YET ANOTHER CONSTRUCTION OF THE CENTRAL EXTENSION OF THE LOOP GROUP.

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Abstract. We give a characterisation of central extensions of a Lie group $G$ by $C^\times$ in terms of a differential two-form on $G$ and a differential one-form on $G \times G$. This is applied to the case of the central extension of the loop group.

1. Introduction

Let $G$ and $A$ be groups. A central extension of $G$ by $A$ is another group $\hat{G}$ and a homomorphism $\pi: \hat{G} \to G$ whose kernel is isomorphic to $A$ and in the center of $\hat{G}$. Note that because $A$ is in the center of $\hat{G}$ it is necessarily abelian. We will be interested ultimately in the case that $G = \Omega(K)$ the loop group of all smooth maps from the circle $S^1$ to a Lie group $K$ with pointwise multiplication but the theory developed applies to any Lie group $G$.

2. Central extension of groups

Consider first the case when $G$ is just a group and ignore questions of continuity or differentiability. In this case we can choose a section of the map $\pi$. That is a map $s: G \to \hat{G}$ such that $\pi(s(g)) = g$ for every $g \in G$. From this section we can construct a bijection

$$\phi: A \times G \to \hat{G}$$

by $\phi(g, a) = \iota(a)s(g)$ where $\iota: A \to \hat{G}$ is the identification of $A$ with the kernel of $\pi$. So we know that as a set $\hat{G}$ is just the product $A \times G$. However as a group $\hat{G}$ is not generally a product. To see what it is note that $\pi(s(g)s(h)) = \pi(s(g))\pi(s(h)) = gh = \pi(s(gh))$ so that $s(g)s(h) = c(g,h)s(gh)$ where $c: G \times G \to A$. The bijection $\phi: A \times G \to \hat{G}$ induces a product on $A \times G$ for which $\phi$ is a homomorphism. To calculate this product we note that

$$\phi(a, g)\phi(b, h) = \iota(a)s(g)\iota(b)s(h) = \iota(ab)s(g)s(h) = \iota(ab)c(g,h)s(gh).$$

Hence the product on $A \times G$ is given by $(a, g) \star (b, h) = (abc(g,h)gh)$ and the map $\phi$ is a group isomorphism between $\hat{G}$ and $A \times G$ with this product.

Notice that if we choose a different section $\hat{s}$ then $\hat{s} = sh$ were $h: G \to A$.

It is straightforward to check that if we pick any $c: G \times G \to A$ and define a product on $A \times G$ by $(a, g) \star (b, h) = (abc(g,h)gh)$ then this is an associative product.

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if and only if $c$ satisfies the cocycle condition
$$c(g, h) c(gh, k) = c(g, hk) c(h, k)$$
for all $g$, $h$ and $k$ in $G$.

If we choose a different section $\tilde{s}$ then we must have $\tilde{s} = s d$ for some $d: G \to A$. If $\tilde{c}$ is the cocycle determined by $\tilde{s}$ then a calculation shows that
$$c(g, h) = \tilde{c}(gh) d(g) d(h)^{-1} d(h)^{-1}.$$  

We have now essentially shown that all central extensions of $G$ by $A$ are determined by cocycles $c$ modulo identifying two that satisfy the condition $[\tilde{]}$. Let us recast this result in a form that we will see again in this talk.

Define maps $d_i: G^{p+1} \to G^p$ by
$$d_i(g_1, \ldots, g_{p+1}) = \begin{cases} (g_2, \ldots, g_{p+1}), & i = 0, \\ (g_1, \ldots, g_{i-1} g_i g_{i+1}, \ldots, g_{p+1}), & 1 \leq i \leq p-1, \\ (g_1, \ldots, g_p), & i = p. \end{cases}$$

If $M^p(G; A) = \text{Map}(G^p, A)$ then we define $\delta: M^p(G; A) \to M^{p-1}(G; A)$ by $\delta(c) = (c \circ d_1)(c \circ d_2)^{-1}(c \circ d_3) \ldots$. This satisfies $\delta^2 = 1$ and defines a complex
$$M^0(G; A) \xrightarrow{\delta} M^1(G; A) \xrightarrow{\delta} M^2(G; A) \xrightarrow{\delta} \ldots$$

The cocycle condition can be rewritten as $\delta(c) = 1$ and the condition that two cocycles give rise to the same central extension is that $c = \tilde{c} \delta(d)$. If we define
$$H^p(G; A) = \frac{\text{kernel } \delta: M^p(G; A) \to M^{p-1}(G; A)}{\text{image } \delta: M^{p+1}(G; A) \to M^p(G; A)}$$
then we have shown that central extensions of $G$ by $A$ are classified by $H^2(G; A)$.

3. Central extensions of Lie groups

In the case that $G$ is a topological or Lie group it is well-known that there are interesting central extensions for which no continuous or differentiable section exists. For example consider the central extension
$$\mathbb{Z}_2 \to SU(2) = \text{Spin}(3) \to SO(3)$$
of the three dimensional orthogonal group $SO(3)$ by its double cover $\text{Spin}(3)$. Here $SU(2)$ is known to be the three sphere but if a section existed then we would have $SU(2)$ homeomorphic to $\mathbb{Z}_2 \times SO(3)$ and hence disconnected.

From now on we will concentrate on the case when $A = \mathbb{C}^\times$. Then $\hat{G} \to G$ can be thought of as a $\mathbb{C}^\times$ principal bundle and a section will only exist if this bundle is trivial. The structure of the central extension as a $\mathbb{C}^\times$ bundle is important in what follows so we digress to discuss them in more detail.

3.1. $\mathbb{C}^\times$ bundles. Let $P \to X$ be a $\mathbb{C}^\times$ bundle over a manifold $X$. We denote the fibre of $P$ over $x \in X$ by $P_x$. Recall $[\text{2}]$ that if $P$ is a $\mathbb{C}^\times$ bundle over a manifold $X$ we can define the dual bundle $P^*$ as the same space $P$ but with the action $p^*g = (pg^{-1})^*$ and, that if $Q$ is another such bundle, we can define the product bundle $P \otimes Q$ by $(P \otimes Q)_x = (P_x \times Q_x)/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by $(p, q)w = (pw, qw^{-1})$. We denote an element of $P \otimes Q$ by $p \otimes q$ with the understanding that $(pw) \otimes q = p \otimes (qw) = (p \otimes q)w$ for $w \in \mathbb{C}^\times$. It is straightforward to check that $P \otimes P^*$ is canonically trivialised by the section $x \mapsto p \otimes p^*$ where $p$ is any point in $P_x$. 

If $P$ and $Q$ are $\mathbb{C}^\times$ bundles on $X$ with connections $\mu_P$ and $\mu_Q$ then $P \otimes Q$ has an induced connection we denote by $\mu_P \otimes \mu_Q$. The curvature of this connection is $R_P + R_Q$ where $R_P$ and $R_Q$ are the curvatures of $\mu_P$ and $\mu_Q$ respectively. The bundle $P^*$ has an induced connection whose curvature is $-R_P$.

Recall the maps $d_i : G^p \to G^{p-1}$ defined by $[3]$. If $P \to G^p$ is a $\mathbb{C}^\times$ bundle then we can define a $\mathbb{C}^\delta$ bundle over $G^{p+1}$ denoted $\delta(P)$ by

$$\delta(P) = \pi_1^{-1}(P) \otimes \pi_2^{-1}(P)^* \otimes \pi_3^{-1}(P) \otimes \ldots.$$  

If $s$ is a section of $P$ then it defines $\delta(s)$ a section of $\delta(P)$ and if $\mu$ is a connection on $P$ with curvature $R$ it defines a connection on $\delta(P)$ which we denote by $\delta(\mu)$. To define the curvature of $\delta(\mu)$ let us denote by $\Omega^q(G^p)$ the space of all differentiable $q$ forms on $G^p$. Then we define a map

$$\delta : \Omega^q(G^p) \to \Omega^q(G^{p+1})$$

by $\delta = \sum_{i=0}^p d_i^*$, the alternating sum of pull-backs by the various maps $d_i : G^{p+1} \to G^p$. Then the curvature of $\delta(\mu)$ is $\delta(R)$. If we consider $\delta(\delta(P))$ it is a product of factors and every factor occurs with its dual so $\delta(\delta(P))$ is canonically trivial. If $s$ is a section of $P$ then under this identification $\delta\delta(s) = 1$ and moreover if $\mu$ is a connection on $P$ then $\delta\delta(\mu)$ is the flat connection on $\delta\delta(P)$ with respect to $\delta(\delta(s))$.

### 4. Central Extensions

Let $G$ be a Lie group and consider a central extension

$$\mathbb{C}^\times \to \hat{G} \xrightarrow{\pi} G.$$  

Following Brylinski and McLaughlin [2] we think of this as a $\mathbb{C}^\times$ bundle $\hat{G} \to G$ with a product $M : \hat{G} \times \hat{G} \to \hat{G}$ covering the product $m = d_1 : G \times G \to G$.

Because this is a central extension we must have that $M(pz, qw) = M(p, q)zw$ for any $p, q \in P$ and $z, w \in \mathbb{C}^\times$. This means we have a section $s$ of $\delta(P)$ given by

$$s(g, h) = p \otimes M(p, q) \otimes q$$

for any $p \in P_g$ and $q \in P_h$. This is well-defined as $pw \otimes M(pw, qz) \otimes qz = pw \otimes M(p, q)(wz)^{-1} \otimes qz = p \otimes M(p, q) \otimes q$. Conversely any such section gives rise to an $M$.

Of course we need an associative product and it can be shown that $M$ being associative is equivalent to $\delta(s) = 1$. To actually make $\hat{G}$ into a group we need more than multiplication we need an identity $\hat{e} \in \hat{G}$ and an inverse map. It is straightforward to check that if $e \in G$ is the identity then, because $M : G_e \times G_e \to G_e$, there is a unique $\hat{e} \in \hat{G}_e$ such that $M(\hat{e}, \hat{e}) = \hat{e}$. It is also straightforward to deduce the existence of a unique inverse.

Hence we have the result from [3] that a central extension of $G$ is a $\mathbb{C}^\times$ bundle $P \to G$ together with a section $s$ of $\delta(P) \to G \times G$ such that $\delta(s) = 1$. In [3] this is phrased in terms of simplicial line bundles. Note that this is a kind of cohomology result analogous to that in the first section. We have an object (in this case a $\mathbb{C}^\times$ bundle) and $\delta$ of the object is ‘zero’ i.e. trivial as a $\mathbb{C}^\times$ bundle.

For our purposes we need to phrase this result in terms of differential forms. We call a connection for $\hat{G} \to G$, thought of as a $\mathbb{C}^\times$ bundle, a connection for the central extension. An isomorphism of central extensions with connection is an isomorphism of bundles with connection which is a group isomorphism on the total
space $\hat{G}$. Denote by $C(G)$ the set of all isomorphism classes of central extensions of $G$ with connection.

Let $\mu \in \Omega^2(\hat{G})$ be a connection on the bundle $\hat{G} \to G$ and consider the tensor product connection $\delta(\mu)$. Let $\alpha = s^*(\delta(\mu))$. We then have that
\[
\delta(\alpha) = (\delta(s^*))(\delta(\mu)) = (1)^*(\delta^2(\mu)) = 0
\]
as $\delta^2(\mu)$ is the flat connection on $\delta^2(P)$. Also $d\alpha = s^*(d\delta(\mu)) = \delta(R)$.

Let $\Gamma(G)$ denote the set of all pairs $(\alpha, R)$ where $R$ is a closed, $2\pi i$ integral, two form on $G$ and $\alpha$ is a one-form on $G \times G$ with $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$.

We have constructed a map $C(G) \to \Gamma(G)$. In the next section we construct an inverse to this map by showing how to define a central extension from a pair $(\alpha, R)$. For now notice that isomorphic central extensions with connection clearly give rise to the same $(\alpha, R)$ and that if we vary the connection, which is only possible by adding on the pull-back of a one-form $\eta$ from $G$, then we change $(\alpha, R)$ to $(\alpha + \delta(\eta), R + d\eta)$.

4.1. Constructing the central extension. Recall that given $R$ we can find a principal $\mathbb{C}^\times$ bundle $P \to G$ with connection $\mu$ and curvature $R$ which is unique up to isomorphism. It is a standard result in the theory of bundles that if $P \to X$ is a bundle with connection $\mu$ which is flat and $\pi_1(X) = 0$ then $P$ has a section $s: X \to P$ such that $s^*(\mu) = 0$. Such a section is not unique of course it can be multiplied by a (constant) element of $\mathbb{C}^\times$. As our interest is in the loop group $G$ which satisfies $\pi_1(G) = 0$ we shall assume, from now on, that $\pi_1(G) = 0$. Consider now our pair $(R, \alpha)$ and the bundle $P$. As $\delta(R) = d\alpha$ we have that the connection $\delta(w) - \pi^*(\alpha)$ on $\delta(P) \to G \times G$ is flat and hence (as $\pi_1(G \times G) = 0$) we can find a section $s$ such that $s^*(\delta(w)) = \alpha$.

The section $s$ defines a multiplication by
\[
s(p, q) = p \otimes M(p, q)^* \otimes q.
\]
Consider now $\delta(s)$ this satisfies $\delta(s)^*(\delta(\delta(w))) = \delta(s^*(\delta(w))) = \delta(\alpha) = 0$. On the other hand the canonical section 1 of $\delta(\delta(P))$ also satisfies this so they differ by a constant element of the group. This means that there is a $w \in \mathbb{C}^\times$ such that for any $p$, $q$ and $r$ we must have
\[
M(M(p, q), r) = wM(p, M(q, r)).
\]
Choose $p \in \hat{G}_e$ where $e$ is the identity in $G$. Then $M(p, p) \in \hat{G}_e$ and hence $M(p, p) = pz$ for some $z \in \mathbb{C}^\times$. Now let $p = q = r$ and it is clear that we must have $w = 1$.

So from $(\alpha, R)$ we have constructed $P$ and a section $s$ of $\delta(P)$ with $\delta(s) = 1$. However $s$ is not unique but this is not a problem. If we change $s$ to $s' = sz$ for some constant $z \in \mathbb{C}^\times$ then we have changed $M$ to $M' = Mz$. As $\mathbb{C}^\times$ is central multiplying by $z$ is an isomorphism of central extensions with connection. So the ambiguity in $s$ does not change the isomorphism class of the central extension with connection. Hence we have constructed a map
\[
\Gamma(G) \to C(G)
\]
as required. That it is the inverse of the earlier map follows from the definition of \( \alpha \) as \( s^*(\delta(\mu)) \) and the fact that the connection on \( P \) is chosen so its curvature is \( R \).

5. Conclusion: Loop groups

In the case where \( G = L(K) \) there is a well known expression for the curvature \( R \) of a left invariant connection on \( L(K) \) — see [5]. We can also write down a 1-form \( \alpha \) on \( L(K) \times L(K) \) such that \( \delta(R) = d\alpha \) and \( \delta(\alpha) = 0 \). We have:

\[
R(g)(gX, gY) = \frac{1}{4\pi^2} \int_{S^1} \langle X, \partial_\theta Y \rangle d\theta
\]

\[
\alpha(g_1, g_2)(g_1X_1, g_2X_2) = \frac{1}{4\pi^2} \int_{S^1} \langle X_1, (\partial_\theta g_2)\partial_\theta g_2^{-1} \rangle d\theta.
\]

Here \( \langle , \rangle \) is the Killing form on \( \mathfrak{k} \) normalised so the longest root has length squared equal to 2 and \( \partial_\theta \) denotes differentiation with respect to \( \theta \in S^1 \). Note that \( R \) is left invariant and that \( \alpha \) is left invariant in the first factor of \( G \times G \). It can be shown that these are the \( R \) and \( \alpha \) arising in [3].

In [4] we apply the methods of this talk to give an explicit construction of the ‘string class’ of a loop group bundle and relate it to earlier work of Murray on calorons.

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