Persistence of Heterodimensional Cycles

Dongchen Li\textsuperscript{1} and Dmitry Turaev\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of Padova
\textsuperscript{2}Department of Mathematics, Imperial College London

November 23, 2021

Abstract. A heterodimensional cycle is an invariant set of a dynamical system consisting of two hyperbolic periodic orbits with different dimensions of their unstable manifolds and a pair of orbits that connect them. For systems which are at least $C^2$, we show that bifurcations of a coindex-1 heterodimensional cycle within a generic 2-parameter family always create robust heterodimensional dynamics, i.e., chain-transitive sets which contain coexisting orbits with different numbers of positive Lyapunov exponents and persist for an open set of parameter values. In particular, we solve the so-called $C^r$-stabilization problem for the coindex-1 heterodimensional cycles in any regularity class $r = 2, \ldots, \infty, \omega$. The results are based on the observation that arithmetic properties of moduli of topological conjugacy of systems with heterodimensional cycles determine the emergence of Bonatti-Diaz blenders.

1 Introduction

In this paper, we prove the $C^r$-stabilization property for heterodimensional cycles of co-index 1. As we explain below, the result implies the ubiquity of heterodimensional dynamics, which is, in our opinion, one of the most basic properties of non-hyperbolic multidimensional dynamical systems with chaotic behavior.

We call chaotic dynamics multidimensional if there exist orbits with more than one positive Lyapunov exponent. In the absence of hyperbolicity, it is natural to expect that orbits with different numbers of positive Lyapunov exponents coexist, see more discussion e.g. in [23]. The first example of such sort was given by Abraham and Smale in [1]. They constructed an open region in the space of $C^1$-diffeomorphisms where each diffeomorphism has hyperbolic orbits with different dimensions of unstable manifolds within the same transitive set. More examples followed, see e.g. [15–17, 24, 31, 37, 38], with a general theory developed by Bonatti and Díaz in [9]. We use

Definition 1. Let a smooth dynamical system (a diffeomorphism or a flow) have two compact, transitive, uniformly-hyperbolic invariant sets $\Lambda_1$ and $\Lambda_2$. We say that $f$ has a heterodimensional cycle involving $\Lambda_1$ and $\Lambda_2$ if

- the dimensions $d_1$ and $d_2$ of unstable manifolds of $\Lambda_1$ and $\Lambda_2$ are different; and
the unstable sets $W^u(\Lambda_1)$ and $W^u(\Lambda_2)$ intersect the stable sets $W^s(\Lambda_2)$ and, respectively, $W^s(\Lambda_1)$. The difference $|d_1 - d_2|$ is called the co-index of the heterodimensional cycle.

Particularly, we pay the most of attention to the case where the two sets $\Lambda_1$ and $\Lambda_2$ are trivial, i.e., the heterodimensional cycle is associated with just two hyperbolic periodic orbits (see Figure 1). When we do not specify which hyperbolic sets are involved in a heterodimensional cycle, we also say that the system exhibits heterodimensional dynamics.

Due to the difference between the dimensions of the unstable manifolds, the heterodimensional intersections can be fragile. Indeed, consider a diffeomorphism of a $d$-dimensional manifold. We have $\dim W^u(\Lambda_1) = d_1$ and $\dim W^s(\Lambda_2) = d - d_2$, so the space spanned by the tangents to $W^u(\Lambda_1)$ and $W^s(\Lambda_2)$ at any of their intersection points has, in the case $d_1 < d_2$, dimension less than the dimension $d$ of the full space. This means that every particular heteroclinic intersection of $W^u(\Lambda_1)$ and $W^s(\Lambda_2)$ is non-transverse and can be removed by an arbitrarily small perturbation.

However, it may happen that, whenever such a connection is destroyed, a new one arises. In this case the heterodimensional dynamics are called robust. Recall that uniformly-hyperbolic sets continue uniquely when the dynamical system varies continuously in the $C^1$ topology.

**Definition 2.** We say that a system exhibits $C^1$-robust heterodimensional dynamics if it has a heterodimensional cycle involving two hyperbolic sets $\Lambda_1$ and $\Lambda_2$ and there exists a $C^1$-neighborhood $U$ of the original dynamical system such that every system from $U$ has a heterodimensional cycle involving the hyperbolic continuations of $\Lambda_1$ and $\Lambda_2$.

This was the case in the original Abraham-Smale example and in the other examples we mentioned. In [11], it was shown that any diffeomorphism with a heterodimensional cycle of co-index 1 can be arbitrarily well approximated, in the $C^1$ topology, by a diffeomorphism with a $C^1$-robust heterodimensional dynamics of co-index 1. This is called the stabilization property.

The result gives a characterization of the topological structure of the set of systems with heterodimensional dynamics of co-index 1: this set is the $C^1$-closure of its $C^1$-interior. However, the
construction of [11] uses, in an essential way, the Franks’ lemma which is only valid in the $C^1$ topology. As a result, the $C^1$-small perturbations proposed in [11, 13] are large in $C^r$ for any $r > 1$. This raises the question whether a $C^r$-stabilization of heterodimensional dynamics is possible\footnote{That the $C^1$ bifurcation theory of heterodimensional cycles cannot be straightforwardly translated to the $C^r$-case is illustrated by the result of [3, 4] which shows that the dynamics created by $C^2$-small or $C^3$-small perturbations of partially-hyperbolic heterodimensional cycles of co-index 1 can be very much different from what can be achieved by $C^1$-small perturbations. The reason is that the dynamics in the central direction are restricted by the signs of the second derivative and the Schwarzian derivative of the one-dimensional transition map; this signature can be changed by $C^1$-small perturbations but not by $C^3$-small perturbations.}. In this paper, we answer this question by proving the following

**Theorem 1.** Any dynamical system of class $C^r$ ($r = 2, \ldots, \infty, \omega$) having a heterodimensional cycle of co-index 1 associated with a pair of periodic orbits can be $C^r$-approximated by a system which has $C^1$-robust heterodimensional dynamics.

**Remark 1.** A partial case of Theorem 1 can be derived from the result in [18] about renormalization near heterodimensional cycles of three-dimensional diffeomorphisms with two saddle-foci. For two-dimensional endomorphisms under the partial hyperbolicity condition, the result of Theorem 1 is Theorem B in [7].

It is well-known that every point in a transitive, uniformly hyperbolic set $\Lambda$ is a limit point of hyperbolic periodic points (with the same dimension of the unstable manifold as $\Lambda$), and the $C^r$-closure of invariant manifolds of these periodic points contains the stable and unstable manifolds of $\Lambda$, see e.g. [27, Theorem 6.4.15]. Hence, whenever we have a heterodimensional cycle involving two hyperbolic sets, we can always obtain, by an arbitrarily small perturbation, a heterodimensional cycle associated with periodic orbits. Thus, Theorem 1 implies

**Corollary 1.** Any dynamical system of class $C^r$ ($r = 2, \ldots, \infty, \omega$) having a heterodimensional cycle of co-index 1 can be $C^r$-approximated by a system which has $C^1$-robust heterodimensional dynamics.

We stress that the result holds true, in particular, in the real-analytic case ($r = \omega$): given a real-analytic dynamical system on a real-analytic manifold we consider any complex neighborhood $\mathcal{M}$ of this manifold such that the system admits a holomorphic extension on $\mathcal{M}$; then the $C^\omega$-topology in Theorem 1 and Corollary 1 is the topology of uniform convergence on compacta in $\mathcal{M}$. In fact, we obtain Theorem 1 from its “constructive version”, Theorem 2, which can be directly applied for showing that the stabilization of heterodimensional dynamics holds true in much more restrictive settings - it can be achieved by polynomial perturbations, perturbations which keep various sorts of symmetry, etc..

The proof of Theorem 1 is constructive in the following sense. First, we bring a given heterodimensional cycle into general position (as defined in Sections 2.2 and 2.3) - this can be done by an arbitrarily small $C^r$-perturbation of any heterodimensional cycle. Then, we embed our system $f$ into a finite-parameter family of perturbations $f_\varepsilon$ with at least 2 parameters. We formulate certain explicit conditions in Section 2.4 which define an open and dense set in the space of $C^r$-families $f_\varepsilon$ such that $f_0 = f$. An arbitrary family from this set is called a proper unfolding of $f$. Theorem 1 follows immediately from our main result,

**Theorem 2.** Let $f$ have a generic heterodimensional cycle of co-index 1 associated with two periodic orbits, and let $f_\varepsilon$ be a proper unfolding of $f$. Then, arbitrarily close to $\varepsilon = 0$ in the space of parameters
there exist open regions where \( f_\varepsilon \) has \( C^1 \)-robust heterodimensional dynamics\(^2\).

As depicted in Figure 2, one distinguishes three main cases here: saddle, saddle-focus, and double-focus, depending on whether the central multipliers are real or complex (see Section 2.1 for the precise definition). The proper unfolding is defined differently, and the theorem is proved differently, in these three cases. Moreover, the result of the theorem should also be interpreted differently. Namely, in the saddle-focus and double-focus cases, the robust heterodimensional dynamics given by Theorem 2 are always associated with hyperbolic sets which are homoclinically related to the continuations of the two periodic orbits in the original heterodimensional cycle. However, in the saddle case, whether this homoclinic relation holds depends on the type of heterodimensional cycle, as described in Section 2.5.

![Figure 2: Three cases of a heterodimensional cycle involving two hyperbolic fixed points of a three-dimensional diffeomorphism. The central multipliers corresponding to the stable manifold of the left fixed point and to the unstable manifold of the right fixed point are both real in the saddle case (a), are one real and one complex in the saddle-focus case (b), and are both complex in the double-focus case (c).](image)

Dynamical systems coming from scientific applications usually have a form of finite-parameter families of differential equations or maps. The genericity/propriety conditions of Theorem 2 are explicit, and checking them requires only a finite amount of computations with a finite number of periodic and heteroclinic orbits. Thus, Theorem 2 provides a universal tool for detecting and demonstrating the robust heterodimensional dynamics in multidimensional systems.

In a series of papers [28–30] we have shown that heterodimensional cycles emerge due to several types of homoclinic bifurcations. In fact, in the spirit of [26, 41, 42], one can conjecture that coindex-1 heterodimensional cycles can appear, with very few exceptions, in any homoclinic/heteroclinic bifurcation whose effective dimension allows it, i.e., when the dynamics of the map under consideration are not reduced to a two-dimensional invariant manifold and the map is not area-contracting\(^3\). We believe this is true, so Theorem 2 allows for establishing the presence of robust heterodimensional dynamics whenever a non-hyperbolic chaotic behavior with more than one positive Lyapunov exponent (the “hyperchaos” in the terminology of [34]) is observed.

In particular, it was also shown in [28–30] that the coindex-1 heterodimensional cycles can be a part of a pseudohyperbolic chain-transitive attractor which appears in systems with Shilnikov loops [19, 36, 43] or after a periodic perturbation of the Lorenz attractor [44]. It, thus, follows from Theorems

\(^2\) As the family \( f_\varepsilon \) is of class \( C^r \), small \( \varepsilon \) correspond to \( C^r \)-small perturbations of \( f \). Thus, this theorem implies Theorem 1 indeed.

\(^3\) To have a heterodimensional cycle we need saddles with different dimensions of unstable manifolds, and these conditions are obviously necessary. The conjecture is that they should also be sufficient for the birth of a heterodimensional cycle in most situations.
1 and 2 that the attractor in such systems remains heterodimensional for an open set of parameter values.

An important feature of robust heterodimensional dynamics is the robust presence of orbits with a zero Lyapunov exponent. In particular, the result of [25] implies, for parametric families described by our Theorem 2, the existence of open sets of parameter values where a generic system has an ergodic invariant measure with at least one zero Lyapunov exponent, i.e., the dynamics for such parameter values are manifestly non-hyperbolic.

For a dense set of parameter values from such regions the system has a non-hyperbolic periodic orbit. Bifurcations of such periodic orbits depend on the coefficients of the nonlinear terms of the Taylor expansion of the first-return map restricted to a center manifold. The degeneracy in the nonlinear terms increases complexity of the bifurcations. It follows from [3, 4] that once the so-called “sign conditions” are imposed on a heterodimensional cycle, the regions of robust heterodimensional dynamics given by Theorem 1 contain a $C^r$-dense set of systems having infinitely degenerate (flat) non-hyperbolic periodic orbits. This fact also leads to the $C^\infty$-genericity (for systems from these regions) of a superexponential growth of the number of periodic orbits and the so-called $C^r$-universal dynamics, see [3, 4].

The basic object in the theory of robust heterodimensional dynamics was introduced by Bonatti and Diaz in [9]. It is called “blender” and can be defined as follows.

**Definition 3.** Consider a dynamical system (a diffeomorphism or a smooth flow) $f$ on a smooth manifold $\mathcal{M}$, and let $f$ have a compact, transitive, uniformly-hyperbolic invariant set $\Lambda$. Let the dimension of its stable manifold be equal to $d_s$ when $f$ is a discrete-time dynamical system (a diffeomorphism) or $d_s + 1$ when $f$ is a flow. The set $\Lambda$ is called a center-unstable blender of co-index $k$ if there exists a $C^1$-open neighborhood $\mathcal{U}$ of $f$ and a $C^1$-open set $\mathcal{D}$ of smooth embeddings of a $(d_s - k)$-dimensional disk into $\mathcal{M}$ such that, for any system from $\mathcal{U}$, each disk $D \in \mathcal{D}$ intersects the unstable manifold of the hyperbolic continuation of $\Lambda$. The set $\Lambda$ is a center-stable blender if it is a center-unstable blender for the system obtained from $f$ by the reversal of time.

It is immediate by this definition that, if $\Lambda_1$ and $\Lambda_2$ are compact, transitive, uniformly-hyperbolic invariant sets with $\dim W^s(\Lambda_1) - 1 = \dim W^s(\Lambda_2)$, and if $\Lambda_1$ is a co-index 1 center-unstable blender, then when $W^u(\Lambda_2)$ has a transverse intersection with $W^s(\Lambda_1)$, and a disc from the collection $\mathcal{D}$ from Definition 3 is a subset of $W^s(\Lambda_2)$, the system exhibits $C^1$-robust heterodimensional dynamics. Thus, the existence of a blender (as a part of a heteroclinic cycle) is, essentially, a reformulation of the existence of the robust heterodimensional dynamics.

There are several geometric constructions of blenders, see e.g. [8–10, 32]. The dynamics of blenders we find for the first-return map near a heterodimensional cycle can be modelled (in some approximation, see formula (70)) by an iterated function system composed of a large collection of affine maps of an interval (therefore, we expect that the so-called parablenders, introduced by Berger, can also be implemented in this case, cf. [5–7]).

We describe a class of heterodimensional cycles for which a blender emerges in an arbitrarily small neighborhood of the cycle whenever the value of a certain modulus (a continuous invariant) of topological conjugacy is irrational (see e.g. Theorem 3 for the saddle case), or, when several moduli
are rationally independent (see Theorem 9 for the saddle-focus and double-focus cases). The result, in particular, implies that when the value of these moduli changes, \textit{blenders are born continuously out of the heterodimensional cycle}\textsuperscript{4}. We also show that a generic unfolding of any heterodimensional cycle creates heterodimensional cycles of the “blender-producing” class, thus proving creation of blenders by a generic perturbation of an arbitrary heterodimensional cycle of co-index 1.

2 Stabilization of heterodimensional dynamics in finite-parameter families

In this section we give a precise formulation of the results, which, in particular, imply Theorem 2. We consider the discrete and continuous-time cases. For both cases we define local maps and transition maps near the heterodimensional cycle (see Section 2.2). After that the proofs are solely based on the analysis of these maps and hence hold for both cases simultaneously.

We start with a more precise description of heterodimensional cycle. Let \( f \) be a \( C^r \)-diffeomorphism of a \( d \)-dimensional manifold or a \( C^r \)-flow of a \((d+1)\)-dimensional manifold, where \( d \geq 3 \) and \( r = 2, \ldots, \infty, \omega \). Let \( f \) have a heterodimensional cycle \( \Gamma \) of co-index 1 associated to two hyperbolic periodic orbits \( L_1 \) and \( L_2 \) with \( \dim W^u(L_2) = \dim W^u(L_1) + 1 \). Along with the orbits \( L_1 \) and \( L_2 \), the heterodimensional cycle \( \Gamma \) consists of two heteroclinic orbits \( \Gamma^0 \in W^u(L_1) \cap W^s(L_2) \) and \( \Gamma^1 \in W^u(L_2) \cap W^s(L_1) \). Due to the difference in the dimensions of \( W^u(L_1) \) and \( W^u(L_2) \), the intersection \( W^u(L_1) \cap W^s(L_2) \) is non-transverse and can be removed by a small perturbation. We call the orbit \( \Gamma^0 \) a \textit{fragile heteroclinic orbit}. On the other hand, the intersection \( W^s(L_1) \cap W^u(L_2) \) at the points of the orbit \( \Gamma^1 \) is assumed to be transverse and it gives a smooth one-parameter family of heteroclinic orbits. We call them \textit{robust heteroclinic orbits}. See Figure 1 for an illustration.

Our goal is to show how \( C^1 \)-robust heterodimensional dynamics emerge in a small neighborhood of the cycle \( \Gamma \). The mechanisms for that depend on the type of the heterodimensional cycle, as described in detail below.

2.1 Local maps near periodic orbits

In the discrete-time case, \( f \) is a diffeomorphism. Let \( O_1 \) and \( O_2 \) be some points of the orbits \( L_1 \) and \( L_2 \). We take a small neighborhood \( U_{0j} \) of the point \( O_j \), \( j = 1, 2 \), and consider the first-return map \( F_j \) in this neighborhood: \( F_j = f^{\tau_j} \) where \( \tau_j \) is the period of \( O_j \) (see Figure 3).

In the continuous-time case, the system \( f \) is a flow generated by some vector field. In this case we take some points \( O_1 \in L_1 \) and \( O_2 \in L_2 \) and let \( U_{0j} \) \( (j = 1, 2) \) be small \( d \)-dimensional (i.e., of co-dimension 1) cross-sections transverse at \( O_j \) to the vector field of \( f \). Let \( F_j \) be the first-return map (the Poincare map) to the cross-section \( U_{0j} \) (see Figure 3).

In both cases \( O_j \) is a hyperbolic fixed point of \( F_j \): \( F_j(O_j) = O_j \). The multipliers of \( O_j \) are defined as the eigenvalues of the derivative of \( F_j \) at \( O_j \). The hyperbolicity means that no multipliers are equal

\textsuperscript{4} One can see here a parallel to Gonchenko’s theory of a homoclinic tangency, which relates dynamics near a homoclinic tangency - the structure of hyperbolic sets, the existence of infinitely many sinks - to arithmetic properties of moduli of topological and \( \Omega \)-conjugacy [20, 21].
to 1 in the absolute value; we assume that $d_j < d$ multipliers of $O_j$ lie outside the unit circle and $(d - d_j)$ multipliers lie inside. By our co-index 1 assumption

$$d_2 = d_1 + 1.$$  

We denote the multipliers of $O_j$, $j = 1, 2$, as $\lambda_{j,d-d_j}, \ldots, \lambda_{j,1}, \gamma_{j,1}, \ldots, \gamma_{j,d_j}$ and order them as follows:

$$|\lambda_{j,d-d_j}| \leq \cdots \leq |\lambda_{j,1}| < 1 < |\gamma_{j,1}| \leq |\gamma_{j,2}| \leq \cdots \leq |\gamma_{j,d_j}|.$$  

(1)

We call the largest in the absolute value multipliers inside the unit circle center-stable multipliers and those nearest to the unit circle from the outside are called the center-unstable multipliers. The rest of the multipliers $\lambda$ and $\gamma$ are called strong-stable and, respectively, strong-unstable.

It is important whether the center-stable multipliers of $O_1$ and center-unstable multipliers of $O_2$ are real or complex. Note that by adding an arbitrarily small perturbation, if necessary, we can always bring the multipliers into general position. In our situation, this means that we can assume that $O_1$ has only one center-stable multiplier $\lambda_{1,1}$ which is real and simple, or a pair of simple complex conjugate center-stable multipliers $\lambda_{1,1} = \lambda_{1,2}^*$; we also can assume that $O_2$ has either only one center-unstable multiplier $\gamma_{2,1}$ which is real and simple, or a pair of simple complex conjugate center-unstable multipliers $\gamma_{2,1} = \gamma_{2,2}^*$.

Accordingly, we distinguish three main cases.

- **Saddle case**: here $\lambda_{1,1}$ and $\gamma_{2,1}$ are real and simple, i.e., we have $|\lambda_{1,2}| < |\lambda_{1,1}|$ and $|\gamma_{2,1}| < |\gamma_{2,2}|$.

- **Saddle-focus case**: here either

$$\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega}, \omega \in (0, \pi), \text{ and } \gamma_{2,1} \text{ is real},$$

where $\lambda > |\lambda_{1,3}|$ and $|\gamma_{2,1}| < |\gamma_{2,2}|$, or

$$\gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega}, \omega \in (0, \pi), \text{ and } \lambda_{1,1} \text{ is real},$$

where $\gamma < |\gamma_{2,3}|$ and $|\lambda_{1,2}| < |\lambda_{1,1}|$. Note that the second option is reduced to the first one by the inversion of time. Therefore, we assume below that in the saddle-focus case $\lambda_{1,1}$ is complex.
and $\gamma_{2,1}$ is real.

- Double-focus case: here
  \[
  \lambda_{1,1} = \lambda_{1,2}^+ = \lambda e^{i\omega_1}, \quad \omega_1 \in (0, \pi), \quad \text{and} \quad \gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega_2}, \quad \omega_2 \in (0, \pi),
  \]
  where $\lambda > |\lambda_{1,3}|, \quad 0 < \gamma < |\gamma_{2,3}|$.

  Below, we denote $\lambda_{1,1}$ and $\gamma_{2,1}$ by $\lambda$ and $\gamma$ if they are real. Let $d_{cs}$ denote the number of the center-stable multipliers of $O_1$ and $d_{cu}$ be the number of the center-unstable multipliers of $O_2$. We have $d_{cs} = d_{cu} = 1$ in the saddle case, $d_{cs} = 2, d_{cu} = 1$ in the saddle-focus case, and $d_{cs} = d_{cu} = 2$ in the double-focus case.

  Recall (see e.g. [23, 39]) that the first-return map $F_1$ near the point $O_1$ has a $d_1$-dimensional local unstable manifold $W_{loc}^u(O_1)$ which is tangent at $O_1$ to the eigenspace corresponding to the multipliers $\gamma_{1,1}, \ldots, \gamma_{1,d_1}$ and $(d - d_1)$-dimensional local stable manifold $W_{loc}^s(O_1)$ which is tangent at $O_1$ to the eigenspace corresponding to the multipliers $\lambda_{1,1}, \ldots, \lambda_{1,d-d_1}$. In $W_{loc}^u(O_1)$ there is a $(d - d_1 - d_{cs})$-dimensional strong-stable $C^r$-smooth invariant manifold $W_{loc}^{ss}(O_1)$ which is tangent at $O_1$ to the eigenspace corresponding to the strong-stable multipliers $\{\gamma_{1,1}, \ldots, \gamma_{1,d-d_1-d_{cs}}\}$. This manifold is a leaf of the strong-stable $C^r$-smooth foliation $\mathcal{F}_{ss}$ of $W_{loc}^s(O_1)$. There also exists a $(d_1 + d_{cs})$-dimensional extended-unstable invariant manifold $W_{loc}^{ue}(O_1)$ corresponding to the center-stable multipliers and the multipliers $\gamma_{1,1}, \ldots, \gamma_{1,d_1}$. Such manifold is not unique but all of them contain $W_{loc}^u(O_1)$ and are tangent to each other at the points of $W_{loc}^u(O_1)$ (see Figure 4).

Similarly, the first-return map $F_2$ near the point $O_2$ has a $d_2$-dimensional local unstable manifold $W_{loc}^u(O_2)$ and $(d - d_2)$-dimensional local stable manifold $W_{loc}^s(O_2)$. In $W_{loc}^u(O_2)$ there is a $(d_2 - d_{cu})$-dimensional strong-unstable invariant manifold $W_{loc}^{uu}(O_2)$ which is tangent at $O_1$ to the eigenspace corresponding to the strong-unstable multipliers $\{\gamma_{2,1}, \ldots, \gamma_{2,d_2}\}$. This manifold is a leaf of the strong-unstable $C^r$-smooth foliation $\mathcal{F}_{uu}$ of $W_{loc}^u(O_2)$. There also exists a $(d - d_2 + d_{cu})$-dimensional extended-stable invariant manifold $W_{loc}^{se}(O_2)$ corresponding to the center-unstable multipliers and the multipliers $\lambda_{2,1}, \ldots, \lambda_{2,d-d_2}$. Any two such manifolds contain $W_{loc}^*(O_2)$ and are tangent to each other at the points of $W_{loc}^u(O_2)$.

### 2.2 Transition maps and geometric genericity conditions

For each of the heteroclinic orbits $\Gamma^0$ and $\Gamma^1$, a transition map between neighborhoods of $O_1$ and $O_2$ is defined, as follows.

Let $f$, first, be a diffeomorphism. Take four points $M_1^+ \in \Gamma^1 \cap W_{loc}^s(O_1)$, $M_1^- \in \Gamma^0 \cap W_{loc}^u(O_1)$, $M_2^+ \in \Gamma^0 \cap W_{loc}^s(O_2)$, and $M_2^- \in \Gamma^1 \cap W_{loc}^u(O_2)$. Note that $M_2^-$ and $M_1^+$ belong to the same robust heteroclinic orbit $\Gamma^1$ and $M_1^-$ and $M_2^+$ belong to the same fragile heteroclinic orbit $\Gamma^0$. Thus, there exist positive integers $n_1$ and $n_2$ such that $f^{n_1}(M_1^+) = M_2^+$ and $f^{n_2}(M_2^-) = M_1^+$. We define the transition maps from a small neighborhood of $M_1^+$ to a small neighborhood of $M_2^+$ and from a small neighborhood of $M_2^-$ to a small neighborhood of $M_1^+$ as, respectively, $F_{12} = f^{n_1}$ and $F_{21} = f^{n_2}$ (see Figure 4).

When $f$ is a flow, we take the points $M_1^+ \in \Gamma^1 \cap W_{loc}^s(O_1)$ and $M_1^- \in \Gamma^0 \cap W_{loc}^u(O_1)$ on the
cross-section $U_{01}$ and the points $M_{2}^{+} ∈ Γ^{0} ∩ W_{loc}^{s}(O_{2})$ and $M_{2}^{-} ∈ Γ^{1} ∩ W_{loc}^{u}(O_{2})$ on the cross-section $U_{02}$. Then the transition map $F_{12}$ is defined as the map by the orbits of the flow which start in the cross-section $U_{01}$ near $M_{1}^{−}$ and hit the cross-section $U_{02}$ near the point $M_{2}^{+}$, and the transition map $F_{21}$ is defined as the map by the orbits of the flow which start in the cross-section $U_{02}$ near $M_{2}^{−}$ and hit the cross-section $U_{01}$ near the point $M_{1}^{+}$. By the definition, $F_{12}(M_{1}^{−}) = M_{2}^{+}$ and $F_{21}(M_{2}^{−}) = M_{1}^{+}$.

We can now precisely describe the genericity conditions which we impose on the heterodimensional cycle $Γ$.

**GC1.** Simplicity of the fragile heteroclinic: $F_{12}^{-1}(W_{loc}^{sE}(O_{2})) ∩ W_{loc}^{u}(O_{1})$ at the point $M_{1}^{−}$ and $F_{12}(W_{loc}^{uE}(O_{1})) ∩ W_{loc}^{s}(O_{2})$ at $M_{2}^{+}$;

**GC2.** Simplicity of the robust heteroclinic: the leaf of $F^{uu}$ at the point $M_{2}^{−}$ is not tangent to $F_{21}^{-1}(W_{loc}^{s}(O_{1}))$ and the leaf of $F^{ss}$ at the point $M_{1}^{+}$ is not tangent to $F_{21}(W_{loc}^{u}(O_{2}))$; and

**GC3.** $Γ^{1} ∩ (W^{ss}(O_{1}) ∪ W^{uu}(O_{2})) = ∅$, i.e., $M_{1}^{+} ∉ W^{ss}(O_{1})$ and $M_{2}^{-} ∉ W^{uu}(O_{2})$.

Figure 4 provides an illustration. The manifolds involved in these conditions depend continuously (as $C^{1}$-manifolds) on $f$ in the $C^{r}$-topology. This implies that these conditions are $C^{r}$-open (condition GC1 is $C^{r}$-open in the class of systems with the heterodimensional cycle). It is also standard that one can always achieve the fulfillment of GC1 and GC2 by adding an arbitrarily small (in $C^{r}$) perturbation to $f$ (in the smooth case one adds a local perturbation to $f$; in the analytic case one uses the scheme described in [14, 22]). In case condition GC3 is not fulfilled, we do not need to perturb the system: for the same system $f$ we can always find, close to $Γ^{1}$, another robust heteroclinic orbit which satisfies GC3. To see this, note that condition GC2 ensures that the line $ℓ_{1} = W_{loc}^{s}(O_{1}) ∩ F_{21}(W_{loc}^{u}(O_{2}))$ (corresponding to robust heteroclinics, see Figure 4) is not tangent to $W^{ss}(O_{1}) ∪ F_{21}(W_{loc}^{uu}(O_{2}))$, so we can always shift the position of the point $M_{1}^{+}$ on this line and hence the position of $M_{2}^{−} = F_{21}^{-1}M_{1}^{+}$.
2.3 Local partial linearization and the fourth genericity condition.

There is one last genericity condition, which is different for the saddle case and the other cases. To state it precisely, let us introduce $C^r$-coordinates $(x, y, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2-d_1-d_x}$ in $U_{01}$ such that the local stable and unstable manifolds get straightened near $O_1$:

$$W_{loc}^{s}(O_1) = \{ y = 0 \}, \quad W_{loc}^{u}(O_1) = \{ x = 0, z = 0 \},$$

and the extended-unstable manifold $W_{loc}^{uu}(O_1)$ is tangent to $\{ z = 0 \}$ when $x = 0, z = 0$ (see Section 3.1). Moreover, the leaves of the foliation $\mathcal{F}^{ss}$ are also straightened and are given by $\{ x = \text{const}, y = 0 \}$. In particular, we have

$$W_{loc}^{ss}(O_1) = \{ x = 0, y = 0 \}.$$

We also introduce $C^r$-coordinates $(u, v, w) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_1-d_u}$ in $U_{02}$ such that the local stable and unstable manifolds are straightened near $O_2$:

$$W_{loc}^{s}(O_2) = \{ u = 0, w = 0 \}, \quad W_{loc}^{u}(O_2) = \{ v = 0 \},$$

the extended-stable manifold $W_{loc}^{ss}(O_2)$ is tangent to $\{ w = 0 \}$ when $u = 0, v = 0$, and the leaves of the foliation $\mathcal{F}^{uu}$ are also straightened and are given by $\{ u = \text{const}, v = 0 \}$, so

$$W_{loc}^{uu}(O_2) = \{ u = 0, v = 0 \}.$$

We restrict the choice of the coordinates by the further requirement (which can always be fulfilled, see e.g. [23]) that the first-return maps $F_1$ and $F_2$ act linearly on center-stable and, respectively, center-unstable coordinates. Namely, if we restrict these maps on $W_{loc}^{s}(O_1) = \{ y = 0 \}$ and, respectively, $W_{loc}^{u}(O_2) = \{ v = 0 \}$ and use the notation $F_1|_{W_{loc}^{s}(O_1)}: (x, z) \mapsto (\bar{x}, \bar{z})$ and $F_2|_{W_{loc}^{u}(O_2)}: (u, w) \mapsto (\bar{u}, \bar{w})$, then we have

Saddle case: \( \bar{x} = \lambda x \) and \( \bar{u} = \gamma u \);

Saddle-focus case: \( \bar{x} = \lambda \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} x \) and \( \bar{u} = \gamma u \); \hspace{1cm} (2)

Double-focus case: \( \bar{x} = \lambda \begin{pmatrix} \cos \omega_1 & \sin \omega_1 \\ -\sin \omega_1 & \cos \omega_1 \end{pmatrix} x \) and \( \bar{u} = \gamma \begin{pmatrix} \cos \omega_2 & \sin \omega_2 \\ -\sin \omega_2 & \cos \omega_2 \end{pmatrix} u. \)

We denote, in these coordinates, $M_1^{+} = (x^+, 0, z^+)$ and $M_2^{-} = (u^-, 0, w^-)$ (so condition GC3 reads as $x^+ \neq 0$ and $u^- \neq 0$).

Recall that $F_2$ takes a point with coordinates $(u, v, w)$ into a point with coordinates $(x, y, z)$, where $x$ and $u$ are the center-stable and center-unstable coordinates near the points $O_1$ and $O_2$, respectively. By condition GC2, the line $\ell_1$ is not tangent to the foliation $\mathcal{F}^{ss}$ and the line $\ell_2 = F_2^{-1}\ell_1$ is not tangent to the foliation $\mathcal{F}^{uu}$. In the saddle case this means that these curves are transverse to these foliations (see Figure 4), so they are parameterized by the coordinates $x$ (the line $\ell_1$) and $u$ (the
line \( \ell_2 \). Therefore, as \( F_{21} |_{\ell_2} \) acts as a diffeomorphism \( \ell_2 \to \ell_1 \), we have

\[
b = \frac{\partial x}{\partial u} \bigg|_{M_2^\pm} \neq 0.
\]

(3)

We introduce the quantity \( \alpha := bu^- / x^+ \) and require

**GC4.1** (saddle case).

\[|\alpha| \neq 1.\]  

(4)

Note that in the saddle case conditions GC1, GC2 are equivalent (see [41]) to the requirement that the heteroclinic cycle \( \Gamma \) is a partially-hyperbolic set with the 1-dimensional central direction field tangent to the center-stable eigenvector at \( O_1 \) and the center-unstable eigenvector at \( O_2 \). As we show, \( \alpha \) determines the behavior in the central direction: the first-return maps near \( \Gamma \) are contracting in the central direction when \( |\alpha| < 1 \) and expanding when \( |\alpha| > 1 \) (see Lemma 1). Note that \( \alpha \) is an invariant of smooth coordinate transformations which keep the foliations \( \mathcal{F}^{ss} \) and \( \mathcal{F}^{uu} \) locally straightened and the action of the local maps \( F_1 \) and \( F_2 \) in the central direction linear, as in (2). Indeed, any such transformation is linear in the central directions in a small neighborhoods \( U_{01} \) and \( U_{02} \) of the points \( O_1 \) and \( O_2 \), i.e., the coordinates \( x \) and \( u \) are only multiplied to some constants \( c_x \) and \( c_u \). As a result, the coefficient \( b \) is replaced by \( bc_x / c_u \), and \( x^+ \) and \( u^- \) are replaced by \( c_x x^+ \) and \( c_u u^- \), so \( \alpha \) remains unchanged. Similarly, the invariant \( \alpha \) does not depend on the choice of the points \( M_1^+ \) and \( M_2^- \) on the given heteroclinic orbit \( \Gamma^1 \).

In the saddle-focus and double-focus cases, the partial hyperbolicity is not assumed, and no condition similar to GC4.1 is needed. However, we need another condition:

**GC4.2** (saddle-focus and double-focus cases). When the center-stable multipliers \( \lambda_{1,1} \) and \( \lambda_{1,2} \) are complex and \( x \in \mathbb{R}^2 \), the \( x \)-vector component of the tangent to \( \ell_1 \) at the point \( M_1^+ \) is not parallel to the vector \( x^+ \) (see Figure 5). When the center-unstable multipliers \( \gamma_{2,1} \) and \( \gamma_{2,2} \) are complex and \( u \in \mathbb{R}^2 \), the \( u \)-vector component of the tangent to \( \ell_2 \) at the point \( M_2^- \) is not parallel to the vector \( u^+ \).

[Figure 5: An illustration of condition GC4.2. The vector \((x_1^+, x_2^+)\) is not parallel to the \( x \)-vector component (the dashed line) of the tangent of \( \ell_1 \).]

Like in condition GC4.1, coordinate transformations that keep the action of the local maps \( F_1 \) and \( F_2 \) in \( x \) and \( u \) linear are also linear in \( x \) and \( u \), respectively. This immediately implies that Condition GC4.2 is invariant with respect to the choice of the linearizing coordinates. It also does not depend
on the choice of the points $M_1^+$ and $M_2^-$. 

The heterodimensional cycles satisfying conditions GC1-GC4.1.2 will be further called \textit{generic}.

2.4 Finite-parameter unfoldings

The perturbations we use to prove Theorem 1 are done within finite-parameter families $f_\varepsilon$ which we assume to be of class $C^r$ ($r = 2, \ldots, \infty, \omega$) jointly with respect to coordinates and parameters $\varepsilon$.

Let $f_0 = f$; for any sufficiently small $\varepsilon$ the hyperbolic points $O_1$ and $O_2$ exist and depend smoothly on $\varepsilon$. The corresponding multipliers also depend smoothly ($C^{r-1}$) on $\varepsilon$. We define

$$\theta(\varepsilon) = -\frac{\ln |\lambda|}{\ln |\gamma|}. \quad (5)$$

In the saddle-focus and double-focus cases, an important role is also played by the frequencies $\omega(\varepsilon)$ and, respectively, $\omega_{1,2}(\varepsilon)$. The values of $\theta$ as well as $\omega_{1,2}$ are moduli of topological conjugacy of diffeomorphisms with generic heterodimensional cycles (see [33, 40]).

The local stable and unstable manifolds of $O_1, O_2$ also depend smoothly on $\varepsilon$, as well as their images by the transition maps $F_{12}$ and $F_{21}$. The fragile heteroclinic $\Gamma^0$ is not, in general, preserved when $\varepsilon$ changes. To determine whether the fragile heteroclinic disappears or not, one introduces a \textit{splitting parameter} $\mu$, a continuous functional such that for any system $g$ from a small $C^r$-neighborhood of $f$ the absolute value of $\mu(g)$ equals to the distance between $W^u_{loc}(O_2)$ and $F_{12}(W^s_{loc}(O_1))$; the fragile heteroclinic persists for those $g$ for which $\mu(g) = 0$. The codimension-1 manifold $\mu = 0$ separates the neighborhood of the system $f$ into two connected components; we define $\mu$ such that it changes sign when going from one component to the other. For parametric families, we define the splitting parameter as $\mu_\varepsilon = \mu(f_\varepsilon)$. It depends smoothly on $\varepsilon$; the fragile heteroclinic persists when $\mu_\varepsilon = 0$.

A one-parameter family $f_\varepsilon$ will be called a \textit{generic one-parameter unfolding} of $f$ if $\frac{d\mu}{d\varepsilon} \neq 0$. This means that we can make a smooth change of parameters such that $\mu = \varepsilon$.

We also need to consider families depending on two or more parameters, i.e., $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$. We call the family $f_\varepsilon$ a \textit{proper unfolding}, if $\frac{d\mu}{d\varepsilon} \neq 0$ (so the set $\mu_\varepsilon = 0$ forms a smooth codimension-1 manifold $\mathcal{H}_0$ in the space of parameters $\varepsilon$) and, the following conditions hold for the subfamily corresponding to $\varepsilon \in \mathcal{H}_0$:

- in the saddle case, $\frac{d\theta}{d\varepsilon} \neq 0$, where the derivative is taken over $\varepsilon \in \mathcal{H}_0$ (this implies that we can make a smooth change of parameters in the family $f_\varepsilon$ such that $\mu = \varepsilon_1$ and $\theta = \varepsilon_2$);
- in the saddle-focus case, the condition is that the functions $\theta(\varepsilon)$ and $\frac{1}{2\pi}\omega(\varepsilon)$ and 1 are linearly independent in a neighborhood of $\varepsilon = 0$ on $\mathcal{H}_0$;
- in the double-focus case, the condition is the linear independence of $\theta(\varepsilon)$, $\frac{1}{2\pi}\omega_1(\varepsilon)$, $\frac{1}{2\pi}\omega_2(\varepsilon)\theta(\varepsilon)$ and 1 in a neighborhood of $\varepsilon = 0$ on $\mathcal{H}_0$.\footnote{Note that in the double-focus case the problem is symmetric with respect to the time reversal and interchange of $O_1$ and $O_2$. This operation changes $\theta$ to $\theta^{-1}$ and interchanges $\omega_1$ and $\omega_2$, thus keeping the linear independence condition intact.}

5
Note that we only need *rational* linear independence in the propriety conditions for the saddle-focus and double-focus case. However, we do not include this requirement in the definition to make the class of proper families open.

With the above definitions, the formulation of our main result, Theorem 2 as given in Section 1, is now complete. The proof goes differently in different cases: for the saddle case the theorem follows from the results described in Sections 2.5 and 2.6, and in the saddle-focus and double focus case it follows from the results of Section 2.7.

### 2.5 Three types of heterodimensional cycles in the saddle case

In the saddle case, the proof of Theorem 2 is most involved: not because of technicalities, but because the dynamics emerging at perturbations of the generic heterodimensional cycles in the saddle case depend very essentially on the type of the cycle. According to that, we introduce 3 types of the heterodimensional cycles in the saddle case, as follows.

First, note (by counting dimensions) that in the saddle case condition GC1 implies that the intersection of $F_{12}^{-1}(W^{sE}_{loc}(O_2))$ and $W^{uE}_{loc}(O_1)$ is a smooth curve, which we denote as $\ell^0$ (see Figure 6). At $\varepsilon = 0$, this curve goes through the point $M_1^-$ and its image $F_{12}\ell^0$ goes through the point $M_2^+$. The tangent space $T_{M_1^-}\ell^0$ lies in $T_{M_1^-}W^{sE}_{loc}(O_1) = \{ z = 0 \}$ and, by GC1, $T_{M_1^-}\ell^0 \not\subset T_{M_1^-}W^{u}_{loc}(O_1) = \{ x = 0, z = 0 \}$, which implies that $T_{M_1^-}\ell^0$ has a non-zero projection to the $x$-axis. Thus, the curve $\ell^0$ is parameterized by the coordinate $x$. Similarly, the curve $F_{12}\ell^0$ is parameterized by coordinate $u$. The restriction of $F_{12}$ to $\ell^0$ is a diffeomorphism, so

$$a = \frac{du}{dx} \bigg|_{M_1^-} \neq 0. \quad (6)$$

![Figure 6: Condition GC1 in straightened coordinates.](image)

We say that a heterodimensional cycle $\Gamma$ in the saddle case is of

- **type I**, if there exist points $M_1^+(x^+, 0, z^+) \in \Gamma \cap U_{01}$ and $M_2^-(u^-, 0, w^-) \in \Gamma \cap U_{02}$ such that $ax^+u^- > 0$;
- **type II**, if there exist points $M_1^+(x^+, 0, z^+) \in \Gamma \cap U_{01}$ and $M_2^-(u^-, 0, w^-) \in \Gamma \cap U_{02}$ such that $ax^+u^- < 0$;
• type III, if there exist points $M_1^+ \in \Gamma^1 \cap U_{01}$ and $M_2^- \in \Gamma^1 \cap U_{02}$ for which $ax^+u^- > 0$ and another pair of points $M_1^+ \in \Gamma^1 \cap U_{01}$ and $M_2^- \in \Gamma^1 \cap U_{02}$ for which $ax^+u^- < 0$.

Type-I cycles correspond to twisted cycles of [13] and type-II cycles to non-twisted cycles. The cycle of type III is, by definition, a cycle which is simultaneously of type I and type II. Like in condition GC4.1, one shows that the sign of $ax^+u^-$ is independent of the choice of coordinates which keep the action of the local maps $F_1$ in the neighborhood $U_{01}$ of $O_1$ and $F_2$ in the neighborhood $U_{02}$ of $O_2$ linear in the central coordinates $x$ and $u$. Thus, the above definition is coordinate independent.

Notice that $\theta$ is determined by a pair of points $M_1^-$ and $M_2^+$ on the fragile heteroclinic $\Gamma^0$, while $x^+$ and $u^-$ are coordinates of points on the robust heteroclinic $\Gamma^1$. By (2) (the saddle case), if the central multipliers $\lambda$ and $\gamma$ are positive, the local maps $F_1$ and $F_2$ multiply $x^+$ and $u^-$ to positive factors, so the sign of $ax^+u^-$ is independent of the choice of the points $M_1^+$ and $M_2^-$ on $\Gamma^1$ in this case. Similarly, it does not depend on the choice of the points $M_1^-$ and $M_2^+$ on $\Gamma^0$. On the other hand, if at least one of the central multipliers is negative, the sign of $x^+u^-$ changes when one replaces the pair $(M_1^+, M_2^-)$ by the pair of points $(F_1(M_1^+), M_2^-)$ or the pair of points $(M_1^+, F_2^{-1}(M_2^-))$ on the same orbit $\Gamma^1$. Thus, a generic heterodimensional cycle has either type I or type II, and not type III, if and only if both central multipliers are positive, and it has type III if and only if at least one of the central multipliers is negative.

### 2.6 Main results for the saddle case

The key observation in our proof of Theorem 2 in the saddle case and the fundamental reason behind the emergence of robust heterodimensional dynamics is given by the following result proven in Section 3.2.

**Theorem 3.** In the saddle case, in any neighborhood of a generic heterodimensional cycle $\Gamma$ of type I (including type III) for which the value of $\theta = -\ln |\lambda| / \ln |\gamma|$ is irrational, there exists a co-index 1 blender, center-unstable with index 5 $d_1$ if $|\alpha| < 1$ or center-stable with index $d_2$ if $|\alpha| > 1$.

The result holds true for systems $f$ of class at least $C^2$. The blender is not the one constructed in [9] by means of a $C^1$-small but not $C^2$-small perturbation of $f$. We do not perturb $f$, but give explicit conditions for the existence of the blender. Moreover, the (at least) $C^2$ regularity is important for the proof, and it is not clear whether Theorem 3 holds when $f \in C^1$. Namely, it is a priori possible that there could exist systems $f \in C^1$ for which a neighborhood of a heterodimensional cycle of type I does not contain a blender even when $\theta$ is irrational.

Next theorem tells us when the blender of Theorem 3 is activated, i.e., when it gets involved in a heterodimensional cycle with another hyperbolic set and robust heterodimensional dynamics emerge. By definition, the blender exists for any system $C^1$-close to $f$.

**Theorem 4.** Let $\Gamma$ be a generic type-I cycle and let $\theta$ be irrational. Consider a sufficiently small $C^r$-neighborhood $V$ of $f$ such that the blender given by Theorem 3 persists for any system $g \in V$. Let $\mu$ be a splitting functional.

---

6 Here, the index of a hyperbolic set is the dimension of its unstable manifold if $f$ is a map and the dimension of the unstable manifold minus 1 if $f$ is a flow.
• In the case $|\alpha| < 1$, there exist constants $C_1 < C_2$ such that for all sufficiently large $m \in \mathbb{N}$ any system $g \in \mathcal{V}$ which satisfies $\mu \gamma^m \in [C_1, C_2]$ has $C^1$-robust heterodimensional dynamics involving the index-$d_1$ cu-blender $\Lambda^c_{\mu}$ of Theorem 3 and a non-trivial, index-$d_2$ hyperbolic set containing $O_2$.

• In the case $|\alpha| > 1$, there exist constants $C_1 < C_2$ such that for all sufficiently large $k \in \mathbb{N}$ any system $g \in \mathcal{V}$ which satisfies $\mu \lambda^{-k} \in [C_1, C_2]$ has $C^1$-robust heterodimensional dynamics involving the index-$d_2$ cs-blender $\Lambda^{cs}_{\mu}$ of Theorem 3 and a non-trivial, index-$d_1$ hyperbolic set containing $O_1$.

The theorem is proven in Section 3.3 (see Proposition 4), as well as Theorems 5 - 7 below. Note that the cases $|\alpha| < 1$ and $|\alpha| > 1$ are reduced to each other by reversion of time and the interchange of the points $O_1$ and $O_2$. Theorem 4 immediately implies Theorem 2 in the case of type-I cycles. Indeed, in a proper unfolding of $f$ we can, by an arbitrarily small increment make $\theta$ irrational while keeping $\mu = 0$, and then put $\mu$ to an interval corresponding to the $C^1$-robust heterodimensional dynamics.

We also show (see Proposition 4) that there exist intervals of $\mu$ for which the index-$d_1$ blender $\Lambda^c_{\mu}$ is homoclinically related to $O_1$ if $|\alpha| < 1$ or the index-$d_2$ blender $\Lambda^{cs}_{\mu}$ is homoclinically related to $O_2$ if $|\alpha| > 1$. Recall that two transitive hyperbolic sets of the same index are homoclinically related if they are involved in a transverse heteroclinic cycle, i.e., the unstable manifold of each set has a transverse intersection with the stable manifold of the other set. If the blender is homoclinically related to a saddle $O_1$ or $O_2$ and, simultaneously, forms a robust heterodimensional cycle involving the other saddle, this would give robust heterodimensional dynamics involving both these saddles. However, the following result shows that if the central multipliers $\lambda$ and $\gamma$ are both positive, this does not happen.

Let $f$ have a generic heterodimensional cycle $\Gamma$ of type I with $\lambda > 0$ and $\gamma > 0$, i.e., it is not type-III (we do not insist now that $\theta$ is irrational). Let $U$ be a small neighborhood of $\Gamma$.

**Theorem 5.** One can choose the sign of the splitting functional $\mu$ such that for every system $g$ from a small $C^r$-neighborhood of $f$

• in the case $|\alpha| < 1$, for $\mu(g) > 0$, the set of all points whose orbits lie entirely in $U$ consists of a hyperbolic set $\Lambda$ of index $d_1$ (it includes the orbit $L_1$ of $O_1$), of the orbit $L_2$ of the periodic point $O_2$, and of heteroclinic orbits corresponding to the transverse intersection of $W^s(L_2)$ with $W^u(\Lambda)$, so there are no heterodimensional dynamics in $U$;
  if $\mu(g) \leq 0$, then no orbit in the unstable manifold of $L_1$ stays entirely in $U$, except for $L_1$ itself and, at $\mu(g) = 0$, the fragile heteroclinic $\Gamma^0$, so $L_1$ cannot be a part of any heterodimensional cycle in $U$ when $\mu(g) < 0$;

• in the case $|\alpha| < 1$, for $\mu(g) < 0$, the set of all points whose orbits lie entirely in $U$ consists of a hyperbolic set $\Lambda$ of index $d_2$ (it includes the orbit $L_2$ of $O_2$), of the orbit $L_1$ of the periodic point $O_1$, and of heteroclinic orbits corresponding to the transverse intersection of $W^s(L_1)$ with $W^u(\Lambda)$, so there are no heterodimensional dynamics in $U$;
  for $\mu(g) \geq 0$, no orbit in $W^s(L_2)$ stays entirely in $U$ except for $L_2$ itself and, at $\mu(g) = 0$, the fragile heteroclinic $\Gamma^0$, so $L_2$ cannot be a part of any heterodimensional cycle in $U$ when $\mu(g) > 0$. 

15
This situation changes if the type-I cycle is accompanied by a type-II cycle in the following sense.

Definition 4. We say that two generic heterodimensional cycles associated with $O_1$ and $O_2$ are tied if they share the same fragile heteroclinic, and the robust heteroclinic orbits $\Gamma^1$ and $\tilde{\Gamma}^1$ belonging to the corresponding cycles $\Gamma$ and $\tilde{\Gamma}$ intersect the same leaf of $F^{ss}$ or $F^{uu}$. Specifically, there exists a pair of points $M^+_1 = (x^+, 0, z^+) \in \Gamma^1 \cap W^s_{loc}(O_1)$ and $\tilde{M}^+_1 = (\tilde{x}^+, 0, \tilde{z}^+) \in \tilde{\Gamma}^1 \cap W^s_{loc}(O_1)$ such that $x^+ = \tilde{x}^+$ or a pair of points $M^-_2 = (u^-, 0, u^-) \in \Gamma^1 \cap W^u_{loc}(O_2)$ and $\tilde{M}^-_2 = (\tilde{u}^-, 0, \tilde{u}^-) \in \tilde{\Gamma}^1 \cap W^u_{loc}(O_2)$ such that $u^- = \tilde{u}^-$. 

The existence of tied cycles is a $C^2$-open property in the set of systems for which the fragile heteroclinic is preserved. Indeed, if for a system $f$ we have two points $M^+_1 \in \Gamma^1$ and $\tilde{M}^+_1 \in \tilde{\Gamma}^1$ lying in a common leaf $l^{ss}$ of the foliation $F^{ss}$, then there are curves $\ell_1$ and $\tilde{\ell}_1$ containing these points, which correspond to the transverse intersection of $W^u(O_2)$ and $W^s_{loc}(O_1)$ and which, by condition GC2, are transverse to the leaf $l^{ss}$ of $F^{ss}$. The transversality implies that a $C^r$-small perturbation of $f$ does not destroy this double intersection in $l^{ss}$. The same is true if we have a double intersection with a leaf of $F^{uu}$.

Theorem 6. Let a system $f$ have a generic type-I cycle $\Gamma$ tied with a generic type-II cycle $\tilde{\Gamma}$; assume that $\theta$ is irrational. Then, for any generic one-parameter unfolding $f_\mu$ there exists a converging to $\mu = 0$ sequence of intervals $I_j$ such that $f_\mu$ at $\mu \in I_j$ has $C^1$-robust heterodimensional dynamics involving the blender given by Theorem 3 and a non-trivial transitive hyperbolic set; of these two hyperbolic sets, the one with index $d_1$ is homoclinically related to the point $O_1$ and the one with index $d_2$ is homoclinically related to $O_2$.

Observe that a type-III cycle is, by definition, a cycle of type I and II, and, obviously, it is tied with itself. Hence, applying the above theorem, we obtain

Corollary 2. Let $\Gamma$ be a generic cycle with real central multipliers $\lambda$ and $\gamma$, at least one of which is negative. If $\theta = -\ln|\lambda|/\ln|\gamma|$ is irrational, then for any generic one-parameter unfolding $f_\mu$ of class $C^r$ ($r = 2, \ldots, \infty, \omega$) there exist converging to zero intervals of $\mu$ corresponding to $C^1$-robust heterodimensional dynamics involving non-trivial transitive hyperbolic sets, one of which contains the point $O_1(\mu)$ and the other contains the point $O_2(\mu)$.

Remark 2. Tied cycles also occur when $O_1$ or $O_2$ have a transverse homoclinic. For example, let us have a generic heterodimensional cycle $\Gamma$ with a fragile heteroclinic $\Gamma^0$ and a robust heteroclinic $\Gamma^1$. Assume the central multipliers are real, and let $M' \in W^u_{loc}(O_2)$ be a point of transverse intersection of the stable and unstable manifolds of $O_2$. If we take a small piece of $W^u_{loc}(O_2)$ around $M'$, its forward images converge to the entire unstable manifold of $O_2$. Therefore, some of them must intersect transversely the strong-stable leaf of the point $M^+_1 \in \Gamma^1 \cap W^s_{loc}(O_1)$ (as this leaf intersects $W^s_{loc}(O_2)$ transversely at the point $M^+_1$ by condition GC2). The orbit of the intersection point is a robust heteroclinic $\tilde{\Gamma}^1$, and the corresponding cycle $\tilde{\Gamma}$ is tied with $\Gamma$. By construction, the orbit $\tilde{\Gamma}^1$ has a point in $W^u_{loc}(O_2)$ close to the homoclinic point $M'$. Therefore, its $u$-coordinate is close to the $u$-coordinate $u'$ of $M'$. Therefore, if $u^- u' < 0$, i.e., the homoclinic point $M'$ and the point $M^-_2$ of $\Gamma^1$ lie in $W^u_{loc}(O_2)$ on opposite sides from $W^u_{loc}(O_2)$, then the tied cycles $\Gamma$ and $\tilde{\Gamma}$ have different types, and Theorem 6 is applicable.

Theorem 2 for type-II cycles is inferred from Theorem 6 by means of the following result.
Theorem 7. Let a system \( f \) of class \( C^r \) \((r = 2, \ldots, \infty, \omega)\) have a generic type-II cycle \( \Gamma \) with irrational \( \theta \). For any generic one-parameter unfolding \( f_\mu \) there exists a sequence \( \mu_j \to 0 \) such that \( f_\mu \) at \( \mu = \mu_j \) has a pair of tied heterodimensional cycles \( \Gamma_{j,I} \) and \( \Gamma_{j,II} \) of type I and type II, which are associated

- with \( O_1(\mu) \) and an index-\( d_2 \) saddle \( O_2'(\mu) \) which is homoclinically related to \( O_2(\mu) \) if \(|\alpha| < 1\); or
- with \( O_2(\mu) \) and an index-\( d_1 \) saddle \( O_1'(\mu) \) which is homoclinically related to \( O_1(\mu) \) if \(|\alpha| > 1\).

In the case \( r \geq 3 \), it follows from our computations that if we extend \( f_\mu \) to any proper, at least two-parameter unfolding \( f_\varepsilon \) of \( f \), the same family \( f_\varepsilon \) will give a proper unfolding for the newly obtained cycles \( \Gamma_{j,I} \) and \( \Gamma_{j,II} \). Since \( f_\varepsilon \) is proper for the cycles \( \Gamma_{j,I} \) and \( \Gamma_{j,II} \), one can always find the values of \( \varepsilon \) for which the value of \( \theta \) for these cycles is irrational. Hence, applying Theorem 6, we obtain the result of Theorem 2 when \( f_\varepsilon \) is at least \( C^3 \). In the \( C^2 \)-case, the difficulty is that we use, for every parameter value, the coordinates which linearize the action of the local maps \( F_1 \) and \( F_2 \) in the central direction. It is known that in the \( C^2 \)-case the linearizing coordinate transformation is, in general, not smooth with respect to parameters, so our technique does not allow to compute derivatives with respect to \( \varepsilon \) which enter the definition of a proper unfolding. Instead, we use continuity arguments to show in the \( C^2 \)-case that, still, for the tied cycles \( \Gamma_{j,I} \) and \( \Gamma_{j,II} \) the value of \( \theta \) can be made irrational and the splitting parameter for these cycles can be pushed, by a small change of \( \varepsilon \), inside the intervals described by Theorem 6. Altogether, this gives us the result of Theorem 2 for type-II cycles in the following form.

Corollary 3. Let \( \Gamma \) be a generic type-II cycle, and let \( f_\varepsilon \) be a proper, at least two-parameter unfolding. Arbitrarily close to \( \varepsilon = 0 \) there exist open regions in the parameter space for which the corresponding system \( f_\varepsilon \) has \( C^1 \)-robust heterodimensional dynamics associated to a blender and a non-trivial transitive hyperbolic set; one of these sets contains point \( O_1 \) and the other contains point \( O_2 \).

As we see, the emergence of heterodimensional dynamics depends strongly on the arithmetic properties of moduli of topological equivalence \( \theta \) and \( \omega \)'s. The following result shows that in the saddle case we have a clear dichotomy: while in the case of irrational \( \theta \) we have highly non-trivial dynamics and bifurcations in any neighborhood of the heterodimensional cycle \( \Gamma \), in the case of rational \( \theta \) the dynamics in a small neighborhood of \( \Gamma \) are quite simple, in general.

Theorem 8. Let a system \( f \) of class \( C^r \) \((r \geq 2)\) have a generic heterodimensional cycle \( \Gamma \), and let the central multipliers be real, \(|\lambda| < 1 \) and \(|\gamma| > 1 \). Let \( \theta = -\ln |\lambda|/\ln |\gamma| \) be rational, i.e., \(|\gamma| = |\lambda|^{-p/q} \) for some coprime integers \( p > 0, q > 0 \). Suppose the following conditions are fulfilled:

\[
|\alpha| \neq |\gamma|^{|s^2/2|} \quad \text{for} \quad s \in \mathbb{Z}, \tag{7}
\]

\[
\left| \frac{u}{ax^t} \right| \not\in Q \left\{ |\gamma|^{s} \left( 1 - \frac{\lambda^t}{1 - \gamma^{-n}} \right) \right\}_{s \in \mathbb{Z}, l \in \mathbb{N}, n \in \mathbb{N}}. \tag{8}
\]

Let \( U \) be a sufficiently small neighborhood of \( \Gamma \) and let \( \mathcal{N} \) be the set of all orbits that lie entirely in \( U \). Then, at \( \mu = 0 \), the set \( \mathcal{N} \) is the union of \( L_1, L_2, \Gamma^0 \), and the orbits of transverse intersection of \( W^u(L_2) \) with \( W^s(L_1) \) near \( \Gamma^1 \).

For any generic one-parameter unfolding \( f_\mu \), for any small \( \mu \neq 0 \), either
• $N$ is comprised by $L_2$, an index-$d_1$ uniformly-hyperbolic compact set $\Lambda_1$ which contains $L_1$, and by transverse heteroclinic connections between $W^u(L_2)$ and $W^s(\Lambda_1)$, while no heteroclinic connection between $W^u(\Lambda_1)$ and $W^s(L_2)$ exists, or

• $N$ is comprised by $L_1$, an index-$d_2$ uniformly-hyperbolic compact set $\Lambda_2$ which contains $L_2$, and by transverse heteroclinic connections between $W^u(\Lambda_2)$ and $W^s(L_1)$, while no heteroclinic connection between $W^u(L_1)$ and $W^s(\Lambda_2)$ exists.

The proof of this theorem is given in Section 3.4. Notice that, for fixed values of $\lambda$ and $\gamma$, conditions (7),(8) are fulfilled for all $ab$ and $\frac{ax+u−}{\omega}$ except for a countable, nowhere dense set of values. Thus, the simplicity of dynamics at rational $\theta$ is indeed quite generic. It also follows from this theorem and Theorem 3 that whenever we have a heterodimensional cycle $\Gamma$ of type I, if we change $\theta$ without destroying $\Gamma$, the blender that forms at irrational $\theta$’s immediately departs from $\Gamma$, so that for each rational $\theta$ a sufficiently small “blender-free” neighborhood of $\Gamma$ emerges.

2.7 Main result for the case of complex multipliers

In the remaining saddle-focus and double-focus cases, we obtain Theorem 2 from Theorem 9.

**Theorem 9.** Let $f$ have a heterodimensional cycle $\Gamma$ and let central multipliers be $\lambda_{1,1} = \lambda_{1,2} = \lambda e^{i\omega}$ and real $\gamma_{2,1} = \gamma$ (the saddle-focus case) or $\lambda_{1,1} = \lambda_{1,2} = \lambda e^{i\omega_1}$ $\gamma_{2,1} = \gamma_{2,2} = \gamma e^{i\omega_2}$ (the double-focus case). Assume, in the saddle-focus case, that the numbers $\theta = -\ln |\lambda|/\ln |\gamma|$, $\frac{\omega_1}{2\pi}$ and 1 are rationally independent. In the double-focus case, assume that $\theta$, $\frac{\omega_1}{2\pi}$, $\frac{\omega_2}{2\pi}$, and 1 are rationally independent. Then, in any neighborhood of $\Gamma$, the system $f$ has a $C^1$-robust heterodimensional dynamics associated to a cu-blender and a cs-blender.

Moreover, in the double-focus case, the point $O_1$ is homoclinically related to the cu-blender and the point $O_2$ is homoclinically related to the cs-blender, and the same holds true for any $C^r$-close system $g$. In the saddle-focus case, the point $O_2$ is homoclinically related to the cu-blender for the system $f$ and for any $C^r$-close system $g$, and there exist intervals $I_j$ converging to $\mu = 0$ such that if $\mu(g) \in I_j$, then the point $O_2$ is homoclinically related to the cs-blender.

**Remark 3.** By the inversion of time, we deduce a similar result for the case where $\gamma_{2,1} = \gamma_{2,2} = \gamma e^{i\omega}$ and $\lambda_{1,1} = \lambda$ is real. Namely, we obtain that the point $O_2$ is homoclinically related to a cs-blender for the system $f$ and for any $C^r$-close system $g$, and that there exist intervals $I_j$ converging to $\mu = 0$ such that if $\mu(g) \in I_j$, then the point $O_1$ is homoclinically related to the cu-blender.

This theorem is proved in Section 4. Since in a proper unfolding of $f$ the rational independence conditions of Theorem 9 are achieved by arbitrarily small changes of parameters, the claim of Theorem 2 follows immediately.

2.8 Local stabilization of heterodimensional cycles

The above results can be interpreted in terms of the local stabilization problem for heterodimensional cycles.
Definition 5. A heterodimensional cycle $\Gamma$ associated to two saddle periodic orbits $L_1$ and $L_2$ is called locally $C^r$-stabilizable if for any neighborhood $U$ of $\Gamma$ the system $f$ belongs to the $C^r$-closure of a $C^1$-open set where systems having a heterodimensional cycle which lies in $U$ and is associated to the continuations of $L_1$ and $L_2$ are $C^r$-dense.

This definition is a version of the one introduced in [12, 13] where results on $C^1$-stabilizability were established. In particular, one can infer from [12, 13] that type-I cycles are not locally $C^1$-stabilizable, so they are not $C^r$-stabilizable for any $r \geq 1$. As an immediate consequence of Corollaries 2 and 3 and Theorem 9 we obtain a complement to this result:

Theorem 10. Except for type-I cycles, all other co-index 1 heterodimensional cycles associated to a pair of hyperbolic periodic orbits are locally $C^r$-stabilizable for any $r = 2, \ldots, \infty, \omega$.

If we replace at least one of the periodic orbits in the heterodimensional cycle by a non-trivial, transitive, compact hyperbolic set $\Lambda$, then even if all periodic orbits in $\Lambda$ have central multipliers real and positive, one can, in the generic situation, find a periodic orbit in $\Lambda$ such that its homoclinic points accumulate to it from both sides in the central direction. Then, by Remark 2 and Theorem 6 we obtain

Theorem 11. Consider a heterodimensional cycle involving two transitive compact hyperbolic sets. It is locally $C^r$-stabilizable for any $r = 2, \ldots, \infty, \omega$ if at least one of these sets is not a single periodic orbit.

3 The saddle case

In this section, we consider the case of real central multipliers and prove Theorems 3-8 and Corollary 3.

3.1 First-return maps

Recall that we consider the behavior in a small neighborhood of the heterodimensional cycle $\Gamma$. We take periodic points $O_1$ and $O_2$ in $\Gamma$ and consider, if $f$ is a discrete dynamical system, small neighborhoods $U_{01}$ and $U_{02}$ of $O_1$ and $O_2$. If $f$ is a flow, then $U_{01}$ and $U_{02}$ are small codimension-1 cross-section to the flow through the points $O_1$ and $O_2$. In both cases, the local maps $F_1$ and $F_2$ act on $U_{01}$ and $U_{02}$, which are defined by the orbits of the system $f$ near the orbits $L_1$ and $L_2$ of the points $O_1$ and $O_2$. We take a pair of points $M_1^- \in U_{01}$ and $M_2^+ \in U_{02}$ on the fragile heteroclinic $\Gamma^0$ and a pair of points $M_2^- \in U_{02}$ and $M_1^+ \in U_{01}$ on the robust heteroclinic $\Gamma^1$. The orbits near the fragile heteroclinic define the transition map $F_{12}$ from a small neighborhood of $M_1^-$ in $U_{01}$ to a small neighborhood of $M_2^+$ in $U_{02}$, and the orbits near the robust heteroclinic define the transition map $F_{21}$ from a small neighborhood of $M_2^-$ in $U_{02}$ to a small neighborhood of $M_1^+$ in $U_{01}$. Our immediate goal is to analyze first-return maps near the heterodimensional cycle, which take points from a small neighborhood $M_1^+$ in $U_{01}$ back to a vicinity of $M_1^+$. Such map is the composition $T_{k,m} = F_{21} \circ F_{m}^m \circ F_{12} \circ F_{k}^k$ where $k$ and $m$ are sufficiently large positive integers such that $F_k^k$ takes points from a small neighborhood of $M_1^+$ in $U_{01}$ to a small neighborhood of $M_1^-$ in $U_{01}$ and $F_m^m$ takes points from a small neighborhood of $M_2^+$ in $U_{02}$ to a small neighborhood of $M_2^-$ in $U_{02}$.
First, we discuss necessary estimates for \( F_1^k \) and \( F_2^p \). By [23], one can choose local coordinates \((x, y, z) \in \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1-1}\) in \( U_{01} \) such that \( O_1 \) is at the origin and the map \( F_1 \) takes the form

\[
\begin{align*}
\bar{x} &= \lambda x + g_1(x, y, z), \\
\bar{y} &= P_1 y + g_2(x, y, z), \\
\bar{z} &= P_2 z + g_3(x, y, z),
\end{align*}
\]

where \( \lambda = \lambda_{1,1} \), and the eigenvalues of the matrices \( P_1 \) and \( P_2 \) are \( \gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{1,d_1} \) and \( \lambda_{1,2} \ldots \lambda_{1,d-d_1} \), respectively (see (1)). The functions \( g_{1,2,3} \) vanish along with their first derivatives at the origin and satisfy the identities

\[
g_{1,3}(0, y, 0) = 0, \quad \tilde{g}_2(x, 0, z) = 0, \quad \tilde{g}_1(x, 0, z) = 0, \quad \frac{\partial \tilde{g}_{1,3}}{\partial x}(0, y, 0) = 0,
\]

for all sufficiently small \( x, y, \) and \( z \). As discussed in Section 2, the first two identities in (10) imply that the local manifolds \( W_{loc}^s(O_1) \) and \( W_{loc}^u(O_1) \) are straightened and given by the equations \( \{y = 0\} \) and \( \{x = 0, z = 0\} \), respectively. The third identity shows that the strong-stable foliation \( F^{ss} \) (it enters condition GC2) is straightened, its leaves are given by equations \( \{x = \text{const}, y = 0\} \), and the map \( F_1 \) restricted to \( W_{loc}^s(O_1) \) is linear in \( x \). The forth identity implies that the extended unstable manifold \( W_{loc}^u(O_1) \) (this manifold enters condition GC1) is tangent to \( z = 0 \) at the points of \( W_{loc}^u(O_1) \).

Similarly, we introduce coordinates \((u, v, w) \in \mathbb{R} \times \mathbb{R}^{d-d_1-1} \times \mathbb{R}^{d_1}\) in \( U_{02} \) with \( O_2 \) at the origin such that \( F_2 \) takes the form

\[
\begin{align*}
\hat{u} &= \gamma u + \hat{g}_1(u, v, w), \\
\hat{v} &= Q_1 v + \hat{g}_2(u, v, w), \\
\hat{w} &= Q_2 w + \hat{g}_3(u, v, w),
\end{align*}
\]

where \( \gamma = \gamma_{2,1} \), and the eigenvalues of the matrices \( Q_1 \) and \( Q_2 \) are \( \lambda_{2,2}, \lambda_{2,3}, \ldots, \lambda_{2,d-d_1-1} \) and \( \gamma_{2,2}, \ldots, \gamma_{2,d_1+1} \), respectively (see (1)). The functions \( \hat{g}_{1,2,3} \) vanish along with their first derivatives at the origin and satisfy

\[
\begin{align*}
\hat{g}_{1,3}(0, v, 0) = 0, \quad \hat{g}_2(u, 0, w) = 0, \quad \hat{g}_1(u, 0, w) = 0, \quad \frac{\partial \hat{g}_{1,3}}{\partial u}(0, v, 0) = 0,
\end{align*}
\]

for all sufficiently small \( u, v \) and \( w \).

According to [23], the transformations that bring \( F_1 \) to form (9) and \( F_2 \) to form (9) are of class \( C^r \), so we do not loose regularity when using these coordinates. Moreover, these coordinate transformations depend uniformly-continuously on the system \( f \). When we consider parametric families \( f_\varepsilon \), and \( f \) is analytic or \( C^\infty \) with respect to coordinates and parameters, the coordinate transformations are also analytic or \( C^\infty \) with respect to \( \varepsilon \). In the case of finite smoothness \( r \), we lose, in general, two derivatives with respect to \( \varepsilon \). Namely, by [23], the second derivative of the transformation is \( C^{r-2} \)-smooth with respect to both the coordinates and parameters. Therefore, the functions \( P_{1,2} \) and \( Q_{1,2} \) in (9) and (9) are \( C^{r-2} \)-functions of \( \varepsilon \), and the functions \( g_{1,2,3} \) and \( \hat{g}_{1,2,3} \), as well as their derivatives with respect to \( (x, y, z) \) or \( (u, v, w) \) up to order 2, are \( C^{r-2} \)-functions of the coordinates and \( \varepsilon \). If \( r \geq 3 \), this gives us at least 1 continuous derivative with respect to \( \varepsilon \). In the \( C^2 \) case, we can only assume continuity of \( P_{1,2}, Q_{1,2} \) with respect to \( \varepsilon \), the same goes for \( g_{1,2,3}, \hat{g}_{1,2,3} \), and their derivatives with respect to the coordinates. The eigenvalues \( \lambda \) and \( \gamma \) do not depend on coordinate transformations; as they are the eigenvalues of the first derivative of \( F_1 \) or \( F_2 \) (at \( O_1 \) or, respectively, \( O_2 \)), they are at least \( C^1 \) with
respect to $\varepsilon$ in any case.

Take any point $(x, y, z)$ in $U_{01}$ and let $(\tilde{x}, \tilde{y}, \tilde{z}) = F^k(x, y, z)$. It is known (see e.g. [2, 35]) that the value of $(\tilde{x}, \tilde{y}, \tilde{z})$ is uniquely defined by $(x, \tilde{y}, \tilde{z})$ for all $k \geq 0$. By Lemma 7 of [23], when identities (10) are fulfilled, the relation between the coordinates can be written as

\[
\begin{align*}
\tilde{x} &= \lambda^k x + p_1(x, \tilde{y}, z), \\
y &= p_2(x, \tilde{y}, z), \\
\tilde{z} &= p_3(x, \tilde{y}, z),
\end{align*}
\]

(13)

where

\[
||p_{1,3}||_{C^1} = o(\lambda^k), \quad ||p_2||_{C^1} = o(\tilde{\gamma}^{-k}),
\]

for some constant $\tilde{\gamma} \in (1, |\gamma_{1,1}|)$. These estimates are uniform for all systems $C^2$-close to $f$; when we consider parametric families $f_\varepsilon$, the functions $p_{1,2,3}$ depend on $\varepsilon$ uniformly-continuously, along with their first derivatives with respect to $(x, \tilde{y}, z)$. In the case $r \geq 3$, we have the same $o(\lambda^k)$ and $o(\tilde{\gamma}^{-k})$ estimates for the derivatives of $p_{1,2}$ and, respectively, $p_3$ with respect to parameters $\varepsilon$, see [23].

Likewise, for any $(u, v, w) \in U_{02}$ we have $(\tilde{u}, \tilde{v}, \tilde{w}) = F^m(u, v, w)$ if and only if

\[
\begin{align*}
u &= \gamma^{-m}u + q_1(\tilde{u}, v, w), \\
\tilde{v} &= q_2(\tilde{u}, v, w), \\
w &= q_3(\tilde{u}, v, w),
\end{align*}
\]

(15)

where

\[
||q_{1,3}||_{C^1} = o(\gamma^{-m}), \quad ||q_2||_{C^1} = o(\tilde{\lambda}^m),
\]

(16)

for some constant $\tilde{\lambda} \in (|\lambda_{2,1}|, 1)$; the functions $q_{1,2,3}$ depend uniformly-continuously on the system $f$, and the estimates (16) hold uniformly for all systems $C^2$-close to $f$. When $r \geq 3$, estimates (16) also hold for the derivatives with respect to parameters $\varepsilon$.

Now, we consider the transition maps $F_{12}$ and $F_{21}$. We use the following notation for the coordinates of the points $M_{1,2}^\pm$:

\[
M_{1}^+ = (x^+, 0, z^+), \quad M_{1}^- = (0, y^-, 0), \quad M_{2}^+ = (0, v^+, 0), \quad M_{2}^- = (u^-, 0, w^-).
\]

We can write the Taylor expansion of the transition map $F_{12} : (\tilde{x}, \tilde{y}, \tilde{z}) \mapsto (u, v, w)$ near $M_{1}^-$ as

\[
\begin{align*}
u &= a'_1 + a'_{11} \tilde{x} + a'_{12} (\tilde{y} - y^-) + a'_{13} \tilde{z} + O(\tilde{x}^2 + (\tilde{y} - y^-)^2 + \tilde{z}^2), \\
v - v^+ &= a'_2 + a'_{21} \tilde{x} + a'_{22} (\tilde{y} - y^-) + a'_{23} \tilde{z} + O(\tilde{x}^2 + (\tilde{y} - y^-)^2 + \tilde{z}^2), \\
w &= a'_3 + a'_{31} \tilde{x} + a'_{32} (\tilde{y} - y^-) + a'_{33} \tilde{z} + O(\tilde{x}^2 + (\tilde{y} - y^-)^2 + \tilde{z}^2),
\end{align*}
\]

(17)

where $a'_i$ and $a'_{ij}$ ($i, j = 1, 2, 3$) are some constants. Since $F_{12}(M_{1}^-) = M_{2}^+$ for the system $f$, it follows that $a'_{1,2,3}$ vanish, but when we perturb $f$, these coefficients can become non-zero (though small).

Recall that $W^s_{loc}(O_2)$ is tangent at $M_{2}^+$ to $w = 0$ and $W^u_{loc}(O_1)$ is given by $(x_k = 0, z_k = 0)$. Thus, the transversality condition GC1 writes as the uniqueness of the (trivial) solution to the system

\[
u = a'_{12} (\tilde{y} - y^-), \quad v - v^+ = a'_{22} (\tilde{y} - y^-), \quad 0 = a'_{32} (\tilde{y} - y^-),
\]

i.e., $a'_{32}$ is invertible. It follows that $\tilde{y} - y^-$ can be expressed, from the last equation of (17), as a
smooth function of \((w, \tilde{x}, \tilde{z})\), so the map \(F_{12}\) can be written in the so-called “cross-form” as

\[
\begin{align*}
  u &= \mu + ax + a_{12}w + a_{13}\tilde{z} + O(\tilde{x}^2 + w^2 + \tilde{z}^2), \\
  v - \hat{v}^+ &= a_{21}\tilde{x} + a_{22}w + a_{23}\tilde{z} + O(\tilde{x}^2 + w^2 + \tilde{z}^2), \\
  y - \hat{y}^- &= a_{31}\tilde{x} + a_{32}w + a_{33}\tilde{z} + O(\tilde{x}^2 + w^2 + \tilde{z}^2),
\end{align*}
\] (18)

where the coefficients \(\hat{v}^+, \hat{y}^-, \mu, a, \) and \(a_{ij}\) change uniformly continuously when the system \(f\) is perturbed, and for the original system \(f\), we have \(\hat{v}^+ = v^+, \hat{y}^- = y^-, \) and \(\mu = 0\). Since it does not cause ambiguity, in further references to (18) we use \(v^+\) and \(y^-\) instead of \(\hat{v}^+\) and \(\hat{y}^-\). Note that the coefficient \(a \neq 0\) is exactly the derivative defined in (6).

Since \(W^u_{loc}(O_1)\) is given by \((\tilde{x} = 0, \tilde{z} = 0)\) and \(W^s_{loc}(O_2)\) is given by \((u = 0, w = 0)\), a fragile heteroclinic corresponding to the intersection of \(F_{12}(W^u_{loc}(O_1))\) and \(W^s_{loc}(O_2)\) persists for a perturbation of \(f\) if and only if \(\mu = 0\). More precisely, \(\mu\) is the \(u\)-coordinate of the point of the intersection of \(F_{12}(W^u_{loc}(O_1))\) with \(w = 0\). This intersection is transverse by condition GC1, from which one infers that the ratio of \(|\mu|\) to the distance between \(F_{12}W^u_{loc}(O_1)\) and \(W^s_{loc}(O_2)\) tends (uniformly in some \(C^2\)-neighborhood of \(f\)) to a finite non-zero limit value when \(\mu \to 0\). Recall that we defined the splitting parameter \(\mu(f),\) whose absolute value equal the distance between \(F_{12}W^u_{loc}(O_1)\) and \(W^s_{loc}(O_2)\) and which enters Theorems 4-7 and 8. By scaling the variable \(u\), we can always obtain

\[
\lim_{\mu \to 0} \frac{\hat{\mu}}{\mu(f)} = 1, \quad \text{or} \quad \hat{\mu} = \mu + o(\mu).
\] (19)

If \(r \geq 3\), then when we consider parametric families \(f_s\), the coefficients of (18) are at least \(C^1\) with respect to \(\varepsilon\). When the family is generic or proper, \(\mu = \mu(f_s)\) is one of the parameters, and we have (for \(r \geq 3\))

\[
\frac{\partial \mu}{\partial \mu} \bigg|_{\mu=0} = 1.
\]

It follows that in this case we can make a smooth change of parameters such that \(\hat{\mu} = \mu\).

The Taylor expansion of the other transition map \(F_{21} : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (x, y, z)\), which is defined for \((\tilde{u}, \tilde{v}, \tilde{w})\) near \(M_{2} = (u^-, 0, w^-)\) and takes values \((x, y, z)\) near \(M_{1} = (x^+, 0, z^+)\), is given by

\[
\begin{align*}
  x - x^+ &= b_{11}(\tilde{u} - u^-) + b_{12}\tilde{v} + b_{13}(\tilde{w} - w^-) + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + (\tilde{w} - w^-)^2), \\
  y &= b_{21}(\tilde{u} - u^-) + b_{22}\tilde{v} + b_{23}(\tilde{w} - w^-) + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + (\tilde{w} - w^-)^2), \\
  z - z^+ &= b_{31}(\tilde{u} - u^-) + b_{32}\tilde{v} + b_{33}(\tilde{w} - w^-) + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + (\tilde{w} - w^-)^2),
\end{align*}
\] (20)

where \(b_{ij}\) are some constants. Arguing as for the map \(F_{12}\) above, one can use the assumption that \(F_{21}^{-1}(W^s_{loc}(O_1)) \cap \mathcal{F}^{uu}\) from condition GC2 to deduce that \(\det b_{23} \neq 0\) (note that the leaf of the foliation \(\mathcal{F}^{uu}\) through \(M_2^-\) is given by \((\tilde{u} = u^-, \tilde{v} = 0)\) and \(W^s_{loc}(O_1)\) is given by \(y = 0\) here). Consequently, \(F_{21}\) can be written in the following cross-form:

\[
\begin{align*}
  x - x^+ &= b(\tilde{u} - u^-) + b_{12}\tilde{v} + b_{13}y + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + y^2), \\
  \tilde{w} - w^- &= b_{21}(\tilde{u} - u^-) + b_{22}\tilde{v} + b_{23}y + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + y^2), \\
  z - z^+ &= b_{31}(\tilde{u} - u^-) + b_{32}\tilde{v} + b_{33}y + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + y^2),
\end{align*}
\] (21)
where $b$ is the derivative defined by (3). All the coefficients in (21) change uniformly continuously when the system $f$ is perturbed; if $r \geq 3$, then they are at least $C^1$ with respect to the perturbation parameters $\varepsilon$. Note that since $b \neq 0$, we can express $\tilde{u} - u^-$ as a function of $(x - x^+, \tilde{v}, y)$ and rewrite (21) as

\[
\begin{align*}
\tilde{u} - u^- &= b^{-1}(x - x^+ - b_{13}y) + O(||\tilde{v}|| + (x - x^+)^2 + y^2), \\
\tilde{w} - w^- &= O(||x - x^+|| + ||\tilde{v}|| + ||y||), \\
\tilde{z} - z^+ &= b_{31}b^{-1}(x - x^+ - b_{13}y) + b_{33}y + O(||\tilde{v}|| + (x - x^+)^2 + y^2).
\end{align*}
\]  

(22)

Combining (13) and (18), we obtain a formula for the map $F_1 \circ F_1^k$. Namely, by substituting the last equation of (18) ($\tilde{y}$ as a function of $\tilde{x}$, $w$, and $\tilde{z}$) into the first and the last equations of (13), we can express $\tilde{x}$ and $\tilde{z}$ as functions of $(x, z, w)$:

\[
\tilde{x} = \lambda^k x + o(\lambda^k), \quad \tilde{z} = o(\lambda^k).
\]

After that, we substitute these formulas into the rest of equations (13),(18) and find that there exist smooth functions

\[
\begin{align*}
\tilde{h}_1(x, z, w) &= O(||w||) + o(\lambda^k), \quad \tilde{h}_2(x, z, w) = o(\gamma^{-k}), \quad \tilde{h}_3(x, z, w) = O(||w|| + |\lambda|^k),
\end{align*}
\]

such that, for sufficiently large $k$, a point $(x, y, z)$ from a small neighborhood of $M_1^+$ is taken by the map $F_1 \circ F_1^k$ to a point $(u, v, w)$ in a small neighborhood of $M_2^+$ if and only if

\[
\begin{align*}
u &= \tilde{\mu} + a\lambda^k x + \tilde{h}_1(x, z, w) \\
y &= \tilde{h}_2(x, z, w), \quad v - v^+ = \tilde{h}_3(x, z, w).
\end{align*}
\]  

(23)

Similarly, combining (15) and (22) yields that there exist smooth functions

\[
\begin{align*}
\tilde{h}_{0i}(x - x^+, \tilde{y}) = O((x - x^+)^2 + y^2) \quad (i = 1, 2), \quad \tilde{h}_{1,3}(x, v, y) = o(\gamma^{-m}), \quad \tilde{h}_2(x, v, y) = o(\lambda^m),
\end{align*}
\]

(24)

such that, for sufficiently large $m$, a point $(u, v, w)$ from a small neighborhood of $M_2^+$ is taken by the map $F_2 \circ F_2^m$ to a point $(x, y, z)$ in a small neighborhood of $M_1^+$ if and only if

\[
\begin{align*}
u &= \gamma^{-m}(u^- + b^{-1}(x - x^+ - b_{13}y + \tilde{h}_{01}(x - x^+, y))) + \tilde{h}_1(x, v, y), \\
z - z^+ &= b_{31}b^{-1}(x - x^+ - b_{13}y) + b_{33}y + \tilde{h}_{02}(x - x^+, y) + \tilde{h}_2(x, v, y), \\
w &= \tilde{h}_3(x, v, y).
\end{align*}
\]  

(25)

Let us now consider the first-return map $T_{k,m} = F_2 \circ F_2^m \circ F_1 \circ F_1^k$ for sufficiently large positive integers $k$ and $m$ that takes a point $(x, y, z)$ from a small neighborhood of $M_1^+$ to its image $(\bar{x}, \bar{y}, \bar{z})$. To obtain a formula for this map, we replace $(x, y, z)$ in (25) by $(\bar{x}, \bar{y}, \bar{z})$ and combine it with (23). More specifically, from the system comprised by the last equations of (23) (for $v$) and (25) (for $w$) we can express $v$ and $w$ as smooth functions of $(x, \bar{x}, z, \bar{y})$. By substituting these functions into the rest of the equations (23) and (25), we find that there exist functions

\[
\begin{align*}
h_1(x, \bar{x}, z, \bar{y}) = o(\lambda^k) + o(\gamma^{-m}), \quad h_2(x, \bar{x}, z, \bar{y}) = o(\lambda^m), \quad h_3(x, \bar{x}, z, \bar{y}) = o(\gamma^{-k}),
\end{align*}
\]
such that $T_{k,m}(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$ if and only if
\[
\begin{align*}
\bar{x} - x^+ & = b \gamma^m \mu - bu^- + ab \lambda^k \gamma^m x + b_{13}\bar{y} - \hat{h}_{01}(\bar{x} - x^+, \bar{y}) + \gamma^m h_1(x, \bar{x}, z, \bar{y}), \\
\bar{z} - z^+ & = b_{31} b^{-1}(x - x^+ - b_{13}\bar{y}) + b_{33}\bar{y} + \hat{h}_{02}(\bar{x} - x^+, \bar{y}) + h_2(x, \bar{x}, z, \bar{y}), \\
y & = h_3(x, \bar{x}, z, \bar{y}).
\end{align*}
\] (26)

After the coordinate transformation
\[
\begin{align*}
X &= x - x^+ - b_{13}y, & Y &= y, \\
Z &= z - z^+ - b_{31} b^{-1}(x - x^+ - b_{13}y) - b_{33}y - \hat{h}_{02}(x - x^+, y)
\end{align*}
\] (27)
the map $T_{k,m}$ assumes the form
\[
\begin{align*}
\hat{X} &= b \gamma^m \mu + ab \lambda^k \gamma^m x^+ - bu^- + ab \lambda^k \gamma^m X + \hat{\phi}_0(\hat{X}, \hat{Y}) + \gamma^m \hat{\phi}_1(X, \hat{X}, \hat{Y}, Z), \\
Y &= \hat{\phi}_2(X, \hat{X}, \hat{Y}, Z), & Z &= \hat{\phi}_3(X, \hat{X}, \hat{Y}, Z),
\end{align*}
\] (28)
where
\[\hat{\phi}_0 = O(\bar{X}^2 + \bar{Y}^2),\] (29)
i.e., $\hat{\phi}_0$ vanishes at $(\hat{X}, \hat{Y}) = 0$ along with the first derivatives, and
\[
\|\hat{\phi}_1\|_{C^1} = o(\lambda^k) + o(\gamma^{-m}), \quad \|\hat{\phi}_2\|_{C^1} = o(\hat{\gamma}^{-k}), \quad \|\hat{\phi}_3\|_{C^1} = o(\hat{\lambda}^m).\] (30)

By construction, these estimates are uniform for all systems $C^2$-close to $f$, and when the system is at least $C^3$-smooth, the same estimates are also true for the first derivatives with respect to parameters $\varepsilon$. We note also that in the coordinates (27) we have the local stable manifold of $O_1$ and the image of the local unstable manifold of $O_2$ straightened:
\[
W_{loc}^s(O_1) : \{Y = 0\} \quad \text{and} \quad F_{21}(W_{loc}^u(O_2)) : \{Z = 0\}\] (31)
(one can see this from that the equation for $F_{21}(W_{loc}^u(O_2))$ in the $(x, y, z)$ coordinates can be obtained by taking the limit $m \to +\infty$ in (25)).

We stress that the first-return map must take points from a small neighborhood of $M_1^+$ in $U_{01}$ to a small neighborhood of $M_1^+$, which corresponds to small values of $(X, Y, Z)$ and $(\hat{X}, \hat{Y}, \hat{Z})$. We further denote this neighborhood where we want the first-return maps to be defined by
\[
\Pi = [-\delta, \delta] \times [-\delta, \delta]^{d_1} \times [-\delta, \delta]^{d-d_1-1}.
\] (32)
For $(X, \hat{Y}, Z) \in \Pi$, we can assure that $Y$ and $Z$ are small in (28) by taking $k$ and $m$ sufficiently large. However, to have $\hat{X}$ small (we will need $\hat{X} = O(\delta)$) one needs some additional restriction on possible values of $k$ and $m$.

In the next Section we will consider the first-return maps near the unperturbed cycles $\Gamma$, which corresponds to $\mu = \hat{\mu} = 0$ in (28) (see (19)). In this case, the $O(\delta)$ smallness of both $\hat{X}$ and $X$ in the first equation of (28) implies that the map $T_{k,m}$ at $\mu = 0$ acts from $\Pi$ to an $O(\delta)$-neighborhood of $M_1^+$.
in $U_{\theta_1}$ only when here is a certain balance between $k$ and $m$, namely
\begin{equation}
abla \lambda^k \gamma^m = \alpha + O(\delta),
\end{equation}
where $\alpha = bu^-/x^+ \neq 0$ is the quantity introduced in condition (2.3).

Thus, $\lambda^k \gamma^m$ must be uniformly bounded in this case, and hence $\hat{\phi}_1 = o(\gamma^{-m})$ in (30). Consequently, the derivative of the right-hand side of the first equation of (28) with respect to $\bar{X}$ is of order $O(\delta) + o(1)_{k,m \to \infty}$, so $\bar{X}$ can be expressed as a function of $(X, \bar{Y}, Z)$. Therefore, for sufficiently large $k, m$ such that (33) is satisfied, the formula (28) for $T_{k,m}$ at $\mu = 0$ implies that for a point $(X, Y, Z) \in \Pi$ we have $(\bar{X}, \bar{Y}, \bar{Z}) = T_{k,m}(X, Y, Z)$ if and only if the points are related by the cross-map $T_{k,m}^\times : (X, \bar{Y}, Z) \mapsto (\bar{X}, Y, \bar{Z})$ given by
\begin{equation}
\bar{X} = ab\lambda^k \gamma^m x^+ - bu^- + ab\lambda^k \gamma^m X + \phi_1(X, \bar{Y}, Z), \\
Y = \phi_2(X, \bar{Y}, Z), \\
Z = \phi_3(X, \bar{Y}, Z),
\end{equation}
where
\begin{equation}
\phi_1 = O(\delta^2) + o(1)_{k,m \to \infty}, \\
\frac{\partial \phi_1}{\partial (X, Y, \bar{Z})} = O(\delta) + o(1)_{k,m \to \infty}, \\
\|\phi_2\|_{C^1} = o(\gamma^{-k}), \\
\|\phi_3\|_{C^1} = o(\lambda^m).
\end{equation}

The following result characterizes the action of the derivative $DT_{k,m}$ of the maps $T_{k,m}$ at $\mu = 0$. We will further use the notation $(\Delta X, \Delta Y, \Delta Z)$ for vectors in the tangent space to $\Pi$.

**Lemma 1.** Let $\mu = 0$. Given any $K < 1$, for all sufficiently small $\delta$ and large $(k, m)$ satisfying (33), the cone fields on $\Pi$
\begin{equation}
C^{cu} = \{(\Delta X, \Delta Y, \Delta Z) : \|\Delta Z\| \leq K(\|\Delta X\| + \|\Delta Y\|)\},
\end{equation}
\begin{equation}
C^{uu} = \{(\Delta X, \Delta Y, \Delta Z) : \max\{\|\Delta X\|, \|\Delta Z\|\} \leq K\|\Delta Y\|\},
\end{equation}
are forward-invariant in the sense that if a point $M \in \Pi$ has its image $\bar{M} = T_{k,m}(M)$ in $\Pi$, then the cone at $M$ is mapped into the cone at $\bar{M}$ by $DT_{k,m}$; and the cone fields
\begin{equation}
C^{cs} = \{(\Delta X, \Delta Y, \Delta Z) : \|\Delta Y\| \leq K(\|\Delta X\| + \|\Delta Z\|)\},
\end{equation}
\begin{equation}
C^{ss} = \{(\Delta X, \Delta Y, \Delta Z) : \max\{\|\Delta X\|, \|\Delta Y\|\} \leq K\|\Delta Z\|\},
\end{equation}
are backward-invariant in the sense that if a point $\bar{M} \in \Pi$ has its pre-image $M = T_{k,m}^{-1}(\bar{M})$ in $\Pi$, then the cone at $\bar{M}$ is mapped into the cone at $M$ by $DT_{k,m}^{-1}$. Moreover, vectors in $C^{cu}$ and, if $|\alpha| > 1$, also in $C^{cs}$ are expanded by $DT_{k,m}$; vectors in $C^{ss}$ and, if $|\alpha| < 1$, also in $C^{cs}$ are contracted by $DT_{k,m}$.

**Proof.** Let us establish the backward invariance of $C^{ss}$ and $C^{cs}$. Take $\bar{M} = (\bar{X}, \bar{Y}, \bar{Z}) \in \Pi$ such that $T_{k,m}^{-1}(\bar{M}) = M \in \Pi$ for some $(k, m)$. Take a vector $(\Delta X, \Delta Y, \Delta Z)$ in the tangent space at the point $\bar{M}$. Let $(\Delta X, \Delta Y, \Delta Z) = DT_{k,m}^{-1}(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z})$. 

25
It follows from (28) and (35) that

\[
\Delta \bar{X} = (\alpha + O(\delta) + o(1)_{k,m \to \infty}) \Delta X + (O(\delta) + o(1)_{k,m \to \infty}) \Delta \bar{Y} + (O(\delta) + o(1)_{k,m \to \infty}) \Delta Z, \\
\Delta \bar{Y} = o(\hat{\gamma}^{-k}) \Delta X + o(\hat{\gamma}^{-k}) \Delta \bar{Y} + o(\hat{\gamma}^{-k}) \Delta Z, \\
\Delta \bar{Z} = o(\hat{\lambda}^m) \Delta X + o(\hat{\lambda}^m) \Delta \bar{Y} + o(\hat{\lambda}^m) \Delta Z.
\]

Thus, there exists a constant $C$ such that if $k$ and $m$ are large enough, then

\[
(|\alpha| - C\delta)|\Delta X| \leq |\Delta \bar{X}| + C\delta\|\Delta \bar{Y}\| + C\delta\|\Delta Z\|, \quad (41) \\
|\Delta \bar{X}| \leq (|\alpha| + C\delta)|\Delta X| + C\delta\|\Delta \bar{Y}\| + C\delta\|\Delta Z\|, \quad (42) \\
\|\Delta \bar{Y}\| = o(\hat{\gamma}^{-k})(|\Delta X| + \|\Delta \bar{Y}\| + \|\Delta Z\|), \quad (43) \\
\|\Delta \bar{Z}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta \bar{Y}\| + \|\Delta Z\|). \quad (44)
\]

Let $(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z}) \in \mathcal{C}^{cs}$. Since $K < 1$, we have

\[
\max\{|\Delta \bar{X}|, \|\Delta \bar{Y}\|\} \leq \|\Delta \bar{Z}\|.
\]

Now, it follows from (44) that

\[
\|\Delta \bar{Z}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta Z\|) \quad (45)
\]

and, hence,

\[
|\Delta \bar{X}| + \|\Delta \bar{Y}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta Z\|).
\]

We substitute these estimates into (41) and (43) and obtain

\[
|\Delta X| = O(\delta)\|\Delta Z\| \quad \text{and} \quad \|\Delta Y\| = o(\hat{\gamma}^{-k})\|\Delta Z\|,
\]

i.e., for any fixed choice of the constant $K$, if $k$ and $m$ are large enough and $\delta$ is small enough, the vector $(\Delta X, \Delta Y, \Delta Z)$ lies in $\mathcal{C}^{cs}$ at the point $M$, as required. Equation (45) implies the contraction in $\mathcal{C}^{cs}$ if $m$ is large enough.

Similar arguments are applied when $(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z}) \in \mathcal{C}^{cs}$. Here, we have

\[
\|\Delta \bar{Y}\| \leq |\Delta \bar{X}| + \|\Delta \bar{Z}\|.
\]

Substituting this into (42) and (44) gives

\[
|\Delta \bar{X}| \leq (|\alpha| + O(\delta))|\Delta X| + O(\delta)\|\Delta Z\|, \\
\|\Delta \bar{Z}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta Z\|),
\]

for a sufficiently small $\delta$ and sufficiently large $m$, and, hence,

\[
\|\Delta \bar{Y}\| \leq (|\alpha| + 1)|\Delta X| + \|\Delta Z\|.
\]

We substitute the last estimate into (43) and obtain

\[
\|\Delta \bar{Y}\| = o(\hat{\gamma}^{-k})(|\Delta X| + \|\Delta Z\|),
\]

i.e., for any fixed choice of the constant $K$, if $k$ is large enough, the vector $(\Delta X, \Delta Y, \Delta Z)$ lies in $\mathcal{C}^{cs}$. 

26
at the point \( M \), as required. Equation (46) implies the contraction in \( C^a \) if \(|\alpha| < 1 \) and \( m \) is large enough and \( \delta \) is small enough.

The proof of the forward invariance of \( C^{uu} \) and \( C^{cu} \) is done in the same way, as everything is symmetric here with respect to change of \( T_{k,m} \) to \( T_{k,m}^{-1} \).

3.2 Blenders near heterodimensional cycles

Now we prove Theorem 3. We do not perturb the system \( f \) in this theorem, i.e., the fragile heteroclinic is not split, so \( \mu = 0 \) in (23). We consider generic type-I cycles, which means that \(|\alpha| = |bu^-/x^+| \neq 1 \) and \( ax^+u^- > 0 \) in (28). We also assume that \( \theta = -\ln |\lambda|/\ln |\gamma| \) is irrational.

We consider first the case where \(|\alpha| < 1 \) and start with finding hyperbolic sets in the small neighborhood \( \Pi \), which can be candidates for blenders. Consider the set of pairs of sufficiently large integers \((k, m)\):

\[
P_N = \{ (k, m) : k > N, m > N \text{ and } |ab\lambda^k \gamma^m x^+ - bu^-| \leq \frac{2}{3}(1 - |\alpha|)\delta \}.
\]

Note that this set is non-empty for any \( N \) and \( \delta > 0 \): because \( \theta = -\ln |\lambda|/\ln |\gamma| \) is irrational, the set \( \lambda^k \gamma^m \) is dense among positive reals, hence, since \( abx^+ \) and \( bu^- \) have the same sign by the assumption of the theorem, \( ab\lambda^k \gamma^m x^+ \) can be made as close as we want to \( bu^- \) for arbitrarily large \( k \) and \( m \).

By construction, for every \((k, m) \in P_N\), the estimate (33) is satisfied. Therefore, for these \((k, m)\) the relation between the coordinates \((X, Y, Z) \in \Pi\) of a point in the domain of definition of the first-return map \( T_{k,m} \) and the coordinates \((\bar{X}, \bar{Y}, \bar{Z})\) of its image by \( T_{k,m} \) is given by (34). Since \(|\alpha| < 1\), comparing (47) with the \(X\)-equation in (34), one sees that for every \((k, m) \in P_N\), with \( N \) sufficiently large and \( \delta \) sufficiently small, we have \( \bar{X} \in [-\delta, \delta] \).

Thus, for any such \((k, m)\) we find that the cross-map \( T_{k,m}^\times : (X, \bar{Y}, Z) \mapsto (\bar{X}, Y, \bar{Z}) \) given by (34) satisfies

\[
T_{k,m}^\times(\Pi) \subset \Pi.
\]

It also follows from \(|\alpha| < 1\) that the cross-map is contracting on \( \Pi \), i.e.,

\[
\left\| \frac{\partial (X, Y, \bar{Z})}{\partial (X, Y, Z)} \right\| < 1.
\]

By Shilnikov lemma on the fixed point in a direct product of metric spaces (Theorem 6.2 in [35]), these two facts immediately imply

**Lemma 2.** For any sequence \( \{(k_s, m_s)\}_{s \in \mathbb{Z}} \) of pairs \((k, m)\) from \( P_N \), there exists a unique sequence of points \( \{M_s = (X_s, Y_s, Z_s)\} \in \Pi \) such that \( M_{s+1} = T_{k_s, m_s}M_s \) for every \( s \).

**Proof.** By definition of the cross-map, a sequence of points \( \{M_s = (X_s, Y_s, Z_s)\} \in \Pi \) satisfies

\[
M_{s+1} = T_{k_s, m_s}M_s
\]

if and only if

\[
(X_{s+1}, Y_{s+1}, Z_{s+1}) = T_{k_s, m_s}^\times(X_s, Y_{s+1}, Z_s).
\]
Thus, the sought sequence \( \{(X_s, Y_s, Z_s)\}_{s \in \mathbb{Z}} \) is a fixed point of the map \( \{(X_s, Y_s, Z_s)\}_{s \in \mathbb{Z}} \mapsto \{(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s)\}_{s \in \mathbb{Z}} \) (acting on the space of sequences of points in \( \Pi \)) which is defined by the rule
\[
(\tilde{X}_{s+1}, \tilde{Y}_s, \tilde{Z}_{s+1}) = T_{k,m}^\times(X_s, Y_{s+1}, Z_s).
\]

Obviously, this map is a contraction (because each of the maps \( T_{k,m}^\times \) is a contraction on \( \Pi \)), so the fixed point indeed exists and is unique.

We will call the sequence \( \{(k_s, m_s)\}_{s \in \mathbb{Z}} \) the coding of the point \( M_0 \). Lemma 2 establishes the existence of an invariant set which is in one-to-one correspondence with the set of all codings formed from \((k, m) \in \mathcal{P}_N\). It follows immediately from Lemma 1 that this set is hyperbolic and, also, partially-hyperbolic with a 1-dimensional central (weakly-hyperbolic) direction. Particularly, we have

**Lemma 3.** Let \(|a| < 1\) and \( J \) be any finite subset of \( \mathcal{P}_N \). Then, there exists a closed hyperbolic set \( \Lambda_J \) of index \( d_1 \) near the heterodimensional cycle \( \Gamma \) such that it is in one-to-one correspondence with the set of codings \( \{(k_s, m_s) \in J\}_{s \in \mathbb{Z}} \).

**Remark 4.** The intersection \( \Lambda_J \cap \Pi \) is located in a finite union of ‘horizontal strips’ in \( \Pi \). Indeed, one sees from (34) that the domains of \( T_{k,m} \) with \((k, m) \in \mathcal{P}_N\) are given by the strips:
\[
\sigma_{k,m} = \{(X, Y, Z) \mid X \in [-\delta, \delta], Y \in \phi_2(X, [-\delta, \delta], Z), Z \in [-\delta, \delta]\}.
\]

(49) Since \( \phi_2 = o(\gamma^{-k}) \), these strips accumulate on \( \{Y = 0\} \) and strips corresponding to different \((k, m)\) are disjoint. By construction, we have
\[
\Lambda_J \cap \Pi \subset \bigcup_{(k,m) \in J} \sigma_{k,m} =: \Sigma_J.
\]

(50) Every point whose backward orbit stays in \( W^s(\Lambda_J) \), and every point whose forward orbit stays in \( \Sigma_J \) lies in \( W^s(\Lambda_J) \). This property holds for any \( C^1\)-close system since the hyperbolic set persists under \( C^1\)-small perturbations.

**Remark 5.** For every point \( M \in \Lambda_J \), we can define its local stable manifold as a connected piece of \( W^s(\Lambda_J) \cap \Pi \) through \( M \), and the local unstable manifold as a connected piece of \( W^u(\Lambda_J) \cap \Pi \) through \( M \). If \( M \in \sigma_{k,m} \), then \( W^u_{loc}(M) \subset \sigma_{k,m} \). Moreover, since the tangent to \( W^s_{loc}(M) \) at any point lies in the stable cone \( C^s \), it follows that \( W^s_{loc}(M) \) is a horizontal surface of the form \( Y = \xi_M^s(X, Z) \), where the smooth function \( \xi_M^s \) is defined for all \( X \in [-\delta, \delta], Z \in [-\delta, \delta]^{d-d_1} \). Similarly, \( W^u_{loc}(M) \) is a vertical surface of the form \( (X, Z) = \xi_M^u(Y) \), where \( \xi_M^u \) is defined for all \( Y \in [-\delta, \delta]^{d_1} \). In particular, it follows from (31) that for each \( M \) there exist transverse intersections
\[
W^u_{loc}(M) \cap W^s_{loc}(O_1) \neq \emptyset \quad \text{and} \quad W^s_{loc}(M) \cap F_{21}(W^u_{loc}(O_2)) \neq \emptyset.
\]

Now let us find a subset \( J \) such that the corresponding hyperbolic set \( \Lambda_J \) is a cu-blender.

**Definition 6.** Consider a cube
\[
Q = \{(X, Y, Z) \in I \times D_X \times D_Y\}
\]
for some closed interval $I \subset \mathbb{R}$ and closed balls $D_Y \subset \mathbb{R}^{d_1}$ and $D_Z \subset \mathbb{R}^{d-d_1-1}$. A $(d-d_1-1)$-dimensional surface $S$ is said to cross $Q$ if the intersection $S \cap Q$ is given by $(X,Y) = s(Z)$ where $s$ is a smooth function defined on $D_Z$; it crosses $Q$ properly with respect to a cone field $C$ if the tangent spaces of $S \cap Q$ lie in $C$.

When constructing a cu-blender, we define a cube $\Pi' \subset \Pi$ and take as discs $D$ from Definition 3 surfaces $S \cap \Pi'$ where $S$ cross $\Pi'$ properly with respect to the cone field $C^{\text{ss}}$ given by Lemma 1 (with an appropriate $K$ in its definition to be determined). We will show that any such surface $S$ intersects the unstable manifold $W^u(\Lambda_{\mathcal{A}})$ (and this property persists at $C^1$-small perturbations of the system).

Recall that by Remark 4, any point whose entire backward orbit by $T_{k,m}$ stays in $\Sigma_{\mathcal{A}}$ belongs to $W^u(\Lambda_{\mathcal{A}})$. Therefore, we will have $S \cap W^u(\Lambda_{\mathcal{A}}) \neq \emptyset$ by showing below that any $S$ that intersects $\Pi'$ properly contains points whose entire backward orbit stays in $\Sigma_{\mathcal{A}}$.

Take any $q < (1 - |\alpha|)/2$, and define

$$\delta' = q\delta. \quad (51)$$

**Lemma 4.** There exists a finite subset $\mathcal{J}_q \subset \mathcal{P}_N$ such that, if a $(d-d_1-1)$-dimensional surface $S$ properly crosses

$$\Pi' := \Pi \cap \{-\delta' \leq X \leq \delta'\} \quad (52)$$

properly with respect to

$$C^{\text{ss}} = \{(\Delta X, \Delta Y, \Delta Z) : \max(\|\Delta X\|, \|\Delta Y\|) < \frac{q|\alpha|}{4}\|\Delta Z\|)\}, \quad (53)$$

then one can find a pair $(k,m) \in \mathcal{J}_q$ such that the pre-image $T_{k,m}^{-1}(S)$ contains a surface crossing $\Pi'$ properly with respect to $C^{\text{ss}}$.

See Figure 7 for an illustration. This lemma shows that given any properly crossing surface $S$, there is a sequence of surfaces defined by $S_{i+1} = T_{k_i,m_i}^{-1}(S_i) \cap \Pi'$ with $S_0 := S$ and some sequence $(k_i,m_i) \in \mathcal{J}_q$. By construction, this gives a sequence of nested closed sets

$$\hat{S}_i := T_{k_{i-1},m_{i-1}} \circ \cdots \circ T_{k_0,m_0}(S_i) \subset S.$$

The intersection of all $\hat{S}_i$ contains a point $M$ whose backward orbit stays in $\Sigma_{\mathcal{J}_q}$ defined by (50). (In fact, this point is unique due to the contraction in Z-directions.) Hence, we have $M \in W^u(\mathcal{J}_q)$, where $\mathcal{J}_q$ is the hyperbolic set given by Lemma 3 associated with the set $\mathcal{J}_q$. Obviously, Lemma 4 holds for any $C^1$-close system for the same set $\mathcal{J}_q$. Therefore, we indeed obtain a cu-blender.

Since the cone field $C^{\text{ss}}$ is backward-invariant by Lemma 1, the pre-image of any surface proper with respect to $C^{\text{ss}}$ is still proper. Hence, in order to prove Lemma 4 we only need to show that any properly crossing surface has a pre-image which crosses $\Pi'$. The key procedure in controlling the pre-image of the surface $S$ is to show that the $X$-coordinates of its points always lie in the domain $\sigma_{k,m}$ of some return map $T_{k,m}$. This is ensured by the so-called covering property which we prove below. After that, we finish the proof of Lemma 4.

Denote the length of an interval $E$ by $|E|$, and the affine part of the $X$-equation in (34) by

$$R_{k,m}(X) := ab\lambda^k\gamma^m x^+ - bu^- + ab\lambda^k\gamma^m X. \quad (54)$$
Figure 7: The blue regions depict intersections $\Pi' \cap T_{k,m}(\Pi')$ with $(k,m) \in \mathcal{J}_{\delta'}$. For any surface (or curve in this three dimensional figure) crossing $\Pi'$ properly, it must intersect one of the three blue regions. Hence its pre-image contains a piece crossing $\Pi'$.

**Lemma 5.** Let $|\alpha| < 1$. There exists, for all sufficiently large $N$ and sufficiently small $\delta$, a finite subset $\mathcal{J}_{\delta'} = \{(k_j, m_j)\}_{j=1}^n \subset \mathcal{P}_N$ such that the intervals $E_j := R_{k_j, m_j}([-\delta', \delta'])$ satisfy

$$
\bigcup_{j=1}^n \text{int}(E_j) \supset [-\delta', \delta']
$$

and

$$
|E_j \cap E_{j+1}| > \frac{\delta' |\alpha|}{2}, \quad j = 1, \ldots, n-1.
$$

**Proof.** One notes that since $\theta$ is irrational, the set of numbers $k\theta - m$ where $k$ and $m$ are even positive integers is dense in $\mathbb{R}$. Thus, given any $\rho \in \mathbb{R}$, we can find a sequence $\{(k_i(\rho), m_i(\rho))\}_{i \in \mathbb{N}}$ of pairs of positive even integers such that $k_i(\rho), m_i(\rho) \to \infty$ and

$$
k_i(\rho)\theta - m_i(\rho) \to - \ln \left( \frac{bu^- + \delta \rho}{abx^+} \right) \ln^{-1} |\gamma|
$$

as $i \to \infty$ (the logarithm is defined for small $\delta$ since $ax^+u^- > 0$ and $b \neq 0$ by our assumptions). Since $k_i$ and $m_i$ are even, we have $|\lambda|^{k_i} = \lambda^{k_i}$ and $|\gamma|^{m_i} = \gamma^{m_i}$ even when $\lambda$ or $\gamma$ are negative, so we obtain that

$$
\frac{1}{\delta}(ab\lambda^{k_i(\rho)}\gamma^{m_i(\rho)}x^+ - bu^-) \to \rho,
$$

as $i \to \infty$. Observe that, if $\delta$ is sufficiently small, then for any

$$
\rho \in \left( \frac{-2}{3}(1 - |\alpha|), \frac{2}{3}(1 - |\alpha|) \right) =: I_{hyp},
$$

we have

$$
\mathcal{P}_N(\rho) := \{(k_i(\rho), m_i(\rho)) : k_i(\rho), m_i(\rho) > N\} \subset \mathcal{P}_N.
$$

In what follows we construct $\mathcal{J}_{\delta'}$ by taking pairs $(k, m)$ from $\mathcal{P}_N(\rho)$.

Recall $\alpha = bu^-/x^+$. Note from (54) and (57) that, for any $(k_i(\rho), m_i(\rho)) \in \mathcal{P}_N(\rho)$, we have

$$
R_{k_i(\rho), m_i(\rho)}(X) = \rho \delta + \left( \alpha + \frac{\rho \delta}{x^+} \right) X + o(1)_{i \to \infty}.
$$
Consequently, the end points of $E_{k_i, m_i} = R_{k_i, m_i}([-\delta', \delta])$ are

$$a_i = \rho \delta - \delta' \left( \alpha + \frac{\rho \delta}{x^+} \right) + o(1)_{i \to \infty} \quad \text{and} \quad b_i = \rho \delta + \delta' \left( \alpha + \frac{\rho \delta}{x^+} \right) + o(1)_{i \to \infty}. $$

The length of each $E_{k_i, m_i}$ satisfies

$$|E_{k_i, m_i}| = |b_i - a_i| = 2\delta' \left| \alpha + \frac{\rho \delta}{x^+} \right| + o(1)_{i \to \infty} > \delta'|\alpha|. \quad (60)$$

for all sufficiently large $i$ and sufficiently small $\delta$. The centres of $E_{k_i, m_i}$ are given by

$$X_i^c = \frac{|a_i - b_i|}{2} = \rho \delta + o(1)_{i \to \infty}. \quad (61)$$

Let $n$ be the smallest integer not less than $4/|\alpha| + 1$. Then, by (60) and (61), one can choose $n$ different values $\rho_j$ ($j = 1, \ldots, n$) of $\rho$ such that the intervals $E_{k_i(\rho_j), m_i(\rho_j)}$ (with fixed large $i$) cover $[-\delta', \delta']$ with the overlaps required by (56). Specifically, we can take

$$\rho_j = -\frac{\delta'}{\delta} + \frac{(j-1)2\delta'}{(n-1)\delta}, \quad j = 1 \ldots n,$$

and $\rho_j \in I_{hyp}$ by (51) and (58). Now fix $i$ large enough so that $(k_i(\rho_j), m_i(\rho_j)) \in P_N$, and we obtain the desired set $J_{\vartheta} = \{(k^3, m^3) := k_i(\rho_j), m_i(\rho_j)\}$. \hfill \Box

**Proof of Lemma 4.** As we explained before Lemma 5, it suffices to show that for any proper surface $S$ there exists a pair $(k, m) \in J_{\vartheta}$ such that $T_{k, m}^{-1}(S)$ crosses $\Pi'$. As $S$ crosses $\Pi'$, it is a graph of a smooth function $s = (s_X, s_Y) : [-\delta, \delta]^d \to [-\delta', \delta'] \times [-\delta, \delta]^{d_1}$. We need to find some $(k, m) \in J_{\vartheta}$ such that, for any $Z \in [-\delta, \delta]^{d-d_1-1}$, there exist $X \in [-\delta', \delta'], Y \in [-\delta, \delta]^{d_1}$ and $\bar{Z} \in [-\delta, \delta]^{d-d_1-1}$ which satisfy $(X, Y, Z) = T_{k, m}^{-1}(s_X(\bar{Z}), s_Y(\bar{Z}), \bar{Z})$.

By formulas (34) and (54), this is equivalent to solving the system of equations

$$s_X(\bar{Z}) = R_{k, m}(X) + \phi_1(X, s_Y(\bar{Z}), Z), \quad (62)$$

$$Y = \phi_2(X, s_Y(\bar{Z}), Z), \quad (63)$$

$$\bar{Z} = \phi_3(X, s_Y(\bar{Z}), Z). \quad (64)$$

According to the estimates (35), $Y$ lies in $[-\delta, \delta]^{d_1}$ for all sufficiently large $k$ as required; also, $\bar{Z}$ can be expressed from the last equation as a function of $(X, Z)$:

$$\bar{Z} = \tilde{\phi}_3(X, Z)$$

where $\|\tilde{\phi}_3\|_{C^1} = o(\lambda^m)$. Hence, to solve the above system it suffices to find $X \in [-\delta', \delta']$ satisfying

$$R_{k, m}(X) - s_X(\tilde{\phi}_3(X, Z)) + \tilde{\phi}_1(X, Z) = 0, \quad (65)$$

where

$$\tilde{\phi}_1(X, Z) = \phi_1(X, s_Y(\tilde{\phi}_3(X, Z)), Z) = O(\delta^2) + o(1)_{k, m \to \infty}. \quad (66)$$
We claim that one can choose \((k, m) \in \mathcal{J}_{d'}\) such that for any fixed \(Z \in [-\delta, \delta]^{d-d_1-1}\), the left hand side of (65) takes both positive and negative values when \(X\) runs over \([-\delta', \delta']\) (the lemma then follows by the intermediate value theorem).

Let us prove this claim. On one hand, the proper crossing with respect to \(C^s\) means that the range of \(s_X\) lies in \([-\delta', \delta']\), and, according to (53), the total change of \(s_X\) is bounded by

\[
\max_{Z_1, Z_2 \in [-\delta, \delta]^{d-d_1-1}} \{ |s_X(Z_1) - s_X(Z_2)| \} < \frac{\delta'|\alpha|}{4}.
\]

Since \(\tilde{\phi}_3([-\delta', \delta'], Z) \subset [-\delta, \delta]^{d-d_1-1}\) for large \(m\), we obtain that \(s_X(\tilde{\phi}_3([-\delta', \delta'], Z))\) lies in some interval in \([-\delta', \delta']\) of length less than \(\delta'|\alpha|/4\).

On the other hand, Lemma 5 shows that the interiors of the intervals \(R_{k,m}([-\delta', \delta'])\) with \((k, m) \in \mathcal{J}_{d'}\) cover the interval \([-\delta', \delta']\) with overlaps larger than \(\delta'|\alpha|/2\). It follows that there exists a pair \((k, m) \in \mathcal{J}_{d'}\) satisfying

\[
s_X(\tilde{\phi}_3([-\delta', \delta'], Z)) \subset \text{int}(R_{k,m}([-\delta', \delta']))
\]

for any fixed \(Z \in [-\delta, \delta]^{d-d_1-1}\). By continuity, the difference \(R_{k,m}(X) - s_X(\tilde{\phi}_3(X, Z))\) will change its sign when \(X\) runs the interval \([-\delta', \delta']\). The above analysis also shows that \(R_{k,m} - s_X = O(\delta)\), which, along with (66), implies that it is sufficient to prove the claim without the \(\tilde{\phi}_1\) term when \(\delta\) is sufficiently small and \(k, m\) are sufficiently large.

As discussed before, the preceding lemma gives us a cu-blender of index \(d_1\) when \(|\alpha| < 1\), and this blender can be taken arbitrarily close to the heterodimensional cycle by taking \(N\) of \(\mathcal{P}_N\) sufficiently large. When \(|\alpha| > 1\), one gets a cs-blender of index \(d_1 + 1\) due to the symmetry of the problem. More specifically, we make time reversal and consider the first-return map defined by \(F_{21}^{-1} \circ F_1^{-k} \circ F_{12}^{-1} \circ F_2^{-m} : (u, v, w) \mapsto (\bar{u}, \bar{v}, \bar{w})\) in some cube around \(M_2\). After that, all the lemmas in this Section hold for the new return map and we obtain a cu-blender of index \((d - d_1 - 1)\). After reversing time back, this becomes a cs-blender of index \(d_1 + 1\) for our original system \(f\). This completes the proof of Theorem 3.

In fact, what has been achieved so far is more than Theorem 3. We can give a detailed description for the blenders found here according to their activating pairs defined below. For an invariant cone field \(C\), denote by \(\dim(C)\) the largest possible dimension of a linear subspace of the tangent space that can be contained in \(C\) (at each point where \(C\) is defined).

**Definition 7.** A pair \((Q, C)\) consisting of a cube and a cone field is called an activating pair for a cu-blender \(\Lambda\) if \(\dim(C) = \dim W^s(\Lambda) - 1\) and if any surface \(S\) crossing \(Q\) properly with respect to \(C\) intersects \(W^u(\Lambda)\). The pair \((Q, C)\) is an activating pair for a cs-blender \(\Lambda\) if \(\dim(C) = \dim W^u(\Lambda) - 1\) and any surface \(S\) crossing \(Q\) properly with respect to \(C\) intersects \(W^s(\Lambda)\). The cube \(Q\) is called an activating domain.

Denoting \(\Lambda^{cu} := \Lambda_{\mathcal{J}_{d'}}\), we immediately get

**Proposition 1.** If \(|\alpha| < 1\), then there exists arbitrarily close to \(\Gamma\) an index-\(d_1\) cu-blender \(\Lambda^{cu}\) with an activating pair \((\Pi', C^s)\).
When $|\alpha| > 1$, one can get an activating pair for the cs-blender immediately by using the above-mentioned first-return map $F_{21}^{-1} \circ F_1^{-k} \circ F_{12}^{-1} \circ F_2^{-m}$. However, this activating pair has an activating domain near $O_2$. Below, instead of considering the above composition, we make computations directly for the maps $T_{k,m}^{-1}$ and get an activating pair with the same activating domain $\Pi'$ as before, which will be convenient for the later use.

**Proposition 2.** If $|\alpha| > 1$, then there exists, arbitrarily close to $\Gamma$, an index-$(d_1 + 1)$ cs-blender $\Lambda^{cs}$ with an activating pair $(\Pi', C^{uu})$, where

$$C^{uu} = \{(\Delta X, \Delta Y, \Delta Z) : \max\{\|\Delta X\|, \|\Delta Z\|\} \leq \frac{q|\alpha|^{-1}}{4} \|\Delta Y\|\}. \quad (67)$$

**Proof.** Inverting the first equation in (34), we obtain the following formula for the map $T_{k,m}^{-1}$:

\begin{align*}
X &= (\alpha^{-1} + O(\delta))\bar{X} + (a\lambda^k \gamma^m)^{-1}u^+ - x^+ + \psi_1(\bar{X}, \bar{Y}, \bar{Z}), \\
Y &= \psi_2(\bar{X}, \bar{Y}, \bar{Z}), \\
Z &= \psi_3(\bar{X}, \bar{Y}, \bar{Z}),
\end{align*}

where

\begin{align*}
\psi_1 &= O(\delta^2) + o(1)_{k,m \to \infty}, \\
\left\|\psi_2\right\|_{C^1} &= o(\gamma^{-k}), \\
\left\|\psi_3\right\|_{C^1} &= o(\hat{\lambda}^m).
\end{align*}

(69)

Observe that this formula has the same form as (34), with the replacement of $\alpha$ by $\alpha^{-1}$ and the term $(ab\lambda^k \gamma^m x^+ - bu^-)$ by the term $((a\lambda^k \gamma^m)^{-1}u^- - x^+)$. So, since $|\alpha| < 1$, we obtain the result by repeating the same arguments we used for the case $|\alpha| < 1$ - we only need to replace the cone field $C^{ss}$ by $C^{uu}$, as we work with the inverse map $T_{k,m}^{-1}$. \hfill \Box

Note that in proving all the results in this section, we did not use the full strength of the estimates in (35). We essentially used the fact that the functions $\phi$ along with their first derivatives go to 0 as $\delta \to 0$ and $k, m \to \infty$. Additionally, we used the fact that $\phi_1 = o(\delta)$ (hence $\tilde{\phi}_1 = o(\delta)$ in (66)) in the last line in the proof of Lemma 4. With the above observation, We finish this Section by the following summary which will be used later for the saddle-focus and double-focus cases.

**Proposition 3.** Consider a system of maps $\{T_n\}$ such that for all small $(X, Y, Z)$ and $(\bar{X}, \bar{Y}, \bar{Z})$ we have $(\bar{X}, \bar{Y}, \bar{Z}) = T_n(X, Y, Z)$ if and only if

\begin{align*}
\bar{X} &= A_n X + B_n + \phi_1(X, \bar{Y}, Z; n), \\
Y &= \phi_2(X, \bar{Y}, Z; n), \\
\bar{Z} &= \phi_3(X, \bar{Y}, Z; n),
\end{align*}

(70)

where

1. the constant coefficients $A_n$ are such that for some constants $L_1$ and $L_2$ independent of $n$, either $0 < L_1 < |A_n| < L_2 < 1$, or $1 < L_1 < |A_n| < L_2 < \infty$;

2. the set of constant terms $\{B_n\}$ is dense in a neighbourhood of zero; and
3. The functions \( \phi \) are defined on a cube \( \Pi = [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta] \) for some small \( \delta > 0 \); the functions \( \phi_{2,3} \) along with their first derivatives tend to 0 as \( \delta \to 0 \) and \( n \to \infty \), while \( \phi_1 = o(\delta) + O(1)_{n \to \infty} \) and the derivative of \( \phi_1 \) is \( o(1)_{\delta \to 0} + o(1)_{n \to \infty} \).

Then, decreasing \( \delta \) if necessary, there exists arbitrarily close to the origin a cu-blender if \( |A_n| \) is bounded between zero and 1, or a cs-blender if bounded between 1 and infinity. Moreover, the blender has an activating pair \((Q, C)\), where

\[
Q = [-q\delta, q\delta] \times [-\delta, \delta] \times [-\delta, \delta] \subset \Pi
\]

for some \( q \in (0, 1) \), and \( C \) is a field of cones around \( Z \)-coordinates if the blender is center-unstable, or around \( Y \)-coordinates if it is center-stable.

### 3.3 Local stabilization of heterodimensional cycles

In this section we prove Theorems 4 - 7 and Corollary 3. We conduct all the proofs only for the \( |\alpha| < 1 \) case, – the \( |\alpha| > 1 \) case is dealt with by using the symmetry argument (i.e., by considering the system obtained by the reversion of time) except for Theorem 4, where Proposition 2 is additionally used. Depending on the situations, we embed \( f \) into one- or two-parameter families that generically and properly unfold the heterodimensional cycle of \( f \). In the remaining part of this paper, we will denote the continuations of hyperbolic objects (e.g. \( O_{1,2} \) and \( \Lambda^u \)) after a small perturbation by the same letters and omit the term ‘continuation’.

#### 3.3.1 Proof of Theorem 4

In the next two lemmas, we investigate the iterations of \( W^u_{loc}(O_1) \) and \( W^u_{loc}(O_2) \). Whether their iterates intersect the region ‘reserved’ for the emergence of blenders is crucial for the local stabilization of heterodimensional cycles.

**Lemma 6.** Define the intervals

\[
I^u_m = \left( \gamma^{-m}u - \frac{1}{2}b^{-1}\gamma^{-m}\delta', \gamma^{-m}u + \frac{1}{2}b^{-1}\gamma^{-m}\delta' \right).
\]

If \( m \) is sufficiently large and \( \delta \) is sufficiently small, then for \( \mu \in I^u_m \) the image \( S^u_m := F_{21} \circ F_2 \circ F_{12}(W^u_1) \) of

\[
W^u_1 := W^u_{loc}(O_1) \cap \{ \| y - y^- \| = \delta \}
\]

is a ‘vertical’ surface of the form \( (X, Z) = s(Y) \) for some smooth function \( s \). The surface \( S^u_m \) crosses the cube \( \Pi' \) (defined in (52)) properly with respect to the cone field \( C^u \) defined in (67). In particular, \( S^u_m \) intersects \( W^u_{loc}(O_1) : \{ Y = 0 \} \) transversely, i.e., \( O_1 \) has a transverse homoclinic orbit when \( \mu \in I^u_m \).

**Proof.** By formulas (18) and (25), for any \((0, y, 0) \in W^u_1\), we have \( F_{21} \circ F^u_2 \circ F_{12}(0, y, 0) = (\bar{x}, \bar{y}, \bar{z}) \) if
and only if
\[
\begin{align*}
\ddot{x} - x^+ &= b\gamma^m\ddot{\mu} - bu^- + b_{13}\dot{y} + O((b\gamma^m\ddot{\mu} - bu^-)^2 + \dot{y}^2) + o(1)_{m\to\infty}, \\
\ddot{z} - z^+ &= b_{31}\gamma^m\ddot{\mu} - b_{33}u^- + b_{33}\dot{y} + O((b\gamma^m\ddot{\mu} - bu^-)^2 + \dot{y}^2) + o(1)_{m\to\infty},
\end{align*}
\]
which after the coordinate transformation (27) recasts as
\[
\begin{align*}
\ddot{X} &= b\gamma^m\ddot{\mu} - bu^- + O((b\gamma^m\ddot{\mu} - bu^-)^2 + \dot{Y}^2) + o(1)_{m\to\infty}, \\
\ddot{Z} &= O((b\gamma^m\ddot{\mu} - bu^-)^2 + \dot{Y}^2) + o(1)_{m\to\infty}.
\end{align*}
\] (73)

By (71) we have
\[
|b\gamma^m\ddot{\mu} - bu^-| < \delta'/2,
\]
which for \(\|\dot{Y}\| \leq \delta\) implies
\[
|\ddot{X}| < \frac{\delta'}{2} + O(\delta^2) + o(1)_{m\to\infty} < \delta' \quad \text{and} \quad \|\ddot{Z}\| = O(\delta^2) + o(1)_{m\to\infty} < \delta.
\]

This means that \(S^u_m\) crosses \(\Pi'\). One also finds from (73) that
\[
\frac{\partial(\ddot{X}, \ddot{Z})}{\partial Y'} = O(\delta) + o(1)_{m\to\infty},
\]
which can be made sufficiently small so that the tangent spaces of \(S^u_m \cap \Pi\) lie in \(C^{uu}\). So, the crossing is also proper with respect to \(C^{uu}\).

**Lemma 7.** Define intervals
\[
I^s_k = \left(-a\lambda^k x^+ - \frac{1}{2}a\lambda^k|\delta'|, -a\lambda^k x^+ + \frac{1}{2}a\lambda^k|\delta'|\right).
\] (74)

If \(k\) is sufficiently large, then for \(\ddot{\mu} \in I^s_k\), the pre-image \(S^s_k := F^{-k}_{12} \circ F^{-1}_{12}(W^s_2)\) of
\[
W^s_2 := W^s_{\text{loc}}(O_2) \cap \{\|v - v^+\| \leq \delta\}
\] (75)
is a ‘horizontal’ surface of the form \((X, Y) = s(Z)\) for some smooth function \(s\). The surface \(S^s_k\) crosses \(\Pi'\) properly with respect to the cone field \(C^{ss}\) defined in (53). In particular, \(S^s_k\) intersects \(F_{21}(W^u_{\text{loc}}(O_2)) : \{Z = 0\}\) transversely (see (31)), i.e., \(O_2\) has a transverse homoclinic orbit when \(\ddot{\mu} \in I^s_k\).

**Proof.** According to (18), the preimage \(F^{-1}_{12}(W^s_2)\) is given by
\[
\begin{align*}
\ddot{x} &= -a^{-1}\dot{\mu} - a_{13}a_{11}^{-1}\ddot{z} + O((\ddot{z}^2 + \dot{\mu}^2), \\
\ddot{y} &= y^- - a_{31}a^{-1}\dot{\mu} + (a_{33} - a_{13}a_{31}a^{-1})\ddot{z} + O(\ddot{z}^2 + \dot{\mu}^2),
\end{align*}
\]
By (13), we find in coordinates (27) the surface \(F^{-k}_{12} \circ F^{-1}_{12}(W^s_2) \cap \Pi\) as
\[
\begin{align*}
X &= -a^{-1}\lambda^{-k}(\ddot{\mu} + O(\dot{\mu}^2)) - x^+ + o(1)_{k\to\infty}, \\
Y &= o(\hat{\gamma}^{-k}),
\end{align*}
\] (76)
where \( o(\cdot) \) and \( O(\cdot) \) terms are functions of \((\hat{\mu}, Z)\), and the first derivatives satisfy
\[
\frac{\partial (X, Y)}{\partial Z} = o(1)_{k \to \infty}.
\] (77)

This surface crosses \( \Pi' \) if \(|X| < \delta' \) and \(|Y| < \delta\). One readily finds from (76) that this happens when \( k \) is sufficiently large and
\[
| - a^{-1} \lambda^{-k} \hat{\mu} - x^+ | < \frac{\delta'}{2},
\]
which gives the intervals (74). By (77), the crossing is proper with respect to \( C^{ss} \) for large \( k \).

One can summarize the above results as follows.

**Proposition 4.** Let the cycle \( \Gamma \) have type I and let \( \theta \) be irrational.

- Let \( |\alpha| < 1 \). Then the blender \( \Lambda^{cu} \) has the same index as \( O_1 \) and, when \( \hat{\mu} \in I^u_\alpha \), the saddle \( O_1 \) is homoclinically related to \( \Lambda^{cu} \). If \( \hat{\mu} \in I^s_\alpha \), then there exists a persistent non-transverse heteroclinic connection between \( W^u(\Lambda^{cu}) \) and \( W^s(O_2) \) and a transverse heteroclinic intersection of \( W^u(O_2) \) and \( W^s(\Lambda^{cu}) \), i.e., there exist heterodimensional dynamics involving \( \Lambda^{cu} \) and a non-trivial hyperbolic set containing \( O_2 \).

- Let \( |\alpha| > 1 \). Then the blender \( \Lambda^{cs} \) has the same index as \( O_2 \) and, when \( \hat{\mu} \in I^u_\alpha \), the saddle \( O_2 \) is homoclinically related to \( \Lambda^{cs} \). If \( \hat{\mu} \in I^s_\alpha \), then there exists a persistent non-transverse heteroclinic connection between \( W^u(O_1) \) and \( W^s(\Lambda^{cs}) \) and a transverse heteroclinic intersection of \( W^u(\Lambda^{cs}) \) and \( W^s(O_2) \), i.e., there exist heterodimensional dynamics involving \( \Lambda^{cs} \) and a non-trivial hyperbolic set containing \( O_1 \).

**Proof.** Assume \( |\alpha| < 1 \). By Remark 5, the local stable manifold of \( \Lambda^{cu} \) intersects transversely (for \( \mu = 0 \), hence for all small \( \mu \); see Figure 8) any vertical surface that crosses \( \Pi \) properly with respect to \( C^u \), so it intersects transversely \( F_{21} \circ F_{12}^m \circ F_{12}(W^u_{loc}(O_1)) \) when \( \hat{\mu} \in I^u_\alpha \) (by Lemma 6). Remark 5 also gives us the existence of a transverse intersection of \( W^u_{loc}(\Lambda^{cu}) \) and \( W^s_{loc}(O_1) \), which proves that \( O_1 \) and \( \Lambda^{cu} \) are homoclinically related for \( \hat{\mu} \in I^u_\alpha \).

Now, let \( \hat{\mu} \in I^s_\alpha \). With the formula for \( F_{21}(W^u_{loc}(O_2)) \) in (31), the existence of a non-trivial hyperbolic set containing \( O_2 \) is given by Lemma 7 (the existence of a transverse homoclinic to \( O_2 \)). Also by Lemma 7, and by Proposition 1, we immediately find that \( W^s(O_2) \) activates \( \Lambda^{cu} \), meaning the existence of a persistent non-transverse heteroclinic connection (see Figure 9). It remains to note that the non-empty transverse intersection of \( W^s(\Lambda^{cu}) \) with \( W^u(O_2) \) is given by Remark 5.

When \( |\alpha| > 1 \), we inverse the time and use the blender \( \Lambda^{cs} \) given by Proposition 2 with the activating pair \( (\Pi', C^{uu}) \). After that, the proof is the same.

Recall that \( \hat{\mu} \sim \mu \) by (19), and all coefficients in formulas (71), (74) for the intervals \( I^u_\alpha \) and \( I^s_\alpha \) depend continuously on the system. Thus, there exists \( \kappa > 0 \) such that for any system \( g \) in a small neighborhood of \( f \), we have \( \hat{\mu} \in I^u_\alpha \) if
\[
\mu(g)^{\gamma(g)^m} \in [u_0 - \frac{1}{2} b_0^{-1} \delta', \kappa, u_0 + \frac{1}{2} b_0^{-1} \delta' - \kappa].
\]
Figure 8: For a type-I heterodimensional cycle with $x^+, u^- > 0$, there exist positive $\hat{\mu}$ values such that some forward iterate of $W^u_{\text{loc}}(O_1)$ intersects $W^s(\Lambda_{cu})$ while the backward iterates of $W^s_{\text{loc}}(O_2)$ leave the neighborhood of the cycle. Here the green planes in $\Pi'$ represent the local stable manifolds of points in the blender $\Lambda_{cu}$.

Figure 9: For a type-I heterodimensional cycle with $x^+, u^- > 0$, there exist negative $\hat{\mu}$ values such that some backward iterate of $W^s_{\text{loc}}(O_2)$ intersects $W^u(\Lambda_{cu})$ and the forward iterates of $W^u_{\text{loc}}(O_1)$ leave the neighborhood of the cycle.

and $\hat{\mu} \in I^u_m$ if

$$\mu(g)\lambda(g)^k \in [-a_0 x_0^+ \frac{-1}{2} a_0 \delta' + \kappa, -a_0 x_0^+ \frac{1}{2} a_0 \delta' - \kappa],$$

where $a_0, b_0, x_0^+$, and $u_0^-$ are the values of the corresponding coefficients for $f$. So, Proposition 4 implies Theorem 4 immediately.

\begin{proof}

\end{proof}

Remark 6. Note that Lemmas 6 and 7 hold, without any change, for type-II cycles as well. Therefore, one can formulate the following version of Proposition 4, which will be used in the proof of Theorem 6 in Section 3.3.3:

Suppose the system $f$ with a generic heterodimensional cycle (of type I or II) has an index-$d_1$ center-unstable blender $\Lambda_{cu}$ with the activating pair $(\Pi', C^{\alpha})$. Moreover, let the properties described in Remark 5 hold for $\Lambda_{cu}$ – namely, $W^u_{\text{loc}}(\Lambda_{cu})$ intersects transversely any vertical surface that crosses $\Pi$ properly with respect to $C^u$, and $W^u_{\text{loc}}(\Lambda_{cu})$ intersects transversely $W^s(O_1)$. Then, when $\hat{\mu} \in I^u_m$, the index-$d_1$
saddle $O_1$ is homoclinically related to $Λ^{cu}$, and when $\hat{\mu} \in I^+_k$, there exist persistent heterodimensional dynamics involving $Λ^{cu}$ and a non-trivial hyperbolic set containing $O_2$.

Analogously, if the system has an index-$d_2$ center-stable blender $Λ^{cs}$ with the activating pair $(π', C^{cu})$, and if $W^{u}_{loc}(Λ^{cs})$ intersects transversely any horizontal surface that crosses $Π$ properly with respect to $C^{s}$, and $W^{s}_{loc}(Λ^{cw})$ intersects transversely $W^{u}(O_2)$, then, when $\hat{\mu} \in I^+_m$, the saddle $O_2$ is homoclinically related to $Λ^{cs}$, and when $\hat{\mu} \in I^+_m$, there exist persistent heterodimensional dynamics involving $Λ^{cs}$ and a non-trivial hyperbolic set containing $O_1$.

### 3.3.2 Proof of Theorem 5

By the assumption of type-I cycle, we have $\lambda > 0, γ > 0$ and $ax^+u^- > 0$ at $\mu = 0$, and hence at all sufficiently small $\mu$ values. We further assume that $u^- > 0$ at $\mu = 0$, which means that $ax^+ > 0$ too. It is clear from the proof that when $u^- < 0$, all results remain true after reversing the sign of $\mu$.

We first prove the second part of the statement describing the situation at $\mu \leq 0$. It suffices to show that, at $\mu \geq 0$, the image $F^m \circ F_{12}^1(W^{u}_{loc}(O_1))$ does not enter the domain of $F_{21}$, i.e., it is outside a small neighborhood of $M^-_2 = (u^-0, w^-)$ for any sufficiently large $m$. Since $W^{u}_{loc}(O_1)$ is given by the equation $(x = 0, z = 0)$, this is equivalent (see (18) and (15)) to showing that the system of equations

$$
\begin{align*}
    u &= \hat{\mu} + a_{12}w + O(w^2), \\
    v &= a_{22}w + O(w^2), \\
    y - y^- &= a_{32}w + O(w^2),
\end{align*}
$$

does not have a solution $(\hat{u}, \hat{v}, \hat{w})$ near $(u^-0, w^-)$ for small $(y - y^-)$ and $w$, for any sufficiently large $m$. A straightforward computation reduces the above system to

$$
\hat{u} = γ^m\hat{\mu} + o(1)_{m \to \infty}.
$$

Since we have here $γ > 0$, the above equation together with the relation (19) implies that $\hat{u} \leq 0$ for $\mu \leq 0$ and sufficiently large $m$. Thus, by the assumption $u^- > 0$, the point $(\hat{u}, \hat{v}, \hat{w})$ never enters a small neighborhood of $(u^-0, w^-)$ indeed.

Now we switch to the case $\mu > 0$. First, note that like above, one can easily see (e.g. from equation (76) for $F^{-k} \circ F_{12}^1(W^{s}_{loc}(O_2))$) that no point of $W^{s}_{loc}(O_2)$ has an orbit that lies entirely in $U$ for $\mu > 0$. In particular, no heteroclinic orbits of intersection of $W^{u}(O_1)$ and $W^{s}(O_2)$ can lie entirely in $U$. Thus, given an orbit $O$ that lies entirely in $U$, we have two possibilities: ether $O$ is one of the orbits $L_{1,2}$ of the points $O_{1,2}$, or it intersects the $δ$-neighborhood $Π$ of the heteroclinic point $M^+_1$ in $U_{01}$. Let $M_s$ be the consecutive points of intersection of $O$ with $Π; we have $M_{s+1} = T_{k_s, m_s}M_s$, where $T_{k_s, m_s}$ is the first-return map given by (28) and $(k_s, m_s)$ are positive integers. The sequence $M_s$ can be infinite in both directions, infinite in one direction, or finite; let us consider the case where the sequence $(M_s)_{s \in \mathbb{Z}}$ is infinite in both directions first.

Let us show that the set $\hat{Λ}$ comprised by such orbits $O$ is indeed a hyperbolic set with index-$d_1$ when $\mu > 0$ (equivalently, when $\hat{\mu} > 0$, see (19)). It follows from (28) and $M_{s+1} \in Π$ that

$$
γ^{m_s}\hat{\mu} + ax^+λ^{k_s}γ^{m_s} = u^- + O(δ).
$$

(78)
Recall that we assumed \( \gamma > 0, \lambda > 0, ax^+ > 0, u^- > 0, \) and \( \mu \geq 0 \), so we obtain
\[
0 < ax^+ \lambda^k \gamma^{ms} < u^- + O(\delta).
\] (79)

In particular, the numbers \((k_s, m_s)\) are always such that \( \lambda^{k_s} \gamma^{ms} \) is uniformly bounded. Therefore, by the same argument we used to deduce formula (34) from (28) at \( \mu = 0 \), we obtain for our case the following formula for \( T_{k_s, m_s} : (X, Y, Z) \mapsto (\tilde{X}, \tilde{Y}, \tilde{Z}) \) (where we use coordinates (27)):
\[
\begin{align*}
\tilde{X} &= b\gamma^{ms} \mu + ab\lambda^{k_s} \gamma^{ms} x^+ - bu^- + ab\lambda^{k_s} \gamma^{ms} X + \tilde{\phi}_1(X, \tilde{Y}, Z), \\
\tilde{Y} &= \tilde{\phi}_2(X, \tilde{Y}, Z), \\
\tilde{Z} &= \tilde{\phi}_3(X, \tilde{Y}, Z),
\end{align*}
\] (80)

where
\[
\begin{align*}
\tilde{\phi}_1 &= O(\delta^2) + o(1)_{k, m \to \infty}, \\
\frac{\partial \tilde{\phi}_1}{\partial (X, Y, Z)} &= O(\delta) + o(1)_{k, m \to \infty}, \\
\|\tilde{\phi}_2\|_{C^1} &= O(\delta^{-k}), \\
\|\tilde{\phi}_3\|_{C^1} &= O(\delta^m).
\end{align*}
\]

Note that by multiplying \(|b/x^+|\) to each term in (79), leads to
\[
|ab\lambda^{k_s} \gamma^{ms}| < |\alpha| + O(\delta) < 1
\] (81)

for all sufficiently small \( \delta \). So, we have exactly the same formula (40) for the derivative of \( T_{k_s, m_s} \) (one should only replace \( \alpha \) by \( ab\lambda^{k_s} \gamma^{ms} \)). Also note that by shrinking the size of \( U \), the values \( k_s, m_s \) can be made arbitrarily large. Now, we have the result of Lemma 1, which along with (81), gives us the existence of invariant cone fields \( C^s \) and \( C^u \), which immediately implies that \( \tilde{\Lambda} \) is a hyperbolic set with index-\(d_1\) (all maps \( T_{k_s, m_s} \) contract in \((X, Z)\) and expand in \(Y\)).

The results analogous to Remarks 4 and 5 also hold. In particular, every forward orbit that intersects \( \Pi \) infinitely many times must belong to \( W^u_{loc}(\tilde{\Lambda} \cap \Pi) \), while every backward orbit that intersects \( \Pi \) infinitely many times must belong to \( W^u_{loc}(\tilde{\Lambda} \cap \Pi) \). Also, \( W^u_{loc}(\tilde{\Lambda} \cap \Pi) \) intersects \( W^s_{loc}(O_1) \) transversely, while \( W^u_{loc}(\tilde{\Lambda} \cap \Pi) \) intersects \( F_{21}(W^u_{loc}(O_2)) \) transversely.

We can now go through various cases where the sequence \( M_s \) of consecutive intersections of \( O \) with \( \Pi \) is not infinite in both directions.

1. \( \{M_s\}_{s=-\infty}^{0} \) is infinite backwards. As we just explained, this may happen only when \( M_0 \in W^u_{loc}(\tilde{\Lambda} \cap \Pi) \cap W^s_{loc}(O_1) \) or \( M_0 \in W^u_{loc}(\tilde{\Lambda} \cap \Pi) \cap W^s(O_2) \). The latter case does not happen since no point of \( W^s_{loc}(O_2) \) has an orbit entirely contained in \( U \) for \( \mu > 0 \), as we mentioned. In the former case, \( M_0 \) is a hyperbolic point of index \( d_1 \) (since \( O \) and \( \tilde{\Lambda} \) have the same index \( d_1 \) and the intersection of \( W^u_{loc}(\tilde{\Lambda} \cap \Pi) \) and \( W^s_{loc}(O_1) \) is transverse). We include all such orbits into the set \( \tilde{\Lambda} \) of Theorem 5, along with the set \( \tilde{\Lambda} \) and the orbit \( L_1 \) (and orbits in \( W^u(L_1) \) which we consider below).

2. \( \{M_s\}_{s=0}^{+\infty} \) is infinite forwards. This happens when \( M_0 \in W^s_{loc}(\tilde{\Lambda} \cap \Pi) \cap F_{21} \circ F_{2}^m \circ F_{12}(W^u_{loc}(O_1)) \) for some large \( m \) or when \( M_0 \in W^s_{loc}(\tilde{\Lambda} \cap \Pi) \cap F_{21}(W^u_{loc}(O_2)) \). The latter case, as the intersection is transverse, is in the complete agreement with the statement of the Theorem. In the former case, \( M_0 \) is a hyperbolic point of index \( d_1 \) because the intersection of \( W^s_{loc}(\tilde{\Lambda} \cap \Pi) \) and \( F_{21} \circ F_{2}^m \circ F_{12}(W^u_{loc}(O_1)) \) is transverse. Indeed, the surface \( F_{21} \circ F_{2}^m \circ F_{12}(W^u_{loc}(O_1)) \) is given by equation
and of positive even integers satisfying $k \in \mathbb{Z}$. From that, exactly like in Lemma 6, one obtains that the tangent spaces of $F_{21} \circ F_2 \circ F_{12}(W^u_{loc}(O_1))$ lie in $C^{uu}$, hence the transversality with $W^s_{loc}(\tilde{\Lambda} \cap \Pi)$.

3. The sequence $\{M_s\}$ is finite. This happens when $M_0 \in W^s(O_1) \cap F_{21}(W^u_{loc}(O_2))$ or $M_0 \in W^s(O_1) \cap F_{21} \circ F_2 \circ F_{12}(W^u_{loc}(O_1))$. Since the tangents to $W^s_{loc}(O_1)$ : $\{Y = 0\}$ lie in in the backward-invariant cone field $C^s$, all its pre-images by $T_{k_s,m_s}$ also have tangents in $C^s$, hence the intersections of $W^s(O_1)$ with $F_{21}(W^u_{loc}(O_2))$ and $F_{21} \circ F_2 \circ F_{12}(W^u_{loc}(O_1))$ are transverse.

In all cases we have the agreement with the statement of the theorem.

\[\square\]

### 3.3.3 Proof of Theorem 6

As can be seen from the proof of Theorem 5, the main reason that prevents the blender from being simultaneously homoclinically related to both $O_1$ and $O_2$ is that, when one saddle activates the blender, the iterates of the local stable or unstable manifold of the other will leave the small neighborhood $U$ of $\Gamma$ (see Figures 8 and 9). In what follows, we show that, if there exists a type-II cycle tied with $\Gamma$, then the leaving manifold can return to $U$ by following the robust heteroclinic orbit of the type-II cycle.

Let $\hat{\Gamma} = L_1 \cup L_2 \cup \Gamma^0 \cup \hat{\Gamma}^1$ be a heterodimensional cycle sharing the fragile heteroclinic orbit $\Gamma^0$ with $\Gamma$. By definition, any generic one-parameter unfolding of $\Gamma$ is also a generic one for $\hat{\Gamma}$. These two cycles have the same transition map $F_{12}$, and the local maps $F_1$ and $F_2$ are the same. Denote by $\hat{F}_{21}$ the transition map from a neighborhood of $\hat{M}_2 = (\hat{u}^-, 0, \hat{w}^-) \in \hat{\Gamma}^1$ to a neighborhood of $\hat{M}_1 = (\hat{x}^+, 0, \hat{z}^+) \in \hat{\Gamma}^1$. The map $\hat{F}_{21}$ has the same form as (21), just one needs to replace coefficients $b_{ij}$ by some $\hat{b}_{ij}$ and $b$ by $\hat{b} \neq 0$.

Let us mark the values of the coefficients $a, \hat{b}, \lambda, \gamma, \hat{u}^-, \hat{x}^+$ at $\mu = 0$ by the subscript “0” (these coefficients depend continuously on $\mu$; they are smooth in $\mu$ when the smoothness class of $f_\mu$ is at least $C^3$; in fact, $\lambda$ and $\gamma$ remain smooth in $\mu$ in the $C^2$-case too).

**Lemma 8.** Let $\hat{\Gamma}$ be a generic type-II cycle (i.e., $a_0\hat{x}_0^+ \hat{u}_0^- < 0$ and $|\hat{b}_0\hat{u}_0^- / \hat{x}_0^+| \neq 1$). Consider a generic one-parameter unfolding $\{f_\mu\}$ of $\hat{\Gamma}$. Assume $\theta_0$ is irrational and take a sequence $\{(k_j, m_j)\}$ of pairs of positive even integers satisfying $k_j, m_j \to \infty$ and

\[
a_0 \hat{b}_0 \lambda_0^{k_j} \gamma_0^{-m_j} \to -\alpha_0 = -\frac{\hat{b}_0 \hat{u}_0^-}{\hat{x}_0^+},
\]

as $j \to \infty$. There exists a sequence of values $\{\mu_j\}$ satisfying

\[
\mu_j = -a_0 \hat{x}_0^+ \lambda_0^{k_j} + o(\lambda_0^{k_j}) = \hat{u}_0^- \gamma_0^{-m_j} + o(\gamma^{-m_j}),
\]

such that the system $f_{\mu_j}$ has an orbit $\Gamma^{0, new}$ of heteroclinic intersection of $W^s(O_1)$ with $W^s(O_2)$. The heteroclinic connection $\Gamma^{0, new}$ splits when $\mu$ varies in an $o(\mu_j)$-interval around $\mu_j$.

**Remark 7.** The newly created heterodimensional cycle $\Gamma^{new} = L_1 \cup L_2 \cup \Gamma^{0, new} \cup \hat{\Gamma}^1$ is generic. Indeed, one only needs to check that $\Gamma^{0, new}$ satisfies condition GC1 (the other genericity conditions
hold because of the genericity of the cycle $\Gamma$). Condition GC1 holds for $\Gamma^{0,\text{new}}$ because $\Gamma^{\text{new}}$ is a partially-hyperbolic set with a one-dimensional center direction. This is true because $\text{cl}(\Gamma^{0,\text{new}})$ lies in a small neighborhood of $\tilde{\Gamma}$ which is a compact partially-hyperbolic set with a one-dimensional center direction (by the genericity assumption), and the partial hyperbolicity of a compact invariant set is inherited by every closed invariant set in its neighborhood.

Proof of Lemma 8. We create the secondary (fragile) heteroclinic orbit $\Gamma^{0,\text{new}}$ by finding values of $\mu$ at which $W_{\text{loc}}^s(O_2)$ intersects $\tilde{W}_1^u := F_{12} \circ F_1^k \circ \tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{\text{loc}}^u(O_1))$.

By putting $\tilde{x} = 0$, $\tilde{z} = 0$ in (85), we find that the image $F_{12}(W_{\text{loc}}^u(O_1))$ near the point $M_2^{+}$ is given by

$$u = \tilde{\mu} + O(w), \quad v = v^{+} + O(w). \quad (84)$$

Let us now find an equation for $\tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{\text{loc}}^u(O_1))$. We use formula (25) (with the coefficients $b$, $b_{13}$, $u^-$, and $x^+$ replaced by $\tilde{b}$, $\tilde{b}_{13}$, $\tilde{u}^-$, and $\tilde{x}^+$, and with modified functions $\tilde{h}$) for the map $\tilde{F}_{21} \circ F_2^m$ from a small neighborhood of $M_2^{+}$ to a small neighborhood of $M_1^{-}$. By substituting equation (84) into this formula, we obtain (see (24)) that $\tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{\text{loc}}^u(O_1))$ is given by

$$x = \tilde{x}^+ + \tilde{b}(\gamma^m \tilde{\mu} - \tilde{u}^-) + O(\|y\| + (\gamma^m \tilde{\mu} - \tilde{u}^-)^2) + o(1)_{n \to +\infty},$$

$$z - z^+ = O(\|y\| + (\gamma^m \tilde{\mu} - \tilde{u}^-)^2) + o(1)_{n \to +\infty}, \quad (85)$$

where $m$ must be such that $\gamma^m \tilde{\mu}$ must be sufficiently close to $\tilde{u}^-$. 

By substituting (85) in (23), we, finally, obtain the equation for $\tilde{W}_1^u := F_{12} \circ F_1^k \circ \tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{\text{loc}}^u(O_1))$:

$$u = \tilde{\mu} + \tilde{a} \lambda^k \gamma^m \tilde{\mu} + a \lambda^k \tilde{x}^+ - \tilde{a} \lambda^k \tilde{u}^+ + O(w) + \lambda^k O((\gamma^m \tilde{\mu} - \tilde{u}^-)^2) + o(\lambda^k), \quad (86)$$

$$v = v^+ + O(w) + O(\lambda^k).$$

The heteroclinic orbit $\Gamma^{0,\text{new}}$ corresponds to an intersection of $\tilde{W}_1^u$ with $W_{\text{loc}}^s(O_2)$, which corresponds to letting $(u, w) = 0$ in (86), i.e.,

$$0 = \tilde{\mu} + \tilde{a} \lambda^k \gamma^m \tilde{\mu} + a \lambda^k \tilde{x}^+ - \tilde{a} \lambda^k \tilde{u}^+ + \lambda^k O((\gamma^m \tilde{\mu} - \tilde{u}^-)^2) + o(\lambda^k). \quad (87)$$

We will look for solutions $\mu = O(\lambda^k)$ for large enough and even $(k, m)$ from the sequence $\{k_j, m_j\}$ satisfying (82), i.e.,

$$m - k \theta_0 \to \frac{1}{\ln |\gamma_0|} \ln \left| \frac{\tilde{u}^-}{a_0 \tilde{x}_0^+} \right|. \quad (88)$$

Since $\lambda$ and $\gamma$ depend smoothly on $\mu$, we have, in particular, Lipshitz dependence of $\theta = -\ln |\lambda| / \ln |\gamma|$ on $\mu$, implying

$$\theta = \theta_0 + O(\mu) = \theta_0 + O(\lambda^k) \quad (89)$$

(recall that the 0 subscript stands for the value at $\mu = 0$). Since $k$ and $m$ are even, we have

$$\lambda^k \gamma^m = |\gamma|^m |\mu - k \theta_0| + O(k \lambda^k) \frac{\gamma^m}{\gamma_0} \left| \frac{\tilde{u}^-}{a_0 \tilde{x}_0^+} + o(1)_{k, m \to \infty}; \quad (90)$$

note that we used here Lipshitz dependence of $\theta$ on $\mu$, but we only used continuity of the $\gamma$’s dependence.
implies that the blender near the type-I cycle $\Gamma$ is homoclinically related to one of the saddles

which, upon substitution into (91), gives us the blender at

Since $\tilde{b}_0/\tilde{x}_0^+ = \tilde{a}_0 \neq 1$ (as given by the genericity assumption $\text{GC}4.1$ for the cycle $\tilde{\Gamma}$), we find that the secondary heteroclinic connection $\Gamma^{0,\text{new}}$ exists when

for large enough even $(k_j, m_j)$ defined by (82). By (19), this can be rewritten as

Note that $\gamma$ and $\lambda$ in this formula (as well as the $o(1)$-terms) depend on $\mu$, so this is an implicit relation on $\mu_j$; however, since the dependence on $\mu$ is continuous, it is obvious that for all sufficiently large $j$ such $\mu_j$ exist. Thus, we have the existence of the sought fragile heteroclinic connection $\Gamma^{0,\text{new}}$ for some $\mu_j \in \text{int} (\Delta_j)$ where the interval $\Delta_j$ corresponds to

for some $\kappa_j \to 0$. By construction, if $\kappa_j$ goes to 0 sufficiently slowly, then $\Gamma^{0,\text{new}}$ necessarily splits when $\mu$ gets out of $\Delta_j$.

We finish the proof of the lemma by recalling that $\lambda$ and $\gamma$ depend smoothly on $\mu$, hence

which, upon substitution into (92), gives (83).

Since the cycle $\tilde{\Gamma}$ is tied with $\Gamma$, we have $x^+ = \tilde{x}^+$ or $u^- = \tilde{u}^-$. It follows immediately from formula (91) that if $x^+ = \tilde{x}^+$, then $\hat{\mu}_j$ lies in the interval $I_m^k$ defined in Lemma 7, and if $u^- = \tilde{u}^-$, then $\hat{\mu}_j$ lies in the interval $I_m^k$ defined in Lemma 6. In any case, Proposition 4 implies that the blender which exists by Theorem 3 near the type-I cycle $\Gamma$ is homoclinically related to one of the saddles $O_{1,2}$ when the system has the secondary fragile heteroclinic orbit $\Gamma^{0,\text{new}}$ of Lemma 8.

Theorem 3 gives us the blender at $\mu = 0$, which persists for all small $\mu$, and satisfies conditions of Remark 6. The homoclinic connection to the saddle $O_1$ or $O_2$ also persists for all $\mu$ corresponding to the interval $I_m^k$ or $I_k^k$, i.e., it persists for all $\mu$ from the intervals $\Delta_j$ defined by (93). Let us show that there are sub-intervals inside $\Delta_j$ where the blender has a homoclinic connection to the other saddle too.

Indeed, when $\mu$ varies from one end of $\Delta_j$ to the other, the heteroclinic orbit $\Gamma^{0,\text{new}}$ splits.
Therefore, we can apply Remark 6 to the heterodimensional cycle \( \Gamma^{\text{new}} = L_1 \cup L_2 \cup \Gamma^0,\text{new} \cup \Gamma^1 \) at \( \mu = \mu_j \) (that is, system \( f \) in Remark 6 is our system \( f_{\mu_j} \); the coefficient \( \hat{\mu} \) in Remark 6 measures the distance between \( \tilde{W}^u \) with \( W^u_{\text{loc}}(O_2) \) when \( \Gamma^0,\text{new} \) splits; note that Remark 6 is applicable because \( \Gamma^{\text{new}} \) is generic, see Remark 7). As a result, we find that there are intervals of \( \mu \) values inside \( \Delta_j \) which correspond to a homoclinic connection between the blender and \( O_1 \), as well as intervals which correspond to a homoclinic connection between the blender and \( O_2 \) (the connections can be transverse, or persistent heterodimensional). In any case, we have found intervals of \( \mu \) values for which the blender is connected to both saddles \( O_1 \) and \( O_2 \); see Figure 10. This finishes the proof of Theorem 6.

![Diagram](image)

Figure 10: A pair of tied heterodimensional cycles of type I and II with \( M^+ \) and \( \tilde{M}^+ \) lying in the same strong-stable leaf and with \( x^+ > 0, u^- > 0 \). There exist negative \( \mu \) values such that some backward iterate of \( W^u_{\text{loc}}(O_2) \) intersects \( W^u(\Lambda^{cu}) \), and the forward iterate \( F_{12} \circ F_{21} \circ F_{12} \circ F_{21} \circ F_{12} \circ W^u_{\text{loc}}(O_1) =: F_{12}^{\text{new}}(W^u_{\text{loc}}(O_1)) \) intersects \( W^s_{\text{loc}}(O_2) \) producing a secondary heterodimensional cycle. After a further change in \( \mu \), so that the newly obtained fragile heteroclinics splits to the right (i.e., \( F_{12}^{\text{new}}(W^u) \) lies in \( \{ u > 0 \} \)), one can find \( m' \) satisfying \( F_{21}^{m'} \circ F_{12}^{\text{new}} \cap F_{21}^{m'}(W^s_{\text{loc}}(O_1)) \neq \emptyset \) such that the iterate \( F_{21} \circ F_{21}^{m'} \circ F_{12}^{\text{new}}(W^u) \) crosses \( \Pi' \) vertically, and, therefore, it intersects \( W^s(\Lambda^{cu}) \). As a result, the two saddles \( O_1 \) and \( O_2 \) get homoclinically related to the blender at the same time.

**Remark 8.** Note that in the proof of this theorem we only used the smoothness with respect to \( \mu \) in 2 places: when we claimed the Lipshitz dependence of \( \theta \) on \( \mu \) (see (89)), and when we inferred (83) from (92). However, we did not use formula (83) for \( \mu_j \) when we derived Theorem 6 from Lemma 8, and used only relation (92) for \( \mu_j \). The existence of such values \( \mu_j \) is established for any continuous family in which \( \Gamma^0 \) splits, i.e., for which the splitting functional is not constantly zero and changes sign. Therefore, Theorem 6 remains intact, for example, for any one-parameter family going through \( f \) such that

- the splitting functional \( \mu \) changes sign,
- \( \theta \) stays constant (irrational).

Even more, the result still holds if we replace the one-parameter unfolding by any connected set of systems which contains \( f \) and satisfies the above conditions – the intervals \( I_j \) in the formulation of Theorem 6 should be replace by open subsets \( I_j \) converging to \( f \).
3.3.4 Proofs of Theorem 7 and Theorem 2 for type-II cycles

Theorem 6 immediately shows that cycles of type III can be locally stabilized. In this Section we prove the same for type-II cycles in the following way. We let $|\alpha| < 1$ (the case $|\alpha| > 1$ is reduced to this one by the time-reversal). Then, we show that a generic one-parameter unfolding of a type-II cycle gives rise to a pair of tied cycles of type I and type II associated to $O_1$ and to a new saddle periodic point $O_2'$ of index $d_1 + 1$. Next, we compute the value of the modulus $\theta$ for the new cycles: $\theta' := -\ln |\lambda|/\ln |\gamma'|$, where $\gamma'$ is the central multiplier of $O_2'$, and show that by taking $\theta$ of the original cycle as a second parameter, we can make $\theta'$ irrational, so that Theorem 6 becomes applicable.

**Proof of Theorem 7.** Assume $|\alpha| < 1$. Since $\Gamma$ is a cycle of type II, Lemma 8 is applicable to it (just replace $\hat{\Gamma}$ by $\Gamma$ in the formulation of the lemma, and remove the tildes from the coefficients in (82) and (83)). This gives us that for a sequence $\{(k_j, m_j)\}$ of pairs of positive even integers satisfying $k_j, m_j \to \infty$ and

$$a_0 b_0 \lambda_0^{k_j} \gamma_0^{m_j} \to -\frac{b_0 \mu_0}{x_0^*} 
$$

as $j \to \infty$, there exists a sequence of values $\{\mu'_j\}$

$$\mu'_j = -a_0 x_0^* \lambda_0^{k_j} + o(\lambda_0^{k_j}) \n$$

such that the system $f_{\mu'_j}$ has a secondary heterodimensional cycle associated to $O_1$ and $O_2$. By (19), we also have

$$\hat{\mu}'_j = \hat{\mu}(\mu'_j) = -a_0 x_0^* \lambda_0^{k_j} + o(\lambda_0^{k_j}) \n$$

for the coefficient $\hat{\mu}$ in formula (28) for the first-return map near $\Gamma$.

Denote $m'_j = m_j + m^*$ for some fixed even integer $m^*$ (to be determined below). In what follows, we find for each sufficiently large $j$ a fixed point of the first-return map $T_{k_j, m'_j}$ for the values of $\mu$ which are $o(\mu'_j)$-close to $\mu'_j$. Note first that due to the smooth dependence of $\lambda$ and $\gamma$ on the parameter $\mu$, we obtain from (95), (94) that

$$\lambda(\mu_j)^{k_j} = (\lambda_0 + O(\mu_j))^{k_j} = \lambda_0^{k_j} + O(k_j \lambda_0^{k_j} \mu_j) = \lambda_0^{k_j} (1 + O(k_j \lambda_0^{k_j})), \n$$

$$\gamma(\mu_j)^{m_j} = (\gamma_0 + O(\mu_j))^{m_j} = \gamma_0^{m_j} + O(m_j \gamma_0^{m_j} \mu_j) = \gamma_0^{m_j} (1 + O(m_j \gamma_0^{m_j})). \n$$

Thus,

$$\lambda^{k_j} \gamma^{m'_j} \to -\frac{a_0 \gamma^{m^*}}{x_0^*}, \n$$

so, in particular, $\lambda^{k_j} \gamma^{m'_j}$ stays uniformly bounded as $j \to +\infty$.

With this, one notices that the term $\gamma^{m^*} \hat{\phi}_1$ in formula (28) for the first-return map $T_{k, m}$ is $o(1)_{j \to +\infty}$ when $k = k_j$ and $m = m'_j$. Thus, we can write the map $T_{k_j, m'_j}$ as

$$\begin{align*}
\hat{X} &= b \gamma^{m'_j} \hat{\mu} + abx^{k_j} \gamma^{m'_j} x^+ - bu + abx^{k_j} \gamma^{m'_j} X + O(\hat{X}^2 + \hat{Y}^2) + o(1)_{j \to +\infty}, \\
\hat{Y} &= o(\gamma^{-k_j}), \quad \hat{Z} = o(\lambda^{m'_j}),
\end{align*} \n$$

(99)
see (29),(30). It follows that the fixed point \((X_j, Y_j, Z_j)\) of \(T_{k_j,m'_j}\) satisfies
\[
X_j = \frac{(-(a\lambda^k)^{-1}\mu'_j + (a\lambda^k\gamma^m)\gamma^{-1} - x^+ + O(X_j^2) + o(1))_{k_j,m'_j \to \infty}}{1 - (ab\lambda^k\gamma^m)\gamma^{-1}}, \tag{100}
\]
\[
Y_j = O(\hat{\gamma}^{-k_j}), \quad Z_j = O(\hat{\lambda}^m).
\]

Thus, by taking \(m^*\) sufficiently large and \(\hat{\delta}\) sufficiently small, we obtain that \(|X_j| < \hat{\delta}\) for all sufficiently large \(j\) as required.

We need to verify that \((X_j, Y_j, Z_j) \in \Pi\). It suffices to show that \(|X_j| < \delta\) for all large \(k_j\) and \(m'_j\).

Substituting (96), (97), and (98) into the first equation of (100) yields
\[
X_j = x_{\infty} + o(1)_{j \to \infty}, \tag{101}
\]
where
\[
x_{\infty} = -x_0^+ \gamma_0^{-m^*} + O(\gamma_0^{-2m^*}). \tag{102}
\]

Thus, by taking \(m^*\) sufficiently large and \(\hat{\delta}\) sufficiently small, we obtain that \(|X_j| < \hat{\delta}\) for all sufficiently large \(j\), as required.

We have shown that at \(\mu = \mu'_j\) the system has a generic heterodimensional cycle associated to \(O_1\) and \(O_2\) and the first-return map \(T_{k_j,m'_j}\) has a fixed point in \(\Pi\) given by (101) for any sufficiently large \(m^*\). We denote such point as \(O'_{2}\). By (99), the map \(T_{k_j,m'_j}\) strongly contracts in \(Z\) and strongly expands in \(Y\). Also, since
\[
|ab\lambda^k\gamma^m| = \left|\frac{b_0\mu_0\gamma_0^m}{x_0^+}\right| + o(1)_{j \to \infty} \gg 1,
\]
the map is expanding in \(X\) if \(m^*\) is taken large enough. Thus, the point \(O'_{2}\) is a saddle of index \((d_1+1)\).

Arguing as in the proof of Lemma 1, one finds, on the set of points whose images under \(T_{k_j,m'_j}\) belong to \(\Pi\), a forward-invariant unstable cone field around the \((X,Y)\)-space and a forward-invariant strong-unstable cone-field around the \(Y\)-space, and also, on the set of points whose pre-images belong to \(\Pi\), a backward-invariant stable cone field around the \(Z\)-space. This implies that \(W^u_{loc}(O'_{2})\) and \(W^s_{loc}(O'_{2})\) are given by \(Z = Z_j + w^u_j(X,Y)\) and \((X,Y) = (X_j,Y_j) + w^s_j(Z)\), respectively, for a smooth function \(w^u_j\) defined for \((X,Y) \in [-\delta, \delta] \times [-\delta, \delta]^{d_1}\) and a smooth function \(w^s_j\) defined for \(Z \in [-\delta, \delta]^{d-d_1-1}\). Also, \(W^u_{loc}(O'_{2})\) contains the strong-unstable manifold \(W^u_{loc}(O'_{2})\) of the form \(\{X = X_j + w^u_j(Y), Z = Z_j + w^u_j(X,Y)\}\) where the smooth function \(w^u_j\) is defined for \(Y \in [-\delta, \delta]^{d_1}\). Note that it immediately follows from (99) that in the limit \(j \to \infty\) we have
\[
W^u_{loc}(O'_{2}) \to \{Z = 0\}, \quad W^s_{loc}(O'_{2}) \to \{X = x_{\infty}, Y = 0\}, \quad W^u_{loc}(O'_{2}) \to \{X = x_{\infty}, Z = 0\}. \tag{103}
\]
in the \(C^1\)-topology.

Now, since \(F_{21}(W_{loc}^u(O'_{2}))\) is given by \(\{Z = 0\}\) (see (31)), we obtain that it intersects \(W^s_{loc}(O'_{2})\) transversely, and, since the value \(\mu'_j\) lies in the interval \(I^1_{k_j}\) given by Lemma 7, it follows that \(F_{12}^{-k_j} \circ F_{12}^{-1}(W^s_{loc}(O'_{2}))\) is a horizontal surface that crosses \(\Pi\) properly, so it intersects \(W^s_{loc}(O'_{2})\) transversely. Thus, \(O'_{2}\) is homoclinically related to \(O_2\).

This, in particular, implies that \(O'_{2}\) has transverse homoclinics. Let us show that in \(W^u_{loc}(O'_{2})\)
there exists a homoclinic point \( M' \) of transverse intersection of \( W^u_{\text{loc}}(O_2) \) with \( W^s(O_2) \), such that \( M' \not\in W^u_{\text{Loc}}(O_2) \). By (103), we just need to show that the \( x \)-coordinate of the homoclinic point \( M' \) can be kept bounded away from \( x_\infty \). Since \( W^u_{\text{Loc}}(O_2) \) intersects \( F_3(1)(W^u_{\text{loc}}(O_2)) \) transversely, the pre-images \( F_2^{-m} \circ F_2^{-1}(W^s_{\text{loc}}(O_2)) \) accumulate (in \( C^1 \)) to \( W^s_{\text{loc}}(O_2) \) as \( m \to +\infty \). Hence, we can choose the homoclinic point \( M' \) as close as we want to a point of transverse intersection of \( W^u_{\text{loc}}(O_2) \) with the pre-image \( F_1^{-k_j} \circ F_{12}^{-1}(W^s_{\text{loc}}(O_2)) \). This pre-image is given by the equation

\[
x = -\frac{1}{a} \mu \lambda^{-k_j} + o(1)_{k_j \to +\infty}, \quad y = o(1)_{k_j \to +\infty},
\]

as follows from substitution of \( (u = 0, w = 0) \) (the equation of \( W^s_{\text{loc}}(O_2) \)) into equation (23) for the map \( F_{12} \circ F_1 \). In the coordinates (27), we get

\[
X = -\frac{1}{a} \mu \lambda^{-k_j} - x^+ + o(1)_{k_j \to +\infty}.
\]

This implies, by (96),(97), that for any \( \mu \) which is \( o(\mu'_j) \)-close to \( \mu = \mu'_j \) the \( x \)-coordinate of the intersection with \( W^u_{\text{loc}}(O_2) \) is \( o(1)_{j \to +\infty} \). Hence, the \( x \)-coordinate of the homoclinic point \( M' \) can be made as close to zero as we want, i.e., it is bounded away, as claimed, from \( x_\infty \) (which is non-zero by (102)).

Now, let us construct a heteroclinic intersection of \( W^u(O_1) \) and \( W^s(O_2) \). The secondary heterodimensional cycle at \( \mu = \mu'_j \) includes an orbit of a non-transverse intersection between \( W^u(O_1) \) and \( W^s(O_2) \). This orbit splits as \( \mu \) varies in an interval of size \( o(\mu'_j) \) (by Lemma 8). Since \( O_2 \) and \( O'_2 \) are homoclinically related, the invariant manifold \( W^s(O'_2) \) accumulates on \( W^s(O_2) \), which means that when \( \mu \) varies, an orbit of a non-transverse intersection of \( W^u(O_1) \) and \( W^s(O'_2) \) emerges. Let it happen at \( \mu = \mu_j \); it is \( o(\mu'_j) \)-close to \( \mu'_j \). Note that the orbit \( \Gamma_0^{d_j} \) of the fragile heteroclinic intersection of \( W^u(O_1) \) and \( W^s(O'_2) \) at \( \mu = \mu_j \) satisfies condition GC1 by the partial-hyperbolicity argument as in Remark 7.

To finish the proof, we need to find a pair of orbits \( \Gamma_{j,I}^{1} \) and \( \Gamma_{j,II}^{1} \) of a transverse intersection of \( W^u(O'_2) \) and \( W^s(O_1) \) such that the corresponding heterodimensional cycles \( \Gamma_{j,I} = L_1 \cup L'_2 \cup \Gamma_j^0 \cup \Gamma_{j,I}^1 \) and \( \Gamma_{j,II} = L_1 \cup L'_2 \cup \Gamma_j^0 \cup \Gamma_{j,II}^1 \) (where \( L'_2 \) is the orbit of the periodic point \( O'_2 \)) are generic, tied to each other, and have different types (I and II).

We, first, notice that \( W^s_{\text{loc}}(O_1) : \{ Y = 0 \} \) transversely intersects the local strong-unstable manifold \( W^u_{\text{loc}}(O'_2) \) (indeed, by (103), \( W^u_{\text{loc}}(O'_2) \) is \( C^1 \)-close to \( \{ X = x_\infty, Z = 0, Y \in [-\delta, \delta[\alpha_1] \} \)). Let \( M \) be the point of intersection. The local unstable manifold \( W^u_{\text{loc}}(O'_2) \) is divided by \( W^u_{\text{loc}}(O'_2) \) into two connected components. It follows from the transversality of the intersection of \( W^u_{\text{loc}}(O'_2) \) with \( W^u_{\text{loc}}(O_1) \) that the intersection of \( W^s_{\text{loc}}(O_1) \) with \( W^u_{\text{loc}}(O'_2) \) near \( M \) is a curve that goes from one component to another while crossing \( W^u_{\text{loc}}(O'_2) \) at \( M \). Choose a point \( \tilde{M} \) on this curve such that \( \tilde{M} \) is close to \( M \) and \( \tilde{M} \) lies in a different component of \( W^u_{\text{loc}}(O'_2) \setminus W^u_{\text{loc}}(O'_2) \) from the homoclinic point \( M' \in W^u_{\text{loc}}(O'_2) \cap W^s(O'_2) \).

The orbit of \( \tilde{M} \) is an orbit of transverse intersection of \( W^u(L'_2) \) and \( W^s(L_1) \). The genericity condition GC2 is satisfied by this orbit due to partial hyperbolicity of every orbit lying in a small neighborhood of the original generic cycle \( \Gamma \) (see Remark 7). Condition GC3 is satisfied because we take \( \tilde{M} \neq M \), i.e., \( \tilde{M} \) is not in \( W^u_{\text{loc}}(O'_2) \). However, we take it sufficiently close to \( W^u_{\text{loc}}(O'_2) \), so the corresponding coefficient \( u^- \) is sufficiently close to zero, making the genericity condition GC4.1 fulfilled too. The generic heterodimensional cycle comprised by the orbit of \( M \), the periodic orbits
$L_1$ and $L_2$ and the fragile heteroclinic $\Gamma'_0$ at $\mu = \mu_j$ is the sought cycle $\Gamma_{j,I}$ or $\Gamma_{II,j}$ (depending on whether it is type I or type II). By Remark 2, the fact that $M$ and the homoclinic point $M'$ belong to different components of $W^u_{loc}(O'_2) \setminus W^s_{loc}(O'_2)$ implies that this cycle is tied with a cycle of a different type. So, we have the sought pair of heterodimensional cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$, and hence the result of the theorem, once we show the genericity of the heterodimensional cycle given by Remark 2.

The orbit of transverse intersection of $W^s(O_1)$ and $W^u(O'_2)$ given by Remark 2 intersects $W^u_{loc}(O'_2)$ at some point $M''$ close to the homoclinic point $M'$. By the partial hyperbolicity argument, we have the genericity condition GC2; condition GC3 holds because $M' \not\in W^u_{loc}(O'_2)$. Let us establish the last genericity condition GC4.1. The orbit of the point $M''$ goes close to the orbit of $M'$ and gets back to a small neighborhood of $O'_2$ (since $M'$ is homoclinic). After that the orbit spends a long time near $L'_2$, which corresponds to a large number of iterations of the first-return map $T_{k_j,m_j,m'}$ near $O_2$, before getting to $W^s_{loc}(O_1)$. The iterations near $O_2$ create a very large expansion in the central direction, i.e., the expansion factor $b$ in (3) gets very large for this orbit, making $|a| \gg 1$ in (2.3). Thus, the genericity condition GC4.1 is fulfilled for this orbit, provided $M''$ is chosen close enough to $M'$.

We can now finish the proof of Theorem 2 for the case of real central multipliers.

Proof of Corollary 3. In a proper unfolding $f_x$ of a heterodimensional cycle of type II, we fix the values of all parameters except for $\mu$ and $\theta$ and show that an arbitrary small change of $\mu$ and $\theta$ can lead the system into the region of robust heterodimensional dynamics involving $O_1$ and $O_2$. So, we may from the very beginning consider a two-parameter family $f_{\mu,\theta}$ where $\mu$ varies in a small neighborhood of zero and $\theta$ varies in a neighborhood of some $\theta = c_0$. By Theorem 7, on any line $\theta = constant$ we have

$$
\mu = \mu_j = -a_0x_0^+ + \lambda_0^{k_j}(1 + o(1))_{j \to +\infty}
$$

for which the system has a tied pair of cycles of different types involving $O_1$ and a periodic point $O'_2$, homoclinically related to $O_2$. The coefficients $a_0$, $x_0^+$, $\lambda_0$ correspond to $\mu = 0$ and depend continuously on the parameter $\theta$; the $o(1)$ term also depends continuously on $\theta$ and the rate with which it tends to zero is uniform with respect to $\theta$.

Moreover, as $\mu$ varies within an interval of size $o(\mu_j)$ the fragile heteroclinic connection splits. It follows that there exists a small $\kappa > 0$ such that for any connected set $L$ in the $(\mu, \theta)$-plane with a point at the line $\mu = \mu_j^- = \lambda_0^{k_j}(-a_0x_0^+ - \kappa)$ and another point at the line $\mu = \mu_j^+ = \lambda_0^{k_j}(-a_0x_0^+ + \kappa)$ there is a point in $L$ corresponding to the tied pair of cycles involving $O_1$ and $O'_2$. Moreover, when we move within $L$ from $\mu = \mu_j^-$ to $\mu = \mu_j^+$, the fragile heteroclinic connection which is shared by these cycles is split. Therefore, we immediately get the result (the existence of robust heterodimensional dynamics involving $O_1$ and $O'_2$ - hence, $O_2$) by applying the version of Theorem 6 given in Remark 8, if we can choose the connected set $L$ such that the modulus $\theta' = -\ln |\lambda|/\ln |\gamma'|$ stays equal to a certain irrational constant everywhere on $L$ (recall that we denote as $\gamma'$ the central multiplier of the point $O'_2$).

For that, we just need to show that arbitrarily close to $c_0$ there exist constants $c_-$ and $c_+$ such that, if $j$ is large enough,

$$
\sup_{\mu \in [\mu_j^-, \mu_j^+]} \theta'(\mu, \theta = c^-) < \inf_{\mu \in [\mu_j^-, \mu_j^+]} \theta'(\mu, \theta = c^+),
$$

(104)
Let us compute the central multiplier $\gamma'$ of $O_2'$. It is, up to $o(1) j\to+\infty$ terms, the derivative $\left.\frac{d\Delta}{dx}\right|_{x=x_0}$ of the map (99) at $O_2'$. This gives us

$$ \gamma' = ab\lambda^{k_2} \gamma^{m_2'}(1 + O(\delta)) + o(1) j\to+\infty, $$

or, according to (98),

$$ \gamma' = -\frac{b_{0}u_{0}}{x_{0}} \gamma^{m_2}(1 + O(\delta)) + o(1) j\to+\infty. $$

This gives

$$ \theta' = \frac{1}{m^*} \gamma' + O((m^*)^{-2}), \quad (105) $$

so, by taking $m^*$ sufficiently large, we obtain (104).

\[\square\]

### 3.4 Heterodimensional cycles with rational $\theta$. Proof of Theorem 8

Note that an orbit that lies entirely in the small neighborhood $U$ of the cycle $\Gamma$ either belongs to the stable or unstable manifold of $L_1$ or $L_2$, or intersects the neighborhood $\Pi$ of the heteroclinic point $M_1^+$ in $U_{01}$ infinitely many times both forwards and backwards in time. If such orbit exists, then for any two consecutive intersection points of the orbit with $\Pi$, the second one is the image of the first one under the map $T_{k,m}$ given by (28) for some pair of integers $(k, m)$. Moreover, $k$ and $m$ have to be large enough if $U$ is small.

**Lemma 9.** Let $f_{\mu}$ be a generic one-parameter unfolding of the system $f_0$ with a generic heterodimensional cycle involving two saddles, such that $\theta(f_0) = p/q$ is rational. Assume condition (8) is satisfied. Let $(k_1, m_1), (k_2, m_2)$ be two different pairs of integers such that for some small $\mu$, for some points $M_1(X_1, Y_1, Z_1) \in \Pi$ and $M_2(X_2, Y_2, Z_2) \in \Pi$ we have $T_{k_1,m_1}(M_1) = (\bar{X}_1, \bar{Y}_1, \bar{Z}_1) =: \bar{M}_1 \in \Pi$ and $T_{k_2,m_2}(M_2) = (\bar{X}_2, \bar{Y}_2, \bar{Z}_2) =: \bar{M}_2 \in \Pi$. Then either $m_1 = m_2 = m$ and

$$ \lambda^{k_1,2} = O(\delta) \gamma^{-m}, \quad (106) $$

or $k_1 = k_2 = k$ and

$$ \gamma^{-m_1,2} = O(\delta) \lambda^{k}. \quad (107) $$

**Proof.** By (28), conditions $T_{k_1,m_1}(M_1) = \bar{M}_1$ and $T_{k_2,m_2}(M_2) = \bar{M}_2$ give

$$ \dot{\mu} + a\lambda^{k_1} X_1 + a\lambda^{k_1} x^+ - \gamma^{-m_1} u^- + \gamma^{-m_1} O((X_1)^2 + (Y_1)^2) + o(\lambda^{k_1}) + o(\gamma^{-m_1}) = \gamma^{-m_1} b^{-1}\bar{X}_1, $$

$$ \dot{\mu} + a\lambda^{k_2} X_2 + a\lambda^{k_2} x^+ - \gamma^{-m_2} u^- + \gamma^{-m_2} O((X_2)^2 + (Y_2)^2) + o(\lambda^{k_2}) + o(\gamma^{-m_2}) = \gamma^{-m_2} b^{-1}\bar{X}_2. \quad (108) $$

Conditions $M_{1,2} \in \Pi$ and $\bar{M}_{1,2} \in \Pi$ imply $|X_{1,2}| < \delta$, $|\dot{X}_{1,2}| < \delta$, $|\bar{Y}_{1,2}| < \delta$. Thus, it follows from (108) that the system

$$ \dot{\mu} + a\lambda^{k_1} (x^+ + K_1\delta) - \gamma^{-m_1} u^- = \gamma^{-m_1} b^{-1} C_1\delta, $$

$$ \dot{\mu} + a\lambda^{k_2} (x^+ + K_2\delta) - \gamma^{-m_2} u^- = \gamma^{-m_2} b^{-1} C_2\delta, \quad (109) $$

must have a solution $(\mu, k_1, m_1, K_1, C_1, k_2, m_2, K_2, C_2)$ with $|K_{1,2}| < 1$, $|C_{1,2}| < 1$. Subtracting the
Since and rewrite (second equation of (109)) from the first one, yields

\[ \lambda^k - \lambda x^+ + K^2 = \gamma^{-m_1} \frac{C_1 \delta + bu^-}{ab(x^+ + K^1 \delta)} - \gamma^{-m_2} \frac{C_2 \delta + bu^-}{ab(x^+ + K_1 \delta)}. \]  

(110)

Recall that \( \lambda, \gamma, a, b, x^+ \), and \( u^- \) depend on \( \mu \). Let us indicate their values at \( \mu = 0 \) by the subscript \( \mu \). Since the multipliers \( \lambda \) and \( \gamma \) depend smoothly on \( \mu \), we have

\[ \lambda^k = (\lambda_0 + O(\mu))^k = \lambda_0^k + O(k \lambda_0^k \mu), \quad \gamma^{-m} = (\gamma_0 + O(\mu))^{-m} = \gamma_0^{-m} + O(m \gamma_0^{-m} \mu). \]  

(111)

Using (19), (111), one can estimate \( \mu \) from (109) as

\[ \mu = O((\lambda_0)^{k_1} + |\gamma_0|^{-m_1}) \quad \text{and} \quad \mu = O((\lambda_0)^{k_2} + |\gamma_0|^{-m_2}). \]  

(112)

Substituting this into (111), and also using the continuous dependence of all the coefficients on \( \mu \), we rewrite (110) as

\[ \lambda_0^k - \lambda_0 x^+ + K_0 \delta = (1 + o(1)) = \gamma_0^{-m_1} \frac{C_1 \delta + bu^-}{a_0 b_0(x^+ + K_1 \delta)}(1 + o(1)) - \gamma_0^{-m_2} \frac{C_2 \delta + bu^-}{a_0 b_0(x_0^+ + K_1 \delta)}(1 + o(1)), \]

where \( o(1) \) denotes terms that tend to zero as \( k_1, 2 \to \infty, m_1, 2 \to \infty \).

It is obvious, that this equation is not solvable for sufficiently small \( \delta \) and sufficiently large \( k_1 \neq k_2 \) and \( m_1 \neq m_2 \), unless the quantity \( u^-/(a_0 x_0^+) \) is a limit point of the set

\[ \left\{ \frac{\lambda_0^k - \lambda_0}{\gamma_0^{-m_1} - \gamma_0^{-m_2}} : k_1, 2 \in \mathbb{N}, m_1, 2 \in \mathbb{N}, k_1 \neq k_2, m_1 \neq m_2 \right\}. \]  

(113)

Since

\[ |\lambda_0| = |\gamma_0|^{-\theta_0}, \]  

(114)

and \( \theta_0 \) is a rational number \( p/q \) (with \( p \) and \( q \) coprime), we obtain, assuming \( k_1 > k_2 \) and \( m_1 > m_2 \), that

\[ \left| \frac{\lambda_0^k - \lambda_0}{\gamma_0^{-m_1} - \gamma_0^{-m_2}} \right| = \left| \frac{\lambda_0^{k_1} - \lambda_0^{k_2}}{\gamma_0^{-m_1} - \gamma_0^{-m_2}} \right| = \left| \frac{\gamma_0^{-m_1} - \gamma_0^{-m_2}}{\gamma_0^{-m_1} - \gamma_0^{-m_2}} \right| = \left| \gamma_0 \right|^{-\frac{\theta_0}{1 - \gamma_0^{-m}}} \frac{1 - \lambda_0^{k_1} - \lambda_0^{k_2}}{1 - \gamma_0^{-m_1 - m_2}}. \]

This implies that the absolute values of the limit points of the set (113) form the set cl \( \left\{ \left| \gamma_0 \right|^{\frac{\theta_0}{1 - \gamma_0^{-m}}} \right\}_{s \in \mathbb{Z}, l \in \mathbb{N}, n \in \mathbb{N}} \) and, therefore, by (8), we find that \( u^-/(a_0 x_0^+) \) is not a limit point of the set (113).

This shows that (109) can have a solution (for sufficiently small \( \delta \) and large \( k_1, 2, m_1, 2 \)) only if \( k_1 = k_2 \) or \( m_1 = m_2 \). If \( m_1 = m_2 = m \), we have

\[ \lambda^k - \lambda x^+ + K_2 \delta = O(\delta) \gamma^{-m} \]

from (110). This implies (106) if \( k_1 \neq k_2 \). If \( m_1 \neq m_2 \) and \( k_1 = k_2 = k \), we have

\[ \gamma^{-m_1} \frac{C_1 \delta + bu^-}{ab(x^+ + K_1 \delta)} - \gamma^{-m_2} \frac{C_2 \delta + bu^-}{ab(x^+ + K_1 \delta)} = O(\delta) \gamma^k. \]
Note that this lemma can be extended also to the case of infinite \( k_{1,2} \) or \( m_{1,2} \). Indeed, it is easy to see that \( k_i = \infty \) corresponds to \( M_i \in W^{s}_{loc}(O_1) \) and \( \bar{M}_i \) being a point of intersection of \( W^{s}(L_1) \) with \( \Pi \); more specifically, \( \bar{M}_i \subset F_{21} \circ F^{m}_{2} \circ F_{12}(W^{s}_{loc}(O_1)) \). The case \( m_i = \infty \) corresponds to \( M_i \) being a point of intersection of \( W^{s}(L_2) \) with \( \Pi \), i.e., \( M_i \subset F^{-k_i}_{1} \circ F^{1}_{12}(W^{s}_{loc}(O_2)) \).

The case \( k_1 = m_1 = \infty \) corresponds to the orbit of the fragile heteroclinic intersection of \( F_{12}(W^{s}_{loc}(O_1)) \) with \( W^{s}_{loc}(O_2) \) (so we may think of \( M_i \in W^{s}_{loc}(O_1) \) and \( \bar{M}_i \in F_{21}(W^{s}_{loc}(O_2)) \)). Such intersection exists at \( \mu = 0 \). Note that in this case we have \( k_2 = m_2 = \infty \) in Lemma 9 (if we assume \( m_2 \neq m_1 \), then \( k_2 = k_1 = 1 \) and, by (107), we get \( \gamma^{-m_2} = 0 \), i.e., \( m_2 = \infty \), and if we assume \( k_2 \neq k_1 \), then \( m_2 = m_1 = \infty \) and, by (106), \( \lambda \gamma^2 = 0 \), i.e., \( k_2 = \infty \) too). This means that no other orbits in \( U \) can intersect \( \Pi \) in this case except for the orbits corresponding to the transverse intersection of \( F_{12}(W^{s}_{loc}(O_2)) \) with \( W^{s}_{loc}(O_1) \). This gives us the result of Theorem 8 at \( \mu = 0 \).

At \( \mu \neq 0 \), let there exist an orbit \( \mathcal{O} \) in \( U \), different from \( L_1, L_2 \) or heteroclinic orbits corresponding to the intersection of \( F_{21}(W^{s}_{loc}(O_2)) \) with \( W^{s}_{loc}(O_1) \) (we call \( \mathcal{O} \) a non-exceptional orbit). Then, \( \mathcal{O} \) intersects \( \Pi \) in a sequence of points \( M_s \) such that \( M_{s+1} = T_{k_s,m_s}M_s \). If this sequence is infinite, then \((k_s,m_s)\) are finite integers. If the sequence is finite from the left, then either the most left point \( M_{s_1} \in F_{21}(W^{s}_{loc}(O_2)) \cap \Pi \), or \( M_{s_1} \in F_{21} \circ F^{m}_{2} \circ F_{12}(W^{s}_{loc}(O_1)) \cap \Pi \) for some finite \( m \) – in this case we define \( k_{s_1-1} = \infty \), \( m_{s_1-1} = m \). Similarly, if this sequence is finite from the right, then either the most right point \( M_{s_r} \in W^{s}_{loc}(O_1) \cap \Pi \), or \( M_{s_r} \in F^{-k}_{1} \circ F^{1}_{12}(W^{s}_{loc}(O_2)) \cap \Pi \) for some finite \( k \) – in this case we define \( m_{s_r} = \infty \), \( k_{s_r} = k \).

By Lemma 9, we have three possibilities.

1. The first possibility is that all \((k_s,m_s) = (k,m)\) are finite and the same for all \( s \) and all non-exceptional orbits. In this case \( T_{k,m}(\Pi) \) intersects \( \Pi \), which implies, by the first equation of (28), that \( \mu = O((\lambda|k| + |\gamma|-m)) \). By (111), this implies \( \mu = O((\lambda|k| + |\gamma|^{-m})) \), hence

\[
\lambda^k \gamma^m = \lambda^k(1+O(k|\lambda|)+O(k|\gamma|^{-m})) \gamma^m((1+O(m|\gamma|^{-m})+O(m|\lambda|)) = \lambda^{k_m} \gamma^{m}(1+O(1,k_m \to \infty)) + o(1,k_m \to \infty).
\]

Thus, using (114), we obtain

\[
ab\lambda^k \gamma^m = ab\lambda^k \gamma^m(1+O(1,k_m \to \infty)) + o(1,k_m \to \infty) = ab\gamma^m \chi^{\frac{m_k-k_m}{\gamma}}(1+O(1,k_m \to \infty)) + o(1,k_m \to \infty).
\]

By (7), we obtain that \(|ab\lambda^k \gamma^m|\) stays bounded away from 1. Since this is, up to small corrections, the derivative \( d\bar{X}/dX \) in (28), and since we have a strong contraction in \( Z \) and a strong expansion in \( Y \), the hyperbolicity of \( T_{k,m} \) follows, if \( k, m \) are sufficiently large and the neighborhood \( U \) is sufficiently small.

By the hyperbolicity of \( T_{k,m} \), it can have only one fixed point and it is the only orbit of \( T_{k,m} \) that never leaves \( \Pi \). Thus, in the case under consideration, the orbit \( L \) of \( M_0 \) is a hyperbolic periodic orbit, and any other orbits in \( U \) must lie in the stable or unstable manifold of \( L_{1,2} \). This includes the orbits \( L_1 \) and \( L_2 \) themselves, as well as orbits of transverse intersection of \( W^{s}(L_2) \) with \( W^{s}(L_1) \) or \( W^{u}(L) \) and of \( W^{s}(L) \) with \( W^{u}(L_1) \). At the same time, no orbits from \( W^{s}(L_1) \setminus L_1 \) or \( W^{u}(L_2) \setminus L_2 \) can lie entirely in \( U \) in this case (as this would correspond to infinite \( k \) or \( m \)). This is in a complete
agreement with the statement of the theorem: if \( L \) is of index \( d_1 \), then the hyperbolic set \( \Lambda_1 \) from the formulation of the theorem is the union of \( L_1 \), \( L \), and the heteroclinic orbits corresponding to the intersection of \( W^u(L) \) with \( W^s(L_1) \); otherwise, we have the set \( \Lambda_2 \) comprised by \( L_2 \), \( L \), and the heteroclinic orbits corresponding to the intersection of \( W^u(L_2) \) with \( W^s(L) \).

- The second possibility is that all \( m_s = m \) are finite and the same for all \( s \) and all non-exceptional orbits, while some \( k_s \) are different. In this case, \( |ab\lambda^k\gamma^m| < 1 \) by (106), where \( k \) is the minimal value of \( k_s \) taken over all non-exceptional orbits. Thus, the derivative \( d\overline{X}/dX \) in (28) is small. This means that the maps \( T_{k_s,m_s} \) are all hyperbolic, with strong contraction in \((X, Z)\) and strong expansion in \( Y \), and the set of all non-exceptional orbits which are not in \( W^u(L_2) \) is uniformly-hyperbolic of index \( d_1 \). The orbits of this set can stay for no more than \( m \) consecutive iterations of the local map \( F_2 \) in a neighborhood of \( O_2 \), i.e., this set is at a non-zero distance from \( L_2 \). The union of this set and \( L_1 \) is the hyperbolic set \( \Lambda_1 \) from the formulation of the theorem. Note also that the finiteness of \( m \) implies that no orbit in \( W^s(L_2) \setminus L_2 \) can lie entirely in \( U \) in this case.

- The last possibility is that all \( k_s = k \) are finite and the same for all \( s \) and all non-exceptional orbits, while some \( m_s \) are different. In this case, \( |ab\lambda^k\gamma^m| > 1 \) by (107), where \( m \) is the minimal value of \( m_s \) taken over all non-exceptional orbits. Thus, the derivative \( d\overline{X}/dX \) in (28) is very large, so all the maps \( T_{k_s,m_s} \) are hyperbolic, with strong expansion in \((X, Y)\) and strong contraction in \( Z \). The set of all non-exceptional orbits which are not in \( W^s(L_1) \) is uniformly-hyperbolic of index \( d_2 \), and stays at a non-zero distance from \( L_1 \) (the union of this set with \( L_2 \) is the set \( \Lambda_2 \) from the formulation of the theorem). The finiteness of \( k \) implies that no orbit in \( W^u(L_1) \setminus L_1 \) can lie entirely in \( U \) in this case.

In all three cases we have a complete agreement with the statement of the theorem.

\( \square \)

4 The case of complex multipliers

Below, we consider the case where at least one of the central multipliers \( \lambda_{1,1} \) and \( \gamma_{2,1} \) is complex, and prove Theorem 9. We study the corresponding first return maps and reduce them to the form given by Proposition 3, and, therefore, obtain blenders. With the rotation brought by the complex multipliers, we can then easily stabilize the heterodimensional cycles using versions of Lemmas 6 and 7.

Let us start with the saddle-focus case. We assume that

\[
\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega}, \quad \omega \in (0, \pi), \quad \text{and} \quad \gamma := \gamma_{2,1} \quad \text{is real},
\]

where \( \lambda > |\lambda_{1,3}| \) and \( |\gamma_{2,1}| < |\gamma_{2,2}| \). As mentioned in the introduction, the other case (where \( \lambda_{1,1} \) is real while \( \gamma_{2,1} \) is complex) can be reduced to this one by the time reversal.

We use the same coordinates near \( O_2 \) as in the saddle case so that the formulas for the local map \( F_2 \) remains the same (see (15)). Let us now introduce coordinates \((x_1, x_2, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1-2}\)
near $O_1$ such that the local map $F_1$ takes the form (see e.g. [23])

$$\begin{align*}
\bar{x}_1 &= \lambda x_1 \cos k\omega + \lambda x_2 \sin k\omega + g_1(x_1, x_2, y, z), \\
\bar{x}_2 &= -\lambda x_1 \sin k\omega + \lambda x_2 \cos k\omega + g_2(x_1, x_2, y, z), \\
\bar{y} &= P_1 y + g_3(x_1, x_2, y, z), \\
\bar{z} &= P_2 z + g_4(x_1, x_2, y, z),
\end{align*}$$

(115)

where we do not indicate the dependence on parameters for simplicity. The eigenvalues of the matrices $P_1$ and $P_2$ are $\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{1,d_1}$ and $\lambda_1, \ldots, \lambda_{d-d_1}$, respectively. The functions $g$ vanish along their first derivatives at the origin, and satisfy

$$g_{1,2,4}(0, 0, y, 0) = 0, \quad g_3(x_1, x_2, 0, z) = 0, \quad g_{1,2}(x_1, x_2, 0, z) = 0, \quad \frac{\partial g_{1,2,4}}{\partial (x_1, x_2)}(0, 0, y, 0) = 0,$$

for all sufficiently small $x_1, x_2, y$ and $z$. Similar to the saddle case, in these coordinates the local manifolds $W^r_{\text{loc}}(O_1)$ and $W^u_{\text{loc}}(O_1)$ are given by $\{y = 0\}$ and $\{x_1 = 0, x_2 = 0, z = 0\}$; the leaves of the strong-stable foliation have the form $\{(x_1, x_2) = \text{const}, y = 0\}$; the restriction of the map to $W^s_{\text{loc}}(O_1)$ is linear in $x$. This is the same coordinate system as described in Section 2.

By Lemma 7 of [23], for any point $(x_1, x_2, y, z)$ in $U_{01}$, we have $(\bar{x}_1, \bar{x}_2, \bar{y}, \bar{z}) = F_1^k(x_1, x_2, y, z)$ if and only if

$$\begin{align*}
\bar{x}_1 &= \lambda^k x_1 \cos k\omega + \lambda^k x_2 \sin k\omega + p_1(x_1, x_2, \bar{y}, \bar{z}), \\
\bar{x}_2 &= -\lambda^k x_1 \sin k\omega + \lambda^k x_2 \cos k\omega + p_2(x_1, x_2, \bar{y}, \bar{z}), \\
y &= p_3(x_1, x_2, \bar{y}, \bar{z}), \\
\bar{z} &= p_4(x_1, x_2, \bar{y}, \bar{z}),
\end{align*}$$

(116)

where

$$||p_{1,2,4}||_{C^1} = o(\lambda^k), \quad ||p_3||_{C^1} = o(\gamma^{-k}),$$

for some constant $\gamma \in (1, |\gamma_{1,1}|)$, and these estimates are uniform for all systems $C^2$-close to $f$.

Let us now define the transition maps $F_{12}$ and $F_{21}$ from a neighborhood of $M^-_1 = (0, 0, y^-, 0)$ to a neighborhood of $M^+_2 = (0, v^+, 0)$ and, respectively, from a neighborhood of $M^+_2 = (u^-, 0, w^-)$ to a neighborhood of $M^-_1 = (x^+, x^+, x^+, 0, z^+)$. Arguing like in the saddle case, by the first part of condition GC1 (that $F_{12}^{-1}(W^E_{\text{loc}}(O_2)) \cap W^u_{\text{loc}}(O_1)$ at $M^-_1$), the transition map $F_{12} : (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{z}) \mapsto (u, v, w)$ can be written as follows (the dots refer to the second and higher order terms in the Taylor expansion):

$$\begin{align*}
u &= \hat{\mu} + a_{12} x_1 + a_{12} \bar{x}_2 + a_{13} \bar{z} + a_{14} w + \ldots, \\
v - v^+ &= a_{21} \bar{x}_1 + a_{22} \bar{x}_2 + a_{23} \bar{z} + a_{24} w + \ldots, \\
\bar{y} - y^- &= a_{31} \bar{x}_1 + a_{32} \bar{x}_2 + a_{33} \bar{z} + a_{34} w + \ldots, \\
w &= a_{41} \bar{x}_1 + a_{42} \bar{x}_2 + a_{43} \bar{z} + a_{44} w + \ldots, \quad (117)
\end{align*}$$

where the relation between $\hat{\mu}$ and $\mu$ is the same as in the saddle case and is given by (19). The second part of GC1 (that $F_{12}(W^u_{\text{loc}}(O_1)) \cap W^s_{\text{loc}}(O_2)$ at $M^+_2$) implies that

$$a_{11}^2 + a_{12}^2 \neq 0. \quad (118)$$

Similarly, the condition GC2 shows that the other transition map $F_{21} : (\bar{u}, \bar{v}, \bar{w}) \mapsto (x_1, x_2, y, z)$
can be written as
\begin{align}
  x_1 - x_1^+ &= b_{11}(\hat{\mu} - u^-) + b_{12} \hat{v} + b_{13} y + \ldots, \\
  x_2 - x_2^+ &= b_{21}(\hat{\mu} - u^-) + b_{22} \hat{v} + b_{23} y + \ldots, \\
  \hat{w} - w^- &= b_{31}(\hat{\mu} - u^-) + b_{32} \hat{v} + b_{33} y + \ldots, \\
  z - z^+ &= b_{41}(\hat{\mu} - u^-) + b_{42} \hat{v} + b_{43} y + \ldots,
\end{align}

with $b_{11}^2 + b_{21}^2 \neq 0$. By rotating the $x$-coordinates to a small angle, if necessary, we can assume
\begin{equation}
  b_{11} \neq 0.
\end{equation}

Thus, the above formula can be rewritten as
\begin{align}
  \hat{u} - u^- &= b_{11}^{-1}(x_1 - x_1^+ - b_{13} y) + O(\|\hat{v}\| + (x_1 - x_1^+)^2 + y^2), \\
  x_2 - x_2^+ &= b_{21}^{-1} b_{11}^{-1} (x - x^+ - b_{13} y) + b_{23} y + O(\|\hat{v}\| + (x_1 - x_1^+)^2 + y^2), \\
  \hat{w} - w^- &= O(|x_1 - x_1^+| + \|\hat{v}\| + \|y\|), \\
  z - z^+ &= b_{41}^{-1} b_{11}^{-1} (x - x^+ - b_{13} y) + b_{43} y + O(\|\hat{v}\| + (x_1 - x_1^+)^2 + y^2). 
\end{align}

Now we can find a formula for the first-return map in the same way as in the saddle case. Namely, combining (116) and (117) yields the analogue of (23):
\begin{align}
  u &= \hat{\mu} + a_{11} \lambda^k x_1 \cos k\omega + a_{11} \lambda^k x_2 \sin k\omega - a_{12} \lambda^k x_1 \sin k\omega + a_{12} \lambda^k x_2 \cos k\omega + \tilde{h}_1(x_1, x_2, z, w) \\
  y &= \tilde{h}_2(x_1, x_2, z, w), \quad v - v^+ = \tilde{h}_3(x_1, x_2, z, w),
\end{align}

where
\begin{align}
  \tilde{h}_1(x_1, x_2, z, w) &= O(\|w\|) + o(\lambda^k), \quad \tilde{h}_2(x_1, x_2, z, w) = o(\gamma^{-k}), \quad \tilde{h}_3(x_1, x_2, z, w) = O(\|w\| + |\lambda|^k).
\end{align}

Combining (15) and (121) yields the analogue of (25)
\begin{align}
  u &= \gamma^{-m} (u^- + b_{11}^{-1}(x_1 - x_1^+ - b_{13} y + \hat{h}_0_1(x_1 - x_1^+, y))) + \hat{h}_1(x_1, v, y), \\
  x_2 - x_2^+ &= b_{21} b_{11}^{-1} (x - x^+ - b_{13} y) + b_{23} y + \hat{h}_0_2(x_1 - x_1^+, y) + \hat{h}_2(x_1, v, y), \\
  z - z^+ &= b_{41} b_{11}^{-1} (x - x^+ - b_{13} y) + b_{43} y + \hat{h}_0_3(x_1 - x_1^+, y) + \hat{h}_3(x_1, v, y), \\
  w &= \hat{h}_4(x_1, v, y),
\end{align}

where
\begin{align}
  \hat{h}_0_i(x_1 - x_1^+, \hat{y}) &= O((x_1 - x_1^+)^2 + y^2) \quad (i = 1, 2, 3), \quad \hat{h}_1_4(x_1, v, y) = o(\gamma^{-m}), \quad \hat{h}_2_3(x_1, v, y) = o(\lambda^m).
\end{align}

Finally, combining (122) and (123), we obtain the following form for the first return map $T_{k,m} := \ldots$
\[ F_{21} \circ F_{2}^m \circ F_{12} \circ F_{1}^{k} : (x_1, x_2, y, z) \mapsto (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{z}): \]

\[
\begin{align*}
\bar{x}_1 - x_1^+ &= b_{11} \gamma^m \mu + b_{11} \lambda^k \gamma^m \left((a_{11} x_1 + a_{12} x_2) \cos k \omega + (a_{11} x_2 - a_{12} x_1) \sin k \omega - u^- + b_{13} \bar{y}\right) \\
&\quad - \hat{h}_{01}(\bar{x}_1 - x_1^+, \bar{y}) + \gamma^m h_1(x_1, \bar{x}_1, x_2, \bar{y}, z), \\
\bar{x}_2 - x_2^+ &= b_{21} b_{11}^{-1}(x - x^+ - b_{13} \bar{y}) + b_{23} \bar{y} + \hat{h}_{02}(x_1 - x_1^+, y) + h_2(x_1, \bar{x}_1, x_2, \bar{y}, z), \\
\bar{z} - z^+ &= b_{41} b_{11}^{-1}(x - x^+ - b_{13} \bar{y}) + b_{43} \bar{y} + \hat{h}_{03}(\bar{x}_1 - x_1^+, \bar{y}) + h_3(x_1, \bar{x}_1, x_2, \bar{y}, z) \\
y &= h_4(x_1, \bar{x}_1, x_2, \bar{y}, z),
\end{align*}
\]

(124)

where

\[ h_{1,2} = o(\lambda^k) + o(\gamma^{-m}), \quad h_3 = o(\hat{\gamma}^{-k}), \quad h_4 = o(\hat{\lambda}^m). \]

We further do computations with the first-return map \( T_{k,m} \) only at \( \mu = 0 \), so we will omit the term \( b \gamma^m \mu \) in formula (124). Make the coordinate transformation

\[
\begin{align*}
X_1 &= x_1 - x^+ - b_{13} y, & Y &= y, \\
X_2 &= \delta^{-\frac{1}{2}} \left(x_2 - x_2^+ - b_{21} b_{11}^{-1}(x - x^+ - b_{13} y) - b_{23} y - h_{02}(x_1 - x_1^+, y)\right), \\
Z &= z - z^+ - b_{41} b_{11}^{-1}(x - x^+ - b_{13} y) - b_{43} y - h_{03}(x_1 - x_1^+, y).
\end{align*}
\]

(125)

The first-return map acquires the form

\[
\begin{align*}
\hat{X}_1 &= A_{k,m} X_1 + B_{k,m} + \delta^{\frac{1}{2}} \hat{O}(\lambda^k \gamma^m) X_2 + \hat{\phi}_{01}(\hat{X}_1, \hat{Y}) + \gamma^m \hat{\phi}_1(X_1, \hat{X}_1, X_2, \hat{Y}, Z), \\
\hat{X}_2 &= \delta^{\frac{1}{2}} \hat{\phi}_2(X_1, \hat{X}_1, X_2, \hat{Y}, Z), \\
\hat{Y} &= \hat{\phi}_3(X_1, \hat{X}_1, X_2, \hat{Y}, Z), \\
\hat{Z} &= \hat{\phi}_4(X_1, \hat{X}_1, X_2, \hat{Y}, Z),
\end{align*}
\]

(126)

where

\[ \hat{\phi}_{0i} = O(\hat{X}_1^2 + \hat{Y}^2) \quad (i = 1, 2), \quad \| \hat{\phi}_1 \|_{C^1} = o(\lambda^k) + o(\gamma^{-m}), \quad \| \hat{\phi}_{2,4} \|_{C^1} = o(\hat{\lambda}^m), \quad \| \hat{\phi}_3 \|_{C^1} = o(\hat{\gamma}^{-k}), \]

(127)

and

\[
\begin{align*}
A_{k,m} &= \lambda^k \gamma^m \left((a_{11} b_{21} - a_{12} b_{11}) \sin k \omega + (a_{11} b_{11} + a_{12} b_{21}) \cos k \omega\right) = \lambda^k \gamma^m A \sin(k \omega + \eta_1), \\
B_{k,m} &= \lambda^k \gamma^m b_{11}(a_{11} x_2^+ - a_{12} x_1^+) \sin k \omega + b_{11} (a x_1^+ + a_{12} x_2^+) \cos k \omega - u^- \\
&= \lambda^k \gamma^m B \sin(k \omega + \eta_2) - b_{11} u^-.
\end{align*}
\]

(128)

Note the factor \( \delta^{\frac{1}{2}} \) in front of \( X_2 \) in (126) which appears because we scale \( x_2 \) to \( \delta^{\frac{1}{2}} \) in (125). Note also that the coefficients

\[ A = \sqrt{(a_{11}^2 + a_{12}^2)(b_{11}^2 + b_{21}^2)}, \quad B = |b_{11}| \| x^+ \| \sqrt{a_{11}^2 + a_{12}^2} \]

are non-zero by (118), (120) and because \( x^+ \neq 0 \) due to the genericity condition GC3. The phases \( \eta_{1,2} \) are defined by

\[ A \sin \eta_1 = a_{11} b_{11} + a_{12} b_{21}, \quad A \cos \eta_1 = a_{11} b_{21} - a_{12} b_{11}, \quad \frac{B}{b_{11}} \sin \eta_2 = a_{11} x_1^+ + a_{12} x_2^+, \quad \frac{B}{b_{11}} \cos \eta_2 = a_{11} x_2^+ - a_{12} x_1^+, \]

54
which implies, because \( b_{11}/b_{21} \neq x_1^+/x_2^+ \) due to the genericity condition \( \textbf{GC4.2} \) (see (119)), that
\[
\tan \eta_1 \neq \tan \eta_2. \tag{129}
\]

Similarly to the saddle case, we consider pairs \((k,m)\) such that the maps \( T_{k,m} \) take
\[
\Pi = [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_1} \times [-\delta, \delta]^{d-d_1-1} \tag{130}
\]
into itself, which implies that
\[
\lambda^k \gamma^m B \sin(k \omega + \eta_2) = b_{11} u^- + O(\delta). \tag{131}
\]
We will consider only such \( k \) for which \( \sin(k \omega + \eta_2) \) stays bounded away from 0. This, along with (131), implies that \( \lambda^k \gamma^m \) is uniformly bounded. In particular, the term \( \gamma^m \hat{\phi}_1 \) in (126) tends to zero as \( k, m \to \infty \). This allows to express \( \bar{X}_1 \) as a function of \((X_1, X_2, Z, \bar{Y})\) from the first equation of (126), and thus get rid of the dependence on \( \bar{X}_1 \) in the right-hand side of (126). Thus, we rewrite formula (126) for \( T_{k,m} \) as
\[
\bar{X}_1 = A_{k,m} X_1 + B_{k,m} + \phi_1(X_1, X_2, Z, \bar{Y}),
\bar{X}_2 = \phi_2(X_1, X_2, Z, \bar{Y}), \quad \bar{Z} = \phi_4(X_1, X_2, Z, \bar{Y}),
Y = \phi_3(X_1, X_2, Z, \bar{Y}), \tag{132}
\]
where
\[
\phi_1 = O(\delta^\frac{3}{2}) + o(1)_{k,m \to \infty}, \quad \frac{\partial \phi_1}{\partial (X_1, X_2, Y, Z)} = O(\delta^\frac{3}{2}) + o(1)_{k,m \to \infty},
\|
\phi_2, 4\|_{C^1} = o(\lambda^m), \quad \|
\phi_3\|_{C^1} = o(\gamma^k). \tag{133}
\]
We have such estimates for \( \phi_2 \) because in the \( \bar{X}_2 \)-equation in (126) the coefficient \( \delta^{-\frac{3}{2}} \) can be absorbed by \( \hat{\phi}_2 \). Indeed, we always first take \( \delta \) sufficiently small and then take the limit of \( k, m \) sufficiently large.

Formula (132) represents the first-return maps in the form used in Proposition 3 (where one should choose \((X_2, Z)\) as a new \( Z \)-variable). We check in Section 4.1 that these maps indeed satisfy conditions of this proposition, thus establishing the existence of the blenders. Before doing that, we further restrict the choice of \((k,m)\) by the requirement that \( \sin(k \omega + \eta_1) \) stays bounded away from zero. Then the constants \( A_{k,m} \) in (132) stay bounded away from zero, and we notice that the map (132) assumes, upon setting \( X = X_1 \) and \( Z^{\text{new}} = (X_2, Z) \), the same form as (34) (with the only difference being a slightly worse estimate for \( \phi_1 \), which does not affect the further results). This gives us the following analogue of Lemma 1 for the saddle-focus case:

**Lemma 10.** Let \( \mu = 0 \). Given any \( K > 0 \), one can choose \( \delta \) sufficiently small, such that for all sufficiently large \((k,m)\) such that \( \sin(k \omega + \eta_1) \) and \( \sin(k \omega + \eta_2) \) stay bounded away from zero, the cone fields on \( \Pi \)
\[
C^{\text{cu}} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \| (\Delta X_2, \Delta Z) \| \leq K (|\Delta X_1| + \| \Delta Y \|) \}, \tag{134}
C^{\text{uu}} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \max \{|\Delta X_1|, \| (\Delta X_2, \Delta Z) \| \} \leq K \| \Delta Y \| \}, \tag{135}
\]
are forward-invariant in the sense that if a point \( M \in \Pi \) has its image \( \hat{M} = T_{k,m}(M) \) in \( \Pi \), then the cone at \( M \) is mapped into the cone at \( \hat{M} \) by \( DT_{k,m} \); the cone fields
\[
C^{\text{cs}} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \| \Delta Y \| \leq K (|\Delta X_1| + \| (\Delta X_2, \Delta Z) \|) \}, \tag{136}
\]
\[ C^{ss} = \{ (\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \max\{|\Delta X_1|, \|\Delta Y\|\} \leq K\|\Delta X_2, \Delta Z\| \}, \]  

are backward-invariant in the sense that if a point \( \bar{M} \in \Pi \) has its pre-image \( M = T_{-1}^{k,m}(\bar{M}) \) in \( \Pi \), then the cone at \( \bar{M} \) is mapped into the cone at \( M \) by \( DT_{-1}^{k,m} \). Moreover, vectors in \( C^{cu} \) and, if \( |A_{k,m}| > 1 \), also in \( C^{cu} \) are expanded by \( DT^{k,m} \); vectors in \( C^{ss} \) and, if \( |A_{k,m}| < 1 \), also in \( C^{cs} \) are contracted by \( DT^{k,m} \).

### 4.1 Co-existence of mutually activating blenders

Here we prove the first part of Theorem 9 for the saddle-focus case.

**Proposition 5.** If \( \theta, \omega/\pi \) and 1 are rationally independent, then, at \( \mu = 0 \), there exist, arbitrarily close to the heterodimensional cycle \( \Gamma \), a cu-blender \( \Lambda^{cu} \) with an activating pair \( (\Pi^{\prime}, C^{ss}) \) and a cs-blender \( \Lambda^{cs} \) with an activating pair \( (\Pi^{\prime}, C^{uu}) \) such that \( W^{s}_{loc}(\Lambda^{cs}) \) activates \( \Lambda^{cu} \) and \( W^{u}_{loc}(\Lambda^{cu}) \) activates \( \Lambda^{cs} \), while \( W^{u}_{loc}(\Lambda^{cs}) \) intersects \( W^{s}_{loc}(\Lambda^{cu}) \) transversely. Here

\[ \Pi' = [-q\delta, q\delta] \times [-\delta, \delta] \times [-\delta, \delta]^d_1 \times [-\delta, \delta]^{d_1-d_2} \]  

(138)

for some fixed \( q \in (0, 1) \), and \( C^{ss}, C^{uu} \) are given by Lemma 10 with some sufficiently small \( K \).

**Proof.** The rational independence condition implies that the set

\[ \{(k\theta + m, k\frac{\omega}{2\pi} - p)\}_{k,m \in \mathbb{N}, p \in \mathbb{Z}} \]  

(139)

is dense in \( \mathbb{R}^2 \). So, given any \((s, t) \in \mathbb{R}^2\), one can find a sequence \( \{(k_n, m_n, p_n)\} \) with \( k_n, m_n \to \infty \) such that \( m_n \) are even and

\[ -k_n\theta + m_n \to t \quad \text{and} \quad k_n\frac{\omega}{2\pi} - p_n \to s. \]  

(140)

In fact, we take a sufficiently large fixed value of \( t \) and let \( s \) depend on \( n \) so that

\[ k_n\frac{\omega}{2\pi} - p_n - s_n \to 0 \]

and the sequence \( s_n \) is dense in a sufficiently small closed interval \( \Delta \) such that the sequence

\[ |\gamma|^t B \sin(2\pi s_n + \eta_2) - b_{11} u^- \]

is dense in a small interval around zero. Moreover, by (129), we can always choose \( \Delta \) such that

\[ 0 < L_1 < \frac{|bu^- A \sin(2\pi s + \eta_1)|}{B \sin(2\pi s + \eta_2)} < L_2 < 1, \]  

(141)

for all \( s \in \Delta \) with some constants \( L_1, L_2 \), and such that

\[ 1 < L_3 < \frac{|bu^- A \sin(2\pi s + \eta_1)|}{B \sin(2\pi s + \eta_2)} < L_4 < \infty \]  

(142)

for all \( s \in \Delta \) with some constants \( L_3, L_4 \).
For the corresponding sequences \( \{(k_n, m_n, p_n)\} \), the values of \( \sin(2\pi k_n + \eta_1) \) and \( \sin(2\pi k_n + \eta_2) \) get bounded away from zero for all sufficiently large \( n \). Since \( m_n \) are even, we have \( \gamma^{m_n} = |\gamma|^{m_n} \) and, since \( \lambda > 0 \), we have \( \lambda^{k_n \gamma^{m_n}} = |\gamma|^{m_n - k_n \delta} \to |\gamma|^\ell \). This is a finite number, hence the first-return maps \( T_n := T_{k_n, m_n} \) can be represented in the form (132), and the cone lemma (Lemma 10) holds.

The coefficients \( B_{k_n, m_n} \) given by (128) are dense in a small interval around zero, and, by passing to a subsequence if necessary, we have that \( B_{k_n, m_n} \) lie in this interval for all \( n \). Since by (128) one can write

\[
A_{k_n, m_n} = \frac{(B_{k_n, m_n} + b_1 u^-) A \sin(k \omega + \eta_1)}{B \sin(k \omega + \eta_2)},
\]

inequalities (141) and (142) imply that the coefficients \( A_{k_n, m_n} \) stay bounded away from 0, 1 and infinity, and, depending on the choice of the interval \( \Delta \), remain all smaller than 1 in the absolute value or all larger than 1 in the absolute value.

Then, by setting \( X = X_1 \) and \( Z^{new} = (X_2, Z) \), one immediately sees that conditions of Proposition 3 are satisfied by the maps \( T_n \), which gives us both the cs- and cu-blender (by appropriate choices of the interval \( \Delta \)).

What remains is to show the homoclinic relation between the two blenders. Similarly to Remark 5, for every point \( M_1 \in \Lambda^{cu} \), we define its local stable manifold as a connected piece of \( W^s(\Lambda^{cu}) \cap \Pi \) through \( M_1 \), and the local unstable manifold as a connected piece of \( W^u(\Lambda^{cu}) \cap \Pi \) through \( M_1 \). By Lemma 10, the tangent space of \( W^s_{loc}(M_1) \) at any point lies in the stable cone \( C^s \), so \( W^s_{loc}(M_1) \) is given by an equation \( Y = \xi^s_{M_1}(X_1, X_2, Z) \) where \( \xi^s_{M_1} \) has its derivative small and is defined on \( [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_1 - 2} \). Similarly, the manifold \( W^{cs}_{loc}(M_1) \) is the graph of \( (X_1, X_2, Z) = \xi^{cs}_M(Y) \) defined for \( Y \in [-\delta, \delta]^{d_1} \) and the tangent space of \( W^{cs}_{loc}(M_1) \) at any point lies in \( C^{cs} \). Hence, \( W^{cs}_{loc}(M_1) \) crosses \( \Pi' \) properly with respect to \( C^{cs} \) (see Definition 6), i.e., it activates \( \Lambda^{cs} \).

Similarly, for every point \( M_2 \in \Lambda^{cs} \), the manifold \( W^{cs}_{loc}(M_2) \) is the graph of \( (X_1, Y) = \xi^{cs}_{M_2}(X_1, Y) \) defined for \( (X_1, Y) \in [-\delta, \delta] \times [-\delta, \delta]^{d_1} \) and its tangents lie in \( C^{cs} \). It immediately follows that it has a non-empty transverse intersection with \( W^{cs}_{loc}(M_1) \) for any point \( M_1 \in \Lambda^{cs} \). The manifold \( W^{cs}_{loc}(M_2) \) is the graph of \( (X_1, Y) = \xi^{cs}_M(X_2, Z) \) defined for \( (X_2, Z) \in [\delta, \delta] \times [-\delta, \delta]^{d - d_1 - 2} \), and its tangents lie in \( C^{cs} \). Hence, it crosses \( \Pi' \) properly with respect to \( C^{cs} \), i.e., it activates \( \Lambda^{cu} \).

**Remark 9.** It follows from the arguments at the end of this proof that \( W^s(\Lambda^{cu}) \) intersects transversely \( W^s_{loc}(O_1) \) (i.e., the manifold \( \{Y = 0\} \)), while \( W^s(\Lambda^{cu}) \) intersects transversely any manifold which crosses \( \Pi \) properly with respect to \( C^{cu} \). Similarly, \( W^s(\Lambda^{cs}) \) intersects transversely \( F_{21}(W^{cs}_{loc}(O_2)) \) (i.e., the manifold \( \{X_2 = 0, Z = 0\} \) obtained by taking \( m \to \infty \) in (123)), and \( W^u(\Lambda^{cs}) \) intersects transversely any manifold which crosses \( \Pi \) properly with respect to \( C^{cs} \).

### 4.2 Local stabilization of the heterodimensional cycle in the saddle-focus case

In this section we prove the second part of Theorem 9 for the saddle-focus case. As in the saddle case, we investigate the iterates of the local invariant manifolds \( W^{cs}_{loc}(O_1) \) and \( W^{cs}_{loc}(O_2) \).

**Lemma 11.** Let \( q, \Pi', C^{cu} \) be given by Proposition 5. Define the intervals

\[
I^u_m = \left( \gamma^{-m} u^- - \frac{1}{2} |b^{-1} \gamma^{-m}| q \delta, \gamma^{-m} u^- + \frac{1}{2} |b^{-1} \gamma^{-m}| q \delta \right).
\]
Take sufficiently small $\delta$. Then, for every sufficiently large $m$, if $\mu \in I_m^u$, then the image $S_m^u := F_{21} \circ F_2 \circ F_{12}(W_{loc}^u(O_1))$ is a surface of the form $(X_1, X_2, Z) = s(Y)$ for some smooth function $s$. The surface $S_m^u$ crosses the cube $\Pi'$ properly with respect to the cone field $C^{uu}$.

In particular, $S_m^u$ intersects $W_{loc}^s(O_1) : \{Y = 0\}$ transversely, i.e., $O_1$ has a transverse homoclinic orbit.

**Proof.** Since $W_{loc}^u(O_1)$ near $M_-^u$ has equation $(\tilde{x}, \tilde{z}) = 0$, it follows from formulas (117), (15) and (119), that the image $S_m^u$ is given by

\[
x_1 - x_1^+ = b_{11} \gamma^m \hat{\mu} - b_{11} u^- + b_{13} y + O((\gamma^m \hat{\mu} - u^-)^2 + y^2) + o(1)_{m \to \infty},
\]

\[
x_2 - x_2^+ = b_{21} \gamma^m \hat{\mu} - b_{21} u^- + b_{23} y + O((\gamma^m \hat{\mu} - u^-)^2 + y^2) + o(1)_{m \to \infty},
\]

\[
z - z^+ = b_{41} \gamma^m \hat{\mu} - b_{41} u^- + b_{43} y + O((\gamma^m \hat{\mu} - u^-)^2 + y^2) + o(1)_{m \to \infty},
\]

which, after the transformation (125), recasts as

\[
X_1 = b_{11} \gamma^m \hat{\mu} - b u^- + O((\gamma^m \hat{\mu} - u^-)^2 + Y^2) + o(1)_{m \to \infty},
\]

\[
X_2 = -b\frac{1}{2}O((\gamma^m \hat{\mu} - u^-)^2 + Y^2) + \delta^{-\frac{1}{2}} o(1)_{m \to \infty},
\]

\[
Z = O((\gamma^m \hat{\mu} - u^-)^2 + Y^2) + o(1)_{m \to \infty}
\]

(144)

By (143) we have

\[|b \gamma^m \hat{\mu} - b u^-| < \frac{q\delta}{2},\]

which for $Y \in [-\delta, \delta]^d$ implies

\[|X_1| < \frac{q\delta}{2} + O(\delta^2) + o(1)_{m \to \infty} < q\delta \quad \text{and} \quad \|(X_2, Z)\| = O(\delta^2) + \delta^{-\frac{1}{2}} o(1)_{m \to \infty} < \delta,
\]

where we first take $\delta$ sufficiently small and then take $m$ sufficiently large (and we do the same whenever terms like $\delta^{-\frac{1}{2}} o(1)_{m \to \infty}$ appear). This means that $S_m^u$ crosses $\Pi'$. One also finds from (144) that

\[
\frac{\partial(X_1, X_2, Z)}{\partial Y} = O(\delta^2) + \delta^{-\frac{1}{2}} o(1)_{m \to \infty},
\]

which can be made sufficiently small so that the tangent spaces of $S_m^u \cap \Pi$ lie in $C^{uu}$. So, the crossing is also proper with respect to $C^{uu}$. \qed

**Lemma 12.** Let $\Pi'$ and $C^{ss}$ be given by Proposition 5. There exists a sequence $\{j\} \to \infty$ such that, at $\mu = 0$, the pre-image $S_j^s := F_1^{-k_j} \circ F_{12}^{-1}(W_{loc}^s(O_2))$ is a surface of the form $(X_1, Y) = s(X_2, Z)$ for some smooth function $s$. The surface $S_j^s$ crosses $\Pi'$ properly with respect to the cone field $C^{ss}$.

In particular, $S_j^s$ intersects $F_{21}(W_{loc}^u(O_2)) : \{X_2 = 0, Z = 0\}$ transversely, i.e., $O_2$ has a transverse homoclinic orbit.

**Proof.** Since $W_{loc}^u(O_2)$ near the point $M_2^+$ is given by $(u, w) = 0$, it follows from formula (117) that the pre-image $F_{12}^{-1}(W_{loc}^u(O_2))$ at $\mu = 0$ is given by

\[0 = a_{11} x_1 + a_{12} x_2 + a_{13} z + O(x_1^2 + x_2^2 + z^2),\]

58
implies that \( \Lambda \) for the saddle-focus case. Let \( \Lambda \) such that the local map

\[
F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

where the eigenvalues of the matrices \( A \) and \( B \) are \( \lambda > 0 \) and \( \mu < 0 \), respectively.

4.3 The double-focus case

We finish the paper by considering the case \( \lambda = \lambda^* = \lambda e^{i\omega_1}, \omega_1 \in (0, \pi) \) and \( \gamma = \gamma^* = \gamma e^{i\omega_2}, \omega_2 \in (0, \pi) \), where \( \lambda > |\lambda_{1,3}|, \gamma < |\gamma_{2,3}| \). We no longer need to split the heterodimensional cycle to get robust heterodimensional dynamics, so below we write formulas only for the unperturbed system (e.g. \( \mu = 0 \)).

We use the same coordinates near \( O_1 \) as in the saddle-focus case, so the local map \( F_1 \) is given by (115) with replacing \( \omega \) by \( \omega_1 \). Near \( O_2 \) we introduce coordinates \((u_1, u_2, v, w) \in \mathbb{R} \times \mathbb{R}^{d-d_1-1} \times \mathbb{R}^{d_1-1}\) such that the local map \( F_2 \) assumes the form

\[
\begin{align*}
\tilde{u}_1 &= \gamma u_1 \cos k \omega_2 + \gamma u_2 \sin k \omega_2 + \tilde{g}_1(u_1, u_2, v, w), \\
\tilde{u}_2 &= -\gamma u_1 \sin k \omega_2 + \gamma u_2 \cos k \omega_2 + \tilde{g}_2(u_1, u_2, v, w), \\
\tilde{v} &= Q_1 v + \tilde{g}_3(u_1, u_2, v, w), \\
\tilde{w} &= Q_2 w + \tilde{g}_4(u_1, u_2, v, w),
\end{align*}
\]

where the eigenvalues of the matrices \( Q_1 \) and \( Q_2 \) are \( \lambda_{2,1}, \ldots, \lambda_{2,d-d_1-1} \) and \( \gamma_{2,3} \ldots \gamma_{1,d_1+1} \), respectively.
Here the functions \( \hat{g} \) satisfy

\[
\begin{align*}
\hat{g}_{1,2,4}(0,0,v,0) &= 0, \\
\hat{g}_3(u_1,u_2,0,w) &= 0, \\
\hat{g}_{1,2}(u_1,u_2,0,w) &= 0, \\
\frac{\partial \hat{g}_{1,2,4}}{\partial (u_1,u_2)}(0,0,v,0) &= 0
\end{align*}
\]

for all sufficiently small \( u_1,u_2,v \) and \( w \). Similar to the saddle case, in these coordinates the local manifolds \( W^u_{\text{loc}}(O_2) \) and \( W^s_{\text{loc}}(O_2) \) are given by \( \{ v = 0 \} \) and \( \{ u_1 = 0,u_2 = 0,w = 0 \} \); the leaves of the strong-unstable foliation have the form \( \{ (u_1,u_2) = \text{const}, v = 0 \} \); and the restriction of the map to \( W^u_{\text{loc}}(O_2) \) is linear in \( u \). This is the same coordinate system as discussed in Section 2.

By Lemma 7 of [23], the above coordinates can be chosen such that for any point \((u_1,u_2,v,w) \in U_{02}\), we have \((\tilde{u}_1,\tilde{u}_2,\tilde{v},\tilde{w}) = \tilde{F}_2^m(u_1,u_2,v,w)\) if and only if

\[
\begin{align*}
\tilde{u}_1 &= \gamma^{-m} u_1 \cos m\omega_2 + \gamma^{-m} u_2 \sin m\omega_2 + q_1(\tilde{u}_1,\tilde{u}_2,\tilde{v},\tilde{w}), \\
\tilde{u}_2 &= -\gamma^{-m} u_1 \sin m\omega_2 + \gamma^{-m} u_2 \cos m\omega_2 + q_2(\tilde{u}_1,\tilde{u}_2,\tilde{v},\tilde{w}), \\
\tilde{v} &= q_3(\tilde{u}_1,\tilde{u}_2,\tilde{v},\tilde{w}), \\
\tilde{w} &= q_4(\tilde{u}_1,\tilde{u}_2,\tilde{v},\tilde{w}),
\end{align*}
\]

where

\[
\|q_{1,2,4}\|_{C^1} = o(\gamma^{-m}), \quad \|q_3\|_{C^1} = o(\lambda^m),
\]

for some constant \( \lambda \in (1,|\lambda_{2,1}|) \), and these estimates are uniform for all systems \( C^2 \)-close to \( f \).

We now take the heteroclinic points

\[
M^+_1 = (x_1^+,x_2^+,0,z^+), \quad M^-_1 = (0,0,y^-,0), \quad M^+_2 = (0,0,v^+,0), \quad M^-_2 = (u_1^-,u_2^-,0,w^-).
\]

By the genericity condition GC3,

\[
x^+ \neq 0, \quad u^- \neq 0.
\]

By making a linear rotation of the coordinates \( u \), if necessary, we can always achieve

\[
u_1^- \neq 0,
\]

which will be our standing assumption.

The transition map \( F_{12} : (\tilde{x}_1,\tilde{x}_2,\tilde{y},\tilde{z}) \mapsto (u_1,u_2,v,w) \) from a neighborhood of \( M^-_1 \) to a neighborhood of \( M^+_2 \) is given by

\[
\begin{align*}
\tilde{u}_1 &= a_{11}' \tilde{x}_1 + a_{12}' \tilde{x}_2 + a_{13}' (\tilde{y} - y^-) + a_{14}' \tilde{z} + \ldots, \\
\tilde{u}_2 &= a_{21}' \tilde{x}_1 + a_{22}' \tilde{x}_2 + a_{23}' (\tilde{y} - y^-) + a_{24}' \tilde{z} + \ldots, \\
\tilde{v} - v^+ &= a_{34}' \tilde{x}_1 + a_{32}' \tilde{x}_2 + a_{33}' (\tilde{y} - y^-) + a_{34}' \tilde{z} + \ldots, \\
\tilde{w} &= a_{41}' \tilde{x}_1 + a_{42}' \tilde{x}_2 + a_{43}' (\tilde{y} - y^-) + a_{44}' \tilde{z} + \ldots,
\end{align*}
\]

where dots denote the second and higher order terms in the Taylor expansion. Note that here \( \dim y = \dim w + \dim u_2 \) and condition GC1 means that \( \det(a_{23}',a_{13}') \neq 0 \). So, we can rewrite the formula as

\[
\begin{align*}
\tilde{u}_1 &= a_{11} \tilde{x}_1 + a_{12} \tilde{x}_2 + a_{13} \tilde{z} + a_{14} u_2 + a_{15} w + \ldots, \\
\tilde{v} - v^+ &= a_{21} \tilde{x}_1 + a_{22} \tilde{x}_2 + a_{23} \tilde{z} + a_{24} u_2 + a_{25} w + \ldots, \\
\tilde{y} - y^- &= a_{31} \tilde{x}_1 + a_{32} \tilde{x}_2 + a_{33} \tilde{z} + a_{34} u_2 + a_{35} w + \ldots
\end{align*}
\]
where \( a_{11}^2 + a_{12}^2 \neq 0 \) by the second part of GC1. Similarly, it follows from condition GC2 that the transition map \( F_{21} : (\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \mapsto (x_1, x_2, y, z) \) from a neighborhood of \( M_2^- \) to a neighborhood of \( M_4^+ \) is given by

\[
\begin{align*}
    x_1 - x_1^+ &= b_{11}(\tilde{u}_1 - u_1^-) + b_{12}\tilde{v} + b_{13}y + \ldots, \\
    x_2 - x_2^+ &= b_{21}(\tilde{u}_1 - u_1^-) + b_{22}\tilde{v} + b_{23}y + \ldots, \\
    z - z^+ &= b_{31}(\tilde{u}_1 - u_1^-) + b_{32}\tilde{v} + b_{33}y + \ldots, \\
    \tilde{u}_2 - u_2^- &= b_{41}(\tilde{u}_1 - u_1^-) + b_{42}\tilde{v} + b_{43}y + \ldots, \\
    \tilde{v} - w^- &= b_{51}(\tilde{u}_1 - u_1^-) + b_{52}\tilde{v} + b_{53}y + \ldots,
\end{align*}
\]

(148)

where \( b_{21}^2 + b_{12}^2 \neq 0 \).

Slightly different from the previous cases, here we work with the first-return map

\[
F_{k,m} := F_{2}^m \circ F_{12} \circ F_{1}^k \circ F_{21} : (\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \mapsto (\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w})
\]

(149)

defined in a small neighbourhood of \( M_2^- \).

**Lemma 13.** Suppose that \( \cos m\omega_2 \) and \( (\cos m\omega_2 + a_{14} \sin m\omega_2) \) are both bounded away from zero. Then, for a point \((\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \) in a small neighborhood of \( M_2^- \), we have \((\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) = F_{k,m}(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \) if and only if

\[
\begin{align*}
(\cos m\omega_2 + a_{14} \sin m\omega_2)\tilde{u}_1 &= \lambda^k \gamma^m(C_k(\tilde{u}_1 - u_1^-) + D_k) + (a_{14} \cos m\omega_2 - \sin m\omega_2)\tilde{u}_2 \\
&\quad + \gamma^m h_{01}(\tilde{u}_1 - u_1^-) + \gamma^m h_{1}(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}), \\
\tilde{u}_2 - u_2^- &= b_{41}(\tilde{u}_1 - u_1^-) + b_{42}\tilde{v} + h_{02}(\tilde{u}_1 - u_1^-) \tilde{v} + h_{2}(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}), \\
\tilde{v} - w^- &= h_4(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}),
\end{align*}
\]

(150)

where

\[
C_k = (a_{11}b_{11} + a_{12}b_{21}) \cos k\omega_1 + (a_{11}b_{21} - a_{12}b_{11}) \sin k\omega_1, \\
D_k = (a_{11}x_1^+ + a_{12}x_2^+) \cos k\omega_1 + (a_{11}x_2^+ - a_{12}x_1^+) \sin k\omega_1,
\]

(151)

and

\[
\begin{align*}
    h_{01} &= O((\tilde{u}_1 - u_1^-)^2 + ||\tilde{v}||), & h_{0i} &= O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) & (i = 2, 3), \\
    h_1 &= o(\lambda^k) + o(\gamma^{-m}), & h_{2,3} &= o(\gamma^{-k}), & h_4 &= o(\lambda^m).
\end{align*}
\]

**Proof.** Consider first the composition \( F_{i}^k \circ F_{21} \). Substituting the \( y \)-equation of (116) into the first three equations in (148), one expresses \( x_1, x_2, z \) as functions of \( \tilde{u}_1, \tilde{v}, \tilde{y} \). Substituting these expressions together with the \( y \)-equation of (116) into the remaining equations in (148) leads to \( \tilde{u}_2, \tilde{w} \) as functions of \( \tilde{u}_1, \tilde{v}, \tilde{y} \). Then, combining (148) with the newly obtained equations for \( x_1, x_2, z, \tilde{u}_2, \tilde{w} \) yields that for a point \((\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \) in a small neighborhood of \( M_2^- \) we have \((\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z}) = F_{i}^k \circ F_{21}(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \) if
and only if

\[ \begin{align*}
\tilde{x}_1 &= \lambda^k(x_1^+ + b_{11}(\tilde{u}_1 - u_1^-) + b_{12}(\tilde{v})) \cos k\omega_1 + \lambda^k(x_2^+ + b_{21}(\tilde{u}_1 - u_1^-) + b_{22}(\tilde{v})) \sin k\omega_1 \\
&\quad + \lambda^kO((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_1(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
\tilde{x}_2 &= -\lambda^k(x_1^+ + b_{11}(\tilde{u}_1 - u_1^-) + b_{12}(\tilde{v})) \cos k\omega_1 + \lambda^k(x_2^+ + b_{21}(\tilde{u}_1 - u_1^-) + b_{22}(\tilde{v})) \cos k\omega_1 \\
&\quad + \lambda^kO((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_2(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
\tilde{u}_2 - u_2^- &= b_{41}(\tilde{u}_1 - u_1^-) + b_{42}(\tilde{v}) + O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_3(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
\tilde{w} - w_2^- &= b_{51}(\tilde{u}_1 - u_1^-) + b_{52}(\tilde{v}) + O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_4(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
\tilde{z} &= \tilde{h}_5(\tilde{u}_1, \tilde{v}, \tilde{y}),
\end{align*} \]

where \( \|\tilde{h}_{1,2,5}\|_{C^1} = o(\lambda^k) \) and \( \|\tilde{h}_{3,4}\|_{C^1} = o(\gamma^{-k}). \)

We proceed to find a formula for \( F_2^m \circ F_{12} \). Substituting the \( v \)-equation in (147) into the \( u_2 \)- and \( w \)-equations in (146), one obtains \( u_2 \) and \( w \) as functions of \( \tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z} \). Substituting these into the remaining equations, leads to \( u_1 \) and \( \tilde{v} \) as functions of \( \tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z} \). So we have the following relations:

\[ \begin{align*}
u_1 &= \gamma^{-m}\tilde{u}_1 \cos m\omega_2 + \gamma^{-m}\tilde{u}_2 \sin m\omega_2 + q_1'(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
u_2 &= \gamma^{-m}\tilde{u}_1 \sin m\omega_2 + \gamma^{-m}\tilde{u}_2 \cos m\omega_2 + q_2'(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
\tilde{v} &= q_3'(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
w &= q_4'(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}),
\end{align*} \]

where \( \|q_1',q_2',q_3',q_4'\|_{C^1} = o(\gamma^{-m}) \) and \( \|q_5',q_6'\|_{C^1} = o(\lambda^m) \).

Now, since \( \cos m\omega_2 \) is bounded away from zero, one can express \( \tilde{u}_1 \) as a function of \( u_1, \tilde{u}_2, \tilde{w}, \tilde{v} \) from the \( u_1 \)-equation. Substituting the result into the equations for \( u_2, \tilde{v} \) and \( w \), we obtain

\[ \begin{align*}
u_1 &= \gamma^{-m}\tilde{u}_1 \cos m\omega_2 = u_1 - \gamma^{-m}\tilde{u}_2 \sin m\omega_2 + q_1'(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
u_2 &= -\gamma^{-m}\tilde{u}_1 \sin m\omega_2 + \gamma^{-m}\tilde{u}_2 \cos m\omega_2 + q_2'(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
\tilde{v} &= q_3'(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
w &= q_4'(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}),
\end{align*} \]

where \( \|q_1',q_2',q_3',q_4'\|_{C^1} = o(\gamma^{-m}) \) and \( \|q_5',q_6'\|_{C^1} = o(\lambda^m) \).

Recall that by assumption \( \cos m\omega_2 + a_{14} \sin m\omega_2 \) is also bounded away from zero. Then, substituting the above \( u_2 \)-equation into the \( u_1 \)-equation in (147) yields

\[ \left( 1 + \frac{a_{14} \sin m\omega_2}{\cos m\omega_2} \right) u_1 = a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} + \frac{a_{14}\gamma^{-m}\tilde{u}_2}{\cos m\omega_2} + a_{15}w + O(x_1^2 + x_2^2 + z^2 + \tilde{u}_2^2 + w^2). \]

Combining this with (153) and the \( \tilde{y} \)-equation in (147), yields that for a point \((\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z})\) in a small neighborhood of \( M_1^{-} \) we have \((\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) = F_2^m \circ F_{12}(\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z})\) if and only if

\[ \begin{align*}
\gamma^{-m}(\cos m\omega_2 + a_{14} \sin m\omega_2)\tilde{u}_1 &= \gamma^{-m}(a_{14} \cos m\omega_2 - \sin m\omega_2)\tilde{u}_2 + a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} \\
&\quad + O(x_1^2 + x_2^2 + z^2) + \tilde{h}_1(\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{u}_2, \tilde{w}), \\
\tilde{y} - y^- &= O(|\tilde{x}_1| + |\tilde{x}_2| + ||\tilde{z}||) + \tilde{h}_2(\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{u}_2, \tilde{w}), \\
\tilde{v} &= \tilde{h}_3(\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{u}_2, \tilde{w}),
\end{align*} \]
where \( \| \hat{h}_1 \|_{C^1} = o(\gamma^{-m}), \quad \| \hat{h}_2 \|_{C^1} = O(\gamma^{-m}), \quad \| \hat{h}_3 \|_{C^1} = o(\hat{\lambda}^m). \)

Replacing \( \tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w} \) by \( \tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w} \) in the above formula, and combining it with (152) yields (150).

Denote
\[
C_k = C \sin(k\omega_1 + \eta_1), \quad D_k = D \sin(k\omega_1 + \eta_2)
\]
and
\[
\cos m\omega_2 + a_{14} \sin m\omega_2 = \sqrt{1 + a_{14}^2 \cos(m\omega_2 + \eta_3)}, \quad a_{14} \cos m\omega_2 - \sin m\omega_2 = -\sqrt{1 + a_{14}^2 \sin(m\omega_2 + \eta_3)}
\]
in formula (150). Since \( a_{11}^2 + a_{12}^2 \neq 0, b_{21}^2 + b_{12}^2 \neq 0, \) and \( x^+ \neq 0 \) by conditions GC1-3, we have
\[
C \neq 0, \quad D \neq 0.
\]
Moreover, by condition GC4.2,
\[
\tan \eta_1 \neq \tan \eta_2,
\]
see (151).

Introduce the coordinate transformation
\[
U_1 = \tilde{u}_1 - u_1^- \quad \delta \tilde{U}_2 = \tilde{u}_2 - u_2^- - b_{41}(\tilde{u}_1 - u_1^-) - b_{42} \tilde{v},
\]
\[
\delta \tilde{V} = \tilde{v}, \quad W = \tilde{w} - w^- - b_{51}(\tilde{u}_1 - u_1^-) - b_{52} \tilde{v},
\]
and consider the restriction of \( F_{k,m} \) to
\[
\Pi = [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d-d_1-1} \times [-\delta, \delta]^{d_1-1}
\]
in the new coordinates. Then, formula (150), along with (156), (157), implies that for a point \( (U_1, U_2, V, W) \in \Pi \) we have \( (\tilde{U}_1, \tilde{U}_2, \tilde{V}, \tilde{W}) = F_{k,m}(U_1, U_2, V, W) \) if and only if
\[
(1 + b_{41} \tan(m\omega_2 + \eta_3)) \tilde{U}_1 = \frac{\lambda^k \gamma^m C \sin(k\omega_1 + \eta_1)U_1 + \lambda^k \gamma^m D \sin(k\omega_1 + \eta_2)}{\sqrt{1 + a_{14}^2 \cos(m\omega_2 + \eta_3) + \tan(m\omega_2 + \eta_3)u_1^-}} - u_1^- - \phi_1(U_1, \tilde{U}_2, V, W),
\]
\[
U_2 = \phi_2(U_1, \tilde{U}_2, V, W), \quad \tilde{V} = \phi_3(U_1, \tilde{U}_2, V, W), \quad W = \phi_4(U_1, \tilde{U}_2, V, W),
\]
where
\[
\phi_1 = \lambda^k \gamma^m (O(\delta \tilde{U}_1^2) + o(1))_{k,m \to \infty} + O(\delta \tilde{U}_1^2) + o(1)_{k,m \to \infty},
\]
\[
\frac{\partial \phi_1}{\partial (U_1, U_2, V, W)} = \lambda^k \gamma^m (O(\delta \tilde{U}_1^2) + o(1))_{k,m \to \infty} + O(\delta \tilde{U}_1^2) + o(1)_{k,m \to \infty}
\]
\[
\phi_{2,4} = O(\delta \tilde{U}_1^2) + o(\tilde{\gamma}^{-k}), \quad \frac{\partial \phi_{2,4}}{\partial (U_1, U_2, V, W)} = O(\delta \tilde{U}_1^2) + o(\tilde{\gamma}^{-k}), \quad \| \phi_3 \|_{C^1} = o(\hat{\lambda}^m).
\]

For the proof of the next proposition, we will consider only such \( m \) that \( \sin(m\omega_2 + \eta_3) \) is close to zero. In this case, the condition of Lemma 13 that \( \cos m\omega_2 \) and \( (\cos m\omega_2 + a_{14} \sin m\omega_2) \) are both bounded away from zero are automatically satisfied, as follows from (157).
Proposition 6. Let the system $f$ have a heterodimensional cycle $\Gamma$ of double-focus type, and $\theta, \omega_1/\pi, \theta \omega_2/\pi$ and 1 are rationally independent. There exist, arbitrarily close to $\Gamma$, a cu-blender with an activating pair $(\Pi', C^{ss})$ and a cs-blender with an activating pair $(\Pi', C^{uu})$ where

$$\Pi' = [-q\delta, q\delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_1-1} \times [-\delta, \delta]^{d_1-1}$$

(163)

for some fixed $q \in (0, 1)$, and

$$C^{ss} = \{ (\Delta U_1, \Delta U_2, \Delta V, \Delta W) : \max\{ |\Delta U_1|, ||(\Delta U_2, \Delta W)|| \} \leq K ||\Delta V|| \},$$

$$C^{uu} = \{ (\Delta U_1, \Delta U_2, \Delta V, \Delta W) : \max\{ |\Delta U_1|, ||\Delta V|| \} \leq K ||(\Delta U_2, \Delta W)|| \}$$

for some sufficiently small $K$. The blenders are homoclinically related to each other.

Proof. Denote

$$A_{k,m} = \lambda^{k\gamma^m} \frac{C}{\sqrt{1+a_1^2}} \sin(k\omega_1 + \eta_1),$$

$$B_{k,m} = \lambda^{k\gamma^m} \frac{D}{\sqrt{1+a_1^2}} \sin(k\omega_1 + \eta_2) - \frac{a_1}{\sqrt{1+a_1^2}}.$$ 

(164)

We see that if we replace

$$U_1 \to X_1, \quad (U_2, W) \to Y, \quad V \to (X_2, Z),$$

(165)

then the first-return map (161) takes the same form as the first-return map (132) in the saddle-focus case, provided we consider the values of $(k, m)$ such that $\sin(m\omega_2 + \eta_3) \to 0$ and $\lambda^{k\gamma^m}$ stays bounded. Moreover, formulas (164) coincide with formulas (128) for the saddle-focus case if we rename the constants $A = \frac{C}{\sqrt{1+a_1^2}}, B = \frac{D}{\sqrt{1+a_1^2}}$, and replace $\frac{a_1}{\sqrt{1+a_1^2}}$ by $b_{11}u^-$, and $\omega_1$ by $\omega$.

Thus, we obtain Proposition 6 in the same way as Proposition 5, if we show that given any $(s, t) \in \mathbb{R}^2$, one can find a sequence $\{(k_n, m_n, p_n)\}$ with $k_n, m_n \to \infty$ such that conditions (140) are satisfied, and

$$\sin(m_n\omega_2 + \eta_3) \to 0$$

(we do not need the evenness of $m_n$ here, as $\gamma > 0$ in the double-focus case).

These requirements are equivalent to the existence of a sequence of integers $\{(k_n, m_n, p_n, l_n)\}$ such that

$$k_n\theta - m_n \to -t \quad k_n \frac{\omega_1}{2\pi} - p_n \to s \quad k_n \frac{1}{\pi} \theta \omega_2 - l_n \to \frac{t\omega_2 + \eta_3}{\pi}.$$ 

This is guaranteed by our assumption that $\theta, \omega_1/(2\pi), \theta \omega_2/(2\pi)$, and 1 are rationally independent. \(\square\)

To complete the proof of Theorem 9 we need to show that the two periodic points $O_1$ and $O_2$ are homoclinically related to the cu-blender and, respectively, the cs-blender obtained in the above proposition.

Lemma 14. Let $\Pi'$ and $C^{uu}$ be given by Proposition 6. There exists a sequence $\{m_j\}$ of positive integers with $m_j \to \infty$ such that the image $S^{u}_j := F_2^{m_j} \circ F_12(W_{loc}(O_1)) \cap \Pi$ is a surface of the form $(U_1, V) = s(U_2, W)$ for some smooth function $s$. The surface $S^{u}_j$ crosses $\Pi'$ properly with respect to the cone field $C^{uu}$. 

64
Lemma 15. Let $\Pi'$ and $C^{ss}$ be given by Proposition 6. There exists a sequence $\{k_j\}$ of positive integers with $k_j \to \infty$ such that the pre-image $S_j^u := F_{21}^{-1} \circ F_{1}^{-k_j} \circ F_{12}^{-1}(W_{\text{loc}}^s(O_2)) \cap \Pi$ is a surface of the form $(U_1, U_2, W) = s(V)$ for some smooth function $s$. The surface $S_j^u$ crosses $\Pi'$ properly with respect to the cone field $C^{ss}$.

Proof. By (147), the pre-image $F_{12}^{-1}(W_{\text{loc}}^s(O_2))$ satisfies

$$0 = a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} + O(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{z}^2),$$

$$\tilde{y} - y^- = a_{31}\tilde{x}_1 + a_{32}\tilde{x}_2 + a_{33}\tilde{z} + O(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{z}^2).$$

(166)

Substitute the equations for $\tilde{x}_1, \tilde{x}_2, \tilde{z}$ from formula (152) for the map $F_{12}^k \circ F_{21}$ into the second equation above. This yields $\tilde{y} = y^- + O(\lambda^k)$ as a function of $(\tilde{u}_1, \tilde{v})$. Combining the first equation above with (152) and using the new expression for $\tilde{y}$, we obtain the following equation for $F_{21}^{-1} \circ F_{1}^{-k} \circ F_{12}^{-1}(W_{\text{loc}}^s(O_2))$ (see (151),(156)):

$$\tilde{u}_1 - u^-_1 = -\frac{D\sin(k\omega + \eta_2)}{C\sin(k\omega_1 + \eta_1)} + O((\tilde{u}_1 - u^-)^2 + ||\tilde{v}||) + o(1)_{k \to \infty},$$

$$\tilde{u}_2 - u^-_2 = b_{41}(\tilde{u}_1 - u^-_1) + b_{42}\tilde{v} + O((\tilde{u}_1 - u^-)^2 + \tilde{v}^2) + o(\gamma^{-k}),$$

$$\tilde{w} - w^-_2 = b_{51}(\tilde{u}_1 - u^-_1) + b_{52}\tilde{v} + O((\tilde{u}_1 - u^-)^2 + \tilde{v}^2) + o(\gamma^{-k}),$$

where the small terms are functions of $\tilde{u}_1$ and $\tilde{v}$. Since $\omega_1$ is irrational, we can choose the sequence of values of $k \to \infty$ such that $\sin(k\omega_1 + \eta_1)$ stays bounded away from zero.
Now, after solving $\bar{u}_1$ from the first equation, we apply the coordinate transformation (159) and obtain the intersection $F_{21}^{-1} \circ F_1^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2)) \cap \Pi$ as

$$
U_1 = \frac{D \sin(k\omega_1 + \eta_2)}{C \sin(k\omega_1 + \eta_1)} + O(\delta^{\frac{1}{2}} V) + o(1)_{k \to \infty},
$$

$$
U_2 = O(\delta^{-\frac{1}{2}} U_1^2 + \delta^{\frac{1}{2}} |V|) + o(1)_{k \to \infty}, \quad W = O(U_1^2 + \delta |V|) + o(1)_{k \to \infty}.
$$

After comparing this with (145) using the correspondence (165), one just follows the proof of Lemma 12.

One easily sees from (159) that $W_{loc}^u(O_2) = \{V = 0\}$, and, after additionally taking $k \to \infty$ in (152), that the intersection $F_{21}^{-1}(W_{loc}^s(O_1)) \cap \Pi$ is the graph of some smooth function $(U_2, W) = s(U_1, V)$ satisfying $s = O(U_1^2 + \delta V^2)$ and defined on $[-\delta, \delta] \times [-\delta, \delta]^{d-d_1-1}$.

Therefore, since the above two lemmas are completely analogous to Lemmas 11 and 12, one obtains the homoclinic relations between $O_1$ and the cs-blender, and between $O_2$ and the cu-blender in the same way as it is shown in the end of Section 4.2.

5 Acknowledgments

The authors are grateful to Katsutoshi Shinohara and Pierre Berger for important discussions during the preparation of this paper. The research of D. Turaev was supported by Leverhulme Trust, by the grants 19-11-00280 and 19-71-10048 of RSF, by the Mathematical Center at the Lobachevsky University of Nizhny Novgorod, and by the grant 075-15-2019-1931 of Russian Ministry of Science and Higher Education. The research of D. Li was supported by these funding until March 31, 2020. Starting from April 1, 2020, D. Li has been supported by the ERC project 677793 StableChaoticPlanetM.

References

[1] R. Abraham and S. Smale, Nongenericity of $\Omega$-stability, Global Analysis I, Proc. Symp. Pure Math. AMS, 14 (1970), 5-8.

[2] V. S. Afraimovich and L. P. Shilnikov, On singular sets of Morse-Smale systems, Trudy Moskov. Mat., 28 (1973), 181-214.

[3] M. Asaoka, K. Shinohara and D. V. Turaev, Degenerate behavior in non-hyperbolic semigroup actions on the interval: fast growth of periodic points and universal dynamics, Mathematische Annalen, 368:3-4 (2017), 1277–1309.

[4] M. Asaoka, K. Shinohara and D. V. Turaev, Fast growth of the number of periodic points arising from heterodimensional connections, arXiv:1808.07218.

[5] P. Berger, S. Crovisier, E. Pujals, Iterated functions systems, blenders, and parablenders, Conference of Fractals and Related Fields, (2015), 57-70.

[6] P. Berger, Generic family with robustly infinitely many sinks, Inventiones mathematicae, 205:1 (2016), 121-172.
[7] P. Berger, S. Crovisier, E. Pujals, Germ-typicality of the coexistence of infinitely many sinks, *arXiv:2103.16697*.

[8] J. Bochi, C. Bonatti and L. J. Díaz, Robust criterion for the existence of nonhyperbolic ergodic measures, *Comm. Math. Phys.*, **344** (2016), 751-795.

[9] C. Bonatti and L. J. Díaz, Persistent transitive diffeomorphisms, *Annals of Mathematics*, **143:2** (1996), 357-396.

[10] C. Bonatti, L. J. Díaz and M. Viana, *Dynamics Beyond Uniform Hyperbolicity*, Springer, Berlin, Heidelberg, New York, 2000.

[11] C. Bonatti and L. J. Díaz, Robust heterodimensional cycles and $C^1$-generic dynamics, *Journal of the Institute of Mathematics of Jussieu*, **7:3** (2008), 469-525.

[12] C. Bonatti and L. J. Díaz, Fragile cycles, *JDE*, **252** (2012), 4176-4199.

[13] C. Bonatti, L. J. Díaz and S. Kiriki, Stabilization of heterodimensional cycles, *Nonlinearity*, **25:4** (2012), 931-960.

[14] H. W. Broer and F. M. Tangerman, From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems, *Ergodic Theory and Dynamical Systems*, **6(3)** (1986), 345 - 362.

[15] L. J. Díaz and J. Rocha, Non-connected heterodimensional cycles: bifurcation and stability, *Nonlinearity*, **5** (1992), 1315-1341.

[16] L. J. Díaz, Robust nonhyperbolic dynamics and heterodimensional cycles, *Ergodic Theory and Dynamical Systems*, **15** (1995), 291-315.

[17] L. J. Díaz, Persistence of cycles and nonhyperbolic dynamics at the unfolding of heteroclinic bifurcations, *Ergodic Theory and Dynamical Systems*, **8** (1995), 693-715.

[18] L. J. Díaz and A. Pérez, Hénon-Like Families and Blender-Horseshoes at Nontransverse Heterodimensional Cycles, *Internat. J. Bifur. Chaos Appl.*, **29:3** (2019).

[19] S. V. Gonchenko, A. Kazakov and D. V. Turaev, Wild pseudohyperbolic attractor in a four-dimensional Lorenz system, *Nonlinearity*, **34:4** (2021), 2018-2048.

[20] S. V. Gonchenko and L. P. Shilnikov On dynamical systems with structurally unstable homoclinic curves, *Sov. Math. - Dokl.*, **33** (1986), 234-238.

[21] S. V. Gonchenko and L. P. Shilnikov, Invariants of Ω-conjugacy of diffeomorphisms, *Ukrainian Math. J.*, **42** (1990), 134-140.

[22] S. V. Gonchenko, D. V. Turaev and L. P. Shilnikov, Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps, *Nonlinearity*, **20** (2007), 241-275.

[23] S. V. Gonchenko, L. P. Shilnikov and D. V. Turaev, On dynamical properties of multidimensional diffeomorphisms from Newhouse regions, *Nonlinearity*, **21** (2008), 923–972.

[24] A. S. Gorodetski and Yu. S. Ilyashenko, Certain properties of skew products over a horseshoe and a solenoid, *Proc. Steklov Inst. Math.*, **231** (2000), 90–112.
[25] A. S. Gorodetski and L. J. Díaz, Non-hyperbolic ergodic measures for non-hyperbolic homoclinic classes, *Ergodic Th. Dyn. Sys.*, 29 (2009), 1479 - 1513.

[26] N. Gourmelon, Steps towards a classification of $C^r$-generic dynamics close to homoclinic points, *arXiv:1410.1758*, (2009).

[27] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications)*, Cambridge University Press, Cambridge, 1995.

[28] D. Li, Homoclinic bifurcations that give rise to heterodimensional cycles near a Saddle-focus equilibrium, *Nonlinearity*, 30 (2016), 173-206.

[29] D. Li and D. V. Turaev, Existence of heterodimensional cycles near Shilnikov loops in systems with a $Z_2$ symmetry, *Discrete Conti. Dynam. Sys.*, 37:8 (2017), 4399-4437.

[30] D. Li and D. V. Turaev, Persistent heterodimensional cycles in periodic perturbations of Lorenz-like attractors, *Nonlinearity*, 33 (2020), 971–1015.

[31] R. Mañé, Contributions to the stability conjecture, *Topology*, 17 (1978), 386–396.

[32] M. Nassiri and E. Pujals, Robust transitivity in Hamiltonian dynamics, *Annales scientifiques de l’École Normale Supérieure*, 45:2 (2012), 191-239.

[33] S. E. Newhouse, J. Palis and F. Takens, Bifurcations and stability of families of diffeomorphisms, *Publications Mathématiques de l’IHÉS*, 57 (1983), 5-71.

[34] O. E. Rössler, An equation for hyperchaos, *Physics Letters A*, 71 (1979), 155-157.

[35] L. P. Shilnikov, On a Poincaré-Birkhoff problem, *Math. USSR Sbornik*, 3 (1967), 91-102.

[36] L. P. Shilnikov, A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type, *Math. USSR Sbornik*, 10 (1970), 91-102.

[37] M. Shub, Topological transitive diffeomorphism on $T^4$, *Lect. Notes in Math.*, 206 (1971), 39.

[38] R.C. Simon, A 3-dimensional Abraham-Smale example, *Proc. A.M.S.*, 34:2 (1972), 629–630.

[39] L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev and L. O. Chua, *Methods Of Qualitative Theory In Nonlinear Dynamics (Part I)*, 2nd World Sci.-Singapore, New Jersey, London, Hong Kong, 2001.

[40] S. J. van Strien, *One parameter families of vector fields*, PhD Thesis, Utrecht, 1982.

[41] D. V. Turaev, On dimension of non-local bifurcational problems, *International Journal of Bifurcation and Chaos*, 6 (1996), 919-948.

[42] D. V. Turaev, Richness of chaos in the absolute Newhouse domain, *Proceedings of the International Congress of Mathematicians 2010*.

[43] D. V. Turaev and L. P. Shilnikov, An example of a wild strange attractor, *Math. USSR Sbornik*, 189 (1998), 291-314.

[44] D. V. Turaev and L. P. Shilnikov, Pseudohyperbolicity and the problem on periodic perturbations of Lorenz-type attractors, *Doklady Mathematics*, 77:1 (2008), 17-21.