HARDY-TYPE INEQUALITIES FOR DUNKL OPERATORS

ANDREI VELICU

Abstract. In this paper we study various variants of the Hardy inequality for Dunkl operators, including the classical inequality, $L^p$ inequalities, the Hardy inequality for domains, an improved Hardy inequality, as well as the Rellich inequality and a special case of the Caffarelli-Kohn-Nirenberg inequality.

1. Introduction

The classical Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla f|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, dx$$

is one of the most important results in analysis. It has seen an incredible development from its beginnings in Hardy’s papers, having been refined and extended to various settings; see [3], [18], [8] and references therein for an overview of the topic.

Here we begin a systematic study of Hardy’s inequality and its variants for Dunkl operators. As far as we can see, Hardy type inequalities for Dunkl operators have not been explicitly studied. In [11], the authors proved a Pitt’s inequality which implies the Hardy’s inequality for fractional powers of the Dunkl laplacian. In [6], the ground state representation method was used to obtain Hardy’s inequality for fractional powers of the Dunkl-Hermite operators.

Apart from the classical Hardy’s inequality with sharp constant, we also study in this paper $L^p$ inequalities which hold for small coefficients $\gamma$ and the Hardy inequality on domains with weight depending on the distance to the boundary. As in the classical case, the sharp constant in the Hardy inequality is not achieved and we prove here an improved Hardy inequality using a method based on spherical h-harmonics. Two Hardy-type inequalities are also discussed, the Rellich inequality (with sharp constant), and a special case of the Caffarelli-Kohn-Nirenberg inequality. More background on these variants is given in each corresponding section.

The paper is organised as follows. In section 2 we briefly introduce the theory of Dunkl operators and spherical h-harmonics. In section 3 we discuss the classical form of Hardy’s inequality for Dunkl operators, and in section 4 we prove the $L^p$ inequality. Section 5 discusses Hardy’s inequality on $G$-invariant domains. In section 6 we prove an improved Hardy inequality and as a corollary we deduce the Poincaré inequality for Dunkl operators. Finally, section 7 contains two Hardy-type results: the Rellich inequality and the Caffarelli-Kohn-Nirenberg inequality.
2. Preliminaries

In this section we will present a very quick introduction to Dunkl operators. For more details see the survey papers [21] and [2].

A root system is a finite set \( R \subset \mathbb{R}^N \setminus \{0\} \) such that \( R \cap \alpha \mathbb{R} = \{-\alpha, \alpha\} \) and \( \sigma_\alpha(R) = R \) for all \( \alpha \in R \). Here \( \sigma_\alpha \) is the reflection in the hyperplane orthogonal to the root \( \alpha \), i.e.,

\[
\sigma_\alpha x = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha.
\]

The group generated by all the reflections \( \sigma_\alpha \) for \( \alpha \in R \) is a finite group, and we denote it by \( G \).

Let \( k : R \to [0, \infty) \) be a \( G \)-invariant function, i.e., \( k(\alpha) = k(g\alpha) \) for all \( g \in G \) and all \( \alpha \in R \). We will normally write \( k_\alpha = k(\alpha) \) as these will be the coefficients in our Dunkl operators. We can write the root system \( R \) as a disjoint union \( R = \bigoplus_{p=1}^P R_p \), and we call \( R_p \) a positive subsystem; this decomposition is not unique, but the particular choice of positive subsystem does not make a difference in the definitions below because of the \( G \)-invariance of the coefficients \( k_\alpha \).

From now on we fix a root system in \( \mathbb{R}^N \) with positive subsystem \( R_+ \). We also assume without loss of generality that \( |\alpha|^2 = 2 \) for all \( \alpha \in R \). For \( i = 1, \ldots, N \) we define the Dunkl operator on \( C^1(\mathbb{R}^N) \) by

\[
T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} k_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.
\]

We will denote by \( \nabla_k = (T_1, \ldots, T_N) \) the Dunkl gradient, and \( \Delta_k = \sum_{i=1}^N T_i^2 \) will denote the Dunkl laplacian. Note that for \( k = 0 \) Dunkl operators reduce to partial derivatives, and \( \nabla_0 = \nabla \) and \( \Delta_0 = \Delta \) are the usual gradient and laplacian.

We can express the Dunkl laplacian in terms of the usual gradient and laplacian using the following formula:

\[
\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k_\alpha \left[ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} \cdot \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right].
\]

The weight function naturally associated to Dunkl operators is

\[
w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha}.
\]

This is a homogeneous function of degree

\[
\gamma := \sum_{\alpha \in R_+} k_\alpha.
\]

We will work in spaces \( L^p(\mu_k) \), where \( d\mu_k = w_k(x) \, dx \) is the weighted measure; the norm of these spaces will be written simply \( \| \cdot \|_p \). With respect to this weighted measure we have the integration by parts formula

\[
\int_{\mathbb{R}^N} T_i(f)g \, d\mu_k = -\int_{\mathbb{R}^N} fT_i(g) \, d\mu_k.
\]
If one of the functions $f, g$ is $G$-invariant, then we have the Leibniz rule

$$T_i(fg) = fT_ig + gT_if.$$ 

In general we have

$$T_i(fg)(x) = T_if(x)g(x) + f(x)T_ig(x) - \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{(f(x) - f(\sigma_\alpha x))(g(x) - g(\sigma_\alpha x))}{\langle \alpha, x \rangle}.$$ 

An important function associated with Dunkl operators is the Dunkl kernel $E_k(x, y)$, defined on $\mathbb{C}^N \times \mathbb{C}^N$, which acts as a generalisation of the exponential and is defined, for fixed $y \in \mathbb{C}^N$, as the unique solution $Y = E_k(\cdot, y)$ of the equations

$$T_iY = y_iY, \quad i = 1, \ldots, N,$$

which is real analytic on $\mathbb{R}^N$ and satisfies $Y(0) = 1$. Another definition of the Dunkl exponential can be given in terms of the intertwining operator $V_k$ which connects Dunkl operators to usual derivatives via the relation

$$T_iV_k = V_k\partial_i.$$ 

The Dunkl exponential can then be equivalently defined as

$$E_k(x, y) = V_k \left( e^{\langle \cdot, y \rangle} \right)(x).$$

The following growth estimates on $E_k$ are known: for all $x \in \mathbb{R}^N$, $y \in \mathbb{C}^N$ and all $\beta \in \mathbb{Z}_+^N$ we have

$$|\partial^\beta_y E_k(x, y)| \leq |x|^{|\beta|} \max_{g \in G} e^{\text{Re}(gx, y)}.$$ 

It is then possible to define a Dunkl transform on $L^1(\mu_k)$ by

$$D_k(f)(\xi) = \frac{1}{M_k} \int_{\mathbb{R}^N} f(x)E_k(-i\xi, x) \, d\mu_k(x), \quad \text{for all } \xi \in \mathbb{R}^N,$$

where

$$M_k = \int_{\mathbb{R}^N} e^{-|x|^2/2} \, d\mu_k(x)$$

is the Macdonald-Mehta integral. The Dunkl transform extends to an isometric isomorphism of $L^2(\mu_k)$; in particular, the Plancherel formula holds. When $k = 0$ the Dunkl transform reduces to the Fourier transform.

If $N + 2\gamma > 2$, then for any $f \in C_c^\infty(\mathbb{R}^N)$ we have the following Sobolev inequality

$$\|f\|_q \leq C \|\nabla_k f\|_2,$$

where $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$. The value of the sharp constant in this inequality has been established in [22].
Spherical \( h\)-harmonics. We will briefly introduce \( h\)-harmonics; our presentation here is based on [7] and we invite the interested reader to this reference for more details. An \( h\)-harmonic polynomial of degree \( n \) is a homogeneous polynomial \( p \) of degree \( n \) that satisfies
\[
\Delta k p = 0.
\]
Spherical \( h\)-harmonics (or just \( h\)-harmonics) of degree \( n \) are then defined to be restrictions of \( h\)-harmonic polynomials of degree \( n \) to the sphere \( S^{N-1} \). Let \( \mathcal{H}_n^N \) be the space of \( h\)-harmonics of degree \( n \); this is a finite-dimensional space and denote its dimension by \( d \). Moreover, the space \( L^2(S^{N-1}, w_k(\xi) d\xi) \) is the orthogonal direct sum of the spaces \( \mathcal{H}_n^N \), for \( n = 0, 1, 2, \ldots \). Let
\[
Y_i^n \quad \text{for } i = 1, \ldots, d(n)
\]
be an orthonormal basis of \( \mathcal{H}_n^N \). In spherical polar coordinates \( x = r\xi \), for \( r \in [0, \infty) \) and \( \xi \in S^{N-1} \), we can write the Dunkl laplacian as
\[
\Delta k = \frac{\partial^2}{\partial r^2} + \frac{N + 2 \gamma - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{k,0},
\]
where \( \Delta_{k,0} \) is a generalisation of the Laplace-Beltrami operator on the sphere, and it only acts on the \( \xi \) variable. Then the spherical harmonics are eigenvalues of \( \Delta_{k,0} \), i.e.,
\[
\Delta_{k,0} Y = -n(n + N + 2\gamma - 2) Y =: \lambda_n Y, \quad \text{for all } Y \in \mathcal{H}_n^N.
\]
The \( h\)-harmonic expansion of a function \( f \in L^2(\mu_k) \) is given by
\[
f(r\xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} f_{n,i}(r) Y_i^n(\xi),
\]
where
\[
f_{n,i}(r) = \int_{S^{N-1}} f(r\xi) Y_i^n(\xi) w_k(\xi) \, d\sigma(\xi),
\]
and \( \sigma \) is the surface measure on the sphere \( S^{N-1} \).

3. HARDY’S INEQUALITY

In this section we look at the classical form of Hardy’s inequality for the Dunkl operators, providing firstly an elementary proof based mainly on integration by parts. A more general result for fractional powers of the laplacian can be obtained from a variant of Pitt’s inequality from [11]. This method relies heavily on the theory of Dunkl transform.

Theorem 3.1 (Hardy inequality). The following inequality holds for all \( f \in C_0^\infty(\mathbb{R}^N) \) for which the integral on the right hand side is finite:
\[
\int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_k \geq \left( \frac{N + 2\gamma - 2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^2} \, d\mu_k.
\]
Moreover, if \( n + 2\gamma \neq 2 \) the constant in this inequality is sharp.

Proof. For any \( s \in \mathbb{R} \) we have
\[
\int_{\mathbb{R}^N} \left| \nabla_k f + s \frac{x}{|x|^2} f \right|^2 \, d\mu_k(x) \geq 0.
\]
Expanding out, this implies that

\[(3.1) \quad \int_{\mathbb{R}^N} |\nabla f|^2 \, d\mu_k + 2s \int_{\mathbb{R}^N} \frac{x \cdot \nabla f}{|x|^2} \, f \, d\mu_k + s^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k \geq 0.\]

Using integration by parts we can compute the middle terms as follows

\[I := \int_{\mathbb{R}^N} \frac{x \cdot \nabla f}{|x|^2} \, f \, d\mu_k = -N \sum_{i=1}^{N} \int_{\mathbb{R}^N} T_i \left( \frac{f(x_i)}{|x|^2} \right) \, d\mu_k \]

\[= -N \int_{\mathbb{R}^N} \left[ \frac{x_i}{|x|^2} T_i f + \frac{T_i(x_i)}{|x|^2} - 2 \frac{x^2}{|x|^4} f - \sum_{\alpha \in R_+} k_\alpha \frac{\alpha_\alpha^2}{|x|^2} (f(x) - f(\sigma_\alpha x)) \right] \, d\mu_k.\]

Using the fact that \(T_i(x_i) = 1 + \sum_{\alpha \in R_+} k_\alpha \alpha_\alpha^2\) and \(|\alpha|^2 = 2\), this implies

\[I = -I - (N + 2\gamma - 2) \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k + 2 \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} \frac{f^2(x) - f(x)f(\sigma_\alpha x)}{|x|^2} \, d\mu_k.\]

In the last term we use the elementary inequality \(-XY \geq -\frac{1}{2}X^2 - \frac{1}{2}Y^2\) to obtain

\[\int_{\mathbb{R}^N} \frac{f^2(x) - f(x)f(\sigma_\alpha x)}{|x|^2} \, d\mu_k \geq \frac{1}{2} \int_{\mathbb{R}^N} \frac{f^2(x)}{|x|^2} \, d\mu_k - \frac{1}{2} \int_{\mathbb{R}^N} \frac{f^2(\sigma_\alpha x)}{|x|^2} \, d\mu_k = 0.\]

The last step follows by change of variables \(y = \sigma_\alpha x\) from which it can be seen that the two integrals are indeed equal.

Thus, we have obtained that

\[I \geq \frac{N + 2\gamma - 2}{2} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k,\]

and replacing this in (3.1), we have that for all \(s \in \mathbb{R}\) the following holds

\[s^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k - s(N + 2\gamma - 2) \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k + \int_{\mathbb{R}^N} |\nabla f|^2 \, d\mu_k \geq 0.\]

Since this holds for all \(s \in \mathbb{R}\), then we must have

\[(N + 2\gamma - 2) \left( \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k \right)^2 - 4 \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} \, d\mu_k \cdot \int_{\mathbb{R}^N} |\nabla f|^2 \, d\mu_k \leq 0,\]

from which Hardy’s inequality follows immediately.

To check that the constant \((N + 2\gamma - 2)^2\) is sharp we consider first the case \(N + 2\gamma > 2\).

Define, for each \(n = 1, 2, \ldots\), the radial function \(f_n(x) = h_n(|x|)\), where

\[h_n(r) = \begin{cases} \frac{1}{c_n} & \text{if } r \leq 1 \\ \frac{r}{c_n} & \text{if } r > 1 \end{cases},\]

where \(c_n = -\frac{1}{n} - \frac{N + 2\gamma - 2}{2}\). Then we have

\[\int_{\mathbb{R}^N} |\nabla f_n|^2 \, d\mu_k = \int_0^\infty (h_n'(r))^2 r^{N + 2\gamma - 1} \, dr = \frac{\frac{2}{c_n^2}}{\left( \frac{N + 2\gamma - 2}{2} \right)^2} \rightarrow \frac{(N + 2\gamma - 2)^2}{4} \]

as \(n \to \infty\).
as \( n \to \infty \).

The case \( N + 2\gamma < 2 \) is similar to the above except for now
\[
h_n(r) = \begin{cases} \frac{1}{c_n} r^{c_n} & \text{if } r \leq 1 \\ \frac{1}{c_n} & \text{if } r > 1, \end{cases}
\]
and \( c_n = \frac{1}{n} - \frac{N+2\gamma-2}{2} \).

As mentioned above, a more general result can be obtained from the following Pitt’s inequality
\[ C(s) \| |\xi|^{-s} D_k(f)(\xi) \|_2 \leq \| |x|^s f(x) \|_2 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^N), \]
which holds for \( 0 \leq s < N/2 + \gamma \), with sharp constant
\[ C(s) = 2^s \frac{\Gamma\left(\frac{1}{2}(N/2 + \gamma + s)\right)}{\Gamma\left(\frac{1}{2}(N/2 + \gamma - s)\right)} \]
By Plancherel’s formula we have that \( \|(-\Delta)^{s/2} f\|_2 = \| |\xi|^s D_k(f)(\xi) \|_2 \), so the above inequality can be restated as
\[ C(s) \| |x|^{-s} f(x) \|_2 \leq \|(-\Delta)^{s/2} f\|_2 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^N). \]
Using the fact that fractional laplacian is self-adjoint (which can be checked using Fubini’s theorem) we obtain the following Hardy’s inequality.

**Theorem 3.2.** For all \( 0 \leq s < N/2 + \gamma \) and all \( f \in \mathcal{S}(\mathbb{R}^N) \) we have
\[ C(s)^2 \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^{2s}} \, d\mu_k \leq \langle (-\Delta)^{s} f, f \rangle, \]
where the constant is sharp.

In particular, for \( s = 1 \) we recover Theorem 3.1

### 4. \( L^p \) Hardy Inequality

In this section we will prove the Hardy inequality in weighted \( L^p \) spaces which holds for small \( \gamma \).

**Theorem 4.1.** Let \( p > 1 \) such that \( p \neq N + 2\gamma \). Let \( f \in C_0^\infty(\mathbb{R}^N); \) if \( p > N + 2\gamma \), assume in addition that \( f(0) = 0 \). Then we have
\[
\int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} \, d\mu_k \leq C \int_{\mathbb{R}^N} |\nabla_k f|^p \, d\mu_k.
\]

**Proof.** We will use polar coordinates \( x = r\omega \), where \( r = |x| \) and \( \omega \in S^{N-1} \). We have
\[
\int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} \, d\mu_k = \int_{S^{N-1}} \int_0^\infty r^{N+2\gamma-p-1} |f(r\omega)|^p \, dr \, d\sigma(\omega).
\]
Using integration by parts, we obtain
\[
I(\omega) := \int_0^\infty r^{N+2\gamma-p-1} |f(r\omega)|^p \, dr
\]
\[
= - \int_0^\infty \frac{r^{N+2\gamma-p}}{N+2\gamma-p} \frac{d}{dr}(|f(r\omega)|^p) \, dr \, d\sigma(\omega) + \left[ \frac{r^{N+2\gamma-p}}{N+2\gamma-p} |f(r\omega)|^p \right]_0^\infty \, d\sigma(\omega).
\]
The last term vanishes thanks to the assumptions on \( f \). For the rest, we have

\[
I(\omega) \leq \frac{p}{|N + 2\gamma - p|} \int_0^{\infty} r^{N+2\gamma-p}|f(r\omega)|^{p-1} \left| \frac{df}{dr}(r\omega) \right| dr \sigma(\omega)
\]

\[
\leq \frac{p}{|N + 2\gamma - p|} \left( \int_0^{\infty} r^{N+2\gamma-1} \left| \frac{df}{dr}(r\omega) \right|^p dr \right)^{1/p} \left( \int_0^{\infty} r^{N+2\gamma-p+1}|f(r\omega)|^p dr \right)^{1/p'}.
\]

Here we used Hölder’s inequality and \( p' \) is the Hölder conjugate of \( p \). This implies that

\[
I(\omega) \leq \left( \frac{p}{|N + 2\gamma - p|} \right)^p \int_0^{\infty} r^{N+2\gamma-1} \left| \frac{df}{dr}(r\omega) \right| dr.
\]

The equality \( r \frac{df}{dr}(r\omega) = x \cdot \nabla f(x) \) is well known. We then have

\[
r \frac{df}{dr}(r\omega) = x \cdot \nabla f(x) - x \cdot \sum_{\alpha \in R_+} k_\alpha f(x) - f(\sigma_\alpha x) \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}
\]

\[
= x \cdot \nabla f(x) - \sum_{\alpha \in R_+} k_\alpha (f(x) - f(\sigma_\alpha x))
\]

Applying this result to \( \Box \), we have

\[
I(\omega) \leq \left( \frac{p}{|N + 2\gamma - p|} \right)^p \int_0^{\infty} r^{N+2\gamma-1-\frac{1}{p}} \left| x \cdot \nabla f - \sum_{\alpha \in R_+} k_\alpha (f(x) - f(\sigma_\alpha x)) \right|^p dr
\]

\[
\leq \frac{p^p(|R_+| + 2)^{p-1}}{|N + 2\gamma - p|^p} \int_0^{\infty} r^{N+2\gamma-1-\frac{1}{p}} \left[ x \cdot \nabla f \right]^p + \frac{\gamma p |f|^p}{|x|^p} \sum_{\alpha \in R_+} k_\alpha^p |f(\sigma_\alpha x)|^p \right) dr.
\]

Integrating this inequality over the sphere \( S^{N-1} \), we obtain

\[
\int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} \, d\mu_k \leq \frac{p^p(|R_+| + 2)^{p-1}}{|N + 2\gamma - p|^p} \left[ \int_{\mathbb{R}^N} \frac{|x \cdot \nabla f(x)|^p}{|x|^p} \, d\mu_k + 2\gamma p \int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} \, d\mu_k \right].
\]

This shows that for \( \gamma \) small enough such that

\[
2\gamma p \frac{p^p(|R_+| + 2)^{p-1}}{|N + 2\gamma - p|^p} < 1,
\]

we have

\[
\int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} \, d\mu_k \leq C \int_{\mathbb{R}^N} \frac{|x \cdot \nabla f(x)|^p}{|x|^p} \, d\mu_k \leq C \int_{\mathbb{R}^N} |\nabla f|^p \, d\mu_k,
\]

which is the desired Hardy’s inequality.

\[ \square \]

5. Hardy’s inequality on domains

In this section we will consider Hardy’s inequalities on domains \( \Omega \subset \mathbb{R}^N \) involve the distance \( d_\Omega(x) = d(x, \partial \Omega) \) from points \( x \in \mathbb{R}^N \) to the boundary \( \partial \Omega \). These inequalities have been studied extensively in the case of usual partial derivatives, when they take the form

\[
\int_\Omega |\nabla f|^2 \, dx \geq C(\Omega) \int_\Omega \frac{|f|^2}{d_\Omega(x)^2} \, dx,
\]

for a constant \( C(\Omega) \) that depends on the domain \( \Omega \). The geometry of the domain \( \Omega \) plays an important role here. This inequality was characterised by Maz’ya in terms of \( p \)-capacity.
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In terms of optimal constant, if \( \partial \Omega \) is convex, inequality (5.1) with best constant \( C_p \Omega \) was established in [15] and [16]. The case \( \partial \Omega \) not convex is more complicated; some results for planar domains were obtained by Ancona in [1] using Koebe’s theorem, and this was further generalised by Laptev and Sobolev in [13].

Here we will prove the Dunkl equivalent of this inequality for \( G \)-invariant convex domains \( \Omega \). Before stating this theorem, we collect some information about the distance function \( d_\Omega \) in the following Proposition.

**Proposition 5.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open subset such that \( \partial \Omega \neq \emptyset \). The following hold true.

(i): The function \( d_\Omega \) is differentiable at a point \( x \in \Omega \) if and only if there exists a unique point \( y \in \partial \Omega \) such that \( d_\Omega(x) = |x - y| \). If this holds, then \( \nabla d_\Omega(x) = \frac{x - y}{|x - y|} \). In particular, \( |\nabla d_\Omega| = 1 \) wherever defined.

(ii): The set of points where \( d_\Omega \) is not differentiable has Lebesgue measure zero.

(iii): Assume that \( \Omega \) is convex. Then \( \Delta d_\Omega \leq 0 \) in the sense of distributions, i.e.,

\[
\int_{\Omega} d_\Omega(x) \Delta \varphi(x) \, dx \leq 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega), \varphi \geq 0.
\]

We are now ready to state and prove the main result of this section, a Hardy inequality on \( G \)-invariant domains for Dunkl operators.

**Theorem 5.2.** Let \( \Omega \subset \mathbb{R}^N \) be a convex \( G \)-invariant domain with \( \partial \Omega \neq \emptyset \), and define \( d_\Omega(x) = d(x, \partial \Omega) \), the Euclidean distance from \( x \) to the boundary of \( \Omega \). Then we have the inequality

\[
\int_{\Omega} |\nabla_k f|^2 \, d\mu_k \geq \frac{1}{4} \int_{\Omega} \frac{f^2(x)}{d_\Omega^2(x)} \, d\mu_k.
\]

**Proof.** First we show that the distance function \( d_\Omega \) is \( G \)-invariant. Indeed, let \( x \in \Omega \) and \( g \in G \). By definition, there exists a sequence \( (y_n) \subset \partial \Omega \) such that

\[
|x - y_n| \to d(x, \partial \Omega) = d_\Omega(x) \quad \text{as } n \to \infty.
\]

By the \( G \)-invariance of the domain \( \Omega \) we have \( (gy_n) \subset \partial \Omega \), and since

\[
d_\Omega(gx) \leq |gy_n - gx| = |y_n - x| \quad \text{for all } n \geq 1,
\]

by taking \( n \to \infty \) we obtain \( d_\Omega(gx) \leq d_\Omega(x) \). The reverse inequality is proved similarly (or take \( gx \) instead of \( x \), and \( g^{-1} \) instead of \( g \)). Hence we have \( d_\Omega(x) = d_\Omega(gx) \).

Fix a constant \( a \in \mathbb{R} \) and consider the following operator in \( L^2(\mu_k) \)

\[
D := (-\Delta_k + a \frac{\nabla_k d_\Omega}{d_\Omega}) \cdot (\Delta_k + a \frac{\nabla_k d_\Omega}{d_\Omega}) \geq 0.
\]

We compute

\[
Df = -\Delta_k f - a \sum_{i=1}^N T_i \left( \frac{\nabla_i d_\Omega}{d_\Omega} f \right) + a \frac{\nabla_k d_\Omega \cdot \nabla_k f}{d_\Omega} + a^2 \frac{|\nabla_k d_\Omega|^2}{d_\Omega^2}.
\]
Here we used the fact that $d_\Omega$ is $G$-invariant, so $\nabla_k d_\Omega = \nabla d_\Omega$. Because of $G$-invariance of $d_\Omega$ we can also use the Leibniz rule to get

$$T_i \left( \frac{\partial_i d_\Omega}{d_\Omega} f \right) = -\left( \frac{\partial_i d_\Omega}{d_\Omega} \right)^2 f + \frac{T_i (f \partial_i d_\Omega)}{d_\Omega}$$

$$= -\left( \frac{\partial_i d_\Omega}{d_\Omega} \right)^2 f + \frac{1}{d_\Omega} \left[ \partial_i d_\Omega T_i f + f \partial_i^2 d_\Omega + \sum_{\alpha \in R_+} k_\alpha \alpha_i f(\sigma_\alpha x) \frac{\partial_i d_\Omega(x) - \partial_i d_\Omega(\sigma_\alpha x)}{\langle \alpha, x \rangle} \right].$$

Thus

$$Df = -\Delta_k f + (a + a^2) \frac{\nabla_k d_\Omega}{d_\Omega}^2 f - a \frac{\Delta d_\Omega}{d_\Omega} f$$

$$- a \frac{1}{d_\Omega} \sum_{i=1}^N \sum_{\alpha \in R_+} k_\alpha \alpha_i f(\sigma_\alpha x) \frac{\partial_i d_\Omega(x) - \partial_i d_\Omega(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$ 

In order to simplify the double sum in this expression of $Df$, we note that

$$\partial_i (d_\Omega \circ \sigma_\alpha)(x) = \sum_{j=1}^N (\delta_{ij} - \alpha_i \alpha_j) \partial_j d_\Omega(\sigma_\alpha x),$$

where $\delta_{ij}$ is the Kronecker symbol. But we also have $d_\Omega = d_\Omega \circ \sigma_\alpha$, so in particular $\partial_i d_\Omega(x) = \partial_i (d_\Omega \circ \sigma_\alpha)(x)$, and thus

$$\alpha_i \sum_{j=1}^N \alpha_j \partial_j d_\Omega(\sigma_\alpha x) = 0$$

for almost every $x \in \mathbb{R}^N$. This holds for any $i = 1, \ldots, N$, so it is enough to choose $i$ such that $\alpha_i \neq 0$ to deduce that

$$\sum_{j=1}^N \alpha_j \partial_j d_\Omega(\sigma_\alpha x) = 0 \quad \text{a.e.}$$

Therefore

$$Df = -\Delta_k f + (a + a^2) \frac{\nabla_k d_\Omega}{d_\Omega}^2 f - a \frac{\Delta d_\Omega}{d_\Omega} f - a \frac{1}{d_\Omega} \sum_{i=1}^N \sum_{\alpha \in R_+} k_\alpha \alpha_i f(\sigma_\alpha x) \frac{\partial_i d_\Omega(x) - \partial_i d_\Omega(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$ 

Hence, using also the fact that $|\nabla d| = |\nabla d|$ = 1 (see Proposition 5.1.1), this implies

$$0 \leq \int_{\Omega} Df(x) f(x) \, d\mu_k$$

$$= \int_{\Omega} |\nabla_k f|^2 \, d\mu_k + (a + a^2) \int_{\Omega} \frac{|f|^2}{d_\Omega} \, d\mu_k - a \int_{\Omega} \frac{\Delta d_\Omega}{d_\Omega} |f|^2 \, d\mu_k$$

$$- a \sum_{\alpha \in R_+} k_\alpha \int_{\Omega} \frac{1}{d_\Omega} \sum_{i=1}^N \alpha_i f(\sigma_\alpha x) \frac{\partial_i d_\Omega(x) - \partial_i d_\Omega(\sigma_\alpha x)}{\langle \alpha, x \rangle} f(x) \, d\mu_k.$$
To compute the last integral we perform a change of variables \( y = \sigma_x x \) and using the fact that the measure \( \mu_k \) is invariant under \( \sigma_x \), we have
\[
\int_{\Omega} \frac{1}{d_{\Omega}(x)} \sum_{i=1}^{N} \alpha_i f(\sigma_x x)^{\langle \alpha, x \rangle} f(x) d\mu_k = - \int_{\Omega} \frac{1}{d_{\Omega}(y)} f(y)^{\langle \sigma_x y \rangle} \sum_{i=1}^{N} \alpha_i \frac{\partial \sigma_x y}{\langle \alpha, y \rangle} d\mu_k = 0,
\]
where we used (5.2) again.

Going back to the above, we have proven that
\[
\int_{\Omega} |\nabla f|^2 d\mu_k + (a + a^2) \int_{\Omega} \frac{1}{d_{\Omega}^2} d\mu_k - a \int_{\Omega} \frac{\Delta d_{\Omega}}{d_{\Omega}} |f|^2 d\mu_k \geq 0.
\]
Since the domain \( \Omega \) is assumed to be convex, we have by Proposition 5.1.iii that \( \Delta d_{\Omega} \leq 0 \) in a distributional sense. Thus, choosing \( a = -\frac{1}{4} \), we finally have
\[
\int_{\Omega} |\nabla f|^2 d\mu_k \geq \frac{1}{4} \int_{\Omega} \frac{1}{d_{\Omega}^2} d\mu_k,
\]
as required. \( \square \)

**Remark.** The above does not show that the constant \( \frac{1}{4} \) is sharp. This remains an open question.

### 6. Improved Hardy’s Inequalities

It is a well known fact that the best constant in the classical Hardy inequality is not achieved, i.e., there is no \( f \neq 0 \) such that
\[
\int_{\mathbb{R}^N} |\nabla f|^2 = \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2}.
\]
Based on this observation, improved Hardy inequalities were proved, where the error in the classical Hardy inequality is bounded from below, usually by a term involving a suitable potential \( V \). More precisely, inequalities of the following form are studied
\[
(6.1) \quad \int_{\Omega} |\nabla f|^2 dx - \frac{(N-2)^2}{4} \int_{\Omega} \frac{f^2}{|x|^2} dx \geq \int_{\Omega} |V| f^2 dx.
\]
The first such result was proved by Brezis and Vazquez in [4] where it was used in the study of singular extremal solutions of a semilinear elliptic equation. In this paper the above inequality is proved for constant potential \( V \) that depends on the domain \( \Omega \). The proof is based on a symmetrisation argument. However, the authors noticed that even in this case the best constant is not achieved, so they posed the question of whether the improvement appearing on the right hand side of inequality (6.1) is just the first term of a series. This was answered positively by Filippas and Tertikas in [9], where such a construction can be seen. Similar improved inequalities have also been found for other Hardy type inequalities, for example of the type discussed in the previous section.

In this section we will prove a similar improved Hardy’s inequality for Dunkl operators using a method similar to [4] based on spherical h-harmonics.
Theorem 6.1. Let $X(t) = (1 - \log t)^{-1}$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\delta = \sup_{x \in \Omega} |x|$. Then there exists a constant $C > 0$ such that, for any $f \in C^\infty_0(\Omega)$ we have the inequality
\[
\int_{\Omega} \left| \nabla_k f \right|^2 \, d\mu_k \geq C \left( \int_{\Omega} \left| f \right|^q X^{1+q/2} \left( \frac{|x|}{\delta} \right) \, d\mu_k \right)^{2/q},
\]
where $q = \frac{2(N+2\gamma)}{N+2\gamma - 2}$ is the Sobolev coefficient.

We will need the following Lemma, which is itself a weighted Hardy inequality in one dimension. For a comprehensive treatment of such inequalities, see [18] and [17]. The following Lemma is a direct consequence of Theorem 1.3.2/3 in [17].

Lemma 6.2. Let $q \geq 2$ and $\delta > 0$. Then there exists a constant $C > 0$ such that the following inequality holds for all $g \in C^\infty_0(0, \delta)$
\[
\int_0^\delta \left| t g'(t) \right|^2 \, dt \geq C \int_0^\delta \frac{|g(t)|^q}{t} X \left( \frac{t}{\delta} \right)^{1+\frac{q}{2}} \, dt.
\]

Proof of Theorem 6.1. We note first that it is enough to prove the result in the case when $\Omega$ is the ball $B_\delta$ of radius $\delta$ centred at the origin. Indeed, for a general $\Omega$ we have $\Omega \subset B_{3\delta}$, so the result for $B_\delta$ implies in particular the inequality for $\Omega$.

Consider the $h$-harmonic expansion of a function $f \in L^2(\mu_k)$
\[
f(r\xi) = \sum_{n=0}^\infty \sum_{i=1}^{d(n)} f_{n,i}(r)Y_i^n(\xi).
\]
The functions $f_{n,i}$, given by [20], are defined on $\mathbb{R}_+$, but with a slight abuse of notation we will also see them as radial functions on $\mathbb{R}^N$ by identifying $f_{n,i}(x) = f_{n,i}(|x|)$. Using the formula [20] for the Dunkl laplacian, we have
\[
\int_{\Omega} \left| \nabla_k f \right|^2 \, d\mu_k = - \int_{\Omega} f \Delta_k f \, d\mu_k = - \int_{\Omega} f \left[ \frac{\partial^2 f}{\partial r^2} + \frac{N+2\gamma-1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_{k,0} f \right] \, d\mu_k.
\]

Using the orthogonality of the $h$-harmonics $\{Y_i^n\}$, as well as the fact that $Y_i^n$ are eigenfunctions of the operator $\Delta_{k,0}$ with eigenvalues $\lambda_n$, it follows that
\[
\int_{\Omega} \left| \nabla_k f \right|^2 \, d\mu_k
\]
\[
= - \sum_{n=0}^\infty \sum_{i=1}^{d(n)} \int_0^\delta \left[ f_{n,i}(r) \frac{\partial^2 f_{n,i}}{\partial r^2}(r) + \frac{N+2\gamma-1}{r} f_{n,i}(r) \frac{\partial f_{n,i}}{\partial r}(r) + \lambda_n \frac{1}{r^2} f^2_{n,i}(r) \right] r^{N+2\gamma-1} \, dr
\]
\[
= - \sum_{n=0}^\infty \sum_{i=1}^{d(n)} \int_0^\delta \left[ f_{n,i}(x) \Delta_k f_{n,i}(x) + \lambda_n \frac{f^2_{n,i}(x)}{|x|^2} \right] \, dx
\]
\[
= \sum_{n=0}^\infty \sum_{i=1}^{d(n)} \int_0^\delta \left[ \left| \nabla_k f_{n,i}(x) \right|^2 - \lambda_n \frac{f^2_{n,i}(x)}{|x|^2} \right] \, dx.
\]
Let $\Lambda := \left(\frac{N+2\gamma-2}{2}\right)^2$ denote the Hardy inequality constant. From the above, we have
\[
\int_{\Omega} \left[ |\nabla_k f|^2 - \Lambda \frac{f^2}{|x|^2} \right] \, d\mu_k = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} I_{n,i},
\]
where
\[
I_{n,i} := \int_{\Omega} \left[ |\nabla_k f_{n,i}|^2 - (\Lambda + \lambda_n) \frac{f_{n,i}^2}{|x|^2} \right] \, d\mu_k.
\]

When $n > 0$, it can be checked by rearranging the terms that Hardy’s inequality implies the following
\[
I_{n,i} \geq \frac{\lambda_n}{\lambda_n - \Lambda} \int_{\Omega} \left[ |\nabla_k f_{n,i}|^2 - \lambda_n \frac{f_{n,i}^2}{|x|^2} \right] \, d\mu_k.
\]

Thus, we have that
\[
\sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} I_{n,i} \geq C_1 \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_{\Omega} \left[ |\nabla_k f_{n,i}|^2 - \lambda_n \frac{f_{n,i}^2}{|x|^2} \right] \, d\mu_k = C_1 \int_{\Omega} |\nabla_k (f - f_{0,1})|^2 \, d\mu_k,
\]
where $C_1 = \min_{n \geq 1} \frac{\lambda_n}{\lambda_n - \Lambda} > 0$. Using the Sobolev inequality, we have that
\[
\int_{\Omega} |\nabla_k (f - f_{0,1})|^2 \, d\mu_k \geq C_2 \left( \int_{\Omega} |f - f_{0,1}|^q \, d\mu_k \right)^{2/q} \geq C_2 \left( \int_{\Omega} |f - f_{0,1}|^q X^{1+q/2} \left( \frac{|x|}{\delta} \right) \, d\mu_k \right)^{2/q},
\]
for a constant $C_2 > 0$. Here, in the second inequality, we used the fact that $X$ is bounded above by 1.

When $n = 0$, we have that $d(0) = 1$ and $c_0 = 0$, so
\[
I_{0,1} = \int_{\Omega} \left[ |\nabla_k f_{0,1}|^2 - \Lambda \frac{f_{0,1}^2}{|x|^2} \right] \, d\mu_k.
\]

Let $u(r) = r^{(N+2\gamma-2)/2} f_{0,1}$ so after an easy computation we find that
\[
I_{0,1} = \int_{\Omega} |x|^{-(N+2\gamma-1)} \left[ -u(|x|)u'(|x|) + |x|u'(|x|)^2 \right] \, d\mu_k(x).
\]

Using polar coordinates we then have
\[
I_{0,1} = p(B_1) \int_0^\delta \left[ -u(r)u'(r) + ru(r)^2 \right] \, dr = p(B_1) \int_0^\delta ru(r)^2 \, dr,
\]
where $p(B_1) = \int_{\partial B_1} w_k(\theta) \, d\sigma(\theta)$, and in the last equality we used the fact that $u(0) = u(\delta) = 0$. Applying Lemma 6.2 this implies
\[
I_{0,1} \geq p(B_1) C_3 \left( \int_0^\delta |u|^q X^{1+q/2} \left( \frac{r}{\delta} \right) \, dr \right)^{2/q} = p(B_1)^{1-2/q} C_3 \left( \int_{\Omega} |f_{0,1}|^q X^{1+q/2} \left( \frac{|x|}{\delta} \right) \, d\mu_k \right)^{2/q}.
\]
Finally, from (6.2) we obtain
\[
\int \left[ |\nabla_k f|^2 - C \frac{f^2}{|x|^2} \right] \, d\mu_k \geq I_{0,1} + C_1 \int |\nabla_k (f - f_{0,1})|^2 \, d\mu_k \\
\geq C \left( \int |f|^{q} X^{1+q/2}(|x|) \, d\mu_k \right)^{2/q},
\]
for a constant \( C > 0 \), where in the last line we used (6.3), (6.4), and the triangle inequality in the space \( L^q(\mu_k) \).

□

The following Corollary, which is a Dunkl equivalent of the original result of Brezis and Vazquez, is very important because it establishes a Poincaré inequality for Dunkl operators.

**Corollary 6.3.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Then there exists a constant \( C(\Omega) > 0 \) such that for any \( f \in C^\infty_0(\Omega) \) we have the inequality
\[
\int_{\Omega} |\nabla_k f|^2 \, d\mu_k \geq C(\Omega) \int_{\Omega} f^2 \, d\mu_k.
\]

**Proof.** This follows from the previous Theorem and Hölder’s inequality applied to the function \( f^2 X^{1+\frac{q}{2}} \in L^{q/2} \) and \( X^{-\frac{q}{2}} \in L^{q/(q-2)} \). □

**Corollary 6.4** (Poincaré inequality). Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Then there exists a constant \( C(\Omega) > 0 \) such that for any \( f \in C^\infty_0(\Omega) \) we have the inequality
\[
\int_{\Omega} |\nabla_k f|^2 \, d\mu_k \geq C(\Omega) \int_{\Omega} f^2 \, d\mu_k.
\]

7. Other Hardy-type inequalities

In this section we present two results closed related to the Hardy inequality: the Rellich inequality and a Caffarelli-Kohn-Nirenberg inequality.

7.1. **The Rellich Inequality.** This classical inequality, first proved by Rellich in [19] (see also [20]), states that for all \( f \in C^\infty_0(\mathbb{R}^N \setminus \{0\}) \) we have
\[
\int_{\mathbb{R}^N} |\Delta f|^2 \, dx \geq \frac{N^2(N - 4)^2}{16} \int_{\mathbb{R}^N} \frac{f^2}{|x|^4} \, dx,
\]
where the constant is sharp.

Here we prove the Dunkl analogue of this inequality. Our proof below uses the method of spherical h-harmonics already employed above to obtain an improved Hardy inequality, and it is similar in style to the original proof of Rellich.

**Theorem 7.1** (Rellich inequality). Suppose that \( N + 2\gamma \neq 2 \). Then, for any \( f \in C^\infty_0(\mathbb{R}^N \setminus \{0\}) \), we have the inequality
\[
\int_{\mathbb{R}^N} |\Delta_k f|^2 \, d\mu_k \geq \frac{(N + 2\gamma)^2(N + 2\gamma - 4)^2}{16} \int_{\mathbb{R}^N} \frac{f^2}{|x|^4} \, d\mu_k.
\]

The constant in this inequality is sharp.
Proof. Consider the expansion of $f$ in terms of spherical h-harmonics

$$f(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} f_{n,i}(r) Y_n^i(\xi).$$

We then have

$$\Delta_k f(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \left[ f_{n,i}''(r) + \frac{N + 2\gamma - 1}{r} f_{n,i}'(r) + \frac{\lambda_n}{r^2} f_{n,i}(r) \right] Y_n^i(\xi).$$

In order to simplify the computations below, we introduce the notation $N := N + 2\gamma$. Thus, from the orthogonality properties of $\{Y_{n,i}\}$ we have

$$\int_{\mathbb{R}^N} (\Delta_k f)^2 \, d\mu_k = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_{\mathbb{R}^N} \left[ f_{n,i}''(r) + \frac{N - 1}{r} f_{n,i}'(r) + \frac{\lambda_n}{r^2} f_{n,i}(r) \right]^2 r^{N-1} \, dr.$$

Expanding the brackets and computing the terms containing products of mixed derivatives using integration by parts (recall that $f_{n,i}$ has compact support away from 0), we have

$$\int_{\mathbb{R}^N} (\Delta_k f)^2 \, d\mu_k = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_{\mathbb{R}^N} \left[ (f_{n,i}'')^2 r^{N-1} + (N - 2\lambda_n - 1)(f_{n,i}')^2 r^{N-3} \right.$$

$$\left. + \lambda_n(\lambda_n - 2(N - 4)) f_{n,i}^2 r^{N-5} \right] \, dr.$$

Fix now some $n = 0, 1, \ldots$, and some $i = 1, \ldots, d(n)$, and define

$$g_{n,i} = r^{(N-4)/2} f_{n,i}.$$

To simplify notation, let $u = f_{n,i}$ and $v = g_{n,i}$; let also $c = N - 2\lambda_n - 1$. We then have

$$u'(r) = \frac{4 - \frac{N}{2}}{2} u^{(2-N)/2} + r^{(4-N)/2} v'$$

$$u''(r) = \frac{(4 - \frac{N}{2})(2 - \frac{N}{2})}{4} r^{-(N-2)/2} v + (4 - N)r^{(2-N)/2} v' + r^{(4-N)/2} v''.$$
Thus, using these relations and integrating by parts the terms involving mixed products of derivatives of $v$, we have

$$\int_0^\infty \left[ (f''_{n,i})^2 r^{\kappa-1} + (\kappa - 2\lambda_n - 1)(f'_{n,i})^2 r^{\kappa-3} \right] \, dr = \int_0^\infty \left[ (u''_{n,i})^2 r^{\kappa-1} + c(u')^2 r^{\kappa-3} \right] \, dr$$

$$= \int_0^\infty \left[ (v''_{n,i})^2 r^3 + \left( (4 - \kappa) v' + \frac{(4 - \kappa)^2}{4} \left( \frac{2 - \kappa}{4} + c \right) v^2 r^{-1} \right) \right. \right.$$

$$\left. + 2 \frac{4 - \kappa}{2} \left( \frac{4 - \kappa}{2} \right) + c \right) v v' \right.$$

$$\left. + 2 \frac{(4 - \kappa)(2 - \kappa)}{4} v v'' r + 2(4 - \kappa) v' v'' r^2 \right] \, dr$$

$$= \int_0^\infty \left[ (v''_{n,i})^2 r^3 + \left( c + \frac{(4 - \kappa)(2 - \kappa)}{2} \right) v' r^2 + \frac{(4 - \kappa)^2}{4} \left( \frac{2 - \kappa}{4} + c \right) v^2 r^{-1} \right] \, dr$$

$$= \frac{\kappa^2 (4 - \kappa)^2}{16} \int_0^\infty v^2 r^{-1} \, dr$$

$$+ \int_0^\infty \left[ (v''_{n,i})^2 r^3 + \left( \frac{1}{2} (\kappa - 2)^2 + 1 - 2\lambda_n \right) v' r^2 - \lambda_n \frac{(4 - \kappa)^2}{2} v^2 r^{-1} \right] \, dr.$$

It then follows that

$$\int_0^\infty \left[ (f''_{n,i})^2 r^{\kappa-1} + (\kappa - 2\lambda_n - 1)(f'_{n,i})^2 r^{\kappa-3} + \lambda_n (\lambda_n - 2(N - 4)) f_{n,i}^2 r^{N-5} \right] \, dr \geq \frac{\kappa^2 (4 - \kappa)^2}{16} \int_0^\infty v^2 r^{-1} \, dr + \int_0^\infty \left[ (v''_{n,i})^2 r^3 + A v' r^2 + B v^2 r^{-1} \right] \, dr,$$

where

$$A = \frac{1}{2} (\kappa - 2)^2 + 1 - 2\lambda_n = 2n(n + \kappa - 2) + \frac{1}{2} (\kappa - 2)^2 + 1$$

$$B = \lambda_n (\lambda_n - 2(N - 4)) - \lambda_n \frac{(4 - \kappa)^2}{2} = n(n + \kappa - 2) \left[ n(n + \kappa - 2) + \frac{\kappa}{2} (\kappa - 4) \right].$$

It is then clear that $A \geq 0$ for all $n = 0, 1, \ldots$ without any restrictions on $\kappa$, whilst $B > 0$ for all $n$ as long as $\kappa \neq 2$ (which is why we made this assumption). Finally, we have obtained

$$\int_0^\infty \left[ (f''_{n,i})^2 r^{\kappa-1} + (\kappa - 2\lambda_n - 1)(f'_{n,i})^2 r^{\kappa-3} + \lambda_n (\lambda_n - 2(N - 4)) f_{n,i}^2 r^{N-5} \right] \, dr \geq \frac{\kappa^2 (4 - \kappa)^2}{16} \int_0^\infty v^2 r^{-1} \, dr + \frac{\kappa^2 (4 - \kappa)^2}{16} \int_0^\infty f_{n,i}^2 r^{N-5} \, dr.$$

Adding these up for all $n = 0, 1, \ldots$ and $i = 1, \ldots, d(n)$, from the above, and reconstructing $f$ back from its spherical h-harmonics components, we have obtained that

$$\int_{R^N} (\Delta_k f)^2 \, d\mu_k \geq \frac{(N + 2\gamma)^2 (N + 2\gamma - 4)^2}{16} \int_{R^N} \frac{f^2}{|x|^2} \, d\mu_k,$$

as required.
To check that the constant is sharp we can use a similar example as in the classical case. More precisely, for $n = 3, 4, \ldots$ let $f_n(x) = |x|^{2 - \frac{2}{N} - \frac{2}{n}} h_n(|x|)$, where

$$h_n : [0, \infty) \to [0, 1]$$

is such that

$$h_n(r) = \begin{cases} 
0 & \text{if } r \leq 1 \\
1 & \text{if } 2 \leq r \leq n \\
0 & \text{if } r \geq 2n,
\end{cases}$$

with derivatives satisfying

$$|h_n'| \leq \frac{c_1}{n} \quad \text{and} \quad |h''_n| \leq \frac{c_2}{n^2}$$

for some constants $c_1, c_2 > 0$ (an explicit such $h_n$ can be found in [20]). Then we can compute

$$\int_{\mathbb{R}^N} |\nabla f_n|^2 \, d\mu_k = C_1 + p(B_1) \frac{(N + 2\gamma)^2(N + 2\gamma - 4)^2}{16} \int_2^{2n} \frac{1}{r} \, dr$$

and

$$\int_{\mathbb{R}^N} \frac{f_n^2}{|x|^4} \, d\mu_k = C_2 + p(B_1) \int_2^{2n} \frac{1}{r} \, dr,$$

where $C_1$ and $C_2$ can be bounded by constants that do not depend on $n$. Thus

$$\lim_{n \to \infty} \frac{\int_{\mathbb{R}^N} |\nabla f_n|^2 \, d\mu_k}{\int_{\mathbb{R}^N} \frac{f_n^2}{|x|^4} \, d\mu_k} = \frac{(N + 2\gamma)^2(N + 2\gamma - 4)^2}{16}.$$

\[ \square \]

7.2. The Caffarelli-Kohn-Nirenberg Inequality. The Caffarelli-Kohn-Nirenberg inequality [5] is

$$\left( \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} \, dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^N} \frac{|\nabla f|^r}{|x|^{ra}} \, dx \right)^{\theta/r} \left( \int_{\mathbb{R}^N} \frac{|f|^q}{|x|^{qb}} \, dx \right)^{(1-\theta)/q}.$$

In [5] necessary and sufficient conditions on the parameters $p, q, r, a, b, c, \theta$ are given for which the above inequality holds for all $f \in C_c^\infty(\mathbb{R}^N)$. We will prove here the Dunkl analogue of a particular case of this inequality, corresponding to the values $r = 2$ and $\theta = 1$. In this special case the inequality was known before the work of Caffarelli, Kohn and Nirenberg, see for example [12], and it is sometimes known as the Hardy-Sobolev inequality as it generalises both these results (which correspond to the values $a = 0$, $b = 0$, and $a = 0$, $b = 1$, respectively). Best constants are also known, see [17, Corollary 4.8]. Our proof below is inspired by the method of [10].

Theorem 7.2. Let $a \leq b \leq a + 1$ and $p = \frac{2(N + 2\gamma)}{N + 2\gamma - 2 + 2(\alpha - \alpha)}$, and suppose that $a < \frac{N + 2\gamma}{2}$. Then, for any $f \in C_c^\infty(\mathbb{R}^N)$ we have the inequality

$$\int_{\mathbb{R}^N} \frac{|
abla_k f|^2}{|x|^{2a}} \, d\mu_k \geq C_{a,b} \left( \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pb}} \, d\mu_k \right)^{2/p},$$

where $C_{a,b} > 0$ is a constant.
Proof. The strategy of the proof is to establish the inequality in the end cases $b = a + 1$ and $b = a$ separately, and then to interpolate between these two cases to obtain the result in full generality.

Step 1. Suppose $b = a + 1$, so $p = 2$. We begin by considering the function $u = \frac{f}{|x|^a}$ so

$$T_i u = T_i f - \frac{x_i}{|x|^{a+2}} f.$$ \hfill (7.2)

Then we have

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k = \int_{\mathbb{R}^N} \left| \nabla_k u + a \frac{x}{|x|^{a+2}} f \right|^2 \, d\mu_k$$
\hfill (7.3)

$$= \int_{\mathbb{R}^N} |\nabla_k u|^2 \, d\mu_k + a^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} \, d\mu_k + 2a \int_{\mathbb{R}^N} \frac{f}{|x|^{a+2}} \nabla_k u \, d\mu_k.
$$

Let $\epsilon > 1$. Applying the inequality $2xy \geq -\epsilon x^2 - \frac{1}{\epsilon} y^2$, we can estimate the last term on the right hand side of the previous equality

$$2a \int_{\mathbb{R}^N} \frac{f}{|x|^{a+2}} \nabla_k u \, d\mu_k \geq -\epsilon a^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} \, d\mu_k + \frac{1}{\epsilon} \int_{\mathbb{R}^N} |\nabla_k u|^2 \, d\mu_k.
$$

Plugging this in (7.3), we have obtained

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k \geq a^2 (1 - \epsilon) \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} \, d\mu_k + (1 - \frac{1}{\epsilon}) \int_{\mathbb{R}^N} |\nabla_k u|^2 \, d\mu_k.
$$

Applying Hardy’s inequality to the last term on the right hand side of the above inequality, we have

$$\int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k \geq a^2 (1 - \epsilon) \left( 1 - \frac{1}{\epsilon} \right) \left( N + 2\gamma - 2 \right)^2 \int_{\mathbb{R}^N} \frac{f^2}{|x|^{2a+2}} \, d\mu_k.
$$

This holds for all $\epsilon > 1$ and since $a > \frac{N+2\gamma-2}{2}$, we can choose for example $\epsilon = \frac{N+2\gamma-2}{2a}$ to obtain a positive constant.

Step 2. Suppose now that $a = b$. In this case $p = q := \frac{2(N+2\gamma)}{N+2\gamma-2}$, the Sobolev coefficient. Using the Sobolev inequality we have

$$\left( \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pa}} \, d\mu_k \right)^{\frac{2}{p}} \leq C \int_{\mathbb{R}^N} \left| \nabla_k \left( \frac{f}{|x|^a} \right) \right|^2 \, d\mu_k.
$$

Using (7.2), we obtain

$$\left| \nabla_k \left( \frac{f}{|x|^a} \right) \right|^2 = \left| \nabla_k f - a \frac{x}{|x|^{a+2}} f \right|^2 \leq 2 \left| \nabla_k f \right|^2 + 2a^2 \frac{f^2}{|x|^{2a+2}}.
$$

Thus, from the last two relations it follows that

$$\left( \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^{pa}} \, d\mu_k \right)^{\frac{2}{p}} \leq 2Ca^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^{2a+2}} \, d\mu_k + 2C \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k$$
\hfill (7.4)

$$\leq 2C(a^2C_{a,a+1}^{-1} + 1) \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k,$$

where we used the previous step.
Step 3. We now look at the case $a < b < a + 1$. As above, let $q = \frac{2(N+2\gamma)}{N+2\gamma-2}$ be the Sobolev coefficient. We have $2 < p < q$, so there exists $\theta \in (0, 1)$ such that

$$p = 2\theta + q(1-\theta),$$

so

$$b = a + \frac{\theta(N + 2\gamma - 2)}{N + 2\gamma - 2},$$

and also

$$pb = 2(a+1)\theta + qa(1-\theta).$$

Then, using Hölder’s inequality, we obtain

$$\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^b} \, d\mu_k = \int_{\mathbb{R}^N} \frac{|f|^{2\theta+q(1-\theta)}}{|x|^{2(a+1)\theta+qa(1-\theta)}} \, d\mu_k$$

$$\leq \left( \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^{2(a+1)}} \, d\mu_k \right)^\theta \left( \int_{\mathbb{R}^N} \frac{|f|^q}{|x|^{qa}} \, d\mu_k \right)^{1-\theta}.$$  

Using the two steps above, this implies

$$\int_{\mathbb{R}^N} \frac{|f|^p}{|x|^b} \, d\mu_k \leq C_{a,a+1}^{-\theta} \left( \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k \right)^{p/2},$$

as required. This completes the proof. \qed

Remark. One could prove a more general Caffarelli-Kohn-Nirenberg inequality of the form

$$\left( \int_{\mathbb{R}^N} \frac{|f|^p}{|x|^b} \, d\mu_k \right)^{1/p} \leq C \left( \int_{\mathbb{R}^N} \frac{|\nabla_k f|^2}{|x|^{2a}} \, d\mu_k \right)^{\theta/2} \left( \int_{\mathbb{R}^N} \frac{|f|^q}{|x|^{qc}} \, d\mu_k \right)^{(1-\theta)/q},$$

which holds for all $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, subject to the assumption that

$$\frac{1}{p} - \frac{b}{N + 2\gamma} = \theta \left( \frac{1}{2} - \frac{a + 1}{N + 2\gamma} \right) + (1-\theta) \left( \frac{1}{q} - \frac{c}{N + 2\gamma} \right),$$

where $b = (1-\theta)c + \theta d$, for parameters $p, q, a, c, d, \theta \in \mathbb{R}$ such that $p > 0, q > 1, \theta \in [0, 1]$, and such that all integrals above are finite. This could be achieved by interpolating using Hölder’s inequality between the case $\theta = 0$ (which is trivial as $p = q, b = c$, and both sides reduce to $\||x|^{-k}f\|_p$), and the case $\theta = 1$, which was done in the previous Theorem. However, this only works for a more restricted and rather complicated range of $\theta$ depending on $p$ and $q$.

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Andrei Velicu, Department of Mathematics, Imperial College London, Huxley Building, 180 Queen’s Gate, London SW7 2AZ, UK

E-mail address: a.velicu15@imperial.ac.uk