A Simple and Optimal Policy Design with Safety against Heavy-tailed Risk for Multi-armed Bandits

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We design new policies that ensure both worst-case optimality for expected regret and light-tailed risk for regret distribution in the stochastic multi-armed bandit problem. Recently, Fan and Glynn (2021b) showed that information-theoretically optimized bandit algorithms suffer from some serious heavy-tailed risk; that is, the worst-case probability of incurring a linear regret slowly decays at a polynomial rate of $1/T$, as $T$ (the time horizon) increases. Inspired by their results, we further show that widely used policies (e.g., Upper Confidence Bound, Thompson Sampling) also incur heavy-tailed risk; and this heavy-tailed risk actually exists for all “instance-dependent consistent” policies. With the aim to ensure safety against such heavy-tailed risk, starting from the two-armed bandit setting, we provide a simple policy design that (i) has the worst-case optimality for the expected regret at order $\tilde{O}(\sqrt{T})$ and (ii) has the worst-case tail probability of incurring a linear regret decay at an optimal exponential rate $\exp(-\Omega(\sqrt{T}))$. Next, we improve the policy design and analysis to the general $K$-armed bandit setting. We provide explicit tail probability bound for any regret threshold under our policy design. Specifically, the worst-case probability of incurring a regret larger than $x$ is upper bounded by $\exp(-\Omega(x/\sqrt{KT}))$. We also enhance the policy design to accommodate the “any-time” setting where $T$ is not known a priori, and prove equivalently desired policy performances as compared to the “fixed-time” setting with known $T$. A brief account of numerical experiments is conducted to illustrate the theoretical findings. Our results reveal insights on the incompatibility between consistency and light-tailed risk, whereas indicate that worst-case optimality on expected regret and light-tailed risk on regret distribution are compatible.

Key words: multi-armed bandit, worst-case optimality, instance-dependent consistency, heavy-tailed risk

1. Introduction

The stochastic multi-armed bandit (MAB) problem is a widely studied problem in the domain of sequential decision-making under uncertainty, with many applications such as online advertising, recommendation systems, clinical trials, financial portfolio design, etc. It also has valuable theoretical insights exhibiting the exploration-exploitation trade-offs. For policy design and analysis, much of the MAB literature uses the metric of maximizing the expected cumulative reward, or...
equivalently minimizing the expected regret (where regret is defined as the difference between the cumulative reward obtained by always pulling the best arm and by executing a policy that does not a priori know the reward distributions). The optimality of a policy is often characterized through its expected regret’s rate (order of dependence) on the experiment horizon $T$.

In this paper, however, we show that renowned policies, such as the standard Upper Confidence Bound (UCB) policy (Auer et al. 2002), the Successive Elimination (SE) policy (Even-Dar et al. 2006) and the Thompson Sampling (TS) policy (Russo et al. 2018), that are designed to enjoy optimality in terms of expected regret can incur a “heavy-tailed risk”. That is, the distribution of the regret has a heavy tail — the probability of incurring a linear regret slowly decays at a polynomial rate $\Omega(poly(1/T))$ as $T$ tends to infinity. In contrast, a “light-tailed” risk means that the probability of a policy incurring a linear regret decays at an exponential rate $\exp(-\Omega(T^{\beta}))$ for some $\beta > 0$. The heavy-tailed risk can be undesired when an MAB policy is used in applications that are sensitive to tail risks, including but not limited to finance, healthcare, supply chain, etc. In fact, understanding heavy-tailed risks and their associated disruptions in the aforementioned applications have been a keen focus in the literature; see Bouchaud and Georges (1990), Bouchaud et al. (2000), Chopra and Sodhi (2004), Embrechts et al. (2013), Sodhi and Tang (2021) for example.

Our work is primarily motivated by an attempt to answer the following question. Is it possible to design a policy that on one hand enjoys optimality under some classical expected regret notion, whereas on the other hand has light-tailed risk?

1.1. Our Contributions

We argue that instant-dependent consistency and light-tailed risk are incompatible. Built upon the analysis and results in Fan and Glynn (2021b), we find that a wide range of policies, including the standard UCB and SE policy, and the TS policy, suffer from heavy-tailed risk. More precisely, each of these three policies incurs a linear regret with probability $\Omega(poly(1/T)) = \exp(-O(\ln T))$. More broadly, any instant-dependent consistent policy cannot be light-tailed: if an instant-dependent consistent policy has the probability of incurring a linear regret decay as $\exp(-f(T))$, then $f(T)$ must be $o(T^{\beta})$ as $T \to +\infty$ for any $\beta > 0$. Motivated by this finding, we further make policy design and analysis contributions as follows.

We show that worst-case optimality and light-tailed risk can co-exist. Starting from the two-armed bandit setting, we provide a new policy design and prove that it enjoys both the worst-case optimality $\tilde{O}(\sqrt{T})$ for the expected regret and the light-tailed risk $\exp(-\Omega(\sqrt{T}))$ for the regret distribution. We also show that such exponential decaying rate of the tail probability is optimal within the class of worst-case optimal policies. Our policy design builds upon the idea of confidence bounds, and constructs different bonus terms compared to the standard ones to ensure safety against heavy-tailed risk.
We extend our results from the two-armed bandit to the general $K$-armed bandit and characterize the tail probability bound for any regret threshold in an explicit form and through a non-asymptotic way. By further improving our policy design, we show that the worst-case probability of incurring a regret larger than $x$ is bounded approximately by $\exp(-\Omega(x/\sqrt{KT}))$ for any $x > 0$. Finally, we enhance the policy design to accommodate the “any-time” setting where $T$ is not known a priori, as a more challenging setting compared to the “fixed-time” setting where $T$ is known a priori. We design a policy for the “any-time” setting and prove that the policy enjoys an equivalently desired exponential decaying tail and optimal expected regret as in the “fixed-time” setting.

Our policy design resolves two issues that create a large regret: (i) spending too much time before correctly discarding a sub-optimal arm and (ii) wrongly discarding the optimal arm due to under-estimation (pointed out by Fan and Glynn (2021b)). Despite of the simplicity of our proposed policy design, the associated proof techniques are novel and may be useful for broader analysis on regret distribution and tail risk. Our result also partially answers an open question raised in Lattimore and Szepesvári (2020) for the stochastic MAB problem. Finally, a brief account of experiments are conducted to illustrate our theoretical findings.

1.2. Related Work
Our work builds upon the vast literature of designing and analyzing policies for the stochastic MAB problem and its various extensions. Comprehensive reviews can be found in Bubeck and Cesa-Bianchi (2012), Russo et al. (2018), Slivkins (2019), Lattimore and Szepesvári (2020). A standard paradigm for obtaining a near-optimal regret is to first fix some confidence parameter $\delta > 0$. Then a “good event” is defined such that good properties are retained conditioned on the event (for example, in the stochastic MAB problem, the good event is such that the mean of each arm always lies in the confidence bound). Then one can obtain both high-probability and worst-case expected regret bounds through careful analysis on the good event. It is known that the stochastic MAB problem has the following regret bound: for any fixed $\delta \in (0, 1)$, the regret bound of UCB is bounded by $O(\sqrt{KT\ln(T/\delta)})$ with probability at least $1 - \delta$. Or equivalently speaking, the probability of incurring a $\Omega(\sqrt{KT\ln(T/\delta)})$ regret is bounded by $\delta$. However, the parameter $\delta$ must be an input parameter for the policy. We will discuss this issue in more details in Section 3. In Section 17.1 of Lattimore and Szepesvári (2020), an open question is asked: Is it possible to design a single policy such that the worst-case probability of incurring a $\Omega(\sqrt{KT\ln(1/\delta)})$ regret is bounded by $\delta$ for any $\delta > 0$ and any $K$-armed bandit problem with 1-subgaussian stochastic rewards? We partially answer this question by designing a policy such that for any $\delta > 0$, the probability of incurring a

$$\Omega\left(\frac{\sqrt{KT\ln(T/\delta)}}{\sqrt{\ln T}}\right)$$
regret is bounded by $\delta$. We note that there has been a related result in the adversarial bandit setting (see, e.g., Neu (2015), Lattimore and Szepesvári (2020)). It is shown that for the $K$-armed bandit problem with adversarial rewards uniformly in $[0, 1]$, there exists a single policy EXP3-IX such that the worst-case probability of incurring a

$$\Omega \left( \sqrt{KT \ln(K/\delta)} / \sqrt{\ln K} \right)$$

regret is bounded by $\delta$ for any $\delta > 0$. The difference between this result and ours are two-folds. From the policy design prospective, the idea behind EXP3-IX is to use exponential weight, while the idea behind our policy is to use a modified confidence bound that is designed to handle the stochastic setting. From the model setting perspective, in the adversarial setting, rewards are assumed to be uniformly bounded, while in the stochastic setting, the magnitude of a single reward is uncontrollable. In fact, naively reducing the bound under the adversarial setting into one under the stochastic setting will make the new bound sub-optimal on the order of $\ln(1/\delta)$. This point is discussed in detail in Section 4.

There has been not much work on understanding the tail risk of stochastic bandit algorithms. Two earlier works are Audibert et al. (2009), Salomon and Audibert (2011) and they study the concentration properties of the regret around the instance-dependent mean $O(\ln T)$. They show that in general the regret of the policies concentrate only at a polynomial rate. That is, the probability of incurring a regret of $c(\ln T)^p$ (with $c > 0$ and $p > 1$ fixed) is approximately polynomially decaying with $T$. Different from our work, the concentration in their work is under an instance-dependent environment, and so such polynomial rate might be different across different instances. Nevertheless, their results indicate that standard bandit algorithms generally have undesirable concentration properties. Recently, Ashutosh et al. (2021) shows that an online learning policy with the goal of obtaining logarithmic regret can be fragile, in the sense that a mis-specified risk parameter (e.g., the parameter for subgaussian noises) in the policy can incur an instance-dependent expected regret polynomially dependent on $T$. They then focus on robust algorithm development to circumvent the issue. Note that their goal is to handle mis-specification related with risk, but still the task is to minimize the expected regret.

Our work is inspired by Fan and Glynn (2021b), who first provided a rigorous formulation to analyze heavy-tailed risk for bandit algorithms and showed that for any information-theoretically optimized bandit policy, the probability of incurring a linear regret is very heavy-tailed: at least $\Omega(1/T)$. They additionally showed that optimized UCB bandit designs are fragile to mis-specifications and they modified UCB algorithms to ensure a desired polynomial rate of tail risk, which makes the algorithms more robust to mis-specifications. Built upon their analysis, we show an incompatibility
result. That is, a large family of policies — all policies that are consistent — suffer from heavy-tailed risk (see Section 1.1). Further, we propose a simple and new policy design that leads to both light-tailed risk (tail bound exponentially decaying with $\sqrt{T}$) and worst-case optimality (expected regret bounded by $\tilde{O}(\sqrt{T})$).

Recently, there are some works analyzing the distributional behaviour of UCB and TS policies by considering the diffusion approximations (see, e.g., Araman and Caldentey (2021), Wager and Xu (2021), Fan and Glynn (2021a), Kalvit and Zeevi (2021)). These works typically consider asymptotic limiting regimes that are set such that the gaps between arm means shrink with the total time horizon. We do not consider limiting regimes but instead consider the original problem setting with general parameters (e.g., gaps). We study how the tail probability decays with $T$ under original environments without taking the gaps to zero. Another line of works closely related with ours involve solving risk-averse formulations of the stochastic MAB problem (see, e.g., Sani et al. (2012), Galichet et al. (2013), Zimin et al. (2014), Vakili and Zhao (2016), Cassel et al. (2018), Khajonchotpanya et al. (2021)). Compared to standard stochastic MAB problems, the main difference in their works is that arm optimality is defined using formulations other than the expected value, such as mean-variance criteria and (conditional) value-at-risk. These formulations consider some single metric that is different compared to the expected regret. From the formulation perspective, our work is different in the sense that we develop policies that on one hand maintain the classical worst-case optimal expected regret, whereas simultaneously achieve light-tailed risk bound. The policy design and analysis in our work are therefore also different from the literature and might be of independent interest.

1.3. Organization and Notations

The rest of the paper is organized as follows. In Section 2, we discuss the setup and introduce the key concepts: light-tailed risk, instance-dependent consistency, worst-case optimality. In Section 3, we show a result on the incompatibility between light-tailed risk and consistency, and show the compatibility between light-tailed risk and worst-case optimality via a new policy design. In Section 4, we consider the general $K$-armed bandit model and show a precise regret tail bound for our new policy design. We detail the proof road-map and how to further improve the design. In Section 5, we present numerical experiments. Finally, we conclude in Section 6. All detailed proofs are left to the supplementary material.

Before proceeding, we introduce some notations. Throughout the paper, we use $O(\cdot)$ ($\tilde{O}(\cdot)$) and $\Omega(\cdot)$ ($\tilde{\Omega}(\cdot)$) to present upper and lower bounds on the growth rate up to constant (logarithmic) factors, and $\Theta(\cdot)$ ($\tilde{\Theta}(\cdot)$) to characterize the rate when the upper and lower bounds match up to constant (logarithmic) factors. We use $o(\cdot)$ to present strictly dominating upper bounds. In addition, for any $a,b \in \mathbb{R}$, $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. For any $a \in \mathbb{R}$, $a_+ = \max\{a,0\}$. We use $[N] = \{1, \cdots, N\}$. 
2. The Setup

Fix a time horizon of $T$ and the number of arms as $K$. Throughout the paper, we assume that $T \geq 3$, $K \geq 2$, and $T \geq K$. In each time $t \in [T]$, based on all the information prior to time $t$, the decision maker (DM) pulls an arm $A_t \in [K]$ and receives a reward $r_{t,A_t}$. More specifically, let $H_t = \{ A_1, r_{1,A_1}, \ldots, A_{t-1}, r_{t-1,A_{t-1}} \}$ be the history prior to time $t$. When $t = 1$, $H_1 = \emptyset$. At time $t$, the DM adopts a policy $\pi_t : H_t \mapsto A_t$ that maps the history $H_t$ to an action $A_t$, where $A_t$ follows a discrete probability distribution on $[K]$. The environment then independently samples a reward $r_{t,A_t} = \theta_{A_t} + \epsilon_{t,A_t}$ and reveals it to the DM. Here, $\theta_{A_t}$ is the mean reward of arm $A_t$, and $\epsilon_{t,A_t}$ is an independent zero-mean noise term. We assume that $\epsilon_{t,A_t}$ is $\sigma$-subgaussian. That is, there exists a $\sigma > 0$ such that for any time $t$ and arm $k$,

$$\max \{ \Pr(\epsilon_{t,k} \geq x), \Pr(\epsilon_{t,k} \leq -x) \} \leq \exp(-x^2/(2\sigma^2)).$$

Let $\theta = (\theta_1, \ldots, \theta_K)$ be the mean vector. Let $\theta_* = \max\{\theta_1, \cdots, \theta_K\}$ be the optimal mean reward among the $K$ arms. Note that DM does not know $\theta$ at the beginning, except that $\theta \in [0,1]^K$. The empirical regret of the policy $\pi = (\pi_1, \cdots, \pi_T)$ under the mean vector $\theta$ over a time horizon of $T$ is defined as

$$\hat{R}_\theta^\pi(T) = \theta_* \cdot T - \sum_{t=1}^T (\theta_{A_t} + \epsilon_{t,A_t}).$$

Let $\Delta_k = \theta_* - \theta_k$ be the gap between the optimal arm and the $k$th arm. Let $n_{t,k}$ be the number of times arm $k$ has been pulled up to time $t$. That is, $n_{t,k} = \sum_{s=1}^t 1\{ A_s = k \}$. For simplicity, we will also use $n_k = n_{T,k}$ to denote the total number of times arm $k$ is pulled throughout the whole time horizon. We define $t_k(n)$ as the time period that arm $k$ is pulled for the $n$th time. Define the pseudo regret as

$$R_\theta^\pi(T) = \sum_{k=1}^K n_k \Delta_k$$

and the genuine noise as

$$N_\pi^\sigma(T) = \sum_{t=1}^T \epsilon_{t,A_t} = \sum_{k=1}^K n_k \sum_{m=1}^{t_k(n)} \epsilon_{t_k(m),k}.$$ 

Then the empirical regret can also be written as $\hat{R}_\theta^\pi(T) = R_\theta^\pi(T) - N_\pi^\sigma(T)$. The following simple lemma gives the mean and the tail probability of the genuine noise $N_\pi^\sigma(T)$. Intuitively, it shows when bounding the mean or the tail probability of the empirical regret, one only need to consider the pseudo regret term. We will make it more precise when we discuss the proof of main theorems.

\textbf{Lemma 1.} We have $\mathbb{E}[N_\pi^\sigma(T)] = 0$ and

$$\max \{ \Pr(N_\pi^\sigma(T) \geq x), \Pr(N_\pi^\sigma(T) \leq -x) \} \leq \exp\left(\frac{-x^2}{2\sigma^2T}\right).$$
2.1. Light-tailed Risk, Instance-dependent Consistency, Worst-case Optimality

Now we describe concepts that are needed to formalize the policy design and analysis.

1. **Light-tailed risk.** A policy is called light-tailed, if for any constant $c > 0$, there exists some $\beta > 0$ and constant $C > 0$ such that

$$\limsup_{T \to +\infty} \frac{\ln \left\{ \sup_{\theta} \mathbb{P} \left( \hat{R}_\theta^*(T) > cT \right) \right\}}{T^\beta} \leq -C.$$ 

Note that here, we allow $\beta$ and $C$ to be dependent on $c$. In brief, a policy has light-tailed risk if the probability of incurring a linear regret can be bounded by an exponential term of polynomial $T$:

$$\sup_{\theta} \mathbb{P}(\hat{R}_\theta^*(T) \geq cT) = \exp(-\Omega(T^\beta))$$

for some $\beta > 0$. If a policy is not light-tailed, we say it is heavy-tailed.

**Remark 1.** We clarify that conventionally, a distribution is called “lighted-tailed” when its moment generating function is finite around a neighborhood of zero. Our definition of “light-tailed” emphasizes the boundary between heavy and light to separate polynomial rate of decaying versus exponential-polynomial rate of decaying, which is aligned with but technically different from the conventional definition of “lighted-tailed”. For example, for regret random variables $R(T)$ indexed by $T$, when $T$ is large, if $\mathbb{P}(R(T) > T/2) \sim T^{-\beta}$ for some positive $\beta$, then its distribution is heavy-tailed in both our definition and the conventional definition. If $\mathbb{P}(R(T) > T/2) \sim \exp(-T^\beta)$ for $\beta \in (0, 1)$, then its distribution is lighted-tailed in our definition and is heavy-tailed in the conventional definition. If $\mathbb{P}(R(T) > T/2) \sim \exp(-T^\beta)$ for $\beta \geq 1$, then its distribution is lighted-tailed in both our definition and the conventional definition. Therefore, when we claim safety against heavy-tailed risk, it indicates tail distribution lighter than any polynomial rate of decay.

2. **Instance-dependent consistency.** A policy is called consistent, if for any underlying true mean vector $\theta$, the policy has that

$$\limsup_{T \to +\infty} \mathbb{E} \left[ \frac{\hat{R}_\theta^*(T)}{T^\beta} \right] = 0$$

holds for any $\beta > 0$. In brief, a policy is consistent if the expected regret can never be polynomially growing in $T$ for any fixed instance.

3. **Worst-case optimality.** A policy is said to be worst-case optimal, if for any $\beta > 0$, the policy has that

$$\limsup_{T \to +\infty} \sup_{\theta} \mathbb{E} \left[ \frac{\hat{R}_\theta^*(T)}{T^{1/2+\beta}} \right] = 0.$$
In brief, a policy is worst-case optimal if the worst-case expected regret can never be growing in a polynomial rate faster than $T^{1/2}$. Note that here we adopt a relaxed definition of optimality, in the sense that we do not clarify how the regret scale with the number of arms $K$ compared to that in literature. The notion of worst-case optimality in this work focuses on the dependence on $T$. For example, a policy with worst-case regret $O(poly(K)\sqrt{T} \cdot poly(lnT))$ is also optimal by our definition.

It is well known that for the stochastic MAB problem, one can design algorithms to achieve both instance-dependent consistency and worst-case optimality. Among them, two types of policies are of prominent interest: Successive Elimination (SE) and Upper Confidence Bound (UCB). We list the algorithm paradigms in Algorithm 1 and 2. The bonus term $\text{rad}(n)$ is typically set as

$$\text{rad}(n) = \sigma \sqrt{\frac{\eta \ln T}{n}}$$

with $\eta > 0$ being some tuning parameter.

| Algorithm 1 Successive Elimination |
|-----------------------------------|
| 1: $\mathcal{A} = [K]$, $t \leftarrow 0$. |
| 2: while $t < T$ do |
| 3: Pull each arm in $\mathcal{A}$ once. $t \leftarrow t + |\mathcal{A}|$. |
| 4: Eliminate any $k \in \mathcal{A}$ from $\mathcal{A}$ if |
| $\exists k' : \hat{\mu}_{t,k'} - \text{rad}(n_{t,k'}) > \hat{\mu}_{t,k} + \text{rad}(n_{t,k})$ |
| 5: end while |

| Algorithm 2 Upper Confidence Bound |
|-----------------------------------|
| 1: $\mathcal{A} = [K]$, $t \leftarrow 0$. |
| 2: while $t < T$ do |
| 3: $t \leftarrow t + 1$. |
| 4: Pull the arm with the highest UCB: |
| $\arg \max_k \{ \hat{\mu}_{t-1,k} + \text{rad}(n_{t-1,k}) \}$. |
| 5: end while |

SE maintains an active action set, and for each arm in the action set, it maintains a confidence interval. After pulling each arm in the action set, SE updates the action set by eliminating any arm whose confidence interval is strictly dominated by others. As a comparison, UCB does not shrink the active action set. It always pulls the arm with the highest upper confidence bound. These two algorithms are somewhat similar, in the sense that they both utilize confidence intervals to guide the actions.

3. The Basic Case: Two-armed Bandit

We start from the simple two-armed bandit setting. The general multi-armed setting is deferred to the next section. We first show that some standard policies considered widely in the literature are heavy-tailed. The result reveals an incompatibility between instant-dependent consistency and light-tailed risk. Then we show how to add a simple twist to standard confidence bound based policies to obtain light-tailed risk. Moreover, we show that our design leads to an optimal tail decaying rate for all policies that enjoy worst-case optimal order of expected regret.
3.1. Instance-dependent Consistency Causes Heavy-tailed Risk

**Theorem 1.** If a policy is instance-dependent consistent, then it can never be light-tailed. Moreover, if $\pi = \text{SE}$ or UCB with

$$\text{rad}(n) = \sigma \sqrt{\frac{\eta \ln T}{n}}$$

(1)

for some $\eta > 0$, we have the following stronger argument. For any $c > 0$, there exists $C_\pi > 0$ such that

$$\limsup_{T \to +\infty} \frac{\ln \left\{ \sup_{\theta} \mathbb{P} \left( \hat{R}_\theta(T) > cT \right) \right\}}{\ln T} \geq -C_\pi.$$  

(2)

Theorem 1 suggests that a consistent policy must have a risk tail heavier than an exponential one. To prove Theorem 1, we use a refined version of the change of measure argument originally introduced by Fan and Glynn (2021b). We consider two environments with $\theta = (1/2, 1)$ and $\tilde{\theta} = (1/2, 0)$. The noise $\epsilon$ is always Gaussian with standard deviation $\sigma$. If $\pi$ is consistent, then the probability of incurring a linear (pseudo) regret under $\theta$ is at least $\mathbb{P}_\theta(n_2 < (1 - 2c)T) \geq \mathbb{P}_\tilde{\theta}(n_2 < o(T^\beta))$ (here $c \in (0, 1/2)$, $\beta \in (0, 1)$). However, $\mathbb{P}_\pi(n_2 < o(T^\beta)) \to 1$ since $\pi$ is consistent. Intuitively speaking, if we want a policy to be adaptive enough to handle different instances, then the policy will be sensitive to risk. Moreover, for the family of confidence bound related policies (SE and UCB), the standard bonus term will lead to a tail polynomially dependent on $T$. This polynomial dependency stems solely from the fact that when $\pi = \text{SE}$ or UCB, for any fixed $\theta$, we have

$$\limsup_{T \to +\infty} \frac{\mathbb{E} \left[ \hat{R}_\theta(T) \right]}{\ln T} < +\infty.$$  

(3)

We note that Theorem 1 is general enough to include other families of policies aside from SE and UCB. One example is the Thompson Sampling (TS) policy. It has been established that $\pi = \text{TS}$ with Beta prior has the property (3) (see, e.g., Theorem 1 and 2 in Agrawal and Goyal (2012)). Our proof suggests that (2) also holds for $\pi = \text{TS}$.

We need to remark on the difference between Theorem 1 and high-probability bounds in the stochastic MAB literature. It has been well-established that UCB or SE with

$$\text{rad}(n) = \sigma \sqrt{\frac{\eta \ln(1/\delta)}{n}}$$

achieves $\tilde{O}(\sqrt{T \cdot \text{polylog}(T/\delta)})$ regret with probability at least $1 - \delta$. Such design also leads to a consistent policy. However, the bound holds only for fixed $\delta$. In fact, one can see that the bonus design is dependent on the confidence parameter $\delta$. If $\delta = \exp(-\Omega(T^\beta))$ with $\beta > 0$, then the scaling speed of the regret with respect to $T$ can only be greater than $1/2$, which is sub-optimal.
As a comparison, in our problem, ideally we seek to find a single policy such that it achieves $O(\sqrt{T \cdot \text{polylog}(T/\delta)})$ regret for any $\delta > 0$.

Up till now, we make two core observations. First, from standard stochastic MAB results, consistency and optimality can hold simultaneously. Second, from Theorem 1, consistency and light-tailed risk are always incompatible. Then a natural question arises: Can we design a policy that enjoys both optimality and light-tailed risk? If we can, then can we make the tail risk decay with $T$ in an optimal rate? We answer these two questions with an affirmative “yes” in the next section.

### 3.2. A New Policy Design Achieving Light-tailed Risk and Worst-case Optimality

In this section, we propose a new policy design that achieves both light-tailed risk and worst-case optimality. The design is very simple. We still use the idea of confidence bounds, but instead of setting the bonus as (1), we set

$$\text{rad}(n) = \sigma \sqrt{\eta T \ln T} / n$$

with $\eta > 0$ being a tuning parameter. Theorem 2 gives performance guarantees for the mean and the tail probability of the empirical regret when $\pi = \text{SE}$.

**Theorem 2.** For the two-armed bandit problem, the SE policy with $\eta \geq 4$ and the bonus term being (4) satisfies the following two properties.

1. $\sup_{\theta} \mathbb{E}[\hat{R}_{\theta}(T)] = O(\sqrt{T \ln T})$.
2. For any $c > 0$ and any $\alpha \in (1/2, 1]$, we have

$$\sup_{\theta} \mathbb{P}(\hat{R}_{\theta}^*(T) \geq cT^\alpha) = \exp(-\Omega(T^{\alpha-1/2})).$$

The first item in Theorem 2 means that with the modified bonus term, the worst case regret is still bounded by $O(\sqrt{T \ln T})$, which is the same as the regret bounds for SE and UCB with the standard bonus term (1). The second item shows that the tail probability of incurring a $\Omega(T^\alpha)$ regret ($\alpha > 1/2$) is exponentially decaying in $\Omega(T^{\alpha-1/2})$, and thus the policy is light-tailed. The detailed proof of Theorem 2 is provided in the supplementary material. The illustrative road-map of the proof is delegated to Section 4, where we provide the proof idea for a theorem that is a strict generalization of Theorem 2. Here, we give some intuition on the new bonus design. Our new bonus term inflates the standard one by a factor of $\sqrt{T/n}$. This means our policy is more conservative than the traditional confidence bound methods, especially at the beginning. In fact, one can observe that for the first $\Theta(\sqrt{T})$ time periods, our policy consistently explores between arm 1 and 2, regardless of the environment. A naturally corollary is that our policy is never “consistent”, and this is reasonable following Theorem 1. However, the bonus term (4) decays at a faster rate on the number of pulling times $n$ compared to (1). This means as the experiment goes on, the policy...
leans towards exploitation. We need to stress that this is not the same as the explore-then-commit policy, which is well-known to achieve a sub-optimal $\Theta(T^{2/3})$ regret.

The following theorem shows that the risk tail in Theorem 2 is not improvable. That is, if the policy $\pi$ is worst-case optimal, then for fixed $\alpha \in (1/2, 1]$, the exponent of $\alpha - 1/2$ is tight.

**Theorem 3.** Let $c \in (0, 1/2)$. Consider the 2-armed bandit problem with gaussian noise. Let $\pi$ be a worst-case optimal policy. That is, for any $\alpha > 1/2$,

$$\lim_{T} \sup_{\theta} \mathbb{E}[\hat{R}_\theta^\pi(T)] = 0.$$  

Then for any $\alpha \in (1/2, 1]$,

$$\lim_{T} \ln \left\{ \sup_{\theta} \mathbb{P}(\hat{R}_\theta^\pi(T) \geq cT^\alpha) \right\} / T^{\beta} = 0$$

holds for any $\beta > \alpha - 1/2$.

Theorem 3 also relies on the change of measure argument appeared in the proof of Theorem 1. However, there are two important differences: we only have worst-case optimality rather than consistency, and the regret threshold $cT^\alpha$ is in general not linear in $T$. Therefore, we need to take care of constructing the specific “hard” instance when doing the change of measure. The detailed proof is delegated to the supplementary material.

4. **The General Case: Multi-armed Bandit**

In this section, we provide step-by-step extensions to our previous results in Section 3 to the general multi-armed bandit setting. We first give a direct extension where the bonus term is set as (4). It turns out that such bonus design only yields a $\tilde{O}(K\sqrt{T})$ expected regret, and so we study how to achieve the optimal dependence on both $K$ and $T$ by slightly modifying the design. Finally, we relax the assumption of knowing $T$ a priori and give an any-time policy that enjoys an equivalent tail probability bound as compared to the fixed-time case.

4.1. **The Direct Extension**

We first provide a generalization of our previous tail probability bound in Theorem 2 from the following aspects: (a) a general $K$-armed bandit setting; (b) an analysis for UCB aside from SE; (c) a detailed characterization of the tail bound for any fixed regret threshold.

**Theorem 4.** For the $K$-armed bandit problem, $\pi = \text{SE}$ or $\pi = \text{UCB}$ with

$$\text{rad}(n) = \sigma \sqrt{nT \ln T}$$

satisfies the following two properties.
1. If $\eta \geq 4$, then $\sup_\theta \mathbb{E}\left[\hat{R}_\theta^*(T)\right] \leq 4K + 4K\sigma\sqrt{\eta T \ln T}$.

2. If $\eta > 0$, then for any $x > 0$, we have

$$\sup_\theta \mathbb{P}(\hat{R}_\theta^*(T) \geq x) \leq \exp\left(-\frac{x^2}{2K^2}\right) + 2K \exp\left(-\frac{(x - 2K - 4K\sigma\sqrt{\eta T \ln T})^2}{32\sigma^2 K^2 T}\right) + K^2 T \exp\left(-\frac{x\sqrt{\eta T \ln T}}{8K\sigma T}\right).$$

**Proof Idea.** We provide a road-map of proving Theorem 4. The expected regret bound is proved using standard techniques. That is, we define the good event to be such that the mean of each arm always lies in the confidence bounds throughout the whole time horizon. Conditioned on the good event, the regret of each arm is bounded by $O(\sqrt{T \ln T})$, and thus the total expected regret is $O(K \sqrt{T \ln T})$.

The proof of the tail bound requires more effort. Without loss of generality, we assume arm 1 is optimal. We first illustrate the proof for $\pi = \text{SE}$.

1. We use

$$\sup_\theta \mathbb{P}(\hat{R}_\theta^*(T) \geq x) \leq \mathbb{P}\left(N^*(T) \leq -x/\sqrt{K}\right) + \sup_\theta \mathbb{P}\left(R_\theta^*(T) \geq x(1 - 1/\sqrt{K})\right)$$

The term with the genuine noise can be easily bounded using Lemma 1. We are left to bound the tail of the pseudo regret. By a union bound, we observe that

$$\mathbb{P}\left(R_\theta^*(T) \geq x(1 - 1/\sqrt{K})\right) \leq \sum_{k \neq 1} \mathbb{P}(n_k \Delta_k \geq x/(K + \sqrt{K})) \leq \sum_{k \neq 1} \mathbb{P}(n_k \Delta_k \geq x/(2K)) \quad (5)$$

Thus, we reduce bounding the sum of the regret incurred by different arms to bounding that by a single sub-optimal arm.

2. For any $k \neq 1$, we define

$$S_k = \{\text{Arm 1 is not eliminated before arm } k\}.$$  

With a slight abuse of notation, we let $n_0 = \lceil x/(2K\Delta_k)\rceil - 1$. Consider the case when both $n_k \Delta_k \geq x/(2K)$ and $S_k$ happen. This corresponds to the risk of spending too much time before correctly discarding a sub-optimal arm. Then arm 1 and $k$ are both not eliminated after each of them being pulled $n_0$ times. This indicates

$$\hat{\mu}_{1(n_0),1} - \frac{\sigma\sqrt{\eta T \ln T}}{n_0} \leq \hat{\mu}_{1(n_0),k} + \frac{\sigma\sqrt{\eta T \ln T}}{n_0}$$

The probability of this event can be bounded using concentration of subgaussian variables, which yields the second term in the tail probability bound in Theorem 4. We note that the choice of $n_0$ is important. Also, at this step, even if we replace our new bonus term by the standard one, the bound still holds.
3. Now consider the situation when both $n_k \Delta_k \geq x/(2K)$ and $\bar{S}_k$ happen. This corresponds to the risk of wrongly discarding the optimal arm. Then after some phase $n$, the optimal arm 1 is eliminated by some arm $k'$, while arm $k$ is not eliminated. Note that $k = k'$ does not necessarily hold when $K > 2$. As a consequence, we have the following two events hold simultaneously:

$$\hat{\mu}_{k'}(n), k' - \frac{\sigma \sqrt{\eta T \ln T}}{n} \geq \hat{\mu}_{1}(n), 1 + \frac{\sigma \sqrt{\eta T \ln T}}{n} \quad \text{and} \quad \hat{\mu}_{k}(n), k + \frac{\sigma \sqrt{\eta T \ln T}}{n} \geq \hat{\mu}_{1}(n), 1 + \frac{\sigma \sqrt{\eta T \ln T}}{n}.$$ 

The first event leads to

$$\text{Mean of some noise terms} \geq \frac{2\sigma \sqrt{\eta T \ln T}}{n} + \frac{\Delta_{k'}}{n} \geq \frac{2\sigma \sqrt{\eta T \ln T}}{n}.$$ 

The second inequality leads to

$$\text{Mean of some noise terms} \geq \Delta_k \geq \frac{x}{2KT}.$$ 

Now comes the trick to deal with an arbitrary $n$. We bound the probabilities of the two events separately and take the minimum of the two probabilities ($\mathbb{P}(A \cap B) \leq \min\{\mathbb{P}(A), \mathbb{P}(B)\}$ ($\forall A, B$)). Then such minimum can be further bounded using the basic inequality $\min\{a, b\} \leq \sqrt{ab}$ ($\forall a, b \geq 0$). This makes the probability bounded by $\exp(-\Omega(\Delta_k \sqrt{T}))$, which yields the last term in Theorem 4. We note that at this step, the $\sqrt{T}/n$ design in our new bonus term plays a crucial role. The standard bonus term (1) does not suffice to get an exponential bound.

We next illustrate the proof for $\pi = \text{UCB}$, enlightened by our proof for $\pi = \text{SE}$. The proof is in fact simpler. We use the first step in the proof for $\pi = \text{SE}$. For fixed $k$, we also take the same $n_0 = \lfloor x/(2K \Delta_k) \rfloor - 1$. The difference here is that we do not need to define the event $S_k$. When arm $k$ is pulled for the $(n_0 + 1)$th time, by the design of the UCB policy, there exists some $n$ such that

$$\mu_1 + \sum_{m=1}^{n} \epsilon_{t_1(m), 1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \leq \mu_k + \sum_{m=1}^{n_0} \epsilon_{t_k(m), k} + \frac{\sigma \sqrt{\eta T \ln T}}{n_0}.$$ 

Now comes the trick. The event is included by a union of two events described as follows:

$$\sum_{m=1}^{n_0} \epsilon_{t_k(m), k} + \frac{\sigma \sqrt{\eta T \ln T}}{n_0} \geq \frac{\Delta_k}{2} \quad \text{and} \quad \exists n : \sum_{m=1}^{n} \epsilon_{t_1(m), 1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \leq -\frac{\Delta_k}{2}.$$ 

The probability of each of the two events can be bounded using similar techniques adopted when $\pi = \text{SE}$. In fact, it is implicitly shown in our proof that UCB can yield better constants than SE. We still need to emphasize that when bounding the second event, similar to the argument for $\pi = \text{SE}$, the choice of our new bonus term is crucial.

**Remarks.** Some remarks for Theorem 4 are as follows.
1. For the regret bound, we note that compared to the optimal $\tilde{\Theta}(\sqrt{KT})$ bound, we have an additional $\sqrt{K}$ term. We should point out that the additional $\sqrt{K}$ term is not surprising under the bonus term (4). An intuitive explanation is as follows. Compared to the bonus term (1), we widen the bonus term by a factor of $\sqrt{T/n}$. Among the $K - 1$ arms, there must exist an arm such that it is pulled no more than $T/K$ times throughout the whole time horizon. That is, the bonus term of this arm is always inflated by a factor of at least $\sqrt{K}$. The standard regret bound analysis will, as a result, lead to an additional $\sqrt{K}$ factor compared to the optimal regret bound $\tilde{\Theta}(\sqrt{KT})$.

2. For the tail bound, from our proof road-map, one can see that the tail bound in Theorem 4 is also valid for the pseudo regret $\sup_\theta \mathbb{P}(R_{\bar{\theta}}^\pi(T) \geq x)$. To get a neat form of the tail bound, one can notice that the last term in the bound can be written as

$$K \exp \left( -x \frac{\eta \ln T - 8\sigma K \sqrt{T \ln (KT)}}{8\sigma K \sqrt{T}} \right) \leq K \exp \left( -\frac{(x - 16K \sigma \sqrt{1/\eta} \cdot T \ln T) \sqrt{\eta \ln T}}{8\sigma K \sqrt{T}} \right).$$

Since the tail probability has a trivial upper bound of 1, the last term can be replaced by

$$K \exp \left( -\frac{(x - 16K \sigma \sqrt{1/\eta} \cdot T \ln T) \sqrt{\eta \ln T}}{8\sigma K \sqrt{T}} \right).$$

Therefore, if we let

$$y = \frac{x - 2K - 16\sigma K \sqrt{(\eta \vee 1/\eta) T \ln T}}{8\sigma K \sqrt{T}},$$

then for any $x \geq 0$, we get a neat form

$$\sup_\theta \mathbb{P}(R_{\bar{\theta}}^\pi(T) \geq x) \leq \exp(-y^2) + K \exp(-y^2) + K \exp(-y\sqrt{\eta \ln T}) \leq 4K \exp(-y^2 \wedge y \sqrt{\eta \ln T}).$$

One can observe that for any $\eta > 0$, our policy always yields a $O(\sqrt{T})$ expected regret (although with a constant larger than that in the first result in Theorem 4). In fact, notice that for any $x > 0$

$$\mathbb{E}[\hat{R}_\theta^\pi(T)] = \mathbb{E}[R_\theta^\pi(T)] \leq x + \mathbb{P}(R_\theta^\pi(T) \geq x) \cdot T.$$

If we let $x = 2K + C \sigma K \sqrt{(\eta \vee 1/\eta) T \ln T}$ with the absolute constant $C$ being moderately large, then $\mathbb{P}(R_\theta^\pi(T) \geq x) \cdot T = O(1)$. As a result, the worst-case regret becomes

$$O \left( K \sqrt{(\eta \vee 1/\eta) T \ln T} \right).$$

This observation shows that our policy design is not sensitive to the parameter $\eta$ regarding the growth rate on $T$, as opposed to the standard UCB policy with (1), where a very small $\eta$ can possibly make the UCB policy no longer enjoy a $O(\sqrt{T})$ worst-case regret. We believe there is work
in the literature that has precisely documented that a very small $\eta$ may possibly make the UCB policy no longer enjoy a $O(\sqrt{T})$ worst-case regret, but we have not been able to identify one. For completeness, we summarize this point with a proof in the supplementary material. In fact, we show that when $\eta$ is very small, the regret for either SE or UCB is lower bounded by $\tilde{\Omega}(T^{1-2\eta})$, the order of which can be arbitrarily close to 1.

4.2. Optimal Expected Regret

A natural question is whether we can improve the regret bound in Theorem 4 to $\tilde{\Theta}(\sqrt{KT})$ and get a probability bound of

$$\ln \left\{ \sup_{\theta} \mathbb{P}(\hat{R}_\theta(T) \geq x) \right\} = -\Omega \left( \frac{x}{\sqrt{KT}} \right)$$

for large $x$. By slightly modifying the bonus term (4), we give a “yes” to this question in Theorem 5.

**Theorem 5.** For the $K$-armed bandit problem, $\pi = \text{SE}$ or $\pi = \text{UCB}$ with

$$\text{rad}(n) = \sigma \sqrt{\frac{\ln T}{n}} \cdot \max \left\{ \sqrt{\frac{\eta_1 T}{nK}}, \sqrt{\eta_2} \right\}$$

(6)

satisfies the following two properties.

1. If $\eta_1, \eta_2 \geq 4$, then $\sup_\theta \mathbb{E}[R_\theta] \leq 4K + 8\sigma \sqrt{\max\{\eta_1, \eta_2\} KT \ln T}$.
2. If $\eta_1 > 0, \eta_2 \geq 0$, then for any $x > 0$, we have

$$\sup_\theta \mathbb{P}(\hat{R}_\theta(T) \geq x) \leq \exp \left( -\frac{x^2}{8K \sigma^2 T} \right) + 4K \exp \left( -\frac{(x - 2K - 8\sigma \sqrt{(\eta_1 \lor \eta_2) KT \ln T})^2}{128\sigma^2 KT} \right)$$

$$+ 2K^2 T \exp \left( -\frac{(x - 2K) \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right).$$

**Proof Idea.** We provide a brief road-map for proving Theorem 5. The basic idea is similar to that for Theorem 4, but directly applying the analysis will still lead to the same bound in Theorem 4. Here we emphasize the key difference in the analysis. Simply speaking, the main challenge is to reduce the $K$ factor into a $\sqrt{K}$ factor. To address the challenge, we define the (random) arm set as

$$\mathcal{A}_0 = \{k \neq 1 : n_k \leq 1 + T/K\}.$$ 

The bound for the expected regret is then proved using standard techniques, but by considering arms in or not in $\mathcal{A}_0$ separately. We stress that the standard techniques are feasible only when $\eta_1$ and $\eta_2$ are both not too small. Otherwise, it is not valid to show that the good event (the mean of each arm always lies in the confidence bounds throughout the whole time horizon) happens with
high probability. To obtain better bounds when \( \eta_1 \) and \( \eta_2 \) are small, we should resort to the tail bound.

To prove the tail bound, instead of using (5), we take

\[
P \left( R_\theta^*(T) \geq x(1-1/2\sqrt{K}) \right)
\]

\[
= P \left( \sum_{k \in A_0} n_k \Delta_k + \sum_{k \notin A_0} n_k \Delta_k \geq x(1-1/2\sqrt{K}) \right)
\]

\[
\leq P \left( \sum_{k \in A_0} (n_k - 1) \Delta_k + \sum_{k \notin A_0} (n_k - 1) \Delta_k \geq x(1-1/2\sqrt{K}) - K \right)
\]

\[
\leq P \left( \left( \bigcup_{k \in A_0} \left\{ (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K} \right\} \right) \bigcup \left( \bigcup_{k \notin A_0} \left\{ (n_k - 1) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T} \right\} \right) \right)
\]

\[
\leq \sum_{k \neq 1} \left( P \left( (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K}, \ k \in A_0 \right) + P \left( (n_k - 1) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T}, \ k \notin A_0 \right) \right)
\]

\[
\leq \sum_{k \neq 1} \left( (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K}, \ k \in A_0 \right) + \sum_{k \neq 1} \left( \Delta_k \geq \frac{x - 2K}{4T}, \ k \notin A_0 \right)
\]

That is, when \( k \in A_0 \), we consider the event that \( n_k \Delta_k = \Omega(x/K) \); when \( k \notin A_0 \), we consider the event that \( \Delta_k = \Omega(x/T) \). In this way, in each event we consider, \( \Delta_k \) is guaranteed to be \( \Omega(x/T) \), and when \( k \in A_0 \), \( \Delta_k \) enjoys a possibly better lower bound. Combined with the bonus design of \( \sqrt{T/K}/n \), we can get an exponential \(-\Omega(x/\sqrt{KT})\) term for the tail probability. Detailed derivation are left to the supplementary material. We note that if we do as in (5), we can only have \( \Delta_k = \Omega(x/KT) \), yielding an exponential \(-\Omega(x/K\sqrt{T})\) term.

**Remarks.** Some remarks are as follows. For the tail bound, we can do a similar thing to get a neat form as in Theorem 4. If we let \( \eta_1 = \eta > 0 \), \( \eta_2 \in [0, \eta] \) and

\[
y = \frac{\left( x - 2K - 32\sigma \sqrt{(\eta \lor 1/\eta)KT \ln T} \right)_+}{16\sigma \sqrt{KT}},
\]

then for any \( x \geq 0 \), we have

\[
\sup_{\theta} P(R_\theta^*(T) \geq x) \leq 8K \exp \left( -y^2 \land y \sqrt{\eta \ln T} \right).
\]

There are two observations:

1. For any \( \eta > 0 \), our policy always yields a \( O(\sqrt{KT \ln T}) \) expected regret (although with a constant larger than that in the first result in Theorem 5). In fact, notice that for any \( x > 0 \)

\[
\mathbb{E}[\hat{R}_\theta^*(T)] = \mathbb{E}[R_\theta^*(T)] \leq x + P(R_\theta^*(T) \geq x) \cdot T.
\]
If we let \( x = 2K + C\sigma \sqrt{(\eta \lor 1/\eta)KT\ln T} \) with the absolute constant \( C \) being moderately large, then \( \mathbb{P}(R_\pi^*(T) \geq x) \cdot T = O(1) \). As a result, the worst-case regret becomes

\[
O \left( \sqrt{(\eta \lor 1/\eta)KT\ln T} \right).
\]

We need to stress that \( \eta^2 \geq 0 \) does not have much effect in obtaining the light tail, and it is OK to take \( \eta^2 = 0 \). Nevertheless, the bonus design (6) indeed incorporates our new design with the standard one (1).

2. If we set \( \eta = 1 \) and

\[
\delta = 8K \exp \left( -\left( y - \sqrt{\ln T} \right) + \sqrt{\ln T} \right) \geq 8K \exp \left( -y^2 \land y\sqrt{\ln T} \right)
\]

Then one can see that for any \( \delta > 0 \), with probability at least \( 1 - \delta \), the regret of our policy is bounded by

\[
O \left( \sigma \sqrt{KT} \left( \sqrt{\ln T} + \frac{\ln (8K/\delta)}{\sqrt{\ln T}} \right) \right) = O \left( \sigma \sqrt{KT} \frac{\ln (T/\delta)}{\sqrt{\ln T}} \right).
\]

This partially answers the open question in Section 17.1 of Lattimore and Szepesvári (2020) (we’ve mentioned it Section 1.2) up to a logarithmic factor.

We note that a naive way to reduce the high-probability bound in the literature under the adversarial setting (see Section 1.2) into one under the stochastic setting is as follows. For simplicity, we assume \( \sigma = 1 \). First, a simple union bound suggests that with probability at least \( 1 - 2T \exp(-C^2/2) \), all the rewards are bounded within \([-C, 1 + C]\). Then applying the known result under the adversarial setting, one knows that for any \( \delta' > 0 \), with probability at least \( 1 - \delta' - T \exp(-C^2/2) \), the regret in the stochastic setting is bounded by

\[
O \left( C\sqrt{KT} \frac{\ln (K/\delta')}{\sqrt{\ln K}} \right).
\]

Now let \( \delta = \delta' + 2T \exp(-C^2/2) \), then with probability at least \( 1 - \delta \) the regret is bounded by

\[
O \left( \sqrt{\ln \left( \frac{T}{\delta - \delta'} \right)} \sqrt{KT} \frac{\ln (K/\delta')}{\sqrt{\ln K}} \right).
\]

However, such bound has a dependence of approximately \( \ln(1/\delta)^{3/2} \) on \( \delta \), which is sub-optimal.

### 4.3. From Fixed-time to Any-time

Finally, we enhance the policy design to accommodate the “any-time” setting where \( T \) is not known a priori, as a more challenging setting compared to the “fixed-time” setting where \( T \) is known a priori. We design a policy for the “any-time” setting and prove that the policy enjoys an equivalently desired exponential decaying tail and optimal expected regret as in the “fixed-time” setting. That is, our any-time policy enjoys a similar tail bound in Theorem 5. In the following, we use \( \text{rad}_t(n) \) to denote the bonus term at time \( t \).
Theorem 6. For the K-armed bandit problem, \( \pi = \text{UCB} \) with

\[
\operatorname{rad}_t(n) = \sigma \sqrt{\eta (1 \lor \ln(Kt)) / n\sqrt{K}}
\]

satisfies the following property: fix any \( \eta > 0 \), for any \( x > 0 \), we have

\[
\sup_{\theta} \mathbb{P}(\hat{R}_\theta^n(T) \geq x) \leq \exp\left(-\frac{x^2}{8K\sigma^2T}\right) + 2KT^2 \exp\left(-\frac{(x - 2K - 16\sigma\sqrt{2\eta KT\ln T})^2}{512\sigma^2KT}\right) + 2KT^3 \exp\left(-\frac{(x - 2K) + \sqrt{\eta\ln T}}{16\sigma\sqrt{KT}}\right).
\]

It is clear that for any \( \eta > 0 \), the policy in Theorem 6 always yields an expected regret of

\[
O(\sigma \sqrt{(\eta \lor 1/\eta)KT\ln T}).
\]

The reason is the same as that for Theorem 5. Another remark is that Theorem 6 only involves the UCB policy. In fact, the SE policy can always fail under an any-time bonus design. This is because SE will never pull an arm if this arm was eliminated previously. Therefore, even in the basic 2-armed setting, at the beginning when \( t \) is small compared to \( T \), the behaviour of SE with \( \operatorname{rad}_t(n) \) can be nearly as worse as that of SE with (1): it may eliminate the optimal arm with a probability heavy-tailed in \( T \). On the contrary, in UCB, arms are always activated, and so the gradually time-increasing nominator in the bonus term will take effect and prevents the optimal arm from being discarded forever.

The bonus design \( \operatorname{rad}_t(n) \) in Theorem 6 can be approximately regarded as replacing the \( T \) term in (6) with \( t \). We use \( 1 \lor \ln(Kt) \) instead of \( \ln t \) primarily out of convenience for analysis. The basic idea of proving Theorem 6 is also similar to that for Theorem 5, but requires more delicate formulas. The main challenge here stems from the unfixed \( t \) in the bonus term. In the proof of Theorem 5, in each event we consider, \( \Delta_k = \Omega(x/T) \). However, such lower bound may not be large enough, and the tail probability of wrongly discarding the optimal arm can only be bounded by

\[
\exp\left(-\Omega(\Delta_k \sqrt{t/K})\right),
\]

which is not a meaningful bound with an uncontrolled \( t \). Also, it is not clear whether a \( \sqrt{\ln T} \) term can be produced in the last term of the tail bound (the probability of wrongly discarding the optimal arm) under an any-time bonus design, which is essential to obtain an expected \( O(\sqrt{T\ln T}) \) regret bound. Both issues show that we need to rectify the set \( \mathcal{A}_0 \) and the formula (7) such that \( \Delta_k \) enjoys a possibly better bound depends on \( t_k \), and that \( \ln t_k \) is connected with \( \ln T \) in the analysis. This involves several tricks, as we will discuss next. Fix a time horizon of \( T \), we define \( t_k = t_k(n_{T,k}) \) as the last time period we pull arm \( k \), and define

\[
\mathcal{A}_1 = \left\{ k \neq 1 : n_k \leq 1 + \frac{t_k^{3/4}T^{1/4}}{K} \right\}
\]
to replace $A_0$, and instead of (7), we take
\[
P\left(R_0^\pi(T) \geq x(1-1/2\sqrt{K})\right) \\
= P\left(\sum_{k \in A_1} n_k \Delta_k + \sum_{k \notin A_1} n_k \Delta_k \geq x(1-1/2\sqrt{K})\right) \\
\leq P\left(\sum_{k \in A_1} (n_k-1) \Delta_k + \sum_{k \notin A_1} (n_k-1) \Delta_k \geq x(1-1/2\sqrt{K}) - K\right) \\
\leq P\left(\bigcup_{k \in A_1} \left\{ (n_k-1) \Delta_k \geq \frac{x-2K}{4K} \right\} \bigcup \bigcup_{k \notin A_1} \left\{ (n_k-1) \Delta_k \geq \frac{(n_k-1)(x-2K)}{4\sqrt{t_k T}} \right\}\right) \\
\leq \sum_{k \neq 1} \left( P\left((n_k-1) \Delta_k \geq \frac{x-2K}{4K}, k \in A_1\right) + P\left((n_k-1) \Delta_k \geq \frac{(n_k-1)(x-2K)}{4\sqrt{t_k T}}, k \notin A_1\right)\right) \\
\leq \sum_{k \neq 1} \left( P\left((n_k-1) \Delta_k \geq \frac{x-2K}{4K}, k \in A_1\right) + P\left(\Delta_k \geq \frac{x-2K}{4\sqrt{t_k T}}, k \notin A_1\right)\right)
\]

The correctness of the second inequality above stems from the fact that
\[
\sum_{k} n_k t_k \sqrt{t_k} = O(\sqrt{T}).
\]

The specific form in $A_1$ allows us to ensure $\Delta_k = \Omega(x/\sqrt{t_k T})$, and meanwhile derive the additional $\sqrt{\ln T}$ factor in the the last term of the tail bound. Details are all left to the supplementary material. We also need to stress that since $n_k$ and $t_k$ are both random variables, when bounding the probabilities, we must use a union bound to cover through all possible pairs $(n_k, t_k)$. This is the reason why we have an additional $T^2$ factor before the exponential tail.

5. Numerical Experiments

In this section, we provide numerical experiments results. We first consider a two armed-bandit problem with $\theta = (0.2, 0.8), \sigma = 1, T = 500$ and Gaussian noise. We test four policies: SE and UCB with the classical bonus design described in (1), and SE\_new and UCB\_new with the proposed new bonus design in (4). The tuning parameter has 4 choices: $\eta \in \{0.1, 0.2, 0.4, 0.8\}$. For each policy and $\eta$, we run 5000 simulation paths and for each path we collect the cumulative reward. We provide the empirical mean for the cumulative reward in Table 1. We also plot the empirical distribution (histogram) for a policy’s cumulative reward in Figure 1.

Table 1 shows that, SE\_new (or UCB\_new) achieves empirical mean for the cumulative reward as high as that SE (or UCB) can achieve. The highest empirical mean for the cumulative reward that can be achieved by SE\_new (or UCB\_new) with various choices of $\eta$ is comparable to the highest empirical mean that can be achieved by SE (or UCB). We note that there is no direct implication
by comparing the four different algorithms at the same value of $\eta$, because the algorithms use the parameter $\eta$ in different ways and in different format of the bonus term. For example, for some value of $\eta$, SE has a higher empirical mean for the cumulative reward compared to SE\textsubscript{new}, whereas for some other value of $\eta$, SE has a smaller empirical mean compared to SE\textsubscript{new}. There is no direct implication by fixing a value of $\eta$ and comparing different algorithms. However, Figure 1 shows that, compared to SE, SE\textsubscript{new} has much lower probability of incurring a low cumulative reward. The implication is that (i) in terms of the empirical mean of cumulative reward, SE\textsubscript{new} is as good as SE; (ii) in terms of the risk of incurring a low cumulative reward, SE\textsubscript{new} is much better (i.e., lower risk) than SE. The same implication holds analogously for the comparison between UCB\textsubscript{new} and UCB. Indeed, one can observe that for both SE and UCB with (1), there is a significant part of distribution around 100, indicating a significant risk of incurring a linear regret, especially when $\eta$ is small. In contrast, with the new design (4), the reward is highly concentrated for every $\eta > 0$ with almost no tail risk of getting a low total reward. Particularly, when $\eta = 0.1$ or $\eta = 0.2$, UCB\textsubscript{new} achieves both high empirical mean and light-tailed distribution.

![Figure 1: Empirical distribution for the cumulative reward; bottom two are new proposed policies](image)

Next, we consider a 4-armed bandit problem with $\theta = (0.2, 0.4, 0.6, 0.8), \sigma = 1, T = 500$ and Gaussian noise. We test four policies: SE and UCB with the classical bonus design described in (1), and SE\textsubscript{new} and UCB\textsubscript{new} with the proposed new bonus design in (4). The tuning parameter has 4 choices: $\eta \in \{0.1, 0.2, 0.4, 0.8\}$. For each policy and $\eta$, we run 5000 simulation paths and for each path we collect the cumulative reward. We plot the empirical distribution (histogram) for
a policy’s cumulative reward in Figure 2. We also report the empirical mean in Table 2. Indeed, one can observe that for both SE and UCB with (1), there is a significant part of distribution around 200 and 300, which means that with a non-negligible probability the two policies always pull arm 2 or 3, incurring a linear regret. Such phenomenon becomes more significant when $\eta$ is small. In contrast, with the new design (4), the reward is highly concentrated for every $\eta > 0$. Particularly, when $\eta = 0.1$, either SE new or UCB new achieves both high empirical mean and light-tailed distribution. When $\eta$ is relatively large, e.g., $\eta = 0.8$, the empirical mean is not very satisfactory. But this is consistent with our analysis in Theorem 4, which indicates an additional $\sqrt{K}$ factor compared to the optimal $\tilde{O}(\sqrt{KT})$ expected regret, if $\eta$ is not scaled by a factor of $\sqrt{K}$ as in (6).

### Table 1

| Policy | $\eta$ | 0.1   | 0.2   | 0.4   | 0.8   |
|--------|--------|-------|-------|-------|-------|
| SE     |        | 311.60| 336.46| 375.53| 374.69|
| UCB    |        | 349.68| 359.68| 377.17| 390.23|
| SE_new |        | 388.16| 376.69| 354.25| 309.58|
| UCB_new|        | 393.27| 387.48| 377.72| 360.69|

**Table 1** Empirical mean for the cumulative reward

### Figure 2

Empirical distribution for the cumulative reward; bottom two are new proposed policies

6. Conclusion

In this work, we consider the stochastic multi-armed bandit problem with a joint goal of minimizing the worst-case expected regret and obtaining light-tailed probability bound of the regret distribu-
We demonstrate that light-tailed risk and instance-dependent consistency are incompatible, and show that light-tailed risk and worst-case optimality can co-exist through a simple new policy design. We also discuss generalizations of our results and show how to achieve the optimal rate dependence on both the number of arms $K$ and the time horizon $T$ with or without knowing $T$.

A possible future direction is to generalize our results to other bandit settings such as the linear bandit problem. We suspect that such generalization may be highly non-trivial because the analysis here may not be directly applicable to the linear bandit setting. Note that in the stochastic multi-armed bandit problem, the feature vectors of different arms are orthogonal with one another, but in the general linear bandit setting, the estimation of the unknown parameter is entangled with uncontrollable arm feature vectors. It is also not clear how the equation (5) can be refined to accommodate the linear setting and prevent the appearance of $K$. Nevertheless, we hope our results and analysis in this paper may bring about new insights on understanding and alleviating the tail risk of learning algorithms under a stochastic environment.

References

Agrawal S, Goyal N (2012) Analysis of thompson sampling for the multi-armed bandit problem. Conference on learning theory, 39–1 (JMLR Workshop and Conference Proceedings).

Araman VF, Caldentey R (2021) Diffusion approximations for a class of sequential testing problems. arXiv preprint arXiv:2102.07030.

Ashutosh K, Nair J, Kagrecha A, Jagannathan K (2021) Bandit algorithms: Letting go of logarithmic regret for statistical robustness. International Conference on Artificial Intelligence and Statistics, 622–630 (PMLR).

Audibert JY, Munos R, Szepesvári C (2009) Exploration–exploitation tradeoff using variance estimates in multi-armed bandits. Theoretical Computer Science 410(19):1876–1902.

Auer P, Cesa-Bianchi N, Fischer P (2002) Finite-time analysis of the multiarmed bandit problem. Machine learning 47(2):235–256.

Bouchaud JP, Georges A (1990) Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. Physics reports 195(4-5):127–293.

Bouchaud JP, Potters M, et al. (2000) Theory of financial risks: from statistical physics to risk management (Cambridge University Press).
Simchi-Levi, Zheng and Zhu: Optimal Multi-armed Bandits Policies with Light-tailed Risk

Bubeck S, Cesa-Bianchi N (2012) Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *arXiv preprint arXiv:1204.5721*.

Cassel A, Mannor S, Zeevi A (2018) A general approach to multi-armed bandits under risk criteria. *Conference On Learning Theory*, 1295–1306 (PMLR).

Chopra S, Sodhi M (2004) Supply-chain breakdown. *MIT Sloan management review* 46(1):53–61.

Embrechts P, Klüppelberg C, Mikosch T (2013) *Modelling extremal events: for insurance and finance*, volume 33 (Springer Science & Business Media).

Even-Dar E, Mannor S, Mansour Y, Mahadevan S (2006) Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. *Journal of machine learning research* 7(6).

Fan L, Glynn PW (2021a) Diffusion approximations for thompson sampling. *arXiv preprint arXiv:2105.09232*.

Fan L, Glynn PW (2021b) The fragility of optimized bandit algorithms. *arXiv preprint arXiv:2109.13595*.

Galichet N, Sebag M, Teytaud O (2013) Exploration vs exploitation vs safety: Risk-aware multi-armed bandits. *Asian Conference on Machine Learning*, 245–260 (PMLR).

Kalvit A, Zeevi A (2021) A closer look at the worst-case behavior of multi-armed bandit algorithms. *Advances in Neural Information Processing Systems* 34.

Khajonchotpanya N, Xue Y, Rujeerapaiboon N (2021) A revised approach for risk-averse multi-armed bandits under cvar criterion. *Operations Research Letters* 49(4):465–472.

Lattimore T, Szepesvári C (2020) *Bandit algorithms* (Cambridge University Press).

Neu G (2015) Explore no more: Improved high-probability regret bounds for non-stochastic bandits. *Advances in Neural Information Processing Systems* 28.

Russo DJ, Van Roy B, Kazerouni A, Osband I, Wen Z, et al. (2018) A tutorial on thompson sampling. *Foundations and Trends® in Machine Learning* 11(1):1–96.

Salomon A, Audibert JY (2011) Deviations of stochastic bandit regret. *International Conference on Algorithmic Learning Theory*, 159–173 (Springer).

Sani A, Lazaric A, Munos R (2012) Risk-aversion in multi-armed bandits. *Advances in Neural Information Processing Systems* 25.

Slivkins A (2019) Introduction to multi-armed bandits. *arXiv preprint arXiv:1904.07272*.

Sodhi MS, Tang CS (2021) Supply chain management for extreme conditions: research opportunities. *Journal of Supply Chain Management* 57(1):7–16.

Vakili S, Zhao Q (2016) Risk-averse multi-armed bandit problems under mean-variance measure. *IEEE Journal of Selected Topics in Signal Processing* 10(6):1093–1111.

Wager S, Xu K (2021) Diffusion asymptotics for sequential experiments. *arXiv preprint arXiv:2101.09855*. 
Zimin A, Ibsen-Jensen R, Chatterjee K (2014) Generalized risk-aversion in stochastic multi-armed bandits. 

arXiv preprint arXiv:1405.0833.
Supplementary Material

Proof of Theorem 1.
We consider the environment where the noise \( \epsilon \) is gaussian with standard deviation \( \sigma \). Let \( \theta_1 = 1/2 \). Let \( \theta = (\theta_1, \theta_2) \) and \( \tilde{\theta} = (\theta_1, \tilde{\theta}_2) \), where \( \theta_2 = \theta_1 + \frac{1}{2} \) and \( \tilde{\theta}_2 = \theta_1 - \frac{1}{2} \). Let \( c' \in (c, 1/2) \). Define

\[
E_T = \left\{ |\hat{\mu}_{T,2} - \tilde{\theta}_2| \leq \delta \right\}
\]

where \( \delta > 0 \) is a small number, and

\[
F_T = \{ n_2 \leq f(T) \}.
\]

Here, \( f(T) > 0 \) is a strictly increasing function such that

\[
\limsup_{T} \frac{f(T)}{T} < 1 - 2c'.
\]

We will detail how \( f(T) \) should be chosen under different conditions in the last step of the proof. Then there exists \( T_0 \) such that \( f(T) < (1 - 2c')T \) for any \( T > T_0 \). Now we fix any \( T > T_0 \). Under the environment \( \tilde{\theta} \), we have

\[
\mathbb{P}_\theta^* (\tilde{F}_T) = \mathbb{P}_\theta^* (n_2 > f(T)) \leq \frac{\mathbb{E}_\theta^*[n_2]}{f(T)} \leq \frac{2\mathbb{E}[R^*_\theta(T)]}{f(T)} = \frac{2\mathbb{E}[\tilde{R}^*_\theta(T)]}{f(T)}.
\]

Combined with the weak law of large numbers, we have

\[
\liminf_{T} \mathbb{P}_\theta^* (E_T, F_T) \geq 1 - \limsup_{T} \frac{2\mathbb{E}[\tilde{R}^*_\theta(T)]}{f(T)}.
\]

(8)

Notice that

\[
\begin{align*}
\mathbb{P} \left( \tilde{R}^*_\theta(T) \geq cT \right) \\
\geq \mathbb{P} (R^*_\theta(T) \geq cT, N'_{\theta}(T) \geq -(c' - c)T) \\
= \mathbb{P} (R^*_\theta(T) \geq cT) - \mathbb{P} (R^*_\theta(T) \geq cT, N'_{\theta}(T) < -(c' - c)T) \\
\geq \mathbb{P} (R^*_\theta(T) \geq cT) - \mathbb{P} (N'_{\theta}(T) < -(c' - c)T) \\
\geq \mathbb{P} (R^*_\theta(T) \geq cT) - \exp \left( \frac{-(c' - c)^2T}{2\sigma^2} \right)
\end{align*}
\]

The last inequality holds from Lemma 1. Now

\[
\begin{align*}
\mathbb{P} (R^*_\theta(T) \geq cT) \\
\geq \mathbb{P}_\theta^* (n_1 \geq 2c'T) \\
\geq \mathbb{P}_\theta^* (n_2 \leq (1 - 2c')T) \\
\geq \mathbb{P}_\theta^* (n_2 \leq f(T)) \\
\geq \mathbb{P}_\theta^* (E_T, F_T) \\
= \mathbb{E}_\theta^* [\mathbb{I} \{ E_T, F_T \}] \\
= \mathbb{E}_\theta^* \left[ \exp \left( \sum_{n=1}^{n_2} \frac{(X_{12(n),2} - \tilde{\theta}_2)^2 - (X_{12(n),2} - \theta_2)^2}{2\sigma^2} \right) \mathbb{I} \{ E_T, F_T \} \right]
\end{align*}
\]
Then from (9), we have

\[ f(T) = \frac{\exp\left(\frac{\hat{\theta}_2^2 - \theta_2^2}{2\sigma^2} + \frac{(\theta_2 - \hat{\theta}_2)\delta}{\sigma^2}\right)}{\exp\left(\frac{\hat{\theta}_2^2 - \theta_2^2}{2\sigma^2} + \frac{(\theta_2 - \hat{\theta}_2)(\delta - \hat{\delta})}{\sigma^2}\right)} \cdot \text{I}\left\{E_T, F_T\right\} \]

Therefore,

\[
\liminf_T \frac{\ln\left\{\sup_{\theta'} \mathbb{P}\left(\hat{R}_{\theta'}^*(T) \geq cT\right)\right\}}{f(T)} \geq \liminf_T \frac{\ln\left\{\exp(-f(T)(1/2\sigma^2 + \delta/\sigma^2))\mathbb{P}_\theta^a(E_T, F_T) - \exp\left(-\frac{(\hat{\theta}_2 - \theta_2)^2}{2\sigma^2}\right)\right\}}{f(T)}.
\]

Now assume that \( \pi \) is consistent. Then we set \( f(T) = T^{\beta} \) with \( \beta \in (0, 1) \). From (8), we have

\[
\liminf_T \mathbb{P}_\theta^a(E_T, F_T) \geq 1 - \limsup_T \frac{2\mathbb{E}[\hat{R}_{\theta}^*(T)]}{T^{\beta}} = 1.
\]

Then from (9), we have

\[
\liminf_T \frac{\ln\left\{\sup_{\theta'} \mathbb{P}(\hat{R}_{\theta'}^*(T) \geq cT)\right\}}{T^{\beta}} \geq -(1/2\sigma^2 + \delta/\sigma^2).
\]

Since \( \delta > 0 \) is arbitrary, we have

\[
\liminf_T \frac{\ln\left\{\sup_{\theta'} \mathbb{P}(\hat{R}_{\theta'}^*(T) \geq cT)\right\}}{T^{\beta}} \geq -1/2\sigma^2.
\]

Note again that \( \beta > 0 \) is arbitrary. Now let \( 0 < \beta' < \beta \), we have

\[
\liminf_T \frac{\ln\left\{\sup_{\theta'} \mathbb{P}(\hat{R}_{\theta'}^*(T) \geq cT)\right\}}{T^{\beta'}} = \liminf_T \frac{\ln\left\{\sup_{\theta'} \mathbb{P}(\hat{R}_{\theta'}^*(T) \geq cT)\right\}}{T^{\beta'}} \cdot \liminf_T T^{\beta' - \beta} \geq -1/2\sigma^2 \cdot 0 = 0.
\]

Now assume that \( \pi \) satisfies

\[
\limsup_T \frac{\mathbb{E}[\hat{R}_{\theta}^*(T)]}{\ln T} = c_n \sigma^2 < +\infty.
\]

for any \( \theta \). Note that when \( \pi = \text{SE} \) and \( \pi = \text{UCB} \) with the bonus term (1), the property above always holds.

Let \( f(T) = 4c_n \sigma^2 \ln T \). From (8), we have

\[
\liminf_T \mathbb{P}_\theta^a(E_T, F_T) \geq 1 - \limsup_T \frac{2\mathbb{E}[\hat{R}_{\theta}^*(T)]}{4c_n \sigma^2 \ln T} = 1/2.
\]

Then from (9), we have

\[
\liminf_T \frac{\ln\left\{\sup_{\theta'} \mathbb{P}(\hat{R}_{\theta'}^*(T) \geq cT)\right\}}{4c_n \sigma^2 \ln T} \geq -(1/2\sigma^2 + \delta/\sigma^2).
\]
Since $\delta > 0$ is arbitrary, we have
\[
\liminf_t \frac{\ln \left\{ \sup_{\theta'} \mathbb{P}(\hat{R}_\theta^*(T) \geq cT) \right\}}{4c_\pi \sigma^2 \ln T} \geq -1/2\sigma^2.
\]
Let $C_\pi = 2c_\pi$, we have
\[
\liminf_t \frac{\ln \left\{ \sup_{\theta'} \mathbb{P}(\hat{R}_\theta^*(T) \geq cT) \right\}}{\ln T} \geq -\frac{C_\pi}{\sigma^2}.
\]
\[\square\]

**Proof of Theorem 2.** Without loss of generality, we assume $\theta_1 > \theta_2$. We prove the results one by one. Since the environment $\theta$ is fixed, we will write $\mathbb{P}(E)$ instead of $\mathbb{P}_\theta(\pi, E)$.

1. From Lemma 1,
\[
E[\hat{R}_\theta^*(T)] = E[R_\theta^*(T)] = E[n_2] \cdot \Delta_2.
\]
Let $G$ be the event such that
\[
G = \{ \mu_k \in \text{CI}_{t,k}, \ \forall (t,k) \}.
\]
Then
\[
\mathbb{P}(\bar{G}) \leq \sum_{(t,k)} \mathbb{P}(\mu_k \notin \text{CI}_{t,k}) \leq 2 \sum_{n=1}^T 2 \exp(-2\frac{\eta T \ln T}{n}) \leq 4T^{1-2\eta}.
\]
Thus,
\[
E[n_2] = E[n_2 | G] \mathbb{P}(G) + E[n_2 | \bar{G}] \mathbb{P}(\bar{G}) \leq E[n_2 | G] + T \cdot \mathbb{P}(\bar{G}) \leq E[n_2 | G] + 4T^{2-2\eta} \leq E[n_2 | G] + 4.
\]

With a slight abuse of notation, we let $t$ be the largest time period such that arm 2 is pulled but subsequently not eliminated from $A$. Then under $G$, we have
\[
\mu_1 - 2\sigma \sqrt{\eta T \ln T} n_{t,2} - 1 \leq \mu_1 - 2\sigma \sqrt{\eta T \ln T} n_{t,1} \leq \hat{\mu}_{t,1} - \sigma \frac{\sqrt{\eta T \ln T}}{n_{t,1}} \leq \hat{\mu}_{t,2} + \sigma \frac{\sqrt{\eta T \ln T}}{n_{t,2} - 1} \leq \mu_2 + 2\sigma \frac{\sqrt{\eta T \ln T}}{n_{t,2} - 1}.
\]
Therefore,
\[
n_{t,2} \leq 1 + 4\sigma \frac{\sqrt{\eta T \ln T}}{\Delta_2}
\]
and thus,
\[
n_2 \leq 2 + 4\sigma \frac{\sqrt{\eta T \ln T}}{\Delta_2}.
\]
As a result,
\[
E[R_\theta^*(T)] \leq 2\Delta_2 + 4\sigma \sqrt{\eta T \ln T} + 4 = O(\sqrt{T \ln T}).
\]
2. We have
\[ \mathbb{P}(\hat{R}_\theta(T) \geq cT^\alpha) \leq \mathbb{P}(R_\theta(T) \geq cT^\alpha/2) + \mathbb{P}(N_\theta(T) \leq -cT^\alpha/2) \]

From Lemma 1, the second term can be bounded as
\[ \mathbb{P}(N_\theta(T) \leq -cT^\alpha/2) \leq \exp\left( -\frac{c^2T^{2\alpha}}{2\sigma^2T} \right) = \exp\left( -\frac{c^2T^{2\alpha-1}}{2\sigma^2} \right). \tag{10} \]

We are left to bound \( \mathbb{P}(R_\theta(T) \geq cT^\alpha/2) \). Let \( S \) be the event defined as
\[ S = \{ \text{Arm 1 is never eliminated throughout the whole time horizon} \} \]

Then
\[ \bar{S} = \{ \exists t \text{ such that arm 1 is eliminated at time } t \} \]

So
\[ \mathbb{P}(R_\theta(T) \geq cT^\alpha/2) = \mathbb{P}(R_\theta(T) \geq cT^\alpha/2, S) + \mathbb{P}(R_\theta(T) \geq cT^\alpha/2, \bar{S}) \]

Let \( T \) be such that
\[ cT^\alpha \geq \max\{4, 16\sigma\sqrt{T\ln T}\} \]

Let \( n_0 = \left[ cT^\alpha/2\Delta_2 \right] - 1 \), then
\[ n_0 \geq cT^\alpha/4\Delta_2. \]

Also, if \( R_\theta(T) \geq cT^\alpha/2 \), we must have
\[ T \geq cT^\alpha/2\Delta_2, \]

which means \( \Delta_2 \geq cT^{\alpha-1}/2 \). We have
\[ \mathbb{P}(R_\theta(T) \geq cT^\alpha/2, S) \]
\[ = \mathbb{P}(n_2 \geq cT^\alpha/2\Delta_2, S) \]
\[ \leq \mathbb{P}(n_2 \geq n_0 + 1, \text{arm 1 and 2 are pulled in turn for } (n_0 + 1) \text{ times}) \]
\[ \leq \mathbb{P}(\text{arm 1 and 2 are pulled in turn for } n_0 \text{ times and arm 1 and 2 are both not eliminated}) \]
\[ \leq \mathbb{P}\left( \mu_1 - \frac{\sigma\sqrt{T\ln T}}{n_0} \leq \mu_2 + \frac{\sigma\sqrt{T\ln T}}{n_0}, \sum_{m=1}^{n_0} \epsilon_{f_1(m),1} + \frac{\sigma\sqrt{T\ln T}}{n_0} \geq \Delta_2 \right) \]
\[ = \mathbb{P}\left( \mu_1 - \frac{\sum_{m=1}^{n_0} \epsilon_{f_1(m),1} + \sigma\sqrt{T\ln T}}{n_0} \leq \mu_2 + \frac{\sum_{m=1}^{n_0} \epsilon_{f_2(m),2} + \sigma\sqrt{T\ln T}}{n_0} \right) \]
\[ \leq \mathbb{P}\left( \sum_{m=1}^{n_0} \epsilon_{f_1(m),1} \geq \frac{\Delta_2}{2} - \frac{\sigma\sqrt{T\ln T}}{n_0} \right) + \mathbb{P}\left( \sum_{m=1}^{n_0} \epsilon_{f_2(m),2} \geq \frac{\Delta_2}{2} - \frac{\sigma\sqrt{T\ln T}}{n_0} \right) \]
\[ \leq 2 \exp\left( -n_0 \left( \frac{\Delta_2}{2} - \frac{\sigma\sqrt{T\ln T}}{n_0} \right)^2 / 2\sigma^2 \right) \]
\[= 2 \exp \left( -n_0 \Delta_2^2 \left( 1 - \frac{8 \sigma \sqrt{nT \ln T}}{cT^\alpha} \right)^2 / 2\sigma^2 \right)\]
\[\leq 2 \exp \left( -n_0 \Delta_2^2 / 8\sigma^2 \right)\]
\[\leq 2 \exp \left( -cT^\alpha \cdot cT^{\alpha-1} / 128\sigma^2 \right)\]
\[= 2 \exp \left( -c^2 T^{2\alpha-1} / 128\sigma^2 \right). \quad (11)\]

Meanwhile,
\[
P(R^\alpha(T) \geq cT^\alpha / 2, \tilde{\delta})
\leq P \left( \exists n \leq T/2 : \hat{\mu}_{t_1(n),1} + \frac{\sigma \sqrt{nT \ln T}}{n} < \hat{\mu}_{t_2(n),2} - \frac{\sigma \sqrt{nT \ln T}}{n} \right)
\leq P \left( \exists n \leq T/2 : \mu_1 + \frac{\sum_{m=1}^n \epsilon_{t_1(m),1} \sigma \sqrt{nT \ln T}}{n} < \mu_2 + \frac{\sum_{m=1}^n \epsilon_{t_2(m),2} \sigma \sqrt{nT \ln T}}{n} \right)
\leq \sum_{n=1}^{\lfloor T/2 \rfloor} P \left( \frac{\sum_{m=1}^n \epsilon_{t_2(m),2} - \epsilon_{t_1(m),1}}{n} > \Delta_2 + \frac{2\sigma \sqrt{nT \ln T}}{n} \right)
\leq \sum_{n=1}^{\lfloor T/2 \rfloor} \left( P \left( \sum_{m=1}^n \epsilon_{t_2(m),2} > \Delta_2 + \frac{2\sigma \sqrt{nT \ln T}}{n} \right) + P \left( \sum_{m=1}^n \epsilon_{t_1(m),1} > \Delta_2 + \frac{\sigma \sqrt{nT \ln T}}{n} \right) \right)
\leq 2 \sum_{n=1}^{\lfloor T/2 \rfloor} \exp \left( -n \left( \frac{\Delta_2}{2} + \frac{\sigma \sqrt{nT \ln T}}{n} \right)^2 / 2\sigma^2 \right)
\leq T \exp \left( -2n\Delta_2 \frac{\sigma \sqrt{nT \ln T}}{n} / 2\sigma^2 \right)
\leq T \exp(-\sigma \cdot cT^{\alpha-1} \cdot \sqrt{n \ln T} / 2\sigma^2)
= \exp \left( -cT^{\alpha-1/2} \sqrt{\ln T - \sigma \ln T} / 4\sigma \right)
\leq \exp \left( -cT^{\alpha-1/2} / 16\sigma \right). \quad (12)\]

Note that the equations above hold for any instance \( \theta \). Combining (10), (11), (12) yields
\[
\sup_{\theta} P(R^\alpha_{\theta}(T) \geq cT^\alpha) \leq 4 \exp \left( \frac{cT^{\alpha-1/2}}{16\sigma} \right).
\]

**Proof of Theorem 3.**

We consider the environment where the noise \( \epsilon \) is Gaussian with standard deviation \( \sigma \). Fix any \( \alpha > 1/2 \). Let \( \theta_1 = 1/2 \). Let \( \theta(T) = (\theta_1, \theta_2(T)) \) and \( \tilde{\theta}(T) = (\theta_1, \tilde{\theta}_2(T)) \), where \( \theta_2(T) = \theta_1 + \frac{1}{2T^{1-\alpha}} \sigma \) and \( \tilde{\theta}_2(T) = \theta_1 - \frac{1}{2T^{1-\alpha}} \sigma \).

Let \( \gamma \in (3/2 - \alpha, \min\{1, \beta + 2 - 2\alpha\}) \).

Such \( \gamma \) always exists because \( \beta + 2 - 2\alpha > \alpha - 1/2 + 2 - 2\alpha = 3/2 - \alpha \) and \( 3/2 - \alpha < 3/2 - 1/2 = 1 \). For notation simplicity, we will write \( \theta \) (\( \tilde{\theta} \)) instead of \( \theta(T) \) (\( \tilde{\theta}(T) \)), but we must keep in mind that \( \theta \) (\( \tilde{\theta} \)) is dependent on \( T \). Define
\[
E_T = \left\{ |\hat{\mu}_{T,2} - \tilde{\theta}_2| \leq \delta \right\}
\]
where $\delta > 0$ is a small number, and

$$F_T = \{n_2 \leq T^\gamma\}.$$  

Then under the environment $\hat{\theta}$, we have

$$\Pr(\hat{R}_n^\gamma(T) = \hat{R}_n^\gamma(n_2 > T^\gamma)) \leq \frac{\E_{\theta}[n_2]}{T^\gamma} \leq E[\hat{R}_n^\gamma(T)] \leq \sup_{\theta^*} E[\hat{R}_n^\gamma(T)] \to 0$$

as $T \to +\infty$. Combined with the weak law of large numbers, we have

$$\lim inf_{\nu} \Pr_{\theta^*}(E_T, F_T) = 1.$$  

Let $c' \in (c, 1/2)$. There exists $T_0$ such that $(1 - 2c')T > T^\gamma$ for any $T > T_0$. Fix $T > T_0$. Notice that

$$\Pr(\hat{R}_n^\gamma(T) \geq c'T^\alpha)$$

$$\geq \Pr(R_n^\gamma(T) \geq c'T^\alpha, N^\alpha(T) \geq -(c' - c)T^\alpha)$$

$$= \Pr(R_n^\gamma(T) \geq c'T^\alpha) - \Pr(R_n^\gamma(T) \geq c'T^\alpha, N^\alpha(T) < -(c' - c)T^\alpha)$$

$$\geq \Pr(R_n^\gamma(T) \geq c'T^\alpha) - \Pr(N^\alpha(T) < -(c' - c)T^\alpha)$$

$$\geq \Pr(R_n^\gamma(T) \geq c'T^\alpha) - \exp \left(-\frac{(c' - c)^2T^{2\alpha - 1}}{2\sigma^2}\right)$$

The last inequality holds from Lemma 1. Now

$$\Pr(R_n^\gamma(T) \geq c'T^\alpha)$$

$$\geq \Pr(n_1 \geq 2c'T)$$

$$\geq \Pr(n_2 \leq (1 - 2c')T)$$

$$\geq \Pr(n_2 \leq T^\gamma)$$

$$\geq \Pr(E_T, F_T)$$

$$= \E_{\theta^*} \left\{ \delta \{E_T, F_T\} \right\}$$

$$= \E_{\theta^*} \left[ \exp \left( \frac{\sum_{n=1}^{n_2} (X_{12(n), 2} - \tilde{\theta}_2)^2 - (X_{12(n), 2} - \tilde{\theta}_2)^2}{2\sigma^2} \right) \delta \{E_T, F_T\} \right]\}

$$= \E_{\theta^*} \left[ \exp \left( \frac{n_2 \left( \tilde{\theta}_2 - \tilde{\theta}_2 \right)^2 + (\theta_2 - \tilde{\theta}_2)\tilde{\theta}_T, 2}{\sigma^2} \right) \delta \{E_T, F_T\} \right]\}

$$\geq \E_{\theta^*} \left[ \exp \left( n_2 \left( \tilde{\theta}_2 - \tilde{\theta}_2 \right)^2 + (\theta_2 - \tilde{\theta}_2)(\tilde{\theta}_2 - \delta)\right) \delta \{E_T, F_T\} \right]\}

$$= \E_{\theta^*} \left[ \exp \left( n_2 \left( \tilde{\theta}_2 - \tilde{\theta}_2 \right)^2 + (\theta_2 - \tilde{\theta}_2)^2 \right) \delta \{E_T, F_T\} \right]\}

$$= \exp \left( (\tilde{\theta}_2 - \tilde{\theta}_2)^2 \right) \delta \{E_T, F_T\}$$

Notice that

$$\gamma + 2\alpha - 2 < 2\alpha - 1, \quad \gamma + 2\alpha - 2 \leq \gamma + \alpha - 1,$$
and $\delta > 0$ can be arbitrary. Therefore,

$$\liminf_T \frac{\ln \left\{ \sup_{\theta} \mathbb{P}(\hat{R}_\theta^\pi(T) \geq cT^\alpha) \right\}}{T^\beta} \geq \liminf_T \frac{-T^{\gamma+2\alpha-2}/2\sigma^2}{T^\beta} = 0.$$ 

Since $\ln \left\{ \sup_{\theta} \mathbb{P}(R_\theta^\pi(T) \geq cT^\gamma) \right\} \leq 0$ always holds, we obtain the result. \hfill \qed

**Proof of Theorem 4.** Without loss of generality, we assume $\theta_1 = \theta_*$. We prove the results one by one.

1. From Lemma 1,

$$\mathbb{E}[R_\theta^\pi] = \mathbb{E}[\hat{R}_\theta^\pi] = \sum_{k=2}^K \mathbb{E}[n_k] \cdot \Delta_k.$$ 

Let $G$ be the event such that

$$G = \{\mu_k \in \text{CI}_{t,k}, \forall (t,k)\}.$$ 

Then

$$\mathbb{P}(\bar{G}) \leq \sum_{(t,k)} \mathbb{P}(\mu_k \notin \text{CI}_{t,k}) \leq K \sum_{n=1}^T 2 \exp\left(-\frac{\eta T \ln T}{2n}\right) \leq 2KT^{1-\eta/2}.$$ 

Thus,

$$\mathbb{E}[R_\theta^\pi] = \sum_{k=2}^K (\mathbb{E}[n_k|G] \mathbb{P}(G) + \mathbb{E}[n_k|\bar{G}] \mathbb{P}(\bar{G})) \Delta_k$$

$$\leq \sum_{k=2}^K \mathbb{E}[n_k|G] + T \cdot \mathbb{P}(\bar{G})$$

$$\leq \sum_{k=2}^K \mathbb{E}[n_k|G] + 2KT^{2-\eta/2}$$

$$\leq \sum_{k=2}^K \mathbb{E}[n_k|G] + 2K.$$ 

(a) Let $\pi = \text{SE}$. Fix any arm $k \neq 1$. We let $t'_k$ be the largest time period such that we have traversed all the arms in $A$, and meanwhile arm $k$ is not eliminated from $A$. Then $n_k = n_{t'_k,k} + 1$. When doing the elimination after $t_k$, arm 1 and $k$ are both pulled $n_{t'_k,k}$ times. Under $G$, we have

$$\mu_1 - 2\sigma \frac{\sqrt{\eta T \ln T}}{n_{t'_k,k}} \leq \mu_1 - 2\sigma \frac{\sqrt{\eta T \ln T}}{n_{t'_k,k}} \leq \hat{\mu}_{t'_k,1} - \sigma \frac{\sqrt{\eta T \ln T}}{n_{t'_k,1}} \leq \hat{\mu}_{t'_k,k} + \sigma \frac{\sqrt{\eta T \ln T}}{n_{t'_k,k}} \leq \mu_k + 2\sigma \frac{\sqrt{\eta T \ln T}}{n_{t'_k,k}}.$$ 

Therefore,

$$n_{t'_k,k} \leq 1 + 4\sigma \frac{\sqrt{\eta T \ln T}}{\Delta_k}$$

and thus,

$$n_k \leq 2 + 4\sigma \frac{\sqrt{\eta T \ln T}}{\Delta_k}.$$ 

As a result,

$$\mathbb{E}[\hat{R}_\theta^\pi(T)] \leq 2 \sum_{k=2}^K \Delta_k + 4 \sum_{k=2}^K \sigma \frac{\sqrt{\eta T \ln T}}{\Delta_k} + 2K \leq 4K + 4K \sigma \sqrt{\eta T \ln T}.$$
(b) Let \( \pi = \text{UCB} \). Fix any arm \( k \neq 1 \). We let \( t_k \) be the largest time period such that arm \( k \) is pulled. Then \( n_k = n_{t_k,k} = n_{t_k-1,k} + 1 \). Under \( G \), we have
\[
\mu_1 \leq \hat{\mu}_{t_k-1,1} + \sigma \frac{\sqrt{\eta T \ln T}}{n_{t_k-1,1}} \leq \hat{\mu}_{t_k-1,k} + \sigma \frac{\sqrt{\eta T \ln T}}{n_{t_k-1,k}} \leq \mu_k + 2\sigma \frac{\sqrt{\eta T \ln T}}{n_{t_k-1,k}}
\]
Therefore,
\[
n_{t_k-1,k} \leq 2\sigma \frac{\sqrt{\eta T \ln T}}{\Delta_k}
\]
and thus,
\[
n_k \leq 1 + 2\sigma \frac{\sqrt{\eta T \ln T}}{\Delta_k}.
\]
As a result,
\[
\mathbb{E}[\tilde{R}_{\theta}(T)] \leq \sum_{k=2}^{K} \Delta_k + 2 \sum_{k=2}^{K} \sigma \sqrt{\eta T \ln T} + 2K \leq 3K + 2K \sigma \sqrt{\eta T \ln T}.
\]
2. We have
\[
P(\tilde{R}_{\theta}(T) \geq x) \leq 2 \sum_{k=2}^{K} \mathbb{P}(n_k \Delta_k \geq x/(K + \sqrt{K})) + \mathbb{P}(N_{\theta}^*(T) \leq -x/\sqrt{K})
\]
From Lemma 1, the second term can be bounded as
\[
\mathbb{P}(N_{\theta}^*(T) \leq -x/K) \leq \exp\left( -\frac{x^2}{2K\sigma^2T} \right).
\]
We are left to bound \( \mathbb{P}(R_{\theta}^*(T) \geq x(1 - 1/\sqrt{K})) \).

(a) Let \( \pi = \text{SE} \). For any \( k \neq 1 \), let \( S_k \) be the event defined as
\[
S_k = \{ \text{Arm 1 is not eliminated before arm } k \}.
\]
Then
\[
\tilde{S}_k = \{ \text{Arm 1 is eliminated before arm } k \}.
\]
So
\[
P(R_{\theta}^*(T) \geq x(1 - 1/\sqrt{K}))
\]
\[
\leq \sum_{k=2}^{K} \mathbb{P}(n_k \Delta_k \geq x/(K + \sqrt{K}))
\]
\[
= \sum_{k=2}^{K} \mathbb{P}(n_k \Delta_k \geq x/2K, S_k) + \mathbb{P}(n_k \Delta_k \geq x/2K, \tilde{S}_k).
\]
Let \( x > 0 \). Fix any \( k \neq 1 \). With a slight abuse of notation, we let \( n_0 = \lceil x/2K\Delta_k \rceil - 1 \), then
\[
n_0 \geq x/2K\Delta_k - 1 \geq (x - 2K)/2K\Delta_k.
\]
Also, if \( n_k \Delta_k \geq x/2K \), we must have
\[
T \geq x/2K\Delta_k,
\]
which means $\Delta_k \geq x/2KT$. By the definition of $n_0$, arm $k$ is not eliminated after being pulled $n_0$ times. So under $S_k$, after arm 1 being pulled $n_0$ times, it is still in the active set. We have

$$P(n_k \Delta_k \geq x/2K, S_k)$$

$$= P(n_k \geq x/2K \Delta_k, S_k)$$

$$\leq P(\operatorname{arm} 1 \text{ and } k \text{ are both not eliminated after each of them being pulling } n_0 \text{ times})$$

$$\leq P \left( \hat{\mu}_{1(n_0)} + \frac{\sigma \sqrt{\eta T \ln T}}{n_0} \leq \hat{\mu}_{k(n_0,k)} + \frac{\sigma \sqrt{\eta T \ln T}}{n_0} \right)$$

$$= P \left( \mu_1 - \sum_{m=1}^{n_0} \epsilon_{1(m),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n_0} \leq \mu_k + \sum_{m=1}^{n_0} \epsilon_{k(m),k} + \frac{\sigma \sqrt{\eta T \ln T}}{n_0} \right)$$

$$\leq \sum_{m=1}^{n_0} \left( \epsilon_{1(m),1} - \epsilon_{k(m),k} \right) \geq \Delta_k - \frac{2\sigma \sqrt{\eta T \ln T}}{n_0}$$

$$\leq 2 \exp \left( -n_0 \left( \frac{\Delta_k}{2} - \frac{\sigma \sqrt{\eta T \ln T}}{n_0} \right)^2 / 2\sigma^2 \right)$$

$$= 2 \exp \left( -n_0 \Delta_k^2 \left( 1 - 2\frac{\sigma \sqrt{\eta T \ln T}}{n_0 \Delta_k} \right)^2 / 8\sigma^2 \right)$$

$$\leq 2 \exp \left( -\frac{(x - 2K)^+}{4KT} \left( 1 - 4K^2 \frac{\sigma \sqrt{\eta T \ln T}}{x - 2K} \right)^2 / 8\sigma^2 \right)$$

$$\leq 2 \exp \left( -\frac{(x - 2K)^+}{32\sigma^2 K^2 T} \right) \cdot (14)$$

In the following, we bound $P(n_k \Delta_k \geq x/2K, \bar{S}_k)$. Suppose that after $n$ phases, arm 1 is eliminated by arm $k'$ ($k'$ is not necessarily $k$). By the definition of $\bar{S}_k$, arm $k$ is not eliminated. Therefore, we have

$$\hat{\mu}_{k,(n),k'} - \frac{\sigma \sqrt{\eta T \ln T}}{n} \geq \hat{\mu}_{1(n),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n}, \ \ \hat{\mu}_{k(n),k} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \geq \hat{\mu}_{1(n),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \tag{15}$$

holds simultaneously. The first inequality holds because arm 1 is eliminated. The second inequality holds because arm $k$ is not eliminated. Now for fixed $n$,

$$P \left( \text{(15) happens; } \Delta_k \geq \frac{x}{2KT} \right)$$

$$\leq \sum_{k' \neq 1} \left( P \left( \sum_{m=1}^{n_0} \epsilon_{k(m),k'} \geq \frac{\sigma \sqrt{\eta T \ln T}}{n} \right) + P \left( \sum_{m=1}^{n_0} \epsilon_{k(m),1} \geq -\frac{\sigma \sqrt{\eta T \ln T}}{n} \right) \right)$$

$$= \sum_{k' \neq 1} \left( P \left( \sum_{m=1}^{n_0} \epsilon_{k(m),k'} \geq \frac{x}{4KT} \right) + P \left( \sum_{m=1}^{n_0} \epsilon_{k(m),1} \geq -\frac{x}{4KT} \right) \right)$$
\[ \leq 2K \exp \left( -\frac{\eta T \ln T}{2n} \right)^2 \times 2K \exp \left( -\frac{n x^2}{32 \sigma^2 K^2 T^2} \right) \]
\[ = 2K \exp \left( -\frac{\eta T \ln T}{2n} \vee \frac{n x^2}{32 \sigma^2 K^2 T^2} \right) \]
\[ \leq 2K \exp \left( -\frac{x \sqrt{\eta \ln T}}{8 \sigma K \sqrt{T}} \right) \]

Therefore,
\[ \mathbb{P}(n_k \Delta_k \geq x/2K, S) \]
\[ = \mathbb{P}(\exists n \leq T/2 : (15) \text{ happens}; n_k \Delta_k \geq x/2K) \]
\[ \leq \sum_{n=1}^{\lfloor T/2 \rfloor} \mathbb{P} \left( (15) \text{ happens}; \Delta_k \geq \frac{x}{2KT} \right) \]
\[ \leq KT \exp \left( -\frac{x \sqrt{\eta \ln T}}{8 \sigma K \sqrt{T}} \right). \tag{16} \]

Note that the equations above hold for any instance \( \theta \). Combining (13), (14), (16) yields
\[ \sup_{\theta} \mathbb{P}(\tilde{R}_\theta^*(T) \geq x) \]
\[ \leq \exp \left( -\frac{x^2}{2K \sigma^2 T} \right) + 2K \exp \left( -\frac{(x - 2K - 4K \sigma \sqrt{\eta T \ln T})^2}{32 \sigma^2 K^2 T^2} \right) \]
\[ + K^2 T \exp \left( -\frac{x \sqrt{\eta \ln T}}{8 \sigma K \sqrt{T}} \right) \]

(b) Let \( \pi = \text{UCB} \). From (a), we know that
\[ \mathbb{P} \left( R_\theta^*(T) \geq x(1 - 1/\sqrt{K}) \right) \leq \sum_{k=2}^{K} \mathbb{P}(n_k \Delta_k \geq x/(K + \sqrt{K})) \leq \sum_{k=2}^{K} \mathbb{P}(n_k \Delta_k \geq x/2K). \]

Let \( x > 0 \). Fix \( k \neq 1 \). With a slight abuse of notation, we let \( n_0 = \lfloor x/2K \Delta_k \rfloor - 1 \). Remember that \( t_k(n_0 + 1) \) is the time period that arm \( k \) is pulled for the \( (n_0 + 1) \)th time. We emphasize again that \( \Delta_k \geq x/(2KT) \). Then
\[ \mathbb{P}(n_k \Delta_k \geq x/2K) \]
\[ = \mathbb{P}(n_k \geq x/2K \Delta_k) \]
\[ \leq \mathbb{P} \left( \sum_{m=1}^{n_0} \epsilon_{t_k(m),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n_k(n_0 + 1) - 1} \right) \]
\[ \leq \mathbb{P} \left( \sum_{m=1}^{n_0} \epsilon_{t_k(m),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \right) \geq \Delta_k \]
\[ \leq \mathbb{P} \left( \sum_{m=1}^{n_0} \epsilon_{t_k(m),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \right) \geq \Delta_k \]
\[ \leq \mathbb{P} \left( \sum_{m=1}^{n_0} \epsilon_{t_k(m),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \right) \geq \frac{\Delta_k}{2} \]
\[ + \mathbb{P} \left( \exists n \in [T] : \sum_{m=1}^{n} \epsilon_{t_k(m),1} + \frac{\sigma \sqrt{\eta T \ln T}}{n} \right) \geq \frac{\Delta_k}{2} \]
\[ \leq \exp \left( -\frac{(x - 2K - 4K \sigma \sqrt{\eta T \ln T})^2}{32 \sigma^2 K^2 T} \right) \times T \exp \left( -\frac{x \sqrt{\eta \ln T}}{2 \sigma K T} \right). \tag{17} \]
The last inequality holds from (14) and concentration of subgaussian variables. Note that the equations above hold for any instance \(\theta\). Combining (13), (17) yields
\[
\sup_{\theta} \mathbb{P}(\hat{R}_{\theta}^{\pi}(T) \geq x) \\
\leq \exp\left(- \frac{x^2}{2K\sigma^2T}\right) + K \exp\left(- \frac{(x - 2K - 4K\sqrt{\eta T \ln T})^2}{32\sigma^2KT}\right) + K^2T \exp\left(- \frac{x\sqrt{\eta T \ln T}}{8\sigma KT}\right).
\]
\(\square\)

**Remark:** SE or UCB with (1) may lead to a sub-optimal regret when \(\eta\) is too small.

Let \(\theta = (1, 0)\) and \(\sigma = 1\) with independent Gaussian noise. We first consider \(\pi = \text{SE}\). The probability that arm 1 is eliminated after the first phase is
\[
\mathbb{P}\left(\bar{\mu}_{1,1} + \sqrt{\eta \ln T} < \bar{\mu}_{2,2} - \sqrt{\eta \ln T}\right) \\
= \mathbb{P}\left(\epsilon_{1,1} - \epsilon_{2,2} < -1 - 2\sqrt{\eta \ln T}\right) \\
\geq \frac{1}{\sqrt{2}\pi} \frac{1}{1 + (1/\sqrt{2} + \sqrt{2}\eta \ln T)^2} \exp\left(- (1/\sqrt{2} + \sqrt{2}\eta \ln T)^2/2\right) \\
= \Theta\left(\frac{T^{-\eta}}{\sqrt{\ln T}}\right).
\]
The inequality holds because for a standard normal variable \(X\), it is established that
\[
\mathbb{P}(X > t) \geq \frac{1}{\sqrt{2}\pi} \frac{t}{1 + t^2} \exp(-t^2/2)
\]
Therefore, the expected regret is at least
\[
\Theta\left(\frac{T^{-\eta}}{\sqrt{\ln T}}\right) \cdot (T - 2) = \Theta\left(\frac{T^{1-\eta}}{\sqrt{\ln T}}\right).
\]
If \(\eta\) is very small, then apparently the regret is sub-optimal.

Now we consider \(\pi = \text{UCB}\). The probability that arm 1 is pulled only once is
\[
\mathbb{P}\left(\forall 2 \leq t \leq T : \bar{\mu}_{1,1} + \sqrt{\eta \ln T} < \bar{\mu}_{t,2} + \sqrt{\eta \ln T / t - 1}\right) \\
\geq \mathbb{P}\left(\forall 2 \leq t \leq T : \bar{\mu}_{1,1} + \sqrt{\eta \ln T} < \bar{\mu}_{t,2}\right) \\
\geq \mathbb{P}\left(\{\epsilon_{1,1} < -1 - 2\sqrt{\eta \ln T}\} \cap \bigg\{\forall 1 \leq t < T : \sum_{s=2}^{t+1} \epsilon_{s,2} > -1 - t\sqrt{\eta \ln T}\bigg\}\right) \\
= \mathbb{P}\left(\{\epsilon_{1,1} < -2 - 2\sqrt{\eta \ln T}\} \cap \bigg\{\forall 1 \leq t < T : \sum_{s=2}^{t+1} \epsilon_{s,2} > -1 - t\sqrt{\eta \ln T}\bigg\}\right)
\]
We have
\[
\mathbb{P}\left(\{\epsilon_{1,1} < -2 - 2\sqrt{\eta \ln T}\}\right) \\
\geq \frac{1}{\sqrt{2}\pi} \frac{2 + 2\sqrt{\eta \ln T}}{1 + (2 + 2\sqrt{\eta \ln T})^2} \exp\left(- (2 + 2\sqrt{\eta \ln T})^2/2\right) \\
= \Theta\left(\frac{T^{-2\eta}}{\sqrt{\ln T}}\right)
\]
We use a martingale argument and the optional sampling theorem to bound the second probability. Define 
\[ Z_t = \sum_{s=2}^{t+1} \epsilon_{s,2} \]. Define the stopping time
\[ \tau = \inf_t \{ Z_t \leq -1 - t \sqrt{\eta \ln T} \} \]
Then
\[ \Pr \left( \forall 1 \leq t < T : \sum_{s=2}^{t+1} \epsilon_{s,2} > -1 - t \sqrt{\eta \ln T} \right) = \Pr(\tau \geq T) \]
For fixed \( T \), \( \tau \wedge (T - 1) \) is finite. Notice that
\[ \exp(-2\sqrt{\eta \ln T}Z_t - 2\eta T \cdot t) \]
is a martingale with mean 1. By the optional sampling theorem, we have
\[
1 = \mathbb{E} \left[ \exp(-2\sqrt{\eta \ln T}Z_{\tau \wedge (T-1)} - 2\eta T \cdot (\tau \wedge (T-1))) \right] \\
\geq \mathbb{E} \left[ \exp(-2\sqrt{\eta \ln T}Z_{\tau} - 2\eta T \cdot \tau) \mathbb{1}_{\{\tau < T\}} \right] \\
\geq \exp(2\sqrt{\eta \ln T}) \Pr(\tau < T)
\]
Therefore, the second probability is bounded by
\[ 1 - \exp(-2\sqrt{\eta \ln T}). \]
The expected regret is at least
\[
\Theta \left( \frac{T - 2n}{\sqrt{\ln T}} \right) \cdot (1 - \exp(-2\sqrt{\eta \ln T})) \cdot (T - 2) = \Omega \left( \frac{T^{1-2\eta}}{\sqrt{\ln T}} \right).
\]

**Proof of Theorem 5.** Without loss of generality, we assume \( \theta_1 = \theta_* \). We prove the results one by one.

1. From Lemma 1,
\[
\mathbb{E}[R^*_n] = \mathbb{E}[\hat{R}^*_n] = \sum_{k=2}^K \mathbb{E}[n_k] \cdot \Delta_k.
\]
Let \( G \) be the event such that
\[ G = \{ \mu_k \in \text{Cl}_{t,k}, \forall (t,k) \}. \]
Then
\[ \Pr(G) \leq \sum_{(t,k)} \Pr(\mu_k \notin \text{Cl}_{t,k}) \leq K \sum_{n=1}^T 2 \exp(-\frac{\eta T \ln T}{2n}) \leq 2K T^{1-\eta/2}. \]
Thus,
\[
\mathbb{E}[R^*_n] = \sum_{k=2}^K (\mathbb{E}[n_k | G] \Pr(G) + \mathbb{E}[n_k | \bar{G}] \Pr(\bar{G})) \Delta_k \\
\leq \sum_{k=2}^K \mathbb{E}[n_k | G] \Delta_k + T \cdot \Pr(G)
\]
\[
\sum_{k=2}^{K} E[n_k|G] \Delta_k + 2KT^{2-\eta/2} \\
\leq \sum_{k=2}^{K} E[n_k|G] \Delta_k + 2K.
\]

Define the (random) arm set
\[
A_0 = \left\{ k \neq 1 : n_k \leq 1 + \frac{T}{K} \right\}
\]
as the set of arms that are pulled no more than \(1 + \frac{T}{K}\) times. Then
\[
E[R^*_\pi] \leq E\left[ \sum_{k \in A_0} n_k \Delta_k | G \right] + E\left[ \sum_{k \notin A_0} n_k \Delta_k | G \right] + 2K
\]

(a) Let \(\pi = SE\). Fix any \(k \neq 1\). We let \(t'_k\) be the largest time period such that we have traversed all the arms in \(A\), and meanwhile arm \(k\) is not eliminated from \(A\). Then \(n_k = n_{t'_{k},k}\) or \(n_k = n_{t'_{k},k} + 1\). When doing the elimination after \(t_k\), arm 1 and \(k\) are both pulled \(n_{t'_{k},k}\) times. Under \(G\), we have
\[
\mu_1 - 2\text{rad}(n_{t'_{k},k}) \leq \hat{\mu}_{t'_{k},1} - \text{rad}(n_{t'_{k},k}) \leq \hat{\mu}_{t'_{k},k} + \text{rad}(n_{t'_{k},k}) \leq \mu_k + 2\text{rad}(n_{t'_{k},k}).
\]

— Fix any \(k \in A_0\). Then under \(G\), we have
\[
\Delta_k \leq 4\text{rad}(n_{t'_{k},k}) \leq 4\sigma \frac{\sqrt{(\eta_1 \lor \eta_2)T \ln T}}{\sqrt{Kn_{t'_{k},k}}}
\]

Therefore,
\[
n_{t'_{k},k} \leq 4\sigma \frac{\sqrt{(\eta_1 \lor \eta_2)T \ln T}}{\sqrt{K} \Delta_k}
\]

and thus,
\[
n_k \leq 2 + 4\sigma \frac{\sqrt{(\eta_1 \lor \eta_2)T \ln T}}{\sqrt{K} \Delta_k}
\]

As a result,
\[
E\left[ \sum_{k \in A_0} n_k \Delta_k | G \right] \leq 2E\left[ \sum_{k \in A_0} \Delta_k \right] + 4E\left[ \sum_{k \in A_0} \sigma \frac{\sqrt{(\eta_1 \lor \eta_2)T \ln T}}{\sqrt{K}} \right]
\]
\[
\leq 2E[|S|] + 4\sigma E[|S|] \sqrt{(\eta_1 \lor \eta_2)T \ln T}.
\]

— Fix any \(k \notin A_0\). Then under \(G\), we have
\[
\Delta_k \leq 4\text{rad}(n_{t'_{k},k}) \leq 4\sigma \frac{\sqrt{(\eta_1 \lor \eta_2) \ln T}}{\sqrt{n_{t'_{k},k}}}
\]

Therefore,
\[
\Delta_k \leq 4\sigma \frac{\sqrt{(\eta_1 \lor \eta_2) \ln T}}{\sqrt{n_{t'_{k},k}}}
\]

and thus,
\[
\Delta_k \leq 4\sigma \frac{\sqrt{(\eta_1 \lor \eta_2) \ln T}}{\sqrt{\max\{n_k - 2, 0\}}}
\]
As a result,
\[
\mathbb{E} \left[ \sum_{k \notin A_0} n_k \Delta_k \Big| G \right] \leq 2 \mathbb{E} \left[ \sum_{k \in A_0} \Delta_k \right] + 4 \mathbb{E} \left[ \sum_{k \notin A_0} \sigma \sqrt{(\eta_1 \lor \eta_2) \max\{n_k - 2, 0\} \ln T} \right]
\]
\[
\leq 2(K - \mathbb{E}[|S|]) + 4 \mathbb{E} \left[ \sum_{k \notin A_0} \sigma \sqrt{(\eta_1 \lor \eta_2)n_k \ln T} \right]
\]
\[
\leq 2(K - \mathbb{E}[|S|]) + 4 \sigma \sqrt{(\eta_1 \lor \eta_2)(K - \mathbb{E}[|S|])T \ln T}.
\]

Now we have
\[
\mathbb{E}[R^*_g] \leq \mathbb{E} \left[ \sum_{k \in S} n_k \Delta_k \Big| G \right] + \mathbb{E} \left[ \sum_{k \notin A_0} n_k \Delta_k \Big| G \right] + 2K
\]
\[
\leq 4K + 4 \sigma \sqrt{T \ln T} \left( \sqrt{(\eta_1 \lor \eta_2) \mathbb{E}[|S|]} \right) + \sqrt{(\eta_1 \lor \eta_2)(K - \mathbb{E}[|S|])T \ln T}
\]
\[
\leq 4K + 8 \sigma \sqrt{(\eta_1 \lor \eta_2)KT \ln T}.
\]

(b) Let \( \pi = \text{UCB} \). With a slight abuse of notation, we let \( t_k \) be the largest time period such that arm \( k \) is pulled. Then \( n_k = n_{t_k,k} = 1 + n_{t_k-1,k} \). Under \( G \), we have
\[
\mu_1 \leq \hat{\mu}_1 + \text{rad}(n_{t_k-1,1}) \leq \hat{\mu}_k + \text{rad}(n_{t_k-1,k}) \leq \mu_k + 2 \text{rad}(n_{t_k-1,k}).
\]
— Fix any \( k \in A_0 \). Then under \( G \), we have
\[
\mu_1 \leq \mu_k + 2 \sigma \frac{\sqrt{(\eta_1 \lor \eta_2)T \ln T}}{\sqrt{K(n_{t_k,k} - 1)}}.
\]
Therefore,
\[
n_k = n_{t_k,k} \leq 1 + 2 \sigma \frac{\sqrt{(\eta_1 \lor \eta_2)T \ln T}}{\sqrt{K} \Delta_k}.
\]
As a result,
\[
\mathbb{E} \left[ \sum_{k \in A_0} n_k \Delta_k \Big| G \right] \leq \mathbb{E} \left[ \sum_{k \in A_0} \Delta_k \right] + 2 \mathbb{E} \left[ \sum_{k \notin A_0} \sigma \sqrt{(\eta_1 \lor \eta_2)T \ln T} \right]
\]
\[
\leq \mathbb{E}[|S|] + 2 \sigma \sqrt{(\eta_1 \lor \eta_2)T \ln T}.
\]
— Fix any \( k \notin A_0 \). Then under \( G \), we have
\[
\mu_1 \leq \mu_k + 2 \sigma \frac{\sqrt{(\eta_1 \lor \eta_2)\ln T}}{\sqrt{n_{t_k,k} - 1}}.
\]
Therefore,
\[
\Delta_k \leq 2 \sigma \frac{\sqrt{(\eta_1 \lor \eta_2)\ln T}}{\sqrt{n_{t_k,k} - 1}}
\]
and thus,
\[
\Delta_k \leq 2 \sigma \frac{\sqrt{(\eta_1 \lor \eta_2)\ln T}}{\sqrt{\max\{n_k - 1, 0\}}}.
\]
As a result,
\[
\mathbb{E} \left[ \sum_{k \notin A_0} n_k \Delta_k \big| G \right] \leq \mathbb{E} \left[ \sum_{k \notin A_0} \Delta_k \right] + 2 \mathbb{E} \left[ \sum_{k \notin A_0} \sigma \sqrt{(\eta_1 \lor \eta_2) \max\{n_k - 2, 0\} \ln T} \right]
\]
So

Then

From Lemma 1, the second term can be bounded as

Now we have

\[ E[R^*_o] \leq E \left[ \sum_{k \in S} n_k \Delta_k \mid G \right] + E \left[ \sum_{k \in A_0 \setminus \{1\}} n_k \Delta_k \mid G \right] + 2K \]

\[ \leq 3K + 2\sigma \sqrt{T \ln T} \left( \sqrt{(\eta_1 \lor \eta_2) \frac{E[|S|]}{\sqrt{K}}} + \sqrt{(\eta_1 \lor \eta_2)(K - E[|S|])} \right) \]

\[ \leq 4K + 4\sigma \sqrt{(\eta_1 \lor \eta_2) K T \ln T}. \]

2. Let \( x \geq 2K \). We have

\[ P(\tilde{R}^*_o(T) \geq x) \leq P \left( R^*_o(T) \geq x(1 - 1/2\sqrt{K}) \right) + P \left( N^*_o(T) \leq -x/2\sqrt{K} \right) \]

From Lemma 1, the second term can be bounded as

\[ P \left( N^*_o(T) \leq -x/2\sqrt{K} \right) \leq \exp \left( -\frac{x^2}{8K\sigma^2 T} \right). \]  

(18)

We are left to bound \( P \left( R^*_o(T) \geq x(1 - 1/2\sqrt{K}) \right) \).

(a) Let \( \pi = \text{SE} \). For any \( k \neq 1 \), let \( S_k \) be the event defined as

\[ S_k = \{ \text{Arm 1 is not eliminated before arm } k \}. \]

Then

\[ \tilde{S}_k = \{ \text{Arm 1 is eliminated before arm } k \}. \]

So

\[ P \left( R^*_o(T) \geq x(1 - 1/2\sqrt{K}) \right) \]

\[ = P \left( \sum_{k \in A_0} n_k \Delta_k + \sum_{k \notin A_0} n_k \Delta_k \geq x(1 - 1/2\sqrt{K}) \right) \]

\[ \leq P \left( \sum_{k \in A_0} (n_k - 1) \Delta_k + \sum_{k \notin A_0} (n_k - 1) \Delta_k \geq x(1 - 1/2\sqrt{K}) - K \right) \]

\[ \leq P \left( \bigcup_{k \in A_0} \left\{ (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K} \right\} \right) + \sum_{k \notin A_0} P \left( \left( n_k - 1 \right) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T}, \ k \notin A_0 \right) \]

The reason that the first inequality holds is as follows. To prove it, we only need to show that the following cannot holds:

\[ (n_k - 1) \Delta_k < \frac{x - 2K}{4K}, \ \forall k \in A_0; \quad (n_k - 1) \Delta_k < \frac{(n_k - 1)(x - 2K)}{4T}, \ \forall k \notin A_0. \]
If not, then we have
\[
\sum_{k \neq 1} (n_k - 1)\Delta_k = \sum_{k \in \mathcal{A}_0} (n_k - 1)\Delta_k + \sum_{k \notin \mathcal{A}_0} (n_k - 1)\Delta_k
\]
\[
< \frac{(x - 2K)|\mathcal{A}_0|}{4K} + \frac{x - 2K}{4}
\]
\[
\leq \frac{x - 2K}{4} + \frac{x - 2K}{4}
\]
\[
= \frac{x - 2K}{2} \leq x(1 - 1/2\sqrt{K}) - K.
\]

Therefore,
\[
\mathbb{P}\left(R^*_n(T) \geq x(1 - 1/\sqrt{K})\right)
\]
\[
\leq \sum_{k \neq 1} \left(\mathbb{P}\left((n_k - 1)\Delta_k \geq \frac{x - 2K}{4K}, k \in \mathcal{A}_0\right) + \mathbb{P}\left((n_k - 1)\Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T}, k \notin \mathcal{A}_0\right)\right)
\]
\[
= \sum_{k \neq 1} \mathbb{P}\left((n_k - 1)\Delta_k \geq \frac{x - 2K}{4K}, k \in \mathcal{A}_0, S_k\right) + \sum_{k \neq 1} \mathbb{P}\left((n_k - 1)\Delta_k \geq \frac{x - 2K}{4K}, k \in \mathcal{A}_0, S_k\right)
\]
\[
+ \sum_{k \neq 1} \mathbb{P}\left((n_k - 1)\Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T}, k \notin \mathcal{A}_0, S_k\right) + \sum_{k \neq 1} \mathbb{P}\left(\Delta_k \geq \frac{x - 2K}{4T}, k \notin \mathcal{A}_0, S_k\right)
\]

Fix $k \neq 1$. Now for each $k$, we consider bounding the four terms separately.

$-k \in \mathcal{A}_0$. With a slight abuse of notation, we let $n_0 = \left\lceil\frac{x - 2K}{4K}\right\rceil \leq n_k - 1$. Also,
\[
\Delta_k \geq \frac{x - 2K}{4K} \geq \frac{(x - 2K)K}{4KT} = \frac{x - 2K}{4T}.
\]

We have
\[
\mathbb{P}\left((n_k - 1)\Delta_k \geq \frac{x - 2K}{4K}, k \in \mathcal{A}_0, S_k\right)
\]
\[
\leq \mathbb{P}\left(\hat{\mu}_{t_1(n_0), 1} - \text{rad}(n_0) \leq \hat{\mu}_{t_k(n_0), k} + \text{rad}(n_0)\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
= \mathbb{P}\left(\mu_1 - \frac{\sum_{m=1}^{n_0} \sigma_{t_1(m), 1}}{n_0} - \text{rad}(n_0) \leq \mu_k + \frac{\sum_{m=1}^{n_0} \sigma_{t_k(m), k}}{n_0} + \text{rad}(n_0)\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
= \mathbb{P}\left(\frac{\sum_{m=1}^{n_0} \sigma_{t_1(m), 1} - \sigma_{t_k(m), k}}{n_0} \geq \Delta_k - 2\text{rad}(n_0)\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
\leq \mathbb{P}\left(\frac{\sum_{m=1}^{n_0} \sigma_{t_1(m), 1}}{n_0} \geq \Delta_k \right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
+ \mathbb{P}\left(\frac{\sum_{m=1}^{n_0} \sigma_{t_k(m), k}}{n_0} \geq \Delta_k - 2\text{rad}(n_0)\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
\leq 2\exp\left(-n_0 \left(\frac{\Delta_k}{2} - \text{rad}(n_0)\right)^2 /2\sigma^2\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
= 2\exp\left(-n_0 \left(\frac{\Delta_k}{2} - \frac{\sqrt{(\eta_1 \lor \eta_2)KT}}{n_0\sqrt{K}}\right)^2 /2\sigma^2\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
= 2\exp\left(-n_0 \Delta_k^2 \left(1 - \frac{2\sqrt{(\eta_1 \lor \eta_2)KT}}{n_0\Delta_k \sqrt{K}}\right)^2 /8\sigma^2\right) \mathbb{I}\left\{\Delta_k \geq \frac{x - 2K}{4T}\right\}
\]
\[
\leq 2\exp\left(-\frac{(x - 2K)^2}{16KT} \left(1 - \frac{8\sqrt{(\eta_1 \lor \eta_2)KT} \ln T}{x - 2K}\right)^2 /8\sigma^2\right)
\]
\[
\leq 2 \exp \left( -\frac{(x - 2K - 8\sigma \sqrt{\eta_1} \sqrt{\eta_2} KT \ln T)^2}{128\sigma^2 KT} \right).
\]

Then we bound \( \mathbb{P}(n_k \Delta_k \geq (x - 2K)/4K, k \in \mathcal{A}_0, \bar{S}_k) \). Suppose that after \( n \) phases, arm 1 is eliminated by arm \( k' \) (\( k' \) is not necessarily \( k \)). By the definition of \( \bar{S}_k \), arm \( k \) is not eliminated. Therefore, we have

\[
\hat{\mu}_{t_k(n),k'} - \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \leq \hat{\mu}_{t_1(n),1} + \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}}, \quad \hat{\mu}_{t_k(n),k} + \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \geq \hat{\mu}_{t_1(n),1} + \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}}
\]

holds simultaneously. The first inequality holds because arm 1 is eliminated. The second inequality holds because arm \( k \) is not eliminated. Now for fixed \( n \),

\[
\mathbb{P} \left( (20) \text{ happens}; \Delta_k \geq \frac{x - 2K}{4T} \right)
\]

\[
\leq \mathbb{P} \left( \exists k': \hat{\mu}_{t_k(n),k'} - \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \leq \hat{\mu}_{t_1(n),1} + \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}}, \quad \hat{\mu}_{t_k(n),k} + \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \geq \hat{\mu}_{t_1(n),1} + \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \right)
\]

\[
\wedge \mathbb{P} \left( \sum_{m=1}^{n} (\epsilon_{t_k(n),k'} - \epsilon_{t_1(n),1}) \geq \frac{2\sigma \eta_1 T \ln T}{n \sqrt{K}} \right)
\]

\[
\wedge \mathbb{P} \left( \sum_{m=1}^{n} (\epsilon_{t_k(n),k} - \epsilon_{t_1(n),1}) \geq \frac{2\sigma \eta_1 T \ln T}{n \sqrt{K}} \right)
\]

\[
\leq \sum_{k' \neq 1} \mathbb{P} \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(n),k'}}{n} \geq \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \right) + \mathbb{P} \left( \frac{\sum_{m=1}^{n} \epsilon_{t_1(n),1}}{n} \leq \frac{-\frac{\sigma \eta_1 T \ln T}{n \sqrt{K}}}{8T} \right)
\]

\[
\wedge \mathbb{P} \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(n),k}}{n} \geq \frac{\sigma \eta_1 T \ln T}{n \sqrt{K}} \right) + \mathbb{P} \left( \frac{\sum_{m=1}^{n} \epsilon_{t_1(n),1}}{n} \leq \frac{-\frac{\sigma \eta_1 T \ln T}{n \sqrt{K}}}{8T} \right)
\]

\[
\leq 2K \exp \left( -\frac{\eta_1 T \ln T}{2nK} \right) \wedge 2K \exp \left( -\frac{n(x - 2K)^2}{128\sigma^2 T^2} \right)
\]

\[
= 2K \exp \left( -\frac{\eta_1 T \ln T}{2nK} \wedge \frac{n(x - 2K)^2}{128\sigma^2 T^2} \right)
\]

\[
\leq 2K \exp \left( -\frac{(x - 2K) \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right)
\]

Therefore,

\[
\mathbb{P}(n_k \Delta_k \geq (x - 2K)/4K, \bar{S}_k, k \in \mathcal{A}_0)
\]

\[
= \mathbb{P}(\exists n \leq T/2: (20) \text{ happens}; n_k \Delta_k \geq (x - 2K)/4K, k \in \mathcal{A}_0)
\]

\[
\leq \sum_{n=1}^{\lfloor T/2 \rfloor} \mathbb{P} \left( (20) \text{ happens}; \Delta_k \geq \frac{x - 2K}{4T} \right)
\]

\[
\leq KT \exp \left( -\frac{(x - 2K) \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right).
\]

\[\neg k \notin \mathcal{A}_0.\] With a slight abuse of notation, we let \( n_0 = \lfloor \frac{T}{2} \rfloor \leq n_k - 1 \). Also,

\[
\Delta_k \geq \frac{x - 2K}{4T}.
\]

We have

\[
\mathbb{P} \left( (n_k - 1) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T}, k \notin \mathcal{A}_0, S_k \right)
\]
Then we bound $P((n_k - 1)\Delta_k \geq (n_k - 1)(x - 2K)/4T, k \notin \mathcal{A}_0, \tilde{S}_k)$. The procedure is nearly the same as in the case where $k \in \mathcal{A}_0$. Suppose that after $n$ phases, arm 1 is eliminated by arm $k'$ ($k'$ is not necessarily $k$). By the definition of $\tilde{S}_k$, arm $k$ is not eliminated. Therefore, we have

$$
\mu_{t_k(n), k'} - \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \geq \hat{\mu}_{t_1(n), 1} + \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}}, \quad \hat{\mu}_{t_k(n), k} + \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \geq \hat{\mu}_{t_1(n), 1} + \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}}
$$

(23) holds simultaneously. The first inequality holds because arm 1 is eliminated. The second inequality holds because arm $k$ is not eliminated. Now for fixed $n$,

$$
P \left( \text{(23) happens; } \Delta_k \geq \frac{x - 2K}{4T}, k \notin \mathcal{A}_0 \right)$$

$$
\leq P \left( \exists k': \hat{\mu}_{t_k(n), k'} - \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \geq \hat{\mu}_{t_1(n), 1} + \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \right)
\land P \left( \hat{\mu}_{t_k(n), k} + \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \geq \hat{\mu}_{t_1(n), 1} + \frac{\sqrt{\eta_1 T \ln T}}{n\sqrt{K}} ; \Delta_k \geq \frac{x - 2K}{4T} \right)
$$

$$
\leq P \left( \exists k': \frac{\sum_{m=1}^{n} \epsilon_{t_k'(m), k'} - \epsilon_{t_1(m), 1}}{n} \geq \frac{2\sigma \sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \right)
\land P \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(m), k} - \epsilon_{t_1(m), 1}}{n} \geq \Delta_k ; \Delta_k \geq \frac{x - 2K}{4T} \right)
$$

$$
\leq \sum_{k' \neq 1} \left( P \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k'(m), k'}}{n} \geq \frac{\sigma \sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \right) + P \left( \frac{\sum_{m=1}^{n} \epsilon_{t_1(m), 1}}{n} \leq -\frac{\sigma \sqrt{\eta_1 T \ln T}}{n\sqrt{K}} \right) \right)
\land \left( P \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(m), k}}{n} \geq \frac{(x - 2K)^+}{8T} \right) + P \left( \frac{\sum_{m=1}^{n} \epsilon_{t_1(m), 1}}{n} \leq -\frac{(x - 2K)^+}{8T} \right) \right)
$$

$$
\leq 2K \exp \left( -\frac{\eta_1 T \ln T}{2nK} \right) \land 2K \exp \left( -\frac{n(x - 2K)^2_+}{128\sigma^2 T^2} \right)
$$

$$
= 2K \exp \left( -\frac{\eta_1 T \ln T}{2nK} \lor \frac{n(x - 2K)^2_+}{128\sigma^2 T^2} \right)
$$

$$
\leq 2K \exp \left( -\frac{(x - 2K)^+ \sqrt{\eta_1 T \ln T}}{16\sigma \sqrt{K T}} \right)
$$
Therefore,
\[ \mathbb{P}( (x_k - 1) \Delta_k \geq (n_k - 1)(x - 2K)/4T, k \notin A_0 ) \]
\[ = \mathbb{P}( (x_k - 1) \Delta_k \geq (n_k - 1)(x - 2K)/4T, k \notin A_0 ) \]
\[ \leq \sum_{n=1}^{[T/2]} \mathbb{P}( (23) \text{ happens}; \Delta_k \geq \frac{x - 2K}{4T} ) \]
\[ \leq KT \exp \left( -\frac{(x - 2K) + \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right). \] (24)

Note that the equations above hold for any instance \( \theta \). Combining (18), (19), (21), (22), (24) yields
\[ \sup_{\theta} \mathbb{P}( R^\theta_0(T) \geq x ) \]
\[ \leq \exp \left( -\frac{x^2}{8K\sigma^2 T} \right) + 4K \exp \left( -\frac{(x - 2K - 8\sqrt{\eta_1 \ln T})^2}{128\sigma^4 KT} \right) + 2K^2 T \exp \left( -\frac{(x - 2K) + \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right) \]

(b) From (a), we have
\[ \mathbb{P}( R^\theta_0(T) \geq x(1 - 1/\sqrt{K}) ) \]
\[ \leq \sum_{k \neq 1} \left( \mathbb{P}( (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K}, k \in A_0 ) + \mathbb{P}( (n_k - 1) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4T}, k \notin A_0 ) \right). \]

Fix \( k \neq 1 \). Now for each \( k \), we consider bounding the two terms separately.

\[ -k \in A_0. \] With a slight abuse of notation, we let \( n_0 = \lceil \frac{x - 2K}{4K\Delta_k} \rceil \leq n_k - 1 \). Remember that \( t_k(n_0 + 1) \) is the time period that arm \( k \) is pulled for the \( (n_0 + 1) \)th time. We have
\[ \mathbb{P}( (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K}, k \in A_0 ) \]
\[ = \mathbb{P}( n_k \geq 1 + \frac{x - 2K}{4K\Delta_k}, k \in A_0 ) \]
\[ \leq \mathbb{P}( \mu_k + \sum_{t=1}^{n_0} \epsilon_{t_{k}(m),m} + \epsilon_{t_{k}(m),n} + \epsilon_{t_{k}(m),k} + \epsilon_{t_{k}(m),l} \leq \mu_k + \sum_{t=1}^{n_0} \epsilon_{t_{k}(m),m} + \epsilon_{t_{k}(m),n} + \epsilon_{t_{k}(m),k} + \epsilon_{t_{k}(m),l} \right) \]
\[ \leq \mathbb{P}( \exists n \in [T] : \mu_k + \sum_{t=1}^{n_0} \epsilon_{t_{k}(m),m} + \epsilon_{t_{k}(m),n} + \epsilon_{t_{k}(m),k} + \epsilon_{t_{k}(m),l} \right) \]
\[ \leq \mathbb{P}( \exists n \in [T] : \sum_{t=1}^{n_0} \epsilon_{t_{k}(m),m} + \epsilon_{t_{k}(m),n} + \epsilon_{t_{k}(m),k} + \epsilon_{t_{k}(m),l} \right) \]
\[ \leq \mathbb{P}( \sum_{t=1}^{n_0} \epsilon_{t_{k}(m),m} + \epsilon_{t_{k}(m),n} + \epsilon_{t_{k}(m),k} + \epsilon_{t_{k}(m),l} \right) \]
\[ \leq \mathbb{P}( \sum_{t=1}^{n_0} \epsilon_{t_{k}(m),m} + \epsilon_{t_{k}(m),n} + \epsilon_{t_{k}(m),k} + \epsilon_{t_{k}(m),l} \right) \]
\[ \leq \exp \left( -\frac{(x - 2K) + \sqrt{\eta_1 \ln T}}{2n \sqrt{KT}} \right) + T \exp \left( -\frac{(x - 2K) + \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right). \] (25)

The last inequality holds from (19) and (21).
Fix a time horizon of $T$. We write $t_k = t_k(n_{T,k})$ as the last time that arm $k$ is pulled throughout the $T$ time periods. By the definition of $n_k$ and $t_k$, the following event happens w.p. 1:

$$\hat{\mu}_{t_k-1,1} + \text{rad}_{t_k} (n_{t_k-1,1}) \leq \hat{\mu}_{t_k-1,k} + \text{rad}_{t_k} (n_{t_k-1,k})$$

Define

$$A_1 = \left\{ k \neq 1 : n_k \leq 1 + \frac{3^4 T^{1/4}}{K} \right\}.$$

Fix $x \geq 2K$. We have

$$P \left( R^*_\theta (T) \geq x(1 - 1/2\sqrt{K}) \right)$$

$$= P \left( \sum_{k \in A_1} n_k \Delta_k + \sum_{k \notin A_1} n_k \Delta_k \geq x(1 - 1/2\sqrt{K}) \right)$$

$$\leq P \left( \sum_{k \in A_1} (n_k - 1) \Delta_k + \sum_{k \notin A_1} (n_k - 1) \Delta_k \geq x(1 - 1/2\sqrt{K}) - K \right)$$

$$\leq P \left( \bigcup_{k \in A_1} \left\{ (n_k - 1) \Delta_k \geq \frac{x - 2K}{4T} \right\} \bigcup \bigcup_{k \notin A_1} \left\{ (n_k - 1) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{4\sqrt{t_k}T} \right\} \right)$$

The last inequality holds from (22) and (24). Note that the equations above hold for any instance $\theta$. Combining (18), (25), (26) yields

$$\sup_{\theta} P(R^*_\theta (T) \geq x)$$

$$\leq \exp \left( -\frac{x^2}{8K\sigma^2T} \right) + 4K \exp \left( -\frac{(x - 2K - 8\sigma \sqrt{(\eta_1 \lor \eta_2) KT \ln T})^2}{128\sigma^2KT} \right) + 2K^2 \exp \left( -\frac{(x - 2K) + \sqrt{\eta_1 \ln T}}{16\sigma \sqrt{KT}} \right).$$

Proof of Theorem 6. Fix a time horizon of $T$. We write $t_k = t_k(n_{T,k})$ as the last time that arm $k$ is pulled throughout the $T$ time periods. By the definition of $n_k$ and $t_k$, the following event happens w.p. 1:

$$\hat{\mu}_{t_k-1,1} + \text{rad}_{t_k} (n_{t_k-1,1}) \leq \hat{\mu}_{t_k-1,k} + \text{rad}_{t_k} (n_{t_k-1,k})$$

Define

$$A_1 = \left\{ k \neq 1 : n_k \leq 1 + \frac{3^4 T^{1/4}}{K} \right\}.$$
Thus, if not, then we have
\( t_k \) because before up to time \( t_k \).

In fact, to bound \( \sum \Delta_k \), we can assume \( 0 = t_{h_0} < t_{k_1} < t_{k_2} < \cdots \). Then we have
\( t_{k_i} \geq n_{k_1} + \cdots + n_{k_i} \)

because before up to time \( t_{k_i} \), arms \( k_1, \cdots, k_i \) have been pulled completely, and after time \( t_{k_i} \) none of them will be pulled. Thus,
\[
\sum_{k \in A_1} n_k \frac{t_{k_i}}{\sqrt{t_{k_i}}} = \sum_{i=1}^{|A_1|} t_{k_i} - t_{k_{i-1}} \leq 2 \sum_{i=1}^{|A_1|} \frac{t_{k_i} - t_{k_{i-1}}}{\sqrt{t_{k_i}} + \sqrt{t_{k_{i-1}}}} = 2 \sqrt{t_{k_{|A_1|}}} \leq 2 \sqrt{T}.
\]

Now fix \( k \neq 1 \). For each \( k \), we consider bounding the two terms separately.

- \( k \in A_1 \). Remember that \( n_k \) is the last time period that arm \( k \) is pulled. Then from \( k \in A_1 \), we know
  \[
  \frac{t_k^{3/4} T^{1/4}}{K} \geq n_k - 1 \geq \frac{x - 2K}{4K \Delta_k}
  \]

Thus,
\[
\Delta_k \geq \frac{x - 2K}{4t_k^{3/4} T^{1/4}}.
\]

We have
\[
P \left( (n_k - 1) \Delta_k \geq \frac{x - 2K}{4K}, \ k \in A_1 \right)
= P \left( n_k \geq 1 + \frac{x - 2K}{4K \Delta_k}, \ k \in A_1 \right)
= P \left( \bar{\mu}_{t_k - 1, 1} + \text{rad}_{t_k} (n_k - 1, 1) \leq \bar{\mu}_{t_k - 1, k} + \text{rad}_{t_k} (n_k - 1, k); \frac{t_k^{3/4} T^{1/4}}{K} \geq n_k - 1 \geq \frac{x - 2K}{4K \Delta_k} \right)
= P \left( \mu_1 + \sum_{m=1}^{n_k - 1} \frac{\epsilon_{t_k (m), 1}}{n_{t_k - 1, 1}} + \text{rad}_{t_k} (n_k - 1, 1) \leq \mu_k + \sum_{m=1}^{n_k - 1} \frac{\epsilon_{t_k (m), k}}{n_{k - 1}} + \text{rad}_{t_k} (n_k - 1) \right);
\]
\[
\frac{t_k^{3/4}T^{1/4}}{K} \geq n_k - 1 \geq \frac{x - 2K}{4K\Delta_k}
\]
\[
\leq \mathbb{P} \left( \exists n \in [T] : \mu_1 + \frac{\sum_{m=1}^{n} \epsilon_{t_k(m),k}}{n} + \text{rad}_k(n) \leq \mu_k + \frac{\sum_{m=1}^{n-1} \epsilon_{t_k(m),k}}{n-1} + \text{rad}_k(n-1) ; \right)
\]
\[
\frac{t_k^{3/4}T^{1/4}}{K} \geq n_k - 1 \geq \frac{x - 2K}{4K\Delta_k}
\]
\[
\leq \sum_{n_0 : T} \sum_{K,n_0 \leq t^{3/4}T^{1/4}} \mathbb{P} \left( \exists n \in [T] : \left( \frac{\sum_{m=1}^{n_0} \epsilon_{t_k(m),k}}{n_0} + \text{rad}_k(n_0) \right) - \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(m),1}}{n} + \text{rad}_k(n) \right) \geq \frac{x - 2K}{4Kn_0} \right)
\]

Note that here \(n_k\) and \(t_k\) are both random variables, so we need to decompose the probability by \(n_k - 1 = n_0\) and \(t_k = t\) through all possible \((n_0, t)\). Now for any \(n_0\) and \(t\) such that \(K,n_0 \leq t^{3/4}T^{1/4}\), we have
\[
\mathbb{P} \left( \exists n \in [T] : \left( \frac{\sum_{m=1}^{n_0} \epsilon_{t_k(m),k}}{n_0} + \text{rad}_k(n_0) \right) - \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(m),1}}{n} + \text{rad}_k(n) \right) \geq \frac{x - 2K}{4Kn_0} \right)
\]
\[
\leq \mathbb{P} \left( \sum_{m=1}^{n_0} \epsilon_{t_k(m),k} + \text{rad}_k(n_0) \geq \frac{x - 2K}{8Kn_0} \right) + \mathbb{P} \left( \exists n \in [T] : \frac{\sum_{m=1}^{n} \epsilon_{t_k(m),1}}{n} + \text{rad}_k(n) \leq -\frac{x - 2K}{8Kn_0} \right)
\]
\[
\leq \mathbb{P} \left( \sum_{m=1}^{n_0} \epsilon_{t_k(m),k} + \text{rad}_k(n_0) \geq \frac{x - 2K}{8Kn_0} \right) + \sum_{n=1}^{T} \mathbb{P} \left( \sum_{m=1}^{n} \epsilon_{t_k(m),1} \geq \frac{\sqrt{n\ln(K,t)}}{8Kn_0} \right)
\]
\[
\leq \exp \left( -\frac{n_0}{8Kn_0} \left( \frac{x - 2K}{\sqrt{n_0\ln(K,t)}} \right)^2 \right) + \sum_{n=1}^{T} \exp \left( -\frac{(x - 2K) + \sqrt{n\ln(T)}}{8\sigma\sqrt{KT}} \right)
\]
\[
\leq \exp \left( -\frac{1}{8Kn_0} \left( \frac{x - 2K}{\sqrt{K}} \right)^2 \right) + T \exp \left( -\frac{(x - 2K) + \sqrt{\ln(T)}}{8\sigma\sqrt{KT}} \right)
\]
\[
\leq \exp \left( -\frac{K}{T} \left( \frac{x - 2K}{8K} \right)^2 \right) + T \exp \left( -\frac{(x - 2K) + \sqrt{\ln(T)}}{8\sigma\sqrt{KT}} \right)
\]
\[
\leq \exp \left( -\frac{(x - 2K - 8\sigma\sqrt{\ln(T)}}{128\sigma^2KT} \right) + T \exp \left( -\frac{(x - 2K) + \sqrt{\ln(T)}}{8\sigma\sqrt{KT}} \right).
\]

Note that here we use the fact that for any \(1 \leq t \leq T\),
\[
\frac{\ln(2t)}{\sqrt{t}} \geq \frac{1}{4} \frac{\ln T}{\sqrt{T}}.
\]

So we have
\[
\mathbb{P} \left( (n_k - 1)\Delta_k \geq \frac{x - 2K}{4K} , \quad k \in A_1 \right)
\]
\[
\leq \sum_{n_0 : T} \sum_{K,n_0 \leq t^{3/4}T^{1/4}} \mathbb{P} \left( \exists n \in [T] : \left( \frac{\sum_{m=1}^{n_0} \epsilon_{t_k(m),k}}{n_0} + \text{rad}_k(n_0) \right) - \left( \frac{\sum_{m=1}^{n} \epsilon_{t_k(m),1}}{n} + \text{rad}_k(n) \right) \geq \frac{x - 2K}{4Kn_0} \right)
\]
\[
\leq T^2 \exp \left( -\frac{(x - 2K - 8\sigma\sqrt{\ln(T)}}{128\sigma^2KT} \right) + T^3 \exp \left( -\frac{(x - 2K) + \sqrt{\ln(T)}}{8\sigma\sqrt{KT}} \right).
\]

• \(k \notin A_1\). Remember that \(t_k\) is the last time period that arm \(k\) is pulled. Then from \(k \notin A_1\), we know
\[
t_k \geq n_k \geq 1 + \frac{t_k^{3/4}T^{1/4}}{K}.
\]
We have
\[
\mathbb{P} \left( (n_k - 1) \Delta_k \geq \frac{(n_k - 1)(x - 2K)}{8\sqrt{t_k T}}, k \notin A_t \right) \\
= \mathbb{P} \left( \Delta_k \geq \frac{x - 2K}{8\sqrt{t_k T}}, n_k \geq 1 + \frac{t_k^{3/4}T^{1/4}}{K} \right)
\]
\[
= \mathbb{P} \left( \mu_{t_k-1} + \text{rad}_t(n_{t_k-1}) \leq \mu_{t_k-1} + \text{rad}_t(n_k - 1); \Delta_k \geq \frac{x - 2K}{8\sqrt{t_k T}}, n_k \geq 1 + \frac{t_k^{3/4}T^{1/4}}{K} \right)
\]
\[
= \mathbb{P} \left( \mu_1 + \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n_{t_k-1}} + \text{rad}_t(n_{t_k-1}) \leq \mu_k + \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),k}}{n_k - 1} + \text{rad}_t(n_k - 1); \Delta_k \geq \frac{x - 2K}{8\sqrt{t_k T}}, n_k \geq 1 + \frac{t_k^{3/4}T^{1/4}}{K} \right)
\]
\[
\leq \mathbb{P} \left( \exists n \in [T] : \mu_1 + \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n} + \text{rad}_t(n) \leq \mu_k + \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),k}}{n_k - 1} + \text{rad}_t(n_k - 1); \Delta_k \geq \frac{x - 2K}{8\sqrt{t_k T}}, n_k - 1 \geq 1 + \frac{t_k^{3/4}T^{1/4}}{K} \right)
\]
\[
\leq \sum_{K_n \geq \frac{t^{3/4}T^{1/4}}{16}} \mathbb{P} \left( \exists n \in [T] : \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n} + \text{rad}_t(n_0) \right) - \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),k}}{n_k - 1} + \text{rad}_t(n) \right) \geq \frac{x - 2K}{8\sqrt{t T}} \right)
\]

Note that here \(n_k\) and \(t_k\) are both random variables, so we need to decompose the probability by \(n_k - 1 = n_0\) and \(t_k = t\) through all possible \((n_0, t)\). Now for any \((n_0, t)\) such that \(t \geq n_0\) and \(K n_0 \geq t^{3/4}T^{1/4}\), we know that
\[
K t \geq K n_0 \geq T^{1/4},
\]
and so
\[
\ln(K t) \geq \frac{1}{4} \ln T.
\]

We have
\[
\mathbb{P} \left( \exists n \in [T] : \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n_0} + \text{rad}_t(n_0) \right) - \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),k}}{n_k} + \text{rad}_t(n) \right) \geq \frac{x - 2K}{8\sqrt{t T}} \right)
\]
\[
\leq \mathbb{P} \left( \mathbb{P} \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n_0} + \text{rad}_t(n_0) \geq \frac{x - 2K}{16\sqrt{t T}} \right) + \mathbb{P} \left( \frac{x - 2K}{16\sqrt{t T}} \cdot \mathbb{P} \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n_0} + \text{rad}_t(n) \leq \frac{x - 2K}{16\sqrt{t T}} \right) \right) \right)
\]
\[
\leq \mathbb{P} \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n_0} + \text{rad}_t(n_0) \geq \frac{x - 2K}{16\sqrt{t T}} \right) + \mathbb{P} \left( \sum_{m=1}^{n_k-1} \frac{\epsilon_{t_k(m),1}}{n_0} + \text{rad}_t(n) \geq \frac{x - 2K}{16\sqrt{t T}} \right)
\]
\[
\leq \exp \left( -\frac{\left( x - 2K \right)_+ \sqrt{2\eta \ln(K T)}}{8n_0 \sqrt{K}} \right) + \exp \left( -\frac{\left( x - 2K \right)_+ \sqrt{2\eta \ln(T)}}{16n_0 \sqrt{K}} \right)
\]
\[
\leq \exp \left( -\frac{\left( x - 2K \right)_+ \sqrt{2\eta \ln(T)}}{512\sigma^2 KT} \right) + T \exp \left( -\frac{\left( x - 2K \right)_+ \sqrt{\eta \ln(T)}}{16\sigma \sqrt{K T}} \right).
\]
So we have
\[
P\left((n_k - 1)\Delta_k \geq \frac{(n_k - 1)(x - 2K)}{8\sqrt{t_k T}}, \ k \notin A_1\right)
\leq \sum_{n_0, t_i} P\left(\exists n \in [T]: \left(\sum_{m=1}^{n_0} \epsilon_{1i,m} \frac{k}{n_0} + \text{rad}_i(n_0)\right) - \left(\sum_{m=1}^{n} \epsilon_{1i,m} \frac{1}{n} + \text{rad}_i(n)\right) \geq \frac{x - 2K}{8\sqrt{tT}}\right)
\leq T^2 \exp\left(-\frac{(x - 2K - 16\sigma\sqrt{2}\eta KT)^2}{512\sigma^2 KT}\right) + T^3 \exp\left(-\frac{(x - 2K)\sqrt{\eta \ln T}}{16\sigma \sqrt{KT}}\right)
\] (28)
Note that all the equations above hold for any instance \(\theta\). Combining (18), (27), (28) yields
\[
\sup_{\theta} P(R^*_{\pi}(T) \geq x)
\leq \exp\left(-\frac{x^2}{8K\sigma^2 T}\right) + 2KT^2 \exp\left(-\frac{(x - 2K - 16\sigma\sqrt{2}\eta KT \ln T)^2}{512\sigma^2 KT}\right) + 2KT^3 \exp\left(-\frac{(x - 2K)\sqrt{\eta \ln T}}{16\sigma \sqrt{KT}}\right).
\]