \int_{\mathbb{R}^3} e^{i2^j x \xi} \partial^\mu \left( e^{-t2^j |\xi|^2} a(\xi) \varphi(\xi) \right) d\xi \right|
\]
\[
\lesssim 2^{(3+m)j} \int_{\text{supp } \varphi} e^{-t2^j |\xi|^2} \left( t^4 2^{8j} + 1 \right) d\xi
\]
\[
\lesssim 2^{(3+m)j} e^{-ct2^j},
\]
where we used $e^{-t2^j |\xi|^2} 2^{2\alpha N_j} \leq C_N e^{-ct2^j}$ for $\xi \in \text{supp } \varphi$. Hence we obtain that
\[
|K_j(t, x)| \lesssim 2^{mj} e^{-ct2^j} 2^{3j} (1 + 2^j |x|)^{-1},
\]
which concludes the desired estimate (2.5). \qed
Lemma 2.3. Let \( \psi \) be defined in (II), \( M \psi \) be the Hardy-Littlewood maximal function of \( \psi \) and \( \theta = (M \psi)^2 \). For any \((k, h) \in \mathbb{Z}^3 \times \mathbb{Z}^3\) and \(\min\{|h|, |k|, |h+k|\} \geq 6\), we denote

\[
F(t, \tau, x; k, h) = a_1(D)e^{(t-\tau)\Delta} (a_2(D)e^{\tau \Delta} \psi_k a_3(D)e^{\tau \Delta} \psi_{h}) (x) \quad (2.6)
\]

where \(a_\ell(D)\) are Fourier multipliers with homogeneous symbols \(a_\ell(\xi)\) of degree \(\ell \geq 0\) \((\ell = 1, 2, 3)\), \(\psi_k\) is either \(\cos(kx)\psi(x)\) or \(\sin(kx)\psi(x)\) and \(\psi_h\) is either \(\cos(hx)\psi(x)\) or \(\sin(hx)\psi(x)\). Then there exist positive constants \(c\) and \(C\) such that

\[
|a_1(D)e^{t\Delta} \psi_{h}(x)| \leq C|h|^{m_1} e^{-c|h|^2} (M\psi)(x) \quad (2.7)
\]

and

\[
|F(t, \tau, x; k, h)| \leq C\frac{|k+h|^{|m_1|} |k|^{m_2} |h|^{m_3}}{e^{c(t-\tau)|k+h|^2+e\tau(|k|^2+|h|^2)}} (M\theta)(x). \quad (2.8)
\]

Proof. We first recall that \( \psi \) satisfies: \( \hat{\psi}(\xi) \geq 0 \), \( \text{supp} \hat{\psi} \subset B_{\frac{1}{4}}(0) \) and \( \|\hat{\psi}\|_{L^1} = 1 \). For any \( h \in \mathbb{Z}^3 \) and \( |h| > 2^2 \), we have \( \left\lfloor \log_2 |h| \right\rfloor \geq 2 \). Furthermore, we get

\[
\frac{7}{8} \left\lfloor \log_2 |h| \right\rfloor + 2^{-1} < |h| < \frac{5}{4} \left\lfloor \log_2 |h| \right\rfloor + 1 - 2.
\]

Similar to Remark 2.1 (iii) and (iv), we get

\[
a_1(D)e^{t\Delta} \psi_{h}(x) = a_1(D)e^{t\Delta} (\Delta_{\left\lfloor \log_2 |h| \right\rfloor + 1}) \psi_{h}(x). \quad (2.9)
\]

Therefore, (2.7) follows immediately from (2.6), (2.9) and \( |\psi_{h}(x)| \leq |\psi(x)| \), i.e.

\[
|a_1(D)e^{t\Delta} \psi_{h}(x)| \lesssim \sum_{\sigma = 0, 1} |a_1(D)e^{t\Delta} \Delta_{\left\lfloor \log_2 |h| \right\rfloor + \sigma} \psi_{h}(x)|
\]

\[
\lesssim |h|^{m_1} e^{-c|h|^2} (M\psi)(x).
\]

Again by applying Lemma 2.2 repeatedly to \( f = a_2(D)e^{\tau \Delta} \psi_{k} \), \( f = a_3(D)e^{\tau \Delta} \psi_{h} \) and \( f = a_2(D)e^{\tau \Delta} \psi_{k} a_3(D)e^{\tau \Delta} \psi_{h} \), we can prove the desired estimate (2.8).

Lemma 2.4. Let \( \mu \geq 0 \) and \( \ell = 0, 1 \). We have the following estimates

\[
\sum_{s=1}^{r} e^{-\frac{\ell}{2}|k_s|^2} t^\mu |k_s|^{2\mu} |k_s-\ell| \lesssim \left( \sum_{s=1}^{r} e^{-\frac{\ell}{2}|k_s|^2} |k_s-\ell|^2 \right)^{\frac{\mu}{2}}. \quad (2.10)
\]

Proof. Noticing that \( e^{-\frac{\ell}{2}|k_s|^2} t^\mu |k_s|^{2\mu} \lesssim 1 \) and squaring the left side of (2.10) we get

\[
\left( \sum_{s=1}^{r} e^{-\frac{\ell}{2}|k_s|^2} |k_s-\ell|^2 \right)^2 \lesssim \sum_{s=1}^{r} e^{-\frac{\ell}{2}|k_s|^2} |k_s-\ell|^2 + \sum_{s=1}^{r} e^{-\frac{\ell}{2}|k_s|^2} |k_s-\ell| (|k_s-\ell| + \cdots + |k_1-\ell|))
\]

\[
\lesssim \sum_{s=1}^{r} e^{-\frac{\ell}{2}|k_s|^2} |k_s-\ell|^2,
\]

which concludes the desired estimates. \( \square \)
3 Analysis of ill-posedness

In this section, we will prove "norm inflation" of the NS equation in $\dot{F}_{\infty}^{-1,q}$ with $q > 2$. Following the ideas in [1], we rewrite the solution to the NS equations as a summation of the first approximation terms, the second approximation terms and remainder terms, i.e.

$$u = u_1 - u_2 + y,$$

where $u_1 = e^{t\Delta}u_0$ and $u_2 = B(u_1, u_1)$. Moreover, the remainder terms satisfy the following integral equations:

$$y = G_0 + G_1 - G_2,$$

on $(0, \infty)$ with the initial conditions $y(0) = 0$, $G_0 = B(u_2, u_1 - u_2) + B(u_1, u_2)$ and $G_1 = B(y, u_2 - u_1) + B(u_2 - u_1, y)$, $G_2 = B(y, y)$.

In the following, we will establish the a-priori estimates for $u_0$, $u_1$, $u_2$ and $y$. Precisely, In Subsection 3.1 we estimate the small upper bounds of $u_0$ and $e^{t\Delta}u_0$; In Subsection 3.2, we prove both upper bound and lower bound of $u_2$; In Subsection 3.3, we prove the upper bound of $y$; In Subsection 3.4, we complete the proof of Theorem 1.3.

3.1 Estimates for initial data and the first approximation terms

In this subsection, we will estimate $u_0$ and $e^{t\Delta}u_0$.

Lemma 3.1. For any initial $u_0$ defined in (2.1) and any $q \geq 2$, we obtain that

$$\|u_0\|_{\dot{F}_{\infty}^{-1,q}} + \|e^{t\Delta}u_0\|_{\dot{F}_{\infty}^{-1,q}} \lesssim Q r^{\frac{1}{q} - \frac{1}{2}},$$

for some absolute constant $c > 0$.

Proof. In view of the construction of $u_0$ and (2.4), we get

$$u_0 = \frac{Q}{\sqrt{T}} \sum_{\ell \in \mathbb{Z}} \Delta_\ell f_\ell, \quad e^{t\Delta}u_0 = \frac{Q}{\sqrt{T}} \sum_{\ell \in \mathbb{Z}} \Delta_\ell e^{t\Delta}f_\ell.$$

By Definition 1.1 and (2.4), we have

$$\|u_0\|_{\dot{F}_{\infty}^{-1,q}} \lesssim \frac{Q}{\sqrt{T}} \left\| \left( \sum_{s=1}^{r} |k_s|^{-q} |f_{j_s}(\cdot)|^q \right)^\frac{1}{q} \right\|_{L^\infty_x},$$

$$\|e^{t\Delta}u_0\|_{\dot{F}_{\infty}^{-1,q}} \lesssim \frac{Q}{\sqrt{T}} \left\| \left( \sum_{s=1}^{r} |k_s|^{-q} |e^{t\Delta}f_{j_s}(\cdot)|^q \right)^\frac{1}{q} \right\|_{L^\infty_x}.$$
Applying (2.5) to (2.4), recalling the properties of \( \psi(x) \) and definitions of \( \Psi_j \) \((j = 1, 2, 3, 4)\) in (H), we have the following point-wise estimates

\[
|\cos(k_s x)\Psi_1(x)| + |\cos(k'_s x)\Psi_3(x)| \lesssim (M(\nabla \psi))(x),
\]

\[
|\sin(k_s x)\Psi_2(x)| + |\cos(k'_s x)\Psi_4(x)| \lesssim (M\psi)(x),
\]

where \( M\psi \) and \( M(\nabla \psi) \) are the Hardy-Littlewood maximal functions of \( \psi \) and \( \nabla \psi \), respectively. Hence it follows from (2.4)-(2.5), (3.4) and Hardy-Littlewood theorem \cite{[12] Chapter 1, p.13} that

\[
(3.4) \lesssim \frac{Q}{\sqrt{T}} \left( \left( \sum_{s=1}^{r} \frac{1}{q_s} \|M\psi\|_{L^\infty} + \left( \sum_{s=1}^{r} |k_s|^{-q} \right) \frac{1}{q} \|M(\nabla \psi)\|_{L^\infty} \right) \right)
\]

\[
\lesssim \frac{Q}{\sqrt{T}} (r^{\frac{1}{q}} \|\psi\|_{L^\infty} + \|\nabla \psi\|_{L^\infty})
\]

\[
\lesssim \frac{Q}{\sqrt{T}} (r^{\frac{1}{q}} \|\hat{\psi}\|_{L^q_{\xi}} + \|\hat{\xi}\|_{L^q_{\xi}}).
\]

Thus (3.3) follows immediately from \( \|\hat{\psi}\|_{L^q_{\xi}} = 1 \) and \( \|\hat{\xi}\|_{L^q_{\xi}} \sim 1 \).

**Lemma 3.2.** For any \( T > 0 \), \( u_0 \) given in (2.1), we obtain that

\[
sup_{0 < t < T} t^\frac{1}{T} \|e^{t\Delta} u_0\|_{L^\infty_x} \lesssim \frac{T}{r}.
\]

**Proof.** In view of the initial data construction in (2.1), by making use of (2.4) and \( \|M\psi\|_{L^\infty} + \|M(\nabla \psi)\|_{L^\infty} \lesssim 1 \) as well as \( sup_{t > 0} \sum_{s=1}^{r} e^{-ct2^{2js}} t^{\frac{1}{T}} 2^{js} \lesssim 1 \), we have

\[
\sup_{0 < t < T} t^\frac{1}{T} \|e^{t\Delta} u_0\|_{L^\infty_x} \lesssim \frac{Q}{\sqrt{T}} \sup_{t > 0} \sum_{s=1}^{r} e^{-ct2^{2js}} t^{\frac{1}{T}} \left( 2^{js} \|M\psi\|_{L^\infty} + \|M(\nabla \psi)\|_{L^\infty} \right)
\]

\[
\lesssim \frac{T}{r}.
\]

Therefore, we obtain the desired estimate. \( \Box \)

Next we need to estimate the norm \( \|e^{t\Delta} u_0\|_{X_T} \), which is defined as follows

\[
\|e^{t\Delta} u_0\|_{X_T} = \left( \sup_{x_0 \in \mathbb{R}^3, 0 < R < T} \frac{1}{|B_R(x_0)|} \int_0^R \int_{B_R(x_0)} |e^{t\Delta} u_0(x)|^2 dt dx \right)^{\frac{1}{2}}.
\]

In particular,

\[
\|e^{t\Delta} u_0\|_{X_T} \leq \|e^{t\Delta} u_0\|_{L^2_T L^\infty_x}.
\]

**Lemma 3.3.** For any \( T > 0 \), \( u_0 \) is given in (2.1), for any \( 0 \leq N_0 \leq r \) we have

\[
\|e^{t\Delta} u_0\|_{X_T} \lesssim \frac{Q}{\sqrt{T}} (T^{\frac{1}{r}} |k_r - N_0| + \sqrt{N_0}).
\]
Proof. By the construction of $u_0$, it suffices to estimate $\frac{Q}{\sqrt{T}} e^{t\Delta} f_\ell$. Using (2.4),
\[ \|f_\ell\|_{L^\infty_T} \lesssim |k_\ell| \quad \text{for } \ell = k, \text{ and } s \in \{1, 2, \ldots, r\}, \quad \text{and } \|f_\ell\|_{L^\infty_T} = 0 \quad \text{for other } \ell, \]
we have
\[ \frac{Q}{\sqrt{T}} \left| \sum_{\ell \in \mathbb{Z}} e^{t\Delta} f_\ell \right|_{X_T} \lesssim \frac{Q}{\sqrt{T}} \left| \sum_{\ell \in \mathbb{Z}} e^{t\Delta} f_\ell \right|_{L^3_T L^\infty_x} \lesssim \frac{Q}{\sqrt{T}} \left| \sum_{s=1}^r e^{-c t^2 |k_s|^2} 2^{js}\right|_{L^2_T}. \] (3.5)

It follows from (2.10) and $|k_\ell| = 2^{js}$ that for any $N_0 \in [1, r] \cap \mathbb{N}$,
\[ \frac{Q}{\sqrt{T}} \left( \sum_{s=1}^r |k_s|^2 e^{-\frac{2\pi^2}{4 t^2 |k_s|^2}} \right)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{T}} \left( \sum_{s=1}^r \int_0^T |k_s|^2 e^{-\frac{2\pi^2}{4 t^2 |k_s|^2}} dt \right)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{T}} \left( \sum_{s=1}^r \int_0^T |k_s|^2 e^{-\frac{2\pi^2}{4 t^2 |k_s|^2}} dt \right)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{T}} (T^{\frac{1}{2}} |k_{r-N_0}| + \sqrt{N_0}), \] (3.6)
where we used \[ \int_0^T |k_s|^2 e^{-\frac{2\pi^2}{4 t^2 |k_s|^2}} dt \lesssim \min\{1, T|k_s|^2\} \quad \text{and} \quad \sum_{s=1}^r |k_s|^2 \lesssim |k_{r-N_0}|^2. \]

By checking the estimate (3.6) for the case $N_0 = 0$, we know that $\|e^{t\Delta} u_0\|_{X_T} \to 0$ as $T \to 0$. Similarly by checking the estimate (3.6) for the case $N_0 = r$ again, we observe that the best upper bound of $\|e^{t\Delta}\|_{X_T}$ is actually $cQ$, which is not good enough to bound the remainder $y$ (defined in (3.1)). Therefore, we need to analyze their contributions by using the time-step-division method introduced in [1].

Let \[ |k_\ell|^{-2} = T_0 < T_1 < T_2 < \cdots < T_\beta = |k_0|^{-2}, \] (3.7)
where $\beta = Q^3$, $T_\alpha = |k_{r_\alpha}|^{-2}$, $r_\alpha = r - \alpha Q^{-3} r$ and $\alpha = 0, 1, 2, \ldots, \beta$.

Lemma 3.4. Assume that $u_0$ satisfies (2.1). Then we have
\[ \left\| e^{t\Delta} u_0 \chi_{[T_\alpha, T_{\alpha+1}]}(t) \right\|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{T}} (1 + \sqrt{r Q^3}). \] (3.8)

Proof. By the construction of initial data $u_0$, we have
\[ e^{t\Delta} u_0 \chi_{[T_\alpha, T_{\alpha+1}]}(t) = \frac{Q}{\sqrt{T}} \sum_{\ell \in \mathbb{Z}} \chi_{[T_\alpha, T_{\alpha+1}]}(t) e^{t\Delta} f_\ell. \]

Similar to (3.5) and (3.6), we get
\[ \left\| \sum_{\ell \in \mathbb{Z}} \chi_{[T_\alpha, T_{\alpha+1}]}(t) e^{t\Delta} f_\ell \right\|_{X_{T_{\alpha+1}}} \lesssim \left\| \chi_{[T_\alpha, T_{\alpha+1}]}(t) \sum_{s=1}^r e^{-c t |k_s|^2} 2^{js} \right\|_{L^2_{T_{\alpha+1}}} . \] (3.9)
Applying \((2.11)\) to \((3.9)\), then using Fubini theorem and the following facts
\[
\int_{T_0}^{T_0+1} e^{-\frac{1}{2}|k|_s^2|k|_s^2} dt \lesssim \min \left\{ T_{a+1}|k|_s^2, 1, e^{-\frac{1}{2}T_{a+1}|k|_s^2} \right\},
\]
we get
\[
(3.9) \lesssim \left( \sum_{s=1}^{r} \min\{T_{a+1}|k|_s^2, 1, e^{-\frac{1}{2}T_{a+1}|k|_s^2}\} \right)^{\frac{1}{2}} \lesssim 1 + \sqrt{r}Q^{-3},
\]
which can conclude the desired \((3.8)\).

The following result is a consequence of Lemma \(3.4\) since \(\sum_{s=1}^{r} e^{-cT_0|k|_s^2} \lesssim 1.\)

**Corollary 3.5.** For any \(T > T_\beta = |k_0|^{-2} = 2^{-2m_0}\), we have
\[
\| e^{t\Delta} u_0 \chi_{[T_\beta,T]}(t) \|_{X_T} \lesssim Qr^{-\frac{1}{2}}.
\]

3.2 Estimates for the second approximation terms

We start this subsection by making some preliminary calculations. In order to study the bilinear form \(u_2 = B(e^{t\Delta} u_0, e^{r\Delta} u_0)\), we first split the second approximation terms \(u_2\) into
\[
u_2 = u_{20} + u_{21} + u_{22},
\]
where
\[
\begin{align*}
u_{20} &= Q^2 \frac{r}{r} \sum_{s=1}^{r} \int_{0}^{t} e^{(t-\tau)\Delta} \mathbb{P} \text{div} \left( e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_s} \right) d\tau, \\
u_{21} &= Q^2 \frac{r}{r} \sum_{s=1}^{r} \sum_{l=1}^{s-1} \int_{0}^{t} e^{(t-\tau)\Delta} \mathbb{P} \text{div} \left( e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_l} \right) d\tau, \\
u_{22} &= Q^2 \frac{r}{r} \sum_{l=1}^{r} \sum_{s=1}^{l-1} \int_{0}^{t} e^{(t-\tau)\Delta} \mathbb{P} \text{div} \left( e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_l} \right) d\tau.
\end{align*}
\]

3.2.1 Analysis of \(u_{20}\). To obtain the lower bound of \(u_{20}\), we need to calculate the exactly expressions of \(u_{20}\) and figure out which part plays the key role. From \((2.4)\) and \((H)\), we observe that
\[
e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_s} = e^{\tau \Delta} (\cos(k_s x) \Psi_1 + \cos(k_s' x) \Psi_3) \otimes e^{\tau \Delta} (\cos(k_s x) \Psi_1 + \cos(k_s' x) \Psi_3) \\
+ |k_s| e^{\tau \Delta} (\cos(k_s x) \Psi_1 + \cos(k_s' x) \Psi_3) \otimes e^{\tau \Delta} (\sin(k_s x) \Psi_4 - \sin(k_s x) \Psi_2) \\
+ |k_s| e^{\tau \Delta} (\sin(k_s x) \Psi_4 - \sin(k_s x) \Psi_2) \otimes e^{\tau \Delta} (\cos(k_s x) \Psi_1 + \cos(k_s' x) \Psi_3) \\
+ |k_s|^2 e^{\tau \Delta} (\sin(k_s' x) \Psi_4 - \sin(k_s x) \Psi_2) \otimes e^{\tau \Delta} (\sin(k_s' x) \Psi_4 - \sin(k_s x) \Psi_2) \\
:= L_{s1} + L_{s2} + L_{s3} + L_{s4}.
\]
Noticing that $L_{s1}$, $L_{s2}$ and $L_{s3}$ are lower order of $|k_s|^2$, hence it suffices to estimate

$$L_{s4} = |k|^2 e^\tau \Delta \left( \sin(k_s^x)\Psi_4 - \sin(k_s^x)\Psi_2 \right) \otimes e^\tau \Delta \left( \sin(k_s^x)\Psi_4 - \sin(k_s^x)\Psi_2 \right),$$

where we remark that $L_{s4}$ plays the key role in obtaining the best lower bound of $u_{20}$. To be more precisely, by plugging

$$\sin(k_s^x)\Psi_4 = \frac{e^{ik_s^x}x}{2i} - e^{-ik_s^x}x$$

into (3.13), we can rewrite $L_{s4}$ into the following four parts:

$$\begin{align*}
J_{s1} &= e^\tau \Delta (e^{ik_s^x}x) \Psi_4 + e^{-ik_s^x}x \Psi_2) \otimes e^\tau \Delta (e^{ik_s^x}x + e^{-ik_s^x}x)
\end{align*}$$

such that

$$L_{s4} = J_{s1} + J_{s2} + J_{s3} + J_{s4}. \tag{3.14}$$

Correspondingly, we can write that

$$\begin{align*}
&u_{200} = \frac{Q^2}{4r} \sum_{s=1}^r |k_s|^2 \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \div (J_{s3} + J_{s4}) d\tau,
&u_{201} = -\frac{Q^2}{4r} \sum_{s=1}^r |k_s|^2 \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \div (J_{s1} + J_{s2}) d\tau, \tag{3.15}
&u_{202} = \frac{Q^2}{r} \sum_{s=1}^r o(|k_s|^2) \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \div (L_{s2} + L_{s3}) d\tau.
\end{align*}$$

In what follows, we will spend a lot of effort to deal with $u_{200}$ and get the desired lower bound. For any $x_0 \in \mathbb{R}^3$, recalling the definition of $\Psi_2, \Psi_4$ in (H) and denoting $\nu := k_s^x + k_s^y = (2^3, 0, 0)$, we have

$$\begin{align*}
u_{200}(x_0, t) = C_n \frac{Q^2}{r} \sum_{s=1}^r \int_{\mathbb{R}^3} \int_0^t |k_s|^2 e^{ix_0 \xi - (t-\tau) |\xi|^2} \mathbb{P} \div \int_0^t \int_0^t \left( \xi - \eta - k_s^x \right) \left( \xi - \eta - k_s^y \right) A_s + \left( \xi - \eta - k_s^x \right) \left( \xi - \eta - k_s^y \right) B \right) d\tau d\eta d\xi, \tag{3.16}
\end{align*}$$

where $C_n$ is a positive constant depending only on dimension $n$, $\mathbb{P} \div = i \xi \cdot$, $\mathbb{P}$ is a real-valued vector function whose $jl$-th component is $\delta_{jl} - \frac{\xi_j \xi_l}{|\xi|^2}$, and

$$A_s = \begin{pmatrix} 2 & 2^4 |k_s|^2 & 0 \\ 2 |k_s|^2 & 2^7 |k_s|^4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad D_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2^3 |k_s|^3 \\ 1 & 2^3 |k_s|^3 & 0 \end{pmatrix}. \tag{3.17}$$
Next we prove the lower bound of $u_{200}$ in critical space $\dot{F}_{\infty}^{-1,q} \to L^\infty$ and the upper bound of $u_{200}$ in $BMO^{-1}$. Particularly, the lower bound of $u_{200}$ plays a crucial role in the proof of norm inflation. To obtain such bounds, we will use Fourier analysis methods. Due to the vector-valued nature of velocity field and the divergence free condition, we not only need to explore each of the three components but also need to analyze the action of Leray projection operator $P$. Furthermore, we need to figure out which of the three components is the largest one that produces norm inflation.

**Lemma 3.6.** (Lower/upper bound) For $|k_1|^2 \ll T \ll 1$ and $1 \leq q \leq \infty$, we have

\[
\|u_{200}(T)\|_{\dot{F}_{\infty}^{-1,q}} \gtrsim Q^2,
\]

\[
\sup_{0 < t < T} \|u_{200}\|_{L^\infty} + \|u_{200}\|_{X_T} \lesssim T^{1/2} Q^2.
\]

**Proof.** We divide the proof of (3.18) and (3.19) into the following three steps.

**Step 1.** Recall that $\nu := k_s + k'_s = (2^{3/2}, 0, 0)$. From assumptions (H) and (3.14)–(3.16) we have

\[
\text{supp} \hat{u}_{200} \subset B_{1/2}(\nu) \cup B_{1/2}(-\nu) \cup B_{1/2}(0) \subset B_9(0).
\]

Hence for any $t > 0$, we have $u_{200}(:t) \in C^2(\mathbb{R}^3)$. Furthermore, by $\dot{F}_{\infty}^{-1,q} \hookrightarrow \dot{B}_{\infty}^{-1,\infty}$,

\[
\|u_{200}\|_{\dot{F}_{\infty}^{-1,q}} \gtrsim \|u_{200}\|_{\dot{B}_{\infty}^{-1,\infty}} \gtrsim \|u_{200}\|_{L^\infty}.
\]

follows from

\[
\|u_{200}\|_{L^\infty} \lesssim \sum_{j \leq 3} 2^j 2^{-j} \|\Delta_j u_{200}\|_{L^\infty} \lesssim \sup_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j u_{200}\|_{L^\infty} = \|u_{200}\|_{\dot{B}_{\infty}^{-1,\infty}}.
\]

We refer readers to [9, Chapter 5] to see more information about the equivalence of the definition of Besov spaces.

**Step 2.** Considering the arguments in Step 1, it suffices to prove that

\[
u_{200}(x_0, t) \gtrsim Q^2
\]

at some point $x_0$, for instance, here we chose $x_0 = (\pi/4, 0, 0)$. Once we prove (3.22), then combining $u_{200} \in C^2(\mathbb{R}^3)$ with (3.21), we obtain that

\[
\|u_{200}\|_{\dot{F}_{\nu_0}^{-\alpha, r}} \gtrsim \|u_{200}\|_{L^\infty} \gtrsim Q^2,
\]

which is the desired (3.18).

To prove (3.22), we first recall from (3.12) that $u_{20}$ is real-valued which shows that the imaginary parts of $u_{200}$, $u_{201}$ and $u_{202}$ cancels. Therefore, it suffices to bound the real part of $u_{200}(x_0, t)$ with $x_0 = (-\pi/4, 0, 0)$. By (3.16), we set

\[
\Gamma_s = C_n |k_s|^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_0^t \left(\sin \frac{\pi \xi_1}{2}\right) e^{-(t-\tau)|\xi|^2 - \tau(|\xi-\eta|^2 + |\eta|^2) \frac{1}{\nu_s}(\xi)}
\]

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\[
\xi \cdot \left( \hat{\psi}(\xi - \eta - k_s') \hat{\psi}(\eta + k_s') A_s + \hat{\psi}(\xi - \eta - k_s) \hat{\psi}(\eta + k_s) B \\
+ \hat{\psi}(\xi - \eta - k_s') \hat{\psi}(\eta - k_s) D_s + \hat{\psi}(\xi - \eta + k_s') \hat{\psi}(\eta + k_s) D_s \right) d\tau d\eta d\xi. \tag{3.23}
\]

It is clear that
\[
(\text{Re} u_{200})(x_0, t) = \frac{Q^2}{4r} \sum_{s=1}^{r} \Gamma_s.
\]

It is also clear that the last two terms in (3.23) are identical since substituting \((\xi, \eta)\) by \((-\xi, -\eta)\) yields the same results. Furthermore, \(A_s \to A\) and \(D_s \to D\) as \(m_0 \to \infty\), where
\[
A = \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \tag{3.24}
\]

In order to compute the third component of \(\Gamma_s\), for any \(\frac{1}{|k_s|} \ll t \ll 1\), recalling that \(\nu = (2^3, 0, 0)\), then by changing variables, we obtain that as \(m_0 \to \infty\),
\[
\int_{\mathbb{R}^3} \int_{0}^{t} |k_s|^2 \sin \frac{\pi \xi}{2} \frac{\overline{\mathcal{P}(\xi)} \xi \cdot A_s}{e^{\pi |\xi|^2 + \tau (|\xi - \eta|^2 + |\eta|^2 - |\xi|^2)}} \hat{\psi}(\xi - \eta - k_s') \hat{\psi}(\eta + k_s') d\tau d\eta d\xi \\
\longrightarrow \int_{\mathbb{R}^3} \int_{0}^{t} \frac{\sin \frac{\pi \xi}{2} \xi \cdot A_s}{e^{\pi |\xi|^2} |\xi|^2} \hat{\psi}(\xi - \eta - k_s') \hat{\psi}(\eta + k_s') d\xi \eta d\xi; \tag{3.25}
\]
\[
\int_{\mathbb{R}^3} \int_{0}^{t} |k_s|^2 \sin \frac{\pi \xi}{2} \frac{\overline{\mathcal{P}(\xi)} \xi \cdot B}{e^{\pi |\xi|^2 + \tau (|\xi - \eta|^2 + |\eta|^2 - |\xi|^2)}} \hat{\psi}(\xi - \eta - k_s) \hat{\psi}(\eta + k_s) d\tau d\eta d\xi \\
\longrightarrow \int_{\mathbb{R}^3} \int_{0}^{t} \frac{\sin \frac{\pi \xi}{2} \xi \cdot B}{e^{\pi |\xi|^2} |\xi|^2} \hat{\psi}(\xi - \eta - k_s) \hat{\psi}(\eta + k_s) d\xi \eta d\xi; \tag{3.26}
\]
\[
\int_{\mathbb{R}^3} \int_{0}^{t} |k_s|^2 \sin \frac{\pi \xi}{2} \frac{\overline{\mathcal{P}(\xi)} \xi \cdot 2D_s}{e^{\pi |\xi|^2 + \tau (|\xi - \eta|^2 + |\eta|^2 - |\xi|^2)}} \hat{\psi}(\xi - \eta - k_s') \hat{\psi}(\eta - k_s) d\tau d\eta d\xi \\
\longrightarrow \int_{\mathbb{R}^3} \int_{0}^{t} \frac{\sin \frac{\pi (\xi_1 + \xi_2^2)}{2}}{e^{\pi |\xi + \nu|^2} |\xi + \nu|^2} \hat{\psi}(\xi - \eta) \hat{\psi}(\eta) d\xi \eta. \tag{3.27}
\]

Plugging the above three limits and \(\| \hat{\psi} \|_{L_t^1} = 1\) as well as \(\text{supp} \hat{\psi} \subset B_{\frac{1}{2}}(0)\) into (3.23), for any \(2^{-2m_0} \ll t \leq 2^{-6}\) and any \(k_s\), if \(m_0\) is large enough, then we get \(\Gamma_s \sim 1\). As a consequence,
\[
\| u_{200}(\cdot, t) \|_{L_x^\infty} \geq \| (\text{Re} u_{200})(\cdot, t) \|_{L_x^\infty} \geq \frac{Q^2}{r} \sum_{s=1}^{r} \Gamma_s \sim Q^2.
\]

**Step 3.** It remains to prove \(\| u_{200} \|_{X_T} \lesssim Q^2 T^\frac{1}{4} \) and \(\sup_{0 \leq t < T} t^\frac{1}{4} \| u_{20,0} \|_{L_x^\infty} \). Using Hausdorff–Young’s inequality, we have
\[
\| u_{200} \|_{L_t^\frac{1}{2} L_x^\infty} + \sup_{0 \leq t < T} t^\frac{1}{2} \| u_{200} \|_{L_t^\infty} \lesssim \| \tilde{u}_{200} \|_{L_t^\frac{1}{4} L_x^1} + T^\frac{1}{2} \| \tilde{u}_{200} \|_{L_t^\frac{1}{4}}.
\]

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By checking the proof of Step 2, it is easy to show $\sup_{t>0} |\hat{u}_{2,0}(\xi, t)| \lesssim Q^2 (\hat{\psi} * \hat{\psi})(\xi)$. Consequently, applying Young’s inequality to $|\hat{u}_{2,0}|$ yields

$$\|\hat{u}_{2,0}\|_{L^1_\xi} \lesssim Q^2 \|\hat{\psi}\|_{L^1_\xi} \|\hat{\psi}\|_{L^\infty} \lesssim Q^2$$

and $\|\hat{u}_{2,0}\|_{L^2_\xi L^1_\xi} \lesssim T^{\frac{1}{2}} Q^2$.

Hence we finish the whole proof.

Now we prove the following estimates for $u_{201}$ and $u_{202}$.

**Lemma 3.7.** For any $q > 2$, $0 < t < T \ll 1$ and large enough $|k_s|$ and $r$, we have

$$\|u_{201}\|_{L^\infty_T \dot{F}^{1,q}_\infty} + \sup_{0 < t < T} t^{\frac{3}{4}} \|u_{201}\|_{L^\infty_t} + \|u_{201}\|_{X_T} \lesssim \frac{Q^2}{\sqrt{r}},$$

(3.28)

$$\|u_{202}\|_{L^\infty_T \dot{F}^{1,q}_\infty} + \sup_{0 < t < T} t^{\frac{3}{2}} \|u_{202}\|_{L^\infty_t} + \|u_{202}\|_{X_T} \lesssim o(Q^2).$$

(3.29)

**Proof.** First deal with the norm $\|u_{201}\|_{L^\infty_T \dot{F}^{1,q}_\infty}$. Noticing from (2.2) and (2.3) that

$$\sup \hat{I}_{s1} \subset B_\frac{r}{2}(2k_s') \cup B_\frac{r}{2}(-2k_s) \cup B_\frac{r}{2}(k_s'-k_s),$$

$$\sup \hat{I}_{s2} \subset B_\frac{r}{2}(-2k_s') \cup B_\frac{r}{2}(2k_s) \cup B_\frac{r}{2}(k_s'-k_s').$$

Hence by $L^\infty \hookrightarrow BMO$, $BMO^{-1} \hookrightarrow \dot{F}^{1,q}_\infty$ and isomorphism as well as boundedness of $\mathbb{P}$ in homogeneous Tribel-Lizorkin spaces, for any $0 < t < T$ we get

$$\|u_{201}(t)\|_{\dot{F}^{1,q}_\infty} \lesssim \frac{Q^2}{r} \left( \sum_{s=1}^r (k_s)^2 \right) \int_0^t e^{(t-\tau)\Delta} (I_{s1} + I_{s2}) d\tau \|_{BMO}$$

$$\lesssim \frac{Q^2}{r} \left( \sum_{s=1}^r (k_s)^2 \right) \int_0^t e^{(t-\tau)\Delta} (I_{s1} + I_{s2}) d\tau \|_{L^\infty_t}.$$  (3.30)

For any $0 \leq \tau \leq t$, by applying (2.3) of Lemma 2.3 to (3.30) we have

$$|e^{(t-\tau)\Delta} (I_{s1} + I_{s2})(x)| \lesssim e^{-ct|k_s|^2} (M\theta)(x) \quad \text{where } \theta = (M\psi)^2.$$  (3.31)

Plugging (3.31), $\|M\theta\|_{L^\infty_x} \lesssim 1$ and $\sup_{t>0} \sum_{s=1}^r t|k_s|^2 e^{-ct|k_s|^2} \lesssim 1$ into (3.30) we have

$$\|u_{201}\|_{\dot{F}^{1,q}_\infty} \lesssim \frac{Q^2}{r} \sum_{s=1}^r t|k_s|^2 e^{-ct|k_s|^2} \|M\theta\|_{L^\infty_x} \lesssim \frac{Q^2}{r}.$$  (3.32)

To estimate $t^{\frac{3}{2}} \|u_{2,1}\|_{L^\infty_x}$, by using (3.31) and $\sup_{t>0} \sum_{s=1}^r t^{\frac{3}{2}}|k_s|^3 e^{-ct|k_s|^2} \lesssim 1$ we get

$$t^{\frac{3}{2}} \|u_{2,1}\|_{L^\infty_x} \lesssim \frac{Q^2}{r} \sum_{s=1}^r t^{\frac{3}{2}}|k_s|^3 e^{-ct|k_s|^2} \|M\theta\|_{L^\infty_x} \lesssim \frac{Q^2}{r}.$$  (3.33)
To estimate \( \|u_{2,1}\|_{X_T} \), by using (3.31) and (2.10) with \( \mu = 1 \) and \( \ell = 0 \) we get

\[
\|u_{2,1}\|_{X_T} \lesssim \|u_{2,1}\|_{L_T^2 L_x^\infty} \lesssim \frac{Q^2}{r} \| \sum_{s=1}^r t |k_s|^3 e^{-ct|k_s|^2} \|_{L_T^2} \| M \theta \|_{L_x^\infty} \\
\lesssim \frac{Q^2}{r} \left( \sum_{s=1}^r |k_s|^2 e^{-ct|k_s|^2} \right)^{\frac{1}{2}} \|_{L_T^2} \lesssim \frac{Q^2}{\sqrt{r}}. \tag{3.34}
\]

Combining (3.30) and (3.32)–(3.34), we finish the proof of (3.28).

To estimate \( u_{202} \), we recall that similar to (3.13),

\[
u_{202} \sim \frac{Q^2}{r} \sum_{s=1}^r a(|k_s|^2) \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \text{div}(L_{1s} + L_{2s} + L_{3s}) d\tau,
\]

where \( a(|k_s|^2) \) is lower order term of \( |k_s|^2 \) satisfying \( 0 \leq \frac{a(|k_s|^2)}{|k_s|^2} \leq 2^{-m_0} \). Similar to \( L_{4s} \), for large enough \( |k_0| \) and \( r \), we can prove the desired estimate. \( \square \)

### 3.2.2 Analysis of \( u_{21} \) and \( u_{22} \)

**Lemma 3.8.** For any \( q > 2 \) and \( i = 1, 2 \), we have

\[
\|u_{2i}\|_{L_T^\infty \dot{F}_x^{-1, q}} + \sup_{0 < t < T} t^{\frac{1}{2}} \|u_{2i}\|_{L_x^\infty} + \|u_{2i}\|_{X_T} \lesssim Q^2 r^{-1}. \tag{3.35}
\]

**Proof.** It suffices to estimate \( u_{21} \). To estimate \( \|u_{21}\|_{L_T^\infty \dot{F}_x^{-1, q}} \), recalling from the support of \( \hat{f}_{js} \) and \( |k_l| = j_l \neq j_s = |k_s| \), we get

\[
\text{supp} \mathcal{F} (e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_l}) \subset B_2^4 (\pm k_s \pm k_l) \cup B_2^4 (\pm k_s \pm k'_l).
\]

Note that when \( s > \ell, |\pm k'_s \pm k_l| \sim |k_s| \sim |\pm k_s \pm k'_l| \), thus the support of \( \hat{u}_{21} \) is far away from origin which ensures that \( \mathbb{P} \) is well-defined and no singularity arguments for \( \mathbb{P} \) are involved. Moreover, \( \mathbb{P} \) is a bounded operator in \( BMO \). By \( L_T^\infty \rightarrow BMO, \ BMO^{-1} \rightarrow \dot{F}_x^{-1, q} \), isomorphism, Lemma 2.3 and (H), we get

\[
\|u_{21}\|_{L_T^\infty \dot{F}_x^{-1, q}} \lesssim \frac{Q^2}{r} \left\| \sum_{s=1}^r \sum_{l=1}^{s-1} \int_0^t e^{(t-\tau)\Delta} (e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_l}) d\tau \right\|_{L_T^\infty BMO} \\
\lesssim \frac{Q^2}{r} \left\| \sum_{s=1}^r \sum_{l=1}^{s-1} \int_0^t e^{(t-\tau)\Delta} (e^{\tau \Delta} f_{j_s} \otimes e^{\tau \Delta} f_{j_l}) d\tau \right\|_{L_T^\infty L_x^\infty} \\
\lesssim \frac{Q^2}{r} \sup_{t > 0} \left( \sum_{s=1}^r \sum_{l=1}^{s-1} \int_0^t e^{-ct|k_s|^2} d\tau \right) \\
\|M((|k_s|M\psi + M(\nabla\psi)((|k_l|M\psi + M(\nabla\psi)))\|_{L_x^\infty}
\]

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\[
\lesssim \frac{Q^2}{r} \sup_{t > 0} \sum_{s=1}^{r} \sum_{l=1}^{s-1} e^{-ct|k_s|^2} |k_s||k_l| \\
\lesssim \frac{Q^2}{r},
\]  
(3.36)

where in the fourth inequalities we used the following simple fact:

\[
\left\| M((|k_s| M\psi + M(\nabla\psi)(|k_l| M\psi + M(\nabla\psi))) \right\|_{L^\infty_x} \lesssim |k_s||k_l|. 
\]  
(3.37)

Next to the norm of \( \|u_{21}\|_{X_T} \). By using Lemmas 2.3 and 2.4, then we have

\[
\|u_{21}\|_{X_T} \lesssim \|u_{21}\|_{L^2_T L^\infty_x} \lesssim \frac{Q^2}{r} \left\| \sum_{s=1}^{r} \sum_{l=1}^{s-1} |k_s|^2 |k_l| e^{-ct|k_s|^2} \right\|_{L^2_T} \\
\lesssim \frac{Q^2}{r} \left( \sum_{s=1}^{r} |k_s-1|^2 e^{-\frac{t}{2}|k_s|^2} \right)^{\frac{1}{2}} \| \lesssim \frac{Q^2}{r}. 
\]  
(3.38)

At last, we estimate \( t^\frac{1}{2}\|u_{21}\|_{L^\infty_x} \). By using Lemmas 2.3 and 2.4, we have

\[
\sup_{0 < t < T} t^\frac{1}{2}\|u_{21}\|_{L^\infty_x} \lesssim \frac{Q^2}{r} \sum_{s=1}^{r} \sum_{l=1}^{s-1} t^\frac{3}{2} |k_s|^2 |k_l| e^{-ct|k_s|^2} \lesssim \frac{Q^2}{r}. 
\]  
(3.39)

Thus combining (3.36), (3.38) and (3.39), we finish the proof of (3.35).

3.3 Estimates of remainder \( y \)

In this subsection, we use iteration arguments to prove the \textit{a-priori} estimate for remainder \( y \). Recall that \( y \) satisfy the integral equations (3.2), i.e.

\[
y = G_0 + G_1 - G_2
\]

with initial condition \( y|_{t=0} = 0 \), \( G_0 = B(u_2, u_1) + B(u_1, u_2) - B(u_2, u_2) \) and

\[
G_1 = B(y, u_2 - u_1) + B(u_2 - u_1, y), \quad G_2 = B(y, y).
\]

From Lemma 3.4 we observe that in order to obtain more accurate decay estimate for \( y \), it suffices to split \( u_1 \), \( u_2 \) and \( y \) into two terms, e.g.

\[
\begin{align*}
\begin{cases}
u_1 = u_1 \chi_{[0,T_\alpha]}(t) + u_1 \chi_{[T_\alpha, T_{\alpha+1}]}(t), \\
u_2 = u_2 \chi_{[0,T_\alpha]}(t) + u_2 \chi_{[T_\alpha, T_{\alpha+1}]}(t), \\
y = y \chi_{[0,T_\alpha]}(t) + y \chi_{[T_\alpha, T_{\alpha+1}]}(t),
\end{cases}
\end{align*}
\]

Plugging the above decompositions of \( u_1, u_2 \) and \( y \) into \( G_0, G_1 \) and \( G_2 \), we have the following iteration rules which play an important role in controlling \( y \).
Lemma 3.9. If \( y \) solves system (3.2), then for any \( \alpha = 0, 1, 2, \cdots, Q^3 \) and for large enough \( r \) and \( |k_0| \) we have
\[
\|y\|_{X_{T_{\alpha+1}}} \lesssim Q^{\alpha+3}(r^{-\frac{1}{2}} + |k_0|^{-1}).
\] (3.40)

Moreover, for any \( T > |k_0|^{-2} \), we have
\[
\|y\|_{X_T} \lesssim Q^3(r^{-\frac{1}{2}} + T^{\frac{1}{2}}) + Q^{Q^3+3}(r^{-\frac{1}{2}} + |k_0|^{-1}).
\] (3.41)

Proof. Applying 1.11 to (3.2), we have the following bilinear estimates:
\[
\|y\|_{X_{T_{\alpha+1}}} \lesssim \|u_2\|_{X_{T_{\alpha+1}}} (\|u_1\|_{X_{T_{\alpha+1}}} + \|u_2\|_{X_{T_{\alpha+1}}}) + (\|u_1\|_{X_{T_{\alpha+1}}} + \|u_2\|_{X_{T_{\alpha+1}}})\|y\|_{X_{T_0}} + \|y\|_{X_{T_{\alpha+1}}},
\] (3.42)

where in the second inequality we used
\[
\|y\|_{X_{T_0}} + \|y\|_{X_{T_{\alpha+1}}} \lesssim \|y\|_{X_{T_{\alpha+1}}} \text{ and } \|u_2\|_{X_{T_0}} + \|u_2\|_{X_{T_{\alpha+1}}} \lesssim \|u_2\|_{X_{T_{\alpha+1}}}.
\]

Recalling that for any \( 1 \leq \alpha \leq \beta, T_\alpha \leq T_\beta \). Then from Lemmas 3.2–3.6 we get
\[
\|u_2\|_{X_{T_{\alpha+1}}} \lesssim Q^3(r^{-\frac{1}{2}} + |k_0|^{-1}) + Q\|y\|_{X_{T_0}} + Q^{-\frac{1}{2}}\|y\|_{X_{T_{\alpha+1}}} + \|y\|_{X_{T_{\alpha+1}}},
\] (3.43)

Plugging (3.43) in (3.42), and assuming that \( r > Q^{10}, T_\beta = |k_0|^{-2} < Q^{-5} \), we have
\[
\|y\|_{X_{T_{\alpha+1}}} \lesssim Q^3(r^{-\frac{1}{2}} + |k_0|^{-1}) + Q\|y\|_{X_{T_0}} + Q^{-\frac{1}{2}}\|y\|_{X_{T_{\alpha+1}}} + \|y\|_{X_{T_{\alpha+1}}}. (3.44)
\]

Similarly, when \( T > T_\beta \), by splitting \([0, T]\) into \([0, T_\beta]\) and \([T_\beta, T]\), then using Corollary 3.5 and (3.42)–(3.44), we get
\[
\|y\|_{X_{T_0}} \lesssim Q^3(r^{-\frac{1}{2}} + T^{\frac{1}{2}}) + Q\|y\|_{X_{T_\beta}} + Q^{-\frac{1}{2}}\|y\|_{X_{T}} + \|y\|_{X_{T}}.
\] (3.45)

Lemma 3.2 ensures that \( \|y\|_{X_{T_0}} \) can be small since \( T_0 = |k_r|^{-2} \) and
\[
\|u_1\|_{X_{T_0}} \lesssim \frac{Q}{\sqrt{T}_0} T_0^{\frac{1}{2}} |k_r| \lesssim \frac{Q}{\sqrt{T}}
\]

which can be arbitrarily small if \( r \) is large enough. Thus iteration argument can be applied to (3.44)–(3.45). Iterating (3.44) and (3.45) give the desired results. \( \square \)

Making use of (1.11) and Lemma 3.9, we obtain the following estimate.

Corollary 3.10. For any \( q > 2 \), sufficiently large \( r \) and \( |k_0| \) such that \( r \gg Q^2 Q^{3+4}, |k_0| \gg Q^{-Q^2-2} \) and \( |k_0|^{-2} < T \ll Q^{-2} \), we have
\[
\|y(T)\|_{\tilde{F}_{\infty}^{-1, q}} \ll Q^2.
\] (3.46)
Proof. From (3.2), we notice that
\[ y(T) = G_0(T) + G_1(T) - G_2(T) \]
and \( G_i(T) \) are several bilinear terms. By (1.11) and (1.12), we obtain that
\[
\| y(T) \|_{F^{-1,q}_\infty} \lesssim \| y(T) \|_{F^{-\alpha,2}_\infty} \lesssim \| y \|_{L^\infty T F^{-\alpha,2}_\infty} \lesssim \| y \|_{X T} \|
\]
and
\[
\| y(T) \|_{F^{-1,q}_\infty} \lesssim \| y \|_{X T} (\| u_1 \|_{X_T} + \| u_2 \|_{X_T}) + \| y \|_{X_T} (\| u_1 \|_{X_T} + \| u_2 \|_{X_T}) + \| y \|_{X_T}^2.
\]

Applying Lemmas 3.2, 3.7–3.9 to the above inequality, we have
\[
\| y(T) \|_{F^{-1,q}_\infty} \lesssim (Q^2 r^{-\frac{1}{3}} + Q^2 T^{-\frac{1}{3}}) (Q + Q^2 r^{-\frac{1}{3}} + Q^3 T^{-\frac{1}{3}})
\]
\[
+ (Q + Q^2 r^{-\frac{1}{3}} + Q^3 T^{-\frac{1}{3}}) \left( Q^3 (r^{-\frac{1}{3}} + T^{-\frac{1}{3}}) + Q^3 + (r^{-\frac{1}{3}} + |k_0|^{-1}) \right)
\]
\[
+ \left( Q^3 (r^{-\frac{1}{3}} + T^{-\frac{1}{3}}) + Q^3 + (r^{-\frac{1}{3}} + |k_0|^{-1}) \right)^2 \lesssim Q^2.
\]

Hence we prove the desired result.

3.4 Proof of Theorem 1.3

In this subsection, combining the results proved in Subsections 3.1–3.4, we are ready to prove the norm inflation of the Navier-Stokes equations.

Proof of Theorem 1.3 Combining the equalities (3.1) and (3.11), the estimates (3.3), (3.28), (3.35), (3.18) and (3.46), we have
\[
\| u(T) \|_{F^{-1,q}_\infty} \geq \| u_{200}(T) \|_{F^{-1,q}_\infty} - \| u_1(T) \|_{F^{-1,q}_\infty}
\]
\[
- \sum_{\ell=1}^{2} (\| u_{20\ell}(T) \|_{F^{-1,q}_\infty} + \| u_{2\ell}(T) \|_{F^{-1,q}_\infty}) - \| y(T) \|_{F^{-1,q}_\infty}
\]
\[
\gtrsim Q^2 \left( 1 - Q^{-1} r^{-\frac{1}{3}} - r^{-\frac{1}{2}} - o(1) \right) \gtrsim Q^2,
\]
where \( 0 < o(1) \ll \frac{1}{2}, r \gg Q^{2Q^3} \) and \( |k_0|^{-2} < T \ll Q^{-2} \). Hence we finish the proof.

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