Quasi-Local Strange Metal

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Abstract

One of the key factors that determine the fates of quantum many-body systems in the zero temperature limit is the competition between kinetic energy that delocalizes particles in space and interaction that promotes localization. While one dominates over the other in conventional metals and insulators, exotic states can arise at quantum critical points where none of them clearly wins. Here we present a novel metallic state that emerges at an antiferromagnetic (AF) quantum critical point in the presence of one-dimensional Fermi surfaces embedded in space dimensions three and below. At the critical point, interactions between particles are screened to zero in the low energy limit at the same time the kinetic energy is suppressed in certain spatial directions to the leading order in a perturbative expansion that becomes asymptotically exact in three dimensions. The resulting dispersionless and interactionless state exhibits distinct quasi-local strange metallic behaviors due to a subtle dynamical balance between screening and infrared singularity caused by spontaneous reduction of effective dimensionality. The strange metal, which is stable near three dimensions, shows enhanced fluctuations of bond density waves, d-wave pairing, and pair density waves.
The richness of exotic zero-temperature states in condensed matter systems\cite{1, 2} can be attributed to quantum fluctuations driven by kinetic energy and interaction which can not be simultaneously minimized due to the uncertainty principle. In conventional metals, kinetic energy plays the dominant role, and interactions only dress electrons into quasiparticles which survive as coherent excitations in the absence of instabilities\cite{3, 4}. The existence of well defined quasiparticle excitations is the cornerstone of Landau Fermi liquid theory\cite{5}, which successfully explains a large class of metals. However, the Fermi liquid theory breaks down at the verge of spontaneous formation of order in metals\cite{6–8}. Near continuous quantum phase transitions, new metallic states can arise as quantum fluctuations of order parameter destroy the coherence of quasiparticles through interactions that persist down to the zero energy limit\cite{9, 10}. Systematic understanding of the resulting strange metallic states is still lacking, although there exist some examples whose universal behaviors in the low energy limit can be understood within controlled theoretical frameworks\cite{11–15}.

Antiferromagnetic (AF) quantum phase transition commonly arises in strongly correlated systems including electron doped cuprates\cite{16}, iron pnictides\cite{17} and heavy fermion compounds\cite{18}. In two space dimensions, it has been shown that the interaction between the AF mode and itinerant electrons qualitatively modify the dynamics of the system at the critical point\cite{19, 20}. A recent numerical simulation shows a strong enhancement of superconducting correlations near the AF critical point\cite{21}. However, the precise nature of the putative strange metallic state has not been understood yet due to a lack of theoretical control over the strongly coupled theory that governs the critical point\cite{22}.

In this article, based on a controlled expansion, we show that a novel quantum state arises at the AF quantum critical point in metals that support one-dimensional Fermi surface through a non-trivial interplay between kinetic energy and interactions. To the lowest order in the perturbative expansion that becomes asymptotically exact at low energies in three dimensions, we find that quasiparticles are destroyed even though the interaction between electrons and the AF mode is screened to zero in the low energy limit. This unusual behavior is possible as the system develops an infinite sensitivity to the interaction through the kinetic energies that become dispersionless in certain spatial directions. The dynamical balance between vanishing kinetic energy and interactions results in a stable \textit{quasi-local strange metal} which supports incoherent single-particle excitations and enhanced correlations for various competing orders.

**Model and dimensional regularization.** We first consider two space dimensions. Although

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FIG. 1: A two-dimensional Fermi surface where the shaded (unshaded) region represents occupied (unoccupied) states in momentum space. The hot spots on the Fermi surface are denoted as (red) dots. The (green) arrows represent the AF wavevector $\vec{Q}_{\text{AF}}$.

the specific lattice is not crucial for the following discussion, we consider the square lattice with the nearest and next-nearest neighbor hoppings. For electron density close to half-filling (one electron per site), the system supports a Fermi surface shown in Fig. 1. The minimal theory that describes the AF critical point in the two dimensional metal includes the collective AF fluctuations that are coupled to electrons near the hot spots, which are the set of points on the Fermi surface connected by the AF wavevector [19, 20, 22]. In this paper we consider the collinear AF order with a commensurate wavevector that is denoted as arrows in Fig. 1. If the AF order is incommensurate or non-collinear, the critical theory is modified from the one for the collinear AF order with a commensurate wavevector. As will be shown later, the simplest case we consider here already has quite intricate structures.

The action for the commensurate AF mode and the electrons near the hot spots reads

$$S = \sum_{n=1}^{4} \sum_{m=\pm} \sum_{\sigma=\uparrow,\downarrow} \int \frac{d^3k}{(2\pi)^3} \psi^{(m)*}_{n,\sigma}(k) \left[ ik_0 + e^m_n(k) \right] \psi^{(m)}_{n,\sigma}(k)$$

$$+ \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ q_0^2 + c^2|q|^2 \right] \vec{\phi}(q) \cdot \vec{\phi}(q)$$

$$+ g_0 \sum_{n=1}^{4} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ \vec{\phi}(q) \cdot \psi^{(+)*}_{n,\sigma}(k+q) \vec{\tau}_{\sigma,\sigma'} \psi^{(-)}_{n,\sigma'}(k) + \text{c.c.} \right]$$
In this coordinate, the energy dispersions of the fermions near the hot spots can be written as
\[ e_n^{(\pm)}(\vec{k}) = \mp e_0(\vec{k}) = \mp v k_x \pm k_y, \]
where \( \vec{k} \) represents deviation of momentum away from each hot spot. It is noted that local curvature of the Fermi surface can be ignored because the \( k \)-linear terms dominate at low energies. The component of Fermi velocity parallel to \( \vec{Q}_{AF} \) at each hot spot is set to be unity up to sign by rescaling \( \vec{k} \). \( v \) measures the component of the Fermi velocity perpendicular to \( \vec{Q}_{AF} \). If \( v \) was zero, the hot spots connected by \( \vec{Q}_{AF} \) would be perfectly nested. \( \vec{\phi}(q) \) represents three components of boson field which describes the fluctuating AF order parameter carrying frequency \( q_0 \) and momentum \( \vec{Q}_{AF} + \vec{q} \). \( \vec{\tau} \) represents the three generators of the \( SU(2) \) group. \( c \) is the velocity of the AF collective mode. \( g_0 \) is the Yukawa coupling between the collective mode and the electrons near the hot spots, and \( u_0 \) is the quartic coupling between the collective modes. \( v, c, g_0, u_0 \) are genuine parameters of the theory, which can not be removed by redefinition of momentum or fields.

In two dimensions, the perturbative expansion in \( g_0, u_0 \) fails because the couplings grow rapidly as the length scale is increased under the renormalization group (RG) flow. Although the growth of the couplings is tamed by screening, it is hard to follow the RG flow because the flow will stop (if it does) outside the perturbative window. In higher dimensions, the growth of the couplings becomes slower. Therefore we aim to tune the space dimension such that the balance between the slow growth of the couplings and the screening stabilizes the interacting theory at weak coupling. Here we increase the co-dimension of the Fermi surface while fixing its dimension to be one. A mere increase of co-dimension of the Fermi surface introduces a non-locality in the kinetic energy[23]. In order to keep locality of the theory, we introduce two-component spinors[14], by combining fermion fields on opposite sides of the Fermi surface, \( \Psi_{1,\sigma} = (\psi_{1,\sigma}^{(\pm)}, \psi_{3,\sigma}^{(\pm)})^T \), \( \Psi_{2,\sigma} = (\psi_{2,\sigma}^{(\pm)}, \psi_{4,\sigma}^{(\pm)})^T \), \( \Psi_{3,\sigma} = (\psi_{1,\sigma}^{(-)}, -\psi_{3,\sigma}^{(+)}), \Psi_{4,\sigma} = (\psi_{2,\sigma}^{(-)}, -\psi_{4,\sigma}^{(+)})^T \) and writing the kinetic term of the fermions as
\[ S_0 = \sum_{n=1}^4 \sum_{\sigma=\uparrow,\downarrow} \int \frac{d^3 k}{(2\pi)^3} \langle \bar{\Psi}_{n,\sigma}(k) \vec{i} \gamma_0 k_0 + i \gamma_1 \varepsilon_n(\vec{k}) \rangle \Psi_{n,\sigma}(k), \]
where \( \gamma_0 = \sigma_y, \gamma_1 = \sigma_x, \vec{\varepsilon}_n(\vec{k}) = \vec{\varepsilon}_1(\vec{k}), \vec{\varepsilon}_2(\vec{k}), \vec{\varepsilon}_3(\vec{k}), \vec{\varepsilon}_4(\vec{k}), \vec{\varepsilon}_5(\vec{k}) \) and \( \vec{\varepsilon}_6(\vec{k}) \). Now we add \( (d - 2) \) extra dimensions which are perpendicular to the Fermi surface. We also generalize the
SU(2) group to SU(Nc), and introduce Nf flavors of fermion to write a general theory,

\[ S = \sum_{n=1}^{4} \sum_{\sigma=1}^{Nc} \sum_{j=1}^{Nf} \int dk \bar{\Psi}_{n,\sigma,j}(k) \left[ i \gamma \cdot K + i \gamma_{d-1} \epsilon_n(\vec{k}) \right] \Psi_{n,\sigma,j}(k) \]

\[ + \frac{1}{4} \int dq \left[ |Q|^2 + c^2 |\vec{q}|^2 \right] \text{Tr} \left( \Phi(-q) \Phi(q) \right) \]

\[ + i g \mu^{(3-d)/2} \sqrt{N_f} \sum_{n=1}^{N_c} \sum_{\sigma,\sigma'=1}^{N_f} \sum_{j=1}^{N_f} \int dk dq \left[ \bar{\Psi}_{n,\sigma,j}(k + q) \Phi_{\sigma,\sigma'}(q) \gamma_{d-1} \Psi_{n,\sigma',j}(k) \right] \]

\[ + \frac{\mu^{3-d}}{4} \int dk dk dq \left[ u_1 \text{Tr} \left( \Phi(k_1 + q) \Phi(k_2 - q) \right) \text{Tr} \left( \Phi(k_1) \Phi(k_2) \right) \right] + u_2 \text{Tr} \left( \Phi(k_1 + q) \Phi(k_2 - q) \Phi(k_1) \Phi(k_2) \right) \]  \hspace{1cm} (2)

Here \( dk \equiv \frac{d^{d+1}k}{(2\pi)^d} \) and \( k = (\mathbf{K}, \vec{k}) \) is \((d + 1)\)-dimensional vector. \( \vec{k} = (k_x, k_y) \) represents the original two-dimensional momentum and \( \mathbf{K} = (k_0, k_1, \ldots, k_{d-2}) \) includes frequency and momentum components along the \((d - 2)\) new directions present in \( d > 2 \). \( (\Gamma, \gamma_{d-1}) \) with \( \Gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{d-2}) \) represent \((d - 1)\)-dimensional gamma matrices that satisfy the Clifford algebra, \( \{ \gamma_\mu, \gamma_\nu \} = 2I \delta_{\mu\nu} \) with \( \text{Tr} [I] = 2 \). \( \Psi_{n,\sigma,j} \) with \( \sigma = 1, 2, \ldots, N_c \) and \( j = 1, 2, \ldots, N_f \) is in the fundamental representation of \( SU(N_c) \) spin group and \( SU(N_f) \) flavor group. \( \Phi(q) = \sum_{a=1}^{N^2_c - 1} \phi^a(q) \tau^a \) is a matrix field where \( \tau^a \)'s are the \( SU(N_c) \) generators with \( \text{Tr} \left[ \tau^a \tau^b \right] = 2 \delta^{ab} \). In the Yukawa interaction, \( (n, \bar{n}) \) represent pairs of hot spots connected by \( \vec{Q}_{AF} : \bar{I} = 3 \), \( \bar{2} = 4, 3 = 1, \bar{4} = 2 \). \( \mu \) is an energy scale introduced for the Yukawa coupling and the quartic couplings which have the scaling dimensions \((3 - d)/2 \) and \((3 - d)\) respectively. For \( N_c \leq 3 \), \( \text{Tr} [\Phi^4] = \frac{1}{2} \left( \text{Tr} [\Phi^2] \right)^2 \) and \( u_2 \) is not an independent coupling. In this case, it is convenient to set \( u_2 = 0 \) without loss of generality. For \( N_c \geq 4 \), however, \( u_1 \) and \( u_2 \) are independent, and one should keep both of them. It is straightforward to check that Eq. (1) is reproduced from Eq. (2) once we set \( d = 2 \), \( N_c = 2 \) and \( N_f = 1 \).

The action supports one-dimensional Fermi surfaces embedded in \( d \)-dimensional momentum space. The fermions have energy, \( E_n(k_1, \ldots, k_{d-2}, \vec{k}) = \pm \sqrt{\sum_{i=1}^{d-2} k_i^2 + \epsilon_n(\vec{k})^2} \) which disperses linearly in the \((d - 1)\)-dimensional space perpendicular to the line node defined by \( k_i = 0 \) for \( 1 \leq i \leq d - 2 \) and \( \epsilon_n(\vec{k}) = 0 \). To understand the physical content of the dimensional regularization, it is useful to consider the theory at \( d = 3 \). With the choice of \( (\gamma_0, \gamma_1, \gamma_2) = (\sigma_y, \sigma_z, \sigma_x) \) and identifying \( k_1 = k_z \), the kinetic energy for \( \Psi_{1,\sigma} \) and \( \Psi_{3,\sigma} \) is written as \( H_0 = (vk_x \pm k_y) \left[ \psi_{1,\sigma,j}^{(\pm)} \psi_{1,\sigma,j}^{(\pm)*} - \psi_{3,\sigma,j}^{(\pm)} \psi_{3,\sigma,j}^{(\pm)*} \right] \mp k_z \left[ \psi_{1,\sigma,j}^{(\pm)} \psi_{3,\sigma,j}^{(\pm)*} + h.c. \right] \). The kinetic energy for \( \Psi_{2,\sigma} \) and \( \Psi_{4,\sigma} \) can be obtained by 90° rotation. The first term gives patches of locally flat two-
FIG. 2: One-dimensional Fermi surfaces embedded in the three dimensional momentum space. The locally flat patches of two-dimensional Fermi surface near the hot spots are gapped out by the $p_z$-wave charge density wave carrying momentum $2\vec{k}_F$ except for the line nodes at $k_z=0$.

FIG. 3: A patch of Fermi surface created by a $p_z$-wave CDW with momentum $2\vec{k}_F$. The center of the patch is pinched due to the CDW order that vanishes linearly in $p_z$. If the antiferromagnetic order parameter connects the pinched points, the low energy effective theory for the critical point becomes Eq. (2) in three dimensions.

dimensional Fermi surface. The second term describes a $p_z$-wave charge density wave (CDW)
that gaps out the two-dimensional Fermi surface to leave line nodes at \( k_z = 0 \), as is shown in Fig. 2. The full action in Eq. (2) describes the AF transition driven by electrons near the hot spots on the line nodes.

We emphasize that Eq. (2) is not just a mathematical construction. The theory in three space dimensions can arise at the AF quantum critical point in the presence of \( p_z \)-wave CDW of momentum \( 2k_F \). If local curvature of the underlying Fermi surface is included, the dispersion near a pair of points on the Fermi surface connected by the momentum \( 2k_F \) can be written as \( \epsilon(\vec{k}) = \pm k_x + \gamma_1 k_y^2 + \gamma_2 k_z^2 \), where \( k_x \) is chosen to be perpendicular to the Fermi surface, and \( \gamma_i \) represent the local curvatures of the Fermi surface. The \( p_z \)-wave CDW leads to the spectrum, \( \mathcal{E} \) which is determined by

\[
\begin{vmatrix}
  k_x + \gamma_1 k_y^2 + \gamma_2 k_z^2 - \mathcal{E} & k_z \\
  -k_x + \gamma_1 k_y^2 + \gamma_2 k_z^2 - \mathcal{E} & k_z
\end{vmatrix} = 0.
\]

This results in a pinched Fermi surface located at \( \gamma_1 k_y^2 + \gamma_2 k_z^2 = \sqrt{k_x^2 + k_z^2} \) as is shown in Fig. 3. If the antiferromagnetic ordering wave vector connects the pinched points, the low energy effective theory for the phase transition is precisely described by Eq. (2) in three dimensions. Because the curvature is irrelevant at low energies, the pinched Fermi surfaces can be regarded as Fermi lines near the hot spots. Similar field theory can also arise at an orbital selective antiferromagnetic quantum critical point in three-dimensional semi-metal as is discussed in Appendix A.

The action in general dimensions respects the \( U(1) \) charge conservation, the \( SU(N_c) \) spin rotation, the \( SU(N_f) \) flavor rotation, the 90° space rotation in \((k_x, k_y)\), the reflections, and the time-reversal symmetries. For \( N_c = 2 \), the pseudospin symmetry, which rotates \( \Psi_{n,\sigma,j}(k) \) into \( i\sigma^{(u)}_{\sigma,\sigma'} \gamma_0 \Psi^T_{n,\sigma',j}(-k) \), is present[22]. The action in Eq. (2) is also invariant under the \( SO(d-1) \) rotation in \( K \). In Appendix B we provide further details on symmetry.

The theory in \( 2 \leq d \leq 3 \) continuously interpolates the physical theories which describe the AF critical points in \( d = 2 \) and 3. Because the couplings are marginal in three dimensions, we consider \( d = 3 - \epsilon \) and expand around three space dimension using \( \epsilon \) as a small parameter. We use the field theoretic renormalization group scheme to compute the beta functions which govern the RG flow of the renormalized velocities and coupling constants.

By embedding the one-dimensional Fermi surface in higher dimensions, the density of state (DOS) is reduced to \( \rho(E) \sim E^{d-2} \). As is the case for the usual dimensional regularization scheme for relativistic field theories, the reduced DOS tames quantum fluctuations at low energies and allows us to access low energy physics in a controlled way. Of course, there is no guarantee that the physics obtained near \( d = 3 \) is continuously extrapolated all the way to \( d = 2 \) because of the
possibility that some operators that are irrelevant near \( d = 3 \) become relevant to drive instability near \( d = 2 \). However, it is our very goal to systematically examine the potential instability as dimension is lowered toward \( d = 2 \), for which we first need to establish the existence of stable fixed point at \( d = 3 \), which can be realized on its own.

**Strange metal fixed point.** We include one-loop quantum corrections to obtain the beta functions for the velocities and couplings (see Appendices C and D for computational details),

\[
\frac{dv}{dl} = -\left(\frac{N_c^2 - 1}{2N_cN_f\pi^2}\right) z v \frac{g^2}{c} h_2(v, c),
\]

\[
\frac{dc}{dl} = -\frac{zg^2}{8\pi^2} \left[ \frac{c}{v} \pi - \frac{2(N_c^2 - 1)}{N_cN_f} [h_1(v, c) - h_2(v, c)] \right] ,
\]

\[
\frac{dg}{dl} = \frac{z}{2} g \left[ \epsilon - \frac{g^2}{4\pi u} - \frac{g^2}{4\pi^3 N_cN_f c} \{2(N_c^2 - 1)\pi h_2(v, c) - h_3(v, c)\} \right] ,
\]

\[
\frac{du_1}{dl} = \frac{zu_1}{2c^2\pi^2} \left[ c^2\pi \left( 2\pi\epsilon - \frac{g^2}{v} \right) + \frac{(N_c^2 - 1)}{N_fN_c} c g^2 \{h_1(v, c) - h_2(v, c)\} - (N_c^2 + 7) u_1 - \frac{2(N_c^2 - 3)}{N_c} u_2 - 3 \left( 1 + \frac{3}{N_c^2} \right) \frac{u_2^2}{u_1} \right] ,
\]

\[
\frac{du_2}{dl} = \frac{zu_2}{2c^2\pi^2} \left[ c^2\pi \left( 2\pi\epsilon - \frac{g^2}{v} \right) + \frac{(N_c^2 - 1)}{N_fN_c} c g^2 \{h_1(v, c) - h_2(v, c)\} - 12 u_1 - \frac{2(N_c^2 - 9)}{N_c} u_2 \right] .
\]

Here \( l \) is the logarithmic length scale. \( z = \left[ 1 - \frac{(N_c^2 - 1)}{4N_cN_f\pi^2} \frac{g^2}{c} \left\{ h_1(v, c) - h_2(v, c) \right\} \right]^{-1} \) is the dynamical critical exponent that determines the scaling dimension of \( K \) relative to \( k \). \( h_i(v, c) \) are given by \( h_1(v, c) = \int_0^1 dx \sqrt{c^2 + (1 + v^2 - c^2)x^2} \), \( h_2(v, c) = c^2 \int_0^1 dx \sqrt{c^2 + (1 + v^2 - c^2)x^2} \) and \( h_3(v, c) = \sqrt{\frac{1}{\sqrt{c^2 + (1 + v^2 - c^2)x^2}}} \) with

\[
\zeta(v, c, x_1, x_2, \theta) = \frac{2v}{c} (x_1 \cos^2 \theta + x_2 \sin^2 \theta) + (1 - x_1 - x_2) \left[ vc \cos^2 (\theta + \pi/4) + \frac{c}{v} \sin^2 (\theta + \pi/4) \right] .
\]

The RG flow of the couplings and the velocities is shown in Fig. 4a. We first examine the RG flow in the subspace of \( g = 0 \). At the Gaussian fixed points (\( u_1 = g = 0 \) with \( v, c \neq 0 \)), the theory is free. With \( u_1 \neq 0 \) and \( u_2 = 0 \), the theory flows to the \( O(N_c^2 - 1) \) Wilson-Fisher (WF) fixed points at \( u_1^* = \frac{4\pi^2\lambda}{N_c^2 + 4\pi^2} \) with dynamical critical exponent \( z = 1 \). For \( N_c \geq 4 \), one also needs to consider \( u_2 \). As \( u_2 \) is turned on, the \( O(N_c^2 - 1) \) WF fixed points become unstable and it shows a run-away flow[24] which suggests a first-order phase transition.

In the presence of Yukawa coupling, a stable low energy fixed point arises. If some components of the velocities were not allowed to flow, the theory could flow to a fixed point with finite
FIG. 4: One-loop RG flow of the couplings and velocities for $N_c = 2$, $N_f = 1$ and $\epsilon = 0.01$. We set $u_2 = 0$, which can be done without loss of generality for $N_c < 4$. The solid lines denote flows in the three-dimensional space of the parameters shown in the figure. The dashed lines represent flows within the subspace of $g = 0$. (a) The whole manifold of $g = u_1 = 0$ represents the non-interacting (Gaussian) fixed points parameterized by $(v, c)$. Once $u_1$ is turned on at the Gaussian fixed points (denoted by circles), the theory flows to the Wilson-Fisher fixed points (squares). As the Yukawa coupling is introduced, couplings and velocities flow to the stable fixed point at $g^* = u_i^* = v^* = c^* = 0$. (b) RG flow of the ratios of the parameters for the same values of $N_c$, $N_f$ and $\epsilon$ as in (a). The Yukawa coupling measured in the unit of $\sqrt{v}$ and the ratio of the two velocities remain non-zero at the stable fixed point.
couplings[25]. However, the full RG flow is more complicated because of running velocities. As $g$ is turned on, it initially grows as is expected from the fact that it is relevant below $d = 3$. As $g$ grows, the fermions at different hot spots are mixed with each other through quantum fluctuations. As a result, the hot spots become increasingly nested at low energies: $v$ flows to zero as $1/l$ for $\epsilon > 0$ and as $1/log(l)$ for $\epsilon = 0$ in the low energy limit. The dynamical nesting of the fermionic band, in turn, modifies the AF mode in two important ways. First, the boson becomes increasingly slow in the $q_x, q_y$ directions because the collective mode can decay into dispersionless particle-hole pairs near the nested hot spots. As a result, $c$ decreases toward zero, leading to emergent locality in the $(x, y)$ space. Second, quantum fluctuations become more and more efficient in screening the interactions due to the abundant low energy density of states supported by the nested Fermi surface and the dispersionless boson. In summary of the RG flow, i) $g$ induces dynamical nesting, renormalizing $v, c$ to smaller values, ii) smaller $v, c$ make screening more efficient, making $g, u_i$ smaller. This cycle of negative feedback leaves no room for a coexistence of the kinetic terms $(v, c)$ and the interactions $(g, u_i)$. It has only one fate down the road of RG flow: mutual destruction. To the one-loop order, all of $v, c, g, u_i$ eventually flow to zero in the low energy limit at and below three dimensions if initial values of $u_i$’s are not too large in magnitude.

The new interactionless and quasi-dispersionless fixed point is distinct from the Gaussian fixed point which is dominated by the kinetic energy. Unlike at the Gaussian fixed point, the kinetic energy and the interactions maintain ‘a balance of power’ along the path to their demise. This can be seen from the fact that the ratios defined by

$$w \equiv \frac{v}{c}, \quad \lambda \equiv \frac{g^2}{v}, \quad \kappa_i \equiv \frac{u_i}{c^2}$$

flow to a stable fixed point,

$$w^* = \frac{N_c N_f}{N_c^2 - 1}, \quad \lambda^* = \frac{4\pi(N_c^2 + N_c N_f - 1)}{N_c^2 + N_c N_f - 3} \epsilon, \quad \kappa_i^* = 0$$

in the $c \to 0$ limit as is shown in Fig. 4b. At the fixed point, the dynamical critical exponent is renormalized to $z = 1 + \frac{\lambda^*}{8\pi}$ to the leading order in $\lambda$. The non-trivial quantum correction to $z$ implies that the effect of interaction is not gone even though $g, u_i$ vanish in the low energy limit. This is due to the emergent locality associated with the dynamical nesting of the Fermi surface and the dispersionless bosonic spectrum. The IR singularity supported by the locality makes the system infinitely susceptible to interaction, leading to finite quantum corrections even with vanishing interactions. The fixed point is stable for general $N_c$ and $N_f$, and small perturbations of
\(w, \lambda, \kappa\) away from Eq. (9) die out in the low energy limit. In particular, the \(\phi^4\) vertices acquire an anomalous dimension and become irrelevant at the new fixed point. The one-loop fixed point is exact at \(d = 3\) because higher order terms are systematically suppressed by \(\lambda\) and \(\kappa\), which flow to zero in the low energy limit. For \(d < 3\), \(\kappa, c, v\) can receive higher-loop corrections to become nonzero at the fixed point. The details on higher-loop contributions can be found in Appendix E.

If the initial value of \(\kappa_1\) is sufficiently large and negative, \(\kappa_1\) runs away to \(-\infty\), potentially driving a first-order transition. The stable fixed point in Eq. (9) and the run-away flow is separated by an unstable fixed point at \(\kappa^*_1 = -\frac{4\pi^2\epsilon}{(N^2_c + 1)(N^2_c + N_cN_f - 3)}\), \(\kappa^*_2 = 0\) with the same values of \(w^*\) and \(\lambda^*\) as in Eq. (9). The unstable fixed point, which can be realized at a multi-critical point, describes a state distinct from the state described by the stable fixed point in Eq. (9). The two fixed points are distinguished by the different ways the couplings and velocities approach the origin.

**Physical properties.** The existence of the stable low energy fixed point implies scale invariance of the Green’s function in the limit \(k_x, k_y, |K|\) go to zero with \(k_y/k_x, |K|/|k_x|\) fixed at the second order phase transition. Here we focus on the Green’s function near the hot spot 1+ in Fig. 1. The Green’s function near other hot spots can be obtained by applying reflection or \(90^\circ\) rotation. In the scaling limit, the fermion Green’s function takes the form,

\[
G(k) = \frac{1}{|k_y|^{1-2\tilde{\eta}_\psi}} \tilde{G} \left( \frac{|K|}{|k_y|^{1/2}} \right),
\]

where \(\tilde{\eta}_\psi \sim O(\epsilon^2)\) is the net anomalous dimension which vanishes to the linear order in \(\epsilon\) and \(\tilde{G}(x)\) is a universal function. Because \(v\) flows to zero logarithmically in the low energy limit, the dependence on \(k_x\) is suppressed as \(\frac{k_y}{\log(1/k_x)}\) for \(d < 3\) and as \(\frac{k_y}{\log(\log(1/k_x))}\) at \(d = 3\) in the scaling limit. The dynamical critical exponent is non-trivial even to the linear order in \(\epsilon\) for \(d < 3\). As a result, the spectral function shows a power-law distribution in energy instead of a delta function peak, exhibiting a non-Fermi liquid behavior. At \(d = 3\), we have \(z = 1\) as in Fermi liquid. However, the Green’s function is modified by logarithmic corrections compared to that of the Fermi liquid due to \(\lambda\) which flows to zero logarithmically. This is a marginal Fermi liquid[26]. Since the boson velocity also flows to zero in the same fashion \(v\) flows to zero, the boson Green’s function becomes independent of \(k_x, k_y\) in the scaling limit up to corrections that are logarithmically suppressed,

\[
D(k) = \frac{C}{|K|^{2-2\tilde{\eta}_\phi}},
\]

where \(C\) is a constant and \(\tilde{\eta}_\phi \sim O(\epsilon^2)\). This *quasi-local strange metal* supports non-quasiparticle excitations which are dispersionless along \(x, y\) directions in the scaling limit. Here the effec-
tive space dimension becomes dynamically reduced as a result of quantum fluctuations. The quasi-local behaviors associated with extreme velocity anisotropies were reported in nodal semimetals[27, 28]. Local critical behaviors with $z = \infty$ also arise in the dynamical mean-field approximation for the Kondo lattice model[29] and from gravitational constructions[30–32]. The present quasi-local state is distinct from the earlier examples in that it is a stable zero temperature state which supports extended Fermi surface with a finite $z$.

The quasi-local strange metal is stable at the one-loop order which becomes exact in the $d \rightarrow 3$ limit. As one approaches $d = 2$, higher order corrections become important. The theory at $d = 2$ remains strongly coupled even in the large $N_c$ and/or large $N_f$ limit. One possibility is that the quasi-local strange metal becomes unstable towards an ordered state below a critical dimension. To identify the channels that may become unstable at $d = 2$, we examine charge density wave (CDW) and superconducting (SC) correlations that are enhanced by quantum fluctuations[21, 22]. In principle, particle-hole or particle-particle fluctuations between un-nested patches of Fermi surface may drive an instability if the coupling is strong at the lattice scale[34]. However, those operators that connect nested patches receive strongest quantum corrections.

In the spin-singlet CDW channel, the set of operators

$$O_{CDW}^{\pm} = \int dk \left[ (\bar{\Psi}_{1,\sigma,j} \Psi_{1,\sigma,j} + \bar{\Psi}_{3,\sigma,j} \Psi_{3,\sigma,j}) \pm (\bar{\Psi}_{2,\sigma,j} \Psi_{2,\sigma,j} + \bar{\Psi}_{4,\sigma,j} \Psi_{4,\sigma,j}) \right]$$

which describes a $p_y$-wave and a $p_x$-wave CDW, respectively, with momentum $2\vec{k}_F$ is most strongly enhanced. These CDW operators, which are pseudospin singlets for $N_c = 2$, break the reflection symmetry and represent bond density waves without on-site modulation of charge. This is different from the bond density wave order which forms a pseudospin doublet with the d-wave pairing order[35]. In the SC channel, we focus on the representation that is symmetric in $SU(N_f)$ and anti-symmetric in $SU(N_c)$, which reduces to the spin-singlet SC order for $N_c = 2, N_f = 1$. There are two sets of equally strong SC fluctuations. The first set of operators describes the $d_{x^2-y^2}$-wave and $g$-wave pairings with zero momentum[36, 37], while the second set of operators describes $s$-wave and $d_{xy}$-wave pairings with finite momentum, $2\vec{k}_F[38, 39]$. The attractive interaction for the pairing is mediated by the commensurate spin fluctuations that scatter a pair of electrons from one hot spot to another hot spot. Due to the nesting, the finite momentum pairing is as strong as the conventional zero momentum pairing to the one-loop order. The propensity for finite momentum pairing may lead to exotic superconducting states in two dimensions[40, 41]. If the quasi-local strange metal is unstable toward a competing order at low temperature in two
dimensions, the strange metallic behaviors predicted in Eqs. (10) and (11) can show up within a finite temperature window whose range can be made parametrically large by tuning $N_c$ and $N_f$[42]. For more details on the computation of the anomalous dimensions for the CDW and SC orders, please see Appendix F.

Within the perturbative regime that we explore in this paper, the anomalous dimensions for various susceptibilities associated with ‘hot’ electrons near the hot spots remain small. Because hot spots are only points in momentum space, thermodynamic and transport properties are dominated by cold electrons which exhibit Fermi liquid behaviors. For example, the specific heat will be proportional to $T^{2-\epsilon}$ to the leading order of temperature $T$, and the conductivity is expected to be dominated by cold electrons[33]. As one approaches $d = 2$, the contribution from hot electrons may, in principle, dominate over the contribution from the cold electrons as the anomalous dimensions become larger. Moreover, the behavior of cold electrons may also deviate from those of Fermi liquid far away from three dimensions, as the coupling between cold electrons and collective modes, which is irrelevant in the perturbative regime, becomes strong near $d = 2$[33]. In this case, non-Fermi liquid behavior may show up even for the thermodynamic and transport properties of cold electrons. However, we can not address this issue in a controlled manner because it requires strong coupling which lies outside the perturbative window.

**Conclusion.** We show that a novel strange metallic state emerges at the AF quantum critical point in a metal that supports one-dimensional Fermi surface based on a perturbative expansion which gives the exact low energy fixed point in three dimensions. Even though the interaction is screened to zero in the low energy limit, dynamical reduction of the effective dimensionality drives the system into a strange metallic state, which supports partially dispersionless incoherent single-particle excitations along with enhanced superconducting and charge density wave fluctuations. The present theory continuously interpolates between the three dimensional theory for one dimensional Fermi surfaces and two dimensional metals. The three-dimensional theory can arise at the AF quantum critical point in the presence of $p_z$-wave CDW, which is described by a stable quasi-local marginal Fermi liquid. Our formalism also provides a way to access potential instabilities of the non-Fermi liquids that arise at the AF quantum critical points below three dimensions as $\epsilon$ increases.
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Appendix A: A three-dimensional lattice model for a related field theory

![Diagram of a three-dimensional lattice model](image)

**FIG. 5:** (a) Flux lattice in the $XY$-plane. The green (dark) and the red (light) disks represent sites $A$ and $B$, respectively. (b) Three dimensional arrangement of the $A$ and $B$ sites.

In this section we construct a three dimensional lattice model in which a field theory similar to the one considered in the main text can be realized. We consider a tetragonal lattice where staggered fluxes pierce through unit plaquettes. A gauge is chosen such that the hopping $t_z$ along the $z$ direction is real. The nearest neighbor hoppings in the $XY$-plane are written as $\{t_1 e^{i\phi}, t_2 e^{i\phi}\}$ ($\{t_1 e^{-i\phi}, t_2 e^{-i\phi}\}$) in the two orthogonal directions along (against) the arrows, as is shown in Figs. 5a and 5b. Here the magnitudes of staggered flux per plaquette are $(4\phi, 2\phi, 2\phi)$ in the three planes.

In the coordinate system shown in Figs. 5a and 5b, the lattice vectors become $\vec{a}_1 = (1, 0, 0)$, $\vec{a}_2 = (0, 1, 0)$, $\vec{a}_3 = \frac{1}{2}(1, 1, 1)$, where the distance between nearest neighbor $A$ sites in the $XY$-
plane is chosen to be 1. The reciprocal vectors are given by \( \vec{b}_1 = 2\pi(1, 0, -1), \vec{b}_2 = 2\pi(0, 1, -1), \vec{b}_3 = 4\pi(0, 0, 1) \).

\[ \text{FIG. 6: The (blue) curves represent the Fermi lines embedded in the three dimensional momentum space for Eq. (A3) with } t_1 = 0.6t_2. \text{ The (green) arrows represent the antiferromagnetic ordering vector with } \vec{Q} = (\pi, \pi, 0). \text{ The shaded regions represent the } k_z = \pm \pi \text{ planes. There are four distinct hot spots connected by } \vec{Q} \text{ denoted by } n = 1, 2, 3, 4. \]

The tight binding Hamiltonian with the nearest neighbor hoppings becomes

\[
H = -\sum_{\vec{k}} \left[ \mathcal{D}(\vec{k}) \ c_A^\dagger(\vec{k}) \ c_B(\vec{k}) + \text{h.c.} \right], \tag{A1}
\]

where \( c_{A(B)} \) is the destruction operator for electrons at \( A \) (\( B \) sites), and

\[
\mathcal{D}(\vec{k}) = 2 \left[ \cos(\phi) \left\{ t_+ \cos \left( \frac{k_x}{2} \right) \cos \left( \frac{k_y}{2} \right) + t_- \sin \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) \right\} + t_z \cos \left( \frac{k_z}{2} \right) \right] \\
+ 2i \sin(\phi) \left\{ t_+ \sin \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) + t_- \cos \left( \frac{k_x}{2} \right) \cos \left( \frac{k_y}{2} \right) \right\}, \tag{A2}
\]

with \( t_{\pm} = t_1 \pm t_2 \). The Hamiltonian is diagonal in spin indices, and we have suppressed the spin indices in the electron operators. The \( 2 \times 2 \) Hamiltonian supports a particle-hole symmetric band with the dispersion \( E(\vec{k}) = \pm \left| \mathcal{D}(\vec{k}) \right| \) at half filling. With the choice of \( 0 < t_1 < t_2 \) and \( \phi = \frac{\pi}{2} \), one obtains one-dimensional Fermi surfaces (or \textit{Fermi lines}) located at

\[
k_z = \pm \pi, \\
\sin \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) + \frac{t_-}{t_+} \cos \left( \frac{k_x}{2} \right) \cos \left( \frac{k_y}{2} \right) = 0. \tag{A3}
\]
The Fermi lines embedded in the three-dimensional momentum space are shown in Fig. 6.

We assume that there exists an electron-electron interaction which drives the semi-metal into an antiferromagnetic state. In particular, we consider an orbital selective antiferromagnetic order, where electrons in the bonding and anti-bonding states within an unit cell have opposite spins, which then modulate with momentum $(\pi, \pi, 0)$ in space. This is illustrated in Fig. 7. If the phase transition is continuous, the critical spin fluctuations associated with the order strongly interact with electrons on the Fermi lines connected by the ordering vector $\vec{Q} = (\pi, \pi, 0)$ as is shown in Fig. 6. In this case, there exist four distinct hot spots connected by the ordering vector. As is considered in the main text, the minimal theory that describes the quantum critical point includes the electronic excitations near the hot spots and the critical antiferromagnetic mode that is coupled with electrons through the Yukawa coupling,

\[
\mathcal{S}_{\text{eff}} = \int \frac{d^4k}{(2\pi)^4} \sum_n \bar{\psi}_n(k) \left[ i k_0 \gamma_0 + i \gamma_1 k_z + i \gamma_2 \epsilon_n(k) \right] \psi_n(k) + i \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(q) \cdot \left[ \bar{\psi}_{1,s}(k + q) \gamma_2 \sigma_{s,s'} \psi_{4,s'}(k) + \bar{\psi}_{2,s}(k + q) \gamma_2 \sigma_{s,s'} \psi_{3,s'}(k) \right] + \text{h.c.} \\
+ \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left[ q_0^2 + c_q^2 q_x^2 + c_q^2 q_y^2 \right] \Phi(-q) \cdot \Phi(q) + \frac{u_0}{4!} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \left[ \Phi(q_1 + q_2) \cdot \Phi(q_2) \right] \left[ \Phi(q_3 - q_2) \cdot \Phi(q_3) \right].
\]

FIG. 7: A real space pattern for the orbital selective antiferromagnetic order (a) on the $XY$ plane and (b) in the full three dimensional lattice.
Here \( \Psi_n(\vec{k}) \equiv \begin{pmatrix} c_A(\vec{K}_n + \vec{k}) \\ c_B(\vec{K}_n + \vec{k}) \end{pmatrix} \) with \( n = 1, 2, 3, 4 \) denotes electrons near the \( n \)-th hot spot with

\[
\vec{K}_1 = (\pm \pi, 0, \pm \pi), \quad \vec{K}_2 = (0, \pm \pi, \pm \pi), \quad \vec{K}_3 = (\mp \pi, 0, \pm \pi), \quad \vec{K}_4 = (0, \mp \pi, \pm \pi),
\]

and \( \bar{\Psi}_n \equiv \Psi_n^\dagger \gamma_0 \) with \( \gamma_0 = \sigma_z, \gamma_1 = \sigma_y, \) and \( \gamma_2 = \sigma_x \). \( \varepsilon_n(\vec{k}) \) is the linearized dispersion around each hot spot,

\[
\varepsilon_1(\vec{k}) = t_- k_x - t_+ k_y, \quad \varepsilon_2(\vec{k}) = -t_+ k_x + t_- k_y, \\
\varepsilon_3(\vec{k}) = -t_- k_x + t_+ k_y, \quad \varepsilon_4(\vec{k}) = t_+ k_x - t_- k_y.
\]

We have scaled away \( t_z \) dependence by absorbing it in the \( \hat{z} \)-component of momentum. \( \tilde{\Phi}(q) \) represents the critical fluctuations of the SDW order parameter.

Unlike the action in Eq. (2), Eq. (A4) lacks the \( C_4 \) symmetry in the \( XY \)-plane and the \( SO(2) \) rotation symmetry in the \( k_0 - k_z \) plane. This results in velocity anisotropy for the bosons. However, we expect that this theory also flows to a quasi-local fixed point similar to the one discussed in the main text[43].

**Appendix B: Symmetry**

In this section, we elaborate on the symmetries of the action in Eq. (2). The internal symmetry is \( U(1)^2 \times SU(N_c) \times SU(N_f)^2 \) associated with charge, spin and flavor conservations. There are two \( U(1) \)'s and two \( SU(N_f) \)'s because the charge and flavor are conserved within the two sets of hot spots (\( \{1, 3\} \) and \( \{2, 4\} \)) separately. Besides the internal symmetry, the action has \( \pi/2 \) rotation and reflection symmetries under which the spinors transform as is shown in the Table I. In \( d > 2 \), the \( SO(d-1) \) spacetime rotational symmetry is present. Under \( SO(d-1) \) rotation, \( k_\mu \) and \( \bar{\Psi}_{n,\sigma,j} \gamma_\mu \Psi_{n,\sigma,j} \) form vectors for \( \mu = 0, 1, \ldots, d-2 \). For \( N_c = 2 \), there also exists a pseudo-spin symmetry under which the super-spinor, \( \chi_{n,\sigma,j}(k) = \left( \Psi_{n,\sigma,j}(k), iT^{(y)}_{\sigma,\sigma'} \gamma_0 \bar{\Psi}_{n,\sigma',j}(-k) \right)^T \) transforms as \( \chi_{n,\sigma,j} \mapsto U \chi_{n,\sigma,j} \), where \( U \) represents \( SU(2) \) matrix that acts on the particle-hole space.

**Appendix C: Renormalization group analysis**

In this section, we describe the method that is used to compute the beta functions for the velocities and couplings. In order to incorporate quantum corrections, we renormalize the theory
where ε in the minimal subtraction scheme, the counter terms only include contributions that are divergent symmetries guarantee that the counter terms take the following form,

\[
\begin{array}{|c|c|c|c|}
\hline
\Psi_1(k) & \Psi_2(k_{R/s/2}) & -i\gamma_0\Psi_3(k_{R_s}) & i\gamma_0\gamma_{d-1}\Psi_3(k_{R_y}) \\
\Psi_2(k) & \gamma_{d-1}\Psi_1(k_{R/s/2}) & i\gamma_0\gamma_{d-1}\Psi_4(k_{R_x}) & -i\gamma_0\Psi_4(k_{R_y}) \\
\Psi_3(k) & \Psi_4(k_{R/s/2}) & i\gamma_0\Psi_1(k_{R_x}) & i\gamma_0\gamma_{d-1}\Psi_1(k_{R_y}) \\
\Psi_4(k) & -\gamma_{d-1}\Psi_3(k_{R/s/2}) & i\gamma_0\gamma_{d-1}\Psi_2(k_{R_x}) & i\gamma_0\Psi_2(k_{R_y}) \\
\hline
\end{array}
\]

TABLE I: Table of spinors obtained by applying the spatial π/2 rotation and reflections in the x and y directions accompanied by reflections in k_1, k_2, \ldots, k_{d-2}. Under the three space symmetries, the energy-momentum vector k = (k_0, k_1, \ldots, k_{d-2}, k_x, k_y) is transformed to k_{R/s/2} = (k_0, k_1, \ldots, k_{d-2}, -k_y, k_x), k_{R_x} = (k_0, -k_1, \ldots, -k_{d-2}, k_x, k_y) and k_{R_y} = (k_0, -k_1, \ldots, -k_{d-2}, k_x, -k_y), respectively. The spin and flavor indices are suppressed.

by ‘tuning’ the parameters in the action in Eq. (2) such that the physical observables become insensitive to the UV cut-off scale. This amounts to adding counter terms that remove UV divergences in the quantum effective action order by order in the couplings. The internal and spacetime symmetries guarantee that the counter terms take the following form,

\[
S_{CT} = \sum_{n=1}^{4} \sum_{\sigma=1}^{N_c} \sum_{j=1}^{N_f} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\Psi}_{n,\sigma,j}(k) \left[ iA_1 \Gamma \cdot K + iA_3 \gamma_{d-1} \varepsilon_n \left( \frac{k}{A_2} \right) \right] \Psi_{n,\sigma,j}(k) + \frac{1}{4} \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \left[ A_4 |Q|^2 + A_5 c^2 |q|^2 \right] \text{Tr} \left[ \Phi(-q) \Phi(q) \right] + \frac{iA_6}{\sqrt{N_f}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \left[ \Psi_{n,\sigma,j}(k+q) \Phi_{\sigma,\sigma'}(q) \gamma_{d-1} \Psi_{n,\sigma',j}(k) \right] + \frac{iA_7 u_1 \text{Tr} \left[ \Phi(k_1 + q) \Phi(k_2) \right] \text{Tr} \left[ \Phi(k_2) \right] + \mathcal{A}_8 u_2 \text{Tr} \left[ \Phi(k_1 + q) \Phi(k_2 - q) \Phi(k_1) \Phi(k_2) \right] },
\]

where \( \varepsilon_1(\vec{k}; v) = v k_x + k_y, \varepsilon_2(\vec{k}; v) = -k_x + v k_y, \varepsilon_3(\vec{k}; v) = v k_x - k_y, \) and \( \varepsilon_4(\vec{k}; v) = k_x + v k_y. \)

In the minimal subtraction scheme, the counter terms only include contributions that are divergent in the \( \epsilon \to 0 \) limit,

\[
A_n \equiv A_n(v, c, g, u; \epsilon) = \sum_{m=1}^{\infty} \frac{Z_{n,m}(v, c, g, u)}{\epsilon^m},
\]

where \( Z_{n,m}(v, c, g, u) \) are finite functions of the couplings in the \( \epsilon \to 0 \) limit. The renormalized action is given by the sum of the original action and the counter terms, which can be expressed in
terms of bare fields and bare couplings,

\[
\mathcal{S}_{\text{ren}} = \sum_{n=1}^{d+1} \int \frac{d^{d+1}k_B}{(2\pi)^{d+1}} \Psi_{B_{n,\sigma,j}}(k_B) \left[ i\Gamma \cdot k_B + i\gamma_{d-1}\varepsilon_n(k_B; v_B) \right] \Psi_{B_{n,\sigma,j}}(k_B) + \frac{1}{4} \int \frac{d^{d+1}q_B}{(2\pi)^{d+1}} \left[ |Q_B|^2 + c_B^2 |q_B|^2 \right] \text{Tr} \left[ \Phi_B(-q_B) \Phi_B(q_B) \right] + \frac{i g_B}{\sqrt{N_f}} \sum_{n=1}^{d+1} \int \int d^d\vec{q} \sum_{j=1}^{N_f} \int d^d\vec{k} \sum_{\sigma, \sigma'} \frac{1}{(2\pi)^{d+1}} \left[ \Psi_{B_{n,\sigma,j}}(k_B + q_B) \Phi_{B_{B,\sigma',j}}(q_B) \gamma_{d-1} \Psi_{B_{B,\sigma',j}}(k_B) \right] + \frac{1}{4} \int \frac{d^{d+1}k_{1B} k_{2B} k_{3B}}{(2\pi)^{d+1}} \int d^d\vec{b} \sum_{\sigma, \sigma'} \left[ u_1 \text{Tr} \left[ \Phi_B(k_{1B} + q_B) \Phi_B(k_{2B} - q_B) \right] \right] \text{Tr} \left[ \Phi_B(k_{1B}) \Phi_B(k_{2B}) \right] + u_2 \text{Tr} \left[ \Phi_B(k_{1B} + q_B) \Phi_B(k_{2B} - q_B) \Phi_B(k_{1B}) \Phi_B(k_{2B}) \right].
\]

(C3)

Here the renormalized quantities are related to the bare ones through

\[
\begin{align*}
K &= Z_{\tau}^{-1} K_B, \\
\Psi_{n,\sigma,j}(k) &= Z_\psi^{-1/2} \Psi_{B_{n,\sigma,j}}(k_B), \\
v &= Z_3 \frac{Z_2}{Z_1} v_B, \\
g &= \frac{Z_\phi Z_\tau^{2(d-1)}}{Z_6} \mu^{-3-d} g_B, \\
u_2 &= \frac{Z_\phi Z_\tau^{3(d-1)}}{Z_8} \mu^{-(3-d)} u_2 B, \\
\Phi(q) &= Z_\phi^{-1/2} \Phi_B(q_B), \\
c &= \frac{Z_\phi Z_\tau^{d-1}}{Z_5} c_B, \\
u_1 &= \frac{Z_\phi Z_\tau^{3(d-1)}}{Z_7} \mu^{-(3-d)} u_1 B, \\
Z_{\tau} &= Z_3, Z_\psi = Z_1 Z_{\tau}^{-d}, \text{ and } Z_\phi = Z_4 Z_{\tau}^{-(d+1)} \text{ with } Z_n = 1 + A_n. \text{ We use the freedom of choosing an overall scale to fix the scaling dimension of } \vec{k} \text{ to be 1. The renormalized Green's function defined through}
\end{align*}
\]

\[
\langle \Psi(k_1) \ldots \Psi(k_f) \bar{\Psi}(k_{f+1}) \ldots \bar{\Psi}(k_{2f}) \Phi(q_1) \ldots \Phi(q_b) \rangle
\]

\[
= G^{(f,b)}(k_i, q_j; v, c, g, u, \mu) \delta^{d+1} \left( \sum_{i=1}^{f} (k_i - k_{i+f}) + \sum_{j=1}^{b} q_j \right)
\]

(C5)

satisfies the renormalization group equation,

\[
\begin{align*}
\left[ z \left( K_i \cdot \nabla K_i + Q_j \cdot \nabla Q_j \right) + \left( \vec{k} \cdot \nabla \vec{k} + \vec{q} \cdot \nabla \vec{q} \right) - \beta_v \frac{\partial}{\partial v} - \beta_c \frac{\partial}{\partial c} - \beta_g \frac{\partial}{\partial g} - \beta_u \frac{\partial}{\partial u} \right] + 2f \left( \frac{d+2}{2} - \eta_\psi \right) + b \left( \frac{d+2}{2} - \eta_\phi \right) - (z (d-1) + 2) \right] G^{(f,b)}(k_i, q_j; v, c, g, u, \mu) = 0.
\end{align*}
\]

(C6)
Here the dynamical critical exponent and the anomalous dimensions of the fields are given by
\[ z = 1 + \frac{\partial \ln Z_u}{\partial \ln \mu}, \eta_\psi = \frac{1}{2} \frac{\partial \ln Z_\psi}{\partial \ln \mu}, \eta_\phi = \frac{1}{2} \frac{\partial \ln Z_\phi}{\partial \ln \mu}, \]
and the beta functions that describe the flow of the parameters with increasing energy scale are given by \( \beta_v = \frac{\partial v}{\partial \ln \mu}, \beta_c = \frac{\partial c}{\partial \ln \mu}, \beta_g = \frac{\partial g}{\partial \ln \mu}, \beta_u = \frac{\partial u}{\partial \ln \mu} \). The set of coupled equations for the critical exponents and the beta functions can be rewritten as

\[
\mathcal{Z_3} [(d-1)(z-1) + 2\eta_\psi] - \mathcal{Z_3}' = 0,
\]
\[
\mathcal{Z_1} [d(z-1) + 2\eta_\psi] - \mathcal{Z_1}' = 0,
\]
\[
\mathcal{Z_4} [(d+1)(z-1) + 2\eta_\phi] - \mathcal{Z_4}' = 0,
\]
\[
\mathcal{Z_2} [\beta_v - v \{(d-1)(z-1) + 2\eta_\psi\}] + v\mathcal{Z_2}' = 0,
\]
\[
\mathcal{Z_5} [2\beta_c - c \{(d-1)(z-1) + 2\eta_\phi\}] + c\mathcal{Z_5}' = 0,
\]
\[
\mathcal{Z_6} \left[ \beta_g - g \left\{ \frac{3-d}{2} + 2(d-1)(z-1) + 2\eta_\psi + \eta_\phi \right\} \right] + g\mathcal{Z_6}' = 0,
\]
\[
\mathcal{Z_7} [\beta_{u_1} - u_1 \{- (3-d) + 3(d-1)(z-1) + 4\eta_\psi\}] + u_1\mathcal{Z_7}' = 0,
\]
\[
\mathcal{Z_8} [\beta_{u_2} - u_2 \{- (3-d) + 3(d-1)(z-1) + 4\eta_\phi\}] + u_2\mathcal{Z_8}' = 0,
\]
which solve to give

\[
z = \left[ 1 + \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (Z_{1,1} - Z_{3,1}) \right]^{-1},
\]
\[
\eta_\psi = -\frac{\epsilon}{2} z \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (Z_{1,1} - Z_{3,1}) + \frac{1}{2} z \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (2Z_{1,1} - 3Z_{3,1}),
\]
\[
\eta_\phi = -\frac{\epsilon}{2} z \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (Z_{1,1} - Z_{3,1}) + \frac{1}{2} z \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (4Z_{1,1} - 4Z_{3,1} - Z_{4,1}),
\]
\[
\beta_v = z v \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (Z_{2,1} - Z_{3,1}),
\]
\[
\beta_c = \frac{1}{2} z c \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (2Z_{1,1} - 2Z_{3,1} - Z_{4,1} + Z_{5,1}),
\]
\[
\beta_g = -z g \left[ \frac{\epsilon}{2} + \frac{1}{2} \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (2Z_{3,1} + Z_{4,1} - 2Z_{6,1}) \right],
\]
\[
\beta_{u_1} = -z u_1 \left[ \epsilon - \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (2Z_{1,1} - 2Z_{3,1} - 2Z_{4,1} + Z_{7,1}) \right],
\]
\[
\beta_{u_2} = -z u_2 \left[ \epsilon - \left( \frac{1}{2} g \partial g + u_i \partial u_i \right) (2Z_{1,1} - 2Z_{3,1} - 2Z_{4,1} + Z_{8,1}) \right].
\]
FIG. 8: One-loop Feynman diagrams that contribute to the quantum effective action. Solid (wiggly) lines represent the fermion (boson) propagator. Cubic vertices represent the Yukawa coupling, \( g \). In (e), each quartic vertex can be either \( u_1 \) and \( u_2 \).

The counter terms can be computed order by order in the loop expansion. We include the contributions from the one-loop diagrams shown in Fig. 8. The computations of the diagrams are discussed in the next section of this supplementary material. Here we summarize the final results,

\[
\begin{align*}
Z_{1,1} &= -\frac{(N_c^2 - 1)}{4\pi^2 N_c N_f} \frac{g^2}{c} h_1(v, c), \\
Z_{2,1} &= \frac{(N_c^2 - 1)}{4\pi^2 N_c N_f} \frac{g^2}{c} h_2(v, c), \\
Z_{3,1} &= -Z_{2,1}, \\
Z_{4,1} &= -\frac{1}{4\pi} \frac{g^2}{v}, \\
Z_{5,1} &= 0, \\
Z_{6,1} &= -\frac{1}{8\pi^3 N_c N_f} \frac{g^2}{c} h_3(v, c), \\
Z_{7,1} &= \frac{1}{2\pi^2 c^2} \left[ (N_c^2 + 7)u_1 + 2 \left( 2N_c - \frac{3}{N_c} \right) u_2 + 3 \left( 1 + \frac{3}{N_c^2} \right) \frac{u_2^2}{u_1} \right], \\
Z_{8,1} &= \frac{1}{2\pi^2 c^2} \left[ 12u_1 + 2 \left( N_c - \frac{9}{N_c} \right) u_2 \right],
\end{align*}
\]

(C16)
where $h_i(v, c)$ are defined in the main text. This gives the beta functions and the dynamical critical exponent shown in Eqs. (3)-(7) and below. The anomalous dimensions of the fields are given by

$$
\eta_{\psi} = z \left( \frac{N_c^2 - 1}{8\pi N_c N_f} \right) g^2 c \left[ \epsilon \left\{ h_1(v, c) - h_2(v, c) \right\} - \left\{ 2h_1(v, c) - 3h_2(v, c) \right\} \right], \quad (C17)
$$

$$
\eta_{\phi} = z \left( \frac{g^2}{8\pi} \right) \left[ \frac{c}{v} - (4 - \epsilon) \frac{(N_c^2 - 1)}{\pi N_c N_f} \left\{ h_1(v, c) - h_2(v, c) \right\} \right]. \quad (C18)
$$

It is noted that the beta functions used in the main text describe the flow of the couplings with increasing length scale, which is defined to be $\partial g / \partial l \equiv -\beta_g$.

Because all $v, c, g, u_i$ flow to zero in the low energy limit as discussed in the main text, it is more convenient to consider the ratios of the couplings, $w = \frac{u}{v}, \lambda = \frac{\pi}{2} g$ and $\kappa_i = \frac{\partial}{\partial l}$. The beta functions for the ratios are given by

$$
\frac{\partial w}{\partial l} = \frac{zw\lambda}{8\pi} \left[ 1 - \frac{2w(N_c^2 - 1)}{\pi N_c N_f} \left\{ h_1(wc, c) + h_2(wc, c) \right\} \right], \quad (C19)
$$

$$
\frac{\partial \lambda}{\partial l} = z \lambda \left\{ \epsilon - \frac{\lambda}{4\pi} \left[ 1 - \frac{w}{\pi N_c N_f} h_3(wc, wc) \right] \right\}, \quad (C20)
$$

$$
\frac{\partial \kappa_1}{\partial l} = z \kappa_1 \left\{ \epsilon - \frac{\lambda}{4\pi} - \left( \frac{N_c^2 + 7}{2\pi^2} \right) \kappa_1 - \frac{6}{\pi^2} \kappa_1 \right\}, \quad (C21)
$$

$$
\frac{\partial \kappa_2}{\partial l} = z \kappa_2 \left\{ \epsilon - \frac{\lambda}{4\pi} - \left( \frac{N_c^2 - 9}{\pi^2} \right) \kappa_2 \right\}. \quad (C22)
$$

By using $\lim_{c \to 0} h_1(wc, c) = \frac{\pi}{2}, \lim_{c \to 0} h_2(wc, c) = 0$ and $\lim_{c \to 0} h_3(wc, c) = \frac{2\pi^2}{1+w}$, it can be shown that the beta functions for $c, w, \lambda$ and $\kappa_i$ simultaneously vanish at the attractive fixed point given in Eq. (9). At the fixed point, the dynamical critical exponent and the anomalous dimensions are given by

$$
z = 1 + \frac{N_c^2 + N_c N_f - 1}{2(N_c^2 + N_c N_f - 3)} \epsilon,
$$

$$
\eta_{\psi} = \eta_{\phi} = -\frac{N_c^2 + N_c N_f - 1}{2(N_c^2 + N_c N_f - 3)} \epsilon \quad (C23)
$$

to the leading order in $\epsilon$. Both the dynamical critical exponent and the anomalous dimensions modify the scaling of the renormalized Green’s function as can be checked from Eq. (C6). As a result, the two-point functions in Eqs. (10) and (11) are controlled by the net anomalous dimensions defined by $\tilde{\eta}_\psi = \eta_\psi + \frac{(z-1)(2-\epsilon)}{4}, \tilde{\eta}_\phi = \eta_\phi + \frac{(z-1)(2-\epsilon)}{4}$, which vanish to the linear order in $\epsilon$. It is expected that there will be non-trivial anomalous dimensions for the two-point functions beyond the one-loop level[14]. Higher-point correlation functions exhibit non-trivial anomalous dimensions even to the linear order in $\epsilon$ because the quantum corrections are not canceled in Eq. (C6) for $2f + b > 2$. 

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Appendix D: Computation of one-loop diagrams

In this section, we outline the computations of the one-loop Feynman diagrams that result in Eq. (C16). We will use $\delta S$ to denote the contributions to the quantum effective action, and $S_{CT}$ to denote the counter terms that are needed to cancel the UV divergent pieces in $\delta S$ in the $\epsilon \to 0$ limit.

1. Fermion self energy

The quantum correction to the fermion self-energy from the diagram in Fig. 8a is

$$\delta S^{(2,0)} = \frac{2g^2}{N_f} \left( N_c - \frac{1}{N_c} \right) \sum_{n=1}^{N_c} \sum_{\sigma=1}^{N_f} \sum_{j=1}^{N_f} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\Psi}_{n,\sigma,j}(k) \Upsilon^{(n)}_{(2,0)}(k) \Psi_{n,\sigma,j}(k),$$

(D1)

where

$$\Upsilon^{(n)}_{(2,0)}(k) = \int \frac{d^{d-1}Q}{(2\pi)^{d-1}} \frac{d^2 q}{(2\pi)^2} \gamma_{d-1} G_n(k+q) \gamma_{d-1} D(q),$$

(D2)

and the bare Green’s functions are given by

$$G_n(k) = -i \frac{\Gamma \cdot K + \gamma_{d-1} \varepsilon_n(k)}{|K|^2 + \varepsilon_n^2(k)},$$

(D3)

$$D(q) = \frac{1}{|Q|^2 + c^2 |q|^2}.$$  

(D4)

After the integrations over $q$ and $Q$, Eq. (D2) can be expressed in terms of a Feynman parameter,

$$\Upsilon^{(n)}_{(2,0)}(k) = \frac{i}{4\pi^{(d+1)/2}} \Gamma \left( \frac{3-d}{2} \right) \int_0^1 dx \sqrt{\frac{1-x}{c^2 + x(1 + \nu^2 - c^2)}}$$

$$\times \left[ x(1-x) \left( |K|^2 + \frac{c^2 \varepsilon_n^2(k)}{c^2 + x(1 + \nu^2 - c^2)} \right)^{-\frac{3-d}{2}} \left[ K \cdot \Gamma - \frac{c^2 \varepsilon_n(k) \gamma_{d-1}}{c^2 + x(1 + \nu^2 - c^2)} \right] \right].$$

(D5)

The UV divergent part in the $\epsilon \to 0$ limit is given by

$$\Upsilon^{(n)}_{(2,0)}(k) = \frac{i}{8\pi^2 c \epsilon} \left[ h_1(v, c) K \cdot \Gamma - h_2(v, c) \varepsilon_n(k) \gamma_{d-1} \right],$$

where $h_1(v, c) = \int_0^1 dx \sqrt{\frac{1-x}{c^2 + (1 + \nu^2 - c^2)x}}$, $h_2(v, c) = c^2 \int_0^1 dx \sqrt{\frac{1-x}{(c^2 + (1 + \nu^2 - c^2)x)}}$. This leads to the one-loop counter term for the fermion self-energy,

$$S_{CT}^{(2,0)} = -i \frac{g^2}{4\pi^2 c \epsilon} \frac{N_c^2 - N_c}{N_f N_c} \sum_{n=1}^{N_c} \sum_{\sigma=1}^{N_f} \sum_{j=1}^{N_f} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}$$

$$\times \left[ h_1(v, c) K \cdot \Gamma - h_2(v, c) \varepsilon_n(k) \gamma_{d-1} \right].$$

(D6)
\[ \Psi_{n,\sigma,j}(k) \left[ h_1(v,c) \mathbf{K} \cdot \Gamma - h_2(v,c) \varepsilon_n(k) \gamma_{d-1} \right] \Psi_{n,\sigma,j}(k). \]  (D6)

2. Boson self energy

The boson self energy in Fig. 8b is given by

\[ \delta S^{(0,2)} = -2g^2 \mu^{3-d} \sum_a \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \Upsilon^{(0,2)}(q) \phi^a(-q)\phi^a(q), \]  (D7)

where

\[ \Upsilon^{(0,2)}(q) = \frac{1}{2} \sum_n \int \frac{d^{d-1}K}{(2\pi)^{d-1}} \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ \gamma_{d-1} G_n(k + q) \gamma_{d-1} G_n(k) \right]. \]  (D8)

We first integrate over \( \vec{k} \). Using the Feynman parameterization, we write the resulting expression as

\[ \Upsilon^{(0,2)}(q) = \frac{1}{2\pi \nu} \int_0^1 dx \int \frac{d^{d-1}K}{(2\pi)^{d-1}} \frac{d^2k}{2} \left[ x(1 - x) \right]^{-\frac{1}{2}} \frac{\mathbf{K} \cdot (\mathbf{K} + Q)}{x |\mathbf{K} + Q|^2 + (1 - x) |\mathbf{K}|^2}. \]  (D9)

The quadratically divergent term is the mass renormalization, which is automatically tuned away at the critical point in the present scheme. The remaining correction to the kinetic energy of the boson becomes \( \Upsilon^{(0,2)}(q) = -\frac{|Q|^2}{16\pi \nu \epsilon} \) up to finite terms. Accordingly we add the following counter term,

\[ S_{CT}^{(0,2)} = -\sum_a \frac{g^2}{8\pi \nu \epsilon} \int \frac{d^{d+1}q}{(2\pi)^{d+1}} |Q|^2 \phi^a(-q)\phi^a(q). \]  (D10)

3. Yukawa vertex correction

The diagram in Fig. 8c gives rise to the vertex correction in the quantum effective action,

\[ \delta S^{(2,1)} = i \frac{g^3 \mu^{3-d}}{N_c N_f^{3/2}} \sum_{a,n} \sum_{j,\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \phi^a(q) \Psi_{n,\sigma,j}(k + q) \tau^a_{\sigma,\sigma'} \Upsilon^{(n)}_{(2,1)}(k, q) \Psi_{n,\sigma',j}(k), \]  (D11)

where

\[ \Upsilon^{(n)}_{(2,1)}(k, q) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{d^2p}{(2\pi)^2} \gamma_{d-1} G_n(p + q + k) \gamma_{d-1} G_n(p + k) \gamma_{d-1} D(p). \]  (D12)
Here we use the identity for the $SU(N_c)$ generators, $\sum_{a=1}^{N_c^2-1} \tau^a \tau^b \tau^a = -\frac{2}{N_c} \tau^b$. The UV divergent part in the $\epsilon \to 0$ limit, which can be extracted by setting all external frequency and momenta to zero except $K$, is given by

$$\Upsilon_{(2,1)}^{(n)}(K) = \frac{\Gamma(3)}{4\pi^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{2\pi} d\theta \int_0^{\pi} dR \int \frac{d^{d-1}P}{(2\pi)^{d-1}} \frac{|P|^2 - \varepsilon_n(\bar{p})\varepsilon_n(p)}{|P|^2 + c^2|\bar{p}|^2} \frac{|P|^2 - \varepsilon_n(\bar{p})\varepsilon_n(p)}{|P|^2 + \varepsilon_n(\bar{p})^2} \frac{|P|^2 + \varepsilon_n(\bar{p})^2}{|K + P|^2 + \varepsilon_n(\bar{p})^2}.$$  

(D13)

We introduce two Feynman parameters to combine the denominators in the above expression. In new coordinates $(R, \theta)$ defined by $\varepsilon_n(\bar{p}) = \sqrt{2v} R \cos \theta$ and $\varepsilon_n(\bar{p}) = \sqrt{2v} R \sin \theta$, Eq. (D13) is rewritten as

$$\Upsilon_{(2,1)}^{(n)}(K) = \frac{\gamma_{d-1}}{4\pi^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{2\pi} d\theta \int_0^{\infty} dR R \int \frac{d^{d-1}P}{(2\pi)^{d-1}} \frac{|P|^2 - v R^2 \sin(2\theta)}{\left(|P|^2 + 2(x_1 + x_2)K \cdot P + M^2(v, c, x_1, x_2, K, R, \theta)\right)^3},$$

(D14)

where $M^2(v, c, x_1, x_2, K, R, \theta) = (x_1 + x_2)|K|^2 + 2R c \zeta(v, c, x_1, x_2, \theta)$ with $\zeta(v, c, x_1, x_2, \theta) = \frac{2v}{c}(x_1 \cos^2 \theta + x_2 \sin^2 \theta) + (1 - x_1 - x_2)(vc \cos^2(\theta + \pi/4) + \frac{v}{c} \sin^2(\theta + \pi/4))$. Integrating over $P$ and $R$, we obtain $\Upsilon_{(2,1)}^{(n)}(K) = \frac{\gamma_{d-1}}{16\pi^3} h_3(v, c) + \mathcal{O}(\epsilon^0)$, where

$$h_3(v, c) = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{2\pi} d\theta \left[ \frac{1}{\zeta(v, c, x_1, x_2, \theta)} - \frac{v \sin 2\theta}{c \zeta^2(v, c, x_1, x_2, \theta)} \right].$$

It is noted that the UV divergent part of $\Upsilon_{(2,1)}^{(n)}$ is independent of $n$. From this, we identify the counter term for the Yukawa vertex,

$$S_{CT}^{(2,1)} = -i \frac{g^3 \mu^{3-d/2}}{8\pi^3 N_c N_f^{3/2}} h_3(v, c) \sum_{a,n} \sum_{j,s,j'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \phi^a(q) \cdot \left[ \bar{\Psi}_{\bar{n},s,j}(k + q) \gamma^a_{s,s'} \gamma_{d-1} \Psi_{n,s',j}(k) \right].$$

(D15)

4. $\phi^4$ vertex corrections

There are two one-loop diagrams for the quartic vertex. The quantum correction from Fig. 8d is given by

$$\delta S_1^{(0,4)} = \frac{1}{4} \frac{g^4 \mu^{2(d-2)}}{N_f^2} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1}^{N_f^2-1} \int \frac{d^{d+1}q_1}{(2\pi)^{d+1}} \frac{d^{d+1}q_2}{(2\pi)^{d+1}} \frac{d^{d+1}q_3}{(2\pi)^{d+1}} \frac{d^{d+1}q_4}{(2\pi)^{d+1}} \delta(q_1 + q_2 + q_3 + q_4)$$

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\[ \times \Upsilon_{(0,4);1}(q_1, q_2, q_3) \text{ Tr} \left[ \tau^{a_1} \tau^{a_2} \tau^{a_3} \tau^{a_4} \right] \phi^{a_1}(q_1) \phi^{a_2}(q_2) \phi^{a_3}(q_3) \phi^{a_4}(q_4), \]

where

\[ \Upsilon_{(0,4);1}(q_1, q_2, q_3) = \sum_n \int_0^{d+1} \frac{d k}{(2\pi)^d+1} \text{ Tr} \left[ \gamma_{d-1} G_n(q_1 + k) \gamma_{d-1} G_n(q_1 + q_2 + k) \gamma_{d-1} G_n(q_1 + q_2 + q_3 + k) \gamma_{d-1} G_n(k) \right]. \]

When \( Q_i = 0 \), the above expression becomes

\[ \Upsilon_{(0,4);1}(q_1, q_2, q_3) = \sum_n \int_0^{d+1} \frac{d k}{(2\pi)^d+1} \text{ Tr} \left[ \frac{1}{\varepsilon_n(\vec{q}_1 + \vec{k}) + \gamma_{d-1} \vec{K} \cdot \vec{\Gamma}} + \frac{1}{\varepsilon_n(\vec{q}_1 + \vec{q}_2 + \vec{k}) + \gamma_{d-1} \vec{K} \cdot \vec{\Gamma}} \varepsilon_n(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{k}) + \gamma_{d-1} \vec{K} \cdot \vec{\Gamma} \right]. \]

The matrix \( \gamma_{d-1} \vec{K} \cdot \vec{\Gamma} \) has an eigenvalue \( i|\vec{K}| \) or \( -i|\vec{K}| \). Since the Green’s functions in the trace involve the common matrix, they always have poles on one side in the complex plane of \( k_x \) (\( k_y \)) for \( n = 1, 3 \) (\( n = 2, 4 \)). This is because \( \varepsilon_n(\vec{k}) \) and \( \varepsilon_n(\vec{k}) \) have the same velocity in the \( k_x \) (\( k_y \)) direction for \( n = 1, 3 \) (\( n = 2, 4 \)). As a result, the integration over \( \vec{k} \) vanishes when the external \( Q_i \)'s are zero. Therefore no counter term is generated from Fig. 8d.

The diagram in Fig. 8e represents three different terms which are proportional to \( u_1^2 \), \( u_1 u_2 \) and \( u_2^2 \). It is straightforward to compute the counter terms to obtain

\[ S^{(0,4)}_{CT} = \frac{\mu^3}{8 \pi^2 c^2} \varepsilon \int_0^{d+1} \frac{d q_1}{(2\pi)^d+1} \frac{d q_2}{(2\pi)^d+1} \frac{d q_3}{(2\pi)^d+1} \frac{d q_4}{(2\pi)^d+1} \delta(q_1 + q_2 + q_3 + q_4) \left\{ \left[ (N_e^2 + 7) u_1^2 + 2 \left( 2N_e - \frac{3}{N_e} \right) u_1 u_2 + 3 \left( 1 + \frac{3}{N_e^2} \right) u_2^2 \right] \text{ Tr} \left[ \Phi(q_1) \Phi(q_2) \right] \text{ Tr} \left[ \Phi(q_3) \Phi(q_4) \right] \right\} \]

\[ + 12 u_1 u_2 + 2 \left( N_e - \frac{9}{N_e} \right) u_2^2 \right] \text{ Tr} \left[ \Phi(q_1) \Phi(q_2) \Phi(q_3) \Phi(q_4) \right]. \]
have only $\Psi_{n,\sigma}$ and $\bar{\Psi}_{\bar{n},\bar{\sigma}}$ for a fixed $n$. Consider a general $L$-loop diagram that involves hot spots $n=1,3$ with $V_g$ Yukawa vertices and $V_u$ quartic vertices,

$$I \sim g^{V_g} u_1^{V_u} \int \left[ \prod_{i=1}^{L} dp_i \right] \prod_{l=1}^{I_f} \left( \frac{1}{\Gamma \cdot K_l + \gamma_{d-1} \left[ \nu k_{l,x} + (-1)^{n_l-1} \frac{1}{2} k_{l,y} \right]} \right) \prod_{m=1}^{I_b} \left( \frac{1}{|q_m|^2 + c^2 |\bar{q}_m|^2} \right).$$

(E1)

Here both $u_1$ and $u_2$ are loosely denoted as $u_i$ because the power counting is equivalent for the two. $k_l = (K_l, \vec{k}_l)$ and $q_m = (Q_m, \vec{q}_m)$ represent the momenta that go through the fermion and boson propagators, respectively. They are linear superpositions of the internal momenta $p_i$ and external momenta. $n_l$ is either 1 or 3. Once $x$-components of all momenta are scaled by $1/v$, one has

$$I \sim g^{V_g} u_1^{V_u} \int \left[ \prod_{i=1}^{L} dp_i \right] \prod_{l=1}^{I_f} \left( \frac{1}{\Gamma \cdot K_l + \gamma_{d-1} \left[ k_{l,x} + (-1)^{n_l-1} \frac{1}{2} k_{l,y} \right]} \right) \prod_{m=1}^{I_b} \left( \frac{1}{|Q_m|^2 + \frac{q_{m,x}^2}{w^2} + c^2 q_{m,y}^2} \right).$$

(E2)

The integrations of the internal momenta are well defined in the $v,c \rightarrow 0$ limit with fixed $w$ as far as each loop contains at least one fermion propagator. The exceptions are the loops that are solely made of the boson propagators for which the $y$-momentum integration is UV divergent for $c = 0$. The UV divergence is cut-off at $q_{m,y} \sim 1/c$ for each boson loops. If there are $L_b$ boson loops, the entire diagram goes as

$$I \sim g^{V_g} u_1^{V_u} \frac{v^{L_c L_b}}{w^{V_u}} = \lambda \frac{v^{2-E}}{2} \frac{v^{V_u}}{w^{V_u}} g^{(E-2)} c^\delta,$$

(E3)

where $E$ is the number of external lines and $\delta = V_u - L_b \geq 0$. Here we used the identity $L = V_g + 2V_u + 2-E$.

Eq. (E3) implies that the multiplicative renormalizations for the kinetic energy ($E = 2$), the Yukawa coupling ($E = 3$) and the quartic vertices ($E = 4$) defined in Eq. (C2) go as

$$\mathcal{A}_1, \ldots, \mathcal{A}_5 \sim \lambda \frac{v^{V_g}}{2} \frac{v^{V_u}}{w^{V_u}} c^\delta,$$

$$\mathcal{A}_6 \sim \lambda \frac{v^{V_u-1}}{2} \frac{v^{V_u}}{w^{V_u}} c^\delta,$$

$$\mathcal{A}_7 \sim \lambda \frac{v^{V_u-2}}{2} \frac{v^{V_u}}{w^{V_u}} c^\delta \frac{g^2}{u_7},$$

$$\mathcal{A}_8 \sim \lambda \frac{v^{V_u-2}}{2} \frac{v^{V_u}}{w^{V_u}} c^\delta \frac{g^2}{u_8}.$$
\[ A_1, \ldots, A_6 \] remain finite in the limit \( v, c, g, u_i \) go to zero with fixed \( \lambda, w, \kappa_i \). Therefore, higher-loop corrections in \( A_1, \ldots, A_6 \) are systematically suppressed by powers of \( \lambda, \kappa_i \). On the other hand, \( A_7 \) and \( A_8 \) are proportional to \( \lambda^\delta \kappa_i^{V_u^{-1}} w^{-V_u+1} c^{\delta-1} \). Since the higher-loop quantum corrections with \( \delta \geq 1 \) are obviously suppressed, we will focus on the contributions with \( \delta = 0 \) in \( A_7 \) and \( A_8 \) which go as \( 1/c \) in the \( c \to 0 \) limit. Only diagrams with \( \delta = 0 \) are the ones that do not contain \( u_i \). At the one-loop order, there is one such diagram for the \( \phi^4 \) vertices, Fig. 8d. Due to a chiral structure that is present in the one-loop diagram, it vanishes as is shown in Sec. D4. Higher-loop diagrams with \( \delta = 0 \) do not vanish in general. For example, the two-loop diagrams in Fig. 9 generate quantum corrections for \( \kappa_i \) which are order of \( \lambda^3/c \). At \( d = 3 \), these higher-loop corrections are still vanishingly small in the low energy limit. This is because \( \lambda \) vanishes as \( 1/l \) while \( c \) vanishes only as \( 1/\log(l) \) in the \( l \to \infty \) limit, where \( l \) is the logarithmic length scale. Since all higher-loop corrections are suppressed at \( d = 3 \), the one-loop beta functions become asymptotically exact in the low energy limit where \( \lambda, \kappa_i \) vanish along with \( v, c \). For \( d < 3 \), the higher-loop quantum corrections to \( \kappa_i \) grow as \( c \) becomes small with \( \lambda \neq 0 \). This suggests that \( c \) should be stabilized at a nonzero value once higher-loop corrections are included. In particular, \( \kappa_i \) enters into the beta function of \( c \) at the three-loop and higher orders. It is expected that the feedback of \( \kappa_i \) will stabilize \( c \) at a nonzero value in the low energy limit. Once the velocity becomes nonzero, \( \kappa_i \) will flow to a nonzero and finite value at the fixed point. Because of the continuity from the exact \( d = 3 \) fixed point, not only \( c, v, \kappa_i, \lambda \) but also \( \lambda^n/c \) with \( n \geq 3 \) at the fixed point in \( d = 3 - \epsilon \) should go to zero in the \( \epsilon \to 0 \) limit. Therefore, higher order corrections including the corrections to \( \kappa_i \) with \( \delta = 0 \) are systematically suppressed, and the expansion is controlled for small \( \epsilon \).
Appendix F: Enhancement of superconducting and charge density wave fluctuations

In this section, we compute the anomalous dimensions of the superconducting (SC) and charge density wave (CDW) operators that are enhanced at the strange metallic fixed point.

1. Anomalous dimension

We consider an insertion of a fermion bilinear,

\[ S_\rho = \rho \mu \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \tilde{\Psi}_{n,\sigma,j}(k) \Omega_{n,\sigma,j;n',\sigma',j'}(k') \Psi_{n',\sigma',j'}(k). \]  

(F1)

Here \( \rho \) is a dimensionless source. \( \tilde{\Psi}_{n,\sigma,j}(k) \) is either \( \bar{\Psi}_{n,\sigma,j}(k) \) or \( \Psi_{n,\sigma,j}^T(-k) \) depending on whether the operator creates particle-hole or particle-particle excitations. \( \Omega_{n,\sigma,j;n',\sigma',j'} \) is a matrix that specifies the momentum, spin and flavor quantum numbers of the insertion. The UV divergence in the quantum effective action coming from the insertion is canceled by a counter term of the same form,

\[ S_\rho^{CT} = \rho \mu A_\rho \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \tilde{\Psi}_{n,\sigma,j}(k) \Omega_{n,\sigma,j;n',\sigma',j'} \Psi_{n',\sigma',j'}(k) \]  

with

\[ A_\rho = \sum_m \frac{Z_\rho m}{e^m}. \]

The renormalized insertion can be written as

\[ S_{\rho;ren} = \rho_B \int \frac{d^{d+1}k_B}{(2\pi)^{d+1}} \tilde{\Psi}_{B,n,\sigma,j}(k) \Omega_{B,n,\sigma,j;n',\sigma',j'} \Psi_{B,n',\sigma',j'}(k), \]  

(F2)

where the renormalized source \( \rho \) is related to the bare source \( \rho_B \) as

\[ \rho = \mu^{-1} \frac{Z_\rho^{d-1}}{Z_\rho} \rho_B \]  

with \( Z_\rho = 1 + A_\rho \). From this, one can obtain the beta function for the source,

\[ \frac{d\rho}{dl} = \rho \left[ 1 + \gamma_\rho \right], \]  

(F3)

where \( \gamma_\rho = z \left( \frac{\partial}{\partial g} + u_i \partial_{u_i} \right) (Z_{3,1} - Z_{\rho,1}) \) is the anomalous dimension of the source. The larger the anomalous dimension of the source is, the stronger the enhancement is.

2. Superconducting channel

Here we examine the superconducting channels described by the pairing vertices of the form,

\[ S_{A,\tilde{\Omega}}^{(\pm)} = \mu V \sum_{j,\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left[ \left\{ \Psi_{1,\sigma,j}^T(-k)A_{\sigma,\sigma'} \tilde{\Omega} \Psi_{1,\sigma',j}(k) + \Psi_{3,\sigma,j}^T(-k)A_{\sigma,\sigma'} \tilde{\Omega} \Psi_{3,\sigma',j}(k) \right\} \right. \]

\[ + \left. \left\{ \Psi_{2,\sigma,j}^T(-k)A_{\sigma,\sigma'} \tilde{\Omega} \Psi_{2,\sigma',j}(k) + \Psi_{4,\sigma,j}^T(-k)A_{\sigma,\sigma'} \tilde{\Omega} \Psi_{4,\sigma',j}(k) \right\} \right]. \]  

(F4)
FIG. 10: Cooper pair wavefunctions represented by the vertices (a) $S_{A,\gamma_{d-1}}^{(+)}$ and (b) $S_{A,\gamma_{d-1}}^{(-)}$. A wiggly line connecting two momenta $k_1$ and $k_2$ represents a Cooper pair made of electrons at those momenta. The dashed wiggly lines are intended to represent the relative minus sign in the Cooper pair wavefunction relative to the ones connected by the solid lines. The Cooper pair created by $S_{A,\gamma_{d-1}}^{(+)} (S_{A,\gamma_{d-1}}^{(-)})$ undergoes four (two) phase winding under $2\pi$ rotation, which correspond to $g (d_{x^2-y^2})$ wave pairing.

Here $V$ is a source for the pairing operator. $A_{\sigma,\sigma'}$ is an anti-symmetric matrix, which represents the spin-singlet pairing for the case of $N_c = 2$. $\tilde{\Omega}$ is a $2 \times 2$ matrix that acts on the Dirac indices. Among all possible $\tilde{\Omega}$, we find the channels with $\tilde{\Omega} = \gamma_{d-1}$ and $I$ are most strongly enhanced. Therefore we will focus on these channels in the rest of the section. $S_{A,\gamma_{d-1}}^{(+)} (S_{A,\gamma_{d-1}}^{(-)})$ describes the $g$-wave ($d_{x^2-y^2}$-wave) pairing with zero net momentum of Cooper pairs as is illustrated in Fig. 10. $S_{A,I}^{(+)} (S_{A,I}^{(-)})$ describes the Cooper pairs with non-zero net momentum $2\vec{k}_F$ in the $s$-wave ($d_{xy}$-wave) channel as is shown in Fig. 11.

The one-loop quantum correction to the SC insertion is given by

$$\delta S_{A,\tilde{\Omega}}^{(\pm)} = N_V \mu^{4-d} V \int \frac{d^{d+1}k}{(2\pi)^{d+1}} g^2 \sum_{j,\sigma,\sigma'} \left\{ \Psi_{1,\sigma,j}^T (-k) A_{\sigma,\sigma'} Y_{\tilde{\Omega}}^{(1)} (k) \Psi_{1,\sigma',j} (k) + \Psi_{3,\sigma,j}^T (-k) A_{\sigma,\sigma'} Y_{\tilde{\Omega}}^{(3)} (k) \Psi_{3,\sigma',j} (k) \right\}$$

$$\pm \left\{ \Psi_{2,\sigma,j}^T (-k) A_{\sigma,\sigma'} Y_{\tilde{\Omega}}^{(2)} (k) \Psi_{2,\sigma',j} (k) + \Psi_{4,\sigma,j}^T (-k) A_{\sigma,\sigma'} Y_{\tilde{\Omega}}^{(4)} (k) \Psi_{4,\sigma',j} (k) \right\},$$

(F5)
FIG. 11: Cooper pair wavefunctions represented by the vertices (a) $S_{A,I}^{(+)}$ and (b) $S_{A,I}^{(-)}$. A wiggly line that ends on a hot spot represents a Cooper pair made of electrons from that hot spot. Therefore, the Cooper pair carries non-zero momenta, $2\tilde{k}_F$. The Cooper pair created by $S_{A,I}^{(+)}$ ($S_{A,I}^{(-)}$) undergoes zero (two) phase winding under $2\pi$ rotation, which correspond to $s$ ($d_{xy}$) wave pairing.

where

$$\Upsilon^{(n)}_{\Omega}(k) = \int \frac{d^{d+1}q}{(2\pi)^{d+1}} D(q) \gamma_{d-1}^T G_{\Omega}^T (-k - q) \gamma_{d-1}$$  \hspace{1cm} (F6)$$

and $N_V = \frac{2(N_{c+1})}{N_{c-1}}$. Using $\gamma_0^T = -\gamma_0$ and $\gamma_i^T = \gamma_i$, for $i = 1, 2, \ldots, (d-1)$, we obtain

$$\Upsilon^{(n)}_{\Omega}(K) = \int \frac{d^{d+1}q}{(2\pi)^{d+1}} [ (K_0 + Q_0)\gamma_0 - \sum_{\nu=1}^{d-2}(K_\nu + Q_\nu)\gamma_\nu + \varepsilon_n(\vec{q})\gamma_{d-1} ] \gamma_{d-1} \gamma_{d-1} \gamma_{d-1} \gamma_{\tilde{\Omega}} \gamma_{d-1} \gamma_{d-1} \gamma_{d-1} \gamma_{d-1}$$

$$[(\vec{Q})^2 + c^2 |\vec{q}|^2] [ |\vec{K} + \vec{Q}|^2 + \varepsilon_n^2(\vec{q})]^2$$

when $\tilde{k} = 0$. Changing coordinates from $(q_x, q_y)$ to $(R, \theta)$ with $\varepsilon_n(\vec{q}) = \sqrt{2vR \cos \theta}$ and $\varepsilon_n(\vec{q}) = \sqrt{2vR \sin \theta}$, one can perform the integrations over $R$ and $Q$ using the Feynman parameterization to obtain

$$\Upsilon^{(n)}_{\Omega}(K) = \frac{1}{16\pi^3 c \varepsilon} \tilde{\Omega} h_{SC}(v, c) + \mathcal{O}(\varepsilon^0),$$

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where \( h_{SC}(v, c) = \frac{2\pi}{c} \int_0^1 dx \int_0^{2\pi} d\theta \frac{\sin^2 \theta}{\zeta_1(v, x, \theta)} \) with
\[
\zeta_1(v, x, \theta) = \frac{2v}{c} x \cos^2 \theta + (1 - x) \left[ \frac{c}{v} \sin^2 \left( \theta + \frac{\pi}{4} \right) + vc \cos^2 \left( \theta + \frac{\pi}{4} \right) \right].
\]

Note that \( h_{SC}(v, c) \) is same for \( \tilde{\Omega} = \gamma_{d-1}, I \). Therefore, we add the counter term,
\[
S^{(+)}_{A,\Omega, CT} = -\mu V \frac{N_v}{16\pi^2} \frac{g^2}{c} h_{SC}(v, c) \sum_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}
\times \left[ \{ \Psi_{1,\sigma}(v, \sigma') \tilde{\Omega} \Psi_{1,\sigma'}(k) + \Psi_{3,\sigma}(v, \sigma') \tilde{\Omega} \Psi_{3,\sigma'}(k) \} \right.
\pm \left. \{ \Psi_{2,\sigma}(v, \sigma') \tilde{\Omega} \Psi_{2,\sigma'}(k) + \Psi_{4,\sigma}(v, \sigma') \tilde{\Omega} \Psi_{4,\sigma'}(k) \} \right],
\]
which gives the anomalous dimension of the source, \( \gamma_V = \frac{N_v}{8\pi^2} \frac{g^2}{c} \frac{1}{2\pi} h_{SC}(v, c) - (N_c - 1)h_2(v, c) \) for the four vertices, \( S^{(+)}_{A,\gamma_{d-1}} \) and \( S^{(+)}_{A,\tilde{\Omega}} \). At the quasi-local strange metal fixed point, we have \( \lim_{c \to 0} h_2(w^2 c, c) = 0 \) and \( \lim_{c \to 0} h_{SC}(w^2 c, c) = \pi^2 \), and the anomalous dimension becomes \( \gamma_V = \frac{\lambda^c}{8\pi(N_c - 1)} \).

It is interesting that the finite momentum pairing is as strong as the zero momentum pairing. This is a consequence of the nesting, which allows a pair of electrons to stay on the Fermi surface as they are scattered from one hot spot to another. The attractive interaction is mediated by the commensurate spin fluctuations which scatter a pair of electrons in one hot spot to another. The attractive interaction is mediated by the pseudospin transformation. The one-loop quantum correction is given by
\[
\delta S^{(\pm)} = -N_p \rho^{(\pm)} 4 - d \frac{g^2}{(2\pi)^{d+1}} \left[ \left\{ \bar{\Psi}_{1,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{1,\sigma,j}(k) + \bar{\Psi}_{3,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{3,\sigma,j}(k) \right\} \right.
\pm \left. \left\{ \bar{\Psi}_{4,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{4,\sigma,j}(k) + \bar{\Psi}_{2,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{2,\sigma,j}(k) \right\} \right],
\]

3. Charge density wave channel

Here we compute anomalous dimensions for CDW operators of the form,
\[
S^{(+)}_{\rho} = \rho^{(\pm)} \mu \sum_{j,\sigma} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left[ \left\{ \bar{\Psi}_{1,\sigma,j}(k) \Psi_{1,\sigma,j}(k) + \bar{\Psi}_{3,\sigma,j}(k) \Psi_{3,\sigma,j}(k) \right\} \right.
\pm \left. \left\{ \bar{\Psi}_{4,\sigma,j}(k) \Psi_{4,\sigma,j}(k) + \bar{\Psi}_{2,\sigma,j}(k) \Psi_{2,\sigma,j}(k) \right\} \right],
\]

\( S^{(+)}_{\rho}(S^{(-)}_{\rho}) \) describes \( p_y \)-wave \( (p_x \)-wave) CDW which carries momentum \( 2k_F \) as is shown in Fig. 12. These operators are pseudospin singlets for \( N_c = 2 \) and has no SC counterpart connected by the pseudospin transformation. The one-loop quantum correction is given by
\[
\delta S^{(\pm)}_{\rho} = -N_p \rho^{(\pm)} 4 - d \frac{g^2}{(2\pi)^{d+1}} \left[ \left\{ \bar{\Psi}_{1,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{1,\sigma,j}(k) + \bar{\Psi}_{3,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{3,\sigma,j}(k) \right\} \right.
\pm \left. \left\{ \bar{\Psi}_{4,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{4,\sigma,j}(k) + \bar{\Psi}_{2,\sigma,j}(k) \Gamma_{\rho}(k) \Psi_{2,\sigma,j}(k) \right\} \right].
\]
FIG. 12: Wavefunctions of the particle-hole pairs created by the vertices (a) $S_\rho^+(\pm)$ and (b) $S_\rho^-(\pm)$. An arrow from $k_2$ to $k_1$ represents a particle-hole pair created by $i(c_{k_1}^* c_{k_2} - c_{k_2}^* c_{k_1})$, where $c_k$ is the electron field at momentum $k$ with spin and flavor indices suppressed. $S_\rho^+(\pm)$ and $S_\rho^-(\pm)$ is odd under the $y$ ($x$) reflection, while both of them preserve time-reversal.

$$\pm \left\{ \bar{\Psi}_{2,\sigma,j}(k) \Upsilon_\rho^{(2)}(k) \Psi_{2,\sigma,j}(k) + \bar{\Psi}_{4,\sigma,j}(k) \Upsilon_\rho^{(4)}(k) \Psi_{4,\sigma,j}(k) \right\}, \quad (F10)$$

where

$$\Upsilon_\rho^{(n)}(k) = \int \frac{d^{d+1}q}{(2\pi)^{d+1}} D(q) \gamma_{d-1} G_n(k + q) G_n(k + q) \gamma_{d-1} \quad (F11)$$

and $N_\rho = \frac{2}{N_f} \left( N_c - \frac{1}{N_c} \right)$. From a straightforward calculation, we identify the counter term

$$S_{\rho;CT}^{(\pm)} = -\rho^{(\pm)} \mu \frac{N_\rho}{16\pi^3 \epsilon} \frac{g^2}{c} h_{CDW}(v, c) \sum_{j,\sigma} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left[ \left\{ \bar{\Psi}_{1,\sigma,j}(k) \Psi_{1,\sigma,j}(k) + \bar{\Psi}_{3,\sigma,j}(k) \Psi_{3,\sigma,j}(k) \right\} \right]$$

$$\pm \left\{ \bar{\Psi}_{4,\sigma,j}(k) \Psi_{4,\sigma,j}(k) + \bar{\Psi}_{2,\sigma,j}(k) \Psi_{2,\sigma,j}(k) \right\}, \quad (F12)$$

where $h_{CDW}(v, c) = \int_0^1 dx \int_0^{2\pi} d\theta \left[ \frac{1}{\zeta_1(v, c, x, \theta)} + \frac{2v}{c} \frac{\cos^2 \theta}{\zeta_2(v, c, x, \theta)} \right]$. From this we find the anomalous dimension of the CDW source, $\gamma_{CDW}^{(\pm)} = \frac{N_\rho}{16\pi^3} \frac{g^2}{c} \left[ h_{CDW}(v, c) - 2\pi h_2(v, c) \right]$. At the fixed point, we have $\lim_{c \to 0} h_{CDW}(w^* c, c) = 2\pi^2$ and the anomalous dimension becomes $\gamma_{CDW}^{(\pm)} = \frac{\lambda^*}{4\pi}$.

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