Kummer surfaces for primality testing

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Abstract
We show how to use the arithmetic of the Kummer surface associated to the Jacobian of a hyperelliptic curve to study the primality of integers of the form $4m^2 \cdot 5^n - 1$.

Keywords: Primality, Jacobians, Hyperelliptic Curves, Kummer Surface

1. Introduction

This paper is motivated by an open question stated by Abatzoglou, Silverberg, Sutherland and Wong in [1] (Remark 4.13) asking for a primality test algorithm using higher dimensional Abelian varieties such as Jacobians of genus 2 curves.

Using genus 1 elliptic curves, Gross [17] developed a primality test for Mersenne integers. Further, Denomme and Savin [6] used complex multiplication of an elliptic curve to construct primality tests for different families of integers, later generalized using various one-dimensional group schemes by Gurevich and Kunyavskiǐ [18]. For genus 2 Adleman and Huang [2] developed a technique using Jacobian varieties of genus 2 curves to improve the heuristics on the success of the Goldwasser-Kilian primality test that uses random elliptic curves.

In this paper we use the Kummer surface associated to the Jacobian $J$ of the hyperelliptic curve $y^2 = x^5 + h$ to study the primality of integers of the form

$$\lambda_{m,n} := 4m^2 \cdot 5^n - 1, \quad m, n \in \mathbb{Z}_{\geq 1}.$$ 

First note that if $n$ is even, then $\lambda_{m,n}$ is the difference of two squares, hence composite. Therefore, we will assume that $n$ is odd. We will also assume that $5 \nmid m$, as we can always reach this situation by dividing out a power of 5 from $m$ and adjusting $n$ accordingly, without changing the value of $\lambda_{m,n}$.

This paper consists of two parts. In the first, we will explain the algorithm from a theoretical point of view. The basic idea is that when $\lambda_{m,n}$ is prime, the group of rational points of the Jacobian $J/J_{\lambda_{m,n}}$ of the curve $H$ given by $y^2 = x^5 + h$ is a cyclic $\mathbb{Z}[^{\sqrt{5}}]$-module of known order, and we can construct explicitly $[^{\sqrt{5}}] \in \text{End}_{\mathbb{Q}_{\lambda_{m,n}}}(J)$. If $\lambda_{m,n}$ is not prime we can still consider the
scheme $\mathcal{J}/\mathcal{S}$ and construct $[\sqrt{5}] \in \text{End}_S(\mathcal{J})$ where $\mathcal{S} := \text{Spec } \mathbb{Z}/\lambda_{m,n} \mathbb{Z}$ as we will see in the first section. With this, we choose some base point $Q \in \mathcal{J}$, and study the integer $\inf\{k : [\sqrt{5}]^k P = 0 \text{ in } \mathcal{J}(\mathbb{Z}/\lambda_{m,n} \mathbb{Z})\}$ where $P = 4m^2Q$ to determine the primality or compositeness of $\lambda_{m,n}$.

In the second part, we describe how to make this primality test explicit. We use the Kummer Surface $\mathcal{K}$ associated to $\mathcal{J}$. We will see that $\mathcal{K}$ is a simpler geometrical and arithmetical object compared to $\mathcal{J}$, which preserves the necessary information to determine compositeness or primality of $\lambda_{m,n}$.

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2. Theory

As in the introduction, we fix numbers $n, m \in \mathbb{Z}_{\geq 1}$ with $n$ odd and $m$ not divisible by 5, and we consider the number

$$\lambda_{m,n} = 4m^25^n - 1.$$ 

We also fix some non-zero integer $h$, and we consider the hyperelliptic curve $\mathcal{H}$ defined by $y^2 = x^5 + h$. Our primality test uses that the Jacobian of this curve admits real multiplication by $\sqrt{5}$. We often need to consider the reduction of $\mathcal{H}$ and its Jacobian modulo $\lambda_{m,n}$, even in cases where this number might be composite, and so we use the language of schemes. Therefore, in the rest of this section, we always view $\mathcal{H}$ as a projective curve (or more precisely, an arithmetic surface) over the scheme $\mathcal{S} := \text{Spec } \mathbb{Z}[\frac{1}{10\sqrt{5}}, \sqrt{5}]$. Over this curve, $\mathcal{J}$ is smooth with complete, geometrically connected curves as fibers, and so its relative Jacobian $\mathcal{J} := \text{Jac}(H/S)$ is well-defined. The following proposition shows that $\text{End}_S(\mathcal{J}) = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, and in particular we have a $[\sqrt{5}]$-map on $\mathcal{J}$ defined over $S$.

**Proposition 2.1.** Consider $\omega = \frac{dx}{y}$ as a differential on $\mathcal{J}$. There is a unique ring isomorphism $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \xrightarrow{\sim} \text{End}_S(\mathcal{J})$, $\alpha \mapsto [\alpha]$ such that $[\alpha]^*\omega = \alpha\omega$ for all $\alpha \in \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

**Proof.** In [4], Chapter 15, it is proven that the Jacobian of $\mathcal{H}_{\mathbb{Q}}$ is simple, and has endomorphism ring $\mathbb{Z}[\zeta]$, where $\zeta \in \mathbb{Q}$ is a fifth root of unity. The action of $\zeta$ on $\mathcal{J}$ is induced by the map $\mathcal{H}_{\mathbb{Q}} \rightarrow \mathcal{H}_{\mathbb{Q}}$ sending $(x, y)$ to $(\zeta x, y)$. From this description, one easily computes that $[\zeta]^*\omega = \zeta\omega$, and so the same holds for all $\alpha \in \mathbb{Z}[\zeta]$. We have a canonical morphism of rings $\phi : \text{End}_S(\mathcal{J}) \rightarrow \text{End}_{\mathbb{Q}}(\mathcal{J}_{\mathbb{Q}}) = \mathbb{Z}[\zeta]$, and this morphism is injective. Thus we need to show that $\phi$ has image $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. 

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Let $R$ be the image of $\phi$. First we show that $R \subset \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. Indeed, the above morphism $\phi$ sends an endomorphism $E$ of $\mathcal{J}$ to the eigenvalue of $\omega$ under the action of $E$ on the cotangent space at the origin. If $E$ is defined over $S$, then so are both $\omega$ and $E^*\omega$, and so the eigenvalue of $\omega$ is defined over $S$ as well. Thus, we get $\phi(E) \in \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ as claimed.

Now we show that $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \subset R$. For this, let $S' = \text{Spec} \mathbb{Z}\left[\frac{1}{10h}, \zeta\right]$. The canonical map $S' \to S$ is an (unramified) étale covering of degree 2, hence Galois, and the Galois action is given by $\zeta \mapsto \zeta^4$. Note that $\text{End}_{S'}(\mathcal{J}_{S'}) = \mathbb{Z}[\zeta]$, since the endomorphism $\zeta$ is defined over $S'$. Since we have that $1 + \frac{\sqrt{5}}{2} = \pm(\zeta + \zeta^4) + 1$, (the sign depends on the choice of $\zeta$) and the right hand side is clearly Galois invariant, the theory of Galois descent for endomorphisms on abelian varieties shows that the endomorphism of $\mathcal{J}_{S'}$ corresponding to $1 + \frac{\sqrt{5}}{2}$ descends to an endomorphism of $\mathcal{J}$ over $S$. Hence $1 + \frac{\sqrt{5}}{2} \in R$, and we are done.

Note that over $\mathbb{Z}[\frac{1}{10h}, \sqrt{5}]$ we may factor $\lambda_{m,n}$ as $\lambda_{m,n} = (2m\sqrt{5^n} + 1)(2m\sqrt{5^n} - 1)$. These two factors are coprime over $\mathbb{Z}[\frac{1}{10h}, \sqrt{5}]$ because their difference is a unit.

**Lemma 2.2.** Assume that $h$ and $\lambda_{m,n}$ are coprime. The canonical map
\[
\frac{\mathbb{Z}}{\lambda_{m,n}\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}[\frac{1}{10h}, \sqrt{5}]}{(2m\sqrt{5^n} - 1)}
\]
is an isomorphism of rings, and its inverse is given by the map
\[
\frac{\mathbb{Z}[\frac{1}{10h}, \sqrt{5}]}{(2m\sqrt{5^n} - 1)} \longrightarrow \frac{\mathbb{Z}}{\lambda_{m,n}\mathbb{Z}}
\quad \sqrt{5} \mapsto 2m \cdot 5^{(n+1)/2}, \quad \frac{1}{10h} \mapsto m^2 \cdot 5^{n-1}h^{-1}
\]

**Proof.** Easy computation.

**Remark 2.3.** In what follows we will always assume that $\lambda_{m,n}$ is coprime with $h$, and identify $\mathbb{Z}/\lambda_{m,n}\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{10h}, \sqrt{5}]/(2m\sqrt{5^n} - 1)$. In practice this just means that we have selected a 'canonical' square root of 5 in $\mathbb{Z}/\lambda_{m,n}\mathbb{Z}$, namely $2m \cdot 5^{(n+1)/2}$. In particular, the base changes of $\mathcal{H}$ and $\mathcal{J}$ to $\mathbb{Z}/\lambda_{m,n}\mathbb{Z}$ are well-defined: formally, they are the base change of $\mathcal{H}$ and $\mathcal{J}$ via the map $\text{Spec}(\mathbb{Z}/\lambda_{m,n}\mathbb{Z}) \to S$ corresponding to the ideal of $\mathbb{Z}[\frac{1}{10h}, \sqrt{5}]$ generated by $2m\sqrt{5^n} - 1$. We will denote these base changes by $\mathcal{H}_{\lambda_{m,n}}$ and $\mathcal{J}_{\lambda_{m,n}}$. Note that this base change depends on a choice, as we could as well have chosen the ideal corresponding to $2m\sqrt{5^n} + 1$. 

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Base change gives a canonical map $\mathbb{Z} \left[ \frac{1 + \sqrt{5}}{2} \right] \to \text{End}_{\mathbb{Z}/\lambda_m, n\mathbb{Z}}(J_{\lambda_m, n})$. In other words, for any $\alpha \in \mathbb{Z} \left[ \frac{1 + \sqrt{5}}{2} \right]$ we have a canonical endomorphism $[\alpha]$ of $J_{\lambda_m, n}$ defined over $\mathbb{Z}/\lambda_m, n\mathbb{Z}$. In what follows, the endomorphism $[\sqrt{5}]$ will play an important role, and the main consequence of the results above is that we can make a consistent choice of these endomorphisms, defined over $\mathbb{Z}/\lambda_m, n\mathbb{Z}$, for each choice of $m$ and $n$.

We will now study the structure of the group $J(\mathbb{Z}/\lambda_m, n\mathbb{Z})$ in the case where $\lambda_m, n$ is prime. We start with the 2-torsion.

**Proposition 2.4.** Suppose $\lambda_m, n$ is prime. Then $J[2](\mathbb{F}_{\lambda_m, n}) \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

*Proof.* We know that $J[2](\mathbb{F}_{\lambda_m, n}) \subset J(\mathbb{F}_{\lambda_m, n})$ consists of divisor classes $D - 2\infty$ where $D$ is a the sum of a pair of distinct Weierstrass points of $\mathcal{H}$ and $D$ is fixed under the action of the absolute Galois group of $\mathbb{F}_{\lambda_m, n}$. Since $\gcd(\lambda_m, n - 1, 5) = 1$, there is a unique $\alpha \in \mathbb{F}_{\lambda_m, n}$ with $\alpha^5 = -h$. Then the Weierstrass points of $\mathcal{H}$ are the points $\infty$ at infinity and the points of the form $(\zeta^j, 0)$ for $0 \leq j \leq 4$. Exactly two of these Weierstrass points are defined over $\mathbb{F}_{\lambda_m, n}$, namely $(\alpha, 0)$ and $\infty$. The other four are defined over the quadratic extension of $\mathbb{F}_{\lambda_m, n}$, because $\zeta$ lies there. Since $\lambda_m, n \equiv 4 \mod 5$, $\zeta$ and $\zeta^4$ are Galois conjugate, as are $\zeta^2$ and $\zeta^3$. Hence, the Galois action fixes the points $\infty$ and $(\alpha, 0)$, it interchanges the pair $(\zeta \alpha, 0)$ and $(\zeta^4 \alpha, 0)$, and it interchanges the pair $(\zeta^2 \alpha, 0)$ and $(\zeta^3 \alpha, 0)$. Therefore there are exactly three unordered pairs of Weierstrass points stable under the Galois action, namely

$\{(\alpha, 0), \infty\}, \{\{\zeta \alpha, 0\}, \{\zeta^4 \alpha, 0\}\}$ and $\{\{\zeta^2 \alpha, 0\}, \{\zeta^3 \alpha, 0\}\}$.

This means that the group $J(\mathbb{F}_{\lambda_m, n})$ has exactly three points of order 2. Together with the identity element, this shows that $\#J[2](\mathbb{F}_{\lambda_m, n}) = 4$, and so $J[2](\mathbb{F}_{\lambda_m, n}) \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2)$. \hfill $\Box$

Observe that we can obtain explicitly the $\mathbb{F}_{\lambda_m, n}$-rational zero $\alpha$ of $x^5 + h \in \mathbb{F}_{\lambda_m, n}[x]$ as follows. We know that there is a $d \in \mathbb{Z}$ such that the map $x \mapsto (x^5)^d$ defined over $\mathbb{F}_{\lambda_m, n}$ is the identity map. By Fermat’s little theorem, $d$ satisfies $5d \equiv 1 \mod (\lambda_m, n - 1)$ and $\lambda_m, n - 1 = 4m^2 \cdot 5^n - 2$. To calculate $d$, let $N = 2m^2 \cdot 5^n - 1$ and write $\lambda_m, n - 1 = 2N$. Using the Chinese Remainder Theorem we evaluate $5^{-1}$ with the isomorphism $\tau : \mathbb{Z}/2N\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, using the fact that $5^{-1} \equiv 2m^2 \cdot 5^{n-1} \mod N$ and $5$ is odd. Hence, $\tau(5^{-1}) = (1, 2m^2 \cdot 5^{n-1}) = (1, 0) + (0, 2m^2 \cdot 5^{n-1})$ and therefore:

$d = 5^{-1} = \tau^{-1}(1, 0) + \tau^{-1}(0, 2m^2 \cdot 5^{n-1}) = N + 2m^2 \cdot 5^{n-1} = 12m^2 \cdot 5^{n-1} - 1$.

Using this we have that $x^{5d} = x$ in $\mathbb{F}_{\lambda_m, n}$, and particularly if $x = -h$, we obtain:

$$\alpha = (-h)^d = (-h)^{12m^2 \cdot 5^{n-1} - 1}.$$  

Proposition 2.4 allow us to deduce the full group structure of $J(\mathbb{F}_{\lambda_m, n})$. For an abelian group $G$ and a prime $p$, we will denote by $G[p^\infty]$ its subgroup of
elements whose order is a power of \( p \).

**Proposition 2.5.** Assume that \( \lambda_{m,n} \) is prime. Then we have
\[
J(F_{\lambda_{m,n}}) \cong (\mathbb{Z}/4m^25^n\mathbb{Z})^2
\]
as abelian groups.

**Proof.** For the proof, we will write \( \mathcal{H}, J \) and \( F \) for \( \mathcal{H}_{\lambda_{m,n}}, J_{\lambda_{m,n}} \) and \( F_{\lambda_{m,n}} \) respectively. First we calculate the zeta function of \( \mathcal{H} \). We refer to a paper by Tate and Shafarevich [19] in which they give an explicit description for the numerator of the zeta function of the curve \( C/F_p \) given by \( y^2 = x^3 + \delta \) in the case \( \mu = \text{lcm}(e,f)|p^k + 1 \) for some \( k \). In our case \( p = \lambda_{m,n} = 4m^2 \cdot 5^n - 1, \mu = 10 \) and \( k = 1 \). By [19] the numerator of the zeta function of \( \mathcal{H}/F \) is in this case given by \( \lambda_{m,n}^2T^4 + 2\lambda_{m,n}T^2 + 1 \), which tells us the characteristic polynomial \( \chi_{\mathcal{H}}(T) \) of Frobenius of \( J \) equals \( T^4 + 2\lambda_{m,n}T + \lambda_{m,n}^2 = (T^2 + \lambda_{m,n})^2 \). With this information, we obtain
\[
\# J(F) = \chi_{\mathcal{H}}(1) = 16m^4 \cdot 5^{2n}.
\]

A finite abelian group is the product of its Sylow subgroups for all primes dividing the order of the group. Therefore, to show that \( J(F) \) and \( (\mathbb{Z}/4m^25^n\mathbb{Z})^2 \) are isomorphic as groups, it is sufficient to show that they have the same \( p \)-Sylow group for all primes. We first look at the odd primes. Let \( p \) be odd, and let \( k \) be the \( p \)-adic valuation of \( \# J(F) \) (i.e. the integer \( k \) such that \( p^k \) divides \( \# J(F) \) but \( p^{k+1} \) does not). Since \( \# J(F) = 16m^45^{2n} \) is a square, \( k \) is even. Lemma 3.1 of [20], together with the factorization \( \chi_{\mathcal{H}}(T) = (T^2 + \lambda_{m,n})^2 \), tells us that
\[
J(F)[p^\infty] = (\mathbb{Z}/p^{k/2}\mathbb{Z})^2
\]
as expected.

The case \( p = 2 \) requires more work. First, Proposition 2.4 shows that \( J(F)[2] = (\mathbb{Z}/2\mathbb{Z})^2 \). Therefore, we get that
\[
J(F)[2^\infty] \cong (\mathbb{Z}/2^a\mathbb{Z}) \times (\mathbb{Z}/2^b\mathbb{Z})
\]
for certain \( a, b \geq 1 \). Without loss of generality, we may assume that \( a \geq b \). We want to prove that \( a = b \). Suppose towards a contradiction that \( a > b \). Let
\[
\phi: J(F)[2^\infty] \to (\mathbb{Z}/2^a\mathbb{Z}) \times (\mathbb{Z}/2^b\mathbb{Z})
\]
be an isomorphism of groups. Let \( P \in J(F)[2^\infty] \) be an arbitrary element of order \( 2^a \). Then \( \phi(P) \) is of the form \((s,t)\) with \( s \in \mathbb{Z}/2^a\mathbb{Z} \) of order \( 2^a \) and \( t \in \mathbb{Z}/2^b\mathbb{Z} \) arbitrary. Then \( 2^{a-1}s \in \mathbb{Z}/2^a\mathbb{Z} \) has order 2, and is therefore equal to \( 2^{a-1} \). And \( 2^{a-1}t = 0 \), since \( a - 1 \geq b \). Hence, we see that \( \phi(2^{a-1}P) = (2^{a-1}s, 2^{a-1}t) = (2^{a-1},0) \), which is independent of the choice of \( P \). Since \( \phi \) is an isomorphism, we conclude that \( 2^{a-1}P = 2^{a-1}Q \) for any two points \( P \) and \( Q \) in \( J(F) \) of order \( 2^a \).
Now consider the endomorphism $\theta = \left[ \frac{1 + \sqrt{5}}{2} \right] \in \text{End}_F(J)$ that we know exists by Proposition 2.1. A short computation shows that $\theta \circ (\theta - 1) = \text{id}_J$. In particular $\theta$ and $\theta - 1$ are both automorphisms. This implies that $\theta$ preserves the order of elements of $J$, and also that $\theta$ has no non-trivial fixed points (because otherwise $\theta - 1$ would not be injective). Now let $P \in J(F)$ be a point of order $2^a$. Then also $\theta(P)$ is a point of $J(F)$ of order $2^a$. By the above independence, we now have that

$$2^{a-1}P = 2^{a-1}\theta(P) = \theta(2^{a-1}P).$$

But $2^{a-1}P \neq 0$, so $2^{a-1}P$ is a non-trivial fixed point of $\theta$, which is not possible. This contradiction shows that $a > b$ is not possible. We conclude that $a = b$, so that $J(F)\{2\infty\} \cong (\mathbb{Z}/2^a\mathbb{Z})^2$.

The result follows.

We will also need to understand the action of $[\sqrt{5}]$ on the 5-power torsion.

**Proposition 2.6.** Assume that $\lambda_{m,n}$ is prime and that $n > 1$. Consider $J(F_{\lambda_{m,n}})[5\infty]$ as a $\mathbb{Z}[\sqrt{5}]$-module via the map $[\sqrt{5}]$ as above. Then we have

$$J(F_{\lambda_{m,n}})[5\infty] = 4m^2 \cdot J(F_{\lambda_{m,n}}) \cong \mathbb{Z}[\sqrt{5}]/(\sqrt{5}^{2n})$$

as $\mathbb{Z}[\sqrt{5}]$-modules.

**Proof.** For the proof, we will write $H$, $J$ and $F$ for $H_{\lambda_{m,n}}$, $J_{\lambda_{m,n}}$ and $F_{\lambda_{m,n}}$ respectively.

The first claimed equality follows directly from Proposition 2.4. Moreover, that same proposition tells us that

$$J(F)[5\infty] = (\mathbb{Z}/5^n\mathbb{Z})^2$$

as abelian groups. In particular, the endomorphism $[5^n] = [\sqrt{5}]^{2n}$ acts as the zero map on $J(F)[5\infty]$, so we may view $J(F)$ as a module over $\mathbb{Z}[\sqrt{5}]/(\sqrt{5}^{2n})$. This ring is an artinian principal ideal ring (its ideals are of the form $(\sqrt{5}^k)$ for $k = 0, \ldots, 2n$), and the structure theorem for modules over such rings shows that any finitely generated module over such a ring is a product of cyclic modules. Hence, we may write

$$J(F)[5\infty] = \prod_{i=1}^r \mathbb{Z}[\sqrt{5}]/(\sqrt{5}^{e_i})$$

for certain integers $e_1 \geq \cdots \geq e_r \geq 1$. To get the number of elements correct, we need that $e_1 + \cdots + e_r = 2n$. Since $[5^{n-1}] = [\sqrt{5}]^{2n-2}$ does not act as the zero map on $J(F)[5\infty]$, we need that $e_1 \geq 2n - 1$. Since $n > 1$, we have $e_1 \geq 3$, and we see that the factor $\mathbb{Z}[\sqrt{5}]/(\sqrt{5}^{e_1})$ contains exactly 24 elements of order 5. But from the structure of $J(F)[5\infty]$ as abelian group, we know that in total it contains 24 elements of order 5. Hence we have $r = 1$, and so $e_1 = 2n$, and the result follows.

$\square$
Proof. Let \( n, m \in \mathbb{Z} \) with \( n \) odd and \( 5 \nmid m \) s.t. \( m^2 < \frac{(\sqrt{5}^r - 1)^4 + 1}{4^5} \). Set as before \( \lambda_{m,n} = 4m^25^n - 1 \), and assume that \( \gcd(\lambda_{m,n}, h) = 1 \). Let \( Q \in \mathcal{J}(\mathbb{Z}/\lambda_{m,n}\mathbb{Z}) \) be any point, and define \( P = 4m^2 \cdot Q \). Let

\[
\inf \{ k : [\sqrt{5}]^kP = 0 \text{ in } \mathcal{J}(\mathbb{Z}/\lambda_{m,n}\mathbb{Z}) \} \in \mathbb{N} \cup \{ \infty \}.
\]

1. If \( r > 2n \), then \( \lambda_{m,n} \) is composite.
2. If \( 4 \cdot \log_5(\sqrt{\lambda_{m,n}} + 1) < r \leq 2n \), then either
   \( 2a \) \( \lambda_{m,n} \) is prime, or
   \( 2b \) there is a prime \( p|\lambda_{m,n} \) such that \( [\sqrt{5}]^{r-1}P = 0 \text{ mod } p \).
3. If \( r \leq 4 \cdot \log_5(\sqrt{\lambda_{m,n}} + 1) \), then either
   \( 3a \) \( \lambda_{m,n} \) is composite, or
   \( 3b \) \( \lambda_{m,n} \) is prime and there exists a point \( Q' \in \mathcal{J}(\mathbb{Z}/\lambda_{m,n}\mathbb{Z}) \) with \( Q = [\sqrt{5}]^{2n-r}(Q') \).

On the other hand the Hasse-Weil inequality gives

\[
\#\mathcal{J}(\mathbb{F}_p) \leq (\sqrt{p} + 1)^4.
\]

Comparing these inequalities, we find that \( p > \sqrt{\lambda_{m,n}} \). This is possible only if \( p = \lambda_{m,n} \), and in particular \( \lambda_{m,n} \) is prime.

3. Suppose \( \lambda_{m,n} \) is prime. By Proposition 2.6, we have \( \mathcal{J}(\mathbb{F}_{\lambda_{m,n}}) = \mathbb{Z}[\sqrt{5}]/(\sqrt{5}^{2n}) \) as \( \mathbb{Z}[\sqrt{5}] \)-modules. Since \( [\sqrt{5}]P = 0 \) and \( [\sqrt{5}]^{r-1}P \neq 0 \), there is a point \( P' \in \mathcal{J}(\mathbb{F}_{\lambda_{m,n}}) \) with \( P = [\sqrt{5}]^{2n-r}P' \). Choose integers \( a \) and \( b \) such that \( 4m^2a + 5^n b = 1 \) (such integers exist because \( 4m^2 \) and \( 5^n \) are coprime). Then define \( Q' = aP' + [\sqrt{5}]^r(bQ) \in \mathcal{J}(\mathbb{F}_{\lambda_{m,n}}) \). Then we have

\[
[\sqrt{5}]^{2n-r}Q' = [\sqrt{5}]^{2n-r}aP' + 5^n bQ = aP + 5^n bQ = (4m^2a + 5^n b)Q = Q
\]

as needed. \( \square \)
3. Implementation

In this section, we describe how to implement Theorem 2.7 as an algorithm to test primality of numbers of the form \( \lambda_{m,n} = 4m^2 \cdot 5^n - 1 \). The algorithm depends on the auxiliary data of the hyperelliptic curve \( \mathcal{H} \): \( y^2 = x^5 + h \) with \( h \in \mathbb{Z} \) and a base point \( Q_0 = (\alpha, \beta) \in \mathcal{H}(\mathbb{Q}) \) whose image in \( \mathcal{J} \) has infinite order (e.g., \( h = -\alpha^5 + \beta^2 \) with \( \alpha, \beta \in \mathbb{Z} \), \( h \nmid \lambda_{m,n} \) and \( [(\alpha, \beta) - \infty] \in \mathcal{J}(\mathbb{Q}) \) of infinite order). This algorithm consists of three parts. First one has to compute an explicit expression for the \( \sqrt{5} \) morphism of the Jacobian \( \mathcal{J} \) of \( \mathcal{H} \). Secondly, one has to compute the expression \( P_0 = 4m^2 \cdot Q_0 \in \mathcal{J}(\mathbb{Q}) \). Finally, one has to apply \( \sqrt{5} \) iteratively on \( P_0 \) and compare the result to the cases in Theorem 2.7. Note that the first step does not depend on \( m \) and \( n \), and the second step does not depend on \( n \). Hence, for a fixed choice of \( m \) and a fixed choice of the auxiliary data, one only has to perform steps 1 and 2 once, and the output of these steps can then be used to test primality of \( \lambda_{m,n} \) for any value of \( n \). This is important, because the first two steps are reasonably time and resource intensive, and require computations in the Jacobian of \( \mathcal{H} \) which requires specialized mathematical software like MAGMA. The third step, on the other hand, consists of applying explicit polynomials repeatedly to an explicit vector of numbers, and therefore can be done in general purpose programming languages like Python.

In order to do explicit computations with elements of the Jacobian, one needs a way of explicitly representing elements of \( \mathcal{J} \). One way of doing this would be to embed \( \mathcal{J} \) in projective space. The curve \( \mathcal{H} \) embeds into \( \mathbb{P}^8 \), see [16] for explicit formulas. (If \( \mathcal{H} \) did not have a rational Weierstrass point, one would even need \( \mathbb{P}^{15} \), see [15].) Unfortunately, this large number of coordinates turns out to be impractical computationally. Another option is to use the fact that elements of \( \mathcal{J}(\mathbb{Q}) \) are represented by divisors of the form \( P_1 + P_2 - 2 \cdot \infty \) for some \( P_1, P_2 \in \mathcal{H}(\mathbb{Q}) \). A common system that uses this is Mumford coordinates, see [5]. However, in this system one often has to distinguish between divisors based on whether 0, 1 or 2 of the points \( P_1 \) and \( P_2 \) are equal to \( \infty \). In our case, this leads to complicated formulas involving these different cases.

To avoid these difficulties, we work not with points on the Jacobian \( \mathcal{J} \), but with points on the associated Kummer surface \( \mathcal{K} = \mathcal{J}/([\pm 1]) \) given by modding out the involution on \( \mathcal{J} \). This object is not an algebraic group anymore, because addition is not well-defined. However, each endomorphism of \( \mathcal{J} \) as abelian variety descends to an endomorphism of \( \mathcal{K} \). In particular, there is still a \( \sqrt{5} \) map \( \mathcal{K} \to \mathcal{K} \). Moreover, \( \mathcal{K} \) embeds as a quartic surface in \( \mathbb{P}^3 \), so we can represent points on \( \mathcal{K} \) with four coordinates. A nice additional benefit of using the Kummer surface is that the \( \sqrt{5} \) endomorphism on \( \mathcal{K} \) is defined already over \( \mathbb{Q} \), rather than over \( \mathbb{Q}(\sqrt{5}) \), so the formulas we find involve only rational numbers. See [4, Chapter 3] and [13, Section 5] for more background on the Kummer surface and its embedding into \( \mathbb{P}^3 \). For the rest of the section, we fix the quotient map \( \kappa: \mathcal{J} \to \mathcal{K} \) and the embedding \( \iota: \mathcal{K} \to \mathbb{P}^3 \) as defined in [14, Section 2]; these maps are implemented in MAGMA.
3.1. Computation of \([\sqrt{5}]\)

Because we consider \(K\) as embedded in \(\mathbb{P}^3\), the morphism \([\sqrt{5}] : K \to K\) can be written in the form

\[
\hat{\phi} : K \subset \mathbb{P}^3 \to K \subset \mathbb{P}^3
\]

\[
P := [x_0 : x_1 : x_2 : x_3] \mapsto [\hat{\phi}_0(P) : \hat{\phi}_1(P) : \hat{\phi}_2(P) : \hat{\phi}_3(P)]
\]

where the \(\hat{\phi}_i\) are homogeneous polynomials of some degree \(N \geq 1\) (it seems that in our case, we can always take \(N = 5\)). Note that these polynomials are not uniquely determined, for two reasons: one can multiply the four polynomials by a constant, and one can add to each \(\hat{\phi}_i\) an arbitrary homogeneous polynomial of degree \(N\) that vanishes identically on \(K\).

To determine explicit polynomials \(\hat{\phi}_i\), we use an interpolation strategy. That is, we first generate a large number of pairs of points \((P, Q)\) with \(P, Q \in K\) such that \(Q = [\sqrt{5}]P\), and then we solve a linear system of equations to obtain the coefficients of the \(\hat{\phi}_i\). Roughly, the steps are the following.

1. Generate a sufficiently large set \(S \subset J(\mathbb{Q}(\zeta))\) using \(Q_0\) and the action of \(\zeta\) on \(J\). For example \(S = \{[a + b\zeta + c\zeta^2 + d\zeta^3]Q_0\} \) for \(a, b, c, d \in \{-B, \ldots, B\}\) for some sufficiently large integer \(B\). If \(P\) and \(-P\) are both in \(S\), drop one of them.

2. Calculate the pairs \((P, [\sqrt{5}]P) \in J(\mathbb{Q}(\zeta))^2\) for each \(P \in S\).

3. Calculate the set \(T = \{[\kappa(P), \kappa([\sqrt{5}]P)) : P \in S\}\), as a subset of \((\mathbb{P}^3)^2\).

4. If the degree \(N\) of the polynomials expressing \(\hat{\phi} : K \to K\) is known, construct a projective system of linear equations \(\mathcal{L}\) using the set \(T\). Use this system to deduce the coefficients of four homogeneous polynomials of degree \(N\) that express \(\hat{\phi} : K \to K\) in \(\mathbb{P}^3\). If \(N\) is unknown, choose a large enough \(N\).

5. Remove any common factors in the \(\hat{\phi}_i\) (in case \(N\) was larger than needed).

6. Check the validity of the \(\hat{\phi}_i\) with a generic point computation.

We now explain these steps in more detail.

For step 1, we use the implementation of Jacobians in MAGMA [2]. In this computer algebra system, points on the Jacobian are represented in Mumford coordinates. The idea of this representation is to encode the divisor class \([x_1, y_1] + (x_2, y_2) - 2\infty\) as a pair of polynomials \(\langle u(X), v(X) \rangle\) such that \(u(x_1) = 0\), \(v(x_1) = y_1\), \(\deg u \leq 2\) and \(\deg v \leq 1\). For generic choices of \((x_i, y_i)\), these are given explicitly by

\[
\langle u(X), v(X) \rangle := \langle X^2 - AX + B, CX + D \rangle
\]

\[
= \langle X^2 - (x_1 + x_2)X + x_1x_2, \frac{y_1 - y_2}{x_1 - x_2}X + \frac{x_2y_1 - x_1y_2}{x_1 - x_2} \rangle.
\]

The other non-generic cases \([x_1, y_1] - \infty\), \([2(x_1, y_1) - 2\infty]\) and \([0]\) are represented respectively by \(\langle X - x_1, y_1 \rangle\), \(\langle X^2 - (x_1 + x_2)X + x_1x_2, \frac{X'}{2y_1}X - \frac{X'}{2y_2}x_1 + \right)\).
coefficients in $Q$ usual with the point $(\alpha, \beta)$. For step 1, we start with the point $S$ that the coefficients of the $\hat{\phi}$ for some unknown coefficients $a_{i,\mu}$. For each pair $(\mathbf{v}, \mathbf{w}) \in T$, writing $\mathbf{v} = (v_0, v_1, v_2, v_3)$ and $\mathbf{w} = (w_0, w_1, w_2, w_3)$, we have the relation

$$[\hat{\varphi}_0(\mathbf{v}) : \hat{\varphi}_1(\mathbf{v}) : \hat{\varphi}_2(\mathbf{v}) : \hat{\varphi}_3(\mathbf{v})] = [w_0 : w_1 : w_2 : w_3]$$

as points of $\mathbb{P}^3$. Thus, for each pair $(\mathbf{v}, \mathbf{w})$ there is a non-zero constant $\lambda_{\mathbf{v}}$ such that

$$\sum_{\mu \in \mathcal{M}} a_{i,\mu} \cdot \mu(\mathbf{v}) = \lambda_{\mathbf{v}} w_i$$

for some $\lambda_{\mathbf{v}}$. In our case, it seems that $N = 5$ always works.
holds for \( i = 0, 1, 2, 3 \). This defines a linear system of equations in \( 4 \cdot \#m + \#T \) unknowns (namely the \( a_{i,\mu} \) and the \( \lambda_\nu \)) with a total of \( 4 \cdot \#T \) linear relations between them.

To solve this system in MAGMA, we construct the matrix \( M \) given by \( M_{j,k} := \mu_k(\nu_j) \) where \( m = \{ \mu_k \}_{k=1,2,\ldots,\#m} \) and \( T = \{ (\nu_j, \omega_j) \}_{j=1,2,\ldots,\#T} \). Furthermore, build the diagonal matrices \( \Delta^i \) using the \( i \)-th coordinate of each image point \( \omega_j = ((w_j)_0, (w_j)_1, (w_j)_2, (w_j)_3) \), that is, \( \Delta^i_{j,j} = (w_j)_i \) and \( \Delta^i_{j,j'} = 0 \) if \( j \neq j' \). With this we obtain the system:

\[
\mathcal{L} := \begin{bmatrix}
M & 0 & 0 & 0 & -\Delta^0 \\
0 & M & 0 & 0 & -\Delta^1 \\
0 & 0 & M & 0 & -\Delta^2 \\
0 & 0 & 0 & M & -\Delta^3 
\end{bmatrix}.
\]

The kernel of \( \mathcal{L} \) gives the coefficients \( a_{i,\mu} \) and the constants \( \lambda_\nu \) satisfying Equation 6. But note that not every solution is useful, as there will be solutions where one or more of the \( \hat{\phi}_i \) are identically zero on \( K \). For example, there is always the trivial solution where all variables are zero, but this solution is never useful. One should choose an element of the kernel for which each \( \hat{\phi}_i \) is non-zero on \( K \). This can be read off from the \( \lambda_\nu \). These should be non-zero. Note that \( [\sqrt{5}] \) and \( [-\sqrt{5}] \) induce the same map \( K \to K \), so a Galois argument shows that \( [\sqrt{5}] : K \to K \) is defined over \( \mathbb{Q} \). Hence, one should get a solution whose coefficients are in \( \mathbb{Q} \). After scaling, these coefficients can be taken to be coprime integers.

For step 5, one should check that the \( \hat{\phi}_i \) are coprime. If not, a common factor can be divided out. This happens only if \( N \) is chosen too large.

Step 6 is a check to ensure that the polynomials \( \hat{\phi}_1 \) indeed represent \([\sqrt{5}]\). This check is necessary, because it is theoretically possible that the set of points \( S \) used for the interpolation is not ‘generic’ enough: if all points of \( S \) happen to map into a curve on \( K \) of low degree, then the equations \( \hat{\phi}_i \) are only guaranteed to be correct on this curve and not on all of \( K \). Therefore, we check that the polynomials \( \hat{\phi}_i \) act correctly on a generic point on \( K \).

To do this, we then consider the hyperelliptic curve \( \mathcal{H}_F \) over \( F := \mathbb{Q}(\mathcal{J}) \) given by the equation \( y^2 = x^5 + h \), its Jacobian \( \mathcal{J}_F \) and its associated Kummer surface \( K_F \). By construction there is a generic point \( P = [(x_1, y_1) + (x_2, y_2) - 2\infty] \in \mathcal{J}_F(F) \). Using Equation 6 we can then compute \( Q = [\sqrt{5}]P \in \mathcal{J}_F(F) \). With this we can check that

\[
[\hat{\phi}_0(\kappa(P)) : \hat{\phi}_1(\kappa(P)) : \hat{\phi}_2(\kappa(P)) : \hat{\phi}_3(\kappa(P))] = \kappa(Q)
\]

as points in \( \mathbb{P}^3(F) \). If this is the case, then the polynomials \( \hat{\phi}_i \) correctly represent the action of \([\sqrt{5}]\). If not, one has to start with a larger set \( S \) in step 1.

The way to construct the function field of \( F \) of \( \mathcal{J} \) in MAGMA is by building field of fractions:

\[
F = \text{Frac} \left( \mathbb{Q}(\zeta)[A, B, C, D] / (\Psi_1, \Psi_2) \right).
\]
3.2. Computation of $P$ behaviour of the canonical height. Thus, one expects that $P$.

Then the canonical height of $\mathbf{v}$ for the iterative application of $m$.

height approximately 16

value around $\exp(16m)$. For $m = 1$ we expect coordinates of size $10^{4.1}$, and we find

$$\kappa(4P) = [2624400 : -3559904 : 1744784 : 4190401]$$

with indeed 8 digits as expected. For $m = 2$, we expect coordinates of size
10^{114.3}$, and we find

$$\kappa(16P) = \left[ 4046394696885304072483789465384168714445705653541548795862 : 
3510553111202524305692345962145080299913005912965945600 : 
1517458072687990649583893248327329169920989863562377996111 : 
862113150391066601241233474677857409335081678175317998016 : 
-519775702244808047789789255873896726222838826011524190361 
7027886730157406556798917146366584295088827451720640256 : 
770605931674056814506337558989036249242473883610475236262 
20042074686653408771113007590496528284727186389858585601 \right]$$

which has coordinates with 114 decimal digits each. For $m = 3$, one expects 579 digits per coordinate, and indeed we get coordinates of this size. This illustrates how quickly the size of the starting vector grows with $m$: the number of digits grows with the fourth power of $m$. It also illustrates that one should choose $Q_0 = (\alpha, \beta)$ in such a way that its canonical height is small, in order to obtain a small starting vector $\kappa(P_0)$.

3.3. The iteration

Using the explicit representations of $\sqrt{5} : \mathcal{K} \to \mathcal{K}$ and of $\kappa(P_0) = 4m^2\kappa(Q_0)$ obtained in the previous subsections, we can implement Theorem 2.7 as an algorithm. This step does not need specialized mathematical software, and can be done in a general purpose programming language like C or Python. The procedure in pseudocode is shown in Algorithm 1.

We explain the steps. Line numbers 2 through 8 check that $h$ and $\lambda = \lambda_{m,n}$ are coprime, as this is used in the theory (if $h$ and $\lambda$ are not coprime, then the hyperelliptic curve $H$ will fail to have good reduction at some primes dividing $\lambda$). Of course, if this finds a non-trivial factor of $\lambda$ then we are immediately done: $\lambda$ is composite. So the only inconclusive case here occurs if $h$ is a multiple of $\lambda$. Of course this should not happen in practice, as $h$ is usually taken small while $\lambda$ is big.

From line 9 on the core of the algorithm starts. We begin the iteration by taking the point $\kappa(P_0)$, considered as a vector consisting of four coprime integers, and take each component modulo $\lambda$. This is $v_0$. After this, we recursively compute $v_r$ by applying the polynomials $\hat{\phi}_i$ to $v_{r-1}$ and reducing the result modulo $\lambda$ again. This computes the image of $\kappa([\sqrt{5}]^rP_0)$ modulo $\lambda$. At each step, we check whether $[\sqrt{5}]^rP_0 = 0$ in $\mathcal{J}$ by checking if $v_r$ is the point $(0 : 0 : 0 : 1)$ projectively. This is because the map $\kappa : \mathcal{J} \to \mathcal{K}$ sends 0 to $(0 : 0 : 0 : 1)$, and 0 is the only point in $\mathcal{J}$ mapping to $(0 : 0 : 0 : 1)$.

If after $2n$ iterations the point $(0 : 0 : 0 : 1)$ is not reached, then $\lambda$ is composite, per Theorem 2.7.1. We return this result in line 19. If we reach $(0 : 0 : 0 : 1)$ in at most $4\log_5(\sqrt{\lambda} + 1)$ steps, then the primality test is inconclusive: this is case 3 of Theorem 2.7. In all other cases, we are in the second case of Theorem 2.7 and we have to decide between cases 2a and 2b of
that theorem. This means we have to check whether $v_r$ is projectively equal to $(0 : 0 : 0 : 1)$ modulo some prime $p$ dividing $\lambda$. This is done by computing the $\gcd$ $d[i]$ of the first three components of $v_r$ with $\lambda$. If $d[0]$, $d[1]$ or $d[2]$ is a non-trivial factor of $\lambda$, then $\lambda$ is not prime. Note that it is possible that one or two of the $d[i]$ are equal to $\lambda$, corresponding to $v_r[i]$ being zero modulo $\lambda$, but it is not possible that all three are equal to $\lambda$. Therefore, if none of the $d[i]$ give a non-trivial factor of $\lambda$, then at least one of the $d[i]$ is 1, and so $v_r$ is not $(0 : 0 : 0 : 1)$ modulo any primes $p$ dividing $\lambda$. By Theorem 2.7.2, we then conclude that $\lambda$ is prime.

Algorithm 1: Primality test for $\lambda_{m,n} := 4m^25^n - 1$.

INPUT: $m, n \in \mathbb{N}$ with $m^2 < \frac{(\sqrt{5^n} - 1)^4 + 1}{4\sqrt{5^n}}$, $h$, polynomials $\hat{\varphi}_0, \ldots, \hat{\varphi}_3$ obtained from Subsection 3.1, $\kappa(P_0)$ from Subsection 3.2

OUTPUT: prime if $\lambda_{m,n} = 4m^25^n - 1$ is prime, composite if not prime or unknown if the primality test is inconclusive

1. $\lambda := 4m^25^n - 1$
2. $d := \gcd(h, \lambda)$
3. if $1 < d < \lambda$ then
   return composite (factor $d$)
end
4. if $d = \lambda$ then
   return unknown /* Choose a different $(\alpha, \beta)$ such that $\lambda \nmid h$ */
end
5. $v_0 := \kappa(P_0) \mod \lambda$
6. reached_identity := false
7. for $r = 1, \ldots, 2n$ do
   $v_r := (\hat{\varphi}_0(v_{r-1}), \hat{\varphi}_1(v_{r-1}), \hat{\varphi}_2(v_{r-1}), \hat{\varphi}_3(v_{r-1})) \mod \lambda$
   if $v_r[0] = v_r[1] = v_r[2] = 0$ then
      reached_identity := true
      break
   end
end
8. if reached_identity = false then
   return composite
end
9. if $r > \frac{4\log(\sqrt{\lambda_{m,n}} + 1)}{\log(5)}$ then
   for $i = 0, \ldots, 2$ do
      $d[i] := \gcd(v_r[i], \lambda)$
      if $1 < d[i] < \lambda$ then
         return composite (factor $d[i]$)
      end
   end
end
10. return prime
11. return unknown /* retry with another $\alpha, \beta$ (rebuild $\hat{\varphi}$ and $\kappa(P_0)$) */
Our example implementation used Python 3.6.4 in Darwin 18.7.0 x86_64 (macOS Mojave 10.14.6) Intel Core M 1.2 GHZ. The results are in the following table which are easily verifiable.

| m  | n < 500 where \(4m^25^n - 1 < (\sqrt{5^n} - 1)^4\) such that \(\lambda_{m,n}\) is prime | Time  | \(Q_0\) | \(h\) |
|----|-------------------------------------------------------------------------------------|-------|--------|------|
| 1  | 3, 9, 13, 15, 25, 39, 69, 165, 171, 209, 339                                          | 62.8s | (1,2)  | 3, 10* |
| 3  | 7, 39                                                                              | 60.1s | (-1,1) | 2     |
| 7  | 39, 53                                                                             | 62.6s | (2,1)  | 31    |
| 11 | 19, 55, 89, 91, 119, 123, 177, 225, 295                                           | 63.0s | (-1,3) | 10    |

Table 1: Implementation example

The mark * means that for that \(n\) the test returned unknown, however, with another choice of \(h\) the primality was determined successfully.

All the \(\kappa(P_0)\) points for each \(m \in \{1, 3, 7, 11\}\) and each curve \(y^2 = x^5 + h\) where \(h \in \{2, 3, 10, 31\}\) can be found in [11] (Python). Further, explicit \([\sqrt{5}]\) endomorphisms for each \(h\) can be found in [9]. Furthermore, a MAGMA script to generate other choices of \(\kappa(P_0)\) for different \(m\) using other curves is in [10]. Their respective explicit \([\sqrt{5}]\) endomorphisms (or other endomorphisms) are calculated using [8]. CSVs with other low-height vectors for several \(m\) is in [12].

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