Revisited Swanson’s Hamiltonian

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We notice Swanson’s Hamiltonian $H = w a^\dagger a + \alpha a^2 + \beta (a^\dagger)^2$, eventhough non-hermitian in nature can allow real eigenvalues only when it is transformed to equivalent hermitian operator. Further we find real spectra of this oscillator admit only two possible conditions satisfying the condition $w \geq \alpha + \beta$ under the relation: $\Omega^2 = w^2 - 4\alpha\beta$

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I.Introduction

Since the development of quantum mechanics, it is commonly known that hermitian operators yield real eigenvalues [1]

$$H = H^\dagger$$

However the concept of real spectra took a new turn, after Bender and Boettecher[2] proposed a concept: $\mathcal{PT}$ invariant nature on quantum operator. Here $\mathcal{P}$ - operator
stands for parity operator and $\mathcal{T}$ stands for time-reversal operator. An operator reflecting $\mathcal{PT}$-invariant behaviour are non-hermitian in behaviour i.e

$$H \neq H^\dagger$$  \hspace{1cm} (2)

Probably a simplest model on non-hermitian operator is Swanson’s Hamiltonian\[3\]

$$H = w(a^\dagger a + \frac{1}{2}) + \alpha a^2 + \beta (a^\dagger)^2$$ \hspace{1cm} (3)

where the parameters $w; \alpha \neq \beta$ are real. Swanson found that spectra of this oscillator are real if and only if

$$\Omega^2 = w^2 - 4\beta\alpha \gg 0$$ \hspace{1cm} (4)

and

$$w \gg \alpha + \beta$$ \hspace{1cm} (5)

and the energy spectrum is given by

$$E_n = (n + \frac{1}{2})\Omega$$ \hspace{1cm} (6)

Under the condition two possible cases arise:

**case – I**

For

$$w \gg \alpha + \beta$$ \hspace{1cm} (7)

the corresponding Hermitian Hamiltonian is \[4\]

$$H = \frac{1}{2}[(w - \alpha - \beta)p^2 + \frac{(w^2 - 4\alpha\beta)}{(w - \alpha - \beta)x^2}]$$ \hspace{1cm} (8)

whose eigen-spectrum is

$$E_n = \Omega(n + \frac{1}{2})$$ \hspace{1cm} (9)

**case – II**

$$w = \alpha + \beta$$ \hspace{1cm} (10)
It is interesting to note that many authors [5-10] have studied this oscillator from different angles and none of them has addressed the case-II. Hence the aim of this paper is to address this point. However before addressing this point we feel to address this system from perturbation theory [11] to gather some information regarding the constants $\alpha, \beta$. After gathering the information we will suitably address this system using similarity transformation [4,12].

II. Non-Hermitian Perturbation Theory

Standard perturbation theory [1,11] can be suitable to address Swanson’s Hamiltonian if one incorporates the non-hermitian property of operators. Let

$$H = H_D + H_N$$

(11)

where $H_D$ is

$$H_D = w(a^\dagger a + \frac{1}{2})$$

(12)

and

$$H_N = \alpha a^2 + \beta (a^\dagger)^2$$

(13)

The corresponding eigenvalue relation of unperturbed Hamiltonian is

$$H_D|\phi_n> = [w(a^\dagger a + \frac{1}{2})]|\phi_n> = (n + \frac{1}{2})|\phi_n>$$

(14)

The energy eigenvalue relation can be written as [11]

$$E_n = E_n^{(0)} + E_n^{(2)} + E_n^{(4)} + \ldots$$

(15)

Here odd correction terms will be zero. Explicitly

$$E_n^{(0)} = <n|H_D|n>_w = (n + \frac{1}{2})$$

(16)

$$E_n^{(2)} = \sum_m <n|H_N|m><m|H_N|n>_w \frac{E_n^{(0)} - E_m^{(0)}}{E_n^{(0)} - E_m^{(0)}}$$

(17)

$$E_n^{(4)} = \sum_{m,q,r} <n|H_N|m><m|H_N|q><q|H_N|r><r|H_N|n>_w \frac{E_n^{(0)} - E_m^{(0)})(E_n^{(0)} - E_q^{(0)})(E_n^{(0)} - E_r^{(0)})}{(E_n^{(0)} - E_m^{(0)})(E_n^{(0)} - E_q^{(0)})(E_n^{(0)} - E_r^{(0)})}$$

(18)
On explicit calculation one will see that series becomes convergent if and only if \( \frac{\alpha}{w} + \frac{\beta}{w} \) must be less than one. If this condition is violated then the infinite series diverges.

**III. Similarity Transformation**

Similarity transformation\([4,12]\) has a specific role in quantum mechanics while dealing with spectral analysis. Many of the spectral properties which can hardly be visualised in direct study are easily visualised using similarity transformation. The importance of similarity transformation lies with the problem under investigation. Interestingly problems involving non-hermitian in behaviour are easily studied. Mathematically let

\[
H|\phi > = E|\phi > \quad (19)
\]

and

\[
|\Psi > = S|\phi > \quad \implies \quad < \Psi| =< \phi |S^{-1}
\]

\[
S^{-1}S = SS^{-1} = I \quad (21)
\]

then

\[
SHS^{-1} = h \quad (22)
\]

Hence it is obvious that

\[
< \Psi|h|\Psi > = \epsilon = E \quad (23)
\]

Here wave function \( |\phi > \) corresponds to spectrum of the original hamiltonian \( H \). In fact spectrum of Hamiltonian \( H \) is very difficult to visualise.

**II. Swanson’s Hamiltonian**

Consider the Hamiltonian

Now consider that\([12]\)

\[
S_1 = e^{(a^\dagger)^2} \quad (24)
\]

then it is easy to show that

\[
h = S_1HS_1^{-1} = (w - 2\alpha)(a^\dagger a + \frac{1}{2})\alpha a^2 + (\alpha + \beta - w)(a^\dagger)^2 \quad (25)
\]
If \( w = \alpha + \beta \), then the relation can be written as

\[
h = (\beta - \alpha)(a^\dagger a + \frac{1}{2}) + \alpha a^2
\]  

(26)

Here the Hamiltonian is still non-hermitian and energy spectrum is [13]

\[
\epsilon_n = (\beta - \alpha)(n + \frac{1}{2})
\]  

(27)

Now we use a second transformation as [14]

\[
S_2 = e^{-\frac{\alpha x^2}{2(\beta - 2\alpha)}}
\]  

(28)

The resulting Hamiltonian [14]

\[
S_2hS_2^{-1} = h' = \frac{1}{2}[(\beta - 2\alpha)p^2 + \frac{(\beta - \alpha)^2}{(\beta - 2\alpha)}x^2]
\]  

(29)

One can see that \( h' \) is purely hermitian in nature and has the same energy spectrum

\[
\epsilon_n = (\beta - \alpha)(n + \frac{1}{2})
\]  

(30)

V. Conclusion

We notice that Swanson’s Hamiltonian can only yield real spectra if \( w \geq \alpha + \beta \). In other words SW under suitable transformation can be converted to Hermitian operator which can be viewed as Harmonic Oscillator under suitable frequency of oscillation.

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