Minimum error discrimination between similarity transformed quantum states

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May 31, 2011

Abstract

Using the known necessary and sufficient conditions for minimum error discrimination (MED), first it is shown that a Helstrom family of ensembles is equivalent to these conditions and then by a convex combination of the initial states (the states which we try to discriminate them) and the corresponding conjugate states, a more suitable and convenient form for the MED conditions is extracted, so that optimal set of measurements and corresponding optimal success probability of discrimination can be determined. Then, using the introduced identity, MED between $N$ similarity transformed equiprobable quantum states is investigated. As a special case, MED between the so called group covariant or symmetric states is considered.

Keywords: Minimum error discrimination (MED), Helstrom family of ensembles, similarity transformed quantum states, POM, group covariant states

PACS Index: 01.55.+b, 02.10.Yn

1 Introduction

The theory of quantum information and communication concerns the transmission of information using quantum states and channels. The transmission party encodes a message onto a set of quantum states $\rho_i$ with prior probability $p_i$ for each state $\rho_i$. The task of the receiving party is to decode the received message, i.e., finding the best measurement strategy based upon the knowledge of the signal states and their prior probabilities. One possibility is to choose the strategy that minimizes the probability of error. In order that a set of probability operator measure (POM) minimizes the error probability of a detection, it must satisfy a

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known set of necessary and sufficient conditions [1]-[7]. Using the no-signaling principle, an upper bound to the success probability in the minimum-error state discrimination has been given [8]. The problem of minimum error probability discrimination of symmetric quantum states has been studied in [9,10,11]. For \( N \) symmetric pure states occurring with equal prior probabilities, the optimal minimum error measurement is the square root measurement (SRM) [12]. In this paper, first we show that the known necessary and sufficient conditions for MED [5] are equivalent to a Helstrom family of ensembles [1], i.e., the optimal conditions for MED are fulfilled by a Helstrom family of ensembles. Then, we extract a suitable identity from the MED conditions which is more convenient for the study of mixed quantum states discrimination with optimal success probability. By using the introduced identity, we study MED between \( N \) equiprobable similarity transformed qudit states (quantum states in \( d \)-dimensional Hilbert space) \( \rho_1,\ldots,\rho_N \), defined by \( \rho_i = U_i \rho_1 U_i^{-1} \), so that the unitary operators \( U_1 \equiv I_d, U_2,\ldots,U_N \) generate a (finite or continuum) subgroup of the unitary group \( U(d) \). It is shown that, in the case in which \( U_i \)'s generate a non-abelian subgroup, the state space is irreducible (none of the states have invariant components under the action of \( U_i \)) and the optimal discrimination can be achieved. In the case that \( U_i \)'s generate an abelian subgroup, for instance rotations about a fixed axis, the state space is reducible and although there is no closed-form formula in general case but the procedure can be applied in each case in accordance to that case.

2 Minimum error discrimination between quantum states and Helstrom family of ensembles

In general, the measurement strategy is described in terms of a set of non-negative-definite operators called the probability operator measure (POM). The measurement outcome labeled by \( i \) is associated with the element \( \Pi_i \) of POM that has all the eigenvalues either positive or zero. The POM elements must add up to the identity operator, i.e., \( \sum_i \Pi_i = 1 \). The probability that the receiver will observe the outcome \( i \) given that the transmitted signal is \( \rho_j \) is given by \( p(i|j) = Tr(\Pi_i \rho_j) \). Assume that given states \( \rho_1,\rho_2,\ldots,\rho_N \) have prior probabilities \( p_1, p_2,\ldots,p_N \), respectively (\( p_i \geq 0, \sum_i p_i = 1 \)). It follows that the probability for correctly identifying states \( \rho_i \) is given by

\[
p = 1 - p_{error} = \sum_{i=1}^{N} p_i Tr(\rho_i \Pi_i),
\]

where \( p_{error} \) is the error probability.

The necessary and sufficient condition that lead to the minimum-error probability is known to be [5]

\[
\sum_{i=1}^{N} p_i \Pi_i \rho_i - p_j \rho_j \geq 0 \quad \forall \ j = 1,\ldots,N.
\]

While these conditions do give us a starting point for finding minimum error POM’s, they do not themselves provide a great insight into either the form of minimum error measurement strategies, or into how error probability depends on the set of possible states. For this we
must examine the solutions to these conditions, and there are not many solutions. The essential difficulty in solving the conditions directly is that all of the variables \( \Pi_k \) appear in each condition, and they are not independent variables.

In the following we use the fact that, the inequality (2) indicates

\[
\sum_{i=1}^{N} p_i \rho_i \Pi_i - p_j \rho_j = \alpha_j \tau_j, \quad \forall \quad j = 1, \ldots, N. \tag{3}
\]

where \( \alpha_j \geq 0 \) and \( \tau_j \)'s are positive operators. By taking the trace of both sides of Eqs. (3) and using (1), we obtain \( \alpha_j = p_{\text{opt}} - p_j \) (In [14] it has indicated that \( p_{\text{opt}} \geq p_j \)). If we postmultiply Eq. (3) by \( \Pi_j \) and summing up over \( j \), we get

\[
\sum_{j=1}^{N} (p_{\text{opt}} - p_j) \tau_j \Pi_j = 0.
\]

Because both \( \tau_j \) and \( \Pi_j \) are positive operators it follows that \( (p_{\text{opt}} - p_j) \tau_j \Pi_j = 0 \) for every \( j = 1, \ldots, N \). This indicates that \( \tau_j \Pi_j = 0 \) for all \( j = 1, \ldots, N \). Moreover, in order that the optimal measurement operators \( \Pi_j \) can be constructed, the states \( \tau_j \) must be possess at least one zero eigenvalue, i.e., \( \tau_j \)s are not full rank. By denoting the term \( \sum_{i=1}^{N} p_i \rho_i \Pi_i \) by \( \mathcal{M} \), the necessary and sufficient conditions for minimum-error probability, i.e., the conditions (3) take the following form

\[
\mathcal{M} = p_j \rho_j + (p_{\text{opt}} - p_j) \tau_j, \quad \forall \quad j. \tag{4}
\]

For the case that all given states \( \rho_i, i = 1, \ldots, N \) have equal prior probability \( p_i = \frac{1}{N} \), we have

\[
\mathcal{M} = \frac{1}{N} \rho_j + (p_{\text{opt}} - \frac{1}{N}) \tau_j, \quad \forall \quad j. \tag{5}
\]

Although, we will use the identity (4) - equivalent to necessary and sufficient conditions (2) - in order to discriminate quantum states with minimum error, in the following we show that a Helstrom family of ensembles is necessary and sufficient for realizing a minimum error measurement, i.e., the necessary and sufficient conditions (2) are fulfilled by a Helstrom family of ensembles.

### 2.1 Equivalence between optimality conditions and Helstrom family of ensembles

First let us to recall the definition of a Helstrom family of ensembles. A set of \( N \)-numbers \( \{\tilde{p}_i, \rho_i; 1 - \tilde{p}_i, \tau_i\}_{i=1}^{N} \) is called a weak Helstrom family (of ensembles) if there exist \( N \)-numbers of binary probability discriminations \( \{\tilde{p}_i, 1 - \tilde{p}_i\}_{i=1}^{N} (0 < \tilde{p}_i \leq 1) \) and states \( \{\tau_i \in s\}_{i=1}^{N} \) satisfying

\[
p = \frac{p_i}{\tilde{p}_i} \leq 1 \tag{6}
\]

and

\[
p_i \rho_i + (p - p_i) \tau_i = p_j \rho_j + (p - p_j) \tau_j, \tag{7}
\]

for any \( i, j = 1, \ldots, N \) [13]. We assume that a prior probability distribution satisfies \( p_i \neq 0, 1 \) in order to remove trivial cases. \( p \) and \( \tau_i \) are called Helstrom ratio and conjugate state to \( \rho_i \), respectively. Multiplying (7) with \( \Pi_i \), taking the sum over \( i \) and then taking the trace of both sides leads to

\[
Tr\mathcal{M} + \sum_{i=1}^{N} (p - p_i) Tr(\tau_i \Pi_i) = p. \tag{8}
\]
Thus we have

\[ p_{opt} \leq p. \]  

(9)

The observables \( \{\Pi_i\}_{i=1}^N \) satisfy \( p_{opt} = p \) if \( (p - p_i) Tr(\tau_i \Pi_i) = 0, \ i = 1, \ldots, N \). In this case, the observables \( \{\Pi_i\}_{i=1}^N \) give an OM to discrimination of states \( \{\rho_i\}_{i=1}^N \), and we call the family \( \{\bar{\rho}_i, \rho_i; 1 - \bar{\rho}_i, \tau_i\}_{i=1}^N \) Helstrom family of ensembles [13].

Now, we show the equivalence between optimality conditions (2) and Helstrom family of ensembles. To see that Helstrom family of ensembles is sufficient to minimize the error let us postmultiply (7) by \( \Pi_i \). If we consider \( (p - p_i) \tau_i \Pi_i = 0 \) then

\[ p_i \rho_i \Pi_i = p_j \rho_j \Pi_i + (p - p_j) \tau_j \Pi_i \]  

(10)

Summing up over \( i \) and using the completeness condition \( \sum_{i=1}^N \Pi_i = I \) for probability operators, we obtain

\[ \mathcal{M} = p_j \rho_j + (p - p_j) \tau_j, \ \forall \ j \]  

(11)

which is the same condition (2) or equivalently (4).

To show that Helstrom family of ensembles is necessary condition, we take the trace of Eqs. (3) and obtain \( \alpha_j = p - p_j \). Then, by subtracting Eqs. (3) for different values of \( j \), the Eq. (7) is followed. If we postmultiply Eq. (11) by \( \Pi_j \) and summing up over \( j \), we get just

\[ \sum_{j=1}^N (p - p_j) \tau_j \Pi_j = 0. \]  

Because both \( \tau_j \) and \( \Pi_j \) are both positive operators it must then be the case that \( (p - p_j) \tau_j \Pi_j = 0 \).

3 Minimum error discrimination (MED) between similarity transformed quantum states

Let \( \{U_1 = I_d, U_2, \ldots, U_N\} \) be a generating set for a (finite or continuum) subgroup of the unitary group \( U(d) \). Now, we consider the states \( \rho_i \) and the corresponding conjugate states \( \tau_i \) as

\[ \rho_i = U_i \rho_1 U_i^{-1}, \ \tau_i = U_i \tau_1 U_i^{-1}, \ \forall \ i = 1, 2, \ldots, N. \]  

(12)

We will refer to such states as similarity transformed states. Then, we have

\[ \mathcal{M} = U_i \left[ \frac{1}{N} \rho_1 + (p - \frac{1}{N}) \tau_1 \right] U_i^{-1} = U_i \mathcal{M} U_i^{-1}, \ i = 1, \ldots, N. \]

This indicates that \( \mathcal{M} \) commutes with the generators \( U_i \), and so commutes with all elements of the subgroup \( G \) generated by \( U_i \), i.e., we have

\[ U(g)\mathcal{M} = \mathcal{M}U(g), \ \forall \ g \in G, \]  

(13)

where, \( U \) is a \( d \times d \) unitary representation of \( G \). A consequence of the above equation is that \( \mathcal{M} \) is a diagonal matrix. To see this, we first consider that the representation \( U \) be irreducible (\( U \) is called as irreducible if it has no non-trivial - other than \( \{0\} \) and the hole Hilbert space- invariant subspaces). Then, the well known Schur’s first lemma in the group representation theory [15] implies that \( \mathcal{M} \) is a constant multiple of the identity matrix, i.e., \( \mathcal{M} = \mu I_d \) for some complex number \( \mu \in \mathbb{C} \). If \( U \) be reducible, one can decompose it as
nonequivalent irreducible components $V_i$. For instance, assume that $U$ be a direct sum of two nonequivalent irreducible representations $V_1$ and $V_2$. Then, the commutativity relation (13) can be written as

$$U(g)M = \begin{pmatrix} V_1(g) \\ V_2(g) \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_1(g) \\ V_2(g) \end{pmatrix} = M U(g), \ \forall \ g \in G,$$

which indicates that

$$V_1(g)A = AV_1(g), \ V_2(g)D = DV_2(g), \ V_1(g)B = BV_2(g), \ V_2(g)C = CV_1(g), \ \forall \ g \in G. \tag{14}$$

Now, from the Schur’s first lemma, we conclude that $A = \alpha I_1$ and $D = \beta I_2$ ($I_1$ and $I_2$ are identity matrices of dimension $dimV_1$ and $dimV_2$, respectively). From the relations (14), and the fact that $V_1$ and $V_2$ are nonequivalent irreducible representations, the Schur’s second lemma implies that $B = C = 0$ where, $0$ denotes zero matrix. The proof for diagonality of $M$ in general case in which $U$ is decomposed to more than two irreducible representations, is similar.

In order to $\Pi_i$’s form a set of POM, we need to have the completeness condition $\sum_{i=1}^{N} \Pi_i = I$. Assume that $\Pi_i = \lambda_i \Pi_i'$ for some positive complex number $\lambda_i$ and positive operator $\Pi_i'$ so that, the condition $\Pi_i \tau_i = 0$ implies $\Pi_i' \tau_i = 0$ for all $i = 1, \ldots, N$. The operators $\Pi_i'$ can be obtained from a positive operator $\Pi_i'$ via the same similarity transform which defines the states $\rho_i$ and the corresponding conjugate states $\tau_i$, i.e., we consider

$$\Pi_i' = U_i \Pi_i' U_i^{-1}, \quad i = 1, 2, \ldots, N. \tag{15}$$

Therefore, it is sufficient to choose $\Pi_i'$ perpendicular to $\tau_i$ in order to have $\Pi_i' \tau_i = 0$ for all $i$. Then, we must have

$$\sum_{i=1}^{N} \Pi_i = \sum_{i=1}^{N} \lambda_i \Pi_i' = I, \tag{16}$$

with $\sum_{i=1}^{N} \lambda_i = 1$; That is, the convex hull of the operators $\Pi_i'$, $i = 1, \ldots, N$ must conclude the identity operator $I$. Also, the completeness condition (16) leads to the following fact

$$\sum_{i=1}^{N} \lambda_i Tr(\Pi_i') = \sum_{i=1}^{N} \lambda_i Tr(\Pi_i) = d \quad \rightarrow \quad Tr(\Pi_i') = d \tag{17}$$

where, we have used the fact that $Tr(\Pi_i') = Tr(U_i \Pi_i' U_i^{-1}) = Tr(\Pi_i')$ for all $i$.

Now, by choosing suitable operator $\Pi_i'$ perpendicular to $\tau_i$, and using (1), one can obtain the optimal success probability of discrimination between quantum states $\rho_i$, $i = 1, 2, \ldots, N$ with equal prior probability $p_i = \frac{1}{N}$, as follows

$$p_{opt} = \frac{1}{N} \sum_{i=1}^{N} \lambda_i Tr(\Pi_i' \rho_i) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i Tr(\Pi_i' \rho_1) = \frac{1}{N} Tr(\Pi_i' \rho_1). \tag{18}$$

The above result shows that the positive coefficients $\lambda_i$ in the convex combination (16) can be chosen arbitrarily in a way that $\sum_i \lambda_i = 1$, i.e., the optimal POM set $\{\Pi_i = \lambda_i \Pi_i', \ i = 1, 2, \ldots, N\}$ with equal prior probability $p_i = \frac{1}{N}$, is a suitable one.
1, 2, \ldots, N\} satisfying the optimality condition (3) - provided that $\Pi'_\tau = 0$ - is not unique.

A. The irreducible case

In the case that the subgroup generated by $U_i$, $i = 1, 2, \ldots, N$ is a non-abelian subgroup of $U(d)$, the only operator which can be invariant under the action of representation $U$ is multiple of identity operator. Therefore, from (13) we have $\mathcal{M} = \mu I$ and so the identity (5) is written as the following resolution of identity

$$\mathcal{M} = \mu I = \frac{1}{N} \rho_i + (p_{\text{opt}} - \frac{1}{N}) \tau_i, \quad \forall \ i = 1, \ldots, N. \quad (19)$$

Taking the trace of both sides of (19), we find

$$d\mu = \frac{1}{N} + (p_{\text{opt}} - \frac{1}{N}) = p_{\text{opt}} \quad \Rightarrow \quad \mu = \frac{p_{\text{opt}}}{d}. \quad (20)$$

Now, in order to consider optimal discrimination between the states $\rho_i$ in (12), we assume that $\rho_1 = \sum_{i=1}^{d} a_i |\psi_i^{(1)}\rangle\langle \psi_i^{(1)}|$ be mixed state. Then, the resolution of identity (19) implies that $\tau_1$ is also diagonal in the bases $|\psi_i^{(1)}\rangle$. In the case that $\rho_1$ is full rank ($a_i \neq 0$ for all $i = 1, 2, \ldots, d$), the state $\tau_1$ can be written as $\tau_1 = \sum_i b_i |\psi_i^{(1)}\rangle\langle \psi_i^{(1)}|$ so that at least one of the coefficients $b_i$ is zero. Then, by using (19) and (20), we have

$$\frac{p_{\text{opt}}}{d} \left( \sum_{i=1}^{d} |\psi_i^{(1)}\rangle\langle \psi_i^{(1)}| \right) = \frac{1}{N} \left( \sum_{i=1}^{d} a_i |\psi_i^{(1)}\rangle\langle \psi_i^{(1)}| \right) + (p_{\text{opt}} N - 1) \sum_{i=1}^{d} b_i |\psi_i^{(1)}\rangle\langle \psi_i^{(1)}|,$$

so that

$$p_{\text{opt}} = \frac{1}{N} [a_i + (p_{\text{opt}} N - 1) b_i].$$

Since, at least one of the coefficients $b_i$, say $b_1$, is zero, the above relation leads to the following result

$$p_{\text{opt}} = \frac{d}{N} a_{\text{max}} \quad (21)$$

where, $a_{\text{max}}$ is the largest eigenvalue of $\rho_1$. From the fact that $a_{\text{max}} \geq \frac{1}{d}$, it is seen that $p \geq \frac{1}{N}$. It should be noticed that, in the case that all $a_i$’s are distinct, only one of the coefficients $b_i$, say $b_1$, must be zero so that $\Pi'_1$ will be given as $\Pi'_1 = d |\psi_1^{(1)}\rangle\langle \psi_1^{(1)}|$ and consequently the set of POMs $\Pi_i$ are pure, i.e., $\Pi_1 = \frac{d}{N} |\psi_1^{(1)}\rangle\langle \psi_1^{(1)}|.$

Now, let there be $m$ independent eigenvectors of $\rho_1$ having the same maximum eigenvalue $a_{\text{max}}$. Then, $m$ eigenvalues of $\tau_1$ must be zero. Denoting these eigenvalues by $b_{i_1}, \ldots, b_{i_m}$, the operator $\Pi'_1$ will be written as

$$\Pi'_1 = \alpha_1 |\psi_{i_1}^{(1)}\rangle\langle \psi_{i_1}^{(1)}| + \ldots + \alpha_m |\psi_{i_m}^{(1)}\rangle\langle \psi_{i_m}^{(1)}|,$$

where $\alpha_i$’s are arbitrary complex numbers with $\sum_{i=1}^{m} \alpha_i = d$. Then, using (13) we obtain the same result (21) for the optimal success probability.

For instance, in the case $d = 2$, we consider $\rho_i = U_i \rho_1 U_i^{-1}$ with

$$\rho_1 = \frac{1}{2} (I + a \hat{n} \hat{\sigma}) = \frac{1}{2} (I + \frac{1}{2} + n)\langle + n| + \frac{1}{2} - n\langle - n|$$
with $|a| < 1$, $|+n\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ e^{i\varphi} \end{array} \right)$ and $|-n\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -e^{i\varphi} \end{array} \right)$. Then, we have $\tau_i = U_i \tau_1 U_i^{-1}$ with $\tau_1 = \frac{1}{2} (I - \hat{n}.\vec{\sigma}) = | -n \rangle \langle -n |$ and, the optimal success probability reads as

$$p_{\text{opt}} = \frac{1 + a}{N}. \quad (22)$$

The above result has been obtained in Ref. [14] via the Helstrom family of ensemble.

Now, assume that $\rho_1$ has rank $r < d$, i.e., $a_i = 0$ for $i = r + 1, \ldots, d$. Then, the completeness relation (19) can be written as

$$p_{\text{opt}} d \left( \sum_{i=1}^{r} |\psi_i^{(1)} \rangle \langle \psi_i^{(1)} | + \sum_{i=r+1}^{d} |\phi_i^{(1)} \rangle \langle \phi_i^{(1)} | \right) =$$

$$\frac{1}{N} \left[ \sum_{i=1}^{r} a_i |\psi_i^{(1)} \rangle \langle \psi_i^{(1)} | + (p_{\text{opt}} N - 1) \left( \sum_{i=1}^{r} b_i |\psi_i^{(1)} \rangle \langle \psi_i^{(1)} | + \sum_{i=r+1}^{d} b_i |\phi_i^{(1)} \rangle \langle \phi_i^{(1)} | \right) \right],$$

so that

$$p_{\text{opt}} d = \frac{1}{N} [a_i + (p_{\text{opt}} N - 1)b_i], \quad i = 1, \ldots, r$$

and

$$p_{\text{opt}} d = \frac{1}{N} (p_{\text{opt}} N - 1)b_i, \quad i = r + 1, \ldots, d.$$

Therefore, we obtain the same result (21).

In the special case in which the initial state $\rho_1$ is pure, i.e., we have $a_i = 0$ for all $a_i$ except for one, say $a_l$, which is equal to one, the result (21) leads to the following optimal success probability

$$p_{\text{opt}} = \frac{d}{N}. \quad (23)$$

**B. The reducible case**

In the case that the subgroup generated by $U_i$, $i = 1, \ldots, N$ is an abelian subgroup of $U(d)$, invariance of $\mathcal{M}$ under the action of $U$ (Eq. (13)) implies that $\mathcal{M}$ is diagonal but not in general proportional to the identity matrix. Therefore, in order to determine the set of optimal POM in this case, one can consider that the state $\rho_1$ be diagonal, so that the identity (20) will be lead to the fact that $\tau_1$ is also diagonal. Then, similar to the irreducible case, one can obtain the suitable $\tau_1$ which fulfill the minimum error condition (20). Bellow, we consider an example in details, in order to clarify the method.

Let us consider the abelian subgroup $SO(2)$ of the rotation group $SO(3)$ which is generated by a rotation operator as $\exp(-i\theta \hat{n}.\vec{J})$, that rotates a spin-$j$ state by $\theta$ with respect to the $\vec{J}$-axis. Based on the rotation picture, the ensembles of $\rho_k$ can be constructed as follows. The states $\rho_k$ for $k = 1, \ldots, N$, that we wish to discriminate among are:

$$\rho_1 = \frac{1}{d} (I + 2a\hat{n}.\vec{J}),$$

$$\rho_k = U_k \rho_1 U_k^{-1}, \quad (24)$$
where $U_k = \exp\left(\frac{2\pi i (k-1)}{N} J_z\right)$ is a rotation of magnitude $\frac{2\pi (k-1)}{N}$ about the $z$-axis. The states $\tau_k$ are constructed similarly via $\tau_1 = \frac{1}{d}(I + 2b\hat{n} \cdot \vec{J})$.

We note that in order to $\rho_1$ be a density matrix, its eigenvalues $\lambda_m$ given by $\lambda_m = \frac{1+2am}{d}$ for $-j \leq m \leq j$ must be non-negative, i.e., we have $a \leq \frac{1}{2j}$. Also, it should be noticed that, $\tau_1$ is not full rank and so, its minimum eigenvalue is zero which indicates that $b = \frac{1}{2j}$.

Now, we want to obtain the upper bound $p$ for the optimal success probability. To this end, we note that in this case, although $\mathcal{M}$ is invariant under the action of operators $U_k$, but it is not proportional to the identity matrix. In this case, the invariance of $\mathcal{M}$ under rotations about the $z$-axis implies that, it has the following form

$$\mathcal{M} = \alpha I_d + \beta J_z,$$

where the constants $\alpha$ and $\beta$ must be calculated. Then, for a given mixed state $\rho_1$ as in (24) and the corresponding $\tau_e$ as

$$\tau_1 = \frac{1}{d}(I + 2b\hat{n} \cdot \vec{J}),$$

with $b = \frac{1}{2j} = \frac{1}{d-1}$, we have

$$\mathcal{M} = \frac{1}{N} \{\rho_1 + (Np - 1)\tau_1\} =$$

$$\frac{1}{Nd} \{NpI_d + 2(an_x + (Np-1)bn'_x)J_x + 2(an_y + (Np-1)bn'_y)J_y + 2(an_z + (Np-1)bn'_z)J_z\}.$$  (26)

Comparing (26) by (25) results

$$\alpha = \frac{p}{d}, \quad \beta = 2\frac{an_z + (Np-1)bn'_z}{Nd}, \quad an_x + (Np-1)bn'_x = an_y + (Np-1)bn'_y = 0.$$  (27)

From the above equation, we have $\frac{n'_x}{n'_y} = \frac{n_x}{n_y}$ or $\cot \phi' = \cot \phi$ so that $\phi' = \pi + \phi$. In the other hand, we have the condition $\beta = 0$ for the case that $\rho_1$ is in the subspace spanned by $J_x$ and $J_y$ (invariant subspace under the rotations about the $z$ axes), i.e., for the case that we have $n_z = 0$. Then, Eq.(27) implies that $n'_z = \cos \theta' = 0$, or $\theta' = \pi/2$. Therefore, we have $n'_x = \sin \theta' \cos \phi' = -\cos \phi$ and Eq.(27) leads to

$$p_{opt} = \frac{1}{N}(1 + \frac{a \sin \theta}{b}).$$  (28)

By substituting $b = \frac{1}{d-1}$, the optimal success probability is given by

$$p_{opt} = \frac{1}{N}[1 + a(d-1) \sin \theta].$$  (29)

As an another example, let us consider the equiprobable qubit states $\rho_1, \rho_2, \ldots, \rho_N$ with the corresponding Bloch vectors

$$a_j = (a \sin \theta \cos \phi_j, a \sin \theta \sin \phi_j, a \cos \theta), \quad j = 1, \ldots, N.$$  (30)
which share a common latitude of a ball with the radius equal to \( a \). Except for \( \theta = \pi/2 \), the states are reducible, i.e., none of the states are invariant under the rotations about \( z \)-axis. By choosing \( \phi_1 = 0 \), we have
\[
\rho_j = U_j \rho_1 U_j^{-1}, \quad U_j = \begin{pmatrix} e^{-i\phi_j/2} & 0 \\ 0 & e^{i\phi_j/2} \end{pmatrix} = e^{-i\phi_j \sigma_z/2}, \quad j = 2, \ldots, N.
\]

It is seen, the operators \( U_j \), for \( j = 1, 2, \ldots, N \) generate the infinite rotation group about \( z \)-axis, i.e., we choose finite number of non symmetric states from infinite states- which can be produce in this way- and discriminate them with minimum error probability. In the qubit case, \( \tau_j \)'s are pure and so can be taken as \( \tau_j = 1/2(I + b_j \sigma) \) with \( |b_j| = 1 \). Similar to the previous example, by substituting \( \mathcal{M} = \alpha I_2 + \beta \sigma_z \) and using the identity \( (5) \), one can obtain the optimal success probability of discrimination as follows
\[
p_{\text{opt}} = \frac{1}{N} (1 + a \sin \theta),
\]
which is special case \( d = 2 \) of the result \( (29) \). This example has solved in Ref. \[14\] via the Helstrom family of ensemble and using the convex optimization method.

**MED between group covariant states**

In the special case in which the states \( \rho_i \) are labeled by all of the elements of a group \( G \) (denoted by \( \rho_g \) called also group covariant states or symmetric states, the set of optimal POVM \( \{ \Pi'_g, g \in G \} \) is determined uniquely in terms of \( \{ \Pi'_g, g \in G \} \). To see this, assume that the states \( \rho_g \) are defined as
\[
\rho_g = U(g) \rho_e U(g)^{-1}, \quad \tau_g = U(g) \tau_e U(g)^{-1}, \quad \forall \ g \in G
\]
where, \( U \) is an irreducible unitary representation of \( G \). These states are sometimes called group covariant quantum states or symmetric states.

It should be noticed that, the irreducible representation associated with the group elements belonging to the center (the center of a group is defined as a set of elements commuting with all of the group elements, i.e., \( Z = \{ g' \in G; g' g' = g' g' \forall g \in G \} \) is multiple of identity matrix, that is we have \( U(g) = e^{i\phi(g)} I \) for all \( g \in Z \). Therefore, one can consider the groups \( G \) with trivial center \( Z = \{ e \} \) or- in the case of groups with non-trivial center- the quotient group \( G/Z \) instead of \( G \) and parameterize the initial states \( \rho_i \) and POVM set \( \Pi'_g \), with elements of \( G/Z \).

In order to see that, in this special case, the coefficients \( \lambda_g \) in \( (16) \) are determined uniquely, we denote \( \sum_{g \in G} \Pi'_g \) by \( \Pi \). Then, one can write
\[
U(g') \Pi U(g')^{-1} = \sum_{g \in G} U(g') \Pi'_g U(g')^{-1} = \sum_{g \in G} U(g' g) \Pi'_g U(g' g)^{-1} =
\]
\[
\sum_{g'' \in G} U(g'') \Pi'_g U(g'')^{-1} = \sum_{g'' \in G} \Pi'_{g''} = \Pi, \quad \forall \ g' \in G.
\]

Due to the ineducability of group representation \( U \), the Schur’s lemma implies that \( \Pi \) must be multiple of identity operator, i.e., we have
\[
\Pi = \sum_{g \in G} \Pi'_g = \alpha I.
\]
Taking the trace of both sides and using \((17)\), we obtain

\[
|G| \text{Tr}_{\Pi} = \alpha d, \quad \rightarrow \quad \alpha = |G|.
\]

Therefore, we have

\[
\frac{1}{|G|} \sum_{g \in G} \Pi_g = I \quad \text{so that the set of optimal POVM is given by} \quad \Pi_g = \frac{1}{|G|} \Pi_g.
\]

**Conclusion**

The minimum error discrimination (MED) between quantum mixed states was studied where, it was shown that the known necessary and sufficient conditions for MED [5] are fulfilled by a Helstrom family of ensembles. Then, a more convenient and suitable identity as a convex combination of the initial states - which we try to discriminate between them- and their corresponding conjugate states was extracted from the MED conditions in order to obtain optimal set of measurements and corresponding optimal success probability of discrimination. The introduced identity was applied for MED between the similarity transformed quantum states. Finally, as a special case, MED between the group covariant or symmetric states was considered.

**References**

[1] C. W. Helstrom, *Quantum Detection and Estimation theory*, New York: Academic, (1976).

[2] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, Amsterdam: North-Holland, (1982).

[3] Y. C. Eldar, A. Megretski, and G. C. Verghese, IEEE Trans. Inform. Theory 49, 1007 (2003).

[4] Y. C. Eldar, A. Megretski, and G. C. Verghese, IEEE Trans. Inform. Theory 50, 1198 (2004).

[5] S. M. Barnett and S. Croke, J. Phys. A: Math. Theor. 42, 062001, (2009).

[6] K. Hunter, in Proceedings of The Seventh International Conference on Quantum Communication, Measurement and Computing (QCMC04), Vol. 734 of American Institute of Physics Conference Series (AIP, 2004), 8386.

[7] Yuen, H., Kennedy, R., and Lax, M., IEEE Trans. Inf. Theory, IT-21, 125 (1975).

[8] W. Y. Hwang and J. Bae, J. Math. Phys. 51, 022202, (2010).

[9] A. Assalini, G. Cariolaro, and G. Pierobonar, Phys. Rev. A 81, 012315, (2010).

[10] S. M. Barnett, Phys. Rev. A 64, 030303(R), (2001).

[11] C. L. Chou and L. Y. Hsu, Phys. Rev. A 68, 042305, (2003).
[12] S. M. Barnett and S. Croke, Adv. Opt. Photon. 1, 238, (2009).

[13] G. Kimura, T. Miyadera, and H. Imai, Phys. Rev. A 79, 062306, (2009).

[14] M.A. Jafarizadeh, Y. Mazhari, M. Aali, Quantum Inf Process, DOI 10.1007/s11128-010-0185-y, (2010).

[15] A. W. Joshi, *Elements of group theory for physicists*, New Age International (P) Limited, Publishers, (1997).