Theta-terms in non-linear sigma-models

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We trace the origin of $\theta$-terms in non-linear $\sigma$-models as a nonperturbative anomaly of current algebras. The non-linear $\sigma$-models emerge as a low energy limit of fermionic $\sigma$-models. The latter describe Dirac fermions coupled to chiral bosonic fields. We discuss the geometric phases in three hierarchies of fermionic $\sigma$-models in spacetime dimension $(d + 1)$ with chiral bosonic fields taking values on $d$, $d + 1$, and $d + 2$-dimensional spheres. The geometric phases in the first two hierarchies are $\theta$-terms. We emphasize a relation between $\theta$-terms and quantum numbers of solitons.

I. INTRODUCTION

Non-linear $\sigma$-models describe the low energy dynamics of Goldstone bosons emerging as a result of a symmetry breaking. One particularly important source of this phenomenon is a chiral symmetry breaking due to an interaction between Dirac fermions and a chiral bosonic field. This interaction is described by models of current algebras, fermionic $\sigma$-models. Examples are considered below.

In case of chiral symmetry breaking, the non-linear $\sigma$-models are determined not only by the pattern of symmetry breaking, but also by global characteristics of a configurational space of chiral fields. In most interesting cases configurational space is not simply connected. There are spacetime configurations of a chiral field which can not be continuously deformed one into another. An adiabatic motion along a noncontractible closed path in a configurational space leads to a geometric phase acquired by the wave function. In the action of a non-linear $\sigma$-model, the geometric phase emerges as a $\theta$-term. It is a topological invariant of a mapping of a spacetime manifold $M$ into a target space $G$ of a chiral field. Although a $\theta$-term does not appear in equations of motions, its dramatic impact on dynamics of Goldstone bosons is needless to emphasize.

The $\theta$-terms reflect the anomalous character of the chiral current algebra. Some aspects of chiral anomalies can be studied perturbatively. For instance, a fermionic current, induced by a soliton can be obtained in a regular gradient expansion. The $\theta$-terms are more involved. They defy a perturbative analysis and are often referred to as a global (nonperturbative) anomaly.

The $\theta$-terms appear in different physical situations. A particularly interesting one is when a noncontractible spacetime configuration is a world trajectory of a soliton carrying a fermionic number. In this case a $\theta$-term is linked to quantum numbers of solitons. It is assumed that all $d$ spatial dimensions are compactified to a sphere $M = S^d$. The homotopy group $\pi(M, G) = \pi_d(G) \neq 0$ is not zero. Then solitons correspond to the homotopy classes of the mapping $S^d \rightarrow G$. Interaction with fermions induces a fermionic number localized on a soliton. This internal quantum number of a soliton converts into rotational quantum numbers: spin, statistics and isospin. Intuitively it is appealing that a fermionic number and spin or statistics of a soliton are two faces of the same phenomenon. In fact, a soliton acquires a unit fermionic number, one expects it to become a spin $1/2$ fermion. In spite of this, the fermionic number and spin or statistics of a soliton appear differently in the a non-linear $\sigma$-model. The fermionic number is assigned by the term $NA_{\mu}J_{\mu}$, where $A_{\mu}$ is an abelian gauge field, $J_{\mu}$ is a topological current of a soliton, and $N$ is a number of flavors of a current algebra. In its turn the spin and statistics are described by a $\theta$-term. If the homotopy group $\pi(M; G)$ is nonzero, one can consider an adiabatic $2\pi$-rotation of a soliton around a given axis as an example of a noncontractible spacetime configuration. The $\theta$-term of the non-linear $\sigma$-model represents in this case the geometric phase obtained in this process.

In this paper we intend to clarify some aspects of nonperturbative anomalies of current algebras and $\theta$-terms in non-linear $\sigma$-models. We consider two hierarchies of fermionic $\sigma$-models in spatial dimensions $d = 0, 1, 2, 3, \cdots$ which generate $\theta$-terms. The target space of the first hierarchy is a $d$-dimensional sphere $S^d$. It admits solitons $\pi_d(G) = \pi_d(S^d) = Z$. The target space of the second hierarchy is $S^{d+1}$. There are no solitons in this case $\pi_d(S^{d+1}) = 0$.

Most of non-linear $\sigma$-models we discuss have important physical applications, some of which will be described elsewhere.

We show how $\theta$-terms emerge as nonperturbative anomalies of fermionic $\sigma$-models and how they can be
obtained from perturbative anomalies. In particular, we show that a non-linear $\sigma$-model of current algebra which supports solitons and has noncontractible spacetime paths always contains a $\theta$-term with a fixed value of $\theta = N\pi$.

In the next three sections we summarize the results of the paper. We write the fermionic models for two hierarchies in Sec.I, and list the corresponding non-linear $\sigma$-models obtained via gradient expansion in Sec.II. The generalization for arbitrary dimensions is discussed in Sec.III. We sketch the computations of anomalous terms for the first and second hierarchies in Secs.IV-VI. The Sec.VII is a summary.

II. FERMIONIC $\sigma$-MODELS

A. The first hierarchy of fermionic $\sigma$-models in spatial dimensions $d = 1, 2, 3$ is

\[(1 + 1): L_1 = \bar{\psi} (i\tilde{D} + im(\Delta_1 + i\gamma_5\Delta_2))\psi, \quad \Delta_1^2 + \Delta_2^2 = 1, \quad (1)\]

\[(2 + 1): L_2 = \bar{\psi} (i\tilde{D} + im\vec{n}\vec{\tau})\psi, \quad \vec{n}^2 = 1, \quad (2)\]

\[(3 + 1): L_3 = \bar{\psi} (i\tilde{D} + im(\pi_0 + i\gamma_5\vec{n})\psi), \quad \pi_0^2 + \vec{n}^2 = 1. \quad (3)\]

Here and thereon we use Euclidian formulation and assume a mass to be positive $m > 0$. A Dirac fermion $\psi$ has a flavor running from 1 to $N$ and $D = \gamma_{\mu}(\partial_{\mu} - iA_{\mu})$. In dimensions $(2+1)$ and $(3+1)$ the fermion is also an $SU(2)$ doublet and $\vec{n}$ are Pauli matrices acting in isospace. Fermions interact with chiral fields which take values on spheres $S^d$ (target space). Hereafter we reserve a different (historically motivated) notations for 2-, 3-, and 4-dimensional unit vectors. In one spatial dimension the chiral field is a phase $\Delta_1 + i\Delta_2 = e^{i0}$. In $d = 2$ the chiral field is a 3-dimensional unit vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$. Correspondingly the chiral field in three spatial dimensions is a 4-dimensional unit vector $(\vec{n}, \pi_0)$. It can also be considered as an element of $SU(2)$ group $g = g_{0} + i\vec{n}\vec{\tau}$.

Physical applications of fermionic models (1-3) on spheres are known. In $d = 1$ it is familiar Peierls-Fröhlich model of polyacetylene. The model in $d = 3$ appears in the past as a model for nuclear forces and recently has been used in the QCD context. All these models also emerged in the context of topological superfluids.

In every model of this hierarchy a compactified coordinate space $S^d$ matches the target space of chiral fields, also $S^d$. Therefore, every model supports solitons with an arbitrary (integer) topological charge $Q$. The topological charge $Q$ labels homotopy classes $\pi_d(S^d) = Z$ and can be written as a spatial integral $Q = \int d^dx J_0$ of a zeroth component of a topological current $J_\mu$. In dimensions $d = 1, 2, 3$ topological currents are:

\[J_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu \phi, \quad (4)\]

\[J_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \vec{n} \cdot \partial_\nu \vec{n} \times \partial_\lambda \vec{n}, \quad (5)\]

\[J_\mu = \frac{1}{12\pi^2} \epsilon_{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\nu \pi_a \partial_\lambda \pi_b \partial_\rho \pi_c \pi_d \]

\[= \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\rho} \text{tr}(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)(g^{-1}\partial_\rho g). \quad (6)\]

On smooth configurations of chiral fields topological currents are identically conserved $\partial_\mu J_\mu = 0$. We discuss two different types of spacetime boundary conditions: a compactification of the spacetime to a sphere $S^{d+1}$, and more physical one – a compactification of the space to a sphere $S^d$ at every moment of time and periodic (anti-periodic) boundary conditions in Euclidian time. In other words, $M = S^d \times S^1$. In the latter case a configurational space is divided into disconnected “topological sectors” characterized by total number of solitons $Q$.

In every model of the first hierarchy there are processes which wrap a spacetime $M = S^d \times S^1$ over a target space $S^d$. We consider simple examples of such processes. In $d = 1$ it is a translation of a soliton around the spatial ring; in $d > 1$ it is e.g., a $2\pi$-rotation of a soliton around a chosen axis. These spacetime configurations belong to a nontrivial homotopy classes of $\pi(M; G)$. These classes can be labeled by an integer $H_d$ (in addition to $Q$), which for our simple processes appears to be equal to the topological charge of the soliton $H_d = Q$.

If the spacetime is compactified into a sphere $S^{d+1}$, a configuration with a given number of solitons at any time does not exist. Noncontractible spacetime configurations are more complicated. They consist, e.g., of the creation of a soliton-antisoliton pair, rotation of a soliton around its axis, and subsequent annihilation of the pair. This process belongs to a nontrivial homotopy class of $\pi_{d+1}(S^d)$. The corresponding homotopy groups are $\pi_2(S^1) = 0$, $\pi_3(S^2) = Z$, and $\pi_{d+1}(S^d) = Z_2$ for $d > 2$.

B. Next we consider the second hierarchy of fermionic $\sigma$-models with the target space $S^{d+1}$. It starts from the model in $d = 0$ (quantum mechanics):
(0 + 1): \[ L_0 = \bar{\psi}(i\hat{D} + im(\tau_3 \cos \nu + \Delta_1 \tau_i \sin \nu))\psi, \quad \Delta_i^2 = 1, \] (1 + 1): \[ L_1 = \bar{\psi}(i\hat{D} + im(\cos \nu + i\gamma_5 \vec{n}\vec{\tau}\sin \nu))\psi, \quad \vec{n}^2 = 1, \] (2 + 1): \[ L_2 = \bar{\psi}(i\hat{D} + im(\cos \nu + \pi_i \Gamma_i \sin \nu))\psi, \quad \pi_i^2 = 1. \]

Here \( \nu \) is some constant, \( \Gamma_i \) and \( \Gamma_5 \) are 4 \times 4 Dirac matrices: \( \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}, \quad \Gamma_5 = -\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \). Fermions in \( d = 0, 1 \) are 2-component isospinors of \( SO(3) \) and in \( d = 2 \) are 4-component isospinors of \( SO(4) \). The chiral fields do not have solitons \( \pi_d(S^{d+1}) = 0 \), but do have noncontractible spacetime configurations according to the homotopy classes \( \pi_{d+1}(S^{d+1}) = \mathbb{Z} \).

### III. Non-Linear \( \sigma \)-Models

An action for a chiral field which appears as a result of an integration over fermions is a non-linear \( \sigma \)-model \( W_d = -\ln \int \exp(-\int dx L_d) D\bar{\psi}D\psi \). The non-linear \( \sigma \)-models can be systematically studied in \( 1/m \) (gradient) expansion. Then anomalies are given by dimensionless terms of the action of a non-linear \( \sigma \)-model. They give the only contribution to the imaginary part of Euclidian action.

Below we list the leading terms of non-linear \( \sigma \)-models for the fermionic models of the Sec.\( \mathbb{I} \). They are the main result of the paper. We start from the first hierarchy (\( \mathbb{III} \)).

**A.** In (1+1):

\[
W_1 = -N \ln \text{Det}(i\hat{D} + ime^{i\gamma_5\phi}) = iN \int d^2x A_\mu J_\mu + i\pi NH_1 + N \int d^2x \frac{1}{8\pi} (\partial_\mu \phi)^2. \tag{10}
\]

Hereafter we omit higher order terms in \( 1/m \).

The geometric phase \( \pi NH_1 \) is a \( \theta \)-term with \( \theta = N\pi \). If the spacetime is compactified to a sphere \( S^2 \), the geometric phase vanishes unless \( \phi \) is singular (spacetime vortices). For the spacetime compactified into a torus \( T^2 = S^1 \times S^1 \), we have \( \pi(T^2, S^1) = \mathbb{Z} \times \mathbb{Z} \). There are two integer numbers to characterize spacetime configurations. One of them is a topological charge \( Q = \oint dx \frac{\partial_\phi}{2\pi} \) while the other is a temporal winding number \( \oint dt \frac{\partial_\phi}{2\pi} \). The geometric phase in this case is nontrivial even for non-singular configurations. It is a product of 1-cycles over space and time:

\[
H_1[\phi] = Q \oint dt \frac{\partial_\phi}{2\pi}. \tag{11}
\]

In (2+1):

\[
W_2 = -N \ln \text{Det}(i\hat{D} + im\vec{n}\vec{\tau}) = iN \int d^3x A_\mu J_\mu + i\pi NH_2[\vec{n}] + N \frac{m}{8\pi} \int d^3x (\partial_\mu \vec{n})^2. \tag{12}
\]

Here \( \pi NH_2 \) is a \( \theta \)-term with \( \theta = \pi N \). The integer \( H_2 \) is a homotopy class of a map of the spacetime to the target space \( S^2 \). In the case when the spacetime is compactified to a sphere \( S^3 \), \( H_2 \) is the Hopf number. It can be any integer \( \pi_3(S^2) = \mathbb{Z} \). A geometric interpretation of the Hopf number is well known (see e.g., Refs.\( \mathbb{III} \)). It is a linking number of world lines of two different values \( \vec{n}_1 \) and \( \vec{n}_2 \) of \( \vec{n} \)-field. The Hopf number can be explicitly written in terms of \( SU(2) \) matrix \( U(x) \) which rotates the vector \( \vec{n} \) to the chosen (third) axis: \( \vec{n}\vec{\tau} = U^{-1} \tau_3 U \). Then the Hopf number is a degree of mapping of (2+1) spacetime into \( SU(2) \):

\[
H_2[\vec{n}] = \frac{\epsilon^{\mu\nu\lambda}}{24\pi^2} \int d^3x \text{tr}(U^{-1} \partial_\mu U)(U^{-1} \partial_\nu U)(U^{-1} \partial_\lambda U). \tag{13}
\]

If the spacetime is compactified into \( M = S^2 \times S^1 \), the topology of the mapping \( M \to S^2 \) is more complicated. The homotopy classes are labeled by two integers \( (Q, H_2) \). The first is a topological charge of solitons. It is the same at all times. The second integer \( H_2 \) is a topological invariant of spacetime configurations. \( H_2 \) is generalizing the Hopf invariant, defined for a map \( S^3 \to S^2 \).

In general, the homotopy classes of maps \( S^2 \times S^1 \to S^2 \) do not form a group. However, they do form a group in a sector with a fixed topological charge. In a sector with zero topological charge and if the vector \( \vec{n}(x = \infty, t) \) on the space infinity does not depend on time, \( H_2 \) is the Hopf invariant. It is a linking number of world trajectories of
two arbitrarily chosen points $\vec{n}_1$ and $\vec{n}_2$ of the target space. However, in a sector with non-zero topological charge some links can be unlabeled with the help of a continuous deformation of the vector at space infinity. To construct a proper topological invariant one has to consider the angle of rotation of $\vec{n}(x = \infty, t)$ around one of the vectors $\vec{n}_1$ or $\vec{n}_2$ as it is illustrated in Ref. $\text{[3]}$. The new invariant, however, is intrinsically ambiguous and is defined modulo $2Q$ only. There is a deep mathematical theorem, which says that in a sector with topological charge $Q$ there are only $2Q$ homotopy classes, and that they form a finite Abelian group $Z_{2Q}$ $\text{[3]}$. In a sector with $Q$ solitons a $2\pi k$ rotation of a single soliton with $k = 0, \cdots, 2Q - 1$ is a representative spacetime configuration of the $k$-th homotopy class. If $\phi(t)$ is an angle of rotation of a soliton of charge $Q$ around fixed axis, the invariant $H_2$ is given by the same formula $\text{[3]}$ as in $(1+1)$ case. It is a product of the spatial 2-cycle $Q$ and the temporal 1-cycle. A more detailed discussion is planned elsewhere.

Finally in $(3+1)$:

\[
W_3 = -N \ln \text{det}(i\hat{D} + img^{\gamma_5}) = iN \int d^3x A_\mu J_\mu + i\pi NH_3[g] + N \frac{F_\pi^2}{4} \int d^3x \text{tr}(\partial_\mu g^{-1} \partial_\mu g),
\]

where $F_\pi^2 = \frac{1}{2\pi}m^2 \ln \frac{\Lambda}{m}$ with $\Lambda$ being ultraviolet cutoff and we use notation $g^{\gamma_5} = \frac{1 + \gamma_5}{2}g + \frac{1 - \gamma_5}{2}g^{-1}$. The integer $H_3$ is a topological invariant of the map of the spacetime into the target space $S^3$. There are only two homotopy classes $\pi_4(S^3) = \mathbb{Z}_2$, so that $H_3 = 0$ or 1. Geometric phase $\pi N H_3$ is the $\theta$-term with $\theta = N \pi$.

In general, the finite number of homotopy classes gives a restriction for admissible values of $\theta$. The wave function $e^{i\theta W}$ forms a single-valued representation of the homotopy group only if $\theta$ is a multiple of $2\pi/\Lambda$, where $\Lambda$ is a dimension of the homotopy group. In $(3+1)$ and higher dimensional models of this hierarchy $\pi_{d+1}(S^d) = \mathbb{Z}_2$. This limits the value of $\theta$ to multiples of $\pi$.

This restriction is of a particular interest in $(2+1)$ case. In a sector with $Q$ solitons the allowed values of $\theta$ are multiples of $\pi/Q$. Even in the case when the spacetime is compactified to a sphere, where there is no formal restriction on a value of $\theta$, solitons and antisolitons cannot be treated as true particles, unless $\theta$ is quantized in units of $\pi/Q$.

If one does not want to restrict a $\sigma$-model to a sector with a given number of solitons, the only allowed values of $\theta$ are multiples of $\pi$ (see also Ref. $\text{[3]}$).

**B.** Let us now list the non-linear $\sigma$-models of the second hierarchy (Ref. $\text{[4]}$).

$$(0+1) : \quad W_0 = N \int dt \frac{\sin^2 \nu}{8\pi} (\partial_\mu \phi)^2 - i\theta \Omega_1 [\phi],$$

$$\Omega_1 [\phi] = \oint_c \partial_\mu \phi \frac{dt}{2\pi}.$$  

This is the Lagrangian of a plane quantum rotator moving around magnetic flux $\theta = \pi N(1 - \cos \nu)$ i.e., $(0+1)$-dimensional $O(2)$ non-linear $\sigma$-model with $\theta$-term.

$$(1+1) : \quad W_1 = N \int d^2x \frac{\sin^2 \nu}{4\pi} (\partial_\mu \vec{n})^2 - i\theta \Omega_2 [\vec{n}],$$

$$\Omega_2 [\vec{n}] = \int d^2x \frac{1}{8\pi} \epsilon_{\mu\nu\lambda\kappa \alpha \beta \gamma \delta} n_\alpha \partial_\nu n_\beta \partial_\lambda n_\gamma.$$  

This is the famous $(1+1)$, $O(3)$ non-linear $\sigma$ model with $\theta = N(2\nu - \sin 2\nu)$. For $\nu = \pi/2$ we find $\theta = N \pi$ and this is the non-linear $\sigma$-model with topological term which was used to describe the effective action for spin-$N/2$ chain $\text{[4]}$.

$$(2+1) : \quad W_2 = N \int d^3x \frac{m \sin^2 \nu}{4\pi} (\partial_\mu \pi_1)^2 - i\theta \Omega_3 [\pi_1],$$

$$\Omega_3 [\pi_1] = \int d^3x \frac{1}{16\pi^2} \epsilon^{\mu\nu\lambda} \epsilon^{ijkl} \pi_1 \partial_\mu \pi_2 \partial_\nu \pi_3 \partial_\lambda \pi_4.$$  

This is a $(2+1)$, $O(4)$ non-linear $\sigma$-model with $\theta = N\pi(1 - \frac{1}{4}\cos \nu + \frac{1}{4}\cos 3\nu)$.

The anomalous part $A_\mu J_\mu$ in (Ref. $\text{[2]}$) with $J_\mu$ from (Ref. $\text{[3]}$) has been computed in Ref. $\text{[3]}$ for $d = 1, 2, 3$ respectively. The geometric phase in $(1+1)$ on a spacetime compactified into a torus $\text{[3]}$ appeared in a context of bosonization on general Riemann surfaces (see Ref. $\text{[3]}$). The geometric phase of $\text{[3]}$ has been considered in Ref. $\text{[4]}$.

The geometric phase in $(3+1)$ is related to the so-called $SU(2)$ anomaly and can be found in Ref. $\text{[4]}$. $O(3)$ non-linear $\sigma$-model at $\nu = \pi/2$ has been obtained from $(1+1)$ fermionic model $\text{[3]}$ in Ref. $\text{[4]}$.

Below we also mention one more (third) hierarchy, with the target space $S^{d+2}$. These models do not have solitons $\pi_d(S^{d+2}) = 0$, and they do not have $\theta$-terms either: $\pi_{d+1}(S^{d+2}) = 0$. Their geometric phases are perturbative Wess-Zumino terms due to $\pi_{d+2}(S^{d+2}) = \mathbb{Z}$. 

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IV.  $\sigma$-MODELS IN HIGHER DIMENSIONS

Obviously both hierarchies may be continued to higher dimensions. Their topological properties are “inherited” from the ones in lower dimensions due to the stability of homotopy groups $\pi_n(S^n)$ are the same for all $n \geq k + 2$. To write the models explicitly we introduce the following notations. Let $\vec{V} = (V_1, \ldots, V_l)$ be a unit vector $\sum V_i^2 = 1$ which takes values on $(l - 1)$-sphere $S^{l-1}$ and $\Gamma_{l+1}^{(2k+1)} \cdots, \Gamma_{2k+1}^{(2k+1)}$ are $2^k \times 2^k$ Hermitian Dirac matrices (representation of Clifford algebra with $2k + 1$ generators). We denote

$$V^{(l)} = \sum_{i=1}^{l} V_i \Gamma_{i}^{(l)}; \quad l = \text{odd},$$

$$V^{(l)} = V_l + i\gamma_5 \sum_{i=1}^{l-1} V_i \Gamma_{i}^{(l-1)}; \quad l = \text{even}.$$  

Here $V^{(l)}$ is a matrix with unit square $V^{(l)}(V^{(l)})^T = 1$. In these notations the hierarchy of fermionic models (1-9) with the target space $S^d$ in $d + 1$ dimensions is

$$L_d = \bar{\psi}(i\dot{D} + i m V^{(d+1)})\psi.$$  

The non-linear $\sigma$-model for this hierarchy is

$$W_d = iN \int d^{d+1}x A_{\mu} J_{\mu} + i\pi N H_d + N m^{d-1} \alpha_d \int d^{d+1}x (\partial_{\mu} \vec{V})^2,$$  

where $\alpha_d$ is a constant (non-universal for $d > 3$, $\alpha_d \sim (\pi/m)^{d-3}$), $\alpha_3 = \frac{1}{4\pi^2} \ln \frac{\Lambda}{m}$. Topological invariant $H_d$ takes values 0 and 1 for $d \geq 3$ according to two homotopy classes $\pi_d(S^d) = Z_2$ and $J_{\mu}$ is a topological current

$$J_{\mu} = \frac{1}{d! \text{Area}(S^d)} \epsilon^{\mu\nu_1 \cdots \mu_d \nu_{d+1} \cdots \nu_{d+1}} V_a \partial_{\mu_{\nu_1}} V_a \cdots \partial_{\mu_{d+1}} V_{a_{d+1}}.$$  

Here $\text{Area}(S^d) = \frac{2\pi^{d+1}}{\Gamma(d+1)}$ is an area of unit $d$-sphere. Zero component of topological current integrated over space is a topological charge of configuration $Q = \int d^d x J_0$ — the degree of mapping $S^d \rightarrow S^d$.

The hierarchy of fermionic models (10-13) with the target space $S^{d+1}$ is given by

$$L_d = \bar{\psi}(i\dot{D} + i m V^{(d+3)})\psi,$$  

with a condition

$$V_{d+3} = \cos \nu, \quad \sum_{i=1}^{d+2} V_i^2 = \sin^2 \nu.$$  

If $\vec{v}$ is a unit $d + 2$-vector ($\vec{v}^2 = 1$) defined as $\vec{v} \sin \nu \equiv (V_1, \ldots, V_{d+2})$ and matrix $v$ is defined as $v \equiv \sum_{i=1}^{d+2} v_i \Gamma_i$, then this hierarchy can also be written as

$$L_d = \bar{\psi}(i\dot{D} + i m (\cos \nu \Gamma_{d+3} + \sin \nu \nu))\psi; \quad d + 1 = \text{odd},$$

$$L_d = \bar{\psi}(i\dot{D} + i m e^{i\nu \nu})\psi; \quad d + 1 = \text{even}.$$  

The non-linear $\sigma$-model for this hierarchy is

$$W_d = -i\theta \Omega_d + N m^{d-1} 2\alpha_d \sin^2 \nu \int d^{d+1}x (\partial_{\mu} \vec{v})^2.$$  

The $\theta$-term $\Omega_d$ is a degree of mapping $S^{d+1} \rightarrow S^{d+1}$:

$$\Omega_d = \frac{1}{(d+1)! \text{Area}(S^{d+1})} \int d^{d+1}x e^{\nu_1 \cdots \nu_{d+1} \epsilon a_{a+1} \cdots a_{d+1}} v_a \partial_{\mu_1} v_{a_1} \cdots \partial_{\mu_{d+1}} v_{a_{d+1}}$$  

and coefficient $\theta$ is proportional to a fraction of the area of $d + 2$-sphere $S^{d+2}$ cut off by latitude $\nu$

$$\theta = 2\pi N \int_0^\nu \frac{\sin d z \; dz}{\int_0^\nu \sin d z \; dz}.$$
V. $\theta$-TERM AND QUANTUM NUMBERS OF SOLITON

We start from the first hierarchy \([1],[3]\). It is well known that a soliton in these models acquires a fermionic charge due to a chiral anomaly. It is described by the first term in non-linear $\sigma$-models \([1],[12],[14]\). Varying over the gauge field, we obtain a fermionic current induced by a soliton

$$j_\mu = -i \frac{\delta}{\delta A_\mu} W_d = NJ_\mu.$$  \hspace{1cm} (30)

This result means that a Hamiltonian of a fermionic model taken in a static soliton background have an additional $NQ$ levels with negative energy $E < 0$ compared to the one in a topologically flat background. Computations of the induced current are perturbative and known \([14]\). We write $L = \bar{\psi} D \psi$ and $j_\mu = i N \frac{\delta}{\delta A_\mu} N \text{Tr} \ln D = i N \text{Tr} \{ \gamma_\mu (D^\dagger D)^{-1} D^\dagger \}$. Expanding the denominator in gradients of chiral fields

$$j_\mu = mN \text{Tr} \left\{ \gamma_\mu \frac{1}{-\partial^2 + m^2} \left( m\partial V^{(d+1)} + \frac{1}{1 - \partial^2 + m^2} \right)^d (V^{(d+1)})^t \right\},$$  \hspace{1cm} (31)

and calculating the trace, we obtain the result \([30]\) with topological current \([22]\). For dimensions $d = 1, 2, 3$ this gives \([14]\). This is the first nonvanishing term in the gradient expansion for current. One can use \([31]\) only for large size ($\gg 1/m$) solitons. If the soliton is small, the gradient expansion is not applicable: some levels cross $E = 0$ and soliton becomes uncharged \([14]\).

The leading regular term of effective actions \([10],[12],[14]\) can be obtained in a similar fashion by varying over a chiral field:

$$\delta W_d = - N \text{Tr} \left\{ i m \delta V^{(d+1)} (D^\dagger D)^{-1} D^\dagger \right\} = N m^2 \text{Tr} \left\{ \delta V^{(d+1)} \frac{1}{-\partial^2 + m^2} \right\} \text{Tr} \ln (D^\dagger D)^{-1} D^\dagger.$$  \hspace{1cm} (32)

The calculation of the $\theta$-term is more complicated due to its nonperturbative nature. Below we employ the method suggested in Ref.\([14]\). We illustrate it on the example of \((2+1)\) theory \([3]\) and choose the spacetime to be compactified $S^2 \times S^1$. We specify $\vec{n}(\vec{x}, t) = e^{-\frac{i}{\tau_0} H_0 \tau_3} \vec{n}_0 e^{\frac{i}{\tau_0} H_3 \tau_3}$ to be a charge $Q$ soliton slowly rotating around “third” axis given by $\vec{n}_0 (\vec{x} = \infty)$ and performing total $2\pi$-angle rotation $\varphi(T) - \varphi(0) = 2\pi$ in time $T$. Then we compare the value of the determinant of the Dirac operator $\text{Det}(D) = \text{Det}(D^\dagger)$ with the one for a static, nonrotating soliton $\vec{n}_0 (\vec{x})$. According to the Sec.\([11]\), we expect that a difference between topological terms $H_2$ for these two configurations is $Q$. Then the ratio of the determinants is $e^{i\theta_{\vec{n}_0}}$ which gives the value of the coefficient $\theta$ in front of $\theta$-term. Other (regular) terms are negligible as soon rotation is adiabatically slow.

To facilitate computation it is tempting to make a gauge transformation $\psi \rightarrow e^{-\frac{i}{2} \varphi \tau_3} \psi'$. Then the transformed Dirac operator $iD + \frac{1}{2} \gamma_0 \varphi \tau_3 + im\vec{n}_0 \vec{\sigma}$ depends on time derivative of $\varphi$ and is ready for a gradient expansion. This approach, however, misses the geometric phase. The resolution of this puzzle is typical for an anomalous calculus. The gauge transformation we performed is anomalous. It is not single-valued: a change of $\varphi$ by $2\pi$ changes the sign of $\psi$ if $Q$ is odd. This transformation changes the antiperiodic boundary condition $\psi(0) = -\psi(T)$ to the periodic one $\psi'(0) = -e^{\frac{i}{2} \varphi} \psi'(T) = (-1)^{Q+1} \psi'(T)$ by creating a flux $\pi$ through the temporal loop. The situation may be improved by making an additional Abelian gauge transformation $\psi' \rightarrow e^{\frac{i}{2} \varphi} \psi''$. Then $D \rightarrow iD + \frac{1}{2} \gamma_0 \varphi \tau_3 + \frac{1}{2} \gamma_0 \varphi + im\vec{n}_0 \vec{\sigma}$. Now boundary conditions for $\psi''$ are the same as for $\psi$ and we can expand the transformed Dirac operator in $\varphi$. At this point we notice that $\frac{1}{2} \hat{\varphi}$ enters Dirac operator the same way as constant in space gauge potential $A_0$. Then the variation of an effective action over $\varphi$ can be computed the same way as it has been done for the calculation of an induced current. The result is expressed in terms of the induced charge. We have

$$-i \frac{\delta W_d}{\delta \varphi} = i N \text{Tr} \left\{ \frac{1 + \tau_3}{2} \gamma_0 (D^\dagger D)^{-1} D^\dagger \right\} = i N \text{Tr} \left\{ \frac{1}{2} \gamma_0 (D^\dagger D)^{-1} D^\dagger \right\} = -i \frac{\delta W_d}{2 \delta A_0} = \frac{N}{2} \int d^2 x J_0.$$  \hspace{1cm} (32)

Thus we obtain an important result: an angular momentum (spin) of a soliton $I = -\frac{\delta}{\delta \varphi} W_d$ is equal to the half of the topological charge of a soliton times the degeneracy of fermionic states.
This gives us the topological term $\theta H_2$ in (32) with $\theta = N\pi$.

A computation of the geometric phase for a configuration other than a rotating soliton is technically involved. It is not necessary though due to a topological nature of the geometric phase. It is a topological invariant and does not change within a homotopy class. We may, therefore consider a particular configuration—2 rotation of a soliton of charge $Q$ to find a $\theta$-angle, a coefficient in front of the topological invariant. Generalization of these arguments to higher dimensions is straightforward.

These arguments clearly relate the fermionic number of a soliton with its angular momentum (33) and, therefore, its statistics. If solitons are charged they also have nonzero angular momentum (and corresponding statistics). The latter translates into the $\theta$-term in a non-linear $\sigma$-model. The value of $\theta$ in this term is determined by the charge of soliton. Since the fermionic charge in our models is an integer the value of $\theta$ is always a multiple of $\pi$. Contrary the $\theta$-terms in the second hierarchy can get any value.

Despite different dimensions of models belonging to the first hierarchy, statistical properties of solitons do not depend on a dimension of the spacetime. Here is a brief list of them.

(i) A soliton with a topological charge $Q$ carries a fermionic number $NQ$.

(ii) A soliton of unit topological charge in a model with odd (even) number of flavors $N$ has a half-integer (integer) spin. To see this in spatial dimensions 2 and 3 in a spacetime compactified into a sphere, one might consider an adiabatic process when soliton-antisoliton pair is created, the soliton is rotated by $2\pi$-angle, and then the pair is annihilated. This process corresponds to a spacetime configuration from nontrivial homotopy class. The value of $\theta$-term in the action for this spacetime configuration is $N\pi$ and the wave function of the entire system changes its sign under $2\pi$-rotation. We conclude that this soliton has a spin $N/2$. We notice that although the $\theta$-term vanishes modulo $2\pi$ at even $N$ its effect does not disappear. A soliton in this case has an integer spin which is equal to $N/2$.

(iii) A process when two solitons interchange their spatial positions also changes the topological invariant by one and the geometric phase by $N\pi$. This leads to a conclusion that a soliton is a fermion (boson) in case $N$ is odd (even).

(iv) In the toroidal space geometry (spatial dimensions are compactified into a torus) the $\theta$-term changes momentum quantization rule from $P = \frac{2\pi}{\theta}n$ to $P = \frac{2\pi}{\theta}(n + \frac{NQ}{2})$ in the sector with $Q$ solitons. This change is nontrivial if $NQ$ is odd.

(v) The $\theta$-term can be also interpreted locally as a Berry phase of an adiabatically rotating soliton. The total phase acquired by the vacuum state as a result of $2\pi$-rotation is the $\theta$-term.

VI. WESS-ZUMINO AND $\theta$-TERMS

Noncontractible spacetime configurations of the second hierarchy are not linked to any particle-like states. The analysis of $\theta$-terms in this case is simpler. They may be obtained through a reduction from the Wess-Zumino perturbative anomaly. The construction is following. Let us increase the target space $S^{d+1}$ of the models to $S^{d+2}$. Then all spacetime configurations become contractible $\pi_{d+1}(S^{d+2}) = 0$. It does not mean, however, that a geometric phase vanishes. It exists due to a nonzero homotopy group $\pi_{d+2}(S^{d+2}) = Z$ (See Ref. [23]). In this case the geometric phase is perturbative and is known as Wess-Zumino term. Under the reduction back to $S^{d+1}$, the Wess-Zumino term converts into a nonperturbative $\theta$-term. To carry out this procedure, we consider the third hierarchy of fermionic models with the target space $S^{d+2}$. They are the same as (24) but with no constraint (24) (a constant $\nu$ becomes a dynamic field):

$$D = i\hat{D} + imV,$$

where we adopt the notation $V^{(d+3)} = V$.

In dimensions $d = 0, 1$ these are familiar models:

$$(0 + 1): \quad D = iD + im\vec{n}\vec{\tau}, \quad (35)$$

$$(1 + 1): \quad D = i\hat{D} + im(\nu + i\gamma_5\vec{\tau}). \quad (36)$$

We shall compute the perturbative anomaly for the third hierarchy (24) and then enforce the condition (24).

Varying over $V$, we have $\delta W = -N\text{Tr} \left( im\delta V(D^\dagger D)^{-1}D^\dagger \right)$. Expanding $(D^\dagger D)^{-1}$ in gradients of $V$ we obtain for an imaginary part

$$\delta \Im W = iNK\int_{S^{d+1}} d^{d+1}x \text{Tr} \left( \delta V(\partial V)^{d}V^\dagger \right). \quad (37)$$
Here $K = \int \frac{d^{d+1}p}{(2\pi)^d} \frac{m^{d+3}}{(\rho + m^2)^{d+1}}$ and trace is taken over both Lorentzian and isospace gamma-matrices. Integration is performed over the spacetime $S^{d+1}$. The method to restore the action of non-linear $\sigma$-model is standard. Introduce a parameter $\xi$ such that $\mathcal{V}(x, \xi)$ continuously interpolates between constant $\mathcal{V}(x, \xi = 0) = (0, V_{d+1})$ and given spacetime configuration $\mathcal{V}(x, \xi = 1) = \mathcal{V}(x)$. The field $\mathcal{V}(x, \xi)$ is therefore defined on a disk $\mathcal{B}$, which boundary $\partial \mathcal{B} = S^{d+1}$ is the spacetime of our model. Take a Jacobian of a map of $d + 2$-sphere $(x, \xi)$ to $d + 2$-sphere $\mathcal{V}(x, \xi)$ and integrate it over the disk:

$$3mW = -2\pi N \Gamma[\mathcal{V}],$$

$$\Gamma[\mathcal{V}] = \frac{1}{(d + 2)! \text{Area}(S^{d+2})} \int_B d^{d+2}x \epsilon^{\mu_1 \cdots \mu_{d+2}} \epsilon^{a_1 \cdots a_{d+2}} V_{\alpha} \partial_{\alpha_1} V_{a_1} \cdots \partial_{\alpha_{d+2}} V_{a_{d+2}}.$$ 

Since the integrand is a full derivative the result of the integration, depends only on the physical field defined on the boundary of the disk. Its variation gives (37). Adding a leading term of the gradient expansion of the real part of $-N \text{Tr} \ln D$ we obtain the non-linear $\sigma$-models for the third hierarchy (34):

$$W = \frac{F^2}{2} \int d^{d+1}x (\partial_\mu \mathcal{V})^2 - 2\pi i N \Gamma[\mathcal{V}].$$

These are Wess-Zumino models. In $(0+1)$ it is an action for a spin $N/2$ (see e.g., 24-26)

$$W_0 = \frac{N}{8m} \int dt (\dot{n})^2 - 2\pi i N \int d^2x \frac{1}{8\pi} \epsilon^{\mu \nu \alpha} \partial_\mu \vec{n} \times \partial_\nu \vec{n}.$$ 

In $(1+1)$ it is $SU(2)$ level $N$ WZ-model of conformal field theory 24-26.

$$W_1 = \frac{N}{8\pi} \int d^2x \text{Tr} (\partial_\mu g^{-1} \partial_\mu g)$$

$$- 2\pi i N \int d^3x \frac{1}{24\pi^2} \epsilon^{\mu \nu \lambda} \text{Tr} (g^{-1} \partial_\mu g)(g^{-1} \partial_\nu g)(g^{-1} \partial_\lambda g).$$

Now let us perform a reduction of the target space $(d + 2)$-sphere of the Wess-Zumino models to $S^{d+1}$ by imposing constraint (24). It brings us back to the models of interest (15,17,27,29). The condition (24) embeds a $(d+1)$-sphere into a $(d+2)$-sphere as a latitude $\nu$ section. Under this condition, $\mathcal{V}$ on the boundary of the disk $(x, \xi = 1)$ (physical spacetime) takes values on $S^{d+1}$. In this case the Wess-Zumino term is equal to a degree of a mapping $S^{d+1} \rightarrow S^{d+1}$ times the fraction of the volume of $S^{d+2}$ cut off by the latitude corresponding to the angle $\nu$. This factor is the value of $\theta$ in the models (15,17,27,29). It is given by (29). In particular at $\nu = \pi/2$ the value of the $\theta$-angle is $N\pi$.

VII. SUMMARY

To summarize the results of the paper: we discussed connections between global properties of chiral current algebras and geometric phases in non-linear $\sigma$-models. To understand these connections the mappings of three different manifolds to the manifold (target space) of the chiral field are essential. These manifolds are: (i) space, (ii) spacetime, and (iii) a disk which boundary is a spacetime.

We considered three hierarchies of models of Dirac fermions in $(d+1)$-dimension coupled with chiral boson field. The chiral fields take values on $d$, $d + 1$, and $d + 2$-dimensional spheres respectively. Each hierarchy provides a different physical origin of a geometric phase. Most of the models in dimensions $d = 1, 2, 3$ have important physical applications.

The current algebras with a $d$-dimensional sphere as a target space (first hierarchy) support solitons—non-trivial homotopy classes of spatial configurations of the chiral field. Solitons carry an integer fermionic charge. We showed (see also Ref. 14) that the induced actions for the chiral field (non-linear $\sigma$-models) necessarily possess $\theta$-terms. These terms represent homotopy classes of $d+1$-dimensional spacetime configurations. The value of $\theta$-angle is tightly related to the charge of solitons. It is equal to $\pi$ times the fermionic number of the soliton with a unit topological charge. The $\theta$-term is a geometric phase reflecting a spin and statistics of the soliton. The spin appears to be equal a half of the fermionic number, thus establishing a relation between a fermionic number and spin (statistics) of a soliton.

The current algebra with $S^{d+1}$ target space (the second hierarchy) does not have solitons. All spatial configurations are contractible. However, spacetime configurations are not. As a result there is a $\theta$-term in the non-linear $\sigma$-model.
Contrary to the first hierarchy, a $\theta$-angle is not restricted and may take any value, depending on the value of the parameter in the fermionic model. We obtained it as a result of a reduction from current algebras on $d+2$-spheres.

Finally for the current algebras on $d+2$-dimensional spheres (the third hierarchy) both spatial and spacetime configurations are contractible. As a result, there is no $\theta$-term in the non-linear $\sigma$-model. However, the geometric phase exists due to nontrivial extensions of spacetime configurations to a $d+2$-dimensional sphere. This geometric phase is the WZ-term with a coefficient equal to the number of flavors of fermions. The non-linear $\sigma$-models in this case can be obtained by a regular gradient expansion.

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Homotopy groups $\pi_{n+k}(S^n)$ are the same for $n \geq k + 2$ (sometimes even for smaller $n$). For example $\pi_n(S^n) = Z$ for any $n$, $\pi_{n+1}(S^n) = Z_2$ for $n > 2$, but it is $Z$ for $n = 2$ and $0$ for $n = 1$, etc. See e.g., Ref. 11.

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