The existence and space-time decay rates of strong solutions to Navier-Stokes Equations in weighed $L^\infty(|x|^\gamma dx) \cap L^\infty(|x|^\beta dx)$ spaces

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Abstract: In this paper, we prove some results on the existence and space-time decay rates of global strong solutions of the Cauchy problem for the Navier-Stokes equations in weighed $L^\infty(\mathbb{R}^d, |x|^\gamma dx) \cap L^\infty(\mathbb{R}^d, |x|^\beta dx)$ spaces.

§1. Introduction

This paper studies the Cauchy problem of the incompressible Navier-Stokes equations (NSE) in the whole space $\mathbb{R}^d$ for $d \geq 2$,

$$\begin{aligned}
\partial_t u &= \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0,
\end{aligned}$$

which is a condensed writing for

$$\begin{aligned}
1 \leq k \leq d, \quad &\partial_t u_k = \Delta u_k - \sum_{l=1}^{d} \partial_l(u_l u_k) - \partial_k p, \\
\sum_{l=1}^{d} \partial_l u_l &= 0, \\
1 \leq k \leq d, \quad &u_k(0, x) = u_{0k}.
\end{aligned}$$

The unknown quantities are the velocity $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ of the fluid element at time $t$ and position $x$ and the pressure $p(t, x)$.

There is an extensive literature on the existence and decay rate of strong solutions of the Cauchy problem for NSE. Maria E. Schonbek \cite{1} established the decay of the homogeneous $H^m$ norms for solutions to NSE in two dimensions. She showed that if $u$ is a solution to NSE with an arbitrary $u_0 \in H^m \cap L^1(\mathbb{R}^2)$ with $m \geq 3$ then

\[\|D^\alpha u\|_2^2 \leq C_\alpha(t+1)^{-(|\alpha|+1)} \text{ and } \|D^\alpha u\|_{\infty} \leq C_\alpha(t+1)^{-(|\alpha|+\frac{1}{2})} \text{ for all } t \geq 1, \alpha \leq m.\]

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They showed that if \( u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), (d \leq p < \infty) \) and \( \|u_0\|_1 + \|u_0\|_p \) is small enough then there is a unique solution \( u \in BC([0, \infty); L^1 \cap L^p) \), which satisfies decay property
\[
\sup_{t>0} t^\frac{d}{2} (\|u\|_\infty + t^\frac{1}{2} \|Du\|_\infty + t^\frac{3}{2} \|D^2u\|_\infty) < \infty.
\]

Kato [3] studied strong solutions in the spaces \( L^q(\mathbb{R}^d) \) by applying the \( L^q - L^p \) estimates for the semigroup generated by the Stokes operator. He showed that there is \( T > 0 \) and a unique solution \( u \), which satisfies
\[
t^\frac{1}{2}(1-\frac{d}{q}) u \in BC([0,T); L^q), \text{ for } d \leq q \leq \infty,
\]
\[
t^\frac{1}{2}(2-\frac{d}{q}) \nabla u \in BC([0,T); L^q), \text{ for } d \leq q \leq \infty,
\]
as \( u_0 \in L^d(\mathbb{R}^d) \). He showed that \( T = \infty \) if \( \|u_0\|_{L^q(\mathbb{R}^d)} \) is small enough.

In 2005, Okihiro Sawada [5] obtained the decay rate of solution to NSE with initial data in \( L^d(\mathbb{R}^d) \). They showed that if \( \|u_0\|_{L^d(\mathbb{R}^d)} \) is small enough then there is a unique solution \( u \), which satisfies
\[
t^\frac{d}{2}(1+|\alpha|+2\alpha_0-\frac{d}{q}) D_x^\alpha D_t^{\alpha_0} u \in BC([0, \infty); L^q), \text{ for } q \geq d,
\]
\[
t^\frac{d}{2}(2+|\alpha|-\frac{d}{q}) D_x^\alpha p \in BC([0, \infty); L^q), \text{ for } q \geq d,
\]
where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d \) and \( \alpha_0 \in \mathbb{N} \). \( D_x^\alpha \) denotes \( \partial_x^{\alpha} = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d} \), \( D_t^{\alpha_0} = \partial^{\alpha_0}/\partial t^{\alpha_0} \).

In 2005, Okihiro Sawada [5] obtained the decay rate of solution to NSE with initial data in \( H^\frac{d}{2}-1(\mathbb{R}^d) \). He showed that every mild solution in the class
\[
u \in BC([0,T); \dot{H}^\frac{d}{2}-1) \text{ and } t^\frac{d}{2}(\frac{d}{2}-\frac{d}{q}) u \in BC([0,T); \dot{H}^\frac{d}{2}-1),
\]
for some \( T > 0 \) and \( p \in (2, \infty) \) satisfies
\[
\|u(t)\|_{\dot{H}^\frac{d}{2}-1+\alpha} \leq K_1 (K_2 \tilde{\alpha} t^{-\frac{\tilde{\alpha}}{q}}) \text{ for } \tilde{\alpha} := \alpha + \frac{d}{2} - \frac{d}{q}
\]
where constants \( K_1 \) and \( K_2 \) depend only on \( d, p, M_1, \) and \( M_2 \) with \( M_1 = \sup_{0 < t < T} \|u(t)\|_{\dot{H}^\frac{d}{2}-1} \) and \( M_2 = \sup_{0 < t < T} t^\frac{d}{2}(\frac{d}{2}-\frac{d}{q}) \|u(t)\|_{\dot{H}^\frac{d}{2}-1} \).

The time-decay properties are therefore well understood. However, there are few results on the spatial decay properties. Farwing and Sohr [7] showed a class of weighted \( |x|^{\alpha} \) weak solutions with second derivatives in space...
variables and one order derivatives in time variable in $L^s([0, +\infty); L^q)$ for $1 < q < 3/2, 1 < s < 2$ and $0 \leq 3/q + 2/s - 4 \leq \alpha < \min\{1/2, 3 - 3/q\}$ in the case of exterior domains. In [10], they also showed that there exists a class of weak solutions satisfying

$$
\| |x|^{\frac{\alpha}{2}} u\|_2^2 + \int_0^t \| |x|^{\frac{\alpha}{2}} \nabla u\|_2^2 \, dt \leq \begin{cases} 
C(u_0, f, \alpha) & \text{if } 0 \leq \alpha < \frac{1}{2}, \\
C(u_0, f, \alpha', \alpha) t^{\frac{q}{2} - \frac{1}{4}} & \text{if } \frac{1}{2} \leq \alpha < \alpha' < 1, \\
C(u_0, f)(t^{1/4} + t^{1/2}) & \text{if } \alpha = 1.
\end{cases}
$$

While in [11], a class of weak solutions

$$(1 + |x|^2)^{1/4} u \in L^\infty([0, +\infty); L^p(\mathbb{R}^3))$$

was constructed for $6/5 \leq p < 3/2$, which satisfies

$$
\| |x|^{\frac{\alpha}{2}} u\|_2^2 + \int_0^t \| |x|^{\frac{\alpha}{2}} \nabla u\|_2^2 \, dt \leq C(u_0, f)(t^{1/4} + t^{1/2}).
$$

In 2002 Takahashi [9] studied the existence and space-time decay rates of global strong solutions of the Cauchy problem for the Navier-Stokes equations in the weighted $L^\infty(\mathbb{R}^d, (1 + |x|)^\beta dx)$ spaces. Takahashi showed that if $u_0$ satisfies

$$
|\langle e^{t\Delta} u_0 \rangle(x)\rangle < \delta(1 + |x|)^{-\beta}, \quad |\langle e^{t\Delta} u_0 \rangle(x)\rangle < \delta(1 + t)^{-\beta},
$$

with sufficiently small $\delta$, then NSE has a global mild solution $u$ such that

$$
|u(x, t)| \leq C(1 + |x|)^{-\beta}, \quad |u(x, t)| \leq C(1 + t)^{-\frac{\beta}{2}},
$$

where $\beta$ is restricted by the condition $1 \leq \beta \leq d + 1$.

Takahashi also showed that if

$$
|u_0(x)| \leq c(1 + |x|)^{-\beta} \quad \text{for some } 0 < \beta \leq d,
$$

then

$$
|\langle e^{t\Delta} u_0 \rangle(x)\rangle \leq c(1 + |x|)^{-\beta}, \quad |\langle e^{t\Delta} u_0 \rangle(x)\rangle \leq c(1 + t)^{-\frac{\beta}{2}}.
$$

In this paper, we discuss the existence and space-time decay rates of global strong solutions of the Cauchy problem for the Navier-Stokes equations in the weighted $L^\infty(\mathbb{R}^d, |x|^\gamma dx) \cap L^\infty(\mathbb{R}^d, |x|^\beta dx)$ spaces. The spaces $L^\infty(\mathbb{R}^d, |x|^\gamma dx) \cap L^\infty(\mathbb{R}^d, |x|^\beta dx)$ are more general than the spaces $L^\infty(\mathbb{R}^d, (1 + |x|)^\beta dx)$. In particular, $L^\infty(\mathbb{R}^d, |x|^\gamma dx) \cap L^\infty(\mathbb{R}^d, |x|^\beta dx) = L^\infty(\mathbb{R}^d, (1 + |x|)^\beta dx)$ when $\gamma = 0$, and so this result improves the previous one.

The content of this paper is as follows: in Section 2, we state our main
theorems after introducing some notations. In Section 3, we first prove the some estimates concerning the heat semigroup with the Helmholtz-Leray projection and some auxiliary lemmas. Finally, in Section 4, we will give the proof of the main theorems.

§2. Statement of the results

Now, for \( T > 0 \), we say that \( u \) is a mild solution of NSE on \([0, T]\) corresponding to a divergence-free initial datum \( u_0 \) when \( u \) solves the integral equation

\[
u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}P\nabla.(u(\tau,.) \otimes u(\tau,.))d\tau.
\]

Above we have used the following notation: for a tensor \( F = (F_{ij}) \) we define the vector \( \nabla.F \) by \((\nabla.F)_i = \sum_{d=1}^d \partial_j F_{ij} \) and for two vectors \( u \) and \( v \), we define their tensor product \((u \otimes v)_{ij} = u_i v_j \). The operator \( P \) is the Helmholtz-Leray projection onto the divergence-free fields

\[
(Pf)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k,
\]

where \( R_j \) is the Riesz transforms defined as

\[
R_j = \frac{\partial_j}{\sqrt{-\Delta}} \text{ i.e. } \hat{R_j}g(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).
\]

The heat kernel \( e^{t\Delta} \) is defined as

\[
e^{t\Delta}u(x) = (4\pi t)^{-d/2}e^{-|x|^2/4t} \ast u(x).
\]

For a space of functions defined on \( \mathbb{R}^d \), say \( E(\mathbb{R}^d) \), we will abbreviate it as \( E \) and we do not distinguish between the vector-valued and scalar-value spaces of functions. Throughout the paper, we sometimes use the notation \( A \lesssim B \) as an equivalent to \( A \leq CB \) with a uniform constant \( C \). The notation \( A \simeq B \) means that \( A \lesssim B \) and \( B \lesssim A \). Let \( \beta \geq 0 \), we define the space \( L^\infty(|x|^\beta dx) := L^\infty(\mathbb{R}^d, |x|^\beta dx) \) which is made up by the measurable functions \( u \) such that

\[
\|u\|_{L^\infty(|x|^\beta dx)} := \operatorname{esssup}_{x \in \mathbb{R}^d} |x|^\beta |u(x)| < +\infty.
\]

Now we can state our result
Theorem 1. Assume that \( d \geq 1 \), and \( 0 \leq \gamma \leq 1 \leq \beta < d \). Then for all \( f \in L^\infty(|x|\gamma dx) \cap L^\infty(|x|\beta dx) \) we have

\[
\sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^{\tilde{\gamma} t^{(\gamma - 5)}} + |x|^{\alpha t^{\frac{1}{2} (1 - \alpha)}} + |x|^\beta t^{\frac{1}{2} (\beta - \beta)} |e^{t \Delta} f| \right) \lesssim \|f\|_{L^\infty(|x|\gamma dx)} + \|f\|_{L^\infty(|x|\beta dx)}
\]

for \( 0 \leq \tilde{\gamma} \leq \gamma, 0 \leq \alpha \leq 1 \), and \( 0 \leq \tilde{\beta} \leq \beta \).

Theorem 2. Let \( 0 \leq \gamma \leq 1 \leq \beta < d \) be fixed, then for all \( \tilde{\gamma}, \alpha \), and \( \tilde{\beta} \) satisfying

\[
0 \leq \tilde{\gamma} \leq \gamma, \tilde{\beta} \geq 0, \beta - 2 < \tilde{\beta} \leq \beta, 0 < \alpha < 1, \text{ and } \beta - \tilde{\beta} - 1 < \alpha < d - \tilde{\beta},
\]

there exists a positive constant \( \delta_{\gamma, \tilde{\gamma}, \alpha, \beta, \tilde{\beta}} \) such that for all \( u_0 \in L^\infty(|x|\gamma dx) \cap L^\infty(|x|\beta dx) \) with \( \text{div}(u_0) = 0 \) satisfying

\[
\sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^{\tilde{\gamma} t^{\frac{1}{2} (\gamma - 5)}} + |x|^{\alpha t^{\frac{1}{2} (1 - \alpha)}} + |x|^\beta t^{\frac{1}{2} (\beta - \beta)} |e^{t \Delta} u_0| \right) \leq \delta_{\gamma, \tilde{\gamma}, \alpha, \beta, \tilde{\beta}}.
\]

NSE has a global mild solution \( u \) on \( (0, \infty) \times \mathbb{R}^d \) such that

\[
\sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^\gamma + t^{\tilde{\gamma}} + |x|^\beta + t^{\tilde{\beta}} \right) |u(x, t)| < +\infty.
\]

Remark 1. Our result improves the previous result for \( L^\infty(\mathbb{R}^d, (1 + |x|)^\beta dx) \). This space, studied in [9], is a particular case of the space \( L^\infty(|x|\gamma dx) \cap L^\infty(|x|\beta dx) \) when \( \gamma = 0 \). Furthermore, we prove that Takahashi’s result holds true under a much weaker condition on the initial data. Indeed, from Lemma 4 and Theorem 11 it is easily seen that the condition (2) of Theorem 2 is weaker than the condition (1).

Remark 2. We invoke Theorem 1 to deduce that if \( u_0 \in L^\infty(|x|\gamma dx) \cap L^\infty(|x|\beta dx) \) and \( \|u_0\|_{L^\infty(|x|\gamma dx)} + \|u_0\|_{L^\infty(|x|\beta dx)} \) is small enough then the condition (2) of Theorem 2 is valid.

Theorem 3. Let \( 1 \leq \beta < d \) be fixed, then for all \( \alpha \) satisfying \( 0 < \alpha < 1 \), there exists a positive constant \( \delta_{\alpha, d} \) such that for all \( u_0 \in L^\infty(|x|dx) \cap L^\infty(|x|\beta dx) \) with \( \text{div}(u_0) = 0 \) satisfying

\[
\sup_{x \in \mathbb{R}^d, t > 0} |x|^\alpha t^{\frac{1}{2} (1 - \alpha)} |e^{t \Delta} u_0| \leq \delta_{\alpha, d},
\]

NSE has a global mild solution \( u \) on \( (0, \infty) \times \mathbb{R}^d \) such that

\[
\sup_{x \in \mathbb{R}^d, t > 0} \left( |x| + t^{\frac{1}{2}} \right) |u(x, t)| < +\infty
\]
and

\[ \sup_{x \in \mathbb{R}^d, 0 < t < T} |x|^\beta |u(x, t)| < +\infty, \text{ for all } T \in (0, \infty). \]

**Remark 3.** We invoke Theorem 1 to deduce that if \( u_0 \in L^\infty(|x|\,dx) \) and \( \|u_0\|_{L^\infty(|x|\,dx)} \) is small enough then the condition (4) of Theorem 3 is valid.

§3. Some auxiliary results

In this section we establish some auxiliary lemmas. We first prove a version of Young’s inequality type for convolutions in \( L^\infty(|x|\,dx) \) spaces.

**Lemma 1.** Assume that \( d \geq 1, 0 < \alpha < d, 0 < \beta < d \) and \( \alpha + \beta > d \). Then for all \( f \in L^\infty(|x|^\alpha \,dx) \) and for all \( g \in L^\infty(|x|^\beta \,dx) \) we have

\[ \|f * g\|_{L^\infty(|x|^\alpha + \beta - d \,dx)} \lesssim \|f\|_{L^\infty(|x|^\alpha \,dx)} \|g\|_{L^\infty(|x|^\beta \,dx)}. \]

**Proof.** Since \( f * g \) is bilinear on \( L^\infty(|x|^\alpha \,dx) \times L^\infty(|x|^\beta \,dx) \), we may assume \( \|f\|_{L^\infty(|x|^\alpha \,dx)} = \|g\|_{L^\infty(|x|^\beta \,dx)} = 1 \). We have

\[
(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\,dy = \int_{|y| < \frac{|x|}{2}} + \int_{\frac{|x|}{2} < |y| < \frac{3|x|}{4}} + \int_{|y| > \frac{3|x|}{4}} = I_1 + I_2 + I_3.
\]

From

\[ |f(x)| \leq |x|^{-\alpha}, \text{ and } |g(x)| \leq |x|^{-\beta}, \]

we get

\[ |I_1| \leq \int_{|y| < \frac{|x|}{4}} \frac{dy}{|x - y|^\alpha |y|^\beta} \leq 2^\alpha \int_{|y| < \frac{|x|}{4}} \frac{dy}{|y|^\beta} \lesssim \frac{1}{|x|^\alpha + \beta - d}. \]

\[ |I_2| \leq \int_{\frac{|x|}{4} < |y| < \frac{3|x|}{4}} \frac{dy}{|x - y|^\alpha |y|^\beta} \leq 2^\beta \int_{\frac{|x|}{4} < |y| < \frac{3|x|}{4}} \frac{dy}{|y|^\alpha} \lesssim \frac{1}{|x|^\alpha + \beta - d}. \]

\[ |I_3| \leq \int_{|y| > \frac{3|x|}{4}} \frac{dy}{|x - y|^\alpha |y|^\beta} \leq 3^\alpha \int_{|y| > \frac{3|x|}{4}} \frac{dy}{|y|^\alpha} \lesssim \frac{1}{|x|^\alpha + \beta - d}. \]

We thus obtain

\[ |(f * g)(x)| \lesssim \frac{1}{|x|^\alpha + \beta - d}. \]

The proof Lemma 1 is complete.\( \square \)

We now deduce the \( L^\infty(|x|^\alpha \,dx) - L^\infty(|x|^\beta \,dx) \) estimate for the heat semigroup.
Lemma 2. Assume that $d \geq 1$ and $0 \leq \gamma \leq \beta < d$. Then for all $f \in L^\infty(|x|^{\beta}dx)$ we have
\[
\|e^{t\Delta}f\|_{L^\infty(|x|^{\gamma}dx)} \lesssim t^{-\frac{\beta}{2}(\beta-\gamma)}\|f\|_{L^\infty(|x|^{\beta}dx)}, \text{ for } t > 0. \tag{5}
\]

**Proof.** We have
\[
(e^{t\Delta}f)(x) = \int_{\mathbb{R}^d} \frac{1}{t^{d/2}} E\left(\frac{x-y}{\sqrt{t}}\right) f(y) dy, \text{ where } E(x) = (4\pi)^{-d/2} e^{-\frac{|x|^2}{4}}.
\]
Recall the simate
\[
t^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} \lesssim |x|^{-\alpha} t^{-\frac{d}{2}(d-\alpha)}, \text{ for } 0 \leq \alpha \leq d. \tag{6}
\]
We first consider the case $0 < \gamma < \beta$. From the inequality (5) and Lemma 1, we have
\[
\|(e^{t\Delta}f)(x)\| \lesssim \int_{\mathbb{R}^d} \frac{\|f\|_{L^\infty(|x|^{\beta}dx)}}{t^{d/2(\beta-\gamma)|x-y|^{d-\beta}|y|^\beta}} dy \lesssim t^{-\frac{\beta}{2}(\beta-\gamma)|x|^{-\gamma}}\|f\|_{L^\infty(|x|^{\beta}dx)}.
\]
This proves (5).

We consider the case $0 = \gamma < \beta$. Applying Proposition 2.4 (b) in ([6], pp. 20) and note that $|x|^{-\beta} \in L^{\frac{d}{d-\beta},\infty}$
\[
|e^{t\Delta}f(x)| \lesssim t^{-\frac{d}{2}} \left\| \frac{1}{t^{d/2}} E\left(\frac{x-y}{\sqrt{t}}\right) \right\|_{L^{\frac{d}{d-\beta},\infty}} \|f\|_{L^{\frac{d}{d-\beta},\infty}} \lesssim t^{-\frac{d}{2}} \|E\|_{L^{\frac{d}{d-\beta},\infty}} \|f\|_{L^\infty(|x|^{\beta}dx)}.
\]
This proves (5).

Suppose finally that $0 \leq \gamma = \beta$. We have
\[
\int_{\mathbb{R}^d} \frac{1}{t^{d/2}} E\left(\frac{x-y}{\sqrt{t}}\right) f(y) dy = \int_{|y|<\frac{|x|}{2}} + \int_{|y|>\frac{|x|}{2}} = I_1 + I_2.
\]
From the inequality (6), we have
\[
|I_1| \lesssim \|f\|_{L^\infty(|x|^{\beta}dx)} \int_{|y|<\frac{|x|}{2}} |x-y|^{-d}|y|^{-\beta} dy \leq \|f\|_{L^\infty(|x|^{\beta}dx)} \left(\frac{|x|}{2}\right)^{-d} \int_{|y|<\frac{|x|}{2}} |y|^{-\beta} dy \simeq \|f\|_{L^\infty(|x|^{\beta}dx)} |x|^{-\beta}.
\]
\[
|I_2| \leq \|f\|_{L^\infty(|x|^{\beta}dx)} \int_{|y|>\frac{|x|}{2}} \frac{1}{t^{d/2}} E\left(\frac{x-y}{\sqrt{t}}\right) |y|^{-\beta} dy \leq \|f\|_{L^\infty(|x|^{\beta}dx)} \left(\frac{|x|}{2}\right)^{-\beta} \int_{y \in \mathbb{R}^d} \frac{1}{t^{d/2}} E\left(\frac{y}{\sqrt{t}}\right) dy = C \|f\|_{L^\infty(|x|^{\beta}dx)} |x|^{-\beta},
\]

where
\[ C = 2^\beta \int_{y \in \mathbb{R}^d} E(y) dy < +\infty. \]

Therefore,
\[ |e^{t\Delta} f(x)| \lesssim \|f\|_{L^\infty(|x|^\beta dx)} |x|^{-\beta}. \]

The proof of Lemma 2 is complete. \[ \square \]

We now deduce the \( L^\infty(|x|^{\gamma} dx) - L^\infty(|x|^\beta dx) \) estimate for the operator \( e^{t\Delta} \nabla \). As shown in [6], the kernel function \( F_t \) of \( e^{t\Delta} \nabla \) satisfies the following inequalities
\[ F_t(x) = t^{-\frac{d+1}{2}} F\left(\frac{x}{\sqrt{t}}\right), \quad |F(x)| \lesssim \frac{1}{(1 + |x|)^{d+1}}, \quad (7) \]
\[ |F_t(x)| \lesssim |x|^{-\alpha} t^{-\frac{1}{2}(d+1-\alpha)}, \quad \text{for } 0 \leq \alpha \leq d + 1. \quad (8) \]

By using the inequalities (7) and (8) and arguing as in the proof of Lemma 2, we can easily prove the following lemma.

**Lemma 3.** Assume that \( d \geq 1 \) and \( 0 \leq \gamma \leq \beta < d \). Then for all \( f \in L^\infty(|x|^\beta dx) \) we have
\[ \|e^{t\Delta} \nabla f\|_{L^\infty(|x|^{\gamma} dx)} \lesssim t^{-\frac{1}{2}(\beta+1-\gamma)}\|f\|_{L^\infty(|x|^\beta dx)}, \quad \text{for } t > 0. \]

**Lemma 4.** Let \( 0 \leq \gamma < \beta \leq d \). Assume that \( f \in S'(\mathbb{R}^d) \) and satisfies the following inequality
\[ \sup_{x \in \mathbb{R}^d, t > 0} (|x|^\gamma + |x|^\beta)(e^{t\Delta} f)(x) = C < +\infty, \quad (9) \]
then
\[ f \in L^\infty(|x|^{\gamma} dx) \cap L^\infty(|x|^\beta dx) \]
and
\[ \text{esssup}_{x \in \mathbb{R}^d} (|x|^\gamma + |x|^\beta) |f(x)| \leq C. \quad (10) \]

**Proof.** Since \( \frac{1}{\lambda |x|^\gamma} \in L^{\frac{d}{\beta} \infty} \cap L^{\frac{d}{\gamma} \infty} \) and \( L^{\frac{d}{\beta} \infty} \cap L^{\frac{d}{\gamma} \infty} \subset L^q \) for all \( q \) satisfying \( \frac{d}{\beta} < q < \frac{d}{\gamma} \), it follows that \( e^{t\Delta} f \in L^\infty(0, \infty; L^q) \) for all \( q \in \left(\frac{d}{\beta}, \frac{d}{\gamma}\right) \), by a compactness theorem in Banach space, there exists a sequence \( t_k \) which converges to 0 such that \( e^{t_k\Delta} f \) converges weakly to \( f' \) in \( L^q \) with \( f' \in L^q \). Since \( e^{t\Delta} \) is a continuous semigroup on \( S'(\mathbb{R}^d) \), it follows that \( f = f' \in L^q \). Since \( e^{t\Delta} \) is a continuous semigroup on \( L^q(\mathbb{R}^d), (1 \leq q < \infty) \), we get
\[ \lim_{k \to \infty} \|e^{t_k\Delta} f - f\|_{L^q} = 0, \quad \text{for } q \in \left(\frac{d}{\beta}, \frac{d}{\gamma}\right). \]
Therefore, there exists a subsequence \( t_{kj} \) of the sequence \( t_k \) such that
\[
\lim_{j \to \infty} (e^{t_{kj} \Delta} f)(x) = f(x) \text{ for almost everywhere } x \in \mathbb{R}^d.
\]  
(11)
The inequality (10) is deduced from equalities (9) and (11).

**Remark 4.** We invoke Lemma 4 for \( \gamma = 0 \) and Lemma 2 for \( \gamma = \beta \) to deduce that the condition (11) of Takahashi on the initial data is equivalent to the condition
\[
\|u_0\|_{L^\infty((1+|x|)^\beta dx)} \leq \delta.
\]

**Lemma 5.** Let \( \gamma, \theta \in \mathbb{R} \) and \( t > 0 \), then
(a) If \( \theta < 1 \) then
\[
\int_0^{t^\frac{1}{2}} (t - \tau)^{-\gamma} \tau^{-\theta} d\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^{t^\frac{1}{2}} (1-\tau)^{-\gamma} \tau^{-\theta} d\tau < \infty.
\]
(b) If \( \gamma < 1 \) then
\[
\int_0^t (t - \tau)^{-\gamma} \tau^{-\theta} d\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^1 (1-\tau)^{-\gamma} \tau^{-\theta} d\tau < \infty.
\]
(c) If \( \gamma < 1 \) and \( \theta < 1 \) then
\[
\int_0^t (t - \tau)^{-\gamma} \tau^{-\theta} d\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^1 (1-\tau)^{-\gamma} \tau^{-\theta} d\tau < \infty.
\]
The proof of this lemma is elementary and may be omitted.

Let us recall the following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [6], p. 227).

**Theorem 4.** Let \( E \) be a Banach space, and \( B : E \times E \to E \) be a continuous bilinear map such that there exists \( \eta > 0 \) so that
\[
\|B(x, y)\| \leq \eta \|x\| \|y\|,
\]
for all \( x \) and \( y \) in \( E \). Then for any fixed \( y \in E \) such that \( \|y\| \leq \frac{1}{\eta} \), the equation \( x = y - B(x, x) \) has a unique solution \( x \in E \) satisfying \( \|x\| \leq \frac{1}{2\eta} \).

§ 4. Proofs of Theorems 1, 2, and 3

In this section we will give the proofs of Theorems 1, 2, and 3. We now need eight more lemmas. In order to proceed, we define an auxiliary space
Let $\alpha, \beta,$ and $T$ be such that $0 \leq \alpha \leq \beta < d, 0 < T \leq +\infty$, we define the auxiliary space $K^{\beta}_{\alpha,T}$ which is made up by the measurable functions $u(t, x)$ such that
\[
\operatorname{esssup}_{x \in \mathbb{R}^d, 0 < t < T} |x|^\alpha t^{\frac{1}{2}((\beta - \alpha))} |u(x, t)| < +\infty.
\]
The auxiliary space $K^{\beta}_{\alpha,T}$ is equipped with the norm
\[
\|u\|_{K^{\beta}_{\alpha,T}} := \operatorname{esssup}_{x \in \mathbb{R}^d, 0 < t < T} |x|^\alpha t^{\frac{1}{2}((\beta - \alpha))} |u(x, t)|.
\]
We rewrite Lemma 2 as follows

**Lemma 6.** Assume that $d \geq 1$ and $0 \leq \alpha \leq \beta < d$. Then for all $f \in L^\infty(|x|^\beta dx)$ we have $e^{\Delta f} \in K^{\beta}_{\alpha,T}$ and $\|e^{\Delta f}\|_{K^{\beta}_{\alpha,T}} \leq C\|f\|_{L^\infty(|x|^\beta dx)}$, where $C$ is a positive constant independent of $T$.

**Lemma 7.** Assume that $d \geq 1$ and $0 \leq \alpha \leq \beta < d$. Then
\[
K^{\beta}_{\alpha,T} \subset K^{\beta}_{\beta,T} \cap K^{\beta}_{0,T}.
\]
The proof of this lemma is elementary and may be omitted.

**Lemma 8.** Assume that $d \geq 1, T < +\infty$, and $0 \leq \alpha \leq \beta \leq \tilde{\beta} < d$. Then
\[
K^{\beta}_{\alpha,T} \subset K^{\beta}_{\beta,T} \subset K^{\beta}_{\tilde{\beta},T}.
\]
The proof of this lemma is elementary and may be omitted.

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u, v)(t)$ defined by
\[
B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta} P \nabla \cdot (u(\tau) \otimes v(\tau)) d\tau. \tag{12}
\]

**Lemma 9.** Let $\beta, \tilde{\beta}, \hat{\beta}$, and $\alpha$ be such that
\[
0 \leq \beta < d, \tilde{\beta} > \beta - 2, 0 \leq \beta \leq \beta, 0 < \alpha < 1, \beta - \tilde{\beta} - 1 < \alpha < d - \tilde{\beta},
\]
\[
0 \leq \tilde{\beta} \leq \beta, \text{ and } \alpha + \tilde{\beta} - 1 < \beta \leq \alpha + \hat{\beta}.
\]
Then the bilinear operator $B$ is continuous from $K^{1}_{\alpha,T} \times K^{\beta}_{\beta,T}$ into $K^{\beta}_{\beta,T}$ and the following inequality holds
\[
\|B(u, v)\|_{K^{\beta}_{\beta,T}} \leq C\|u\|_{K^{1}_{\alpha,T}} \|v\|_{K^{\beta}_{\beta,T}}, \tag{13}
\]
where $C$ is a positive constant independent of $T$. 

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Proof. Since $B(.,.)$ is bilinear on $K_{\alpha,T}^1 \times K_{\beta,T}^\beta$, we may assume $\|u\|_{K_{\alpha,T}^1} = \|v\|_{K_{\beta,T}^\beta} = 1$. From

$$|(u \otimes v)| \leq |y|^{-\alpha+\beta} t^{-\frac{1}{2}(1-\alpha+\beta-\delta)}$$

by using Lemma 3, we have

$$|e^{(t-s)\Delta} B \nabla (u \otimes v)| \leq |x|^{-\hat{\beta}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha+\beta-\delta)} t^{\frac{1}{2}(1-\alpha+\beta-\delta)}}$$

then applying Lemma 5 (c), we get

$$|B(u,v)| \leq |x|^{-\hat{\beta}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha+\beta-\delta)} t^{\frac{1}{2}(1-\alpha+\beta-\delta)}} ds \simeq |x|^{-\hat{\beta}} t^{-\frac{1}{2}(\beta-\delta)}.$$ 

This proves Lemma 9.

Note that since $\alpha > \beta - \hat{\beta} - 1$ and $\hat{\beta} > \alpha + \hat{\beta} - 1$, it follows that the conditions

$$\frac{1-\alpha+\beta-\delta}{2} < 1 \quad \text{and} \quad \frac{1+\alpha+\beta-\delta}{2} < 1$$

are valid. So we can apply Lemma 5 (c). \(\square\)

Lemma 10. Assume that NSE has a mild solution $u \in K_{\hat{\alpha},T}^1$ for some $\hat{\alpha} \in (0,1)$ with initial data $u_0 \in L^\infty(|x|dx)$ then $u \in K_{\alpha,T}^1$ for all $\alpha \in [0,1]$.

Proof. From $u = e^{\Delta} u_0 + B(u,u)$, applying Lemmas 6 and 9 with $\beta = 1$ and $\alpha = \hat{\beta} = \hat{\alpha}$, we get $u \in K_{\beta,T}^1$ for all $\beta \in (\hat{\alpha} - (1 - \hat{\alpha}), 2\hat{\alpha}) \cap [0,1]$. Applying again Lemmas 6 and 9 with $\beta = 1$, $\alpha = \hat{\alpha}$, and $\hat{\beta} \in (\hat{\alpha} - (1 - \hat{\alpha}), 2\hat{\alpha}) \cap [0,1]$ to get $u \in K_{\beta,T}^1$ for all $\hat{\beta} \in (\hat{\alpha} - (1 - \hat{\alpha}), 2\hat{\alpha}) \cap [0,1]$. By induction, we get $u \in K_{\beta,T}^1$ for all $\hat{\beta} \in (\hat{\alpha} - n(1 - \hat{\alpha}), (n + 1)\hat{\alpha}) \cap [0,1]$ with $n \in \mathbb{N}$. Since $\hat{\alpha} \in (0,1)$, it follows that there exists sufficiently large $n$ satisfying

$$(\hat{\alpha} - n(1 - \hat{\alpha}), (n + 1)\hat{\alpha}) \supset [0,1].$$

This proves Lemma 10. \(\square\)

Lemma 11. Let $\beta$ be a fixed number in the interval $[0,d)$. Assume that NSE has a mild solution $u \in \bigcap_{\alpha \in [0,1]} K_{\alpha,T}^1 \cap K_{\beta,T}^\beta$, for some $\beta \in [0,\beta] \cap (\beta - 2, \beta]$ with initial data $u_0 \in L^\infty(|x|^{\beta}dx)$, then $u \in K_{\beta,T}^\beta$ for all $\beta \in [0,\beta] \cap (\beta - 1, \beta + 1]$.

Proof. We first prove that $u \in K_{\beta,T}^\beta$ for all $\beta \in [0,\beta] \cap (\beta - 1, \beta + 1)$. Let $\alpha_1$ and $\alpha_2$ be such that

$$\max\{\beta - \beta - 1, \beta - \beta + 1, 0\} < \alpha_1 < 1$$

Then...
and
\[
\max\{\hat{\beta} - \tilde{\beta}, 0\} < \alpha_2 < \min\{1, \hat{\beta} - \tilde{\beta} + 1\}.
\]

We split the integral given in (12) into two parts coming from the subintervals \((0, \frac{t}{2})\) and \((\frac{t}{2}, t)\)
\[
B(u, u)(t) = \int_0^{\frac{t}{2}} e^{(t-\tau)\Delta} \mathbb{P} \nabla.(u \otimes u) d\tau + \int_{\frac{t}{2}}^t e^{(t-\tau)\Delta} \mathbb{P} \nabla.(u \otimes u) d\tau = I_1 + I_2.
\]

Since \(u \in \bigcap_{\alpha \in [0,1]} K_{\alpha,T}^{\beta_1}\), it follows that
\[
|u(x, t)| \lesssim |x|^{-\alpha_1 t^{-\frac{1}{2}}(1-\alpha_1)}, \quad (14)
\]
and since \(u \in K^{\beta}_{\beta,T}\), it follows that
\[
|u(x, t)| \lesssim |x|^{-\beta t^{-\frac{1}{2}}(\beta-\tilde{\beta})}. \quad (15)
\]

From the inequalities (14) and (16), and Lemma 3, we get
\[
|e^{(t-\tau)\Delta} \mathbb{P} \nabla.(u \otimes u)| \lesssim \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\beta-\beta)} t^{\frac{1}{2}(1-\alpha_1+\beta-\beta)}}.
\]

Then applying Lemma 5 (a), we have
\[
|I_1| \lesssim |x|^{-\beta} \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\beta-\beta)} t^{\frac{1}{2}(1-\alpha_1+\beta-\beta)}} ds \simeq |x|^{-\beta} t^{-\frac{1}{2}(\beta-\tilde{\beta})}. \quad (17)
\]

Then applying Lemma 5 (b), we have
\[
|I_2| \lesssim |x|^{-\beta} \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_2+\beta-\beta)} t^{\frac{1}{2}(1-\alpha_2+\beta-\beta)}} ds \simeq |x|^{-\beta} t^{-\frac{1}{2}(\beta-\tilde{\beta})}. \quad (18)
\]

From the inequalities (17) and (18), we get \(B(u, u) \in K^{\beta}_{\beta,T}\), and from \(u = e^{t\Delta} u_0 + B(u, u)\) and Lemma 6 we have \(u \in K^{\beta}_{\beta,T}\). This proves the result.
We now prove \( u \in K^\beta_{\tilde{\beta},T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \beta) \). Indeed, if \( \tilde{\beta} > \beta - 1 \) then \( u \in K^\beta_{\tilde{\beta},T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \tilde{\beta} + 1) = [0, \beta] \cap (\tilde{\beta} - 1, \beta) \) and so the lemma is proved. In the case \( \tilde{\beta} \leq \beta - 1 \), in exactly the same way, since \( u \in K^\beta_{\tilde{\beta},T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \tilde{\beta} + 1) \), it follows that \( u \in K^\beta_{\tilde{\beta},T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \tilde{\beta} + 2) = [0, \beta] \cap (\tilde{\beta} - 1, \beta) \). Therefore the proof of Lemma 11 is complete. \( \square \)

**Lemma 12.** Assume that NSE has a mild solution \( u \in \bigcap_{\alpha \in [0,1]} K^1_{\alpha,T} \bigcap_{\beta \in [\tilde{\beta},\beta]} K^\beta_{\tilde{\beta},T} \) for some \( \tilde{\beta} \in [0, \beta] \) with initial data \( u_0 \in L^\infty(|x|^\beta \, dx) \). Then \( u \in K^\beta_{\tilde{\beta},T} \) for all \( \tilde{\beta} \in [0, \beta] \).

**Proof.** We first prove that \( u \in K^\beta_{\tilde{\beta},T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \tilde{\beta}) \).

We split the integral given in (12) into two parts coming from the subintervals \((0, \frac{1}{2})\) and \((\frac{1}{2}, t)\)

\[
B(u, u)(t) = \int_0^{\frac{1}{2}} e^{(t-s)\Delta} \nabla \cdot (u \otimes u) \, ds + \int_{\frac{1}{2}}^t e^{(t-s)\Delta} \nabla \cdot (u \otimes u) \, ds = I_1 + I_2.
\]

Let \( \alpha_1 \) be such that \( 0 < \alpha_1 < 1 \). Since \( u \in K^1_{\alpha_1,T} \bigcap K^\beta_{\tilde{\beta},T} \), it follows that

\[
|u(x, t)| \lesssim |x|^{-\alpha_1} t^{-\frac{1}{4}(1-\alpha_1)}, \quad (19)
\]

\[
|u(x, t)| \lesssim |x|^{-\beta}. \quad (20)
\]

From the inequalities (19) and (20), and Lemma 3 we get

\[
|e^{(t-s)\Delta} \nabla \cdot (u \otimes u)| \lesssim |x|^{-\beta} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\beta-\tilde{\beta})} t^{\frac{1}{2}(1-\alpha_1)}}.
\]

Then applying Lemma 3 (a), we have

\[
|I_1| \lesssim |x|^{-\tilde{\beta}} \int_0^{\frac{1}{2}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\beta-\tilde{\beta})} t^{\frac{1}{2}(1-\alpha_1)}} \, ds \approx |x|^{-\tilde{\beta}} t^{-\frac{1}{2}(\beta-\tilde{\beta})}. \quad (21)
\]

Since \( u \in K^1_{0,T} \bigcap K^\beta_{\tilde{\beta},T} \), it follows that

\[
|u(x, t)| \lesssim t^{-\frac{1}{2}} \text{ and } |u(x, t)| \lesssim |x|^{-\tilde{\beta}} t^{-\frac{1}{2}(\beta-\tilde{\beta})}. \quad (22)
\]

From the inequality (22), and Lemma 3 we get

\[
|e^{(t-s)\Delta} \nabla \cdot (u \otimes u)| \lesssim |x|^{-\tilde{\beta}} \frac{1}{(t-s)^{\frac{1}{2}(1+\beta-\tilde{\beta})} t^{\frac{1}{2}(1+\beta-\tilde{\beta})}}.
\]
Then applying Lemma 5 (b), we obtain
\[
|I_2| \lesssim |x|^{-\tilde{\beta}} \int_0^t \frac{1}{\frac{3}{4} (t - s)^{\frac{1}{2}} (1+\beta - \gamma) t^\frac{1}{2} (1+\beta - \gamma)} ds \simeq |x|^{-\tilde{\beta}} t^{-\frac{1}{2} (\beta - \tilde{\beta})}. \tag{23}
\]

From the inequalities (21) and (23), we get \( B(u, u) \in K^\beta_{\tilde{\beta}, T} \). From \( u = e^{t\Delta} u_0 + B(u, u) \) and Lemma 6 we deduce \( u \in K^\beta_{\tilde{\beta}, T} \). This proves the result. Therefore, we get \( u \in K^\beta_{\tilde{\beta}, T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \beta] \).

We now prove that \( u \in K^\beta_{\tilde{\beta}, T} \) for all \( \tilde{\beta} \in [0, \beta] \). Indeed, in exactly the same way, since \( u \in K^\beta_{\tilde{\beta}, T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 1, \beta] \), it follows that \( u \in K^\beta_{\tilde{\beta}, T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - 2, \beta] \). By induction, we get \( u \in K^\beta_{\tilde{\beta}, T} \) for all \( \tilde{\beta} \in [0, \beta] \cap (\tilde{\beta} - n, \beta] \) with \( n \in \mathbb{N} \). However, there exists a sufficiently large number \( n \) satisfying \( \tilde{\beta} - n < 0 \) and therefore \( u \in K^\beta_{\tilde{\beta}, T} \) for all \( \tilde{\beta} \in [0, \beta] \). The proof of Lemma 12 is complete.

**Lemma 13.** Let \( 0 \leq \beta < d \) be fixed, then for all \( \alpha \) and \( \tilde{\beta} \) satisfying
\[
\tilde{\beta} \geq 0, 0 < \alpha < 1, \beta - 2 < \beta \leq \beta, \text{ and } \beta - \tilde{\beta} - 1 < \alpha < d - \tilde{\beta},
\]
there exists a positive constant \( \delta_{\alpha, \beta, \tilde{\beta}, d} \) such that for all \( u_0 \in L^\infty(|x| dx) \cap L^\infty(|x|^\beta dx) \) with \( \text{div}(u_0) = 0 \) satisfying
\[
\sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^{\alpha} t^{\frac{1}{2}(1-\alpha)} + |x|^\beta t^{\frac{1}{2}(\beta - \tilde{\beta})} \right) |e^{t\Delta} u_0| \leq \delta_{\alpha, \beta, \tilde{\beta}, d}, \tag{24}
\]

NSE has a global mild solution \( u \) on \((0, \infty) \times \mathbb{R}^d\) such that
\[
\sup_{x \in \mathbb{R}^d, t > 0} \left( |x| + t^{\frac{1}{2}} + |x|^\beta + t^{\frac{\beta}{2}} \right) |u(x, t)| < +\infty. \tag{25}
\]

**Proof.** Applying Lemma 9 we deduce that the bilinear operator \( B \) is bounded from \( K^1_{\alpha, \infty} \times K^1_{\alpha, \infty} \) into \( K^1_{\alpha, \infty} \) and from \( K^1_{\alpha, \infty} \times K^\beta_{\tilde{\beta}, \infty} \) into \( K^\beta_{\tilde{\beta}, \infty} \). Therefore, the bilinear operator \( B \) is bounded from
\[
(K^1_{\alpha, \infty} \cap K^\beta_{\tilde{\beta}, \infty}) \times (K^1_{\alpha, \infty} \cap K^\beta_{\tilde{\beta}, \infty}) \text{ into } (K^1_{\alpha, \infty} \cap K^\beta_{\tilde{\beta}, \infty}),
\]
where the space \( K^1_{\alpha, \infty} \cap K^\beta_{\tilde{\beta}, \infty} \) is equipped with the norm
\[
\|u\|_{K^1_{\alpha, \infty} \cap K^\beta_{\tilde{\beta}, \infty}} := \max\{\|u\|_{K^1_{\alpha, \infty}}, \|u\|_{K^\beta_{\tilde{\beta}, \infty}}\}.
\]
Applying Theorem 4 to the bilinear operator $B$, we deduce that there exists a positive constant $\delta_{\alpha,\beta,\tilde{\beta},d}$ such that for all $u_0 \in L^\infty(|x|dx) \cap L^\infty(|x|^\beta dx)$ with $\text{div}(u_0) = 0$ satisfying

$$ \left\| e^{t\Delta}u_0 \right\|_{K^\beta_{\alpha,\infty} \cap K^\beta_{\tilde{\beta},\infty}} \leq \delta_{\alpha,\beta,\tilde{\beta},d}, $$

then NSE has a unique mild solution $u$ satisfying

$$ u \in K^\beta_{\alpha,\infty} \cap K^\beta_{\tilde{\beta},\infty}. $$

Applying Lemmas 10, 11, and 12, we get $u \in K^\beta_{\tilde{\beta},\infty}$ for all $\tilde{\beta} \in [0, \beta]$ and $u \in K^\beta_{\alpha,\infty}$ for all $\alpha \in [0, 1]$. The proof of Lemma 13 is now complete.

**Proof of Theorem 1**

Since $|x| \leq |x|^\gamma + |x|^\beta$, it follows that

$$ \left\| f \right\|_{L^\infty(|x|dx)} \leq \left\| f \right\|_{L^\infty(|x|^\gamma dx)} + \left\| f \right\|_{L^\infty(|x|^\beta dx)}. $$

From Lemma 2 we have

$$ |x|^\alpha t^{\frac{1}{\gamma}(1-\alpha)}|e^{t\Delta}u_0| \leq \left\| f \right\|_{L^\infty(|x|dx)} \leq \left\| f \right\|_{L^\infty(|x|^\gamma dx)} + \left\| f \right\|_{L^\infty(|x|^\beta dx)}, $$

$$ |x|^\gamma t^{\frac{1}{\gamma}(\gamma-\tilde{\gamma})}|e^{t\Delta}u_0| \leq \left\| f \right\|_{L^\infty(|x|^\gamma dx)} + \left\| f \right\|_{L^\infty(|x|^\beta dx)}.$$}

This proves Theorem 1.

**Proof of Theorem 2**

Since $L^\infty(|x|dx) \subset L^\infty(|x|^\gamma dx) \cap L^\infty(|x|^\beta dx)$, it follows that $u_0 \in L^\infty(|x|dx)$. Applying Lemma 13 there exists a positive constant $\delta_{\alpha,\beta,\tilde{\beta},d}$ such that if

$$ \sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^\alpha t^{\frac{1}{\gamma}(1-\alpha)} + |x|^\beta t^{\frac{1}{\gamma}(\beta-\tilde{\beta})} \right) |e^{t\Delta}u_0| \leq \delta_{\alpha,\beta,\tilde{\beta},d}, $$

NSE has a global mild solution $u$ on $(0, \infty) \times \mathbb{R}^d$ such that

$$ \sup_{x \in \mathbb{R}^d, t > 0} \left( |x| + t^{\frac{1}{\gamma}} + |x|^\beta + t^{\frac{\beta}{\gamma}} \right) |u(x, t)| < +\infty. $$

Applying Lemma 13 for $\beta = \gamma$ then there exists a positive constant $\delta_{\alpha,\gamma,\tilde{\gamma},d}$ such that if

$$ \sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^\alpha t^{\frac{1}{\gamma}(1-\alpha)} + |x|^\gamma t^{\frac{1}{\gamma}(\gamma-\tilde{\gamma})} \right) |e^{t\Delta}u_0| \leq \delta_{\alpha,\gamma,\tilde{\gamma},d}, $$

$$ \sup_{x \in \mathbb{R}^d, t > 0} \left( |x| + t^{\frac{1}{\gamma}} + |x|^\gamma + t^{\frac{\gamma}{\gamma}} \right) |u(x, t)| < +\infty. $$

Applying Lemma 13 for $\beta = \tilde{\gamma}$ then there exists a positive constant $\delta_{\alpha,\beta,\tilde{\beta},d}$ such that if

$$ \sup_{x \in \mathbb{R}^d, t > 0} \left( |x|^\alpha t^{\frac{1}{\gamma}(1-\alpha)} + |x|^\gamma t^{\frac{1}{\gamma}(\gamma-\tilde{\gamma})} \right) |e^{t\Delta}u_0| \leq \delta_{\alpha,\gamma,\tilde{\gamma},d}, $$

$$ \sup_{x \in \mathbb{R}^d, t > 0} \left( |x| + t^{\frac{1}{\gamma}} + |x|^\gamma + t^{\frac{\gamma}{\gamma}} \right) |u(x, t)| < +\infty. $$
NSE has a global mild solution \( u \) on \( (0, \infty) \times \mathbb{R}^d \) such that
\[
\sup_{x \in \mathbb{R}^d, t > 0} (|x| + t^\gamma + |x|^{\gamma} + t^\gamma) |u(x, t)| < +\infty.
\]

Therefore, if \( u_0 \) satisfies the following inequality
\[
\sup_{x \in \mathbb{R}^d, t > 0} (|x|^{\gamma} t^{\theta(\gamma)} + |x|^\alpha t^{\theta(1-\alpha)} + |x|^\beta t^{\theta(\beta-\tilde{\beta})}) |e^{t\Delta} u_0| \leq \min\{\delta_{a,\beta,d}, \delta_{a,\gamma,d} \}
\]
NSE has a global mild solution \( u \) on \( (0, \infty) \times \mathbb{R}^d \) such that (3).

The proof of Theorem 2 is complete.

Proof of Theorem 3

Applying Lemma 9 we deduce that the bilinear operator \( B \) is bounded from \( K_{1,\infty}^1 \times K_{1,\infty}^1 \) into \( K_{1,\infty}^1 \). Applying Theorem 4 to the bilinear operator \( B \), we deduce that there exists a positive constant \( \delta_{a,d} \) such that for all \( u_0 \in L^\infty(|x|dx) \) with \( \text{div}(u_0) = 0 \) satisfying
\[
\|e^{t\Delta} u_0\|_{K_{1,\infty}^1} \leq \delta_{a,d},
\]
then NSE has a unique mild solution \( u \) satisfying \( u \in K_{1,\infty}^1 \). Applying Lemma 10, we have \( u \in \bigcap_{\alpha \in [0,1]} K_{1,\infty}^1 \).

We prove that \( u \in K_{\beta,T}^\gamma \) for all \( T \in (0, \infty) \). Indeed, let \( \gamma \) be such that \( \gamma \in [1, \beta] \cap (\alpha, \alpha + 2) \). Applying Lemma 8, we have \( u \in K_{1,T}^\gamma \), then using Lemmas 11, we get \( u \in K_{1,T}^\gamma \), in exactly the same way, using again Lemmas 11 since \( u \in K_{1,T}^\gamma \) for \( \gamma \in [1, \beta] \cap (\alpha, \alpha + 2) \), it follows that \( u \in K_{1,T}^\gamma \) for \( \gamma \in [1, \beta] \cap (\alpha, \alpha + 4) \). By induction, we get \( u \in K_{1,T}^\gamma \) for \( \gamma \in [1, \beta] \cap (\alpha, \alpha + 2n) \). However, there exists a sufficiently large number \( n \) satisfying \( \alpha + 2n > \beta \) and therefore \( u \in K_{1,T}^\gamma \). The proof of Theorem 3 is complete.

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