SOME STATISTICS ON RESTRICTED 132 INVOLUTIONS

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Abstract

In [GM] Guibert and Mansour studied involutions on \( n \) letters avoiding (or containing exactly once) 132 and avoiding (or containing exactly once) an arbitrary pattern on \( k \) letters. They also established a bijection between 132-avoiding involutions and Dyck word prefixes of same length. Extending this bijection to bilateral words allows to determine more parameters; in particular, we consider the number of inversions and rises of the involutions onto the words. This is the starting point for considering two different directions: even/odd involutions and statistics of some generalized patterns. Thus we first study generating functions for the number of even or odd involutions on \( n \) letters avoiding (or containing exactly once) 132 and avoiding (or containing exactly once) an arbitrary pattern \( \tau \) on \( k \) letters. In several interesting cases the generating function depends only on \( k \) and is expressed via Chebyshev polynomials of the second kind. Next, we consider other statistics on 132-avoiding involutions by counting an occurrences of some generalized patterns, related to the enumeration according to the number of rises.

1. Introduction

The aim of this paper is to give analogies of enumerative results on certain classes of permutations characterized by pattern-avoidance in the symmetric group \( S_n \). In \( I_n = \{ \pi \in S_n : \pi = \pi^{-1} \} \) we identify classes of restricted involutions with enumerative properties analogous to results for permutations. More precisely, we study generating functions for the number of even or odd involutions in \( I_n \) avoiding (or containing exactly once) 132, and avoiding (or containing exactly once) an arbitrary permutation \( \tau \in S_k \). Moreover we consider statistics of some generalized patterns. In the remainder of this section we present a brief account of earlier works which motivated our investigation, basic definitions used throughout the paper, and the organization of this paper.

1.1. Background. In this subsection we present some classical statistics on permutations, pattern avoidance for permutations, some earlier results, generalized patterns, and pattern avoidance for involutions.

Let \( \pi \in S_n \). The number of inversions of \( \pi \) is given by \( |\{(i,j) : \pi_i > \pi_j, \ 1 \leq i < j \leq n\}| \). We say \( \pi \) is an even permutation [respectively; odd permutation] if \( \pi \) is a permutation together with even [respectively; odd] number of inversions. We say two consecutive elements \( \pi_j \) and \( \pi_{j+1} \) form a rise [respectively; descent] if \( \pi_j < \pi_{j+1} \) [respectively; \( \pi_j > \pi_{j+1} \) ] where \( 1 \leq j \leq n - 1 \). We say \( \pi_j \) is a right-to-left maximum [respectively; left-to-right minimum] if \( \pi_j > \pi_i \) [respectively; \( \pi_j < \pi_i \) ] for all
\( j < i \leq n \) [respectively; \( 1 \leq i < j \)] where \( 1 \leq j \leq n \). The number of \textit{fixed points} of \( \pi \) is given by 
\[ |\{ j : \pi_j = j, 1 \leq j \leq n \}|. \]

Let \( \pi \in \mathfrak{S}_n \) and \( \tau \in \mathfrak{S}_k \) be two permutations. We say that \( \pi \) \textit{contains} \( \tau \) if there exists a subsequence \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that \((\pi_{i_1}, \ldots, \pi_{i_k})\) is order-isomorphic to \( \tau \); in such a context \( \tau \) is usually called a \textit{pattern}. We say that \( \pi \) \textit{avoids} \( \tau \), or is \( \tau \)-\textit{avoiding}, if such a subsequence does not exist. The set of all \( \tau \)-avoiding permutations in \( \mathfrak{S}_n \) is denoted \( \mathfrak{S}_n(\tau) \). For an arbitrary finite collection of patterns \( T \), we say that \( \pi \) \textit{avoids} \( T \) if \( \pi \) avoids any \( \tau \in T \); the corresponding subset of \( \mathfrak{S}_n \) is denoted \( \mathfrak{S}_n(T) \). For example, \( 34521 \in \mathfrak{S}_5(132) \) whereas \( 31542 \notin \mathfrak{S}_5(132) \) because it contains two subsequences (that is \( 354, 154, 152, \) and \( 142 \)) order-isomorphic to \( 132 \).

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns \( \tau_1, \tau_2 \). This problem was solved completely for \( \tau_1, \tau_2 \in \mathfrak{S}_3 \) (see [S]), for \( \tau_1 \in \mathfrak{S}_3 \) and \( \tau_2 \in \mathfrak{S}_4 \) (see [W2]), and for \( \tau_1, \tau_2 \in \mathfrak{S}_4 \) (see [Km and references therein]). Several recent papers [CW, MV1, Cv, MV2, MV3, MV4] deal with the case \( \tau_1 \in \mathfrak{S}_3, \tau_2 \in \mathfrak{S}_k \) for various pairs \( \tau_1, \tau_2 \). Another natural question is to study permutations avoiding \( \tau_1 \) and containing \( \tau_2 \) exactly \( t \) times. Such a problem for certain \( \tau_1, \tau_2 \in \mathfrak{S}_3 \) and \( t = 1 \) was investigated in [RWZ, MV1, Cv, MV3, MV4].

In [RS], Babson and Steingrimsson introduced \textit{generalized patterns} that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In this context, we write a classical pattern with dashes between any two adjacent letters of the pattern, say 1243, as 1-2-4-3, and if we write, say 124-3, then we mean that if this pattern occurs in permutation \( \pi \), then the letters in the permutation \( \pi \) that correspond to 1 and 2 are adjacent. For example, the permutation \( \pi = 3542176 \) has only one occurrence of the pattern 12-43, namely the subsequences 354; whereas \( \pi \) has two occurrences of the pattern 1-2-4-3, namely the subsequence \( 35421 \) and \( 3476 \). Claesson \([C]\) presented a complete solution for the number of permutations avoiding any single 3-letters generalized pattern with exactly one adjacent pair of letters. Claesson and Mansour \([CM]\) (see also \([M1, M2, M3]\)) presented a complete solution for the number of permutations avoiding any double 3-letters generalized patterns with exactly one adjacent pair of letters. Besides, Kitaev \([K]\) investigated simultaneous avoidance of two or more 3-letters generalized patterns without internal dashes.

An \textit{involution} \( \pi \) is a permutation such that \( \pi = \pi^{-1} \) (permutation equivalent to its inverse). Some authors considered involutions avoiding patterns. In particular, someones studied the enumeration of involutions in \( \mathfrak{S}_n(12 \ldots k) \) in general \([R, G]\) or for some \( k \) \([G-P]\) because this pattern is directly connected to Young tableaux of bounded height by Robinson-Schensted algorithm \([R, S]\). Moreover, some other ones \([G, G-P]\) considered several sets of avoiding involutions enumerated by the Motzkin numbers. It already remains a connected open problem: in \([G]\) conjectures that \( \mathfrak{J}_n(1432) \) is also enumerated by the \( n \)th Motzkin number.

1.2. Basic tools. In this subsection we present some tools we will use later as Chebyshev polynomials, Dyck word prefixes and bilateral words, and generating trees. As an example, we also present a sorting algorithm.

\textit{Chebyshev polynomials of the second kind} (in what follows just Chebyshev polynomials) are defined by
\begin{equation}
U_r(\cos \theta) = \frac{\sin(r + 1)\theta}{\sin \theta}
\end{equation}
for \( r \geq 0 \). The Chebyshev polynomials satisfy the following recurrence \( U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \) for \( r \geq 2 \) together with \( U_0(t) = 1 \) and \( U_1(t) = 2t \). Evidently, \( U_r(x) \) is a polynomial of degree \( r \) in \( x \) with integer coefficients. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [R]). Apparently, for the first time the relation between restricted permutations and Chebyshev polynomials was discovered by Chow and West in [CW], and later by Mansour and Vainshtein [MV1, MV2, MV3, MV4] and Krattenthaler [Kr]. These results are related to a rational function

\[
R_k(x) = \frac{U_{k-1} \left( \frac{1}{\sqrt{x}} \right)}{\sqrt{x}U_k \left( \frac{1}{\sqrt{x}} \right)}
\]

for all \( k \geq 1 \). For example, \( R_1(x) = 1 \), \( R_2(x) = \frac{1}{\sqrt{x}} \), and \( R_3(x) = \frac{1 - \sqrt{x}}{2x} \). It is easy to see that for any \( k \), \( R_k(x) \) is rational in \( x \) and satisfies the following equation (see [MV1, MV3, MV4])

\[
R_k(x) = \frac{1}{1 - xR_{k-1}(x)}.
\]

For all \( k \geq 0 \), we define

\[
R'_k(x) = \frac{1}{2}(R_k(x) + R_k(-x)) \quad \text{and} \quad R''_k(x) = \frac{1}{2}(R_k(x) - R_k(-x)).
\]

Let \( L \) be a set of letters; the set of all words on \( L \) we denote by \( L^* \). The length of a word \( w \) we denote by \( |w| \) is its number of letters. The number of occurrences of the letter \( l \in L \) in a word \( w \) we denote by \( |w|_l = |\{w_i : w_i = l\}| \). We denote by \( \varepsilon \) the empty word (of length 0).

**Dyck word prefixes** are words \( w \in \{x, \bar{x}\}^* \) such that for all \( w = w'w'' \), \( |w'|_x \geq |w'|_{\bar{x}} \). The set of all Dyck word prefixes we denote by \( P_{x, \bar{x}} \). For example, \( xxx, xxx\bar{x}, xx\bar{x}x, xx\bar{x}\bar{x}, x\bar{x}xx \), and \( x\bar{x}x\bar{x} \) are all the Dyck word prefixes of length 4. Dyck word prefixes in \( P_{x, \bar{x}} \) of length \( n \) are enumerated by the central binomial coefficient \( a_n = \binom{n}{\lfloor n/2 \rfloor} \) for all \( n \geq 0 \). A Dyck word prefix corresponds to the beginning of a Dyck word.

**Dyck words** are Dyck word prefixes \( w \in \{x, \bar{x}\}^* \) such that \( |w|_x = |w|_{\bar{x}} \). For example, \( xxxx \) and \( xx\bar{x}\bar{x} \) are all the Dyck words of length 4. Dyck words of length \( 2n \) are enumerated by \( C_n = \frac{1}{n+1} \binom{2n}{n} \) the \( n \)th Catalan number whose generating function is \( C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \).

**Bilateral words** are words in \( \{x, \bar{x}\}^* \) such that \( |w|_x = |w|_{\bar{x}} \) or \( |w|_x = |w|_{\bar{x}} - 1 \). The set of all bilateral words we denote by \( B_{x, \bar{x}} \). For example, \( xxxx, xx\bar{x}x, xx\bar{x}\bar{x}, xxxx, xx\bar{x}x, \) and \( x\bar{x}xx \) are all the bilateral words of length 4.

We note \( \Xi \) the well known bijection which is an example of a result due to Chottin and Cori [CC] (see also [GM], just before Theorem 2.4) between a Dyck word prefix \( w_0xw_1x \ldots xw_p \in P_{x, \bar{x}} \) where \( w_i \) is a Dyck word for all \( 0 \leq i < p \) and the bilateral word \( w_0\bar{x}w_1\bar{x} \ldots \bar{x}w_{(p-1)/2} \bar{x}w_{(p+1)/2} xw_{(p+3)/2} \bar{x} \ldots xw_p \in B_{x, \bar{x}} \) of the same length. For example, the Dyck word prefix \( xxx\bar{x}x\bar{x}xx\bar{x}x\bar{x}x \) is in bijection by \( \Xi \) with the bilateral word \( xxx\bar{x}x\bar{x}xx\bar{x}x\bar{x}x \). Moreover, the words of length 4 given in the previous paragraph are respectively in bijection. So, the number of Dyck word prefixes of length \( n \) is equal to the number of bilateral words of length \( n \) trivially enumerated by the central binomial coefficient \( a_n \).

Following [CGHK] (see also several PHD-thesis [G, S, P, W]), a generating tree of a set of objects is a tree subject to the conditions that each object of length \( n \) appears once and only once on a vertex of level \( n \) and that the edges correspond to a manner to grow the objects. In order to characterize a generating tree by a succession system we associate to each object a label such that any two nodes
have the same label if their subtrees are isomorphic. Therefore it suffices to specify the label of the root and a set of succession rules explaining how to derive from the label of a parent the labels of all of its children. For example, in [GM, Theorem 2.4], Guibert and Mansour established a bijection called $\Phi$ between $\mathcal{J}_n(132)$ and Dyck word prefixes in $P_{x,\tau}$ of length $n$. These objects can both be characterized by a succession system whose root is (0), and whose succession rules are $(0) \leadsto (1)$ and $(p) \leadsto (p + 1), (p - 1)$ if $p \geq 1$. They also stated that the number of fixed points of the involution corresponds to the difference between the number of letters $x$ and $\tau$ into the Dyck word prefix. So, they established [GM, Corollary 2.5] that the number of 132-avoiding involutions of length $n$ having $p$ fixed points with $0 \leq p \leq n$ (and $p$ is odd if and only if $n$ is odd) is the ballot (or Delannoy) number $a_{n,p} = \left(\frac{n}{2}\right) - \left(\frac{n + p + 1}{2}\right)$.

One very nice proof that $|\mathcal{S}_n(132)| = C_n$ and that the numbers of 132-avoiding permutations of length $n$ having $s$ right-to-left maxima is equal to the ballot number $a_{2n-s-1,s-1} = \binom{2n-s-1}{n-1} - \binom{2n-s-1}{n}$ is due to Knuth [Kn]. It consists in using a sorting algorithm (the output is the mirror of the identity permutation) with one stack (verifying its elements decrease from the top to the basis). This defines a bijection with these one stack-sortable permutations (by taking successively the last element, the previous one, until the first one) of length $n$ having $s$ right-to-left maxima and Dyck words (coding the movements of insertion/deletion on the stack) of length $2n$ having $s$ primitive Dyck words (that is the Dyck paths enumerated according to the number of times they touch the horizontal axis). For example, to 43512 $\in \mathcal{S}_5(132)$ having two right-to-left maxima, 5 and 2, corresponds the Dyck word $x_{x_{x_{x_{x_{x}}}x_{x_{x_{x_{x_{x}}}}}}}$ (insert 2, insert 1, delete 1, delete 2, insert 5, insert 3, delete 3, insert 4, and then delete 4 and 5 to empty the stack), concatenating the two primitive Dyck words $xx_{x_{x_{x_{x_{x}}}}}x_{x_{x_{x_{x_{x}}}}}$ and $xx_{x_{x_{x_{x_{x}}}}}x_{x_{x_{x_{x_{x}}}}}$.

We also encounter several times the $n$th Fibonacci number $F_n$ with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ whose generating function is $F(x) = \frac{1}{1-x-x^2}$.

1.3. Organization of the paper. In Section 2 we establish that the correspondence $\Phi \circ \Xi$ between 132-avoiding involutions and bilateral words allows to determine more parameters. In particular, we consider the number of inversions (and also the number of even or odd involutions) and the number of rises of the involutions onto the words.

In Section 3 we consider the four cases of even or odd involutions avoiding (or containing exactly once) 132 and avoiding (or containing exactly once) an arbitrary pattern $\tau$ on $k$ letters. In several interesting cases the generating function depend only on $k$ and is expressed via Chebyshev polynomials of the second kind.

Finally, in Section 4 we present other statistics on 132-avoiding involutions by considering the distribution according to the number of occurrences of some generalized patterns. In particular, we relate some of these results to Subsection 2.4 because the number of rises of a permutation is, evidently, given by the number of occurrences of the generalized pattern 12.

2. Restricted 132 involutions: number of inversions, and number of rises

This section presents refinements of bijections $\Phi$ and $\Psi$ given in [GM]. This allows us to determine the number of even [respectively; odd] involutions in $\mathcal{J}_n$ avoiding (or containing exactly once) 132. Besides, we determine the number of involutions in $\mathcal{J}_n(132)$ having exactly $r$ rises.

2.1. Refinements of bijection $\Phi$. In this subsection, following to [GM], we establish the correspondence $\Phi \circ \Xi$ between 132-avoiding involutions and bilateral words which allows to determine more
parameters. In particular, we consider the number of inversions and of rises of the involutions onto the words. So we recall the bijection \( \Phi \) given in \cite{GM}, Theorem 2.4.

Let \( \pi \in \mathcal{I}_n(132) \) having \( p \) fixed points. We have \( \pi = \pi' \pi'' \pi''' \) with \( |\pi'| = \frac{n-p}{2} \) (\( \pi' \) has no fixed points and constitutes cycles with \( \pi'' \) or \( \pi''' \)), \( \pi'' \) does not contain fixed point (\( \pi''' \) constitutes cycles with \( \pi' \)) and \( \pi(x) = x \) (x is the first fixed point and \( \pi''' \) constitutes cycles with \( \pi' \) and/or contains fixed points). We obtain two involutions in \( \mathcal{I}_{n+1}(132) \) from \( \pi \); the first one is given by inserting a fixed point between \( \pi' \) and \( \pi'' \), and the second one (if and only if \( \pi \) has at least one fixed point) is given by modifying the first fixed point \( x \) by a cycle starting between \( \pi' \) and \( \pi'' \).

Let \( w \in P_{x,\pi} \) of length \( n \) such that \( |w|_x - |w|_\pi = p \). So we have \( w = w_0 x w_1 x \ldots x w_p \) where \( w_i \) is a Dyck word for all \( 0 \leq i \leq p \). We obtain two Dyck word prefixes of length \( n + 1 \) from \( w \): \( xw \) and \( xw_0 x w_1 \ldots x w_p \) (the latter if \( p > 0 \)).

Of course, the same construction can be applied to bilateral words in bijection by \( \Xi \) with Dyck word prefixes. We obtain two bilateral words of length \( n + 1 \) from \( w_0 \Xi w_1 \ldots \Xi w_{(p+1)/2} x \ldots x w_p \) in \( B_{x,\pi} \) of length \( n: \Xi w_0 \Xi w_1 \ldots \Xi w_{[p/2]} x \ldots x w_p \) and \( xw_0 \Xi w_1 \ldots \Xi w_{[p/2]+1} x \ldots x w_p \) (the latter if \( p > 0 \)).

The three generating trees for 132-avoiding involutions, Dyck word prefixes, and bilateral words are characterized by the same succession system given in \cite{GM}, between Theorem 2.4 and Corollary 2.5:

\[
(2.1) \quad \begin{cases} 
(0) \\ (0) \\ (p) \sim (1) \\ (p) \sim (p+1), (p-1) \quad \text{if } p \geq 1 
\end{cases}
\]

Definition 2.1. Let \( w = w_0 x w_1 x w_2 x \ldots x w_p \) a Dyck word prefix where \( w_i \) is a Dyck word for all \( 0 \leq i \leq p \). We define \( i(w) \) by \( i(\varepsilon) = 0 \) and

\[
i(w) = \begin{cases} 
|w_1 w_2 \ldots w_p| + i(w_1 x w_2 x \ldots x w_p) \\
|w_0' w_0'' w_1 w_2 \ldots w_p| + |w_0'' w_1 w_2 \ldots w_p| + i(w_0' w_0'' w_1 x w_2 x \ldots x w_p) \\
\quad \text{if } w_0 = x w_0\, x w_0' \\
\quad \text{where } w_0' \text{ and } w_0'' \text{ are Dyck words.}
\end{cases}
\]

In such a context \( i(w) \) can be called the number of right Dyck steps of \( w \).

Of course, this statistic on Dyck word prefixes is similarly defined on bilateral words.

Theorem 2.2. Let \( \pi \) be an 132-avoiding involution and \( w \) be the bilateral word in correspondence by \( \Phi \circ \Xi \). Moreover, if \( \pi \) has length \( n \), \( p \) fixed points, \( i \) inversions, and \( r \) rises, then \( w \) has length \( n \), \([p+1)/2]\) minimal nonpositive height, \( i = i(w) \) right Dyck steps, and \( r = |wx|_x + |w|_\pi \) double identical consecutive steps.

Proof. In \cite{GM} proved that Succession system 2.1 established a one-to-one correspondence between involutions \( \pi \in \mathcal{I}_n(132) \) having \( p \) fixed points and Dyck word prefixes \( w = w_0 x w_1 x \ldots x w_p \) of length \( n \) where \( w_i \) is a Dyck word for all \( 0 \leq i \leq p \). So it is sufficient to consider the two others parameters, inversions and rises. We show that Succession system 2.1 can be extended to include these two parameters. We consider the decomposition \( \pi = \pi' \pi'' \pi''' \) (x is the first fixed point and \( \pi'' \) contains only cycles connected to \( \pi' \)).
First, we consider the number of inversions $i$ of involutions that is the number of right Dyck steps of the Dyck word prefixes (or equivalently bilateral words) and we obtain the succession system:

\[
\begin{align*}
(2.2) \quad \{(0,0) & \sim (1, i + n - p) \\
(0, i) & \sim (p + 1, i + n - p), (p - 1, i + n - p + 1) \quad \text{if } p \geq 1 \}
\end{align*}
\]

Of course, if $n = 0$, we have $i = 0$ for the empty involution and word.

The involution obtained by inserting a fixed point between $\pi'$ and $\pi''$ has $i$ inversions unchanged from $\pi$ and $\frac{n - p}{2}$ more for the new fixed point and $\frac{n - p}{2}$ more for $\pi'$ whereas the involution obtained by transforming the first fixed point $x$ by a cycle starting between $\pi'$ and $\pi''$ has $i$ inversions unchanged from $\pi$ and $\frac{n - p}{2}$ more for $\pi'$ and $1 + \frac{n - p}{2}$ more for the new element between $\pi'$ and $\pi''$.

We now consider the Dyck word prefix (or equivalently the bilateral word). By Definition 2.1 we have

\[
\text{Theorem 2.3.} \quad \text{The number of even } \pi \text{ inversions in } \mathcal{I}_n(132) \text{ is given by}
\]

\[
\text{We now consider the bilateral word. If } xw \text{ becomes 0 for the case where } w \text{ starts by } \pi \text{ [respectively; } x],
\]

for the case where $p$ increases and

\[
i(xw) = |w_0w_1 \ldots w_p| + i(w) = n - p + i(w)
\]

and

\[
i(xw_0\pi w_1 w_2 x \ldots w_p) = |w_0\pi w_1 w_2 \ldots w_p| + |w_1 w_2 \ldots w_p| + i(w_0 w_1 x w_2 x \ldots w_p)
\]

\[
= 1 + n - p + |w_1 w_2 \ldots w_p| + i(w) - |w_1 w_2 \ldots w_p|
\]

\[
= 1 + n - p + i(w)
\]

for the case where $p$ decreases.

Secondly, we consider the number of rises $r$ of involutions that is the number of double identical consecutive steps of bilateral words. We must add an extra technical parameter $b$ (an indicator 0 or 1) and we obtain the succession system:

\[
(2.3) \quad \begin{align*}
(0,0,0) & \sim (1, r + b, 1) \\
(0, r, b) & \sim (p + 1, r + b, 1), (p - 1, r + 1 - b, 0) \quad \text{if } p \geq 1
\end{align*}
\]

with $b = 1$ [respectively; 0] if $\pi'' = \varepsilon$ [respectively; $\neq \varepsilon$] and $b = 1$ [respectively; 0] if $w_0 = \varepsilon$ [respectively; $\neq \varepsilon$] that is $w$ starts by $\pi$ [respectively; $x$].

Of course, if $n = 0$, we have $r = 0$ and $b = 0$ for the empty involution and word.

If $\pi''$ is [respectively; is not] empty and $\pi$ has $r$ rises then the involution obtained by inserting a fixed point has $r + 1$ [respectively; $r$] rises (one rise [respectively; descent] more just after the new fixed point) and $b$ is still equal to 1 whereas the involution obtained by transforming the first fixed point by a cycle has $r$ [respectively; $r + 1$] rises (one descent [respectively; rise] more just after [respectively; before] the new element belonging to the new cycle) and $b$ becomes 0.

We now consider the bilateral word. If $w_0$ is [respectively; is not] empty and $w$ is such that $|wx|_{xx} + |w|_{xx} = r$, then the new bilateral word has $r + 1$ [respectively; $r$] double identical consecutive steps (it starts by $xx$ [respectively; $xw_0 = xxw_0$]) and $b$ is still equal to 1 for the case where $p$ increases whereas the new bilateral word has $r$ [respectively; $r + 1$] double identical consecutive steps (it starts by $xw_1$ [respectively; $xw_0w_1$]) and $b$ becomes 0 for the case where $p$ decreases.

Let $a'(n, p, i)$ be the number of involutions in $\mathcal{I}_n(132)$ having $p$ fixed points and $i$ inversions. Immediately, we deduce from Succession system (2.2) that

\[
a'(n, p, i) = a'(n - 1, p - 1, i + p - n) + a'(n - 1, p + 1, i + p - n + 1)
\]

together with $a'(0, 0, 0) = 1$.

**Theorem 2.3.** The number of even [respectively; odd] involutions in $\mathcal{I}_n(132)$ is given by
Moreover, if we assume that the first involution obtained by inserting a fixed point of the same parity (it has \(i + n - p\) inversions) and a second involution obtained by transforming the first fixed point by a cycle of a different parity (it has \(i + n - p + 1\) inversions).

If \(n = 2k\) then the number of even involutions in \(I_n(132)\) is equal to the number of odd involutions in \(I_n(132)\), which is equal to \(|I_{n-1}(132)| = a_{2k-1} = (2k-1)\). If \(n = 4k + 1\) then the number of even involutions in \(I_n(132)\) is equal to \(|I_{n-1}(132)| = a_{4k} = (4k)\), whereas the number of odd involutions in \(J_n(132)\) is equal to the number of involutions in \(I_n(132)\) excepted those having no fixed points; that is, \(a_{4k} - a_{4k,0} = (4k)\). If \(n = 4k + 3\) then the number of even involutions in \(J_n(132)\) is equal to the number of involutions in \(I_{n-1}(132)\) excepted those having no fixed points; that is, \(a_{4k+2} - a_{4k+2,0} = (4k+2)\), whereas the number of odd involutions in \(J_n(132)\) is equal to \(|I_{n-1}(132)| = a_{4k+2} = (4k+2)\). \(\square\)

One can remark that, from Succession system 2.2, we immediately have for the number of even involutions in \(I_n(132)\)

\[
\sum_{j=0}^{\lfloor n/4 \rfloor} \{ \pi \in I_n(132) : \pi \text{ has } 4j + (n \mod 4) \text{ fixed points} \} = \left(\frac{n-1}{2\lfloor (n+1)/4 \rfloor}\right)
\]

and for the number of odd involutions in \(I_n(132)\)

\[
\sum_{j=0}^{\lfloor (n-2)/4 \rfloor} \{ \pi \in I_n(132) : \pi \text{ has } 4j + ((n+2) \mod 4) \text{ fixed points} \} = \left(\frac{n-1}{1+2\lfloor (n-2)/4 \rfloor}\right).
\]

Let \(a''(n, p, r, b)\) be the number of involutions \(\pi \in I_n(132)\) having \(p\) fixed points and \(r\) rises and such that its decomposition \(\pi = \pi''x\pi''\) (\(x\) is the first fixed point and \(\pi''\) contains only cycles connected to \(\pi'\)) leads to \(\pi'' = \varepsilon\) (respectively; \(\neq \varepsilon\)) if and only if \(b = 1\) (respectively; \(0\)). Immediately, we deduce from Succession system 2.3 that

\[
a''(n, p, r, b) = a''(n - 1, p + 1 - 2b, r, 1 - b) + a''(n - 1, p + 1 - 2b, r, 1 - b)
\]

together with \(a''(0, 0, 0, 0) = 1\).

**Theorem 2.4.** The number of involutions in \(I_n(132)\) having \(r\) rises is given by

\[
\left(\begin{array}{c}
\frac{n}{2} \\
\frac{(r+1)/2}{2}
\end{array}\right) \left(\begin{array}{c}
\frac{n-1}{2} \\
\frac{r/2}{2}
\end{array}\right).
\]

**Proof.** Let \(w\) be the bilateral word in bijection with an involution in \(I_n(132)\) having \(r\) rises. So \(|w| = n\) and \(|wx|_{xx} + |w|_{xx} = r\). But \(w\) can be regarded as two words \(u\) and \(v\) on \(\{x, x\}^*\) obtained in such a way: \(u\) [respectively; \(v\)] consists in all the letters following an \(x\) [respectively; \(x\)] in \(wx\) (taken in the same order). For example, if \(w = \overline{r}xx\overline{x}\) then \(u = xx\overline{x}\) and \(v = xx\). This mapping is bijective: we can retrieve \(w\) from \(u\) and \(v\) because at each step we choose either the next letter of \(u\) or \(v\), and there is only one way to start. Note also that \(|u| = \lfloor n/2 \rfloor\) and \(|v| = \lfloor (n+1)/2 \rfloor\), and that \(v\) is ended by \(x\). Moreover, if \(n\) is even we have \(|u|_{x} = \lfloor (r + 1)/2 \rfloor\) and \(|v|_{x} = \lfloor r/2 \rfloor\) otherwise \(n\) is odd and we have \(|u|_{x} = \lfloor r/2 \rfloor\) and \(|v|_{x} = \lfloor (r + 1)/2 \rfloor\) such that \(\lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor - 1\). So the formula holds in each case. \(\square\)
Remark 2.5. An involution in $\mathcal{I}_n(132)$ having $r$ rises with $0 \leq r < n$ has exactly $n - r$ left-to-right minima (that is one plus the number of descents).

Proof. First, let $e$ be a left-to-right minimum (but not the first one which is the first element). By definition of a left-to-right minimum, all the elements left to $e$ are greater than $e$, and so the element just left to $e$ constitutes a descent.

Secondly, consider a descent in an 132-avoiding involution $\pi$ that is $e' > e$ and $\pi^{-1}(e') + 1 = \pi^{-1}(e)$. In order to avoid 132, all the elements left to $e'$ must be greater than $e$. So $e$ is a left-to-right minimum. Moreover, the first element of the involution is always a left-to-right minimum. □

We complete this subsection by relating these results to [G, Theorem 4.6] (see also [DGG]). Indeed, Guibert [G] established that all the sets (and also the others obtained by symmetry operations as inverse, mirror and complement) $\mathcal{S}_{n+1}(1234, 1243, 1423, 4123, \mathcal{S}_{n+1}(1324, 1342, 1432, 4132)$, $\mathcal{S}_{n+1}(1234, 2134, 2413, 4213)$, $\mathcal{S}_{n+1}(2314, 2413, 3142, 3241)$, $\mathcal{S}_{n+1}(1234, 1324, 2134, 2314)$, $\mathcal{S}_{n+1}(1324, 2134, 2314, 3124)$, $\mathcal{S}_{n+1}(1324, 2134, 3124, 3214)$, $\mathcal{S}_{n+1}(1324, 3124, 3214)$, $\mathcal{S}_{n+1}(1324, 3214, 3124)$, $\mathcal{S}_{n+1}(1324, 2341, 3142, 3241)$, and $\mathcal{S}_{n+1}(1324, 2341, 2431, 3241)$ are in bijection with bilateral words of length $2n$ enumerated by $a_{2n} = (2^n)$. Thus, by using the correspondence $\Phi \circ \Xi$, all these sets of permutations of length $n+1$ avoiding four patterns of length 4 are in bijection with $\mathcal{I}_{2n}(132)$. Moreover, Guibert [G] also stated that $|\mathcal{S}_{n+1}(1342, 2341, 2431, 3241)| = a_{2n}$.

2.2. Refinements of bijection $\Psi$. In this subsection, following to [GM], we study the bijection $\Psi$ between involutions containing 132 exactly once (of length $n$ and having almost one fixed point) and 132-avoiding involutions (of length $n - 2$ and having the same number of fixed points) according to one more parameter that is the number of inversions. So we recall the bijection $\Psi$ given in [GM, Theorem 4.1].

Let $\pi \in \mathcal{I}_n$ be an involution containing 132 exactly once having $p$ fixed points with $1 \leq p \leq n$. We have $\pi = \pi' x \pi'' y \pi'''$ with $\pi(x) = x$, $\pi(y) = z$ and $1 + x = y < z$ such that $xyz$ is the only subsequence of type 132 into $\pi$. In order to obtain $\sigma$ an involution in $\mathcal{I}_{n-2}(132)$ having also $p$ fixed points in bijection with $\pi$ by $\Psi$, we replace the subsequence $xyz$ by a fixed point between $\pi''$ and $\pi'''$.

Theorem 2.6. Let $\pi$ be an involution containing 132 exactly once and let $\sigma$ be the 132-avoiding involution in bijection by $\Psi$. If $\pi$ has length $n$, $p$ fixed points with $1 \leq p \leq n$, and $i$ inversions, then $\sigma$ has length $n - 2$, $p$ fixed points, and $i - 2n + 2p + 3$ inversions.

Proof. Let $\pi = \pi' x \pi'' y \pi'''$ with $\pi(x) = x$, $\pi(y) = z$ and $1 + x = y < z$ and let $\sigma = \sigma' \sigma'' \sigma'''$ with $\sigma(t) = t$ and $\sigma(j) \neq j$ for all $1 \leq j < t$ such that $\sigma'$, $\sigma''$, $\sigma'''$ corresponds to $\pi'$, $\pi''$, $\pi'''$; respectively. Moreover, $\pi'$ contains only cycles connected to $\pi''$ or $\pi'''$, $\pi''$ contains only cycles connected to $\pi'$, and $\pi'''$ contains either cycles connected to $\pi'$ or fixed points; thus, we have $n - p = 2(|\pi'| + 1) = 2x$.

The number of inversions in $\pi$ but not in $\sigma$ is the sum of $|\pi'|$ because all the elements of $\pi'$ are greater than $x$, $|\pi' - |\pi'''|)$ because all the elements of $\pi'$ connected to $\pi'''$ are greater than $z$ (but all the elements of $\pi'$ connected to $\pi''$ are smaller than $z$), $|\pi'|$ because all the elements of $\pi'$ are greater than $y$, $|\pi''|$ because $x$ is greater than all the elements of $\pi''$ (but $x < z$ and $x < y$), $|\pi' - |\pi''|$ because $x$ is greater than all the elements of $\pi'''$ excepted its fixed points, $|\pi''| + 1$ because $z$ is greater than all the elements of $\pi''$ and than $y$, $|\pi' - |\pi'''|$ because $z$ is greater than all the elements of $\pi'''$ excepted its fixed points, $0$ (all the elements of $\pi''$ are smaller than $y$), and $|\pi' - |\pi'''|$ because $y$ is greater than all the elements of $\pi'''$ excepted its fixed points.
The number of inversions in $\sigma$ but not in $\pi$ is the sum of $|\pi'| - |\pi''|$ because all the elements of $\sigma'$ connected to $\sigma''$ are greater than $t$ (but all the elements of $\sigma'$ connected to $\sigma''$ are smaller than $t$), 0 (all the elements of $\sigma''$ are smaller than $t$), and $|\pi'|-|\pi''|$ because $t$ is greater than all the elements of $\sigma''$ excepted its fixed points.

Thus, $\sigma$ has $i = (6|\pi'| + 1 - 2|\pi''|) + (2|\pi'|-2|\pi''|) = i - 4|\pi'| - 1 = i - 2n + 2p + 3$ inversions. \hfill $\square$

**Theorem 2.7.** The number of even [respectively; odd] involutions containing 132 exactly once of length $n$ is given by

$$\left(1 + 2\frac{n-3}{(n-1)/4}\right)\left[\text{respectively; } \frac{n-3}{2(n-5)/4}\right]$$

for all $n \geq 5$ [respectively; $n \geq 3$].

**Proof.** By Theorem 2.4 we trivially deduce that an even [respectively; odd] involution containing 132 once corresponds by $\Psi$ to an odd [respectively; even] 132-avoiding involution because the parity of the number of inversions of the two involutions is different. Moreover, Theorem 2.3 gives the number of odd [respectively; even] 132-avoiding involutions according to the length.

Finally, [GM, Corollary 2.4] enumerates 132-avoiding involutions according to the number of fixed points. In particular, the number of 132-avoiding involutions without fixed points of length $2n$ is $C_n$ the $n$th Catalan number (indeed $\Phi$ sets these involutions in bijection with Dyck words).

Thus, the number of even [respectively; odd] involutions containing 132 once of length $n$ is $\left(1 + 2\frac{n-3}{(n-1)/4}\right)$ [respectively; $\left(2\frac{n-3}{(n-5)/4}\right)$] the number of odd [respectively; even] 132-avoiding involutions of length $n - 2$ minus $C_{2j-1}$ [respectively; $C_{2j}$] if $n = 4j$ [respectively; $n = 4j + 2$] with $j \geq 1$ the number of 132-avoiding involutions without fixed points of (even) length. \hfill $\square$

### 3. Restricted 132-even (odd) involutions

In this section we study, by block decompositions approach (see [MV2, MV4]), generating functions for the number of even or odd involutions on $n$ letters avoiding (or containing exactly once) 132 and avoiding (or containing exactly once) an arbitrary pattern $\tau$ on $k$ letters. The core of this approach initiated by Mansour and Vainshtein [MV2] lies in the study of the structure of 132-avoiding permutations, and permutations containing a given number of occurrences of 132.

In several interesting cases the generating function depends only on $k$ and is expressed via Chebyshev polynomials of the second kind.

This section is organized in four subsections as corresponding to the four cases of even or odd involutions avoiding (or containing exactly once) 132 and avoiding (or containing exactly once) $\tau$.

**3.1. Avoiding 132 and another pattern.** Let $I_\tau^e(n)$ [respectively; $I_\tau^o(n)$] denote the number of even [respectively; odd] involutions in $\mathcal{I}_n(132, \tau)$, and let $I_\tau^e(x) = \sum_{n \geq 0} I_\tau^e(n)x^n$ [respectively; $I_\tau^o(x) = \sum_{n \geq 0} I_\tau^o(n)x^n$] be the corresponding generating function. We denote by $I_\tau(n)$ the number of involutions in $\mathcal{I}_n(132, \tau)$, and let $I_\tau(x)$ be the corresponding generating function. In fact, for all $\tau$

$$I_\tau(x) = I_\tau^e(x) + I_\tau^o(x).$$

The following proposition is the base of all the other results in this section, which holds immediately from definitions.
Proposition 3.1. Let \( \pi \in I_n(132) \) be an even [respectively; odd] involution such that \( \pi_j = n \). Then holds one of the following assertions:

1. \( \pi_n = n \);
2. \( \pi = (\beta, n, \gamma, \delta, j) \) where \( 1 \leq j \leq n/2 \), \( \delta = \beta^{-1} \) and \( \beta \) avoids 132 such that,
   (2.1) if \( j \) even, then \( \gamma \) is an 132-avoiding even [respectively; odd] involution of length \( n - 2j \);
   (2.2) if \( j \) odd, then \( \gamma \) is an 132-avoiding odd [respectively; even] involution of length \( n - 2j \).

Our present aim is to find the generating functions \( I^e_\pi(x) \) and \( I^o_\pi(x) \), and since that we need the following lemma.

Lemma 3.2. Let \( \{y_n\}_{n \geq 0} \) and \( \{z_n\}_{n \geq 0} \) be two sequences and the corresponding generating functions we denote by \( Y(x) \) and \( Z(x) \); respectively. Then

\[
\begin{align*}
\sum_{n \geq 0} \sum_{j=0}^{\lfloor n/4 \rfloor} y_{2j+1} z_{n-4j} x^n &= \frac{(Y(x^2) - Y(-x^2))Z(x)}{2x^2}, \\
\sum_{n \geq 0} \sum_{j=0}^{\lfloor n/4 \rfloor} y_{2j} z_{n-4j} x^n &= \frac{(Y(x^2) + Y(-x^2))Z(x)}{2}.
\end{align*}
\]

3.1.1. Pattern \( \tau = \emptyset \). The first interesting example is \( \tau = \emptyset \) which presented in [SS].

Theorem 3.3. (Simion and Schmidt [SS, Proposition 5]) The generating function for the number of even involutions in \( I_n(132) \) is given by

\[
I^e_\emptyset(x) = \frac{2(1 - x) - x^2(C(x^2) - C(-x^2))}{2(1 - x + x^2C(-x^2))(1 - x - x^2C(x^2))}.
\]

Proof. By Proposition 3.1, we have exactly three possibilities for block decomposition of an arbitrary \( \pi \) even involution in \( I_n(132) \). Let us write an equation for \( I^e_\emptyset(x) \). The contribution of the first decomposition above is \( x I^e_\emptyset(x) \). By Lemma 3.2, the contribution of the second and the third decomposition above are \( \frac{1}{2}x^2(C(x^2) - C(-x^2))I^e_\emptyset(x) \) and \( \frac{1}{2}x^2(C(x^2) + C(-x^2))I^e_\emptyset(x) \); respectively. Therefore,

\[
I^e_\emptyset(x) = 1 + xI^e_\emptyset(x) + \frac{1}{2}x^2(C(x^2) - C(-x^2))I^e_\emptyset(x) + \frac{1}{2}x^2(C(x^2) + C(-x^2))I^e_\emptyset(x),
\]

where 1 for the empty involution. Hence, solving the obtained linear equation together with \( \emptyset \) we get the desired result. \( \square \)

3.1.2. Pattern \( \tau = 12 \ldots k \). Let us start by the following example.

Example 3.4. Using Proposition 3.1 we have

\[
I^e_{12}(n) = I^e_{12}(n - 2), \quad I^o_{12}(n) = I^o_{12}(n - 2).
\]

Besides \( I^e_{12}(0) = I^e_{12}(1) = 1 \) and \( I^o_{12}(0) = I^o_{12}(1) = 0 \), so

\[
I^e_{12}(x) = \frac{1 + x}{1 - x^4}, \quad I^o_{12}(x) = \frac{x^2(1 + x)}{1 - x^4}.
\]

The case of varying \( k \) is more interesting. As an extension of Example 3.4 let us consider the pattern \( \tau = 12 \ldots k \).
Theorem 3.5. For all $k \geq 1$,
\[
I^r_{12\ldots k}(x) = \sum_{j=0}^{k-1} \left( x^j \left( 1 + x^2 R^r_{k-1-j}(x^2) I_{k-j}(x) \right) \prod_{i=j}^{k} R_i(x^2) \right),
\]
where $I_{12\ldots k}(x) = \frac{1}{x U_k(\frac{1}{2x})} \sum_{j=0}^{k-1} U_j \left( \frac{1}{2x} \right)$ (see [GM, Theorem 2.8]).

Proof. By Proposition 3.1 with the use of the generating function for the number of permutations in $\mathfrak{S}_n(132,12\ldots k)$ given by $R_k(x)$ (see [CW]), we have exactly three possibilities for block decomposition of an arbitrary even involution in $\mathfrak{S}_n(132)$. Let us write an equation for $I^r_{12\ldots k}(x)$ with use of Lemma 3.2. The contribution of the first, second, and the third decomposition above is $x I^r_{12\ldots(k-1)}(x)$, $x^2 R^r_{k-1}(x^2) I^r_{12\ldots k}(x)$, and $x^2 R^r_{k-1}(x^2) I^r_{12\ldots k}(x)$; respectively. Therefore, by definitions of $R^r_k(x)$ and $R^r_k(x)$ and by Identities 1.3 and 3.1 we get for all $k \geq 3$,
\[
I^r_{12\ldots k}(x) = R_k(-x^2) \left( 1 + x^2 R^r_{k-1}(x^2) I_k(x) \right) + x R_k(-x^2) I^r_{12\ldots(k-1)}(x).
\]
Hence, by the principle of induction on $k$ and by [GM, Theorem 2.8] together with Example 3.3 we have the desired result. \qed

Example 3.6. By Theorem 3.5 and Example 3.4 it is easy to see that
\[
I^r_{123}(x) = \frac{1 + x + x^2 - 2 x^4}{1 - 4 x^4},
\]
and then for all $n \geq 0$,
\[
(I^r_{1234}(4n), I^r_{1234}(4n + 1), I^r_{1234}(4n + 2), I^r_{1234}(4n + 3)) = (F_{4n-2}, F_{4n+1}, F_{4n+2}, F_{4n+3}),
\]
where $F_n$ is the $n$th Fibonacci number.

We can also establish these results for $k = 3, 4$ and 5 by a combinatorial approach. In [GM, Subsection 2.1, formula (2)] it is established that involutions avoiding both 132 and 12\ldots $k$ can be characterized by the succession system:
\[
\begin{align*}
(0) & \sim (1) \\
(0) & \sim (p + 1), (p - 1) \quad 1 \leq p \leq k - 2 \\
(p) & \sim (k - 1), (k - 2) \quad k \geq 3
\end{align*}
\]
and three simple bijections are stated involutions avoiding both 132 and 12\ldots $k$ (for $k = 3, 4, 5$) and some words [respectively; $\{a, b\}$ or $\{a, b\}^*$, $\{a, b, c\}$ $a$ or $\{a, b, c\} a \cup b \{a, b, c\}^*$].

It is clear that $|\mathfrak{S}_n(132,12\ldots k)|$ for $k = 3, 4, 5$ according to the number of fixed points $p$ is $(n$ and $p$ have same parity) is $2^{(n-1)/2}$ for $0 \leq p \leq 2$ and $k = 3$, $F_{n-2}$ [respectively; $F_{n-1}$] for $p = 0$ or 3 [respectively; 1 or 2] and $k = 4$, and $3^{n/2-1}$ [respectively; $\frac{3^{(n-1)/2}+1}{2}$ or $\frac{3^{(n-1)/2}+1}{2}$] for $p = 2$ [respectively; $p = 0$ or 1, or $p = 3$ or 4] and $k = 5$.

So we now deduce from Succession system 2.2 that the number of even [respectively; odd] involutions avoiding both 132 and 12\ldots $k$ of length $n = 4l + 1, 4l + 2, 4l + 3, 4l + 4$ with $l \geq 0$ is: $2^{2l}, 2^{2l}$, $0, 2^{2l+1}$ [respectively; $0, 2^{2l+1}, 2^{2l+1}$] for $k = 3$, $F_{4l}, F_{4l+1}, F_{4l+1}, F_{4l+2}$ [respectively; $F_{4l-1}, F_{4l}, F_{4l+2}, F_{4l+3}$] for $k = 4$, and $3^{2l+1}$, $3^{2l+1}, 3^{2l+1}$ [respectively; $\frac{3^{2l}+1}{2}, \frac{3^{2l}+1}{2}, \frac{3^{2l}+1}{2}$] for $k = 5$.

Moreover, we can see this split between even and odd involutions onto the words in bijection. For $k = 3$, the words of $\{a,b\}^n$ or $\{a,b\}^n$ (in bijection with involutions respectively of length $2n$ or...
2n + 1) corresponds for the even [respectively; odd] part to the words of \( a \{a, b\}^{2k} \cup b \{a, b\}^{2k+1} \) [respectively; \( a \{a, b\}^{2k+1} \cup b \{a, b\}^{2k} \) for any \( k \geq 0 \). For \( k = 4 \), the words of \( \{a, b\}^4 \) corresponds for the even [respectively; odd] part to the words of \( a \{a, b\}^{4k} \cup a \{a, b\}^{4k+1} \cup b^2 \{a, b\}^{4k+1} \cup b^2 \{a, b\}^{4k+2} \) [respectively; \( a \{a, b\}^{4k+2} \cup a \{a, b\}^{4k+3} \cup b^2 \{a, b\}^{4k+3} \cup b^2 \{a, b\}^{4k} \)]. We do not give the languages for \( k = 5 \) because their expressions are not so simple.

3.1.3. Pattern \( \tau = 2134 \ldots k \). Using the argument proof of Theorem \( \ref{thm:3.5} \) together with \( I_{21}^c(x) = \frac{1}{1-x} \) we get

**Theorem 3.7.** For all \( k \geq 2 \),

\[
I_{2134 \ldots k}(x) = \sum_{j=0}^{k-3} x^j \left( 1 + x^2 R_{k-j}^c(1) \right) \prod_{i=j}^{k} R_i(-x^2) + x^k R_2(x) \prod_{j=3}^{k} R_j(-x^2).
\]

where \( I_{2134 \ldots k}(x) = \frac{1}{x-t_k\left(\frac{1}{2x}\right)} \cdot \sum_{j=0}^{k-1} U_j \left( \frac{1}{2x} \right) \) (see \( \text{GM, Theorem 2.10} \)).

3.1.4. Pattern \( \tau = (d + 1, d + 2, \ldots, k, 1, 2, \ldots, d) \). Let us start by the following example.

**Example 3.8.** By Proposition \( \ref{prop:3.4} \) it is easy to see that

\[
I_{231}^e(x) = \frac{x^4 - x^3 + x^2 - x + 1}{(1-x)^2(1+x^2)}.
\]

The case of varying \( k \) is more interesting. As an extension of Example \( \ref{example:3.8} \) let us consider the pattern \( \tau = \langle k, d \rangle = (d + 1, d + 2, \ldots, k, 1, 2, \ldots, d) \).

**Theorem 3.9.** For all \( 1 \leq d \leq k/2 \),

\[
I_{\langle k, d \rangle}^e(x) = \frac{R_{k-d-1}(-x^2)}{1 - x R_{k-d-1}(-x^2)} \left( 1 + x^2 (R_{k-d-1}(-x^2) - R_{d-1}(-x^2)) I_{12 \ldots (k-d)}^e(x) + (R_{d-1}^e(-x^2) - R_{k-d-1}^e(-x^2)) I_{12 \ldots (k-d)}^e(x) + R_{k-d-1}^e(x^2) I_{\langle k, d \rangle}^e(x) \right),
\]

where \( I_{12 \ldots i}(x) \) is given by \( \text{GM, Theorem 2.10} \) and \( I_{\langle k, d \rangle}^e(x) \) is given by \( \text{GM, Theorem 2.14} \).

**Proof.** By Proposition \( \ref{prop:3.1} \) we have exactly three possibilities for block decomposition an arbitrary even involution \( \tau \in \mathcal{I}_n(132) \). Let us write an equation for \( I_{\langle k, d \rangle}^e(x) \) by using Lemma \( \ref{lemma:3.2} \). The contribution of the first above decomposition above is \( x I_{\langle k, d \rangle}^e(x) \). To find the contribution of the second and the third decompositions above let us consider two cases. First, if \( \gamma \) avoids \( 12 \ldots (k-d) \), then \( \beta \) and \( \delta \) avoid \( 12 \ldots (k-d-1) \), so the generating function for this number of even involutions is given by

\[
x^2 R_{k-d-1}^e(-x^2) I_{12 \ldots (k-d)}^e(x) + x^2 R_{k-d-1}^e(-x^2) I_{12 \ldots (k-d)}^e(x).
\]

Secondly, if \( \gamma \) contains \( 12 \ldots (k-d) \), then \( \beta \) and \( \delta \) avoid \( 12 \ldots (d-1) \), so the generating function for this number of even involutions is given by

\[
x^2 R_{d-1}^e(-x^2)(I_{\langle k, d \rangle}^e(x) - I_{12 \ldots (k-d)}^e(x)) + x^2 R_{d-1}^e(-x^2)(I_{\langle k, d \rangle}^e(x) - I_{12 \ldots (k-d)}^e(x)).
\]
We mentioned that, the generating function for the number of even involutions in $\mathcal{I}_n(132, < k, d >) \sum_{n} \sum_{k} \sum_{d} \mathcal{I}_n(x, y, z)$ is given by $I_{<k,d>}(x) - I_{<k-1,d>}(x)$. Therefore,

\[
I_{<k,d>}(x) = 1 + \sum_{k=2}^{\infty} I_{<k,d>}(x) + x^2 R_{k-1}^e(x^2) I_{<k-1,d>}(x) + x^2 R_{k-2}^e(x^2) I_{<k-1,d>}(x) + \ldots
\]

Hence, by Identities [3.1, 3.4 and 3.3] we get the desired result.

3.2. Avoiding 132 and containing another pattern. Let $I_{<r>}(n)$ [respectively; $I_{<r>}(n)$] denote the number of even [respectively; odd] involutions in $\mathcal{I}_n(132)$ such containing $\tau$ exactly $r$ times, and let $I_{<r>}(x) = \sum_{n=0}^{\infty} I_{<r>}(n)x^n$. Hence, by Identities [3.1, 3.4 and 3.3] we get the desired result.

\[
I_{<r>}(x) = I_{<r>}(x) + I_{<r>}(x).
\]

3.2.1. Pattern $\tau = 12\ldots k$. Now let us start by the following example.

Example 3.10. Proposition 3.1 yields

\[
I_{12;1}(x) = x^2 + x^2 I_{12;1}^p(x), \quad I_{12;1}^p(x) = x^2 I_{12;1}^p(x),
\]

which means that $I_{12;1}(x) = \frac{x^2}{1-x^2}$ and $I_{12;1}^p(x) = \frac{x^4}{1-x^2}$.

The case of varying $k$ is more interesting. As an extension of Example 3.10, let us consider the pattern $\tau = 12\ldots k$.

Theorem 3.11. For all $k \geq 2$,

\[
I_{12\ldots k;1}(x) = \sum_{j=0}^{k-1} x^{k-j} R_{j+1}^e(x^2) \prod_{i=j+2}^{k} R_i(-x^2) U_{j+2} \left( \frac{1}{2x} \right).
\]

Proof. By Proposition 3.1, we have exactly three possibilities for block decompositions an arbitrary even involution in $\mathcal{I}_n(132)$. Similarly as proof of Theorem 3.7, we have

\[
I_{12\ldots k;1}(x) = x I_{12\ldots (k-1);1}(x) + x^2 R_{k-1}^e(x^2) I_{12\ldots k;1}(x) + x^2 R_{k-2}^e(x^2) I_{12\ldots k;1}(x).
\]

hence, by using Identities [1.3, 1.4, and 3.3] together with [GM, Theorem 3.2], and then using the principle of induction on $k$ together with Example 3.10, we get the desired result.

Example 3.12. Theorem 3.11 yields $I_{1234;1}(x) = \frac{x^3(2x+1)}{(1+3x^2+x^2)(1-3x^2+x^2)}$. In other words, for all $n \geq 0$ we have $I_{1234;1}(n) = F_{n-2}$ if $n/4$ is a positive integer number otherwise $I_{1234;1}(n) = 0$, where $F_m$ is the $m$th Fibonacci number.

3.2.2. Pattern $\tau = 2134\ldots k$. Similarly as Theorem 3.5 and Theorem 3.11 together with using [GM, Theorem 3.3] we have the following result.

Theorem 3.13. For all $k \geq 2$,

\[
I_{2134\ldots k;1}(x) = (1 - x^2) \sum_{j=0}^{k} x^{k-j} R_{j+1}^e(x^2) \prod_{i=j+2}^{k} R_i(-x^2) U_j \left( \frac{1}{2x} \right).
\]
Example 3.14. Theorem 3.13 yields $I_{231;1}^x(n) = \frac{x^6(2-x^4)}{(1+3x^2+x^4)(1-3x^2+x^4)}$. In other words, for all $n \geq 0$ we have $I_{231;1}^x(n) = F_{n-3}$ if $n/4$ is a positive integer number otherwise $I_{231;1}^x(n) = 0$, where $F_m$ is the $m$th Fibonacci number.

3.2.3. Pattern $\tau = 23\ldots k1$. Now, let us consider another interesting case where $\tau = 23\ldots k1$.

Theorem 3.15. For all $k \geq 3$,

$$I_{23\ldots k1;1}^x(x) = \frac{x^3}{1-x} I_{12\ldots (k-2);1}^x(x).$$

Proof. By Proposition 3.1 it is easy to see that

$$I_{23\ldots k1;1}^x(x) = xI_{23\ldots k1;1}^x(x) + x^2 g(x),$$

where $g(x)$ the generating function for the number of odd involutions in $\mathcal{I}_n(132, 23\ldots k1)$ such containing $12\ldots(k-1)$ exactly once. On the other hand, also by Proposition 3.1 we get

$$g(x) = xI_{12\ldots (k-2);1}^x(x).$$

Hence the theorem holds.

As a remark, to find an explicit formula for $I_{23\ldots k1;1}^x(x)$ see Theorem 3.13, Theorem 3.11 and Identity 3.3.

Example 3.16. Theorem 3.13 and Theorem 3.11 yield $I_{231;1}^x(x) = 0$ and $I_{2341;1}^x(x) = \frac{x^7}{(1-x)(1-x^3)}$.

3.3. Containing 132 once and avoiding another pattern. Let $J_e^x(n)$ [respectively; $J_o^x(n)$] denote the number of even [respectively; odd] involutions in $\mathcal{I}_n(\tau)$ such containing 132 exactly once, let $J_e^x(x) = \sum_{n \geq 0} J_e^x(n)x^n$ [respectively; $J_o^x(x) = \sum_{n \geq 0} J_o^x(n)x^n$] be the corresponding generating function. Also, we denote by $J_r(x)$ the generating function for the number of involutions in $\mathcal{I}_n(\tau)$ containing 132 exactly $r$ times. So we have

$$J_r(x) = J_e^x(x) + J_o^x(x).$$

The following proposition is the base of all the other results in this section, which holds immediately from definitions.

Proposition 3.17. Let $\pi \in \mathcal{I}_n$ be an even involution contains 132 exactly once such that $\pi_j = n$. Then holds one of the following assertions:

1. $\pi_n = n$;
2. $\pi = (\beta, n, \gamma, \delta, j)$ where $1 \leq j \leq n/2$, $\delta = \beta^{-1}$, and $\beta$ avoids 132, such that
   (2.1) if $j$ even, then $\gamma$ is an even involution contains 132 exactly once of length $n-2j$,
   (2.2) if $j$ odd, then $\gamma$ is an odd involution contains 132 exactly once of length $n-2j$;
3. $\pi = (\beta, m, 2m+1, \gamma, m+1)$ where $n = 2m+1$ and $\gamma = \beta^{-1}$ such that $m$ even and $\beta \in \mathcal{S}_{m-1}(132)$.
3.3.1. Pattern $\tau = \emptyset$. The first interesting case is where $\tau = \emptyset$ which is analogue to Theorem 3.3.

**Theorem 3.18.** The generating function for the number of involutions in $\mathcal{I}_n$ containing 132 exactly once is given by

$$J_{132}^\circ(x) = \frac{x(1 - 2x + x\sqrt{1 - 4x^2} + x\sqrt{1 + 4x^2} - \sqrt{1 - 4x^2}\sqrt{1 + 4x^2})}{(1 - 2x + \sqrt{1 + 4x^2})(1 - 2x + \sqrt{1 - 4x^2})}.$$ 

In other words, for all $n \geq 1$,

$$J_{132}^\circ(n) = \frac{1}{2} C_{(n-2)/2} \left( \frac{1}{2} [n/2](3 + (-1)^{n+1}) - 1 - (-1)^{\binom{n}{2}} \right),$$

where $C_m$ is the $m$th Catalan number.

**Proof.** Proposition 3.17 we have exactly four possibilities for block decomposition an arbitrary even involution in $\mathcal{I}_n$ containing 132 exactly once. Let us write an equation for $J_{132}^\circ(x)$ by using Lemma 3.2. The contribution of the first, the second, the third, and the fourth decompositions above are $xJ_{132}^\circ(x)$, $\frac{x^2}{2}(C(x^2) - C(-x^2))J_{132}^\circ(x)$, $\frac{x^2}{2}(C(x^2) + C(-x^2))J_{132}^\circ(x)$, and $\frac{x^3}{2}(C(x^2) - C(-x^2))$; respectively. Therefore,

$$J_{132}^\circ(x) = xJ_{132}^\circ(x) + \frac{x^2}{2}(C(x^2) - C(-x^2))J_{132}^\circ(x) + \frac{x^2}{2}(C(x^2) + C(-x^2))J_{132}^\circ(x) + \frac{x^3}{2}(C(x^2) - C(-x^2)).$$

Hence, by Identity 3.4 and [GM, Theorem 4.4] we get the desired result. \hfill \square

3.3.2. Pattern $\tau = 12\ldots k$. Let us start by the following example.

**Example 3.19.** By Proposition 3.17 it is easy to see $J_{12}^\tau(x) = 0$.

As extension of Example 3.19 let us consider the pattern $\tau = 12\ldots k$.

**Theorem 3.20.** For all $k \geq 2$,

$$J_{12\ldots k}^\tau(x) = \sum_{j=3}^{k} \left( x^{k+1-j} \left[ R_{j-1}^\circ(x^2) + \frac{R_{j-1}^\circ(x^2)}{U_j(2x)} \sum_{i=1}^{j-2} U_i(1/(2x)) \right] \prod_{i=j}^{k} R_i(-x^2) \right).$$

**Proof.** Similarly as the argument proof of Theorem 3.18 together with using the fact that the generating function for the number of permutations in $\mathcal{S}_n(132, 12\ldots m)$ is given by $R_m(x)$ (see [CW]), we get

$$J_{12\ldots k}^\tau(x) = xJ_{12\ldots (k-1)}^\tau(x) + x^2R_{k-1}^\circ(132)(x)J_{12\ldots k}^\tau(x) + x^3R_{k-1}^\circ(132)(x)J_{12\ldots k}^\tau(x) + x^3R_{k-1}^\circ(132)(x).$$

Therefore, by Identities 3.3 and 3.4 we have

$$J_{12\ldots k}^\tau(x) = xR_k(-x^2)(x^2)R_{k-1}^\circ(132)(x) + xR_{k-1}^\circ(132)(x)J_{12\ldots k}^\tau(x) + J_{12\ldots (k-1)}^\tau(x)).$$

Hence, by the principle of induction on $k$ together with Example 3.19 and [GM, Theorem 4.6] we get the desired result. \hfill \square

**Example 3.21.** Theorem 3.21 for $k = 4$ yields $J_{1234}^\tau(n) = F_{n-2}$ where $n = 5, 6, 9, 10, 13, 14, \ldots$, otherwise $J_{1234}^\tau(n) = 0$, where $F_m$ is the $m$th Fibonacci number.
3.3. Pattern \( \tau = 2134 \ldots k \) or \( \tau = 23 \ldots k1 \). Similarly as the argument proof of Theorem 3.20 together with using the fact that the generating function for 132-avoiding permutations such avoiding 2134 \ldots k (or 23 \ldots k1) is given by \( R_k(x) \) (see [MV3]), and use [GM, Theorems 4.7 and 4.10] (expressions for \( J_{2134 \ldots k}(x) \) and \( J_{23 \ldots k1}(x) \); respectively) we obtain other cases \( \tau = 2134 \ldots k \) and \( \tau = 23 \ldots k1 \).

**Theorem 3.22.**

\[
J_{2134 \ldots k}(x) = xR_k(-x^2)\left( x^2 R_{k-1}^e(x^2) + xR_{k-1}^o(x^2)J_{2134 \ldots k}(x) + J_{2134 \ldots (k-1)}^e(x) \right),
\]

with \( J_{213}^e(x) = \frac{x^5}{1-4x} \), where \( J_{213\ldots m}(x) = \frac{x[x^2U_2\left( \frac{x}{2} \right) + \sum_{m=2}^{\infty} U_j\left( \frac{x}{2^j} \right)]}{U_m\left( \frac{x}{2} \right)} \) (see [GM, Theorem 4.7]).

**Theorem 3.23.** For all \( k \geq 2 \),

\[
J_{23 \ldots k1}^e(x) = \frac{x^2}{1-x} \left[ xR_{k-2}^e(x^2) + R_{k-2}^o(x^2)J_{12 \ldots (k-1)}(x) - R_{k-2}(-x^2)J_{12 \ldots (k-1)}^e(x) \right],
\]

where \( J_{12 \ldots m}(x) = \frac{x}{U_m\left( \frac{x}{2} \right)} \sum_{j=1}^{m-2} U_j\left( \frac{1}{2^j} \right) \) (see [GM, Theorem 4.6]).

3.4. Containing 132 and another pattern once. Let \( J_{\tau,r}^e(n) \) [respectively; \( J_{\tau,r}^o(n) \)] denote the number of even [respectively; odd] involutions in \( \mathfrak{S}_n \) such containing 132 exactly once and containing \( \tau \) exactly \( r \) times, let \( J_{\tau,r}^e(n) = \sum_{n \geq 0} J_{\tau,r}^e(n)x^n \) [respectively; \( J_{\tau,r}^o(n) = \sum_{n \geq 0} J_{\tau,r}^o(n)x^n \)] be the corresponding generating function.

**Example 3.24.** By Proposition 3.17 it is easy to see that \( J_{123}^e(x) = 0 \) and \( J_{213}^e(x) = 0 \). So the first interesting case can be examined in this case is when \( \tau = 123, 213 \), or 231. But \( J_{123}^{e,1}(x) = J_{231}^{e,1}(x) = 0 \) and \( J_{213}^{e,1}(x) = -\frac{x^6}{1-4x} \).

Once again, as extension of Example 3.24 let us consider the patterns \( \tau = 12 \ldots k \), \( \tau = 23 \ldots k1 \), and \( \tau = 2134 \ldots k \). Similarly as the arguments proofs in the last subsection together with [GM, Theorems 5.1, 5.2 and 5.4] we obtain as the following.

**Theorem 3.25.** For all \( k \geq 2 \),

\[
J_{12 \ldots k1}^e(x) = J_{23 \ldots k1}^e(x) = 0.
\]

**Theorem 3.26.** For all \( k \geq 2 \),

\[
J_{2134 \ldots k1}^e(x) = (1 - x^2) \sum_{j=2}^{k-1} \frac{x^{k+2-j}R_j^e(x^2) \prod_{i=j+1}^{k} R_i(-x^2)}{U_{j+1}\left( \frac{1}{2^j} \right)}.
\]

4. Statistics of generalized patterns on 132-avoiding involutions

In the current section let us consider the case of generalized patterns (see Subsection 1.1 for their definitions), and let us study some statistics on 1-3-2-avoiding involutions. We relate some of these results to the enumeration of 1-3-2-avoiding involutions according to the length and the number of rises (given in Subsection 2.4).

Robertson, Wilf and Zeilberger [RWZ] showed a simple continued fraction that records the joint distribution of the patterns 1-2 and 1-2-3 on 1-3-2-avoiding permutations. Recently, generalization of this theorem given, by Mansour and Vainshtein [MV1], by Krattenthaler [K], by Jani and Rieper
and by Brändén, Claesson and Steingrimsson [BCS]. Mansour [M3] generalizes the main result of [BCS] by replacing generalized patterns with classical patterns.

An another analogue of these results is to replace 1-3-2-avoiding permutations with 1-3-2-avoiding involutions.

First of all, let us define the set of all 1-3-2-avoiding involutions of all sizes with the empty permutation by $I(1,3,2)$, and the set of all 1-3-2-avoiding permutations of all sizes with the empty permutation by $S(1,3,2)$. Now, let us start by following proposition [GM, Proposition 2.1] which is the base of all the other results in the following subsections.

**Proposition 4.1.** (Guibert and Mansour [GM, Proposition 2.1]) Let $\pi \in \mathfrak{S}_n(1,3,2)$ be an involution such that $\pi_j = n$. Then holds one of the following assertions:

1. for $1 \leq j \leq \lfloor n/2 \rfloor$, $\pi = (\beta, n, \gamma, \delta, j)$, where
   1.1 $\beta$ is a 1-3-2-avoiding permutation of the numbers $n-j+1, \ldots, n-2, n-1$,
   1.2 $\delta$ is a 1-3-2-avoiding permutation of the numbers $1, \ldots, j-2, j-1$ such that $\delta \cdot \beta$ is the identity permutation of $S_{j-1}$,
   1.3 $\gamma$ is a 1-3-2-avoiding involution of the numbers $j+1, j+2, \ldots, n-j-1, n-j$;
2. for $j = n$, $\pi = (\beta, n)$ where $\beta$ is an involution in $\mathfrak{S}_{n-1}(1,3,2)$.

4.1. Counting an occurrences of 1-2-3-$\cdots$-$k$. Let us define

$$C_I(x_1, x_2, \ldots) = \sum_{\pi \in \mathfrak{I}(1,3,2)} \prod_{k \geq 1} x_k^{\sharp 1-2-\cdots-k(\pi)},$$

$$C_S(x_1, x_2, \ldots) = \sum_{\pi \in \mathfrak{S}(1,3,2)} \prod_{k \geq 1} x_k^{\sharp 1-2-\cdots-k(\pi)},$$

where $\sharp \tau(\pi)$ is the number of occurrences of $\tau$ in $\pi$.

**Theorem 4.2.** The generating function $C_I(x_1, x_2, \ldots)$ given by

$$C_I(x_1, x_2, \ldots) = \frac{1 + x_1 C_I(x_1 x_2, x_2 x_3, \ldots)}{1 - x_1^2 C_S(x_1^2 x_2^2, x_2^2 x_3^2, \ldots)},$$

where (see [BCS, Theorem 1])

$$C_S(x_1, x_2, \ldots) = \frac{1}{1 - x_1 C_S(x_1 x_2, x_2 x_3, \ldots)}.$$

**Proof.** In [BCS, Theorem 1] proved

$$C_S(x_1, x_2, \ldots) = \frac{1}{1 - x_1 C_S(x_1 x_2, x_2 x_3, \ldots)}.$$

On the other hand, by Proposition 4.1 it is easy to see for $k \geq 1$,

$$\sharp 1-2-3-\cdots-k(\pi', n) = \sharp 1-2-3-\cdots-k(\pi') + \sharp 1-2-3-\cdots-(k-1)(\pi'),$$

and

$$\sharp 1-2-3-\cdots-k(\pi', n, \beta, \pi'', j) = 2 \cdot \sharp 1-2-3-\cdots-k(\pi') + 2 \cdot \sharp 1-2-3-\cdots-(k-1)(\pi') + \sharp 1-2-3-\cdots-k(\beta).$$

Let us write an equation for the generating function $C_I(x_1, x_2, \ldots)$. The contribution of the first case is $x_1 C_I(x_1 x_2, x_2 x_3, \ldots)$, and of the second case is $x_1^2 C_I(x_1 x_2, \ldots) C_S(x_1^2 x_2^2, x_2^2 x_3^2, \ldots)$ (see Lemma 3.2). Hence

$$C_I(x_1, x_2, \ldots) = 1 + x_1 C_I(x_1 x_2, x_2 x_3, \ldots) + x_1^2 C_I(x_1 x_2, \ldots) C_S(x_1^2 x_2^2, x_2^2 x_3^2, \ldots),$$
Example 4.3. It is easy to see by Theorem 4.2 that, the number of involutions in \( I_n(1\text{-}3\text{-}2) \) containing 1-2 exactly once is given by \( \frac{1}{2}(1 + (-1)^n) \), and the number of involutions in \( I_n(1\text{-}3\text{-}2) \) containing 1-2 exactly twice is given by \( \frac{1}{4}(2n - 3 - (-1)^n) \).

An application for Theorem 4.2 we get

\[
\sum_{\pi \in I(1\text{-}3\text{-}2)} x^{\#\pi} = \frac{1}{1 - x - x^2} = \frac{2}{1 - 2x + \sqrt{1 - 4x^2}},
\]

which means that the number of 1-3-2-avoiding involutions in \( I_n \) is given by the central binomial coefficient \( a_n = \binom{n}{n/2} \) (see [SS]).

Let \( s_\pi \) be the number of right-to-left maxima of \( \pi \in S(132) \); so by [BCS, Proposition 5] we get

\[ s_\pi = \sharp 1(\pi) - \sharp 1\text{-}2(\pi) + \sharp 1\text{-}2\text{-}3(\pi) - \cdots . \]

An application for Theorem 4.2 with [BCS, Theorem 1] \( (C_S(x, 1, 1, \ldots) = C(x)) \) we get

\[
\sum_{\pi \in I(1\text{-}3\text{-}2)} x^{\#\pi} y^{s_\pi} = \frac{1 + xyC_f(x, 1, 1, \ldots)}{1 - x^2y^2CS(x^2, 1, 1, \ldots)},
\]

and \( C_f(x, 1, 1, \ldots) = \frac{1}{1 - x - x^2CS(x^2)} \). Hence,

\[
\sum_{\pi \in I(1\text{-}3\text{-}2)} x^{\#\pi} y^{s_\pi} = \sum_{j \geq 0} x^{2j} y^{2j} + \sum_{j \geq 0} \frac{x^{2j+1} C_f^j(x^2)}{1 - x - x^2CS(x^2)} y^{2j+1}.
\]

We can also prove this last result \( \sum_{\pi \in I(1\text{-}3\text{-}2)} x^{\#\pi} y^{s_\pi} \) from an entire combinatorial way (for convenience, we put \( |\pi| = n \) and \( s_\pi = s \).

Theorem 4.4. The number of 1-3-2-avoiding involutions of length \( n \) having \( s \) right-to-left maxima with \( 1 \leq s \leq n \) is:

\[
\begin{align*}
0 & \quad \text{if } n \text{ is odd and } s \text{ is even}, \\
\binom{n-1-s/2}{n/2} - \binom{n-1-s/2}{n/2} & \quad \text{if } n \text{ and } s \text{ are even}, \\
\binom{n-1-(s-1)/2}{n/2} & \quad \text{if } s \text{ is odd}.
\end{align*}
\]

Remark 4.5. Let \( \pi \) be an 1-3-2-avoiding involution of length \( n \) having \( s \) right-to-left maxima (with \( 1 \leq s \leq n \)) namely \( m_1, m_2, \ldots, m_s \) with \( 0 = m_0 < m_1 < m_2 < \cdots < m_{s-1} < m_s = n \). Thus, we have the following facts.

1. All the elements of \( |m_{i-1}, m_i| \) are located between \( m_{i+1} \) and \( m_i \).
2. \( \pi^{-1}(m_i) = m_{s+1-i} \) for all \( 1 \leq i \leq s \).
3. If \( s \) is even, then \( \pi \) has no fixed point, else, and if \( s \neq 1 \), we have that \( \pi \) has all its fixed points between \( m_{\frac{s+1}{2}} \) (excluded) and \( m_{\frac{s+1}{2}} \) (included because it is a fixed point), and that \( m_{\frac{s+1}{2}} + m_{\frac{s+1}{2}} - n = n \).
Proof. (1) is immediately deduced from the definition of a right-to-left maximum and from the pattern 1-3-2 we must avoid. Thus \( \pi^{-1}(m_i) = |m_{i-1}, n| = n - m_{i-1} \) for all 1 \( \leq i \leq s \).

(2) is deduced from (1), from that \( \pi^{-1}(n) = m_1 \) and because \( \pi \) is an involution. The proof is successively stated for \( i = 1, 2, \ldots, ([s + 1]/2) \).

(3) is deduced from (1), (2) and Proposition 4.1. The last fact is given by a simple calculus: \( m_{\pi^{-1}(m_{i+1})} = \pi^{-1}(m_{\pi^{-1}(m_{i+1})}) = n - m_{\pi^{-1}(m_{i+1})} \).

Lemma 4.6. There is a bijection between 1-3-2-avoiding involutions of length 2\( k \) having \( 2l + 1 \) right-to-left maxima and 1-3-2-avoiding involutions of length \( 2k + 1 \) having 2\( l + 3 \) right-to-left maxima, with \( 0 \leq l < k \).

Proof. Let \( \pi \in \mathcal{I}_{2k}(1-3-2) \) having 2\( l + 1 \) right-to-left maxima which are \( m_1, m_2, \ldots, m_{2l+1} \) with 0 = \( m_0 < m_1 < m_2 < \cdots < m_{2l} < m_{2l+1} = 2k \). \( \pi \) has two or more fixed points because one of them is a right-to-left maximum by (3) of Remark 4.3 and the length is even. Let \( x \) be the penultimate fixed point of \( \pi \) (the last one is \( m_{l+1} \)). In order to avoid 1-3-2 and by (1) of Remark 4.3, we have that the elements of \( \pi \) located between \( m_{l+2} \) and \( m_{l+1} \) forms a factor \( \pi' x \pi'' \) with \( |\pi'| = |\pi''| = m_{l+1} - x - 1 \) and all the elements of \( \pi' \), \( \pi'' \) belong respectively to \( [x, m_{l+1}], [m_{l+1} + m_l - x, x[ \) and \( ]m_l, m_{l+1} + m_l - x] \) that is all the elements of \( \pi' \) are connected to all the elements of \( \pi'' \). Thus, we obtain \( \sigma \) by changing into \( \pi \) the fixed point \( m_{l+1} \) by a cycle starting between \( \pi' \) and \( \pi'' \). \( \sigma \) avoids 1-3-2, has length 2\( k + 1 \) and has 2\( l + 3 \) right-to-left maxima which are \( m_1, m_2, \ldots, m_l, m_{l+1} + m_l - x \) (instead of \( m_{l+1} \)), \( x + 1 \) (new one), \( m_{l+1} + 1 \) (new one), \( m_{l+2} + 1 \), \( m_{l+3} + 1 \), \( \ldots \), \( m_{2l+1} + 1 \). This mapping is clearly bijective.

For example, \( \pi = 19 18 20 17 15 13 14 10 9 8 11 12 6 7 5 16 4 2 1 3 \) of length 20 (\( k = 10 \)) having 5 right-to-left maxima (\( l = 2 \) and \( m_1 = 3, m_2 = 4, m_3 = 16, m_4 = 17, m_5 = 20 \)) with \( x = 12 \), \( \pi' = 15 13 14, \pi'' = 10 9 8 11, \pi''' = 6 7 5 \), and \( \sigma = 20 19 21 18 16 14 15 17 11 10 9 12 13 6 7 5 8 4 2 1 3 \) of length 21 having 7 right-to-left maxima (3, 4, 8, 13, 17, 18, 21) are in bijection.

Lemma 4.7. There is a bijection between 1-3-2-avoiding involutions of length 2\( k + 1 \) having 2\( l + 1 \) right-to-left maxima and 1-3-2-avoiding involutions of length 2\( k + 2 \) having 2\( l + 2 \) or 2\( l + 3 \) right-to-left maxima, with 0 \( \leq l \leq k \).

Proof. Let \( \pi \in \mathcal{I}_{2k+1}(1-3-2) \) having 2\( l + 1 \) right-to-left maxima which are \( m_1, m_2, \ldots, m_{2l+1} \) with 0 = \( m_0 < m_1 < m_2 < \cdots < m_{2l} < m_{2l+1} = 2k + 1 \). There are two cases to consider depending on the number of fixed points of \( \pi \).

The first case consists in considering \( \pi \) having one fixed point which is \( m_{l+1} \). In order to avoid 1-3-2 and because there is only one fixed point, we have that the elements of \( \pi \) located between \( m_{l+2} \) and \( m_{l+1} \) forms a factor \( \pi' x \pi'' \) with \( |\pi'| = |\pi''| \) such that all the elements of \( \pi' \) are connected to all the elements of \( \pi'' \). Thus, \( \sigma \) the involution of even length is obtained by changing into \( \pi \) the fixed point \( m_{l+1} \) by a cycle starting between \( \pi' \) and \( \pi'' \). \( \sigma \) avoids 1-3-2, has length 2\( k + 2 \), has no fixed point and has 2\( l + 2 \) right-to-left maxima which are \( m_1, m_2, \ldots, m_l, m_{l+1} + m_l \) (instead of \( m_{l+1} \)), \( m_{l+1} + 1 \) (new one), \( m_{l+2} + 1, m_{l+3} + 1, \ldots, m_{2l+1} + 1 \). This mapping is clearly bijective.

The second case consists in considering \( \pi \) having more than two fixed points, and we apply exactly the same bijection given in Lemma 4.6. The involution \( \sigma \) we obtain avoids 1-3-2, has length 2\( k + 2 \) and has 2\( l + 3 \) right-to-left maxima.
As an example of the first case, \( \pi = 18 17 19 14 15 16 11 10 12 8 7 9 13 4 5 6 2 1 3 \) of length 19 \((k = 9)\) having 5 right-to-left maxima \((l = 2 \text{ and } m_1 = 3, m_2 = 6, m_3 = 13, m_4 = 16, m_5 = 19)\) with \( \pi' = 11 10 12, \pi'' = 8 7 9 \) and one fixed point \((13)\), and \( \sigma = 19 18 20 15 16 17 12 11 13 14 8 7 9 10 4 5 6 2 1 3 \) of length 20 having 6 right-to-left maxima \((3, 6, 10, 14, 17, 20)\) and no fixed point are in bijection.

Moreover, as an example of the second case, \( \pi = 12 13 10 9 6 5 7 8 4 3 11 12 \) of length 13 \((k = 6)\) having 3 right-to-left maxima \((l = 1 \text{ and } m_1 = 2, m_2 = 11, m_3 = 13)\) with \( x = 8, \pi' = 10 9, \pi'' = 6 5 7, \pi''' = 4 3 \) and 3 fixed points \((7, 8, 11)\) and \( \sigma = 13 14 11 10 12 7 6 9 4 3 5 1 2 \) of length 14 having 5 right-to-left maxima \((2, 5, 9, 12, 14)\) and 2 fixed points \((8, 9)\) are in bijection.

**Proof of Theorem 4.4**

Let \( \pi \in \mathcal{J}_n(1-3-2) \) having \( s \) right-to-left maxima with \( 1 \leq s \leq n \).

By (3) of Remark 4.3 we immediately obtain that there is no \( \pi \) such that \( n \) is odd and \( s \) is even. Trivially, if \( \pi \) has only one right-to-left maximum \((s = 1)\), the \( n - 1 \) first elements constitute an 1-3-2-avoiding involution. They are enumerated by \( a_{n-1} = \binom{n-1}{\lfloor (n-1)/2 \rfloor} \).

By (3) of Remark 4.3 we deduce when \( n \) and \( s \) are even that for \( \pi = \pi'\pi'' \) with \(|\pi'| = |\pi''|\) and \( \pi'' \in \mathcal{S}_{n/2}(1-3-2) \) having \( s/2 \) right-to-left maxima. This mapping is trivially bijective. Moreover, the number of 1-3-2-avoiding permutations according to the length and to the number of right-to-left maxima is given by the ballot numbers (see the sorting algorithm with one stack \([k_n]\) described in the end of Subsection 1.2).

So, the formula is established from a combinatorial way for the case of an even number of right-to-left maxima and for the special case of one right-to-left maximum. Two simple bijections given in Lemma 4.4 and Lemma 4.7 allow to compute for the case of an odd \((\text{and greater than} 1)\) number of right-to-left maxima, but we now state its formula from a combinatorial way.

First, we prove that all the arguments established for 1-3-2-avoiding involutions hold for Dyck word prefixes that is the number involutions in \( \mathcal{J}_n(1-3-2) \) having \( 2l \) or \( 2l + 1 \) right-to-left maxima is equal to the number of Dyck word prefixes of length \( n \) ended by \( x\overline{x}l \) which are also Dyck words or not respectively. Of course, a Dyck word prefix of odd length cannot be also a Dyck word. Let \( w = w'x\overline{x}l \) be a Dyck word prefix of length \( n \). Trivially, if \( l = 0 \), \( w = w'x \) is in bijection with \( w' \) a Dyck word prefix of length \( n - 1 \). It is well known that all Dyck words of length \( n = 2k \) ended by \( x\overline{x}l \) are enumerated by the ballot number \( \binom{2k-l-1}{k-1} \). Moreover, if \( w \) is not also a Dyck word then \( w \) is in bijection with a Dyck word prefix \( w\overline{x}l \) of length \( n + 1 \) such that if \( n = 2k \) then \( w\overline{x}l \) is not also a Dyck word whereas if \( n = 2k + 1 \) then \( w\overline{x}l \) is also a Dyck word if and only if \(|w| = |w'| + 1\).

We now complete the proof by establishing the number of Dyck word prefixes \( w = w'x\overline{x}l \) of length \( 2k + 1 \) (which are not Dyck words) is \( \binom{2k-l}{k-1} \). Let \( w' = wxv \) such that \( wx\overline{x}l \) is a Dyck word \((\text{so of even length, thus like } u)\). Let \( u' \) be the bilateral word in bijection with \( u \) by \( \Xi \). Thus, \( u'xv \) is any word of \( \{x, \overline{x}\}^* \) of length \( 2k \) with \( k \) letters \( x \) (and \( k-l \) letters \( \overline{x} \)). Conversely, let \( w'' \) be a word of \( \{x, \overline{x}\}^* \) with \(|w''| = 2k-l \) and \(|w''|_x = k \) (and \(|w''|_{\overline{x}} = k-l \)). There exists a nonnegative integer \( m \) such that \( w'' = w_0xw_1s\ldots s_{m}xw_{m+l}1x\ldots xw_{2m+l} \) where \( w_i \) is a Dyck word for all \( 0 \leq i \leq 2m+l \). Let \( u' = w_0xw_1\overline{x}\ldots\overline{x}w_m\overline{xw_{m+1}}\ldots\overline{xw_{2m}} \) and \( v = w_{2m+1}xw_{2m+2}x\ldots xw_{2m+l} \). So, by applying bijection \( \Xi \) on \( u' \), we have \( u = w_0xw_1\overline{x}\ldots\overline{xw_{m}xw_{m+1}}\ldots\overline{xw_{2m}} \). Thus, we obtain \( w = u'xv\overline{x}l \).

4.2. Counting an occurrences of 12-3-...-k. Now, let us define

\[
D_I(x_1, x_2, \ldots) = \sum_{\pi \in \mathcal{J}_{n}(1-3-2)} x_1^{11(\pi)} \prod_{k \geq 2} x_k^{12\ldots-k(\pi)},
\]

\[
D_S(x_1, x_2, \ldots) = \sum_{\pi \in \mathcal{S}_{n}(1-3-2)} x_1^{11(\pi)} \prod_{k \geq 2} x_k^{12\ldots-k(\pi)}.
\]
Theorem 4.8. The generating function $D_I(x_1, x_2, \ldots)$ is given by
\[
D_I(x_1, x_2, \ldots) = \frac{1 + x_1 - x_1 x_2 + x_1 x_2 D_I(x_1, x_2 x_3, x_3 x_4, \ldots)}{1 - x_1^2 + x_1^2 x_2^2 - x_1^2 x_2^2 D_S(x_1^2, x_2^2 x_3^2, x_3^2 x_4^2, \ldots)}.
\]
where (see [M3, Theorem 1])
\[
D_S(x_1, x_2, x_3, x_4, \ldots) = \frac{1}{1 - x_1 + x_1 x_2 - x_1 x_2 D_S(x_1, x_2 x_3, x_3 x_4, \ldots)}.
\]

Proof. In [M3, Theorem 1] proved
\[
D_S(x_1, x_2, x_3, x_4, \ldots) = \frac{1}{1 - x_1 + x_1 x_2 - x_1 x_2 D_S(x_1, x_2 x_3, x_3 x_4, \ldots)}.
\]
On the other hand, by Proposition [1] we have exactly two block decompositions for an arbitrary 1-3-2-avoiding involution. The contribution of the first case is, if $n = 1$ we get $x_1$, otherwise we have $x_1 x_2 D_I(x_1, x_2 x_3, x_3 x_4, \ldots) - 1)$. The contribution of the second case, if $\beta$ is empty (see Proposition [1]) we get $x_1^2 D_I(x_1, x_2, \ldots)$, otherwise we have (see Lemma [5,2]) $x_1^2 x_2^2 (D_S(x_1^2, x_2^2 x_3^2, x_3^2 x_4^2, \ldots) - 1) D_I(x_1, x_2, \ldots).$ Therefore
\[
D_I(x_1, x_2, \ldots) = 1 + x_1 - x_1 x_2 + x_1 x_2 D_I(x_1, x_2 x_3, x_3 x_4, \ldots) + x_1^2 D_I(x_1, x_2, \ldots) + \frac{x_1^2 x_2^2 D_I(x_1, x_2, \ldots)}{D_S(x_1^2, x_2^2 x_3^2, x_3^2 x_4^2, \ldots) - 1 - 1).
\]
The rest is easy to check.

Example 4.9. By [M3, Proposition 3.1] we get
\[
f(x, y) := D_S(x^2, y^2, 1, 1, \ldots) = \frac{-1 + x^2 - \sqrt{1 - 2x^2 + x^4 + 4x^2 y^2}}{2x^2 y^2}.
\]
So by Theorem 4.8 we have
\[
\sum_{\pi \in \mathcal{I}(1-3-2)} x^{\vert \pi \vert} y^{12(\pi)} = \frac{1 + x - xy}{1 - x^2 - xy + x^2 y^2 - x^2 y^2 f(x, y)}.
\]
For example, the number of involutions in $\mathcal{I}_n(1-3-2)$ containing exactly once a generalized pattern 12 is given by $\frac{1}{8}(2n - 1 + (1 - 1)^n)$, and the number of involutions in $\mathcal{I}_n(1-3-2)$ containing exactly twice a generalized pattern 12 is given by $\frac{1}{8}(2n(n - 2) + 1 - (1)^n)$.

Of course this last result for $\sum_{\pi \in \mathcal{I}(1-3-2)} x^{\vert \pi \vert} y^{12(\pi)}$ is equivalent to Theorem 2.4 enumerating the number of involutions in $\mathcal{I}_n(1-3-2)$ having $r$ rises (or equivalently having $n - r$ left-to-right minima by Remark 2.3) with $\vert \pi \vert = n$ and with $12(\pi) = r$.

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