Birational rigidity and $\mathbb{Q}$-factoriality of a singular double quadric

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Abstract. We prove birational rigidity and calculate the group of birational automorphisms of a nodal $\mathbb{Q}$-factorial double cover $X$ of a smooth three-dimensional quadric branched over a quartic section. We also prove that $X$ is $\mathbb{Q}$-factorial provided that it has at most 11 singularities; moreover, we give an example of a non-$\mathbb{Q}$-factorial variety of this type with 12 simple double singularities.

1. Introduction

One of the popular problems of birational geometry is to find all Mori fibrations (see [15] for definition) that are birationally equivalent to a given Mori fibration $\mathcal{X} \to T$, and to compute the group of birational automorphisms Bir($\mathcal{X}$) of a variety $\mathcal{X}$. The cases when there are few structures of Mori fibrations on $\mathcal{X}$, for example, when there is only one — up to a natural equivalence (see below) — structure of Mori fibration, are of special interest.

A Mori fibration $\mathcal{X} \to T$ is called birationally rigid if for any birational map $\chi : \mathcal{X} \dasharrow \mathcal{X}'$ to another Mori fibration $\mathcal{X}' \to T'$ there is a birational selfmap $\gamma : \mathcal{X} \dasharrow \mathcal{X}$ and a birational map $\sigma : T \dasharrow T'$ such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\chi \circ \gamma} & \mathcal{X}' \\
\downarrow & & \downarrow \\
T & \xrightarrow{\sigma} & T'
\end{array}
\]

Moreover, it is required that the map $\chi \circ \gamma$ restricts to an isomorphism on a general fiber. Informally, this definition means that $\mathcal{X}$ cannot be transformed into a principally different Mori fibration. A birationally rigid Fano variety $X$ with Picard number $\rho(X) = 1$ such that its group of birational automorphisms Bir($X$) coincides with the subgroup Aut($X$) of biregular ones is called birationally superrigid (see [6] or [18]).

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Fano varieties of low degree (a double cover of $\mathbb{P}^3$ branched over a sextic, a quartic and a double cover of a quadric branched over a divisor of degree 4 — see [11]) give examples of birationally rigid varieties with relatively simple groups of birational selfmaps. Birational superrigidity of a smooth quartic was proved in [13]; a proof of birational superrigidity of a smooth double cover of $\mathbb{P}^3$ branched over a sextic and birational rigidity of a smooth double cover of a quadric branched over a divisor of degree 4 (together with the calculation of its group of birational automorphisms) may be found in [12] and in [14]. The case of a quartic with one simple double point was considered in [17] (where its birational rigidity was proved and a group of its birational automorphisms was calculated); the case of a $\mathbb{Q}$-factorial double cover of $\mathbb{P}^3$ branched over a sextic with arbitrary number of simple double points was studied in [5] (this variety appeared to be birationally superrigid); the case of a $\mathbb{Q}$-factorial quartic with arbitrary number of simple double points was considered in [16] (this variety was proved to be birationally rigid and the generators of its groups of birational automorphisms were found). In [9] birational rigidity of a double cover of a quadric branched over a divisor of degree 4 with one simple double point was proved and the group of its birational selfmaps was calculated. The main goal of this paper is to prove the following statement that continues this series of results.

Let $Q \subset \mathbb{P}^4$ be a smooth quadric, $W \subset \mathbb{P}^4$ — a three-dimensional quartic, $S = W \cap Q$, $X$ — a double cover of $Q$ branched over $S$; $X$ is a Fano variety of degree $\deg X = 4$, and the double cover structure is given by the map $\varphi_{-K_X} : X \to Q \subset \mathbb{P}^4$.

**Theorem 1.1.** Assume that the variety $X$ is $\mathbb{Q}$-factorial and has only simple double singularities. Then the following holds

1. $X$ is birationally rigid.
2. Let $B \subset X$ be a line (i.e. a curve of anticanonical degree 1) such that $\varphi_{-K_X}|(B) \not\subset S$. Then there is a birational involution $\tau_B$ associated to $B$ (see Example 4.2 for an explicit description); these involutions together with the subgroup $\text{Aut}(X)$ generate the group $\text{Bir}(X)$.
3. Involutions $\tau_B$ are independent in the group $\text{Bir}(X)$ (i.e. there are no relations on them, except for the trivial relations $\tau_B^2 = 1$). Equivalently, there exists an exact sequence

$$1 \longrightarrow F(\{\tau_B\}) \longrightarrow \text{Bir}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1,$$

where $F(\{\tau_B\})$ denotes a free group generated by all involutions $\tau_B$ associated to the lines $B$ such that $\varphi_{-K_X}|(B) \not\subset S$. 


Remark 1.2. The group Aut(X) of a general variety X contains only one nontrivial element — an involution δ of a double cover \( \varphi|_{-K_X} \).

Remark 1.3. For X to have only simple double singularities it is necessary and sufficient that S has only simple double singularities. One may assume that W is smooth along S and is tangent to Q at the singular points of S.

Remark 1.4. The assumption that Q is smooth is important — a double cover of a cone over a smooth quadric surface branched over a quartic section (not passing through a vertex) is a much more complicated variety. To start with, it is not \( \mathbb{Q} \)-factorial. It has two small resolutions; each of them covers a corresponding small resolution of a quadric cone and is fibered into Del Pezzo surfaces of degree 2 (these fibrations arise from two pencils of planes on a quadric cone). On the other hand, all structures of Mori fibrations on this variety are these two (see [8]).

Theorem 1.1 is proved by the method of maximal singularities (see [18] or [6]). In section 2 we introduce the necessary notions (as well as the notations used throughout the paper), and after that we prove statements 1 and 2 of Theorem 1.1 in sections 3 and 4. The proof of statement 3 of Theorem 1.1 is contained in section 5.

Since Theorem 1.1 holds only for \( \mathbb{Q} \)-factorial varieties, it seems natural to ask which conditions guarantee \( \mathbb{Q} \)-factoriality of X. In section 6 we’ll prove the following statement (note that an analogous statement was established for a sextic double solid in [5] and for a quartic in [3]; see also [19] for details about \( \mathbb{Q} \)-factorial quartics).

Proposition 1.5. If the number of singular points of X is at most 11, then X is \( \mathbb{Q} \)-factorial. On the other hand, there exist non-\( \mathbb{Q} \)-factorial varieties of this type with 12 simple double points.

All varieties are assumed to be defined over the field of complex numbers. The degrees are always calculated with respect to the anticanonical linear system. We’ll use the notations introduced in this section throughout the work.

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2. Preliminaries on the method of maximal singularities

In this section we briefly describe main constructions of the method of maximal singularities and introduce the necessary notation (for a detailed treatment see [18] or [6]). All relevant definitions may be found, for example, in [15].
Let $V$ be a (three-dimensional) Fano variety with terminal singularities and Picard number $\rho(V) = 1$, and let $V' \to S'$ be a Mori fibration (in a more general setting one may assume that $V$ is also a Mori fibration over an arbitrary base $S$, but we don’t need it). Let $\chi : V \dasharrow V'$ be a rational map. There is an algorithm (known as Sarkisov program) that decomposes the map $\chi$ into a composition of elementary maps (links) of four types (see [6] or [15]). Choose a very ample divisor $H'$ on $V'$ and let $H = \chi^{-1}_*|H'|$ (note that $H$ has no fixed components). Let $\mu$ be a (rational) number such that $H \subset |-\mu K_V|$. The Nöether–Fano inequality (see [12], [6], [15] or [18]) claims that if in this setting $\chi$ is not an isomorphism then the pair $(V, 1/\mu H)$ is not canonical. Next, one can prove that there is an extremal contraction $g : \tilde{V} \to V$ such that discrepancy of an exceptional divisor of $g$ with respect to the pair $(V, cH)$, where $c < \frac{1}{\mu}$ is a canonical threshold of the pair $(V, H)$, is zero (such divisor is called a maximal singularity, and its center on $V$ is called a maximal center). One can check that there is a link of type I or II starting with this contraction and decreasing a properly defined “degree” of the map $\chi$, all arising varieties have only terminal singularities and all divisorial contractions (as well as nonbirational ones) are extremal contractions in the sense of the ordinary Minimal Model Program (see [6] or [15] for details).

The previous statements imply the following: to prove that $V$ cannot be transformed to another Mori fibration (i.e. is birationally rigid) it suffices to check that there are no maximal centers on $V$ except those that are associated with birational automorphisms of $V$, and to describe all birational selfmaps $\chi : V \dasharrow V$ it is sufficient to classify all maximal centers. In the next two sections we are going to do this for the variety $X$ described in section 1. We are also going to use the notations introduced in the current section throughout the paper.

3. Excluding maximal centers: points

The absence of the points that are maximal centers is easily implied by the standard statements.

**Theorem 3.1** ($4n^2$-inequality, see [18] or [6]). Let $V$ be a threedimensional variety such that the linear system $|-K_V|$ is free; let $H \subset |-\mu K_V|$. Consider a smooth point $x \in V$ and an intersection $Z = D_1D_2$ of two general divisors from $H$. If $x$ is a maximal center (for $H$), then $\text{mult}_x(Z) > 4\mu^2$.

**Corollary 3.2.** A smooth point cannot be a maximal center on $X$. 

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**Proof.** Indeed, if a smooth point $x$ is a maximal center, then for general $D_1, D_2 \in \mathcal{H}$ we have

$$4\mu^2 < \operatorname{mult}_x D_1 D_2 \leq K_X D_1 D_2 = \mu^2 \deg X = 4\mu^2,$$

a contradiction.

To exclude singular points we’ll need the following theorem.

**Theorem 3.3** (see, for example, [2]). Let $\dim V \geq 3$; let $x \in V$ be a simple double point and $D$ — an effective (\(\mathbb{Q}\)-)divisor such that the pair $(V,D)$ is not canonical at $x$. Then $\operatorname{mult}_x D > 1$.

**Corollary 3.4.** A singular point cannot be a maximal center on $X$.

**Proof.** Assume that a singular point $x \in X$ is a maximal center; recall that all singular points of $X$ are simple double points. Let $g : \tilde{X} \to X$ be a blow-up of the point $x$ with an exceptional divisor $E$; let $L \subset \mathbb{Q}$ be a general line passing through $\varphi|_{-K_X}(x)$, $L' = \varphi^{-1}_{-K_X}(L)$, and $\tilde{L}'$ — a preimage of $L'$ under the map $g$. Note that $L'$ is singular at $x$, and $EL' = 2$. Let $D \in \frac{1}{\mu} \mathcal{H}$ be a general divisor, $g^*(\mu D) = g^{-1}_* (\mu D) + \nu E$. By Theorem 3.3 we have $\nu > \mu$. Hence

$$2\mu = \mu DL' = g^*(\mu D)\tilde{L}' \geq \nu E \tilde{L}' \geq 2\nu > 2\mu,$$

a contradiction. □

**Remark 3.5.** Similar statements were proved in [9], but the references to $4n^2$-inequality and Theorem 3.3 make the proofs much easier.

**4. Excluding maximal centers: curves**

We’ll use the following description of low-degree curves on the variety $X$ (recall that the degrees are always calculated with respect to the anticanonical linear system).

**Lemma 4.1** (see, for example, [12]). Let $Z \subset X$ be a curve not contained in the ramification divisor, $\deg Z \leq 3$, $\varphi|_{-K_X}(Z) = Y \subset \mathbb{Q}$. Then one of the following cases occur:

1. $\deg Z = 1$, $Y$ is a line tangent to $S$ at two (possibly coinciding) points (it is also possible that in one or both of these points is just a singular point of $S$), $\varphi|_{-K_X}\big|_Z : Z \to Y$ is an isomorphism, $\varphi^{-1}_{-K_X}(Y) = Z \cup \delta(Z)$.

2. a) $\deg Z = 2$, $Z$ is a smooth curve, $p_a(Z) = 1$, $Y$ is a line intersecting $S$ at two different points.

b) $\deg Z = 2$, $Z$ is a curve with one double point, $p_a(Z) = 1$, $Y$ is a line tangent to $S$ at a single point.
c) \( \deg Z = 2 \), \( Z \) is a smooth curve, \( Y \) is a conic tangent to \( S \) at four points (some of them may coincide), \( \varphi|_{-K_X}|_Z : Z \to Y \) is an isomorphism, \( \varphi^{-1} \cdot K_X = Z \cup \delta(Z) \).

3. \( \deg Z = 3 \), \( Y \) is a twisted cubic, \( \varphi|_{-K_X}|_Z : Z \to Y \) is an isomorphism.

Let us describe an involution associated to a line on \( X \) not contained in the ramification divisor.

**Example 4.2** (see. [12] or [9]). Let \( B \subset X \) be a line not contained in the ramification divisor (as in case 1 of Lemma 4.1), \( B' = \delta(B) \). The linear system \( |-K_X - B - B'| \) gives a rational map \( \psi : X \dashrightarrow \mathbb{P}^2 \); its general fiber is an elliptic curve — a preimage of a general line on \( Q \) intersecting the line \( \varphi|_{-K_X}|(B) \). This map may be regularized by passing to a variety \( \tilde{X} \) that is a subsequent blow-up of \( B \) and of a strict transform of \( B' \). The preimage \( E' \) of the curve \( B' \) gives a section of the elliptic fibration \( \psi : \tilde{X} \to \mathbb{P}^2 \). Reflection with respect to this section gives a biregular involution on an open subset of \( \tilde{X} \) that gives rise to a biregular involution of \( \tilde{X} \) and a birational involution of \( X \). The action of this involution on \( \text{Pic}(X) \) is computed in [12]. Note that unlike the involutions of a singular quartic (cf. [16] or [17]) the involution \( \tau_B \) is constructed uniformly regardless to the number of singular points on \( B \) (moreover, there may be no singular points on \( B \) at all).

From the point of view of Sarkisov program the involution \( \tau_B \) is a link of type II such that any decomposition of the map \( \chi \) starts with \( \tau_B \) provided that the curve \( B \) is a maximal center for \( \mathcal{H} \). Now we’ll check that no other curves except the lines not contained in the ramification divisor cannot be maximal centers. Since neither smooth nor singular points of \( X \) can be maximal centers (see Corollary 3.2 and Corollary 3.4), according to section 2 this will imply that there are no birational maps between \( X \) and other Mori fibrations, and that any birational selfmap \( \chi : X \dashrightarrow X \) can be decomposed into a composition of the maps described in Example 4.2.

**Lemma 4.3.** Let \( Z \subset X \) be a curve not contained in the ramification divisor, and \( \deg Z > 1 \). Then \( Z \) is not a maximal center.

**Proof.** Let multiplicity of \( \mathcal{H} \) in the curve \( Z \) be equal to \( \nu \). Assume that \( \nu > \mu \). By a standard computation we get \( \deg Z < \deg X = 4 \). Consider several possible cases depending on what the curve \( Z \) is (we label them as in Lemma 4.1). By \( H \) we’ll always denote a general member of the anticanonical system.
Case 2a) The curve \( Z \) does not pass through the singular points of \( X \), and we may use, for example, the following computation from [12]. Let \( f : X' \to X \) be a blow-up of the curve \( Z, E = f^{-1}(Z) \). Then
\[
0 \leq (\mu f^*H - \nu E)^2(f^*H - E) = 4\mu^2 - 4\mu\nu,
\]
a contradiction.

Case 2b) The curve \( Z \) either does not pass through singular points of \( X \), or passes through only one of them, and this point is the singular point \( x_0 \) of \( Z \). Let \( g : \tilde{X} \to X \) be a blow-up of the point \( x_0 \), \( f : X' \to \tilde{X} \) — a blow-up of a strict transform \( \tilde{Z} \) of the curve \( Z, E_0 = g^{-1}(x_0), E = f^{-1}(\tilde{Z}) \). Let us define the multiplicity \( \nu_0 = \text{mult}_{x_0}H \) of the linear system \( H \) in the (possibly singular) point \( x_0 \) by an equality
\[
g^*H = g_*^{-1}H + (\text{mult}_{x_0}H)E_0;
\]
if \( x_0 \) is a smooth point this is the multiplicity in the usual sense. If \( x_0 \) is smooth on \( X \), then
\[
0 \leq (\mu(fg)^*H - \nu_0 f^*E_0 - \nu E)^2((fg)^*H - f^*E_0 - E) = 4\mu^2 - 4\mu\nu - (\nu_0 - 2\nu)^2 < 0,
\]
a contradiction. If \( x_0 \) is a singular point of \( X \), then
\[
0 \leq (\mu(fg)^*H - \nu_0 f^*E_0 - \nu E)^2((fg)^*H - f^*E_0 - E) = 4\mu^2 - 4\mu\nu - 2(\nu_0 - \nu)^2 < 0,
\]
a contradiction.

Case 2c) Let \( Z' = \delta(Z) \), and \( \text{mult}_{Z'}H = \nu' \). Assume that \( Z \) passes through \( k \) singular points of \( X, 0 \leq k \leq 4 \) (note that these points must be contained in \( Z \cap Z' \)). Let \( U \) be a general anticanonical divisor passing through \( Z \) (and hence through \( Z' \)), and \( \mathcal{H}|_U = \nu Z + \nu' Z' + C \). Then on the surface \( U \) the following equalities hold: \( Z'^2 = -2 + \frac{k}{2}, Z'Z = 4 - \frac{k}{2} \). Hence for a general \( D \in \mathcal{H} \) we have
\[
2\mu = DZ' \geq (4 - \frac{k}{2})\nu - (2 - \frac{k}{2})\nu',
\]
that implies \( \nu' > \nu > \mu \). The latter contradicts the fact that the degree of a curve that is a maximal center on \( X \) cannot exceed 3.

Case 3. Let \( Z \) pass through \( k \) singular points of \( X \), say, \( x_1, \ldots, x_k \). Since \( Z \) is not contained in the ramification divisor, we have \( 0 \leq k \leq 6 \). Let \( g : \tilde{X} \to X \) be a blow-up of the points \( x_1, \ldots, x_k \), \( f : X' \to \tilde{X} \) — a blow-up of a strict transform \( \tilde{Z} \) of the curve \( Z, E_i = g^{-1}(x_i), E = \)
Lemma 4.4 (see [12], [5]). Let $Z \subset X$ be a curve contained in the ramification divisor. Then $Z$ is not a maximal center.

Proof. Assume that $\text{mult}_Z \mathcal{H} > \mu$. Let $p$ be a general point of the curve $Z$; let $L \subset \mathcal{Q}$ be a line tangent to $S$ at the point $\varphi_{|-K_X|}(p)$, $\tilde{L} = \varphi_{|-K_X|}^{-1}(L)$. Then $\tilde{L}$ is singular at $p$, and generality of $p$ implies that $\tilde{L} \not\subset \text{Bs} \mathcal{H}$. Hence for a general $D \in \mathcal{H}$ we have

$$2 \mu = \mu \deg \tilde{L} = \tilde{L} D \geq \text{mult}_p \tilde{L} \text{mult}_p \mathcal{H} \geq 2 \text{mult}_Z \mathcal{H} > 2 \mu,$$

a contradiction. \qed

5. Relations

To prove that involutions $\tau_B$ are independent in the group $\text{Bir}(X)$ it suffices to check that two lines $Z_1$ and $Z_2$ cannot appear simultaneously as maximal centers. Let $\text{mult}_{Z_1} \mathcal{H} = \nu_1$, $\text{mult}_{Z_2} \mathcal{H} = \nu_2$. Assume that $\nu_1 > \mu$, $\nu_2 > \mu$; we are going to obtain a contradiction in all the possible cases (cf. [12] and [14]).

Lemma 5.1. The lines $Z_1$ and $Z_2$ cannot intersect in a single point.

Proof. Assume that the point $x_0$ is the only intersection point of the lines $Z_1$ and $Z_2$. Assume in addition that there are no other singularities of $X$ on the curves $Z_i$ except a possible singularity at $x_0$ (other cases are treated in a similar way but require more computations). Let $g : \tilde{X} \to X$ be a blow-up of the point $x_0$; let $f : X' \to \tilde{X}$ be a blow-up of the strict transforms $\tilde{Z}_1$, $\tilde{Z}_2$ of the curves $Z_1$ and $Z_2$, $E_0 = g^{-1}(x_0)$, $E_1 = f^{-1}(\tilde{Z}_1)$, $E_2 = f^{-1}(\tilde{Z}_2)$, $\nu_0 = \text{mult}_{x_0} \mathcal{H}$. If $x_0$ is nonsingular on
Let a quartic \( X \) be given by an equation \( f(x) = 0 \), where \( f \) is a (general) form of degree \( 2 \). Then the image of the divisor \((f)\) is a singular point of \( X \), then

\[
0 \leq (μ(fg)^*H - ν_0f^*E_0 - ν_1E_1 - ν_2E_2)^2(2(fg)^*H - f^*E_0 - E_1 - E_2) = \\
= 8μ^2 - 2μ(ν_1 + ν_2) + 2ν_0(ν_1 + ν_2) - 4(ν_1^2 + ν_2^2) - ν_0^2 = \\
= (8μ^2 - 2μ(ν_1 + ν_2) - 2(ν_1^2 + ν_2^2)) - \frac{1}{2}(ν_0 - 2ν_1)^2 - \frac{1}{2}(ν_0 - 2ν_2)^2 < 0,
\]
a contradiction. If \( x_0 \) is a singular point of \( X \), then

\[
0 \leq (μ(fg)^*H - ν_0f^*E_0 - ν_1E_1 - ν_2E_2)^2(2(fg)^*H - f^*E_0 - E_1 - E_2) = \\
= 8μ^2 - 2μ(ν_1 + ν_2) + 2ν_0(ν_1 + ν_2) - 3(ν_1^2 + ν_2^2) - 2ν_0^2 = \\
= (8μ^2 - 2μ(ν_1 + ν_2) - 2(ν_1^2 + ν_2^2)) - (ν_0 - ν_1)^2 - (ν_0 - ν_2)^2 < 0,
\]
a contradiction.

Lemma 5.2. The lines \( Z_1 \) and \( Z_2 \) cannot intersect in two points (or, equivalently, \( Z_2 \neq δ(Z_1) \)).

Proof. Assume that they do. Let \( p \) be a general point of the curve \( Z_1 \), \( \bar{L} \subset Q - \) a general line passing through the point \( \varphi_{-K_X}(p) \), \( \bar{L} = \varphi_{-K_X}^{-1}(L) \). Then \( L \cap Z_1 = p, L \cap Z_2 = δ(p), \) and \( \bar{L} \not\subset BsH \). Hence for a general \( D \in H \) we have

\[
2μ = μ deg \bar{L} = \bar{L}D \geq mult_p H + mult_δ(p) H \geq ν_1 + ν_2 > 2μ,
\]
a contradiction. □

Lemma 5.3. \( Z_1 \cap Z_2 \neq \emptyset \)

Proof. Assume that \( Z_1 \cap Z_2 = \emptyset \). Note that there is a one-parameter family of lines \( Q \) that intersect \( \varphi_{-K_X}(Z_1) \) and \( \varphi_{-K_X}(Z_2) \). Let \( L \) be a general line of this family, \( \bar{L} = \varphi_{-K_X}^{-1}(L) \). Then \( \bar{L} \not\subset BsH \). Hence for a general \( D \in H \) we have

\[
2μ = μ deg \bar{L} = \bar{L}D \geq ν_1 + ν_2 > 2μ,
\]
a contradiction. □

6. \( \mathbb{Q} \)-factoriality

In this section we prove Proposition 1.3

Let us start with an example of non-\( \mathbb{Q} \)-factorial variety \( X \).

Example 6.1. Let a quartic W be given by an equation \( f_2(x)^2 + f_1(x)f_3(x) = 0 \), where \( f_i \) is a (general) form of degree \( i \). Then the preimage of the divisor \((f_i = 0) \subset Q \) under the map \( \varphi_{-K_X} \) splits into a union of two divisors of degree 2 that implies non-factoriality of \( X \).
(and hence non-$\mathbb{Q}$-factoriality as well, since the latter is equivalent to the former in the case of nodal threefolds).

The variety $X$ has 12 singular points: these are 12 intersection points of the hypersurfaces $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ and $Q$ in $\mathbb{P}^4$.

Note that this variety may be also described in a different way. Consider a cone $K \subset \mathbb{P}^5$ with a vertex $P$ over a smooth three-dimensional quadric $Q \subset \mathbb{P}^4$ and a general cubic $C$ passing through $P$. Let $X' = K \cap C$; let $\pi : X' \dashrightarrow Q$ be a projection of $X'$ from the point $P$. The birational map $\pi$ gives rise to a birational morphism $\tilde{\pi} = \pi \circ \sigma : \tilde{X} \to Q$, where $\sigma : \tilde{X} \to X'$ is a blow-up of $X'$ at the point $P$; the latter morphism is a composition $\tilde{X} \xrightarrow{\phi} X \xrightarrow{\varphi} Q$, where $\phi$ is a contraction of 12 lines on $X'$ passing through $P$ and $\varphi$ is a double cover. The variety $X$ has 12 simple double singularities; since $\phi$ is a small contraction, $X$ is not $\mathbb{Q}$-factorial.

Remark 6.2 (cf. [16, Example 6] and [4, Example 1.21]). Consider a general complete intersection $Y$ of a quadric $K$ and a cubic $C$ in $\mathbb{P}^5$, such that $Y$ has one simple double singularity $P$. Then $Y$ contains 12 lines $l_1, \ldots, l_{12}$ passing through $P$; all these lines are contained in a common tangent space $\tilde{L}$ to $K$ and $C$ at $P$ (the generality condition lets us assume that both $K$ and $C$ are smooth). A projection $\pi : Y \dashrightarrow \mathbb{P}^4$ from the point $P$ is a composition of a blow-up $\sigma : \tilde{Y} \to Y$ of the point $P$ and a contraction $\phi : \tilde{Y} \to Z$ of the preimages of the 12 lines. It is easy to see that the image $Z$ of the variety $Y$ under this projection is a (non-$\mathbb{Q}$-factorial) quartic, containing a quadric surface — an image of an exceptional divisor of the blow-up — and having simple double singularities at the points $\pi(l_i)$.

One can check that the quartic $Z$ contains two quadric surfaces and has two (projective) small resolutions; one of them is the variety $\tilde{Y}$ described above, and the other is a variety $\tilde{Y}'$ that is also obtained as a blow-up of a singular complete intersection $Y'$ of a quadric and a cubic in $\mathbb{P}^5$ in its unique singular (simple double) point $P'$ (see [16, Example 6] or [4, Example 1.21]). The varieties $Y$ and $Y'$ are not isomorphic in general, but there is a natural birational map $\tilde{\Delta} : Y \dashrightarrow Y'$.

When the quadric $K$ degenerates to a cone over a nonsingular three-dimensional quadric $Q$, and the cubic $C$ passes through a vertex $P$ of the cone $K$, the variety $Z$ obtains a structure of a double cover of the quadric $Q$ (see the second description in Example 6.1); the corresponding variety $Y$ has a birational involution $\tilde{\delta} : Y \dashrightarrow Y$ arising from an involution of a double cover $\delta : Z \to Z$.

Note also that a simpler example of a non-$\mathbb{Q}$-factorial quartic with simple double singularities — a general quartic containing a plane —
does not degenerate to a non-\(\mathbb{Q}\)-factorial double quadric with the same number of simple double points.

Now let the divisor \(S\) be singular at the points \(p_1, \ldots, p_s, s < 12\).

In the case of a nodal threefold \(\mathbb{Q}\)-factoriality is equivalent to factoriality; on the other hand, factoriality of a nodal Fano threefold \(X\) is equivalent to a topological condition \(\text{rk} H^2(X, \mathbb{Z}) = \text{rk} H_4(X, \mathbb{Z})\) (in our case \(\text{rk} H^2(X, \mathbb{Z}) = \rho(X) = 1\)). Hence to prove \(\mathbb{Q}\)-factoriality of \(X\) it suffices to check that for a small resolution \(h : \tilde{X} \to X\) we have \(\rho(\tilde{X}) = s + 1\). To check this it is sufficient to show that, using the terminology of \([7]\), the defect of \(X\) equals zero (cf. the proof of Theorem 2 in \([7]\), and also \([5]\)). The latter condition means that the singular points \(p_1, \ldots, p_s\) impose independent conditions on the hypersurfaces of degree 3 in \(\mathbb{P}^4\), i.e. that for any point \(p_i, 1 \leq i \leq s\), there is a cubic hypersurface \(D_i \subset \mathbb{P}^4\) such that \(p_i \not\in D_i\) and \(p_j \in D_i\) for \(j \neq i\). In the remaining part of the section we check this condition for our variety \(X\).

We may assume \(s = 11\). We’ll need the following theorem, proved in \([10]\).

**Theorem 6.3.** The points \(p_1, \ldots, p_{11} \in \mathbb{P}^n\) impose independent conditions on forms of degree \(d\) if each linear subspace of dimension \(k\) contains at most \(dk + 1\) of the points \(p_1, \ldots, p_{11}\).

In the remaining part of the section we’ll check the assumptions of Theorem 6.3 and derive Proposition 1.5 from Theorem 6.3.

**Lemma 6.4** (see \([19]\) Corollary 2.5]). Let \(Y \subset \mathbb{P}^3\) be an irreducible quadric; let \(p_1, \ldots, p_{10}, q \in Y\) be such points that no line contains 4 of the points \(p_1, \ldots, p_{10}\), no conic contains 7 and no twisted cubic contains 10 of them. Then there is a divisor \(D \in \mathcal{O}_{\mathbb{P}^3}(3)|_Y\) passing through \(p_1, \ldots, p_{10}\) and not passing through \(q\).

**Lemma 6.5.** A line contains at most 3 points of \(p_1, \ldots, p_{11}\).

**Proof.** Assume that 4 points, say, \(p_1, \ldots, p_4\), lie on a line \(l\). Then \(l \subset Q\), and since \(W\) is tangent to \(Q\) at \(p_1, \ldots, p_4\), it follows that \(W\) is tangent to \(l\) at \(p_1, \ldots, p_4\), and hence \(l \subset W\). Choose homogeneous coordinates \((x_0 : \ldots : x_4)\) so that \(Q\) is given by an equation \(1/2(x_0^2 + \ldots + x_4^2) = 0\), and \(l\) is given by equations \(x_2 = x_3 = x_4 = 0\). Then \(W\) is defined by an equation

\[
x_2 C_2(x) + x_3 C_3(x) + x_4 C_4(x) = 0,
\]

where \(\deg C_i = 3\). The divisor \(S\) is singular at the points of \(l\) where the matrix of partial derivatives

\[
\begin{pmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 \\
0 & 0 & C_2 & C_3 & C_4
\end{pmatrix}
\]

is not invertible.
has rank 1. In these points the minors \( x_0C_2 \) and \( x_1C_2 \) must vanish. Since there are at least 4 such points, \( C_2(x) \) is identically zero along \( l \). Similar arguments show that \( C_3 \) and \( C_4 \) also vanish along \( l \), and \( S \) has non-isolated singularities, that contradicts our assumptions. □

Lemma 6.6. A plane contains at most 6 of the points \( p_1, \ldots, p_{11} \).

Proof. Assume that 7 points, say, \( p_1, \ldots, p_7 \), lie in a plane \( L \). If \( L \) intersects \( Q \) along a reducible conic, then there are 4 collinear points among \( p_1, \ldots, p_7 \) that contradicts Lemma 6.5. Hence the conic \( L \cap Q \) is nonsingular, and we may assume that \( Q \) is given by an equation \( 1/2(x_0^2 + \ldots + x_4^2) = 0 \), \( L \) is given by equations \( x_3 = x_4 = 0 \), and \( W \) — by an equation

\[
x_3C_3(x) + x_4C_4(x) + Q(x)\tilde{Q}(x) = 0,
\]

where \( \deg C_i = 3 \), \( \deg \tilde{Q} = 2 \), and \( Q(x) = 0 \) is an equation of the quadric \( Q \). The divisor \( S \) is singular at the points of \( L \cap Q \) where the matrix of partial derivatives

\[
\begin{pmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 \\
x_0\tilde{Q} & x_1\tilde{Q} & x_2\tilde{Q} & C_3 + x_3\tilde{Q} & C_4 + x_4\tilde{Q}
\end{pmatrix}
\]

has rank 1. In these points the minors \( x_0(C_3 + x_3\tilde{Q}) - x_3x_0\tilde{Q} = x_0C_3 \) and \( x_1(C_3 + x_3\tilde{Q}) - x_3x_1\tilde{Q} = x_1C_3 \) must vanish. Since there are at least 7 such points, \( C_3 \) and, similarly, \( C_4 \) must vanish along \( L \cap Q \); hence \( S \) has non-isolated singularities, that is a contradiction. □

Lemma 6.7. A twisted cubic contains at most 9 of the points \( p_1, \ldots, p_{11} \).

Proof. Assume that there is a twisted cubic \( \Gamma \) containing 10 of the points \( p_1, \ldots, p_{11} \). The twisted cubics contained in \( Q \) form two orbits with respect to the action of \( \text{Aut}(Q) = \text{PSO}(5) \): one of them consists of the twisted cubics \( \gamma \) such that their linear span \( L(\gamma) \) intersects \( Q \) by a nonsingular quadric surface, and the other consists of the twisted cubics \( \gamma \) such that \( L(\gamma) \) intersects \( Q \) by a quadric cone. In particular, we may assume that in appropriate homogeneous coordinates \((x_0 : \ldots : x_4)\) the curve \( \Gamma \) is given by equations

\[
\begin{align*}
x_1^2 &= x_0x_2, & x_2^2 &= x_1x_3, \\
x_0x_3 &= x_1x_2, & x_4 &= 0,
\end{align*}
\]

and \( Q \) is given either by equation \( x_0x_3 - x_1x_2 + x_4^2 = 0 \), or by equation \( x_1^2 - x_0x_2 + x_3x_4 = 0 \). The quartic \( W \) is given by

\[
(x_0x_3 - x_1x_2)Q_1 + (x_1^2 - x_0x_2)Q_2 + (x_2^2 - x_1x_3)Q_3 + x_4C = 0,
\]

\( \Box \)
where $\deg Q_i = 2$, $\deg C = 3$. The divisor $S$ is singular at the points of $\Gamma$ where the matrix $M$ of partial derivatives has rank 1. In either case one can check that if $M$ has rank 1 at 10 points on $\Gamma$, then it has rank 1 along $\Gamma$, i.e. $S$ has non-isolated singularities, a contradiction. □

**Lemma 6.8.** Assume that 10 of the points $p_1, \ldots, p_{11}$ are contained in a three-dimensional space. Then the points $p_1, \ldots, p_{11}$ impose independent conditions on the hypersurfaces of degree 3.

**Proof.** Let $p_1, \ldots, p_{10} \in H \simeq \mathbb{P}^3$. If $p_{11} \in H$, we apply Lemma 6.4 to a quadric $Y = Q \cap H$ (choosing different points $p_i$ as the distinguished point $q$; the assumptions of Lemma 6.4 hold due to Lemma 6.5, Lemma 6.6 and Lemma 6.7). If $p_{11} \notin H$, then we may choose a cone $K$ over a divisor $D_{11} \in O(3)|_Y$ passing through $p_1, \ldots, p_{10}$, so that $p_{11} \notin K$. On the other hand, Lemma 6.4 implies that for any point $p_i$, $1 \leq i \leq 10$, there is a divisor $D_i \in O(3)|_Y$ passing through all the points $p_1, \ldots, p_{10}$ except $p_i$; a cone with a vertex $p_{11}$ over such divisor passes through all the points $p_1, \ldots, p_{11}$ except $p_i$. □

**Completion of the proof of 1.5.** Let us check that the points $p_1, \ldots, p_{11}$ impose independent conditions on the forms of degree 3 in $\mathbb{P}^4$. In the notations of Theorem 6.3 we have $n = 4$, $d = 3$. No 4 of the points $p_1, \ldots, p_{11}$ are collinear by Lemma 6.5, no 7 of them are coplanar due to Lemma 6.6. By Lemma 6.8 we may assume that no 10 of the points $p_1, \ldots, p_{11}$ lie in a three-dimensional space. Hence the assumptions of Theorem 6.3 hold, and we are done. □

**References**

[1] E. Bese, *On the spanningness and very ampleness of certain line bundles on the blow-ups of $\mathbb{P}_C^2$ and $\mathbb{F}_r$*, Math. Ann. 262 (1983), 225–238.

[2] I. A. Cheltsov, *Birationally rigid Fano varieties*, Uspekhi Mat. Nauk, 2005, 60, 5 (365), 71–160; English transl.: Russian Mathematical Surveys, 2005, 60, 5, 875–965.

[3] I. Cheltsov, *Non-rational nodal quartic threefolds*, Pacific J. of Math., 226 (2006), 1, 65–82.

[4] I. Cheltsov, *Points in projective spaces and applications*, arXiv:math.AG/0511578 (2006).

[5] I. Cheltsov, J. Park, *Sextic double solids*, arXiv:math.AG/0404452 (2004).

[6] A. Corti, *Singularities of linear systems and 3-fold birational geometry*, L.M.S. Lecture Note Series 281 (2000), 259–312.

[7] S. Cynk, *Defect of a nodal hypersurface*, Manuscripta Math. 104 (2001), 325–331.

[8] M. M. Grinenko, *Birational automorphisms of a 3-dimensional double cone*, Mat. Sb., 1998, 189, 7, 37–52; English transl.: Sb. Math., 1998, 189, 991–1007.
[9] M. M. Grinenko, Birational automorphisms of a three-dimensional double quadric with an elementary singularity, Mat. Sb., 1998, 189, 1, 101–118; English transl.: Sb. Math., 1998, 189, 97–114.
[10] D. Eisenbud, J.-H. Koh, Remarks on points in a projective space, Commutative algebra, Berkeley, CA (1987), MSRI Publications 15, Springer, New York, 157–172.
[11] V. A. Iskovskikh, Anticanonical models of three-dimensional algebraic varieties, Itogi Nauki Tekh. Sovrem. Probl. Mat., vol. 12, Moscow, VINITI, 1979, 59–157; English transl.: J. Soviet Math., 13 (1980), 745–814.
[12] V. A. Iskovskikh, Birational automorphisms of three-dimensional algebraic varieties, Itogi Nauki Tekh. Sovrem. Probl. Mat., vol. 12, Moscow, VINITI, 1979, 159–235; English transl.: J. Soviet Math., 13 (1980), 815–867.
[13] V. A. Iskovskikh, Yu. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Mat. Sb., 1971, 86, 1, 140–166; English transl.: Math. USSR-Sb., 1971, 15, 1, 141–166.
[14] V. A. Iskovskikh, A. V. Pukhlikov, Birational automorphisms of multidimensional algebraic varieties, Itogi Nauki Tekh. Sovrem. Probl. Mat., vol. 19, Moscow, VINITI, 2001, 5–139.
[15] K. Matsuki. Introduction to the Mori program. Universitext, Springer, 2002.
[16] M. Mella, Birational geometry of quartic 3-folds II: the importance of being Q-factorial, Math. Ann. 330 (2004), 107–126.
[17] A. V. Pukhlikov, Birational automorphisms of three-dimensional quartic with an elementary singularity, Mat. Sb., 1988, 135, 4, 472–496; English transl.: Math. USSR-Sb., 1989, 63, 457–482.
[18] A. Pukhlikov, Essentials of the method of maximal singularities, L.M.S. Lecture Note Series 281 (2000), 73–100.
[19] C. A. Shramov, Q-factorial quartic threefolds, Sbornik Mathematics, 198 (2007), 8, 1165–1174.

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