The Harmonic Oscillator in the Plane and the Jordan-Schwinger Algebras

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The direct and indirect Lagrangian representations of the planar harmonic oscillator have been discussed. The reduction of these Lagrangians in their basic forms characterising either chiral, or pseudo-chiral oscillators have been given. A Hamiltonian analysis, showing its equivalence with the Lagrangian formalism has also been provided. Finally, we show that the chiral and pseudo-chiral modes act as dynamical structures behind the Jordan-Schwinger realizations of the SU(2) and SU(1,1) algebras. Also, the SU(1,1) construction found here is different from the standard Jordan-Schwinger form.

The inverse problem of variational calculus is to construct the Lagrangian from the equations of motion. Different Lagrangian representations are obtained from the direct and indirect approaches. In the direct representation as many variables are introduced as there are in the equations of motion. The equation of motion corresponding to a coordinate \( q \) is related with the variational derivative of the action with respect to the same coordinate. Whereas, in the indirect representation, the equation of motion is supplemented by its time-reversed image. The equation of motion with respect to the original variable then corresponds to the variational derivative of the action with respect to the image coordinate and vice versa. The linear harmonic oscillator, whose direct representation is completely familiar, can also be solved in the inverse representation. The importance of the harmonic oscillator problem in physics hardly needs to be emphasised. A thorough analysis of the different Lagrangian representations is thus interesting in its own right. Note that due to the doubling of degrees of freedom associated with the indirect approach, the bidimensional oscillator appear automatically. Thus a comparison of the different approaches can only been done in the context of the two-dimensional (planar) oscillator.

The planar harmonic oscillator model serves as a prototype in the Landau problem of the motion of a charged particle in an external magnetic field and thus is instrumental in the theory of the Quantum Hall Effect. Now the usual Lagrangian of the harmonic oscillator in the plane can be viewed as a combination of the left and right handed chiral oscillators. The later point has been exploited in to discuss the noncommutativity of the Landau problem. Various facets of the two-dimensional oscillator are then worth exploring from different angles.

A beautiful piece of scenary that may be added here is the Jordan-Schwinger (J-S) construction of the SU(2) algebra from a pair of independent harmonic oscillator algebras. Such constructions can be generalised to groups with polynomial algebras. A physical interpretation of these algebraic constructions in terms of dynamical structures is still lacking.

In the present paper we consider the problem of the harmonic oscillator in the plane from the above mentioned points of views. The organisation of the paper is as follows. In section 1 we review the direct and indirect Lagrangian constructions of the linear harmonic oscillator. Section 2 contains our analysis of the bidimensional oscillator in the Lagrangian formalism. The reduction of the planar oscillator to their basic forms characterising chiral and pseudo-chiral oscillators corresponding to direct and indirect representations is discussed here. This is followed by a Hamiltonian analysis in section 3 where an equivalence with the Lagrangian formalism is also established. The connection of the J-S realizations of SU(2) and SU(1,1) algebras with the elementary modes of the bidimensional oscillator - the chiral and pseudo-chiral oscillators, respectively - is discussed in section 4. Section 5 contains the concluding remarks.

Section 1: Direct and indirect Lagrangian representations of the harmonic oscillator
The equation of motion satisfied by the harmonic oscillator is
\[ \ddot{x} + \omega^2 x = 0 \] (1)

To find the Lagrangian in the direct method we write (1) in the self adjoint form
\[ \frac{d}{dt} \dot{x} + \omega^2 x = 0 \] (2)

The variation of the action \( S \) must then be
\[ \delta S = \int_{t_1}^{t_2} dt \left[ \frac{d}{dt} \dot{x} + \omega^2 x \right] \delta x \] (3)

so that the variational derivative of \( S \) w.r.t. \( x \) gives the required equation of motion (1). By discarding a surface term we get from (3)
\[ \delta S = -\delta \left( \int_{t_1}^{t_2} dt \left( \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 \right) \right) \] (4)

From (4) we can easily identify the Lagrangian
\[ L_D = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 \] (5)

of the one dimensional harmonic oscillator.

In the indirect approach we consider (1) along with its time - reversed copy
\[ \ddot{y} + \omega^2 y = 0 \] (6)

and write the variation of the action as
\[ \delta S = \int_{t_1}^{t_2} dt \left[ \left( \frac{d}{dt} \dot{x} + \omega^2 x \right) \delta y + \left( \frac{d}{dt} \dot{y} + \omega^2 y \right) \delta x \right] \] (7)

From (5), equation (6) is obtained by varying \( S \) with \( y \) whereas (5) follows from varying \( S \) with \( x \). Since the equations of motion for \( x \) and \( y \) follow as Euler - Lagrange equations for \( y \) and \( x \) respectively, the method is called the indirect method. Now, starting from (5) we can deduce
\[ \delta S = -\delta \int_{t_1}^{t_2} dt \left[ \dot{x} \dot{y} - \omega^2 xy \right] \] (8)

It is then possible to identify
\[ L_I = \dot{x} \dot{y} - \omega^2 xy \] (9)

as the appropriate Lagrangian in the indirect representation. The Lagrangian (5) can be written in a suggestive form by the substitution of the hyperbolic coordinates \( x_1 \) and \( x_2 \) defined by
\[ x = \frac{1}{\sqrt{2}} (x_1 + x_2) \]
\[ y = \frac{1}{\sqrt{2}} (x_1 - x_2) \] (10)

We find that the Lagrangian \( L_I \) becomes
\[ L_I = \frac{1}{2} \dot{x}_1^2 - \frac{\omega^2}{2} \dot{x}_1^2 - \frac{1}{2} \dot{x}_2^2 + \frac{\omega^2}{2} \dot{x}_2^2 \] (11)

The above Lagrangian can be expressed in a notationally elegant form [10]
\[ L_I = \frac{1}{2} g_{ij} \dot{x}_i \dot{x}_j - \frac{\omega^2}{2} g_{ij} x_i x_j \] (12)

by introducing the pseudo - Euclidean metric \( g_{ij} \) given by \( g_{11} = -g_{22} = 1 \) and \( g_{12} = 0 \).

Section 2 : Elementary modes of the bidimensional oscillator in direct and indirect representations

The Lagrangians given by equation (12) represent the combination of two one dimensional oscillators, the equations of motion of which are given by (5) and (8), in the indirect representation. If we substitute \( g_{ij} \) in (12) by \( \delta_{ij} \) then the Lagrangian \( L_I \) is mapped to
\[ L_I \rightarrow L_I \] (13)

Relabeling \( x = x_1 \) and \( y = x_2 \) and then using (5) we see that (13) is the Lagrangian of the bidimensional oscillator in the direct representation. Thus the mapping
\[ g_{ij} \rightarrow \delta_{ij} \] (14)

allows us to transform the indirect Lagrangian of the two - dimensional oscillator in the hyperbolic coordinates to the direct Lagrangian in the usual coordinates. The converse is also true. This helps us to find the elementary modes in the indirect representation from the known result for the direct Lagrangian (13).
The elementary modes of the bidimensional oscillator \([13]\) are 
\[ L_\pm = \pm \omega \epsilon_{ij} x_i \dot{x}_j - \omega^2 x_i x_i \] (15)

This can be demonstrated by the soldering formalism which has found applications in various contexts - duality symmetric electromagnetic actions were constructed \([1]\); implications in higher dimensional bosonization were discussed \([2]\); the doublet structure in topologically massive gauge theories was revealed \([3]\); a host of phenomena in two dimensions were analysed \([4]\). We start from a simple sum

\[ L(y, z) = L_+(y) + L_-(z) \] (16)

and consider the gauge transformation

\[ \delta y_i = \delta z_i = \Lambda_i(t) \] (17)

where \(\Lambda_i\) are some arbitrary functions of time. Under these transformations the change in \(L\) is given by

\[ \delta L(y, z) = \delta L_+(y) + \delta L_-(z) = \Lambda_i \left( J_i^+(y) + J_i^-(z) \right) \] (18)

where the currents are,

\[ J_i^\pm(x) = 2 \left( \pm \omega \epsilon_{ij} \dot{x}_j - \omega^2 x_i \right) \] (19)

The idea is to iteratively modify \(L(y, z)\) by suitably introducing auxiliary variables such that the new Lagrangian is invariant under the transformations \([17]\). To this end an auxiliary field \(B_i\) transforming as \([17]\),

\[ \delta B_i = \Lambda_i \] (20)

is introduced and a modified Lagrangian is constructed as

\[ L(y, z, B) = L(y, z) - B_i (J_i^+(y) + J_i^-(z)) - 2 \omega^2 B_i B_i \] (21)

This Lagrangian is now invariant under \([17]\) and \([24]\). Since the variable \(B_i\) has no independent dynamics, it is eliminated by using its equation of motion. The residual Lagrangian no longer depends on \(y\) or \(z\) individually but only on the difference \(y - z\). Writing this difference as \(x\), the residual Lagrangian reproduces \([13]\).

The essence of the soldering procedure can be understood also in the following alternative way. Use \(x_i = y_i - z_i\) in \(L(y, z)\) to eliminate \(z_i\) so that

\[ L(y, x) = - 2 \omega \epsilon_{ij} y_i \dot{x}_j - \omega x_j \dot{x}_j - 2 \omega^2 \left( y_i y_i - y_i z_i + \frac{1}{2} x_i x_i \right) \] (22)

Since there is no kinetic term for \(y_i\) it is really an auxiliary variable. Eliminating \(y_i\) from \(L(y, x)\) by using its equation of motion we directly arrive at \([13]\). Note that the opposite chirality of the elementary Lagrangians are crucial in the cancellation of the time derivative of \(y\) in \([24]\) which in turn is instrumental in the success of the soldering method.

The bidimensional Lagrangian \([13]\) is invariant under

\[ x_i \rightarrow x_i + \theta \epsilon_{ij} x_j \] (23)

which is nothing but an infinitesimal rotation in the \(x_1, x_2\) plane. The chiral oscillators \([13]\) are also separately invariant under the SU(2) transformation \([24]\). Corresponding to this symmetry the angular momentum

\[ J = \epsilon_{ij} x_i p_j \] (24)

is conserved where \(p_i = -\omega \epsilon_{ij} x_j\) is the canonical momentum conjugate to \(x_i\). The spectrum of \(J\) is given by

\[ J_\pm = \pm \frac{\tilde{H}}{\omega} \] (25)

where \(\tilde{H} = \omega^2 (x_1^2 + x_2^2)\) is the Hamiltonian following from \(L_\pm\). The chiral oscillators thus manifest dual aspects of the symmetry of \([13]\). This is why the ‘plus’ and ‘minus’ type of oscillators given by \([13]\) are interpreted as the left and right handed chiral oscillators.

Now, in view of the mapping \([14]\) existing between \([12]\) and \([13]\) one can construct the elementary modes of \([12]\) as has been done for \([13]\). By exactly a similar approach as detailed above one can show that the elementary modes

\[ L_\pm = \pm i \omega \epsilon_{ij} x_i \dot{x}_j - \omega^2 g_{ij} x_i x_j \] (26)

can be soldered to yield \([13]\). Of course the form \([26]\) is suggested by \([13]\), on account of the mapping \([14]\). Note, however, that there is a factor of \(i\) in the Lagrangians \([26]\) which makes them complex valued. Such Lagrangians have recently appeared in a new canonical formulation of the damped harmonic oscillator problem \([10]\).
Note that the composite Lagrangian (12) and also the pieces (26) are invariant under the transformation
\[ x_i \rightarrow x_i + \theta \sigma_{ij} x_j \]  
(27)
where \( \sigma \) is the first Pauli matrix. The Noether charges following from (26) are
\[ C_\pm = \pm \tilde{H} \omega \]  
(28)
where \( \tilde{H} = \omega^2 (x_1^2 - x_2^2) \) is the Hamiltonian following from \( L_\pm \). The transformations (27) belong to the SU(1,1) group. The doublets (26) have opposite 'charges' w.r.t. the SU(1,1) transformations in the plane. They may be aptly called as the pseudo-chiral oscillators. The pseudo-chiral oscillators thus manifest dual aspects of the SU(1,1) symmetry of (13).

Section 3 : Hamiltonian analysis

In the previous section we have seen that the Lagrangian of the bidimensional oscillator in the usual coordinates is the synthesis of the chiral oscillators (15) whereas the indirect Lagrangian in the hyperbolic coordinate can be viewed as a coupling of the independent pseudochiral doublet (26). This can also be understood from the Hamiltonian approach.

From the Lagrangian (13) we can construct the Hamiltonian by a Legendre transformation
\[ H_D = \frac{1}{2} p_i p_i + \frac{\omega^2}{2} x_i x_i \]  
(29)
where \( p_i \) are canonically conjugate to \( x_i \). \( H_D \) is manifestly the sum of two one-dimensional Harmonic oscillators,
\[ H_D = H_1 + H_2 \]  
(30)
where
\[ H_i = \frac{1}{2} p_i^2 + \frac{\omega^2}{2} x_i^2; \quad (i = 1, 2) \]  
(31)

We will show that the individual pieces \( H_i \) are the Hamiltonians of the left and right chiral oscillators. Consider the Lagrangian \( L_+ \) of the left chiral oscillator (the first one of (13)). This is already in the first order form and we can read off the Hamiltonian directly
\[ H_+ = \omega^2 (x_1^2 + x_2^2) \]  
(32)
with the symplectic algebra
\[ \{x_i, x_j\} = -\frac{1}{2\omega} \delta_{ij} \]  
(33)
From (33) we find that \( 2\omega x_1 \) is canonically conjugate to \( x_2 \). Now by a canonical transformation to the set \( (x, p_x) \) defined by
\[ x_1 = \frac{1}{\sqrt{2}} \frac{p_x}{\omega} \quad ; \quad x_2 = \frac{1}{\sqrt{2}} x \]  
(34)
the Hamiltonian (32) becomes
\[ H_+ = \left( \frac{p_x^2}{2} + \frac{\omega x_2^2}{2} \right) \]  
(35)
The above Hamiltonian coincides with \( H_1 \) of (30). Similarly we can prove that the piece \( H_2 \) of (30) follows from \( L_- \) of (13). The reduction of the bidimensional oscillator in the direct representation as a doublet of the chiral oscillators is thus also established in the Hamiltonian approach.

We then consider the indirect representation (12). The Hamiltonian obtained from (12) is explicitly
\[ H_I = \frac{1}{2} p_1 ^2 + \frac{1}{2} \omega^2 x_1^2 \quad - \quad \frac{1}{2} p_2 ^2 + \frac{1}{2} \omega^2 x_2^2 \]  
(36)
Note that it is equivalent to the difference of two one-dimensional oscillators. Making a canonical transformation
\[ p_\pm = \frac{1}{\sqrt{2}} p_1 \pm \frac{\omega}{\sqrt{2}} x_2 \quad ; \quad x_\pm = \frac{1}{\sqrt{2}} x_1 \pm i \frac{1}{\sqrt{2}} \omega p_2 \]  
(37)

it is possible to write
\[ H_I = \mathcal{H}_+ + \mathcal{H}_- \]  
(38)
where
\[ \mathcal{H}_\pm = \frac{1}{2} p_\pm ^2 + \frac{1}{2} \omega^2 x_\pm^2 \]  
(39)
The price one has to pay is that the canonical variables \( x_\pm \) and \( p_\pm \) are no longer real. As a result the Hamiltonians \( \mathcal{H}_\pm \) are not hermitian. Note, however, that
\[ H_{\pm}^\dagger = H_{\mp} \]  

so that the hermiticity of \( H_I \) is preserved. One can prove that

\[ \eta H_{\pm} \eta^{-1} = H_{\mp}^\dagger \]  

where \( \eta = PT \). The above condition follows from the observation that under \( PT \) transformation

\[ \eta x_i \eta^{-1} = g_{ij} x_j, \quad \eta p_i \eta^{-1} = -g_{ij} p_j \]  

Thus

\[ \eta x_{\pm} \eta^{-1} = x_\dagger \quad \text{and} \quad \eta p_{\pm} \eta^{-1} = -p_\dagger \]  

Basing on (41) it is possible to build a consistent quantum mechanics \([10]\). Conservation of probability is ensured by the following redefinition of the scalar product

\[ \langle \langle \psi | \phi \rangle \rangle = \int d\tau \tilde{\psi} \phi \]  

where \( \tilde{\psi} \) is the \( \eta \)-transformed wavefunction. Operators satisfying (41) are called generalised hermitian (g-hermitian) operators \([15]\). One can define the g-hermitian adjoint of an operator \( O \) by

\[ \tilde{O} = \eta^{-1} O^\dagger \eta \]  

Then the idea of g-hermiticity can easily be seen as a generalisation of the usual idea of hermiticity. Recently this idea has resurfaced again \( [16] \) where the term pseudo hermiticity is used to denote g-hermiticity.

We can now show that the Hamiltonians \( H_{\pm} \) follow from the Lagrangians \( [17] \) in the same way as we have shown that the Hamiltonians \( H_1 \) and \( H_2 \) of equation (30) follow from the Lagrangians of the chiral oscillators \( [13] \). Hence we show by hamiltonian analysis that in the indirect representation the bidimensional oscillator is reduced to a combination of two pseudo-chiral oscillators.

Section 4: The chiral and the pseudo-chiral oscillators as the dynamical structures behind the Jordan-Schønberger realizations of SU(2) and SU(1,1) algebras

We begin with the Jordan-Schønberger (J-S) realization of the SU(2) algebra. Define two sets of operators \((a, a^\dagger)\) and \((b, b^\dagger)\) which satisfy the bosonic algebra

\[ [a, a^\dagger] = [b, b^\dagger] = 1 \]  

The algebras are independent;

\[ [a, b^\dagger] = [a, b] = 0 \]  

Using the \( a \) and \( b \) set of operators one can construct

\[ J_z = \frac{1}{2} (a^\dagger a - b^\dagger b) \]
\[ J_+ = J_z + i J_y = a^\dagger b \]
\[ J_- = J_z - i J_y = ab^\dagger \]  

The operators defined by (48) satisfy the SU(2) algebra

\[ [J_z, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_z \]  

These operators constitute the J-S realization of SU(2).

The J-S realization of the SU(2) algebra is constructed from two independent oscillator algebras. It is now pretty suggestive that the chiral oscillators \( [15] \) constitute a dynamical model for the construction of (48). Indeed, from (25) we observe that corresponding to one quantum of excitation of the left handed chiral oscillator (energy \( \omega \)) the angular momentum projection is \( \frac{1}{2} \) whereas that for the right handed one it is \( -\frac{1}{2} \). A physical system corresponding to these excitations is also there, namely an electron rotating clockwise or anticlockwise about the direction of the external magnetic field. The operators of (48) can be explicitly constructed by the angular momentum addition rules. Using (25) and the expression of the harmonic oscillator Hamiltonian in terms of the creation and annihilation operators we can write

\[ J_{az} = \frac{1}{2} a^\dagger a \]
\[ J_{bz} = -\frac{1}{2} b^\dagger b \]  

Note that we have deducted the zero-point energy. The \( z \) component of the total angular momentum is

\[ J_z = J_{az} + J_{bz} \]  

which is the \( J_z \) given by equation (48). Again the total angular momentum operator squared is given by
\[ J^2 = j(j + 1) \]
\[ = \left[ \frac{1}{2} \left( a^\dagger a + b^\dagger b \right) \right] \left[ \frac{1}{2} \left( a^\dagger a + b^\dagger b \right) + 1 \right] \] (52)

Now using the relation
\[ J^2 = J_x^2 + J_y^2 + J_z^2 \]
\[ = J_x^2 + \frac{1}{2} (J_+ J_- + J_- J_+) \] (53)

and the expression for \( J_z \) we can deduce the expressions of \( J_\pm \) given in (48). The identification of (15) as the dynamical structures behind the J - S algebraic construction is thus fully established.

The chiral oscillators carry dual aspects of SU(2) and provide dynamical structures for the J - S construction of SU(2). Clearly, the pseudo chiral oscillators (26) are candidates for SU(1,1) as they also serve as the ‘spin’ doublets with respect to SU(1,1) transformations. We introduce
\[ a = \sqrt{\frac{\omega}{2}} \left( x_+ + i p_+ \right) \] (54)
and
\[ b = \sqrt{\frac{\omega}{2}} \left( x_- + i p_- \right) \] (55)

The g - hermitian conjugates of \( a \) and \( b \) are obtained from the definition (12) as \( \tilde{a} \) and \( \tilde{b} \) respectively where
\[ \tilde{a} = \eta^{-1} a^\dagger \eta \] (56)
\[ \tilde{b} = \eta^{-1} b^\dagger \eta \] (57)

From (54) and (55) we get using (13)
\[ \tilde{a} = \sqrt{\frac{\omega}{2}} \left( x_+ - i p_+ \right) \] (58)
and
\[ \tilde{b} = \sqrt{\frac{\omega}{2}} \left( x_- - i p_- \right) \] (59)

It is easy to prove the algebra
\[ [a, \tilde{a}] = [b, \tilde{b}] = 1 \] (60)
and
\[ [a, \tilde{b}] = [a, b] = 0 \] (61)

remembering that \((x_+, p_+)\) and \((x_-, p_-)\) are independent canonical pairs. Let us then construct
\[ J_z = \frac{1}{2} (\tilde{a}a - \tilde{b}b) \]
\[ J_+ = \tilde{a}b \]
\[ J_- = -\tilde{b}a \] (62)

where,
\[ J_\pm = J_x \pm i J_y \] (63)

One can easily verify that these operators satisfy
\[ [J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2 J_z \] (64)

which is nothing but the SU(1,1) algebra. The construction (62) is then a realization of the SU(1,1) algebra based on the algebra (13) and (14). The operators \( a, \tilde{a}(b, \tilde{b}) \) are respectively the annihilation and creation operators belonging to the plus (minus) type pseudo - chiral oscillators (26). The corresponding Hamiltonians has been shown to be \( \mathcal{H}_\pm \) given by (19). These Hamiltonians can be diagonalized in terms of the operators \( a \) and \( b \) as
\[ \mathcal{H}_+ = \omega \tilde{a}a \] (65)
and
\[ \mathcal{H}_- = \omega \tilde{b}b \] (66)

where, again, we have subtracted the zero point energy. Now, in view of the relation (18), the construction (62) can be interpreted in terms of the pseudo - chiral oscillators just as (18) was interpreted in terms of the chiral oscillators. Note, however, a difference of minus sign in the last equation of (62) and (48). This is due to the specific form of the Casimir operator of SU(1,1)
\[ J^2 = J_x^2 - \frac{1}{2} (J_+ J_- + J_- J_+) \]
\[ = \left[ \frac{1}{2} (\tilde{a}a + \tilde{b}b) \right] \left[ \frac{1}{2} (\tilde{a}a + \tilde{b}b) + 1 \right] \] (67)

Comparing the above with the well known form of the Casimir operator of SU(2) ( see equation (53)) , we can understand the difference, mentioned above, between (62) and (48). The structural similarity between the expressions of the Casimir operator in terms of the basic variables in (52) and (67) is remarkable. This reveals again the parallel between our constructions of
SU(2) and SU(1,1) algebras based on the dynamical structures of the chiral or pseudo - chiral oscillators. Using (63) we can explicitly determine $J_x$, $J_y$ and $J_z$ from (62) as
\begin{align*}
J_x &= \frac{1}{2} (\tilde{a}b - \tilde{b}a) \\
J_y &= -\frac{i}{2} (\tilde{a}b + \tilde{b}a) \\
J_z &= \frac{1}{2} (\tilde{a}a - \tilde{b}b)
\end{align*}
(68)

Using the expressions (54), (55),(58) and (59) we find that $J_x$ is hermitian whereas $J_y$ and $J_z$ are anti-hermitian. It is possible to construct a realization of the SU(1,1) algebra consisting of hermitian operators only from (68) by the following mapping
\begin{align*}
J_x &\rightarrow J_y \\
J_y &\rightarrow iJ_z \\
J_z &\rightarrow -iJ_x
\end{align*}
(69)

Explicitly,
\begin{align*}
J_x &= \frac{i}{2} (\tilde{a}a - \tilde{b}b) \\
J_y &= \frac{1}{2} (\tilde{a}b - \tilde{b}a) \\
J_z &= -\frac{1}{2} (\tilde{a}b + \tilde{b}a)
\end{align*}
(70)

That the mapping (69) preserves the SU(1,1) algebra can be seen from (64). Alternatively, one can check it directly from (70).

At this point it is instructive to compare our representation (62) with the usual J - S realization of SU(1,1). The later is given by
\begin{align*}
J_x &= \frac{1}{2} (a^\dagger a + bb^\dagger) \\
J_+ &= J_x + iJ_y = a^\dagger b^\dagger \\
J_- &= J_x - iJ_y = ab
\end{align*}
(71)

This representation is based on the algebras given by (49) and (17), which are two independent harmonic oscillator algebras. Note that, in contrast to (72), (71) cannot be interpreted in terms of independent dynamical structures. This can be seen very simply by writing the Casimir operator from (71)
\begin{align*}
C &= J_x^2 - \frac{1}{2} (J_+J_- + J_-J_+) \\
&= \frac{1}{4} (a^\dagger a - b^\dagger b)^2 - \frac{1}{2} (a^\dagger a + b^\dagger b + 1)
\end{align*}
(72)

Clearly this cannot be factorised as (52). On the other hand, the Casimir operator obtained from our realization (72) factorises properly. The realization (62) is therefore fundamentally different from the usual one (equation (71)).

\section*{Section 5 : Conclusions}

The one dimensional harmonic oscillator is known to provide a fundamental basis for solving a great variety of problems. Here we have analysed the ramifications leading from the oscillator in the plane, arguably whose best known appearance is in the context of the planar motion of an electron moving under the influence of a constant perpendicular magnetic field (Landau problem). The fundamental modes of the 2 - d oscillator were studied, both in the direct and indirect representations. These modes were termed as the chiral and pseudo - chiral oscillators, respectively. An equivalence between the Lagrangian and the Hamiltonian formulations was established. Finally it was shown that the basic operators defining the chiral (pseudo - chiral) operators occurred in the Jordan - Schwinger construction of SU(2) (SU(1,1)) algebras. This provided a dynamical realization of the algebras.

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