ON ENTANGLED INFORMATION AND QUANTUM CAPACITY.

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Abstract. The pure quantum entanglement is generalized to the case of mixed compound states on an operator algebra to include the classical and quantum encodings as particular cases. The true quantum entanglements are characterized by quantum couplings which are described as transpose-CP, but not Completely Positive (CP), trace-normalized linear positive maps of the algebra.

The entangled (total) information is defined in this paper as a relative entropy of the conditional (the derivative of the compound state with respect to the input) and the unconditional output states. Thus defined the total information of the entangled states leads to two different types of the entropy for a given quantum state: the von Neumann entropy, or c-entropy, which is achieved as the supremum of the information over all c-entanglements and thus is semi-classical, and the true quantum entropy, or q-entropy, which is achieved at the standard entanglement. The q-capacity, defined as the supremum over all entanglements, coincides with the topological entropy. In the case of the simple algebra it doubles the c-capacity, coinciding with the rank-entropy. The conditional q-entropy based on the q-entropy, is positive, unlike the von Neumann conditional entropy, and the q-information of a quantum channel is proved to be additive.

1. Introduction

Quantum theory, which celebrated its 100 years anniversary last December, gives new possibilities for transmission of information which cannot be explained in the framework of information theory based on the classical (Kolmogorovian) probability theory. These possibilities are due to the entanglements, the specifically quantum (q-) correlations which were first studied by Schrödinger who introduced this term in his analysis of EPR paradox (for more details of this history see the anniversary review paper [1]).

Many authors have recently suggested to use the entanglements for quantum information processes in quantum computation, quantum teleportation, and quantum cryptography [2, 3, 4]. The mathematical study of entanglement as a special type of quantum correlations from an operational point of view has been initiated in [5, 6]. In these papers the entangled mutual information was introduced as the von Neumann entropy of the entangled compound state related to the product of marginal states in the sense of Lindblad, Araki and Umegaki relative entropy [6, 7, 8]. The corresponding quantum mutual information leads to an entropy bound for quantum capacity, the additivity of which is not obvious for non-trivial quantum channels.

Date: July 2, 2000.
1991 Mathematics Subject Classification. Quantum Information.
Key words and phrases. Entanglements, Compound States, Dimensional Entropy and Quantum Information.

This work was partially supported by the Royal Society grant for UK–Japan collaboration.
In this paper we will use another possibility to define the entangled mutual entropy, based on the alternative definition of relative entropy first introduced in \cite{10,13}. As it was proved in \cite{13} our relative entropy is larger than the LAU relative entropy, so that based on it quantum mutual information will give a larger entropy bound for quantum capacity.

We are going to prove that this bound for quantum capacity is in a sense additive, so that there is no need to consider under certain conditions this mutual information for the powers of quantum channel in order to guarantee that this entropy bound gives the real upper bound for long quantum block encodings. As far as we know this is first such measure of quantum capacity, although it is larger of \cite{5,6} and all other earlier suggested measures.

For the benefit of reader we repeat all needed definitions and notations related to the entanglement from \cite{5,6} in the first part of this paper. As in these papers we shall use the word entanglement in the generalized sense including the classical (c-) correlations as c-entanglements, and calling non-classical correlations as true quantum entanglements.

We shall show that any compound state can be achieved by a generalized entanglement, and the classically (c-) entangled states of c-q encodings and q-c decodings can be achieved by d-entanglements, the diagonal c-entanglements for these disentangled states. The pure orthogonal disentangled compound states are most informative among the c-entangled states in the sense that the maximum of mutual information over all c-entanglements is achieved on the extreme c-entangled states as the von Neumann entropy $S(\varsigma)$ of a given normal state $\varsigma$. Thus the maximum of mutual entropy over all classical couplings, described by c-entanglements of a classical probe systems $A$ to the system $B$, is bounded by the c-capacity $C = \log \text{rank}B$, where rank$B$ is the dimensionality of a maximal Abelian subalgebra $A \subset B$.

We prove that the truly entangled states are most informative in the sense that the maximum of mutual entropy over all entanglements to the quantum system $B$ is achieved by an extreme entanglement of the probe system $A = B$, called standard for a given $\varsigma$. The mutual information for such extreme q-compound state defines another type of entropy, the q-entropy $H(\varsigma) \leq 2S(\varsigma)$. The maximum of mutual entropy over all quantum couplings, described by true quantum entanglements of probe systems $A$ to the system $B$ is bounded by $C_q = \log \text{dim}B$.

In this paper we consider the case of a discrete decomposable W*-algebra $B$ for which the results are achieved by relatively simple proofs. The purely quantum case of a simple algebra $B = L(H)$, for which some proofs are rather obvious, will be also published elsewhere.

2. Compound States and Entanglements

Let $H$ denote a separable Hilbert space of quantum system, and $L(H)$ be the algebra of all linear operators $B : H \rightarrow H$ having the Hermitian adjoints $B^\dagger$ on $H$. In order to include the classical discrete systems as a particular quantum case, we shall fix a decomposable subalgebra $B \subseteq L(H)$ of bounded observables $B \in B$ of the block-diagonal form $B = \{ B(i) \delta^k \}$, where $B(i) \in L(H_i)$ are arbitrary bounded operators in Hilbert subspaces $H_i$, corresponding to an orthogonal decomposition $H = \oplus_i H_i$. 

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A normal state on $B$ is a positive linear functional $\zeta : B \to \mathbb{C}$ which can be expressed as

$$\zeta(B) = \text{Tr}_G \chi^\dagger B \chi = \text{Tr} B \sigma, \quad B \in B.$$  

Here $G$ is another separable Hilbert space, $\chi$ is a Hilbert-Schmidt operator from $G$ to $H$, $\chi^\dagger$ is its adjoint $H \to G$, and $\text{Tr}_G$ (or simply $\text{Tr}$ if there is no ambiguity) denotes trace in $G$ (or in $H$). This $\chi$ is called the amplitude operator (just amplitude if $G$ is one dimensional space $\mathbb{C}$, $\chi = \psi \in H$ with $\chi^\dagger \chi = ||\psi||^2 = 1$ in which case $\chi^\dagger$ is the functional $\psi^\dagger$ from $H$ to $\mathbb{C}$). If the normal state $\zeta$ is pure on the decomposable algebra $B$, then the density operator $\sigma = \chi \chi^\dagger$ is uniquely defined as one dimensional projector $P = \psi \psi^\dagger \in B$. For mixed states the $\sigma$ in (2.1) may not be unique, but it is uniquely defined as a positive trace one operator ($\sigma \geq 0$, $\text{Tr} \sigma = 1$) by an additional condition $\sigma \in B$, and is called in this case the probability operator on $B$, denoted as $\sigma = p_B$.

The amplitude operator is not unique, however it is defined uniquely up to a unitary transform $\chi^\dagger \mapsto U \chi^\dagger$ in $G$ as a probability amplitude by the additional condition $\chi \chi^\dagger \in B$. Such a $\chi$ always exists as square root of the decomposable probability operator $P_B = \oplus P_B (i) \in B$ with the components $P_B (i) \in \mathcal{L}(H_i)$ normalized as

$$p(i) = \text{Tr}_{H_i} P_B (i) \geq 0, \quad \sum_i p(i) = 1.$$  

We denote by $B_* \subset B$ the predual space to $B$ identified with the Banach subspace $\mathcal{T}(H) \cap B = \oplus \mathcal{T}(H_i)$ of trace class operators $\sigma = \oplus \sigma_i$, where $\sigma_i \in \mathcal{T}(H_i)$. The probability operators (the unique densities) $P_A$, $P_B$ of the states $\rho$, $\zeta$ on different algebras $A \subseteq \mathcal{L}(G)$, $B \subseteq \mathcal{L}(H)$ will be usually denoted by the variables $\rho \in A_*$ and $\sigma \in B_*$ corresponding to their Greek variations $\rho$ and $\zeta$.

In general, $G$ is not one dimensional, the dimensionality $\dim G$ must not be less than $\text{rank} \rho$, the dimensionality of the range $\text{ran} \chi^\dagger$ of the density operator $\tilde{\rho} = \chi^\dagger \chi$ coinciding with rank $\sigma$ of the probability operator $\sigma = \chi \chi^\dagger$. This implies that the rank $A$ of any discretely decomposable subalgebra $A \subseteq \mathcal{L}(G)$ having $\rho$ as the probability operator $\rho \in A$ must not be less than $\text{rank} \sigma$ if $\sigma = \chi \chi^\dagger$.

We can always equip $H$ (and we will equip the other Hilbert spaces) with an isometric involution $J = J^\dagger$, $J^2 = I$, having the properties of complex conjugation on $H$,

$$J \sum \lambda_j \eta_j = \sum \bar{\lambda}_j J \eta_j, \quad \forall \lambda_j \in \mathbb{C}, \eta_j \in H,$$

with respect to which the fixed density $\sigma$ is invariant, $J \sigma J = \sigma$, as a real element of an invariant Abelian subalgebra $A = JAJ$ of $\mathcal{L}(H)$. The latter can also be expressed as the symmetricity property $\zeta = \zeta$ of the state $\zeta(B) = \text{Tr} B \sigma$ given by the real Hermitian and so symmetric density operator $\sigma = \sigma$ on $H$ with respect to the complex conjugation $B = JBJ$ and the tilde operation (transposition) $\tilde{B} = JB^\dagger J$ on $B$. One can always assume that $J$ is the standard complex conjugation in an eigen-representation of $\sigma$, and take the maximal Abelian subalgebra $A \subset \mathcal{L}(H)$ of all diagonal operators in this basis.

Given the amplitude operator $\chi$, one can define not only the states $\zeta$ by $\sigma = \chi \chi^\dagger$ on the algebra $B$ but also a pure compound state $\pi$ on the algebra of all bounded operators on the tensor product Hilbert space $G \otimes H$ by
\[ \psi (A \otimes B) = \text{Tr}_\mathcal{H} B \chi \tilde{A} \chi^\dagger = \text{Tr}_\mathcal{G} \chi^\dagger B \chi \tilde{A} \]

Thus defined \( \psi \) is uniquely extended by linearity to a normal state on the algebra \( \mathcal{L} (\mathcal{G} \otimes \mathcal{H}) \) generated by all the linear combinations \( C = \sum \lambda_j A_j \otimes B_j \) due to \( \psi (I \otimes I) = \text{Tr} \chi^\dagger \chi = 1 \) and

\[
\psi (C^\dagger C) = \sum_{i,k} \lambda_i \lambda_k \text{Tr}_\mathcal{G} \tilde{A}_i \chi^\dagger \chi^\dagger B_i B_k \chi
\]

\[ = \sum_{i,k} \lambda_i \lambda_k \text{Tr}_\mathcal{G} \tilde{A}_i \chi^\dagger B_i B_k \chi \tilde{A}_k = \text{Tr}_\mathcal{G} X^\dagger X \geq 0, \]

where \( X = \sum_j \lambda_j B_j \chi \tilde{A}_j \).

This compound state \( \psi \) is pure on \( \mathcal{L} (\mathcal{G} \otimes \mathcal{H}) \), and it is entangled unless its marginal state \( \varsigma \) is also pure. Indeed, \( \psi \) corresponds to the amplitude \( \psi \in \mathcal{G} \otimes \mathcal{H} \) defined by an involution \( J \) in \( \mathcal{G} \) as

\[ (\zeta \otimes \eta)^\dagger \psi = \eta^\dagger J \zeta, \quad \forall \zeta \in \mathcal{G}, \eta \in \mathcal{H} . \]

This definition implies

\[ \psi^\dagger (A \otimes B) \psi = \text{Tr} B \chi J A^\dagger J \chi^\dagger \quad \forall A \in \mathcal{L} (\mathcal{G}), B \in \mathcal{L} (\mathcal{H}) \]

as it can be easily seen for \( A = \zeta \eta^\dagger, B = \eta \eta^\dagger \), and \( \zeta \) and \( \eta \) are the marginals of \( \psi \) defined as

\[ (2.2) \quad \psi^\dagger (I \otimes I) \psi = \text{Tr}_\mathcal{G} A \sigma, \quad \psi^\dagger (A \otimes I) \psi = \text{Tr}_\mathcal{G} A \rho. \]

As follows from the next theorem, any pure compound state

\[ \psi (A \otimes B) = \psi^\dagger (A \otimes B) \psi, \quad A \in \mathcal{A}, B \in \mathcal{B} \]

given on the decomposable \( \mathcal{A} \otimes \mathcal{B} \) by a probability amplitude \( \psi \in \mathcal{G} \otimes \mathcal{H} \) with \( \psi \psi^\dagger \in \mathcal{A} \otimes \mathcal{B} \), can be achieved as described by a unique entanglement of its marginal states \( \rho \) and \( \varsigma \).

**Theorem 1.** Let \( \psi : \mathcal{A} \otimes \mathcal{B} \to \mathbb{C} \) be a compound state

\[ (2.3) \quad \psi (A \otimes B) = \text{Tr}_\mathcal{F} v^\dagger (A \otimes B) v, \]

defined by an amplitude operator \( v : \mathcal{F} \to \mathcal{G} \otimes \mathcal{H} \) on a separable Hilbert space \( \mathcal{F} \) into the tensor product Hilbert space \( \mathcal{G} \otimes \mathcal{H} \) with

\[ vu^\dagger \in \mathcal{A} \otimes \mathcal{B}, \quad \text{Tr}_\mathcal{F} v^\dagger v = 1. \]

Then this state is achieved by an entangling operator \( \chi : \mathcal{G} \to \mathcal{F} \otimes \mathcal{H} \) as

\[ (2.4) \quad \psi (A \otimes B) = \text{Tr}_\mathcal{F} \chi^\dagger (I \otimes B) \chi^\dagger = \text{Tr}_\mathcal{G} \chi^\dagger (I \otimes B) \chi \tilde{A} \]

of the states \( (2.2) \) with \( \rho = J \chi^\dagger \chi J \) and \( \sigma = \text{Tr}_\mathcal{F} \chi \chi^\dagger \), where \( \chi \) is an operator \( \mathcal{G} \to \mathcal{F} \otimes \mathcal{H} \) satisfying the conditions

\[ \text{Tr}_\mathcal{F} \chi \tilde{A} \chi^\dagger \subset \mathcal{B}, \quad \chi^\dagger (I \otimes B) \chi \subset \mathcal{A}. \]

The amplitude operator \( \chi \) is uniquely defined by \( \chi U = v \), where

\[ (2.5) \quad (\zeta \otimes \eta)^\dagger \chi \zeta = (J \eta \otimes \eta)^\dagger \chi J \zeta, \quad \forall \zeta \in \mathcal{F}, \zeta \in \mathcal{G}, \eta \in \mathcal{H}, \]

up to a unitary transformation \( U \) of the minimal space \( \mathcal{F} = \text{ran} v^\dagger \) equipped with an isometric involution \( J \).
Proof. Without loss of generality we can assume that the space $F$ is a subspace of $\ell^2(\mathbb{N})$ for the diagonal representation of $v^\dagger v$ equipped with the standard complex conjugation $C$ just as the space $G$ is a diagonal representation of $\chi^\dagger \chi$. In these canonical basises of $F$ and $G$ the amplitude operator $\chi = \sum \chi(n) \langle n |$ can be defined as the block-matrix $\sum |k\rangle \otimes \chi_k(n) \langle n|$ transposed to $\sum |n\rangle \otimes \chi_k(n) \langle k|$, where the amplitudes $\psi_k(n) \in H$ are given by the matrix elements $\eta^\dagger \chi_k(n) = (|n\rangle \otimes \eta^\dagger) v|k\rangle$:

$$\text{Tr}_G \tilde{A} \chi^\dagger (I \otimes B) \chi = \sum_{n,m} \langle n|\tilde{A}|m\rangle \chi^\dagger_k(m) B \chi_k(n)$$

$$= \sum_{n,m} \chi^\dagger_k(m) \langle m|A |n\rangle B \chi_k(n) = \text{Tr}_F \eta^\dagger (A \otimes B) \eta v.$$ 

In any other ortho-normal basis $\{\xi_k\} \subset F$ the involution $J : F \rightarrow F$ satisfying $J\xi_k = \xi_k$ is defined as $U^\dagger C U$, and $v = \sum |n\rangle \otimes \psi_k(n) \xi_k^\dagger = \chi U$, where $U = \sum |k\rangle \xi_k^\dagger$. The isometric transformation $U$ of $\{\xi_k\}$ into the canonical basis $\{|k\rangle\} \subset \ell^2(\mathbb{N})$ is real in the sense $\tilde{U} := CUJ = U$, and thus $\tilde{U} := C U^\dagger J = U^\dagger$. Hence amplitude operator $\chi : G \rightarrow F \otimes H$ which was defined above by the transposition of $vU^\dagger = v\tilde{U} \equiv \chi$, is equivalent to $\tilde{v}$: $\chi = (U \otimes I) \tilde{v}$. Thus

$$J \chi^\dagger \chi J = \text{Tr}_H \chi v^\dagger = \rho, \quad \text{Tr}_F \chi v^\dagger = \text{Tr}_G \eta^\dagger = \sigma.$$ 

Moreover, it satisfies the conditions (2.3) since $\omega = \chi v^\dagger \in A \otimes B$:

$$J \chi^\dagger (I \otimes B) \chi J = \text{Tr}_H (I \otimes B) \omega \in A, \quad \text{Tr}_F \chi \tilde{A} \chi^\dagger = \text{Tr}_G (A \otimes I) \omega \in B.$$ 

The uniqueness up to the $U$ follows from the obvious isometricity of the families

$$\left\{ \sum_k |k\rangle \eta^\dagger \psi_k(n) : n \in \mathbb{N}, \eta \in H \right\}, \quad \left\{ \sum_k \eta^\dagger \psi_k(n) \xi_k^\dagger : n \in \mathbb{N}, \eta \in H \right\}$$

of vectors $(I \otimes \eta^\dagger) \chi(n)$ in $F \subseteq \ell^2(\mathbb{N})$ and of $(|n\rangle \otimes \eta^\dagger) v$ in $F^\dagger$ which follows from

$$\text{Tr}_G |n\rangle \langle n| \chi^\dagger (I \otimes \eta^\dagger) \chi = \text{Tr}_F \eta^\dagger (|n\rangle \otimes \eta^\dagger) v.$$ 

Thus they are unitary equivalent in the minimal space $F$. So the entangling operator $\chi$ is defined in the minimal $F$ up to unitary equivalence corresponding to the unitary operator $U$ in $F$ intertwining the involutions $C$ and $J$. $\blacksquare$

Note that the entangled state (2.4) is written as

$$\varpi (A \otimes B) = \text{Tr}_H B \pi^* (A) = \text{Tr}_G A \pi (B),$$ 

where the operator $\pi^* (A) = \text{Tr}_F \chi \tilde{A} \chi^\dagger \in B$, bounded by $\|A\| \sigma \in B_\sigma$, is in the predual space $B_\sigma$ for any $A \in L(G)$, and

$$(2.6) \quad \pi (B) = J \chi^\dagger (I \otimes B) \chi J = \chi^\dagger (I \otimes B) \chi$$

is in $A_\sigma$ as a trace-class operator in $G$, bounded by $\|B\| \rho \in A_\sigma$. The linear map $\pi$ is written in the Steinspring form ([14]) of the normal completely positive map $B \mapsto \pi (B)$, while $\pi^* : A \rightarrow B_\sigma$ is written in the Kraus form ([15]) of the normal CP map $A \mapsto \pi^* (\tilde{A})$ in the canonical orthonormal basis $|k\rangle$ of $F \subseteq \ell^2(\mathbb{N})$:

$$\pi^* (A) = \sum_k (|k\rangle \otimes I) \chi \tilde{A} \chi^\dagger (|k\rangle \otimes I).$$
A linear map \( \pi : B \rightarrow A \) is called completely positive (CP) if the operator matrix
\[
\pi (B) = \pi (\{B_{ik}\}) = [\pi (B_{ik})]
\]
is positive (in the sense of non-negative definiteness) for every positive operator-matrix \( B = [B_{ik}] \) (which is thus Hermitian, \( B_{ik} = B_{ki} \)). But the defined in (2.6) \( \dagger \)-map \( \pi (B^\dagger) = \pi (B) \dagger \) is not necessarily CP but tilde-completely positive (TCP) in the sense that the map \( B \rightarrow \pi^\dagger (B) \) given by the transposed operator-matrix
\[
\pi^\dagger (B) := [J \pi (B_{ik}) \dagger J] = [J \pi (B_{ki}) J] \equiv \pi^\dagger (B')
\]
is positive (in the sense of non-negative definiteness) for every positive operator-matrix \( B_{ik} = [B_{ki}] \). Obviously every tilde-positive \( \pi \rightarrow B \rightarrow A \) as \( \pi^\dagger (B) J \) is positive for every positive \( B \), but it is not necessarily CP even if it is TCP unless \( A \) (or \( B \)) is Abelian. The TCP maps can be obtained simply by partial tracing
\[
(2.7) \quad \pi^*(A) = \text{Tr}_G ((A \otimes I) \omega), \quad \pi (B) = \text{Tr}_H ((I \otimes B) \omega).
\]
in terms of the compound density operator \( \omega = \nu \nu^\dagger \) for the entangled state
\[
\varpi (A \otimes B) = \text{Tr} (A \otimes B) \omega.
\]

**Definition 1.** The normal TCP map \( \pi : B \rightarrow A_\ast \) (and its dual map \( \pi^* : A \rightarrow B_\ast \)) normalized to a probability operator \( \rho = \pi (I) \) as \( \text{Tr}_G \pi (I) = 1 \) (to \( \sigma = \pi^* (I) \) as \( \text{Tr}_H \pi^* (I) = 1 \)) is called coupling of the state \( \zeta \) on \( B \) to \( \varrho \) (or generalized entanglement of the state \( \varrho \) on \( A \) to \( \zeta \)). The coupling \( \pi \) (or entanglement \( \pi^* \)) is called true quantum if it is not CP, i.e. if there exists a positive operator-matrix \( B = [B_{ik}] \) with \( B_{ik} \in B \) for which \( \pi (B) = [\pi (B_{ik})] \) is not positive. The coupling (entanglement)
\[
\pi = \pi^\sigma = \pi^* \quad \mathrm{by}
\]
\[
(2.8) \quad \pi^\sigma (B) = \sigma^{1/2} \hat{B} \sigma^{1/2}, \quad B \in B
\]
of the state \( \varrho = \zeta \) on the algebra \( A = B \) is called standard for the system \( (B, \zeta) \).

Note that the standard entanglement is true as soon as the reduced algebra \( B_\sigma = E_\sigma B E_\sigma \) on the support \( E_\sigma = E_\sigma H \) of the state \( \zeta \) is not Abelian. (Here \( E_\sigma \) is the minimal orthoprojector \( E \in A \) with \( \zeta (E) = 1 \).) In the case of the simple algebra \( B = \mathcal{L} (H) \) it is obvious as \( \pi^\sigma \) restricted to \( B_\sigma \) is the composition of the non-degenerated multiplication \( B_\sigma \ni B \mapsto \sigma^{1/2} B \sigma^{1/2} \) (which is CP) and the transposition \( \hat{B} = JB^\dagger J \) on \( B_\sigma \) (which is TCP but not CP if \( \dim E_\sigma > 1 \)). The standard compound state
\[
\varpi^\sigma (A \otimes B) = \text{Tr}_H B \sigma^{1/2} \hat{A} \sigma^{1/2} = \text{Tr}_H A \sigma^{1/2} \hat{B} \sigma^{1/2}
\]
on the algebra \( B \otimes B \) is pure in this case, given by the amplitude \( \nu \simeq |\sigma^{1/2} \rangle \equiv \psi \), where \( |\sigma^{1/2} \rangle = \hat{\chi} \) with \( \chi = \sigma^{1/2} \) and \( \hat{\chi} \) defined in (2.5) as \( (\zeta \otimes \eta) \dagger \hat{\chi} = \eta \dagger \chi J \zeta \). In particular, any pure compound state is truly entangled if \( \text{rank} \rho = \text{rank} \sigma \) is not one because \( \pi^* (A) = \chi \hat{A} \chi^\dagger \) can be decomposed as
\[
\chi \hat{A} \chi^\dagger = J \sigma^{1/2} U^\dagger J A J U \sigma^{1/2} J = \pi^\varrho \left( U^\dagger \hat{A} U \right),
\]
where \( U : \sigma^{1/2} J \eta \mapsto J \chi^\dagger \eta \) is a unitary operator from \( E_\sigma \) onto the support of \( \rho \) in \( G \) with nonabelian \( B_\sigma = U \dagger AU = \mathcal{L} (E_\sigma) \).
In the general case of a discretely decomposable \((\mathcal{B}, \varsigma)\) with the density operator 
\[ \sigma = \oplus \sigma (i) \]
having more than one components \(\sigma (i) = \sigma _i, p (i)\) with nonzero probability \(p (i) = \text{Tr} \sigma (i)\) and positive trace one \(\sigma _i, \in \mathcal{T} \left( \mathcal{H}_i \right)\), the standard compound state is mixed, described by the decomposable density operator
\[ \omega_q = \oplus _{i,j} p (j) \delta _j^i | \sigma _j^{1/2} \rangle \langle \sigma _j^{1/2} |, \quad A, B \in \mathcal{B} \]
with zero components \(\omega_q (i,j) = \delta _j^i p (j) \omega_j\) at \(i \neq j\), corresponding to the pure compound states \(\omega_j = \psi_j \psi_j^\dagger\). The amplitudes \(v_j \simeq | \sigma _j^{1/2} \rangle \equiv \psi_j \in \mathcal{H}_j \otimes \mathcal{H}_j\) define the orthogonal decomposition
\[ v_q = \oplus _{i,j} p (j) | \sigma _j^{1/2} \rangle \delta _j^i | i \rangle = \oplus _{i,j} \psi (i) \delta _j^i | i \rangle \]
of the standard amplitude operator \(v_q : \mathcal{F} \to \oplus \mathcal{H}_i \otimes \mathcal{H}_j\) on \(\mathcal{F} = \ell^2 (\mathbb{N})\) with the components
\[ v_q (i,j) = v_q (j) \delta _j^i, \quad v_q (j) = \psi (j) \langle i |, \]
where \(\psi (j) = p (j) | \sigma _j^{1/2} \rangle \psi_j\). It corresponds to the block-diagonal entangling operator \(\chi = [\chi (j) \delta _j^i]\) with
\[ \chi (j) = | j \rangle \otimes \sigma (j)^{1/2} = \tilde{v}_q (j) \].

The so called separable compound states, which are given by convex combinations
\[ \omega_c (A \otimes B) = \sum _n \rho _n (A) \varsigma _n (B) \mu (n) \]
of the product states \(\rho _n \otimes \varsigma _n\), are obviously not true entangled as the corresponding map
\[ \pi ^{\varsigma} (B) = \sum _n \varsigma _n (B) \rho _n \mu (n) \]
is both CP and TCP.

3. Quantum Entropy via Entanglements

As we shall prove in this section, the most informative for a quantum system \((\mathcal{B}, \varsigma)\) is the standard entanglement \(\pi _q = \pi ^\delta = \pi ^\varsigma\) to the probe system \((\mathcal{A}^0, \theta _0) = (\mathcal{B}, \varsigma)\), described in \((\ref{2.8})\).

Let us consider the entangled information and quantum entropies of states by means of the entangled compound states. To define the quantum information, we need to apply a quantum version of the relative entropy to compound state on the algebra \(\mathcal{A} \otimes \mathcal{B}\). In classical information theory the relative entropy is defined as the expectation of the logarithm of the derivation of the state \(\omega\) with respect to a reference measure \(\phi\). The relative entropy measures the information divergence of the state \(\omega\) with respect to \(\phi\). This information divergence is equal to the expectation of \(\ln \omega \phi^{-1} = \ln \omega - \ln \phi\) in the state \(\omega\), where \(\omega\) and \(\phi\) are the densities of \(\omega\) and \(\phi\) with respect to Lebesgue or any other appropriate measure (with respect to which \(\phi\) is invertible, \(\phi^{-1} \phi = I = \phi \phi^{-1}\)). Such defined, it should be better called the relative information rather than entropy: indeed many authors reserve the term relative entropy for the expectation of \(-\ln \omega \phi^{-1}\) such that it coincides with Boltzmann entropy if \(\phi = 1\).
In quantum case, however,
\[ \ln \omega^{-1} \neq \ln \omega - \ln \phi \neq \ln \phi^{-1} \omega \]
for noncommuting density operators \( \omega \) and \( \phi \), thus giving different possibilities for definition of relative entropy for quantum state \( \varpi \) with respect to \( \varphi \). In [10] we investigated the possibility for entangled mutual information based on the most common definition of quantum relative entropy (information) as the quantum expectation of the difference \( \ln \omega - \ln \phi \), but it was hard to prove the additivity of the corresponding estimate for quantum capacity of the nontrivial channels.

Here we take another choice
\[ (3.1) \]
\[ R(\varpi : \varphi) = \text{Tr} \log \omega^{1/2} \phi^{-1} \omega^{1/2} \]
suggested in [10] for quantum relative entropy of the state \( \varpi \) an algebra \( M \) with respect to a weight \( \varphi \), given by a positive invertible operator \( \phi \in M \). Note that this quantum information divergence is well defined as suggested in [10, 13] we investigated the possibility for entangled mutual information based on the most common definition of quantum relative entropy (information) as the quantum expectation of the difference \( \ln \omega - \ln \phi \), but it was hard to prove the additivity of the corresponding estimate for quantum capacity of the nontrivial channels.

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\[ (3.1) \]
\[ R(\varpi : \varphi) = \text{Tr} \log \omega^{1/2} \phi^{-1} \omega^{1/2} \]
suggested in [10] for quantum relative entropy of the state \( \varpi \) an algebra \( M \) with respect to a weight \( \varphi \), given by a positive invertible operator \( \phi \in M \). Note that this quantum information divergence is well defined as
\[ R(\varpi : \varphi) = \text{Tr} \varphi \eta \left( \phi^{-1} \omega \right) = \text{Tr} \varphi v^\dagger v \log v^\dagger \phi^{-1} v, \]
where \( \eta(x) = x \log x \), \( \omega = vv^\dagger \) and
\[ \eta \left( \phi^{-1} \omega \right) = \phi^{-1/2} \eta \left( \phi^{-1/2} \omega \phi^{-1/2} \right) \phi^{1/2} \]
as soon as the quantum Radon-Nicodim derivative \( \phi^{-1/2} \omega \phi^{-1/2} \) of the state \( \varpi \) with respect to \( \varphi \) is defined as a positive operator in the Hilbert space. This definition can be extended to any state \( \varpi \) absolutely continuous with respect to \( \varphi \) [12], i.e. if \( \varpi(E) = 0 \) for the maximal null-orthoprojector \( E \phi = 0 \) (otherwise the entropy is infinite by definition). As proved in [14], it gives a larger relative entropy than the expectation of \( \ln \omega - \ln \phi \), and it has a positive value \( R(\varpi : \varphi) \in [0, \infty] \) if the states are equally normalized, say (as usually) \( \text{Tr} \omega = 1 = \text{Tr} \phi \).

The most important property of the information divergence \( R \) is its monotonicity property [10], i.e. nonincrease of the divergence \( R(\varpi_0 : \varphi_0) \) after the application of the pre-dual \( K_\ast \) of a normal completely positive unital map \( K : M \to M^0 \) to the state \( \varpi_0 \) and \( \varphi_0 \) on a von Neumann algebra \( M^0 \):
\[ (3.2) \]
\[ \varpi = \varpi_0 K, \varphi = \varphi_0 K \Rightarrow R(\varpi : \varphi) \leq R(\varpi_0 : \varphi_0). \]
A quantum statistical morphism \( K \) can only decrease their information divergence; it can even be made zero by \( \varpi_0 K = \varphi_0 K \).

Let \( \pi : A \to A_\omega \) be an entanglement of the state \( \varrho \) corresponding to the density operator \( \rho = \pi(I) \). We shall define the entangled entropy (or, better, entangled information) \( E(\pi) \) as the relative entropy (information) \( E(\pi) \) of the achieved compound state \( \varpi = \varpi_0 K, \varphi = \varphi_0 K \) with respect to the weight \( \varphi = \varrho \otimes \text{Tr} \) corresponding to the density operator \( \phi = \rho \otimes I \):
\[ (3.3) \]
\[ E(\pi) = \text{Tr} \varrho \log (\rho \otimes I)^{-1} \omega^{1/2} = \text{Tr} \varrho v^\dagger v \log v^\dagger (\rho \otimes I)^{-1} v, \]
where \( \rho^{-1} \) is quasiinverse to \( \rho \) in the case of \( \text{rank} \rho \neq \text{dim} \varrho \).

If \( \omega = \psi \psi^\dagger \) is one dimensional orthoprojector (corresponding to a pure state on the decomposable algebra \( A \otimes B \) with \( v \simeq \psi \in G \otimes \mathcal{H} \)), then \( v^\dagger v = 1 \), and
\[ v^\dagger (\rho \otimes I)^{-1} v = \text{Tr} \chi \rho^{-1} \chi^\dagger = \text{Tr} \rho^{1/2} \rho^{-1} \rho^{-1/2} = \text{rank} \rho. \]
In this case \( E(\pi) = \ln \text{rank} \rho \geq 0 \), where \( \text{rank} \rho = \text{rank} \varrho \) is the dimensionality of the range \( \text{ran} \chi \) or \( \text{ran} \chi^\dagger \). Thus for quantum entanglements \( \pi \) corresponding to pure \( \omega \)
but mixed $\rho = \chi^\dagger \chi$ and $\sigma = \chi \chi^\dagger$ (with rank $> 1$) the entangled entropy is strictly positive. However it might be negative as it is in the case of an Abelian $\mathcal{A}$ when

$$\pi (B) = \sum |n\rangle \zeta_n (B) \langle n| \mu (n) = \pi^d (B).$$

In this case $\omega = \sum |n\rangle \sigma \langle n|, \rho = \sum |n\rangle \mu (n) \langle n|$, and $E (\pi) = -S (\pi)$, where

$$S (\pi) = -\text{Tr} \omega \ln \frac{\omega}{\rho \otimes I} = \sum \mu (n) S (\frac{\zeta (n)}{\mu (n)}) \geq 0$$

is the mean of the conditional von Neumann entropy

$$S (\zeta_n) = -\text{Tr} \sigma_n \ln \sigma_n$$
on $\mathcal{B}$ corresponding to $\sigma_n = \sigma (n) / \mu (n)$. We can call the (mixed) state $\omega$ essentially disentangled (or separable) if $E (\pi) \leq 0$; if $\omega$ is pure and $E (\pi) > 0$, it is truly entangled as the entangling map $\pi$ is obviously not CP in this case.

The total entangled information $I (\pi)$ is defined as the sum of the von Neumann entropy

$$S (\zeta) = -\text{Tr} \sigma \ln \sigma$$

corresponding to the state $\zeta (B) = \text{Tr} \pi (B)$ and the entangled entropy (information) $E (\pi)$:

$$I (\pi) = \text{Tr} \omega \left( \frac{1}{\rho \otimes I} \ln \omega - \ln (I \otimes \sigma) \right).$$

Note that $I (\pi) \geq 0$, but in general $I (\pi) \neq I (\pi^*)$ unless $\rho \otimes I$ commutes with $\omega$ as in the case of $d$-entanglement or Abelian $\mathcal{A}$ when the total information $I (\pi) = I (\pi^*)$ coincides with the mutual information $I_{A:B} (\pi) = I_{B:A} (\pi^*)$ defined in [3, 4]. In the classical case when both algebras $\mathcal{A}$ and $\mathcal{B}$ are Abelian, $I (\pi)$ coincides with the Shannon mutual information $I_{A:B} (\pi)$.

The following proposition follows from the monotonicity property (3.4) of the relative entropy on $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ with respect to the predual $K_\ast (\omega_0) = \omega_0 (K \otimes I)$ to the ampliation $K \otimes I$ of a normal completely positive unital map $K : \mathcal{A} \to \mathcal{A}^0$.

**Proposition 1.** Let $\pi : \mathcal{B} \to \mathcal{A}$ be an entanglement of $(\mathcal{B}, \zeta)$ of a state $\zeta (B) = \text{Tr} \pi (B), B \in \mathcal{B}$ to $(\mathcal{A}, \varrho)$ with the density operator $\rho = \pi (I)$, and $\pi_0 : \mathcal{A}^0 \to \mathcal{B}_x$ be an entanglement defining $\pi$ by the composition $\pi^* = \pi_0 K$ with a normal completely positive unital map $K : \mathcal{A} \to \mathcal{A}^0$. Then $E (\pi) \leq E (\pi^0)$, where $\pi^0 = \pi_0^\dagger r_0$, and thus $I (\pi) \leq I (\pi^0)$. In particular, for any separable $\pi = \pi^s$ where $\pi^s$ is the convex combination \( \sum c_n \mu (n) = \zeta \), there exists a not less informative entanglement $\pi^d : \mathcal{B} \to \mathcal{A}^0$ with the same $\zeta (B) = \text{Tr} \pi^d (B)$ and Abelian $\mathcal{A}^0$, and the standard entanglement $\pi^0 = \pi^s$ to $\varrho_0 = \zeta$ with $\mathcal{A}^0 = \mathcal{B}$ is the maximal one in the sense that for any entanglement $\pi$ there exists not less informative $q$-entanglement with the same $(\mathcal{B}, \zeta)$.

**Proof.** The first proposition follows from the monotoncity property (3.4) applied to the ampliation $K (A \otimes B) = K (A) \otimes B$ of the CP map $K$ from $\mathcal{A} \to \mathcal{A}^0$ to $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}^0 \otimes \mathcal{B}$, with the compound state $K_\ast (\omega_0) = K_\ast (K \otimes I) \left( I \right)$ denotes the identity map $\mathcal{B} \to \mathcal{B}$ corresponding to the entanglement $\pi^* = \pi_0 K$ and $K_\ast (\varrho_0) = \varrho \otimes \zeta$ with $\varrho = \varrho_0 K$ corresponding to $\omega_0 = \varrho_0 \otimes \zeta$.

If $\pi$ is separable entanglement (2.10), $\pi_e = \pi^*$ can be decomposed as

$$\pi_e (A) = \sum_n \tilde{\varrho}_n (A) \sigma_n \mu (n) = \pi_0 (K (A)).$$
Here $K (A) = \sum \lvert n \rangle g_n (A) \langle n \rvert$ is a normal unital CP map on $\mathcal{A}$ into the Abelian algebra $\mathcal{A}^0$ of diagonal operators on $\mathcal{G}^0 = \ell^2 (\mathbb{N})$, and $\pi^0 = \pi^0_\delta = \pi^d$ is the diagonalizing entanglement (3.4).

The inequality (3.4) can be also applied to the standard entanglement corresponding to the compound state (2.6) on $\mathcal{B} \otimes \mathcal{B} = \oplus_{i,j} \mathcal{B} (i) \otimes \mathcal{B} (j)$, where $\mathcal{B} (i) = \mathcal{L} (\mathcal{H}_i)$. It is described by the density operator

$$\pi = \frac{1}{\dim A} \sum_{i,j} \pi_{ij} \psi_i \psi_j^\dagger \rho (i),$$

where $\rho_B (i,j) = \delta_{i,j} \omega_{ij} \rho (i)$ is concentrated on the diagonal $\oplus_i \mathcal{B} (i) \otimes \mathcal{B} (i)$ of $\mathcal{B} \otimes \mathcal{B}$. The amplitudes $\psi_i \in \mathcal{H}_i \otimes \mathcal{H}_i$ are defined in (2.5) as $\psi_i = |\sigma_i^{1/2}\rangle$ by the components $\chi_0 (i) = |i\rangle \otimes \sigma (i)^{1/2}$ of the standard entangling operator $\chi_0$ on $\mathcal{G}_0 = \mathcal{H}$ into $\ell^2 (\mathcal{N}) \otimes \mathcal{H}$. Indeed, any entanglement $\pi^* (A) = \text{Tr}_X \chi A^\dagger$ as a normal CP map $\mathcal{A} \to \mathcal{B}$ normalized to the density operator $\sigma = \text{Tr}_X \chi^\dagger$ can be represented as the composition $\pi_0 K$ of the standard entanglement $\pi^0 = \pi^q$ on $(\mathcal{A}^0, \mathcal{G}_0) = (\mathcal{B}, \varsigma)$ and a normal unital CP map $K : \mathcal{A} \to \mathcal{B}$. The CP map $K$ is defined by $\sigma^{1/2} K (A) \sigma^{1/2} = \pi^* (\tilde{A})$. It has the form

$$K (A) = \text{Tr}_X A^\dagger X, \quad A \in \mathcal{A}$$

where $X$ is an operator $\mathcal{F}_- \otimes \mathcal{H} \to \mathcal{G}$, $\text{Tr}_X X^\dagger X = I$ defining the entangling operator $\chi = (I^- \otimes \chi_0) X^\dagger$ for $\pi$. Thus the standard entanglement $\pi^q (B) = \sigma^{1/2} B \sigma^{1/2}$ corresponds to the maximal mutual information.

Note that the supremum of the information gain (3.6) over all c-entanglements to the system $(\mathcal{B}, \varsigma)$ is the von Neumann entropy (3.3). It is achieved on any extreme entanglement $\pi^q$ with an Abelian $\mathcal{A}^0$, given by a decomposition $\varsigma = \sum c_n \mu_\varsigma (n)$ into pure states $\varsigma_n$. For example, by a Schatten decomposition $\sigma = \sum_n \lvert n \rangle \langle n \rvert$, corresponding to $c_n (B) = \langle n \rvert B \lvert n \rangle$ and $\mu_\varsigma (n) = \nu_\varsigma (n)$. The maximal value of $\text{rank} \mathcal{B}$ of the von Neumann entropy on the algebra $\mathcal{B}$ is restricted by $\ln \text{dim} \mathcal{B}$ as $\text{dim} \mathcal{B}^{1/2} \leq \text{rank} \mathcal{B} \leq \text{dim} \mathcal{B}$.

**Definition 2.** The maximal total information

$$H (\varsigma) = \sup_{\pi(1) = \varsigma} I (\pi) = I (\pi_\varsigma),$$

achieved on $\mathcal{A}^0 = \mathcal{B}$ by the standard q-entanglement $\pi^q (B) = \sigma^{1/2} B \sigma^{1/2}$ for a fixed state $\varsigma (B) = \text{Tr}_X B \sigma$, is called q-entropy of the state $\varsigma$. The difference

$$H (\pi) = H (\varsigma) - I (\pi)$$

is called the q-conditional entropy on $\mathcal{B}$ with respect to $\mathcal{A}$.

Obviously, $H (\pi)$ is positive in contrast to $S (\varsigma) - I (\pi) = - E (\pi)$ which is positive as the averaged conditional entropy $S (\pi)$ in the case of Abelian $\mathcal{A}$, but which can achieve also the negative value

$$S (\varsigma) - H (\varsigma) = - \ln \text{dim} \sigma$$

the following theorem states.

**Theorem 2.** Let $\mathcal{B}$ be a discrete decomposable algebra on $\mathcal{H} = \oplus_i \mathcal{H}_i$, with the state $\varsigma = \oplus_i \omega (i)$ given by normal states $\varsigma_i$ on $\mathcal{L} (\mathcal{H}_i)$, and $\mathcal{C} \subseteq \mathcal{B}$ be its center with the
state \( p = \varphi|\mathcal{C} \) induced by the probability distribution \( p(i) \). Then the \( q \)-entropy is given by

\[
H(q) = S(q) + \sum_i p(i) \ln \operatorname{rank} \sigma_i = H(p) + \sum_i H(\sigma_i) p(i),
\]

where \( H(p) = -\sum_i p(i) \ln p(i) = S(p) \), and

\[
H(\sigma_i) = \ln \operatorname{rank} \sigma_i - \text{Tr} \sigma_i, \sigma_i \ln \sigma_i = S(\sigma_i) + \ln \operatorname{rank} \sigma_i.
\]

It is positive, \( H(\sigma) \in [0, \infty) \), and if the reduced algebra \( B_\sigma = E_\sigma B E_\sigma \) is finite dimensional, it is bounded, with the maximal value \( H(\sigma^0) = \ln \dim B_\sigma \), achieved on the tracial \( \sigma^0 = (\dim E_\sigma)^{-1} E_\sigma \), with \( E_\sigma = E_\sigma B \).

\textbf{Proof.} The \( q \)-entropy \( H(\sigma) \) is the supremum (3.8) of the mutual information (3.6) which is achieved on the standard entanglement, corresponding to the density operator (3.7) of the standard compound state (2.6) with \( \mathcal{A} = \mathcal{B} \), \( \rho = \sigma \). Thus

\[
H(\sigma) = E(\pi^q) + S(\sigma),
\]

where

\[
E(\sigma^p) = \sum_i \psi(i) \psi(i)^\dagger \ln \psi(i)^\dagger (\sigma(i) \otimes 1)^{-1} \psi(i) = \sum_i p(i) \ln \operatorname{rank} \sigma_i,
\]

as \( \sigma(i) = \sigma p(i), v_q(i) = \psi(i)(i), \psi(i) = |\sigma^1/2\rangle p(i)^1/2, \) and

\[
\psi(i)^\dagger (\sigma(i) \otimes I)\psi(i) = \langle \sigma^1/2 | (\sigma^{-1} \otimes I) | \sigma^1/2 \rangle = \text{Tr} \sigma(i)^1/2 \sigma^{-1} \sigma(i)^1/2 = \operatorname{rank} \sigma_i.
\]

Decomposing the von Neumann entropy as

\[
S(\sigma) = \sum_i (S(\sigma_i) - \ln p(i)) p(i),
\]

we obtain the corresponding decomposition for \( q \)-entropy

\[
H(\sigma) = \sum_i (H(\sigma_i) - \ln p(i)) p(i),
\]

where \( H(\sigma_i) = \ln \operatorname{rank} \sigma_i + S(\sigma_i) \). Due to \( 0 \leq S(\sigma_i) \leq \ln \operatorname{rank} \sigma_i \), each \( H(\sigma_i) \) is positive, and it is bounded by \( 2 \ln \operatorname{rank} \sigma_i = \ln \dim B_i \), where we took into account that \( B_i = E_\sigma \mathcal{L}(H_i) E_\sigma = \mathcal{L}(E_i) \) has the squared dimensionality \( \dim E_i = \operatorname{rank} \sigma_i \) of \( E_i = E_\sigma H_i \). This gives the dimensional bound

\[
H(\sigma) \leq \ln \sum_i (\dim E_i)^2 = \ln \dim B_\sigma
\]

for the \( q \)-entropy of a state on the reduced algebra \( B_\sigma = \oplus B_i \). Actually this boundary is achievable, as well as dimensional capacity

\[
C_q(\mathcal{B}) = \sup_p \sum_i p(i) \left( 2 \sup_{\sigma_i} S(\sigma_i) - \ln p(i) \right) = -\inf_p \sum_i p(i) (\ln p(i) - 2 \ln \dim H_i) = \ln \dim \mathcal{B},
\]

of the algebra \( \mathcal{B} \) (in case of finitedimensional \( \mathcal{B} \)). Here we used the fact that the supremum of von Neumann entropies

\[
S(\sigma_i) = -\sum_i \text{Tr} \sigma_i \ln \sigma_i
\]
for the simple algebras $B(i) = \mathcal{L}(H_i)$ with $\dim B(i) = (\dim H_i)^2 < \infty$ is achieved on the tracial density operators $\sigma_i = (\dim H_i)^{-1} I_i \equiv \sigma_i^\otimes$, and the infimum of the relative entropy

$$I(p; p^\otimes) = \sum_i p(i) (\ln p(i) - \ln p^\otimes(i)),$$

where $p^\otimes(i) = \dim B(i) / \dim B$, is zero, achieved at $p = p^\otimes$. \[\Box\]

4. QUANTUM CHANNEL AND ITS Q-CAPACITY

Let $H_1$ be a Hilbert space describing a quantum input system and $H$ describe its output Hilbert space. A quantum channel is an affine operation sending each input state defined on $H_1$ to an output state defined on $H$ such that the mixtures of states are preserved. A deterministic (noiseless) quantum channel is defined by a linear isometry $Y: H_1 \to H$ with $Y^\dagger Y = I^1$ (I$^1$ is the identity operator in $H_1$) such that each input state vector $\eta_1 \in H_1, \|\eta_1\| = 1$, is transmitted into an output state vector $\eta = Y\eta_1 \in H, \|\eta\| = 1$. The orthogonal mixtures $\sigma_1 = \sum_n \sigma_1(n)$ of the pure input states $\sigma_1(n) = \eta_1(n)\eta_1(n)^\dagger$ are sent into the orthogonal mixtures $\sigma = \sum_n \sigma(n)$ of the corresponding pure states $\sigma(n) = Y\sigma_1(n)Y^\dagger$.

A noisy quantum channel sends pure input states $\varsigma_1$ on the algebra $B^1 = \mathcal{L}(H_1)$ into mixed ones $\varsigma = \Lambda^* (\varsigma_1)$ given by the predual $\Lambda_\pi = \Lambda^*|B_\pi^1$ to a normal completely positive unital map $\Lambda : B \to B^1$,

$$\Lambda(B) = \text{Tr}_{\pi} Y^\dagger BY, \quad B \in B,$$

where $Y$ is a linear operator from $H_1 \otimes F_+$ to $H$ with $\text{Tr}_{\pi} Y^\dagger Y = I$, and $F_+$ is a separable Hilbert space of quantum noise in the channel. Each input mixed state $\varsigma_1$ is transmitted into an output state $\varsigma = \varsigma_1\Lambda$ given by the density operator

$$\Lambda^* (\sigma_1) = Y (\sigma_1 \otimes I^+) Y^\dagger \in B_\pi,$$

for each density operator $\sigma_1 \in B_\pi^1$, where $I^+$ is the identity operator in $F_+$.

The input entanglements $\pi^1 = A \to B_\pi^1$ dual to $\pi_1 : B^1 \to A$ will be denoted as $\pi^1 = \kappa = \pi^*_1$. They define the quantum-quantum correspondences (q-encodings) of probe systems ($A, \varrho$) with the density operator $\rho = \kappa^* (I^1)$, to the input ($B^1, \varsigma_1$) of the channel $\Lambda$ with $\sigma_1 = \kappa (I)$. If $K : A \to A_0$ is a normal completely positive unital map

$$K(A) = \text{Tr}_{\pi} X^\dagger AX, \quad A \in A,$$

where $X$ is a bounded operator $F_- \otimes G_0 \to G$ with $\text{Tr}_{\pi} X^\dagger X = I^0$, the compositions $\kappa = \pi_0^1 K, \pi = \Lambda^* \kappa$ are the entanglements of the probe system ($A, \varrho$) with the channel input ($B^1, \varsigma_1$) and to the output ($B, \varsigma$) via this channel. The state $\varrho = \varrho_0 K$ is given by

$$K^* (\rho_0) = X (I^- \otimes \rho_0) X^\dagger \in A_\pi$$

for each density operator $\rho_0 \in A_0^0$, where $I^-$ is the identity operator in $F_-$. The resulting entanglement $\pi = \Lambda^* K$ defines the compound state $\omega = \omega_{01} (K \otimes \Lambda)$ on $A \otimes B$ with

$$\omega_{01} (A^0 \otimes B^1) = \text{Tr} \tilde{A}^0 \pi_0^1 (B^1) = \text{Tr} v_{01}^\dagger (A^0 \otimes B^1) v_{01}$$

on $A^0 \otimes B^1$. Here $v_{01} : F_{01} \to G_0 \otimes H_1$ is the amplitude operator, uniquely defined by the input compound state $\omega_{01} \in A_0^0 \otimes B_1^1$ up to a unitary operator $U^0$ on $F_{01}$,
and the effect of the input entanglement $\kappa$ and the output channel $\Lambda$ can be written in terms of the amplitude operator of the state $\varpi$ as

$$v = (X \otimes Y) \left( I^- \otimes v_{01} \otimes I^+ \right) U$$

up to a unitary operator $U$ in $\mathcal{F} = \mathcal{F}_- \otimes \mathcal{F}_{01} \otimes \mathcal{F}_+$. Thus the density operator $\omega = vu^\dagger$ of the input-output compound state $\varpi$ is given by $\varpi_{01}(K \otimes \Lambda)$ with the density

$$\omega_{01} = vu_{01}^\dagger$$

where $\varpi_{01} = v_{01}^\dagger v_{01}$.

Let $\mathcal{K}_q^1$ be the set of all normal TCP maps $\kappa : \mathcal{A} \to \mathcal{B}^1$ with any probe algebra $\mathcal{A}$, normalized as $\text{Tr}_\kappa(I) = 1$, and $\mathcal{K}_q^1(\varsigma_1)$ be the subset of all $\kappa \in \mathcal{K}_q^1$ with $\kappa(I) = \varsigma_1$. Each $\kappa \in \mathcal{K}_q^1(\varsigma_1)$ can be decomposed as $\kappa_0 K$, where $\kappa_0^* = \pi_1^q = \kappa_0 \in \mathcal{K}_q(\varsigma_1)$ is the standard entanglement on $(\mathcal{A}^0, \varrho_0) = (\mathcal{B}^1, \varsigma_1)$, and $K$ is a normal unital CP map $\mathcal{A} \to \mathcal{B}^1$. Further let $\mathcal{K}_c^1$ be the set of all $c$-entanglements $\kappa$ described by $\kappa(A) = \sum_n \varrho_n(A) \sigma_1(n)$, i.e., $\kappa^* = \pi_1^c$ are convex combinations (**) on $\mathcal{B}_1$, and $\mathcal{K}_c(\varsigma_1)$ denotes the subset of $\mathcal{K}_c$ corresponding to a fixed $\kappa(I) = \varsigma_1$. Each $\kappa \in \mathcal{K}_c(\varsigma_1)$ can be represented as $\kappa = \kappa_0 K$, where $\kappa_0^* = \pi_1^c$ is an extreme $d$-entanglement of an Abelian $\mathcal{A}^0$ to $\mathcal{B}^1$, by a proper choice of the CP map $K : \mathcal{A} \to \mathcal{B}^1$.

Now, let us maximize the entangled mutual entropy for a given quantum channel $\Lambda$ (and a fixed input state $\varsigma_1$) by means of the above two types of quantum (true) and classical (not true) entanglements $\kappa$. The entangled entropy (***) was defined in the previous section by the derivative of the probability operator $\omega$ of the corresponding compound state $\varpi$ on $\mathcal{A} \otimes \mathcal{B}$ with respect to the density operator $\rho \otimes I$ of the product $\varphi = \rho \otimes \text{Tr}_\mathcal{H}$. In each case

$$\varpi = \varpi_{01}(K \otimes \Lambda), \quad \varphi = \varrho_0 K \otimes \text{Tr},$$

where $K$ is a CP map $\mathcal{A} \to \mathcal{A}^0 = \mathcal{B}^1$, $\varpi_{01}$ is one of the corresponding extreme compound states $\varpi_{q1}$, $\varpi_{d1}$ on $\mathcal{B}^1 \otimes \mathcal{B}^1$, and $\varrho_0 (\mathcal{A}^0) = \varpi_{01}(\mathcal{A}^0 \otimes I^1)$.

**Proposition 2.** The entangled information achieves the following maximal values

$$E_q(\varsigma_1, \Lambda) := \sup_{\kappa \in \mathcal{K}_q(\varsigma_1)} E(\kappa^* \Lambda) = E(\pi_1^q \Lambda),$$

$$E_c(\varsigma_1, \Lambda) := \sup_{\kappa \in \mathcal{K}_c(\varsigma_1)} E(\kappa^* \Lambda) = -S(\varsigma_1, \Lambda).$$

Here $S(\varsigma_1, \Lambda)$ is the minimal von Neumann mean conditional entropy

$$S(\varsigma_1, \Lambda) = \sup_{\pi_d^B} S(\pi_d^B \Lambda) \equiv S(\pi_1^d \Lambda),$$

which is achieved on an extreme (optimal) diagonalizing map $\pi_d^B = \pi_1^d$ with $\text{Tr}\pi_1^d = \varsigma_1$ for all $B \in \mathcal{B}_1$. The total entangled information achieves respectively the following maximal values

$$I_q(\varsigma_1, \Lambda) := \sup_{\kappa \in \mathcal{K}_q(\varsigma_1)} I(\kappa^* \Lambda) = S(\varsigma_1 \Lambda) + E(\pi_1^q \Lambda),$$

$$I_c(\varsigma_1, \Lambda) := \sup_{\kappa \in \mathcal{K}_c(\varsigma_1)} I(\kappa^* \Lambda) = S(\varsigma_1 \Lambda) - S(\pi_1^d \Lambda).$$
They are ordered as

\[ E_q (\varsigma_1, \Lambda) \geq E_c (\varsigma_1, \Lambda), \quad l_q (\varsigma_1, \Lambda) \geq l_c (\varsigma_1, \Lambda) \]

**Proof.** Owing to the monotonicity

\[ R (\varpi_{01} (K \otimes 1) : \varrho_0 K \otimes \text{Tr}) \leq R (\varpi_{01} (I \otimes \Lambda) : \varrho_0 \otimes \text{Tr}) , \]

the supremum of over all \( \kappa \in K_q (\varsigma_1) \) is achieved on the standard entanglement \( B_1 \rightarrow A^0 \) given by \( \kappa^* = \pi^d_1 \equiv \kappa^0 \). Due to the same reason the supremum over all \( c \)-entanglements \( \kappa \in K_c (\varsigma_1) \) coincides with the supremum over all normal unital maps \( \kappa_0 \) on an Abelian \( A^0 \) satisfying the condition \( \kappa_0 (I^0) = \sigma_1 \). However the entangled information \( E_0 (\pi^d_1 \Lambda) \) for the not true entanglements \( \kappa_0^* = \pi^d_1 \equiv \kappa^0 \) is not positive, coinciding with the minus the averaged conditional von Neumann entropy \( S (\pi^d_1 \Lambda) \). The minimum of \( S (\pi^d_1 \Lambda) \) over all diagonalizing maps \( \pi^d_1 \) is achieved on an optimal pure \( d \)-entanglement \( \kappa^0 = \pi^d_1 \) on \( (B^1, \varsigma_1) \).

The same arguments apply also for the total informations \( I (\pi) = E (\pi) + S (\varsigma) \), however the suprema \( l_q (\varsigma, \Lambda) \) and \( l_c (\varsigma, \Lambda) \) can be obtained now straightforward as \( S (\varsigma) \) does not depend on the input entanglements with a fixed \( \kappa (I) = \sigma_1 \). The inequalities (4.3) simply follow from \( K_q (\varsigma_1) \supseteq K_c (\varsigma_1) \).

**Definition 3.** The suprema

\[ C_q (\Lambda) := \sup_{\kappa \in K_q_1} I (\kappa^* \Lambda) = \sup_{\varsigma_1} l_q (\varsigma_1, \Lambda) , \]

\[ C_c (\Lambda) := \sup_{\kappa \in K_c_1} I (\kappa^* \Lambda) = \sup_{\varsigma_1} l_c (\varsigma_1, \Lambda) , \]

are called the \( q \)- and \( c \)-capacities respectively for the quantum channel defined by a normal unital CP map \( \Lambda : B \rightarrow B^1 \).

Obviously the capacities (4.4) satisfy the inequalities

\[ C_c (\Lambda) \leq C_q (\Lambda) . \]

**Theorem 3.** Let \( \Lambda (B) = Y^+ BY \) be a unital CP map \( B \rightarrow B^1 \) describing a quantum deterministic (noiseless) channel. Then

\[ l_c (\varsigma_1, \Lambda) = S (\varsigma_1), \quad l_q (\varsigma_1, \Lambda) = H (\varsigma_1) , \]

and thus in this case

\[ C_c (\Lambda) = \ln \text{rank} B^1, \quad C_q (\Lambda) = \ln \dim B^1 \]

**Proof.** It was proved in the previous section for the case of the identity channel \( \Lambda = I \), and thus it is also valid for any isomorphism \( \Lambda \) described by a unitary operator \( Y \). In the case of non-unitary \( Y \) we can use the identity

\[ \text{Tr} Y (\sigma_1 \otimes I^+) Y^+ \ln Y (\sigma_1 \otimes I^+) Y^+ = \text{Tr} S (\sigma_1 \otimes I^+) \ln S (\sigma_1 \otimes I^+) + \ln S (\sigma_1 \otimes I^+) , \]

where \( S = Y^+ Y \). Due to this \( S (\varsigma_1, \Lambda) = -\text{Tr} S (\sigma_1 \otimes I^+) \ln S (\sigma_1 \otimes I^+) \). However in the case of the noiseless channel \( I^+ = 1 \), \( S = I \), and thus

\[ S (\varsigma) = S (\varsigma_1, \Lambda) = S (\varsigma_1) . \]
Moreover, as \( v = (X \otimes Y) (I^- \otimes v_{01}) \), \( v^\dagger v = (I^- \otimes v_{01})^\dagger (R \otimes I) (I^- \otimes v_{01}) = v^\dagger v_1 \), where \( R = X^\dagger X \), \( v_1 = (X \otimes I) (I^- \otimes v_{01}) \), and

\[
v^\dagger (\rho \otimes I)^{-1} v = (I^- \otimes v_{01})^\dagger (X^\dagger \rho^{-1} X \otimes I) (I^- \otimes v_{01}) = v^\dagger (\rho \otimes I)^{-1} v_1,\]

where \( \rho = X (I^- \otimes \rho_0) X^\dagger \). Hence

\[
E (\pi_1 \Lambda) = \text{Tr}_X v^\dagger_1 \ln v_1 \}
\]

where \( \pi_1 = \kappa^\ast \). Thus \( C_q (A) = \sup_{\varsigma, I } H (\varsigma_1) \), \( C_c (A) = \sup_{\varsigma_1} S (\varsigma_1) \) due to

\[
\sup_{\kappa \in \mathcal{K}_q (\varsigma_1)} E (\kappa^\ast \Lambda) = H (\varsigma_1), \quad \sup_{\kappa \in \mathcal{K}_c (\varsigma_1)} E (\kappa^\ast \Lambda) = S (\varsigma_1) .
\]

Therefore \( C_q (\Lambda) = \ln \dim B^1 = C_q (B^1) \), \( C_c (\Lambda) = \ln \text{rank} B^1 = C_c (B^1) \).

In order to consider block entanglements let us introduce the product systems \((B^\otimes n, \varsigma^\otimes n)\) on the tensor product \( \mathcal{H}^\otimes n = \bigotimes_{i=1}^n \mathcal{H}_i \) of identical spaces \( \mathcal{H}_i = \mathcal{H} \), and the product channels \( \Lambda^\otimes n : B^\ast_n \to B^\otimes n \), the preduals of the normal unital CP product maps

\[
\Lambda^\otimes n : B^\otimes n \to B^\ast_n = \bigoplus_{i=1}^n B^1_i, \quad \Lambda^\otimes n (B^\otimes n) = \Lambda (B)^\otimes n ,
\]

where \( B_i = B^1 \) for all \( i \). Obviously

\[
S (\varsigma^\otimes n) = n S (\varsigma), \quad C_c (B^\otimes n) = n C_c (B),
\]

\[
H (\varsigma^\otimes n) = n H (\varsigma), \quad C_q (B^\otimes n) = n C_q (B).
\]

However it not obvious that this additivity should take place for

\[
I^\circ_q (\varsigma_1, \Lambda) = I_c (\varsigma^\otimes n), \quad C^0_n (\Lambda) = C_c (\Lambda^\otimes n), \quad I^\circ_n (\varsigma_1, \Lambda) = I_q (\varsigma^\otimes n), \quad C^q_n (\Lambda) = C_q (\Lambda^\otimes n).
\]

the suprema of the mutual information \( l (\kappa^\ast \Lambda^\otimes n) \) over the set \( \mathcal{K}_q (\varsigma^\otimes n) \) of all input entanglements \( \kappa : A \to B^\ast_n \) with any probe algebra \( A \), normalized as \( \kappa (I) = \sigma^\otimes n_1 \), and with any such normalization respectively, \( \kappa \in \mathcal{K}_q ^\ast \). It is easily seen, by applying the monotonicity property (2.2) with respect to the normal unital CP map \( K : A \to A^0 = B^\otimes n \) in the decomposition \( \kappa = \pi_q K \), where

\[
\pi_q (A^0) = (\sigma^\otimes n_1)^{1/2} \tilde{A}^0 (\sigma^\otimes n_1)^{1/2}, \quad A^0 \in B^\otimes n,
\]

that the quantities \( I^\circ_q (\varsigma_1, \Lambda) \) are additive:

\[
I^\circ_q (\varsigma_1, \Lambda) = n l_q (\varsigma_1), \quad C^q_n (\Lambda) = n C_q (\Lambda).
\]

Note that if the supremum \( \sup_{\varsigma_n} l_q (\varsigma_n, \Lambda^\otimes n) \) is taken over all states \( \varsigma_n \in B^\ast_n \) but not just over \( \varsigma^\otimes n_1 \), it might be possible to achieve more than \( n C_q (\Lambda) \). However for the classical entanglements this additivity cannot be proved even in the case of \( \varsigma_n = \varsigma^\otimes n_1 : \)

\[
I^\circ_c (\varsigma_1, \Lambda) \geq n l_c (\varsigma_1), \quad C^c_n (\Lambda) \geq n C_c (\Lambda).
\]

The superadditivity implies that the quantities \( i^\circ_c = I^\circ_c / n, c^\circ_n = C^c_n / n \) have the limits

\[
i_c (\varsigma_1) = \lim_{n \to \infty} \frac{1}{n} I^\circ_c (\varsigma_1) \geq l_c (\varsigma_1), \quad c_c (\Lambda) = \lim_{n \to \infty} \frac{1}{n} C^c_n (\Lambda) \geq C_c (\Lambda).
\]
which are usually taken as the bounds of classical information and capacity [17, 18] for a quantum channel Λ. As it has been recently proved in [19] under a certain regularity condition, the upper bound C(Λ) is indeed asymptotically achievable by long block classical-quantum encodings. Note that the c-quantities iₙ, Cₙ are not easy to evaluate for each n, but they all are bounded by the corresponding q-quantities iₙ = 1/n lₙ = lₙ(ς₁, Λ) and cₙ = lim 1/n Cₙ(ςₙ, Λ):

\[ iₙ(ς₁, Λ) ≤ lₙ(ς₁, Λ), \quad cₙ(Λ) ≤ cₙ(Λ). \]

In order to measure the "real" entangled information of quantum channels "without the classical part", another quantity, the "coherent information" was introduced in [3]. It is defined in our notations as

\[ I_{ς₁}(ς₁, Λ) = lₙ(ς₁, Λ) - S(ς₁), \quad Cₙ^{ς₁}(Λ) = sup I_{ς₁}(ς₁, Λ). \]

Obviously iₙ(ς₁, Λ) ≤ E(ς₁, Λ), and Cₙ^{ς₁}(Λ) ≤ Cₙ^{ς₁}(Λ) = sup, E(ς₁, Λ). The supremum Cₙ^{ς₁}(Λ) = Cₙ^{ς₁} (ς₁, Λ) over all states ς₁ⁿ ∈ ⊗ⁿ B₁ in general is not additive but superadditive, and the coherent capacity is defined as the limit

\[ c_{ς₁}(Λ) = \lim_{n→∞} \frac{1}{n} Cₙ^{ς₁}(Λ) ≤ \lim_{n→∞} \frac{1}{n} Cₙ(Λ) = c_{ς₁}(Λ). \]

Obviously this capacity has the bounds cₙ(Λ) ≤ cₙ(Λ) ≤ cₙ(Λ) due to Eₙ(ς₁ⁿ, Λ⊗ⁿ) ≤ Iₙ(ς₁ⁿ, Λ⊗ⁿ) ≥ 0 for each input state ς₁.

Thus the entangled information lₙ(ς₁, Λ) for a single channel corresponding to the standard entanglement κ gives upper bound of all other informations and is good analog of the corresponding classical quantities.

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