AN ANALYSIS OF RICK LOCKYER’S “OCTONION VARIANCE SIEVE”

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Abstract. In “Octonion Algebra and its Connection to Physics” [16] an algorithm on octonions is brought forward for description of physical law, the “octonion variance sieve process”. This paper expresses the used algorithm in symbolic form, and highlights the structure between the “function”, “distance”, and “algebraic invariant” concepts therein. An alternative description in terms of derivation algebras is shown.

1. Introduction

Maxwell electromagnetism has been expressed various times on octonionic algebras (e.g. [1, 2, 3, 4]), and octonionic Dirac equations or spinors (e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]) promise usefulness of octonions across all fundamental forces in physics. A recent proposal, the “octonion variance sieve” in [16] introduces a class of functions on the octonions that are invariant under a new algorithm. A “law of octonion algebraic invariance” recovers the general electromagnetic action when assigning the electromagnetic field to certain octonion functions, and modeling dynamic interaction through octonionic differential operators.

The current paper restates this octonion variance sieve in a more symbolic form and highlights structural similarities between concepts used therein. Generalization to connection to classical electrodynamics (as then treated in [16]) or a future nonassociative quantum theory (as in [13]) will not be handled here.

2. Octonions

The octonions \(O\) are the highest-dimensional normed division algebra. They supply \(\mathbb{R}^8\) with a multiplicative norm \(|·| : \mathbb{R}^8 \to \mathbb{R}_+\), where for any \(a, b \in O\) there is \(|a||b| = |ab|\). The multiplicative inverse for any number other than 0 is unique, \(\forall a \in O/\{0\} \exists b, ab = 1\). Using eight orthogonal vectors in \(\mathbb{R}^8\) as octonion basis, \(b_0 := \{1, i_1, \ldots, i_7\}\), an octonion is described through real coefficients \(a := (a_0, \ldots, a_7)\). “Addition” is the vector space addition, “multiplication” distributes over addition and is described by basis element relations \(i_1^2 = 1, i_n^2 = -1 (n = 1, \ldots, 7)\), and a set of seven associative anticommutative ordered triplets \(t_{O[N]}\):

\[ i_l i_m = \epsilon_{lmn} i_n \text{ for all } \{l, m, n\} \in t_{O[N]} \]

These associative triplets then fully describe an octonion algebra, e.g.:

\[ t_{O[8]} := \{\{1, 2, 3\}, \{7, 6, 1\}, \{5, 7, 2\}, \{6, 5, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 4, 7\}\}. \]

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Even permutations within a triplet do not change the algebra and are choice of notation only. A total of 16 sets of triplets, \( t_{0[N]} \), \( N = 0, \ldots, 15 \) now generates equivalent octonion multiplication rules \( \mathbb{O}[N] \), under the class that allows for odd permutations within some of the triplets. The \( \mathbb{O}[N] \) are called equivalent algebras for short in this paper. Notation will be abbreviated for the \( t_{0[N]} \) by using the octonion index numbers from \( t_{0[0]} \) (equation \( \text{(2.3)} \)) as a reference, and then indicating whether permutations within each triplet are of even \((+\)) or odd \((-\)) parity. For example:

\[
(2.4) \quad t_{0[1]} := \{+, +, -, -, +, -, -\} \\
\equiv \{\{1, 2, 3\}, \{7, 6, 1\}, \{5, 2, 7\}, \{6, 3, 5\}, \{1, 4, 5\}, \{2, 6, 4\}, \{3, 7, 4\}\}.
\]

### 3. Automorphisms

Four duality automorphisms, \( T_0, \ldots, T_3 \), act on the multiplication rules \( \mathbb{O}[N] \) that make up equivalent octonion algebras:

\[
\text{(3.1)} \quad T_n : \{\mathbb{O}[N]\} \rightarrow \{\mathbb{O}[N]\}, \quad n \in \{0, 1, 2, 3\}, \\
T_n T_n = (id), \\
\{id, T_n\} \cong \mathbb{Z}_2.
\]

\( \mathbb{Z}_2 \) is the cyclic group with two elements. Acting on the \( t_{0[N]} \), the \( T_n \) either leave the parity of a permutation triplet unchanged, \((id)\), or swap it, \((sw)\):

\[
\text{(3.2)} \quad T_0 := \{(id), (id), (id), (id), (sw), (sw), (sw), (sw)\}, \\
T_1 := \{(sw), (sw), (sw), (sw), (id), (id), (id), (id)\}, \\
T_2 := \{(id), (sw), (id), (sw), (sw), (id), (sw), (sw)\}, \\
T_3 := \{(id), (id), (sw), (sw), (id), (sw), (sw), (sw)\}.
\]

Whereas \( T_0 \) changes the parity of three triplets, the \( \{T_1, T_2, T_3\} \) each change the parity of four triplets. \( T_0 \) transitions between what is called “left-” and “right-handed” octonion multiplication rules \( \text{[10]} \), that are “not isomorphic” in the sense that they cannot be transformed into one another through transformation of the basis vectors in \( \mathbb{R}^8 \) alone. Instead, \( T_0 \) is an algebra isomorphism that transitions between opposite algebras of different chirality \( \text{[5]} \). The combined \( T_0 T_1 \) inverts the sign of all seven nonreal octonion basis elements and corresponds to complex conjugation.

For a select \( n \), the pair \( \{id, T_n\} \) forms the two element cyclic group \( \mathbb{Z}_2 \) under repeat application. The possible unique combinations of the \( \{T_1, T_2, T_3\} \) form the set

\[
\text{(3.3)} \quad \{T_1, T_2, T_3, T_1 T_2, T_1 T_3, T_2 T_3, T_1 T_2 T_3\}
\]

which transitions between octonions \( \mathbb{O}[N] \) of the same chirality. It can be graphed in the Fano plane, where three automorphisms lay on each line such that the combination of any two automorphisms yields the third one (figure \text{[3.1]}). Together with the identity element, \((id)\), this forms the group \( \mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

All possible combinations of the \( \{T_n\} \) acting on \( t_{0[0]} \) then generate the 16 triplet sets \( t_{0[N]} \) for the \( \mathbb{O}[N] \) respectively. By choice, the chirality of all seven triplets in \( t_{0[0]} \) is chosen to be positive and written in abbreviated form as “+”. An explicit
Figure 3.1. All unique automorphisms from repeat application of the \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\} can be graphed in the Fano plane (left), where the product of each two automorphisms on a line yields the third. Together with identity (id) this forms the group \(\mathbb{Z}_3^2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) (right).

\[
\begin{align*}
t_{\mathrm{O}[0]} & := \{+, +, +, +, +, +, +\} = (\text{id}) t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[1]} & := \{+, +, -, -, -, -, -\} = \mathcal{T}_3 t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[2]} & := \{+, -, +, -, -, +, +\} = \mathcal{T}_2 t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[3]} & := \{+, -, -,-, +, +, +\} = \mathcal{T}_1 t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[4]} & := \{-, -, -, -,-, +, +\} = \mathcal{T}_1 t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[5]} & := \{-, -, +, +, -,-, +\} = \mathcal{T}_1 \mathcal{T}_3 t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[6]} & := \{-, +, -, -,-, +, +\} = \mathcal{T}_1 \mathcal{T}_2 t_{\mathrm{O}[0]}, \\
t_{\mathrm{O}[7]} & := \{-, +, +, -,-, +, +\} = \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 t_{\mathrm{O}[0]}. 
\end{align*}
\]

These correspond to the eight left-handed multiplication tables from [16]. The right-handed ones are then obtained from:

\[
\begin{align*}
t_{\mathrm{O}[N+8]} & := \mathcal{T}_0 t_{\mathrm{O}[N]}. 
\end{align*}
\]

Overall, this concept is identical to the group action \(T\) from [5] (equation 30 therein). Octonions that are here mapped through \(\mathcal{T}_0\) are called “opposite algebra” in [5] (equation 33 therein), and correspond to octonionic spinors of opposite chirality. The structure of octonion algebra and its relation to \(\mathbb{Z}_3^2\) and Hadamard transforms is also investigated in [17].

4. Functions, distances, and “octonion variance sieve”

This section now defines a set of 16 polynomial functions, \(A(f) := \{f[N]\}\), that use octonion multiplication rules \(\mathrm{O}[N]\) on a given polynomial \(f\). A certain linear
superposition of these functions, the Hadamard transform, will yield 16 “distances” $B(f)$ (corresponding to the 14 “distances” and two “invariants” from [16]). Applying the same superposition rule again on the distances yields the original functions, making them dual to each other. Furthermore, the automorphisms on the distances $B(f)$ are similar to the automorphisms on the octonion rules $O[N]$ used to generate the functions $A(f)$.

Let $f \in P$ be polynomial with finite number of arguments, $A$ the functor that turns the polynomial into a polynomial function in $\mathbb{R}^8$, then $f[N]$ are the functions that use the multiplication rule $O[N]$ for multiplication:

$$\{f[N]\} := \left\{ \mathbb{R}^8 \otimes \ldots \otimes \mathbb{R}^8 \to \mathbb{R}^8, f[N] \in O[N] \right\},$$

$$A : P \to \left\{ \mathbb{R}^8 \otimes \ldots \otimes \mathbb{R}^8 \to \mathbb{R}^8 \right\}.$$

An $f[N]$ can then be represented by the resultant vector made from general coefficients of the function’s parameters. Because octonions are a normed division algebra, with unique multiplicative inverses and free from zero-divisors or nilpotents (except 0), knowledge of all coefficients from a general octonion product allows to uniquely identify the multiplication rule $t_{O[N]}$ used.

Given a polynomial $f$, all functions $f[N]$ form the set $A(f)$:

$$A(f) := \{f[N]\}, \quad N = 0, \ldots, 15.$$

The automorphisms $S^A$ on $A(f)$ follow directly from the construction of the $t_{O[N]}$ above, as the group of repeat application of the $T_n$ and identity (id) on the associative triplets $t_{O[N]}$:

$$S^A : A(f) \to A(f),$$

$$S^A \cong \{(id), T_0\} \times \{(id), T_1\} \times \{(id), T_2\} \times \{(id), T_3\} = \mathbb{Z}_2^4.$$

Distances $B(f)$ are now constructed from linear superposition of the 16 functions in $A(f)$. Left- and right-octonions (with $N < 8$ and $N \geq 8$ respectively) will not be treated separately as in [16], instead they will be handled as one set here. A sign matrix is defined:

$$b_{jk} := (-1)^{j \wedge k}, \quad j, k = 0, \ldots, 15,$$

where $j \wedge k$ is logical “and” from bitwise representation of the $j$ and $k$. The distances $B(f)$ then are the Hadamard transforms $H_4$ on the functions:

$$g[k] := \frac{1}{4} \sum_{j=0}^{15} b_{jk} f[j], \quad k = 0, \ldots, 15,$$

$$B(f) := \{g[k]\} = \left\{ \frac{1}{4} \sum_{j=0}^{15} b_{jk} f[j] \right\} = H_4(f[N]).$$

---

1. On a sidenote, summing over results obtained from different multiplication rules is also part of the two dimensional “W space” [18].
2. Except of course for the trivial case where no octonion multiplication occurred at all, such as e.g. $f(a_0, a_1) = a_0 + a_1$. 

Since
\[
\left(4.6\right) \frac{1}{4} \sum_{k=0}^{15} b_{kl} g[k] = \frac{1}{16} \sum_{j,k=0}^{15} b_{jk} b_{kl} f[j] = \frac{1}{16} \sum_{j,k=0}^{15} (-1)^{j \wedge k} (-1)^{k \wedge l} f[j] = \frac{1}{16} \sum_{j,k=0}^{15} (-1)^{j \wedge (j+l)} f[j] = \frac{1}{16} \sum_{j,k=0}^{15} b_{(j+l)k} f[j] = f[l].
\]

the sets of distances \(B(f)\) and functions \(A(f)\) are dual to each other, related through linear superposition using the sign matrix \(b_{jk}\). This dualism is a property of the Hadamard transform in general, as it is its own inverse.

Rows and columns in the \(b_{jk}\) correspond to functions \(f[k]\) and distances \(g[j]\). The set of rows \(\{b_{jk}, k \text{ fixed}\}\) can be constructed from a 16 element vector \(b_{j0}\) and four duality morphisms \(\{T^b_n\}\) acting on the sign of \(b_{j0}\):
\[
\left(4.7\right) \quad \begin{align*}
b_{j0} & = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\
T^b_0 & := (+, +, +, +, +, +, +, +, +, +, +, +, +, +, +), \\
T^b_1 & := (+, +, +, +, +, +, +, +, +, +, +, +, +, +, +), \\
T^b_2 & := (+, +, +, +, +, +, +, +, +, +, +, +, +, +, +), \\
T^b_3 & := (+, +, +, +, +, +, +, +, +, +, +, +, +, +, +).
\end{align*}
\]

All possible combinations of the \(\{T^b_n\}\) and identity \((\text{id})\) on \(b_{j0}\) generate the set of rows \(\{b_{jk}, k \text{ fixed}\}\) (and since \(b_{jk} = b_{kj}\) the similar reasoning applies to the columns \(\{b_{j,k}, j \text{ fixed}\}\)). The actual order of rows (or columns) is not significant. Simultaneously swapping any two rows and columns with same indices always yields another Hadamard transform.

Given a set of functions \(\{f[N]\}\), the automorphisms \(S^B\) on the distances \(B(f)\) then are:
\[
\left(4.8\right) \quad S^B = \left\{ (\text{id}), T^b_0 \right\} \times \left\{ (\text{id}), T^b_1 \right\} \times \left\{ (\text{id}), T^b_2 \right\} \times \left\{ (\text{id}), T^b_3 \right\} = \mathbb{Z}_2^4.
\]

This makes the automorphisms \(S^A\) on the octonion functions \(A(f)\) similar to the automorphisms on the distances \(B(f)\),
\[
\left(4.9\right) \quad S^A \sim S^B.
\]

With the definitions from above, the “octonion variance sieve” from [16] is equivalent to computing the \(g[j]\). An “algebraic invariant” is then built from functions \(f[0]\) such that:
\[
\left(4.10\right) \quad g[j] = 0 \text{ for } j > 0.
\]

5. ALGEBRA OF DERIVATIONS

The algebra of derivations \(\mathfrak{der}(A)\) in \(A \in \{\mathbb{H}, \mathbb{O}\}\) is the Lie algebra that corresponds to the automorphism Lie group over \(A\):
\[
\left(5.1\right) \quad \mathfrak{der}(A) = \begin{cases} \mathfrak{so}(3) & (A = \mathbb{H}, \text{ Aut}(\mathbb{H}) \cong \text{SO}(3)), \\ \mathfrak{g}_2 & (A = \mathbb{O}, \text{ Aut}(\mathbb{O}) \cong G_2). \end{cases}
\]

The \(\mathfrak{der}(A)\) have features similar to a derivative operator:
\[
\left(5.2\right) \quad \mathfrak{der}(A) := \left\{ D : A \rightarrow A, \text{ where } D(ab) = D(a)b + aD(b), a, b \in A \right\}.
\]
Any \( u, v \in \mathbb{A} \) form a \( \mathfrak{der} (\mathbb{A}) \) as \cite{19, 20}:

\[
D_{u,v} (a) := \begin{bmatrix} [u,v] & a \\ a, u, v & \in & \mathbb{A} & , & \mathbb{A} & \in & \{ \mathbb{H}, \mathbb{O} \} \end{bmatrix}.
\]

It is now shown that changing between equivalent octonion algebras from above leaves such algebra of derivations invariant exactly in their associative subspaces. Since \( D_{u,v} (a) \) is linear in its arguments \( \{a, u, v\} \) it is sufficient to examine relations where \( a, u, v \) are basis elements from \( b_2 := \{1, i_1, \ldots, i_7\} \). For real \( u, v \) or \( a \) the commutator brackets \([u, v], a\) and associator \((uv) a − u (va))\) are always zero. This becomes the trivial case. If the \( \{a, u, v\} \) are part of an associative triplet then the associator is zero. This is always the case for quaternions, where changing the order of all multiplications does not change the remaining commutator brackets:

\[
[u, v], a = uva − vua − auv + avu
= avu − auv − vua + uva
\text{for} \{a, u, v\} \in \mathbb{H}.
\]

All quaternion algebras therefore have the same algebra of derivations \( \mathfrak{der} (\mathbb{H}) = \mathfrak{so} (3) \).

In the remaining case the \( \{a, u, v\} \) form an antiassociative triplet. The three elements are pairwise anticommutative and each element is also anticommutative with the product of the other two. This yields:

\[
\begin{align*}
[u, v], a & = (uv) a − (vu) a − a (uv) + a (vu) − 3 (uv) a + 3u (va) \\
& = (uv) a + (uv) a + (uv) a + (uv) a − 3 (uv) a − 3 (uv) a \\
& = −2 (uv) a.
\end{align*}
\]

By inspecting the algebra automorphisms \( T_n \) on the set of equivalent octonion algebras \cite{21} there are no two basis element triplets that simultaneously retain or change parity across all four \( T_n \). This means that there are no two triplets that contain \( \{u, v\} \) and \( \{uv, a\} \), respectively, such that the product \( (uv) a \) would remain unaltered in all 16 algebras mapped by the \( T_n \). Therefore, the \( D_{u,v} (a) \) can only yield the same algebra for all 16 equivalent octonion multiplications \( \mathbb{O} [N] \) in its associative subspace \( \mathfrak{so} (3) \subset \mathfrak{g}_2 \):

\[
D_{u,v} (a) [N] = D_{u,v} (a) [0] \text{ for } N = 0, \ldots, 15
\]

\[
\iff\quad D_{u,v} (a) [0] \in \mathfrak{der} (\mathbb{H}) .
\]

This is also the case if one generalizes \( a \in \mathbb{O} \) to polynomial functions \( A (f) = \{ f [N]\} \). The \(-2 (uv) a \) term from equation (5.5) cannot be made to vanish from any term in a polynomial function \( f [N] \) that doesn’t associate with \( u \) and \( v \) since octonions are free from zero-divisors and nilpotents. The \( f [N] \) therefore must be contained in the subalgebra of quaternionic polynomials \( h : \mathbb{H} \rightarrow \mathbb{H} \) which include \( u \) and \( v \) as well, to have the same derivation algebra for any \( N \):

\[
\begin{align*}
D_{u,v} (f) [N] & = D_{u,v} (f) [0] \text{ for } N = 0, \ldots, 15 \\
\iff\quad \{ f [N]\} \in \{ h : \mathbb{H}_{u,v} \rightarrow \mathbb{H}_{u,v}\} \}; u, v \in \mathbb{H}_{u,v}.
\end{align*}
\]
With this, “algebraic invariance” condition from equation (4.10) can be written as:

\[ \text{der} (f) \subseteq \text{der} (\mathbb{H}). \]

For octonionic differential expressions \( \hat{D} f \) an operator \( \hat{D} \) may exist such that:

\[ \text{der} (\hat{D} f) \subseteq \text{der} (\mathbb{H}). \]

(5.8)

(5.9)

If the “octonion variance sieve process” is indeed as claimed [16], then this does not necessarily require \( f \) to be quaternionic.

**Remark 1.** If true, this construction could relate to exotic \( \mathbb{R}^4 \) spaces. Solutions for equation (5.9) are generally quaternionic, which are built over a set that is homeomorphic to \( \mathbb{R}^4 \). However, due to nonassociativity of octonion multiplication \( \hat{D} f \), the various spaces \( \text{der} (\hat{D} f) \) might not necessarily be diffeomorphic to \( \text{der} (\mathbb{H}) \).

The construction that led to equation (5.9) requires nonassociativity of a normed division algebra, uniquely satisfied by octonions, which makes its solutions specific to spaces over \( \mathbb{R}^4 \). More investigation is needed to better understand the claim and its consequences, if true.

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**Glossary of symbols**

- **(f)**: An ordered set of orthogonal vectors \( \{1, i_1, \ldots, i_7\} \in \mathbb{R}^8 \).
- **\( \mathbb{O} [N] \)**: An octonion algebra with a given \( b_0 \) as basis and one of 16 multiplication rules indexed with \( N = 0, \ldots, 15 \).
- **\( t_{\mathbb{O} [N]} \)**: The seven associative triplets of basis elements from \( \mathbb{O} [N] \).
- **\( f [N] \)**: Polynomial functions on \( \mathbb{R}^8 \), \( f [N] : \mathbb{R}^8 \times \ldots \times \mathbb{R}^8 \to \mathbb{R}^8 \), that use \( \mathbb{O} [N] \) for multiplication.
- **\( A (f) \)**: Given a polynomial \( f \), the set of functions \( A (f) := \{f [N]\}, N = 0, \ldots, 15 \).
- **\( \mathcal{T}_0, \ldots, \mathcal{T}_3 \)**: Duality automorphisms on the multiplication rules, \( \mathcal{T}_n : \{ \mathbb{O} [N] \} \to \{ \mathbb{O} [N] \}, \mathcal{T}_n \mathcal{T}_n = (\text{id}) \) (for \( n = 0, 1, 2, 3 \)).
- **\( S^A \)**: The automorphisms on \( A (f) \), i.e.: \( S^A : A (f) \to A (f) \).
- **\( b_{jk} \)**: A \( 16 \times 16 \) sign matrix, \( b_{jk} := (-1)^{j+k}; j, k = 0, \ldots, 15 \).
- **\( H_4 \)**: Hadamard transform generated by the \( b_{jk} \).
- **\( g [N] \)**: A “distance” function obtained from linear superposition of the \( f [N] \), using \( b_{jk} \) as coefficients.
- **\( B (f) \)**: Given a polynomial \( f \), the set of distance functions \( B (f) := H_4 (A (f)) = \{g [N]\}, N = 0, \ldots, 15 \).
- **\( \mathcal{T}_0^b, \ldots, \mathcal{T}_3^b \)**: Duality automorphisms on the rows (or columns) of \( b_{jk} \).
The automorphisms on $B(f)$, i.e.: $S^B : B(f) \to B(f)$.

$\text{der}(\mathbb{A})$ Algebra of derivations over $\mathbb{A}$, e.g. $\text{der}(\mathbb{H}) = \mathfrak{so}(3)$, $\text{der}(\mathbb{O}) = \mathfrak{g}_2$.

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